LONG MEMORY AND FINANCIAL MARKET BUBBLE DYNAMICS IN AFFINE STOCHASTIC DIFFERENTIAL EQUATIONS WITH AVERAGE FUNCTIONALS

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Abstract. In this paper we consider the growth, large fluctuations and memory properties of an affine stochastic functional differential equation with an average functional where the contributions of the average and instantaneous terms are parameterised. An asymptotic analysis of the solution of this equation is conducted for all values of the parameters of the equation. When solutions are recurrent, we show that the autocovariance function of the solution decays at a polynomial rate, even though the solution is asymptotically equal to another asymptotically stationary process whose autocovariance function decays exponentially. It is shown that when solutions grow, they do so at either a polynomial or exponential rate in time depending on the sign of a parameter of the model, modulo some exceptional parameter sets. On these exceptional sets, solutions are recurrent on the real line with large fluctuations consistent with the Law of the Iterated logarithm, or exhibit subexponential yet superpolynomial growth.

1. Introduction and overview

In this paper, we consider the asymptotic behaviour of an affine scalar stochastic functional differential equation where the average of the process over its entire history appears on the right-hand side. Accordingly, we study

\[ dX(t) = \left( aX(t) + b \frac{1}{1+1} \int_{-1}^{t} X(s) ds \right) dt + \sigma dB(t), \quad t \geq 0, \]  

where \( X \) is given by the continuous function \( \psi \), defined on \([-1, 0] \), \( B \) is a standard one-dimensional Brownian motion and \( \sigma \neq 0 \). Here \( a \) and \( b \) are real parameters. There is a unique strong solution of (1.1) which is a Gaussian process. The goal of the paper is to describe for all pairs of the parameters \( a \) and \( b \) the asymptotic behaviour of the paths, as well as information about the autocovariance function of \( X \) in the case that the solution is recurrent on \( \mathbb{R} \).

1.1. Organisation of results and methods of proof. This model was studied in \([3]\), where the authors considered the case \( a > 0 \). Under this condition the solution \( X \) was shown to grow at a well-defined exponential rate, with a polynomial correction. Specifically, the rate of growth is given by

\[ \lim_{t \to \infty} \frac{X(t)}{e^{at} b/a} = C, \quad \text{a.s.} \]  

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where $C$ is an almost surely finite and Gaussian distributed random variable. The results in [3] rely on the theory of admissibility of linear deterministic Volterra operators.

While the present article establishes some new results concerning the case when $a > 0$, for the most part it is concerned with the case when $a \leq 0$, where the solution need not have a well-defined growth rate but rather may fluctuate. This behaviour is not wholly unexpected; in the case when $a < 0$ and $b = 0$, for example, the solution of (1.1) is an asymptotically stationary Ornstein–Uhlenbeck process, while when $a = 0$ and $b = 0$, it is a scaled standard Brownian motion.

A complete asymptotic dynamical picture of the solution $X$ is determined for all real values of $a$ and $b$ in the paper. Our analysis shows that there are only three principal regions in the ‘$a - b$’ parameter space, within which the process $X$ undergoes different pathwise asymptotic behaviour. For clarity we provide a bifurcation diagram of the parameter space:

- In Theorem 3.1, corresponding to $a < 0$ and $a + b \leq 0$, the solution $X$ is asymptotically equal to an Ornstein–Uhlenbeck process and has oscillations of magnitude described by
  \[
  \limsup_{t \to \infty} \frac{X(t)}{\sqrt{2 \log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t \to \infty} \frac{X(t)}{\sqrt{2 \log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.}
  \]
- In Theorem 4.1, corresponding to $a < 0$ and $a + b > 0$, the solution $X$ tends to plus or minus infinity at a polynomial rate
  \[
  \lim_{t \to \infty} \frac{X(t)}{t^{-(1 + \frac{b}{a})}} = C, \quad \text{a.s.} \tag{1.3}
  \]
  where $C$ is an almost surely finite proper random variable.
- In Theorem 4.2, corresponding to $a > 0$, the solution $X$ is shown to obey (1.2).
- In Theorem 4.3, corresponding to $a = 0$ and $b > 0$, the solution grows at a rate which is faster than the polynomial growth of (1.3) yet slower than the exponential growth given by (1.2).
- In Theorem 4.4, corresponding to $a = 0$ and $b < 0$, the solution $X$ is recurrent on $\mathbb{R}$ and its largest fluctuations are described by a result reminiscent of the Law of the Iterated Logarithm.

In analysing the solution of the stochastic equation it is helpful first to ask how the underlying deterministic equation behaves asymptotically. This deterministic equation is attained from (1.1) by letting $\sigma = 0$. The solution of this underlying equation (which corresponds to the mean of $X$) may be expressed in terms
of confluent hypergeometric, modified Bessel and Bessel functions. Properties of these special functions are well–documented, c.f. e.g. [1, 15, 16]. An associated differential resolvent may also be decomposed in terms of these special functions. In Theorems 4.1, 4.2 and 4.3 the asymptotic behaviour of the solution $X$ may then be shown to mirror that of the deterministic equations, i.e. the asymptotic rates of growth or decay of the solutions of the deterministic equations are preserved under the addition of a stochastic perturbation. However, as Theorem 3.1 demonstrates the stochastic perturbation can for particular values of the parameters produce asymptotic behaviour which is distinct from that of the solution of the associated deterministic equation. The analysis is achieved via this decomposition of the resolvent and a variation of parameters formula.

Many of the asymptotic results concern pathwise behaviour. However, many of the growth results also hold true in mean or in mean square. Furthermore, in the main case where there are fluctuations (i.e., when $a < 0$ and $a + b \leq 0$), we show that the autocovariance function of the process $X$ decays at a polynomial rate in time, i.e. for any fixed $t > 0$, 

$$\lim_{\Delta \to \infty} \frac{\gamma_t(\Delta)}{\Delta^{-1-a}} = c_t \in (0, \infty),$$

where $\gamma_t(\cdot) = \text{Cov}[X(t), X(t + \Delta)]$. Thus $X$ may be viewed as possessing long memory, in the sense that for any fixed $t$,

$$\int_0^\infty \gamma_t,\gamma_t(\Delta) d\Delta = +\infty, \quad a < 0, \quad b > 0, \quad a + b < 0.$$ 

This result is all the more striking as Theorem 3.1 proves that $X$ is asymptotically equal to a process whose autocovariance function decays exponentially quickly, i.e. a “short memory” process. Moreover, it can be shown that $X$ is transiently non–stationary, and has limiting autocovariance function equal to that of the stationary Ornstein Uhlenbeck process to which it converges pathwise. We comment more on the this result in the next section.

1.2. Motivation for the work. One of the motivations of this work is to develop a parameterised stochastic functional differential equation whose asymptotic behaviour is completely characterised, as such an equation can act as a test equation for simulation methods for SFDEs. Another mathematical motivation is to demonstrate that the general approach of admissibility theory developed in [3] can generate the same results as the special function theory outlined here (at least in some cases), thus supporting the conjecture that it can prove a sharp tool in studying the asymptotic behaviour of linear, quasilinear or affine stochastic functional differential equation.

However, one of the main interests in examining this equation is to gain insight into some features of price dynamics in inefficient financial markets. First, we argue that (1.1) may be considered as a simple model of such a market. Suppose that there is a class of technical analysts who compare the current returns of a risky asset with the average of historical returns. This leads to an instantaneous excess demand of

$$\alpha \left( X_1(t) - \frac{1}{1+t} \int_{-1}^t X_1(s) \, ds \right)$$

per unit time at time $t$. A class of feedback traders compare the returns to a reference level $\bar{X}$, leading to an instantaneous excess demand of

$$\beta(X_1(t) - \bar{X})$$

per unit time at time $t$. Unplanned demand by the traders arises from "news", where the news in each period is independent of that in previous periods. The
contribution of this news to overall excess demand is $\sigma(B(t_2) - B(t_1))$ over the

time interval $[t_1, t_2]$, where $B$ is a standard one-dimensional Brownian motion. If we

presume that returns respond linearly to the excess demand of the market, then

$X_1(t) = \psi_1(t)$ for $t \in [-1, 0]$. The price of the risky asset at time

$t \geq 0$ is denoted by $S(t)$ and defined by

$$dS(t) = \mu S(t) \, dt + S(t) \, dX_1(t), \quad t \geq 0$$

with $S(0) = s_0$. Now define $X(t) = X_1(t) - \bar{X}$ for $t \geq 0$ and $\psi(t) = \psi_1(t) - \bar{X}$ for

$t \in [-1, 0]$. Then $X$ obeys (1.1) with $a = \alpha + \beta$ and $b = -\alpha$.

Motivation and literature for such models, as well as alternative inefficient market

models may be found in [6], in which a market with finite memory is considered.

In common with [6], in this work $X_1$ can grow to plus or minus infinity, with both

events being possible. In terms of the mathematics, this happens if and only if

- $a > 0$;
- $a < 0$ and $a + b > 0$;
- $a = 0$ and $b > 0$.

From an economic perspective, the first case corresponds to the situation where

the feedback traders chase trends ($\alpha > 0$) and either dominate the fundamental

investors, who have mean–reverting expectations about price movements ($\alpha + \beta > 0$,

$\beta < 0$) or both classes of agents have trend chasing type expectations ($\alpha > 0$,

$\beta > 0$). The other two cases, while interesting mathematically, are less likely

within the scope of the model: the second case requires $\beta > 0$, which implies that

fundamental investors are bullish about higher than average returns, but $\alpha + \beta < 0$,

which indicates these investors dominate the technical traders, who now have mean

reverting expectations about returns. Nonetheless, this case serves to demonstrate

that if at least one of the investor classes believes that high and rising returns are

a signal of higher returns in the future, and that that class of agent dominates,

then bubbles are likely outcomes. The third case occurs if the two classes of traders

have equal strength, $\beta + \alpha = 0$, with the technical traders having mean reverting

expectations, and the fundamental investors being bullish about higher than average

returns ($\alpha < 0$, $\beta > 0$).

In all these cases, the limiting random variable is path dependent, so it follows

that the initial behaviour of the market determines whether there is a bubble or a

crash. This picture is consistent with the mechanism proposed for the formation

of bubbles with those formed in models of mimetic contagion, first introduced by

Orl´eon [15].

From a modelling and time series perspective, the behaviour in the ”non–bubble”

case when $a < 0$ and $a + b \leq 0$, or $a = 0$ and $b < 0$ is also of interest. The former

corresponds to the situation where $\alpha + \beta < 0$, $\beta \leq 0$, in which the fundamental

investors have mean reverting expectations, and either dominate the technical

investors (if they have trend chasing expectations) or the technical traders also have

mean reverting expectations themselves. In this case as we observed the size of the

largest fluctuations of the process is given by $\sigma/\sqrt{2|\alpha + \beta|}$. Thus as the process

is actually mean reverting in this scenario it is in the interests of the trend chasing

traders to ensure that $\alpha + \beta$ is as close to zero as possible so that the the process

undergoes as large fluctuations as possible. This phenomenon is observed in financial

markets, i.e. when there is a large proportion of uninformed investors in a market

then the volatility of the market tends to be higher than in their absence c.f. e.g.
De Long et al. [11]. If however the uninformed investors where to force $\alpha + \beta > 0$ then this, as already observed, will result in the formation of an uncontrollable bubble.

The case when $a = 0$ and $b < 0$ is consistent with solutions obeying the law of the iterated logarithm, and so may be roughly associated with Gaussian processes that are non-stationary, but possess stationary increments. However, in the former case, not only (as we have already pointed out) is $X$ asymptotically indistinguishable from an asymptotically stationary process, it can be shown that $X$ itself is asymptotically stationary (or transiently non-stationary), i.e.

$$
\lim_{t \to \infty} \text{Cov}(X(t), X(t + \Delta)) = \gamma(\Delta),
$$

for some function $\gamma : \mathbb{R} \to \mathbb{R}$. Moreover this limiting autocovariance, as a function of $\Delta$, decays exponentially and so is indicative of a short memory process.

At the same time, we have already seen that when $t$ is fixed and $\Delta \to \infty$, then $\Delta \to \text{Cov}(X(t), X(t + \Delta))$ tends to zero at a polynomial rate, and is indeed non-integrable when $b > 0$. In a sense therefore, the process exhibits “long-memory” and “short-memory” characteristics. Of course, it is not unheard of that reversing the order of these limits leads to different answers, and while this is an interesting mathematical example of this phenomenon, it is otherwise not noteworthy. However, given that there is considerable debate among empiricists in finance concerning the presence or absence of long memory in certain financial time series, it is interesting to note that this paper presents an asymptotically stationary process in a (highly simplified, indeed unrealistic) market model, which also possesses somewhat ambiguous memory properties.

1.3. Organisation of the paper and mathematical preliminaries. The article is organised as follows. In this section (Section 1.3), we formally introduce the equation under scrutiny and define some notation. Section 2 gives a detailed description of the decomposition of the solution of the deterministic equation into special functions, and in particular details the differing functions which are used depending on the values of $a$ and $b$. In order to make our presentation self-contained, various properties of these functions which are needed in the analysis of the asymptotic behaviour, are listed. Section 3 deals with recurrent dynamics of $X$, with Subsection 3.1 giving results on the almost sure pathwise asymptotic behaviour of the process, while Subsection 3.2 discusses the memory properties when $X$ has these recurrent dynamics. Section 4 gives results concerning transient dynamical behaviour of the process. Proofs of the results are deferred to Section 5.1.

Let $\mathbb{R}$ denote the real numbers. If $x \in \mathbb{R}$, then $[x]$, or the ceiling of $x$ is the smallest integer greater than or equal to $x \in \mathbb{R}$. The Wronskian for two any functions $x_1$ and $x_2$, is defined as $W(t) = x_1(t)x'_2(t) - x_1'(t)x_2(t)$. We define the Gamma function $\Gamma : \mathbb{C} \to \mathbb{C}$ according to $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s} ds$ for $\Re(z) > 0$.

When $\Re(z) \leq 0$, $\Gamma(z)$ is defined by analytic continuation. We employ the standard Landau notation: if $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{R}$, we write $f = O(g)$ as $|z| \to \infty$ if there exist $z_0 > 0$ and $M > 0$ such that $|f(z)| \leq M|g(z)|$ for all $|z| > z_0$, while $f \sim g$ as $z \to z_0$ is equivalent to $\lim_{z \to z_0} f(z)/g(z) = 1$.

Let us fix a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}(t)\}_{t \geq 0}$ satisfying the usual conditions and let $B = \{B(t) : t \geq 0\}$ be a one-dimensional Brownian motion adapted to $\{\mathcal{F}(t)\}_{t \geq 0}$ on this space. The probability measure induces an expectation $\mathbb{E}$ in the usual manner, in the sense that if $Y$ is an $\mathcal{F}$-measurable random variable such that $\int_0^\infty Y(\omega) \, dp(\omega) < +\infty$, then $\mathbb{E}[Y] = \int_0^\infty Y(\omega) \, dp(\omega)$. In this paper, the abbreviation a.s. stands for “almost sure” or “almost surely”.

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\section*{Linear SFDE with Average Functional}
We consider the affine scalar stochastic functional differential equation with an average functional
\[
dX(t) = \left(ax(t) + \frac{b}{1 + t}\int_{-1}^{t} X(s) \, ds\right) \, dt + \sigma \, dB(t), \quad t \geq 0; \tag{1.4a}
\]
\[
X(t) = \psi(t), \quad t \in [-1, 0], \tag{1.4b}
\]
Here \(\sigma > 0\), \(a, b \in \mathbb{R}\) and \(\psi \in C([-1,0], \mathbb{R})\). Then by Berger and Mizel \cite{8} or Mao \cite[Theorem 2.3.1]{13} there is a unique continuous adapted process which obeys (1.4), hereinafter referred to as the solution of (1.4) and denoted \(X\). There is also a unique continuous solution of
\[
x'(t) = ax(t) + \frac{b}{1 + t}\int_{-1}^{t} x(s) \, ds, \quad t \geq 0, \tag{1.5a}
\]
\[
x(t) = \psi(t), \quad t \in [-1, 0]. \tag{1.5b}
\]
The differential resolvent \(r\) associated with (1.5) is defined according to
\[
\frac{\partial r}{\partial t}(t, s) = ar(t, s) + \frac{b}{1 + t}\int_{s}^{t} r(u, s) \, du, \quad t > s; \tag{1.6a}
\]
\[
r(t, s) = 0, \quad t < s; \quad r(s, s) = 1. \tag{1.6b}
\]
Then with \(x\) being the solution of (1.5), the solution of (1.4) has a variation of parameters representation.

**Lemma 1.1.** Suppose that \(\psi \in C([-1,0]; \mathbb{R})\). Let \(X\) be the unique solution of (1.4), \(x\) the unique solution of (1.5) and \(r\) the unique solution of (1.6). Then \(X\) is a Gaussian process and obeys
\[
X(t) = x(t) + \sigma \int_{0}^{t} r(t, s) \, dB(s), \quad t \geq 0. \tag{1.7}
\]
A proof of the validity of this representation is provided in Section 5.

Using the representation (1.7) for \(X\), we deduce formulae for the mean and autocovariance of \(X\). By considering for \(t \geq 0\) fixed and \(\tau \geq 0\) the process
\[
M(\tau) = \int_{0}^{\tau} r(t, s) \, dB(s), \quad \tau \geq 0,
\]
we can see that \(M\) is a martingale and moreover a Gaussian process, so therefore \(X(t) = x(t) + M(t)\) is Gaussian distributed. Since \(\mathbb{E}[M(\tau)^2] < +\infty\) for all \(\tau \geq 0\), we have that \(\mathbb{E}[M(\tau)] = 0\) for all \(\tau \geq 0\), and hence \(\mathbb{E}[M(t)] = 0\). Hence
\[
\mathbb{E}[X(t)] = x(t), \quad t \geq 0. \tag{1.8}
\]
Since \(\mathbb{E}[X(t)^2]\) is finite for all \(t \geq 0\), it follows that \(\text{Cov}(X(t), X(t + \Delta))\) is well-defined for all \(t \geq 0\) and \(\Delta \geq 0\). We also see that
\[
\text{Cov}(X(t), X(t + \Delta)) = \sigma^2 \mathbb{E}[M(t)M(t + \Delta)]
\]
\[
= \sigma^2 \mathbb{E}\left[\int_{0}^{t+\Delta} r(t, s) \chi_{[0,t]}(s) \, dB(s) \int_{0}^{t+\Delta} r(t + \Delta, s) \, dB(s)\right].
\]
Considering \(t\) and \(\Delta\) as fixed, we may apply Itô’s isometry to obtain the variance of
\[
V_1 := \int_{0}^{t+\Delta} r(t + \Delta, s) \, dB(s), \quad V_2 := \int_{0}^{t+\Delta} r(t, s) \chi_{[0,t]}(s) \, dB(s)
\]
and
\[
\int_{0}^{t+\Delta} \{r(t, s) \chi_{[0,t]} + r(t + \Delta, s)\} \, dB(s) = V_1 + V_2.
\]
and using the fact that \(2 \text{Cov}(V_1, V_2) = \text{Var}[V_1 + V_2] - \text{Var}[V_1] - \text{Var}[V_2]\), we obtain

\[
\text{Cov}(X(t), X(t + \Delta)) = \sigma^2 \int_0^t r(t, s)r(t + \Delta, s) \, ds, \quad t \geq 0, \quad \Delta \geq 0. \tag{1.9}
\]

We have already seen that mean and resolvent obey functional differential equations involving an average functional. This also holds true for the autocovariance function, and the result is recorded below.

**Proposition 1.** Suppose that \(\psi \in C([−1, 0]; \mathbb{R})\). Let \(X\) be the unique solution of (1.4) and \(r\) the unique solution of (1.6). Fix \(t \geq 0\) and define

\[
\gamma_t(\Delta) := \sigma^2 \int_0^t r(t, s)r(t + \Delta, s) \, ds, \quad \Delta \geq -t. \tag{1.10}
\]

If \(\Delta \geq 0\), then \(\gamma_t(\Delta) = \text{Cov}(X(t), X(t + \Delta))\) and,

\[
\gamma_t'(\Delta) = a\gamma_t(\Delta) + \frac{b}{1 + t + \Delta} \int_{-t}^{\Delta} \gamma_t(w) \, dw, \quad \Delta \geq 0, \tag{1.11}
\]

\[
\gamma_t'(\Delta) = a\gamma_t(\Delta) + \frac{b}{1 + t + \Delta} \int_{-t}^{\Delta} \gamma_t(w) \, dw + \sigma^2 r(t, t + \Delta), \quad -t \leq \Delta < 0. \tag{1.12}
\]

This result is proven in Section 5.1. The differential equation (1.11) may be thought of as a Yule–Walker–type representation of the autocovariance function.

In this work, we could equally have studied the equation

\[
dX(t) = \left(aX(t) + \frac{b}{t} \int_0^t X(s) \, ds\right) \, dt + \sigma \, dB(t), \quad t \geq 0; \quad X(0) = \xi.
\]

However, this equation is more delicate to analyse, on account of the potential singularity in the average functional at \(t = 0\). We obviate such complications by considering an equation with an initial history on a non–trivial compact interval. Taking this to be \([-1, 0]\) leads to (1.4).

### 2. Formulae and Asymptotic Behaviour of Solutions of (1.5) and (1.6)

The solution of (1.5) can be rewritten as the solution of an initial value problems for a second–order differential equation. The equation is

\[
x''(t) + \left(\frac{1}{1 + t} - a\right) x'(t) - \frac{a + b}{1 + t} x(t) = 0, \quad t \geq 0; \tag{2.1a}
\]

\[
x(0) = \psi(0), \quad x'(0) = a\psi(0) + b \int_{-1}^{0} \psi(s) \, ds. \tag{2.1b}
\]

There are three cases to consider: \(a < 0, a > 0\) and \(a = 0\). We discuss each case and their subcases, conditioned by \(b\), in turn. In the case when \(b = 0\), the stochastic differential equation (1.4) reduces to an Ornstein–Uhlenbeck SDE, and so the behaviour of \(x, r\), and indeed \(X\), are well–understood. Therefore, we exclude the case \(b = 0\) from our analysis. In the exposition below the asymptotic behaviour of the solution of (6.2.1) is deduced from the known asymptotic behaviour of certain functions. It is here observed however that a general theory concerning the asymptotic behaviour of linear second order equations with analytic coefficients may be found in e.g. [15] Ch. 7.1 and 7.2.
2.1. \( a < 0 \). When \( a < 0 \), the solution of (2.1) can be expressed in terms of two linearly independent confluent hypergeometric functions, according to:

\[
x(t) = c_1 r_1(t) + c_2 r_2(t) \quad \text{for } a < 0 \text{ and } b/a \not\in \{1, 2, \ldots\}
\]

(2.2)

where

\[
r_1(t) = e^{at} U(-b/a, 1, -a(1 + t)), \quad r_2(t) = e^{at} M(-b/a, 1, -a(1 + t)).
\]

Here \( U(\alpha, \beta, \cdot) \) and \( M(\alpha, \beta, \cdot) \) are two linearly independent solutions of Kummer’s differential equation which is given by

\[
z w''(z) + (\beta - z) w'(z) - \alpha w(z) = 0,
\]

where \( \alpha \) and \( \beta \) are real and \( z \) a complex number. \( M \) is sometimes referred to as Kummer’s function of the first kind or a confluent hypergeometric function, while \( U \) is sometimes called the Tricomi confluent hypergeometric function. See \([16] \) Chapter 13.2.1 and following sections.

To see that \( r_1 \) and \( r_2 \) are solutions of (2.1a), observe that as \( z \mapsto U(\alpha, \beta, z) \) is a solution of Kummer’s equation then \( t \mapsto U(-b/a, 1, -a(1 + t)) \) satisfies

\[
-a(1 + t)U''(-b/a, 1, -a(1 + t)) + (1 + a(1 + t))U'(-b/a, 1, -a(1 + t)) + \frac{b}{a} U(-b/a, 1, -a(1 + t)) = 0.
\]

Therefore

\[
r_1(t) + \left( \frac{1}{1 + t} - a \right) r_1'(t) - a + \frac{b}{1 + t} r_1(1 + t) = 0,
\]

as required. A similar calculation shows that \( r_2 \) is a solution of (2.1a).

