Sparse least squares solutions of multilinear equations

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ABSTRACT
In this paper, we propose a sparse least squares (SLS) optimization model for solving multilinear equations, in which the sparsity constraint on the solutions can effectively reduce storage and computation costs. By employing variational properties of the sparsity set, along with differentiation properties of the objective function in the SLS model, the first-order optimality conditions are analysed in terms of the stationary points. Based on the equivalent characterization of the stationary points, we propose the Newton Hard-Threshold Pursuit (NHTP) algorithm and establish its locally quadratic convergence under some regularity conditions. Numerical experiments conducted on simulated datasets including cases of Completely Positive(CP)-tensors and symmetric strong M-tensors illustrate the efficiency of our proposed NHTP method.

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1. Introduction
Multilinear equations (also known as tensor equations) have a wide range of applications in engineering and scientific computing such as data mining, numerical partial differential equations, tensor complementarity problems and high-dimensional statistics [1]. A recent line of research has been focused on numerical algorithms for solving multilinear systems with various coefficient tensors, see, e.g. CP-tensors [1], strong M-tensors [2–6] and other structured tensors [7–11]. Furthermore, as the data dimension grows in practical problems, the sparsity constraint on the solutions turns to be a reasonable choice to effectively alleviate the ‘curse of dimensionality’ in systems of multilinear equations.

As the direct and accurate characterization of the entry-wise sparsity in vectors, the so-called $\ell_0$-norm (i.e. the number of non-zero entries in the vector) is non-convex and discontinuous. Thus, the problem of finding sparse solutions of multilinear equations in the sense of least squares is generally NP-hard in computational complexity. Little can be found in this research direction, except the work [12] in which the strong M-tensor coefficient and non-negative right-hand side vector are considered. In this special setting, they showed that the sparsest solution to the corresponding multilinear equations can be obtained by solving the $\ell_1$-norm (i.e. the sum of absolute values of all entries) relaxation problem. However, for general multilinear equations with sparsity constraint, little can be found to our best

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knowledge. This motivates us to consider the sparse least squares (SLS) optimization model for solving multilinear equations.

The SLS model is actually a special case of cardinality constrained optimization (CCO) problems. Existing algorithms for CCO can be roughly classified into two categories. The first category is the ‘relaxation’ method by using continuous and/or convex surrogates of the involved $\ell_0$-norm, see. e.g. [13–16]. The other category is the ‘greedy’ type method by tackling the involved $\ell_0$-norm directly. Typical algorithms include the matching pursuit algorithm [17], the iterative projection algorithm [18–21], the hard threshold pursuit algorithm [22], just name a few. Particularly, some second-order methods are proposed. For example, Yuan et al. [23] and Bahmani et al. [24] realized that restricted Newton steps can be employed in the underlying subproblems in subspaces to improve the algorithm performance. Recently, Zhou et al. [25] proposed a Newton Hard Thresholding Pursuit (NHTP) algorithm with quadratic convergence rate under some regularity assumptions. Its superior numerical performance and theoretical convergence inspire us to develop NHTP for solving the SLS model.

The main contributions of this paper are three-fold. First, the SLS model is proposed to solve multilinear equations with the cardinality constraint. Second, the optimality of the SLS model, and the regularity properties of the objective functions are elaborated, which serve as the crucial theoretical guarantees for designing the Newton-type algorithm in the sequel. Third, the NHTP is developed for solving SLS and its locally quadratic convergence is established.

The remainder of the paper is organized as follows. In Section 2, we review the related tensor basics and notations. In Section 3, we analyse the first-order optimality conditions for the proposed SLS optimization model and regularity properties of the objective function in theory, and then design a quadratically convergent NHTP in algorithm. In Section 4, we report some numerical results to verify the proposed NHTP algorithm, by comparing with the existing homotopy algorithm. Concluding remarks are drawn in Section 5.

2. Preliminaries

For any positive integer $n$, denote $[n] := \{1, 2, \ldots, n\}$, $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq 0\}$. We denote $\mathcal{A} = (a_{i_1i_2 \cdots i_m})$ as an $m$th order $n$-dimensional tensor with $i_j \in [n], j \in [m]$, and then the linear space of all $m$th order $n$-dimensional tensors as $\mathbb{R}^{[m,n]} := \mathbb{R}^{n \times n \times \cdots \times n}$. For any $\mathcal{A} \in \mathbb{R}^{[m,n]}$, we call $\mathcal{A}$ a symmetric tensor if its entries remain unchanged under any permutation of the indices and denote the set of all symmetric tensors in $\mathbb{R}^{[m,n]}$ as $\mathbb{S}^{[m,n]}$. If there exists a positive integer $r$ and $u^{(k)} \in \mathbb{R}_+^n$ such that $\mathcal{A} = \sum_{k=1}^r (u^{(k)})^m$, where $(u^{(k)})^m = (u^{(k)}_{i_1} \cdots u^{(k)}_{i_m})$, then $\mathcal{A}$ is called a Completely Positive tensor (CP-tensor). Furthermore, if $\text{Span}\{u^{(1)}, u^{(2)}, \ldots, u^{(r)}\} = \mathbb{R}^n$, then $\mathcal{A}$ is called a Strong Completely Positive tensor (SCP-tensor). We denote $\mathbb{C}P^{[m,n]}$ and $\mathbb{S}C{P}^{[m,n]}$ as the sets of all $m$th order $n$-dimensional CP-tensors and SCP-tensors, respectively.

