Hamiltonian systems and semiclassical dynamics for interacting spins in QED

L. Amour and J. Nourrigat
Université de Reims, France

Abstract
In this article, we consider fixed spin $-1/2$ particles interacting through the quantized electromagnetic field in a constant magnetic field. We give approximate evolutions of coherent states. This uses spins-photon classical Hamiltonian mechanics. These approximations enable to derive that the approximate average fields and spins follow Maxwell-Bloch equations with a current density coming from spins. In addition, we obtain a law concerning the evolution of the approximate average number of photons. Next, we provide stationary points of the spins-photon Hamiltonian when the spin particles belong to an orthogonal plane to the constant magnetic field. This allows for the construction of quasimodes with spins colinear to the constant magnetic field. Finally, a quasimode with an arbitrary high accuracy is built up and its first order radiative correction is computed.

Keywords: Semiclassical analysis, spins interaction, quantum electrodynamics, evolution of coherent states, quasimodes, photons number, Maxwell equations, Bloch equations, spin-photon classical Hamiltonian, Ising model.

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Contents

1 The model. 2
2 Photons and spins Hamiltonian mechanics. 7
3 Approximate evolution of coherent states. 11
4 Applications. Photons number, fields and spins. 18
5 Approximate evolution laws 19
6 Fixed points and quasimodes. 22
1 The model.

The aim of this work is to initiate a semiclassical study for a Hamiltonian operator modelling the interaction between quantized electromagnetic field and \( N \) fixed spin-1/2 particles in a constant magnetic field. To achieve that goal, a Hamiltonian system is defined, playing the same role for this Hamiltonian operator as the usual system for the Schrödinger operator. This Hamiltonian is first recalled below in (14) and (15) (see Reuse [16], Hübner-Spohn [11], Dereziński-Gérard [8] Sections 3.2-3.3). The Hilbert space associated with this Hamiltonian is the completed tensor product \( \mathcal{H}_{ph} \otimes \mathcal{H}_{sp} \).

The Hilbert space \( \mathcal{H}_{ph} \) for photons may be viewed as the symmetrized Fock space \( \mathcal{F}_{s}(H_{\mathbb{C}}) \) associated with the complexified of some real Hilbert space \( H \) defined by Lieb-Loss [12]. This space \( H \) is the space of mappings \( f = (f_1, f_2, f_3) \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) with \( f_j \) belonging in \( L^2(\mathbb{R}^3) \), taking real values and satisfying,

\[
 k_1 f_1(k) + k_2 f_2(k) + k_3 f_3(k) = 0 \quad \text{a.e.}
\] (1)

This space is equipped with the norm,

\[
 |f|^2 = \sum_{j=1}^{3} \int_{\mathbb{R}^3} |f_j(k)|^2 dk.
\] (2)

The Fock space \( \mathcal{F}_{s}(H_{\mathbb{C}}) \) definition is reminded in Section 6. For each \( h > 0 \), the space \( \mathcal{H}_{ph} \) may also be viewed as a \( L^2(B, \mu_{B,h/2}) \) space where \( B \) is some Banach space related to \( H \) and containing \( H \), and \( \mu_{B,h/2} \) is some gaussian measure with variance \( h/2 \) on the Borel \( \sigma- \) algebra of \( B \). The hypotheses that should be fulfilled by \( B \) are recalled in [11] or [3]. When these assumptions are satisfied then \((i, H, B)\) is called Wiener space where \( i \) the injection from \( H \) into \( B \).

The space \( \mathcal{H}_{sp} \) for the spin particles is denoted by \((\mathbb{C}^2)^{\otimes N}\).

In the space \( \mathcal{H}_{ph} \), the definition of the model involves three kinds of operators: the number operator \( N \), the free photons energy operator \( H_{ph} \) and operators at each point \( x \in \mathbb{R}^3 \) associated with the three components of the magnetic field. These operators are denoted by \( B_m(x) \), \( 1 \leq m \leq 3 \), and for the electric field, it is denoted by \( E_m(x) \), \( 1 \leq m \leq 3 \). Each of these operators is depending on the semiclassical parameter \( h > 0 \) which is sometimes not explicitly written.

Within the Fock space formalism, the number operator \( N \) and the free photons Hamiltonian \( H_{ph} \) are defined by,

\[
 N = d\Gamma(I), \quad H_{ph} = hd\Gamma(M),
\] (3)
\( M \) being the multiplication operator by \( \omega(k) = |k| \) with domain \( D(M) \subset H \), \( d\Gamma \) is the standard operator (see \[13\] or \[91\] below) and \( h > 0 \) is the semiclassical parameter. These equalities classically define selfadjoint operators (see \[15\]).

Using Wiener spaces formalism, the Wick symbol (see \[1\][2][3]) of the number operator \( N \) is \((q, p) \mapsto \left( |q|^2 + |p|^2 \right) / 2h \). The Wick symbol of the operator \( H_\phi \) is

\[
H_\phi(q, p) = \frac{1}{2} \int_{\mathbb{R}^3} |k| \left[ |q(k)|^2 + |p(k)|^2 \right] dk. 
\]

With the notations of \[1\], the operators \( B_m(x) \) and \( E_m(x) \) are associated with functions on \( H^2 \), denoted by \((q, p) \mapsto B_m(x, q, p)\) and \((q, p) \mapsto E_m(x, q, p)\), being the Weyl and Wick symbols of these operators. These symbols are continuous linear forms on \( H^2 \) and are denoted by,

\[
a_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \sin(k \cdot x) \frac{k \wedge e_m}{|k|}, \\
b_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \cos(k \cdot x) \frac{k \wedge e_m}{|k|},
\]

where \( \chi \) is a function belonging to \( \mathcal{S}(\mathbb{R}) \) vanishing in a neighborhood of the origin and \( (e_1, e_2, e_3) \) is the canonical basis of \( \mathbb{R}^3 \). The fact that \( \chi \) vanishes near 0 is used to apply results of \[3\] only involved in Lemma 3.6 and it is also used in the construction of a quasimode in Section 6. The link between the symbol \( B_m(x, p, q) \) and the operator \( B_m(x) \) may be defined either by the Weyl calculus studied in \[1\] and \[2\] or with the Segal field formalism.

In \[2\], an operator denoted by \( Op_{weyl}^h(F) \) is associated with any continuous linear form \( F \) on \( H^2 \), for any \( h > 0 \) which, in its initial definition, maps some space \( D \) into itself and is thereafter extended to a bounded operator from the space \( W_1 \) defined in Section 3 into \( \mathcal{H}_\phi \). When setting

\[
F_{a,b}(q, p) = a \cdot q + b \cdot p,
\]

the operator \( Op_{weyl}^h(F) \) is usually called Segal field and is denoted by \( \Phi_S(a + ib) \) (see \[15\]). The point in \[1\] and \[2\] is to be able to define operators \( Op_{weyl}^h(F) \) for others functions \( F \) which are not linear, but this is only used in Lemma 3.6. With these notations, one has,

\[
B_m(x) = Op_{weyl}^h(F_{a_m(x), b_m(x)}) = \Phi_S(a_m(x) + ib_m(x)).
\]
In the following, the operator is denoted by $B_m(x)$ and its symbol is denoted by $B_m(x, \cdot)$. Concerning the symbol $E_m(x, q, p)$ of the operator $E_m(x)$, it is related to $B_m(x, q, p)$ through the helicity operator $J$ mapping $H^2$ into $H^2$ and defined by,

$$J(q, p)(k) = \left( \frac{k \wedge q(k)}{|k|}, \frac{k \wedge p(k)}{|k|} \right), \quad k \in \mathbb{R}^3 \setminus \{0\}. \quad (10)$$

The symbol $E_j(x, q, p)$ is defined by,

$$E_j(x, q, p) = -B_j(x, J(q, p)). \quad (11)$$

Operators in $H_{sp}$ use in particular Pauli matrices $\sigma_j$ ($1 \leq j \leq 3$),

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12)$$

For all $\lambda \leq N$ and for any $j \leq 3$, $\sigma_j^{[\lambda]}$ denotes the following operator in $H_{sp}$,

$$\sigma_j^{[\lambda]} = I \otimes \cdots \sigma_j \otimes \cdots I, \quad (13)$$

where $\sigma_j$ is located at the $\lambda^{th}$ position.

We assume that there are $N$ fixed spin$-1/2$ particles at points $x_\lambda$ in $\mathbb{R}^3$ ($1 \leq \lambda \leq N$). Denoting the constant magnetic field by $\beta = (\beta_1, \beta_2, \beta_3)$, the system constituted with these particles and the quantized magnetic field is governed by the operator in $H_{ph} \otimes H_{sp}$ defined by,

$$H(h) = H_0 + hH_{int}, \quad H_0 = H_{ph} \otimes I, \quad (14)$$

where

$$H_{int} = \sum_{\lambda=1}^{N} \sum_{j=1}^{3} (\beta_j + B_j(x_\lambda)) \otimes \sigma_j^{[\lambda]}. \quad (15)$$

It is recalled in [3] (Section 4) that it defines a selfadjoint operator with domain $D(H_{ph}) \otimes H_{sp}$.

In order to describe the same interaction in the semiclassical limit, a symplectic manifold is introduced in Section 2, namely $H^2 \times \Omega$, where $H$ is the space of Lieb-Loss and $\Omega$ is an orbit of the coadjoint representation of the group $SU(2^N)$. On this symplectic manifold, we introduce a Hamiltonian system modelling the interaction between photons and spins in the semiclassical regime. Let us mention a similarity with the Hamiltonian system of Bolte-Glaser [5] modelling the evolution of a particle whose spin interacts with a given magnetic field (semiclassical limit of Pauli equations). In Section 2, an existence and uniqueness result for weak and strong solutions to the Hamiltonian system is given.
A linear form \( \pi_a \) on the Lie algebra \( \mathcal{G} \) of the Lie group \( SU(2^N) \) is associated with each element \( a \) in \( \mathcal{H}_{sp} \) defined by,

\[
< \pi_a, Y > = i^{-1} < Y a, a >, \quad Y \in \mathcal{G}.
\]  

The unit sphere of \( \mathcal{H}_{sp} \) is thus a circle bundle above the coadjoint orbit \( \Omega \).

In (40), with each solution \((q(t), p(t), \omega(t))\) of the Hamiltonian system and with any \( a_0 \) such that \( \pi_{a_0} = \omega(0) \), it is associated a flow \( a(t) \) in \( \mathcal{H}_{sp} \) satisfying \( \pi_{a(t)} = \omega(t) \), for all \( t \).

The first application of this Hamiltonian system is the study of the approximation as \( h \) goes to 0 of evolutions of coherent states. Coherent states may be very simply defined as elements of \( F_{s}(H_{C}) \). For all \( X = (a, b) \) in \( H^{2} \) and for any \( h > 0 \), \( \Psi_{X,h} \) denotes the corresponding coherent state, element of \( F_{s}(H_{C}) \) defined by,

\[
\Psi(a,b),h = \sum_{n \geq 0} \frac{e^{-\frac{|a+b|^{2}}{(2h)^{n/2}}}}{n!} (a + ib) \otimes \cdots \otimes (a + ib).
\]  

If \( H_{ph} \) and \( L^{2}(B, \mu_{B,h/2}) \) are identified then the coherent state \( \Psi_{(a,b),h} \) is explicitly written in (45) below.

