Trisecting the 9-vertex complex projective plane

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Abstract

In this paper we will give a short and direct proof that Wolfgang Kühnel’s 9-vertex simplicial complex \( CP^2_9 \) is homeomorphic to \( CP^2 \), the complex projective plane. The idea of our proof is to recall the trisection of \( CP^2 \) into 3 bi-disks and then to see this trisection inside a symmetry-breaking subdivision of \( CP^2_9 \). After giving the proof we will elaborate on the construction and sketch an explicit homeomorphism.

1 Introduction

A \( k \)-simplex is a \( k \)-dimensional convex polytope with \( k + 1 \) vertices. For \( k = 0, 1, 2, 3 \) respectively, a \( k \)-simplex is usually called a vertex, edge, triangle, tetrahedron. When \( k \) is not important, a \( k \)-simplex is just called a simplex.

A simplicial complex is a finite collection \( \mathcal{C} \) of simplices, all in an ambient Euclidean space, such that

- If \( S \in \mathcal{C} \) and \( S' \) is a sub-simplex of \( S \) then \( S' \in \mathcal{C} \).
- If \( S, T \in \mathcal{C} \) then \( S \cap T \) is either empty or in \( \mathcal{C} \).

Informally, the simplices in a simplicial complex fit together cleanly, without crashing through each other. The support \( |\mathcal{C}| \) of \( \mathcal{C} \) is the union of all the simplices in \( \mathcal{C} \). Often we blur the distinction between \( \mathcal{C} \) and \( |\mathcal{C}| \) and think of a simplicial complex as a union of simplices.

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A simplicial complex may be described with no mention of the ambient space containing it, but there is always the understanding that in principle one can find an isomorphic complex in some Euclidean space. To give a pertinent example, let $\mathbb{R}P^2_6$ be the quotient of the regular icosahedron by the antipodal map. This simplicial complex has 6 vertices, 15 edges, and 10 faces. One can reconstruct $\mathbb{R}P^2_6$ in $\mathbb{R}^5$ by fixing some 5-simplex $\Sigma \subset \mathbb{R}^5$, the convex hull of vertices $v_1, ..., v_6$, then mapping vertex $k$ of $\mathbb{R}P^2_6$ to $v_k$ and extending linearly.

The left side of Figure 1 shows another incarnation of $\mathbb{R}P^2_6$. In this picture, the outer edges of the hexagon are supposed to be identified according to the labels. The complex $\mathbb{R}P^2_6$ is called a 6-vertex triangulation of the real projective plane $\mathbb{R}P^2$ because its support is homeomorphic to $\mathbb{R}P^2$. This triangulation has the fewest number of vertices amongst triangulations of $\mathbb{R}P^2$, so it is called a minimal triangulation of $\mathbb{R}P^2$. It is in fact the unique minimal triangulation of $\mathbb{R}P^2$. (Smaller examples like the quotient of the regular octahedron by the antipodal map fail to be simplicial complexes.)

Here are some other examples related to minimal triangulations.

- The boundary of a tetrahedron is the unique 4-vertex minimal triangulation of the 2-sphere. More generally, the boundary of a $(k + 1)$ simplex is the unique minimal triangulation of the $k$-sphere.

- If you identify the opposite sides of the dotted hexagon on the right side of Figure 1, you get the unique minimal triangulation $T^2_7$ of the 2-torus. $T^2_7$ has 14 triangles, 21 edges, and 7 vertices.
• In 1980, W. Kühnel discovered $CP^9_2$, the unique 9-vertex minimal triangulation of the complex projective plane $CP^2$. This triangulation has 36 4-simplices and a symmetry group of order 54.

• In 1992, U. Brehm and W. Kühnel [BK] defined $HP^2_{15}$ (and two variants), a 15-vertex simplicial complex with 490 8-simplices. In 2019, D. Gorodkov [G] proved that $HP_{15}$ and the variants are PL homeomorphic to the quaternionic projective plane $HP^2$.

• So far it an open question as to whether there is a 27-vertex triangulation of $OP^2$, the octonionic (a.k.a. Cayley) projective plane.

• The minimal triangulations of $RP^3$ and $RP^4$ respectively have 11 and 16 vertices. See [D],

• In 2021, K. Adiprasito, S. Avvakumov, R. Karasev [AAK] proved that real projective space can be triangulated using a sub-exponential number of simplices.

