On the number of variables in undecidable superintuitionistic propositional calculi

Grigoriy V. Bokov
Department of Mathematical Theory of Intelligent Systems
Lomonosov Moscow State University
Moscow, Russian Federation
E-mail: bokov@intsys.msu.ru

April 15, 2015

Abstract

In this paper, we construct an undecidable three-variable superintuitionistic propositional calculus, i.e., a finitely axiomatizable extension of the intuitionistic propositional calculus with axioms containing only 3 variables. We also show that no a two-variable propositional calculus can derive all intuitionistic tautologies. Particularly, there is no an undecidable superintuitionistic propositional calculus using axioms in 2 variables.

1 Introduction

Decidability is the most important property of propositional calculi, it means that the set of their derivable formulas (or theorems) can be effectively determined. A natural question is how to separate classes of decidable and undecidable calculi. On the other hand, since undecidable propositional calculi can be used as a base for obtaining “negative” solutions to various algorithmic problems, it is of interest to find the simplest possible calculi of that class. There are many possible ways to separate decidable and undecidable calculi. A significant and simplest way is to describe the number of variables in their axioms.

In 1949, Linial and Post [10] found a first undecidable propositional calculus. In 1975, Hughes and Singletary [9] proved that there is undecidable propositional calculi with axioms containing 3 variables. In 1976, Hughes [8] constructed an undecidable implicational propositional calculus using axioms in 2 variables. The final solution was found by Gladstone in 1979. In [7] he proved that every one-variable propositional calculus is decidable.

A first undecidable superintuitionistic propositional calculus was built in 1978 by Shehtman [14, 15]. Axioms of this calculus are containing 7 variables. Later Chagrov in 1994 [4] did the same using axioms in only 4 variables. In [5, Sections 16.9] he noted that for 2 and 3 variables this question is open.

In [6] Gladstone proved that the following formula

\[ A = (p \to q) \to ((q \to r) \to (p \to r)) \]
is not derivable from the set of all two-variable tautologies by modus ponens and substitution. Since \( A \) is the intuitionistic tautology, therefore no a two-variable propositional calculus can derive all intuitionistic tautologies. If we combine this with the Gladstone result for one-variable propositional calculi, we get that there is no an undecidable superintuitionistic propositional calculus with axioms containing less than 3 variables. The aim of this paper is to construct an undecidable three-variable superintuitionistic propositional calculus.

This paper is organized as follows. In the next section we introduce the basic terminology and notation. In Section 3 we state and prove our main result. Finally, in Section 4 we give some concluding remarks and discuss further researches.

2 Definitions

In this section, we recall definitions of the intuitionistic propositional calculus and Kripke semantics. For more details we refer the reader to [5].

First, we introduce some notation. Let us consider the language consisting of an infinite set of propositional variables \( \mathcal{V} \), brackets, and the signature \( \Sigma = \{ \bot, \land, \lor, \rightarrow \} \), where \( \bot \) is the constant symbol, \( \land, \lor \) and \( \rightarrow \) are binary connectives. Letters \( p, q, r, x, y \), etc., are used to denote propositional variables. We define \( \neg, \leftrightarrow \) and \( \top \) as the usual abbreviations:

\[
\neg A := A \rightarrow \bot, \quad A \leftrightarrow B := (A \rightarrow B) \land (B \rightarrow A), \quad \top = \neg \bot.
\]

**Propositional formulas** or **\( \Sigma \)-formulas** are built up from the signature \( \Sigma \), propositional variables from \( \mathcal{V} \), and brackets in the usual way. For example, the following notations

\[
x, \enspace \neg A, \enspace (A \land B), \enspace (A \lor B), \enspace (A \rightarrow B)
\]

are formulas. Capital letters \( A, B, C \), etc., are used to denote propositional formulas. Throughout the paper, we will omit the outermost parentheses in formulas and parentheses assuming the customary priority of connectives: we assume \( \neg \) to connect formulas more strongly than \( \land \) and \( \lor \), which in turn are stronger than \( \rightarrow \).

By a **propositional calculus** or a **\( \Sigma \)-calculus** we mean a finite set \( \mathcal{P} \) of \( \Sigma \)-formulas referred to as **axioms** together with two rules of inference:

1) **modus ponens**

\[
A, \enspace A \rightarrow B \vdash B,
\]

2) **substitution**

\[
A \vdash \sigma A,
\]

where \( \sigma A \) is the substitution instance of \( A \), i.e., the result of applying the substitution \( \sigma \) to the formula \( A \).

Denote by \( [\mathcal{P}] \) the set of derivable (or provable) formulas of a calculus \( \mathcal{P} \). A **derivation** in \( \mathcal{P} \) is defined from the axioms and the rules of inference in the usual way. The statement that a formula \( A \) is derivable from \( \mathcal{P} \) is denoted by \( \mathcal{P} \vdash A \).

Let us introduce the following pre-order relation on the set of all propositional calculus. We write \( \mathcal{P}_1 \leq \mathcal{P}_2 \) (or, equivalently, \( \mathcal{P}_2 \geq \mathcal{P}_1 \)) if each derivable formula of \( \mathcal{P}_1 \) is also derivable from \( \mathcal{P}_2 \), i.e., if \( [\mathcal{P}_1] \subseteq [\mathcal{P}_2] \). We write \( \mathcal{P}_1 \sim \mathcal{P}_2 \) and say that two calculi \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are **equivalent** if \( [\mathcal{P}_1] = [\mathcal{P}_2] \). Finally, we write \( \mathcal{P}_1 < \mathcal{P}_2 \) if \( [\mathcal{P}_1] \subsetneq [\mathcal{P}_2] \).

An **intuitionistic Kripke frame** is a pair \( \mathfrak{F} = \langle W, R \rangle \) consisting of a nonempty set \( W \) and a partial order \( R \) on \( W \), which is reflexive, transitive and antisymmetric, i.e., \( \mathfrak{F} \) is just a
It is well known that \( \mathcal{F} = \mathcal{A} \) say that \( \mathcal{A} \) instead of (for all \( w \) if \( \mathcal{A} \) which is read as \( \mathcal{A} \)).

A valuation in an intuitionistic frame \( \mathfrak{F} = \langle W, R \rangle \) is a map \( \mathfrak{V} \) associating with each propositional variable \( p \in \mathcal{V} \) some (possibly empty) subset \( \mathfrak{V}(p) \) of \( W \) such that, for every \( w \in \mathfrak{V}(p) \) and every \( w' \in W \), \( w \leq_R w' \) implies \( w' \in \mathfrak{V}(p) \).

An intuitionistic Kripke model is a pair \( \mathcal{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle \), where \( \mathfrak{F} \) is an intuitionistic frame and \( \mathfrak{V} \) is a valuation in \( \mathfrak{F} \).