Let us fix \( X_{0} = (q_{0}, p_{0}) \) in \( H^{2} \) and \( a_{0} \) in the unit sphere of \( \mathcal{H}_{sp} \). Our goal is to give an approximation when \( h \) goes to 0 of

\[
u_{h}(t) = \exp^{-\frac{1}{h}H(h)}\left( \Psi_{X_{0},h} \otimes a_{0} \right).
\]  

**Theorem 1.1.** Let \( H(h) \) be the Hamiltonian defined in (14)-(15). Let \( X_{0} = (q_{0}, p_{0}) \) belongs to \( H^{2} \) and \( a_{0} \) in \( \mathcal{H}_{sp} \) with a unit norm. Let \((q(t), p(t), \omega(t)) = (X(t), \omega(t))\) be the weak solution to the Hamiltonian system (31)-(32)-(35) with initial data \((q_{0}, p_{0}, \pi_{a_{0}})\). Let \( a(t) \) be the solution to the differential system (40) with initial data \( a_{0} \). Set,

\[
\varphi(t) = \int_{0}^{t} \left[ \frac{1}{2} \left( p(s) \cdot q'(s) - q(s) \cdot p'(s) \right) - H(q(s), p(s), \omega(s)) \right] ds,
\]  

where \( H(q, p, \omega) \) is the Hamiltonian function defined in (30). Let \( u_{h}(t) \) be defined in (18) and

\[
v_{h}(t) = \exp^{\varphi(t)}\left( \Psi_{X(t),h} \otimes a(t) \right).
\]  

Then

\[
\|u_{h}(t) - v_{h}(t)\| \leq C(t)h^{1/2}.
\]  

The above norm is the one of \( \mathcal{H}_{ph} \otimes \mathcal{H}_{sp} \) and the above constant \( C(t) \) is bounded on every compact set of \( \mathbb{R} \).

This result is proved in Section 3 also using a Sobolev space norm being necessary for applications.
One may compare these results to those in finite dimension of Combescure Robert [7] (also see [17] and Hagedorn [10]).

There is here a different approach which is somewhat close to the one of Bolte-Glaser [5]. Even if the spin terms are not part of what one may consider as the principal symbol, these are taken into account in the Hamiltonian system. This enables the Hamiltonian to model the interaction between photons and matter and not only the free evolution of the photons.

We are next interested with the time evolution of observables. For example, with the initial data of Theorem 1.1, we consider the average number of photons at time $t$ given by $< N u_h(t), u_h(t) >$. One of the points of Theorem 1.1 (adapted with Sobolev spaces) is to approximate the average number of photons by $< N u_h(t), v_h(t) >$ with an error of $O(h^{1/2})$. This also holds true for the three components of the electric field and the magnetic field at each point $x \in \mathbb{R}^3$ and at any time $t$. This remains still valid for the three components of the spin (viewed as a vector in $\mathbb{R}^3$) of each particle at time $t$. The point of these approximations is that, these average values of these observables taken on coherent states can be explicitly computed. These computations enable to describe approximate laws for the evolution of the above observables. For the case of the magnetic field, one finds Maxwell equations with a divergence free current, without charge, modelling the spin (see (67)-(70)). For the case of spins viewed as vectors in $\mathbb{R}^3$, one recovers the equations of F. Bloch [4] (1946) (see 73). For the evolution of the average number of photons, one finds a time evolution law, may be new, that one may express as follows if there is a unique spin particle. If the photons of the coherent states are in a given circularly polarized state then then the derivative of the the average number of photons at time $t$ is equal to the scalar product of the spin at this time with the electric field at the point where the particle is located, multiplied by $+1$ or $-1$ according to the circular polarization direction (see (74)). It is clearly an approximate law, but one may expect to give more precise laws with quantum corrections in a next article. These points are detailed in Sections 4 and 5.

Next, we consider fixed points of the Hamiltonian system. We show that, if there are $N$ spin$-1/2$ particles then there are $2^N$ fixed points of the Hamiltonian system provided that all the particles are located in the same plane orthogonal to the constant magnetic field. Energy levels are given in (79)-(80). In order to give them a more physical meaning, the function $\chi$ in (6)(7) has to tend to 1, which is possible since the hypothesis on $\chi$ vanishing at the origin is useless here. Taking the limit $\chi$ goes to 1, while subtracting a common term going to infinity to all energy levels, one then finds for critical energy levels,

$$E = h|\beta| \sum_{\lambda=1}^{N} \varepsilon_{\lambda} + \frac{\hbar^2}{8\pi} \sum_{\lambda \neq \mu} \frac{\varepsilon_{\lambda} \varepsilon_{\mu}}{|x_\lambda - x_\mu|^3},$$

(22)
where \( \varepsilon_{\lambda} = \pm 1 \) depending if the spin \( \lambda \) of the considered critical point is orientated in one direction or the other. We still suppose that the points \( x_{\lambda} \) are in the same plane orthogonal to the constant field. 

One recognizes the Ising model.

Finally, we construct a quasimode, which may be have a connection with the ground state whose existence is proved in [11] or [8] (for massive photons) and [9] (for massless photons). See [13] for a construction of quasimodes with coherent states. The quasimode is constructed in Section 6 with the Fock formalism under the form,

\[
u_h \sim u_0 + h^{1/2}u_1 + hu_2 + \cdots \]

where the \( u_j \) are elements of \( H_{ph} \otimes H_{sp} \) and independent on \( h \) (if considered as elements of the Fock space). There are no hypotheses on the position of the particles for this quasimode. For all \( x \) in \( \mathbb{R}^3 \), one has, assuming that the constant magnetic filed is \( \beta = (0, 0, |\beta|) \),

\[
\langle (B_m(x) \otimes I)(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) \rangle = h(2\pi)^{-3} N \sum_{\lambda=1}^{N} \int_{\mathbb{R}^3} |\chi(k)|^2 \cos(k \cdot (x_{\lambda} - x)) \frac{(k \wedge e_m) \cdot (k \wedge e_3)}{|k|^2} \, dk.
\]

The corresponding electric field is zero. Denoting by \( B(x) \) the vector with the above three components, it is seen that,

\[
\text{div} B = 0, \quad \text{rot} B = -he_3 \wedge \text{grad} \Phi(x), \quad \Phi(x) = \sum_{\lambda=1}^{N} \rho(x - x_{\lambda}), \quad \rho(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(k)|^2 \cos(k \cdot x) \, dk.
\]

One notes the role of the function \( \chi \): the spin is not exactly a point-like particle. One recovers the stationary Maxwell equations in the first order approximation of the magnetic field associated with the quasimode. The approximate eigenvalue has an asymptotic expansion \( \lambda(h) \sim \lambda_1 h + \lambda_2 h^2 + \cdots \) and we have explicit formulas for the first two terms.

### 2 Photons and spins Hamiltonian mechanics.

Let us begin by defining the symplectic manifold of our system.

Let \( G = SU(2^N) \) be the group of unitary matrices in \( H_{sp} \). The Lie algebra \( \mathcal{G} \) of this group is the algebra of traceless antihermitian maps. For all \( a \) in the unit sphere of \( H_{sp} \), let \( \pi_a \) be the linear form on \( \mathcal{G} \) defined in [16]. Let \( \Omega \subset \mathcal{G}^* \) be the set of all linear forms \( f \) written as \( \pi_a \) with \( a \) in the unit sphere of \( H_{sp} \). When
\( a \) is any element of this sphere, one notes that \( \Omega \) is the orbit \( \pi_a \) of the coadjoint representation, that is to say, the set of linear forms on \( G \) written as,
\[
G \ni Y \mapsto i^{-1} \langle Yga, ga \rangle,
\]
(26)
with \( g \in G \). It is known that \( \Omega \) is a symplectic manifold. For all \( Y \) in \( G \), let \( \varphi_Y \) be the function on \( \Omega \) defined by \( \varphi_Y(\omega) = \langle \omega, Y \rangle \). Then, the Poisson bracket of \( \varphi_Y \) and \( \varphi_Z \) in \( G \) satisfies,
\[
\{ \varphi_Y, \varphi_Z \} = h^{-1} \varphi_{[Y,Z]}.
\]
(27)

When \( H \) is the Hilbert space of Lieb-Loss defined in Section 1, the space \( H^2 \) is equipped with the symplectic form defined by,
\[
\sigma((q,p),(q',p')) = (q',p) - (q,p').
\]
(28)
The symplectic manifold of our system is,
\[
V = H^2 \times \Omega.
\]
(29)
The Hamiltonian function \( H_{ph}(q,p) \) for photons used here is defined in (4). The functions \( B_m(x,q,p) \) and the operators \( \sigma_m^{(\lambda)} \) are defined in Section 1. We denote by \( \beta = (\beta_1, \beta_2, \beta_3) \) the constant magnetic field.

The Hamiltonian function for photons in interaction with \( N \) spin-1/2 particles fixed at the points \( x_\lambda \) (\( 1 \leq \lambda \leq N \)) is,
\[
H(q,p,\omega) = H_{ph}(q,p) + h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} (\beta_m + B_m(x_\lambda, q, p)) \langle \omega, i\sigma_m^{(\lambda)} \rangle >, \quad (q,p) \in H^2, \quad \omega \in \Omega.
\]
(30)
The above bracket denotes the duality between \( G' \) and \( G \).

Consequently, a solution \((q(t), p(t), \omega(t))\) to the Hamiltonian system should satisfy,
\[
\frac{dq(t)}{dt} = Mp(t) + h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} b_m(x_\lambda) \langle \omega(t), i\sigma_m^{(\lambda)} \rangle > \]
(31)
\[
\frac{dp(t)}{dt} = -Mq(t) - h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} a_m(x_\lambda) \langle \omega(t), i\sigma_m^{(\lambda)} \rangle >.
\]
(32)
where \( a_m(x) \) and \( b_m(x) \) are defined in (10) and (7). Here \( M \) denotes the multiplication operator by \( |k| \) with domain \( D(M) \subset H \). For the third Hamilton equation, one defines for all \((q,p)\) in \( H^2 \) an operator
The element $iT(q,p)$ belongs to $\mathcal{G}$. Thus, the Hamiltonian function is written as,

$$H(q,p,\omega) = H_{ph}(q,p) + h < \omega, iT(q,p) >, \quad (q,p) \in H^2, \quad \omega \in \Omega. \quad (34)$$

The third Hamilton equation is then written as, for all $Y \in \mathcal{G}$, with (27),

$$\frac{d}{dt} < \omega(t), Y > = \frac{d}{dt} \phi_Y(\omega(t)) = \{H, \phi_Y\}(\omega(t)) = \{h\phi_{iT(q(t),p(t))}, \phi_Y\}(\omega(t))$$

$$= \phi_{i[T(q(t),p(t)),Y]}(\omega(t)).$$

In other words, one should have, for all $Y$ in $\mathcal{G}$,

$$\frac{d}{dt} < \omega(t), Y > = \sum_{N=1}^{N} \sum_{m=1}^{3} (\beta_m + B_m(x_\lambda, q(t), p(t))) < \omega(t), [i\sigma^{|\lambda|}_m, Y] >. \quad (35)$$

Set, for all $(q,p)$ in $H^2$ and for all $t \in \mathbb{R}$,

$$\chi_t(q,p) = (q_t,p_t), \quad \begin{cases} q_t(k) = \cos(t|k|)q(k) + \sin(t|k|)p(k) \\ p_t(k) = -\sin(t|k|)q(k) + \cos(t|k|)p(k) \end{cases}. \quad (36)$$

We say that a function $t \mapsto (q(t), p(t), \omega(t)) = (X(t), \omega(t))$, continuous from $\mathbb{R}$ into $H \times H \times \mathcal{G}^*$, is a weak solution to the Hamiltonian system if,

$$X(t) = \chi_t(X(0)) + \int_0^t \chi_{t-s}(Q(s), P(s)) ds, \quad (34)$$

$$Q(s) = h \sum_{N=1}^{N} \sum_{m=1}^{3} b_m(x_\lambda) < \omega(s), i\sigma^{|\lambda|}_m >, \quad P(s) = -h \sum_{N=1}^{N} \sum_{m=1}^{3} a_m(x_\lambda) < \omega(s), i\sigma^{|\lambda|}_m >$$

and if, for all $Y$ in $\mathcal{G}$,

$$< \omega(t), Y > = < \omega(0), Y > + \sum_{N=1}^{N} \sum_{m=1}^{3} \int_0^t (\beta_m + B_m(x_\lambda, q(s), p(s))) < \omega(s), [i\sigma^{|\lambda|}_m, Y] > ds.$$

We call a strong solution, a map $t \mapsto (q(t), p(t), \omega(t))$ continuous from $\mathbb{R}$ to $D(M) \times D(M) \times \mathcal{G}^*$, $C^1$ from $\mathbb{R}$ to $H \times H \times \mathcal{G}^*$, satisfying (31) (32) and (35).

