Extremal graphs for $\alpha$-index

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Abstract. Let $N(G)$ be the number of vertices of the graph $G$. Let $P_l(B_i)$ be the tree obtained from the path $P_l$ and the trees $B_1, B_2, ..., B_l$ by identifying the root vertex of $B_i$ with the $i$-th vertex of $P_l$. Let $V_m^l = \{ P_l(B_i) : N(P_l(B_i)) = n; N(B_i) \geq 2; l \geq m \}$. In this paper, we determine the tree that has the largest $\alpha$-index among all the trees in $V_m^l$.

Keywords: Caterpillar, diameter, distance, index, tree.

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Grafos extremales para $\alpha$-índice

Resumen. Sea $N(G)$ el número de vértices del grafo $G$. Sean $P_l(B_i)$ los árboles obtenidos del camino $P_l$ y los árboles $B_1, B_2, ..., B_l$, identificando el vértice raíz de $B_i$ con el $i$-th vértice de $P_l$. Sea $V_m^l = \{ P_l(B_i) : N(P_l(B_i)) = n; N(B_i) \geq 2; l \geq m \}$. En este artículo determinamos el árbol que tiene el $\alpha$-índice más grande entre todos los árboles en $V_m^l$.

Palabras clave: Oruga, diámetro, distancia, índice, árbol.

1. Introduction

Let $G$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is $d(v)$ or simply $d_v$. We denote by $N(G)$ the number of vertices of the graph $G$. A graph $G$ is bipartite if there exists a partitioning of $V(G)$ into disjoint, nonempty sets $V_1$ and $V_2$ such that the end vertices of each edge in $G$ are in distinct sets $V_1$ and $V_2$. In this case $V_1$, $V_2$ are referred as a bipartition of $G$. A graph $G$ is a complete bipartite graph if $G$ is bipartite with bipartition $V_1$ and $V_2$, where each vertex in $V_1$ is connected to all the vertices in $V_2$. If $G$ is a complete bipartite graph and $N(V_1) = p$ and $N(V_2) = q$, the graph $G$ is written as $K_{p,q}$. The Laplacian matrix of $G$ is the $n \times n$ matrix $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the matrices adjacency and diagonal of vertex degrees of $G$ [7], [8], and [12], respectively. It is well known that $L(G)$ is a positive semi-definite matrix and that $(0, e)$ is an eigenpair of $L(G)$ where $e$ is the
all ones vector. The matrix \( Q(G) = A(G) + D(G) \) is called the signless Laplacian matrix of \( G \) (see \cite{4}, \cite{5}, and \cite{6}). The eigenvalues of \( A(G) \), \( L(G) \) and \( Q(G) \) are called the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of \( G \), respectively. The matrices \( Q(G) \) and \( L(G) \) are positive semidefinite. (see \cite{21}). The spectra of \( L(G) \) and \( Q(G) \) coincide if and only if \( G \) is a bipartite graph, (see \cite{2}, \cite{4}, \cite{7}, and \cite{8}). The largest eigenvalue \( \mu_1 \) of \( L(G) \) is the Laplacian index of \( G \), the largest eigenvalue \( q_1(G) \) of \( Q(G) \) is known as the signless Laplacian index of \( G \) and the largest eigenvalue \( \lambda_1(G) \) of \( A(G) \) is the adjacency index or index of \( G \) \cite{3}.

In \cite{13}, it was proposed to study the family of matrices \( A_\alpha(G) \) defined for any real number \( \alpha \in [0, 1] \) as

\[
A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G).
\]

Since \( A_0(G) = A(G) \) and \( 2A_{1/2}(G) = Q(G) \), the matrices \( A_\alpha(G) \) can underpin a unified theory of \( A(G) \) and \( Q(G) \). In this paper, the eigenvalues of the matrices \( A_\alpha(G) \) are called the \( \alpha \)-eigenvalues of \( G \). We write \( \rho_\alpha(G) \) for the spectral radii of the matrices \( A_\alpha(G) \) and are called the \( \alpha \)-indices of \( G \). The \( \alpha \)-eigenvalue set of \( G \) is called \( \alpha \)-spectrum of \( G \). The spectrum of a matrix \( M \) will be denoted by \( \text{Sp}(M) \).

Let \( [l] \) denote the set \( \{1, 2, ..., l\} \). Given a rooted graph, define the level of a vertex to be equal to its distance to the root vertex increased by one. A generalized Bethe tree is a rooted tree in which vertices at the same level have the same degree. Throughout this paper \( \{B_i : i \in [l]\} \) is a set of generalized Bethe trees. Let \( P_l \) be a path of \( l \) vertices.

In this paper, we study the tree \( P_l\{B_i : i \in [l]\} \) obtained from \( P_l \) and \( B_1, B_2, ..., B_l \), by identifying the root vertex of \( B_i \) with the \( i \)-th vertex of \( P_l \) where each \( B_i \) has order greater than or equal to 2. For brevity, we write \( P_l(B_i) \) instead of \( P_l\{B_i : i \in [l]\} \). Let

\[
\mathcal{V}_n^m = \{P_l(B_i) : \text{N}(P_l(B_i)) = n; \text{N}(B_i) \geq 2; l \geq m\}.
\]

![Figure 1. The complete caterpillar \( P_5(K_{1,2}, K_{1,1}, K_{1,3}, K_{1,2}) \).](image)

In a graph, a vertex of degree at least 2 is called an internal vertex, a vertex of degree 1 is a pendant vertex and any vertex adjacent to a pendant vertex is a quasi-pendant vertex. We recall that a caterpillar is a tree in which the removal of all pendant vertices and incident edges results in a path. We define a complete caterpillar as a caterpillar in which each internal vertex is a quasi-pendant vertex.

A complete caterpillar \( P_l(K_{1,p_i}) \) is a graph obtained from the path \( P_l \) and the stars \( K_{1,p_1}, ..., K_{1,p_l} \) by identifying the root of \( K_{1,p_i} \) with the \( i \)-th vertex of \( P_l \) where \( p_i \geq 1 \) for all \( i \in [l] \) (see Fig. 1 for an example). Let \( q \in [l] \). Let \( A_q \) be the complete caterpillar \( P_l(K_{1,p_q}) \), where \( p_q = n - 2l + 1 \) and \( p_i = 1 \) for all \( i \neq q \).

