ON THE THEORY OF DISCONTINUOUS SOLUTIONS TO SOME STRONGLY DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT

It is studied the Cauchy problem for the equations of Burgers’ type but with bounded dissipation flux

\[ u_t + f(u)_x = Q(u)_x, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \]

where \( Q' > 0, \max_{x \in \mathbb{R}} |Q(s)| < +\infty \). Such equation degenerates to hyperbolic one as the velocity gradient tends to infinity. Thus the discontinuous solutions are permitted. In the paper the definition of the generalized solution is given and the existence theorem is established in the classes of functions close to ones of bounded variation. The main feature of used a priori estimates is the fact that one needs to estimate only \( Q(u_x) \) which allows to have in fact arbitrary local growth of the velocity gradient. The uniqueness theorem is proven for essentially narrower class of piecewise smooth functions with regular behavior of discontinuity lines.

1 Introduction

It is studied the Cauchy problem to the equations of the following type

\[ \mathcal{L}u \equiv u_t + f(u)_x - Q(u)_x = 0 \quad (1) \]

in the strip \((t, x) \in \Pi_T \equiv [0, T] \times \mathbb{R}\) with initial conditions

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \quad (2) \]

where \( \max_{x \in \mathbb{R}} |u_0(x)| = M < +\infty \).

We will consider the problem of existence of generalized solutions \( u(t, x) \in L^1(\Pi_T) \) to the equation (1) from the class \( BV_{C^1}(\mathbb{R}) \) for almost all \( t \in [0, T] \); through \( BV_{C^1}(\mathbb{R}) \) one denotes the set of such functions \( g(x) \in BV_{loc}(\mathbb{R}) \) \( \cap \)
that \( g \in C^1 \) on the set of full measure in \( \mathbb{R} \). The initial function \( u_0(x) \) satisfies the following conditions: 1) there exists such compact set \( K \) that \( u_0(x) \in C^2(K \setminus \{x_i\}) \), where \( \{x_i\} \) — the finite number of points; 2) \( u_0'(x_i \pm 0) = 0 \); 3) \( u_0(x) \in W^2_1(\mathbb{R} \setminus K) \), \( Q(u_0'(x)) \in W^1_1(\mathbb{R} \setminus K) \). (Thus the initial function can have discontinuities in finite number of points.) One will say that \( u_0(x) \) belongs to the class \( BV^C_1(\mathbb{R}) \).

We will also consider the uniqueness problem but with respect to essentially narrower class of functions \( K \). Roughly speaking this is the class of piecewise smooth functions with sufficiently regular behavior of discontinuity lines (the exact definition will be given in § 2). Such restriction to the class of functions has most likely the technical character because of arising difficulties in the proof of general uniqueness theorem. Our proof is based on the essential fact that for (1) one can define the analog of the concept of characteristic lines (in case of hyperbolic equations) — level lines of the function \( u(t, x) \).

Assume that functions \( f, Q \) satisfy the following conditions: \( f \in C^1 \), \( Q \in C^2 \), \( f(0) = Q(0) = 0 \), \( Q'(\infty) \equiv Q_{-\infty} = \text{const} < 0 \), \( Q(-\infty) \equiv Q_{+\infty} = \text{const} > 0 \). Let us also introduce the notations

\[
\bar{Q} \equiv \max(|Q_{-\infty}|, |Q_{+\infty}|), F(M) \equiv \max_{|s| \leq M} |f'(s)|, Q_1 \equiv \max_{s \in \mathbb{R}} Q'(s).
\]

The equation (1) actually is the generalization to the case of bounded dissipation flux \( Q \) of known Burgers equation \( f(u) \equiv u^2/2, Q(s) \equiv \varepsilon s \) which was introduced as simplest turbulence model (see, for example, [1]). Such generalization arose relatively recently in the problems of nonlinear diffusion, phase transitions theory and generalization of Navier-Stokes equations (see, for example, [13]–[15]). The equation (1) is the simplest model which describes the interaction between the nonlinear convective transport and dissipation process when the dissipation flux is bounded. As \(|u_x| \to \infty\) the equation (1) becomes the first order equation \( u_t + f(u)_x = 0 \). So it is natural to expect the appearance of discontinuities in the generalized solutions. The fact of emerging of the hyperbolic properties in solutions to parabolic problems was studied, for example, in [4] and called there strong degeneracy. The degenerate parabolic equations were intensively studied, see, for example, the review [5] and references therein. But as a rule only continuous solutions and their properties were considered. In the paper [18] the developed theory included discontinuous solutions, but these solutions appeared only when the term with higher derivatives turned identically to zero for some range of values of \( t, x, u \). So actually there was no relation between the propagation of discontinuities and viscosity terms. The peculiarity of (1) follows from the presence of discontinuities and their effective interaction
with viscous terms. This interaction generates some kind of 'boundary layer' in the vicinity of the discontinuity.

Formally it is possible to use the corresponding theory for hyperbolic equations [10], [6] to study the equation (1). The advantage to formulate the notion of generalized solution in terms of integral inequality gives at once the validity of entropy conditions for discontinuities and as a consequence the uniqueness theorem. But to prove the existence theorem it is necessary to have the additional smoothness which in general one does not have [10]. From the other hand the presence of viscosity however degenerate most likely provides the validity of entropy conditions for discontinuities. So the integral inequality is not necessary and the main difficulty will be the proof of general uniqueness theorem.

The equation (1) and its more complicated variant when the function $Q$ is non-monotone was studied in [7], [8]. There were proven the existence and uniqueness theorems when one has no discontinuities and were shown a number of numerical calculations by the small viscosity method to illustrate the qualitative behavior of the solutions. In the recent work [3] it was proven that in the case of non-monotone as well as monotone function $Q$ the discontinuities really emerged. There were also new series of numerical calculations which were based on the splitting method.

Let us briefly outline the contents of the paper. In §2 one formulates the definition of the notion of generalized solution and some preliminaries are shown to justify introduced definition. §3 is devoted to the proof of the existence theorem by small viscosity method. To prove the theorem one needs only a priori estimates for $Q(u_x)$ and not for $u_x$ itself. This is the main feature of used estimates and allows us to have almost arbitrary local growth of the velocity gradient. In §4 the uniqueness theorem is proved for the functions from the class $\mathcal{K}$ providing that Oleinik’s condition $E$ of [11] is true.

\section{The formulation of basic results}

\subsection{The definition of generalized solution and main theorems}

Let us first give the definition of generalized solution to the problem (1), (2).

\textbf{DEFINITION 1} \textit{Bounded measurable function $u(t,x)$ will be called the generalized solution to the problem (1), (2) in $\Pi_T$ iff:}
1) there exists such set $E \subset [0, T]$, $\text{mes } E = 0$ that as $t \in [0, T] \setminus E$ the function $u(t, x) \in BV_{C^1}(\mathbb{R})$ is defined a.e. in $\mathbb{R}$ and there exists

$$\lim_{h \to 0} \int_{[0, 1] \setminus E(x)} Q\left(\frac{u(t, x + h) - u(t, x - h)}{2h}\right) = Q_{\text{lim}}(t, x)$$

for every $x \in \mathbb{R}$, where $\text{mes } X(x) = 0$, $Q_{\text{lim}}(t, x) \in BV_{\text{loc}}(\mathbb{R})$ and is continuous with respect to $x$;

2) for an arbitrary function $\varphi \in C_0^\infty(\Pi_T)$ the following integral identity holds

$$\int_{\Pi_T} \left\{ u(t, x)\varphi_t + f(u(t, x))\varphi_x - Q_{\text{lim}}(t, x)\varphi_x \right\} dxdt = 0 ;$$

3) for every segment $[a, b] \subset \mathbb{R}$

$$\lim_{t \to 0} \int_{[0, T] \setminus E} |u(t, x) - u_0(x)|dx = 0 .$$

Define now the functional classes $K_0$ and $K$ which we will use for the proof of uniqueness theorem.