The survey article by B. Datta [D] has a wealth of information about minimal triangulations up to the year 2007 and a large number of references.

The subject of this paper is $CP^9_2$. In [KB], Kühnel and T. Banchoff establish many interesting properties of $CP^9_2$ and give a rather intricate proof that $CP^9_2$ really is homeomorphic to $CP^2$. Since [KB], there has been a lot of work done trying to understand $CP^9_2$ from various points of view. In particular, there are a number of proofs that $CP^9_2 \cong CP^2$, and also a number of proofs that $CP^9_2$ is the only minimal triangulation of $CP^2$. See the article by B. Morin and M. Yoshida [MY] for a survey of these proofs. See also the paper by B. Bagchi and B. Datta [BD].

The purpose of this paper is to give a new and very nice proof that $CP^9_2 \cong CP^2$. The basic idea of the proof here is to recall the trisection of $CP^2$ into 3 bi-disks, and then to see this trisection inside a symmetry-breaking subdivision of $CP^9_2$. The construction is perfectly compatible with an easier version that works for $RP^6_2$, so I will explain that as well.

The picture developed here is related to the 10-vertex triangulation $CP^2_{10}$ of $CP^2$ that in [BK] is constructed by building outward from $T^2_7$. Indeed Denis Gorodkov, in a private communication, explained to me how one can find a “path” from $CP^2_9$ to $CP^2_{10}$ using the subdivision idea and then something akin to bi-stellar flips. (I’ll let Denis tell this story elsewhere if he wants to, but see the end of §2 for a hint.)
My proof also has a close kinship with the “red-white-blue discussion” in §1.3 of the M.P.I.M. preprint by Morin and Yoshida that is the precursor to [MY] (and has the same title). This discussion is, in turn, related to Figure 8 in [KB]. Morin and Yoshida describe the red-white-blue discussion as a “topological insight” but they don’t really push it forward into a proof. I think that my picture is very similar, but clarified by the special subdivision.

The approach here possibly could shed light on Gorodkov’s result that $HP_{15}^2 \cong HP^2$. The same subdivision and trisection ideas go through for $HP_{15}^2$ almost verbatim, and I can see computationally that each of the 3 sub-complexes is shellable and therefore PL homeomorphic to an 8-ball. However, the high dimensional topology involved in analyzing $HP_{15}^2$ makes a direct topological analysis of the whole complex formidable. For instance, the sub-complex that plays the role of $T_7^2$ has 288 6-simplices. A key step in extending the proof here to $HP_{15}^2$ would be showing that this 288-monster is homeomorphic to $(S^3 \times S^3 \times S^3)/S^3$ in a 3-fold symmetric way.

Here is an outline of the paper.

- In §2 I will give the analogous version of my proof for $RP_{6}^2$. This case is quite concrete and one can see the idea (in Figure 2) at a glance.

- In §3 I will recall the trisection of $CP^2$ and discuss a few key properties of the central torus in this decomposition.

- In §4 I will describe $CP_{9}^2$ and then explain my symmetry-breaking subdivision. The construction parallels the real case.

- In §5 I will find the trisection inside the subdivision and construct a homeomorphism $h : CP_{9}^2 \to CP^2$ which respects the trisections. Figures 3 and 4 below are the key to the analysis.

- In §6 I will explain how one can see the real case of the construction inside the complex case. This analysis leads to a refinement of $h$ and gives the full power of our main result, Theorem 5.1.

- In §7 I will sketch how to make $h$ completely explicit.

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2 The Real Case

$\mathbb{RP}^2$ is the space of scale equivalence classes of nonzero vectors in $\mathbb{R}^3$. We denote the equivalence class of $(x_1, x_2, x_3) \in \mathbb{R}^3$ by $[x_1 : x_2 : x_3] \in \mathbb{RP}^2$.

We have the trisection $\mathbb{RP}^2 = \beta_1 \cup \beta_2 \cup \beta_3$, where $\beta_j$ is the set where $\max(|x_1|, |x_2|, |x_3|) = |x_j|$. Points in $\beta_1$ may be written uniquely in the form $[1 : x_2 : x_3]$, with $|x_2|, |x_3| \leq 1$. Thus $\beta_1$ is a square. So are $\beta_2$ and $\beta_3$. Each intersection $\beta_i \cap \beta_j$ is a pair of opposite edges, and the triple intersection is a union of the 4 points $[\pm 1 : \pm 1 : \pm 1]$. If we interpret $\mathbb{RP}^2$ as the quotient of a cube by the antipodal map, then the 3 quotient faces are $\beta_1, \beta_2, \beta_3$.

