The construction of Frobenius manifolds from KP tau-functions

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Abstract

Frobenius manifolds (solutions of WDVV equations) in canonical coordinates are determined by the system of Darboux–Egoroff equations. This system of partial differential equations appears as a specific subset of the \( n \)-component KP hierarchy. KP representation theory and the related Sato infinite Grassmannian are used to construct solutions of this Darboux–Egoroff system and the related Frobenius manifolds. Finally we show that for these solutions Dubrovin’s isomonodromy tau-function can be expressed in the KP tau-function.

1 Introduction

In the beginning of the 90’s in the physics literature on two-dimensional field theory a remarkable and amazingly rich system of partial differential equations emerged. Roughly speaking, this system describes the conditions for a function \( F = F(t) \) of the variable \( t = (t^1, t^2, \ldots, t^n) \) such that the third-order derivatives define structure constants of an associative algebra. These equations are commonly known as the Witten–Dijkgraaf–E. Verlinde–H. Verlinde (WDVV) equations \([21], [5]\). From the geometric point of view the WDVV equations describe the conditions defining a Frobenius manifold. This concept of Frobenius manifold was introduced and extensively studied by Dubrovin, whose lecture notes \([3]\) constitute the primary reference for Frobenius manifolds and many of their applications. The lecture notes of Manin \([16]\) are also a very good general reference. Frobenius manifolds have appeared in a wide range of settings, including quantum cohomology \([15]\), Gromov–Witten invariants, unfolding of singularities, reflection groups and integrable systems. Thus Frobenius manifolds (WDVV equations) are relevant in describing some

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deep geometrical phenomena. So it is expected that these Frobenius manifold equations are rather difficult to solve. Suprisingly some exact explicit solutions of this system of nonlinear equations do exist.

The WDVV equations first appeared in 2D topological field theory. It was derived as a system of equations for so-called primary free energy. According to an idea of Witten the procedure of coupling to gravity should be described in terms of an integrable hierarchy of partial differential equations. In this context Witten–Kontsevich [22],[14] proved that the partition function is a particular tau-function of the KdV hierarchy. For general 2D topological field theories the corresponding integrable hierarchies are not known.

The connection of Frobenius manifolds with integrable systems has been the subject of many investigations. For instance Dubrovin (see e.g. [3],[37], §6) made extensive study of Frobenius manifolds in relation to semi-classical approximations (dispersionless limit, Witham averaging) of integrable hierarchies of partial differential equations. Here also tau-functions emerge, but their representation theoretical meaning remains unclear and underexposed. Recently tau-functions also reappear in studying one-loop approximations [6],[8].

The particular class of semisimple Frobenius manifolds may be effectively studied in the so-called canonical coordinates. In these coordinates Frobenius manifolds are determined by the classical Darboux–Egoroff equations, a system of differential equations, playing a major part in many investigations in classical differential geometry.

It is observed that these Darboux–Egoroff equations are a special case of the $n$-component KP hierarchy. This observation enables us to study Frobenius manifolds in the context of the KP hierarchy. In particular this implies that we have the machinery from the representation theory for the KP hierarchy at our disposal and may take advantage of it to produce solutions. This is the subject of the present paper.

The paper is devoted to the construction of Frobenius manifolds by considering the WDVV equations in the context of the KP hierarchy and to construct solutions in terms of appropriate classes of tau-functions emerging in the representation theory of the KP hierarchy.

We summarize the contents of the paper. In §2 we explain the construction of the semi-infinite wedge representation of the group $GL_{\infty}$ and write down the condition for the $GL_{\infty}$-orbit $O_m$ of the highest weight vector $|m\rangle$. The resulting equation is called the KP hierarchy in the fermionic picture. Moreover we briefly discuss the formulation within Sato’s Grassmannian. §3 is devoted to bosonization of the fermionic picture. We express the fermionic fields in terms of bosonic fields and determine the conditions for elements of orbits $O_m$ in bosonic terms. Using the so-called boson-fermion correspondence we reformulate in §4 the KP hierarchy in the bosonic setting. Introducing formal pseudodifferential operators we obtain Sato’s equation, another reformulation of the KP hierarchy. In §5, the central part of the paper, we construct solutions of the Darboux–Egoroff system by considering this system as a special case of the Sato equation and applying the results described in the previous sections and furthermore by introducing appropriate well-chosen tau-functions. The relevance of the orthogonal group is briefly explained. Using the KP wave function corresponding to all solutions of §5, we construct in §6 specific eigenfunctions that determine the Frobenius manifold. We find an expression for the flat coordinates and express Dubrovin’s isomonodromy tau-function in terms of the KP tau-function. Finally in §7 as an illustration we describe the simplest example in all detail.

For notations and general background we refer to Dubrovin [3] and Kac and van de Leur [11].
2 The semi-infinite wedge representation of the group $GL_\infty$ and Sato’s Grassmannian

Consider the infinite complex matrix group

$$GL_\infty = \{ A = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0 \}$$

and its Lie algebra

$$gl_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | \text{ all but a finite number of } a_{ij} \text{ are } 0 \}$$

with bracket $[a, b] = ab - ba$. The Lie algebra $gl_\infty$ has a basis consisting of matrices $E_{ij}$, $i, j \in \mathbb{Z} + \frac{1}{2}$, where $E_{ij}$ is the matrix with a 1 on the $(i,j)$-th entry and zeros elsewhere. Let $C_\infty = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis $\{v_j\}_{j \in \mathbb{Z} + \frac{1}{2}}$. Both the group $GL_\infty$ and its Lie algebra $gl_\infty$ act linearly on $C_\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk}v_i.$$ 

The well-known semi–infinite wedge representation is constructed as follows [12] (see also [13] and [11]). The semi–infinite wedge space $F = \Lambda_2^\infty C_\infty$ is the vector space with a basis consisting of all semi–infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \ldots$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. We can now define representations $R$ of $GL_\infty$ and $r$ of $gl_\infty$ on $F$ by

$$R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots,$n

$$r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k-1} \wedge av_{i_k} \wedge v_{i_k+1} \wedge \cdots. \quad (2.1)$$

These equations are related by the usual formula:

$$\exp(r(a)) = R(\exp a) \text{ for } a \in gl_\infty.$$ 

In order to perform calculations later on, it is convenient to introduce a larger group

$$\overline{GL}_\infty = \{ A = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ with } i \geq j \text{ are } 0 \}$$

and its Lie algebra

$$\overline{gl}_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | \text{ all but a finite number of } a_{ij} \text{ with } i \geq j \text{ are } 0 \}.$$ 

Both $\overline{GL}_\infty$ and $\overline{gl}_\infty$ act on a completion $\overline{C_\infty}$ of the space $C_\infty$, where

$$\overline{C_\infty} = \{ \sum_j c_jv_j | c_j = 0 \text{ for } j >> 0 \}.$$ 

It is easy to see that the representations $R$ and $r$ extend to representations of $\overline{GL}_\infty$ and $\overline{gl}_\infty$ on the space $F$.

The representation $r$ of $gl_\infty$ and $\overline{gl}_\infty$ can be described in terms of wedging and contracting operators in $F$ (see e.g. [12], [13]). Let $v_j^*$ be the linear functional on $\overline{C_\infty}$ defined
by \( \langle v^*_i, v_j \rangle := v^*_i(v_j) = \delta_{ij} \) and let \( \mathbb{C}^{\infty*} = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}v^*_j \) be the restricted dual of \( \mathbb{C}^\infty \), then for any \( w \in \mathbb{C}^\infty \), we define a wedging operator \( \psi^+[w] \) on \( F \) by
\[
\psi^+[w](v_{i_1} \wedge v_{i_2} \wedge \cdots) = w \wedge v_{i_1} \wedge v_{i_2} \cdots.
\tag{2.2}
\]
Let \( w^* \in \mathbb{C}^{\infty*} \), we define a contracting operator
\[
\psi^-[w^*](v_{i_1} \wedge v_{i_2} \wedge \cdots) = \sum_{s=1}^{\infty} (-1)^{s+1}(w^*, v_{i_s})v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots.
\tag{2.3}
\]
For simplicity we write
\[
\psi^+_j = \psi^+[v_{-j}], \quad \psi^-_j = \psi^-[v^*_j] \quad \text{for } j \in \mathbb{Z} + \frac{1}{2},
\tag{2.4}
\]
These operators satisfy the following relations \((i, j) \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)\):
\[
\psi^\lambda_i \psi^\mu_j + \psi^\mu_j \psi^\lambda_i = \delta_{\lambda, -\mu} \delta_{i, -j},
\]

hence they generate a Clifford algebra, which we denote by \( \mathbb{C}\ell \).

Introduce the following elements of \( F \) \((m \in \mathbb{Z})\):
\[
|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots.
\]
It is clear that \( F \) is an irreducible \( \mathbb{C}\ell \)-module generated by the vacuum \( |0\rangle \) such that
\[
\psi^\pm_j |0\rangle = 0 \quad \text{for } j > 0.
\]
It is straightforward that the representation \( r \) is given by the following formula:
\[
r(E_{ij}) = \psi^+_i \psi^-_j.
\tag{2.5}
\]
Define the charge decomposition
\[
F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}
\tag{2.6}
\]
by letting
\[
\text{charge } |0\rangle = 0 \quad \text{and charge } \psi^\pm_j = \pm 1.
\tag{2.7}
\]
It is clear that the charge decomposition is invariant with respect to \( r(g\ell_\infty) \) (and hence with respect to \( R(GL_\infty) \)). Moreover, it is easy to see that each \( F^{(m)} \) is irreducible with respect to \( g\ell_\infty \) (and \( GL_\infty \)). Note that \( |m\rangle \) is its highest weight vector, i.e.
\[
r(E_{ij})|m\rangle = 0 \quad \text{for } i < j,
\]
\[
r(E_{ii})|m\rangle = 0 \quad (\text{resp. } = |m\rangle) \quad \text{if } i > m \quad (\text{resp. if } i < m).
\]
Let \( w \in F \), we define the Annihilator space \( \text{Ann}(w) \) of \( w \) as follows:
\[
\text{Ann}(w) = \{ v \in \mathbb{C}^\infty | v \wedge w = 0 \}.
\tag{2.8}
Notice that \( \text{Ann}(w) \neq 0 \), since \( v_j \in \text{Ann}(w) \) for \( j << 0 \). This Annihilator space for perfect (semi-infinite) wedges \( w \in F(m) \) is related to the \( GL_\infty \)-orbit
\[
\mathcal{O}_m = R(GL_\infty)|m > \subset F(m)
\]
of the highest weight vector \( |m \rangle \) as follows. Let \( A = (A_{ij})_{i,j \in \mathbb{Z}} \in GL_\infty \), denote by \( A_j = \sum_{i \in \mathbb{Z}} A_{ij} v_i \) then by (2.8)
\[
\tau_m = R(A)|m \rangle = A_{m-\frac{1}{2}} A_{m-\frac{3}{2}} \cdots,
\]
with \( A_{-j} = v_{-j} \) for \( j >> 0 \). Notice that since \( \tau_m \) is a perfect (semi-infinite) wedge
\[
\text{Ann}(\tau_m) = \sum_{j < m} \mathbb{C} A_j \subset \mathbb{C}^\infty.
\]
The following theorem also characterizes the group orbit for a proof see [12] [13]:

**Theorem 2.1** Let \( \tau_m \in F(m) \), then \( \tau_m \in \mathcal{O}_m \) if and only if \( \tau_m \) satisfies the (fermionic) \( KP \) hierarchy:
\[
\sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^+ \tau_m \otimes \psi_{-k} \tau_m = 0.
\]

It is obvious from the construction that if \( w \in \mathbb{C}^\infty \) and \( \tau_m \in \mathcal{O}_m \) that \( w \wedge \tau_m \in \mathcal{O}_{m+1} \). In fact one has the following useful lemma.

