Second-order asymptotics for convolution of distributions with light tails

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Abstract In this paper, asymptotic behaviors of convolutions of distributions belonging to two classes of distributions with light tails are considered, respectively. The precise second-order tail asymptotics of the convolutions are derived under the condition of second-order regular variation.

Key words Convolution; Light tail; Second-order approximation; Second-order regular variation.

AMS 2000 subject classification Primary 62E20, 60G50; Secondary 60F15, 60F05.

1 Introduction

Let $X$ and $Y$ be two independent and nonnegative random variables with distribution $F$ and $G$, respectively. The distribution of the sum $X + Y$, written as $F * G$, is called the convolution of $F$ and $G$.

A distribution $F$ on $[0, \infty)$ is said to belong to the class $\mathcal{L}_\alpha$ for some $\alpha \geq 0$, if its right tail satisfies

$$F(t) = 1 - F(t) > 0 \text{ for all } t > 0 \text{ and the relation}$$

$$\lim_{t \to \infty} \frac{F(t - u)}{F(t)} = c^\alpha u$$

holds for all $u$. For $\alpha = 0$, the class $\mathcal{L}_0$ reduces to the well-known class of long-tailed distributions. Clearly, the class $\mathcal{L}_\alpha$ is related to the class $RV_{-\alpha}$ of regularly varying functions with exponent $-\alpha$ by the fact that

$$F \in \mathcal{L}_\alpha \quad \text{if and only if} \quad F(\ln t) \in RV_{-\alpha}.$$

Applying Karamata’s representation theorem for regularly varying functions (see Bingham et al. (1987) and Cline (1986)), we know that $F \in \mathcal{L}_\alpha$ if and only if $F(t)$ can be expressed as

$$F(t) = c(t) \exp \left( - \int_0^t \alpha(y)\,dy \right)$$

with $\lim_{t \to \infty} c(t) = c > 0$ and $\lim_{t \to \infty} \alpha(t) = \alpha.$

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An important subclass of \( \mathcal{L}_\alpha \) is the class of convolution equivalent distributions \( \mathcal{S}_\alpha \). We say that \( F \) belongs to the class \( \mathcal{S}_\alpha \) if \( F \in \mathcal{L}_\alpha \) and the limit
\[
\lim_{t \to \infty} \frac{F * F(t)}{F(t)} = 2m_F(\alpha)
\]
exist and is finite, where the constant \( m_F(\alpha) = \int_0^\infty e^{\alpha u} dF(u) \); see Chistyakov (1964) and Chover et al. (1973a, 1973b), Cline (1987). The class \( S := \mathcal{S}_0 \) is called the class of subexponential distributions.

Properties of those mentioned classes have been extensively investigated by many researchers and have been applied to many fields. Embrechts and Goldie (1980) proved that \( \mathcal{L}_\alpha \) is closed under convolution, and gave sufficient conditions for the statements \( F \in \mathcal{S} \Leftrightarrow F * G \in \mathcal{S} \) and \( F * G \in \mathcal{S} \Leftrightarrow pF + (1 - p)G \in \mathcal{S} \) with some (all) \( p \in (0, 1) \). Some more results on \( \mathcal{S}_\alpha \) are presented by Embrechts and Goldie (1982). For \( F, G \in \mathcal{L}_\alpha \), Cline (1986) investigated the relationship between \( F * G \) and its components \( F \) and \( G \). Further, Cline (1987) derived some results on distribution tails of random sums from \( \mathcal{S}_\alpha \), and gave more closure and factorization properties for distributions belonging to \( \mathcal{L}_\alpha \) and \( \mathcal{S}_\alpha \). Recently, Zachary and Foss (2006) derived the asymptotic of tail distribution of convolution of distributions belonging to \( \mathcal{S}_\alpha \), and further gave an distributional asymptotic for the supremum of a random walk with increments in \( \mathcal{S}_\alpha \). By using \( h \)-insensitivity function, more properties of convolutions of long-tailed and subexponential distributions were investigated by Foss et al. (2009). For more studies on random sums, random walk and queue theory related to the classes \( \mathcal{L}_\alpha \) and \( \mathcal{S}_\alpha \), we refer to Albin (2008), Borovkov (1976), Foss (2007), Pakes (2007), Shimura and Watanabe (2005), Zachary (2004).

With motivation from Cline (1986), in this paper we are interested in the second-order tail asymptotic expansions of convolutions of the following two classes of distributions:
\[
\mathcal{F}(t) = e^{-\alpha t + \chi(t)}, \quad \chi(t) \in RV_\rho, \ 0 < \rho < 1
\]
and
\[
\mathcal{F}(t) = b(t)e^{-\alpha t}, \quad \mathcal{G}(t) = c(t)e^{-\alpha t}, \quad b(t) \in RV_\beta, c(t) \in RV_\gamma, \beta, \gamma \in \mathbb{R}.
\]
Clearly, \( F \) and \( G \) given by (1.4) and (1.5) not only belong to \( \mathcal{L}_\alpha \), but also are Weibull-type distributions. For such distributions \( F \) and \( G \), the first-order tail asymptotics of \( F * G \) have been derived by Cline (1986).

In recent literature, more and more researchers focus on the second-order asymptotic behaviors for the sake of understanding precisely the tail behaviors of risks, ruin probabilities and random summation. Hua and Joe (2011) obtained the second-order approximation of the conditional tail expection \( \text{CTE}_p(X) \) for risk \( X \) with its survival function having the property of the second-order regular variation. Degen et al. (2010) and Mao and Hu (2013) derived the second-order approximations of the risk concentrations \( \text{CVar}_R(p) \) and \( \text{CCTE}(p) \), respectively. Baltrūnas (2005) investigated second-order behaviour of ruin probabilities with subexponential claim-size. Lin (2012) established second-order asymptotics for ruin probabilities in a renewal risk model with heavy-tailed claims. For the convolutions of the regularly varying distributions, the second-order or higher-order tail asymptotics have been discussed by Borovkov and Borovkov (2001), Geluk et al. (1997), Hashorva et al. (2014) and among others. Barbe and McCormick (2008) and Lin (2014) respectively established the second-order tail asymptotics of convolution for rapidly varying subexponential distributions and second order subexponential distributions. For related works on the higher-order asymptotics of random...
sum, see, e.g., Geluk (1996), Omey and Willekens (1986, 1987), Albrecher et al. (2010). To the best of our knowledge, there are no studies on higher-order expansions for convolution of distributions belonging to \( L_\alpha \) with \( \alpha > 0 \). The main goal of this paper is to investigate the second-order asymptotics of convolution of light tail distributions \( F, G \) which satisfied (1.4) and (1.5). In order to get the desired results, we assume that \( \chi(t), b(t), c(t) \) in (1.4) and (1.5) are second-order regularly varying functions.

The rest of this paper is organized as follows. In Section 2 some preliminary concepts and results of second-order regularly varying functions are presented, which will be used to prove the main results. The main results and some illustrating examples are given in Section 3. All proofs are deferred to Section 4.

2 Preliminaries

For analysis on tail asymptotics of convolutions of \( F \) and \( G \) satisfied (1.4) and (1.5), the theory of regularly variation on survivor functions plays an important role. We refer to de Haan and Ferreira (2006) and Resnick (1987) for standard references on regular variation.

Definition 1. A measurable function \( \chi : \mathbb{R}_+ \to \mathbb{R} \) that is eventually positive is regularly varying at \( \infty \) with some \( \alpha \in \mathbb{R} \) (written \( \chi \in RV_{\alpha} \)) if for any \( u > 0 \),

\[
\lim_{t \to \infty} \frac{\chi(tu)}{\chi(t)} = u^\alpha.
\]

(2.1)

We call \( \alpha \) the index of variation. If \( \alpha = 0 \), \( \chi(t) \) is said to be slowly varying at \( \infty \).

The following result is the famous Karamata’s Theorem.

Lemma 1. (de Haan and Ferreira, 2006, Theorem B.1.5; Resnick, 1987, Karamata’s Theorem 0.6) Suppose that \( \chi \in RV_{\alpha} \) for some \( \alpha \in \mathbb{R} \) and \( t_0 > 0 \) such that \( \chi(t) \) is positive for \( t > t_0 \). If \( \alpha \geq -1 \), then

\[
\lim_{t \to \infty} \frac{t \chi(t)}{\int_0^t \chi(s)ds} = \alpha + 1.
\]

(2.2)

If \( \alpha < -1 \), or \( \alpha = -1 \) and \( \int_0^\infty \chi(s)ds < \infty \), then

\[
\lim_{t \to \infty} \frac{t \chi(t)}{\int_t^\infty \chi(s)ds} = -\alpha - 1.
\]

(2.3)

Conversely, if (2.2) holds with \(-1 < \alpha < \infty \), then \( \chi \in RV_{\alpha} \); if (2.3) holds with \(-\infty < \alpha < -1 \), then \( \chi \in RV_{\alpha} \).

Karamata’s Theorem described the effect of integration on a regularly varying function. When an \( \alpha \)-varying function is differentiated, the associated property was investigated by Proposition 0.7 of Resnick (1987) which is cited as follows.

Lemma 2. (Resnick, 1987, Proposition 0.7) Suppose \( \chi : \mathbb{R}_+ \to \mathbb{R}_+ \) is absolutely continuous with density \( \chi' \) so that \( \chi(x) = \int_0^x \chi'(t)dt \). If

\[
\lim_{t \to \infty} \frac{t \chi'(t)}{\chi(t)} = \alpha,
\]

(2.4)

then \( \chi \in RV_{\alpha} \). Conversely, if \( \chi \in RV_{\alpha} \), \( \alpha \in \mathbb{R} \), and \( \chi' \) is monotone then (2.4) holds and if \( \alpha \neq 0 \), then \( (\text{sgn} \alpha) \chi'(x) \in RV_{\alpha - 1} \).
The following definition of the second-order regular variation comes from de Haan and Ferreira (2006) and Geluk et al. (1997).

**Definition 2.** A measurable function $\chi : \mathbb{R}_+ \to \mathbb{R}$ that is eventually positive is said to be of second-order regularly variation with the first-order parameter $\alpha \in \mathbb{R}$ and the second-order parameter $\rho \leq 0$, denoted by $\chi \in 2RV_{\alpha, \rho}$, if there exists some ultimately positive or negative function $A(t)$ with $\lim_{t \to \infty} A(t) = 0$ such that

$$
\lim_{t \to \infty} \frac{\chi(tx)}{A(t)} - x^\alpha = o(x^{\rho - 1}), \quad x > 0.
$$

(2.5)

Here, $x^{\rho - 1}$ is interpreted as $\log x$ when $\rho = 0$, $A(t)$ is referred as auxiliary function of $\chi$, and $\rho$ governs the speed of convergence in (2.1).

