MOTIVIC CELL STRUCTURES FOR SPHERICAL VARIETIES

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Abstract. In this note, we give a general method to obtain unstable motivic cell structures, following Wendt’s application [Wen10] of the Białynicki-Birula algebraic Morse theory. We then apply the method to spherical varieties, with special attention to the case of rank 1, to obtain unstable motivic cell structures after a finite number of $\mathbb{P}^1$-suspensions. This refines the toolkits of Dugger–Isaksen and Wendt.

We give an explicit proof (on page 10) of

**Theorem 1.** Let $k$ be a field and $X$ a spherical $k$-variety, then $X$ is motivic stably cellular in the sense of Dugger–Isaksen.

**Corollary.** The Voevodsky motive of a spherical variety is mixed Tate.

Dugger and Isaksen chose to work mostly with stable cellularity because products of unstably cellular spaces are not necessarily unstably cellular (see remark 22). We refine this investigation by taking only a finite amount of suspensions.

**Definition.** Let $X$ be a motivic space. If the simplicial suspension $\Sigma^k X$ is motivic unstably cellular, we say $X$ is $k$-suspended cellular.

To gain more control on cell structures, we prove (on page 10)

**Theorem 2.** Let $X$ be a smooth $S$-variety and let $X \hookrightarrow \bar{X}$ be a closed immersion into a smooth $S$-variety $\bar{X}$ with complement $D = \bigcup_{i=1}^n D_i$ a divisor with $n$ irreducible smooth components $D_i$. If $\bar{X}$ is $n$-suspended cellular and each $D_i$ is atacc (see definition 20), then $X$ is $(n+k)$-suspended cellular.

**Corollary (Theorem 35).** Let $X$ be a homogeneous space under a split reductive group $G$ with a $G$-equivariant completion $X \hookrightarrow \bar{X}$ such that the complement $D$ is irreducible (a two-orbit completion). Then $X$ is 1-suspended cellular.

**Conjecture.** A two-orbit completable homogeneous space is unstably cellular.

This is known for affine split quadrics $AQ_{2n} = SO_{n,n+1}/SO_n \times SO_{n+1}$ by [ADF17, Theorem 2.2.5], $AQ_{2n-1} = SO_{n,n}/SO_n \times SO_n$ by a well-known elementary argument (lemma 11) and quaternionic projective space $HP^n = Sp_{2n+2}/Sp_2 \times Sp_{2n}$ by [Voe16, Theorem 4.4.8] (with the idea coming from Panin–Walter [PW10]). In future work, we intend to discuss the other two-orbit completions, like the Cayley plane $OP^2$, in more detail.

On page 8, we prove

**Theorem 3.** Let $\mathcal{A} = \{L_i\}_{i=0}^r$ an arrangement of $r + 1$ linear subspaces $L_i \hookrightarrow \mathbb{A}^n$ of a fixed affine space $\mathbb{A}^n$. Then the complement $X := \mathbb{A}^n \setminus \bigcup \mathcal{A}$ is $r$-suspended cellular.

**Corollary.** A rank $r$ split torus $T = G_m^r$ is $(r-1)$-suspended cellular.
Convention. We use the notation $Z \hookrightarrow X$ for a closed immersion of $Z$ into $X$ and $U \hookrightarrow X$ for an open immersion of $U$ into $X$ throughout.

Two results not due to the author were not easily available in the literature, which is why we provide a proof here: the folklore lemma 6 that affine Zariski bundles are $\mathbb{A}^1$-equivalences and Wendt’s lemma 7 that vector bundle projections are sharp maps.

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1. Motivic Spaces

We model motivic spaces $\text{Spc}(S)$ with simplicial presheaves on smooth finite type schemes over a Noetherian base scheme $S$, with the $\mathbb{A}^1$-local Nisnevich-local injective model structure. We always consider pointed motivic spaces. This model category is well explained by Dugger [Dug01] and Dugger–Hollander–Isaksen [DHI04]. The idea to approach a homotopy theory of algebraic varieties in this way was introduced by Morel and Voevodsky [MV99], building on work of Jardine.

Definition 4. A space $X \in \text{Spc}(S)$ such that the basepoint $S \to X$ is an $\mathbb{A}^1$-equivalence is called $\mathbb{A}^1$-contractible.

Warning. A space $X$ which is $\mathbb{A}^1$-contractible need not be an affine space itself. An ample supply of quasi-affine non-affine varieties which are $\mathbb{A}^1$-contractible is given by Asok and Doran [AD07], [AD08]. Duboulouz and Fasel gave examples of smooth affine threefolds over fields of characteristic 0 which happen to be $\mathbb{A}^1$-contractible but are not isomorphic to affine spaces [DF18].

1.1. Bundles on Schemes. One source of $\mathbb{A}^1$-equivalences are affine bundle projections:

Definition 5. For $B$ a motivic space, a Zariski locally trivial fiber bundle $p : E \to B$ with fibers $p^{-1}(b)$ isomorphic to $\mathbb{A}^n$ is called an affine bundle on $B$.

