I. INTRODUCTION

The application of replica theory [1] to the study of random combinatorial optimization problems (RCOPs) has a long tradition that started more than thirty years ago with the seminal works by Orland [2] and Mézard and Parisi [3]. It became immediately clear that methods borrowed from the theory of disordered systems are very effective to study the average properties of ensembles of RCOPs. Exact results have been obtained, e.g., for the average optimal cost (AOC) of mean-field versions of many RCOPs, such as the matching problem [3], the travelling salesman problem [4], K-SAT problems [5], graph partitioning [6], matching enumeration in sparse graphs [7], constraint least square problems [8], and many others. Using the replica approach, expressions for the finite-size corrections to the AOC have also been obtained [9]. The parallel success of the cavity method [1, 10] inspired message-passing algorithms for the same problem. Such an investigation is of methodological interest beyond the analysis of the specific problem. In this paper, we study the deviations from the typical optimal cost in a particular RCOP, the random-link matching problem. We will study both the small fluctuations around the AOC and the large deviations from it. In the matching problem we assume that the weight \( w_{ij} \) is i.i.d random variables, distributed according to a probability density function \( \rho(w) \). Using the replica theory, in Ref. [3] it has been proven that, if \( \lim_{w \to 0} \rho(w) = 1 \), then

\[
\bar{C} := \lim_{N \to +\infty} \frac{1}{N} \ln \mathbb{E} \left[ \min_{\mathbf{M}} C_N[\mathbf{M}] \right] = \frac{\zeta(2)}{2},
\]

where we have denoted the average over all possible realizations by \( \mathbb{E} \), and \( \zeta(z) \) is the Riemann zeta function. The calculation was performed introducing a partition function

\[
Z_w(\beta) := \sum_{\{m_{ij}\}} \prod_{i=1}^{2N} \left( \sum_{j=1}^{2N} m_{ij} = 1 \right) e^{-\beta N C_N[\mathbf{M}]},
\]

where the indicator function \( \mathbb{I}(\bullet) \) is equal to one if its argument is true, zero otherwise. From the expression above, the replicated free-energy can be derived

\[
\Phi(n, \beta) := -\lim_{N \to +\infty} \frac{1}{\beta n N} \ln \mathbb{E} \left[ Z_n^w(\beta) \right].
\]

The functional \( \Phi(n, \beta) \) has been obtained, in the replica symmetric hypothesis, in Ref. [3] and it is equal to

\[
-\beta n \Phi(n, \beta) = -\frac{\beta}{2} \sum_{p=1}^{n} \frac{\Gamma(n+1)}{\Gamma(p) \Gamma(n-p+1)} q_p^2 + 2 \ln \left[ \int \int \frac{dx \, dy}{2\pi} (\text{sign}(y))^n \exp \left( i \eta x + \sum_{p=1}^{\infty} \frac{x^p q_p}{p!} \right) \right]
\]

where the order parameters \( q_p \) have to be specified using the saddle-point condition

\[
\frac{\partial \Phi(n, \beta)}{\partial q_p} = 0 \quad p \in \mathbb{N}.
\]
The asymptotic AOC is then recovered as the value of Φ in the zero temperature limit, taking the number of replicas n going to zero,

$$\tilde{C} = \lim_{\beta \to +\infty} \Phi(n, \beta).$$  \hspace{1cm} (7)

Here and in the following, we denote by $C_\beta := \min_M C_N[M]$ the instance-dependent optimal cost, and by $G_N(C)$ its distribution.

Our computation of the large deviation function for the random-link matching problem starts exactly from Eq. (6a). It is well known, indeed, that the replicated average free-energy $n \Phi(n, \beta)$ contains information not only on the AOC, but also on the fluctuations of the AOC [13]. In particular, using Eq. (5) it is possible to show that $-n \beta \Phi(n, \beta)$ is the cumulant generating function of the optical cost and different from zero, i.e., $\kappa$ is the cumulant generating function of $\ln Z_w(\beta)$ [14]. This fact has been used, for example, by Parisi and Rizzo [15], to extract the large deviation function in the Sherrington-Kirkpatrick model and confirm the anomalous scaling of fluctuations of the free energy in the RSB phase predicted, near the critical temperature, by Crisanti et al. [14], on the basis of a previous result by Kondor [16]. In the present paper, we are interested in the fluctuation of the “ground state free-energy” $C_\beta$ in the random-link matching problem, and therefore we have to take $\beta \to +\infty$. The cumulant generating function of the optimal cost is then obtained as

