Stabilization of Capacitated Matching Games

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Introduction
Introduction

- We consider a graph $G = (V, E)$ with edge weights $w$ and vertex capacities $c$.
- We say $M \subseteq E$ is a $c$-matching if $|M \cap \delta(v)| \leq c_v$ for all $v \in V$.
- $\nu_c(G) = \max \left\{ \sum_{e \in M} w_e : M \text{ is a } c\text{-matching in } G \right\}$
- $\nu_f^c(G) = \max \left\{ \sum_{e \in E} w_e x_e : \sum_{e \in \delta(v)} x_e \leq c_v \ \forall v \in V, \ 0 \leq x \leq 1 \right\}$
- We call a graph $G$ stable if $\nu_c(G) = \nu_f^c(G)$.

$$
\begin{align*}
\nu_c(G) &= 2 = \nu_f^c(G)
\end{align*}
$$
Network bargaining games

Network bargaining games were first introduced by Kleinberg and Tardos (2008), as a generalization of Nash’s 2-player bargaining solution (1950).

- $V$: players
- $E$: potential deals with value $w$
- A player $v$ can enter in $c_v$ deals: a set of deals is a $c$-matching $M$
- Players decide how to split the value of their deal: $z_{uv} + z_{vu} = w_{uv}$ if $uv \in M$ and $z_{uv} = z_{vu} = 0$ otherwise
- Stable solution if all players are satisfied

$$\alpha_u(M, z) = \max_{v : uv \in E \setminus M} \left( w_{uv} - 1_{d_v = c_v} \min_{vw \in M} z_{vw} \right)$$

**Theorem (Bateni, Hajiaghayi, Immorlica, Mahini (2010))**

There exists a stable solution for the network bargaining game on $G$ if and only if $G$ is stable.
The stabilization problem: minimally modify a graph to turn it into a stable one.

Previously studied modifications:

- Edge removal [Biró et al. (2014)] [Bock et al. (2015)] [Koh, Sanità (2020)]
- Vertex removal [Ito et al. (2017)] [Ahmadian et al. (2018)] [Koh, Sanità (2020)]
- Edge and vertex addition [Ito et al. (2017)]
- Increasing edge weights [Chandrasekaran et al. (2019)]
The stabilization problem - vertex removal

A vertex-stabilizer is a set \( S \subseteq V \) such that \( G \setminus S \) is stable (\( \nu_c(G \setminus S) = \nu_f^c(G \setminus S) \)).

**Vertex-stabilizer problem:** given a graph \((G, w, c)\), find a min-cardinality vertex-stabilizer

**\( M \)-vertex-stabilizer problem:** given a graph \((G, w, c)\) and a max-weight \( c \)-matching \( M \), find a min-cardinality vertex-stabilizer among those avoiding \( M \)

**Generalized \( M \)-vertex-stabilizer problem:** given a graph \((G, w, c)\) and an arbitrary \( c \)-matching \( M \), find a min-cardinality vertex-stabilizer \( S \) among those for which \( M \) is a max-weight \( c \)-matching in \( G \setminus S \)

| \((G, w, c)\) | vertex-stabilizer | \( M \)-vertex-stabilizer | generalized \( M \)-vertex-stabilizer |
|----------------|-------------------|---------------------------|--------------------------------------|
| \( w = 1, c = 1 \) | \( P \) [IKK\textsuperscript{+}17][AHS18] | \( P \) [AHS18] | tight 2-approx [KS20] |
| \( w \geq 0, c = 1 \) | \( P \) [KS20] | \( P \) [KS20] | tight 2-approx [KS20] |
| \( w \geq 0, c \geq 0 \) | tight \(|V|\)-approx | \( P \) | tight 2-approx [this work] |
(Generalized) M-vertex-stabilizer
Natural idea for an algorithm:

- Transform the given graph and $c$-matching into an auxiliary unit-capacity instance using a reduction from [Farczadi, Georgiou, Könemann (2013)].
- Apply the algorithm from [Koh, Sanità (2020)] to the unit-capacity auxiliary instance.