As we are chiefly interested in the long–run behaviour of \( X \) it is necessary to have information on the asymptotic behaviour of both \( U \) and \( M \). This is given by \([11] \) 13.1.4 & 13.1.8], or

\[
M(\alpha, \beta, t) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^{\alpha t} t^{\alpha - \beta}[1 + O(t^{-1})], \quad \text{as } t \to \infty,
\]

(2.3a)

(2.3b)

\[
U(\alpha, \beta, t) = t^{-\alpha}[1 + O(t^{-1})], \quad \text{as } t \to \infty.
\]

This immediately gives asymptotic information about \( r_1 \) and \( r_2 \):

\[
r_1(t) \sim e^{at} |a|^{-b/a} b/a, \quad \text{as } t \to \infty,
\]

(2.4)

\[
r_2(t) \sim \frac{1}{\Gamma(-b/a)} e^{-a |a|^{-b/a} - b/a}, \quad \text{as } t \to \infty
\]

(2.5)

To determine the asymptotic behaviour of \( x \), we need values for \( c_1 \) and \( c_2 \) in (2.2) in terms of the initial conditions of (2.1a). As usual, by using (2.1b), these
values are obtained by solving
\[ c_1 r_1(0) + c_2 r_2(0) = \psi(0), \quad c_1 r'_1(0) + c_2 r'_2(0) = a\psi(0) + b \int_{-1}^{0} \psi(s) \, ds. \quad (2.6) \]

Clearly, these values can be expressed in terms of the Wronskian of \( r_1 \) and \( r_2 \), evaluated at \( t = 0 \), as well as the derivatives of \( r_1 \) and \( r_2 \). Since \( r_1 \) and \( r_2 \) depend on \( M \) and \( U \), it is of value to have a general formula for the Wronskian and the derivatives of \( U \) and \( M \). A formula for the Wronskian, \( W \), of \( M \) and \( U \) is given by [16, 13.2.34]:
\[ W(M(\alpha, \beta, z), U(\alpha, \beta, z)) = -\Gamma(\beta)z^{-\beta}e^\beta/\Gamma(\alpha). \quad (2.7) \]

Expressions for the derivatives of \( U \) and \( M \) are given by [16, 13.3.15 & 13.3.22]:
\[ M'(\alpha, \beta, z) = \frac{\alpha}{\beta}M(\alpha + 1, \beta + 1, z), \quad U'(\alpha, \beta, z) = -\alpha U(\alpha + 1, \beta + 1, z). \quad (2.8) \]

Using these results, we obtain the following formulae for \( c_1 \) and \( c_2 \):
\[ c_1 = \Gamma(-\frac{b}{a})e^{\frac{b}{a}}\left(\psi(0)M(1 - \frac{b}{a}, 2, -a) - \int_{-1}^{0} \psi(s) \, ds M(-\frac{b}{a}, 1, -a)\right), \]
\[ c_2 = \Gamma(-\frac{b}{a})e^{\frac{b}{a}}\left(\psi(0)U(1 - \frac{b}{a}, 2, -a) + \int_{-1}^{0} \psi(s) \, ds U(-\frac{b}{a}, 1, -a)\right). \quad (2.9) \]

We now consider the case when \( b/a \in \{1, 2, \ldots \} \). As alluded to earlier, in this case \( t \mapsto M(-b/a, 1, -a(1 + t)) \) and \( t \mapsto U(-b/a, 1, -a(1 + t)) \) are linearly dependent, and therefore the representation \((2.2)\) for \( x \) is not valid. It is however known that \( t \mapsto U(-b/a, 1, -a(1 + t)) \) is a polynomial in \( |a(1 + t)| \) of degree \( b/a \). We even have an explicit formula for this polynomial. Indeed, for \( n \in \{0, 1, 2, \ldots \}, \) we have from [16, 13.2.7] that
\[ U(-n, 1, z) = (-1)^n \sum_{j=0}^{n} \frac{(n!)^2}{(n-j)!(j!)^2}(-z)^j. \quad (2.10) \]

Note that \( z \mapsto U(-n, 1, z) \) is analytic, and so its (at most \( b/a \)) zeros are isolated. Therefore, the zeros of the real–valued polynomial \( t \mapsto U(-b/a, 1, -a(1 + t)) \) are also isolated.

Suppose now we take \( r_1(t) = e^{\alpha t}U(-b/a, 1, -a(1 + t)) \) for \( t \geq 0 \). We know from standard theory (cf., e.g. [9]) that there exists a second solution, \( \tilde{r}_2 \), of \((2.1a)\) which is linearly independent of \( r_1 \). Next, by Abel’s Theorem (cf., e.g. [9] Ch.3.3.2), the Wronskian of \( r_1 \) and \( \tilde{r}_2 \), which is associated with \((2.1a)\) obeys
\[ W(a, b, t) = W(a, b, 0)e^{\alpha t}(1 + t)^{-1}, \quad t \geq 0, \]
where \( W(a, b, 0) = r_1(0)\tilde{r}_2'(0) - r_1'(0)\tilde{r}_2(0) \neq 0. \)

This expression is equivalent to
\[ r_1(t)\tilde{r}_2'(t) - r_1'(t)\tilde{r}_2(t) = W(a, b, 0)e^{\alpha t}(1 + t)^{-1}, \quad t \geq 0. \]

We now wish to find a representation for \( \tilde{r}_2 \) which allows us to deduce its asymptotic properties.

Notice that because \( r_1 \) has finitely many zeros, it must have a maximal real zero. Let \( t_1 = 1 + \max\{0, \sup\{t \in \mathbb{R} : r_1(t) = 0\}\} \), where we define \( \sup\{t \in \mathbb{R} : r_1(t) = 0\} = -\infty \) if \( r_1(t) \neq 0 \) for all \( t \geq 0 \). Then for \( t \geq t_1 \) we have
\[ \tilde{r}_2(t) - \frac{r_1'(t)}{r_1(t)}\tilde{r}_2(t) = W(a, b, 0)e^{\alpha t}(1 + t)^{-1}r_1(t), \quad t \geq t_1. \quad (2.11) \]

Since \( r_1(t) \neq 0 \) for all \( t \geq t_1 \), we have that \( t \mapsto r_1'(t)/r_1(t) \) and \( t \mapsto e^{\alpha t}(1 + t)^{-1}/r_1(t) \) are continuous on \([t_1, \infty)\), and therefore we may solve \((2.11)\) for \( \tilde{r}_2 \) to obtain the
following representation for $\tilde{r}_2$ on $[t_1, \infty)$:

$$\tilde{r}_2(t) = r_1(t) \frac{\tilde{r}_1(t)}{r_1(t)} + W(a, b, 0) r_1(t) \int_{t_1}^{t} \frac{e^{as}(1+s)^{-1}}{r_1(s)} ds, \quad t \geq t_1. \quad (2.12)$$

Since $t_1$ exceeds the maximal zero of $r_1$, the integral on the right hand side of (2.12) is well-defined for $t \geq t_1$. Moreover, using l'Hôpital's rule together with (2.3b) or (2.10), one may show that

$$\lim_{t \to \infty} t^{1+\frac{b}{a}} \tilde{r}_2(t) = W(a, b, 0)|a|^{-1-\frac{b}{a}}, \quad a < 0, \quad b/a \notin \{1, 2, \ldots\}. \quad (2.13)$$

Note that this recovers the asymptotic behaviour of $r_2$ above in (2.5) the case $a < 0$ and $b/a \notin \{1, 2, \ldots\}$.

It is also useful to determine some asymptotic information about $\tilde{r}_2$. Notice that $r_1(t) \sim e^{atb/a}|a|^{b/a}$ as $t \to \infty$. Also we have

$$\frac{r_1'(t)}{r_1(t)} - a = \frac{r_1'(t) - ar_1(t)}{r_1(t)} = \frac{-aU'(-b/a, 1, -a(1+t))}{U(-b/a, 1, -a(1+t))}, \quad t \geq t_1,$$

so using the fact that $t \mapsto U(-b/a, 1, -a(1+t))$ is a polynomial of degree $b/a \in \mathbb{N}$, we have that $\lim_{t \to \infty} r_1'(t)/r_1(t) = a$. By (2.13), it follows that there is $t_1' > 0$ such that $\tilde{r}_2(t) \neq 0$ for all $t \geq t_1'$. Let $t_1' = \max(t_1, t_1)$. Then we may rewrite (2.11) for $t \geq t_1'$ to get

$$\frac{\tilde{r}_2'(t)}{\tilde{r}_2(t)} = r_1'(t) r_1(t) + W(a, b, 0) e^{at} (1+t)^{-1} \frac{W(a, b, 0)}{r_1(t) \tilde{r}_2(t)}.$$

Using the fact that $r_1(t) \sim e^{atb/a}|a|^{b/a}$ as $t \to \infty$ together with (2.13) shows that the second term has limit $|a| = -a$, and therefore

$$\lim_{t \to \infty} \frac{\tilde{r}_2'(t)}{\tilde{r}_2(t)} = 0. \quad (2.14)$$

Finally, we see that the solution of (2.1) is given by

$$x(t) = \tilde{c}_1 r_1(t) + \tilde{c}_2 \tilde{r}_2(t), \quad t \geq 0, \quad \text{for } a < 0 \text{ and } b/a \in \{1, 2, \ldots\} \quad (2.15)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are found using (2.11). Note that $\tilde{c}_2$ is known entirely in terms of $r_1$ and its dependence on $\tilde{r}_2$ is solely through the value of the Wronskian, because

$$\tilde{c}_2 = \frac{1}{W(a, b, 0)} \left( b \psi(0) U(-b/a, 2, |a|) + b \int_{-1}^{0} \psi(s) ds U(-b/a, 1, |a|) \right).$$

Note also that for $b = 0$, (2.15) reduces to $x(t) = \psi(0)e^{at}$.

We now turn our attention to the representation of the resolvent $r$ defined by (1.6). In a manner similar to the treatment of the solution $x$ of (1.5), it can be shown for every fixed $s \geq 0$, the solution $t \mapsto r(t,s) =: r_s(t)$ of the resolvent equation (1.6) is also the solution of the second order differential equation

$$r_s''(t) + \left( \frac{1}{1 + t} - a \right) r'_s(t) - \frac{a + b}{1 + t} r_s(t) = 0, \quad t \geq s, \quad (2.16)$$

with initial conditions $r_s(s) = 1$ and $r_s'(s) = a$. It is to be noted that (2.16) is the same differential equation as (2.1a) apart from the fact that the argument of the solution is restricted to the interval $[s, \infty)$, a subinterval of the interval of existence of the equation (2.1a). Therefore, $r(t,s) = r_s(t)$ can be represented as a linear combination of the linearly independent solutions of (2.1a) according to

$$r(t,s) = \begin{cases} d_1(s) r_1(t) + d_2(s) \tilde{r}_2(t), & t \geq s \geq 0, \quad a < 0, \quad b/a \notin \{1, 2, \ldots\}, \\
\tilde{d}_1(s) r_1(t) + \tilde{d}_2(s) \tilde{r}_2(t), & t \geq s \geq 0, \quad a < 0, \quad b/a \in \{1, 2, \ldots\}. \end{cases} \quad (2.17)$$

The multipliers $d_1$, $d_2$ etc are $s$-dependent, because initial data for the problem (2.16) is specified at $s$. Considering first the non-degenerate case when $b/a \notin \{1, 2, \ldots\}$
\[ d_1(s)_r(s) + d_2(s)_r(s) = 1, \quad d_1(s)_r(s) + d_2(s)_r(s) = a. \] (2.18)

From these equations, and using (2.7) and (2.8), we obtain the formulae
\[ d_1(s) = \Gamma(-\frac{b}{a})(1 + s)e^{a b} M(1 - \frac{b}{a}, 2, -a(1 + s)), \] (2.19)
\[ d_2(s) = \Gamma(-\frac{b}{a})(1 + s)e^{a b} U(1 - \frac{b}{a}, 2, -a(1 + s)). \] (2.20)

Using the fact that \( \Gamma(1 - b/a) = -b/a \Gamma(-b/a) \) and employing (2.3), we get
\[ d_1(s) \sim b \frac{\Gamma(-\frac{b}{a})}{\Gamma(1 - \frac{b}{a})} |a|^{-1 - \frac{b}{a}} e^{-a s} s^{-\frac{b}{a}}, \quad \text{as } s \to \infty, \] (2.21)
\[ d_2(s) \sim \Gamma(-\frac{b}{a})e^{a b}|a|^{-1} s^{\frac{b}{a}}, \quad \text{as } s \to \infty. \] (2.22)

In the degenerate case when \( b/a \in \{1, 2, \ldots \} \), we have
\[ \tilde{d}_1(s) = \frac{r_2(s) - ar_2(s)}{W(a, b, 0)e^{a s}(1 + s)^{-\frac{b}{a}}} = -\frac{a}{W(a, b, 0)} r_2(s)(1 + s)e^{-a s} \left( 1 + \frac{1}{-a r_2(s)} \right), \]
\[ \tilde{d}_2(s) = \frac{r_1(s) - ar_1(s)}{W(a, b, 0)e^{a s}(1 + s)^{-\frac{b}{a}}} = \frac{1}{W(a, b, 0)} b(1 + s)U(1 - \frac{b}{a}, 2, -a(1 + s)). \]

We notice by (2.13) and (2.14) that
\[ \tilde{d}_1(s) \sim |a|^{-\frac{b}{a}} s^{-\frac{b}{a}} e^{-a s}, \quad \text{as } s \to \infty, \] (2.23)

which mirrors the asymptotic behaviour for \( d_1 \) in (2.21) in the non–degenerate case.

As to the asymptotic behaviour of \( d_2 \), we may use (2.3b) to obtain
\[ \tilde{d}_2(s) \sim \frac{1}{W(a, b, 0)} b|a|^{-1} s^{\frac{b}{a}}, \quad \text{as } s \to \infty, \] (2.24)

and so \( \tilde{d}_2 \) has the same asymptotic behaviour as \( d_2 \) given in (2.22) in the non–

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Using the fact that \( \text{Cov}(X(t), X(t + \Delta)) \) obeys (1.9) for \( t \geq 0 \) and \( \Delta \geq 0 \), and \( r(t, s) \) is given by (2.17), we have
\[ \text{Cov}(X(t), X(t + \Delta)) = \begin{cases} 
c_1 r_1(t + \Delta) + c_2, & a < 0, \quad b/a \not\in \{1, 2, \ldots \}, 
c_1 r_1(t + \Delta) + \tilde{c}_2 r_2(t + \Delta), & a < 0, \quad b/a \in \{1, 2, \ldots \}, 
\end{cases} \] (2.25)

for \( t \geq 0 \) and \( \Delta \geq 0 \), where
\[ c_{1,t} = \sigma^2 \int_0^t r(t, s)d_1(s) ds, \quad c_{2,t} = \sigma^2 \int_0^t r(t, s)d_2(s) ds, \] (2.26)
and
\[ \tilde{c}_{1,t} = \sigma^2 \int_0^t r(t, s)\tilde{d}_1(s) ds, \quad \tilde{c}_{2,t} = \sigma^2 \int_0^t r(t, s)\tilde{d}_2(s) ds. \] (2.27)

In order that certain limiting constants in our analysis are non–zero, we find it useful to employ the following integral representation of \( U \):
\[ U(\alpha, \beta, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tu^\alpha}(1 + u)^{\beta - \alpha - 1} du, \quad \alpha > 0. \] (2.28)

It appears as [10] 3.4.4.
2.2. $a > 0$. When $a > 0$, the solution of (2.1a) can be expressed in terms of confluent hypergeometric functions, according to:

$$x(t) = c_3r_3(t) + c_4r_4(t)$$

for $a > 0$ and $b/a \notin \{-1, -2, \ldots\}$ (2.29)

where

$$r_3(t) = U(1 + \frac{b}{a}, 1, a(1 + t)), \quad r_4(t) = M(1 + \frac{b}{a}, 1, a(1 + t)).$$

(2.30)

Using (2.30), we get

$$r_3(t) \sim a^{-1 - \frac{b}{2}}t^{-1 - \frac{b}{2}}, \quad \text{as } t \to \infty, \; a > 0,$$

(2.31)

and using (2.30), we obtain

$$r_4(t) \sim \frac{1}{1 + \frac{b}{a}}e^{\frac{b}{a}t}t^\frac{b}{2}, \quad \text{as } t \to \infty, \; a > 0, \; b/a \notin \{-1, -2, \ldots\}$$

(2.32)

The initial conditions (2.1b) can be used to determine $c_3$ and $c_4$; the relevant formulae are:

$$c_3 = \Gamma(1 + \frac{b}{a})e^{-a} \left( b\psi(0)M(1 + \frac{b}{a}, 2, a) - b \int_{-1}^{0} \psi(s) \, ds \, M(1 + \frac{b}{a}, 1, a) \right),$$

$$c_4 = \Gamma(1 + \frac{b}{a})e^{-a} \left( a\psi(0)U(1 + \frac{b}{a}, 2, a) + b \int_{-1}^{0} \psi(s) \, ds \, U(1 + \frac{b}{a}, 1, a) \right).$$

(2.33)

One may verify, as before, that $r_3$ and $r_4$ solve (2.1a). In the determination of these formulae for $c_3$ and $c_4$, we have used the fact that one may deduce from Kummer’s differential equation the identities [16] 13.3.13 & 13.3.14, which are

$$(\alpha + 1)zM(\alpha + 2, \beta + 2, z) + (\beta + 1)(\beta - z)M(\alpha + 1, \beta + 1, z) - \beta(\beta + 1)M(\alpha, \beta, z) = 0$$

(2.34)

$$(\alpha + 1)zU(\alpha + 2, \beta + 2, z) + (z - \beta)U(\alpha + 1, \beta + 1, z) - U(\alpha, \beta, z) = 0.$$  

(2.35)

Moreover, letting $\beta \to 0$ in (2.34) and (2.35) gives

$$(\alpha + 1)zM(\alpha + 2, 2, z) - zM(\alpha + 1, 1, z) = 0,$$

(2.36)

$$(\alpha + 1)zU(\alpha + 2, 2, z) + zU(\alpha + 1, 1, z) - U(\alpha + 1, 2, z) = 0.$$  

(2.37)

as [16] 13.2.5 in conjunction with [16] 5.2.1 gives $\lim_{\beta \to 0} \beta M(\alpha, \beta, z) = azM(\alpha + 1, 2, z)$ and [16] 13.2.11 gives $U(\alpha, 0, z) = zU(\alpha + 1, 2, z)$.

Again, for certain values of $a$ and $b$ (i.e., if $-b/a \in \{1, 2, 3, \ldots\}$), the two functions on the right-hand side of (2.2) are no longer linearly independent. Nevertheless the second-order equation (2.1a) has two linearly independent solutions $r_3$ (still given by (2.30)) and $\tilde{r}_4$, and so the solution of (1.5) obeys

$$x(t) = \tilde{c}_3r_3(t) + \tilde{c}_4\tilde{r}_4(t), \quad t \geq 0, \quad \text{for } a > 0 \text{ and } b/a \notin \{-1, -2, \ldots\}.$$  

(2.38)

By (2.31), $r_3(t) > 0$ for all $t$ sufficiently large. Therefore we may define $t_2 = 1 + \max(0, \sup \{t \in \mathbb{R} : r_3(t) = 0\})$, where $\sup \{t \in \mathbb{R} : r_3(t) = 0\} := -\infty$ if $r_3(t) \neq 0$ for all $t \geq 0$. By considering the Wronskian of $r_3$ and $\tilde{r}_4$ for $t \geq t_2$, we have

$$\tilde{r}_4'(t) - \frac{r_4'(t)}{r_3(t)} \tilde{r}_4(t) = W(a, b, 0) \frac{e^{at}(1 + t)^{-1}}{r_3(t)}, \quad t \geq t_2,$$

(2.39)

where $W(a, b, 0) \neq 0$ is the Wronskian of $r_3$ and $\tilde{r}_4$ at $t = 0$.

(2.39) yields the representation

$$\tilde{r}_4(t) = r_3(t) \frac{\tilde{r}_4(t_2)}{r_3(t_2)} + W(a, b, 0)r_3(t) \int_{t_2}^{t} \frac{e^{as}(1 + s)^{-1}}{r_3^2(s)} \, ds, \quad t \geq t_2.$$
for \( \tilde{r}_4 \). By means of l'Hôpital's rule and (2.31) we can deduce from this representation for \( \tilde{r}_4 \) that

\[
\lim_{t \to \infty} e^{-at} t^{-\frac{b}{a}} \tilde{r}_4(t) = \mathcal{W}(a, b, 0)a^{\frac{b}{a}}.
\]  

(2.40)

This is consistent with the asymptotic behaviour we established for \( r_4 \) in (2.32).

It is also useful to determine some asymptotic information about \( \tilde{r}_4 \). Notice that \( t \mapsto U(1 + \frac{b}{a}, 1, a(1 + t)) \) is a polynomial of degree \(-1 - b/a \in \mathbb{N} \), and so \( \lim_{t \to \infty} \tilde{r}_4(t) = 0 \). By (2.40), it follows that \( \tilde{r}_4(t) \neq 0 \) for all \( t \geq t_3 \). Letting \( t_4 = \max(t_2, t_3) \), we rewrite (2.39) for \( t \geq t_4 \) to get

\[
\frac{\tilde{r}_4(t)}{r_4(t)} = \frac{r_3(t) + \mathcal{W}(a, b, 0)e^{at}(1 + t)^{-1}}{r_3(t)\tilde{r}_4(t)}.
\]

Using the fact that \( r_3(t) \sim a^{-1-\frac{b}{a}}t^{-1-b/a} \) as \( t \to \infty \) together with (2.40) shows that the second term has limit \( a \), and therefore

\[
\lim_{t \to \infty} \frac{\tilde{r}_4(t)}{r_4(t)} = a.
\]  

(2.41)

Since \( r_3 \) and \( \tilde{r}_4 \) are linearly independent, we can use the representation (2.38) for \( x \) to find \( \tilde{c}_3 \) and \( \tilde{c}_4 \) such that the initial conditions of (2.1b) (or (1.5)) are satisfied. In particular, \( \tilde{c}_4 \) can be expressed according to

\[
\tilde{c}_4 = \frac{1}{\mathcal{W}(a, b, 0)} \left( a\psi(0) U(1 + \frac{b}{a}, 2, a) + b \int_{-1}^{0} \psi(s)ds U(1 + \frac{b}{a}, 1, a) \right).
\]

An argument, which is identical in all respects to that used to deduce the representation (2.17) of the solution \( r \) of the resolvent equation (1.6) in the case when \( a < 0 \), can be used to justify the formulae

\[
r(t, s) = \begin{cases} 
\tilde{d}_3(s) r_3(t) + \tilde{d}_4(s) r_4(t), & a > 0, \quad b/a \notin \{-1, -2\}, \\ 
\tilde{d}_3(s) r_3(t) + \tilde{d}_4(s) \tilde{r}_4(t), & a > 0, \quad b/a \in \{-1, -2\}, \end{cases}
\]  

(2.42)

Conditions for \( \tilde{d}_3 \) and \( \tilde{d}_4 \), and for \( \tilde{d}_3 \) and \( \tilde{d}_4 \), are obtained from the initial conditions (1.6b) and (2.7), just as was done to obtain the equations (2.18) for \( d_1 \) and \( d_2 \) in the case when \( a < 0 \). Solving the corresponding equations to (2.18), we obtain

\[
\tilde{d}_3(s) = \Gamma(1 + \frac{b}{a}) e^{-a(1+s)}(1 + s) bM(1 + \frac{b}{a}, 2, a(1+s)),
\]

\[
\tilde{d}_4(s) = \Gamma(1 + \frac{b}{a}) e^{-a(1+s)}(1 + s) aU(1 + \frac{b}{a}, 2, a(1+s)).
\]  

(2.43)

Proceeding in the same manner in the degenerate case when \( b/a \in \{-1, -2, ...\} \) yields the expressions

\[
\tilde{d}_3(s) = \frac{\tilde{r}_4(s) - a\tilde{r}_4(s)}{\mathcal{W}(a, b, 0)e^{as}(1 + s)^{-1}} = \frac{a}{\mathcal{W}(a, b, 0)} \tilde{r}_4(s)(1 + s)e^{-as}\left(1 + \frac{1}{-a} \tilde{r}_4(s)\right),
\]

\[
\tilde{d}_4(s) = -\frac{\tilde{r}_4(s) - ar_3(s)}{\mathcal{W}(a, b, 0)e^{as}(1 + s)^{-1}} = \frac{1}{\mathcal{W}(a, b, 0)} e^{-as}(1 + s)aU(1 + \frac{b}{a}, 2, a(1+s)).
\]

We now turn our attention to the asymptotic behaviour of \( d_3, d_4 \) etc. Using (2.3), we can show that

\[
d_3(s) \sim ba^{b/a-1}s^{b/a}, \quad \text{as } s \to \infty,
\]

(2.44)

\[
d_4(s) \sim \Gamma(1 + b/a)e^{-a-b/s}a^{-b/a}s^{b/a}e^{-as}, \quad \text{as } s \to \infty.
\]  

(2.45)

In the degenerate case when \( b/a \in \{-1, -2, ...\} \), we may use (2.40) and (2.41) to establish that

\[
\tilde{d}_3(s) = o(s^\frac{b}{a}+1), \quad \text{as } s \to \infty.
\]  

(2.46)
expressed in terms of modified Bessel functions can be written in the form which appear as \([1, 9.7.1 & 9.7.2]\), for example.

\[ I_\nu = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi} \Gamma(\frac{\nu}{2})} e^{\nu t} e^{-x} \]

where \(I_\nu\) and \(K_\nu\) are two linearly independent solutions of modified Bessel’s equation

\[ z^2 w''(z) + zw'(z) - (z^2 + \nu^2)w(z) = 0, \]

with \(\nu\) a real parameter. See e.g. [16, Chapter 10.25.1] for details.

\(c_5\) and \(c_6\) in (2.48) can be found using the initial conditions (2.1b) or (1.5b). Doing this yields the formulae

\[ c_5 = 2 \left( \psi(0) \sqrt{\nu} I_1(2\sqrt{\nu}) + b \int_{-1}^{0} \psi(s) ds \right) \]

\[ c_6 = 2 \left( \psi(0) \sqrt{\nu} I_1(2\sqrt{\nu}) - b \int_{-1}^{0} \psi(s) ds \right) \]

In finding these expressions for \(c_5\) and \(c_6\), we have exploited the fact that the Wronskian of \(I_\nu\) and \(K_\nu\) obeys the identity

\[ W\{K_\nu(z), I_\nu(z)\} = 1/z \]

(2.51)

which appears as [16, 10.28.2], for example) and the derivatives of \(I_0\) and \(K_0\) obey

\[ I_0'(z) = I_1(z), \quad K_0'(z) = -K_1(z). \]

(2.52)

(cf., e.g. [16, 10.29.3]). We will also employ in the sequel the asymptotic behaviour of \(I_\nu\) and \(K_\nu\). The relevant results are

\[ I_\nu(t) = \frac{e^t}{\sqrt{2\pi t}} \{1 + O(t^{-1})\}, \quad K_\nu(t) = \frac{\pi}{2t} e^{-t} \{1 + O(t^{-1})\}, \quad \text{as } t \to \infty. \]

(2.53)

which appear as [11, 9.7.1 & 9.7.2], for example.

