One-point theta functions for vertex operator algebras

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Abstract

One-point theta functions for modules of vertex operator algebras (VOAs) are defined and studied. These functions are a generalization of the character theta functions studied by Miyamoto and are deviations of the classical one-point functions for modules of a VOA. Transformation laws with respect to the group $SL_2(\mathbb{Z})$ are established.

Keywords: vertex operator algebras; modular invariance

1 Introduction

As the surge in interest of modular functions surrounding Monstrous Moonshine continued in the late 1980s and early 1990s, mathematical attention quickly expanded to a number theoretic study of the more general $n$-point functions associated to vertex operator algebras (VOAs). Zhu’s celebrated (partial) solution [17] to the modularity of such functions consisted of developing recursion formulas enabling $n$-point functions to be written as a combination of classical elliptic and $(n - 1)$-point functions, thus reducing the problem to the study of 1-point functions. The latter of which, characters (or graded dimensions) associated to modules of VOAs are a special case. Motivated to express transformation properties of trace functions with automorphisms using only ordinary modules, Miyamoto [14] studied a deviation of characters for the modules of a VOA and developed their transformation laws with respect to the group $SL_2(\mathbb{Z})$. By exploiting these transformation laws, a number of works have been developed pertaining to elliptic genera [6], Jacobi forms [9, 10], and modular-invariance relations between shifted VOAs and orbifold theory [16], among others. It is the aim of this paper to develop a 1-point analogue of Miyamoto’s theta functions and deduce their transformation laws with respect to $SL_2(\mathbb{Z})$. This generalizes work of Miyamoto [14] and Yamauchi [16], and helps pave a way for studying 1-point functions involving multiple variables.

For a VOA $(V, Y(\cdot, \cdot), 1, \omega)$ of central charge $c$, the vertex operator $Y(v, z) := \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}$ identifies infinitely many endomorphisms $v(n)$ to an element $v \in V$. The endomorphism $L(0)$ defined by setting $L(n) := \omega(n + 1)$ supplies any ordinary $V$-module
$M^\nu$ with an nonnegative integer grading $M^\nu = \bigoplus_{n \geq 0} M^\nu_{\lambda_n + n}$, where $\lambda_n$ is the conformal weight of $M^\nu$ and $M^\nu_{\lambda_n + n} = \{ w \in M^\nu \mid L(0)w = (\lambda_n + n)w \}$. Here, and throughout the paper, we assume $V$ is rational and $C_2$-cofinite. Rationality implies $V$ has finitely many inequivalent irreducible modules, which we denote $V_1, \ldots, V_N$, and each of these possess such an $L(0)$-grading [5, Theorem 8.1].

For elements $J, K \in V_1$, Miyamoto [14] introduced functions of the form

$$\Phi_\nu \left( (J, K), \tau \right) := \text{tr}_{M^\nu} e^{2\pi i (J(0) + \frac{1}{2}J^3)} \sum_{n=0}^N \frac{q^n}{(q^n)^{1/24}} \tilde{\tau}^n,$$

(1)

where $\langle \cdot, \cdot \rangle$ is a bilinear form associated to $V$ (see [12] for more details about this form). Suppose (i) $V$ is a rational, $C_2$-cofinite VOA, (ii) $J, K \in V_1$ and satisfy $\alpha(n) \beta = \delta_{n,1} \langle \alpha, \beta \rangle 1$ for $\alpha, \beta \in \{ J, K \}$ and integers $n \geq 0$, (iii) $J(0)$ and $K(0)$ act semisimply on $V$ with eigenvalues in $\mathbb{C}$, (iv) $J$ and $K$ are quasi-primary, that is $L(1)J = 0$ and $L(1)K = 0$, and (v) each function $\Phi_j$, $1 \leq j \leq N$, converges for all $\tau \in \mathbb{H}$. Under these assumptions, Miyamoto proves that for each $V$-module $M^\nu$ and every $\gamma = \left( \frac{a}{c} \frac{b}{d} \right) \in \text{SL}_2(\mathbb{Z})$, there are scalars $A^\gamma_{\nu,j}$ such that

$$\Phi_\nu \left( (J, K), \frac{a\tau + b}{c\tau + d} \right) = \sum_{j=1}^N A^\gamma_{\nu,j} \Phi_j \left( (bK + dJ, aK + cJ), \tau \right).$$

(2)

This result generalizes and deviates from previous works. Most substantially, taking $(J, K) = (0, 0)$ collapses to the level of characters, or the $n = 1$ case with $a_1 = 1$ considered by Zhu [17, Theorem 5.3.2]. In fact, the scalars $A^\gamma_{\nu,j}$ are precisely the $S(\gamma, \nu, j)$ found in Zhu’s theorem. We also mention the occurrence of this result with $K = 0$ in the theory of vertex operator superalgebras [1, Theorem 5.4].

To unravel the more general story of 1-point functions and their deviations, we must introduce the zero mode $o(v) := v(\text{wt } v - 1)$ of a homogeneous element $v \in V_{\text{wt } v}$, where $\text{wt } v$ denotes the weight of $v$ with respect to the $L(0)$-grading. We extend this definition linearly and note $o(v)$ is the unique endomorphism associated to $v$ which preserves the grading induced by $o(\omega) = L(0)$. Additionally, it is necessary to discuss the relationship between the original structure of a VOA $(V, Y(\cdot, \cdot), 1, \omega)$ and the change of coordinate VOA $(V, Y(\cdot, \cdot), 1, \tilde{\omega})$ considered by Zhu [17, Theorem 4.2.1]. By setting $\tilde{\omega} := \omega - \frac{c}{24} 1$ and $Y(\cdot, z) := Y(e^{2\omega(z)}v, e^z - 1) := \sum_{n \in \mathbb{Z}} Y[v^n] z^{-n-1}$ as in [4], $V$ acquires a different VOA structure with grading $V = \bigoplus_{n \in \mathbb{Z}} V[n]$, where $V[n]$ are now eigenspaces with respect to the $n = 0$ case of the operators $L[n] = \tilde{\omega}[n + 1]$. If $v$ is homogeneous with respect to $L[0]$, we denote its weight by $\text{wt } v$ for $v \in V_{\text{wt } v}$. We refer the reader to Section 2 below and [4, 17] for more details about these different VOA structures.

While Miyamoto’s theorem continues to be used, a 1-point analogue has thus far been lacking. Indeed, Miyamoto’s results can be interpreted as the $v = 1$ case of the functions

$$\Phi_j(v : (J, K), \tau) := \text{tr}_{M^j} e^{2\pi i (o(J) + \frac{1}{2}J^3)} o(v) q^{o(\tilde{\omega}) + o(K) + \frac{J^3}{2}} q^{\frac{J^3}{2}},$$

(3)

and the primary purpose of this paper is to develop transformation laws for functions of this type for any $v \in V$. To state our main result, we first define functions $\Phi_{j,\ell}$ for integers $\ell \geq 0$

\footnote{See the Main Theorem on page 233 of [14]. The same theorem on page 223 contains a typo.}
and \( v \in V \) by
\[
\Phi_{j,\ell}(v : (J, K), \tau) := \frac{1}{\ell!} \Phi_j((J + \tau K)[1]^{\ell}v : (J, K), \tau). \tag{4}
\]

Under precisely the same assumptions (i)-(v) above applied to the functions (3), we obtain the following theorem.

**Theorem 1.1.** Suppose (i) \( V \) is a rational, \( C_2 \)-cofinite VOA, (ii) \( J, K \in V_1 \) and satisfy \( \alpha(n)\beta = \delta_{n,1}(\alpha, \beta)I \) for \( \alpha, \beta \in \{J, K\} \) and integers \( n \geq 0 \), (iii) \( J(0) \) and \( K(0) \) act semisimply on \( V \) with eigenvalues in \( \mathbb{C} \), (iv) \( J \) and \( K \) satisfy \( L(1)J = L(1)K = 0 \), and (v) each function \( \Phi_j, 1 \leq j \leq N \), converges for all \( \tau \in \mathbb{H} \). Then for any \( v \in V_{[\mathfrak{wt}]v} \) and \( \gamma = (\frac{a}{c}, \frac{b}{d}) \in SL_2(\mathbb{Z}) \), we have
\[
\Phi_j \left( v : (J, K), \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{\mathfrak{wt}[v]} \sum_{k=1}^{N} A_{j,k}^\gamma \Phi_k \left( e^{cJ[1]+\frac{a\tau + b}{c\tau + d}+K[1]}v : (bK + dJ, aK + cJ), \tau \right), \tag{5}
\]
where the constants \( A_{j,k}^\gamma \) are precisely those that arise in [17, Theorem 5.3.2]. In other words,
\[
\Phi_j \left( v : (J, K), \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{\mathfrak{wt}[v]} \sum_{k=1}^{N} A_{j,k}^\gamma \sum_{\ell=0}^{\mathfrak{wt}[v]} \Phi_{k,\ell} \left( v : (bK + dJ, aK + cJ), \tau \right) \left( \frac{c}{c\tau + d} \right)^\ell. \tag{6}
\]

Additionally, ignoring convergence and condition (v), these transformation rules hold so long as \( J(0)v = K(0)v = 0 \).

