Form Factors and Correlation Functions of the Stress–Energy Tensor in Massive Deformation of the Minimal Models $(E_n)_1 \otimes (E_n)_1 / (E_n)_2$

C. Acerbi$^{1,2}$, G. Mussardo$^{1,2,3}$ and A. Valleriani$^{1,2}$

$^1$International School for Advanced Studies
$^2$Istituto Nazionale di Fisica Nucleare
$^3$International Centre of Theoretical Physics
34013 Trieste, Italy

Abstract

The magnetic deformation of the Ising Model, the thermal deformations of both the Tricritical Ising Model and the Tricritical Potts Model are governed by an algebraic structure based on the Dynkin diagram associated to the exceptional algebras $E_n$ (respectively for $n = 8, 7, 6$). We make use of these underlying structures as well as of the discrete symmetries of the models to compute the matrix elements of the stress–energy tensor and its two–point correlation function by means of the spectral representation method.
1 Introduction

Important progress has recently been achieved in the computation of correlation functions for integrable models, defined either as lattice systems or as continuum theories. In addition to the well-established results on the spin–spin correlation function in the Ising model away from the critical temperature \([1, 2, 3, 4]\), correlation functions of several important statistical integrable models have been obtained by means of different techniques, such as those discussed in the references \([1, 2]\), for instance. Furthermore, for those models which present relativistic invariance and for which the exact \(S\)-matrix is known, a powerful method to compute the correlation functions is provided by the Form Factor (FF) approach originally proposed in \([7, 8]\). This approach has proved to be extremely efficient because it leads to fast convergent series for the correlators, as confirmed for instance in \([12, 13, 14]\). One of the most remarkable results achieved by means of the FF approach is the solution of the long-standing problem of the computation of the spin–spin correlator of the Ising Model in a Magnetic Field at \(T = T_c\) (IMMF) \([13]\). The aim of this paper is to extend the analysis of reference \([13]\) to two statistical models which are very closely related to the IMM, namely those relative to the thermal deformation of the Tricritical Ising model (TIM) and the Tricritical 3–state Potts model (TPM). The dynamics of all these systems are ruled by an algebraic structure related to the exceptional algebras \(E_n\).

In fact, the magnetic deformation of the Ising model highlights its underlying \(E_8\) structure as well as the thermal deformation of the TIM and of the TPM, which highlights respectively the \(E_7\) and the \(E_6\) structures of these models. In this paper we are concerned with the determination of the FFs of one of the most important fields of the above mentioned models, i.e. the stress–energy tensor \(T_{\mu\nu} (x)\). The matrix elements of this operator are particularly simple to determine for several reasons. Firstly, all the matrix elements of the components of this tensor can be expressed in terms of the FFs of just one scalar operator, namely its trace \(\Theta (x)\). This operator, in turn, is related to the operator \(\Phi (x)\) which deforms the conformal action by the relationship

\[
\Theta (x) = 2\pi \lambda (2 - 2\Delta_\Phi) \Phi (x) ,
\]

where \(\Delta_\Phi\) is the conformal dimension of the field \(\Phi (x)\). The stress–energy tensor is furthermore characterized by its conservation law \(\partial^\mu T_{\mu\nu}(x) = 0\) and by its obvious relation to the total energy and momentum of the system. These additional pieces of information are sufficient to uniquely identify the operator \(\Theta (x)\) and then, to reconstruct its correlation functions\(^1\). Some simple tests, based on certain sum rules, may be performed to check the

\(^1\)Identification of other local operators of an integrable theory may be in general more difficult. Although we are still lacking a general answer to this problem, nevertheless the series of papers \([10, 11]\) already yields some remarkable theoretical results.
efficiency of the approach. The first check involves the zeroth moment of the two-point correlation function \( G(x) = \langle \Theta(x)\Theta(0) \rangle \), a quantity which is related to the amplitude \( U \) of the free energy per unit volume \( f_s \sim -Um^2 \). Conversely, this amplitude can also be obtained independently by means of the Thermodynamics Bethe Ansatz \([19, 20]\). The second check is based on the second moment of \( G(x) \) which is proportional to the difference of the central charges \( c_{uv} \) and \( c_{ir} \) of the conformal field theories arising in the ultraviolet and infrared limits \([21]\). An important point illustrated by these comparisons is that the spectral representation series of the correlation function is saturated with a high degree of accuracy by its first terms. This implies that the analytic efforts needed to obtain an accurate determination of the correlation function are simplified enormously\(^2\).

The paper is organized as follows: in the next section we briefly review the bootstrap approach to integrable models based on the exact \( S \)-matrix formulation. In Section 3 and 4 we compute the first Form Factors of the stress–energy tensor of the perturbed Tricritical Ising Model and Tricritical Potts models respectively (the computations relative to the Ising model in a magnetic field may be found in the original reference \([13]\)). Our conclusions are presented in Section 5. The paper also contains two appendices: in the first one we gather useful mathematical formulas used in our computations, while the second one contains the three–particle FF results of the TPM.

### 2 General Features of Integrable Quantum Field Theories

In order to clarify the basic principles of the bootstrap approach to the solution of an integrable Quantum Field Theory and to set up the notation, in this section we endeavor to present a concise picture of the conceptual framework on which such approach is based and its most important consequences. Our starting point is the theory of integrable scattering processes, as originally developed in \([22, 23]\).

#### 2.1 Perturbed CFT and Factorized Scattering Theory

The statistical systems analized in this article are particular integrable massive deformations of the first minimal unitary models of Conformal Field Theories (CFT), namely the Ising model, the Tricritical Ising Model and the Tricritical 3–state Potts Model. The Ising Model is deformed along the magnetic direction whereas the deformation of the other two models is along their thermal direction. The three models belong to the minimal unitary

\(^2\)It is worth to notice that this remarkable behaviour of the spectral series has been also observed for massless models \([14]\).
series of CFT and therefore, at their critical point they may be simply realized in terms of a coset construction based on the affine $SU(2)$ algebra \cite{27}. However, it is well known that at criticality they also admit an alternative realization in terms of the coset constructions $(E_n)_1 \otimes (E_n)_1 / (E_n)_2$ based on the exceptional algebras $E_n$: the Ising Model is associated to the exceptional algebra $E_8$, the Tricritical Ising Model to $E_7$ and the Tricritical 3–state Potts Model to $E_6$. As shown in \cite{23, 24, 30, 31, 36}, the advantage of considering such alternative realizations of the three models at criticality becomes clear once they are perturbed away from the fixed points along particular directions in the scaling region. Since we will have the opportunity to come back and comment on this point in the next sections, where we consider in more detail each model, here we prefer to keep our discussion as general as possible and focus our attention on the common features of a typical integrable perturbed CFT. For such a theory, the action can be formally written as

$$A = A_{\text{CFT}} + g \int d^2 x \Phi(x),$$

where $\Phi(x)$ is the relevant operator that moves the system away from criticality. It breaks the original conformal invariance and therefore introduces a mass scale in the theory. For integrable deformations, the dynamics of the system is supported by the presence of an infinite number of conserved charges (this may be argued by means of the counting argument proposed in \cite{23}). One of the consequences of the quantum integrability is that the scattering processes of the massive excitations of the theory are completely elastic. Since production processes are absent in the dynamics of such integrable QFT, their general $S$–matrix element is diagonal in the number of external ($in$ and $out$) particles involved in the collision and moreover is completely factorized into the product of two–particle amplitudes. Hence, to know the on–mass–shell properties of such class of QFT we have simply to determine the two–particle $S$–matrices. For the class of models which we will consider in this paper, there is an additional simplification, i.e. all particles can be unambiguously distinguished on the basis of their different quantum numbers (this is certainly the case for theories with a non degenerate mass spectrum). Under this circumstance, the scattering processes are completely diagonal and the two–particle scattering amplitudes are simply defined by the equation

$$| A_a(\theta_a) A_b(\theta_b) \rangle_{\text{out}} = S_{ab}(\theta_{ab}) | A_a(\theta_a) A_b(\theta_b) \rangle_{\text{in}},$$

where $\theta_{ab} = \theta_a - \theta_b$ and we have used, as usual, $\theta_a$ to parametrize the dispersion relation of the particle $A_a$, i.e.

$$(E, p) = (m_a \cosh \theta_a, m_a \sinh \theta_a).$$

In terms of $\theta_{ab}$, the Mandelstam variable $s$ reads $s = m^2 + m^2 + 2 m_a m_b \cosh \theta_{ab}$. As functions of the variable $\theta_{ab}$ (from now on simply denoted by $\theta$), $S_{ab}(\theta)$ are analytic
functions, with possible poles on the imaginary axis $0 \leq \theta \leq i\pi$. They satisfy the functional equations
\[
S_{ab}(\theta) S_{ab}(-\theta) = 1 ,
\]
expressing the unitarity condition and the crossing symmetry of the theory, respectively. The general solution of (2.3) can be written in terms of an arbitrary product of the functions
\[
s_\alpha(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\pi \alpha)}{\sinh \frac{1}{2}(\theta - i\pi \alpha)} ,
\]
where $-1 \leq \alpha \leq 1$. The parameter $\alpha$ is related to the position of the pole of $s_\alpha(\theta)$ which is located at $\theta = i\pi \alpha$. For those models with a non–degenerate mass spectrum (in which all particles are self–conjugate), the functional space of the solution of (2.3) is instead spanned by an arbitrary product of the crossing symmetric functions
\[
f_\alpha(\theta) \equiv s_\alpha(\theta) s_\alpha(i\pi - \theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi \alpha)}{\tanh \frac{1}{2}(\theta - i\pi \alpha)} .
\]
The simple poles of $f_\alpha$ are located at the crossing symmetric points $\theta = i\pi \alpha$ and $\theta = i\pi (1 - \alpha)$.

For a theory with a degenerate mass spectrum the general expression of the two–particle $S$–matrix element may be written as
\[
S_{ab}(\theta) = \prod_{\alpha \in A_{ab}} s_\alpha(\theta)^{p_\alpha} ,
\]
while for a theory with a non–degenerate mass spectrum we have
\[
S_{ab}(\theta) = \prod_{\alpha \in A_{ab}} f_\alpha(\theta)^{p_\alpha} .
\]
In both cases, the exponents $p_\alpha$ denote the multiplicities of the corresponding poles identified by the indices $\alpha$. For those models which present an underlying algebraic structure related to a Dynkin diagram, as for instance the three models investigated in this paper, the labels $\alpha \in A_{ab}$ are integer multiples of $1/h$ where $h$ is the Coxeter number of the associated Lie algebra. The complete two–particle $S$–matrices of the Tricritical Ising and Potts Models in a thermal perturbation (originally computed in [30, 31, 36]) are reported for convenience in Tables (2) and (7) (the one relative to the IMMF may be found in the original reference [23]).

The two–particle elastic $S$–matrices considered in this paper present quite a rich pattern of poles in the complex $\theta$ plane. According to the analysis carried out by several
groups [31, 38, 39, 40], the simple and higher odd–order poles are associated to the presence of bound states appearing as intermediate virtual particles in the scattering processes. Viceversa, all even–order poles are simply due to intermediate multi–scattering processes with no one–particle singularities. For a simple pole of $S_{ab}(\theta)$ at $\theta = iu_{ab}^c$ corresponding to a bound state $A_c$, we can compute both the mass of the bound state by means of the equation

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c,$$

and the on–mass–shell three–point coupling constant $\Gamma_{ab}^c$ by taking the residue on the pole

$$-i \lim_{\theta \to iu_{ab}^c} (\theta - iu_{ab}^c) S_{ab}(\theta) = (\Gamma_{ab}^c)^2.$$ (2.9)

In the case of a double pole of the $S$–matrix placed at $\theta = i\pi\alpha$, the associated residue is given by

$$-i \lim_{\theta_{ab} \to i\varphi} (\theta_{ab} - i\varphi)^2 S_{ab}(\theta) = (\Gamma_{ab}^c)^2 S_c(i\gamma),$$ (2.10)

where $\gamma = \pi - u_{ed}^h - u_{ed}^h$. It is beyond our scope to discuss here in more detail the multi–scattering interpretation relative to the analytic structure of the $S$–matrix. The interested reader may consult the original literature quoted above for a complete account of the pole structure of the $S$–matrix. Some examples will be however provided in the next sections to enlighten the analytic structure of matrix elements of local operators.

