ON THE FUNDAMENTAL MASS–PERIOD FUNCTIONS OF EXTRASOLAR PLANETS

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ABSTRACT

Employing a catalog of 175 extrasolar planets (exoplanets) detected by the Doppler-shift method, we constructed the independent and coupled mass–period functions. It is the first time in this field that the selection effect is considered in the coupled mass–period functions. Our results are consistent with those of Tabachnik and Tremaine in 2002, with the major difference that we obtain a flatter mass function but a steeper period function. Moreover, our coupled mass–period functions show that about 2.5% of stars would have a planet with mass between Earth Mass and Neptune Mass, and about 3% of stars would have a planet with mass between Neptune Mass and Jupiter Mass.

Key words: methods: data analysis – methods: numerical – methods: statistical – planetary systems

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1. INTRODUCTION

The mass (size) function, i.e., the differential form for the number of objects as a function of mass (size),

\[ f_M(m) = \frac{dN}{dm} , \quad (1) \]

has been an important physical property to be investigated in many fields of astronomy such as galaxies, stars, asteroids, and also dust grains. The importance lies in that this function is related with the formation and evolution of particular types of objects. Due to the fact that the mass function can be studied either through observational techniques or theoretical calculations, numerous research projects have been done on this subject.

For stars, the initial mass function (IMF) is the distribution of stellar masses from one star formation event in a given volume of space. Although the star-forming conditions vary with the environment, the measured IMF appears to be universal and can be modeled by a power law,

\[ f_M(m) = \frac{dN}{dm} = c_\star m^{-\alpha_\star} , \quad (2) \]

where \( c_\star \) is a normalization constant and \( \alpha_\star = 2.35 \) for the well-known Salpeter IMF. According to Kroupa et al. (1993),

\[
\begin{align*}
 f_M(m) &= c_1 m^{-4.5} & \text{for } m > 1 \, \text{M}_\odot, \\
 f_M(m) &= c_2 m^{-2.2} & \text{for } 0.5 \, \text{M}_\odot < m < 1 \, \text{M}_\odot, \\
 f_M(m) &= c_3 m^{-1.2} & \text{for } m < 0.5 \, \text{M}_\odot,
\end{align*}
\]

where \( m \) is the star’s mass, and \( c_1, c_2, c_3 \) are constants to be determined by the total number of stars in the considered system.

Moreover, there is a new development in astronomy wherein more than 300 planets have been detected around solar-type stars. The discovery has led to a new era in the study of planetary systems and thus triggered many interesting or controversial results of theoretical works (Jiang & Ip 2001; Kinoshita & Nakai 2001; Armitage et al. 2002; Ji et al. 2003; Boss 2005; Jiang & Yeh 2007; Rice et al. 2008; Ji et al. 2009). For example, many discovered exoplanets have extremely short orbital periods. It is likely that they are formed at larger radial distances and migrate to the current locations later. However, because the migration timescale is too short, the rapid inward type I migration caused by disk–core interactions poses a serious issue.

At the time when there were only about 70 detected exoplanets, Tabachnik & Tremaine (2002) first used the maximum likelihood method to determine the mass and period functions with the assumption of two independent power laws. This work is, in fact, the only one that takes into account the selection effect and has intentions to obtain the fundamental mass–period functions. We have to note that planetary functions are different from the definition of stellar IMF in that planetary functions are constructed through the data of exoplanets from all different systems.

Without considering the selection effect, Zucker & Mazeh (2002) calculated the linear correlation coefficient between mass and period and concluded that the mass–period correlation exists. The study of correlations within sub-groups in the cluster analysis done by Jiang et al. (2006) and Marchi (2007) also confirms the mass–period correlation. Thus, strictly speaking, the mass and period functions are not two independent power laws. Motivated by the above results, Jiang et al. (2007) employed an algorithm to generate a pair of positively correlated \( \beta \)-distributed random variables in order to construct a coupled mass–period function, and was the first one in the field of exoplanets to include the correlation into the construction of mass–period functions.

Moreover, through a nonparametric approach, Jiang et al. (2009) further constructed new coupled mass–period functions from 279 exoplanets and presented two main statistical results: (1) confirming the deficit of massive close-in planets; and (2) discovering that more massive planets have larger ranges of possible semimajor axes. However, due to the fact that the selection effect is not considered in the above study, it is unclear how strong statement (2) is. Yeh et al. (2009) argued that the planets larger than \( 1 \, \text{M}_J \) should be all within the detection limit, and thus implied that (2) should be a statistically valid statement.

Therefore, in order to address the above problem, it would be a great success if the coupled mass–period function can be obtained while the selection effect is considered simultaneously. It is this goal that motivates this present work. The coupled mass–period function will be constructed by the same statistic method as in Jiang et al. (2009) and the selection effect would...
be considered through the similar procedure used in Tabachnik & Tremaine (2002). However, as this work is an extension of Tabachnik & Tremaine (2002), power laws would be employed as standard forms of mass–period functions. In order to consider the selection effect, only the exoplanets detected by the Doppler-shift method will be included in our sample.

We first constructed a reference-based catalog in Section 2, and discuss the normalization problem in Section 3. Then, the independent mass–period function is constructed in Section 4. In Section 5, mass–period correlations are studied. The coupled mass–period function is established in Section 6, and the fraction of stars with planets is discussed in Section 7. We conclude the paper in Section 8.

2. A REFERENCE-BASED CATALOG

The most well-known catalog of exoplanets is the one maintained by Jean Schneider (Schneider Catalog hereafter), i.e., http://exoplanet.eu/catalog.php. This catalog is updated frequently when new detections are reported in refereed papers or conference proceedings. It is also notable that Butler et al. (2006a) updated the orbital solutions and compiled a list of 172 exoplanets. In order to extend the work of Tabachnik & Tremaine (2002) with more updated data, we constructed here a new exoplanet catalog, in which all samples were discovered by the Doppler-shift method, as listed in Appendix A (Table A). A major principle in constructing this catalog is that all exoplanets to be included in our catalog must have been reported as new detections in papers of refereed journals. In order to establish such a reference-based catalog, we searched and reviewed many published papers. We intended to make sure that these papers did report new discoveries, and to check in which observational survey the results belong to. Thus, all the references listed in our catalog are the papers that reported new detections.

In our catalog, the Column 1 is the data-set identity and the Column 2 is the name of the observational survey. Columns 3 and 4 are the reference papers and the papers’ corresponding identities. The Column 5 lists the number of exoplanets discovered in that corresponding paper. Finally, Columns 6, 7, and 8 give the name, the projected mass $M$, and the orbital period $P$ of exoplanets. Most planets’ $M$ and $P$ are obtained from the Schneider Catalog. The values in Butler et al. (2006a) are used when they are missing in the Schneider Catalog. We have to note that the exoplanet HD 154345b in Ref. (E-1) is removed here due to its extraordinarily larger period, 10900 days.

Furthermore, we found that some exoplanets were detected by more than one group around the same time, and they could have been reported as new discoveries in two different papers. We checked these and made a list in Table 2. However, they are still repeatedly listed in Appendix A (Table A), as our catalog was constructed based on published papers. Moreover, some papers studied more than one exoplanet, among which only one planet is a new discovery. In those papers, additional planets were included for the purpose of comparison. We also carefully examined these kinds of papers, so that the reference papers in the catalog are exactly the papers that discover those listed exoplanets.