Let \( T_{n,d} \) be the class of all trees on \( n \) vertices and diameter \( d \). Let \( P_m \) be a path on \( m \) vertices and \( K_{1,p} \) be a star on \( p + 1 \) vertices.

In \cite{20} the authors prove that the tree in \( T_{n,d} \) having the largest index is the caterpillar \( P_{d,n-d} \) obtained from \( P_{d+1} \) on the vertices 1, 2, ..., \( d + 1 \) and the star \( K_{1,n-d-1} \) identifying the root of \( K_{1,n-d-1} \) with the vertex \( \lceil \frac{d+1}{2} \rceil \) of \( P_{d+1} \). In \cite{10}, for \( 3 \leq d \leq n - 4 \), the first
of a particular, it is shown that if spectral radius among all connected graphs with diameter $d$.

As applications, we determine the unique graph with maximum $A_\alpha$-spectral radius among all connected graphs with diameter $d$, and determine the unique graph with minimum $A_\alpha$-spectral radius among all connected graphs with given clique number.

In [14] the authors gives several results about the spectral radius of graphs on $n$ vertices.

In [16] the authors determine the unique complete caterpillars that minimize and maximize the $\alpha$-index among all complete caterpillars on $n$ vertices.

Numerical experiments suggest us that the complete caterpillars were initially studied in [18] and [19]. In particular, in [18] the authors determine the unique complete caterpillars that minimize and maximize the algebraic connectivity (second smallest Laplacian eigenvalue) among all complete caterpillars on $n$ vertices and diameter $m+1$. Below we summarize the result corresponding to the caterpillar attaining the largest algebraic connectivity.

**Theorem 1.1** ([18] Theorems 3.3 and 3.6.). Among all caterpillars on $n$ vertices and diameter $m+1$, the largest algebraic connectivity is attained by the caterpillar $A_{\frac{m+1}{n}}$.

**Theorem 1.2** (Abreu, Lenes, Rojo [1]). Let $\alpha = 0, 1/2$. Let $G$ be a complete caterpillar on $n$ vertices and diameter $m+1$. Then,

$$\rho_\alpha(G) \leq \rho_\alpha(A_{\frac{m+1}{n}}),$$

with equality if, and only if, $G \cong A_{\frac{m+1}{n}}$.

Numerical experiments suggest us that $A_{\frac{m+1}{n}}$ is also the tree attaining the largest $\alpha$-index in the class $V^m_n$. In this paper we prove that this conjecture is true; we come up with a bound for the whole family $A_\alpha(G)$, which implies the result of Abreu, Lenes, and Rojo. This is organized as follows. In Section 2, we introduce trees obtained of the path $P_l$ and the trees $B_1, B_2, ..., B_l$ by identifying the root vertex of $B_i$ with the $i$-th vertex of $P_l$ and give a reduction procedure for calculating their $\alpha$-spectra, thereby extending the main results of [16]. In the Section 3, we determine the graph that maximize the $\alpha$-index in $V^m_n$. We finish the section maximizing the $\alpha$-index among all the unicyclic connected graphs on $n$ vertices.

### 2. The $\alpha$-eigenvalues of $P_l(B_i)$

Given a generalized Bethe tree $B_i$ with $k_i$ levels and an integer $j \in [k_i]$, we write $n_{i,k_i-j+1}$ for the number of vertices at level $j$ and $d_{i,k_i-j+1}$ for their degree. In particular, $d_{i,1} = 1$ and $n_{i,k_i} = 1$. Further, for any $j \in [k_i-1]$, let $m_{i,j} = n_{i,j}/n_{i,j+1}$. Then, for any $j \in [k_i-2]$, we see that

$$n_{i,j} = (d_{i,j+1} - 1)n_{i,j+1},$$

and, in particular,

$$n_{i,k_i} = d_{i,k_i} = m_{i,k_i-1}.$$
Figure 2. Labelling the tree $P_{l}(B_i)$.

For $i \in [l]$, it is worth pointing out that $m_{i,1}, \ldots, m_{i,k_{i}-1}$ are always positive integers, and that $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,k_{i}}$. We label the vertices of $P_{l}(B_i)$ as in [16]. (See figure 2).

Recall that the Kronecker product $C \otimes E$ of two matrices $C = (c_{i,j})$ and $E = (e_{i,j})$ of sizes $m \times m$ and $n \times n$, is an $mn \times mn$ matrix defined as 

$$(C \otimes E)(F \otimes H) = (CF \otimes EH),$$

and

$$(C \otimes E)^{T} = C^{T} \otimes E^{T},$$

which hold for any matrices of appropriate sizes.

We write $I_{l}$ for the identity matrix of order $l$ and $j_{l}$ for the column $l$-vector of ones. For $i \in [l]$, let $s_{i} = \sum_{j=1}^{k_{i}-2} n_{i,j}$ and $D_{i}$ be the matrix of order $s_{i} \times l$ defined by

$$D_{i}(p, q) = \begin{cases} 1, & \text{if } q = i \text{ and } s_{i} + 1 \leq p \leq s_{i} + n_{i,k_{i}-1}, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $\beta = 1 - \alpha$, and assume that $P_{l}(B_i)$ is a tree labeled as described above. It is not hard to see that the matrix $A_{\alpha}(P_{l}(B_i))$ can be represented as a symmetric block tridiagonal matrix

$$\begin{bmatrix} X_{1} & 0 & \cdots & 0 & \beta D_{1} \\ 0 & X_{2} & \ddots & \beta D_{2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & X_{l} & \beta D_{l} \\ \beta D_{1}^{T} & \beta D_{2}^{T} & \cdots & \beta D_{l}^{T} & X_{l+1} \end{bmatrix},$$

where, for $i \in [l]$, the matrix $X_{i}$ is the block tridiagonal matrix:

$$\begin{bmatrix} \gamma_{i,1} I_{n_{i,1}} & \beta I_{n_{i,2}} \otimes j_{m_{i,1}} \\ \beta I_{n_{i,2}} \otimes j_{m_{i,1}}^{T} & \gamma_{i,2} I_{n_{i,2}} & \beta I_{n_{i,3}} \otimes j_{m_{i,2}} \\ & \ddots & \ddots & \ddots \\ & & \ddots & \gamma_{i,k_{i}-2} I_{n_{i,k_{i}-1}} & \beta I_{n_{i,k_{i}-2}} \otimes j_{m_{i,k_{i}-2}} \\ & & & \beta I_{n_{i,k_{i}-1}} \otimes j_{m_{i,k_{i}-2}}^{T} & \gamma_{i,k_{i}-1} I_{n_{i,k_{i}-1}} \end{bmatrix}. \]
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and

$$X_{l+1} = \begin{bmatrix}
\gamma_{1,k_1} + \alpha & \beta & & & \\
\beta & \gamma_{2,k_2} + 2\alpha & \beta & & \\
& \ddots & \ddots & \ddots & \\
& & \beta & \gamma_{l-1,k_{l-1}} + 2\alpha & \beta \\
& & & \beta & \gamma_{l,k_l} + \alpha
\end{bmatrix},$$

where

$$\gamma_{i,j} = \alpha d_{i,j}.$$

Let’s define the polynomials $P_0(\lambda), P_1(\lambda), ..., P_l(\lambda)$ and $P_{i,j}(\lambda)$ for $i \in [l]$ and $j \in [k_i]$ as follows:

**Definition 2.1.** For $i \in [l]$ and $j \in [k_i]$, let

$$\gamma_{i,j} = \alpha d_{i,j}.$$

For $i \in [l]$, let

$$P_{i,0}(\lambda) = 1, P_{i,1}(\lambda) = \lambda - \alpha,$$

and for $i \in [l]$ and $j = 2, 3, ..., k_i - 1$, let

$$P_{i,j}(\lambda) = (\lambda - \gamma_{i,j})P_{i,j-1}(\lambda) - \beta^2 m_{i,j-1}P_{i,j-2}(\lambda).$$  \hfill (1)

Moreover, let

$$P_1(\lambda) = (\lambda - \gamma_{1,k_1} - \alpha)P_{1,k_1-1}(\lambda) - \beta^2 n_{1,k_1-1}P_{1,k_1-2}(\lambda),$$

$$P_l(\lambda) = (\lambda - \gamma_{l,k_l} - \alpha)P_{l,k_l-1}(\lambda) - \beta^2 n_{l,k_l-1}P_{l,k_l-2}(\lambda),$$

and

$$P_i(\lambda) = (\lambda - \gamma_{i,k_i} - 2\alpha)P_{i,k_i-1}(\lambda) - \beta^2 n_{i,k_i-1}P_{i,k_i-2}(\lambda),$$  \hfill (2)

for $i = 2, 3, ..., l - 1$.

**Theorem 2.2.** The characteristic polynomial $\phi(\lambda)$ of $A_\alpha(P_i(B_i))$ satisfies

$$\phi(\lambda) = P(\lambda) \prod_{i=1}^{m} \prod_{j=1}^{k_i-1} P_{i,j}^{n_{i,j-1}} P_{i,j}^{n_{i,j+1}}(\lambda),$$  \hfill (3)

where

$$P(\lambda) = \begin{bmatrix}
P_1(\lambda) & -\beta P_{1,k_1-1}(\lambda) \\
-\beta P_{2,k_2-1}(\lambda) & \ddots & \ddots \\
& \ddots & \ddots & -\beta P_{l-1,k_{l-1}-1}(\lambda) \\
& & -\beta P_{l,k_l-1}(\lambda) & P(\lambda)
\end{bmatrix}.$$
Proof. Write \(|A|\) for the determinant of a square matrix \(A\). To prove 3, we shall reduce \(\phi(\lambda) = |I - A_0(P_l(B_l))|\) to the determinant of an upper triangular matrix. For a start, note that

\[
\phi(\lambda) = \begin{vmatrix}
X_1(\lambda) & 0 & \cdots & 0 & -\beta D_1 \\
0 & X_2(\lambda) & \cdots & -\beta D_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X_l(\lambda) & -\beta D_l \\
-\beta D_1^T & -\beta D_2^T & \cdots & -\beta D_l^T & X_{l+1}(\lambda)
\end{vmatrix},
\]

where, for \(i \in [l]\), the matrix \(X_i(\lambda)\) given by,

\[
P_{i,1}(\lambda)I_{m_1} - \beta I_{n_2} \otimes \mathbf{j}_{m_1} \\
-\beta I_{n_2} \otimes \mathbf{j}_{m_1}^T \quad (\lambda - \gamma_{i,2})I_{n_1} - \beta I_{n_3} \otimes \mathbf{j}_{m_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\beta I_{n_{k_i-1}} \otimes \mathbf{j}_{m_{k_i-2}} \quad (\lambda - \gamma_{i,k_i-1})I_{n_{k_i-1}}
\]

and

\[
X_{l+1}(\lambda) = \begin{vmatrix}
\lambda - \gamma_{1,k_1} - \alpha & -\beta \\
-\beta & \lambda - \gamma_{2,k_2} - 2\alpha & -\beta \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\lambda - \gamma_{l-1,k_{l-1}} - 2\alpha & -\beta \\
-\beta & \lambda - \gamma_{l,k_l} - \alpha
\end{vmatrix}.
\]

Let \(\lambda \in \mathbb{R}\) be such that \(P_{i,j}(\lambda) \neq 0\) for any \(i \in [l]\) and \(j \in [k_i - 1]\); set \(P_{i,j} = P_{i,j}(\lambda)\). For each \(i \in [l]\) and for all \(j \in [k_i - 2]\), multiplying the \(j\)-th row of \(X_i(\lambda)\) inserted in \(\phi(\lambda)\) by \(\frac{\beta P_{i,j}}{P_{i,j}} \otimes \mathbf{j}_{m_j}\) and add it to the next row. Since

\[
\lambda - \gamma_{i,j+1} - \frac{\beta m_{i,j} P_{i,j-1}}{P_{i,j}} = \frac{(\lambda - \gamma_{i,j+1}) P_{i,j} - \beta^2 m_{i,j} P_{i,j-1}}{P_{i,j}} = \frac{P_{i,j+1}}{P_{i,j}} P_{i,j},
\]

we obtain,

\[
\phi(\lambda) = \begin{vmatrix}
Y_1(\lambda) & 0 & \cdots & 0 & -\beta D_1 \\
0 & Y_2(\lambda) & \cdots & -\beta D_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & Y_l(\lambda) & -\beta D_l \\
0 & 0 & \cdots & 0 & Y_{l+1}(\lambda)
\end{vmatrix},
\]

where, for \(i \in [l]\), the matrix \(Y_i(\lambda)\) is given by

\[
P_{i,1}I_{n_1} - \beta I_{n_2} \otimes \mathbf{j}_{m_1} \\
\frac{P_{i,2}}{P_{i,1}} I_{n_2} - \beta I_{n_3} \otimes \mathbf{j}_{m_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\beta I_{n_{k_i-1}} \otimes \mathbf{j}_{m_{k_i-2}} \\
-\frac{P_{i,k_i-1}}{P_{i,k_i-2}} I_{n_{k_i-1}}
\]

