On The Isoperimetric Spectrum of Graphs

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Abstract  
In this paper we introduce the $k$’th isoperimetric constant of a directed graph as the minimum of the mean outgoing normalized flows from a given set of $k$ disjoint subsets of the vertex set of the graph. In this direction we show that the second isoperimetric constant in the general setting coincides with (the mean version of) the classical Cheeger constant of the graph, while for the rest of the spectrum we show that there is a fundamental difference between the $k$’th isoperimetric constant and the number obtained by taking the minimum over all $k$-partitions. In this regard, we define the concept of a supergeometric graph by proving a Federer-Fleming-type theorem and analyzing the parameters through the corresponding functional definition. We also study the relationships of the isoperimetric spectrum to the classical spectrum of Laplacian eigenvalues, by proving a generalized Cheeger-type inequality as well as generalized Courant-Hilbert inequalities in the context of graph no-homomorphism theorems.

1 Introduction

The chapter on isoperimetric numbers and Cheeger-type inequalities is a classic in geometric analysis and has been considered from many different aspects and points of view [3, 6, 7, 19, 34, 35]. The study of such concepts in the discrete case, although more recent, has also been a center of attention mainly because of its many diverse connections to important problems of the century, both applied and theoretical in nature [1, 9, 11, 12, 13, 22, 27, 29, 33, 38, 39].

Let us recall (e.g. see [39]) the definition of the classical Cheeger constant of a Markov chain $K$ on a connected simple base-graph $G = (V(G), E(G))$, with a stationary distribution $\pi$, as

$$\varsigma(K, \pi) \overset{\text{def}}{=} \min_{A \subseteq V(G), \pi(A) \leq 1/2} \frac{\partial(Q)}{\pi(Q)},$$

where $\partial(Q)$ is the boundary of the set $Q$.
where
\[ \partial(Q) = \frac{1}{2} \sum_{u \in Q \land v \not\in Q} (K(u, v)\pi(u) + K(v, u)\pi(v)), \]
and the not so common mean version as follows,
\[ \iota(K, \pi) = \min_{Q \subseteq V(G)} \frac{\partial(Q)}{2\pi(Q)(1 - \pi(Q))}. \tag{1} \]

Our objectives in this article are manifold, in the sense that, on the one hand, we intend to present a unified and concise treatment of the subject in the discrete case for the general case of directed graphs, containing both geometric and graph theoretic approaches as well as results (e.g. see [8, 25] for the background). In this direction, one of our main contributions is to concentrate on the mean version of the isoperimetric number, and introduce its extensions as a set of constants called the isoperimetric spectrum, in juxtaposition to the classical spectrum consisting of the Laplacian eigenvalues.

One main reason for this shift of interest is the fact that, in our opinion, the mean spectrum of the Laplacian whose \(k\)'th element is the (arithmetic mean) of the first \(k\) eigenvalues of the Laplacian operator, seems to be much more well-behaved than the classical spectrum (e.g. see [31]), which is intuitively quite natural because of the smoothing property of the mean operation. This may, in a way, present a fair chance of a better study of the spectrum and, in this regard, the generalized mean version of the isoperimetric constant plays the central role as the most natural \(L^1\) counterpart.

Another important aspect of considering the mean version is the fact that, as far as we could verify, it presents the best and the most natural framework to generalize the isoperimetric constant, and what is in our opinion a bit of a surprise, is that the new definition seems to be well-defined (e.g. in the sense that it satisfies a generalized co-area formula) only when it is defined as the mean over disjoint subsets of the space (which does not necessarily constitute a partition). The difference between the two definitions based on taking the minimum over the partitions or disjoint sets, although disguised in the case of the classical Cheeger constant (i.e. when we deal with 2-partitions), seems to be inherently nontrivial in general, and will be our main motivation to define the concept of a supergeometric graph.

On the other hand, naturally, pursuing this line of thought, we analyze the isoperimetric spectrum, both from analytic and graph theoretic points of view, and we prove a Federer-Fleming-type theorem as well as Cheeger-type inequalities connecting these parameters and the classical spectrum of eigenvalues in different levels. Also, as a byproduct, it is shown that generalized Cheeger inequalities at the \(k\)'th level seem to be strongly related to the concept of a nodal domain and the corresponding Courant-Hilbert-type results, as it must be expected (e.g. see [32, 41] for the background).

The contents of Section 2 are mostly classic and are mainly presented to fix a common language and notation as well as to record a couple of basic results for further reference.

In Section 3 we introduce and investigate the generalized isoperimetric numbers.
associated to a graph. In this regard, after proving some basic properties in Section 3.1, we prove a Federer-Fleming-type theorem in Section 3.2 to fix an equivalent functional definition. In Section 3.3 we consider the combinatorial kernel related to the natural random walk on the base graph, and in Section 3.4 we show that all the parameters involved are equivalent for a class of graphs, we call supergeometric.

Section 4 is devoted to the study of eigenspaces where we study the relationships between the isoperimetric and the mean spectrum of a graph, in which we show that a generalization of the classical Cheeger inequality is naturally valid but is intrinsically related to the concept of a nodal domain. In this approach we also prove generalized Courant-Hilbert inequalities in the context of graph no-homomorphism results. Section 5 contains our concluding remarks.

2 Preliminaries

In this section we go through some basic definitions and facts that will be used later. In what follows \( \mathbb{R} \) and \( \mathbb{R}^+ \) are the sets of real and nonnegative real numbers, respectively, and for any real number \( x \in \mathbb{R} \) we define

\[
(x)^+ \overset{\text{def}}{=} \begin{cases} x & x > 0 \\ 0 & x \leq 0. \end{cases}
\]

For an \( n \)-list of real numbers (repetition is allowed) as \((\zeta_1, \zeta_2, \ldots, \zeta_n)\), the mean \( n \)-list is denoted by \((\overline{\zeta_1}, \overline{\zeta_2}, \ldots, \overline{\zeta_n})\), where

\[
\overline{\zeta_k} \overset{\text{def}}{=} \frac{1}{k} \sum_{i=1}^{k} \zeta_i.
\]

Hereafter, \( \chi_A \) denotes the characteristic function of a set \( A \) and we adopt the notations \( I_k \overset{\text{def}}{=} \{k, k+1, \ldots, n\} \). Also, \( I_n \overset{\text{def}}{=} I_1 \).

2.1 Function spaces

If \( X \) is a set then \( \mathcal{F}(X) \) stands for the set of all real functions \( f : X \rightarrow \mathbb{R} \), and similarly, \( \mathcal{F}^+(X) \overset{\text{def}}{=} \{ f : X \rightarrow \mathbb{R}^+ \} \). Also, if \( \omega : X \rightarrow \mathbb{R}^+ - \{0\} \) is a weight function then we define the inner product \( <\cdot, \cdot>_{\omega} \) and the norm \( \|\cdot\|_{p,\omega} \) on \( \mathcal{F}(X) \) as

\[
<f, g>_{\omega} \overset{\text{def}}{=} \sum_{x \in X} f(x)g(x)\omega(x), \quad \|f\|_{p,\omega} \overset{\text{def}}{=} \left( \sum_{x \in X} |f(x)|^p \omega(x) \right)^{\frac{1}{p}},
\]

respectively, where we usually use the subscript \( \omega \) to refer to the product structure (e.g. \( \mathcal{F}_\omega(X) \)). Two functions \( f, g \in \mathcal{F}_\omega(X) \) are said to be orthogonal with respect to \( \omega \), i.e. \( f \perp_{\omega} g \), whenever \( <f, g>_{\omega} = 0 \).

For any \( f \in \mathcal{F}(X) \), \( \text{supp}(f) \) stands for the set \( \{v \in V(G) \mid f(v) \neq 0\} \). Also, for any subset \( A \subseteq \text{Domain}(f) \subseteq X \) the restriction of \( f \) to \( A \) is denoted by \( f|_A \), i.e.,

\[
f|_A(x) \overset{\text{def}}{=} \begin{cases} f(x) & x \in A \\ 0 & x \notin A. \end{cases}
\]
Moreover, for any real function \( f \), the functions \( f^+ \) and \( f^- \) stand for the positive and negative parts of \( f \), respectively; and consequently,

\[
f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.
\]

For any two functions (or vectors) \( f, g \), we write \( f \leq g \) if

\[
\forall \ v \in V(G) \quad f(v) \leq g(v).
\]

Also, we write \( f < g \) if \( f \leq g \) and \( f \neq g \).

### 2.2 Graphs and kernels

The main objective of this section is to introduce a common language of graphs and kernels that is accessible to both graph theorists and experts in functional analysis and also benefits from all aspects of the two points of view.

Throughout the paper, a graph \( G = (V(G), E(G)) \) is always assumed to be a finite directed graph (possibly with loops and without multiple edges), where \( E(G) \subseteq V(G) \times V(G) \). Similarly, an undirected graph \( \overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G})) \) is a finite set \( V(\overrightarrow{G}) \) along with a set of undirected edges \( E(\overrightarrow{G}) \), each element of which is a subset of \( V(\overrightarrow{G}) \) whose size is less than or equal to 2. When it is clear from the context, by abuse of notation, we use the same symbol \( uv \) both for the directed edge \((u, v) \in V(G) \times V(G)\) of a directed graph and also for a simple edge \( \{u, v\} \) of an undirected graph.