$$-\alpha \Phi(\alpha) := \lim_{\beta \to +\infty} \frac{\ln E[e^{-\alpha N C_\beta}]}{N} = \lim_{\beta \to +\infty} \frac{1}{N} \sum_{k=1}^{\infty} (-1)^k \frac{\kappa_k \alpha^k}{k!},$$  \hspace{1cm} (8)

where $\kappa_k$ is the k-th cumulant of the random variable $NC_\beta = \beta^{-1} \ln Z_w$ [14]. In particular, if $\lim_{N^{-1} \beta \to 0} E[(C_N - C)^2] = \sigma^2 N^{-1}$, i.e., small fluctuations of the optimal cost are Gaussian. The Cramér function of $\gamma_N(C)$ is obtained as the Legendre transform of $\alpha \Phi(\alpha)$,

$$L(C) := \lim_{\beta \to +\infty} \frac{\ln \gamma_N(C)}{\beta} = \tilde{\Phi}(\tilde{\alpha}) - \tilde{\alpha} C,$$  \hspace{1cm} (9a)

with $\tilde{\alpha}$ such that

$$C = \frac{\partial [\alpha \Phi(\alpha)]}{\partial \alpha} \bigg|_{\alpha = \tilde{\alpha}}.$$  \hspace{1cm} (9b)

In the following, we will derive the exact value of $\sigma^2$, proving that the small fluctuation of the AOC are indeed Gaussian, and we will obtain an expression for the function $\Phi(\alpha)$ that we will solve numerically.

II. SMALL FLUCTUATIONS

Let us start from the computation of the variance of the optimal cost. The small $\alpha$ expansion of $\Phi(\alpha)$ up to $o(\alpha^2)$ terms will provide us the first and the second cumulant of the optimal cost, i.e., the AOC and its variance. Due to the fact that we are performing an expansion around $\alpha = 0$, it is useful to recall the expression of the saddle-point value of the order parameter $q_0 \equiv Q_p$ in this particular case, i.e., for $n \to 0$ and $\beta \to +\infty$. It has been shown in Ref. [3] that, introducing the function

$$G_0(t) := \int_{-\infty}^{\infty} e^{-\Gamma(t)} dt \Rightarrow G_0(x) = \ln (1 + e^{2x}).$$  \hspace{1cm} (10)

the saddle point condition for $n \to 0$ and $\beta \to +\infty$ reads

$$G_0(x) = 2 \int_{-\infty}^{\infty} e^{-\Gamma(t)} dt \Rightarrow G_0(x) = \ln (1 + e^{2x}).$$  \hspace{1cm} (11)

To obtain the expansion of $\Phi(\alpha)$, let us start from the double integral appearing in the argument of the logarithm in Eq. (6a), to be evaluated on the saddle-point $Q_p$ for small $\alpha$. We have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\eta (-i\eta)^n e^{i n x + \sum_{p=1}^{\infty} \frac{s_p Q_p}{\beta}} = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\eta (-i\eta) e^{i n x + \sum_{p=1}^{\infty} \frac{s_p Q_p}{\beta}}.$$  \hspace{1cm} (12)

Using now the representation of the logarithm

$$\ln(x) = \int_{0}^{+\infty} \frac{e^{-t} - e^{-xt}}{t} dt$$

we can write

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\eta (-i\eta)^n e^{i n x + \sum_{p=1}^{\infty} \frac{s_p Q_p}{\beta}} =$$

$$= 1 + \alpha \int_{-\infty}^{\infty} \left[ \theta(-h) - e^{-G_0(h)} \right] dh$$

$$+ \frac{\alpha^2}{2} \int_{-\infty}^{\infty} dh_1 dh_2 \left[ \theta(-h_1) \theta(-h_2) - \theta(-h_1) e^{-G_0(h_2)} - \theta(-h_2) e^{-G_0(h_1)} + e^{-G_0(\max(h_1, h_2))} \right] + o(\alpha^2)$$

$$= 1 + \frac{\xi(2)}{4} \alpha^2 + o(\alpha^2).$$  \hspace{1cm} (13)

