On the Laplace transform of absolutely monotonic functions

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Abstract
We obtain necessary and sufficient conditions on a function in order that it be the Laplace transform of an absolutely monotonic function. Several closely related results are also given.

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1 Introduction and results
Before stating our main results let us recall the notions of absolute and of complete monotonicity. A function \( \varphi : [0, \infty) \to \mathbb{R} \) is called absolutely monotonic if it is infinitely often differentiable on \([0, \infty)\) and \( \varphi^{(k)}(x) \geq 0 \) for all \( k \geq 0 \) and all \( x \geq 0 \). It is well-known that an absolutely monotonic function \( \varphi \) on \([0, \infty)\) has an extension to an entire function with the power series expansion

\[
\varphi(z) = \sum_{n=0}^{\infty} a_n z^n,
\]

where \( a_n \geq 0 \) for all \( n \geq 0 \). The Laplace transform of \( \varphi \) is defined exactly when \( \varphi \) extends to an entire function of at most exponential type zero, meaning that \( \varphi \) has the following property. For any given \( \epsilon > 0 \) there exists a positive constant \( C_\epsilon \) such that \( |\varphi(z)| \leq C_\epsilon e^{\epsilon |z|} \) for all \( z \). We remark that the Laplace transform of an entire function of exponential type is defined on a half-line and is called the Borel transform of the entire function. It is related to the so-called indicator diagram of a function of exponential type, see [2].
A function \( f : (0, \infty) \rightarrow \mathbb{R} \) is called completely monotonic if \( f \) is infinitely often differentiable on \((0, \infty)\) and \((-1)^n f^{(n)}(x) \geq 0 \) for all \( n \geq 0 \) and all \( x > 0 \). Bernstein’s theorem states that \( f \) is completely monotonic if and only if there exists a positive measure \( \mu \) on \([0, \infty)\) such that \( t \mapsto e^{-xt} \) is integrable w.r.t. \( \mu \) for all \( x > 0 \) and
\[
f(x) = \int_0^\infty e^{-xt} \, d\mu(t).
\]
The class of Stieltjes functions is an important subclass of the completely monotonic functions: A function \( f : (0, \infty) \rightarrow \mathbb{R} \) is a Stieltjes function if
\[
f(x) = \int_0^\infty \frac{d\mu(t)}{x + t} + c,
\]
where \( \mu \) is a positive measure on \([0, \infty)\) making the integral converge for \( x > 0 \) and \( c \geq 0 \). It is well known that the class of Stieltjes functions can be described as the Laplace transforms of completely monotonic functions. Widder characterized this class as follows: \( f \) is a Stieltjes function if and only if the function \((x^k f(x))^{(k)}\) is completely monotonic for all \( k \geq 0 \). (See [8], and also [7].)

In order to motivate our results let us consider the following example. It is easy to show that \( H(x) = x^{-1} e^{1/x} \) satisfies
\[
H_k(x) \equiv (-1)^k (x^k H(x))^{(k)} = x^{-(k+1)} e^{1/x}, \quad x > 0.
\]
Hence \( H_k \) is completely monotonic for all \( k \geq 0 \), being a product of completely monotonic functions. We also have
\[
H(x) = \int_0^\infty e^{-xt} h(t) \, dt,
\]
where \( h(t) = \sum_{n=0}^\infty (n!)^{-2} t^n \) is absolutely monotonic. This example indicates that there must be an analogue of Widder’s characterization mentioned above for Laplace transforms of absolutely monotonic functions and we present it in Theorem 1.1 below. In our analogue it will become clear that the situation is different than in Widder’s characterization.

**Theorem 1.1** The following properties of a function \( f : (0, \infty) \rightarrow \mathbb{R} \) are equivalent.

(i) There is an absolutely monotonic function \( \varphi : [0, \infty) \rightarrow \mathbb{R} \) such that
\[
f(x) = \mathcal{L}(\varphi)(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt, \quad x > 0.
\]
(ii) There is a sequence \( \{a_n\} \), with \( a_n \geq 0 \), such that we have for all \( n \geq 0 \)

\[
f(x) = \sum_{k=1}^{n} \frac{a_k}{x^k} + R_n(x), \quad x > 0
\]

where \( R_n \) is a completely monotonic function of order \( n \).

