1 Introduction and statement of the results

In this paper we shall present a construction of Alexandrov-embedded complete surfaces $M$ in $\mathbb{R}^3$ with finitely many ends and finite topology, and with nonzero constant mean curvature (CMC). This construction is parallel to the well-known original construction by Kapouleas [3], but we feel that ours somewhat simpler analytically, and controls the resulting geometry more closely. On the other hand, the surfaces we construct have a rather different, and usually simpler, geometry than those of Kapouleas; in particular, all of the surfaces constructed here are noncompact, so we do not obtain any of his immersed compact examples. The method we use here closely parallels the one we developed recently [8] to study the very closely related problem of constructing Yamabe metrics on the sphere with $k$ isolated singular points, just as Kapouleas’ construction parallels the earlier construction of singular Yamabe metrics by Schoen [15].

The original examples of noncompact CMC surfaces were those in the one-parameter family of rotationally invariant surfaces discovered by Delaunay in 1841 [2]. One extreme element of this family is the cylinder; the ‘Delaunay surfaces’ are periodic, and the embedded members of this family interpolate between the cylinder and an infinite string of spheres arranged along a common axis. The family continues beyond this, but the elements now are immersed, and we shall not consider them here. We are mostly interested in surfaces which are Alexandrov embedded. This condition, by definition, means that although the surface may be immersed, the immersion extends to one of the solid handlebody bounded by the surface. The CMC surfaces we construct here have Alexandrov embedded ends; if the minimal $k$-noids out of which they are built (as we describe below) are Alexandrov embedded, then the whole surfaces satisfy this condition.

The rôle of Delaunay surfaces in the theory of complete CMC surfaces is analogous to the rôle of catenoids (and planes) in the study of complete minimal surfaces of finite total curvature. For example, just as any complete minimal surface with two ends must be a catenoid [6], it was proved by Meeks [14] and Korevaar, Kusner and Solomon [15] that any constant mean curvature surface with at most two ends is necessarily a...

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Delaunay surface. A rather more remarkable theorem, paralleling the fact that any end of a complete minimal surface of finite total curvature must be asymptotic to a catenoid or a plane, is the fact that any end of a CMC surface of the type we consider must be asymptotic to one of these rotationally symmetric Delaunay surfaces (and in particular, must be cylindrically bounded).

The fact that such CMC surfaces exist in abundance was proved, as noted above, by Kapouleas [3] in 1987. More recently, using much softer methods based on the Schwarz reflection principle, Grosse-Brauckmann has constructed families of CMC surfaces with \( k \) ends and with \( k \)-fold dihedral symmetry [5].

The general analysis of the moduli space of CMC surfaces was considered by the first author, Kusner and Pollack [7] (essentially merely translating the analogous results in [10] for the singular Yamabe problem). The basic result is that the moduli space \( \mathcal{M}_{g,k} \) of \( k \)-ended surfaces with genus \( g \) is a locally real analytic variety of virtual dimension 3\( k \) (before dividing out by the action of the group of Euclidean motions). This virtual dimension is attained at any point \( \Sigma \in \mathcal{M}_{g,k} \) where a certain analytic nondegeneracy criterion is satisfied. This condition will be explained in some detail below. It seems very difficult to decide whether any of the surfaces constructed by Kapouleas satisfy this nondegeneracy criterion. This was one motivation for the present work, because the solutions we construct do satisfy it. We note finally that recently, Kusner, Grosse-Brauckmann and Sullivan gave a heuristic determination of the moduli space \( \mathcal{M}_{0,3} \); this work strongly suggests that all elements of this space are nondegenerate. In work in progress, they expect to make their arguments rigorous.

We now state our main result in more detail. This result is simply that CMC surfaces may be constructed out of certain building blocks in a specified and controlled manner. There are only two types of building blocks: Delaunay surfaces (or more precisely, halves of Delaunay surfaces) and minimal \( k \)-noids. The former we have already encountered; on the other hand, a minimal \( k \)-noid is by definition a complete minimal surface \( \Sigma \) of finite total curvature and with \( k \) ends. Denote the moduli space of minimal \( k \)-noids of genus \( g \) by \( \mathcal{H}_{g,k} \). This space has been studied by Perez and Ros [13], and they prove the result corresponding to that of [10] that this space is real analytic of virtual dimension 3\( k \).

Elements of various of these spaces have been shown to exist by the classical Weierstrass method, and more recently elements have been constructed for \( g \) very large by Kapouleas by his gluing methods [4]. Again there is a notion of nondegeneracy of such surfaces, and a surface \( \Sigma \) is a smooth point in its corresponding moduli space precisely when it satisfies this nondegeneracy condition. Fortunately, the Weierstrass representation gives sufficient information in some cases to establish the existence of nondegenerate minimal \( k \)-noids. Quite recently, Cosin and Ros [11] have proven the existence of nondegenerate minimal \( k \)-noids with specified weight parameters (in the sense described later here). Finally, as noted earlier, any end of an element \( \Sigma \in \mathcal{H}_{g,k} \) is asymptotic to the end of a plane or a catenoid. We define \( \mathcal{H}_{g,k}^0 \) to be the subset of \( \mathcal{H}_{g,k} \) consisting of nondegenerate elements \( \Sigma \) with all ends of \( \Sigma \) catenoidal, rather than planar.

We may now state the main

**Theorem 1** Fix any \( \Sigma_0 \in \mathcal{H}_{g,k}^0 \). Then there exists a family of CMC surfaces \( \Sigma_\varepsilon \in \mathcal{M}_{g,k} \) constructed by gluing half-Delaunay surfaces onto each end of the dilated surface \( \varepsilon\Sigma_0 \). The surfaces \( \Sigma_\varepsilon \) have the property that for any \( R > 0 \), the dilated surface \( \varepsilon^{-1}\Sigma_\varepsilon \) restricted
to the ball \( B_R(0) \) converges in the \( C^\infty \) topology to the restriction of \( \Sigma_0 \) to \( B_R(0) \). In addition, \( \Sigma_\varepsilon \) are regular points of \( \mathcal{M}_{g,k} \).

Our proof has some novel features. Rather than finding solutions as perturbations off of degenerating families of approximate solutions, as has been common in such constructions, we instead find (infinite dimensional) families of CMC surfaces as normal graphs over each of the component pieces, the half-Delaunay surfaces and a truncated \( k \)-noid. By studying the Cauchy data of the functions producing these normal graphs, and prove that we may match the Cauchy data from the inner piece (the graph over the \( k \)-noid) with that from the ends, thus one advantage of this procedure is that more of the technical complications caused by the nonlinearities are avoided than would be otherwise.

The plan of this paper is as follows. We first discuss the Delaunay surfaces in some detail, collecting and proving various technical properties concerning them that we require later, specifically those concerned with their behavior in the singular limit, as they approach the bead of spheres. This is followed by the analysis of the Jacobi (i.e. linearized CMC) operator, especially in this singular limit, for the half-Delaunay surfaces. We then use this to discuss the full family of CMC surfaces in a neighborhood of these rotationally invariant surfaces, as usual, keeping careful track of the behavior in the limit. We then turn to a discussion of minimal surfaces with \( k \) catenoidal ends, i.e. the \( k \)-noids, briefly reviewing their geometry and then treating the relevant aspects of the linear analysis of their Jacobi operator. After that we can approach the family of CMC surfaces obtained as normal graphs over suitable truncations of these \( k \)-noids. At last we can put all of this together and prove that it is possible to match the Cauchy data, and so obtain the proof of the main theorem.

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2 Notation, conventions and definitions

We recall some basic facts about the geometry of immersed surfaces, and review various ways the equations for constant mean curvature may be specified. Some good references for this material are the book by Osserman [12] and the survey article by Wente [17].

Suppose that \( \Sigma \) is given as the image of a regular immersion \( x: \mathcal{U} \rightarrow \mathbb{R}^3 \). Here \( \mathcal{U} \) is an open set in \( \mathbb{R}^2 \) with coordinates \( u = (u_1, u_2) \). The unit normal to \( \Sigma \) is defined to be

\[
\nu(u_1, u_2) = \frac{\partial u_1 \mathbf{x} \wedge \partial u_2 \mathbf{x}}{\|\partial u_1 \mathbf{x} \wedge \partial u_2 \mathbf{x}\|},
\]

and the components of the first and second fundamental forms \( g \) and \( B \) are then

\[
E = \langle \partial_{u_1} \mathbf{x}, \partial_{u_1} \mathbf{x} \rangle, \quad F = \langle \partial_{u_1} \mathbf{x}, \partial_{u_2} \mathbf{x} \rangle, \quad G = \langle \partial_{u_2} \mathbf{x}, \partial_{u_2} \mathbf{x} \rangle \\
L = \langle \partial^2_{u_1 u_1} \mathbf{x}, \nu \rangle, \quad M = \langle \partial^2_{u_1 u_2} \mathbf{x}, \nu \rangle, \quad N = \langle \partial^2_{u_2 u_2} \mathbf{x}, \nu \rangle.
\]

The principal curvatures \( k_1 \) and \( k_2 \) are the eigenvalues of \( B \) relative to \( g \). The mean curvature is defined to be the sum (not the average) of the principal curvatures, \( H \equiv k_1 + k_2 \), and the Gauss curvature is their product, \( K \equiv k_1 k_2 \).
We shall almost always be using orthogonal parameterizations (that is to say, parameterizations for which $F = 0$), in which case the formulae for $H$ and $K$ reduce to:

$$H = \frac{L + N}{E}, \quad K = \frac{LN - M^2}{E^2}.$$  

The important equations of surface theory are the Gauss and Codazzi equations, which link the intrinsic and extrinsic geometry of $\Sigma$. Rather than write these down in general, we consider the special case where the parameterization is isothermal, so that $E = G \equiv e^{2\omega}$ and $F = 0$. Let $u = u_1 + iu_2$ be the corresponding complex coordinate, and define the Hopf differential

$$\phi(u) \, du^2 = \left( \frac{1}{2}(L - N) - iM \right) \, du^2.$$  \hspace{1cm} (2)

The principal curvatures are then given by

$$k_1 = \frac{H}{2} - |\phi|e^{-2\omega}, \quad k_2 = \frac{H}{2} + |\phi|e^{-2\omega}.$$  

If $\Sigma$ is CMC, so $H$ is constant, then the Codazzi equations are equivalent to the holomorphy of this differential. The Gauss equation is simply

$$\Delta \omega + \frac{H^2}{4} e^{2\omega} - |\phi|^2 e^{-2\omega} = 0.$$  \hspace{1cm} (3)

3 Delaunay surfaces

We now make a detailed study of the first of the basic building blocks we use later, the Delaunay surfaces of revolution.

3.1 Definition and basic equations

The Delaunay surfaces mentioned in the introduction are surfaces of revolution, and so we use cylindrical coordinates. In particular, if the axis of rotation is the vertical one, and if $t$ is a linear coordinate along this axis and $\theta$ is the angular variable around it, then we consider surfaces $\Sigma$ given by the parametrization

$$\mathbf{x}(t, \theta) = (\rho(t) \cos \theta, \rho(t) \sin \theta, t).$$  \hspace{1cm} (4)

The condition that such a surface has constant mean curvature 1 gives an ordinary differential equation for the function $\rho(t)$, and solutions of this equation correspond to the Delaunay surfaces.

To obtain this ODE, first note that the unit normal of $\Sigma$ at $\mathbf{x}(t, \theta)$ is

$$\nu(t, \theta) = \frac{1}{\sqrt{1 + \rho_t^2}} (-\cos \theta, -\sin \theta, \rho_t),$$

where subscripts denote derivatives, and then that the metric tensor and second fundamental form are given by

$$g = (1 + \rho_t^2) \, dt^2 + \rho^2 \, d\theta^2, \quad B = -\frac{\rho_t}{\sqrt{1 + \rho_t^2}} \, dt^2 + \frac{\rho}{\sqrt{1 + \rho_t^2}} \, d\theta^2.$$  \hspace{1cm} (5)
It follows that the mean curvature is given by the expression
\[
H = -\rho_tt(1 + \rho_t^2)^{-3/2} + \rho^{-1}(1 + \rho_t^2)^{-1/2},
\]
and so the condition \( H = 1 \) becomes the equation
\[
\rho_tt - \frac{1}{\rho}(1 + \rho_t^2) + (1 + \rho_t^2)^{3/2} = 0.  
\]

There are two special solutions of (7) that can be determined immediately. The first is the constant solution \( \rho_1(t) \equiv 1 \), the cylindrical graph of which is the cylinder of radius 1. The other, \( \rho_0(t) = \sqrt{4 - (t - 2)^2}, |t - 2| \leq 2 \), corresponds to the sphere of radius 2 centered at \((0, 0, 2)\). The singular limit of the Delaunay surfaces mentioned in the introduction corresponds to the periodic extension of \( \rho_0(t) \) to all of \( \mathbb{R} \).

For \( 0 < \varepsilon < 1 \), we define \( \rho_\varepsilon(t) \) to be the solution of (7) which attains its minimum value \( \rho_\varepsilon(0) = \varepsilon \) at \( t = 0 \). By differentiating, we see that if \( \rho \) is a solution of (7) then
\[
H(\rho, \rho_t) \equiv \rho^2 - \frac{2\rho}{\sqrt{1 + \rho_t^2}},
\]
is constant. In particular, \( H(\rho_\varepsilon, (\rho_\varepsilon)_t) = \varepsilon(\varepsilon - 2) < 0 \). Introduce the new parameter \( \tau \) by \( \tau^2 = \varepsilon(2 - \varepsilon) \), so that \( \varepsilon = 1 - \sqrt{1 - \tau^2} \) and \( 0 < \tau < 1 \) as well. We then deduce immediately the

**Proposition 1** The solution \( \rho_\varepsilon \) of (7) with \( \rho_\varepsilon(0) = \varepsilon, \partial_t \rho_\varepsilon(0) = 0 \) is periodic and varies between the limits
\[
\varepsilon \equiv 1 - \sqrt{1 - \tau^2} \leq \rho_\varepsilon \leq 1 + \sqrt{1 - \tau^2}.
\]
In particular, \( \rho_\varepsilon(t) \leq 2 \) for all \( t \) and \( \varepsilon \).

These solutions and their translates constitute the (embedded) Delaunay family; the surfaces determined by them, as well as their images under Euclidean motions, are the Delaunay surfaces.

To simplify notation, we often drop the subscript \( \varepsilon \) (which should not be confused with the standard partial derivative notation). We also introduce a new parameterization, changing both the independent and dependent variables, which simplifies the study of the \( \rho_\varepsilon \). A change of independent variable corresponds to the introduction of a function \( t = k(s) \), which should be a diffeomorphism of \( \mathbb{R} \) onto itself. the function \( k \) is chosen so that the corresponding parameterization
\[
(s, \theta) \longrightarrow (\rho(k(s)) \cos \theta, \rho(k(s)) \sin \theta, k(s)),
\]
is isothermal. This corresponds to the condition
\[
(1 + \rho_t^2) k_s^2 = \rho^2. 
\]
With the initial condition \( k(0) = 0 \) and noting that \( k_s > 0 \), then \( \rho \) uniquely determines \( k \). Also, \( k_s \neq 0 \) (so long as \( \rho \neq 0 \), which is always the case here), and so using the periodicity of \( \rho \) we see that \( k \) must be a diffeomorphism. Now, use the parameter \( \tau \in (0, 1) \) from above and define the function \( \sigma(s) \) by
\[
\rho(k(s)) = \tau e^{\sigma(s)}. 
\]
A brief calculation shows that
\[ \rho_t = \frac{\sigma_s}{\sqrt{1 - \sigma_s^2}}, \quad 1 + \rho_t^2 = \frac{1}{1 - \sigma_s^2}, \quad \text{and} \quad \rho_{tt} = \frac{\sigma_{ss}}{\tau e^\sigma (1 - \sigma_s)^2}. \] (10)

In terms of the new variable \( s \) and function \( \sigma \), the first and second fundamental forms are now
\[ g = \tau^2 e^{2\sigma} \left( ds^2 + d\theta^2 \right), \quad B = -\frac{\sigma_{ss} \tau e^\sigma}{\sqrt{1 - \sigma_s^2}} ds^2 + \tau e^\sigma \sqrt{1 - \sigma_s^2} d\theta^2, \] (11)
and (\ref{eq:14}) becomes
\[ \sigma_{ss} + \tau e^\sigma \sqrt{1 - \sigma_s^2} - (1 - \sigma_s^2) = 0. \] (12)

We can now see that this parameterization is indeed simpler.

**Proposition 2** The function \( \sigma \) defined by (9) satisfies the equation
\[ \sigma_{ss} + \frac{\tau^2}{2} \sinh 2\sigma = 0, \] (13)
and in fact
\[ \sigma_s^2 + \tau^2 \cosh^2 \sigma = 1. \] (14)

Conversely, if \( \sigma \) satisfies (14) for some \( 0 < \tau < 1 \) (and hence also (13)), with \( \tau^2 \cosh^2 \sigma(0) = 1 \), and if \( t = k(s) \), where \( k(0) = 0 \) and
\[ k(s) = \frac{\tau^2}{2} (1 + e^{2\sigma}), \] (15)
then \( \rho(t) \equiv \tau e^{\sigma(s)} \) satisfies (3) and \( \rho(0) = \varepsilon \) where \( \varepsilon = 1 - \sqrt{1 - \tau^2} \).

**Proof:** Let \( \tilde{\sigma} = \sigma + \log \tau \); since \( \sigma \) solves (13) then \( \tilde{\sigma} \) solves
\[ \tilde{\sigma}_{ss} + e^{\tilde{\sigma}} \sqrt{1 - \tilde{\sigma}_s^2} - (1 - \tilde{\sigma}_s^2) = 0. \] (16)

The function
\[ \tilde{H}(\tilde{\sigma}, \tilde{\sigma}_s) = 2e^{\tilde{\sigma}} \sqrt{1 - \tilde{\sigma}_s^2} - e^{2\tilde{\sigma}} \]
is a positive constant, say \( \gamma \), when \( \tilde{\sigma} \) is a solution of (10). Evaluating at \( s = 0 \), since \( \sigma_s(0) = \tilde{\sigma}_s(0) = 0 \) and \( e^{\tilde{\sigma}(0)} = \tau e^{\sigma(0)} = \varepsilon \), we have \( \gamma = 2\varepsilon - \varepsilon^2 = \tau^2 \). Reverting back to \( \sigma \), this implies (14), and hence (13) by differentiation.

The converse, that starting from \( \sigma(s) \) and \( \tau \), then defining \( t = k(s) \) as in the statement of the Proposition, the corresponding function \( \rho(t) \) satisfies (\ref{eq:14}) is a straightforward calculation which we leave to the reader. Notice that then Gauss and Codazzi equations are automatically fulfilled. \( \square \)

**Remark 1** Translating back to the notation of §2, the log of the conformal factor \( \omega \) and the norm of the coefficient function of the Hopf differential are given by
\[ \omega = \sigma + \log \tau \quad \text{and} \quad |\phi| = \frac{\tau^2}{2}, \]
cf. (4) and (3).

Henceforth, the functions \( \rho(t), \sigma(s) \) and \( k(s) \) will always be related in the manner dictated by this Proposition; furthermore, the dependence on the parameters \( \varepsilon \) and \( \tau \) will not always be written explicitly, but we shall use them interchangeably. We shall call either of these parameters the necksize of the corresponding Delaunay solution.
3.2 Uniform estimates for Delaunay solutions in the singular limit

In this section we present a series of technical lemmata regarding the behavior of various quantities associated to the Delaunay solutions as $\varepsilon$ (or $\tau$) tends to zero. Some of the estimates below are easier to obtain for $\rho$ and some for $\sigma$, and we shall use these functions interchangeably. We first estimate the period of $\sigma$; the corresponding estimate for $\rho$ is not required later so we merely state it and refer to [3] for its proof. Then we obtain some simple ‘global’ estimates for $\rho$, which are rather weak, but frequently useful, as well as a corresponding simple estimate for $\sigma$. Finer estimates for $\rho$ when $t$ is not too large then lead to a good comparison between the variables $s$ and $t$.

Proposition 3 Let $S_\varepsilon$ and $T_\varepsilon$ denote the periods of $\sigma$ and $\rho$, respectively. Then as functions of $\tau$ and $\varepsilon$,

\[ S_\varepsilon = -4 \log \tau + O(1) = -2 \log \varepsilon + O(1) \]

and

\[ T_\varepsilon = 4 + \tau^2 \log(1/\tau) + O(\tau^2) = 4 + 2 \varepsilon \log(1/\varepsilon) + O(\varepsilon). \]

Proof: As stated above, we only check the statement about $S_\varepsilon$. First, using (14), we see

\[ \frac{1}{4} S_\varepsilon = \int_0^\sigma(0) \frac{1}{1 - \tau^2 \cosh^2 x} \, dx. \]

Expand the denominator into exponentials and change variables, setting $u = e^x$. Then, letting

\[ A(\tau) \equiv e^{\sigma(0)} = \frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 1} = \frac{\tau}{2} + O(\tau^3), \]

this becomes

\[ \frac{1}{4} S_\varepsilon = \int_{A(\tau)}^1 \frac{1}{u^2 - \frac{1}{4} \tau^2 (u^2 + 1)^2} \, du. \]

Changing variables once again, reduces this to an integral of the form

\[ \int_{\tau/2}^{1-O(\tau^2)} \frac{1}{\sqrt{u^2 - \tau^2/4}} \, dv. \]

Finally, this last integral may be computed explicitly, and equals $- \log \tau + O(1)$. The estimate for $S_\varepsilon$ in terms of $\varepsilon$ follows from the relationship between $\varepsilon$ and $\tau$. \qed

To place the next result into context, note that one limiting solution of the basic equation (6) is $\rho_0(t) = \sqrt{4 - t^2}$, the equation for the sphere. Also, the catenoid of necksize $\varepsilon$ (which is a solution of the equation corresponding to (6) when $H = 0$) is given as a cylindrical graph by the function $\rho_c(t) = \varepsilon \cosh(t/\varepsilon)$. The function $\rho(t)$ may be compared to each of these solutions.
**Proposition 4** For any $\varepsilon \in (0, 1)$, the Delaunay solution $\rho(t) = \rho_{\varepsilon}(t)$ satisfies the following bounds

\[
\varepsilon \leq \rho(t) \leq \varepsilon \cosh(t/\varepsilon) \tag{17}
\]

\[
1 + \rho_t^2 \leq \frac{1}{\varepsilon^2} \rho^2 \quad \text{(comparison with the equation of a catenoid)} \tag{18}
\]

\[
\rho^2(1 + \rho_t^2) \leq 4 \quad \text{(comparison with the equation of a sphere)} \tag{19}
\]

for any $t \in \mathbb{R}$.