Let \( \mathcal{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle \) be an intuitionistic Kripke model and \( w \) be a point in the frame \( \mathfrak{F} = \langle W, R \rangle \). By induction on the construction of a formula \( A \) we define a relation \( (\mathcal{M}, w) \models A \), which is read as \( A \) true at \( w \) in \( \mathcal{M} \):

\[
(\mathcal{M}, w) \not\models \bot \quad \iff \quad w \in \mathfrak{V}(p);
\]

\[
(\mathcal{M}, w) \models A \land B \quad \iff \quad (\mathcal{M}, w) \models A \text{ and } (\mathcal{M}, w) \models B;
\]

\[
(\mathcal{M}, w) \models A \lor B \quad \iff \quad (\mathcal{M}, w) \models A \text{ or } (\mathcal{M}, w) \models B;
\]

\[
(\mathcal{M}, w) \models A \rightarrow B \quad \iff \quad \text{for all } w' \in W \text{ such that } w \leq_R w',
\]

\[
(\mathcal{M}, w') \models A \text{ implies } (\mathcal{M}, w') \models B.
\]

From the definition it follows that

\[
(\mathcal{M}, w) \models \top \quad \iff \quad w \in \mathfrak{V}(p);
\]

\[
(\mathcal{M}, w) \models \neg A \quad \iff \quad \text{for all } w' \in W \text{ such that } w \leq_R w', (\mathcal{M}, w') \not\models A.
\]

If \( (\mathcal{M}, w) \models A \) does not hold, i.e., \( (\mathcal{M}, w) \not\models A \), we say that \( A \) is refuted at the point \( w \) in \( \mathcal{M} \).

We say that \( A \) is true in a model \( \mathcal{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle \) defined on a frame \( \mathfrak{F} = \langle W, R \rangle \) if \( (\mathcal{M}, w) \models A \) for all \( w \in W \); if \( A \) is true in \( \mathcal{M} \), we write \( \mathcal{M} \models A \). We say that \( A \) is true in a frame \( \mathfrak{F} = \langle W, R \rangle \) if \( A \) is true in every model based on \( \mathfrak{F} \); if \( A \) is true in \( \mathfrak{F} \), we write \( \mathfrak{F} \models A \). We say that \( A \) is true at the point \( w \) in frame \( \mathfrak{F} \) if \( (\mathcal{M}, w) \models A \) for every model \( \mathcal{M} \) defined on \( \mathfrak{F} \); if \( A \) is true at the point \( w \) in frame \( \mathfrak{F} \), we write \( (\mathfrak{F}, w) \models A \). If \( \mathcal{M} \) is fixed we write \( w \models A \) instead of \( (\mathcal{M}, w) \models A \).

We define the intuitionistic propositional calculus \( \text{Int} \) as the following set of axioms:

\[
\begin{align*}
(\to_1) & \quad p \to (q \to p) \\
(\to_2) & \quad (p \to (q \to r)) \rightarrow ((p \to q) \to (q \to r)) \\
(\land_1) & \quad p \land q \to p \\
(\land_2) & \quad p \land q \to q \\
(\land_3) & \quad p \to (q \to p \land q) \\
(\lor_1) & \quad p \to p \lor q \\
(\lor_2) & \quad q \to p \lor q \\
(\lor_3) & \quad (p \to r) \rightarrow ((q \to r) \to (p \lor q \to r)) \\
(\neg_1) & \quad (p \to q) \rightarrow ((p \to \neg q) \to \neg p) \\
(\neg_2) & \quad p \to (\neg p \to q)
\end{align*}
\]

It is well known that

\[
\text{Int} \vdash A \quad \iff \quad \mathfrak{F} \models A, \text{ for every Kripke frame } \mathfrak{F}.
\]

3
By a superintuitionistic propositional calculus we mean a finitely axiomatizable extension of \( \text{Int} \), i.e., a propositional calculus obtained from \( \text{Int} \) by adding a finite set of new axioms. If \( M \) is a finite set of propositional formulas, then a propositional calculus obtained from \( \text{Int} \) by adding new axioms \( M \) is denoted by \( \text{Int} + M \). Let us call a propositional formulas \( A \) superintuitionistic if \( \text{Int} + A > \text{Int} \). Since

\[
\text{Int} + \{A_1, \ldots, A_n\} \sim \text{Int} + A_1 \wedge \ldots \wedge A_n,
\]

we can assume that a superintuitionistic propositional calculus is a calculus \( \text{Int} + A \) for some superintuitionistic propositional formula \( A \).

3 Main result

Our main result is the following theorem.

**Theorem 3.1.** There is a three-variable superintuitionistic propositional formula \( A \) such that \( \text{Int} + A \) is undecidable.

First, we recall what a Minsky machine is and encode configurations of a Minsky machine by superintuitionistic propositional formulas. Next, we construct a Kripke model refuting all codes of derivable configurations. Finally, we encode instructions of a Minsky machine \( M \) by a single superintuitionistic formula \( A_M \) and formally reduce the configuration problem of \( M \) to the derivation problem of a superintuitionistic propositional calculus \( \text{Int} + A_M \).

3.1 Minsky machine

There are many algorithmic formalisms to prove the undecidability of a propositional calculus \[3\]. For example, the undecidability of a calculus contained in the classical \[1\], intuitionistic \[2\] propositional calculi or in another subcalculi \[3\] can be easily proved by using tag systems. But for calculi, which contain the intuitionistic propositional calculus, this is very hard \[12, 16\]. For this reason, in order to prove the undecidability of superintuitionistic propositional calculi we will use an algorithmic formalism which is called Minsky machines \[11\]. In \[5\] Chagrov mentioned that it is the most convenient for being simulated by modal and intuitionistic formulas.

In accordance with \[5\] we define a Minsky machine as a finite set of instructions for transforming triples \( \langle s, m, n \rangle \) of natural numbers, called configurations, where \( s \) is the number of the instruction to be executed at the next step (referred to as the current machine state), and \( m, n \in \mathbb{N} \). Each instruction has one of the following four forms:

\[
\begin{align*}
    s \mapsto \langle t, 1, 0 \rangle, & \quad s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle, \\
    s \mapsto \langle t, 0, 1 \rangle, & \quad s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle,
\end{align*}
\]

where \( s, t, u \) are the machine states. Note that all Minsky machines are assumed to be deterministic, i.e., they may not contain distinct instructions with the same numbers.

As an example, let us consider the applying of first two instructions. The instruction

\[
    s \mapsto \langle t, 1, 0 \rangle
\]

\[1\] We assume that \( \mathbb{N} = \{0, 1, 2, \ldots\} \).
transforms \( \langle s, m, n \rangle \) into \( \langle t, m + 1, n \rangle \), and the instruction

\[
s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle
\]

transforms \( \langle s, m, n \rangle \) into \( \langle t, m - 1, n \rangle \) if \( m > 0 \) and into \( \langle u, m, n \rangle \) if \( m = 0 \). The meaning of the others is defined analogously.

Let \( \mathcal{M} \) be a Minsky machine, then the notation \( \langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle \) means that the configuration \( \langle t, k, l \rangle \) is obtained from \( \langle s, m, n \rangle \) by applying an instruction of machine \( \mathcal{M} \) once. We write \( \langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle \) if the configuration \( \langle t, k, l \rangle \) is obtained from \( \langle s, m, n \rangle \) by applying instructions of machine \( \mathcal{M} \) in finitely many steps (possibly, in 0 steps). Particularly, we always have \( \langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle s, m, n \rangle \).