Lemma 6. Let $p : E \to B$ be an affine bundle of rank $n$, with $B$ smooth. Then $p$ is an $\mathbb{A}^1$-weak equivalence.

Proof. By definition of a bundle, $B$ admits a Zariski cover $\mathcal{U} = \{U_i \to B\}_{i \in I}$ and there are isomorphisms $\varphi_i$ over $U_i$ from $U_i \times \mathbb{A}^n$ to $p^{-1}U_i$. Choose (for convenience of stating the proof) a well-order on $I$. For every word $\alpha = (i_1, \ldots, i_k)$ of length $k$ over $I$ we let $\varphi_\alpha$ be the restriction of $\varphi_i$ to $U_\alpha := U_{i_1} \times_B \cdots \times_B U_{i_k}$ for $i = \min(\alpha)$. As the Čech nerve of a cover is defined as $\check{C}^k(\mathcal{U}) = \coprod_{|\alpha|=k} U_\alpha$, we get an isomorphism between $p^*\check{C}^k(\mathcal{U})$ and $\check{C}^k(\mathcal{U}) \times \mathbb{A}^n$ over $\check{C}^k(\mathcal{U})$, by disjoint union of the $\varphi_\alpha$. 
By definition of the $\mathbb{A}^1$-weak equivalences, the projections $\check{C}^k(U) \times \mathbb{A}^n \to \check{C}^k(U)$ are $\mathbb{A}^1$-weak equivalences, and so is the morphism $p^*\check{C}^k(U) \to \check{C}^k(U)$. As any degree-wise weak equivalence of simplicial objects is a weak equivalence, $p^*\check{C}^*(U) \to \check{C}^*(U)$ is an $\mathbb{A}^1$-weak equivalence. By definition, $p^*\check{C}^*(U) = \check{C}^*(p^*U)$, where $p^*U := \{p^*U_i \to E\}_{i \in I}$ is the induced Zariski cover of $E$. As the homotopy colimit of a Čech nerve is the space covered [DHI04, Theorem 1.2], we get a commutative diagram in which we know that all morphisms except possibly $p$ are $\mathbb{A}^1$-weak equivalences:

\[
\begin{array}{ccc}
\text{hocolim}(p^*\check{C}^*(U)) & \longrightarrow & \text{hocolim}(\check{C}^*(U)) \\
\downarrow & & \downarrow \\
E & \longrightarrow & B
\end{array}
\]

From the diagram we see that $p$ is also an $\mathbb{A}^1$-weak equivalence. \qed

**Lemma 7** (Wendt). Let $S$ be either a field or a Dedekind ring with perfect residue fields and $R$ a smooth $S$-algebra. For $E$ and $B$ two $R$-varieties and $p: E \to B$ a rank $n$ vector bundle projection, the underived pullback $p^*$ preserves homotopy colimits (for $\mathbb{A}^1$-local weak equivalences). For a diagram $D \in \text{Spc}(R)/B$ we can compute

\[\text{hocolim} p^*D \simeq p^* \text{hocolim} D.\]

**Proof.** Denote $E \to B$ the associated frame bundle of $E \to B$, which is a $\text{GL}_n$-principal bundle. Under the assumptions on $R$, the classifying space $B\text{Sing}_{\mathbb{A}^1} \text{GL}_n$ is $\mathbb{A}^1$-local due to [AHW17, Theorem 5.1.3 and the proof of Theorem 5.2.3]. As in [Wen11, Proof of Theorem 4.6] for the case $G = \text{GL}_n$, there is an $\mathbb{A}^1$-local fiber sequence $\text{GL}_n \to E \to B$, so in particular a simplicial fiber sequence. By Rezk’s theorem [Rez98, Theorem 4.1., (1) $\leftrightarrow$ (3)] a map $E \to B$ that induces a simplicial fiber sequence is sharp. Since the $\mathbb{A}^1$-local injective model category is right proper, pullback along a sharp map preserves homotopy colimits [Rez98, Proposition 2.7]. Zariski-locally, we can recover the (homotopy) pullback along $E \to B$ from the pullback along $E \to B$, hence globally the homotopy pullback is given by the underived pullback along $E \to B$. To see the last claim, compute the homotopy pullbacks of $X \to B$ by fibrant replacement of $X \to B$. \qed

**1.2. Motivic Spheres.**

**Definition 8.** For $p, q \in \mathbb{Z}$ with $p \geq q$, the motivic space

\[\mathcal{G}^{p,q} := (\mathbb{G}_{m,S})^{\wedge q} \wedge \mathcal{G}^{p-q} \in \text{Spc}(S)\]

is called a *motivic sphere*. Here $\mathcal{G}^{p-q} := (\mathcal{G}^1)^{\wedge p-q} \in \text{sSet}$ is a simplicial sphere, where $\mathcal{G}^1 := \Delta^1/\partial\Delta^1$ (pointed by $\partial\Delta^1$) and $\mathbb{G}_{m,S} := \mathbb{G}_m \times_{\mathbb{Z}} S$ is the multiplicative group scheme over $S$ (pointed by the unit), where $\mathbb{G}_m(R) = R^\times$ for any ring $R$.