For a given graph \( G \), we use the natural notation, \( \overrightarrow{G} \), for its symmetric directed base graph i.e. \( \overleftrightarrow{G} \equiv V(G) \) and

\[
(uv \in E(\overrightarrow{G}) \text{ and } vu \in E(\overrightarrow{G})) \iff (uv \in E(G) \text{ or } vu \in E(G)).
\]

Moreover, for a given graph \( G \), \( \overrightarrow{G} \) stands for its symmetric undirected base graph i.e. \( \overleftrightarrow{G} \equiv V(G) \) and

\[
uv \in E(\overrightarrow{G}) \iff (uv \in E(G) \text{ or } vu \in E(G)).
\]

Note that for an undirected graph \( \overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G})) \) we may think of any simple edge \( uv \) as a subset \( \{u, v\} \subseteq V(\overrightarrow{G}) \). With this interpretation \( \overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G})) \) is a directed graph obtained from replacing any simple edge \( uv \in E(\overrightarrow{G}) \) by two directed edges \( uv \in E(\overrightarrow{G}) \) and \( vu \in E(\overrightarrow{G}) \). Note that there is a one to one correspondence between undirected graphs and symmetric directed graphs, where the undirected presentation can be interpreted as a more compact version of expressing the same data.

Given any \( n \times n \) matrix \( K \) whose rows and columns are indexed by the elements of an \( n \)-set \( V \), in general, one can construct a graph \( G_K = (V, E) \) where

\[
uv \in V \quad \iff \quad K(u, v) \neq 0.
\]

Then, it is clear that from this point of view, the concept of a weighted graph contain the same data as the concept of a matrix, and moreover, symmetric graphs as well as undirected graphs correspond to the concept of symmetric matrices.
In this paper, we are mainly concerned about the relationship between the properties of the kernel and the connectivity of the corresponding symmetric base graph. We continue this line of thought in the next section.

As for a couple of more notations, the set $D_n(G)$ is defined to be the set of all $n$-sets $\{Q_1, \ldots, Q_n\}$ with $\emptyset \neq Q_i \subseteq V(G)$ for all $1 \leq i \leq n$ such that for every pair $1 \leq i < j \leq n$ we have $Q_i \cap Q_j = \emptyset$. The set of $n$-partitions of a graph $G$, which is denoted by $P_n(G)$, is the subset of $D_n(G)$ that contains all partitions $\{Q_1, \ldots, Q_n\}$ for which $\cup_{i=1}^n Q_i = V(G)$. For two given subsets $X, Y$ of $V(G)$ we define,

$$\overrightarrow{E}(X, Y) \overset{\text{def}}{=} \{e = uv \in E(G) \mid u \in X \& v \in Y\}.$$ 

Also, for a subset $Q \subseteq V(G)$ we define,

$$\overrightarrow{E}(Q) \overset{\text{def}}{=} \overrightarrow{E}(Q, Q^c), \quad \overrightarrow{E}(Q^c) \overset{\text{def}}{=} \overrightarrow{E}(Q^c, Q),$$

and

$$\overrightarrow{E}(Q) \overset{\text{def}}{=} \overrightarrow{E}(Q) \cup \overrightarrow{E}(Q).$$

### 2.3 Markov kernels and the energy space

In this section we collect some basic facts about the theory of finite Markov chains with a graph theoretic emphasis (e.g. for more on this see [1, 17, 39]). Hereafter, given a graph $G$, we assume that $K$ is the kernel of a Markov chain on this graph and $\pi$ is a nowherezero stationary distribution, i.e. $\pi K = \pi$ and $\pi(v) \neq 0$ for all $v \in V(G)$.

In this setting, $\phi(u, v) \overset{\text{def}}{=} K(u, v)\pi(u)$ defines a nowherezero flow on $G$. Also, for any $Q \subseteq V(G)$ we define

$$\pi(Q) \overset{\text{def}}{=} \sum_{u \in Q} \pi(u), \quad \overrightarrow{\partial}(Q) \overset{\text{def}}{=} \sum_{uv \in E(Q)} \phi(u, v),$$

and $\overrightarrow{\partial}^c(Q)$ analogously. Note that since $\phi$ is a flow, for every nonvoid $Q \subseteq V(G)$, we have

$$\overrightarrow{\partial}(Q) = \overrightarrow{\partial}(Q) = \overrightarrow{\partial}^c(Q^c).$$

Sometimes, by abuse of notation, we may write $\overrightarrow{\partial}(Q)$ for simplicity. (Note that for Riemannian manifolds $\partial(Q) = Vol(Borderay(Q))$ has the same property as a trivial flow with only a nonzero component to the complement.) Within the same setup we consider the flow

$$\overrightarrow{\phi}(u, v) \overset{\text{def}}{=} \frac{1}{2}(\phi(u, v) + \phi(v, u)),$$

on $\overrightarrow{G}$, coming from the kernel

$$\overrightarrow{K}(u, v) \overset{\text{def}}{=} \frac{1}{2}(K + K^*) = \frac{1}{2} \left( K(u, v) + \frac{K(v, u)\pi(v)}{\pi(u)} \right),$$

\[5\]
with the same stationary distribution \( \pi \). Also, we consider two linear Laplacian operators on \( F_{\pi}(G) \) as follows,

\[
\overrightarrow{\Delta} \overset{\text{def}}{=} \text{id} - K \quad \text{and} \quad \Delta \overset{\text{def}}{=} \text{id} - K^*,
\]

where \( \text{id} \) is the identity operator. It is clear that \( K \) and \( \Delta \) are self-adjoint operators on \( F_{\pi}(G) \) by definition, while \( K \) and \( \overrightarrow{\Delta} \) may not be necessarily so, unless \( K = K^* \) and \( \overrightarrow{\Delta} = \Delta \). Hence, when \( |V(G)| = n \), one may order all real eigenvalues of \( K \) and \( \Delta \) as

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n,
\]

(2)

and

\[
0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,
\]

(3)

respectively. (At times we may use superscripts as \( \lambda_1^G \) or \( \alpha_2^K \) to refer to the graph or the kernel when details are clear from the context.)

Also, it is a well-known fact (Perron-Frobenius theorem) that for a strongly connected graph \( G \), the eigenspace corresponding to the eigenvalue \( 0 = \lambda_1^G \) is one dimensional and is generated by the constant vector \( 1 \). Moreover, for any self-adjoint matrix \( A \), by Courant–Fischer variational principle (see [39]) and for any \( 1 \leq k \leq n \) one may write

\[
\lambda_k^A = \min_{W_k} \max_{\varnothing \neq f \in W_k} \left\{ \frac{< Af, f >}{\| f \|^2} \right\} = \max_{W_k} \min_{\varnothing \neq f \in W_k} \left\{ \frac{< Af, f >}{\| f \|^2} \right\},
\]

(4)

in which

\[
W_k \overset{\text{def}}{=} \{ W \mid \dim(W) \geq k \}, \quad W_k^\perp \overset{\text{def}}{=} \{ W \mid \dim(W^\perp) \leq k \},
\]

and \( < f, g > \) is the inner product of the space. In the rest of this paper we will adopt the following important assumption for the notations.

**Assumption:** Hereafter, all operators (e.g. \( \nabla, \nabla, \Delta \)) are assumed to be related to a fixed kernel \( K \) on a base graph \( G = (V(G), E(G)) \) with a nowherezero stationary distribution (i.e. invariant measure) \( \pi \). Also, we always assume the orderings mentioned in 2 and 3.

In what follows, we will be working with the measure spaces \( (V(G), \pi) \) and \( (E(G), \phi) \) for any directed graph \( G \) (that may have loops) as well as \( (\overline{V(G)}, \pi) \) and \( (\overline{E(G)}, \phi) \) for the corresponding undirected graph, (note that the last case also covers the case of simple graphs). In our notations the subscript determines the function space under consideration (e.g. \( F_{\phi}(G) \) stands for the set of all real functions defined on \( E(G) \), the set of edges of a given graph \( G \), equipped with an inner product weighted by \( \phi \)).

It is instructive to note that considering the inner-product space equipped with the weighted inner-product \( < \cdot, \cdot >_{\pi} \) has the advantage of reflecting parts of the global structural properties of the base graph in the spectrum of the corresponding Laplacian operator, while this is not necessarily true when one uses the ordinary inner-product of \( \mathbb{R}^n \). (Section 3.3 demonstrates how this framework also covers the classical case.)
2.4 Gradients, energy and their properties

Given a graph \( G \), one may define the directed, classical, and symmetric gradients, respectively, as follows,

- \( \nabla : \mathcal{F}_\pi(G) \rightarrow \mathcal{F}_\phi(G) \) as \( \nabla f(uv) \overset{\text{def}}{=} (f(u) - f(v))^+ \).
- \( \nabla : \mathcal{F}_\pi(G) \rightarrow \mathcal{F}_\phi(G) \) as \( \nabla f(uv) \overset{\text{def}}{=} f(u) - f(v) \).
- \( \nabla : \mathcal{F}_\pi(G) \rightarrow \mathcal{F}_\phi(G) \) as \( \nabla f(uv) \overset{\text{def}}{=} |f(u) - f(v)| \).

**Lemma 1.** For a given graph \( G \), the classical gradient, \( \nabla \), is a linear operator and has an adjoint \( \nabla^* : \mathcal{F}_\phi(G) \rightarrow \mathcal{F}_\pi(G) \) defined as

\[
\nabla^* f(u) \overset{\text{def}}{=} \frac{1}{\pi(u)} \left( \sum_{uv} f(uv)\phi(u,v) - \sum_{vu} f(vu)\phi(v,u) \right).
\]

Moreover, \( 2\Delta = \nabla^* \nabla \).

**Proof.** It is easy to check that

\[
(\Delta f)(u) = \frac{1}{2} \nabla^* \nabla f(u) = \frac{1}{\pi(u)} \sum_{v \in V(G)} \nabla f(uv) \overline{\phi}(u,v) = \frac{1}{\pi(u)} \sum_{v \in V(G)} (f(u) - f(v)) \overline{\phi}(u,v).
\]

Also, note that,

\[
< 2\Delta f, g >_\pi = < \nabla^* \nabla f, g >_\pi = < \nabla f, \nabla g >_\pi,
\]

holds for all \( f, g \in \mathcal{F}_\pi(G) \).

