Extremal Projectors of \( q \)-Boson Algebras

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Abstract

We define the extremal projector of the \( q \)-boson Kashiwara algebra \( B_q(\mathfrak{g}) \) and study their basic properties. Applying their properties to the representation theory of the category \( \mathcal{O}(B_q(\mathfrak{g})) \), whose objects are “upper bounded” \( B_q(\mathfrak{g}) \)-modules, we obtain its semi-simplicity and the classification of simple modules.

1 Introduction

In [3], we studied the so-called \( q \)-boson Kashiwara algebra, in particular, a kind of the \( q \)-vertex operators and their 2 point functions. We found therein some interesting object \( \Gamma \). But at that time we did not reveal its whole properties, as “Extremal Projectors”. Tolstoy, V.N., et.al., introduced the notion of “Extremal Projectors” for Lie (super)algebras and quantum (super) algebras, and made extensive study of their properties and applied it to the representation theory, (see [2], [6] and the references therein). In the present paper, we shall re-define the extremal projector for the \( q \)-boson algebras, clarify their properties and apply it to the representation theory of \( q \)-boson algebras.

To be more precise, let \( \{ e''_i, f_i, q^h | i \in I, h \in P^* \} \) be the generators of the \( q \)-boson algebra \( B_q(\mathfrak{g}) \). The extremal projector \( \Gamma \) is an element in \( \hat{B}_q(\mathfrak{g}) \)(some completion of \( B_q(\mathfrak{g}) \)) which satisfies the following:

\[
e''_i \Gamma = \Gamma f_i = 0, \quad \Gamma^2 = \Gamma, \quad \sum_k a_k \Gamma b_k = 1,
\]

for some \( a_k \in B_q^+(\mathfrak{g}) \) and \( b_k \in B_q^- (\mathfrak{g}) \) (see Theorem 5.2). Let \( \mathcal{O}(B) \) be the category of ‘upper bounded’ \( B_q(\mathfrak{g}) \)-modules (see Sect 3). By using the above properties of \( \Gamma \), we shall show that the category \( \mathcal{O}(B) \) is semi-simple and classify its simple modules.

In [1], Kashiwara gave the projector \( P \) for \( q \)-boson algebra of \( \mathfrak{sl}_2 \)-case in order to define the crystal base of \( U_q^- (\mathfrak{g}) \). He uses it to show the semi-simplicity of \( \mathcal{O}(B_q(\mathfrak{sl}_2)) \). So our \( \Gamma \) is a generalizations of his projector \( P \) to arbitrary Kac-Moody algebras.

The organization of this article is as follows; In Sect.2, we review the definitions of the quantum algebras and the \( q \)-boson Kashiwara algebras and their properties. In Sect.3, we introduce the category of modules of the \( q \)-boson algebras \( \mathcal{O}(B) \), which we treat in the sequel. In Sect.4, we review so-called Drinfeld Killing form and by using it we define some element \( C \) in the tensor product of \( q \)-boson algebras, which plays a significant role of studying extremal projectors. In Sect.5, we define extremal projectors for the \( q \)-boson algebras and involve...
their important properties. In the last section, we apply it to show the semi-simplicity of the category $\mathcal{O}(B)$ and classify the simple modules in $\mathcal{O}(B)$. In [3] we gave the proof of its semisimplicity, but there was a quite big gap. Thus, the last section would be devoted to an erratum for it. We can find an elementary proof of the semisimplicity of the category $\mathcal{O}(B)$ in e.g. [7].

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2 Quantum algebras and $q$-boson Kashiwara algebras

We shall define the algebras playing a significant role in this paper. First, let $\mathfrak{g}$ be a symmetric Kac-Moody algebra over $\mathbb{Q}$ with a Cartan subalgebra $t$, \{\(\alpha_i \in t^*\)\} the set of simple roots and \{\(h_i \in t\)\} \{\(i \in I\)\} the set of coroots, where $I$ is a finite index set. We define an inner product on $t^*$ such that \((\alpha_i, \lambda) = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)\) for $\lambda \in t^*$. Set $Q = \oplus_i \mathbb{Z} \alpha_i$, $Q_+ = \oplus_i \mathbb{Z}_{\geq 0} \alpha_i$, and $Q_- = -Q_+$. We call $Q$ a root lattice. Let $P$ a lattice of $t^*$ i.e. a free $\mathbb{Z}$-submodule of $t^*$ such that $t^* \cong \mathbb{Q} \otimes Z P$, and $P^* = \{h \in t^{|(h, P) \subset Z}\}$. Now, we introduce the symbols \(\{e_i, e''_i, f_i, f'_i (i \in I), q^h (h \in P^*)\}\). These symbols satisfy the following relations:

\[
 q^0 = 1, \quad q^h q^{h'} = q^{h+h'},
\]

\[
 q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \tag{2.1}
\]

\[
 q^h e''_i q^{-h} = q^{(h, \alpha_i)} e''_i, \tag{2.2}
\]

\[
 q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i, \tag{2.3}
\]

\[
 q^h f'_i q^{-h} = q^{-(h, \alpha_i)} f'_i, \tag{2.4}
\]

\[
 |e_i, f_j| = \delta_{i,j} (t_i - t_i^{-1})/(q_i - q_i^{-1}), \tag{2.5}
\]

\[
 e''_i f_j = q_i^{(h, \alpha_j)} f_j e''_i + \delta_{i,j}, \tag{2.6}
\]

\[
 f'_i e_j = q_i^{(h, \alpha_j)} e_j f'_i + \delta_{i,j}, \tag{2.7}
\]

\[
 1^{-(h, \alpha_j)} \sum_{k=0} (-1)^k x_i^{(h, \alpha_j) - k} = 0, (i \neq j), \tag{2.8}
\]

for \(x_i = e_i, e''_i, f_i, f'_i\).