3. THE NORMALIZATION PROBLEM

In addition to the selection effect, the number of observed stars in a survey, $N_s$, is one of the key parameters in understanding the probability that a star could host an exoplanet with particular orbital properties. Unfortunately, we found that the exact numbers of stars in surveys were not clearly mentioned normally. In most cases, an approximate number of target stars in a continuous long-term survey might be given, but the exact number of observed stars at that time when a new exoplanet was detected was usually not stated. For example, in Butler et al. (2003), it is written as “The Keck survey includes about 650 main-sequence and subgiant stars...” and later in the same section, it is stated as “200 stars have one or more Keck observations but have been subsequently dropped from the program...” In this case, neither 650 nor 450 can be used as the $N_s$ here.

Fortunately, Lineweaver & Grether (2003) obtained the numbers of target stars in surveys carefully and listed the results in their Table 4. In fact, they also estimated the number of repeated stars in different surveys, and finally determined the total number of target stars to be 1812 in these surveys. The numbers of detected planets in the corresponding surveys are also shown in their Table 3, and the total number is 122. Therefore, we decide to set $N_{ratio} = N_s/N_r = 122/1812 = 0.0673$ in our paper. This value of $N_s/N_r$ would be used when we determine the value of the normalization constant $c$ through the related equations, and also as a way to obtain $N_s$ in a particular survey for a given number of detected exoplanet $N$.

4. INDEPENDENT MASS-PERIOD FUNCTIONS

In this section, the method and results based on the assumption of independent mass and period power-law functions will be described.

4.1. The Method

The procedures in this section, i.e., the analytical approach, the choice of parameters, etc., follow exactly that of Tabachnik & Tremaine (2002). Here we describe the method in a self-consistent way. However, please note that we simply consider the projected (minimum) mass $M$, which satisfies $M = M_{real} \sin i$, where $M_{real}$ is the real physical mass of the exoplanet and $i$ is the orbital inclination angle, in all calculations in this paper. That is, the mass $M$ in this paper means the minimum mass. The probability, $dp$, that a single star has an exoplanet with mass $M$ and orbital period $P$ in the range, $[M, M + dM]$, $[P, P + dP]$, is given by the product of independent power laws on $M$ and $P$ as

$$dp = c \left[ \frac{M}{M_0} \right]^{-\alpha} \left[ \frac{P}{P_0} \right]^{-\beta} \frac{dM \, dP}{M \, P},$$

where $c$, $\alpha$, and $\beta$ are constants to be determined, and $M_0 = 1.5M_J$ and $P_0 = 90$ days. We assume that there are $N$ exoplanets in the data set, and let

$$x_i = \ln \left( \frac{M_i}{M_0} \right),$$

$$y_i = \ln \left( \frac{P_i}{P_0} \right) \quad \text{for} \quad 1 \leq i \leq N,$

where $M_i$ and $P_i$ are the mass and orbital period of one particular exoplanet. According to Equations (2) and (6) of Tabachnik & Tremaine (2002), the value of $x_i - y_i/3$ shall satisfy

$$x_i - y_i/3 \geq \ln \left[ \frac{K_P}{28.4 \, \text{m/s}} \right] - \ln \left[ \frac{M_0}{M_J} \right] + \frac{1}{3} \ln \left[ \frac{P_0}{1 \, \text{yr}} \right],$$

$$\text{(6)}$$
Results of Independent Mass–Period Functions: Estimates of $\alpha$, $\beta$, $c$, and the most likely $K_D$ and $P_{\text{max}}$ for the Data in Each Survey

| Survey    | Data Set | $N$ | $\alpha$ | $\beta$ | $c \times 10^3$ | $K_D$ (m s$^{-1}$) | $P_{\text{max}}(\text{yr})$ |
|-----------|----------|-----|----------|---------|-----------------|-------------------|------------------|
| Lick (A)  | 7        | -0.401 ± 1.051 | -0.144 ± 0.388 | 4.466 ± 4.915 | 19.794 | 14.296 |
| Coralie (B) | 38   | 0.332 ± 0.186 | -0.393 ± 0.101 | 2.969 ± 0.372 | 21.032 | 5.759 |
| Elodie (C) | 14   | -0.473 ± 0.827 | -0.343 ± 0.291 | 2.327 ± 5.876 | 21.400 | 7.921 |
| HARPS (D)  | 23   | 0.081 ± 0.146 | 0.097 ± 0.146 | 1.733 ± 4.189 | 1.128 | 2.844 |
| N2K (E)   | 14   | -0.071 ± 0.425 | 0.077 ± 0.196 | 3.320 ± 4.057 | 12.998 | 3.507 |
| Keck (F)  | 48   | 0.131 ± 0.135 | -0.326 ± 0.095 | 1.362 ± 0.243 | 2.9304 | 9.184 |
| AAPS (G)  | 23   | 0.296 ± 0.262 | -0.558 ± 0.202 | 1.778 ± 0.715 | 10.318 | 8.181 |
| Other (H) | 17   | 0.168 ± 0.350 | -0.341 ± 0.192 | 3.896 ± 3.154 | 28.189 | 5.957 |
| Single    | 175  | -0.143 ± 0.040 | -0.124 ± 0.044 | 1.316 ± 0.080 | 1.128 | 14.296 |

Therefore, in the $x$–$y$ space, the exoplanet probability $dp$ can be expressed as

$$dp = c(x^\alpha - y^\beta) dx dy,$$

and the expected number of exoplanets in the area $dx dy$ in a survey of $N_s$ stars can be written as

$$n(x, y) dx dy = N_s dp = cN_s e^{-\alpha x} e^{-\beta y} dx dy.$$
and the absolute values of precision on the measurement of radial velocities. In general, are about as small as those in the Coralie Survey due to the high precision on the measurement of radial velocities. HARPS Survey has 23 planets, but the error bars as the Coralie Survey has 38 planets and the Keck Survey has 48 planets. Lick telescopes are used in that paper. We found that, among these surveys, the Coralie and Keck results have smaller error bars than those in the Other Survey because both data came from the Keck and Lick telescopes in Fischer et al. (2001) and the reason is that we put the planets in Fischer et al. (2001) into the likelihood function (Jiang et al. 2007). This set of α and β with error bars is also shown in Figure 1.

In order to show the distributions implied by these results and to be compared with the samples’ histograms, we take integrations for the function \( n(x,y) \), and thus the number of planets with masses between \( M_1 \) and \( M_2 \) and periods between \( P_1 \) and \( P_2 \) is given by

\[
N(M_1, M_2; P_1, P_2) = \int_{\ln P_1/\ln P_0}^{\ln P_2/\ln P_0} \int_{\ln M_1/\ln M_0}^{\ln M_2/\ln M_0} n(x, y) dx dy
\]

where the error bars are also estimated via the bootstrap algorithm (Jiang et al. 2007). This set of α and β with error bars is also shown in Figure 1.