As in the cases when \(a < 0\) or \(a > 0\), the solution to the resolvent equation (1.6) can be represented as the sum of products of functions in \(t\) and \(s\). Indeed, \(r(t,s)\) can be written in the form

\[ r(t,s) = d_5(s)r_5(t) + d_6(s)r_6(t), \quad t \geq s \geq 0, \text{ for } a = 0 \text{ and } b > 0. \]

(2.54)

As in e.g. (2.1b), \(d_5\) and \(d_6\) may be found by solving a pair of linear simultaneous equations formulated from (1.6b). This leads to the formulae

\[ d_5(s) = 2\sqrt{b(s+1)} K_1(2\sqrt{b(s+1)}), \quad d_6(s) = 2\sqrt{b(s+1)} I_1(2\sqrt{b(s+1)}), \]

(2.55)

by making use of the identities (2.51) and (2.52).
In the case when \( a = 0 \) and \( b < 0 \), it turns out that the solution of (2.1a) can be expressed in terms of Bessel functions. Indeed, we have
\[
x(t) = c_7 r_7(t) + c_8 r_8(t) \quad \text{for } t \geq 0, \text{ when } a = 0 \text{ and } b < 0
\]
where
\[
r_7(t) = J_0(2\sqrt{-b(t+1)}), \quad r_8(t) = Y_0(2\sqrt{-b(t+1)})
\]
and \( J_\nu \) and \( Y_\nu \) are two linearly independent solutions of Bessel’s Equation
\[
z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0,
\]
where \( \nu \) is a real parameter (cf., e.g. [16] Chapter 10.2.1 for details). \( J_\nu \) and \( Y_\nu \) are referred to as the Bessel functions of the first kind and second kind respectively. We remark that the Bessel functions are oscillatory, convergent to zero and real–valued for positive arguments. Moreover as the argument \( z \to +\infty \), \( Y_\nu(z) \) and \( J_\nu(z) \) share the same amplitude, and are out of phase by \( -\pi \) (cf., e.g. [16, pp.242, Ch.7.5.1]. We make this precise in (2.60) below. One may verify by direct calculation that \( r_7 \) and \( r_8 \) are linearly independent solutions of (2.1a).

From (2.56) and (2.10), we can find expressions for the constants \( c_7 \) and \( c_8 \). In fact, one obtains
\[
c_7 = \pi \left( \psi(0) \sqrt{|b|} Y_1(2\sqrt{|b|}) - b \int_{-1}^{0} \psi(s) ds J_0(2\sqrt{|b|}) \right), \quad (2.58)
\]
\[
c_8 = \pi \left( \psi(0) \sqrt{|b|} J_1(2\sqrt{|b|}) + b \int_{-1}^{0} \psi(s) ds J_0(2\sqrt{|b|}) \right). \quad (2.59)
\]

In deducing these formulae, we have used the fact that the Wronskian of \( J_\nu \) and \( Y_\nu \) obeys
\[
W\{ J_\nu(z), Y_\nu(z) \} = 2/(\pi z)
\]
(cf., e.g., [16] 10.5.2) and also that the derivatives of \( J_\nu \) and \( Y_\nu \) obey
\[
J_\nu'(z) = -J_{\nu+1}(z), \quad Y_\nu'(z) = Y_{\nu+1}(z)
\]
(cf., e.g. [16] 10.6.3]. In asymptotic analysis of the solution of the stochastic equation, we will need information about the asymptotic behaviour of \( J_\nu(t) \) and \( Y_\nu(t) \) as \( t \to \infty \). The required asymptotic information is furnished by [11] 9.2.1, 9.2.2, 9.2.5, 9.2.6, which we record now for convenience:
\[
J_\nu(t) = \sqrt{2/(\pi t)} \{ \cos(t - \frac{1}{2} \nu \pi - \frac{1}{4} \pi) + O(t^{-1}) \}, \quad \text{as } t \to \infty, \quad (2.60a)
\]
\[
Y_\nu(t) = \sqrt{2/(\pi t)} \{ \sin(t - \frac{1}{2} \nu \pi - \frac{1}{4} \pi) + O(t^{-1}) \}, \quad \text{as } t \to \infty. \quad (2.60b)
\]
Once again the solution to the resolvent equation (1.6) can be written as a sum of products of functions depending on \( t \) and \( s \). Indeed, \( r(t,s) \) can be written in the form
\[
r(t,s) = d_7(s)r_7(t) + d_8(s)r_8(t), \quad t \geq s \geq 0, \quad a = 0, \quad b < 0, \quad (2.61)
\]
and expressions for \( d_7 \) and \( d_8 \) may be obtained from this representation and (1.6b). This yields
\[
d_7(s) = \pi \sqrt{|b|(1+s)} Y_1(2\sqrt{|b|(s+1)}), \quad d_8(s) = \pi \sqrt{|b|(s+1)} J_1(2\sqrt{|b|(s+1)}), \quad (2.62)
\]
upon use of the identities for the Wronskian of \( J_0 \) and \( Y_0 \) and formulae for the derivatives of \( J_0 \) and \( Y_0 \).
3. Recurrent Asymptotic Behaviour

3.1. Pathwise asymptotic stationary behaviour. The asymptotic behaviour of (1.4) in the case when \( a < 0 \) and \( a + b < 0 \) is very similar to the Ornstein–Uhlenbeck process \( U \) given by

\[
dU(t) = aU(t) \, dt + \sigma \, dB(t), \quad t \geq 0; \quad U(0) = 0.
\]

There is a unique continuous adapted process which obeys (3.1) and it is given by

\[
U(t) = e^{at} \int_0^t \sigma e^{-as} \, dB(s), \quad t \geq 0.
\]

**Theorem 3.1.** Let \( a < 0 \) and \( a + b \leq 0 \). Suppose that \( \psi \in C([-1,0]; \mathbb{R}) \). Let \( X \) be the unique continuous adapted process which obeys (1.4) and let \( U \) be the unique continuous adapted process which obeys (3.1). Then:

(i) \( X \) obeys

\[
\limsup_{t \to \infty} X(t) = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t \to \infty} X(t) = -\frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.} \tag{3.3}
\]

(ii) In the case that \( a + b < 0 \), we have

\[
\lim_{t \to \infty} \{ X(t) - U(t) \} = 0, \quad \text{a.s.} \tag{3.4}
\]

and

\[
\lim_{t \to \infty} \frac{1}{1 + t} \int_{-1}^t X(s) \, ds = 0, \quad \text{a.s.} \tag{3.5}
\]

(iii) In the case that \( a + b = 0 \), we have

\[
\lim_{t \to \infty} \{ X(t) - U(t) \} = L, \quad \text{a.s.} \tag{3.6}
\]

where \( L \) is a proper Gaussian random variable with mean and variance given by

\[
\mathbb{E}[L] = b^2 \Gamma(-\frac{b}{a}) \left( \int_{-1}^0 \psi(u) \, du \right) \int_0^\infty U(1 - \frac{b}{a}, 2, -a(1 + s)) \, ds
\]

\[
+ b^2 \Gamma(-\frac{b}{a}) \psi(0) \int_0^\infty e^{au} \int_u^\infty U(1 - \frac{b}{a}, 2, -a(1 + s)) \, ds \, du,
\]

\[
\text{Var}[L] = \sigma^2 \int_0^\infty e^{-2au} \left( \int_u^\infty e^{aw} \int_w^\infty b^2 \Gamma(-\frac{b}{a}) U(1 - \frac{b}{a}, 2, -a(1 + s)) \, ds \, dw \right)^2 \, du,
\]

and

\[
\lim_{t \to \infty} \frac{1}{1 + t} \int_{-1}^t X(s) \, ds = L, \quad \text{a.s.} \tag{3.7}
\]

The result (3.3) shows that, when \( a < 0 \) and \( a + b < 0 \), the sample mean of the process \( X \) tends to zero, i.e. the fluctuations of \( X \), which are of order \( \sqrt{\log t} \), occur symmetrically about zero. The result (3.7) however shows that, when \( a < 0 \) and \( a + b = 0 \), the fluctuations of \( X \) occur about the level \( L \) (which is random and so will be different for each sample path).

It is of interest to ask if we provide an upper bound on the a.s. rate of convergence of \( X - U \) to zero when \( a + b < 0 \). Of course the case when \( a + b = 0 \) is excluded, because in that case \( X - U \) tends to a non–trivial limit. We show that in all cases, the bound on the closeness decays polynomially.

**Theorem 3.2.** Let \( a < 0 \) and \( a + b < 0 \). Suppose that \( \psi \in C([-1,0]; \mathbb{R}) \). Let \( X \) be the unique continuous adapted process which obeys (1.4) and let \( U \) be the unique continuous adapted process which obeys (3.1). Then:
(i) If $a + b < 0$ and $2b + a > 0$, then
$$
\limsup_{t \to \infty} \frac{|X(t) - U(t)|}{t^{-1-\frac{b}{2}}} \in [0, \infty), \quad \text{a.s.}
$$

(ii) If $2b + a < 0$ and $2b + a > 0$, then
$$
\limsup_{t \to \infty} \frac{|X(t) - U(t)|}{t^{-1/2} \sqrt{\log \log t}} \in [0, \infty), \quad \text{a.s.}
$$

(iii) If $2b + a = 0$, then
$$
\limsup_{t \to \infty} \frac{|X(t) - U(t)|}{t^{-1/2} \log t \sqrt{\log \log t}} \in [0, \infty), \quad \text{a.s.}
$$

While we conjecture that these estimates are sharp, i.e. the limits superior in Theorem 3.2 are positive, such an analysis would involve, amongst other things, a sharper analysis of the leading order terms in the expansions in (2.3), as well as lower estimates of certain integrals in the proof. Such analysis goes beyond the scope of the present work.

3.2. Asymptotic behaviour of the autocovariance function. Theorem 3.1 asserts that $X$ is a Gaussian process which is asymptotically close to the asymptotically stationary Gaussian process $U$ (for $b = 0$, $X$ is itself an Ornstein-Uhlenbeck process). Since $U$ is given by (1.4), its autocovariance function may be shown to obey

$$
\text{Cov}(U(t), U(t + \Delta)) = \sigma^2 e^{a\Delta} e^{2at} \int_0^t e^{-2as} ds = e^{a\Delta} \sigma^2 \frac{1}{2|a|} (1 - e^{2at}).
$$

Therefore, for each fixed $t > 0$ we have $\Delta \mapsto \text{Cov}(U(t), U(t + \Delta))$ decays exponentially to zero as $\Delta \to \infty$. It is therefore reasonable to expect that the autocovariance function of $X$ defined by (1.9) to behave according to $\lim_{\Delta \to \infty} \text{Cov}(X(t), X(t + \Delta)) = 0$ for every $t \geq 0$. However, as is shown below, although $X(t) - U(t) \to 0$ as $t \to \infty$ a.s., for each fixed $t > 0$, the autocovariance $\Delta \mapsto \text{Cov}(X(t), X(t + \Delta))$ decays polynomially to zero as $\Delta \to \infty$.

We have already seen in (2.25) that it is possible to represent the autocovariance function in terms of $r_1, r_2, d_1, d_2$ etc. Using the information about the asymptotic behaviour of these functions, we can readily how rapidly the autocovariance function decays in the time lag $\Delta$.

Theorem 3.3. Suppose that $a < 0$ and $a + b \leq 0$. Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let $X$ be the unique continuous adapted process which obeys (1.4). Let $t \geq 0$ be fixed. Then

$$
\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t + \Delta))}{\Delta^{-(1+\frac{b}{2})}} = c_t(a, b), \quad (3.8)
$$

where $c_t = c_t(a, b)$ is given by

$$
c_t(a, b) = \sigma^2 b|a|^{-1-b/a} \int_0^t r(t, s)(1 + s)^{1 - b/a, 2, -a(1 + s)} ds. \quad (3.9)
$$

Hence the process $X$ defined by (1.4) is a long memory process (i.e., for each fixed $t$, $\int_0^\infty \text{Cov}(X(t), X(t + \Delta)) d\Delta = +\infty$) when $a < 0$, $b > 0$ and $a + b < 0$.

In the case when $a + b = 0$, the covariance does not tend to zero as $\Delta \to \infty$; instead

$$
\lim_{\Delta \to \infty} \text{Cov}(X(t), X(t + \Delta)) = c_t(a, b). \quad (3.10)
$$

In the special case $a < 0$ and $b = 0$, equation (1.4) reduces to an Ornstein-Uhlenbeck equation and hence its autocovariance function is decays exponentially. This is consistent with the result of Theorem 3.3 because the value of $c_t$ is zero in
This leads us to question under what conditions will the limit obtained in Theorem 3.3 be nonzero.

Proposition 2. Let \( b > 0 \). Then \( \text{Cov}(X(t), X(t + \Delta)) > 0 \) for all \( \Delta > 0 \).

Proposition 3. If \( a < 0, b > 0 \) and \( a + b < 0 \), then the limiting constant in (3.9) obeys \( c_t(a, b) > 0 \).

The case when \( b < 0 \) is more delicate to analyse. However, it can be shown that if \( t \) is sufficiently large, then \( c_t(a, b) \) is negative. We can also show that \( c_t(a, b) \to 0 \) as \( t \to \infty \) in the case when \( b > 0 \) and that \( c_t(a, b) \to -\infty \) as \( t \to \infty \) in the case that \( b < 0 \). We also see that \( \lim_{t \to \infty} c_t(a, b) \) is nontrivial in the case when \( a + b = 0 \), and its limit will be of interest later in this section. Accordingly, the asymptotic behaviour of \( c_t \) is recorded in the next result.

Proposition 4. Suppose that \( a < 0 \) and \( a + b \leq 0 \) and let \( c_t(a, b) \) be defined by (3.9).

(a) If \( b < 0 \) and \( a + b < 0 \), then
\[
\lim_{t \to \infty} \frac{c_t(a, b)}{t^{b/a}} = \sigma^2 b |a|^{-3} \left( \frac{|b|}{|a|} + 1 \right) < 0,
\]
and so \( c_t \to -\infty \) as \( t \to \infty \).

(b) If \( b > 0 \) and \( a + b < 0 \), then \( c_t \to 0 \) as \( t \to \infty \). Furthermore

(i) If \( 2b + a > 0 \), then
\[
\lim_{t \to \infty} \frac{c_t(a, b)}{t^{-b/a-1}} = \sigma^2 b^2 \left( 1 - \frac{b}{a} \right)^2 - \frac{2}{b/a} \left( 1 - \frac{b}{a} \right) \int_0^\infty (1 + s)^2 U^2 - \frac{1}{a} \left( 1 + s \right) ds > 0;
\]

(ii) If \( 2b + a = 0 \), then
\[
\lim_{t \to \infty} \frac{c_t(a, b)}{t^{-1/2} \log t} = \sigma^2 \frac{1}{4} |a|^{-2} > 0;
\]

(iii) If \( 2b + a < 0 \), then \( c_t \) obeys (3.11) with the limit on the righthand side being positive.

(c) If \( a + b = 0 \), then
\[
\lim_{t \to \infty} c_t(a, b) = \sigma^2 \frac{b^2}{|a|^{2+2b/a}} \int_0^\infty (1 + s)^2 U(1 - b/a, 2, |a|(1 + s))^2 ds.
\]

In Theorem 3.3 we held the starting time, \( t \), fixed and observed the behaviour of the auto-covariance function as the time lag, \( \Delta \) tended to infinity. However it is perhaps more typical, when testing for long memory (c.f. e.g. [3]), to fix the time lag and let the starting time tend to infinity. It is then observed that this limiting auto-covariance function depends only on the time lag \( \Delta \) (so that the process is transiently non-stationary) and the limiting autocovariance function is integrable over \( \Delta \), so that \( X \) does not have long memory.

Theorem 3.4. Suppose that \( a < 0 \) and \( a + b \leq 0 \). Suppose that \( \psi \in C([-1, 0]; \mathbb{R}) \). Let \( X \) be the unique continuous adapted process which obeys (1.4). Then, for all \( \Delta \geq 0 \),

(a) If \( a + b < 0 \), then
\[
\lim_{t \to \infty} \text{Cov}(X(t), X(t + \Delta)) = \frac{\sigma^2}{2|a|} e^{a\Delta}.
\]
(b) If $a + b = 0$, then

$$\lim_{t \to \infty} \text{Cov}(X(t), X(t + \Delta)) = \frac{\sigma^2}{2|a|} e^{a\Delta} + \sigma^2 \frac{b^2}{|a|^{2+2\sigma}} \int_{0}^{\infty} (1 + s)^2 U(1 - \frac{b}{a}, 2, |a|(1 + s))^2 ds. \quad (3.16)$$

It is interesting to remark that the differing rates of decay of the autocovariance function recorded for the solution of (1.4) when $a < 0$ and $a + b < 0$ in the limits (3.15) and (3.8) are not generally seen in autonomous affine differential equations.

We show below for asymptotically stationary scalar affine SFDEs which are either finite delay or of Volterra type, that one is in a position to characterise short or long memory by means of a single limiting autocovariance function. Therefore, in the case of autonomous affine equations, it does not matter whether one takes $\Delta \to \infty$ or $t \to \infty$: as both limits leads to the same function, both give the same classification to the process as being short or long memory.

To make this claim more precise, and to find notation to connect the behaviour of the autocovariance function of the solution of (1.4) with autocovariance functions of solutions of such autonomous affine SFDEs, and to also contrast these behaviours, we start by examining, for example, the solution $X$ of an affine SFDE with finite delay. Such a process $X$ would be the solution of

$$dX(t) = L(X_t) dt + \sigma dB(t), \quad t \geq 0; \quad X(t) = \psi(t), \quad t \in [-\tau, 0], \quad (3.17)$$

where $L : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is a linear functional and $\psi \in C([-\tau, 0]; \mathbb{R})$. Suppose that $r$ is the differential resolvent is given by

$$r'(t) = L(r_t), \quad t > 0; \quad r(0) = 1; \quad r(t) = 0 \text{ for } t \in [-\tau, 0).$$

We now summarise the situation in the following claim.

**Remark 1.** If $X$ is the solution of (3.17), and the differential resolvent $r$ associated with the drift of (3.17) obeys $r(t) \to 0$ as $t \to \infty$ and $r(t)$ is of one sign for all $t$ sufficiently large, then there are functions $\gamma$ and $c$ such that

$$\lim_{t \to \infty} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma(\Delta)} = 1, \quad (3.18a)$$

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma(\Delta)} = ct, \quad (3.18b)$$

$$\lim_{t \to \infty} ct = 1. \quad (3.18c)$$

A similar result pertains to Volterra equations with slowly decaying autocovariance function. For instance, if $X$ is the solution of

$$dX(t) = \left(-aX(t) + \int_{0}^{t} k(t - s)X(s) ds\right) dt + \sigma dB(t), \quad t \geq 0; \quad X(0) = \xi, \quad (3.19)$$

and we suppose that $k$ is a continuous, positive and integrable function. Let the differential resolvent $r$ be the solution of

$$r'(t) = -ar(t) + \int_{0}^{t} k(t - s)r(s) ds, \quad t \geq 0; \quad r(0) = 1.$$

**Remark 2.** Suppose that $k$ is a positive, continuous and integrable function which is subexponential and asymptotic to a decreasing function, and moreover obeys $a > \int_{0}^{\infty} k(s) ds$. Then the autocovariance function of the solution $X$ of (3.19) obeys (3.18).
We are now in a position to compare and contrast the situation with (3.18), which pertains for solutions of affine autonomous equations. For the average equation the autocovariance function obeys

\[
\begin{align*}
\lim_{{t \to \infty}} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma_1(\Delta)} &= 1, \quad (3.20a) \\
\lim_{{\Delta \to \infty}} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma_2(\Delta)} &= c_t, \quad (3.20b) \\
\lim_{{t \to \infty}} c_t &= \begin{cases} 
0, & b > 0, \\
-\infty, & b < 0
\end{cases} \quad (3.20c)
\end{align*}
\]

where \(\gamma_1(\Delta) = \sigma^2/|a| \cdot e^{\alpha \Delta} \) and \(\gamma_2(\Delta) = \Delta^{-(1+b/a)}\). Therefore, the situation in (3.20) differs from the case in (3.18), because there are two different rates of decay in \(\Delta\) in (3.20a) and (3.20b) and the function \(c_t\) in (3.20c) does not tend to a non-trivial finite limit as \(t \to \infty\).

Theorem 3.3 part (a) is consistent with Theorem 3.1 part (b), because in the case when \(a + b < 0\), the latter result shows that \(X\) is pathwise asymptotic to a process whose limiting autocovariance function is given in part (a). The result of part (b) is also consistent with Theorem 3.1 because when \(a + b = 0\), we know from part (c) of Theorem 3.1 that the solution is asymptotic to \(U\) plus a non-trivial limiting random variable, whose presence is suggested by the form of the limiting autocovariance function in part (b).

It is tempting to remark that when \(b > 0\), Proposition 4 part (a) may be thought of as partly reconciling the differing asymptotic behaviour of \(\text{Cov}(X(t), X(t + \Delta))\) recorded in Theorem 3.3 and 3.4 according as to whether \(\Delta \to \infty\) or \(t \to \infty\). This is because \(c_t(a, b) \to 0\) as \(t \to \infty\), so that the “long memory” recorded in (3.8) becomes ever weaker as the start time \(t\) becomes greater, and therefore becomes closer to the “short memory” or exponential decay in \(\Delta\) in the limiting autocovariance function determined in part (a) of Theorem 3.4.

This heuristic explanation of the reconciliation of the asymptotic behaviour of the autocovariance must however be taken with caution. In particular, in the case when \(b < 0\), it is harder to forward with equal confidence the same explanation as to the differing asymptotic behaviour recorded in Theorem 3.3 and 3.4. In this case, Proposition 4 part (b) shows that \(c_t(a,b) \to -\infty\) as \(t \to \infty\), suggesting that the polynomial decay in the autocovariance function given in (3.8) tends to become stronger as the start time is chosen to be very large. On the other hand, the fact that \(|c_t|\) has power law growth which is less rapid as \(t \to \infty\) (at a rate \(t^{b/a}\) according to (3.11)) compared to the power law decay of \(\text{Cov}(X(t), X(t + \Delta))\) as \(\Delta \to \infty\) (which is at the rate \(\Delta^{-(1+b/a)}\)) may point to a weakening overall correlation.

One situation in which it does not seem to matter in which order limits are taken is when \(a + b = 0\). Taking the limit as \(\Delta \to \infty\) in (3.16) leads to

\[
\lim_{{\Delta \to \infty}} \lim_{{t \to \infty}} \text{Cov}(X(t), X(t + \Delta)) = \sigma^2 \frac{b^2}{|a|^{2+2\xi}} \int_0^\infty (1+s)^{2} U(1-b/a, 2, |a|(1+s))^{2} ds.
\]

On the other hand, by (3.10) and (3.14) we have that

\[
\lim_{{t \to \infty}} \lim_{{\Delta \to \infty}} \text{Cov}(X(t), X(t + \Delta)) = \sigma^2 \frac{b^2}{|a|^{2+2\xi}} \int_0^\infty (1+s)^{2} U(1-b/a, 2, |a|(1+s))^{2} ds,
\]

so the limits are equal.

3.3. Non-stationary asymptotic behaviour. In the case when \(a < 0\) and \(b < 0\), we have already seen that the solution of (1.4) is asymptotically stationary, and when \(a > 0\) and \(b < 0\), the solution exhibits a.s. exponential growth. Therefore, we expect to see intermediate asymptotic behaviour on the boundary of these two
parameter regions, where $a = 0$ and $b < 0$. In broad terms, we can establish that the solution behaves in some ways like a standard Brownian motion, in the sense that the solution is a Gaussian process which has asymptotically vanishing mean, variance which grows linearly in time, and experiences a.s. large fluctuations which satisfy the Law of the iterated logarithm.

**Theorem 3.5.** Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let $X$ be the unique continuous adapted process which obeys $(1.4)$. If $a = 0$ and $b < 0$, then $\mathbb{E}[X(t)] \to 0$ as $t \to \infty$ and
\[
\lim_{t \to \infty} \frac{\text{Var}[X(t)]}{t} = \frac{1}{3}\sigma^2.
\]

We now state the result which deals with the magnitude of the large fluctuations of $X$.

**Theorem 3.6.** Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let $X$ be the unique continuous adapted process which obeys $(1.4)$. If $a = 0$ and $b < 0$, then
\[
\limsup_{t \to \infty} \frac{X(t)}{\sqrt{2t \log \log t}} = \frac{1}{\sqrt{3}}\sigma, \quad \liminf_{t \to \infty} \frac{X(t)}{\sqrt{2t \log \log t}} = -\frac{1}{\sqrt{3}}\sigma, \quad \text{a.s.}
\]

**Remark 3.** Both Theorems 3.5 and 3.6 show that, asymptotically, $X$ has behavior somewhat akin to standard Brownian motion. In particular it is observed that the limiting constant in Theorem 3.5 is the square of that in Theorem 3.6. We are then drawn to conjecture that the increments of $X$, under the hypotheses of Theorems 3.5 and 3.6, are asymptotically stationary.

### 4. Transient Asymptotic Behaviour

From (1.7) we see that as $X$ depends upon $x$, we then expect the asymptotic behaviour of $X$ to also depend upon $x$, especially in the case when $|x(t)| \to \infty$ as $t \to \infty$. This arises in two main situations: when $a < 0$ and $a + b > 0$, and when $a > 0$. We deal with the first of these cases first, and establish that $|X(t)| \to \infty$ as $t \to \infty$ like a power of $t$. In fact, $X$ can tend to $+\infty$ or to $-\infty$, each with positive probability. Moreover, the choice of which limit is attained depends on the path of the Brownian motion driving $X$, with the increments of $B$ earlier in the path generally proving to be more influential in deciding which limit is attained.

The key to the proof of this result, and to the others in this Section, hinge on the representation of the solution $X$ of (1.4) in terms of the resolvent $r$ and mean $x$, as well as the asymptotic analysis of these functions given in Section 2.