**Lemma 2.1** Let \( \tau_m \in \mathcal{O}_m \), \( w \in \mathbb{C}^\infty \) and \( w^* \in \mathbb{C}^{\infty*} \). If \( \psi^+[w] \tau_m \neq 0 \) (resp. \( \psi^-[w^*] \tau_m \neq 0 \)), then \( \psi^+[w] \tau_m \in \mathcal{O}_{m+1} \) (resp. \( \psi^-[w^*] \tau_m \in \mathcal{O}_{m-1} \)).

**Proof.** We only have to prove the statement for \( \psi^-[w^*] \tau_m \). Let \( \psi^-[w^*] \otimes \psi^-[w^*] \) act on (2.10), then we obtain
\[
0 = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^+ \psi^-[w^*] \tau_m \otimes \psi^-[w^*] \tau_m - \sum_{k \in \mathbb{Z} + \frac{1}{2}} \tau_m \otimes \langle w^*, v_{-k} \rangle \psi^-_{-k} \psi^-[w^*] \tau_m
\]
\[
= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^+ \psi^-[w^*] \tau_m \otimes \psi_{-k} \psi^-[w^*] \tau_m - \tau_m \otimes \psi^-[w^*] \psi^-[w^*] \tau_m
\]
Since the last term is clearly zero we obtain the desired result. \( \square \)

Choose a positive integer \( n \) and relabel the basis vectors \( v_i \) as follows. Define for \( j \in \mathbb{Z}, 1 \leq j \leq n, k \in \mathbb{Z} + \frac{1}{2} \):
\[
v_k^{(j)} = v_{nk-\frac{1}{2}(n-2j+1)},
\]
and identify
\[
v_k^{(j)} = t^{-k-\frac{1}{2}} e_j,
\]
where \( e_j \), \( 1 \leq j \leq n \), is a basis of \( \mathbb{C}^n \). We can thus write the vectors \( A_\ell \) in (2.9) as
\[
A_\ell = A_\ell(t) = \sum_{j=1}^n \left( \sum_{i \in \mathbb{Z} + \frac{1}{2}} A_{ni-\frac{1}{2}(n-2j+1)} t^{-i-\frac{1}{2}} \right) e_j,
\]
hence as a vector in $H = (\mathbb{C}[t, t^{-1}])^n$. In this way we can identify $\text{Ann}(\tau_m)$ with a subspace $W_{\tau_m} = \sum_{j < m} \mathbb{C}A_j(t)$ of the space $H$ and hence with a point in an infinite (polynomial) Grassmannian $Gr$. A point of $Gr$ is a linear subspace of $H$ and hence with a point in an infinite (polynomial) Grassmannian $Gr$. A point of $Gr$ is a linear subspace of $H$.

$$H_\ell := \sum_{j=1}^n \sum_{i=\ell}^\infty \mathbb{C}t^i e_j$$

for $\ell \gg 0$. Now $Gr = \cup_{m \in \mathbb{Z}} Gr_m$ (disjoint union) with

$$Gr_m = \{W \in Gr | H_\ell \subset W \text{ and } \dim W/H_\ell = \ell n + m \text{ for } \ell \gg 0\},$$

and we can construct a canonical map

$$\phi : \mathcal{O}_m \to Gr_m, \quad \phi(\tau_m) = W_{\tau_m} := \sum_{i < m} \mathbb{C}A_i(t).$$

It is clear that $\phi(|mn\rangle) = H_{-m}$ and that $\phi$ is surjective with fibers $\mathbb{C}^\times$. This construction is due to Sato [S].

3 The boson-fermion correspondence

The relabeling of the $v_i$’s given by (2.11) induces a relabeling of the $\psi_{\pm j}^\pm$’s, viz.,

$$\psi_{\pm j}^\pm = \psi_{nk \pm \frac{1}{2}(n-2j+1)}^\pm.$$

Notice that with this relabeling we have:

$$\psi_{\pm j}^\pm |0\rangle = 0 \text{ for } k > 0.$$

Besides the charge decomposition, we also introduce an energy decomposition defined by

$$\text{energy } |0\rangle = 0, \text{ energy } \psi_{\pm j}^\pm = -k.$$ (3.1)

Note that energy on $F$ is never negative. Introduce the fermionic fields ($z \in \mathbb{C}^\times$) by

$$\psi_{\pm j}^\pm(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_{\pm j}^\pm_k z^{-k-\frac{1}{2}},$$ (3.2)

and bosonic fields ($1 \leq i, j \leq n$) by

$$\alpha^{(ij)}(z) = \sum_{k \in \mathbb{Z}} \alpha^{(ij)}_k z^{-k-1} =: \psi^{+(i)}(z)\psi^{-(j)}(z) :$$ (3.3)

where $:\ :$ stands for the normal ordered product defined in the usual way ($\lambda, \mu = +$ or $-$):

$$\psi_{\ell}^{\lambda(i)} \psi_{k}^{\mu(j)} := \begin{cases} 
\psi_{\ell}^{\lambda(i)} \psi_{k}^{\mu(j)} & \text{if } \ell \geq k, \\
-\psi_{\ell}^{\mu(j)} \psi_{k}^{\lambda(i)} & \text{if } \ell < k. 
\end{cases}$$ (3.4)
One checks (using e.g. the Wick formula) that the operators $\alpha_{ij}^{(k)}$ satisfy the commutation relations of the affine algebra $gl_n(C)^\wedge$ with central charge 1, i.e.:

$$[\alpha_{ij}^{(p)}, \alpha_{k\ell}^{(q)}] = \delta_{jk}\alpha_{p+q}^{(i\ell)} - \delta_{ij}\alpha_{p+q}^{(kj)} + p\delta_{ij}\delta_{jk}\delta_{p,-q},$$ (3.5)

and that

$$\alpha_{(i)}^{(k)}|m\rangle = 0 \text{ if } k > 0 \text{ or } k = 0 \text{ and } i < j.$$ (3.6)

The operators $\alpha_{(i)}^{(i)} \equiv \alpha_{(i)}^{(i)}$ satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by $\dagger$:

$$[\alpha_{(i)}^{(i)}, \alpha_{(j)}^{(j)}] = h\delta_{ij}\delta_{k,-\ell},$$ (3.7)

and one has

$$\alpha_{(i)}^{(i)}|m\rangle = 0 \text{ for } k > 0.$$ (3.8)

It is easy to see that restricted to $gl_n(C)^\wedge$, $F(0)$ is its basic highest weight representation (see [11]).

In order to express the fermionic fields $\psi_{\pm(i)}(z)$ in terms of the bosonic fields $\alpha^{(i)}(z)$, we need some additional operators $Q_i$, $i = 1, \ldots , n$, on $F$. These operators are uniquely defined by the following conditions:

$$Q_i|0\rangle = \psi_{\pm(i)}|0\rangle, \quad Q_i\psi_{\pm(j)} = (-1)^{\epsilon_{ij}+1}\psi_{\pm(j)}Q_i.$$ (3.9)

They satisfy the following commutation relations:

$$Q_iQ_j = -Q_jQ_i \text{ if } i \neq j, \quad [\alpha_{(i)}^{(i)}, Q_j] = \delta_{ij}\delta_{k0}Q_j.$$ (3.10)

**Theorem 3.1** ([3], [4])

$$\psi_{\pm(i)}(z) = Q_i^{\pm1}z^{\pm\alpha_0^{(i)}} \exp(\mp\sum_{k<0} \frac{1}{k}\alpha_{(k)}^{(i)}z^{-k}) \exp(\mp\sum_{k>0} \frac{1}{k}\alpha_{(k)}^{(i)}z^{-k}).$$ (3.11)

**Proof.** See [17].

The operators on the right-hand side of (3.11) are called vertex operators. They made their first appearance in string theory (cf. [7]).

We can describe now the $n$-component boson-fermion correspondence. Let $\mathbb{C}[x]$ be the space of polynomials in indeterminates $x = \{x_{ki}^{(i)}\}, \quad k = 1, 2, \ldots , i = 1, 2, \ldots , n$. Let $L$ be a lattice with a basis $\delta_{1}, \ldots , \delta_{n}$ over $\mathbb{Z}$ and the symmetric bilinear form $(\delta_{i}|\delta_{j}) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. Let

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \leq j. \end{cases}$$ (3.12)

Define a bimultiplicative function $\varepsilon : L \times L \to \{\pm1\}$ by letting

$$\varepsilon(\delta_{i}, \delta_{j}) = \varepsilon_{ij}.$$ (3.13)
Let \( \delta = \delta_1 + \ldots + \delta_n, \) \( M = \{ \gamma \in L \mid (\delta|\gamma) = 0 \}, \) \( \Delta = \{ \alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, n, i \neq j \}. \) Of course \( M \) is the root lattice of \( \mathfrak{sl}_n(\mathbb{C}) \), the set \( \Delta \) being the root system.

Consider the vector space \( \mathbb{C}[L] \) with basis \( e^\gamma, \gamma \in L, \) and the following twisted group algebra product:

\[
e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha + \beta}.
\]

Let \( B = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[L] \) be the tensor product of algebras. Then the \( n \)-component boson-fermion correspondence is the vector space isomorphism

\[
\sigma : F \rightarrow B,
\]

given by

\[
\sigma(\alpha^{(i_1)}_{-m_1} \ldots \alpha^{(i_s)}_{-m_s} Q_1^{k_1} \ldots Q_n^{k_n} |0\rangle) = m_1 \ldots m_s x^{(i_1)}_{m_1} \ldots x^{(i_s)}_{m_s} \otimes e^{k_1 \delta_1 + \ldots + k_n \delta_n}.
\]

The transported charge and energy then will be as follows:

\[
\text{charge } p(x) \otimes e^\gamma = (\delta|\gamma),
\]

\[
\text{energy } x^{(i_1)}_{m_1} \ldots x^{(i_s)}_{m_s} \otimes e^\gamma = m_1 + \ldots + m_s + \frac{1}{2}(\gamma|\gamma).
\]

We denote the transported charge decomposition by

\[
B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}.
\]

The transported action of the operators \( \alpha^{(i)}_{m} \) and \( Q_j \) looks as follows:

\[
\begin{aligned}
\sigma \alpha^{(j)}_{m} \sigma^{-1}(p(x) \otimes e^\gamma) &= mx^{(j)}_{m} p(x) \otimes e^\gamma, \text{ if } m > 0, \\
\sigma \alpha^{(j)}_{m} \sigma^{-1}(p(x) \otimes e^\gamma) &= \frac{\partial p(x)}{\partial x_m} \otimes e^\gamma, \text{ if } m > 0, \\
\sigma \alpha^{(j)}_{0} \sigma^{-1}(p(x) \otimes e^\gamma) &= (\delta|\gamma)p(x) \otimes e^\gamma, \\
\sigma Q_j \sigma^{-1}(p(x) \otimes e^\gamma) &= \varepsilon(\delta_j, \gamma)p(x) \otimes e^{\gamma + \delta_j}.
\end{aligned}
\]

The transported action of the fermionic fields is as follows:

\[
\sigma \psi^{\pm(j)}(z) \sigma^{-1} = e^{\pm \delta_j z^{\pm \delta_j}} \exp(\pm \sum_{k=1}^{\infty} x^{(j)}_k) \exp(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial x^{(j)}_k} z^{-k})
\]

