Data-driven aggregation in circular deconvolution

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Abstract

In a circular deconvolution model we consider the fully data driven density estimation of a circular random variable where the density of the additive independent measurement error is unknown. We have at hand two independent iid samples, one of the contaminated version of the variable of interest, and the other of the additive noise. We show optimality, in an oracle and minimax sense, of a fully data-driven weighted sum of orthogonal series density estimators. Two shapes of random weights are considered, one motivated by a Bayesian approach and the other by a well known model selection method. We derive non-asymptotic upper bounds for the quadratic risk and the maximal quadratic risk over Sobolev-like ellipsoids of the fully data-driven estimator. We compute rates which can be obtained in different configurations for the smoothness of the density of interest and the error density. The rates (strictly) match the optimal oracle or minimax rates for a large variety of cases, and feature otherwise at most a deterioration by a logarithmic factor. We illustrate the performance of the fully data-driven weighted sum of orthogonal series estimators by a simulation study.

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1 Introduction

In a circular convolution model one objective is to estimate non-parametrically the density of a random variable taking values on the unit circle from observations blurred by an additive noise. Here we show optimality, in an oracle and minimax sense, of a fully data-driven weighted sum of orthogonal series estimators (OSE’s). Two shapes of random weights are considered, one motivated by a Bayesian approach and the other by a well known model selection method. Circular data are met in a variety of applications, such as data representing a direction on a compass in handwriting recognition (Bahlmann [2006]) and in meteorology (Carnicero et al. [2013]), or anything from opinions on a political compass to time reading on a clock face (Gill and Hangartner [2010]) in political sciences. The non-parametric density estimation in a circular deconvolution model has been considered for example in Comte and Taupin [2003], Efromovich [1997], Johannes and Schwarz [2013], while Schluttenhofer and Johannes [2020a,b], for example, study minimax testing. For an overview of convolutional phenomenons met in other models the reader may refer to Meister [2009].

Throughout this work we will tacitly identify the circle with the unit interval $[0, 1)$, for notational convenience. Let $Y := X + \varepsilon - \lfloor X + \varepsilon \rfloor = X + \varepsilon \mod 1$ be the observable contaminated random variable and $g$ its density. If we denote by $f$ and $\varphi$ the respective circular densities of the random variable of interest $X$ and of the additive and independent noise $\varepsilon$, then, we have

$$g(y) = (f \oslash \varphi)(y) := \int_{[0,1)} f((y - s) - \lfloor y - s \rfloor) \varphi(s) \, ds, \quad y \in [0, 1),$$

such that $\oslash$ stands for the circular convolution. Therefore, the estimation of $f$ is called a circular deconvolution problem.

We highlight hereafter that, thanks to the convolution theorem, an estimator of the circular density $f$ is usually based on the Fourier transforms of $\varphi$, and $g$ which may be estimated from the data. For any complex number $z$, denote $\overline{z}$ its complex conjugate, and $|z|$ its modulus. Let $L^2 := L^2([0, 1))$ be the Hilbert space of square integrable complex-valued functions defined on $[0, 1)$ endowed with the usual inner product $\langle h_1, h_2 \rangle_{L^2} = \int_{[0,1)} h_1(x)\overline{h_2(x)} \, dx$, and associated norm $\|\cdot\|_{L^2}$. Each $h \in L^2$ admits a representation as discrete Fourier series $h = \sum_{j \in \mathbb{Z}} [h]_j e_j$ with respect to the exponential basis $\{e_j\}_{j \in \mathbb{Z}}$, where $[h]_j := \langle h, e_j \rangle_{L^2}$ is the $j$-th Fourier coefficient of $h$, and $e_j(x) := \exp(-i2\pi j x)$ for $x \in [0, 1)$, and a square root $i$ of $-1$.

In this work we suppose that $f$, $\varphi$, and hence $g$, belong to the subset $\mathcal{D}$ of real-valued Lebesgue densities in $L^2$. We denote the expectation associated with $g$ and $\varphi$ by $E_g$, and $E_\varphi$ respectively. We note that $[g]_0 = 1$, and $E_g[e_j(-Y)] = [g]_j = [\overline{g}]_{-j}$ for any $j \in \mathbb{Z}$ as it is the case for any density. The key to our analysis is the convolution theorem which states that, in a
circular model, \( g = \varphi \ast f \) holds if and only if \( [g]_j = [\varphi]_j \cdot [f]_j \) for all \( j \in \mathbb{Z} \). Therefore and as long as \( [\varphi]_j \neq 0 \) for all \( j \in \mathbb{Z} \), which is assumed from now on, we have

\[
f = e_0 + \sum_{|j| \in \mathbb{N}} [\varphi]^{-1}_j [g]_j e_j \quad \text{with } [g]_j = \mathbb{E}_g[e_j(-Y)] \text{ and } [\varphi]_j = \mathbb{E}_\varphi[e_j(-\varepsilon)]. \quad (1.1)
\]

Note that an analogous representation holds in the case of deconvolution on the real line with compactly supported \( X \)-density, i.e. when the error term \( \varepsilon \), and hence \( Y \), take their values in \( \mathbb{R} \). In this situation, the deconvolution density still admits a discrete representation as in (1.1), but involving the characteristic functions of \( \varphi \) and \( g \) rather than their discrete Fourier coefficients. For a more detailed study of the Fourier analysis of probability distributions, the reader is referred, for example, to Brémaud [2014], Chapter 2.

In this paper we do not know neither the density \( g = f \ast \varphi \) of the contaminated observations nor the error density \( \varphi \), but we have at our disposal two independent samples of independent and identically distributed (iid.) random variables of size \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \), respectively:

\[
Y_i \sim g, \quad i \in [n] := [1, n] \cap \mathbb{N}, \quad \text{and} \quad \varepsilon_i \sim \varphi, \quad i \in [m]. \quad (1.2)
\]

In this situation, for each dimension parameter \( k \in \mathbb{N} \) an OSE of \( f \) is given by

\[
\hat{f}_k := e_0 + \sum_{|j| \in \mathbb{N}} [\varphi]^{-1}_j [g]_j e_j, \quad \text{with } [g]_j := n^{-1} \sum_{i \in [n]} e_j(-Y_i),
\]

\[
[\varphi]^{-1}_j := [\varphi]^{-1}_j \mathbb{1}_{\{|[\varphi]_j|^{\geq 1/m}\}} \text{ and } [\varphi]_j := m^{-1} \sum_{i \in [m]} e_j(-\varepsilon_i). \quad (1.3)
\]

The threshold using the indicator function \( \mathbb{1}_{\{|[\varphi]_j|^{\geq 1/m}\}} \), accounts for the uncertainty caused by estimating \( [\varphi]_j \) by \( [\varphi]_j \). It corresponds to \( [\varphi]_j \)'s noise level as an estimator of \( [\varphi]_j \) which is a natural choice (cf. Neumann [1997], p. 310f.). Thanks to the properties of the sequences \( ([g]_j)_{j \in \mathbb{Z}} \), and \( ([\varphi]^{-1}_j)_{j \in \mathbb{Z}} \), for any \( k \in \mathbb{N} \), the estimator \( \hat{f}_k \) is a real valued function integrating to 1. It is not necessarily positive valued, however, one might project the estimator on \( \mathcal{D} \), leading to an even smaller quadratic error. Nevertheless \( \hat{f}_k \) depends on a dimension parameter \( k \) whose choice essentially determines the estimation accuracy.

In Johannes and Schwarz [2013], a minimax criterion is used to formulate optimality. It is shown that, by choosing the dimension parameter properly, the maximal risk of an OSE as in (1.3) reaches the lower bound over Sobolev-like ellipsoids. However, the optimal choice of the dimension depends on the unknown ellipsoids. A fully data-driven selection based on a penalised contrast method is proposed and it is shown to yield minimax optimal rates for a large family of such ellipsoids. This selection procedure is inspired by the work of Barron et al. [1999], which was applied in the case of known error density by Comte and Taupin [2003]. For an extensive overview of model selection by penalised contrast, the reader may refer to Massart [2007]. More precisely, Johannes and Schwarz [2013] introduce an upper
bound $\hat{M}$ for the dimension parameter, and penalties $(\text{pen}_k^\hat{\varphi})_{k \in [\hat{M}]}$, depending on the samples $(Y_i)_{i \in [n]}$, and $(\varepsilon_i)_{i \in [n]}$, but neither on $f$ nor $\varphi$. Then, the fully data-driven estimator is defined as
\[
\hat{f}_k := e_0 + \sum_{|j| \in [\hat{k}]} [\varphi_j]^*[g_j] e_j \quad \text{with } \hat{k} := \arg \min_{k \in [\hat{M}]} \{-\|\hat{f}_k\|_{L^2}^2 + \text{pen}_k^\hat{\varphi}\}.
\] (1.4)

The empirical upper bound $\hat{M}$ proposed in Johannes and Schwarz [2013] is technically rather involved and more importantly simulations suggest that it leads to values which are often much too restrictive.

Here, rather than a data-driven selection of a dimension parameter, we propose to sum the OSE’s with positive data-driven weights adding up to one. Namely, given for each $k \in [n]$, the OSE’s as in (1.3), and a random weight $w_k \in [0, 1]$, we consider the convex sum
\[
\hat{f}_w = \sum_{k \in [n]} w_k \hat{f}_k, \quad \text{with } \sum_{k \in [n]} w_k = 1.
\] (1.5)

Introducing the model selection weights,
\[
\hat{w}_k := 1_{\{k = \hat{k}\}}, \quad k \in [n], \quad \text{with } \hat{k} := \arg \min_{k \in [n]} \{-\|\hat{f}_k\|_{L^2}^2 + \text{pen}_k^\hat{\varphi}\}
\] (1.6)

allows us to consider the model selected estimator $\hat{f}_k = \hat{f}_{\hat{w}} = \sum_{k \in [n]} \hat{w}_k \hat{f}_k$ as a data-driven weighted sum, avoiding a restrictive empirical upper bound $\hat{M}$ as in (1.4).

We study a second shape of random weights, motivated by a Bayesian approach in the context of an inverse Gaussian sequence space model and its iterative extension respectively described in Johannes et al. [2020] and Loizeau [2020]. For some constant $\eta \in \mathbb{N}$ we define Bayesian weights
\[
\hat{w}_k := \frac{\exp(-\eta n \{-\|\hat{f}_k\|_{L^2}^2 + \text{pen}_k^\hat{\varphi}\})}{\sum_{l=1}^n \exp(-\eta n \{-\|\hat{f}_l\|_{L^2}^2 + \text{pen}_l^\hat{\varphi}\})}, \quad k \in [n].
\] (1.7)

Note that in (1.6) and (1.7) the quantity $\|\hat{f}_k\|_{L^2}^2 = \sum_{j=-k}^{k} |[\varphi_j]^*[g_j]|^2$ can be calculated from the data without any prior knowledge about the error density $\varphi$. Thereby, as the sequence of penalties $(\text{pen}_k^\hat{\varphi})_{k \in [n]}$ given in bellow (3.6) does not involve any prior knowledge neither of $f$ nor $\varphi$, the weights in (1.6) and (1.7) are fully data-driven.

Let us emphasise the role of the parameter $\eta$ used in (1.7). If $\hat{k}$ as in (1.6) minimises uniquely the penalised contrast function, then it is easily seen that for each $k \in [n]$ the Bayesian weight $\hat{w}_k$ converges to the model selection weight $\hat{w}_k$ as $\eta \to \infty$. We shall see that the fully data-driven weighted sum $\hat{f}_w$ with Bayesian weights $w = \hat{w}$ or model selection weights $w = \hat{w}$ yields minimax optimal convergence rates over Sobolev-like ellipsoids. Thus, the theory presented here does not give a way to chose the parameter $\eta$. However, simulations
suggest that the Bayesian weights lead to more stable results as it is often recorded in the field of estimator aggregation.

The shape of the weighted sum \( \hat{f}_w \) is similar to the form studied in the estimator aggregation literature. Aggregation in the context of regression problems is considered, for instance, in Bellec and Tsybakov [2015], Dalalyan and Tsybakov [2008, 2012], Rigollet et al. [2012], Tsybakov [2014]), while Rigollet and Tsybakov [2007] study density estimation. Traditionally, the aggregation of a family of arbitrary estimators is performed through an optimisation program for the random weights, and the goal is to compare the convergence rate of the aggregation to the one of the best estimator in the family. Here, while we restrict ourselves to OSE’s, their number is as large as the sample size. The random weights are given explicitly without an optimisation program and do not rely on a sample splitting. In addition, we allow for a degenerated cases where one OSE receives all the weight of the sum.

This paper is organised as follows. In section 2 assuming that the error density \( \varphi \) is known, we introduce a family of OSE’s. We briefly recall the oracle and minimax theory before introducing model selection and Bayesian weights respectively similar to (1.6), and (1.7), which still depend on characteristics of the error density. The weighted sum of the OSE’s is thus only partially data-driven. We derive non-asymptotic upper bounds for the quadratic risk and the maximal quadratic risk over Sobolev-like ellipsoids of the partially data-driven estimator. In section 3, dismissing the knowledge of the density \( \varphi \) an additional sample of the noise is observed. Choosing the weights in (1.6), and (1.7) fully data-driven we derive non-asymptotic upper risk bounds for the now fully data-driven weighted sums of OSE’s. In sections 2 and 3 we compute rates which can be obtained in different configurations for the smoothness of the density of interest \( f \) and the error density \( \varphi \). The rates (strictly) match the optimal oracle or minimax rates for a large variety of cases, and feature otherwise at most a deterioration by a logarithmic factor. We illustrate in section 4 the reasonable performance of the fully data-driven weighted sum of OSE’s by a simulation study. All technical proofs are deferred to the Appendix.

2 Partially data-driven aggregation: known error density

Notations. Throughout this section the error density \( \varphi \in D \) is known. Therefore, given an iid. \( n \)-sample \( (Y_i)_{i \in [n]} \) from \( g = f \otimes \varphi \) we denote by \( \mathbb{E}^n_{f,\varphi} \) the expectation with respect to their joint distribution \( \mathbb{P}^n_{f,\varphi} \). The estimation of the unknown circular density \( f \) is based on a dimension reduction which we briefly elaborate first. Given the exponential basis \( \{e_j, j \in \mathbb{Z}\} \) and a dimension parameter \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) we have the subspace \( \mathbb{U}_k \) spanned by the \( 2k+1 \) basis functions \( \{e_j, j \in [-k, k]\} \) at our disposal. For abbreviation, we denote by \( \Pi_k \) and \( \Pi_k^\perp \) the orthogonal projections on \( \mathbb{U}_k \) and its orthogonal complement \( \mathbb{U}_k^\perp \) in \( L^2 \), respectively. For
each $h \in L^2$ we consider its orthogonal projection $h_k := \Pi_k h$ and its associated approximation error $\|h_k - h\|_{L^2} = \|\Pi_k h\|_{L^2}$. Note that for any density $p \in \mathcal{D} \cap L^2$ holds $\Pi_k p = p - e_0$ and we define $b_k(p) := (\Phi_k(p))_{k \in \mathbb{N}_0} \in \mathbb{R}^{N_0}$ with
\begin{equation}
1 \geq b_k(p) := \|\Pi_k^+ p\|_{L^2} / \|\Pi_k^- p\|_{L^2} \quad \text{ (with the convention } 0/0 = 0) \tag{2.1}
\end{equation}
where $\lim_{k \to \infty} b_k(p) = 0$ due to the dominated convergence theorem.

**Risk bound.** Keeping in mind that the error density satisfies $\|\varphi\|_k > 0$ for all $k \in \mathbb{Z}$, we define $\Phi_k = (\Phi_k)_{k \in \mathbb{N}_0} \in \mathbb{R}^{N_0}$, and, for any $x \in \mathbb{R}^{N_0}$ we introduce $\Sigma_k^x = (\Sigma_k^{x})_{k \in \mathbb{N}_0} \in \mathbb{R}^{N_0}$ with
\begin{equation}
\Sigma_0^x := 0, \quad \Sigma_k^x := k^{-1} \sum_{j \in [k]} x_j; \quad \text{ and } \quad 1 \leq \Phi_k := \|\varphi\|_k - 2 = \|\varphi\|_{k-1} - 2. \tag{2.2}
\end{equation}
We define the OSE’s in the present case similarly to (1.3) by
\begin{equation}
\tilde{f}_k = e_0 + \sum_{j \in [k]} [\varphi]_j^{-1} [\hat{g}]_j e_j. \tag{2.3}
\end{equation}
By elementary calculations for each $k \in \mathbb{N}_0$ the risk of $\tilde{f}_k$ in (2.3) satisfies
\begin{equation}
\mathbb{E}_{p,q} \|\tilde{f}_k - f\|^2_{L^2} + n^{-1}\|\Pi_k^+ f\|^2_{L^2} = 2n^{-1} k \Sigma_k^x \|\Pi_k^+ f\|^2_{L^2} b_k^2(f). \tag{2.4}
\end{equation}

The quadratic risk in the last display depends on the dimension parameter $k$ and hence by selecting an optimal value it will be minimised, which we formulate next. For a sequence $(a_k)_{k \in \mathbb{N}_0}$ of real numbers with minimal value in a set $A \subseteq \mathbb{N}_0$ we set $\arg \min \{a_k : k \in A\} := \min\{k \in A : a_k, \forall j \in A\}$. For any non-negative sequence $x := (x_k)_{k \in \mathbb{N}_0}, y := (y_k)_{k \in \mathbb{N}_0}$ and each $k \in \mathbb{N}_0$ define
\begin{equation}
R_n^k(x, y) := [x_k \vee n^{-1} k y_k] := \max \{x_k, n^{-1} k y_k\}, \quad
k_n^0(x, y) := \arg \min \{R_n^k(x, y), k \in \mathbb{N}_0\} \quad \text{ and } \quad
R_n^k(x, y) := \min \{R_n^k(x, y), k \in \mathbb{N}_0\} = R_n^{k_n^0(x, y)}(x, y). \tag{2.5}
\end{equation}

**Remark 2.1.** Here and subsequently, our upper bounds of the risk derived from (2.4) make use of the definitions (2.5), for example, replacing the sequences $x$ and $y$ by $b_\alpha^x(f)$ and $\Sigma_\alpha^x$, respectively. However, in what follows the sequence $x$ and $y$ is always monotonically non-increasing and non-decreasing, respectively, with $x_0 \leq 1 \leq y_0$ and $\lim_{k \to \infty} x_k = 0 = \lim_{k \to \infty} y_k^{-1}$. In this situations by construction hold $k_n^0(x, y) \in [n]$ and $R_n^k(x, y) \geq n^{-1}$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} R_n^k(x, y) = 0$. For the latter observe that for each $\delta > 0$ there is $k_\delta \in \mathbb{N}$ and $n_\delta \in \mathbb{N}$ such that $x_{k_\delta} \leq \delta$ and $k_\delta y_{k_\delta} n^{-1} \leq \delta$, $R_n^k(x, y) \leq R_n^{k_\delta}(x, y) \leq \delta$, for all $n \geq n_\delta$. We shall use those elementary findings in the sequel without further reference. \hfill \square
Throughout the paper we shall distinguish for the deconvolution density \( f \) and hence its associated sequence \( b_1 := b_1(f) \in \mathbb{R}^{N_0} \) of approximation errors as in (2.1) the two cases: (p) there is \( K \in \mathbb{N} \) with \( b_K = 0 \) and \( b_{K-1} > 0 \) (with the convention \( b_{-1} := 1 \)), and (np) for all \( K \in \mathbb{N} \) holds \( b_K > 0 \). Let us stress, that for any monotonically non-decreasing sequence \( y \) with \( y_0 \geq 1 \), the order of the rate \( (\mathcal{R}_n(b_1, y))_{n \in \mathbb{N}} \) defined in (2.5) with \( x \), replaced by \( b \), in case (p) and (np) is parametric and non-parametric, respectively. More precisely, in case (p) it holds \( k_n(b, y) = K \) and \( \mathcal{R}_n(b, y) = n^{-1} Ky_K \) for all \( n > K y_K / b_{K-1} \), while in case (np) holds \( \lim_{n \to \infty} k_n(b, y) = \infty \) and \( \lim_{n \to \infty} n \mathcal{R}_n(b_1, y) = \infty \).

**Oracle optimality.** Coming back to the identity (2.4) and exploiting the definition (2.5) with \( x \) and \( y \), respectively, replaced by \( b_2 := b_2(f) \) and \( \Sigma^\bullet_2 \) as in (2.2) it follows immediately

\[
\inf \{ \mathbb{E}_{\tilde{f}_k} \| \tilde{f}_k - f \|^2_{L^2} , k \in \mathbb{N} \} \leq \mathbb{E}_{\tilde{f}_k} \| \tilde{f}_k(b_2, \Sigma^\bullet_2) - f \|^2_{L^2} \leq 2 [1 \vee \| \Pi_2 f \|^2_{L^2}] \mathcal{R}_n(b_2, \Sigma^\bullet_2). \tag{2.6}
\]

On the other hand with \( [a \wedge b] := \min(a, b) \) for \( a, b \in \mathbb{R} \) from (2.4) we also conclude

\[
\inf \{ \mathbb{E}_{\tilde{f}_k} \| \tilde{f}_k - f \|^2_{L^2} , k \in \mathbb{N} \} \geq \left( [\| \Pi_2 f \|^2_{L^2} \wedge 2] - \frac{\| \Pi_2 f \|^2_{L^2}}{n \mathcal{R}_n(b_2, \Sigma^\bullet_2)} \right) \mathcal{R}_n(b_2, \Sigma^\bullet_2). \tag{2.7}
\]

For each \( n \in \mathbb{N} \) combining (2.6) and (2.7) \( \mathcal{R}_n(b_2, \Sigma^\bullet_2) \), \( k_n(b_2, \Sigma^\bullet_2) \) and \( \tilde{f}_k(b_2, \Sigma^\bullet_2) \), respectively, is an oracle rate, oracle dimension and oracle optimal estimator (up to a constant), if the leading factor on the right hand side in (2.7) is uniformly in \( n \) bounded away from zero. Note that \( \mathcal{R}_n(b_2, \Sigma^\bullet_2) \) is in case (np) always an oracle rate, while in case (p) whenever \( KS^\bullet_K > [1 \vee \frac{1}{2} \| \Pi_2 f \|^2_{L^2}] \).

**Aggregation.** We call aggregation weights any \( w := (w_k)_{k \in [n]} \in [0, 1]^n \) defining on the set \( [n] \) a discrete probability measure \( \mathbb{P}_w(\{ k \}) := w_k, k \in [n] \). We consider here and subsequently a weighted sum \( \tilde{f}_w := \sum_{k \in [n]} w_k \tilde{f}_k \) of the orthogonal series estimators defined in (2.3). Clearly, the coefficients \( ([\tilde{f}_w])_{j \in \mathbb{Z}} \) of \( \tilde{f}_w \) satisfy \( [\tilde{f}_w]_j = 0 \) for \( |j| > n \), and for any \( |j| \in [n] \) holds \( [\tilde{f}_w]_j = \sum_{k \in [n]} w_k [\tilde{f}_k]_j = \mathbb{P}_w ([|j|, n]) |[\varphi]^{-1}_j| [\varphi]_j \). We note that by construction \( [\tilde{f}_w]_0 = 1, [\tilde{f}_w]_{-j} = [\tilde{f}_w]_j \) and \( 1 \geq [\tilde{f}_w]_j \). Hence, \( \tilde{f}_w \) is real and integrates to one, however, it is not necessary non-negative. Our aim is to prove an upper bound for its risk \( \mathbb{E}_{\tilde{f}_w} \| \tilde{f}_w - f \|^2_{L^2} \) as well its maximal risk over Sobolev-like ellipsoids. For arbitrary aggregation weights and penalty sequence the next lemma establishes an upper bound for the loss of the aggregated estimator. Selecting suitably the weights and penalties this bound provides in the sequel our key argument.

**Lemma 2.2.** Consider a weighted sum \( \tilde{f}_w \) with arbitrary aggregation weights \( w \) and non-
negative penalty terms \(\text{pen}_k^\Phi\) for any \(k_-, k_+ \in \mathbb{N}\). For any \(k_-, k_+ \in \mathbb{N}\) holds

\[
\|\tilde{f}_\tilde{w} - f\|_{L^2}^2 \leq \frac{8}{7} \text{pen}_k^\Phi + 2\|\Pi_l^f f\|_{L^2}^2 b_k^\circ (f) + 2\|\Pi_l^f f\|_{L^2}^2 \mathbb{E}(|k-|) + \frac{8}{7} \sum_{k \in [k_+, n]} \text{pen}_k^\Phi w_k \mathbb{I}\{\|\tilde{f}_\tilde{w} - f_k\|_{L^2} < \text{pen}_k^\Phi / 7\} + \frac{8}{7} \sum_{k \in [k_+, n]} \text{pen}_k^\Phi \mathbb{I}\{\|\tilde{f}_\tilde{w} - f_k\|_{L^2} \geq \text{pen}_k^\Phi / 7\}. \tag{2.8}
\]

Remark 2.3. Keeping (2.8) in mind let us briefly outline the principal arguments of our aggregation strategy. Selecting the values \(k_+\) and \(k_-\) close to the oracle dimension \(k_n^\circ(b_n^\circ, \Sigma_n^\circ)\) the first two terms in the upper bound of (2.8) are of the order of the oracle rate. On the other hand the weights are in the sequel selected such that the third and fourth are negligible with respect to the oracle rate, while the choice of the penalties allows as usual to bound the deviation of the last two terms by concentration inequalities.