**Proof:** It is clear from (14) that $\sigma$ is monotone increasing on $[0, S_{\varepsilon}/2]$ and monotone decreasing on $[S_{\varepsilon}/2, S_{\varepsilon}]$. Correspondingly, $\rho$ is monotone increasing on $[0, T_{\varepsilon}/2]$ and monotone decreasing on $[T_{\varepsilon}/2, T_{\varepsilon}]$. In particular, its value at 0 is its absolute minimum, so the lower bound in (17) is valid.

Now multiply (6) by $2\rho_t(1 + \rho_t^2)^{-1}$ and integrate to get

\[
\log(1 + \rho_t^2(t)) = 2\log(\rho(t)/\varepsilon) - 2\int_0^t \rho(u) \sqrt{1 + \rho_t^2(u)} \, du. \tag{20}
\]

Since $\rho_t \geq 0$ for $t \in [0, T_{\varepsilon}/2]$, we get (18). Next, because $\rho \geq \varepsilon$, we may write $\rho(t) = \varepsilon \cosh(w(t)/\varepsilon)$. Inserting this into (18) leads to the inequality $w_t^2 \leq 1$. Since $w(0) = 0$, we conclude that $w(t) \leq t$ for all $t \in [0, T_{\varepsilon}/2]$ and the second part of (17) follows by periodicity.

Since the final estimate (19) is not required later, we shall not prove it here. \(\square\)

Now we come to the more refined estimates for $\rho$.

**Proposition 5** For any $\varepsilon \in (0, 1)$ and $t \in \mathbb{R}$, the Delaunay solution $\rho(t) = \rho_{\varepsilon}(t)$ satisfies

\[
|\rho(t) - \varepsilon \cosh(t/\varepsilon)| \leq c \varepsilon^2 e^{3|t|/\varepsilon} \quad \text{and} \quad |\rho_t(t) - \sinh(t/\varepsilon)| \leq c \varepsilon e^{3|t|/\varepsilon}.
\]

These estimates are nontrivial only when $|t| \leq \varepsilon^2 \log(c/\varepsilon)$ for some constant $c > 0$.

**Proof:** From (17) and (18) we get

\[
1 + \rho_t^2(t) \leq \frac{\rho^2(t)}{\varepsilon^2} \leq \cosh^2(t/\varepsilon),
\]

and hence $|ho_t(t)| \leq \sinh(t/\varepsilon)$ for any $t > 0$. Now use this in (20) to obtain for $t > 0$

\[
-2\int_0^t \sinh(u/\varepsilon) \cosh(u/\varepsilon) \, du \leq \log(\varepsilon^2 (1 + \rho_t^2(t)))(t) \leq 0,
\]

where the last inequality follows from (18); equivalently,

\[
\rho^2(t) \left( \exp(-\varepsilon \sinh^2(t/\varepsilon)) - 1 \right) \leq \varepsilon^2 (1 + \rho_t^2(t)) - \rho^2(t) \leq 0.
\]

As we did above, set $\rho = \varepsilon \cosh(w/\varepsilon)$. Then since we already know that $w(t) \leq t$, and since $u \to \coth u$ is monotone decreasing, we conclude that

\[
\frac{\cosh^2(w(t)/\varepsilon)}{\sinh^2(w(t)/\varepsilon)} \geq \frac{\cosh^2(t/\varepsilon)}{\sinh^2(t/\varepsilon)}.
\]
The previous inequality now implies that

$$ \left( \exp(-\varepsilon \sinh^2(t/\varepsilon)) - 1 \right) \frac{\cosh^2(t/\varepsilon)}{\sinh^2(t/\varepsilon)} \leq \left( w_t^2(t) - 1 \right) \leq 0. $$

Since $e^{-x} - 1 \geq -x$ for all $x \geq 0$, the left side of this last inequality is certainly larger than or equal to $-\varepsilon \cosh^2(t/\varepsilon)$. Hence, we finally obtain

$$ \left( 1 - \varepsilon \cosh^2(t/\varepsilon) \right)^{1/2} \leq w_t(t) \leq 1, \quad (21) $$

for all $t \in [0, -\varepsilon/2 \log \varepsilon]$. Now integrate this inequality, using that $(1 - x)^{1/2} \geq 1 - x$ when $0 < x < 1$ and $\cosh^2 x \leq e^{2|x|}$ for all $x \in \mathbb{R}$, to get

$$ -\frac{\varepsilon^2}{2} e^{2t/\varepsilon} \leq -\varepsilon^2 \int_0^{t/\varepsilon} \cosh^2(s) ds \leq w_{\varepsilon}(t) - t \leq 0. $$

We conclude that for $0 < t < -\varepsilon/2 \log \varepsilon$, we have

$$ \varepsilon \cosh(t/\varepsilon - \frac{\varepsilon}{2} e^{2t/\varepsilon}) \leq \rho(t) \leq \varepsilon \cosh(t/\varepsilon). $$

The estimate for $\rho$ follows at once. To get the estimate for $\rho_t$, use (21) and the relationship $\rho_t(t) = w_t(t) \sinh(w(t))/\varepsilon$.

We can finally give a quantitative estimate for the relationship between the variables $t$ and $s$, or equivalently, for the function $k(s)$.

**Proposition 6** For $|s| < S_\varepsilon/8$, the function $t = k(s)$ admits the expansion

$$ k(s) = \varepsilon s + \frac{\varepsilon^2}{8} e^{2s} + O(\varepsilon^3 \log \varepsilon), $$

uniformly as $\varepsilon \to 0$.

**Proof:** It suffices to consider the case $t = k(s) \geq 0$. The estimate for $\rho$ from the last Proposition implies

$$ \rho(k(s))^2 = \frac{1}{4} \varepsilon^2 e^{2k/\varepsilon} + O(\varepsilon^2) + O(\varepsilon^3 e^{4k/\varepsilon}) + O(\varepsilon^4 e^{6k/\varepsilon}). $$

The errors here are all of size no greater than $O(\varepsilon^2)$ precisely when $|k| \leq -\varepsilon/4 \log \varepsilon$ (up to an additive constant). Assuming that $s_0 > 0$ is chosen so that this bound is satisfied, then by the definition of $\tau$ and $\sigma$,

$$ \tau^2 e^{2\sigma(s)} = \frac{\varepsilon^2}{4} e^{2k(s)/\varepsilon} + O(\varepsilon^2). $$

Now recall the definition of $k$ via its derivative from Proposition 3.

$$ k_s = \frac{\tau^2}{2} e^{2\sigma} + \frac{\tau^2}{2} e^{2\sigma} = \varepsilon + \frac{\varepsilon^2}{8} e^{2k/\varepsilon} + O(\varepsilon^2). $$

Since $k \leq (\varepsilon/4) \log(1/\varepsilon)$, we obtain $e^{2k/\varepsilon} \leq e^{-1/2}$, and so $k_s = \varepsilon + O(\varepsilon^{3/2})$ in this range. Integrate to get $k = \varepsilon s + O(\varepsilon^{3/2})$ for $0 \leq s \leq s_0$. From this equation, we see that
\[ k \leq (\varepsilon/4) \log(1/\varepsilon) \] provided \( s \leq S_\varepsilon/8 \), so that we may take \( s_0 \) to be this last value. Now use this formula for \( k \) in terms of \( s \) in the estimate for \( k_s \) above to get that

\[ k_s(s) = \varepsilon + \frac{\varepsilon^2}{8} e^{2s} + O(\varepsilon^3 \log \varepsilon), \]

for \( 0 \leq s \leq S_\varepsilon/8 \). Integrating this, at last, gives the estimate of the Proposition.

Collecting the results of Proposition 5 and the result of Proposition 6, we obtain:

**Proposition 7** There exists a constant \( c > 0 \) independent of \( \varepsilon \) such that the following inequalities hold

\[ \tau e^\sigma \geq c \varepsilon^{3/4}, \quad \tau^2 \cosh(2\sigma) \leq c \varepsilon^{1/2}, \quad \text{if} \quad s \in \left[ \frac{S_\varepsilon}{8}, \frac{3S_\varepsilon}{8} \right], \]

and

\[ \tau^2 \cosh(2\sigma) \leq c \varepsilon^2 e^{2s}, \quad \text{if} \quad s \in \left[ \frac{3S_\varepsilon}{8}, \frac{S_\varepsilon}{2} \right]. \]

**Proof:** Recall that \( S_\varepsilon = -2 \log \varepsilon + O(1) \). It follows from Proposition 5 that

\[ k(s) = \varepsilon s + O(\varepsilon^{3/2}) \quad \text{if} \quad s \in \left[ 0, \frac{S_\varepsilon}{8} \right]. \]

Therefore, using Proposition 5, we obtain the expansion

\[ \tau e^\sigma = \varepsilon \cosh s + O(\varepsilon^{3/2} e^s) \quad \text{and} \quad \tau e^{-\sigma} = \frac{2}{\cosh^2 s} + O(\varepsilon^{1/2} e^{-s}), \quad (22) \]

if \( s \in [0, S_\varepsilon/8] \). Since \( \sigma \) is increasing in \( [0, S_\varepsilon/2] \), we conclude that

\[ \tau e^{\sigma(s)} \geq \tau e^{\sigma(S/8)} \geq c \varepsilon^{3/4}, \]

for all \( s \in [S_\varepsilon/8, S_\varepsilon/2] \). Similarly, since \( |\sigma| \) is increasing in \( [0, S_\varepsilon/4] \) and decreasing in \( [S_\varepsilon/4, S_\varepsilon/2] \), we get

\[ \tau^2 \cosh(2\sigma(s)) \leq \tau^2 \cosh(2\sigma(S_\varepsilon/8)) \leq c \varepsilon^{1/2}, \]

for all \( s \in [S_\varepsilon/8, 3S_\varepsilon/8] \). Finally, since we always have

\[ \sigma(s) = -\sigma(S_\varepsilon/2 - s), \]

it follows at once that, for all \( s \in [S_\varepsilon/2 - S_\varepsilon/8, S_\varepsilon/2] \), we have

\[ \tau^2 \cosh(2\sigma(s)) = \tau^2 \cosh(2\sigma(S_\varepsilon/2 - s)) \leq c e^{2s - S} \leq \varepsilon^2 e^{2s}. \]

This ends the proof of the Proposition.

We finally come to some simple estimates for \( \sigma(s) \). The first is that

\[ \tau^2 \cosh 2\sigma \leq 2 - \tau^2. \quad (23) \]

This follows trivially from multiplying \( \cosh(2\sigma) = 2 \cosh^2 \sigma - 1 \) by \( \tau^2 \) and applying (14).
Next, define $\xi(s) \equiv \tau \cosh \sigma(s)$. This function is periodic of period $S_\varepsilon/2$, attains its maximum value $\sup \xi = 1$ at $s = 0$, and its minimum $\inf \xi = \tau$ at $s = S_\varepsilon/4$. In addition, it is a solution of the equation
\[ \xi_{ss} = (1 + \tau^2)\xi - 2\xi^3, \tag{24} \]
which satisfies
\[ \xi_s^2 = (\xi^2 - \tau^2)(1 - \xi^2). \tag{25} \]

**Proposition 8** Suppose that $s_\ell$ is any sequence of real numbers, and that $\tau_\ell \to 0$. Let $\sigma_\ell$ denote the function $\sigma$ when $\tau = \tau_\ell$, we define $\xi_\ell(s) = \tau_\ell \cosh \sigma_\ell(s + s_\ell)$ and $\tilde{\xi}_\ell(s) = \tau_\ell^2 \cosh(2\sigma_\ell)(s + s_\ell)$. Then there exists an $s_0 \in \mathbb{R}$ and subsequences of the $\xi_\ell$ and $\tilde{\xi}_\ell$ which either converge uniformly to 0 or else converge respectively to $1/\cosh(s + s_0)$ and $2/\cosh^2(s + s_0)$, uniformly on compact subsets in $\mathbb{R}$.

**Proof**: Since $(\xi_\ell)^2 = (\xi_\ell^2 - \tau_\ell^2)(1 - \xi_\ell^2)$, and $|\xi_\ell| \leq 1$, we see that $\xi_\ell$ is bounded in $C^1(\mathbb{R})$. Using (24) we see that $\xi_\ell$ is bounded in $C^2(\mathbb{R})$. This allows us to extract a subsequence which converges uniformly on compact subsets of $\mathbb{R}$ to a solution of
\[ \xi_{ss} = (1 + \tau^2)\xi - 2\xi^3 \]
which satisfies
\[ \xi_s^2 = \xi^2(1 - \xi^2). \]
For $\xi_\ell$, the claim follows since the only solutions of these equations are $\xi \equiv 0$ or $\xi = 1/\cosh(s + s_0)$ for some $s_0 \in \mathbb{R}$. Finally, for $\tilde{\xi}_\ell$, it is sufficient to notice that
\[ \tilde{\xi}_\ell = 2\xi_\ell^2 - \tau_\ell^2, \]
and the claim follows. $\Box$

Because $\xi(s)$ attains its supremum at $s = 0$ we next conclude that

**Corollary 1** As $\varepsilon \to 0$ the families of functions $\tau \cosh \sigma(s)$ and $\tau^2 \cosh(2\sigma(s))$ converge to $1/\cosh s$ and $2/\cosh^2 s$, respectively, uniformly on compact sets.

In fact, we may improve the range on which the convergence in this last Corollary takes place.

**Corollary 2** As $\varepsilon \to 0$,
\[ \tau \cosh \sigma(s) = 1/\cosh s + O(\varepsilon^{1/2}), \quad \tau^2 \cosh 2\sigma(s) = 2/\cosh^2 s + O(\varepsilon^{1/2}), \]
uniformly for $|s| \leq S_\varepsilon/8$.

**Proof**: Note that $\xi(s) = \partial_s k(s)/\rho(k(s))$. The estimates here follow from inserting the estimates for $k(s)$ and $\rho(t)$ from Propositions 3 and 5 above. $\Box$

Finally, since $\xi(s)$ is decreasing on $[0, S_\varepsilon/4]$ and increasing on $[S_\varepsilon/4, S_\varepsilon/2]$, we obtain using the previous Corollary, the

**Proposition 9** For all $\eta > 0$, there exists an $\varepsilon_0 \in (0, 1)$ and an $s_0 > 0$ such that whenever $0 < \varepsilon \leq \varepsilon_0$ and $N/2 S_\varepsilon + s_0 \leq s \leq (N + 1)/2 S_\varepsilon - s_0$ for some $N \in \mathbb{Z}$, then
\[ \xi(s) = \tau \cosh \sigma \leq \eta \quad \text{and} \quad \tilde{\xi} = \tau^2 \cosh(2\sigma) \leq \eta. \]
4 The Jacobi operator on degenerating Delaunay surfaces

In this section we first give an explicit expression for the linearization of the mean curvature operator about any one of the Delaunay surfaces, and then proceed to develop its Fredholm theory on weighted Hölder spaces. This theory was already developed for weighted Sobolev spaces in [7], and the results are essentially identical. In particular, we need to find spaces on which this Jacobi operator is surjective. As usual, we also need this surjectivity with as good control as possible as the necksize shrinks. The results and proofs here are very close to those in [8].

4.1 The Jacobi operator

Recall from the last section the cylindrical parameterization $x_\varepsilon(t, \theta)$ for the Delaunay surface $\Sigma_\varepsilon$ of necksize $\varepsilon$, and the corresponding expression for its unit normal $\nu_\varepsilon$. Given any function $w(t, \theta)$ on $\Sigma_\varepsilon$, its normal graph

$$x_w(t, \theta) = x_\varepsilon(t, \theta) + w(t, \theta)\nu(t, \theta)$$

(26)
gives a regular parametrization of a surface $\Sigma_w$, provided $w$ is sufficiently small. In terms of the coefficients of the first and second fundamental forms of this surface, the nonlinear operator we are interested in takes the form

$$N(w) = 1 - L_w G_w - 2 M_w F_w + N_w E_w$$

(27)

It is well known that the linearization $L_\varepsilon$ of $N$ at $w = 0$, which is usually called the Jacobi operator for $\Sigma_\varepsilon$, is given by

$$L_\varepsilon = \Delta_{\Sigma_\varepsilon} + |A_{\Sigma_\varepsilon}|^2.$$

(28)

In terms of the parameterization above, this may be written as

$$L_\varepsilon = \frac{1}{\rho \sqrt{1 + \rho_t^2}} \partial_t \left( \frac{\rho}{\sqrt{1 + \rho_t^2}} \partial_t \right) + \frac{1}{\rho^2} \partial_\theta^2 + \frac{\rho^2 \rho_t^2 + (1 + \rho_t^2)^2}{\rho^2 (1 + \rho_t^2)^3}.$$

(29)

This looks complicated, but fortunately, becomes simpler in the $(s, \theta)$ coordinate system introduced above. Now

$$L_\varepsilon = \frac{1}{\tau^2 e^{2\sigma}} \left( \partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma) \right).$$

(30)

Removing the factor $(\tau^2 e^{2\sigma})^{-1}$, it will be sufficient to study the operator

$$L_\varepsilon w = \partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma).$$

(31)

Our main goal now is to study the boundary problem

$$\begin{cases}
L_\varepsilon w = f & \text{in} \ [s_0, +\infty) \times S^1 \\
w = \phi(\theta) & \text{on} \ \{s_0\} \times S^1,
\end{cases}$$

(32)
uniformly down to \( \varepsilon = 0 \). Because of the rotational invariance of the operator \( L_\varepsilon \), we may introduce the eigenfunction decomposition with respect to the cross-sectional Laplacian \( \partial_\theta^2 \). In this way we obtain operators
\[
L_{\varepsilon,j} = \partial_\theta^2 + (\tau^2 \cosh 2\sigma - j^2), \quad j \in \mathbb{Z}.
\]
Since we wish to deal only with real-valued functions, we shall use the eigenfunctions \( \chi_j(\theta) = (1/\sqrt{\pi}) \cos(j\theta) \) for \( j > 0 \), \( \chi_j(\theta) = (1/\sqrt{\pi}) \sin(j\theta) \) for \( j < 0 \), and \( \chi_0(\theta) = 1/\sqrt{2\pi} \). It will frequently be useful to separate out the operators corresponding to the indices \( j = -1, 0, 1 \) from the rest, and we shall often use the notation \( L'_\varepsilon \) to refer to the projection of the operator acting on these three components, and \( L''_\varepsilon \) to refer to the operator acting on all the others together. This division is natural because, by (23), the term of order zero in \( L_{\varepsilon,j} \) is strictly negative when \( |j| > 1 \), and so the estimates for \( L''_\varepsilon \) follow easily from the maximum principle, but this is false when \( |j| \leq 1 \) and \( \tau \) is small.

