Nonassociativity, Dirac monopoles and Aharonov-Bohm effect

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The Aharonov-Bohm (AB) effect for the singular string associated with the Dirac monopole carrying an arbitrary magnetic charge is studied. It is shown that the emerging difficulties in explanation of the AB effect may be removed by introducing nonassociative path-dependent wave functions. This provides the absence of AB effect for the Dirac string of nonquantized magnetic monopole.

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Nowadays interest in the Dirac monopole problem is growing in connection with the ‘fictitious’ monopoles that are similar to the ‘real’ magnetic monopoles, however, appearing in the context of the Berry phase. Recently, the experimental results providing evidence for the ‘fictitious’ magnetic monopole in the crystal-momentum space has been reported in relation to the anomalous Hall effect. Besides of the anomalous Hall effect, these type of magnetic monopoles emerges in trapped A-type atoms, anisotropic spin systems, noncommutative quantum mechanics, etc., and may carry an arbitrary ‘magnetic’ charge.

It is known that the Dirac quantization condition

$$2\mu = n, \ n \in \mathbb{Z},$$

where $\mu = e q$, electric charge being $e$ and magnetic charge $q$ (we set $\hbar = c = 1$), does not follow from the quantum-mechanical consideration alone. Any treatment uses some additional assumptions that may not be physically inevitable. For instance, in Dirac’s theory ‘quantization of magnetic charge’ follows from the requirement of the wave function be single-valued. However, single-valuedness is not one of the fundamental principles of quantum mechanics, and having multi-valued wave functions may be allowed until it does not affect the algebra of observables.

Other well-known topological and geometrical arguments in behalf of Dirac quantization rule is based on employing classical fibre bundle theory. In this approach the Dirac monopole is treated as the Hopf bundle $U(1)$ over $S^2$. A realization of the Dirac monopole in this way implies that there exists the division of space into overlapping regions $\{U_i\}$ with non-singular vector potential being defined in $\{U_i\}$ and yielding the correct monopole magnetic field in each region. On the triple overlap $U_i \cap U_j \cap U_k$ it holds

$$\exp(i(q_{ij} + q_{jk} + q_{ki})) = \exp(i4\pi \mu)$$

(1)

where $q_{ij}$ are the transition functions such that $U_i \cap U_j \rightarrow U(1)$. The consistency condition, which is equivalent to the associativity of the group multiplication, requires

$$q_{ij} + q_{jk} + q_{ki} = 0 \mod 2\pi.$$
ment yields the Dirac quantization condition.
This raises the question of whether the absence of AB effect for the Dirac string is compatible with arbitrariness of magnetic monopole charge. In this Letter we show that the response is affirmative, but it requires employment of nonassociative structures like quasigroups and loops.

Magnetic monopole preliminaries. – Since any choice of the vector potential $\mathbf{A}$ being compatible with a magnetic field $\mathbf{B} = \frac{q}{r^3}$ of Dirac monopole must have the singular string, one has $\mathbf{B} = \nabla \times \mathbf{A}$ locally, but not globally, and this implies

$$\mathbf{B} = \text{rot} \mathbf{A} + \mathbf{h}$$  (2)

where $\mathbf{h}$ is the magnetic field of the Dirac string $C$.

There is an ambiguity in the definition of the vector potential. For instance, Dirac introduced the vector potential as

$$\mathbf{A}_n = q \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{n} \cdot \mathbf{r})}$$  (3)

where the unit vector $\mathbf{n}$ determines the direction of string (hereafter denoted by $S_n$), which passes from the origin of coordinates to $\infty$, and

$$\mathbf{h}_n = 4\pi q \mathbf{n} \int_0^\infty \delta^3(\mathbf{r} - \mathbf{n}\tau) d\tau.$$  (4)

Schwinger’s choice is 31:

$$\mathbf{A}^{SW} = \frac{1}{2}(\mathbf{A}_n + \mathbf{A}_{-n}),$$  (5)

with the string being propagated from $-\infty$ to $\infty$. Both vector potentials yield the same magnetic monopole field, however the quantization is different. The Dirac condition is $2\mu = p$, while the Schwinger one is $\mu = p$, $p \in \mathbb{Z}$.