(iii) The function \( (-1)^k(x^k f(x))^{(k)} \) is completely monotonic for all \( k \geq 0 \).

(iv) The function \( (-1)^k(x^k f(x))^{(k)} \) is non-negative for all \( k \geq 0 \).

(v) We have \( f(x) \geq 0 \) and \((x^k f(x))^{(2k-1)} \leq 0\) for all \( k \geq 1 \).

If the Laplace transform \( f \) of an absolutely monotonic function is defined on a half line then \( f \) has the properties (ii), (iii), (iv) and (v) above. Moreover, (iii), (iv) and (v) are equivalent for a function \( f \) defined on a half line \((a, \infty)\).

A function \( f : (0, \infty) \rightarrow \mathbb{R} \) is called completely monotonic of order \( \alpha > 0 \) if \( x^\alpha f(x) \) is completely monotonic. These functions have been studied in different contexts and in particular as remainders in asymptotic formulae, see [4] and [5].

To show that (iii) implies (i) in Theorem 1.1 we characterize those functions \( f \) that satisfy (iii) up to some given positive integer, in terms of the properties of the representing measure of \( f \) itself. More specifically, we shall prove theorems 1.3 and 1.4 below. Before stating these results we recall some classes introduced in [5].

**Definition 1.2** Let \( A_0 \) denote the set of positive Borel measures \( \sigma \) on \([0, \infty)\) such that \( \int_0^\infty e^{-xt} \, d\sigma(s) < \infty \) for all \( x > 0 \), let \( A_1(\sigma) \) denote the set of functions \( t \mapsto \sigma([0, t]) \), where \( \sigma \in A_0 \), and for \( n \geq 2 \), let \( A_n(\sigma) \) denote the set of \( n-2 \) times differentiable functions \( \xi : [0, \infty) \rightarrow \mathbb{R} \) satisfying \( \xi^{(j)}(0) = 0 \) for \( j \leq n-2 \) and \( \xi^{(n-2)}(t) = \int_0^t \sigma([0, s]) \, ds \) for some \( \sigma \in A_0 \).

These classes are simply the fractional integrals of positive integer order of a Borel measure.

**Theorem 1.3** Let \( f : (0, \infty) \rightarrow \mathbb{R} \) and \( k \geq 1 \). If \((-1)^j(x^j f(x))^{(j)}\) is completely monotonic for \( j = 0, \ldots, k \) then

\[
f(x) = \int_0^\infty e^{-xt} p(t) \, dt,
\]

where for some measures \( \sigma, \sigma_0, \ldots, \sigma_{k-1} \in A_0 \) we have \( t^{k-1} p(t) \in A_k(\sigma) \), \( t^j p^{(j)}(t) \in A_1(\sigma_j) \) for \( 0 \leq j \leq k-1 \) and where the measure

\[
d\sigma_k(t) = t \, d\sigma_{k-1}(t) - (k-1) \sigma_{k-1}([0, t]) \, dt
\]
is positive. Furthermore,
\[ (-1)^j (x^j f(x))^{(j)} = \int_0^\infty e^{-xt} t^j p^{(j)}(t) \, dt, \quad \text{for } j \leq k - 1, \text{ and} \]
\[ (-1)^k (x^k f(x))^{(k)} = \int_0^\infty e^{-xt} d\sigma_k(t). \]

This theorem has a converse.

