REPRESENTATION THEORY OF SUPERCONFORMAL ALGEBRAS AND THE KAC-ROAN-WAKIMOTO CONJECTURE

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Abstract. We study the representation theory of the superconformal algebra $W_k(g, f_\theta)$ associated with a minimal gradation of $g$. Here, $g$ is a simple finite-dimensional Lie superalgebra with a non-degenerate, even supersymmetric invariant bilinear form. Thus, $W_k(g, f_\theta)$ can be one of the well-known superconformal algebras including the Virasoro algebra, the Bershadsky-Polyakov algebra, the Neveu-Schwarz algebra, the Bershadsky-Knizhnik algebras, the $N=2$ superconformal algebra, the $N=4$ superconformal algebra, the $N=3$ superconformal algebra and the big $N=4$ superconformal algebra. We prove the conjecture of V. G. Kac, S.-S. Roan and M. Wakimoto for $W_k(g, f_\theta)$. In fact, we show that any irreducible highest weight character of $W_k(g, f_\theta)$ at any level $k \in \mathbb{C}$ is determined by the corresponding irreducible highest weight character of the Kac-Moody affinization of $g$.

1. Introduction

Suppose that the following are given: (i) a simple finite-dimensional Lie superalgebra $g$ with non-degenerate, even supersymmetric invariant bilinear form, (ii) a nilpotent element $f$ in the even part of $g$ and (iii) a level $k \in \mathbb{C}$ of the Kac-Moody affinization $\hat{g}$ of $g$. Then, the corresponding $W$-algebra $W_k(g, f)$ can be constructed using the method of quantum BRST reduction. This method was first introduced by B. L. Feigin and E. V. Frenkel in the case that $g$ is a Lie algebra and $f$ is its principal nilpotent element, and it was recently extended to the general case by V. G. Kac, S.-S. Roan and M. Wakimoto.

In this paper we study the representation theory of those $W$-algebras for which the nilpotent element $f$ is equal to the root vector $f_\theta$ corresponding to the lowest root $-\theta$ of $g$. V. G. Kac, S.-S. Roan and M. Wakimoto showed that these $W$-algebras $W_k(g, f_\theta)$ are quite different from the standard $W$-algebras associated with principal nilpotent elements and noteworthy because they include almost all the superconformal algebras so far constructed to this time by physicists such as the Virasoro algebra, the Bershadsky-Polyakov algebra, the Neveu-Schwarz algebra, the Bershadsky-Knizhnik algebras, the $N=2$ superconformal algebra, the $N=4$ superconformal algebra, the $N=3$ superconformal algebra and the big $N=4$ superconformal algebra.

Let $O_k$ be the Bernstein-Gel'fand-Gel'fand category of $\hat{g}$ at level $k \in \mathbb{C}$. Let $M(\lambda)$ be the Verma module of $\hat{g}$ with highest weight $\lambda$ and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$. The method of quantum BRST reduction gives a family of functors $V \rightsquigarrow H^i(V)$, depending on $i \in \mathbb{Z}$, from $O_k$ to the category of $W_k(g, f)$-modules. Here, $H^\bullet(V)$ is the BRST cohomology of the corresponding

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quantum reduction. In the case that \( g \) is a Lie algebra and \( f \) is its principal nilpotent element, E. V. Frenkel, V. G. Kac and M. Wakimoto used this functor in their construction of the “minimal” series presentations of \( \mathcal{W}_k(g, f) \), and they conjectured that \( H^\bullet(L(\lambda)) \) is irreducible (or zero) for an admissible weight \( \lambda \). This conjecture was extended by V. G. Kac, S.-S. Roan and M. Wakimoto to the general case, in which they conjectured that, for an admissible weight \( \lambda \), the irreducibility of \( H^\bullet(L(\lambda)) \) holds for a general pair \((g, f)\) consisting of a Lie superalgebra \( g \) and a nilpotent element \( f \) (see Conjecture 3.1B of Ref. [17]).

As a continuation of the present author’s previous work [1] [2], in which the conjecture of E. V. Frenkel, V. G. Kac and M. Wakimoto was proved (completely for the “−” case and partially for the “+” case), we prove the conjecture of V. G. Kac, S.-S. Roan and M. Wakimoto for \( \mathcal{W}_k(g, f_\theta) \): Actually we prove even stronger results, showing that the representation theory of \( \mathcal{W}_k(g, f_\theta) \) is controlled by \( \hat{g} \) in the following sense:

**Main Theorem.** For arbitrary level \( k \in \mathbb{C} \), we have the following:

1. (Theorem 6.7.1) We have \( H^i(V) = \{0\} \) with \( i \neq 0 \) for any \( V \in \text{Obj}_k \).
2. (Theorem 6.7.4) Let \( L(\lambda) \in \text{Obj}_k \) be the irreducible \( \hat{g} \)-module with highest weight \( \lambda \). Then, \( (\lambda, \alpha_0^+) \in \{0, 1, 2, \ldots\} \) implies \( H^0(L(\lambda)) = \{0\} \). Otherwise, \( H^0(L(\lambda)) \) is isomorphic to the irreducible \( \mathcal{W}_k(g, f_\theta) \)-module with the corresponding highest weight.

Main Theorem (1) implies, in particular, that the correspondence \( V \rightsquigarrow H^0(V) \) defines an exact functor from \( \text{Obj}_k \) to the category of \( \mathcal{W}_k(g, f_\theta) \)-modules, defining a map between characters. On the other hand, every irreducible highest weight representation of \( \mathcal{W}_k(g, f_\theta) \) is isomorphic to \( H^0(L(\lambda)) \) for some \( \lambda \) (see Section 6). It is also known that \( H^0(M(\lambda)) \) is a Verma module over \( \mathcal{W}_k(g, f_\theta) \) [19]. Hence, from the above results, it follows that the character of any irreducible highest weight representation of \( \mathcal{W}_k(g, f_\theta) \) is determined by the character of the corresponding irreducible \( \hat{g} \)-module \( L(\lambda) \). We remark that Main Theorem (2) is consistent with the computation of V. G. Kac, S.-S. Roan and M. Wakimoto of the Euler-Poincaré character of \( H^\bullet(L(\lambda)) \).

This paper is organized as follows. In Section 2 we collect the necessary information regarding the affine Lie superalgebra \( \hat{g} \). In this setting, a slight modification is need for the \( A(1, 1) \) case, which is summarized in Appendix. In Section 3 we recall the definition of the BRST complex constructed by V. G. Kac, S.-S. Roan, and M. Wakimoto. As explained in Ref. [17], their main idea in generalizing the construction of B. L. Feigin and E. V. Frenkel [10] [11] was to add the “neutral free superfermions” whose definition is given at the beginning of that section. Although the \( \mathcal{W} \)-algebra \( \mathcal{W}_k(g, f) \) can be defined for an arbitrary nilpotent element \( f \), the assumption \( f = f_\theta \) simplifies the theory in many ways. This is also the case that the above-mentioned well-known superconformal algebras appear, as explained in Ref. [17]. In Section 4 we derive some basic but important facts concerning the BRST cohomology under the assumption \( f = f_\theta \). In Section 5 we recall the definition of the \( \mathcal{W} \)-algebra \( \mathcal{W}_k(g, f) \) and collect necessary information about its structure. In Section 6 we present the parameterization of irreducible highest weight representations of \( \mathcal{W}_k(g, f_\theta) \) and state our main results (Theorems 6.7.1 and 6.7.4). The most important part of the proof is the computation of the BRST cohomology \( H^\bullet(M(\lambda)^+) \)
associated with the dual $M(\lambda)^*$ of the Verma module $M(\lambda)$. This is carried out in Section 7 by introducing a particular spectral sequence. The argument used here is a modified version of that given in Ref. [1], where we proved the vanishing of the cohomology associated to the original quantum reduction formulated by B. L. Feigin and E. V. Frenkel [10, 11, 13].

The method used in this paper can also be applied to general $\mathcal{W}$-algebras, with some modifications. Our results for that case will appear in forthcoming papers.

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2. Preliminaries

2.1. Let $g$ be a complex, simple finite-dimensional Lie superalgebra with a non-degenerate, even supersymmetric invariant bilinear form $(.,.)$. Let $(e, x, f)$ be a $\mathfrak{sl}_2$-triple in the even part of $g$ normalized as

$$[e, f] = x, \ [x, e] = e, \ [x, f] = -f. \tag{1}$$

Further, let $g = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} g_j$, $g_j = \{ u \in g \mid [x, u] = ja \} \tag{2}$

be the eigenspace decomposition of $g$ with respect to $\text{ad} \ x$.

2.2. Define $g^f$ as the centralizer of $f$ in $g$, so that $g^f = \{ u \in g \mid [f, u] = 0 \}$. Then, we have $g^f = \sum_{j \leq 0} g^f_j$, where $g^f_j = g^f \cap g_j$. Similarly, we define $g^e = \{ u \in g \mid [e, u] = 0 \} = \sum_{j \geq 0} g^f_j$.

2.3. Define the following:

$$g_{\geq 1} = \bigoplus_{j \geq 1} g_j, \quad g_{> 0} = \bigoplus_{j > 0} g_j.$$

Then, $g_{> 0} = g_{\geq 1} \oplus g_{\frac{1}{2}}$, and $g_{\geq 1}$ and $g_{> 0}$ are both nilpotent subalgebras of $g$. Similarly define $g_{\geq 0}$, $g_{\leq 0}$, $g_{< 0}$ and $g_{\leq -1}$.

2.4. Define the character $\bar{\chi}$ of $g_{\geq 1}$ by

$$\bar{\chi}(u) = (f|u), \quad \text{where} \ u \in g_{\geq 1}. \tag{3}$$

Then, $\bar{\chi}$ defines a skew-supersymmetric even bilinear form $\langle .|\cdot \rangle_{\text{ne}}$ on $g_{\frac{1}{2}}$ through the formula

$$\langle u|v \rangle_{\text{ne}} = \bar{\chi}([u, v]). \tag{4}$$

Note that $\langle .|\cdot \rangle_{\text{ne}}$ is non-degenerate, as guaranteed by the $\mathfrak{sl}_2$-representation theory. Also, we have the property

$$\langle u|[a, v] \rangle_{\text{ne}} = \langle [u, a]|v \rangle_{\text{ne}} \quad \text{for} \ a \in g^f_0, \ u, v \in g_{\frac{1}{2}}. \tag{5}$$
2.5. Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a Cartan subalgebra of $\mathfrak{g}$ containing $x$, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. It is known that the root space $\mathfrak{g}^\alpha$, with $\alpha \in \Delta$, is one-dimensional except for $A(1,1)$ (see [10]). For this reason, in the case of $A(1,1)$, a slight modification is needed in the following argument, which is summarized in Appendix. Now, let $\Delta_j = \{ \alpha \in \Delta | \langle \alpha, x \rangle = j \}$ with $j \in \frac{1}{2} \mathbb{Z}$. Then, $\Delta = \bigcup_{j \in \frac{1}{2} \mathbb{Z}} \Delta_j$, and $\Delta_0$ is the set of roots of $\mathfrak{g}_0$. Let $\Delta_{0+}$ be a set of positive roots of $\Delta_0$. Then, $\Delta_+ = \Delta_{0+} \cup \Delta_{0-}$ is a set of positive roots of $\mathfrak{g}$, where $\Delta_{0+} = \bigcup_{j>0} \Delta_j$. This gives the triangular decompositions

\begin{equation}
\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+, \quad \mathfrak{g}_0 = n_{0-} \oplus \mathfrak{h} \oplus n_{0+}.
\end{equation}

Here, $n_+ \overset{\text{def}}{=} \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $n_{0+} \overset{\text{def}}{=} \sum_{\alpha \in \Delta_{0+}} \mathfrak{g}_\alpha$ and analogously for $n_-$ and $n_{0-}$.

2.6. Let $u \mapsto u^t$ be an anti-automorphism of $\mathfrak{g}$ such that $e^t = f$, $f^t = e$, $x^t = x$, $\mathfrak{g}_\alpha^t = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$ and $(u^t|v^t) = (v|u)$ for $u, v \in \mathfrak{g}$. We fix the root vectors $u_\alpha \in \mathfrak{g}_\alpha$, where $\alpha \in \Delta$, such that $(u_\alpha, u_{-\alpha}) = 1$ and $u^t_\alpha = u_{-\alpha}$ with $\alpha \in \Delta_+$.

2.7. Let $p(\alpha)$ be the parity of $\alpha \in \Delta$ and $p(v)$ be the parity of $v \in \mathfrak{g}$.

2.8. Let $\hat{\mathfrak{g}}$ be the Kac–Moody affinization of $\mathfrak{g}$. This is the Lie superalgebra given by

\begin{equation}
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} K \oplus \mathbb{C} D
\end{equation}

with the commutation relations

\begin{equation}
[u(m), v(n)] = [u, v](m + n) + m \delta_{m+n,0}(u|v) K,
\end{equation}

\begin{equation}
[D, u(m)] = mu(m), \quad [K, \hat{\mathfrak{g}}] = 0
\end{equation}

for $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Here, $u(m) = u \otimes t^m$ for $u \in \mathfrak{g}$ and $m \in \mathbb{Z}$.

The invariant bilinear form $(\cdot | \cdot)$ is extended from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by stipulating $(u(m)|v(n)) = (u|v) \delta_{m+n,0}$ with $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \mathbb{C} K \oplus \mathbb{C} D) = 0$, $(K, K) = (D, D) = 0$ and $(K, D) = (D, K) = 1$.

2.9. Define the subalgebras

\begin{equation}
L_{\mathfrak{g}_{\geq 1}} = \mathfrak{g}_{\geq 1} \otimes \mathbb{C}[t, t^{-1}], \quad L_{\mathfrak{g}_{\geq 0}} = \mathfrak{g}_{\geq 0} \otimes \mathbb{C}[t, t^{-1}] \subset \hat{\mathfrak{g}}.
\end{equation}

Similarly, define $L_{\mathfrak{g}_{\leq 0}}$, $L_{\mathfrak{g}_{< 0}}$, $L_{\mathfrak{g}_{< 0}}$ and $L_{\mathfrak{g}_{\leq -1}}$.