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and
\[
Y_{l+1}(\lambda) = \begin{bmatrix}
\frac{p_1}{P_{1,k_1-1}} & -\beta & & & & \\
-\beta & \frac{p_2}{P_{2,k_2-1}} & -\beta & & & \\
& \ddots & \ddots & \ddots & & \\
& & & \frac{p_{l-1}}{P_{l-1,k_{l-1}-1}} & -\beta & \\
& & & & \frac{p_l}{P_{l,k_l-1}} & 
\end{bmatrix}.
\]

Thereby,
\[
\phi(\lambda) = \prod_{i=1}^{l+1} |Y_i(\lambda)| = |Y_{l+1}(\lambda)| \prod_{i=1}^{l} P_{i,1}^n \left( \frac{P_{i,2}}{P_{i,1}} \right)^{n_{i,2}} \left( \frac{P_{i,3}}{P_{i,2}} \right)^{n_{i,3}} \cdots \left( \frac{P_{i,k_i-2}}{P_{i,k_i-3}} \right)^{n_{i,k_i-2}} \left( \frac{P_{i,k_i-1}}{P_{i,k_i-2}} \right)^{n_{i,k_i-1}}.
\]

where
\[
|Y_{l+1}(\lambda)| = \frac{1}{\prod_{i=1}^{l+1} P_{i,k_i-1}} \begin{vmatrix}
P_1 & -\beta P_{1,k_1-1} & & & \\
-\beta P_{2,k_2-1} & P_2 & -\beta P_{2,k_2-1} & & \\
& \ddots & \ddots & \ddots & \\
& & -\beta P_{l-1,k_{l-1}-1} & P_{l-1} & -\beta P_{l-1,k_{l-1}-1} \\
& & & -\beta P_{l,k_l-1} & P_l
\end{vmatrix}.
\]

Hence
\[
|\lambda - A_{\alpha}(P_i(B_i))| = P(\lambda) \prod_{i=1}^{l} \prod_{j=1}^{n_{i,k_i-1}} P_{i,j}^{n_{i,j} - n_{i,j+1}}(\lambda).
\]

Thus, the equality (3) is proved whenever \(P_i(\lambda) \neq 0\) for any \(i \in [l]\) and \(j \in [k_i - 1]\). Since for any \(i \in [l]\) and \(j \in [k_i - 1]\) the polynomials \(P_{i,j}(\lambda)\) have finitely many roots, the equality (3) is verified for infinitely many value of \(\lambda\). The proof is complete.

**Definition 2.3.** For \(i \in [l]\) and \(j \in [k_i - 1]\), let \(T_{i,j}\) be the \(j \times j\) leading principal submatrix of the \(k_i \times k_i\) symmetric tridiagonal matrix
\[
T_i = \begin{bmatrix}
\alpha d_{i,1} & \beta \sqrt{d_{i,2} - 1} \\
\beta \sqrt{d_{i,2} - 1} & \alpha d_{i,2} \\
\beta \sqrt{d_{i,k_i-1} - 1} & \alpha d_{i,k_i-1} \\
\beta \sqrt{d_{i,k_i-1} - 1} & \alpha d_{i,k_i-1} \\
\beta \sqrt{d_{i,k_i} - 1} & \alpha d_{i,k_i} \\
\beta \sqrt{d_{i,k_i} - 1} & \alpha d_{i,k_i} \\
\end{bmatrix},
\]
where \(\beta = 1 - \alpha, c = 2\) for \(i \in [l - 1]\) and \(c = 1\) for \(i = 1\) and \(i = l\).
Since $d_s > 1$ for all $s = 2, ..., j$, each matrix $T_j$ has nonzero codiagonal entries and it is known that its eigenvalues are simple. Using the well known three-term recursion formula for the characteristic polynomials of the leading principal submatrices of a symmetric tridiagonal matrix and the formulas (1) and (2), one can easily prove the following assertion:

**Lemma 2.4.** Let $\alpha \in [0, 1)$. Then

$$|\lambda I - T_{ij}| = P_{i,j}(\lambda)$$

and

$$|\lambda I - T_i| = P_i(\lambda),$$

for any $i \in [l]$ and $j \in [k_i - 1]$.

Let $\tilde{A}$ be the matrix obtained from a matrix $A$ by deleting its last row and last column. Moreover, for $i, j \in [r]$, let $E_{i,j}$ be the $k_i \times k_j$ matrix with $E_{i,j}(k_i, k_j) = 1$ and zeroes elsewhere. We recall the following Lemma.

**Lemma 2.5 ([17]).** For $i, j \in [r]$, let $C_i$ be a matrix of order $k_i \times k_i$ and $\mu_{i,j}$ be arbitrary scalars. Then,

$$
\begin{pmatrix}
C_1 & \mu_{1,2}E_{1,2} & \cdots & \mu_{1,r-1}E_{1,r-1} & \mu_{1,r}E_{1,r} \\
\mu_{2,1}E_{2,1}^T & C_2 & \cdots & \cdots & \mu_{2,r}E_{2,r} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\mu_{r,1}E_{r,1}^T & \mu_{r,2}E_{r,2} & \cdots & \mu_{r,r-1}E_{r,r-1} & \mu_{r,r}E_{r,r}
\end{pmatrix} =
\begin{pmatrix}
|C_1| & \mu_{1,2}C_2 & \cdots & \mu_{1,r-1}C_{r-1} & \mu_{1,r}C_r \\
\mu_{2,1}C_1 & |C_2| & \cdots & \cdots & \mu_{2,r}C_r \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\mu_{r,1}C_1 & \mu_{r,2}C_2 & \cdots & \mu_{r,r-1}C_{r-1} & \mu_{r,r}C_r
\end{pmatrix}
$$

From now on, for $i \in [l - 1]$, by $F_i$ we denote the matrix of order $k_i \times k_{i+1}$ whose entries are 0, except for the entry $F_i(k_i, k_{i+1}) = 1$.