### 4.2 Jacobi fields

A deeper reason for the separation into low and high eigencomponents in the Jacobi operator becomes apparent when one examines the Jacobi fields, i.e. the solutions of
\[
\rho \partial_t \phi + \text{intrinsic term} = 0.
\]
In fact, we may determine these solutions explicitly in terms of the function \( \rho(t) \) solves \( L_{\varepsilon,j} \phi_j = 0 \). It turns out that the solutions for this problem when \( j = 0, \pm 1 \) may be determined explicitly in terms of the functions \( \rho \) or \( \sigma \); in fact, these Jacobi fields correspond to quite explicit one-parameter families of CMC surfaces of which \( \Sigma_\varepsilon \) is an element. To exhibit these, first note that any smooth one-parameter family \( \Sigma(\eta) \) of CMC surfaces, with \( \Sigma(0) = \Sigma_\varepsilon \), will have differential at \( \eta = 0 \) which is a Jacobi field on \( \Sigma_\varepsilon \). (This is meant in the sense that \( \Sigma(\eta) \) should be written, for small \( \eta \), as a normal graph over \( \Sigma(0) \).) This is possible over any fixed compact set of \( \Sigma_\varepsilon \) for some nontrivial range of values of \( \eta \) which might diminish to zero as the compact set grows. However, this is sufficient to make sense of the derivative at \( \eta = 0 \). The one-parameter families of CMC surfaces here are simple to describe: the first two families, corresponding to the two different solutions of \( L_{\varepsilon,0}\phi = 0 \), arise from varying the neksize parameter \( \varepsilon \), and translating the \( t \)-variable, i.e., translating along the axis of \( \Sigma_\varepsilon \). We denote the associated Jacobi fields by \( \Psi^{0,\pm}_\varepsilon \) and \( \Psi^{1,\pm}_\varepsilon \), respectively. The other families arise from either translating or rotating the axis of \( \Sigma_\varepsilon \), so that one such translation and one such rotation will correspond to solutions \( \Psi^{0,\pm}_\varepsilon \) and \( \Psi^{1,\pm}_\varepsilon \) of \( L_{\varepsilon,j} \) for \( j = 1 \), while the translation and rotation in the orthogonal direction corresponds to solutions for \( j = -1 \). In fact, we may determine these solutions explicitly in terms of the function \( \rho(t) \). Let us write
\[
\Psi^{\pm}_\varepsilon(t, \theta) \equiv \Phi^{\pm}_\varepsilon(t) \chi_j(\theta).
\]
We obtain :

**Proposition 10** The coefficient functions \( \Phi^{\pm}_\varepsilon(t) \) of the Jacobi fields \( \Psi^{\pm}_\varepsilon(t, \theta) \) for \( \Sigma_\varepsilon \) for \( j = -1, 0, 1 \), are given by the formulæ:
\[
\begin{align*}
\Phi^{0,+}_\varepsilon(t) &= \rho_t / \sqrt{1 + \rho_t^2}, & \Phi^{0,-}_\varepsilon(t) &= -\partial_\varepsilon \rho / \sqrt{1 + \rho_t^2}, \\
\Phi^{1,+}_\varepsilon(t) &= \Phi^{1,+}_\varepsilon(t), & \Phi^{1,-}_\varepsilon(t) &= \Phi^{1,-}_\varepsilon(t), \\
&= -1 / \sqrt{1 + \rho_t^2}, & \rho_t &= -(t + \rho_0) / \sqrt{1 + \rho_t^2}.
\end{align*}
\]
**Proof:** First consider the families given by translations. Suppose that the function \( w \) is chosen locally so that its normal graph is a (small) translation of magnitude \( d \) of \( \Sigma_\varepsilon \) along the \( x \) axis. Thus, for some value \( t' \) near to \( t \) and some value of \( \theta' \) near to \( \theta \),

\[
\begin{align*}
\rho(t) \cos \theta + d &= \rho(t') \cos \theta' - w(t', \theta') \cos \theta' \\
\rho(t) \sin \theta &= \rho(t') \sin \theta' - w(t', \theta') \sin \theta' \\
t &= t' + w(t', \theta') \rho_t(t') ,
\end{align*}
\]

This system is equivalent to

\[
\rho^2(t) = \rho^2(t') - 2w(t', \theta') \rho(t') + 2dw(t', \theta') \cos \theta' + w^2(t', \theta') - 2d \rho \cos \theta' + d^2
\]

and

\[
t = t' + w(t', \theta') \rho_t(t') .
\]

After inserting the value of \( t \) from this second equation into the first and collecting the lowest order terms we get

\[
w(t', \theta') = -d \cos \theta' /(1 + \rho^2(t')) + \text{higher order terms}.
\]

Recalling that the normal of \( \Sigma_\varepsilon \) at \( x_\varepsilon(t', \theta') \) is

\[
\nu(t', \theta') = \frac{1}{\sqrt{1 + \rho^2(t')}}(-\cos \theta', -\sin \theta', \rho(t')) ,
\]

we get the stated expression for \( \Phi_{\varepsilon}^{1,+} \). The expression for \( \Phi_{\varepsilon}^{-1,+} \), corresponding to translations along the \( y \) axis, is derived in an identical manner.

In fact, nearly identical arguments work in all other cases as well. The relevant systems of equations are

\[
\begin{align*}
\rho(t) \cos \theta &= \rho(t') \cos \theta' - w(t', \theta') \cos \theta' \\
\rho(t) \sin \theta &= \rho(t') \sin \theta' - w(t', \theta') \sin \theta' \\
t + d &= t' + w(t', \theta') \rho_t(t'),
\end{align*}
\]

for the translation of size \( d \) along the \( z \) axis,

\[
\begin{align*}
\rho_{z+d}(t) \cos \theta &= \rho_{z}(t') \cos \theta' - w(t', \theta') \cos \theta' \\
\rho_{z+d}(t) \sin \theta &= \rho_{z}(t') \sin \theta' - w(t', \theta') \sin \theta' \\
t &= t' + w(t', \theta')(\rho_{z})(t'),
\end{align*}
\]

for the variation of necksize, and

\[
\begin{align*}
(\cos d)\rho(t) \cos \theta + (\sin d) t &= \rho(t') \cos \theta' - w(t', \theta') \cos \theta' \\
\rho(t) \sin \theta &= \rho(t') \sin \theta' - w(t', \theta') \sin \theta' \\
(\cos d) t - (\sin d)\rho(t) \cos \theta &= t' + w(t', \theta') \rho_t(t'),
\end{align*}
\]

for a rotation of size \( d \) of the \( z \)-axis toward the \( x \)-axis, and similarly for the rotation toward the \( y \)-axis.

The calculations proceed as in the first case, and we leave the details to the reader. \( \square \)
Corollary 3  The expressions for these Jacobi fields in terms of the functions \( \sigma(s) \) and \( k(s) \) and the parameter \( \tau \) are

\[
\begin{align*}
\Phi_{\varepsilon}^{0,+}(s) &= \sigma_s, \\
\Phi_{\varepsilon}^{0,-}(s) &= \frac{\sqrt{1-\tau^2}}{\tau} \sigma_s \partial_\tau k - \sqrt{1-\tau^2} e^{\sigma} \cosh \sigma (1 + \tau \partial_\tau \sigma), \\
\Phi_{\varepsilon}^{1,+}(s) &= \Phi_{\varepsilon}^{-1,+}(s) = -\tau \cosh \sigma, \\
\Phi_{\varepsilon}^{1,-}(s) &= \Phi_{\varepsilon}^{-1,-}(s) = -k \tau (\cosh \sigma + \sigma_s(s) e^{\sigma}).
\end{align*}
\]

The proof involves simply inserting the expressions for \( \rho, t \) and \( \varepsilon \) in terms of \( \sigma, k \) and \( \tau \) into the previous formulæ.

We shall require later the limits of these Jacobi fields as \( \varepsilon \) tends to zero.

Proposition 11  Let \( I \subset \mathbb{R} \) be any compact interval. Then the following limits exist uniformly for \( s \in I \):

\[
\begin{align*}
\lim_{\varepsilon \to 0} \Phi_{\varepsilon}^{0,+}(\varepsilon s) &= \tanh s, & \lim_{\varepsilon \to 0} \Phi_{\varepsilon}^{0,-}(\varepsilon s) &= -(1 - s \tanh s), \\
\lim_{\varepsilon \to 0} \Phi_{\varepsilon}^{1,+}(\varepsilon s) &= \lim_{\varepsilon \to 0} \Phi_{\varepsilon}^{-1,+}(\varepsilon s) = -\frac{1}{\cosh s}, \\
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Phi_{\varepsilon}^{1,-}(\varepsilon s) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Phi_{\varepsilon}^{-1,-}(\varepsilon s) = -(\frac{s}{\cosh s} + \sinh s).
\end{align*}
\]

Proof: We have used the variable \( s \) in the statement of the theorem because by Proposition 3, \( \varepsilon s/t \to 1 \) uniformly for \( s \in I \). The limits may be calculated using either the estimates for \( \rho \) from Proposition 5, or else the expressions for these Jacobi fields from Corollary 3 in terms of \( \sigma \), and then using the limiting behaviour of \( \sigma(s) \) as determined in Corollary 1.

The Jacobi fields we have considered so far, \( \Psi_{\varepsilon}^{j,\pm}, j = 0, \pm 1 \), are all either bounded (in fact periodic) or linearly growing (because both \( k \) and \( \partial_\tau k \) are linearly growing). There are of course, two linearly independent solutions of the equation \( L_{\varepsilon,j} \phi = 0 \) for all \( j \) with \( |j| > 1 \) as well. It is proved in [7], following [10], that there exists a discrete sequence of positive numbers \( \gamma_j \to \infty, |j| > 1 \), with \( \gamma_{-j} = \gamma_j \), and for each \( j \) a solution \( \Phi_{\varepsilon}^{j,+}(s) \) of \( L_{\varepsilon,j} \phi = 0 \) such that

\[ e^{\pm \gamma_j s} \Phi_{\varepsilon}^{j,\pm}(s), \]

are periodic functions of \( s \). In particular,

\[ |\Phi_{\varepsilon}^{j,+}(s)| \leq ce^{-\gamma_j s}, \quad |\Phi_{\varepsilon}^{j,-}(s)| \leq ce^{\gamma_j s}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{33} \]

In fact, \( \Phi_{\varepsilon}^{j,-}(s) = \Phi_{\varepsilon}^{j,+}(-s) \). Because of Corollary 3, it is natural to define \( \gamma_0 = \gamma_{\pm 1} = 0 \).

While there are analogous conclusions were we to be using the independent variable \( t \) instead of \( s \), the values of these ‘indicial exponents’ \( \gamma_j = \gamma_j(\varepsilon) \) would behave in a less desirable way as \( \varepsilon \) tends to zero.
Proposition 12 For any $\eta > 0$, there exists an $\varepsilon_0 > 0$ such that when $0 < \varepsilon \leq \varepsilon_0$, the numbers $\gamma_j$ satisfy $\gamma_j \geq \sqrt{4 - \eta}$ for $|j| > 1$.

Proof: The $\Phi_\varepsilon^\pm(s)$ are homogeneous solutions for the ordinary differential operator $-\partial_s^2 + Q_j$, where $Q_j(s) = -\tau^2 \cosh 2\sigma(s) + j^2$. We are trying to estimate the exponential growth rate of solutions for this operator. It follows from (23) that

$$\tau^2 \leq \tau^2 \cosh 2\sigma \leq 2 - \tau^2.$$  

Thus, we see that $Q_j \geq j^2 - 2 + \tau^2$. In particular, for $|j| \geq 3$, $Q_j > 4$. In this case, the result is clear: $e^{\pm 2s}$ are supersolutions for the operator, and homogeneous solutions bounded by these supersolutions may be constructed by the shooting method, as described next for the slightly more complicated case when $j = \pm 2$.

Let $w$ be the unique decreasing solution of $(-\partial_s^2 + Q_2)w = 0$ with $w(0) = 1$. This solution may be constructed as a limit, as $s_1 \rightarrow \infty$, of solutions $w_{s_1}$ of this equation on $[0, s_1]$ with $w_{s_1}(0) = 1$, $w_{s_1}(s_1) = 0$. Since $e^{-\sqrt{2+\tau^2}s}$ is a supersolution for this operator and dominates $w_{s_1}$ at the endpoints, $s = 0$ and $s = s_1$, it gives an upper bound for $w_{s_1}$ on the whole interval $0 \leq s \leq s_1$. Thus the limit as $s_1 \rightarrow \infty$, which we call $w$, exists and is also bounded by this same function. Next, let $\bar{Q}_2 = (2/\cosh^2 s) + 4$, and define $\bar{w}$ to be the unique solution of $(-\partial_s^2 + \bar{Q}_2)\bar{w} = 0$ with $\bar{w}(0) = 1$ which is decreasing on $[0, \infty)$. Since $\bar{Q}_2$ converges exponentially to $4$ as $|s| \rightarrow \infty$, we know that $\bar{w}(s) \sim C_+ e^{-2s}$ as $s \rightarrow +\infty$; because it decreases as $s \rightarrow +\infty$ and because this equation has no global bounded solutions, we deduce that $\bar{w}(s) \sim C_- e^{-2s}$ as $s \rightarrow -\infty$.

Next, for any fixed $s_0 > 0$, $Q_2$ converges in $C^\infty$ to $\bar{Q}_2$ on the interval $[-s_0, s_0]$ as $\tau \rightarrow 0$; we claim that $w \rightarrow \bar{w}$ uniformly, along with all its derivatives, on this interval too. To establish this, it suffices to show that the Cauchy data of $\bar{w}$ at $s = 0$ converges to that of $w$. Indeed, using the equations satisfied by both $w$ and $\bar{w}$ we get

$$w \partial_s^2 \bar{w} - \bar{w} \partial_s^2 w = (\bar{Q}_2 - Q_2) w \bar{w}.$$  

An integration by parts leads to

$$w_s(0) \bar{w}(0) - w(0) \bar{w}_s(0) = \int_0^{+\infty} (\bar{Q}_2 - Q_2) w \bar{w} ds.$$  

Thanks to Corollary 4 [9] and also thanks to the fact that we already know that $\bar{w}(s) \sim C_+ e^{-2s}$ as $s \rightarrow +\infty$ and that $|w| \leq ce^{-\sqrt{2+\tau^2}s}$, we see that the right hand side of this equality tends to $0$ as $\tau$ tends to $0$, and so

$$\lim_{\tau \rightarrow 0} w_s(0) = \bar{w}_s(0).$$  

The claim then follows at once using Corollary 4 [9] once again.

By the Bloch wave theoretic construction of solutions of operators with coefficients periodic of period $S_c/2$, we may write $w(s) = e^{-\tau_2 s} p(s)$, where $p(s + S_c/2) = e^{\lambda} p(s)$ for some $\lambda \in \mathbb{R}$. Since $p(s)$ is real valued and strictly positive, actually $\lambda = 0$. We wish to show that this exponent $\gamma_2$ converges to the exponential rate of decrease $-2$ corresponding to $\tau = 0$, or more quantitatively, that for any fixed $\eta > 0$, we have $\gamma_2 > \sqrt{4 - \eta} \equiv \gamma$, so long as $\tau$ is sufficiently close to $0$.  

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To estimate this exponential rate of decrease, it suffices to show that for some fixed $s_0 > 0$,  
\[ w \left( \frac{1}{2} S_\varepsilon - s_0 \right) \leq C e^{-\gamma \frac{S_\varepsilon}{2} w(-s_0)} , \]  
where the constant $C$ is independent of $\tau$. We do this in two steps.

First we choose $s_0$ sufficiently large so that $2 / \cosh^2 s_0 < \eta$; also, using Proposition 9, we can assume that $2 \tau^2 \cosh^2 \sigma \leq \eta$ on $[s_0, S_\varepsilon/2 - s_0]$ for all $\tau$ small enough. We have noted that $\bar{w} \sim C \pm e^{-2s}$ as $s \to \pm \infty$, and this implies in particular that, for $s > 0$
\[ \bar{w}(s) \leq C e^{-4s} \bar{w}(-s), \]
for some $C > 0$ independent of $s$. Moreover, increasing the value of $s_0$, if necessary, we may assume that
\[ \bar{w}(s) \leq -(2 - \eta/4) \bar{w}(s_0). \]
By the uniform convergence of $w$ to $\bar{w}$, we deduce that, for $\tau$ small enough
\[ w(s_0) \leq 2C e^{-4s_0} w(-s_0). \]  
(35)
and also
\[ w(s_0) \leq -(2 - \eta/2) w(s_0). \]
Next, on the interval $[s_0, S_\varepsilon/2 - s_0]$, the potential $Q_2$ is bounded from below by $\gamma^2 \equiv 4 - \eta$, and so we may use $\tilde{w}(s) = e^{-\gamma(s-s_0)} w(s_0)$ as a supersolution here. Indeed $w(s_0) = \tilde{w}(s_0)$ and
\[ w_s(s_0) \leq -(2 - \eta/2) w(s_0) = -(2 - \eta/2) \tilde{w}(s_0) = \frac{\sqrt{4 - \eta}}{2} \tilde{w}(s_0) \leq \bar{w}(s_0), \]
provided $\tau$ is chosen small enough. We conclude that $w \leq \bar{w}$ on this whole interval. Thus, we get in particular the estimate
\[ w \left( \frac{1}{2} S_\varepsilon - s_0 \right) \leq e^{-\gamma \frac{1}{2} S_\varepsilon} \bar{w}(s_0). \]
Putting (35) together with this last inequality, we get that
\[ w \left( \frac{1}{2} S_\varepsilon - s_0 \right) \leq C e^{2s_0(\gamma - 2)} e^{-\gamma \frac{S_\varepsilon}{2} w(-s_0)} \leq C e^{-\gamma \frac{S_\varepsilon}{2} w(-s_0)}. \]
This gives the desired estimate, and the proof is complete.

### 4.3 Mapping properties of the Jacobi operator

To fully analyze the problem (32), we must study the mapping properties of the Jacobi operator $L_\varepsilon$, both for fixed $\varepsilon > 0$ and uniformly down to $\varepsilon = 0$. To state this result it is first necessary to define appropriate function spaces on which the Jacobi operator will act; these are exponentially weighted Hölder spaces. This is one of the main places where the difference between the independent variables $s$ and $t$ is seen: it is possible to obtain good mapping properties on spaces of this type, defined either in terms of the $s$ or $t$ variables, for fixed $\varepsilon$, but it is impossible to obtain the uniform behaviour down to $\varepsilon = 0$ when using $t$. Fortunately, this uniformity does occur when using $s$, and so from now on, unless saying explicitly otherwise, this choice of independent variable will be used in the sequel.

The definition of the weighted Hölder spaces is the natural one:
**Definition 1** Parametrize $\mathbb{R}^+ \times S^1$ by the variables $(s, \theta)$. For each $r \in \mathbb{N}$ and $0 < \alpha < 1$ and $s \in \mathbb{R}$, let

$$|w|_{r, \alpha, [s, s+1]},$$

denote the usual $C^{r, \alpha}$ Hölder norm on the set $[s, s+1] \times S^1$. Then for any $\mu \in \mathbb{R}$ and $s_0 \in \mathbb{R}$,

$$C^r_{\mu, \alpha}([s_0, +\infty) \times S^1) = \{ w \in C^r_{\text{loc}}([s_0, +\infty) \times S^1) \text{ and } \|w\|_{r, \alpha, \mu} = \sup_{s \geq s_0} e^{-\mu s} |w|_{r, \alpha, [s, s+1]} < \infty \}.$$

In particular, the function $e^{\mu s}$ is in $C^r_{\mu, \alpha}([s_0, +\infty) \times S^1)$.

Recall now the splitting of $L_\varepsilon$ into $L'_\varepsilon$ and $L''_\varepsilon$, corresponding to the operator induced on the eigenspaces with $|j| \leq 1$ and $|j| > 1$, respectively. $\Pi'$ and $\Pi''$ are the projectors onto the corresponding subspaces. This will often be abbreviated by letting $\Pi' w = w'$, and so on. The main result of this section is the

**Proposition 13** Fix $\mu$ with $1 < \mu < 2$. Then there exists an $\varepsilon_0 > 0$, depending only on $\mu$, such that whenever $\varepsilon \in (0, \varepsilon_0)$, there exists a unique solution $w \in C^{2, \alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1)$ of the problem

$$\begin{cases}
L_\varepsilon w = f & \text{in } (S_\varepsilon/8, \infty) \times S^1 \\
\Pi'' w = \phi'' & \text{on } \{S_\varepsilon/8\} \times S^1,
\end{cases}$$

for $f \in C^{0, \alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1)$ and $\phi'' \in \Pi''(C^{2, \alpha}(S^1))$. The solution of the homogeneous Dirichlet problem, when $\phi'' = 0$, will be denoted $w = G_\varepsilon(f)$, while the Poisson operator, which gives the solution when $f = 0$, will be denoted by $w = P_\varepsilon(\phi'')$. The linear maps

$$G_\varepsilon : C^{0, \alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1) \to C^{2, \alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1),$$

$$\varepsilon^{\mu/4} P_\varepsilon : \Pi''(C^{2, \alpha}(S^1)) \to C^{2, \alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1),$$

are bounded uniformly for all $\varepsilon \in (0, \varepsilon_0)$.

**Proof:** The proof of the existence of $G_\varepsilon$ and $P_\varepsilon$ and of their uniformity is accomplished in a number of steps. Solutions are constructed on each eigenspace of the Laplacian on $S^1$, and the cases where $|j| \leq 1$ must be treated somewhat differently than the others.

We shall give the proof of this result in a slightly more general context where the boundary point $S_\varepsilon/8$ is replaced by $s_0$ arbitrarily chosen in $\mathbb{R}$.

Fix $f$ and $\phi$ in the appropriate function spaces. We decompose $w = w' + w''$, $f = f' + f''$, then we must solve

$$\begin{cases}
L'_\varepsilon w' = f' & \text{for } s > s_0 \\
L''_\varepsilon w'' = f'' & \text{for } s > s_0 \\
w'' = \phi'' & \text{for } s = s_0.
\end{cases}$$

(37)

Notice that, no boundary conditions are imposed on $w'$ at $s = s_0$. We will also need to decompose

$$w'(s, \theta) = \frac{1}{\sqrt{\pi}} w_{-1}(s) \sin \theta + \frac{1}{\sqrt{2\pi}} w_0(s) + \frac{1}{\sqrt{\pi}} w_1(s) \cos \theta,$$
and

\[ f'(s, \theta) = \frac{1}{\sqrt{\pi}} f_{-1}(s) \sin \theta + \frac{1}{\sqrt{2\pi}} f_0(s) + \frac{1}{\sqrt{\pi}} f_1(s) \cos \theta. \]

**Step 1:** We first consider the problem where \( \phi'' = 0 \). Thus \( f \in C^{\alpha, \mu}_{-\mu} \), and multiplying by a suitable factor, we may assume that

\[ ||f'||_{0, \alpha, -\mu} + ||f''||_{0, \alpha, -\mu} = 1. \]

In this step, we only consider the restriction of the problem to the high eigencomponents. We first show that for every \( s_1 > s_0 \) there is a unique solution of

\[
\begin{cases}
L''w'' = f'' & \text{in } (s_0, s_1) \times S^1 \\
w'' = 0 & \text{on } \{s_0\} \times S^1 \\
w'' = 0 & \text{on } \{s_1\} \times S^1.
\end{cases}
\]  

(38)

The existence of \( w'' \) follows from a standard variational argument using the energy functional

\[ E(w) = \int_{s_0}^{s_1} \int_{S^1} (|\partial_s w|^2 + |\partial_\theta w|^2 - \tau^2 \cosh(2\sigma) |w|^2 + f''w) \, ds \, d\theta. \]

Using the fact that \( \forall |j| > 1, j^2 - \tau^2 \cosh(2\sigma) > 2 \), we see that when we restrict the domain of \( E \) to the span of the eigenfunctions \( \chi_j(\theta) \) with \( |j| > 1 \), this functional is convex and proper, and the existence of a unique minimizer for it, which we denote by \( w'' \), is then immediate.