2.10. Fix the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ in the standard way. Then, we have

\begin{align*}
\hat{\mathfrak{h}} &= \mathfrak{h} \oplus \mathbb{C} K \oplus \mathbb{C} D, \\
\hat{\mathfrak{n}}_- &= \mathfrak{n}_- \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes \mathbb{C}[t^{-1}] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t^{-1}] t^{-1}, \\
\hat{\mathfrak{n}}_+ &= \mathfrak{n}_- \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t].
\end{align*}

Let $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C} \Lambda_0 \oplus \mathbb{C} \delta$ be the dual of $\hat{\mathfrak{h}}$, where, $\Lambda_0$ and $\delta$ are dual elements of $\mathfrak{K}$ and $\mathfrak{D}$, respectively. Next, let $\hat{\Delta}$ be the set of roots of $\hat{\mathfrak{g}}$, $\hat{\Delta}_+$ the set of positive roots, and $\hat{\Delta}_-$ the set of negative roots. Then, we have $\hat{\Delta}_- = -\hat{\Delta}_+$. Further, let $\hat{Q}$ be the root lattice and $\hat{Q}_+ = \sum_{\alpha \in \hat{\Delta}_+} \mathbb{Z}_{\geq 0} \alpha \subset \hat{Q}$. We define a partial ordering $\mu \leq \lambda$ on $\hat{\mathfrak{h}}^*$ by $\lambda - \mu \in \hat{Q}_+$. 
2.11. For an $\mathfrak{h}$-module $V$, let $V^\lambda$ be the weight space of weight $\lambda$, that is, $V^\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. If all the weight spaces $V^\lambda$ are finite-dimensional, we define the graded dual $V^*$ of $V$ by
\[
V^* = \bigoplus_{\lambda \in \mathfrak{h}^*} \text{Hom}_\mathbb{C}(V^\lambda, \mathbb{C}) \subset \text{Hom}_\mathbb{C}(V, \mathbb{C}).
\]

2.12. Throughout this paper, $k$ represents a complex number. Let $\mathfrak{h}_k^*$ denote the set of weights of $\hat{\mathfrak{g}}$ of level $k$:
\[
\mathfrak{h}_k^* = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, K \rangle = k\}.
\]

Also, let $\mathcal{O}_k$ be the full subcategory of the category of left $\hat{\mathfrak{g}}$-modules consisting of objects $V$ such that
\begin{enumerate}
\item[(1)] $V = \bigoplus_{\lambda \in \mathfrak{h}_k^*} V^\lambda$ and $\dim \mathbb{C} V^\lambda < \infty$ for all $\lambda \in \mathfrak{h}_k^*$;
\item[(2)] there exists a finite set $\{\mu_1, \ldots, \mu_r\} \subset \mathfrak{h}_k^*$ such that $\lambda \in \bigcup_i (\mu_i - \mathfrak{O}_V)$ for any weight $\lambda$ with $V^\lambda \neq \{0\}$.
\end{enumerate}

Then, $\mathcal{O}_k$ is an abelian category. Let $M(\lambda) \in \text{Obj} \mathcal{O}_k$, with $\lambda \in \mathfrak{h}_k^*$, be the Verma module with highest weight $\lambda$. That is, $M(\lambda) = U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{h}}) C_\lambda$, where $C_\lambda$ is the one-dimensional $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{h}}$-module on which $\hat{\mathfrak{h}}$ acts trivially and $\hat{\mathfrak{g}}$ acts as $\langle \lambda, h \rangle \mathrm{id}$. Let $v_\lambda$ be the highest weight vector of $M(\lambda)$. Next, let $L(\lambda) \in \text{Obj} \mathcal{O}_k$ be the unique simple quotient of $M(\lambda)$.

2.13. The correspondence $V \rightsquigarrow V^*$ defines the duality functor in $\mathcal{O}_k$. Here, $\hat{\mathfrak{g}}$ acts on $V^*$ as $(af)(v) = f(a'v)$, where $a \mapsto a'$ is the antiautomorphism of $\hat{\mathfrak{g}}$ defined by $u(m) = u(m')$ (with $u \in \mathfrak{g}, m \in \mathbb{Z}$), $K^t = K$ and $D^t = D$. We have $L(\lambda)^* = L(\lambda)$ for $\lambda \in \mathfrak{h}_k^*$.

2.14. Let $\mathcal{O}_k^\Delta$ be the full subcategory of $\mathcal{O}_k$ consisting of objects $V$ that admit a Verma flag, that is, a finite filtration $V = V_0 \supset V_1 \supset \cdots \supset V_r = \{0\}$ such that each successive subquotient $V_i/V_{i+1}$ is isomorphic to some Verma module $M(\lambda_i)$ with $\lambda_i \in \mathfrak{h}_k^*$. The category $\mathcal{O}_k^\Delta$ is stable under the operation of taking a direct summand. Dually, let $\mathcal{O}_k^\Lambda$ be the full subcategory of $\mathcal{O}_k$ consisting of objects $V$ such that $V^* \in \text{Obj} \mathcal{O}_k^\Delta$.

2.15. For $\lambda \in \mathfrak{h}_k^*$, let $\mathcal{O}_k^{\leq \lambda}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects $V$ such that $V = \bigoplus_{\mu \leq \lambda} V^\mu$. Then, $\mathcal{O}_k^{\leq \lambda}$ is an abelian category and stable under the operation of taking (graded) dual. Also, every simple object $L(\mu) \in \text{Obj} \mathcal{O}_k^{\leq \lambda}$, with $\mu \leq \lambda$, admits a projective cover $P_{\leq \lambda}(\mu)$ in $\mathcal{O}_k^{\leq \lambda}$, and hence, every finitely generated object in $\mathcal{O}_k^{\leq \lambda}$ is an image of a projective object of the form $\bigoplus_{i=1}^r P_{\leq \lambda}(\mu_i)$ for some $\mu_i \in \mathfrak{h}_k^*$. Indeed, as in the Lie algebra case (see, e.g., Ref. [27]), $P_{\leq \lambda}(\mu)$ can be defined as an indecomposable direct summand of
\[
U(\hat{\mathfrak{g}}) \otimes U(\hat{\mathfrak{h}}) \tau_{\leq \lambda} \left( U(\hat{\mathfrak{h}}) \otimes U(\hat{\mathfrak{g}}) C_\mu \right)
\]
that has $L(\mu)$ as a quotient. Here, $\tau_{\leq \lambda}(V) = V/\bigoplus_{\nu \in \mathfrak{h}_k^*, \nu > \lambda} V^\nu$, and $C_\mu$ is a one-dimensional $\hat{\mathfrak{h}}$-module on which $h \in \mathfrak{h}$ acts as $\mu(h) \mathrm{id}$. Note that $P_{\leq \lambda}(\mu) \in \text{Obj} \mathcal{O}_k^\Lambda$. 

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Moreover, the Bernstein-Gelfand-Gelfand reciprocity holds:

\[ [P_{\leq \lambda}(\mu) : M(\mu')] = [M(\mu') : L(\mu)] \quad \text{(with } \mu, \mu' \leq \lambda). \]

Here, \([P_{\leq \lambda}(\mu) : M(\mu')]\) is the multiplicity of \(M(\mu')\) in the Verma flag of \(P_{\leq \lambda}(\mu)\), and \([M(\mu') : L(\mu)]\) is the multiplicity of \(L(\mu)\) in the local composition factor (see Ref. [15]) of \(M(\mu')\). Dually, \(P_{\leq \lambda}(\mu) = P_{\leq \lambda}(\mu)^*\) is the injective envelope of \(L(\mu)\) in \(\mathcal{O}^Z_\lambda\). In particular, \(V \in \text{Obj} \mathcal{O}^Z_\lambda\) is a submodule of an injective object of the form \(\bigoplus_{i=1}^r I_{\leq \lambda}(\mu_i)\) for some \(\mu_i \in \hat{h}^*\) if its dual \(V^*\) is finitely generated.

3. The Kac-Roan-Wakimoto construction I: the BRST complex

3.1. Define a character \(\chi\) of \(Lg_{\geq 1}\) by

\[ \chi(u(m)) = (f(1)|u(m)) = \bar{\chi}(u)\delta_{m,-1} \quad \text{for } u \in g_{\geq 1}, m \in \mathbb{Z}. \]

Let \(\ker \chi \subset U(Lg_{\geq 1})\) be the kernel of the algebra homomorphism \(\chi : U(Lg_{\geq 1}) \to \mathbb{C}\). Define \(I_\chi = U(Lg_{\geq 0})\ker \chi\). Then, \(I_\chi\) is a two-sided ideal of \(U(Lg_{\geq 0})\). Next, define

\[ N(\chi) = \frac{U(Lg_{\geq 0})}{I_\chi}. \]

Now, let \(\Phi_\alpha(n)\), with \(n \in g_{> 0}\) and \(n \in \mathbb{Z}\), denote the image of \(u(n) \in Lg_{> 0}\) in the algebra \(N(\chi)\). With a slight abuse of notation, we write \(\Phi_{\alpha}(n)\) as \(\Phi_{\alpha}(n)\) for \(\alpha \in \Delta_2^{\dagger}\) and \(n \in \mathbb{Z}\). Then, the superalgebra \(N(\chi)\) is generated by \(\Phi_{\alpha}(n)\), where \(\alpha \in \Delta_2^{\dagger}\) and \(n \in \mathbb{Z}\), with the relations

\[ [\Phi_{\alpha}(m), \Phi_{\beta}(n)] = \langle u_\alpha|u_\beta\rangle_{ne}\delta_{m+n,-1} \quad \text{for } \alpha, \beta \in \Delta_2^{\dagger}, m, n \in \mathbb{Z}. \]

The elements \(\{\Phi_{\alpha}(n)\}\) are called the neutral free superfermions (see Example 1.2 of Ref. [17]).

Let \(\{u_\alpha\}_{\alpha \in \Delta_2^{\dagger}}\) be the basis of \(g_2^{\dagger}\) dual to \(\{u_\alpha\}_{\alpha \in \Delta_2^{\dagger}}\) with respect to \(|\, |\rangle_{ne}\), that is, \(\langle u_\alpha | w^\beta\rangle_{ne} = \delta_{\alpha, \beta}\). We set \(\Phi^\alpha(n) = \Phi_{\alpha^\dagger}(n)\) for \(\alpha \in \Delta_2^{\dagger}\) and \(n \in \mathbb{Z}\), and thus,

\[ [\Phi_{\alpha}(m), \Phi^\beta(n)] = \delta_{\alpha, \beta}\delta_{m+n,-1} \quad \text{with } \alpha, \beta \in \Delta_2^{\dagger}, m, n \in \mathbb{Z}. \]

3.2. Let \(F_{ne}(\chi)\) be the irreducible representations of \(N(\chi)\) generated by the vector \(1_\chi\) with the property

\[ \Phi_{\alpha}(n)1_\chi = 0 \quad \text{for } \alpha \in \Delta_2^{\dagger} \text{ and } n \geq 0. \]

Note that \(F_{ne}(\chi)\) is naturally a \(Lg_{> 0}\)-module through the algebra homomorphism \(Lg_{> 0} \ni u(m) \mapsto \Phi_{\alpha}(m) \in N(\chi)\).

There is a unique semisimple action of \(\mathfrak{h}\) on \(F_{ne}(\chi)\) such that the following hold:

\[ h1_\chi = 0 \quad \text{for } h \in \mathfrak{h}, \]

\[ \Phi_{\alpha}(n)F_{ne}(\chi)^{\lambda} \subset F_{ne}(\chi)^{\lambda + \alpha + n\delta} \quad \text{for } \alpha \in \Delta_2^{\dagger}, n \leq -1 \text{ and } \lambda \in \hat{h}^*. \]

Lemma 3.2.1. We have \(\Phi_{\alpha}(n)F_{ne}(\chi)^{\lambda} \subset \sum_{\beta \in \Delta_2^{\dagger} \mid \chi(\langle u_\alpha | u_\beta\rangle) \neq 0} F_{ne}(\chi)^{\lambda - \beta + (n+1)\delta} \) for \(\alpha \in \Delta_2^{\dagger}, n \geq 0 \text{ and } \lambda \in \hat{h}^*\).
Proof. By definition we have
\[\Phi_\alpha(n)\mathcal{F}(\chi)^\lambda \subset \mathcal{F}(\chi)^{\lambda-\alpha+(n+1)\delta}\] for \(n \geq 0\)
(see \(^{19}\)). But \(\Phi_\alpha(n) = \sum_{\beta \in \Delta_\lambda} \bar{\chi}([u_\beta, u_\alpha])\Phi_\delta(n)\) for \(\alpha \in \Delta_\lambda\). \(\square\)

3.3. Let \(\mathcal{C}(Lg_{g,0})\) be the Clifford superalgebra (or the charged free superfermions) associated with \(Lg_{g,0} \oplus (Lg_{g,0})^*\) and its natural bilinear from. The superalgebra \(\mathcal{C}(Lg_{g,0})\) is generated by \(\psi_\alpha(n)\) and \(\psi^\alpha(n)\), where \(\alpha \in \Delta_{g,0}\) and \(n \in \mathbb{Z}\), with the relations
\[
[\psi_\alpha(m), \psi^\beta(n)] = \delta_{\alpha,\beta}\delta_{m+n,0},
[\psi_\alpha(m), \psi_\beta(m)] = [\psi^\alpha(m), \psi^\beta(n)] = 0,
\]
where the parity of \(\psi_\alpha(n)\) and \(\psi^\alpha(n)\) is reverse to \(u_\alpha\).

3.4. Let \(\mathcal{F}(Lg_{g,0})\) be the irreducible representation of \(\mathcal{C}(Lg_{g,0})\) generated by the vector \(1\) that satisfies the relations
\[
\psi_\alpha(n)1 = 0 \quad \text{for} \quad \alpha \in \Delta_{g,0} \quad \text{and} \quad n \geq 0,
\]
and \(\psi^\alpha(n)1 = 0 \quad \text{for} \quad \alpha \in \Delta_{g,0} \quad \text{and} \quad n > 0.\)
The space \(\mathcal{F}(Lg_{g,0})\) is graded; that is, \(\mathcal{F}(Lg_{g,0}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^i(Lg_{g,0})\), where the degree is determined from the assignments \(\deg 1 = 0\), \(\deg \psi_\alpha(n) = -1\) and \(\deg \psi^\alpha(n) = 1\), with \(\alpha \in \Delta_{g,0}\) and \(n \in \mathbb{Z}\).

There is a natural semisimple action of \(\hat{h}\) on \(\mathcal{F}(Lg_{g,0})\), namely, \(\mathcal{F}(Lg_{g,0}) = \bigoplus_{\lambda \in \hat{h}^*} \mathcal{F}(Lg_{g,0})^\lambda\). This is defined by the relations \(h1 = 0\) \((h \in \hat{h})\), \(\psi_\alpha(n)\mathcal{F}(Lg_{g,0})^\lambda \subset \mathcal{F}(Lg_{g,0})^{\lambda+\alpha+n\delta}\), \(\psi^\alpha(n)\mathcal{F}(Lg_{g,0})^\lambda \subset \mathcal{F}(Lg_{g,0})^{\lambda-\alpha+n\delta}\), where \(\alpha \in \Delta_{g,0}\) and \(n \in \mathbb{Z}\).

3.5. For \(V \in \text{Obj}_{\mathcal{O}_k}\), define
\[C(V) = \text{def} \mathcal{F}(\chi) \otimes \mathcal{F}(Lg_{g,0}) = \sum_{i \in \mathbb{Z}} C^i(V),\]
where \(C^i(V) = \text{def} \mathcal{F}(\chi) \otimes \mathcal{F}^i(Lg_{g,0}).\)

Let \(\hat{h}\) act on \(C(V)\) by the tensor product action. Then, we have \(C(V) = \bigoplus_{\lambda \in \hat{h}^*} C(V)^\lambda\), where \(C(V)^\lambda = \sum_{\mu_1 + \mu_2 + \mu_3 = \lambda} V^{\mu_1} \otimes \mathcal{F}(\chi)^{\mu_2} \otimes \mathcal{F}(Lg_{g,0})^{\mu_3}\). Note that we also have
\[C(V) = \bigoplus_{\mu \leq \lambda} C(V)^\mu\]
with an object \(V\) of \(\text{Ob}_{\mathcal{O}_k}\).

3.6. Define the odd operator \(d\) on \(C(V)\) by
\[d = \text{def} \sum_{\alpha \in \Delta_{g,0}} (-1)^{p(\alpha)} (u_\alpha(-n) + \Phi_{u_\alpha}(-n))\psi^\alpha(n)\]
\[+ \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{g,0} \atop k + l + m = 0} (-1)^{p(\alpha)p(\gamma)} ([u_\alpha, u_\beta], u_{-\gamma})\psi^\alpha(k)\psi^\beta(l)\psi^\gamma(m).\]
Then, we have
\[d^2 = 0, \quad dC^i(V) \subset C^{i+1}(V).\]
Thus, \((C(V), d)\) is a cohomology complex. Define
\[
H^i(V) = H^i(C(V), d) \text{ with } i \in \mathbb{Z}.
\]

**Remark 3.6.1.** By definition, we have
\[
H^\bullet(V) = H^\bullet_\infty(L\mathfrak{g}_{>0}, V \otimes F^{\text{ne}}(\chi)),
\]
where \(H^\bullet_\infty(L\mathfrak{g}_{>0}, V)\) is the semi-infinite cohomology \([9]\) of the Lie superalgebra \(L\mathfrak{g}_{>0}\) with coefficients in \(V\).