**Theorem 4.1.** Suppose that $a < 0$, $a + b > 0$. Suppose also that $\psi \in C([-1, 0]; \mathbb{R})$. Let $X$ be the unique continuous adapted process which obeys $(1.4)$. Then
(a) There exists an $\mathcal{F}^B(\infty)$ measurable normal random variable $C$ such that
\[
\lim_{t \to \infty} \frac{X(t)}{t^{-(1+\frac{b}{a})}} = C, \quad \text{a.s.}
\]

(b) $C$ is given by
\[
C = |a|^{-1-\frac{b}{a}} \left\{ \psi(0) U \left( 1 - \frac{b}{a}, 2, |a| \right) + \int_{-1}^{0} \psi(s) ds U \left( -\frac{b}{a}, 1, |a| \right) \right\}
+ \sigma \int_{0}^{\infty} \frac{b}{|a|^{1+\frac{b}{a}}} (1 + s) U \left( 1 - \frac{b}{a}, 2, |a|(1 + s) \right) dB(s).
\]
there is a positive probability of each of the events
an exponential rate, with a power law correction growth factor. Once again,
function and so is itself eventually positive. Thus we have
\[ C \]
Suppose that
the unique continuous adapted process which obeys
\[ (1.4) \]
\[ \text{random variable, because} \]
s
The mean and variance of \( C \) are given by
\[ \mathbb{E}[C] = |a|^{-1 + \frac{b}{a}} b \left\{ \psi(0) U \left( 1 - \frac{b}{a}, 2, |a| \right) + \int_{-1}^{0} \psi(s) ds U \left( \frac{b}{a}, 1, |a| \right) \right\}, \quad (4.2) \]
\[ \text{Var}[C] = \sigma^2 \frac{b^2}{|a|^{2 + 2\frac{b}{a}}} \int_{0}^{\infty} (1 + s)^2 U \left( 1 - \frac{b}{a}, 2, |a|(1 + s)^2 \right) ds > 0. \quad (4.3) \]
\[ \text{The mean and variance of} X \text{ obey} \]
\[ \lim_{t \to \infty} \frac{\mathbb{E}[X(t)]}{t^{-1 + \frac{b}{a}}} = \mathbb{E}[C], \quad \lim_{t \to \infty} \frac{\text{Var}[X(t)]}{t^{-2 - 2\frac{b}{a}}} = \text{Var}[C]. \]

Once the formula (4.3) is established, it is clear that \( C \) is a proper Gaussian random variable, because \( s \mapsto U \left( 1 - \frac{b}{a}, 2, |a|(1 + s)^2 \right) \) is asymptotic to a positive function and so is itself eventually positive. Thus we have \( C \neq 0 \) a.s.

In the case when \( a > 0 \), we show that \( X \) grows to plus or minus infinity at an exponential rate, with a power law correction growth factor. Once again, there is a positive probability of each of the events \( \{\lim_{t \to \infty} X(t) = +\infty\} \) and \( \{\lim_{t \to \infty} X(t) = -\infty\} \) occurring.

**Theorem 4.2.** Suppose that \( a > 0 \). Suppose also that \( \psi \in C([-1, 0]; \mathbb{R}) \). Let \( X \) be the unique continuous adapted process which obeys (1.4).

(a) There exists a \( \mathcal{F}^{\mathbb{B}}(\infty) \) normal random variable \( C \) such that
\[ \lim_{t \to \infty} \frac{X(t)}{e^{at^{b/a}}} = C, \quad \text{a.s.} \quad (4.4) \]

(b) \( C \) is given by
\[ C = a^{\frac{b}{a}} \left\{ a \psi(0) U \left( 1 + \frac{b}{a}, 2, a \right) + b \int_{-1}^{0} \psi(s) ds U \left( 1 + \frac{b}{a}, 1, a \right) \right\} \]
\[ + \sigma a^{1 + \frac{b}{a}} \int_{0}^{\infty} e^{-as}(1 + s) U \left( 1 + \frac{b}{a}, 2, a(1 + s) \right) dB(s). \]

(c) The mean and variance of \( C \) are given by
\[ \mathbb{E}[C] = a^{\frac{b}{a}} \left\{ a \psi(0) U \left( 1 + \frac{b}{a}, 2, a \right) + b \int_{-1}^{0} \psi(s) ds U \left( 1 + \frac{b}{a}, 1, a \right) \right\}, \quad (4.5) \]
\[ \text{and} \]
\[ \text{Var}[C] = \sigma^2 a^{2 + 2\frac{b}{a}} \int_{0}^{\infty} e^{-2as}(1 + s)^2 U \left( 1 + \frac{b}{a}, 2, a(1 + s)^2 \right) ds > 0. \]

(d) The mean and variance of \( X \) obey
\[ \lim_{t \to \infty} \frac{\mathbb{E}[X(t)]}{e^{at^{b/a}}} = \mathbb{E}[C], \quad \lim_{t \to \infty} \frac{\text{Var}[X(t)]}{e^{2at^{2b/a}}} = \text{Var}[C]. \]

It can be seen from part (b) of Theorem 4.2 that the limiting random variable in (4.4) is a linear functional of (the increments of) the Brownian motion \( B \). The formula for \( \mathbb{E}[C] \), given in part (c) of Theorem 4.2 is discussed [3], where it is shown that in certain regions of the parameter space \( \mathbb{E}[C] \) is non-zero and hence the continuous random variable \( C \) is non-zero almost surely. While part (a) is also dealt with in [3] we present an alternative method of proof in this paper, with the chief difference being that a simpler formula for \( C \) is attained in this paper from the the variation of parameters representation (rather using an admissibility approach as in [3]).

In the \( ab \)-parameter space the line \( a = 0 \) and \( b > 0 \) is bordered by a region wherein \( X \) undergoes polynomial growth (covered by Theorem 4.1) and a region of
exponential growth (which is described by Theorem 4.2). As neither the representation (2.2) nor (2.20) of the resolvent \( r \) are valid on this line, it therefore seems somewhat apt that \( X \) should have a rate of faster than polynomial yet slower than exponential growth on this line. A precise asymptotic result is recorded in the next theorem.

**Theorem 4.3.** Suppose that \( a = 0 \) and \( b > 0 \). Suppose also that \( \psi \in C([-1, 0]; \mathbb{R}) \). Let \( X \) be the unique continuous adapted process which obeys (1.4). Then

(a) There exists an \( \mathcal{F}_t \infty \) measurable normal random variable \( C \) such that

\[
\lim_{t \to \infty} \frac{X(t)}{t^{-1/4}e^{2\sqrt{bt}}} = C, \quad \text{a.s.}
\]

(b) \( C \) is given by

\[
C = \frac{1}{b^{1/4}\sqrt{\pi}} \left( \psi(0)\sqrt{b}K_1(2\sqrt{b}) + b \int_{-1}^{0} \psi(s) ds K_0(2\sqrt{b}) \right)
+ \frac{\sigma b^{1/4}}{\sqrt{\pi}} \int_{0}^{\infty} \sqrt{s+1}K_1(2\sqrt{b(s+1)}) dB(s),
\]

where \( K_0 \) and \( K_1 \) are modified Bessel functions of the second kind.

(c) The mean and variance of \( C \) are given by

\[
\mathbb{E}[C] = \frac{1}{\sqrt{\pi}} \left( \psi(0)b^{1/4}K_1(2\sqrt{b}) + b^{3/4} \int_{-1}^{0} \psi(s) ds K_0(2\sqrt{b}) \right),
\]

\[
\text{Var}[C] = \frac{\sigma^2 b^{1/2}}{\pi} \int_{0}^{\infty} (s+1)K_1^2(2\sqrt{b(s+1)}) ds > 0.
\]

(d) The mean and variance of \( X \) obey

\[
\lim_{t \to \infty} \frac{\mathbb{E}[X(t)]}{t^{-1/4}e^{2\sqrt{bt}}} = \mathbb{E}[C], \quad \lim_{t \to \infty} \frac{\text{Var}[X(t)]}{t^{-1/2}e^{4\sqrt{bt}}} = \text{Var}[C].
\]

We see from part (c) that \( C \) has positive variance, so we have that \( C \neq 0 \) a.s. Therefore the limit in part (a) is nontrivial a.s.

**Remark 4.** If one scales (1.7) by \( r_2 \) then we have

\[
X(t)/r_2(t) = x(t)/r_2(t) + \sigma \int_{0}^{t} H(t, s) dB(s)
\]

where \( H(t, s) = r(t, s)/r_2(t) \). Under the hypothesis of Theorem 4.1, it is immediate from Theorem 4 in [4] that as the stochastic integral \( \int_{0}^{t} H(t, s) dB(s) \) converges to \( C \) almost surely then the convergence must take place in mean square also. Similarly each of the results of Theorems 4.2, 4.3, 4.6 for almost sure convergence hold true for mean square convergence also.

5. **Proofs from Sections 1.5 and 3.2**

5.1. **Proof of Lemma 1.1.** Existence and uniqueness of the solution of (1.4) is known from general theory of SFDEs, c.f. e.g. [8, 13]. Thus we need only demonstrate that the representation (1.7) satisfies the SFDE (1.4).

Firstly observe that the resolvent equation, (1.6), may be re-expressed as

\[
r(t, s) = 1 + a \int_{s}^{t} r(u, s) du + \int_{s}^{t} \frac{b}{1+u} \int_{u}^{\infty} r(w, s) dw du, \quad t \geq s.
\]
Defining $Z = X - x$, we have that $Z$ obeys

$$Z(t) = a \int_0^t Z(s) ds + \int_0^t \frac{b}{1 + s} \int_0^s Z(u) du ds + \sigma B(t), \quad t \geq 0, \quad (5.1a)$$

and

$$Z(t) = 0, \quad t \in [-1, 0]. \quad (5.1b)$$

From the definition of $Z$ it is apparent that demonstrating the validity of (1.7) is equivalent to showing that $Z$ obeys

$$Z(t) = \sigma \int_0^t \int_0^s r(s, w) dB(w), \quad t \geq 0. \quad (5.2)$$

Let $Z^*(t) = \sigma \int_0^t r(t, s) dB(s), \ t \geq 0$ and so $Z^*(0) = 0$ as required. Now using the stochastic Fubini theorem

$$\begin{align*}
a \int_0^t Z^*(s) ds + \int_0^t \frac{b}{1 + s} \int_0^s Z^*(u) du ds + \sigma B(t) \\
&= a \sigma \int_0^t \int_0^s r(s, w) dB(w) ds \\
&\quad + \int_0^t \frac{b}{1 + s} \int_0^s \sigma \int_{w=0}^u r(u, w) dB(w) du ds + \sigma B(t) \\
&= \sigma \int_0^t \left( a \int_0^t r(t, s) ds + \int_0^t \frac{b}{1 + s} \int_0^s r(u, w) du ds \right) dB(w) + \sigma B(t) \\
&= \sigma \int_0^t \left( r(t, w) - 1 \right) dB(w) + \sigma B(t) = \sigma \int_0^t r(t, w) dB(w) = Z^*(t).
\end{align*}$$

As $Z$ is the unique solution of (5.1) we have $Z = Z^*$ and hence $X$ has the representation (1.7).

### 5.2. Proof of Proposition 1

Let $t \geq 0$ and $\Delta \geq 0$. Differentiating (1.10) with respect to $\Delta$, using (1.6a), and by exchanging the order of integration and decomposing the integral, we get

$$\begin{align*}
\gamma_t(\Delta) &= \sigma^2 \int_0^t r(t, s) \frac{\partial}{\partial \Delta} r(t + \Delta, s) ds \\
&= \sigma^2 a \int_0^t r(t, s) r(t + \Delta, s) ds + \sigma^2 \frac{b}{1 + t + \Delta} \int_0^t \int_s^{t+\Delta} r(t, s) r(u, s) du ds \\
&= a \gamma_t(\Delta) + \frac{b \sigma^2}{1 + t + \Delta} \int_0^t \int_0^u r(t, s) r(u, s) ds du \\
&\quad + \frac{b \sigma^2}{1 + t + \Delta} \int_t^{t+\Delta} \int_0^t r(t, s) r(u, s) ds du.
\end{align*}$$

Next, because $r(u, s) = 0$ for $0 \leq w < s$, we see that $\int_0^u r(t, s) r(u, s) ds = \int_0^u r(t, s) r(u, s) ds$ for $u \in [0, t]$. Hence the two integrals on the right hand side can be combined. By making the substitution $w = u - t$, and then splitting the
By (2.4) and (2.5) we have that
\[ \int_0^t r(t, s) r(u, s) ds du \]
get
\[ \gamma_t(\Delta) = a \gamma_t(\Delta) + \frac{b \sigma^2}{1 + t + \Delta} \int_0^{t+\Delta} r(t, s) r(u, s) ds du \]
\[ = a \gamma_t(\Delta) + \frac{b \sigma^2}{1 + t + \Delta} \int_0^t r(t, s) r(t+w, s) ds dw \]
\[ + \frac{b \sigma^2}{1 + t + \Delta} \int_{t}^{t+\Delta} r(t, s) r(t+w, s) ds dw \]
\[ = a \gamma_t(\Delta) + \frac{b \sigma^2}{1 + t + \Delta} \int_0^{w+t} \gamma_t(w) dw \]
\[ + \frac{b \sigma^2}{1 + t + \Delta} \int_{t}^{t+\Delta} r(t, s) r(t+w, s) ds dw, \]
where we have used the definition of \( \gamma_t(w) \) at the last step. It now suffices to show that the last integral is zero. We first decompose it according to
\[ \int_0^{w+t} \int_{-t}^t r(t, s) r(t+w, s) ds dw \]
\[ = \int_0^t \int_{-t}^t r(t, s) r(t+w, s) ds dw + \int_0^t \int_{0}^{w+t} r(t, s) r(t+w, s) ds dw \]
\[ = \int_{-t}^t \int_0^{w+t} r(t, s) r(t+w, s) ds dw \]
where the last integral is zero as when \( w > 0, r(t, s) = 0 \) for \( s \in (t, t+w] \). Since \( t \geq 0 \) and \( w \in [-t, 0] \), we have that \( s \in (t+w, t] \) in the remaining integral and therefore \( r(t+w, s) = 0 \). Thus,
\[ \int_0^{w+t} \int_{-t}^t r(t, s) r(t+w, s) ds dw = 0, \]
which proves (1.11).

For \( t \geq 0 \) and \(-t \leq \Delta \leq 0\), we prove (1.12) in a similar manner to (1.11). However, since \( \Delta \in [-t, 0] \), we can show that \( \gamma_t \) can be written in the form
\[ \gamma_t(\Delta) = \sigma^2 \int_0^{t+\Delta} r(t, s) r(t+\Delta, s) ds, \quad \Delta \in [-t, 0]. \]
The function on the righthand side is differentiable with respect to \( \Delta \) on \((-t, 0)\), because \( \Delta \mapsto r(t+\Delta, s) \) is differentiable on \((-t, 0)\). Now, differentiating with respect to \( \Delta \), we get
\[ \gamma'_t(\Delta) = \sigma^2 \int_0^{t+\Delta} r(t, s) \frac{\partial}{\partial \Delta} r(t+\Delta, s) ds + \sigma^2 r(t+\Delta) r(t+\Delta, t+\Delta), \quad \Delta \in (-t, 0), \]
and proceeding in a manner similar to the proof of (1.11) above, we establish (1.12).

5.3. Proof of Theorem 3.3 In the case when \( b/a \notin \{1, 2, \ldots, \} \), from (2.25), we have
\[ \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+b/a)}} = c_1 r_1(t+\Delta) \frac{\Delta^{-(1+b/a)}}{\Delta^{-(1+b/a)}} + c_2 r_2(t+\Delta) \frac{r_2(t+\Delta)}{\Delta^{-(1+b/a)}} \left( \frac{t+\Delta}{\Delta} \right)^{-(1+b/a)}. \]
By (2.4) and (2.5) we have that
\[ \lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+b/a)}} = c_{2,t} \frac{1}{\Gamma(-b/a)} e^{-a} |a|^{-1-b/a}. \]
Since $c_{2,t}$ is given by (2.26) and $d_2$ by (2.20), we obtain
\[ \lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+\frac{2}{a})}} = c_t(a,b) \]
where $c_t$ is given by (3.9). The proof in the case when $b/a \in \{1, 2, \ldots\}$ proceeds in the same manner, making use of (2.4) and (2.13) to obtain
\[ \lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+\frac{b}{a})}} = \tilde{c}_{2,t}(a,b,0)|a|^{-1-b/a}. \]
From this and the formula for $\tilde{c}_{2,t}$ in (2.27) we obtain the desired representation.

5.4. **Proof of Proposition 2**. Since Cov$(X(t), X(t+\Delta)$ obeys (1.9) for $t \geq 0$ and $\Delta \geq 0$, we see that it suffices to show that $r(t,s) > 0$ for all $t \geq s > 0$.

To this end, fix $s > 0$ and write $r_s(t) = r(t,s)$ for $t \geq s$. Then (1.6a) and (1.6b) are equivalent to
\[ r_s(t) = ar_s(t) + b \frac{1}{1+t} \int_s^t r_s(u) \, du, \quad t \geq s; \quad r_s(s) = 1. \]
Note that $r_s \in C^1(s, \infty)$. Hence there exists some $\epsilon > 0$ such that $r(t,s) > 0$ for $t \in (s, s+\epsilon)$. Suppose there exists a minimal $t_0 > s$ such that $r_s(t_0) = 0$, but $r_s(t) > 0$ for $s < t \leq t_0$. Then $r_s'(t_0) \leq 0$ and
\[ 0 \geq r_s'(t_0) = ar_s(t_0) + b \frac{1}{1+t} \int_0^{t_0} r_s(u) \, du = b \frac{1}{1+t} \int_s^{t_0} r_s(u) \, du > 0, \]
a contradiction. Hence $r(t,s) = r_s(t) > 0$ for all $t \geq s$, and so Cov$(X(t), X(t+\Delta)) > 0$ for all $t > 0$ and $\Delta \geq 0$.

5.5. **Proof of Proposition 3**. Since $a < 0$, by Theorem 3.3 we have that $c_t(a,b)$ obeys (3.9). In the proof of Proposition 2 we showed that $r(t,s) > 0$ for all $t \geq s > 0$. Therefore, to show that $c_t(a,b) > 0$ for all $t > 0$, by examining the integral in (3.9), it suffices to show that $U(1-b/a,2,|a|(1+t)) > 0$ for $t \geq 0$. Since $a < 0$ and $b > 0$, we have $1-b/a > 0$, so by the integral representation (2.28) we have
\[ U(1-b/a,2,-a(1+t)) = \frac{1}{\Gamma(1-\frac{b}{a})} \int_0^\infty e^{(1+t)s} s^{-\frac{b}{a}} (1+s)^\frac{b}{a} \, ds, \quad \text{for } t \geq 0. \]
Thus $U(1-b/a,2,-a(1+t)) > 0$ for all $t \geq 0$ and $a < 0 < b$, and the claim is proven.

5.6. **Proof of Proposition 4**. Suppose that $b/a \notin \{1, 2, \ldots\}$. We estimate the asymptotic behaviour of $c_t$ in (3.9) by substituting $r(t,s) = r_1(t) d_1(s) + r_2(t) d_2(s)$ and estimating the asymptotic behaviour of each resulting integral in
\[ c_t/|a|^2 (1-b/a)^{-1-b/a} = r_1(t) \int_0^t d_1(s)(1+s)U(1-b/a,2,-a(1+s)) \, ds + r_2(t) \int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s)) \, ds. \]
We start with the first integral in (5.3). By (2.3b) we have that
\[ (1+s)U(1-b/a,2,-a(1+s)) \sim |a|^{b/a-1}s^{b/a} \text{ as } s \to \infty. \]
Therefore by (2.21) we have that
\[ d_1(s)(1+s)U(1-b/a,2,-a(1+s)) \sim \frac{1}{|a|} e^{-as} \text{ as } s \to \infty. \]
Using the fact that \( a < 0 \), by (2.4) we get
\[
\begin{align*}
    r_1(t) \int_0^t d_1(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds & \sim |a|^{b/a - 2} t^{b/a}, \quad \text{as } t \to \infty. \\
\end{align*}
\]

(5.5)

In the case when \( b/a \in \{1, 2, \ldots\} \), \( c_t \) is given by
\[
    c_t/(a^2b|a|^{-1-b/a}) = r_1(t) \int_0^t \tilde{d}_1(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds
    + \tilde{r}_2(t) \int_0^t \tilde{d}_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds. 
\]

(5.6)

Again, we estimate the asymptotic behaviour of the first integral. By (5.4) and (2.23), we have that
\[
    \tilde{d}_1(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \sim |a|^{-1}e^{-as}, \quad \text{as } s \to \infty.
\]

Using the fact that \( a < 0 \) and that \( r_1 \) obeys (2.4), we get
\[
    r_1(t) \int_0^t \tilde{d}_1(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim |a|^{b/a - 2} t^{b/a}, \quad \text{as } t \to \infty. 
\]

(5.7)

We next prepare estimates of the integrand in the second integral in (5.3) and (5.6). When \( b/a \notin \{1, 2, \ldots\} \), we use (2.22) and (5.4) to obtain
\[
    d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \sim \Gamma(-\frac{b}{a})e^{\alpha}b|a|^{2b/a - 2} s^{2b/a - 1} \quad \text{as } s \to \infty. 
\]

(5.8)

When \( b/a \in \{1, 2, \ldots\} \), we use (2.24) and (5.4) to obtain
\[
    \tilde{d}_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \sim \frac{1}{\mathcal{W}(a, b, 0)} b|a|^{2b/a - 2} s^{2b/a - 1} \quad \text{as } s \to \infty. 
\]

(5.9)

We now prove part (a). If \( b < 0 \) and \( b/a \notin \{1, 2, \ldots\} \), we have that \( 2b/a > 0 \), so using (5.8)
\[
    \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim \Gamma(-\frac{b}{a})e^{\alpha}b|a|^{2b/a - 2} s^{2b/a - 1} \frac{1}{2b/a + 1}, 
\]

as \( t \to \infty \). Therefore by (2.5), as \( t \to \infty \), we have that
\[
    r_2(t) \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim b|a|^{b/a - 3} \frac{1}{2b/a + 1} t^{b/a}. 
\]

(5.10)

In the case that \( b < 0 \) and \( b/a \in \{1, 2, \ldots\} \) using (5.9) gives
\[
    \int_0^t \tilde{d}_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim \frac{1}{\mathcal{W}(a, b, 0)} b|a|^{2b/a - 2} s^{2b/a - 1} \frac{1}{2b/a + 1}, 
\]

as \( t \to \infty \). Therefore by (2.13) we have that
\[
    \tilde{r}_2(t) \int_0^t \tilde{d}_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim b|a|^{b/a - 3} \frac{1}{2b/a + 1} t^{b/a}, \quad \text{as } t \to \infty. 
\]

(5.11)

Examining (5.10) and (5.11), we see that the second integrals on the righthand sides of (5.3) and (5.6) have the same asymptotic behaviour. Similarly, by (5.5) and (5.7), we see that the first integrals on the righthand sides of (5.3) and (5.6) have the same asymptotic behaviour. Hence, if \( b < 0 \), we have that
\[
    \frac{c_t}{a^2b|a|^{-1-b/a}} \sim |a|^{b/a - 2} \left( b|a|^{-1} \frac{1}{2b/a + 1} + 1 \right) t^{b/a}, \quad \text{as } t \to \infty, 
\]

which implies (3.11).

We now prove part (b). In this case \( b > 0 \). Therefore, \( b/a \notin \{1, 2, \ldots\} \), so we estimate the asymptotic behaviour of each integral on the right hand side of
In particular, the estimate \(5.5\) holds for the first integral. To analyse the asymptotic behaviour of the second term, we must consider three subcases: \(2b/a < -1, 2b/a = -1\) and \(2b/a > -1\).

**Case 1:** \(2b/a < -1\). If \(2b/a < -1\), by \(5.8\) we have

\[
\lim_{t \to \infty} \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds = \Gamma(-\frac{b}{a})e^{a}b \int_0^\infty (1 + s)^2 U(1 - b/a, 2, -a(1 + s))^2 \, ds,
\]

where we have used \(2.20\) to obtain the formula for the limit. Hence by \(2.5\) we have

\[
r_2(t) \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim b|a|^{-b/a-1} \int_0^\infty (1 + s)^2 U(1 - b/a, 2, -a(1 + s))^2 \, ds \cdot t^{-b/a-1} \text{ as } t \to \infty.
\]

Since \(2b/a < -1\), we have that \(b/a < -1 - b/a < 0\), so using the last estimate, \(5.5\) and \(5.7\) we have \(5.12\). Notice also that \(c_t \to 0\) as \(t \to \infty\).

**Case 2:** \(2b/a = -1\). If \(2b/a = -1\), by \(5.8\) and \(2.5\) we have

\[
r_2(t) \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim b|a|^{b/a-3}t^{-1/2} \log t = \frac{1}{2} |a|^{-5/2}t^{-1/2} \log t, \quad \text{as } t \to \infty.
\]

Using this estimate, \(5.3\) and \(5.5\), together with the fact that \(b/a = -1/2\), we have \(3.13\). Notice also that \(c_t \to 0\) as \(t \to \infty\).

**Case 3:** \(2b/a > -1\). If \(2b/a > -1\), then by \(5.8\) and \(2.5\) we have

\[
r_2(t) \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) \, ds \sim b|a|^{b/a-3} t^b/a \frac{1}{2b/a + 1} \text{ as } t \to \infty.
\]

Using this estimate, \(5.3\) and \(5.5\), we have \(3.11\). Since \(b > 0\) and \(a < 0\), we have \(c_t \to 0\) as \(t \to \infty\).

