Physical and mathematical simulation of bridge segment oscillations

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Abstract. On the basis of known mathematical models describing vibrations in the gas flow of a bluff body with one degree of freedom, a model of vibrations of a body with two degrees of freedom is obtained. The Krylov–Bogolyubov method is applied. Equations for slowly varying amplitudes and phase shift of vibrations are obtained. It turned out that the differential equations written for the squares of dimensionless amplitudes of translational and rotational vibrations coincide with the well-known Lotka–Volterra equations describing competition between two species of animals that eat the same food. In a wind tunnel, the change of modes of vibrations of the bridge segment predicted by a mathematical model is studied.

1. Introduction
Bluff elastic or elastically fixed bodies can perform translational or rotational vibrations under the influence of the wind. The source of vibrations of long bodies could be periodically shedding chain of vortices, if the frequency of the vortices is close to the natural frequency of vibrations of the structure. In this paper, we consider oscillations called galloping [1]. We assume that the frequency of periodic vortex shedding is much higher than the natural frequency of structure vibrations. Therefore, the aerodynamic forces resulting from the vortices are averaged and do not affect the significantly slow oscillation process during galloping. In most works on bridge vibrations, instabilities leading to galloping were considered to determine the critical wind speed in accordance with the linear dynamic theory [2].

The quasi-steady model is proposed in [3] for description of the translational galloping of a bluff body. This model is based on the assumption that the aerodynamic forces acting on the body depend only on the relative velocity of the flow and on the angles describing the orientation of the body relative to the velocity vector of the air flow. For a transverse flow of air around a long body that oscillates in a direction perpendicular to the flow, the normal aerodynamic force acting in the direction of motion depends only on the instantaneous angle of attack \( \alpha \). The coefficient of normal aerodynamic force \( c_y \) depending on the angle of attack \( \alpha \) can be determined in a wind tunnel in experiments with a fixed body. In [3], the dependence \( c_y(\alpha) \) of the square prism was approximated by a fifth-order polynomial. Later [4] it is found that the model works better if we use a seventh-order polynomial approximation. The quasi-steady model was widely used later to describe the translational galloping of rectangular cylinders of various proportions [5], cylinders with a triangular cross-section [6], a diamond-shaped cross-section [7], and prisms of small aspect ratio [8].

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The model of vibrations of an elastically fixed body with two degrees of freedom was developed in [9] and [10]. The authors of these works made attempts to extend the quasi-steady model of galloping to the rotational vibrations of two bluff bodies: a square prism and an angular profile. The difficulty of applying these models is that different points of a rotating body have different velocities and, therefore, the instantaneous angles of attack of these points are different. It is impossible to choose a characteristic point to determine the instantaneous angle of attack.

2. Mathematical model of translational and rotational vibrations of a bluff body

Let the body be fixed on an elastic suspension. In a fixed reference frame, it can move along the axis Z, perpendicular to the velocity of the incoming flow \( \mathbf{v} \). The body is inclined at an angle \( \theta \) to the horizon. The relative velocity of the body center of mass is the sum of two perpendicular vectors. The first vector is directed along the X axis, its absolute value is \( v \). The second vector is the velocity of the body center of mass along the Z axis. Its value is \( \dot{z} \), where \( z \) is the coordinate of the center of mass of the body, the dot above the symbol here and further denotes differentiation with respect to time. The angle of attack \( \alpha \) (the angle between the axis X and the axis of the wind frame \( X_w \)) and the relative velocity of the body \( v_r \) will be determined by the formulas: \( \alpha = \theta - \arctan(\dot{z}/v) \), \( v_r = v \left( 1 + (\dot{z}/v)^2 \right)^{0.5} \approx v \left( 1 + 0.5 (\dot{z}/v)^2 \right) \). We consider that the angles of attack are small and use approximate expressions: \( \alpha = \theta - (\dot{z}/v) \), \( v_r = v \).

The coefficient \( c_z \) of the force acting along the Z axis is approximated by a polynomial of degree \( n \) with respect to the angle of attack \( \alpha \). We assume that the coefficient \( A_0 \) for \( \alpha^0 \) is zero, since only the average value of the coordinate \( z = z_0 \) depends on this term. The equation of motion is written as:

\[
m\ddot{z} + r_1 \dot{z} + k_1 z = (1/2)s\rho_0 v^2 \sum_{i=1}^{n} A_i \alpha^i,
\]

where \( m \) is body mass, \( r_1 \) is coefficient of friction in an elastic suspension, \( k_1 \) is reduced suspension stiffness, \( s \) is characteristic body area and \( \rho_0 \) is air density. Let’s introduce the notation:

\[
\mu_1 = -s\rho_0 b A_1/(2m), \quad \eta_1 = r_1/m, \quad \omega_1^2 = k_1/m, \quad \nu_1 = \eta_1 b/\mu_1,
\]

where \( b \) is the characteristic body size. Parameter \( \mu_1 \) is small.