4.3. Multiple Surveys

The analysis of individual surveys is generalized to Multiple Surveys here. In this analysis, different domains are considered for different surveys in Equation (16). Thus, from Equation (16), the likelihood function \( L_j \) for survey \( j \) is

\[
L_j = \prod_{i=1}^{N_j} n_j(x_{i,j}, y_{i,j}) \exp\left[-\int_{D_j} n_j(x, y) dx dy\right],
\]

where \( N_j \) is the number of discovered planets for the survey \( j \), and

\[
n_j(x, y) = c N_j e^{-ax-\beta y} = \frac{c N_j}{N_{\text{ratio}}} e^{-ax-\beta y}.
\]

Thus, the log-likelihood is given by

\[
\ln L = \sum_{j=1}^{J} \ln L_j,
\]

where \( J \) is total number of considered surveys, and we have \( J = 8 \) here. When \( \ln L \) is maximized, the estimates for the parameters are obtained as

\[
\alpha = -0.099 \pm 0.055, \quad (21)
\]

\[
\beta = -0.129 \pm 0.040, \quad (22)
\]

and

\[
c = (2.423 \pm 0.154) \times 10^{-3}, \quad (23)
\]

where the error bars are also estimated via the bootstrap algorithm (Jiang et al. 2007).

5. MASS–PERIOD CORRELATION COEFFICIENTS

In addition to the mass and period histograms and distributions shown in Figure 2, the exoplanets’ locations in the mass–period space, i.e., \( M–P \) space, are often presented, as the mass and orbital period are the most important physical parameters of exoplanets. Figure 3 is the distribution of 175 exoplanets in logarithmic space, \( x–y \) space, where \( x, y \) are as defined previously. The region enclosed by four solid lines is the Domain D, where all 175 planets are included. In order to investigate the

\[
\text{Table 2}
\]

Repeated Exoplanets in the Catalog

| S.No. | Planet | Reference ID |
|-------|--------|--------------|
| 1     | HD102117 | (G-12) and (D-5) |
| 2     | HD 196050 | (B-1) and (G-2) |
| 3     | HD 216437 | (B-1) and (G-2) |
| 4     | HD 52265 | (B-4) and (F-16) |
| 5     | HD 192263 | (B-7) and (F-7) |
| 6     | HD 168443c | (B-11) and (F-10) |
| 7     | HD 33636 | (C-2) and (F-9) |
| 8     | HD 37124c | (F-1) and (F-12) |
| 9     | HD 92788 | (B-1) and (H-1) |

Figure 1. Estimators of the exponents, α and β, with error bars for all surveys listed in Table 1, and also the results of Multiple Surveys.

Figure 2(a) and (b).
strength of mass–period correlations, the Spearman rank–order correlation coefficients $\rho_S$ are calculated, as shown in Table 3.

In general, the values of $\rho_S$ are larger than 0.5 in five individual surveys and the Single Imaginary Survey of all 175 planets. These results imply a strong positive correlation (Cohen 1988). Because there is a mass–period correlation, we shall consider the mass–period coupling and construct coupled mass–period functions in this paper.

On the other hand, as pointed in Jiang et al. (2006) and in Marchi (2007), the correlation of each group of exoplanets identified in the clustering analysis might be linked with the physical mechanisms more easily. The overall correlations could be more difficult to explain. A recent result by Marchi et al. (2009) is a good example of two dominant groups of close-in exoplanets that can be explained by two physical mechanisms successfully. This is an interesting and important topic that we would like to further investigate in the future. However, the main point in this section is to re-confirm that there is indeed a mass–period correlation for our selected samples, and thus it is necessary to construct the coupled mass–period functions in this paper.

### 6. COUPLED MASS–PERIOD FUNCTIONS

We consider the mass–period coupling and construct coupled mass–period functions here. We define variables $x = \ln(M/M_0)$, $y = \ln(P/P_0)$, and their probability density functions as

$$f_X(x) = \frac{\alpha e^{-\alpha x}}{e^{-\alpha x_{\min}} - e^{-\alpha x_{\max}}}, \quad x_{\min} \leq x \leq x_{\max},$$

$$f_Y(y) = \frac{\beta e^{-\beta y}}{e^{-\beta y_{\min}} - e^{-\beta y_{\max}}}, \quad y_{\min} \leq y \leq y_{\max}.$$  

Based on the copula modeling method introduced in Jiang et al. (2009), the coupled probability density function $f_{XY}(x, y)$ is

$$f_{XY}(x, y) = \frac{\partial^2 C(F_X(x), F_Y(y); \theta)}{\partial F_X \partial F_Y} f_X(x) f_Y(y),$$

$$= \frac{-\theta(e^{-\theta} - 1)e^{-\theta F_X(x)}e^{-\theta F_Y(y)}}{[e^{-\theta} - 1 + (e^{-\theta F_X(x)} - 1)(e^{-\theta F_Y(y)} - 1)]^2} f_X(x) f_Y(y),$$

(27)
Thus, \( u_1 \equiv F_X(x) \equiv \frac{e^{-\alpha x}}{e^{-\alpha x_{\text{min}}} - e^{-\alpha x_{\text{max}}}}, \quad x_{\text{min}} < x < x_{\text{max}}, \) (29)

where the function \( C(F_X(x), F_Y(y); \theta) \) is given by

\[
C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln \left[ 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right],
\]

where \( u_1 \) is the integral of \( f_X(x) \), and \( u_2 \) is the integral of \( f_Y(y) \). Thus,

\[
u_2 \equiv F_Y(y) \equiv \frac{e^{-\beta y}}{e^{-\beta y_{\text{min}}} - e^{-\beta y_{\text{max}}}}, \quad y_{\text{min}} < y < y_{\text{max}}.
\]

The dependence parameter \( \theta (-\infty < \theta < \infty) \) can be positive, zero, and negative, corresponding to the positive dependence, independence, and negative dependence between two variables \( x \) and \( y \), respectively. When \( \theta \) approaches zero, the term \( \partial^2C / \partial F_X \partial F_Y \) would approach to one, and \( f_{XY}(x, y) = f_X(x)f_Y(y) \). Thus, \( \partial^2C / \partial F_X \partial F_Y \) is called the coupling factor in this paper as it controls the \( x-y \) dependence.
The function $C(u_1, u_2; \theta)$ in Equation (28) is called the Frank copula function. In fact, there are many available copula functions, and the reason why we choose this one is that it is more flexible as it allows to have negative, zero, and positive correlations.