The set $\mathcal{G}$ is equipped with the scalar product,

$$X \cdot Y = -\text{Tr}(XY), \quad X \in \mathcal{G}, \quad Y \in \mathcal{G} \quad (37)$$

and the dual $\mathcal{G}^*$ is endowed with the associated Euclidean structure.
Theorem 2.1. For all \((q_0, p_0, \omega_0)\) in \(H \times H \times G^*\), there exists a unique weak solution \((q(t), p(t), \omega(t))\) on \(\mathbb{R}\) to the system, satisfying \(q(0) = q_0\), \(p(0) = p_0\) and \(\omega(0) = \omega_0\). One has,

\[
|\omega(t)| = |\omega(0)|, \quad |X(t)| \leq |X(0)| + C|t||\omega(0)|. \tag{38}
\]

If \(q_0\) and \(p_0\) belong to \(D(M)\) then this solution is a strong solution.

Proof. Since the \(\chi_t\) is a continuous group according to [18] (also see [14]) then for any \((q_0, p_0, \omega_0)\) in \(H \times H \times G^*\) there is a local in time and unique weak solution on some interval \([0, T)\). Let us show that (38) holds on \([0, T)\). For each \(Y\) in \(G\), we denote by \(\text{ad}^* Y\) the adjoint mapping of \(X \mapsto \text{ad} Y (X) = [Y, X]\). Equality (34) is written as,

\[
\omega'(t) = \text{ad}^* (i^{-1}T(q(t), p(t)))\omega(t).
\]

Since the mapping \(\text{ad}(i^{-1}T(q, p))\) is antiselfadjoint, this is also true for \(\text{ad}^*(i^{-1}T(q, p))\). Besides, the mapping \(\chi_t\) is unitary. Inequality (38) then follows. According to Pazy [14] (Corollary 2.3) or [6], one concludes that \(T = +\infty\). If \(q_0\) and \(p_0\) belongs to \(D(M)\) then the solution is strong by [6].

\[\square\]

Proposition 2.2. If \((q_1(t), p_1(t), \omega_1(t))\) and \((q_2(t), p_2(t), \omega_2(t))\) are two weak solutions then one has,

\[
|X_1(t) - X_2(t)| + |\omega_1(t) - \omega_2(t)| \leq A \left[|X_1(0) - X_2(0)| + |\omega_1(0) - \omega_2(0)|\right] e^{B|t|}, \tag{39}
\]

where \(A\) is a constant and \(B\) a function of \(|\omega_1(0)| + |\omega_2(0)|\).

Proof. One first observes that,

\[
|X_1(t) - X_2(t)| \leq |X_1(0) - X_2(0)| + A \int_0^t |\omega_1(s) - \omega_2(s)| \, ds
\]

\[
|\omega_1(t) - \omega_2(t)| \leq |\omega_1(0) - \omega_2(0)| + C \int_0^t |X_1(s) - X_2(s)| \, ds,
\]

where \(C\) is a function of \(|\omega_1(0)| + |\omega_2(0)|\). Inequality (39) then follows by Gronwall Lemma.

\[\square\]

Proposition 2.3. Let \((q(t), p(t), \omega(t))\) be a weak solution to the system. Then \(\omega(t)\) lies in the orbit of \(\omega(0)\) of the coadjoint representation.

Proof. There exists a function \(g(t)\) taking values in \(SU(\mathcal{H}_{sp})\) satisfying,

\[
g'(t) = iT(q(t), p(t))g(t), \quad g(0) = I,
\]

10
where $T(q, p)$ is defined in (33). One sees that, for all $Y$ in $G$,

$$< \omega(t), Y > = < \omega(0), g(t)^* Y g(t) > .$$

Consequently, $\omega(t)$ belongs to the coadjoint orbit of $\omega(0)$.

□

Since the solution $(q(t), p(t), \omega(t))$, a priori belonging to $H^2 \times G^*$, remains in our symplectic manifold then system (31) (32) (35) is now called Hamiltonian system. We associate a differential equation in $H_{sp}$ with any weak solution $(q(t), p(t), \omega(t))$ to the Hamiltonian system,

$$a'(t) = -iT(q(t), p(t)) a(t) + i < T(q(t), p(t)) a(t), a(t) > \frac{a(t)}{|a(t)|^2}.$$  

(40)

When $a(0)$ belongs to the unit sphere of $H_{sp}$, one sees that $a(t)$ remains in that sphere and if $\pi_a(0) = \omega(0)$, one has $\pi_a(t) = \omega(t)$, for all $t$.

3 Approximate evolution of coherent states.

We shall derive Theorem 1.1. Note that in view of applications, the norm of $u_h(t) - v_h(t)$ has to be estimated, where $u_h(t)$ and $v_h(t)$ are defined in (18) and (20), not only in the $H_{ph} \otimes H_{sp}$ norm, but also in a Sobolev spaces norm related to the number operator $N$.

We denote by $W_m$ the domain of the operator $N^{m/2}$, endowed with the norm,

$$||u||_{W_m}^2 = < (I + 2hN)^m u, u > .$$  

(41)

We shall prove the result below which is stronger than Theorem 1.1.

**Theorem 3.1.** Let $H(h)$ be the Hamiltonian defined in (14) (15). Set $X_0 = (q_0, p_0)$ in $H^2$. Let $a_0$ belongs to $H_{sp}$ with norm 1. Let $(q(t), p(t), \omega(t)) = (X(t), \omega(t))$ be the weak solution to the Hamiltonian system (31) (32) (35) with initial data $(q_0, p_0, \pi_{a_0})$. Let $a(t)$ be the solution to the differential system (40) with initial data $a_0$. Let $u_h(t)$ and $v_h(t)$ be the elements defined in (18) and (20).

Then

$$||u_h(t) - v_h(t)||_{W_2 \otimes H_{sp}} \leq C(t)h^{1/2}.$$  

(42)

The above constant $C(t)$ is bounded on all compact sets of $\mathbb{R}$.
The proof uses the five following Lemmas.

**Lemma 3.2.** Let \( t \mapsto X(t) = (q(t), p(t)) \) be a \( C^1 \) map from \( \mathbb{R} \) to \( H^2 \). Then, the function \( t \mapsto \Psi_{q(t), p(t), h} \) is \( C^1 \) from \( \mathbb{R} \) into \( \mathcal{H}_{ph} \) and using notation (8),
\[
\hbar \frac{\partial}{\partial t} \Psi_{q(t), p(t), h} = \text{Op}_\text{weyl} h (F_{q'(t), p'(t)}) \Psi_{q(t), p(t), h} + \gamma(t) \Psi_{q(t), p(t), h},
\]
with
\[
\gamma(t) = - \left( q(t) \cdot q'(t) + p(t) \cdot p'(t) \right) + \frac{i}{2} \left( q(t) \cdot p'(t) - p(t) \cdot q'(t) \right).
\]

**Proof.** The \( C^1 \) property comes from [8] (Proposition 2.4 (iii)). See also Theorem X.41 (d) of [15]. We identify \( H_{ph} \) with \( L^2(B, \mu_{B,h}) \). In [1] or in [2] (Section 2.2), a function \( \ell_a \) defined almost everywhere on \( B \) is associated with any element \( a \) in the complexified space of \( H \), satisfying for all \((q, p)\) in \( H^2 \),
\[
\Psi_{q,p,h}(u) = e^{\frac{1}{\hbar} \ell_{(q+ip)}(u)} - \frac{i}{2} q |q|^2 - \frac{i}{2} q p,
\]
and such that, for all \( A \) and \( B \) in \( H \), one has with the notation [8], (11), formula (137),
\[
(\text{Op}_\text{weyl} h F_{A,B}) = \ell_{A+iB}(u) + \frac{\hbar}{i} B \cdot \frac{\partial}{\partial u},
\]
where \( \ell_{A+iB}(u) \) stands for the multiplication by the function \( \ell_{A+iB}(u) \). The derivation a priori has a meaning only on cylindrical functions but one may extend the equality and shows that,
\[
(\text{Op}_\text{weyl} h F_{A,B}) \Psi_{q,p,h}(u) = \left[ \ell_{A+iB}(u) + \frac{1}{i} B \cdot (q + ip) \right] \Psi_{q,p,h}(u).
\]

The Lemma then follows. \( \square \)

Let us recall ([15]) that, for all \( X \) in \( H^2 \), there exists a unitary transform \( V(X) \) (depending on \( h \)) satisfying,
\[
\Psi_{X,h} = V(X) \Psi_{0,h}
\]
and such that, for all continuous linear form \( L \) on \( H^2 \),
\[
V(X)^* \left( \text{Op}_h \text{weyl}(L) - L(X) \right) V(X) = \text{Op}_h \text{weyl}(L)
\]
and verifying, for all \( X = (q, p) \) in \( D(M)^2 \),
\[
V(X)^* H_{ph} V(X) = H_{ph} + \text{Op}_h \text{weyl}(F_{M,X}) + H_{ph}(X).
\]
Lemma 3.3. Let $M : H \to H$ be the multiplication operator by $\omega(k) = |k|$ with $D(M) \subset H$. Then one has,

$$H_{ph}\Psi_{X,h} = Op^w_{ph}(F_{MX})\Psi_{X,h} - H_{ph}(X)\Psi_{X,h}. \quad (51)$$

In addition, for all $(q, p)$ in $D(M)^2$,

$$[H_{ph}, Op^w_{h}(F_{q,p})] = i\hbar Op^w_{h}(F_{-M_p,M_q}). \quad (52)$$

Proof. Equality (51) follows from (48)(50) and (49) applied to $L = F_{MX}$ together with the fact that $H_{ph}\Psi_{0,h} = 0$ and $F_{MX}(X) = 2H_{ph}(X)$. Equality (52) is standard.

Lemma 3.4. Let $F$ be a continuous linear form on $H^2$ satisfying $F(-p, q) = iF(q, p)$ for all $(q, p)$ in $H^2$. Then, one has for all $X$ in $H^2$,

$$Op^w_{h}(F)\Psi_{X,h} = F(X)\Psi_{X,h}. \quad (53)$$

Proof. Let $V(X)$ be the unitary operator satisfying (48) and (49). In particular,

$$Op^w_{h}(F)\Psi_{X,h} = F(X)\Psi_{X,h} + V(X)Op^w_{h}(F)\Psi_{0,h}. \quad (54)$$

Under the hypothesis of the Lemma, one has $Op^w_{h}(F)\Psi_{0,h} = 0$ since $Op^w_{h}(F)$ is an annihilation operator.