**Theorem 1.4** Let \( k \geq 1 \) be given. Suppose that
\[ f(x) = \int_0^\infty e^{-xt} p(t) \, dt, \]
where \( t^j p^{(j)}(t) \in A_1(\mu_j) \) for all \( j \leq k - 1 \) and that
\[ d\mu_k(t) = t \, d\mu_{k-1}(t) - (k - 1) \mu_{k-1}([0, t]) \, dt \]
is a positive measure. Then \((-1)^j (x^j f(x))^{(j)}\) is completely monotonic for
\( j \leq k \), and
\[ (-1)^j (x^j f(x))^{(j)} = \int_0^\infty e^{-xt} t^j p^{(j)}(t) \, dt, \quad j \leq k - 1, \]
\[ (-1)^k (x^k f(x))^{(k)} = \int_0^\infty e^{-xt} d\sigma_k(t). \]

Sokal [7] introduced for \( \lambda > 0 \) the operators
\[ T_{n,k}^\lambda(f)(x) \equiv (-1)^n x^{-(n+\lambda-1)} \left(x^{k+n+\lambda-1} f^{(n)}(x))^k, \quad n, k \geq 0 \]
and showed that \( f \) is of the form
\[ f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t) \, dt, \quad x > 0 \]
for some completely monotonic function \( \varphi \) if and only if \( T_{n,k}^\lambda(f)(x) \geq 0 \) for
all \( x > 0 \), and \( n, k \geq 0 \). It turns out that the corresponding result when \( \varphi \)
is absolutely monotonic is relatively simple. We give it in Theorem 1.5.

**Theorem 1.5** Let \( \lambda > 0 \) be given. The following properties of a function
\( f : (0, \infty) \to \mathbb{R} \) are equivalent.

(i) There is an absolutely monotonic function \( \varphi : [0, \infty) \to \mathbb{R} \) such that
\[ f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t) \, dt, \quad x > 0. \]
(ii) The function \((-1)^k(x^{k+\lambda-1} f(x))^{(k)}\) is completely monotonic for all \(k \geq 0\).

(iii) The function \((-1)^k(x^{k+\lambda-1} f(x))^{(k)}\) is non-negative for all \(k \geq 0\).

The proof of this theorem follows from Theorem 1.1 by noticing

\[
f(x) = \int_0^\infty e^{-x t} t^{\lambda-1} \varphi(t) \, dt
\]

for some absolutely monotonic function \(\varphi\) of exponential type zero if and only if

\[
x^{\lambda-1} f(x) = \int_0^\infty e^{-x t} \psi(t) \, dt
\]

for some absolutely monotonic function \(\psi\) of exponential type zero. Indeed, the relationship between the functions \(\varphi\) and \(\psi\) is:

\[
\varphi(t) = \sum_{n=0}^\infty a_n t^n \Leftrightarrow \psi(t) = \sum_{n=0}^\infty \frac{a_n \Gamma(n+\lambda)}{n!} t^n.
\]

2 Proof of Theorem 1.1

Proof that (i) implies (ii): We have \(\varphi(t) = \sum_{k=0}^\infty a_k t^k\), with \(a_k \geq 0\) and get by interchanging integration and summation

\[
f(x) = \int_0^\infty e^{-x t} \varphi(t) \, dt = \sum_{k=0}^\infty \frac{a_k k!}{x^{k+1}} = \sum_{k=0}^{n-1} \frac{a_k k!}{x^{k+1}} + \frac{1}{x^n} \sum_{k=n}^\infty \frac{a_k k!}{x^{k-n}},
\]

where the right-most sum is completely monotonic of order \(n\).

Proof that (ii) implies (iii): This follows from the relation

\[
(-1)^k(x^k f(x))^{(k)} = (-1)^k(x^k R_k(x))^{(k)},
\]

since \(x^k R_k(x)\), and hence \((-1)^k(x^k R_k(x))^{(k)}\) is completely monotonic.

Proof that (iii) implies (i): Let \(k \geq 2\) be given. By Theorem 1.3 we have

\[
f(x) = \int_0^\infty e^{-x t} q_k(t) \, dt,
\]

where \(q_k\) is \(k-2\) times differentiable on \((0, \infty)\) with \(q_k^{(j)}(t) \geq 0\) for all \(j \leq k-2\) and all \(t > 0\). Furthermore, \(q_k\) is continuous on \([0, \infty)\), and it does not depend on \(k\).
Proof that (iii) implies (v): The identity

\[-(x^k f(x))^{(2k-1)} = (-1)^{k-1}((-1)^k (x^k f(x))^{(k)})^{(k-1)}\]

shows that \(-(x^k f(x))^{(2k-1)}\) is completely monotonic, hence non-negative. □