**Example 9.** There is an $\mathbb{A}^1$-homotopy equivalence

\[(\mathbb{P}^1, 0) \xrightarrow{\simeq} \mathcal{G}^{2,1},\]

that is, an isomorphism in the homotopy category of $\text{Spc}(S)$ [MV99, Example 3.2.18]. One can see this directly by writing $\mathbb{P}^1 = X \cup Y$ with $X = \mathbb{P}^1 \setminus \{0\}$.
and $Y = \mathbb{P}^1 \setminus \{\infty\} \xrightarrow{\sim} X$, so that $X \times_{\mathbb{P}^1} Y = \mathbb{A}^{1} \setminus \{0\} \simeq S^{1,1}$ and $X$ and $Y$ are both $\mathbb{A}^{1}$-contractible.

**Example 10** ([MV99, Example 3.2.20]). Affine space without origin is a motivic sphere:

$$\mathbb{A}^{n} \setminus \{0\} \simeq S^{2n-1,n}$$

For odd-dimensional split quadrics, there is a well-known elementary argument to see that they are motivic spheres:

**Lemma 11.** Let $AQ_{2n-1} := \{(x, y) \in \mathbb{A}^{n} \times \mathbb{A}^{n} \mid \sum_{i=1}^{n} x_{i}y_{i} = 1\}$ (considered as affine algebraic variety over $\mathbb{Z}$), then $\pi: AQ_{2n-1} \twoheadrightarrow \mathbb{A}^{n} \setminus \{0\}$ given by $(x, y) \mapsto y$ is a rank $n-1$ affine bundle and over any base scheme $S$ there is an isomorphism

$$AQ_{2n-1} \xrightarrow{\sim} S^{2n-1,n}.$$  

**Proof.** We can cover $\mathbb{A}^{n} \setminus \{0\}$ by the varieties $U_{i} := \{y_{i} \neq 0\}$, over which $\pi^{-1}(U_{i}) = \{(x, y) \in \mathbb{A}^{n} \times \mathbb{A}^{n} \mid y_{i} \neq 0, \sum_{j=1}^{n} x_{j}y_{j} = 1\}$ can be rewritten as

$$\pi^{-1}(U_{i}) = \left\{ (x, y) \in \mathbb{A}^{n} \times \mathbb{A}^{n} \mid y_{i} \neq 0, x_{i} = y_{i}^{-1}\left(1 - \sum_{j=1, j \neq i}^{n} x_{j}y_{j}\right) \right\}$$

so there are isomorphisms

$$\pi^{-1}(U_{i}) \xrightarrow{\sim} \mathbb{A}^{n-1} \times U_{i}, \quad (x, y) \mapsto ((x_{1}, \ldots, \tilde{x}_{i}, \ldots, x_{n}), y).$$

For fixed $y \in \mathbb{A}^{n} \setminus \{0\}$, the equation for $x_{i}$ is linear in $x$ with $x_{i}$ removed, which guarantees that the inverse map is linear in the $\mathbb{A}^{n-1}$-component.

Now apply [lemma 6](#) and [example 10](#). \[\square\]

### 2. Motivic Cell Structures

**Definition 12** (Dugger and Isaksen [DI05, Definition 2.1]). Let $\mathcal{M}$ be a pointed model category and $\mathcal{A} \subset \text{Ob} \mathcal{M}$ a set of objects. The class of $\mathcal{A}$-cellular objects in $\mathcal{M}$ is defined as the smallest class of objects containing $\mathcal{A}$ that is closed under weak equivalence and contains all homotopy colimits over diagrams whose objects are all $\mathcal{A}$-cellular.

**Definition 13.** For the pointed model category of pointed motivic spaces $\text{Spc}(S)$ let $\mathcal{A} := \{S^{p,q} \mid p, q \in \mathbb{N}, p \geq q\}$ be the set of motivic spheres. The $\mathcal{A}$-cellular objects in $\text{Spc}(S)$ are called motivically cellular. A motivic space $X$ with $k$-fold simplicial suspension $\Sigma^{k}X$ motivically cellular is called $k$-suspended cellular. For motivic spectra with $\mathcal{A}^{s} := \{S^{p,q} \mid p, q \in \mathbb{Z}\}$, we call $\mathcal{A}^{s}$-cellular objects stably motivically cellular. A motivic space $X \in \text{Spc}(S)$ with $\Sigma^{\infty}X$ stably motivically cellular is also called stably motivically cellular.

**Definition 14.** Given a morphism $f: X \to Y$ of motivic spaces, we define the homotopy cofiber $\text{hocofib}(f)$ as the homotopy colimit of the solid-lines diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \searrow & \\
& & \ast \xrightarrow{\sim} \text{hocofib}(f)
\end{array}
$$
We call a sequence $X \xrightarrow{f} Y \rightarrow Z$ in the homotopy category of $\text{Spc}(S)$ a homotopy cofiber sequence if the sequence is isomorphic to $X \xrightarrow{f} Y \rightarrow \text{hocofib}(f)$ in the homotopy category.