**DEFINITION 2** One will say that piecewise smooth function $u(t, x)$ belongs to the class $K_0$ iff for every $T > 0$ as $0 < t < T$ the following conditions hold:

i). At any point of discontinuity except finite number of points there exist one-sided limits $u(t, x_i(t) \pm 0) \equiv u^\pm(t, x_i(t))$, $(i = 1, \ldots, N)$, $u^- \neq u^+$. (Thus the discontinuity lines can intersect only at finite number of points.)

ii). There exists such $\delta > 0$ that for every line of discontinuity or non-smoothness $x_i(t)$, $(i = 1, \ldots, N_1)$ the equation $u^\pm(t, x_i(t)) = c = \text{const}$ has only finite number of solutions for almost every $|c| \in [0, \delta]$.

**DEFINITION 3** One will say that the function $u(t, x)$ belongs to the class $K$ iff for every $T > 0$ as $0 < t < T$ the following conditions hold:

i). The function $u(t, x) \in C^2(\mathbb{R}^2)$ everywhere except finite number of lines $x_i(t)$, $i = 1, \ldots, N$ which themselves belong to the class $C^2$. Moreover $\sup_{[0, T]} |u(t, R)| \to 0$ as $|R| \to 0$. 

4
ii). At every discontinuity point except finite number of them there exist one-sided limits \( u(t, x_i(t) \pm 0) \equiv u^\pm, \ u^- \neq u^+ \).

iii). The function \( \hat{Q}(t, x) \equiv Q(u_x(t, x)) \) as 
\[ (t, x) \in \mathbb{R}^2 \setminus \bigcup_{1 \leq i \leq N, t \in \mathbb{R}^+} (t, x_i(t)) \], otherwise \( \hat{Q}(t, x) \equiv Q_{-\infty} \) as \( u^- > u^+ \) and \( \hat{Q}(t, x) \equiv Q_{+\infty} \) as \( u^- < u^+ \), is continuous. Moreover \( \sup_{[0, T]} |\hat{Q}(t, x)| \to 0 \) as \( |R| \to 0 \).

iv). The difference of any two functions from the class \( \mathcal{K} \) belongs to the class \( \mathcal{K}_0 \).

REMARK 2.1 Let us note that the integral identity 2) and properties of the function \( Q \) from the Definition 1 imply the Hugoniot condition at the discontinuity lines \( y(t) \) for the functions belonging to the class \( \mathcal{K} \):

\[ \dot{y}(t) = \frac{f(u^+(t, y(t))) - f(u^-(t, y(t)))}{u^+(t, y(t)) - u^-(t, y(t))}. \]

Remind also necessary for us condition \( E \) from the paper [11]. Let us introduce the notation

\[ l(u) \equiv f(u^-) + (u - u^-) \frac{f(u^+)}{u^+ - u^-}. \]

DEFINITION 4 One will say that the generalized solution \( u(t, x) \) from the class \( \mathcal{K} \) to the problem (4), (3) satisfies the condition \( E \) iff at every point of discontinuity of the function \( u(t, x) \) except finite number of points the following condition holds: if \( u^- > u^+ \) then \( l(u) \geq f(u) \) for \( u \in [u^+, u^-] \); if \( u^- < u^+ \) then \( l(u) \leq f(u) \) for \( u \in [u^-, u^+] \).

The main result of the present paper is the proof of the following theorems.

THEOREM 2.1 For the Cauchy problem (4), (3) there exists the generalized solution \( u(t, x) \) in the sense of Definition 1.

THEOREM 2.2 If the generalized solution \( u(t, x) \) to the Cauchy problem (4), (3) belongs to the class \( \mathcal{K} \) and satisfies the condition \( E \) then it is unique.

Let us now study some particular solutions to the equation (4) to demonstrate the validity of Definition 1.
2.2 The traveling wave solution.

Suppose that \( f'' > 0 \) and let seek the solution of (1) in the form
\[
    u(t, x) = b(x - st) \equiv b(\xi),
\]
where \( b(-\infty) = b_- = \text{const}, b(+\infty) = b_+ = \text{const}, b' \to 0 \) as \( |\xi| \to \infty \). After the substitution to (1) one has
\[
    -sb'(\xi) + f(b(\xi))' - Q(b'(\xi))' = 0,
\]
i.e.
\[
    Q(b') = f(b) - f(b_-) - s(b - b_-) \equiv \hat{f}(b).
\]
Hence
\[
    s = \frac{f(b_+) - f(b_-)}{b_+ - b_-}; \int_{b_0}^{b} \frac{dB}{Q^{-1}(f(B) - f(b_-) - s(B - b_-))} = \xi,
\]
(4)
where \( b_0 \) is some arbitrary constant.

Suppose \( b_- > b_+ \) and \( m \equiv \min_{b \in [b_-, b_+]} \hat{f}(b) \). Then one encounters two cases:

1) If \( |m| \leq |Q_{-\infty}| \) then formula (4) gives the continuous solution to the equation (1) in the form (3), for definiteness it is set that \( b_0 = (b_+ + b_-)/2 \).

2) If \( |m| > |Q_{-\infty}| \) then, denoting through \( b_2 < b_1 \) such values of \( b \) that \( \hat{f}(b_i) = m, i = 1, 2 \), one obtains the discontinuous solution to the equation (1) in the form (3) by formula (4): \( b_0 = b_1 \) as \( \xi < 0 \), \( b_0 = b_2 \) as \( \xi > 0 \). Thus there is the discontinuity of strength \( b_1 - b_2 \) at point \( \xi = 0 \).

So we discover that in general as \( b_- - b_+ \) is sufficiently large the discontinuous traveling wave can exist. But formula (4) shows that the function \( \hat{Q}(\xi) = Q(b') \) as \( \xi \neq 0 \), \( \hat{Q}(\xi) = Q_{-\infty} \) as \( \xi = 0 \), is continuous. Thus the hyperbolic properties (emerging of discontinuities) and parabolic properties (smoothness of the graph of the function on the plane \((\xi, b)\)) are combined.

Further, the function \( \hat{Q}(\xi) \) has the oscillation of the strength \( |Q_{-\infty}| \) in the traveling wave because \( Q(b'(-\infty)) = 0, Q(b'(0)) = Q_{-\infty}, Q(b'(\infty)) = 0 \). Suppose that the typical singularities of (1) are similar to the traveling wave. Than the function \( Q(u_x) \) is continuous at the discontinuity but oscillates. As one has the set of discontinuities one has the set of oscillations as well. Suppose at some moment of time \( t_0 \) we have the countable number of discontinuities within the finite segment. Then we have the oscillations of \( \sin(1/x) \) type and \( Q(u_x) \) is now discontinuous. But at any moment of time \( t_0 + \Delta \tau \) after arbitrarily small time interval \( \Delta \tau \) these oscillations disappear. As two
discontinuities merge the continuity of \( Q(u_x) \) with respect to \( t \) fails because instantaneously single oscillation arises instead of two ones.

Finally in case \( b_- < b_+ \) formula (4) shows that the solution of traveling wave type does not exist.