where $q$ is transcendental over $\mathbb{Q}$ and we set \(q_i = q^{(\alpha_i, \alpha_i)/2}, t_i = q_i^{h_i}, [n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1}), [n]_i! = \prod_{k=1}^n [k]_i\) and $X_i^{(n)} = X_i^n/[n]_i!$.

Now, we define the algebras $B_q(\mathfrak{g}), \overline{B}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$. The algebra $B_q(\mathfrak{g})$ (resp. $\overline{B}_q(\mathfrak{g})$) is an associative algebra generated by the symbols \(\{e''_i, f_i\}_i \in I\) (resp. \(\{e_i, f'_i\}_i \in I\)) and \(q^h (h \in P^*)\) with the defining relations (2.1), (2.2), (2.4), (2.7) and (2.9) (resp. (2.1), (2.2), (2.4), (2.8) and (2.9)) over $Q(q)$. The algebra $U_q(\mathfrak{g})$ is the usual quantum algebra generated by the symbols \(\{e_i, f_i\}_i \in I\) and \(q^h (h \in P^*)\) with the defining relations (2.1), (2.2), (2.4), (2.7) and (2.9) over $Q(q)$. We shall call algebras $B_q(\mathfrak{g})$ and $\overline{B}_q(\mathfrak{g})$ the $q$-boson Kashiwara algebras ([K1]). Furthermore, we define their subalgebras

\[
 T = \langle q^h | h \in P^* \rangle = B_q(\mathfrak{g}) \cap \overline{B}_q(\mathfrak{g}) \cap U_q(\mathfrak{g}),
\]
Proposition 2.2

We shall use the abbreviated notations $U, B, \overline{B}, B', \ldots$ for $U_q(g), B_q(g), \overline{B}_q(g), B'_q(g), \ldots$ if there is no confusion.

For $\beta = \sum m_i \alpha_i \in Q_+$ we set $|\beta| = \sum m_i$ and

$$U_{\pm \beta}^h = \{ u \in U^\pm | q^h u q^{-h} = q^{(h, \beta)} u \ (h \in P^+) \},$$

and call $|\beta|$ a height of $\beta$ and $U_{\pm \beta}^h$ (resp. $U^h$) a weight space of $U^+$ (resp. $U^-$) with a weight $\beta$ (resp. $-\beta$). We also define $B_{\pm \beta}^h$ and $\overline{B}_{\pm \beta}$ by the similar manner.

**Proposition 2.1**

(i) We have the following algebra homomorphisms:

- $\Delta : U \rightarrow U \otimes U, \Delta^{(r)} : B \rightarrow B \otimes U, \Delta^{(l)} : \overline{B} \rightarrow U \otimes \overline{B}$ and $\Delta^{(b)} : U \rightarrow \overline{B} \otimes B$ given by

  $$\Delta(q^h) = \Delta^{(r)}(q^h) = \Delta^{(l)}(q^h) = \Delta^{(b)}(q^h) = q^h \otimes q^h,$$

- $$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,$$

- $$\Delta^{(r)}(e_i^\prime) = (q_i - q_i^{-1}) \otimes t_i^{-1} e_i + e_i^\prime \otimes t_i^{-1}, \quad \Delta^{(r)}(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,$$

- $$\Delta^{(l)}(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta^{(l)}(f_i^\prime) = (q_i - q_i^{-1}) t_i f_i \otimes 1 + t_i \otimes f_i^\prime,$$

- $$\Delta^{(b)}(e_i) = t_i \otimes \frac{t_i e_i^\prime}{q_i - q_i^{-1}} + e_i \otimes 1, \quad \Delta^{(b)}(f_i) = 1 \otimes f_i + \frac{t_i^{-1} f_i^\prime}{q_i - q_i^{-1}} \otimes t_i^{-1},$$

and extending these to the whole algebras by the rule: $\Delta(xy) = \Delta(x) \Delta(y)$ and $\Delta^{(i)}(xy) = \Delta^{(i)}(x) \Delta^{(i)}(y)$ ($i = r, l, b$).

(ii) We have the following anti-isomorphisms $S : U \rightarrow U$ and $\varphi : \overline{B} \rightarrow B$ given by

$$S(e_i) = -t_i^{-1} e_i, \quad S(f_i) = -f_i t_i, \quad S(q^h) = q^{-h},$$

$$\varphi(e_i) = -\frac{1}{q_i - q_i^{-1}} e_i^\prime, \quad \varphi(f_i^\prime) = -(q_i - q_i^{-1}) f_i, \quad \varphi(q^h) = q^{-h},$$

and extending these to the whole algebras by the rule: $S(xy) = S(y) S(x)$ and $\varphi(xy) = \varphi(y) \varphi(x)$. Here $S$ is called an anti-pode of $U$. We also denote $\varphi_{|_{\geq}} = \varphi_{|_{B_{\geq}}}$ by $\varphi$.