The configuration problem for a Minsky machine \( \mathcal{M} \) and a configuration \( \langle s, m, n \rangle \) is, given a configuration \( \langle t, k, l \rangle \), to determine whether \( \langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle \).

**Theorem 3.2** (Minsky, [11]). **There exist a Minsky machine \( \mathcal{M} \) and a configuration \( \langle s, m, n \rangle \) for which the configuration problem is undecidable.**

Let \( \mathcal{M} \) be a Minsky machine and \( \langle s_0, m_0, n_0 \rangle \) a configuration for which the configuration problem is undecidable.

### 3.2 Encoding of Configurations

Let \( p, q \) and \( r \) be three distinct propositional variables. Now we define some propositional formulas using only three variables \( p, q, r \), which encode configurations of Minsky machines. Note that some basic ideas of defining these formulas was found in [5] and [13].

First, let us define the following groups of propositional formulas constructed from variables \( p, q \) and \( r \).

**Group \((C)\):**

\[
\begin{align*}
C_1 &= \neg\neg r \to r, \\
C_2 &= \neg\neg r \lor \neg r;
\end{align*}
\]

**Groups \((A^0)\) and \((B^0)\):**

\[
\begin{align*}
A_{-1}^0 &= C_1, \\
B_{-1}^0 &= A_{-1}^0 \to C_2, \\
A_0^0 &= B_{-1}^0 \to A_{-1}^0 \lor \neg\neg r, \\
B_0^0 &= A_0^0 \to A_{-1}^0 \lor B_{-1}^0, \text{ and} \\
A_{i+1}^0 &= B_{i-1}^0 \to A_{i-1}^0 \lor B_{i-2}^0, \\
B_i^0 &= A_i^0 \to A_{i-1}^0 \lor B_{i-1}^0, \text{ for all } i \geq 1;
\end{align*}
\]

**Groups \((A^1)\) and \((B^1)\):**

\[
\begin{align*}
A_{-2}^1 &= C_1, \\
B_{-2}^1 &= C_1, \\
A_{-1}^1 &= \neg\neg p \to p, \\
B_{-1}^1 &= \neg\neg p \lor \neg p, \text{ and} \\
A_i^1 &= C_2 \land B_{i-1}^1 \to C_1 \lor A_{i-1}^1 \lor B_{i-2}^1, \\
B_i^1 &= C_2 \land A_{i-1}^1 \to C_1 \lor A_{i-2}^1 \lor B_{i-1}^1, \text{ for all } i \geq 0;
\end{align*}
\]
Groups \((A^2)\) and \((B^2)\):

\[
\begin{align*}
A^2_{-2} &= C_2, \\
B^2_{-2} &= C_2, \\
A^2_{-1} &= \neg q \rightarrow q, \\
B^2_{-1} &= \neg q \lor \neg q, \text{ and} \\
A^2_i &= C_1 \land B^2_{i-1} \rightarrow C_2 \lor A^2_{i-1} \lor B^2_{i-2}, \\
B^2_i &= C_1 \land A^2_{i-1} \rightarrow C_2 \lor A^2_{i-2} \lor B^2_{i-1}, \text{ for all } i \geq 0;
\end{align*}
\]

Note that the groups \((A^0)\), \((B^0)\) contain only one variable \(r\), \((A^1)\), \((B^1)\) contain only variables \(r\), \(p\), and \((A^2)\), \((B^2)\) contain only variables \(r\), \(q\). Now we define formulas encoding configurations of the Minsky machine \(M\).

**Group \((E)\):**

\[
E_{s,m,n} = A^0_{s+1} \land B^0_{s+1} \land A^1_{m+1} \land B^1_{m+1} \land A^2_{n+1} \land B^2_{n+1} \rightarrow A^0_s \lor B^0_s \lor A^1_m \lor B^1_m \lor A^2_n \lor B^2_n,
\]

for all \(s,m,n \geq 0\). The formula \(E_{s,m,n}\) is called the code of a configuration \(\langle s,m,n \rangle\).

Denote by \((A)\) and \((B)\) the following sets of formulas:

\[
\begin{align*}
(A) &= (A^0) \cup (A^1) \cup (A^2), \\
(B) &= (B^0) \cup (B^1) \cup (B^2),
\end{align*}
\]

and by \(M\) the set of formulas:

\[M = (A) \cup (B) \cup (C) \cup (E).\]

### 3.3 Kripke model refuting codes of derivable configurations

In this section, we construct a Kripke model \(\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{W} \rangle\) refuting all formulas from \(M\), i.e., for every formula from \(M\), there exists an unique maximal point, at which this formula is refuted.

First, let us define the following equivalence relation \(\sim_\mathcal{M}\) on the set of all configurations \(\{\langle s,m,n \rangle \mid s,m,n \geq 0\}\):

\[
\langle s,m,n \rangle \sim_\mathcal{M} \langle t,k,l \rangle \iff \langle s,m,n \rangle \xrightarrow{\mathcal{M}} \langle t,k,l \rangle \text{ and } \langle t,k,l \rangle \xrightarrow{\mathcal{M}} \langle s,m,n \rangle.
\]

Denote by \([s,m,n]\) the equivalence class of a configuration \(\langle s,m,n \rangle\):

\[
[s,m,n] = \{ \langle t,k,l \rangle \mid \langle s,m,n \rangle \sim_\mathcal{M} \langle t,k,l \rangle \}.
\]

The set of all equivalence classes of relation \(\sim_\mathcal{M}\) is denoted by \(\mathcal{E}_\mathcal{M}\).

Let us define the relation \(\xrightarrow{\mathcal{M}}\) on the set of equivalence classes \(\mathcal{E}_\mathcal{M}\):

\[
[s,m,n] \xrightarrow{\mathcal{M}} [t,k,l] \iff \langle s,m,n \rangle \xrightarrow{\mathcal{M}} \langle t,k,l \rangle.
\]

Greek letters \(\alpha, \beta, \gamma,\) etc., are used to denote equivalence classes. Denote by \(\alpha_0\) the equivalence class of the initial configuration \(\langle s_0,m_0,n_0 \rangle\), i.e., \(\alpha_0 = [s_0,m_0,n_0]\).

Now we define a Kripke frame \(\mathfrak{F} = \langle W, R \rangle\) as follows. Let

\[
W = \bigcup_{i \geq -1, j \in \{0,1,2\}} \{ a^i_j, b^i_j, c^i_j, d^j \} \cup \bigcup_{\alpha \in \mathcal{E}_\mathcal{M}} \{ e_\alpha \}.
\]
To define the accessibility relation $R$ on $W$, we consider the following groups of relations:

**Group $R^i_d$, $i \geq -1$, $j \in \{0, 1, 2\}$**:

- $R^i_d = \{ \langle a^i_{-1}, c_j \rangle, \langle b^i_{-1}, c_j \rangle, \langle b^i_{-1}, d_j \rangle \}$, 
- $R^0_0 = \{ \langle a^0_0, a^0_{-1} \rangle, \langle a^0_0, d_0 \rangle, \langle b^0_0, a^0_{-1} \rangle, \langle b^0_0, b^0_{-1} \rangle \}$, 
- $R^0_0 = \{ \langle a^1_0, a^1_{-1} \rangle, \langle a^1_0, a^0_{-1} \rangle, \langle b^1_0, b^1_{-1} \rangle, \langle b^1_0, a^0_{-1} \rangle \}$, 
- $R^0_2 = \{ \langle a^2_0, a^2_{-1} \rangle, \langle a^2_0, b^0_{-1} \rangle, \langle b^2_0, b^2_{-1} \rangle, \langle b^2_0, b^0_{-1} \rangle \}$, and
- $R^0_1 = \{ \langle a^1_i, a^1_{i-2} \rangle, \langle b^1_i, a^1_{i-2} \rangle, \langle b^1_i, b^1_{i-1} \rangle \}$ for all $i \geq 1$, 
- $R^i_j = \{ \langle a^i_j, a^i_{j-1} \rangle, \langle b^i_j, b^i_{j-1} \rangle \}$ for all $i \geq 1$, $j \in \{1, 2\}$.

