Gauge Invariant Regularization of Quantum Field Theory on the Light-Front

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Abstract

Gauge invariant regularization of quantum field theory in the framework of Light-Front (LF) Hamiltonian formalism via introducing a lattice in transverse coordinates and imposing boundary conditions in LF coordinate $x^-$ for gauge fields on the interval $|x^-| \leq L$ is considered. The remaining ultraviolet divergences in the longitudinal momentum $p_-$ are removed by gauge invariant finite mode regularization. We find that LF canonical formalism for the introduced regularization does not contain usual most complicated second class constraints connecting zero and nonzero modes of gauge fields. The described scheme can be used either for the regularization of conventional gauge theory or for gauge invariant formulation of effective low-energy models on the LF. The lack of explicit Lorentz invariance in our approach leads to difficulty with defining the vacuum state. We discuss this difficulty, particularly, in the connection with the problem of taking the limit of continuous space.

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1. Introduction

Canonical formulation of field theory in Light-Front (LF) coordinates $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, $x^1, x^2$, where $x^0, x^1, x^2, x^3$ are Lorentz coordinates, was proposed by Dirac [1]. The $x^+$ plays the role of time, and canonical quantization is carried out on a hypersurface $x^+ = \text{const}$. The advantage of this scheme is connected with the positivity of the momentum $P_-$ (translation operator along $x^-$ axis), which becomes quadratic in fields on the LF. As a consequence the lowest eigenstate of the operator $P_-$ is both physical vacuum and the "mathematical" vacuum of perturbation theory. Using Fock space over this vacuum one can solve stationary Schroedinger equation with Hamiltonian $P_+$ (translation operator along $x^+$ axis) to find the spectrum of bound states. The problem of describing the physical vacuum, very complicated in usual formulation with Lorentz coordinates, does not appear here. Such approach, based on solving Schroedinger equation on the LF, is called LF Hamiltonian approach. It attracts attention for a long time as a possible mean for solving Quantum Field Theory nonperturbatively.

While giving essential advantages, the application of LF coordinates in Quantum Field Theory leads to some difficulties. The hyperplane $x^+ = \text{const}$ is a characteristic surface for relativistic differential field equations. It is not evident without additional investigation that the quantization on such a hypersurface generates a theory equivalent to one quantized in usual way in Lorentz coordinates [2–8]. This is essential, in particular, because of special divergences at $p_- = 0$ appearing in LF quantization scheme. Beside of conventional ultraviolet regularization one has to apply special regularization of these divergences. Usually the following simplest prescriptions of such regularization are considered:

(a) cutoff of the momenta $p_-$

$$|p_-| \geq \varepsilon, \quad \varepsilon > 0; \quad (1)$$

(b) cutoff of the coordinate $x^-$

$$-L \leq x^- \leq L \quad (2)$$

with periodic boundary conditions in $x^-$ for all fields.

The prescriptions (a) and (b) are convenient for Hamiltonian approach, but both of them break Lorentz invariance, and the prescription (a) breaks also the gauge invariance.
The problem of constructing a LF Hamiltonian which generates a theory equivalent to original Lorentz and gauge invariant one turned out to be rather difficult. Nevertheless it can be solved perturbatively to all orders for non-gauge field theories [9, 10] and also for gauge theories (including QCD) [11–14] via gauge-noninvariant regularization procedure (under the prescription (a)) using special methods to restore the symmetries in the limit of removed regularization.

The regularization prescription (b) discretizes the spectrum of the operator $P_-$ ($p_- = \pi n/L$, where $n$ is an integer). This formulation is called sometimes "Discretized Light Cone Quantization (DLCQ)" [15]. Fourier components of fields, corresponding to $p_- = 0$ (and usually called "zero modes") turn out to be dependent variables and are to be expressed in terms of nonzero modes via solving constraint equations (constraints) [16–19]. These constraints are usually very complicated due to their nonlinearity in fields, and solving of them is a difficult problem. Moreover, in quantum theory one encounters the uncertainty in ordering of operators in the expressions for these constraints.