**Lemma 2.6.** Let $r = \sum_{i=1}^{l} k_i$. Let $M(P_i(B_i))$ be the symmetric matrix of order $n \times n$ defined by

$$
\begin{pmatrix}
T_1 & \beta F_1 \\
\beta F_1^T & T_2 & \ddots \\
\vdots & \ddots & \ddots & \beta F_{l-1} \\
\beta F_{l-1}^T & \cdots & \beta F_{l-1} & T_l
\end{pmatrix}.
$$

Then,

$$|\lambda - M(P_i(B_i))| = P(\lambda).$$
The characteristic polynomial of the matrix $M(P_l(B_i))$ is given by
\[
\begin{vmatrix}
\lambda - T_1 & -\beta F_1 \\
-\beta F_1^T & \lambda - T_2 \\
& \ddots \\
& & -\beta F_{l-1} \\
& & & \lambda - T_l
\end{vmatrix}.
\]

From Lemma 2.5, we have that $|\lambda I - M(P_l(B_i))|$ is given by
\[
\begin{vmatrix}
|\lambda - T_1| & -\beta |\lambda - T_1| \\
-\beta |\lambda - T_2| & |\lambda - T_2| \\
& \ddots \\
& & -\beta |\lambda - T_{l-1}| \\
& & & |\lambda - T_l|
\end{vmatrix}.
\]

Since $\lambda I - T_i = \lambda - T_{i,k_i-1}$ for $i \in [l]$, by Lemma 2.4, the proof is complete.

Theorem 2.2, Lemma 2.4, Lemma 2.6, and the interlacing property for the eigenvalues of hermitian matrices yield the following summary statement:

**Theorem 2.7.** Let $\alpha \in [0,1)$. Then:

1. The $\alpha$-spectrum of $P_l(B_i)$ is
   \[
   \bigcup_{i=1}^{l} \bigcup_{j=1}^{k_i-1} Sp(T_{i,j}) \cup Sp(M(P_l(B_i))); \]

2. The multiplicity of each eigenvalue of $T_{i,j}$ as an $\alpha$-eigenvalue of $P_l(B_i)$ is $n_{i,j} - n_{i,j+1}$, if $i \in [l]$ and $j \in [k_i - 1]$, and is 1 if $i \in [l]$ and $j = k_i$;

3. $\rho_\alpha(P_l(B_i))$ is the largest eigenvalue of $M(P_l(B_i))$;

4. $\rho_\alpha(P_l(B_i)) > \alpha$.

**3. The $\alpha$-index of graphs**

In Theorem 2.7, we characterize the $\alpha$-eigenvalues of the trees $P_l(B_i)$ obtained from path $P_l$ and the generalized Bethe trees $B_1, B_2, ..., B_l$ obtained identifying the root vertex of $B_i$ with the $i$-th vertex of $P_l$. This is the case for the caterpillars $P_l(K_{1,p_i})$ in which the path is $P_l$ and each star $K_{1,p_i}$ is a generalized Bethe tree of 2 levels. From Theorem 2.7, we get
Lemma 3.1. Let $\alpha \in [0, 1)$. Then:

1. the $\alpha$-spectrum of $P_l(K_{1, p_1})$ is formed by $\alpha$ with multiplicity $\sum_{i=1}^l p_i - l$, and the eigenvalues of the $2l \times 2l$ irreducible nonnegative matrix

$$M(P_l(K_{1, p_1})) = \begin{bmatrix} T(p_1) & \beta E \\ \beta E & S(p_2) & \beta E \\ & \ddots & \ddots \\ & & \ddots & S(p_{l-1}) & \beta E \\ & & & \beta E & T(p_l) \end{bmatrix},$$

where

$$T(x) = \begin{bmatrix} \alpha & \beta \sqrt{x} \\ \beta \sqrt{x} & \alpha(x+1) \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ; S(x) = T(x) + \alpha E,$$

2. $\rho_{\alpha}(P_l(K_{1, p_1}))$ is the largest eigenvalue of $M(P_l(K_{1, p_1}))$;

3. $\rho_{\alpha}(P_l(K_{1, p_1})) > \alpha$.

Let $t(\lambda, x)$ and $s(\lambda, x)$ be the characteristic polynomials of the matrices $T(x)$ and $S(x)$, respectively. That is,

$$t(\lambda, x) = \lambda^2 - \alpha(x+2)\lambda + \alpha^2(x+1) - \beta^2 x$$

and

$$s(\lambda, x) = \lambda^2 - \alpha(x+3)\lambda + \alpha^2(x+2) - \beta^2 x.$$

Then,

$$s(\lambda, x) - t(\lambda, x) = \alpha(\alpha - \lambda).$$

The notation $|A|$ will be used to denote the determinant of the matrix $A$ of order $l \times l$. The next result is an immediate consequence of the Lemma 2.5.