Theorem 1.1 establishes a type of quasi-modular form transformation property. Cleaner modular transformation laws similar to those satisfied by theta functions arise when \( K[1]v = J[1]v = 0 \). When \( (J, K) = (0, 0) \), Theorem 1.1 collapses to the main theorem in [17], while the \( v = 1 \) case gives the main result of [14]. Meanwhile, the condition \( (J, K) = (J, 0) \) with \( o(J) \) having rational eigenvalues produces a case of the modular transformations in [4]. Other relevant results include Theorem 9.13 in [7], which is a generalization of a special case of Theorem 1.1, as well as the similar, but independently developed Theorem 1.2 in [2].

Additionally, we note how Theorem 1.1 relates with another formulation of VOA theta functions due to Yamauchi [16]. By restricting attention to the case when \( o(J) \) and \( o(K) \) have rational eigenvalues, Yamauchi develops a generalization of Miyamoto’s functions to incorporate automorphisms of \( V \), as well as certain 1-point elements. Utilizing theory related to shifted VOAs and orbifolds, he also establishes convergence on \( \mathbb{H} \) for these functions by invoking work in [4]. Specifically, for an element \( v \in V_{[\mathfrak{wt}]v} \), the 1-point insertion \( o(\sum_{s=0}^{\infty} p_s(J(1), J(2), \ldots)v) \) is considered, where \( p_s(J(1), J(2), \ldots) \) are Schur polynomials defined by
\[
z^{-J(0)}\Delta_J(z) := \exp \left( -\sum_{n=1}^{\infty} \frac{J(n)}{n} (-z)^{-n} \right) =: \sum_{s=0}^{\infty} p_s(J(1), J(2), \ldots)z^s. \tag{7}
\]
Yamauchi employs results due to Li [11], where \( \Delta_J(z) \) is introduced and used to establish isomorphisms among twisted \( V \)-modules. It can be seen that

\[
e^{J[1]} = z^{-J(0)} \Delta_J(1) = \sum_{s=0}^{\infty} p_s(J(1), J(2), \ldots).
\]

Along with results such as Proposition 9 in [13], this helps illuminate the connection between shifted VOA structures, orbifold theory, and the elliptic and modular form properties of 1-point functions.

Set

\[
\Psi_j(v : (J, K), \tau) := \Phi_j(e^{K[1]}v : (J, K), \tau).
\]

Using Theorem 1.1 we obtain the following result analogous to the \((g, h) = (1, 1)\) case of the main result of [16], Theorem 1.1, but allowing for complex eigenvalues of \(o(J)\) and \(o(K)\).

**Corollary 1.2.** Assume the same conditions as in Theorem 1.1. Then for any \(v \in V_{\text{wt}[v]}\) and \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\), we have

\[
\Psi_{\nu}(v : (J, K), \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{\text{wt}[v]} \sum_{k=1}^{N} A^\gamma_{j,k} \Psi_k(v : (bK + dJ, aK + cJ), \tau),
\]

where the constants \(A^\gamma_{j,k}\) are precisely those that arise in [17, Theorem 5.3.2].

Both Theorem 1.1 and Corollary 1.2 are proved in Section 3. We recall they assume convergence. However, as noted before, some situations of convergence are known. For example, a statement of convergence for functions with \(v = 1\) is made in Proposition 1.8 of [6]. Therefore, \(\Phi_j\) of any \(v\) corresponding to the derivative with respect to \(\tau\) of \(\Phi_j(1 : (J, K), \tau)\) also converge on this domain. Meanwhile, as mentioned above, the convergence of the functions \(\Psi_j\) for all \(v\) on \(\mathbb{H}\) when \(o(J)\) and \(o(K)\) have rational eigenvalues is in [16].

In Section 4 we explore an example and, using Theorem 1.1, establish another proof of the quasi-modular properties for a partial derivative of the Jacobi theta functions \(\theta_j\). Additionally, just as in the case of Miyamoto’s original work, the case \(K = 0\) is of particular importance, and the transformation laws of Theorem 1.1 can be exploited to establish a portion of the transformation laws for (quasi) Jacobi forms for strongly regular VOAs. This will be explored elsewhere.

## 2 Preliminaries and notation

### 2.1 Elliptic functions

Let \(q_x\) denote \(e^{2\pi ix}\) for a variable \(x\). Define the functions \(P_k(\tau, z)\) for \(k \geq 1\) by

\[
P_k(\tau, z) := \frac{1}{(k-1)!} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^{k-1} q_z^n}{1 - q^n},
\]

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These functions, when multiplied by \((2\pi i)^k\), are the functions \(P_k(q, q)\) in [14, 17]. For 
\[\gamma = \left(\frac{a}{c}, \frac{b}{d}\right) \in \text{SL}_2(\mathbb{Z}) \text{ and } k \geq 3,\] 
they have the transformation properties 
\[P_k \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k P_k(\tau, z).\]

Meanwhile, for \(k = 1, 2\), we have 
\[P_1 \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)P_1(\tau, z) + \frac{c\tau + d}{2} - cz - \frac{1}{2}\]
and 
\[P_2 \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^2P_2(\tau, z) - \frac{c(c\tau + d)}{2\pi i}.

### 2.2 Weight one elements of a VOA

The relationship between modes of an element \(u \in V\) under the original VOA structure and 
that of the change of coordinate VOA is given by 
\[u[m] = m! \sum_{i \geq m} c(\text{wt} u, i, m) u(i),\] 
where \(c(\text{wt} u, i, m)\) are defined by the coefficients of the series 
\[m! \sum_{i \geq m} c(\text{wt} u, i, m)x^i := (\ln(1 + x))^{m}(1 + x)^{\text{wt} u - 1}\] 
expanded in the variable \(x\) (see, for example, [17, Lemma 4.3.1]). Since \(\ln(1 + x) = -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}\), we have \(c(1, i, 1) = -\frac{(-1)^i}{i}\) and combining with (11) we find (8) holds as stated in the introduction. Additionally, (11) implies \(L[0] = L(0) + \sum_{i \geq 0} k_i L(i)\) for some scalars \(k_i\). It follows that if \(u \in V_1\) and \(L(1)u = 0\), then \(u \in V_1\) as well. We also note that for \(u \in V_1\), (12) gives \(c(1, i, 0) = \delta_{i,0}\) and thus \(u[0] = u(0)\). Suppose \(u, v \in V_1\) and 
\(u(k)v = \delta_{k,1}(u,v)1 \in V_0\) for \(k \geq 0\). Since (12) gives \(c(1, 1, 1) = 1\) under this assumption, 
(11) implies \(u[k]v = \delta_{k,1}u(1)v = \delta_{k,1}(u,v)1\). Additionally, we have for all \(s, t \geq 0\) that
\[u[s]v[t] = v[t]u[s] + [u[s], v[t]] = v[t]u[s] + \sum_{k \geq 0} \binom{s}{k} (u[k]v)[s + t - k] = v[t]u[s].\] 

As these simple results will be referenced later, we collect them in the following lemma.

**Lemma 2.1.** Suppose \(v_1, \ldots, v_n \in V_1\) and for every \(1 \leq i, j \leq n\) we have 
(a) \(v_i(k)v_j = \delta_{k,1}(v_i, v_j)1\) for \(k \geq 0\), and 
(b) \(v_1, \ldots, v_n\) are quasi-primary (and thus primary), that is \(L(1)v_j = 0\).

Then for any \(1 \leq i, j \leq n\) and \(v \in V\),
1. \(v_j(0)v = v_j[0]v,\)
2. \(v_j \in V_{[1]},\)
3. \(v_i[k]v_j = \delta_{k,1}(v_i, v_j)1\) for \(k \geq 0\), and
4. \(v_i[s]v_j[t] = v_j[t]v_i[s]\) for all \(s, t \geq 0\).
3 Proofs of theorems

The proof of Theorem 1.1 follows the ideas developed by Miyamoto in [14]. In particular, we are ultimately interested in the transformation properties for expressions of the form

$$\sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{\ell_1! \ell_2!} \text{tr}_{M^\nu} o(J)^{\ell_1} o(\tau K)^{\ell_2} o(v) q^{o(\tilde{w})}$$

under the action of $SL_2(\mathbb{Z})$ (see Subsection 3.5 below), where $M^\nu$ is a $V$-module from the list $\{M^1, \ldots, M^N\}$ discussed in the introduction. Before undertaking this, however, we accumulate some necessary results in Subsections 3.1–3.4. Moreover, before considering $o(J)^{\ell_1}$ and $o(K)^{\ell_2}$, we instead consider arbitrary elements satisfying the assumptions of Lemma 2.1.

For the entirety of Section 3, we assume $v_1, \ldots, v_n$ satisfy the assumptions of Lemma 2.1 and $\psi$ is an arbitrary grade-preserving endomorphism on the underlying vector space. Set

$$S^\nu(\psi; z_1, \ldots, z_n, \{v, x\}, \tau) := \text{tr}_{M^\nu} \psi Y \left( q_{z_1}^{o(\omega)} v_1 \right) \cdots Y \left( q_{z_n}^{o(\omega)} v_n, q_x \right) q^{o(\tilde{w})}, \quad (14)$$

where again $q_w = e^{2\pi i w}$ for a variable $w$. Many of the upcoming subsections begin with new notation which will carry over to subsequent subsections.