### 3.7. Decompose \(d\) as \(d = d^x + d^{st}\), where
\[
d^x = \sum_{\alpha \in \Delta^t} (x)^{p(\alpha)} \Phi_\alpha(n) \phi^\alpha(-n) + \sum_{\alpha \in \Delta^t} (-1)^{p(\alpha)} \chi(u_\alpha(-1)) \phi^\alpha(1)
\]
and \(d^{st} = d - d^x\). Then, by Lemma 3.2.1 we have
\[
d^{x} C(V)^\lambda \subset \sum_{\alpha \in \Delta^t} C(V)^{\lambda - \alpha + \delta}, \quad d^{st} C(V)^\lambda \subset C(V)^\lambda
\]
for all \(\lambda\). Therefore, by [22], it follows that
\[
(d^x)^2 = (d^{st})^2 = \{d^x, d^{st}\} = 0.
\]

**Remark 3.7.1.** We have
\[
H^\bullet(C(V), d^{st}) = H^\bullet_\infty(L\mathfrak{g}_{>0}, V \otimes F^{\text{ne}}(\chi_0)),
\]
where \(F^{\text{ne}}(\chi_0)\) is the \(L\mathfrak{g}_{>0}\)-module associated with the trivial character \(\chi_0\) of \(L\mathfrak{g}_{>1}\) defined similarly to \(F^{\text{ne}}(\chi)\).

### 3.8. Define
\[
\text{Def} \quad D^W = x + D \in \hat{\mathfrak{h}},
\]
Here, \(x\) is the semisimple element in the \(\mathfrak{sl}_2\)-triple, as in Subsection 2.1. Set
\[
\hat{t} = \mathfrak{h}^* + CD^W \subset \hat{\mathfrak{h}}.
\]
Let \(\hat{t}^*\) be the dual of \(\hat{t}\). For \(\lambda \in \hat{t}^*\), let \(\xi_\lambda \in \hat{t}^*\) denote its restriction to \(\hat{t}\).

### 3.9. Let \(V \in \text{Obj}\mathcal{O}_\mathfrak{k}\) and let
\[
C(V) = \bigoplus_{\xi \in \hat{t}^*} C(V)_{\xi}, \quad C(V)_\xi = \sum_{\lambda \in \hat{t}^*} C(V)^\lambda
\]
be the weight space decomposition with respect to the action of \(\hat{t} \subset \hat{\mathfrak{h}}\). Here and throughout, we set
\[
M_{\xi} = \{m \in M \mid tm = \langle \xi, t \rangle m \text{ for } \forall t \in \hat{t} \}\}
\]
for a \(t\)-module \(M\). By [20], we see that
\[
dC(V)_\xi \subset C(V)_{\xi} \text{ for any } \xi \in \hat{t}^*.
\]
Hence, the cohomology space \(H^\bullet(V)\) decomposes as
\[
H^\bullet(V) = \bigoplus_{\xi \in \hat{t}^*} H^\bullet(V)_{\xi}, \quad H^\bullet(V)_{\xi} = H^\bullet(C(V)_{\xi}, d).
\]
Note that the weight space $C(V)_{\xi}$, with $\xi \in \hat{\tau}^*$, is not finite dimensional in general because $[\hat{t}, e(-1)] = 0$.

**Remark 3.9.1.** As discussed in Remark 5.3.1, the operator $D^W$ is essentially $-L(0)$, where $L(0)$ is the zero-mode of the Virasoro field, provided that $k \neq -\hbar^\vee$.

4. **The assumption $f = f_\theta$**

4.1. The gradation (2) is called *minimal* if

$$g = g_{-1} \oplus g_{-\frac{1}{2}} \oplus g_0 \oplus g_{\frac{1}{2}} \oplus g_1, \quad g_{-1} = \mathbb{C} f \quad \text{and} \quad g_1 = \mathbb{C} e.$$ 

As shown in Section 5 of Ref. [19], in this case, one can choose a root system of $g$ so that $e = e_\theta$ and $f = f_\theta$, which are the roots vectors attached to $\theta$ and $-\theta$, where $\theta$ is the corresponding highest root.

The condition (32) simplifies the theory in many ways. *In this section we assume that $f = f_\theta$ and the condition (32) is satisfied.* Also, we normalize $(\mid \ )$ as $(\theta \mid \theta) = 2$.

4.2. From the $\mathfrak{sl}_2$-representation theory, we have

$$g^f = g_{-1} \oplus g_{-\frac{1}{2}} \oplus g_0 \oplus g_{\frac{1}{2}} \oplus g_1,$$



$$g^f_0 = n_{0, -} \oplus \mathfrak{h}^f \oplus n_{0, +}.$$ 

In particular,

$$\mathfrak{h} = \mathfrak{h}^f \oplus \mathbb{C} x, \quad n_{-} \subset g^f,$$

and we have the exact sequence

$$0 \rightarrow \mathbb{C} a_0 \oplus \mathbb{C} A_0 \rightarrow \hat{\mathfrak{h}}^* \rightarrow \hat{\tau}^* \rightarrow 0$$

Here, $a_0 = \delta - \theta$. Therefore, for $\lambda, \mu \in \hat{h}^*_L$,

$$\xi_\lambda = \xi_\mu \text{ if and only if } \lambda \equiv \mu \text{ (mod } a_0).$$

Let $\hat{Q}^L_+ \subset \hat{\tau}^*$ be the image of $\hat{Q}_+ \subset \hat{\mathfrak{h}}^*$ in $\hat{\tau}^*$. Then, by (36) we have

$$\langle \eta, D^W \rangle \geq 0 \quad \text{for all } \eta \in \hat{Q}^L_+.$$ 

Define a partial ordering on $\hat{\tau}^*$ by $\xi \leq \xi' \iff \xi' - \xi \in \hat{Q}^L_+$. Then, for $\lambda, \mu \in \hat{h}^*$, we have the property

$$\xi_\lambda \leq \xi_\mu \text{ if } \lambda \leq \mu.$$ 

In particular,

$$V = \bigoplus_{\xi \leq \xi_\lambda} V_\xi \text{ for an object } V \text{ of } \mathcal{O}_k^{\leq \lambda}.$$
4.3. Let \( \widehat{\mathfrak{g}} = \bigoplus_{\eta \in \mathfrak{g}} (\widehat{\mathfrak{g}})_\eta \) be the weight space decomposition with respect to the adjoint action of \( \hat{\mathfrak{k}} \). Then, we have
\[
(\widehat{\mathfrak{g}})_0 = \hat{\mathfrak{h}} \oplus \mathbb{C} e(-1) \oplus \mathbb{C} f(1)
\]
(recall that \( e = e_0 \) and \( f = f_0 \)).

Let \( V \in \mathcal{O}_k \). Then, each weight space \( V_\xi \), where \( \xi \in \hat{\mathfrak{t}}^* \), is a module over \( (\widehat{\mathfrak{g}})_0 \).

Also, we have
\[
V_\xi = \sum_{\mu, \eta \in \hat{\mathfrak{h}}^*_k \ (\text{mod} \ n_0)} V_\mu \text{ for } \lambda \in \hat{\mathfrak{h}}^*_k.
\]

Let \( (\mathfrak{sl}_2)_0 \cong \mathfrak{sl}_2 \) denote the subalgebra of \( (\widehat{\mathfrak{g}})_0 \) generated by \( e(-1) \) and \( f(1) \).

4.4. Let \( \mathcal{O}(\mathfrak{sl}_2) \) be the Bernstein-Gel’fand-Gel’fand category \[3\] of the Lie algebra \( \mathfrak{sl}_2 = \langle e, x, f \rangle \) (defined by the commutation relations \[11\]). That is, \( \mathcal{O}(\mathfrak{sl}_2) \) is the full subcategory of the category of left \( \mathfrak{sl}_2 \)-modules consisting of modules \( V \) such that (1) \( V \) is finitely generated over \( \mathfrak{sl}_2 \), (2) \( e \) is locally nilpotent on \( V \), (3) \( x \) acts semisimply on \( V \).

Let \( \mathcal{O}_k \) be the full subcategory of \( \mathcal{O}_k \) consisting of objects \( V \) such that each \( V_\xi \), with \( \xi \in \hat{\mathfrak{t}}^* \), belongs to \( \mathcal{O}(\mathfrak{sl}_2) \) (viewed as a module over \( (\mathfrak{sl}_2)_0 \cong \mathfrak{sl}_2 \)). It is clear that \( \mathcal{O}_k \) is abelian.

**Lemma 4.4.1.**

1. Any Verma module \( M(\lambda) \), with \( \lambda \in \hat{\mathfrak{h}}^*_k \), belongs to \( \mathcal{O}_k \).
2. Any simple module \( L(\lambda) \), with \( \lambda \in \hat{\mathfrak{h}}^*_k \), belongs to \( \mathcal{O}_k \).
3. Any object of \( \mathcal{O}_k^\Delta \) belongs to \( \mathcal{O}_k \).
4. Any object of \( \mathcal{O}_k^\gamma \) belongs to \( \mathcal{O}_k \).

**Proof.** (1) Certainly, on \( M(\lambda) \), \( f(1) \) is locally nilpotent and \( h_0 = [f(1), e(-1)] \) acts semisimply. We have to show that each \( M(\lambda)_\xi \), with \( \xi \in \hat{\mathfrak{t}}^* \), is finitely generated over \( (\mathfrak{sl}_2)_0 \). Clearly, this follows from (37), (39) and the PBW theorem. (2), (3) These assertions follow from the first assertion. (4) The category \( \mathcal{O}(\mathfrak{sl}_2) \) is closed under the operation of taking (graded) dual. Hence, \( \mathcal{O}_k \) is closed under the operation of taking (graded) dual. Therefore (4) follows from the third assertion.

Let \( \mathcal{O}_k^{<\lambda} \) be the full subcategory of \( \mathcal{O}_k \) consisting of objects that belong to both \( \mathcal{O}_k \) and \( \mathcal{O}_k^{\leq \lambda} \). Then, by Lemma 4.4.1, \( P_{\leq \lambda}(\mu) \) and \( I_{\leq \lambda}(\mu) \) \( (\mu \leq \lambda) \) belong to \( \mathcal{O}_k^{<\lambda} \).

The following proposition asserts that every object of \( \mathcal{O}_k \) can be obtained as an injective limit of objects of \( \mathcal{O}_k \).

**Proposition 4.4.2.** Let \( V \) be an object of \( \mathcal{O}_k \). Then, there exits a sequence \( V_1 \subset V_2 \subset V_3 \ldots \) of objects of \( \mathcal{O}_k \) such that \( V = \bigcup_i V_i \).

**Proof.** Note that, because each projective module \( P_{\leq \lambda}(\mu) \) belongs to \( \mathcal{O}_k \), so too do finitely generated objects. Let \( \{0\} = V_0 \subset V_1 \subset V_2 \subset V_3 \ldots \) be a highest weight filtration of \( V \), so that \( V = \bigcup_i V_i \), and each successive subquotient \( V_i / V_{i-1} \) is a highest weight module. In particular, each \( V_i \) is finitely generated, and hence belongs to \( \mathcal{O}_k \).

**Lemma 4.4.3.** Let \( \xi \in \hat{\mathfrak{t}}^* \). Then, for any object \( V \) of \( \mathcal{O}_k^{<\lambda} \), with \( \lambda \in \hat{\mathfrak{h}}^*_k \), there exits a finitely generated submodule \( V' \) of \( V \) such that \( (V/V')_{\xi'} = \{0\} \) if \( \xi' \geq \xi \), where \( \xi' \in \hat{\mathfrak{t}}^* \).
Lemma 4.5.1. Let \( \mathcal{P} = \{v_1, v_2, \ldots\} \) be a set of generators of \( V \) such that (1) each \( v_i \) belongs to \( V^{\mu_i} \) for some \( \mu_i \in \hat{h}^* \), and (2) if we set \( V_i = \sum_{r=1}^i U(\hat{g})v_r \) (and \( V_0 = \{0\} \)), then each successive subquotient \( V_i/V_{i-1} \) is a (nonzero) highest weight module with highest weight \( \mu_i \). (Therefore, \( V_1 \subset V_2 \subset \ldots \) is a highest weight filtration of \( V \).) Then, by definition, we have \( \mathcal{P} \{j \geq 1 \mid \mu_j = \mu \} \leq [V : L(\mu)] \) for \( \mu \in \hat{h}^* \). Next, let \( \mathcal{P}_{\geq \xi} = \{v_j \in \mathcal{P} \mid \xi_{\mu_j} \geq \xi\} \subset \mathcal{P} \). Then, by the definition of \( \mathcal{O}_k \), \( \mathcal{P}_{\geq \xi} \) is a finite subset of \( \mathcal{P} \). The assertion follows, because \( V' = \sum_{\nu \in \mathcal{P}_{\geq \xi}} U(\hat{g})v \subset V \) satisfies the desired properties. \( \Box \)

4.5. Observe that we have the following:

(43) \( \mathcal{F}^{ne}(\chi) = \bigoplus_{\xi \leq 0} \mathcal{F}^{ne}(\chi)_{\xi} \), \( \dim_{\mathbb{C}} \mathcal{F}^{ne}(\chi)_{\xi} < \infty \) (\( \forall \chi \)), \( \mathcal{F}^{ne}(\chi)_{0} = \mathbb{C}1_{\chi} \),

(44) \( \mathcal{F}(Lg_{>0}) = \bigoplus_{\xi \leq 0} \mathcal{F}(Lg_{>0})_{\xi}, \mathcal{F}(Lg_{>0})_{0} = \mathbb{C}1 \oplus \mathbb{C}\psi(1) 1 \)

Thus, if \( V = \bigoplus_{\xi' \leq \xi} \hat{V}_{\xi'} \), for some \( \xi' \in \hat{r}^* \), then \( C(V) = \bigoplus_{\xi' \leq \xi} C(V)_{\xi'} \). Hence we have the following assertion.

Lemma 4.5.1. Let \( V \) be an object of \( \mathcal{O}_k \). Suppose that \( V = \bigoplus_{\xi' \leq \xi} V_{\xi'} \) for some \( \xi \in \hat{r}^* \). Then, \( H^\bullet(V) = \bigoplus_{\xi' \leq \xi} H^\bullet(V)_{\xi'} \). In particular, \( H^\bullet(V) = \bigoplus_{\xi \leq \xi} H^\bullet(V)_{\xi} \) for \( V \in \text{Obj} \mathcal{O}^{\leq}_{k} \).

Lemma 4.5.2. Let \( \xi \in \hat{r}^* \). Then, for any object \( V \) of \( \mathcal{O}^{\leq}_{k} \), where \( \lambda \in \hat{h}^* \), there exists a finitely generated submodule \( V' \) of \( V \) such that \( H^\bullet(V)_{\xi} = H^\bullet(V')_{\xi} \).

Proof. Let \( V' \) be as in Lemma 4.4.3 Then, from the exact sequence \( 0 \to V' \to V \to V/V' \to 0 \), we obtain the long exact sequence

(45) \( \cdots \to H^{i-1}(V/V') \to H^i(V') \to H^i(V) \to H^i(V/V') \to \cdots \)

Clearly, the restriction of (45) to the weight space \( \xi \) remains exact. The desired result then follows from Lemma 4.4.3 and Lemma 4.5.1. \( \Box \)

4.6. Here and throughout, we identify \( \mathcal{F}(Lg_{>0})_{0} \) with the exterior power module \( \Lambda(\mathbb{C}e(-1)) \) by identifying \( \psi(1) \) with \( e(-1) \) (see (44)). Let \( \mathbb{C}_{\chi} \) be the one-dimensional module over the commutative Lie algebra \( \mathbb{C}e(-1) \) defined by the character \( \chi(\mathbb{C}e(-1)) \), that is, the one-dimensional \( \mathbb{C}e(-1) \)-module on which \( e(-1) \) acts as the identity operator. Also, let \( V \) be an object of \( \mathcal{O}_k \). Then, \( \hat{V}_{\xi} \otimes \mathbb{C}_{\chi} \), where \( \xi \in \hat{r}^* \), is a module over \( \mathbb{C}e(-1) \) by the tensor product action.