### 2.3 The example of the solution with discontinuous initial data

Consider the equation (1) and let \( f \equiv 0 \). Suppose the function \( Q(s) \) is such that \( Q(s) = -Q(-s) \), \( Q'(s) \sim K|s|^{-\beta} \) as \( |s| \to \infty; \beta > 0, K > 0 \). Suppose also that we are given with the initial step-like function

\[
 u(0, x) = \begin{cases} 
 u \equiv \text{const} > 0 & \text{as } x < 0 \\
 0 & \text{as } x > 0 
\end{cases}
\]

and let seek the solution of (1) with \( f \equiv 0 \) in self-similar form

\[
 u(t, x) = \begin{cases} 
 u - \sqrt{t} h(z) & \text{as } x < 0 \\
 \sqrt{t} h(z) & \text{as } x > 0 
\end{cases}
\]

(5)

where \( z = x/\sqrt{t} \). Then for the function \( h(z) \) one obtains the following ordinary differential equation

\[
 h(z) - zh'(z) = 2 [Q(h'(z))]'
\]

with the condition

\[
 |h(z)| \to 0 \quad |z| \to \infty.
\]

We are interested in the behavior of the solution in the neighborhood of the point \( z = 0 \). Representing there the unknown function \( h(z) \) in the form

\[
 h(z) = h(0) + Az^\alpha + \ldots,
\]

we obtain that \( \alpha = (\beta - 2)/(\beta - 1) \) as \( \beta > 2 \). It is easy to see that \( 0 < \alpha < 1 \) and initial discontinuity is preserved some time (after the time moment \( t^* = (u/2h(0))^2 \) the discontinuity disappears and evolution loses self-similar character (4)) but its graph has certain smoothness in a sense that defined above function \( \hat{Q} \) remains continuous.

From the other hand let us note that generally speaking another self-similar regime with \( \alpha = 2 \) exists for the same initial data; then the function \( \hat{Q} \) is discontinuous at point \( x = 0 \) but this discontinuity is removable. The analogous situation arises in case \( f \neq 0 \) as well. Namely, let us find the
solution in Kruzhkov’ sense \[6\] of the following equation (the function \(f\) is convex)
\[
  u_t + f(u)_x = 0
\]
with the same as above step-like initial data. We obtain the discontinuous solution with constant values of \(u\) on the left and right of discontinuity line. It can be easily seen that this solution satisfies also items 2) and 3) of Definition 1 for the generalized solutions to equation (4). But the solution does not satisfy item 1) because the corresponding function \(\hat{Q}\) is discontinuous (although the discontinuity is yet removable). Thus the restriction of continuity to the function \(Q_{\text{lim}}\) from item 1) of Definition 1 is necessary to prove the uniqueness theorem.

3 The existence of generalized solutions.

Let us approximate the equation (1) by the uniformly parabolic equation
\[
  L_\varepsilon u \equiv u_t + f(u)_x - Q(u_x)_x - \varepsilon u_{xx} = 0,
\]
where \(\varepsilon > 0\) can be taken arbitrary small.

Multiplying (3) by the function \(\varphi \in C^\infty_0(\Pi_T)\) and integrating with respect to \(\Pi_T\) one immediately gets as \(\varepsilon \to 0\) the integral identity 2) from the Definition 1 providing \(|u|\) is bounded and appropriate convergence as \(\varepsilon \to 0\) are valid.

Let us now study the problem (3), (2). Suppose in addition \(f, Q \in C^3\) and \(u_0(x) \in C^4(\Omega)\) for every domain \(\Omega \subset \mathbb{R}\). Then it is known [4] that under supposed requirements there exists unique classical solution \(u^\varepsilon(t, x)\) to the problem (3), (2) and \(u^\varepsilon(t, x) \in C^{4,2}(\bar{Q}_T)\) for every bounded cylinder \(Q_T \subset \Pi_T\). Further we will get the uniform with respect to \(\varepsilon\) estimates for the solutions to the problem (3), (2) which do not depend on the additional smoothness of \(f, Q\) and \(u_0(x)\). Thus our results will be valid for the initial data pointed out in (2).

Applying maximum principle [4] one immediately obtains
\[
  |u^\varepsilon(t, x)| \leq M \equiv \max_{\mathbb{R}} |u_0|.
\]
Consider the linear equation
\[
  Lz \equiv z_t + (a(t, x, \varepsilon)z)_x - (b(t, x, \varepsilon)z_x)_x = F(t, x)
\]
in \(\Pi_T\), where \(\varepsilon\) is a parameter, \(b \geq 0, a, b\) are bounded and continuous with their first derivatives with respect to \(x\) in every cylinder \(Q_T \subset \Pi_T\) (not necessary uniformly with respect to \(\varepsilon\)), \(F(t, x) \in C(\bar{\Pi}_T)\).
Let us formulate the following theorem which is direct generalization of corresponding statement from \[17\].

THEOREM 3.1
1) Suppose \( Q_r \equiv \{(t, x) : |x| < r, 0 < t < T\} \) and \( z(t, x) \) is classical solution to the equation (8) in \( \Pi_T \). Then for any \( \varphi(t, x) \geq 0, \varphi \in C^\infty, \varphi \) equals zero outside of the cylinder \( Q_r \), the following equality takes place

\[
\int \int_{Q_r} L^* \varphi |z| dx dt + \int_{\mathbb{R}} \varphi(T, x)|z(T, x)| dx \leq \int_{\mathbb{R}} \varphi(0, x)|z(0, x)| dx + \int \int_{Q_r} |F| dx dt,
\]

(9)

where

\[
L^* \varphi = - (\varphi_t + a(t, x, \varepsilon)\varphi_x + (b(t, x, \varepsilon)\varphi_x)_x).
\]

(10)

2) If additionally one suppose that \( z(0, x) \in L^1(\mathbb{R}), F(t, x) \in L^1(\Pi_T) \) then

\[
\int_{\mathbb{R}} |z(t, x)| dx \leq \int_{\mathbb{R}} |z(0, x)| dx + \int \int_{\Pi_T} |F(t, x)| dx dt
\]

(11)

for every \( t \in [0, T] \).

REMARK 3.1 We formulate this generalization only in one-dimensional case which we need here. It is clearly seen that the similar assertion is true in multi-dimensional case as well.

THEOREM 3.2 Suppose \( u^\varepsilon(t, x) \) is classical solution to the problem (3), (4) in \( \Pi_T \). Then the family of functions \( \{u^\varepsilon(t, x)\} \) is compact in \( L^1(\Pi_T) \).

PROOF. Differentiating (3) with respect to \( x \) or with respect to \( t \) and applying Theorem 3.1 one obtains the following estimates

\[
\int_{\mathbb{R}} |u^\varepsilon| dx \leq \int_{\mathbb{R}} |u_0| dx \equiv M_0
\]

\[
\int_{\mathbb{R}} |u^\varepsilon_x| dx \leq \int_{\mathbb{R}} |u'_0| dx \equiv M_1
\]

(12)
\[
\int_{\mathbb{R}} |u_\varepsilon^t| dx \leq \int_{\mathbb{R}} |\varepsilon u_0'' + Q(u_0)' - f(u_0)'| dx ,
\]
where 'prime' denotes the differentiation with respect to \( x \). Taking into account the conjecture of sufficient smoothness of initial function \( u_0(x) \) one gets the \( L_1 \)-compactness of the family \( \{u_\varepsilon(t, x)\} \).

\[\square\]

**COROLLARY 3.1**

\[
\int_{\mathbb{R}} |Q(u_x^\varepsilon)| dx + \varepsilon \int_{\mathbb{R}} |u_{xx}^\varepsilon| dx \leq M_2 = \text{const}
\]

uniformly with respect to \( \varepsilon \).