3.1 Step one

The first step is to express the $(n+1)$-point functions (14) as linear combinations of functions $P_k(\tau, z)$ and VOA trace functions of $n$ zero modes and one vertex operator.

For an element $v_i \in \{v_1, \ldots, v_n\}$, set $\phi(v_i) := \phi(v_i, x - z_i, \tau) = \sum_{m_i \geq 1} P_{m_i} (x - z_i, \tau) v_i [m_i - 1]$. Since $v_i[s]v_i[t] = v_j[s]v_j[t]$, we observe that $\phi(v_i)\phi(v_j) = \phi(v_j)\phi(v_i)$ for all $1 \leq i, j \leq n$.

For a set $U$, let $I(U)$ denote the set of all elements $\sigma \in \text{Sym}(U)$ such that $\sigma^2 = 1$. Here $\text{Sym}(U)$ denotes the symmetric group with identity 1 of the set $U$. In the case $U = \{1, \ldots, n\}$, we often write $I(n)$ in place of $I(\{1, \ldots, n\})$. For $\sigma \in \text{Sym}(U)$ set

$$m(\sigma) := \{ i \in U \mid \sigma(i) \neq i \} \quad \text{and} \quad f(\sigma) := \{ i \in U \mid \sigma(i) = i \}.$$

Finally, set $x_i := x - z_i$ and $z_{i,j} := z_i - z_j$.

**Proposition 3.1.** Suppose $v_1, \ldots, v_n$ satisfy the assumptions of Lemma 2.1, $v \in V$, and $[\psi, v_j(m)] = 0$ for each $1 \leq j \leq n$ and $m \in \mathbb{Z}$. For $n \geq 1$ we have

$$S^\nu(\psi; z_1, \ldots, z_n, \{v, x\}, \tau) = \sum_{\sigma \in I(n)} \prod_{j < \sigma(j)} \langle v_j, v_{\sigma(j)} \rangle P_2 \left( z_{\sigma(j)}, \tau \right) \sum_{U \subseteq f(\sigma)} S^\nu \left( \prod_{r \in U} o(v_r); \left\{ \left( \prod_{s \in f(\sigma) \setminus U} \phi(v_s) \right) v, x \right\}, \tau \right).$$
Proof. Throughout this proof we suppress the notation relying on the module \(M^n\). That is, we write \(S\) and \(\text{tr}\) instead of \(S^n\) and \(\text{tr}M^n\), respectively. Without loss of generality we may assume \(v \in V_{\text{wt}}v\). For \(k \in \mathbb{Z}\), a similar calculation as in the proof of Proposition 4.1 in [14] gives

\[
S(\psi v_1(k)q^{-k}_{z_1}; z_2, \ldots, z_n, \{v, x\}, \tau) = kq_{z_j,1}^k \sum_{j=2}^n \langle v_j, v_1 \rangle S(\psi; z_2, \ldots, \hat{z}_j, \ldots, z_n, \{v, x\}, \tau) + q_{z_1}^k \sum_{i \geq 0} \binom{k}{i} S(\psi; z_2, \ldots, z_n, \{v_1(i)v, x\}, \tau) + q^k S(\psi v_1(k)q^{-k}_{z_1}; z_2, \ldots, z_n, \{v, x\}, \tau),
\]

where \(\hat{X}\) denotes the omission of the term \(X\). Then for \(k \neq 0\), using

\[
\sum_{i \geq 0} \binom{k}{i} v_1(i) = \sum_{m \geq 0} \frac{(k + 1 - \text{wt } v_1)^m}{m!} v_1[m],
\]

(which can be deduced from (11) and (12)) we have

\[
S(\psi v_1(k)q^{-k}_{z_1}; z_2, \ldots, z_n, \{v, x\}, \tau) = \sum_{j=2}^n \langle v_j, v_1 \rangle kq_{z_j,1}^k S(\psi; z_2, \ldots, \hat{z}_j, \ldots, z_n, \{v, x\}, \tau) + \sum_{m \geq 1} \frac{k^m q_{z_1}^k}{m!} S(\psi; z_2, \ldots, z_n, \{v_1[m]v, x\}, \tau).
\]

Therefore, we find

\[
S(\psi; z_1, z_2, \ldots, z_n, \{v, x\}, \tau) = S(\psi v_1(0); z_2, \ldots, z_n, \{v, x\}, \tau) + \sum_{k \in \mathbb{Z} \setminus \{0\}} S(\psi v_1(k)q^{-k}_{z_1}; z_2, \ldots, z_n, \{v, x\}, \tau) = S(\psi v_1(0); z_2, \ldots, z_n, \{v, x\}, \tau) + \sum_{j=2}^n \langle v_1, v_j \rangle P_2(z_{j,1}, \tau) S(\psi; z_2, \ldots, \hat{z}_j, \ldots, z_n, \{v, x\}, \tau) + \sum_{m \geq 1} P_m(x_1, \tau) S(\psi; z_2, \ldots, z_n, \{v_1[m - 1]v, x\}, \tau).
\]

Repeating the steps gives the desired result. \(\square\)

Note that rearranging the order of the vertex operators in the previous theorem leads to a different, but similar result (see Lemma 8.5 in [4], for example).

### 3.2 Step two

The next step is to incorporate the action of \(SL_2(\mathbb{Z})\) into the the terms of Proposition 3.1. We do this by utilizing Zhu’s modularity theorem for \(n\)-point functions [17]. More
precisely, we use the following $g = h = 1$ case of Assertion 2 in the proof of Theorem 4.10 in [16], which generalizes the 1-point modularity result of Dong, Li, and Mason [4] to $n$-point functions. When applied to our situation, this result states there are scalars $A_{\nu,k}^\gamma$ for each $\gamma = (a b \ c d) \in \text{SL}_2(\mathbb{Z})$ such that for any $v \in V_{\text{wt}(v)}$ we have

$$
S^\nu \left( \psi; \frac{z_1}{c^\tau + d}, \ldots, \frac{z_n}{c^\tau + d}, \left\{ v, \frac{x}{c^\tau + d} \right\}; \frac{a^\tau + b}{c^\tau + d} \right) = (c^\tau + d)^{\text{wt}(v) + n} \sum_{k=1}^N A_{\nu,k}^\gamma S_k \left( \psi; z_1, \ldots, z_n, \left\{ v, x \right\}, \tau, z \right).
$$

(15)

Throughout the remainder of Section 3, for $\gamma = (a b \ c d) \in \text{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$ we set

$$
\gamma \tau := \frac{a^\tau + b}{c^\tau + d}, \quad \hat{\gamma} \tau := a^\tau + b, \quad \text{and} \quad \hat{\gamma} \tau := c^\tau + d.
$$

As we fix $\gamma$, we will typically write $A_k^\gamma$ in place of $A_{\nu,k}^\gamma$.

For a subset $U$ of a set $W$, let

$$
D_\sigma := \prod_{j < \sigma(j)} \langle v_j, v_{\sigma(j)} \rangle \hat{\gamma} \tau P_2 (z_{\sigma(j), j}, \tau),
$$

$$
E_\sigma := \prod_{j < \sigma(j)} \langle v_j, v_{\sigma(j)} \rangle \left( \hat{\gamma} \tau P_2 (z_{\sigma(j), j}, \tau) - \frac{c \hat{\gamma} \tau}{2\pi i} \right),
$$

$$
F_U^W := \prod_{s \in W \setminus U} \sum_{m_s \geq 1} P_{m_s} (x_s, \tau) v_s [m_s - 1],
$$

$$
G_U^W := \prod_{s \in W \setminus U} \sum_{m_s \geq 1} P_{m_s} \left( \frac{x_s}{\hat{\gamma} \tau}, \gamma \tau \right) v_s [m_s - 1], \quad \text{and}
$$

$$
H_U^W := \prod_{s \in W \setminus U} \left( A_\gamma (x_s, \tau) v_s [0] - \frac{c}{2\pi i} v_s [1] + \hat{\gamma} \tau \sum_{m_s \geq 1} P_{m_s} (x_s, \tau) v_s [m_s - 1] \right),
$$

where

$$
A_\gamma (x_s, \tau) := \frac{\hat{\gamma} \tau}{2} - c x_s - \frac{1}{2}.
$$

For $j \geq 1$ and a nested set of subsets

$$
U_\ell \subseteq f(\sigma_{\ell-1}) \subseteq U_{\ell-1} \subseteq \cdots \subseteq f(\sigma_2) \subseteq U_2 \subseteq f(\sigma_1) \subseteq U_1 = U_0 = \{1, \ldots, n\},
$$

set $X_{U_\ell}^{\sigma_{j-1}} := X_{U_\ell}^{f(\sigma_{j-1})}$ for $X = F, G, H$. In this notation we also set $H_{U_1}^{\sigma_0} = G_{U_1}^{\sigma_0} = F_{U_1}^{\sigma_0} = 1$. By (13), it follows that $F_{U_\ell}^{\sigma_{i-1}} F_{U_j}^{\sigma_{j-1}} = F_{U_{\ell-i}}^{\sigma_{i-1}} F_{U_{j-i}}^{\sigma_{j-1}}$, $G_{U_\ell}^{\sigma_{i-1}} G_{U_j}^{\sigma_{j-1}} = G_{U_{\ell-i}}^{\sigma_{i-1}} G_{U_{j-i}}^{\sigma_{j-1}}$, and $H_{U_\ell}^{\sigma_{i-1}} H_{U_j}^{\sigma_{j-1}} = H_{U_{\ell-i}}^{\sigma_{i-1}} H_{U_{j-i}}^{\sigma_{j-1}}$ for any $1 \leq i, j \leq \ell$. Moreover, let $U_{j-1} := f(\sigma_{j-1}) \setminus U_j$ denote the complement of $U_j$ in $f(\sigma_{j-1})$, and $|U_j|$ be the number of elements in $U_j$.