Given $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $b \in \mathbb{R}^n$, multilinear equations can be expressed as $\mathcal{A} x^{m-1} = b$ with $(\mathcal{A} x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_1i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$ for $i \in [n]$. For any $d \in [m-1]$, $\mathcal{A} x^{m-d} = \mathbb{R}^{[d,n]}$ with entries $(\mathcal{A} x^{m-d})_{i_1 \cdots i_d} = \sum_{i_{d+1}, \ldots, i_m=1}^n a_{i_1 \cdots i_d i_{d+1} \cdots i_m} x_{i_{d+1}} \cdots x_{i_m}$ for $i_j \in [n], j \in [d]$. We call $\lambda$ an eigenvalue of $\mathcal{A}$ and $x$ a corresponding eigenvector if $x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in \mathbb{R}$ satisfy $\mathcal{A} x^{m-1} = \lambda x^{m-1}$ with $x^{m-1} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^\top \in \mathbb{R}^n$ [26]. The
Table 1. A list of notation.

| Notation | Definition |
|----------|------------|
| \text{supp}(x) | := \{i \in [n] : x_i \neq 0\} the support set of x. |
| \|x\|_0 | := \ell\{i \in [n] : x_i \neq 0\} is the l_0-norm of x |
| x_{(i)} | the i-th largest element of x. |
| \Gamma^* | := \text{supp}(x^*) |
| T | the subset of \{1, 2, \ldots, n\}. |
| | the cardinality of T. |
| T^c | the complementary set of T. |
| x_T | the sub vector of x containing elements indexed on T. |
| \nabla_T f(x) | := (\nabla f(x))_T |
| \nabla^2_T f(x) | := \nabla^2 f(x)_T |
| \nabla^2 f(x) | := \nabla^2 f(x) |
| \|x\| | the submatrix of the Hessian matrix with rows are indexed by T. |
| \|x\|_F | the cardinality of f. |
| \|A\| | the Euclidean norm of the vector x. |
| \|A\|_F | the Frobenius norm of the tensor A \in \mathbb{R}^{[m,n]}, i.e. \|A\|_F := \sqrt{\sum_{i,j=1}^n (a_{ij})^2}. |
| J(x) | := \{j \in [n] : |j| = s, \text{supp}(x) \subseteq J\}. |
| Q_2(s) | := \{|T| \subseteq [n] : |T| \leq 2s, \text{supp}(x) \subseteq T\}. |

The spectral radius of A is defined by \( \rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} \) [27]. Let \( I \in \mathbb{R}^{[m,n]} \) be an identity tensor (i.e. the diagonal entries are 1, and the other entries are 0), a tensor A is called an M-tensor if there exists a non-negative tensor B and a positive real number \( s \geq \rho(B) \) such that \( A = sI - B \); if \( s > \rho(B) \), A is a strong M-tensor. For convenience, notations that will be used in the paper are listed in Table 1.

3. Main results

In this section, we consider the following sparse least squares (SLS) optimization problem of multilinear equations:

\[
\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2} \|Ax^{m-1} - b\|^2, \quad \text{s.t. } \|x\|_0 \leq s, \tag{1}
\]

where \( A \in S^{[m,n]} \), the integer \( s \in [1, n] \) is a prescribed upper bound that controls the sparsity of x. Note that f is differentiable of any order. By virtue of the symmetry of the tensor A, the gradient and the Hessian matrix of f at any \( x \in \mathbb{R}^n \) take the forms of

\[
\nabla f(x) = (m - 1)Ax^{m-2}(Ax^{m-1} - b), \tag{2}
\]

\[
\nabla^2 f(x) = (m - 1)(m - 2)Ax^{m-3}(Ax^{m-1} - b) + (m - 1)^2(Ax^{m-2})^2. \tag{3}
\]

3.1. Optimality analysis

This subsection is dedicated to the discussion of several differential properties of the objective function f and the optimality conditions for problem (1), all of which will provide theoretical guarantees for the design of the Newton-type algorithm in next subsection.

**Definition 3.1** ([25]): Suppose that \( f : \mathbb{R}^n \mapsto \mathbb{R} \) is a twice continuously differentiable function. Let \( M_{2s}(x) := \sup_{y \in \mathbb{R}^n} \{ \langle y, \nabla^2 f(x)y \rangle : \text{supp}(x) \cup \text{supp}(y) \leq 2s, \|y\| = 1 \} \) and \( m_{2s}(x) := \inf_{y \in \mathbb{R}^n} \{ \langle y, \nabla^2 f(x)y \rangle : \text{supp}(x) \cup \text{supp}(y) \leq 2s, \|y\| = 1 \} \) for any s-sparse vector x.
(1) We say $f$ is $M_{2s}$-RSS if there exists a constant $M_{2s} > 0$ such that $M_{2s}(x) \leq M_{2s}$.
(2) We say $f$ is $m_{2s}$-RSC if there exists a constant $m_{2s} > 0$ such that $m_{2s}(x) \geq m_{2s}$.
(3) We say $f$ is locally restricted Hessian Lipschitz continuous at $x$ if there exists a constant $L_f > 0$ and a neighbourhood $N(x) := \{ z \in \mathbb{R}^n : \text{supp}(x) \subseteq \text{supp}(z), \| z \|_0 \leq s \}$ such that
\[
\| \nabla^2_f(y) - \nabla^2_f(z) \| \leq L_f \| y - z \|, \quad \forall y, z \in N(x)
\]
for any index set $T$ satisfying $|T| \leq s$ and $T \supseteq \text{supp}(x)$.

