Large-$N_c$ sum rules for charmed baryons at subleading orders

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Abstract

Sum rules for the low-energy constants of the chiral SU(3) Lagrangian with charmed baryons of spin $J^P = 1/2^+$ and $J^P = 3/2^+$ baryons are derived from large-$N_c$ QCD. We consider the large-$N_c$ operator expansion at subleading orders for current-current correlation functions in the charmed baryon-ground states for two scalar and two axial-vector currents.

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I. INTRODUCTION

The dependence of the charmed baryon masses on the up, down and strange quark masses encodes useful information on the coupled-channel interaction dynamics of the Goldstone bosons with such baryon states \[1–8\]. Lattice QCD simulations for the baryon masses at unphysical quark masses are particularly useful \[9–14\] since they complement the well known values for the masses of the charmed baryon ground states at physical quark masses \[15\].

An accurate flavour SU(3) chiral extrapolation of the baryon ground states with zero charm content was established in a series of works \[16–20\]. Based on the chiral Lagrangian formulated with spin 1/2 and 3/2 fields the available lattice data on the baryon masses was reproduced and accurate predictions for the size of the low-energy parameters relevant at \(N^3\)LO were made \[20\]. The success of such analyses relies on two crucial ingredients. First, the chiral expansion is formulated in terms of physical meson and baryon masses rather than bare masses as is requested by traditional chiral perturbation theory (\(\chi PT\)). Second, the flood of low-energy constants that arises at subleading orders is tamed by sum rules for the latter as they arise in the limit of a large number of colors \(N_c\) in QCD \[20, 21\]. The large-\(N_c\) sum rules provide a large parameter reduction that allowed fits at \(N^3\)LO to the lattice data set that are significant. A corresponding program was started for the charmed baryons \[22\]. At present, however, the large-\(N_c\) sum rules for the charmed baryons are derived at leading order only \[22\]. It is the purpose of this work to close this gap and establish such sum rules accurate to subleading orders in the \(1/N_c\) expansion. This will pave the way to accurate chiral extrapolation studies of the charmed baryon masses.

The desired sum rules can systematically be derived from QCD by a study of current-current correlation functions in the baryon ground states. We study matrix elements of current-current correlation functions in the charmed baryon states \[21, 23\]. The technology developed in \[24–26\] will be applied. The implications of heavy-quark symmetry on the counter terms was worked out already using a suitable multiplet representation of the charmed baryons \[22, 27, 29\].
II. CHIRAL DYNAMICS FOR CHARMED BARYONS

The chiral dynamics for the charmed baryon fields is most economically deduced from an effective chiral Lagrangian that is based on power counting rules. We consider here the flavour antisymmetric anti-triplet and the flavour symmetric sextet fields $B_{[3]}$, $B_{[6]}$ and $B_{[6]}^\mu$ with $J^P = \frac{1}{2}^+$ and $J^P = \frac{3}{2}^+$ quantum numbers. The chiral Lagrangian consists of all possible interaction terms, formed with the baryon fields and the conventional chiral blocks $U_\mu$ and $\chi_{\pm}$ that include the Goldstone boson fields $\Phi$ as well as the classical source functions, $s, p$ and $v_\mu, a_\mu$ of QCD \[30\]. Derivatives of the fields must be included in compliance with the local chiral SU(3) symmetry which in turn requests the covariant derivative $D_\mu$ to act on the various flavour multiplet fields as follows

$$(D_\mu U_\nu)^a_b = \partial_\mu U^a_{\nu,b} + \Gamma^a_{\mu,l} U^l_{\nu,b} - \Gamma^l_{\mu,b} U^{a,l},$$

$$(D_\mu B_{[6]}^{ab}) = \partial_\mu B^{ab}_{[6]} + \Gamma^a_{\mu,l} B_{[6]}^{b,l} + \Gamma^b_{\mu,l} B_{[6]}^{a,l},$$

$$(D_\mu B_{[3]}^{ab}) = \partial_\mu B_{[3]}^{ab} + \Gamma^a_{\mu,l} B_{[3]}^{b,l} + \Gamma^b_{\mu,l} B_{[3]}^{a,l},$$

with the chiral connection $\Gamma_\mu = -\Gamma^{\dagger}_\mu$ given by

$$\Gamma_\mu = \frac{1}{2} \epsilon^{-1} \frac{s}{f} \left[ \partial_\mu - i (v_\mu + a_\mu) \right] e^{+i \frac{s}{f} \frac{\Phi}{f}} + \frac{1}{2} e^{+i \frac{s}{f} \frac{\Phi}{f}} \left[ \partial_\mu - i (v_\mu - a_\mu) \right] e^{-i \frac{s}{f} \frac{\Phi}{f}},$$

$$U_\mu = \frac{1}{2} u^\dagger (\partial_\mu e^{i \frac{s}{f} \frac{\Phi}{f}}) u^\dagger - \frac{i}{2} u^\dagger (v_\mu + a_\mu) u^\dagger + \frac{i}{2} u (v_\mu - a_\mu) u^\dagger,$$

$$u = e^{i \frac{s}{f} \frac{\Phi}{f}}.$$ (2)

The various hadron fields can be decomposed into their isospin multiplet components

$$\Phi = \tau \cdot \pi (140) + \alpha^\dagger \cdot K(494) + K(494)^\dagger \cdot \alpha + \eta(547) \lambda_8,$$

$$\sqrt{2} B_{[3]} = \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c (2470) - \frac{1}{\sqrt{2}} \Xi_c^T (2470) \cdot \alpha + i \tau_2 \Lambda_c (2284),$$

$$\sqrt{2} B_{[6]} = \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c (2580) + \frac{1}{\sqrt{2}} \Xi_c^T (2580) \cdot \alpha + \Sigma_c (2455) \cdot \tau i \tau_2 + \frac{\sqrt{2}}{3} \left( 1 - \sqrt{3} \lambda_8 \right) \Omega_c (2704),$$

$$\sqrt{2} B_{[6]}^\mu = \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c^\mu (2645) + \frac{1}{\sqrt{2}} \Xi_c^T \Omega_c^\mu (2645) \cdot \alpha + \Sigma_c^\mu (2520) \cdot \tau i \tau_2 + \frac{\sqrt{2}}{3} \left( 1 - \sqrt{3} \lambda_8 \right) \Omega_c^\mu (2770),$$

$$\alpha^\dagger = \frac{1}{\sqrt{2}} (\lambda_4 + i \lambda_5, \lambda_6 + i \lambda_7), \quad \tau = (\lambda_1, \lambda_2, \lambda_3),$$ (3)

where the matrices $\lambda_i$ are the standard Gell-Mann generators of the SU(3) algebra.