The introduction of space-time lattice for gauge-invariant regularization of nonabelian gauge theories is well known [20]. Gauge invariant regularization in continuous space-time is also known [21] but it seems not suitable for the LF quantization. For the LF formulation only the lattice in transverse coordinates $x^1, x^2$ is used. In this formulation it is convenient to define variables so as to have the action polynomial in these variables [22–24]. Such a regularization is not Lorentz invariant, and one can only hope that the Lorentz invariance can be restored in continuous space limit. Nevertheless many attempts to apply LF Hamiltonian formulation with the prescription (b), combined with transverse space lattice, are undertaken (for ”color dielectric” type models [27–30]). In all of these works zero modes of fields are thrown out, so that, in fact, gauge invariance is violated.

In the present paper we consider canonical LF formulation of gauge theories, regularized in gauge-invariant way. To achieve this goal we introduce transverse space lattice, discretize the momentum $p_-$ according to the prescription (b) (with all zero modes of fields included) and apply the so called ”finite mode” ultraviolet regularization in $p_-$. The last means a cutoff in eigen values of covariant derivative operator $D_-$ in the expansion of lattice field variables in eigen functions of this operator. These field variables are lattice modification of transverse components of usual gauge fields. They are described by complex matrices, defined on lattice links. Only such variables
admit mentioned above "finite mode" regularization (for fermion fields analogous method was applied in [25, 26]).

It is interesting that in the framework of this formulation one can avoid complicated canonical 2nd-class constraints, usually present in canonical LF formalism in continuous space. This greatly simplifies canonical quantization procedure. However the absence of explicit Lorentz invariance of the regularization scheme makes the investigation of the connection with conventional Lorentz-covariant formulation difficult. In particular, there is a problem of the description of quantum vacuum as common lowest eigen state of both operators \( P_- \) and \( P_+ \). This question is discussed at the end of this paper.

2. Gauge-invariant action on the transverse lattice

At first we introduce particular ultraviolet regularization via a lattice in transverse coordinates \( x^1, x^2 \) and choose variables so as to have the action, which is polynomial in these variables [22, 23]. Furthermore, we use the described gauge-invariant regularization (b) of singularities at \( p_- \to 0 \) and gauge-invariant ultraviolet cutoff in modes of covariant derivative \( D_- \) (then ultraviolet regularization of the theory is complete). For simplicity we consider \( U(N) \) theory of pure gauge fields although the generalization to \( SU(N) \)-theory or a theory with fermions is not difficult.

The components of gauge field along continuous coordinates \( x^+, x^- \) can be taken without a modification and related to the sites of the lattice. Transverse components are described by complex \( N \times N \) matrices \( M_k(x), k = 1, 2 \). Each matrix \( M_k(x) \) is related to the link directed from the site \( x - e_k \) to the site \( x \). The transverse vector \( e_k \) connects two neighbouring sites on the lattice being directed along the positive axis \( x^k \) (\(|e_k| \equiv a\)), see fig. 1. The matrix

\[
\begin{align*}
  x - e_k & \quad M_k(x) \quad x \\
  \end{align*}
\]

Fig. 1.

\( M_k^+(x) \) is related to the same link but with opposite direction, see fig. 2. In the following for the index \( k \) the usual rule of summation on repeated indices is not used, and, where it is necessary, the symbol of a sum is indicated.

The elements of these matrices are considered as independent variables.
This makes the action polynomial. For any closed directed loop on the lattice we can construct the trace of the product of matrices $M_k(x)$ sitting on the links and order from the right to the left along this loop. For example the expression

$$\text{Tr} \left\{ M_2(x)M_1(x-e_2)M_2^+(x-e_1)M_1^+(x) \right\}$$  \hspace{1cm} (3)$$

is related to the loop shown in fig. 3.

