EXISTENCE OF CANONICAL MODELS FOR KAWAMATA LOG TERMINAL PAIRS

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Abstract. We prove that a Kawamata log terminal pair has the canonical model.

1. Introduction

We work over an algebraically closed field of characteristic zero.

Our main result is the existence of canonical models for Kawamata log terminal pairs.

Theorem 1.1. Let \((X/Z, B)\) be a Kawamata log terminal pair with the Kodaira dimension \(\kappa_i(X/Z, K_X + B) \geq 0\). Then, \((X/Z, B)\) has the canonical model.

If \(B\) is a \(\mathbb{Q}\)-divisor, then Theorem 1.1 is \([\text{BCHM10}, \text{Corollary 1.1.2}]\). In this paper, we prove it for the general case. The idea of proof is to reduce Theorem 1.1 to \([\text{BCHM10}, \text{Theorem 1.2}]\), by a canonical bundle formula of Fujino-Mori type for \(\mathbb{R}\)-divisors (cf. \([\text{FM00}]\)).

Theorem 1.2. Let \(f : X \to Y\) be a contraction of normal varieties and \((X, B)\) be a klt pair such that \(\kappa_i(X/Y, K_X + B) = 0\). Then, there exists a commutative diagram

\[
\begin{array}{ccc}
(X', B') & \xrightarrow{\pi} & (X, B) \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\phi} & Y
\end{array}
\]

which consists of birational models \(\pi : X' \to X\) and \(\phi : Y' \to Y\), such that:

1. \(K_{X'} + B' = \pi^*(K_X + B) + E\) where \(E\) is exceptional/\(X\) and \(B', E \geq 0\) have no common components.
2. \(K_{X'} + B' \sim_{\mathbb{R}} f'^*(K_{Y'} + B_{Y'} + M_{Y'}) + R\) where \(R \geq 0\) and \((Y', B_{Y'} + M_{Y'})\) is a \(g\)-klt generalised pair with the moduli \(b\)-divisor \(M\).
3. \(\kappa(X'/Y', R^h) = 0\) and \(R^v\) is very exceptional/\(Y'\), where \(R^h\) (resp. \(R^v\)) denotes the horizontal (resp. vertical) part over \(Y'\).

One can easily generalise the above theorem to log canonical pairs. See Remark 3.3.
2. Preliminaries

In this section we collect definitions and some important results. Throughout this paper all varieties are quasi-projective over a fixed algebraically closed field of characteristic zero and a divisor refers to an $R$-Weil divisor unless stated otherwise.

2.1. Notations and definitions. We collect some notations and definitions. We use standard definitions of Kawamata log terminal (klt, for short) pair and sub-klt pair (for example, see [Hu20, Section 2.1]).

Contractions. In this paper a contraction refers to a proper morphism $f: X \to Y$ of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. In particular, $f$ has connected fibres. Moreover, if $X$ is normal, then $Y$ is also normal. A birational map $\pi: X \dashrightarrow Y$ is a birational contraction if the inverse of $\pi$ does not contract divisors. Note that $\pi$ is not necessarily a morphism unless stated otherwise.

Very exceptional divisors. Let $f: X \to Y$ be a dominant morphism from a normal variety to a variety, $D$ a divisor on $X$, and $Z \subset X$ a closed subset. We say $Z$ is horizontal over $Y$ if $f(Z)$ dominates $Y$, and we say $Z$ is vertical over $Y$ if $f(Z)$ is a proper subset of $Y$.

Suppose $f$ is a contraction of normal varieties. Recall that a divisor $D$ is very exceptional if $D$ is vertical and for any prime divisor $P$ on $Y$ there is a prime divisor $Q$ on $X$ which is not a component of $D$ but $f(Q) = P$, i.e. over the generic point of $P$ we have $\text{Supp} f^*P \not\subseteq \text{Supp}D$.

If $\text{codim} f(D) \geq 2$, then $D$ is very exceptional. In this case we say $D$ is $f$-exceptional.

Generalised pairs. For the basic theory of generalised polarised pairs (generalised pairs for short) we refer to [BZh16, Section 4]. Below we recall some of the main notions and discuss some basic properties.

A generalised sub-pair consists of

- a normal variety $X$ equipped with a proper morphism $X \to Z$, 
- an $R$-divisor $B$ on $X$, and 
- a b-$R$-Cartier b-divisor over $X$ represented by some projective birational morphism $\overline{\phi}: \overline{X} \to X$ and $R$-Cartier divisor $\overline{M}$ on $X$ such that $\overline{M}$ is nef/$Z$ and $K_X + B + M$ is $R$-Cartier, where $M := \phi_*\overline{M}$.