Similarly, the first term in Eq (6a) can be written as

$$\frac{\beta}{2} \sum_{p=1}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(p) \Gamma(n - p + 1)} \frac{Q_p^2}{p} =$$

$$= \frac{\alpha}{2} \sum_{p=1}^{\infty} (-1)^{p-1} \left[ 1 - \frac{\alpha}{\beta} H_{p-1} + o\left(\frac{\alpha}{\beta}\right) \right] Q_p^2$$

$$= \frac{\alpha}{2} \sum_{p=1}^{\infty} (-1)^{p-1} H_{p-1} Q_p^2 + o(\alpha^2).$$  \hspace{1cm} (14)
where $H_p$ is the $p$th harmonic number. If we now use the fact that $H_{p-1} = \sum_{p=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-p+1)} Q_p^2$, then

$$
\lim_{n,\beta \to \alpha} \frac{\beta}{2} \sum_{p=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(p)\Gamma(n-p+1)} Q_p^2 = \alpha \left( \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2}\right) \alpha + o(\alpha), \quad \text{valid for } \alpha < 0,
$$

Collecting all results we get

$$
\Phi(\alpha) = \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2}\alpha + o(\alpha), \quad \text{valid for } \alpha < 0.
$$

implying that small fluctuations of the optimal cost in the random-link matching problem are Gaussian, with a variance given by

$$
E[(C_N - \bar{C})^2] = \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2}\alpha + o\left(\frac{1}{N}\right), \quad \text{valid for } N \to \infty.
$$

Recently Wästlund [17] has derived the variance of the random-link assignment problem with exponentially distributed random weights on a complete bipartite graph. His result, obtained using a purely probabilistic approach, coincides with Eq. (17), apart from a global factor 4, due to the fact that the optimal cost of the assignment problem is twice the optimal cost of the matching problem for $N \to \infty$. In our calculation, only the assumption $\lim_{w \to 0} \rho(w) = 1$ for the weight probability density function has been used.

### III. LARGE DEVIATIONS VIA REPLICA

In the previous Section we have performed a small $\alpha$ expansion to extract the variance of the optimal cost $C_N$ for $N \gg 1$. To get instead the large deviation function $L(C)$, we have to keep $\alpha$ finite. We follow two different approaches and we compare then our results with numerical simulations. Let us start from the replica method. Following our general recipe, we derive the large deviation function starting from Eq. (6a), writing down the saddle-point equation in the $\beta \to +\infty$ limit, taking $n\beta = \alpha$ fixed. We start considering $\alpha < 0$ (we will relax this assumption in our cavity calculation). Using the fact that, for $n \in (-1,0)$,

$$
\int (-in)^n e^{in\eta} d\eta = -\frac{2\Gamma(n+1)\sin(n\pi)}{(-x)^{n+1}} \theta(-x), \quad \text{valid for } \alpha < 0,
$$

the argument of the logarithm can be re-written as

$$
\int \left[ \int \frac{dx}{2\pi} (-in)^n \exp \left( inx + \sum_{p=1}^{\infty} \frac{x^n Q_p}{p!} \right) \right] = -\frac{\beta(n+1)\sin(n\pi)}{\pi} \int_{-\infty}^{+\infty} e^{-n\beta t - G_{n,\beta}(t)} dt,
$$

the saddle-point equation (6b) becoming

$$
\beta p \left( \frac{n}{p} \right) Q_p = 2 \left( -1 \right)^p \int_{-\infty}^{+\infty} e^{-n\beta t - G_{n,\beta}(t)} dt \quad \Rightarrow \quad \int_{-\infty}^{+\infty} K_{n,\beta}(x + t) e^{-n\beta t - G_{n,\beta}(t)} dt = -2 \frac{\beta(n+1)\sin(n\pi)}{\pi} \int_{-\infty}^{+\infty} e^{-n\beta t - G_{n,\beta}(t)} dt,
$$

where the function

$$
K_{n,\beta}(u) := \sum_{p=1}^{\infty} \left( \frac{n}{p} \right) \frac{\beta p(p+1)!}{(p+1)^2}
$$