We claim that there exists a constant \( \varepsilon_0 > 0 \) and a constant \( C = C(\mu) > 0 \), independent of \( s_0 < s_1 \) and \( \varepsilon \in (0, \varepsilon_0) \), such that

\[ \sup_{\theta \in S^1} \| e^{\mu s} |w''(s, \theta)| \|_{C(\mu)}. \]

Assuming that the claim is already proven, we can choose a sequence \( s_{1,i} \) tending to \( +\infty \) and build \( w''_{s_{1,i}} \) the corresponding solutions of (38). The uniform bound above allows us to extract from the sequence \( w''_{s_{1,i}} \) a subsequence which converges to a solution \( w'' \) of

\[
\begin{cases}
L''w'' = f'' & \text{in } (s_0, +\infty) \times S^1 \\
w'' = 0 & \text{on } \{s_0\} \times S^1,
\end{cases}
\]

which satisfies

\[ \sup_{\theta \in S^1} \| e^{\mu s} |w''(s, \theta)| \|_{C(\mu)}. \]

and then, by classical elliptic estimates, that

\[ ||w''||_{2, \alpha, -\mu} \leq c(\mu), \]

for some constant \( c(\mu) > 0 \) independent of \( s_0 \in \mathbb{R} \) and \( \varepsilon \in (0, \varepsilon_0) \).

The claim is proved by contradiction. By assumption, we have \( e^{i\pi s} |f''(s, \theta)| \leq 1 \) for \( s_0 \leq s \leq s_1, \theta \in S^1 \). If the assertion were not true, then there would exist sequences of
numbers \( s_{0,i}, s_{1,i}, \) functions \( f_i'' \), Delaunay parameters \( \varepsilon_i \) and corresponding solutions \( u''_{s,i} \) such that
\[
A_i \equiv \sup_{\theta \in S^1, s_{0,i} \leq s \leq s_{1,i}} e^{\mu s} |u''_{s,i}(s, \theta)| \to \infty,
\]
and
\[
\sup_{\theta \in S^1, s_{0,i} \leq s \leq s_{1,i}} e^{\mu s} |f_i''(s, \theta)| \leq 1.
\]
Suppose that this maximum, for each \( i \), is attained at some point \((s_i, \theta_i)\), and define
\[
\tilde{w}_{i}(s, \theta) = A_i^{-1} e^{\mu s} u''_{s_{1,i}}(s + s_i, \theta), \quad \tilde{f}_i''(s, \theta) = A_i^{-1} e^{\mu s} f''(s + s_i, \theta).
\]
Then
\[
\sup_{\theta \in S^1, s_{0,i} - s_i \leq s \leq s_{1,i} - s_i} e^{\mu s} |\tilde{w}_{i}(s, \theta)| = 1,
\]
and this supremum is attained on \( \{0\} \times S^1 \), while \( \tilde{f}_i'' \to 0 \) in norm. Furthermore,
\[
I_{\varepsilon_i} \tilde{w}_{i}'' = \partial^2_s \tilde{w}_{i}'' + \partial^2_{\theta} \tilde{w}_{i}'' + \tau_i^2 \cosh(2\sigma_i(s + s_i))\tilde{w}_{i}'' = \tilde{f}_i''
\]
on \([s_{0,i} - s_i, s_{1,i} - s_i] \times S^1 \).

Passing to a subsequence if necessary, we assume that \( s_{0,i} - s_i \) converges to \( v_1 \in \mathbb{R} \cup \{-\infty\} \) and \( s_{1,i} - s_i \) converges to \( v_2 \in \mathbb{R} \cup \{+\infty\} \). By using the result of Proposition 8, the bounds above, as well as those provided by elliptic estimates, we can also assume that \( \tilde{w}_{i}'' \) converges, along with all its derivatives, over any compact subset of \((v_1, v_2) \times S^1 \) (including endpoints if either is finite) to a function \( \tilde{w}'' \), which satisfies \( e^{\mu s} |\tilde{w}''| \leq 1 \) over this set, is nonvanishing (because of the normalization of \( \tilde{w}_{i}'' \) at \( s = 0 \)), and which solves one of the following equations:
\[
\partial^2_s \tilde{w}'' + \partial^2_{\theta} \tilde{w}'' + \tau^2 \cosh(2\sigma(s + s_i))\tilde{w}'' = 0, \quad (39)
\]
for some \( \varepsilon \in (0, \varepsilon_0) \) and \( s \in \mathbb{R} \),
\[
\partial^2_s \tilde{w}'' + \partial^2_{\theta} \tilde{w}'' = 0, \quad (40)
\]
or
\[
\partial^2_s \tilde{w}'' + \partial^2_{\theta} \tilde{w}'' + \frac{2}{\cosh^2(s + s_i)} \tilde{w}'' = 0, \quad \text{for some } s \in \mathbb{R}, \quad (41)
\]
on \([v_1, v_2] \times S^1 \). In addition, if either \( v_1 \) or \( v_2 \) is finite, then \( \tilde{w}'' \) vanishes at that endpoint.

We must analyze a few cases, depending on the values of \( v_1 \) and \( v_2 \) and which of the equations above is satisfied by \( \tilde{w}'' \). The goal in each case is to show that \( \tilde{w}'' \) must, in fact, vanish identically, which would be a contradiction.

The point, in all cases, is that we wish to multiply the appropriate equation for \( \tilde{w}'' \) by \( |\tilde{w}''| \) and integrate by parts, to obtain
\[
\int_{v_1}^{v_2} \int_{S^1} |\partial_s \tilde{w}''|^2 + |\partial_{\theta} \tilde{w}''|^2 - \tau^2 \cosh(2\sigma(s + s_i)) |\tilde{w}''|^2 \, ds \, d\theta = 0,
\]
and
\[
\int_{v_1}^{v_2} \int_{S^1} |\partial_s \tilde{w}''|^2 + |\partial_{\theta} \tilde{w}''|^2 \, ds \, d\theta = 0.
\]
or
\[
\int_{v_1}^{v_2} \int_{S^1} |\partial_s \tilde{w}''|^2 + |\partial_\theta \tilde{w}''|^2 - \frac{2}{\cosh^2(s+s')} |\tilde{w}''|^2 \, ds \, d\theta = 0.
\]
In each of these three cases we see that the integrand is positive, because we always have
the inequality
\[
\int_{v_1}^{v_2} |\partial_\theta \tilde{w}''|^2 \, ds \geq 4 \int_{v_1}^{v_2} |\tilde{w}''|^2 \, ds,
\]
and so we would conclude that \( \tilde{w}'' \equiv 0 \), which is a contradiction.

To make this argument work, it suffices to show that the boundary terms in the
integration by parts vanish. When either \( v_1 \) or \( v_2 \) is finite, this is immediate from the
Dirichlet conditions at that boundary, so it remains to show that if either \( v_1 \) or \( v_2 \) is
infinite, then \( \tilde{w}'' \) decays exponentially in that direction. Any unbounded solution of (40)
on a half-line must grow at least at the rate \( e^{2|s|} \), which would violate the condition
\( e^{\mu s} |\tilde{w}''| \leq 1 \), so we see that \( \tilde{w}'' \) must decrease exponentially in this case. The same
argument works when \( \tilde{w}'' \) satisfies (41) because solutions of that equation have the same
asymptotic rates of growth or decay as solutions of (40). Finally, if \( \tilde{w}'' \) satisfies (39), we
first choose \( \eta > 0 \) such that \( \sqrt{4-\eta} > \mu \), then, we apply Proposition 12, which states
that any unbounded solution must grow at least at the rate \( e^{\sqrt{4-\eta}|s|} \) provided \( \varepsilon \) is less
than, say, \( \varepsilon_0 \). Therefore, we can eliminate the possibility of exponential growth. This
ends the proof of the claim.

**Step 2:** We now consider the cases when \( |j| \leq 1 \). The argument when \( j = \pm 1 \) is
almost identical to the one for \( j = 0 \), so we shall just consider the latter case, commenting
on the end on the very minor changes that need to be made. Thus, recalling that we are
no longer requiring any boundary conditions, we wish to find a solution to the problem
\[
L_{\varepsilon,0} w_0 \equiv \partial_s^2 w_0 + \varepsilon^2 \cosh(2\sigma) w_0 = f_0 \quad \text{in } [s_0, \infty), \tag{42}
\]
with the desired decay property at infinity. We find this solution again as a limit of functions \( w_* \) solutions of \( L_{\varepsilon,0} w_* = f_0 \) on \([s_0,s_1] \), where now \( w_*(s_1) = \partial_s w_*(s_1) = 0 \). For convenience, we choose a \( C^{0,\alpha} \)
extension of \( f_0 \), vanishing when \( s < s_0 - 1 \), say, and consider the solution \( w_* \) for this extended right hand side, now defined on \((-\infty,s_1]\).

As in Step 1, we claim that there exists a constant \( C = C(\mu) \), independent of \( s_0, s_1 \)
and \( \varepsilon \), such that
\[
\sup_{s \in (-\infty,s_1]} e^{\mu s} |w_*(s)| \leq C.
\]
Once this claim is proved, the arguments of the proof are identical to those in Step 1, so
we shall omit them.

Again this is proved by contradiction. First, note that when \( s < s_0 - 1 \), \( w_*(s) \) is a linear
combination of the Jacobi fields \( \phi_{\varepsilon}^{0,\pm} \), hence is at most linearly growing. If the assertion
were false, there would exist sequences \( f_{0,i} \), \( s_{0,i} \), \( s_{1,i} \), \( \varepsilon_i \), and \( w_{*,i} \) such that
\[
A_i \equiv \sup_{s \in (-\infty,s_{1,i})} e^{\mu s} |w_{*,i}| \to \infty, \quad A_i \equiv \sup_{s \in (-\infty,s_{1,i})} e^{\mu s} |f_{0,i}| \leq 1.
\]
If this maximum is attained at \((s_i, \theta_i)\), \(s_i \in (-\infty, s_{1,i})\), then we rescale the functions and translate the independent variable by \(s_i\) to obtain a solution of

\[
\frac{d^2 \tilde{w}_i}{ds^2} + \tau_i^2 \cosh(2\sigma_i) \tilde{w}_i = \tilde{f}_{0,i},
\]

in \((-\infty, s_{1,i} - s_i]\) which satisfies

\[
\sup_{s \in (-\infty, s_{1,i} - s_i]} e^{\mu s} |\tilde{w}_i(s)| = 1,
\]

while \(\tilde{f}_{0,i}\) tends to zero in norm.

Passing to a subsequence, we obtain in the limit a nontrivial solution \(\tilde{w}\) of one the following equations:

\[
\frac{d^2 \tilde{w}}{ds^2} + \tau^2 \cosh(2\sigma(s + \bar{s})) \tilde{w} = 0, \quad \text{for some} \quad \varepsilon \in (0, \varepsilon_0) \quad \text{and} \quad \bar{s} \in \mathbb{R},
\]

\[
\frac{d^2 \tilde{w}}{ds^2} + \frac{2}{\cosh^2(s + \bar{s})} \tilde{w} = 0, \quad \text{for some} \quad \bar{s} \in \mathbb{R},
\]

or

\[
\frac{d^2 \tilde{w}}{ds^2} = 0,
\]

over some interval \((-\infty, v]\), and in each case, \(|\tilde{w}(s)| \leq e^{-\mu s}\) in \((-\infty, v]\).

Clearly \(v\) cannot be finite, because if it were then \(\tilde{w}\) would have to satisfy \(\tilde{w}(v) = \partial_s \tilde{w}(v) = 0\), which would imply that it would vanish identically.

Now, for each of the three equations we know that there are no exponentially decreasing solutions; for the second and third equations this is obvious, while for the first it follows because we know the family of solutions explicitly. However, since we know that \(\tilde{w}\) does decay exponentially as \(s \to \infty\), we again would have to conclude that it vanishes identically, and this is a contradiction.

When \(j = \pm 1\), the changes that need to be made in this argument are minor. For example, when \(j = 1\), for all \(s_1 > s_0\), the solution \(w_*\) is defined as before to be the solution of

\[
L_{\varepsilon,1} w_* \equiv \partial_s^2 w_* - w_* + \tau^2 \cosh(2\sigma)w_* = f_1 \quad \text{in} \quad [s_0, s_1),
\]

which satisfies \(w_*(s_1) = \partial_s w_*(s_1) = 0\) and where \(f_1\) has been extended by 0 in \((-\infty, s_0 - 1]\).

And to establish its uniform bound, we proceed by contradiction. In this case, however, the limiting equations are now

\[
\frac{d^2 \tilde{w}}{ds^2} - \tilde{w} + \tau^2 \cosh(2\sigma(s + \bar{s})) \tilde{w} = 0, \quad \text{for some} \quad \varepsilon \in (0, \varepsilon_0) \quad \text{and} \quad \bar{s} \in \mathbb{R},
\]

\[
\frac{d^2 \tilde{w}}{ds^2} - \tilde{w} + \frac{2}{\cosh^2(s + \bar{s})} \tilde{w} = 0, \quad \text{for some} \quad \bar{s} \in \mathbb{R},
\]

or

\[
\frac{d^2 \tilde{w}}{ds^2} - \tilde{w} = 0,
\]
on \((-\infty, v] \times S^1\), with boundary condition \(\tilde{w}(v) = \partial_s \tilde{w}(v) = 0\) if \(v\) is finite, and where \(|\tilde{w}(s)| \leq e^{-\mu s}\) for all \(s \in (-\infty, v]\).

Once again, \(v\) cannot be finite, but now the equations do admit exponentially decreasing solutions at \(\pm \infty\). However, all such solutions decay no faster than \(e^{-s}\), whereas we have assumed that \(1 < \mu < 2\), so once again we obtain a contradiction.

**Step 3:** Finally consider the problem when \(f = 0\) and \(\phi'' \neq 0\). We may as well assume that \(\|\phi''\|_{2, \alpha} = 1\). Let \(\eta(s)\) be a smooth cutoff function equal to 1 for \(s \leq 0\) and vanishing for \(s \geq 1\). Then

\[
L''_w w = 0, \quad w(s_0, \theta) = \phi''(\theta),
\]

is equivalent to

\[
L''_w \tilde{w} = -L''_w(\eta(s - s_0)\phi''(\theta)), \quad w(s_0, \theta) = 0,
\]

which has already been solved in Step 1. Moreover, since

\[
\|\eta\phi''\|_{0, \alpha, -\mu} \leq c e^{\mu s_0},
\]

it follows from Step 1 that

\[
\|w\|_{2, \alpha, -\mu} \leq c e^{\mu s_0},
\]

as we wished. This completes the proof in all cases.

**Corollary 4** Fix \(1 < \mu < 2\). Then there exists a constant \(c > 0\) and an \(\varepsilon_0 > 0\), depending only on \(\mu\), such that for \(0 < \varepsilon < \varepsilon_0\), we have

\[
\|(P_\varepsilon - P_0)(\phi'')\|_{2, \alpha, -\mu} \leq c \varepsilon^{-\mu/4} \left(\varepsilon^{1/2} + \varepsilon^{(6-3\mu)/4}\right) \|\phi''\|_{2, \alpha}.
\]

Here, if \(\phi'' \in \Pi''\left(C^{2, \alpha}(S^1)\right)\), the function \(P_0(\phi'')\) is the unique solution in \(C^{2, \alpha}\left([S_\varepsilon/8, \infty) \times S^1\right)\) of the problem

\[
\begin{aligned}
\Delta w &= 0 & \text{in} & & [S_\varepsilon/8, \infty) \times S^1 \\
w &= \phi'' & \text{on} & & \{S_\varepsilon/8\} \times S^1.
\end{aligned}
\]

**Proof:** Write \(w_\varepsilon = P_\varepsilon \phi''\) and \(w_0 = P_0 \phi''\). If \(w_\varepsilon = w_0 + h\), then \(L_\varepsilon h = -(\tau^2 \cosh(2\sigma)) w_0\) and \(\Pi''h(S_\varepsilon/8, \theta) = 0\), and so \(h = -G_\varepsilon(\tau^2 \cosh(2\sigma) w_0)\). We first estimate

\[
\|h\|_{2, \alpha, -\mu} \leq c \|\tau^2 \cosh(2\sigma) w_0\|_{0, \alpha, -\mu}.
\]

Using

\[
\|w_0\|_{0, \alpha, [s, s+1]} \leq c e^{-2(s-S_\varepsilon/8)} \|\phi''\|_{0, \alpha} \leq c \varepsilon^{-\frac{1}{2}} e^{-2s} \|\phi''\|_{0, \alpha},
\]

we bound this by

\[
c \varepsilon^{-\frac{1}{2}} \left(\sup_{s \geq S_\varepsilon/8} e^{(\mu-2)s} \|\tau^2 \cosh 2\sigma\|_{0, \alpha, [s, s+1]}\right) \|\phi''\|_{0, \alpha}.
\]

When \(S_\varepsilon/8 \leq s \leq 3S_\varepsilon/8\), we know from Proposition 7 that

\[
\tau^2 \cosh(2\sigma) + |\partial_s \tau^2 \cosh(2\sigma)| \leq \varepsilon^{\frac{1}{2}}.
\]

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Therefore
\[ \varepsilon^{-\frac{1}{2}} e^{(\mu - 2)s} || \tau^2 \cosh 2\sigma \|_{0,\alpha,[s,s+1]} \leq c \varepsilon^{(\mu - 2)s} \frac{\varepsilon}{8} = c \varepsilon^{\frac{1}{2}} e^{-\mu/4}. \]

Next, when \( 3S\varepsilon/8 \leq s \leq S\varepsilon/2 \), we know, still from Proposition [\( \square \)], that we may estimate
\[ \tau^2 \cosh 2\sigma + |\partial_s \tau^2 \cosh 2\sigma| \leq c \varepsilon^2 e^{2s}, \]
hence
\[ \varepsilon^{-\frac{1}{2}} e^{(\mu - 2)s} || \tau^2 \cosh 2\sigma \|_{0,\alpha,[s,s+1]} \leq c \varepsilon^{-\frac{1}{2}} e^{2-\mu} = c \varepsilon^{(6-3\mu)/4} e^{-\mu/4}. \]
Finally, for \( S\varepsilon/2 \leq s \) we use the fact that \( \tau^2 \cosh 2\sigma \leq 2 \), and proceed as before. This proves the Corollary. \( \square \)

5 CMC surfaces near to a half Delaunay surface

In this section we construct by perturbation methods the full space of CMC surfaces near to a fixed (half) Delaunay surface \( D \) of necksize \( \varepsilon \), as usual controlling the behaviour as \( \varepsilon \to 0 \). Assume that \( D \) has the parametrization
\[ \mathbf{x}(s,\theta) = (\tau e^{\sigma(s)} \cos \theta, \tau e^{\sigma(s)} \sin \theta, k(s)), \]
where \( \tau \in (0, 1) \), \( \sigma \) and \( k \) are as in Proposition [\[ \square \]]. The unit normal at \( \mathbf{x}(s,\theta) \) is defined to be
\[ \nu(s,\theta) = (-\tau \cosh \sigma(s) \cos \theta, -\tau \cosh \sigma(s) \sin \theta, \sigma(s)). \]
Therefore, surfaces which may be written as normal graphs over \( D \) admit the parametrization
\[ \mathbf{x}_w(s,\theta) = \mathbf{x}(s,\theta) + w(s,\theta)\nu(s,\theta), \]
for some sufficiently small function \( w \) on \( D \). We denote by \( D_w \) the surface obtained in this way. The components of its metric tensor are
\[ E_w = \tau^2 (e^\sigma - \sinh \sigma w)^2 + w_s^2, \quad F_w = w_s w_\theta, \]
and
\[ G_w = \tau^2 (e^\sigma - \cosh \sigma w)^2 + w_\theta^2. \]
The components of the second fundamental form are considerably less simple. In computing the following, we use that we always have the bounds \( \tau e^\sigma \leq 2 \), \( \tau^2 \sinh^2 \sigma \leq 1 \) and \( \sigma_s^2 \leq 1 \). After substantial work, we find that
\[ \sqrt{E_w G_w - F_w^2} L_w = \tau^3 e^{3\sigma} \left( \tau \sinh \sigma + P_1 \left( \frac{w}{\tau e^\sigma}, \frac{\nabla w}{\tau e^\sigma}, \frac{\nabla^2 w}{\tau e^\sigma} \right) \right), \]
where \( P_1 \) is some polynomial (of degree at most 3) without any constant term, the coefficients of which are functions of \( s \) and such that they and their derivatives are bounded uniformly in \( s \) and \( \varepsilon \). In a similar manner we derive that
\[ \sqrt{E_w G_w - F_w^2} M_w = \tau^3 e^{3\sigma} P_2 \left( \frac{w}{\tau e^\sigma}, \frac{\nabla w}{\tau e^\sigma}, \frac{\nabla^2 w}{\tau e^\sigma} \right) \]
\[
\sqrt{E_w G_w - F_w^2} N_w = \tau^3 \epsilon^3 \sigma \left( \tau \cosh \sigma + P_3 \left( \frac{w}{\tau e^{\sigma}}, \frac{\nabla w}{\tau e^{\sigma}}, \frac{\nabla^2 w}{\tau e^{\sigma}} \right) \right),
\]
where \(P_2\) and \(P_3\) have the same properties as \(P_1\).