A generalised sub-pair is a generalised pair if $B$ is effective. We usually refer to the sub-pair by saying $(X/Z, B + M)$ is a generalised sub-pair with data $\overline{M}$ or with the moduli b-divisor $M$, where $M$ is represented by $\overline{M}$. We will use standard definitions of b-divisors, generalised singularities and log minimal models (for example, see [Hu20, Section 2.1]).

2.2. Iitaka dimension and Iitaka fibration. In this subsection we introduce the notion of invariant Iitaka dimension and invariant Iitaka fibration.

Recall the following definitions of Iitaka dimension, which is a birational invariant integer given by the growth of the quantity of sections.

Definition 2.1 (Invariant Iitaka dimension). Let $X$ be a normal projective variety, and $D$ be an $R$-Cartier divisor $D$ on $X$. We define the invariant Iitaka dimension of $D$, denoted by $\kappa_i(X, D)$, as follows (see also [Fuj-book17, Definition 2.5.5]): If there is
an \(\mathbb{R}\)-divisor \(E \geq 0\) such that \(D \sim_{\mathbb{R}} E\), set \(\kappa_\iota(X,D) = \kappa(X,E)\). Here, the right hand side is the usual Iitaka dimension of \(E\). Otherwise, we set \(\kappa_\iota(X,D) = -\infty\). We can check that \(\kappa_\iota(X,D)\) is well-defined, i.e., when there is \(E \geq 0\) such that \(D \sim_{\mathbb{R}} E\), the invariant Iitaka dimension \(\kappa(Y,D)\) does not depend on the choice of \(E\). By definition, we have \(\kappa_\iota(X,D) \geq 0\) if and only if \(D\) is \(\mathbb{R}\)-linearly equivalent to an effective \(\mathbb{R}\)-divisor.

Let \(X \to Z\) be a projective morphism from a normal variety to a variety, and let \(D\) be an \(\mathbb{R}\)-Cartier divisor on \(X\). Then the relative invariant Iitaka dimension of \(D\), denoted by \(\kappa_\iota(X/Z,D)\), is defined by \(\kappa_\iota(X/Z,D) = \kappa_\iota(X,D|_F)\), where \(F\) is a very general fibre (i.e. the fibre over a very general point) of the Stein factorisation of \(X \to Z\). Note that the value \(\kappa_\iota(X,D|_F)\) does not depend on the choice of \(F\) (see [HH19, Lemma 2.10]).

For basic properties of the invariant Iitaka dimension, we refer to [HH19, Remark 2.8].

**Definition 2.2** (Invariant Iitaka fibration). Let \(X\) be a normal variety projective over \(Z\), and \(D\) be an \(\mathbb{R}\)-Cartier divisor on \(X\) with \(\kappa_\iota(X/Z,D) \geq 0\). Pick an \(\mathbb{R}\)-Cartier divisor \(E \geq 0\) such that \(D \sim_{\mathbb{R}} E/Z\). Then there exists a contraction \(\phi : X' \to Y\) of smooth varieties such that for all sufficiently large integers \(m > 0\), the rational maps \(\phi_m : X \dasharrow Y_m\) given by \(f^*f_*\mathcal{O}_X([mE])\) are birationally equivalent to \(\phi\), that is, there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow{\phi_m} & & \downarrow{\phi} \\
Y_m & \xleftarrow{\varphi_m} & Y
\end{array}
\]

of rational maps \(\phi_m, \varphi_m\) and a contraction \(\pi\), where the horizontal maps are birational, \(\dim Y = \kappa_\iota(X,D)\), and \(\kappa(X'/Y, f^*E) = 0\). Such a fibration is called an Iitaka fibration of \(D\). It is unique up to birational equivalence.