appears. Observing now that (see Appendix)

$$
\lim_{n,\beta \to \alpha} K_{n,\beta}(u) = \theta(u) \left( \frac{1}{\alpha} - u \right),
$$

the saddle-point equation can be written as

$$
G_{\alpha}(x) = \lim_{n,\beta \to \alpha} G_{n,\beta}(x) = -\frac{2\alpha}{Z_{\alpha}} \int_{-\infty}^{+\infty} \left( x + t - \frac{1}{\alpha} \right) e^{-\alpha t - G_{\alpha}(t)} dt,
$$

where we have introduced

$$
Z_{\alpha} := -\alpha \int_{-\infty}^{+\infty} e^{-\alpha t - G_{\alpha}(t)} dt.
$$

The equation above implies that $\lim_{x \to +\infty} x^{-1} G(x) = 2$ for any value of $\alpha$, i.e., $G_{\alpha}(x) \sim 2x$ for large $x$. Assuming $\alpha > -2$, this also implies that $\lim_{x \to -\infty} G_{\alpha}(x) = 0$. By consequence, the integral appearing in the expression of $Z_{\alpha}$ converges for $\alpha > -2$ only and diverges otherwise. Using the saddle-point equation (20) in Eq. (6a) we finally get

$$
\alpha \Phi(\alpha) = \frac{\alpha}{Z_{\alpha}} \int_{-\infty}^{+\infty} G_{\alpha}(t) e^{-\alpha t - G_{\alpha}(t)} dt - 2 \ln Z_{\alpha}.
$$

Eq. (23a) can be solved numerically for a given value of $\alpha$, allowing then to evaluate $\alpha \Phi(\alpha)$ in Eq. (23c), whose Legendre transform is the desired large deviation function $L(C)$. Using the properties of $G_{\alpha}$ derived above, it can be seen that $\lim_{x \to -2} -\alpha \Phi(\alpha) = -\infty$: the presence of such a singularity gives us information on the large $C$ behavior of the Cramér function, i.e., it implies that $\lim_{x \to -\infty} C^{-1} L(C) = 2$. As anticipated, the expressions above have been derived assuming $-2 < \alpha < 0$. Eq. (23c). To get an expression that can be prolonged to positive values of $\alpha$ we will use the cavity method.
IV. LARGE DEVIATIONS VIA CAVITY

The equation for $\Phi(\alpha)$ given by the replica method for $\alpha \in (-2,0]$ can be also obtained using the cavity method and actually extended to positive values of $\alpha$. The starting point is the cavity condition for the occupancy of an edge in the random-link matching problem. In particular, in the cavity approach, each edge $(i,j)$ is associated to its weight $w_{ij}$ and to two cavity fields, $\phi_i$ and $\phi_j$ on its vertices, containing information on the rest of the graph, in such a way that the occupancy $m_{ij}$ of the the edge is distributed as

$$P(m_{ij}) = \exp \left[ -\beta m_{ij} (Nw_{ij} - \phi_i - \phi_j) \right] / \left[ 1 + \exp \left[ -\beta (Nw_{ij} - \phi_i - \phi_j) \right] \right]. \quad (24)$$

In the $\beta \to +\infty$ limit, an edge is occupied if, and only if, $Nw_{ij} < \phi_i + \phi_j$. At zero temperature and in the large $N$ limit, the cavity fields satisfy the following equation [4, 18, 19]

$$\phi_0 = \min_{k \in \partial 0} (Nw_{k0} - \phi_k). \quad (25)$$

Here $\partial 0$ is the set of neighbors of the node 0. The mate node $i^*$ of 0 is such that

$$i^* = \arg \min_{k \in \partial 0} (Nw_{k0} - \phi_k) \quad (26)$$

In Refs. [4, 18, 19] the previous equation have been studied and solved, and the AOC predicted by the cavity method coincides with the one obtained using the replica approach.