The equation that \(D_w\) has mean curvature 1 is
\[
L_w G_w - 2M_w F_w + N_w E_w - (E_w G_w - F_w^2) = 0. \tag{44}
\]
This is a rather complicated nonlinear elliptic equation for \(w\) which we shall not write out in full. Notice that it is satisfied when \(w = 0\). Using the previous formula for the coefficients of the first and second fundamental forms, we find that its Taylor expansion about \(w = 0\) is
\[
L_\epsilon w = \tau e^\sigma Q \left( \frac{w}{\tau e^\sigma}, \frac{\nabla w}{\tau e^\sigma}, \frac{\nabla^2 w}{\tau e^\sigma} \right), \tag{45}
\]
where
\[
L_\epsilon w = w_{ss} + w_{\theta\theta} + \tau^2 \cosh(2\sigma) w,
\]
and \(Q\) is again a polynomial (now of higher order) without any constant or linear terms, the coefficients of which have partial derivatives bounded uniformly in \(s\) and \(\epsilon\). We also write, for brevity,
\[
Q(w) \equiv \tau e^\sigma Q \left( \frac{w}{\tau e^\sigma}, \frac{\nabla w}{\tau e^\sigma}, \frac{\nabla^2 w}{\tau e^\sigma} \right).
\]

Given \(\phi'' \in \Pi''(C^{2,\alpha}(S^1))\), we would like to solve the boundary value problem
\[
\begin{cases}
L_\epsilon w = Q(w) & \text{in } [S_{\epsilon}/8, +\infty) \times S^1 \\
\Pi'' w = \phi'' & \text{on } \{S_{\epsilon}/8\} \times S^1.
\end{cases} \tag{46}
\]
Let \(w_\epsilon\) be the unique solution in \(C^{2,\alpha}_{-\mu}([S_{\epsilon}/8, +\infty) \times S^1)\), \(1 < \mu < 2\), of
\[
\begin{cases}
L_\epsilon w_\epsilon = 0 & \text{in } [S_{\epsilon}/8, +\infty) \times S^1 \\
\Pi'' w_\epsilon = \phi'' & \text{on } \{S_{\epsilon}/8\} \times S^1,
\end{cases}
\]
which is given by Proposition \[13\]. Setting \(w = w_\epsilon + v\), then we would like to find \(v \in C^{2,\alpha}_{-\mu}([S_{\epsilon}/8, +\infty) \times S^1)\) such that
\[
\begin{cases}
L_\epsilon v = Q(w_\epsilon + v) & \text{in } [S_{\epsilon}/8, +\infty) \times S^1 \\
\Pi'' v = 0 & \text{on } \{S_{\epsilon}/8\} \times S^1.
\end{cases}
\]
Notice that, it is sufficient to find a fixed point of the mapping
\[
\mathcal{K}(v) = G_\epsilon Q(w_\epsilon + v), \tag{47}
\]
at least when \(\epsilon\) is sufficiently small.
Proposition 14 There exists a constant $c_0 > 0$ such that if $||\phi''||_{2,\alpha} \leq c_0 \varepsilon^{3/4}$, then

$$||G_\varepsilon(Q(w_\varepsilon))||_{2,\alpha,-\mu} \leq c \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2$$

and

$$||G_\varepsilon(Q(w_\varepsilon + v) - Q(w_\varepsilon + v_1))||_{2,\alpha,-\mu} \leq \frac{1}{2} ||v_2 - v_1||_{2,\alpha,-\mu},$$

for all $v_1, v_2$ in $B_{c_0} \equiv \{v : ||v||_{2,\alpha,-\mu} \leq c_0 \varepsilon^{(3-\mu)/4}\}$. Thus, $\mathcal{K}$ is a contraction mapping on the ball $B_{c_0}$ into itself. Consequently, $\mathcal{K}$ has a unique fixed point $v$ in this ball.

Proof: We shall use that

$$||w_\varepsilon||_{2,\alpha,[s,s+1]} \leq c_0 \varepsilon^{3/4} e^{\mu(S_\varepsilon/8-s)} ||\phi''||_{2,\alpha} \leq c c_0 \varepsilon^{3/4} e^{\mu(S_\varepsilon/8-s)}$$

and also that

$$||v||_{2,\alpha,[s,s+1]} \leq c_0 \varepsilon^{3/4} e^{\mu(S_\varepsilon/8-s)},$$

for $v \in B_{c_0}$.

First consider $s$ in the range $[S_\varepsilon/8, 7S_\varepsilon/8]$. Here, from Proposition 8, we get

$$\tau e^{\sigma(s)} \geq c \varepsilon^{3/4}.$$ 

Together with the fact that all derivatives of $\sigma$ are bounded, this gives

$$\left(\frac{\tau e^\sigma}{\tau e^\sigma} \right)^j \leq c \varepsilon^{-(j-1)/4} ||w_\varepsilon||_{2,\alpha,[s,s+1]} \leq c c_0^{j-2} e^{\mu(S_\varepsilon/8-s)} \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2$$

for any integer $j \geq 2$. Thus, we already have obtained that

$$e^{\mu S} ||Q(w_\varepsilon)||_{0,\alpha,[s,s+1]} \leq c e^{2\mu(S_\varepsilon/8-s)+\mu S} \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2 \leq c c_0^2 \varepsilon^{(3-\mu)/4}.$$ 

Similarly

$$e^{\mu S} ||Q(w_\varepsilon + v) - Q(w_\varepsilon + v_2)||_{0,\alpha,[s,s+1]}$$

can be estimated by the sum of products of $e^{\mu S} ||v_1 - v_2||_{2,\alpha,[s,s+1]}$ with various terms of the form

$$||(w_\varepsilon + v_1)^j(w_\varepsilon + v_2)^j \tau^j e^{-j-j'} ||_{0,\alpha,[s,s+1]}, \quad \text{where } j + j' \geq 1.$$ 

Each of these can be bounded by $c (c_0 \varepsilon^{3/4})^{j+j'} e^{-(3/4)(i+j) e^{(3-\mu)/4}(S_\varepsilon/8-s)} \leq c c_0^{j+j'}$, and so can be made as small as desired.

For $s \geq 3S_\varepsilon/4$, we have $s - S_\varepsilon/8 \geq 3S_\varepsilon/4 \geq c - (3/2) \log \varepsilon$, and we will simply use the fact that

$$\tau e^{\sigma(s)} \geq \tau e^{\sigma(0)} = \varepsilon.$$ 

Arguing as before, we get first that

$$e^{\mu S} ||Q(w_\varepsilon)||_{0,\alpha,[s,s+1]} \leq c e^{\mu S} \varepsilon^{3/4} e^{-3/4} ||\phi''||_{2,\alpha}^2 \leq c c_0 \varepsilon^{3/4} e^{-(1/4)(3\mu/2)}.$$
and the quantity on the right in parentheses can be made as small as desired when \( \varepsilon \) is sufficiently small. Furthermore

\[
e^{\mu s}||Q(w_\varepsilon + v_1) - Q(w_\varepsilon + v_2)||_{0,\alpha,[s,s+1]} \leq c_0 \varepsilon^{-1} e^{\mu(S_\varepsilon/8 - s)} e^{\mu s} ||v_2 - v_1||_{2,\alpha,[s,s+1]}\]

\[
\leq (c_0 \varepsilon^{-1/4 + 3\mu/2}) ||v_2 - v_1||_{2,\alpha,-\mu},
\]

and again the coefficient can be made as small as desired when \( \varepsilon \) is chosen small enough.

Putting the estimates in these two domains together, and using that \( G_\varepsilon \) is bounded, we have now checked all the conditions necessary to ensure that \( K \) is a contraction mapping. Therefore there is a unique element \( v \in B_{c_0} \) such that \( K(v) = v \), and the proof is complete.

Examining this proof more carefully, we also obtain the

**Corollary 5** There exists a constant \( c_0 > 0 \) and an \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0,\varepsilon_0) \) and for any \( \phi'' \in \Pi''(C^{2,\alpha}(S^1)) \) with \( ||\phi''||_{2,\alpha} \leq c_0 \varepsilon^{3/4} \), the problem (46) has a unique solution \( w \). The mapping

\[
\Pi''(C^{2,\alpha}(S^1)) \ni \phi'' \longrightarrow w \in C^{2,\alpha}_{-\mu}([S_\varepsilon/8, \infty) \times S^1),
\]

is continuous and the solution \( w \) satisfies the estimates

\[
||w||_{2,\alpha,-\mu} \leq c \varepsilon^{-\mu/4} (||\phi''||_{2,\alpha} + \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2),
\]

(50)

and

\[
||(w - \Pi''w)(S_\varepsilon/8, \cdot)||_{2,\alpha} + ||\partial_\nu (w - \Pi''w)(S_\varepsilon/8, \cdot)||_{1,\alpha} \leq c \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2.
\]

(51)

Finally, if \( w_0 = P_0(\phi'') \in C^{2,\alpha}_{-\mu}([S_\varepsilon/8, +\infty) \times S^1) \) as in Corollary 4, then

\[
||w - w_0||_{2,\alpha,-\mu} \leq c \varepsilon^{-\mu/4} \left((\varepsilon^{1/2} + \varepsilon^{-3\mu/4}) ||\phi''||_{2,\alpha} + \varepsilon^{-3/4} ||\phi''||_{2,\alpha}^2\right).
\]

(52)

**Proof:** The solution \( w \) is a sum \( w_\varepsilon + v \) and we already know that \( ||w_\varepsilon||_{2,\alpha,-\mu} \leq c ||\phi''||_{2,\alpha} \). For fixed \( \phi'' \), the map \( K \) is actually a contraction on the balls of radius a constant times \( \varepsilon^{-(\mu + 3)/4} ||\phi''||_{2,\alpha}^2 \), and so the norm of \( v \) is at most this large. And this gives (54). The second estimate (51) follows by evaluating at \( s = S_\varepsilon/8 \). Finally, for (52), we write

\[
||w - w_0||_{2,\alpha,-\mu} \leq ||w_\varepsilon - w_0||_{2,\alpha,-\mu} + ||v||_{2,\alpha,-\mu},
\]

and use Corollary 4. \( \square \)

## 6 k-noids

The second type of component in our construction of CMC surfaces are a somewhat restricted class of minimal surfaces of finite total curvature with \( k \) ends, or as we shall call them, \( k \)-noids. In this brief section we discuss some of the global and asymptotic aspects of the geometry and topology of \( k \)-noids, and in the next, discuss Jacobi operators on these surfaces and their compact truncations.
It is well-known that any \( k \)-noid \( \Sigma \) has finite topology, and in fact is conformally equivalent to the complement of a finite number of points in a compact Riemann surface \( \bar{\Sigma} \), i.e. \( \Sigma = \bar{\Sigma} \setminus \{ p_1, \ldots, p_k \} \). As in the introduction, we denote the space of \( k \)-noids of genus \( g \) by \( \mathcal{H}_{g,k} \). When \( g > 0 \), \( \mathcal{H}_{g,1} \) and \( \mathcal{H}_{g,2} \) are empty, while \( \mathcal{H}_{0,1} \) and \( \mathcal{H}_{0,2} \) contain only the plane and catenoid, respectively. The standard catenoid \( \mathcal{C}_1 \) is a surface of revolution, given in cylindrical coordinates by the parametrization

\[
\mathbf{x}(s, \theta) = (\cosh s \cos \theta, \cosh s \sin \theta, s).
\]

This is a conformal parametrization, and the unit normal, metric tensor and second fundamental forms are given by

\[
\nu(s, \theta) = \frac{1}{\cosh s}(-\cos \theta, -\sin \theta, \sinh s),
\]
\[
g = \cosh^2 s (ds^2 + d\theta^2), \quad A = -ds^2 + d\theta^2.
\]

In particular, the mean curvature vanishes, and the catenoid is minimal.

We shall be discussing the space of moduli of \( k \)-noids. Just as with CMC surfaces, it is possible to determine the moduli space explicitly in the simplest case, when \( k = 2 \). In fact, the only complete minimal surfaces in \( \mathbb{R}^3 \) with two ends are images of the standard catenoid \( \mathcal{C}_1 \) by rigid motions and homotheties. While we shall frequently not distinguish between \( \mathcal{C}_1 \) and its translates or rotations, it will be important to keep track of the homothety factor. The dilation of \( \mathcal{C}_1 \) by the factor \( a \) will be denoted \( \mathcal{C}_a \), and has the parametrization

\[
\mathbf{x}^{(a)}(s, \theta) = (a \cosh s \cos \theta, a \cosh s \sin \theta, as).
\]

Thus any element of \( \mathcal{H}_{0,2} \) is given as a rigid motion of some \( \mathcal{C}_a \). The metric tensor and second fundamental forms for this parametrization are

\[
g_a = a^2 \cosh^2 s (ds^2 + d\theta^2), \quad A_a = a(-ds^2 + d\theta^2).
\]

The plane and catenoid provide the asymptotic models for the ends of any \( k \)-noid: the basic structure theorem for \( k \)-noids states that an end of any \( k \)-noid may be written as a normal graph of a decaying function over an end of some suitably translated, rotated plane or dilated catenoid. The corresponding ends will then be referred to as planar or catenoidal. Only \( k \)-noids with all ends catenoidal will be used in our construction; henceforth this will always be assumed.

Using this asymptotics theorem, we may assign a dilation, or weight, parameter \( a_\ell \) to each end \( E_\ell \) of \( \Sigma \in \mathcal{H}_{g,k} \), \( \ell = 1, \ldots, k \), signifying that that end is the normal graph over (some translated and rotated copy of) \( \mathcal{C}_{a_\ell} \). This is analogous to the necksize parameters of the ends of CMC surfaces. This defines, at least in neighbourhoods of the moduli space where some ordering of the ends is fixed, a map \( \mathcal{H}_{g,k} \to \mathbb{R}^k \).

Fix \( \Sigma \in \mathcal{H}_{g,k} \). We describe the parametrization of the ends more carefully. Assume that \( \Sigma \) has been rotated and translated so that the the end \( E_\ell \) is asymptotic to the model \( \mathcal{C}_{a_\ell} \). By definition, there is a function \( w \), defined on \( \mathcal{C}_{a_\ell} \cap \{ s \geq s_\ell \} \), such that \( E_\ell \) is parametrized by

\[
\mathbf{x}_w(s, \theta) \equiv \mathbf{x}_{a_\ell}(s, \theta) + w(s, \theta)\nu(s, \theta), \quad s \geq s_\ell.
\]
This gives a canonical cylindrical coordinate system \((s, \theta)\) on \(E_\ell\), which we will always use. The function \(w\) is assumed \text{a priori} only to decay, but in fact admits an asymptotic expansion
\[
w(s, \theta) \sim \sum_{|j| > 1} a_j \chi_j(\theta) e^{-js}, \quad \text{as } s \to \infty.
\]

7 The Jacobi operator on \(k\)-noids

Continuing our treatment of analysis on \(k\)-noids paralleling that on Delaunay surfaces, we now consider the Jacobi operator \(L = L_\Sigma\), which is the linearization of the mean curvature operator \(M\) over \(\Sigma\). This is
\[
L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2,
\]
where the term of order zero is the squared norm of the second fundamental form of \(\Sigma\).

7.1 Mapping properties of \(L\) and Jacobi fields

Just as for Delaunay surfaces, we require detailed knowledge of the mapping properties of \(L\), first over all of \(\Sigma\), then in a later section for the Dirichlet problem on certain (deformations of) compact truncations \(\Sigma_\varepsilon\) of \(\Sigma\), and finally uniformly as \(\varepsilon \to 0\).

The analysis for \(L\) over the complete surface \(\Sigma\) is based on the fact that the ends have good asymptotic models. In fact, using the canonical cylindrical coordinates on each end \(E_\ell\), we see that the Jacobi operator for the model catenoid there is
\[
L_{a_\ell} = a_\ell^{-2} \cosh^{-2} s \left( \partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s} \right), \quad (54)
\]
and so the true Jacobi operator is equal, as \(s \to \infty\) in \(E_\ell\), to the sum of this model operator and a correction term, which is a second order operator each coefficient of which decays at least like \(e^{-4s}\).

We let \(L\) act on the weighted Hölder spaces \(C^{r,\alpha}_\mu(\Sigma)\), where \(\phi\) is in this space if it is locally in \(C^{r,\alpha}(\Sigma)\) and on each end may be written as \(e^{\mu s}\psi\) where \(\psi \in C^{r,\alpha}(\mathbb{R}_+^+ \times S^1_\theta)\). The basic mapping properties for \(L\) are summarized in the

**Proposition 15** The operator
\[
L : C^{2,\alpha}_\mu(\Sigma) \longrightarrow C^{0,\alpha}_{\mu-2}(\Sigma),
\]
is Fredholm provided \(\mu \notin \mathbb{Z}\). In addition, \(L\) is surjective on \(C^{2,\alpha}_\mu(\Sigma)\) if and only if it is injective on \(C^{2,\alpha}_{\mu-2}(\Sigma)\).

The drop of two in the weight parameter comes from the factor \((\cosh(a_\ell s))^{-2}\) in the expression for \(L\) on \(E_\ell\). This sort of result is fairly standard by now; it may be proved by constructing local parametrices, or solution operators, for the model operators on each of the ends \(E_\ell\) using explicit ODE techniques on each of the cross-sectional eigenspaces, joining these to a parametrix for the interior compact region, and finally using standard perturbation techniques and Fredholm theory.
This Proposition leads naturally to the issue of determining the values of the weight parameters $\mu$ for which $L$ is surjective or injective. Although for a given $k$-noid $\Sigma$ this may be quite difficult to determine, the following condition is essential for the moduli space theory:

**Definition 2** A $k$-noid $\Sigma$ is called nondegnerate if its Jacobi operator $L$ is surjective on $C^{2,\alpha}_\mu(\Sigma)$ whenever $\mu > 1$, $\mu \neq 2, 3, \ldots$, or equivalently, whenever there are no anomalous decaying Jacobi fields and so $L$ is injective on $C^{2,\alpha}_{-\mu}(\Sigma)$ for $\mu > 1$.

We cannot preclude the existence of Jacobi fields in $C^{2,\alpha}_\mu(\Sigma)$ for $|\mu| \leq 1$, and in fact these always exist, at least locally on each end, for geometric reasons. They may be exhibited explicitly on the catenoid: just as for the Delaunay surfaces, solutions of $Lw = 0$ corresponding to the eigenvalues of the cross-sectional Laplacian with $|j| \leq 1$ arise from translations, rotations and dilations (which substitute for changes in Delaunay parameter):

**Proposition 16** The Jacobi fields

$$
\Psi^{0,+}(s) = \tanh s, \quad \Psi^{0,-}(s) = s \tanh s - 1,
$$
correspond to vertical translation along the axis of the catenoid, and change of the dilation parameter $a$, respectively. The Jacobi fields

$$
\Psi^{1,+}(s, \theta) = \psi^+(s) \cos \theta, \quad \Psi^{-1,+}(s, \theta) = \psi^+(s) \sin \theta,
$$
correspond to horizontal translations in the $x_1$ and $x_2$ directions, while

$$
\Psi^{1,-}(s, \theta) = \psi^-(s) \cos \theta, \quad \Psi^{-1,-}(s, \theta) = \psi^-(s) \sin \theta,
$$
correspond to rotations about the $x_2$-axis and $x_1$-axis, respectively. Here

$$
\psi^+(s) = \frac{1}{\cosh s} \quad \psi^-(s) = \frac{s}{\cosh s} + \sinh s.
$$

**Proof:** As with the analogous statement in the Delaunay case, these Jacobi fields may be computed by finding the parametrizations of the one-parameter family of minimal surfaces in each case and differentiating to get the deformation vector field, the inner product of which with the unit normal of $C_a$ yields the appropriate expression. We leave the details to the reader. $\blacksquare$

Jacobi fields asymptotic to these exist on the ends of any $k$-noid. Let $\tilde{\Sigma}$ denote some fixed truncation $\Sigma_{\epsilon_0}$ of the $k$-noid $\Sigma$, and let $E_1, \ldots, E_k$ denote the components of $\Sigma \setminus \tilde{\Sigma}$. These are in one to one correspondence with the ends of $\Sigma$, and are minimal surfaces with boundary.

**Proposition 17** On each end $E_\ell$ of $\Sigma$, there exists a six-dimensional space of functions $\Psi^{j,\pm}_\ell$, $j = 0, \pm 1$, such that each $L \Psi^{j,\pm}_\ell = 0$, and which are asymptotic to the corresponding model Jacobi fields $\Psi^{j,\pm}$ for the catenoid $C_{a_\ell}$ modelling $E_\ell$ in the sense that

$$
\left| \Psi^{j,+}_\ell(s, \theta) - \Psi^{j,+}(s, \theta) \right| \leq C e^{-(j+2)s},
$$

$$
\left| \Psi^{j,-}_\ell(s, \theta) - \Psi^{j,-}(s, \theta) \right| \leq C e^{(j-2)s}.
$$
Proof: These new Jacobi fields are produced by the same geometric process, namely forming the families of minimal surfaces with boundary, $E_\ell(\eta)$, by translating, rotating or dilating $E_\ell$, and then differentiating with respect to the parameter $\eta$ at $\eta = 0$, and taking the inner product of the resulting vector field along $E_\ell$ with the unit normal. The statement about asymptotics is obtained from the fact that $E_\ell$ is a normal graph over $C_{a_i}$ of a function $\phi_\ell$ which decays like $e^{-2s}$. □

7.2 Moduli space theory

Following these preliminaries, we now briefly sketch the moduli space theory for $k$-noids. This was developed by Perez and Ros [13] at around the same time that the very similar moduli space theory was set down for solutions of the singular Yamabe problem and CMC surfaces in [10] and [7], using slightly different (but equivalent) methods. The parallels between the three problems are discussed carefully in [9]. We state the results following these latter three papers. For $\Sigma \in H_{g,k}$, define the $6k$-dimensional space

$$W = \bigoplus_{\ell=1}^k W_\ell,$$

where $W_\ell = \{ \eta_\ell \Psi^\pm_j, j = 0, \pm 1 \}$.

and where $\eta_\ell$ is a cutoff function vanishing on $\tilde{\Sigma}$ and equalling one outside of a slight enlargement of this truncation on the end $E_\ell$. At least around non-degenerate $k$-noids, the moduli space theory is based on the implicit function theorem. For this one requires surjectivity of $L$ on some geometrically natural function spaces, but unfortunately the spaces on which $L$ is surjective in Proposition 15 have positive exponential weight, hence are ill-suited for the nonlinear operator. To remedy this one uses the following more refined linear result.