The simple but important statement of Lemma 1 in a way presents the symmetrization process of constructing the undirected symmetric graph \( \overline{G} \) from a given graph \( G \), in an analytic sort of way. In other words, starting from a kernel \( K \) on a base graph \( G \), and considering the operators \( \nabla \) and \( \nabla^* \), one may construct the symmetric Laplacian operator as \( 2\Delta = \nabla^* \nabla \) that introduces a new kernel whose base graph is \( \overline{G} \). Also, a classical and interesting fact is that if one starts from a graph \( G \) and considers the conservation of energy as Kirchhoff’s node and loop laws, then one finds a Poisson’s equation relating the current and voltage (i.e. potential) functions whose basic operator is the symmetric Laplacian \( \Delta \) on \( \overline{G} \). Hence, in this sense, conservation of energy leads to symmetric models.

With this background, one of our main objectives can be described as finding methods that can reflect some of the connectivity properties of \( G \) in its related symmetric model \( \overline{G} \), through the self-adjoint or symmetric operators defined on it. Therefore, it is natural to concentrate on well-behaved or induced operators on \( \overline{G} \) (e.g. \( \nabla : \mathcal{F}_\pi(\overline{G}) \rightarrow \mathcal{F}_\phi(\overline{G}) \)) and consider their relationships to those of \( G \). The following lemma summarizes some of the basic properties of these operators for further reference.

**Lemma 2.** For any given graph \( G \), and \( f \in \mathcal{F}_\pi(G) \),
\[ a \) \| \nabla f \|_{1, \phi} = \frac{1}{2} \| \nabla f \|_{1, \phi} = \| \nabla f \|_{1, \pi}. \]

\[ b \) \| \overline{\nabla} f \|_{1, \phi} = \| \nabla f^+ \|_{1, \phi} + \| \nabla f^- \|_{1, \phi}. \]

\[ c \) \frac{1}{2} \| \nabla f \|_{2, \phi}^2 = \| \nabla f \|_{2, \pi}^2 = \langle \Delta f, f \rangle = \langle \overline{\Delta} f, f \rangle. \]

**Proof.** For (a) note that \( \phi \) is a flow on \( E(G) \) and consequently,

\[ \| \overline{\nabla} f \|_{1, \phi} = \sum_{u, v \in V(G)} (f(u) - f(v))^+\phi(u, v) = \frac{1}{2} \sum_{u, v \in V(G)} |f(u) - f(v)| \phi(u, v) = \| \nabla f \|_{1, \pi}. \]

Equality in (b) is clear. Also, (c) follows from Lemma 1 and the following equalities,

\[ \langle I - K \rangle f, f \rangle = \langle I - \frac{1}{2}(K + K^*) \rangle f, f \rangle = \| \nabla f \|_{2, \pi}^2 = \sum_{u, v \in E(G)} |f(u) - f(v)|^2 \phi(u, v). \]

In the rest of this paper, given any graph \( G \), we will be working within the frameworks \( [G, (V(G), \pi), (E(G), \phi), \overline{\nabla}, \nabla, \Delta, \overline{\Delta}] \) and \( \overline{[G, (V(G), \pi), (E(G), \overline{\phi}), \overline{\nabla}, \overline{\nabla}, \Delta, \overline{\Delta}]} \) for the corresponding undirected graph. In this regard, we have chosen an inner product space that is tightly related to the global properties of the base graph through the stationary distribution \( \pi \), where the corresponding symmetric Laplacian (hopefully) reflects some of these global properties through its spectrum (e.g. see [14, 15, 17]). This approach, in our opinion, is the most important application of invariant measures in the contexts and marks the distinction between the classical approach through ordinary inner product an the one based on the the global properties through an invariant measure.

Clearly, in this approach, one needs some relations between the energy (Dirichlet) norms and different norms of the operators to construct the necessary connections needed. The following two lemmas demonstrate the most basic relationships.

**Lemma 3.** For every \( f \in \mathcal{F}_x(G) \) we have

\[ a \) \| \nabla f \|_{1, \phi} \leq \| \nabla f \|_{2, \phi}. \]

\[ b \) \| \overline{\nabla} f \|_{1, \phi} \leq \| \nabla f \|_{1, \pi} \leq \frac{\sqrt{2}}{2} \| \nabla f \|_{2, \pi}. \]

**Proof.** Since \( \phi \) is a flow we have \( \sum_{u, v \in V(G)} \phi(u, v) = 1 \), and consequently, by Cauchy-Schwartz inequality,

\[ \sum_{u, v \in V(G)} |f(u) - f(v)|^2 \phi(u, v) = \left( \sum_{u, v \in V(G)} |f(u) - f(v)|^2 \phi(u, v) \right) \left( \sum_{u, v \in V(G)} \phi(u, v) \right) \geq \left( \sum_{u, v \in V(G)} |f(u) - f(v)| \phi(u, v) \right)^2. \]
Lemma 4. For every $f \in \mathcal{F}_x(G)$ we have,

\begin{align*}
a) \quad & \frac{\|\nabla f\|_{2,\phi}}{\|f\|_{2,\pi}} \leq 2 \frac{\|\nabla f\|_{2,\phi}}{\|f\|_{1,\pi}}. \\
b) \quad & \frac{\|\nabla f\|_{2,\phi}}{\|f\|_{2,\pi}} = \frac{\|\nabla f\|_{2,\phi}}{\|f\|_{2,\pi}} \leq \sqrt{2} \frac{\|\nabla f\|_{2,\phi}}{\|f\|_{2,\pi}}.
\end{align*}

Proof. The proof is clear by Cauchy-Schwartz inequality and the following,

\begin{align*}
\left(\frac{\sqrt{2} \|\nabla f\|_{2,\pi}}{\|f\|_{2,\pi}}\right)^2 &= \sum_{u,v \in V(G)} \frac{|f(u) - f(v)|^2 \phi(u,v)}{\sum_{u \in V(G)} |f(u)|^2 \pi(u)} \cdot \frac{\sum_{u,v \in V(G)} |f(u) + f(v)|^2 \phi(u,v)}{\sum_{u,v \in V(G)} |f(u)|^2 \pi(u)} \\
&\geq \frac{\left(\sum_{u,v \in V(G)} |f(u)^2 - f(v)|^2 \phi(u,v)\right)^2}{4 \left(\sum_{u \in V(G)} |f(u)|^2 \pi(u)\right)^2} = \left(\frac{\|\nabla f\|_{2,\phi}}{\|f\|_{1,\pi}}\right)^2.
\end{align*}

\[\blacksquare\]

3 The isoperimetric spectrum

In this section we concentrate on the isoperimetric constant and its generalization. In this regard, our point of view is to consider a generalization that is, firstly, well-behaved computationally, and secondly, can present a good relation to the classical eigenvalues.

Throughout the section $K$ is the kernel of a fixed Markov chain on the base graph as before, and $\pi$ is a nowherezero stationary distribution for this kernel. It is a well-known fact from random-matrix theory and the recent literature that the behaviour of the classical spectrum of the Laplacian operator is quite hard to predict and, as a matter of fact, is related to some deep problems in contemporary mathematic \[22\]. In our opinion, one possible approach in this direction is to analyze a smooth function of the spectrum, that in a way contains a fair amount of data, rather than the eigenvalues themselves. Naturally, the most simple candidate for such a function can be considered to be the arithmetic mean, and consequently, there seems to be a fair chance that the behaviour of the mean-spectrum, whose $k^{th}$ element is the mean of the first $k$ eigenvalues, be more well-behaved than the spectrum itself. We should also mention the results of J. B. Hiriart-Urruty and D. Ye \[31\] that, in a sense, justifies this approach.

Therefore, based on the above-mentioned approach we will focus on the mean version of the isoperimetric constant and will generalize it as the most natural $L^1$ counterpart of the mean eigenvalue. It is instructive to note that this generalization leads to a definition for the $k^{th}$ isoperimetric number which is based on taking a minimum over all $k$-disjoint subsets of the ground-space, rather than its $k$-partitions, and also
satisfies a Federer-Fleming-type theorem (Theorem 1). This difference, although disguised in the classical case \( k = 2 \), seems to be quite nontrivial in general and will be our main motivation for the definition of a supergeometric graph.

It is not hard to check that there is a straightforward translation of almost all results of this section to the case of compact Riemannian manifolds (considering appropriate modifications).

3.1 The isoperimetric constant

In what follows we introduce the generalized isoperimetric number (in the mean case), and we investigate some of its basic properties. To begin we define a couple of function spaces as follows.

**Definition 1.** We define the space of *unit positive functions* as,
\[
\mathcal{U}^+_n(G) \overset{\text{def}}{=} \{ f \mid f \neq 0 \text{ and } f \in \mathcal{F}^+_n(G) \text{ and } \| f \|_{1,n} = 1 \}.
\]
Also, a class of functions \( \{ f_i \}_{i=1}^n \) is called *positive orthonormal*, if for all \( 1 \leq i \leq n \) we have \( f_i \in \mathcal{U}^+_n(G) \) and moreover, for all pairs of indices \( i \neq j \) we have \( f_i \perp \pi f_j \). In this regard we define,
\[
\mathcal{O}^+_n(G) \overset{\text{def}}{=} \{ \{ f_i \}_{i=1}^n \mid \{ f_i \}_{i=1}^n \text{ is positive orthonormal} \}.
\]
\[
\tilde{\mathcal{O}}^+_n(G) \overset{\text{def}}{=} \{ \{ f_i \}_{i=1}^n \in \mathcal{O}^+_n(G) \mid \{ \supp(f_i) \}_{i=1}^n \in \mathcal{P}^+_n(G) \}.
\]

Now, we define the generalized isoperimetric numbers as,

**Definition 2.** Given a graph \( G \) and a kernel \( K \), the \( n \)th *isoperimetric constant* of \( G \) (with respect to \( K \)) is defined as follows
\[
\iota_n(G, K) \overset{\text{def}}{=} \min_{\{ Q_i \}_{i=1}^n \in \mathcal{D}^+_n(G)} \frac{1}{n} \left( \sum_{i=1}^n \tilde{\partial}(Q_i) \pi(Q_i) \right).
\]
Also, considering the partitions, we define the following related constant,
\[
\tilde{\iota}_n(G, K) \overset{\text{def}}{=} \min_{\{ Q_i \}_{i=1}^n \in \mathcal{P}^+_n(G)} \frac{1}{n} \left( \sum_{i=1}^n \tilde{\partial}(Q_i) \pi(Q_i) \right).
\]

We exclude the kernel \( K \), when it is fixed or is clear from the context.