Let us introduce some notation and give two simple lemmas. For a sufficiently smooth function \(f : (0, \infty) \to \mathbb{R}\) we define

\[f_j(x) = (-1)^j (x^j f(x))^{(j)}.\]  (1)

**Lemma 2.1** We have

\[-xf'_k(x) = f_{k+1}(x) + (k+1)f_k(x) \geq 0\]

so \(-f'_k(x) \geq 0\). If \((-1)^n f_k^{(n)}(x) \geq 0\) for some \(n \geq 0\) and all \(k \geq 0\) then by Lemma 2.2

\[x(-1)^{n+1} f_k^{(n+1)}(x) = (-1)^n \left(f_{k+1}^{(n)}(x) + (k + n + 1) f_k^{(n)}(x)\right) \geq 0,\]

and in this way it follows that \(f_k\) is completely monotonic. □

Before completing the proof of Theorem 1.1 we need another lemma.

**Lemma 2.3** Suppose that \(f : (0, \infty) \to \mathbb{R}\) satisfies \(f(x) \geq 0\) and

\[(x^k f(x))^{(2k-1)} \leq 0\] for all \(k \geq 1\).

Then

\[\lim_{x \to \infty} x^k f^{(k)}(x) = 0, \quad \text{for all } k \geq 0,\]  (2)

and

\[\lim_{x \to \infty} (x^k f(x))^{(\nu)} = 0 \quad \text{for all } \nu \geq k \geq 0.\]  (3)

Proof: Since \(f(x) \geq 0\) and \(xf(x)\) is decreasing on \((0, \infty)\) we have that

\[\lim_{x \to +\infty} xf(x) = B \geq 0.\]
The following identity is given in [8, Lemma 3.11]:

\[ x^{k-1} (x^k f(x))^{(2k-1)} = (x^{2k-1} f^{(k-1)}(x))^{(k)}. \]  

(4)

Hence by assumption \((x^{2k-1} f^{(k-1)}(x))^{(k)} \leq 0\), and therefore there exists a constant \(c_k\) such that

\[ (x^{2k-1} f^{(k-1)}(x))^{(k-1)} \leq c_k \quad \text{for all } x \in [1, \infty) \]

Integrating this on \([1, x]\) we have that

\[ (x^{2k-1} f^{(k-1)}(x))^{(k-2)} \leq O(x), \quad x \to \infty \]

and repeating the same process we end up with

\[ x^{2k-1} f^{(k-1)}(x) \leq O(x^{k-1}), \quad x \to \infty, \]

that is

\[ f^{(k-1)}(x) \leq O(x^{-k}), \quad x \to \infty. \]

We next show by a Tauberian argument that these one-sided estimates yield (2). Put \(g(x) = f(x) - Bx^{-1}\). Then \(g(x) = o(x^{-1})\) and

\[ g''(x) = f''(x) - 2Bx^{-3} \leq O(x^{-3}), \quad x \to \infty. \]

By [9, Theorem 4.4, page 193] we conclude

\[ f'(x) + Bx^{-2} = g'(x) = o(x^{-2}), \]

and in particular \(xf'(x) \to 0\) as \(x \to \infty\). This process is continued: \(g^{(m)}(x) \leq O(x^{-4})\) and \(g'(x) = o(x^{-2})\) gives \(g''(x) - 2Bx^{-3} = g''(x) = o(x^{-3})\), so that \(x^2 f''(x) \to 0\) for \(x \to \infty\). In this way (2) follows. The second assertion follows from the first and Leibniz’ rule.

\[ \square \]