We obtain the following triangular decomposition of the $q$-boson Kashiwara algebra;

**Proposition 2.2**

The multiplication map defines an isomorphism of vector spaces:

$$B_q^-(g) \otimes T \otimes B_q^+(g) \xrightarrow{\sim} B_q(g)_{u_1 \otimes u_2 \otimes u_3} \mapsto u_1 u_2 u_3.$$
Proposition 4.1 ([4],[5])

4 Bilinear forms and elements

In Sect.6, we shall also show that $O$ are naturally extend for such completions.

We define weight completions of $L^{(1)} \otimes \cdots \otimes L^{(m)}$, where $L^{(i)} = B$ or $U$. (See [T])

Proof. By [I, (3.1.2)], we have

$$
e_i^m f_j^{(m)} = \begin{cases} 
\sum_{i=0}^{\min(n,m)} q_i 2^{nm+(n+m)i-i(i+1)/2} \left[ \frac{\min(n,m)}{i} \right] f_j^{(m-i)} e_i^n, & \text{if } i = j, \\
q_i^{nm(h_i,\alpha_j)} j f_j^{(m)} e_i^n, & \text{otherwise.}
\end{cases}
$$

By this formula and the standard argument, we can show the proposition. □

We define weight completions of $B^{(1)} \otimes \cdots \otimes B^{(m)}$ for $B$,

\begin{align*}
\hat{L}^{(1)} \otimes \cdots \otimes \hat{L}^{(m)} &= \lim_{\longleftarrow} L^{(1)} \otimes \cdots \otimes L^{(m)}/(L^{(1)} \otimes \cdots \otimes L^{(m)})L^{+,l},
\end{align*}

where $L^{+,l} = \oplus_{|\beta_1|+\cdots+|\beta_m| \geq 2L^{(1)} \oplus \cdots \oplus L^{(m)} \oplus L^{+,l} \beta_m}$. (Note that $U \cong U^- \otimes T \otimes U^+$ and $B \cong B^- \otimes T \otimes B^+$.) The linear maps $\Delta, \Delta^\ast, S$, $\varphi$, multiplication, $e.t.c.$ are naturally extend for such completions.

3 Category $O(B)$

Let $O(B)$ be the category of left $B$-modules such that

(i) Any object $M$ has a weight space decomposition $M = \oplus_{\lambda \in P} M_{\lambda}$ where $M_{\lambda} = \{ u \in M \mid q^h u = q^{(h,\lambda)} u \}$ for any $h \in P^*$.

(ii) For any element $u \in M$ there exists $l > 0$ such that $e_{i_1}^n \cdots e_{i_l}^n u = 0$ for any $i_1, i_2, \cdots, i_l \in I$.

The similar category $O(B^\gamma)$ for $B_q(q)^\gamma$ is introduced in [I], which is defined with the above condition (ii). In [I], Kashiwara mentions that the category $O(B^\gamma)$ is semi-simple though he does not give an exact proof. Here we give a proof of the semi-simplicity of $O(B)$ in Sect 6.

Here for $\lambda \in P$ we define the $B$-module $H(\lambda)$ by $H(\lambda) := B/I\lambda$ where the left ideal $I\lambda$ is defined as

$$I\lambda := \sum_i B e_i^\nu + \sum_{h \in P^*} B(q^h - q^{(h,\lambda)}).$$

In Sect.6, we shall also show that $\{ H(\lambda) | \lambda \in P \}$ is the isomorphism class of simple modules.

4 Bilinear forms and elements $C$

Proposition 4.1 ([4],[5])

(i) There exists the unique bilinear form

$$\langle \ , \ \rangle : U^\geq \times U^\leq \rightarrow Q(q),$$

satisfying the following:

$$\langle x, y_1 y_2 \rangle = \langle \Delta(x), y_1 \otimes y_2 \rangle, \quad (x \in U^\geq, y_1, y_2 \in U^\leq),$$

$$\langle x, y_1 y_2 \rangle = \langle x, y_2 y_1 \rangle, \quad (x, y_1, y_2 \in U^\geq),$$

$$\langle e_i, y \rangle = \delta_{i,1} y, \quad (i = 1, \cdots, n),$$

$$\langle x, e_i \rangle = 0, \quad (x \in U^\leq, i = 1, \cdots, n).$$

Furthermore, the bilinear form is symmetric.
\[(x_1 x_2, y) = (x_2 \otimes x_1, \Delta(y)), \quad (x_1, x_2 \in U^\geq, y \in U^\leq),\]
\[(q^n, q^{h'}) = q^{-\langle h| h' \rangle}} \quad (h, h' \in P^*),\]
\[(T, f_1) = \langle e_i, T \rangle = 0,\]
\[\langle e_i, f_j \rangle = \delta_{ij} / (q_i^{-1} - q_i),\]

where \(\langle \cdot, \cdot \rangle\) is an invariant bilinear form on \(t\).

