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Loop Groupoids, Gerbes, and Twisted Sectors on Orbifolds.

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Abstract. The purpose of this paper is to introduce the notion of loop groupoid \( LG \) associated to a groupoid \( G \). After studying the general properties of \( LG \), we show how this notion provides a very natural geometric interpretation for the twisted sectors of an orbifold \( [7] \), and for the inner local systems introduced by Ruan \([14]\) by means of a natural generalization of the concept holonomy of a gerbe.

1. Introduction

1.1. The study of the free loop space \( \mathcal{L}M \) of a space \( M \) without singularities (say for example an smooth manifold) has proved very important in geometry, topology, representation theory and string theory. It is a very natural question that of generalizing this notion to more general spaces, like an orbifold or the leaf space of a foliation. The purpose of this paper is to put forward the notion of the Loop Groupoid \( LG \) associated to a given groupoid \( G \), and to show how this notion provides a very natural framework to understand geometrically several notions that have appeared recently in the theory of the Chen-Ruan cohomology of orbifolds, and also in the work of Freed, Hopkins and Teleman on the Verlinde Algebra \([5]\). Some of the constructions that we propose work just as well in the case of the leaf space of a foliation.

We should point out that in the case in which the groupoid \( G \) represents an orbifold, the coarse moduli space of \( LG \) is precisely the loop space of an orbifold defined by Chen in his work on orbispace \([3]\).

1.2. Let us briefly describe the contents of this paper. In section 2 we collect some facts about groupoids and set the notations used in the rest of the paper. In this paper by the word groupoid we mean a topological category in which every morphism has an inverse. In most of the paper we will suppose that our groupoids are étale and proper. In section 3 we define the loop groupoid associated to a given groupoid. This is topological category but it is far from being étale. In section 4 we have an algebraic interlude showing how this theory looks in the case in which the groupoid \( G \) is simply a finite group. Here we also make some remarks relating this results to the loop group and compact lie groups, and the work of Freed, Hopkins and Teleman mentioned before. If our groupoid represents a smooth manifold \( M \) then the loop groupoid represents the free loop space \( \mathcal{L}M \),
this is proved in section 5. In this section we also explain how this groupoid is useful in understanding how the holonomy of a gerbe on $M$ induces a line bundle on $LM$, and therefore a one dimensional representation of the loop groupoid $LM$.

In section 6 we consider groupoids representing orbifolds. We show that the twisted sectors $14$ of an orbifold can be understood as the $S^1$ in variant subgroupoid of the loop groupoid. This helps us to put all the story together and show how an inner local system in the sense of Ruan $14$ is simply the natural generalization of the holonomy line bundle of a gerbe that we have defined before in $8$.

In this paper we try to be reasonably self-contained, but we will use at several points the results of $8$, and this paper can be thought of as a continuation of that one. We strongly recommend $10$ for an introduction to the subject, including the original references. We refer the reader to $10, 8$ for the motivations and the details of those results.

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2. Groupoids

In this section we will briefly summarize the definition of groupoids as well as gerbes over groupoids; for a more detailed exposition we recommend the reader to see $9, 4, 8$.

A groupoid $G$ in great generality is a category in which every morphism is invertible. Here we will only consider small categories.

We will denote by $G_0$ and $G_1$ the set of objects and morphism respectively, and the structure maps by:

$$
\begin{array}{cccc}
G_1 & \times_s & G_1 & \overset{m}{\rightarrow} G_1 \\
\downarrow & & \downarrow & \\
G_0 & \overset{s}{\rightarrow} & G_0 & \overset{e}{\rightarrow} G_1
\end{array}
$$

where $s$ and $t$ are the source and the target maps of morphisms, $m$ is the composition of two of them whenever the target of the first equals the source of the second, $i$ gives us the inverse morphism and $e$ assigns the identity arrow to every object. We will also call arrows the morphisms of the groupoid.

The groupoid will be called topological (smooth) if the sets $G_1$ and $G_0$ and the structure maps belong to the category of topological spaces (smooth manifolds). In the case of a smooth groupoid we will also require that the maps $s$ and $t$ must be submersions, so that $G_1 \times_s G_1$ is also a manifold.

A topological (smooth) groupoid is called étale if the source and target maps $s$ and $t$ are local homeomorphisms (local diffeomorphisms). For an étale groupoid we will mean a topological étale groupoid. We will always denote groupoids by letters of the type $G, H, S$. We will work with topological groupoids but the reader can think of them as smooth if he prefers.

Due to a fundamental result of Moerdijk and Pronk $11, 13$ we know that orbifolds are related to a special kind of étale groupoids, they have the peculiarity that the anchor map $(s, t) : G_1 \rightarrow G_0 \times G_0$ is proper, groupoids with this property are called proper. The theorem of Moerdijk and Pronk states that the category of orbifolds is equivalent to a quotient category of the category of proper étale groupoids after inverting Morita equivalence. We explain this in more detail now. Whenever
we write orbifold, we will choose a proper étale smooth groupoid representing it (up to Morita equivalence.)

A morphism of groupoids $\Psi : H \to G$ is a pair of maps $\Psi_i : H_i \to G_i$, $i = 0, 1$ such that they commute with the structure maps. The maps $\Psi_i$ will be continuous (smooth) depending on which category we are working on.

The morphism $\Psi$ is called Morita if the following square is a cartesian square.

$$
\begin{array}{ccc}
H_1 & \xrightarrow{\psi_1} & G_1 \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
H_0 \times H_0 & \xrightarrow{\psi_0 \times \psi_0} & G_0 \times G_0
\end{array}
$$

Two (abstract) groupoids $G$ and $H$ are Morita equivalent if there exist another groupoid $K$ with Morita morphisms $G \xleftarrow{\sim} K \xrightarrow{\sim} H$. We will write $BG$ to denote the classifying space of the groupoid $G$ \cite{15}. If two groupoids $G$ and $H$ are Morita equivalent we write $G \simeq H$, in this case the classifying spaces are homotopy equivalent $BG \simeq BH$.

On working with topological groupoids there is a further requirement for a morphism to be called Morita, namely given the morphism $\Phi : H \to G$ we require the map $s\pi_2 : H_0 \times H_0 \to G_0 \times G_0 \to U(1)$ to be an open surjection. If this is the case for a morphism $\Phi$ between topological étale groupoids, then we necessarily have that the maps $\Phi_1$ and $\Phi_0$ are both étale (see \cite{13}). As this paper only deals with topological groupoids, whenever we mention a Morita morphism we will require this additional condition.

If $G$ is a group, we will write $G$ for the groupoid $\ast \times G \Rightarrow \ast$ where $m(g, h) = gh$.

All the group actions will be carried out from the right. A morphism from $H$ to $G$ determines a unique principal $G$ bundle over a groupoid $H$ (cf. 4.1.5 \cite{8}).

A gerbe $(L, \theta)$ over an étale groupoid $G$ (in most cases it will be an orbifold) is a complex line bundle $L$ over $G_1$ (not over $G$) satisfying the following conditions:

- $i^*L \cong L$
- $\pi_1^*L \otimes \pi_2^*L \otimes m^*i^*L \cong 1$
- $\theta : G_1 \times_s G_1 \to U(1)$ is a 2-cocycle

where $\pi_i$’s are projections and $\theta$ is a trivialization of the line bundle. The cohomology class $[L] \in H^2(G, \mathbb{C}^*)$ of $\theta$ is called the characteristic class of the gerbe $([\mathbb{C}^*]$ is the sheaf of $\mathbb{C}^*$ valued functions over the groupoid $G$).

A connection on a gerbe $(L, \theta)$ over an étale groupoid $G$ consist of a real valued 0-form $\theta \in \Omega^0(G_1 \times_s G_1)$, a 1-form $A \in \Omega^1(G_1)$, a 2-form $F \in \Omega^2(G_0)$ and a 3-form $K \in \Omega^3(G_0)$ satisfying:

- $K = dF$
- $\pi_2^2F - \pi_1^2F = dA$ and
- $\pi_1^1A + \pi_2^2A + m^*i^*A = -\sqrt{-1}\theta^{-1}d\theta$

A connection is flat if the curvature $K$ vanishes.

3. The Loop Groupoid

3.1. We would like to define the loop groupoid associated to a groupoid in a similar manner to the definition of the free loop space associated to a manifold. Therefore we want the loop groupoid to be a category whose objects are $\text{Hom}(S^1, G)$ in the category of orbifolds. Let us remember that the category of orbifolds is
the same as the category of proper étale topological groupoids modulo Morita equivalence. Therefore we must first assign a groupoid to $S^1$ and then consider all Morita equivalent groupoids. This amounts in a concrete language to consider finer and finer covers of the circle. Moreover, this allows us to give the loop groupoid a natural topology. We will do so in a very concrete manner by taking certain “well behaved” open covers of the circle.