The main goal of this work is to derive correlations amongst the low-energy parameters of the chiral Lagrangian as they follow from a $1/N_c$ expansion. For that purpose we consider
QCD’s axial-vector and scalar currents,
\[ A^{(a)}_{\mu}(x) = \bar{\Psi}(x) \gamma_{\mu} \gamma_5 \frac{\lambda_a}{2} \Psi(x), \quad S^{(a)}(x) = \bar{\Psi}(x) \frac{\lambda_a}{2} \Psi(x), \quad (4) \]
in baryon matrix elements, where we recall their definitions in terms of the Heisenberg quark-field operators $\Psi(x)$. With $\lambda_a$ we denote the Gell-Mann flavour matrices supplemented with a singlet matrix $\lambda_0 = \sqrt{2/3} \mathbf{1}$. Given the chiral Lagrangian, it is well defined how to derive the contribution of a given term to such matrix elements. The classical matrices of source functions, $a_\mu$ and $s$, enter the chiral Lagrangian via the building blocks
\[ U_\mu = \frac{i}{2 f} \partial_\mu \Phi - i a_\mu + \cdots, \quad \chi_+ = \frac{1}{2} (u \chi_0 u + u^\dagger \chi_0 u^\dagger) = 2 B_0 s + \cdots, \quad (5) \]
where for notational simplicity in the following we put $B_0 = 1/2$.

We recall all terms in the chiral Lagrangian that are relevant in a chiral extrapolation of the baryon masses at $N^3\text{LO}$. Altogether we recalled $34 + 16 = 50$ distinct low-energy constants, which have to be correlated. There are 16 symmetry breaking counter terms
\[ \mathcal{L}_\chi = c_{1,33} \text{tr} (\bar{B}_3 B_3) \text{tr} \left( \chi_+^2 \right) + c_{2,33} \text{tr} (\bar{B}_3 B_3) \left( \text{tr} \chi_+ \right)^2 
+ c_{3,33} \text{tr} (\bar{B}_3 \chi_+ B_3) \text{tr} \left( \chi_+ \right) + c_{4,33} \text{tr} (\bar{B}_3 \chi_+^2 B_3) 
+ c_{1,66} \text{tr} (\bar{B}_6 B_6) \text{tr} \left( \chi_+^2 \right) + c_{2,66} \text{tr} (\bar{B}_6 B_6) \left( \text{tr} \chi_+ \right)^2 
+ c_{3,66} \text{tr} (\bar{B}_6 \chi_+ B_6) \text{tr} \left( \chi_+ \right) + c_{4,66} \text{tr} (\bar{B}_6 \chi_+^2 B_6) + c_{5,66} \text{tr} (\bar{B}_6 \chi_+ B_6 \chi_+^T) 
+ c_{1,36} \text{tr} (\bar{B}_6 \chi_+ B_3 + \text{h.c.}) \text{tr} \left( \chi_+ \right) + c_{2,36} \text{tr} (\bar{B}_6 \chi_+^2 B_3 + \text{h.c.}) 
- e_{1,66} \text{tr} (\bar{B}_6^\mu g_{\mu\nu} B_6^\nu) \text{tr} \left( \chi_+^2 \right) - e_{2,66} \text{tr} (\bar{B}_6^\mu g_{\mu\nu} B_6^\nu) \left( \text{tr} \chi_+ \right)^2 
- e_{3,66} \text{tr} (\bar{B}_6^\mu \chi_+ g_{\mu\nu} B_6^\nu) \text{tr} \left( \chi_+ \right) - e_{4,66} \text{tr} (\bar{B}_6^\mu g_{\mu\nu} \chi_+^2 B_6^\nu) 
- e_{5,66} \text{tr} (\bar{B}_6^\mu g_{\mu\nu} \chi_+ B_6^\nu \chi_+^T), \quad (6) \]
which contribute to the baryon masses at tree-level. Not that as compared to [22] we dropped the flavour redundant term proportional to $c_{3,36}$. The symmetry breaking counter terms contribute to the current-current correlation function of two time-ordered scalar currents
\[ S^{ab}(q) = i \int d^4 x \ e^{-i q x} \mathcal{T} S^{(a)}(x) S^{(b)}(0), \quad (7) \]
in the baryon states. We consider singlet and octet components with $a, b = 0, \cdots, 8$.

In addition there is a class of 34 symmetry conserving two-body counter terms that contribute to the baryon masses at the one-loop level. Following [21, 22, 31] the symmetry
conserving counter term are classified according to their Dirac structure.

\[ \mathcal{L}^{(2)} = \mathcal{L}^{(2)} + \mathcal{L}^{(S)} + \mathcal{L}^{(V)} + \mathcal{L}^{(A)} + \mathcal{L}^{(T)}. \]  

(8)

A complete list relevant at second order was constructed in [22] with

\[
\mathcal{L}^{(S)} = -g_{0,[33]}^{(S)} \text{tr} (\bar{B}_{[3]} B_{[3]}) \text{tr} (U_\mu U^\mu) - g_{D,[33]}^{(S)} \text{tr} (\bar{B}_{[3]} \{ U_\mu, U^\mu \} B_{[3]}) \\
- g_{0,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} B_{[6]}) \text{tr} (U_\mu U^\mu) - g_{1,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} U^\mu B_{[6]} U^\mu) \\
- g_{D,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} \{ U_\mu, U^\mu \} B_{[6]}) - g_{D,[36]}^{(S)} \text{tr} (\bar{B}_{[6]} \{ U_\mu, U^\mu \} B_{[3]} + \text{h.c.}) \\
+ h_{0,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} g_{B\mu} B_{[6]}^\nu) \text{tr} (U_\alpha U^\alpha) + h_{1,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu B_{[6]}^\nu) \text{tr} (U_\nu U^\nu) \\
+ h_{2,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} g_{B\mu} \{ U^\alpha, U_\alpha \} B_{[6]}^\nu) + h_{3,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu \{ U_\mu, U^\nu \} B_{[6]}^\nu) \\
+ h_{4,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu g_{B\mu} U^\alpha B_{[6]}^\nu U^\mu T^\alpha) + \frac{1}{2} h_{5,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu U_\nu B_{[6]}^\nu U^\mu T^\mu + \bar{B}_{[6]} U^\mu B_{[6]} U^\mu U^\mu T^\mu) ,
\]