We are now in position to establish the necessary lemmas.
Lemma 3.2. Let \( t_1, \ldots, t_{|U_\ell|} \) denote the elements of \( U_\ell \). Then for every \( U_\ell, \ell \geq 1 \), we have

\[
S^\nu \left( 1; \frac{z_{t_1}}{\gamma_{T}}, \ldots, \frac{z_{t_{|U_\ell|}}}{\gamma_{T}}, \left\{ G_{U_\ell}^{\sigma_{\ell-1}} G_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots G_{U_2}^{\sigma_1} v, \frac{x}{\gamma_{T}} \right\}, \gamma_T \right)
= \sum_{k=1}^{N} A_k^\nu S^k \left( 1; z_{t_1}, \ldots, z_{t_{|U_\ell|}}, \left\{ H_{U_\ell}^{\sigma_{\ell-1}} H_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots H_{U_2}^{\sigma_1} v, x \right\}, \tau \right).
\]

Proof. We first note that

\[
S^\nu \left( 1; \frac{z_{t_1}}{\gamma_{T}}, \ldots, \frac{z_{t_{|U_\ell|}}}{\gamma_{T}}, \left\{ G_{U_\ell}^{\sigma_{\ell-1}} G_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots G_{U_2}^{\sigma_1} v, \frac{x}{\gamma_{T}} \right\}, \gamma_T \right)
= S^\nu \left( 1; \frac{z_{t_1}}{\gamma_{T}}, \ldots, \frac{z_{t_{|U_\ell|}}}{\gamma_{T}}, \left\{ \prod_{i=2}^\ell \prod_{s_i \in U_{i-1}} \sum_{m_{s_i} \geq 1} P_{m_{s_i}} \left( \frac{x_{s_i}}{\gamma_{T}}, \gamma_T \right) v_{s_i} [m_{s_i} - 1] \right\}, \gamma_T \right)
= S^\nu \left( 1; \frac{z_{t_1}}{\gamma_{T}}, \ldots, \frac{z_{t_{|U_\ell|}}}{\gamma_{T}}, \left\{ \prod_{i=2}^\ell \prod_{s_i \in U_{i-1}} \left( \frac{x_{s_i}}{\gamma_{T}} - c(x - z_{s_i}) - \frac{1}{2} \right) v_{s_i} [0] \right\}
- \frac{c_{-\sigma_0}}{2\pi i} v_{s_i} [1] + \sum_{m_{s_i} \geq 1} \hat{z}_{s_i}^{m_{s_i}} P_{m_{s_i}} (x_{s_i}, \tau) v_{s_i} [m_{s_i} - 1] \right\}, \gamma_T \right).
\]

where the index \( i \) begins at 2 since \( G_{U_1}^{\sigma_0} = 1 \). Since the functions \( S^k \) are linear and each component being summed is comprised of an element of weight (cf. 2. of Lemma 2.1)

\[
\text{wt} \left[ \prod_{i=2}^\ell \left( \prod_{s_i \in U_{i-1}} v_{s_i} [m_{s_i} - 1] \right) \right] = \text{wt}[v] + \sum_{i=2}^\ell \left[ U_{i-1} \right] - \sum_{i=2}^\ell \sum_{s_i \in U_{i-1}} m_{s_i},
\]

it follows from (15) that (16) becomes

\[
\sum_{k=1}^{N} A_k^\nu S^k \left( 1; z_{t_1}, \ldots, z_{t_{|U_\ell|}}, \left\{ H_{U_\ell}^{\sigma_{\ell-1}} H_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots H_{U_2}^{\sigma_1} v, x \right\}, \tau \right).
\]

This completes the proof. \( \square \)

Lemma 3.3. For \( U_{\ell-1} \) such that \( |U_{\ell-1}| \geq 0 \), we have

\[
S^\nu \left( \prod_{r \in U_{\ell-1}} o(v_r); \left\{ G_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots G_{U_2}^{\sigma_1} v, \frac{x}{\gamma_{T}} \right\}, \gamma_T \right)
= \sum_{k=1}^{N} A_k^\nu \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} D_{\sigma_{\ell-1}} \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} S^k \left( \prod_{r \in U_\ell} \gamma_{\tau} o(v_r); \left\{ F_{U_\ell}^{\sigma_{\ell-1}} H_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots H_{U_2}^{\sigma_1} v, x \right\}, \tau \right)
- \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} E_{\sigma_{\ell-1}} \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} \sum_{U_\ell \neq U_{\ell-1}} S^\nu \left( \prod_{r \in U_\ell} o(v_r); \left\{ G_{U_\ell}^{\sigma_{\ell-1}} G_{U_{\ell-1}}^{\sigma_{\ell-2}} \cdots G_{U_2}^{\sigma_1} v, \frac{x}{\gamma_{T}} \right\}, \gamma_T \right).
\]
Proof. Utilizing Proposition 3.1 twice, with a use of Lemma 3.2 in between, we find

\[
\sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} \prod_{j < \sigma_{\ell-1}(j)} (v_j, v_{\sigma_{\ell-1}(j)}) P_2 \left( \frac{z_{\sigma_{\ell-1}(j)} j}{\bar{\gamma}_r}, \gamma \right) \\
\times \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) G_{\sigma_{\ell-2}} U_{\ell-1} \cdots G_{\sigma_{1}} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
= S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) G_{\sigma_{\ell-2}} U_{\ell-1} \cdots G_{\sigma_{1}} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
= \bar{\gamma}_{U_{\ell-1}} [U_{\ell-1}] + \text{wt}[v] \sum_{k=1}^{N} A_k^v S_k \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) H_{\ell-1} U_{\ell-1} \cdots H_{1} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
\times \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) H_{\ell-1} U_{\ell-1} \cdots H_{1} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right). 
\]

(17)

Isolating the piece associated to \( \sigma_{\ell-1} = 1 \) and \( U_\ell = U_{\ell-1} \) (so that \( U_{\ell-1}^\ell = \emptyset \)) in the left side of (17), we have

\[
S^v \left( \prod_{r \in U_{\ell-1}} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) G_{\sigma_{\ell-2}} U_{\ell-1} \cdots G_{\sigma_{1}} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
= \bar{\gamma}_{U_{\ell-1}} [U_{\ell-1}] + \text{wt}[v] \sum_{k=1}^{N} A_k^v \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} \prod_{j < \sigma_{\ell-1}(j)} (v_j, v_{\sigma_{\ell-1}(j)}) P_2 \left( \frac{z_{\sigma_{\ell-1}(j)} j}{\bar{\gamma}_r}, \gamma \right) \\
\times \sum_{U_\ell \subseteq f(\sigma_{\ell-1})} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) H_{\ell-1} U_{\ell-1} \cdots H_{1} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
- \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} \prod_{j < \sigma_{\ell-1}(j)} (v_j, v_{\sigma_{\ell-1}(j)}) P_2 \left( \frac{z_{\sigma_{\ell-1}(j)} j}{\bar{\gamma}_r}, \gamma \right) \\
\times \sum_{U_\ell \subseteq f(\sigma_{\ell-1}) \cap U_{\ell-1} \neq U_{\ell-1}} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) G_{\sigma_{\ell-2}} U_{\ell-1} \cdots G_{\sigma_{1}} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
= \bar{\gamma}_{\text{wt}[v]} \sum_{k=1}^{N} A_k^v \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} D_{\sigma_{\ell-1}} \sum_{U_\ell \subseteq f(\sigma_{\ell-1}) \cap U_{\ell-1} \neq U_{\ell-1}} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) F_{\ell-1} U_{\ell-1} \cdots H_{1} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right) \\
- \sum_{\sigma_{\ell-1} \in I(U_{\ell-1})} E_{\sigma_{\ell-1}} \sum_{U_\ell \subseteq f(\sigma_{\ell-1}) \cap U_{\ell-1} \neq U_{\ell-1}} S^v \left( \prod_{r \in U_\ell} o(v_r); \left\{ \left( \prod_{s \in U_{\ell-1}^j} \phi \left( v_s, \frac{x_s}{\bar{\gamma}_r} \right) \right) G_{\sigma_{\ell-1}} U_{\ell-1} \cdots G_{\sigma_{1}} U_2, \frac{x}{\bar{\gamma}_r} \right\}, \gamma \right),
\]
where we used that $|m(\sigma_{\ell-1})| + |U_{\ell}| + |U_{\ell-1}^{\ell-1}| = |U_{\ell-1}|$ for each $\sigma_{\ell-1}$. 