For some constant \(\eta \in \mathbb{N}\), we consider either Bayesian weights

\[
\tilde{w}_k := \frac{\exp(-\eta m\{|\tilde{f}_k\|_{L^2}^2 + \text{pen}_k^\Phi\})}{\sum_{l \in [N]} \exp(-\eta m\{|\tilde{f}_l\|_{L^2}^2 + \text{pen}_l^\Phi\})}, \quad k \in [1, n]. \tag{2.9}
\]

or model selection weights

\[
\tilde{w}_k := \mathbb{I}_{\{k = \bar{k}\}}, \quad k \in [n], \quad \text{with } \bar{k} := \arg \min_{k \in [n]} \{-|\tilde{f}_k|_{L^2}^2 + \text{pen}_k^\Phi\} \tag{2.10}
\]

respectively similar to the ones defined in (1.7) and (1.6). Until now we have not specified the sequence of penalty terms. For a sequence \(x_n \in \mathbb{R}^{N_0}\) and \(k \in \mathbb{N}_0\) define

\[
x_{(k)} := \max\{x_j : j \in [0, k]\}, \quad \lambda_k^\circ := \frac{|\log(kx_{(k)}\vee (k+2))|^2}{|\log(k+2)|^2}, \quad \text{and } \Lambda_k^\circ := \lambda_k^\circ x_{(k)}. \tag{2.11}
\]

Given \(\Phi_n \in \mathbb{R}^{N_0}\) as in (2.2) and a numerical constant \(\Delta > 0\) we use

\[
\text{pen}_k^\Phi := \Delta k\Lambda_k^\circ n^{-1}, \quad k \in \mathbb{N}. \tag{2.12}
\]

as penalty terms. For the theoretical results below we need that the numerical constant satisfies \(\Delta \geq 84\). However, for a practical application this values is generally too large and a suitable constant might be chosen by preliminary calibration experiments see Baudry et al. [2012].

We derive bounds for the risk of the weighted sum estimator \(\tilde{f}_\tilde{w}\) with Bayesian weights and the model selected estimator \(\tilde{f}_\tilde{w} = \tilde{f}_\tilde{w}\) by applying Lemma 2.2. From definition (2.5) replacing \(x, y\), respectively, by \(b_n^\circ = b_n^\circ (f)\) and \(\Lambda_n^\circ\) we consider \(\mathcal{R}_n^\circ(b_n^\circ, \Lambda_n^\circ)\) for each \(n, k \in \mathbb{N}\). Note that by construction (2.2) and (2.11), we have \(\mathcal{R}_n^\circ(b_n^\circ, \Lambda_n^\circ) \geq \mathcal{R}_n^\circ(b_n^\circ, \Sigma_n^\circ)\) for all \(n \in \mathbb{N}\). We denote in the sequel by \(\mathcal{C}\) an universal finite numerical constant with value changing possibly from line to line.
Proposition 2.4. Consider an aggregation \( \tilde{f}_w = \sum_{k \in [n]} w_k \hat{f}_k \) using either Bayesian weights \( w := \tilde{w} \) as in (2.9) or model selection weights \( w := \tilde{w} \) as in (2.10) and penalties \( (\text{pen}_k^*)_{k \in [n]} \) as in (2.12) with numerical constant \( \Delta \geq 84 \). Let \( k_g := \lfloor 3(400)^2 \| [g] \|_2^2 \rfloor \).

(p) Assume there is \( K \in \mathbb{N}_0 \) with \( 1 > b_{(K-1)}(f) > 0 \) and \( b_K(f) = 0 \). If \( K = 0 \) we set \( c_f := 0 \) and \( c_f := \frac{4\Delta}{\| \Pi_0 f \|_2^2 \| \Pi_{(k-1)} f \|_2^2} \), otherwise. For \( n \in \mathbb{N} \) let \( k_n^* := \max \{ k \in [n] : n > c_f k \Lambda_n^* \} \), if the defining set is not empty, and \( k_n^* := \lfloor k_g \log(2 + n) \rfloor \) otherwise. There is a finite constant \( C_{f,\varphi} \) given in (B.24) depending only on \( f \) and \( \varphi \) such that for all \( n \in \mathbb{N} \) holds

\[
\mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|_2^2 \leq C \| \Pi_0 f \|_2^2 [n^{-1} \vee \exp\left( -\frac{\lambda_{k_n^*}^{f,\varphi}}{k_g} \right)] + \mathcal{C}_{f,\varphi} n^{-1}. \tag{2.13}
\]

(np) Assume that \( b_k(f) > 0 \) for all \( k \in \mathbb{N} \). There is a finite constant \( C_{f,\varphi} \) given in (B.10) depending only on \( f \) and \( \varphi \) such that for all \( n \in \mathbb{N} \) holds

\[
\mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|_2^2 \leq C \| \Pi_0 f \|_2^2 \left[ 1 \wedge \rho_n^\varphi(b_k^2, \Lambda_n^*) \right] + \mathcal{C}_{f,\varphi} n^{-1}
\]

with \( \rho_n^\varphi(b_k^2, \Lambda_n^*) := \min_{k \in [n]} \left\{ \left[ \mathcal{R}_{f}^\varphi(b_k^2, \Lambda_n^*) \vee \exp\left( -\frac{\lambda_{k_n^*,\varphi}}{k_g} \right) \right] \right\}. \tag{2.14}

Corollary 2.5. Let the assumptions of Proposition 2.4 be satisfied.

(p) If in addition (A1) there is \( n_{f,\varphi} \in \mathbb{N} \) such that for all \( n \geq n_{f,\varphi} \) holds \( \lambda_{k_n^*,\varphi} k_n^* \geq k_g \log n \), then there is a constant \( C_{f,\varphi} \) depending only on \( f \) and \( \varphi \) such that for all \( n \in \mathbb{N} \) holds

\[
\mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|_2^2 \leq \mathcal{C}_{f,\varphi} n^{-1}.
\]

(np) If in addition (A2): there is \( n_{f,\varphi} \in \mathbb{N} \) such that \( k_n^* := k_n^\varphi(b_k^2, \Lambda_n^*) \) as in (2.5) for all \( n \geq n_{f,\varphi} \) satisfies \( k_n^\varphi \lambda_{k_n^*,\varphi} \geq k_g \log \mathcal{R}_{f}^\varphi(b_k^2, \Lambda_n^*) \), then there is a constant \( \mathcal{C}_{f,\varphi} \) depending only on \( f \) and \( \varphi \) such that \( \mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|_2^2 \leq \mathcal{C}_{f,\varphi} \mathcal{R}_{f}^\varphi(b_k^2, \Lambda_n^*) \) for all \( n \in \mathbb{N} \).

Illustration 2.6. Here and subsequently, we use for two strictly positive sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) the notation \( a_n \sim b_n \) if the sequence \( (a_n/b_n)_{n \in \mathbb{N}} \) is bounded away both from zero and infinity. We illustrate the last results considering usual behaviours for the sequences \( b \) and \( \Phi \). Regarding the error density \( \varphi \) we consider for \( a > 0 \) the following two cases (o) \( \Phi_k \sim k^{2a} \) and (s) \( \Phi_k \sim \exp(k^{2a}) \). The error density \( \varphi \) is called ordinary smooth in case (o) and super smooth in case (s), and it holds, respectively, (o) \( k_n^* \sim n^{1/(2a+1)} \) and \( k_n^* \lambda_{k_n^*,\varphi} \sim n^{1/(2a+1)} \), and (s) \( k_n^* \sim (\log n)^{1/(2a)} \) and \( k_n^* \lambda_{k_n^*,\varphi} \sim (\log n)^{2+1/(2a)} \). Clearly in both cases (A1) holds true and hence employing Corollary 2.5 (p) the aggregated estimator attains the parametric rate. On the other hand, for (np) we use for the deconvolution density \( f \) as particular specifications (o) \( \| f \|_k^2 \sim k^{-2p-1} \) and (s) \( \| f \|_k^2 \sim k^{2p-1} \exp(-k^{2p}) \) with \( p > 0 \).
To calculate the order in the last table we used that the dimension parameter $k_n := k_n^o(b^2, \Lambda^o)$ satisfies $[\text{o-o}] k_n^o \sim n^{1/(2p+2a+1)}$ and $\lambda_n^o k_n^o \sim n^{1/(2p+2a+1)}$, $[\text{o-s}] k_n^o \sim (\log n)^{1/(2a)}$ and $\lambda_n^o k_n^o \sim (\log n)^{2a+1/(2a)}$, and $[\text{s-o}] k_n^o \sim (\log n)^{(a+1)/p}$ and $\lambda_n^o k_n^o \sim (\log n)^{1/(2p)}$. We note that in each of the three cases the order of $\mathcal{R}_n^o(b^2, \Lambda^o)$ and the order of the oracle rate $\mathcal{R}_n^o(b^2, \Lambda^o)$ coincide. Moreover, the additional assumption (A2) in Corollary 2.5 (np) is satisfied in case $[\text{o-o}]$ and $[\text{o-s}]$, but in $[\text{s-o}]$ only with $p < 1/2$. Consequently, in this situations due to Corollary 2.5 (np) the partially data-driven aggregation is oracle optimal (up to a constant). Otherwise, the upper bound $\rho_n^o(b^2, \Lambda^o)$ in Proposition 2.4 (2.14) faces a deterioration compared to the rate $\mathcal{R}_n^o(b^2, \Lambda^o)$. In case $[\text{s-o}]$ with $p \geq 1/2$ setting $k_n^* := k_n^o |\log \mathcal{R}_n^o(b^2, \Lambda^o)| \sim (\log n)$ the upper bound $\rho_n^o(b^2, \Lambda^o) \leq \mathcal{R}_n^o(b^2, \Lambda^o) \sim (\log n)^{2a+1/n} n^{-1}$ features a deterioration at most by a logarithmic factor $(\log n)^{(2a+1)/(1-1/(2p))}$ compared to the rate $\mathcal{R}_n^o(b^2, \Lambda^o)$. \hfill \Box

Minimax optimality. Rather than considering for each $k \in \mathbb{N}$ the risk of the OSE $\hat{f}_k$ for given $f$ and $\varphi$ we shall measure now its accuracy by a maximal risk over pre-specified classes of densities determining a priori conditions on $f$ and $\varphi$, respectively. For an arbitrary positive sequence $x, y \in \mathbb{R}^n_+$ and $h \in L^2$ we write shortly $\|h\|_2^2 := \sum_{j \in \mathbb{Z}} |h_j|^2$. Given strictly positive sequences $\mathbf{f} = (f_k)_{k \in \mathbb{N}_0}$ and $\mathbf{s} = (s_k)_{k \in \mathbb{N}_0}$, and constants $r, d \geq 1$ we define

$$\mathcal{F}_r := \{p \in \mathcal{D} : \|p\|_1 / l \leq r\} \quad \text{and} \quad \mathcal{E}_d^r := \{p \in \mathcal{D} : d^{-1} \leq s_j |p| |j| \leq d, \forall j \in \mathbb{Z}\}.$$ 

Here and subsequently, we suppose the following minimal regularity conditions are satisfied.

Assumption (A3). The sequences $\mathbf{f}, s^{-1}$ are monotonically non-increasing with $f_0 = 1 = s_0$, $\lim_{k \to \infty} f_k = 0 = \lim_{k \to \infty} s_k^{-1}$ and $\sum_{k \in \mathbb{N}_0} f_k / s_k = \|f / s\|_1 < \infty$.

We shall emphasize that for $k \in \mathbb{N}_0$, $f \in \mathcal{F}_r$ and $\varphi \in \mathcal{E}_d^r$ hold $\|\Pi_2 f\|_2^2 b_k^2(f) \leq r f_k$ and $1 / d \leq \Sigma_k / \Sigma_k^* \leq d$ with $\Sigma_k^* = k^{-1} \sum_{j \in \mathbb{Z}} s_j$, which we use in the sequel without further reference. Exploiting again the identity (2.4) and the definition (2.5) with $x$ and $y$, respectively, replaced by $\mathbf{f}$ and $\Sigma_k^*$ it follows for all $k, n \in \mathbb{N}$

$$\sup \left\{ \mathcal{E}_d^r, \varphi \| f_k - f \|_2^2 : f \in \mathcal{F}_r, \varphi \in \mathcal{E}_d^r \right\} \leq (2d + r) \mathcal{R}_n^{\mathbf{f}}(\mathbf{f}, \Sigma^*) . \tag{2.15}$$

The upper bound in the last display depends on the dimension parameter $k$ and hence by choosing an optimal value $k_n^o(\mathbf{f}, \Sigma^*)$ the upper bound will be minimised. From (2.15) we
deduce that \( \sup \left\{ \mathbb{E}_{F,r}^n \| \tilde{f}_{k_0}(\ell, n) - f \|_{L^2}^2 : f \in F_r, \varphi \in \mathcal{E}^d \right\} \leq (2d + r) \mathcal{R}_n^\circ(f, \Sigma_r^*) \) for all \( n \in \mathbb{N} \). On the other hand Johannes and Schwarz [2013] have shown that for all \( n \in \mathbb{N} \)

\[
\inf_f \sup \left\{ \mathbb{E}_{F,r}^n \| \tilde{f} - f \|_{L^2}^2 : f \in F_r, \varphi \in \mathcal{E}^d \right\} \geq C \mathcal{R}_n^\circ(f, \Sigma_r^*),
\]

where \( C > 0 \) and the infimum is taken over all possible estimators \( \tilde{f} \) of \( f \). Consequently, \((\mathcal{R}_n^\circ(f, \Sigma_r^*))_{n \in \mathbb{N}}, (k_0^n(f, \Sigma_r^*))_{n \in \mathbb{N}}\) and \((\tilde{f}_{k_0^n}(f, \Sigma_r^*))_{n \in \mathbb{N}}\), respectively, is a minimax rate, a minimax dimension and minimax optimal estimator (up to a constant).

**Aggregation.** Exploiting Lemma 2.2 we derive now bounds for the maximal risk of the aggregated estimator \( \tilde{f}_w \) using either Bayesian weights \( w := \tilde{w} \) as in (2.9) or model selection weights \( w := \tilde{w} \) as in (2.10). Keeping the definition (2.11) in mind we use in the sequel that for any \( \varphi \in \mathcal{E}^d \) and \( k \in \mathbb{N}_0 \) hold

\[
(1+\log d)^{-2} \leq \lambda_k^s/\lambda_k^r \leq (1+\log d)^2 \quad \text{and} \quad \zeta_d := d(1+\log d)^2 \geq \lambda_k^s \Phi(k)/(\lambda_k^r \sigma_k) \geq \zeta_d^{-1}.
\]

It follows for all \( k, n \in \mathbb{N}, f \in F_r \) and \( \varphi \in \mathcal{E}^d \) immediately

\[
r \mathcal{R}_n^k(f, \Lambda_r^s) \geq \| \Pi^\varphi_{k} f \|_{L^2}^2 b_k^4(f) \quad \text{and} \quad \Delta \zeta_d \mathcal{R}_n^k(f, \Lambda_r^s) \geq \text{pen}_k^s.
\]

Note that by construction \( \mathcal{R}_n^\circ(f, \Lambda_r^s) \geq \mathcal{R}_n^\circ(f, \Sigma^*) \) for all \( n \in \mathbb{N} \).

**Proposition 2.7.** Consider an aggregation \( \tilde{f}_w \) using either Bayesian weights \( w := \tilde{w} \) as in (2.9) or model selection weights \( w := \tilde{w} \) as in (2.10) and penalties \( (\text{pen}_k^s)_{k \in [n]} \) as in (2.12) with numerical constant \( \Delta \geq 84 \). Let (A3) be satisfied and set \( k_0 := \lfloor 3(400)^2 r \zeta_d \| f/\|_f \| \ell^1 \rfloor \).

There is a constant \( C_{fs}^d \) given in (B.31) depending only on \( F_r \) and \( \mathcal{E}^d \) such that for all \( n \in \mathbb{N} \)

\[
\sup \left\{ \mathbb{E}_{F,r}^n \| \tilde{f}_w - f \|_{L^2}^2 : f \in F_r, \varphi \in \mathcal{E}^d \right\} \leq C (r + \zeta_d) \rho_n^s(f, \Lambda_r^s) + C_{fs}^d n^{-1}
\]

with \( \rho_n^s(f, \Lambda_r^s) := \min_{k \in [n]} \left\{ \mathcal{R}_n^k(f, \Lambda_r^s) \vee \exp \left( -\lambda_k^s \right) \right\} \). (2.19)

**Corollary 2.8.** Let the assumptions of Proposition 2.7 be satisfied. If in addition (A2') there is \( n_{fs} \in \mathbb{N} \) such that \( k_0^n := k_0^n(f, \Lambda_r^s) \) as in (2.5) satisfies \( k_0^n \lambda_n^{r_2} \geq k_{fs} \left| \log \mathcal{R}_n^\circ(f, \Lambda_r^s) \right| \) for all \( n \geq n_{fs} \), then there is a constant \( C_{fs}^d \) depending only on the classes \( F_r \) and \( \mathcal{E}^d \) such that

\[
\sup \left\{ \mathbb{E}_{F,r}^n \| \tilde{f}_w - f \|_{L^2}^2 : f \in F_r, \varphi \in \mathcal{E}^d \right\} \leq C_{fs}^d \mathcal{R}_n^\circ(f, \Lambda_r^s) \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Illustration 2.9.** We illustrate the last results considering usual configurations for \( f \) and \( s \).

| \( f_k \) | \( s_k \) | \( \mathcal{R}_n^\circ(f, \Sigma^*) \) | \( \mathcal{R}_n^\circ(f, \Lambda_r^s) \) | \( \rho_n^s(f, \Lambda_r^s) \) |
|---|---|---|---|---|
| \([o-o]\) | \( k^{-2p} \) | \( k^{2a} \) | \( n^{-2p/2a+1} \) | \( n^{-2p/2p+2a+1} \) |
| \([o-s]\) | \( k^{-2p} \) | \( k^{2a} \) | \( \log(n)^{-p/a} \) | \( \log(n)^{-p/a} \) |
| \([s-o]\) | \( e^{-2p/2} \) | \( k^{2a} \) | \( \log(n)^{2a+1/2p} n^{-1} \) | \( \log(n)^{2a+1/2p} n^{-1} \) |

\( p < 1/2, \quad \log(n)^{2a+1/2p} n^{-1} \) : \( p > 1/2. \)
We note that in each of the three cases the order of $\mathcal{R}_n^0(f, \Lambda^*)$ coincide with the order of the minimax rate $\mathcal{R}_n^0(f, \Sigma^*)$. Moreover, the additional assumption $(A2')$ in Corollary 2.8 is satisfied in case $\{0,0\}$ and $\{0,\mathbf{s}\}$, but in $\{\mathbf{s},\mathbf{o}\}$ only with $p < 1/2$. Consequently, in this situations due to Corollary 2.8 the partially data-driven aggregation is minimax optimal (up to a constant). Otherwise, the upper bound $\rho_n^0(f, \Lambda^*)$ in Proposition 2.7 (2.19) faces a deterioration compared to $\mathcal{R}_n^0(f, \Lambda^*)$, e.g. in case $\{\mathbf{s} \circ \mathbf{o}\}$ with $p \geq 1/2$ by a logarithmic factor $(\log n)^{(2a+1)(1-1/(2p))}$.

\section{Data-driven aggregation: unknown error density}

In this section we dispense with any knowledge about the error density $\varphi$. Instead we assume two independent sample $(Y_i)_{i \in [n]}$ and $(\varepsilon_i)_{i \in [m]}$ as in (1.2). We denote by $\mathbb{E}_{f, \varphi}^n$, $\mathbb{E}_{f, \varphi}^m$ and $\mathbb{E}^m$ the expectation with respect to their joint distribution $\mathbb{P}_{f, \varphi}^n$, and marginals $\mathbb{P}_{f, \varphi}$, and $\mathbb{P}^m$, respectively.

**Risk bound.** Exploiting the independence assumption, the risk of the orthogonal series estimators $\hat{f}_k$ given in (1.3) can be decomposed for each $n, m, k \in \mathbb{N}$ as follows

\[
\mathbb{E}_{f, \varphi}^{n,m} \| \hat{f}_k - f \|^2_{L^2} = n^{-1} \sum_{|j| \in [k]} \Phi_j (1 - ||g_j||^{2}) \mathbb{E}^m (||[\hat{\varphi}_j]^+ [\varphi_j]||^2) + ||\Pi_b^f ||_{L^2} b^2_k(f) \\
+ \sum_{|j| \in [k]} ||f_j||^2 \mathbb{E}^m (||[\hat{\varphi}_j] - [\varphi_j]||^2) + \sum_{|j| \in [k]} ||f_j||^2 \mathbb{E}^m (||[\varphi_j]|^2 < 1/m). \tag{3.1}
\]

Exploiting Lemma A.6 in the appendix we control the deviations of the additional terms estimating the error density. Therewith, setting $||\Pi_b^f ||_{L^2}^2 = 2 \sum_{j \in \mathbb{N}} ||f_j||^2 [1 \wedge \Phi_j/m]$, selecting $k_n^0 := k_n(b^2, \Sigma^*)$ as in (2.5) with $\mathcal{R}_n^0(b^2, \Sigma^*) = \mathcal{R}_n^0(b^2, \Sigma^*)$ it follows for all $n, m \in \mathbb{N}$

\[
\mathbb{E}_{f, \varphi}^{n,m} \| \hat{f}_{k_n^0} - f \|^2_{L^2} \leq (||\Pi_b^f ||_{L^2}^2 + 8) \mathcal{R}_n^0(b^2, \Sigma^*) + 8(C + 1) ||\Pi_b^f ||_{L^2}^2 \tag{3.2}
\]

**Remark 3.1.** \textit{Note that $||\Pi_b^f ||_{L^2}^2 = 0$ implies $||\Pi_b^f ||_{L^2}^2 \Phi/m = 0$, while for $||\Pi_b^f ||_{L^2}^2 > 0$ holds $||\Pi_b^f ||_{L^2}^2 \Phi/m \geq ||\Pi_b^f ||_{L^2}^2 m^{-1}$, and hence any additional term of order $n^{-1} + m^{-1}$ is negligible with respect to $\mathcal{R}_n^0(f) + ||\Pi_b^f ||_{L^2}^2 \Phi/m$, since $\mathcal{R}_n^0(f) \geq n^{-1}$. On the other hand if $||f||_{B^2} < \infty$ then $||\Pi_b^f ||_{L^2}^2 \Phi/m \leq m^{-1} ||f||_{B^2}^2$. Consequently, in case (p) the order of the upper bound is parametric in both sample sizes, i.e., $\mathbb{E}_{f, \varphi}^{n,m} \| \hat{f}_{k_n^0} - f \|^2_{L^2} \leq C_{f, \varphi} (n \wedge m)^{-1}$ for all $n, m \in \mathbb{N}$ and a finite constant $C_{f, \varphi} > 0$ depending on $f$ and $\varphi$ only.} We shall further
emphasise that in case \( n = m \) for any density \( f \) and \( \varphi \) it holds

\[
\| \Pi^\perp_j f \|_{1, \Lambda^2}^2 = \sum_{|j| \leq k^*_n(j)} |[f]_j|^2 [1 \wedge n^{-1} \Phi_j] + \sum_{|j| > k^*_n(j)} |[f]_j|^2 [1 \wedge n^{-1} \Phi_j] \\
\leq \| \Pi^\perp_j f \|_{L^2}^2 \frac{c \sum_{k^*_n} k^*_n}{n} + \| \Pi^\perp_j f \|_{L^2}^2 \frac{b^2_n}{n} \leq 2 \| \Pi^\perp_j f \|_{L^2}^2 \mathcal{R}_n^\Lambda (b^2, \Sigma^\Phi) \tag{3.3}
\]

which in turn implies \( \mathbb{E}_{f, \varphi} \| \hat{f}_{k_n} - f \|_{L^2}^2 \leq C(1 \vee \| \Pi^\perp_j f \|_{L^2}^2) \mathcal{R}_n^\Lambda (b^2, \Sigma^\Phi) \). In other words, the estimation of the unknown error density \( \varphi \) is negligible whenever \( n \leq m \).