(ii) The bilinear form \(\langle \cdot, \cdot \rangle\) enjoys the following properties:
\[
\langle qx^n, yq^{h'} \rangle = q^{-\langle h| h' \rangle} \langle x, y \rangle , \quad \text{for } x \in U^\geq, \ y \in U^\leq, \ h, h' \in P^*, \quad (4.1)
\]

For any \(\beta \in Q_+\), \(\langle \cdot, \cdot \rangle_{U^+_\beta \times U^-_{-\beta}}\) is non-degenerate and \(\langle U^+_\gamma, U^-_{-\delta} \rangle = 0\), if \(\gamma \neq \delta\). \(\quad (4.2)\)

We call this bilinear form the Drinfeld-Killing form of \(U\).

For \(\beta = \sum_i m_i \alpha_i \in Q_+\) \((m_i \geq 0)\), set \(k_\beta := \prod_i t_i^{m_i}\) and let \(\{x^\beta_r\}_r\) be a basis of \(U^+\beta\) and \(\{y^{-\beta}_r\}_r\) be the dual basis of \(U^-_{-\beta}\) with respect to the Drinfeld-Killing form. We denote the canonical element in \(U^+_{\beta} \otimes U^-_{-\beta}\) with respect to the Drinfeld-Killing form by
\[
C_\beta := \sum_r x^\beta_r \otimes y^{-\beta}_r.
\]

We set
\[
\mathcal{C} := \sum_{\beta \in Q_+} (1 \otimes k_\beta^{-1})(1 \otimes S^{-1})(C_\beta) \in U^+ \hat{\otimes} U^- = U^+ \hat{\otimes} B^- . \quad (4.3)
\]

The element \(\mathcal{C}\) satisfies the following relations:

**Proposition 4.2**

(i) For any \(i \in I\), we have
\[
(t_i^{-1} \otimes e''_i) \mathcal{C} = \mathcal{C}(t_i^{-1} \otimes e''_i + (q_i - q_i^{-1})t_i^{-1} e_i \otimes 1), \quad (4.4)
\]
\[
(f_i \otimes t_i^{-1} + 1 \otimes f_i)(\varphi \otimes 1(C)) = (\varphi \otimes 1(C))(f_i \otimes t_i^{-1}). \quad (4.5)
\]

Here note that \((4.4)\) is the equation in \(U_q(\mathfrak{g}) \hat{\otimes} B_q(\mathfrak{g})\) and \((4.5)\) is the equation in \(B_q(\mathfrak{g}) \otimes B_q(\mathfrak{g})\).

(ii) The element \(\mathcal{C}\) is invertible and the inverse is given as
\[
\mathcal{C}^{-1} = \sum_{\beta \in Q_+} q^{-(\beta, \beta)}(k_\beta \otimes k_\beta^{-1})(S^{-1} \otimes S^{-1})(C_\beta) \quad (4.6)
\]

Proof. The proof of \((4.3)\) has been given in \([3, 6.2]\). Thus, let us show \((4.4)\).

For that purpose, we need the following lemma;

**Lemma 4.3** For \(\beta \in Q_+\), let \(C_{\beta} := \sum_r x^\beta_r \otimes y^{-\beta}_r\) be the canonical element in \(U^+_{\beta} \otimes U^-_{-\beta}\) as above and set \(C'_{\beta} := (1 \otimes S^{-1})(C_\beta)\). Then for any \(\beta \in Q_+\) and \(i \in I\), we have
\[
[t_i^{-1} \otimes e''_i, (1 \otimes k_{\beta + \alpha_i}^{-1})(C'_{\beta + \alpha_i})] = (1 \otimes k_{\beta}^{-1})(C'_\beta)(t_i^{-1} e_i \otimes (q_i - q_i^{-1})) \in U_q(\mathfrak{g}) \otimes B_q(\mathfrak{g}), \quad (4.7)
\]

where we use the identification \(B_q^-(\mathfrak{g}) = U_q^-(\mathfrak{g})\).
Proof. Applying $\langle \cdot, z \rangle \otimes 1$ on the both sides of (4.3) where $z \in U_{-\beta - \alpha_i}$, we obtain

$$
(\langle \cdot, z \rangle \otimes 1)(\text{L.H.S. of (4.3)}) = \sum_r (t_i^{-1} x_r x_i^{-\alpha_i}, z) \otimes e_i'' k_{\beta + \alpha_i}^{-1} S^{-1}(y_r^{-\beta - \alpha_i})
$$

$$
- \langle x_r x_i^{-\alpha_i}, z \rangle \otimes k_{\beta + \alpha_i}^{-1} S^{-1}(y_r^{-\beta - \alpha_i}) e_i''
$$

$$
= q^{-(\alpha_i, \beta + \alpha_i)} e_i'' k_{\beta + \alpha_i}^{-1} S^{-1}(z) - k_{\beta + \alpha_i}^{-1} S^{-1}(z) e_i''
$$

$$
= k_{\beta + \alpha_i}^{-1} (e_i'' S^{-1}(z) - S^{-1}(z) e_i'').
$$

$$
(\langle \cdot, z \rangle \otimes 1)(\text{R.H.S. of (4.7)}) = \sum_r (x_r x_i^{-\alpha_i}, z) \otimes (q_i - q_i^{-1}) k_{\beta + \alpha_i}^{-1} S^{-1}(y_r^{-\beta})(4.8)
$$