Next result concerns the Drees-type inequalities for $RV$ functions and $2RV$ functions which establishes uniform inequalities.

**Lemma 3.** (de Haan and Ferreira, 2006, Theorem B.1.10, Theorem 2.3.9; Drees, 1998) If $\chi \in RV_{\alpha}$ with $\alpha \in \mathbb{R}$, for each $\varepsilon, \delta > 0$, there is a $t_0 = t_0(\varepsilon, \delta)$ such that for $t, tx \geq 0$,

$$
\left| \frac{\chi(tx)}{\chi(t)} - x^\alpha \right| \leq \varepsilon x^\alpha \max \left( x^\delta, x^{-\delta} \right).
$$

(2.6)

Further, if $\chi \in 2RV_{\alpha, \rho}$ with auxiliary function $A(t)$ and $\rho \leq 0$, then for any $\varepsilon, \delta > 0$, there exist an auxiliary function $A_1(t)$, $A_1(t) \sim A(t)$ as $t \to \infty$, and $t_0 = t_0(\varepsilon, \delta) > 0$ such that for all $t, tx > t_0$,

$$
\left| \frac{\chi(tx)}{\chi(t)} - x^\alpha (x^\rho - 1) \right| \leq \varepsilon x^{\alpha + \rho} \max \left( x^\delta, x^{-\delta} \right).
$$

(2.7)

To end this section, we provide the following nice representation of $2RV_{\alpha, \rho}$ with $\rho < 0$ given by Hua and Joe (2011).

**Lemma 4.** (Hua and Joe, 2011) Let $\alpha \in \mathbb{R}$, $\rho < 0$ and $|A(t)| \in RV_\rho$. Then $\chi \in 2RV_{\alpha, \rho}$ with auxiliary function $A(t)$ if and only if there exists a constant $a > 0$ such that

$$
\chi(t) = at^a \left( 1 + \frac{1}{\rho} A(t) + o(A(t)) \right)
$$

as $t \to \infty$.

### 3 Main results

In this section, we provide the main results. For $F(t) = e^{-\alpha t + \chi(t)}$ given by (1.3), the second-order tail asymptotic of $F_F$ are presented in the following theorem by assuming that $\chi(t) \in 2RV_{\rho, \rho_1}$, $0 < \rho < 1$, $\rho + \rho_1 < 0$.

**Theorem 1.** Let $F(t) = e^{-\alpha t + \chi(t)}$ with $\alpha > 0$ and $\chi(t)$ is eventually differentiable such that $\chi'(t)$ is nonincreasing eventually. Assume that $\chi(t) \in 2RV_{\rho, \rho_1}$ with $0 < \rho < 1$, $\rho + \rho_1 < 0$ and auxiliary function $A_1(t)$. Then, for large $t$ we have

$$
\frac{\chi^2(t)}{tF^2(t)} = \frac{\alpha}{2} \sqrt{\frac{\pi}{\rho(1 - \rho)}} \frac{2^{\rho - 5} \sqrt{\pi(\rho - 2)(\rho - 3)}}{(\rho(1 - \rho))^{3/2} \chi(t)} - \frac{1}{2} \sqrt{\frac{\rho \pi}{1 - \rho}} \frac{\chi(t)}{t} + o \left( A_1(t) \chi(t) + \frac{1}{\chi(t)} + \frac{\chi^2(t)}{t} \right).
$$

(3.1)
Remark 1. For \( F(t) = e^{-\alpha t + x(t)} \), we have \( m_F(\alpha) = \infty \) which implies that \( F \in \mathcal{L}_\alpha \) but \( F \not\in S_\alpha \).

Example 1. Let risk \( X \) have distribution tail \( F(x) = e^{-\alpha t + x(t)} \) with \( x(t) = \alpha b t + o(A_1(t)) \), \( b > 0 \), \(|A_1(t)| \in RV_{-2} \). Then by Lemma 4, \( x(t) \in 2RV_{\frac{\beta}{\gamma} - 2} \), and the second-order expansion \( 3.1 \) leads to

\[
F * F(t) = t^2 F^2 \left( \frac{1}{2} \right) \left( 2^{\frac{1}{4}} \left( \frac{\alpha \pi}{b} \right)^{\frac{1}{2}} - 2^{-\frac{1}{4}} \left( \frac{\pi}{\alpha b} \right)^{\frac{1}{2}} \left( 1 + \frac{15}{8ab^2} \right) t^{-\frac{1}{2}} + o \left( t^{-\frac{1}{2}} \right) \right)
\]

for large \( t \).

Theorem 2 provides the second-order expansions of \( F * G \) where \( F \) and \( G \) are given by \( 1 \) with \( b(t) \in 2RV_{\beta_2}, c(t) \in 2RV_{\gamma_3}, \beta, \gamma \in \mathbb{R}, \rho_2, \rho_3 \leq 0 \). These results are more complex than results derived in Theorem 1. In order to state the main result Theorem 2 clearly, we first present the following proposition.

**Proposition 1.** Let \( F, G \in \mathcal{L}_\alpha, \alpha > 0 \). Assume that \( b(t) = e^{\alpha t} F(t) \in 2RV_{\beta_2} \) with auxiliary function \( A_2(t) \) for \( \rho_2 \leq 0, \beta \in \mathbb{R} \), and \( c(t) = e^{\alpha t} G(t) \in RV_{\gamma}, \gamma \in \mathbb{R} \). Then

(i) if \( \gamma \leq -1 \), for large \( t \) we have

\[
\int_0^t F(t - u) dG(u) = F(t) M_1(\beta, \gamma, t)
\]

with

\[
M_1(\beta, \gamma, t) = \begin{cases} 
\int_0^t c(u) du \left( \alpha + \frac{\alpha c(t)}{\int_0^t c(u) du} (\beta \int_0^t (1 - u)^{\beta - 1} (\log u) du + (\log 2) (1 - 2^{-\beta})) \right) \\
+ \frac{1}{\int_0^t c(u) du} \left( \beta \int_0^t (1 - u)^{\beta - 1} (\log u) du + (\log 2) (1 - 2^{-\beta}) \right) \right) \\
+ o \left( \frac{1}{\int_0^t c(u) du} + A_2(t) \right), & \gamma = -1 \text{ with } m_G(\alpha) = \infty; \\
m_G(\alpha) - \alpha \beta t^{-1} \int_1^t uc(u) du + o \left( t^{-1} \int_1^t uc(u) du + A_2(t) \right), & \gamma = -2 \text{ with } m_G(\alpha) < \infty; \\
m_G(\alpha) - \beta t^{-1} \int_1^\infty uc(u) du + o \left( t^{-1} \int_1^\infty uc(u) du + A_2(t) \right), & \gamma \leq -2 \text{ with } m_G(\alpha) < \infty.
\end{cases}
\]

(ii) if \( \gamma > -1 \), suppose \( c(t) \in 2RV_{\gamma_3} \) with auxiliary function \( A_3(t) \) and \( \rho_3 \leq 0, \gamma + \rho_3 + 1 > 0 \), then

\[
M_1(\beta, \gamma, t) = \begin{cases} 
tc(t) \left( \alpha \int_0^t (1 - u)^{\beta - 2} u^{\gamma} du + \frac{\alpha}{\rho_2} \int_0^t ((1 - u)^{\rho_2} - 1) (1 - u)^{\beta} u^{\gamma} du A_2(t) \right) \\
+ \left( \beta \int_0^t (1 - u)^{\beta - 1} u^{\gamma} (2u - 1) du - 2(1 + \gamma) \int_0^t (1 - u)^{\beta} u^{\gamma} du \right) t^{-1} \\
+ \alpha \left( \frac{2 - \rho_3}{\rho_3 (\gamma + \rho_3 + 1)} \int_0^t (1 - u)^{\beta - 1} u^{\gamma + 1} ((2u)^{\rho_3} - 1) du \right. \\
\left. + \frac{2 - \rho_3 (\gamma + 1) - \gamma - \rho_3 - 1}{\rho_3 (\gamma + \rho_3 + 1)} \int_0^t (1 - u)^{\beta} u^{\gamma} du \right) A_3(t) + o \left( t^{-1} + A_2(t) + A_3(t) \right)
\end{cases}
\]

for large \( t \).
Theorem 2. Let $F, G \in \mathcal{L}_\alpha$, $\alpha > 0$. Assume that $b(t) = e^{\alpha t}F(t) \in 2R_{\beta, \rho_2}$ with auxiliary function $A_2(t)$ for $\rho_2 \leq 0$, $\beta \in \mathbb{R}$, and $c(t) = e^{\alpha t}G(t) \in 2R_{\gamma, \rho_3}$ with auxiliary function $A_3(t)$ for $\rho_3 \leq 0$, $\gamma \in \mathbb{R}$. Then

$$F \ast G(t) = F(t)M_1(\beta, \gamma, t) + G(t)M_2(\beta, \gamma, t)$$

(3.5)

for large $t$, where $M_1(\beta, \gamma, t)$ is given by Proposition [I] and $M_2(\beta, \gamma, t)$ is given as follows:

(i) if $\beta \leq -1$, for large $t$ we have

$$M_2(\beta, \gamma, t) = \begin{cases} \int_0^t b(u)du \left( \alpha + \frac{\alpha b(t)}{\int_0^t b(u)du} \left( \gamma \int_0^u (1-u)\gamma^{-1}(\log u)du + (\log 2)(1-2^{-\gamma}) \right) \\ + \frac{1}{\int_0^t b(u)du} \left( \frac{1}{\rho_2 \gamma} + A_3(z) \right) \right), & \beta = -1 \text{ with } m_F(\alpha) = \infty; \\ m_F(\alpha) - \alpha \int_0^t b(u)du + o \left( \int_0^t b(u)du + A_3(z) \right), & \beta = -1 \text{ with } m_F(\alpha) < \infty; \\ m_F(\alpha) + \alpha \left( \int_0^t (1-u)\gamma^{-1}u^\beta du + \frac{1}{\rho_2} (1+\beta) \right) t^\beta + o(t^\beta + A_3(z)), & 2 < \beta < -1; \\ m_F(\alpha) + \beta - 1 \int_0^t bu(u)du + o \left( t^{-1} \int_0^t bu(u)du + A_3(z) \right), & \beta = -2 \text{ with } \int_0^\infty bu(u)du = \infty; \\ m_F(\alpha) - \gamma t^{-1} \int_0^\infty u e^\alpha uF(u) + o(t^{-1} + A_3(t)), & \beta \leq -2 \text{ with } \int_0^\infty bu(u)du < \infty. \end{cases}$$