We will now determine the second part of the boson-fermion correspondence, i.e., we want to determine \( \sigma(\tau_m) \), where \( \tau_m \) is given by (2.9). Since all spaces \( F^{(m)} \) give a similar representation of \( gl_\infty \), we will restrict our attention to the case that \( m = 0 \) and we write \( \tau \) instead of \( \tau_0 \). We will generalize the proof of Theorem 6.1 of [13]. For this purpose we have to introduce elements \( \Lambda^{(j)}_\ell \in gl_\infty, 1 \leq j \leq n, \ell \in \mathbb{N} \), by

\[
\Lambda^{(j)}_\ell = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{nk-\frac{1}{2}(n-2j+1), nk+\ell-\frac{1}{2}(n-2j+1)}.
\]

Notice that \( \Lambda^{(j)}_\ell = (\Lambda^{(j)}_1)_\ell \), \( r(\Lambda^{(j)}_\ell) = \alpha^{(j)}_\ell \) and that \( \exp \Lambda^{(j)}_\ell \in Gl_\infty \). With the relabeling \( |0\rangle \) becomes

\[
|0\rangle = v^{(n)}_{-\frac{1}{2}} \wedge v^{(n-1)}_{-\frac{1}{2}} \wedge \ldots \wedge v^{(1)}_{-\frac{1}{2}} \wedge v^{(n)}_{-\frac{1}{2}} \wedge v^{(n-1)}_{-\frac{1}{2}} \wedge \ldots ,
\]
and
\[ Q_i Q_j^{-1} |0⟩ = (-)^{n-j} v_i^{(j)} v_{i-n}^{(n)} \wedge v_{i-1}^{(n-1)} \wedge \cdots \wedge v_{i-j+1}^{(j+1)} \wedge v_{i-j}^{(j-1)} \wedge v_{i-j+1}^{(1)} \wedge v_{i-j}^{(n)} \wedge v_{i-j+1}^{(n-1)} \wedge \cdots . \]

We now want to determine \( σ(τ) \), where
\[ τ = R(A) |0⟩ = A_{-\frac{1}{2}} \wedge A_{-\frac{3}{2}} \wedge A_{-\frac{5}{2}} \wedge \cdots , \text{ with } A_{-p} = v_{-p} \text{ for all } p > P >> 0. \quad (3.21) \]

Let \( σ(τ) = \sum_{α ∈ M} τ_α(x)e^α \), we want to compute
\[ σ \left( R \left( \exp \left( \sum_{j=1}^{∞} \sum_{k=1}^{∞} y_k^{(j)} A_k^{(j)} \right) \right) \right) = \exp \left( \sum_{j=1}^{∞} \sum_{k=1}^{∞} y_k^{(j)} \frac{∂}{∂x_k} \right) \sum_{α ∈ M} τ_α(x)e^α . \]

Now let \( F_α(y) \) denote the coefficient of \( 1 \otimes e^α \) in this expression, then
\[ F_α(y) = \exp \left( \sum_{j=1}^{∞} \sum_{k=1}^{∞} y_k^{(j)} \frac{∂}{∂x_k} \right) τ_α(x)|_{x=0} = τ_α(x + y)|_{x=0} = τ_α(y). \]

So \( τ_α(y) \) is the coefficient of \( 1 \otimes e^α \) in
\[ σ \left( R \left( \exp \left( \sum_{j=1}^{∞} \sum_{k=1}^{∞} y_k^{(j)} A_k^{(j)} \right) A \right) |0⟩ \right) . \]

Now let \( α = \sum_{j=1}^{n} k_j δ_j \) then
\[ 1 \otimes e^α = σ(Q_1^{k_1} Q_2^{k_2} \cdots Q_n^{k_n} |0⟩) , \]

hence \( τ_α(y) \) is the coefficient of
\[ R \left( \exp \left( \sum_{j=1}^{∞} \sum_{k=1}^{∞} y_k^{(j)} A_k^{(j)} \right) A \right) |0⟩ = R \left( \left( \sum_{j=1}^{∞} \sum_{k=0}^{∞} S_k(y^{(j)} A_k^{(j)}) \right) A \right) |0⟩ \

\[ = R \left( \sum_{ℓ<0} \sum_{j=1}^{n} \sum_{q=±\frac{1}{2}}^{∞} \sum_{k=0}^{∞} A_{n(q+k)-\frac{5}{2}(n-2j+1),n(q)-\frac{5}{2}(n-2j+1)} S_k(y^{(j)}) E_{nq-\frac{5}{2}(n-2j+1),ℓ} \right) |0⟩ , \]

where \( S_k(y) \) are the elementary Schur functions defined by \( \sum_{k∈Z} S_k(y) z^k = \exp(∑_{k=1}^{∞} y_k z^k) \).

Using formula (4.48) of [13], i.e.,
\[ R(A) |0⟩ = \sum_{j_{>\frac{1}{2}} > j_{>\frac{3}{2}} > j_{>\frac{5}{2}} > \cdots} \det \left( A_{j_{>\frac{1}{2}},j_{>\frac{3}{2}},j_{>\frac{5}{2}},\cdots} \right) v_{j_{>\frac{1}{2}}} \wedge v_{j_{>\frac{3}{2}}} \wedge v_{j_{>\frac{5}{2}}} \wedge \cdots , \]

where \( A_{j_{>\frac{1}{2}},j_{>\frac{3}{2}},j_{>\frac{5}{2}},\cdots} \) denotes the matrix located at the intersection of the rows \( j_{>\frac{1}{2}}, j_{>\frac{3}{2}}, j_{>\frac{5}{2}}, \cdots \)
and the columns \( -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots \) of the matrix \( A \), we can calculate \( τ_α(y) \) if we can determine \( Q_1^{k_1} Q_2^{k_2} \cdots Q_n^{k_n} |0⟩ \) as a perfect simple wedge. This is in general quite complicated, so we assume for the moment that
\[ Q_1^{k_1} Q_2^{k_2} \cdots Q_n^{k_n} |0⟩ = λ_α v_{j_{>\frac{1}{2}}} \wedge v_{j_{>\frac{3}{2}}} \wedge v_{j_{>\frac{5}{2}}} \wedge \cdots , \]
with \( j_q = -q \) for all \( q > Q > 0 \) and \( \lambda_\alpha = \pm 1 \), then

\[
\tau_\alpha(y) = \lambda_\alpha \det \left( \sum_{\ell < 0} \sum_{r=j_{-\frac{1}{2}}}^{\ell} \sum_{\ell+\frac{1}{2} = j_{-\frac{3}{2}}}^{\ell} \cdots \sum_{n\in \mathbb{Z}+\frac{1}{2}} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} A_{r+nk,\ell} S_k(y^{(j)}) \right) E_{r,\ell} \right).
\]

Finally notice that this is in fact only a finite determinant of size \( R = \max(P, Q) \), hence we have determined

**Proposition 3.1** Let \( A = (A_{i,j})_{i,j\in \mathbb{Z}+\frac{1}{2}} \in GL_\infty \) be such that \( A_{ij} = \delta_{ij} \) for \( j < -P \) then \( \sigma(R(A)|0) = \sum_{\alpha \in M} \tau_\alpha(x)e^\alpha \). Assume that \( \alpha = \sum_{j=1}^{n} k_j \delta_j \) and suppose that

\[
Q_1^{k_1} Q_2^{k_2} \cdots Q_n^{k_n}|0\rangle = \lambda_\alpha v_{j_{-\frac{1}{2}}} \wedge v_{j_{-\frac{3}{2}}} \wedge v_{j_{-\frac{5}{2}}} \wedge \cdots ,
\]

with \( j_{-\frac{1}{2}} > j_{-\frac{3}{2}} > j_{-\frac{5}{2}} \cdots \) and \( j_q = -q \) for all \( q > Q > 0 \) and \( \lambda_\alpha = \pm 1 \), then

\[
\tau_\alpha(x) = \lambda_\alpha \det \left( \sum_{R<\ell < 0} \sum_{r=j_{-\frac{1}{2}}}^{\ell} \sum_{\ell+\frac{1}{2} = j_{-\frac{3}{2}}}^{\ell} \cdots \sum_{n\in \mathbb{Z}+\frac{1}{2}} 1 < j_{-\frac{5}{2}} \cdots 1 \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\infty} A_{r+nk,\ell} S_k(x^{(j)}) \right) E_{r,\ell} \right),
\]

where \( R = \max(P, Q) \). In particular if \( 1 \leq i < j \leq n \) and \( \alpha = 0, \delta_i = -\delta_j, \delta_j = \delta_i \), respectively, then \( \lambda_0 = 1, \lambda_{0} - \delta_{j} = (-1)^{n-j}, \lambda_{j} - \delta_{i} = (-1)^{n-i+1} \) and \( (j_{-\frac{1}{2}}, j_{-\frac{3}{2}}, \cdots) = (-1, -\frac{3}{2}, \cdots), (j_{-\frac{1}{2}}, j_{-\frac{3}{2}}, \cdots) = (j, j_{-\frac{1}{2}}, j_{-\frac{3}{2}}, \cdots) \), respectively.

### 4 The KP hierarchy as a dynamical system

Using the isomorphism \( \sigma \) we can reformulate the KP hierarchy (2.10) in the bosonic picture. We start by observing that (2.10) can be rewritten as follows:

\[
\text{Res}_{z=0} \sum_{j=1}^{n} \psi_{+}^{(j)}(z) \tau \otimes \psi_{-}^{(j)}(z) \tau = 0, \quad \tau \in F^{(0)}.
\]

Here and further \( \text{Res}_{z=0} \sum_{j=0}^{n} f_j z^j \) (where \( f_j \) are independent of \( z \)) stands for \( f_{-1} \). Notice that for \( \tau \in F^{(0)}, \sigma(\tau) = \sum_{\gamma \in M} \tau_\gamma(x)e^\gamma \). Here and further we write \( \tau_\gamma(x)e^\gamma \) for \( \tau_\gamma \otimes e^\gamma \). Using Theorem 3.1, equation (1.1) turns under \( \sigma \otimes \sigma : F \otimes F \rightarrow \mathbb{C}[x', x''] \otimes (\mathbb{C}[L'] \otimes \mathbb{C}[L'']) \) into the following equations, which we call the \( n \)-component KP hierarchy. Let \( 1 \leq a, b \leq n, \alpha, \beta \in M \):

\[
\begin{align*}
\text{Res}_{z=0} \sum_{j=1}^{n} \varepsilon(\delta_j, \alpha + \delta_a - \beta + \delta_b) z^{(\delta_j)\alpha + \delta_a - \beta + \delta_b - 2\delta_f} \\
\times \exp \left( \sum_{k=1}^{\infty} \left( x_k^{(j)} - x_k^{(j')} \right) z^k \right) \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(j)}} - \frac{\partial}{\partial x_k^{(j')}} \right) z^{-k} \right) \\
\tau_{\alpha + \alpha_a} (x') \tau_{\beta - \alpha_a} (x'') = 0 \quad (\alpha, \beta \in M).
\end{align*}
\]
Define the support of $\tau$ by $\text{supp} \tau = \{ \alpha \in M | \tau_{\alpha} \neq 0 \}$, then for each $\alpha \in \text{supp} \tau$ we define the (matrix valued) wave functions

$$V^\pm(\alpha, x, z) = (V_{ij}^\pm(\alpha, x, z))_{i,j=1}^n \quad (4.3)$$

as follows:

$$V_{ij}^\pm(\alpha, x, z) := \varepsilon(\delta_j, \alpha + \delta_i) z^{(\delta_j|\pm\alpha + \alpha_{ij})} \times \exp(\pm \sum_{k=1}^\infty x_k^{(j)} z^{-k}) \exp(\mp \sum_{k=1}^\infty \frac{\partial}{\partial x_k^{(j)}} z^{-k}) \tau_{\alpha \pm \alpha_{ij}}(x) / \tau_{\alpha}(x) \quad (4.4)$$