**Example 15.** Projective space $\mathbb{P}^n$ carries a motivic cell structure, as there exists a homotopy cofiber sequence (compare [DI05, Proposition 2.13])

$\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1} \rightarrow \mathbb{P}^n$

and $\mathbb{A}^n \setminus \{0\}$ is a motivic sphere $\mathbb{S}^{2n-1,n}$ (up to $\mathbb{A}^1$-homotopy equivalence [DI05, Example 2.11] for a proof of this claim first made by Morel and Voevodsky [MV99, Example 3.2.20]). This homotopy cofiber sequence yields a distinguished triangle in the derived category of motives

$M(\mathbb{P}^{n-1}) \rightarrow M(\mathbb{P}^n) \rightarrow 1(n)[2n] \rightarrow$

and one can show that the attaching map $\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ is 0 at the level of motives for weight reasons, hence there is a decomposition

$M(\mathbb{P}^n) = \bigoplus_{i=0}^n 1(i)[2i].$

However, even on the level of spectra, in the stable motivic homotopy category, the attaching map is non-trivial and $\bigvee_{i=0}^n \mathbb{S}^{2n,i}$ is a different motivic space with the same motivic decomposition as $\mathbb{P}^n$.

**Lemma 16.** If a motivic space $X$ admits a stable motivic cell structure, its Voevodsky motive is of mixed Tate type.

**Proof.** This follows directly from $M(\mathbb{S}^{p,q}) = 1 \oplus 1(q)[p]$ and the fact that the homotopy colimits defining the cell structure can be written as homotopy coequalizer and homotopy coproduct, which translates directly to distinguished triangles of mixed motives. \qed

2.1. Motivic Thom Spaces.

**Theorem** (Homotopy Purity, Morel and Voevodsky [MV99, Thm. 3.2.23]). For a closed immersion $\iota : Z \hookrightarrow X$ with open complement $U \hookrightarrow X$, there is a natural homotopy cofiber sequence of pointed motivic spaces

$U \rightarrow X \rightarrow \text{Th}(N_\iota)$

where $\text{Th}(N_\iota)$ denotes the Thom space of the normal bundle $N_\iota$ of $\iota$, which is defined using the zero section $Z \hookrightarrow N_\iota$ as $\text{Th}(N_\iota) := N_\iota/(N_\iota \setminus Z) := \text{hocofib}(N_\iota \setminus Z \hookrightarrow N_\iota)$.

**Warning.** It is not necessarily true that a Thom space over a motivically cellular base is again motivically cellular (it is not known whether counterexamples exist or whether we simply lack a proof). This is also unknown for stable motivic cell structures.

**Remark 17.** For this reason, and also to be able to describe a cell structure explicitly, it is highly desirable to trivialize normal bundles. Thom spaces over trivial bundles are just suspensions ([MV99, Proposition 3.2.17]):

$\text{Th}(\mathbb{A}^n \times B \rightarrow B) = B_+ \wedge \mathbb{S}^{2n,n}.$

The following are tools to trivialize vector bundles.
Theorem (Quillen–Suslin). Let $R$ be a smooth finite type algebra over a Dedekind ring. Then all algebraic vector bundles on $\mathcal{A}_R^n$ are extended from $\text{Spec}(R)$.

This is beautifully explained in Lam’s book [Lam06, Theorem III.1.8].

It has been used to obtain a generalization, which one may see as a corollary to the vector bundle classification of Asok–Hoyois–Wendt:

Lemma 18. There are no non-trivial vector bundles on a smooth affine finite type $\mathcal{A}^1$-contractible variety $X$ over a Dedekind ring with perfect residue fields or a field.

Proof. Let $f : \text{Spec}(k) \xrightarrow{\sim} X$ be the isomorphism in the homotopy category of motivic spaces $\mathcal{H}o \text{Spec}(k)$ given by contractibility. From [AHW17, Theorem 5.2.3], we know

$$\{\text{rank } r \text{ vector bundles on } X\} / \sim \xrightarrow{\sim} [X, Gr_{\mathcal{A}^1}] \xrightarrow{f_*} [*, Gr_{\mathcal{A}^1}] = 1. \quad \square$$

This fails already for smooth non-affine quasi-affine varieties that are $\mathcal{A}^1$-contractible, where one can give infinitely many counterexamples [ADF17, Corollary 4.3.9].

Corollary 19. If $V \hookrightarrow M$ is a codimension $c$ closed immersion of smooth varieties over a smooth finite type $\mathcal{Z}$-algebra $R$, and the complement $M \setminus V$ is an $\mathcal{A}^1$-contractible smooth affine $R$-variety, and $V$ is the total space of a vector bundle $V \to M'$ with $M'$ an $\mathcal{A}^1$-contractible $R$-variety, $M$ is a motivic sphere $\mathcal{S}^{c, c}$.