(ii) if $\beta > -1$ with $\beta + \rho_2 + 1 > 0$, for large $t$ we have

$$M_2(\beta, \gamma, t) = \begin{cases} tb(t) \left( \alpha \int_0^t (1-u)\gamma^{-1}u^\beta du + \frac{\alpha}{\rho_3} \int_0^t ((1-u)\rho_3 - 1)(1-u)^\gamma u^\beta duA_3(t) \\ + \left( \gamma \int_0^t (1-u)\gamma^{-1}u^\beta (2u-1)du - 2(1+\beta) \int_0^t (1-u)^\gamma u^\beta du + 2^{\beta-\gamma} \right) t^{-1} \\ + \frac{2^{\beta-\gamma}}{\rho_2 (\beta + \rho_2 + 1)} \int_0^t (1-u)^\gamma u^\beta du(2u-1) \\ + \frac{2^{\beta-\gamma}(\beta + 1) - \beta - \rho_2 - 1}{\rho_2 (\beta + \rho_2 + 1)} \int_0^t (1-u)^\gamma u^\beta du A_2(t) + o \left( t^{-1} + A_2(t) + A_3(t) \right) \right), & 2 < \beta < -1; \\ m_F(\alpha) - \gamma t^{-1} \int_0^\infty u e^\alpha uF(u) + o(t^{-1} + A_3(t)), & \beta \leq -2 \text{ with } \int_0^\infty bu(u)du < \infty. \end{cases}$$

(3.7)

Remark 2. For $F(t) = b(t)e^{-t\alpha}, b(t) \in R_{\beta, \beta} \in \mathbb{R}$, it is clearly that $F \in \mathcal{L}_\alpha$ for all $\beta \in \mathbb{R}$. Note that $F \in S_\alpha$ if and only if $\beta \leq -1$ with $m_F(\alpha) < \infty$. So Theorem also derives the second-order tail asymptotics of convolution of distributions from a subclass of $S_\alpha$.

Example 2. Suppose $X$ and $Y$ are nonnegative random variables with distribution $F$ and $G$, respectively. Let $F(t) = b(t)e^{-t\alpha}$ and $G(t) = c(t)e^{-t\alpha}$ with $b(t) = t^{-3}(1 + t^{-\alpha}A_2(t) + o(A_2(t)))$, $A_2(t) \in R_{-4}$, $c(t) = \alpha^{\gamma-1} t^{-3} \Gamma(\alpha) 1 + (\zeta t^{-1} + o(t^{-1}))$, $\alpha > 0$, $\zeta > 1$. Here $\Gamma(\cdot)$ denotes the gamma function. From Lemma [I], it follows that $b(t) \in R_{-3-4}$, $c(t) \in R_{\zeta-1-1}$ which imply that $m_F(\alpha) < \infty$, $m_G(\alpha) = \infty$. Then second-order expansion leads to

$$F \ast G(t) = F(t) \left( m_F(\alpha) - (\zeta - 1)t^{-1} \int_0^\infty u e^\alpha uF(u) + o(t^{-1}) \right),$$

$$F \ast F(t) = 2F(t) \left( m_F(\alpha) + 3t^{-1} \int_0^\infty u e^\alpha uF(u) + o(t^{-1}) \right).$$

(3.8)
and
\[
G \ast G(t) = \frac{2\alpha \zeta^{-1}}{\Gamma(\zeta)} t^{\zeta} C(t) \left[ \alpha \int_0^{\frac{1}{2}} (1 - y)^{-1} y^{\zeta-1} dy + \left( \zeta - 1 \right) \int_0^{\frac{1}{2}} (1 - y)^{-2} y^{\zeta-1} dy + 2^{2(\zeta - 1)} \right] t^{-1} + o(t^{-1})
\]
for large \( t \).

4 \hspace{1cm} \text{Proofs}

The aim of this section is to prove our main results. In order to prove the main results, we give some auxiliary lemmas first. Without loss of generality, we assume that the auxiliary functions of \(2RV\) functions are positive eventually in the following proofs.

**Lemma 5.** Let \( \chi(t) \in 2RV_{\rho, \rho_1} \) with auxiliary function \( A_1(t) \) for \( 0 < \rho < 1, \rho_1 \leq 0 \). Assume that \( \chi' \) is nonincreasing eventually. Then for large \( t \), we have
\[
\left| \frac{t \chi'(t)}{\chi(t)} - \rho \right| \leq c \max(A_1^{c_1}(t), A_1^{1-c_1}(t)),
\]
where \( c \) and \( c_1 \) are positive constants with \( 0 < c_1 < 1 \).

**Proof.** First we consider the case of \( \rho_1 < 0 \). Note that
\[
1 + \rho x + \frac{\rho(\rho - 1)x^2}{2} < (1 + x)^\rho < 1 + \rho x,
\]
(4.2)
\[
1 - \rho x + (1 - c_2^{-1})^{\rho - 1} \frac{\rho(\rho - 1)x^2}{2} < (1 - x)^\rho < 1 - \rho x,
\]
(4.3)
\[
1 + \rho_1 x < (1 + x)^{\rho_1} < 1 + \rho_1 x + \frac{\rho_1(\rho_1 - 1)x^2}{2}
\]
(4.4)
and
\[
1 - \rho_1 x < (1 - x)^{\rho_1} < 1 - \rho_1 x + (1 - c_2^{-1})^{\rho_1 - 1} \frac{\rho_1(\rho_1 - 1)x^2}{2}
\]
(4.5)
for \( 0 < \rho < 1, \rho_1 < 0, 0 < x < c_2^{-1}, 1 < c_2 < 1/\left(1 - (2 - 2\rho)^{\frac{1}{2}}\right) \). By using (2.7), (4.2)-(4.5), we can get
\[
\frac{\chi'(t)}{\chi(t)} \leq \frac{\int_{t(1-A_1^{c_1}(t))}^{t} \chi'(y) dy}{A_1^{c_1}(t) \chi(t)}
\]
\[
\leq A_1^{c_1}(t) \left(1 - (1 - A_1^{c_1}(t))^\rho - A_1(t) \left(1 - A_1^{c_1}(t))^\rho \frac{(1 - A_1^{c_1}(t))\rho_1 - 1}{\rho_1} - \varepsilon (1 - A_1^{c_1}(t))^{\rho + \rho_1 - \delta} \right) \right)
\]
\[
\leq \rho - (1 - c_2^{-1})^{\rho - 1} \frac{\rho(\rho - 1)}{2} A_1^{c_1}(t) + A_1(t) (1 - A_1^{c_1}(t))^{\rho} \left(1 - (1 - c_2^{-1})^{\rho_1 - 2} \frac{\rho_1 - 1}{2} A_1^{c_1}(t) \right)
\]
\[
+ \varepsilon A_1^{1-c_1}(t) (1 - A_1^{c_1}(t))^{\rho + \rho_1 - \delta}
\]
for large \( t \). Similarly,
\[
\frac{\chi'(t)}{\chi(t)} \geq \rho + \frac{\rho(\rho - 1)}{2} A_1^{c_1}(t) + A_1(t) (1 + A_1^{c_1}(t))^{\rho} \left(1 + \frac{\rho_1 - 1}{2} A_1^{c_1}(t) \right) - \varepsilon A_1^{1-c_1}(t) (1 + A_1^{c_1}(t))^{\rho + \rho_1 + \delta}
\]
for large \( t \), which implies
\[
\left| \frac{t \chi'(t)}{\chi(t)} - \rho \right| \leq c \max(A_1^{c_1}(t), A_1^{1-c_1}(t))
\]
with \( 0 < c_1 < 1, \ c > 0 \).

The arguments for the case of \( \rho = 0 \) are similar.

**Lemma 6.** Let \( \mathcal{F}(t) = e^{-at + \chi(t)} \) such that \( \chi'(t) \) is nonincreasing eventually. Suppose \( \chi(t) \in 2RV_{\rho, \rho_1} \) with auxiliary function \( A_1(t) \) and \( 0 < \rho < 1, \ \rho + \rho_1 < 0 \). Then

\[
\frac{\chi(t)}{t} \int_{f(2t)}^{\chi(t)} \frac{F(2t-u) dF(u)}{tf^2(t)} = \frac{\alpha}{2} \sqrt{\frac{\pi}{\rho(1-\rho)}} - \frac{\alpha \sqrt{\pi(\rho-2)(\rho-3)}}{32(1-\rho)\chi(t)} - \frac{1}{2} \sqrt{\frac{\rho^2 \chi(t)}{1-\rho}} \left( \frac{1}{2t} + o \left( \frac{1}{\chi(t)} + \frac{\chi(t)}{t} \right) \right) \tag{4.6}
\]

for large \( t \), where \( f(t) = 1/\chi'(t) \).

**Proof.** From Lemma 2 we have \( \lim_{t \to \infty} t \chi'(t)/\chi(t) = \rho \), which implies that \( \chi(t) \) is increasing eventually. For sufficiently large \( t \) and \( A \) satisfied \( f(2t) < A < t \), we can get

\[
\int_{\chi(t)\chi A}^{\chi(t)\chi A} \left( \alpha - \chi' \left( t - \frac{ty}{\chi^2(t)} \right) \right) \exp \left( \chi \left( t + \frac{ty}{\chi^2(t)} \right) + \chi \left( t - \frac{ty}{\chi^2(t)} \right) - 2\chi(t) \right) dy \\
\leq \frac{\chi(t)}{t} \int_0^A \left( \alpha - \chi'(s) \right) \exp \left( \chi (2t - s) + \chi (s) - 2\chi(t) \right) ds \\
\leq \left( \max_{s \in (0,A)} e^{\chi(s)} \right) \alpha A \frac{\chi(t)}{t} e^{(2t-2)\chi(t)} - \left( \min_{s \in (0,A)} \chi'(s) e^{\chi(s)} \right) A \frac{\chi(t)}{t} e^{(2t-2)\chi(t)} \\
\leq \left( \max_{s \in (0,A)} e^{\chi(s)} \right) \alpha A \frac{\chi(t)}{t} e^{(1-c)(2^\rho-2)\chi(t)} - \left( \min_{s \in (0,A)} \chi'(s) e^{\chi(s)} \right) A \frac{\chi(t)}{t} e^{(1-c)(2^\rho-2)\chi(t)} \\
= o \left( A_1(t) \chi(t) + \chi^{-1}(t) + \frac{\chi^2(t)}{t} \right) \tag{4.7}
\]

and

\[
\int_{\chi(t)\chi A}^{\chi(t)\chi A} \left( \alpha - \chi' \left( t - \frac{ty}{\chi^2(t)} \right) \right) \exp \left( \chi \left( t + \frac{ty}{\chi^2(t)} \right) + \chi \left( t - \frac{ty}{\chi^2(t)} \right) - 2\chi(t) \right) dy \\
\leq \left( \alpha - \chi' \left( t(1-c_1^{-1}) \right) \right) \left( 1 - c_1^{-1} - A \chi(t) \right) \exp \left( \chi (2t - A) + \chi (t(1-c_1^{-1})) - 2\chi(t) \right) \\
\leq \left( \alpha - \chi' \left( t(1-c_1^{-1}) \right) \right) \left( 1 - c_1^{-1} - A \chi(t) \right) \exp \left( (1-\varepsilon) \left( 2^\rho + (1-c_1^{-1})\rho - 2 \right) \chi(t) \right) \\
= o \left( A_1(t) \chi(t) + \chi^{-1}(t) + \frac{\chi^2(t)}{t} \right), \tag{4.8}
\]

where \( 1 < c_2 < 1 / \left( 1 - (2^\rho) \right) \).