Lemma 4.6.1. Let \( V \in \text{Obj} \mathcal{O}^{\leq}_{k}, \) with \( \lambda \in \hat{h}^* \). Then, we have

\[
H^i(V)_{\xi_\lambda} = \begin{cases} H_{-i}(\mathbb{C}e(-1), V_{\xi_\lambda} \otimes \mathbb{C}_{\chi}) & (i = 0, -1), \\ \{0\} & (\text{otherwise}). \end{cases}
\]

Proof. Because \( V \) is an object of \( \mathcal{O}^{\leq}_{k} \), we have

\[
C(V)_{\xi_\lambda} = V_{\xi_\lambda} \otimes \mathcal{F}(Lg_{>0})_{0} (= V_{\xi_\lambda} \otimes \Lambda(\mathbb{C}e(-1))).
\]

Next, observe that

\[
d_{C(V)_{\xi_\lambda}} = \bar{d} = e(-1)\psi(1) + \psi(1).
\]
From this, it follows that, for $V \in \text{Obj} \mathcal{O}_{\hat{h}}^\leq \lambda$, the subcomplex $(C(V)_{\xi_i}, d)$ is identically the Chevalley complex for calculating the (usual) Lie algebra homology $H_*((C(-1), V_{\xi_i} \otimes \mathbb{C})$ (with the opposite grading).

4.7. Recall $\mathfrak{sl}_2 = \{e, x, f\}$. Let $M_{\mathfrak{sl}_2}(a) \in \text{Obj} \mathcal{O}(\mathfrak{sl}_2)$ be the Verma module of $\mathfrak{sl}_2$ with highest weight $a \in \mathbb{C}$ and $L_{\mathfrak{sl}_2}(a)$ be its unique simple quotient. Here, the highest weight is the largest eigenvalue of $2x$ (see [1]).

Let $\mathbb{C}_{\lambda -}$ be the one-dimensional $\mathbb{C}f$-module on which $f$ acts as the identity operator.

**Proposition 4.7.1.**

1. For $a \in \mathbb{C}$, $H_i(Cf, M_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\lambda -}) = \begin{cases} \mathbb{C} & (i = 0), \\
\{0\} & (i = 1). \end{cases}$

2. For $a \in \mathbb{C}$, $H_i(Cf, \bar{L}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\lambda -}) = \begin{cases} \mathbb{C} & (i = 0 \text{ and } a \notin \{0, 1, 2, \ldots\}), \\
\{0\} & \text{(otherwise)}. \end{cases}$

3. For $a \in \mathbb{C}$, $H_i(Cf, M_{\mathfrak{sl}_2}(a)^* \otimes \mathbb{C}_{\lambda -}) = \begin{cases} \mathbb{C} & (i = 0), \\
\{0\} & (i = 1). \end{cases}$

4. For any object $V$ of $\mathcal{O}(\mathfrak{sl}_2)$, we have $H_i(Cf, V \otimes \mathbb{C}_{\lambda -}) = \{0\}.$

5. For any object $V$ of $\mathcal{O}(\mathfrak{sl}_2)$, we have $\dim_{\mathbb{C}} H_0(Cf, V \otimes \mathbb{C}_{\lambda -}) < \infty.$

**Proof.** (1) Since $\bar{M}_{\mathfrak{sl}_2}(a)$ is free over $\mathbb{C}f$, so is $\bar{M}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\lambda -}. (2)$ The case in which $a \notin \{0, 1, 2, \ldots\}$ follows from the first assertion. Otherwise, $\bar{L}_{\mathfrak{sl}_2}(a)$ is finite dimensional. Hence, $f$ is nilpotent on $\bar{L}_{\mathfrak{sl}_2}(a). But this implies that the corresponding Chevalley complex is acyclic, by the argument of Theorem 2.3 of Ref. [13]. (3) The case in which $a \notin \{0, 1, 2, \ldots\}$ follows from the first assertion. Otherwise, we have the following exact sequence in $\mathcal{O}(\mathfrak{sl}_2)$:

$$0 \to L_{\mathfrak{sl}_2}(a) \to \bar{M}_{\mathfrak{sl}_2}(a)^* \to \bar{M}_{\mathfrak{sl}_2}(-a - 2) \to 0.$$ 

This induces the exact sequence

$$0 \to L_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\lambda -} \to M_{\mathfrak{sl}_2}(a)^* \otimes \mathbb{C}_{\lambda -} \to \bar{M}_{\mathfrak{sl}_2}(-a - 2) \otimes \mathbb{C}_{\lambda -} \to 0.$$ 

Hence, the assertion is obtained from the first and the second assertions by considering the corresponding long exact sequence of the Lie algebra homology. (4) Recall that $\mathcal{O}(\mathfrak{sl}_2)$ has enough injectives, and each injective object $I$ admits a finite filtration such that each successive quotient is isomorphic to $\bar{M}_{\mathfrak{sl}_2}(a)^*$ for some $a \in \mathbb{C}$. Therefore, the third assertion implies that $H_1(Cf, I \otimes \mathbb{C}_{\lambda -}) = \{0\}$ for any injective object $I$ in $\mathcal{O}(\mathfrak{sl}_2)$ (cf. Theorem 8.2 of [1]). For a given $V \in \text{Obj} \mathcal{O}(\mathfrak{sl}_2)$, let $0 \to V \to I \to V/I \to 0$ be an exact sequence in $\mathcal{O}(\mathfrak{sl}_2)$ such that $I$ is injective. Then, from the associated long exact sequence, it is proved that $H_1(Cf, V \otimes \mathbb{C}_{\lambda -}) = \{0\}.$ (4) By the third assertion the correspondence $V \mapsto H_0(Cf, V \otimes \mathbb{C}_{\lambda -})$ defines an exact functor from $\mathcal{O}(\mathfrak{sl}_2)$ to the category of $\mathbb{C}$-vector spaces. Because the assertion follows from the first assertion for Verma modules, it also holds for any projective object $P$ of $\mathcal{O}(\mathfrak{sl}_2)$, since $P$ has a (finite) Verma flag. This completes the proof, as $\mathcal{O}(\mathfrak{sl}_2)$ has enough projectives. 

4.8. For $\lambda \in \hat{h}^*$, let

$$|\lambda| = v_\lambda \otimes 1 \in C(M(\lambda)).$$
Then, $d(\lambda) = 0$, and thus $|\lambda|$ defines an element of $H^0(M(\lambda))$. Again with a slight abuse of notation, we denote the image of $|\lambda|$ under the natural map $C(M(\lambda)) \to C(L(\lambda))$ by $|\lambda|$. Also, let

\[(46) \quad |\lambda|^* = v^*_\lambda \otimes 1 \in H^0(M(\lambda)^*),\]

where $v^*_\lambda$ is the vector of $M(\lambda)^*$ dual to $v_\lambda$.

**Proposition 4.8.1.** For any $\lambda \in T^*$, we have the following:

1. $H^i(M(\lambda))_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda| & (\text{if } i = 0), \\ \{0\} & (\text{otherwise}). \end{cases}$

2. $H^i(L(\lambda))_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda| & (\text{if } i = 0 \text{ and } (\lambda, \alpha_0^\vee) \notin \{0, 1, 2, \ldots\}), \\ \{0\} & (\text{otherwise}). \end{cases}$

3. $H^i(M(\lambda)^*)_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda|^* & (\text{if } i = 0), \\ \{0\} & (\text{otherwise}). \end{cases}$

**Proof.** Observe that $F$ space of $\mathfrak{g}$, is finite dimensional. Moreover, if $\xi \neq 0$, then $H^i(\mathfrak{g}) = \{0\}$. Thus, $H^i(\mathfrak{g})_{\xi_\lambda} = \{0\}$ if $\lambda \neq 0$. Hence, from (40) and Proposition 4.8.1, we obtain the following assertion.

4.9. Let $V$ be an object of $\hat{O}_k$. For a given $\xi \in T^*$, consider the (usual) Lie algebra homology $H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\xi)$. It is calculated using the Chevalley complex $(V \otimes \Lambda(\mathfrak{g}(-1)), \partial)$, (see Section 4.6). The action of $\partial$ on $V$ commutes with $\partial$. Thus, there is a natural action of $\partial$ on $H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi)$:

$$H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi) = \bigoplus_{\xi \in T^*} H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi)_{\xi}.$$  

By definition, we have

\[(47) \quad H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi)_\xi = H_*(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi) \quad \text{for} \ \xi \in T^*.
\]

Hence, from (40) and Proposition 4.8.1, we obtain the following assertion.

**Proposition 4.9.1.** Let $V$ be an object of $\hat{O}_k$. Then, we have the following:

1. $H_1(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi) = \{0\}$.
2. $H_0(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi) = \bigoplus_{\lambda \leq \xi_\lambda} H_0(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi)_{\xi}$ if $V \in \text{Obj} \hat{O}_k^{\leq \lambda}$ with $\lambda \in \hat{T}^*$.
3. Each weight space $H_0(\mathfrak{g}(-1), V \otimes \mathbb{C}_\chi)_{\xi}$, $\xi \in T^*$, is finite dimensional.

4.10. We end this section with the following important proposition.

**Proposition 4.10.1.** For any object $V$ of $\hat{O}_k$, each weight space $H^*(V)_{\xi}$, where $\xi \in T^*$, is finite dimensional. Moreover, if $V \in \text{Obj} \hat{O}_k^{\leq \lambda}$, with $\lambda \in \hat{T}^*$, then $H^i(V)_{\xi} = \{0\}$ if $\frac{1}{2}||i| > (\xi_\lambda - \xi, D^W)$ for $i \in \mathbb{Z}$.

**Proof.** We may assume that $V \in \text{Obj} \hat{O}_k^{\leq \lambda}$ for some $\lambda \in \hat{T}^*$. Decompose $\mathcal{F}(Lg_{>0})$ as $\mathcal{F}(Lg_{>0}) = \mathcal{F}(Lg_{>0}/\mathfrak{g}(-1)) \otimes \Lambda(\mathfrak{g}(-1))$, where $\mathcal{F}(Lg_{>0}/\mathfrak{g}(-1))$ is the sub-space of $\mathcal{F}(Lg_{>0})$ spanned by the vectors $\psi_{\alpha_1}(m_1) \ldots \psi_{\alpha_r}(m_r) \psi^{\beta_1}(n_1) \ldots \psi^{\beta_s}(n_s) 1$, where $\psi_{\alpha}(m)$ is the $(\alpha, m)$-weight space of $\mathfrak{g}$...
with \( \alpha_i, \beta_i \in \Delta_{>0}, m_i \leq \begin{cases} -2 & (\text{if } \alpha_i = \theta), \\ -1 & (\text{otherwise}), \end{cases} n_i \leq 0 \). Then, we have

\[
F^n(L\mathfrak{g}_{>0}) = \sum_{i-j=n} F^i(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) \otimes \Lambda^j(C\mathfrak{e}(-1)),
\]

where \( F^i(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) = F(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) \cap F^j(L\mathfrak{g}_{>0}) \).

Now, define

\[
(48) \quad G^n C^n(V) = V \otimes F^{ne}(\chi) \otimes \sum_{i-j=n, i \geq p} F^i(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) \otimes \Lambda^j(C\mathfrak{e}(-1)) \subset C^n(V).
\]

Then, we have

\[
C^n(V) = G^n C^n(V) \supset G^{n+1} C^n(V) \supset G^{n+2} C^n(V) = \{0\},
\]

\[
d G^n C^n(V) \subset G^n C^{n+1}(V).
\]

The corresponding spectral sequence, \( E_r \Rightarrow H^*(V) \), is the Hochschild-Serre spectral sequence (more precisely, the semi-infinite, Lie superalgebra analogue of this spectral sequence) for the ideal \( C\mathfrak{e}(-1) \subset L\mathfrak{g}_{>0} \). By definition, we have the isomorphism

\[
E^p_1 = H^{-q}(C\mathfrak{e}(-1), V \otimes F^{ne}(\chi) \otimes F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)),
\]

because the complex \( \sum_p G^p C(V)/G^{p+1} C(V), d \) is identical to the corresponding Chevalley complex. By Proposition 4.9.1 (1), we have

\[
(49) \quad E^p_1 \cong \begin{cases} H_0(C\mathfrak{e}(-1), V \otimes C\chi) \otimes F^{ne}(\chi) \otimes F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) & (q = 0) \\ \{0\} & (q \neq 0) \end{cases}
\]

as \( \hat{\mathfrak{i}} \)-modules for any \( p \).

Next, observe that

\[
F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1)) = \bigoplus_{\xi \leq 0} F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1))_\xi,
\]

\[
\dim_C F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1))_\xi < \infty \quad \text{for any } \xi,
\]

\[
F^p(L\mathfrak{g}_{>0}/C\mathfrak{e}(-1))_\xi = \{0\} \text{ unless } (\xi, \mathfrak{D}^W) \leq \frac{1}{2}|p|.
\]

Hence, from \( 48 \), Proposition 4.9.1 and (49), it follows that

\[
(50) \quad E^{p,0}_1 = \bigoplus_{\xi \leq 0} (E^{p,0}_1)_\xi, \quad \dim_C (E^{p,0}_1)_\xi < \infty \quad \text{for any } \xi.
\]

The assertion is thus proved, as our filtration is compatible with the action of \( \hat{\mathfrak{i}} \).

\[ \square \]

5. The Kac-Roan-Wakimoto construction II: the \( \mathcal{W} \)-algebra construction of superconformal algebras

In this section we recall the definition of the \( \mathcal{W} \)-algebra \( \mathcal{W}_k(\mathfrak{g}, f) \) and collect necessary information about its structure.
5.1. Let \( V_k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t] \otimes \mathbb{C}K \otimes \mathbb{C}D)} \mathbb{C}_k \in \text{Ob}_k \) be the universal affine vertex algebra associated with \( \mathfrak{g} \) at a given level \( k \in \mathbb{C} \). Here, \( \mathbb{C}_k \) is the one-dimensional representation of \( \mathfrak{g} \otimes \mathbb{C}[t] \otimes \mathbb{C}K \otimes \mathbb{C}D \) on which \( \mathfrak{g} \otimes \mathbb{C}[t] \otimes \mathbb{C}D \) acts trivially and \( K \) acts as \( k \) \text{id}. Hence, \( V_k(\mathfrak{g}) \) is a quotient of \( M(k\Lambda_0) \) as a \( \hat{\mathfrak{g}} \)-module. It is known that the space \( V_k(\mathfrak{g}) \) has a natural vertex (super)algebra structure, and the space

\[
C(V_k(\mathfrak{g})) = V_k(\mathfrak{g}) \otimes F^\text{ne}(\chi) \otimes F(L_{g>0})
\]

also has a natural vertex (super)algebra structure (see Ref. [17] for details). Let \( |0\rangle = (1 \otimes 1) \otimes 1 \otimes 1 \) be the canonical vector. Also, let \( Y(v, z) \in \text{End}(C(V_k(\mathfrak{g}))[z, z^{-1}]) \) be the field corresponding to \( v \in C(V_k(\mathfrak{g})) \). Then, by the definition,

\[
Y(v(-1)|0\rangle, z) = v(z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1} \quad \text{for} \quad v \in \mathfrak{g},
\]

\[
Y(\Phi_\alpha(-1)|0\rangle, z) = \Phi_\alpha(z) = \sum_{n \in \mathbb{Z}} \Phi_\alpha(n)z^{-n-1} \quad \text{for} \quad \alpha \in \Delta_\mathbb{Z}.
\]

\[
Y(\psi_\alpha(-1)|0\rangle, z) = \psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n)z^{-n-1} \quad \text{for} \quad \alpha \in \Delta_{>0},
\]

\[
Y(\psi_{-\alpha}(0)|0\rangle, z) = \psi_{-\alpha}(z) = \sum_{n \in \mathbb{Z}} \psi_{-\alpha}(n)z^{-n} \quad \text{for} \quad \alpha \in \Delta_{>0}.
\]

5.2. Define

\[
W_k(\mathfrak{g}, f) = \text{def} \ H^0(V_k(\mathfrak{g})).
\]

Then, \( Y \) descends to the map

\[
Y : W_k(\mathfrak{g}, f) \to \text{End} \ W_k(\mathfrak{g}, f)[[z, z^{-1}]],
\]

because, as shown in Ref. [17], the following relation holds:

\[
[d, Y(v, z)] = Y(dv, z) \quad \text{for all} \quad v \in C(V_k(\mathfrak{g})).
\]

Therefore, \( W_k(\mathfrak{g}, f) \) has a vertex algebra structure. The vertex algebra \( W_k(\mathfrak{g}, f) \) is called the \( W \)-algebra associated with the pair \( (\mathfrak{g}, f) \) at level \( k \). By definition, the vertex algebra \( W_k(\mathfrak{g}, f) \) acts naturally on \( H^i(V) \), where \( V \in \mathcal{O}_k \), \( i \in \mathbb{Z} \). Thus, we obtain a family of functors \( V \rightsquigarrow H^i(V) \), depending on \( i \in \mathbb{Z} \), from \( \mathcal{O}_k \) to the category of \( W_k(\mathfrak{g}, f) \)-modules.