Proof. Assume that $M'$ is smooth affine as well. Using lemma 18, the vector bundle $V \to M'$ is trivial, so the total space $V$ is also smooth affine $\mathcal{A}^1$-contractible. Using lemma 18 again, the normal bundle $N_i \to V$ is trivial. The Thom space of a trivial bundle of rank $r$ over a base $B$ is $\mathcal{A}^1$-homotopy equivalent to $(\mathbb{P}^1)^{\wedge r} \wedge B$. We conclude by using homotopy purity, which hands us a homotopy cofiber sequence

$$M \setminus V \to M \to \text{Th}(N_i).$$

Contractibility of $M \setminus V$ implies that $M \to \text{Th}(N_i)$ is a weak equivalence, so $M \simeq (\mathbb{P}^1)^{\wedge c} \wedge \mathcal{S}^0 \simeq \mathcal{S}^{2c, c}$.

Now if $M'$ is not smooth affine, $N_i \to V$ may be non-trivial. Since $V \to M'$ is still a weak equivalence, $V$ is still $\mathcal{A}^1$-contractible, hence $\text{Th}(N_i) \simeq (\mathbb{P}^1)^{\wedge c} \wedge \mathcal{S}^0$, as Thom spaces are invariant under $\mathcal{A}^1$-equivalence by definition. \square

Definition 20. For a variety $N$, a Zariski cover $U = (U_i \hookrightarrow N)_{i \in I}$ (with $N = \bigcup_{i \in I} U_i$) is called totally cellular if the Čech nerve $\check{C}^\bullet(U)$ is a simplicial object in cellular varieties. It is called totally contractible if the $U_\alpha = \bigcap_{j \in J} U_j$ for each $J \subseteq I$ are $\mathcal{A}^1$-contractible. It is called totally affinely contractible if there are affine bundles $\check{U}_\alpha \to U_\alpha$ with affine total spaces $\check{U}_\alpha \cong \mathcal{A}_{mn}$, compatible with the simplicial structure on $\check{C}^\bullet(U)$ (assembling to an affine bundle $\check{C}^\bullet(\check{U}) \to \check{C}^\bullet(U)$). A variety that admits a totally affinely contractible (Zariski) cover is called atacc for short.

The definition of total cellularity was made in the stable context by Dugger and Isaksen [DI05, Definition 3.7], see also [DI05, Lemma 3.8].

From the definitions follows immediately
Proposition 21. A totally contractible Zariski cover is totally cellular and a totally affinely contractible Zariski cover is totally contractible. A variety admitting a totally cellular Zariski cover (in particular, an atac variety) is unstably cellular.

Remark 22. While smash products of unstably cellular spaces are again unstably cellular, Dugger and Isaksen already noticed [DI05, Example 3.5] that it is in general hard to show whether a cartesian product of cellular spaces is unstably cellular. Since it is easy to show that cartesian products of stably cellular spaces are stably cellular, they only prove that Thom spaces of bundles over a totally cellular base are stably cellular [DI05, Corollary 3.10]. As we are interested in unstable cell structures on spaces which are iterated Thom spaces, we need a stronger statement: theorem 25.

Lemma 23. Let \( p: E \to B \) be a vector bundle and \( B' \to B \) an \( \mathbb{A}^1 \)-weak equivalence. Then there exists a weak equivalence of Thom spaces \( \text{Th}(p) \sim \to \text{Th}(p') \).

Proof. Since vector bundle projections are sharp (lemma 7), the morphism \( E \times_B B' \to E \) is a weak equivalence. Let \( s \) be the zero section of \( p \) and \( s' \) the zero section of the base change \( p': E \times_B B' \to B' \). By construction of \( s' \), we get a weak equivalence of \( E \times_B B' \setminus s'(B') \) with \( E \setminus s(B) \). We proved that the diagrams whose homotopy colimits are \( \text{Th}(p) \) respectively \( \text{Th}(p') \) are weakly equivalent.

Corollary 24. Let \( p: E \to B \) be a rank \( n \) vector bundle and \( B' \to B \) an affine bundle with \( B' \cong \mathbb{A}^n \) (as varieties). Then \( \text{Th}(p) \sim \to B_+ \wedge \mathbb{S}^{2n,n} \cong \mathbb{S}^{2n,n} \).

Proof. We use lemma 6 to apply lemma 23 and then remark 17.

Theorem 25. Let \( p: E \to B \) be an algebraic vector bundle of rank \( r \) and \( U = (U_i \hookrightarrow B)_{i \in I} \) a totally affinely contractible Zariski cover of \( B \), all defined over a ring \( R \) which is smooth and finite type over a Dedekind ring with perfect residue fields or a field. Then \( B, E \) and \( \text{Th}(p) \) are unstably cellular.