The trisection has 3-fold symmetry. The map $\Sigma : (x_1, x_2, x_3) \to (x_2, x_3, x_1)$ permutes the sets $\beta_1, \beta_2, \beta_3$. In terms of the cube, $\Sigma$ rotates around the appropriate long diagonal. $\mathbb{RP}_6^2$ has a very similar 3-fold symmetry: The permutation $S = (123)(456)$ acts as a rotational symmetry of $\mathbb{RP}_6^2$.

We add a new vertex $[123]$ at the center of the triangle $(1, 2, 3)$, and also new vertices $[12], [13], [23]$ at the centers of the corresponding edges.

![Figure 2: A subdivision of $\mathbb{RP}_6^2$ into 18 triangles.](image)

Using the new vertices, we divide the central triangle of $\mathbb{RP}_6^2$ into 6 triangles and we subdivide each of the adjacent triangles in half. The subdivision has $18 = 3 \times 6$ triangles, with each having exactly one vertex from the set $\{1, 2, 3\}$. For $j = 1, 2, 3$ we let $B_k$ be the subset of 6 new triangles having $k$ for a vertex. The sets $B_1, B_2, B_3$ are colored green, red, blue in Figure 2.

This is now the trisection, and there is a clear homeomorphism from this subdivided complex to $\mathbb{RP}^2$ which maps $B_j$ to $\beta_j$ and conjugates $S$ to $\Sigma$.

Incidentally, a related approach would be to add only $[123]$ and then to replace the sides $(1, 2), (2, 3), (3, 1)$ with the sides $([123], 6), ([123], 5)$ and $([123], 4)$. Gorodkov’s “path” from $\mathbb{CP}_9^2$ to $\mathbb{CP}_2^{10}$ is a more elaborate complex-number analogue of this.
3 The Smooth Trisection: Complex Case

The complex projective plane $\mathbb{CP}^2$ is defined just as $\mathbb{RP}^2$ but with respect to the field $\mathbb{C}$ of complex numbers. We denote points in $\mathbb{CP}^2$ by $[z_1 : z_4 : z_7]$. The variable names will line up with the notation for $\mathbb{CP}^2$. We have the trisection $\mathbb{CP}^2 = \beta_1 \cup \beta_4 \cup \beta_7$, where $\beta_j$ is defined just as in the real case, using the complex norm in place of the absolute value. This time, $\beta_j$ is the product of 2 unit disks. The bi-disks $\beta_1, \beta_4, \beta_7$ have disjoint interiors and are permuted by the same map $\Sigma$ as defined in the real case.

The boundary $\partial \beta_1$ is a 3-sphere, and it decomposes into the solid tori $\beta_{14}$ and $\beta_{17}$. Here $\beta_{ij} = \beta_i \cap \beta_j$. To see that $\beta_{14}$ is a solid torus, note that $\beta_{14}$ consists of points of the form $[1 : u : z]$ with $|z| \leq |u| = 1$ and is therefore the product of the unit disk and the unit circle. The central torus $\beta_{147} = \beta_{14} \cap \beta_{17} = \beta_1 \cap \beta_4 \cap \beta_7$ consists of points where $|z_1| = |z_4| = |z_7|$. We discuss $\beta_{147}$ in more detail, with a view towards seeing it inside $\mathbb{CP}_{9}^2$.

Hexagonal Structure: Let $R^3_0 \subset R^3$ denote the plane of points whose coordinates sum to 0. Let $H = R^3_0 \cap [-1, 1]^3$. The vertices of this regular hexagon are the permutations of $(1, -1, 0)$. Let $\overline{H}$ be the flat torus obtained by identifying the opposite sides of $H$ by translations. The translation vectors are the cyclic permutations of $(1, 1, -2)$. The map $(x_1, x_4, x_7) \to [x_1^* : x_4^* : x_7^*]$, $x^* = e^{2\pi i z}$ induces a homeomorphism $\overline{H} \to \beta_{147}$. The main point behind this fact is that $[1^* : 1^* : (-2)^*] = [0^* : 0^* : 0^*]$, etc. We equip $\beta_{147}$ with the metric which makes $\overline{H} \to \beta_{147}$ an isometry.