This does not work if both algorithms are used as a black-box.
Issues when using the unit-capacity algorithm

**Issue 1:** the algorithm suggest to remove a vertex that cannot be removed.

**Solution:** show that only removing $t$ is enough, or that the graph cannot be stabilized.
Issues when using the unit-capacity algorithm

Issue 2: the algorithm suggest to remove two vertices, while one could be enough. → 2-approximation algorithm.

Solution: 
- $M$-vertex-stabilizer problem: use traceback operation to get an exact algorithm.
- Generalized $M$-vertex-stabilizer problem: satisfied with 2-approximation.
Theorem

The $M$-vertex stabilizer problem can be solved in polynomial time, and the generalized $M$-vertex stabilizer problem admits an efficient 2-approximation.

→ Use the auxiliary construction, and apply the unit-capacity algorithm keeping in mind issues 1 and 2.
Vertex-stabilizer
Theorem

The vertex-stabilizer problem is NP-complete, and no efficient $|V|^{1-\varepsilon}$-approximation exists for any $\varepsilon > 0$, unless $P = NP$. This is true even when all edges have weight one.

Trivial $|V|$-approximation: remove all vertices.

We will give an approximation preserving reduction from the following problem:

**Minimum independent dominating set problem:** given a graph $G = (V, E)$, compute a minimum-cardinality subset $S \subseteq V$ that is independent and dominating.
Reduction

Let $G = (V, E)$.

Let $G'$ be the union of the following gadgets:

- $\Gamma_v$ for every $v \in V$:

- $\Gamma^i_e$ for every $e = uv \in E$ and $i \in \{1, \ldots, |V|\}$:

Claim: $G$ has an independent dominating set of size at most $k$ if and only if $G'$ has a vertex-stabilizer of size at most $k$. 
Cooperative matching games
Cooperative matching games

• $V$: players
• players can distribute a total value of $\nu^c(G)$
• $y \in \mathbb{R}^V_{\geq 0}$: allocation vector
  $\rightarrow y_v$ is the value assigned to player $v$
• stable solution if every subset of players is satisfied:
  $$\text{if } \sum_{v \in S} y_v \geq \nu^c(G[S]) \text{ for all } S \subseteq V$$

• core: set of all stable solutions of total value $\nu^c(G)$

\[ \nu^c(G') = 2 \]
Equivalence

(i) $G$ is stable
(ii) there exists a stable solution for the network bargaining game on $G$
(iii) there exists a solution in the core of the cooperative matching game on $G$

**Theorem (Kleinberg, Tardos (2008), Deng, Ibaraki, Nagamochi (1999))**

In unit-capacity graphs (i), (ii) and (iii) are equivalent.

Does this generalize to capacitated graphs?

- Bateni et al. (2010): (i) and (ii) are equivalent, and (i) implies (iii).
- Biró et al. (2016), Gerstbrein, Sanità, Verberk (2022): (iii) does not imply (i).

\[
\begin{align*}
\text{(1, 1, 1, 0)} & \in \text{core} \\
\nu^c(G) & = 3 \\
\nu_f^c(G) & = 3.5
\end{align*}
\]
Conclusion & future work
Conclusion & future work

- Results for the (generalized) $M$-vertex-stabilizer problem extend to the capacitated case:
  - The $M$-vertex-stabilizer problem is polynomial time solvable.
  - The generalized $M$-vertex-stabilizer problem admits an efficient 2-approximation.
- Results for the vertex-stabilizer problem do not extend to the capacitated case:
  - While the problem is polynomial time solvable in the unit-capacity case, there cannot be an efficient $|V|^{1-\epsilon}$-approximation in the capacitated case.
  - There is a trivial $|V|$-approximation for the vertex-stabilizer problem.

Future

- Stabilizing capacitated graphs by reducing the capacity of vertices.
- Stabilizing capacitated graphs by removing edges.
- Stabilizing cooperative matching games in capacitated instances.