**Example 1.** As the first example let us consider the case \( n = 2 \).
\[
\tilde{\iota}_2(G) = \min_{\{ Q_i \}_{i=1}^2 \in \mathcal{P}^+_2(G)} \frac{1}{2} \left( \frac{\tilde{\partial}(Q_1)}{\pi(Q_1)} + \frac{\tilde{\partial}(Q_2)}{\pi(Q_2)} \right).
\]
\[
= \min_{Q \subseteq V(G)} \frac{1}{2} \left( \frac{\tilde{\partial}(Q)}{\pi(Q)} + \frac{\tilde{\partial}(Q^c)}{\pi(Q^c)} \right)
= \min_{Q \subseteq V(G)} \frac{1}{2} \left( \frac{\tilde{\partial}(Q)}{\pi(Q)} + \frac{\tilde{\partial}(Q)}{(1 - \pi(Q))} \right)
= \min_{Q \subseteq V(G)} \frac{\tilde{\partial}(Q)}{2\pi(Q)(1 - \pi(Q))}.
\]
which is the (mean version) of the classical Cheeger constant. Soon we will prove that
\( \tilde{\iota}_2(G) = \iota_2(G) \) (see Corollary \( \Box \)(b)) that justifies our definition for the isoperimetric
number in the classical case.

Some basic properties of the isoperimetric number will follow from the following
technical proposition.

**Proposition 1.** For any graph \( G \) (and a given kernel \( K \) on it),

a) If \( \{Q_i\}_{i \in I_n} \in \mathcal{D}_n(G) \) and we define \( \{Q^*\} \equiv (\cup_{i \in I_n} Q_i)^c \), and for every \( j \in I_n \)
\[
S^n_j = \frac{1}{n} \left( \frac{\partial (Q_j \cup Q^*)}{\pi(Q_j \cup Q^*)} + \sum_{i \in I_n \setminus \{j\}} \frac{\partial (Q_i)}{\pi(Q_i)} \right).
\]

then
\[
\min_{j \in I_n} S^n_j \leq \frac{1}{n(1 + (n - 1)\pi(Q^*))} \left( (n - 2)\partial(Q^*) + (1 + (n - 2)\pi(Q^*)) \sum_{i=1}^n \frac{\partial(Q_i)}{\pi(Q_i)} \right).
\]

b) If \( \{Q_i\}_{i \in I_{n+1}} \in \mathcal{D}_{n+1}(G) \), and we define \( \{Q^*\} \equiv (\cup_{i \in I_{n+1}} Q_i)^c \), and for every pair of indices \( \{j, k\} \subseteq I_{n+1} \)
\[
T^n_{j,k} = \frac{1}{n} \left( \frac{\partial (Q_j \cup Q_k)}{\pi(Q_j \cup Q_k)} + \sum_{i \in I_{n+1} \setminus \{j, k\}} \frac{\partial (Q_i)}{\pi(Q_i)} \right).
\]

then
\[
\min_{\{j,k\} \subseteq I_{n+1}} T^n_{j,k} \leq \frac{\partial(Q^*)}{n^2(1 - \pi(Q^*))} + \frac{(n - 1)}{n^2} \sum_{i \in I_{n+1}} \frac{\partial(Q_i)}{\pi(Q_i)}.
\]

**Proof.** The proof is based on the fact that a weighted mean of a set of numbers is
greater than or equal to the minimum of the set. For part (a) let \( w_j \equiv \pi(Q_j \cup Q^*) \)
and note that
\[
\sum_{j=1}^n w_j = 1 + (n - 1)\pi(Q^*).
\]

Also,
\[
n \sum_{j=1}^n w_j S^n_j = \sum_{j=1}^n \frac{\partial (Q_j \cup Q^*)}{\pi(Q_j \cup Q^*)} + \sum_{j=1}^n \sum_{i \in I_n \setminus \{j\}} \pi(Q_j \cup Q^*) \frac{\partial(Q_i)}{\pi(Q_i)}
\]
\[
= (n - 2)\partial(Q^*) + \sum_{j=1}^n \partial(Q_j)
\]
\[
+ \sum_{j=1}^n \frac{(1 + (n - 2)\pi(Q^*) - \pi(Q_j))}{\pi(Q_j)} \frac{\partial(Q_j)}{\pi(Q_j)}
\]
\[
= (n - 2)\partial(Q^*) + \sum_{j=1}^n \frac{(1 + (n - 2)\pi(Q^*))}{\pi(Q_j)} \partial(Q_j).
\]

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Similarly, for part (b) let \( w_{j,k} \overset{\text{def}}{=} \pi(Q_j \cup Q_k) \) and note that
\[
\sum_{1 \leq j < k \leq n+1} w_{j,k} = n(1 - \pi(Q^*)).
\]

Also,
\[
n \sum_{1 \leq j < k \leq n+1} \sum_{i \in I_{n+1} - \{j,k\}} \pi(Q_j \cup Q_k) \frac{\delta(Q_i)}{\pi(Q_i)} = \delta(Q^*) + (n-1) \sum_{i=1}^{n} \frac{\delta(Q_i)}{\pi(Q_i)}
\]
\[
+ \sum_{i=1}^{n} (n-1)(1 - \pi(Q_i) - \pi(Q^*)) \frac{\delta(Q_i)}{\pi(Q_i)}
\]
\[
= \delta(Q^*) + (n-1)(1 - \pi(Q^*)) \sum_{i=1}^{n} \frac{\delta(Q_i)}{\pi(Q_i)}
\].

**Corollary 1.** For any graph \( G \) (and a given kernel \( K \) on it) and for all \( 1 \leq n \leq |V(G)| \) we have,

a) \( 0 \leq \tilde{\iota}_n(G) - \iota_n(G) \leq \frac{1}{n} \).

b) \( \tilde{\iota}_2(G) = \iota_2(G) \).

c) \( \tilde{\iota}_n(G) \leq (1 - \frac{1}{n^2}) \tilde{\iota}_{n+1}(G) \).

d) \( \iota_n(G) \leq \iota_{n+1}(G) \).

**Proof.** The left hand inequality of part (a) is clear by definitions. Applying Proposition \( \Pi(a) \) and noting that \( \tilde{\iota}_n(G) \leq S^n_j \) we have,
\[
\tilde{\iota}_n(G) \leq \min_{j \in \mathcal{I}_n} S^n_j \leq \frac{1}{n} \left( 1 + \sum_{i=1}^{n} \frac{\delta(Q_i)}{\pi(Q_i)} \right) \leq \frac{1}{n} + \iota_n(G).
\]

Also, for the second part note that,
\[
\tilde{\iota}_2(G) \leq \min_{j \in \{1,2\}} S^2_j \leq \frac{1}{2(1 + \pi(Q_3))} \left( \sum_{i=1}^{2} \frac{\delta(Q_i)}{\pi(Q_i)} \right),
\]
which yields \( \tilde{\iota}_2(G) \leq \iota_2(G) \). Now applying the first inequality of part (a) we have \( \tilde{\iota}_2(G) = \iota_2(G) \).

For the next part, apply Proposition \( \Pi(b) \) with \( Q^* = \emptyset \). Hence,
\[
\tilde{\iota}_n(G) \leq \min_{\{j,k\} \in \mathcal{I}_{n+1}} T^n_{j,k} \leq \left( \frac{n^2 - 1}{n^2} \right) \left( \frac{1}{n+1} \sum_{i \in \mathcal{I}_{n+1}} \frac{\delta(Q_i)}{\pi(Q_i)} \right),
\]

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which proves part (c).

For part (d), let \( \{Q_i\}_{i \in \mathcal{I}_{n+1}} \in \mathcal{D}_{n+1}(G) \) be chosen such that,

\[
\ell_{n+1}(G) = \frac{1}{n+1} \sum_{i \in \mathcal{I}_{n+1}} \frac{-\partial(Q_i)}{\pi(Q_i)},
\]

and

\[
\frac{-\partial(Q_i)}{\pi(Q_i)} \leq \cdots \leq \frac{-\partial(Q_{n+1})}{\pi(Q_{n+1})}.
\]

Then clearly,

\[
\ell_n(G) \leq \frac{1}{n} \sum_{i=1}^{n} \frac{-\partial(Q_i)}{\pi(Q_i)} \leq \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{-\partial(Q_i)}{\pi(Q_i)} = \ell_{n+1}(G).
\]

\[\square\]

### 3.2 A Federer-Fleming-type theorem

Our basic aim in this section is to find a functional definition through proving a Federer-Fleming type theorem. In this direction, given a kernel \( K \) and a nowherezero stationary distribution \( \pi \), let us consider the following parameters which are naturally related to the constants \( \ell_n(G) \) and \( \tilde{\ell}_n(G) \),

\[
\gamma_n(G, K) \overset{\text{def}}{=} \min_{(f_i^0)^n \in \mathcal{O}_n^+(G)} \frac{1}{n} \left( \sum_{i=1}^{n} \|\nabla f_i\|_{1,\phi} \right).
\]

\[
\tilde{\gamma}_n(G, K) \overset{\text{def}}{=} \min_{(f_i^0)^n \in \mathcal{O}_n^+(G)} \frac{1}{n} \left( \sum_{i=1}^{n} \|\nabla f_i\|_{1,\phi} \right).
\]

As usual we exclude the kernel when it is fixed or is clear from the context.