The recurrence relation for the cavity fields can be used, however, to extract information on the fluctuations and evaluate $\alpha \Phi(\alpha)$. For the sake of simplicity, let us start from a different version of the problem, i.e., the random-link matching problem on a sparse graph, and in particular on a Bethe lattice topology, and let us follow the approach of Rivoire [12] for the study of large deviation on sparse topologies. In this case, we are interested in solving our problem on a graph having $2N$ vertices, each one of them having coordination $z$: we will later take the limit $z \to 2N - 1$. Taking this limit might sound dangerous, because we apply a result obtained for a sparse topology to a dense one. However, the random-link matching problem is an “effectively sparse” problem: given a fully-connected topology, the probability that a given node is connected, in the optimal matching, to its $n$th nearest neighbor is exponentially small in $n$ [19]. We will denote by $L_z(C)$ the large deviation function for random-link matching problem on the Bethe lattice, so that $L_z(C) = \lim_{z \to +\infty} L_z(C)$.

To obtain an expression for it, we proceed as usual in the cavity approach, i.e., starting from an intermediate graph having $z$ randomly chosen (cavity) nodes with coordination $z - 1$, and all the other nodes with coordination $z$.

Let us denote by $\hat{L}_z$ the large deviation function corresponding to the random-link matching problem on such a topology.

We can recover the “correct” Bethe lattice topology in two ways. We can, for example, add a node to the $z$ cavity nodes.

The optimal cost will be shifted by a certain amount $N^{-1} \epsilon$, in such a way that the probability density function of the optimal cost satisfies the equation

$$e^{-(N+1/2)\hat{L}_z(C)} = \int e^{-NL_z\left(\frac{N+\epsilon z}{N}\right)} p_\epsilon(\epsilon) d\epsilon \leq e^{-N\hat{L}_z(C) + \frac{z}{2} C} \int e^{-\alpha N z p_\epsilon(\epsilon)} d\epsilon, \quad (27)$$

where $\alpha := -\partial C \hat{L}_z(C)$ and $p_\epsilon(\epsilon)$ is the distribution of the energy-shift $\epsilon$ due to a vertex addition.

Another possibility is to add $z/2$ edges.

We obtain in this case the following relation

$$e^{-NL_z(C)} = \left( \prod_{k=1}^{z/2} \int p_\epsilon(\epsilon_k) d\epsilon_k \right) e^{-NL_z(C - \sum_k \epsilon_k)} \leq e^{-N\hat{L}_z(C) \left( \int p_\epsilon(\epsilon) e^{-\alpha N \epsilon} d\epsilon \right)^{z/2}}, \quad (28)$$

where $p_\epsilon(\epsilon)$ is the distribution of the energy-shift $\epsilon$ due to an edge addition. Taking the ratio of the two expressions
above, we obtain an equation for $L_z(C)$ at the leading order in $N$, namely
\[
\alpha \Phi(\alpha) \equiv L_z(C) + \alpha C
\]
\[
= \lim_{N \to +}\frac{1}{N} \left[ \ln \left( \int p_{v}(\epsilon) e^{-\alpha N \epsilon} d\epsilon \right) - 2 \ln \left( \int e^{-\alpha N \epsilon} p_{v}(\epsilon) d\epsilon \right) \right].
\]  
(29)

The previous quantity provides us $\alpha \Phi(\alpha)$ on the Bethe lattice. Taking $z = 2N - 1 \approx 2N$ we obtain the expression for our case. To evaluate the previous quantity, let us introduce the joint distribution $p_{v}(\phi, \epsilon)$ of the cavity field entering in the added node and of the energy shift, such that $p_{v}(\epsilon) = \int p_{v}(\phi, \epsilon) d\phi$, and the reweighted distribution of the cavity field given by
\[
p_{v}(\phi) := \frac{1}{Z_{\alpha}} \int e^{-\alpha N \epsilon} p_{v}(\phi, \epsilon) d\epsilon,
\]  
(30)

with $Z_{\alpha}$ proper normalization constant. In our case
\[
p_{v}(\phi, \epsilon) \equiv p_{v}(\epsilon) \delta \left( \phi \frac{N}{N} - \epsilon \right),
\]

because the cost shift due to the addition of a node coincides with the incoming cavity field, see Eq. (25). If we denote by
\[
\pi_{\alpha}(u) := \int_{0}^{+\infty} p_{v}(\hat{w} - u) d\hat{w},
\]