**Theorem 3.2:** Suppose that $x^*$ is an optimal solution of (1). There exists $\delta_0 > 0$ such that for any $x \in N(x^*, \delta_0) := \{ x \in \mathbb{R}^n : \text{supp}(x^*) \subseteq \text{supp}(x), \| x \|_0 \leq s, \| x - x^* \| < \delta_0 \}$, $f$ is locally restricted Hessian Lipschitz continuous at $x^*$, and the Lipschitz constant is given by
\[
L_f = 2(m - 1)(m - 2)(2m - 3)\| A \|_F^2(\| x^* \| + \delta_0)^{2m-5} + (m - 1)(m - 2)(m - 3)\| b \| \| A \|_F^2(\| x^* \| + \delta_0)^{m-4}.
\]

**Proof:** For any $x, y \in N(x^*, \delta_0)$, we have $\| x \| \leq \| x^* \| + \delta_0$, $\| y \| \leq \| y^* \| + \delta_0$, then
\[
\| \nabla^2_f(x) - \nabla^2_f(y) \| = \| (m - 1)(m - 2)Ax^{m-3}(Ax^{m-1} - b) + (m - 1)^2(Ax^{m-2})^2
\]
\[
- (m - 1)(m - 2)Ay^{m-3}(Ay^{m-1} - b) + (m - 1)^2(Ay^{m-2})^2 \| \leq (m - 1)(m - 2)\| Ax^{m-3}(Ax^{m-1} - b) - Ay^{m-3}(Ay^{m-1} - b) \| 
\]
\[
+ (m - 1)^2(\| Ax^{m-1} \|^2 - (Ay^{m-2})^2).\]

For simplicity, denote
\[
c1 := \| Ax^{m-3}(Ax^{m-1} - b) - Ay^{m-3}(Ay^{m-1} - b) \| 
\]
\[
\leq \| Ax^{m-3}Ax^{m-1} - Ay^{m-3}Ay^{m-1} \| + \| Ax^{m-3} - Ay^{m-3} \|_F \| b \| 
\]
\[
\leq \| Ax^{m-3} \|_F \| Ax^{m-1} - Ay^{m-1} \| + \| Ay^{m-1} \| \| Ax^{m-3} - Ay^{m-3} \|_F 
\]
\[
+ \| Ax^{m-3} - Ay^{m-3} \|_F \| b \|.\]
\[
c2 := (\| Ax^{m-2} \|^2 - (Ay^{m-2})^2) 
\]
\[
= \| (Ax^{m-2})(Ax^{m-2}) - (Ax^{m-2})(Ay^{m-2}) + (Ax^{m-2})(Ay^{m-2}) 
\]
\[
- (Ay^{m-2})(Ay^{m-2}) \| 
\]
\[
\leq \| Ax^{m-2} \| \| Ax^{m-2} - Ay^{m-2} \| + \| Ay^{m-2} \| \| Ax^{m-2} - Ay^{m-2} \|.\]

We can see that
\[
\| Ax^{m-1} - Ay^{m-1} \|
\]
\[
= \| Ax \cdots x - Ax \cdots y + Ax \cdots y + \cdots + Ay \cdots y - Ay \cdots y \|
\]
\[
\leq \| Ax \cdots (x - y) \| + \| Ax \cdots (x - y)y \| + \cdots + \| A(x - y)y \cdots y \|
\]
\[
\leq (\| A \|_F \| x \|^{m-2} + \| A \|_F \| x \|^{m-3} \| y \| + \cdots + \| A \|_F \| y \|^{m-2}) \| x - y \|
\]
\[
\leq (m - 1)\| A \|_F(\| x^* \| + \delta_0)^{m-2} \| x - y \|.\]
Similarly, one has
\[
\|Ax^{m-2} - Ay^{m-2}\| \leq (m - 2)\|A\|_F(\|x^*\| + \delta_0)^{m-3}\|x - y\|,
\]
\[
\|Ax^{m-3} - Ay^{m-3}\| \leq (m - 3)\|A\|_F(\|x^*\| + \delta_0)^{m-4}\|x - y\|.
\]

Plugging the above three inequalities into c1 and c2 yields
\[
c1 \leq 2(m - 2)\|A\|_F^2(\|x^*\| + \delta_0)^{2m-5}\|x - y\|
+ (m - 3)\|b\|\|A\|_F(\|x^*\| + \delta_0)^{m-4}\|x - y\|,
\]
\[
c2 \leq 2(m - 2)\|A\|_F^2(\|x^*\| + \delta_0)^{2m-5}\|x - y\|.
\]

Direct manipulations lead to
\[
\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq (m - 1)(m - 2)c1 + (m - 1)^2c2 = L_f\|x - y\|,
\]
where \(L_f\) is given in (4). This completes the proof. \(\blacksquare\)

With the similar proof skills, we have the following property.