Aggregation. Introducing aggregation weights \( w \) consider an aggregation \( \hat{f}_w = \sum_{k \in [n]} w_k \hat{f}_k \) of the orthogonal series estimators \( \hat{f}_k, k \in \mathbb{N} \), defined in (1.3) with coefficients \( ([\hat{f}_w])_{j \in \mathbb{Z}} \) satisfying \( \hat{f}_w \equiv 0 \) for \( |j| > n \), and \( \mathbb{E}_{f, \varphi} ([\hat{f}_w])_{j \in \mathbb{Z}} \) for any \( |j| \in [n] \). We note that again by construction \( \| \hat{f}_w \|_0 = 1 \), \( \| \hat{f}_w \|_j = \| \hat{f}_w \|_j \) and \( 1 \geq \| \hat{f}_w \|_j \). Hence, \( \hat{f}_w \) is real and integrates to one, however, it is not necessarily non-negative. Our aim is to prove an upper bound for its risk \( \mathbb{E}_{f, \varphi} \| \hat{f}_w - f \|_{L^2}^2 \) and its maximal risk \( \sup \{ \mathbb{E}_{f, \varphi} \| \hat{f}_{\varphi} - f \|_{L^2}^2 : f \in \mathcal{F}, \varphi \in \mathcal{F}_f \} \). Here and subsequently, we denote \( \hat{f}_{\varphi} := \sum_{j=-k}^{k} [\hat{\varphi}]_j \hat{e}_j = \sum_{j=-k}^{k} [\hat{\varphi}]_j \hat{e}_j \) for \( k \in \mathbb{N} \). For arbitrary aggregation weights and penalties, the next lemma establishes an upper bound for the loss of the aggregated estimator. Selecting the weights and penalties suitably, it provides in the sequel our key argument.

**Lemma 3.2.** Consider an weighted sum \( \hat{f}_w \) with arbitrary aggregation weights \( w \) and non-negative penalty terms \( \{pen^\Phi_k\}_{k \in [n]} \). For any \( k_-, k_+ \in [n] \) holds

\[
\| \hat{f}_w - f \|_{L^2}^2 \leq 3 \| \hat{f}_{k_+} - \hat{f}_{k_-} \|_{L^2}^2 + 3 \| \Pi^\perp_j f \|_{L^2}^2 b^2_{k_-} (f) \\
\leq 3 \| \Pi^\perp_j f \|_{L^2}^2 \mathbb{E}_{f, \varphi} ([k_- \varphi]) + \frac{3}{\eta} \sum_{l \in [k_+, n]} pen^\Phi_l \mathbb{1}_{\{\| \hat{f}_l - f \|_{L^2}^2 < pen^\Phi_l \}} \\
+ \frac{3}{\eta} \sum_{l \in [k_+, n]} \left( |\hat{f}_l - f|_{L^2}^2 - pen^\Phi_l / \eta \right)_+ + \frac{3}{\eta} \sum_{l \in [k_+, n]} pen^\Phi_l \mathbb{1}_{\{\| \hat{f}_l - f \|_{L^2}^2 \geq pen^\Phi_l / \eta \}} \\
+ 6 \sum_{j \in [n]} \| [\hat{\varphi}]_j \|_{L^2}^4 - \| [\varphi]_j \|_{L^2}^4 + 2 \sum_{j \in [n]} 1_{\{|\varphi|_j^2 < 1/m \}} |[f]_j|^2 \tag{3.4}
\]

**Remark 3.3.** The upper bound in (3.4) is similar to (2.8) apart of the last two which are controled again by Lemma 3.6. However, in order to control the third and fourth term we replace in both weights and penalties the quantities \( \Phi \), which are not anymore known in advance, by their natural estimators.

We consider either for some constant \( \eta \in \mathbb{N} \) Bayesian weights \( (\hat{\varphi}_k)_{k \in [n]} \) as in (1.7) or model selection weights \( (\hat{\varphi}_k)_{k \in [n]} \) as in (1.6). Given \( \hat{\Phi} = (\hat{\Phi}_j)_{j \in \mathbb{Z}} \), with

\[
\hat{\Phi}_j := |[\varphi]_j^+|^2 - |[\varphi]_{j-1}^+|^2 = |[\varphi]_{j-1}^-|^2 1_{\{|[\varphi]_j^2| \geq 1/m \}} ; \tag{3.5}
\]
If $\Lambda_k^\phi$ as in (2.11) with $x$, replaced by $\hat{\Phi}$, and a numerical constant $\Delta > 0$ we use

$$\text{pen}_k^\phi := \Delta k \Lambda_k^\phi n^{-1}, \quad k \in \mathbb{N}$$

as penalty terms.

**Theorem 3.4.** Consider a weighted sum $\hat{f}_w = \sum_{k \in [n]} w_k \hat{f}_k$ using either Bayesian weights $w := \hat{w}$ as in (1.7) or model selection weights $w := \hat{w}$ as in (1.6) and penalties $(\text{pen}_k^\phi)$, as in (3.6) with numerical constant $\Delta \geq 84$. Let $k_g := [3(400)^2||[g]||^2_w]$ and for $m \in \mathbb{N}$ set $k_m^* := \max\{k \in [m] : 289 \log(k + 2) \lambda_k^\phi \Phi(k) \leq m\}$, if the defining set is not empty, and $k_m^* := \lceil k_g \log(2 + m) \rceil$ otherwise.

(p) Assume there is $K \in \mathbb{N}_0$ with $1 \geq b_{(K-1)}(f) > 0$ and $b_K(f) = 0$. If $K = 0$ we set $c_f := 0$

and $c_f := \frac{104\Delta}{\|\hat{f}\|_2^2b_{(K-1)}(f)}$, otherwise. For $n \in \mathbb{N}$ let $k_n^* := \max\{k \in [n] : n > c_f k_m^*\}$, if the defining set is not empty, and $k_n^* := \lceil k_g \log(2 + n) \rceil$ otherwise. There is a constant $C_{f,\varphi}$ given in (C.15) depending only on $f$ and $\varphi$ such that for all $n, m \in \mathbb{N}$ holds

$$\mathbb{E}_{f,\varphi} \| \hat{f}_w - f \|^2_{L^2} \leq C \| \Pi_{f,\varphi} \|_{L^2}^{-1} \exp \left( -\frac{\lambda_n^\phi \lambda_m^\phi \lambda_n k_m^*}{k_g} \right) + C_{f,\varphi} (n \wedge m)^{-1}. \quad (3.7)$$

(np) Assume $b_k(f) > 0$ for all $k \in \mathbb{N}$ and consider $\rho_n^\phi(b_k^2, \Lambda_k^\phi)$ as in (2.14). There is a constant $C_{f,\varphi}$ given in (C.11) depending only on $f$ and $\varphi$ such that for all $n, m \in \mathbb{N}$ holds

$$\mathbb{E}_{f,\varphi} \| \hat{f}_w - f \|^2_{L^2} \leq C \{ 1 \vee \| \Pi_{f,\varphi} \|_{L^2}^2 \} \rho_n^\phi(b_k^2, \Lambda_k) + \rho_n^\phi(f, \Phi) \} + C_{f,\varphi} (n \wedge m)^{-1} \quad \text{with} \quad \rho_n^\phi(f, \Phi) := \| \Pi_{f} \|_{L^2}^2 \vee \| \Pi_{f,\varphi} \|_{L^2}^2 \left[ b_{k_m^*}^2(f) \vee \exp \left( -\frac{\lambda_n^\phi \lambda_m^\phi k_m^*}{k_g} \right) \right]. \quad (3.8)$$

**Corollary 3.5.** Let the assumptions of Theorem 3.4 be satisfied and in addition (A4) there is $m_{f,\varphi} \in \mathbb{N}$ such that $\lambda_n^\phi k_m^* \geq k_g \log m$ for all $m \geq m_{f,\varphi}$.

(p) If (A1) as in Corollary 2.5 and (A4) hold true, then there is a constant $C_{f,\varphi}$ depending only on $f$ and $\varphi$ such that for all $n, m \in \mathbb{N}$ holds $\mathbb{E}_{f,\varphi} \| \hat{f}_w - f \|^2_{L^2} \leq C_{f,\varphi} (n \wedge m)^{-1}$.

(np) If (A2) as in Corollary 2.5 and (A4) hold true, then there is a constant $C_{f,\varphi}$ depending only on $f$ and $\varphi$ such that $\mathbb{E}_{f,\varphi} \| \hat{f}_w - f \|^2_{L^2} \leq C_{f,\varphi} (R_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2 \| \Pi_{f,\varphi} \|_{L^2}^2 + b_{k_m^*}^2(f))$ for all $n, m \in \mathbb{N}$ holds true.

**Illustration 3.6.** Consider the cases (o) and (s) for the error density $\varphi$ as in Illustration 2.6, where in both cases Corollary 2.5 (A1) holds true (cf. Illustration 2.6 (o) and (s)). Moreover Corollary 3.5 (A4) is satisfied, since (o) $k_m^* \sim (m / \log m)^{1/(2a)}$ and $k_m^* \lambda_n^\phi \sim (m / \log m)^{1/(2a)}$, and (s) $k_m^* \sim (\log m)^{1/(2a)}$ and $k_m^* \lambda_n^\phi \sim (\log m)^{2+1/(2a)}$. Therefore, employing Corollary 3.5 (p) the fully data-driven aggregation attains the parametric rate. For (np) due to Corollary 3.5 the risk of the fully data-driven aggregated estimator has the order $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$, $\mathcal{R}_n^\phi(b_k^2, \Lambda_k^\phi) + \| \Pi_{f,\varphi} \|_{L^2}^2$.
if (A2) and (A4) are satisfied and \( b_{k_m}^2(f) \) is negligible with respect to \( \| \Pi_0^l f \|_{\Lambda_{\Phi}/m}^2 \). The upper bound \( \rho_m^o(f, \Phi_k) \) in Theorem 3.4 (3.8) faces otherwise a deterioration compared to \( \| \Pi_0^l f \|_{\Lambda_{\Phi}/m}^2 \) which we illustrate considering the cases in Illustration 2.6. Note that the other upper bound \( \rho_m^o(b^2, \Lambda^*_{\Phi}) \) in Theorem 3.4 (3.8) already appears in Proposition 2.4 and has been discussed in Illustration 2.6. Therefore, we state below the order of the additional term \( \rho_m^o(f, \Phi_k) \) only.

| Order | \( ||f||_{k,l}^2 \) | \( \Phi_k \) | \( ||\Pi_0^l f||_{\Lambda_{\Phi}/m}^2 \) | \( \rho_m^o(f, \Phi_k) \) |
|-------|----------------|---------|------------------|----------------|
| [o-o] | \( k^{-2p-1} \) | \( k^{2a} \) | \( m^{-p/a} \) | \( (m/ \log m)^{-p/a} : p < a \) |
| [o-s] | \( k^{-2p-1} \) | \( e^{k^{2a}} \) | \( m^{-1} \) | \( (m/ \log m)^{-1} : p = a \) |
| [s-o] | \( k^{-2p-1} e^{-k^{2a}} \) | \( k^{2a} m^{-1} \) | \( m^{-1} \) | \( | \log m |^{-p/a} : p > a \) |

Combining the Illustrations 2.6 and 3.6 the fully data-driven aggregation attains the rate \( R_n^o(b^2, \Sigma^*) + \| \Pi_0^l f \|_{\Lambda_{\Phi}/m}^2 \) in case [o-s], [o-o] with \( p \geq a \), and [s-o] with \( p \leq 1/2 \). In case [o-o] with \( p < a \) and [s-o] with \( p > 1/2 \) its rate features a deterioration compared to \( R_n^o(b^2, \Sigma^*) + \| \Pi_0^l f \|_{\Lambda_{\Phi}/m}^2 \) by a logarithmic factor \( (\log m)^{p/a} \) and \( (\log n)^{2a+1}(1-1/(2p)) \), respectively.

**Minimax optimality.** For \( m \in \mathbb{N} \) setting \( \|f,(1 \land s_j/m)\|_\infty := \sup \{ f_j(1 \land s_j/m) : j \in \mathbb{N}_0 \} \) it holds \( \| \Pi_0^l f \|_{\Lambda_{\Phi}/m}^2 \leq crd (R_n^o(f, \Sigma^*) + \|f,(1 \land s_j/m)\|_\infty) \) for all \( f \in \mathcal{F}^l \) and \( \varphi \in \mathcal{F}^d \). Exploiting again the upper bound (3.2) and the definition (2.5) for all \( k, m \in \mathbb{N} \) follows immediately

\[
\sup \{ E_{\varphi}^{m,n} \| \hat{f}_k - f \|_{L^2}^2 : f \in \mathcal{F}_m, \varphi \in \mathcal{E}^d \} \leq c \varphi d (R_n^o(f, \Sigma^*) + \|f,(1 \land s_j/m)\|_\infty).
\]

The upper bound in the last display depends on the dimension parameter \( k \) and hence by choosing an optimal value \( k_n^o(f, \Sigma^*) \) as in (2.5) the upper bound will be minimised and it holds \( \sup \{ E_{\varphi}^{m,n} \| \hat{f}_{k_n^o(f, \Sigma^*)} - f \|_{L^2}^2 : f \in \mathcal{F}_m, \varphi \in \mathcal{E}^d \} \leq c \varphi d (R_n^o(f, \Sigma^*) + \|f,(1 \land s_j/m)\|_\infty) \). On the other hand Johannes and Schwarz [2013] have shown that for all \( n, m \in \mathbb{N} \)

\[
\inf_{\varphi} \sup \{ E_{\varphi}^{m,n} \| \hat{f} - f \|_{L^2}^2 : f \in \mathcal{F}_m, \varphi \in \mathcal{E}^d \} \geq C \|f,(1 \land s_j/m)\|_\infty,
\]

where \( C > 0 \) and the infimum is taken over all possible estimators \( \hat{f} \) of \( f \). Consequently, combining (2.16) and the last lower bound \( (R_n^o(f, \Sigma^*) + \|f,(1 \land s_j/m)\|_\infty)_{n,m \in \mathbb{N}}, (k_n^o(f, \Sigma^*))_{n \in \mathbb{N}} \) and \( (\hat{f}_{k_n^o(f, \Sigma^*)})_{n \in \mathbb{N}} \), respectively, is a minimax rate, a minimax dimension and minimax optimal estimator (up to a constant).

**Aggregation.** By applying Lemma 3.2 we derive bounds for the maximal risk defined of the fully data-driven aggregation.
Theorem 3.7. Consider an aggregation \( \hat{f}_w \) using either Bayesian weights \( w = \hat{w} \) as in (1.7) or model selection weights \( w := \tilde{w} \) as in (1.6) and penalties \((\text{pen}_k^s)_k \in [n] \) as in (3.6) with numerical constant \( \Delta \gg 84 \). Let (A3) be satisfied and \( \rho_0^s(\hat{f}, \mathcal{A}_t^s) \) as in (2.19). Set \( k_{fs} := [3(400)^2 r \zeta_d \| \hat{f}/s_\|_{1,1}] \) and for \( m \in \mathbb{N} \), \( k_{m}^s := \max \{ k \in \mathbb{Z}^+ : 289 \log(k+2) \zeta_d \lambda_k^s s_k \leq m \} \), if the defining set is not empty, and \( k_{m}^s := \lfloor k_{fs} \log(2+m) \rfloor \) otherwise. Then there is a constant \( C_{fs}^d \) given in (C.23) depending only on the classes \( \mathcal{F}_r^s \) and \( \mathcal{E}_r^d \) such that for all \( n, m \in \mathbb{N} \)

\[
\sup \left\{ \mathbb{E}_{f, r}^n \| \hat{f}_w - f \|_{L_2}^2 : f \in \mathcal{F}_r^s, \varphi \in \mathcal{E}_r^d \right\} \leq C(r+\zeta_d) \left( \rho_0^n(\hat{f}, \mathcal{A}_t^s) + \rho_m^n(\hat{f}, s_\|_{1,1}) \right) + C_{fs}^d (n \wedge m)^{-1} \quad \text{with} \quad \rho_m^n(\hat{f}, s_\|_{1,1}) := \| \hat{f}, (1 \wedge s_\|_{1,1}) \|_{\infty} \vee \hat{f}_m \vee \exp \left( \frac{-\lambda_m^s k_{m}^s}{k_{fs}} \right). \tag{3.9}
\]

Corollary 3.8. Let the assumptions of Theorem 3.7 be satisfied. If (A2') as in Corollary 2.8 and in addition (A4') there is \( m_{fs} \in \mathbb{N} \) such that \( \lambda_{m_{fs}}^s k_{m_{fs}}^s \geq k_{fs} \log m \) for all \( m \geq m_{fs} \), then there is a constant \( C_{fs}^d \) depending only on the classes \( \mathcal{F}_r^s \) and \( \mathcal{E}_r^d \) such that for all \( n, m \in \mathbb{N} \) holds \( \sup \left\{ \mathbb{E}_{f, r}^n \| \hat{f}_w - f \|_{L_2}^2 : f \in \mathcal{F}_r^s, \varphi \in \mathcal{E}_r^d \right\} \leq C_{fs}^d (R_n^o(\hat{f}, \mathcal{A}_t^s) + \| \hat{f}, (1 \wedge s_\|_{1,1}) \|_{\infty} + \hat{f}_m) \).

Illustration 3.9. We have discussed the order of \( \rho_0^n(\hat{f}, \mathcal{A}_t^s) \) appearing in Theorem 3.7 for typical configurations in Illustration 2.9, thus we state below the order of the additional term only.

| Order | \( \bar{f}_k \) | \( s_k \) | \( \| \hat{f}, (1 \wedge s_\|_{1,1}) \|_{\infty} \) | \( \rho_m^n(\hat{f}, s_\|_{1,1}) \) |
|-------|-------|-------|-----------------|-------------------|
| [o-o]  | \( k^{-2p} \) | \( k^{2a} \) | \( m^{-p/a} \) | \( m/\log m \}_{p/a} \quad : p \leq a, \quad m^{-1} \quad : p > a. \)
| [o-s]  | \( k^{-2p} \) | \( e^{k_{2a}} \) | \( \log m \}_{p/a} \quad : p \leq a, m^{-1} \quad : p > a. \)
| [s-o]  | \( e^{-k_{2p}} \) | \( k^{2a} \) | \( m^{-1} \) | \( m^{-1} \) |

Note that in all cases the additional assumption (A4') in Corollary 3.8 is satisfied (as in Illustration 3.6), and hence \( \rho_0^n(\hat{f}, s_\|_{1,1}) \) is of order \( \| \hat{f}, (1 \wedge s_\|_{1,1}) \|_{\infty} + \hat{f}_m \). Moreover, in case [o-o], [o-s] and [s-o] holds \( \hat{f}_{k_m} \sim (m/\log m)^{-p/a} \), \( \hat{f}_{k_m} \sim (\log m)^{-p/a} \) and \( \hat{f}_{k_m} \sim \exp(-(m/\log m)^{p/a}) \), respectively. Consequently, \( \hat{f}_{k_m} \) is negligible compared to \( \| \hat{f}, (1 \wedge s_\|_{1,1}) \|_{\infty} \) in case [o-s] and [s-o], but in [o-o] for \( p > a \) only. Combining Illustrations 2.9 and 3.9 the fully data-driven aggregation attains the minimax rate in case [o-s] with \( p > 0 \), [o-o] with \( p > a \) and [s-o] with \( p \leq 1/2 \), while in case [o-o] with \( p \leq a \) and [s-o] with \( p > 1/2 \) its rate features a deterioration by a logarithmic factor \( (\log m)^{p/a} \) and \( (\log n)^{(2a+1)(1-1/(2p))} \), respectively, compared to the minimax rate. \( \square \)

4 Simulation study

Let us illustrate the performance of the fully data-driven weighted sum of OSE’s either with model selection (1.6) or Bayesian (1.7) \( (\eta = 1) \) weights or by a simulation study. As a first
step, we calibrate the constant $\Delta$ appearing in the penalty (3.6). Indeed, Theorems 3.4 and 3.7 stipulate that any choice $\Delta \geq 84$ ensures optimal rates but this is not a necessary condition and the constant obtained this way is often too large. Hence, we select a value minimising a Bayesian empirical risk obtained by repeating 1000 times the procedure as described hereafter. We randomly pick a noise density $\varphi$ and a density of interest $f$, respectively, from a family of wrapped asymmetric Laplace distributions and a family of wrapped normal distributions. For the noise density the location parameter is uniformly-distributed in $[0, 1]$ and both the scale, and the asymmetry parameter follow a $\Gamma$ distribution with shape 0.5 and scale 1. For the density of interest the mean is again uniformly-distributed in $[0, 1]$, while the standard deviation has a $\Gamma$ distribution with shape 9 and scale 0.5. Next we generate a sample $(\varepsilon_i)_{i\in[m]}$ of size $m = 5000$ from $\varphi$, and a sample $(Y_i)_{i\in[n]}$ of size $n = 500$ from $g = f \ast \varphi$. We use them to construct the estimators $\hat{f}_w$, and $\hat{f}_\bar{w}$ as in Theorem 3.4 for a range of values of $\Delta$. Finally, we compute and store the $L^2$-loss of each estimator obtained this way. Given the result of the 1000 repetitions, for each value of the constant $\Delta$ we use the sample of $L^2$-losses to compute estimators of the mean squared error and the quantiles of the distribution of the $L^2$-loss. Finally, we select and fix from now on a value of $\Delta$ that minimises the empirical mean squared error. The results of this procedure are reported in fig. 1. Using the calibrated constants and samples of size $n = m = 1000$ in fig. 2 we depict a realisation of the weighted sum estimators with Bayesian or model selection weights. In this example, $f$ is a mixture of two von Mises distributions and $\varphi$ is a wrapped asymmetric Laplace distribution. The two estimators estimate the true density properly and behave similarly, we investigate next if there can be a significant performance difference.

In the remaining part of this section we illustrate the numerical performance of the weighted sum estimators and their dependence on the sample sizes $n$ and $m$ by reporting the Bayesian empirical risk obtained by repeating 100 times a procedure described next. In opposite to above we randomly pick a noise density $\varphi$ and a density of interest $f$, respectively, from a
family of wrapped normal distributions and a family of wrapped asymmetric Laplace distributions. For the noise density the mean and concentration parameters are uniformly distributed in [0, 1]. For the density of interest, location parameter is uniformly-distributed in [0, 1], and both the scale, and the asymmetry parameters follow a $\Gamma$ distribution with shape 1 and scale 5. Note that the families differ from the ones used to calibrate the constant $\Delta$. Next we generate a sample $(\varepsilon_i)_{i \in [m]}$ of size $m = 1000$ from $\varphi$, and a sample $(Y_i)_{i \in [n]}$ of size $n = 1000$ from $g = f \otimes \varphi$. For a range of subsamples with different samples sizes we construct the estimators $\hat{f}_\varepsilon$, and $\hat{f}_w$ as in Theorem 3.4 and compute their $L^2$-losses. Given the results of the 100 repetitions, for the different values of $n$ and $m$ we use the sample of $L^2$-losses to compute estimators of the mean squared error and the quantiles of the distribution of the $L^2$-loss. The evolution of the $L^2$-loss for the weighted sum estimator with Bayesian weights or model selection weights, and their ratio, is represented in fig. 3, when the sample sizes $n$ and $m$ vary. In figs. 3a and 3b, both empirical errors decrease nicely as $n$ and $m$ increase. In fig. 3c, it seems like, on smaller sample sizes the estimator with Bayesian weights performs better than the one with model selection weights, while the opposite happens for larger samples. In fig. 4 more attention is given to the spread of the $L^2$-loss around its empirical mean. The three columns (from left to right) refer to the estimator with Bayesian or model selection weights, and their ratio. In the first row (figs. 4a to 4c) the noise sample size is fixed at $m = 500$ and in each graph the sample-size $n$ increases from 100 to 1000. In the second row (figs. 4d to 4f) both sample have the same size $m = n$ which again in each graph increases from 100 to 1000. In the last row (figs. 4g to 4i) the size of the noisy sample is fixed at $n = 500$ and in each graph the sample-size $m$ of the noise increases from 100 to 1000. These graphics show that the distribution of the $L^2$-losses is skewed. However, in all cases, both estimators behave reasonably.

The simulations were performed with the R software, using the libraries 'circular', 'ggplot2', 'reshape2', 'foreach', and 'doParallel'. (see Agostinelli and Lund [2017], Corporation
Figure 3: Empirical Bayesian risk for Bayesian (a) or model selection (b) weights and their ratio (c) over 100 replicates as a function of the sample size $n$ (abscissa) and $m$ (ordinate).

Figure 4: Empirical convergence rate for weighted sum estimators with Bayesian and model selection weights, and their ratio
and Weston [2019], Microsoft and Weston [2020], R Core Team [2018], Wickham [2007, 2016)). All the scripts are available upon request to the authors.

Appendix

A Preliminaries

This section gathers technical results. The next result is due to Johannes et al. [2020].