**PROOF.** The result follows from the estimates (12), equation (6) and inequality \( Q' > 0 \).

\[\square\]

Further for our convenience we omit superscript \( \varepsilon \) in case it does not influence the clearness of presentation.

Differentiating (6) with respect to \( x \) one obtains the equation for \( v \equiv u_x \)

\[
v_t + (f'(u)v)_x = (Q(v)_x + \varepsilon v_x)_x .
\]

(14)

Denote through \( v^c \) the cut-off function for \( v \)

\[
v^c \equiv \begin{cases} 
c, & v \geq c \\
v, & -c \leq v \leq c \\
-c, & v \leq -c ,
\end{cases}
\]

and through \( Q_c(s) \) the following function

\[
Q_c(s) \equiv \begin{cases} 
Q(c + 1) , & s \geq c + 1 \\
Q^+(s) , & c \leq s \leq c + 1 \\
Q(s) , & -c \leq s \leq c \\
Q^-(s) , & -c - 1 \leq s \leq -c \\
Q(-c - 1) , & s \leq -c - 1 ,
\end{cases}
\]

where \( Q^+ \) and \( Q^- \) are chosen such that \( Q_c(s) \in C^2(\mathbb{R}) \) and \( Q'_c(s) > 0 \) as \( -c - 1 < s < c + 1 \).

Below the sign of \( \int \) denotes the integration with respect to \( \mathbb{R} \) and \( \mathbb{J} \) denotes the integration with respect to the whole strip \( \Pi_T \).
THEOREM 3.3 Suppose \( v(t, x) \equiv u_x(t, x) \) is the classical solution of the equation (14) in \( \Pi_T \). Then
\[
\int_0^T \int_{\mathbb{R}} (v_x^c)^2 \, dx \, dt \leq K(c, T, M, M_1, \bar{Q})
\]
uniformly with respect to \( \varepsilon > 0 \).

PROOF. Multiply (14) on \( Q_c(v) \eta \), where \( \eta(x) \geq 0, \eta \in S \) (the space of rapidly decreasing at infinity functions), and integrate with respect to \( \mathbb{R} \). Then integrating by parts one obtains
\[
\int v_t Q_c(v) \eta \, dx - \int f'(u) v [Q_c(v) \eta]_x \, dx = - \int (Q(v) + \varepsilon v)_x [Q_c(v)_x \eta + Q_c(v) \eta'] \, dx.
\]
Denote through \( \hat{Q}_c(s) \) the primitive of the function \( Q_c(s) \), i.e. \( \hat{Q}_c'(s) = Q_c(s) \). Then
\[
\frac{d}{dt} \int \hat{Q}_c(v) \eta \, dx - \int f'(u) v Q_c(v)_x \eta \, dx - \int f'(u) v Q_c(v) \eta' \, dx = - \int (Q(v)_x + \varepsilon v_x) Q_c(v)_x \eta \, dx - \int (Q(v)_x + \varepsilon v_x) Q_c(v) \eta' \, dx.
\]
Now let us integrate with respect to the segment \([0, T]\)
\[
\int \hat{Q}_c(v) \eta(x) \, dx \bigg|_0^T - \int \int f'(u) v Q_c(v)_x \eta \, dx \, dt - \int \int f'(u) v Q_c(v) \eta' \, dx \, dt = - \int \int Q(v)_x Q_c(v)_x \eta \, dx \, dt - \varepsilon \int \int v_x Q_c(v)_x \eta \, dx \, dt - \int \int (Q(v)_x + \varepsilon v_x) Q_c(v) \eta' \, dx \, dt ;
\]
\[
\int \int Q'(v) Q_c(v)_x^2 \eta \, dx \, dt + \varepsilon \int \int Q'_c(v) v_x^2 \eta \, dx \, dt = - \int \int (Q(v)_x + \varepsilon v_x) Q_c(v) \eta' \, dx \, dt + \int \int f'(u) v Q_c(v) \eta' \, dx \, dt +
\]
\[
\int\int f'(u)vQ_c(v)\eta\,dx\,dt - \int\hat{Q}_c(v)\eta\,dx \bigg|_0^T.
\]

Further, taking into account that \( Q'_c \geq 0; \ Q'_c(v) = 0 \) as \(|v| \geq c + 1\), and applying to the third summand on the right hand side the inequality \( 2ab \leq a^2 + b^2 \) one has

\[
\int\int Q'(v)Q'_c(v)v_x^2\eta\,dx\,dt \leq \bar{Q}\int\int |Q(v)_x + \varepsilon v_x|\eta'\,dx\,dt +
\]

\[
\int\int \frac{Q'_c(v)}{Q'(v)}v^2x\eta\,dx\,dt 
\]

Now take \( \eta(x) = e^{\lambda \sqrt{1 + x^2}} \) and using the inequality \(|\eta'| \leq \lambda \eta\) one gets when \( \lambda \) tends to zero (accounting of (12), (13))

\[
\frac{1}{2}\int\int Q'_cQ'(v)v_x^2\eta\,dx\,dt + 2\bar{Q}\int |u'_0|\eta\,dx.
\]

Further,

\[
\int\int f'(u)vQ_c(v)\eta\,dx\,dt - \int\hat{Q}_c(v)\eta\,dx \bigg|_0^T.
\]

Further, taking into account that \( Q'_c \geq 0; \ Q'_c(v) = 0 \) as \(|v| \geq c + 1\), and applying to the third summand on the right hand side the inequality \( 2ab \leq a^2 + b^2 \) one has

\[
\int\int Q'(v)Q'_c(v)v_x^2\eta\,dx\,dt \leq \bar{Q}\int\int |Q(v)_x + \varepsilon v_x|\eta'\,dx\,dt +
\]

\[
\int\int \frac{Q'_c(v)}{Q'(v)}v^2x\eta\,dx\,dt 
\]

Now take \( \eta(x) = e^{\lambda \sqrt{1 + x^2}} \) and using the inequality \(|\eta'| \leq \lambda \eta\) one gets when \( \lambda \) tends to zero (accounting of (12), (13))

\[
\frac{1}{2}\int\int Q'_cQ'(v)v_x^2\eta\,dx\,dt + 2\bar{Q}\int |u'_0|\eta\,dx.
\]

Further,

\[
\int\int f'(u)vQ_c(v)\eta\,dx\,dt - \int\hat{Q}_c(v)\eta\,dx \bigg|_0^T.
\]
\[
\frac{1}{2} \int |v|Q'(v)Q'_c(v)v_x^2dxdt \leq \left( \frac{F(M)^2}{2}K_0(c)T + 2\bar{Q} \right) \int |u'_0|dx .
\]

\[\square\]

**THEOREM 3.4** For every \( \eta \in C_0^\infty(\mathbb{R}) \) the sequence of the functions \( \int Q(u_x^\varepsilon(t,x))\eta(x)dx \) is compact in \( L_1([0,T]) \).