**Definition 3.1.1.** Let $\mathcal{F}$ be the family of all finite sets of the unit circle $S^1$. We identify $\mathcal{F}$ with the family of all finite sets of $(0, 1]$ using the exponential map. In other words the set $\{q_1, q_2, \ldots, q_n, q_0\} \subset (0, 1]$ (always written in increasing order $q_1 < q_2 < \cdots < q_n < q_0$) will be identified with the ordered $n$-tuple of points in the circle given by $\{e^{2\pi i q_k} | k = 1, \ldots, n, 0\}$. To every element $\{q_1, q_2, \ldots, q_n, q_0\} \in \mathcal{F}$ and every $\epsilon > 0$ we will associate a unique open cover of the circle $\mathcal{V}(\{q_k\}, \epsilon)$ defined by the sets $V_\epsilon := (q_i - \epsilon, q_i + \epsilon)$. We denote by $\mathcal{F}(\epsilon)$ the family of all such covers. Notice that 1 is always in $V_0$. Since for every $\epsilon > 0$, $\mathcal{F} \cong \mathcal{F}(\epsilon)$ as sets, then $\mathcal{F}(\epsilon)$ acquires a partial ordering (that we call refinement) induced by that on $\mathcal{F}$ given by inclusion of sets. In addition we say that a cover $\mathcal{V}(p_1, p_2, \ldots, p_m, p_0; \epsilon)$ is a refinement of $\mathcal{V}(q_1, q_2, \ldots, q_n, q_0; \epsilon')$ when $\epsilon \leq \epsilon'$ and $\{q_1, q_2, \ldots, q_n, q_0\} \subset \{p_1, p_2, \ldots, p_m, p_0\}$ as sets. Given any two covers of types $\mathcal{F}(\epsilon)$ and $\mathcal{F}(\epsilon')$ that are related (one being refinement of the other) then there is a canonical embedding of the open sets of the coarser cover into the ones of the finer. Finally we will denote by $W$ the set of all covers $\mathcal{V}(q_1, q_2, \ldots, q_n, q_0; \epsilon) \in \mathcal{F}(\epsilon)$ for some $\epsilon$ so that $\epsilon$ is small enough to make all triple intersections of elements of $\mathcal{V}(q_1, q_2, \ldots, q_n, q_0; \epsilon')$, $V_i \cap V_j \cap V_k = \emptyset$. The set $W$ is called the set of admissible covers of $S^1$. The remarks of this paragraph amount to a definition of a partial order in $W$ called refinement. This partial order has the property that given any two elements $\mathcal{V}(p_1, p_2, \ldots, p_m, p_0; \epsilon) \in W$ and $\mathcal{V}(q_1, q_2, \ldots, q_n, q_0; \epsilon') \in W$ then there is an element of $W$ refining both at the same time (take for example $\mathcal{V}(\{q_1, q_2, \ldots, q_n, q_0\} \cup \{p_1, p_2, \ldots, p_m, p_0\}; \min(\epsilon, \epsilon')) \in W$).

Now we will define a unique groupoid associated to the circle $S^1$ and a cover of it (and therefore one such groupoid for every element of $W$.) Let $\{V_\alpha\}_{0 \leq \alpha \leq n}$ be such a cover of $S^1$ and let $W = \{W_\beta\}_\beta$ be the pullback of this cover under the map $\mathbb{R} \xrightarrow{e^{2\pi i \cdot}} S^1$ and let’s call by $\mathbb{R}^W$ the groupoid associated to this cover, in other words $\mathbb{R}^W := \bigsqcup_{(\beta_1, \beta_2)} W_{\beta_1} \cap W_{\beta_2}$ and $\mathbb{R}^W_0 := \bigsqcup_\beta W_\beta$.

If we call by $\{W^n\}_n$ the inverse image of the set $V_\alpha$ by the exponential map, we can define an action of $\mathbb{Z}$ in the following way: $\mathbb{R}^W_0 \times \mathbb{Z} \rightarrow \mathbb{R}^W_0$, $(x, W_{\beta\gamma}, l) \mapsto (x + l, W_{\beta\gamma}^l)$ where $x \in W_{\beta\gamma}$ and $x + l \in W_{\beta\gamma}^l$.

**Definition 3.1.2.** Let $S^1_W$ be the groupoid $\mathbb{R}^W_0 \times \mathbb{Z}$.
with maps
\[ s((x, W_{\beta\gamma}), l) = (x, W_{\beta}), \quad t((x, W_{\beta\gamma}), l) = (x + l, W'_{\gamma}) \]
\[ e(x, W_{\beta}) = ((x, W_{\beta\beta}), 0), \quad i((x, W_{\beta\gamma}), l) = ((x + l, W'_{\beta\gamma}), -l) \]
\[ m \left[ ((x, W_{\beta\gamma}), l), ((x + l, W'_{\gamma\sigma}), k) \right] = ((x, W_{\beta\sigma}), l + k) \]

3.2. It is an easy exercise to check that \( S^1_W \) is Morita equivalent to \( S^1 \) where its groupoid structure is the trivial one, i.e. \( S^1 \Rightarrow S^1 \) with the source and the target maps equal to the identity. Moreover, for another cover \( W' \) defined in the same way as \( W \) that refines it, there is a natural morphism of groupoids \( \rho^W_{W'} : S^1_{W'} \to S^1_W \) which also turns out to be Morita.

**Definition 3.2.1.** For \( G \) a topological groupoid and an open cover \( W \) of the circle, the loop groupoid \( LG(W) \) associated to \( G \) and the open cover \( W \) will be defined by the following data:

- **Objects** \( (LG(W))_0 \): Morphisms \( S^1_W \to G \)
- **Morphisms** \( (LG(W))_1 \): For two elements in \( (LG(W))_0 \), say \( \Psi, \Phi : S^1_W \to G \), a morphism (arrow) from \( \Psi \) to \( \Phi \) is a map \( \Lambda : \mathbb{R}^W_0 \times \mathbb{Z} \to G_1 \) that makes the following diagram commute

\[
\begin{array}{ccc}
\mathbb{R}^W_0 \times \mathbb{Z} & \xrightarrow{\Lambda} & G_1 \\
\downarrow{s \times t} & & \downarrow{s \times t} \\
\mathbb{R}^W_0 \times \mathbb{R}^W_0(\Psi_0, \Phi_0) & \to & G_0 \times G_0
\end{array}
\]

and such that for \( r \in \mathbb{R}^W_0 \times \mathbb{Z} \)

\[ \Lambda(r) = \Psi_1(r) \cdot \Lambda(\text{es}(r)) = \Lambda(\text{es}(r)) \cdot \Phi_1(r). \]

The composition of morphisms is defined pointwise, in other words, for \( \Lambda \) and \( \Omega \) with

\[ \Psi \xrightarrow{\Lambda} \Phi \xrightarrow{\Omega} \Gamma \]

we set

\[ \Omega \circ \Lambda(\text{es}(r)) := \Lambda(\text{es}(r)) \cdot \Omega(\text{es}(r)) \]

and

\[ \Omega \circ \Lambda(r) := \Omega \circ \Lambda(\text{es}(r)) \cdot \Gamma(r) = \Psi(r) \cdot \Omega \circ \Lambda(\text{et}(r)) \]

The last equation comes from the following set of equalities:

\[ \Omega \circ \Lambda(\text{es}(r)) \cdot \Gamma(r) = \Lambda(\text{es}(r)) \cdot \Omega(\text{es}(r)) \cdot \Gamma(r) = \Lambda(\text{es}(r)) \cdot \Phi(r) \cdot \Omega(\text{et}(r)) = \Psi(r) \cdot \Lambda(\text{et}(r)) \cdot \Omega(\text{et}(r)) = \Psi(r) \cdot \Omega \circ \Lambda(\text{et}(r)) \]

The groupoid \( LG(W) \) can be given the compact-open topology to make it a topological groupoid. To do this consider the following identities satisfied by any arrow \( \Lambda : \Psi \to \Phi \)

\[ \Psi_1(r) = \Lambda(r) \cdot \Lambda(\text{et}(r)))^{-1}, \]
\[ \Psi_0(x) = s(\Lambda(e(x))), \]
\[ \Phi_1(r) = \Lambda(\text{es}(r))^{-1} \cdot \Lambda(r), \]
\[ \Phi_0(x) = t(\Lambda(e(x))). \]

This identities imply that \( \Lambda \) determines its source \( \Psi \) and its target \( \Phi \). Let \( \Lambda^o = \Lambda \circ e \).

Then from the previous identities is easy to see that from the pair \( (\Psi_1, \Lambda^o) \) we can recover \( \Lambda \). Therefore we can see the set of all such \( \Lambda \)'s, namely \( \text{LG}(W)_1 \) as a subspace of the space of continuous maps \( \text{Map}(\mathbb{R}^W_1, G_1) \times \text{Map}(\mathbb{R}^W_0, G_1) \). In this way the spaces \( \text{LG}(W)_1 \) and \( \text{LG}(W)_0 \) inherit the compact-open topology, making the groupoid \( \text{LG}(W) \) into a topological one.

From now on we restrict our attention to admissible covers of the circle. We would like that two morphisms \( \Psi_i : S^1 \to G (i = 1, 2) \), be equivalent if there exists a refinement \( W \in \mathcal{W} \) of \( W_1 \in \mathcal{W} \) and \( W_2 \in \mathcal{W} \) such that the following diagram of morphisms commute

\[
\begin{array}{ccc}
S^1_W & \overset{\rho^W_{W_1}}{\longrightarrow} & S^1_{W_2} \\
\downarrow_{\rho^W_{W_2}} & & \downarrow_{\Psi_1} \\
S^1_{W_2} & \underset{\Psi_2}{\longrightarrow} & G
\end{array}
\]

This can be achieved using a limit construction. For \( W' \) a refinement of \( W \) there is a natural monomorphism of topological groupoids

\[ \text{LG}(W) \hookrightarrow \text{LG}(W') \]

(here is where we use the partial order in \( \mathcal{W} \)) this allows us to define

**Definition 3.2.2.** The loop groupoid \( \text{LG} \) of the topological groupoid \( G \) is defined as the monotone union (colimit) of the groupoids \( \text{LG}(W) \) where \( W \) runs over the set \( \mathcal{W} \) of admissible covers

\[ \text{LG} := \lim_{\mathcal{W}} \text{LG}(W). \]

In this way the loop groupoid is naturally endowed with a topology, becoming a topological groupoid.

