BOOTSTRAPPING WHITTLE ESTIMATORS

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ABSTRACT. Fitting parametric models by optimizing frequency domain objective functions is an attractive approach of parameter estimation in time series analysis. Whittle estimators are a prominent example in this context. Under weak conditions and the (realistic) assumption that the true spectral density of the underlying process does not necessarily belong to the parametric class of spectral densities fitted, the distribution of Whittle estimators typically depends on difficult to estimate characteristics of the underlying process. This makes the implementation of asymptotic results for the construction of confidence intervals or for assessing the variability of estimators, difficult in practice. This paper proposes a frequency domain bootstrap method to estimate the distribution of Whittle estimators which is asymptotically valid under assumptions that not only allow for (possible) model misspecification but also for weak dependence conditions which are satisfied by a wide range of stationary stochastic processes. Adaptions of the bootstrap procedure developed to incorporate different modifications of Whittle estimators proposed in the literature, like for instance, tapered, de-biased or boundary extended Whittle estimators, are also considered. Simulations demonstrate the capabilities of the bootstrap method proposed and its good finite sample performance. A real-life data analysis also is presented.

1. INTRODUCTION

Optimizing frequency domain objective functions is an attractive approach of fitting parametric models to observed time series. Let $X_1, X_2, \ldots, X_n$ be a time series stemming from a stationary process $\{X_t, t \in \mathbb{Z}\}$ and assume that this process possesses a spectral density $f$. Let $\mathcal{F}_\theta$ be a family of parametric spectral densities, where $f_\theta \in \mathcal{F}_\theta$ is determined by a $m$-dimensional parameter vector $\theta$ and suppose that we are interested in fitting to $X_1, X_2, \ldots, X_n$ a model from the class $\mathcal{F}_\theta$. Notice that we do not assume $f \in \mathcal{F}_\theta$, that is, we allow for the practical important case of model misspecification where the parametric model class considered does not necessarily contain the true spectral density $f$.

Several approaches for selecting a frequency domain objective function and consequently for developing a frequency domain procedure to fit parametric models exist; we refer here

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to Taniguchi (1987) and to Dahlhaus and Wefelmeyer (1996) for examples. In this context, Whittle estimators play an important role. This is due to the fact that Whittle estimators are computationally fast and they are obtained via minimizing a frequency domain approximation of the (Gaussian) log likelihood function. Moreover, for a variety of models and under different assumptions, Whittle estimators are asymptotically normal, asymptotically equivalent to the exact maximum likelihood estimators and asymptotically efficient (in Fisher-sense) if \( f \in \mathcal{F}_\theta \); see Sykulski et al. (2019) and Subba Rao and Yang (2020) for a recent discussion of the related literature. Furthermore and even if \( f \notin \mathcal{F}_\theta \), Whittle estimators retain certain efficiency properties; Dahlhaus and Wefelmeyer (1996). Despite these nice properties, however, the limiting distribution of Whittle’s estimators is affected by characteristics of the process which make the implementation of asymptotic results for assessing their variability or for constructing confidence intervals, difficult in practice. To elaborate, allowing for possible model misspecification and avoiding restrictive structural assumptions for the underlying process class, like for instance linearity assumptions, the limiting distribution of Whittle estimators typically depends on the parametric spectral density from the class \( \mathcal{F}_\theta \) which “best fits” the data, say \( f_{\theta_0} \), the true spectral density \( f \) as well as the entire fourth order cumulant structure of the underlying process \( \{X_t; t \in \mathbb{Z}\} \).

Estimation of the last mentioned quantity is a rather difficult problem.

In situations like the above, bootstrapping may offer an alternative to classical large sample approximations. Bootstrapping Whittle estimators has been discussed in the literature under certain structural assumptions on the underlying process \( \{X_t, t \in \mathbb{Z}\} \). It is typically assumed that \( \{X_t, t \in \mathbb{Z}\} \) is a linear process driven by i.i.d. innovations and that Kolmogorov’s formula holds true; see Dahlhaus and Janas (1996) and Kim and Nordman (2013). Such structural assumptions lead to a simplification of the limiting distribution of Whittle estimators and consequently of the features of the underlying linear process that the bootstrap procedure has to appropriately mimic in order to be consistent. To elaborate, notice first that, under linearity assumptions, the problem of estimating the variance \( \sigma^2 \) of the i.i.d. innovations driving the linear process can be separated from the problem of estimating the remaining coefficients of the parameter vector, denoted by \( \tau \in \mathbb{R}^{m-1} \), i.e., \( \theta = (\sigma^2, \tau) \). Second and more importantly, under the linearity assumption, the limiting distribution of the Whittle estimator of the parameter part \( \tau \), does not depend on the fourth order cumulants of the process; see Section 2 for details. Therefore, if one is solely interested in estimating the distribution of the Whittle estimator of the innovation free part \( \tau \), then standard frequency domain bootstrap procedures which generate pseudo periodogram ordinates that are independent across frequencies, can successfully be applied. In this context, the multiplicative bootstrap, see Hurvich an Zeger (1987), Franke and Härdle (1992) and Dahlhaus and Janas (1996), or the local periodogram bootstrap, Paparoditis and Politis (1999), are consistent. However, the pseudo periodogram ordinates generated by the aforementioned bootstrap procedures are independent across frequencies. Therefore, these bootstrap procedures are not able to imitate the covariance structure of the periodogram ordinates which is responsible for the fact that the fourth structure of the process shows up in the limiting distribution of Whittle estimators. As a consequence, these procedures fail in all cases where the fourth order cumulants of the process affect the distribution of interest.
In this paper we present a frequency domain, hybrid bootstrap procedure for Whittle estimators which is valid under weak assumptions on the underlying process \( \{X_t, t \in \mathbb{Z}\} \) and which also covers the practical important case where the true spectral density \( f \) does not necessarily belong to the parametric class \( \mathcal{F}_\theta \). The procedure consists of two main parts: A multiplicative frequency domain bootstrap part and a part based on the convolution of resampled periodograms of subsamples. The latter is an adaption to the frequency domain of the convolved subsampling idea proposed in Tewes et al. (2019). The two parts contribute differently and complementary in estimating the distribution of interest. The multiplicative part of the bootstrap procedure is used to estimate all features of the distribution of Whittle estimators including the parts of the limiting covariance matrix that depend on the second order structure of the underlying process and of the parametric model fitted to the time series at hand. However, the components of the covariance matrix of these estimators that depend on the fourth order structure of the underlying process \( \{X_t, t \in \mathbb{Z}\} \), are estimated using the part of the bootstrap procedure which is based on the convolution of resampled periodograms of subsamples. Putting the two parts together in an appropriate way, leads to a bootstrap procedure which is asymptotically valid under conditions on the dependence structure of the process \( \{X_t, t \in \mathbb{Z}\} \) which go far beyond linearity and at the same time appropriately captures the effects of (possible) model misspecification on the distribution of interest.

The frequency domain, hybrid approach proposed in this paper and which uses convolution of resampled periodograms of subsamples together with the multiplicative periodogram bootstrap, is related to the proposal of Meyer et al. (2020). However and additional to differences in the technical tools used to establish bootstrap consistency, the merging of the multiplicative and of the convolved part of the bootstrap procedure presented here, is different, more involved and tailormade for Whittle estimators. Furthermore, we show, how the described bootstrap procedure can appropriately be modified, respectively, extended to incorporate several modifications of Whittle estimators which have been proposed in order to improve the finite (small) sample bias of these estimators. This concerns tapered Whittle estimators, Dahlhaus (1988), so-called de-biased Whittle estimators, Sykulski et al. (2019), and improvements of Whittle’s quasi Gaussian likelihood approximation based on boundary corrected periodograms; see Subba Rao and Yang (2020). The corresponding extensions of the frequency domain bootstrap proposed in this context are novel and of interest on their own.

The paper is organized as follows. In Section 2 we review some basic results on Whittle estimators which are important for our subsequent development of the bootstrap. The basic frequency domain bootstrap procedure introduced is presented in Section 3. Section 4 is devoted to the derivation of theoretical results and establishes consistency of the bootstrap. Section 5 deals with the extensions of the basic bootstrap procedure in order to incorporate the aforementioned modifications of standard Whittle estimators. Section 6 discusses some issues related to the practical implementation of the bootstrap algorithm, presents simulations which investigate the finite sample performance of the new bootstrap method and make comparisons with the asymptotic Gaussian approximation, respectively,
the multiplicative periodogram bootstrap. A real-life data application also is discussed. Auxiliary lemmas as well as proofs of the main results are deferred to Section 7.

2. Whittle Estimators

In his PhD thesis Whittle introduced a frequency domain approximation of the log-likelihood function of a stationary Gaussian time series (cf. Whittle (1951) and Whittle (1953)). This approximation can be written as

\[ l_n(\theta) = 2n \log(2\pi) + \sum_{j \in F_n} \left\{ \log f(\lambda_{j,n}) + \frac{I_n(\lambda_{j,n})}{f(\lambda_{j,n})} \right\}, \]

where \( I_n(\lambda_{j,n}) \), denotes the periodogram of the time series \( X_1, X_2, \ldots, X_n \) evaluated at the Fourier frequency \( \lambda_{j,n} = \frac{2\pi j}{n} \in F_n \) and \( F_n = \{-[(n-1)/2], \ldots, [n/2]\} \) is the set of Fourier frequencies. Ignoring the first additive term and approximating the integral over the set \( G(n) = \{-N, -N+1, \ldots, -1, 1, \ldots, N\} \), where \( N = \lfloor n/2 \rfloor \), Whittle’s approximation to the log-likelihood function used in this paper, is given by

\[ D_n(\theta, I_n) = \frac{1}{n} \sum_{j \in G(n)} \left\{ \log f_\theta(\lambda_{j,n}) + \frac{I_n(\lambda_{j,n})}{f_\theta(\lambda_{j,n})} \right\}. \]

A minimizer \( \hat{\theta}_n \) of

\[ \theta \mapsto D_n(\theta, I_n), \]

is called a Whittle estimator of \( \theta \). Note that (2.1) can be considered as a Riemann sum approximation of

\[ D(\theta, I_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \log f_\theta(\lambda) + \frac{I_n(\lambda)}{f_\theta(\lambda)} \right\} d\lambda. \]

Let

\[ D(\theta, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \log f_\theta(\lambda) + \frac{f(\lambda)}{f_\theta(\lambda)} \right\} d\lambda \]

and assume that

\[ \theta_0 = \arg\min_{\theta \in \Theta} D(\theta, f), \]

exists and is unique. \( \hat{\theta}_n \) is then an estimator of \( \theta_0 \) and \( f_{\theta_0} \) denotes the spectral density from the parametric family \( F_\theta \) which best fits the spectral density \( f \) of the underlying process \( \{X_t, t \in \mathbb{Z}\} \) in the sense of minimizing the divergence measure (2.4). Note that if \( f \not\in F_\theta \), then the best approximating parametric spectral density \( f_{\theta_0} \) from the class \( F_\theta \), clearly depends on the particular divergence measure \( D(\theta, f) \) associated with Whittle’s approximation of the log-likelihood function. Observe that \( \theta_0 \) which minimizes \( D(\theta, f) \) is the same as the one which minimizes the so-called Kullback-Leibler information divergence. The later is given for Gaussian processes by

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \log \frac{f_\theta(\lambda)}{f(\lambda)} + \frac{f(\lambda)}{f_\theta(\lambda)} - 1 \right\} d\lambda; \]
see Dahlhaus and Wefelmeyer (1996). Different modifications of Whittle’s likelihood approximation \((2.3)\) have been proposed in the literature in order to improve the finite sample behavior and more specifically the bias of the estimator \(\hat{\theta}_n\). We mention here the tapered Whittle likelihood proposed by Dahlhaus (1988), the debiased Whittle likelihood proposed by Sykulski et al. (2019), the boundary corrected and the hybrid Whittle likelihood approximation proposed by Subba Rao and Yang (2020). For the sake of a better presentation, however, we will first focus on the standard Whittle likelihood approximation \((2.1)\), respectively, \((2.3)\). Later on, we will elaborate on how to appropriately modify the basic bootstrap procedure proposed in order to take into account the aforementioned modifications/extensions of Whittle estimators.