\[
\mathcal{L}^{(V)} = -\frac{1}{2} g_{0,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha (D^\beta B_{[3]}) \text{tr} (U_\beta U^\alpha) + \text{h.c.}) \\
- \frac{1}{2} g_{1,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha U_\beta (D^\beta B_{[3]}) U^\alpha T^\alpha + \bar{B}_{[3]} i \gamma^\alpha U_\alpha (D^\beta B_{[3]}) U^\beta + \text{h.c.}) \\
- \frac{1}{2} g_{D,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha \{ U_\alpha, U_\beta \} (D^\beta B_{[3]}) + \text{h.c.}) \\
- \frac{1}{2} g_{D,[36]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha \{ U_\alpha, U_\beta \} (D^\beta B_{[3]}) - (D^\beta \bar{B}_{[6]}) i \gamma^\alpha \{ U_\alpha, U_\beta \} B_{[3]} + \text{h.c.}) \\
- \frac{1}{2} g_{0,[66]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha (D^\beta B_{[6]})) \text{tr} (U_\beta U^\beta) + \text{h.c.}) \\
- \frac{1}{4} g_{1,[66]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha U_\beta (D^\beta B_{[6]}) U^\alpha T^\alpha + \bar{B}_{[6]} i \gamma^\alpha U_\alpha (D^\beta B_{[6]}) U^\beta + \text{h.c.}) \\
- \frac{1}{2} g_{D,[66]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha \{ U_\alpha, U_\beta \} (D^\beta B_{[6]}) + \text{h.c.}) \\
+ \frac{1}{2} h_{0,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{B\mu} i \gamma^\alpha (D^\beta B_{[6]}^\nu) \text{tr} (U_\alpha U^\beta) + \text{h.c.}) \\
+ \frac{1}{4} h_{1,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{B\mu} i \gamma^\alpha U_\beta (D^\beta B_{[6]}^\nu) U^\alpha T^\alpha + \bar{B}_{[6]}^\mu g_{B\mu} i \gamma^\alpha U_\alpha (D^\beta B_{[6]}^\nu) U^\beta + \text{h.c.}) \\
+ \frac{1}{2} h_{2,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{B\mu} i \gamma^\alpha \{ U_\alpha, U_\beta \} (D^\beta B_{[6]}^\nu) + \text{h.c.}) ,
\]

\[
\mathcal{L}^{(A)} = f_{0,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 B_{[6]} \text{tr} (U_\nu U^\mu) + \text{h.c.}) \\
+ f_{1,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\nu B_{[6]} U^\mu + \bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\nu B_{[6]} U^\nu + \text{h.c.}) \\
+ f_{D,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{ U_\mu, U_\nu \} B_{[6]} + \text{h.c.}) + f_{F,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{ U_\mu, U_\nu \} B_{[6]} + \text{h.c.}) \\
+ f_{1,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\mu B_{[3]} U^\nu + \bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\mu B_{[3]} U^\mu + \text{h.c.}) \\
+ f_{D,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{ U_\mu, U_\nu \} B_{[3]} + \text{h.c.}) + f_{F,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{ U_\mu, U_\nu \} B_{[3]} + \text{h.c.}) ,
\]

\[
\mathcal{L}^{(T)} = -g_{F,[33]}^{(T)} \text{tr} (\bar{B}_{[3]} i \sigma^{\alpha \beta} \{ U_\alpha, U_\beta \} B_{[3]}) - g_{1,[36]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha \beta} U_\alpha B_{[3]} U^\beta + \text{h.c.}) \\
- g_{F,[36]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha \beta} \{ U_\alpha, U_\beta \} B_{[3]} + \text{h.c.}) - g_{F,[66]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha \beta} \{ U_\alpha, U_\beta \} B_{[6]}).
\]
\[ + h_{F,[6]}^{(T)} \text{tr} \left( \bar{B}_{[6]} B_{[6]} g_\mu \sigma^{\alpha \beta} \left[ U_\alpha, U_\beta \right] B_{[6]} \right), \] (9)

where possible further terms are redundant owing to flavour identities or the on-shell conditions of spin-\( \frac{3}{2} \) fields with \( \gamma_\mu B_{[6]}^\mu = 0 \) and \( \partial_\mu B_{[6]}^\mu = 0 \). As compared to [22] we further streamlined the notations and dropped the flavour redundant terms proportional to \( g_{1,[36]}^{(S)} \) and \( g_{1,[36]}^{(V)} \).

The symmetry conserving parameters contribute to the current-current correlation function of two time-ordered axial-vector currents

\[ A_{\mu \nu}^{ab}(q) = i \int d^4x e^{-iq \cdot x} \mathcal{T} A_\mu^{(a)}(x) A_\nu^{(b)}(0), \] (10)

in the baryon states.

The specific form of the matrix elements of the current-current correlation functions (7) and (10) was already worked out in the previous work [22]. The matrix elements are detailed in the flavour SU(3) limit where the physical baryon states are specified by the momentum \( p \) and the flavour indices \( i, j = 1, 2, 3 \).
The low-energy constants recalled in (6) and (8) can be analyzed systematically in the $1/N_c$ expansion [21–23, 25, 26]. Leading order results have already been worked out in [22]. Here we extend these results to the next accuracy level.

The large-$N_c$ operator expansion is performed in terms of a complete set of static and color-neutral one-body operators that act on effective baryon states rather than the physical states $|p, ij\pm, S, \chi\rangle$. In our case the physical and effective baryon states

$$|p, ij\pm, S, \chi\rangle, \quad |ij\pm, S, \chi\rangle,$$

are specified by the momentum $p$ and the flavour indices $i, j, k = 1, 2, 3$. The spin $S$ and the spin-polarization are $\chi = 1, 2$ for the spin one-half ($S = 1/2$) and $\chi = 1, \cdots, 4$ for the spin three-half states ($S = 3/2$). The flavour sextet and the anti-triplet are discriminated by their symmetric (index $+$) and anti-symmetric (index $-$) behaviour under the exchange of $i \leftrightarrow j$. At leading order in the $1/N_c$ expansion all considered baryon states are mass degenerate. The generic form of the operator expansion takes the form

$$\langle \bar{p}, mn\pm, \bar{S}, \bar{\chi} | O_{QCD} | p, kl\pm, S, \chi \rangle = \sum_{n=0}^{\infty} c_n(\bar{p}, p) \langle mn\pm, \bar{S}, \bar{\chi} | O_{\text{static}}^{(n)} | kl\pm, S, \chi \rangle. \quad (12)$$

It is important to note that unlike the physical baryon states, the effective baryon states do not depend on the momentum $p$. All dynamical information is moved into appropriate coefficient functions $c_n(\bar{p}, p)$. The contributions on the right-hand-side of (12) can be sorted according to their relevance at large $N_c$.