**Theorem 1.** For any graph \( G \) (and a given kernel \( K \) on it) and for all \( 1 \leq n \leq |V(G)| \) we have

\[
\ell_n(G) = \gamma_n(G).
\]

**Proof.** On the one hand, by considering characteristic functions of sets we have,

\[
\ell_n(G) \geq \gamma_n(G).
\]

On the other hand, let \( \{f_i^0\}_i \in \mathcal{O}_n^+(G) \) be a class of functions with \( R_i^0 = \text{Range}(f_i) \) such that \( \bigcup_{i \in \mathcal{I}_n} R_i = \{r_1, r_2, \ldots, r_{t+1}\} \) and

\[
0 = r_0 < r_1 < r_2 < \ldots < r_{t+1}.
\]

For every \( i, j \in \mathcal{I}_n \) define \( s_i^0 = r_{i+1} - r_i \), \( F_{i,j}^0 = \{v \mid f_i(v) > r_j\} \), \( \pi_{i,j}^0 = \pi(F_{i,j}) \) and \( \partial_{i,j}^0 = \partial(F_{i,j}) \). Also, define

\[
t_i^0 = \min\{j \mid 0 \leq j \leq t \text{ and } F_{i,j} \neq \emptyset\}.
\]
Note that since $f_i \neq 0$ for all $i$, the right hand set is nonvoid and consequently for any such $i$ we have $t_i \geq 0$. Moreover, by definitions and the co-area formula we have,

$$
1 = \|f_i\|_{1,\phi} = \sum_{j=0}^{t_i} s_j \pi_{i,j}, \quad \|\nabla f_i\|_{1,\phi} = \sum_{j=0}^{t_i} s_j \partial_{i,j},
$$

and moreover,

$$
\forall (Q_1, Q_2, \ldots, Q_n) \subseteq (\{F_{i,j}\}_{0}^{t_1}, \{F_{2,j}\}_{0}^{t_2}, \ldots, \{F_{n,j}\}_{0}^{t_n}), \quad \tau_n(G) \leq \frac{1}{n} \left( \sum_{k=1}^{n} \frac{\partial(Q_k)}{\pi(Q_k)} \right).
$$

For simplicity we index these inequalities by $(j_1, j_2, \ldots, j_n) \subseteq (I_1^0 \times \cdots \times I_n^0)$ if $Q_k = F_{k,j_k}$ for all $1 \leq k \leq n$. Hence, for any such index $(j_1, j_2, \ldots, j_n)$,

$$
n \tau_n(G) \prod_{k=1}^{n} \pi_{k,j_k} \leq \sum_{k=1}^{n} \left( \prod_{i \neq k} \pi_{i,j_i} \right) \partial_{k,j_k}.
$$

Now, if we multiply both sides of such an inequality by $\prod_{k=1}^{n} s_{j_k}$ and sum over all indices $(j_1, j_2, \ldots, j_n) \subseteq (I_1^0 \times \cdots \times I_n^0)$ we get,

$$
n \tau_n(G) \prod_{k=1}^{n} \sum_{j=0}^{t_k} s_j \pi_{k,j} = n \tau_n(G) \prod_{k=1}^{n} \sum_{j=0}^{t_k} s_j \pi_{k,j} \pi_{i,j_i} \partial_{i,j_i}
$$

$$
\leq \sum_{(j_1, j_2, \ldots, j_n)} \sum_{k=1}^{n} s_{j_k} \left( \prod_{i \neq k} \pi_{i,j_i} \right) \partial_{k,j_k}
$$

$$
= \sum_{k=1}^{n} \sum_{j_k=1}^{t_k} s_{j_k} \left( \sum_{j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_n} \left( \prod_{i \neq k} \pi_{i,j_i} \right) \partial_{k,j_k} \right)
$$

$$
= \sum_{k=1}^{n} \sum_{j_k=1}^{t_k} s_{j_k} \left( \prod_{i \neq k} \pi_{i,j_i} \partial_{k,j_k} \right)
$$

$$
= \sum_{k=1}^{n} \sum_{j_k=1}^{t_k} s_{j_k} \partial_{k,j_k}
$$

$$
= \sum_{k=1}^{n} \|\nabla f_i\|_{1,\phi},
$$

which shows that $\tau_n(G) \leq \gamma_n(G)$, and the theorem is proved. ■
3.3 The combinatorial kernel

Comparison method is a well-known tool in the theory of Markov chains that is based on a basic lemma of Diaconis and Saloff-Coste (e.g. see [20, 21, 39]). It has already been shown that this method can be considered in a more general setting of graph homomorphisms where one can effectively use a combination of well-known results in graph theory and the lemma (e.g. see [14, 15, 16, 17]). In this section and also in Section 4.2 we will pursue this line of thought to present a combinatorial approach toward the main problems.

To start, let us recall that a homomorphism $\sigma$ from a graph $G$ to a graph $H$ is a map $\sigma : V(G) \rightarrow V(H)$ such that $u \sim v$ implies $\sigma(u) \sim \sigma(v)$. $\text{Hom}(G, H)$, $\text{Hom}^0(G, H)$ and $\text{Hom}^0(G, H)$ denote the sets of ordinary, onto (vertices) and onto-edges homomorphisms from $G$ to $H$, respectively (e.g. see [30] for the background).

Let $K_G$ be a kernel on the graph $G$ with a nowherezero stationary distribution $\pi_G$. Then we define,

**Definition 3.**

- $\phi_G(u, v) \overset{\text{def}}{=} K_G(u, v)\pi_G(u)$,
- $\pi_G^M = \max_{u \in V(G)} \pi_G(u)$ and $\pi_G^m = \min_{u \in V(G)} \pi_G(u)$,
- $\phi_G^M = \max_{u \neq v \in V(G)} \phi_G(u, v)$,
- $\phi_G^m = \min_{u \neq v \in V(G)} \{\phi_G(u, v) \mid \phi_G(u, v) \neq 0\}$,
- $M_\sigma = \min_{x, y \in V(H)} \{|E(\sigma^{-1}(x), \sigma^{-1}(y))| \mid E(\sigma^{-1}(x), \sigma^{-1}(y)) \neq \emptyset\}$,
- $M^\sigma = \max_{x, y \in V(H)} |E(\sigma^{-1}(x), \sigma^{-1}(y))|$, $\sigma^\sigma = \min_{x \in V(H)} |\sigma^{-1}(x)|$, $S_\sigma = \max_{x \in V(H)} |\sigma^{-1}(x)|$,
- For two given kernels on $G$ and $H$, we define

$$\tau(G, H) \overset{\text{def}}{=} \frac{\max_{x \neq y \in E(G)} \phi_G(x, y)}{\min_{u \neq v \in E(H)} \phi_H(u, v)}$$

The following theorem is a summary of what one may prove using the comparison technique.
Theorem 2. (see [16, 17]) Let $G$ and $H$ be two graphs with $n = |V(G)| ≥ |V(H)| = m$ such that $π_G$ and $π_H$ are nowhere zero stationary distributions of the corresponding kernels $K_G$ and $K_H$, respectively.

a) If $σ ∈ \text{Hom}^v(G, H)$, then for all $1 ≤ k ≤ m$,
$$λ_k^G ≤ \frac{M^v}{S^v} \frac{\overline{Φ}_G^m}{\overline{Φ}_H^m} π_H^m λ_k^H$$
and
$$t_k^G ≤ \frac{M^v}{S^v} \frac{\overline{Φ}_G^m}{\overline{Φ}_H^m} t_k^H$$

b) If $σ ∈ \text{Hom}^e(G, H)$, then for all $1 ≤ k ≤ m$,
$$λ_{n-m+k}^G ≥ \frac{M^e}{S^e} \frac{\overline{Φ}_G^m}{\overline{Φ}_H^m} λ_k^H$$

It is clear that one may compare different kernels on a graph $G$ by considering the identity homomorphism on $G$.

Now, we consider symmetric graphs and combinatorial kernels on them. To begin, let $G = (V, E)$ be a graph for which $d_u$ is the out-degree of the vertex $u ∈ V$, and $d_G^{\text{max}}$ is the maximum out-degree of $G$. Define the kernel $K_c$ on $G$ as,
$$K_c(u, v) = \begin{cases} \frac{1}{d_G^{\text{max}}} & uv ∈ E \text{ & } u ≠ v \\ \frac{d_G^{\text{max}} - d_u}{d_G} & u = v \text{ & } u \nrightarrow u \\ \frac{d_G^{\text{max}} - d_u + 1}{d_G} & u = v \text{ & } u \rightarrow u \\ 0 & \text{otherwise} \end{cases}$$

and note that $π = [\frac{1}{|V|}, \ldots, \frac{1}{|V|}]$ is a nowhere zero stationary distribution for $K_c$.

Also, it is easy to check that
$$Δ_c = id - K_c = \frac{1}{d_G^{\text{max}}} (D_G - A_G),$$

where $D_G$ is a diagonal matrix whose diagonal entries correspond to vertex out-degrees and $A_G$ is the following adjacency matrix,
$$A_G(u, v) = \begin{cases} 1 & uv ∈ E \text{ & } vu ∈ E \\ 0 & uv ∉ E \text{ & } vu ∉ E \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Hereafter, the operator $Δ_G \overset{\text{def}}{=} d_G^{\text{max}} Δ_c$ is called the combinatorial Laplacian of $G$, and it is clear by definition that the spectrum of $Δ_G$ lies within the interval $[0, d_G^{\text{max}}]$. The following proposition shows that, in a sense, the combinatorial Laplacian is our best choice for a large family of graphs.
Proposition 2. Let $G$ be a graph on $n$ vertices, such that $\text{Aut}(G)$, the group of automorphisms of $G$, acts transitively on both $V(G)$ and $E(G)$. If $K$ is an arbitrary kernel on $G$, and $\lambda^c_k$ is the $k$'th eigenvalue of the corresponding combinatorial Laplacian, then

\[ a) \quad \lambda^K_2 \leq \lambda^c_2, \quad \text{and} \quad \lambda^K_n \geq \lambda^c_n, \]

\[ b) \quad \text{for all } 1 \leq k \leq n \text{ we have } \iota_k(G, K) \leq \iota_k(G, K_c). \]

Proof. Let $|\text{Aut}(G)| = t$ and for each $\sigma \in \text{Aut}(G)$ define

\[ \sigma(K)(u, v) \overset{\text{def}}{=} K(\sigma(u), \sigma(v)). \]

For part (a) by Courant–Fischer variational principle,

\[
\lambda^K_2 = \frac{1}{t} \sum_{\sigma \in \text{Aut}(G)} \lambda^K_2^\sigma(K) \\
= \frac{1}{t} \sum_{\sigma \in \text{Aut}(G)} \min_{\sigma \neq 1} \left\{ \frac{\langle \sigma(\Delta)f, f \rangle_{\sigma(\pi)}^{\sigma(\pi)}}{\|f\|^2} \right\} \\
\leq \frac{1}{t} \sum_{\sigma \in \text{Aut}(G)} \left\{ \frac{\langle \sigma(\Delta)g, g \rangle_{\sigma(\pi)}}{\|g\|^2} \right\} \\
\leq \frac{\langle \Delta_g, g \rangle}{\|g\|^2} = \lambda^c_2,
\]

where $g$ is an eigenfunction of $\lambda^c_2$. The next inequality also follows similarly.