**Remark 3.2.3.** Unless stated otherwise the groupoids we will focus our attention on will be orbifolds (smooth, étale and proper groupoids) although some of the results that follow can be generalized.

Now let’s see a property of the arrows in the loop groupoid of an étale groupoid \( G \). This paragraph justifies us to interpret the loop groupoid \( \text{LG} \) of an (orbifold) étale proper groupoid \( G \) as an infinite dimensional orbifold groupoid.

**Lemma 3.2.4.** Let \( G \) be an étale groupoid. An arrow \( \Lambda \in \text{LG}_1 \) between \( \Psi \) and \( \Phi \) is determined completely by \( \Psi \) and the image \( \Lambda^o(x) \) of a marked point \( x \in \mathbb{R}^W_1 \). Here \( \Lambda^o = \Lambda \circ e \) is the restriction of \( \Lambda \) to the identity arrows of \( G \).

**Proof.** First is clear that \( \Lambda \) is completely determined by \( \Lambda^o \) because

\[ \Lambda(r) = \Psi(r) \cdot \Lambda^o(t(r)). \]

What this lemma is saying is that the image of any other point \( y \in \mathbb{R}^W_0 \) is uniquely determined by \( \Lambda^o(x) \) and the morphism \( \Psi \). Let’s suppose without loss of generality that \( x \in V_1 \) and \( y \in V_2 \), both sets in \( W \) such that \( V_{12} := V_1 \cap V_2 \neq \phi \). As \( G \) is étale the image of \( V_1 \) under \( \Lambda \) is determined by the point \( \Lambda^o(x) \) and \( \Psi_0(V_1) \); this
because we can find a neighborhood $U_1$ of $\Lambda^o(V_1)$ in $G_1$ homeomorphic to $s(U_1)$ which contains $\Psi_0(V_1)$. Now if $z \in U_{12}$, labeling it $z_1$ when we see it in $V_1$ and $z_2$ when in $V_2$, then

$$\Lambda^o(z_2) = \Psi_1(z)^{-1}\Lambda^o(z_1)\Psi_1(z).$$

Applying the same procedure as before, the set $\Lambda(V_2)$ is determined by $\Psi_0(V_2)$ and $\Lambda^o(z_2)$; and hence it determines $\Lambda^o(y)$. □

This allows us to get the following:

**Corollary 3.2.5.** Let $\Lambda_0$ and $\Lambda_1$ be arrows joining $\Psi$ and $\Phi$. If $\Lambda_0(x) = \Lambda_1(x)$ then $\Lambda_0 = \Lambda_1$.

**Corollary 3.2.6.** If $G$ is an étale proper groupoid, then the loop groupoid $LG$ is étale and the local isotropy at any object $\Psi$ is finite.

**Proof.** The arrows between two morphisms are determined by the image of only one point $\Lambda(x)$ in $G_1$. If the source and the target of $\Lambda$ are equal, then the same is true for the arrow $\Lambda(x)$. As the isotropy in $G$ is finite, the result follows. This implies that the source map of the loop groupoid is étale. □

We say then that $LG$ represents an infinite dimensional orbifold.

### 3.3.

In order to motivate this definitions let’s consider the case when $R_1 = R_0 = R$. Let $\Psi : S^1 \to G$ be a morphism, then it is determined by the image of $R \times \{1\}$ under $\Psi_1$; clearly $\Psi_1(x,0) = es(\Psi_1(x,1))$, $\Psi_0(x) = s(\Psi_1(x,1))$ and $\Psi_0(x + 1) = t(\Psi_1(x,1))$, also using the fact that

$$(x, n) = (x, 1) \cdot (x + 1, 1) \cdots (x + n - 1, 1)$$

we obtain

$$\Psi_1(x, n) = \Psi_1(x, 1) \cdot \Psi_1(x + 1, 1) \cdots \Psi_1(x - n + 1, 1).$$

The use of the word loop is justified because the function $\Psi_0|_{[x,x+1]}$ gives us a path from the source to the target of $\Psi_1(x,1)$

and when we consider the coarse moduli space of the groupoid the path determined by $\Psi_0|_{[0,1]}$ becomes a loop. The morphisms of $LG$ relate the loops that in the coarse moduli space are identified, so for $\Lambda$ an arrow from $\Psi$ to $\Phi$ we have the following
3.4. Now we set to the task of showing that $LG$ doesn’t depend on the Morita equivalence class of $G$. In particular $L$ is a functor from groupoids to groupoids.

**Definition 3.4.1.** A morphisms of groupoids $F : H \to G$ induces naturally a morphism between the loop groupoids $LF : LH \to LG$ in the following way:

- **Objects:**
  \[ (LH)_0 \xrightarrow{(LF)_0} (LG)_0 \]
  \[ \Psi \quad \overset{F \circ \Psi}{\longleftarrow} \]

- **Morphisms:** For $\Lambda \in LH$, an arrow between $\Psi$ and $\Phi$, we define $(LF)_1(\Lambda) := F_1 \circ \Lambda$ as an arrow between $(LF)_0(\Psi)$ and $(LF)_0(\Phi)$

\[ \mathbb{Z} \times \mathbb{R}_1 \overset{A}{\longrightarrow} H_1 \underset{F_1}{\longrightarrow} G_1 \]
\[ \mathbb{R}_0 \times \mathbb{R}_0 \overset{(\Psi_0, \Phi_0)}{\longrightarrow} H_0 \times H_0 \overset{(f_0, f_0)}{\longrightarrow} G_0 \times G_0 \]

The morphism $LF$ is continuous because the limit commutes with the composition of $F$.

**Lemma 3.4.2.** If $F : H \to G$ is a Morita morphism of étale groupoids then the induced morphism $LF : LH \to LG$ is also Morita.

**Proof.** First we need to check that the following square is a fibered product

\[ (LH)_1 \xrightarrow{(LF)_1} (LG)_1 \]
\[ (LH)_0 \times (LH)_0 \xrightarrow{((LF)_0, (LF)_0)} (LG)_0 \times (LG)_0 \]

So for $\Psi, \Phi \in (LH)_0$ and $\Omega \in (LG)_1$ with

\[ s(\Omega) = F \circ \Psi \quad t(\Omega) = F \circ \Phi \]

we need to show that there is only one $\Lambda \in (LH)_1$ such that $(LF)_1(\Lambda) = \Omega$, $s(\Lambda) = \Psi$ and $t(\Lambda) = \Phi$. Let’s work pointwise; for $a \in e(\mathbb{R}_1)$ let

\[ r := \Omega(a) \quad x := s(\Psi_1(r)) \quad y := s(\Phi_1(r)) \]

then $s(r) = F_0(x)$ and $t(r) = F_0(y)$; as $F$ is a Morita morphism then there is only one element $w \in H_1$ such that $s(w) = x$, $t(w) = y$ and $F_1(w) = r$; then $\Lambda(a) := w$. So we have a well defined map $\Lambda : \mathbb{R}_1 \times \mathbb{Z} \to H_1$ with the required properties. As
the maps \( F_0 \) and \( F_1 \) are étale then it is easy to see that \( \Lambda \) is continuous. We still have to prove that for \( \alpha \in \mathbb{R}_0 \times \mathbb{Z} \) we have

\[
\Lambda(es(\alpha)) \cdot \Phi_1(\alpha) = \Psi_1(\alpha) \cdot \Lambda(et(\alpha))
\]

Let \( v := \Psi_1(\alpha) \cdot \Lambda(et(\alpha)) \cdot \Phi(\alpha)^{-1} \cdot \Lambda(es(\alpha))^{-1} \), as \( \Omega \) belongs to \((\text{LG})_1\) then \( F_1(v) = es(F_1(v)) \), but \( es(v) \) also maps to the same morphism under \( F_1 \); then by the properties of the Morita morphism we get that \( v = es(v) \).

Now we’ll see that the map \( LF_0 : LH_0 \to \text{LG}_0 \) is surjective. Let \( W \) be a cover associated to the tuple \( \{ q_1, \ldots, q_n, q_0 \} \) and \( \Psi : S^1_W \to G \) an object in \( \text{LG} \). Let \( V_i = (q_i - \epsilon, q_{i+1} + \epsilon) \) be an open set of \( W \) and \( K_i \subset G_0 \) the image under \( \Psi_0 \) of the compact set \([q_i, q_{i+1}]\). As the map \( F_0 \) is étale, there are a finite number of open sets \( U^i_j \subset H_0, 0 \leq j \leq l_i \) diffeomorphic to their image under \( F_0 \), i.e. \( U^i_j \cong F_0(U^i_j) \), such that the sets \( F_0 = \{ U^i_j \} \) cover \( K_i \) (\( K_i \) is compact). This is done in such a way that we can partition the interval \([q_i, q_{i+1}]\) in \( l_i + 1 \) pieces \( q_i = p^i_0 < p^i_1 < \cdots < p^i_{l_i} = q_{i+1} \) such that

\[
\Psi_0([p^i_j, p^i_{j+1}]) \subset F_0(U^i_j).
\]

We do the same for \( i \) in \( 0 \leq i \leq n \) and we take the cover \( W' \) associated to the tuple given by the \( p^i_j \)'s. Define \( \Phi_0 : \mathbb{R}^{W'}_0 \to H_0 \) by

\[
\Phi_0 : (p^i_j - \epsilon', p^i_{j+1} + \epsilon') \to H_0
\]

\[
x \mapsto (F_0|_{F_0(U^i_j)})^{-1} \circ \Psi_0 \circ \rho^{W'}_\alpha(x).
\]

The construction of \( \Phi_1 \) is straight forward using the fact that the map \( F \) is a Morita equivalence, i.e. the image of \( \Phi_0 \) and the cartesian square of \( 2.0.1 \) defines it uniquely. Even though \( F \circ \Phi \) is not equal to \( \Psi \), it is clear that both are in the same class of the limit.

\[\square\]

3.5. We can see that the loop groupoid defined in this way is an immense object. We want this definition to be more manageable, so in some cases, depending on whether our groupoid \( G \) has some additional structure we will see, in the following sections, that we can considerably simplify the construction. Notice as well that it is closely related to the path constructions of \[2, 12\].