Symmetries: The 3 fixed points of $\Sigma$ lie in $\beta_{147}$ and correspond to the points on $\overline{H}$ represented by the center and vertices of $H$. The fixed point set of coordinatewise complex conjugation, which we call $\mathcal{C}$, is $\mathbb{RP}^2$. Note that $\mathbb{RP}^2 \cap \beta_{147} = \{[\pm 1 : \pm 1 : \pm 1]\}$. These points correspond to the center of $H$ and to the centers of the edges of $H$.

A Contractible Loop: The line in $R^3_0$ where $x_1 = x_4$ bisects $H$ and contains the midpoints of a pair of opposite sides. This line gives rise to a geodesic loop in $\overline{H}$ we call $a_{14}$ in Figure 4 below. The corresponding loop $a_{14} \subset \beta_{147}$ is given by $\{[1 : 1 : u] | |u| = 1\}$. The loop $\alpha_{14}$ is contractible in $\beta_{14}$: It bounds the disk in $\beta_{14}$ consisting of points $[1 : 1 : z]$ with $|z| \leq 1$. 


4 The Complex and its Subdivision

The vertices of $CP^2_9$ are labeled 1, ..., 9. Here are 16 of the 36 4-simplices of $CP^2_9$ listed on p. 15 of [KB].

- $15289 \ 12389 \ 13689 \ 45289 \ 42389 \ 43689$
- $14256 \ 14356 \ 14259 \ 14368$
- $14726 \ 14768 \ (14783 \ 14735 \ 14759 \ 14792)$

Comparing our list to [KB], we have sometimes permuted the vertices so as to highlight the indices 1, 4, 7. The other 24 4-simplices are orbits of the first 12 under the action of the fundamental permutation:

$$S = (147)(258)(369).$$

For instance, 14726 has orbit $14726 \rightarrow 14759 \rightarrow 14783$. The four simplices in parentheses are listed for the sake of making our tetrahedron list below more transparent. In [KB] the authors exhibit a symmetry group of order 54 acting on $CP^2_9$. For us, one other special element of this group is the symmetry $T = (23)(56)(89)$.

Let $[ij]$ be the midpoint of the edge $i \leftrightarrow j$. Let $[ijk]$ be the center of the triangle $ijk$. Let the rank of a simplex be the number of vertices which belong to the set $\{1, 4, 7\}$. Our list above goes by rank. Parallel to the real case, we divide each rank $k$ simplex into $k!$ smaller simplices, as follows: The rank 1 simplices are untouched. The rank 2 simplex $14abc$ divides into

$$1[14]abc \quad 4[14]abc,$$

and likewise with the indices 1, 4, 7 permuted. The rank 3 simplex $147ab$ divides into

$$1[14][147]ab \quad 1[17][147]ab \quad 4[14][147]ab \quad 4[47][147]ab \quad 7[17][147]ab \quad 7[47][147]ab.$$

We replace our original 36 simplices with the subdivided simplices. Since there are respectively 18, 12, 6 simplices of rank 1, 2, 3 we get a total of

$$(1, 2, 6) \cdot (18, 12, 6) = 78 = 3 \times 26$$

new simplices. (The rank 1 simplices count as “new”.) Each new simplex has exactly one vertex from the set $\{1, 4, 7\}$. 7
5 The Combinatorial Trisection

We have $\mathbb{CP}_9^2 = B_1 \cup B_4 \cup B_7$, where $B_j$ is the union of the 26 new simplices having $j \in \{1, 4, 7\}$ as a vertex. Each $B_j$ is the cone to vertex $j$ of $\partial B_j$. Hence $B_i$ and $B_j$ have disjoint interiors for $i \neq j$. Here is our main result.

Theorem 5.1 There is a homeomorphism $h : \mathbb{CP}_9^2 \to \mathbb{CP}^2$ with the following properties:

1. $h$ maps vertices $1, 4, 7$ respectively to $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.
2. $h$ maps $B_j$ to $\beta_j$ for $j = 1, 4, 7$.
3. $h$ conjugates $S$ to $\Sigma$.
4. $h$ conjugates $T$ to $T$.