Also, to prove part (b), for all $1 \leq k \leq n$,

\[
iota_k(G, K) = \frac{1}{t} \sum_{\sigma \in \text{Aut}(G)} \iota_k(G, \sigma(K)) \\
= \frac{1}{t} \sum_{\sigma \in \text{Aut}(G)} \min_{\{f_i\}^{k}_i \in \mathcal{O}_k^+(G)} \frac{1}{k} \left( \sum_{i=1}^{k} \|\nabla f_i^\sigma\|_{1, \sigma(\phi)} \right) \\
\leq \frac{1}{kt} \sum_{i=1}^{k} \sum_{\sigma \in \text{Aut}(G)} \|\nabla g_i^\sigma\|_{1, \sigma(\phi)} \\
\leq \frac{1}{k} \sum_{i=1}^{k} \|\nabla g_i\|_{1, \phi_c} = \iota_k(G, K_c),
\]

where $\{g_i\}^{k}_i \in \mathcal{O}_k^+(G)$ is a set of function that give rise to the isoperimetric constant $\iota_k(G, K_c)$.

3.4 Geometric graphs

By Corollary 1, for any given graph $G$ with a kernel $K$ and a nowherezero stationary distribution $\pi$ on it, one can talk about the isoperimetric spectrum,

\[ 0 = \iota_1(G, K) \leq \iota_2(G, K) \leq \ldots \leq \iota_{|V(G)|}(G, K) = 1. \]
On the other hand, by definitions and Theorem 1 for any given graph $G$ and for all $1 \leq n \leq |V(G)|$ the following inequality holds,

$$\iota_n(G, K) = \gamma_n(G, K) \leq \tilde{\gamma}_n(G, K) \leq \tilde{i}_n(G, K),$$

that motivates the following definition.

**Definition 4.** A graph $G$ is said to be $n$-geometric with respect to a kernel $K$, if $\iota_n(G, K) = \tilde{i}_n(G, K)$. A graph $G$ is said to be supergeometric with respect to a kernel $K$, if it is $n$-geometric with respect to $K$, for every $2 \leq n \leq |V(G)|$.

As an immediate consequence of the definition and Corollary 1 we deduce that $\iota_2(G, K) = \gamma_2(G, K) = \tilde{\gamma}_2(G, K) = \tilde{i}_2(G, K)$ and consequently, any strongly connected graph is 2-geometric (with respect to any given kernel).

**Example 2.** A supergeometric graph.

An easy observation is that for any graph and with respect to any kernel,

$$|Q| = 1 \Rightarrow \frac{\vartheta(Q)}{\pi(Q)} = 1.$$

This, for instance, shows that all graphs on a set of 4 vertices are supergeometric. As a second example, we show that the complete graph $K_n$ is supergeometric with respect to the natural random walk defined by

$$K_r(u, v) \overset{\text{def}}{=} \begin{cases} \frac{1}{n-1} & u \neq v \\ 0 & u = v. \end{cases}$$

Note that, for any pair of vertices $u, v \in V(K_n)$, we have $\phi(u, v) = \tilde{\phi}(u, v) = \frac{1}{n(n-1)}$. Consequently, for any $\{Q_i\}_1^t \in D_t(K_n)$, with $|Q_i| = n_i$,

$$\iota_t(K_n) = \frac{nt - \sum_i n_i}{n - 1},$$

which is clearly minimized when $\{Q_i\}_1^t \in P_t(K_n)$, in which case

$$\iota_t(K_n) = \tilde{i}_t(K_n) = \frac{n}{n-1}(t-1).$$

**Example 3.** A 2-geometric graph that is not 3-geometric.

Let $G$ be a simple graph on a set of vertices $\{y\} \cup (\bigcup_{i=1}^3 A_i)$ where $A_i$’s are disjoint sets that do not contain $y$ and $|A_i| = n$ for all $i = 1, 2, 3$. Also, $E(G)$ is defined such that for $i = 1, 2, 3$ the central vertex $y$ of degree 3 is connected to the fixed vertices $x_i \in A_i$ and the rest of edges form three copies of the complete graph $K_n$ on the sets $A_i$. We claim that when $n$ is sufficiently large then $G$ is not 3-geometric with respect to the natural random walk $K_r(G)$.
First, note that $|V(G)| = 3n+1$ and $|E(G)| = \frac{3}{2}(n^2 - n + 2)$. Also, for the stationary distribution $\pi$ we have,

$$
\pi(y) = \frac{1}{n^2 - n + 2}, \quad \pi(x_i) = \frac{n}{3(n^2 - n + 2)} \quad (i = 1, 2, 3),
$$

$$
\pi(x) = \frac{n - 1}{3(n^2 - n + 2)} \quad (x \neq y \& x \neq x_i, \quad i = 1, 2, 3);
$$

which shows that for sufficiently large $n$ and $u, v \in A_i$ we have

$$
\pi(u) \simeq \frac{n}{3(n^2 - n + 2)}, \quad \phi(u, v) \simeq \frac{1}{3(n^2 - n + 2)}.
$$

Now, it is not hard to see (by a simple case-study) that for sufficiently large $n$

$$
\iota_3(G, K_c) = \frac{1}{3} \left( \sum_{i=1}^{3} \frac{\partial(A_i)}{\pi(A_i)} \right) \simeq \frac{1}{n^2},
$$

where for any other choice of three disjoint subsets, including 3-partitions where $y$ must be included in one of the subsets, we obtain a larger number, and consequently, $\iota_3(G, K_c) < \tilde{\iota}_3(G, K_c)$. ♣

**Problem 1.** Discuss the case $\tilde{\iota}_n(G) \leq \iota_{n+1}(G)$.

**Problem 2.** Is it true that any Cayley graph is combinatorially supergeometric?

## 4 Connections to eigenspaces

In this section we are going to prove generalized versions of Cheeger and Courant-Hilbert inequalities for the isoperimetric spectrum of a graph $G$. To begin let us recall an interesting variational principle due to Ky Fan.

**Theorem A.** (e.g. see [2]) Ky Fan’s minimum principle

Let $A \in \text{End}(V)$ be a self-adjoint matrix operating on the $\nu$-dimensional inner-product space $V$, and let

$$
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu,
$$

be the set of eigenvalues of $A$ ordered in an increasing order. Then for any $1 \leq n \leq \nu$ we have,

$$
\overline{\lambda}_n = \frac{1}{n} \left( \min_{U \in \mathbb{R}^{n \times \nu}, \quad U^* = \text{id}_n} \text{tr}(UAU^*) \right),
$$

where $U$ is an arbitrary $n \times \nu$ matrix, $\text{id}_n$ is the $n \times n$ identity matrix, and $(\text{tr})$ is the trace function.

Note that another way of expressing the Ky Fan’s result is that, subject to the same conditions of Theorem [A]

$$
\overline{\lambda}_n = \frac{1}{n} \left( \min_{f_1, f_2, \ldots, f_n} \sum_{i=1}^{n} \frac{\langle Af_i, f_i \rangle}{\|f_i\|^2} \right).
$$
Therefore, considering Theorem 1 and Lemma 1, it is clear that the parameters
\( \gamma_n(G) = \iota_n(G) \), \( \tilde{\gamma}_n(G) \) and \( \lambda_n \) satisfy similar variational expressions but over different spaces of functions.

Before we proceed any further, it is instructive to recall one more useful basic lemma (for more on the history and corrections see [11, 15, 18, 24, 26, 41]).

**Theorem B.** [24] (Duval-Reiner Lemma)
Let \( T \) be a self-adjoint linear operator on the finite inner product space \( F_\omega(X) \). If \( f \in F_\omega(X) \), and \( \{Q_j\}_{j \in \mathcal{J}_k} \in \mathcal{P}_k(X) \) is a class of subsets such that \( f_i \overset{\text{def}}{=} f|_{Q_i} \) for every \( 1 \leq i \leq k \). Then, for any set of scalar coefficients \( c_i \)'s \( (1 \leq i \leq k) \), the real number \( \zeta \) and \( g \overset{\text{def}}{=} \sum_{i=1}^k c_i f_i \) we have,

\[
<T(g), g>_{\omega} - \zeta \|g\|^2_{2,\omega} = \sum_{i=1}^k c_i^2 <(T(f) - \zeta f), f_i>_{\omega}
\]

\[
- \frac{1}{2} \sum_{i,j=1}^k (c_i - c_j)^2 <T(f_j), f_i>_{\omega}.
\]

Let \( K \) be the kernel of an ergodic Markov chain on the (strongly connected) base graph \( G \), and let \( f \) be a real function such that \( K^n(f) \leq f \). It is easy to check that for any natural number \( n \) we have \( K^n(f) \leq f \), and also one can check that \( h = \lim_{n \to \infty} K^n(f) \) is a constant vector, every element of which is the weighted mean of \( f \) with respect to the stationary distribution \( \pi \). Note that \( h \) is a harmonic function (eigenfunction of the eigenvalue 1) and an annihilator of the Laplacian \( \Delta = I - K \).