The effective baryon states $|ij\pm, \chi\rangle$ have a mean-field structure that can be generated in terms of effective quark operators $q$ and $Q$ for the light and heavy species respectively. A corresponding complete set of color-neutral one-body operators may be constructed in terms of the very same static quark operators

$$\begin{align*}
1 &= q^\dagger (1 \otimes 1 \otimes 1) q, \\
J_i &= q^\dagger \left( \frac{\sigma_i}{2} \otimes 1 \otimes 1 \right) q, \\
T^a &= q^\dagger \left( 1 \otimes \frac{\lambda_a}{2} \otimes 1 \right) q, \\
G_i^a &= q^\dagger \left( \frac{\sigma_i}{2} \otimes \frac{\lambda_a}{2} \otimes 1 \right) q, \\
1_Q &= Q^\dagger (1 \otimes 1) Q, \\
J^i_Q &= Q^\dagger \left( \frac{\sigma_i}{2} \otimes 1 \right) Q, \quad (13)
\end{align*}$$

with static operators $q = (u, d, s)^T$ and $Q = c$ of the up, down, strange and charm quarks. With $\lambda_a$ we denote the Gell-Mann matrices supplemented with a singlet matrix.
\( \lambda_0 = \sqrt{2/3} \). Here we use a redundant notation with

\[
T^0 = \sqrt{\frac{1}{6}} \mathbb{1}, \quad G_i^0 = \sqrt{\frac{1}{6}} J_i,
\]

which will turn useful when analyzing matrix elements of scalar currents.

In the sum of (12) there are infinitely many terms one may write down. The static operators \( \mathcal{O}_{\text{static}}^{(n)} \) are finite products of the one-body operators \( J_i, T^a \) and \( G_i^a \). Terms that break the heavy-quark spin symmetry are exclusively caused by the heavy-spin operator

\[
J_Q^i \sim \frac{1}{M_Q},
\]

with the heavy-quark mass \( M_Q \). In contrast the counting of \( N_c \) factors is intricate since there is a subtle balance of suppression and enhancement effects. An \( r \)-body operator consisting of the \( r \) products of any of the spin and flavour operators receives the suppression factor \( N_c^{-r} \). This is counteracted by enhancement factors for the flavour and spin-flavour operators \( T^a \) and \( G_i^a \) that are produced by taking baryon matrix elements at \( N_c \neq 3 \). Altogether this leads to the effective scaling laws \cite{25, 26}

\[
J_i \sim \frac{1}{N_c}, \quad T^a \sim N_c^0, \quad G_i^a \sim N_c^0.
\]  

According to (16) there is an infinite number of terms contributing at a given order in the \( 1/N_c \) expansion. Taking higher products of flavour and spin-flavour operators does not reduce the \( N_c \) scaling power. A systematic \( 1/N_c \) expansion is made possible by a set of operator identities \cite{21, 25, 26}, that allows a systematic summation of the infinite number of relevant terms. This can be summarized into two reduction rules:

- All operator products in which two flavour indices are contracted using \( \delta_{ab} \), \( f_{abc} \) or \( d_{abc} \) or two spin indices on \( G \)'s are contracted using \( \delta_{ij} \) or \( \varepsilon_{ijk} \) can be eliminated.

- All operator products in which two flavour indices are contracted using symmetric or antisymmetric combinations of two different \( d \) and/or \( f \) symbols can be eliminated.

The only exception to this rule is the antisymmetric combination \( f_{acg} d_{bch} - f_{bcg} d_{ach} \).

As a consequence the infinite tower of spin-flavour operators truncates at any given order in the \( 1/N_c \) expansion. We can now turn to the \( 1/N_c \) expansion of the baryon matrix elements of the QCD’s axial-vector and scalar currents. In application of the operator reduction rules, the baryon matrix elements of time-ordered products of the current operators are
expanded in powers of the effective one-body operators according to the counting rule \[15,16\] supplemented by the reduction rules. In contrast to Jenkins \[26\] we consider the ratio \(N_l/N_c = 1 - 1/N_c\) not as a suppression factor. The strength of the spin-symmetry breaking terms we estimate with \(1/M_Q \sim 1/N_c\). In the course of the construction of the various structures, parity and time-reversal transformation properties are taken into account.

All what is needed in any practical application of the \(1/N_c\) expansion is the action of any of the one-body operators introduced in \[13\] on the effective mean-filed type baryon states \(|ij, \chi\rangle\). In fact it suffices to provide results at the physical value \(N_c = 3\), for which a complete list was already generated in \[22\]. We exemplify such results with

\[
J^k_Q |ij, \frac{1}{2}, \chi\rangle = -\frac{1}{6} \sigma^{(k)}_{\chi\chi} |ij, \frac{1}{2}, \chi\rangle + \frac{1}{\sqrt{3}} S^{(k)}_{\chi\chi} |ij, \frac{3}{2}, \chi\rangle,
\]
\[
J^k_Q |ij, \frac{3}{2}, \chi\rangle = \frac{1}{2} (\bar{S} \sigma^{(k)} \bar{S}^\dagger)_{\chi\chi} |ij, \frac{3}{2}, \chi\rangle + \frac{1}{\sqrt{3}} \sigma^{(k)}_{\chi\chi} |ij, \frac{1}{2}, \chi\rangle,
\]
\[
J^k_Q |ij, \frac{1}{2}, \chi\rangle = \frac{1}{2} \sigma^{(k)}_{\chi\chi} |ij, -\frac{1}{2}, \chi\rangle,
\]
\[
J^k |ij, \frac{1}{2}, \chi\rangle = \frac{2}{3} \sigma^{(k)}_{\chi\chi} |ij, \frac{1}{2}, \chi\rangle - \frac{1}{\sqrt{3}} S^{(k)}_{\chi\chi} |ij, \frac{3}{2}, \chi\rangle,
\]
\[
J^k |ij, \frac{3}{2}, \chi\rangle = (\bar{S} \sigma^{(k)} \bar{S}^\dagger)_{\chi\chi} |ij, \frac{3}{2}, \chi\rangle - \frac{1}{\sqrt{3}} S^{(k)}_{\chi\chi} |ij, \frac{1}{2}, \chi\rangle,
\]
\[
J^k |ij, -\frac{1}{2}, \chi\rangle = 0,
\]

(17)

where we apply the spin matrices \(\sigma^{(k)}\) and \(S^{(k)}\) in the convention as used in \[22\]. Note that an error in the action of the heavy-spin operator \(J^k_Q\) on the \(|ij, \frac{1}{2}, \chi\rangle\) states is corrected here in \[17\]. We affirm that now with \[17\] the relations

\[
[J_i, J_j] = i \epsilon_{ijk} J_k, \quad [J^i_Q, J^j_Q] = i \epsilon_{ijk} J^k_Q, \quad \{J^i_Q, J^j_Q\} = \frac{3}{2} \mathbb{1}_Q,
\]
\[
[J^i_Q, J_j] = 0, \quad [J^i_Q, G^a_j] = 0, \quad [J^i_Q, T^a] = 0,
\]

(18)

hold if matrix elements in the charmed baryon states as introduced in \[11\] are taken. The latter were affected by the error made in \[22\]. Further corrections for matrix elements of the anti-commutator of two one-body operators are considered in the Appendix.
IV. TWO SCALAR CURRENTS IN CHARMED BARYON MATRIX ELEMENTS