For $z \in U_{-\beta - \alpha_i}$ we can define $v \in U_{-\beta}$ uniquely by

$$
\Delta(z) = 1 \otimes z + f_i \otimes v t_i^{-1} + \cdots.
$$

By the property of the Drinfeld Killing form, we have

$$
\langle x_r x_i^{-1}, \Delta(z) \rangle = \langle e_i \otimes x_r x_i^{-1}, \Delta(z) \rangle
$$

$$
= \langle e_i \otimes x_r x_i^{-1}, 1 \otimes z + f_i \otimes v t_i^{-1} + \cdots \rangle
$$

$$
= \langle e_i, f_i \rangle \langle x_r x_i^{-1}, v t_i^{-1} \rangle
$$

$$
= \frac{q_i^2}{q_i - q_i^{-1}} \langle x_r^\beta, v \rangle.
$$

Thus,

$$
\text{R.H.S. of (4.8)} = -q_i^{-2} k_{\beta}^{-1} S^{-1}(v).
$$

(4.9)

Here in order to complete the proof of Lemma 4.3 let us show;

$$
e_i'' S^{-1}(z) - S^{-1}(z) e_i'' = -q_i^{-2} t_i S^{-1}(1). (4.10)
$$

Without loss of generality, we may assume that $z$ is in the form $z = f_1 f_2 \cdots f_k \in U_{-\beta - \alpha_i}$ ($\beta + \alpha_i = \alpha_{i_1} + \cdots + \alpha_{i_k}$). For $\beta = \sum_m m_j \alpha_j$, we shall show by the induction on $m_i$ for fixed $i \in I$.

If $m_i = 0$, $z$ is in the form $z = z' f_i z''$ where $z'$ and $z''$ are monomials of $f_j$'s not including $f_i$. By $S^{-1}(f_j) = -t_j f_j$ and $e_i''(t_j f_j) = (t_j f_j) e_i'' (i \neq j)$ we have

$$
e_i'' S^{-1}(z') = S^{-1}(z') e_i'', \quad e_i'' S^{-1}(z'') = S^{-1}(z'') e_i''.
$$

(4.11)

Hence, we obtain

$$
e_i'' S^{-1}(z) = S^{-1}(z'') (e_i'' t_i f_i) S^{-1}(z') = S^{-1}(z'') (-t_i f_i e_i'' - q_i^{-2} t_i S^{-1}(z'))
$$

$$
= S^{-1}(z'') (-t_i f_i S^{-1}(z')) e_i'' - q_i^{-2} S^{-1}(z'') t_i S^{-1}(z')
$$

$$
= S^{-1}(z'') S^{-1}(f_i) S^{-1}(z') e_i'' - q^{(\beta'' - \alpha_i, \alpha_i)} t_i S^{-1}(z' z''),
$$

where $\beta'' = wt(z'')$. Therefore, for $m_i = 0$, we have

$$
\text{L.H.S. of (4.11)} = -q^{(\beta'' - \alpha_i, \alpha_i)} t_i S^{-1}(z' z'').
$$
In the case \(m_i = 0\) we can easily obtain \(v = q^{(\beta'',\alpha_i)}z'z''\) and then
\[
\text{R.H.S. of (4.10)} = -q^{(\beta''-\alpha_i,\alpha_i)}t_iS^{-1}(z'z'') = \text{L.H.S. of (4.10)}
\]
Thus, the case \(m_i = 0\) has been shown.

Suppose that \(m_i > 0\). we divide \(z = z'z''\) such that \(m'_i < m_i\) and \(m''_i < m_i\) where \(m'_i\) (resp. \(m''_i\)) is the number of \(f_i\) including in \(z'\) (resp. \(z''\)). Writing
\[
\Delta(z') = 1 \otimes z' + f_i \otimes v't_i^{-1} + \cdots,
\]
\[
\Delta(z'') = 1 \otimes z'' + f_i \otimes v''t_i^{-1} + \cdots,
\]
and calculating \(\Delta(z'z'')\) directly, we obtain
\[
v = z'v'' + q^{(\beta'',\alpha_i)}v'z''.
\]  
(4.12)

By the hypothesis of the induction,
\[
e''_iS^{-1}(z) = e''_iS^{-1}(z'')S^{-1}(z') = (S^{-1}(z'')e''_i - q_{i}^{-2}t_iS^{-1}(v''))S^{-1}(z')
\]
\[
= S^{-1}(z'')(e''_iS^{-1}(z') - q_{i}^{-2}t_iS^{-1}(z'v''))
\]
\[
= S^{-1}(z'')(S^{-1}(z')e''_i - q_{i}^{-2}t_iS^{-1}(v')) - q_{i}^{-2}t_iS^{-1}(z'v'')
\]
\[
= S^{-1}(z''e''_i - q_{i}^{-2}t_i(S^{-1}(z'v'') + q^{(\beta'',\alpha_i)}S^{-1}(v'z''))
\]
\[
= S^{-1}(z)e''_i - q_{i}^{-2}t_iS^{-1}(v).
\]

Note that in the last equality, we use (1.12). Now, we have completed to show Lemma 4.3.