In order to derived the desired result, we need some more precise inequalities as follows:

\[
1 + \rho x + \frac{\rho(\rho-1)}{2} x^2 + \frac{\rho(\rho-1)(\rho-2)}{3!} x^3 + \frac{\rho(\rho-1)(\rho-2)(\rho-3)}{4!} x^4 \\
< (1 + x)^\rho \\
< 1 + \rho x + \frac{\rho(\rho-1)}{2} x^2 + \frac{\rho(\rho-1)(\rho-2)}{3!} x^3 + \frac{\rho(\rho-1)(\rho-2)(\rho-3)}{4!} x^4 + \frac{\rho(\rho-1)(\rho-2)(\rho-3)(\rho-4)}{5!} x^5 
\]
and
\[
1 - \rho x + \frac{\rho(\rho - 1)}{2} x^2 - \frac{\rho(\rho - 1)(\rho - 2)}{3!} x^3 + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{4!} x^4
\]
\[
- (1 - c_2^{-1})^\rho \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)(\rho - 4)}{5!} x^5
\]
\[
< (1 - x)^ho
\]
\[
< 1 - \rho x + \frac{\rho(\rho - 1)}{2} x^2 - \frac{\rho(\rho - 1)(\rho - 2)}{3!} x^3 + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{4!} x^4
\]
(4.10)
for \(0 < x < c_2^{-1}\).

Combining with (4.10), (4.12), and (4.16), we have
\[
\rho(\rho - 1)y^2 + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{12} y^4 \left[ \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 + \varepsilon} - \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 - \varepsilon} \right]
\]
\[
\leq \chi \left( t + \frac{yt}{\chi(t)} \right) - 2\chi(t)
\]
\[
\leq \rho(\rho - 1)y^2 + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{12} y^4 \left[ \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 + \varepsilon} - \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 - \varepsilon} \right]
\]
(4.11)
for large \(t\) and \(0 < y < c_2^{-1}\chi(t)\).

Note that \(\rho(\rho - 1)(\rho - 2)(\rho - 3)y^4/(12\chi(t)) + \rho(\rho - 1)(\rho - 2)(\rho - 3)(\rho - 4)y^5/\left(5!\chi(t)^2\right) < 0\) for \(0 < \rho < 1\), \(0 < y < c_2^{-1}\chi(t)\). By using (4.11) and the following equality
\[
1 + x < e^x < 1 + x + \frac{x^2}{2}, \quad x < 0,
\]
(4.12)
we can get
\[
\int_0^{\frac{1}{c_2} \chi(t)} \exp \left( \chi \left( t + \frac{yt}{\chi(t)} \right) + \chi \left( t - \frac{yt}{\chi(t)} \right) - 2\chi(t) \right) dy
\]
\[
\leq \exp \left( \varepsilon \left( 1 + (1 - c_2^{-1})^{\rho + p_1 - \varepsilon} \right) A_1(t)\chi(t) \right)
\]
\[
\times \left[ \int_0^{\frac{1}{c_2} \chi(t)} \left( 1 + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{12} y^4 \left[ \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 + \varepsilon} - \left( 1 + \frac{y}{\chi(t)} \right)^{\rho + p_1 - \varepsilon} \right] \right) \exp (\rho(\rho - 1)y^2) dy
\]
\[
= \int_0^{\frac{1}{c_2} \chi(t)} \exp (\rho(\rho - 1)y^2) dy + \frac{\rho(\rho - 1)(\rho - 2)(\rho - 3)}{12\chi(t)} \int_0^{\frac{1}{c_2} \chi(t)} y^4 \exp (\rho(\rho - 1)y^2) dy + o(\chi^{-1}(t) + A_1(t)\chi(t))
\]
and
\[
\int_0^{\frac{1}{c_2} \chi(t)} \exp \left( \chi \left( t + \frac{ty}{\chi(t)} \right) + \chi \left( t - \frac{ty}{\chi(t)} \right) - 2\chi(t) \right) dy
\]
\[
\exp\left(-\varepsilon (1 + (1 - c_2^{-1})^\rho + \rho \varepsilon) A_1(t) \chi(t)\right) \int_0^{\frac{1}{\sqrt{2}} \chi^\frac{1}{2}(t)} \left( 1 + \frac{\rho (\rho - 1)(\rho - 2)(\rho - 3)}{12} y^4 \right) \chi(t) dy
\]

\[
- (1 - c_2^{-1})^{\rho - 5} \frac{\rho (\rho - 1)(\rho - 2)(\rho - 3)(\rho - 4)}{5!} \chi^\frac{4}{3}(t) + \frac{1}{2^{\rho + 1}} \left( \rho + \rho_1 \right) \left( \rho + \rho_1 - 1 \right) \left( 1 + (1 - c_2^{-1})^{\rho + \rho_1 - 2} \right)
\]

\[
- \rho (\rho - 1) \left( 1 + (1 - c_2^{-1})^{\rho - 2} \right) y^2 A_1(t) \right) \exp (\rho (\rho - 1) y^2) dy
\]

\[
= \int_0^\infty \exp (\rho (\rho - 1) y^2) dy + \frac{\rho (\rho - 1)(\rho - 2)(\rho - 3)}{12 \chi(t)} \int_0^\infty y^4 \exp (\rho (\rho - 1) y^2) dy + o \left( \chi^{-1}(t) + A_1(t) \chi(t) \right)
\]

for large \( t \), if \( \rho + \rho_1 < 0 \). Hence,

\[
\int_0^{\frac{1}{\sqrt{2}} \chi^\frac{1}{2}(t)} \exp \left( \chi \left( t + \frac{yt}{\chi^\frac{1}{2}(t)} \right) + \chi \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) - 2\chi(t) \right) dy
\]

\[
= \int_0^\infty \exp (\rho (\rho - 1) y^2) dy + \frac{\rho (\rho - 1)(\rho - 2)(\rho - 3)}{12 \chi(t)} \int_0^\infty y^4 \exp (\rho (\rho - 1) y^2) dy + o \left( \chi^{-1}(t) + A_1(t) \chi(t) \right)
\]

\[
= \frac{1}{2} \sqrt{\frac{\pi}{\rho (1 - \rho)}} \frac{\sqrt{\pi} (\rho - 2)(\rho - 3)}{32 (\rho (1 - \rho))^{\frac{3}{2}}} + o \left( \chi^{-1}(t) + A_1(t) \chi(t) \right) \tag{4.13}
\]

for large \( t \).

Note that max \( (A_1^c(t), A_1^{1-c}(t)) = o \left( \chi^{-\frac{1}{2}}(t) \right) \) as \( t \to \infty \), if \( -\rho/(2\rho_1) < c_1 < 1 + \rho/(2\rho_1) < 1 \) and \( \rho + \rho_1 < 0 \). Due to (2.10), (2.20), (10), (1.13) and (4.13), we obtain

\[
\int_0^{\frac{1}{\sqrt{2}} \chi^\frac{1}{2}(t)} \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) \chi' \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) - \rho \chi \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) \frac{\chi \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right)}{1 - \frac{y}{\chi^\frac{1}{2}(t)}} \chi(t) dy
\]

\[
\leq \left( A_1^c \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) + A_1^{1-c} \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right) \right) \left( 1 - \frac{y}{\chi^{\frac{1}{2}}(t)} \right)^{\rho - 1} \left( 1 + \varepsilon \left( 1 - \frac{y}{\chi^{\frac{1}{2}}(t)} \right)^{-\varepsilon} \right)
\]

\[
\times \exp \left( t + \frac{yt}{\chi^\frac{1}{2}(t)} + t - \frac{yt}{\chi^\frac{1}{2}(t)} - 2\chi(t) \right) dy
\]

\[
= o \left( \chi^{-\frac{1}{2}}(t) \right)
\]

with \( -\rho/(2\rho_1) < c_1 < 1 + \rho/(2\rho_1) < 1 \), and

\[
\int_0^{\frac{1}{\sqrt{2}} \chi^\frac{1}{2}(t)} \left( \frac{\chi \left( t - \frac{yt}{\chi^\frac{1}{2}(t)} \right)}{1 - \frac{y}{\chi^\frac{1}{2}(t)}} \chi(t) \right) - 1 + \frac{(\rho - 1)y}{\chi^\frac{1}{2}(t)} \exp \left( t + \frac{yt}{\chi^\frac{1}{2}(t)} + t - \frac{yt}{\chi^\frac{1}{2}(t)} - 2\chi(t) \right) dy
\]

\[
\leq \int_0^{\frac{1}{\sqrt{2}} \chi^\frac{1}{2}(t)} \left( 1 - c_2^{-1} \right)^{\rho - 3} \left( \frac{\rho - 1)(\rho - 2)}{2} y^2 \right) + A_1(t) \left( 1 - \frac{y}{\chi^\frac{1}{2}(t)} \right) \left( 1 - \frac{y}{\chi^\frac{1}{2}(t)} \right)^{\rho - 1 + 1}
\]