We turn to a derivation of large-\(N_c\) sum rules for the chiral-symmetry breaking low-energy constants introduced in \([3]\). They contribute to the time-ordered product of two scalar currents as evaluated in the baryon states. At NLO in the \(1/N_c\) expansion we find the relevance of 11 operators

\[
\langle \bar{p}, \, mn_{\pm}, \, S, \, \bar{\chi} \, | \, S^{ab}(q) \, | \, p, \, kl_{\pm}, \, S, \, \chi \rangle = \langle \, mn_{\pm}, \, S, \, \bar{\chi} \, | \, O^{ab} \, | \, kl_{\pm}, \, S, \, \chi \rangle,
\]

\[
O^{ab} = \hat{c}_1 \delta_{a0} \delta_{b0} 1 + \hat{c}_2 \delta_{ab} 1 + \hat{c}_3 (T_a \delta_{b0} + \delta_{a0} T_b) + \hat{c}_4 d_{abe} T_e + \hat{c}_5 \{T_a, T_b\}
\]

\[
+ \hat{c}_6 d_{abe} \{J^i, G^i_a\} + \hat{c}_7 \left( \{J^i, G^i_a\} \delta_{b0} + \delta_{a0} \{J^i, G^i_b\} \right)
\]

\[
+ \hat{c}_8 \left( \{J^i, \{T_a, G^i_b\}\} + \{J^i, \{T_b, G^i_a\}\} \right)
\]

\[
+ \hat{c}_9 d_{abe} \{J^i_Q, G^i_a\} + \hat{c}_{10} \left( \{J^i_Q, G^i_a\} \delta_{b0} + \delta_{a0} \{J^i_Q, G^i_b\} \right)
\]

\[
+ \hat{c}_{11} \left( \{J^i_Q, \{T_a, G^i_b\}\} + \{J^i_Q, \{T_b, G^i_a\}\} \right) + \cdots,
\]

where only the first five operators that are required at LO were considered previously in \([22]\). We find six additional terms either involving the spin operators \(J_i\) or \(J^i_Q\). The three sums in (19) run over \(e = 1, \ldots, 8\).

The operator truncation (19) can be matched to the tree-level Lagrangian \([6]\). For this the matrix elements of the operators in (19) are derived in the Appendix and \([22]\). Altogether we claim the identifications as detailed in Tab. I.

At LO with \(\hat{c}_{6-11} = 0\) there are \(16 - 5 = 11\) sum-rules. No spin-symmetry breaking operator \(J^i_Q\) has to be considered at this accuracy level. In turn we recover the eight heavy-spin symmetry relations in (59) from \([22]\). Additional relations arise from the large-\(N_c\) considerations. We correct two misprints in (60) of \([22]\). Altogether the following set of sum rules arises

\[
c_{n,[66]} = e_{n,[66]} \quad \text{for} \quad n = 1, \ldots, 5,
\]

\[
c_{n,[36]} = 0 \quad \text{for} \quad n = 1, 2,
\]

\[
c_{1,[33]} = c_{1,[66]} + \frac{1}{2} c_{5,[66]}, \quad c_{2,[33]} = c_{2,[66]} - \frac{1}{2} c_{5,[66]},
\]

\[
c_{3,[33]} = c_{3,[66]} + 2 c_{5,[66]}, \quad c_{4,[33]} = c_{4,[66]} - 2 c_{5,[66]}.
\]

(20)
TABLE I. The symmetry breaking two-body counter terms as introduced in (6) are correlated by the large-$N_c$ operators (19).

At NLO all 11 operators in (15) turn relevant and we have $16 - 11 = 5$ sum rules

\[
c_1, [66] = \frac{1}{\sqrt{3}} \left( c_3, [66] - e_3, [66] \right), \qquad c_2, [66] = e_2, [66],
\]
\[
c_2, [66] = \frac{1}{\sqrt{3}} \left( c_4, [66] - e_4, [66] \right),
\]
\[
3 c_1, [33] + c_2, [33] + c_4, [33] = 3 c_1, [66] + c_2, [66] + c_4, [66] - c_5, [66],
\]
\[
3 e_1, [66] + e_4, [66] - e_5, [66] = 3 c_1, [66] + c_4, [66] - c_5, [66].
\]
V. TWO AXIAL CURRENTS IN CHARMED BARYON MATRIX ELEMENTS

We study the time-ordered product of two axial-vector currents. The large-$N_c$ operator expansion was already worked out in [21] at leading order. Matrix elements in charmed baryons were derived in [22]. Here we consider and derive the implication for the chiral two-body interactions introduced [8] at subleading orders in this expansion. At NLO there are 19 distinct operators to be considered

\[
\langle \bar{p}, mn_\pm, \bar{S}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_\pm, S, \chi \rangle = (mn_\pm, \bar{S}, \bar{\chi} | O_{ij}^{ab} | kl_\pm, S, \chi ) ,
\]

\[
O_{ij}^{ab} = -\delta_{ij} \left[ \hat{g}_1 \delta_{ab} \mathbb{1} + \hat{g}_2 d_{abc} T_c + \hat{g}_3 \{ T_a, T_b \} + \hat{g}_4 d_{abc} \{ J^k, G^k_c \} \right.
\]

\[
+ \hat{g}_5 \left( \{ J^k, \{ T_a, G^k_b \} \} + \{ J^k, \{ G^k_a, T_b \} \} \right) + \hat{g}_6 d_{abc} \{ J^k, G^k_c \}
\]

\[
+ \hat{g}_7 \left( J^k_1 \{ T_a, G^k_b \} + J^k_2 \{ G^k_a, T_b \} \right) \right] + \hat{g}_8 \left( \{ G^a_i, G^b_i \} + \{ G^a_i, G^b_i \} \right) + \hat{g}_9 \left( \{ G^a_i, G^b_i \} - \{ G^a_i, G^b_i \} \right) \]

\[
+ \hat{g}_{20} d_{abc} \left( \{ G^a_{ij}, J^k_c \} + \{ J^k_c, G^a_{ij} \} \right) + \hat{g}_{21} d_{abc} \left( \{ G^a_{ij}, J^k_c \} - \{ J^k_c, G^a_{ij} \} \right) \]

\[
+ \epsilon_{ijk} f_{abc} \left( \hat{g}_{10} G^k_c + \hat{g}_{11} \{ J^k_c, T_c \} + \hat{g}_{12} \{ J^k_Q, T_c \} \right) \]

\[
+ \hat{g}_{22} i f_{abc} \left( \{ J^k_c, G^a_{ij} \} - \{ G^a_{ij}, J^k_c \} \right) + \hat{g}_{23} i f_{abc} \left( \{ J^k_Q, G^a_{ij} \} - \{ G^a_{ij}, J^k_Q \} \right) \]