**PROOF.** Let us prove uniform boundedness and equicontinuity in \( L_1([0,T]) \) of the sequence of functions mentioned above:

i). \[
\int_0^T \left| \int Q(u_x^\varepsilon)\eta(x)dx \right| dt =
\]

\[
\int_0^T dx \int_0^1 |Q'(\theta u_x^\varepsilon)d\theta u_x^\varepsilon\eta(x)| dt \leq \int_0^T \left| \int_0^1 Q'(\theta u_x^\varepsilon)d\theta \right| |u_x^\varepsilon||\eta|dxdt \leq \max_\mathbb{R} Q' \cdot T \cdot \max_\mathbb{R} |\eta| \cdot \int |u_x^\varepsilon|dx \leq Q_1 \cdot T \cdot \max_\mathbb{R} |\eta| \cdot \int |u'_0|dx ;
\]

ii). \[
I \equiv \int_0^T \left| \int [Q(u_x^\varepsilon(t+\Delta t,x)) - Q(u_x^\varepsilon(t,x))] \eta(x)dx \right| \leq \int_0^T \left\{ \left| \int [Q(u_x^\varepsilon(t+\Delta t,x)) - Q_c(u_x^\varepsilon(t+\Delta t,x))] \eta(x)dx \right| + \right.
\]

\[
\left. \left| \int [Q_c(u_x^\varepsilon(t+\Delta t,x)) - Q_c(u_x^\varepsilon(t,x))] \eta(x)dx \right| + \right.
\]

\[
\left. \left| \int [Q_c(u_x^\varepsilon(t,x)) - Q(u_x^\varepsilon(t,x))] \eta(x)dx \right| \right\} dt .
\]
For an arbitrary small $\Delta > 0$ choose sufficiently large $c > 0$ such that

$$|Q(v) - Q_c(v)| \leq \frac{\Delta}{3T \int |\eta| dx}.$$  

This is possible because $Q$ and $Q_c$ tend to the constants at infinity.

Further using the equation (14) one has

$$I \leq \frac{2\Delta}{3} + \int_0^T \int dx \eta(x) \int_t^{t+\Delta t} Q_c(u^\varepsilon_x)_x d\tau \left| dt = \right.$$ 

$$\frac{2\Delta}{3} + \int_0^T \int_0^{t+\Delta t} d\tau \int dx Q'_c(v) \left[ (Q(v)_x + \varepsilon v_x)_x - (f'(u) v)_x \right] \eta(x) dx \left| dt = \right.$$ 

$$\frac{2\Delta}{3} + \int_0^T \int_0^{t+\Delta t} d\tau \int dx Q'_c(v)_x [Q(v)_x + \varepsilon v_x - f'(u) v] \eta(x) dx +$$ 

$$\int_0^{t+\Delta t} d\tau \left\{ \int dx Q'_c(v) [Q(v)_x + \varepsilon v_x - f'(u) v] \eta'(x) dx \right\} \leq$$ 

$$\frac{2\Delta}{3} + \int_0^T \int_0^{t+\Delta t} dt \int dx |\eta(x)| \left\{ \left| Q''_c(v) (Q'(v) + \varepsilon) v_x^2 \right| +$$ 

$$|Q''_c(v)v_x f'(u)v| + \max |\eta'| \cdot \max Q'_c \cdot \int [\|Q(v)_x + \varepsilon v_x\| + F(M)v] dx \cdot T\Delta t.$$  

Hence making the change of variables $\tau' = \tau$, $t' = -t + \tau$ and using Theorem 3.3 we get

$$I \leq \frac{2\Delta}{3} + \int_0^{\Delta t} dt' \int_{t'}^{T+t'} d\tau' \int_{|v| \leq c+1} dx |\eta(x)| \left\{ \left| Q''_c(v) (Q'(v) + \varepsilon) v_x^2 \right| +$$ 

14
\[
\left| Q''_c(v) v_x f'(u) v \right| + \Delta t \cdot |\eta'||\max \cdot \max Q'_c \cdot (M_2 + F(M)M_1) \leq \\
\frac{2\Delta}{3} + \int_0^{\Delta t} \int_0^{T+1} \left[ \max_{|v| \leq c+1} Q''_c(v)(Q'(v) + 1) \right] \int dx |\eta(x)| (v_{x+1}^c)^2 + \\
F(M) \left[ \max_{|v| \leq c+1} \max Q''_c(v) \right] \int dx |\eta(x)| (v_{x+1}^c) + \Delta t \cdot \max |\eta'| \cdot \max \leq \\
\frac{2\Delta}{3} + \Delta t \cdot \left[ \max_{|v| \leq c+1} Q''_c(v) (Q'(v) + 1) \right] \cdot \max |\eta| \cdot K(c+1, T+1, M, \bar{Q}, M_1) + \\
\Delta t \cdot F(M) \cdot \left[ \max_{|v| \leq c+1} Q''_c(v) \right] \frac{1}{2} \left[ K(c+1, T+1, M, \bar{Q}, M_1) + \int \eta^2(x) dx \right] + \\
\Delta t \cdot \max |\eta'| \cdot \max \leq \frac{2\Delta}{3} + \Delta t \cdot \max \leq \\
\text{Now as } c \text{ is fixed let choose } \Delta t \text{ in such a way that } \Delta t \cdot \max \leq \Delta/3, \text{ i.e. } I \leq \Delta. \]

\[ \square \]

**LEMMA 3.1** There exists such countable family of functions \( \eta_n \in C_0^\infty(\mathbb{R}) \), that for every \( \eta \in C_0^\infty(\mathbb{R}) \) and every \( \delta > 0 \) there exists such \( \eta_n \) that \( \sup_{\mathbb{R}} |\eta_n - \eta| \leq \delta \).

**PROOF.** Consider the family of all polynomials \( P_k(x), x \in \mathbb{R} \) with rational coefficients. Suppose

\[
P_{kl} = \begin{cases} 
0 & , \ |x| > l, l \in N \\
P_k(x) & , \ |x| < l, l \in N 
\end{cases} 
\]

and for \( m \in \mathbb{N} \)

\[
P_{klm} = \frac{1}{h} \int_{\mathbb{R}} \omega \left( \frac{x-y}{h} \right) P_{kl}(y) dy , \quad h = \frac{1}{m} , \]

(15)

15
where $\omega \in C_0^\infty$, $\omega \geq 0$, $\int_R \omega(y)dy = 1$. The family $\{P_{klm}(x)\} \subset C_0^\infty$ and obviously countable. Consider some arbitrary function $\eta(x) \in C_0^\infty(\mathbb{R})$, suppose $\text{supp } \eta(x) \subset [-l_1, l_1], l_1 \in \mathbb{N}$. Then for every $\delta > 0$ for the segment $[-l_1, l_1]$ by the aid of Weierstrass theorem there exists the polynomial with rational coefficients $P_{k_1}(x)$ such that $\sup_{[-l_1, l_1]} |P_{k_1}(x) - \eta(x)| \leq \delta/5$.

Let us extend $P_{k_1}(x)$ by zero outside the interval $(-l_1, l_1)$ and denote obtained function through $P_{k_1l_1}(x)$. Then there exists such $m_1 \in \mathbb{N}$ that $|P_{k_1l_1m_1}(x) - P_{k_1l_1}(x)| \leq 4\delta/5$, where $P_{k_1l_1m_1}(x)$ is defined with respect to the formula (13). Hence $|P_{k_1l_1m_1}(x) - \eta(x)| \leq \delta$.

$\square$

**THEOREM 3.5** There exists such subsequence $\{\varepsilon_k\}$ that the sequence $Q(u_{x^k}(t, x))$ converges in $L^1(\Pi_T)$ to the function $Q_{lim}(t, x)$. Moreover for almost all $t_\ast \in [0, T]$ the sequence $Q(u_{x^k}(t_\ast, x))$ converges in $L^1(\mathbb{R})$ to the function $Q_{lim}(t_\ast, x) \in L^1(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$.

