SOME RELATIVE STABLE CATEGORIES ARE COMPACTLY GENERATED

MATTHEW GRIME AND PETER JØRGENSEN

Abstract. Let $G$ be a finite group. The stable module category of $G$ has been applied extensively in group representation theory. In particular, it has been used to great effect that it is a triangulated category which is compactly generated.

Let $H$ be a subgroup of $G$. It is possible to define a stable module category of $G$ relative to $H$. It too is a triangulated category, but no non-trivial examples have been known where this relative stable category was compactly generated.

We show here that the relative stable category is compactly generated if the group algebra of $H$ has finite representation type. In characteristic $p$, this is equivalent to the Sylow $p$-subgroups of $H$ being cyclic.

The study of localizations of triangulated categories has a rich and varied heritage arising from the work of Adams, Bousfield, Brown, Thomason, and others. Neeman further developed these theories, and showed that to bring the full power of such arguments to bear one needs a compactly generated category.

Localization techniques were brought to the attention of the representation theory world in Rickard’s [10], where they were applied to the stable module category which is easily shown to be compactly generated.

The stable module category is not the only triangulated quotient of the module category that one meets in representation theory. In [4] Carlson, Peng, and Wheeler note that one can adapt Rickard’s work to relative stable module categories. However, not much is known about the structure of these categories. In particular no non-trivial examples have been given which are known to be compactly generated.

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In this note we prove the following theorem on the relative stable module category $\text{StMod}_H(kG)$:

**Theorem.** Let $k$ be an algebraic closure of $\mathbb{Z}/p$. Let $G$ be a finite group, $H$ a subgroup of $G$. If $kH$ has finite representation type, then $\text{StMod}_H(kG)$ is compactly generated.

We will keep the assumptions on $k$, $G$, and $H$ for the rest of the paper. All modules will be left-modules. Recall that $kH$ has finite representation type precisely if the Sylow $p$-subgroups of $H$ are cyclic, see [2, thm. VI.3.3]. Let us remind the reader of the construction of $\text{StMod}_H(kG)$.

We will denote the class of $H$-projective $kG$-modules by $H$-Proj; this is the class of all summands of modules induced up from $kH$. It is an additive subcategory of $\text{Mod}(kG)$. We use $H$-Proj to define the triangulated categories $\text{StMod}_H(kG)$ and $K(H$-Proj).

The relative stable module category $\text{StMod}_H(kG)$ is $\text{Mod}(kG)$ modulo the morphisms that factor through objects of $H$-Proj. The category $K(H$-Proj) is the homotopy category of complexes of objects of $H$-Proj.

By $\text{Tate}_H(kG)$ we denote the collection of complexes $Q$ of $H$-Proj-modules for which the restriction $\text{Res}_G^H(Q)$ to $H$ is split exact. We may think of $\text{Tate}_H(kG)$ either as a triangulated subcategory of $K(H$-Proj), or as a full subcategory of $C(H$-Proj), the category of complexes of objects of $H$-Proj and chain maps.

**Remark 1.** If $X$ is in $\text{Tate}_H(kG)$ then $X$ is exact and splits into short exact sequences

$$0 \to Z^n(X) \to X^n \to Z^{n+1}(X) \to 0$$

which become split exact upon restriction to $H$. In particular, it is easy to show that $Z^n(X) \to X^n$ is an $H$-Proj-preenvelope and $X^n \to Z^{n+1}(X)$ is an $H$-Proj-precovar.

Several variants of the following result are well known, see for instance [3, thm. 2.3] and [6, thm. 9.6.4].

**Proposition 2.** View $\text{Tate}_H(kG)$ as a full subcategory of $K(H$-Proj). There is a triangulated equivalence of categories

$$\text{Tate}_H(kG) \simeq \text{StMod}_H(kG)$$

given by $X \mapsto Z^0(X)$.

**Proof.** We can clearly view $Z^0$ as a functor $C(H$-Proj) $\to \text{Mod}(kG)$. Viewing $\text{Tate}_H(kG)$ as a full subcategory of $C(H$-Proj), we hence have
a functor

\[ Z^0 : \text{Tate}_H(kG) \to \text{Mod}(kG). \]  

Let \( X \to Y \) be a chain map of complexes from \( \text{Tate}_H(kG) \). There is an induced homomorphism \( Z^0(X) \to Z^0(Y) \). The chain map is null homotopic if and only if the induced homomorphism factors through a module from \( H\)-Proj, that is, if and only if the induced homomorphism becomes 0 in \( \text{StMod}_H(kG) \). This holds by a lifting argument using Remark [1] cf. [3, proof of lem. 2.2].

Hence the functor from equation (1) induces a faithful functor

\[ Z^0 : \text{Tate}_H(kG) \to \text{StMod}_H(kG) \]  

where \( \text{Tate}_H(kG) \) is now viewed as a full subcategory of \( K(H\text{-Proj}) \).