### 2.2 Correlation Functions and Form Factors

One of the approaches that has proved to be extremely efficient in the computation of correlation functions for statistical models away from criticality is the spectral representation method. For integrable models, this approach has been originally proposed in [7, 8] and further analysed and applied by different groups [7-18]. The simplest example which illustrates this approach is given by the computation of the two–point functions (higher–point functions being determined similarly). In the spectral representation approach, the two–point function of a local operator $\varphi(x)$ may be expressed as

$$\langle \varphi(x) \varphi(0) \rangle = \sum_{n=0}^{\infty} \int_{\theta_1 > \theta_2 > \cdots > \theta_n} \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n} \left| F_{a_1, \ldots, a_n}(\theta_1, \ldots, \theta_n) \right|^2 e^{-|x|} \sum_{k=1}^n m_k \cosh \theta_k,$$ (2.11)

where

$$F_{a_1, \ldots, a_n}(\theta_1, \ldots, \theta_n) \equiv \langle 0 | \varphi(0) | A_{a_1}^\dagger(\theta_1) \cdots A_{a_n}^\dagger(\theta_n) \rangle.$$ (2.12)

The above matrix elements are the so–called Form Factors (FF) and, as we will briefly discuss below, their computation can be performed once the exact $S$–matrix and the bound state structure of the theory are known. Inserted into (2.11), they give rise to
fast convergent series both in the infrared region \(|x| \to \infty\), with a corresponding exponential decay, and in the ultraviolet limit \(|x| \to 0\) where the correlation functions show power–law behaviours.

The FF satisfy the so–called Watson equations, given by

\[
F^{\varphi}_{a_1,\ldots,a_i+1,\ldots,a_n} (\theta_1, \ldots, \theta_i+1, \ldots, \theta_n) = S_{a_i+1,a_i} (\theta_i - \theta_{i+1}) F^{\varphi}_{a_1,\ldots,a_i+1,\ldots,a_n} (\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n),
\]

\[
F^{\varphi}_{a_1,a_2,\ldots,a_n} (\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = F^{\varphi}_{a_2,\ldots,a_n,a_1} (\theta_2, \ldots, \theta_n, \theta_1).
\] (2.13)

Among the solutions of these equations, there are those (called minimal solutions) characterized by the property that they have neither poles nor zeros in the strips \(\text{Im} \theta_{ij} \in (0,2\pi)\). By using the factorization properties of the underlying scattering theory, the minimal solution associated to a generic FF may be easily expressed in terms of the minimal two–particle FFs \(F_{ab}^{\text{min}}(\theta)\) by

\[
F_{a_1,a_2,\ldots,a_n}^{\text{min}} (\theta_1, \theta_2, \ldots, \theta_n) = \prod_{1 \leq i < j \leq n} F_{a_i,a_j}^{\text{min}} (\theta_i - \theta_j).
\] (2.14)

where the explicit expressions of \(F_{ab}^{\text{min}}(\theta)\) are given for theories with degenerate mass spectrum by

\[
F_{ab}^{\text{min}}(\theta) = \left(-i \sinh \frac{\theta}{2}\right)^{\delta_{a,b}} \prod_{\alpha \in A_{ab}} h_\alpha(\theta)^{p_\alpha},
\] (2.15)

while for theories with non–degenerate mass spectrum by

\[
F_{ab}^{\text{min}}(\theta) = \left(-i \sinh \frac{\theta}{2}\right)^{\delta_{a,b}} \prod_{\alpha \in A_{ab}} g_\alpha(\theta)^{p_\alpha}.
\] (2.16)

The definition and the properties of the functions \(h_\alpha(\theta)\) and \(g_\alpha(\theta)\) are collected in Appendix A.

The minimal expression of the FFs does not carry any dependence on the specific operator we are considering, as it must be, since the monodromy properties derive from the \(S–\)matrix alone. To characterize the different operators and to take into account the dynamical pattern of bound states of the theory, let us consider in more detail the analytic structure of the FFs, starting our discussion from the occurrence of their poles. Their pattern may be very complicated for the multi–scattering processes of the theory. There are however two classes of simple order poles in the FF which have a simple and natural origin [8]. The first class is that of kinematical poles relative to particle–antiparticle singularities at the relative rapidity \(\theta = i\pi\) with the corresponding residue given by

\[
-i \lim_{\theta \to i\pi} F_{a,a_1,\ldots,a_n} (\tilde{\theta} + i\pi, \theta, \theta_1, \ldots, \theta_n) = \left(1 - \prod_{1}^{n} S_{aa} (\theta - \theta_i)\right) F_{a_1,\ldots,a_n} (\theta_1, \ldots, \theta_n).
\] (2.17)
The second class of simple order poles of the FFs which admit a simple explanation is that associated to bound state singularities. Namely, whenever $A_a(\theta_a)$ and $A_b(\theta_b)$ form a bound state $A_c(\theta_c)$ for the value $\theta_{ab} = iu_{ab}^c$ of their relative rapidity, then all the matrix elements $F^\varphi_{a,b,a_1,...,a_n}(\theta_a, \theta_b, \theta_1, \ldots, \theta_n)$ involving the two particles $A_a(\theta_a)$ and $A_b(\theta_b)$ will have as well a simple order pole at the same position, with the residue ruled by the on–shell three–point coupling constant $\Gamma^c_{ab}$, i.e. (see Figure 1)

$$-i \lim_{\theta_{ab} \to iu_{ab}^c} (\theta_{ab} - iu_{ab}^c) F^\varphi_{a,b,a_1,...,a_n}(\theta_a, \theta_b, \theta_1, \ldots, \theta_n) = \Gamma^c_{ab} F^\varphi_{c,a_1,...,a_n}(\theta_c, \theta_1, \ldots, \theta_n). \quad (2.18)$$

In addition to the above classes of simple poles, the FFs may present poles of higher order relative to the underlying multi–scattering processes, as recently clarified in [13].

A key point to understand the rich analytic structure of the matrix elements is to initially analyse the two–particle FFs. Following the analysis of [13] (see also [16]), the two–particle FFs can be conveniently written as

$$F^\varphi_{ab}(\theta) = Q^\varphi_{ab}(\theta) \frac{F_{min}^{ab}(\theta)}{D_{ab}(\theta)}, \quad (2.19)$$

where $D_{ab}$ takes into account its poles structure and $Q^\varphi_{ab}$ is a polynomial in $\cosh \theta$ which carries the dependence on the operator $\varphi$.

The polynomials $D_{ab}(\theta)$ are determined from the poles of the $S$–matrix. The analysis of ref. [13] gives the following simple rules for determining them in the case of non–degenerate theories:

$$D_{ab}(\theta) = \prod_{\alpha \in A_{ab}} \left( P_{\alpha}(\theta) \right)^{i_{\alpha}} \left( P_{1-\alpha}(\theta) \right)^{j_{\alpha}}, \quad (2.20)$$

$$i_{\alpha} = n + 1, \quad j_{\alpha} = n, \quad \text{if} \quad p_{\alpha} = 2n + 1;$$

$$i_{\alpha} = n, \quad j_{\alpha} = n, \quad \text{if} \quad p_{\alpha} = 2n, \quad (2.21)$$

where $A_{ab}$ and $p_{\alpha}$ are defined in eq. (2.7). The functions

$$P_{\alpha}(\theta) \equiv \frac{\cos \pi \alpha - \cosh \theta}{2 \cos^2 \frac{\pi \alpha}{2}} \quad (2.22)$$

give a suitable parametrization of the pole at $\theta = i\pi \alpha$. The above prescription can be also generalized to degenerate theories. In fact, referring to equation (2.4), one can write

$$D_{ab}(\theta) = \prod_{\alpha \in A_{ab}} \left( P_{\alpha}(\theta) \right)^{i_{\alpha}}, \quad (2.23)$$

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3 The reason is that, by factorization, FFs with higher number of particles inherit their pole structure from the analytic structure of the two–particle channels. Moreover, the two–particle FFs play an important role in the theory since they provide the “initial conditions” needed for solving the recursive functional equations (2.17) and (2.18).
\[ i_\alpha = n + 1, \quad \text{if} \quad p_\alpha = 2n + 1 \quad s\text{-channel pole;} \]
\[ i_\alpha = n, \quad \text{if} \quad p_\alpha = 2n + 1 \quad t\text{-channel pole;} \]
\[ i_\alpha = n, \quad \text{if} \quad p_\alpha = 2n. \quad (2.24) \]

where it is convenient to distinguish between poles associated to the direct \(s\)-channel and those relative to the crossed \(t\)-channel. As we will show in the sequel, the above rules play an essential role for implementing the bootstrap program for the computation of the FFs in the TIM and in the TPM. Let us quote at this point the equations which will be often employed in the next sections. Those are: (a) the residue equations at a simple order pole that corresponds to a bound state

\[ -i \lim_{\theta \to i u_{ab}} (\theta - i u_{ab}^c) F_{ab}^\varphi(\theta) = \Gamma_{ab}^c F_{c}^\varphi, \quad (2.25) \]

(see Figure 1); (b) the residue equations relative to a simple order pole induced by a double pole in the \(S\)-matrix

\[ -i \lim_{\theta_{ab} \to i \varphi} (\theta_{ab} - i \varphi) F_{ab}(\theta_{ab}) = \Gamma_{ab}^c \Gamma_{ce}^e F_{ce}(i \gamma), \quad (2.26) \]

where \(\gamma = \pi - u_{a\bar{d}}^d - u_{\bar{b}d}^b\) (see Figure 2) and finally, (c) the residue equations relative to a double order pole induced by a third order pole in the corresponding \(S\)-matrix (see Figure 3 where \(\varphi = u_{ab}^f\))

\[ \lim_{\theta_{ab} \to i u_{ab}^f} (\theta_{ab} - i u_{ab}^f) F_{ab}(\theta_{ab}) = i \Gamma_{ad}^c \Gamma_{bd}^e \lim_{\theta_{ce} \to i u_{ce}^f} (\theta_{ce} - i u_{ce}^f) F_{ce}(\theta_{ce}) = -\Gamma_{ad}^c \Gamma_{bd}^e \Gamma_{ce}^f F_{f}. \quad (2.27) \]

After having considered the pole structure of the two–particle FFs, let us concentrate our attention on the polynomial \(Q_{ab}^\varphi(\theta)\) in the numerator of \(2.19\). In contrast to \(D_{ab}(\theta)\), which is only fixed by the \(S\)-matrix singularities, the polynomials \(Q_{ab}^\varphi(\theta)\) depend, on the contrary, on the operator \(\varphi(x)\) and may be used to characterize it. An upper bound on the maximal degree of the polynomials \(Q_{ab}^\varphi(\theta)\) has been derived in [13]. Briefly stated, the argument consists in looking at the large energy limit of the FF and relating it to the conformal properties of the corresponding operator \(\varphi(x)\). Denoted by \(\Delta_\varphi\) its conformal weight and by \(y_\varphi\) the real quantity defined by

\[ \lim_{|\theta_i| \to \infty} F_{a_1,\ldots,a_n}^\varphi(\theta_1,\ldots,\theta_n) \sim e^{y_\varphi |\theta_i|} \]

we have \[13\]

\[ y_\varphi \leq \Delta_\varphi. \quad (2.28) \]

Taking into account the degree of the factor \(F_{ab}^{\min}/D_{ab}(\theta)\) in the two–particle FF \(2.19\) by also using eq. \(A.8\), it is easy to translate the inequality \(2.28\) into an upper bound on the degree of the polynomial \(Q_{ab}^\varphi(\theta)\).
In what follows we will consider the FFs of the trace $\Theta(x)$ of the stress–energy tensor. In this case we have additional constraints for the corresponding polynomial $Q_{ab}(\theta)$. In fact, the conservation law $\partial^\mu T_{\mu\nu}(x)$ satisfied by the stress–energy tensor implies that the FFs of the trace $\Theta(x)$ must contain the kinematical polynomial $P^2 = (p_1 + \cdots + p_n)^2$, with the exception of the FFs with two identical particles. For the two–particles FFs, this property can be expressed by means of the following factorization

$$Q_{ab}^\Theta(\theta) = \left( \cosh \theta + \frac{m_a^2 + m_b^2}{2m_a m_b} \right)^{1-\delta_{ab}} P_{ab}(\theta),$$  

where

$$P_{ab}(\theta) \equiv \sum_{k=0}^{N_{ab}} a_{ab}^k \cosh^k \theta,$$  

The degree $N_{ab}$ in (2.30) may be determined by implementing the inequality (2.28). In this way, the problem is reduced to determine the coefficients $a_{ab}^k$ of the polynomials $P_{ab}$. This goal can be achieved by applying the residue equations together with the normalization condition of the two–particle FF, which are expressed by

$$F_{ab}^\Theta(i\pi) = 2\pi m_a^2.$$  

The above conditions prove in general sufficient or even redundant in number, to fix all the coefficients of the polynomial $P_{ab}(\theta)$.