Proposition 18 Suppose $\Sigma$ is nondegenerate. Fix $\mu$ with $1 < \mu < 2$. Then the mapping

$$L : C^{2,\alpha}_\mu(\Sigma) \oplus W \longrightarrow C^{0,\alpha}_{\mu-2}(\Sigma), \quad (55)$$

is surjective. Its nullspace, $B \equiv B_\Sigma$ (which we call the bounded nullspace) is $3k$-dimensional.

The proof is essentially identical to the one in [10] and [7], although the linear theory here is more elementary than the analysis on asymptotically periodic ends in those papers. The dimension count for $B$ is obtained by a relative index theorem (which is essentially equivalent to the Riemann-Roch theorem).

To make sense of the mean curvature operator $N$ on elements of the domain space in (55), we use that elements of $W$ correspond to geometric motions. Thus $N(u', \tilde{u})$ calculates the mean curvature of the normal graph of the function $\tilde{u} \in C^{2,\mu}_\Sigma$ over the surface $\Sigma_{u'}$ obtained by slightly deforming the ends of $\Sigma$ in the manner prescribed by the components of $u' \in W$. (More specifically, one considers an ‘exponential map’ from a neighbourhood of 0 in $W$ to a space of surfaces deforming $\Sigma$, such that the derivatives of the families of surfaces $\Sigma(\lambda u')$, for $u' \in W$, at $\lambda = 0$ equal $u'$.) Proposition 18 then states that the differential of this map $N$ is surjective at $(0, 0)$ when $\Sigma$ is nondegenerate, and so the first part of the following is a trivial application of the standard implicit function theorem:
Corollary 6. In the neighbourhood of any one of its nondegenerate points, the moduli space $H_{g,k}$ of $k$-noids of genus $g$ is a real analytic manifold of dimension $3k$. In the neighbourhood of an arbitrary point, it has the structure of a locally defined (possibly singular) real analytic variety.

Perez and Ros note that the second part of this result follows from the general theory of Weierstrass representations of $k$-noids. It may also be proved in a manner more consistent with the first part by the Kuranishi method, as in [7], cf. also [9].

8 Truncated $k$-noids and their deformations

In this section we first introduce the compact truncations $\Sigma_\epsilon$ which fill out the $k$-noid $\Sigma$ as $\epsilon \to 0$. These are the building blocks occupying the central portion of the surfaces we shall construct. The reason for introducing them is that there are no surfaces of mean curvature one which may be written as normal graphs over all of $\Sigma$, but there are many which are graphs over any one of the $\Sigma_\epsilon$. Next we study a natural boundary problem for the Jacobi operator on these compact surfaces and analyze its behaviour as $\epsilon$ tends to zero. Finally, we introduce a finite dimensional family of deformations of the Jacobi operator, corresponding to the elements of $W$, and show that the preceding linear analysis carries over to the operators in this family.

8.1 The Jacobi operator on truncated $k$-noids

We start by defining the truncations $\Sigma_\epsilon$. Recall that each end $E_\ell$ of $\Sigma$ admits the parametrization

$$x_\ell(s, \theta) = a_\ell (\cosh s \cos \theta + O(e^{-3s}), \cosh s \sin \theta + O(e^{-3s}), s + O(e^{-2s})). \quad (56)$$

We simply define $\Sigma_\epsilon$ to be the union of the compact piece $K$ of $\Sigma$ and the portion of each of the ends up to $s = S_\epsilon/8$ (which is of the order $-\frac{1}{4} \log \epsilon$).

Preliminary to the nonlinear analysis, in the next section, of the family of surfaces of constant mean curvature one which are normal graphs over all of $\Sigma_\epsilon$, we shall require information about a certain inhomogeneous boundary problem for the Jacobi operator $L$ on $\Sigma_\epsilon$, in particular its solvability and the uniformity of this solution with respect to $\epsilon$.

Before stating this boundary problem precisely, we digress briefly. Set

$$X(\Sigma_\epsilon) \equiv C^{2,\alpha}(\Sigma_\epsilon).$$

If $u \in X(\Sigma_\epsilon)$, then we let $(C)_\epsilon(u)$ denote its Cauchy data on $\partial \Sigma_\epsilon$. Thus

$$(C)_\epsilon(u) \in Z(\partial \Sigma_\epsilon) \equiv C^{2,\alpha}(\partial \Sigma_\epsilon) \oplus C^{1,\alpha}(\partial \Sigma_\epsilon).$$

Since

$$C^{2,\alpha}(\Sigma_\epsilon) = C^{2,\alpha}(\Sigma)|_{\Sigma_\epsilon};$$

we may use the restriction of $\| \cdot \|_{2,\alpha,\mu}$ as a norm on $X(\Sigma_\epsilon)$.

For later use we also record that this space admits a decomposition

$$X(\Sigma_\epsilon) = W \oplus X(\Sigma_\epsilon)'', \quad 32$$
where, by a slight abuse of notation, $W$ here represents the restrictions to $\Sigma_\varepsilon$ of elements of the true ‘global’ deficiency space, and

$$X(\Sigma_\varepsilon)'' = \left\{ w \in X(\Sigma_\varepsilon) : w|_{E_\varepsilon} \equiv w_\ell(s, \theta) \in \text{span} \{ \chi_j(\theta) \}_{|j| \geq 2} \text{ for } s_\ell \leq s \leq S_\varepsilon/8 \right\}. $$

If $X$ is any of the space of functions we consider on all of $\Sigma$, for example $C^2_\alpha(\Sigma)$ or $C^{-2,\alpha}_\varepsilon(\Sigma)$, then we let $X''$ denote the finite codimension subspace defined in the analogous manner (omitting only the restriction $s \leq S_\varepsilon/8$ in the definition). Most commonly, $X$ will be $C^{-2,\alpha}_\varepsilon(\Sigma)$, and so we write

$$X_W(\Sigma) = C^2_\varepsilon(\Sigma) \oplus W,$$

for brevity. Notice that the bounded nullspace $B$ is a subspace of $X_W(\Sigma)$.

Next, if $v \in X(\Sigma_\varepsilon)$, denote its Cauchy data at $\partial \Sigma_\varepsilon$ by $C_v(\varepsilon) \in Z(\partial \Sigma_\varepsilon)$. Similarly, if $v \in X_W(\Sigma)$, and its decomposition is written $v = w + v''$, then we regard its component $w \in W$ as its Cauchy data at infinity, and write $w = C_0(v)$.

There is a natural (weak) symplectic structure on $Z(\partial \Sigma_\varepsilon)$ given by

$$\omega_\varepsilon(\phi, \psi) \equiv \int_{\partial \Sigma_\varepsilon} (\phi_0 \psi_1 - \psi_0 \phi_1) \, d\sigma,$$

if $\phi = (\phi_0, \phi_1), \psi = (\psi_0, \psi_1) \in Z(\partial \Sigma_\varepsilon)$, and where $d\sigma$ is the length form of $\partial \Sigma_\varepsilon$. (Note that $\omega_\varepsilon$ does not induce an isomorphism between $Z$ and its dual; fortunately, this is unimportant for our purposes.) If $u, v \in X(\Sigma_\varepsilon)$ and $\phi = C_\varepsilon(u), \psi = C_\varepsilon(v)$, then by Stokes’ theorem,

$$\omega_\varepsilon(\phi, \psi) = \int_{\partial \Sigma_\varepsilon} \left( \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \right) \, d\sigma = \int_{\Sigma_\varepsilon} (L_\varepsilon u - u(L_\varepsilon v)).$$

(57)

Here, of course, $\nu$ the appropriately oriented unit normal of $\partial \Sigma_\varepsilon$. The expression on the right does not depend on the particular extensions $u$ and $v$ of $\phi$ and $\psi$. However, fixing $u, v \in X_W(\Sigma)$, then this expression has a limit as $\varepsilon \to 0$ because now both $Lu$ and $L_\varepsilon v$ decay exponentially on the ends of $\Sigma$. In fact, write

$$u = u' + u'' \quad \text{where} \quad u' = \sum_{\ell,j, \pm} u_{\ell j}^{j, \pm} \Psi_{\ell j}^{j, \pm}, \quad u'' \in X(\Sigma)'',$$

and similarly $v = v' + v''$; then

$$\lim_{\varepsilon \to 0} \omega_\varepsilon(\phi, \psi) \equiv \omega(\phi, \psi) = \frac{1}{2} \sum_{\ell = 1}^k \sum_{j = 0, \pm 1} \left( u_{\ell j}^{j, +} v_{\ell j}^{j, +} - u_{\ell j}^{j, -} v_{\ell j}^{j, -} \right).$$

(58)

Abusing notation in our customary manner, we identify $\omega(\phi, \psi)$ with $\omega(u', v')$ and even with $\omega(u, v)$; this is the induced symplectic form on $W$, which by this computation is the ‘standard’ one with respect to the basis $\{ \Psi_{\ell j}^{j, \pm} \}$.

From (57), if $u, v \in B$, then $\omega(u, v) = 0$. Since $\dim B = \frac{1}{2} \dim W$, we conclude that $B$ is Lagrangian in the symplectic vector space $(W, \omega)$. From Corollary 3, $B$ is identified with $T_{\Sigma} \mathcal{H}_{g, k}$, and so we have shown that this moduli space inherits at least the
infinitesimal structure of a Lagrangian submanifold of some larger symplectic manifold. This was discussed both in [14] and in [10], [7].

We now set up the boundary problem for the Jacobi operator on the surfaces \( \Sigma_\varepsilon \). That the mapping

\[
L : \{ u \in C^{2,\alpha}(\Sigma_\varepsilon) : u|_{\partial\Sigma_\varepsilon} = 0 \} \longrightarrow C^{0,\alpha}(\Sigma_\varepsilon),
\]

is surjective when \( \varepsilon \) is sufficiently small is fairly easy to establish. Unfortunately, the norm of the inverse is not uniformly bounded as \( \varepsilon \to 0 \). This happens for a good reason: the range of the inverse may be too close to the restriction of the bounded nullspace \( B \) to \( \Sigma_\varepsilon \). Therefore we impose a boundary condition, the corresponding solution operator for which does have the uniformity we need later.

First select a \( 3k \)-dimensional subspace \( \tilde{B} \subset W \) which is Lagrangian with respect to \( \omega|_W \), and which is transverse to \( B \). There are many ways to do this, of course, but the choice is irrelevant! The space \( (B \oplus \tilde{B}, \omega) \) is then a \( 6k \)-dimensional dimensional symplectic subspace of \( Z(\partial\Sigma_\varepsilon) \), which is a small perturbation of \( W \) when \( \varepsilon \) is small.

Continuing our practice of splitting into low and high eigen spaces, we write \( B \oplus \tilde{B} \) as \( Z(\partial\Sigma_\varepsilon) \) because, up to an error which decreases with \( \varepsilon \), it corresponds to the span in \( Z(\partial\Sigma_\varepsilon) \) of the eigenfunctions \( \{ \chi_j(\theta) \}_{|j| \leq 1} \). For its complement we use \( Z(\partial\Sigma_\varepsilon)^\prime \equiv [C^{2,\alpha}(\partial\Sigma_\varepsilon) \oplus C^{1,\alpha}(\partial\Sigma_\varepsilon)]^\prime \), the span of the eigenfunctions \( \{ \chi_j(\theta) \}_{|j| > 1} \). Thus if \( \phi \in Z(\partial\Sigma_\varepsilon) \) then \( \phi = \phi^\prime + \phi^\prime\prime \) with \( \phi^\prime \in Z(\partial\Sigma_\varepsilon)^\prime \), \( \phi^\prime\prime \in Z(\partial\Sigma_\varepsilon)^\prime\prime \), and these components split further as

\[
\phi^\prime = \phi_B + \phi_B^\prime \in B \oplus \tilde{B} \quad \text{and} \quad \phi^\prime\prime = (\phi_0^\prime, \phi_1^\prime) \in C^{2,\alpha}(\Sigma_\varepsilon) \oplus C^{1,\alpha}(\Sigma_\varepsilon).
\]

Finally, define the projection

\[
\Pi_B : Z(\partial\Sigma_\varepsilon) \longrightarrow B
\]

\[
\phi \longrightarrow \phi_B,
\]

which has nullspace \( \tilde{B} \oplus [C^{2,\alpha}(\partial\Sigma_\varepsilon) \oplus C^{1,\alpha}(\partial\Sigma_\varepsilon)]^\prime \). Both \( \Pi_B \) and the space on which it acts depend on \( \varepsilon \), but we omit this from the notation for simplicity. We let \( \Pi_{B,0} \) denote the projection sending \( \phi \in Z(\partial\Sigma_\varepsilon) \) to \( \phi_B + \phi_0^\prime \).

Finally, for \( f \in C^{0,\alpha}_\mu(\Sigma) \) and \( \phi_0^\prime \in C^{2,\alpha}(\partial\Sigma_\varepsilon) \), consider the boundary problem

\[
\begin{align*}
Lu & = f|_{\Sigma_\varepsilon} \quad \text{in} \quad \Sigma_\varepsilon \\
\Pi_{B,0}(C_\varepsilon(u)) & = \phi_0^\prime \quad \text{on} \quad \partial\Sigma_\varepsilon.
\end{align*}
\]

**Proposition 19** There exists an \( \varepsilon_0 > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then (59) has a unique solution \( u \in X(\Sigma_\varepsilon) \). Furthermore, there exists a constant \( c > 0 \), independent of \( \varepsilon \in (0, \varepsilon_0) \), such that

\[
\|u\|_{2,\alpha,\mu} \leq c \left( \varepsilon^{\mu/4} \|\phi_0^\prime\|_{2,\alpha} + \|f\|_{0,\alpha,\mu-2} \right).
\]
Proof: To start, use a bounded extension operator in the $C^{2,\alpha}(\partial \Sigma_\varepsilon) \to C^{2,\alpha}_\mu(\Sigma)$ in the usual way to reduce to the case where $\phi_0'' = 0$.

Our first claim is that when $\varepsilon$ is sufficiently small, the only solution of the problem (54) with $f = 0$ and $\phi_0'' = 0$ is $u = 0$. Granting this for the moment, then because the range of $\Pi_B$ is Lagrangian with respect to the forms $\omega_\varepsilon$, the problem (59) is self-adjoint, hence the inhomogeneous problem (59) has a unique solution.

Our second claim is that there exists a constant $c > 0$, independent of $\varepsilon > 0$ such that

$$\|u\|_{2,\alpha,\mu} \leq c\|f\|_{2,\alpha,\mu-2}. \quad (60)$$

The proposition follows from these two claims. As will be seen from the following argument, the proof of the first claim is a special case, or at least follows directly from, the proof of the second claim. Therefore, we concentrate on proving the estimate (60), and may be proved by contradiction as well. If it were false, then we could pass to a subsequence to obtain a limit $\bar{u}_\ell$ as well. Assume that the maximum of $e^{-\mu s}|\bar{u}_\ell|$ occurs at some point $p_\ell = (s_\ell, \theta_\ell)$. Then $s_\ell > s_m$, where $s_m$ gives the cylindrical coordinate of the ‘inner’ boundary of $E_m$. Define now

$$\bar{w}_\ell(s, \theta) = e^{-\mu s_\ell}w_\ell(s + s_\ell, \theta),$$

which is defined on $(s_m - s_\ell, S_\varepsilon/8 - s_\ell) \times S^1$ and satisfies $e^{-\mu s}|\bar{w}_\ell| \leq 1$ on this domain with the supremum of 1 attained at $(0, \theta_\ell)$, and of course vanishes at both boundary components of its domain of definition.

Pass to a subsequence to obtain a limit $\bar{w} \in C^{2,\alpha}_\mu$ which is defined on $(\zeta^-, \zeta^+) \times S^1$ for some $\zeta^\pm \in \mathbb{R} \cup \{\pm \infty\}$, and satisfies Dirichlet conditions at either of the boundaries if they remain finite, i.e. if $|\zeta^\pm| < \infty$. If $s_\ell$ had remained bounded, so $\zeta_- > -\infty$, then $L\bar{w} = 0$ on the infinite end $[-\zeta_-, \infty) \times S^1$ and vanishes at $s = -\zeta_-$, so that $\bar{w}$ would be in the nullspace of $L$, which is a contradiction. Therefore $s_\ell \to \infty$, $\zeta_- = -\infty$, and $(\partial_\theta^2 + \partial_s^2)\bar{w} = 0$. But solutions of this equation are sums of exponentials, multiplied by eigenfunctions on the cross-section, and one checks readily that no there is no solution such that $|\bar{w}| \leq e^{\mu s_\ell}$ either on all of $\mathbb{R}$ or on $(-\infty, \zeta^+)$. This gives a contradiction again and our claim is proved.

By subtracting off $\chi \bar{u}_\ell$ from $u_\ell$, where $\chi$ is smooth, vanishes for $s \leq s_m$ and equals one for $s \geq s_m + 1$ on each $E_m$, we have now reduced to the case where $f_\ell$ is compactly
supported. We do not change the names of these adjusted functions, and still assume that they satisfy (51).

Now we continue in a very similar fashion as before. Select a smooth positive function \( d \) on \( \Sigma \) which agrees on each end with the coordinate function \( s \), so that

\[
\sup_{\Sigma} e^{-\mu d}|u_\ell| = 1.
\]

This supremum is attained at some point \( p_\ell \in \Sigma_{\epsilon_\ell} \).

If \( p_\ell \) were to tend to infinity, then we could assume that it did so within one end \( E_m \). We then apply an argument identical to the one just above to reach a contradiction.

The final case is when \( p_\ell \) remains in some fixed compact set in \( \Sigma \), which means that we can extract a subsequence converging to some function \( u \in C^{2,\alpha}_\mu(\Sigma) \) such that \( Lu = 0 \) and \( u \not\equiv 0 \). Hence \( u \) is an element of the bounded nullspace \( B \) (and so in particular, lies in \( W \oplus C^{2,\alpha}_\mu(\Sigma) \)). Let \( C_0(u) = \phi \in B \subset W \). Then, choose any \( \psi \in \tilde{B} \) and an extension \( v \in W + C^{2,\alpha}_\mu(\Sigma) \) of it to \( \Sigma \) so that \( \psi = C_0(v) \), by definition, and \( Lv \) is compactly supported.

Then we compute that

\[
\omega(\phi, \psi) = \int_{\Sigma} ((Lu)v - u(Lv))
\]

\[
= \lim_{\ell \to \infty} \int_{\Sigma_{\epsilon_\ell}} ((Lu_\ell)v - u_\ell(Lv)) = \lim_{\ell \to \infty} \omega_{\epsilon_\ell}(C_{\epsilon_\ell}(u_\ell), C_{\epsilon_\ell}(v)) = 0.
\]

The first equality is obvious, while for the second one, the compact supports of \( Lu_\ell = f_\ell \) and \( Lv \) ensure the convergence. The third is again obvious, while the final equality holds because \( \tilde{B} \) is Lagrangian, so the contribution from \( Z' \) is zero, while the contribution from \( Z'' \) tends to zero with \( \epsilon \). But \( \phi \in B \) while \( \psi \) is an arbitrary element of \( \tilde{B} \), and the symplectic pairing between \( B \) and \( \tilde{B} \) is nondegenerate. This proves that \( \phi \) and hence \( u \) vanishes identically, which is a contradiction.

The proof is complete in all cases. \( \square \)

### 8.2 Deformations of \( \Sigma_\epsilon \)

Now we shall take up the task of defining slightly different truncations of the scaled surface \( \epsilon \Sigma \), which we shall call \( \Sigma_\epsilon, P \) (the \( P \) here refers to a parameter set which we shall define below), which will be more convenient later. In the next subsection we shall also consider the Jacobi operators which correspond to writing nearby surfaces as graphs using vector fields which are small deformations of the normal vector field on \( \Sigma_\epsilon, P \).