\[
\leq \varepsilon A_1(t) \left( 1 - \frac{y}{\chi^\frac{1}{2}(t)} \right)^{\rho + \rho_1 - 1 - \varepsilon} \exp \left( t + \frac{yt}{\chi^\frac{1}{2}(t)} + t - \frac{yt}{\chi^\frac{1}{2}(t)} - 2\chi(t) \right) dy
\]

\[
= o \left( \chi^{-\frac{1}{2}}(t) \right)
\]
as $t \to \infty$, which implies that

$$\int_0^\infty \chi'(t) \left( t - \frac{yt}{\chi^2(t)} \right) \exp \left( \chi \left( t + \frac{yt}{\chi^2(t)} \right) + \chi \left( t - \frac{yt}{\chi^2(t)} \right) - 2\chi(t) \right) dy$$

$$= \chi(t) \int_0^\infty \frac{1}{t} \chi'(t) \frac{t \chi' \left( t - \frac{yt}{\chi^2(t)} \right)}{\chi(t)} \exp \left( \chi \left( t + \frac{yt}{\chi^2(t)} \right) + \chi \left( t - \frac{yt}{\chi^2(t)} \right) - 2\chi(t) \right) dy$$

$$= \rho \int_0^\infty \exp \left( \rho(\rho - 1)y^2 \right) dy \frac{\chi(t)}{t} + \rho(1 - \rho) \int_0^\infty y \exp \left( \rho(\rho - 1)y^2 \right) dy \frac{\chi^2(t)}{t} + o \left( \frac{\chi^2(t)}{t} \right)$$

$$= \frac{1}{2} \sqrt{\frac{\rho(1 - \rho)}{\rho - 1}} \chi(t) \frac{\chi}{t} + \frac{\chi^2(t)}{2t} + o \left( \frac{\chi^2(t)}{t} \right)$$

for large $t$.

Combining (4.7)-(4.8) and (4.13)-(4.14), we have

$$\chi^2(t) \int_{f(2t)}^\infty F(2t - u) dF(u) \frac{\chi^2 \left( \frac{t}{2} \right)}{t^2}$$

$$= \int_0^\infty \left( \alpha - \chi' \left( t - \frac{yt}{\chi^2(t)} \right) \right) \exp \left( \chi \left( t + \frac{yt}{\chi^2(t)} \right) + \chi \left( t - \frac{yt}{\chi^2(t)} \right) - 2\chi(t) \right) dy$$

$$= \alpha \int_0^\infty \frac{\pi}{\rho(1 - \rho)} - \frac{\alpha \sqrt{\pi(\rho - 2)(\rho - 3)}}{32(\rho(1 - \rho))^\frac{3}{2}} \frac{\chi^2(t)}{t} - \frac{\chi^2(t)}{2t} + o \left( A_1(t) \chi(t) + \frac{1}{\chi(t)} + \frac{\chi(t)}{t} \right)$$

for large $t$, which complete the proof. $\square$

**Lemma 7.** Under the conditions of Lemma 6 we have

$$\frac{\chi^2 \left( \frac{t}{2} \right)}{t^2} = \frac{2^{\frac{1}{2}} \sqrt{\rho(1 - \rho)} \chi^2(t)}{\alpha \sqrt{\pi} t} (1 + o(1))$$

as $t \to \infty$.

**Proof.** From (2.7), (4.9) and (4.10), it follows that

$$\rho(\rho - 1) \frac{4\chi \left( \frac{t}{2} \right)}{t^2} u^2 + \frac{4 \rho(\rho - 1)(\rho - 2)(\rho - 3)}{3} \frac{\chi \left( \frac{t}{2} \right)}{t^4} u^4 - \frac{4 \rho(\rho - 1)(\rho - 2)(\rho - 3)(\rho - 4)}{16} \frac{\chi \left( \frac{t}{2} \right)}{t^5} u^5$$

$$+ A_1 \left( \frac{t}{2} \right) \chi \left( \frac{t}{2} \right) \left( \left( 1 + \frac{2u}{t} \right)^{\rho} \left( 1 + \frac{2u}{t} \right)^{\rho_1} - 1 \right) - \frac{A_1 \left( \frac{t}{2} \right)}{\rho_1} \left( 1 + \frac{2u}{t} \right)^{\rho_1} \left( 1 - \frac{2u}{t} \right)^{\rho_1}$$

$$\leq \chi \left( \frac{t}{2} + u \right) + \chi \left( \frac{t}{2} - u \right) - 2 \chi \left( \frac{t}{2} \right)$$

$$\leq \rho(\rho - 1) \frac{4\chi \left( \frac{t}{2} \right)}{t^2} u^2 + \frac{4 \rho(\rho - 1)(\rho - 2)(\rho - 3)}{3} \frac{\chi \left( \frac{t}{2} \right)}{t^4} u^4 + \frac{4 \rho(\rho - 1)(\rho - 2)(\rho - 3)(\rho - 4)}{16} \frac{\chi \left( \frac{t}{2} \right)}{t^5} u^5$$

$$+ A_1 \left( \frac{t}{2} \right) \chi \left( \frac{t}{2} \right) \left( \left( 1 + \frac{2u}{t} \right)^{\rho} \left( 1 + \frac{2u}{t} \right)^{\rho_1} - 1 \right) - \frac{A_1 \left( \frac{t}{2} \right)}{\rho_1} \left( 1 + \frac{2u}{t} \right)^{\rho_1} \left( 1 - \frac{2u}{t} \right)^{\rho_1}$$

$$+ \varepsilon \left( 1 + \frac{2u}{t} \right)^{\rho + \rho_1 + \varepsilon} + \varepsilon \left( 1 - \frac{2u}{t} \right)^{\rho + \rho_1 - \varepsilon}$$

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for $0 < u < t/(2c_2)$, $1 < c_2 < 1/ \left(1 - (2 - 2\rho)\frac{s}{t}\right)$. Combining with (4.12), we have

\[
\int_0^{\frac{t}{2}} \exp \left(\chi \left(\frac{t}{2} + u\right) + \chi \left(\frac{t}{2} - u\right) - 2\chi \left(\frac{t}{2}\right)\right) du \\
\leq \exp \left( -\varepsilon \left(1 + (1 - c_2^{-1})^{\rho + \rho_1 - 1}\right) - \frac{2 + (1 + c_2^{-1})^{\rho + \rho_1}}{\rho_1} \right) A_1 \left(\frac{t}{2}\right) \chi \left(\frac{t}{2}\right) \\
\times \frac{t\chi^{-\frac{s}{2}} \left(\frac{t}{2}\right)}{2\sqrt{2}\rho(1 - \rho)} \int_0^{\frac{x^{\frac{s}{2}} \left(\frac{t}{2}\right)}{c_2(2c_2 - 1 - \rho)}} \exp \left(-\frac{s^2}{2} - \frac{(\rho - 2)(\rho - 3)}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} s^4 - \frac{(\rho - 2)(\rho - 3)(\rho - 4)}{480\sqrt{2}\rho(1 - \rho)} \chi^2(t) s^5\right) ds \\
\leq \frac{(1 + O \left(A_1 \left(\frac{t}{2}\right) \chi \left(\frac{t}{2}\right)\right))}{2\sqrt{2}\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} \int_0^{\frac{x^{\frac{s}{2}} \left(\frac{t}{2}\right)}{c_2(2c_2 - 1 - \rho)}} \exp \left(-\frac{s^2}{2} - \frac{(\rho - 2)(\rho - 3)}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} s^4 - \frac{(\rho - 2)(\rho - 3)(\rho - 4)}{480\sqrt{2}\rho(1 - \rho)} \chi^2(t) s^5\right) ds \\
+ \frac{1}{2} \left(\frac{(\rho - 2)(\rho - 3)}{48\rho(1 - \rho)}\right) \chi \left(\frac{t}{2}\right) \sqrt{\pi t} \chi^2(t) \\
= \frac{1}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} (1 + o(1))
\]
and

\[
\int_0^{\frac{t}{2}} \exp \left(\chi \left(\frac{t}{2} + u\right) + \chi \left(\frac{t}{2} - u\right) - 2\chi \left(\frac{t}{2}\right)\right) du \\
\geq \exp \left(-\varepsilon \left(1 + (1 - c_2^{-1})^{\rho + \rho_1 - 1}\right) - \frac{2 + (1 + c_2^{-1})^{\rho + \rho_1}}{\rho_1} \right) A_1 \left(\frac{t}{2}\right) \chi \left(\frac{t}{2}\right) \\
\times \frac{t\chi^{-\frac{s}{2}} \left(\frac{t}{2}\right)}{2\sqrt{2}\rho(1 - \rho)} \int_0^{\frac{x^{\frac{s}{2}} \left(\frac{t}{2}\right)}{c_2(2c_2 - 1 - \rho)}} \exp \left(-\frac{s^2}{2} - \frac{(\rho - 2)(\rho - 3)}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} s^4 + \frac{(1 - c_2^{-1})^{\rho - 5}}{480\sqrt{2}\rho(1 - \rho)} \chi^2(t) s^5\right) ds \\
= \frac{1}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} (1 + o(1))
\]
for large $t$, if $0 < \rho < 1$ and $\rho + \rho_1 < 0$. So

\[
\int_0^{\frac{t}{2}} \exp \left(\chi \left(\frac{t}{2} + u\right) + \chi \left(\frac{t}{2} - u\right) - 2\chi \left(\frac{t}{2}\right)\right) du = \frac{\sqrt{\pi t}}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} (1 + o(1)) \quad (4.16)
\]
for large $t$.

By arguments similar to (4.7), (4.8), (4.14), we can get

\[
\int_0^{\frac{t}{2}} \chi' \left(\frac{t}{2} - u\right) \exp \left(\chi \left(\frac{t}{2} + u\right) + \chi \left(\frac{t}{2} - u\right) - 2\chi \left(\frac{t}{2}\right)\right) du = \frac{1}{2} \sqrt{\frac{\rho \pi}{1 - \rho}} \chi^\frac{s}{2} \left(\frac{t}{2}\right) (1 + o(1))
\]
and

\[
\int_0^{\frac{t}{2}} \left(\alpha - \chi' \left(\frac{t}{2} - u\right)\right) \exp \left(\chi \left(\frac{t}{2} + u\right) + \chi \left(\frac{t}{2} - u\right) - 2\chi \left(\frac{t}{2}\right)\right) du = o \left(\frac{t}{\chi^\frac{s}{2}(t)}\right)
\]
for large $t$. Combining with (4.16), we have

\[
\int_0^{\frac{t}{2}} \mathcal{F}(t - u)dF(u) = \frac{\alpha \sqrt{\pi t}}{4\rho(1 - \rho)\chi \left(\frac{t}{2}\right)} (1 + o(1)) \\
= \frac{2\sqrt{\pi t}}{\sqrt{\rho(1 - \rho)\chi \left(\frac{t}{2}\right)}} (1 + o(1))
\]
for large $t$, which deduces the desired result.