As mentioned in the introduction, there is now a strong evidence that the spectral series based on the FFs are fastly convergent. For the correlation functions of the stress–energy tensor, one way to test this convergence is to employ two sum rules satisfied by the moments $M_p$ of the two–point function of $\Theta(x)$, defined by

$$M_p = \int d^2 x \, |x|^p \langle \Theta(x)\Theta(0) \rangle.$$  

The first sum rule is relative to the bulk free energy $f \sim -U m^2$, where the amplitude $U$ is related to the zero–moment $M_0$ by

$$U = \frac{1}{16\Delta_\Phi} \frac{1}{\pi^2 m^2} M_0,$$  

with $m$ the lightest mass of the theory. The second sum rule relates the second moment $M_2$ to the central charge $c$ of the original CFT, according to the formula [21]

$$c = \frac{3}{4\pi} M_2.$$  

By inserting the spectral representation of the two–point function $\langle \Theta(x)\Theta(0) \rangle$, both moments can be expressed as a series on the number of the intermediate particles with an increasing value of their center–of–mass energy. In this way we can test the fast convergence of the spectral representation by comparing the truncated values of the series of the moments $M_0$ and $M_2$, with the known value of the central charge $c$ and the exact value of $U$ computed by means of the Thermodynamic Bethe Ansatz [19, 20].
3 Form Factors of the Energy Operator for the Thermal Perturbation of the Tricritical Ising Model

The Tricritical Ising model is the second model in the minimal unitary conformal series with central charge $c = 7/10$ and four relevant fields [26]. The microscopic formulation of the model, its conformal properties and its scaling region nearby the critical point have been discussed in several papers (see, for instance [30, 31, 32, 33, 34, 35]). In the following we give a short review of the features of the TIM which are most relevant to the FF approach to integrable massive models.

3.1 Generalities of the TIM

The Tricritical Ising model may be regarded as the universality class of the Landau–Ginzburg $\Phi^6$–theory

$$L = (\nabla \Phi)^2 + g_6 \Phi^6 + g_4 \Phi^4 + g_3 \Phi^3 + g_2 \Phi^2 + g_1 \Phi$$  

(3.1)

at its critical point $g_1 = g_2 = g_3 = g_4 = 0$ [37]. This Lagrangian describes the continuum limit of microscopic models with a tricritical point, among them the Ising model with annealed vacancies, with an Hamiltonian given by [32, 33]

$$\mathcal{H} = -\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j t_i t_j - \mu \sum_i t_i.$$  

(3.2)

$\beta$ is the inverse temperature, $\mu$ the chemical potential, $\sigma_i = \pm 1$ the Ising spins and $t_i = 0, 1$ is the vacancy variable. The model has a tricritical point $(\beta_0, \mu_0)$ related to the spontaneous symmetry breaking of the $Z_2$ symmetry. At the critical point $(\beta_0, \mu_0)$, the TIM can be described by the following scaling fields: the energy density $\epsilon(z, \overline{z})$ with anomalous dimensions $(\Delta, \overline{\Delta}) = (1/10, 1/10)$, the vacancy operator or subleading energy operator $t(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (3/5, 3/5)$, the irrelevant field $\epsilon''$ with $(\Delta, \overline{\Delta}) = (3/2, 3/2)$, the magnetization field (or order–parameter) $\sigma(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (3/80, 3/80)$, and the so–called subleading magnetization operator $\alpha(z, \overline{z})$ with anomalous dimensions $(7/16, 7/16)$. With respect to the $Z_2$ symmetry of the spin model, the spin operators are odd while the energy operator, the vacancy operator and the irrelevant field $\epsilon''$ are even.

A peculiar feature of the TIM is the presence of another infinite dimensional symmetry in addition to the Virasoro algebra, i.e. a hidden $W$–algebra based on the $E_7$ root system. This is related to the equivalent construction of the TIM in terms of the coset model $(\hat{E}_7)_1 \otimes (\hat{E}_7)_1 / (\hat{E}_7)_2$. Let us briefly recall the coset formulation at the critical point [27].
From the theory of Kac–Moody algebras, the central charge of a CFT constructed on an affine Lie algebra $G$ at level $k$ is given by

$$c_G = \frac{k |G|}{k + h_G},$$

where $|G|$ is the dimension of the algebra and $h_G$ the dual Coxeter number. The unitarity condition for the CFT restricts the highest weight representations $| \lambda \rangle$ which can appear at the level $k$. Denoting with $\omega$ the highest root, the allowed representations $| \lambda \rangle$ at the level $k$ must satisfy

$$\frac{2 \omega \cdot \lambda}{\omega^2} \leq k,$$

and their dimension is given by

$$\Delta_\lambda = \frac{C_\lambda / \omega^2}{k + h_G},$$

where $C_\lambda$ is the quadratic Casimir in the representation $\{ \lambda \}$. Using a subgroup $H \subset G$, one can construct a CFT on the coset group $G/H$, with a central charge equal to

$$c_{G/H} = c_G - c_H = \frac{k |G|}{k + h_G} - \frac{k |H|}{k + h_H}.$$

Its representations $\psi^k$ are simply obtained by the decomposition of the Hilbert space

$$| c_G, \lambda_G \rangle = \bigoplus_k \left[ | c_{G/H}, \psi^k_{G/H} \rangle \otimes | c_H, \lambda_H^k \rangle \right].$$

In the case of the TIM, $h = 18$ and eq. (3.6) gives $c = \frac{7}{10}$. At level $k = 1$, the possible representations are the identity 1 and the representation $\Pi_6$ with scaling dimension $0$ and $\frac{3}{4}$ respectively

$$(E_7)_1 \rightarrow \{1, \Pi_6\} = \{0, \frac{3}{4}\}.$$

Their components $(n_1, n_2, \cdots, n_7)$ ($n_i$ integer) with respect to the simple roots of $E_7$ are given by [28]

$$1 \rightarrow (0, 0, 0, 0, 0, 0),$$

$$\Pi_6 \rightarrow (0, 0, 0, 0, 0, 1, 0).$$

At the level $k = 2$, the representations are given by

$$(E_7)_2 \rightarrow \{1, \Pi_1, \Pi_2, \Pi_5, \Pi_6\} = \{0, \frac{9}{16}, \frac{21}{16}, \frac{7}{5}, \frac{57}{80}\},$$

with the corresponding fundamental weights

$$\Pi_1 \rightarrow (1, 0, 0, 0, 0, 0, 0),$$

$$\Pi_2 \rightarrow (0, 1, 0, 0, 0, 0),$$

$$\Pi_5 \rightarrow (0, 0, 0, 1, 0, 0).$$
\(\Pi_1\) is the adjoint representation. Using eq. (3.7), the scaling dimensions of the TIM are recovered by the decomposition

\[
(0)_1 \times (0)_1 = [(0)_{TIM} \otimes (0)_{2}] + [(\frac{1}{10})_{TIM} \otimes (\Pi_1)_{2}] + [(\frac{6}{10})_{TIM} \otimes (\Pi_5)_{2}],
\]
\[
(0)_1 \times (\frac{3}{4})_1 = [(\frac{7}{10})_{TIM} \otimes (\Pi_2)_{2}] + [(\frac{3}{10})_{TIM} \otimes (\Pi_6)_{2}],
\]
\[
(\frac{3}{4})_1 \times (\frac{3}{4})_1 = (\frac{3}{2})_{TIM} \otimes (0)_{2}.
\]  

(3.12)

The off–critical perturbation considered in this paper is the one given by the leading energy operator \(\epsilon(z, \bar{z})\) of conformal weights \((\frac{1}{10}, \frac{1}{10})\). Note that this operator is associated to the adjoint of \(E_7\). According to the analysis of \([29]\), this leads to a structure of the off–critical system deeply related to the root system of \(E_7\), as we briefly recall in the following.

First of all, the off–critical massive model shares the same grading of conserved currents as the Affine Toda Field Theory constructed on the root system of \(E_7\), i.e. the spins of the higher conserved currents are equal to the exponents of the \(E_7\) algebra modulo its Coxeter number \(h = 18\), i.e.

\[
s = 1, 5, 7, 9, 11, 13, 17 \pmod{18}.
\] 

(3.13)

The presence of these higher conserved currents implies the elasticity of the scattering processes of the massive excitations. To compute the mass spectrum and the scattering amplitudes, it is important to observe that, according to the sign of the coupling constant \(g\) in (2.1), this perturbation drives the system either in its high–temperature phase or in its low–temperature phase. While in the latter phase we have a spontaneously symmetry breaking of the \(Z_2\) symmetry of the underlying microscopic spin system, in the former phase the \(Z_2\) symmetry is a good quantum number and therefore can be used to label the states. In the low–temperature phase, the massive excitations are given by kink states and bound state thereof, in the high–temperature phase we have instead ordinary particle excitations. The two phases are related by a duality transformation and therefore we can restrict our attention to only one of them, which we choose to be the high–temperature phase. In this phase, the massive excitations are given by seven self–conjugated particles \(A_1, \ldots, A_7\) with mass

\[
m_1 = M(g),
\]
\[
m_2 = 2 m_1 \cos \frac{5\pi}{18} = (1.28557\ldots) m_1,
\]
\[
m_3 = 2 m_1 \cos \frac{\pi}{9} = (1.87938\ldots) m_1,
\]
\[
m_4 = 2 m_1 \cos \frac{\pi}{18} = (1.96961\ldots) m_1,
\]
\[
m_5 = 2 m_2 \cos \frac{\pi}{18} = (2.53208\ldots) m_1.
\] 

(3.14)
\[ m_6 = 2 m_3 \cos \frac{2\pi}{9} = (2.87938..) m_1, \]
\[ m_7 = 4 m_3 \cos \frac{\pi}{18} = (3.70166..) m_1. \]

The dependence of the mass scale \( M \) on the coupling constant \( g \) has been computed in \[20\]
\[ M(g) = C g^{\frac{2}{3}}, \quad (3.15) \]
where
\[ C = \left[ 4 \pi^2 \gamma\left(\frac{1}{3}\right) \gamma\left(\frac{2}{3}\right) \gamma\left(\frac{4}{9}\right) \right]^{\frac{1}{3}} \frac{2 \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{19}{9}\right)} = 3.745372836\ldots, \quad (3.16) \]
where \( \gamma(x) \equiv \Gamma(x) / \Gamma(1-x) \). The mass ratios are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra \( E_7 \) \[39\] and therefore the particles \( A_i \) may be put in correspondence with the following representations of \( E_7 \) (here identified by their dimensions)

\[
\begin{align*}
A_1 & \rightarrow 56, \\
A_2 & \rightarrow 133, \\
A_3 & \rightarrow 912, \\
A_4 & \rightarrow 1539, \\
A_5 & \rightarrow 8645, \\
A_6 & \rightarrow 27664, \\
A_7 & \rightarrow 365750.
\end{align*}
\]

(3.17)

The exact \( S \)-matrix of the model is given by the minimal \( S \)-matrix of the Affine Toda Field Theory based on the root system of \( E_7 \). It has been calculated in \[30, 31\] and is listed for convenience in Table 2. The structure of the bound states may be written in a concise way by grouping the particle states into two triplets and one singlet states \[31\]
\[
\begin{align*}
(Q_1, Q_2, Q_3) & \equiv (A_6, A_3, A_1), \\
(K_1, K_2, K_3) & \equiv (A_2, A_4, A_7), \\
(N) & \equiv (A_5).
\end{align*}
\]

(3.18)

The first triplet consists of the \( Z_2 \) odd particles whereas the other triplet and the singlet are made of \( Z_2 \) even particles. The “bootstrap fusions” involving \([N]\) and \([N,K_i]\) form closed subsets
\[
\begin{align*}
N \cdot N &= N, \\
K_A \cdot K_{A+1} &= K_A + N, \\
K_A \cdot K_A &= K_A + K_{A+1} + N.
\end{align*}
\]

(3.19)

Including the first triplet, we obtain the following algebra
\[
\begin{align*}
K_A \cdot Q_A &= Q_{A+1}, \\
K_A \cdot Q_{A-1} &= Q_{A-1} + Q_{A+1}, \\
Q_A \cdot Q_A &= K_{A-1} + K_{A+1}, \\
Q_A \cdot Q_{A+1} &= K_A + K_{A-1} + N, \\
N \cdot Q_A &= Q_{A-1} + Q_{A+1}.
\end{align*}
\]

(3.20)
It has been observed that these bootstrap fusions are a subset of the tensor product decomposition of the associate representations of $E_7$.