**PROOF.** Let us take an arbitrary function $\eta_n$ from Lemma 3.1, then according to the Theorem 3.4 it is possible to choose such subsequence $\{\varepsilon_k^{(n)}\}$ that there exists the set $\mathcal{E}_n, \text{mes } \mathcal{E}_n = 0$ such that $\int Q\left(u_{x^k}^{(n)}\right) \eta_n(x)dx$ converges for every $t \in [0, T] \setminus \mathcal{E}_n$. With the help of diagonal process it is possible to choose required subsequence $\varepsilon_k$ in such a way that $\int Q\left(u_{x^k}\right) \eta_n(x)dx$ converges for every $n$ and all $t \in [0, T] \setminus \bigcup \mathcal{E}_n$.

Further for an arbitrary $\eta(x) \in C_0^\infty(\mathbb{R})$ by Lemma 3.1 choose such $n$ that $\int |\eta_n - \eta|dx \leq \Delta/(4\bar{Q})$, then

$$I \equiv \left| \int Q\left(u_{x^k_1}\right) \eta(x)dx - \int Q\left(u_{x^k_2}\right) \eta(x)dx \right| \leq$$

$$\left| \int Q\left(u_{x^k_1}\right) (\eta(x) - \eta_n(x)) dx \right| + \left| \int (Q\left(u_{x^k_1}\right) - Q\left(u_{x^k_2}\right)) \eta_n(x)dx \right| +$$

$$\left| \int Q\left(u_{x^k_2}\right) (\eta_n(x) - \eta(x)) dx \right| \leq 2\bar{Q} \int |\eta_n - \eta|dx +$$

$$\left| \int (Q\left(u_{x^k_1}\right) - Q\left(u_{x^k_2}\right)) \eta_n(x)dx \right| .$$

Now for fixed $n$ let choose such $\varepsilon > 0$ that as $\varepsilon_k_1 < \varepsilon, \varepsilon_k_2 < \varepsilon$ the second integral in the last inequality does not exceed $\Delta/2$. So the sequence $Q\left(u_{x^k}\right)$
converges weakly for almost all \( t \in [0, T] \) and therefore it converges in \( L_1 \) because of estimate (13). Taking into account the boundedness of the function \( Q \), by Lebesgue theorem one obtains the convergence in \( L_1(\Pi_T) \). Taking into account (13) it is clear that \( \lim_{\varepsilon_k \to 0} Q(u_{x}^{\varepsilon_k}) \in BV_{\text{loc}}(\mathbb{R}) \). □

**LEMMA 3.2** Suppose one has the sequence of integrable functions \( a_n(t) \geq 0 \) and \( \int_0^T a_n(t)dt \leq C(T) = \text{const} \). Then \( A_k(t) \equiv \inf_{n \geq k} a_n(t) < +\infty \) for almost all \( t \in [0, T] \).

**PROOF.** One has \( A_k(t) = a_k(t), \int_0^T A_k(t)dt \leq C(T), A_k(t) \leq A_{k+1}(t) \). By the B. Levi theorem \( A_k(t) < +\infty \) for almost all \( t \in [0, T] \). □

**LEMMA 3.3** Consider an arbitrary sequence of numbers \( c_n \to +\infty \). Then for almost all \( t \in [0, T] \) there exists such subsequence \( \varepsilon_{p}(t) \to 0 \) that
\[
\int_{\mathbb{R}} (v_{x}^{c_n,\varepsilon_{p}}(t, x))^2 dx < +\infty \tag{16}
\]
uniformly with respect to \( \varepsilon_{p}(t) \).

**PROOF.** Consider an arbitrary sequence \( c_n \to +\infty \). According to the theorem 3.3 one has
\[
\int_0^T \int_{\mathbb{R}} (v_{x}^{c_n,\varepsilon})^2 dxdt < K(\cdot, c_n),
\]
where through \( (\cdot) \) one denotes the dependence on arguments we do not worry about at the moment. By Lemma 3.2 there exist the sets \( \mathcal{E}_n, \text{mes } \mathcal{E}_n = 0 \) such that for every \( t \in [0, T] \setminus \mathcal{E}_n \)
\[
V_\alpha(t) \equiv \inf_{0 < \varepsilon < \alpha} \int_{\mathbb{R}} (v_{x}^{c_n,\varepsilon})^2 dx < +\infty \tag{17}
\]
Then for \( t \in \mathcal{T} \equiv [0, T] \setminus \bigcup_n \mathcal{E}_n \) with the aid of diagonal process one obtains (14) from (17). □
**THEOREM 3.6** The function $Q_{\lim}(t, x)$ is continuous with respect to $x$ for almost all $t$.

**PROOF.** According to Lemma 3.3 for almost all $t$ there exists such subsequence $\varepsilon_p(t)$ that the estimate (16) is valid. For such $t$ let us check the equicontinuity with respect to $x$ (dropping in the notations index $p$ and argument $t$)

\[
I \equiv \sup_{x \in K \subseteq \mathbb{R}} \left| Q \left( u_x^\varepsilon(t, x + \Delta x) \right) - Q \left( u_x^\varepsilon(t, x) \right) \right| \leq \\
\sup_{x \in K \subseteq \mathbb{R}} \left| Q \left( u_x^\varepsilon(t, x + \Delta x) \right) - Q_c \left( u_x^\varepsilon(t, x + \Delta x) \right) \right| + \\
\sup_{x \in K \subseteq \mathbb{R}} \left| Q_c \left( u_x^\varepsilon(t, x + \Delta x) \right) - Q_c \left( u_x^\varepsilon(t, x) \right) \right| + \\
\sup_{x \in K \subseteq \mathbb{R}} \left| Q_c \left( u_x^\varepsilon(t, x) \right) - Q \left( u_x^\varepsilon(t, x) \right) \right| \equiv I_1 + I_2 + I_3.
\]

Choose $c = c_n$ in such a way that $I_1 + I_3 \leq 2\Delta/3$; and by Hölder inequality

\[
I_2 = \sup_{x \in K \subseteq \mathbb{R}} \left| \int_x^{x+\Delta x} \frac{\partial}{\partial \xi} Q_c \left( u_x^\varepsilon(t, \xi) \right) d\xi \right| = \sup_{x \in K \subseteq \mathbb{R}} \left| \int_x^{x+\Delta x} Q_c' \left( u_x^\varepsilon(t, \xi) \right) v_x^\varepsilon d\xi \right| 
\]

\[
Q_1 \sup_{x \in K \subseteq \mathbb{R}} \left( \int_x^{x+\Delta x} \left| v_x^c \right| d\xi \right) \leq Q_1 \Delta x^{1/2} \sup_{x \in K \subseteq \mathbb{R}} \left( \int_x^{x+\Delta x} \left| v_x^c \right|^2 d\xi \right)^{1/2} = \\
Q_1 \Delta x^{1/2} \left( \int_{\mathbb{R}} \left| v_x^c \right|^2 dx \right)^{1/2} \leq \text{const}(c_n) \cdot \Delta x^{1/2}.
\]

Now taking $\Delta x$ small enough to provide that right hand side is less than $\Delta/3$ we get necessary equicontinuity and assertion of Theorem 3.6. \hfill \Box

**THEOREM 3.7** For almost all $t \in [0, T]$ \n
\[
Q_{\lim}(t, x) = \lim_{h \to 0} Q \left( \frac{u(t, x + h) - u(t, x - h)}{2h} \right). 
\]

(18)
PROOF. It follows from the proof of the Theorem 3.6 that for almost all \( t \in [0, T] \) there exists uniformly converging (for each \( t \) its own) subsequence. Let us fix the value of \( t \in [0, T] \) and denote such subsequence again through \( \{ u^\varepsilon(t, x) \} \).