the distribution of the cavity field can be rewritten as
\[
p_{v}(\phi) = \frac{1}{Z_{\alpha}} \int p_{v}(\phi, \epsilon) e^{-\alpha N \epsilon} d\epsilon = \frac{p_{v}(\phi / N) e^{-\alpha \phi}}{Z_{\alpha}}
\]
\[= \frac{2 e^{-\alpha \phi} \pi_{\alpha}(\phi)}{Z_{\alpha}} \left( 1 - \frac{1}{N} \int_{-\infty}^{\phi} \pi_{\alpha}(\chi) d\chi \right)^{2N-1},
\]  
(31)

where we have used Eq. (25) to express $p_{v}$ in terms of $\pi_{\alpha}$. In the $N \to +\infty$ limit we obtain an equation for $p_{v}$,
\[
p_{v}(\phi) = \frac{2 e^{-\alpha \phi} \pi_{\alpha}(\phi)}{Z_{\alpha}} \exp \left( -2 \int_{-\infty}^{\phi} \pi_{\alpha}(\chi) d\chi \right),
\]  
(32)

Moreover, because of Eq. (24), the energy cost due to the addition of an edge
\[
p_{e}(\epsilon) = \int_{0}^{-\phi_{1}} d\phi_{1} \int_{-\phi_{2}}^{r} d\phi_{2} p_{e}(\phi_{1}) p_{e}(\phi_{2})
\]
\[\times \int_{0}^{+\infty} dw \rho(w) \delta \left( \epsilon - \min \left\{ 0, \frac{\phi_{1} - \phi_{2}}{N} \right\} \right),
\]  
(33)

and therefore, by means of an integration by parts, we get for large $N$
\[
\alpha \Phi(\alpha) = \alpha C + L(C) = -2 \ln \mathcal{Z}_{\alpha}
\]
\[+ \frac{\alpha}{Z_{\alpha}} \int d\phi_{1} d\phi_{2} p_{e}(\phi_{1}) p_{e}(\phi_{2})
\]
\[\times \int_{0}^{+\infty} dw \rho(w) e^{-\alpha \left( w - \phi_{1} - \phi_{2} \right)} (\phi_{1} + \phi_{2} - w).
\]  
(34)

Note that, up to now, no assumptions have been made on the range of values of $\alpha$, except the implicit ones about the fact that the quantities above are well defined and convergent: the cavity expression can be used therefore for both positive and negative values of $\alpha$, provided that the involved quantities are finite.

It can be seen that the expression in Eq. (34) is equivalent to the one given in Eq. (23c) in its range of validity. Indeed, introducing the function
\[
G_{\alpha}(\phi) := 2 \int_{0}^{+\infty} \hat{w} p_{v}(\hat{w} - \phi) d\hat{w},
\]  
(35)

Eq. (32) simplifies as
\[
p_{v}(\phi) = \frac{1}{Z_{\alpha}} \frac{dG_{\alpha}(\phi)}{d\phi} e^{-\alpha \phi - G_{\alpha}(\phi)},
\]  
(36)

and therefore we can write, using Eq. (35), a self-consistent equation for the function $G_{\alpha}(\phi)$ that is found to be identical to Eq. (23a), proving that the function $G_{\alpha}$ introduced here is the same appearing in the replica approach. Repeating the arguments presented in the replica derivation, we obtain that the expressions are finite for $\alpha > -2$. Substituting Eq. (36) in Eq. (34), simple manipulations give us the same expression presented in Eq. (23c), the main difference being that, in the obtained formula,
\[
\mathcal{Z}_{\alpha} = \int e^{-\alpha \phi} p_{v}(\phi) d\phi = \int e^{-\alpha \phi} \frac{dG_{\alpha}(\phi)}{d\phi} e^{-G_{\alpha}(\phi)} d\phi.
\]  
(37)

If we further restrict ourselves to negative values of $\alpha$, an integration by parts allow us to write $\mathcal{Z}_{\alpha}$ in the same form given in Eq. (23a), proving the equivalence of the two approaches and implicitly providing us a prolongation of the replica result.