Observe that \( X \) can be viewed as a relative Tate resolution of \( Z^0(X) \). Hence the functor from (2) is also full, since any homomorphism of modules can be lifted to the relative Tate resolutions; this is again a lifting argument using Remark [1].

To conclude that the functor from (2) is an equivalence of categories, all that is needed is to see that it is essentially surjective. But each \( kG \)-module \( m \) has a relative Tate resolution \( X \), so indeed, \( m \cong Z^0(X) \) for some \( X \). Note that we can construct such an \( X \) by splicing a left-\( H\)-Proj-resolution and a right-\( H\)-Proj-resolution of \( m \). These resolutions become split exact upon restriction to \( H \) because this is true for \( H\)-Proj-precovers and -preenvelopes. \( \square \)

**Definition 3.** If \( kH \) has finite representation type, then \( y \) will be the direct sum of its indecomposable finitely generated modules, and \( x = \text{Ind}_{H}^{G}(y) \) the induced module over \( kG \).

**Remark 4.** In the case of the definition, note that \( H\)-Proj = \( \text{Add}(x) \). Note also that \( x \) can be viewed as a complex concentrated in degree zero. As such, it is in \( K(H\text{-Proj}) \).

**Lemma 5.** We have

\[ \text{Tate}_H(kG) = x^\perp = \{ Q \in K(H\text{-Proj}) \mid \text{Hom}_{K(H\text{-Proj})}(\Sigma^n x, Q) = 0 \text{ for each } n \} \]

in \( K(H\text{-Proj}) \).

**Proof.** Let \( Q \) be in \( K(H\text{-Proj}) \). Then

\[
\text{Hom}_{K(H\text{-Proj})}(\Sigma^n x, Q) = \text{Hom}_{K(kG)}(\Sigma^n \text{Ind}_{H}^{G}(y), Q) \\
\cong \text{Hom}_{K(kH)}(\Sigma^n y, \text{Res}_{H}^{G}(Q)) \\
= (\ast)
\]
by adjointness, since Ind\(_G^H(y) = kG \otimes_{kH} y\) while Res\(_H^G\) restricts \(kG\)-modules to \(kH\)-modules. If \((\ast)\) is 0 then so is

\[
\text{Hom}_{K(kH)}(\Sigma^n m, \text{Res}_H^G(Q))
\]

for each \(m\) in Mod\((kH)\), since Mod\((kH)\) equals Add\((y)\) by [1, cor. 4.8] because \(kH\) has finite representation type. But if this expression is 0 for each \(m\) and each \(n\), then Res\(_H^G(Q)\) is null homotopic by an easy argument; that is, \(Q\) is in Tate\(_H(kG)\). □

**Proposition 6.** If \(kH\) has finite representation type, then \(K(H\text{-Proj})\) is compactly generated.

**Proof.** Since \(k\) is countable, \(kG\) has pure global dimension \(\leq 1\) by [8, thm. 11.21]. The finite dimensional algebra \(kG\) is certainly coherent, and \(x\) is a finitely generated \(kG\)-module.

By [3] sec. 4, (1)], the category \(K(\text{Add} x)\) is compactly generated. But this category is \(K(H\text{-Proj})\) by Remark [4]. □

**Corollary 7.** If \(kH\) has finite representation type, then Tate\(_H(kG)\) is compactly generated.

**Proof.** The category \(K(H\text{-Proj})\) is compactly generated by Proposition [6] and Tate\(_H(kG) = x^\perp\) by Lemma [5].

But \(x\) is a compact object of \(K(H\text{-Proj})\), as follows for instance from the formula

\[
\text{Hom}_{K(H\text{-Proj})}(x, -) \cong H^0 \text{Hom}_{kG}(x, -)
\]

since \(x\) is finitely generated over \(kG\).

So Tate\(_H(kG)\) is the right perpendicular category of a compact object, so it is compactly generated by [7, prop. 1.7(1)]. □

Finally, the theorem from page [2] follows.

**Theorem 8.** If \(kH\) has finite representation type, then StMod\(_H(kG)\) is compactly generated.

**Proof.** Combine Proposition [2] with Corollary [7]. □

**Remark 9.** It is not clear that our methods can be used to compute a set of compact generators of StMod\(_H(kG)\).

To do so, we would need to find a set of compact generators of the category Tate\(_H(kG)\) and then use the equivalence \(Z^0\). By unravelling the proof of [7] prop. 1.7(1)], it can be seen that the compact generators of Tate\(_H(kG)\) would come by taking a set of compact generators of
K(H-Proj) and applying the left adjoint to the inclusion of \( \text{Tate}_H(kG) \) into \( K(H\text{-Proj}) \). This left adjoint is constructed by Neeman in [9], but the construction is infinite and does not obviously lend itself to concrete computations.

It would be interesting to find a procedure whereby a set of compact generators of \( \text{StMod}_H(kG) \) could be computed.

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Jørgensen: School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom

E-mail address: peter.jorgensen@ncl.ac.uk

URL: http://www.staff.ncl.ac.uk/peter.jorgensen