3.6. The loop groupoid \( \text{LG} \) has a natural action of \( \mathbb{R} \) (that actually descends to an action of \( S^1 \) given by rotating the loop). Let’s recall that the groupoid \( S^1 \) is subordinated to a cover \( W \) of \( \mathbb{R} \); for \( z \in \mathbb{R} \), let \( W^z \) be the cover of \( \mathbb{R} \) obtained from \( W \) after shifting by \( z \). To be more explicit \( W^z = \{ V \mid V \in W \} \) where \( V^z = \{ x + z \mid x \in V \} \). Using this new cover we can define the action of \( \mathbb{R} \) into \( \text{LG} \).

**Definition 3.6.1.** For \( z \in \mathbb{R} \) and \( \Psi \in (\text{LG})_0 \), let \( \Psi^z \) be the morphism

\[
\Psi^z : S^1_{W^z} \to G
\]

defined in the following way

\[
\Psi^z_1((x, V^z_\alpha), l) := \Psi((x - z, V_\alpha), l) \\
\Psi^z_0(x, V^z_\alpha) := \Psi_0(x - z, V_\alpha)
\]

Obtaining an action of \( \mathbb{R} \) into \( \text{LG} \)

\[
(\text{LG})_0 \times \mathbb{R} \xrightarrow{\Psi^z} (\text{LG})_0 \\
(\Psi, z) \mapsto \Psi^z
\]
We will denote by \( \text{LG}^\mathbb{R} \) the subgroupoid of \( \text{LG} \) that is fixed under the action of \( \mathbb{R} \). The great generality that we are allowing in our definition of the loop groupoid may persuade us that \( \text{LG}^\mathbb{R} \) is very complicated groupoid. This is not the case and we identify it now. Every morphism in \( \text{LG}^\mathbb{R} \) is constant when is seen in the coarse moduli space of \( G \), but in \( G \) several objects map into that constant point. The groupoid \( \text{LG}^\mathbb{R} \) is taking into account all that information. The following technical lemmas will make more explicit the idea just mentioned.

**Lemma 3.6.2.** Let \( \Psi \) be an object in \( \text{LG}^\mathbb{R} \), then \( \Psi_1 \) and \( \Psi_0 \) are locally constant.

**Proof.** Let \( V \in W \) and \( z \in \mathbb{R} \) such that \( V \cap V^z \neq \emptyset \). As \( V \cap V^z \) belongs to a common refinement of \( W \) and \( W^z \) and \( \Psi = \Psi^z \) then we must have that

\[
\Psi_0(x, V) = \Psi_0^z(x, V^z) = \Psi_0(x - z, V)
\]

where the first equality is the condition of being a fixed element under the action of \( \mathbb{R} \), and the second is just the definition of \( \Psi^z \). Then we can see that \( \Psi_0 \) is constant in all \( V \). For \( \Psi_1 \) the same argument is applied. \( \square \)

**Lemma 3.6.3.** Let \( \Psi \) be in \( \text{LG}^\mathbb{R} \), then there exist another object \( \Phi \) in \( \text{LG}^\mathbb{R} \) with

\[
\Phi : S^1_{\mathbb{R}} \to G
\]

defined over the trivial cover of \( \mathbb{R} \) such that there is an arrow \( \Lambda \) joining \( \Psi \) and \( \Phi \).

**Proof.** The idea underlying in this lemma is the fact that all the objects in the image of \( \Psi_0 \) are related by arrows in \( G_1 \).

Let’s fix the point \((0, V_0)\) in \( \mathbb{R}^n \) (as we said before, \( V_0 \) always contains the marked point of the circle); for \((y, U)\) another point in \( \mathbb{R}^n \) there exist open sets \( V = U_0, U_1, \ldots, U_n = U \) in the cover such that \( U_i \cap U_{i+1} \neq \emptyset \); let \( y_i \) be any point in \( U_{i,i+1} := U_i \cap U_{i+1} \). As \( \Psi \) is a locally constant morphism

\[
\Psi_0(y_i, U_i) = \Psi_0(y_{i+1}, U_i)
\]

and \( \Psi_1((y_i, U_{i,i+1}), 0) \) is an arrow between

\[
\Psi_0(y_i, U_i) \quad \text{and} \quad \Psi_0(y_i, U_{i+1})
\]

Thus

\[
(3.6.1) \quad \text{Ar}_{y_i,U}^{x,V} := \Psi_1((y_0, U_{0,1}), 0) \cdots \Psi_1((y_{n-1}, U_{n-1,n}), 0)
\]

is an arrow in \( G_1 \) joining \( \Psi_0(x, V) \) and \( \Psi_0(y, U) \). It is worth pointing out that because of the properties of the groupoids the map \( \text{Ar}_{y_i,U}^{x,V} \) is independent of the chosen sets \( U_i \); is also clear that \( \text{Ar} \) only depends on the open sets \( V \) and \( U \), so we can suppress the points \( x \) and \( y \); \( \text{Ar}_{V,U}^V = \text{Ar}_{V,U}^x \). Let \( \Phi : S^1_{W} \to G \) be the morphism defined in the following way (as the morphism is locally constant we will define it on the open sets):

\[
\Phi_0(U) := \Psi_0(V)
\]

\[
(3.6.2) \quad \Phi_1(U_{\alpha\beta}, l) := \text{Ar}_{U_{\alpha\beta}}^V \cdot \Psi_1(U_{\alpha\beta}, l) \cdot (\text{Ar}_{U_{\beta}}^V)^{-1}
\]

Replacing \( \text{Ar} \) when \( l = 0 \) we get that

\[
\Phi_1(U_{\alpha\beta}, 0) = \Psi_1(V, 0) = e\Psi_0(V)
\]

and from these it follows that

\[
\Phi_1((U_{\alpha\beta}, l) = \Psi_1(V, l).
\]
So we can make $\Phi$ to be defined on the trivial cover of $\mathbb{R}$, $\Phi : S^1_\mathbb{R} \to G$

$$\Phi_0(y) := \Psi_0(V) \quad \text{and} \quad \Phi_1(y, l) := \Psi_1(V, l)$$

Now we need to construct an arrow $\Lambda$ between $\Psi$ and $\Phi$, and it is very clear how it should be defined:

$$\Lambda(U_\alpha, 0) := A r_{U_\alpha}^V$$

then equation 3.6.2 gives explicitly the condition that $\Lambda$ has to fulfill.

So, for the groupoid $LG^\mathbb{R}$ we can restrict our attention to the morphisms $\Phi : S^1_\mathbb{R} \to G$ defined on the trivial cover of $\mathbb{R}$, but we just proved that they are locally constant; so we get

**Theorem 3.6.4.** The groupoid $LG^\mathbb{R}$ is Morita equivalent to the groupoid $\text{Hom}(\mathbb{Z}, G)$ (with the natural topology) whose objects are morphisms $\phi : \mathbb{Z} \to G$ with $s \phi(1) = t \phi(1) = \phi(0)$ and whose arrows $\phi \xrightarrow{\lambda} \psi$ are maps $\lambda : \mathbb{Z} \to G_1$ with

$$\lambda(0) \cdot \psi_1(n) = \phi_1(n) \cdot \lambda(0) = \lambda(n)$$

**Proof.** From $\Phi : S^1_\mathbb{R} \to G$ obtained in the previous lemma, which is locally constant, we can define $\phi : \mathbb{Z} \to G$ as $\phi_1(n) := \Phi_1(x, n)$ and $\phi_0 := \Phi(x)$ for any $x$ in $\mathbb{R}$. From the fact that $s \Phi_1(0, 1) = \Phi_0(0) = \Phi_0(1) = t \Phi_1(0, 1)$ we get that $s \phi(1) = t \phi(1) = \phi(0)$.

For an arrow $\phi \xrightarrow{\lambda} \psi$, with $\Psi$ another object in $LG^\mathbb{R}$ as in previous lemma, we can define $\lambda(n) := \Lambda(x, n)$ and clearly we get what we needed.

The Morita equivalence follows from the fact that we are getting rid of extra objects and arrows, the proof is straightforward. \(\square\)

This description of $LG^\mathbb{R}$ matches the one of what is known as the inertia groupoid.