**Theorem 3.3:** Suppose that \(x^*\) is an optimal solution of (1). There exists \(\delta_1 > 0\) such that for any \(x \in N_3(x^*, \delta_1)\), \(f(x)\) is strongly smooth, and its strong smoothness constant is given by
\[
M_{2s} = (m - 1)(2m - 3)\|A\|_F^2(\|x^*\| + \delta_1)^{2m-4}
+ (m - 1)(m - 2)\|b\|\|A\|_F(\|x^*\| + \delta_1)^{m-3}.
\]

Next we discuss RSC of \(f\) under the following assumption.

**Assumption 3.1:** For any \(T \in Q_{2s}(x^*)\), \(\nabla^2_T f(x^*)\) is positive definite.

Observe that problem (1) reduces to the Compressed Sensing (CS) problem when \(m = 2\), and in this circumstance, Assumption 3.1 holds iff \(A\) is 2s-regular \([18]\). For illustration purpose, we give the following example in which Assumption 3.1 holds with \(m > 2\).

**Example 3.4:** Let \(u_1 = ((-1)^m, 1, \ldots, 1)^\top \in \mathbb{R}^n\), \(u_2 = ((-1)^{m-1}, 1, \ldots, 1)^\top \in \mathbb{R}^n\), \(A = u_1^m + u_2^m \in S^{m,n}\), \(b = u_1 + (-1)^{m-1}u_2 \in \mathbb{R}^n\), \(s = 1\), where \(m > 2\) is a given positive
integer. It is easy to verify that $x^* = e_1 \in \mathbb{R}^n$ is an optimal solution of problem (1), since

$$A(x^*)^{m-1} = (u_1^T x^*)^{m-1} u_1 + (u_2^T x^*)^{m-1} u_2 = u_1 + (-1)^{m-1} u_2 = b,$$

and $\|x^*\|_0 = s = 1$. Now we claim that Assumption 3.1 holds at $x^*$ in two cases. Note that

$$A(x^*)^{m-2} = (u_1^T x^*)^{m-2} u_1 u_1^T + (u_2^T x^*)^{m-2} u_2 u_2^T = (-1)^m u_1 u_1^T + u_2 u_2^T.$$ 

**Case I:** $m$ is a positive even integer. It follows that

$$A(x^*)^{m-2} = u_1 u_1^T + u_2 u_2^T = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2 & \cdots & 2 \end{bmatrix},$$

Thus, for any $T \in Q_{2s}(x^*) = \{\{1\}, \{1, 2\}, \{1, 3\} \cdots \{1, n\}\}$, we have

$$0 < \nabla^2_T f(x^*) = \begin{cases} 4 (m - 1)^2, & \text{if } T = \{1\}, \\ (m - 1)^2 \begin{bmatrix} 4 & 0 \\ 0 & 4(n - 1) \end{bmatrix}, & \text{if } T \in \{\{1, 2\}, \ldots, \{1, n\}\}. \end{cases}$$

**Case II:** $m$ is a positive odd integer. Note that

$$A(x^*)^{m-2} = -u_1 u_1^T + u_2 u_2^T = \begin{bmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, for any $T \in Q_{2s}(x^*) = \{\{1\}, \{1, 2\}, \{1, 3\} \cdots \{1, n\}\}$,

$$0 < \nabla^2_T f(x^*) = \begin{cases} 4 (n - 1) (m - 1)^2, & \text{if } T = \{1\}, \\ (m - 1)^2 \begin{bmatrix} 4(n - 1) & 0 \\ 0 & 4 \end{bmatrix}, & \text{if } T \in \{\{1, 2\}, \ldots, \{1, n\}\}. \end{cases}$$

Under Assumption 3.1, the desired RSC is achieved as stated in the following theorem.

**Theorem 3.5:** Suppose that $x^*$ is an optimal solution of (1) and Assumption 3.1 holds. There exists $\delta_0 > \delta_2 > 0$ and $m_{2s} > 0$ such that $\forall x \in \mathcal{N}_s(x^*, \delta_2), f(x)$ is RSC, that is

$$m_{2s} I \preceq \nabla^2_T f(x), \quad \forall T \in Q_{2s}(x^*),$$

where $I$ is the identity matrix.

**Proof:** It is known from Theorem 3.2 that $\nabla^2 f(x)$ is Lipschitz continuous for $x \in \mathcal{N}_s(x^*, \delta_0)$, thus for any $T \in Q_{2s}(x^*)$ and $x \in \mathcal{N}_s(x^*, \delta_2)$, $\nabla^2_T f(x)$ is Lipschitz continuous. Since $\nabla^2_T f(x^*)$ is positive definite, there exists $\delta_T > 0$ such that $\forall x \in \mathcal{N}_s(x^*, \delta_T)$, $\nabla^2_T f(x)$ is also positive definite, and $m_T I \preceq \nabla^2_T f(x)$ with $m_T = \min\{\lambda_{\min}(\nabla^2_T f(x)) | x \in \mathcal{N}_s(x^*, \delta_T)\}$. 