Further, the expected number of exoplanets in the area $dx\,dy$ in a survey of $N_s$ stars is

$$n(x, y)\,dx\,dy = cN_s f_{XY}(x, y)\,dx\,dy,$$

where the parameter $c$ is a constant to be determined. For a given function $n(x, y)$, the number of planets with masses between $M_1$ and $M_2$ and periods between $P_1$ and $P_2$ is determined by

$$N_{[M_1, M_2] \cap [P_1, P_2]} = \int_{\ln(P_2/P_1)}^{\ln(P_1/P_0)} \int_{\ln(M_2/M_0)}^{\ln(M_1/M_0)} n(x, y)\,dx\,dy$$

$$= cN_s \int_{\ln(P_2/P_0)}^{\ln(P_1/P_0)} \int_{\ln(M_2/M_0)}^{\ln(M_1/M_0)} f_{XY}(x, y)\,dx\,dy.$$

On the other hand, the likelihood function is

$$L = \prod_{i=1}^{N} n(x_i, y_i) \exp \left[ -\int_D n(x, y)\,dx\,dy \right],$$

$$= \prod_{i=1}^{N} cN_s f_{XY}(x_i, y_i) \times \exp \left[ -cN_s \int_{y_{min}}^{y_{max}} \int_{x+y/3}^{x+y/3} f_{XY}(x, y)\,dx\,dy \right].$$

After the derivation shown in Appendix B, $\ln L$ is finally expressed as a function of $\alpha$, $\beta$, and $\theta$. The maximum likelihood method is used to simultaneously estimate the parameters $\alpha$, $\beta$, and $\theta$ through the full log likelihood $\ln L$. The estimates of $c$, $\alpha$, $\beta$, and $\theta$ for each survey are listed in Table 4. Moreover, as in the procedure in Section 4, here we also generalize the result to the case of Multiple Surveys and the result is at the bottom of Table 4. The bootstrap method is also used to get error bars. Figure 4 shows the values of $\alpha$ and $\beta$ with error bars.
bars. The result of the Single Imaginary Survey of 175 planets gives $\alpha = -0.187 \pm 0.034$, $\beta = -0.133 \pm 0.041$, so the mass (period) function has a power index $-0.813 \pm 0.034$ ($-0.867 \pm 0.041$). Moreover, the result of Multiple Surveys gives $\alpha = -0.038 \pm 0.080$, $\beta = -0.137 \pm 0.044$, and thus the mass (period) power index is $-0.962 \pm 0.080$ ($-0.863 \pm 0.044$). On the other hand, the mass (period) power index obtained in Tabachnik & Tremaine (2002) is $-1.11 \pm 0.10$ ($-0.73 \pm 0.06$). Thus, considering the error bars, our results are consistent with those in Tabachnik & Tremaine (2002). However, our mean values imply a flatter mass function but a slightly steeper period function. We hope to use the future data to investigate whether this new result remains valid.

In Figure 5(a), from the result of the Multiple Surveys, the coupled probability density function in $x$–$y$ space, $f_{XY}(x, y)$, is shown as a three-dimensional plot. The corresponding contour is in Figure 5(b). To visualize it in a realistic space, the above two are transformed to be in $M$–$P$ space and shown in Figures 6(a) and (b), where the coupled probability density function in $M$–$P$ space, $f_{MP}(M, P)$, is defined by

$$f_{MP}(M, P) = f_{XY}(x, y)|J|,$$

where $J$ is the Jacobian determinant $\partial x/\partial M \times \partial y/\partial P$. As we know from Equation (27), the mass–period coupling is primarily determined by the coupling factor. In order to visualize it, the three-dimensional and color contour plots in $x$–$y$ space are shown in Figures 7(a) and (b). It shows that the coupling factor is bigger when $x$ and $y$ have the same sign, i.e., both positive or both negative. This implies that each group of exoplanets would have its own strength of mass–period correlations if the exoplanets are clustered into a few groups. For example, Marchi et al. (2009) performed clustering analysis on exoplanets, and found the existence of two types of close-in planets. They discovered that these two types of planets have very different distributions of semi-major axes. Moreover, Marchi et al. (2009) also proposed possible mechanisms to produce these two different types of exoplanets.

Figures 8(a) and (b) show the coupled mass–period probability density function (pdf), $f_{XY}(x, y)$, for a few given masses or periods. For the purpose of comparison, the pdfs without
that a star could host a planet with masses between $M$ and periods between $P$. We used the symbol $\text{Prob}_S$ and $\text{Prob}_P$ for the independent mass–period functions. (The long dashed curve is for $y = \ln(100 \text{ days}/P_0)$ (short dashed curve), and $y = \ln(150 \text{ days}/P_0)$ (long dashed curve). (d) The independent mass functions of $y = \ln(1 \text{ day}/P_0)$ (solid curve), $y = \ln(50 \text{ days}/P_0)$ (dotted curve), $y = \ln(100 \text{ days}/P_0)$ (short dashed curve), and $y = \ln(150 \text{ days}/P_0)$ (long dashed curve).)

Figure 8. Mass and period functions in $x$–$y$ space for the results of Multiple Surveys. (a) The period functions of $x = \ln(1 M_J/M_0)$ (solid curve), $x = \ln(5 M_J/M_0)$ (dotted curve), $x = \ln(10 M_J/M_0)$ (short dashed curve), and $x = \ln(15 M_J/M_0)$ (long dashed curve). (b) The mass functions of $y = \ln(1 \text{ day}/P_0)$ (solid curve), $y = \ln(10 \text{ days}/P_0)$ (dotted curve), $y = \ln(100 \text{ days}/P_0)$ (short dashed curve), and $y = \ln(150 \text{ days}/P_0)$ (long dashed curve). (c) The independent period functions of $x = \ln(1 M_J/M_0)$ (solid curve), $x = \ln(5 M_J/M_0)$ (dotted curve), $x = \ln(10 M_J/M_0)$ (short dashed curve), and $x = \ln(15 M_J/M_0)$ (long dashed curve). (d) The independent mass functions of $y = \ln(1 \text{ day}/P_0)$ (solid curve), $y = \ln(50 \text{ days}/P_0)$ (dotted curve), $y = \ln(100 \text{ days}/P_0)$ (short dashed curve), and $y = \ln(150 \text{ days}/P_0)$ (long dashed curve).

7. THE FRACTION OF STARS WITH PLANETS

From both theoretical and observational points of view, it is important to know what fraction of stars would have planets. If we divide $N_{[M_1,M_2][P_1,P_2]}$ by $N$, in Equation (24), the probability that a star could host a planet with masses between $M_1$ and $M_2$ and periods between $P_1$ and $P_2$ is given by

$$\text{Prob} = \frac{c}{\alpha^\beta} \left\{ \left( \frac{M_2}{M_0} \right)^{-\alpha} - \left( \frac{M_1}{M_0} \right)^{-\alpha} \right\} \left\{ \left( \frac{P_2}{P_0} \right)^{-\beta} - \left( \frac{P_1}{P_0} \right)^{-\beta} \right\}$$  \hspace{1cm} (35)

for the independent mass–period function.

As the power indexes obtained from the results of the Single Imaginary Survey are different from those from the Multiple Surveys, we calculated both results using Equation (35). We used the symbol $\text{Prob}_S$ to represent the results of the Single Imaginary Survey and used $\text{Prob}_P$ for the results of the Multiple Surveys.

On the other hand, for coupled mass–period functions, we have

$$\text{Prob}_{CP} = \frac{1}{N_s} \int_{\ln(P_1/P_0)}^{\ln(P_2/P_0)} \int_{\ln(M_1/M_0)}^{\ln(M_2/M_0)} n(x, y) dxdy$$

$$= \int \int c f_X(x, y) dxdy$$

$$= c \int \int [e^{-(\theta e^{-\theta} - 1)e^{-\theta f_X(x)} - e^{-\theta f_Y(y)}} - (e^{-\theta f_Y(y)})]^2 x f_X(x) f_Y(y) dxdy.$$  \hspace{1cm} (36)

Similarly, we used the symbol $\text{Prob}_{CS}$ to represent the results of the Single Imaginary Survey and used $\text{Prob}_{CM}$ for the results of the Multiple Surveys in the case of coupled mass–period functions.