The proof is complete. \hfill $\square$

**Lemma 8.** Assume that $b(t) \in 2R\beta_{\rho,2}$ with auxiliary function $A_2(t)$ and $\rho_2 \leq 0$, $\beta \in \mathbb{R}$, and $c(t) \in 2R\gamma_{\rho,3}$ with auxiliary function $A_3(t)$ and $\rho_3 \leq 0$, $\gamma \in \mathbb{R}$. Then $b(t)c(t) \in 2R\beta_{\gamma+1,\max(\rho_2,\rho_3)}$.

**Proof.** From Lemma 3 it follows that

\[
\frac{b(tx)c(tx)}{b(t)c(t)} = x^{\beta+\gamma} \left( 1 + A_2(t) \frac{x^\rho_2 - 1}{\rho_2} + A_3(t) \frac{x^\rho_3 - 1}{\rho_3} + o(A_2(t) + A_3(t)) \right)
\]

(4.17)

for large $t$ and fixed $x > 0$, which implies the desired result. \hfill $\square$

**Lemma 9.** Let $G \in L_\alpha$ and $\mathcal{C}(t) = \int_0^t e^{\alpha u}dG(u)$ with $\alpha > 0$. Assume that $c(t) = e^{\alpha t}G(t) \in 2R\gamma_{\rho,3}$ with auxiliary function $A_3(t)$ for $\rho_3 \leq 0$, $\gamma > -1$, $\gamma + \rho_3 + 1 > 0$, then $\mathcal{C}(t) \in 2R\gamma+1,\max(\rho_3,-1)$.

**Proof.** By using Lemma 3 for every $\varepsilon > 0$ and $x, y > 0$, there exists $t_0 = t_0(\varepsilon)$ such that all $tx, txy \geq t_0$,

\[
\left| \int_0^1 \frac{\left( \frac{c(tx)}{c(t)} - y^\gamma \right)}{A_3(tx)} dy - \int_0^1 y^\gamma y^\rho_3 - 1 \rho_3 dy \right| \\
\leq \int_0^1 \left| \frac{\left( \frac{c(tx)}{c(t)} - y^\gamma \right)}{A_3(tx)} - y^\gamma \frac{y^\rho_3 - 1}{\rho_3} \right| dy \\
\leq \varepsilon \int_0^1 y^{\gamma+\rho_3-\varepsilon} dy \\
= \frac{\varepsilon}{\gamma + \rho_3 + 1 - \varepsilon} \left( 1 - \left( \frac{t_0}{tx} \right)^{\gamma+\rho_3+1-\varepsilon} \right),
\]

which implies

\[
\frac{1}{\gamma + 1} - A_3(tx) \left( \frac{\varepsilon}{\gamma + \rho_3 + 1 - \varepsilon} - \frac{1}{\rho_3(\gamma + 1)} \left( \frac{t_0}{tx} \right)^{\gamma+1} \right) - A_3(tx) \left( \frac{1}{\gamma + 1} \left( \frac{t_0}{tx} \right)^{\gamma+1} \right)
\]

\[
\leq \int_0^t c(y)dy \\
\leq \frac{1}{\gamma + 1} - A_3(tx) \left( \frac{\varepsilon}{\gamma + \rho_3 + 1 - \varepsilon} - \frac{1}{\rho_3(\gamma + \rho_3 + 1)} \left( \frac{t_0}{tx} \right)^{\gamma+\rho_3+1} \right) + \int_0^t c(y)dy \\
\]

for large $t$.

Note that $\mathcal{C}(t) = 1 - c(t) + \alpha \int_0^t c(u)du$. Combining with Lemma 3 for large $t$ we have

\[
-\frac{\gamma + 1}{\alpha t} x^\gamma \left( 1 + A_3(t) \left( \frac{x^\rho_3 - 1}{\rho_3} + \varepsilon x^\rho_3 \max \left( x^\varepsilon, x^{-\varepsilon} \right) \right) \right) + \frac{\gamma + 1}{\alpha t c(t)} \left( 1 + A_3(t) \left( \frac{x^\rho_3 - 1}{\rho_3} - \varepsilon x^\rho_3 \max \left( x^\varepsilon, x^{-\varepsilon} \right) \right) \right) \times \left[ 1 - \left( \frac{t_0}{tx} \right)^{\gamma+1} \right]
\]

\[
- A_3(t) x^\rho_3 \left( 1 + \varepsilon \max \left( x^\varepsilon, x^{-\varepsilon} \right) \right) \left( \frac{1}{\gamma + \rho_3 + 1} + \frac{\varepsilon(\gamma + 1)}{\gamma + \rho_3 + 1 - \varepsilon} - \frac{1}{\rho_3} \left( \frac{t_0}{tx} \right)^{\gamma+1} \right)
\]

\[13\]
Lemma 10. Let $F, G \in L_\alpha$, $\alpha > 0$. Assume that $b(t) = e^{\alpha t} F(t) \in 2RV_{\rho_2}$ with auxiliary function $A_2(t)$ and $\rho_2 \leq 0$, $\beta \in \mathbb{R}$, and $c(t) = e^{\alpha t} G(t) \in 2RV_{\gamma, \rho_3}$ with auxiliary function $A_3(t)$ and $\rho_3 \leq 0$, $\gamma > -1$, $\gamma + \rho_3 + 1 > 0$. Then

$$\int_0^t F(t-u)dG(u) = \alpha \int_0^\frac{t}{\alpha+1} (1-u)^\beta u^\gamma du + \frac{\alpha}{\rho_2} \int_0^\frac{t}{\alpha+1} ((1-u)^\rho_2 - 1)(1-u)^\beta u^\gamma du A_2(t)$$

$$+ \left( \beta \int_0^\frac{t}{\alpha+1} (1-u)^{\beta-1} u^\gamma (2u - 1) du - 2(1+\gamma) \int_0^\frac{t}{\alpha+1} (1-u)^\beta u^\gamma du \right) t^{-1}$$

$$+ \alpha \left( \frac{2^{\rho_3 - \beta}}{\rho_3(\gamma + \rho_3 + 1)} \int_0^\frac{t}{\alpha+1} (1-u)^{\beta-1} u^\gamma (2u)^{\rho_3} - 1) du$$

$$+ \frac{2^{\rho_3 - \beta} (2u)^{\rho_3 - 1} - \gamma - \rho_3 - 1}{\rho_3(\gamma + \rho_3 + 1)} \int_0^\frac{t}{\alpha+1} (1-u)^\beta u^\gamma du \right) A_3(t) + o\left(t^{-1} + A_2(t) + A_3(t)\right)$$

for large $t$.

Proof. Note that $C(t) = \int_0^t e^{\alpha u}dG(u)$. From (4.18), (4.19) and the dominated convergence theorem, we have

$$\int_0^\frac{t}{\alpha+1} \left( \frac{C(ut)}{C(\frac{t}{2})} \right) - (2u)^{\gamma + 1} du$$

$$= \frac{\frac{\alpha}{\gamma + 1} \frac{t}{2} C(\frac{t}{2})}{C(\frac{t}{2})} \int_0^\frac{t}{\alpha+1} \left( \frac{C(ut)}{\gamma + 1} \frac{t}{2} C(\frac{t}{2}) - (2u)^{\gamma + 1} \right) du + \left( \frac{\frac{\alpha}{\gamma + 1} \frac{t}{2} C(\frac{t}{2})}{C(\frac{t}{2})} - 1 \right) \int_0^\frac{t}{\alpha+1} (1-u)^{\beta-1} (2u)^{\gamma + 1} du$$
\[
\left(1 + \frac{A_3 \left(\frac{t}{2}\right)}{\gamma + \rho_3 + 1} + \frac{2(\gamma + 1)}{\alpha t} + o \left(\frac{1}{t} + A_3(t)\right)\right) \left[-\frac{2(\gamma + 1)}{\alpha t} \int_0^{\frac{t}{2}} (1 - u)^{\beta-1}(2u)^{\gamma} du + A_3 \left(\frac{t}{2}\right) \left(\frac{1}{\gamma + \rho_3 + 1} \int_0^{\frac{t}{2}} (1 - u)^{\beta-1}(2u)^{\gamma+1} du - \frac{A_3 \left(\frac{t}{2}\right)}{\gamma + \rho_3 + 1} + \frac{2(\gamma + 1)}{\alpha t} + o \left(\frac{1}{t} + A_3(t)\right)\right) + \right. \\
\left. \int_0^{\frac{t}{2}} (1 - u)^{\beta-1}(2u)^{\gamma+1} du \left(\frac{A_3 \left(\frac{t}{2}\right)}{\gamma + \rho_3 + 1} + \frac{2(\gamma + 1)}{\alpha t} + o \left(\frac{1}{t} + A_3(t)\right)\right) \right)
\]

for large \( t \). So

\[
\int_0^{\frac{t}{2}} (1 - \frac{u}{t})^{\beta} e^{\alpha u} dG(u) = (\gamma + 1)^{2^{\gamma+1}} \int_0^{\frac{t}{2}} (1 - u)^{\beta} u^{\gamma} du
\]

\[
= \int_0^1 (1 - \frac{u}{t})^{\beta} e^{\alpha u} dG(u) - (1 - \frac{1}{t})^{\beta} C(1) - \beta \int_0^{\frac{t}{2}} (1 - u)^{\beta-1}(2u)^{\gamma+1} du
\]

\[
+ \beta \int_0^{\frac{t}{2}} (1 - u)^{\beta-1} \left(\frac{C(ut)}{C \left(\frac{t}{2}\right)} - (2u)^{\gamma+1}\right) du
\]

\[
= A_3(t) \frac{2^{\gamma-\rho_3+1}(\gamma + 1)}{\rho_3(\gamma + \rho_3 + 1)} \int_0^{\frac{t}{2}} (1 - u)^{\beta-1}(2u)^{\gamma+1} du
\]

\[
+ \frac{2^{\gamma+1}(\gamma + 1)}{\alpha t} \int_0^{\frac{t}{2}} (1 - u)^{\beta-1} u^{\gamma}(2u - 1) du + o \left(\frac{1}{t} + A_3(t)\right), \tag{4.20}
\]
due to \( C(t) \in RV_{\gamma+1} \) and \( \gamma + \rho_3 + 1 > 0 \).

