Eigenvalues and diagonal elements

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Abstract A basic theorem in linear algebra says that if the eigenvalues and the diagonal entries of a Hermitian matrix $A$ are ordered as $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and $a_1 \leq a_2 \leq ... \leq a_n$, respectively, then $\lambda_1 \leq a_1$. We show that for some special classes of Hermitian matrices this can be extended to inequalities of the form $\lambda_k \leq a_{2k-1}$, $k = 1, 2, ..., \lceil \frac{n}{2} \rceil$.

Keywords Hermitian matrix · Majorization · Nonnegative matrix · Laplacian matrix of graph

Let $A$ be an $n \times n$ complex Hermitian matrix. The eigenvalues and the diagonal entries of $A$ are real numbers, and we enumerate them in increasing order as

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n,$$

and

$$a_1 \leq a_2 \leq ... \leq a_n,$$

respectively. Various inequalities relating these two $n$-tuples are known and are much used in matrix analysis. For example, we have

$$\lambda_1 \leq a_1 \quad \text{and} \quad \lambda_n \geq a_n. \quad (1)$$

These are subsumed in the majorization relations due to I. Schur: for $1 \leq k \leq n$

$$\sum_{j=1}^{k} \lambda_j \leq \sum_{j=1}^{k} a_j, \quad (2)$$

with equality when $k = n$. This is a complete characterization of two $n$-tuples that could be the eigenvalues and diagonal entries of a Hermitian matrix. In general, there are no further relations between individual $\lambda_j$ and $a_k$.

However, for large and interesting subsets of Hermitian matrices, it might be possible to find such extra relations. In [1], the authors consider eigenvalues of matrices associated with graphs. Let $G$ be a simple weighted graph on $n$ vertices and let $A$ be the signless Laplacian matrix associated with $G$. Then, it is shown in [1] that $\lambda_2 \leq a_3$. This result is extended to other classes in [3]. One of these is the class $P$ of Hermitian matrices whose off-diagonal
entries are nonnegative. (In particular, this includes symmetric entrywise nonnegative matrices.) It is shown in [3] that if \( A \in \mathcal{P} \), then \( \lambda_2 \leq a_3 \).

In this note we consider, in addition to the class \( \mathcal{P} \), another class \( \mathcal{I} \) consisting of Hermitian matrices all whose off-diagonal entries are purely imaginary. We show that the inequality \( \lambda_2 \leq a_3 \) is valid for \( A \in \mathcal{I} \) as well. The proof we give works for both the classes \( \mathcal{P} \) and \( \mathcal{I} \). Then we show that much more is true for the class \( \mathcal{I} \). We show that in this case, the inequality \( \lambda_{n-1} \geq a_{n-2} \) also holds. Further, for all \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \) we have \( \lambda_k \leq a_{2k-1} \). We construct examples to show that neither of these results is true for the class \( \mathcal{P} \).

**Theorem 1** Let \( A \) be an \( n \times n \) Hermitian matrix whose off-diagonal entries are either all nonnegative real numbers or all purely imaginary numbers. Then

\[
\lambda_2 \leq a_3. \tag{3}
\]

In case the off-diagonal entries are all purely imaginary, we also have

\[
\lambda_{n-1} \geq a_{n-2}. \tag{4}
\]

For the second class of matrices in Theorem 1, we can go further:

**Theorem 2** Let \( A \) be an \( n \times n \) Hermitian matrix whose off-diagonal entries are all purely imaginary. Then, for \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \),

\[
\lambda_k \leq a_{2k-1} \quad \text{and} \quad \lambda_{n-k+1} \geq a_{n-2k+2}. \tag{5}
\]

We remark that in both (1) and (5) the second inequality follows from the first by considering \(-A\) in place of \( A \). Similarly (4) follows from (3). The argument cannot be used for the class \( \mathcal{P} \).

Our proofs rely upon two basic theorems of matrix analysis. Let \( \lambda_j(A) \), \( 1 \leq j \leq n \), denote the eigenvalues of a Hermitian matrix enumerated in the increasing order. Weyl’s inequality says that if \( A \) and \( B \) are two \( n \times n \) Hermitian matrices, then

\[
\lambda_j(A + B) \leq \lambda_j(A) + \lambda_n(B), \quad 1 \leq j \leq n. \tag{6}
\]

Cauchy’s interlacing principle says that if \( A_r \) is an \( r \times r \) principal submatrix of \( A \), then

\[
\lambda_j(A) \leq \lambda_j(A_r), \quad 1 \leq j \leq r. \tag{7}
\]

See Chapter III of [2] for this and other facts used here.