**Remark 5.2.1.**

1. If \( \mathfrak{g} \) is a Lie algebra and \( f \) is a regular nilpotent element of \( \mathfrak{g} \), then \( W_k(\mathfrak{g}, f) \) is identical to \( W_k(\mathfrak{g}) \), the \( W \)-algebra defined by B. L. FeĂgin and E. V. Frenkel [10].

2. V. G. Kac, S.-S. Roan and M. Wakimoto gave a more general definition of \( W \)-algebras (see Ref. [17] for details).

5.3. As shown in Ref. [17], the vertex algebra \( W_k(\mathfrak{g}, f) \) has a superconformal algebra structure provided that the level \( k \) is non-critical, i.e., that \( k + h^\vee \neq 0 \). Here, \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \). Let \( L(z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \) be the corresponding Virasoro field. The explicit form of \( L(z) \) is given in Ref. [17]. If \( f = f_0 \), its central charge is given by

\[
e(k) = \frac{k \text{dim} \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.
\]
Let
\[ S(z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-2} = 2(k + h^\vee) L(z). \]
Then, \( S(z) \) is well-defined for any level \( k \).

**Remark 5.3.1.** Let \( \hat{\Omega} \) be the universal Casimir operator [15] of \( \hat{\mathfrak{g}} \) acting on \( V \in \mathcal{O}_k \). Then, we have
\[ S(0) + 2(k + h^\vee) \mathcal{D}^W = \hat{\Omega} \]
on \( H^*(V) \).

5.4. Let
\[ J^{(v)}(z) = \sum_{n \in \mathbb{Z}} J^{(v)}(n) z^{-n-1} = v(z) + \sum_{\beta, \gamma \in \Delta_{>0}} (-1)^{p(\gamma)} [v, u_\beta] u_{-\gamma} \psi_\gamma(z) \psi^\beta(z), \]
for \( v \in \mathfrak{g}_{\leq 0} \).

Also, let \( C_k(\mathfrak{g}) \) be the subspace of \( C(V_k(\mathfrak{g})) \) spanned by the vectors
\[ J^{(u_1)}(m_1) \cdots J^{(u_p)}(m_p) \Phi_{\alpha_1}(n_1) \cdots \Phi_{\alpha_q}(n_q) \psi^{\beta_1}(s_1) \cdots \psi^{\beta_r}(s_r) |0\]
with \( u_i \in \mathfrak{g}_{\leq 0}, \alpha_i \in \Delta_\pm, \beta_i \in \Delta_{>0}, m_i, n_i, s_i \in \mathbb{Z} \). As shown in Ref. [19], \( C_k(\mathfrak{g}) \) is a vertex subalgebra and a subcomplex of \( C(V_k(\mathfrak{g})) \). Moreover, it was proved that
\[ W_k(\mathfrak{g}, f) = H^0(C_k(\mathfrak{g}), d) \]
as vertex algebras. This follows from the tensor product decomposition of the complex \( C(V_k(\mathfrak{g})) \) (see Ref. [19] for details).

5.5. Let
\[ \hat{\mathfrak{g}}^f = \mathfrak{g}^f \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \]
be the affine Lie superalgebra of \( \mathfrak{g}^f \) with respect to the 2-cocycle \( (\ , \ )^2 \), defined by
\[ (u \otimes t^m, v \otimes t^n)^2 = \begin{cases} m \delta_{m,n} ((k + h^\vee)(u|v) - \frac{1}{2} \text{str}_{\mathfrak{g}_0} (\text{ad} u)(\text{ad} v)) & \text{(if } u, v \in \mathfrak{g}_0), \\ 0 & \text{(otherwise)}. \end{cases} \]

Also, let \( V_k^2(\mathfrak{g}^f) \) be the corresponding universal vertex affine algebra:
\[ V_k^2(\mathfrak{g}^f) = U(\hat{\mathfrak{g}}^f) \otimes_{U(\mathfrak{g}^f \otimes \mathbb{C}[t] \oplus \mathbb{C})} \mathbb{C}. \]
Then, the correspondence
\[ v \otimes t^n \to J^{(v)}(n) \text{ for } v \otimes t^n \in \hat{\mathfrak{g}}^f \]
defines a \( V_k^2(\mathfrak{g}^f) \)-module structure on \( C(V), V \in \mathcal{O}_k \). In particular, we have the following embedding of vertex algebras:
\[ V_k^2(\mathfrak{g}^f) \hookrightarrow C_k(\mathfrak{g}) \subset C(V_k(\mathfrak{g})). \]

**Theorem 5.5.1** (V. G. Kac and M. Wakimoto: Theorem 4.1 of Ref. [19]). There exists a filtration
\[ \{0\} = F_{-1} \mathcal{W}_k(\mathfrak{g}, f) \subset F_0 \mathcal{W}_k(\mathfrak{g}, f) \subset F_1 \mathcal{W}_k(\mathfrak{g}, f) \subset \ldots \]
of $W_k(\mathfrak{g}, f) = H^0(C_k(\mathfrak{g}), d)$ such that

$$W_k(\mathfrak{g}, f) = \bigcup_p F_p W_k(\mathfrak{g}, f),$$

where the left-hand-side of (61) denotes the span of the vectors,

$$Y_n(v)w \quad (v \in F_p W_k(\mathfrak{g}, f), \ w \in F_q W_k(\mathfrak{g}, f), n \in \mathbb{Z}),$$

and such that the map (60)

$$V_k^f(\mathfrak{g}^f) \cong \text{gr}_{\mathfrak{g}^f} W_k(\mathfrak{g}, f) = \bigoplus F_p W_k(\mathfrak{g}, f)/F_{p-1}W_k(\mathfrak{g}, f)$$

as vertex algebras and $\hat{\mathfrak{g}}$-modules.

**Remark 5.5.2.** Actually, stronger results were proved by V. G. Kac and M. Wakimoto [19]. Specifically, it was shown that $H^i(V_k(\mathfrak{g})) = H^i(C_k(\mathfrak{g}), d) = \{0\}$ $(i \neq 0)$. Their proof is based on the argument given in Ref. [12]. Further, the explicit form of $W_k(\mathfrak{g}, f)$ was obtained for the case $f = f_\theta$.

For $v \in \mathfrak{g}^f$, let $W^{(v)} \subset W_k(\mathfrak{g}, f)$ be the cocycle in $C_k(\mathfrak{g})$ corresponding to $v(-1)|0\rangle \in V_k^f(\mathfrak{g}^f)$. Let us write

$$Y(W^{(v)}, z) = \sum_{n \in \mathbb{Z}} W^{(v)}(n)z^{-n-1}. \quad (62)$$

Then, by Theorem 5.5.1, the following map defines an isomorphism of $\hat{\mathfrak{g}}$-modules:

$$U(\mathfrak{g}^f \otimes \mathbb{C}[t^{-1}]) \to W_k(\mathfrak{g}, f), \quad u_1(n_1) \cdots u_r(n_r) \to W^{(u_1)}(n_1) \cdots W^{(u_r)}(n_r)|0\rangle.$$

We remark that the map of Theorem 5.5.1 is now described by

$$F_p W_k(\mathfrak{g}, f) \quad (63)$$

$$= \text{span}\{W^{(u_1)}(n_1) \cdots W^{(u_r)}(n_r)|0\rangle \mid r \geq 0, \ n_i \in \mathbb{Z}, \ u_i \in \mathfrak{g}^f_{-s_i}, \ \sum_{i=1}^r s_i \leq p}\}.$$

**Remark 5.5.3.**

1. Let $V \in \mathcal{O}_k$. Then, we have

$$W^{(v)}(n)H^\bullet(V)_{\xi} \subset H^\bullet(V)_{\xi+n} \quad (\text{if } v(n) \in (\mathfrak{g}^f)_n).$$

Thus, in particular,

$$[D^W, W^{(v)}(n)] = (n-j)W^{(v)}(n) \quad \text{if } v \in \mathfrak{g}_{-j}^f$$

in $\text{End}(H^\bullet(V))$.

2. $W^{(f)}(n)$ coincides with $S(n-1)$ up to a nonzero multiplicative factor.

6. **Irreducible highest weight representations and their characters**

We assume that $f = f_\theta$ and the condition (62) is satisfied for the remainder of the paper.
6.2.1 defines the action of denote the Verma module with highest weight \( \phi \) module as a module over \( W \) if the above vectors respond to a highest weight module \( V \).

Lemma 6.1.1. The operator \( \hat{W}^{(u)}(n) \), with \( u(n) \in (\hat{g}^f)^+ \), is locally nilpotent on \( H^*(V) \), for \( V \in \text{Obj} \mathcal{O}_k \).

6.2. A \( \mathcal{W}_k(\mathfrak{g}, f) \)-module \( V \) is called \( \hat{Q}^+_1 \)-gradable if it admits a decomposition \( V = \bigoplus_{\xi \in \hat{Q}^+_1} V[-\xi] \) such that \( W^{(v)}(n)V[-\xi] \subset V[-\xi + \eta] \) for \( v(n) \in (\hat{g}^f)^n, \xi \in \hat{Q}^+_1 \) and \( \forall \eta \in \hat{Q}^1 \). A \( \hat{Q}^+_1 \)-gradable \( \mathcal{W}_k(\mathfrak{g}, f) \)-module \( V \) is called a highest weight module with highest weight \( \phi \in \hat{h}_W^* \) if there exists a non-zero vector \( v_\phi \) (called a highest weight vector) such that

\[ V \text{ is generated by } v_\phi \text{ over } \mathcal{W}_k(\mathfrak{g}, f), \]

\[ W^{(u)}(n)v_\phi = 0 \quad \text{if } u \otimes t^n \in (\hat{g}^f)^+, \]

\[ W^{(h)}(0)v_\phi = \phi(h)v_\phi \quad \text{if } h \in \mathfrak{h}^f, \]

\[ S(0)v_\phi = \phi(f \otimes t)v_\phi. \]

Let \( B = \{ b_j \mid j \in J \} \) be a PBW basis of \( U((\hat{g}^f)^-) \) of the form

\[ b_j = (u_{j_1} \otimes t^{n_{j_1}}) \ldots (u_{j_k} \otimes t^{n_{j_k}}). \]

Then, a highest weight module is spanned by the vectors

\[ W^{(b_j)} = W^{(u_{j_1})}(n_{j_1}) \ldots W^{(u_{j_k})}(n_{j_k})v_\phi. \]

A highest weight module \( V \) with highest weight vector \( v_\phi \) is called a Verma module if the above vectors \( W^{(b)} \), with \( b \in B \), forms a basis of \( V \) (see Ref. [19]). Let \( M(\phi) \) denote the Verma module with highest weight \( \phi \in \hat{h}_W^* \).

Remark 6.2.1. Let

\[ F_pM(\phi) = \text{span}\{ W^{(u_1)}(n_1) \ldots W^{(u_r)}(n_r)v_\phi \mid r \geq 0, n_i \in \mathbb{Z}, u_i \in \mathfrak{g}^f_{-n_i}, \sum_{i=1}^r s_i \leq p \}. \]
Theorem 6.5.1. For any object $V$ in $Obj\mathcal{O}_k^\wedge$, we have $H^i(V) = \{0\}$ ($i \neq 0$). In particular, we have $H^i(P_{\leq \lambda}(\mu)) = \{0\}$ ($i \neq 0$) for any $\lambda, \mu \in \hat{h}_k^*$ such that $\mu \leq \lambda$.

Proof. By Theorem 6.3.1, the assertion can be shown by induction applied to the length of a Verma flag of $V$ in $Obj\mathcal{O}_k^\wedge$ (cf. Theorem 8.1 of Ref. [1]). \hfill \Box

Theorem 6.5.2. For any object $V$ in $\mathcal{O}_k$ we have $H^i(V) = \{0\}$ for all $i > 0$.

Proof. We may assume that $V \in Obj\mathcal{O}_k^{\leq \lambda}$ for some $\lambda \in \hat{h}_k^*$. Also, by Proposition 4.4.2, we may assume that $V \in Obj\mathcal{O}_k^{\geq \lambda}$, since the cohomology functor commutes with injective limits. Clearly, it is sufficient to show that $H_i(V) = \{0\}$ ($i > 0$) for each $\xi \in \hat{\mathbb{T}}$. By Lemma 15.5.2 for a given $\xi$, there exists a finitely generated submodule $V' \subseteq V$ such that

$$H^i(V) = H^i(V') \quad \text{for all } i \in \mathbb{Z}.$$
Because $V'$ is finitely generated, there exists a projective module $P$ of the form
\[ \bigoplus_{i=1}^r P_{\lambda_i}(\mu_i) \] and an exact sequence $0 \to N \to P \to V \to 0$ in $\mathcal{O}^\leq_k$. Considering the corresponding long exact sequence, we obtain
\[ \cdots \to H^i(P) \to H^i(V) \to H^{i+1}(N) \to H^{i+1}(P) \to \cdots \]
Hence, it follows that $H^i(V') \cong H^{i+1}(N)$ for all $i > 0$, by Proposition 6.5.1. Therefore, we find
\[ H^i(V)_\xi \cong H^{i+1}(N)_\xi \quad \text{for all } i > 0, \]
by (69). Then, because $N \in \text{Obj}\mathcal{O}^\leq_k$, we can repeat this argument to find, for each $k > 0$, some object $N_k$ of $\mathcal{O}^\leq_k$ such that
\[ H^i(V)_\xi \cong H^{i+k}(N_k)_\xi \quad \text{for all } i > 0. \]
But by Proposition 4.10.1, this implies that $H^i(V)_\xi = \{0\}$ for $i > 0$.

6.6.

Theorem 6.6.1. For any $\lambda \in \hat{h}^*_k$, we have $H^i(M(\lambda)^*) = \{0\}$ for $i \neq 0$.

Theorem 6.6.1 is proved in Subsection 7.17.

Theorem 6.6.2. Suppose that $(\lambda, \alpha_0^\vee) \notin \{0, 1, 2, \ldots\}$. Then, any nonzero $W_k(g, f)$-submodule of $H^0(M(\lambda)^*)$ contains the canonical vector $|\lambda|^*$ of $H^0(M(\lambda)^*)$.

Theorem 6.6.2 is proved together with Theorem 7.16.1.

Remark 6.6.3. Theorem 6.6.2 holds without any restriction on $\lambda$. Indeed, it can be seen from Corollary 6.7.3 and (the proof of) Theorem 6.7.2 that if $(\lambda, \alpha_0^\vee) \in \{0, 1, 2, \ldots\}$, then $H^0(M(\lambda)^*) \cong H^0(M(r_0 \circ \lambda)^*)$, where $r_0$ is the reflection corresponding to $\alpha_0$.