In this section I will construct a non-explicit homeomorphism $h$ which has the first 3 properties but not necessarily the fourth. This should satisfy a reader who just wants to see why $\mathbb{CP}_9^2 \simeq \mathbb{CP}^2$. In §6, I will give a more refined version of $h$ which has the fourth property. In §7 I will sketch how to make $h$ explicit.

The first thing we do is list the tetrahedra in $B_{14} = B_1 \cap B_4$. We will derive this tetrahedron list from the simplex list above. The reader might want to check that this actually works, so for convenience we repeat the simplex list here:

- $15289$ $12389$ $13689$ $45289$ $42389$ $43689$
- $14256$ $14356$ $14259$ $14368$
- $14726$ $14768$ ($14783$ $14735$ $14759$ $14792$)

Now for the derivation. We get 13 tetrahedra contained in $B_{14}$ by subdividing the simplices on our list above and omitting 1 or 4. The tetrahedra are listed in a way that corresponds to the simplices.

- $5289$ $2389$ $3689$
- $[14]256$ $[14]356$ $[14]259$ $[14]368$
- $[14][147]26$ $[14][147]68$ $[14][147]83$ $[14][147]35$ $[14][147]59$ $[14][147]92$
The images of these 13 tetrahedra under $S^{-1}$ lie in $B_{17}$ and are totally distinct from the ones above. This accounts for all 26 tetrahedra in $\partial B_1$. Hence, the 13 above are the complete list of tetrahedra comprising $B_{14}$, and moreover $B_{14}$ and $B_{17}$ have disjoint interiors.

**Lemma 5.2** $B_{14}$ is a solid torus.

**Proof:** Write $B_{14} = B'_{14} \cup B''_{14}$, where $B'_{14}$ is the union of the first 3 tetrahedra above and $B''_{14}$ is the union of the last ten. $B'_{14}$ is a 3-ball because it is the join of the path 5236 with the segment 89, and $B''_{14}$ is a 3-ball because it is the cone to vertex [14] of $\partial B''_{14}$, a 10-triangle triangulation of the 2-sphere.

Figure 3 below shows $\partial B'_{14}$ and $\partial B''_{14}$. Each one is drawn as the union of 2 combinatorial hexagons glued along their boundaries according to the labels. $B'_{14} \cap B''_{14}$ is the union of the 2 disjoint grey triangles 259 and 368. Topologically, we get $B_{14}$ by gluing two 3-balls together along a pair of disjoint disks in their boundaries. The orientations of the gluings are such that the result is a solid torus (as opposed to the so-called solid Klein bottle, a nontrivial disk bundle over the circle).

![Figure 3: $\partial B'_{14}$ and $\partial B''_{14}$. Glue the hex boundaries together.](image)

We get the triangulation of $B_{147} = \partial B_{14}$ by gluing the two triangulations from Figure 3 along the grey triangles. Figure 4 shows the universal cover of the triangulation. We get back to $B_{147}$ by gluing the opposite sides of the big hexagon by translations. This triangulation of $B_{147}$ is exactly $T^2_7$. Note that
$S$ acts on $B_{147}$ fixing $[147], [258], [369]$, points which respectively correspond to the center and vertices of the hexagon, just as in the smooth case.

![Figure 4: The universal covering of the triangulation of $B_{147}$.

From all this structure we see that (after suitably scaling) there is an isometry $h_{147} : B_{147} \to \beta_{147}$ which conjugates $S$ to $\Sigma$ and which maps the green loop $a_{14}$ to $\alpha_{14}$. The labels of the hexagon vertices, such as $(1, -1, 0)$, indicate precisely how the hexagon here lines up with the one described in connection with the central torus of $CP^2$. Note that $a_{14}$ is contractible in $B_{14}$ because $a_{14} \subset B''_{14}$, and recall that $\alpha_{14}$ is contractible in $\beta_{14}$. Hence $h_{147}$ extends to a homeomorphism $h_{14} : B_{14} \to \beta_{14}$.