\[
+ \frac{1}{4} (\bar{p} + p)_i (\bar{p} + p)_j \left[ \hat{g}_{13} \delta_{ab} \mathbb{1} + \hat{g}_{14} d_{abc} T_c + \hat{g}_{15} \{ T_a, T_b \} \right.
\]

\[
+ \hat{g}_{16} d_{abc} \{ J^k_c, G^a_{ij} \} + \hat{g}_{17} \left( \{ J^k_c, \{ T_a, G^k_b \} \} \right) \left. + \{ J^k_c, \{ G^a_{ij}, T_b \} \} \right) \]

\[
+ \hat{g}_{18} d_{abc} \{ J^k_Q, G^a_{ij} \} + \hat{g}_{19} \left( \{ J^k_Q, \{ T_a, G^k_b \} \} \right) \left. + \{ J^k_Q, \{ G^a_{ij}, T_b \} \} \right) \right] + \cdots , \quad (22)
\]

where we focus on the space components of the correlation function. In [22] we have $q = \bar{p} - p$ and $a, b = 1, \cdots, 8$. In addition, we consider terms only that arise in the small-momentum expansion and that are required for the desired matching with [8]. The dots in (22) represent additional terms that are further suppressed in the $1/N_c$ expansion or for small 3-momenta $p$ and $\bar{p}$.

In the previous work [22] only seven leading order operators were considered. The ansatz of this work is reproduced with

\[
\hat{g}_1 = \frac{1}{3} \hat{g}_2 , \quad \hat{g}_{4-7} = 0 , \quad \hat{g}_{11-12} = 0 , \quad \hat{g}_{13} = \frac{1}{3} \hat{g}_{14} , \quad \hat{g}_{16-23} = 0 . \quad (23)
\]
TABLE II. The symmetry conserving two-body counter terms as introduced in (8) are correlated by the large-$N_c$ operators (22) with $\hat{g}_{n,[ab]}^{(V)} = 2 g_{n,[ab]}/(M_{[a]}^{(1/2)} + M_{[b]}^{(1/2)})$ and $\hat{h}_{n,[aa]}^{(V)} = h_{n,[aa]}^{(V)}/M_{[a]}^{(3/2)}$. We use also $\hat{g}_{2-5} = \hat{g}_2 + \hat{g}_3 + 2 \hat{g}_4 + 4 \hat{g}_5$. In the flavour SU(3) chiral limit we use here three distinct charm baryon masses $M_{[3]}^{(J=1/2)}$, $M_{[6]}^{(J=1/2)}$ and $M_{[6]}^{(J=3/2)}$. Only the combinations $g_{D,[33]}^{(V)} - 2 g_{1,[33]}^{(V)}$ and $g_{0,[33]}^{(V)} + g_{1,[33]}^{(V)}$ can be matched at leading order in a non-relativistic expansion.

An application of the results of our Appendix leads to the matching result as detailed in Tab. II.

From the operator analysis (22), we obtain $33 - 7 = 26$ sum rules. We do not reproduce all sum rules as considered first in (22). In our analysis we unravel two misprint in (62) of
As argued already in [22] this does not contradict the systematics of the large-
operator expansion. Though the operator analysis is not predicting such a feature, it can not exclude
confirm the result of [22] that the combination of the heavy spin-symmetry sum rules as
where the two identities in the fourth line of (24) were not presented correctly in [22]. We
f3
[22]. A complete list of relation reads
\[
\begin{align*}
\hat{g}_{D,[36]}^{(S)} &= h_{2,[66]}^{(S)} - h_{4,[66]}^{(S)} - h_{5,[66]}^{(S)}, &
\hat{g}_{0,[33]}^{(S)} &= \frac{1}{2} \left( h_{4,[66]}^{(S)} + h_{5,[66]}^{(S)} \right), \\
g_{0,[66]}^{(S)} &= g_{D,[36]}^{(S)} = 0, &
\hat{g}_{1,[66]}^{(S)} &= h_{4,[66]}^{(S)} + \frac{1}{3} h_{5,[66]}^{(S)}, &
\hat{g}_{D,[66]}^{(S)} &= h_{2,[66]}^{(S)}, \\
h_{0,[66]}^{(S)} &= h_{1,[66]}^{(S)} = h_{3,[66]}^{(S)} = 0, \\
\hat{g}_{0,[33]}^{(V)} &= -\hat{g}_{1,[33]}^{(V)} + \frac{1}{2} \hat{h}_{1,[66]}^{(V)}, &
\hat{g}_{D,[33]}^{(V)} &= 2 \hat{g}_{1,[33]}^{(V)} - \hat{h}_{1,[66]}^{(V)} + \hat{h}_{2,[66]}^{(V)}, \\
\hat{g}_{0,[66]}^{(V)} &= \hat{g}_{D,[36]}^{(V)} = \hat{h}_{0,[66]}^{(V)}, &
\hat{g}_{1,[66]}^{(V)} &= \hat{h}_{1,[66]}^{(V)}, &
\hat{g}_{D,[66]}^{(V)} &= \hat{h}_{2,[66]}^{(V)}, \\
n_{[1,36]}^{(A)} &= -\sqrt{3} \hat{g}_{F,[36]}^{(T)} + \frac{1}{2} \hat{h}_{5,[66]}^{(S)}, &
\hat{f}_{F,[36]}^{(A)} &= -2 \hat{h}_{F,[66]}^{(T)}, &
\hat{f}_{D,[36]}^{(A)} &= \hat{f}_{0,[66]}^{(A)} = \hat{f}_{D,[66]}^{(A)} = 0, \\
\hat{f}_{1,[36]}^{(A)} &= \frac{1}{2} \sqrt{3} \hat{h}_{5,[66]}^{(S)}, &
\hat{f}_{F,[36]}^{(A)} &= -\frac{2}{\sqrt{3}} \hat{h}_{F,[66]}^{(T)}, &
\hat{f}_{D,[36]}^{(A)} &= \hat{f}_{D,[36]}^{(A)} = 0, \\
\hat{g}_{F,[33]}^{(T)} &= 0, &
\hat{g}_{D,[36]}^{(T)} &= -\frac{1}{\sqrt{3}} \hat{h}_{F,[66]}^{(T)}, &
\hat{g}_{F,[36]}^{(T)} &= \frac{2}{3} \hat{h}_{F,[66]}^{(T)}, &
\hat{g}_{D,[36]}^{(T)} &= \frac{2}{3} \hat{h}_{F,[66]}^{(T)},
\end{align*}
\]

where the two identities in the fourth line of [22] were not presented correctly in [22]. We
confirm the result of [22] that the combination of the heavy spin-symmetry sum rules as
summarized in (41) of [22] with the large-$N_c$ sum rules [24] does lead to one extra relation

\[
h_{5,[66]}^{(S)} = 0.
\]  

As argued already in [22] this does not contradict the systematics of the large-$N_c$ operator
expansion. Though the operator analysis is not predicting such a feature, it can not exclude
it.