Proof that (v) implies (iv): We show that \((-1)^k(x^k f(x))^{(k)} \geq 0\) for all \(k \geq 0\). Let \(k \geq 1\) be given. By (v), the function \((x^k f(x))^{(2k-2)}\) is decreasing on \((0, \infty)\) and by Lemma 2.3 we see that \(\lim_{x \to \infty} (x^k f(x))^{(2k-2)} = 0\). Therefore, the function \((-1)^2(x^k f(x))^{(2k-2)} = (x^k f(x))^{(2k-2)}\) must be non-negative. This means in turn that the function \((x^k f(x))^{(2k-3)}\) is increasing on \((0, \infty)\) and by Lemma 2.3 we see that \(\lim_{x \to \infty} (x^k f(x))^{(2k-3)} = 0\). Therefore, the function \((-1)^3(x^k f(x))^{(2k-3)} = -(x^k f(x))^{(2k-3)}\) must be non-negative. This argument is continued until we conclude that the function \((-1)^k(x^k f(x))^{(2k-k)} = (-1)^k(x^k f(x))^{(k)}\) must be non-negative.
3 Proof of Theorem 1.3 and Theorem 1.4

Let us start by noticing some consequences of Lemma 2.1 and Lemma 2.2, for the functions in (1).

Corollary 3.1 We have \((-xf_{j-1}(x))' = f_j(x) + (j - 1)f_{j-1}(x)\) for \(j \geq 1\).

This follows immediately from Lemma 2.1.

Corollary 3.2 Let \(k \geq 1\). For any \(j \in \{0, \ldots, k - 1\}\) we have

\[
(-1)^j x^j f^{(k-1)}(x) = \sum_{l=0}^{j} a_{j,l} f^{(k-j-1)}_l(x),
\]

where \(a_{j,l} \geq 0\).

Proof: For \(j = 0\) the assertion clearly holds, with \(a_{0,0} = 1\). Assume now that the assertion holds for \(j\) and \(j \leq k - 2\). Then

\[
(-1)^{j+1} x^{j+1} f^{(k-1)}(x) = -x \sum_{l=0}^{j} a_{j,l} f^{(k-j-1)}_l(x).
\]

Now, \(-x f^{(k-1-j)}_l(x) = f^{(k-j-2)}_{l+1}(x) + (k - 1 - j + l)f^{(k-j-2)}_l(x)\) by Lemma 2.2 so

\[
(-1)^{j+1} x^{j+1} f^{(k-1)}(x) = \sum_{l=0}^{j} a_{j,l} \left( f^{(k-j-2)}_{l+1}(x) + (k - 1 - j + l)f^{(k-j-2)}_l(x) \right)
\]

and the proof is complete. \(\square\)

Lemma 3.3 Assume that \(f_j\) is completely monotonic for all \(j \in \{0, \ldots, k\}\). Then

(a) \(xf_j(x)\) is completely monotonic for \(0 \leq j \leq k - 1\);

(b) \((-1)^{j-1} x^j f^{(j-1)}(x)\) is completely monotonic for \(1 \leq j \leq k\).

Proof: To prove (a) we notice that the function \(xf_j(x)\) is non-negative since \(f_j\) is completely monotonic. By Corollary 3.1,

\[
-(xf_j(x))' = f_{j+1}(x) + jf_j(x),
\]

so that \(-(xf_j(x))'\) is completely monotonic.
To prove (b) we use Corollary 3.2:

\[ (-1)^j x^{j-1} f^{(j-1)}(x) = \sum_{l=0}^{j-1} a_{j-1,l} f_l(x), \]

from which it follows that \((-1)^j x^{j-1} f^{(j-1)}(x)\) is completely monotonic. Furthermore,

\[ (-1)^j x^j f^{(j-1)}(x) = x \sum_{l=0}^{j-1} a_{j-1,l} f_l(x) = \sum_{l=0}^{j-1} a_{j-1,l} x f_l(x), \]

and for \(l \leq j - 1\), the function \(xf_l(x)\) is completely monotonic by (a). □

Proof of Theorem 1.3: By (b) of Lemma 3.3, \((-1)^k x^{k-1} f^{(k-1)}(x)\) is completely monotonic of order \(k\), so

\[ (-1)^{k-1} f^{(k-1)}(x) = \int_0^\infty e^{-xt} p_k(t) \, dt, \]

where \(p_k \in \mathcal{A}_k(\sigma)\). By the same lemma, \(xf(x)\) is also completely monotonic, so that \(f\) is of the form

\[ f(x) = \int_0^\infty e^{-xt} p(t) \, dt \]

for some \(p \in \mathcal{A}_1(\sigma_0)\). Hence \((-1)^{k-1} f^{(k-1)} = \mathcal{L}(t^{k-1} p(t))\) so that \(t^{k-1} p(t) = p_k(t) \in \mathcal{A}_k(\sigma)\).