Lemma 3.5. For any $(a, b)$ and each $X$ in $H^2$, one has,

$$\|\left( Op^w_{h}(F_{a,b}) - F_{a,b}(X) \right)\Psi_{X,h} \|_{W_2} \leq C(1 + |X|)^2h^{1/2}. \quad (55)$$

Proof. One uses the operator $V(X)$ satisfying (48) and (49). It satisfies,

$$Op^w_{h}(F)\Psi_{X,h} = F(X)\Psi_{X,h} + V(X)Op^w_{h}(F)\Psi_{0,h}. \quad (56)$$

One has, for all $u$ in $W_2$

$$\|u\|_{X_2} \leq \|u\| + 2\hbar\|Nu\|.$$ 

It is sufficient to show that,

$$2\hbar\|NV(X)Op^w_{h}(F)\Psi_{0,h}\| \leq C(1 + |X|)^2h^{1/2}. \quad (57)$$
One has,
\[2\hbar N = \sum_k \Op_{h}^{\text{weyl}}(A_{2,k})\Op_{h}^{\text{weyl}}(A_{1,k})\]
with
\[A_{1,k}(q, p) = \langle q + ip, e_k \rangle, \quad A_{2,k}(q, p) = \langle e_k, q + ip \rangle.\]
One applies (49) with \(L = A_{1,k}\) and then with \(L = A_{2,k}\) to estimate the left hand side of (57). One gets,
\[2\hbar N V(X)\Op_{h}^{\text{weyl}}(F) = \sum_k A_{1,k}(X)A_{2,k}(X)V(X)\Op_{h}^{\text{weyl}}(F) + A_{1,k}(X)V(X)\Op_{h}^{\text{weyl}}(A_{2,k})\Op_{h}^{\text{weyl}}(F)\]
\[+ A_{2,k}(X)V(X)\Op_{h}^{\text{weyl}}(A_{1,k})\Op_{h}^{\text{weyl}}(F) + V(X)\Op_{h}^{\text{weyl}}(A_{2,k})\Op_{h}^{\text{weyl}}(A_{1,k})\Op_{h}(F).\]
Since \(\Op_{h}^{\text{weyl}}(A_{1,k})\) is an annihilation operator, one has
\[\Op_{h}^{\text{weyl}}(A_{1,k})\Op_{h}^{\text{weyl}}(F)\Psi_{0,h} = \left[\Op_{h}^{\text{weyl}}(A_{1,k}), \Op_{h}^{\text{weyl}}(F)\right]\Psi_{0,h} = \hbar i \{A_{1,k}, F\}\Psi_{0,h}.\]
One sees \(\sum_k A_{1,k}(X)A_{2,k}(X) = |X|^2/(2\hbar)\). Set
\[G_X(q, p) = \sum_k \left(A_{1,k}(X)A_{2,k}(q, p) + A_{2,k}(X)A_{1,k}(q, p)\right),\]
\[H_X(q, p) = \frac{1}{\hbar} \sum_k \{A_{1,k}, F\} A_{2,k}(q, p).\]
One has,
\[2\hbar N V(X)\Op_{h}^{\text{weyl}}(F)\Psi_{0,h} = |X|^2V(X)\Op_{h}^{\text{weyl}}(F)\Psi_{0,h} + V(X)\Op_{h}^{\text{weyl}}(G_X)\Op_{h}^{\text{weyl}}(F)\Psi_{0,h}\]
\[+ hV(X)\Op_{h}^{\text{weyl}}(H_X)\Psi_{0,h}.\]
One classically has,
\[\|\Op_{h}^{\text{weyl}}(F)\Psi_{0,h}\| \leq \sqrt{2}|F|h^{1/2},\]
\[\|\Op_{h}^{\text{weyl}}(G_X)\Op_{h}^{\text{weyl}}(F)\Psi_{0,h}\| \leq 4h|G_X||F|.\]
Consequently,
\[2\hbar \|NV(X)\Op_{h}^{\text{weyl}}(F)\Psi_{0,h}\| \leq |X|^2\sqrt{2}|F|h^{1/2} + 4h|G_X||F| + \sqrt{2}h^{3/2}|H_X|.\]
One then deduces (57) and then the Lemma follows. □
Lemma 3.6. There exists $C > 0$ such that, for all $f$ in $W_1$ and for any $(a, b)$ in $H^2$,
\[ \|O_p^{\text{weyl}}(F_{a,b})f\| \leq C(|a| + |b|)\|f\|_{W_1}. \] (58)

One has for all $f$ in $W_m \otimes H_{sp}$ and for any $t \in \mathbb{R}$,
\[ \|e^{-\frac{i}{\hbar}H(t)}f\|_{W_m \otimes H_{sp}} \leq C(t)\|f\|_{W_m \otimes H_{sp}}. \] (59)

**Proof.** The first inequality is standard. For the second one, one uses the following factorization of \[ (Theorem 1.1), \]
\[ e^{-\frac{i}{\hbar}H(t)} = e^{-i\frac{t}{\hbar}H_0}U_{\text{red}}^{\text{red}}(t), \]
where $H_0 = H_{ph} \otimes I$ and $H_{ph}$ is the free photons Hamiltonian operator. Since $H_{ph}$ commutes with the number operator, one gets for all $f$ in $W_m$,
\[ \|e^{-\frac{i}{\hbar}H_0}f\|_{W_m \otimes H_{sp}} \leq C(t)\|f\|_{W_m \otimes H_{sp}}. \]

One now uses the Weyl calculus developed in \[ \text{[1]} \] and \[ \text{[2]}. \] It is shown in \[ \text{[3]} \] that $U_{\text{red}}^{\text{red}}(t)$ is associated by this Weyl calculus with a function $H^2 \ni (q, p) \mapsto U(t, q, p, h)$ taking matrices values in $\mathcal{L}(H_{sp})$. As a function of $(q, p)$, it belongs to the space $S_{\text{mat}}^\infty(\mathcal{B}, |t|\varepsilon(t))$ introduced in Definition 3.1 of \[ \text{[3]} \] associated with a Hilbertian basis $\mathcal{B}$ of $H$ constructed in Section 2.2 of \[ \text{[3]} \] and with a sequence $\varepsilon(t)$ defined in Proposition 5.1 of \[ \text{[3]} \]. In addition, this function is bounded in this space (in the sense of Definition 3.1) as $h$ is running over $(0, 1)$ and $t$ in compact sets of $\mathbb{R}$. According to Proposition 2.8 of \[ \text{[2]} \], these properties imply that,
\[ \|U_{\text{red}}^{\text{red}}(t)f\|_{W_m \otimes H_{sp}} \leq C(t)\|f\|_{W_m \otimes H_{sp}}, \]
where the constant $C(t)$ is bounded on all compact sets of $\mathbb{R}$. Proposition 2.8 actually proves the analog of this inequality for the space $W_1$ but the proof for $W_m$ is identical. Inequality (59) then follows.

\[ \square \]

**End of the proof of Theorem 3.1.** Let $(q(t), p(t), \omega(t))$ be a strong solution to the Hamiltonian system \[ \text{[31]} \text{[32]} \text{[33]}, \] with $(q(0), p(0), \omega(0))$ in $D(M) \times D(M) \times \Omega$. Set $z_h(t)$ the function taking values in $H_{ph}$ defined by,
\[ z_h(t) = \Psi_{q(t),p(t),h}. \] (60)

From Lemma 3.2 one may write,
\[ \hbar \frac{\partial z_h(t)}{\partial t} = O_p^{\text{weyl}}(F_{q'(t),p'(t)})z_h(t) + \gamma(t)z_h(t) \]
where $\gamma(t)$ is defined by (34). From Lemma 3.3 one sees,

$$H_{ph} z_h(t) = Op_h^{wep}(F_{Mq(t),Mp(t)}) z_h(t) - H_{ph}(q(t),p(t)) z_h(t).$$

From (31) and (32),

$$F_{-p'(t),q'(t)} = F_{Mq(t),Mp(t)} + \hbar \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \frac{B_m(x_\lambda,\cdot)}{\cdot} \omega(t), i\sigma_m^{[\lambda]} > .$$

The function $G = F_{q'(t),p'(t)} + iF_{-p'(t),q'(t)}$ satisfies the hypothesis of Lemma 3.4. Therefore,

$$Op_h^{wep}(G) z_h(t) = G(q(t),p(t)) z_h(t).$$

In consequence,

$$\hbar \frac{\partial z_h(t)}{\partial t} + i H_{ph} z_h(t) = -i \hbar \sum_{\lambda=1}^{N} \sum_{m=1}^{3} < \omega(t), i\sigma_m^{[\lambda]} > B_m(x_\lambda,\cdot) z_h(t) + i \lambda(t) z_h(t)$$

with, according to the notation (3),

$$\lambda(t) = (t) + F_{q'(t),p'(t)}(q(t),p(t)) + i F_{-p'(t),q'(t)}(q(t),p(t)) - i H_{ph}(q(t),p(t)).$$

From (41), one obtains,

$$\lambda(t) = \frac{1}{2}(p(t) \cdot q'(t) - q(t) \cdot p'(t)) - H_{ph}(q(t),p(t)).$$

Set $w_h(t) = e^{i\varphi(t)} z_h(t)$ where $\varphi(t)$ is defined in (10) and $z_h(t)$ is defined in (30). One gets from (10),

$$\varphi'(t) = \frac{1}{2}(p(t) \cdot q'(t) - q(t) \cdot p'(t)) - H(q(t),p(t),\omega(t))$$

where $H(q,p,\omega)$ is the total Hamiltonian function defined in (30). Consequently,

$$\hbar \frac{\partial w_h(t)}{\partial t} + i H_{ph} w_h(t) = -i \hbar \sum_{\lambda=1}^{N} \sum_{m=1}^{3} < \omega(t), i\sigma_m^{[\lambda]} > B_m(x_\lambda,\cdot) w_h(t) + i \mu(t) w_h(t) \quad (61)$$

$$\mu(t) = p(t) \cdot q'(t) - q(t) \cdot p'(t) - H_{ph}(q(t),p(t)) - H(q(t),p(t),\omega(t)). \quad (62)$$

From (41), (30), (31) and (32),

$$\mu(t) = -\hbar \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \beta_m < \omega(t), i\sigma_m^{[\lambda]} > . \quad (63)$$

Let $a(t)$ be the solution to (30) with initial data $a_0$. Let

$$v_h(t) = w_h(t) \otimes a(t) = e^{i\varphi(t)} \Psi_{q(t),p(t),h} \otimes a(t).$$
Since \( \pi_{a_0} = \omega(0) \), and then \( \pi_{a(t)} = \omega(t) \), one sees from (10),
\[
< \omega(t), i\sigma_m^{[N]} > = < \sigma_m^{[N]} a(t), a(t) > . \tag{64}
\]

Therefore, equality (10) together with (61) give,
\[
h \frac{d}{dt} v_h(t) + iH(h)v_h(t) = hS(t,h)
\]
with
\[
S(t,h) = -i \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \left( B_m(x_{\lambda}) - B_m(x_{\lambda}, q(t), p(t)) \right) \left( < \sigma_m^{[N]} a(t), a(t) > - I \otimes \sigma_m^{[N]} \right) v_h(t).
\]

Applying Lemma 3.5 with the function \( F(q, p) = B_m(a_{\lambda}, q, p) \), one concludes that,
\[
\|S(t,h)\|_{W_2 \otimes H_{sp}} \leq C(t)h^{1/2},
\]
where the constant \( C(t) \) is bounded on all compact sets of \( \mathbb{R} \). One clearly has,
\[
\frac{d}{dt} e^{-\frac{it}{h}H(h)} \left( \Psi_{X_0, h} \otimes a_0 \right) = -\frac{i}{h} H(h) e^{-\frac{it}{h}H(h)} \left( \Psi_{X_0, h} \otimes a_0 \right).
\]

Thus,
\[
v_h(t) - e^{-\frac{it}{h}H(h)} \left( \Psi_{X_0, h} \otimes a_0 \right) = \int_0^t e^{-\frac{i(t-s)}{h}H(h)} S(s, h) ds.
\]

From Lemma 3.6 one deduces (12) if \( (q_0, p_0) \) lies in \( D(M)^2 \). Now suppose that \( (q_0, p_0) \) only belongs to \( H^2 \). There exists a sequence \( (q_j, p_j) \) in \( D(M)^2 \) converging to \( (q_0, p_0) \) in \( H^2 \). Set \( (q_j(t), p_j(t), \omega_j(t)) \) the corresponding strong solution and \( a_j(t) \) the solution to (10). Inequality (12) is proved for this solution. In view of Proposition 2.2 the sequence \( (q_j(t), p_j(t)) \) tends to \( (q(t), p(t)) \) in \( H^2 \). In addition, \( a_j(t) \) tends to \( a(t) \) in \( H_{sp} \). From Proposition 2.4 (iii) in [8], \( \Psi_{q_j(t), p_j(t), h} \) tends to \( \Psi_{q(t), p(t), h} \) in \( W_2 \). According to Lemma 3.6,
\[
\lim_{j \to \infty} \left\| e^{-\frac{i}{h}H(h)} \left( \Psi_{q_j, p_j, h} \otimes a_0 \right) - e^{-\frac{i}{h}H(h)} \left( \Psi_{q, p, h} \otimes a_0 \right) \right\|_{W_2 \otimes H_{sp}} = 0.
\]

Denoting \( \varphi_j(t) \) the action integral analogous to (19) and corresponding to the initial data \( (q_j, p_j, \pi_{a_0}) \), one also has,
\[
\lim_{j \to \infty} \left\| e^{\frac{i}{h}\varphi_j(t)} \Psi_{q_j(t), p_j(t), h} \otimes a(t) - e^{\frac{i}{h}\varphi(t)} \Psi_{q(t), p(t), h} \otimes a(t) \right\|_{W_2 \otimes H_{sp}} = 0.
\]

Inequality (12) then follows for the weak solution.
4 Applications. Photons number, fields and spins.