**Group $R_{s,m,n}$, $s, m, n \geq 0$, $a_0 \xrightarrow{M} [s, m, n]$**:

- $R_{s,m,n} = \{ \langle e_{[s,m,n]}, a^0_s \rangle, \langle e_{[s,m,n]}, b^0_s \rangle, \langle e_{[s,m,n]}, a^1_m \rangle, \langle e_{[s,m,n]}, b^1_m \rangle, \langle e_{[s,m,n]}, a^2_n \rangle, \langle e_{[s,m,n]}, b^2_n \rangle \}$.

Let 

$$R' = \bigcup_{i \geq -1, j \in \{0, 1, 2\}} R^i_j \cup \bigcup_{s,m,n \geq 0; a_0 \xrightarrow{M} [s, m, n]} R_{s,m,n} \cup \bigcup_{\alpha, \beta \in E_M; a_0 \xrightarrow{M} [s, m, n]} \{ \langle e_\alpha, e_\beta \rangle \}.$$

We take as $R$ the reflexive and transitive closure of $R'$.

Let us define a valuation $\mathcal{V}$ of the Kripke model $\mathcal{M} = \langle \mathfrak{F}, \mathcal{V} \rangle$ in the following way:

- $\langle \mathcal{M}, w \rangle \not\models r \iff w \leq_R a^0_{-1}$ or $w \leq_R d_0$;
- $\langle \mathcal{M}, w \rangle \not\models p \iff w \leq_R a^1_{-1}$ or $w \leq_R d_1$;
- $\langle \mathcal{M}, w \rangle \not\models q \iff w \leq_R a^2_{-1}$ or $w \leq_R d_2$. 
The model $\mathcal{M}$ is depicted on Figure 1. Now we prove some basic semantic properties of the Kripke model $\mathcal{M}$.

**Lemma 3.3.** Let $w$ be a world of $\mathcal{M}$, then

$$w \not\models A_i^i \iff w \leq_R a_i^i,$$

$$w \not\models B_i^j \iff w \leq_R b_i^j$$

for all $i \geq -1$ and $j \in \{0, 1, 2\}$.

**Proof.** By induction on $i \geq -1$. The basis of induction consists of two cases: $i = -1$ and $i = 0$.

**Induction base: $i = -1$.** Let $x_0 = r$, $x_1 = p$, and $x_2 = q$.

If $w \not\models A_{-1}^0$, then there exists a point $w' \geq_R w$ such that $w' \models \neg x_j$ and $w' \not\models x_j$. By definition of the valuation $\mathcal{V}$, we have either $w' \leq_R a_{-1}^1$ or $w' \leq_R d_j$. Since $w' \models \neg x_j$, therefore for all point $w'' \geq_R w'$ there is a point $w''' \geq_R w''$ such that $w''' \models x_j$. It is clear that $w'' \not\models d_j$. Hence, $w \leq_R a_{-1}^1$.

If $w \not\models B_{-1}^1$, then there exist points $w' \geq_R w$ and $w'' \geq_R w$ such that $w' \models x_j$ and $w'' \models \neg x_j$. By definition of the valuation $\mathcal{V}$, we have $w' \not\models a_{-1}^1$, $w' \not\models d_j$, and $w'' = d_j$. If $w' \leq_R c_j$ or $w' \not\models d_j$ for some $j' \in \{0, 1, 2\} \setminus \{j\}$, then $w' \leq_R e_{[s,m,n]}$ for some $s,m,n \geq 0$ and therefore $w \leq_R b_{-1}^1$. Otherwise, $w' = c_j$. It is easily seen that $w \leq_R b_{-1}^1$ if $j \neq 0$. Let $j = 0$, so there is a point $w''' \geq_R w$ such that $w''' \leq_R c_0$, $w''' \leq_R d_0$, and $w''' \models A_{-1}^0$. Thus, $w''' = b_{-1}^0$ and therefore $w \leq_R b_{-1}^0$ for $j = 0$.

**Induction base: $i = 0$.** We need to consider two subcases: $j = 0$ and $j = 1$. It is clear that the subcase $j = 2$ is identical to the subcase $j = 1$.

1) Let $j = 0$. If $w \not\models A_0^0$, then there exists a point $w' \geq_R w$ such that $w' \not\models A_0^0$, $\neg x_0$ and $w' \models R a_{-1}^1$ and $w' \not\models R b_{-1}^0$. Since $w' \not\models \neg x_0$, there is a point $w'' \geq_R w'$ such that $w'' \models \neg x_0$. It is clear that $w'' = d_0$. It is easily seen that the model $\mathcal{M}$ contains only one point $a_0^0$ satisfied the following condition: $w' \leq_R a_0^0$, $w' \leq_R d_0$ and $w' \not\models R b_{-1}^0$. Hence, $w \leq_R a_0^0$.

If $w \not\models B_0^0$, then there exists a point $w' \geq_R w$ such that $w' \not\models A_0^0$, $A_0^0$ and $w' \models R b_{-1}^0$. Then $w' \leq_R a_{-1}^1$, $w' \leq_R b_{-1}^1$ and $w' \not\models R a_0^0$. It is evident that the model $\mathcal{M}$ contains only one point $b_0^0$ satisfied this condition. Therefore, $w \leq_R b_0^0$.

2) Let $j = 1$. If $w \not\models A_1^0$, then there exists a point $w' \geq_R w$ such that $w' \not\models A_1^0$, $A_1^1$ and $w' \models C_2, B_{-1}^1$. By above, $w' \leq_R a_0^1$, $w' \leq_R a_{-1}^1$, $w' \not\models R b_{-1}^1$, and $w' \not\models R b_{-1}^1$. It is easily shown that $a_0^1$ is a unique maximal point of model $\mathcal{M}$, which is satisfied this condition. So, we have that $w \leq_R a_0^1$. Similarly, if $w \not\models B_1^0$, then $w \leq_R b_1^0$.

**Induction step:** assume that $i \geq 1$. Without loss of generality, we can consider the case $j = 1$. The cases $j = 0$ and $j = 2$ are proved by analogy.