Let us briefly review the main ideas involved in deriving the limiting distribution of the estimator \(\hat{\theta}_n\) and which are important for our subsequent discussion of the bootstrap. Consider \((2.3)\) and \((2.4)\), assume that \(f_\theta\) is sufficiently smooth with respect to \(\theta\) and recall that \(\theta_n\) and \(\theta_0\) satisfy the score equations

\[
\frac{\partial}{\partial \theta} D_n(\theta, I_n) \bigg|_{\theta = \hat{\theta}_n} = 0 \quad \text{and} \quad \frac{\partial}{\partial \theta} D(\theta, f) \bigg|_{\theta = \theta_0} = 0,
\]

respectively. Using a linear approximation of \(\frac{\partial}{\partial \theta} D_n(\theta, I_n)\bigg|_{\theta = \hat{\theta}_n}\) around \(\theta_0\) we get, taking into account \((2.6)\), that

\[
0 = \frac{\partial}{\partial \theta} D_n(\theta_0, I_n) + \frac{\partial^2}{\partial \theta \partial \theta^\top} D_n(\theta_0, I_n) (\hat{\theta}_n - \theta_0) + R_n.
\]

Notice that for simplicity, the notation \(\frac{\partial}{\partial \theta} D_n(\theta_0, I_n)\) for \(\frac{\partial}{\partial \theta} D_n(\theta, I_n)\bigg|_{\theta = \theta_0}\) has been used with an analogue notation for the matrix of second order partial derivatives \(\frac{\partial^2}{\partial \theta \partial \theta^\top} D_n(\theta_0, I_n)\). Provided that the remainder \(R_n\) is \(o_P(n^{-1/2})\) and that \(\frac{\partial^2}{\partial \theta \partial \theta^\top} D_n(\theta_0, I_n)\) is invertible, the following basic expression is then obtained,

\[
\sqrt{n}(\hat{\theta}_n - \theta_0) = \left( \frac{\partial^2}{\partial \theta \partial \theta^\top} D_n(\theta_0, I_n) \right)^{-1} \sqrt{n} \int_{-\pi}^{\pi} g_{\theta_0}(\lambda) (I_n(\lambda) - f(\lambda)) d\lambda + o_P(1).
\]

Here \(g_{\theta_0}(\lambda)\) is a \(m\)-dimensional vector of scores, the \(j\)th element of which is given by

\[
g_{j,\theta_0}(\lambda) = \frac{1}{2\pi f_{\theta_0}(\lambda)} \frac{\partial}{\partial \theta_j} \log f_{\theta_0}(\lambda) = -\frac{1}{2\pi} \frac{\partial}{\partial \theta_j} f_{\theta_0}^{-1}(\lambda),
\]

\(j = 1, 2, \ldots, m\) and \(f_{\theta}^{-1} = 1/f_{\theta}\). Equation \((2.7)\) suggests that the distribution of Whittle’s estimator \(\hat{\theta}_n\), can be well approximated by the product of the inverse of the \(m \times m\) random matrix \(W_n = \left( \frac{\partial^2}{\partial \theta_j \partial \theta_k} D_n(\theta_0, I_n) \right)_{j,k=1,2,\ldots,m}\) with the \(m\)-dimensional vector of integrated periodograms \(\sqrt{n} \int_{-\pi}^{\pi} g_{j,\theta_0}(\lambda) (I_n(\lambda) - f(\lambda)) d\lambda, j = 1, 2, \ldots, m\)\(^\top\).

Based on expression \((2.7)\), asymptotic theory for Whittle estimators has been developed in the literature under a variety of assumptions on the dependence structure of the underlying process. In early papers Walker (1964) and Hannan (1973) derived asymptotic normality under linearity assumptions while Hosoya (1979) also allowed for long range dependence.

For a more recent review of the related literature as well as derivations of asymptotic normality based on a physical dependence measure, we refer to Shao (2010). It is typically shown, that,

\[
\sqrt{n}(\hat{\theta}_n - \theta_0) \overset{D}{\to} \mathcal{N}(0, W^{-1}(V_1 + V_2)W^{-1}),
\]
as \( n \to \infty \), where the \( m \times m \) matrices \( W, V_1 \) and \( V_2 \) are given by
\[
W = \left( \frac{\partial^2}{\partial \theta_j \partial \theta_k} D(\theta_0, f) \right)_{j,k=1,2,\ldots,m},
\]
\[
V_1 = \left( 4\pi \int_{-\pi}^{\pi} g_{j,\theta_0}(\lambda) g_{k,\theta_0}(\lambda) f^2(\lambda) d\lambda \right)_{j,k=1,2,\ldots,m}
\]
and
\[
V_2 = \left( 2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_{j,\theta_0}(\lambda_1) g_{k,\theta_0}(\lambda_2) f_4(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 \right)_{j,k=1,2,\ldots,m}.
\]
Observe that \( W \) and \( V_1 \) only depend on the parametric and the true spectral densities, that is on \( f_{\theta_0} \) and \( f \), while the matrix \( V_2 \) depends on \( f_{\theta_0} \) and on the fourth order cumulant spectral density \( f_4 \) of the underlying process \( \{X_t, t \in \mathbb{Z} \} \). The latter is defined as
\[
f_4(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{(2\pi)^3} \sum_{h_1, h_2, h_3 \in \mathbb{Z}} \text{cum}(X_0, X_{h_1}, X_{h_2}, X_{h_3}) \exp^{-i(h_1 \lambda_1 + h_2 \lambda_2 + h_3 \lambda_3)},
\]
where \( \text{cum}(X_0, X_{h_1}, X_{h_2}, X_{h_3}) \) denotes the fourth order cumulant of the process \( \{X_t; t \in \mathbb{Z} \} \), cf. Rosenblatt (1985). Notice that if \( f = f_{\theta_0} \), that is, if the spectral density of the process belongs to the parametric family \( F_{\theta} \) and the model is correctly specified, then \( V_1 = 2W \) and the covariance matrix of the limiting Gaussian distribution (2.8) is given by \( W^{-1}(2I_m + V_2W^{-1}) \), where \( I_m \) denotes the \( m \times m \) unit matrix.

The matrix \( V_2 \) in general does not entirely disappear even if the underlying process is linear, that is, if \( X_t \) is generated as \( X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j} \), where \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \) and the \( \varepsilon_i \)'s are zero mean, i.i.d. innovations with variance \( \sigma^2 > 0 \). However, in this case, the matrix \( V_2 \) simplifies considerably. To elaborate, recall that under the assumption of linearity and of validity of Kolmogorov’s formulae, (see Blockwell and Davis (1991), Ch. 5.8), \( \theta = (\sigma^2, \tau) \) where \( \tau \in \mathbb{R}^{m-1} \) is free of the innovation variance \( \sigma^2 \). The dependence on the fourth order moment structure of Whittle estimators disappears then if one is solely interested in the distribution of the estimators of the part \( \tau \) of the parameter vector. This is due to the fact that for linear processes, the spectral density \( f_{\theta}(-\cdot) \) factorizes as \( f_{\theta}(-\cdot) = h_{\tau}(-\cdot)\sigma^2/(2\pi) \), where the function \( h_{\tau}(-\cdot) \) depends on \( \tau \) only. Then, and since for the same class of processes, \( f_4(\lambda_1, \lambda_2, -\lambda_2) = (2\pi)^{-1} \eta_4 f(\lambda_1) f(\lambda_2) \), with \( \eta_4 = E(\varepsilon_1^4/\sigma^4 - 3) \), the rescaled fourth order cumulants (kurtosis) of the i.i.d. innovations, we get that,
\[
2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_{j,\tau_0}(\lambda_1) g_{k,\tau_0}(\lambda_2) f_4(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2
\]
\[
= \eta_4 \int_{-\pi}^{\pi} g_{j,\tau_0}(\lambda_1) f(\lambda_1) d\lambda_1 \int_{-\pi}^{\pi} g_{k,\tau_0}(\lambda_2) f(\lambda_2) d\lambda_2 = 0.
\]
The last equality follows since the score function implies \( \int_{-\pi}^{\pi} g_{s,\tau_0}(\lambda) f(\lambda) d\lambda = 0 \) for every \( s = 1, 2, \ldots, m-1 \), where \( g_{s,\tau_0}(-\cdot) \) denotes the partial derivative of \(-h_{\tau}^{-1}(-\cdot)/(2\pi)\) with respect to the \( s \)-th variable of the \( m-1 \) dimensional vector \( \tau \), evaluated at \( \tau = \tau_0 \). Thus the \((m-1) \times (m-1)\) submatrix of \( V_2 \) which corresponds to the elements of the vector \( \tau \) only, consists of zeros. Notice that this simplification does not hold true for the components of the matrix \( V_2 \) which are affected by the estimator of \( \sigma^2 \); see Dahlhaus and Janas (1996) for more details.
Now, a close look at the derivations leading to the covariance formulae $W^{-1}(V_1 + V_2)W^{-1}$ of the limiting Gaussian distribution (2.8), reveals that the term $V_2$ is solely due to the weak and asymptotically vanishing covariance of the periodogram ordinates across frequencies. Recall the basic expression of the covariance of the periodogram for nonzero Fourier frequencies $|\lambda_{j,n}| \neq |\lambda_{k,n}|$,

\begin{equation}
\text{cov}(I_n(\lambda_{j,n}), I_n(\lambda_{k,n})) = \frac{1}{n} f_4(\lambda_{j,n}, \lambda_{k,n}, -\lambda_{k,n})(1 + o(1)) + O(n^{-2}).
\end{equation}

Summing up these covariances over all frequencies $|\lambda_{j,n}| \neq |\lambda_{k,n}|$ in the set $G(n)$, leads to a non-vanishing contribution to the limiting distribution of Whittle estimators as expressed by the matrix $V_2$. As already mentioned, frequency domain bootstrap procedures which generate independent periodogram ordinates, like for instance the multiplicative periodogram bootstrap, can not imitate the weak dependence structure (2.9). Therefore, and besides the special case $f_4 = 0$, e.g. Gaussian time series, such procedures do not appropriately capture the term $V_2$ and they fail in consistently estimating the distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$.

3. The basic bootstrap procedure

Our goal is to develop a consistent bootstrap estimator of the distribution of $L_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ without imposing restrictive structural assumptions on the process $\{X_t, t \in \mathbb{Z}\}$ and allowing at the same time for the case of model misspecification. Toward this goal, we propose a hybrid, frequency domain bootstrap procedure which builds upon the multiplicative periodogram bootstrap and appropriately extends it, in order to overcome its limitations for the class of Whittle estimators. The procedure consists of two main parts. The first part is based on the multiplicative bootstrap approach as proposed by Franke and Härdle (1992) and Dahlhaus and Janas (1996). It is used to estimate all features of the distribution of $L_n$ including the parts of the covariance matrix which do not depend on the fourth order characteristics of the process $\{X_t, t \in \mathbb{Z}\}$ and in particular the matrices $W$ and $V_1$. This is done in Step 2 and Step 3 of the following algorithm. However, and as already mentioned, since the multiplicative periodogram bootstrap generates independent pseudo periodogram ordinates, it can not be used to imitate the fourth order characteristics of the process which affect the distribution of $L_n$ and more specifically the matrix $V_2$.

The second part of our bootstrap procedure corrects for this shortcoming. This is achieved by generating pseudo periodograms of subsamples of length $b$, $b < n$, using randomly selected sets of appropriately defined frequency domain residuals. The advantage of these pseudo periodograms of subsamples is that they retain the weak dependence structure of the periodogram across the Fourier frequencies corresponding to the subsamples. They can, therefore, be used to consistently estimate the missing part $V_2$. This is done in Step 4 and Step 5 of the bootstrap algorithm presented below. Putting these two parts together in an appropriate way, finally, leads in Step 6, to a consistent estimator of the distribution of the random sequence $\sqrt{n}(\hat{\theta}_n - \theta_0)$ of interest.

The following algorithm implements the above ideas and is the basic hybrid frequency domain bootstrap procedure proposed in this paper.
Step 1: Calculate Whittle’s estimator \( \hat{\theta}_n \).

Step 2: Let \( \hat{f} \) be a nonparametric estimator of \( f \). For \( j = 1, 2, \ldots, N \), generate

\[
I_n^*(\lambda_{j,n}) = \hat{f}(\lambda_{j,n}) \cdot U_j^*,
\]

where \( U_j^* \) are i.i.d. standard exponential distributed random variables. Set \( I_n^*(\lambda_{j,n}) = I_n^*(-\lambda_{j,n}) \) for \( j = -1, -2, \ldots, -N \).