**Lemma 2.3.** The definition above is well-defined and independent of the choice of \(E\).

**Proof.** By compactification, we may assume \(Z\) is projective, and hence \(X, Y\) projective. The definition is well-defined by [Nak04, II.3.14]. Let \(\phi' : X' \to Y'\) be a relative Iitaka fibration over \(Z\) associated to an \(\mathbb{R}\)-Cartier divisor \(E' \geq 0\) such that \(D \sim_{\mathbb{R}} E'/Z\). Pick a very general closed point \(y' \in Y'\). By [Fuj-book17, Proof of Lemma 2.5.6], for any sufficiently large positive integer \(m\), there is an injection

\[
H^0(f'^{-1}(y'), \mathcal{O}_X([mE]_{f'^{-1}(y')}) \hookrightarrow H^0(f^{-1}(y'), \mathcal{O}_X([m+1]E'_{f^{-1}(y')}) \simeq k.
\]

We infer that the image of \(f'^{-1}(y')\) under \(\phi_m\) is a point. Therefore, by the rigidity lemma [Nak04, II.1.12], \(\phi'\) induces a birational map \(\psi_m : Y' \dasharrow Y_m\) such that \(\phi_m \circ \pi = \psi_m \circ \phi'\), which completes the proof. \(\square\)

**Canonical models.** Recall that, given a a proper morphism \(h : X \to Z\) from a normal variety to a variety, an \(\mathbb{R}\)-Cartier divisor \(D\) is semi-ample over \(Z\) if there exist a proper surjective morphism \(g : X \to Y\) over \(Z\) and an ample/Z divisor \(D_Y\) of \(Y\) such that \(D \sim_{\mathbb{R}} g^*D_Y\).

**Remark 2.4** ([Hu20, Lemma 2.5.5]). Notation as above, let \(D\) be an \(\mathbb{R}\)-Cartier divisor.
(1) $D$ is semi-ample if and only if $D$ is a convex combination of semi-ample $\mathbb{Q}$-divisors.

(2) Let $D'$ be another $\mathbb{R}$-Cartier divisor. If $D, D'$ are semi-ample, then so is $D + D'$.

Given an $\mathbb{R}$-linear system $|D/Z|_\mathbb{R}$, we say a divisor $E \geq 0$ is contained in the fixed part of $|D/Z|_\mathbb{R}$ if, for every $B \in |p^*D/Z|_\mathbb{R}$, then $B \geq E$.

**Definition 2.5** ([BCHM10, Definitions 3.6.5 and 3.6.7]). Let $h : X \to Z$ be a projective morphism of normal quasi-projective varieties and let $D$ be a $\mathbb{R}$-Cartier divisor on $X$.

(1) We say that a birational contraction $f : X \to X'$ over $Z$ is a semi-ample model of $D$ over $Z$, if $f$ is $D$-non-positive, $X'$ is normal and projective over $Z$ and $D' = f_*D$ is semi-ample over $Z$.

(2) We say that $g : X \to Y$ is the ample model of $D$ over $Z$, if $g$ is a rational map over $Z$, $Y$ is normal and projective over $Z$ and there is an ample divisor $H$ over $Z$ on $Y$ such that if $p : W \to X$ and $q : W \to Y$ resolve $g$ then $q$ is a contraction morphism and we may write $p^*D \sim q^*H + E/Z$, where $E \geq 0$ is contained in the fixed part of $|p^*D/Z|_\mathbb{R}$. By [BCHM10, Lemma 3.6.6], the ample model is unique up to isomorphism.

(3) (Canonical model.) If $(X, B)$ is a klt pair and $D = K_X + B$, then we say $Y$ is the canonical model of $(X, B)$ over $Z$.

### 2.3. Klt-trivial fibrations

Recall that the discrepancy $b$-divisor $A = A(X, B)$ of a pair $(X, B)$ is the $b$-divisor of $X$ with the trace $A_Y$ defined by the formula

$$K_Y = f^*(K_X + B) + A_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. By the definition, we have $\mathcal{O}_X([A(X, B)]) = \mathcal{O}_X$ when $(X, B)$ is klt (see [Fuj12, Lemma 3.19]).

**Definition 2.6** ([Hu20, Definition 2.21]). Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. A $\mathbb{K}$-klt-trivial fibration $f : (X, B) \to Y$ consists of a contraction $f : X \to Y$ of normal varieties and a sub-pair $(X, B)$ satisfying the following properties:

1. $(X, B)$ is sub-klt over the generic point of $Y$;
2. $\text{rank}_f \mathcal{O}_X([A(X, B)]) = 1$;
3. There exists an $\mathbb{R}$-Cartier divisor $D$ on $Y$ such that $K_X + B \sim_\mathbb{K} f^*D$.