Define $h_{17} = \Sigma^{-1} \circ h_{14} \circ S$ and $h_{47} = \Sigma \circ h_{14} \circ S^{-1}$. This gives us homeomorphisms $h_{17} : B_{17} \to \beta_{17}$ and $h_{47} : B_{47} \to \beta_{47}$. The maps $h_{ij}$ all agree on $B_{147}$ because $h_{147}$ conjugates $S$ to $\Sigma$. The union

$$h = h_{14} \cup h_{17} \cup h_{47} : \partial B_1 \cup \partial B_4 \cup \partial B_7 \to \partial \beta_1 \cup \partial \beta_4 \cup \partial \beta_7$$

is a homeomorphism which respects the individual pieces and their intersections. Since $B_j$ and $\beta_j$ are cones over $\partial B_j$ and $\partial \beta_j$ we can extend $h$, by coning, to a homeomorphism from $CP^2_9 = B_1 \cup B_4 \cup B_7$ to $CP^2 = \beta_1 \cup \beta_4 \cup \beta_7$. 

10
6 The Extra Symmetry

The fixed set of $T = (23)(56)(89)$ is a copy of $\mathbb{R}P^2_6$. The 6 vertices are 1, 4, 7, [23], [89], [56]. If we rename these vertices $\hat{1}$, $\hat{2}$, $\hat{3}$, $\hat{4}$, $\hat{5}$, $\hat{6}$ we get the same combinatorial pattern as in Figure 1. Our subdivision of $\mathbb{C}P^2_9$ induces the same subdivision as in Figure 2. In particular, the intersection $\mathbb{R}P^2_6 \cap B_{14}$ is a union of 3 edges which together make 2 line segments, namely

\[ \hat{4}\hat{5} = [23][89] \subset B'_{14}, \quad 6[1\hat{2}], [1\hat{2}][1\hat{2}3] = [56][14], [14][147] \subset B''_{14}. \]

The map $h_{147}$ conjugates $T$ to $T$. Figure 5 below indicates how $h_{147}$ maps the fixed points of $T$ in $B_{147}$ to the fixed points of $T$ in $\beta_{147}$.

Now we explain how to choose our homeomorphism $h$ so that it conjugates $T$ to $T$. Figure 5 below illustrates the following 4 disks.

- Let $D'_{14}$ be the cone to $[[23][89]]$ of the loop $a'_{14}$ shown in Figure 4.
- Let $D''_{14}$ be the cone to [14] of the loop $a_{14}$ shown in Figure 4.
- Let $\Delta'_{14}$ be the disk $[1 : -1 : z]$ with $|z| \leq 1$. We think of $\Delta'_{14}$ as the cone to $[1 : -1 : 0]$ of $\partial \Delta'_{14}$.
- Let $\Delta''_{14}$ be the disk $[1 : 1 : z]$ with $|z| \leq 1$. We think of $\Delta''_{14}$ as the cone to $[1 : 1 : 0]$ of $\partial \Delta''_{14}$.

Let $Y_{14}$ be the component of $B_{14} - D'_{14} - D''_{14}$ that contains the point [259]. Let $\Upsilon_{14}$ be the component of $\beta_{14} - \Delta'_{14} - \Delta''_{14}$ which contains the point $[1 : 1 : i]$. Both $Y_{14}$ and $\Upsilon_{14}$ are solid balls. $T$ interchanges $Y_{14}$ with (the closure of) $B_{14} - Y_{14}$, the other component of $B_{14} - D'_{14} - D''_{14}$. Likewise $T$ interchanges $\Upsilon_{14}$ with (the closure of) $\beta_{14} - \Upsilon_{14}$.

![Figure 5: One component of $X_{14}$ or of $\chi_{14}$, depending on the label choice.](image-url)
When we use the top label of each pair, Figure 5 shows \(Y_{14}\). (Think about cutting a pink-frosted grey donut in half.) The pink boundary is half of \(B_{147}\). The left grey disk is \(D'_{14}\) and the right grey disk is \(D''_{14}\). The map \(T\) acts as rotation by 180 degrees about the purple line bisecting the grey disks. The intersection of the purple line with the grey disks is the part lying in our copy of \(RP^2\). When we use the bottom labels, Figure 5 shows the same things for \(\Upsilon_{14}\).