The following theorem is a consequence of Theorem 6.6.1, which can be proved in the same manner as Theorem 8.1 of Ref. 1.

Theorem 6.6.4. For a given $\lambda \in \hat{h}^*_k$, $H^i(I_{\leq \lambda}(\mu)) = \{0\} \ (i \neq 0)$ for any $\lambda, \mu \in \hat{h}^*_k$ such that $\mu \leq \lambda$.

Using Theorem 6.6.4, the following assertion can be proved in the same manner as Theorem 8.5.1.

Theorem 6.6.5. For any object $V$ of $\mathcal{O}_k$ we have $H^i(V) = \{0\}$ for all $i < 0$.

6.7. Main results. From Theorems 6.6.2 and 6.6.5, we obtain the following results.

Theorem 6.7.1. Let $k$ be an arbitrary complex number. We have $H^i(V) = \{0\} \ (i \neq 0)$ for any object $V$ in $\mathcal{O}_k$.

Remark 6.7.2. By Theorem 6.7.1 we have, in particular, $H^i(L(\lambda)) = \{0\} \ (i \neq 0)$ for each $\lambda \in \hat{h}^*$. This was conjectured by V. G. Kac, S.-S. Roan and M. Wakimoto 17 in the case that $\lambda$ is admissible.

Corollary 6.7.3. For any $k \in \mathbb{C}$, the correspondence $V \sim H^0(V)$ defines an exact functor from $\mathcal{O}_k$ to the category of $W_k(g, f)$-modules.

Theorem 6.7.4. Let $k$ be an arbitrary complex number and let $\lambda \in \hat{h}^*_k$. If $(\lambda, \alpha_0^\vee) \in \{0, 1, 2, \ldots\}$, then $H^0(L(\lambda)) = \{0\}$. Otherwise $H^0(L(\lambda))$ is an irreducible $W_k(g, f)$-module that is isomorphic to $L(\phi)$. 
Remark by the character of the corresponding \( \hat{\nu} \) from (74), Theorem 6.3.1 and Theorem 6.7.4 that the character of Because the correspondence (74) is generated over \( W_k(\mathfrak{g}, f) \) by the image \( \pi(\langle \lambda \rangle) \) of the highest weight vector \( |\lambda\rangle \in H^0(M(\lambda)) \). Thus it follows that \( H^0(L(\lambda)) = \{0\} \) if and only if \( \pi(\langle \lambda \rangle) \neq \{0\} \). But \( \pi(\langle \lambda \rangle) \neq \{0\} \) if and only if \( H^0(L(\lambda))\xi_\lambda \neq \{0\} \), because \( \dim_\mathbb{C} H^0(M(\lambda))\xi_\lambda = 1 \), by Proposition 4.8.1(1). Hence, by Proposition 4.8.1(2), it follows that \( H^0(L(\lambda)) \) is nonzero if and only if \( \langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \ldots \} \).

Next, suppose that \( \langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \ldots \} \), so that \( H^0(L(\lambda)) \neq \{0\} \). Also, let \( N \) be a nonzero submodule of \( H^0(L(\lambda)) \). As \( L(\lambda) \) is a submodule of \( M(\lambda)^* \), \( H^0(L(\lambda)) \) is also a submodule of \( H^0(M(\lambda)^*) \), by the exactness of the functor \( H^0(?) \) (Corollary 6.7.3). Hence, \( N \) is a submodule of \( H^0(M(\lambda)^*) \). But then, by Theorem 6.6.2 it follows that \( \pi(\langle \lambda \rangle) \in N \). Therefore \( N \) must coincide with the entire space \( H^0(L(\lambda)) \). We have thus shown that \( H^0(L(\lambda)) \) is irreducible. Finally, we have to show (in the case \( k = -h^\vee \)) that \( \mathbb{N}(\phi_\lambda) = H^0(N(\lambda)) \) (\( \mathbb{N}(\phi) \) is defined in Section 6.4). Clearly, \( H^0(N(\lambda)) \) is stable under the action of \( D^W \). Thus, \( \mathbb{N}(\phi_\lambda) \supset H^0(N(\lambda)) \). In addition, \( \mathbb{N}(\phi_\lambda) \subset H^0(N(\lambda)) \), by the irreducibility of \( H^0(L(\lambda)) = H^0(M(\lambda))/H^0(N(\lambda)) \), which we have just proved. This completes the proof. \( \square \)

Remark 6.7.5. Theorem 6.7.4 was conjectured by V. G. Kac, S.-S. Roan, M. Wakimoto \[17\] in the case of an admissible weight \( \lambda \).

6.8. The characters. For an object \( V \) of \( \mathcal{O}_k \), define the formal characters
\[
\text{ch} V = \sum_{\lambda \in \mathfrak{h}^*} e^{\lambda} \dim_\mathbb{C} V^\lambda,
\]
\[
\text{ch} H^0(V) = \sum_{\xi \in \mathfrak{h}^*} e^{\xi} \dim_\mathbb{C} H^0(V)_\xi.
\]

In addition, for \( \lambda, \mu \in \mathfrak{h}^* \) define an integer \( [L(\lambda) : M(\mu)] \) by
\[
\text{ch} L(\lambda) = \sum_{\mu \in \mathfrak{h}^*} [L(\lambda) : M(\mu)] \text{ch} M(\mu).
\]

Then, by the exactness of the functor \( H^0(?) \) (see Corollary 6.7.3), we have
\[
\text{ch} H^0(L(\lambda)) = \sum_{\mu \in \mathfrak{h}^*} [L(\lambda) : M(\mu)] \text{ch} H^0(M(\mu)).
\]

Because the correspondence \( \mathfrak{h}^*_k \ni \lambda \mapsto \phi_\lambda \in \mathfrak{h}^*_W \) (see (13)) is surjective, it follows from (74), Theorem 6.3.1 and Theorem 6.7.4 that the character of any irreducible highest weight representation \( L(\phi) \) of \( W_k(\mathfrak{g}, f_k) \) at any level \( k \in \mathbb{C} \) is determined by the character of the corresponding \( \mathfrak{g} \)-module \( L(\lambda) \).

Remark 6.8.1.

(1) For an admissible weight \( \lambda \), the character formula (74) was conjectured by V. G. Kac, S.-S. Roan, M. Wakimoto \[17\].
(2) By Theorem 6.7.1, it follows that (74) can be obtained by calculating the Euler-Poincaré character of $H^n(L(\lambda))$ (see Ref. 17 for details).

(3) In the case that $g$ is a Lie algebra and $k \neq -h^\vee$, the number $[L(\lambda) : M(\mu)]$ is known. (It can be expressed in terms of the Kazhdan-Lusztig polynomials. The most general formula is given in Ref. 21).

(4) In the case that $g = \mathfrak{sp}(2|1)$, $W_k(g, f_\theta)$ is the Neveu-Schwarz algebra.

All minimal series representations of the Neveu-Schwarz algebra (see, e.g., Ref. 13) can be obtained from the admissible $\mathfrak{sp}(2|1)$-modules 18, as explained by V. G. Kac, S.-S. Roan and M. Wakimoto 17.

(5) In the case that $g = \mathfrak{sl}(2|1)$, $W_k(g, f_\theta)$ is the $N = 2$ superconformal algebra.

The minimal series representations of the $N = 2$ superconformal algebra (cf. Refs. 3, 23, 24) can be obtained from the admissible $\mathfrak{sl}(2|1)$-modules 18, as explained by V. G. Kac, S.-S. Roan and M. Wakimoto 17.

(6) For further examples and references, see Refs. 17, 19, 20.

7. The computation of $H^*_\bullet(M(\lambda)^*)$

In this section, we prove Theorems 6.6.1 and 6.6.2. Specifically, we compute $H^*_\bullet(M(\lambda)^*)$, with $\lambda \in \hat{h}_\lambda^\ast$. This is done by using a spectral sequence which we define in Subsection 7.11. It is a version of the Hochschild-Serre spectral sequence for the subalgebra $\mathbb{C}e(-1) \oplus g_{>0} \otimes \mathbb{C}[t] \subset Lg_{>0}$.

7.1. As in [14], let

$$N(\chi-) = U(Lg_{<0})/U(Lg_{<0}) \ker \chi_-.$$  

Here, $\ker \chi_- \subset U(Lg_{\leq -1})$ is the kernel of the character $\chi_-$ of $Lg_{\leq -1}$, defined by

$$\chi_-(u(m)) = (e(-1)|u(m)) \text{, where } u \in g_{\leq -1} \text{ and } m \in \mathbb{Z}.$$  

Let $\Phi_\mu(n)$, with $u \in g_{<0}$ and $n \in \mathbb{Z}$, denote the image of $u(n) \in Lg_{<0}$ in $N(\chi_-)$. As above, we set $\Phi_{-\alpha}(n) = \Phi_{\alpha,-}(n)$ with $\alpha \in \Delta_\frac{1}{2}$ and $n \in \mathbb{Z}$. Then, the correspondence $\Phi_\mu(n) \mapsto \Phi_{-\alpha}(-n)$, with $\alpha \in \Delta_\frac{1}{2}$ and $n \in \mathbb{Z}$, defines the anti-algebra isomorphism

$$N(\chi) \cong N(\chi_-).$$

7.2. Let $F^\text{ne}(\chi_-)$ be the irreducible representation of $N(\chi_-)$ generated by a vector $1_{\chi_-}$ such that $\Phi_{-\alpha}(n)1_{\chi_-} = 0$ for $\alpha \in \Delta_\frac{1}{2}$ and $n \geq 1$. As in the case considered above, we define a semisimple action of $\hat{h}$ on $F^\text{ne}(\chi_-)$ by $h1_{\chi_-} = 0$, $\Phi_{-\alpha}(n)F^\text{ne}(\chi_-)^\lambda \subset F^\text{ne}(\chi_-)^{\lambda - \alpha + n\delta}$, with $h \in \hat{h}$, $\alpha \in \Delta_\frac{1}{2}$, $n \leq 0$, and $\lambda \in \hat{h}^\ast$. Then, $F^\text{ne}(\chi_-) = \bigoplus_{\xi \in \hat{t}} F^\text{ne}(\chi_-)_\xi$ and $\dim F^\text{ne}(\chi_-)_\xi < \infty$ for all $\xi$.

7.3. There exists a unique bilinear form

$$\langle \cdot | \cdot \rangle^\text{ne} : F^\text{ne}(\chi) \times F^\text{ne}(\chi_-) \to \mathbb{C}$$

such that $\langle 1_{\chi}|1_{\chi_-}\rangle^\text{ne} = 1$ and $\langle \Phi_\mu(m)v|u^\prime \rangle^\text{ne} = \langle v|\Phi_{-\alpha}(-m)v^\prime \rangle^\text{ne}$, where $v \in F^\text{ne}(\chi)$, $v^\prime \in F^\text{ne}(\chi_-)$, $\alpha \in \Delta_\frac{1}{2}$ and $m \in \mathbb{Z}$. It is easy to see that this form is non-degenerate. Indeed, its restriction on $F^\text{ne}(\chi)_\xi \times F^\text{ne}(\chi_-)_\xi$, with $\xi \in \hat{t}^\ast$, is non-degenerate. Hence, we have

$$F^\text{ne}(\chi) = F^\text{ne}(\chi_-)^\ast.$$
since each space $F_{\text{nc}}(\chi_-)\xi$, with $\xi \in \hat{\frak h}$, decomposes into a finite sum of finite-dimensional weight spaces $F_{\text{nc}}(\chi_-)^\lambda$.

7.4. Let $\mathcal{C}(L_{\mathcal{G},<0})$ be the Clifford superalgebra associated with $L_{\mathcal{G},<0} \oplus (L_{\mathcal{G},<0})^*$ and its natural bilinear form. It is generated by the elements $\psi_{-\alpha}(n)$ and $\psi^{-\alpha}(n)$ with $\alpha \in \Delta_{<0}$ and $n \in \mathbb{Z}$ which satisfy the relations $[\psi_{-\alpha}(m), \psi^{-\beta}(n)] = \delta_{\alpha, \beta}\delta_{m+n, 0}$. Here the parity of $\psi_{-\alpha}(n)$ and $\psi^{-\alpha}(n)$ is reverse to $u_{-\alpha}$. We have an anti-algebra isomorphism $\mathcal{C}(L_{\mathcal{G},<0}) \cong \mathcal{C}(L_{\mathcal{G},<0})$ defined by $\psi_{\alpha}(m) \mapsto (-1)^{p(\alpha)}\psi_{-\alpha}(-m)$ and $\psi^{\alpha}(m) \mapsto \psi^{-\alpha}(-m)$, where $\alpha \in \Delta_{>0}$ and $m \in \mathbb{Z}$.

7.5. Let $\mathcal{F}(L_{\mathcal{G},<0})$ be the irreducible representation of $\mathcal{C}(L_{\mathcal{G},<0})$ generated by the vector $1_-$ with the properties $\psi_{-\alpha}(n)1_- = 0$, where $\alpha \in \Delta_{>0}, n \geq 1$, and $\psi^{-\alpha}(n)1_- = 0$, where $\alpha \in \Delta_{>0}, n \geq 0$. As above, we have a natural action of $\hat{\frak h}$ on $\mathcal{F}(L_{\mathcal{G},<0})$.

There exists a unique bilinear form

$$\langle \cdot, \cdot \rangle^{\text{ch}} : \mathcal{F}(L_{\mathcal{G},>0}) \times \mathcal{F}(L_{\mathcal{G},<0}) \to \mathbb{C},$$

which is non-degenerate on $\mathcal{F}(L_{\mathcal{G},>0})^\lambda \times \mathcal{F}(L_{\mathcal{G},<0})^\lambda$, with $\lambda \in \hat{\frak h}^*$, such that $(1|1_-)^{\text{ch}} = 1, \langle \psi_{\alpha}(n)v|v'\rangle^{\text{ch}} = (-1)^{p(\alpha)}\langle v|\psi_{-\alpha}(-n)v'\rangle^{\text{ch}}$ and $\langle \psi^{\alpha}(n)v|v'\rangle^{\text{ch}} = \langle v|\psi^{-\alpha}(-n)v'\rangle^{\text{ch}}$, where $v \in \mathcal{F}(L_{\mathcal{G},>0}), v' \in \mathcal{F}(L_{\mathcal{G},<0}), \alpha \in \Delta_{>0}$ and $n \in \mathbb{Z}$. Hence, we have

$$\mathcal{F}(L_{\mathcal{G},>0}) = \mathcal{F}(L_{\mathcal{G},<0})^*.$$  

7.6. Let

$$C_-(V) = V \otimes F_{\text{nc}}(\chi_-) \otimes \mathcal{F}(L_{\mathcal{G},<0})$$

with $V \in \text{Obj}_k$. Then, $C_-(V) = \bigoplus_{\lambda \in \hat{\frak h}^*} C_-(V)^\lambda$ with respect to the diagonal action of $\hat{\frak h}$. By [8] and [10], we have

$$C(V^*) = C_-(V)^*$$

for $V \in \text{Obj}_k$ as $\mathbb{C}$-vector spaces. Here again, $^*$ is defined by [11]. Under the identification [31], we have

$$dg(v) = g(d_-v) \quad (g \in C(V^*), v \in C(V),)$$

where

$$d_- = \sum_{\alpha \in \Delta_{>0}} \left(-1\right)^{p(\alpha)}(u_{-\alpha})(n) + \Phi_{u_{-\alpha}}(n).$$

Clearly, we have $d_-^2 = 0$. Also, $d_-$ decomposes as

$$d_- = d_-^\mathfrak{c} + d_-^\mathfrak{st},$$

$$\langle d_-^\mathfrak{c}, d_-^\mathfrak{c} \rangle = \langle d_-^\mathfrak{st}, d_-^\mathfrak{st} \rangle = 0,$$

where

$$d_-^\mathfrak{c} = \sum_{\alpha \in \Delta_{>0}} \left(-1\right)^{p(\alpha)}\Phi_{-\alpha}(n)\psi^{-\alpha}(-n) + \sum_{\alpha \in \Delta_{>0}} \left(-1\right)^{p(\alpha)}\chi_-(u_{-\alpha}(1))\psi^{-\alpha}(-1)$$

and $d_-^\mathfrak{st} = d_- - d_-^\mathfrak{c}$. 