Our first aim here is to study the average number of photons at time $t$ knowing that the initial state of the system at $t = 0$ is $\Psi_{X_0, h} \otimes a_0$.

**Theorem 4.1.** Let $u_h(t)$ be defined in (18) where $X_0 = (q_0, p_0) \in H^3$ and let $a_0$ belonging to the unit sphere of $\mathcal{H}_{sp}$. Set $(q(t), p(t), \omega(t))$ the solution to the Hamiltonian system with initial data $(q_0, p_0, \pi_{a_0})$. Then one has,

$$\left| < 2h(N \otimes I) \ u_h(t), u_h(t) > - (q(t))^2 + |p(t)|^2 \right| \leq C(t)h^{1/2},$$

where the constant $C(t)$ is bounded on all compact sets of $\mathbb{R}$.

**Proof.** Let $u_h(t)$ and $v_h(t)$ be defined in (18) and (20). One has,

$$\left| < (N \otimes I) \ u_h(t), u_h(t) > - < (N \otimes I) \ v_h(t), v_h(t) > \right| \leq 2\| (N \otimes I)(u_h(t) - v_h(t)) \|.$$

Since the Wick symbol of $N$ is $X = (q, p) \mapsto (|q|^2 + |p|^2)/2h$, one sees,

$$< (N \otimes I) \ v_h(t), v_h(t) > = < N\Psi_{q(t), p(t), h}, \Psi_{q(t), p(t), h} > = \frac{|q(t)|^2 + |p(t)|^2}{2h}.$$

Besides, from (41)

$$\left\| (N \otimes I)(u_h(t) - v_h(t)) \right\| \leq \frac{1}{2h} \left\| u_h(t) - v_h(t) \right\|_{W^2_{\mathcal{H}_{sp}}}.$$

The Proposition is then completed according to Theorem 5.1.

\[\square\]

Our second aim is to study the average value of the components of the electric and magnetic fields at each point $x$ of $\mathbb{R}^3$ and at time $t$, knowing that the initial state of the system at $t = 0$ is $\Psi_{X_0, h} \otimes a_0$.

**Theorem 4.2.** Let $u_h(t)$ be defined in (18) where $X_0 = (q_0, p_0) \in H^3$ and take $a_0$ in the unit sphere of $\mathcal{H}_{sp}$. Set $(q(t), p(t), \omega(t))$ the solution to the Hamiltonian system with initial data $(q_0, p_0, \pi_{a_0})$. Then one has,

$$\left| < (B_m(x) \otimes I) \ u_h(t), u_h(t) > - B_m(x, q(t), p(t)) \right| \leq C(t)h^{1/2}$$

where the constant $C(t)$ is bounded on all compact sets $\mathbb{R}$. This remains also valid for the electric field.

**Proof.** Let $u_h(t)$ and $v_h(t)$ be defined in (18) and (20). One sees,

$$\left| < (B_m(x) \otimes I) \ u_h(t), u_h(t) > - < (B_m(x) \otimes I) \ v_h(t), v_h(t) > \right| \leq 2\| (B_m(x) \otimes I)(u_h(t) - v_h(t)) \|.$$

18
Since the Wick symbol of $B_m(x)$ is $B_m(x, p, p)$, one has gets,

$$< (B_m(x) \otimes I) v_h(t), v_h(t) > = < B_m(x) \Psi_{q(t), p(t), h}, \Psi_{q(t), p(t), h} > = B_m(x, q(t), p(t)).$$

Besides, from Lemma 3.6

$$\| (B_m(x) \otimes I) (u_h(t) - v_h(t)) \| \leq C \| u_h(t) - v_h(t) \|_{W_1 \otimes H_{sp}}.$$

The Proposition is then complete in view of Theorem 3.1.

□

Our next aim is to study the average value of the spins of the particle $s$ at time $t$, knowing that the initial state of the system at time $t = 0$ equals to $\Psi_{X_0, h} \otimes a_0$.

**Theorem 4.3.** Let $u_h(t)$ be defined in (18) where $X_0 = (q_0, p_0) \in H^3$ and let $a_0$ lying in the unit sphere of $H_{sp}$. Let $(q(t), p(t), \omega(t))$ be the solution to Hamiltonian system with initial data $(q_0, p_0, \pi a_0)$. Let $a(t)$ be the solution to the differential system (40) with initial data $a_0$. Then one has, for all $\lambda \leq N$ and for all $m \leq 3$,

$$| < (I \otimes \sigma_m^{[\lambda]}) u_h(t), u_h(t) > - < \sigma_m^{[\lambda]} a(t), a(t) > | \leq C(t) h^{1/2}$$

where the above constant $C(t)$ is bounded on all compact sets of $\mathbb{R}$.

The proof is identical to those of Theorems 4.1 and 4.2.

## 5 Approximate evolution laws

Theorems 4.1, 4.2 and 4.3 enable to write (semiclassical) approximate laws for the evolution of the electromagnetic field average, the spin averages and the photons number average. We recover classical laws for fields, laws going back to Bloch [4] (1946) for spins and we state a perhaps new law for the photons number average.

Fix $(q(t), p(t), \omega(t))$ a solution to the Hamiltonian system and $a(t)$ a solution to (40). Set,

$$B_{appr}^m(x, t) = B_m(x, q(t), p(t)), \quad E_{appr}^m(x, t) = E_m(x, q(t), p(t))$$

and also, using (64),

$$S_m^{[\lambda, appr]}(t) = < \sigma_m^{[\lambda]} a(t), a(t) > = < \omega(t), i \sigma_m^{[\lambda]} > .$$
Also set, \( \mathbf{B}^{\text{appr}}(x, t) = (B_1^{\text{appr}}(x, t), B_2^{\text{appr}}(x, t), B_3^{\text{appr}}(x, t)) \), and similarly for \( \mathbf{E}^{\text{appr}}(x, t) \) and \( \mathbf{S}^{[\lambda, \text{appr}]}(t) \).

**Fields evolution.** First, one remarks that,

\[
\text{div}\mathbf{B}^{\text{appr}}(x, t) = \text{div}\mathbf{E}^{\text{appr}}(x, t) = 0.
\]  

(67)

We shall show that,

\[
\frac{\partial}{\partial t} \mathbf{B}^{\text{appr}}(x, t) = -\text{rot}\mathbf{E}^{\text{appr}}(x, t)
\]  

(68)

\[
\frac{\partial}{\partial t} \mathbf{E}^{\text{appr}}(x, t) = \text{rot}\mathbf{B}^{\text{appr}}(x, t) + h \sum_{\lambda=1}^{N} \mathbf{S}^{[\lambda, \text{appr}]}(t) \wedge \text{grad}\rho(x - x_\lambda),
\]  

(69)

with,

\[
\rho(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(k)|^2 \cos(k \cdot x) dk.
\]  

(70)

Using definition (11) of \( E_m(x, p, p) \) together with the fact that \( J \) is antisymmetric, one may write

\[
E_m(x, q, p) = \alpha_m(x) \cdot q + \beta_m(x) \cdot p,
\]

where \( \alpha_m(x) \) and \( \beta_m(x) \) are the elements of \( H \) defined by,

\[
\alpha_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \sin(k \cdot x) k \wedge (k \wedge e_m),
\]  

(71)

\[
\beta_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \cos(k \cdot x) k \wedge (k \wedge e_m).
\]  

(72)

One uses the Hamiltonian system noticing that, for all \( q \) and \( p \) in \( D(M) \),

\[
\mathbf{E}(x, Mp, -Mq) = \text{rot}\mathbf{B}(x, q, p), \quad \mathbf{B}(x, Mp, -Mq) = -\text{rot}\mathbf{E}(x, q, p).
\]

One then deduces equality (68) and the fact that,

\[
\frac{\partial}{\partial t} \mathbf{E}^{\text{appr}}(x, t) = \text{rot}\mathbf{B}^{\text{appr}}(x, t) + \mathbf{F}(x, t),
\]

where

\[
K_{mn}(x, x_\lambda) = \alpha_m(x) \cdot b_n(x_\lambda) - \beta_m(x) \cdot a_n(x_\lambda),
\]

and the two others are obtained by circular permutation. Thus, one gets (69). This approximate evolution law is Maxwell laws, with zero charge and a divergence free current modelling the spin.
Evolution of spins. From (66) and (35),

\[
\frac{d}{dt} S_m^{[\lambda, \text{appr}]} (t) = \frac{d}{dt} \langle \omega(t), i \sigma_m^{[\lambda]} \rangle = \sum_{\mu=1}^{N} \sum_{n=1}^{3} (\beta_n + B_n(x, q(t), p(t))) \langle \omega(t), i \sigma_n^{[\mu]} \rangle > .
\]

The commutator vanishes if \( \mu \neq \lambda \). Using Pauli matrices commutation formulas, for example \( [\sigma_3, \sigma_1] = 2i\sigma_2 \), and then using again (66), one obtains,

\[
\frac{d}{dt} S_m^{[\lambda, \text{appr}]} (t) = 2(\beta + B_{\text{appr}}(x, t)) \wedge S_m^{[\lambda, \text{appr}]} (t).
\] (73)

These are Bloch equations [4] (1946).

Approximate evolution of the photons number average. Set,

\[
N_{\text{appr}} (t) = \frac{|q(t)|^2 + |p(t)|^2}{2\hbar}.
\]

In view of the Hamiltonian system, this approximate average evolves according to the following law,

\[
\frac{d}{dt} N_{\text{appr}} (t) = \sum_{\lambda=1}^{N} \beta \sum_{m=1}^{3} B_m(x, q(t), p(t)) < \omega(t), i \sigma_m^{[\lambda]} > .
\]

From (66), \( < \omega(t), i \sigma_m^{[\lambda]} > = S_m^{[\lambda, \text{appr}]} (t) \). One may give a more physical meaning to \( B_m(x, q(t), p(t)) \) using the helicity operator \( J \) from \( H^2 \) to \( H^2 \) defined in (10). For all \((q, p) \in H^2 \), set \( F(q, p) = (-p, q) \).

