Fluid modes of axisymmetric neutron stars

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Abstract. Oscillation modes of axisymmetric neutron stars (NSs) are computed in the framework of general relativity (GR) and by use of linearized perturbation theory. Applying the Cowling approximation, the mode calculation forms an eigenvalue problem which is then solved using polytropic equations of state. We compare the resulting pressure modes with the literature and find satisfying agreement. This work is the first step towards calculating the complete oscillation mode spectrum of rapidly rotating neutron stars.

1. Introduction

Just as every physical body, neutron stars have their own distinct set of modes at which they oscillate when excited. Physical phenomena that can lead to excitations are the birth of a NS, after a supernova explosion, where the (proto-)NS is hot and rapidly rotating, as well as the infall of matter onto an accreting NS which forms a binary system with a main-sequence companion. In order for oscillations to become observable, a mechanism is needed to increase their amplitudes, preferably for a significant amount of time. Instabilities can do that work and play an important role in NS-physics. Friedman and Schutz [4] proved that several modes can become unstable for any rotation rate.

Oscillations have been detected in the sun as well as other stars helping us understand their consistence. For NSs, though, a challenging observational window is just about to open: gravitational waves. After their prediction by the general theory of relativity and indirect indications [10], huge efforts are currently undertaken in order to detect such signals from earth-based observatories. The known and dense distribution of pulsars (believed to be NSs) together with their relativistic compactnesses automatically places them among the possible sources. The goal in this case is not just their detection but also the extraction of information about the physics at such high densities, the so-called asteroseismology [1].

The importance of the NS oscillation spectrum for astrophysics gave rise to numerous works towards calculating their properties analytically as well as numerically (for a review see [6] or [5]). Perturbation theory is a useful tool for calculating deviations from equilibrium. The early studies used simplified physics, such as Newtonian gravity, no-rotation and incompressible fluid approximations. Improved approaches introduced the slow-rotation approximation and polytropic equations of state, while the relativistic Cowling approximation [2] has proved to give up to 80% accuracy of the fully relativistic results [11] at a significantly lower numerical effort. Recently, 3D relativistic time-evolutions were achieved, and simulations in rapid rotation combined with relativistic gravity are yielding first results. Still, the full oscillation spectrum...
of NSs is unresolved. Towards this goal, we present here the pressure modes of rapidly rotating NSs and will deal with inertial modes in a forthcoming paper.

2. Problem set-up
Linearized perturbation theory has proved to describe well oscillations of small amplitude. Here we restrict ourselves to the Cowling approximation, neglecting the metric perturbations. This will prevent us from calculating the damping time of the various modes but will still reveal their frequencies and overall picture (see also section 1). The background metric we use is given by

\[ ds^2 = -e^\gamma r^2 dt^2 + e^{2\alpha} \left( dr^2 + r^2 d\theta^2 \right) + e^{\gamma - \rho} r^2 \sin^2 \theta (d\phi - \omega dt)^2. \]  

and is kept fixed. The 4-velocity of a NS rotating with angular velocity \( \Omega \) is \( u^\alpha = U^0 \{ 1, 0, 0, \Omega \} \), and we write down its perturbation as \( \delta u_\alpha = \frac{1}{\rho_0^2} \{ -\Omega f_3, f_1, f_2, f_3 \} e^{im\phi} \). As a fourth variable we use the pressure perturbation, related to the perturbed total energy-density by \( \delta p = \frac{dp}{d\epsilon} \delta \epsilon \equiv C_s^2 \delta \epsilon \), where \( C_s \) is the speed of sound. The equations that we need to solve arise from the perturbed form of the conservation of energy-momentum (equations of motion for the fluid):

\[ \delta \left( T^\nu_{\mu,\nu} \right) = g^{\nu\lambda} (\delta T_{\mu\kappa,\nu} - \delta T_{\mu\nu} \Gamma^\rho_{\nu\kappa} - \Gamma^\rho_{\mu\nu} \delta T_{\rho\kappa}) = 0 \]  

In order to solve this system we additionally need to define the properties of the fluid. It is a good approximation to use an ideal fluid, whose energy-momentum tensor is described by \( T_{\mu\nu} = p g_{\mu\nu} + (p + \epsilon) U_\mu U_\nu \) and a polytropic equation of state: \( p = k \times \rho_0^\gamma \). For \( \gamma = 2, k = 100 \) and central energy-density of \( \epsilon_c = 3 \times 10^{15} \text{ g cm}^{-3} \) (model poly-2), the resulting NS has a mass of \( M = 1.05 M_\odot \) and a radius of \( R = 8.78 \text{ km} \), values consistent with observations [8].

2.1. Numerical Analysis
The system of eqs. 2 yields in general four independent first order in time and space partial differential equations. We perform mode calculations, which means assuming harmonic time-dependence \( e^{i\sigma t} \) of all variables and solving for the frequency. Numerically we treat the problem as an eigenvalue one: discretize the system of equations at every grid point (including the boundaries by making use of the appropriate conditions there) and solve the whole set simultaneously. This is done using the Q-R algorithm (provided by the NETLIB libraries).