These two strings belong a family $\{S^\kappa_n\}$ of weighted strings, which magnetic field is given by 32

$$\mathbf{h}^\kappa_n = \frac{1 + \kappa}{2} \mathbf{h}_n + \frac{1 - \kappa}{2} \mathbf{h}_{-n}$$  (6)

where $\kappa$ is the weight of a semi-infinite Dirac string. The respective vector potential reads

$$\mathbf{A}^\kappa_n = \frac{1 + \kappa}{2} \mathbf{A}_n + \frac{1 - \kappa}{2} \mathbf{A}_{-n}.$$  (7)

Note that two arbitrary strings $S^\kappa_n$ and $S^\lambda_n$ are related by gauge transformation:

$$\mathbf{A}^\prime_n = \mathbf{A}^\kappa_n + d\chi,$$  (8)

and the corresponding wave function transforms as follows: $\psi' = \exp(-i\chi)\psi$. In fact, in Dirac’s approach and subsequent development of the monopole theory the wave functions are considered as having a non-integrable (or path-dependent) phase factor 11 12 32.

Aharonov-Bohm effect, Dirac’s string and nonassociative algebra of observables. – Let a coherent beam of electrons with charge $e$ is incoming from $-\infty$. The beam is split at the point $P$ in two parts, passing by an infinite long solenoid (Fig. 1). In spite of the fact that the magnetic field $\mathbf{B}$ outside the solenoid is equal to zero, it produces an interference effect at the point $Q$ of the screen, where the beams are brought together 30.

Following Mandelstam 32, let us consider the path-dependent wave function

$$\Psi(\mathbf{r}, \gamma) = e^{i\alpha_{1}(\mathbf{r}, \gamma)}\psi(\mathbf{r}),$$  (9)

where $\psi(\mathbf{r})$ denotes the wave function in the absence of the magnetic field, $\mathbf{A} = 0$, and $\alpha_{1}(\mathbf{r}, \gamma) = e \int_{P}^Q \mathbf{A} \cdot d\mathbf{r}$, the integration is performed along the path $\gamma$ connecting $\mathbf{r}$ and $\mathbf{r}'$: $\mathbf{r} \rightarrow \mathbf{r}'$.

The total wave function at the point $Q$ of the screen is the superposition of the wave functions along the both paths

$$\Psi_Q = \Psi_1(\mathbf{r}, \gamma_1) + \Psi_2(\mathbf{r}, \gamma_2) = e^{i\alpha_{1}(\mathbf{r}, \gamma_1)}\psi_1(\mathbf{r}) + e^{i\alpha_{1}(\mathbf{r}, \gamma_2)}\psi_2(\mathbf{r}) = e^{i\alpha_{1}(\mathbf{r}, \gamma_2)}(e^{ie\oint \mathbf{A} \cdot d\mathbf{r}}\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})),$$  (10)

where $\alpha_{1}(\mathbf{r}, \gamma_1)$ and $\alpha_{1}(\mathbf{r}, \gamma_2)$ are equal to $-e \int_{P}^Q \mathbf{A} \cdot d\mathbf{r}$ along the paths of the first and second beam respectively.

A relative phase shift $\Delta\varphi$ is given by

$$\Delta\varphi = e \oint_{\gamma} \mathbf{A} \cdot d\mathbf{r} = e\Phi,$$  (11)

where the integration is performed along the closed path $\gamma = \gamma_1 \cup \gamma_2$, $\Phi$ being the total magnetic flux through the solenoid. The condition for the absence of observable AB effect is $e\Phi = 2\pi n$, $n \in \mathbb{Z}$.

Let us assume that the beam passes in the upper half of the space divided by the plane $z = 0$. Then the contribution of the string $S^\kappa_n$ to the relative phase shift of the wave function at the point $Q$ is found to be

$$\Delta\varphi = 2\pi(1 + \kappa)\mu$$  (12)

and, if the beam passes in the lower half-space ($z < 0$), one has

$$\Delta\varphi = 2\pi(\kappa - 1)\mu$$  (13)