Step 3: Let

\[
\hat{\theta}_n^* = \arg\min_{\theta \in \Theta} D_n(\theta, I_n^*) \quad \text{and} \quad \hat{\theta}_0 = \arg\min_{\theta \in \Theta} D_n(\theta, \hat{f}).
\]

Define

\[
V_{1,n}^* = Var^*(M_n^*) \quad \text{and} \quad W_n^* = \left( \frac{\partial^2}{\partial \theta \partial \theta^\top} D_n(\theta, I_n^*) \right|_{\theta = \hat{\theta}_0},
\]

where

\[
M_n^* = \frac{2\pi}{\sqrt{n}} \sum_{j \in G(n)} g_{\hat{\theta}_0}(\lambda_{j,n})(I_n^*(\lambda_{j,n}) - \hat{f}(\lambda_{j,n})),
\]

and \( g_{\hat{\theta}_0}(\lambda) \) the \( m \)-dimensional vector

\[
g_{\hat{\theta}_0}(\lambda) = \left( g_{j,\hat{\theta}_0}(\lambda) = -\frac{1}{2\pi} \frac{\partial}{\partial \theta_j} \hat{f}^{-1}(\lambda) \big|_{\theta = \hat{\theta}_0}, \quad j = 1, 2, \ldots, m \right)^\top.
\]

Calculate the pseudo random vector \( Z_n^* \) defined by

\[
Z_n^* = (V_{1,n}^*)^{-1/2} W_n^* \sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_0).
\]

Step 4: Select an integer \( b < n \) and generate \( k = \lfloor n/b \rfloor \) pseudo periodograms pseudo periodograms \( I_b^{(\ell)}(\lambda_{j,b}), \ell = 1, 2, \ldots, k \), where

\[
I_b^{(\ell)}(\lambda_{j,b}) = \hat{f}(\lambda_{j,b}) \cdot U_b^{(\ell)}(\lambda_{j,b}).
\]

Here, \( U_b^{(\ell)}(\lambda_{j,b}) = I_b^{(\ell)}(\lambda_{j,b})/\hat{f}(\lambda_{j,b}) \), where

\[
I_b^{(\ell)}(\lambda_{j,b}) = \frac{1}{2\pi} \left| \sum_{s=1}^b X_{i_\ell+s-1} e^{-is\lambda_{j,b}} \right|^2
\]

and \( i_\ell, \ell = 1, 2, \ldots, k \), are i.i.d. random variables uniformly distributed on the set \( \{1, 2, \ldots, n-b+1\} \). Furthermore,

\[
\tilde{f}(\lambda_{j,b}) = \frac{1}{n-b+1} \sum_{\ell=1}^{n-b+1} I_b^{(\ell)}(\lambda_{j,b}),
\]

with \( I_b^{(\ell)}(\lambda) = (2\pi)^{-1} \left| \sum_{s=1}^b X_{i_\ell+s-1} e^{-is\lambda} \right|^2 \). Calculate the pseudo random variables \( M_n^+ \) as

\[
M_n^+ = \sqrt{kb} \frac{1}{k} \sum_{\ell=1}^k \frac{2\pi}{b} \sum_{j \in G(b)} g_{\hat{\theta}_0}(\lambda_{j,b})(I_b^{(\ell)}(\lambda_{j,b}) - \tilde{f}(\lambda_{j,b})).
\]

Step 5: Calculate the \( m \times m \) matrix \( V_{2,n}^+ \) as

\[
V_{2,n}^+ = \Sigma_n^+ - C_n^+,
\]
where $\Sigma_n^+ = \text{Var}^*(M_n^+)$, $C_n^+ = (c_{n}^{(r,s)})_{r,s=1,2,\ldots,m}$ and the elements $c_{n}^{(r,s)}$ are given by

$$c_{n}^{(r,s)} = \frac{8\pi^2}{b} \sum_{j \in \mathcal{G}(b)} g_{r,\hat{\theta}_0}(\lambda_{j,b})g_{s,\hat{\theta}_0}(\lambda_{j,b})\hat{f}(\lambda_{j,b})^2 \times \left(\frac{1}{n-b+1} \sum_{i=1}^{n-b+1} \frac{f_b^{(i)}(\lambda_{j,b})^2}{f_b(\lambda_{j,b})^2} - 1\right).$$

**Step 6:** Approximate the distribution of $L_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ by the distribution of

$$L_n^* = (W_n^*)^{-1}(V_{1,n}^* + V_{2,n}^+)^{1/2} \cdot Z_n^*.$$ 

Several aspects of the above bootstrap algorithm are clarified by the following series of remarks and comments.

**Remark 3.1.**

(i) Observe that in order to appropriately capture the effect of model misspecification, i.e., the case $f \notin F_\theta$, a nonparametric estimator $\hat{f}$ is used in Step 2 to generate the pseudo periodogram ordinates $I_n^*$ and not the estimated parametric spectral density $f_{\hat{\theta}_n}$.

(ii) The estimator $\hat{\theta}_0$ in Step 3 is defined in a way which imitates the properties of $\theta_0$. This estimator also delivers the appropriate centering of the bootstrap estimator $\hat{\theta}_n^*$, that is $\sqrt{n}(\hat{\theta}_n^* - \theta_0)$ is used as a bootstrap analogue of $\sqrt{n}(\hat{\theta}_n - \theta_0)$.

(iii) As the proof of Theorem 4.1 shows,

$$\sqrt{n}(\hat{\theta}_n^* - \theta_0) \stackrel{D}{\rightarrow} \mathcal{N}(0, W^{-1}V_1W^{-1}),$$

in probability. Furthermore, Lemma 7.2 of Section 7 shows that $W_n^* \stackrel{P}{\rightarrow} W$ and $V_{1,n}^* \stackrel{P}{\rightarrow} V_1$. These facts imply that the limiting distribution of the standardized pseudo random variable $Z_n^* = (V_{1,n}^*)^{-1/2}W_n^*\sqrt{n}(\hat{\theta}_n^* - \theta_0)$ appearing at the end of Step 3, has covariance matrix the $m \times m$ identity matrix.

**Remark 3.2.**

(i) In Step 4 the resampled periodograms of subsamples of length $b$, i.e., $I_{b}^{(t)}(\lambda_{j,b})$, are obtained by using the same spectral density estimator $\hat{f}$ as in Step 2 but evaluated at the Fourier frequencies $\lambda_{j,b}$ corresponding to the length $b$ of the subsamples. Furthermore, in order to generate $I_{b}^{(t)}(\lambda_{j,b})$, $j = 1,2,\ldots,B$, the estimated spectral density $\hat{f}(\lambda_{j,b})$ is multiplied with the entire set of frequency domain residuals, denoted by $U_{b}^{(t)}(\lambda_{j,b})$, $j = 1,2,\ldots,B$. Because these residuals are obtained by rescaling the entire set of periodogram ordinates of a subsample, they retain for the Fourier frequencies $\lambda_{j,b}$ of the subsample, the weak dependence structure of the periodogram. Notice that the particular rescaling of these residuals applied, ensures that $E^*(U_{b}^{(t)}(\lambda_{j,b})) = 1$, that is $E^*(I_{b}^{(t)}(\lambda_{j,b})) = \hat{f}(\lambda_{j,b})$. 
(ii) As Lemma 7.2 of Section 7 shows, \( V_{2,n}^+ \xrightarrow{P} V_2 \). This implies that \( V_{1,n}^* + V_{2,n}^+ \xrightarrow{P} V_1 + V_2 \), in probability. That is, the pseudo random variables \( M_{n}^+ \) based on the convolved periodograms generated in Step 4 of the above algorithm, are solely used to estimate the part \( V_2 \) of the covariance matrix of the distribution of \( L_n \). As already mentioned, this part can not be captured by the distribution of \( \sqrt{n} (\hat{\theta}_n - \theta_0) \) generated in Step 3 due to the independence of the pseudo periodograms \( I_{n}^*(\lambda_{j,n}) \) across the Fourier frequencies \( \lambda_{j,n} \).

(iii) The statistic \( \tilde{f} \) appearing in Step 4 is itself a nonparametric estimator of the spectral density \( f \) the properties of which have been investigated in the literature; see Dahlhaus (1985) and the references therein. However, we use in this step the estimator \( \hat{f} \) for generating the pseudo periodograms \( I_{n}(\ell) \) in order to ensure that the same spectral density estimator is used here as in Step 2 of the bootstrap procedure.

Remark 3.3. To understand the motivation behind the displayed equation in Step 6, notice that as the proof of Theorem 4.1 shows and by Lemma 7.2 of Section 7, we have that \( Z_{n}^* \xrightarrow{P} N(0, I_m) \), in probability. Furthermore and by the same lemma, it holds true that

\[
(W_n^*)^{-1}(V_{1,n}^* + V_{2,n}^+)^{1/2} \xrightarrow{P} W^{-1}(V_1 + V_2)^{1/2}.
\]

These results imply that \( L_n \) defined in Step 6 satisfies \( L_n^* \xrightarrow{P} N(0, W^{-1}(V_1 + V_2) W^{-1}) \), in probability, which coincides with the limiting distribution of the random sequence \( L_n = \sqrt{n} (\hat{\theta}_n - \theta_0) \).

Remark 3.4. The \( m \times m \) matrix \( W_n^* \) containing the second order partial derivatives may be difficult to calculate in some situations. In this case an additional step can be included in the above procedure the aim of which will be to directly estimate \( W_n^* \). To elaborate, let \( \Sigma_n^* = \operatorname{Var}^* (\sqrt{n} (\hat{\theta}_n^* - \theta_0) ) \) and recall the definition of \( V_{1,n}^* \) in Step 3. The matrix \( W_n^* \) can then be estimated by

\[
\hat{W}_n^* = V_{1,n}^* (\Sigma_n^* \cdot V_{1,n}^*)^{-1/2}.
\]

By the property \( \Sigma_n^* \xrightarrow{P} W^{-1}V_1W^{-1} \), we have by Lemma 7.2 of Section 7 that

\[
V_{1,n}^* (\Sigma_n^* \cdot V_{1,n}^*)^{-1/2} \xrightarrow{P} V_1 (W^{-1}V_1W^{-1})^{-1/2} = V_1 (W^{-1}V_1)^{-1} = W.
\]

That is, \( \hat{W}_n^* \) given in (3.1) consistently estimates \( W \) and can, therefore, be used to replace \( W_n^* \) in the bootstrap algorithm.

4. Bootstrap Validity

In this section we establish the asymptotic validity of the bootstrap procedure proposed. Toward this goal we need to impose some conditions on the dependence structure of the process \( \{X_t, t \in \mathbb{Z}\} \) as well as on the smoothness properties of the functions and of the spectral densities involved. These conditions are summarized in the following assumptions.

Assumption 1:
i) The process \( \{X_t, t \in \mathbb{Z}\} \) has mean zero, is eighth-order stationary, i.e., the joint cumulants up to eighth-order, \( \text{cum}(X_t, X_{t+h_1}, \ldots, X_{t+h_7}) \), do not depend on \( t \) for any \( h_1, h_2, \ldots, h_7 \in \mathbb{Z} \). Furthermore, \( \sum_{h \in \mathbb{Z}} |h| |\text{cum}(X_0, X_h)| < \infty \), \( \inf_{\lambda \in [0, \pi]} f(\lambda) > 0 \),

\[
\sum_{h_1, h_2, h_3 \in \mathbb{Z}} (|h_1| + |h_2| + |h_3|) |\text{cum}(X_0, X_{h_1}, X_{h_2}, X_{h_3})| < \infty,
\]

and

\[
\sum_{h_1, \ldots, h_7 \in \mathbb{Z}} |\text{cum}(X_0, X_{h_1}, \ldots, X_{h_7})| < \infty.
\]

(ii) The sequence of Whittle estimators \( \{\sqrt{n}(\hat{\theta}_n - \theta_0), n \in \mathbb{N}\} \), satisfies (2.8).

The above assumptions on the dependence structure of the process \( \{X_t, t \in \mathbb{Z}\} \) are rather weak and cover a wide range of processes considered in the literature; see among others, Rosenblatt (1985), Doukhan and León (1989), Wu and Shao (2004) for summability properties of cumulants for processes satisfying different weak dependent conditions. The requirement of eighth-order stationarity seems inavoidable taking into account the fact that our derivations include calculations of the variance of time averaged products of periodograms of subsamples at different frequencies. Notice that beyond the summability requirements of Assumption 1 and in order to be as flexible as possible, we do not impose any further conditions on the moment or on the dependence structure of the underlying process. Instead, we directly require that the asymptotic normality of the sequence \( \{\sqrt{n}(\hat{\theta}_n - \theta_0), n \in \mathbb{N}\} \), as stated in Assumption 1(ii), holds true. This also covers a wide range of processes satisfying a variety of weak dependence conditions; see the discussion before equation (2.8) in Section 2.

Assumption 2:

(i)

\[
\mathcal{F}_\theta = \{f_\theta, \theta \in \Theta, \inf_{\theta \in \Theta} \inf_{\lambda \in [-\pi, \pi]} f_\theta(\lambda) \geq \delta > 0\},
\]

where \( \Theta \) is a compact subset of \( \mathbb{R}^m \) and \( \theta_0 \) defined in (2.5) is unique and belongs to the interior of \( \Theta \).

(ii) \( D(\theta, f) \) is twice differentiable in \( \theta \in \Theta \) under the integral sign.

(iii) \( f_\theta(\lambda) \) is continuous at any \( (\lambda, \theta) \in [-\pi, \pi] \times \Theta \).

(iv) The first and second order partial derivatives of \( f_\theta^{-1}(\cdot) = 1/f_\theta(\cdot) \) with respect to \( \theta \) are continuous at any \( (\lambda, \theta) \in [-\pi, \pi] \times \Theta \).

(v) The \( m \times m \) matrix \( W = \left( \frac{\partial^2}{\partial \theta_j \partial \theta_k} D(\theta_0, f) \right)_{j,k=1,\ldots,m} \) is non singular.

Assumption 2 specifies the conditions imposed on the family \( \mathcal{F}_\theta \) of parametric spectral densities considered. Assumption 2(i) requires that the spectral densities \( f_\theta \in \mathcal{F}_\theta \) are bounded from below away from zero for all frequencies \( \lambda \in [-\pi, \pi] \) and for all \( \theta \in \Theta \). This part of the assumption as well as the smoothness properties of \( f_\theta \) imposed in part (ii) to part (v), are standard and common in the literature; see Taniguchi (1987) and Dahlhaus and Wefelmeyer (1996).
Our next assumption specifies the required consistency properties of the nonparametric spectral density estimator $\hat{f}$ used to generate the pseudo periodograms $I_n^*$ and $I_b^{(\ell)}$. It is a standard requirement of uniform consistency.