**Proof of Theorem 1** If \( P \) is a permutation matrix, then the increasingly ordered eigenvalues and diagonal entries of \( PAP^T \) are the same as those of \( A \). So, for simplicity, we may assume that the diagonal entries of \( A \) are in increasing order. Let

\[
A_3 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{bmatrix}
\]

be the top-left \( 3 \times 3 \) submatrix of \( A \). (Note \( a_{jj} = a_j \) is our notation.) Decompose

\[
A_3 = D_3 + M_3 \tag{8}
\]

where \( D_3 \) is the diagonal part and \( M_3 \) the off-diagonal part of \( A_3 \). By Weyl’s inequality

\[
\lambda_2(A_3) \leq \lambda_2(M_3) + \lambda_3(D_3) = \lambda_2(M_3) + a_3. \tag{9}
\]

Note that \( \det M_3 = 2Rea_{12}a_{23}a_{13} \). So, under the hypothesis of Theorem 1, \( \det M_3 \geq 0 \). We also have \( \text{tr} M_3 = 0 \). These two conditions imply that we must have \( \lambda_2(M_3) \leq 0 \). For, if \( \lambda_3(M_3) \geq \lambda_2(M_3) > 0 \), then the condition \( \text{tr} M_3 = 0 \) forces \( \lambda_1(M_3) \) to be negative. But this is impossible if \( \det M_3 \geq 0 \). So, from (9) we see that \( \lambda_2(A_3) \leq a_3 \). Then, by the interlacing principle (7), we have \( \lambda_2(A) \leq a_3 \).

Here we should observe that the only property of \( M_3 \) we used was that \( \det M_3 \geq 0 \). Thus the conclusion of Theorem 1 is valid for some other matrices not included in the classes \( \mathcal{P} \) or \( \mathcal{I} \).
Proof of Theorem 2 Let \( A_r \) be the top \( r \times r \) principal submatrix of \( A \). Decompose \( A_r \) as
\[
A_r = D_r + M_r
\]
where \( D_r \) is diagonal and \( M_r \) off-diagonal. The matrix \( iM_r \) is a real skew-symmetric matrix. So, the nonzero eigenvalues of \( iM_r \) are purely imaginary and occur in conjugate pairs. Thus the nonzero eigenvalues of \( M_r \) occur in \( \pm \) pairs. This shows that
\[
\lambda_k(M_r) \leq 0 \quad \text{for} \quad 1 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor. \tag{10}
\]
Now let \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \). Using, successively, the interlacing principle, Weyl’s inequality and (10), we get
\[
\lambda_k(A) \leq \lambda_k(A_{2k-1}) \leq \lambda_k(M_{2k-1}) + a_{2k-1} \leq a_{2k-1}.
\]
\( \square \)

We now give two examples to show why for the case of matrices with nonnegative off-diagonal entries we have to be content just with inequality (3). Let \( A \) be the \( 4 \times 4 \) matrix
\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}.
\]
The \( 4 \times 4 \) matrix \( E \) all whose entries are equal to one has eigenvalues \((4, 0, 0, 0)\). So the matrix \( A = E - I \) has eigenvalues \((3, -1, -1, -1)\). Thus \( \lambda_3 = -1 \), and the inequality (4) does not hold in this case. Further, this example shows that in the general case, not only the inequality (4) might fail, \( \lambda_{n-1} \) could be smaller than \( a_j, j = 1, 2, \ldots, n \).

Let \( B \) be the \( 5 \times 5 \) matrix
\[
B = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]
Then \( B = S^2 + S^3 \), where \( S \) is the shift matrix
\[
S = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
The eigenvalues of \( S \) are the fifth roots of 1. Using this one readily sees that the eigenvalues of \( B \) are \( 2, 2 \cos \frac{2\pi}{5} \) and \( 2 \cos \frac{4\pi}{5} \), the first of these with multiplicity one and the latter two with multiplicities two each. In particular, \( \lambda_3 > 0 \) and the assertion \( \lambda_3 \leq a_5 \) in the first inequality (5) does not hold in this case.

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