Since \( J \) and \( F \) are commuting, one sees that \( J^{-1} F \) is selfadjoint in \( H^2 \) and that its square equals to the identity. Its eigenvalues are +1 and −1. Denote by \( E_+ \) and \( E_- \) the corresponding eigenspaces. The sum of these subspaces is \( H^2 \). One denotes by \( \Pi_+ \) and \( \Pi_- \) the corresponding projections. One has for all \( Z = (q, p) \),

\[
(\Pi_{+} Z)(k) = \frac{1}{2} [Z(k) + (JZ)(k)].
\]

The subspaces \( E_+ \) and \( E_- \) are respectively corresponding to the right and left circular polarization. One has \( \mathcal{F} \Pi_{\pm} Z = \pm J \Pi_{\pm} Z \). Consequently,

\[
\frac{d}{dt} N_{\text{appr}} (t) = \sum_{\lambda=1}^{N} \beta \sum_{j=1}^{3} (B_j(x, \Pi_+ (q(t), p(t))) - B_j(x, \Pi_- (q(t), p(t)))) S_j^{[\lambda, \text{appr}]} (t).
\]

Therefore, from (11),

\[
\frac{d}{dt} N_{\text{appr}} (t) = \sum_{\lambda=1}^{N} \beta \sum_{j=1}^{3} (E_j(x, \Pi_- (q(t), p(t))) - E_j(x, \Pi_+ (q(t), p(t)))) S_j^{[\lambda, \text{appr}]} (t).
\] (74)
6 Fixed points and quasimodes.

We shall first give a characterization of the fixed points of the Hamiltonian system. For all \( u \) in the unit sphere of \( \mathcal{H}_{sp} \), define \((q_h(u), p_h(u))\) as the point in \( H^2 \) by,

\[
M q_h(u) = -h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} a_m(x_{\lambda}) \ < \sigma_m^{(\lambda)} u, u >, \tag{75}
\]

\[
M p_h(u) = -h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} b_m(x_{\lambda}) \ < \sigma_m^{(\lambda)} u, u >, \tag{76}
\]

where \( a_m(x_{\lambda}) \) and \( b_m(x_{\lambda}) \) are elements of \( H \) defined in (6) and (7). Since the function \( \chi \) in (6) and (7) belongs to \( S(\mathbb{R}) \) then this element \((q_h(u), p_h(u))\) exists in \( H^2 \). Even if \( \chi \) is not vanishing in a neighborhood of the origin, the function \( k \mapsto a_m(x_{\lambda}(k)/|k| \) belongs to \( H \).

Proposition 6.1. Let \((q, p)\) be in \( H^2 \) and \( a \) in the unit sphere of \( \mathcal{H}_{sp} \). The following properties are equivalent.

(i) The point \((q, p, \pi a)\) is a fixed point of the Hamiltonian system.

(ii) One has \(q = q_h(a)\), \(p = p_h(a)\) and \(a\) is an eigenvector of the operator \( T(q, p) \) defined in (33).

Proof. If (i) is satisfied then equalities (31) and (32) show that \(q = q_h(a)\) and \(p = p_h(a)\). In view of (35) one has,

\[
< [T(q, p), X] a, a > = 0, \quad \forall X \in G.
\]

Set \( b \) in \( \mathcal{H}_{sp} \) orthogonal to \( a \). There exists \( X \in G \) satisfying \( Xa = b \). To check that, it is sufficient to choose an orthonormal basis \((e_1, \ldots, e_p)\) (\( p = 2^N \)) of \( \mathcal{H}_{sp} \) satisfying \( a = e_1 \) and \( b = \mu e_2 \), with \( \mu \in \mathbb{R} \) and to choose \( X \) such that \( Xe_1 = -\mu e_2 \), \( Xe_2 = -\mu e_1 \) and \( Xe_j = 0 \) if \( j \geq 3 \). One deduces,

\[
< [T(q, p), X] a, a > = 2\Re < T(q, p)a, b > = 0.
\]

Replacing \( b \) by \( ib \), it is concluded that \( < T(q, p)a, b > = 0 \) for all vectors \( b \) orthogonal to \( a \). This means that \( T(q, p)a \) is colinear to \( a \) and therefore \( a \) is an eigenvector of \( T(q, p) \).

We shall now identify the fixed points of the Hamiltonian system. The constant magnetic field \( \beta \) is supposed to be non zero. Two unitary vectors \( b_0 \) and \( b_1 \) of \( \mathbb{C}^2 \) are chosen satisfying,

\[
\sum_{m=1}^{3} \beta_m \sigma_m b_0 = -|\beta| b_0, \quad \sum_{m=1}^{3} \beta_m \sigma_m b_1 = +|\beta| b_1. \tag{77}
\]
For all $E \subset \{1, \ldots, N\}$, $a_E$ denotes the following element,

$$a_E = a_1 \otimes \cdots \otimes a_N, \quad a_j = \begin{cases} b_1 & \text{if } j \in E \\ b_0 & \text{if } j \notin E \end{cases}. \quad (78)$$

The eigenvalues of the operator $T(0, 0)$ defined in (33) are $\mu_p = (2p - N)|\beta|$ ($0 \leq p \leq N$). The eigenspaces corresponding to $\mu_p$ is spanned by the $a_E$ with $|E| = p$.

**Proposition 6.2.** The constant magnetic field $\beta$ is assumed to be non vanishing. Let $a$ be in the unit sphere of $H_{sp}$ and not belonging to the union of circles $e^{i\theta}a_E$ ($E \subset \{1, \ldots, N\}, \theta \in \mathbb{R}$). Then, for small enough $h$, there exists no point $(q, p)$ in $H^2$ such that $(q, p, \pi_a)$ is a stationary point of the Hamiltonian system.

**Proof.** Suppose that there is a point $(q, p)$ in $H^2$ such that $(q, p, \pi_a)$ is a stationary point. We necessary have $q = q_h(a)$ and $p = p_h(a)$ (defined in (75) and (76)). In view of Proposition 6.1 the point $a$ is an eigenvector of $T(q_h(a), p_h(a))$ defined in (33). Equalities (33)(75) and (76) show that,

$$\|T(q_h(a), p_h(a)) - T(0, 0)\| \leq C h.$$

Since $a$ is not an eigenvector of $T(0, 0)$ then there is a contradiction for $h$ small enough.

\[ \square \]

Conversely, we shall associate fixed points of the Hamiltonian system with elements $a_E$. However, a geometrical hypothesis concerning the position of spin particles is needed.

**Proposition 6.3.** The constant magnetic field $\beta$ is assumed to be non vanishing and all the particles are supposed to be located at points $x_\lambda$ in the same orthogonal plane to $\beta$. Set $a = a_E$ in the unit sphere of $H_{sp}$ defined by (77) (78) with $E \subset \{1, \ldots, N\}$.

Then, for all $h > 0$, there exists $(q_h, p_h)$ in $H^2$ such that $(q_h, p_h, \pi_a)$ is a fixed point of the Hamiltonian system. Assuming that $\beta = (0, 0, |\beta|)$ and setting $\varepsilon_\lambda = 1$ if $\lambda \in E$ and $\varepsilon_\lambda = -1$ otherwise, the energy level of the fixed point is given by,

$$H(q_h(a), p_h(a), \pi_a) = h|\beta| \sum_{\lambda=1}^{N} \varepsilon_\lambda - \frac{h^2}{2} \sum_{\lambda, \mu \leq N} C_{33, \lambda, \mu}^{[\lambda, \mu]} \varepsilon_\lambda \varepsilon_\mu, \quad (79)$$

with

$$C_{33, \lambda, \mu}^{[\lambda, \mu]} = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(|k|)|^2 \cos(k \cdot (x_\lambda - x_\mu)) \frac{k^2 + k_\mu^2}{|k|^2} dk. \quad (80)$$

23
Proof. One may suppose that $\beta = (0, 0, |\beta|)$. Set $(q_h(a), p_h(a))$ the point of $H^2$ defined by (63) and (70). The operator $T(q_h(a), p_h(a))$ defined in (83) is written as,

$$
T(q_h(a), p_h(a)) = |\beta| \sum_{\lambda=1}^{N} \sigma_3^{[\lambda]} - h \sum_{\lambda, \mu \leq N} \sum_{m,n \leq 3} C_{mn}^{[\lambda, \mu]} < \sigma_{n}^{[\mu]} a, a > \sigma_{m}^{[\lambda]},
$$

with

$$
C_{mn}^{[\lambda, \mu]} = a_m(x_{\lambda}) \cdot (M^{-1}a_n(x_{\mu})) + b_m(x_{\lambda}) \cdot (M^{-1}b_n(x_{\mu})),
$$

where the $a_m(x)$ and $b_m(x)$ are defined in (6) and (7). Consequently,

$$
C_{mn}^{[\lambda, \mu]} = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(|k|)|^2 \cos(k \cdot (x_{\lambda} - x_{\mu})) \frac{(k \wedge e_m) \cdot (k \wedge e_n)}{|k|^2} dk.
$$

If $a = a_E$ then one has $< \sigma_{n}^{[\mu]} a, a > = 0$ excepted if $n = 3$. If $n = 3$ then one has $C_{mn}^{[\lambda, \mu]} = 0$ excepted if $m = 3$. Indeed, if $n = 3$ and $m \neq 3$, it is sufficient to replace $k_3$ by $-k_3$ to see that $C_{mn}^{[\lambda, \mu]} = 0$ since $\cos(k \cdot (x_{\lambda} - x_{\mu}))$ is independent of $k_3$, since $\chi$ is radial and since $(k \wedge e_m) \cdot (k \wedge e_3)$ is an odd function of $k_3$ when $m \neq 3$. Therefore,

$$
T(q_h(a), p_h(a)) = |\beta| \sum_{\lambda=1}^{N} \sigma_3^{[\lambda]} - h \sum_{\lambda, \mu \leq N} C_{33}^{[\lambda, \mu]} < \sigma_3^{[\mu]} a, a > \sigma_3^{[\lambda]}.
$$

Thus, $a = a_E$ is an eigenvector of $T(q_h(a), p_h(a))$ with an eigenvalue being,

$$
\lambda_h = |\beta|(2|E| - N) - h \sum_{\lambda, \mu \leq N} C_{33}^{[\lambda, \mu]} \epsilon_\lambda \epsilon_\mu.
$$

Proposition 6.1 shows that $(q_h(a), p_h(a), \pi_a)$ is a fixed point of the Hamiltonian system. From (4) and (30)-(33), one has,

$$
H(q_h(a), p_h(a), \pi_a) = \frac{h}{2} \left[ < T(0,0)a, a > + < T(q_h(a), p_h(a))a, a > \right].
$$

Then (70) is derived.

\[ \square \]

In order to give a more physical meaning to (80), $\chi$ has to tend to 1. The fact that $\chi$ is vanishing in a neighborhood of the origin is not used here (it is used in Sections 3-5 and in the below construction of the quasimode). One may then take $\chi(|k|) = \chi_\varepsilon(|k|) = \varphi(|k|\varepsilon)$, where $\varphi$ is a function $C^\infty$ with compact support which equals to 1 in a neighborhood of 0. Explicitly writing the parameter $\varepsilon$ as an index, one then has,

$$
H_\varepsilon(q_h(a), p_h(a), \pi_a) = h|\beta| \sum_{\lambda=1}^{N} \varepsilon_\lambda - \frac{h^2}{2} \sum_{\lambda, \mu \leq N} F_\varepsilon(x_{\lambda} - x_{\mu}) \epsilon_\lambda \epsilon_\mu,
$$

24
\[ F_\varepsilon(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\varphi(|k|)|^2 \cos(k \cdot x) \frac{k^2 + k^2_3}{|k|^2} \, dk. \]

For all non zero \( x \), one sees that,
\[
\lim_{\varepsilon \to 0} F_\varepsilon(x) = - (\partial^2_1 + \partial^2_2) \frac{1}{4\pi |x|}.
\]

In particular, if \( x_3 = 0 \),
\[
\lim_{\varepsilon \to 0} F_\varepsilon(x) = - \frac{1}{4\pi |x|^3}.
\]

Set \( C_\varepsilon \) the sum of the terms corresponding to \( \lambda = \mu \) in (79). This sum is independent of \( E \). One obtains,
\[
\lim_{\varepsilon \to 0} H_\varepsilon(q_h(a_E), p_h(a_E), \pi_{a_E}) - C_\varepsilon = h|\beta| \sum_{\lambda=1}^{N} \varepsilon_\lambda + \frac{h^2}{8\pi} \sum_{\lambda \neq \mu} |\varepsilon_\lambda \varepsilon_\mu| |x_\lambda - x_\mu|^3.
\]

We shall now associate quasimodes with points \( a_E (E \subset \{1, \ldots, N\}) \). These quasimodes are of low precision if \( E \neq \emptyset \) and of arbitrary high accuracy if \( E = \emptyset \).