Remark 7.6.1. By Theorem 2.3 of [13], the complex \((C_-(V), d_-)\) is acyclic for any \(V \in \text{Obj} \mathcal{O}_k\), since \(f(1)\) is locally nilpotent on \(V\) (see Remark 7.5.2).

7.7. It is clear that \(C_-(V_k(g))\) possesses a natural vertex algebra structure. The correspondences \(v(n) \mapsto v'(-n), \psi_\alpha(n) \mapsto (-1)^{p(\alpha)} \psi_\alpha(-n), \psi_\alpha^\ast(n) \mapsto \psi^\ast(-n), \Phi_\alpha(n) \mapsto \Phi_\alpha(-n)\) extend to the anti-algebra homomorphism

\[ (85) \]

\[ t : \mathcal{U}(C(V_k(g))) \to \mathcal{U}(C_-(V_k(g))), \]

where \(\mathcal{U}(C(V_k(g)))\) and \(\mathcal{U}(C_-(V_k(g)))\) are universal enveloping algebras of \(C(V_k(g))\) and \(C_-(V_k(g))\) respectively in the sense of Ref. [14]. Note that we have the relations \(d_- = d^t, d^t = (d^\ast)^t\) and \(d_-^\ast = (d^\ast)^t\).

7.8. For \(v \in \mathfrak{g}\), define

\[ J_{-}^{(v)}(z) = \sum_{n \in \mathbb{Z}} J_{-}^{(v)}(n) z^{-n-1} \]

\[ \text{def} \quad v(z) + \sum_{\alpha, \beta \in \Delta, \Delta_0} (-1)^{p(\gamma)} \psi_\gamma(v, u_{\alpha, \beta}) : \psi_\gamma(z) \psi^{-\beta}(z) :, \]

where \(\psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n) z^{-n}\) and \(\psi^{-\alpha}(z) = \sum_{n \in \mathbb{Z}} \psi^{-\alpha}(z) z^{-n-1}\) with \(\alpha \in \Delta_0\). Then, we have

\[ (86) \]

\[ J^{(v)}(n)^t = J_{-}^{(v')}(-n) \quad \text{for} \quad v \in \mathfrak{g}^t, \quad n \in \mathbb{Z}. \]

7.9. Let \(\lambda \in \hat{\mathfrak{h}}^*_k\) and let \(C_-(\lambda)\) be the subspace of \(C_-(M(\lambda))\) spanned by the vectors

\[ J_{-}^{(u_1)}(m_1) \ldots J_{-}^{(u_p)}(m_p) \Phi_{-\alpha_1}(n_1) \ldots \Phi_{-\alpha_q}(n_q) \psi^{-\beta_1}(s_1) \ldots \psi^{-\beta_{\gamma}}(s_{\gamma}), \]

with \(u_i, \alpha_i \in \Delta, \beta_i \in \Delta_0\) and \(m_i, n_i, s_i \in \mathbb{Z}\), where \(|\lambda\rangle_\ast\) is the canonical vector \(v_{\lambda} \otimes 1_{\chi} \otimes 1_{-}\). Then, \(d_- C_-(\lambda) \subset C_-(\lambda), \) i.e., \(C_-(\lambda)\) is a subcomplex of \(C_-(M(\lambda)).\)

The inclusion \(C_-(\lambda) \hookrightarrow C_-(M(\lambda))\) induces the surjection

\[ (87) \]

\[ C_-(M(\lambda)) \to C_-(\lambda). \]

Let the differential \(d\) act on \(C_-(\lambda)^\ast\) as \((dg)(v) = g(d_- v)\) with \(g \in C_-(\lambda)^\ast, \) \(v \in C_-(\lambda).\) Then, the space \(H^i(C_-(\lambda)^\ast, d)\), where \(i \in \mathbb{Z}\), is naturally a module over \(W_k(g, f) = H^0(C_k(g))\), and \([\mathcal{S}]\) induces the homomorphism

\[ (88) \]

\[ H^\ast(M(\lambda)^\ast) \to H^\ast(C_-(\lambda)^\ast, d) \]

of \(W_k(g, f)\)-modules. The action of \(W_k(g, f)\) on \(H^i(C_-(\lambda)^\ast, d)\) is described by

\[ (89) \]

\[ (W^{(v)}(n)g)(v) = g(W^{(v')}(n)v), \]

where \(W^{(v')}(n)\) is the image of \(W^{(v)}(n) \in H^0(\mathcal{U}(C_k(g))), \) ad \(d)\) under the map \([\mathcal{S}].\)

The following proposition can be shown in the same manner as Proposition 6.3 of [14].

Proposition 7.9.1. For any \(\lambda \in \hat{\mathfrak{h}}^*_n\), the map \([\mathcal{S}]\) is an isomorphism of \(W_k(g, f)\)-modules and \(\mathfrak{g}\)-modules:

\[ H^\ast(M(\lambda)^\ast) \cong H^\ast(C_-(\lambda)^\ast, d). \]

Below we compute the cohomology \(H^\ast(C_-(\lambda)^\ast) = H^\ast(C_-(\lambda)^\ast, d).\)
7.10. Here we employ the notation of the previous subsections. First, note that

\[ C_-(\lambda) = \bigoplus_{\xi \leq \lambda} C_-(\lambda)_\xi. \]

Also, observe that the subcomplex \( C_-(\lambda)_{\xi\lambda} \subset C_-(\lambda) \) is spanned by the vectors

\[ J^{(e_\mu)}_-(-1)^{|\lambda|} \varphi^{\theta}(-1)_{\lambda}, \quad J^{(e_\mu)}_-(-1)^n \varphi^{\theta}(-1)_{\lambda}, \]

with \( n \in \mathbb{Z}_{\geq 0} \). (Hence, \( C_-(\lambda)_{\xi\lambda} \) is infinite dimensional.) Let

\[ G_p C_-(\lambda)_{\xi\lambda} = \sum_{\rho \in \mathfrak{h}^* \subset \rho \geq p} C_-(\lambda)_{\rho\lambda} \subset C_-(\lambda)_{\xi\lambda} \quad \text{for } p \leq 0. \]

Thus, \( G_p C_-(\lambda)_{\xi\lambda} \) is a subspace of \( C_-(\lambda)_{\xi\lambda} \) spanned by the vectors

\[ J^{(e_\mu)}_-(-1)^n \varphi^{\theta}(-1)_{\lambda}, \quad J^{(e_\mu)}_-(-1)^n \varphi^{\theta}(-1)_{\lambda}, \quad \text{with } n \geq -p. \]

Now, the following assertion is clear.

**Lemma 7.10.1.** The space \( C_-(\lambda)_{\xi\lambda}/G_p C_-(\lambda)_{\xi\lambda} \) is finite dimensional for each \( p \leq 0 \).

Define \( G_p C_-(\lambda) \), with \( p \leq 0 \), as the subspace of \( C_-(\lambda) \) spanned by the vectors

\[ J^{(m_1)}_- \ldots J^{(n_p)}_- \Phi_{-\alpha_1}(n_1) \ldots \Phi_{-\alpha_q}(n_q) \varphi^{-\beta_1}(s_1) \ldots \varphi^{-\beta_r}(s_r)v, \]

with \( n_i \in \mathbb{Z}_{\geq 0}, \alpha_i \in \Delta_{\pm}, \beta_i \in \Delta_{\geq 0}, m_i, n_i, s_i \in \mathbb{Z} \) and \( v \in G_p C_-(\lambda)_{\xi\lambda} \). Then, we have

\[ \cdots \subset G_p C_-(\lambda) \subset G_{p+1} C_-(\lambda) \subset \cdots \subset G_0 C_-(\lambda) = C_-(\lambda), \]

\[ \bigcup_p G_p C_-(\lambda) = \{0\}, \]

\[ d_- G_p C_-(\lambda) \subset G_p C_-(\lambda). \]

The following Lemma is easily proven.

**Lemma 7.10.2.**

1. The subspace \( G_p C_-(\lambda) \), with \( p \leq 0 \), is preserved under the action of the operators \( J^{(n)}_- \varphi \) with \( u \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}, \Phi_{-\alpha}(n) \) with \( \alpha \in \Delta_{\pm}, n \in \mathbb{Z} \) and \( \varphi^{-\alpha}(n) \) with \( \alpha \in \Delta_{\geq 0}, n \in \mathbb{Z} \).
2. For each \( p \leq 0 \), \( C_-(\lambda)/G_p C_-(\lambda) \) is a direct sum of finite dimensional weight spaces \( (C_-(\lambda)/G_p C_-(\lambda))_{\xi} \), with \( \xi \in \hat{\mathfrak{h}}^* \).

7.11. For a semisimple \( \hat{\mathfrak{h}} \)-module \( X \), we define

\[ D(X) = \bigoplus_{\xi} \text{Hom}_C(X_{\xi}, C). \]

Next, let

\[ G^p C_-(\lambda)^* = (C_-(\lambda)/G_p C_-(\lambda))^* \subset C_-(\lambda)^* \quad \text{for } p \leq 0, \]

where * is defined by \[ 11 \]. Then, by Lemma 7.10.2 we have

\[ G^p C_-(\lambda)^* = D(C_-(\lambda)/G_p C_-(\lambda)). \]
and \( G^pC_-(\lambda)^* \) is a \( C_k(\mathfrak{g}) \)-submodule of \( C_-(\lambda)^* \). Also, by (91), we have
\[
\cdots \supset G^pC_-(\lambda)^* \supset G^{p+1}C_-(\lambda)^* \supset \cdots \supset G^0C_-(\lambda)^* = \{0\},
\]
(94)
\[C_-(\lambda)^* = \bigcup_p G^pC_-(\lambda)^*,\]
d\(G^pC_-(\lambda)^* \subset G^pC_-(\lambda)^*\).

Therefore we obtain the corresponding spectral sequence \( E_r \Rightarrow H^*(C_-(\lambda)^*) = H^*(M(\lambda)^*) \). By definition,
\[
E_1^{s,t} = H^q(\text{gr}_G C_-(\lambda)^*, d),
\]
(95)
where \( \text{gr}_G C_-(\lambda)^* = \sum_p G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^* \). Moreover, because our filtration is compatible with the action of \( \mathfrak{t} \), each \( E_r \) is a direct sum of \( \mathfrak{t} \)-weight spaces:
\[
E_r = \bigoplus_{\xi \in \mathfrak{t}^*} (E_r)_\xi.
\]
In particular, we have
\[
(E_r)_\xi \Rightarrow H^*(C_-(\lambda)^*)_\xi = H^*(M(\lambda)^*)_\xi \quad \text{for each } \xi \in \hat{\mathfrak{t}}^*.
\]
Below we compute this spectral sequence.

7.12. Consider the exact sequence
\[
0 \to G_{p+1}C_-(\lambda)/G_pC_-(\lambda) \to C_-(\lambda)/G_pC_-(\lambda) \to C_-(\lambda)/G_{p+1}C_-(\lambda) \to 0,
\]
where \( p \geq -1 \). This induces the exact sequence
\[
0 \to G^{p+1}C_-(\lambda)^* \to G^pC_-(\lambda)^* \to (G_{p+1}C_-(\lambda)/G_pC_-(\lambda))^* \to 0.
\]
Therefore
\[
G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^* = (G_{p+1}C_-(\lambda)/G_pC_-(\lambda))^* = D(G_{p+1}C_-(\lambda)/G_pC_-(\lambda)).
\]
(97)
Here, the last equality follows from Lemma 7.10.2 (2). Again by Lemma 7.10.2 (2), we have the following proposition.

**Proposition 7.12.1.** We have
\[
H^i(G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^*) = D(H^{-i}(G_{p+1}C_-(\lambda)/G_pC_-(\lambda))
\]
for each \( i \) and \( p \).

**Remark 7.12.2.** It is not the case that \( H^i(C_-(\lambda)^*) = D(H^{-i}(C_-(\lambda))) \).

7.13. Consider the subcomplex
\[
\text{gr}_G C_-(\lambda)^* \subset G^pC_-(\lambda)^* \overset{\text{def}}{=} \sum_p G_pC_-(\lambda)/G_{p-1}C_-(\lambda).
\]
By definition, we have
\[
\text{gr}_G C_-(\lambda)^* \otimes \mathfrak{t}^* = \bigoplus_p G_pC_-(\lambda)^*/G_{p-1}C_-(\lambda)^*_{\xi},
\]
and \( d_\xi \) acts trivially on \( \text{gr}_G C_-(\lambda)^* \) (see (90)). Thus,
\[
(\text{gr}_G C_-(\lambda)^*_{\xi}, d_-) = (C_-(\lambda)^*_{\xi}, d_\xi)
\]
(98)
as complexes. In particular, 
(99) \[ (E_1^{−q})_{\xi_λ} = H^q(\text{gr}^G C_{−}(\lambda)_{\xi_λ}, d) = H^q(C_{−}(\lambda)_{\xi_λ}^*, d^{\text{det}}), \]

because the complex \((C_{−}(\lambda)_{\xi_λ}, d^{\text{det}})\) is a direct sum of finite-dimensional subcomplexes \((C_{−}(\lambda)_{\xi_λ}^\mu, d^{\text{det}})\), with \(\mu \in \hat{h}^*\).

7.14. Consider the complex \(G_0 C_{−}(\lambda)/G_{−1} C_{−}(\lambda) = C_{−}(\lambda)/G_{−1} C_{−}(\lambda)\).

Let \(\overline{\lambda}\) be the image of \(|\lambda\rangle_−\) in \(C_{−}(\lambda)/G_{−1} C_{−}(\lambda)\). Then, the following hold:

\[ (C_{−}(\lambda)/G_{−1} C_{−}(\lambda))_{\xi_λ} = C[\overline{\lambda}], \]
\[ J^{(v)}_{−}(n)|\overline{\lambda}\rangle = 0 \quad \text{for} \quad v(n) \in Lg_{≥ 0} \cap \hat{g}_+, \]
\[ \psi^{−\alpha}(n)|\overline{\lambda}\rangle = 0 \quad \text{for} \quad \alpha \in \Delta_{>0}, n ≥ 0, \]
\[ \Phi^{−\alpha}(n)|\overline{\lambda}\rangle = 0 \quad \text{for} \quad \alpha \in \Delta_{≥1}, n ≥ 1, \]
\[ J^{(−1)}_{−}(\overline{\lambda}) = \psi^{−\theta}(−1)|\overline{\lambda}\rangle = 0, \]
\[ J^{(h)}_{−}(0)|\overline{\lambda}\rangle = \langle \lambda, h|\overline{\lambda}\rangle \quad \text{for} \quad h \in \hat{h}. \]

**Lemma 7.14.1.** We have 
\[ \text{gr}^G C_{−}(\lambda) = \bigoplus_{\mu \in \hat{h}^*} (C_{−}(\mu)/G_{−1} C_{−}(\mu)) \otimes C_{−}(\lambda)_{\xi_λ}^\mu \]
as complexes, where \(C_{−}(\lambda)_{\xi_λ}^\mu = (C_{−}(\lambda)_{\xi_λ}^\mu, d^{\text{det}}).\)

**Proof.** Define a linear map \(\bigoplus_{\mu \in \hat{h}^*} (C_{−}(\mu)/G_{−1} C_{−}(\mu)) \otimes \text{gr}^G C_{−}(\lambda)_{\xi_λ}^\mu \rightarrow \text{gr}^G C_{−}(\lambda)\)
by 
\[ J^{(u_1)}_{−}(m_1) \ldots J^{(u_r)}_{−}(m_p)\Phi_{−\alpha_1}(n_1) \ldots \Phi_{−\alpha_q}(n_q)\psi^{−\beta_1}(s_1) \ldots \psi^{−\beta_r}(s_r)|\mu\rangle \otimes v \]
\[ \mapsto J^{(u_1)}_{−}(m_1) \ldots J^{(u_r)}_{−}(m_p)\Phi_{−\alpha_1}(n_1) \ldots \Phi_{−\alpha_q}(n_q)\psi^{−\beta_1}(s_1) \ldots \psi^{−\beta_r}(s_r)v, \]
where \(u_i \in g_{≥ 0}, \alpha \in \Delta_{≥1}, \beta_i \in \Delta_{>0}, m_i, n_i, s_i \in \mathbb{Z}\) and \(v \in \text{gr}^G C_{−}(\lambda)_{\xi_λ}^\mu\). Then, it can be verified this is an isomorphism of complexes (cf. Proposition 6.12 of Ref. [H]). \(\square\)

7.15. Let 
\[ (\hat{g}^\sim)_{−} \overset{\text{def}}{=} n_{0, −} \otimes \mathbb{C}[t^{−1}] \oplus (\hat{h}^\sim \oplus n_{0, +} \oplus \mathfrak{g}_{≥ 1}) \otimes \mathbb{C}[t^{−1}]^{-1} \oplus \mathfrak{g}_1 \otimes \mathbb{C}[t^{−1}]t^{−2}. \]
The proof of the following assertion is the same as that of Theorem 4.1 of Ref. [H].