It follows from the Eqs. 113, 114 the absence of AB effect when $(1 + \kappa)\mu$ and $(1 - \kappa)\mu$, both are integers. In the case of $\kappa \neq 0$, this yields immediately the following conditions: $2\mu \in \mathbb{Z}$, that is the celebrated Dirac rule, and quantization of the string weight, $2\mu \in \mathbb{Z}$. If $\kappa = 0$, one obtains the Schwinger quantization condition, $\mu \in \mathbb{Z}$. 
At first sight the existence of the magnetic monopoles with an arbitrary magnetic charge is in contradiction with the AB experiment. To clarify issue let us recall that the relative phase shift (11) arises as result of the parallel translation of wave function along the contour $\gamma$ surrounding the Dirac string. It is known that for the string group $Q$ the Jacobi identity continues to fail. Thus, we see that the Jacobi identity fails and for the finite translations on has [13, 33]

$$\Psi(\gamma_1, \gamma_2) = e^{i\alpha_3(\gamma_1, \gamma_2)}\Psi(\gamma_1)\Psi(\gamma_2),$$

where $\alpha_3$ is the so-called three cocycle; $\alpha_3 = 4\pi\mu$ mod $2\pi\mathbb{Z}$, if the monopole is enclosed by the simplex with vertices $(r, r + a, r + a + b, r + a + b + c)$ and zero otherwise [33]. For the Dirac quantization condition being satisfied $\alpha_3 = 0$ mod $2\pi\mathbb{Z}$, and (14) provides an associative representation of the translations, in spite of the fact that the Jacobi identity continues to fail. Thus, we see that the AB effect requires more careful analysis, if one assumes existence of an arbitrary monopole charge.

The emerging difficulties in explanation of the AB effect may be removed by introducing nonassociative path-dependent wave function $\Psi(r; \gamma)$. Here we consider the simplified version of the approach developed in [24, 25, 26].

Let us start with the nonassociative generalization of the gauge transformations known as gauge loop $Q$, which is related to the so-called string group [25, 34]. The string group, denoted as String $\mathcal{M}$, is the group of all paths $\gamma: [0, 1] \rightarrow \text{Diff} \mathcal{M}$, where Diff $\mathcal{M}$ denotes the diffeomorphism group on $\mathcal{M} = R^3 \setminus \{0\}$ such that $r \mapsto r(t) = r_\gamma(t)$, $t \in [0, 1]$ and $\gamma(0) =$ identity; the group composition is defined as $\gamma_2(\gamma_1) = \gamma_2(\gamma_1) \circ \gamma_2(t)$.

Now let $f_\gamma$ be the map:

$$f_\gamma: r \mapsto U_{g_\gamma} = \exp(i\alpha_1(r; \gamma)) \in Q,$$

where the integration is performed along a path connecting a point $r$ with a point $r' = r_\gamma(1)$, and $A(r)$ is the vector potential.

The product of two elements $U_{g_{\gamma_1}}, U_{g_{\gamma_2}} \in Q$ is defined as follows:

$$U_{g_{\gamma_1}} \ast U_{g_{\gamma_2}} = U_{g_{\gamma_1}g_{\gamma_2}}, U_{g_{\gamma_1}g_{\gamma_2}} \in Q \quad (17)$$

$$g_{\gamma_1}g_{\gamma_2} = \alpha_1(r; \gamma_1) + \alpha_1(r_1; \gamma_2) + \sigma(C, \Sigma),$$

where $r_1 = r_\gamma(1)$ and $\sigma(C, \Sigma)$ being the contribution of the Dirac string:

$$\sigma = e \int_{\Sigma} h \cdot dS \quad (18)$$

is not zero if and only if the string $C$ crosses the surface $\Sigma$. The surface is parameterized as $r(t, s) = r_\gamma(t)g_\gamma(s)$ with $0 \leq s \leq 1$, and the vertices are $(r, r_1, r_2)$, where $r_1 = r_\gamma(1)$ and $r_2 = r_\gamma(2)$.

For the triple-product we obtain the following result:

$$U_{g_{\gamma_1}} \ast (U_{g_{\gamma_2}} \ast U_{g_{\gamma_3}}) = e^{i\alpha_3(r; \gamma_1, \gamma_2)}(U_{g_{\gamma_1}} \ast U_{g_{\gamma_2}}) \ast U_{g_{\gamma_3}}$$

where $\alpha_3 = 4\pi\mu$ is the three-cocycle. As is easy to see, if the Dirac quantization condition is fulfilled, the operation $\ast$ is associative and the local group $Q$ becomes the gauge group $U(1)$.

The nonassociative path dependent wave function is given by

$$\Psi(r; \gamma) = e^{i\alpha_1(r; \gamma)}\Psi(r), \quad e^{i\alpha_1(r; \gamma)} \in \text{QU}(1), \quad (19)$$

and the realization of gauge loop $Q$ in the space of the wave functions $\Psi(r; \gamma)$ is defined as follows:

$$U_{g_{\gamma_1}} \ast \Psi(r; \gamma) = e^{i\alpha_1(r_\gamma; \gamma_1) + \sigma(C, \Sigma)}\Psi(r', \gamma_1), \quad (20)$$

where $U_{g_{\gamma_1}} = \exp(i\alpha_1(r; \gamma_1)) \in Q$, and $r' = r_\gamma(1)$.