For \( n = 3 \):

\[
\ddot{z} + \omega_1^2 z = \mu_1 \frac{v^2}{b} F \left( \frac{\dot{z}}{v}, \theta \right),
\]

\[
F = \frac{\dot{z}}{v} \left[ 1 - \frac{v_1}{v} + \frac{A_2}{A_1} \left( \frac{\dot{z}}{v} - 2\theta \right) + \frac{A_3}{A_1} \left( \frac{\dot{z}^2}{v^2} - 3 \frac{\dot{z}}{v} \theta + 3 \theta^2 \right) - \left( \theta - \frac{A_2}{A_1} \theta^2 + \frac{A_3}{A_1} \theta^3 \right) \right].
\]

If we kept a third-degree term \((\dot{z}/v)^3/3\) in the expression for \( \alpha = \theta - \arctan(\dot{z}/v) \) then it would be included in the expression for \( F \), in which the term \((A_3/A_1)(\dot{z}/v)^3\) is present. The experiment gives \( A_3/A_1 >> 1/3 \) (see Table 1 below). Thus, it is possible to neglect the term \((\dot{z}/v)^3/3\).

For the simulation of rotational vibrations, the model of vibrations of a cylinder of small aspect ratio [11] was taken as a basis. The equation of rotational vibrations of a body with the moment of inertia \( I_y \) around the axis \( Y \) passing through the center of mass is:

\[
I_y \ddot{\theta} + r_2 \dot{\theta} + k_2 \theta = s b^2 \rho_0 v^2 \frac{m_0}{2} m_{\dot{\alpha}} (1 - \delta \alpha^2) \dot{\alpha},
\]

where \( r_2 \) is the coefficient of friction in the suspension, \( k_2 \) is the stiffness of the elastic suspension, \( m_{\dot{\alpha}} \) is the rotational derivative of the pitch moment, \( \delta \) is a parameter that is determined in an experiment with rotational vibrations.
Using the notation

\[ \mu_2 = \frac{s \rho_0 b^3 m_\alpha}{(2 I_y)}, \quad \eta_2 = \frac{r_2}{I_y}, \quad \omega_\mu^2 = \frac{k_2}{I_y}, \quad v_2 = \eta_2 b / \mu_2, \]

we obtain the equation of rotational vibrations:

\[ \ddot{\theta} + \omega_\mu^2 \theta = \mu_2 \frac{v}{b} \left[ 1 - \frac{\delta}{v} \left( \frac{\dot{\zeta}}{v} \right)^2 + 2 \delta \frac{\dot{\zeta}}{v} \right] \left( \theta + \frac{\omega_\mu^2 \dot{\zeta}}{v} \right) - \mu_2 v_2 \dot{\theta} / b. \]

The equation for rotational oscillations contains a term that includes \( \dot{\alpha} \). Differentiating the expression \( \alpha = \theta - (\dot{z} / v) \) with respect to time gives \( \dot{\alpha} = \ddot{\theta} - \frac{\dot{z}^2}{v} \). The expression for the second derivative is found from equation (1): \( \ddot{z} = -\omega_\mu^2 z \). We neglect the right-hand side of the equation (1), because it contains small parameter \( \mu_1 \). Thus, in equation (2), we discard the second-order term of smallness with coefficient \( \mu_1 \mu_2 \).

The system of equations (1) and (2) describes two coupled oscillators. The nonlinear terms on the right-hand sides of the equations are multiplied by the small parameters \( \mu_1 \) and \( \mu_2 \). Therefore, the Krylov–Bogolyubov method can be used to solve the system [12]. We assume that the frequency ratio \( \omega_1 / \omega_2 \) is not equal to 1.