Tabachnik & Tremaine (2002) estimated the expected number of planets per star for a given period and mass range through their best results of multiple surveys, which is in fact equivalently defined as $\text{Prob}_M$ here. When $M_1 = M_J$, $M_2 = 10 M_J$, $P_1 = 2$ days, and $P_2 = 10$ yr = 3650 days, Tabachnik & Tremaine (2002) found $\text{Prob}_M = 0.036$ and concluded that 4% of solar-type stars have a planet in the above ranges. For the same given ranges, we obtain $\text{Prob}_S = 0.02618$, $\text{Prob}_M = 0.04667$, $\text{Prob}_{CS} = 0.02909$, and $\text{Prob}_{CM} = 0.02273$. Thus, our results of the Multiple Surveys are similar to those in Tabachnik & Tremaine (2002). Moreover, the estimated probabilities from the coupled mass–period functions are smaller but still consistent with those in Tabachnik & Tremaine (2002). In Tabachnik & Tremaine (2002), the case with $M_1 = 0.003 M_J$ (i.e., Earth Mass), $M_2 = 10 M_J$, $P_1 = 2$ days, and $P_2 = 10$ yr = 3650 days is also estimated and has a probability of 0.18. However, our results show that $\text{Prob}_S = 0.06406$, $\text{Prob}_M = 0.1264$, $\text{Prob}_{CS} = 0.06492$, and $\text{Prob}_{CM} = 0.0736$. Note that the result with $M_1 = 0.003 M_J$ (i.e., the Earth Mass) is an extrapolation as this small mass is out of the mass range of 175 samples. Under the assumption that the smaller planets shall follow the...
that, in 175 samples, the smallest mass trend of our mass–period function, and the fact that the Earth-Mass planet does exist in our universe, i.e., the solar system, this extrapolation shall lead to a good estimation.

On the other hand, Naef et al. (2005) also gave estimations about the fractions of star with planets more massive than 0.5 $M_J$ within three periods: 0.7% for period < 5 days, 4% for period < 1500 days, and 7.3% for period < 3900 days. Correspondingly, using our samples and equations here, for $M_1 = 0.5M_J$, $M_2 = M_{\text{max}}$, $P_1 = P_{\text{min}}$, and $P_2 = 5$ days (Note that, in 175 samples, the smallest mass $M_{\text{min}} = 0.0158$, the largest mass $M_{\text{max}} = 18.39$, the smallest period $P_{\text{min}} = 1.328$, the largest period $P_{\text{max}} = 5218$, and the units are $M_J$ and days), we obtained the fraction of stars with planets to be less than 1%. For $M_1 = 0.5M_J$, $M_2 = M_{\text{max}}$, $P_1 = P_{\text{min}}$, and $P_2 = 1500$ days, we obtained the fraction of stars with planets to be about 3%–6%. Moreover, for $M_1 = 0.5M_J$, $M_2 = M_{\text{max}}$, $P_1 = P_{\text{min}}$, and $P_2 = 3900$ days, the results are about 4%–8%.

Finally, we are interested in the possibility of having a planet with mass between Earth Mass and Neptune Mass for any period, so we set $M_1 = 0.003$, $M_2 = 0.05$, $P_1 = P_{\text{min}}$, and $P_2 = P_{\text{max}}$, and obtain $\text{Prob}_S = 1.604\%$, $\text{Prob}_M = 3.649\%$, $\text{Prob}_CS = 1.263\%$, and $\text{Prob}_{CM} = 2.542\%$. We are also interested in the possibility between Neptune Mass and Jupiter Mass, for any possible period, thus we set $M_1 = 0.05$, $M_2 = 1$, $P_1 = P_{\text{min}}$, and $P_2 = P_{\text{max}}$, and obtain $\text{Prob}_S = 2.589\%$, $\text{Prob}_M = 5.182\%$, $\text{Prob}_CS = 2.319\%$, and $\text{Prob}_{CM} = 3.021\%$. All the above mentioned results calculated by the equations in this paper are listed in Table 5.

### Table 5

| Case | $M_1$ | $M_2$ | $P_1$ | $P_2$ | $\text{Prob}_S$ | $\text{Prob}_M$ | $\text{Prob}_CS$ | $\text{Prob}_{CM}$ |
|------|-------|-------|-------|-------|----------------|----------------|----------------|----------------|
| 1    | 1.00  | 0.10  | 2.00  | 3.65  | 0.02618        | 0.04667        | 0.02909        | 0.02273        |
| 2    | 0.005 | 10.00 | 2.00  | 3.65  | 0.06406        | 0.12640        | 0.06492        | 0.0736         |
| 3    | 0.5   | 18.39 | 1.328 | 5.00  | 0.00453        | 0.00790        | 0.00154        | 0.00169        |
| 4    | 0.5   | 18.39 | 1.328 | 1500  | 0.03527        | 0.06255        | 0.03095        | 0.02661        |
| 5    | 0.5   | 18.39 | 1.328 | 3900  | 0.04290        | 0.07631        | 0.04196        | 0.03658        |
| 6    | 0.003 | 0.05  | 1.328 | 5218  | 0.01604        | 0.03649        | 0.01263        | 0.02542        |
| 7    | 0.05  | 1.00  | 1.328 | 5218  | 0.02589        | 0.05182        | 0.02319        | 0.03021        |

### 8. CONCLUDING REMARKS

In this paper, several steps have been taken to establish the fundamental mass–period functions of exoplanets. First of all, by reading a great number of published papers, we constructed a reference-based catalog of 175 exoplanets, which were all discovered through the Doppler-shift method. Employing this catalog, we determine the independent mass–period functions for individual surveys and also for the case of Multiple Surveys. Moreover, the coupled mass–period functions are also constructed for both individual survey and Multiple Surveys.

The selection effects of surveys are considered in all the results, and thus it is the first time in this field that the selection effect is considered in the coupled mass–period functions. Our results are consistent with those in Tabachnik & Tremaine (2002) with the main differences that our results imply a flatter mass functions but a steeper period function.

On the other hand, our coupled mass–period functions are used to predict the possible fractions of stars in given mass and period ranges. Our results are consistent with previous works and show that about 2.5% of stars would have a planet with mass between Earth Mass and Neptune Mass, and about 3% of stars would have a planet with mass between Neptune Mass and Jupiter Mass.