Notation as above, we set

$$b_P = \max\{t \in \mathbb{R}|(X, B + tf^*P) \text{ is sub-lc over the generic point of } P\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where $P$ runs over prime divisors on $Y$. Then it is easy to see that $B_Y$ is well defined since $b_P = 1$ for all but a finite number of prime divisors and it is called the discriminant divisor. Furthermore, we set

$$M_Y = D - K_Y - B_Y.$$
and call $M_Y$ the moduli divisor. Note that if $\mathbb{K} = \mathbb{Q}$, thanks to the important result [Amb04, Theorem 2.5] obtained by the theory of variations of Hodge structure, the moduli b-divisor $M$ of a $\mathbb{Q}$-klt-trivial fibration is $\mathbb{Q}$-b-Cartier and b-nef. Hence $K + B$ is $\mathbb{R}$-b-Cartier.

The arguments for next lemma are taken from [Hu20].

**Lemma 2.7** ([Hu20, Lemma 2.22]). Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration. Then, $B$ is a convex combination of $\mathbb{Q}$-divisors $B_i$ such that $f : (X, B_i) \to Y$ is $\mathbb{Q}$-klt-trivial. Moreover, if $(X, B)$ is sub-klt, then we can choose $B_i$ so that $(X, B_i)$ is sub-klt for each $i$.

**Proof.** Replacing $X$ we may assume it is smooth. Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration, $\varphi = \prod_{i=1}^k \varphi_i^{a_i}$ be an $\mathbb{R}$-rational function so that $K_X + B + (\varphi) = f^* D$. Let $\mathcal{V} \subset \text{CDiv}_{\mathbb{R}}(Y)$ be a finite dimensional rational linear subspace containing $D$, $\mathcal{L} \subset \text{CDiv}_{\mathbb{R}}(X)$ be a rational polytope containing $B$ such that, for every $\Delta \in \mathcal{L}$, we have $(X, \Delta)$ is a sub-pair which is sub-klt over the generic point of $Y$. Now we consider the rational polytope

$$\mathcal{P} := \{ \Delta \in \mathcal{L} | \Delta + \sum_{i=1}^k \mathbb{R}(\varphi_i) \text{ intersects } f^* \mathcal{V} \}$$

For every $\Delta \in \mathcal{P}$, we have further $K_X + \Delta \sim_{\mathbb{R}} 0/Y$. It is obvious that $B \in \mathcal{P}$.

It suffices to show that, there exists a convex combination $B = \sum_{j} r_j B_j$ of $\mathbb{Q}$-divisors $B_j \in \mathcal{P}$ with rank$f_* \mathcal{O}_X([A(X, B)]) = 1$. To this end, pick a log resolution $\pi : \overline{X} \to X$ of $(X, \sum_{j} \Gamma_j)$ where every element of $\mathcal{P}$ is supported by $\sum_{j} \Gamma_j$. Note that the proofs of [Fuj12, Lemmas 3.19 and 3.20] are still valid for $\mathbb{R}$-sub-boundaries. Hence, by shrinking $Y$, we may assume $(X, \Delta)$ is sub-klt for every $\Delta \in \mathcal{P}$, and we have

$$f_* \mathcal{O}_X([A(X, \Delta)]) = f_* \pi_* \mathcal{O}_{\overline{X}}(\sum_i [a_i] A_i)$$

where $K_{\overline{X}} = \pi^* (K_X + \Delta) + \sum a_i A_i$. Consider the rational sub-polytope

$$\mathcal{Q} = \{ \Delta \in \mathcal{P} | [A(X, \Delta)]_{\mathbb{R}} \leq [A(X, B)]_{\mathbb{R}} \}.$$

Then, for any $B_j \in \mathcal{Q}$, we have rank$f_* \mathcal{O}_X([A(X, B_j)]) = 1$ which completes the first assertion. The last statement is obvious. \hfill $\Box$

**Lemma 2.8.** Let $f : (X, B) \to Y$ be an $\mathbb{R}$-klt-trivial fibration from a sub-klt pair, $B_Y$ be the discriminant divisor and $M_Y$ be the moduli divisor. Then, there exists a b-divisor $M$ satisfying:

1. The trace $M_Y = M_Y$.
2. $(Y, B_Y + M_Y)$ is a g-sub-klt generalised pair with the moduli b-divisor $M$.

**Proof.** Replacing $X$, we may assume it is smooth. By Lemma 2.7, there exists a convex combination of $B = \sum_i r_i B_i$ of $\mathbb{Q}$-divisors such that $f : (X, B_i) \to Y$ is $\mathbb{Q}$-klt-trivial. Let $\mathcal{P} \subset \text{CDiv}_{\mathbb{R}}(X)$ be the polytope defined by $B_i$’s. For any prime divisor $P$ on $Y$, we set the function $b_p$ on $\mathcal{P}$:

$$b_p(\Delta) = \max \{ t \in \mathbb{R} | (X, \Delta + tf^* P) \text{ is sub-lc over the generic point of } P \}.$$
We note that the $b_P$ is piecewisely affine and gives a rational polyhedral decomposition of $\mathcal{P}$. Also note that there are only finitely many $P$ such that $b_P$ is not identically one on $\mathcal{P}$. Therefore, there exists a rational sub-polytope $Q$ containing $B$ such that $b_P$ is affine on $Q$, for any prime divisor $P$. In particular, replacing $B_i$'s and $r_i$’s, we have $B_Y = \sum_i r_iB_{Y,i}$ and $M_Y = \sum_i r_iM_{Y,i}$, where $B_{Y,i}, B_{Y,i}$ are discriminant divisors and $M_Y, M_{Y,i}$ are moduli divisors of $f : (X, B) \to Y, f : (X, B_i) \to Y$ respectively. Letting $M = \sum_i r_iM_i$, where $M_i$ is the moduli b-divisor of $f : (X, B_i) \to Y$ for each $i$, we conclude the lemma by [Amb04, Theorem 2.5].

**Lemma 2.9.** Let $f : (X, B) \to Y$ be a contraction of normal varieties from a klt pair $(X, B)$. Suppose $K_X + B \sim_{R/Y} R/Y$ where $R \geq 0$, and $\kappa(X/Y, R) = 0$, then

$$\text{rank} f_*\mathcal{O}_X([\mathcal{A}(X, B - R)]) = 1.$$ 

**Proof.** Let $\pi : X' \to X$ be a log resolution of $(X, B)$ and write $\Delta = B - R$ and $K_{X'} = \pi^*(K_X + \Delta) + \sum i a_iA_i$. By [Fuj12, Proof of Lemmas 3.19 and 3.20], we have $f_*\mathcal{O}_X([\mathcal{A}(X, \Delta)]) = (f \circ \pi)_*\mathcal{O}_X(\sum [a_i]A_i)$. Because we have

$$\text{Supp} \sum_i [a_i]A_i \subseteq \text{Supp} \pi^* R \bigcup \text{Ex}(\pi),$$

we deduce $\kappa(X'/Y, \sum_i [a_i]A_i) = 0$ and hence the lemma.

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3. Existence of canonical models

**Lemma 3.1.** Let $f : X \to Y$ be a contraction of normal varieties over $Z$, $D$ be an $R$-Cartier divisor on $X$ and $D_Y$ be an $R$-Cartier divisor on $Y$. We suppose that:

- $D \sim R \ f^*D_Y + E/Z$ for some divisor $E \geq 0$, such that $\kappa(X/Y, E^h) = 0$ and $E^v$ is very exceptional$/Y$, where $E^h$ (resp. $E^v$) denotes the horizontal (resp. vertical) part over $Y$.
- There is a semi-ample model of $D_Y/Z$.

Then, there exists the ample model of $D/Z$.

**Proof.** We first reduce the lemma to the case $D_Y$ is semi-ample$/Z$. Let $\varphi : Y' \dashrightarrow Y'$ be the birational contraction to a semi-ample model of $D_Y/Z$, and $p : Y' \to Y$ and $q : Y \to Y'$ which resolve $\varphi$. We write $D_{Y'}$ for the birational transform of $D_Y$ and $p^*D_Y = q^*D_{Y'} + F$ where $F \geq 0$ is exceptional$/Y'$. Pick a resolution $\pi : X \to X$ such that the induced map $\pi : X \dashrightarrow Y$ is a morphism. We write $\overline{D} = \pi^*D, \overline{E} = \pi^*E + \pi^*F$, and $\overline{D} \sim (q \circ \pi)^*D_Y + \overline{E}$. If we denote by $\overline{E^h}$ and $\overline{E^v}$ the horizontal and vertical part over $Y'$, then one can easily verify that $\kappa(\overline{X}/Y', \overline{E^h}) = 0$, and $\overline{E^v} = \pi^*E^v + \pi^*F$ is very exceptional$/Y'$. Replacing $X, Y$ with $\overline{X}, Y'$ and the other data accordingly, we may assume $D_{Y'}$ is semi-ample$/Z$.