Note that $E_{\sigma_{\ell-1}}$ and $D_{\sigma_{\ell-1}}$ are both 1 when $|U_{\ell-1}| = 0, 1$.

### 3.3 Step three

The third step uses the symmetry of the equations developed in the previous steps to show that every term with a function $P_k$ vanishes, while terms only involving the modular anomalies of $P_1$ and $P_2$ remain. To help accomplish this, we make note of a result in [14, Lemma 4.1].

**Lemma 3.4.** Suppose $|m(\sigma)| = 2p$. Then

$$
\sum_{\sigma_1 + \cdots + \sigma_\ell = \sigma} (-1)^t E_{\sigma_1} E_{\sigma_2} \cdots E_{\sigma_\ell} = (-1)^p E_{\sigma}.
$$

We may now prove the following lemma.

**Lemma 3.5.** We have

$$
\sum_{\sigma \in I(n)} \sum_{\sigma_1, \sigma_2 \in I(n)} (-1)^{\frac{|m(\sigma_1)|}{2}} E_{\sigma_1} D_{\sigma_2} \sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma)\backslash W|} S_k \left( \prod_{r \in U} \tilde{\gamma}_r o(v_r); \left\{ F_W^W H_W^\sigma v, x \right\}, \tau \right)
$$

$$
= \sum_{\sigma \in I(n)} \left( \prod_{j<\sigma(j)} \left( \frac{c_{\tau r} \langle v_j, v_{\sigma(j)} \rangle}{2\pi i} \right) \right) \sum_{U \subseteq f(\sigma)} S_k \left( \prod_{r \in U} \tilde{\gamma}_r o(v_r); \left\{ \left( \prod_{s \in f(\sigma) \setminus U} -B_s \right) v, x \right\}, \tau \right),
$$

where $B_s = B(v_s, x_s) := A_s(x_s, \tau) v_s[0] - \frac{c}{2\pi i} v_s[1]$.

**Proof.** By linearity, we can break the proof into two parts. Set $Q_{\sigma} := \prod_{j<\sigma(j)} (-\frac{c_{\tau r}}{2\pi i})$ and $R_{\sigma} := \prod_{j<\sigma(j)} \tilde{\gamma}_r P_2(z_{\sigma(j)}, j, \tau)$. For $\sigma, \sigma_1, \sigma_2 \in I(n)$ such that $\sigma_1 + \sigma_2 = \sigma$, we may consider decompositions of $\sigma_1$ as $\sigma_3 + \sigma_4 = \sigma_1$ with $\sigma_3, \sigma_4 \in I(n)$, so that

$$
\sum_{\sigma_1, \sigma_2 \in I(n)} (-1)^{\frac{|m(\sigma_1)|}{2}} E_{\sigma_1} D_{\sigma_2} = \sum_{\sigma_3 + \sigma_4 = \sigma} (-1)^{\frac{|m(\sigma_1)|}{2}} (-1)^{\frac{|m(\sigma_2)|}{2}} Q_{\sigma_3} R_{\sigma_4} R_{\sigma_2} \prod_{j<\sigma(j)} \langle v_j, v_{\sigma(j)} \rangle
$$

$$
= \sum_{\sigma_3 + \sigma_4 = \sigma} (-1)^{\frac{|m(\sigma_1)|}{2}} Q_{\sigma_3} \prod_{j<\sigma(j)} \langle v_j, v_{\sigma(j)} \rangle \sum_{\sigma_2 + \sigma_4 = \sigma'} (-1)^{\frac{|m(\sigma_2)|}{2}} R_{\sigma_2 + \sigma_4}. \tag{18}
$$

Suppose $m(\sigma') \neq \emptyset$. For $\sigma' = (s_1, s_2) \cdots (s_{2\ell-1}, s_{2\ell})$, we have there are $\binom{\ell}{r}$ possible many $\sigma_4$ with $\sigma_2 + \sigma_4 = \sigma'$ such that $|m(\sigma_4)| = 2r$. For such $\sigma'$, we have

$$
\sum_{\sigma_2 + \sigma_4 = \sigma'} (-1)^{\frac{|m(\sigma_2)|}{2}} R_{\sigma_2 + \sigma_4} = \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} R_{\sigma'} = 0
$$

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since \( \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} = 0 \) for \( \ell > 0 \). In the case \( m(\sigma') = \emptyset \), continuing the calculation in (18) gives

\[
\sum_{\sigma_1, \sigma_2 \in \mathcal{I}(n)} (1) = \sum_{\sigma \subseteq \mathcal{I}(n)} (-1)^{|m(\sigma)|/2} \prod_{j < \sigma(j)} \left( -\frac{c_{\gamma_f}}{2\pi i} \right) \prod_{j < \sigma(j)} \langle v_j, v_{\sigma(j)} \rangle
\]

This proves the first part.

For the second part, set \( \hat{\phi}(v_s) := \bar{\gamma}_{f} \phi(v_s, x_s, \tau) \). Fix \( \sigma \) and consider

\[
\sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus W|} F^W_U H^\sigma_W.
\]

Note that \( (f(\sigma) \setminus W) \cup (W \setminus U) = f(\sigma) \setminus U \). Then

\[
\sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus W|} F^W U H^\sigma_W = \sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus W|} \prod_{r \in W \setminus U} \hat{\phi}(v_r) \prod_{s \in f(\sigma) \setminus W} \left( B_s + \hat{\phi}(v_s) \right)
\]

For fixed \( X_1 \subseteq f(\sigma) \setminus W \subseteq f(\sigma) \setminus U \), the product \( \prod_{r \in X_2} \hat{\phi}(v_r) \) is the same regardless of the \( W \), so long as \( U \subseteq W \) and \( X_1 \subseteq f(\sigma) \setminus W \). Therefore, we are interested in counting how many ways we can choose \( W \) so that \( U \subseteq W \) and \( X_1 \subseteq f(\sigma) \setminus W \). Since there are \( |f(\sigma)| - |X_1| - |U| = |f(\sigma) \setminus U| - |X_1| \) many ways to choose \( W \) which are not in \( U \), there are \( \binom{|f(\sigma) \setminus U| - |X_1|}{j} \) many ways to choose \( W \) so that \( |W| = |U| + j \). Then \( (-1)^{|f(\sigma) \setminus W|} = (-1)^{|f(\sigma) \setminus U|} (-1)^j \), and for the fixed \( X_1 \) we find the total number of \( \prod_{s \in X_1} B_s \prod_{r \in X_2} \hat{\phi}(v_r) \) terms in (19) is

\[
(-1)^{|f(\sigma) \setminus U|} \sum_{j=0}^{|f(\sigma) \setminus U| - |X_1|} (-1)^j \binom{|f(\sigma) \setminus U| - |X_1|}{j}.
\]

However, this sum equals 0 so long as \( |f(\sigma) \setminus U| - |X_1| > 0 \). In the case \( |f(\sigma) \setminus U| - |X_1| = 0 \),
we have $X_1 = f(\sigma) \setminus U$, and continuing the calculation in (19) we find

$$
\sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus W|} F^W_U H^\sigma_W = \sum_{U \subseteq W \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus W|} \prod_{s \in X_1} B_s \prod_{r \in X_2} \tilde{\phi}(v_r)
$$

$$
= \sum_{U \subseteq f(\sigma)} (-1)^{|f(\sigma) \setminus U|} \prod_{s \in f(\sigma) \setminus U} B_s = \sum_{U \subseteq f(\sigma)} \prod_{s \in f(\sigma) \setminus U} (-B_s).
$$

This completes the proof. \qed

### 3.4 Step four

Here we combine the previous steps to obtain the modular transformation properties for the functions of an isolated product of $n$ zero modes.