**Definition 3.6.5.** The inertia groupoid $\wedge G$ is defined in the following way:

- Objects ($\wedge G)_0$: Elements $v \in G_1$ such that $s(v) = t(v)$.
- Morphisms ($\wedge G)_1$: For $v, w \in (\wedge G)_0$ an arrow $v \xrightarrow{\alpha} w$ is an element $\alpha \in G_1$ such that $v \cdot \alpha = \alpha \cdot w$

It is known that the inertia groupoid in the case of an orbifold matches with what is commonly known in the literature by twisted sectors (see [14]), thus this is a natural way to define them. We conclude this section with the following statements:

**Proposition 3.6.6.** The groupoid $LG^\mathbb{R}$ is Morita equivalent to the inertia groupoid $\wedge G$.

**Proof.** Follows from the fact that the groupoids $\wedge G$ and $\text{Hom}(\mathbb{Z}, G)$ are clearly diffeomorphic (homeomorphic). \(\square\)

And by using the result of the appendix we get.

**Corollary 3.6.7.** For an orbifold $G$, the groupoid $LG^\mathbb{R}$ is Morita equivalent to the twisted sectors $\tilde{\Sigma}^1 G$ of $G$. 

4. Finite Groups and Lie Groups.

4.1. Even for the case in which our groupoid $G$ happens to be a finite group the theory is quite interesting. For $G$ a finite group acting on the connected space $X$ and $G$ the groupoid associated to the global quotient $[X/G]$ one does not need to consider the morphisms from $S^1$ to $G$ with all possible open covers of $\mathbb{R}$. As the space $X$ is connected, for any $\Psi : S^1 \to G$ there exist another morphism $\Phi : S^1 \to G$ subordinated to the trivial cover of the reals (i.e. $\mathbb{R}_1 = \mathbb{R}_0 = \mathbb{R}$) and an arrow in $(LG)_1$ relating the two.

The idea here is that we are getting rid of unnecessary arrows as well as objects without changing the relevant information of the groupoid, so we have

**Lemma 4.1.1.** Let $G = [X/G]$ be the groupoid associated to the global quotient $X/G$ for $G$ a finite group and $X$ a space, then for the loop groupoid it suffices to take morphisms from $S^1$ to $G$ associated to the trivial cover of $\mathbb{R}$.

**Proof.** The argument follows the lines of the proof of lemma 3.6.2 although in this case is a bit more complicated. For every $\Psi \in L[X/G]$ we are going to associate a morphism subordinated to the trivial cover as follows. Let $W = \{V_i\}_{0 \leq i \leq n}$ be the cover of the circle on which $\Psi$ is defined and $\{U^j_i\}_{0 \leq j \leq n, i \in \mathbb{Z}}$ the cover of the reals (in the same way that is done at the beginning of chapter 3). To make the notation less cumbersome we will write $U^j_{i+1} := U^j_i \cap U^j_{i+1}$ and $U^j_{n+1} := U^j_n \cap U^j_{n+1}$. We will define another morphism $\Psi'$ subordinated to the same cover $W$ that will differ to $\Psi$ only in the information relevant to the open sets $\{U^j_i\}_{j \in \mathbb{Z}}$ (we want that $\Psi'_0$ could be defined on $U^j_0 \cup U^j_1$). We are going to abuse notation and we will define $\Psi'$ not point by point but by defining the images of the sets $U^j_i$'s.

Let $g_j \in G$ be such that $\Psi'_1(U^j_0, 0) \subset X \times \{g_j\}$, and $\pi_i : X \times G \to X, G$ the projections on the first and second coordinates respectively. Let’s define $\Psi'$ as follows:

\[
\begin{align*}
\Psi'_1(U^j_i, k) &:= \Psi_1(U^j_i, k) \text{ for } i \neq 1, j, k \in \mathbb{Z} \\
\Psi'_0(U^j_i) &:= \Psi_0(U^j_i) \cdot g_j^{-1} \\
\Psi'_1(U^j_{0,1}, 0) &:= (\pi_1(\Psi'_1(U^j_{0,1}, 0)), \text{id}) \\
\Psi'_1(U^j_{1,2}, 0) &:= (\pi_1(\Psi'_1(U^j_{1,2}, 0)) \cdot g_j^{-1}, g_j \pi_2(\Psi'_1(U^j_{1,2}, 0))) \\
\Psi'_1(U^j_1, 1) &:= (\pi_1(\Psi'_1(U^j_1, 1)) \cdot g_j^{-1}, g_j \pi_2(\Psi'_1(U^j_1, 1)) g_j^{-1})
\end{align*}
\]

The rest of the maps are determined by the previous ones. Now, as in $\Psi$ the image of $\Psi'_1(U^j_{0,1}, 0)$ lies in $X \times \{\text{id}\}$ we could say that $\Psi'$ is subordinated to the cover $W' = W - \{V_0, V_1\} \cup \{V_0 \cup V_1\}$.

We could do this process $n$ times and we obtain a morphism $\Psi'$ that is subordinated to the cover given by the sets $U^j := U^j_0 \cup U^j_1 \cup \cdots \cup U^j_n$ (note that $U^j = (j - \epsilon, j + 1 + \epsilon)$). As before let’s denote the set $U^j \cap U^{j+1}$ by $U^{j,j+1}$. Let $k_i, h_i \in G$ be such that $\Psi'_1(U^{j-i,i}, 0) \subset X \times k_i$ for $i > 0$ and $\Psi'_1(U^{-(i-1),-i}, 0) \subset X \times h_i$ for
i > 0; we can define \( \bar{\Psi} \) as follows:

\[
\begin{align*}
\bar{\Psi}_0(U^0) & := \Psi'_0(U^0) \\
\bar{\Psi}_0(U^i) & := \Psi'_0(U^i)k_i^{-1}k_{i-1}^{-1}\cdots k_1^{-1} \text{ for } i > 0 \\
\bar{\Psi}_0(U^i) & := \Psi'_0(U^i)h_i^{-1}h_{i-1}^{-1}\cdots h_1^{-1} \text{ for } i < 0 \\
\bar{\Psi}_1(U^{i-1}, i, 0) & := (\pi_1(\Psi'_1(U^{i-1}, i, 0)))k_i^{-1}k_{i-1}^{-1}\cdots k_1^{-1}, id) \text{ for } i > 0 \\
\bar{\Psi}_1(U^{i+1}, i, 0) & := (\pi_1(\Psi'_1(U^{i+1}, i, 0)))h_i^{-1}h_{i-1}^{-1}\cdots h_1^{-1}, id) \text{ for } i < 0 \\
\bar{\Psi}_1(U^i, 1) & := (\bar{\Psi}_0(U^i), k_i\cdots k_1\pi_2(\Psi'_1(U^i, 1)))k_i^{-1}\cdots k_1^{-1}) \text{ for } i \geq 0 \\
\bar{\Psi}_1(U^i, -1) & := (\bar{\Psi}_0(U^i), h_i\cdots h_1\pi_2(\Psi'_1(U^i, -1)))h_i^{-1}\cdots h_1^{-1}) \text{ for } i \leq 0
\end{align*}
\]

In this way \( \bar{\Psi} \) is subordinated to the trivial cover of the circle and is clear by lemma 3.2.4 that there is only one arrow \( \Lambda \) between \( \Psi \) and \( \bar{\Psi} \) (note that for \( 0 \in U^0_0, \Psi_0(0) = \bar{\Psi}_0(0) \)).

\( \square \)

4.2. For example if \( G = \overline{G} \) for \( G \) a finite groupoid (i.e. \( X = * \)) and a morphism \( \Phi_1 : x \times \mathbb{Z} \to G, \Phi_0 : x \to \ast \) the maps \( \Phi_0 \) and \( \Phi_1(n, \_ ) \) are all constant. So we only need to consider the morphisms of the groupoid \( \overline{G} \) to \( \overline{G} \), which is precisely \( \text{Hom}(\mathbb{Z}, G) \). It’s also easy to see that for any \( h \in G \) there is an arrow between \( \rho \in \text{Hom}(\mathbb{Z}, G) \) and \( \rho^h \in \text{Hom}(\mathbb{Z}, G) \) where \( \rho^h(n) = h^{-1}\rho(n)h \). Hence

**Lemma 4.2.1.** Let \( G \) be a finite group. Then the loop groupoid \( \mathbb{L}G \) of \( \overline{G} \) is Morita equivalent to the groupoid \( [\text{Hom}(\mathbb{Z}, G)]/G \) where \( G \) acts on \( \text{Hom}(\mathbb{Z}, G) \) by conjugation.

So \( \mathbb{L}G \) can also be seen as the groupoid \( G \times G \Rightarrow G \) where \( s(g, h) = g \) and \( t(g, h) = h^{-1}gh \), in other words \( G \) acts on itself by conjugation. It is clear that the groupoids

\[
G \times G \quad \overset{M}{\underset{G}{\cong}} \quad \bigsqcup_{(g)} \left( \star \times C(g) \quad \overset{\ast}{\underset{\star}{\cong}} \right)
\]

are Morita equivalent, where the right hand side runs over the conjugacy classes of elements in \( G \) and \( C(g) \) is the centralizer of \( g \).