Set \( \delta_2 := \min\{\delta_0, \{\delta_T\}_{T \in \mathcal{Q}_2(s^*)}\} \) and \( m_{2s} := \min_{T \in \mathcal{Q}_2(s^*)} \{m_T\} \), therefore we have that \( \nabla^2_T f(x) \) is positive definite and then \( m_{2s} I \leq \nabla^2_T f(x) \) holds. \( \blacksquare \)

**Definition 3.6:** Given \( \eta > 0 \) and \( x^* \in \mathbb{R}^n \), we say \( x^* \) is an \( \eta \)-stationary point of (1) if

\[
x^* \in \mathcal{P}_s(x^* - \eta \nabla f(x^*)),
\]

where \( \mathcal{P}_s(x) := \arg\min_z \{\|x - z\| : \|z\|_0 \leq s\} \) is the sparse projection operator.

Using the definition of \( \mathcal{P}_s \) and Lemma 2.2 in [18], the equivalent definition of the \( \eta \)-stationary point is given in the following lemma.

**Lemma 3.7:** The \( s \)-sparse vector \( x^* \) is an \( \eta \)-stationary point of (1) if and only if

\[
\begin{cases}
    A_{\cdot, \Gamma^*, \ldots, \Gamma^*} (x^*)^{m-2} (A_{\cdot, \Gamma^*, \ldots, \Gamma^*} (x^*)^{m-1} - b) = 0, & \text{if } \|x^*\|_0 < s, \\
    \|A_{\cdot, \Gamma^*, \ldots, \Gamma^*} (x^*)^{m-2} (A_{\cdot, \Gamma^*, \ldots, \Gamma^*} (x^*)^{m-1} - b)\|_{\infty} \leq \frac{|x^*|_{(i)}}{\eta}, & \text{if } \|x^*\|_0 = s.
\end{cases}
\]

The following first-order necessary optimality condition of problem (1) can be obtained according to [18, Theorem 2.2].

**Theorem 3.8:** Let \( \eta < 1/M_{2s} \), with \( M_{2s} \) defined in Theorem 3.3. If \( x^* \) is an optimal solution of (1), then the following fixed point equation holds, that is,

\[
x^* = \mathcal{P}_s(x^* - \eta \nabla f(x^*)).
\]

In order to deal with the non-differentiability of \( \mathcal{P}_s \), we follow the reformulation scheme from [25] and rewrite the optimality condition of (6) as follows.

**Theorem 3.9:** Given \( \eta > 0 \), \( x^* \) satisfies the fixed point Equation (6) if and only if

\[
F_\eta(x^*; T) := \left[ \frac{\nabla_T f(x^*)}{x^*_{T^c}} \right] = \left[ \frac{((m - 1)A(x^*)^{m-2}(A(x^*)^{m-1} - b))_T}{x^*_{T^c}} \right] = 0
\]

for any index set \( T \in T(x^*; \eta) \), where

\[
T(x^*; \eta) := \{ T \subseteq [n] : |T| = s, |u^*_i| \geq |u^*_j|, \forall i \in T, \forall j \in T^c \}
\]

with \( u^* := x^* - \eta \nabla f(x^*) \). Moreover, \( T(x^*; \eta) = J_s(x^*) \).

**Proof:** The first part follows readily from [25, Lemma 4], and the ‘moreover’ part follows from [28, Theorem 3]. \( \blacksquare \)
To solve the smooth equation system \((7)\) for a given index set \(T \in \mathcal{T}(x^*; \eta)\), we next investigate the non-singularity of the Jacobian matrix \(\nabla F_\eta(x; T)\) in a neighbourhood of \(x^*\), where

\[
\nabla F_\eta(x; T) = \begin{bmatrix} \nabla^2_{f}(x) & \nabla^2_{f, T}(x) \\ 0 & I_{n-s} \end{bmatrix}.
\]

Apparently, the non-singularity of \(\nabla F_\eta(x; T)\) is equivalent to that of \(\nabla^2_{f}(x)\). Thus, it suffices to show that \(\nabla^2_{f}(x)\) is non-singular for any \(T \in \mathcal{T}(x; \eta)\) when \(x\) is sufficiently close to \(x^*\). Let \(x^*\) be an optimal solution of (1). Set

\[
\delta_3 := \min_{k \in \Gamma^*} \{ \|x_k^*\| - \eta \|\nabla(\Gamma^*)f(x^*)\|_\infty \}/(\sqrt{2}(1 + \beta M_{2s})).
\]

It follows from Lemma 3.7 that \(\delta_3 > 0\).

**Lemma 3.10:** Let \(x^*\) be an optimal solution of (1) and \(\delta^* := \min\{\delta_1, \delta_2, \delta_3\}\). Then for any \(x \in \mathcal{N}_s(x^*, \delta^*)\), we have

\[
\mathcal{T}(x; \eta) \subseteq \mathcal{T}(x^*; \eta) \quad \text{and} \quad \Gamma^* \subseteq \text{supp}(x) \cap T, \quad \forall T \in \mathcal{T}(x; \eta).
\]

In particular, if \(\|x^*\|_0 = s\), \(\{\Gamma^*\} = \{\text{supp}(x)\} = \mathcal{T}(x; \eta) = \mathcal{T}(x^*; \eta)\).

**Proof:** Denote \(\Gamma = \text{supp}(x)\), \(q = \nabla f(x)\). For any \(x \in \mathcal{N}_s(x^*, \delta^*)\), we know that \(\nabla f(x)\) is Lipschitz continuous from the proof in Theorem 3.3, that is, \(\|q - q^*\| \leq M_{2s}\|x - x^*\|\) with \(M_{2s}\) given in (5). Together with Lemma 2 of [28], we can obtain the desired assertions.