**Proposition 3.6.** With the notation and assumptions above, along with requiring $v_s(0)v = 0$ for all $1 \leq s \leq n$, we have

$$
S^\nu \left(\prod_{r=1}^n o(v_r); \left\{ v, \frac{x}{\gamma_T} \right\}, \gamma_T \right) = \tilde{\gamma}_T \prod_{k=1}^N A_k \sum_{\sigma \in f(\sigma)} \left( \prod_{j < \sigma(j)} \left( \frac{c_{j\sigma} \langle v_j, v_{\sigma(j)} \rangle}{2\pi i} \right) \right)
$$

$$
\times \sum_{U \subseteq f(\sigma)} S^k \left( \prod_{r \in U} \tilde{\gamma}_r o(v_r); \left\{ \left( \prod_{s \in f(\sigma) \setminus U} \frac{c_{rs}}{2\pi i}v_s \right) \right\}, \gamma_T \right).
$$

**Proof.** Using Lemma 3.3 applied to $U_1 = \{1, \ldots, n\}$, we find

$$
S^\nu \left(\prod_{r=1}^n o(v_r); \left\{ v, \frac{x}{\gamma_T} \right\}, \gamma_T \right)
$$

$$
= \tilde{\gamma}_T \prod_{k=1}^N A_k \sum_{\sigma_1 \in I(U_1)} \sum_{U_2 \subseteq f(\sigma_1)} D_{\sigma_1} \sum_{U_3 \subseteq f(\sigma_2)} S^k \left( \prod_{r \in U_2} \tilde{\gamma}_r o(v_r); \left\{ F^\sigma_{U_2} v, x \right\}, \gamma_T \right)
$$

$$
- \sum_{\sigma_1 \in I(U_1)} \sum_{U_1 \neq U_2 \subseteq f(\sigma_1)} E_{\sigma_1} \sum_{U_3 \subseteq f(\sigma_2)} S^\nu \left( \prod_{r \in U_2} o(v_r); \left\{ G^\sigma_{U_2} v, \frac{x}{\gamma_T} \right\}, \gamma_T \right).
$$

Let $\ell \geq 1$ be the length of the largest chain of proper subsets

$$
\emptyset = U_\ell \subset U_{\ell-1} \subset \cdots \subset U_1 = \{1, \ldots, n\}
$$

occurring after successive applications of Lemma 3.3. Then reapplying Lemma 3.3 to the last term in (20), and repeating this process on the corresponding term until we reach $U_\ell$,
we obtain
\[
S^\nu \left( \prod_{r=1}^{n} o(v_r); \left\{ v, \frac{x}{\gamma''} \right\}, \gamma \tau \right) \\
= \sum_{p=1}^{l-1} \gamma''' v^p \sum_{k=1}^{N} A_k \sum_{\sigma_{p-1} \in I(U_{p-1}) U_p \subseteq f(\sigma_{p-1}) \sigma_p \in I(U_p) U_{p+1} \subseteq f(\sigma_p)} \sum_{\sigma_{p-1} \subseteq I(U_{p-1}) U_p \subseteq f(\sigma_{p-1}) \sigma_p \in I(U_p) U_{p+1} \subseteq f(\sigma_p)} \sum_{\sigma_{p} \subseteq I(U_p) U_{p+1} \subseteq f(\sigma_p)} \sum_{U_{p+1} \subseteq f(\sigma_p)} \sum_{U_{p} \subseteq f(\sigma_{p-1})} \sum_{U_{p-1} \subseteq f(\sigma_{p-1})} \\
\times (-1)^{p-1} E_{\sigma_1} \cdots E_{\sigma_{p-1}} D_{\sigma_p} S^k \left( \prod_{r \in U_{p+1}} \gamma''' o(v_r); \left\{ F_{U_{p+1}}^\sigma H_{U_p}^\sigma \cdots H_{U_2}^\sigma v, x \right\}, \tau \right) \\
+ \sum_{k=1}^{N} A_k \sum_{\sigma_{p-1} \in I(U_{p-1}) U_p \subseteq f(\sigma_{p-1}) \sigma_p \in I(U_p) U_{p+1} \subseteq f(\sigma_p)} \sum_{\sigma_{\ell-2} \subseteq I(U_{\ell-2}) U_{\ell-1} \subseteq f(\sigma_{\ell-2}) \sigma_{\ell-1} \subseteq I(U_{\ell-1}) U_{\ell-1} \subseteq f(\sigma_{\ell-1}) \sigma_{\ell} \subseteq I(U_{\ell-1}) \emptyset - U_{\ell} \subseteq f(\sigma_{\ell}) \sigma_{p-1} \subseteq I(U_{p-1}) U_p \subseteq f(\sigma_{p-1})} \\
\times (-1)^{l-1} E_{\sigma_1} \cdots E_{\sigma_{\ell-2}} E_{\sigma_{\ell-1}} S^k \left( 1; \left\{ H_{U_{\ell}}^\sigma H_{U_{\ell-2}}^\sigma \cdots H_{U_2}^\sigma v, x \right\}, \tau \right),
\]
(22)

where the last equality also uses Lemma 3.2.

There may be \( \sigma_j = 1 \) in (22). In this case, the condition \( U_{j+1} \neq U_j \) is needed in the sum to ensure an iteration of Lemma 3.3 on the last term. Otherwise, \( U_{j+1} \neq U_j \) by default. Since \( \sigma_j \in I(U_j) \subseteq f(\sigma_{j-1}) \), we have \( m(\sigma_1) \cap \cdots \cap m(\sigma_{p-1}) = \emptyset \). Therefore, \( \sigma_1 + \cdots + \sigma_{p-1} = \sigma' \) for some \( \sigma' \in I(n) \). In fact, for any \( \sigma' \in I(n) \) there are \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{p-1} \in I(n) \) satisfying the conditions in (22) such that \( \tilde{\sigma}_1 + \cdots + \tilde{\sigma}_{p-1} = \sigma' \). Meanwhile, \( \sigma' \) may also be written as \( \sigma' = \psi_1 + \cdots + \psi_i \) for some \( \psi_1, \ldots, \psi_i \in I(n) \) with no additional conditions. Such a reformulation not only affects the construction of \( U_2, \ldots, U_p \), but \( H_{U_p}^\sigma \cdots H_{U_2}^\sigma \) as well.

We want to reformulate (22) in terms of general involutions \( \psi_1, \ldots, \psi_i \) and sets \( U \subseteq W \subseteq f(\sigma') \) which accomplish the same expression as the \( \sigma_1, \ldots, \sigma_{p-1} \) and \( U_1, \ldots, U_{\ell-1} \) that form the chain (21), but without the nested restrictions.

For starters, by Lemma 3.4 we have

\[
\sum_{\psi_1 + \cdots + \psi_i = \sigma'} (-1)^i E_{\psi_1} \cdots E_{\psi_i} = (-1)^{m(\sigma')} E_{\sigma'}. 
\]

Set \( U := U_{p+1} \) and take \( W \) to be the set \( W \subseteq f(\sigma') \) such that

\[
f(\sigma') \setminus W = \bigcup_{j=1}^{p-1} f(\sigma_j) \setminus U_{j+1}. 
\]

Note \( U \subseteq W \subseteq f(\sigma') \) since \( \sigma' \in I(n) \). On one hand, this shows \( H_{U_p}^\sigma \cdots H_{U_2}^\sigma = H_{W}^\sigma \). On the other, there may be many ways to obtain this equality resulting from different Lemma 3.3 iterations. Note, however, that the maximal number of elements to choose such a fixed \( W \subseteq f(\sigma') \) in this way is \( |f(\sigma') \setminus W| \). The number of ways to choose this corresponding to \( j \) iterations of Lemma 3.3 is \( \binom{|f(\sigma') \setminus W|}{j} \).

Both the choices of \( \sigma_1, \ldots, \sigma_p \) and \( U_2, \ldots, U_p \) in (22) contribute to the number of iterations
resulting from Lemma 3.3. Therefore, the \((-1)^{p-1}\) corresponds to the choices \(\psi_1, \ldots, \psi_i\) for \(\psi_1 + \cdots + \psi_i = \sigma_1 + \cdots + \sigma_{p-1}\) and also from the \(W \subseteq f(\sigma')\) corresponding to the \(U_2, \ldots, U_{p-1}\). We can now rewrite (22) as

\[
-\dot{\gamma}_r^{\text{wt}[v]} \sum_{k=1}^{N} A_k^{\nu} \sum_{\sigma \in I(U_1)} \sum_{\sigma' + \sigma'' = \sigma} \sum_{i \geq 1} (-1)^i E_{\psi_1} \cdots E_{\psi_i} D_{\sigma''} \\
\times \sum_{U \subseteq W \subseteq f(\sigma)} \sum_{j \geq 1} (-1)^j S^k \left( \prod_{r \in U} \dot{\gamma}_r o(v_r); \{ F_U W^\sigma_W v, x \}, \tau \right),
\]

where the minus sign in front corresponds to the final iteration which introduces the \(D_{\sigma''}\) and \(F_U W\) terms (and also the \(E_{\sigma''}\) and \(H_U W\) terms which are now contained in the expression corresponding to \(\sigma'' = 1\) and \(U = W = \emptyset\)). Therefore, by the discussion above and the fact \(\sum_{j \geq 1} (-1)^j \binom{N}{j} = -(1)^N\), we find (23) becomes

\[
-\dot{\gamma}_r^{\text{wt}[v]} \sum_{k=1}^{N} A_k^{\nu} \sum_{\sigma \in I(U_1)} \sum_{\sigma' + \sigma'' = \sigma} \sum_{i \geq 1} (-1)^{\binom{\text{int}(\sigma')}{i}} E_{\sigma'} D_{\sigma''} \\
\times \sum_{U \subseteq W \subseteq f(\sigma)} \sum_{j \geq 1} (-1)^j \left( \frac{|f(\sigma)|}{j} \right) S^k \left( \prod_{r \in U} \dot{\gamma}_r o(v_r); \{ F_U W^\sigma_W v, x \}, \tau \right) \\
= \dot{\gamma}_r^{\text{wt}[v]} \sum_{k=1}^{N} A_k^{\nu} \sum_{\sigma \in I(U_1)} \sum_{\sigma_1 + \sigma_2 = \sigma} (-1)^{\binom{\text{int}(\sigma)}{1}} E_{\sigma_1} D_{\sigma_2} \\
\times \sum_{U \subseteq W \subseteq f(\sigma)} \sum_{j \geq 1} (-1)^{\binom{|f(\sigma)|}{j}} S^k \left( \prod_{r \in U} \dot{\gamma}_r o(v_r); \{ F_U W^\sigma_W v, x \}, \tau \right).
\]

Applying Lemma 3.5 under the assumption \(v_{\alpha}(0)v = 0\) gives the desired result. \(\square\)

### 3.5 Step five

Before proving Theorem 1.1, we first establish the following lemma.