3. Axisymmetric stars
For solving the background we use the RNS code of [see 7] which computes numerically the metric quantities \( \alpha, \gamma, \rho, \omega \) as well as the fluid ones \( (p, \epsilon, C_s^2) \). Since we will use equispaced gridding and would like to avoid extrapolations, we restrict ourselves to the normalized coordinates used in RNS: \( s = \frac{r}{r + r_s} \) and \( \mu = \cos \theta \). Since the code itself is of finite precision, we use a background of fixed resolution and use selected parts of it, in order to be able to follow the convergence of our mode calculation.

3.1. Non-rotating limit
In order to get a better understanding of the underlying physics and numerics, we first solve eqs. 2 for no rotation (\( \Omega = 0 \)) which corresponds to the well-studied case of spherically symmetric stars. The system of equations in its time-independent form looks like:

\[
\begin{align*}
\alpha H = \frac{-ic C_s^2}{c^2 \sin^2 \theta} f_3 - \frac{C_s^2}{c^2 \sin \theta} \left\{ \frac{1}{r^2} \left( \frac{\partial f_2}{\partial r} + \frac{\cos \theta}{\sin \theta} f_2 \right) + \frac{\partial f_1}{\partial r} + \left( \frac{3}{2} \frac{\partial \gamma}{\partial r} + 1 \frac{\partial \epsilon}{\partial r} + \frac{r^2}{2} \right) f_1 \right\} \\
\frac{ih}{c^2} f_3 = \frac{-ic}{c^2} H, \quad \frac{ih}{c^2} f_1 = \frac{1}{c^2} \frac{\partial H}{\partial r}, \quad \frac{ih}{c^2} f_2 = -\frac{1}{c^2} \frac{\partial H}{\partial r}
\end{align*}
\]  

(3)
Table 1. The lowest frequency solutions of the system of equations 3 for a polytropic equation of state with $k = 100 \text{km}^2 \text{s}^{-2}$, $\gamma = 2$, several resolutions in the radial direction and a fixed number of $\theta$-points. The third and last columns show the fundamental modes corresponding to $\ell = 2$ and $\ell = 3$, while the intermediate solutions found are numerical artifacts from the central differencing scheme. In the next to last row the extrapolated values for 'infinite' $r$-resolution (see figure 1) are listed, while the last one shows the solutions of the second order system.

| $n_\theta$ | $n_r$ | $f_{\ell=2}$ | $f'_{\ell=2}$ | $f'_{\ell=3}$ | $f_{\ell=3}$ |
|------------|-------|--------------|---------------|---------------|--------------|
| 10         | 5     | 3478         | 3788          | 4796          | 5096         |
| 10         | 3234  | 3413         | 3927          | 4072          |              |
| 20         | 3203  | 3362         | 3784          | 3894          |              |
| 25         | 3199  | 3358         | 3776          | 3883          |              |
| $\infty$   |       | 3192         | 3349          | 3760          | 3858         |
| $2^{nd}$ OS| 3220  | 3868         |               |               |              |

with $H$ representing the pressure perturbation, $\delta p = H(r, \theta)e^{i(m\phi + \sigma t)}$. In this simple case, one variable is enough to describe the whole oscillation problem and one can transform the system of eqs 3 into a single second order equation for $H$. We have solved it numerically and will use the results for comparison. Since, however, for rotating stars, the system of equations can most likely not be turned into a second order one, we try to solve the system 3 as it is. We do that as described in subsection 2.1, by discretizing the set of equations and forming an eigenvalue problem. We use central differences (as for solving the second order equation) and the appropriate conditions at the boundaries: At the radial edges all variables are vanishing – due to the regularity condition at the center, and by definition at the surface; in the angular-direction we apply symmetry with respect to the rotational axis as well as to the equatorial plane. The latter two restrict us to even $m$ and ($\ell - m$), but it is trivial to cover also the other cases. As we choose $m = 2$, we will be working with even $\ell$ perturbations. Eventually we form an eigensystem as: $[A] \times \vec{X} = i\sigma \vec{X}$, with $\vec{X}$ being the eigenvector, consisting of all 4 variable at all grid points.

In table 1 we show the lowest resulting eigenfrequencies for the model poly-2 (see section 2) for a fixed $\theta$-resolution$^1$ and increasing resolution in the radial direction. The computed frequencies show the expected convergence (second order for central differences and first order for one-sided differences) so one can extrapolate the computed values to get an eigenfrequency of 'infinite' resolution (see figure 1). The relative mean deviation of the fitted numbers are of the order or less than 1%. Note that there is an additional systematic error, related to the finite accuracy of the background quantities, of the same order. By comparing the extrapolated results with the ones from solving the second order equation, we are able to identify the fundamental pressure modes for $\ell = 2$ and $\ell = 3$, but close to each frequency, we find additional solutions with similar eigenfunctions. It becomes obvious, from their non-appearance in the well resolved equivalent analysis, that these solutions are just numerical artifacts and the reason for this is probably due to the different behavior of the four eigenfunctions, in contrast to the single eigenfunction of the second order equation. After a numerical trial-and-error procedure we found that with a one-sided differentiation scheme these unphysical modes disappear and we obtain consistent results. We are now ready to deal with rapid rotation.