Proof. Unstable cellularity of \( B \) is proposition 21, cellularity of \( E \) follows from lemma 6, so it remains to show cellularity of the Thom space \( \text{Th}(p) \). We use the morphism \( l: \tilde{C}^\bullet(U) \to B \) which induces a weak equivalence on homotopy colimits, i.e. \( \text{hocolim} (\tilde{C}^\bullet(U)) \simeq B \). The bundle \( q: E \setminus B \to B \) obtained as sub-bundle of \( p \) is a fiber bundle with fiber \( \mathbb{A}^r \setminus \{0\} \). The following diagram commutes:

\[
\begin{array}{ccc}
q^*\tilde{C}^\bullet(U) & \xrightarrow{l^*i} & p^*\tilde{C}^\bullet(U) \\
q^*l & & p^*l \\
E \setminus B & \xrightarrow{i} & E & \xrightarrow{\text{hocolim}(l^*i)} & \text{Th}(p)
\end{array}
\]

The rows are homotopy cofiber sequences. The middle column is a weak equivalence by lemma 7. The left column is also a weak equivalence, as it is the restriction of the middle column and the model structure is proper. (alternatively one could argue that spherical bundle projections are as sharp as vector bundle projections). We inspect the first row more closely. While
the bundle $E$ might not trivialize over $U_i$, its pullback to an affine space $\tilde{U}_i$
(given by the property of $U_i$ being totally affinely contractible) is trivial, by
Quillen–Suslin. The same holds for each $U_\alpha$ with obvious definition of
$\tilde{U}_\alpha$.

By lemma 23 the Thom space of $E|_{U_i}$ is weakly equivalent to the Thom
space of the pulled back bundles $E|_{\tilde{U}_i}$. Now we can form a diagram,
commutative up to homotopy

\[
\begin{array}{ccc}
q^*U_\alpha & \xrightarrow{i|_{U_\alpha}} & p^*U_\alpha \\
\downarrow l^*i & & \downarrow p^*l \\
\mathbb{A}^n \setminus \{0\} \times \tilde{U}_\alpha & \longrightarrow & \mathbb{A}^n \times \tilde{U}_\alpha \longrightarrow & \Sigma_s(\mathbb{A}^n \setminus \{0\}) \\
\end{array}
\]

whose rows are homotopy cofiber sequences and the leftmost two columns
are weak equivalences. Consequently, the last column is a weak equivalence.
This exhibits both $E \setminus B$ and $\text{Th}(p)$ as homotopy colimit over cellular spaces.

We can view $E \setminus B \to B$ obtained by composing $E \setminus B \to E$ with the
bundle projection $p : E \to B$ as an explicit gluing map, as its homotopy
colimit is again $\text{Th}(p)$. □

Corollary 26. Given a sequence $M_i$ of smooth varieties over a ring $R$
which is smooth and finite type over a Dedekind ring with perfect residue fields or a
field $M_n \supset M_{n-1} \supset \cdots \supset M_0 = * \supset M_{-1} = \emptyset$
such that each $M_i$ is atac (definition 20) and rank $r_i$ vector bundles $V_i \to
M_{i-1}$ together with a closed immersion of the total space $V_i \hookrightarrow M_i$
of codimension $c_i$, and each complement $X_i := M_i \setminus V_i$ is $\mathbb{A}^1$-contractible, there exists
an unstable motivic cell structure on each $M_i$.

Proof. We use induction on $i$, with the base case $M_0$ being trivially cellular.
Let $N_i \to V_i$ be the normal bundle of the closed immersion $V_i \hookrightarrow M_i$. As the
complement $X_i$ is $\mathbb{A}^1$-contractible, by homotopy purity, applied as in the
proof of corollary 19, we get a weak equivalence $M_i \to \text{Th}(N_i)$. From our
assumptions, $\text{Th}(N_i)$ carries an unstable cell structure. □

Remark 27. If the ranks $r_i$ in corollary 26 are all 0, this resembles Wendt’s
unstable cell structure on generalized flag varieties using the Bruhat cells [Wen10,
Proposition 3.7].

We now prove that the complements of subspace arrangements are cellular
after a finite amount of suspensions, depending on the number of subspaces.

Proof of theorem 3. The base case is $\mathbb{A}^n$ with a single linear subspace $L_0$
of dimension $k$. By change of basis we move the linear subspace to the first $k$
coordinates so that $\mathbb{A}^n \setminus L_0 \hookrightarrow (\mathbb{A}^k \times \mathbb{A}^{n-k}) \setminus \{0\} = \mathbb{A}^k \times (\mathbb{A}^{n-k} \setminus \{0\})$,
an affine bundle over the motivic sphere $\mathbb{A}^{n-k} \setminus \{0\}$.

By induction over the number of linear subspaces, assume $X := \mathbb{A}^n \setminus \bigcup_{i=1}^n L_i$
is $(n-1)$-suspended cellular. The intersection $L_0 \cap X = L_0 \setminus \bigcup_{i=1}^n (L_0 \cap L_i)$
is the complement of an arrangement of $n$ linear subspaces in the affine space
$L_0$, hence $(n-1)$-suspended cellular by induction assumption.
The normal bundle $N_0 \to L_0 \cap X$ of $L_0 \cap X \hookrightarrow X$ is the restriction of the normal bundle of $L_0 \hookrightarrow \mathbb{A}^n$, hence trivial by Quillen–Suslin. Let $r_0 = n - \dim(L_0)$ be the rank of $N_0$. From this we see that the Thom space $\text{Th}(N_0)$ is $\mathbb{A}^1$-homotopy equivalent to $\Sigma^{2k,k}(L_0 \cap X) = \Sigma^k\big((\mathbb{G}_m)^k \wedge (L_0 \cap X)\big)$, hence $(n - 1 - k)$-suspended cellular.