If $w \not\models A_1^1$, then there exists a point $w' \geq_R w$ such that $w' \not\models C_1, A_{-1}^1, B_{-1}^1$ and $w' \models C_2, B_{-1}^1$. By induction hypothesis, we obtain that $w' \leq_R a_{-1}^1$, $w' \leq_R b_{-1}^1$, and $w' \not\models R b_{-1}^1$. So, $w' = a_1^1$ and $w \leq_R a_1^1$ by definition of the accessibility relation $R$. Analogously, if $w \not\models B_1^1$, then $w \leq_R b_1^1$. The lemma is proved. □

**Lemma 3.4.** Let $w$ be a world of $\mathcal{M}$, then

$$w \not\models E_{s,m,n} \iff w \leq_R e_{[s,m,n]}$$

for all $s,m,n \geq 0$ such that $a_0 \models [s,m,n]$.

The proof is trivial by definition of the accessibility relation $R$.  


3.4 Key formulas

In this section, we consider the key formulas depending on variables \( p, q, r \). First, let us define the following formulas \( F_k = F_k[p, q, x, y] \) and \( G_k = G_k[p, q, x, y] \) in variables \( p, q, x \) and \( y \):

\[
\begin{align*}
F_0 &= p, \\
G_0 &= q,
\end{align*}
\]

\[
\begin{align*}
F_1 &= y \land q \rightarrow x \lor p, \\
G_1 &= y \land p \rightarrow x \lor q, \text{ and}
\end{align*}
\]

\[
\begin{align*}
F_k &= y \land G_{k-1} \rightarrow x \lor F_{k-1} \lor G_{k-2}, \\
G_k &= y \land F_{k-1} \rightarrow x \lor G_{k-1} \lor F_{k-2}, \text{ for all } k \geq 2.
\end{align*}
\]

Now we introduce the following key formulas:

\[
\begin{align*}
F_k^1[p, q] &= F_k[p, q, C_1, C_2], \\
G_k^1[p, q] &= G_k[p, q, C_1, C_2],
\end{align*}
\]

\[
\begin{align*}
F_k^2[p, q] &= F_k[p, q, C_2, C_1], \\
G_k^2[p, q] &= G_k[p, q, C_2, C_1].
\end{align*}
\]

Note that the formulas \( F_k^m \) and \( G_k^m \) are depending on three variables \( p, q, r \), for all \( k \geq 0 \) and \( m \in \{1, 2\} \).

Besides, we define the following auxiliary formulas:

\[
\begin{align*}
P_{i,j} &= (C_2 \rightarrow C_1 \lor A_i^1 \lor B_i^1) \land (C_1 \rightarrow C_2 \lor A_i^2 \lor B_i^2), \\
Q_{i,j} &= (C_2 \rightarrow C_1 \lor A_{i-1}^1 \lor B_i^1) \land (C_1 \rightarrow C_2 \lor A_{i-1}^2 \lor B_i^2),
\end{align*}
\]

for all \( i, j \geq -1 \). The following lemma is describing the basic properties of the key formulas.

**Lemma 3.5.** For all \( i, j \geq -1, k \geq 1 \) and \( m \in \{1, 2\} \),

\[
\begin{align*}
\text{Int} &\vdash F_k^m[P_{i,j}, Q_{i,j}] \leftrightarrow A_{n+k}^m, \\
\text{Int} &\vdash G_k^m[P_{i,j}, Q_{i,j}] \leftrightarrow B_{n+k}^m,
\end{align*}
\]

where

\[
n = \begin{cases} 
  i, & m = 1; \\
  j, & m = 2.
\end{cases}
\]

**Proof.** By induction on \( k \geq 1 \). Without loss of generality, we can assume that \( m = 1 \). The basis of induction consists of two cases: \( k = 1 \) and \( k = 2 \).

**Induction base:** \( k = 1 \). In this case we have

\[
\begin{align*}
F_1^1[P_{i,j}, Q_{i,j}] &= C_2 \land Q_{i,j} \rightarrow C_1 \lor P_{i,j}, \\
G_1^1[P_{i,j}, Q_{i,j}] &= C_2 \land P_{i,j} \rightarrow C_1 \lor Q_{i,j}.
\end{align*}
\]

It can easily be checked that the following derivations holds in \text{Int}:

\[
\begin{align*}
\text{Int} &\vdash C_2 \land B_i^1 \rightarrow C_2 \land Q_{i,j}, \\
\text{Int} &\vdash C_2 \land A_i^1 \rightarrow C_2 \land P_{i,j},
\end{align*}
\]

\[
\begin{align*}
\text{Int} &\vdash C_2 \land A_i^1 \rightarrow C_2 \land P_{i,j}, \\
\text{Int} &\vdash C_1 \lor Q_{i,j} \rightarrow (C_2 \rightarrow C_1 \lor A_{i-1}^1 \lor B_i^1).
\end{align*}
\]

Hence,

\[
\begin{align*}
\text{Int} &\vdash F_1^1[P_{i,j}, Q_{i,j}] \rightarrow A_{i+1}^1, \\
\text{Int} &\vdash G_1^1[P_{i,j}, Q_{i,j}] \rightarrow B_{i+1}^1.
\end{align*}
\]
Conversely, since the formulas $A_{i-1}^1 \rightarrow A_i^1$ and $B_{i-1}^1 \rightarrow B_i^1$ are derivable from $\textbf{Int}$, we have

\[ \text{Int}, \ A_{i+1}^1 \vdash C \vee A_{i-1}^1 \vee B_i^1 \rightarrow (C_2 \rightarrow C \vee A_i^1 \vee B_{i-1}^1), \]

\[ \text{Int}, \ B_{i+1}^1 \vdash C \vee A_{i-1}^1 \vee B_{i-1}^1 \rightarrow (C_2 \rightarrow C \vee A_i^1 \vee B_{i-1}^1) \]

and therefore the following derivations holds in $\textbf{Int}$:

\[ \text{Int}, \ A_{i+1}^1 \vdash C \wedge (C_2 \rightarrow C \vee A_{i-1}^1 \vee B_i^1) \rightarrow P_{i,j}, \]

\[ \text{Int}, \ B_{i+1}^1 \vdash C \wedge (C_2 \rightarrow C \vee A_i^1 \vee B_{i-1}^1) \rightarrow Q_{i,j}. \]

Hence,

\[ \text{Int} \vdash A_{i+1}^1 \rightarrow F_1^1[P_{i,j}, Q_{i,j}], \]

\[ \text{Int} \vdash B_{i+1}^1 \rightarrow G_1^1[P_{i,j}, Q_{i,j}]. \]

**Induction base:** $k = 2$. In this case we have

\[ \text{Int} \vdash F_2^1[P_{i,j}, Q_{i,j}] \leftrightarrow (C_2 \wedge B_{i+1}^1 \rightarrow C \vee A_{i+1}^1 \vee Q_{i,j}), \]

\[ \text{Int} \vdash G_2^1[P_{i,j}, Q_{i,j}] \leftrightarrow (C_2 \wedge A_{i+1}^1 \rightarrow C \vee B_{i+1}^1 \vee P_{i,j}). \]

Furthermore, it follows easily that:

\[ \text{Int} \vdash C_2 \wedge B_i^1 \rightarrow Q_{i,j}, \quad \text{Int} \vdash Q_{i,j} \rightarrow (C_2 \rightarrow C \vee A_{i+1}^1 \vee B_i^1), \]

\[ \text{Int} \vdash C_2 \wedge A_i^1 \rightarrow P_{i,j}, \quad \text{Int} \vdash P_{i,j} \rightarrow (C_2 \rightarrow C \vee A_i^1 \vee B_{i-1}^1). \]

Hence,

\[ \text{Int} \vdash F_2^1[P_{i,j}, Q_{i,j}] \leftrightarrow A_{i+2}^1, \]

\[ \text{Int} \vdash G_2^1[P_{i,j}, Q_{i,j}] \leftrightarrow B_{i+2}^1. \]

**Induction step** is straightforward and left to the reader. The lemma is proved. \(\square\)

### 3.5 Encoding of the Minsky machine

Now we encode instructions of the Minsky machine $\mathcal{M}$ as superintuitionistic formulas such that derivations from $\textbf{Int}$ and these formulas are simulate transformations of $\mathcal{M}$.