By using integration by parts, for \( \gamma > -1 \) we can get

\[
\int_0^{\frac{t}{2}} (1 - \frac{u}{t})^{\beta} \frac{(1 - \frac{u}{t})^{\rho_2 - 1} e^{\alpha u} dG(u)}{C \left(\frac{t}{2}\right)} \gamma + 1 \int_0^{\frac{t}{2}} ((1 - u)^{\rho_2} - 1) (1 - u)^{\beta} u^{\gamma} du \tag{4.21}
\]

and

\[
\int_0^{\frac{t}{2}} (1 - \frac{u}{t})^{\beta+\rho_2-\varepsilon} e^{\alpha u} dG(u) \rightarrow (\gamma + 1)^{2^{\gamma+1}} \int_0^{\frac{t}{2}} (1 - u)^{\beta+\rho_2-\varepsilon} u^{\gamma} du \tag{4.22}
\]
as \( t \to \infty \).

Note that \( \int_0^{\frac{t}{2}} \frac{f(t - u)}{F(t)} dG(u) = \int_0^{\frac{t}{2}} \frac{b'(t - u)}{b(t)} e^{\alpha u} dG(u) \). From Lemma 3 and (4.19)-(4.22), it follows that

\[
\int_0^{\frac{t}{2}} \frac{F(t - u) dG(u)}{F(t) c(t)}
\]

\[
= \alpha \int_0^{\frac{t}{2}} (1 - u)^{\beta} u^{\gamma} du + \left(\beta \int_0^{\frac{t}{2}} (1 - u)^{\beta-1} u^{\gamma}(2u - 1) du - 2(1 + \gamma) \int_0^{\frac{t}{2}} (1 - u)^{\beta} u^{\gamma} du\right) \frac{1}{t}
\]

\[
+ \alpha \left(\frac{\rho_3 - 1 - \rho_3 - 1}{\gamma + \rho_3 + 1} \int_0^{\frac{t}{2}} (1 - u)^{\beta-1} u^{\gamma+1} du + \frac{2^{\gamma+1} - (\gamma + 1) - \gamma - \rho_3 - 1}{\rho_3(\gamma + \rho_3 + 1)} \int_0^{\frac{t}{2}} (1 - u)^{\beta} u^{\gamma} du\right) A_3(t)
\]

\[
+ \frac{\alpha}{\rho_2} \int_0^{\frac{t}{2}} ((1 - u)^{\rho_2} - 1) (1 - u)^{\beta} u^{\gamma} du A_2(t) + o \left(t^{-1} + A_2(t) + A_3(t)\right)
\]

for large \( t \), which complete the proof. \( \square \)
Lemma 11. Let \( F, G \in L_\alpha, \alpha > 0 \). Assume that \( b(t) = e^{\alpha t} F(t) \in 2RV_{\rho_1, \rho_2} \) with auxiliary function \( A_2(t) \) for \( \rho_2 \leq 0, \beta \in R, \) and \( c(t) = e^{\alpha t} G(t) \in RV_\gamma \) for \( \gamma \leq -1 \). For large \( t \), we have

(i) if \( \gamma = -1 \) with \( m_G(\alpha) = \infty \),

\[
\int_0^\beta \frac{F(t-u)dG(u)}{F(t)} \frac{1}{\int_0^\gamma c(u)du} = \alpha + \frac{\alpha tc(t)}{\int_0^\gamma c(u)du} \left( \beta \int_0^\beta (1-u)^{\beta-1}(\log u)du + (\log 2) \left( 1 - 2^{-\beta} \right) \right) + \frac{1}{\int_0^\frac{\gamma}{\beta} c(u)du} + o \left( \frac{1 + tc(t)}{\int_0^\gamma c(u)du} + A_2(t) \right); \tag{4.23}
\]

(ii) if \( \gamma = -1 \) with \( m_G(\alpha) < \infty \),

\[
\int_0^\beta \frac{F(t-u)dG(u)}{F(t)} \frac{1}{\int_0^\gamma c(u)du} = m_G(\alpha) - \alpha \int_0^\infty c(u)du + o \left( \int_0^\gamma c(u)du + A_2(t) \right); \tag{4.24}
\]

(iii) if \(-2 < \gamma < -1\),

\[
\int_0^\beta \frac{F(t-u)dG(u)}{F(t)} \frac{1}{\int_0^\gamma c(u)du} = m_G(\alpha) + \alpha tc(t) \left( \int_0^\beta ((1-u)^{\gamma} - 1) u^\gamma du + \frac{1}{2^{\gamma+1}(\gamma+1)} \right) + o(tc(t) + A_2(t)); \tag{4.25}
\]

(iv) if \( \gamma = -2 \) with \( \int_1^\infty uc(u)du = \infty \),

\[
\int_0^\beta \frac{F(t-u)dG(u)}{F(t)} \frac{1}{\int_0^\gamma c(u)du} = m_G(\alpha) - \alpha t^{-1} \int_1^\beta uc(u)du + o \left( t^{-1} \int_1^\beta uc(u)du + A_2(t) \right); \tag{4.26}
\]

(v) if \( \gamma \leq -2 \) with \( \int_1^\infty uc(u)du < \infty \),

\[
\int_0^\beta \frac{F(t-u)dG(u)}{F(t)} \frac{1}{\int_0^\gamma c(u)du} = m_G(\alpha) - \beta t^{-1} \int_0^\infty uc^{\alpha+1}du + o \left( t^{-1} + A_2(t) \right). \tag{4.27}
\]

Proof. (i). For \( \gamma = -1 \) with \( \int_0^\infty e^{\alpha u}dG(u) = \infty \), we have

\[
\lim_{t \to \infty} \frac{tc(t)}{\int_0^\gamma c(u)du} = 0
\]

by Lemma \[\text{Lemma 1}\]. Combining with Lemma \[\text{Lemma 3}\] for large \( t \) and arbitrary \( \varepsilon > 0 \), we can get

\[
\frac{tc(t)}{\int_0^\gamma c(y)dy} \left( \log 2 + \log u + \varepsilon (2^\varepsilon - u^{-\varepsilon}) \right) - \frac{c(t)}{\alpha \int_0^\gamma c(y)dy} u^{-1} \left( 1 + \varepsilon^2 u^{-\varepsilon} \right) + \frac{1}{\alpha \int_0^\gamma c(y)dy} \leq \frac{C(tu)}{\alpha \int_0^\gamma c(y)dy} - 1
\]

\[
\leq \frac{tc(t)}{\int_0^\gamma c(y)dy} \left( \log 2 + \log u - \varepsilon (2^\varepsilon - u^{-\varepsilon}) \right) - \frac{c(t)}{\alpha \int_0^\gamma c(y)dy} u^{-1} \left( 1 - \varepsilon^2 u^{-\varepsilon} \right) + \frac{1}{\alpha \int_0^\gamma c(y)dy}
\]

if \( 1/t < u < 1/2 \). Then

\[
\beta \int_0^\gamma (1-u)^{\beta - 1} \left( \frac{C(tu)}{\alpha \int_0^\gamma c(y)dy} - 1 \right) du
\]

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for large $t$. Note that
\[
\frac{\alpha \int_{0}^{t} c(y) dy}{C \left( \frac{1}{2} \right)} = 1 - \frac{1}{\alpha \int_{0}^{t} c(u) du} + o \left( \frac{1 + tc(t)}{\int_{0}^{t} c(y) dy} \right)
\]
for large $t$. Hence,
\[
\beta \int_{t}^{1} (1 - u)^{\beta - 1} \left( \frac{C(tu)}{C \left( \frac{1}{2} \right)} - 1 \right) du
= \frac{\alpha \int_{0}^{t} c(y) dy}{C \left( \frac{1}{2} \right)} \beta \int_{t}^{1} (1 - u)^{\beta - 1} \left( \frac{C(tu)}{\alpha \int_{0}^{t} c(y) dy} - 1 \right) du + \left( \frac{\alpha \int_{0}^{t} c(y) dy}{C \left( \frac{1}{2} \right)} - 1 \right) \beta \int_{0}^{t} (1 - u)^{\beta - 1} du
= \frac{tc(t)}{\int_{0}^{t} c(y) dy} \left( \beta \int_{0}^{t} (1 - u)^{\beta - 1} (\log u) du + (\log 2) (1 - 2^{-\beta}) \right) + o \left( \frac{1 + tc(t)}{\int_{0}^{t} c(y) dy} \right).
\]

Since
\[
\int_{t}^{1} (1 - \frac{u}{t})^{\beta} e^{au} dG(u) - 1 = \beta \int_{t}^{1} (1 - u)^{\beta - 1} \left( \frac{C(tu)}{C \left( \frac{1}{2} \right)} - 1 \right) du - \frac{\beta}{t} (1 + o(1)) - \frac{C(1)}{C \left( \frac{1}{2} \right)} + \frac{\beta C(1)}{tC \left( \frac{1}{2} \right)} (1 + o(1))
\]
and
\[
\int_{0}^{1} (1 - \frac{u}{t})^{\beta} e^{au} dG(u) = \frac{C(1)}{C \left( \frac{1}{2} \right)} - \frac{\beta}{tC \left( \frac{1}{2} \right)} \int_{0}^{1} u e^{au} dG(u) (1 + o(1))
\]
for large $t$, we have
\[
\frac{\int_{0}^{t} (1 - \frac{u}{t})^{\beta} e^{au} dG(u)}{C \left( \frac{1}{2} \right)} - 1 = \frac{tc(t)}{\int_{0}^{t} c(y) dy} \left( \beta \int_{0}^{t} (1 - u)^{\beta - 1} (\log u) du + (\log 2) (1 - 2^{-\beta}) \right) + o \left( \frac{1 + tc(t)}{\int_{0}^{t} c(y) dy} \right)
\]
by combining with (4.29).

Noting that
\[
\int_{0}^{t} \frac{1}{2} \frac{(1 - \frac{u}{t})^{\beta - 1 + \epsilon} e^{au} dG(u)}{C \left( \frac{1}{2} \right)} \rightarrow 0
\]
and
\[
\int_{0}^{t} \frac{(1 - \frac{u}{t})^{\beta + \epsilon} e^{au} dG(u)}{C \left( \frac{1}{2} \right)} \rightarrow 1
\]
as $t \rightarrow \infty$. From Lemma 3 and (4.30), it follows that
\[
\frac{b(t-u)}{b(tu)} e^{au} dG(u)
= \int_{0}^{t} \frac{(1 - \frac{u}{t})^{\beta} e^{au} dG(u)}{C \left( \frac{1}{2} \right)} + o(A_2(t))
= 1 + \frac{tc(t)}{\int_{0}^{t} c(y) dy} \left( \beta \int_{0}^{t} (1 - u)^{\beta - 1} (\log u) du + (\log 2) (1 - 2^{-\beta}) \right) + o \left( \frac{1 + tc(t)}{\int_{0}^{t} c(y) dy} + A_2(t) \right)
\]
for large $t$, which implies \((4.23)\).