Finally we prove part (c), or \(3.14\), in the case that \(a + b = 0\). We consider the asymptotic behaviour of the first term on the right hand side of \(5.3\). We can still apply \(5.5\) so that

\[
r_1(t) \int_0^t d_1(s)(1 + s)U(1 - b/a, 2, |a|(1 + s)) \, ds \sim |a|^{b/a-2}t^{b/a} = |a|^{b/a-2}t^{-1}, \quad \text{as } t \to \infty.
\]

Therefore

\[
\lim_{t \to \infty} r_1(t) \int_0^t d_1(s)(1 + s)U(1 - b/a, 2, |a|(1 + s)) \, ds = 0. \quad (5.12)
\]

Since \(a + b = 0\) and \(r_2\) obeys \(2.5\), we have \(r_2(t) \to \frac{1}{\Gamma(-b/a)} e^{-a}|a|^{-1-b/a} \text{ as } t \to \infty\).

Since \(d_2\) is given by \(2.20\), we have that

\[
\int_0^t d_2(s)(1 + s)U(1 - b/a, 2, |a|(1 + s)) \, ds = e^{a}b \Gamma(-\frac{b}{a}) \int_0^t (1 + s)^2 U(1 - b/a, 2, |a|(1 + s)) \, ds.
\]
By (2.3b), we have that \((1 + s)^2 U^2(1 - b/a, 2, |a|(1 + s)) \sim (|a|s)^{2b/a} = (|a|s)^{-2}\) as \(s \to \infty\). Therefore it follows that the integral tends to a finite limit and therefore

\[
\lim_{t \to \infty} r_2(t) \int_0^t d_2(s)(1 + s)U(1 - b/a, 2, -a(1 + s)) ds = |a|^{-1} e^{-b/a} \int_0^\infty (1 + s)^2 U^2(1 - b/a, 2, |a|(1 + s)) ds.
\]

Combining this limit with (5.12) and taking the limit as \(t \to \infty\) in (5.3), we obtain (3.14).

5.7. Proof of Theorem 3.4. Let \(t \geq 0\) and \(\Delta \geq 0\). Suppose first that \(b/a \notin \{1, 2, \ldots\}\). Using (1.9) and (2.17) one obtains

\[
\text{Cov}(X(t), X(t + \Delta)) = \sigma^2 r_1(t) r_1(t + \Delta) \int_0^t d_1^2(s) ds + \sigma^2 r_1(t) r_2(t + \Delta) \int_0^t d_1(s) d_2(s) ds + \sigma^2 r_2(t) r_2(t + \Delta) \int_0^t d_2^2(s) ds + \sigma^2 r_1(t) r_2(t) \int_0^t d_1(s) d_2(s) ds.
\]

Our plan is now to determine the exact asymptotic behaviour of each of the four terms in (5.13) as \(t \to \infty\) (for fixed \(\Delta \geq 0\)). Since \(a < 0\) from (2.21) we have

\[
d_1^2(t) \sim |a|^{-2b/a} e^{-2at} t^{-2b/a}, \quad \text{as } t \to \infty.
\]

Therefore, one can use the last limit and l’Hôpital’s rule to show that

\[
\int_0^t d_1^2(s) ds \sim \frac{1}{2|a|} |a|^{-2b/a} e^{-2at} t^{-2b/a}, \quad \text{as } t \to \infty.
\]

By (2.4), and the above limit, we have

\[
\lim_{t \to \infty} r_1(t) r_1(t + \Delta) \int_0^t d_1(s)^2 ds = e^{a\Delta} \lim_{t \to \infty} \left\{ \frac{r_1(t)}{e^{at} |a|^{b/a} e^{a(t+\Delta)|a|^{b/a}} e^{2at} |a|^{2b/a} b/a(t + \Delta)^{b/a}} \right\} \left\{ \frac{1}{2|a|} \right\} |a|^{-2b/a} e^{-2at} t^{-2b/a} \right\}
\]

\[
\int_0^t d_1(s)^2 ds \sim \frac{1}{2|a|} e^{a\Delta} \left\{ (t + \Delta)^{b/a} , t^{-b/a} \right\} = \frac{1}{2|a|} e^{a\Delta}.
\]

For the second and fourth terms in (5.13), we use (2.21) and (2.22) to get

\[
\int_0^t d_1(s) d_2(s) ds \sim |a|^{-2} e^{at} \Gamma\left(\frac{b}{a}\right) e^{-at}, \quad \text{as } t \to \infty.
\]
Thus, using (2.4) and (2.5), we get
\[
\lim_{t \to \infty} r_1(t) r_2(t + \Delta) \int_0^t d_1(s) d_2(s) \, ds \\
= \lim_{t \to \infty} \left\{ r_1(t) e^{at}|a|^{b/a-b/a-1} \frac{r_2(t + \Delta)}{(b/a-a-1)(t + \Delta)} e^{-a|t|^{b/a-a-1}} \right. \\
\times e^{at}|a|^{b/a-b/a-1} \frac{1}{(b/a-a-1)} e^{-a|t|^{b/a-a-1}} r_1(t) \\
\left. \times |a|^{-2} e^{at} \Gamma(-\frac{b}{a-a} e^{-at} \int_0^t d_1(s) d_2(s) \, ds \right\} \\
= b|a|^{-3} \lim_{t \to \infty} t^{b/a}(t + \Delta)^{-b/a-1} = 0. \tag{5.15}
\]

Similarly, we can show that the fourth term on the righthand side of (5.13) obeys
\[
\lim_{t \to \infty} r_1(t + \Delta) r_2(t) \int_0^t d_1(s) d_2(s) \, ds = 0. \tag{5.16}
\]

Finally, we consider the third term on the righthand side of (5.13). Using (2.22) we have
\[
d_2^2(s) \sim \Gamma(-\frac{b}{a-a} e^{2b/2|a|^{2b/a}} s^{2b/a}, \text{ as } s \to \infty.
\]

If $2b/a < -1$, we have that $d_2^2 \in L^1(0, \infty)$. In the case that $a + b < 0$, we have that $r_2(t) \to 0$ as $t \to \infty$, so
\[
\lim_{t \to \infty} r_2(t) r_2(t + \Delta) \int_0^t d_2^2(s) \, ds = 0. \tag{5.17}
\]

In the case that $2b/a < -1$ and $a + b = 0$, we have from (2.5) that $r_2(t) \to \frac{1}{(b/a-a-1)} e^{-a}$ as $t \to \infty$. Then from (2.20) we have
\[
\lim_{t \to \infty} r_2(t) r_2(t + \Delta) \int_0^t d_2^2(s) \, ds = \frac{1}{\Gamma(-b/a-a)} e^{-2b/a-2} e^{-a} \int_0^\infty d_2^2(s) \, ds \\
= b^2 |a|^{-2b/a-2} \int_0^\infty (1 + s)^2 U(1 + b/a, 2, |a|(1 + s)) \, ds. \tag{5.18}
\]

If $2b/a = -1$, we have that
\[
\int_0^t d_2^2(s) \, ds \sim \Gamma(-\frac{b}{a-a} e^{2b/2|a|^{2b/a}} 2b/a \log t, \text{ as } t \to \infty.
\]

Since $b/a = -1/2$, we have that $r_2(t) \sim kt^{-3/2}$ as $t \to \infty$ for some $k \neq 0$, and therefore (5.17) holds. If $2b/a > -1$, then
\[
\int_0^t d_2^2(s) \, ds \sim \Gamma(-\frac{b}{a-a} e^{2b/2|a|^{2b/a}} t^{2b/a+1} \frac{1}{2b/a}, \text{ as } t \to \infty.
\]

Using (2.5) we have
\[
r_2(t) r_2(t + \Delta) \int_0^t d_2^2(s) \, ds \sim \frac{1}{\Gamma(-b/a-a)} e^{-4b^2/a} \frac{1}{2b/a} t^{-1},
\]
as $t \to \infty$. Hence (5.17) holds.

Next, in the case when $b/a \in \{1, 2, \ldots\}$ and $a + b < 0$, by taking the limit as $t \to \infty$ on both sides of (5.13), using the limits (5.14), (5.15) and (5.16) on the first, second and fourth terms, and (5.17) on the third term on the righthand side of (5.13), we obtain (3.15).

On the other hand, when $a + b = 0$, by taking the limit as $t \to \infty$ on both sides of (5.13), using the limits (5.14), (5.15) and (5.16) on the first, second and fourth
terms, and [5.18] on the third term on the righthand side of [5.13], we obtain (3.16).

For the case when $b/a \in \{1, 2, \ldots\}$, then one decomposes $\text{Cov}(X(t), X(t + \Delta))$ as in (5.13) above but where $r_2, d_1$ and $d_2$ play the role of $r_2, d_1$ and $d_2$. Moreover as can be seen from (2.13), (2.23) and (2.24), $\tilde{r}$, $d_1$ and $d_2$ have the same asymptotic behaviour as $r_2, d_1$ and $d_2$ (to within a multiplicative constant) and so one can deduce the limits (3.15) and (3.16) as before.

5.8. Proof of Remark 2. Since $a > \int_0^\infty k(s) \, ds$, we have that $r$ is in $L^1(0, \infty)$, and moreover that $\int_0^\infty r(s) \, ds = 1/(a - \int_0^\infty k(s) \, ds)$. Therefore, we have that

$$\lim_{t \to \infty} \text{Cov}(X(t), X(t + \Delta)) = \sigma^2 \int_0^\infty r(s) r(s + \Delta) \, ds =: \gamma(\Delta).$$

Next, suppose that $k$ is a subexponential function. Then

$$\lim_{t \to \infty} \frac{r(t)}{k(t)} = \frac{1}{(a - \int_0^\infty k(s) \, ds)^2}.$$ See e.g., [2]. We determine the asymptotic behaviour of $\gamma(\Delta)$ as $\Delta \to \infty$ under the additional assumption that $k$ is asymptotic to a decreasing function. We then have

$$\frac{\gamma(\Delta)}{k(\Delta)} - \sigma^2 \int_0^\infty r(s) \, ds \cdot \frac{1}{(a - \int_0^\infty k(s) \, ds)^2} \cdot \frac{k(s + \Delta)}{k(\Delta)} \
= \sigma^2 \int_0^\infty r(s) \left( \frac{r(s + \Delta)}{k(s + \Delta)} - \frac{1}{(a - \int_0^\infty k(s) \, ds)^2} \right) \frac{k(s + \Delta)}{k(\Delta)} \, ds \
+ \sigma^2 \int_0^\infty r(s) \left( \frac{k(s + \Delta)}{k(\Delta)} - 1 \right) \, ds \cdot \frac{1}{(a - \int_0^\infty k(s) \, ds)^2}.$$

The first term has zero limit as $\Delta \to \infty$. The second term can be shown to have a zero limit as $\Delta \to \infty$ by splitting the integral into over the intervals $[0, T]$ and $(T, \infty)$ for $T > 0$ so large that $\int_T^\infty \int \, ds < \epsilon (a - \int_0^\infty k(s) \, ds)^2$, where $\epsilon > 0$ is taken arbitrarily small. Then, letting $\Delta \to \infty$, we see that the first of these two integrals tends to zero, while for the second using the monotonicity of $k$, the limit superior of the absolute value is less than $2\sigma^2\epsilon$. Letting $\epsilon \to 0$ confirms that

$$\lim_{\Delta \to \infty} \frac{\gamma(\Delta)}{k(\Delta)} = \sigma^2 \frac{1}{(a - \int_0^\infty k(s) \, ds)^2}.$$ Now we fix $t$ and compute the autocovariance function. We have

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma(\Delta)} = \lim_{\Delta \to \infty} \frac{k(\Delta)}{\gamma(\Delta)} \cdot \sigma^2 \int_0^t r(s + \Delta) \, ds \cdot \frac{k(s + \Delta)}{k(\Delta)} \, ds \
= (a - \int_0^\infty k(s) \, ds)^3 \sigma^2 \frac{1}{(a - \int_0^\infty k(s) \, ds)^2} \int_0^t r(s) \, ds.$$ Therefore, we have

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t + \Delta))}{\gamma(\Delta)} = \frac{\int_0^t r(s) \, ds}{\int_0^\infty r(s) \, ds} =: c_t,$$
so clearly $c_t \to 1$ as $t \to \infty$. Therefore the autocovariance function obeys (3.18).
5.9. Proof of Remark 4.1 If \( r(t) \to 0 \) as \( t \to \infty \), it is known that \( r \in L^1(0, \infty) \) and that \( r \) decays to zero exponentially. As a consequence
\[
\lim_{t \to \infty} \Cov(X(t), X(t + \Delta)) = \sigma^2 \int_0^\infty r(s) r(s + \Delta) \, ds =: \gamma(\Delta).
\]
Let us further suppose, for example, that \( r(t) \to 0 \) as \( t \to \infty \). Then there exists \( n \in \mathbb{Z}^+ \) and \( \alpha > 0 \) such that \( r(t)/t^{n-1}e^{-\alpha t} \to C \neq 0 \) as \( t \to \infty \).
We now determine the asymptotic behaviour of \( \gamma(\Delta) \) as \( \Delta \to \infty \). We start by writing
\[
\frac{\gamma(\Delta)}{\Delta^{n-1}e^{-\alpha\Delta}} - \sigma^2 C \int_0^\infty e^{-\alpha s} r(s) \, ds
\]
\[
= \sigma^2 \int_0^\infty e^{-\alpha s} r(s) \left( \frac{r(s + \Delta)}{(s + \Delta)^{n-1}e^{-\alpha(s+\Delta)}} - C \right) \frac{(s + \Delta)^{n-1}}{\Delta^{n-1}} \, ds
\]
\[
+ \left\{ \sigma^2 C \int_0^\infty e^{-\alpha s} r(s) \frac{(s + \Delta)^{n-1}}{\Delta^{n-1}} \, ds - \sigma^2 C \int_0^\infty e^{-\alpha s} r(s) \right\}.
\]
It can then be shown that the limits as \( \Delta \to \infty \) of the two terms on the righthand side is zero, so that
\[
\lim_{\Delta \to \infty} \frac{\gamma(\Delta)}{\Delta^{n-1}e^{-\alpha\Delta}} = \sigma^2 C \int_0^\infty e^{-\alpha s} r(s) \, ds =: c^*.
\]
Considering now the limit when \( \Delta \to \infty \) for \( t \) fixed, we have
\[
\frac{\Cov(X(t), X(t + \Delta))}{\gamma(\Delta)} = \frac{\Cov(X(t), X(t + \Delta))}{\Delta^{n-1}e^{-\alpha\Delta}} \cdot \gamma(\Delta)
\]
\[
= \sigma^2 \int_0^t r(s)e^{-\alpha s} \frac{r(s + \Delta)}{(s + \Delta)^{n-1}e^{-\alpha(s+\Delta)}} \frac{(s + \Delta)^{n-1}}{\Delta^{n-1}} \, ds \cdot \frac{\Delta^{n-1}e^{-\alpha\Delta}}{\gamma(\Delta)}.
\]
Therefore we have
\[
\lim_{\Delta \to \infty} \frac{\Cov(X(t), X(t + \Delta))}{\gamma(\Delta)} = \frac{1}{c^*} \sigma^2 \int_0^t r(s)e^{-\alpha s} \, ds =: c_t.
\]
We see that \( c_t \to 1 \) as \( t \to \infty \). Therefore (3.18) holds.

6. Proof of Results in Section 4

In this section, we give the proofs of the growth rates of \( X \) stated in Section 4.

6.1. Proof of Theorem 4.2 For \( b/a \notin \{-1, -2, \ldots\} \), from (2.29), (2.42) and (1.7), we can write \( X \) according to
\[
X(t) = r_3(t)c_3 + r_4(t)c_4 + \sigma r_3(t) \int_0^t d_3(s) \, dB(s) + \sigma r_4(t) \int_0^t d_4(s) \, dB(s).
\]
(6.1)
We have already deduced the asymptotic behaviour of \( r_3, r_4, d_3 \) and \( d_4 \) in (2.31), (2.32), (2.44) and (2.45). We recapitulate their limiting behaviour now:
\[
r_3(t) \sim a^{-1+\frac{b}{a}1-\frac{\alpha}{2}} t^{-\frac{\alpha}{2}}, \quad r_4(t) \sim \frac{1}{\Gamma(1+\frac{\alpha}{2})} a^{a(1+t)} t^{\frac{\alpha}{2}}, \quad \text{as } t \to \infty,
\]
\[
d_3(s) \sim b a^{-1+\frac{\alpha}{2}} s^{\frac{\alpha}{2}}, \quad d_4(s) \sim \Gamma(1+\frac{b}{a}) a^{-\frac{\alpha}{2}} e^{-a(s+1)} s^{-\frac{\alpha}{2}}, \quad \text{as } s \to \infty.
\]
Dividing across (6.1) by \( r_4(t) \) yields
\[
\frac{X(t)}{r_4(t)} = \frac{r_3(t)}{r_4(t)} c_3 + c_4 + \sigma \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s) \, dB(s) + \sigma \int_0^t d_4(s) \, dB(s).
\]
(6.2)
The asymptotic behaviour of the first and last terms is readily estimated. Since 
\( a > 0, r_3(t)/r_4(t) \to 0 \) as \( t \to \infty \). \( a > 0 \) also implies \( d_4 \in L^2(0, \infty) \). Therefore by the Martingale Convergence Theorem for continuous martingales (cf., e.g., [19 Thm. V.1.8]) we have that

\[
\lim_{t \to \infty} \sigma \int_0^t d_4(s) \, dB(s) = \sigma \int_0^\infty d_4(s) \, dB(s), \quad \text{a.s.}
\]

If

\[
\lim_{t \to \infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_4(s) \, dB(s) = 0, \quad \text{a.s.} \tag{6.3}
\]

then we obtain

\[
\lim_{t \to \infty} \frac{X(t)}{r_4(t)} = c_4 + \sigma \int_0^\infty d_4(s) \, dB(s) = C, \quad \text{a.s.}
\]

By (2.32) we therefore have

\[
\lim_{t \to \infty} \frac{X(t)}{e^{\alpha t^\frac{2}{a}}} = \Gamma(1 + \frac{b}{a})e^{\sigma a^\frac{2}{a}}c_4 + \Gamma(1 + \frac{b}{a})e^{\sigma a^\frac{2}{a}} \int_0^\infty d_4(s) \, dB(s) = C, \quad \text{a.s.}
\]

which implies (4.4) and also part (b), due to the definitions of \( c_4 \) and \( d_4 \) in (2.33) and (2.43) and of \( C \) in part (b).

Moreover, it follows from [20 Ch. 2.13.5, p.304-305] that

\[
\mathbb{E}[C] = \lim_{t \to \infty} \mathbb{E}\left[\frac{X(t)}{e^{\alpha t^\frac{2}{a}}}\right] = \lim_{t \to \infty} \frac{x(t)}{e^{\alpha t^\frac{2}{a}}} = c_4 \Gamma(1 + \frac{b}{a})e^{\sigma a^\frac{2}{a}},
\]

and that

\[
\text{Var}[C] = \lim_{t \to \infty} \text{Var}\left[\frac{X(t)}{e^{\alpha t^\frac{2}{a}}}\right] = \sigma^2 \Gamma^2(1 + \frac{b}{a})e^{2\sigma a^\frac{2}{a}} \int_0^\infty d_4^2(s) \, ds > 0.
\]

These results and (2.33) and (2.43) establish the validity of parts (c) and (d).

All that remains to show is that (6.3) is indeed true. If \( \frac{2b}{a} > -1 \), then \( d_3 \in L^2(0, \infty) \), and the stochastic integral tends to a finite limit by the Martingale Convergence Theorem. Since \( r_3(t)/r_4(t) \to 0 \) as \( t \to \infty \), we obtain

\[
\lim_{t \to \infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s) \, dB(s) = 0, \quad \text{a.s.}
\]

If \( \frac{2b}{a} > -1 \), then \( d_3 \in L^2(0, \infty) \). Indeed, the quadratic variation of \( \int_0^t d_3(s) \, dB(s) \) is given by

\[
v(t) := \int_0^t d_3^2(s) \, ds \sim b^2 a^\frac{2}{a^2} \left(\frac{2b}{a} + 1\right)^{-1} t^{\frac{2b}{a^2} + 1}, \quad \text{as } t \to \infty,
\]

and hence \( \log \log v(t) \sim \log \log t \) as \( t \to \infty \). Therefore the stochastic integral \( \int_0^t d_3(s) \, dB(s) \) obeys the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [19 Exercise V.1.15]), so

\[
\limsup_{t \to \infty} \frac{\int_0^t d_3(s) \, dB(s)}{\sqrt{2v(t) \log \log v(t)}} = -\liminf_{t \to \infty} \frac{\int_0^t d_3(s) \, dB(s)}{\sqrt{2v(t) \log \log v(t)}} = 1, \quad \text{a.s.}
\]

These asymptotic estimates for the stochastic integral and \( v \), together with (2.31) and (2.32) yield

\[
\lim_{t \to \infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s) \, dB(s) = 0, \quad \text{a.s.}
\]

as required. The above argument holds similarly for the case when \( \frac{2b}{a} = -1 \).

The case \( b/a \in \{-1, -2, -3, \ldots\} \) can be dealt with similarly. While we only have the crude estimate (2.46) for the asymptotic behaviour of \( d_3 \), it is nevertheless the
case that the quadratic variation of $\int_0^t \tilde{d}_3(s) dB(s)$ can grow no faster than a power of $t$ as $t \to \infty$ (or indeed may converge as $t \to \infty$). Thus we obtain

$$\lim_{t \to \infty} \frac{r_3(t)}{r_4(t)} \int_0^t \tilde{d}_3(s) dB(s) = 0, \quad \text{a.s.}$$

as before.

6.2. Proof of Theorem 4.1. Since $a < 0$ and $a + b > 0$, we have $b/a \notin \{1, 2, \ldots\}$. Therefore, from (2.2), (2.17) and (1.7) one has,

$$X(t) = r_1(t)c_1 + r_2(t)c_2 + \sigma r_1(t) \int_0^t d_1(s) dB(s) + \sigma r_2(t) \int_0^t d_2(s) dB(s). \quad (6.4)$$

We have already deduced the asymptotic behaviour of $r_1$, $r_2$, $d_1$ and $d_2$ in (2.4), (2.5), (2.21) and (2.22). We recapitulate their limiting behaviour now:

$$r_1(t) \sim e^{at} |a|^{b/2} t^{b/2}, \quad r_2(t) \sim \frac{e^{-a}}{\Gamma(-\frac{b}{a})} |a|^{-1 - \frac{b}{a}} t^{-1 - \frac{b}{a}}, \quad \text{as } t \to \infty,$$

$$d_1(s) \sim |a|^{-\frac{b}{a}} e^{-as} s^{-\frac{b}{a}}, \quad d_2(s) \sim \Gamma(-\frac{b}{a}) b e^{as} |a|^{-1 + \frac{b}{a}} s^{\frac{b}{a}}, \quad \text{as } s \to \infty.$$ 

Dividing across (6.4) by $r_2(t)$ yields

$$\frac{X(t)}{r_2(t)} = \frac{r_1(t)}{r_2(t)} c_1 + c_2 + \sigma \frac{r_1(t)}{r_2(t)} \int_0^t d_1(s) dB(s) + \sigma \int_0^t d_2(s) dB(s). \quad (6.5)$$

The asymptotic behaviour of the first and last terms is readily estimated. Since $a < 0$, we have from (2.4) and (2.5) that $r_1(t)/r_2(t) \to 0$ as $t \to \infty$. Also, since $a < 0$ and $a + b > 0$, we have $2b/a < -2$. Hence $d_2 \in L^2(0, \infty)$ and therefore by the martingale convergence theorem for continuous martingales (cf., e.g., [19, Thm. V.1.8]) we have

$$\lim_{t \to \infty} \int_0^t d_2(s) dB(s) = \int_0^\infty d_2(s) dB(s), \quad \text{a.s.} \quad (6.6)$$

We now examine the asymptotic behaviour of the third term on the righthand side of (6.5). Firstly observe that $\int_0^t d_1(s) dB(s)$ is normally distributed with mean zero and variance given by

$$v_1(t) = \int_0^t d_2^2(s) ds.$$ 

By l’Hôpital’s rule we have

$$v_1(t) \sim \frac{1}{2} |a|^{-1 - \frac{b}{a}} e^{-2at} (1 + t)^{-\frac{b}{a}}, \quad \log \log v_1(t) \sim \log t, \quad \text{as } t \to \infty,$$

and so we have by the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [19, Exercise V.1.15]) that

$$\limsup_{t \to \infty} \frac{\int_0^t d_1(s) dB(s)}{\sqrt{2v_1(t) \log \log v_1(t)}} = \liminf_{t \to \infty} \frac{\int_0^t d_1(s) dB(s)}{\sqrt{2v_1(t) \log \log v_1(t)}} = 1, \quad \text{a.s.}$$

Thus we have

$$\limsup_{t \to \infty} \frac{r_1(t) \int_0^t d_1(s) dB(s)}{\sqrt{\log t}} = \liminf_{t \to \infty} \frac{r_1(t) \int_0^t d_1(s) dB(s)}{\sqrt{\log t}} = \frac{\sigma}{\sqrt{|a|}}. \quad (6.7)$$

Using (6.7), the fact that $\log t/r_2(t) \to 0$ as $t \to \infty$, together with (6.6), we arrive at

$$\lim_{t \to \infty} \frac{X(t)}{r_2(t)} = c_2 + \sigma \int_0^\infty d_2(s) dB(s), \quad \text{a.s.}$$
By (2.5) we therefore obtain
\[ \lim_{t \to \infty} \frac{X(t)}{t^{-\frac{1}{4}}} = \frac{e^{-\alpha}}{\Gamma(-\frac{b}{a})} |a|^{-1-\frac{b}{a}} c_2 + \sigma \frac{e^{-\alpha}}{\Gamma(-\frac{b}{a})} |a|^{-1-\frac{b}{a}} \int_0^\infty d_2(s) \, dB(s) = C, \quad \text{a.s.} \] (6.8)
which implies part (a) and also part (b), due to the definitions of \(c_2\) and \(d_2\) in (2.9) and (2.20) and of \(C\) in part (b).