Lemma 3.2. The characteristic polynomial of $M(P_l(K_{1, p_1}))$ is

$$| t(\lambda, p_1) \beta(\alpha - \lambda) \\ \beta(\alpha - \lambda) s(\lambda, p_2) \beta(\alpha - \lambda) \\ & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & s(\lambda, p_{l-1}) \beta(\alpha - \lambda) & \beta(\alpha - \lambda) t(\lambda, p_l) |.$$ 

For $q \in [l]$, let $A_q$ be the complete caterpillar $P_l(K_{1, p_q})$, where $p_1 = n - 2l + 1$ and $p_i = 1$ for all $i \neq q$. We define

$$r_0(\lambda) = 1, r_1(\lambda) = t(\lambda, 1)$$

and, for $2 \leq q \leq \lfloor \frac{l+1}{2} \rfloor$, we define

$$r_q(\lambda) = \begin{bmatrix} s(\lambda, 1) & \beta(\alpha - \lambda) \\ \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\ & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & s(\lambda, 1) \beta(\alpha - \lambda) & \beta(\alpha - \lambda) t(\lambda, 1) \end{bmatrix}.$$
Let \( \phi_q(\lambda) \) be the characteristic polynomial of \( M(A_q) \), then,
\[
\phi_q(\lambda) = |\lambda I - M(A_q)|.
\]

**Lemma 3.3.** Let \( \alpha \in [0,1) \). Then
\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (a-1)(\alpha\lambda - 2\alpha + 1)(\beta(\lambda - \alpha))^{2q-1}[\alpha r_{m-2q}(\lambda) + \beta(\lambda - \alpha)r_{l-2q-1}(\lambda)]
\]
for all \( q \in \left[ \left\lfloor \frac{l+1}{2} \right\rfloor - 1 \right] \), where \( l \geq 3 \).

**Proof.** By Lemma 3.2, the \((q, q)\)-entry of \( \phi_q(\lambda) = |\lambda I - M(A_q)| \) is \( t(\lambda, a) \) if \( q = 1 \) and \( s(\lambda, a) \) if \( q \neq 1 \). Let \( E_i \cong P_i(K_{1,p_i}) \), where \( p_i = 1 \) for all \( i \in [l] \). Let \( \varphi_s(\lambda) = |\lambda I - M(E_s)| \). From Lemma 3.2, we have
\[
\varphi_s(\lambda) = \begin{vmatrix}
   \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
   s(\lambda, 1) & \beta(\alpha - \lambda) & \ddots \\
   \ddots & \ddots & \ddots & \ddots
\end{vmatrix}.
\]
Since
\[
r_0(\lambda) = 1, r_1(\lambda) = t(\lambda, 1)
\]
and
\[
r_q(\lambda) = \begin{vmatrix}
   s(\lambda, 1) & \beta(\alpha - \lambda) \\
   \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
   \ddots & \ddots & \ddots & \ddots
\end{vmatrix},
\]
for \( q = 2, \ldots, \left\lfloor \frac{l+1}{2} \right\rfloor \); then, expanding along the first row, we obtain
\[
r_q(\lambda) = s(\lambda, 1)r_{q-1}(\lambda) - \beta(\lambda - \alpha)^2r_{q-2}(\lambda).
\]  \hspace{1cm} (4)
Since \( s(\lambda, x) = t(\lambda, x) + \alpha(\alpha - \lambda) \), by linearity on the first column, we have
\[
r_q(\lambda) = \begin{vmatrix}
   t(\lambda, 1) & \beta(\alpha - \lambda) \\
   \beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
   \ddots & \ddots & \ddots
\end{vmatrix} + \alpha(\alpha - \lambda)r_{q-1}(\lambda).
\]
Then,
\[
r_q(\lambda) = \varphi_q(\lambda) + \alpha(\alpha - \lambda)r_{q-1}(\lambda).
\]
Let \( q \in \left[ \left\lfloor \frac{l+1}{2} \right\rfloor - 1 \right] \). We search for the difference \( \phi_q(\lambda) - \phi_{q+1}(\lambda) \). We recall that \((q, q)\)-entry of \( \phi_q(\lambda) = |\lambda I - M(A_q)| \) is \( t(\lambda, a) \) if \( q = 1 \) and \( s(\lambda, a) \) if \( q \neq 1 \). Since
By repeated applications of this process, we conclude that

\[ t(\lambda, a) = t(\lambda, 1) + (1 - a)(\alpha \lambda - 2\alpha + 1) \] and \( s(\lambda, a) = s(\lambda, 1) + (1 - a)(\alpha \lambda - 2\alpha + 1) \), by linearity on the \( q \)-th column, we have

\[
\phi_q(\lambda) = \begin{vmatrix}
 t(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & s(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & t(\lambda, 1) & \beta(\alpha - \lambda) & 0 \\
\end{vmatrix}_{l}
\]

(5)

Applying the recurrence formula (4) to \( r \)

\[
\phi_q(\lambda) = \begin{vmatrix}
 r_{q-1}(\lambda) & 0 \\
0 & r_{l-q}(\lambda) \\
\end{vmatrix}_{l}.
\]

The \((q + 1, q + 1)\)-entry of the determinant of order \( l \) on the second right hand of (5) is \( s(\lambda, 1) \), and since \( s(\lambda, 1) = s(\lambda, a) + (a - 1)(\lambda \alpha - 2\alpha + 1) \), by linearity on the \((q + 1)\)-th column, we obtain

\[
\phi_q(\lambda) = \begin{vmatrix}
 t(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & s(\lambda, 1) & \beta(\alpha - \lambda) \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & s(\lambda, 1) & \beta(\alpha - \lambda) \\
\beta(\alpha - \lambda) & t(\lambda, 1) & \beta(\alpha - \lambda) & 0 \\
\end{vmatrix}_{l} + (1 - a)(\alpha \lambda - 2\alpha + 1)
\]

\[
\begin{vmatrix}
 r_{q-1}(\lambda) & 0 \\
0 & r_{l-q}(\lambda) \\
\end{vmatrix}_{l}.
\]

Thereby,

\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) =
\]

\[
(1 - a)(\alpha \lambda - 2\alpha + 1)
\]

\[
\begin{vmatrix}
 r_{q-1}(\lambda) & 0 \\
0 & r_{l-q}(\lambda) \\
\end{vmatrix}_{l} + (a - 1)(\alpha \lambda - 2\alpha + 1)
\]

\[
\begin{vmatrix}
 r_{q}(\lambda) & 0 \\
0 & r_{l-q-1}(\lambda) \\
\end{vmatrix}_{l}.
\]

Thus,

\[
\phi_q(\lambda) - \phi_{q+1}(\lambda) = (a - 1)(\alpha \lambda - 2\alpha + 1)[r_q(\lambda)r_{m-q-1}(\lambda) - r_{q-1}(\lambda)r_{m-q}(\lambda)].
\]

Applying the recurrence formula (4) to \( r_q(\lambda) \) and \( r_{l-q}(\lambda) \), we obtain