### 3.2 Form Factors of the TIM

After the discussion on the general features of the model, let us consider now the problem of computing the FFs of the operator $\epsilon(x)$ or, equivalently, of the trace $\Theta(x)$ of the stress–energy tensor. To this aim, the $Z_2$ parity of the model is extremely helpful. In fact, because of the even parity of the energy operator, we can immediately conclude that its FF with a $Z_2$–odd (multi–particle) state must vanish. In particular, the one–particle FFs of $\Theta$ for the odd particles are all zero.

To start with the bootstrap procedure, let us consider the two–particle FF relative to the fundamental excitation $A_1$

\[
F_{11}^\Theta(\theta) = \frac{F_{11}^{\text{min}}(\theta)}{D_{11}(\theta)} Q_{11}^\Theta(\theta),
\]

where

\[
F_{11}^{\text{min}}(\theta) = -i \sinh(\theta/2) \, g_{5/9}(\theta) \, g_{1/9}(\theta),
\]

and

\[
D_{11}(\theta) = P_{5/9}(\theta) \, D_{1/9}(\theta).
\]

By using the bound (2.28), we see that the polynomial $Q_{11}^\Theta(\theta)$ reduces just to a constant, which can be easily determined by means of the normalization condition (2.31), i.e. $a_{11}^0 = 2\pi m_1^2$. Thus $F_{11}(\theta)$ is now completely determined and its expression can be used to derive the one–particle FFs $F_2$ and $F_4$. Indeed, the particles $A_2$ and $A_4$ appear as bound state of the particle $A_1$ with itself, the coupling $\Gamma_{11}^2$ and $\Gamma_{11}^4$ being easily determined by the residue equation (2.28). By using then the equation for the bound state poles of the Form Factors (2.29), one gets the desired result (see Table 4).

To proceed further, it is convenient to list the $Z_2$ even states (the only ones giving non–vanishing FFs of the stress–energy tensor) in order of increasing energy, as in Table 3. After computing $F_{22}^\Theta$, $F_5^\Theta$ and $F_{13}^\Theta$, which are obtained by means of the same technique, (i.e. fixing the unknown coefficients of FFs by using the simple pole residue equations), a more interesting computation is represented by the two–particle FF $F_{24}(\theta)$. The corresponding $S$–matrix element displays a double pole and therefore, according to eq. (2.31), we have

\[
F_{24}^\Theta(\theta) = \frac{F_{24}^{\text{min}}(\theta)}{D_{24}(\theta)} Q_{24}^\Theta(\theta),
\]

where

\[
F_{24}^{\text{min}}(\theta) = g_{7/9}(\theta) \, g_{4/9}(\theta) \, g_{1/3}^2(\theta),
\]

(3.24)
and
\[ D_{24}(\theta) = P_{7/9}(\theta) P_{4/9}(\theta) P_{1/3}(\theta) P_{2/3}(\theta) . \] (3.26)

Taking into account the asymptotic behaviour of the FF and eqs. (2.29) and (2.30), we conclude that in this case the polynomial \( P_{24} \) has degree \( N_{24} = 1 \) and therefore \( Q_{24}(\theta) \) reads
\[ Q_{24}(\theta) = \left( \cosh \theta + \frac{m_2^2 + m_4^2}{2m_2m_4} \right) (a_{24}^0 + a_{24}^1 \cosh \theta) . \] (3.27)

To determine the constants \( a_{24}^0 \) and \( a_{24}^1 \), we need at least two linearly independent equations, which are provided by eq. (2.25) on the fusions
\[ (A_2, A_4) \rightarrow A_2 \quad \text{and} \quad (A_2, A_4) \rightarrow A_5 . \] (3.28)

Both \( F_2 \) and \( F_5 \) are known, of course, from previous computations. In this case, the double pole in the \( S \)-matrix provides a non–trivial check for the computation. In fact, we have the process drawn in Figure 2, with the identifications
\[ a = 2, \quad b = 4, \quad d = e = 1 , \]
and respectively
\[ c = 1, \quad \varphi = 2\pi/3, \quad \gamma = \pi/3 , \]
or
\[ c = 3, \quad \varphi = \pi/3, \quad \gamma = \pi/9 . \]

These processes give rise to the corresponding residue equations
\[ -i \lim_{\theta \rightarrow i2\pi/3} (\theta - i2\pi/3) F_{24}^\Theta(\theta) = \Gamma_{21}^1 \Gamma_{41}^1 F_{11}^\Theta(i\pi/3) , \] (3.29)
\[ -i \lim_{\theta \rightarrow i\pi/3} (\theta - i\pi/3) F_{24}^\Theta(\theta) = \Gamma_{21}^3 \Gamma_{41}^1 F_{31}^\Theta(i\pi/9) . \]

which are indeed fulfilled. This example clearly shows the over–determined nature of the bootstrap equations and their internal consistency.

The next FF in order of increasing value of the energy of the asymptotic state is given by the lightest \( Z_2 \) even three–particle state \( | A_1 A_1 A_2 \rangle \). The FF may be parametrized in the following way
\[ F_{112}^\Theta(\theta_a, \theta_b, \theta_c) = \frac{F_{11}^{\min}(\theta_{ab}) F_{12}^{\min}(\theta_{ac}) F_{12}^{\min}(\theta_{bc})}{D_{11}(\theta_{ab}) D_{12}(\theta_{ac}) D_{12}(\theta_{bc})} \frac{Q_{112}^\Theta}{\cosh \theta_{ac} + \cosh \theta_{bc}} , \] (3.30)

where \( F_{11}^{\min} \) and \( D_{11}^{\min} \) are given by equations (3.22) and (3.23), while
\[ F_{12}^{\min}(\theta) = g_{13/18}(\theta) g_{7/18}(\theta) , \] (3.31)
and
\[ D_{12}(\theta) = P_{13/18}(\theta) \ P_{7/18}(\theta) \ . \quad (3.32) \]

We have introduced into (3.30) the term
\[
\frac{1}{\cosh \theta_{ac} + \cosh \theta_{bc}} ,
\]
to take into account the kinematical pole of this FF at \( \theta_a = \theta_b + i\pi \). The polynomial \( Q_{112} \) in the numerator can be further decomposed as
\[ Q_{112}(\theta_a, \theta_b, \theta_c) = P^2 \ P_{112}^\Theta , \quad (3.33) \]
where \( P^2 \) is the kinematical polynomial expressed by
\[ P^2 = 2m_1^2 + m_2^2 + 2m_1^2 \cosh \theta_{ab} + 2m_1m_2(\cosh \theta_{ac} + \cosh \theta_{bc}) . \quad (3.34) \]
The degree of \( P_{112}^\Theta \) can be computed by means of the asymptotic behaviour in the three variables \( \theta_{a,b,c} \) separately. This gives the following results for \( Q \sim \exp [x_i \theta_i] \):
\[ x_a = x_b = 1 \ \text{and} \ \ x_c = 2 . \quad (3.35) \]
Hence, a useful parametrization of the polynomial \( P_{112} \) is given by
\[ P_{112}^\Theta(\theta_a, \theta_b, \theta_c) = p_0 + p_1 \cosh \theta_{ab} + p_2(\cosh \theta_{ac} + \cosh \theta_{bc}) + p_3 \cosh \theta_{ac} \cosh \theta_{bc} , \quad (3.36) \]
where four unknown constants have to be determined through the poles of \( F_{112}^\Theta \). By using the kinematical pole at \( \theta_{ab} = i\pi \) and the bound state poles at \( \theta_{ab} = \frac{i5\pi}{18}, \frac{i\pi}{9} \) and \( \theta_{ac} = \frac{i13\pi}{18}, \frac{i7\pi}{18} \), one obtains a redundant but nevertheless consistent system of five equations in the four unknown \( p_i \) whose solution is given by
\[ p_0 = -p_1 = \frac{p_3}{2} = -39.74991118... \ , \quad p_2 = -198.2424080... \quad (3.37) \]
The other FFs which we have computed correspond to the states listed in Table 3. The values of the one-particle FFs are collected in Table 4, while the results concerning the two-particle computations are encoded in Table 5 via the coefficients \( a_{ab}^k \) of the polynomials \( P_{ab}(\theta) \).

### 3.3 Recursive Equations of Form Factors in the TIM

For sake of completeness, we now illustrate an efficient technique to compute multiparticle FFs. This is based on recursive identities which relate FFs of the type \( F_{1,1,...,1} \) with different (even) numbers of fundamental particles. Once these FFs are known, those relative
to \( Z_2 \) even multi–particle state involving heavier particles may be obtained through bootstrap procedure. In general this way of proceeding is the simplest one as far as FFs with three or more particles are concerned. In order to write down these recursive equations, we can adopt the following parameterization for the \( 2n \)–particles FF \( F_1,1,...,1 \):

\[
F_{1,1,...,1}(\theta_1,\ldots,\theta_{2n}) \equiv F_{2n}(\theta_1,\ldots,\theta_{2n}) = \frac{H_{2n}Q_{2n}(x_1,\ldots,x_{2n})}{\sigma_{2n}^{n-1}} \prod_{i<k} \frac{F^{\text{min}}_{11}(\theta_{ik})}{D_{11}(\theta_{ik})} \frac{1}{x_i + x_k}.
\]

Here and in the following \( \sigma_k(x_1,\ldots,x_{2n}) \) represents the symmetric polynomials of degree \( k \) in the variables \( x_i = e^{\theta_i} \) defined through their generating function

\[
\prod_{k=1}^{m} (x + x_k) = \sum_{j=0}^{m} x^{m-j} \sigma_j(x_1,\ldots,x_m).
\]

(3.39)

\( F_{11} \) and \( D_{11} \) are defined by (3.22) and (3.23) while \( H_n \) is an overall multiplicative constant and \( Q_n \) is a symmetric polynomial in its variables. The factors \( (x_i + x_k)^{-1} \) give a suitable parametrization of the kinematical poles, while the dynamical poles are taken into account by the functions \( D_{11} \)’s.