V. NUMERICAL RESULTS

We have integrated Eq. (34) by means of a population dynamics algorithm, and we have compared our results with the value of $\alpha \Phi(\alpha)$ obtained by numerical simulations. The agreement is very good in the neighborhood of the origin. We have discarded the data points where finite-sample effects appear for larger values of $|\alpha|$; obviously, better estimates can be obtained running an exponentially large number of instances in the size of the system. Finite-size effects still appear in the evaluation.
of the derivative of $\alpha \Phi(\alpha)$ for $\alpha < 0$, larger sizes being closer to the theoretical predictions, see Fig. 1b. Finally, the theoretical prediction near $\alpha = -2$ becomes noisy and less reliable, because of the approaching of the divergence. In this regime, the convergence of the cavity fields distribution fails and an accurate estimation of $\alpha \Phi(\alpha)$ is more challenging.

VI. CONCLUSIONS

Using the replica approach we have evaluated the variance of the average optimal cost for the random-link matching problem on the complete graph, assuming a distribution of the weights such that $\lim_{w \to 0} \rho(w) = 1$. Our result is in agreement with a previously obtained expression by Wästlund, proving that the small fluctuations of the optimal cost around its asymptotic AOC are Gaussian. We have then derived an expression for the Legendre transform $\alpha \Phi(\alpha)$ of the Cramér function $L(C)$ using both the replica theory (for positive cost fluctuations) and the cavity method. The cavity formula, in particular, has been obtained for the random-link matching problem on a generic sparse graph having fixed coordination, it provides a recipe for the numerical evaluation of $\alpha \Phi(\alpha)$. In the fully-connected case, our results also show that $\alpha \Phi(\alpha)$ diverges for $\alpha \to -2$, implying that $\lim_{C \to +\infty} C^{-1} L(C) = 2$.

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Appendix: Derivation of Eq. (22)

To prove the limit in Eq. (22), we can start using the fact that $\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$. The series can be written as

$$
\sum_{p=1}^{\infty} \frac{1}{(\beta p)^2} e^{\beta u} = \frac{1}{\beta} \sum_{p=1}^{\infty} \frac{(-e^{\beta u})^p}{p^p}.
$$

(A.1)
We have to compute the $\beta \to +\infty$ limit at $n\beta = \alpha$ fixed. The coefficient in front of the series has limit
\[
\lim_{\beta \to +\infty} \beta^{-1} \Gamma (-\alpha/\beta) = -1/\alpha.
\]
On the other hand,
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{1}{\Gamma(p-\alpha/\beta)} \left( -e^{\beta u} \right)^p / (p!)^2 + \theta(u) = \lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \left( \frac{\Gamma(p)}{\Gamma(p-\alpha/\beta)} - 1 \right) \left( -e^{\beta u} \right)^p / (p!)^2,
\]
where we have used the fact that
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \frac{(-e^{\beta u})^p}{(p!)^2} = -\theta(u).
\]