Let \( \rho_z, \rho_\theta, \varphi_1, \varphi_2 \) be the slowly varied amplitudes and phase shifts of oscillations:

\[ z = \rho_z \cos \psi_1, \quad \psi_1 = \omega_1 t + \varphi_1, \quad \theta = \rho_\theta \cos \psi_2, \quad \psi_2 = \omega_2 t + \varphi_2. \]

The Krylov-Bogolyubov method gives approximate equations:

\[ \dot{\rho}_z = \frac{\mu_1}{2} \frac{v}{b} \rho_z \left\{ 1 - \frac{v_1}{v} + \frac{3 A_3}{4 A_1} \left[ \left( \frac{\rho_\omega \omega_1}{v} \right)^2 + 2 \rho_\theta^2 \right] \right\}, \]

\[ \dot{\rho}_\theta = \frac{\mu_2}{2} \frac{v}{b} \rho_\theta \left\{ 1 - \frac{v_2}{v} - \frac{\delta}{4} \left[ \frac{\rho_\omega^2}{v} + 2 \left( \frac{\rho_\omega \omega_1}{v} \right)^2 \right] \right\}. \]

New notations are introduced: \( \rho_Z = \rho_z \omega_1 / v, \ c_1 = -3 A_3 / (4 A_1) \) and \( c_2 = \delta / 4, \ \rho_Z \) is the dimensionless amplitude of translational vibrations. It is convenient to switch from equations for amplitudes to equations for squares of amplitudes \( \zeta = \rho_Z^2 \) and \( \Theta = \rho_\theta^2 \):

\[ \dot{\zeta} = \frac{\mu_1}{2} \frac{v}{b} \zeta \left[ 1 - \frac{v_1}{v} - c_1 \left( \zeta + 2 \Theta \right) \right], \]

\[ \dot{\Theta} = \frac{\mu_2}{2} \frac{v}{b} \Theta \left[ 1 - \frac{v_2}{v} - c_2 \left( \Theta + 2 \zeta \right) \right]. \]

The equations coincide with the well-known Lotka–Volterra equations [13] obtained to describe the competition between two species of animals that eat the same food. In contrast to the original Lotka–Volterra equations, the coefficients in the resulting equations (3) and (4) depend on the velocity of the incoming flow. The square of the oscillation amplitude corresponds to the number of animals.

The system of equations (3) and (4) is symmetric with respect to the variables included in the system. Let us assume for certainty that \( v_1 < v_2 \). There are 4 steady solutions of these equations:

1. \( \zeta = 0, \ \Theta = 0 \). The solution is stable in the range \( v < v_1 \)
2. \( \zeta = (v - v_1) / (c_1 v), \ \Theta = 0 \). If \( c_2 / c_1 > 1 / 2 \) then solution is stable in the range \( v > v_1 \). If \( c_2 / c_1 < 1 / 2 \) then solution is stable in the range \( v_1 < v < v_3 \), where \( v_3 = v_2 + 2 c_2 (v_2 - v_1) / (c_1 - 2 c_2) \)
3. \( \zeta = 0, \ \Theta = (v - v_3) / (c_2 v) \). If \( c_2 / c_1 < 2 \) then solution is stable in the range \( v > v_4 \), where \( v_4 = v_2 + c_2 (v_2 - v_1) / (2 c_1 - c_2) \), else the solution is unstable.
4. $\zeta = \frac{[2c_1(v - v_2) - c_2(v - v_1)]/(3c_1c_2v)}{3c_1c_2v}$, $\Theta = \frac{[2c_2(v - v_1) - c_1(v - v_2)]/(3c_1c_2v)}{3c_1c_2v}$. Solution exists in the range $v_4 < v < v_3$. The solution is unstable.

Let us consider the sequence of mode changes at increasing and decreasing of the air flow velocity at $c_2/c_1 < 1/2$. While the velocity of the incoming flow does not exceed the critical value $v < v_1$, there are no oscillations. When the flow velocity increases from $v_1$ to $v_3$, there are translational oscillations. At $v > v_3$ the translational vibrations are replaced by rotational ones. At velocity decreasing the reverse transition to translational oscillations is at the flow rate $v = v_4 < v_3$. Thus, the model predicts hysteresis with increasing and decreasing air flow velocity.

3. Experiment

The mathematical model is tested in the wind tunnel of the Saint Petersburg University. The two models of the bridge segment differ in size and weight. Each model consists of three beams of circular cross-section. The beams are covered with decking. One model is made of wood. Its width $b = 100$ mm, length $L = 700$ mm. The other model is made of metal. Its width $b = 110$ mm, length $L = 780$ mm. All experiments are carried out in the presence of circular end plates with a diameter of 200 mm or 220 mm. The purpose of the end plates is to prevent the flow of air through the ends of the model. It was shown that in the case of modeling the rotational vibrations of a thick plate, the aspect ratio of the plate $L/b$ should be greater than five [14].