By construction \(h_{147}(\partial D'_{14}) = \partial \Delta'_{14}\) and \(h_{147}(\partial D''_{14}) = \partial \Delta''_{14}\). We define \(h_{14}\) on each of \(D'_{14}\) and \(D''_{14}\) by coning over the boundaries. By symmetry this extension conjugates \(T\) to \(T\) and is defined in particular on \(\partial Y_{14}\). Our extension maps the (pink and grey) sphere \(\partial Y_{14}\) to the (pink and grey) sphere \(\partial \Upsilon_{14}\). We now extend to a homeomorphism from the ball \(Y_{14}\) to the ball \(\Upsilon_{14}\) and use the action of \(T\) and \(T\) to extend the homeomorphism to all of \(B_{14}\).

Our improved \(h_{14}\) conjugates \(T\) to \(T\).

The rest of the construction is as above. The union map \(h_{14} \cup h_{17} \cup h_{47}\), defined on \(\partial B_1 \cup \partial B_4 \cup \partial B_7\), conjugates \(T\) to \(T\) because the pairs \((S, T)\) and \((\Sigma, T)\) commute. The final coning process respects \(T\) and \(T\), so the final extension of \(h\) to \(B_1 \cup B_4 \cup B_7\) conjugates \(T\) to \(T\).

Here is where \(h\) sends the vertices:

- \(1 \rightarrow [1 : 0 : 0]\).
- \([14] \rightarrow [1 : 1 : 0]\).
- \([147] \rightarrow [1 : 1 : 1]\).
- \([56] \rightarrow [1 : 1 : -1]\).
- \(2 \rightarrow [1 : e^{4\pi i/7} : e^{12\pi i/7}]\).
- \([259] \rightarrow [1 : e^{4\pi i/7} : 0]\).

The remaining images can be readily deduced from the action of \(S, T, \Sigma, T\). The last two entries require some explanation. The coordinates of the point \(p_2 \in R^2_0\) corresponding to vertex 2 are \((-1, 5, -4)/7\). We found this by solving the equation \(2p_2 - \Sigma^2(p_2) = (-1, 2, -1)\). The choice of where to send \([259]\) is not determined by the construction above, but we might as well make it. The explicit construction below makes this choice, and so it is convenient to list it here.
7 Making the Homeomorphism Explicit

The only non-explicit part of our construction is the extension of the sphere map $h_{147} : \partial Y_{14} \to \partial \Upsilon_{14}$ to the ball map $h_{14} : Y_{14} \to \Upsilon_{14}$. In this section we sketch an explicit extension.

Gluing the opposite blue sides of the parallelogram in Figure 6 gives the cylinder $\partial Y_{14} - D'_{14} - D''_{14}$, drawn pink in Figure 5. The green loops $a_{14}$ and $a_{14}'$ are on the boundary. Figure 6 suggests an explicit foliation of $\partial Y_{14}$ by polygonal loops. Intrinsically these are geodesic bigons in $B_{147}$. The intersections with the blue edges of the triangulation move linearly.

Let the core of $Y_{14}$ be the path with vertices $[14], [259], [[23][89]]$. Figure 6 indicates a piecewise linear correspondence between the loops in the foliation and the points on the core. We cone each loop in the foliation to the corresponding point on the core. Now (after some checking of disjointness) we have a disk foliation of $Y_{14}$ which interpolates between $D'_{14}$ and $D''_{14}$ and respects the partition of $Y_{14}$ into $Y_{14} \cap B'_{14}$ and $Y_{14} \cap B''_{14}$.

We use $h_{147}$ to transfer our foliation to $\partial \Upsilon_{14} - \Delta'_1 - \Delta''_1$. We change coordinates:

$$[1 : u : z] \to (\arg u, z).$$

In these coordinates, $\Upsilon_{14} = [0, \pi] \times D^2$. Let $s$ be the blue line segment in Figure 6 that runs all the way across the cylinder and contains vertex 2. The
intersection \((t, s) = \gamma \cap s\) is a center of symmetry of the loop \(\gamma\). We cone \(\gamma\) to \((t, 0)\). Now (after some checking of disjointness) we have a disk foliation of \(Y_{14}\) which interpolates between \(\Delta'_{14}\) and \(\Delta''_{14}\).

We extend \(h_{147}\) to \(Y_{14}\) by coning, so that it maps the one foliation to the other. This description explicitly determines \(h_{14} : Y_{14} \to Y_{14}\). Again, the rest of the construction is just symmetry and coning, so that final map \(h\) is explicitly defined.

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14