It should be noticed that a product of matrices related to closed loop, consisting of one and the same link passed in both directions, is not identically unity because the matrices $M_k$ are not unitary (see, for example, fig. 4).
The unitary matrices $U(x)$ of gauge transformations act on the $M$ and $M^+$ in the following way:

$$M_k(x) \rightarrow M'_k(x) = U(x)M_k(x)U^+(x - e_k),$$  \hspace{1cm} (4)$$

$$M^+_k(x) \rightarrow M'^+_k(x) = U(x - e_k)M^+_k(x)U^+(x).$$  \hspace{1cm} (5)$$

A trace of the product of the matrices, related to a closed loop along lattice links, is invariant with respect to these transformations. To relate the matrices $M_k$ with usual gauge fields of continuum theory let us write these matrices in the following form:

$$M_k(x) = I + gaB_k(x) + i gA_k(x), \quad B^+_k = B_k, \quad A^+_k = A_k.$$  \hspace{1cm} (6)$$

Then in the $a \rightarrow 0$ limit the fields $A_k(x)$ coincide with transverse gauge field components, and the $B_k(x)$ turn out to be extra (nonphysical) fields which should be switched off in the limit. Below we show how to get this.

The analog of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$  \hspace{1cm} (7)$$

multiplied by $i$, can be defined as follows:

$$G_{+-} = iF_{+-}, \quad F_{+-}(x) = \partial_+ A_-(x) - \partial_- A_+(x) - ig[A_+(x), A_-(x)],$$

$$G_{\pm,k}(x) = \frac{1}{ga} \left[ \partial_\pm M_k(x) - ig \left( A_\pm(x)M_k(x) - M_k(x)A_\pm(x - e_k) \right) \right],$$

$$G_{12}(x) = -\frac{1}{ga^2} \left[ M_1(x)M_2(x - e_1) - M_2(x)M_1(x - e_2) \right].$$  \hspace{1cm} (8)$$

Under gauge transformation these quantities transform as follows:

$$G_{+-}(x) \rightarrow G'_{+-}(x) = U(x)G_{+-}(x)U^+(x),$$

$$G_{\pm,k}(x) \rightarrow G'_{\pm,k}(x) = U(x)G_{\pm,k}(x)U^+(x - e_k),$$

$$G_{12}(x) \rightarrow G'_{12}(x) = U(x)G_{12}(x)U^+(x - e_1 - e_2).$$  \hspace{1cm} (9)$$

We choose a simplest form of the action having correct naive continuum limit:

$$S = a^2 \sum_{x^\perp} \int_{-L}^{L} dx^+ \int dx^- \text{Tr} \left[ G'^+_{+-}G_{+-} + \sum_k \left( G'^{+}_{+k}G_{-k} + G'^{+}_{-k}G_{+k} \right) - G'^{+}_{12}G_{12} \right] + S_m,$$  \hspace{1cm} (10)$$
where the additional term $S_m$ gives an infinite mass to extra fields $B_k$ in $a \to 0$ limit:

$$S_m = -\frac{m^2(a)}{4g^2} \sum_{x_\perp} \int dx^+ \int dx^- \sum_k \text{Tr} \left[ (M^+_k(x)M_k(x) - I)^2 \right] \xrightarrow{a \to 0}$$

$$- \frac{m^2(a)}{4g^2} \int d^2x_\perp \int dx^+ \int dx^- \sum_k \text{Tr} \left( B^2_k \right), \quad m(a) \xrightarrow{a \to 0} \infty. \quad (11)$$

It is supposed that this leads to necessary decoupling of the fields $B_k$.

### 3. Canonical quantization on the Light Front

Let us fix the gauge as follows:

$$\partial_- A_- = 0, \quad A^i_j(x) = \delta^{ij}v^j(x_\perp, x^+). \quad (12)$$

For simplicity below we denote the argument of quantities, not depending on the $x^-$, again by $x$. Let us remark that starting with arbitrary field $A_\mu$, periodic in $x^-$, it is not possible to take zero modes of the $A_-$ equal to zero without a violation of the periodicity. But it is possible to make the $A_-$ diagonal as in the eq.-n (12) [16–19].

Then the action (10) can be written in the form:

$$S = a^2 \sum_{x^\perp} \int dx^+ \int dx^- \left\{ \sum_i \left[ 2F^{ii}_+(x)\partial_+v^i(x) \right] + \frac{1}{(ga)^2} \sum_{i,j} \sum_k \left[ D_-M^{ij}_k(x)\partial_+M^i_j(x) + H.c. \right] + \sum_{i,j} A^i_j(x)Q^{ji}(x) - \mathcal{H}(x) \right\}, \quad (13)$$

where

$$D_-M^{ij}_k(x) \equiv (\partial_- - igv^i(x) + igv^j(x - e_k))M^{ij}_k(x),$$

$$D_-M^{ij}_k^+(x) \equiv (\partial_- + igv^i(x) - igv^j(x - e_k))M^{ij}_k^+(x),$$

$$D_-F^{ij}_+(x) \equiv (\partial_- - igv^i(x) + igv^j(x))F^{ij}_+(x), \quad (14)$$