It is easy to see that equation (4.2) is equivalent to the following bilinear identity:

$$\text{Res}_{z=0} V^+(\alpha, x, z) \ t V^-(\beta, x', z) = 0 \quad \text{for all } \alpha, \beta \in M, \quad (4.5)$$

where $t V$ stands for the transposed of the matrix $V$. Define $n \times n$ matrices $W^\pm(m)(\alpha, x)$ by the following generating series (cf. (4.4)):

$$\sum_{m=0}^\infty W^\pm(m)(\alpha, x)(z) = \varepsilon ij z^{\delta_{ij} - 1} \exp(\mp \sum_{k=1}^\infty \frac{\partial}{\partial x_k^{(j)}} z^{-k}) \tau_{\alpha \pm \alpha_{ij}}(x) / \tau_{\alpha}(x). \quad (4.6)$$

Note that

$$W^{(0)}(\alpha, x) = I_n, \quad (4.7)$$

$$W^{(1)}_{ij}(\alpha, x) = \begin{cases} \varepsilon ij \tau_{\alpha \pm \alpha_{ij}} / \tau_{\alpha} & \text{if } i \neq j \\ -\tau_{\alpha}^{-1} \frac{\partial \tau_{\alpha}}{\partial x_1^{(j)}} & \text{if } i = j, \end{cases} \quad (4.8)$$

We see from (4.4) that $V^\pm(\alpha, x, z)$ can be written in the following form:

$$V^\pm(\alpha, x, z) = \sum_{m=0}^\infty W^\pm(m)(\alpha, x)(z)^{-m} R^\pm(\alpha, \pm z) S^\pm(x, z), \quad (4.9)$$

where

$$R^\pm(\alpha, z) = \sum_{i=1}^n \varepsilon(\delta_i, \alpha) E_{ii}(\pm z)^{\pm(\delta_i|\alpha)}, \quad (4.10)$$

$$S^\pm(x, z) = \sum_{i=1}^n e^{\pm \sum_{j=1}^\infty x_j^{(i)} z^j} E_{ii}.$$

Here $E_{ij}$ stands for the $n \times n$ matrix whose $(i, j)$ entry is 1 and all other entries are zero. Now let $\partial = \sum_{j=1}^n \frac{\partial}{\partial x_j^{(j)}}$, then $V^\pm(\alpha, x, z)$ can be written in terms of formal pseudo-differential operators (see [14] for more details).

$$P^\pm(\alpha) \equiv P^\pm(\alpha, x, \partial) = I_n + \sum_{m=1}^\infty W^\pm(m)(\alpha, x) \partial^{-m}, \quad R^\pm(\alpha) = R^\pm(\alpha, \partial) \quad (4.11)$$
as follows:

\[ V^\pm(\alpha, x, z) = P^\pm(\alpha)R^\pm(\alpha)S^\pm(x, z) \]  

(4.12)

Since obviously

\[ R^-(\alpha, \partial)^{-1} = R^+(\alpha, \partial)^*, \]  

(4.13)

where \( P^* = \sum_k (-\partial)^k t P^{(k)} \) stands for the formal adjoint of \( P = \sum_k P^{(k)} \partial^k \). Moreover one can deduce (see \[11\]) from the bilinear identity (4.5):

\[ (P^+(\alpha, x, \partial)R^+(\alpha - \beta, \partial)P^-(-\beta, x' \partial)^*)_+ = 0 \]  

(4.14)

for any \( \alpha, \beta \in \text{supp} \tau \). Here \( Q^- = Q - Q_+ \), where \( Q_+ \) stands for the differential operator part of \( Q \).

Furthermore, put \( x = x' \), then one deduces from (4.14) with \( \alpha = \beta \) that

\[ P^-(-\alpha) = (P^+(\alpha)^*)^{-1}, \]  

(4.15)

since \( R^\pm(0) = I_n \) and \( P^\pm(\alpha) \in I_n + \text{lower order terms} \). With all these ingredients one can prove the following Lemma:

**Proposition 4.1** Let \( \alpha, \beta \in \text{supp} \tau \), then \( P^+(\alpha) \) satisfies the Sato equations:

\[ \frac{\partial P^+(\alpha)}{\partial x^{(j)}_k} = -(P^+(\alpha)E_{jj} \partial^k P^+(\alpha)^{-1})_+ P^+(\alpha) \]  

(4.16)

and \( P^+(\alpha), P^+(\beta) \) satisfy

\[ (P^+(\alpha)R^+(\alpha - \beta)P^-(-\beta)^*)_+ = 0 \text{ for all } \alpha, \beta \in \text{supp} \tau. \]  

(4.17)

This is another formulation of the \( n \)-component KP hierarchy (see \[11\]). Introduce the following formal pseudo-differential operators \( L(\alpha), C^{(j)}(\alpha) \):

\[ L(\alpha) \equiv L(\alpha, x, \partial) = P^+(\alpha)\partial P^+(\alpha)^{-1}, \]
\[ C^{(j)}(\alpha) \equiv C^{(j)}(\alpha, x, \partial) = P^+(\alpha)E_{jj} P^+(\alpha)^{-1}, \]  

(4.18)

then related to the Sato equation is the following linear system

\[
L(\alpha)V^+(\alpha, x, z) = zV^+(\alpha, x, z), \\
C^{(i)}(\alpha)V^+(\alpha, x, z) = V^+(\alpha, x, z)E_{ii}, \\
\frac{\partial V^+(\alpha, x, z)}{\partial x^{(i)}_k} = (L(\alpha)^k C^{(j)}(\alpha) + V^+(\alpha, x, z)).
\]  

(4.19)

To end this section we write down explicitly some of the Sato equations (4.16) on the matrix elements \( W^{(s)}_{ij} \) of the coefficients \( W^{(s)}(x) \) of the pseudo-differential operator

\[ P = P^+(\alpha) = I_n + \sum_{m=1}^\infty W^{(m)}(x)\partial^{-m}. \]
We shall write $W = W^{(1)}$ and $W_{ij}$ for $W_{ij}^{(1)}$ to simplify notation, then the simplest Sato equation is

$$\frac{\partial P}{\partial x_1^{(k)}} = [\partial E_{kk}, P] + [W, E_{kk}]P. \quad (4.20)$$

In particular we have for $i \neq k$:

$$\frac{\partial W_{ij}}{\partial x_1^{(k)}} = W_{ik}W_{kj} - \delta_{jk}W_{ij}^{(2)}. \quad (4.21)$$

The equation (4.20) is equivalent to the following equation for $V = V^+(\alpha)$:

$$\frac{\partial V}{\partial x_1^{(k)}} = (E_{kk}\partial + [W, E_{kk}])V. \quad (4.22)$$

5 Solutions of the Darboux-Egoroff system

Define

$$\gamma_{ij}(x) = W_{ij}^{(1)}(0, x)|_{x_1^{(i)} = x_1^{(j)} = x_k^{(o)} \text{ for } k > 1}, \quad (5.1)$$

where the $x_1^{(i)}$ for $k > 0$ are chosen to be certain specific but at the moment still unknown constants. From (4.21) we already know that

$$\frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = \gamma_{ik}(x)\gamma_{kj}(x) \quad i \neq k \neq j. \quad (5.2)$$

This is the $n$-wave equation if $i, j, k$ are distinct. The aim of this section is to construct specific $\gamma_{ij}$’s which satisfy

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = 0, \quad (5.3)$$

and

$$\gamma_{ij}(x) = \gamma_{ji}(x). \quad (5.4)$$

In other words we want to find the rotation coefficients $\gamma_{ij}$ for the Darboux–Egoroff system (5.2)-(5.4). Sometimes we will assume an additional equation, viz.

$$\sum_{k=1}^{n} x_1^{(k)} \frac{\partial \gamma_{ij}(x)}{\partial x_1^{(k)}} = -\gamma_{ij}(x), \quad (5.5)$$

which means that $\gamma_{ij}$ has degree $-1$. This equation holds for the so-called semisimple conformal invariant Frobenius manifolds, see [2].

The restriction

$$\sum_{k=1}^{n} \frac{\partial W_{ij}^{(1)}(0, x)}{\partial x_1^{(k)}} = 0, \quad (5.6)$$
is a very natural restriction. If we assume that
\[ \sum_{k=1}^{n} \frac{\partial \tau(x)}{\partial x_1^{(k)}} = 0, \] (5.7)
then this clearly implies (5.6). Notice that one may even assume that \( \sum_{k=1}^{n} \frac{\partial \tau(x)}{\partial x_1^{(k)}} = \lambda \tau(x) \), but since we are in the polynomial case \( \lambda \) must be 0. Equation (5.7) means that \( \tau \) (in the fermionic picture) belongs to the \( GL_n(\mathbb{C}[t,t^{-1}]) \)-loop group orbit or even the \( SL_n(\mathbb{C}[t,t^{-1}]) \)-loop group orbit of \( |0\rangle \) (see [11] for more details). The homogeneous space for this group is in fact the restricted Grassmannian
\[ \overline{Gr} = \{ W \in Gr_0 | \sum_{k=1}^{n} tE_{kk} W \subset W \}. \]

In fact \( \tau \) satisfies (5.7) if and only if
\[ \sum_{k=1}^{n} tE_{kk} W_\tau \subset W_\tau. \] (5.8)

Since equation (5.7) holds for \( \tau \) we do not only find equation (5.6) for \( W^{(1)}(0,x) \), but we find that this equation holds for all \( W^{(s)}(\alpha,x) \)'s and hence
\[ \sum_{k=1}^{n} \frac{\partial P^+(\alpha,x)}{\partial x_1^{(k)}} = 0. \] (5.9)

This means that we do not really have formal pseudo-differential operators, but rather formal matrix-valued Laurent series in \( z^{-1} \). The Sato equation takes the following simple form. Let \( P(z) = P^+(\alpha,x,z) \) then
\[ \frac{\partial P(z)}{\partial x_1^{(j)}} = -(P(z)E_{jj} P(z)^{-1} z^k) - P(z) \]
and the simplest Sato'equation becomes
\[ \frac{\partial P(z)}{\partial x_1^{(k)}} = z[E_{kk}, P(z)] + [W, E_{kk}] P(z). \]

Equation (4.22) turns into
\[ \frac{\partial V(z)}{\partial x_1^{(k)}} = (zE_{kk} + [W, E_{kk}]) V(z), \] (5.10)
where \( V(z) = V^+(\alpha,x,z) \). Define \( X = \sum_{j=1}^{n} x_1^{(j)} E_{jj} \), then
\[ \sum_{j=1}^{n} x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} V(z) = (zX + [W, X]) V(z). \] (5.11)

From now on we will only consider tau-functions that are homogeneous with respect to the energy. Notice that if energy \( \tau = N \), then energy \( \tau_\alpha = N - \frac{1}{2}(\alpha|\alpha) \), in particular
energy $\tau_{\delta_i - \delta_j} = \text{energy } \tau_0 - 1$. Since the energy $x_k^{(j)} = k$, it is straightforward to check that for $\alpha = 0$,

$$L_0 V(z) = z \frac{\partial V(z)}{\partial z}, \quad \text{where}$$

$$L_0 = \sum_{j=1}^{n} k x_k^{(j)} \frac{\partial}{\partial x_k^{(j)}}. \quad (5.12)$$