Proof of Proposition 4.2. If \(\beta \in Q_{+}\) does not include \(\alpha_i\), since \(e''_i\) and \(S^{-1}(z) (z \in U_{-\beta})\) commute with each other by (4.11), we have
\[
(t_i^{-1} \otimes e''_i)(1 \otimes k_{\beta}^{-1})(C_{\beta}) = (1 \otimes k_{\beta}^{-1})(C_{\beta})(t_i^{-1} \otimes e''_i).
\]

Thus, we have
\[
(t_i^{-1} \otimes e''_i)C = C(t_i^{-1} \otimes e''_i)
\]
\[
= \sum_{\gamma \in Q_{+}} (t_i^{-1} \otimes e''_i)(1 \otimes k_{\gamma}^{-1})(C_{\gamma}) - (1 \otimes k_{\gamma}^{-1})(C_{\gamma})(t_i^{-1} \otimes e''_i)
\]
\[
= \sum_{\beta \in Q_{+}} (t_i^{-1} \otimes e''_i)(1 \otimes k_{\beta+\alpha_i}^{-1})(C_{\beta+\alpha_i}) - (1 \otimes k_{\beta+\alpha_i}^{-1})(C_{\beta+\alpha_i})(t_i^{-1} \otimes e''_i)
\]
\[
= \sum_{\beta \in Q_{+}} [t_i^{-1} \otimes e''_i, (1 \otimes k_{\beta+\alpha_i}^{-1})(C_{\beta+\alpha_i})]
\]
\[
= \sum_{\beta \in Q_{+}} (1 \otimes k_{\beta+1}^{-1})(C_{\beta})(q_{i} - q_{i}^{-1})t_i^{-1}e_i \otimes 1 \quad \text{(by Lemma 4.3)}
\]
\[
= C((q_{i} - q_{i}^{-1})t_i^{-1}e_i \otimes 1).
\]

Then we obtain (4.4).

Next, let us show (ii). Set \(\tilde{C} := \sum q^{(\beta,\beta)}(1 \otimes k_{\beta})(S \otimes 1)(C_{\beta})\). By [3 Sect.4], we have \(\tilde{C}^{-1} := \sum q^{(\beta,\beta)}(k_{\beta}^{-1} \otimes k_{\beta})(C_{\beta})\). Here note that
\[
(S^{-1} \otimes S^{-1})(\tilde{C}) = \sum q^{(\beta,\beta)}(1 \otimes S^{-1})\{(1 \otimes k_{\beta})(C_{\beta})\}
\]
Thus, we obtain
\[ C^{-1} = (S^{-1} \otimes S^{-1})(C^{-1}) \]
\[ = \sum q^{(\beta, \beta)}((S^{-1} \otimes S^{-1})(C_{\beta}))(k_{\beta} \otimes k_{\beta}^{-1}) \]
\[ = \sum q^{-(\beta, \beta)}(k_{\beta} \otimes k_{\beta}^{-1})(S^{-1} \otimes S^{-1})(C_{\beta}), \]
and completed the proof of Proposition 4.2.

5 Extremal Projectors

Let \( C \) be as in Sect.4. We define the extremal projector of \( B_q(\mathfrak{g}) \) by
\[ \Gamma := m \circ \sigma \circ (\varphi \otimes 1)(C) = \sum_{\beta \in Q^{+}, r} k_{\beta}^{-1} S^{-1}(q^{-\beta} e^\varphi_{r})(e^\varphi_{r}), \]  
(5.1)
where \( m : \ a \otimes b \mapsto ab \) is the multiplication and \( \sigma : \ a \otimes b \mapsto b \otimes a \) is the permutation.

Here note that \( \Gamma \) is a well-defined element in \( \hat{B}_q(\mathfrak{g}) \).

Example 5.1 ([1, 3]) In \( \mathfrak{sl}_2 \)-case, the following is the explicit form of \( \Gamma \).
\[ \Gamma = \sum_{n \geq 0} q^{\frac{1}{2}n(n-1)}(-1)^n f(n) e^{mn}. \]

Theorem 5.2 The extremal projector \( \Gamma \) enjoys the following properties:

(i) \( e_i \Gamma = 0, \quad \Gamma f_i = 0 \quad (\forall i \in I). \)

(ii) \( \Gamma^2 = \Gamma. \)

(iii) There exists \( a_k \in B_q^-(\mathfrak{g}) (= U_q^- (\mathfrak{g})) \), \( b_k \in B_q^+(\mathfrak{g}) \) such that
\[ \sum_k a_k \Gamma b_k = 1. \]

(iv) \( \Gamma \) is a well-defined element in \( \hat{B}_q^i(\mathfrak{g}). \)