(ii)-(v). Note that

\[ 1 - \beta x + 2^{1-\beta} \beta(\beta - 1)x^2 \leq (1 - x)^\beta \leq 1 - \beta x, \quad 0 \leq \beta \leq 1, \]

\[ 1 - \beta x \leq (1 - x)^\beta \leq 1 - \beta x + 2^{1-\beta} \beta(\beta - 1)x^2, \quad 1 \leq \beta \leq 2 \text{ or } \beta < 0, \]  \hspace{1cm} (4.31)

and

\[ 1 - \beta x \leq (1 - x)^\beta \leq 1 - \beta x + 2^{1-\beta} \beta(\beta - 1)x^2, \quad \beta > 2 \]

for $0 < x < \frac{1}{2}$. We only consider the case of $\beta < 0$. For the rest cases, the arguments are similarly and details are omitted here.

By using integration by parts, we have

\[
\int_1^t \left( 1 - \frac{u}{t} \right)^\beta e^{\alpha u} dG(u) - \int_1^\infty e^{\alpha u} dG(u) \\
= \left( (1 - t^{-1})^\beta - 1 \right) c(1) - 2^{-\beta} c \left( \frac{t}{2} \right) + \alpha \int_1^t \left( (1 - \frac{u}{t})^\beta - 1 \right) c(u) du \\
- \alpha \int_1^\infty c(u) du - \beta \int_1^t \left( 1 - \frac{u}{t} \right)^{\beta - 1} c(u) du
\]  \hspace{1cm} (4.32)

From Lemma 1 and (4.31), we can get

\[
\int_1^t \left( (1 - \frac{u}{t})^\beta - 1 \right) c(u) du = -\beta t^{-1} \int_1^\infty uc(u) du + o(t^{-1}),
\]

\[
\int_1^t \left( 1 - \frac{u}{t} \right)^{\beta - 1} c(u) du = \int_1^\infty c(u) du + o(1)
\]  \hspace{1cm} (4.33)

and

\[
\int_1^\infty c(u) du = o(t^{-1})
\]

for large $t$, if $\gamma \leq -2$ with $\int_1^\infty uc(u) du < \infty$. Then

\[
\int_1^t \left( 1 - \frac{u}{t} \right)^\beta e^{\alpha u} dG(u) - \int_1^\infty e^{\alpha u} dG(u) \\
= -\frac{\beta c(1)}{t} - \frac{\alpha \beta}{t} \int_1^\infty uc(u) du - \frac{\beta}{t} \int_1^\infty c(u) du + o(t^{-1})
\]

\[
= -\frac{\beta}{t} \int_1^\infty uc^{\alpha u} dG(u) + o(t^{-1})
\]  \hspace{1cm} (4.34)

for large $t$.

Since

\[
\int_0^1 \left( (1 - \frac{u}{t})^\beta - 1 \right) e^{\alpha u} dG(u) = -\frac{\beta}{t} \int_0^1 uc^{\alpha u} dG(u)(1 + o(1)),
\]

\[
0 < \left| \int_0^\infty \left( 1 - \frac{u}{t} \right)^\beta \frac{1 - \frac{u}{t}}{\rho_2} e^{\alpha u} dG(u) \right| < 2^{-\beta} \frac{2^{-\beta} - 1}{\rho_2} m_G(\alpha) < \infty
\]
Proof of Theorem 1. For large \( A \) and \( f(t) \) satisfied \( 0 < A < f(t) \) and \( \chi'(A) < \alpha \), we have

\[
\frac{\int_{0}^{f(t)} F(t-u) dF(u)}{\int_{f(t)}^{f(t)} F(t-u) dF(u)} \leq \frac{e^{\chi(t)-2x(t)} \int_{0}^{f(t)} e^{\alpha u} dF(u)}{\int_{f(t)}^{f(t)} F(t-u) e^{\alpha u} dF(u)}
\]

The proof is complete. 

**Proof of Theorem 1.** For large \( t \) and \( A \) satisfied \( 0 < A < f(t) \) and \( \chi'(A) < \alpha \), we have

\[
0 < \int_{0}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta - 2 \gamma} e^{\alpha u} dF(u) < 2^{-\beta - 2 \gamma} m_G(\alpha) < \infty
\]

for \( \beta < 0, \rho_2 \leq 0 \), we can get

\[
\int_{0}^{\frac{t}{2}} \frac{F(t-u)}{F(t)} F(t-u) dG(u) = \int_{0}^{\frac{t}{2}} \frac{b(t-u)}{b(t)} e^{\alpha u} dG(u)
\]

\[
= \int_{0}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta} e^{\alpha u} dG(u) + o(A_2(t))
\]

\[
= m_G(\alpha) + \int_{\frac{t}{2}}^{t} \left( 1 - \frac{u}{t} \right)^{\beta} e^{\alpha u} dG(u) - \int_{0}^{\infty} e^{\alpha u} dG(u)
\]

\[
= m_G(\alpha) - \beta t^{-1} \int_{0}^{\infty} e^{\alpha u} dG(u) + o(t^{-1} + A_2(t))
\]

by combining with (4.31), which complete the proof of case (v).

For \( \gamma = -2 \) with \( \int_{1}^{\infty} uc(u)du = \infty \), we have

\[
\int_{1}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta} c(u)du = -\beta t^{-1} \int_{1}^{\frac{t}{2}} uc(u)du(1 + o(1))
\]

and

\[
\int_{\frac{t}{2}}^{t} \int_{1}^{\frac{t}{2}} c(u)du = o(t^{-1} \int_{1}^{\frac{t}{2}} uc(u)du)
\]

by using (4.31) and Lemma 1. Combining (4.32) and (4.35), we can get

\[
\int_{0}^{\frac{t}{2}} \frac{F(t-u)}{F(t)} dG(u) = m_G(\alpha) - \alpha \beta t^{-1} \int_{1}^{\frac{t}{2}} uc(u)du + o \left( t^{-1} \int_{1}^{\frac{t}{2}} uc(u)du + A_2(t) \right)
\]

for large \( t \), which deduces the result in case (iv).

Similarly, we have

\[
\int_{1}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta} e^{\alpha u} dG(u) - \int_{1}^{\infty} e^{\alpha u} dG(u) = \alpha t c(t) \left( \int_{0}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta} u^\gamma du + \frac{1}{2^{1+\gamma}(1+\gamma)} + o(1) \right)
\]

for \( -2 < \gamma < -1 \), and

\[
\int_{1}^{\frac{t}{2}} \left( 1 - \frac{u}{t} \right)^{\beta} e^{\alpha u} dG(u) - \int_{1}^{\infty} e^{\alpha u} dG(u) = -\alpha \int_{\frac{t}{2}}^{\infty} c(u)du(1 + o(1))
\]

for \( \gamma = -1 \) with \( m_G(\alpha) < \infty \). By using (4.32) and (4.35), we can obtain (4.24) and (4.25), respectively.

The proof is complete. 

**Proof of Theorem 1.** For large \( t \) and \( A \) satisfied \( 0 < A < f(t) \) and \( \chi'(A) < \alpha \), we have
\[ \frac{\chi^{\frac{1}{t}}(t) F^*(t)}{t F^2\left(\frac{t}{2}\right)} \]

\[ = \frac{2\chi^{\frac{1}{2}}(t)}{t F^2\left(\frac{t}{2}\right)} \int_0^{\frac{t}{2}} F(t-u) dF(u) \left(1 + \frac{\sqrt{\pi}}{2} \int_0^{\frac{t}{2}} F(t-u) dF(u)\right) \]

\[ = \frac{\chi^{\frac{1}{2}}(t)}{t F^2\left(\frac{t}{2}\right)} \int_0^{\frac{t}{2}} F(t-u) dF(u) \left(1 + \frac{\int_0^{\frac{t}{2}} f(t) F(t-u) dF(u)}{\int_0^{\frac{t}{2}} F(t-u) dF(u)} \right) \left(1 + \frac{\sqrt{\pi}}{2} \int_0^{\frac{t}{2}} F(t-u) dF(u)\right) \]

\[ = \frac{\alpha}{2} \sqrt{\frac{\pi}{\rho(1-\rho)}} \frac{\sqrt{1-\rho}}{\rho(1-\rho)^{\frac{\rho}{2}}} \frac{1}{\chi(t)} - \frac{1}{2^\rho} \sqrt{\frac{\rho\pi}{1-\rho}} \frac{\chi(t)}{t} + o\left(A_1(t)\chi(t) + \frac{1}{\chi(t)} + \frac{\chi^{1/2}(t)}{t}\right) \]

for large \( t \), which deduces the desired result. \( \square \)

**Proof of Proposition 1.** Combining Lemma 10 and Lemma 11 we can derive the desired results. \( \square \)

**Proof of Theorem 2.** From (4.17), it follows that

\[ F\left(\frac{t}{2}\right) G\left(\frac{t}{2}\right) = 2^{-\beta-\gamma} \left(1 + A_2(t) \frac{2^{-\rho_2} - 1}{\rho_2} + A_3(t) \frac{2^{-\rho_3} - 1}{\rho_3} + o(A_2(t) + A_3(t))\right) G(t) b(t) \]

for large \( t \). By arguments similar to Proposition 1 we have

\[ \int_0^{\frac{t}{2}} G(t-u) dF(u) + \frac{t}{2} G\left(\frac{t}{2}\right) = G(t) M_2(\beta, \gamma, t) \]

for large \( t \), where \( M_2(\beta, \gamma, t) \) is given by (5.6) and (5.7). With the decomposition of convolution tail, for large \( t \) we can get

\[ F^* G(t) = \int_0^{\frac{t}{2}} F(t-u) dG(u) + \int_0^{\frac{t}{2}} G(t-u) dF(u) + F\left(\frac{t}{2}\right) G\left(\frac{t}{2}\right) \]

\[ = F(t) M_1(\beta, \gamma, t) + G(t) M_2(\beta, \gamma, t) \]

with \( M_1(\beta, \gamma, t) \) given by (5.36) and (5.37). The proof is complete. \( \square \)

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