We will now describe a class of homogeneous tau-functions, in the fermionic picture that satisfy (5.7). First choose two positive integers $m_1$ and $m_2$ such that $m_1 + m_2 \leq n$. Next choose $m_1$ positive integers $k_i$, $1 \leq i \leq m_1$ and $m_2$ positive integers $\ell_j$, $1 \leq j \leq m_2$, such that $\sum_{i=1}^{m_1} k_i - \sum_{j=1}^{m_2} \ell_j = 0$. Next choose $m_1$ linearly independent vectors $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $m_2$ linearly independent vectors $b_j = (b_{j1}, b_{j2}, \ldots, b_{jn})$ in $\mathbb{C}^n$ such that

$$(a_i, b_j) = \sum_{k=1}^{n} a_{ik} b_{jk} = 0 \text{ for all } 1 \leq i \leq m_1 \text{ and } 1 \leq j \leq m_2. \quad (5.13)$$

Using lemma 2.1 we construct a $\tau \in \mathcal{O}$, as follows

$$\tau = (\sum_{p} a_{1p} \psi^{+(p)}_{-k_1 + \frac{1}{2}}) (\sum_{p} a_{1p} \psi^{+(p)}_{-k_1 + \frac{1}{2}}) \cdots (\sum_{p} a_{1p} \psi^{+(p)}_{-k_1 + \frac{1}{2}}) (\sum_{p} a_{2p} \psi^{+(p)}_{-k_2 + \frac{1}{2}}) (\sum_{p} a_{2p} \psi^{+(p)}_{-k_2 + \frac{1}{2}}) \cdots$$

$$\sum_{p} a_{2p} \psi^{+(p)}_{-k_2 + \frac{1}{2}} s_{a_{1p} \psi^{+(p)}_{-k_1 + \frac{1}{2}}} \sum_{p} a_{3p} \psi^{+(p)}_{-k_3 + \frac{1}{2}} \cdots \sum_{p} a_{m_1,p} \psi^{+(p)}_{-k_m + \frac{1}{2}} (\sum_{p} b_{1p} \psi^{-(p)}_{-\ell_1 + \frac{1}{2}}) (\sum_{p} b_{1p} \psi^{-(p)}_{-\ell_1 + \frac{1}{2}}) \cdots$$

$$\sum_{p} b_{1p} \psi^{-(p)}_{-\ell_1 + \frac{1}{2}} s_{b_{1p} \psi^{-(p)}_{-\ell_1 + \frac{1}{2}}} \sum_{p} b_{2p} \psi^{-(p)}_{-\ell_2 + \frac{1}{2}} \cdots \sum_{p} b_{m_2,p} \psi^{-(p)}_{-\ell_2 + \frac{1}{2}} (\sum_{p} b_{m_2,p} \psi^{-(p)}_{-\ell_2 + \frac{1}{2}}) (\sum_{p} b_{m_2,p} \psi^{-(p)}_{-\ell_2 + \frac{1}{2}}). \quad (5.14)$$

The point of the Grassmannian $W_\tau$ corresponding to this $\tau$ satisfies (5.8).

The symmetry conditions (5.4) of the $\tau_{ij}$’s are not so natural. Using (4.8), it is equivalent to

$$\tau_{\delta_i - \delta_j} (x_1^{(\ell)}, c_1^{(\ell)}, c_2^{(\ell)}, \ldots) = -\tau_{\delta_j - \delta_i} (x_1^{(\ell)}, c_1^{(\ell)}, c_2^{(\ell)}, \ldots). \quad (5.15)$$

To achieve this result, we define an automorphism $\omega$ on $F$ as follows:

$$\omega(|0\rangle) = |0\rangle$$

$$\omega(|\psi_k^{\pm(i)}\rangle) = c_i^{\pm 1} |\psi_k^{\mp(i)}\rangle, \quad \text{with } 1 \leq i \leq n \text{ and } c_i \in \mathbb{C}^\times. \quad (5.16)$$

We will fix the $c_i$ later all to be equal to 1, but for the moment we keep them arbitrary. This gives

$$\omega(\alpha_k^{(i)}) = -\alpha_k^{(i)} \text{ and } \omega(Q_i^{\pm 1}) = c_i^{\pm 1} Q_i^{\mp 1}. \quad (5.17)$$

Using the boson-fermion correspondence this induces an automorphism on $B$, which we will also denote by $\omega$,

$$\omega(x_k^{(i)}) = -x_k^{(i)} \quad \omega(\frac{\partial}{\partial x_k^{(i)}}) = -\frac{\partial}{\partial x_k^{(i)}}, \quad \omega(\delta_i) = -\delta_i \text{ and } \omega(e^{\pm \delta_i}) = c_i^{\pm 1} e^{\mp \delta_i}. \quad (5.18)$$
Define for \( \alpha = \sum_{j=1}^{n} p_{j} \delta_{i} \in M \), \( c_{\alpha} = \prod_{j=1}^{n} c_{j}^{p_{j}} \), then
\[
\omega \left( \sum_{\alpha \in M} \tau_{\alpha}(x)e^{\alpha} \right) = \sum_{\alpha \in M} c_{\alpha} \tau_{\alpha}(-x)e^{-\alpha}.
\] (5.19)

We now want to find homogeneous tau-functions that satisfy \( \omega(\tau(x)) = \lambda \tau(x) \) for some \( \lambda \in \mathbb{C}^{\times} \). Since \( \omega^{2}(\tau_{0}(x)) = \tau_{0}(x) \), \( \lambda = 1 \) or \(-1\). From (5.19) we deduce that
\[
\tau_{\alpha}(x) = \lambda c_{\alpha} \tau_{-\alpha}(-x)
\] (5.20)
and we want this for \( \alpha \in \Delta \), of course after a specific choice of constants \( x_{k}^{(i)} \)'s for \( k \geq 2 \), to be equal to \(-\tau_{\alpha}(x) \). Since we have assumed that \( \tau \) is homogeneous (in the energy), say that it has energy \( N \), then we can get rid of the \(-x \) in the right-hand side of (5.20) if we put all \( x_{2k}^{(i)} \)'s equal to zero. So define
\[
\overline{\tau}(x) = \tau(x) \big|_{x_{2k}=0},
\] (5.21)
then clearly (5.20) turns into
\[
\overline{\tau}_{\alpha}(x) = \lambda c_{\alpha} \tau_{-\alpha}(-x).
\]

Because this also has to hold for \( \alpha = 0 \), we obtain that \( \lambda = (-1)^{N} \) and hence \( c_{\alpha} = 1 \) for all \( \alpha \in \Delta \). Thus \( c_{i} = 1 \) for all \( 1 \leq i \leq n \) or \( c_{i} = -1 \) for all \( i \), we may choose either of these two cases, for simplicity we choose
\[
c_{i} = 1 \text{ for all } 1 \leq i \leq n.
\]

With all these choices, we have finally that
\[
\omega(\overline{\tau}_{\alpha}(x)) = (-1)^{N-\frac{1}{2}(\alpha|\alpha)} \overline{\tau}_{-\alpha}(x).
\] (5.22)

Return to the tau-functions of the form (5.14). If such a \( \tau \) satisfies (5.22) and it contains a factor \( \sum_{i} a_{\ell_{i}}\psi_{k}^{+}(i) \) for a certain \( \ell \), then it must also contain a factor \( \sum_{j} b_{m_{j}}\psi_{k}^{-}(j) \). Since
\[
\text{energy } \left( \sum_{i} a_{\ell_{i}}\psi_{k}^{+}(i) \right) \left( \sum_{j} b_{m_{j}}\psi_{k}^{-}(j) \right) = -2k \in 2\mathbb{Z} + 1,
\]
we must assume that there exists an \( m \) such that
\[
\omega \left( \left( \sum_{i} a_{\ell_{i}}\psi_{k}^{+}(i) \right) \left( \sum_{j} b_{m_{j}}\psi_{k}^{-}(j) \right) \right) = - \left( \sum_{j} b_{m_{j}}\psi_{k}^{+}(j) \right) \left( \sum_{i} a_{\ell_{i}}\psi_{k}^{-}(i) \right)
\]
\[
= - \left( \sum_{i} a_{\ell_{i}}\psi_{k}^{+}(i) \right) \left( \sum_{j} b_{m_{j}}\psi_{k}^{-}(j) \right).
\]

So
\[
a_{\ell_{i}}b_{m_{j}} = a_{\ell_{j}}b_{m_{i}} \text{ for all } 1 \leq i, j \leq n
\]
and \( b_{m} \) must be a multiple of \( a_{\ell} \). Since the length of such a vector does not matter much (only a scalar multiple of the whole tau-function), we may assume that \( a_{\ell} = b_{m} \) and since also \( (a_{\ell}, b_{\ell}) = 0 \) (see (5.13)), we obtain that \( a_{\ell} \) is an isotropic vector in \( \mathbb{C}^{n} \).

Finally we conclude the following
Proposition 5.1 Let \( m \) be the integer part of \( \frac{n}{2} \). Choose \( m \) linearly independent vectors \( a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in \mathbb{C}^n \) which span a maximal isotropic subspace of \( \mathbb{C}^n \), i.e.

\[
(a_i, a_j) = \sum_{k=1}^{n} a_{ik} a_{jk} = 0 \text{ for all } 1 \leq i, j \leq m.
\]

Choose \( m \) non-negative integers \( k_i, 1 \leq i \leq m \) such that

\[
k_1 \geq k_2 \geq \ldots \geq k_m \geq 0,
\]

then \( \sigma(\tau) = \sum_{\alpha \in M} \tau_\alpha(x) e^\alpha \), with

\[
\tau = \left( \sum_{p} a_{1p} \psi_{-k_1 + \frac{1}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{1p} \psi_{-k_1 + \frac{3}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{2p} \psi_{-k_2 + \frac{1}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{2p} \psi_{-k_2 + \frac{3}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{mp} \psi_{-\frac{1}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{mp} \psi_{-k_1 + \frac{1}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{mp} \psi_{-k_1 + \frac{3}{2}}^{\alpha(p)} \right) \left( \sum_{p} a_{mp} \psi_{-\frac{1}{2}}^{\alpha(p)} \right) 0,
\]

satisfies the \( n \)-component KP hierarchy \([4.2]\) and

\[
\omega(\tau) = (-)^{k_1 + k_2 + \ldots + k_m} \tau.
\]

Moreover

\[
\text{energy } \tau_\alpha(x) = k_1^2 + k_2^2 + \cdots + k_m^2 - \frac{1}{2} (\alpha | \alpha),
\]

\[
\sum_{j=1}^{n} \frac{\partial \tau_\alpha(x)}{\partial x_j^{(j)}} = 0
\]

and

\[
\tau_\alpha(x) = (-)^{\frac{1}{2} (\alpha | \alpha)} \tau_{-\alpha}(x),
\]

where \( \tau \) is defined by \([5.24]\).

Notice that the restriction \((5.23)\) is not essential, but we may assume it without loss of generality. Since the energy is nowhere negative, formula \((5.25)\) gives a restriction for \( \text{supp } \tau \).