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# APPENDIX A

The Exoplanet Catalog

| Data Set | Survey | Reference          | Ref. ID | N  | Planet     | M     | P     |
|----------|--------|--------------------|---------|----|------------|-------|-------|
| (A)      | Lick   | Marcy et al. (2002) | (A-1)   | 1  | 55 Cnc d   | 3.835 | 5218  |
|          |        | Fischer et al. (2002a) | (A-2)   | 1  | 47 Uma c   | 1.34  | 2594  |
|          |        | Butler et al. (1997) | (A-3)   | 3  | 55 Cnc b   | 0.824 | 14.65162 |
|          |        |                    |         |    | Tau Boo b  | 3.9   | 3.3135 |
|          |        |                    |         |    | Ups And b  | 0.69  | 4.61708 |
|          |        | Johnson et al. (2008) | (A-4)   | 2  | Kappa CrB b| 1.8   | 1191  |
|          |        |                    |         |    | HD 167042 b| 1.6   | 416.1 |
| (B)      | Coralie | Mayor et al. (2004) | (B-1)   | 16 | HD 19994   | 2     | 454   |
|          |        |                    |         |    | HD 65216   | 1.21  | 613.1 |
|          |        |                    |         |    | HD 92788   | 3.86  | 377.7 |
|          |        |                    |         |    | HD 111232  | 6.8   | 1143  |
|          |        |                    |         |    | HD 114386  | 0.99  | 872   |
|          |        |                    |         |    | HD 142415  | 1.62  | 386.3 |
|          |        |                    |         |    | HD 147513  | 1     | 540.4 |
|          |        |                    |         |    | HD 196050  | 3     | 1289  |
|          |        |                    |         |    | HD 216437  | 2.1   | 1294  |
|          |        |                    |         |    | HD 216770  | 0.65  | 118.45 |
|          |        |                    |         |    | HD 6434    | 0.48  | 22.09 |
|          |        |                    |         |    | HD 121504  | 0.89  | 64.6  |
|          |        |                    |         |    | HD 83443 b | 0.4   | 2.985625 |
|          |        | Tamuz et al. (2008) | (B-2)   | 2  | HD 4113    | 1.56  | 526.62 |
|          |        |                    |         |    | HD 156846  | 10.45 | 359.51 |
|          |        | Udry et al. (2003) | (B-3)   | 1  | HD 73256   | 1.87  | 2.54858 |
|          |        | Naef et al. (2001a) | (B-4)   | 3  | GJ 3021    | 3.32  | 133.82 |
|          |        |                    |         |    | HD 52265   | 1.13  | 118.96 |
|          |        |                    |         |    | HD 169830 b| 2.88  | 225.62 |
|          |        | Queloz et al. (2000)| (B-5)   | 1  | GJ 86      | 4.01  | 15.766 |
|          |        | Udry et al. (2000) | (B-6)   | 2  | HD 75289   | 0.42  | 3.51  |
|          |        |                    |         |    | HD 130322  | 1.08  | 10.724 |
|          |        | Santos et al. (2000)| (B-7)   | 1  | HD 192263  | 0.72  | 24.348 |
|          |        | Santos et al. (2001)| (B-8)   | 2  | HD 28185   | 5.7   | 383   |
|          |        | Correia et al. (2005)| (B-9)  | 1  | HD 202206 c| 2.44  | 1383.4 |
|          |        | Pepe et al. (2002) | (B-10)  | 2  | HD 108147  | 0.4   | 10.901 |
|          |        |                    |         |    | HD 168746  | 0.23  | 6.403 |
|          |        | Udry et al. (2002) | (B-11)  | 4  | HD 141937  | 9.7   | 653.22 |
|          |        |                    |         |    | HD 162020  | 13.75 | 8.428198 |
|          |        |                    |         |    | HD 168443 c| 18.1  | 1765.8 |
|          |        |                    |         |    | HD 202206 b| 17.4  | 255.87 |
|          |        | Zucker et al. (2004)| (B-12)  | 1  | HD 41004 A | 2.3   | 655   |
|          |        | Santos et al. (2002)| (B-13)  | 1  | HD 41004 B | 18.4  | 1.3283 |
|          |        | Eggenberger et al. (2006)| (B-14) | 1  | HD 142022  | 4.4   | 1923  |
| (C)      | Elodie | Galland et al. (2005)| (C-1)   | 1  | HD 33564   | 9.1   | 388   |
|          |        | Perrier et al. (2003)| (C-2)   | 6  | HD 8574    | 2.23  | 228.8 |
|          |        |                    |         |    | HD 23596   | 7.19  | 1558  |
|          |        |                    |         |    | HD 33636   | 9.28  | 2127.7 |
|          |        |                    |         |    | HD 50554   | 4.9   | 1279  |
|          |        |                    |         |    | HD 106252  | 6.81  | 1500  |
|          |        | Naef et al. (2004) | (C-3)   | 3  | HD 74156 b | 1.88  | 51.65 |
|          |        |                    |         |    | HD 74156 c | 8.03  | 2476  |
|          |        |                    |         |    | HD 145675(14 Her) | 4.64 | 1773.4 |
|          |        | Naef et al. (2003) | (C-4)   | 1  | HD 190360 b| 1.502 | 2891  |
|          |        | Moutou et al. (2006)| (C-5)   | 1  | HD 185269  | 0.94  | 6.838 |
|          |        | da Silva et al. (2006)| (C-6) | 1  | HD 118203  | 2.13  | 6.1335 |
|          |        | Mayor & Queloz (1995)| (C-7)   | 1  | 51 Peg     | 0.468 | 4.23077 |
| (D)      | HARPS  | Udry et al. (2006) | (D-1)   | 1  | HD 4308    | 0.047 | 15.56 |
|          |        | Moutou et al. (2005)| (D-2)   | 3  | HD 2638    | 0.48  | 3.4442 |
| Data Set | Survey | Reference | Ref. ID | N | Planet | M   | P   |
|----------|--------|-----------|---------|---|--------|-----|-----|
|          |        |           |         |   | HD 27894 | 0.62 | 17.991 |
|          |        |           |         |   | HD 63454 | 0.38 | 2.81782 |
| Pepe et al. (2004) | (D-3) | 1 | HD 330075 b | 0.76 | 3.369  |
| Lo Curto et al. (2006) | (D-4) | 1 | HD 212301 | 0.45 | 2.457  |
| Lovis et al. (2005) | (D-5) | 3 | HD 93083 | 0.37 | 143.58 |
|           |        |           |         |   | HD 10193 | 0.3  | 70.46 |
|           |        |           |         |   | HD 102117 | 0.172 | 20.67 |
| Santos et al. (2004) | (D-6) | 1 | HD 160691 d | 0.044 | 9.55  |
| Pepe et al. (2007) | (D-7) | 1 | HD 160691 e | 0.5219 | 310.55 |
| Bonfils et al. (2005) | (D-8) | 1 | Gl 581 b | 0.0492 | 5.3683 |
| Udry et al. (2007) | (D-9) | 2 | Gl 581 c | 0.0158 | 12.932 |
|           |        |           |         |   | Gl 581 d | 0.0243 | 83.6 |
| Bonfils et al. (2007) | (D-10) | 1 | GJ 674 | 0.037 | 4.6938 |
| Melo et al. (2007) | (D-11) | 1 | HD 219828 | 0.066 | 3.8335 |
| Santos et al. (2007) | (D-12) | 1 | HD 171028 | 1.83 | 538  |
| Naef et al. (2007) | (D-13) | 3 | HD 100777 | 1.16 | 383.7 |
|           |        |           |         |   | HD 190647 | 1.9  | 1038.1 |
|           |        |           |         |   | HD 221287 | 3.09 | 456.1 |
| Lovis et al. (2006) | (D-14) | 3 | HD 69830 b | 0.033 | 8.667 |
|           |        |           |         |   | HD 69830 c | 0.038 | 31.56 |
|           |        |           |         |   | HD 69830 d | 0.058 | 197  |
| (E) N2K |        |           |         |   | HIP 14810 b | 3.84 | 6.6742 |
| Wright et al. (2007) | (E-1) | 3 | HIP 14810 c | 0.76 | 95.2914 |
|           |        |           |         |   | HD 154345 b | 2.03 | 10900 |
| Johnson et al. (2006) | (E-2) | 3 | HD 33283 | 0.33 | 18.179 |
|           |        |           |         |   | HD 86081 | 1.5  | 2.1375 |
|           |        |           |         |   | HD 224693 | 0.71 | 26.73 |
| Fischer et al. (2006) | (E-3) | 2 | HD 149143 | 1.33 | 4072  |
|           |        |           |         |   | HD 109749 | 0.28 | 5.24 |
| Fischer et al. (2007) | (E-4) | 5 | HD 11506 | 4.85 | 1280  |
|           |        |           |         |   | HD 125612 | 3.2  | 502  |
|           |        |           |         |   | HD 231701 | 1.78 | 141.6 |
|           |        |           |         |   | HD 170469 | 0.67 | 1145 |
|           |        |           |         |   | HD 17156 b | 3.111 | 21.21725 |
| Sato et al. (2005) | (E-5) | 1 | HD 149026 | 0.36 | 2.8766 |
| Fischer et al. (2005) | (E-6) | 1 | HD 88133 | 0.22 | 3.41 |
| (F) Keck |        |           |         |   | HD 108874 b | 1.36 | 395.4 |
| Butler et al. (2003) | (F-1) | 6 | HD 114729 | 0.82 | 1131.478 |
|           |        |           |         |   | HD 72659 | 2.96 | 3177.4 |
|           |        |           |         |   | HD 128311 b | 2.18 | 448.6 |
|           |        |           |         |   | HD 49674 | 0.115 | 49437 |
|           |        |           |         |   | HD 37124 c | 0.683 | 2295 |
| Marcy et al. (2005) | (F-2) | 5 | HD 183263 | 3.69 | 634.23 |
|           |        |           |         |   | HD 117207 | 2.06 | 2627.08 |
|           |        |           |         |   | HD 188015 | 1.26 | 456.46 |
|           |        |           |         |   | HD 45350 | 1.79 | 890.76 |
|           |        |           |         |   | HD 99492 | 0.109 | 17.0431 |
| Robinson et al. (2007) | (F-3) | 2 | HD 75898 b | 1.48 | 204.2 |
|           |        |           |         |   | HD 5319 b | 1.94 | 675 |
| Butler et al. (1998) | (F-4) | 1 | HD 187123 b | 0.52 | 3.097 |
| Marcy et al. (1999) | (F-5) | 2 | HD 210277 | 1.23 | 442.1 |
| Johnson et al. (2007) | (F-6) | 1 | GJ 317 | 1.2 | 692.9 |
| Vogt et al. (2000) | (F-7) | 6 | HD 10697 | 6.12 | 1077.906 |
|           |        |           |         |   | HD 37124 b | 0.61 | 154.46 |
|           |        |           |         |   | HD 134987 | 1.58 | 260 |
|           |        |           |         |   | HD 177830 | 1.28 | 391 |
|           |        |           |         |   | HD 192263 | 0.72 | 24.348 |
|           |        |           |         |   | HD 22258 | 5.11 | 572 |
| Butler et al. (2006b) | (F-8) | 1 | GJ 849 | 0.82 | 1890 |
| Vogt et al. (2002) | (F-9) | 5 | HD 4203 | 1.65 | 400.944 |
|           |        |           |         |   | HD 4208 | 0.8 | 812.197 |
|           |        |           |         |   | HD 33636 | 9.28 | 2127.7 |