Fix one end \( E_\ell \) of \( \Sigma \), and recall its parametrization (56). The end \( \epsilon E_\ell \) can therefore be paramerizered as

\[
x_\ell(s, \theta) = \epsilon_\ell (\cosh s \cos \theta + O(e^{-3s}), \cosh s \sin \theta + O(e^{-3s}), \epsilon + O(e^{-2s})).
\]

(62)

where here and later, we use the notation

\[
\epsilon_\ell \equiv a_\ell \epsilon.
\]

There is a Delaunay surface \( D_\ell \), with Delaunay parameter \( \epsilon_\ell \), which ‘best fits’ the model catenoid for \( \epsilon E_\ell \) near the region where \( s = S_{\epsilon_\ell}/8 \). It has the parametrization

\[
x_{D_\ell}(s, \theta) = (\tau_\ell e^{\sigma_\ell(s)} \cos \theta, \tau_\ell e^{\sigma_\ell(s)} \sin \theta, k_\ell(s)),
\]

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and has unit normal
\[ \nu_{D_\ell}(s, \theta) = (-\tau_\ell \cosh(\sigma_\ell(s)) \cos \theta, -\tau_\ell \cosh(\sigma_\ell(s)) \sin \theta, (\sigma_\ell)_s(s)). \]

The deformations of \( D_\ell \) are parametrized by translations orthogonal to the axis, translations along this axis, rotations of this axis and finally changes in the Delaunay parameter. We label these by
\[ \mathcal{P}_\ell = (t_1^\ell, t_2^\ell, r_1^\ell, r_2^\ell, d_\ell, \delta^\ell), \]
respectively. All these parameters lie in some small neighborhood of zero. (The \( r \)'s are identified with some small neighborhood of the identity in the space of rotations fixing the \( x_3 \) axis; the exact manner is not important, but to be definite, we suppose that the diffeomorphism is given by the exponential map in \( SO_3 \) orthogonal to the copy of \( SO_2 \) which is the stabilizer of that axis, followed by the projection to \( SO_3/SO_2 \).) The full parameter set for all ends of \( \varepsilon \Sigma_\varepsilon \) is
\[ \mathcal{P} = (\mathcal{P}^1, \ldots, \mathcal{P}^k), \]
We also set
\[ \tilde{t} = (t_1^1, \ldots, t_2^k) \in \mathbb{R}^{2k}, \quad \tilde{r} = (r_1^1, \ldots, r_2^k) \in \mathbb{R}^{2k}, \]
\[ \tilde{d} = (d^1, \ldots, d^k) \in \mathbb{R}^k, \quad \text{and} \quad \tilde{\delta} = (\delta^1, \ldots, \delta^k) \in \mathbb{R}^k, \]
and the rigid motion determined by \((t_1^\ell, t_2^\ell, r_1^\ell, r_2^\ell, d_\ell, \delta^\ell)\) will be denoted \( R_{(t^\ell, r^\ell, d^\ell, \delta^\ell)} \). The norm on these parameter sets which arises naturally below is given by
\[ \|\mathcal{P}\| = \|\tilde{t}, \tilde{r}, \tilde{d}, \tilde{\delta}\| \equiv \varepsilon^{1/4}\|\tilde{t}\| + \varepsilon^{3/4}\|\tilde{r}\| + \|\tilde{d}\| + (\log \frac{1}{\varepsilon}) \|\tilde{\delta}\|. \]

The Delaunay surface associated to the set of (small) deformation parameters \( \mathcal{P}_\ell \) will be denoted \( D_{\mathcal{P}_\ell} \), and its induced parametrization and unit normal will be called \( x_{\mathcal{P}_\ell} \) and \( \nu_{\mathcal{P}_\ell} \), respectively. This surface has Delaunay parameter \( \varepsilon_\ell + \delta^\ell \).

We come now to the main point, which is to write a neighbourhood of \( \varepsilon (E_\ell \cap \partial \Sigma_\varepsilon) \) as a normal graph over each \( D_{\mathcal{P}_\ell} \), and to obtain estimates on the graph function.

**Proposition 20** Fix \( \kappa \in (1, \frac{3}{2}) \). Then, for all parameter sets \( \mathcal{P} \) with \( \|\mathcal{P}\| \leq \varepsilon^\kappa \), there is a diffeomorphism \( \Psi(s, \theta) = (s', \theta') \) from \((-2 + S_{\varepsilon_\ell}/8, S_{\varepsilon_\ell}/8) \times S^1 \) onto its image, satisfying
\[ \|\Psi(s, \theta) - (s, \theta)\| = O(\varepsilon^{\kappa - 1}), \]
and we have
\[ x_\ell(s, \theta) = x_{\mathcal{P}_\ell}(s', \theta') + \hat{w}_0(s', \theta') \nu_{\mathcal{P}_\ell}(s', \theta'), \quad (63) \]
for all \( \varepsilon < \varepsilon_0 \), where \( \varepsilon_0 \) depends only on \( \kappa \). The graph function \( \hat{w}_0 \) here is of the form
\[ \hat{w}_0(s', \theta') = -\frac{1}{\cosh s'}(t_1^\ell \cos \theta' + t_2^\ell \sin \theta') - (r_1^\ell \cos \theta' + r_2^\ell \sin \theta') \varepsilon_\ell \cosh s' + d^\ell + \delta^\ell s' + O(\varepsilon^{3/2} + \varepsilon^{2\kappa - 1}). \]
In other words, we are writing a neighbourhood of \( \epsilon (E_\ell \cap \partial \Sigma_\xi) \) as a normal graph over each of the family of nearby model Delaunay surfaces, up to the reparametrization given by the diffeomorphism \( \Psi \).

**Proof:** This follows from a computation similar to the one we have already done in the proof of Proposition \([\text{10}](\text{ sources })\). Recall that in the range \( s \in [-4 + S_\xi/8, 4 + S_\xi/8] \), we have the expansions

\[
\begin{align*}
k(s) &= \epsilon s + O(\epsilon^{3/2}), \\
\tau c^{\sigma(s)} &= \epsilon \cosh s + O(\epsilon^{5/4}),
\end{align*}
\]

and

\[
\begin{align*}
\tau \cosh \sigma(s) &= \frac{1}{\cosh s} + O(\epsilon^{3/4}), \\
\partial_s \sigma(s) &= 1 + O(\epsilon^{1/2}),
\end{align*}
\]

which follow from \([\text{23}](\text{ source })\). Here and below \( O(\epsilon^\gamma) \) will denote functions of \((s, \theta)\) all derivatives of which are bounded by constant multiples of \( \epsilon^\gamma \).

It will be most convenient to apply the transformation \( R^{-1}_{(\ell', \epsilon', d')} \) to both sides of \([\text{38}](\text{ source })\). On the one hand, from \([\text{50}](\text{ source })\), the parametrization for \( R^{-1}_{(\ell', \epsilon', d')} \) is given by

\[
(s, \theta) \mapsto \left( \epsilon \ell' \cosh s \cos \theta - t_1^{\ell'} + O(\epsilon^{\kappa + 1/4} \log \epsilon), \\
\epsilon \ell' \cosh s \sin \theta - t_2^{\ell'} + O(\epsilon^{\kappa + 1/4} \log \epsilon), \\
(r_1^{\ell'} \cos \theta + r_2^{\ell'} \sin \theta) \epsilon \ell' \cosh s + \epsilon \ell' s - d^{\ell'} + O(\epsilon^{3/2}) \right)
\]

for \( s \) in this range.

On the other hand, \( R^{-1}_{(\ell', \epsilon', d')} \) is parameterized by

\[
(s', \theta') \mapsto \left( \left( (\epsilon \ell + \delta^{\ell}) \cosh s' - \frac{1}{\cosh s} \hat{w}_0(s', \theta') \right) \cosh \theta' + O(\epsilon^{5/4}) + O(\epsilon^{3/4}) \hat{w}_0(s', \theta'), \\
(\epsilon \ell + \delta^{\ell}) \cosh s' - \frac{1}{\cosh s} \hat{w}_0(s', \theta') \right) \sin \theta' + O(\epsilon^{5/4}) + O(\epsilon^{3/4}) \hat{w}_0(s', \theta'), \\
(\epsilon \ell + \delta^{\ell}) s' + \hat{w}_0(s', \theta') + O(\epsilon^{3/2}) + O(\epsilon^{1/2}) \hat{w}_0(s', \theta') \right),
\]

again for \( s' \) in this range.

Equating the third coordinates, we already find that \( \hat{w}_0(s', \theta') = \epsilon (s - s') + O(\epsilon^{\kappa}) \). Assuming that \( |s' - s| \) is at least bounded, this gives \( \hat{w}_0(s', \theta') = O(\epsilon) \). Similar estimates hold for its derivatives. Now, writing out the equality of the three coordinates in turn gives

\[
\begin{align*}
\epsilon \ell' \cosh s \cos \theta &= t_1^{\ell'} + ((\epsilon \ell + \delta^{\ell}) \cosh s' - \frac{1}{\cosh s'} \hat{w}_0(s', \theta')) \cos \theta' + O(\epsilon^{5/4}) \\
\epsilon \ell' \cosh s \sin \theta &= t_2^{\ell'} + ((\epsilon \ell + \delta^{\ell}) \cosh s' - \frac{1}{\cosh s'} \hat{w}_0(s', \theta')) \sin \theta' + O(\epsilon^{5/4}),
\end{align*}
\]

and

\[
(r_1^{\ell'} \cos \theta + r_2^{\ell'} \sin \theta) \epsilon \ell' \cosh s + \epsilon \ell' s - d^{\ell'} = (\epsilon \ell + \delta^{\ell}) s' + \hat{w}_0(s', \theta') + O(\epsilon^{3/2} \log \epsilon).
\]

Using the preliminary estimate on \( \hat{w}_0 \), we conclude that

\[
|s - s'| \leq c \epsilon^{\kappa - 1}, \quad |\theta' - \theta| \leq c \epsilon^{\kappa - 1},
\]

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and then, reinserting this information back into the third equality, that

$$|\hat{w}_0| \leq c\varepsilon^{\kappa},$$

along with its derivatives. The third identity gives

$$\hat{w}_0(s', \theta') = -\varepsilon_\ell (s - s') - (r_1^\ell \cos \theta' + r_2^\ell \sin \theta') \varepsilon_\ell \cosh s' + d^\ell + \delta^s s' + O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}),$$

while from the first two identities we get

$$\varepsilon_\ell (\cosh s - \cosh s') = t_1^\ell \cos \theta + t_2^\ell \sin \theta + O(\varepsilon^{5/4} + \varepsilon^{2\kappa-5/4}).$$

This leads finally to

$$\hat{w}_0(s', \theta') = -\frac{1}{\cosh s}(t_1^\ell \cos \theta' + t_2^\ell \sin \theta') - (r_1^\ell \cos \theta' + r_2^\ell \sin \theta') \varepsilon_\ell \cosh s' + d^\ell + \delta^s s' + O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}),$$

which is the desired expansion.

\[\square\]

We may now define the deformation \(\tilde{\Sigma}_{\varepsilon, \mathcal{P}}\) when the set of deformation parameters \(\mathcal{P}\) satisfies \(\|\mathcal{P}\| \leq \varepsilon^\kappa\). Choose \(\varepsilon_0\) sufficiently small that \(S_{\varepsilon_0}/8 - 2 > s_\ell\) for each \(\varepsilon < \varepsilon_0\). Then for any such \(\varepsilon\), define \(\tilde{\Sigma}_{\varepsilon, \mathcal{P}}\) as the union of the central compact portion of \(\varepsilon \Sigma_{\varepsilon}\) and the portion of each end \(\varepsilon E_\ell\) for \(s_\ell \leq s \leq -1 + S_{\varepsilon_\ell}/8\) and by the graph of

\[(s', \theta') \rightarrow \hat{x}_{\mathcal{P}_\ell}(s', \theta') + \hat{w}_0(s', \theta') \nu_{\mathcal{P}_\ell}(s', \theta'),\]

for \(-2 + S_{\varepsilon_\ell}/8 \leq s \leq S_{\varepsilon_\ell}/8\).

**Remark 2** These definitions are compatible in the region of overlap, and all we have done is to slightly alter the boundary of \(\tilde{\Sigma}_{\varepsilon, \mathcal{P}}\) so that it conforms better to the coordinates \((s', \theta')\).

### 8.3 Deformed Jacobi operators

For any small parameter set \(\mathcal{P}\), we define on the surface \(\tilde{\Sigma}_{\varepsilon, \mathcal{P}}\) a vector field \(\tilde{\nu}\) which is the unit normal vector field away from the boundary, and which is a perturbation of this unit normal near to the boundary. More specifically, write \(\varepsilon E_\ell\) as the graph

\[(s, \theta) \rightarrow \hat{x}_{\mathcal{P}_\ell}(s, \theta) + \hat{w}_0(s, \theta) \nu_{\mathcal{P}_\ell}(s, \theta),\]

for all \(s \in [-2 + S_{\varepsilon_\ell}/8, S_{\varepsilon_\ell}/8]\). Let \(\eta(s)\) be a smooth cutoff function equal to 1 for \(s \leq -3/2\) and vanishing for \(s \geq -1\). Then, for all \(s \in [-2 + S_{\varepsilon}/8, S_{\varepsilon}/8]\), the vector \(\tilde{\nu}(s, \theta)\) is defined to be the unit normal to the surface parameterized by

\[(s, \theta) \rightarrow \mathcal{R}_{(\iota^\ell, \iota^\ell)}(\hat{x}_{\mathcal{D}_\ell}(s, \theta) + \eta(s + S_{\varepsilon_\ell}/8) \hat{w}_0(s, \theta) \nu_{\mathcal{D}_\ell}(s, \theta)),\]

As desired, \(\tilde{\nu}\) is still the unit normal to \(\tilde{\Sigma}_{\varepsilon, \mathcal{P}}\) when \(s \leq -3/2 - S_{\varepsilon_\ell}/8\), and equals \(\nu_{\mathcal{P}_\ell}(s, \theta)\) when \(s \in [-1 + S_{\varepsilon_\ell}/8, S_{\varepsilon_\ell}/8]\).
Any surface near to $\tilde{\Sigma}_{\varepsilon,P}$ may be parameterized by

$$\tilde{\Sigma}_{\varepsilon,P} \ni p \mapsto p + w(p)\tilde{\nu}(p),$$

for some scalar valued function $w$. We need to consider the equation which $w$ must satisfy in order for this surface to have constant mean curvature one, which we shall do in a slightly more general context.

Let $S$ be a regular orientable surface, with unit normal vector field $\nu$. Suppose that $\bar{\nu}$ is another unit vector field along $S$ which is nowhere tangential. By the inverse function theorem, for any $p_0 \in S$ there are neighbourhoods $U$ and $V$ near $(p_0,0)$ in $S \times \mathbb{R}$ and a diffeomorphism $(\phi(p,s),\psi(p,s))$ from $U$ to $V$ such that

$$p + sv(p) = \phi(p,s) + \psi(p,s)\bar{\nu}(\phi(p,s)).$$  \hspace{1cm} (64)

Here $\phi(p,0) = p$ and $\psi(p,0) = 0$. To determine the first order Taylor series of these functions in $s$, differentiate (64) with respect to $s$ and set $s = 0$. This gives

$$\nu(p) = \frac{\partial \phi}{\partial s}(p,0) + \frac{\partial \psi}{\partial s}(p,0)\bar{\nu}(p),$$

and so, taking the normal component of this, we get

$$1 = \frac{\partial \psi}{\partial s}(p,0)\nu(p) \bar{\nu}(p), \quad \text{or} \quad \frac{\partial \psi}{\partial s}(p,0) = 1/(\nu(p) \cdot \bar{\nu}(p)).$$

Hence

$$\psi(p,s) = \frac{s}{\nu(p) \cdot \bar{\nu}(p)} + O(s^2).$$

On the other hand, taking the tangential component and using this expansion of $\psi$ yields

$$0 = \frac{\partial \phi}{\partial s}(p,0) + \frac{s}{\nu(p) \cdot \bar{\nu}(p)}\bar{\nu}_t(p),$$

where $\bar{\nu}_t(p)$ is the tangential component of $\bar{\nu}$. Thus

$$\phi(p,s) = p - \frac{s}{\nu(p) \cdot \bar{\nu}(p)}\bar{\nu}_t(p) + O(s^2).$$

Next, any surface which is $C^2$ close to $S$ can be parameterized either as a normal graph of some function $w$ over $S$, using the vector field $\nu$, or as a graph of a different function $\bar{w}$ using the vector field $\bar{\nu}$. These functions are related by

$$p + w(p)\nu(p) = \bar{p} + \bar{w}(\bar{p})\bar{\nu}(\bar{p}) = \phi(p,w(p)) + \psi(p,w(p))\bar{\nu}(\phi(p,w(p))).$$

Using the expansions above, we see that $\bar{w}(p) = w(p)/(\nu(p) \cdot \bar{\nu}(p)) + O(\|w\|^2)$.

The mean curvature operators on these two functions, which we call $H_{\nu,w}$ and $H_{\bar{\nu},\bar{w}}$, respectively, are related by

$$H_{\bar{\nu},\bar{w}}(\bar{p}) = H_{N,w}(p).$$  \hspace{1cm} (65)

Differentiating this with respect to $\bar{w}$ and setting $\bar{w} = 0$, we get

$$D_{\bar{w}}H_{\bar{\nu},0}(u) = D_wH_{\nu,0}((\bar{\nu} \cdot \nu)u) + (\nabla H_{\nu,0} \cdot \bar{\nu}_t)u,$$  \hspace{1cm} (66)
for any scalar function $u$. In the special case where the surface $S$ has constant mean curvature, this reduces to

$$Lu \equiv D_{w}H_{e,0}(u) = D_{w}H_{e,0}((\nu \cdot \nu) u) \equiv L((\nu \cdot \nu) u).$$

(67)

We apply the previous computation to the present situation. Denote by $\tilde{L}_{\epsilon,P}$ the linearized mean curvature operator about $\tilde{\Sigma}_{e,P}$. Away from $\partial \tilde{\Sigma}_{e,P}$, we have

$$\tilde{L}_{\epsilon,P} = \frac{1}{\epsilon^2} L,$$

where $L = \Delta_{\Sigma} + |A_{\Sigma}|^2$ is the operator we have studied in detail. Near $\partial \tilde{\Sigma}_{e,P}$ the structure of $\tilde{L}_{\epsilon,P}$ is described by the next result, the proof of which follows from the expansions given in Proposition 20.

**Lemma 1** In $\epsilon E_\ell$, we can write

$$\tilde{L}_{\epsilon,P} = \frac{1}{\epsilon^2} L + \hat{L}_{\epsilon,P},$$

where $\hat{L}_{\epsilon,P}$ is a second order linear differential operator whose coefficients are supported in $[-2 + S_{\epsilon}/8, S_{\epsilon}/8] \times S^1$ and are bounded by $\frac{1}{\epsilon^2 e^{2s}} e^{\kappa - 1}$.

Also, following from the same ideas as in §7.1 is the simpler

**Lemma 2** In $\epsilon E_\ell$, the difference

$$\frac{1}{\epsilon^2} L - \frac{1}{\epsilon^2 \cosh^2 s} (\partial_s^2 + \partial_\theta^2)$$

is a second order linear differential operator, the coefficients of which are bounded by a constant times $\epsilon^{-2} e^{-4s}$ in $[s_{\ell}, S_{\epsilon}/8]$.

The proofs of both of these results are left to the reader.

From these lemmas, we can immediately generalize Proposition 13 to the deformed Jacobi operators on the surfaces $\tilde{\Sigma}_{e,P}$.

**Proposition 21** Fix $\mu$ with $1 < \mu < 2$. Then there exists an $\epsilon_0 > 0$, depending only on $\mu$, such that whenever $0 < \epsilon < \epsilon_0$, there exists a unique solution $w \in C_{\mu}^{2,\alpha}(\tilde{\Sigma}_{e,P})$ of the problem

$$\begin{cases}
\tilde{L}_{e,P} w = \frac{1}{\epsilon^2} f & \text{in } \tilde{\Sigma}_{e,P} \\
\Pi'' w = \phi'' & \text{on } \partial \tilde{\Sigma}_{e,P},
\end{cases}$$

(68)

for $f \in C_{\mu-2}^{0,\alpha}(\Sigma_{e})$ and $\phi'' = (\phi''_1, \ldots, \phi''_k) \in \Pi''(C^{2,\alpha}(S^1))^k$. The Green and Poisson operators will be denoted $\tilde{G}_{e,P}$, $\tilde{P}_{e,P}$, respectively. The linear maps

$$\begin{align*}
\tilde{G}_{e,P} : C_{\mu-2}^{0,\alpha}(\tilde{\Sigma}_{e,P}) & \rightarrow C_{\mu}^{2,\alpha}(\tilde{\Sigma}_{e,P}), \\
\epsilon^{-\mu/4} \tilde{P}_{e} : \Pi''(C^{2,\alpha}(S^1))^k & \rightarrow C_{\mu}^{2,\alpha}(\tilde{\Sigma}_{e,P}),
\end{align*}$$

are bounded uniformly as $\epsilon \rightarrow 0$. 