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_{l-q}(\lambda) = [s(\lambda, 1)r_{q-1}(\lambda) - \beta^2(\lambda - \alpha)^2r_{q-2}(\lambda)]r_{l-q-1}(\lambda)
\]

\[
- r_{q-1}(\lambda)[s(\lambda, 1)r_{l-q-1}(\lambda) - \beta^2(\lambda - \alpha)^2r_{l-q-2}(\lambda)].
\]

Then,

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_{l-q}(\lambda) = \beta^2(\lambda - \alpha)^2[r_q(\lambda)r_{l-q-2}(\lambda) - r_{q-2}(\lambda)r_{l-q-1}(\lambda)].
\]

By repeated applications of this process, we conclude that

\[
r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_{l-q}(\lambda) = [\beta(\lambda - \alpha)]^{2(q-1)}[r_1(\lambda)r_{l-2q}(\lambda) - r_{l-2q+1}(\lambda)].
\]

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Hence,
\[
    r_q(\lambda)r_{l-q-1}(\lambda) - r_{q-1}(\lambda)r_{l-q}(\lambda) = 0
\]
\[
    = [\beta(\lambda - \alpha)^{2(q-1)}]t(\lambda, 1)r_{l-2q}(\lambda) + \beta^2(\lambda - \alpha)^2r_{l-2q-1}(\lambda)
\]
\[
    = [\beta(\lambda - \alpha)^{2(q-1)}][\alpha(\lambda - \alpha)r_{l-2q}(\lambda) + \beta^2(\lambda - \alpha)^2r_{l-2q-1}(\lambda)]
\]
\[
    = [\beta(\lambda - \alpha)]^{2q-1}[\alpha r_{l-2q}(\lambda) + \beta^2(\lambda - \alpha)r_{l-2q-1}(\lambda)].
\]

Thus,
\[
    \phi_q(\lambda) - \phi_{q+1}(\lambda) = (a - 1)(\alpha \lambda - 2\alpha + 1)[\beta(\lambda - \alpha)]^{2q-1}[\alpha r_{l-2q}(\lambda) + \beta^2(\lambda - \alpha)r_{l-2q-1}(\lambda)].
\]

Let \(\rho(A)\) be the spectral radius of the square matrix \(A\). From Perron-Frobenius’s Theory for nonnegative matrices [23], if \(A\) is a nonnegative irreducible matrix then \(A\) has a unique eigenvalue equal to its spectral radius and it increases whenever any entry of it increases. Hence, we have the next result.

**Lemma 3.4 (22).** If \(A\) is a nonnegative irreducible matrix and \(B\) is any principal submatrix of \(A\), then \(\rho(B) < \rho(A)\).

Let \(C_{n,l}\) be the class of all complete caterpillars on \(n\) vertices and diameter \(l + 1\). A special subclass of \(C_{n,l}\) is \(A_{n,l} = \{A_1, A_2, ..., A_l\}\), where \(A_q \cong P_l(K_{1, p_q}) \in C_{n,l}\), with \(p_i = 1\) for \(i \neq q\) and \(p_q = n - 2l + 1\). Since \(A_q\) and \(A_{l-q+1}\) are isomorphic caterpillars for all \(q \in \lceil \frac{l+1}{2} \rceil\), the next theorem gives a total ordering in \(A_{n,l}\) by the \(\alpha\)-index.

**Theorem 3.5.** Let \(\alpha \in [0, 1)\). Then
\[
    \rho_\alpha(A_q) < \rho_\alpha(A_{q+1})
\]
for all \(q \in \lceil \frac{l+1}{2} \rceil - 1\), where \(l \geq 3\).

**Proof.** Let \(l \geq 3\). Let \(q \in \lceil \frac{l+1}{2} \rceil - 1\). Let \(\phi_q(\lambda)\) and \(\phi_{q+1}(\lambda)\) be the characteristic polynomials of degrees \(2l\) of the matrices \(M(A_q)\) and \(M(A_{q+1})\), respectively. The matrices \(M(A_q)\) and \(M(A_{q+1})\) are nonnegative irreducible matrices, then its spectral radii are simple eigenvalues.

Let
\[
    \rho_\alpha(A_q) = \mu_1 > \mu_2 \geq \cdots \geq \mu_{2l}
\]
and
\[
    \rho_\alpha(A_{q+1}) = \gamma_1 > \gamma_2 \geq \cdots \geq \gamma_{2l}
\]
be the eigenvalues of the matrices \(M(A_q)\) and \(M(A_{q+1})\), respectively.

By Lemma 3.3, we have
\[
    \phi_q(\lambda) - \phi_{q+1}(\lambda) = \prod_{j=1}^{2l}(\lambda - \mu_j) - \prod_{j=1}^{2l}(\lambda - \gamma_j)
\]
\[
    = (a - 1)(\alpha \lambda - 2\alpha + 1)(\beta(\lambda - \alpha))^{2q-1}
\]
\*
\[
    [\alpha r_{l-2q}(\lambda) + \beta^2(\lambda - \alpha)r_{l-2q-1}(\lambda)].
\]

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We recall that \( r_{l-2q}(\lambda) \) and \( r_{l-2q-1}(\lambda) \) are the characteristic polynomials of the matrices \( \widetilde{M}(E_{l-2q+1}) \) and \( M(E_{l-2q}) \) whose spectral radii are \( \rho(\widetilde{M}(E_{l-2q+1})) \) and \( \rho(M(E_{l-2q})) \), respectively. The matrices \( \widetilde{M}(E_{l-2q+1}) \) and \( M(E_{l-2q}) \) are principal submatrices of \( M(A_q) \).

By Lemma 3.4, \( \rho(\widetilde{M}(E_{l-2q+1})) < \rho_A(A_q) \) and \( \rho(M(E_{l-2q})) < \rho_A(A_q) \).

Hence, \( r_{l-2q}(\rho_A(A_q)) > 0 \) and \( r_{l-2q-1}(\rho_A(A_q)) > 0 \). We claim that \( \rho_A(A_q) < \rho_A(A_{q+1}) \).