We can write the following expansion
\[
\frac{\Gamma(p)}{\Gamma(p-\alpha/\beta)} - 1 = \sum_{k=1}^{\infty} \frac{(-\alpha/\beta)^k}{k!} B_k \left( \hat{\psi}_0(p), \hat{\psi}_1(p), \ldots, \hat{\psi}_{k-1}(p) \right),
\]
where we have introduced the (opposite) polygamma function $\hat{\psi}_n(z) := -\psi_n(z)$
\[
\hat{\psi}_n(z) := \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-tz}}{1-e^{-t}} dt
\]
and the (complete) exponential Bell polynomials
\[
B_k(x_1, \ldots, x_k) := \left( \frac{\partial}{\partial t} \right)^k \exp \left( \sum_{j=1}^{\infty} \frac{x_j t^j}{j!} \right) \bigg|_{t=0}.
\]
The expansion above is obtained applying the Faà di Bruno formula for the derivative of a composed function $\partial_x^p (f \circ g)(x)$ with $f(x) \equiv e^x$ and $g(x) \equiv -\ln \Gamma(x)$. If we now assume $B_0 \equiv 1$, the Bell polynomials satisfy the recurrence relations [20]
\[
B_{k+1}(x_1, \ldots, x_{k+1}) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}(x_1, \ldots, x_{k-i}) x_{i+1}.
\]
We will show now by induction that, for $k \geq 2$,
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} B_k \left( \hat{\psi}_0(p), \ldots, \hat{\psi}_{k-1}(p) \right) \frac{(-e^{\beta u})^p}{(p!)^2} = 0.
\]
For $k = 1$ we have
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \hat{\psi}_0(p) \frac{(-e^{\beta u})^p}{(p!)^2} = \lim_{\beta \to +\infty} \int_0^{\infty} \frac{J_0(2e^{\beta u} t) - 1}{1-e^{-\beta t}} dt = -u\theta(u).
\]
The $k = 2$ case follows straightforwardly. Indeed,
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} B_2 \left( \hat{\psi}_0(p), \hat{\psi}_1(p) \right) \frac{(-e^{\beta u})^p}{(p!)^2} = \lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \hat{\psi}_0^2(p) + \hat{\psi}_1(p) \frac{(-e^{\beta u})^p}{(p!)^2} = -\int_0^{+\infty} dt \int_0^{+\infty} dt \int_0^{+\infty} d\tau \theta(u-t-\tau) = 0.
\]
Let us suppose now that the statement is true for generic $k > 2$ and let us consider it for $k+1$. Then we have
\[
\sum_{p=1}^{\infty} B_{k+1} \left( \hat{\psi}_0(p), \ldots, \hat{\psi}_k(p) \right) \frac{(-e^{\beta u})^p}{(p!)^2} = \sum_{i=0}^{k} \binom{k}{i} \sum_{p=1}^{\infty} B_{k-i} \left( \hat{\psi}_0(p), \ldots, \hat{\psi}_{k-i-1}(p) \right) \hat{\psi}_i(p) \frac{(-e^{\beta u})^p}{(p!)^2}.
\]
The inner sum can be written as
\[
\int_0^{+\infty} \frac{dt}{1 - e^{-\beta t}} \sum_{p=1}^{k+1} \frac{B_{k+1}(\psi_0(p), \ldots, \psi_{k+1-1}(p))}{|p|^2} \left(-e^{\beta(u-t)}\right)^p \frac{1}{(p!)^2}
\] (A.12)

implying that
\[
\sum_{p=1}^{k+1} \frac{B_{k+1}(\psi_0(p), \ldots, \psi_{k+1-1}(p))}{|p|^2} \left(-e^{\beta(u-t)}\right)^p = \sum_{i=0}^{k-2} \binom{k}{i} \int_0^{+\infty} \frac{dt}{1 - e^{-\beta t}} \sum_{p=1}^{k+1} \frac{B_{k+1}(\psi_0(p), \ldots, \psi_{k+1-1}(p))}{|p|^2} \left(-e^{\beta(u-t)}\right)^p + k(-1)^{k-1} \int_0^{+\infty} \frac{dt}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{\psi_0(p)}{\beta} \left(-e^{\beta(u-t)}\right)^p + \frac{e^{\beta u} - 1}{e^{\beta u} + 1} \frac{1}{\beta} \int_0^{+\infty} \frac{dt}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \left(-e^{\beta(u-t)}\right)^p. \tag{A.13}
\]

The last two terms in the previous expression in the $\beta \to \infty$ tend to zero
\[
-k(-1)^{k-1} \int_0^{+\infty} \frac{t^{k-1}}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \frac{\psi_0(p)}{\beta} \left(-e^{\beta(u-t)}\right)^p + (-1)^{k-1} \int_0^{+\infty} \frac{dt}{1 - e^{-\beta t}} \sum_{p=1}^{\infty} \left(-e^{\beta(u-t)}\right)^p = 0 \tag{A.14}
\]

By the induction hypothesis, the remaining $k - 2$ contributions are infinitesimal as well for $\beta \to +\infty$, and the thesis is proved. We can restrict therefore the expansion in Eq. (A.4) to the $k = 1$ term in the $\beta \to +\infty$ hypothesis. We have
\[
\lim_{\beta \to +\infty} \sum_{p=1}^{\infty} \left(\frac{\Gamma(p)}{\Gamma(p - \alpha/\beta)} - 1\right) \frac{(-e^{\beta u})^p}{(p!)^2} = \alpha u^2. \tag{A.15}
\]

The relations above finally give us the asymptotic in Eq. (22).

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