We turn to the central result of our work. At NLO, we have derived $33 - 23 = 10$ novel
sum rules

\[
\begin{align*}
\hat{g}_{D,[36]}^{(S)} &= \frac{1}{\sqrt{3}} \left( \hat{g}_{D,[66]}^{(S)} - \hat{h}_{2,[66]}^{(S)} + \frac{1}{2} \hat{h}_{1,[66]}^{(S)} \right), &
\hat{h}_{3,[66]}^{(S)} &= -\frac{3}{2} \hat{h}_{1,[66]}^{(S)}, \\
3 \hat{g}_{0,[66]}^{(V)} + 2 \hat{g}_{D,[66]}^{(V)} - \hat{g}_{1,[66]}^{(V)} &= 3 \hat{h}_{0,[66]}^{(V)} + 2 \hat{h}_{2,[66]}^{(V)} - \hat{h}_{1,[66]}^{(V)}, &
\hat{g}_{D,[36]}^{(V)} &= \frac{1}{\sqrt{3}} \left( \hat{g}_{D,[66]}^{(V)} - \hat{h}_{2,[66]}^{(V)} \right), \\
3 \hat{g}_{0,[66]}^{(S)} + 2 \hat{g}_{D,[66]}^{(S)} - \hat{g}_{1,[66]}^{(S)} &= 3 \hat{h}_{0,[66]}^{(S)} + 2 \hat{h}_{2,[66]}^{(S)} - \hat{h}_{1,[66]}^{(S)} + \frac{1}{3} \hat{h}_{5,[66]}^{(S)}, \\
\hat{f}_{1,[36]}^{(A)} &= -\sqrt{3} \hat{g}_{1,[36]}^{(T)}, &
\hat{f}_{1,[66]}^{(A)} &= \frac{1}{2} \sqrt{3} \hat{h}_{5,[66]}^{(S)}, &
\hat{f}_{D,[36]}^{(A)} &= \frac{3}{2} \hat{f}_{0,[66]}^{(A)}, \\
\hat{f}_{D,[36]}^{(A)} &= -\frac{1}{\sqrt{3}} \hat{f}_{D,[66]}^{(A)} - \frac{1}{2} \hat{h}_{1,[66]}^{(S)}, &
\hat{g}_{F,[33]}^{(T)} &= -\hat{g}_{F,[66]}^{(T)} + \frac{2}{3} \hat{h}_{F,[66]}^{(T)}, &
\hat{g}_{D,[36]}^{(T)} &= \frac{2}{3} \hat{h}_{F,[66]}^{(T)},
\end{align*}
\]

(26)
VI. SUMMARY

In this work we considered the chiral Lagrangian for charmed baryons based on the flavour SU(3) symmetry. Current-current correlation functions in the baryon states were evaluated at tree-level and analyzed in an expansion of $1/N_c$ at subleading orders. Sum rules for the symmetry breaking and symmetry conserving low-energy constants are systematically derived and presented. We correct various misprints found in previous works and establish novel sum rules that are valid at subleading orders in the $1/N_c$ expansion. Such parameter correlations are of crucial importance in chiral extrapolation studies of the charmed baryon masses at $N^3$LO, but also they determine to a large extent the coupled-channel interaction of the charmed baryons with the set of Goldstone bosons.

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APPENDIX

We consider matrix elements of the symmetric product of two one-body operators \( \mathcal{O} \) in the charmed baryon ground state at \( N_c = 3 \). The generic notation

\[
\langle \mathcal{O} \rangle_{SS}^{ab} = (mn_a, \bar{S}, \bar{\chi} | \mathcal{O} | kl_b, S, \chi ) ,
\]

with \( a, b = \pm \) will be applied. The results are expressed in terms of the flavour structures \( \Lambda^{(a), (rs)_\pm}_{(kl)} \) and \( \Lambda^{(a), (rs)_\pm}_{(kl)} \) and the spin structures \( \sigma_i \) and \( S_i \) as introduced in [22]. We correct and supplement the matrix elements that involve the spin operator \( J_Q \) with

\[
\langle \{ J^i_Q, J^j_Q \} \rangle_{++} = 0 , \quad \langle \{ J^i_Q, J^j_Q \} \rangle_{+} = 0 ,
\]

\[
\langle \{ J^i_Q, T^a \} \rangle_{++} = 0 , \quad \langle \{ J^i_Q, G^a_j \} \rangle_{++} = -\frac{1}{2} \delta_{ij} \delta_{\bar{\chi} \chi} \Lambda^{(a), (mn)_-}_{(kl)}
\]

\[
\langle \{ J^i_Q, T^a \} \rangle_{++} = -\frac{\sqrt{3}}{2} \delta_{\bar{\chi} \chi} \left( \Lambda^{(a), (mn)_+}_{(rs)_+} + \Lambda^{(b), (rs)_+}_{(kl)_+} + \Lambda^{(a), (rs)_-}_{(kl)_+} \right)
\]

\[
\langle \{ J^i_Q, J^j_Q \} \rangle_{++} = -\frac{2}{3} \delta_{ij} \delta_{\bar{\chi} \chi} \delta^{(mn)_+}_{(kl)_+} , \quad \langle \{ J^i_Q, J^j_Q \} \rangle_{+} = \frac{1}{2} \delta_{ij} \delta_{\bar{\chi} \chi} \Lambda^{(a), (mn)_+}_{(kl)_+}
\]

\[
\langle \{ J^i_Q, T^a \} \rangle_{++} = -\frac{1}{3} \delta_{\bar{\chi} \chi} \Lambda^{(a), (mn)_+}_{(kl)_+} , \quad \langle \{ J^i_Q, G^a_j \} \rangle_{++} = -\frac{1}{3} \delta_{ij} \delta_{\bar{\chi} \chi} \Lambda^{(a), (mn)_+}_{(kl)_+}
\]

\[
\langle \{ J^i_Q, T^a \} \rangle_{++} = -\delta_{\bar{\chi} \chi} \left( \Lambda^{(a), (mn)_+}_{(rs)_+} + \Lambda^{(b), (rs)_+}_{(kl)_+} + \Lambda^{(a), (rs)_-}_{(kl)_+} \right)
\]

\[
\langle \{ J^i_Q, J^j_Q \} \rangle_{++} = \frac{1}{\sqrt{3}} \left( 2 S_i \sigma_j - S_j \sigma_i \right) \delta^{(mn)_+}_{(kl)_+} , \quad \langle \{ J^i_Q, J^j_Q \} \rangle_{+} = 0 ,
\]

\[
\langle \{ J^i_Q, T^a \} \rangle_{++} = \frac{2}{\sqrt{3}} \delta_{\bar{\chi} \chi} \Lambda^{(a), (mn)_+}_{(kl)_+}
\]