Proof. It is easy to see (iv) by the explicit forms of the anti-pode \( S \), the anti-isomorphism \( \varphi \) and \( \Gamma \) in \( \text{[1,3]} \). The statement (ii) is an immediate consequence of (i). So let us show (i) and (iii). The formula \( \Gamma f_i = 0 \) has been shown in
Thus, we shall show \( c_i'' \Gamma = 0 \). Here let us write \( C = \sum_k c_k \otimes d_k \), where \( c_k \in U_q^+ (g) \) and \( d_k \in B_q^- (g) \). Thus, we have
\[
\Gamma = \sum_k d_k \varphi (c_k).
\]
The equation (4.4) can be written as follows;
\[
\sum_k t_i^{-1} c_k \otimes e_i'' d_k = \sum_k c_k t_i^{-1} \otimes d_k e_i'' + (q_i - q_i^- 1) c_k t_i^{-1} e_i \otimes d_k.
\]
(5.2)
Applying \( m \circ \sigma \circ (\varphi \otimes 1) \) on the both sides of (5.2), we get
\[
\sum_k e_i'' d_k \varphi (c_k) t_i = \sum_k d_k e_i'' t_i \varphi (c_k) - \sum_k d_k e_i'' t_i \varphi (c_k) = 0,
\]
and then \( e_i'' \Gamma t_i = 0 \), which implies the desired result since \( t_i \) is invertible.

Next, let us see (iii). By the remark in the last section, we can write
\[
C^{-1} = \sum_k b_k' \otimes a_k \in U_q^+ (g) \otimes B_q^- (g).
\]
Then,
\[
1 \otimes 1 = \sum_{j,k} b'_k c_j \otimes a_k d_j.
\]
(5.3)
Applying \( m \circ \sigma \circ (\varphi \otimes 1) \) on the both sides of (5.3), we obtain
\[
1 = \sum_{j,k} a_k d_j \varphi (c_j) \varphi (b'_k) = \sum_k a_k \Gamma \varphi (b'_k).
\]
Here setting \( b_k := \varphi (b'_k) \), we get (iii).

6 Representation Theory of \( \mathcal{O}(B) \)

As an application of the extremal projector \( \Gamma \), we shall show the following theorem;

**Theorem 6.1**

(i) The category \( \mathcal{O}(B) \) is a semi-simple category.

(ii) The module \( H(\lambda) \) is a simple object of \( \mathcal{O}(B) \) and for any simple object \( M \) in \( \mathcal{O}(B) \) there exists some \( \lambda \in P \) such that \( M \cong H(\lambda) \). Furthermore, \( H(\lambda) \) is a rank one free \( B_q^- (g) \)-module.

In order to show this theorem, we need to prepare several things.

For an object \( M \) in \( \mathcal{O}(B) \), set
\[
K(M) := \{ v \in M \mid c_i'' v = 0 \text{ for any } i \in I \}.
\]

**Lemma 6.2** For an object \( M \) in \( \mathcal{O}(B) \), we have
\[
\Gamma \cdot M = K(M)
\]
(6.1)
Proof. By Theorem 5.2(i), we have $e''_i \Gamma = 0$ for any $i \in I$. Thus, it is trivial to see that $\Gamma \cdot M \subset K(M)$. Owing to the explicit form of $\Gamma$, we find that

$$1 - \Gamma \in \sum_i \widehat{B}_q(g) e''_i.$$ 

Therefore, for any $v \in K(M)$ we get $(1 - \Gamma)v = 0$, which implies that $\Gamma \cdot M \supset K(M)$. □

**Lemma 6.3** For an object $M$ in $\mathcal{O}(B)$, we have

$$M = B_q^-(g) \cdot (K(M)) \quad (6.2)$$

Proof. By Theorem 5.2(iii), we have $1 = \sum_k a_k \Gamma b_k$ ($a_k \in B_q^-(g)$, $b_k \in B_q^+(g)$). For any $u \in M$,

$$u = \sum_k a_k (\Gamma b_k u).$$

By Lemma 6.1 we have $\Gamma b_k u \in K(M)$. Then we obtain the desired result. □

**Proposition 6.4** For an object $M$ in $\mathcal{O}(B)$, we have

$$M = K(M) \oplus (\sum_i \text{Im}(f_i)). \quad (6.3)$$

Proof. By (6.2), we get

$$M = K(M) + (\sum_i \text{Im}(f_i)).$$

Thus, it is sufficient to show

$$K(M) \cap (\sum_i \text{Im}(f_i)) = \{0\}. \quad (6.4)$$

Let $u$ be a vector in $K(M) \cap (\sum_i \text{Im}(f_i))$. Since $u \in \sum_i \text{Im}(f_i)$, there exist $\{u_i \in M\}_{i \in I}$ such that $u = \sum_{i \in I} f_i u_i$. By the argument in the proof of Lemma 6.2, we have $\Gamma u = u$ for $u \in K(M)$. It follows from Theorem 5.2(i) that

$$u = \Gamma u = \sum_{i \in I} (\Gamma f_i) u_i = 0,$$

which implies (6.4). □

**Lemma 6.5** If $u, v \in M$ ($M$ is an object in $\mathcal{O}(B)$) satisfies $v = \Gamma u$, then there exists $P \in B_q(g)$ such that $v = Pu$.