Specially, in this case we know by Perron-Frobenius theorem that any such function is in the linear subspace spanned by \( h \), and this shows that the eigenspace of the eigenvalue 1 is quite small, where, on the other hand, the space of such functions can be quite rich, or in a more sophisticated language, one may say that we are passing from the space of martingales to the space of supermartingales [23].

The second important fact is related to the concept of a nodal domain. It is interesting to note that in the continuous case the eigenfunctions of the ordinary Laplacian (say of a compact Riemannian manifold) is always a continuous function (essentially smooth) and by Rolle’s theorem there is always a zero point between any two points with different signs. This fact, in a way, justifies the study of connected components of \( f^{-1}(0) \) (as nodal regions [18, 26, 32]) for any eigenfunction \( f \) in the continuous case. However, when we are dealing with a discontinuous object as a graph, an eigenfunction can have opposite signs on the two endpoints of an edge, where this, on the one hand, makes the whole thing more complex, and on the other hand, it makes the space of eigenfunctions far richer.

**Definition 5.** If \( f \in F_s(G) \) and \( Q \subseteq V(G) \), the pair \( (Q, Q^c) \) is called a bipolar cut-set for \( f \) if for any edge \( uv \in E(Q) \) we have \( f(u)f(v) \leq 0 \). Also, a subset \( Q \) is called a nonnegative (nonpositive) bipolar part of \( f \) if \( f_1 \overset{\text{def}}{=} f|_{Q} \) is a nonnegative (nonpositive) function on \( Q \) and \( (Q, Q^c) \) is a bipolar cut-set for \( f \). A signed part of
$f$ is a subset $Q$ that is either a nonnegative or a nonpositive bipolar part of $f$. Note that in this case $f = f_1 + f_2$ where $f_2 \overset{\text{def}}{=} f|_{Q^c}$ and $f_1 \perp f_2$.

For a given graph $G$ and a real function $f \in \mathcal{F}(G)$, we define

- $\mathcal{P}_f \overset{\text{def}}{=} \{ v \in V(G) \mid f(v) > 0 \}$,
- $\mathcal{N}_f \overset{\text{def}}{=} \{ v \in V(G) \mid f(v) < 0 \}$,
- $\mathcal{O}_f \overset{\text{def}}{=} \{ v \in V(G) \mid f(v) = 0 \}$.

Also, for any subset of vertices $W \subseteq V(G)$, we use the notation $\text{Comp}(W)$ for the set of all connected components of the induced subgraph of $G$ on $W$. We call the components of the subgraphs induced on $\mathcal{P}_f$ and $\mathcal{N}_f$ the strong sign-graphs of $f$ and we use the following notations,

$$\kappa^+(f) \overset{\text{def}}{=} |\text{Comp}(\mathcal{P}_f)|, \quad \kappa^-(f) \overset{\text{def}}{=} |\text{Comp}(\mathcal{N}_f)|, \quad \text{and} \quad \kappa(f) \overset{\text{def}}{=} \kappa^+(f) + \kappa^-(f).$$

Note that any strong sign-graph of $f$ is clearly a signed part of $f$.

For a given real number $\zeta \in \mathbb{R}$, a real function $f \in \mathcal{F}(G)$ is said to be $\zeta$-excessive (resp. $\zeta$-deficient) for the kernel $K$ if $Kf \leq \zeta f$ (resp. $Kf \geq \zeta f$). By abuse of language, an $\zeta$-excessive (resp. $\zeta$-deficient) function for $\Delta$ is just refered to as an $\zeta$-excessive (resp. $\zeta$-deficient) function, if details are clear from the context.

The following is a direct corollary of Theorem [B].

**Corollary 2.** Let $G$ be graph, and $f \in \mathcal{F}(G)$ be a $\zeta$-excessive (resp. $\zeta$-deficient) function for $\Delta$, such that a subset $Q \subseteq V(G)$ is a nonnegative (resp. nonpositive) bipolar part of $f$. Then, assuming $g \overset{\text{def}}{=} f|_Q$ we have,

$$\zeta \geq \frac{\|\nabla g\|^2_{L^2}}{\|g\|^2_{L^2}}.$$  

**Proof.** Obviously, $\{Q, Q^c\} \in \mathcal{P}_2(G)$ and, consequently, the inequality follows by Duval-Reiner Lemma (Theorem [B]).

---

### 4.1 Generalized Cheeger inequalities

The following is the first half of the generalized Cheeger inequality.

**Theorem 3.** For any given graph $G$ we have $\overline{\lambda}_n \leq \iota_n(G)$.

**Proof.** Let $\{Q_i\}_1^n \in \mathcal{D}_n(G)$ be chosen such that

$$\iota_n(G) = \frac{1}{n} \sum_{i=1}^n \frac{\partial(Q_i)}{\pi(Q_i)}.$$

Also, for every $i \in \mathcal{I}_n$ define

$$h_i(u) \overset{\text{def}}{=} \begin{cases} \frac{1}{\pi(Q_i)} & u \in Q_i, \\ 0 & u \not\in Q_i. \end{cases}$$

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Note that for every \( i \in I \) we have \( \| h_i \|_{1,0} \leq 1 \) and moreover,
\[
\frac{\| \nabla h_i \|^2_{2,0}}{\| h_i \|^2_{2,0}} = \frac{\| \nabla h_i \|^2_{2,0}}{\| h_i \|^2_{2,0}} = \frac{\partial (Q_i)}{\pi(Q_i)}.
\]

Hence, by Ky-Fan’s minimum principle we have,
\[
\lambda_n = \min_{f_i \perp \pi} \sum_{i=1}^{n} \langle \Delta f, f_i \rangle \leq \sum_{i=1}^{n} \frac{\| \nabla h_i \|^2_{2,0}}{\| h_i \|^2_{2,0}} = n \iota_n (G).
\]

For the second half we need the following definition.

**Definition 6.** Let \( \Gamma = (\zeta_1, \zeta_2, \ldots, \zeta_n) \) be an \( n \)-list of real numbers. Then an \( n \)-list of real functions \( F = (f_1, f_2, \ldots, f_n) \) on a domain \( X \) along with \( n \) disjoint subsets \( Q = (Q_1, Q_2, \ldots, Q_n) \) such that \( Q_i \subseteq X \), is called a compatible orthogonal set of functions for \( K \), if:

- \( f_i|_{Q_i} \neq 0 \).
- For each \( 1 \leq i \leq n \), the function \( f_i \) is a \( \zeta_i \)-excessive (resp. \( \zeta_i \)-deficient) function with respect to the kernel \( K \) on \( X \).
- For each \( 1 \leq i \leq n \), the subset \( Q_i \) is a nonnegative (resp. nonpositive) bipolar part of \( f_i \).

\[ \blacklozenge \]

**Theorem 4.** Consider a graph \( G \) and let \( \Gamma = (\zeta_1, \zeta_2, \ldots, \zeta_n) \). If \( F = (f_1, f_2, \ldots, f_n) \) along with \( Q = (Q_1, Q_2, \ldots, Q_n) \) is a compatible orthogonal set of functions for \( \Delta \), then
\[ 2 \iota_n (G) \geq \iota_n (G)^2. \]

**Proof.** Let \( 0 \neq g_i \defeq f_i|_{Q_i} \). Then,
\[
\iota_n \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\| \nabla g_i \|^2_{2,0}}{\| g_i \|^2_{2,0}} \geq \frac{1}{2n} \sum_{i=1}^{n} \frac{\| \nabla g_i \|^2_{1,0}}{\| g_i \|^2_{1,0}} \geq \frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\| \nabla g_i \|^2_{1,0}}{\| g_i \|^2_{1,0}} \right)^2 \geq \frac{1}{2} \iota_n (G)^2,
\]
where the first and second inequalities follow from Corollary 2 and Lemma 4 respectively, and the third one is a direct application of Cauchy-Schwartz inequality. \[ \blacksquare \]
Corollary 3. For a given graph $G$, let $f$ be an eigenfunction of $\Delta$ corresponding to the eigenvalue $\lambda$ with $k$ strong sign-graphs. Then
\[ \lambda \geq \frac{1}{2} \iota_k(G)^2. \]

Proof. Use Theorem 4 with
\[ \Gamma \overset{\text{def}}{=} (\lambda, \lambda, \ldots, \lambda), \quad F = (f, f, \ldots, f) \quad \text{and} \quad Q = (Q_1, Q_2, \ldots, Q_k) \]
where each $Q_i$ is a strong sign-graph of $f$. ■

Note that for $\lambda_2$, we have $k \geq 2$ and Cheeger’s inequality,
\[ \frac{1}{2} \lambda_2 \leq \iota_2(G) \leq \sqrt{2} \lambda_2, \]
follows from Theorem 3 and Corollary 3. Now, the generalized Cheeger inequality can be stated as follows,

**Theorem 5.** Consider a kernel $K$ on a base graph $G$. Let $F = (f_2, f_3, \ldots, f_n)$ be a list of eigenfunctions of $\Delta$ for the list of eigenvalues $\Gamma = (\lambda_2, \lambda_3, \ldots, \lambda_n)$, respectively, such that in the list $Q = (Q_2, Q_3, \ldots, Q_n)$ of disjoint sets, $Q_i$ is a strong sign-graph of $f_i$ and $0 \neq f_i|_{Q_i}$ for each $2 \leq i \leq n$. Then,
\[ \frac{n-1}{2n} \iota_n(G)^2 \leq \overline{\lambda}_n \leq \frac{n-1}{n} \iota_n(G). \]

**4.2 Generalized Courant-Hilbert inequalities**

It is clear from Corollary 3 and Theorem 5 that an effective application of generalized Cheeger-type inequalities for higher eigenvalues strongly depends on one’s information about the nodal domains and the strong sign-graphs of the eigenfunctions (or excessive functions in general). This in a way motivates a study of the concept that we formulate as generalized Courant-Hilbert-type inequalities.