The coefficients $A_i$ are determined by measuring the drag and lift forces acting on the model. Drag and lift are measured with balances equipped the wire suspension. During this experiment the model is fixed motionlessly in the test section of the wind tunnel. The coefficients $A_i$ are presented in the Table 1.

| $\theta_0$ | $A_0$ | $A_1$ | $A_2$ | $A_3$ |
|------------|-------|-------|-------|-------|
| 0.08       | 0.278 | -0.45 | -34.5 | 264   |
| 0.09       | 0.268 | -1.06 | -26.0 | 296   |
| 0.10       | 0.253 | -1.50 | -17.0 | 303   |
| 0.11       | 0.234 | -1.76 | -8.06 | 286   |
| 0.12       | 0.213 | -1.84 | 0.01  | 248   |
| 0.13       | 0.193 | -1.78 | 6.64  | 192   |
| 0.14       | 0.173 | -1.60 | 11.4  | 122   |
| 0.15       | 0.155 | -1.35 | 13.9  | 46    |

The parameter $\delta$ is determined in an experiment with the rotational vibrations of a model of a bridge segment made of metal. The model could rotate around an axis perpendicular to the velocity of the incoming flow only. The axis pass through the plane of symmetry of the bridge. The tail holder is attached to the springs. Thus, the model performed rotational oscillations with a constant amplitude in the air flow. The amplitude of the rotational oscillations is determined by the strain gauge method. The method of the experiment is described in the paper [14].

The parameter $\delta$ is determined by the dependence of the square of the amplitude of the steady-state oscillations on the Strouhal number $Sh = fb/v$, where $f$ is the frequency of oscillations. This dependence is approximated by a linear function $\rho_0^2 = a + b Sh$, $\delta = 4a$. Figure 1 shows experimental dependence for several equilibrium angles $\theta_0$.

In the next experiments we simulate both translational and rotational oscillations. The scheme of experiment is presented in Fig. 2. The model could move with six degrees of freedom,
but only translational oscillations along the vertical axis $Z$ and rotational oscillations around the axis $Y$ were observed in the experiment.

Two semiconductor strain gauges register the tension of the two lower springs. The Velleman-PCS500 PC oscilloscope converts the analog output signals of the strain gauges into digital signals and transmits them to the control computer. The reading frequency is 1250 Hz. The duration of recording the readings is 3.3 seconds.

The calibration procedure and subsequent processing of the results allow us to link the amplitudes of the output voltage vibrations with the amplitudes of the rotational and translational vibrations of the bridge segment.

The steady-state oscillations of the bridge segment occurred at small positive values of the angle $\theta_0$. The results of measuring the oscillation amplitudes are presented in Fig. 3 and Fig. 4. At low air velocity, translational oscillations occur with a frequency of 2 Hz in the vertical direction. In accordance with the mathematical model, when the flow velocity exceeds a certain critical value, a transition to rotational oscillations can occur. When we introduce a thin rod into the test section and stop the translational oscillations the rotational vibrations arise and persist when the rod is removed. The frequency of translational oscillations is much less than the frequency of rotational oscillations of 5 Hz. The ranges of existence of the two oscillation modes overlap.

Figure 3 shows the dependence of the square of the amplitudes on $1/v$. The relationship is close to linear one, as the mathematical model predicts. It is possible to find parameters $c_1 = 65.1$ and $c_2 = 3.1$.

The dependences of the dimensionless amplitudes of translational $\rho_Z$ and rotational $\rho_\theta$ oscillations on the air velocity are shown in Fig. 4. In addition, Fig. 4 shows the result of numerical calculation of equations (1) and (2) by the Runge-Kutta method. The calculation assumed that $\delta = 12.4$ and $3A_3/(4A_1) = -57.0$ The last parameter corresponds to the angle $\theta_0 = 0.14$. The main difference between the calculated and experimental results is the difference between the values of the hysteresis ranges. The differences between the experiment and the calculation can be explained by the fact that under the influence of aerodynamic forces,
angle of inclination $\theta$ depends on the velocity of the incoming flow. The mathematical model is also imperfect, especially in terms of rotational oscillations.

4. Conclusions
A mathematical model is proposed that describes the translational and rotational oscillations of a bridge segment in a gas flow. The well-known quasi-stationary model is taken as the basis for the model of translational galloping, and the model of oscillations of a cylinder of small aspect ratio is taken as the basis for modeling rotational oscillations. It is taken into account that the instantaneous angle of attack consists of the inclination of the bridge segment and the angle whose tangent is equal to the ratio of the vertical velocity of the body to the horizontal velocity of the segment. The equations of motion of the bridge segment are reduced to differential equations that coincide with the Lotka–Volterra equations, originally obtained to describe the competition of two species of animals that eat the same food. The model’s predictions are confirmed in a wind tunnel experiment. The model qualitatively describes the change of oscillation modes, including the hysteresis associated with an increase and subsequent decrease in the air flow velocity.

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