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Following the results of section §4.3, we also prove

**Corollary 7** Fix $1 < \mu < 2$. Then there exists a constant $c > 0$ and an $\varepsilon_0 > 0$, depending only on $\mu$, such that for $0 < \varepsilon < \varepsilon_0$, we have

$$||(|\tilde{P}_\varepsilon - \tilde{P}_0)(\phi'')||_{2,\alpha,\mu} \leq c \varepsilon^{\mu/4} \left(\varepsilon^{2-\mu}/4 + \varepsilon^{\kappa-1}\right) ||\phi||_{2,\alpha}.$$ 

Here, if $\phi'' = (\phi_1, \ldots, \phi_k) \in \Pi''(C^{2,\alpha}(S^1))^k$, the function $\tilde{P}_0(\phi'') = \tilde{w}_0$ is defined to be equal to $\eta(s - s_\ell)\tilde{w}_\ell$ on each end $\varepsilon E_\ell$ and 0 elsewhere, where $\eta$ is some cutoff function equal to 1 for $s > 0$ and equal to 0 for $s < 0$ and where $\tilde{w}_\ell$ is the unique solution, in $\left(C_2^{2,\alpha}((-\infty, S_{\varepsilon\ell}/8) \times S^1)\right)^k$, of the problem

\[
\begin{align*}
\Delta \tilde{w}_\ell &= 0 \quad &\text{in} \quad (-\infty, S_{\varepsilon\ell}/8) \times S^1 \\
\tilde{w}_\ell &= \phi''_\ell \quad &\text{on} \quad \{S_{\varepsilon\ell}/8\} \times S^1.
\end{align*}
\]

**Proof:** We start by solving, for each $\ell$,

\[
\begin{align*}
\Delta \tilde{w}_\ell &= 0 \quad &\text{in} \quad (-\infty, S_{\varepsilon\ell}/8) \\
\tilde{w}_\ell &= \phi''_\ell \quad &\text{on} \quad \{S_{\varepsilon\ell}/8\} \times S^1
\end{align*}
\]

There is a unique solution of this equation, which is in $C_2^{2,\alpha}((-\infty, S_{\varepsilon\ell}/8) \times S^1)$, and satisfies

$$||\tilde{w}_\ell||_{2,\alpha,2} \leq c \varepsilon^{1/2} ||\phi''||_{2,\alpha}.$$ 

Now truncate these solutions at $s = s_\ell$; this allows one to define $\tilde{w}_0$ globally on $\tilde{\Sigma}_{\varepsilon,P}$ by setting it equal to 0 elsewhere. From Lemmas 1 and 3 if follows that on each end $\varepsilon E_\ell$, the difference

$$\tilde{L}_{\varepsilon,P} \tilde{w}_0 - \frac{1}{\varepsilon^2 \cosh^2 s} \left(\partial_s^2 + \partial_\theta^2\right),$$

is a second order linear differential operator whose coefficients are sums of terms which are either bounded by a constant times $e^{-2s}$ or are supported in $[-2 + S_{\varepsilon\ell}/8, S_{\varepsilon\ell}/8]$ and bounded by a constant times $e^s$. Using this, we see that

$$||\tilde{L}_{\varepsilon,P} \tilde{w}_0||_{0,\alpha,\mu} \leq c(\varepsilon^{1/2} + \varepsilon^{\kappa-1+\mu/4}) ||\phi''||_{2,\alpha}.$$ 

The result then follows from Proposition 21. \qed

9 CMC surfaces near to the truncated $k$-noids

Just as we already did for Delaunay surfaces, we would like to analyze the family of surfaces which are close to each $\tilde{\Sigma}_{\varepsilon,P}$ and which have constant mean curvature 1. To this end, as in (45), we expand the mean curvature operator to see that, for any $\phi \in C^{2,\alpha}(\partial \tilde{\Sigma}_{\varepsilon,P})$, our problem reduces to solve the following boundary value problem

\[
\begin{align*}
\tilde{L}_{\varepsilon,P}w &= 1 + \tilde{Q}(w) \quad &\text{in} \quad \tilde{\Sigma}_{\varepsilon,P} \\
\Pi''(w) &= \phi'' \quad &\text{on} \quad \partial \tilde{\Sigma}_{\varepsilon,P}
\end{align*}
\]

(70)
Here
\[ \tilde{Q}(w) = \frac{1}{\varepsilon e^s} \tilde{Q} \left( \frac{w}{\varepsilon e^s}, \frac{\nabla w}{\varepsilon e^s}, \frac{\nabla^2 w}{\varepsilon e^s} \right), \] (71)
in each end \( \tilde{E}_\varepsilon \), collects all the terms of order higher than one in \( w \). The function \( \tilde{Q} \) has partial derivatives which are uniformly bounded. Denote by \( \tilde{w}_\varepsilon \) the solution of
\[ \begin{cases} \tilde{L}_{\varepsilon,p} \tilde{w}_\varepsilon = 0 & \text{in } \tilde{\Sigma}_{\varepsilon,p} \\ \Pi'' \tilde{w}_\varepsilon = \phi'' & \text{on } \partial \tilde{\Sigma}_{\varepsilon,p}, \end{cases} \]
which is given by Proposition 21. By the same Proposition, we can also solve
\[ \begin{cases} \tilde{L}_{\varepsilon,p} \tilde{w}_1 = 1 & \text{in } \tilde{\Sigma}_{\varepsilon,p} \\ \Pi'' \tilde{w}_1 = 0 & \text{on } \partial \tilde{\Sigma}_{\varepsilon,p}. \end{cases} \]
We find that
\[ ||\tilde{w}_1||_{2,\alpha,\mu} \leq c \varepsilon^{3/2 + \mu/4}. \]
Setting \( w = \tilde{w}_\varepsilon + \tilde{w}_1 + v \), then it remains to solve
\[ \begin{cases} \tilde{L}_{\varepsilon,p} v = \tilde{G}_\varepsilon (\tilde{w}_\varepsilon + \tilde{w}_1 + v) & \text{in } \tilde{\Sigma}_{\varepsilon,p} \\ \Pi'' v = 0 & \text{on } \partial \tilde{\Sigma}_{\varepsilon,p}. \end{cases} \]
It is sufficient to find a fixed point of the mapping
\[ \tilde{K}(v) \equiv \tilde{G}_\varepsilon, p \tilde{Q}(\tilde{w}_\varepsilon + \tilde{w}_1 + v), \] (72)
when \( \varepsilon \) is sufficiently small.

**Proposition 22** There exists a constant \( c_0 > 0 \) such that if \( ||\phi''||_{2,\alpha} \leq c_0 \varepsilon^{3/4} \), then
\[ ||\tilde{G}_\varepsilon (\tilde{Q}(\tilde{w}_\varepsilon + \tilde{w}_1))||_{2,\alpha,\mu} \leq c \varepsilon^{-3/4} \left( ||\phi''||_{2,\alpha} + \varepsilon^3 \right) \varepsilon^{\mu/4}, \]
and
\[ ||\tilde{G}_\varepsilon (\tilde{Q}(\tilde{w}_\varepsilon + \tilde{w}_1 + v_2) - \tilde{Q}(\tilde{w}_\varepsilon + \tilde{w}_1 + v_1))||_{2,\alpha,\mu} \leq \frac{1}{2} ||v_2 - v_1||_{2,\alpha,\mu}, \]
for all \( v_1, v_2 \) in \( \tilde{B}_{c_0} \equiv \{ v : ||v||_{2,\alpha,\mu} \leq c_0 \varepsilon^{(3+\mu)/4} \} \). Thus, \( \tilde{K} \) is a contraction mapping on the ball \( \tilde{B}_{c_0} \) into itself, and therefore has a unique fixed point \( v \) in this ball.

**Proof** : We use that
\[ ||\tilde{w}_\varepsilon||_{2,\alpha,[s,s+1]} \leq c e^{\mu(s-S_\varepsilon/8)} ||\phi''||_{2,\alpha} \leq c c_0 \varepsilon^{3/4} e^{\mu(s-S_\varepsilon/8)}, \] (73)
and also that
\[ ||\tilde{w}_1||_{2,\alpha,[s,s+1]} \leq c \varepsilon^{3/2} e^{\mu(s-S_\varepsilon/8)}. \] (74)
These estimates imply that on the end \( \varepsilon E_\ell \), for \( s \in [s_\ell, S_\varepsilon/8] \), we have
\[ e^{-\mu s} \left\| \frac{1}{\varepsilon e^s} \left( \frac{\tilde{w}_\varepsilon + \tilde{w}_0}{\varepsilon e^s} \right)^2 \right\|_{2,\alpha,[s,s+1]} \leq c \varepsilon^{-3/4} \left( ||\phi''||_{2,\alpha} + \varepsilon^3 \right) \varepsilon^{\mu/4}, \]
and then, from a Taylor expansion we get
\[ e^{-\mu s}||\tilde{Q}(\tilde{w}_x + \tilde{w}_t)||_{0, \alpha, [s, s+1]} \leq c \varepsilon^{-3/4} \left( ||\phi''||_{2, \alpha}^2 + \varepsilon^3 \right) \varepsilon^{\mu/4} \leq c \varepsilon^{(3-\mu)/4}. \]

On the other hand, on the compact piece, we simply have
\[ e^{-\mu s}||\tilde{Q}(\tilde{w}_x + \tilde{w}_t)||_{0, \alpha, [s, s+1]} \leq c \varepsilon^{-3/4} ||\phi''||_{2, \alpha}^2 \varepsilon^{\mu/4} \leq c \varepsilon^{(3-\mu)/4}. \]

The other estimate follows in the same way, and the proof is complete. \(\square\)

As in section 4.3 we finally obtain

**Corollary 8** There exists a constant \( c_0 > 0 \) and an \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \) and for any \( \phi'' \in \Pi'' (C^{2, \alpha}(S^1))^k \) with \( ||\phi''||_{2, \alpha} \leq c_0 \varepsilon^{3/4} \), the problem (70) has a unique solution \( w \). The mapping
\[ \Pi'' (C^{2, \alpha}(S^1))^k \ni \phi'' \rightarrow w \in C^{2, \alpha}_\mu(\tilde{\Sigma}_c), \]

is continuous and the solution \( w \) satisfies the estimates
\[ ||w||_{2, \alpha, \mu} \leq c \varepsilon^{\mu/4} (\varepsilon^{3/2} + ||\phi''||_{2, \alpha} + \varepsilon^{-3/4} ||\phi''||^2_{2, \alpha}) \] (75)

and
\[ ||(w - \Pi''w)(S_{\varepsilon}/8, \cdot)||_{2, \alpha} + ||\partial_s w - \Pi''w)(S_{\varepsilon}/8, \cdot)||_{1, \alpha} \]
\[ \leq c (\varepsilon^{3/2} + (\varepsilon^{\kappa-1} + \varepsilon^{(2-\mu)/4})) ||\phi''||_{2, \alpha} + \varepsilon^{-3/4} ||\phi''||^2_{2, \alpha}. \] (76)

Finally, if \( \tilde{w}_0 = \tilde{P}_0(\phi'') \in C^{2, \alpha}_\mu(\tilde{\Sigma}_c) \) as in Corollary 3, then
\[ ||w - \tilde{w}_0||_{2, \alpha, \mu} \leq c \varepsilon^{-\mu/4} (\varepsilon^{3/2} + (\varepsilon^{\kappa-1} + \varepsilon^{(2-\mu)/4})) ||\phi''||_{2, \alpha} + \varepsilon^{-3/4} ||\phi''||^2_{2, \alpha}. \] (77)

The proof is identical to that of Corollary 3, and so we omit it.

**10 Matching the Cauchy data**

We have now established that, given any set of parameters \( \mathcal{P} = (\tilde{t}, \tilde{r}, \tilde{d}, \tilde{\delta}) \) satisfying \( \|\mathcal{P}\| \leq \varepsilon^\kappa \), and for any \( \phi'' \in \Pi'' (C^{2, \alpha}(S^1))^k \), we can solve the equations
\[ \begin{cases} 
 L_{\varepsilon t + \delta} w_t = \tilde{Q}(w_t) & \text{in } [S_{\varepsilon}/8, +\infty) \times S^1 \\
 \Pi'' w_t = \phi'' & \text{on } \{S_{\varepsilon}/8\} \times S^1 
\end{cases} \] (78)

and
\[ \begin{cases} 
 \tilde{L}_{\varepsilon, \mathcal{P}} w_{\mathcal{K}} = 1 + \tilde{Q}(w_0) & \text{in } \Sigma_{\varepsilon, \mathcal{P}} \\
 \Pi'' w_{\mathcal{K}} = \phi'' & \text{on } \partial \Sigma_{\varepsilon, \mathcal{P}}, 
\end{cases} \] (79)

when \( \varepsilon \) is sufficiently small. Thus we may define the mappings
\[ \mathcal{S}_\varepsilon : \Pi'' (C^{2, \alpha}(S^1))^k \ni \phi'' \rightarrow \]
\[ (\partial_s w_1(S_{\varepsilon}/8, \cdot), \ldots, \partial_s w_k(S_{\varepsilon}/8, \cdot)) \in (C^{1, \alpha}(S^1))^k, \]

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and
\[ T_{\varepsilon,P} : \Pi'' (C^{2,\alpha}(S^1))^k \ni \phi'' \mapsto \] 
\[ (\partial_s(\tilde{w}_0 + w_k)|_{\varepsilon E_1}(S_{\varepsilon}/8, \cdot), \ldots, \partial_s(\tilde{w}_0 + w_k)|_{\varepsilon E_k}(S_{\varepsilon}/8, \cdot)) \in (C^{1,\alpha}(S^1))^k, \]
where \( \tilde{w}_0 \) is defined in Proposition 24. These would be the Dirichlet-to-Neumann mappings for the two nonlinear problems (78) and (79), save for the fact that the low eigen-components of the Dirichlet data are not specified.

It follows from Corollary 3 and Corollary 8 that, for \( c_0 > 0 \) small enough, these mappings are well defined from the ball of radius \( c_0 \varepsilon^{3/4} \) in \((C^{2,\alpha}(S^1))^k\) into the ball of radius \( c_0 \varepsilon^{3/4} \) in the space \((C^{1,\alpha}(S^1))^k\).

**Proposition 23** There exists a constant \( \tilde{c}_0 > 0 \) such that in the ball of radius \( c_0 \varepsilon^{3/4} \) in \((C^{2,\alpha}(S^1))^k\) there is a unique \( \phi'' \) which satisfies the equation
\[ \Pi'' [\Pi''(\phi'')] = \Pi'' [T_{\varepsilon,P}(\phi'')] \]  
(80)
This solution satisfies
\[ \|\phi''\|_{2,\alpha} \leq c(\varepsilon^{2\kappa-1} + \varepsilon^{3/2}), \]
for some constant \( c > 0 \) independent of \( \varepsilon \).

**Proof:** By Corollaries 3 and Proposition 20, we may approximate these partial Dirichlet-to-Neumann maps by the corresponding maps for the Laplacian on the cylinder \( \mathbb{R} \times S^1 \). More specifically,
\[ \Pi'' [S_{\varepsilon,P}(\phi'')] = (\partial_s w_0^1(S_{\varepsilon}/8, \cdot), \ldots, \partial_s w_0^k(S_{\varepsilon}/8, \cdot)) + O(\varepsilon^{1/2} + \varepsilon^{(6-3\mu)/4})\|\phi''\|_{2,\alpha} + O(\varepsilon^{-3/4})\|\phi''\|_{2,\alpha}^2, \]
and similarly
\[ \Pi'' [T_{\varepsilon,P}(\phi'')] = (\partial_s w_0|_{\varepsilon E_1}(S_{\varepsilon}/8, \cdot), \ldots, \partial_s w_0|_{\varepsilon E_k}(S_{\varepsilon}/8, \cdot)) + O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}) + O(\varepsilon^{\kappa-1} + \varepsilon^{(2-\mu)/4})\|\phi''\|_{2,\alpha} + O(\varepsilon^{-3/4})\|\phi''\|_{2,\alpha}^2, \]
where \( w_0^k = P_{\varepsilon}(\phi''_k) \) and \( \tilde{w}_0 = \tilde{P}_{\varepsilon,P}(\phi'') \). We are using here that \( \Pi'' \tilde{w}_0 = O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}) \). Therefore we must find \( \phi'' \) such that
\[ (\partial_s(w_0^1 - \tilde{w}_0)|_{\varepsilon E_1}(S_{\varepsilon}/8, \cdot), \ldots, \partial_s(w_0^k - \tilde{w}_0)|_{\varepsilon E_k}(S_{\varepsilon}/8, \cdot) = F(\varepsilon, \mathcal{P}, \phi''), \]
where
\[ F = O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}) + O(\varepsilon^{1/2} + \varepsilon^{\kappa-1} + \varepsilon^{(2-\mu)/4})\|\phi''\|_{2,\alpha} + O(\varepsilon^{-3/4})\|\phi''\|_{2,\alpha}^2. \]

To see that this equation has a solution, we first note that the corresponding homogeneous linear problem has a unique solution within the range of \( \Pi'' \), namely \( \phi'' = 0 \). The linear subspaces of Cauchy data for the Laplacian on the two half-cylinders \((-\infty, S_{\varepsilon}/8) \times S^1 \) and \([S_{\varepsilon}/8, \infty) \times S^1 \), restricted to the range of \( \Pi'' \), form a transversal Fredholm pair. This implies that
\[ S_0 - T_{0,\mathcal{P}} \]

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is invertible, where $S_0$ and $T_{0,P}$ are the Dirichlet-to-Neumann operators for these linear problems. It is also true that this operator is a first order pseudodifferential operator. This implies that its inverse is compact. Our problem now reduces to

$$\phi'' = (S_0 - T_{0,P})^{-1} F(\varepsilon, P, \phi'').$$

The operator on the right is compact, by the remarks above. Furthermore, for $\varepsilon$ sufficiently small, it maps the ball of radius $c_0 \varepsilon^{3/4}$ to itself, because all the terms in $F$ decay faster (in $\varepsilon$) than $\varepsilon^{3/4}$. Hence, for $\varepsilon$ sufficiently small, for every choice of parameter set $P$ with $\|P\| \leq \varepsilon^\kappa$, this map must have a unique fixed point.

This proposition allows us to reduce the problem, at last, to a finite dimensional one. For every $\varepsilon$ sufficiently small and $P$ with $\|P\| \leq \varepsilon^\kappa$, associate the unique element $\phi''$ in the range of $\Pi''(C^2,\alpha(S^1))$. The equation

$$S_\varepsilon(\phi'') = T_{\varepsilon,P}(\phi'')$$

reduces to a system of $k$ nonlinear equations of the form

$$-\frac{1}{\cosh S_{\varepsilon\ell}}(t^\ell_1 \cos \theta + t^\ell_2 \sin \theta) - (r^\ell_1 \cos \theta + r^\ell_2 \sin \theta)\varepsilon \cosh S_{\varepsilon\ell} + d^\ell + \delta^\ell S_{\varepsilon\ell},$$

$$\frac{\sinh S_{\varepsilon\ell}}{\cosh^2 S_{\varepsilon\ell}}(t^\ell_1 \cos \theta + t^\ell_2 \sin \theta) - (r^\ell_1 \cos \theta + r^\ell_2 \sin \theta)\varepsilon \sinh S_{\varepsilon\ell} + \delta^\ell$$

$$= O(\varepsilon^{3/2} + \varepsilon^{2\kappa-1}).$$

Because of the restriction on the norm of $P$, and because

$$\varepsilon^{3/2} + \varepsilon^{2\kappa-1} = o(\varepsilon^\kappa),$$

we may again conclude that this equation has a solution. This ends the proof of our result.

11 The nondegeneracy of the solutions

We now show that for $\varepsilon$ sufficiently small, the solutions we have constructed above are nondegenerate in the sense defined in [7]. This condition ensures the smoothness of the moduli spaces $M_{g,k}$ near $\Sigma_\varepsilon$. We begin by recalling this notion of nondegeneracy.

**Definition 3** The constant mean curvature surface $\Sigma_\varepsilon \in M_{g,k}$ is nondegenerate if the linearization of the mean curvature operator about $\Sigma_\varepsilon$ is injective on the function space $C^{2,\alpha}_\delta(\Sigma_\varepsilon)$ for all $\delta < 0$.

Here for $r \in \mathbb{N}$, $\alpha \in [0,1)$ and $\delta \in \mathbb{R}$, $C^{r,\alpha}_\delta(\Sigma_\varepsilon)$ is defined to be the space of functions $\phi \in C^{r,\alpha}(\Sigma_\varepsilon)$ which can be written on each end of $\Sigma_\varepsilon$ as $e^{\delta s}$ times a function $\psi$ with $\psi \in C^{r,\alpha}(\mathbb{R}^+ \times S^1)$.

First notice that it is sufficient to prove that, for $\varepsilon$ small enough, the Jacobi operator $L$ is injective on $C^{2,\alpha}_\delta(\Sigma_\varepsilon)$ for some fixed $\delta \in (-2,-1)$. This is because any decaying solution of $Lu = 0$ must decay exponentially near the $i$th end of $\Sigma_\varepsilon$ at least like $e^{-\gamma_2(\varepsilon_i)s}$,
and by Proposition 12, when $\varepsilon$ is sufficiently small, $2 - \gamma_2(\varepsilon_i)$ is as small as desired, so that $u \in C_{2,\alpha}^{2,\alpha}(\Sigma \varepsilon_i)$.

The proof is by contradiction. Fix $\delta \in (-2, -1)$ and assume that for some sequence of $\varepsilon_k$ tending to 0, the Jacobi operator

$$L_k = \Delta_{\Sigma \varepsilon_k} + |A_{\Sigma \varepsilon_k}|^2$$

on $\Sigma \varepsilon_k$ is not injective on $C_{0,\alpha}^{2,\alpha}(\Sigma \varepsilon_k)$. Then there exists some $w_k \in C_{0,\alpha}^{2,\alpha}(\Sigma \varepsilon_k)$ such that $L_k w_k = 0$ and $w_k \neq 0$.

First normalize $w_k$, multiplying it by a suitable constant, so that $||w_k||_{0,0,\delta}(\Sigma \varepsilon_k) = 1$. Choose a point $y_k \in \Sigma \varepsilon_k$ where the above norm is achieved. Suppose first that some subsequence of the $y_k/\varepsilon_k$ converges to a point $y_0 \in \Sigma_0$. Then we can extract a subsequence of the $w_k$ which converge on every compact of $\Sigma_0$ to a limiting function $w$ globally defined on $\Sigma_0$; $w$ must be nontrivial since we also have $||w||_{0,0,\delta}(\Sigma_0) = 1$. Furthermore, $L_{\Sigma_0} w = 0$. Since we have assumed that $\Sigma_0$ is nondegenerate, we have obtained a contradiction.

If, on the other hand, some subsequence of the $y_k$ satisfies $\lim_{k \to +\infty} |y_k/\varepsilon_k| = +\infty$ then, this implies that, at least for a subsequence, the points $y_k$ are always in the same end, say the $i$th. Therefore, we may write,

$$y_k = x_{\varepsilon_k,i}(s_k + s_{\varepsilon_k,i}, \theta_k),$$

with $s_k$ tending to $+\infty$. By translating back by $s_k + s_{\varepsilon_k,i}$ and multiplying by a suitable constant, we find yet another sequence of solutions, which we again call $w_k$, attaining their maximum at $s = 0$, and which solve the translated equation, which we again write as $L_k w_k = 0$. Here $L_k$ is the linearized mean curvature operator relative to the parameterization given above near the ends. It is straightforward to see that the $w_k$ converge to a nontrivial solution $w$ of one of the following two limiting equations

$$\partial_s^2 w + \partial_{\theta}^2 w = 0, \quad (81)$$

or

$$\partial_s^2 w + \partial_{\theta}^2 w + \frac{2}{\cosh^2(s + \bar{s})} w = 0, \quad \text{for some } \bar{s} \in \mathbb{R}, \quad (82)$$

on $\mathbb{R} \times S^1$. In addition, $w$ is bounded by $e^{\delta s}$. By decomposing into eigenfrequencies we then see that necessarily $w = 0$ which is the desired contradiction.

This covers all cases, so we have showed that the linearization is injective on the appropriate weighted Hölder spaces.

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