Lemma A.1. Given \( n \in \mathbb{N} \) and \( \tilde{f}, f \in L^2 \) consider the families of orthogonal projections \( \{ \tilde{f}_k = \Pi_{k} \tilde{f}, k \in [n] \} \) and \( \{ \tilde{f}_k = \Pi_{k} f, k \in [n] \} \). If \( \| \Pi_{k} \tilde{f} \|_{L^2}^2 = \| \Pi_{k} f \|_{L^2}^2 \) for all \( k \in [n] \), then for any \( l \in [n] \) holds

\[
\begin{align*}
(\mathbf{i}) & \quad \| \tilde{f}_k \|_{L^2}^2 - \| f_k \|_{L^2}^2 \leq \frac{1}{n} \| \tilde{f}_k - f_k \|_{L^2}^2, \\
(\mathbf{ii}) & \quad \| \tilde{f}_k \|_{L^2}^2 - \| f_k \|_{L^2}^2 \leq \frac{\tilde{f}_k}{f_k} \| \tilde{f}_k - f_k \|_{L^2}^2 + \frac{3}{n} \| \Pi_{k} f \|_{L^2}^2 \bigl( \| f_k \|_{L^2}^2 - \| f_k \|_{L^2}^2 \bigr),
\end{align*}
\]

for all \( k \in \mathbb{N} \).

The next assertion provides our key arguments in order to control the deviations of the reminder terms. Both inequalities are due to Talagrand [1996], the formulation of the first part eq. (A.2) can be found for example in Klein and Rio [2005], while the second part eq. (A.3) is based on equation (5.13) in Corollary 2 in Birgé and Massart [1998] and stated in this form for example in Comte and Merlevede [2002].

Lemma A.2. (Talagrand’s inequalities) Let \( (Z_i)_{i \in [n]} \) be independent \( \mathbb{Z} \)-valued random variables and let \( \nu_h = n^{-1} \sum_{i \in [n]} \nu_i Z_i - \mathbb{E} (\nu_i Z_i) \) for \( \nu_i \) belonging to a countable class \( \{ \nu_h, h \in \mathcal{H} \} \) of measurable functions. If the following conditions are satisfied

\[
\sup_{h \in \mathcal{H}} \sup_{z \in \mathbb{Z}} |\nu_h(z)| \leq \psi, \quad \mathbb{E} (\sup_{h \in \mathcal{H}} |\nu_h|) \leq \Psi, \quad \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i \in [n]} \text{Var}(\nu_i Z_i) \leq \tau, \quad (A.1)
\]

then there is an universal numerical constant \( C > 0 \) such that

\[
\begin{align*}
\mathbb{E} (\sup_{h \in \mathcal{H}} |\nu_h|^2 - 6\Psi^2) & \leq C \left[ \frac{\tau}{n} \exp \left( \frac{-n\Psi^2}{6\tau} \right) + \frac{\psi^2}{n^2} \exp \left( \frac{-n\Psi}{100\psi} \right) \right], \quad (A.2) \\
\mathbb{P} (\sup_{h \in \mathcal{H}} |\nu_h|^2 \geq 6\Psi^2) & \leq 3 \left[ \exp \left( \frac{-n\Psi^2}{400\tau} \right) + \exp \left( \frac{-n\Psi}{200\psi} \right) \right]. \quad (A.3)
\end{align*}
\]

Remark A.3. Introduce the unit ball \( \mathbb{B}_k := \{ h \in \mathcal{U}_k : \| h \|_{L^2} \leq 1 \} \) contained in the linear subspace \( \mathcal{U}_k \). Setting \( \nu_h(Y) = \sum_{\{i \in [k] \} \mid \{i \} \| h_j \|_{L^2}^{-1} e_j(-Y) \) we have

\[
\| \tilde{f}_k - f_k \|_{L^2}^2 = \sup_{h \in \mathbb{B}_k} \left\{ \sum_{\{j \in [k] \} \mid \{j \} \| h_j \|_{L^2}^{-1} \left( \frac{1}{n} \sum_{i \in [n]} (e_j(-Y_i) - [g_j, z]) \right) \right\}^2 = \sup_{h \in \mathbb{B}_k} \| \nu_h \|^2.
\]
The last identity provides the necessary argument to link the next Lemmata A.4 and A.5 and Talagrand’s inequalities in Lemma A.2. Note that, the unit ball $B_k$ is not a countable set of functions, however, it contains a countable dense subset, say $\mathcal{H}$, since $L^2$ is separable, and it is straightforward to see that $\sup_{h \in B_k} |\mathcal{P}_n| = \sup_{h \in \mathcal{H}} |\mathcal{P}_n|$. □

The proof of Lemma A.4 given in Johannes and Schwarz [2013] makes use of Lemma A.2 by computing the quantities $\psi, \Psi$, and $\tau$ which verify the three inequalities (A.1). We provide in Lemma A.5 a slight modification of this result following along the lines of the proof of Lemma A.4 in Johannes and Schwarz [2013].

**Lemma A.4.** Let $\Phi_{(k)} = \max\{\Phi_j, j \in [k]\}$, $\lambda_k^\Phi \geq 1$ and $k \Lambda_k^\Phi = \lambda_k^\Phi k \Phi_{(k)}$, then there is a numerical constant $C$ such that for all $n \in \mathbb{N}$ and $k \in [n]$ holds

1. $\mathbb{E}^\nu_{\mathbb{P}}(\|\hat{f}_k - \tilde{f}_k\|_2^2 - 12k \Lambda_k^\Phi n^{-1})_+ \leq C \left[ \frac{\|g\|_4 \Phi_{(k)}}{n} \exp \left( \frac{-\lambda_k^\Phi k}{\|g\|_4} \right) + \frac{2k \Phi_{(k)}}{n^2} \exp \left( \frac{-\sqrt{n \lambda_k^\Phi}}{200} \right) \right]$

2. $\mathbb{P}^\nu_{\mathbb{P}}(\|\hat{f}_k - \tilde{f}_k\|_2^2 \geq 12k \Lambda_k^\Phi n^{-1}) \leq 3 \left[ \exp \left( \frac{-\lambda_k^\Phi k}{200 \|g\|_4} \right) + \exp \left( \frac{-\sqrt{n \lambda_k^\Phi}}{200} \right) \right]$

**Lemma A.5.** Consider $\hat{f}_k - \tilde{f}_k = \sum_{j \in [k]} \tilde{\phi}_j \hat{\phi}_j \cdot [g]_j e_j$. Denote by $\mathbb{E}^\nu_{\tilde{\mathbb{P}}}$ and $\mathbb{P}^\nu_{\tilde{\mathbb{P}}}$ the conditional distribution and expectation, respectively, of $(Y_j)_{j \in [n]}$ given $(\varepsilon_j)_{j \in [n]}$. Let $\Phi_j = |[\phi]_j|^2, \Phi_k = \frac{1}{k} \sum_{j \in [k]} \hat{\phi}_j \hat{\phi}_j, \argmax_j = \lambda_k^\Phi k \hat{\phi}_j (k)$ and $\lambda_k^\Phi \geq 1$. Then there is a numerical constant $C$ such that for all $n \in \mathbb{N}$ and $k \in [n]$ holds

1. $\mathbb{E}^\nu_{\mathbb{P}}(\|\hat{f}_k - \tilde{f}_k\|_2^2 - 12 \lambda_k^\Phi n^{-1})_+ \leq C \left[ \frac{\|g\|_4 \Phi_{(k)}}{n} \exp \left( \frac{-\lambda_k^\Phi k}{\|g\|_4} \right) + \frac{2k \Phi_{(k)}}{n^2} \exp \left( \frac{-\sqrt{n \lambda_k^\Phi}}{200} \right) \right]$

2. $\mathbb{P}^\nu_{\mathbb{P}}(\|\hat{f}_k - \tilde{f}_k\|_2^2 \geq 12 \lambda_k^\Phi n^{-1}) \leq 3 \left[ \exp \left( \frac{-\lambda_k^\Phi k}{200 \|g\|_4} \right) + \exp \left( \frac{-\sqrt{n \lambda_k^\Phi}}{200} \right) \right]$

**Proof of Lemma A.5.** For $h \in B_k$ set $\nu_h(Y) = \sum_{j \in [k]} |[\phi]_j |^2 e_j(Y)$ where $\mathbb{E}^\nu_{\mathbb{P}} \nu_h(Y) = \sum_{j \in [k]} |[\phi]_j |^2 |g|_j$ and $\|\hat{f}_k - \tilde{f}_k\|_2 = \sup_{h \in B_k} |\mathcal{P}_n|^2$ (see Remark A.3). We intend to apply Lemma A.2. Therefore, we compute next quantities $\psi, \Psi$, and $\tau$ verifying the three inequalities required in Lemma A.2. First, we have $\sup_{h \in B_k} \sup_{y \in [0, 1]} |\nu_h(y)|^2 = 2 \sum_{j \in [k]} |\hat{\phi}_j| \leq 2k \hat{\Phi}_{(k)} = \psi^2$. Next, find $\Psi$. Exploiting $\sup_{h \in B_k} |\langle \hat{f}_k - \tilde{f}_k, h \rangle|_2 = \sum_{j \in [k]} |[\phi]_j |^2 |g|_j$ and $\mathbb{E}^\nu_{\mathbb{P}} |[\phi]_j |^2 \leq \frac{1}{n}$, it holds $\mathbb{E}^\nu_{\mathbb{P}} \left( \sup_{h \in B_k} |\langle \hat{f}_k - \tilde{f}_k, h \rangle|_2 \right) \leq 2 \sum_{j \in [k]} \hat{\Phi}_{j/n}^2 \leq 2 \lambda_k^\Phi \leq \psi^2$. Finally, consider $\tau$. Using $\mathbb{E}^\nu_{\mathbb{P}} (e_j(Y_1) e_j(Y_2)) = \langle [g], [g] \rangle$ for each $h \in B_k$

$$\mathbb{E}^\nu_{\mathbb{P}} \nu_h(Y_1) = \sum_{j \in [k]} \langle [g], [g] \rangle = \langle \hat{U}_k \hat{U}_k^* [h], [h] \rangle$$

defining the Hermitian and positive semi-definite matrix $\hat{A} := (\langle [g], [g] \rangle)_{j,j'}$, $j,j' \in \mathbb{Z}$ and the mapping $\hat{U}_k : \mathbb{C}^Z \rightarrow \mathbb{C}^Z$ with $z \mapsto \hat{U}_k z = (z_l \mathbb{I}_{\{[l] \in [k]\}})_{l \in \mathbb{Z}}$. Obviously, $\hat{U}_k$ is an orthogonal
projection from \(\ell^2\) onto the linear subspace spanned by all \(\ell^2\)-sequences with support on the index-set \([-k, -1] \cup [k]\). Straightforward algebra shows

\[
\sup_{h \in B_k} \frac{1}{n} \sum_{i \in [n]} \text{Var}_{Y|e}(\nu_h(Y_i)) \leq \sup_{h \in B_k} \langle U_k, \hat{A}U_k[h], [h] \rangle_{\ell^2} = \sup_{h \in B_k} \|U_k, \hat{A}U_k[h]\|_{\ell^2} \leq \|U_k, \hat{A}U_k\|_s.
\]

where \(\|M\|_s := \sup_{\|x\|_2 \leq 1} \|Mx\|_{\ell^2}\) denotes the spectral-norm of a linear \(M : \ell^2 \to \ell^2\). For a sequence \(z \in \mathbb{C}^Z\) let \(\nabla_z\) be the multiplication operator given by \(\nabla_z x := (z_j x_j)_{j \in \mathbb{Z}}\). Clearly, we have \(U_k, \hat{A}U_k = U_k \nabla_{(g)} + U_k C[g] U_k \nabla_{(g)} U_k\), where \(C[g] := ([g]_{j,j'})_{j,j' \in \mathbb{Z}}\). Consequently,

\[
\sup_{h \in B_k} \frac{1}{n} \sum_{i \in [n]} \text{Var}_{Y|e}(\nu_h(Y_i)) \leq \|U_k \nabla_{(g)} + U_k\|_s \|C[g]\|_s \|U_k \nabla_{(g)} U_k\|_s
\]

\[
= \|U_k \nabla_{(g)} + U_k\|_s^2 \|C[g]\|_s,
\]

where \(\|U_k \nabla_{(g)} + U_k\|_s^2 = \max\{\hat{\Phi}_j, j \in [k]\} = \hat{\Phi}(k)\). For \((C[g]z)_k := \sum_{j \in \mathbb{Z}} [g]_{-k} z_j, k \in \mathbb{Z}\) it is easily verified that \(\|C[g]z\|_2^2 \leq \|\|g\|_2^2 \|z\|_2^2\) and hence \(\|C[g]\|_s \leq \|\|g\|_1\|_s\), which implies

\[
\sup_{h \in B_k} \frac{1}{n} \sum_{i \in [n]} \text{Var}_{Y|e}(\nu_h(Y_i)) \leq \|[g]\|_1 \hat{\Phi}(k) =: \tau.
\]

Replacing in Remark A.3 (A.1) and (A.2) the quantities \(\psi, \Psi\) and \(r\) together with \(k\lambda_k = \lambda_k^k \hat{\Phi}(k)\) gives the assertion (i) and (ii), which completes the proof. \(\square\)

**Lemma A.6.** There is a finite numerical constant \(C > 0\) such that for all \(j \in \mathbb{Z}\) hold \(m^2 \mathbb{E}[|[\varphi]_j| - [\tilde{\varphi}]_j]|^4 \leq C\), (i) \(\mathbb{E}[|[\varphi]_j, [\tilde{\varphi}]_j]|^2| < 4(1 + \Phi/j/m)\), \(m^2 \mathbb{E}[|[\tilde{\varphi}]_j|] < 4\mathbb{E}[|\varphi_j|^2] < 1/m\), (ii) \(m^2 \mathbb{E}[|[\varphi]_j, [\tilde{\varphi}]_j]|^2| < 4\mathbb{E}[|\varphi_j|^2] < 1/m\). Given \(k \in \mathbb{N}\) for all \(j \in [k]\) we have

(iii) \(\mathbb{E}[|[\varphi]_j, [\tilde{\varphi}]_j]|^2 < 1/3\) \(\leq 2 \exp\left(-\frac{m|\varphi_j|^2}{2}\right) < 2 \exp\left(-\frac{m}{72\Phi(j)}\right)\).

**Proof of Lemma A.6.** The elementary properties (i)-(iii) are shown, for example, in Johannes and Schwarz [2013] and the assertion (iv) follows directly from Hoeffding’s inequality. \(\square\)

**Lemma A.7.** Let \(m, k \in \mathbb{N}\) and set \(\tilde{\Omega}_k := \{1/4 \leq \Phi_j / \Phi_j \leq 9/4 : \forall j \in [k]\}\).

(i) If \(\Phi(k) \leq (4/9)m\) then \(\mathbb{P}_{m}(\tilde{\Omega}_k) \leq 2k \exp\left(-\frac{m}{72\Phi(m)}\right)\).

(ii) For \(m_k := \lfloor 9\Phi(k)/4 \rfloor\) holds \(\mathbb{P}_{m_k}(\tilde{\Omega}_k) \leq (555km_k^{-2}m^{-2}) \wedge (12km_k^{-1})\) for all \(m \in \mathbb{N}\).

(iii) If \(m \geq 289 \log(k + 2)\lambda_k \Phi(k)\) then \(\mathbb{P}_{m}(\tilde{\Omega}_k) \leq (11226m^{-2}) \wedge (53m^{-1})\).

**Proof of Lemma A.7.** We start our proof with the observation that for each \(j \in \mathbb{Z}\) with \(\Phi_j \leq (4/9)m\) holds \(\{1/4 \leq [\varphi]_j / [\tilde{\varphi}]_j \leq 1/3\} \subseteq \{1/2 \leq [\varphi]_j / [\tilde{\varphi}]_j \leq 3/2\}\). Consequently, if \(\Phi(k) \leq (4/9)m\) then \(\tilde{\Omega}_k \subseteq \bigcup_{j \in [k]} \{1/4 \leq [\varphi]_j / [\tilde{\varphi}]_j \leq 1/3\}\) and hence (i) follows from Lemma A.6 (iv). Consider (ii). Given \(k \in \mathbb{N}\) and \(m_k := \lfloor 9\Phi(k)/4 \rfloor \in \mathbb{N}\) we distinguish for \(m \in \mathbb{N}\) the cases
(a) $m > m_k$ and (b) $m \in [m_k]$. In case (a) it holds $\Phi_{(k)} \leq (4/9)m$, and hence (i) implies (ii). In case (b) (ii) holds trivially, since $\mathbb{P}^m(U_k^c) \leq m^2 m^{-2} \wedge m_k m^{-1}$. Consider (iii). Since $m \geq 289 \log(k+2)\lambda_k^* \Phi_{(k)} \geq (9/4)\Phi_{(k)}$ from (i) follows

$$m^2 \mathbb{P}^m(U_k^c) \leq km^2 \exp \left( -\frac{m}{144 \Phi_{(k)}} \right) \leq 11226 k \Phi_{(k)}^2 \exp \left( -\frac{m}{144 \Phi_{(k)}} \right) \leq 11226$$

and analogously $m \mathbb{P}^m(U_k^c) \leq 53$, which completes the proof. \hfill \Box

Lemma A.8. Consider for any $l \in \mathbb{N}$ the event $\tilde{\Omega}_l := \{ \frac{1}{4} \leq \Phi^{-1}_j \tilde{f}_j < \frac{9}{4}, \forall j \in [l] \}$. For each $l \in \mathbb{N}$ and $k \in [l]$ setting $\| \Pi_{k} \tilde{f}_n \|_{L^2}^2 := \sum_{j \in [k]} \Phi^{-1}_j \tilde{f}_j \| [f]_j \|_2^2$ hold

(i) $\| \Pi_{k} \tilde{f}_n \|_{L^2}^2 \leq \| \Pi_{k} \tilde{f}_n \|_{L^2}^2$ and $\| \Pi_{k} \tilde{f}_n \|_{L^2}^2 1_{\tilde{\Omega}_l} \geq \frac{1}{4} \| \Pi_{k} \tilde{f}_n \|_{L^2}^2 (\tilde{b}_k^2(f) - b_k^2(f))$.

Moreover, for any $l \in \mathbb{N}$ and $k \in [l]$ hold

(ii) $\tilde{\Phi}_l \leq m, \frac{1}{4} \Phi_l \leq \tilde{\Phi}_l \| 1_{\tilde{\Omega}_l} \leq \frac{9}{4} \Phi_l, \lambda_k^* \geq 1, \frac{50}{100} \lambda_k^* \leq \lambda_k^* 1_{\tilde{\Omega}_l} \leq \frac{49}{50} \lambda_k^*$, and hence

(iii) $\{ \tilde{\Phi}_l < 1 \} = \{ \tilde{\Phi}_l = 0 \}$, and hence $\tilde{\text{pen}}^*_l = \text{pen}^*_l 1_{(\tilde{\Phi}_l \geq 1)}$.

Proof of Lemma A.8. The assertions (i) and (ii) follow by elementary calculations from the definition of the event $\tilde{\Omega}_l$, and we omit the details. Consider (iii). For each $j \in \mathbb{Z}$ holds $\tilde{\Phi}_j = \| [\tilde{\varphi}_j] \|_2^2 = 0$ on the event $\{ \| [\tilde{\varphi}_j] \|_2^2 \leq 1 / m \}$ and $\tilde{\Phi}_j \geq 1$ on the complement $\{ \| [\tilde{\varphi}_j] \|_2^2 \geq 1 / m \}$, since $\| [\tilde{\varphi}_j] \|_2^2 \leq 1$. Consequently, $\{ \tilde{\Phi}_j < 1 \} = \{ \| [\tilde{\varphi}_j] \|_2^2 \leq 1 / m \} = \{ \tilde{\Phi}_j = 0 \}$, which implies (iii), and completes the proof. \hfill \Box

B Proofs of section 2

Proof of Lemma 2.2. We start the proof with the observation that $\tilde{[f]}_0 - [f]_0 = 0$, and for each $j \in \mathbb{Z}$ holds $\tilde{[f]}_j - [f]_j = \tilde{[f]}_{j-1} - [f]_{j-1}$, where $\tilde{[f]}_j, [f]_j = -[f]_j$ for all $|j| > n$ and

$$\| \tilde{[f]}_j - [f]_j \|_{L^2}^2 = \| [\tilde{\varphi}_j] \|_2^2 \| (\tilde{g}_j - [g]_j) \mathbb{P}_c ([j], n) \| - [f]_j \mathbb{P}_c ([j], n) \|$$

for all $|j| \in [n]$.

Consequently, (keep in mind that $\| [\tilde{\varphi}]_j \|_2^2 = \Phi_j$) we have

$$\| \tilde{[f]}_0 - [f]_0 \|_{L^2}^2 \leq \sum_{|j| \in [n]} 2 \{ \Phi_j \| [g]_j - [g]_j \|_{\mathbb{P}_c ([j], n)] \}$$

$$+ \sum_{|j| \in [n]} 2 \| [f]_j \|_{\mathbb{P}_c ([j], n]} + \sum_{|j| > n} \| [f]_j \|_2^2, \quad \text{(B.1)}$$
where we consider the first and the two other terms on the right hand side separately. Considering the first term we split the sum into two parts. Precisely,

\[
\sum_{j \in [n]} \Phi_j \mathbb{P}_w(\|j\|, n) \leq \|\tilde{f}_k - f_k\|_{L^2}^2 + \sum_{l \in [k_+, n]} w_l \|\tilde{f}_l - f_l\|_{L^2}^2 \\
\leq \frac{1}{7} \text{pen}_{k_+} + \sum_{l \in [k_+, n]} \left(\|\tilde{f}_l - f_l\|_{L^2}^2 - \text{pen}_l / 7\right) + \frac{1}{7} \sum_{l \in [k_+, n]} \text{pen}_l \mathbb{1}_{\|\tilde{f}_l - f_l\|_{L^2}^2 < \text{pen}_l / 7} \quad (B.2)
\]

Considering the second and third term we split the first sum into two parts and obtain

\[
\sum_{j \in [n]} |[f_j]|^2 \mathbb{P}_w(\|j\|) + \sum_{j > n} |[f_j]|^2 \\
\leq \sum_{j \in [k_-]} |[f_j]|^2 \mathbb{P}_w(\|j\|) + \sum_{j \in [k_, n]} |[f_j]|^2 + \sum_{j > n} |[f_j]|^2 \\
\leq \|\Pi_{[k_-]} f\|_{L^2}^2 \left(\mathbb{P}_w([k_-]) + b_{k_-}^* \right) \quad (B.3)
\]

Combining (B.1) and (B.2), (B.3) we obtain the assertion, which completes the proof. \(\square\)

**B.1 Proof of Proposition 2.4 and Corollary 2.5**

**Proof of Proposition 2.4.** We present the main arguments to prove Proposition 2.4. The technical details are gathered in Lemmata B.2 to B.5 in the end of this section. Keeping in mind the definition (2.5) and (2.12) here and subsequently we use that

\[
\mathcal{R}_n^k(b^\circ_\kappa, \Lambda_\kappa^\circ) \geq b_{k_+}^* (f) \quad \text{and} \quad \Delta \mathcal{R}_n^k(b^\circ_\kappa, \Lambda_\kappa^\circ) \geq \text{pen}_k^* \quad \text{for all } k \in [n]. \quad (B.4)
\]

For arbitrary \(k_-^\circ, k_+^\circ \in [n]\) (to be chosen suitable below) let us define

\[
k_- := \min \left\{ k \in [k_-^\circ] : \|\Pi_{[k_-]} f\|_{L^2}^2 b_{k_-}^2 (f) \leq \|\Pi_{[k_-]} f\|_{L^2}^2 b_{k_-}^2 (f) + 4 \text{pen}_k^* \right\} \quad \text{and} \quad k_+ := \max \left\{ k \in [k_+^\circ, n] : \text{pen}_k^* \leq 6 \|\Pi_{[k_+]} f\|_{L^2}^2 b_{k_+}^2 (f) + 4 \text{pen}_{k_+}^* \right\} \quad (B.5)
\]

where the defining set obviously contains \(k_-^\circ\) and \(k_+^\circ\), respectively, and hence, it is not empty.