Proof. By the definition of the category of $\mathcal{O}(B)$, there exists $l > 0$ such that $\varphi(x^2_r) u = 0$ for any $r$ and $\beta$ with $|\beta| > l$. Thus, by the explicit form of $\Gamma$ in (5.4), we can write

$$v = \Gamma u = \left( \sum_{|\beta| \leq l, r} k^{-1}_\beta S^{-1}(y_r^\beta) \varphi(x^2_r) u, \right.$$
which implies our desired result.

**Proof of Theorem 6.1.** Let $L \subset M$ be objects in the category $\mathcal{O}(B)$. We shall show that there exists a submodule $N \subset M$ such that $M = L \oplus N$.

Since $K(M)$ (resp. $K(L)$) is invariant by the actin of any $q^p$, we have the weight space decomposition:

$$K(M) = \bigoplus_{\lambda \in P} K(M)_{\lambda} \quad \text{(resp.} \quad K(L) = \bigoplus_{\lambda \in P} K(L)_{\lambda}) \text{).}$$

There exist subspaces $N_{\lambda} \subset K(M)_{\lambda}$ such that $K(M)_{\lambda} = K(L)_{\lambda} \oplus N_{\lambda}$, which is a decomposition of a vector space. Here set $N := \bigoplus_{\lambda} N_{\lambda}$. We have

$$K(M) = K(L) \oplus N.$$  \hspace{1cm} (6.5)

Let us show

$$M = L \oplus B_q(g) \cdot N. \hspace{1cm} (6.5)$$

Since $M = B_q(g) \cdot (K(M)) = B_q(g)(K(L) \oplus N)$, we get $M = L + B_q(g) \cdot N$. Let us show

$$L \cap B_q(g) \cdot N = \{0\}. \hspace{1cm} (6.6)$$

For $v \in L \cap B_q(g) \cdot N$ we have by Theorem 5.2 (iii),

$$v = \sum_k a_k(\Gamma_k v).$$

It follows from $v \in L$ that $\Gamma_k v \in K(L)$, and from $v \in B_q(g) \cdot N$ that $\Gamma_k v \in \Gamma(B_q(g) \cdot N) = N$. These imply

$$\Gamma_k v \in K(L) \cap N = \{0\}.$$

Hence we get $v = 0$ and then (6.6).

Next, let us show (ii). As an immediate consequence of Proposition 2.2 we can see that $H(\lambda)$ is a rank one free $B_q^-(g)$-module.

Let $\pi_\lambda : B_q(g) \to H(\lambda)$ be the canonical projection and set $u_\lambda := \pi_\lambda(1)$. Here we have

$$H(\lambda) = B_q^-(g) \cdot u_\lambda = Q(q) u_\lambda + \sum_i \text{Im}(f_i).$$

It follows from this, Proposition 6.4 and $Q(q) u_\lambda \subset K(H(\lambda))$ that $H(\lambda) = Q(q) u_\lambda \oplus \sum_i \text{Im}(f_i)$ and then

$$\Gamma \cdot H(\lambda) = K(H(\lambda)) = Q(q) u_\lambda. \hspace{1cm} (6.7)$$

In order to show the irreducibility of $H(\lambda)$, it is sufficient to see that for arbitrary $u(\neq 0)$, $v \in H(\lambda)$ there exists $P \in B_q(g)$ such that $v = Pu$. Set $v = Q u_\lambda$ ($Q \in B_q^-(g)$). By Theorem 5.2 (iii), we have

$$u = \sum_k a_k(\Gamma_k u) \neq 0.$$
Then, for some $k$ we have $\Gamma b_k u \neq 0$, which implies that $c\Gamma b_k u = u_\lambda$ for some non-zero scalar $c$. Therefore, by Lemma 6.5, there exists some $R \in B_q(g)$ such that $u_\lambda = Ru$ and then we have

$$v = Qu_\lambda = QRu.$$  

Thus, $H(\lambda)$ is a simple module in $\mathcal{O}(B)$.

Suppose that $L$ is a simple module in $\mathcal{O}(B)$. First, let us show

$$\dim(K(L)) = 1.$$  

(6.8)

For $x, y(\neq 0) \in K(L)$, there exists $P \in B_q(g)$ such that $y = Px$. Since $x \in K(L)$, we can take $P \in B_q^-(g)$. Because $y \in K(L)$ and $K(L) \cap \sum_i \text{Im}(f_i) = \{0\}$, we find that $P$ must be a scalar, say $c$. Thus, we have $y = cx$, which derives (6.8).

Let $u_0$ be a basis vector in $K(L)$. The space $K(L)$ is invariant by the action of any $q^h$ and then, $u_0 \in L_\lambda$ for some $\lambda \in P$. Therefore, since $H(\lambda)$ is a rank one free $B_q^-(g)$-module, the map

$$\phi_\lambda : H(\lambda) \longrightarrow L$$

$$Pu_\lambda \mapsto Pu_0, \quad (P \in B_q^-(g)),$$

is a well-defined non-trivial homomorphism of $B_q^-(g)$-modules. Thus, by Schur’s lemma, we obtain $H(\lambda) \cong L$. □

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