**Assumption 3:** The nonparametric spectral density estimator $\hat{f}$ satisfies
\[
\sup_{\lambda \in [-\pi,\pi]} \left| \hat{f}(\lambda) - f(\lambda) \right| \overset{P}{\to} 0.
\]

Our last assumption concerns the rate at which the subsampling size $b$, involved in the generation of the convolved periodograms of subsamples, is allowed to increase to infinity with the sample size $n$ in order to ensure consistency of the estimator $\hat{V}_{2,n}^+$ used.

**Assumption 4:** $b \to \infty$ as $n \to \infty$ such that $b^3/n \to 0$.

We now state the main result of this paper which establishes consistency of the bootstrap proposal $L_n^*$ defined in Step 6 of the basic bootstrap algorithm and used to estimate the distribution of $L_n$.

**Theorem 4.1.** Let Assumptions 1 to 4 be satisfied. Then, as $n \to \infty$,
\[
\sup_{x \in \mathbb{R}^m} \left| P(L_n^* \leq x | X_1, X_2, \ldots, X_n) - P(L_n \leq x) \right| \to 0,
\]
in probability, where $P(L_n^* \leq \cdot | X_1, X_2, \ldots, X_n)$ denotes the distribution function of the random variable $L_n^*$ given the time series $X_1, X_2, \ldots, X_n$.

5. **Incorporating Whittle Likelihood Modifications**

It has been observed that despite their nice properties, Whittle estimators may behave, in certain, small samples situations, inferior compared to the exact, time domain, maximum likelihood estimators. More specifically, Whittle estimators may be biased in small samples due to errors inherited in Whittle’s frequency domain approximation of the time domain Gaussian maximum likelihood or in cases where the spectral density of the underlying process contains (strong) peaks or the periodogram suffers from the well known blurring or aliasing effects. These drawbacks motivated many researchers to investigate modifications of the basic Whittle likelihood in order to improve the finite sample performance of the estimators. Dahlhaus (1988) proposed and investigated the use of tapered periodograms, while Velasco and Robinson (2000) combined tapering with differencing the time series before obtaining Whittle’s estimators. Sykulski et al. (2019) introduced a de-biased Whittle likelihood to reduce leakage and blurring effects and more recently, Subba Rao and Yang (2020) proposed the boundary corrected Whittle likelihood and also combined this with tapering. In this section we will propose modifications, respectively extensions, of the basic bootstrap algorithm presented in Section 3, which appropriately take into account such modifications of Whittle estimators.

5.1. **Tapered Periodograms.** Applying a data taper to the time series observed, leads to the replacement of the periodogram $I_n(\lambda_{j,n})$ used in Whittle’s likelihood approximation (2.1) by a tapered periodogram, denoted by $I_{n,T}(\lambda_{j,n})$. Let $h_{t,n} = h(t/n)$ be a data taper,
Step 2a: Generate $\varepsilon_{i}^{*}, \varepsilon_{2}^{*}, \ldots, \varepsilon_{n}^{*}$ i.i.d., $\mathcal{N}(0,1)$ distributed random variables and calculate for $\lambda_{s,n} \in \mathcal{F}_{n}$, the normalized finite Fourier transform,

$$Z_{s,n}^{*} = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \varepsilon_{t}^{*} e^{-it\lambda_{s,n}}.$$

Step 2b: For $t = 1, 2, \ldots, n$, calculate the pseudo random variables

$$X_{t}^{*} = \sqrt{\frac{2\pi}{n}} \sum_{\lambda_{s,n} \in \mathcal{F}_{n}} \hat{f}_{T}(\lambda_{s,n}) Z_{s,n}^{*} e^{it\lambda_{s,n}}.$$

Step 2c: For $\lambda_{j,n} \in \mathcal{F}_{n}$, calculate the finite Fourier transform of the tapered pseudo time series $X_{1}^{*}, X_{2}^{*}, \ldots, X_{n}^{*}$, that is,

$$J_{n,T}^{*}(\lambda_{j,n}) = \sum_{t=1}^{n} h_{t,n} X_{t}^{*} e^{-it\lambda_{j,n}}.$$

The tapered pseudo periodogram $I_{n,T}^{*}(\lambda_{j,n})$ is then defined as

$$I_{n,T}^{*}(\lambda_{j,n}) = \frac{1}{2\pi H_{2,n}^{2}} J_{n,T}^{*}(\lambda_{j,n}) \cdot J_{n,T}^{*}(\lambda_{j,n}).$$
Remark 5.1. Notice that if we set $h_{t,n} = 1$ for $t = 1, 2, \ldots, n$, in Step 2c) of the above algorithm, which corresponds to the case of no taper, then we get

$$I_{n,T}^*(\lambda_{j,n}) = \hat{f}(\lambda_{j,n})|Z_{j,n}^*|^2.$$  

The random variables $|Z_{j,n}^*|^2, j = 1, 2, \ldots, N$, are independent, which implies that the periodogram ordinates $I_{n,T}^*(\lambda_{j,n})$ are independent across the frequencies $\lambda_{j,n}, j = 1, 2, \ldots, N$. Furthermore and since the $|Z_{j,n}^*|^2$ also have a standard exponential distribution for every $j$, we get that if $h_{t,n} = 1$ for all $t = 1, 2, \ldots, n$, then the pseudo periodograms $I_{n,T}^*(\lambda_{j,n})$ generated following Step 2a) to Step 2c) have exactly the same properties as the (non tapered) pseudo periodograms $I_n^*(\lambda_{j,n})$ generated in Step 2 of the bootstrap algorithm presented in Section 3.

Now, the tapered pseudo periodograms $I_{b,T}^{(\ell)}$ and $I_{n,T}^*$ can be used in the bootstrap algorithm to approximate the distribution of $\sqrt{n}(\hat{\theta}_{n,T} - \theta_0)$, where $\hat{\theta}_{n,T} = \arg \min_{\vartheta \in \Theta} D_n(\vartheta, I_{n,T}^*)$ and $D_n(\cdot)$ the function given in (2.1). To elaborate, $\hat{\theta}_{n}$ appearing in Step 1 and elsewhere in this algorithm is replaced by the tapered estimator $\tilde{\theta}_{n,T}$. In Step 2 the bootstrap periodogram $I_n^*$ is replaced by $I_{n,T}^*$ and consequently $\tilde{\theta}_{n,T}$ in Step 3 by

$$\tilde{\theta}_{n,T} = \arg \min_{\vartheta \in \Theta} D_n(\vartheta, I_{n,T}^*).$$  

Thus the bootstrap approximation of $\sqrt{n}(\hat{\theta}_{n,T} - \theta_0)$ in the same step is given by $\tilde{I}_n^* = \sqrt{n}(\tilde{\theta}_{n,T}^* - \theta_0)$. Furthermore, the tapered pseudo periodogram $I_{n,T}^*$ is used in the expression of the vector $M_n^*$ in Step 3, while the matrix $W_n^*$ in the same step is calculated using $D_n(\vartheta, I_{n,T}^*)$, that is $W_n^*$ is replaced by

$$W_{n,T}^* = \left( \frac{\partial^2}{\partial \vartheta \partial \vartheta^\top} D_n(\theta, I_{n,T}^*) \bigg|_{\theta = \theta_0} \right).$$  

Finally, in Step 4, the periodogram of the random subsamples $I_{b,T}^{(\ell)}$ in the expression for $M_n^+$ as well as the periodogram $I_{b,T}^{(\ell)}$ in the same step and in Step 5, are replaced by their tapered versions $I_{b,T}^{(\ell)}$ and $I_{b,T}^{(\ell)}$, respectively.

5.2. Debiased Whittle Likelihood. Debiasing the Whittle likelihood has been proposed by Sykulski et al. (2019). The basic idea is to replace the parametric spectral density $f_\theta$ appearing in $D_n(\vartheta, I_n)$ by a smoothed version which equals the expectation of the periodogram $I_n(\lambda)$ under the assumption that $f = f_\theta$. More specifically, the objective function considered by this modification of Whittle’s approximation of the quasi Gaussian likelihood is given by

$$D_n(\vartheta, I_n) = \frac{1}{n} \sum_{j \in \mathcal{G}(n)} \left\{ \log \tilde{f}_\theta(\lambda_{j,n}) + \frac{I_n(\lambda_{j,n})}{\tilde{f}_\theta(\lambda_{j,n})} \right\},$$  

where

$$\tilde{f}_\theta(\lambda_{j,n}) = \int_{-\pi}^{\pi} K_n(w - \lambda_{j,n}) f_\theta(\omega) d\omega.$$
and $K_n(\cdot)$ is the Fejer-kernel,

$$K_n(x) = 1_{\{x=0\}} n/2\pi + 1_{\{x\neq 0\}} \sin^2(nx/2)/(2\pi n \sin^2(x/2)).$$

Notice that if $f = f_0$, then $E(I_n(\lambda_{j,n})) = \tilde{f}_\theta(\lambda_{j,n})$; see for instance, Rosenblatt (1963). That is, in this case, the periodogram $I_n$ appearing in $D^{(db)}(\theta, I_n)$ is an unbiased estimator of $\tilde{f}_\theta$, which justifies the name given to this modification.

In the following steps we summarize the modifications needed in order to adapt the basic bootstrap algorithm presented in Section 3 to imitate the random properties of the estimator $\tilde{\theta}_n$.

**Step I:** Calculate $\tilde{\theta}_n = \arg \min_{\theta \in \Theta} D^{(db)}_n(\theta, I_n)$.

**Step II:** The same as Step 2 of the bootstrap algorithm in Section 3.

**Step III:** Calculate $\tilde{\theta}_n^* = \arg \min_{\theta \in \Theta} D^{(db)}_n(\theta, I_n^*)$, $\tilde{\theta}_0 = \arg \min_{\theta \in \Theta} D^{(db)}_n(\theta, \tilde{f})$ and $\tilde{V}_{1,n}^* = \text{Var}^* M_n^*$, where

$$\tilde{M}_n^* = \frac{2\pi}{\sqrt{n}} \sum_{j \in G(n)} \mathcal{g}_{\tilde{\theta}_0}(\lambda_{j,n}) (I_n(\lambda_{j,n}) - \tilde{f}(\lambda_{j,n}))$$

and $\mathcal{g}_{\tilde{\theta}_0}(\lambda)$ is the $m$-dimensional vector

$$\mathcal{g}_{\tilde{\theta}_0}(\lambda) = \left( \mathcal{g}_{j,\tilde{\theta}_0}(\lambda) = -\frac{1}{2\pi} \frac{\partial}{\partial \theta_j} \tilde{f}_{\theta}^{-1}(\lambda) \right|_{\theta = \tilde{\theta}_0}, \ j = 1, 2, \ldots, m \right)^\top.$$

Calculate

$$\tilde{W}_n^* = \left( \frac{\partial^2}{\partial \theta \partial \theta^\top} D^{(db)}_n(\theta, I_n^*) \right|_{\theta = \tilde{\theta}_0},$$

and the pseudo random vector $\tilde{Z}_n^*$ defined by

$$\tilde{Z}_n^* = (\tilde{V}_{1,n}^*)^{-1/2} \tilde{W}_n^* \sqrt{n}(\tilde{\theta}_n^* - \tilde{\theta}_0).$$

**Step IV:** The same as Step 4 of the bootstrap algorithm in Section 3 but by replacing $M_n^+$ by

$$\tilde{M}_n^+ = \sqrt{k} b \frac{1}{k} \sum_{l=1}^{k} \frac{2\pi}{b} \sum_{j \in G(b)} \mathcal{g}_{\tilde{\theta}_0}(\lambda_{j,b}) (I_{b}^{(l)}(\lambda_{j,b}) - \tilde{f}(\lambda_{j,b})).$$

**Step V:** Calculate the $m \times m$ matrix $\tilde{V}_{2,n}^+ = \tilde{\Sigma}_n^+ - \tilde{C}_n^+$, where $\tilde{\Sigma}_n^+ = \text{Var}^* (\tilde{M}_n^+)$, $\tilde{C}_n^+ = (c_{n}^{(r,s)})_{r,s=1,2,\ldots,m}$ and the elements $c_{n}^{(r,s)}$ given by

$$c_{n}^{(r,s)} = \frac{8\pi^2}{b} \sum_{j \in G(b)} \mathcal{g}_{\tilde{\theta}_0}(\lambda_{j,b}) \mathcal{g}_{\tilde{\theta}_0}(\lambda_{j,b}) \tilde{f}(\lambda_{j,b})^2 
\times \left( \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \frac{f_b^{(t)}(\lambda_{j,b})^2}{f_b(\lambda_{j,b})^2} - 1 \right).$$