**Lemma 3.7.** Suppose \(J(0)v = \alpha v\) and \(K(0)v = \beta v\) for \(\alpha, \beta \in \mathbb{C}\). If \(\alpha \neq 0\) or \(\beta \neq 0\), then \(\Phi_j (v : (J, K), \tau) = 0\) for all \(\tau \in \mathbb{H}\) and \(1 \leq j \leq N\).

**Proof.** Since

\[
\text{tr}_{M_j} e^{2\pi i o(J)} o(v) q^{o(\omega)} + o(K) = q_\alpha \text{tr}_{M_j} o(v) e^{2\pi i o(J)} q^{o(\omega)} + o(K) = q_\alpha q_\beta \text{tr}_{M_j} e^{2\pi i o(J)} o(v) q^{o(\omega)} + o(K),
\]

we have

\[
(1 - q_\alpha q_\beta) \text{tr}_{M_j} e^{2\pi i o(J)} o(v) q^{o(\omega)} + o(K) = 0.
\]

Therefore, \(\Phi_j (v : (J, K), \tau) = 0\) for all \(\tau \in \mathbb{H}\) such that \(\alpha + \tau \beta \notin \mathbb{Z}\). It follows that \(\Phi_j (v : (J, K), \tau) = 0\) on an open ball in \(\mathbb{H}\) and by our assumption that \(\Phi_j\) is convergent on \(\mathbb{H}\), this extends to all of \(\mathbb{H}\). \(\square\)
We are now in position to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We begin by mimicking the proof of the Main Theorem in [14] and fix $s, t \geq 0$ so that $s + t = n$, and assume

$$v_j = \begin{cases} J & \text{if } 1 \leq j \leq s \\ K & \text{if } s + 1 \leq j \leq t. \end{cases}$$

For $\sigma \in I(n)$, partition the set $\Omega = \Omega_n = \{1, \ldots, n\}$ by setting

$$f_1(\sigma) := \{ j \in \Omega \mid j = \sigma(j) \leq s \}, \quad m_{11}(\sigma) := \{ j \in \Omega \mid j < \sigma(j) \leq s \},

f_2(\sigma) := \{ j \in \Omega \mid s < j = \sigma(j) \}, \quad m_{12}(\sigma) := \{ j \in \Omega \mid j \leq s < \sigma(j) \}, \quad m_{22}(\sigma) := \{ j \in \Omega \mid s \leq j < \sigma(j) \}.$$ 

For $p, q, r \in \mathbb{Z}$, set $m_1 := s - 2p - r \geq 0$ and $m_2 := t - 2q - r \geq 0$.

There are $\binom{s}{r}$ and $\binom{t}{m_2}$ many ways to choose $r$ elements from $s$ and $t$ many elements, respectively. Having chosen $r$ many elements from both, there are $\binom{2p+m_1}{m_1}\binom{2p}{p}\binom{t}{r}\binom{2q+m_2}{m_2}\binom{2q}{q}$ many ways to choose $\sigma$ such that $|m_{11}(\sigma)| = 2p$ and $|f_1(\sigma)| = m_1$ on $\{1, \ldots, s\}$, excluding the $r$ many elements in $m_{12}(\sigma)$ since $s - r = 2p + k$. Similarly, there are $\binom{2q+m_2}{m_2}\binom{2q}{q}$ many such $\sigma$ with $|m_{22}(\sigma)| = 2q$ and $|f_2(\sigma)| = m_2$. Finally, there are $r!$ many ways which a $\sigma \in I(n)$ under these restrictions can place the $r$ many elements $j \in m_{12}(\sigma)$ with $j \leq s$ into the $r$ spots above $s$. Therefore, the total number of $\sigma \in I(n)$ with $|m_{11}(\sigma)|/2 = p$, $|m_{12}(\sigma)|/2 = r$, $|m_{22}(\sigma)|/2 = q$, $|f_1(\sigma)| = m_1$, and $|f_2(\sigma)| = m_2$ is

$$\binom{s}{r}\binom{2p+m_1}{m_1}\binom{2p!}{p!2^p}\binom{t}{r}\binom{2q+m_2}{m_2}\binom{2q!}{q!2^q}r! = \frac{(2p + r + m_1)!(2q + r + m_2)!}{m_1!m_2!r!p!q!2^{p+q}}.$$

By Lemma 3.7, we may assume $o(J)v = o(K)v = 0$. Using Proposition 3.6, we note

$$S^\nu \left( o(J)^so(K)^t; \left\{ v, \frac{x}{c\tau + d} \right\}, \frac{a\tau + b}{c\tau + d} \right)$$

$$= \sum_{k=1}^{N} A_k^\nu \sum_{\sigma \in I(s+t)} \left( \frac{\langle J, J \rangle c_{\gamma_\tau}}{2\pi i} \right)^{|m_{11}(\sigma)|} \left( \frac{\langle J, K \rangle c_{\gamma_\tau}}{2\pi i} \right)^{|m_{12}(\sigma)|} \left( \frac{\langle K, K \rangle c_{\gamma_\tau}}{2\pi i} \right)^{|m_{22}(\sigma)|}$$

$$\times \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} S^k \left( [\gamma_\tau o(J)]^i [\gamma_\tau o(K)]^j; \left\{ \frac{c}{2\pi i} J[1] \right\}^{m_1-i} \left\{ \frac{c}{2\pi i} K[1] \right\}^{m_2-j} v, x \right), \tau$$

$$= \sum_{k=1}^{N} A_k^\nu \sum_{p,q,r} \frac{(2p + r + m_1)!(2q + r + m_2)!}{m_1!m_2!r!p!q!2^{p+q}}$$

$$\times \left( \frac{\langle J, J \rangle c_{\gamma_\tau}}{2\pi i} \right)^p \left( \frac{\langle J, K \rangle c_{\gamma_\tau}}{2\pi i} \right)^r \left( \frac{\langle K, K \rangle c_{\gamma_\tau}}{2\pi i} \right)^q$$

$$\times \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} S^k \left( [\gamma_\tau o(J)]^i [\gamma_\tau o(K)]^j; \left\{ \frac{c}{2\pi i} J[1] \right\}^{m_1-i} \left\{ \frac{c}{2\pi i} K[1] \right\}^{m_2-j} v, x \right).$$
Therefore,

\[
S^\nu \left( e^{2\pi i (\omega(J)+\frac{(J,K)}{2})+\gamma'\tau(\omega(K)+\frac{(K,K)}{2})} \right) \left\{ v, \frac{x}{\tau + d} \right\}, \frac{a\tau + b}{\tau + d}
\]

\[
= \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \sum_{\ell_1!\ell_2!\ell_3!\ell_4!} \frac{(2\pi i)^{\ell_1+\ell_2+\ell_3+\ell_4}}{\ell_1!\ell_2!\ell_3!\ell_4!} \left( \frac{\langle J, K \rangle}{2} \right)^{\ell_3} \left( \gamma' \frac{(K,K)}{2} \right)^{\ell_4}
\]

\[
\times \sum_{k=1}^{N} A_k^\nu \sum_{p,q,r,m_1,m_2} \frac{(2p + r + m_1)!(2q + r + m_2)!}{m_1!m_2!r!p!q!2^{p+q}} (\gamma')^{2q+r+m_2}
\]

\[
\times \left( \frac{\langle J, J \rangle}{2\pi i} \right)^p \left( \frac{\langle J, K \rangle}{2\pi i} \right)^r \left( \frac{\langle K, K \rangle}{2\pi i} \right)^q
\]

\[
= \sum_{p,q,r,m_1,m_2,\ell_1,\ell_2} \sum_{\ell_1!\ell_2!\ell_3!\ell_4!} \frac{(2\pi i)^{\ell_1+\ell_2+\ell_3+\ell_4}}{\ell_1!\ell_2!\ell_3!\ell_4!} \left( \frac{\langle J, K \rangle}{2} \right)^{\ell_3} \left( \gamma' \frac{(K,K)}{2} \right)^{\ell_4}
\]

\[
\times \sum_{k=1}^{N} A_k^\nu \left( \gamma' \right)^{2q+r+m_2} \left( \frac{\langle J, J \rangle}{2\pi i} \right)^p \left( \frac{\langle J, K \rangle}{2\pi i} \right)^r \left( \frac{\langle K, K \rangle}{2\pi i} \right)^q
\]