The polynomial \( Q_{2n} \) in the numerator can be factorized as

\[
Q_{2n}(x_1,\ldots,x_{2n}) = \sigma_1 \sigma_{2n-1} P_{2n}(x_1,\ldots,x_{2n}),
\]

(3.40)

since the FF will be proportional to the kinematical term \( P^2 \) relative to the total momentum which can be conveniently written as

\[
P^2 = m_1^2 \frac{\sigma_1 \sigma_{2n-1}}{\sigma_{2n}}.
\]

(3.41)

The Lorentz invariance of the FF requires \( P_{2n} \) to be an homogeneous polynomial with respect to all the \( x_i \)'s of total degree

\[
deg P_{2n} = 4n^2 - 5n,
\]

(3.42)

while the condition (2.28), knowing that \( \Delta_\epsilon = 1/10 \), imposes an upper bound to the degree in a single \( x_i \), given by

\[
deg_{x_i} P_{2n} < 4n - 22/5.
\]

(3.43)

Writing down the most general expression of \( P_{2n} \) as a symmetrical polynomial in the basis of the \( \sigma_k \)'s and taking into account the above conditions, one can determine the relative coefficients by means of the recursive equations. A first set of recursive relations is obtained by plugging the parametrization of \( F_{2n} \) into the equation of kinematical poles (2.17); the polynomial \( Q_n \) are then solution of the recursive equation

\[
Q_{2n+2}(-x,x,x_1,\ldots,x_{2n}) = -i Q_{2n}(x_1,\ldots,x_{2n}) U_{2n}(x|x_i),
\]

(3.44)
where the polynomial $U_{2n}$ is given by

\[
U_{2n}(x|x_i) = \prod_{i=1}^{n} \prod_{\alpha \in A_{11}} (x + e^{-i\pi\alpha} x_i)(x - e^{i\pi\alpha} x_i) + \prod_{i=1}^{n} \prod_{\alpha \in A_{11}} (x - e^{-i\pi\alpha} x_i)(x + e^{i\pi\alpha} x_i).
\] (3.45)

The overall constants $H_n$ have been fixed to be

\[
H_{2n} = 2\pi m_1^2 \left( 16 \prod_{\alpha \in A_1} g_\alpha(0) \frac{\cos^4(\pi\alpha/2)}{\sin(\pi\alpha)} \right)^{-n(n-1)},
\] (3.46)

with $H_2 = 2\pi m_1^2$. Given $Q_{2n}$, eq. (3.44) restricts the form of the polynomial $Q_{2n+2}$, although these equations cannot determine uniquely all its coefficients. In fact, polynomials containing the kernel factor $\prod_{i,j=1}^{2n} (x_i + x_j)$ can be added to a given solution $Q_{2n+2}$ with an arbitrary multiplicative factor, without affecting the validity of eq. (3.44). In order to have a more restrictive set of equations for the coefficients of the polynomials $Q_{2n}$, we employ the recursive equations (2.25). To relate $F_{2n+2}$ and $F_{2n}$, we consider two successive fusions $A_1A_1 \to A_2$ and $A_2A_1 \to A_1$, obtaining the following equations

\[
Q_{2n+2}(-\varphi x, x, \varphi x, x_2, \ldots, x_{2n}) = \phi_n M (\Gamma_{11}^2)^2 x^5 Q_{2n}(x, x_2, \ldots, x_{2n}) P_{2n}(x|x_i)
\] (3.47)

where

\[
M = 4 \cos(5\pi/18) \cos(8\pi/18),
\]

\[
\phi_n = (-1)^{n+1} \exp(-i\pi(10n + 1)/18),
\]

\[
\varphi = \exp(-i4\pi/9),
\]

and

\[
P_{2n}(x|x_i) = \prod_{i=2}^{2n} (x - e^{i8\pi/9} x_i)(x - e^{i5\pi/9} x_i)(x + e^{i\pi/3} x_i)(x + x_i).
\] (3.48)

As an application of the above equations, let us consider the determination of the FF $F_4$. Taking into account eqs. (3.42) and (3.43), we can write the following general parametrization for $P_4$ as

\[
P_4(x_1, \ldots, x_4) = c_1 \sigma_1^2 \sigma_4 + c_2 \sigma_2 \sigma_4 + c_3 \sigma_1 \sigma_2 \sigma_3 + c_4 \sigma_3^2 + c_5 \sigma_2^3.
\] (3.49)

From (3.44), knowing $Q_2 = \sigma_1$, one gets a first set of equations on the $c_i$’s

\[
c_2 = 4 \left( 2 \sin(\pi/9) + \sin(\pi/3) + 2 \sin(4\pi/9) \right),
\]

\[
c_5 = -4 \left( \sin(\pi/9) + \sin(4\pi/9) \right),
\]

\[
c_4 = c_1,
\]

\[
c_3 = c_5 - c_1.
\] (3.50)
The residual freedom in the parameters reflects the presence of kernels of eq. (3.44). Given any solution $Q_4^*$, the space of solutions is spanned by

$$Q_4^* = Q_4^* + \alpha \sigma_1 \sigma_3 \prod_{i,j=1}^{4} (x_i + x_j), \quad \alpha \in \mathbb{C} \quad (3.51)$$

Eq. (3.47) solves this ambiguity giving the last needed equation

$$c_1 = 2 \frac{4 \cos(\pi/18) - 11 \cos(\pi/6) + 12 \cos(5\pi/18) - 8 \cos(7\pi/18)}{3 + 5 \cos(5\pi/9) + \cos(\pi/3) - 3 \cos(\pi/9)}. \quad (3.52)$$

Finally one directly computes $H_4$ from (3.46).

The knowledge of $F_4 = F_{1111}$ allows us to compute through successive applications of (2.25) almost all the FFs we needed in order to reach the required precision of the FF expansion of the correlation function. We have used the obtained FFs to compute the two-point correlation function of $\Theta$ by means of the truncated spectral representation (2.11). A plot of $\langle \Theta(x) \Theta(0) \rangle$ as a function of $|x|$ is drawn in Figure 5. To control the accuracy of this result we have tested the fast convergence of the spectral series on the checks relative to the first two moments of the correlation function eqs. (2.33) and (2.34); the single contributions of each multiparticle state in the two series are listed in Table 3 and the partial sum is compared to the exact known values of the central charge $c$ and of the free energy amplitude $U$. A fast convergence behaviour of the spectral sum is indeed observed and therefore the leading dominant role of the first multiparticle states in eq.(2.11) is established.

4 Form Factors of the Energy Operator in the Thermal Deformed Tricritical Potts Model

In this section we will consider the FF computation for the Quantum Field Theory defined by the leading thermal deformation of the Tricritical 3–state Potts Model (TPM). Our strategy will resemble the one already applied to the TIM, with suitable generalizations in order to deal with this theory of degenerate mass spectrum.

4.1 Generalities of the TPM

The 3–state Potts Model at its tricritical point may be identified with the universality class of a subset of the minimal conformal model $M_{6/7}$ [26]. Its central charge is $c = 6/7$. The model is invariant under the permutation group $S_3$. The group $S_3$ is the semi–direct product of the two abelian groups $Z_2$ and $Z_3$, where the $Z_2$ group may be regarded as a charge conjugation symmetry implemented by the generator $C$. For the generator $\Omega$ of
the $Z_3$ symmetry, we have $\Omega^3 = 1$ and $\Omega C = -C \Omega$. The irreducible representations of $S_3$ could be either singlets, invariant with respect to $\Omega$ ($C$ even or $C$ odd) or $Z_3$ charged doublets.

The off–critical model we are interested in, is obtained by perturbing the fixed point action by means of the leading thermal operator $\epsilon(x)$ with conformal dimension $\Delta = 1/7$. This is a singlet field under both symmetries, $C$ and $\Omega$. Hence, the discrete $S_3$ symmetry of the fixed point is still preserved away from criticality and correspondingly the particle states organize into singlets or doublets. The scattering amplitudes of the massive excitations produced by the thermal deformation of the Tricritical Potts Model are nothing but the minimal $S$–matrix elements of the Affine Toda Field Theory based on the root system of $E_6$ (they have been determined and discussed in references \[30, 36\] and can be found in Table 7). Poles occur at values $i\alpha\pi$ with $\alpha$ a multiple of $1/12$, $12$ being the Coxeter number of the algebra $E_6$. The reason of the $E_6$ structure in the massive model is due both to the equivalent realization of the critical model in terms of the coset $(E_6)_1 \otimes (E_6)_1/(E_6)_2$ and to the fact that the leading energy operator $\epsilon(x)$ is associated to the adjoint representation in the decompostion of the fields \[27\]. Then, once again, one may apply the argument of references \[29\] to conclude that the massive theory inherits the $E_6$ symmetry of the fixed point.

The exact mass spectrum consists in two doublets $(A_l, A_l)$ and $(A_h, A_h)$, together with two singlet particle states $A_L$ and $A_H$ \[30, 36\]. Their mass ratios are given by

$$
\begin{align*}
    m_l &= m_T = M(g), \\
    m_L &= 2 m_l \cos \frac{\pi}{4} = (1.41421..) m_l, \\
    m_h &= m_T = 2 m_l \cos \frac{\pi}{12} = (1.93185..) m_l, \\
    m_H &= 2 m_L \cos \frac{\pi}{12} = (2.73205..) m_l,
\end{align*}
$$

(4.1)

where the mass scale depends on $g$ as \[20\]

$$
M(g) = C \, g^{\frac{\pi}{12}},
$$

(4.2)

and

$$
C = \left[ 4 \pi^2 \gamma\left(\frac{4}{7}\right) \gamma\left(\frac{9}{11}\right) \gamma\left(\frac{5}{2}\right) \gamma\left(\frac{11}{12}\right) \right]^{\frac{\pi}{12}} \frac{2 \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{13}{12}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{11}{12}\right)} = 3.746559718 \ldots.
$$

(4.3)

The above values of the masses are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra $E_6$ and therefore the particles may be associated to the dots of the Dynkin diagram (see Figure 6). Hence, they may be put in correspondence with the following representations of $E_6$ (identified by
their dimensions)

\[
\begin{align*}
A_l & \rightarrow 27 , \\
A_\overline{7} & \rightarrow \overline{27} , \\
A_L & \rightarrow 78 , \\
A_h & \rightarrow 351 , \\
A_{\overline{h}} & \rightarrow \overline{351} , \\
A_H & \rightarrow 2925 .
\end{align*}
\]

(4.4)

By introducing the alternative notation

\[
\begin{align*}
A_l & \rightarrow A_1 , \\
A_\overline{7} & \rightarrow \overline{A}_1 , \\
A_h & \rightarrow A_2 , \\
A_{\overline{h}} & \rightarrow \overline{A}_2 , \\
A_L & \rightarrow B_1 , \\
A_H & \rightarrow B_2 ,
\end{align*}
\]

the bootstrap fusions of this model can be written in the following compact way

\[
\begin{align*}
A_i \times A_i & = \overline{A}_1 + \overline{A}_2 , \\
A_i \times A_{i+1} & = \overline{A}_1 + \overline{A}_2 , \\
A_i \times \overline{A}_i & = B_i , \\
A_i \times \overline{A}_{i+1} & = B_1 + B_2 , \\
A_i \times B_i & = A_1 + A_2 , \\
\overline{A}_i \times B_i & = A_1 + A_2 , \\
A_i \times B_{i+1} & = A_{i+1} , \\
B_i \times B_i & = B_1 + B_2 , \\
B_i \times B_{i+1} & = B_1 + B_2 .
\end{align*}
\]

(4.5)

It is easy to check that the above fusion rules are a subset of the tensor product decom-
positions of the above representations of \( E_6 \).

4.2 Form Factors of the TPM

After a brief description of the model, let us turn our attention to the determination of
the matrix elements of the leading energy operator \( \epsilon(x) \). Our strategy will be similar to
that employed in the case of the TIM. For the TPM, however, we have a more stringent
selection rule coming from the \( Z_3 \) symmetry. Given the even parity of the operator \( \epsilon(x) \)
and its neutrality under the \( Z_3 \) symmetry, the only matrix elements which are different
from zero are those of singlet (multiparticle) states and they are the only contributions
which enter the spectral representation series (2.11). For convenience, the first such states
ordered according to the increasing value of the \( s \)-variable are listed in Table 8. Because of the selection rules, one very soon encounters three- and four-particle states among the first contributions, and therefore, the computation of FFs becomes in general quite involved.

Let us briefly illustrate the most interesting FF computations of this model. As far as one- and two-particles FFs are concerned, we just quote the result of the computations since they are quite straightforward and can be obtained by following the same strategy already adopted for the TIM; the one-particle FFs are given in Table 9, while the coefficients \( a^k_{ab} \) of the polynomials \( P_{ab}(\theta) \) of eq. (2.30) are listed in Table 10. The need to compute several three-particle FFs suggests however to adopt a more systematic technique based on the recursive structure of the FF. The lowest neutral mass state is given in this model by a doublet of conjugated particles \( l \) and \( \bar{l} \). Hence, in order to build useful “fundamental” singlet multiparticle FFs we have to consider recursive equations relating FFs of the kind \( F_{l_1l_2...l_n} \), with an arbitrary number of particle-antiparticle pairs.