**Step VI:** Approximate the distribution of $L_n = \sqrt{n}(\tilde{\theta}_n - \theta_0)$ by the distribution of

$$\tilde{L}_n^* = (\tilde{W}_n^*)^{-1} (\tilde{V}_{1,n}^* + \tilde{V}_{2,n}^*)^{1/2} \cdot \tilde{Z}_n^*.$$
5.3. **Boundary Corrected Whittle Likelihood.** In order to reduce the bias of Whittle estimators caused by boundary effects, Subba Rao and Yang (2021) proposed the so called, boundary corrected Whittle likelihood. The objective function to be minimized in this case is given by

\[
D_n^{(bc)}(\theta, \bar{I}_n) = \frac{1}{n} \sum_{j \in \mathbb{G}(n)} \left\{ \log f_{\theta}(\lambda_{j,n}) + \frac{\bar{I}_n(\lambda_{j,n})}{f_{\theta}(\lambda_{j,n})} \right\},
\]

where \( \bar{I}_n(\lambda_{j,n}) = (2\pi n)^{-1} \mathcal{J}_n(\lambda_{j,n}) \mathcal{J}_n(\lambda_{j,n}) \). Here, \( \mathcal{J}_n(\lambda_{j,n}) = \sum_{t=1}^{n} X_t e^{-i\lambda_{j,n} t} \), is the finite Fourier transform of the time series \( X_1, X_2, \ldots, X_n \) while \( \mathcal{J}_n(\lambda_{j,n}) \) is the boundary extended finite Fourier transform given by \( \mathcal{J}_n(\lambda_{j,n}) = \mathcal{J}_n(\lambda_{j,n}) + \mathcal{J}_n(\lambda_{j,n}) \). The boundary extended finite Fourier transform \( \mathcal{J}_n(\lambda_{j,n}) \) is obtained by calculating the finite Fourier transform of the out of sample extended time series

\[
\ldots, X_{n-1}, \hat{X}_0, X_1, X_2, \ldots, X_n, \hat{X}_{n+1}, \hat{X}_{n+2}, \ldots,
\]

where the pseudo observations \( \hat{X}_t \) are the best linear predictors of the corresponding (not observed) values \( X_t \) based on an AR(p) model. To elaborate, let \( (\hat{\phi}_{s,p}, s = 1, 2, \ldots, p) \) be the vector of Yule-Walker estimators obtained by fitting an AR(p) model to the time series \( X_1 X_2, \ldots, X_n \). Then \( \hat{X}_{n+s} = \sum_{j=1}^{p} \hat{\phi}_{j,p} \hat{X}_{n+s-j} \) with \( \hat{X}_t = X_t \) if \( t \in \{1, 2, \ldots, n\} \) and \( \hat{X}_t = \hat{X}_t \) if \( t > n \). An analogue expression yields for \( \hat{X}_{-s}, s < 0 \). This leads to the expression \( \mathcal{J}_n(\lambda_{j,n}) = \mathcal{J}_n(\lambda_{j,n}) + \mathcal{J}_n(\lambda_{j,n}) \), where the “extension” part \( \mathcal{J}_n(\lambda_{j,n}) \) can be written as

\[
\mathcal{J}_n(\lambda_{j,n}) = \frac{1}{\hat{\phi}_p(\lambda_{j,n})} \sum_{\ell=1}^{p} X_\ell \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s,p} e^{-is\lambda_{j,n}} \]

\[
+ e^{in\lambda_{j,n}} \frac{1}{\hat{\phi}_p(\lambda_{j,n})} \sum_{\ell=1}^{p} X_{n+1-\ell} \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s,p} e^{i(s+1)\lambda_{j,n}},
\]

with \( \hat{\phi}_p(\lambda) = 1 - \sum_{s=1}^{p} \hat{\phi}_{s,p} e^{-is\lambda} \); see Subba Rao and Yang (2021).

In the following we only describe how to modify the first part of the basic bootstrap algorithm, that is Step 1 to Step 3, in order to get replicates of the boundary corrected Whittle estimators \( \tilde{\theta}_n = \arg \min_{\theta \in \Theta} D_n^{(bc)}(\theta, \bar{I}_n) \). The modifications needed for the second part of the basic bootstrap procedure which uses convolved periodograms of subsamples (Step 4 to Step 5), easily follow from those presented for the first part.

*Step 1′*: Calculate Whittle’s estimator \( \tilde{\theta}_n = \arg \min_{\theta \in \Theta} D_n^{(bc)}(\theta, \bar{I}_n) \).

*Step 2*: For \( t = 1, 2, \ldots, n \), calculate the pseudo random variables

\[
X_t^* = \sqrt{\frac{2\pi}{n}} \sum_{\lambda_{s,n} \in \mathbb{F}_n} \mathcal{J}_{n/2}(\lambda_{s,n}) Z_{s,n}^* e^{it\lambda_{s,n}},
\]

where

\[
Z_{s,n}^* = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \varepsilon_t^* e^{-it\lambda_{s,n}}
\]
and \( \varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^* \) are i.i.d., \( \mathcal{N}(0, 1) \) distributed random variables. Calculate
\[
\tilde{J}_n^*(\lambda_j, n) = J_n^*(\lambda_j, n) + \hat{J}_n^*(\lambda_j, n),
\]
where \( J_n^*(\lambda_j, n) = \sum_{t=1}^n X_t^* \exp\{-i\lambda_j n t\} \) and
\[
\hat{J}_n^*(\lambda_j, n) = \frac{1}{\phi_p(\lambda_j, n)} \sum_{\ell=1}^p X_\ell^* \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s, p}^* e^{-is\lambda_j n} + e^{in\lambda_j n} \frac{1}{\phi_p(\lambda_j, n)} \sum_{\ell=1}^p X_{n+1-\ell}^* \sum_{s=0}^{p-\ell} \hat{\phi}_{\ell+s, p}^* e^{i(s+1)s\lambda_j n}.
\]

In the above expression, \( \hat{\phi}_p^*(\lambda) = 1 - \sum_{s=1}^p \hat{\phi}_{s, p}^* e^{-is\lambda} \) and \( \hat{\phi}_{s, p}^*, s = 1, 2, \ldots, p \) is the vector of Yule-Walker estimators obtained by fitting an AR(p) model to the pseudo time series \( X_1^*, X_2^*, \ldots, X_n^* \). Define,
\[
\tilde{I}_n(\lambda_j, n) = \frac{1}{2\pi n} \tilde{J}_n(\lambda_j, n) \hat{J}_n(\lambda_j, n).
\]

**Step 3**: Let
\[
\hat{\theta}_n^* = \arg \min_{\theta \in \Theta} D_n(\theta, \tilde{I}_n).
\]

Define
\[
\tilde{V}_{1, n}^* = \text{Var}^*(\tilde{M}_n^*) \quad \text{and} \quad \tilde{W}_n^* = \left( \frac{\partial^2}{\partial \theta \partial \bar{\theta}} D_n(\theta, \tilde{I}_n) \right)_{\theta = \hat{\theta}_0},
\]
where
\[
\tilde{M}_n^* = \frac{2\pi}{\sqrt{n}} \sum_{j \in \Theta(n)} g_{\hat{\theta}_0}(\lambda_j, n) (\tilde{I}_n(\lambda_j, n) - \tilde{f}(\lambda_j, n)),
\]
and \( g_{\hat{\theta}_0}(\lambda) \) the \( m \)-dimensional vector
\[
g_{\hat{\theta}_0}(\lambda) = \left( g_{j, \hat{\theta}_0}(\lambda) = -\frac{1}{2\pi} \frac{\partial}{\partial \theta_j} \tilde{f}^{-1}(\lambda) \right)_{\theta = \hat{\theta}_0}, \quad j = 1, 2, \ldots, m \).
\]

Calculate the pseudo random vector \( \tilde{Z}_n^* \) defined by
\[
\tilde{Z}_n^* = (\tilde{V}_{1, n}^*)^{-1/2} \tilde{W}_n^* \sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_0).
\]

6. **Practical Implementation and Numerical Results**

6.1. **Choice of bootstrap parameters.** To implement our procedure we need to choose two parameters, the bandwidth involved in obtaining the spectral density estimator \( \hat{f} \) and the blocksize \( b \), used in the convolved part of our procedure. Regarding \( \hat{f} \) we use a kernel type estimator obtained by smoothing the periodogram by means of the Bartlett-Priestley kernel. The bandwidth used for this estimator is obtained using a frequency domain cross validation procedure; see Beltrão and Bloomfield (1987). For the choice of the subsampling parameter \( b \), observe first that, as we will see in the next section, our simulation results seem not to be very sensitive with respect to the choice of this parameter, provided \( b \) not chosen too large with respect to \( n \). Based on this empirical observation we use the following practical rule to select this parameter: \( b = 4 \cdot n^{0.25} \). This rule delivers a subsampling size which is not too large and at the same time also satisfies the conditions of Assumption
4. However, in order to see the sensitivity of the bootstrap approximations obtained with respect to the choice of the subsampling parameter, we present in Section 6.2 results for a large range of values of $b$. The aforementioned rule for choosing $b$ has been applied for obtaining the numerical results presented in Section 6.3 for analyzing the real-life data example.

6.2. Simulations. Let $\mathcal{F}_\theta$ be the family containing the spectral densities of the first order autoregressive processes given by

$$f_\theta(\lambda) = \frac{\sigma^2}{2\pi}(1 + a^2 - 2a\cos(\lambda))^{-1},$$

where $\theta = (\sigma^2, a) \in (0, +\infty) \times (-1, 1)$. To demonstrate the advances of the bootstrap procedure proposed in this paper, we concentrate in the following on the (standard) Whittle estimator of $a$ given by

$$\hat{a}_n = \sum_{j \in \mathcal{G}(n)} I_n(\lambda_{j,n}) \cos(\lambda_{j,n}) / \sum_{j \in \mathcal{G}(n)} I_n(\lambda_{j,n}).$$

Observe that $a_0 = \int_{-\pi}^{\pi} \cos(\lambda)f_{\theta_0}(\lambda)d\lambda/\int_{-\pi}^{\pi} f_{\theta_0}(\lambda)d\lambda = \rho(1)$, where $\rho(1)$ denotes the first order autocorrelation of the “best fitting” AR(1) process; see Section 2. As we have seen, the distribution of $\sqrt{n}(\hat{a}_n - a_0)$ depends on the dependence properties of the underlying process $\{X_t; t \in \mathbb{Z}\}$. In order to demonstrate the finite sample behavior and the capabilities of the bootstrap procedure proposed to approximate the distribution of $\sqrt{n}(\hat{a}_n - a_0)$ for a variety of situations, we consider time series $X_1, X_2, \ldots, X_n$ stemming from the following three processes:

- **Model I:** $X_t = 0.8X_{t-1} + \varepsilon_t$, and i.i.d. innovations $\varepsilon_t \sim \mathcal{N}(0, 1)$.
- **Model II:** $X_t = 0.75X_{t-1} + 0.6X_{t-1} \cdot \varepsilon_{t-1} + \varepsilon_t$, and i.i.d. innovations $\varepsilon_t \sim \text{Laplace}(0, 0.1)$.
- **Model III:** $X_t = \begin{cases} -0.3X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \leq 0 \\ 0.8X_{t-1} + \varepsilon_t & \text{if } X_{t-1} > 0, \end{cases}$ and i.i.d. innovations $\varepsilon_t \sim \text{Laplace}(0, 0.1)$.

Model I is a Gaussian AR(1) model. Model II and Model III are nonlinear and have been considered in Fan and Yao (2005) and Auestad and Tjøstheim (1990), respectively. They have been modified so that they are driven by Laplace (double exponential) distributed innovations with mean zero and parameter 0.1. Notice that fitting a linear AR(1) model to time series stemming from the nonlinear Models II and III, resamples a situation of model misspecification.

Two sample sizes, $n = 50$ and $n = 1000$ are considered in order to investigate the small sample behavior of the bootstrap procedure proposed as well as its consistency behavior when the length of the time series becomes large. In order to see the effects of the convolved subsampling step implemented in Step 4 through generating the pseudo random variables $M^+_n$, we also present results for the bootstrap approximation of $\sqrt{n}(\hat{a}_n - a_0)$ using the multiplicative periodogram bootstrap only, that is, the pseudo random variable $L^*_{n,MB} = \sqrt{n}(\hat{a}_n - \hat{a}_0)$, generated in Step 3 of the bootstrap algorithm of Section 3.
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B = 1000 bootstrap replications have been used in each run and the $d_1$-distances, between the exact distribution and the two bootstrap approximations, that is the multiplicative periodogram bootstrap $L_{n,MB}^*$ and the hybrid periodogram bootstrap $L_n^*$ as generated in Step 6, have been calculated. Notice that for $F$ and $G$ distribution functions, $d_1 = \int_0^1 |F^{-1}(u) - G^{-1}(u)|du$. To estimate the exact distribution of $\sqrt{n}(\hat{a}_n - a_0)$, $R = 10,000$ replications have been used. Figure 1 shows averages of the $d_1$ distances calculated over 500 repetitions for each of the three different time series models and for each of the two sample sizes considered.