**Remark 4.2.2.** The groupoid \( \bigsqcup_{(g)} C(g) \) is also a subgroupoid of \( \mathbb{L}G = \wedge G \).

\[
\bigsqcup_{(g)} \left( \star \times C(g) \quad \overset{\ast}{\underset{\star}{\cong}} \right) \Rightarrow \quad G \times G \quad \overset{\ast}{\underset{\star}{\cong}} \quad \bigsqcup_{(g)} C(g)
\]

Now applying the classifying space functor to the map of lemma 1.2.1 we obtain

**Lemma 4.2.3.** For \( G \) a finite group

\[
BLG \cong \bigsqcup_{(g)} BC(g).
\]

And using a result in algebraic topology \( \square \) p. 512] that says that there exists a natural map

\[
f : \bigsqcup_{(g)} BC(g) \cong \mathcal{L}BG
\]
which is a homotopy equivalence, get that

**Proposition 4.2.4.** For $G$ a finite group, there is a natural homotopy equivalence

$$BLG \simeq LBG.$$ 

4.3. Let’s consider now a gerbe over the groupoid $\mathcal{G}$; from [8, Ex. 6.1.4] we know it consists of a line bundle $L$ over $G$ and a 2-cocycle $\theta : G \times G \to U(1)$ such that

$$L_g^{-1} \cong L_{g^{-1}} \quad \text{and} \quad L_g L_h \cong L_{gh}.$$ 

Using this information we would like to define a line bundle on the inertia groupoid as follows. The arrows in the inertia groupoid of $\mathcal{G}$ relate loops via conjugation. Let $g, h \in G$ where $g$ represents a map $\mathbb{Z} \to \mathcal{G}$ and $h$ an arrow in $(L_G)$ joining $g$ and $h^{-1}gh$.

we have

$$L_g L_h \cong L_{h^{-1}gh}$$

then the map

$$\rho(g, h) = \frac{\theta(g, h)}{\theta(h, h^{-1}gh)}$$

relates the fibers $L_g$ and $L_{h^{-1}gh}$. What is remarkable about this map $\rho$ is that it gives a morphism from $\wedge G$ to $U(1)$.

**Lemma 4.3.1.** Let $\wedge G = G \times G \rightrightarrows G$ be the inertia groupoid of $\mathcal{G}$, then

$$\rho : \wedge G \to U(1)$$

is a morphism of groupoids. Moreover, it produces a line bundle on $\wedge G$ which matches the inner local system of $\mathcal{G}$.

**Proof.** See lemma [5.4.1].

Using the subgroupoid representation $\cup_{(g)} C(g) \to \wedge G$ we can consider the map $\rho$ restricted to the centralizers $C(g)$, we get that

**Proposition 4.3.2.** The map

$$\rho(g, \cdot) : C(g) \to U(1)$$

is a representation that defines a line bundle on the groupoid $C(g)$. This description matches precisely the inner local system defined by Ruan [14] that comes from a discrete torsion $\theta \in H^2(G, U(1))$.

So we can give now a natural definition of some concepts introduced in [14],

**Proposition 4.3.3.** The inner local system associated to a discrete torsion $\theta \in H^2(G, U(1))$ arise from the relation on the fibers of the gerbe associated to $\theta$ over constant loops in $L_G$, given by the arrows of $(L_G)^\mathbb{R}$ as in [1,3.4].
4.4. Let us now briefly consider now the case in which $G$ is a Lie group. The groupoid $H = (G \times G \rightrightarrows G)$ (with $G$ acting on $G$ by conjugation.) is also naturally inside $L_G$.

Consider the trivial $G$-principal bundle $P = G \times S^1 \to S^1$ over the circle. Let $\mathcal{A}$ be the space of connections on $P$, and let $\mathcal{G}$ be the gauge group of automorphisms of $P$.

Let $G$ be the groupoid $\mathcal{A} \times \mathcal{G} \rightrightarrows \mathcal{A}$, where $s(A, g) = A$ and $t(A, g) = g^* A$.

Notice that in this case the gauge group $G$ is equal to the loop group $L_G$ of $G$, namely the group whose elements are ordinary smooth maps $S^1 \to G$ and whose operation is pointwise multiplication. From this we see that since the space $\mathcal{A}$ of connections is contractible then $BH$ is weakly homotopy equivalent to $B L G$. In any case we have the following.

**Proposition 4.4.1.** The groupoid $G = (\mathcal{A} \times L_G \rightrightarrows \mathcal{A})$ is Morita equivalent to the groupoid $H = (G \times G \rightrightarrows G)$.

The Morita equivalence is obtained by taking the holonomy of the connection. In view of this we will call $G$ the inertia groupoid of the compact Lie group $G$. The groupoids $G, H$ and their Morita equivalence are relevant to the relation of the twisted equivariant $K$-theory of $G$ (with respect to the conjugate action) and the Verlinde algebra.

5. Smooth Manifolds

5.1. Now we want to see that this description when carried out on the groupoid associated to a manifold $M$ matches the classical one of the loop space $LM$. Let’s denote by $\mathcal{M}$ and $\mathcal{S}^1$ the groupoids associated to the trivial covers of $M$ and $S^1$ respectively (i.e. $\mathcal{M}_0 = \mathcal{M}_1 = M$ and $\mathcal{S}^1_0 = \mathcal{S}^1_1 = S^1$) and by $\mathcal{M}$ the groupoid associated to some fixed cover of $M$, i.e. if $\{U_\alpha\}_\alpha$ is a cover of $M$ then

$$\mathcal{M}_1 := \bigsqcup_{(\alpha, \beta)} U_{\alpha \beta} \quad \text{and} \quad \mathcal{M}_0 := \bigsqcup_{\alpha} U_\alpha$$

with the natural source an target maps.

The following lemma will unveil the internal structure of $LM$.

**Lemma 5.1.1.** For $\Psi \in (LM)_0$ there exist a unique map $\psi : S^1 \to M$ such that the following diagram commutes

$$\begin{array}{ccc}
S^1 & \xrightarrow{P} & S^1 \\
\downarrow \Psi & & \downarrow \psi \\
\mathcal{M} & \xrightarrow{Q} & \mathcal{M}
\end{array}$$

where $P$ and $Q$ are the natural Morita morphisms induced by the covers of $S^1$ and $M$ respectively.

**Proof.** There is no much choice in how to define $\psi$, we just need to check that is well defined. Take a point $x$ in $S^1$, then every pair of points in $(P_0)^{-1}\{x\}$ are joined by exactly one arrow of $(S^1)_1$, and all those arrows form precisely the set $(P_1)^{-1}\{x\}$ (because $P$ is a Morita morphism). Then, under the map $Q_1 \circ \Psi_1$, the set $(P_1)^{-1}\{x\}$ has to go only one element $m$ in $\mathcal{M}_1$, this because the groupoid $\mathcal{M}$ has only the identity arrows and no relations among them. Hence $\psi(x) = m$ is well
defined. The continuity follows from the fact that \( \Psi \) gives us the local continuity of \( \psi \).

And so, we can get that

**Corollary 5.1.2.** The induced map \( LQ : LM \to LM \) sends \( \Psi \) to \( \psi \), and for any other \( \Psi' \) with \( (LQ)_0(\Psi') = \psi \) then there exist \( \Lambda \) in \( (LM)_1 \) joining \( \Psi \) with \( \Psi' \).

**Proof.** We know from lemma 3.4.2 that \( LQ \) is a Morita morphism; then as \( (LQ)_0(\Psi') = (LQ)_0(\Psi) = \psi \) and the fact that over \( \psi \) there is the identity arrow, there must exist an arrow \( \Lambda \) in \( (LM)_1 \) joining \( \Psi \) with \( \Psi' \).

Let us abuse the notation and denote by \( LM \) the groupoid \( LM \Rightarrow LM \) with object space \( LM \) and only identity arrows.

**Proposition 5.1.3.** The loop groupoid \( LM \) is Morita equivalent to \( LM \).

**Proof.** By lemma 3.4.2 the map \( LQ : LM \to LM \) is Morita. And by the previous lemmas is clear that for any two admissible covers \( W, W' \in W \) the groupoids \( LM(W) \) and \( LM(W') \) are equivalent. Hence for \( LM \) it suffices to take the trivial cover of \( \mathbb{R} \) and then is clear that \( LM \) is equivalent to \( LM \).

We got, as expected, that the loop groupoid \( LM \) carries the same information as the loop space \( LM \); this means that we only need to consider the loops on \( M \), in this case the additional structure introduced by the groupoid representation of the manifold doesn’t give anything new.

### 5.2.

This construction is compatible with the following fact. A morphism of groupoids \( \Psi : S^1 \to M \) induces a continuous map on the classifying spaces

\[
BS^1 \to BM
\]

where \( BS^1 \cong S^1 \) and \( BM \cong M \) by a theorem of Segal [15], thus producing a loop \( \psi : S^1 \to M \).

### 5.3.

Let’s now briefly discuss the holonomy of a gerbe with connection as is done in the work of Hitchin [6]. This will permit us define a line bundle with connection over the loop space of \( M \) and these ideas will be generalized for groupoids.