We intend to combine the upper bound in (2.8) and the bounds considering Bayesian weights \(w = \tilde{w}\) as in (2.9) and model selection weights \(w = \check{w}\) as in (2.10) given in Lemma B.2 and Lemma B.3, respectively. First note, that due to Lemma B.2 (i) we have

\[
\mathbb{E}_{\tilde{w}}^m \mathbb{P}_w([k_-]) \leq 1_{(k_- > 1)} \frac{1}{16} \text{exp} \left( - \frac{3n\Delta_{k_-}^*}{16} \right) \quad \text{and} \quad + 1_{(k_- > 1)} \mathbb{P}_w \left( \|\tilde{f}_{k_-} - f_{k_-}\|_{L^2}^2 \geq \text{pen}_{k_-}^* / 7 \right)
\]

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and, hence from (2.8) for Bayesian weights \( w = \tilde{w} \) as in (2.9) follows immediately

\[
\mathbb{E}_{\small{\mathcal{W}_{n,v}}} \| \tilde{f}_w - f \|_{L^2}^2 \leq \frac{2}{7} \text{pen}_k^\circ + 2\| \Pi_0^\circ f \|_{L^2}^2 b_{k_\circ}^2 (f) + n^{-\frac{12}{7}} + \frac{2}{n\Delta} \| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_+ > 1\}} \exp \left( - \frac{3n\Delta}{14} k_0^\circ \Lambda_k^\circ \right) \\
+ 2 \sum_{k \in [k_+^n, n]} \mathbb{E}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_k - f_k \|_{L^2}^2 - \frac{1}{7} \text{pen}_k^\circ)^+ + \frac{2}{7} \sum_{k \in [k_+^n, n]} \text{pen}_k^\circ \mathbb{P}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_k - f_k \|_{L^2}^2 \geq \frac{1}{7} \text{pen}_k^\circ) \\
+ 2\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} \mathbb{P}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_{k_-} - f_{k_-} \|_{L^2}^2 \geq \frac{1}{7} \text{pen}_{k_-}^\circ) \quad (B.6)
\]

On the other hand for model selection weights \( w = \tilde{w} \) we combine again the upper bound in (2.8) and the bounds given in Lemma B.3. Clearly, due to Lemma B.3 we have \( \mathbb{E}_{\small{\mathcal{W}_{n,v}}} (\| k_- \| = \mathbb{P}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_{k_-} - f_{k_-} \|_{L^2}^2 \geq \text{pen}_{k_-}^\circ / 7) \) and, hence from (2.8) follows immediately

\[
\mathbb{E}_{\small{\mathcal{W}_{n,v}}} \| \tilde{f}_k - f \|_{L^2}^2 \leq \frac{2}{7} \text{pen}_k^\circ + 2\| \Pi_0^\circ f \|_{L^2}^2 b_{k_-}^2 (f) + 2 \sum_{k \in [k_+^n, n]} \mathbb{E}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_k - f_k \|_{L^2}^2 - \frac{1}{7} \text{pen}_k^\circ)^+ + \frac{2}{7} \sum_{k \in [k_+^n, n]} \text{pen}_k^\circ \mathbb{P}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_k - f_k \|_{L^2}^2 \geq \frac{1}{7} \text{pen}_k^\circ) \\
+ 2\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} \mathbb{P}_{\small{\mathcal{W}_{n,v}}} (\| \tilde{f}_{k_-} - f_{k_-} \|_{L^2}^2 \geq \frac{1}{7} \text{pen}_{k_-}^\circ) \quad (B.7)
\]

The deviations of the last three terms in (B.6) and (B.7) we bound in Lemma B.4 by exploiting usual concentration inequalities. Precisely, we obtain

\[
\mathbb{E}_{\small{\mathcal{W}_{n,v}}} \| \tilde{f}_w - f \|_{L^2}^2 \leq \frac{2}{7} \text{pen}_k^\circ + 2\| \Pi_0^\circ f \|_{L^2}^2 b_{k_-}^2 + \mathcal{C} (\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} \exp \left( - \frac{\lambda_k^\circ}{k_g} \right) + \mathcal{C} (\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} + \Phi_{(k_0)} k_g^3) \Phi_{(n_0)} n^{-1} \quad (B.8)
\]

Indeed, combining Lemma B.4 and (B.6) for Bayesian weights we obtain

\[
\mathbb{E}_{\small{\mathcal{W}_{n,v}}} \| \tilde{f}_w - f \|_{L^2}^2 \leq \frac{2}{7} \text{pen}_k^\circ + 2\| \Pi_0^\circ f \|_{L^2}^2 b_{k_-}^2 \\
+ \mathcal{C} (\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} \left( \frac{1}{\eta} \exp \left( - \frac{3n\Delta}{14} k_0^\circ \Lambda_k^\circ \right) + \exp \left( - \frac{\lambda_k^\circ k_0^\circ}{200 \| g \|_{L^1} \|} \right) \right) + \mathcal{C} (\| \Pi_0^\circ f \|_{L^2}^2 1_{\{k_- > 1\}} + \Phi_{(k_0)} k_g^3 \Phi_{(n_0)} n^{-1} \quad (B.9)
\]

Therewith, be using that \( \Lambda_k^\circ > \lambda_k^\circ, 3n\Delta > \frac{1}{200 \| g \|_{L^1} \|} > \frac{1}{k_g} \) (since \( \eta \geq 1 \)) and \( \| g \|_{L^1} \geq \| \| g \|_{L^1} \| = 1 \) from (B.9) follows the upper bound (B.8). Consider secondly model selection weights \( w = \tilde{w} \) as in (2.10). Combining Lemma B.4, 200\( \| g \|_{L^1} \| \leq k_g \) and the upper bound given in (B.7) we obtain (B.8).

From the upper bound (B.8) for a suitable choice of the dimension parameters \( k_0^\circ, k_+ \in \mathbb{N} \) we derive separately the risk bound in the two cases (p) and (np) considered in Proposition 2.4. The tedious case-by-case analysis for (p) is deferred to Lemma B.5 in the end of this section.

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In case \((\text{np})\) with \(k_\nu^2 := k_\nu^0(b_\nu^2, \Lambda_\nu^2) \in [n]\) and \(\mathcal{R}_n^k(b_\nu^2, \Lambda_\nu^2)\) as in (2.5) we set \(k_\nu^0 := k_\nu^0\) and let \(k_\nu^0 \in [n]\). Keeping (B.4) in mind the definition (B.5) of \(k_+\) and \(k_-\) implies \(\text{pen}^*_{k_+} \leq 2(3||\Pi_0^k f||_2^2 + 2\Delta)\mathcal{R}_n^{k_0^+}(b_\nu^2, \Lambda_\nu^2)\) and \(\Pi_0^k f||_2^2, b_\nu^2|_{k_-} \leq ((||\Pi_0^k f||_2^2 + 4\Delta)\mathcal{R}_n^{k_0^-}(b_\nu^2, \Lambda_\nu^2)\) which together with \(\mathcal{R}_n^{k_0^0}(b_\nu^2, \Lambda_\nu^2) \geq \mathcal{R}_n^{k_0^0}(b_\nu^2, \Lambda_\nu^2) = \mathcal{R}_n^{k_0^0}(b_\nu^2, \Lambda_\nu^2) = \min \{\mathcal{R}_n^{k_0^0}(b_\nu^2, \Lambda_\nu^2), k \in \mathbb{N}\} \geq n^{-1}\) and exploiting (B.8) implies the assertion (2.14), that is for all \(k_\nu^0 \in [n]\) holds

\[
\mathbb{E}_{\mathcal{F}_{n, \nu}} [f_{\nu} - f]_2^2 \leq C \left(\|\Pi_0^k f\|_2^2 + 1\right) \left[\mathcal{R}_n^{k_0^0}(b_\nu^2, \Lambda_\nu^2) \vee \exp \left(-\frac{\lambda^*_{k_\nu^0} k_\nu^0}{\text{pen}^*_{k_\nu^0}}\right)\right] + C [\Phi_{k_\nu^0}^2 k_\nu^0 \Phi_{k_\nu^0}^2 + \Phi_{k_\nu^0}^2 n^{-1}], \quad (\text{B.10})
\]

with \(n_o = 15(600)^4\), which completes the proof of Proposition 2.4. \(\square\)

**Proof of Corollary 2.5.** Consider the case (p). Under (A1) for all \(n > n_{f, \Phi}\) we have trivially \(\exp \left(-\lambda^*_{k_\nu^0} k_\nu^0/k_\nu^0\right) \leq n^{-1}\), while for \(n \in [n_{f, \Phi}]\) holds \(\exp \left(-\lambda^*_{k_\nu^0} k_\nu^0/k_\nu^0\right) \leq n_{f, \Phi} n^{-1}\). Thereby, from (2.13) in Proposition 2.4 follows immediately the assertion (p). In case (np) due to (A2) for \(k_\nu^0 := k_\nu^0(b_\nu^2, \Lambda_\nu^2)\) as in (2.5) we have trivially \(\exp \left(-\lambda^*_{k_\nu^0} k_\nu^0/k_\nu^0\right) \leq \mathcal{R}_n^{k_\nu^0}(b_\nu^2, \Lambda_\nu^2)\) while for \(n \in [n_{f, \Phi}]\) holds \(\exp \left(-\lambda^*_{k_\nu^0} k_\nu^0/k_\nu^0\right) \leq 1 \leq n \mathcal{R}_n^{k_\nu^0}(b_\nu^2, \Lambda_\nu^2) \leq n_{f, \Phi} \mathcal{R}_n^{k_\nu^0}(b_\nu^2, \Lambda_\nu^2)\). Thereby, from (2.14) in Proposition 2.4 with \(\mathcal{R}_n^{k_\nu^0}(b_\nu^2, \Lambda_\nu^2) = \min_{k \in [n]} \mathcal{R}_n^{k_\nu^0}(b_\nu^2, \Lambda_\nu^2)\) follows (np), which completes the proof of Corollary 2.5. \(\square\)

Below we state and prove the technical Lemmata B.2 to B.5 used in the proof of Proposition 2.4. The proof of Lemma B.2 is based on Lemma B.1 given first.

**Lemma B.1.** Consider Bayesian weights \(\tilde{\omega}\) as in (2.9). Let \(l \in [n]\).

(i) For all \(k \in [l]\) holds \(\tilde{\omega}_k 1 \{\|f_{\nu} - f_l\|_2^2 \leq \text{pen}^*_{k_\nu^0}\} \leq \exp \left(\eta m \left\{\frac{1}{12} \text{pen}^*_{l} + \frac{3}{2} \||\Pi_0^k f||_2^2, b_\nu^2\|_2(f) - \frac{1}{2} \||\Pi_0^k f||_2^2, b_\nu^2\|_2 + \text{pen}^*_{k_\nu^0}\right\}\right)\).

(ii) For all \(k \in [l, n]\) holds \(\tilde{\omega}_k 1 \{\|f_{\nu} - f_l\|_2^2 \leq \text{pen}^*_{k_\nu^0}\} \leq \exp \left(\eta m \left\{\frac{1}{2} \text{pen}^*_{l} + \frac{3}{2} \||\Pi_0^k f||_2^2, b_\nu^2\|_2 + \text{pen}^*_{k_\nu^0}\right\}\right)\).

**Proof of Lemma B.1.** Given \(k, l \in [n]\) and an event \(\Omega_{kl}\) (to be specified below) it follows

\[
\tilde{\omega}_k 1_{\Omega_{kl}} = \frac{\exp(-\eta m \{\|f_{\nu} - f_l\|_2^2 + \text{pen}^*_{l}\})}{\sum_{l \in [n]} \exp(-\eta m \{\|f_{\nu} - f_l\|_2^2 + \text{pen}^*_{l}\})} 1_{\Omega_{kl}} \leq \exp \left(\eta m \left\{|\|f_{\nu} - f_l\|_2^2 - \|f_{\nu} - f_l\|_2^2 + \text{pen}^*_{l} - \text{pen}^*_{k_\nu^0}\right\}\right) 1_{\Omega_{kl}} \quad (\text{B.11})
\]

We distinguish the two cases (i) \(k \in [l]\) and (ii) \(k \in [l, n]\). Consider first (i) \(k \in [l]\). Due to Lemma A.1 (i) (with \(\tilde{f} := f_{\nu}\) and \(f := f\)) from (B.11) we conclude

\[
\tilde{\omega}_k 1_{\Omega_{kl}} \leq \exp \left(\eta m \left\{\frac{1}{2} \|\tilde{f}_{\nu} - f_l\|_2^2 - \frac{1}{2} \||\Pi_0^k f||_2^2(b_\nu^2(f) - b_\nu^2(f)) + (\text{pen}^*_{l} - \text{pen}^*_{k_\nu^0})\right\}\right) 1_{\Omega_{kl}}
\]
Considering \( \Omega_{l} := \{ 7 \| \tilde{f}_{l} - f_{l}\|_{L_{2}}^{2} < \text{pen}_{l}^{*} \} \) the last bound implies

\[
\tilde{w}_{k} \mathbb{I}_{\{ \| \tilde{f}_{l} - f_{l}\|_{L_{2}}^{2} < \text{pen}_{l}^{*} \}} \leq \exp \left( \eta_{l} \left\{ \mathbb{I}_{\{ \| \tilde{f}_{l} - f_{l}\|_{L_{2}}^{2} < \text{pen}_{l}^{*} \}} - \frac{1}{2} \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} (b_{l}^{*}(f_{l}) - b_{l}^{*}(f_{l})) + (\text{pen}_{l}^{*} - \text{pen}_{l}^{*}) \right\} \right).
\]

Rearranging the terms of the last upper bound we obtain the assertion (i). Consider secondly (ii) \( k \in [l, n] \). From Lemma A.1 (ii) (with \( \hat{f} := \tilde{f}_{n} \) and \( \hat{f} := f \)) and (B.11) follows

\[
\tilde{w}_{k} \mathbb{I}_{\Omega_{k}} \leq \exp \left( \eta_{l} \left\{ \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} \}} - \frac{1}{2} \| \Pi_{0}^{l} f_{k} \|_{L_{2}}^{2} (b_{k}^{*}(f_{k}) - b_{k}^{*}(f_{k})) + (\text{pen}_{k}^{*} - \text{pen}_{k}^{*}) \right\} \right).
\]

Setting \( \Omega_{k} := \{ 7 \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} \} \) and exploiting \( b_{k}^{a}(f) \geq 0 \) we obtain (ii), which completes the proof. \( \square \)

**Lemma B.2.** Consider Bayesian weights \( \tilde{w} \) as in (2.9) and penalties \( (\text{pen}_{k}^{*})_{k \in [n]} \) as in (2.12). For any \( k_{\omega}, k_{\omega}^{\ast} \in [n] \) and associated \( k_{+}, k_{-} \in [n] \) as in (B.5) hold

(i) \( \mathbb{P}_{\omega} ([k_{-}]) \leq \frac{1}{\eta_{k}} \mathbb{I}_{\{ \hat{f}_{k} > 1 \}} \left( - \frac{3n}{\eta} k_{\omega} \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} + \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} \geq \text{pen}_{k}^{*} / \tau \}} \right) \)

(ii) \( \sum_{k \in [k_{+}, n]} \text{pen}_{k}^{*} \tilde{w}_{k} \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} \leq \frac{16}{9} n^{-1} \).

**Proof of Lemma B.2.** Consider (i). Let \( k_{-} \in [k_{\omega}^{\ast}] \) as in (B.5). For the non trivial case \( k_{-} > 1 \) from Lemma B.1 (i) with \( l = k_{\omega}^{\ast} \) follows for all \( k < k_{-} \leq k_{\omega}^{\ast} \)

\[
\tilde{w}_{k} \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} \leq \exp \left( \eta_{l} \left\{ \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} - \frac{1}{2} \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} (b_{k}^{*}(f_{l}) - b_{k}^{*}(f_{l})) + (\text{pen}_{l}^{*} - \text{pen}_{l}^{*}) \right\} \right).
\]

and hence by exploiting the definition (B.5) of \( k_{-} \), that is \( \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} b_{k}^{2} \geq \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} (b_{k_{-}}^{2}) > \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} b_{k_{-}}^{2} + 4 \text{pen}_{k_{-}}^{*} \), we obtain for each \( k \in [k_{-}^{\ast}] \)

\[
\tilde{w}_{k} \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} \leq \exp \left( - \frac{3n}{\eta} \text{pen}_{k_{-}}^{*} \right) \frac{1}{\eta} \text{pen}_{k_{-}}^{*} \cdot \text{pen}_{k_{-}}^{*} \). \]

The last upper bound together with \( \text{pen}_{k}^{*} = \Delta k \mathbb{I}_{nk}^{n} \geq \Delta k \mathbb{I}_{nk}^{n-1} \), \( k \in [n] \), as in (2.11) gives

\[
\mathbb{P}_{\omega} ([k_{-}]) \leq \mathbb{P}_{\omega} ([k_{-}]) \left\{ \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} + \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} \geq \text{pen}_{k}^{*} / \tau \}} \right\} \leq \exp \left( - \frac{3n}{\eta} \text{pen}_{k_{-}}^{*} \right) \sum_{k \in [k_{-}]} \exp (-\eta \Delta k) + \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} \geq \text{pen}_{k}^{*} / \tau \}} \}
\]

which combined with \( \sum_{k \in [\mu]} \exp (-\mu k) \leq 1 \) for any \( \mu > 0 \) implies (i). Consider (ii). Let \( k_{+} \in [k_{\omega}^{\ast}, n] \) as in (B.5). For the non trivial case \( k_{+} < n \) from Lemma B.1 (ii) with \( l = k_{\omega}^{\ast} \) follows for all \( k > k_{+} \geq k_{\omega}^{\ast} \)

\[
\tilde{w}_{k} \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} \leq \exp \left( \eta_{l} \left\{ \mathbb{I}_{\{ \| \tilde{f}_{k} - f_{k}\|_{L_{2}}^{2} < \text{pen}_{k}^{*} / \tau \}} - \frac{1}{2} \| \Pi_{0}^{l} f_{l} \|_{L_{2}}^{2} (b_{k_{+}}^{*}(f_{l}) - b_{k_{+}}^{*}(f_{l})) + (\text{pen}_{k_{+}}^{*} - \text{pen}_{k_{+}}^{*}) \right\} \right).
\]

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Consequently, using $\lambda_k^p = \Delta k\lambda_k^p \Phi(k)n^{-1}$, $k \in \{n\}$, as in (2.11) implies

$$\sum_{k \in \{n\}} \text{pen}_k^{\phi} \widehat{w}_k \mathbf{1}_{\{k \in \{n\}, \|f_k - f_{k+1}\|_2^2 < \text{pen}_k^{\phi} / \eta\}} \leq \exp(\eta \{ - \frac{1}{4} \text{pen}_k^{\phi} \}).$$

Consequently, using $\text{pen}_k^{\phi} = \Delta k\lambda_k^p \Phi(k)n^{-1}$, $k \in \{n\}$, as in (2.11) implies

$$\sum_{k \in \{k+, n\}} \text{pen}_k^{\phi} \widehat{w}_k \mathbf{1}_{\{k \in \{k+, n\}, \|f_k - f_{k+1}\|_2^2 < \text{pen}_k^{\phi} / \eta\}} \leq \Delta n^{-1} \sum_{k \in \{k+, n\}} k\lambda_k^p \Phi(k) \exp\left( - \frac{n}{4} k\lambda_k^p \Phi(k) \right) \quad (B.12)$$

Exploiting that $(\lambda_k^p)^{1/2} = \frac{\log(k\Phi(k)) + (k-2)}{\log(k+2)} \geq 1$, $k\Phi(k) \leq \exp((\lambda_k^p)^{1/2} \log(k+2))$ for each $k \in \mathbb{N}$, $\Delta / 4 \geq 2 \log(3e)$ and $\eta \geq 1$ for all $k \in \mathbb{N}$ holds $\frac{n}{4} k - \log(k+2) \geq 1$. Making further use of the elementary inequality $a \exp(-ab) \leq \exp(-b)$ for $a, b \geq 1$ it follows

$$\lambda_k^p k\Phi(k) \exp\left( - \frac{n}{4} k\lambda_k^p \Phi(k) \right) \leq \lambda_k^p \exp\left( - \frac{n}{4} k\lambda_k^p \Phi(k) + \sqrt{\lambda_k^p \Phi(k)} \log(k+2) \right) \leq \lambda_k^p \exp\left( - \lambda_k^p \left( \frac{n}{4} k - \log(k+2) \right) \right) = \exp\left( - \frac{n}{4} k \right),$$

which with $\sum_{k \in \mathbb{N}} \mu k \exp(-\mu k) \leq 2$ and $\sum_{k \in \mathbb{N}} \mu \exp(-\mu k) \leq 1$ for any $\mu > 1$ implies

$$\sum_{k \in \{k+, n\}} \lambda_k^p k\Phi(k) \exp\left( - \frac{n}{4} k\lambda_k^p \Phi(k) \right) \leq \sum_{k=k+1}^{\infty} \exp\left( - \frac{n}{4} k \right) \leq \frac{16}{\Delta n}.$$

Combining the last bound and (B.12) we obtain assertion (ii), which completes the proof. \hfill \Box

The next result can be directly deduced from Lemma B.2 by letting $\eta \to \infty$. However, we think the direct proof given in Lemma B.3 provides an interesting illustration of the values $k_+, k_- \in \{n\}$ as defined in (B.5).

**Lemma B.3.** Consider model selection weights $\widehat{w}$ as in (2.10) and penalties $(\text{pen}_k^{\phi})_{k \in \{n\}}$ as in (2.12). For any $k_+, k_- \in \{n\}$ and associated $k_+, k_- \in \{n\}$ as in (B.5) hold

(i) $\mathbb{P}^{\text{w}}(\{k_+\}) \mathbf{1}_{\{\|f_{k_+} - f_{k_-}\|_2^2 < \text{pen}_{k_-}^{\phi} / \eta\}} = 0$;  
(ii) $\sum_{k \in \{k_+, n\}} \text{pen}_k^{\phi} \widehat{w}_k \mathbf{1}_{\{k \in \{k_+, n\}, \|f_k - f_{k+1}\|_2^2 < \text{pen}_{k_-}^{\phi} / \eta\}} = 0$.

**Proof of Lemma B.3.** By definition of $\widehat{k}$ it holds $-\|\widehat{f}_k\|_2^2 + \text{pen}_k^{\phi} \leq -\|\widehat{f}_k\|_2^2 + \text{pen}_k^{\phi}$ for all $k \in \{n\}$, and hence

$$\|\widehat{f}_k\|_2^2 - \|\widehat{f}_k\|_2^2 \geq \text{pen}_k^{\phi} - \text{pen}_k^{\phi} \text{ for all } k \in \{n\}. \quad (B.13)$$

Consider (i). Let $k_- \in \{k_-\}$ as in (B.5). For the non trivial case $k_- > 1$ it is sufficient to show, that $\{\widehat{k} \in \{k_-\} \leq \{\|f_k - f_{k+1}\|_2^2 > \text{pen}_{k_+}^{\phi} / \eta\} \}$ holds. On the event $\{\widehat{k} \in \{k_-\} \}$ we have $1 \leq \widehat{k} < k_- \leq k_-$ and thus the definition (B.5) of $k_-$ implies

$$\|\Pi^+_k f\|_2^2 b^2_{k_+}(f) > \|\Pi^+_k f\|_2^2 b^2_{k_+}(f) > \|\Pi^+_k f\|_2^2 b^2_{k_-} (f) + 4 \text{pen}_{k_-}^{\phi}. \quad (B.14)$$
On the other hand side from Lemma A.1 (i) (with \( \hat{f} := \tilde{f}_n \) and \( \hat{f} := f \)) follows
\[
\|\tilde{f}_k\|_{L,2}^2 - \|\hat{f}_k\|_{L,2}^2 \leq \frac{1}{2} \|\hat{f}_k - f_k\|_{L,2}^2 - \frac{1}{2}\|\Pi_d f\|_{L,2}^2 \{b_{\hat{k}}^2(\hat{f}) - b_{\hat{k}}^2(f)\}. \tag{B.15}
\]
Combining, first (B.13) and (B.15), and secondly (B.14) with \( \text{pen}_k^\phi \geq 0 \) we conclude
\[
\frac{1}{2}\|\hat{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi - \text{pen}_k^\phi + \frac{1}{2}\|\Pi_d f\|_{L,2}^2 \{b_{\hat{k}}^2(\hat{f}) - b_{\hat{k}}^2(f)\} > \frac{1}{4}\text{pen}_k^\phi,
\]
and hence \( \{k \in [k_-]\} \subseteq \{\|\tilde{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi /7\} \), which shows (i). Consider (ii). Let \( k_+ \in [k_+, n] \) as in (B.5). For the non trivial case \( k_+ < n \) it is sufficient to show that,
\( \{k \in [k_+, n]\} \subseteq \{\|\tilde{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi /7\} \). On the event \( \{k \in [k_+, n]\} \) holds \( \hat{k} > k_+ \geq k_\phi \) and thus the definition (B.5) of \( k_+ \) implies
\[
\text{pen}_k^\phi \geq \text{pen}_{(k_+, k_+)}^\phi \geq 6\|\Pi_d f\|_{L,2}^2 b_{k_+}^2(\hat{f}) + 4\text{pen}_{k_+}^\phi \tag{B.16}
\]
and due to Lemma A.1 (ii) (with \( \hat{f} := \tilde{f}_n \) and \( \hat{f} := f \)) also
\[
\|\tilde{f}_k\|_{L,2}^2 - \|\hat{f}_k\|_{L,2}^2 \leq \frac{3}{2}\|\hat{f}_k - f_k\|_{L,2}^2 + \frac{3}{2}\|\Pi_d f\|_{L,2}^2 \{b_{\hat{k}}^2(\hat{f}) - b_{\hat{k}}^2(f)\}. \tag{B.17}
\]
Combining, first (B.13) and (B.17), and secondly (B.16) with \( b_{\hat{k}}^2(\hat{f}) \geq 0 \) it follows that
\[
\frac{3}{2}\|\hat{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi - \text{pen}_k^\phi - \frac{3}{2}\|\Pi_d f\|_{L,2}^2 \{b_{\hat{k}}^2(\hat{f}) - b_{\hat{k}}^2(f)\} > \frac{1}{2}\text{pen}_k^\phi
\]
and hence \( \{k \in [k_+, n]\} \subseteq \{7\|\tilde{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi\} \), which shows (ii) and completed the proof. \( \Box \)

**Lemma B.4.** Consider \( (\text{pen}_k^\phi)_{k \in [n]} \) as in (2.12) with \( \Delta \geq 84 \). Let \( k_g := \left\lfloor 3(400\|g\|_1) \right\rfloor \) and \( n_o := 15(600)^4 \). There exists a finite numerical constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) and all \( k_\phi \in [n] \) hold

(i) \( \sum_{k \in [n]} \mathbb{E}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 - \text{pen}_k^\phi /7) \leq Cn^{-1}(\Phi_{(k_g)}k_g + \Phi_{(n_o)}) \);

(ii) \( \sum_{k \in [n]} \text{pen}_k^\phi \mathbb{P}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi /7) \leq Cn^{-1}(\Phi_{(k_g)}k_g + \Phi_{(n_o)}) \);

(iii) \( \mathbb{P}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 \geq \text{pen}_k^\phi /7) \leq C(\exp\left(\frac{-\frac{\lambda_{k_\phi}^\phi k_\phi^\phi}{200\|g\|_1}}{n}\right) + n^{-1}) \).