\[
\langle \{ J^i_Q, G^a_j \} \rangle_{++} = \frac{1}{2 \sqrt{3}} \left( 2 S_i \sigma_j - S_j \sigma_i \right) \Lambda^{(a), (mn)_+}_{(kl)_+}
\]
\[ \langle \{ J^i_Q, \{ T^a, G^b_i \} \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = 0, \]
\[ \langle \{ J^i_Q, J^j \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = \left( \delta_{ij} - S_i S_j^\dagger - S_j S_i^\dagger \right) \delta^{(mn)\pm}_{(kl)\pm}, \quad \langle \{ J^i_Q, J^j_Q \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = \frac{1}{2} \delta_{ij} \delta_{\bar{\chi} \chi} \delta^{(mn)\pm}_{(kl)\pm}, \]
\[ \langle \{ J^i_Q, T^a \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = \left( S_s S^\dagger_i \right) \Lambda^{\pm}_{(kl)\pm}, \]
\[ \langle \{ J^i_Q, G^a_j \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = \frac{1}{2} \left( \delta_{ij} - S_i S_j^\dagger - S_j S_i^\dagger \right) \Lambda^{(a),(mn)\pm}_{(kl)\pm}, \]
\[ \langle \{ J^i_Q, \{ T^a, G^b_i \} \} \rangle_{\frac{3+}{2\frac{3}{2}}}^{++} = \frac{1}{2} \delta_{\bar{\chi} \chi} \left( \Lambda^{(a),(mn)\pm}_{(rs)\pm} \Lambda^{(b),(rs)\pm}_{(kl)\pm} + \Lambda^{(b),(mn)\pm}_{(rs)\pm} \Lambda^{(a),(rs)\pm}_{(kl)\pm} \right), \]  
(28)

where
\[ 2 S_i \sigma_j - S_j \sigma_i = \frac{1}{2} \left( S_i \sigma_j + S_j \sigma_i \right) - \frac{3}{2} i \epsilon_{ijk} S_k. \]  
(29)
[1] M. F. M. Lutz and E. Kolomeitsev, Nucl. Phys. A730, 110 (2004), arXiv:hep-ph/0307233 [hep-ph].

[2] J. Hofmann and M. F. M. Lutz, Nucl. Phys. A763, 90 (2005), arXiv:hep-ph/0507071 [hep-ph].

[3] J. Hofmann and M. F. M. Lutz, Nucl. Phys. A776, 17 (2006), arXiv:hep-ph/0601249 [hep-ph].

[4] C. Garcia-Recio, V. K. Magas, T. Mizutani, J. Nieves, A. Ramos, L. L. Salcedo, and L. Tolos, Phys. Rev. D79, 054004 (2009), arXiv:0807.2969 [hep-ph].

[5] C. E. Jimenez-Tejero, A. Ramos, and I. Vidana, Phys. Rev. C80, 055206 (2009), arXiv:0907.5316 [hep-ph].

[6] J.-X. Lu, Y. Zhou, H.-X. Chen, J.-J. Xie, and L.-S. Geng, Phys. Rev. D92, 014036 (2015), arXiv:1409.3133 [hep-ph].

[7] B. Long, Phys. Rev. D94, 011503 (2016), arXiv:1508.06084 [hep-ph].

[8] J.-X. Lu, H.-X. Chen, Z.-H. Guo, J. Nieves, J.-J. Xie, and L.-S. Geng, Phys. Rev. D93, 114028 (2016), arXiv:1603.05388 [hep-ph].

[9] L. Liu, H.-W. Lin, K. Orginos, and A. Walker-Loud, Phys. Rev. D81, 094505 (2010), arXiv:0909.3294 [hep-lat].

[10] G. Bali, S. Collins, and P. Perez-Rubio, J. Phys. Conf. Ser. 426, 012017 (2013), arXiv:1212.0565 [hep-lat].

[11] R. A. Briceno, H.-W. Lin, and D. R. Bolton, Phys. Rev. D86, 094504 (2012), arXiv:1207.3536 [hep-lat].

[12] C. Alexandrou, J. Carbonell, D. Christaras, V. Drach, M. Gravina, et al., Phys. Rev. D86, 114501 (2012), arXiv:1205.6856 [hep-lat].

[13] Y. Namekawa et al. (PACS-CS Collaboration), Phys. Rev. D87, 094512 (2013), arXiv:1301.4743 [hep-lat].

[14] P. Perez Rubio, PoS ConfinementX, 141 (2012), arXiv:1302.5774 [hep-lat].

[15] K. Nakamura et al. (Particle Data Group), J. Phys. G37, 075021 (2010).

[16] A. Semke and M. F. M. Lutz, Nucl. Phys. A778, 153 (2006), arXiv:nucl-th/0511061.

[17] A. Semke and M. F. M. Lutz, Phys. Rev. D85, 034001 (2012).
[18] A. Semke and M. F. M. Lutz, Phys. Lett. B717, 242 (2012), arXiv:1202.3556 [hep-ph].

[19] M. F. M. Lutz, R. Bavontaweepanya, C. Kobdaj, and K. Schwarz, Phys. Rev. D90, 054505 (2014), arXiv:1401.7805 [hep-lat].

[20] M. F. M. Lutz, Y. Heo, and X.-Y. Guo, (2018), arXiv:1801.06417 [hep-lat].

[21] M. F. M. Lutz and A. Semke, Phys. Rev. D83, 034008 (2011), arXiv:1012.4365 [hep-ph].

[22] M. F. M. Lutz, D. Samart, and Y. Yan, Phys. Rev. D90, 056006 (2014), arXiv:1402.6427 [hep-ph].

[23] M. F. M. Lutz, D. Samart, and A. Semke, Phys. Rev. D84, 096015 (2011), arXiv:1107.1324 [hep-ph].

[24] M. A. Luty and J. March-Russell, Nucl. Phys. B426, 71 (1994), hep-ph/9310369.

[25] R. F. Dashen, E. E. Jenkins, and A. V. Manohar, Phys. Rev. D49, 4713 (1994), hep-ph/9310379.

[26] E. E. Jenkins, Phys. Rev. D54, 4515 (1996), arXiv:hep-ph/9603449 [hep-ph].

[27] T.-M. Yan, H.-Y. Cheng, C.-Y. Cheung, G.-L. Lin, Y. Lin, et al., Phys. Rev. D46, 1148 (1992).

[28] P. L. Cho, Phys. Lett. B285, 145 (1992), arXiv:hep-ph/9203225 [hep-ph].

[29] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gatto, F. Feruglio, et al., Phys. Rept. 281, 145 (1997), arXiv:hep-ph/9605342 [hep-ph].

[30] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985).

[31] M. F. M. Lutz and E. E. Kolomeitsev, Nucl. Phys. A700, 193 (2002).