Suppose \( \rho_A(A_q) \geq \rho_A(A_{q+1}) \). Then \( \rho_A(A_q) \geq \gamma_j \) for all \( j \). Taking \( \lambda = \rho_A(A_q) \) in (6), we obtain

\[
-\phi_{q+1}(\rho_A(A_q)) = -\prod_{j=1}^{2q}(\rho_A(A_q) - \gamma_j)
\]

\[
= (a - \alpha)(\rho_A(A_q) - 2\alpha + 1)(\beta(\rho_A(A_q) - \alpha))^{2q-1}
\]

\[
* \left[ \alpha r_{l-2q}(\rho_A(A_q)) + \beta^2(\rho_A(A_q) - \alpha)r_{l-2q-1}(\rho_A(A_q)) \right].
\]

By Lemma 3.1, \( \rho_A(A_q) > \alpha \). Then \( \alpha \rho_A(A_q) - 2\alpha + 1 > 0 \). Thus,

\[
0 \geq -\prod_{j=1}^{2q}(\rho_A(A_q) - \gamma_j)
\]

\[
= (a - \alpha)(\rho_A(A_q) - 2\alpha + 1)(\beta(\rho_A(A_q) - \alpha))^{2q-1}
\]

\[
* \left[ \alpha r_{l-2q}(\rho_A(A_q)) + \beta^2(\rho_A(A_q) - \alpha)r_{l-2q-1}(\rho_A(A_q)) \right]
\]

\[
> 0.
\]

which is a contradiction. The proof is complete.

\[\square\]

**Lemma 3.6** ([11]). Let \( A \) be a nonnegative symmetric matrix and \( x \) be a unit vector of \( \mathbb{R}^n \). If \( \rho(A) = x^T A x \), then \( A x = \rho(A) x \).
Let $N_G(v)$ be the vertex set adjacent to $v$ in $G$.

**Lemma 3.7** ([24]). Let $\alpha \in [0, 1)$. Let $G$ be a connected graph and $\rho_\alpha(G)$ be the $\alpha$-index of $G$. Let $u, v$ be two vertices of $G$. Suppose $v_1, v_2, \ldots, v_s$, are some vertices of $N_G(v) - (N_G(u) \cup \{u\})$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron’s vector of $A_\alpha(G)$, where $x_i$ corresponds to the vertex $v_i$ for $i \in [s]$. Let

$$G_u \equiv G - vv_1 - \cdots - vv_n + uv_1 + \cdots + uv_s$$

(as shown in Fig. 3). If $x_u \geq x_v$, then $\rho_\alpha(G) < \rho_\alpha(G_u)$.

An immediate consequence of Lemma 3.7 is

**Theorem 3.8.** Let $T \in \mathcal{V}_n^m$. Then

$$\rho_\alpha(T) \leq \rho_\alpha(A_{\lceil \frac{m+1}{2} \rceil}),$$  \hspace{1cm} (7)

where $A_{\lceil \frac{m+1}{2} \rceil} \in \mathcal{A}_{n,m}$. For $\alpha \in [0, 1)$, the bound (7) is attained if, and only if, $T \cong A_{\lceil \frac{m+1}{2} \rceil}$. For $\alpha = 1$, the bound (7) is attained if, and only if, $T \cong A_k$, where $k = 2, \ldots, \lfloor \frac{m}{2} \rfloor$ and $m \geq 3$ or $T \cong A_{\lceil \frac{m+1}{2} \rceil}$, where $m = 2$.

**Proof.** Let $\alpha \in [0, 1)$. Let $T \cong P_l(B_i) \in \mathcal{V}_n^m$. Let $x_1, x_2, \ldots, x_l$ be the vertices of the path $P_l$ in the tree $T$. Let $B_i$ be a tree with $k_i$ levels for all $i \in [l]$. Suppose $T$ has the largest $\alpha$-index in $\mathcal{V}_n^m$.

Suppose $k_i > 2$ for some $2 \leq i \leq l - 1$. Let $u_1, \ldots, u_{s_i}$ be all the vertices in the second level of $B_i$; we can assume without loss of generality that $u_1$ is an internal vertex. Let $w_1, \ldots, w_{r_i}$ be all the vertices in $N_G(u_{s_i}) - \{x_i\}$. Let

$$T_{x_i} \equiv T - u_{s_i}w_1 - \cdots - u_{s_i}w_{r_i} + x_iw_1 + \cdots + x_iw_{r_i},$$

and

$$T_{u_{s_i}} \equiv T - x_{i-1}x_i - x_{i+1}x_i - u_1x_i - \cdots - u_{s_i-1}x_i + x_{i-1}u_{s_i} + x_{i+1}u_{s_i} + u_1u_{s_i} + \cdots + u_{s_i-1}u_{s_i}.$$  

By Lemma 3.7, $\rho_\alpha(T_{x_i}) > \rho_\alpha(T)$ or $\rho_\alpha(T_{u_{s_i}}) > \rho_\alpha(T)$. Moreover, $\rho_\alpha(T_{x_i}) \in \mathcal{V}_n^m$ and $\rho_\alpha(T_{u_{s_i}}) \in \mathcal{V}_n^m$, which is a contradiction. If $i = 1$ or $i = l$, we reason analogously. Then, $k_i = 2$ for all $i \in [l]$. This is, $T \cong P_l(K_{1,p_i})$.

By reasoning analogously we can verify that

$$T \in \mathcal{A}_{n,m}.$$

Let $m \geq 3$. By Theorem 3.5,

$$\rho_\alpha(A_1) < \rho_\alpha(A_2) < \cdots < \rho_\alpha(A_{\lceil \frac{m+1}{2} \rceil}).$$

Then the largest $\alpha$-index is attained by $A_{\lceil \frac{m+1}{2} \rceil}$. For $m = 2$ the result is immediate.

Let $\alpha = 1$; then $A_\alpha = D$, where $D$ is the diagonal matrix of vertex degrees. Let $T \in \mathcal{V}_n^m$. Let $m = 3$; then the maximum degree of $T$ is less than or equal to $n - 2l + 3$. Then, $\rho_\alpha(T) \leq n - 2l + 3 \leq \rho_\alpha(A_k)$ for all $k = 2, \ldots, \lfloor \frac{m+1}{2} \rfloor$. For $m = 2$ is result is immediate.  \hspace{1cm} $\Box$
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