As it is seen from Figure 1, for the case of the Gaussian AR(1) model and for both sample sizes considered, the bootstrap approximations $L_n^*$ and $L_{n,MB}^*$ behave very similar and the corresponding $d_1$ distances are very close to each other. Recall that in this case the distribution of $\sqrt{n}(\hat{a}_n - a_0)$ only depends on the second order characteristics of the underlying AR(1) process and therefore, the multiplicative bootstrap estimation $L_{n,MB}^*$ also provides a consistent estimation of the distribution of $L_n$. As it is also seen, for both sample sizes considered and for the case of the Gaussian AR(1) model, both frequency domain bootstrap procedures outperform the asymptotic Gaussian approximation. This is due to the skewness of the distribution of $\sqrt{n}(\hat{a} - a_0)$ which does not vanish even for $n = 1000$ observations. In the case of the nonlinear models considered, that is for Model II and Model III, the behavior of the multiplicative and of the hybrid bootstrap is very different. Recall that in these cases, the multiplicative bootstrap fails to appropriately capture the fourth order characteristics of the underlying nonlinear processes that affect the distribution of $\sqrt{n}(\hat{a}_n - a_0)$. This leads to a larger $d_1$ distance of the bootstrap estimator $L_{n,MB}^*$ compared to the hybrid bootstrap estimator $L_n^*$. The hybrid bootstrap captures these characteristics and performs much better leading to an overall smaller $d_1$-distance.

For the case $n = 50$, this is true for all block sizes $b$ and for all models considered. Only for Model III and for the sample size of $n = 50$, the behavior of the hybrid periodogram bootstrap gets closer to that of the multiplicative bootstrap when the block size $b$ becomes too large. Finally, for the sample size of $n = 1,000$ observations, the advantages of the hybrid bootstrap procedure are clearly seen in the corresponding exhibits of Figure 1 for all models.

6.3. Periodicity of Annual Sunspot Data. We consider the yearly mean total sunspot numbers from 1700 to 2020 available at www.sidc.be/silso/datafiles. A plot of the corresponding time series consisting of $n = 321$ observations, is shown in Figure 2(a). Our aim is to estimate the main periodicity of this time series and to infer properties of the estimator used by applying the frequency domain bootstrap procedure proposed in this paper. To make things precise, suppose that the stochastic process generating the observed yearly mean sunspot data possesses a spectral density $f$ and assume that a unique frequency $\lambda_{\text{max}} \in (0, \pi)$ exists such that $\lambda_{\text{max}} = \text{arg min}_\lambda f(\lambda)$. We are interested in the main periodicity of the yearly sunspot numbers defined as the parameter $P_X = \frac{2\pi}{\lambda_{\text{max}}}$. One approach to estimate this parameter is to use the class of linear $AR(p)$ process to get an estimate of the frequency $\lambda_{\text{max}}$. To elaborate, suppose that an $AR(p)$ model is fitted to the time series of sunspot numbers and that $\hat{\lambda}_{\text{max},AR}$ is the (unique) frequency
in \((0, \pi)\) defined by \(\hat{\lambda}_{\text{max},AR} = \arg\min_{\lambda} \hat{f}_{AR}(\lambda)\), where \(\hat{f}_{AR}\) is the spectral density of the estimated AR(p) model. The estimator of \(P_X\) obtained following this approach is then defined as \(\hat{P}_X = 2\pi/\hat{\lambda}_{\text{max},AR}\). Observe that consistency of the estimator \(\hat{P}_X\) only requires that \(\hat{\lambda}_{\text{max},AR} \to \lambda_{\text{max}}\), as \(n \to \infty\). This can be achieved if the spectral density, say \(f_L\), to which \(\hat{f}_{AR}\) uniformly converges, that is, \(\sup_{\lambda \in [0, \pi]} |\hat{f}_{AR}(\lambda) - f_L(\lambda)| \to 0\), satisfies \(\lambda_{\text{max},L} = \lambda_{\text{max}}\), where \(\lambda_{\text{max},L} = \arg\min_{\lambda} f_L(\lambda)\). Hence consistency of \(\hat{\lambda}_{\text{max},AR}\) only requires that the limiting spectral density \(f_L\) to which \(\hat{f}_{AR}\) uniformly converges in probability, has its largest peak at the same frequency as the spectral density \(f\) of interest. One way to achieve this, is to allow for the order \(p\) of the AR model fitted, to increase to infinity at some appropriate rate as \(n\) increases to infinity. Under certain conditions it can then be shown that \(f_L = f\), i.e., \(\sup_{\lambda \in [0, \pi]} |\hat{f}_{AR}(\lambda) - f(\lambda)| \to 0\) and that the corresponding estimator \(\hat{P}_X = 2\pi/\hat{\lambda}_{\text{max},AR}\) achieves the rate \(\hat{P}_X = P_X + O_P(p^{3/2}/n^{1/2})\); see Newton and Pagano (1983).

However, an alternative way to consistently estimate \(P_X\) is the following. Suppose that there exists a finite order AR(p) model possessing a spectral density \(f_{AR}\) such that \(\lambda_{\text{max},AR} = \lambda_{\text{max}}\), where \(\lambda_{\text{max},AR} = \arg\min_{\lambda} f_{AR}(\lambda)\) and \(f_{AR}\) denotes the spectral density of the AR(p) process. Then \(\sup_{\lambda \in [0, \pi]} |\hat{f}_{AR}(\lambda) - f(\lambda)| \to 0\) implies \(\hat{P}_X = 2\pi/\hat{\lambda}_{\text{max},AR} \to P_X\) and this estimator converges at the parametric rate \(\hat{P}_X = P_X + O_P(1/n^{1/2})\). Notice that consistency of the described approach, does not rely on the assumption that the AR(p) model correctly describes the entire stochastic structure of the process generating the sunspot data. Not even the entire autocovariance structure of the sunspot time series has to appropriately be captured by the AR(p) model. What is solely required is that the spectral density \(f_{AR}\) of the AR(p) process has its main peak at the same frequency as the spectral density \(f\) of the stochastic process generating the sunspot time series. The AR(p) model is then solely used as a vehicle to construct an estimator of the main periodicity \(P_X\). Moreover, in conjunction with the frequency domain bootstrap, this approach also allows for the investigation of the sampling properties of the estimator \(\hat{P}_X\) and for quantifying the uncertainty associated with estimating the parameter \(P_X\) of interest.

For the yearly sunspot time series, selecting an AR(p) model using Akaike’s Information Criterion (AIC), leads to an AR(9) model. However, and as already mentioned, since we are not interested in parametrizing the entire autocovariance structure of the yearly sunspot numbers but solely in consistently estimating the frequency \(\lambda_{\text{max}}\), an AR(2) model, \(X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t\), also can be used for this purpose. Figure 2(b) demonstrates this by showing the periodogram of the sunspot time series together with the spectral densities of the fitted AR(2) and AR(9) models. Notice that the periodogram of this time series takes its maximum value at the Fourier frequency \(\lambda_{j,n} = 0.090625\), which corresponds to a periodicity of 11.034 years. Fitting an AR(2) model also leads to the estimate \(\hat{\lambda}_{\text{max},AR} = 0.090625\), which corresponds to the same estimate of the main periodicity \(\hat{P}_X = 11.034\). Fitting the AR(9) model leads to the estimates 0.09375 for the frequency \(\lambda_{\text{max}}\) and 10.667 years for the main periodicity \(P_X\). We, therefore, proceed by using the more parsimonious AR(2) model for our analysis. In particular, we apply the frequency domain bootstrap procedure proposed in this paper to generate replicates of
the estimated parameters \( \hat{\theta}_n = (\hat{\sigma}_n^2, \hat{\alpha}_1, \hat{\alpha}_2) \) of the AR(2) model. Clearly and since \( f = f_{AR} \) is not a reasonable assumption in our context, we are in the setting of model misspecification. Using the bootstrap replicates of the estimated parameters, we get replicates of the estimated spectral density of the AR(2) model, say \( \hat{\lambda}_{\text{max,AR}} \). Bootstrap replicates of \( \hat{\lambda}_{\text{max,AR}} \) can then be obtained as \( \hat{\lambda}_{\text{max,AR}}^* = \arg \min_{\lambda} \hat{f}_{\text{AR}}^*(\lambda) \) which lead to bootstrap replicates \( \hat{P}_X^* = 2\pi/\hat{\lambda}_{\text{max,AR}}^* \) of the estimator of the main periodicity \( \hat{P}_X = 2\pi/\hat{\lambda}_{\text{max,AR}} \) used. By repeating these steps a large number, say \( B \), of times, bootstrap estimators of the distribution of \( \hat{\lambda}_{\text{max,AR}} \) and, consequently, of \( \hat{P}_X \), are obtained. Figure 2(c) presents a histogram of \( B = 1,000 \) bootstrap replicates of \( \hat{\lambda}_{\text{max,AR}}^* \) and Figure 2(d) of the corresponding estimates \( \hat{P}_X^* = 2\pi/\hat{\lambda}_{\text{max,AR}}^* \) obtained by using a grid of 500 equidistant frequencies in the interval \((0, \pi)\). The corresponding 95\% confidence interval for the main periodicity of the sunspot time series, based on bootstrap percentages, is then given by \([9.90, 12.98]\).

7. Auxiliary Results and Proofs

To simplify notation we write \( D_n^{(j)}(\hat{\theta}, I_n^*) \), \( \hat{\theta} \in \Theta \), for the \( j \)th derivative of \( D_n(\theta, I_n^*) \) with respect to \( \theta \) evaluated at \( \theta = \hat{\theta} \). We also write \( \| \cdot \| \) for the Euclidean norm in \( \mathbb{R}^m \) and \( \| A \|_F \) for the Frobenius norm of a matrix \( A \in \mathbb{R}^{m \times m} \). Furthermore, at different places we will use the expansion

\[
D_n^{(1)}(\theta, I_n^*) = D_n^{(1)}(\hat{\theta}_0, I_n^*) + D_n^{(2)}(\hat{\theta}_0, I_n^*)(\theta - \hat{\theta}_0) + R_n(\theta)(\theta - \hat{\theta}_0),
\]

where \( R_n(\theta) = D_n^{(2)}(\theta^+, I_n^*) - D_n^{(2)}(\hat{\theta}_0, I_n^*) \) for some \( \theta^+ \in \Theta \) such that \( \| \theta^+ - \hat{\theta}_0 \| \leq \| \theta - \hat{\theta}_0 \| \).

We first establish the following two useful lemmas.

**Lemma 7.1.** If Assumption 2 and Assumption 3 are satisfied, then the following assertions hold true.

(i) \( \hat{\theta}_n \xrightarrow{P} \theta_0 \).

(ii) \( \| \hat{\theta}_n^* - \theta_0 \| \xrightarrow{P} 0 \), in probability.

**Proof.** Consider (i). Since \( \hat{\theta}_0 \in \Theta \) and \( \Theta \subset \mathbb{R}^m \) is compact, \( \{\hat{\theta}_0, n \in \mathbb{N}\} \) is a bounded sequence. By the continuity of \( D(\cdot, f) \) as a function on \( \theta \in \Theta \), it suffices to show that

\[
D(\hat{\theta}_0, f) \xrightarrow{P} D(\theta_0, f).
\]

It yields

\[
|D(\hat{\theta}_0, f) - D(\theta_0, f)| \leq |D(\hat{\theta}_0, f) - D_n(\hat{\theta}_0, f)| + |D_n(\hat{\theta}_0, f) - D_n(\hat{\theta}_0, \hat{f})| + |D_n(\hat{\theta}_0, \hat{f}) - D(\hat{\theta}_0, \hat{f})| + |D(\hat{\theta}_0, \hat{f}) - D(\theta_0, f)|
\]

\[
= \sum_{j=1}^{4} D_{j,n},
\]
where $\bar{\theta}_0 = \text{argmin}_\theta D(\theta, \hat{f})$ and with an obvious notation for $D_{j,n}$, $j = 1, \ldots, 4$. We show that $D_{j,n} \xrightarrow{P} 0$, as $n \to \infty$, for $j = 1, \ldots, 4$. We have

$$D_{1,n} \leq \sup_{\theta \in \Theta} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_\theta(\lambda) d\lambda - \frac{1}{n} \sum_{j \in G(n)} \log f_\theta(\lambda_{j,n}) \right|$$

$$+ \sup_{\theta \in \Theta} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda - \frac{1}{n} \sum_{j \in G(n)} \frac{f(\lambda_{j,n})}{f_\theta(\lambda_{j,n})} \right| = O(n^{-1}),$$

where the last equality follows by the differentiability of $1/f_\theta(\lambda)$ with respect to $\theta$, the boundedness properties of $f_\theta \in F_\theta$ and the fact that $\sup_{\theta \in \Theta} |\partial f_\theta^{-1}(\lambda)/\partial \lambda| \leq 1/\delta$ and Assumption 3, we get

$$D_{2,n} \leq \sup_{\lambda \in [-\pi, \pi]} \left| \hat{f}(\lambda) - f(\lambda) \right| \sup_{\theta \in \Theta} \frac{1}{n} \sum_{j \in G(n)} \frac{1}{f_\theta(\lambda_{j,n})} \xrightarrow{P} 0.$$ 