For the product of $\Psi(r; \gamma_1)$ and $\Psi(r; \gamma_2)$ we obtain

$$\Psi(r, \gamma_1) \ast \Psi(r, \gamma_2) = e^{i\alpha_2(r; \gamma_1, \gamma_2)}\Psi(r, \gamma_1\gamma_2), \quad (21)$$

where

$$\alpha_2(r; \gamma_1, \gamma_2) = e \int_{\Sigma} B \cdot dS = e\Phi|_{\Sigma} \quad (22)$$

$\Phi|_{\Sigma}$ being a magnetic flux through the two-dimensional simplex $\Sigma$ with the vertices are $(r, r_1, r_2)$, where $r_1 = r_\gamma(1)$ and $r_2 = r_\gamma(2)$. Finally, the triple-product obeys

$$\Psi(r, \gamma_1) \ast (\Psi(r, \gamma_2) \ast \Psi(r, \gamma_3)) = e^{i\alpha_3(r; \gamma_1, \gamma_2, \gamma_3)}(\Psi(r, \gamma_1) \ast \Psi(r, \gamma_2)) \ast \Psi(r, \gamma_3). \quad (23)$$

Returning to the AB effect, let us consider the following paths (Fig. 1):

$$\gamma_1(t) : r \rightarrow r + t(a_1 + a_2), \quad (24)$$

$$\gamma_2(t) : r \rightarrow r + t(a_3 + a_4). \quad (25)$$
Now, taking into account the formula (21) one can easily see that the result of Eq. (10) is replaced by

$$\Psi_Q = \Psi_1(\mathbf{r}, \gamma_1) + \Psi_2(\mathbf{r}, \gamma_2) = e^{i\alpha_1(\mathbf{r}, \gamma_1)}\psi_1(\mathbf{r}) + e^{i\alpha_2(\mathbf{r}, \gamma_2)}\psi_2(\mathbf{r}) = e^{i\alpha_1(\mathbf{r}, \gamma_1)}(e^{i\alpha_2(\mathbf{r}, \gamma_2)}\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})),$$  

and a relative phase shift $\Delta \varphi$ is given by

$$\Delta \varphi = \alpha_2(\mathbf{r}; \gamma_1, \gamma_2) = e \int_\Sigma \mathbf{B} \cdot d\mathbf{S} = e \Phi|_\Sigma$$

where $\Phi|_\Sigma$ is the magnetic flux of the monopole through the surface $\Sigma$ bounded by $\gamma_1$ and $\gamma_2$. Evidently, there is no any contribution from the Dirac string to the AB effect.

Note, that for the true AB effect $\Phi|_\Sigma$ is equal to the total magnetic flux $\Phi$ through the solenoid, and we obtain for the relative phase shift $\Delta \varphi$ the same result as in (11),

$$\Delta \varphi = \int_\Sigma \mathbf{B} \cdot d\mathbf{S} = e \oint_\Sigma \mathbf{A} \cdot d\mathbf{r} = e\Phi.$$  

We close with some comments about relevance of this work to nonassociative quantum mechanics. A standard quantum mechanics deals with a linear Hilbert space and associative operators, therefore, the Dirac quantization rule is a necessary condition for the self-consistence of quantum mechanics in the presence of magnetic monopoles. Avoiding this condition implies introducing a nonassociative algebra of observables. Since the nonassociativity produces a serious conflict with a Hilbert space, one must define an nonassociative equivalent to quantum mechanics, may be without Hilbert space and in terms of density matrices alone.

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$\Phi$ is the surface $\Sigma$ bounded by $\gamma_1$ and $\gamma_2$. We close with some comments about relevance of this work to nonassociative quantum mechanics. A standard quantum mechanics deals with a linear Hilbert space and associative operators, therefore, the Dirac quantization rule is a necessary condition for the self-consistence of quantum mechanics in the presence of magnetic monopoles. Avoiding this condition implies introducing a nonassociative algebra of observables. Since the nonassociativity produces a serious conflict with a Hilbert space, one must define an nonassociative equivalent to quantum mechanics, may be without Hilbert space and in terms of density matrices alone.