**Theorem 3.11:** Let \(x^*\) be an optimal solution of (1) and Assumption 3.1 holds. Then for any \(x \in \mathcal{N}_s(x^*, \delta^*)\) and \(T \in \mathcal{T}(x; \eta)\), \(\nabla^2_{f}(x)\) is positive definite.

**Proof:** According to Lemma 3.10 and Theorem 3.9, we know for any \(x \in \mathcal{N}_s(x^*, \delta^*)\), \(\mathcal{T}(x; \eta) \subseteq \mathcal{T}(x^*; \eta) = \mathcal{J}_s(x^*) \subseteq Q_{2s}(x^*)\). Consequently, Theorem 3.5 implies that \(\nabla^2_{f}(x)\) is positive definite for any \(x \in \mathcal{N}_s(x^*, \delta^*)\) and \(T \in \mathcal{T}(x; \eta)\).

### 3.2. NHTP algorithm

In this subsection, we apply the Newton Hard-Threshold Pursuit (NHTP) algorithm for problem (1), and analyse the locally quadratic convergence of the algorithm.

Given \(\eta > 0\), let \(x^k\) be the current iteration point. NHTP firstly chooses one index set \(T_k \in \mathcal{T}(x^k; \eta)\), and then does the Newton step for \(F_\eta(x; T_k)\). Specifically, take the following form for the nonlinear equation \(F_\eta(x; T_k) = 0\) to get the next iteration \(\hat{x}^{k+1}\):

\[
\nabla F_\eta(x^k; T_k)(\hat{x}^{k+1} - x^k) = -F_\eta(x^k; T_k).
\]

Denote the Newton direction at the \(k\)th iteration by \(d^k_N := \hat{x}^{k+1} - x^k\). Substituting it into the above formula yields

\[
\begin{align*}
\nabla^2_{T_k^c} f(x^k)(d^k_N)_{T_k} &= \nabla^2_{T_k^c} f(x^k)x^k_{T_k} - \nabla_{T_k^c} f(x^k), \\
(d^k_N)_{T_k} &= -x^k_{T_k}.
\end{align*}
\]

(8)
Note that $\nabla^2_{T_k} f(x^k)$ is non-singular under the condition in Theorem 3.11. Finally, the Armijo line search strategy is adopted to obtain the $(k+1)$th iteration $x^{k+1} = x^k(\alpha_k)$ and

$$
x^k(\alpha_k) := \left[ \begin{array}{c} x^k_T + \alpha_k(d^k_N)T_k \\ x^k_T + (d^k_N)T_k \end{array} \right] = \left[ \begin{array}{c} x^k_T + \alpha_k(d^k_N)T_k \\ 0 \end{array} \right], \quad \alpha_k > 0,
$$

where $\alpha_k$ is the step length. To measure the distance of the $k$th iteration from the $\eta$-stationary point, we take an accuracy measure as

$$
\text{Tol}_\eta(x^k; T_k) := \|F_\eta(x^k; T_k)\| + \max_{i \in T^*_k} \{ \max(\|\nabla f(x^k)\| - |x^i(\delta)|/\eta, 0) \}.
$$

The framework of NHTP algorithm for solving (1) is summarized in Algorithm 1.

---

**Algorithm 1** NHTP for solving (1)

**Step 0** Initialize $x^0$, choose $\eta, \gamma > 0, \sigma \in (0,1/2), \beta \in (0,1)$ and set $k = 0$.

**Step 1** Choose $T_k \in T(x^k; \eta)$. If $\text{Tol}_\eta(x^k; T_k) = 0$, then stop. Otherwise, go to Step 2.

**Step 2** Calculate the search direction by (8).

**Step 3** Compute $x^{k+1} = x^k(\alpha_k)$, where $\alpha_k = \beta^\ell$ with $\ell$ the smallest integer such that

$$
f(x^k(\beta^\ell)) \leq f(x^k) + \sigma \beta^\ell (\nabla f(x^k), d^k).
$$

**Step 4** Set $k = k + 1$, and go to Step 1.

---

According to Theorems 3.3 and 3.5, we know that $f$ is $M_{2s}$-RSS and $m_{2s}$-RSC for any $x \in N_s(x^*, \delta^*)$. We have the following decent lemma.

**Lemma 3.12:** Let $x^*$ be an optimal solution of (1). Given $\gamma \leq m_{2s}$ and $\eta \leq 1/(4M_{2s})$, then in the neighbourhood $N_s(x^*, \delta^*)$ of $x^*$, we have

$$
(\nabla_{T_k} f(x^k), (d^k_N)T_k) \leq -\gamma \|d^k_N\|^2 + \frac{1}{4\eta} \|x^k_T\|^2.
$$

It can be seen that the restricted Newton direction $d^k_N$ provides a good descent direction on the restricted subspace $x_{T_k^c} = 0$, which ensures the convergence of NHTP algorithm. We select the parameters $\gamma, \sigma$ and $\eta$ in the NHTP algorithm such that

$$
0 < \gamma \leq \min\{1,2M_{2s}\}, \quad 0 < \sigma < 1/2, \quad \text{and} \quad 0 < \beta < 1.
$$