First, let us define the following formulas containing only tree variables $p$, $q$, $r$:

\[ \hat{E}_{s,i,j} = A_{s+1}^0 \wedge B_{s+1}^0 \wedge F_{i+1}^1 \wedge G_{i+1}^1 \wedge F_{j+1}^2 \wedge G_{j+1}^2 \rightarrow A_s^0 \vee B_s^0 \vee F_i^1 \vee G_i^1 \vee F_j^2 \vee G_j^2, \]

\[ \hat{E}_{s,0,s} = A_{s+1}^0 \wedge B_{s+1}^0 \wedge A_1^1 \wedge B_i^1 \rightarrow A_s^0 \vee B_s^0 \vee A_i^1 \vee B_0^1 \vee q, \]

\[ \hat{E}_{s,s,0} = A_{s+1}^0 \wedge B_{s+1}^0 \wedge A_i^1 \wedge B_1^1 \rightarrow A_s^0 \vee B_s^0 \vee p \vee A_i^1 \vee B_0^1, \]

\[ \hat{E}_{s,0,0} = E_{s,0,0}, \]

where $s \geq 0$, $i, j \geq 1$. By Lemma 3.5 we have the following evident lemma.

**Lemma 3.6.** For all $s, m, n \geq 0$,

\[ \text{Int} \vdash E_{s,m,n} \leftrightarrow \begin{cases} \hat{E}_{s,i,j}[P_{m-i,n-j}, Q_{m-i,n-j}], & 1 \leq i \leq m + 1, 1 \leq j \leq n + 1; \\ A_{n+1}^2 \wedge B_{n+1}^2 \rightarrow \hat{E}_{s,0,s}[p, A_n^2 \vee B_n^2], & m = 0, n \geq 1; \\ A_{m+1}^1 \wedge B_{m+1}^1 \rightarrow \hat{E}_{s,s,0}[A_1^1 \vee B_{m}^1, q], & m \geq 1, n = 0; \\ \hat{E}_{s,0,0}, & m = 0, n = 0. \end{cases} \]
Let 
\[ \varphi(x) = \begin{cases} 
  x - 1, & x \geq 1; \\
  0, & x = 0; \\
  0, & x = *.
\end{cases} \]

Now we prove that if the Kripke frame \( \mathfrak{F} \) refutes \( \hat{E}_{s,i,j} \) for \( s, i, j \geq 0 \) then it refutes \( \hat{E}_{s,i,j} \) at a point \( e_{s,m,n} \) for some \( m \geq \varphi(i), n \geq \varphi(j) \) such that \( \alpha_0 \models [s,m,n] \).

**Lemma 3.7.** Given a world \( w \) of a Kripke model \( \mathfrak{M}' = (\mathfrak{F}', \mathfrak{B}') \) such that \( \mathfrak{B}'(r) = \mathfrak{B}(r) \), if \( w \not\models \hat{E}_{s,i,j} \), then \( w \leq_R e_{[s,m,n]} \) for some \( m \geq \varphi(i), n \geq \varphi(j) \) such that \( \alpha_0 \models [s,m,n] \).

**Proof.** If \( w \not\models \hat{E}_{s,i,j} \), then there is a point \( w' \geq_R w \) such that the formulas \( A_0, B_0 \) are refuted at \( w' \) and the formulas \( A_{s+1}, B_{s+1} \) are true at \( w' \). Since \( \mathfrak{B}'(r) = \mathfrak{B}(r) \), we have that \( w \leq_R e_{[s,m,n]} \) for some \( m \geq 0, n \geq 0 \) such that \( \alpha_0 \models [s,m,n] \). In order to prove the lemma it is sufficient to show that \( m \geq i - 1, n \geq j - 1 \) for some \( i \geq 1, j \geq 1 \).

If \( i \geq 1 \), then the formulas \( F_i^1, G_i^1 \) are refuted at \( w' \) and the formulas \( F_{i+1}^1, G_{i+1}^1 \) are true at \( w' \). Now we prove that if \( f_k^1 \) is refuted at a point \( f_k^1 \) and \( G_k^1 \) is refuted at a point \( g_k^1 \), then \( f_k^1 \leq_R a_{k+l-1}^1 \) and \( g_k^1 \leq_R b_{k+l-1}^1 \) for some \( l \geq 0 \). By induction on \( k \geq 1 \).

**Induction base:** \( k = 1 \). In this case, there are points \( w_f \geq_R f_1^1 \) and \( w_g \geq_R g_1^1 \) such that

1. \( C_1 \) is refuted at \( w_f, w_g \), therefore \( w_f \leq_R a_{-1}^0, w_g \leq_R a_{-1}^0 \);
2. \( C_2 \) is true at \( w_f, w_g \), therefore \( w_f \not\models_R b_{-1}^0, w_g \not\models_R b_{-1}^0 \);
3. \( w_f \in \mathfrak{B}'(p) \backslash \mathfrak{B}'(q) \) and \( w_g \in \mathfrak{B}'(p) \backslash \mathfrak{B}'(q) \), therefore \( w_f, w_g \) are incomparable points.

Thus, \( w_f = a_{i'}^1 \) and \( w_g = b_{j'}^1 \) for some \( i', j' \geq 0 \) such that \( |i' - j'| < 2 \).

**Induction base:** \( k = 2 \). In this case, there are points \( w_f \geq_R f_2^1 \) and \( w_g \geq_R g_2^1 \) such that

1. \( F_1^1 \) is refuted at \( w_f \) and \( G_1^1 \) is refuted at \( w_g \), therefore \( w_f \leq_R a_{i'}^1, w_g \leq_R b_{j'}^1 \);
2. \( F_1^1 \) is true at \( w_f \) and \( G_1^1 \) is true at \( w_f \), therefore \( w_f \not\models_R b_{j'}^1, w_g \not\models_R a_{i'}^1 \);
3. \( w_f, w_g \not\models \mathfrak{B}'(p) \cup \mathfrak{B}'(q) \), therefore \( w_f \not= a_{i'}^1, w_g \not= b_{j'}^1 \).

Thus, \( f_w = a_{i''}^1, w_g = b_{j''}^1 \) and \( (w_f, b_{j'}^1), (a_{i'}^1, w_g) \) and \( (w_f, w_g) \) are pairs of incomparable points. So, we have
\[ i' < i'' < j' + 2, \]
\[ j' < j'' < i' + 2. \]

Since \( |i' - j'| < 2 \), it can easily be checked that \( i' = j' = l \) and \( i'' = j'' = l + 1 \) for some \( l \geq 0 \).