$A^i_j(x)$ play the role of Lagrange multipliers,

$$Q^{ji}(x) \equiv 2D_-F^{ji}_+(x) +$$

$$+ \frac{i}{ga^2} \sum_{j'} \sum_k \left[ M^{ij'}_k(x)D_-M^{j'i}_k(x) - M^{j'i}_k(x + e_k)D_-M^i_j(x + e_k) \right] = 0, \quad (15)$$

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are gauge constraints and
\[
\mathcal{H} = \sum_{ij} \left( F_{ij}^{ij+} F_{ij}^{ij-} + G_{1ij}^{ij+} G_{1ij}^{ij-} \right) + \mathcal{H}_m \tag{16}
\]
is the Hamiltonian density. The term \(\mathcal{H}_m\) can be obtained from the expression (11) in standard way.

The constraints can be resolved explicitly by expressing the \(F_{ij}^{ij-}\) in terms of other variables, but zero mode components \(F_{ii}^{ii-}(0)\) can not be found from constraint equations and play the role of independent canonical variables. Zero modes \(Q_{ii}^{ii-}(0)(x^-, x^+)\) of the constraints remain unresolved and are imposed as conditions on physical states:
\[
Q_{ii}^{ii-}(0)(x^-, x^+) |\Psi_{\text{phys}}\rangle = 0. \tag{17}
\]
In order to find a set of independent canonical variables we write Fourier transformation in \(x^-\) of the fields \(M_{ij}^{ij}(x)\) in the following form:
\[
M_{ij}^{ij}(x) = \frac{g}{\sqrt{4L}} \sum_{n=-\infty}^{\infty} \left\{ \Theta \left( p_n + gv^i(x) - gv^j(x - e_k) \right) M_{nk}^{ij}(x^+, x^-) + \Theta \left( -p_n - gv^i(x) + gv^j(x - e_k) \right) M_{nk}^{ij+}(x^-, x^+) \right\} \times \\
\times |p_n + gv^i(x) - gv^j(x - e_k)|^{-1/2} e^{-ip_n x^-}, \tag{18}
\]
where
\[
\Theta(p) = \begin{cases} 
1, & p > 0 \\
0, & p < 0 
\end{cases}, \quad p_n = \frac{\pi}{L} n, \quad n \in \mathbb{Z}. \tag{19}
\]
Due to the gauge (12) this Fourier transformation coincides with the expansion in eigen modes of the operator \(D_-\). Therefore the ultraviolet cutoff in these modes, which we shall apply, reduces to the following condition on the number of terms in the sum (18):
\[
|p_n + gv^i(x) - gv^j(x - e_k)| \leq \frac{\pi}{L} \bar{n}, \tag{20}
\]
where the \(\bar{n}\) is integer parameter of ultraviolet cutoff. Let us stress that this regularization is gauge invariant.

The action can be rewritten in the following form (up to nonessential surface terms):
\[
S = a^2 \sum_{x^\pm} \int dx^+ \left\{ \sum_i 4L F_{ij}^{ij-(0)} \partial_+ v^i + \right. \\
+ \frac{i}{a^2} \sum_{ij} \sum_k \sum_n' M_{nk}^{ij+} \partial_+ M_{nk}^{ij-} + 2L \sum_i A_{ij}^{ii-(0)} Q_{ii}^{ii} - \tilde{\mathcal{H}}(x) \right\}, \tag{21}
\]
where the $\sum'_n$ means that the sum is cut off by the condition (20), and the $\tilde{H}$ is obtained from the $H$ via the substitution of the expression

$$F^{ij}_{+-} = (F^{ij}_{+-} - \delta^{ij} F^{ii}_{+-}(0)) + \delta^{ij} F^{ii}_{+-}(0),$$

(22)

where the $F^{ij}_{+-} - \delta^{ij} F^{ii}_{+-}(0)$ are to be written in terms of $M^{ij}_{nk}, M^{ij+}_{nk}, v^i$ by solving the constraints (15) and using eq. (18). The $F^{ii}_{+-}(0)$ remain independent. The $G^{ij}_{12}$ are also to be expressed in terms of $M^{ij}_{nk}, M^{ij+}_{nk}, v^i$ via the eqns. (8), (18).