\[
\times \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} S^k \left( [\gamma o(J)]^i [\gamma o(K)]^j \right) \left\{ \left[ \frac{c}{2\pi i} J[1] \right]^{m_1-i} \left[ \frac{c}{2\pi i} K[1] \right]^{m_2-j} v, x \right\}, \tau
\]

\[
= \sum_{p,q,r,m_1,m_2,\ell_1,\ell_2} \sum_{\ell_1!\ell_2!\ell_3!\ell_4!} \frac{(2\pi i)^{\ell_1+\ell_2+\ell_3+\ell_4}}{\ell_1!\ell_2!\ell_3!\ell_4!} \left( \frac{\langle J, K \rangle}{2} \right)^{\ell_3} \left( \gamma' \frac{(K,K)}{2} \right)^{\ell_4}
\]

\[
\times \sum_{k=1}^{N} A_k^\nu \left( \frac{\langle J, J \rangle}{2} \right)^p \left( \frac{\langle J, K \rangle}{2} \right)^r \left( \frac{\langle K, K \rangle}{2} \right)^q
\]

\[
\times \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} S^k \left( [\gamma o(J)]^i [\gamma o(K)]^j \right) \left\{ \left[ \frac{c}{2\pi i} J[1] \right]^{m_1-i} \left[ \frac{c}{2\pi i} K[1] \right]^{m_2-j} v, x \right\}, \tau
\]

\[
= \sum_{k=1}^{N} A_k^\nu S^k \left( e^{2\pi i [\omega(J)+\omega(K)]} e^{2\pi i \left( \frac{\langle J, K \rangle}{2} \right) \left( \gamma' + \frac{\gamma'^2}{2\gamma'} + (\frac{\langle J, K \rangle}{2} + 2\langle J, J \rangle + (\frac{\langle J, J \rangle}{2})) \right)} \right)
\]

\[
\left\{ e^{J[1]} e^{\frac{\gamma o(J)}{\gamma' + \gamma'^2}} K[1], v, x \right\}, \tau
\]

\[
= \sum_{k=1}^{N} A_k^\nu S^k \left( e^{2\pi i (\omega(bK+dJ)+\frac{1}{2} bK + dJ, aK+cJ)} q^{(\omega(aK+cJ)+\frac{1}{2} aK + cJ, aA+cJ)} \right), \left\{ e^{J[1]} e^{\frac{a\tau + b}{\tau + d}} K[1], v, x \right\}, \tau
\]
and
\[ \frac{c(a\tau + b)}{c\tau + d} = a - \frac{1}{c\tau + d} \] (24)
for \( \gamma \in \text{SL}_2(\mathbb{Z}) \). It follows that

\[ \Phi_{\nu}\left( v; (J,K), \frac{a\tau + b}{c\tau + d} \right) = \sum_{k=1}^{N} A_k^\nu \Phi_k \left( e^\frac{aJ[1]}{c\tau + d} e^{aK[1]} e^\frac{bK[1]}{c\tau + d} v; (bK + dJ, aK + cJ), \tau \right) \]

\[ \sum_{n=0}^{\text{wt}[v]} \sum_{k=1}^{N} A_k^\nu \Phi_k \left( e^{(aK+cJ)[1]} (-K[1])^n v; (bK + dJ, aK + cJ), \tau \right), \]
proving Theorem 1.1. \( \square \)

We complete this section by proving Corollary 1.2.

**Proof of Corollary 1.2.** Since \( K[1]^{\ell} v \in V_{\text{wt}[v]-\ell} \), equation (5) gives

\[ \Phi_j \left( e^{K[1]} v; (J,K), \frac{a\tau + b}{c\tau + d} \right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \Phi_j \left( K[1]^{\ell} v; (J,K), \frac{a\tau + b}{c\tau + d} \right) \]

\[ = \sum_{\ell=0}^{\infty} (c\tau + d)^{\text{wt}[v]-\ell} \sum_{k=1}^{N} A_j^\gamma A_{j,k}^\nu \Phi_k \left( e^{aK[1]+cJ[1]} (-K[1])^\ell v; (bK + dJ, aK + cJ), \tau \right) \]

\[ = \sum_{\ell=0}^{\infty} (c\tau + d)^{\text{wt}[v]} \sum_{k=1}^{N} A_j^\gamma A_{j,k}^\nu \Phi_k \left( e^{aK[1]+cJ[1]} v; (bK + dJ, aK + cJ), \tau \right) \]

as desired. \( \square \)

### 4 Application: lattice VOAs and theta functions

In this section we use the theory of VOAs, and in particular Theorem 1.1, to provide another proof of the modular transformation laws for derivatives of Jacobi theta functions. In doing so, we also study an example of a one-point theta function for an element of a VOA other than \( v = 1 \).
Let $V = V_{2\mathbb{Z} \alpha}$ be the lattice VOA constructed from the 1-dimensional positive definite even lattice $2\mathbb{Z} \alpha$, where $\langle \alpha, \alpha \rangle = 1$. It is known (see [3]) that $V$ has the four inequivalent irreducible modules

\[ M^0 = V, \quad M^1 = V_{(2\mathbb{Z} + \frac{1}{2}) \alpha}, \quad M^2 = V_{(2\mathbb{Z} + 1) \alpha}, \quad \text{and} \quad M^3 = V_{(2\mathbb{Z} - \frac{1}{2}) \alpha}. \]

Consider the Jacobi theta functions defined as

\[ \vartheta_{hk}(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{h}{2})^2 \tau + 2\pi i (n + \frac{h}{2})(z + \frac{k}{2})}, \]

where $h, k = 0, 1$ (see [15], for example, for more details). The transformation laws with respect to the matrix $S = (\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix})$ are

\[ \vartheta_{hk}(\tau, z) = i^{hk} \vartheta_{hk}(i\tau, \frac{1}{2}z). \]

Recall that $\alpha[n] \alpha = \delta_{n,1} \alpha$ for $n \geq 0$. Set $D_x := \frac{1}{2\pi i} \frac{d}{dx}$ for a variable $x$ and let $\vartheta'(\tau, z) := D_x \vartheta(\tau, z)$. Note that $D_x \Phi_j(1 : \{z \alpha, 0\}, \tau) = \Phi_j(\alpha : \{z \alpha, 0\}, \tau)$ for any $j$. It can be shown that

\[ \vartheta_{hk}(\tau, z) = \eta(\tau)i^{hk} \left( \Phi_h(1 : \{z \alpha, 0\}, \tau) + (-1)^k \Phi_{2+h}(1 : \{z \alpha, 0\}, \tau) \right), \]

where $\eta(\tau)$ is the Dedekind eta-function and transforms as $\eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}} \eta(\tau)$. It also follows that

\[ \vartheta'_{hk}(\tau, z) = \eta(\tau)i^{hk} \left( \Phi_h(\alpha : \{z \alpha, 0\}, \tau) + (-1)^k \Phi_{2+h}(\alpha : \{z \alpha, 0\}, \tau) \right). \]

Meanwhile, the values $A^y_{h,j}, 0 \leq j \leq 3$, from Theorem 1.1 (where we have began our indexing of modules with 0 here) are known when $\gamma$ is $S$ or the matrix $T = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ (see [8, Example 1], for example). The values for the $S$-matrix are $(S_h^j) := (A^S_{h,j}) = (\frac{1}{2} e^{\pi i \gamma h j})$. Using Theorem 1.1, we find

\[ \vartheta'_{hk}(\tau, z) = i^{hk} \eta(\tau) \left( \vartheta_{hk}(\tau, z) + \vartheta_{hk}(\tau, z)^\gamma \right) \]

Additionally, using the values for the $T$-matrix $(T_h^j) := (A^T_{h,j}) = (\delta_{h,j} e^{\pi i h j \frac{m}{n}})$, where $\delta_{h,j}$ is
1 if $h = j$ and 0 otherwise, we find

$$\vartheta'_{hk}(\tau + 1, z) = i^{hk} \eta(\tau + 1) \left[ \Phi_h(\alpha : (z\alpha, 0), \tau + 1) + (-1)^k \Phi_{2+h}(\alpha : (z\alpha, 0), \tau + 1) \right]$$

$$= i^{hk} e^{\frac{i\pi}{4}} \eta(\tau) \sum_{j=0}^{3} \left[ T^j_h \Phi_j(\alpha : (z\alpha, 0), \tau) + (-1)^k T^{j}_{2+h} \Phi_j(\alpha : (z\alpha, 0), \tau) \right]$$

$$= i^{hk} \eta(\tau) \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)^h \left( \Phi_h(\alpha : (z\alpha, 0), \tau) - i^h(-1)^k \Phi_{2+h}(\alpha : (z\alpha, 0), \tau) \right)$$

$$= \delta_{k,0} (\vartheta'_{01} \tau + \delta_{k,1} \vartheta'_{00}(\tau, z)) + \delta_{k,1} \frac{\sqrt{2}}{2} (\vartheta'_{11}(\tau, z) + (-1)^k \vartheta'_{10}(\tau, z)),$$

which establishes the modular transformation laws for derivatives of Jacobi theta functions.

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