**Proposition 7.15.1.** For any \(\lambda \in \hat{h}^*\), we have \(H^i(C_{−}(\lambda)/G_{−1} C_{−}(\lambda)) = \{0\}\) for \(i ≠ 0\) and the following map defines an isomorphism of \(\mathbb{C}\)-vector spaces:
\[ U((\hat{g}^\sim)_{−}) \rightarrow H^0(C_{−}(\lambda)/G_{−1} C_{−}(\lambda)), \]
\[ u_1(n_1) \ldots u_r(n_r) \mapsto W^{(u_1)}_{−}(n_1) \ldots W^{(u_r)}_{−}(n_r)|\overline{\lambda}\rangle. \]
Now recall that \( G^{-1}C_-(\lambda)^* = D(C_-(\lambda)/G_{-1}C_-(\lambda)) \subset C_-(\lambda)^* \) (see (84)). Because \( G^{-1}C_-(\lambda)^* \) is a \( C_k(g)-\)submodule of \( C_-(\lambda)^* \), it follows that \( H^*(G^{-1}C_-(\lambda)^*) \) is a module over \( \mathcal{W}_k(g, f) \). It is clear that

\[
H^*(G^{-1}C_-(\lambda)^*) = \bigoplus_{\xi \leq \lambda} H^*(G^{-1}C_-(\lambda)^*)_\xi \text{ and } H^*(G^{-1}C_-(\lambda)^*)_\xi = \mathbb{C}[\lambda]^*.
\]

Here with a slight abuse of notation we have denoted the vector in \( H^*(G^{-1}C_-(\lambda)^*)_\xi \) dual to \( [\lambda] \) by \( [\lambda]^* \).

**Proposition 7.15.2.**

1. \( H^i(G^{-1}C_-(\lambda)^*) = \{0\} \) for \( i \neq 0 \).
2. Any nonzero \( \mathcal{W}_k(g, f)-\)submodule of \( H^0(G^{-1}C_-(\lambda)^*) \) contains the canonical vector \( [\lambda]^* \in H^0(G^{-1}C_-(\lambda)^*)_\xi \).

**Proof.** (1) It is straightforward to demonstrate using Propositions 7.12.1 and 7.15.1. (2) For a \( \mathcal{W}_k(g, f)-\)module \( M \), let \( S(M) \) be the space of singular vectors:

\[
S(M) \overset{\text{def}}{=} \{ m \in M \mid W^{(n)}(m) = 0 \text{ for all } v(n) \in (g^f)_+ \} \subset M.
\]

Then, by (101), we have \( S(M) \neq \{0\} \) for any nonzero \( \mathcal{W}_k(g, f)-\)submodule \( M \) of \( H^0(G^{-1}C_-(\lambda)^*) \). Hence, it is sufficient to show that

\[
S(H^0(G^{-1}C_-(\lambda)^*)) = \mathbb{C}[\lambda]^*.
\]

Note that, by (101), it is obvious that \( S(H^0(G^{-1}C_-(\lambda)^*)) \supset \mathbb{C}[\lambda]^* \). Therefore, we have only to show

\[
S(H^0(G^{-1}C_-(\lambda)^*)) \subset \mathbb{C}[\lambda]^*.
\]

To demonstrate (103), we make use of the filtration \( \{F_p\mathcal{W}_k(g, f)\} \) of \( \mathcal{W}_k(g, f) \) given in Theorem 5.5.1 described by (83). First, set \( F_{-1}H^0(C_-(\lambda)/G_{-1}C_-(\lambda)) = \{0\} \) and

\[
F_pH^0(C_-(\lambda)/G_{-1}C_-(\lambda)) = \text{span}\{W^{-u_1}(n_1) \cdots W^{-u_r}(n_r) | u_1, \ldots, u_r \in g^f, \sum s_i \leq p\} \text{ for } p \geq 0.
\]

Next, define

\[
F_pH^0(G^{-1}C_-(\lambda)^*) = D(H^0(C_-(\lambda)/G_{-1}C_-(\lambda))/F_{-p}H^0(C_-(\lambda)/G_{-1}C_-(\lambda))) \text{ for } p \leq 1.
\]

(Recall that \( H^0(G^{-1}C_-(\lambda)^*) = D(H^0(C_-(\lambda)/G_{-1}C_-(\lambda))) \)). Then, we have

\[
\cdots \subset F_pH^0(G^{-1}C_-(\lambda)^*) \subset F_{p+1}H^0(G^{-1}C_-(\lambda)^*) \subset \cdots \quad \cdots \subset F_0H^0(G^{-1}C_-(\lambda)^*) \subset F_1H^0(G^{-1}C_-(\lambda)^*) = H^0(G^{-1}C_-(\lambda)^*),
\]

\[
\bigcup F_pH^0(G^{-1}C_-(\lambda)^*) = \{0\},
\]

where \( F_j\mathcal{W}_k(g, f) \cdot F_pH^0(G^{-1}C_-(\lambda)^*) \subset F_{p+q}H^0(G^{-1}C_-(\lambda)^*) \),

where \( F_j\mathcal{W}_k(g, f) \cdot F_pH^0(G^{-1}C_-(\lambda)^*) \) denotes the span of the vectors \( Y_n(v)w \) with \( v \in F_j\mathcal{W}_k(g, f) \) and \( w \in F_pH^0(G^{-1}C_-(\lambda)^*) \) and \( n \in \mathbb{Z} \).

By (103), the corresponding graded space \( \text{gr}_F H^0(G^{-1}C_-(\lambda)^*) \), is a module over \( \text{gr}_F \mathcal{W}_k(g, f) = V^2_k(g^f) \). In particular, \( \text{gr}_F H^0(G^{-1}C_-(\lambda)^*) \) is a module over the Lie
algebra $(\hat{g}^f)_+$. (Observe $(\hat{g}^f)_+$ is generated by the image of $W^{(i)}(n), u(n) \in (\hat{g}^f)_+$.) Moreover, from (the proof of) Proposition 7.15.1 it follows that
\[
\text{gr}_F H^0(G^{-1}C_-(\lambda)^*) \cong D(U(\hat{g}_+)) \text{ as } (\hat{g}^f)_+\text{-modules},
\]
where $(\hat{g}^f)_+$ acts on $D(U(\hat{g}_+))$ by $(u(n)g)(v) = g(u^i(-n)v)$, where $u \in \hat{g}^f, v \in U(\hat{g}_+), g \in D(U(\hat{g}_+))$. Hence, it follows that
\[
H^0((\hat{g}^f)_+, \text{gr}_F H^0(G^{-1}C_-(\lambda)^*)) = \mathbb{C}\text{(the image of }|\lambda|^{*} \text{)}.
\]
But this implies that $S(H^0(G^{-1}C_-(\lambda)^*)) \subset \mathbb{C}|\lambda|^{*}$. This completes the proof. \qed

7.16.

**Theorem 7.16.1.** Suppose that $(\lambda, \alpha_0^0) \notin \{0, 1, 2, \ldots\}$. Then, $H^i(M(\lambda)^*) = \{0\}$ for $i \neq 0$ and $H^0(M(\lambda)^*) = H^0(G^{-1}C_-(\lambda)^*)$ as $W_k(g, f)$-modules. In particular, any nonzero $W_k(g, f)$-submodule of $H^0(M(\lambda)^*)$ contains the canonical vector $|\lambda|^{*}$.

**Proof.** We have $M_{\lambda(k)}((\lambda, \alpha_0^0)) = M_{\lambda(k)}((\lambda, \alpha_0^0))$ by assumption. Also, we have $H^i(C_-(\lambda_{\xi}, d^\omega)) \cong H^i(C_{\xi}, M_{\lambda(k)}((\lambda, \alpha_0^0)))$. This can be demonstrated in the same manner as Lemma 7.1.1. Hence, it follows that
\[
H^i(C_-(\lambda_{\xi}, d^\omega)) = \begin{cases} \mathbb{C} & (i = 0 \text{ and } \mu = \lambda), \\ \{0\} & (\text{otherwise}). \end{cases}
\]
Therefore, by Lemma 7.1.1 and Proposition 7.15.1 we have
\[
H^i(\text{gr}^G C_-(\lambda)) = \begin{cases} H^0(C_-(\lambda)/G_{-1}C_-(\lambda)) & (i = 0), \\ \{0\} & (i \neq 0). \end{cases}
\]
Hence, by Proposition 7.12.1 we have
\[
H^i(\text{gr}^G C_-(\lambda)^*) = \begin{cases} H^0(G^{-1}C_-(\lambda)^*) & (i = 0), \\ \{0\} & (i \neq 0). \end{cases}
\]
This implies that our spectral sequence collapses at $E_1 = E_{\infty}$, and therefore
\[
H^i(M(\lambda)^*) = \begin{cases} H^0(G^{-1}C_-(\lambda)^*) & (i = 0), \\ \{0\} & (i \neq 0). \end{cases}
\]
as vector spaces. But the isomorphism $H^0(G^{-1}C_-(\lambda)^*) \cong H^0(M(\lambda)^*)$ is induced by the $C_k(g)$-module homomorphism $G^{-1}C_-(\lambda)^* \hookrightarrow C_-(\lambda)^*$, and hence it is a $W_k(g, f)$-homomorphism. \qed

7.17. We finally consider the case in which $\lambda$ is a general element of $\hat{h}_k^*$. By Proposition 7.15.1 we have
\[
H^0(G^{-1}C_-(\mu)^*) \cong H^0(G^{-1}C_-(\mu')^*)
\]
as $\mathbb{C}$-vector spaces for any $\mu, \mu' \in \hat{h}^*$. With 7.18, it follows from 7.15, Proposition 7.12.1 and Lemma 7.14.1 that we have the isomorphism
\[
E_1^{\bullet, q} \cong H^0(G^{-1}C_-(\lambda)^*) \otimes H^0(C_-(\lambda)^*_{\xi}, d^\omega)
\]
\[
= H^0(G^{-1}C_-(\lambda)^*) \otimes (E_1^{\ast, q})_{\xi},
\]
of complexes, where the differential $d_1$ acts on the first factor $H^0(G^{-1}C_-(\lambda)^*)$ trivially. This induces the isomorphisms
\[(E_r, d_r) \cong (H^0(G^{-1}C_-(\lambda)^*) \otimes (E_r)_{\xi, 1} \otimes d_r)\]
inductively for all $r \geq 1$. Therefore, we obtain
\[E_\infty \cong H^0(G^{-1}C_-(\lambda)^*) \otimes (E_\infty)_{\xi, 1} \otimes d_r\]
On the other hand, by (100), we have
\[(E_\infty)_{\xi, 1} = H^\bullet(M(\lambda))^*)_{\xi, 1}.\]
Hence, by (111) and Proposition A.8.1(3), we have proved Theorem 6.6.1 as desired.

**Appendix A. The setting for $A(1,1)$**

In this appendix we summarize the change of the setting for the $A(1,1)$ case.

A.1. The setting for $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{sl}(2|2)/CI$ and let $\mathfrak{h}'$ be the Cartan subalgebra of $\mathfrak{g}$ containing $x$. Let $a = \mathfrak{sl}(2|2)/CI$. Then, $\mathfrak{g} = [a, a] \subset a$. Let $\mathfrak{h}$ be the Cartan subalgebra of $a$ containing $\mathfrak{h}'$. (So dim $\mathfrak{h} = 3$.) Then, $[\mathfrak{h}, \mathfrak{g}] = \mathfrak{g}$ and we have $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha = \{ u \in \mathfrak{g} | [h, u] = \langle \alpha, h \rangle u \text{ for all } h \in \mathfrak{h} \}$. Define the set $\Delta$ as the subset of $\mathfrak{h}^*$ consisting of elements $\alpha$ such that $\mathfrak{g}^\alpha \neq \{0\}$. Then, dim $\mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta$. The remaining setting is the same as in the other cases.

A.2. The setting for $\widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g} = \mathfrak{sl}(2|2)/CI$ defined by (7). Let $\widehat{\mathfrak{g}} = \widehat{n}_- \oplus \widehat{\mathfrak{h}}' \oplus \widehat{n}_+$ be the standard triangular decomposition, where $\widehat{\mathfrak{h}}' = \mathfrak{h}' \oplus C\mathfrak{k} \oplus C\mathfrak{d}$ is the standard Cartan subalgebra of $\widehat{\mathfrak{g}}$. Next, define the commutative Lie algebra $\widehat{\mathfrak{h}}$ by $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus C\mathfrak{k} \oplus C\mathfrak{d}$, where $\mathfrak{h}$ is as above.

Then, the action of $\mathfrak{h}$ on $\mathfrak{g}$ naturally extends to the action of $\widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{g}}$. This gives the space
\[(112) \quad \widehat{\mathfrak{g}} = \widehat{n}_- \oplus \widehat{\mathfrak{h}}' \oplus \widehat{n}_+\]
a a Lie superalgebra structure such that $\widehat{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$, and $\widehat{\mathfrak{h}}$ is a Cartan subalgebra of $\widehat{\mathfrak{g}}$. Now define $\widehat{\Delta} \subset \widehat{\mathfrak{h}}^*$ as the set of roots of $\widehat{\mathfrak{g}}$ and $\widehat{\Delta}_+ \subset \widehat{\Delta}$ as the set of positive roots of $\widehat{\mathfrak{g}}$ (according to the triangular decomposition (112)). Also, we define $\widehat{Q} = \sum_{\alpha \in \Delta} \mathbb{Z} \alpha \subset \widehat{\mathfrak{h}}^*$, $\widehat{Q}_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z} \alpha \subset \widehat{Q}$ and a partial ordering $\mu \leq \lambda$ on $\widehat{\mathfrak{h}}^*$ by $\lambda - \mu \in \widehat{Q}_+$. Next, we replace $\widehat{\mathfrak{g}}$-modules by $\widehat{\mathfrak{g}}$-modules. In particular, we define the category $\mathcal{O}_{\widehat{\mathfrak{g}}}$ as the full subcategory of the category of $\widehat{\mathfrak{g}}$-modules satisfying the conditions of Subsection A.2.2

Note that the simple $\widehat{\mathfrak{g}}$-module $L(\lambda)$, with $\lambda \in \widehat{\mathfrak{h}}^*$, is already irreducible as a $\mathfrak{g}$-module. This fact can be seen using the argument of the proof of Theorem 3.7.2. The remaining setting is the same as in the other cases.

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