Furthermore, define the following two parameters as

$$
\bar{\alpha} := \min \left\{ \frac{1 - 2\sigma}{M_{2s}/\gamma - \sigma}, 1 \right\}, \quad \bar{\eta} := \min \left\{ \frac{\gamma(\bar{\alpha}\beta)}{M_{2s}^2}, \bar{\alpha} \beta, \frac{1}{4M_{2s}} \right\}.
$$

**Theorem 3.13:** Suppose the sequence $\{x^k\}$ is generated by Algorithm 1, $x^*$ is an optimal solution of (1) and Assumption 3.1 holds. Let the parameters $\gamma, \sigma$ and $\beta$ satisfy the conditions
in (9), \( \bar{\eta} \) be defined in (10), and \( \eta \leq \bar{\eta} \). If the initial point \( x^0 \) of the NHTP algorithm satisfies \( x^0 \in \mathcal{N}_s(x^*, \delta^*) \), then we have

(i) \( \lim_{k \to \infty} x_k = x^* \).

(ii) The rate of convergence from \( \{x^k\} \) to \( x^* \) is quadratic,

\[
\|x^{k+1} - x^*\| \leq \frac{L_f}{2m_2s} \|x^k - x^*\|^2.
\]

**Proof:** It follows from Theorems 3.2–3.5 that for any \( x \in \mathcal{N}_s(x^*, \delta^*) \), \( f \) is restricted Hessian Lipschitz continuous, \( M_{2s}\)-RSC and \( m_{2s}\)-RSC. Thus the regularity conditions in Zhou et al. [25, Theorem 10] are satisfied, and then the locally quadratic convergence of Algorithm 1 can be established. \( \blacksquare \)

### 4. Numerical experiments

In order to verify the effectiveness of the NHTP algorithm for (1), we conduct numerical experiments for the multilinear equations whose coefficient tensors are CP-tensors and symmetric strong M-tensors, and compare NHTP with the homotopy algorithm proposed in Yan et al. [1]. All numerical examples are implemented on a laptop (2.40 GHz, 16 GB of RAM) by using MATLAB (R2021a).

In our experiments, the parameters in NHTP algorithm are chosen as: \( \sigma = 10^{-4}/2, \beta = 0.5, \gamma = \gamma_k \) by updating \( \gamma_k = 10^{-10} \) if \( x_{T_0}^k = 0 \), otherwise, \( \gamma_k = 10^{-4} \). Parameter \( \eta \) is generated by: \( \eta = \min([\|x_0^{\mathbf{0}}\|]/(10(1 + \max([\|\nabla_T f(x^0)\|])))) \) with \( T = \text{supp}(P_s(x^0)) \). Take \( \text{Tol}_{\eta_k}(x^k; T_k) \leq 10^{-7} \) as the stopping criterion. For the homotopy algorithm, set \( \delta = 0.75, \sigma = 10^{-4}, \tau = 10^{-5}, t_0 = 0.01 \) and take \( \|\mathcal{H}(x^k)\| < 10^{-7} \) as its stopping criterion. We conduct 50 independent experiments for the following two examples, where \( (m, n) \) takes different values, \( s = [0.01n], [0.05n] \).

**Example 4.1 (Random CP-tensors):** Let \( A = \sum_{s=1}^{n} (u(s))^m \in \mathbb{CP}^{[m,n]} \), where the components of \( u(s) (s \in [n]) \) are randomly generated in \([0, 1]\). The true value \( x^* \), \( b \) and the initial point \( x^0 \) are generated by the following Matlab pseudocode:

\[
x^* = \text{zeros}(n, 1), \quad \Gamma = \text{randperm}(n), \quad T_x = \Gamma(1 : s), \quad x^*(T_x) = \text{rand}(s, 1),
\]

\[
b = A(x^*)^{m-1}, \quad e = \text{zeros}(n, 1), \quad e(T_x) = 0.1 * \text{rand}(s, 1), \quad x^0 = x^* + e.
\]

We remark that for the cases of unknown \( x^* \), one can generate an initial point for our NHTP algorithm by using the first-order method (e.g. accelerate proximal gradient algorithm) to solve a convex relaxation version of the SLS optimization model (1). This is a commonly used two stage scheme, such as that adopted by Sun et al. [29].

**Example 4.2 (Random symmetric strong M-tensors generated by 0–1 uniform distribution):** Let \( A = sI - B \in \mathbb{R}^{[m,n]}, B \) be a symmetric tensor, where each element is randomly generated in \([0, 1]\), and \( s = n^{m-1} > \rho \), so \( A \) is a symmetric strong M-tensor. The true value \( x^*, b \) and initial point \( x^0 \) are generated in the same way as in Example 4.1.
The accuracy of the solution obtained by homotopy algorithm is about 10−10, and the sparsity recovery of the solutions performs well. However, the accuracy of the solution obtained by homotopy algorithm is about 10−2 for the case of \( s = [0.05n] \), and the sparsity recovery of the solution performs poor. In terms of running time, NHTP algorithm is significantly superior to homotopy algorithm for the case of \( s = [0.05n] \).

Overall, NHTP algorithm is an efficient and stable algorithm in solving the SLS optimization model for multilinear equations with CP-tensor and symmetric strong M-tensor compared with the homotopy algorithm.