Moreover, it follows from [20, Ch. 2.13.5, p.304-305] that
\[ \mathbb{E}[C] = \lim_{t \to \infty} \frac{\mathbb{E}[X(t)]}{t^{-\frac{1}{4}}} = \lim_{t \to \infty} \frac{x(t)}{t^{-\frac{1}{4}}} = c_2 \frac{e^{-\alpha}}{\Gamma(-\frac{b}{a})} |a|^{-1-\frac{b}{a}}, \]
and that
\[ \text{Var}[C] = \lim_{t \to \infty} \frac{\text{Var}[X(t)]}{t^{-2-\frac{b}{a}}} = \sigma^2 \frac{e^{-2\alpha}}{\Gamma^2(-\frac{b}{a})} |a|^{-2-2\frac{b}{a}} \int_0^\infty d_2^2(s) \, ds > 0 \]
These results and (2.9) and (2.20) establish the validity of parts (c) and (d).

6.3. Proof of Theorem 4.3. From (2.48), (2.54) and (1.7), we can write \(X\) according to
\[ X(t) = r_5(t) c_5 + r_6(t) c_6 + \sigma r_5(t) \int_0^t d_5(s) \, dB(s) + \sigma r_6(t) \int_0^t d_6(s) \, dB(s). \] (6.9)
We can deduce the asymptotic behaviour of \(r_5\), \(r_6\), \(d_5\) and \(d_6\) using (2.53) and (2.49). Hence
\[ r_5(t) \sim \frac{1}{2b^{1/4} \sqrt{\pi}} b^{2/4} t^{-1/4}, \quad \text{as } t \to \infty, \quad r_6(t) \sim \frac{\sqrt{\pi}}{2b^{1/4}} e^{-2\sqrt{\pi} t^{-1/4}}, \quad \text{as } t \to \infty, \]
\[ d_5(s) \sim \sqrt{\pi} b^{1/4} s^{1/4} e^{-2\sqrt{\pi} s^{1/2}}, \quad \text{as } s \to \infty, \quad d_6(s) \sim \frac{1}{\sqrt{\pi}} b^{1/4} s^{1/4} e^{2\sqrt{\pi} s^{1/2}}, \quad \text{as } s \to \infty. \]
Dividing across (6.9) by \(r_5(t)\) yields
\[ \frac{X(t)}{r_5(t)} = c_5 + \frac{r_6(t)}{r_5(t)} c_6 + \sigma \int_0^t d_5(s) \, dB(s) + \sigma \frac{r_6(t)}{r_5(t)} \int_0^t d_6(s) \, dB(s). \] (6.10)
The asymptotic behaviour of the second and third terms is readily estimated. First as \(b > 0\), \(r_6(t)/r_5(t) \to 0\) as \(t \to \infty\). Also, \(d_5 \in L^2(0, \infty)\) and therefore by the martingale convergence theorem for continuous martingales (cf., e.g., [19, Thm. V.1.8]) we have
\[ \lim_{t \to \infty} \int_0^t d_5(s) dB(s) = \int_0^\infty d_5(s) dB(s), \quad \text{a.s.} \] (6.11)
We now examine the asymptotic behaviour of the fourth term on the right-hand side of (6.10). Firstly observe that \(\int_0^t d_6(s) dB(s)\) is normally distributed with mean zero and variance given by
\[ v_3(t) := \int_0^t d_6(s)^2 \, ds. \]
By using l’Hôpital’s rule, the asymptotic behaviour of \(v_3(t)\) as \(t \to \infty\) can be found:
\[ \lim_{t \to \infty} \frac{v_3(t)}{t e^{\frac{v_3(t)}{\sqrt{t}}} \pi} = \frac{1}{2}, \quad \text{and } \lim_{t \to \infty} \frac{\log \log v_3(t)}{\log t} = \frac{1}{2}. \]
Thus by the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [19, Exercise V.1.15]) we have that
\[
\limsup_{t \to \infty} \frac{\int_0^t d_6(s) \, dB(s)}{\sqrt{2} v_3(t) \log \log v_3(t)} = - \liminf_{t \to \infty} \frac{\int_0^t d_6(s) \, dB(s)}{\sqrt{2} v_3(t) \log \log v_3(t)} = 1, \quad \text{a.s.}
\]
Thus we have
\[
\int_0^t d_6(s) \, dB(s) = O \left( t^{1/2} e^{2\sqrt{bt} \sqrt{\log t}} \right), \quad \text{as } t \to \infty.
\]
Therefore
\[
\frac{r_6(t)}{r_5(t)} \int_0^t d_6(s) \, dB(s) = O \left( t^{1/2} e^{-2\sqrt{bt} \sqrt{\log t}} \right), \quad \text{as } t \to \infty,
\]
and so
\[
\lim_{t \to \infty} \frac{r_6(t)}{r_5(t)} \int_0^t d_6(s) \, dB(s) = 0 \quad \text{a.s.} \quad (6.12)
\]
Taking the limit as \( t \to \infty \) in (6.10) and using (6.12) together with (6.11), we arrive at
\[
\lim_{t \to \infty} \frac{X(t)}{r_5(t)} = c_5 + \sigma \int_0^\infty d_5(s) \, dB(s), \quad \text{a.s.}
\]
Using the asymptotic behaviour of \( r_5 \) we therefore obtain
\[
\lim_{t \to \infty} \frac{X(t)}{e^{2\sqrt{bt} t^{-1/4}}} = \lim_{t \to \infty} \frac{X(t)}{r_5(t)} \cdot \frac{r_5(t)}{e^{2\sqrt{bt} t^{-1/4}}} = \frac{1}{2b^{1/4} \sqrt{\pi}} \left( c_5 + \sigma \int_0^\infty d_5(s) \, dB(s) \right) = C, \quad \text{a.s.} \quad (6.13)
\]
which implies part (a) and also part (b), due to the definitions of \( c_5 \) and \( d_5 \) in (2.50) and (2.55) and of \( C \) in part (b).
Moreover, it follows from [20, Ch. 2.13.5, p.304-305] that
\[
\mathbb{E}[C] = \lim_{t \to \infty} \frac{\mathbb{E} \left[ X(t) \right]}{e^{2\sqrt{bt} t^{-1/4}}} = \lim_{t \to \infty} \frac{x(t)}{e^{2\sqrt{bt} t^{-1/4}}} = \frac{1}{2b^{1/4} \sqrt{\pi}} c_5,
\]
and that
\[
\text{Var}[C] = \lim_{t \to \infty} \frac{\text{Var} \left[ X(t) \right]}{e^{4\sqrt{bt} t^{-1/2}}} = \frac{1}{4b^{1/2} \pi} \sigma^2 \int_0^\infty d_5^2(s) \, ds > 0.
\]
These results and (2.50) and (2.55) establish the validity of parts (c) and (d).

7. Proof of Theorem 3.1 and Theorem 3.2

We note that similar asymptotic analysis as that above would give us, for \( a + b \leq 0 \),
\[
\limsup_{t \to \infty} \frac{X(t)}{\sqrt{2} \log t} = \frac{\sigma}{\sqrt{2|a|}}.
\]
We choose however to prove this result via Theorem 3.1 as it provides an interesting result regarding the asymptotic behaviour of the process.
7.1. A preliminary lemma.

**Lemma 7.1.** Let $a < 0$ and $a + b = 0$. Define $H$ by

$$H(t, u) = \int_u^t d_2(s) \frac{b}{1 + s} e^{-au} \int_u^s \sigma e^{au} \, dw \, ds, \quad 0 \leq u \leq t, \quad (7.1)$$

where $d_2$ is as given by (2.20). Define $H_\infty$

$$H_\infty(u) = \frac{\sigma}{|a|} \int_u^\infty d_2(s) \, ds - \frac{\sigma}{|a|} e^{-au} \int_u^\infty e^{as} d_2(s) \, ds, \quad u \geq 0. \quad (7.2)$$

Then

$$\lim_{t \to \infty} \int_0^t H(t, u) \, dB(u) = \int_0^\infty H_\infty(u) \, dB(u), \quad a.s.$$  

**Proof.** The proof of this almost sure convergence result is an application of Theorem 7 in [4]. $H$ simplifies to

$$H(t, u) = \frac{\sigma}{|a|} \int_u^t d_2(s) \, ds - \frac{\sigma}{|a|} e^{-au} \int_u^t e^{as} d_2(s) \, ds.$$  

$H_\infty$ given by (7.2) is well-defined by virtue of (2.22). To estimate the rate of decay of $H_\infty$ to zero, we use (2.22) to get

$$\int_u^\infty d_2(s) \, ds \sim |a|^{-1} e^a u^{-1}, \quad as \to \infty, \quad (7.3a)$$

$$e^{-au} \int_u^\infty e^{as} d_2(s) \, ds \sim |a|^{-2} e^a u^{-2}, \quad as \to \infty. \quad (7.3b)$$

Thus $H_\infty(u) \sim \sigma |a|^{-2} e^a u^{-1}$ as $u \to \infty$ and so $H_\infty \in L^2(0, \infty)$.

We now wish to show that

$$\lim_{t \to \infty} \int_0^{t} (H(t, u) - H_\infty(u))^2 \, du \cdot \log t = 0. \quad (7.4)$$

Define

$$f(t) := \frac{\sigma}{|a|} \int_t^\infty d_2(s) \, ds, \quad g(t) := \frac{\sigma}{|a|} \int_t^\infty e^{as} d_2(s) \, ds.$$  

Then the Cauchy-Schwarz inequality gives

$$\int_0^{t} (H(t, u) - H_\infty(u))^2 \, du = \int_0^{t} (-f(t) + e^{-au} g(t))^2 \, du$$

$$\leq \int_0^{t} 2f(t)^2 \, du + \int_0^{t} 2e^{-2au} g(t)^2 \, du$$

$$= 2f(t)^2 + \frac{1}{2|a|} g(t)^2 (e^{-2at} - 1).$$

The asymptotic relations (7.3) determine completely the asymptotic behaviour of $f$ and $g$, and this, together with the last inequality, gives (7.4).

We show now that there exist $q \geq 0$ and $c_q > 0$ such that

$$\int_0^{t} \left[ \frac{\partial}{\partial t} H(t, u) \right]^2 \, du \leq c_q (1 + t)^{2q}, \quad t \geq 0. \quad (7.5)$$

To do this we estimate according to

$$\int_0^{t} \left[ \frac{\partial}{\partial t} H(t, u) \right]^2 \, du = \frac{\sigma^2}{|a|^2} d_2^2(t) \int_0^t \left( 1 - e^{-au} e^{at} \right)^2 \, du,$$

and using (2.22) we see that $H$ obeys (7.5) for any $q \geq 0$ and $c_q > 0$. Also as $H(t, t) = 0$ for all $t \geq 0$ then all of the conditions of [4, Theorem 7] are satisfied and so we conclude $\lim_{t \to \infty} \int_0^{t} H(t, u) \, dB(u) = \int_0^{\infty} H_\infty(u) \, dB(u)$ a.s. as required. □
7.2. **Proof of Theorem 3.1.** We start by defining a process $Y = \{Y(t) : t \geq -1\}$, which is related to $U$ defined by (3.1). It will be used in proving Theorems 3.1 and 4.1. $Y$ is defined by $Y(t) = \psi(t)$ for $t \in [-1, 0]$ and it obeys

$$dY(t) = aY(t) \, dt + \sigma \, dB(t), \quad t \geq 0. \quad (7.6)$$

Note that (3.3) is an immediate consequence of (3.4) or (3.6) and the fact that

$$\limsup_{t \to \infty} \frac{U(t)}{\sqrt{2 \log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t \to \infty} \frac{U(t)}{\sqrt{2 \log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.} \quad (7.7)$$

Therefore it remains to prove (3.4) and (3.6). Firstly extend $U$ to $[-1, 0)$ by $U(t) = 0$ for $t \in [-1, 0)$. Then for $Y$ defined by (7.6), for $t \geq 0$ we have $\dot{Y}(t) - U(t) = \psi(0)e^{at}$. Therefore $U(t) - Y(t) \to 0$ as $t \to \infty$, a.s. Hence it remains to prove that $X(t) - Y(t) \to 0$ as $t \to \infty$ a.s. in order to establish (3.4) and (3.6).

Define $Z(t) = X(t) - Y(t)$ for $t \geq -1$. Then $Z(t) = 0$ for $t \in [-1, 0]$ and

$$Z'(t) = aZ(t) + b \frac{1}{t+1} \int_0^t Z(s) \, ds + f(t), \quad t > 0, \quad (7.8)$$

where

$$f(t) := b \frac{1}{t+1} \int_{-1}^0 \psi(s) \, ds + b \frac{1}{t+1} \int_0^t Y(s) \, ds, \quad t \geq 0. \quad (7.9)$$

Next we show that $f(t) \to 0$ as $t \to \infty$, a.s. This clearly follows if $\int_0^t Y(s) \, ds/t \to 0$ as $t \to \infty$ a.s. To prove this, note that

$$Y(t) = \psi(0) + a \int_0^t Y(s) \, ds + \sigma B(t), \quad t \geq 0. \quad (7.10)$$

Since $U$ obeys (7.7), $Y(t) - U(t) \to 0$ as $t \to \infty$, $Y$ obeys

$$\limsup_{t \to \infty} \frac{|Y(t)|}{\sqrt{2 \log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.}$$

Therefore by this limit and the strong law of large numbers for standard Brownian motion [13, 2.9.3], we get from (7.10) that $\int_0^t Y(s) \, ds/t \to 0$ as $t \to \infty$ a.s., and therefore that $f(t) \to 0$ as $t \to \infty$ a.s. Indeed, by using the Law of the iterated logarithm for standard Brownian motion [13], for every $\epsilon > 0$, we have

$$\limsup_{t \to \infty} \frac{f(t)}{t^{1/2} \sqrt{2 \log \log t}} = -\liminf_{t \to \infty} \frac{f(t)}{t^{1/2} \sqrt{2 \log \log t}} = \frac{|b| \sigma}{|a|}, \quad \text{a.s.} \quad (7.11)$$

Recalling that the resolvent $r$ obeys (1.6), by applying the conventional variation of constants formula to (7.8), and using (2.17) in the case that $b/a \not\in \{1, 2, \ldots\}$, we get

$$Z(t) = \int_0^t r(t, s) f(s) \, ds = r_1(t) \int_0^t d_1(s) f(s) \, ds + r_2(t) \int_0^t d_2(s) f(s) \, ds \quad (7.12)$$

and hence

$$|Z(t)| \leq |r_1(t)| \int_0^t |d_1(s)||f(s)| \, ds + |r_2(t)| \int_0^t |d_2(s)||f(s)| \, ds. \quad (7.13)$$

The first integral on the right hand side of (7.13) converges to zero using (2.4), (2.21) and (7.11) on application on l’Hôpital’s rule.

It transpires that the limiting behaviour as $t \to \infty$ of the second integral on the right hand side of (7.13) differs according to whether $a + b < 0$ or $a + b = 0$. We consider first the case when $a + b < 0$. Using (2.22) and (7.11) in the case that $2b + a > 0$, there exists an a.s. finite positive random variable $M$ such that

$$\limsup_{t \to \infty} \int_0^t |d_2(s)||f(s)| \, ds \leq \limsup_{t \to \infty} M \int_0^\infty (1 + s)^{\frac{1}{2} - 1/2} \sqrt{\log(\log(e + s))} \, ds < \infty.$$
Hence
\[
\lim_{t \to \infty} \int_0^t d_2(s) f(s) \, ds = \int_0^\infty d_2(s) f(s) \, ds \in (-\infty, \infty) \quad a.s. \tag{7.14}
\]

Since \( r_2 \) obeys (2.5), we have
\[
\lim_{t \to \infty} |r_2(t)| \int_0^t |d_2(s)||f(s)| \, ds = 0, \quad a.s. \tag{7.15}
\]

In the case when \( 2b + a \leq 0 \), we notice from (7.11) that for any \( \epsilon < 1/2 \) that \( f(t)/t^{-1/2+\epsilon} \to 0 \) as \( t \to \infty \) on the a.s. event \( \Omega_1 \), say. Therefore, by the continuity of \( f \) and this relation, there is an a.s. finite and positive random variable \( K_r \) such that \( |f(t, \omega)| \leq K_r(1 + t)^{-1/2+\epsilon} \) for all \( t \geq 0 \). Therefore, by virtue of the continuity of \( r_2, d_2 \) and (2.5) and (2.22), there exists an a.s. finite and positive random variable \( M_r \) such that, for all \( t \geq 0 \), we have
\[
|r_2(t)| \int_0^t |d_2(s)||f(s, \omega)| \, ds \leq M_r(\omega)(1 + t)^{-1 - \frac{b}{a} - \frac{1}{2} - \epsilon} ds \\
\quad \leq M_r(\omega)(1 + t)^{-1 - \frac{b}{a} + \frac{1}{2} + \epsilon} \frac{1}{b/a + 1/2 + \epsilon},
\]

for each \( \omega \in \Omega_1 \), with the last inequality holding because \( b/a - 1/2 + \epsilon > -1 \). Since \( \Omega_1 \) is an a.s. event, we again have (7.15) and so, using this limit and (7.13), we see that \( Z(t) \to 0 \) as \( t \to \infty \) a.s. in the case that \( b/a \notin \{1, 2, \ldots\} \). We can demonstrate that \( Z(t) \to 0 \) as \( t \to \infty \) a.s. in a similar manner when \( b/a \in \{1, 2, \ldots\} \) by using the asymptotic behaviour of \( r_1, \tilde{r}_2, d_1 \) and \( \tilde{d}_2 \). Hence the proof of part (i) is complete.

For the the proof of part (ii), we consider the case \( a + b = 0 \). Recall that \( Y \) can be written in the form
\[
Y(t) = \psi(0)e^{at} + \sigma e^{at} \int_0^t e^{-as} dB(s), \quad t \geq 0.
\]

In this case, we wish to show that \( Z \) tends to a non–trivial limit. Arguing as above, we have that the first integral on the right hand side of (7.12) tends to zero as \( t \to \infty \) a.s. As to the second term on the right hand side of (7.12), by using a stochastic Fubini theorem, it is seen that
\[
\int_0^t d_2(s) f(s) \, ds = \int_0^t \frac{b}{1 + s} d_2(s) ds \int_{-1}^0 \psi(u) \, du \\
\quad + \int_0^t d_2(s) \frac{b}{1 + s} \int_0^s \psi(0)e^{au} \, du \, ds + \int_0^t H(t, u) \, dB(u),
\]

where \( H \) is given by (7.1). The two Riemann integrals on the right–hand side of the above equation converge to finite limits as \( t \to \infty \). Moreover as (7.14) holds therefore the stochastic integral on the right–hand side above converges almost surely. Recalling from (2.5) that \( \lim_{t \to \infty} r_2(t) = e^{-a} |a|^{1-b/a-1} \) in the case when \( a + b = 0 \), and by applying Lemma 7.1, we have that
\[
\lim_{t \to \infty} Z(t) = \frac{e^{-a}}{|a|^{b/a+1}} \int_0^\infty \frac{b}{1 + s} d_2(s) ds \int_{-1}^0 \psi(u) \, du \\
\quad + \frac{e^{-a}}{|a|^{b/a+1}} \int_0^\infty d_2(s) \frac{b}{1 + s} \int_0^s \psi(0)e^{au} \, du \, ds + \int_0^\infty H_\infty(u) \, dB(u), \quad a.s.,
\]

where \( H_\infty \) is given by (7.2). We call the limit on the right hand side \( L \). Therefore \( X(t) - U(t) \to L \) as \( t \to \infty \) a.s. Clearly \( L \) is an \( \mathcal{F}_t(\infty) \)–measurable normal random variable. In order to see that \( L \) is nontrivial, we may use Itô’s isometry to show
that its mean and variance are given by the formulae in the statement of part (ii) of the theorem. Since
\[ \lim_{t \to \infty} \frac{1}{1 + t} \int_0^t U(s) \, ds = 0, \quad \text{a.s.} \]
The proofs of (3.5) and (3.7) are simple consequences of the fact that \( X(t) - U(t) \) tends to the finite limits \( 0 \) and \( L \) as \( t \to \infty \) a.s. in case (i) and (ii) respectively.

7.3. Proof of Theorem 3.2. Let \( Y \) and \( Z \) be as defined in the proof of Theorem 3.1. To attain a bound on the rate of \( X - U \) tending to zero, the integral terms in (7.13) need to be analysed more carefully. From (7.11), and by using the continuity of \( f \), it follows for every \( \omega \) in an almost sure event \( \Omega_1 \) that there exists an a.s. finite and positive random variable \( K = K(\omega) > 0 \) such that such that
\[ |f(t, \omega)| \leq K(\omega)(1 + t)^{-1/2} \sqrt{\log \log(t + e)}, \quad t \geq 0. \]
For the first integral in (7.13), we start by using l’Hôpital’s rule to show that
\[ \lim_{t \to \infty} \int_0^t e^{-as}(1 + s)^{-\frac{a}{2}} \sqrt{\log \log(e + s)} \, ds = \frac{e^{-at}(1 + t)^{-\frac{a}{2}} \sqrt{\log \log(e + t)}}{(1 + t)^{-\frac{a}{2}} \sqrt{\log \log(e + t)}} \in (0, \infty). \]
Therefore, there is \( K_3 > 0 \) such that
\[ \int_0^t e^{-as}(1 + s)^{-\frac{a}{2}} \sqrt{\log \log(e + s)} \, ds \leq K_3 e^{-at}(1 + t)^{-\frac{a}{2}} \sqrt{\log \log(e + t)}, \]
for all \( t \geq 0 \). Now, by using (2.4) and (2.21) and the continuity of \( r_1 \) and \( d_1 \), we have that there exist \( K_1 > 0 \) and \( K_2 > 0 \) such that
\[ |r_1(t)| \leq K_1 e^{a'(1 + t)^{\frac{a}{2}}}, \quad t \geq 0; \quad |d_1(s)| \leq K_2 e^{-as}(1 + s)^{-\frac{a}{2}}, \quad s \geq 0. \]
Therefore for all \( \omega \in \Omega_1 \) and \( t \geq 0 \) we have
\[ |r_1(t)| \int_0^t |d_1(s)||f(s, \omega)|ds \leq K_4(\omega)(1 + t)^{-1/2} \sqrt{\log \log(t + e)}, \]
where \( K_4(\omega) = K_1 K_2 K(\omega) K_3 \). Hence
\[ \limsup_{t \to \infty} \frac{|r_1(t)| \int_0^t |d_1(s)||f(s, \omega)|ds}{(1 + t)^{-1/2} \sqrt{\log \log(1 + t)}} \in [0, \infty), \quad \text{a.s.} \quad (7.16) \]
For the second integral in (7.13), we showed in the proof of Theorem 3.1 that \( \limsup_{t \to \infty} \int_0^t |d_2(s)f(s)|ds < +\infty \) a.s. in the case when \( 2b + a > 0 \). Hence
\[ \limsup_{t \to \infty} \frac{|r_2(t)| \int_0^t |d_2(s)||f(s)|ds}{(1 + t)^{-1/2} \sqrt{\log \log(1 + t)}} \in [0, \infty). \quad (7.17) \]
Moreover in this parameter regime \( -1/2 < -1 - b/a < 0 \), and so comparing the decay rates in (7.16) and (7.17) gives (i).

When \( 2b + a < 0 \), we may use l’Hôpital’s rule to get
\[ \lim_{t \to \infty} \int_0^t (1 + s)^{-\frac{b}{2} - 1/2} \sqrt{\log \log(e + s)} \, ds = (0, \infty). \]
Hence there exists \( K_7 > 0 \) such that
\[ \int_0^t (1 + s)^{-\frac{b}{2} - 1/2} \sqrt{\log \log(e + s)} \, ds \leq K_7(1 + t)^{\frac{b}{2} + 1/2} \sqrt{\log \log(e + t)}, \quad t \geq 0. \]
Since \( r_2 \) and \( d_2 \) obey (2.5) and (2.22), we have that there exist \( K_5 > 0 \) and \( K_6 > 0 \) such that

\[
|r_2(t)| \leq K_5(1 + t)^{-\frac{3}{2}}, \quad t \geq 0; \quad |d_2(s)| \leq K_6(1 + s)^{\frac{3}{2}}, \quad s \geq 0.
\]

Therefore for all \( \omega \in \Omega_1 \) and \( t \geq 0 \) we have

\[
|r_2(t)| \int_0^t |d_2(s)||f(s, \omega)| \, ds \\
\leq K_5 K_6 K(1 + t)^{-1/2} \sqrt{\log \log(e + t)},
\]

and so

\[
\limsup_{t \to \infty} \frac{|r_2(t)| \int_0^t |d_2(s)||f(s)| \, ds}{(1 + t)^{-1/2} \sqrt{\log \log(1 + t)}} \in [0, \infty).
\] (7.18)

Applying (7.17) and (7.18) in (7.13) proves (ii).

In the case \( 2b + a = 0 \), we have the estimate

\[
\lim_{t \to \infty} \int_0^t (1 + s)^{-1/2} \sqrt{\log \log(e + s)} \, ds = 1.
\]

Now following the same procedure as for the proof of part (ii) gives the result.

8. Proof of Theorem 3.6

We begin this section with the statement and proof of some preparatory lemmata. Firstly, we give a discrete version of the Law of the Iterated Logarithm. The following result is stated as Theorem 2 of [21] or Exercise 3 in [10, pp383, Section 10.2].