Suppose \( S_t \equiv \{ x \in \mathbb{R} : |u^\varepsilon(t, x)| \to +\infty \} \). Since \( Q(u^\varepsilon_x) \to Q_{\text{lim}}(t, x) \) and the function \( Q \) is monotone one encounters two cases:

i). \( x_0 \in \mathbb{R} \setminus S_t \)

Since the function \( Q_{\text{lim}}(t, x) \) is continuous there exists such interval \( (a, b) \ni x_0 \) such that \( |u^\varepsilon_x(t, x)| \to +\infty \) for some set of positive measure in \( (a, b) \). As far as \( Q_{\text{lim}}(t, x) \) is continuous one can think for definiteness that \( u^\varepsilon_x(t, x) \to +\infty \) uniformly with respect to \( \varepsilon \). Further

\[
    u^\varepsilon(t, x) = u^\varepsilon(t, x_0) + \int_{x_0}^{x} u^\varepsilon_x(t, x) \, dx ,
\]

i.e. \( |u^\varepsilon| \to +\infty \) for the set of positive measure in some segment \( [a_1, b_1] \subset (a, b) \), \( x_0 \not\in [a_1, b_1] \). But this contradicts the convergence of \( \{ u^\varepsilon \} \) in the space \( L^1 \). Thus we have proved that \( \text{mes } S_t = 0 \).

iii). Suppose for definiteness \( u^\varepsilon_x(t, x_0) \to +\infty \).

Because of continuity of the function \( Q_{\text{lim}}(t, x) \) for all \( x \) from some neighborhood \( U_N(x_0) \) the estimate \( u^\varepsilon_x(t, x) > N \) is true uniformly with respect to \( \varepsilon > 0 \), where \( N > 0 \) is an arbitrary large number. Then

\[
    u^\varepsilon(t, x_0 + h) - u^\varepsilon(t, x_0 - h) = \int_{x_0-h}^{x_0+h} u^\varepsilon_x(t, \xi) \, d\xi \geq N \cdot 2h
\]

uniformly with respect to \( \varepsilon > 0 \). Tending \( \varepsilon \) to 0 one has

\[
    \frac{u(t, x_0 + h) - u(t, x_0 - h)}{2h} \geq N
\]

for almost all sufficiently small \( h > 0 \).

□

To extend our results up to the initial data (2) it is enough to approximate \( u_0 \in BV^+_C \) by smooth initial functions and obtain uniform estimates for
the integrals from (12). So consider some function \( \omega(z) \geq 0, \omega \in [-1, 1], \int \omega(z)dz = 1 \) and averaged functions

\[
u_0^h(x) = \frac{1}{h} \int \omega\left(\frac{x - y}{h}\right) u_0(y)dy.
\]

**Lemma 3.4** Uniformly with respect to \( h \) one has the following estimates

i). \[
\int \left| \left( u_0^h \right)_x \right| dx \leq \text{const}_1
\]

ii). \[
\int \left| Q \left( \left( u_0^h \right)_x \right) \right| dx \leq \text{const}_2
\]

\[
(19)
\]

iii). \[
\int \left| \left( u_0^h \right)_{xx} \right| dx \leq \text{const}_3/h
\]

**Proof.** In view of properties of the function \( u_0(x) \in BV_{C^1} \) it is enough to check inequalities (19) operating within small vicinity of the set of discontinuity points \( \{x_i\} \). In addition without loss of generality it can be considered that \( \{x_i\} \) consists of single point \( x_0 \).

Let introduce the notation \([u] \equiv |u_0(x_0 + 0) - u_0(x_0 - 0)|\). Further

\[
i) \int_{x_0 - \delta}^{x_0 + \delta} \left| \left( u_0^h \right)_x \right| dx = \int_{x_0 - \delta}^{x_0 + \delta} \frac{1}{h} \int_{x - h}^{x + h} \frac{\partial}{\partial y} \omega\left(\frac{x - y}{h}\right) u_0(y)dy \left| dx = \right.
\]

\[
\int_{x_0 - \delta}^{x_0 + \delta} \left. \frac{1}{h} \int_{x - h}^{x} \frac{\partial}{\partial y} \omega\left(\frac{x - y}{h}\right) u_0(y)dy + \frac{1}{h} \int_{x_0}^{x + h} \frac{\partial}{\partial y} \omega\left(\frac{x - y}{h}\right) u_0(y)dy \right| dx =
\]

\[
\int_{x_0 - \delta}^{x_0 + \delta} \frac{1}{h} \omega\left(\frac{x - x_0}{h}\right) (u_0(x_0 - 0) - u_0(x_0 + 0)) -
\]

\[
\int_{x_0 - \delta}^{x_0 + \delta} \frac{1}{h} \int_{x - h}^{x + h} \omega\left(\frac{x - y}{h}\right) u_0'(y)dy \left| dx \leq [u] + 2\delta \max_{x_0 - \delta < x < x_0 + \delta} |u_0'(x)| .
\]
\[ii) \quad I \equiv \int_{x_0 - \delta}^{x_0 + \delta} \left| Q' \left( (u_h^0)_x \right) (u_h^0)_{xx} \right| \, dx = \int_{x_0 - \delta}^{x_0 + \delta} \left( \frac{1}{h} \int_{x-h}^{x+h} \frac{\partial}{\partial x} \omega \left( \frac{x-y}{h} \right) u_0(y) \, dy \right) \cdot \left( \frac{1}{h} \int_{x-h}^{x+h} \frac{\partial^2}{\partial y^2} \omega \left( \frac{x-y}{h} \right) u_0(y) \, dy \right) \, dx. \]

Estimate the second derivative of averaged function

\[\left| (u_0^h)_{xx} \right| = \left| \frac{1}{h} \int_{x-h}^{x+h} \frac{\partial^2}{\partial y^2} \omega \left( \frac{x-y}{h} \right) u_0(y) \, dy \right| = \]

\[\frac{1}{h^2} \omega' \left( \frac{x-x_0}{h} \right) (u_0(x_0 - 0) - u_0(x_0 + 0)) + \]

\[\frac{1}{h} \int_{x-h}^{x_0} \omega \left( \frac{x-y}{h} \right) u''_0(y) \, dy + \frac{1}{h} \int_{x_0}^{x+h} \omega \left( \frac{x-y}{h} \right) u''_0(y) \, dy \leq \frac{1}{h^2} |\omega'| \left( \frac{x-x_0}{h} \right) [u] + \max_{x_0 - \delta \leq x \leq x_0} |u''_0(x)| , \]

taking into account the vanishing of one-sided derivatives of the function \( u_0 \) at discontinuity points. From estimates of item i) one also infers

\[\left| (u_0^h)_x - \frac{1}{h} \omega \left( \frac{x-x_0}{h} \right) [u] \right| \leq \max_{x_0 - \delta \leq x \leq x_0 + \delta} |u'_0(x)| . \]

Therefore

\[\left| Q' \left( (u_h^0)_x \right) - Q' \left( \frac{1}{h} \omega \left( \frac{x-x_0}{h} \right) [u] \right) \right| \leq const_1 \cdot \max_{x_0 - \delta \leq x \leq x_0} |u'_0(x)| \leq const_2 \cdot h . \]

Hence

\[I \leq const \left( [u] + h + \delta \right) + \]
\[
\left| \int_{x_0-\delta}^{x_0+\delta} Q' \left( \frac{1}{h^2} \omega' \left( \frac{x-x_0}{h} \right) [u] \right) \left( \frac{1}{h^2} \omega' \left( \frac{x-x_0}{h} \right) [u] \right) dx \right| \leq \text{const} ([u] + h + \delta) + 2\bar{Q} .
\]

iii). Taking into account the estimates from item ii) one has
\[
\left| \int_{x_0-\delta}^{x_0+\delta} \left( u^h_0 \right)_{xx} dx \right| \leq \frac{\text{const}_1}{h} [u] + \text{const}_2 \cdot \delta .
\]

Now if \( \varepsilon \) and \( \delta \) are taken of order \( h \) then one obtains the estimates (19) and the integrals on the right hand side of (13) can be estimated independently on \( h \).

\[ \square \]

**REMARK 3.2** We can see from the proved Lemma that the number of discontinuities in general should be finite because each discontinuity however its strength put the contribution \( 2\bar{Q} \) to the variation of the function \( Q(u'_0) \). If nevertheless one allows to exist infinite number of discontinuity points than probably it is reasonable to require that the corresponding function \( \hat{Q}(u'_0) \) should be continuous (see § 2.2), i.e. it should have infinite one-sided derivatives.