---

**Table 2.** Numerical results for Example 4.1.

| (m, n) | nnz(N_{th}) | Re(N_{th}) | Time(s)(N_{th}) | Iter(N_{th}) |
|-------|--------------|------------|-----------------|--------------|
| (3, 10) | 1|9 | 7.25e−09 | 6.06e−01 | 0.011 | 0.004 | 5|5 |
| (3, 30) | 1|27 | 5.49e−09 | 2.69e−01 | 0.014 | 0.006 | 5|5 |
| (3, 50) | 2|22 | 1.82e−09 | 2.59e−02 | 0.017 | 0.005 | 6|4 |
| (3, 70) | 1|43 | 8.86e−10 | 2.42e−01 | 0.016 | 0.008 | 5|5 |
| (4, 50) | 3|41 | 9.94e−12 | 4.69e−02 | 0.020 | 0.006 | 6|4 |
| (3, 10) | 1|58 | 4.38e−11 | 4.12e−01 | 0.045 | 0.018 | 5|5 |
| (4, 10) | 4|64 | 2.57e−11 | 9.39e−02 | 0.060 | 0.015 | 7|4 |
| (4, 30) | 1|10 | 2.14e−09 | 2.33e−01 | 0.017 | 0.011 | 5|6 |
| (4, 50) | 2|28 | 5.22e−10 | 6.17e−01 | 0.086 | 0.051 | 5|6 |
| (4, 50) | 4|21 | 8.30e−09 | 1.53e−01 | 0.097 | 0.035 | 6|5 |
| (4, 70) | 1|47 | 3.19e−09 | 8.48e−01 | 0.476 | 0.272 | 6|7 |
| (4, 30) | 3|48 | 9.77e−12 | 1.75e−01 | 0.602 | 0.185 | 7|5 |

**Table 3.** Numerical results for Example 4.2.

| (m, n) | nnz(N_{th}) | Re(N_{th}) | Time(s)(N_{th}) | Iter(N_{th}) |
|-------|--------------|------------|-----------------|--------------|
| (3, 10) | 1|1 | 2.13e−10 | 2.90e−04 | 0.011 | 0.004 | 5|4 |
| (3, 30) | 1|1 | 2.03e−13 | 2.55e−09 | 0.015 | 0.005 | 5|4 |
| (3, 50) | 2|12 | 1.25e−14 | 2.54e−03 | 0.016 | 2.859 | 6|208 |
| (3, 70) | 3|38 | 3.40e−11 | 9.79e−05 | 0.020 | 0.007 | 6|5 |
| (4, 50) | 1|1 | 3.21e−13 | 1.35e−06 | 0.004 | 0.015 | 6|5 |
| (4, 10) | 4|55 | 2.17e−16 | 9.41e−03 | 0.078 | 33.561 | 4|808 |
| (4, 30) | 1|1 | 2.78e−12 | 7.06e−04 | 0.019 | 0.007 | 6|5 |
| (4, 50) | 2|9 | 5.16e−16 | 5.74e−12 | 0.132 | 0.036 | 6|5 |
| (4, 70) | 1|1 | 1.43e−15 | 2.84e−02 | 0.143 | 39.010 | 8|403 |
| (4, 50) | 3|45 | 1.15e−17 | 2.79e−02 | 0.690 | 295.253 | 8|504 |

**Table 4.** Numerical results for Example 4.3.

| (m, n) | nnz(N_{th}) | Re(N_{th}) | Time(s)(N_{th}) | Iter(N_{th}) |
|-------|--------------|------------|-----------------|--------------|
| (3, 10) | 1|1 | 2.13e−10 | 2.90e−04 | 0.011 | 0.004 | 5|4 |
| (3, 30) | 1|1 | 2.03e−13 | 2.55e−09 | 0.015 | 0.005 | 5|4 |
| (3, 50) | 2|12 | 1.25e−14 | 2.54e−03 | 0.016 | 2.859 | 6|208 |
| (3, 70) | 3|38 | 3.40e−11 | 9.79e−05 | 0.020 | 0.007 | 6|5 |
| (4, 50) | 1|1 | 3.21e−13 | 1.35e−06 | 0.004 | 0.015 | 6|5 |
| (4, 10) | 4|55 | 2.17e−16 | 9.41e−03 | 0.078 | 33.561 | 4|808 |
| (4, 30) | 1|1 | 2.78e−12 | 7.06e−04 | 0.019 | 0.007 | 6|5 |
| (4, 50) | 2|9 | 5.16e−16 | 5.74e−12 | 0.132 | 0.036 | 6|5 |
| (4, 70) | 1|1 | 1.43e−15 | 2.84e−02 | 0.143 | 39.010 | 8|403 |
| (4, 50) | 3|45 | 1.15e−17 | 2.79e−02 | 0.690 | 295.253 | 8|504 |
5. Concluding remarks

In this paper, a sparse least squares methods for multilinear equations has been modelled, in which the original $\ell_0$-norm constraint has been imposed to control the sparsity of the solutions. For the formulated SLS optimization problem, we have established its optimality condition, along with several regularity conditions of the objective function. Based on this, an NHTP algorithm has been developed for SLS solutions with locally quadratic convergence. Numerical experiments have verified the superiority of our method comparing with the existing homotopy algorithm. Furthermore, it is worth mentioning that the non-singularity of the corresponding Jacobian matrix may not be guaranteed for the more general tensor cases of model (1). Our future research would be in developing the robust and highly efficient smoothing Newton method [30] for the more general SLS model.

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