By homotopy purity we get a homotopy cofiber sequence

\[
\mathbb{A}^n \setminus \bigcup_{i=0}^n H_i \to \mathbb{A}^n \setminus \bigcup_{i=1}^n H_i \to \text{Th}(N_0) \to \Sigma \left( \mathbb{A}^n \setminus \bigcup_{i=0}^n H_i \right)
\]

which shows that the fourth space is $(n - 1)$-suspended cellular as well. \(\square\)

A rank $r$ split torus is the complement of the $r$ coordinate hyperplanes $\{x_i = 0\} \subset \mathbb{A}^n$, hence we have shown that rank $r$ tori are $(r - 1)$-suspended cellular.

Closely related to cellularity is the notion of linear varieties which comes in several closely related versions (discussed by Janssen, Totaro, Joshua among others). We study two of them here.

**Definition 28.** We call the empty scheme $\emptyset$ and any affine space $\mathbb{A}^n$ a 0-linear variety. Inductively, for $n \in \mathbb{N}$ and $Z$ a $(n - 1)$-linear variety with a closed immersion $Z \hookrightarrow X$ and open complement $U := X \setminus Z$, if either $X$ or $U$ is $(n - 1)$-linear as well, then we call $Z, X, U$ $n$-linear varieties. A variety is called linear if it is $n$-linear for some $n$.

**Lemma 29.** A variety $Z$ that is isomorphic to a union of $r$ hyperplanes in $\mathbb{A}^n$ is $(r - 1)$-linear and the complement in $\mathbb{A}^n$ is $r$-linear.

**Proof.** Assume $Z \hookrightarrow \mathbb{A}^n$ to be a union of $r$ hyperplanes and $H \hookrightarrow \mathbb{A}^n$ another hyperplane. Then $Z \hookrightarrow Z \cup H$ has complement $H \setminus Z$ isomorphic to the union of $r - 1$ hyperplanes. By induction over $r$ we can assume $Z$ and $H \setminus Z$ to be $(r - 1)$-linear, hence $Z \cup H$ is $r$-linear. Since $\mathbb{A}^n$ is 0-linear, it is also $r - 1$-linear, so $\mathbb{A}^n \setminus Z$ is $r$-linear. \(\square\)

**Definition 30.** Let $X$ be a $k$-variety with a filtration $F^i$ such that $F^i \hookrightarrow F^{i+1}$ is a closed immersion with open complement isomorphic to a disjoint union of varieties of the type $\mathbb{A}^n \times \mathbb{G}_m^r$. Then $X$ is called very linear.

**Proposition 31.** Very linear varieties are linear.

**Proof.** Products and disjoint unions of linear varieties are again linear and the previous lemma 29 shows that strata of the filtration are linear. \(\square\)

**Remark 32.** While it is an easy exercise to show that the Voevodsky motive of a linear variety is of mixed Tate type, it is not clear whether linear varieties are cellular, as the Thom spaces involved in a homotopy purity argument are not necessarily cellular.

### 3. Cell Structures for Spherical Varieties

Let $G$ be a split reductive group. We will use the theory of spherical varieties, as detailed in the comprehensive book by Timashev [Tim11]. A spherical variety is a normal algebraic variety with an algebraic $G$-action such that a Borel $B \subset G$ acts with a dense orbit. As a consequence it has only finitely many $B$-orbits.
Special cases of spherical varieties are spherical homogeneous spaces such as affine quadrics, flag varieties $G/P$ and the homogeneous spaces $G/H$ that admit a two-orbit equivariant completion.

**Proposition 33.** $G$-spherical varieties are very linear.

**Proof.** By Rosenlicht [Ros63, Theorem 5, page 119], any $k$-variety homogeneous under $B$ is isomorphic to $\mathbb{G}_m \times r \times \mathbb{A}^n$ for some $r, n$. The $B$-orbit decomposition of a spherical variety $X$ therefore yields a filtration $F^i$ with strata the $B$-orbits, turning $X$ into a very linear variety. □

We now prove that every spherical variety admits stable motivic cell structures (composing ideas from Totaro [Tot14, page 8, section 3 and Addendum] and Carlsson–Joshua [CJ11, Proposition 4.7]).