Lemma 8.1. Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of independent Gaussian random variables where \( X_n \) has mean zero and variance \( \sigma_n^2 \). If \( s_n^2 = \sum_{i=1}^n \sigma_i^2 \to \infty \) as \( n \to \infty \) and \( \sigma_n = o(s_n) \) as \( n \to \infty \), then

\[
\limsup_{n \to \infty} \frac{\sum_{j=1}^n X_j}{\sqrt{2s_n^2 \log \log s_n^2}} = -\liminf_{n \to \infty} \frac{\sum_{j=1}^n X_j}{\sqrt{2s_n^2 \log \log s_n^2}} = 1, \quad a.s.
\]

While the above result is sufficient for the analysis of this article, as Tomkins [21] observes these sufficient conditions may be sharpened. For instance, Hartman [12] requires only \( \limsup_{n \to \infty} \sigma_n/s_n < 1 \) as opposed to \( \sigma_n/s_n \to 0 \) as \( n \to \infty \) in order to prove a discrete Law of the Iterated Logarithm in the Gaussian case.

Lemma 8.2. Let \( b < 0 \). Then the following limits hold:

\[
\lim_{t \to \infty} \frac{\pi \sqrt{|b|} \int_0^t (1 + s)^{1/2} \sin^2 \left( \frac{2 \sqrt{|b|}(s + 1)}{4} - \frac{3}{4} \pi \right) \, ds}{\frac{1}{2} |b|^{1/2}(1 + t)^{3/2}} = 1,
\] (8.1)

and

\[
\lim_{t \to \infty} \frac{\pi \sqrt{|b|} \int_0^t (1 + s)^{1/2} \cos^2 \left( \frac{2 \sqrt{|b|}(s + 1)}{4} - \frac{3}{4} \pi \right) \, ds}{\frac{1}{2} |b|^{1/2}(1 + t)^{3/2}} = 1.
\] (8.2)

While this lemma amounts to little more than integration by parts, it serves as an asymptotic estimate of the rate of growth of the quadratic variation of stochastic integrals to be considered later.
Proof of Lemma 8.3 Consider first the limit (8.1). Making the substitution \( w = 2\sqrt{b}(s + 1) - \frac{3}{4}\pi \) in the integral, we get
\[
\pi \sqrt{|b|} \int_0^t (1 + s)^{1/2} \sin^2 \left(2\sqrt{b}(s + 1) - \frac{3}{4}\pi\right) ds = \frac{\pi}{4|b|} \int_{2\sqrt{|b|(1+t)} - 3\pi/4}^{2\sqrt{|b|(1+t)} - \pi/4} \left(w + \frac{3\pi}{4}\right)^2 \sin^2(w) dw.
\]
Since \( \int_0^x (w + 3\pi/4)^2 \sin^2(w) dw \) can be computed explicitly for \( x \geq 0 \), this leads to
\[
\lim_{x \to \infty} \frac{1}{x^4} \int_0^x (w + 3\pi/4)^2 \sin^2(w) dw = \frac{1}{6},
\]
(8.1) holds. Similar calculations confirm the limit (8.2) \( \square \)

We next introduce functions which correspond to the leading order asymptotic behaviour of \( r_7, r_8, d_7 \) and \( d_8 \). Define the functions, for \( t \geq 0 \)
\[
g_1(t) = \frac{1}{\sqrt{\pi}} |b|^{-1/4}(1 + t)^{-1/4} \cos(2\sqrt{|b|(1 + t)} - \pi/4), \tag{8.3a}
g_2(t) = \sqrt{\pi} |b|^{-1/4}(1 + t)^{1/4} \sin \left(2\sqrt{|b|(1 + t)} - \frac{3}{4}\pi\right), \tag{8.3b}
g_3(t) = \frac{1}{\sqrt{\pi}} |b|^{-1/4}(1 + t)^{-1/4} \sin(2\sqrt{|b|(1 + t)} - \pi/4), \tag{8.3c}
g_4(t) = \sqrt{\pi} |b|^{1/4}(1 + t)^{1/4} \cos \left(2\sqrt{|b|(1 + t)} - \frac{3}{4}\pi\right). \tag{8.3d}
\]
We aim to show that these leading order terms describe a continuous time process which obeys the law of the iterated logarithm along many carefully designed sequences. These sequences will later be used to extrapolate the asymptotic behaviour of the continuous time process to the positive real line.

Lemma 8.3 Fix \( \eta \in [0, \pi/2] \). Define the sequence \( \{t_n : n \in \mathbb{Z}^+\} \) such that
\[
t_0 = 0, \quad t_n = |b|^{-1}(n\pi + \pi/8 + \lfloor \sqrt{|b|}/\pi - 1/8 \rfloor\pi + \eta/2)^2 - 1, \quad n \geq 1.
\]
If \( g_1 - g_4 \) are defined by (8.3), then
\[
\limsup_{n \to \infty} \frac{g_1(t_n) \int_{t_n}^{t_n} g_2(s)dB(s) + g_3(t_n) \int_{t_n}^{t_n} g_4(s)dB(s)}{\sqrt{2T_n} \log \log T_n} = \frac{1}{\sqrt{3}}, \quad a.s., \tag{8.4}
\]
\[
\liminf_{n \to \infty} \frac{g_1(t_n) \int_{t_n}^{t_n} g_2(s)dB(s) + g_3(t_n) \int_{t_n}^{t_n} g_4(s)dB(s)}{\sqrt{2T_n} \log \log T_n} = -\frac{1}{\sqrt{3}}, \quad a.s. \tag{8.5}
\]
Proof of Lemma 8.3 We start by noticing that \( t_n \geq 0 \) for all \( n \geq 0 \) and therefore \( (t_n)_{n \geq 1} \) is an increasing sequence. Note also that
\[
2\sqrt{|b|(t_n + 1)} = 2n\pi + \frac{\pi}{4} + \eta + 2\pi L_b, \tag{8.6}
\]
where
\[
L_b := \lfloor \sqrt{|b|}/\pi - 1/8 \rfloor \geq \sqrt{|b|}/\pi - 1/8 \geq -1/8. \tag{8.7}
\]
Therefore, as \( L_b \in \mathbb{Z} \), we see that we must have \( L_b \) a non–negative integer. For all \( n \in \mathbb{Z} \) let \( \beta = \beta_n \) be the number such that \( \cos(2n\pi + \eta) = \beta \in (0,1] \) and it is to be noted that \( \beta \) does not depend upon \( n \). Then (8.6) implies
\[
\cos(2\sqrt{|b|(1 + t_n) - \pi/4}) = \beta, \quad \text{and hence} \quad \sin(2\sqrt{|b|(1 + t_n) - \pi/4}) = \sqrt{1 - \beta^2}, \tag{8.8}
\]
Our plan now is to establish that
\[ \int_0^{t_n} [g_2(s)g_1(t_n) + g_4(s)g_3(t_n)]dB(s) \]
gives rise to a discrete–time Gaussian martingale, to which Lemma [8.1] can be applied. To do this, we write
\[
\int_0^{t_n} [g_2(s)g_1(t_n) + g_4(s)g_3(t_n)]dB(s)
\]
where we have used (8.8) at the last step. As the last stochastic integral on the right hand side does not depend upon \( n \) in the integrand, we can decompose the integral and apply Lemma [8.1] to it. We therefore define for \( n \geq 1 \)
\[
S_n := \sum_{j=1}^n Y_j, \quad \text{where } Y_j = \int_{t_{j-1}}^{t_j} (s + 1)^{1/4} \sin \left(2\sqrt{|b|(s + 1)} - \frac{3}{4}\pi + \eta \right) dB(s).
\]
Then \( Y_j \) is a Gaussian distributed random variable with mean zero and variance
\[
\sigma_j^2 := \int_{t_{j-1}}^{t_j} (s + 1)^{1/2} \sin^2 \left(2\sqrt{|b|(s + 1)} - \frac{3}{4}\pi + \eta \right) ds,
\]
and \( S_n \) is a Gaussian distributed random variable with mean zero and variance
\[
s_n^2 = \sum_{j=0}^n \sigma_j^2 = \int_0^{t_n} (s + 1)^{1/2} \sin^2 \left(2\sqrt{|b|(s + 1)} - \frac{3}{4}\pi + \eta \right) ds.
\]
We wish to ascertain the rate of growth of both \( \sigma_j^2 \) and \( s_n^2 \). Define
\[
M_{\eta}(t) = \int_0^t (1 + s)^{1/4} \sin(2\sqrt{|b|(1 + s)} - 3\pi/4 + \eta)dB(s), \quad t \geq 0.
\]
Then \( M_{\eta} \) is a continuous martingale and its quadratic variation is given by
\[
\langle M_{\eta} \rangle(t) = \int_0^t (1 + s)^{1/2} \sin^2(2\sqrt{|b|(1 + s)} - 3\pi/4 + \eta) ds, \quad t \geq 0.
\]
Therefore we have that
\[
\langle M_{\eta} \rangle(t) = \frac{1}{4|b|^{3/2}} \int_{2\sqrt{|b| - \frac{3\pi}{4} + \eta}}^{2\sqrt{|b| + \frac{3\pi}{4} + \eta}} \left( w + \frac{3\pi}{4} - \eta \right)^2 \sin^2(w) dw, \quad t \geq 0.
\]
An explicit calculation following exactly the model of Lemma [8.2] shows that
\[
\langle M_{\eta} \rangle(t) \sim \frac{1}{3} t^{3/2}, \quad \text{as } t \to \infty.
\]
We remark that the asymptotic behaviour of the quadratic variation is independent of \( \eta \). Thus, since \( t_n \sim n^2 \pi^2 / |b| \) as \( n \to \infty \), we have that
\[
s_n^2 = \langle M_{\eta} \rangle(t_n) \sim \frac{1}{3} t_n^{3/2} \sim \frac{n^2 \pi^3}{3|b|^{3/2}} \quad \text{as } n \to \infty.
\]
For \( n \geq 1 \), by (8.6) we have
\[
\sigma_n^2 = (M_n)(t_n) - (M_n)(t_{n-1})
\]
\[
= \frac{1}{4|b|^{3/2}} \int_{2|b|(1+n)}^{2|b|(1+n+1)} \left( w + \frac{3\pi}{4} - \eta \right)^2 \sin^2(w) \, dw
\]
\[
\leq \frac{1}{4|b|^{3/2}} \int_{2\pi}^{2\pi+2\eta + 2\pi L_b} \left( w + \frac{3\pi}{4} - \eta \right)^2 \, dw
\]
\[
= \frac{1}{12|b|^{3/2}} \left[ (2\pi + \pi/4 + \eta + 2\pi L_b)^3 - (2(n-1)\pi + \pi/4 + \eta + 2\pi L_b)^3 \right].
\]

Therefore we have that \( \sigma_n^2 = O(n^2) = O(t_n) \) as \( n \to \infty \). Hence \( \lim_{n \to \infty} \sigma_n/s_n = 0 \).

Thus all the conditions of Lemma 8.1 are satisfied and so the discrete Law of the Iterated Logarithm may be applied to \( S_n \) (or equivalently, to \( M_n(t_n) \)). Therefore by (8.9), and by using the fact that
\[
\lim_{n \to \infty} \frac{t_n^{-1/4}}{\sqrt{2t_n \log \log t_n}} \sqrt{2(M_n)(t_n) \log \log (M_n)(t_n)} = \frac{1}{\sqrt{3}},
\]
gives the limit superior in (8.4). The limit inferior in (8.5) may be obtained via a symmetry argument. \( \square \)

Remark 5. Although Lemma 8.3 fixes \( \eta \) in the interval \([0, \pi/2]\), it is apparent from the proof of this lemma that one is free to choose \( \eta \) in any of the non-overlapping intervals \([\pi/2, \pi], [\pi, 3\pi/2] \) or \([3\pi/2, 2\pi)\). The only amendments in the proof that would result from choosing \( \eta \) in these other intervals would be changes in the signs of the cosine and sine terms in (8.8).

Lemma 8.4. Fix \( k \in \mathbb{Z}^+ \). Define the sequence \( \{i_n^{(k)} : n \in \mathbb{Z}^+\} \) by \( t_0^{(k)} = 0 \) and
\[
t_{j}^{(k)} = \frac{1}{|b|} \left( N_j \pi + \left[ \frac{\sqrt{|b|}}{\pi} - \frac{1}{8} \right] \pi + \frac{\eta_{j,k}}{2} + \frac{\pi}{8} \right)^2 - 1, \quad j \geq 1 \quad (8.10)
\]
where
\[
N_j = \left[ \frac{j}{2^{2+k}} \right] - 1, \quad i_j = j - 2^{2+k}N_j - 1, \quad \eta_{j,k} = \frac{i_j \pi}{2^k 2^{2k}},
\]
so that \( i_j \in \{0, 1, \ldots, 2^{2+k} - 1\} \). Then
\[
\limsup_{n \to \infty} \sqrt[2]{\frac{1}{2^k \log \log t_n^{(k)}}} g_1(t_n^{(k)}) f_0^{(k)} g_2(s) dB(s) + g_3(t_n^{(k)}) f_0^{(k)} g_4(s) dB(s) = \frac{1}{\sqrt{3}}, \quad a.s.,
\]
\[
\liminf_{n \to \infty} \sqrt[2]{\frac{1}{2^k \log \log t_n^{(k)}}} g_1(t_n^{(k)}) f_0^{(k)} g_2(s) dB(s) + g_3(t_n^{(k)}) f_0^{(k)} g_4(s) dB(s) = -\frac{1}{\sqrt{3}}, \quad a.s.
\]
where \( g_1, g_2, g_3 \) and \( g_4 \) are as defined in (8.3). Also,
\[
N_j \sim \frac{j}{2^{2+k}}, \quad i_j^{(k)} \sim \frac{1}{|b|} N_j^2 \pi^2 \sim \frac{1}{|b|} \left[ \frac{1}{2^{2+k}} j^2 \pi^2 \right], \quad \text{as } j \to \infty, \quad (8.12)
\]
\[
\Delta_{j}^{(k)} := i_{j+1}^{(k)} - i_{j}^{(k)} \sim \frac{1}{|b|} N_j \frac{\pi^2}{2^k 2}, \quad \Delta_j^{(k)} \sim \frac{1}{|b|} \frac{1}{2^{k+2}} j \pi^2 \quad \text{as } j \to \infty. \quad (8.13)
\]

Proof of Lemma 8.4 Define \( \beta_{j,k}^{(i)} := \cos(\eta_{i,k}^{(j)}) \), where
\[
\eta_{i,k}^{(j)} := \left( i - \frac{\pi}{2} + \frac{j \pi}{2^k 2} \right), \quad i \in \{1, 2, 3, 4\}, \quad j \in \{0, 1, \ldots, 2^k - 1\}.
\]
Now define the following $4 \times 2^k$ sequences. For each $j \in \{0, 1, ..., 2^k - 1\}$, we define for $n \geq 0$

$$
\tau_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_1^{(j,k)}/2)^2 - 1,
$$

$$
T_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_2^{(j,k)}/2)^2 - 1,
$$

$$
\theta_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_3^{(j,k)}/2)^2 - 1,
$$

$$
\Theta_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_4^{(j,k)}/2)^2 - 1.
$$

Notice that each of these sequences is increasing. Then the sequence \(\{\tau_n^{(j,k)}\}_{n \geq 0}\) may be expressed in terms of \(\beta_n^{(1)}\) (which is independent of \(n\)) according to

$$
\beta_n^{(1)} = \cos(\eta_1^{(j,k)}) = \cos\left(2\sqrt{|b|}(\tau_n^{(j,k)} + 1) - \pi/4\right).
$$

Similarly \(T_n^{(j,k)}, \theta_n^{(j,k)}, \Theta_n^{(j,k)}\) may be expressed in terms of \(\beta_n^{(2)}, \beta_n^{(3)}, \beta_n^{(4)}\) respectively.

Define

$$
\bar{Y}(t) := \frac{g_1(t)\int_0^t g_2(s)dB(s) + g_3(t)\int_0^t g_4(s)dB(s)}{\sqrt{2t \log \log t}}, \quad t \geq e^5.
$$

Then, from Lemma 8.3, for each $j \in \{0, 1, ..., 2^k - 1\}$,

$$
\limsup_{n \to \infty} \bar{Y}(\tau_n^{(j,k)}) = -\liminf_{n \to \infty} \bar{Y}(\tau_n^{(j,k)}) = \frac{1}{\sqrt{3}},
$$

on an event of probability one, \(\Omega^{(j,k)}_1\). Using Lemma 8.3 in conjunction with Remark 9 gives

$$
\limsup_{n \to \infty} \bar{Y}(T_n^{(j,k)}) = -\liminf_{n \to \infty} \bar{Y}(T_n^{(j,k)}) = \frac{1}{\sqrt{3}},
$$

$$
\limsup_{n \to \infty} \bar{Y}(\theta_n^{(j,k)}) = -\liminf_{n \to \infty} \bar{Y}(\theta_n^{(j,k)}) = \frac{1}{\sqrt{3}},
$$

$$
\limsup_{n \to \infty} \bar{Y}(\Theta_n^{(j,k)}) = -\liminf_{n \to \infty} \bar{Y}(\Theta_n^{(j,k)}) = \frac{1}{\sqrt{3}}.
$$

on almost sure events, \(\Omega^{(j,k)}_2\), \(\Omega^{(j,k)}_3\) and \(\Omega^{(j,k)}_4\) respectively. Now,

$$
\tau_n^{(0,k)} < \tau_n^{(1,k)} < ... < \tau_n^{(2^k-1,k)} < T_n^{(0,k)} < ... < T_n^{(2^k-1,k)} < \theta_n^{(0,k)} < ... < \theta_n^{(2^k-2,k)}
$$

$$
< \theta_n^{(2^k-1,k)} < \Theta_n^{(0,k)} < ... < \Theta_n^{(2^k-1,k)}
$$

and \(\Theta_n^{(2^k-1,k)} < \tau_n^{(0,k)}\). Observe that the sequence \(\{t_n^{(k)}\}_{n \geq 0}\), defined in the statement of this Lemma, obeys, for $j \geq 1$

$$
t_n^{(j,k)} = \begin{cases} 
\tau_n^{(ij_0,j_1,k)}/N_j + [\sqrt{\ln(\pi+1/8)],} & i_j \in \{0, 1, ..., 2^k - 1\}, \\
\tau_n^{(ij_0,j_1-2^k,k)}/N_j + [\sqrt{\ln(\pi+1/8)],} & i_j \in \{2^k, 3^2k - 1\}, \\
\theta_n^{(ij_0,j_1-2^k,k)}/N_j + [\sqrt{\ln(\pi+1/8)],} & i_j \in \{2^2k, 3^2k - 1\}, \\
\Theta_n^{(ij_0,j_1-3^2k,k)}/N_j + [\sqrt{\ln(\pi+1/8)],} & i_j \in \{3^2k, 4^2k - 1\},
\end{cases}
$$

Hence, defining \(\Omega^{(k)}_5 = \bigcap_{l=1}^{4} \bigcap_{j=0}^{2^k-1} \Omega^{(j,k)}_i\) and noting that \(\Omega^{(k)}_5\) is an almost sure event, we have that

$$
\limsup_{n \to \infty} \bar{Y}(t_n^{(k)}) = -\liminf_{n \to \infty} \bar{Y}(t_n^{(k)}) = \frac{1}{\sqrt{3}}.
$$
on the event $\Omega^{(k)}_{n}$, which is (8.11).

We turn next to determining the asymptotic behaviour of the sequences $N_{j}$, $t_{j}^{(k)}$, $\Delta t_{j}^{(k)}$ as $j \to \infty$. We start with $N_{j}$. By definition, we have $j/(2^{2k}) - 1 \leq N_{j} < j/(2^{2k})$, and thus, $1/(2^{2k}) - 1/j \leq N_{j}/j < 1/(2^{2k})$. Now letting $j$ tend to infinity and we have $\lim_{j \to \infty} N_{j}/j = 1/2^{2k}$. Moreover as $\eta^{(j,k)}$ is bounded we have $\lim_{j \to \infty} \eta^{(j,k)}/j = 0$. Then from the definition of the sequence $\{t_{n}^{(k)}\}_{n \geq 0}$ it follows that

$$t_{j}^{(k)} \sim \frac{1}{|b|} N_{j}^{2} \pi^{2} \sim \frac{1}{|b|} \frac{1}{2^{2k} 2^{2k}} j j, \quad \text{as } j \to \infty.$$ 

In determining the asymptotic behaviour of $\Delta t_{j}^{(k)}$ we first consider the asymptotic behaviour of $\Delta \eta^{(j+1,k)} := \eta^{(j+1,k)} - \eta^{(j,k)}$ for large $j$. From the definition of $\eta^{(j,k)}$ it is trivially true that $\Delta \eta^{(j,k)} = \pi/2^{k}$ whenever $N_{j+1} = N_{j}$. Moreover the only values of $j$ for which $N_{j+1} \neq N_{j}$ are values of the type $j = m.2^{k}$ for $m \in \{1, 2, \ldots\}$. So, if $j \neq m.2^{k}$ and $j \geq 1$, we get

$$\Delta t_{j}^{(k)} = \frac{1}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2} + \frac{\Delta \eta^{(j,k)}}{2} \right)^{2} - 1$$

$$= \frac{1}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2} \right)^{2} + 1$$

$$= \frac{2}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2} \right) \Delta \eta^{(j,k)} + \frac{1}{|b|} \left( \frac{\Delta \eta^{(j,k)}}{2} \right)^{2}.$$ 

Thus,

$$\Delta t_{j}^{(k)} \sim \frac{2}{|b|} N_{j} \pi \Delta \eta^{(j,k)} = \frac{N_{j} \pi^{2}}{|b|2^{2k}} \sim \frac{j \pi^{2}}{|b|2^{2k}}, \quad \text{as } j \to \infty. \quad (8.14)$$

If $j = m.2^{k}$ for $m \in \{1, 2, \ldots\}$, we have $N_{j+1} = N_{j} + 1 = m$ (as we are interested in the asymptotic behaviour of $\eta^{(j,k)}$ for large $j$ we may exclude $m = 0$ from our analysis). In this case,

$$\eta^{(j+1,k)} = \frac{(j + 1 - 2^{k} N_{j+1} - 1) \pi}{2^{k}} = \frac{(m.2^{k+1} + 1 - m.2^{k} - 1) \pi}{2^{k}} = 0$$

while

$$\eta^{(j,k)} = \frac{(j - 2^{k} N_{j} - 1) \pi}{2^{k}} = \frac{(m.2^{k} - (m - 1)2^{k} - 1) \pi}{2^{k}} = \frac{(2^{k} - 1) \pi}{2^{k}}.$$ 

This gives

$$\Delta t_{j}^{(k)} = \frac{1}{|b|} \left( N_{j+1} \pi + L_{b} \pi + \frac{\pi}{8} + \frac{\eta^{(j+1,k)}}{2} \right)^{2} - 1$$

$$= \frac{1}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2} \right)^{2} + 1$$

$$= \frac{1}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} \right)^{2}$$

$$= \frac{1}{|b|} \left( N_{j} \pi + L_{b} \pi + \frac{\pi}{8} \right) \frac{\pi^{2}}{2^{2k} 2^{2k}} + \frac{\pi^{2} 2^{2k} - 1}{|b| 2^{4k}}.$$ 

Thus, as $j = m.2^{k}$,

$$\Delta t_{m.2^{k}}^{(k)} \sim \frac{2}{|b|} m \pi^{2} 2^{2k}, \quad \text{as } m \to \infty. \quad (8.15)$$
Therefore (8.15) together with (8.14) yields (8.13). □

**Lemma 8.5.** Let \( g_1, g_2, g_3 \) and \( g_4 \) be as defined in (8.3). Let

\[
Y(t) := g_1(t) \int_0^t g_2(s)dB(s) + g_3(t) \int_0^t g_4(s)dB(s), \quad t \geq 0.
\]

Then,

\[
\limsup_{t \to \infty} \frac{Y(t)}{\sqrt{2t \log \log t}} = \frac{1}{\sqrt{3}}, \quad \liminf_{t \to \infty} \frac{Y(t)}{\sqrt{2t \log \log t}} = -\frac{1}{\sqrt{3}}, \quad \text{a.s.}
\]

**Proof of Lemma 8.5.** A lower bound on the limit superior may easily be obtained from Lemma 8.4. We have

\[
\limsup_{t \to \infty} \frac{Y(t)}{\sqrt{2t \log \log t}} \geq \limsup_{n \to \infty} \frac{Y(t_n)}{\sqrt{2t_n \log \log t_n}} = \frac{1}{\sqrt{3}}, \quad \text{a.s.,} \quad \text{(8.16)}
\]

where the sequence \( \{t_n\}_{n \in \mathbb{Z}^+} \) is as defined by (8.10) (for ease of notation we omit the \( k \)-dependence). We now turn our attention to obtaining an upper bound.