Table A
(Continued)
| Data Set | Survey | Reference | Ref. ID | Planet | \( M \)  | \( P \) |
|----------|---------|-----------|--------|--------|--------|--------|
|          |         | Marcy et al. (2001a) | (F-10) | HD 68988 | 1.9 | 6.276 |
|          |         | HD 114783 | 0.99 | 501 |
|          |         | Rivera et al. (2005) | (F-11) | HD 168443 c | 18.1 | 1765.8 |
|          |         | HD 50449 | 1.71 | 2582.7 |
|          |         | HD 37124 d | 0.6 | 843.6 |
|          |         | HD 190360 c | 0.057 | 17.1 |
|          |         | HD 108874c | 1.018 | 1605.8 |
|          |         | HD 37124 c | 0.683 | 2295 |
|          |         | HD 217107 c | 2.5 | 3352 |
| Butler et al. (2004) | (F-13) | HD 11964 b | 0.11 | 37.82 |
| Butler et al. (2006a) | (F-14) | HD 66428 | 2.82 | 1973 |
|          |         | HD 99109 | 0.502 | 439.3 |
|          |         | HD 107148 | 0.21 | 48.056 |
|          |         | HD 164922 | 0.36 | 1155 |
|          |         | HD 16141 | 0.23 | 75.56 |
|          |         | HD 46375 | 0.249 | 3.024 |
| Marcy et al. (2000) | (F-15) | GJ 436 | 0.072 | 2.64385 |
|          |         | Butler et al. (2000) | (G-1) | HD 73526 b | 2.9 | 188.3 |
|          |         | HD 76700 | 0.197 | 3.971 |
|          |         | HD 30177 | 9.17 | 2819.654 |
|          |         | HD 2039 | 4.85 | 1192.582 |
| Jones et al. (2002b) | (G-2) | HD 196050 | 3 | 1289 |
|          |         | HD 216437 | 2.1 | 1294 |
|          |         | HD 160691 c | 3.1 | 2986 |
| Jones et al. (2002a) | (G-3) | HD 39091 | 10.35 | 2063.818 |
| Tinney et al. (2002) | (G-4) | HD 142 | 1 | 337.112 |
|          |         | HD 23079 | 2.61 | 738.459 |
| Jones et al. (2006) | (G-5) | HD 187085 | 0.75 | 986 |
| O’Toole et al. (2007) | (G-6) | HD 20782 | 1.8 | 585.86 |
|          |         | HD 23127 | 1.5 | 1214 |
| Butler et al. (2001) | (G-7) | HD 159868 | 1.7 | 986 |
|          |         | HD 160691 b | 1.67 | 654.5 |
| Carter et al. (2003) | (G-8) | HD 70642 | 2 | 2231 |
| Tinney et al. (2001) | (G-9) | HD 179949 | 0.95 | 3.0925 |
| Jones et al. (2003) | (G-10) | Tau Gruis b | 1.49 | 1442.919 |
| Tinney et al. (2006) | (G-11) | HD73526 c | 1.6 | 416.1 |
| Tinney et al. (2005) | (G-12) | HD 176718 | 0.19 | 52.2 |
|          |         | HD 208487 | 0.45 | 123 |
|          |         | HD 102117 | 0.172 | 20.67 |
| (H) Others | Fischer et al. (2001) | (H-1) | HD 12661 b | 2.3 | 263.6 |
|          |         | HD 92788 | 3.86 | 377.7 |
|          |         | HD 38529 b | 0.78 | 14.30 |
| Marcy et al. (2001b) | (H-2) | GJ 876 c | 0.56 | 30.1 |
| Fischer et al. (2003) | (H-3) | HD 40979 | 3.32 | 267.2 |
|          |         | HD 12661 c | 1.57 | 1444.5 |
|          |         | HD 38529 c | 12.7 | 2174.3 |
| Delfosse et al. (1998) | (H-4) | GJ 876 b | 1.935 | 60.94 |
| Cochran et al. (1997) | (H-5) | 16 Cyg B b | 1.68 | 799.5 |
| Sozzetti et al. (2006) | (H-6) | HD 81040 | 6.86 | 1001.7 |
| Naef et al. (2001b) | (H-7) | HD 80666 b | 3.41 | 111.78 |
| Fischer et al. (1999) | (H-8) | HD 195019 A b | 3.7 | 18.20163 |
|          |         | HD 217107 | 1.33 | 7.12689 |
| Butler et al. (1999) | (H-9) | Ups And c | 1.98 | 241.52 |
|          |         | Ups And d et al. | 3.95 | 1274.6 |
| Fischer et al. (2002b) | (H-10) | HD 136118 | 11.9 | 1209 |
| Korzennik et al. (2000) | (H-11) | HD 89744 | 7.99 | 256.605 |
APPENDIX B