**Proposition 6.4.** Fix \((q, p)\) in \( H^2 \) and take \( a \) in the unit sphere of \( H_{sp} \) such that \((q, p, \pi_a)\) is a stationary point of the Hamiltonian system (31)-(32)-(35). Then one has,
\[
\left[ H_{ph} + h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} (\beta_m + B_m(x_\lambda)) < \sigma[\lambda]^m a, a > \right] \Psi_{q, p, h} = H(q, p, \pi_a) \Psi_{q, p, h},
\]
where \( H(q, p, \omega) \) is the Hamiltonian function defined in (30).

Suppose that \( \beta \) is non zero. Set \( E \subset \{1, \ldots, N\} \). Then one has, with a constant \( C > 0 \),
\[
\| (H(h) - (2|E| - N)|\beta|h) (\Psi_{q, p, h} \otimes a_E) \| \leq Ch^{3/2}.
\]

**Proof.** From Proposition 6.1, if \((q, p, \pi_a)\) is a stationary point then \( q \) and \( p \) are in \( D(M) \) and given by (75) and (76). From The Lemma 3.3, setting \( X = (q, p) \),
\[
H_{ph} \Psi_{X, h} = Op_h^{wsp}(F_{MX}) \Psi_{X, h} - H_{ph}(X) \Psi_{X, h}.
\]

One has, from equalities (75) and (76),
\[
F_{MX} = -h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} B_m(x_\lambda, \cdot) < \sigma[\lambda]^m a, a >.
\]

In view of (92) and (93) (with \( q' = p' = 0 \)) and (64), one sees,
\[
-H_{ph}(X) - H(q, p, \pi_a) = -h \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \beta_m < \sigma[\lambda]^m a, a >.
\]
Then (81) is deduced. For (82), one may suppose that $\beta = (0,0,|\beta|)$, and that $(q,p) = (q_h(a_E),p_h(a_E))$. Then (81) gives,

$$[H_{ph} \otimes I + \hbar \sum_{\lambda=1}^{N} (\beta_3 + B_3(x_{\lambda})) \otimes \sigma_3^{[H]}] (\Psi_{q_h(a_E), p_h(a_E), h \otimes a_E}) = H(q_h(a_E), p_h(a_E), \pi_{a_E}) (\Psi_{q_h(a_E), p_h(a_E), h \otimes a_E}).$$

Consequently,

$$(H(h) - H(q_h(a_E), p_h(a_E), \pi_{a_E})) (\Psi_{q_h(a_E), p_h(a_E), h \otimes a_E}) = \hbar \sum_{\lambda=1}^{N} \sum_{m=1}^{2} B_m(x_{\lambda}) \otimes \sigma_3^{[H]} (\Psi_{q_h(a_E), p_h(a_E), h \otimes a_E}).$$

Since $|B_m(x_{\lambda}, q_h(a_E), p_h(a_E))| \leq Ch$ then Lemma 5.5 shows that the norm in the right hand side is $\leq Ch^{3/2}$. Besides, $H(q_h(a_E), p_h(a_E), \pi_{a_E}) = (2|E| - N)|\beta|h + o(h^2)$. Thus (82) is derived.

\[\square\]

The following theorem provides additional precisions when the set $E$ is empty.

**Theorem 6.5.** Suppose that $\beta$ is non zero and that the function $\chi$ in (3) and (7) is vanishing in a neighborhood of 0. With the notations (77) (78), set $a = a_0 \in \mathcal{H}_{sp}$. Then, there exists a sequence of elements of $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ denoted $u_j$ ($j \geq 0$) together with a sequence of real numbers $\lambda_j$ ($j \geq 1$) satisfying,

$$u_0 = \Psi_0 \otimes a_0, \quad \lambda_1 = -N|\beta|$$

and such that,

$$\left\| H(h) - (\lambda_1 h + \cdots + \lambda_{p+1} h^{p+1}) \left[ \sum_{j=0}^{2p} u_j h^j \right] \right\| \leq Ch^{p+3/2}, \quad (84)$$

$$\left\| H(h) - (\lambda_1 h + \cdots + \lambda_{p+1} h^{p+1}) \left[ \sum_{j=0}^{2p+1} u_j h^j \right] \right\| \leq Ch^{p+2}. \quad (85)$$

The above elements $u_j$ are independent of $h$ provided that these are considered as element of the Fock space $\mathcal{F}_s(\mathcal{H}_C)$ and not of $L^2(B, \mu_{B,h/2})$. Indeed, the isomorphism between these two spaces depends on $h$.

**Additional details on Fock space.** In Sections 2 to 5 we have considered that $\mathcal{H}_{ph}$ is a $L^2(B, \mu_{B,h/2})$ space associated with a Wiener space related to $H$. From now on, it is more convenient to suppose that it is the symmetrized Fock space

$$\mathcal{F}_s(\mathcal{H}_C) = \oplus_{m \geq 0} \mathcal{F}_m, \quad (86)$$
where $\mathcal{F}_0 = C$ and $\mathcal{F}_m$ is completion of the $m$-fold symmetric tensor product $H_C \otimes \cdots \otimes H_C$. One may then consider an element of $\mathcal{F}_m$ as a symmetric map $f$ from $(\mathbb{R}^3)^m$ to $(\mathbb{R}^3)^{\otimes m}$ satisfying for all $a_1, \ldots, a_m$ in $\{1, 2, 3\}$ and for all $k_1, \ldots, k_m$ in $\mathbb{R}^3$,
\[
\sum_{j=1}^{3} k_{1,j} f_{j,a_2,\ldots,a_m}(k_1, \ldots, k_m) = 0.
\]
(87)

We use here the notation $k_1 = (k_{1,1}, k_{1,2}, k_{1,3})$. In addition, the components of this function $f$ should be in $L^2(\mathbb{R}^{3m})$, which is defining the norm in $\mathcal{F}_m$. Thus, an element of $\mathcal{F}_s(H_C)$ is a sequence $f = (f_m)_{m \geq 0}$ where $f_m$ is an element of $\mathcal{F}_m$ and one has,
\[
\|f\|^2 = \sum_{m \geq 0} \|f_m\|^2.
\]
(88)

For any $\rho > 0$ and $m \geq 1$, $\mathcal{F}_m(\rho)$ stands for the set of elements $f$ in $\mathcal{F}_m$ satisfying $f(k_1, \ldots, k_m) = 0$ if one of the $|k_j|$ is $\leq \rho$. If $m = 0$, it is agreed that $\mathcal{F}_0(\rho) = \mathcal{F}_0$. It is also agreed that $\mathcal{F}_m = 0$ if $m < 0$. One sets,
\[
\mathcal{F}_{\text{even}}(\rho) = \mathcal{F}_0 \oplus \mathcal{F}_2(\rho) \oplus \mathcal{F}_4(\rho) \oplus \cdots
\]
(89)
\[
\mathcal{F}_{\text{odd}}(\rho) = \mathcal{F}_1(\rho) \oplus \mathcal{F}_3(\rho) \oplus \mathcal{F}_5(\rho) \oplus \cdots.
\]
(90)

These elements here are finite sums.

Let us recall that, if $M : H \to H$ is the multiplication by $\omega(k) = |k|$ and if $f$ is a rapidly decreasing function in $\mathcal{F}_m$ then one has,
\[
(d\Gamma(M)f)(k_1, \ldots, k_m) = (|k_1| + \cdots + |k_m|)f(k_1, \ldots, k_m).
\]
(91)

If $L(q,p) = a \cdot q + b \cdot p$ is a continuous linear form on $H^2$, we associate with it, for all $h > 0$, an unbounded operator in $L^2(B, \mu_{B,h/2})$ denoted $Op_h^{\text{weyl}}(L)$. This operator is defined for smooth cylindrical functions by (140). Via the canonical (Segal) isomorphism $\mathcal{J}_h : \mathcal{F}_s(H_C) \to L^2(B, \mu_{B,h/2})$, this operator becomes,
\[
\mathcal{J}_h^{-1}Op_h^{\text{weyl}}(L)\mathcal{J}_h = \sqrt{h}\Phi_S(a + ib),
\]
(92)

where $\Phi_S(a + ib) : \mathcal{F}_m \to \mathcal{F}_{m+1} \oplus \mathcal{F}_{m-1}$ is a continuous linear operator, called Segal field and defined in [15]. We denote by $\Psi_0$ a unitary element of $\mathcal{F}_0$. One may suppose that $\mathcal{J}_h \Psi_0 = \Psi_{0,h}$ where $\Psi_{0,h}$ is used in the preceding sections.

One denotes by $K(h) = (\mathcal{J}_h^{-1} \otimes I)H(h)(\mathcal{J}_h \otimes I)$ the Hamiltonian considered as an operator in $\mathcal{F}_s(H_C) \otimes \mathcal{H}_sp$. Let us recall that $H_{ph} = h d\Gamma(M)$. One then has,
\[
K(h) = (\mathcal{J}_h^{-1} \otimes I)H(h)(\mathcal{J}_h \otimes I) = hK_1 + h^{3/2}K_{3/2},
\]
(93)
with
\[ K_1 = d\Gamma(M) \otimes I + I \otimes T_0, \quad T_0 = \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \beta_m \sigma_m^{[\lambda]}, \tag{94} \]
\[ K_{3/2} = \sum_{\lambda=1}^{N} \sum_{m=1}^{3} \Phi_S(a_m(x_{\lambda}) + i b_m(x_{\lambda})) \otimes \sigma_m^{[\lambda]} \tag{95} \]

where \( a_m(x_{\lambda}) \) and \( b_m(x_{\lambda}) \) are defined in (6) and (7).

The point with identifying \( H_{ph} \) with \( \mathcal{F}_s(H_C) \) is that, the operator \( K_1 \) and \( K_{3/2} \) are independent of \( h \). Note that the operator \( K_1 \) maps each one of the two spaces \( \mathcal{F}_{even}(\rho) \otimes H_{ph} \) and \( \mathcal{F}_{odd}(\rho) \otimes H_{ph} \) into itself, whereas \( K_{3/2} \) maps each space into the other one. This comes from the definition of the Segal field (15).

The following Lemma is needed for the proof of Theorem 6.5.

**Lemma 6.6.** Set \( u_0 \) and \( \lambda_1 \) defined in (63). Then, for all \( f \) in \( \mathcal{F}_{odd}(\rho) \otimes H_{sp} \), (resp. in \( \mathcal{F}_{even}(\rho) \otimes H_{sp} \)), there exists \( u \) in \( \mathcal{F}_{odd}(\rho) \otimes H_{sp} \), (resp. in \( \mathcal{F}_{even}(\rho) \otimes H_{sp} \)) satisfying,
\[ (d\Gamma(M) \otimes I + I \otimes (T_0 - \lambda_1))u = f - \Pi f, \tag{96} \]
where \( \Pi \) is the orthogonal projection on \( u_0 \). In the following, we use the notation \( u = (K_1 - \lambda_1)^{-1}(f - \Pi f) \).

Notice that \( \Pi f = 0 \) if \( f \) belongs to \( \mathcal{F}_{odd}(\rho) \otimes H_{sp} \).