**Proof of theorem 1.** The filtration $F^i$ by $B$-orbits, as in the proof of proposition 33 has the special property that the $B$-orbits are atacc, hence for each $i$: $F_i \rightarrow F_{i-1}$ the homotopy cofiber sequence homotopy purity

$$F_i \setminus F_{i-1} \rightarrow F^i \rightarrow \text{Th}(N_{i, \ast})$$

has $F_{i-1}$ and $\text{Th}(N_{i, \ast})$ stably cellular by induction. □

**Remark 34.** In the special case of wonderful completions, there is another proof of theorem 1: Let $X$ be a homogeneous spherical $G$-variety which admits a wonderful equivariant completion $\overline{X}$ with boundary $Z$, i.e. $\overline{X}$ is smooth,

$$G/H = X \overline{\hookrightarrow} \overline{X} \hookrightarrow Z,$$

the boundary $Z$ has $r$ irreducible components, where $r$ is the rank of $X$, and there is a unique closed orbit in $Z$, which is the intersection of all irreducible components of $Z$. Furthermore, all open orbits of $Z$ are of lower dimension than $X$ [Tim11, Chapter 5, Definition 30.1].

As $\overline{X}$ is a complete $G$-variety, we can apply the algebraic Morse theory of Białynicky-Birula, as Wendt proved [Wen10, Corollary 3.5], to obtain a stable motivic cell structure on $\overline{X}$. The same applies to $Z$, so by a 2-out-of-3-argument, as in the previous proof, the variety $X$ is stably motivically cellular.

One can control the amount of suspensions one has to perform to obtain an unstable cell structure, provided the boundary is atacc:

**Proof of theorem 2.** The case of $n = 1$ is the situation $X \hookrightarrow Y \leftrightarrow D$ with $D$ irreducible, smooth and atacc. By theorem 25 the space $\text{Th}(N_{D \hookrightarrow Y})$ is unstably cellular, hence $\Sigma^k \text{Th}(N_{D \hookrightarrow Y})$ is unstably cellular. By assumption, $\Sigma^k Y$ is unstably cellular. By taking the $k$-fold suspension on the homotopy purity cofiber sequence and adding the next term, we get

$$\Sigma^k X \rightarrow \Sigma^k Y \rightarrow \Sigma^k \text{Th}(N_{D \hookrightarrow Y}) \rightarrow \Sigma^{k+1} X$$

which shows that $\Sigma^{k+1} X$ is unstably cellular.

For arbitrary $n$ we form $X' := Y \setminus D_n$ and the inclusion $X' \hookrightarrow Y$ is again the $n = 1$ case we just proved, so that $\Sigma^{k+1} X'$ is unstably cellular. The inclusion $X \hookrightarrow X'$ has boundary $D \setminus D_n$, so by induction over $n$ we get that $\Sigma^{k+n} X$ is unstably cellular. □

As a pleasant surprise, the rank 1 situation is particularly well-behaved:
**Theorem 35.** For homogeneous spaces $X$ that admit an equivariant completion $\overline{X}$ with a single closed orbit as boundary (two-orbit wonderful completions), there exists an unstable motivic cell structure on $\Sigma X$.

**Proof.** By the classification of such $X \hookrightarrow \overline{X}$ (due to Ahiezer in characteristics $0$ [Ahi83] and Knop in positive characteristics [Kno14]), we know that $\overline{X}$ is a projective homogeneous space under a reductive group. We apply Wendt’s unstable motivic cell structure [Wen10, Theorem 3.6] arising from the Bruhat decomposition of $\overline{X}$ and $\overline{X} \setminus X$ (both are generalized flag varieties). Since the Bruhat cells of a generalized flag variety are affine spaces, the boundary $\overline{X} \setminus X$ is atac. Now theorem 2 gives the conclusion. □

**Remark 36.** For each such two-orbit completions, there exists a choice of Borel for the reductive group acting transitively on the completion such that the associated Schubert stratification restricts to a Schubert stratification on the boundary. One can compute explicitly the embedding and the complement for each Bruhat cell, where the embedding restricts to a linear embedding of an affine subspace into an affine space. With this observation, one can also use the complements, which are $1$-suspended cellular due to theorem 3, to give a slightly different proof of theorem 35.

**Example 37.** For $H^p = Sp_{2n+2}/Sp_2 \times Sp_{2n}$, the completion is a Grassmannian $Gr(2, 2n + 2)$, with complement the symplectic Grassmannian $SpGr(2, 2n + 2)$ classifying symplectic planes in a $2n + 2$-dimensional vector space with the standard symplectic form. Theorem 35 hands us an unstable motivic cell structure for $\Sigma H^p$:

$$Gr(2, 2n + 2) \to Th\left(N_{SpGr(2, 2n+2)} \rightarrow Gr(2, 2n+2)\right) \rightarrow \Sigma H^p.$$  

As mentioned in the introduction, there also exists a motivic cell structure for $H^p$ unsuspended [Voe16, Theorem 4.4.8].

**Example 38.** For the split octonionic projective plane $OP^2 := F_4 / Spin_9$, the completion is the complex Cayley plane $E_6/P_1$, with complement an $F_4/P_4$. Theorem 35 hands us an unstable motivic cell structure for $\Sigma OP^2$:

$$E_6/P_1 \to Th\left(N_{F_4/P_4}\right) \rightarrow \Sigma OP^2.$$  

Unfortunately, there is still no known construction of a motivic cell structure for OP$^2$ unsuspended. Attempts on a construction are in the author’s thesis [Voe16, Section 4.6].

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