We have the following set of canonically conjugated pairs of independent variables:

$$\{v^i, \quad \Pi^i = 4L a^2 F^{ii}_{+-}(0)\},$$

$$\{M^{ij}_{nk}, \quad iM^{ij+}_{nk}\}.$$  

(23)

In quantum theory these variables become operators which satisfy usual canonical commutation relations:

$$[v^i(x), \Pi^j(x')]_{x^+=0} = \frac{i}{\Delta x} \delta^{ij} \delta_{x,x'},$$

$$[M^{ij}_{nk}(x), M^{ij+}_{nk'}(x')]_{x^+=0} = \delta^{ij} \delta^{ij'} \delta_{nn'} \delta_{kk'} \delta_{x,x'},$$

(24)

the other commutators being equal to zero.

Let us remark that the condition (12) does not fix the gauge completely. In particular, discrete group of gauge transformations, depending on the $x^-$, of the form

$$U^{ij}_{n}(x) = \delta^{ij} \exp \left\{ i \frac{\pi}{\Delta L} n^i(x^\perp) x^- \right\},$$

(25)

where $n^i(x^\perp)$ are integers, remains, and, of course, transformations, not depending on the $x^-$, are allowed. Under the transformations (25) canonical variables change as follows:

$$v^i(x) \rightarrow v^i(x) - \frac{\pi}{g L} n^i(x^\perp), \quad \Pi^i \rightarrow \Pi^i,$$

$$M^{ij}_{nk}(x^\perp) \rightarrow M^{ij}_{nk'}(x^\perp), \quad n' = n + n^i(x^\perp) - n^j(x^\perp - e_k).$$

(26)

Let us write the expression for quantum operators $Q^{ii}_{(0)}(x^\perp, x^+)$, which define the physical subspace of states. We fix the order of the operators in such a way as to relate with classical expression $G^\mu_\nu G^\nu_\mu$ a quantum one of the form:

$$\frac{1}{2} \left( G^\mu_\nu G^\nu_\mu + G^\mu_\mu G^\nu_\nu \right).$$

(27)
We remark that other choices of the ordering do not admit reasonable vacuum solution. Then the operators $Q_{(0)}^{ii}(x^+, x^+)$ have the following form in terms of canonical variables:

$$2LQ_{(0)}^{ii}(x^+, x^+) = -\frac{g}{4a^2} \sum_{j} \sum_{k} \sum_{n} \left[ \varepsilon (p_n + gv^j(x + e_k) - gv^i(x)) \times 
\times \left( M_{nk}^{ij+}(x + e_k)M_{nk}^{ij}(x + e_k) + M_{nk}^{ij}(x + e_k)M_{nk}^{ij+}(x + e_k) \right) - 
- \varepsilon (p_n + gv^i(x) - gv^j(x - e_k)) \left( M_{nk}^{ij+}(x)M_{nk}^{ij}(x) + M_{nk}^{ij}(x)M_{nk}^{ij+}(x) \right) \right] \right], \quad (28)$$

where

$$\varepsilon(p) = \begin{cases} 1, & p > 0 \\ -1, & p < 0 \end{cases} \quad \text{.} \quad (29)$$

One can easily construct canonical operator of translations in the $x^-$:

$$P_{-\text{can}} = \frac{1}{2} \sum_{x^+} \sum_{i,j} \sum_{k} \sum_{n} p_n \varepsilon (p_n + gv^i(x) - gv^j(x - e_k)) \times 
\times \left( M_{nk}^{ij+}(x)M_{nk}^{ij}(x) + M_{nk}^{ij}(x)M_{nk}^{ij+}(x) \right) \right], \quad (30)$$