Thus the Theorem 2.1 — the existence theorem — has been proved.

### 4 On the uniqueness of generalized solutions.

Now let us prove the uniqueness Theorem 2.2.

Suppose there exist two solutions \( u(t, x), v(t, x) \) from the class \( \mathcal{K} \) to the problem (1), (2) in the sense of Definition 1. Consider the difference \( u - v \equiv w \). Then \( w \in \mathcal{K}_0 \) and
\[
\int\int_{\Pi_T} \left\{ (u - v)\varphi_t + [f(u) - f(v)] \varphi_x + [Q_{lim}^u - Q_{lim}^v] \varphi_x \right\} dxdt = 0 , \quad (20)
\]

where \( \varphi \in C_0^\infty (\Pi_T) \) and \( Q_{lim}^u(t, x) \) means the function \( Q_{lim} \) from Definition 1 which corresponds to the function \( u(t, x) \).

Because of the properties i) in Definition 2 of the functions from the class \( \mathcal{K}_0 \) the discontinuity lines of the function \( w \) can intersect only at finite
number of points. Hence the strip $\Pi_T$ can be decomposed in finite number of nonoverlapping domains $O_i$, $i = 1, \ldots, m$ where the function $w \in C^2$. Consider the level lines $w = \alpha = \text{const}, \alpha \in [0, \delta]$, $\delta$ is sufficiently small. With the help of Sard theorem (see, for example, [2]) one infers that for almost every $\alpha \in [0, \delta]$ in each domain $O_i$ the level lines of the function $w$ consist of finite number of regular curves. In consequence of the property $ii)$ of Definition 2 for almost every $\alpha$ the level lines and discontinuity lines intersect only at finite number of points.

It follows from (20) and item $iii)$ of Definition 3 that for the discontinuities Hugoniot conditions hold and for every piecewise $C^1$-contour $\Gamma$ in the strip $\Pi_T$ the following integral equality holds (the orientation of the plane has been chosen as $(x, t)$)

$$
\oint_{\Gamma} ([f(u) - f(v)] - [Q_u^u - Q_v^v]) \ dt - (u - v) \ dx = 0. \tag{21}
$$

Consider the connected components $G_i$, $i = 1, \ldots, m_1$ of the domain $w > \alpha$ (for the connected components $G_i$, $i = m_1 + 1, \ldots, m_1 + m_2$ of the domain $w < \alpha$ all considerations are similar). In view of the decrease at infinity of the functions $u$, $v$, $Q_u^u$, $Q_v^v$ (see properties i), iii) of the Definition 3) one can assume that the domain $w > \alpha$ is bounded. Indeed the integral of type (21) with respect to the straight lines $|x| = R$ tends to zero as $R \to 0$.

Consider any connected component $G_1$, without loss of generality one can take $G_1$, such that $\partial \bar{G}_1 \cap \{t = T\} \neq \emptyset$. So the boundary of the domain $G_1$ will consist of four type of curves:

$a)$. The segments $I_k$, $k = 1, \ldots, l$ of the straight line $t = T$.

$b)$. The segments of the straight line $t = 0$.

$c)$. Level lines $w = \alpha$, $w$ is continuous.

$d)$. The lines of discontinuities of the function $w$, where $w$ crosses the value $\alpha$ step-wise. (Here without loss of generality it can be reckoned that only function $u$ has the discontinuities.)

Apply the formula (21) to the contour $\Gamma = \partial \bar{G}_1$. Then observe the following.

$i)$. The integral with respect to lines of type $a)$. gives $\sum_k \int_{I_k} (u - v) \ dx$.

$ii)$. The integral with respect to lines of type $b)$. gives 0 because of the coincidence of initial values for the functions $u$ and $v$. 


iii). Consider the integral with respect to the line $x_{iii}$ of type c). Without loss of generality it can be assumed that the domain $G_1$ is located on the left of this line. Then one has

$$\int \left\{ f(u) - f(v) - [Q^u_{\lim} - Q^v_{\lim}] - (u - v)\dot{x}_{iii} \right\} dt =$$

$$\int \left\{ (f'(\ldots) - \dot{x}_{iii}) \alpha - [Q^u_{\lim} - Q^v_{\lim}] \right\} dt \geq \alpha \int \left\{ f'(\ldots) - \dot{x}_{iii} \right\} dt ,$$

because at the level line $u = v + \alpha$ (the domain is located on the left) $Q^u_{\lim} = Q(u_x), Q^v_{\lim} = Q(v_x)$ and $u_x \leq v_x$ but the function $Q$ is monotone.

iv). Consider the integral with respect to the line $x_{iv}$ of type d). Without loss of generality again it can be assumed that the domain $G_1$ is located on the left of this line. Since we have on the line that the values of $w$ pass from the domain $w > \alpha$ to the domain $w < \alpha$ as one moves in the positive direction of the axis $x$, we have the discontinuity with $u^- > u^+$ and also $u^- \geq v \geq u^+$. Then

$$\int \left\{ f(u^-) - f(v) - [Q^u_{\lim} - Q^v_{\lim}] - (u^- - v)\dot{x}_{iv} \right\} dt =$$

$$\int \left\{ f(u^-) - f(v) + (v - u^-)\dot{x}_{iv} - [Q^u_{\lim} - Q^v_{\lim}] \right\} dt \geq - \int [Q^u - Q^v] dt$$

in consequence of $E$ condition. However at the discontinuity $u_x \to -\infty$, therefore $Q^u_{\lim} \leq Q^v_{\lim}$ whatever is $Q^v_{\lim}$.

Thus one has $\sum \int (u - v)dx \leq O(\alpha)$. Arguing in a similar way for the domains $w < \alpha$, ultimately infer that $\int_{\mathbb{R}} |u - v|dx \leq O(\alpha)$, whence as $\alpha \to 0$ we have $u = v$ a.e. in $\mathbb{R}$ for every $0 < t < T$.

REMARK 4.1 The following fact encourages one to obtain that the condition $E$ is valid for the generalized in the sense of Definition 1 solution (at the expense of although degenerate but nonzero viscosity). It follows from the formula (1) for the traveling wave solutions that the rise of local discontinuities with the values $b_{-0} > b_{+0}$ (it is not necessary that $b_{-0} = b_-$ or $b_{+0} = b_+$) on the left and right of the discontinuity line correspondingly is possible only providing the following condition is true

$$f(b) - f(b_{-0}) - s(b - b_{-0}) \leq 0 \text{ as } b_{+0} \leq b \leq b_{-0} ; \ s = \frac{f(b_{+0}) - f(b_{-0})}{b_{+0} - b_{-0}} .$$

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24
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