It is not difficult to prove that the perfect wedge \( \tau \), given by \((5.24)\), is also a highest weight vector for the \( W_{1+\infty} \)-algebra generated by

\[
J^{(\ell+1)}(z) = \sum_{k \in \mathbb{Z}} J_k^{(\ell+1)} z^{-k-\ell-1} = \sum_{j=1}^{n} : \psi^{(j)}(z) \frac{\partial^{\ell} \psi^{-(j)}(z)}{\partial z^{\ell}} :, \quad \ell = 0, 1, 2, \ldots,
\]

i.e.,

\[
J_k^{(\ell+1)} \tau = \delta_{k0} c_\ell(k_1, k_2, \ldots, k_m) \tau \quad \text{for } k \geq 0.
\]
Here \( c_\ell \in \mathbb{C} \) only depend on the integers \( k_1, k_2, \ldots, k_m \). This induces the following restriction on \( W_\tau \in G_{R_0} \):

\[
\sum_{j=1}^{n} e^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell E_{jj} W_\tau \subset W_\tau \quad \text{for all } k, \ell = 0, 1, 2, \ldots
\]

If we now rewrite the element (5.24) as a perfect wedge, we can use proposition 3.1 to determine \( \tau_\alpha \) for \( \alpha = 0 \) or \( \alpha \in \Delta \). Add to the vectors \( \alpha_i, 1 \leq i \leq m \) vectors \( \alpha_j, m + 1 \leq j \leq n \) such that they form a basis of \( \mathbb{C}^n \), which satisfies

\[
(a_\ell, a_k) = \delta_{k+\ell,2m+1} + \delta_{k+\ell,4m+2} \text{ for all } 1 \leq k, \ell \leq n.
\]

Define

\[
k_{2m+1-i} = -k_i \text{ for } 1 \leq i \leq m.
\]

Then the \( \tau \) given by (5.24) is up to a scalar multiple equal to the following perfect wedge

\[
A_{-\frac{1}{2}} \wedge A_{-\frac{1}{2}} \wedge A_{-\frac{1}{2}} \wedge \cdots,
\]

with

\[
A_{-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell} = \sum_{j=1}^{n} a_{jq} v_{n\ell}^{(j)} \text{ with } 1 \leq q \leq 2m - 1 \text{ and } -k_1 + \frac{1}{2} \leq \ell \leq k_q - \frac{1}{2},
\]

\[
A_{-(2m)-(k_1+k_2+\cdots+k_{2m-1})-\ell} = \sum_{j=1}^{n} a_{n, j} v_{n\ell}^{(j)} \text{ with } -k_1 + \frac{1}{2} \leq \ell \leq -\frac{1}{2} \text{ this only if } n = 2m + 1,
\]

\[
A_q = v_q \text{ for } q < -nk_1 - k_2 - \cdots - k_{2m-1}.
\]

Now using (2.11), this is equal to

\[
A_{-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell} = \sum_{j=1}^{n} a_{jq} v_{n\ell-n/2(n-2j+1)} \text{ with } 1 \leq q \leq 2m - 1 \text{ and } -k_1 + \frac{1}{2} \leq \ell \leq k_q - \frac{1}{2},
\]

\[
A_{-(2m)-(k_1+k_2+\cdots+k_{2m-1})-\ell} = \sum_{j=1}^{n} a_{n, j} v_{n\ell-n/2(n-2j+1)} \text{ with } -k_1 + \frac{1}{2} \leq \ell \leq -\frac{1}{2} \text{ this only if } n = 2m + 1
\]

\[
A_q = v_q \text{ for } q < -nk_1 - k_2 - \cdots - k_{2m-1}.
\]

Using proposition 3.1, one easily deduces that the to (5.24) corresponding \( \tau_0 \) is given by

\[
\tau_0 = \text{det} \left( \sum_{q=1}^{2m-1} \sum_{i=1}^{n} \sum_{j=1}^{k_i-1} a_{qj} S_{\ell+i}(x^{(j)}) E_{j-n-1/2,-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell-1/2} \right.
\]

\[
+ \delta(-1)^{\alpha_i} \sum_{j=1}^{n} \sum_{i=1}^{k_1-1} \sum_{l=-i}^{l} a_{n, j} S_{\ell+i}(x^{(j)}) E_{j-n-1/2,-(2m)-(k_1+k_2+\cdots+k_{2m-1})-\ell-1/2} \right.
\]

and \( \tau_{\delta_r, -\delta_s} \) for \( 1 \leq r, s \leq n \) is equal to the determinant of \( \tau_0 \), but then with the \((s-n-\frac{1}{2})\)-th row replaced by

\[
\sum_{q=1}^{2m-1} \sum_{l=-i}^{l} a_{qr} S_{\ell}(x^{(r)}) E_{s-n-1/2,-qk_1-(k_1+k_2+\cdots+k_{q-1})-\ell-1/2}
\]

Now change the indices and we obtain
Theorem 5.1 Let \( \tau \) be given by (5.24), and let \( \sigma(\tau) = \sum_{\alpha \in \mathcal{M}} \tau_\alpha(x)e^\alpha \), then up to a common scalar factor
\[
\tau_0 = \det \left( \sum_{q=1}^{2m-1} \sum_{j=1}^{n} \sum_{k_1}^{i} \sum_{\ell=1}^{l-1} a_{qj}S_i - \ell(x^{(j)}) E_{m-j+1,qk_1+(k_1+k_2+\cdots+k_{q-1})-\ell+1} \\
+ \delta(-1)^{n-1} \sum_{j=1}^{n} \sum_{k_1}^{i} \sum_{\ell=1}^{l-1} a_{n,j}S_i - \ell(x^{(j)}) E_{m-j+1,(2m)+(k_1+k_2+\cdots+k_{2m-1})-\ell+1} \right)
\]  
(5.30)
and \( \tau_{\delta_r,\delta_s} \) for \( 1 \leq r, s \leq n \) is equal to the determinant of \( \tau_0 \), but then with the \( (n+1-s) \)-th row replaced by
\[
\sum_{q=1}^{2m-1} \sum_{\ell=0}^{k_q-1} a_{qr}S_i - \ell(x^{(r)}) E_{n+1-s,qk_1+(k_1+k_2+\cdots+k_{q-1})+\ell+1},
\]  
(5.31)
where the \( a_{qr}, 1 \leq r \leq n \), satisfy (5.24) and the \( k_j, m+1 \leq j \leq 2m \) are given by (5.24).

Moreover, the
\[
\overline{\tau}_{rs}(x) = \begin{cases} 
\epsilon_{sr} \frac{\tau_{x-k_1(x)}}{\partial \log \tau_0(x)}, & \text{if } 1 \leq r, s \leq n \text{ and } r \neq s, \\
\frac{\partial \log \tau_0(x)}{\partial x^{(i)}_r}, & \text{if } 1 \leq r, s \leq n \text{ and } r = s,
\end{cases}
\]  
(5.32)
satisfy the Darboux-Egoroff system (5.2)-(5.4). If we define
\[
\gamma_{rs}(x) = \overline{\tau}_{rs}(x)|_{x^{(i)}_k = 0 \text{ for all } k \geq 1},
\]  
(5.33)
then these elements satisfy (5.3)-(5.5).

Let \( f(t) = \sum_i f_i(t)e_i \) and \( g(t) = \sum_i g_i(t)e_i \) be two elements in \( H \), define the following bilinear form
\[
B(f, g) = \text{Res}_{t=0} \sum_{i=1}^n f_i(t)g_i(t).
\]  
(5.34)

Then the orthogonal restricted Grassmannian is
\[
\widehat{Gr} = \{ W \in \overline{Gr} | B(W, W) = 0 \}.
\]  
(5.35)

All \( W \in \widehat{Gr} \) are maximal isotropic subspaces with respect to \( B(\cdot, \cdot) \). This Grassmannian is the homogeneous space for the \( O_n(\mathbb{C}[t, t^{-1}]) \)-loop group. The \( O_n(\mathbb{C}[t, t^{-1}]) \)-orbit of \( |0\rangle \) corresponds exactly to this Grassmannian (see e.g. [18]). Notice that all the \( W_r \)’s corresponding to the tau-functions given by (5.24) exactly satisfy this condition. Hence the tau-functions we have constructed to solve the Darboux-Egoroff system are in fact homogeneous tau-functions in the \( O_n(\mathbb{C}[t, t^{-1}]) \)-orbit of \( |0\rangle \). If we consider the affine Lie algebra \( gl_n(\mathbb{C})^\wedge \) with central charge 1, defined by (3.5), then the special orthogonal Lie algebra \( so_n(\mathbb{C})^\wedge \) is given by
\[
so_n(\mathbb{C})^\wedge = \{ x \in gl_n(\mathbb{C})^\wedge | \omega(x) = x \}.
\]
Recall that \( \omega(\psi_k^{\pm(i)}) = \psi_k^{\mp(i)} \). The Grassmannian \( \widehat{Gr} \) has two connected components, which are distinguished by the parity of the dimension of the kernel of the projection \( W \rightarrow H_0 \).
Depending on the energy of our (homogeneous) tau-function, $\omega(\tau) = (-)^{\text{energy}} \tau\tau$, the space $W_\tau$ belongs to one of these two components.

It is obvious, from the above description and from the construction of the tau-functions given by (5.24), that the orthogonal group $O_n$ acts on these tau-functions and hence on the rotation coefficients. One has

**Proposition 5.2** The orthogonal group $O_n$ acts on the rotation coefficients of Theorem 5.1. Let $X = (X_{ij})_{1 \leq i, j \leq n} \in O_n$, then replacing $a_{ij}$, $1 \leq i, j \leq n$, (even if $a_{ij} = 0$) by

$$\sum_{k=1}^n X_{jk} a_{ik}$$

in (5.30) and (5.31) gives a new solution of the Darboux-Egoroff system.

### 6 Semisimple Frobenius manifolds

Let $\gamma_{ij}(x)$, $1 \leq i, j \leq n$, be a solution of the Darboux-Egoroff system. If we can find $n$ linearly independent vector functions $\psi_j = \psi_j(x) = (\psi_{1j}, \psi_{2j}, \ldots, \psi_{nj})$ such that

$$\frac{\partial \psi_{ij}}{\partial x_1^{(k)}} = \gamma_{ik} \psi_{kj}, \quad k \neq i,$$

$$\sum_{k=1}^n \frac{\partial \psi_{ij}}{\partial x_1^{(k)}} = 0,$$

then they determine under certain conditions (locally) a semisimple (i.e. massive) Frobenius manifold (see [2], [3]).