From the knowledge of \( F_{l_1l_2...l_n} \) obtained as solutions of the recursive equations, we can next derive (by bootstrap fusion) all the three-particle FFs we need in our determination of the correlation function. To write these recursive equations, let us parametrize the FFs as

\[
F_{n(l\bar{l})}(\beta_1, \bar{\beta}_1, \ldots, \beta_n, \bar{\beta}_n) = \frac{H_n Q_n(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)}{(\sigma_m \sigma_n)^{n-1}},
\]

(4.6)

where

\[
\hat{F}_{n(l\bar{l})}(\beta_r - \bar{\beta}_s) \equiv \begin{cases} 
F_{n(l\bar{l})}(\beta_r - \bar{\beta}_s) & \text{if } r \leq s, \\
F_{n(l\bar{l})}(\beta_s - \beta_r) & \text{otherwise}.
\end{cases}
\]

(4.7)

In these expressions \( x_i = e^{\beta_i} \) and \( \sigma_m \) is the symmetrical polynomial of degree \( m \) in the \( x_i \)'s (the quantities \( \bar{x}_i \) and \( \sigma_m \) are analogously defined in terms of the \( \bar{\beta}_i \)'s). The two-particle minimal FFs are given by (see eqs. (2.15) and (2.23))

\[
\frac{F_{n(l\bar{l})}(\beta)}{D_{n(l\bar{l})}(\beta)} = \frac{F_{n(l\bar{l})}(\beta)}{D_{n(l\bar{l})}(\beta)} = \frac{-i \sinh(\beta/2) h_{1/6}(\beta) h_{2/3}(\beta) h_{1/2}(\beta)}{p_{1/6}(\beta) p_{2/3}(\beta)},
\]

(4.8)

\[
\frac{F_{n(l\bar{l})}(\beta)}{D_{n(l\bar{l})}(\beta)} = \frac{F_{n(l\bar{l})}(\beta)}{D_{n(l\bar{l})}(\beta)} = \frac{h_{5/6}(\beta) h_{1/3}(\beta) h_{1/2}(\beta)}{p_{1/2}(\beta)}.
\]

(4.9)

In (4.6), \( H_n \) is just a multiplicative overall factor and \( Q_n \) is a polynomial in its arguments. The latter is the only unknown quantity which can be computed through the recursive equations. The function \( Q_n \) must be a symmetrical polynomial both in the \( x_i \)'s and in
the $\sigma_i$’s separately. Furthermore, it must be symmetric under charge conjugation, i.e. under the simultaneous exchange $x_i \leftrightarrow \overline{x}_i \ (\forall i = 1 \ldots n)$. Hence, it can be parametrized in terms of products of $\sigma$’s and $\overline{\sigma}$’s with suitable coefficients in order to guarantee the self–conjugacy. The factor $P^2$ for this set of particles takes the form

$$\begin{align*}
P^2 &= \frac{(\sigma_n - \sigma_1)(\sigma_1 - \sigma_1)}{\sigma_n \overline{\sigma}_n} \, m_i^2, \\
\end{align*}$$

and, correspondingly $Q_n$ will be factorized as

$$Q_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) = (\sigma_n - \sigma_1)(\sigma_1 + \sigma_1)P_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n).$$

The Lorentz invariance of the FF requires $P_n$ to be an homogeneous polynomial with respect to all the $x$’s and $\overline{x}$’s of total degree

$$\text{deg } P_n = 3n^2 - 4n,$$

while the condition (2.28), knowing that $\Delta = 1/7$, imposes the following upper bound for the degree in a single $x_i$ ($\overline{x}_i$)

$$\text{deg}_x P_n < 3n - 74/21.$$

These conditions drastically restrict the possible form of the polynomials $Q_n$.

Let us write down the form assumed by the kinematical recursive equations by using the parametrization (1.6)

$$Q_{n+1}(-x, x, x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) = i \ x \ U_n(x|x_i, \overline{x}_i) \ Q_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n),$$

where (here $A_{ll} = \{1/6, 2/3, 1/2\}$)

$$U_n(x|x_i, \overline{x}_i) = \prod_{i=1}^{n} \prod_{\alpha \in A_{ll}} (x - e^{i\alpha \overline{x}_i})(x - e^{i\alpha x_i}) - \prod_{i=1}^{n} \prod_{\alpha \in A_{ll}} (x - e^{-i\alpha \overline{x}_i})(x - e^{-i\alpha x_i}).$$

The overall constant is explicitly given by:

$$H_n = 2 \pi m_i^2 \left(2 \tan^2(\pi/6) \tan^2(5\pi/12) \prod_{\alpha \in A_{ll}} g_{\alpha}(0) \sin(\pi \alpha)\right)^{-\frac{n(n-1)}{2}}.$$

However, the equations (1.14) are not in general sufficient to fix all the coefficients of $Q_{n+1}$. A more stringent constraint is obtained by using twice eq. (2.23) in relation with the processes $ll \to \overline{l}$ and $\overline{l}l \to l$. The final equations take a very simple form:

$$Q_{n+1}(\eta \overline{y}, \eta y, \overline{\eta} \overline{y}, \overline{\eta} y, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n) =$$

$$= - (\Gamma_{ll}^7)^{2} \ y \overline{y} \ W_n(y, \overline{y}, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n) Q_n(y, \overline{y}, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n),$$

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where $\eta = e^{i\pi/3}$ and

$$W_n(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) =$$

$$= (x_1 + \bar{x}_1)(x_1 - e^{\frac{7\pi i}{6}}x_1)(x_1 - e^{\frac{\pi i}{6}}x_1)$$

$$\cdot \prod_{i=2}^{n}(x_1 + x_i)(x_1 + \bar{x}_i)(x_1 - e^{\frac{5\pi i}{6}}x_i)(x_1 - e^{\frac{-5\pi i}{6}}x_i)(x_1 - e^{\frac{\pi i}{2}}x_i)(x_1 - e^{\frac{-\pi i}{2}}x_i).$$

Let us now illustrate how this procedure works in the case of $F_{2(l\bar{l})}$. Let us start from $F_{l\bar{l}}$; using eq. (2.31) we easily obtain $Q_1 = 1$ and $H_1 = 2\pi m_l^2$. From eqs. (4.12) and (4.13), the general parametrization for $P_2$ is given by

$$P_2(x_1, \bar{x}_1, x_2, \bar{x}_2) = c_1(\sigma_2^2 + \bar{\sigma}_2^2) + c_2(\sigma_1\sigma_2\bar{\sigma}_1 + \bar{\sigma}_1\sigma_2\sigma_1) +$$

$$+ c_3(\sigma_1^2\bar{\sigma}_2 + \bar{\sigma}_1^2\sigma_2) + c_4 \sigma_1^2\bar{\sigma}_1^2 + c_5 \sigma_2^2\bar{\sigma}_2^2.$$ 

Equation (4.14) gives four equations for the five parameters

$$c_4 = -(3 + \sqrt{3}),$$

$$c_2 - c_3 = -3(2 + \sqrt{3}),$$

$$c_1 - c_2 = 3 + 2\sqrt{3},$$

$$2c_2 + c_5 = -18 - 10\sqrt{3},$$

while eq. (4.17) solve the residual freedom yielding

$$c_1 = -\frac{9 + 5\sqrt{3}}{2},$$

$$c_2 = -\frac{3(5 + 3\sqrt{3})}{2},$$

$$c_3 = -\frac{3(1 + \sqrt{3})}{2},$$

$$c_4 = c_5 = -(3 + \sqrt{3}).$$

Once we have determined $H_1$ and $P_2$, we can obtain $F_{2(l\bar{l})}$ from eqs. (4.6) and (4.11). From this four–particles FF it is also easy to obtain the three–particles FFs $F_{l\bar{l}l}$, $F_{l\bar{L}l}$, $F_{l\bar{L}h}$ applying the residue equation (2.18) at the fusion angles $u_{l\bar{l}}^L$, $u_{l\bar{L}}^L$ and $u_{l\bar{L}}^h$ respectively. The explicit expressions of these three–particle FFs are given in Appendix B.

The FFs calculated for the TPM can be used to estimate the two–point function of the stress–energy tensor whose plot is shown in Figure 7. The convergence of the series may
be checked through the sum–rule tests: the contributions of each multiparticle state are listed in Table 8 where the exact and computed values of $c$ and $U$ are compared. A very fast convergence behaviour is indeed observed which supports the validity of the spectral approach to correlations functions in integrable massive models.

5 Conclusions

In this paper we have applied the Form Factor approach to estimate the correlation functions of the stress–energy tensor in the Tricritical Ising and 3–state Potts models. Both models have been perturbed away from the critical point by means of the leading thermal operator. In our computation, an important role has been played by the discrete symmetries of the two models, a $Z_2$ symmetry for the TIM and a $S_3$ symmetry for the TPM. These symmetries have in fact selected the appropriate particle states entering the spectral representation of the two–point correlator of the stress–energy tensor. This correlator has been plotted in Figure 5 for the TIM and in Figure 7 for the TPM: these plots are expected to be extremely precise in the large distance region ($mR \geq 1$) and sufficiently accurate in the crossover and ultraviolet regions ($mR \leq 1$). Obviously, a definite confirmation of their validity can only be obtained by a comparison with some experimental data or numerical simulations.

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Appendix A

In this appendix we collect some different explicit representations of the functions $g_\alpha(\theta)$ and $h_\alpha(\theta)$ together with some useful functional relations.

Let us start by considering the non–degenerate field theories. In this case, the basic functions $g_\alpha$ needed to build the minimal form factors are obtained as solution of the equations

\[ g_\alpha(\theta) = - f_\alpha(\theta) g_\alpha(-\theta) , \]

\[ g_\alpha(iπ + \theta) = g_\alpha(iπ - \theta) , \]

where

\[ f_\alpha(\theta) = \frac{\tanh \frac{1}{2} (\theta + iπ\alpha)}{\tanh \frac{1}{2} (\theta - iπ\alpha)} . \]

They are called minimal solutions because they do not present neither poles nor zeros in the strip $\text{Im}\theta \in (0, 2\pi)$. They admit several equivalent representations. The first is the integral representation given by

\[ g_\alpha = \exp \left[ 2 \int_0^\infty \frac{dt}{t} \frac{\cosh [(\alpha - 1/2)t]}{\cosh t/2 \sinh t} \sin^2(\hat{\theta}t/2\pi) \right] , \]

where $\hat{\theta} = iπ - \theta$. The analytic continuation of the above expression is provided by the infinite product representation

\[ g_\alpha(\theta) = \prod_{k=0}^{\infty} \left[ \frac{1 + \left( \frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}} \right)^2}{1 + \left( \frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}} \right)^2} \right] \left[ \frac{1 + \left( \frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}} \right)^2}{1 + \left( \frac{\hat{\theta}/2\pi}{k+1+\frac{\alpha}{2}} \right)^2} \right]^{k+1} \]

which explicitly shows the position of the infinite number of poles outside the strip $\text{Im}\theta \in (0, 2\pi)$. Another useful representation particularly suitable for deriving functional equations is the following:

\[ g_\alpha(\theta) = \prod_{k=0}^{\infty} \frac{\Gamma^2 \left( \frac{1}{2} + k + \frac{\alpha}{2} \right) \Gamma^2 \left( 1 + k - \frac{\alpha}{2} \right) \Gamma \left( 1 + k + \frac{\alpha}{2} + i\frac{\hat{\theta}}{2\pi} \right) \Gamma \left( 1 + k - \frac{\alpha}{2} + i\frac{\hat{\theta}}{2\pi} \right)}{\Gamma^2 \left( \frac{1}{2} + k + \frac{\alpha}{2} \right) \Gamma^2 \left( 1 + k - \frac{\alpha}{2} \right) \Gamma \left( 1 + k - \frac{\alpha}{2} + i\frac{\hat{\theta}}{2\pi} \right) \Gamma \left( 1 + k + \frac{\alpha}{2} + i\frac{\hat{\theta}}{2\pi} \right)} , \]

where we have used the notation

\[ \left| \Gamma(a + i\hat{\theta}/2\pi) \right|^2 \equiv \Gamma(a + i\hat{\theta}/2\pi) \Gamma(a - i\hat{\theta}/2\pi) . \]
A representation that is particularly suitable for numerical evaluations is the mixed one

\[
g_\alpha(\theta) = \prod_{k=0}^{N-1} \left[ 1 + \left( \frac{\theta/2\pi}{k+1/2} \right)^2 \right]^{k+1} \times \exp \left[ \frac{2}{t} \int_0^\infty \frac{dt}{\cosh \frac{t}{2} \sinh t} \left( N + 1 - Ne^{-2t} \right) e^{-2Nt} \sin^2 \frac{\theta t}{2\pi} \right].
\]

(A.7)

In this formula \( N \) is an arbitrary integer number which may be adopted to obtain a fast convergence of the integral.