$^1$ More points in the $\theta$ direction do not seem to affect the results and are computationally expensive.
Figure 1. The computed eigenfrequencies (here the $\ell = 2$ $f$-mode for poly-2 using one-sided differences) follow an inverse power law. By extrapolating we find the solution corresponding to infinite radial resolution – though restricted by the finite resolution of the background.

3.2. Rotation

Numerical calculations of oscillation modes for rapidly rotating stars have become feasible, so in order to check our results we turn to axisymmetric fluid modes published by Font et al [3] (hereafter FDGS01). They are using polytropic equations of state of the same type with $\gamma = 2$, but different parameters. We adopt two of their models in order to directly compare results. For $k = 217.856 \, \text{km}^2$ and $c_e = 0.894 \times 10^{15} \, \frac{\text{g}}{\text{cm}^3}$ the star has a gravitational mass of $M = 1.4 \, M_\odot$ and radius $R = 9.59 \, \text{km}$ (BU0). For the same parameters but an additional rotation of $\nu = 348 \, \text{Hz}$ ($\Omega = 2.185 \, \text{kHz}$) the star is slightly more compact, $M = 1.437 \, M_\odot$ and $R = 9.75 \, \text{km}$ (BU1).

Following the one sided differentiation scheme described in the previous session, we calculate the eigenfrequencies for these two models. The system of equations solved (resulting from eqs. 2) are lengthy and not presented here. In table 2 we show the lowest pressure modes for BU0, which we identify as $\ell = 2$ by looking at their eigenfunctions, together with the results of FDGS01. The extrapolated solutions (with a typical error of about 10Hz) agree within a few % with the published numbers, which were calculated for a fixed resolution.

In table 3 we show the same for the rapidly rotating model BU1. In this case we have to be careful when comparing results, since the work of FDGS01 considered axisymmetric perturbations². As known by theory (see eg [9]), rotation causes the $m$-degeneracy to be broken. A mode with frequency $f_0$ in a non-rotating star, changes to $f_i = f_0 - m\nu(1-C_{m\ell}) (+O(\nu^2))$ when rotation is switched on, where $C_{m\ell} = \frac{\int \rho r^2 [2\xi_r \xi_h + \xi_l^2]}{\int \rho r^2 [\xi_r^2 + (l+1)\xi_h^2]}$ can be determined (also numerically) from the mode’s eigenfunction ($\xi_r$, $\xi_h$ represent the displacement vectors in the radial and angular direction). The computed value for the $f$-mode is $C_{12} = 0.24$ which results in a 525Hz reduction of the frequency, and $C_{22} = 0.079$ for the $p^1$-mode, with a resulting 640Hz decrease in frequency.

² For $\Omega = 0$ this does not make a difference.
Table 2. The fundamental (f) and first overtone (p^1) m = 2 pressure oscillations for BU0 as in Table 1 with the last row being this time the results of FDGS01.

| n_θ | n_r | f_{l=2} | p^1_{l=2} |
|-----|-----|---------|-----------|
| 10  | 10  | 2270    | 6106      |
| 20  | 2104| 4945    |
| 25  | 2067| 4909    |
| 50  | 1986| 4999    |
| 100 | 1940| 4375    |
| ∞   | 1897| 4180    |
| FDGS01 | 1846 | 4100 |

Table 3. The same as in Table 2, for BU1, with the results by FDGS01 at the last row being subtracted by mν(1 − C_{nl}) which includes a mean deviation error of ∼ 15Hz due to numerical integration.

| n_θ | n_r | f_{l=2} | p^1_{l=2} |
|-----|-----|---------|-----------|
| 10  | 10  | 1753    | 5086      |
| 20  | 1574| 4042    |
| 25  | 1533| 3947    |
| 50  | 1443| 3540    |
| ∞   | 1350| 3520    |
| FDGS01* | 1330 | 3400 |

(typical errors 0.02 ⇒ 15Hz). The agreement is of the same order as before. The reason for the discrepancy is believed to lie with the finite resolution having a larger impact on the high-order modes.

We have to note that our code is looking for all possible solutions of the eigenvalue problem. The inertial modes are calculated as well, but will be analyzed in a forthcoming paper.

4. Concluding remarks

We formulated the non-radial pulsations problem of axisymmetric NSs in GR, by use of the Cowling approximation, as an eigenvalue problem. We found the standard differentiation scheme (central differences) to be problematic and we used an alternative (one-sided differentiation) numerical scheme to give consistent results. The computed eigenfrequencies of low order m = 2 fluid oscillations were compared with the results of FDGS01 with satisfactory agreement. This was the first step towards calculating the complete spectrum of NS oscillations.

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