Integration by parts yields

\[ f(x) = \int_0^\infty e^{-xt} p(t) \, dt = \frac{\sigma_0([0])}{x} + \frac{1}{x} \int_0^\infty e^{-xt} \, d\sigma_0(t), \]

which implies

\[ f_1(x) = (-xf(x))' = \int_0^\infty e^{-xt} \, d\sigma_0(t). \]

By Lemma 3.3 \(f_1\) is completely monotonic of order 1, so that \(t \, d\sigma_0(t) = \sigma_1([0,t]) \, dt\). This means \(tp'(t) \in \mathcal{A}_1(\sigma_1)\) and \(f_1(x) = \int_0^\infty e^{-xt} tp'(t) \, dt\).

Assume now that we have obtained

\[ t^j p^{(j)}(t) = \sigma_j([0,t]) \]
for some \( j \leq k - 2 \), and \( f_j(x) = \int_0^\infty e^{-xt}t^j p^{(j)}(t) \, dt \). Integration by parts gives us

\[
f_j(x) = \frac{\sigma_j(\{0\})}{x} + \frac{1}{x} \int_0^\infty e^{-xt} \, d\sigma_j(t)
= \frac{\sigma_j(\{0\})}{x} + \frac{1}{x} \int_0^\infty e^{-xt} \left( j t^{j-1} p^{(j)}(t) + t^j p^{(j+1)}(t) \right) \, dt,
\]

so that

\[
(-xf_j(x))' = \int_0^\infty e^{-xt} \left( j t^{j-1} p^{(j)}(t) + t^j p^{(j+1)}(t) \right) \, dt.
\]

But also \( f_{j+1} \) is completely monotonic of order 1 so \( f_{j+1} = \mathcal{L}(q) \), where \( q \in A_1(\sigma_{j+1}) \) for some \( \sigma_{j+1} \) and furthermore

\[
(-xf_j(x))' = f_{j+1}(x) + jf_j(x).
\]

Comparing these two relations we find \( t^{j+1} p^{(j+1)}(t) = q(t) \in A_1(\sigma_{j+1}) \).

Finally,

\[
f_k(x) = (-xf_{k-1}(f))' - (k - 1)f_{k-1}(x)
= - \left( x \int_0^\infty e^{-xt} \sigma_{k-1}([0, t]) \, dt \right)' - (k - 1) \int_0^\infty e^{-xt} \sigma_{k-1}([0, t]) \, dt
= - \left( x \left( \frac{\sigma_{k-1}(\{0\})}{x} + \frac{1}{x} \int_0^\infty e^{-xt} \, d\sigma_{k-1}(t) \right) \right)'
- (k - 1) \int_0^\infty e^{-xt} \sigma_{k-1}([0, t]) \, dt
= \int_0^\infty e^{-xt} \{ t \sigma_{k-1}(t) - (k - 1) \sigma_{k-1}([0, t]) dt \}
\]

and the assumed complete monontonicity forces the representing measure to be positive. \( \square \)

*Proof of Theorem 1.4:* The function \( f_0 = f \) is completely monotonic. Assume next that \( f_j \) is completely monotonic with

\[
f_j(x) = \int_0^\infty e^{-xt}t^j p^{(j)}(t) \, dt,
\]

for some \( j \leq k - 2 \). Since

\[
f_{j+1}(x) = -xf_j'(x) - (j + 1)f_j(x)
= x \int_0^\infty e^{-xt}t^j p^{(j)}(t) \, dt - (j + 1) \int_0^\infty e^{-xt}t^j p^{(j)}(t) \, dt,
\]