**Proof.** One may write,
\[ f = \sum_{E \subset \{1, \ldots, N\}} \sum_{m \geq 0} f_{E,m} \otimes a_E, \]
with \( f_{E,m} \) in \( \mathcal{F}_m(\rho) \), vanishing for even \( m \) (resp. for odd \( m \)). If \( m \geq 1 \), set
\[ u_{E,m}(k_1, \ldots, k_m) = \frac{f_{E,m}(k_1, \ldots, k_m)}{|k_1| + \cdots + |k_m| + 2|\beta||E|}. \]
Since \( f_{E,m} \) vanishes in neighborhood of the origin, this element is well defined, even when \( E \) is empty. If \( m = 0 \) and \( E \neq \emptyset \), set
\[ u_{E,0} = \frac{f_{E,0}}{2|\beta||E|}. \]
Then set,
\[ u = \sum_{m \geq 1} \sum_{E \subset \{1, \ldots, N\}} u_{E,m} \otimes a_E + \sum_{E \subset \{1, \ldots, N\}} u_{E,0} \otimes a_E. \]
This element \( u \) has the stated properties in the Lemma.

\[ \Box \]

Theorem 6.5 follows the next Proposition.
Proposition 6.7. Let $\rho > 0$ such that the function $\chi$ in \eqref{eq:chi} and \eqref{eq:chi2} vanishes for $|k| \leq \rho$. One supposes $\beta \neq 0$. With the notations \eqref{eq:notations}, set $a = a_{0} \in \mathcal{H}_{sp}$. Then there exists a sequence of elements of $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ denoted by $u_{j} (j \geq 0)$ and a sequence of real numbers $\lambda_{j} (j \geq 1)$, such that $u_{0}$ and $\lambda_{1}$ are defined in \eqref{eq:u0} \ and for even $j$,

$$u_{j} \in \mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{sp}$$

(97)

and for odd $j$,

$$u_{j} \in \mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$$

(98)

satisfying, for all integers $p$,

$$\left(K(h) - (\lambda_{1}h + \cdots + \lambda_{p+1}h^{p+1})\right) \sum_{j=0}^{2p} u_{j}h^{\frac{j}{2}} = \sum_{k \geq 1} h^{p+1+(k/2)} f_{2p}^{(p+1+(k/2))},$$

(99)

where the $f_{2p}^{(p+1+(k/2))}$ are elements of $\mathcal{F}_{s}(\mathcal{H}_{\mathbb{C}}) \otimes \mathcal{H}_{sp}$ and

$$\left(K(h) - (\lambda_{1}h + \cdots + \lambda_{p+1}h^{p+1})\right) \sum_{j=0}^{2p+1} u_{j}h^{\frac{j}{2}} = \sum_{k \geq 0} h^{p+2+(k/2)} f_{2p+1}^{(p+2+(k/2))},$$

(100)

where the $f_{2p+1}^{(p+2+(k/2))}$ are elements of $\mathcal{F}_{s}(\mathcal{H}_{\mathbb{C}}) \otimes \mathcal{H}_{sp}$. The sums are finite in right hand sides of (99) \ and (100).

Proof. In writing that the coefficient of $h^{j}$ ($j \leq p+1$) in the left hand side of (100) is zero, one sees that the $u_{j}$ and the $\lambda_{j}$ should satisfy the following equalities,

$$(K_{1} - \lambda_{1})u_{0} = 0,$$

(101)

$$(K_{1} - \lambda_{1})u_{1} + K_{3/2}u_{0} = 0,$$

(102)

$$(K_{1} - \lambda_{1})u_{2} + K_{3/2}u_{1} - \lambda_{2}u_{0} = 0,$$

$$(K_{1} - \lambda_{1})u_{3} + K_{3/2}u_{2} - \lambda_{2}u_{1} = 0.$$ 

More generally, if $m$ is odd, one should have,

$$(K_{1} - \lambda_{1})u_{m} + K_{3/2}u_{m-1} - \lambda_{2}u_{m-2} - \cdots - \lambda_{(m+1)/2}u_{1} = 0$$

(103)

and if $m$ is even,

$$(K_{1} - \lambda_{1})u_{m} + K_{3/2}u_{m-1} - \lambda_{2}u_{m-2} - \cdots - \lambda_{(m/2)+1}u_{0} = 0.$$ 

(104)

Since $d_{\Gamma}(M)\Psi_{0} = 0$ and $(T_{0} - \lambda_{1})a_{0} = 0$ then the elements $u_{0}$ and $\lambda_{1}$ defined in \eqref{eq:u0} satisfy (101). Since the operator $K_{3/2}$ exchanges parity then $K_{3/2}u_{0}$ belongs to $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$. From Lemma 6.6, there exists
u_1 in \mathcal{F}_{\text{odd}}(\rho) \otimes \mathcal{H}_{sp} satisfying (102). Set p \geq 0. Suppose that u_0, \ldots, u_{2p+1} and \lambda_1, \ldots, \lambda_{p+1} satisfying (103) and (104) are constructed. In order to determine u_{2p+2} and \lambda_{p+2}, one applies Lemma 6.6 with

\[ f_{2p+2} = \begin{cases} -K_{3/2}u_{2p+1} + \lambda_2 u_{2p} + \cdots + \lambda_{p+1} u_2 & \text{if } p \geq 1 \\ -K_{3/2}u_1 & \text{if } p = 0 \end{cases}. \quad (105) \]

Since \( K_{3/2} \) exchanges parity then this element lies in \( \mathcal{F}_{\text{even}}(\rho) \otimes \mathcal{H}_{sp} \). One defines \( \lambda_{p+2} \) by

\[ \lambda_{p+2} = -\langle f_{2p+2}, u_0 \rangle. \quad (106) \]

From Lemma 6.6 there exists \( u_{2p+2} \) in \( \mathcal{F}_{\text{even}}(\rho) \otimes \mathcal{H}_{sp} \) satisfying

\[ (K_1 - \lambda_1)u_{2p+2} = f_{2p+2} + \lambda_{p+2}u_0, \]

that is to say, (103) with \( p \) replaced by \( p + 1 \). To obtain \( u_{2p+3} \), one applies Lemma 6.6 with

\[ f_{2p+3} = -K_{3/2}u_{2p+2} + \lambda_2 u_{2p+1} + \cdots + \lambda_{p+2} u_1. \]

This element belongs to \( \mathcal{F}_{\text{odd}}(\rho) \otimes \mathcal{H}_{sp} \) and consequently \( \Pi f_{2p+3} = 0 \). In view of Lemma 6.6 there indeed exists \( u_{2p+3} \) in \( \mathcal{F}_{\text{odd}}(\rho) \otimes \mathcal{H}_{sp} \) satisfying,

\[ (K_1 - \lambda_1)u_{2p+3} = f_{2p+3}. \]

We have indeed constructed the sequences \((u_j)\) and \((\lambda_j)\) satisfying (103) and (104). The stated properties in the Proposition then follow and the proof of Theorem 6.5 is completed.

\[ \square \]

Proof of (23): One has,

\[ \langle B_m(x) \otimes I(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) \rangle = 2h^{1/2} \text{Re} \langle B_m(x) \otimes I u_0, u_1 \rangle. \]

Indeed, \( B_m(x) \) maps any element \( u \) of \( \mathcal{F}_{\text{odd}}(\rho) \) in \( \mathcal{F}_{\text{even}}(\rho) \) and then orthogonally to \( u \). The notation \( \pi_\emptyset \) stands for the orthogonal projection in \( \mathcal{H}_{sp} \) on the vector line spanned by \( a_\emptyset \). One sees,

\[ \langle B_m(x) \otimes I u_0, u_1 \rangle = \langle B_m(x) \otimes I u_0, (I \otimes \pi_\emptyset) u_1 \rangle. \]

According to (102),

\[ (I \otimes \pi_\emptyset) u_1 = - (d\Gamma(M^{-1}) \otimes \pi_\emptyset) K_{3/2} u_0. \]

The operator \( d\Gamma(M^{-1}) \) is well defined when applied to an element of \( \mathcal{F}_{\text{odd}}(\rho) \). From (6.21),

\[ (d\Gamma(M^{-1}) \otimes \pi_\emptyset) K_{3/2} u_0 = - \sum_{\lambda=1}^N d\Gamma(M^{-1}) \Phi_S(a_3(x_\lambda) + ib_3(x_\lambda)) \Psi_0 \otimes a_\emptyset. \]

30
Therefore,
\[ < (B_m(x) \otimes I)u_0, u_1 > = h^{1/2} \sum_{\lambda=1}^{N} < \Phi_S(a_m(x) + ib_m(x))\Psi_0, \ d\Gamma(M^{-1})\Phi_S(a_3(x_\lambda) + ib_3(x_\lambda))\Psi_0 > . \]

Using standard facts concerning Segal fields (see [8]),
\[ < \Phi_S(a_m(x) + ib_m(x))\Psi_0, \ d\Gamma(M^{-1})\Phi_S((a_3(x_\lambda) + ib_3(x_\lambda)))\Psi_0 > = \frac{1}{2} \left(M^{-1}a_3(x_\lambda) - iM^{-1}b_3(x_\lambda)\right) \cdot \left(a_m(x) + ib_m(x)\right). \]

The above scalar product is the one of \( H \). One then deduces (23). Equalities (24) and (25) then follows.

\[ \square \]

**Computation of \( \lambda_2 \).** From (6.32) and (105), one has,
\[ \lambda_2 = - < f_2, u_0 > = < K_{3/2}u_0, u_0 > = < u_1, K_{3/2}u_0 > . \]

With the notations of Lemma 6.6
\[ \lambda_2 = - < (K_1 - \lambda_1)^{-1}K_{3/2}u_0, K_{3/2}u_0 > \]
\[ = - \sum_{\lambda, \mu \leq N} \sum_{m, n \leq 3} C_{\lambda m}^{[\lambda, \mu]} \]
\[ = < (K_1 - \lambda_1)^{-1}\Phi_S(a_m(x_\lambda) + ib_m(x_\lambda))\Psi_0 \otimes \sigma_m^{[\lambda]}a_\theta, \Phi_S(a_n(x_\mu) + ib_n(x_\mu))\Psi_0 \otimes \sigma_n^{[\mu]}a_\theta > . \]

Suppose that \( \beta = (0, 0, |\beta|) \). From the expression of the operator \( (K_1 - \lambda_1)^{-1} \) (proof of Lemma 6.6), the above bracket is zero, excepted if \( m = n = 3 \), or if \( m \leq 2 \), \( n \leq 2 \) and \( \lambda = \mu \). In the first case, one has,
\[ C_{33}^{[\lambda, \mu]} = (2\pi)^{-3} \int_{\mathbb{R}^3} \chi(|k|) \cos(k \cdot (x_\lambda - x_\mu)) \frac{k_1^2 + k_2^2}{|k|^2} \, dk. \]

In the second case, one gets,
\[ C_{mn}^{[\lambda, \mu]} = < \sigma_m^{[\lambda]}a_\theta, \sigma_n^{[\lambda]}a_\theta > = (2\pi)^{-3} \int_{\mathbb{R}^3} \chi(|k|) \frac{|k \wedge e_m| \cdot (k \wedge e_n)}{|k|^2} \, dk. \]

The above integral is zero if \( m \neq n \). Set \( C = C_{11}^{[\lambda, \lambda]} \). From the preceding points,
\[ \lambda_2 = -2NC - \sum_{\lambda, \mu \leq N} C_{33}^{[\lambda, \mu]} . \]
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32
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laurent.amour@univ-reims.fr
LMR EA 4535 and FR CNRS 3399, Université de Reims Champagne-Ardenne, Moulin de la Housse, BP 1039, 51687 REIMS Cedex 2, France.

jean.nourrigat@univ-reims.fr
LMR EA 4535 and FR CNRS 3399, Université de Reims Champagne-Ardenne, Moulin de la Housse, BP 1039, 51687 REIMS Cedex 2, France.