Using the integral representation (A.3), it is easy to establish the asymptotic behaviour of \( g_\alpha \)

\[
g_\alpha(\theta) \sim e^{\|\theta\|/2} \quad \text{for} \quad \theta \to \infty.
\]

(A.8)

The function \( g_\alpha \) is normalized according to

\[
g_\alpha(i\pi) = 1,
\]

(A.9)

and satisfies

\[
g_\alpha(\theta) = g_{1-\alpha}(\theta),
\]

(A.10)

with

\[
g_0(\theta) = g_1(\theta) = -i \sinh \frac{\theta}{2}.
\]

(A.11)

The above functions satisfy the following set of functional equations

\[
g_\alpha(\theta + i\pi)g_\alpha(\theta) = -i \frac{g_\alpha(0)}{\sin \pi \alpha} (\sinh \theta + i \sin \pi \alpha),
\]

(A.12)

\[
g_\alpha(\theta + i\pi \gamma)g_\alpha(\theta - i\pi \gamma) = \frac{g_\alpha(i\pi \gamma)g_\alpha(-i\pi \gamma)}{g_{\alpha+\gamma}(0)g_{\alpha-\gamma}(0)} g_{\alpha+\gamma}(\theta)g_{\alpha-\gamma}(\theta),
\]

(A.13)

\[
g_{1-\alpha}(\theta)g_{\alpha-1}(\theta) = \sinh \frac{\theta}{2} \sinh \frac{i(\alpha - 1)\pi}{2} \sinh \frac{i(\alpha + 1)\pi}{2}.
\]

(A.14)

Let us turn our attention to the field theories with a degenerate mass spectrum. In complete analogy with the previous case, we start our analysis from the minimal solutions of the equations

\[
h_\alpha(\theta) = -s_\alpha(\theta) h_\alpha(-\theta),
\]

(A.15)

\[
h_\alpha(i\pi + \theta) = h_\alpha(i\pi - \theta),
\]

where

\[
s_\alpha(\theta) = \sinh \frac{\theta}{2} \left( \theta + i\pi \alpha \right) \sinh \frac{\theta}{2} \left( \theta - i\pi \alpha \right)
\]

(A.16)
The function $h_\alpha(\theta)$ is explicitly given in terms of the following equivalent representations

$$h_\alpha(\theta) = \exp \left[ 2 \int_0^\infty \frac{dt}{t} \frac{\sinh \left( (1 - \alpha) t \right)}{\sinh^2 t} \sin^2 (\hat{\theta} t / 2\pi) \right], \quad (A.17)$$

$$h_\alpha(\theta) = \prod_{k=0}^{\infty} \left( \frac{1 + \left( \frac{\hat{\theta} k}{\pi n + \frac{\alpha}{2} - \frac{\alpha}{2}} \right)^2}{1 + \left( \frac{\hat{\theta} k}{\pi n + \frac{\alpha}{2} - \frac{\alpha}{2}} \right)^2} \right)^{k+1}, \quad (A.18)$$

$$h_\alpha(\theta) = \prod_{k=0}^{\infty} \frac{\Gamma^2 \left( k + \frac{3}{2} + \frac{\alpha}{2} \right) \Gamma \left( k + 1 - \frac{\alpha}{2} - \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 - \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right)}{\Gamma^2 \left( k + \frac{1}{2} - \frac{\alpha}{2} \right) \Gamma \left( k + \frac{1}{2} + \frac{\alpha}{2} - \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right)} , \quad (A.19)$$

The mixed representation is in this case

$$h_\alpha(\theta) = \prod_{k=0}^{N+1} \left( \frac{1 + \left( \frac{\hat{\theta} k}{\pi n + \frac{\alpha}{2} - \frac{\alpha}{2}} \right)^2}{1 + \left( \frac{\hat{\theta} k}{\pi n + \frac{\alpha}{2} - \frac{\alpha}{2}} \right)^2} \right)^{k+1} \times \exp \left[ 2 \int_0^\infty \frac{dt}{t} (N + 1 - N e^{-2t}) e^{-2Nt} \frac{\sinh \left( (1 - \alpha) t \right)}{\sinh^2 t} \sin^2 (\hat{\theta} t / 2\pi) \right] , \quad (A.20)$$

and the asymptotic behaviour depends on the value of $\alpha$

$$h_\alpha(\theta) \sim e^{(1-\alpha)|\theta|} \text{ for } \theta \to \infty . \quad (A.21)$$

The function $h_\alpha$ is normalized according to

$$h_\alpha(i\pi) = 1 \quad (A.22)$$

and satisfies the following functional equations:

$$h_\alpha(2\pi i - \theta) = h_\alpha(\theta),$$

$$h_0(\theta) = -i \sinh(\theta / 2),$$

$$h_1(\theta) = 1,$$

$$h_{1+\alpha}(\theta) = h_{1-\alpha}(\theta), \quad (A.23)$$

The basic “composition rules” for products of $h_\alpha$’s are:

$$h_\alpha(\theta) h_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta),$$

$$h_\alpha(\theta + i\pi \gamma) h_\alpha(\theta - i\pi \gamma) = \frac{h_\alpha(i\pi \gamma) h_\alpha(-i\pi \gamma)}{h_{\alpha+\gamma}(0) h_{\alpha-\gamma}(0)} h_{\alpha+\gamma}(\theta) h_{\alpha-\gamma}(\theta) \quad (A.24)$$

$$h_\alpha(\theta + i\pi) h_{1-\alpha}(\theta) = \frac{h_{1-\alpha}(0)}{\cosh \left( \frac{i\pi \alpha}{2} \right)} \cosh \frac{\theta - i\pi \alpha}{2}.$$
where the polynomial \( P \) is defined in (2.22) of Section 2.

Finally, since \( f_\alpha(\theta) = s_\alpha(\theta)s_{1-\alpha}(\theta) \), the function \( g_\alpha \) can be obtained from the \( h_\alpha \)’s simply through:
\[
g_\alpha(\theta) = h_\alpha(\theta) h_{1-\alpha}(\theta).
\] (A.25)

**Appendix B**

In this appendix we briefly report the results of the three–particle FFs relevant for our computation in the TPM. These FFs have been derived by applying the residue equations (2.18) to the four–particles FF \( F_{\Theta l l} \), as explained in section 4. In writing their final form, we have extensively used the formulas reported in Appendix A. The two–particle minimal FFs \( F_{\min}^{ab} \) appearing in the expressions which follow are defined by eq. (2.15) while the \( D_{ab} \) factors parametrizing the dynamical poles are defined by eq. (2.23).

The FF \( F_{\Theta l l} \) is obtained from \( F_{\Theta l l} \) through the residue equation at \( u_{l l} = 2i\pi/3 \)
\[
F_{\Theta l l}(\theta_1, \theta_2, \theta_3) = \left( \prod_{i<j} \frac{F_{\min}^{\Theta l l}(\theta_{ij})}{D_{\Theta l l}(\theta_{ij})} \right) \left( 3 m_l^2 + 2 m_l^2 \sum_{i<j} \cosh(\theta_{ij}) \right) \cdot a_{l l l}^0.
\] (B.1)

In this expression one immediately recognizes the “minimal” part, the dynamical poles and the \( P^2 \) polynomial, while the only remaining polynomial in the \( \cosh(\theta_{ij}) \)’s allowed by eq. (2.28) is simply a constant given by
\[
a_{l l l}^0 = -102.3375342 \ldots.
\]

The FF \( F_{\Theta l l} \) is obtained from \( F_{\Theta l l} \) by using eq.(2.18), with \( u_{l l} = i\pi/2 \). Its final expression is given by
\[
F_{\Theta l l}(\theta_1, \theta_2, \theta_3) = \frac{F_{\Theta l l}^{\min}(\theta_{12}) F_{\Theta l l}^{\min}(\theta_{13}) F_{\Theta l l}^{\min}(\theta_{23})}{D_{\Theta l l}(\theta_{12}) D_{\Theta l l}(\theta_{13}) D_{\Theta l l}(\theta_{23})} \cdot \frac{2 m_l^2 + m_L^2 + 2 m_l^2 \cosh(\theta_{12}) + 2 m_L m_l \left( \cosh(\theta_{13}) + \cosh(\theta_{23}) \right)}{\cosh(\theta_{13}) + \cosh(\theta_{23})} \cdot \left( a_{l l l}^0 \left( 1 - \cosh(\theta_{12}) + 2 \cosh(\theta_{13}) \cosh(\theta_{23}) \right) + a_{l l l}^1 \left( \cosh(\theta_{13}) + \cosh(\theta_{23}) \right) \right).
\] (B.2)

This expression also exhibits a kinematical pole due to the presence of a particle–antiparticle pair \( l \bar{l} \). Moreover there is a nontrivial polynomial in the \( \cosh(\theta_{ij}) \)’s with coefficients given by
\[
a_{l l l}^0 = -70.50661963 \ldots,
\]
\[
a_{l l l}^1 = -235.9197474 \ldots.
\]
Finally, applying eq. (2.18) to $F_{llh}^\Theta$ at $u_{llh}^l = i\pi/6$ one obtains

$$F_{llh}^\Theta(\theta_1, \theta_2, \theta_3) = \frac{F_{llh}^{\min}(\theta_1) F_{lh}^{\min}(\theta_2) F_{lh}^{\min}(\theta_3)}{D_{ll}(\theta_1) D_{lh}(\theta_1) D_{lh}(\theta_2)} \cdot$$

$$\cdot \left(2 m_l^2 + m_h^2 + 2 m_l^2 \cosh(\theta_{12}) + 2 m_l m_h \left(\cosh(\theta_{13}) + \cosh(\theta_{23})\right)\right) \cdot$$

$$\cdot \left(a_{llh}^0 + a_{llh}^1 \left(\cosh(\theta_{13}) + \cosh(\theta_{23})\right) + a_{llh}^2 \cosh(\theta_{12}) + a_{llh}^3 \cosh(\theta_{13}) \cosh(\theta_{23})\right)$$

where the coefficients $a_{llh}^k$ are given by

$$a_{llh}^0 = 78134.00044 \ldots ,$$

$$a_{llh}^1 = 72661.45729 \ldots ,$$

$$a_{llh}^2 = 31793.68905 \ldots ,$$

$$a_{llh}^3 = 43430.98692 \ldots .$$
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Table Captions

**Table 1** Particle spectrum, mass ratios and $Z_2$-charges in the TIM.

**Table 2** Two-particle $S$-matrix elements of the TIM; the notation $(x) \equiv f_x/h(\theta)$ has been followed, where $h = 18$ is the Coxeter number of $E_7$ and the function $f_\alpha$ is defined in eq. (A.2). Superscripts label the particles occurring as bound states at the fusion angles $u_{ab}^\alpha = x\pi/h$.

**Table 3** The first $Z_2$-even multiparticle states of the TIM ordered according to the increasing value of the center-of-mass energy and their relative contributions to the spectral sum rules of the central charge $c$ and the free-energy amplitude $U$.

**Table 4** One-particle FFs of the $Z_2$-even particles of the TIM.

**Table 5** Coefficients which enter eq. (2.30) for the lightest two-particle FFs of the TIM.

**Table 6** Particle spectrum, mass ratios and $Z_3$-charges in the TPM.

**Table 7** Two-particle $S$-matrix elements of the TPM. In this case $[x] \equiv s_x/h(\theta)$, where $s_\alpha$ is defined in eq. (A.16) and $h = 12$ is the Coxeter number of $E_6$.