Define \( \tilde{Y}(t) := \sqrt{\pi} \max(0, 1 + t)^{1/4} Y(t) \) for \( t \geq 0 \). Then from Lemma 8.11 we have

\[
\limsup_{n \to \infty} \frac{Y(t_n)}{\sqrt{2t_n^{3/4} \log \log t_n}} = \limsup_{n \to \infty} \frac{\tilde{Y}(t_n)}{\sqrt{2t_n^{3/4} \log \log t_n}} = \frac{\sqrt{\pi} |b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.,} \quad \text{(8.17)}
\]

where the limit superior is taken through the sequence \( \{t_n\}_{n \in \mathbb{Z}^+} \) defined in (8.10) (again for ease of notation we omit the \( k \)-dependence). Now, for \( t_n \leq t \leq t_{n+1} \),

\[
\tilde{Y}(t) = \tilde{Y}(t_n) + \frac{\tilde{Y}(t) - \tilde{Y}(t_n)}{\sqrt{2t_n^{3/4} \log \log t_n}} \left( \sqrt{2t_n^{3/4} \log \log t_n} - \frac{\tilde{Y}(t_n)}{\sqrt{2t_n^{3/4} \log \log t_n}} \right)
\]

and so

\[
\tilde{Y}(t) \leq \sup_{t_n \leq t \leq t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)| + \frac{|\tilde{Y}(t_n)|}{\sqrt{2t_n^{3/4} \log \log t_n}}, \quad t \in [t_n, t_{n+1}]. \quad \text{(8.18)}
\]

We firstly examine the asymptotic behaviour of \( \sup_{t_n \leq t \leq t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)| \). Define

\[
\tilde{Y}_1(t) = \sqrt{\pi} \max(0, 1 + t)^{1/4} g_1(t) \int_0^t g_2(s)dB(s), \quad t \geq 0,
\]

\[
\tilde{Y}_2(t) = \sqrt{\pi} \max(0, 1 + t)^{1/4} g_3(t) \int_0^t g_4(s)dB(s), \quad t \geq 0.
\]

Then \( \tilde{Y}(t) = \tilde{Y}_1(t) + \tilde{Y}_2(t) \) and for \( t \in [t_n, t_{n+1}] \) we have

\[
|\tilde{Y}_1(t) - \tilde{Y}_1(t_n)| \leq |\cos(2\sqrt{|b|(1 + t)} - \pi/4)| \left( \int_{t_n}^t g_2(s)dB(s) - \int_0^{t_n} g_2(s)dB(s) \right)
\]

\[
+ |\cos(2\sqrt{|b|(1 + t)} - \pi/4) - \cos(2\sqrt{|b|(1 + t_{n+1})} - \pi/4)| \left( \int_{t_n}^{t_{n+1}} g_2(s)dB(s) \right)
\]

\[
\leq \int_{t_n}^t g_2(s)dB(s) + |2\sqrt{|b|(1 + t)} - 2\sqrt{|b|(1 + t_{n+1})}| \left( \int_0^{t_n} g_2(s)dB(s) \right)
\]

\[
= \int_{t_n}^t g_2(s)dB(s) + 2\sqrt{|b| \frac{1 + t - (1 + t_n)}{1 + t + (1 + t_n)}} \left( \int_0^{t_n} g_2(s)dB(s) \right),
\]
where the Lipschitz continuity of \( \cos(2\sqrt{|b|(1 + \cdot) - \pi}/4) \) on \( \mathbb{R} \) has been used. A similar inequality can be developed for \( |Y_2(t) - Y_2(t_n)| \) for \( t \in [t_n, t_{n+1}] \). Using the fact that \( |\tilde{Y}(t) - Y(t)| \leq |\tilde{Y}_1(t) - Y_1(t_n)| + |\tilde{Y}_2(t) - Y_2(t_n)| \), we obtain

\[
\sup_{t_n \leq t \leq t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)| \leq \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_2(s)dB(s) \right| + \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_4(s)dB(s) \right| \\
+ 2\sqrt{|b|} \left( \frac{t_{n+1} - t_n}{2\sqrt{1 + t_n}} \right) \left\{ \left| \int_{0}^{t_n} g_2(s)dB(s) \right| + \left| \int_{0}^{t_n} g_4(s)dB(s) \right| \right\},
\]

(8.19)

where we have used the fact that \( 1/(\sqrt{1 + t} + \sqrt{1 + t_n}) \leq 1/(2\sqrt{1 + t_n}) \) for \( t \geq t_n \). We now estimate the order of the largest fluctuations of each term on the right hand side of (8.19). We show that, for \( i \in \{2, 4\} \)

\[
\limsup_{n \to \infty} \frac{\sup_{t_n \leq t \leq t_{n+1}} \int_{t_n}^{t} g_i(s)dB(s)}{t_n^{3/4} \sqrt{\log \log t_n}} = 0, \quad \text{a.s. (8.20)}
\]

Now, let \( \epsilon_n > 0 \). By the martingale time change theorem, for every \( n \), there exists a standard Brownian motion \( \tilde{B}_{i,n} \) such that

\[
P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s)dB(s) \right| \geq \epsilon_n \right] = P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s)dB(s) \right|^2 \geq \epsilon_n \right] \\
= P \left[ \sup_{0 \leq u \leq \int_{t_n}^{t} g_i(s)2ds} \left| \tilde{B}_{i,n}(u) \right| \geq \epsilon_n \right],
\]

Hence there is a Brownian motion \( B^*_{i,n} \) such that

\[
P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s)dB(s) \right| \geq \epsilon_n \right] \\
\leq 2P \left[ \sup_{0 \leq u \leq \int_{t_n}^{t} g_i(s)2ds} B^*_{i,n}(u) \geq \epsilon_n \right] = 2P \left[ \int_{t_n}^{t} g_i(s)2ds \geq \epsilon_n \right] \\
= 4P \left[ \int_{t_n}^{t} g_i(s)2ds \geq \epsilon_n \right] = 4 \left\{ 1 - \Phi \left( \frac{\epsilon_n}{\sqrt{\int_{t_n}^{t} g_i(s)2ds}} \right) \right\}, \quad (8.21)
\]

where we have used the fact that \( \max_{0 \leq s \leq t} W(s) \) has the same distribution as \( |W(t)| \) when \( W \) is a standard Brownian motion, the symmetry of the distribution of a standard Brownian motion, and \( \Phi \) denotes the distribution function of a standard normal random variable. Now,

\[
g_2(t)^2 = \pi |b|^{1/2}(1 + t)^{1/2} \sin^2 \left( 2\sqrt{|b|/(1 + t)} - \frac{3}{4}\pi \right) \leq \pi |b|^{1/2}(1 + t)^{1/2},
\]

\[
g_4(t)^2 = \pi |b|^{1/2}(1 + t)^{1/2} \cos^2 \left( 2\sqrt{|b|/(1 + t)} - \frac{3}{4}\pi \right) \leq \pi |b|^{1/2}(1 + t)^{1/2}.
\]

Thus, by \( (8.12) \) we have

\[
\int_{t_n}^{t_{n+1}} g_i(s)2ds \leq \pi |b|^{1/2}(1 + t_{n+1})^{1/2}(t_{n+1} - t_n) \sim \pi |b|^{1/2}t_n^{1/2} \Delta t_n, \quad \text{as } n \to \infty,
\]

(8.22)
and therefore by (8.12) and (8.13)

\[
\limsup_{n \to \infty} \frac{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}}{N_n} \leq \limsup_{n \to \infty} \frac{\pi^{1/2} |b|^{1/4} t_n^{1/4} (\Delta t_n)^{1/2}}{N_n} = \frac{1}{|b|^{1/2}} \frac{n^{2}}{2^{1/2 + k/2}}.
\]

So letting \( \epsilon_n = \frac{5}{8} \sqrt{\log \log t_n} \), and using the last relation and (8.12) gives

\[
\liminf_{n \to \infty} \frac{\epsilon_n}{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}} \geq C_k (1 + n)^{1/4} \sqrt{\log \log(n + e)} , \quad n \geq 1.
\]

By (8.21), this implies

\[
P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s) dB(s) \right| \geq \epsilon_n \right] \leq 4 \left\{ 1 - \Phi \left( C_k (1 + n)^{1/4} \sqrt{\log \log(n + e)} \right) \right\}, \quad n \geq 1.
\]

Now from [13, Problem 2.9.22],

\[
1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}, \quad x > 0.
\]

Thus, for \( n \geq 1 \)

\[
P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s) dB(s) \right| \geq \epsilon_n \right] \leq \frac{4}{\sqrt{2\pi}} C_k (1 + n)^{1/4} \sqrt{\log \log(n + e)} e^{-\frac{1}{4} C_k^2 (1 + n)^{1/4} \log \log(n + e)}.
\]

Therefore

\[
\sum_{n=0}^{\infty} P \left[ \sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s) dB(s) \right| \geq \epsilon_n \right] < +\infty.
\]

The Borel-Cantelli Lemma then gives that

\[
\limsup_{n \to \infty} \frac{\sup_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t} g_i(s) dB(s) \right|}{\epsilon_n^{5/8} \sqrt{\log \log t_n}} \leq 1, \quad \text{a.s.}
\]

Therefore (8.20) holds. We now show for \( i \in \{2, 4\} \) that

\[
\limsup_{n \to \infty} \frac{\int_{t_n}^{t_{n+1}} g_i(s) dB(s)}{\epsilon_n^{5/8} \sqrt{\log \log t_n}} = \frac{\sqrt{\pi} |b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.,} \quad (8.23)
\]

Define

\[
X_n^{(i)} := \int_{t_{n-1}}^{t_n} g_i(s) dB(s), \quad n \geq 1.
\]
Since for every $t < t_n$ there exists $N(t_1) = t < t_{N(t1) + 1}$, it follows from (8.18) that
$$\frac{\tilde{Y}(t)}{\sqrt{2k^{3/4} \log \log t}} \leq K_{N(t)}.$$ 

Now, by (8.25) and (8.17) we have that
$$\limsup_{n \to \infty} K_n \leq \frac{\sqrt{\pi |b|^{1/4}}}{\sqrt{3}} \left( \frac{\pi}{2k} + 1 \right).$$
and since $N(t) \to +\infty$ as $t \to \infty$, we have
$$\limsup_{t \to \infty} \frac{\tilde{Y}(t)}{\sqrt{2k^{3/4} \log \log t}} \leq \frac{\sqrt{\pi |b|^{1/4}}}{\sqrt{3}} \left( \frac{\pi}{2k} + 1 \right)$$
holding on an almost sure set $\Omega^*$. This result also holds on the almost sure set $\Omega^* = \bigcap_{k \in \mathbb{Z}^+} \Omega_k$ and hence
$$\limsup_{t \to \infty} \frac{\tilde{Y}(t)}{\sqrt{2k^{3/4} \log \log t}} \leq \frac{\sqrt{\pi |b|^{1/4}}}{\sqrt{3}}, \quad \text{a.s.}$$
Since $\tilde{Y}(t) = \sqrt{\pi |b|^{1/4}} (1 + t)^{1/4} Y(t)$, we have that
$$\limsup_{t \to \infty} \frac{Y(t)}{\sqrt{2k \log \log t}} \leq \frac{1}{\sqrt{3}}, \quad \text{a.s.}$$
Combining this upper bound on the limit superior with (8.16) gives the required limit superior.
The limit inferior result may be obtained by considering the process \( Z(t) = -Y(t) \). Then

\[
Z(t) = g_1(t) \int_0^t g_2(s)dW(s) + g_3(t) \int_0^t g_4(s)dW(s), \quad t \geq 0,
\]

where \( W(t) := -B(t) \) is a standard Brownian motion. One then may apply the foregoing argument to deduce that

\[
-\liminf_{t \to \infty} \frac{Y(t)}{\sqrt{2t \log \log t}} = \limsup_{t \to \infty} \frac{-Y(t)}{\sqrt{2t \log \log t}} = \limsup_{t \to \infty} \frac{Z(t)}{\sqrt{2t \log \log t}} = \frac{1}{\sqrt{3}}, \quad \text{a.s.}
\]

as required. □

The proof of Theorem 3.6 can now given. It is chiefly concerned with identifying the leading order terms which contribute to the overall asymptotic behaviour of \( X \). The asymptotic behaviour of these leading order terms are then known from Lemma 8.5.

**Proof of Theorem 3.6** By (1.7), (2.56), and (2.61) the solution \( X \) of (1.4) has the representation

\[
X(t) = r_7(t)c_7 + r_8(t)c_8 + \sigma r_7(t) \int_0^t d_7(s)dB(s) + \sigma r_8(t) \int_0^t d_8(s)dB(s). \quad (8.26)
\]

By (2.57) and (2.60), \( r_7 \) and \( r_8 \) have asymptotic behaviour given by

\[
r_7(t) = \frac{1}{\sqrt{\sigma}}|b|^{-1/4}(1 + t)^{-1/4}\{\cos(2\sqrt{|b|(1 + t)} - \pi/4) + O(t^{-1/2})\}, \quad \text{as } t \to \infty,
\]

\[
r_8(t) = \frac{1}{\sqrt{\sigma}}|b|^{-1/4}(1 + t)^{-1/4}\{\sin(2\sqrt{|b|(1 + t)} - \pi/4) + O(t^{-1/2})\}, \quad \text{as } t \to \infty.
\]

Also by (2.55) and (2.60), \( d_7 \) and \( d_8 \) have asymptotic behaviour given by

\[
d_7(s) = \sqrt{\sigma}|b|^{1/4}(s + 1)^{1/4}\left\{\sin(2\sqrt{|b|}(s + 1) - \frac{3}{4}\pi) + O(s^{-1/2})\right\}, \quad \text{as } s \to \infty,
\]

\[
d_8(s) = \sqrt{\sigma}|b|^{1/4}(s + 1)^{1/4}\left\{\cos(2\sqrt{|b|}(s + 1) - \frac{3}{4}\pi) + O(s^{-1/2})\right\}, \quad \text{as } s \to \infty.
\]

Define the functions \( R_7, R_8, D_7 \) and \( D_8 \) so that, for \( s \geq 0 \) and \( t \geq 0 \) we have

\[
r_7(t) = g_1(t) + R_7(t), \quad r_8(t) = g_3(t) + R_8(t), \quad (8.27a)
\]

\[
d_7(s) = g_2(s) + D_7(s), \quad d_8(s) = g_4(s) + D_8(s), \quad (8.27b)
\]

where \( g_1, g_2, g_3 \) and \( g_4 \) are as defined in (8.3). Notice that \( R_7, R_8, D_7 \) and \( D_8 \) are continuous functions. Since

\[
R_7(t) = O(t^{-3/4}), \quad R_8(t) = O(t^{-3/4}) \quad \text{as } t \to \infty,
\]

\[
D_7(s) = O(s^{-1/4}), \quad D_8(s) = O(s^{-1/4}) \quad \text{as } s \to \infty,
\]

it follows that there exists \( M > 0 \) such that

\[
|R_7(t)| \leq M(1 + t)^{-3/4}, \quad |R_8(t)| \leq M(1 + t)^{-3/4}, \quad t \geq 0,
\]

\[
|D_7(s)| \leq M(1 + s)^{-1/4}, \quad |D_8(s)| \leq M(1 + s)^{-1/4}, \quad s \geq 0.
\]
Next, we decompose $X$ according to

$$
\frac{X(t)}{\sqrt{2t \log \log t}} = \frac{r_7(t) + r_8(t) c_8}{\sqrt{2t \log \log t}} + g_1(t) \int_0^t g_2(s) dB(s) + \frac{g_2(t) \int_0^t g_4(s) dB(s)}{\sqrt{2t \log \log t}}
$$

$$
+ \frac{R_7(t) \int_0^t g_2(s) dB(s)}{\sqrt{2t \log \log t}} + \frac{R_8(t) \int_0^t g_4(s) dB(s)}{\sqrt{2t \log \log t}}
$$

$$
+ \frac{r_7(t) \int_0^t D_7(s) dB(s)}{\sqrt{2t \log \log t}} + \frac{r_8(t) \int_0^t D_8(s) dB(s)}{\sqrt{2t \log \log t}}.
$$

(8.28)

Since $r_7(t) \to 0$ and $r_8(t) \to 0$ as $t \to \infty$, the first term on the righthand–side of (8.28) tends to zero as $t \to \infty$. The asymptotic behaviour of the second and third terms is described by Lemma 8.5. We now proceed to demonstrate that the remaining terms have do not contribute to size of the largest oscillations of $X$. We start by considering the last two terms on the right hand side of (8.28). If $\int_0^\infty D_7(s)^2 ds < \infty$ then because $r_7(t) \to 0$ as $t \to \infty$, we have

$$
\lim_{t \to \infty} \frac{r_7(t) \int_0^t D_7(s) dB(s)}{\sqrt{2t \log \log t}} = 0, \quad \text{a.s.} \quad (8.29)
$$

On the other hand, if $\lim_{t \to \infty} \int_0^t D_7(s)^2 ds = +\infty$, by using the estimate on $D_7$, for all $t \geq 0$ we have

$$
\int_0^t D_7(s)^2 ds \leq M_2 \int_0^t (1 + s)^{-1/2} ds \leq 2M_2 (1 + t)^{1/2}.
$$

Therefore

$$
\limsup_{t \to \infty} \frac{2 \int_0^t D_7(s)^2 ds \log \log \int_0^t D_7(s)^2 ds}{t^{1/2} \log \log t} \leq 4M^2.
$$

Hence by the Law of the iterated logarithm for continuous martingales, we have

$$
\limsup_{t \to \infty} \frac{|r_7(t) \int_0^t D_7(s) dB(s)|}{\sqrt{2t \log \log t}}
$$

$$
= \limsup_{t \to \infty} \frac{\sqrt{2} \int_0^t D_7(s)^2 ds \log \log \int_0^t D_7(s)^2 ds}{\sqrt{2t \log \log t}}
$$

$$
= \limsup_{t \to \infty} \frac{\sqrt{2} \int_0^t D_7(s)^2 ds \log \log \int_0^t D_7(s)^2 ds}{\sqrt{2t \log \log t}}
$$

$$
\leq M \limsup_{t \to \infty} \frac{t^{-1/4} \sqrt{2} \int_0^t D_7(s)^2 ds \log \log \int_0^t D_7(s)^2 ds}{\sqrt{t^{1/2} \log \log t}}
$$

$$
\leq 2M^2 \limsup_{t \to \infty} \frac{\log \log t}{\sqrt{2t \log \log t}} = 0.
$$

Hence (8.29) holds. One may similarly show that

$$
\lim_{t \to \infty} \frac{r_8(t) \int_0^t D_8(s) dB(s)}{\sqrt{2t \log \log t}} = 0, \quad \text{a.s.} \quad (8.30)
$$

To estimate the asymptotic behaviour of the fourth and fifth terms on the right hand side of (8.28), we note from Lemma 8.2, we have that

$$
\lim_{t \to \infty} \frac{\int_0^t g_2^2(s) ds}{t^{3/2}} = \frac{1}{3} |b|^{1/2}, \quad \lim_{t \to \infty} \frac{\int_0^t g_3^2(s) ds}{t^{3/2}} = \frac{1}{3} |b|^{1/2}.
$$
Therefore by the Law of the Iterated Logarithm for continuous martingales we have
\[
\limsup_{t \to \infty} \left| \frac{\int_0^t g_2(s) dB(s)}{\sqrt{2t^{3/4} \log \log t}} \right| = \sqrt{\frac{\pi}{3}} |b|^{1/4}, \quad \text{a.s.}
\]

Therefore, using the estimate on \( R_7 \) we have
\[
\limsup_{t \to \infty} \left| \frac{R_7(t) \int_0^t g_2(s) dB(s)}{\sqrt{2t \log \log t}} \right| \leq M \limsup_{t \to \infty} \frac{t^{-3/4} \left| \int_0^t g_2(s) dB(s) \right|}{\sqrt{2t^{3/4} \log \log t}} \sqrt{2t \log \log t} = M \sqrt{\frac{\pi}{3}} |b|^{1/4} \limsup_{t \to \infty} \frac{\sqrt{2 \log \log t}}{\sqrt{2t \log \log t}} = 0.
\]

Thus,
\[
\lim_{t \to \infty} \frac{R_7(t) \int_0^t g_2(s) dB(s)}{\sqrt{2t \log \log t}} = 0, \quad \text{a.s.} \quad (8.31)
\]

Similarly it may be shown that
\[
\lim_{t \to \infty} \frac{R_6(t) \int_0^t g_4(s) dB(s)}{\sqrt{2t \log \log t}} = 0, \quad \text{a.s.} \quad (8.32)
\]

Then due to (8.29), (8.30), (8.31), (8.32), and Lemma 8.5 by taking the limit superior across (8.28) we get
\[
\limsup_{t \to \infty} \frac{X(t)}{\sqrt{2t \log \log t}} = \frac{\sigma}{\sqrt{3}}, \quad \text{a.s.}
\]

Taking the limit inferior and applying these preparatory estimates along with Lemma 8.5 secures the corresponding limit inferior.

\( \square \)

**Proof of Theorem 3.5.** By (1.9) and (2.61) we have that
\[
\frac{1}{\sigma^2} \text{Var}[X(t)] = \int_0^t r(t,s)^2 ds = r_7(t)^2 \int_0^t d_7(s)^2 ds + 2r_7(t)r_8(t) \int_0^t d_7(s)d_8(s) ds + r_8(t)^2 \int_0^t d_8(s)^2 ds. \quad (8.33)
\]

We deduce the asymptotic behaviour of the terms on the righthand side of (8.33). By the definition of \( d_7 \) we have the identity
\[
\frac{1}{t^{3/2}} \int_0^t d_7(s)^2 ds = \frac{1}{t^{3/2}} \int_0^t g_2^2(s) ds = 2 \frac{1}{t^{3/2}} \int_0^t g_2(s)D_7(s) ds + \frac{1}{t^{3/2}} \int_0^t D_7(s)^2.
\]

By the definition of \( g_2 \) and \( D_7 \), we have that \( g_2(t) = O(t^{1/4}) \) and \( D_7(t) = O(t^{-1/4}) \) as \( t \to \infty \), so the limit as \( t \to \infty \) of the two terms on the right hand side is zero. Since the second term on the left hand side has limit \( \pi |b|^{1/2} / \sqrt{3} \) as \( t \to \infty \), we have
\[
\lim_{t \to \infty} \frac{1}{t^{3/2}} \int_0^t d_7(s)^2 ds = \frac{\pi}{3} |b|^{1/2}. \quad (8.34)
\]

Similarly, we may establish
\[
\lim_{t \to \infty} \frac{1}{t^{3/2}} \int_0^t d_8(s)^2 ds = \frac{\pi}{3} |b|^{1/2}. \quad (8.35)
\]
We determine the asymptotic behaviour of the integral in the second term on the right hand side of (8.33). First, we express \(d_7\) and \(d_8\) in terms of \(g_2, g_4, D_7\) and \(D_8\) to get
\[
\int_0^t d_7(s)d_8(s) \, ds = \int_0^t g_2(s)g_4(s) \, ds + \int_0^t \{g_2(s)D_8(s) + g_4(s)D_7(s) + D_7(s)D_8(s)\} \, ds.
\]
Since \(g_2(t) = O(t^{1/4})\), \(g_4(t) = O(t^{1/4})\), \(D_7(t) = O(t^{-1/4})\) and \(D_8(t) = O(t^{-1/4})\) as \(t \to \infty\), the second integral on the right hand side is of order \(t\) as \(t \to \infty\). Finally,
\[
\int_0^t g_2(s)g_4(s) \, ds = \frac{1}{2\pi} \int_0^t |b|^{1/2}(1 + s)^{1/2} \sin \left(4\sqrt{|b|(1 + s)} - \frac{3}{2}\pi\right) \, ds.
\]
Making a substitution in the integral leads to
\[
\int_0^t |b|^{1/2}(1 + s)^{1/2} \sin \left(4\sqrt{|b|(1 + s)} - \frac{3}{2}\pi\right) \, ds = \frac{1}{32|b|} \int_{4\sqrt{|b|- \pi/2}}^{4\sqrt{|b|+\pi/2}} (u + 3\pi/2)^2 \sin(u) \, du.
\]
Since the last integral can be evaluated exactly, we see that
\[
\int_0^t |b|^{1/2}(1 + s)^{1/2} \sin \left(4\sqrt{|b|(1 + s)} - \frac{3}{2}\pi\right) \, ds = O(t), \quad \text{as } t \to \infty,
\]
so it follows that
\[
\int_0^t d_7(s)d_8(s) \, ds = O(t), \quad \text{as } t \to \infty.
\] (8.36)
We prepare one final estimate; it is on \(r_7^2(t) + r_8^2(t)\) as \(t \to \infty\). First we observe that because \(g_1(t) = O(t^{-1/4})\), \(g_3(t) = O(t^{-1/4})\), \(R_7(t) = O(t^{-3/4})\) and \(R_8(t) = O(t^{-3/4})\) as \(t \to \infty\), it follows that
\[
2g_1(t)R_7(t) + 2g_3(t)R_8(t) + R_7^2(t) + R_8^2(t) = O(t^{-1}), \quad \text{as } t \to \infty.
\]
Therefore
\[
r_7^2(t) + r_8^2(t) = g_1^2(t) + g_3^2(t) + 2g_1(t)R_7(t) + 2g_3(t)R_8(t) + R_7^2(t) + R_8^2(t) = 1/\pi |b|^{1/2}(1 + t)^{-1/2} + O(t^{-1}),
\]
or
\[
\lim_{t \to \infty} \frac{r_7^2(t) + r_8^2(t)}{t^{-1/2}} = \frac{1}{\pi} |b|^{1/2}. \quad (8.37)
\]
Now, we return to estimate the asymptotic behaviour of \(\text{Var}[X(t)]\) in (8.33) using the estimates established above. We start by rewriting the identity (8.33) according to
\[
\frac{1}{\sigma^2 t} \text{Var}[X(t)] = \frac{r_7(t)^2}{t^{-1/2}} \left(\int_0^t \frac{d_7(s)^2 \, ds}{s^{3/2}} - \frac{\pi |b|^{1/2}}{3}\right) + 2 \frac{r_7(t) r_8(t)}{t^{-1/4}} \int_0^t \frac{d_7(s)d_8(s) \, ds}{s} \, t^{1/2} + \frac{r_8(t)^2}{t^{-1/2}} \left(\int_0^t \frac{d_8(s)^2 \, ds}{s^{3/2}} - \frac{\pi |b|^{1/2}}{3}\right) + \frac{r_7(t)^2 + r_8(t)}{t^{-1/2}} \cdot \frac{\pi |b|^{1/2}}{3}.
\]
Since \(r_7(t) = O(t^{-1/4})\) and \(r_8(t) = O(t^{-1/4})\), by (8.34) and (8.35), the first and third terms on the right hand side have each limit zero as \(t \to \infty\). Using these
estimates on \( r_7 \) and \( r_8 \), along with (8.36), confirms that the second term has zero limit as \( t \to \infty \). The fourth term has limit \( \frac{1}{3} \) as \( t \to \infty \), by (8.37), and therefore we have

\[
\lim_{t \to \infty} \frac{\text{Var}[X(t)]}{t} = \frac{1}{3} \sigma^2,
\]

as claimed. \( \square \)

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