Recall from (5.10), that the wave function $V(z) = V^+(0, x, z)$ corresponding to the tau-functions of proposition 3.1 and theorem 5.1 satisfy

$$\frac{\partial V_{ij}(z)}{\partial x_1^{(k)}} = W_{ik} V_{kj}(z), \quad k \neq i,$$

$$\sum_{k=1}^n \frac{\partial V_{ij}(z)}{\partial x_1^{(k)}} = z V_{ij}(z).$$

Comparing (6.1) and (6.2), one would like to take $z = 0$ in (6.2), however this does not make sense. There is a way to use the wave function $V(z)$ to construct the $\psi_{ij}$’s of (6.1). Suppose that we have a tau-function of the form (5.24), with the corresponding $k_q$’s, $1 \leq q \leq n$, (in the case that $n$ is odd, we define $k_n = 0$) and $a_{qj}$’s $1 \leq q, j \leq n$. Let

$$X_q(t) = \sum_{j=1}^n a_{qj} t^{-k_q+1} e_j \in H, \quad 1 \leq q \leq n,$$

then it easy to check that

$$W_\tau + \mathbb{C} X_q(t) \neq W_\tau \quad \text{and} \quad W_\tau + \mathbb{C} t X_q(t) = W_\tau.$$

Hence,

$$\sum_{j=1}^n a_{qj} \psi_{-k_q-\frac{1}{2}}^{+(j)} \tau \neq 0 \quad \text{and} \quad \sum_{j=1}^n a_{qj} \psi_{-k_q+\frac{1}{2}}^{+(j)} \tau = 0.$$
We rewrite this as follows

\[ \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q-1} \psi^{(j)}(z) \tau \neq 0 \quad \text{and} \quad \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} \psi^{(j)}(z) \tau = 0. \] (6.5)

From which we deduce that

\[ \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q-1} \tau \sum_{\ell=1}^{\infty} x^{(j)}_{\ell} z^{-\ell} e^{-\sum_{\ell=1}^{\infty} \frac{a_{qj}^{(j)} \tau^{-\ell}}{a_{qj}^{(j)}}} \tau_{\delta_1-\delta_j}(x) \neq 0 \quad \text{and} \quad \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} \tau \sum_{\ell=1}^{\infty} x^{(j)}_{\ell} z^{-\ell} e^{-\sum_{\ell=1}^{\infty} \frac{a_{qj}^{(j)} \tau^{-\ell}}{a_{qj}^{(j)}}} \tau_{\delta_1-\delta_j}(x) = 0. \]

Dividing this by \( \tau_0(x) \) we obtain

\[ \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}(z) \neq 0 \quad \text{and} \quad \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q} V_{ij}(z) = 0. \] (6.6)

Now define for \( 1 \leq i, q \leq n \)

\[ \Psi_{iq} = \text{Res}_{z=0} \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}(z), \] (6.7)

then it is straightforward to check, using (6.2) and (6.6) that

\[ \frac{\partial \Psi_{ij}}{\partial x^{(k)}_1} = W_{ik'} \Psi_{kj'}, \quad k \neq i, \]

\[ \sum_{k=1}^{n} \frac{\partial \Psi_{ij}}{\partial x^{(k)}_1} = 0. \] (6.8)

Notice that the vector functions \( \Psi_q = (\Psi_{1q}, \Psi_{2q}, \ldots, \Psi_{nq}) \) are “eigenfunctions” of the KP hierarchy which lie in the kernel of \( L \). From all this we finally obtain the following

**Theorem 6.1** Let \( V(z) = V^+(0, x, z) \) be the wave function corresponding to the tau-function of (5.24) with \( a_{qj}, 1 \leq q, j \leq n \) and \( k_{\ell}, 1 \leq \ell \leq 2m \), as given in theorem 5.1 and \( k_n = 0 \) if \( n \) is odd. Denote by

\[ \psi_{iq} = \text{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}^+(0, x, z) \big|_{x_k^{(\ell)}=0} \text{for all } k>1, \] (6.9)

\[ \overline{\psi}_{iq} = \text{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{qj} z^{-k_q-1} V_{ij}^+(0, x, z) \big|_{x_{2k}^{(\ell)}=0} \text{for all } k, \]

where \( 1 \leq q \leq n \) and \( \lambda_q \in \mathbb{C}^\times \). Then these \( \psi_{iq} \)'s satisfy the equations (6.4), with \( \gamma_{ij} \) given
by (5.32) and the formula’s
\[
\eta_{ii} = \psi_{i1}^2,
\]
\[
\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha} \psi_{i\beta},
\]
\[
\frac{\partial t_{\alpha}}{\partial x_{(i)}^1} = \psi_{i1} \psi_{i\alpha},
\]
\[
c_{\alpha\beta\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha} \psi_{i\beta} \psi_{i\gamma}}{\psi_{i1}},
\]
with \( t_{\alpha} = \sum_{\epsilon=1}^{n} \eta_{\alpha\epsilon} t_{\epsilon} \), determine (locally) a massive Frobenius manifold on the domain \( x_{(1)}^i \neq x_{(1)}^1 \) and \( \psi_{11} \psi_{21} \cdots \psi_{n1} \neq 0 \). The \( \psi_{iq} \)'s also satisfy (6.1), but now with the \( \gamma_{ij} \) replaced by \( \gamma_{ij} \) of (5.33). The equations (6.10) for these \( \psi_{ij} \)'s also determine a semisimple Frobenius manifold.

**Proof.** Formula (6.10) is a direct consequence of the following proposition, see [4] (cf. [2] and [3]) for more details. \( \square \)

**Proposition 6.1** Let \( X = \sum_{i=1}^{n} x_{(1)}^i E_{ii} \), \( \Gamma = (\gamma_{ij})_{1 \leq i,j \leq n} \), \( \mathcal{V} = [\Gamma, X] \) and \( \mathcal{V}_k = [\Gamma, E_{kk}] \), then \( \mathcal{V} = (\mathcal{V}_{ij})_{1 \leq i,j \leq n} \) is anti-symmetric and satisfies
\[
\frac{\partial \mathcal{V}}{\partial x_{(k)}^1} = [\mathcal{V}_k, \mathcal{V}]
\]
and also
\[
\mathcal{V} \psi_q = \sum_{j=1}^{n} x_{(1)}^j \frac{\partial \psi_q}{\partial x_{(j)}^1} = k_q \psi_q,
\]
\[
\frac{\partial \psi_q}{\partial x_{(k)}^1} = \mathcal{V}_k \psi_q,
\]
for \( \psi_q = (\psi_{1q}, \psi_{2q}, \ldots, \psi_{nq}) \)

**Proof.** The equation (6.11) follows from (5.2), (5.3) and the fact that \( \Gamma \) is symmetric. We prove (6.12) as follows. Let \( \mathcal{V} \) act \( \psi_q \), Using (5.11) and (6.7) one deduces
\[
\mathcal{V} \psi_q = \sum_{j=1}^{n} x_{(1)}^j \frac{\partial \psi_q}{\partial x_{(j)}^1}.
\]
Since \( \psi_q \) is independent of \( x_{(j)}^k \) for all \( k > 1 \), we can use (5.12), to rewrite this as follows
\[
\sum_{j=1}^{n} x_{(1)}^j \frac{\partial \psi_q}{\partial x_{(j)}^1} = \text{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{jq} z^{-k_q-1} \frac{\partial}{\partial z} \left( V_{ij}^+(0, x, z) \big|_{x_{(j)}^1 = 0 \text{ for all } k>1} \right)
\]
\[
= \text{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{jq} \left( \frac{\partial}{\partial z} z^{-k_q + k_q z^{-k_q-1}} \right) \left( V_{ij}^+(0, x, z) \big|_{x_{(j)}^1 = 0 \text{ for all } k>1} \right)
\]
\[
= k_q \text{Res}_{z=0} \lambda_q \sum_{j=1}^{n} a_{jq} z^{-k_q-1} V_{ij}^+(0, x, z) \big|_{x_{(j)}^1 = 0 \text{ for all } k>1}
\]
\[
= k_q \psi_{ijq}.
\]
The second equation of (6.12) can be proved in a similar way, using (5.10). □

From (6.12) we determine the degrees \( d_1, d_2, \ldots, d_n \) and \( d \) (resp. \( d_F \)) of the corresponding \( t^\alpha \)

\[
d_1 = 1, \quad d_\alpha = 1 + k_1 - k_\alpha, \quad 2 \leq \alpha \leq n, \quad d = -2k_1 \quad \text{and} \quad d_F = 3 + 2k_1. \tag{6.13}
\]

With our choice of \( k_\alpha \) we have

\[
d_\alpha + d_{2m+1-\alpha} = 2 - d, \quad 1 \leq \alpha \leq m \quad \text{and} \quad d_n = 1 + k_1 \quad \text{if} \quad n = 2m + 1 \quad \text{is odd.}
\]

Notice that if we define

\[
\Phi(z) = V(0, x, z)|_{x_{k}^{(i)}=0} \quad \text{for all} \quad k>1,
\]

then \( \Phi(z) \) satisfies

\[
\begin{align*}
z \frac{\partial \Phi(z)}{\partial z} &= \sum_{j=1}^{n} x_1^{(j)} \frac{\partial \Phi(z)}{\partial x_1^{(j)}} = (zX + V)\Phi(z), \\
\frac{\partial \Phi(z)}{\partial x_1^{(k)}} &= (zE_{kk} + V_k)\Phi(z). \tag{6.15}
\end{align*}
\]

**Theorem 6.2** Let \( \Psi = (\psi_{ij})_{1 \leq i,j \leq n} \) and define \( \xi(z) = \Psi \Phi(z) = \eta \Psi^{-1} \Phi(z), \quad \mathcal{U} = \eta \Psi^{-1} X \Psi \eta^{-1}, \mu = -\eta \Psi^{-1} \mathcal{V} \eta^{-1} = \sum_{i=1}^{n} k_i E_{ii}, \quad \eta \Psi^{-1} E_{ii} \Psi \eta^{-1}, \) then \( (\mathcal{U} \eta, \mu \eta + \eta \mu) = 0 \) and

\[
\begin{align*}
\begin{align*}
\frac{\partial \xi(z)}{\partial z} &= (z\mathcal{U} - \mu)\xi(z), \\
\sum_{j=1}^{n} x_1^{(j)} \frac{\partial \xi(z)}{\partial x_1^{(j)}} &= z\mathcal{U} \xi(z), \\
\frac{\partial \xi(z)}{\partial x_1^{(k)}} &= z\Pi_k \xi(z), \\
\frac{\partial \xi(z)}{\partial t^\alpha} &= zC_\alpha \xi(z), \tag{6.16}
\end{align*}
\end{align*}
\]

where \( C_\alpha = \sum_{\beta, \gamma=1}^{n} c_{\alpha \beta}^{\gamma} E_{\beta \gamma}. \)

**Proof.** All formulas except the last one of (6.16) follows immediately from (6.11), (6.12), (6.15) and the fact that \( \Psi \Phi = \eta \). Use the last formula of (6.10), \( c_{\alpha \beta}^{\gamma} = \sum_{i=1}^{n} c_{\beta \alpha i} \eta^{i \gamma} \) and

\[
\frac{\partial x_1^{(i)}}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{ii}}
\]

to rewrite

\[
\begin{align*}
\frac{\partial \xi(z)}{\partial t^\alpha} &= \sum_{i=1}^{n} \frac{\partial x_1^{(i)}}{\partial t^\alpha} \frac{\partial \xi}{\partial x_1^{(i)}} \\
&= z \sum_{i=1}^{n} \frac{\partial x_1^{(i)}}{\partial t^\alpha} \eta \Psi^{-1} E_{ii} \Psi \eta^{-1} \xi \\
&= z \Psi \sum_{i=1}^{n} \frac{\psi_{i\alpha}}{\psi_{ii}} E_{ii} \Psi \eta^{-1} \xi \\
&= zC_\alpha \xi.
\end{align*}
\]
This finishes the proof of the theorem.