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and
\[
\int_0^\infty e^{-xt^{j+1}}p^{(j)}(t)\,dt = \frac{1}{x} \int_0^\infty e^{-xt^{j+1}}p^{(j+1)}(t)\,dt \\
+ (j+1) \frac{1}{x} \int_0^\infty e^{-xt^{j+1}}p^{(j)}(t)\,dt
\]
we obtain
\[
f_{j+1}(x) = \int_0^\infty e^{-xt^{j+1}}p^{(j+1)}(t)\,dt.
\]
Finally,
\[
\int_0^\infty e^{-xt^{k}}p^{(k-1)}(t)\,dt = \int_0^\infty e^{-xt^k}d\mu_{k-1}([0,t])\,dt \\
= \frac{1}{x} \int_0^\infty e^{-xt^k}d\mu_{k-1}([0,t])\,dt + \frac{1}{x} \int_0^\infty e^{-xt}d\mu_{k-1}(t)
\]
and if we combine this with the relation \( f_k(x) = -xf_{k-1}'(x) - kf_{k-1}(x) \) we get
\[
f_k(x) = \int_0^\infty e^{-xt}\{td\mu_{k-1}(t) - (k-1)d\mu_{k-1}([0,t])\} dt,
\]
and hence also \( f_k \) is completely monotonic.

\section{Concluding remarks}

\textbf{Remark 4.1} Suppose that the function \( f \) satisfies condition (iii) of Theorem 1.1 and that there exists a positive integer \( r \geq 2 \) such that \((x^rf(x))' \leq 0\) for all \( x > 0 \). Then \( f \) is completely monotonic of order \( r \), and hence of the form
\[
f(x) = \int_0^\infty e^{-xt}\varphi(t)\,dt,
\]
where \( \varphi \) is absolutely monotonic and satisfies \( \varphi^{(k)}(0) = 0 \) for all \( k \leq r-2 \).

Proof: We shall use the fact that a function \( g \) is completely monotonic if \( g(x) \geq 0, \ g'(x) \leq 0, \) and \(( -1)^mg^{(m)}(x) \geq 0 \) for infinitely many \( m \). See [6, Corollary 1.14, p. 12]. It then suffices to consider \( g(x) = x^rf(x) \). The integral representation follows from (v) of Theorem 1.1.

\textbf{Remark 4.2} Suppose that \( f \) is the Laplace transform of a polynomial \( \varphi \) of degree \( r-1 \) with non-negative coefficients. Then clearly
\[
f(x) = \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{x^{k+1}}
\]
whence

\[ \lim_{x \to 0^+} x^r f(x) = \varphi^{(r-1)}(0) < \infty. \]

Conversely, if the function \( f \) satisfies any condition of Theorem 1.1 and there exists a positive integer \( r \) and a number \( a_r \neq 0 \) such that

\[ \lim_{x \to 0^+} x^r f(x) = a_r \tag{5} \]

then \( f \) is the Laplace transform of a polynomial of degree \( r - 1 \) with non-negative coefficients. Indeed, we have

\[ x^r f(x) = \int_0^\infty e^{-xt} \varphi^{(r)}(t) \, dt + \sum_{j=0}^{r-1} \varphi^{(r-j-1)}(0) x^j, \]

where \( \varphi \) is an absolutely monotonic function. Suppose for a contradiction that \( \varphi^{(r)} \) is not identically zero. Then, since \( \varphi^{(r)} \) is increasing and hence not integrable,

\[ \lim_{x \to 0^+} \int_0^\infty e^{-xt} \varphi^{(r)}(t) \, dt = \int_0^\infty \varphi^{(r)}(t) \, dt = \infty, \]

which contradicts (5). Thus, \( \varphi^{(r)} \) is identically zero and hence \( \varphi \) must be a polynomial of degree \( r - 1 \) with non-negative coefficients.

It follows from the discussion above that if \( f \) is the Laplace transform of an absolutely monotonic function \( \varphi(t) = \sum_{n=0}^{\infty} a_n t^n \) in which infinitely many of the numbers \( a_n \) are strictly positive, then for all positive integers \( k \) we have

\[ \lim_{x \to 0^+} x^k f(x) = \infty, \]

and since consequently all derivatives of \( x^k f(x) \) are unbounded near zero we also have

\[ \lim_{x \to 0^+} (-1)^{n+k} \left( x^k f(x) \right)^{n+k} = \infty \]

for all \( n, k \geq 0 \).