**Proof of Lemma B.4.** We show below that for \( k\Lambda_k^\phi = \Lambda_k^\phi k_\Phi_{(k)} \) with \( \Lambda_k^\phi \geq 1 \) as in (2.11) there is a numerical constant \( C \) such that for all \( n \in \mathbb{N} \) and \( k \in [n] \) hold

(a) \( \sum_{k \in [n]} \mathbb{E}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 - 12k\Lambda_k^\phi /n) \leq Cn^{-1}(\Phi_{(k_g)}k_g + \Phi_{(n_o)}) \);

(b) \( \sum_{k \in [n]} k\Lambda_k^\phi \mathbb{P}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 \geq 12k\Lambda_k^\phi /n) \leq C(\Phi^2_{(k_g)}k^3_g + \Phi^2_{(n_o)}) \);

(c) \( \mathbb{P}_{f, r}^n(\|\tilde{f}_k - f_k\|_{L,2}^2 \geq 12k\Lambda_k^\phi /n) \leq C\left(\exp\left(\frac{-\lambda_{k_\phi}^\phi k_\phi^\phi}{200\|g\|_1}\right) + n^{-1}\right) \).
Since $\log\Phi \geq 12k\Lambda^6n^{-1}$ for all $k \in [n]$ the bounds (a), (b) and (c), respectively, imply immediately Lemma B.4 (i), (ii) and (iii). In the sequel we use without further reference that $k\Phi(k) \leq \exp(\sqrt[k]{\lambda^0_{k}}\log(k+2))$ and $\lambda^0_{k} \geq 1$ for each $k \in \mathbb{N}$. Considering (a) we show that

$$
\sum_{k\in[n]} \Phi(k) \exp\left(\frac{-\lambda^0_{k}}{3||g||_{\ell_1}}\right) \leq 9\Phi(k_g)||g||_{\ell_1} \quad \text{and} \quad \sum_{k\in[n]} \frac{k\Phi(k)}{n} \exp\left(\frac{-\sqrt[n]{n\lambda^0_{k}}}{200}\right) \leq \Phi(n_o)n_o. \quad (B.18)
$$

hold for all $n \in \mathbb{N}$, where a combination of the last bounds and Lemma A.4 (i) implies directly (a). We decompose the first sum in (B.18) into two parts which we bound separately. Exploiting that $\sum_{k\in[n]} \exp(-\mu k) \leq \mu^{-1}$ for any $\mu > 0$ and setting $k_g := \lceil 3(6||g||_{\ell_1})^2 \rceil$ holds

$$
\sum_{k\in[k_g]} \Phi(k) \exp\left(\frac{-\lambda^0_{k}}{3||g||_{\ell_1}}\right) \leq \Phi(k_g) \sum_{k\in[k_g]} \exp\left(\frac{-k}{3||g||_{\ell_1}}\right) \leq \Phi(k_g)3||g||_{\ell_1}. \quad (B.19)
$$

On the other hand for any $k > k_g$ holds $\frac{\sqrt[k]{\lambda^0_{k}}}{6||g||_{\ell_1}} \geq \log(k+2)$ implying $\Phi(k) \exp\left(\frac{-\lambda^0_{k}}{3||g||_{\ell_1}}\right) \leq \exp\left(-\frac{k}{6||g||_{\ell_1}}\right)$ and hence $\sum_{k\in[k_g]} \Phi(k) \exp\left(\frac{-\lambda^0_{k}}{3||g||_{\ell_1}}\right) \leq \sum_{k\in[k_g]} \exp\left(-\frac{k}{6||g||_{\ell_1}}\right) \leq 6||g||_{\ell_1}$. The last bound, (B.19) and $k_g \leq k_g$ imply together the first bound in (B.18). Considering the second bound for $n \in \mathbb{N}$ we distinguish the following two cases, (a) $n > \tilde{n}_o := 15(200)^4$ and (b) $n \in [\tilde{n}_o]$. Firstly, consider (a), where $\sqrt{n} \geq 200\log(n+2)$ and hence

$$
\sum_{k\in[n]} \frac{k\Phi(k)}{n} \exp\left(\frac{-\sqrt[n]{n\lambda^0_{k}}}{200}\right) \leq \sum_{k\in[n]} \frac{1}{n} \exp\left(-\sqrt[k]{\lambda^0_{k}}\frac{\sqrt{n}}{200} - \log(k+2)\right) \leq \sum_{k\in[n]} \frac{1}{n} = 1. \quad (B.20)
$$

Secondly, considering (b) $n \in [\tilde{n}_o]$ holds $\sum_{k\in[n]} \frac{k\Phi(k)}{n} \exp\left(\frac{-\sqrt[n]{n\lambda^0_{k}}}{200}\right) \leq \tilde{n}_o \Phi(\tilde{n}_o) \leq n_o \Phi(n_o)$, since $\Phi(n) \leq \Phi(\tilde{n}_o) \leq \Phi(n_o)$. Combining (B.20) and the last bound for the two cases (a) $n > \tilde{n}_o$ and (b) $n \in [\tilde{n}_o]$ we obtain the second bound in (B.18). Consider (b). We show that

$$
\sum_{k\in[n]} k\lambda^0_{k} \Phi(k) \exp\left(\frac{-\lambda^0_{k}}{200||g||_{\ell_1}}\right) \leq \Phi^2(k_g)k_g^3 \quad \text{and} \quad \sum_{k\in[n]} k\lambda^0_{k} \Phi(k) \exp\left(\frac{-\sqrt[n]{n\lambda^0_{k}}}{200}\right) \leq \Phi^2(n_o)n_o. \quad (B.21)
$$

hold for all $n \in \mathbb{N}$. Combining the last bounds and Lemma A.4 (ii) we obtain (b). We decompose the first sum in (B.21) into two parts which we bound separately. Note that $\log(k\Phi(k)) \leq \frac{1}{e}k\Phi(k)$, and hence $\lambda^0_{k} \leq k\Phi(k)$. Setting $k_g = \lceil 3(400||g||_{\ell_1})^2 \rceil$ holds

$$
\sum_{k\in[k_g]} k\lambda^0_{k} \Phi(k) \exp\left(\frac{-\lambda^0_{k}}{200||g||_{\ell_1}}\right) \leq \lambda^0_{k_g} \Phi(k_g)k_g \sum_{k\in[k_g]} \exp\left(\frac{-k}{200||g||_{\ell_1}}\right) \leq k_g^2\Phi^2(k_g)(200||g||_{\ell_1}) \quad (B.22)
$$

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On the other hand for any \( k \geq 3(400 \|g\|_{\ell^1})^2 \) holds \( k \geq (400 \|g\|_{\ell^1}) \log(k+2) \), and hence \( k - 200 \|g\|_{\ell^1} \log(k+2) \geq 200 \|g\|_{\ell^1} \log(k+2) \) or equivalently, \( \frac{k}{200 \|g\|_{\ell^1}} - \log(k+2) \geq \log(k+2) \geq 1 \), which implies \( k \lambda_k^p \Phi(k) \exp \left( -\lambda_k^p \frac{k}{200 \|g\|_{\ell^1}} \right) \leq (k+2) \exp \left( -\frac{k}{200 \|g\|_{\ell^1}} \right) \). Consequently, exploiting that for any \( \mu > 0 \) holds \( \sum_{k \in \mathbb{N}} (k+2) \exp(-\mu k) \leq \exp(\mu) \mu^{-2} + 2 \mu^{-1} \) we obtain
\[
\sum_{k \in [k_0, n]} k \lambda_k^p \Phi(k) \exp \left( -\frac{n \lambda_k^p}{200} \right) = (200 \|g\|_{\ell^1})^2 + 2(200 \|g\|_{\ell^1})^2. \]

The last bound and (B.22) imply together the first bound in (B.21). Considering the second bound, for \( n \in \mathbb{N} \) we distinguish the following two cases, (a) \( n > n_0 = 15(600)^4 \) and (b) \( n \in [n_0] \). Firstly, consider (a), where \( \sqrt{n} \geq 600 \log(n+2) \), and hence together with \( \lambda_k^p \leq k \Phi(k) \) it follows
\[
\sum_{k \in [n]} k \lambda_k^p \Phi(k) \exp \left( -\frac{n \lambda_k^p}{200} \right) \leq \sum_{k \in [n]} k^2 \Phi_k^p \exp \left( -\frac{n \lambda_k^p}{200} \right) \leq \sum_{k \in [n]} \frac{1}{n} \exp \left( -3 \sqrt{\lambda_k^p (\frac{n}{600} - \log(n+2))} \right) \leq \sum_{k \in [n]} \frac{1}{n} = 1. \tag{B.23} \]

Secondly, consider (b). Since \( n^b \exp(-an^{1/c}) \leq (\frac{cb}{a})^c \) for all \( c > 0 \) and \( a, b \geq 0 \) it follows
\[
\sum_{k \in [n]} k \lambda_k^p \Phi(k) \exp \left( -\frac{n \lambda_k^p}{200} \right) \leq n^2 \lambda_n^p \Phi(n) \exp \left( -\frac{n \lambda_n^p}{200} \right) \leq \Phi_n^p n^3 \exp \left( -\frac{n \lambda_n^p}{200} \right) \leq \Phi_n^p (600)^4 \leq \Phi_n^p n_0^2. \] Combining (B.23) and the last bound for the two cases (a) \( n > n_0 \) and (b) \( n \in [n_0] \) we obtain the second bound in (B.21). Consider (c). Since \( \sqrt{n \lambda_k^p} \geq \sqrt{\frac{n}{200}} \) and \( n \exp(-\frac{n \lambda_k^p}{200}) \leq (200)^2 \) from Lemma A.4 (ii) follows immediately (c), which completes the proof. \( \square \)

**Lemma B.5.** *Let the assumptions of Proposition 2.4 (p) be satisfied. There is a finite numeric constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) with \( n_0 := 15(600)^4 \) holds*
\[
\mathbb{E}_{w,v} \| f_w - f \|_{L^2} \leq C \| \Pi_{\Delta}^f f \|_{L^2} \left[ n^{-1} \vee \exp \left( -\frac{n \lambda_k^p k_n^p}{k_n} \right) \right] + C \left( [1 \vee K] \vee c_f K^2 \Phi_{(K)} \right) \left( \Phi_{(1)}^2 + \| \Pi_{\Delta}^f f \|_{L^2}^2 \right) + K^2 \Phi_{(n)}^p k_n^3 + \Phi_{(n)}^2 n^{-1}. \tag{B.24} \]

**Proof of Lemma B.5.** The proof is based on the upper bound (B.8) which holds for any \( k_0, k_0^p \in [n] \) and associated \( k_-, k_+ \in [n] \) as defined in (B.5). Consider first the case \( K = 0 \), where \( b_0 = 0 \) and hence \( \| \Pi_{\Delta}^f f \|_{L^2} = 0 \). From (B.8) follows
\[
\mathbb{E}_{w,v} \| f_w - f \|_{L^2} \leq \frac{2}{l} \frac{1}{n} \Phi_{(k_0)} + C \Phi_{(k_0)} k_n^3 + \Phi_{(n)}^2 n^{-1}. \tag{B.25} \]

Setting \( k_0^p := 1 \) it follows from the definition (B.5) of \( k_+ \) that \( \Phi_{(k_0)} \leq 4 \Phi_{(1)} \) and \( \lambda_0^p = \lambda_1^p \Phi_{(1)} \leq \Phi_{(1)}^2 \). Thereby (keep in mind \( \Delta \geq 84 \)) (B.26) implies
\[
\mathbb{E}_{w,v} \| f_w - f \|_{L^2} \leq C \left( \Phi_{(1)}^2 + \Phi_{(k_0)}^2 k_n^3 + \Phi_{(n)}^2 \right) n^{-1} \tag{B.26} \]
Consider now \( K \in \mathbb{N} \), and hence \( \|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{(K-1)}^2 > 0 \). Setting \( n_f := [K \lor \lfloor c_f K A^e_K \rfloor] \in \mathbb{N} \) we distinguish for \( n \in \mathbb{N} \) the following two cases, (a) \( n \in [n_f] \) and (b) \( n > n_f \). Firstly, consider (a) with \( n \in [n_f] \), then setting \( k^\ominus := 1, k^e := 1, k^\ominus := 1 \) we have \( k^- = 1, 1 \geq b_1 \) and from the definition (B.5) of \( k_+ \) also \( \text{pen}^e_{k_+} \leq 2(3\|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_1^2 + 2 \text{pen}^e_1) \leq 4\|\Pi_{\perp} f\|_{L^2}^2 + 4\Delta \Phi_1^2 \). Consequently, \( \text{B.29} \) follows

\[
\mathbb{E}^n_{f,v} \|f - f\|_{L^2}^2 \leq 2\|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_K^2 + C\|\Pi_{\perp} f\|_{L^2}^2 \lim_{k \to \infty} \left[ n^{-1} \lor \exp \left( -\frac{\lambda^e_{k_+} k_+}{c_f} \right) \right] + C\left( K^2\Phi_{(K)}^2 + \Phi_{(k_+)}^2 g^3 + \Phi_{(n)}^2 \right)n^{-1}. \tag{B.27}
\]

Moreover, for all \( n \in [n_f] \) with \( n_f = [K \lor \lfloor c_f K A^e_K \rfloor] \) and \( K A^e_K = K A^e_K \Phi_{(K)} \leq K^2\Phi_{(K)} \) holds \( n \leq [K \lor c_f K^2\Phi_{(K)}] \) and thereby,

\[
\mathbb{E}^n_{f,v} \|f - f\|_{L^2}^2 \leq C\left( [K \lor c_f K^2\Phi_{(K)}\Phi_{(1)}] + \|\Pi_{\perp} f\|_{L^2}^2 + \Phi_{(k_+)}^2 k^3 + \Phi_{(n)}^2 \right)n^{-1}. \tag{B.28}
\]

Secondly, consider (b), i.e., \( n > n_f \). Setting \( k^\ominus := K \leq [K \lor \lfloor c_f K A^e_K \rfloor] = n_f \), i.e., \( k^\ominus \in [n] \), it follows \( b^0_{k^\ominus} = 0 \) and the definition (B.5) of \( k_+ \) implies \( \text{pen}^e_{k_+} \leq 4 \text{pen}^e_{k_+} = 4\Delta K A^e_K n^{-1} \leq 4\Delta K^2\Phi_{(K)}^2 n^{-1} \). From (B.8) follows for all \( n > n_f \) thus

\[
\mathbb{E}^n_{f,v} \|f - f\|_{L^2}^2 \leq 2\|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{k^-}^2 + C\|\Pi_{\perp} f\|_{L^2}^2 \lim_{k \to \infty} \left[ n^{-1} \lor \exp \left( -\frac{\lambda_{k^-}^e k^-}{c_f} \right) \right] + C\left( K^2\Phi_{(K)}^2 + \Phi_{(k^-)}^2 g^3 + \Phi_{(n)}^2 \right)n^{-1}. \tag{B.28}
\]

Note that for all \( n > n_f \) holds \( k^\ominus = \max \{ k \in [K,n] : n > c_f K A^e_K \} \), since the defining set containing \( K \) is not empty. Consequently, \( k^\ominus \geq K \) and, hence \( b_{k^\ominus}(f) = 0 \), and \( \text{pen}^e_{k^\ominus} \leq 1 < c_f^{-1} = \frac{\|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{k^-}^2}{4\Delta} \), it follows \( \|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{(K-1)}^2 > 4\Delta k^\ominus A^e_{k^-} n^{-1} = 4 \text{pen}_{k^-}^e + \|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{k^-}^2 \) and trivially \( \|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{K}^2 = 0 < 4 \text{pen}_{k^-}^e + \|\Pi_{\perp} f\|_{L^2}^2 \textbf{b}_{k^-}^2 \). Therefore, setting \( k^\ominus := k^\ominus \leq K \) and hence \( \text{b}_{k^-}^2 = \text{b}_K^2 = 0 \). From (B.28) follows now for all \( n > n_f \) thus

\[
\mathbb{E}^n_{f,v} \|f - f\|_{L^2}^2 \leq C\|\Pi_{\perp} f\|_{L^2}^2 \lim_{k \to \infty} \left[ n^{-1} \lor \exp \left( -\frac{\lambda_{k^-}^e k^-}{c_f} \right) \right] + C\left( K^2\Phi_{(K)}^2 + \Phi_{(k^-)}^2 g^3 + \Phi_{(n)}^2 \right)n^{-1}. \tag{B.29}
\]

Combining (B.27) and (B.29) for \( K \geq 1 \) with (a) \( n \in [n_f] \) and (b) \( n \geq n_f \), respectively, and (B.26) for \( K = 0 \) implies for all \( K \in \mathbb{N}_0 \) and for all \( n \in \mathbb{N} \) the claim (B.24), which completes the proof of Lemma B.5.

\[\Box\]

### B.2 Proof of Proposition 2.7 and Corollary B.8

We present first the main arguments to prove Proposition 2.7 which makes use of Corollary B.6 deferred to the end of this section.
Considering an aggregation $\tilde{f}_w = \sum_{k \in [n]} w_k \tilde{f}_k$ using either Bayesian weights $w := \tilde{w}$ as in (2.9) or model selection weights $w := \tilde{w}$ as in (2.10) we make use of the upper bounds (B.6) and (B.7), respectively. In Corollary B.6 we bound the last three terms in (B.6) and (B.7) uniformly over $\mathbb{F}_r^*$ and $\mathcal{E}_d$. Moreover, we note that the definition (B.5) of $k_+ \text{ and } k_-$ implies $\text{pen}_{k_+}^* \leq (6r + 4\Delta \zeta_d)R_n^{k_+}(f, \Lambda^r)$ and $\|\Pi_k^* f\|_{L_2}^2 b_{k_+}(f) \leq (r + 4\Delta \zeta_d)R_n^{k_+}(f, \Lambda^r)$. Combining (B.6) and (B.7), the last bounds, $\|\Pi_k^* f\|_{L_2}^2 \leq r, \eta \geq 1, \frac{3\Delta \zeta_d k^0 \Lambda^r_{k_+}}{4} \geq \frac{1}{k_{i_1}} \lambda_{k_+} R_{k_+}$ and Corollary B.6 we obtain for all $n \in \mathbb{N}$

$$
\sup \{ \mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|^2_{L_2^2} : f \in \mathbb{F}_r^*, \varphi \in \mathcal{E}_d \} \leq \frac{2}{7}(6r + 4\Delta \zeta_d)R_n^{k_+}(f, \Lambda^r) + 2(r + 4\Delta)R_n^{k_+}(f, \Lambda^r) + Cr \exp \left( \frac{-\lambda_{k_+}^0 k_+^0}{k_{i_1}} \right) + Cn^{-1} \{ r + d^2 (s_{k_{i_1}, k_{j_1}}^2 + s_{n_0}^2) \} .
$$

(B.30)

For $k^0_+ := k^0_+(f, \Lambda^r) \in [n]$ and $R_n^k(f, \Lambda^r)$ as in (2.5) we set $k^0_+ := k^0_+$, then for all $k^0_+ \in [n]$ holds $R_n^{k_+}(f, \Lambda^r) \geq R_n^{k_+}(f, \Lambda^r) = R_n^{k_+}(f, \Lambda^r) = \min \{ R_n^{k_+}(f, \Lambda^r), k \in \mathbb{N} \} \geq n^{-1}$. Combining the last bound and (B.30) implies the assertion (2.19), that is for all $k^0_+ \in [n]$ holds

$$
\sup \{ \mathbb{E}_{f,\varphi} \| \tilde{f}_w - f \|^2_{L_2^2} : f \in \mathbb{F}_r^*, \varphi \in \mathcal{E}_d \} \leq C(r + \zeta_d) \left[ R_n^{k_+}(f, \Lambda^r) \lor \exp \left( \frac{-\lambda_{k_+}^0 k_+^0}{k_{i_1}} \right) \right] + Cn^{-1} \{ r + d^2 (s_{k_{i_1}, k_{j_1}}^2 + s_{n_0}^2) \} .
$$

(B.31)

with $n_0 = 15(600)^d$, which completes the proof of Proposition 2.7.

**Proof of Corollary 2.8.** Under (A2’) for $k^0_+ := k^0_+(f, \Lambda^r) \in [n]$ as in (2.5) holds $\exp \left( -\lambda_{k_+}^0 k_+^0/k_{i_1} \right) \leq R_n^0(f, \Lambda^r)$ while for $n \in [n_{i_1}, d]$ we have $\exp \left( -\lambda_{k_+}^0 k_+^0/k_{i_1} \right) \leq r R_n^0(f, \Lambda^r) \leq n_{i_1} R_n^0(f, \Lambda^r)$.

Thereby, from (2.19) with $R_n^0(f, \Lambda^r) = \min_{k \in [n]} R_n^k(f, \Lambda^r)$ follows immediately the claim, which completes the proof of Corollary 2.8. □

**Corollary B.6.** Consider $(\text{pen}_k^*)_{k \in [n]}$ as in (2.12) with $\Delta \geq 84$. Let $n_0 := 15(600)^d$ and $k_{i_1} := \lceil 3(400)^2 \Delta r \zeta_d \rceil |f, /s, \|_{L_2^2}|$. There exists a finite numerical constant $C > 0$ such that for each $f \in \mathbb{F}_r^*$ and $\varphi \in \mathcal{E}_d$ and for all $n \in \mathbb{N}$ and $k \in [n]$ hold

(i) $\sum_{k \in [n]} \mathbb{E}_{f,\varphi} \| \tilde{f}_k - f_k \|^2_{L_2^2} - \text{pen}_k^* / 7 \leq Cn^{-1} d(s_{k_{i_1}, k_{j_1}} + s_{n_0})$;

(ii) $\sum_{k \in [n]} \text{pen}_k^* \mathbb{P}^n_{f,\varphi} \| \tilde{f}_k - f_k \|^2_{L_2^2} \geq \text{pen}_k^* / 7 \leq Cn^{-1} d^2 (s_{k_{i_1}, k_{j_1}}^2 + s_{n_0}^2)$;

(iii) $\mathbb{P}^n_{f,\varphi} \| \tilde{f}_k - f_k \|^2_{L_2^2} \geq \text{pen}_k^* / 7 \leq C \left( \exp \left( \frac{-\lambda_{k_+}^0 k_+^0}{k_{i_1}} \right) + n^{-1} \right)$.