For simplicity in what follows we are going to use subindices instead of the groupoid notation. Let’s recall what a gerbe with connection means [6, p.7]: For a gerbe defined by a cocycle \( g_{\alpha \beta \gamma} : U_{\alpha \beta \gamma} \to U(1) \), a connection on it consist of a global 3-form \( G \in \Omega^3(M) \), 2-forms \( F_\alpha \in \Omega^2(U_\alpha) \) and 1-forms \( A_{\alpha \beta} \in \Omega^1(U_{\alpha \beta}) \) such that

\[
G|_{U_\alpha} = dF_\alpha \\
F_\beta - F_\alpha = dA_{\alpha \beta} \\
iA_{\alpha \beta} + iA_{\beta \gamma} + iA_{\gamma \alpha} = g_{\alpha \beta \gamma}^{-1}dg_{\alpha \beta \gamma}
\]

Adopting the definition of a gerbe as line bundles \( L_{\alpha \beta} \) on each \( U_{\alpha \beta} \) which in the triple intersection \( L_{\alpha \beta} \cap L_{\beta \gamma} \cap L_{\gamma \alpha} \) get trivialized by a cocycle \( \theta_{\alpha \beta \gamma} \), we have that a connection in this formalism is:

- a connection \( \nabla_{\alpha \beta} \) on \( L_{\alpha \beta} \) such that
- \( \nabla_{\alpha \beta} \theta_{\alpha \beta \gamma} = 0 \)
- a 2-form \( F_\alpha \in \Omega^2(U_\alpha) \) such that on \( U_{\alpha \beta} \), \( F_\beta - F_\alpha = F_{\alpha \beta} \) is the curvature of \( \nabla_{\alpha \beta} \)
We call the closed 3-form $G$ the \textit{curvature} of the gerbe with connection, and we say that the connection gerbe is \textit{flat} if the curvature vanishes. If this is the case, $dF_\alpha = 0$; and using a Leray cover of $M$ (every intersection of open sets is contractible) we can define 1-forms $B_\alpha \in \Omega^1(U_\alpha)$ and 0-forms $f_{\alpha \beta} \in \Omega^0(U_{\alpha \beta})$ such that:

\[
F_\beta - F_\alpha = dA_{\alpha \beta} = d(B_\beta - B_\alpha)
\]

\[
A_{\alpha \beta} - B_\beta + B_\alpha = df_{\alpha \beta}
\]

\[
d(if_{\alpha \beta} + if_{\beta \gamma} + if_{\gamma \alpha} - \log g_{\alpha \beta \gamma}) = 0
\]

Then, as $\log g_{\alpha \beta \gamma}$ is only defined modulo $2\pi i \mathbb{Z}$ we get a collection of constants $c_{\alpha \beta \gamma}/2\pi \in \mathbb{R}/\mathbb{Z}$ that represents a Čech class in $H^2(M, \mathbb{R}/\mathbb{Z})$. This 2-cocycle is called the \textit{holonomy} of the connection.

If the holonomy is trivial, then there are constants $k_{\alpha \beta} \in 2\pi i \mathbb{Z}$ such that $c_{\alpha \beta \gamma} = k_{\alpha \beta} + k_{\beta \gamma} + k_{\gamma \alpha}$ and defining $h_{\alpha \beta} := \exp(if_{\alpha \beta} - k_{\alpha \beta})$ we obtain a \textit{flat trivialization} of the gerbe because $h_{\alpha \beta} h_{\beta \gamma} h_{\gamma \alpha} = g_{\alpha \beta \gamma}$. If we have a second trivialization $h'_{\alpha \beta}$, their difference $l_{\alpha \beta} := h'_{\alpha \beta}/h_{\alpha \beta}$ defines a line bundle $L$. Using that

\[
iB_\beta - iB_\alpha - iA_{\alpha \beta} = d \log h_{\alpha \beta}
\]

\[
iB'_\beta - iB'_\alpha - iA_{\alpha \beta} = d \log h'_{\alpha \beta}
\]

we deduce that

\[
i(B' - B)_{\beta} - i(B' - B)_{\alpha} = d \log l_{\alpha \beta}
\]

which defines a connection on $L$. As $d(B'_\alpha - B_\alpha) = 0$ the curvature of this connection is zero; hence, the difference of two flat trivializations of a gerbe is a flat line bundle.

Applying these ideas to a loop $f : S^1 \to M$ we get that, as $S^1$ is 1-dimensional, the pull back of the gerbe with connection to the circle is flat and has trivial holonomy. If we identify flat trivializations that differ by a flat line bundle with trivial holonomy, and we call it the moduli space of flat trivializations, then to each loop, this moduli space is being acted freely and transitively by the moduli space of flat line bundles $H^1(S^1, \mathbb{R}/\mathbb{Z}) \cong S^1$. Therefore we obtain a principal $S^1$ bundle over $LM$. Now, if we take a path in the loop space

\[
F : [0, 1] \times S^1 \to M
\]

the pull back of the gerbe connection is also flat and has trivial holonomy, but it will give a canonical isomorphism between the moduli space of flat trivializations of the gerbe on $\{0\} \times S^1$ and $\{1\} \times S^1$. In this way we have defined the parallel transport in the principal bundle, and hence a connection on the line bundle over the loop space.

Thus from the curvature 3-form $G$ we obtained a 2-form on the loop space, the curvature of the previous line bundle; this process is known as \textit{transgression} and is what we are going to generalize to orbifolds.

6. Orbifolds

6.1. We are going to analyze first the loop groupoid of a global quotient. For $X = [X/G]$ the groupoid given by the orbifold $X/G$ with $G$ a finite group and $X$ a connected $G$-space, we know from lemma 4.1.1 that we only need to consider morphisms $\Psi : S^1 \to X$ associated to the trivial cover of $\mathbb{R}$. Now, from section 3.3 we have that the morphisms are determined by the image of $\mathbb{R} \times \{1\}$ under $\Psi$, because it lies on one connected component of $X_1$ (i.e. $\Psi_1(\mathbb{R} \times \{1\}) \subset X \times \{g\}$ for some $g \in G$). We know that $\Psi_0(x) = s(\Psi_1(x, 1))$ and $\Psi_0(x + 1) = t(\Psi_1(x, 1))$,
therefore if we consider the pairs \((\psi, g)\) where \(\psi := \Psi^0\) and \(\Psi^1(\mathbb{R} \times \{1\}) \subset X \times \{g\}\) then

\[
\begin{cases}
\text{Morphisms } S^1 \to X \\
\text{associated to the trivial cover.}
\end{cases}
\xleftarrow{\cong}
\begin{cases}
\text{Pairs } (\psi, g) \text{ with } \\
\psi: \mathbb{R} \to X \text{ and } \\
\psi(x) \cdot g = \psi(x + 1)
\end{cases}
\]

An arrow \(\Omega \in (LX)_1\) between \(\Psi\) and \(\Phi\) is a map \(\Omega: \mathbb{R} \times \mathbb{Z} \to X_1\) where the relevant information lies on \(\mathbb{R} \times \{0\}\). As \(X\) is connected \(\Omega(\mathbb{R} \times \{0\}) \subset X \times \{h\}\) for some \(h \in G\), and is easy to check that if \((\psi, g)\) and \((\phi, k)\) are the pairs associated to \(\Psi\) and \(\Phi\) respectively, then

\[
k = h^{-1}gh \quad \text{and} \quad \psi(x) \cdot h = \phi(x).
\]

If we call \(P_g\) the set of all pairs \((\psi, g)\)

\[
P_g = \{(\psi, g)|\psi: \mathbb{R} \to X \& \psi(x) \cdot g = \psi(x + 1)\}
\]

then we can endow the set \(\bigsqcup_g P_g\) with a natural \(G\) action:

\[
\begin{align*}
\bigsqcup_g P_g \times G & \to \bigsqcup_g P_g \\
((\psi, g), h) & \mapsto (\psi \cdot h, h^{-1}gh)
\end{align*}
\]

so we can conclude

**Proposition 6.1.1.** If \(X = [X/G]\) is the quotient of the connected space \(X\) by the action of a finite group \(G\), then the loop groupoid \(LX\) is Morita equivalent to the groupoid

\[
L[X/G] := \left( \bigsqcup_g P_g \right) \times G
\]

given by the action explained before.

**Proof.** The functor \(F : L[X/G] \to L[X/G]\) is given by lemma 4.1.1. The Morita equivalence follows from lemma 3.2.4 and the fact that \(L[X/G]\) is also a subgroupoid of \(L[X/G]\) (with maps associated to the trivial cover as in 4.1.1). \(\square\)

By the same argument as in the case of the finite group \(G\) acting on itself by conjugation, we obtain that

**Corollary 6.1.2.** If \(X = [X/G]\) is the quotient of the connected space \(X\) by the action of a finite group \(G\), then the loop groupoid \(LX\) is Morita equivalent to the groupoid

\[
\bigsqcup_{(g)} \left( \left( P_g \times C(g) \right) \right)
\]

where \((g)\) runs over the conjugacy classes of elements in \(G\).
6.2. Now if we consider the groupoid \((LX)^{\mathbb{R}}\), the elements \((\psi, g)\) in \(P_g\) that are fixed under the action of \(\mathbb{R}\) are the constant functions; so if \(\psi(x) = x_0\) with \(x_0\) a fixed point in \(X\), as

\[
x_0 = \psi(x + 1) = \psi(x) \cdot g = x_0 \cdot g
\]

we have that \(x_0\) belongs to the set \(X^g\) of points in \(X\) fixed by the action of \(g\). Hence \((P_g)^{\mathbb{R}} \cong X^g\), and using the previous corollary

**Proposition 6.2.1.** The groupoid \((LX)^{\mathbb{R}}\) (or inertia groupoid \(\Lambda X\)) is Morita equivalent to the groupoid

\[
\bigsqcup (X^g \times \{g\} \times C(g)) \\
\downarrow
\]

Therefore the groupoid \((LX)^{\mathbb{R}}\) is Morita equivalent to the twisted sectors of the orbifold \(X/G\).

**Remark 6.2.2.** The inertia groupoid \(\Lambda X\) has the following representations:

\[
\bigsqcup (X^g \times \{g\} \times C(g)) \\
\downarrow
\]

\[
\bigsqcup (X^g \times \{g\}) \times G
\]

\[
\downarrow
\]

\[
\bigsqcup (X^g \times \{g\})
\]

where the first groupoid is Morita equivalent and a subgroupoid of the second one.