To establish $D_{3,n} \xrightarrow{P} 0$, it suffices to show that $\sup_{\theta \in \Theta} |D_n(\theta, \hat{f}) - D(\theta, \hat{f})| \xrightarrow{P} 0$. For this we have

$$\sup_{\theta \in \Theta} |D_n(\theta, \hat{f}) - D(\theta, \hat{f})| \leq \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j \in G(n)} \log f_\theta(\lambda_{j,n}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_\theta(\lambda) d\lambda \right|$$

$$+ \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j \in G(n)} \hat{f}(\lambda_{j,n})/f_\theta(\lambda_{j,n}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\lambda)/f_\theta(\lambda) d\lambda \right|$$

$$\leq O(n^{-1}) + \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j \in G(n)} \frac{1}{f_\theta(\lambda_{j,n})} (\hat{f}(\lambda_{j,n}) - f(\lambda_{j,n})) \right|$$

$$+ \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j \in G(n)} \frac{f(\lambda_{j,n})}{f_\theta(\lambda_{j,n})} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\lambda)}{f_\theta(\lambda)} d\lambda \right|$$

$$+ \sup_{\theta \in \Theta} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f_\theta(\lambda)} (f(\lambda) - \hat{f}(\lambda)) d\lambda \right|$$

$$= O(n^{-1}) + O(1) \sup_{\lambda \in [-\pi, \pi]} |\hat{f}(\lambda) - f(\lambda)|,$$

where the last equality follows because the second and the last term of the last bound above is $O(1) \sup_{\lambda \in [-\pi, \pi]} |\hat{f}(\lambda) - f(\lambda)|$ and the third term is $O(n^{-1})$. Finally, $D_{4,n} \xrightarrow{P} 0$ follows from $\sup_{\theta \in \Theta} |D(\theta, \hat{f}) - D(\theta, f)| \xrightarrow{P} 0$, which holds true since

$$\sup_{\theta \in \Theta} |D(\theta, \hat{f}) - D(\theta, f)| \leq \sup_{\theta \in \Theta} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f_\theta(\lambda)} (f(\lambda) - \hat{f}(\lambda)) d\lambda \right|$$

$$\leq O(1) \sup_{\lambda \in [-\pi, \pi]} |\hat{f}(\lambda) - f(\lambda)|.$$

Consider (ii). Recall that by Assumption 3, the matrix $W$ is nonsingular. By Lemma 4.2 of Lahiri (2003), $D_n^{(2)}(\bar{\theta}_0, I_n)$ is nonsingular, if for $\delta > 0$,

$$\|D_n^{(2)}(\bar{\theta}_0, I_n) - W\|_F \leq \delta/\|W^{-1}\|_F,$$
on a set with probability arbitrarily close to one for $n$ large enough. This holds true since $\|D_{n}^{(2)}(\hat{\theta}_{n}, I_{n}^{*}) - W\|_{F} \xrightarrow{P} 0$, in probability, see Lemma 7.2(i). Furthermore, by the same Lemma 4.2, we have, on the same set, that, with probability arbitrarily close to one,

$$\|(D_{n}^{(2)}(\hat{\theta}_{n}, I_{n}^{*}))^{-1}\|_{F} \leq \|W^{-1}\|_{F}/(1 - \delta) = 2\|W^{-1}\|_{F},$$

for $\delta = 1/2$. Recall equation (7.1) and define on the set on which (7.2) holds true the function

$$w(\hat{\theta}_{0} - \theta) = (D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*}))^{-1}\left[D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) + \tilde{R}_{n}(\theta)\right], \quad \theta \in B(\hat{\theta}_{0}, \delta),$$

where $B(\hat{\theta}_{0}, \delta) = \{x \in \mathbb{R}^{m} \|x\| \leq \delta\}$ and

$$\tilde{R}_{n}(\theta) = (D_{n}^{(2)}(\theta_{n}^{+}, I_{n}^{*}) - D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*}))((\theta - \hat{\theta}_{0}),$$

with $\|\hat{\theta}_{0} - \theta_{n}^{+}\| \leq \|\hat{\theta}_{0} - \theta\|$. Using the expansion

$$D_{n}^{(1)}(\theta, I_{n}^{*}) = D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) + D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*})(\theta - \hat{\theta}_{0}) + \tilde{R}_{n}(\theta),$$

we can also express $w(\cdot)$ as

$$w(\hat{\theta}_{0} - \theta) = (D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*}))^{-1}\left[D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) - D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*})(\theta - \hat{\theta}_{0})\right].$$

From this and because, by assumption, $\hat{\theta}_{n}^{*}$ is the unique solution of $D_{n}^{(1)}(\theta, I_{n}^{*}) = 0$, we get that for $\theta = \hat{\theta}_{n}^{*}$ it holds true that $w(\hat{\theta}_{0} - \hat{\theta}_{n}^{*}) = \hat{\theta}_{0} - \hat{\theta}_{n}^{*}$ and this is the unique solution of $w(\hat{\theta}_{0} - \theta) = \hat{\theta}_{0} - \theta$.

We next show that for $n$ large enough and with probability arbitrarily close to one, a constant $C > 0$ exists such that $\|w(\hat{\theta}_{0} - \theta)\| \leq C \log(n)/\sqrt{n}$ if $\|\hat{\theta}_{0} - \theta\| \leq C \log(n)/\sqrt{n}$. Toward this goal, we get using (7.3) and the fact that $D_{n}^{(1)}(\hat{\theta}_{0}, \hat{f}) = 0$, the bound

$$\|w(\hat{\theta}_{0} - \theta)\| \leq \|(D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*}))^{-1}\|_{F}\left(\|D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) - D_{n}^{(1)}(\hat{\theta}_{0}, \hat{f})\| + \|\tilde{R}_{n}(\theta)\|\right).$$

Recall the bound $\|(D_{n}^{(2)}(\hat{\theta}_{0}, I_{n}^{*}))^{-1}\|_{F} \leq 2\|W^{-1}\|_{F}$. Furthermore, for the first term in parentheses on the right hand side of (7.4), we have

$$\|D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) - D_{n}^{(1)}(\hat{\theta}_{0}, \hat{f})\| = \|Y_{n}^{*}\|,$$

where

$$Y_{n}^{*} = \frac{1}{n} \sum_{j \in G(n)} \frac{\partial}{\partial \theta} \frac{1}{f_{\theta}(\lambda_{j,n})}\bigg|_{\theta = \hat{\theta}_{0}} \hat{f}(\lambda_{j,n})(U_{j}^{*} - 1).$$

It yields,

$$nE^{*}\|Y_{n}^{*}\|^{2} = \frac{2}{n} \sum_{j \in G(n)} \left(\frac{\partial}{\partial \theta} \frac{1}{f_{\theta}(\lambda_{j,n})}\bigg|_{\theta = \hat{\theta}_{0}}\right)^{\top} \left(\frac{\partial}{\partial \theta} \frac{1}{f_{\theta}(\lambda_{j,n})}\bigg|_{\theta = \hat{\theta}_{0}}\right) \hat{f}(\lambda_{j,n})^{2} \xrightarrow{P} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta} \frac{1}{f_{\theta}(\lambda)}\bigg|_{\theta = \hat{\theta}_{0}}\right)^{\top} \left(\frac{\partial}{\partial \theta} \frac{1}{f_{\theta}(\lambda)}\bigg|_{\theta = \hat{\theta}_{0}}\right) f(\lambda)^{2},$$

by Assumption 3, the continuity of the derivative and Lemma 7.1(i). This implies by Markov’s inequality, that,

$$P^{*}(\|D_{n}^{(1)}(\hat{\theta}_{0}, I_{n}^{*}) - D_{n}^{(1)}(\hat{\theta}_{0}, \hat{f})\| \geq C \log(n)/\sqrt{n}) \leq \frac{nE\|Y_{n}^{*}\|^{2}}{C^{2}\log^{2}(n)} = O_{P}(1/\log^{2}(n)).$$
Lemma 7.2. Suppose that Assumption 1 to Assumption 4 are satisfied. Then, as $n \to \infty$,

(i) $W_n^* \xrightarrow{P} W,$
(ii) $V_{1,n}^* \xrightarrow{P} V_1,$
(iii) $V_{2,n}^+ \xrightarrow{P} V_2.$

Proof. Notice first that

\begin{equation}
\sup_{\lambda \in [\pi, \pi]} \| g_{\hat{\theta}_n}(\lambda) - g_{\theta_0}(\lambda) \| \xrightarrow{P} 0.
\end{equation}

For the term $\tilde{R}_n(\theta)$ in (7.4) we use the bound

$$\| \tilde{R}_n(\theta) \| \leq (\| M_{1,n} \|_F + \| M_{2,n} \|_F) \| \theta - \tilde{\theta}_0 \|,$$

where

\begin{align*}
M_{1,n} &= \frac{1}{n} \sum_{j \in G(n)} \left( \frac{\partial^2}{\partial \theta \partial \theta} \log f_{\theta}(\lambda_{j,n}) \bigg|_{\theta=\hat{\theta}_n} - \frac{\partial^2}{\partial \theta \partial \theta} \log f_{\theta}(\lambda_{j,n}) \bigg|_{\theta=0} \right) \\
M_{2,n} &= \frac{1}{n} \sum_{j \in G(n)} \left( \frac{\partial^2}{\partial \theta \partial \theta} f_{\theta}(\lambda_{j,n}) \bigg|_{\theta=\hat{\theta}_n} - \frac{\partial^2}{\partial \theta \partial \theta} f_{\theta}(\lambda_{j,n}) \bigg|_{\theta=0} \right) I_n^*(\lambda_{j,n}).
\end{align*}

By the Lipschitz continuity of the second order derivatives, following from Assumption 3, and since $\| \theta_n^+ - \tilde{\theta}_0 \| \leq \| \theta - \tilde{\theta}_0 \|$, we get $\| \tilde{R}_n(\theta) \| \leq C \| \theta - \tilde{\theta}_0 \|$. Hence if $\| \theta - \theta_0 \| \leq C \log(n)/\sqrt{n}$ then, for $n$ large enough and with probability arbitrarily close to one, we have that, $\| w(\theta_0 - \theta) \| \leq C \log(n)/\sqrt{n}$. Consider next the function $g : B(0, 1) \to B(0, 1)$ defined as

$$g(x) = \frac{\sqrt{n}}{C \log(n)} w(C \log(n) x/\sqrt{n}).$$

Notice that $g$ is continuous and that because for $x \in B(0, 1)$, $\| C \log(n) x/\sqrt{n} \| \leq C \log(n)/\sqrt{n}$, we have,

$$\| g(x) \| = \frac{\sqrt{n}}{C \log(n)} \| w(C \log(n) x/\sqrt{n}) \| \leq \frac{\sqrt{n}}{C \log(n)} \cdot \frac{C \log(n)}{\sqrt{n}} = 1.$$

By Bronwer’s fixed point Theorem, see Lahiri (2003), Proposition 4.1, there exists $x_0 \in B(0, 1)$ such that $g(x_0) = x_0$, that is,

$$w(C \log(n) x_0/\sqrt{n}) = \frac{C \log(n)}{\sqrt{n}} x_0.$$

Since $\hat{\theta}_0 - \hat{\theta}_n^*$ is the unique solution of $w(\hat{\theta}_0 - \theta) = \hat{\theta}_0 - \theta$, we have that $\hat{\theta}_0 - \hat{\theta}_n^* = x_0 C \log(n)/\sqrt{n}$, that is,

$$\| \hat{\theta}_0 - \hat{\theta}_n^* \| \leq \frac{C \log(n)}{\sqrt{n}} \| x_0 \| \leq \frac{C \log(n)}{\sqrt{n}}.$$

Hence for $n$ large enough and with probability arbitrarily close to one, we have

$$\| \hat{\theta}_0 - \hat{\theta}_n^* \| = O_P\left(\frac{\log(n)}{\sqrt{n}}\right),$$

which converges to zero, as $n \to \infty$. \hfill \Box
and

\[
\sup_{\lambda \in [\pi, \pi]} \left| \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda) \right|_{\theta = \theta_0} - \left| \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda) \right|_{\theta = \theta_0} \xrightarrow{F} 0.
\]

To see why the above assertions hold true, observe first that by Lemma 4.1(i) we have for \(n\) large enough, \(P(\hat{\theta}_0 \in B(\theta_0, \epsilon)) \geq 1 - \epsilon\), where \(B(\theta_0, \epsilon) = \{ \theta : \|\theta - \theta_0\| \leq \epsilon \} \subset \Theta\). (7.5) and (7.6) follow then because the functions \(g_\theta (\lambda)\) and \(\partial^2 / (\partial \theta \partial \theta^\top) f^{-1}_\theta (\lambda)\) are uniformly continuous on the compact set \([-\pi, \pi] \times B(\theta_0, \epsilon)\).