It remains to check that $E$ is contained in the fixed part of $|D/Z|R$. To this end, pick any $D' \sim R D/Z$. Since $D'_F \sim R D_F$ where $F$ is a general fibre of $f$, we have $E^h$ is contained in the fixed part of $|D/Z|R$. Replacing $D$ with $D - E^h$, we may assume $E$ is vertical and very exceptional$/Y$. Hence, the lemma follows from the Negativity lemma [Bir12, Lemma 3.3].
Remark 3.2. The lemma above also holds when \( f \) is a proper surjective morphism instead of a contraction.

Proof of Theorem 1.2. Since \( \kappa_i(X/Y, K_X + B) = 0 \), by [HH19, Lemma 2.10], there exists an \( \mathbb{R} \)-Cartier divisor \( D \geq 0 \) such that \( K_X + B \sim_{\mathbb{R}} D/Y \). Applying [AK00, Theorem 2.1, Proposition 4.4], there exist birational models \( \pi : (X', \Delta') \to X \), \( \phi : (Y', \Delta_Y') \to Y \) such that the induced morphism \( f' : (X', \Delta') \to (Y', \Delta_Y') \) is toroidal and equidimensional to a log smooth pair. Moreover, writing \( K_X + B' = \pi^*(K_X + B) + E \) as in (1), by [ADK13, Theorem 1.1], we have \( B' \leq \Delta' \) and \( \text{Supp} D' \subseteq \Delta' \) where \( D' = \pi^*D + E \). Hence, there exists an \( \mathbb{R} \)-Cartier divisor \( G \geq 0 \), supported by \( \Delta_Y' \), such that \( D^{\nu} - f'^*G \) is very exceptional/\( Y' \), where \( D^{\nu} \) denotes the vertical/\( Y' \) part. Set \( R = D' - f'^*G \). We see \( R \) satisfies (3).

Finally, by Lemma 2.9, \( f' : (X', \Theta) \to Y' \) is an \( \mathbb{R} \)-klt-trivial fibration, where \( \Theta := B' - R \). Hence, by Lemma 2.8, we apply a canonical bundle formula to obtain \( K_{X'} + \Theta \sim_{\mathbb{R}} f'^*(K_Y + B_Y + M_Y) \), such that \( (Y', B_Y + M_Y) \) is a g-sub-klt generalised pair with the moduli b-divisor \( M \). It remains to check that \( (Y', B_Y + M_Y) \) is g-klt. Indeed, the effectiveness of \( B_Y \) follows from the construction of discriminant divisor. □

Remark 3.3. Since the arguments for Lemmas 2.7, 2.8 and 2.9 are still valid for lc-trivial fibrations and lc pairs, one can easily generalise Theorem 1.2 to lc pairs with the above argument. Note that, in this case, with notation from Theorem 1.2, \( (Y', B_Y + M_Y) \) is a g-lc generalised pair, and it is g-klt if all lc centres of \((X, B)\) are horizontal/\( Y \).

Proof of Theorem 1.1. Take a relative Iitaka fibration \( f : \overline{X} \to Y \) over \( Z \). Replacing \((X, B)\), we may assume \( X = \overline{X} \). By definition, we have \( \kappa_i(X/Y, K_X + B) = 0 \). So, by a canonical bundle formula, there exists a commutative diagram

\[
\begin{array}{ccc}
(X', B') & \xrightarrow{\pi} & (X, B) \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\phi} & Y \\
\end{array}
\]

which consists of birational models \( \pi : X' \to X \), \( \phi : Y' \to Y \), satisfying the conditions listed in Theorem 1.2. Replacing \((X, B), (X', B')\), we have \( K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y) + R \). Since \((Y, B_Y + M_Y)\) is g-klt and \( K_Y + B_Y + M_Y \) is big/\( Z \), \( K_Y + B_Y + M_Y \) has a semi-ample model/\( Z \) by [BZh16, Lemma 4.4(2)]. Because \( R \geq 0 \), \( \kappa(X/Y, R^h) = 0 \) and \( R^v \) is very exceptional/\( Y \), where \( R^h \) (resp. \( R^v \)) denotes the horizontal (resp. vertical) part over \( Y \), by Lemma 3.1, we deduce that \((X/Z, B)\) has the canonical model. □

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