**Table 8** The first $Z_3$-neutral multiparticle states of the TIM ordered according to the increasing value of the center-of-mass energy and their relative contributions to the spectral sum rules of the central charge $c$ and the free-energy amplitude $U$.

**Table 9** One-particle FFs of the $Z_2$-neutral particles of the TPM.

**Table 10** Coefficients which enter eq. (2.30) for the lightest two-particle FFs of the TPM.
| particle | mass/m_1 | Z_2 charge |
|----------|----------|-------------|
| A_1      | 1.00000  | -1          |
| A_2      | 1.28558  | 1           |
| A_3      | 1.87939  | -1          |
| A_4      | 1.96962  | 1           |
| A_5      | 2.53209  | 1           |
| A_6      | 2.87939  | -1          |
| A_7      | 3.70167  | 1           |

**Table 1**
| $a$ | $b$ | $S_{ab}$ |
|-----|-----|---------|
| 1   | 1   | $\frac{2}{3}$ $(10)(2)$ |
| 1   | 2   | $\frac{1}{13}$ $(7)$ |
| 1   | 3   | $\frac{2}{4}$ $(14)(6)$ |
| 1   | 4   | $\frac{1}{17}$ $(11)(3)(9)$ |
| 1   | 5   | $\frac{3}{6}$ $(14)(8)(6)^2$ |
| 1   | 6   | $\frac{4}{5}$ $(16)(12)(4)(10)^2$ |
| 1   | 7   | $\frac{6}{7}$ $(15)(9)(5)^2(7)^2$ |
| 2   | 2   | $\frac{2}{4}$ $(12)(2)$ |
| 2   | 3   | $\frac{1}{15}$ $(11)(5)(9)$ |
| 2   | 4   | $\frac{2}{14}$ $(8)(6)^2$ |
| 2   | 5   | $\frac{2}{17}$ $(13)(3)(7)^2(9)$ |
| 2   | 6   | $\frac{3}{15}$ $(7)^2(5)^2(9)$ |
| 2   | 7   | $\frac{5}{16}$ $(10)^3(4)^2(6)^2$ |
| 3   | 3   | $\frac{2}{14}$ $(2)(8)^2(12)^2$ |
| 3   | 4   | $\frac{1}{15}$ $(5)^2(7)^2(9)$ |
| 3   | 5   | $\frac{1}{16}$ $(10)^3(4)^2(6)^2$ |
| 3   | 6   | $\frac{2}{16}$ $(12)^3(8)(4)^2$ |
| 3   | 7   | $\frac{3}{17}$ $(13)^3(3)^2(7)^4(9)^2$ |

Table 2 (Continued)
| $a$ | $b$ | $S_{ab}$ |
|-----|-----|---------|
| 4   | 4   | $\frac{4}{5} (12) \frac{4}{5} (10)^3 (7) (2)^2$ |
| 4   | 5   | $\frac{2}{7} (15) \frac{4}{7} (13)^3 (7)^3 (9)$ |
| 4   | 6   | $\frac{1}{6} (17) \frac{6}{6} (11)^3 (3)^2 (5)^2 (9)^2$ |
| 4   | 7   | $\frac{4}{5} (16) \frac{5}{5} (14)^3 (6)^4 (8)^4$ |
| 5   | 5   | $\frac{5}{3} (12)^3 (2)^2 (4)^2 (8)^4$ |
| 5   | 6   | $\frac{1}{3} (16) \frac{3}{3} (14)^3 (6)^4 (8)^4$ |
| 5   | 7   | $\frac{2}{7} (17) \frac{4}{7} (15)^3 (11)^5 (5)^4 (9)^3$ |
| 6   | 6   | $\frac{4}{7} (14)^3 (10)^5 (12)^4 (16)^2$ |
| 6   | 7   | $\frac{1}{6} (17) \frac{3}{6} (15)^3 (13)^5 (5)^6 (9)^3$ |
| 7   | 7   | $\frac{2}{7} (16)^3 \frac{5}{7} (14)^5 (12)^7 (8)^8$ |

**Table 2 (Continuation)**
| state     | $s/m_1^2$ | c-series | U-series |
|-----------|-----------|----------|----------|
| $A_2$     | 1.28558   | 0.6450605| 0.0706975|
| $A_4$     | 1.96962   | 0.0256997| 0.0066115|
| $A_1 A_1$ | $\geq 2.00000$ | 0.0182735| 0.0071135|
| $A_5$     | 2.53209   | 0.0032417| 0.0013783|
| $A_2 A_2$ | $\geq 2.57115$ | 0.0032549| 0.0025194|
| $A_1 A_3$ | $\geq 2.87939$ | 0.0012782| 0.0020630|
| $A_2 A_4$ | $\geq 3.25519$ | 0.0003010| 0.0007277|
| $A_1 A_1 A_2$ | $\geq 3.28558$ | 0.0007139| 0.001184|
| $A_7$     | 3.70167   | 0.0000316| 0.0000287|
| $A_3 A_3$ | $\geq 3.75877$ | 0.0000700| 0.0001173|
| $A_2 A_5$ | $\geq 3.81766$ | 0.0000860| 0.0001581|
| partial sum | | 0.6980109| 0.0914150|
| exact value | | 0.7000000| 0.0942097|

Table 3
Table 4

| $F_2^θ$  | 0.9604936853 |
|----------|--------------|
| $F_4^θ$  | -0.4500141924 |
| $F_5^θ$  | 0.2641467199  |
| $F_7^θ$  | -0.0556906385 |

Table 5

| $a_{11}^0$ | 6.283185307  |
|------------|--------------|
| $a_{13}^0$ | 30.70767637  |
| $a_{22}^0$ | 15.09207695  |
| $a_{22}^1$ | 4.70783688   |
| $a_{24}^0$ | 79.32168252  |
| $a_{24}^1$ | 16.15028004  |
| $a_{33}^0$ | 295.3281130  |
| $a_{33}^1$ | 396.9648559  |
| $a_{33}^2$ | 123.8295119  |
| $a_{25}^0$ | 3534.798444  |
| $a_{25}^1$ | 4062.255130  |
| $a_{25}^2$ | 556.5589101  |
| particle | mass/m₀ | Z₃ charge |
|----------|---------|-----------|
| Aₜ       | 1.00000 | e²πi/3    |
| Aₙ       | 1.00000 | e⁻²πi/3   |
| Aₙ       | 1.41421 | 1         |
| Aₙ       | 1.93185 | e²πi/3    |
| Aₙ       | 1.93185 | e⁻²πi/3   |
| Aₙ       | 2.73205 | 1         |

Table 6
\[
\begin{array}{|c|c|}
\hline
a & b & S_{ab} \\
\hline
l & l & \overline{7} [8] [6] \overline{h} [2] \\
7 & 7 & \overline{7} [8] [6] h [2] \\
l & 7 & - [10] L [6] [4] \\
l & L & l [9] [7] h [5] [3] \\
\overline{l} & L & \overline{7} [9] [7] \overline{h} [5] [3] \\
l & h & [9] [7] [5]^{2} [3] \overline{7} h [11] \\
7 & \overline{h} & [9] [7] [5]^{2} [3] l [11] \\
l & \overline{h} & [9] [7]^{2} [5] H [3] [1] \\
7 & h & L [9] [7]^{2} [5] H [3] [1] \\
l & H & h [10] [8]^{2} [6]^{2} [4]^{2} [2] \\
\overline{7} & H & h [10] [8]^{2} [6]^{2} [4]^{2} [2] \\
L & L & - [10] l [8] [6]^{2} [4] H [2] \\
L & h & [10] [8]^{2} [6]^{2} [4]^{2} [2] \\
L & \overline{h} & [10] [8]^{2} [6]^{2} [4]^{2} [2] \\
L & H & L [11] [9]^{2} H [7]^{3} [5]^{3} [3]^{2} [1] \\
h & h & \overline{7} [10] [8]^{3} [6]^{3} [4]^{2} [2]^{2} \\
\overline{h} & \overline{h} & [10] [8]^{3} [6]^{3} [4]^{2} [2]^{2} \\
h & \overline{h} & - [10]^{2} [8]^{2} H [6]^{3} [4]^{3} [2] \\
h & H & l [11] [9] [7]^{4} [5]^{4} [3]^{3} [1] \\
\overline{h} & H & [11] [9]^{3} [7]^{4} [5]^{4} [3]^{3} [1] \\
H & H & - [10]^{3} L [8]^{5} [6]^{6} [4]^{5} [2]^{3} \\
\hline
\end{array}
\]

Table 7
\begin{table}
\centering
\begin{tabular}{|l|l|l|l|}
\hline
state & \(s/m_1^2\) & c–series & u–series \\
\hline
\(A_L\) & 1.41421 & 0.7596531 & 0.0705265 \\
\(A_l A_T\) & \(\geq 2.0000\) & 0.0844238 & 0.0229507 \\
\(A_H\) & 2.73205 & 0.0029236 & 0.001013 \\
\(A_L A_L\) & \(\geq 2.82843\) & 0.0024419 & 0.0019380 \\
\(A_l A_T\) & \(\geq 2.93185\) & 0.0023884 & 0.0016745 \\
\(A_T A_h\) & \(\geq 2.93185\) & 0.0023884 & 0.0016745 \\
\(A_l A_l A_l\) & \(\geq 3.0000\) & 0.0004215 & 0.0004925 \\
\(A_T A_T A_T\) & \(\geq 3.0000\) & 0.0004215 & 0.0004925 \\
\(A_l A_T A_L\) & \(\geq 3.41421\) & 0.00159 & 0.000251 \\
\(A_h A_T\) & \(\geq 3.86370\) & 0.0000504 & 0.0001476 \\
\(A_l A_l A_h\) & \(\geq 3.93185\) & 0.000089 & 0.0002015 \\
\(A_T A_T A_T\) & \(\geq 3.93185\) & 0.000089 & 0.0002015 \\
\(A_l A_T A_T A_T\) & \(\geq 4.0000\) & 0.0000959 & 0.000381 \\
\hline
\textit{partial sum} & & 0.8569765 & 0.1019449 \\
\textit{exact value} & & 0.8571429 & 0.1056624 \\
\hline
\end{tabular}
\caption{Table 8}
\end{table}
Table 9

| \( F_L^{\oplus} \) | 1.261353947 |
| \( F_H^{\oplus} \) | 0.292037405 |

Table 10

| \( a_{lL}^0 \) | 6.283185307 |
| \( a_{LL}^0 \) | 21.76559237 |
| \( a_{LL}^1 \) | 9.199221756 |
| \( a_{lH}^0 \) | 25.22648264 |
| \( a_{h\pi}^0 \) | 414.1182423 |
| \( a_{h\pi}^1 \) | 565.6960386 |
| \( a_{h\pi}^2 \) | 175.0269632 |
Figure Captions

Figure 1 Diagrammatic interpretation of the process responsible for a single-pole in a Form Factor.

Figure 2 Diagrammatic interpretation of the process responsible for a double-pole in a Form Factor.

Figure 3 Diagrammatic interpretation of the process responsible for a triple-pole in a Form Factor (here \( \phi = u_{ab}^f \)).

Figure 4 Dynkin diagram of \( E_7 \) and assignment of the masses to the corresponding dots.

Figure 5 Plot of the correlation function \( \langle \Theta(x)\Theta(0) \rangle/m_1^4 \) versus the scaling variable \( m_1 \vert x \vert \) in the TIM. The spectral series (2.11) includes the FF contributions relative to the multiparticle states in Table 3.

Figure 6 Dynkin diagram of \( E_6 \) and assignment of the masses to the corresponding dots.

Figure 7 Plot of the correlation function \( \langle \Theta(x)\Theta(0) \rangle/m_1^4 \) versus the scaling variable \( m_1 \vert x \vert \) in the TPM. The spectral series (2.11) includes the FF contributions relative to the multiparticle states in Table 8.
Figure 2
Figure 4
Figure 6
Figure 7