**Induction step:** \( k > 2 \). Let the induction assumption be satisfied for all \( 2 \leq k' < k \), then there are points \( w_f \geq_R f_{k'}^1 \) and \( w_g \geq_R g_{k'}^1 \) such that

1. \( F_{k-1}^1, G_{k-2}^1 \) are refuted at \( w_f \), therefore \( w_f \not= R a_{k+l-2}^1, w_f \not= R b_{k+l-3}^1 \);
2. \( G_{k-1}^1, F_{k-2}^1 \) are refuted at \( w_g \), therefore \( w_g \not= R b_{k+l-2}^1, w_g \not= R a_{k+l-3}^1 \);
3. \( G_{k-1}^1 \) is true at \( w_f \) and \( F_{k-1}^1 \) is true at \( w_g \), therefore \( w_f \not= R b_{k+l-2}^1 \) and \( w_g \not= R a_{k+l-2}^1 \).
Thus, \( w_f = a_{k+l-1}^1 \) and \( w_g = b_{k+l-1}^1 \).

Since \( F_i^1 \), \( G_i^1 \) are refuted at \( w' \) and the formulas \( F_{i+1}^1 \), \( G_{i+1}^1 \) are true at \( w' \), we have \( w' \leq_R a_{i+l-1}^1, w' \leq_R b_{i+l-1}^1 \) and \( w' \not\leq_R a_{i+1}^1, w' \not\leq_R b_{i+1}^1 \). Therefore, \( m = i + l - 1 \geq i - 1 \).

If \( j \geq 1 \), then the proof are similar. Hence, \( n \geq j - 1 \). The lemma is proved. \( \square \)

Next, we define the formula \( A(x) \) simulating the instruction \( I \) of the Minsky machine \( M \):

1. If \( I \) is an instruction of the form \( s \mapsto \langle t, 1, 0 \rangle \), then \( A(x) \) is the following formula
   \[
   \hat{E}_{t,2,1} \rightarrow \hat{E}_{s,1,1};
   \]
2. If \( I \) is \( s \mapsto \langle t, 0, 1 \rangle \), then \( A(x) \) is
   \[
   \hat{E}_{t,1,2} \rightarrow \hat{E}_{s,1,1};
   \]
3. If \( I \) is \( s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle \), then \( A(x) \) is
   \[
   (\hat{E}_{t,1,1} \rightarrow \hat{E}_{s,2,1}) \land (\hat{E}_{u,0,*} \rightarrow \hat{E}_{s,0,*});
   \]
4. If \( I \) is \( s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle \), then \( A(x) \) is
   \[
   (\hat{E}_{t,1,1} \rightarrow \hat{E}_{s,1,2}) \land (\hat{E}_{u,0} \rightarrow \hat{E}_{s,0,0}),(\hat{E}_{u,0} \rightarrow \hat{E}_{s,0,0}),
   \]
and the formula \( A(x) \) simulating the behavior of \( M \) itself:
   \[
   A(x) = \bigwedge_{I \in M} A(x).
   \]

**Lemma 3.8.** If \( \mathcal{W}' = \langle \mathcal{F}, \mathcal{W}' \rangle \) such that \( \mathcal{W}'(r) = \mathcal{W}(r) \), then \( \mathcal{W}' \models A(x) \).

**Proof.** In order to prove the lemma it is sufficient to show that
   \[
   \mathcal{W}' \models A(x)
   \]
for each instruction \( I \). We need to consider the following 4 cases.

**Case 1:** \( I \) is an instruction of the form \( s \mapsto \langle t, 1, 0 \rangle \), i.e.,
   \[
   A(x) = \hat{E}_{t,2,1} \rightarrow \hat{E}_{s,1,1}.
   \]
If \( w \not\models \hat{E}_{s,1,1} \) for some point \( w \in W \), then by Lemma 3.7 \( w \leq e_{[s,m,n]} \) for some \( m \geq 0 \) and \( n \geq 0 \) such that \( \alpha_0 \xrightarrow{\vDash} [s, m, n] \). It is clear that \( w' \not\models \hat{E}_{t,2,1} \) for all \( w' \leq_R e_{[t,m+1,n]} \). Since
   \[
   \langle s, m, n \rangle \xrightarrow{\vDash} \langle t, m + 1, n \rangle,
   \]
we have that \( e_{[s,m,n]} \leq_R e_{[t,m+1,n]} \) and therefore \( w \not\models \hat{E}_{t,2,1} \).

**Case 2:** \( I \) is an instruction of the form \( s \mapsto \langle t, 0, 1 \rangle \). The proof is analogous.

**Case 3:** \( I \) is an instruction of the form \( s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle \), i.e.,
   \[
   (\hat{E}_{t,1,1} \rightarrow \hat{E}_{s,2,1}) \land (\hat{E}_{u,0,*} \rightarrow \hat{E}_{s,0,*}).
   \]
It is clear that if \( w \not\models \hat{E}_{s,2,1} \), then \( w \not\models \hat{E}_{t,1,1} \). Let \( w \not\models \hat{E}_{s,0,*} \) for some point \( w \in W \), then by Lemma 3.7 \( w \leq e_{[s,0,n]} \) for some \( n \geq 0 \) such that \( \alpha_0 \xrightarrow{\vDash} [s, 0, n] \). It is easily seen that \( w' \not\models \hat{E}_{u,0,*} \) for all \( w' \leq_R e_{[u,0,n]} \). Since
   \[
   \langle s, 0, n \rangle \xrightarrow{\vDash} \langle u, 0, n \rangle,
   \]
we have that \( e_{[s,0,n]} \leq_R e_{[u,0,n]} \) and therefore \( w \not\models \hat{E}_{u,0,*} \).

**Case 4:** \( I \) is an instruction of the form \( s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle \). The proof is similar.

Thus, \( \mathcal{W}' \models A(x) \) for each instruction \( I \in M \). The lemma is proved. \( \square \)
3.6 Reduction of configuration problem

In this section formally reduce the configuration problem of the Minsky machine $\mathcal{M}$ to the derivation problem of the superintuitionistic propositional calculus $\text{Int} + Ax(\mathcal{M})$.

Lemma 3.9. $\text{Int} + Ax(\mathcal{M}) \vdash E_{t,k,l} \rightarrow E_{s_0,m_0,n_0}$ iff $\langle s_0, m_0, n_0 \rangle \not\vdash_{\text{M}} \langle t, k, l \rangle$.

Proof. If $\text{Int} + Ax(\mathcal{M}) \vdash E_{t,k,l} \rightarrow E_{s_0,m_0,n_0}$, then

$$\mathcal{M} \models E_{t,k,l} \rightarrow E_{s_0,m_0,n_0}$$

for all $\mathcal{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that $\mathfrak{V}(r) = \mathfrak{V}(r)$ by Lemma 3.8. If we recall that $E_{s_0,m_0,n_0}$ is refuted at $e_{[s_0,m_0,n_0]}$, then we obtain that $E_{t,k,l}$ is also refuted at $e_{[s_0,m_0,n_0]}$. Thus, $e_{[s_0,m_0,n_0]} \leq R e_{[t,k,l]}$ and therefore $\langle s_0, m_0, n_0 \rangle \not\models_{\mathcal{M}} \langle t, k, l \rangle$.