Consider (i). We have

\[
W^n_\ast = \frac{1}{n} \sum_{j \in G(n)} \frac{\partial^2}{\partial \theta \partial \theta^\top} \left( \log f_\theta (\lambda_{j,n}) + I^*_n (\lambda_{j,n}) f^{-1}_\theta (\lambda) \right) \bigg|_{\theta = \theta_0} + \frac{1}{n} \sum_{j \in G(n)} \left( \frac{\partial^2}{\partial \theta \partial \theta^\top} \log f_\theta (\lambda_{j,n}) \bigg|_{\theta = \theta_0} - \frac{\partial^2}{\partial \theta \partial \theta^\top} \log f_\theta (\lambda_{j,n}) \bigg|_{\theta = \theta_0} \right) I^*_n (\lambda_{j,n}) + o_P(1)
\]

\[
= \frac{1}{n} \sum_{j \in G(n)} \frac{\partial^2}{\partial \theta \partial \theta^\top} \log f_\theta (\lambda_{j,n}) \bigg|_{\theta = \theta_0} + \frac{1}{n} \sum_{j \in G(n)} \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda) \bigg|_{\theta = \theta_0} I^*_n (\lambda_{j,n}) + o_P(1)
\]

\[
\xrightarrow{P} \frac{\partial^2}{\partial \theta \partial \theta^\top} D(\theta_0, f).
\]

The last equality follows because by (7.6),

\[
\left\| \frac{1}{n} \sum_{j \in G(n)} \left( \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda_{j,n}) \bigg|_{\theta = \hat{\theta}_0} - \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda_{j,n}) \bigg|_{\theta = \theta_0} \right) \right\| _F
\]

\[
\leq \sup_{\lambda \in [-\pi, \pi]} \left\| \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda) \bigg|_{\theta = \hat{\theta}_0} - \frac{\partial^2}{\partial \theta \partial \theta^\top} f^{-1}_\theta (\lambda) \bigg|_{\theta = \theta_0} \right\| _F \times \frac{1}{n} \sum_{j \in G(n)} I^*_n (\lambda_{j,n})
\]

\[
= o_P(1) \cdot O_P(1).
\]

Consider (ii). Recall that \(V^*_1 = \text{Var}^*(M^n_\ast)\) and that \(\text{Cov}(I^*_n (\lambda_{j,n}), I^*_n (\lambda_{k,n})) = \mathbf{1}_{|j|=|k|} \tilde{f}(\lambda_{j,n})^2\).

Hence

\[
\text{Var}^*(M^n_\ast) = \frac{4\pi^2}{n} \sum_{j=1}^N \left( g_{\hat{\theta}_0} (\lambda_{j,n}) g_{\hat{\theta}_0}^\top (\lambda_{j,n}) + g_{\hat{\theta}_0} (\lambda_{j,n}) g_{\hat{\theta}_0}^\top (\lambda_{j,n}) \right) \tilde{f}(\lambda_{j,n})^2
\]

\[
= \frac{8\pi^2}{n} \sum_{j \in G(n)} g_{\theta_0} (\lambda_{j,n}) g_{\theta_0}^\top (\lambda_{j,n}) f(\lambda_{j,n})^2 + o_P(1)
\]

\[
\xrightarrow{P} V_1,
\]

where the last equality follows using the symmetry of \(g_\theta (\lambda)\) with respect to \(\lambda\), assertion (7.5) and Assumption 3.
Consider (iii). We have
\[
\Sigma_n^+ = \frac{b}{k} \sum_{\ell=1}^k \text{Var}^* \left( \frac{2\pi}{b} \sum_{j \in \mathcal{G}(b)} g_{\hat{\theta}_0}^j (\lambda_{j,b}) f_b^{(\ell)}(\lambda_{j,b}) \right)
\]
\[
= \frac{4\pi^2}{b} \sum_{j_1 \in \mathcal{G}(b)} \sum_{j_2 \in \mathcal{G}(b)} g_{\hat{\theta}_0}^j (\lambda_{j_1,b}) g_{\hat{\theta}_0}^{T j} (\lambda_{j_2,b}) \text{Cov}^* \left( f_b^{(i_1)}(\lambda_{j_1,b}), f_b^{(i_2)}(\lambda_{j_2,b}) \right)
\]
\[
= \frac{4\pi^2}{b} \sum_{j_1 \in \mathcal{G}(b)} \sum_{j_2 \in \mathcal{G}(b)} g_{\hat{\theta}_0}^j (\lambda_{j_1,b}) g_{\hat{\theta}_0}^{T j} (\lambda_{j_2,b})
\]
\[
\times \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \left\{ f_b^{(t)}(\lambda_{j_1,b}), f_b^{(t)}(\lambda_{j_2,b}) - \bar{f}(\lambda_{j_1,b}) \bar{f}(\lambda_{j_2,b}) \right\}
\]
\[
= \frac{4\pi^2}{b} \sum_{j_1 \in \mathcal{G}(b)} \sum_{j_2 \in \mathcal{G}(b)} g_{\hat{\theta}_0}^j (\lambda_{j_1,b}) g_{\hat{\theta}_0}^{T j} (\lambda_{j_2,b}) \text{Cov} \left( f_b^{(1)}(\lambda_{j_1,b}), f_b^{(1)}(\lambda_{j_2,b}) \right) + o_P(1)
\]
where the $o_P(1)$ term follows using Lemma 4.1 of Meyer et al. (2020) and Assumption 4. Thus using the covariance properties of the periodogram $f_b^{(1)}(\lambda_{j,b})$ of the subsample $X_1, X_2, \ldots, X_b$ and (7.5), we get that, as $b \to \infty$,
\[
\frac{4\pi^2}{b} \sum_{j_1 \in \mathcal{G}(b)} \sum_{j_2 \in \mathcal{G}(b)} g_{\hat{\theta}_0}^j (\lambda_{j_1,b}) g_{\hat{\theta}_0}^{T j} (\lambda_{j_2,b}) \text{Cov} \left( f_b^{(1)}(\lambda_{j_1,b}), f_b^{(1)}(\lambda_{j_2,b}) \right) \xrightarrow{P} V_1 + V_2.
\]
By the same lemma we also have
\[
(n - b + 1)^{-1} \sum_{t=1}^{n-b+1} \frac{f_b^{(t)}(\lambda_{j_1,b})^2}{\bar{f}(\lambda_{j_1,b})^2} \xrightarrow{P} 2,
\]
which implies using (7.5) again and Assumption 3, that $C_n^+ \xrightarrow{P} V_1$, in probability. The assertion follows then since $V_{2,n}^+ = \Sigma_n^+ - C_n^+$. \qed

**Proof of Theorem 4.1.** In view of the definition of $L_n^*$ and Lemma 7.2, to establish the assertion of the theorem it suffices to show that,
\[
\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_0) \xrightarrow{D} \mathcal{N}(0, W^{-1} V_1 W^{-1}). \tag{7.7}
\]
Since $D_n^{(2)}(\hat{\theta}_0, I_n^*) = W_n^*$ we get by Lemma 7.2(i), that $\| D_n^{(2)}(\hat{\theta}_0, I_n^*) - W_F \|_F \xrightarrow{P} 0$ and that, for $n$ large enough, $D_n^{(2)}(\hat{\theta}_0, I_n^*)$ is nonsingular; see also the proof of Lemma 7.1(ii). Recall that $D_n^{(1)}(\hat{\theta}_n^*, I_n^*) = 0$ and that $D_n^{(1)}(\hat{\theta}_0, \bar{f}) = 0$. We get using the expansion (7.1), that
\[
[I_m + (D_n^{(2)}(\hat{\theta}_0, I_n^*))^{-1} R_n(\hat{\theta}_n^*)] \sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_0)
\]
\[
= -(D_n^{(2)}(\hat{\theta}_0, I_n^*))^{-1} \sqrt{n}(D_n^{(1)}(\hat{\theta}_0, I_n^*) - D_n^{(1)}(\hat{\theta}_0, \bar{f})). \tag{7.8}
\]
Since $(D_n^{(2)}(\hat{\theta}_0, I_n^*))^{-1} \xrightarrow{P} W^{-1}$, in probability, assertion (7.7) follows from expression (7.8), if we show that
\[
- \sqrt{n}(D_n^{(1)}(\hat{\theta}_0, I_n^*) - D_n^{(1)}(\hat{\theta}_0, \bar{f})) \xrightarrow{D} \mathcal{N}(0, V_1), \tag{7.9}
\]
and

\[ R_n(\hat{\theta}_n^*) = o_P(1), \]

\[ (7.10) \]

For \((7.9)\) we have,

\[
-\sqrt{n}(D_n^{(1)}(\bar{\theta}_0, I_n^*) - D_n^{(1)}(\bar{\theta}_0, \bar{f})) = -\frac{1}{\sqrt{n}} \sum_{j \in J(n)} \frac{\partial}{\partial \theta} f_{\theta}(\lambda_{j,n}) \bigg|_{\theta = \theta_0} (I_n^*(\lambda_{j,n}) - \hat{f}(\lambda_{j,n}))
\]

\[
= \frac{1}{\sqrt{n}} \sum_{j=1}^{N} W_{j,n}(U_j^* - 1),
\]

where \(W_{j,n} = (g_{\theta_0}(\lambda_{j,n}) + g_{\theta_0}(\lambda_{j,n}))\hat{f}(\lambda_{j,n})\) and the \(U_j^*\)’s are i.i.d. \((7.9)\) follows then because \(E^*(W_{j,n}(U_j^* - 1)) = 0\), \(\text{Var}^*(W_{j,n}(U_j^* - 1)) \xrightarrow{P} V_1\) by the same arguments as those used in the proof that \(\text{Var}^*(M_n^*) \xrightarrow{P} V_1\) in Lemma 7.2(ii), and because

\[
\sum_{j=1}^{N} E^*\left(\frac{1}{\sqrt{n}} W_{j,n}(U_j^* - 1)\right)^3 = E^*(|U_j^* - 1|^3) \frac{1}{n^{3/2}} \sum_{j=1}^{N} \|W_{j,n}\|^3 = O_P(n^{-1/2}),
\]

verifies Liapunov’s condition.

Consider \(7.10\) and observe that \(R_n(\hat{\theta}_n^*) = D_n^{(2)}(\theta_0, I_n^*) - D_n^{(2)}(\bar{\theta}_0, I_n^*)\) for some \(\theta_0^+ \in \Theta\) such that \(\|\theta_0^+ - \bar{\theta}_0\| \leq \|\theta_0^* - \bar{\theta}_0\|\). Denote by \(d_{j,k}(\hat{\theta})\) the \((j,k)\)th element of the matrix \(D_n^{(2)}(\hat{\theta}, I_n^*)\), that is, \(d_{j,k}(\hat{\theta}) = \frac{\partial^2}{\partial \theta_j \partial \theta_k} D_n(\theta, I^*)|_{\theta = \hat{\theta}}\). Then we have for the \((j,k)\)th element \(r_{j,k}(\hat{\theta}_n^*)\) of the matrix \(R_n(\hat{\theta}_n^*)\),

\[
|r_{j,k}(\hat{\theta}_n^*)| = |d_{j,k}(\theta_0^+) - d_{j,k}(\bar{\theta}_0)|
\]

\[
\leq \sup_{\lambda \in [-\pi,\pi]} \left| \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log f_\theta(\lambda)|_{\theta = \theta_0^+} - \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log f_\theta(\lambda)|_{\theta = \bar{\theta}_0} \right| + \sup_{\lambda \in [-\pi,\pi]} \left| \frac{\partial^2}{\partial \theta_j \partial \theta_k} f_\theta^{-1}(\lambda)|_{\theta = \theta_0^+} - \frac{\partial^2}{\partial \theta_j \partial \theta_k} f_\theta^{-1}(\lambda)|_{\theta = \bar{\theta}_0} \right| \frac{1}{n} \sum_{j \in J(n)} I_n^*(\lambda_{j,n})
\]

\[
= o_P(1),
\]

as \(n \to \infty\), since \(n^{-1} \sum_{j \in J(n)} I_n^*(\lambda_{j,n}) = O_P(1)\), the second order partial derivative functions \(\partial^2/(\partial \theta_j \partial \theta_k) \log f_\theta(\lambda)\) and \(\partial^2/(\partial \theta_j \partial \theta_k) f_\theta^{-1}(\lambda)\) are uniformly continuous of the compact set \([-\pi,\pi] \times \Theta\) and \(\|\theta_0^+ - \bar{\theta}_0\| \leq \|\theta_0^* - \bar{\theta}_0\| \xrightarrow{P} 0\), in probability, by Lemma 7.1(ii).

\[ \square \]

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Figure 1. Average $d_1$-distances between the exact and the bootstrap distribution of the Whittle estimator $\sqrt{n}(\hat{a}_n - a_0)$ for various block sizes $b$. Left column $n=50$, right column, $n=1,000$. First row Model I, second row Model II and third row Model III. The crosses denote the $d_1$-distance of the multiplicative periodogram bootstrap estimation $L^*_n,MB$ and the circles of the hybrid periodogram bootstrap estimation $L^*_n$. The dashed lines with the plus symbol in the first row refer to the average $d_1$-distance of the asymptotic Gaussian approximation.
Figure 2. Reading clockwise from top to bottom: (a) Time series of yearly mean sunspot numbers. (b) Periodogram of the time series with estimated spectral densities of the AR(2) model (solid line) and of the AR(9) model (dashed line), log-scale. (c) Histogram of $B = 1000$ replications of $\hat{\lambda}_{\text{max,AR}}^*$ and (d) histogram of the corresponding replications of the estimated main periodicity $\hat{P}_X^*$, both using the AR(2) model. The estimated values $\hat{\lambda}_{\text{max,AR}}$ and $\hat{P}_X$ are indicated in (c) and (d) by vertical dashed lines.