6.3. From [8, Ex. 7.2.7] we know that a gerbe \(L\) over the groupoid \(\overline{G}\) (with cocycle \(\theta : G \times G \to U(1)\)), as in section 4.3, gives rise to a gerbe \(L_X\) over \(X\) in the natural way, i.e.

\[
L_X = X \times L
\]

\[
X_1 = X \times G
\]

with cocycle

\[
\theta^X : X_2 = X \times G \times G \to U(1)
\]

\[
(x, g, h) \mapsto \theta(g, h)
\]

Defining \(L^X_g := L^X |_{X \times \{g\}}\) is easy to see that these line bundles are trivial, and using the notation of section 4.3 we have that \(L^X_g \cong X \times L_g\). Then is straightforward to generalize the constructions and results of the section just mentioned to the global quotient. If \(\Lambda X\) is the inertia groupoid of \(X\) and \(\rho\) is the map defined in lemma 4.3.1, then

**Proposition 6.3.1.** The map

\[
\rho^X : \bigsqcup (X^g \times \{g\} \times C(g)) \to U(1)
\]

defined in \(X^g \times \{g\} \times C(g)\) by

\[
\rho^X(x, g, h) := \rho(g, h)
\]

is a morphism of groupoids \(\rho^X : \Lambda X \to U(1)\). Therefore it determines a \(C(g)\) equivariant line bundle over \(X^g\) for every \(g\).
Remark 6.3.2. These line bundles precisely determine what Ruan \cite{ruan} called an inner local system of a discrete torsion $\theta \in H^2(G, U(1))$ over a global quotient $X/G$.

6.4. We will now generalize the previous arguments to a general \'{e}tale groupoid. For $G$ an \'{e}tale groupoid, a gerbe over $G$ is determined by a 2-cocycle $\theta \in H^2(G, U(1))$ which means that for $(a, b, c) \in G_3 = G_1 \times_s G_1 \times_s G_1$

$$\theta(a, b)\theta(ab, c) = \theta(a, bc)\theta(b, c);$$

this will give rise to a line bundle on the inertia groupoid.

An object $a$ in the inertia groupoid $(a \in \wedge G_0)$ is an arrow of $G_1$ such that its source and target are equal, so

$$\wedge G_0 = \{a \in G_1 | s(a) = t(a)\}$$

and a morphism $v \in \wedge G_1$ joining two objects $a, b$

is an arrow in $G_1$ such that $a \cdot v = v \cdot b$; in some way $b$ can be seen as the conjugate of $a$ by $v$ because $b = v^{-1} \cdot a \cdot v$. But in order to identify the arrow $v$ we need to keep track of its source $a$, then we can consider the morphisms of $\wedge G$ as

$$\wedge G_1 = \{(a, v) \in G_2 | a \in \wedge G_0\}$$

where $s(a, v) = a$ and $t(a, v) = v^{-1} \cdot a \cdot v$.

Let $\rho : \wedge G_1 \to U(1)$ be the map defined as

$$(a, v) \rho (\frac{\theta(a, v)}{\theta(v, v^{-1}av)})$$

then this maps determines the mentioned line bundle.

**Lemma 6.4.1.** The induced map

$\rho : \wedge G \to U(1)$

is a morphism of groupoids.

**Proof.** Let $a, b, c \in \wedge G_0$, and $v, w \in \wedge G_1$

such that $av = vb$ and $bw = wc$, we just need to prove that $\rho(a, v)\rho(b, w) = \rho(a, vw)$. Using the cocycle condition and noting that $b = v^{-1}av$ and $c = w^{-1}bw$, we get the following set of equalities:

$$\rho(a, vw) = \frac{\theta(a, vw)}{\theta(vw, c)} = \frac{\theta(a, vw) \theta(vb, w) \theta(vc, c)}{\theta(av, w) \theta(vb, w) \theta(vc, c)}$$

$$= \frac{\theta(a, v) \theta(b, w) \theta(v, w)}{\theta(a, v) \theta(b, w) \theta(v, w)}$$

$$= \frac{\theta(a, v) \theta(b, w)}{\theta(v, b) \theta(w, c)} = \rho(a, v)\rho(b, w)$$
The continuity is clear.

So we get an interesting result:

**Theorem 6.4.2.** Let $G$ be an étale groupoid and $\wedge G$ its inertia groupoid. A gerbe over $G$ determines a morphism of groupoids $\rho : \wedge G \to U(1)$; hence a line bundle over the inertia groupoid $\wedge G$.

When the groupoid $G$ is an orbifold its is known that the twisted sectors $\tilde{\Sigma}_1 G$ is a subgroupoid of $\wedge G$ which is at the same time Morita equivalent to it (see Appendix).

Then the morphism $\rho$ induces a line orbibundle over $\tilde{\Sigma}_1 G$ in the natural way.

**Corollary 6.4.3.** A gerbe over an orbifold $G$ (in the sense of [8]) determines an inner local system over the twisted sectors $\tilde{\Sigma}_1 G$.

**Proof.** Calling $\eta : \wedge G_0 \to G_1$ the inclusion, and $\mathcal{L}$ the line bundle over $G_1$ determined by the gerbe, the line bundle on $\wedge G$ is constructed from $\eta^* \mathcal{L}$ and the attaching maps given by $\rho$. The properties of the gerbe [8]:

- $i^* \mathcal{L} \cong \mathcal{L}^{-1}$
- $\pi_1^* \mathcal{L} \cdot \pi_2^* \mathcal{L} \cdot (\text{im})^* \mathcal{L} \cong 1$

clearly imply the conditions in [14, Def. 3.1.6] for an inner local system. □

**7. Appendix**

**7.1.** In this section we will summarize some facts regarding orbifolds and its twisted sectors, and the relation with the groupoid approach. For a more detailed account on orbifolds we recommend [8, 14]; the notation and results that will be used in what follows can be seen in [8, Sect. 5.1].

**7.2.** Let $X$ be an orbifold and $\{(V_p, G_p, \pi_p)\}_{p \in X}$ its orbifold structure, the groupoid $X$ associated to it (cf. [11]) consist of the following information:

- Objects ($X_0$): $\bigsqcup_{p \in X} V_p$.
- Morphisms ($X_1$): an arrow $r : (x_1, V_1) \to (x_2, V_2)$ is an equivalence class of triples $r = [\lambda_1, w, \lambda_2]$ where $w \in W$ for another orbifold chart $(W, H, \rho)$, and the $\lambda_i$’s are embeddings $\lambda_i : W \to V_i$ which are $\phi_i$ equivariant, with $\phi_i : H \to G_i$ monomorphisms;

$$\begin{align*}
(V_1, G_1) & \xrightarrow{(\lambda_1, \phi_1)} (W, H) & (\lambda_2, \phi_2) \\
(V_2, G_2) &
\end{align*}$$

**7.3.** If $r \in \wedge X_0$ is an element in the inertia groupoid, then its source and target must be equal. In particular we would have that $x_1 = x_2 (= x)$ and that $V_1 = V_2 (= V)$. Since the only automorphisms of the orbifold chart $(V, G, \pi)$ come from conjugation, then there exist $g \in G$ such that $\lambda_2 = \lambda_1 \cdot g$ and $\phi_2 = g^{-1} \cdot \phi_1 \cdot g$. Now it’s easy to see that

$$[\lambda_1, w, \lambda_2] = [\text{id}_V, x, \lambda_g]$$

where $\lambda_g : V \to V$, $\lambda_g(y) = y \cdot g$ and $\phi_g : G \to G$ is the map obtained conjugating by $g$. So $x \in V^g$ is a fixed point under the action of $g$ in $V$. In this way we can see
that the objects of the inertia groupoid can be seen as the fixed point sets of the actions of the elements of the group $G_p$ in $V_p$

$$X_0 \cong \bigsqcup_{p \in X} \left( \bigsqcup_{g \in G_p} (V_p)^g \times \{g\} \right)$$

Again, using that the only automorphisms of charts are given by conjugation, we can see that if we restrict our attention to the morphisms that are defined on $Z_p := \bigsqcup_{g \in G_p} (V_p)^g \times \{g\}$ we get the same picture as in proposition 6.2.1. Hence, the inertia groupoid $\wedge X$ restricted $Z_p$ is Morita equivalent to

$$\wedge X|_{\tilde{\Sigma}_1 X} \cong \bigsqcup_{(g) \in G_p} \left( (V_p)^g \times \{g\} \times C_{G_p}(g) \right)$$

and this is the description given in [14] of the twisted sectors $\tilde{\Sigma}_1 X$. Therefore we can conclude that

**Proposition 7.3.1.** Let $X$ be an orbifold whose associated groupoid is $X$. Then the inertia groupoid $\wedge X$ is Morita equivalent to the twisted sectors $\tilde{\Sigma}_1 X$. Moreover, $\tilde{\Sigma}_1 X$ is also a subgroupoid of $\wedge X$:

$$\begin{array}{ccc}
\tilde{\Sigma}_1 X & \xrightarrow{M} & \wedge X \\
\downarrow & & \downarrow \\
\Sigma_1 X & \xrightarrow{M} & \tilde{\Sigma}_1 X
\end{array}$$

**Proof.** It follows directly from the previous discussion and the case of a global quotient (remark 6.2.2). $\square$

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