**Proof of Corollary B.6.** The result follows immediately from (a)-(c) in the proof of Lemma B.4 by using that for all $f \in \mathbb{F}_r^*$, $\varphi \in \mathcal{E}_d$ and $k \in \mathbb{N}$ hold $d^{-1} \lambda_{k_+}^0 \geq \zeta_d^{-1} \lambda_{k_+}$, $\Phi(k) \leq d s_{k}$ and $\| [g] \|^2_{L_1^2} \leq d\| f, s, \|_{\ell_1} \| f \|^2_{L_1^2} \leq rd \| f, s, \|_{\ell_1}$, and we omit the details. □
C Proofs of section 3

Proof of Lemma 3.2. We start the proof with the observation that $[\hat{f}]_{0} - [f]_{0} = 0$ and for each $j \in \mathbb{Z}$ holds $[\hat{f}]_{j} - [f]_{j} = [\hat{f}]_{-j} - [-f]_{j}$, where $[\hat{f}]_{j} - [-f]_{j} = -[f]_{j}$ for all $|j| > n$, and

$$[\hat{f}]_{j} - [f]_{j} = [\varphi]^{+}_{j}([\hat{f}]_{j} - [f]_{j}) + [\varphi]^{+}_{j}([\hat{f}]_{j} - [f]_{j}) + 1_{X_{j}}[f]_{j} + 1_{X_{j}}[f]_{j}, \quad \text{for all } |j| \in |n|$$

with $X_{j} := \{|[\varphi]_{j}|^{2} \geq 1/m\}$ and $X_{j}^{c} := \{|[\varphi]_{j}|^{2} < 1/m\}$. Consequently, we have

$$\|\hat{f}_{w} - f\|_{L^{2}}^{2} \leq 3 \sum_{|j| \in |n|} |[\varphi]_{j}^{+}|^{2} |[\hat{f}]_{j} - [f]_{j}|^{2} P_{w}([|j|, n])$$

$$\quad + 3 \sum_{|j| \in |n|} 1_{X_{j}}[f]_{j}^{2} P_{w}([|j|, n]) + \sum_{|j| > n} |[f]_{j}|^{2}$$

$$\quad + 3 \sum_{|j| \in |n|} |[\varphi]^{+}_{j}|^{2} |[\varphi]_{j} - [\varphi]_{j}|^{2} [f]_{j}^{2} + \sum_{|j| \in |n|} 1_{X_{j}^{c}}[f]_{j}^{2}. \tag{C.1}$$

where we consider the first and the second and third term on the right hand side separately. Considering the first term from $\|\hat{f}_{k} - \hat{f}_{k}\|_{L^{2}}^{2} = \sum_{|j| \in |k|} |[\varphi]_{j}^{+}|^{2} |[\hat{f}]_{j} - [f]_{j}|^{2}$ follows

$$\sum_{|j| \in |k|} |[\varphi]_{j}^{+}|^{2} |[\hat{f}]_{j} - [f]_{j}|^{2} P_{w}([|j|, n])$$

$$\quad \leq \|\hat{f}_{k} - \hat{f}_{k}\|_{L^{2}}^{2} + \sum_{l \in |k|} w_{l} (\|\hat{f}_{l} - \hat{f}_{l}\|_{L^{2}}^{2} - \text{pen}_{l} / 7)_{+}$$

$$\quad + \frac{1}{7} \sum_{l \in |k|} w_{l} \text{pen}_{l} \mathbb{1}_{\{\|\hat{f}_{l} - f_{l}\|_{L^{2}}^{2} \geq \text{pen}_{l} / 7\}} + \frac{1}{7} \sum_{l \in |k|} w_{l} \text{pen}_{l} \mathbb{1}_{\{\|\hat{f}_{l} - f_{l}\|_{L^{2}}^{2} < \text{pen}_{l} / 7\}}. \tag{C.2}$$

Considering the second and third term we split the first sum into two parts and obtain

$$\sum_{|j| \in |n|} 1_{X_{j}}[f]_{j}^{2} P_{w}([|j|, n]) + \sum_{|j| > n} |[f]_{j}|^{2}$$

$$\quad \leq \sum_{|j| \in |k_{-}|} |[f]_{j}|^{2} 1_{X_{j}} P_{w}([|j|, n]) + \sum_{|j| \in |k_{-}, n|} |[f]_{j}|^{2} + 2 \sum_{|j| > n} |[f]_{j}|^{2}$$

$$\quad \leq \|\Pi_{k_{-}} f\|_{L^{2}}^{2} \{P_{w}([k_{-}]) + b_{k_{-}}(f)\}. \tag{C.3}$$

Combining (C.1) and (C.2), (C.3) we obtain the assertion, which completes the proof. \hfill \Box

C.1 Proof of Theorem 3.4 and Corollary 3.5

We present first the main arguments of the proof of Theorem 3.4. More technical details are gathered in Lemmata C.2 to C.5 in the end of this section. Keeping in mind the definitions
(2.12) and (3.6) let us for \( l \in [n] \) introduce the event \( \mathcal{O}_l := \{ 1/4 \leq \Phi_j^{-1} \hat{\Phi}_j \leq 9/4, \forall j \in [l] \} \) and its complement \( \mathcal{O}_l^c \), where due to Lemma A.8 holds \( \frac{1}{30} \text{pen}_k^* \mathbb{1}_{\mathcal{O}_l} \leq \text{pen}_k^* \mathbb{1}_{\mathcal{O}_l^c} \leq 7 \text{pen}_k^* \) for all \( k \in [n] \).

For any \( k_+^\circ, k_-^\circ \in [n] \) (to be chosen suitable below) let us define

\[
\begin{align*}
k_- := \min \left\{ k \in [k_-^\circ] : \| \Pi_k^f f \|^2_{L^2} b_k^*(f) \leq \| \Pi_k^f f \|^2_{L^2} b_{k_-}^*(f) + 104 \text{pen}_k^* \right\} \\
k_+ := \max \left\{ k \in [k_+^\circ, n] : \text{pen}_k^* \leq 6 \| \Pi_k^f \hat{f}_n \|_{L^2}^2 + 4 \text{pen}_k^* \right\}
\end{align*}
\]

where \( \| \Pi_k^f \hat{f}_n \|_{L^2}^2 = \sum_{j \in [k,n]} \hat{\Phi}_j \hat{\Phi}_j^{-1} |\hat{f}|^2 \) and the defining set obviously contains \( k_-^\circ \) and \( k_+^\circ \), respectively, and hence, they are not empty. Note that by construction the random dimension \( k_+ \) is independent of the sample \( (Y_i)_{i \in [n]} \). We intend to combine the upper bound in (3.4) and the bounds given in Lemma C.3.

On the other hand (C.5) holds also true for model selection weights \( w = \hat{w} \) by a combination of the upper bound in (3.4) and the bounds given in Lemma C.3.

The deviations of the last three terms in (C.5) we bound in Lemma C.4, which implies
Bounding the second term and the two sums on the right hand side due to Lemma A.6 implies

\[
\mathbb{E}_{\nu_v}^m \|f_t - f\|_{L^2}^2 \leq 2 \mathbb{E}_{\nu_v}^m \|\hat{f}_{k_+} - \hat{f}_{k_+}\|_{L^2}^2 + 3\|\Pi_\perp f\|_{L^2}^2 \|b_{k_+}^2(f)\|
\]
Due to Lemma A.7 (ii) there is a numerical constant $C$ such that for all $m, k \in \mathbb{N}$ holds $\mathbb{P}_m(\mathcal{U}^c_k) \leq Ck^2\Phi^2(k)$ and hence, $m^2\mathbb{P}_m(\mathcal{U}^c_k) \leq C_k^2\Phi^2(k)$ and $m^2\mathbb{P}_m(\mathcal{U}^c_{n^2}) \leq C_n \Phi^2(n)$. Consequently, there is a numerical constant $C > 0$ such that for all $n, m \in \mathbb{N}$ holds

$$\mathbb{E}_{f,m}^n ||\hat{f}_w - f||_{L^2}^2 \leq 2 \text{pen}\_k + \frac{12}{m^2} ||\Pi^f||_{L^2}^2 ||\mathbf{b}_k^2||_2 (f) + 3 ||\Pi^f||_{L^2}^2 ||\mathbf{b}_k^2||_2 (f)$$

$$+ C ||\Pi^f||_{L^2}^2 1_{(k+1)} (\exp (-\lambda^0 \frac{\Phi^0}{k}) + \mathbb{P}_m(\mathcal{U}^c_k)) + Cm\mathbb{P}_m(\mathcal{U}^c_k)$$

$$+ C ||\Pi^f||_{L^2}^2 1_{(k+1)} (\exp (-\lambda^0 \frac{\Phi^0}{k}) + \mathbb{P}_m(\mathcal{U}^c_k))$$

(keep in mind that $n$ is a numerical constant).

From the upper bound (C.7) for a suitable choice of the dimension parameters $k_1, k_2 \in [n]$ we derive separately the risk bound in the two cases (p) and (np) considered in Theorem 3.4. The tedious case-by-case analysis for (p) is deferred to Lemma C.5 in the end of this section.

In case (np) we distinguish for $m \in \mathbb{N}$ with $m := [289(\log 3)\lambda^0 \Phi(1)]$ the following two cases, (a) $m \in [m_0]$ and (b) $m > m_0$. Consider firstly the case (a) $m \in [m_0]$. We set $k_1 = k_2 = 1$, and hence $k_1 = 1$, $b_1^2 \leq 1$, $\text{pen}_1 \leq \Delta \Phi^2(n) m^{-1}$, $\Phi^2(n) \leq \Phi^2(n)$, $m \leq \Phi^2(n)$ and due to Lemma A.7 (ii) $\mathbb{P}_m(\mathcal{U}_k^c) \leq C \Phi^2(n^2) m^{-2}$. Thereby, from (C.7) for all $n \in \mathbb{N}$ and $m \in [m_0]$ follows

$$\mathbb{E}_{f,m}^n ||\hat{f}_w - f||_{L^2}^2 \leq C ||\Pi^f||_{L^2}^2 1_{(k+1)} (\exp (-\lambda^0 \frac{\Phi^0}{k}) + \mathbb{P}_m(\mathcal{U}^c_k)) + Cm \mathbb{P}_m(\mathcal{U}^c_k)$$

(8)

Consider secondly (b) $m > m_0$ with $k_1 := \max\{k \in [m] : 289 \log(k+2)\lambda^0 \Phi(k) \leq m\}$. For each $k \in [m]$ holds $m \geq 289 \log(k+2)\lambda^0 \Phi(k)$, and thus from Lemma A.7 (iii) follows

$$\mathbb{P}_m(\mathcal{U}^c_k) \leq 11226 m^{-2}.$$ For $k_2 := k_2^*(b^*_2, \Lambda^*_2) \in [n]$ as in (2.5) setting $k_2^* := k_2^* \Lambda^*_2$ and

$$m \mathbb{P}_m(\mathcal{U}_k^c) \leq C m^{-1}, \text{pen}_{k^*_2} \leq \Delta \mathcal{R}_n(b^*_2, \Lambda^*_2)$$

and $b^*_2(f) \leq \mathcal{R}_n(b^*_2, \Lambda^*_2) + \mathbf{b}_{k^*_2}(f)$, (C.7) implies

$$\mathbb{E}_{f,m}^n ||\hat{f}_w - f||_{L^2}^2 \leq C \mathbb{P}^n ||\Pi^f||_{L^2}^2 1_{(k+1)} (\exp (-\lambda^0 \frac{\Phi^0}{k}) + \mathbb{P}_m(\mathcal{U}^c_k))$$

(9)

Let $k^*_2 := \arg\min\{\mathcal{R}_n(b^*_2, \Lambda^*_2) : k \in [n]\}$. Setting $k^*_2 := k^*_2 \wedge k^*_m$ from Lemma A.7 (iii) follows $\mathbb{P}_m(\mathcal{U}^c_k) \leq 53 m^{-1}$, while $k_-$ as in definition (C.4) satisfies

$$||\Pi^f||_{L^2}^2 1_{(k+1)} (\exp (-\lambda^0 \frac{\Phi^0}{k}) + \mathbb{P}_m(\mathcal{U}^c_k)).$$
where \( n^{-1} \leq R_n(b^2, \Lambda^*) \leq R_n^{k_1}(b^2, \Lambda^*) \) by (2.5) and \( \| \Pi_{\phi} f \|^2_{1, \phi/m} \geq \frac{1}{2} \| \Pi_{\phi} f \|^2_{L^2} m^{-1} \) (see Remark 3.1). Thereby, we obtain from (C.9) for all \( n \in \mathbb{N} \) and \( m > m_{\phi} \)

\[
\mathbb{E}^{m,m}_{\omega} \left\| \hat{f}_w - f \right\|_{L^2}^2 \leq C \left[ 1 \lor \| \Pi_{\phi} f \|^2_{L^2} \right] \min_{k \in [n]} \left\{ \left[ R_n^k(b^2, \Lambda^*) \lor \exp \left( \frac{-\lambda_{k}^*}{k_g} \right) \right] + C \right. \\
+ C \| \Pi_{\phi} f \|^2_{L^2} \left[ b^2_{k_m}(f) \lor \exp \left( \frac{-\lambda_{k}^* k_m}{k_g} \right) \right] + C \| \Pi_{\phi} f \|^2_{L^2} \left. \right\} \mathbb{1}_{\{m > m_{\phi}\}} + C \left[ 1 \lor \| \Pi_{\phi} f \|^2_{L^2} \right] \phi_{(k_g)} m^{-1} + C \left( \Phi_{(k_g)}^2 k_g^4 + \Phi_{(m_{\phi})}^2 \right) n^{-1}, \quad \text{(C.10)}
\]

Combining (C.8) and (C.10) for the cases (a) and (b) for all \( n, m \in \mathbb{N} \) holds

\[
\mathbb{E}^{m,m}_{\omega} \left\| \hat{f}_w - f \right\|_{L^2}^2 \leq C \left[ 1 \lor \| \Pi_{\phi} f \|^2_{L^2} \right] \min_{k \in [n]} \left\{ \left[ R_n^k(b^2, \Lambda^*) \lor \exp \left( \frac{-\lambda_{k}^*}{k_g} \right) \right] \mathbb{1}_{\{m > m_{\phi}\}} \right. \\
+ C \| \Pi_{\phi} f \|^2_{L^2} \left[ b^2_{k_m}(f) \lor \exp \left( \frac{-\lambda_{k}^* k_m}{k_g} \right) \right] \mathbb{1}_{\{m > m_{\phi}\}} + C \| \Pi_{\phi} f \|^2_{L^2} \left. \right\} \mathbb{1}_{\{m > m_{\phi}\}} + C \left[ 1 \lor \| \Pi_{\phi} f \|^2_{L^2} \right] \phi_{(k_g)} m^{-1} + C \left( \Phi_{(k_g)}^2 k_g^4 + \Phi_{(m_{\phi})}^2 \right) n^{-1}, \quad \text{(C.11)}
\]

which shows (3.8) and completes the proof of Theorem 3.4.

**Proof of Corollary 3.5.** Consider the case (p). In the proof of Corollary 3.5 we have shown, that under the additional assumption (A1) holds \( \mathbb{E}^{m}_{\omega} \left\| \hat{f}_w - f \right\|_{L^2}^2 \leq C_{f,\phi} n^{-1} \) for all \( n \in \mathbb{N} \). If in addition (A4) is satisfied for \( k_m^{*} \) as in Theorem 3.4, then we have for all \( m > m_{\phi} \) trivially \( \exp \left( \frac{-\lambda_{k}^* k_m^{*}}{k_g} \right) \leq m^{-1} \) while for \( n \in \mathbb{N} \) we have \( \exp \left( \frac{-\lambda_{k}^* k_m^{*}}{k_g} \right) \leq 1 \leq m_{\phi} m^{-1} \). Combining both bounds we obtain the assertion (p). On the other hand side, in case (np) under the additional assumption (A2) holds \( \min_{k \in [n]} \left\{ \left[ R_n^k(f, \Lambda^*) \lor \exp \left( \frac{-\lambda_{k}^*}{k_g} \right) \right] \right\} \leq n f \phi R_n^k(b^2, \Lambda^*) \) (cf. Corollary 2.5 (np)). A combination of the last bound and \( \exp \left( \frac{-\lambda_{k}^* k_m^{*}}{k_g} \right) \leq m_{\phi} m^{-1} \) due to (A4) implies the assertion (np), which completes the proof of Corollary 3.5.

Below we state and prove the technical Lemmata C.2 to C.4 used in the proof of Theorem 3.4. The proof of Lemma C.2 is based on Lemma C.1 given first.

**Lemma C.1.** Consider Bayesian weights \( \hat{w} \) as in (1.7) and let \( l \in [n] \).

(i) For \( \Omega_l := \{ \frac{1}{2} \leq \Phi_{l}^{-1} \hat{f}_j \leq \frac{3}{2}, \ \forall \ j \in [l] \} \) and \( k \in [l] \) holds

\[
\hat{w}_k \mathbb{1}_{\left\{ \| \hat{f}_k - f \|_{L^2}^2 < \text{pen}_k^* \right\}} \mathbb{1}_{\Omega_l} \leq \exp \left( \eta n \left\{ \frac{2}{3} \text{pen}_k^* + \frac{1}{4} \| \Pi_{\phi} f \|^2_{L^2} b^2(f) - \frac{1}{8} \| \Pi_{\phi} f \|^2_{L^2} b^2(f) - \frac{1}{50} \text{pen}_k^* \right\} \right);
\]

(ii) For \( \| \Pi_{\phi} f \|^2_{L^2} = \sum_{j \in [l] \cup [n]} \Phi_{j}^{-1} \hat{f}_j |f_j|^2 \) and \( k \in [l], [n] \) holds

\[
\hat{w}_k \mathbb{1}_{\left\{ \| \hat{f}_k - f \|_{L^2}^2 < \text{pen}_k^* \right\}} \leq \exp \left( \eta n \left\{ - \frac{1}{2} \text{pen}_k^* + \frac{3}{2} \| \Pi_{\phi} f \|^2_{L^2} + \text{pen}_l^* \right\} \right).
\]

**Proof of Lemma C.1.** Given \( k, l \in [n] \) and an event \( \Omega_{ml} \) (to be specified below) it follows

\[
\hat{w}_k \mathbb{1}_{\Omega_{ml}} \leq \exp \left( \eta n \left\{ \| \hat{f}_k \|^2_{L^2} - \| f_k \|^2_{L^2} + (\text{pen}_l^* - \text{pen}_k^*) \right\} \right) \mathbb{1}_{\Omega_{ml}}.
\]

(C.12)
We distinguish the two cases (i) \( k \in [1, l] \) and (ii) \( k \in [l, n] \). Consider first (i) \( k \in [1, l] \).

From (i) in Lemma A.1 (with \( \hat{f} := \hat{f}_n \) and \( \bar{f} := \bar{f}_n \)) follows

\[
\hat{w}_k 1_{\Omega_{st}} \leq \exp \left( \eta n \left\{ \frac{11}{4} \| \hat{f} - \bar{f}_n \|_2^2 - \frac{1}{2} \| \Pi_{st} \hat{f}_n \|_2^2 + (\text{pen}^k - \text{pen}_k^g) \right\} \right) 1_{\Omega_{st}} \tag{C.13}
\]

Setting \( \Omega_{st} := \{ \| \hat{f} - \bar{f}_n \|_2^2 < \text{pen}^k \} \cap \Omega_{st} \) the last bound together with Lemma A.8 (i) and (iii) implies the assertion (i). Consider secondly (ii) \( k \in [l, n] \). From (ii) in Lemma A.1 (with \( \hat{f} := \hat{f}_n \) and \( \bar{f} := \bar{f}_n \)) and (C.12) follows

\[
\hat{w}_k 1_{\Omega_{stk}} \leq \exp \left( \eta n \left\{ \frac{7}{2} \| \hat{f}_k - \bar{f}_n \|_2^2 + \frac{3}{2} \| \Pi_{st} \hat{f}_n \|_2^2 + (\text{pen}^k - \text{pen}_k^g) \right\} \right) 1_{\Omega_{stk}}
\]

Setting \( \Omega_{stk} := \{ \| \hat{f}_k - \bar{f}_n \|_2^2 < \text{pen}^k \} \) the last bound together with Lemma A.8 (i) implies (ii), which completes the proof. \( \square \)

**Lemma C.2.** Consider Bayesian weights \( \hat{w} \) as in (1.7) and penalties \( \{ \text{pen}_k^g \}_{k \in [n]} \) as in (3.6). For any \( k^+_o, k^-_o \in [n] \) and associated \( k_+, k_- \in [n] \) as in (C.4) hold

(i) \( \mathbb{P}_\omega ([k_-]) \leq \frac{59}{36} \frac{1}{\log n} \| k_- \| - 1 \exp \left( - \frac{n \omega}{2} k^+ \Lambda^+ \right) + 1 \{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \} \cup K_{k^+_o} \}

(ii) \( \sum_{k \in [k_+ + n]} \text{pen}_k^g \hat{w}_k 1_{\{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \}} \leq \frac{10}{\eta} n^{-1} \).

**Proof of Lemma C.2.** Consider (i). Let \( k_- \in [k^+_o] \) as in (C.4). For the non trivial case \( k_- > 1 \) from Lemma C.1 (i) with \( l = k^+_o \) follows for all \( k < k_- \leq k^+_o \)

\[
\hat{w}_k 1_{\{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \}} \cap \Omega_{stk} \leq \exp \left( \eta n \{ - \frac{1}{8} \| \Pi_k^+ f \|_2^2 b_k^g (f) + \left( \frac{25}{2} \text{pen}_k^g + \frac{1}{8} \| \Pi_k^+ f \|_2^2 b_k^g (f) \right) - \frac{1}{50} \text{pen}^g \} \right),
\]

and hence by exploiting the definition (C.4) of \( k_- \), that is \( \| \Pi_k^+ f \|_2^2 b_k^g \geq \| \Pi_k^+ f \|_2^2 b_{(k_- - 1)} \) > \( \| \Pi_k^+ f \|_2^2 b_{k^+_o} + 104 \text{pen}_k^g \), we obtain for each \( k \in [k_-] \)

\[
\hat{w}_k 1_{\{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \}} \cap \Omega_{stk} \leq \exp \left( - \frac{1}{2} \eta n \text{pen}_k^g - \frac{1}{50} \eta n \text{pen}^g \right).\]

The last upper bound together with \( \text{pen}^g = \Delta k \Lambda^+ n^{-1} \geq \Delta kn^{-1}, k \in [n] \), as in (2.11) gives

\[
\mathbb{P}_\omega ([k_-]) \leq \exp \left( - \frac{9}{2} n \text{pen}_k^g \right) \sum_{k \in [k_-]} \exp \left( - \frac{n \omega}{50} k \right) + 1 \{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \} \cup K_{k^+_o}
\]

which combined with \( \sum_{k \in \mathbb{R}} \exp (-\mu k) \leq \mu^{-1} \) for any \( \mu > 0 \) implies (i). Consider (ii). Let \( k_+ \in [k^+_o, n] \) as in (C.4). For the non trivial case \( k_+ < n \) from Lemma C.1 (ii) with \( l = k^+_o \) follows for all \( k > k_+ \geq k^+_o \)

\[
\hat{w}_k 1_{\{ \| \hat{f}_k - \bar{f}_n \|_2^2 \geq \text{pen}^g / \gamma \}} \leq \exp \left( \eta n \left\{ - \frac{1}{2} \text{pen}^g + \frac{3}{2} \| \Pi_{k_+} \hat{f}_n \|_2^2 + \text{pen}_k^g \right\} \right),\]

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and hence by employing the definition (C.4) of \( k_+ \), that is, \( \frac{1}{4} \text{pen}^g_k > \frac{1}{4} \text{pen}^g_{k+1} > \text{pen}^g_k + \frac{3}{2} \| \Pi^k_0 \hat{f}_k \|_{L^2} \), we obtain for each \( k \in \{ k_+, n \} \)
\[
\hat{w}_k \mathbb{1}_{\{ \| \hat{f}_k - \hat{f}_k \|_{L^2} < \text{pen}^g_k / \gamma \}} \leq \exp \left( \eta n \left\{ - \frac{1}{4} \text{pen}^g_k \right\} \right).
\]

The last bound together with Lemma A.8 (iii), i.e., \( \text{pen}^g_k = \text{pen}^g_k \mathbb{1}_{\{ \hat{\Phi}(k) > 1 \}} \), implies
\[
\sum_{k \in \{ k_+, n \}} \text{pen}^g_k \hat{w}_k \mathbb{1}_{\{ \| \hat{f}_k - \hat{f}_k \|_{L^2} < \text{pen}^g_k / \gamma \}} \leq \sum_{k \in \{ k_+, n \}} \text{pen}^g_k \exp \left( - \frac{\eta}{4} n \text{pen}^g_k \right)
= \sum_{k \in \{ k_+, n \}} \text{pen}^g_k \exp \left( - \frac{\eta}{4} n \text{pen}^g_k \right) \mathbb{1}_{\{ \hat{\Phi}(k) > 1 \}}
= \Delta n^{-1} \sum_{k \in \{ k_+, n \}} k \lambda^2_k \hat{\Phi}(k) \exp \left( - \frac{\eta}{4} n \text{pen}^g_k \right) \mathbb{1}_{\{ \hat{\Phi}(k) > 1 \}} \tag{C.14}
\]

Comparing the last bound with (B.12) the remainder of the proof of (ii) follows line by line the arguments used to prove of Lemma B.2 (ii) starting by (B.12), and we omit the details, which completes the proof.