Remark 4.3 The generalized hypergeometric series

\[ \varphi(t) = _1F_2(a; b, c; t) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k(c)_k k!} t^k, \quad a > 0, b > 0, c > 0 \]
defines an absolutely monotonic function on \([0, \infty)\). According to [1, p. 115] (see also [3]) its Laplace transform exists for all \(x > 0\) and it is given by the formula

\[ f(x) = \int_0^\infty e^{-xt} \varphi(t) dt = \frac{1}{x} 2F_2\left( a, 1; b, c; \frac{1}{x} \right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n} \frac{1}{x^{n+1}}. \] (6)

Therefore \(f\) has all the properties of Theorem 1.1. Notice that the case \(a = b, c = 1\) gives the example \(H(x) = x^{-1} e^{1/x}\), mentioned in the first section. The case \(a = b, c = \alpha + 1\) gives an example related to the modified Bessel function of the first kind. Formula (6) has the following generalization

\[ \int_0^\infty e^{-xt} t^{\lambda-1} F_2(a; b, c; t) dt = \frac{\Gamma(\lambda)}{x^\lambda} 2F_2(a, \lambda; b, c; \frac{1}{x}) \]

for any \(\lambda > 0\). Therefore the function \(\frac{\Gamma(\lambda)}{x^\lambda} 2F_2(a, \lambda; b, c; \frac{1}{x})\) has all the properties of Theorem 1.5.

In order to give an example of a function satisfying the properties of Theorem 1.3 we describe a more general situation in the proposition below.

**Proposition 4.4** Suppose that \(f\) satisfies the conditions in Theorem 1.1, let \(\lambda > 0\) and define \(F_\lambda(x) = x^{-\lambda} f(x)\). Then:

1. If \(\lambda\) is a positive integer then \(F_\lambda\) satisfies again all the conditions in Theorem 1.1.

2. If \(\lambda\) is not an integer, let \(k = \lfloor \lambda \rfloor\). Then \((-1)^j (x^j F_\lambda(x))^{(j)}\) is completely monotonic for \(j \leq k + 1\), so

\( F_\lambda(x) = \int_0^\infty e^{-xt} p_\lambda(t) dt, \)

with \(t^k p_\lambda(t) \in A_{k+1}(\sigma)\) for some \(\sigma \in A_0\).

Moreover, if \(\lim_{x \to \infty} x f(x) > 0\) then \((-1)^{k+2} (x^{k+2} F_\lambda(x))^{(k+2)}\) is not completely monotonic.

**Proof.** We have \(f(x) = \sum_{n=0}^{\infty} a_n n! x^{-n-1}\) and hence (1) is obvious. To prove (2) notice that

\[ (-1)^j (x^j F_\lambda(x))^{(j)} = (-1)^j \sum_{n=0}^{\infty} a_n n! (x^{j-n-1-\lambda})^{(j)} \]

\[ = \sum_{n=0}^{\infty} a_n n! (n+1+\lambda-j) \cdots (n+\lambda) x^{-n-1-\lambda}. \]
For $j \leq k + 1$ all coefficients in this series are non-negative, and this gives the complete monotonicity. The representation then follows from Theorem 1.3. To obtain the last assertion we get from the formula above
\[
(-1)^{k+2}(x^{k+2}F_\lambda(x))^{(k+2)} = a_0(\lambda - k - 1)(\lambda - k) \cdots \lambda x^{-\lambda-1} + o(x^{-\lambda-1})
\]
for $x \to \infty$. Here, $c_\lambda = (\lambda - k - 1)(\lambda - k) \cdots \lambda$ is negative so that
\[
\lim_{x \to \infty} x^{\lambda+1}(-1)^{k+2}(x^{k+2}F_\lambda(x))^{(k+2)} = a_0c_\lambda < 0,
\]
contradicting complete monotonicity. □

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