In this setting we present such results as general spectral inequalities for the no-homomorphism problem. To begin, let us recall the following related result.

**Theorem C.** [15] For any pair of graphs $G$ and $H$ with $|V(G)| = n$ and $|V(H)| = m$, and for any $1 \leq k \leq m$, if $\sigma \in \text{Hom}^\vee(G, H)$ and $f_k$ is an eigenfunction for the eigenvalue $\lambda_k^H$, then $\max(\lambda_k^G, \lambda_k^G) \leq \frac{M^\sigma}{S^\sigma} \lambda_k^H$.

Note that for any graph $G$ the identity homomorphism $\varrho$ is trivially in $\text{Hom}^\vee(G, G)$ where $M^\varrho = S^\varrho = 1$, and consequently, by the theorem, for any eigenfunction, $f_k$, of the eigenvalue $\lambda_k^G$, we have, $\lambda_k^G \leq \lambda_k^G$, which implies that $\kappa(f_k) \leq k$.

Therefore, other similar results (e.g. as those appeared in [15] in terms of the eigenvalues of the co-Laplacian matrix) can be considered as generalized Courant-Hilbert-type inequalities. In this section we are going to show that similar results can be proved in terms of the eigenvalues of the kernel $K$, even though it is not necessarily a positive definite operator. We start by recalling a couple of definitions (for more details refer to [15, 16, 17]). In what follows we prove a generalized version of Theorem C.
Theorem 6. Let $K_G$ and $K_H$ be two kernels on the graphs $G$ and $H$, with the nowhere zero stationary distributions $\pi$ and $\pi'$, respectively. If $\sigma \in \text{Hom}^\epsilon(G, H)$ and $f$ is an $\zeta$-excessive function for $\Delta_H$, then

$$\lambda_{\kappa+(f)}^G \leq \frac{M^\pi}{S_\sigma} \tau(G, H) \zeta.$$ 

Proof. Let $\{Q_i \mid i = 1, \ldots, \kappa(f)\}$ be the vertex sets of the strong sign-graphs corresponding to the $\zeta$-excessive function $f$. Also, let $f_i \overset{\text{def}}{=} f|_{Q_i}$ for $i = 1, \ldots, \kappa(f)$. By Duval-Reiner Lemma (Theorem 13), for any set of scalar coefficients $c_i$’s ($i = 1, \ldots, \kappa(f)$), the real number $\zeta$ and $g \overset{\text{def}}{=} \sum_{i=1}^{\kappa(f)} c_i f_i$ we have,

$$\begin{align*}
\langle \Delta_H(g), g \rangle_{\pi'} - \zeta \|g\|^2_{2, \pi'} &= \sum_{i=1}^{\kappa(f)} c_i^2 \langle \Delta_H(f_i), f_i \rangle_{\pi'} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{\kappa(f)} (c_i - c_j)^2 \langle \Delta_H(f_j), f_i \rangle_{\pi'}.
\end{align*}$$

Now, consider the subspace $W_f^+$ of all functions $g \overset{\text{def}}{=} \sum_{i=1}^{\kappa(f)} c_i f_i$ for which $c_j = 0$ for any negative strong sign-graph $Q_j$. Note that $\kappa^+(f)$ is the dimension of $W_f^+$ and that for any $g \in W_f^+$ we have $\langle \Delta_H(g), g \rangle_{\pi'} - \zeta \|g\|^2_{2, \pi'} \leq 0$.

Hence, $\frac{\langle \Delta_H(g), g \rangle_{\pi'} }{\|g\|^2_{2, \pi'}} \leq \zeta$ for any such $g$, and consequently,

$$\frac{\langle \Delta_H(g \circ \sigma), g \circ \sigma \rangle_{\pi'} }{\|g \circ \sigma\|^2_{2, \pi}} \leq \tau(G, H) \frac{M^\pi}{S_\sigma} \frac{\langle \Delta_H(g), g \rangle_{\pi'} }{\|g\|^2_{2, \pi'}} \leq \tau(G, H) \frac{M^\pi}{S_\sigma} \zeta,$$

and the theorem follows by Courant-Fischer variational principle. 

In the following theorem we prove another version of Theorem 6 where the kernel is not necessarily a positive definite operator (parts of the proof is mainly based on a known result of Powers [13, 36]).

Theorem 7. Let $K_G$ and $K_H$ be two kernels on the graphs $G$ and $H$, with the nowhere zero stationary distributions $\pi$ and $\pi'$, respectively. For any $\zeta \in \mathbb{R}$,

a) if $\sigma \in \text{Hom}^\epsilon(G, H)$ and $f$ is a $\zeta$-deficient function for the kernel $\overline{K}_H$ on $H$, then

$$\alpha_{\kappa+(f)}^G \geq \frac{M^\pi}{S_\sigma} \tau(H, G)^{-1} \zeta.$$

b) if $H$ is strongly connected, $f$ is a $\zeta$-deficient function for the kernel $\overline{K}_H$ on $H$ for which $\kappa^-(f) \neq 0$, $k \overset{\text{def}}{=} |\text{Comp}(\mathcal{P}_f \cup \mathcal{O}_f)|$ and $\sigma \in \text{Hom}^\epsilon(G, H)$, then

$$\alpha_k^G > \frac{M^\pi}{S_\sigma} \tau(H, G)^{-1} \zeta.$$
Proof. Let \( \overline{K}_H = \begin{pmatrix} B & C \\ E & D \end{pmatrix} \), \( f = \begin{pmatrix} a \\ -b \end{pmatrix} \), where the submatrices \( B \) and \( D \) correspond to the subgraphs induced on \( P_f \cup O_f \) and \( N_f \), respectively, and \( a, b \) are nonnegative vectors. Next, let

\[
B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{pmatrix},
\]

where, for any \( 1 \leq i \leq k \), \( B_i \) is the submatrix corresponding to the induced subgraph \( H_i \in \text{Comp}(P_f \cup O_f) \) and \( a \) and \( C \) are partitioned accordingly. Also, define \( G_i \overset{\text{def}}{=} \sigma^{-1}(H_i) \) as an induced subgraph. The hypothesis \( \overline{K}_H(f) \geq \zeta f \) implies that for all \( 1 \leq i \leq k \) we have \( B_i a_i - C_i b \geq \zeta a_i \). Since \( H \) is a strongly connected graph, no \( C_i \) is identically equal to zero and consequently \( C_i y > 0 \) because \( \kappa^- (f) \neq 0 \) and \( b \) is a strictly positive vector. Hence, for all \( 1 \leq i \leq k \) we have \( \overline{B_i a_i, a_i}_{s^\prime} > \zeta \).

On the other hand, one can check that for any graph \( H \),

\[
\overline{K}_H(f), f > s^\prime = 2 \sum_{u \in E(H)} f(u) f(v) \overline{\phi}_H(u, v).
\]

Therefore, each term of \( \overline{B_i a_i, a_i}_{s^\prime} \) is nonnegative, and since \( \sigma \) is a graph homomorphism each term of the expansion in \( \overline{B_i (a_i \circ \sigma), a_i \circ \sigma}_{s^\prime} \) is also nonnegative. Hence, for any \( 1 \leq i \leq k \), we can compare the Rayleigh quotients \( \overline{B_i (a_i \circ \sigma), a_i \circ \sigma}_{s^\prime} \) and \( \overline{B_i a_i, a_i}_{s^\prime} \) as follows,

\[
\overline{B_i (a_i \circ \sigma), a_i \circ \sigma}_{s^\prime} \geq \tau(H,G)^{-1} \frac{M_x}{S} \overline{B_i a_i, a_i}_{s^\prime} > \tau(H,G)^{-1} \frac{M_x}{S} \zeta.
\]

Similar dual theorems can be proved by either considering \( \kappa^-(f) \) or applying the above theorems to the vector \(-f\). Also, note the trade-off between having larger indices and getting an strict inequality in part (b).

5 Concluding remarks

In this paper we considered the isoperimetric spectrum of directed graphs and we considered its relationships to the classical spectrum of Laplacian eigenvalues. We would like to emphasize that similar studies related to higher order isoperimetric numbers (for the minimum version) already exist and can be found in the current
literature (e.g. see [6, 10, 28]), however, we not only tried to present a concise treatment of the subject in the case of directed graphs with a focus on both analytic and graph-theoretic aspects, but also, we would like to single out the mean-version of the isoperimetric constant, because we believe that it may give rise to a spectrum that is computationally more well-behaved (e.g. see [31] for a patch of strong evidence in this regard).

Considering the properties of the isoperimetric spectrum, it is instructive to highlight the difference between $\iota$ and $\tilde{\iota}$ (disguised in the case of classical Cheeger constant (see Corollary 1)) and the fact that $\iota$ can be described by a functional equivalent definition (Theorem 1). This, compared with a celebrated contribution of Rothaus (see [37]) shows that our definition of supergeometric graphs has got the distinction of characterizing a class of graphs with geometric like properties (that also seems to be related to the concept of symmetry). In this regard we believe that the characterization of supergeometric graphs should be a very fundamental problem.

Also, it is worth mentioning the relationships of the subject to the theory of weakly unitary invariant norms (e.g. see [2]) and convex analysis on Hermitian matrices that may give rise to some generalizations.

A missing section in this paper is the part on the relationships of the isoperimetric spectrum and the classical connectivity parameters defined in graph theory (as edge-connectivity, vertex-connectivity, etc.). For this we just refer to the basic fact that this study naturally leads to the subject of solving boundary value problems on the base graph that is well studied in the existing literature about maximum principle and potential theory on graphs (e.g. see [4, 5, 40] and related articles).

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