**Lemma C.3.** Consider model selection weights \( \tilde{w} \) as in (1.6) and penalties \( \text{pen}^g_k \) \( k \in [n] \) as in (3.6). For any \( k^0_\Delta, k^+ \in [n] \) and associated \( k_+, k_- \in [n] \) as in (C.4) hold
(i) \( \mathbb{P}_n^0 (\{ k_- \} \mathbb{1}_{\{ \| \hat{f}_k - \hat{f}_k \|_{L^2} < \text{pen}^g_k / \gamma \}} \cap \tilde{w}_{k_-} = 0) \);
(ii) \( \sum_{k \in \{ k_+, n \}} \text{pen}^g_k \hat{w}_k \mathbb{1}_{\{ \| \hat{f}_k - \hat{f}_k \|_{L^2} < \text{pen}^g_k / \gamma \}} = 0 \).

**Proof of Lemma C.3.** The assertions can be directly deduced from Lemma C.2 by letting \( \eta \to \infty \) or following line by line the proof of Lemma B.3, and we omit the details.

**Lemma C.4.** Consider \( \text{pen}^g_k \) \( k \in [1, n] \) as in (3.6) with \( \Delta \geq 84 \). Let \( k_g := \lfloor 3(400 \| \| g \|_d \| / \gamma \rfloor^2 \rfloor \)
and \( n_o := 15(600)^4 \). There exists a finite numerical constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) and all \( k^\circ \in [n] \) hold
(i) \( \sum_{k \in [n]} \mathbb{E}_{\nu_{k}} \left( \| \hat{f}_k - \hat{f}_k \|_{L^2} - \text{pen}^g_k / \gamma \right) \leq C n^{-1} \left( 1 \lor \hat{\Phi}(k_g) \lor [1 \lor \hat{\Phi}(n_o)] \right) \);
(ii) \( \sum_{k \in [n]} \text{pen}^g_k \mathbb{P}_{\nu_{k}} (\| \hat{f}_k - \hat{f}_k \|_{L^2} \geq \text{pen}^g_k / \gamma) \leq C n^{-1} \left( 1 \lor \hat{\Phi}(k_g^3) \lor [1 \lor \hat{\Phi}(n_o)] \right) \);
(iii) \( \mathbb{P}_{\nu_{k}} (\| \hat{f}_k - \hat{f}_k \|_{L^2} \geq \text{pen}^g_k / \gamma) \leq C \left( \exp \left( -\frac{\lambda^2_0 \cdot k_{\circ}}{200 \| \| g \|_d } \right) + n^{-1} \right) \).

**Proof of Lemma C.4.** By using Lemma A.5 rather than Lemma A.4 together with \( \{ \hat{\Phi}(l) < 1 \} = \{ \hat{\Phi}(l) = 0 \} \) for all \( l \in \mathbb{N} \) due to Lemma A.8 (ii) the proof follows line by line the proof of Lemma B.4, and we omit the details.
Lemma C.5. Let the assumptions of Theorem 3.4 (p) be satisfied. There is a numerical constant $C$ such that for all $n, m \in \mathbb{N}$ with $n_o := 15(600)^4$ holds

$$
\mathbb{E}^{n,m}_{f,\varphi} \left| \hat{f}_w - f \right|_2^2 \leq C \left\| \Pi_o^f f \right\|_2^2 \left[ n^{-1} \vee m^{-1} \vee \exp \left( -\frac{\lambda_o^k \Delta^k}{k_o} \right) \right] + C \left( 1 \vee K \vee c_f K^2 \Phi_{(K)}^2 \right) \left( \Phi_{(1)}^2 + \left\| \Pi_o^f f \right\|_2^2 \right) + \Phi_{(k_o)}^2 k_o^4 + \Phi_{(n_o)}^2 n^{-1} + C \left( \Phi_{(1)}^2 + K \Phi_{(K)}^2 \right) + \left\| \Pi_o^f f \right\|_2^2 \Phi_{(K)} m^{-1}.
$$

Proof of Lemma C.5. The proof follows along the lines of the proof of Lemma B.5 by using the upper bound (C.7) instead of (B.8) which hold for any $k_o^\circ, k_+^\circ \in \mathbb{N}$ and associated $k_-, k_+ \in \mathbb{N}$ as defined in (C.4) contrarily to (B.5). We present exemplary the case (b) $n > n_f := \left[ K \vee \lfloor c_f K^2 \Phi_{(K)}^2 \right] \text{ with } K \in \mathbb{N}$ and $c_f := \frac{104\Delta}{\left\| \Pi_o^f f \right\|_2^2 \Phi_{(K-1)}}$, and omit the details for the others. Setting $k_+^\circ := K \leq n_f$, i.e., $k_+^\circ \in \mathbb{N}$, it follows $b_{k_+}^o = 0$ and $\text{pen}_{k_+^\circ} = \Delta K \Phi_{(K)} n^{-1} \leq \Delta K^2 \Phi_{(K)} n^{-1}$. From (C.7) follows for all $n > n_f$ thus

$$
\mathbb{E}^{n,m}_{f,\varphi} \left| \hat{f}_w - f \right|_2^2 \leq 3 \left\| \Pi_o^f f \right\|_2^2 b_{k_-}^o (f) + C \left\| \Pi_o^f f \right\|_2^2 \mathbb{I}_{\{k_- > 1\}} \left( \exp \left( -\frac{\lambda_o^k k_o^\circ}{k_o} \right) + \mathbb{E}^m (U_{k_o}) \right) + C m \mathbb{E}^m (U_{K}) + C \left\| \Pi_o^f f \right\|_2^2 \Phi_{(K)} m^{-1}.
$$

Exploiting Lemma A.7 (ii) there is a numerical constant $C$ such that for all $m \in \mathbb{N}$ holds $\mathbb{E}^m (U_{K}) \leq CK^2 \Phi_{(K)} m^{-2}$, which together with $\left\| \Pi_o^f f \right\|_2^2 \Phi_{(K)} m^{-1}$ implies

$$
\mathbb{E}^{n,m}_{f,\varphi} \left| \hat{f}_w - f \right|_2^2 \leq C n^{-1} \left( K^2 \Phi_{(K)}^2 + \Phi_{(k_o)}^2 k_o^4 + \Phi_{(n_o)}^2 n^{-1} \right) + 3 \left\| \Pi_o^f f \right\|_2^2 b_{k_-}^o + C \left\| \Pi_o^f f \right\|_2^2 \mathbb{I}_{\{k_- > 1\}} \left( \exp \left( -\frac{\lambda_o^k k_o^\circ}{k_o} \right) + \mathbb{E}^m (U_{k_o}) \right) + C m^{-1} \left( K^2 \Phi_{(K)}^2 + \left\| \Pi_o^f f \right\|_2^2 \Phi_{(K)} \right) (C.16)
$$

In order to control the terms involving $k_o^\circ$ and $k_-$ we distinguish for $m \in \mathbb{N}$ with $m_{f,\Phi} := \left[ 289 \log(K+2) \lambda_o^k \Phi_{(K)} \right]$ the following two cases, (b-i) $m \in [m_{f,\Phi}]$ and (b-ii) $m > m_{f,\Phi}$. Consider first (b-i) $m \in [m_{f,\Phi}]$. We set $k_o^\circ = 1$ and hence $k_- = 1$. Thereby, with $b_{k_o^\circ}^o (f) \leq 1$, $\log(K+2) \leq \frac{K+2}{2} \leq 2K$, $\lambda_o^k \Phi_{(k_o^\circ)} \leq K^2 \Phi_{(K)}$, and hence $m_{f,\Phi} \leq CK^2 \Phi_{(K)}$, from (C.16) follows for all $m \in [m_{f,\Phi}]$

$$
\mathbb{E}^{n,m}_{f,\varphi} \left| \hat{f}_w - f \right|_2^2 \leq C n^{-1} \left( K^2 \Phi_{(K)}^2 + \Phi_{(k_o^\circ)}^2 k_o^4 + \Phi_{(n_o)}^2 n^{-1} \right) + C m^{-1} \left( K^2 \Phi_{(K)}^2 + \left\| \Pi_o^f f \right\|_2^2 \Phi_{(K)} \right) (C.17)
$$

Consider (b-ii) $m > m_{f,\Phi}$ ensuring the defining set of $k_o^\circ = \max \{ k \in \mathbb{N} : 289 \log(K+2) \lambda_o^k \Phi_{(k)} \leq m \}$ is not empty and $k_o^m \geq K$. For each $k_o^\circ \in [K, k_o^m]$ it follows $\mathbb{E}^m (U_{k_o}) \leq 53 m^{-1}$ due to Lemma A.7 (iii). Since $n > n_f = \left[ K \vee \lfloor c_f K^2 \Phi_{(K)}^2 \right] \text{ with } c_f := \frac{104\Delta}{\left\| \Pi_o^f f \right\|_2^2 \Phi_{(K-1)}}$
the defining set of \( k^*_n = \max\{k \in \{n\} : n > cfk\Lambda^*_k\} \) is not empty and \( k^*_n \geq K \). For each \( k^\circ \in [K, k^*_n] \) we have \( b_{k^\circ}(f) = 0 \), and \( \text{pen}^*_k = k^\circ \Lambda^*_k n^{-1} < c_f^{-1} = \frac{\|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_k^2}{104\Delta} \).

It follows \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_{(K^\circ)} > \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_{k^\circ}^2 + 104\text{pen}^*_k \) and trivially \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_K^2 = 0 < \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_{k^\circ}^2 + 104\text{pen}^*_k \). Therefore, \( k^\circ \) as in \((C.4)\) satisfies \( k^\circ = K \) and hence \( b_{k^\circ} = 0 \).

Finally, setting \( k^\leq := k^*_n \wedge k^\circ \), it follows \( \exp^m(\Upsilon_{k^\leq}) \leq 53n^{-1}, k^\circ = K \) and \( b_{k^\circ} = 0 \). From \((C.16)\) follows for all \( m > m_{f,\Phi} \) and \( n > n_{f,\Phi} \) thus

\[
\begin{align*}
\mathbb{E}^n_{f,\varphi} \|\hat{f_w} - f\|_{L^2}^2 &\leq Cn^{-1}(K^2\Phi^2_{(K)}n^{-1} + \Phi^2_{(k^\circ)}k^\circ_s + \Phi^2_{(n_o)} + \|\Pi_{\hat{f}}^l f\|_{L^2}^2) \\
&\quad + C\|\Pi_{\hat{f}}^l f\|_{L^2}^2 \exp\left(-\frac{\lambda^*_k k^\circ}{\kappa_g}\right) + Cm^{-1}(K\Phi^2_{(K)} + \|\Pi_{\hat{f}}^l f\|_{L^2}^2 \Phi_{(K)}). \quad \text{(C.18)}
\end{align*}
\]

By combining \((C.17)\) and \((C.18)\) for the cases \((b-i) \) \( m \in \{m_{f,\Phi}\} \) and \((b-ii) m > m_{f,\Phi}\) the upper bound \((C.15)\) holds in case \((b)\), i.e., for all \( m \in \mathbb{N} \) and for all \( n > n_{f,\Phi} \), which completes the proof of \textit{Lemma C.5}. \( \square \)

\subsection{C.2 Proof of Theorem 3.7 and Corollary 3.8}

\textbf{Proof of Theorem 3.7}. Keeping \((2.17)\) in mind for all \( f \in \mathbb{F}_f, \varphi \in \mathcal{E}^d \) and \( k, n, m \in \mathbb{N} \) we have \( \|g\|_{L^2} \leq rd \|\mathbf{s}_f, \varphi, n\|_{\alpha}, \) hence \( \kappa_g = [3(400)^2\|g\|_{L^2}^2] \leq [3(400)^2r\zeta_d \|\mathbf{s}_f, \varphi, n\|_{\alpha}] = k_{fs} \) and \( \frac{1}{k_{fs}} \lambda^*_k = \frac{1}{k_{fs}} \lambda^*_k \), \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 \leq r, \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_k^2(f) \leq \text{pen}^*_k \), \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_{k^\circ}^2 \leq \Delta^2_{k^\circ} n_0 \), \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 \Phi_{(m)}/m \leq rd\|f(1 \wedge \mathbf{s}_f/m)\|_{\infty}, \) and \( k_{\varphi} \) as in \((C.4)\) satisfies \( \|\Pi_{\hat{f}}^l f\|_{L^2}^2 b_{k_{\varphi}}^2(f) \leq \text{pen}_k \leq 104\Delta^2_{k^\circ} n_0 \).

Combining the last bounds together with the upper bound \((C.7)\) there is a numerical constant \( C > 0 \) such that uniformly for all \( f \in \mathbb{F}_f, \varphi \in \mathcal{E}^d, n, m \in \mathbb{N} \) and \( k^\circ, k^* \in \{n\} \) holds

\[
\begin{align*}
\mathbb{E}^n_{f,\varphi} \|\hat{f_w} - f\|_{L^2}^2 &\leq 2\Delta^2_{k^\circ} k_k^* k_{\varphi}^*/n + \frac{12^2}{\kappa_{fs}^2} r f_{k^\circ}^* + 3r f_{k^\circ}^* + 312\Delta^2_{k^\circ} n_0 \Lambda^*_{k^\circ} /n \\
&\quad + Cr\left( \exp\left(-\frac{\lambda^*_k k^\circ}{\kappa_g}\right) + \exp^m(\Upsilon_{k^\circ}) \right) + Cm\exp^m(\Upsilon_{k^\circ}) \\
&\quad + Crd\|f(1 \wedge \mathbf{s}_f/m)\|_{\infty} + Cn^{-1}(d^2 s_k^2 k_{fs}^4 + d^2 s_n^2 m + r). \quad \text{(C.19)}
\end{align*}
\]

We distinguish for \( m \in \mathbb{N} \) with \( m_s := \lfloor 289(\log 3)\zeta_d k_k^* s_1 \rfloor \) the two cases, \((a) \) \( m \in \{m_s\} \) and \((b) m > m_s \).

Consider \((a)\). We set \( k^\circ = k^* = 1 \). Since \( \exp^m(\Upsilon_{k^\circ}) \leq C\Phi^2_{(1)} m^{-2} \leq C d^2 s_k^2 m^{-2} \) due to \textit{Lemma A.7 (ii)}, \((C.19)\) implies for all \( n \in \mathbb{N} \) and \( m \in \{m_s\} \)

\[
\begin{align*}
\sup\{\mathbb{E}^n_{f,\varphi} \|\hat{f_w} - f\|_{L^2}^2 : f \in \mathbb{F}_f, \varphi \in \mathcal{E}^d\} &\leq Crd\|f(1 \wedge \mathbf{s}_f/m)\|_{\infty} \\
&\quad + Cn^{-1}(d^2 s_k^2 k_{fs}^4 + d^2 s_n^2 m + r). \quad \text{(C.20)}
\end{align*}
\]

Consider secondly \((b)\). Since \( m > m_s \) the defining set of \( k^*_m := \max\{k \in \{m\} : 289 \log(k + 2) \zeta_d k_k^* s_k \leq m \} \) is not empty. Keeping in mind, that due to \((2.17)\) for all \( \varphi \in \mathcal{E}^d \),
and for each \( k \in [k_m^*] \) holds \( \zeta_d \lambda_k \sigma_k \geq \lambda_k \Phi_k \), and hence \( m \geq 289 \log(k + 2) / \lambda_k \Phi_k \) and \( \mathcal{P}_m(\mathcal{U}_k^c) \leq 11226m^{-2} \) applying Lemma A.7 (iii). For \( k_n^0 := k_n^0(f, \Lambda_1) \in [n] \) as in (2.5) let \( k_+ := k_n^0 \wedge k_m^* \) and hence \( m \mathcal{P}_m(\mathcal{U}_k^c) \leq Cm^{-1} \). Since \( \Lambda_{k_+} k_n^0 / n \leq \mathcal{R}_n^{k_n^0}(f, \Lambda_1) = \mathcal{R}_n^{k_+}(f, \Lambda_1) \) and \( f_{k_+} \leq \mathcal{R}_n^{k_+}(f, \Lambda_1) + f_{k_m^0} \) from (C.19) follows

\[
\sup \left\{ \mathbb{E}_{f, \varphi} \| f_w - f \|^2_{L^2} : f \in \mathcal{F}, \varphi \in \mathcal{E} \right\} \leq (2\Delta \zeta_d + 2r) \mathcal{R}_n^{k_+}(f, \Lambda_1) + \frac{12}{\lambda} rf_{k_m^0} + 312 \Delta \zeta_d k^0 \Lambda_1 / n + C r \left( \exp \left( \frac{-\lambda_n^0 k^0}{k_m^0} \right) + \mathcal{P}_m(\mathcal{U}_k^c) \right) + \frac{C}{m} + \frac{C r d}{m} \| f (1 \land \mathcal{A}_m / m) \|_{\infty} + C n^{-1} \left( d^2 \sigma_{k_1}^2 k_1^4 + d^2 \sigma_{n_a}^2 + r \right) \quad (C.21)
\]

For \( k_n^1 := \arg \min \{ \mathcal{R}_n^k(f, \Lambda_1) : k \in [n] \} \) with \( \mathcal{R}_n^{k_n^1}(f, \Lambda_1) \leq \rho_n^0(f, \Lambda_1) \) let \( k_1^0 := k_n^0 \wedge k_m^* \) and hence \( \mathcal{P}_m(\mathcal{U}_k^c) \leq 53m^{-1} \). Since \( rf_{k_1^0} + \zeta_d \Lambda_{k_1^0} k_1^0 n^{-1} \leq rf_{k_1^0} + (r + \zeta_d) \mathcal{R}_n^{k_1^0}(f, \Lambda_1) \) and \( n^{-1} \leq \mathcal{R}_n^{k_1^0}(f, \Lambda_1) \leq \mathcal{R}_n^{k_n^1}(f, \Lambda_1) \) from (C.21) follows for all \( n, m > m_a \)

\[
\sup \left\{ \mathbb{E}_{f, \varphi} \| f_w - f \|^2_{L^2} : f \in \mathcal{F}, \varphi \in \mathcal{E} \right\} \leq C (r + \zeta_d) \rho_n^0(f, \Lambda_1) + \frac{C r \| f (1 \land \mathcal{A}_m / m) \|_{\infty}}{m} + 5 r \| f_{k_1^0} \wedge \exp \left( \frac{-\lambda_{k_1^0}^0 k_{k_1^0}}{k_m^0} \right) \|_{\infty} + C m^{-1} + C n^{-1} \left( d^2 \sigma_{k_1}^2 k_1^4 + d^2 \sigma_{n_a}^2 \right) \quad (C.22)
\]

Combining (C.20) and (C.22) for the cases (a) and (b) for all \( n, m \in \mathbb{N} \) holds

\[
\sup \left\{ \mathbb{E}_{f, \varphi} \| f_w - f \|^2_{L^2} : f \in \mathcal{F}, \varphi \in \mathcal{E} \right\} \leq C (r + \zeta_d) \rho_n^0(f, \Lambda_1) 1_{\{m > m_a\}} + C r \| f_{k_1^0} \wedge \exp \left( \frac{-\lambda_{k_1^0}^0 k_{k_1^0}}{k_m^0} \right) \|_{\infty} 1_{\{m > m_a\}} + C r d / m \| f (1 \land \mathcal{A}_m / m) \|_{\infty} + C m^{-1} \mathcal{R}_n^{k_1^0}(f, \Lambda_1) + C n^{-1} \left( d^2 \sigma_{k_1}^2 k_1^4 + d^2 \sigma_{n_a}^2 \right) \quad (C.23)
\]

which shows (3.9) and completes the proof of Theorem 3.7. \( \square \)

**Proof of Corollary 3.8.** The proof is similar to the proof of Corollary 2.8 and Corollary 3.5, and we omit the details. \( \square \)

**References**

C. Agostinelli and U. Lund. *R package circular: Circular Statistics (version 0.4-93).* CA: Department of Environmental Sciences, Informatics and Statistics, Ca’ Foscari University, Venice, Italy. UL: Department of Statistics, California Polytechnic State University, San Luis Obispo, California, USA, 2017. URL https://r-forge.r-project.org/projects/circular/.

C. Bahlmann. Directional features in online handwriting recognition. *Pattern Recognition,* 39 (1):115–125, 2006.
A. Barron, L. Birgé, and P. Massart. Risk bounds for model selection via penalization. *Probability Theory and Related Fields*, 113(3):301–413, 1999.

J.-P. Baudry, C. Maugis, and B. Michel. Slope heuristics: overview and implementation. *Statistics and Computing*, 22(2):455–470, 2012.

P. C. Bellec and A. B. Tsybakov. Sharp oracle bounds for monotone and convex regression through aggregation. *Journal of Machine Learning Research*, 16:1879–1892, 2015.

L. Birgé and P. Massart. Minimum contrast estimators on sieves: exponential bounds and rates of convergence. *Bernoulli*, 4(3):329–375, 1998.

P. Brémaud. Fourier analysis of stochastic processes. In *Fourier Analysis and Stochastic Processes*, pages 119–179. Springer, 2014.

J. A. Carnicero, M. C. Ausín, and M. P. Wiper. Non-parametric copulas for circular–linear and circular–circular data: an application to wind directions. *Stochastic environmental research and risk assessment*, 27(8):1991–2002, 2013.

F. Comte and F. Merlevede. Adaptive estimation of the stationary density of discrete and continuous time mixing processes. *ESAIM: Probability and Statistics*, 6:211–238, 2002.

F. Comte and M.-L. Taupin. Adaptive density deconvolution for circular data. Prépublication map5 2003-10, Université Paris Descartes, 2003.

M. Corporation and S. Weston. *doParallel: Foreach Parallel Adaptor for the 'parallel' Package*, 2019. URL https://CRAN.R-project.org/package=doParallel. R package version 1.0.15.

A. Dalalyan and A. B. Tsybakov. Aggregation by exponential weighting, sharp pac-bayesian bounds and sparsity. *Machine Learning*, 72(1-2):39–61, 2008.

A. S. Dalalyan and A. B. Tsybakov. Sparse regression learning by aggregation and langevin monte-carlo. *Journal of Computer and System Sciences*, 78(5):1423–1443, 2012.

S. Efromovich. Density estimation for the case of supersmooth measurement error. *Journal of the American Statistical Association*, 92:526–535, 1997.

J. Gill and D. Hangartner. Circular data in political science and how to handle it. *Political Analysis*, pages 316–336, 2010.

J. Johannes and M. Schwarz. Adaptive circular deconvolution by model selection under unknown error distribution. *Bernoulli*, 19(5A):1576–1611, 2013.
J. Johannes, A. Simoni, and R. Schenk. Adaptive bayesian estimation in indirect gaussian sequence space models. *Annals of Economics and Statistics*, (137):83–116, 2020.

T. Klein and E. Rio. Concentration around the mean for maxima of empirical processes. *The Annals of Probability*, 33(3):1060–1077, 2005.

X. Loizeau. *Hierarchical Bayes and frequentist aggregation in inverse problems*. PhD thesis, 2020.

P. Massart. *Concentration inequalities and model selection*. Ecole d’été de probabilités de Saint-Flour XXXIII – 2003, Lecture Notes in Mathematics 1896. Berlin: Springer, 2007.

A. Meister. *Deconvolution problems in nonparametric statistics*. Lecture Notes in Statistics 193. Berlin: Springer, 2009.

Microsoft and S. Weston. *foreach: Provides Foreach Looping Construct*, 2020. URL https://CRAN.R-project.org/package=foreach. R package version 1.5.0.

M. H. Neumann. On the effect of estimating the error density in nonparametric deconvolution. *Journal of Nonparametric Statistics*, 7:307–330, 1997.

R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2018. URL https://www.R-project.org/.

P. Rigollet and A. B. Tsybakov. Linear and convex aggregation of density estimators. *Mathematical Methods of Statistics*, 16(3):260–280, 2007.

P. Rigollet et al. Kullback–leibler aggregation and misspecified generalized linear models. *The Annals of Statistics*, 40(2):639–665, 2012.

S. Schlüttenhofer and J. Johannes. Adaptive minimax testing for circular convolution. Technical report, arXiv:2007.06388, 2020a.

S. Schlüttenhofer and J. Johannes. Minimax testing and quadratic functional estimation for circular convolution. Technical report, arXiv:2004.12714, 2020b.

M. Talagrand. New concentration inequalities in product spaces. *Inventiones mathematicae*, 126:505–563, 1996.

A. B. Tsybakov. Aggregation and minimax optimality in high-dimensional estimation. In *Proceedings of the International Congress of Mathematicians*, volume 3, pages 225–246, 2014.
H. Wickham. Reshaping data with the reshape package. *Journal of Statistical Software*, 21 (12):1–20, 2007.

H. Wickham. *ggplot2: Elegant Graphics for Data Analysis*. Springer-Verlag New York, 2016.