This expression differs from the physical gauge invariant momentum operator $P_-$ by a term proportional to the constraint. The operator $P_-$ is

$$P_- = \frac{a^2}{2} \sum_{x^+} \sum_{k} \int_{-L}^{L} dx^- \text{Tr} \left( G_{-k}^{+}G_{-k} + G_{-k}G_{-k}^{+} \right) = P_{-\text{can}} + 4La^2 \sum_{x^+} \sum_{i} v^i Q_{(0)}^{ii} =$$

$$= \frac{1}{2} \sum_{x^+} \sum_{i,j} \sum_{k} \sum_{n} \left| p_n + gv^i(x) - gv^j(x - e_k) \right| \times 
\times \left( M_{nk}^{ij+}(x)M_{nk}^{ij}(x) + M_{nk}^{ij}(x)M_{nk}^{ij+}(x) \right) =$$

$$= \sum_{x^+} \sum_{i,j} \sum_{k} \sum_{n} \left| p_n + gv^i(x) - gv^j(x - e_k) \right| \left( M_{nk}^{ij+}(x)M_{nk}^{ij}(x) + \frac{1}{2} \right) \right]. \quad (31)$$

Let us choose a representation of the state space, in which the variables $v^i(x)$ are the multiplication operators. The states are described in this representation by normalizable functionals $F[v]$ of classical functions $v^i(x)$ (in fact by functions, depending on the values of the $v^i$ in different points $x^+$ due to the discreteness of these $x^+$). One can define full space of states as a direct product of the Fock space, in which the $M_{nk}^{ij+}(x)$ and $M_{nk}^{ij}(x)$ play the role of creation and annihilation operators, and the space of functionals $F[v]$. Let
us call $M$-vacuum the states of the form $|0\rangle \cdot F[v]$, where the $|0\rangle$ satisfies the condition

$$M_{nk}^{ij}(x)|0\rangle = 0, \quad (32)$$

and the $F$ is some functional. Arbitrary state can be represented in the form of linear combination of vectors $|\{m\}; F\rangle$ of the form

$$\prod_{x^\perp} \prod_i \prod_j \prod_k \prod_n' \left( M_{nk}^{ij}(x) \right)^{m_{nk}^{ij}(x)} |0\rangle \cdot F[v] \quad (33)$$

with different nonnegative integer functions $m_{nk}^{ij}(x^\perp)$ and functionals $F$. One can define orthonormalized set of such functionals if necessary.

One can see from (31) that the state, corresponding to the absolute minimum of the $P_-$ must satisfy the conditions (32), i.e. to be a $M$-vacuum. The value of the $P_-$ in this state can be written in the form

$$\langle 0; F | P_- | 0; F \rangle =$$

$$= \frac{1}{2} \int \prod_{x^\perp} \prod_i \prod_j \prod_k \prod_n' \left( M_{nk}^{ij}(x) \right)^{m_{nk}^{ij}(x)} |p_n + gv^i(x) - gv^j(x - e_k)| |F[v]|^2. \quad (34)$$

Remind that the $\prod_n'$ denotes the sum in $n$ limited by the condition (20). If one uses this condition and shifts the index $n$ in these sums by integer part of the quantities $(gL(v^i(x) - v^j(x - e_k))/\pi)$, one sees that the dependence on the $v^i$ cancels in sums over $n$ and that the expression (34) does not depend on the $F[v]$ if it is normalized to unity. Thus the momentum $P_-$ has the minimum in all $M$-vacua. One can make the value of the $P_-$ in these vacua equal to zero by subtracting corresponding constant from the operator $P_-.$

Let us show that $M$-vacua are the physical states, i.e. satisfy the condition (17). Indeed, in $M$-vacua this condition looks as follows:

$$\sum_j \sum_k \sum_n' \left[ \varepsilon \left( p_n + gv^j(x + e_k) - gv^i(x) \right) - \varepsilon \left( p_n + gv^i(x) - gv^j(x - e_k) \right) \right] F[v] = 0 \quad (35)$$

and is satisfied for any $F[v]$, because for every link in the sum (35) the numbers of positive and negative values of the $\varepsilon$-functions are equal. For arbitrary basic states (33) analogous conditions have the following form:

$$\sum_j \sum_k \sum_n' \left[ \varepsilon \left( p_n + gv^j(x + e_k) - gv^i(x) \right) m_{nk}^{ij}(x + e_k) - \varepsilon \left( p_n + gv^i(x) - gv^j(x - e_k) \right) m_{nk}^{ij}(x) \right] F[v] = 0. \quad (36)$$
The eigen values $p_-$ of the operator $P_-$ can be found from the equation

$$\sum_{x_+} \sum_{i,j} \sum_k \sum_n^\prime |p_n + g v^i(x) - g v^i(x - e_k)| m_{nk}^{ij}(x) F[v] = p_- F[v] \quad (37)$$

where the functional $F[v]$ is normalized.