As in [3, 4] we can reformulate (6.11) as an \{x_1^{(i)}\}_{1 \leq i \leq n}-dependent commuting Hamiltonian system

$$\frac{\partial \mathcal{V}}{\partial x_1^{(k)}} = \{\mathcal{V}, H_k(\mathcal{V}, X)\},$$

with quadratic Hamiltonians

$$H_i(\mathcal{V}, X) = \frac{1}{2} \sum_{j \neq i} \frac{\mathcal{V}_{ij}\mathcal{V}_{ji}}{x_1^{(i)} - x_1^{(j)}} = \frac{1}{2} \sum_{j \neq i} \gamma_{ij}\gamma_{ji}(x_1^{(i)} - x_1^{(j)}) \quad (6.17)$$

with respect to the standard Poisson bracket on \(so_n\):

$$\{\mathcal{V}_{ij}, \mathcal{V}_{k\ell}\} = \delta_{jk}\mathcal{V}_{i\ell} - \delta_{ik}\mathcal{V}_{j\ell} + \delta_{i\ell}\mathcal{V}_{jk} - \delta_{j\ell}\mathcal{V}_{ik}.$$ 

Now consider the 1-form

$$\sum_{i=1}^{n} H_i(\mathcal{V}, X) dx_1^{(i)}. \quad (6.18)$$

Since it is closed for any such \(\mathcal{V}\) (see [2], [3]), there exists a function \(\tau_I(X)\), the isomonodromy tau-function, such that

$$d \log \tau_I(X) = \sum_{i=1}^{n} H_i(\mathcal{V}, X) dx_1^{(i)}. \quad (6.19)$$

Using (5.2), we rewrite \(H_i(\mathcal{V}, X)\) as follows. Let \(\tilde{\tau}_0(X) = \tau_0(x)|_{x_1^{(i)} = 0}\) for all \(k>1\), then

$$H_i(\mathcal{V}, X) = \frac{1}{2} \sum_{j \neq i} \gamma_{ij}\gamma_{ji}(x_1^{(i)} - x_1^{(j)})$$

$$= \frac{1}{2} \sum_{j \neq i} \frac{\partial \gamma_{ii}}{\partial x_1^{(j)}}(x_1^{(i)} - x_1^{(j)})$$

$$= \frac{1}{2} \sum_{j=1}^{n} x_1^{(i)} \frac{\partial \gamma_{ii}}{\partial x_1^{(j)}} - \frac{1}{2} \sum_{j=1}^{n} x_1^{(j)} \frac{\partial \gamma_{ii}}{\partial x_1^{(i)}}$$

$$= - \frac{1}{2} \sum_{j=1}^{n} x_1^{(j)} \frac{\partial \gamma_{ii}}{\partial x_1^{(j)}}$$

$$= \frac{1}{2} \sum_{j=1}^{n} x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} \left( \frac{\partial}{\partial x_1^{(i)}} (\log \tilde{\tau}_0(X)) \right)$$

$$= \frac{1}{2} \frac{\partial}{\partial x_1^{(i)}} \left( \sum_{j=1}^{n} x_1^{(j)} \frac{\partial}{\partial x_1^{(j)}} (\log \tilde{\tau}_0(X)) \right) - \frac{1}{2} \frac{\partial}{\partial x_1^{(i)}} (\log \tilde{\tau}_0(X))$$

$$= - \frac{1}{2} \frac{\partial}{\partial x_1^{(i)}} (\log \tilde{\tau}_0(X)).$$

Hence

$$d \log \tau_I(X) = - \frac{1}{2} d \log \tilde{\tau}_0(X). \quad (6.20)$$
Dubrovin and Zhang defined in [6] a Gromov-Witten type $G$-function of a Frobenius manifold as follows

$$G = \log \left( \frac{\tau_f}{J} \right), \quad \text{where}$$

$$J = \det \left( \frac{\partial t^a}{\partial x_1^{(i)}} \right) = \log (\psi_{11} \psi_{21} \cdots \psi_{n1}).$$

We can explicitly determine this function in the cases of the Frobenius manifolds corresponding to theorem 6.1.

**Theorem 6.3** Let $\tau$ be given by (5.24) and let $\psi_{i1}$ be defined as in (6.9). Let $\tilde{\tau}_0(X) = \tau_0(x)|_{x_k^{(i)}=0}$ for all $k>1$, i.e.,

$$\tilde{\tau}_0(X) = \det(\sum_{q=1}^{2m-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=1-k_q}^{i} a_{qj} (x_1^{(j)})^{i-\ell} E_{\ell}^{in-j+1,qk_1+(k_1+k_2+\cdots+k_{q-1})-\ell+1}$$

$$+ \delta(-1)^{n,-1} \sum_{j=1}^{n} \sum_{i=1}^{k_1} \sum_{\ell=1}^{i} a_{n,j} (x_1^{(j)})^{i-\ell} E_{\ell}^{in-j+1,(2m)+(k_1+k_2+\cdots+k_{2m-1})-\ell+1})$$

(6.22)

Then up to an additive scalar factor

$$G = \frac{1}{2} \log \tilde{\tau}_0(X) - \frac{1}{24} \log (\psi_{11} \psi_{21} \cdots \psi_{n1}).$$

Moreover,

$$\sum_{j=1}^{n} x_1^{(j)} \frac{\partial G}{\partial x_1^{(j)}} = \gamma G,$$

where

$$\gamma = -\frac{1}{4} \sum_{j=1}^{n} k_j^2 - \frac{nk_1}{24}$$

(6.24)

and

$$\frac{\partial}{\partial x_1^{(i)}} \frac{\partial}{\partial x_1^{(j)}} (\log \tilde{\tau}_0(X)) = -\gamma_{ij}^2 \quad i \neq j,$$

where $\gamma_{ij}$ is defined by formula (5.33).

### 7 An example

In this section we describe the simplest example in more detail. Let $n = 2m$, respectively $n = 2m+1$ if $n$ is even respectively odd. Since the choices of the order of $k_1, k_2 \ldots k_m \in \mathbb{Z}$ is rather arbitrary, we choose for simplicity of notation and calculation $k_1 = -k_n = -1$
and all other \( k_i = 0 \). Hence \( d_1 = 1, d_n = -1, d_\alpha = 0, \alpha \neq 1, n, d = 2 \) and \( d_F = 1 \) Choose vectors \( a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \), such that 

\[
(a_i, a_j) = \delta_{i+j,n+1}.
\]

Then

\[
\tau_0 = \sum_{j=1}^{n} a_{ni}^2 u_i \text{ and } \tau_{\delta_i - \delta_j} = -\tau_{\delta_j - \delta_i} = a_{ni} a_{nj} \text{ for } i < j,
\]

where we use the notation \( u_i = x_{1}^{(i)} \). Hence,

\[
\gamma_{ij} = -\frac{a_{ni} a_{nj}}{\sum_{j=1}^{n} a_{nj}^2 u_i} \text{ for } 1 \leq i, j \leq n
\]

and the wave function is equal to

\[
V(z) = \left( I - \frac{1}{\tau_0} \sum_{i,j=1}^{n} a_{ni} a_{nj} E_{ij} z^{-1} \right) \sum_{\ell=1}^{n} \sum_{k=0}^{\infty} S_k(x^{(\ell)}) E_{\ell\ell} z^k.
\]

From which we deduce that

\[
\psi_{i,1} = -\frac{a_{ni}}{\tau_0}, \quad \psi_{in} = -a_{ni} \left( u_i - \frac{1}{2\tau_0} \sum_{j=1}^{n} a_{nj}^2 u_j \right),
\]

\[
\psi_{ik} = a_{ki} - \frac{a_{ni}}{\tau_0} \sum_{j=1}^{n} a_{kj} a_{nj} u_j \text{ for } k \neq 1, n.
\]

Then using the formula’s (6.10) it is straightforward to check that

\[
t_1 = -\frac{1}{\tau_0}, \quad t_n = \frac{\sum_{j=1}^{n} a_{nj}^2 u_j^2}{2\tau_0}, \quad t_k = -\frac{\sum_{j=1}^{n} a_{kj} a_{nj} u_j}{\tau_0},
\]

and hence that

\[
\psi_{i,1} = a_{ni} t_1, \quad \psi_{in} = a_{ni} (t_n - u_i), \quad \psi_{ik} = a_{ki} + a_{ni} t_k,
\]

\[
\eta_{\alpha,\beta} = \delta_{\alpha+\beta,n+1} \text{ and } t^\ell = t_{n+1-\ell}. \text{ Assume from now on that all } a_{ni} \neq 0. \text{ Since } \eta_{\alpha,\beta} = \delta_{\alpha+\beta,n+1}, \text{ the solution } F(t) \text{ of the WDVV equations is of the form (see [3]):}
\]

\[
F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n+1-\alpha} + f(t^2, t^3, \ldots, t^n).
\]

Since \( d_n = -1, d_\alpha = 0 \) for \( \alpha \neq 1, n \) and \( d_F = 1 \), it suffices to determine \( c_{nnn} \), which is

\[
c_{nnn} = \sum_{i=1}^{n} \frac{a_{ni}^2 (t^1 - u_i)^3}{t^n}.
\]
A straightforward calculation shows that

\[ u_i = t^1 - \frac{1}{a_{ni} t^n} \left( a_{1i} - \sum_{\alpha=2}^{n-1} \left( a_{\alpha i} t^\alpha + \frac{a_{ni}}{2} t^{n+1-\alpha} \right) \right). \]

Hence,

\[ \frac{\partial^3 f}{\partial u_n^3} = \frac{1}{(t^n)^4} \left( \sum_{i=1}^{n} \frac{a_{1i}}{a_{ni}} - \sum_{\alpha=2}^{n-1} \left( \frac{a_{\alpha i} t^\alpha}{a_{ni}} + \frac{1}{2} t^{\alpha n+1-\alpha} \right) \right)^3. \]

and thus

\[ F(t) = \frac{1}{2} (t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n+1-\alpha} - \frac{1}{6t^n} \left( \sum_{i=1}^{n} \frac{a_{1i}}{a_{ni}} - \sum_{\alpha=2}^{n-1} \left( \frac{a_{\alpha i} t^\alpha}{a_{ni}} + \frac{1}{2} t^{\alpha n+1-\alpha} \right) \right)^3. \]

Next we give the \( \xi_{ij} \)'s (\( \alpha \neq 1, n \)):

\[ \begin{align*}
\xi_{1j} &= a_{nj} t^n e^{zu_j}, \\
\xi_{\alpha j} &= (a_{\alpha j} + a_{nj} t^{n+1-\alpha}) e^{zu_j}, \\
\xi_{nj} &= \left( a_{nj} z^{-1} + \frac{1}{t^n} \left( a_{1j} - \sum_{\alpha=2}^{n-1} \left( \frac{a_{\alpha j} t^\alpha}{a_{ni}} + \frac{a_{nj}}{2} t^{\alpha n+1-\alpha} \right) \right) \right) e^{zu_j}. 
\end{align*} \]

One easily sees that \( \xi_{ij} = \frac{\partial h_j}{\partial t^i} \) with

\[ h_j = \frac{a_{nj} t^n}{z} e^{zu_j} = \frac{a_{nj} t^n}{z} \left( t^1 - \frac{1}{a_{nj} t^n} \left( a_{1j} - \sum_{\alpha=2}^{n-1} \left( a_{\alpha j} t^\alpha + a_{nj} t^{n+1-\alpha} \right) \right) \right). \]

To see that this is deformed flat coordinates, we determine

\[ \tilde{\theta}^\alpha = (-)^{\delta_{\alpha 1}} \sum_{j=1}^{n} a_{n+1-\alpha, j} h_j. \]

we find

\[ \begin{align*}
\tilde{t}^1 &= 1 + t^1 z + O(z^2), \\
\tilde{t}^\alpha &= t^\alpha + O(z), \quad \alpha \neq 1, n \\
\tilde{t}^{n} &= t^n z^{-1} + O(z^0). 
\end{align*} \]

Finally we calculate the \( G \)-function of the Frobenius manifold. Notice that \( \tilde{\tau}_0(X) = \tau_0(x) = -\frac{1}{t^n} \) and that

\[ \psi_{11} \psi_{21} \cdots \psi_{n1} = \prod_{i=1}^{n} (a_{ni} t^n). \]

So using theorem 6.3, we obtain that \( \gamma = \frac{n-12}{24} \) and that up to an additive constant

\[ G(t) = \frac{12 - n}{24} \log(t^n). \]
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