Conversely, if $\langle s_0, m_0, n_0 \rangle \not\models_{\mathcal{M}} \langle t, k, l \rangle$, then there exists a finite sequence $\langle s_i, m_i, n_i \rangle$, $0 \leq i \leq \mu$, such that $\langle s_\mu, m_\mu, n_\mu \rangle = \langle t, k, l \rangle$ and

$$\langle s_i, m_i, n_i \rangle \not\models_{\mathcal{M}} \langle s_{i+1}, m_{i+1}, n_{i+1} \rangle$$

for all $i$, $0 \leq i < \mu$. Let $\langle s_{i+1}, m_{i+1}, n_{i+1} \rangle$ be a result of applying of an instruction $I \in \mathcal{M}$. We need to consider the following 4 cases.

**Case 1:** $I$ is an instruction of the form $s \mapsto \langle t, 1, 0 \rangle$. Then $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$. By Lemma 3.6 we have

$$\begin{align*}
\text{Int} & \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \leftrightarrow \hat{E}_{s_{i+1},2,1}[P_{m_i-1,n_i-1}, Q_{m_i-1,n_i-1}], \\
\text{Int} & \vdash E_{s_i,m_i,n_i} \leftrightarrow \hat{E}_{s_i,1,1}[P_{m_i-1,n_i-1}, Q_{m_i-1,n_i-1}].
\end{align*}$$

**Case 2:** $I$ is an instruction of the form $s \mapsto \langle t, 0, 1 \rangle$. The proof is analogous.

**Case 3:** $I$ is an instruction of the form $s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle$. If $m_{i+1} = m_i - 1 \geq 0$ and $n_{i+1} = n_i$. By Lemma 3.6 we have

$$\begin{align*}
\text{Int} & \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \leftrightarrow \hat{E}_{s_{i+1},1,1}[P_{m_i-2,n_i-1}, Q_{m_i-2,n_i-1}], \\
\text{Int} & \vdash E_{s_i,m_i,n_i} \leftrightarrow \hat{E}_{s_i,2,1}[P_{m_i-2,n_i-1}, Q_{m_i-2,n_i-1}].
\end{align*}$$

If $m_{i+1} = m_i = 0$ and $n_{i+1} = n_i$. By Lemma 3.6 we have

$$\begin{align*}
\text{Int} & \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \leftrightarrow \left( A_{n_i+1}^2 \land B_{n_i+1}^2 \rightarrow \hat{E}_{s_{i+1},0,0}[p, A_{n_i}^2 \lor B_{n_i}^2] \right), \\
\text{Int} & \vdash E_{s_i,m_i,n_i} \leftrightarrow \left( A_{n_i+1}^2 \land B_{n_i+1}^2 \rightarrow \hat{E}_{s_i,0,0}[p, A_{n_i}^2 \lor B_{n_i}^2] \right).
\end{align*}$$

Therefore $\text{Int} + Ax(\mathcal{M}) \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \rightarrow E_{s_i,m_i,n_i}$.

**Case 4:** $I$ is an instruction of the form $s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle$. The proof is similar.

Thus, $\text{Int} + Ax(\mathcal{M}) \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \rightarrow E_{s_i,m_i,n_i}$ for all $i$, $0 \leq i < \mu$. The lemma is proved.

Since the configuration problem for the Minsky machine $\mathcal{M}$ and the initial configuration $\langle s_0, m_0, n_0 \rangle$ is undecidable by Theorem 3.2, we have that the derivation problem for the superintuitionistic propositional calculus $\text{Int} + Ax(\mathcal{M})$ is also undecidable. This completes the proof of Theorem 3.1.
4 Conclusion and further research

In this paper, we established that there is an undecidable superintuitionistic propositional calculus using axioms in only 3 variables. Since there is no an undecidable superintuitionistic propositional calculus with axioms containing less than 3 variables, therefore a natural and interesting question is there a superintuitionistic propositional formula $A$ containing less than 3 variables for which the superintuitionistic propositional calculus $\text{Int} + A$ is undecidable. In this respect, we note that Sobolev in 1977 [17] constructed a two-variable superintuitionistic propositional formulas $B$ such that $\text{Int} + B$ is not finitely approximable.

References

[1] Bokov G. V. Completeness problem in the propositional calculus. // Intelligent Systems, vol. 13, no. 1-4, p. 165-182, 2009. (Russian).

[2] Bokov G. V. Undecidability of the problem of recognizing axiomatizations for propositional calculi with implication. // Logic Journal of the IGPL, 2015. (Received 24 July 2014).

[3] Bokov G. V. Undecidable problems for propositional calculi with implication. // Journal of Symbolic Logic, 2015. (Received 03 February 2015).

[4] Chagrov A. Undecidable properties of superintuitionistic logics. // Mathematical Problems of Cybernetics, vol. 5, p. 67-108, 1994. (Russian).

[5] Chagrov A., Zakharyaschev M. Modal Logic. — Clarendon Press, 1997.

[6] Gladstone M. D. On the number of variables in the axioms,. // Notre Dame Journal of Formal Logic, vol. 11, p. 1–15, 1970.

[7] Gladstone M. D. The decidability of one-variable propositional calculi. // Notre Dame Journal of Formal Logic, vol. 20, no. 2, p. 438–450, 1979.

[8] Hughes C. E. Two Variable Implicational Calculi of Prescribed Many-One Degrees of Unsolvability. // Journal of Symbolic Logic, vol. 41, no. 1, p. 39–44, 1976.

[9] Hughes C. E., Singletary W. E. Triadic partial implicational propositional calculi. // Zeitschriftfdr für mathematische Logik und Grundlagen der Mathematik, vol. 21, p. 21–28, 1975.

[10] Linial S., Post E. L. Recursive unsolvability of the deducibility, Tarski’s comleteness, and independence of axioms problems of the propositional calculus. // Bulletin of the American Mathematical Society, vol. 55, p. 50, 1949.

[11] Minsky M. L. Computation: Finite and Infinite Machines. — Upper Saddle River, NJ, USA, Prentice-Hall, Inc., 1967.

[12] Popov S. Nondecidable intermediate calculus. // Algebra and Logic, vol. 20, no. 6, p. 424-461, 1981.
[13] Rybakov M. N. Complexity of intuitionistic and Visser’s basic and formal logics in finitely many variables. // Advances in Modal Logic, p. 393–411, 2006.

[14] Shehtman V. B. An undecidable superintuitionistic propositional calculus. // Soviet Mathematics Doklady, vol. 240, no. 3, p. 549–552, 1978. (Russian).

[15] Shehtman V. B. Undecidable propositional calculi. // Problems of Cybernetics, vol. 75, p. 74–116, 1982. (Russian).

[16] Skvortsov D. P. One superintuitionistic calculus of propositions. // Algebra and Logic, vol. 24, no. 2, p. 119–125, 1985.

[17] Sobolev S. K. On the finite approximability of superintuitionistic logics. // Mathematics of the USSR, vol. 31, p. 257–268, 1977. (Russian).