To define physical vacuum state correctly one must consider not only states, corresponding to the minimum of the operator $P_-$, but also to the minimum of the operator $P_+$. One can try to do this via minimization of the $P_+$ on the $M$-vacua, i.e. on the set of states with $p_- = 0$. The expression $\langle 0 | P_+ | 0 \rangle$, where $|0\rangle$ is the Fock space vacuum w.r.t. the $M_{nk}$, $M_{nk}^+$, depends on the functions $v^i(x)$ (which enter, in particular, into the ”normal contractions” of the operators $M_k$, $M_k^+$) and on the operators $\Pi^i$, canonically conjugated to the $v^i(x)$. The expectation value of this expression is to be minimized on the set of functionals $F[v]$. The resulting functional $F[v]$ must decrease in the vicinity of those values of the $v^i(x)$, for which the operator $D_-$ has zero eigen value, because the Hamiltonian is singular at these values (it is seen, for example, from the expansion (18)).

The vacuum state constructed in such a way strongly differs from the usual vacuum in continuous space theory, because for $M$-vacuum we get zero expectation values of the operators $M_k$, but not of the operators $(M_k - I)/ga = B_k + iA_k$, related to usual gauge fields. Beside of this the condition of the unitarity of the matrices $M_k$ in the continuum limit (or equivalent condition of switching off the nonphysical fields $B_k$) cannot be fulfilled in such a vacuum. This disagreement with conventional theory is caused by the absence of explicit Lorentz invariance in our formulation, that leads to different quantum states, corresponding to the minima of the operators $P_-$ and $P_+$. It is not clear whether these states can be made coinciding at least in the limit of continuous space. This requires further investigation.

Nevertheless the method of the quantization of gauge theories on the LF, described here, can be useful for completely gauge-invariant formulation of some effective models, based on analogous formalism (but without complete gauge invariance due to throwing out of all zero modes of fields and due to the absence of gauge-invariant regularization of ultraviolet divergencies in the $p_-$). Such models are described, for example, in papers [27, 28], where the ideas of papers [29, 30] were developed.
4. Conclusion

In the present paper complete gauge-invariant regularization of gauge field theory in Hamiltonian approach on the LF is given with the help of lattice formalism. This regularization includes the limitation of the space in the $x^−$, $|x^−| \leq L$, the periodic boundary conditions for fields in $x^−$ and the two-dimensional lattice in transverse coordinates $x^⊥$. Also we introduced gauge-invariant regularization of ultraviolet divergencies in LF momentum $p^−$ via the cutoff in modes of covariant derivative $D_−$. Instead of transverse components of gauge vector fields we used complex matrix variables defined on lattice links. In these variables the action can be written in simple form, polynomial in the fields. In comparison with usual field variables in continuous space, these matrix variables contain a part, which reduces to some extra nonphysical fields in continuum limit. The extra fields are to be switched off via adding to the Hamiltonian a ”mass” term for these fields with the ”mass” fastly rising in continuum limit.

We found that in our canonical LF formalism there are no 2nd class constraints, connecting zero modes with other modes. This allowed simple quantization of the theory. The vacuum, defined with respect to the minimum of the operator $P_−$, turned out to be different from that of continuous space formulation, because the minimum of LF momentum $P_−$ does not correspond to the minimum of the operator $P_+$. This is due to the absence of explicit Lorentz invariance in our regularization scheme. However it remains open the question, whether a restoration of Lorentz invariance can be achieved in the limit of continuous space.

Remarkably, our method can be applied to effective semiphenomenological, ”color dielectric” type, models where Lorentz invariance can be partially restored via a modification of the LF Hamiltonian.

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