The derivation of the log likelihood \(\ln L\) of coupled mass–period functions is shown here. As in Tabachnik & Tremaine (2002), the likelihood function is

\[
L = \prod_{i=1}^{N} n(x_i, y_i) \exp \left[ -\int_{D} n(x, y) dx dy \right],
\]

\[
= \prod_{i=1}^{N} c N_i f_{XY}(x_i, y_i) \exp \left[ -c N_i \int_{y_{\text{min}}}^{y_{\text{max}}} \int_{v+y/3}^{\tilde{u}} f_{XY}(x, y) dx dy \right]. \quad (B1)
\]

By the other equations in Section 6, we have

\[
\int_{y_{\text{min}}}^{\tilde{u}} \int_{v+y/3}^{y_{\text{max}}} f_{XY}(x, y) dx dy = \int_{y_{\text{min}}}^{\tilde{u}} \int_{v+y/3}^{y_{\text{max}}} \frac{\partial^2 C(F_X(x), F_Y(y); \theta)}{\partial F_X \partial F_Y} f_X(x) f_Y(y) dx dy,
\]

\[
= \int_{y_{\text{min}}}^{\tilde{u}} \left\{ \frac{\partial C(F_X(x_{\text{max}}), F_Y(y); \theta)}{\partial F_Y} - \frac{\partial C(F_X(v + y/3), F_Y(y); \theta)}{\partial F_Y} \right\} f_Y(y) dy,
\]

\[
= C(F_X(x_{\text{max}}), F_Y(\tilde{u}); \theta) - C(F_X(x_{\text{max}}), F_Y(y_{\text{min}}); \theta) - \int_{y_{\text{min}}}^{\tilde{u}} \frac{\partial C(F_X(x + y/3), F_Y(y); \theta)}{\partial F_Y} f_Y(y) dy,
\]

\[
= F_Y(\tilde{u}) - \int_{y_{\text{min}}}^{\tilde{u}} \left[ e^{-\theta(F_Y(y) + F_X(v+y/3))} - e^{-\theta F_Y(y)} \right] f_Y(y) dy,
\]

\[
\quad (B2)
\]

where \(C(F_X(x_{\text{max}}), F_Y(y_{\text{min}}); \theta) = C(1, 0, \theta) = 0, \) and \(C(F_X(x_{\text{max}}), F_Y(\tilde{u}); \theta) = C(1, F_Y(\tilde{u}; \theta) = F_Y(\tilde{u})\). From Equation (B1) and Equation (B2), we have

\[
\exp \left[ -\int_{D} n(x, y) dx dy \right] = \exp \left\{ -c N_i \left( F_Y(\tilde{u}) - \int_{y_{\text{min}}}^{\tilde{u}} \frac{\left[ e^{-\theta(F_Y(y) + F_X(v+y/3))} - e^{-\theta F_Y(y)} \right] f_Y(y)}{e^{-\theta} + e^{-\theta(F_Y(y) + F_X(v+y/3))} - e^{-\theta F_Y(y)} - e^{-\theta F_X(v+y/3)} f_Y(y) dy} \right) \right\}, \quad (B3)
\]

and the log likelihood is

\[
\ln L = N \ln(c N_i) - \alpha \sum_{i=1}^{N} x_i - \beta \sum_{i=1}^{N} y_i + N \ln \left( \frac{\alpha}{e^{\alpha x_{\text{min}}} - e^{\alpha x_{\text{max}}}} \right) + N \ln \left( \frac{\beta}{e^{-\beta y_{\text{min}}} - e^{-\beta y_{\text{max}}}} \right)
\]

\[
+ \sum_{i=1}^{N} \ln \left( C_{\alpha \beta \gamma}(F_X(x_i), F_Y(y_i); \theta) \right) - c N_i I(\alpha, \beta, u, v), \quad (B4)
\]

where

\[
C_{\alpha \beta \gamma}(F_X(x_i), F_Y(y_i); \theta) = \frac{-\theta(e^{-\theta} - 1)e^{-\theta F_X(x_i)}e^{-\theta F_Y(y_i)}}{[e^{-\theta} - 1 + (e^{-\theta F_X(x_i)} - 1)(e^{-\theta F_Y(y_i)} - 1)]^2}, \quad (B5)
\]

\[
I(\alpha, \beta, u, v) = F_Y(\tilde{u}) - \int_{y_{\text{min}}}^{\tilde{u}} \frac{\left[ e^{-\theta(F_Y(y) + F_X(v+y/3))} - e^{-\theta F_Y(y)} \right] f_Y(y)}{e^{-\theta} + e^{-\theta(F_Y(y) + F_X(v+y/3))} - e^{-\theta F_Y(y)} - e^{-\theta F_X(v+y/3)} f_Y(y) dy}.
\]

\[
\quad (B6)
\]

By \(\partial \ln L / \partial c = 0\), we obtain \(c = N/[N_i I(\alpha, \beta, u, v)]\). It is then substituted into Equation (B4) and we have

\[
\ln L = N \ln(N) - N \ln(I) - \alpha \sum_{i=1}^{N} x_i - \beta \sum_{i=1}^{N} y_i + N \ln \left( \frac{\alpha}{e^{\alpha x_{\text{min}}} - e^{\alpha x_{\text{max}}}} \right)
\]

\[
+ N \ln \left( \frac{\beta}{e^{-\beta y_{\text{min}}} - e^{-\beta y_{\text{max}}}} \right) + \sum_{i=1}^{N} \ln \left( C_{\alpha \beta \gamma}(F_X(x_i), F_Y(y_i); \theta) \right) - N. \quad (B7)
\]
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