ON CONNECTIONS AND THEIR CURVATURES

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Abstract. This paper presents a brief study on connections on fiber, principal and vector smooth bundles as well as some relations with their curvatures.

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1. Introduction

The study of bundles and their ad hoc connections is one of the most important subjects in differential geometry. For example, the concept of vector bundle and linear connection appear on every introductory course in the form of the tangent bundle and the Levi–Civita connection, respectively. Principal bundles also play an important role; for example the frame bundle of a Riemannian manifold reflects different ways to give an orthonormal basis for tangent spaces, not to mention that field theories in modern physics are developed by using principal bundles and principal connections \[1\]. All these objects arise from the more general concepts of fiber bundles and non–linear connections. This work’s aim is to present a brief study on connections on fiber, principal and vector smooth bundles as well as some relations with their curvatures.

This paper corresponds to an extended version of the second section of \[3\], but focused in geometrical relations, not the functional ones and it breaks down as follows: in the second section we present the basics of the theory of fiber bundles and non–linear connections, as well as the relations of the curvature. The third section is about principal connections on principal bundles; while the fourth section is about vector bundles and linear connections. Finally in the appendix A we present some calculations related to the curvature and its relation with covariant derivatives in local coordinates.

2. Non–Linear Connection

A fiber bundle over a manifold \( M \) with model fiber manifold \( \mathcal{F} \) is a manifold \( \mathcal{F}M \) endowed with a smooth projection map \( \pi : \mathcal{F}M \rightarrow M \), which is locally trivializable in the sense that there exists an open covering of \( M \) by open subsets \( U \subset M \), each of which allows for a diffeomorphism

\[ \Phi : U \times \mathcal{F} \rightarrow \pi^{-1}(U) \]

such that \( \pi \circ \Phi : U \times \mathcal{F} \rightarrow U \) equals the projection to \( U \). The preimage of a point \( p \in M \) under \( \pi \) is called the fiber of the bundle over \( p \), it is a submanifold \( \mathcal{F}_pM := \pi^{-1}(p) \subset \mathcal{F}M \) of the total space \( \mathcal{F}M \) diffeomorphic to the model fiber \( \mathcal{F} \). Morphisms between two fiber
bundles are smooth maps $\varphi : \mathcal{F}M \rightarrow \hat{\mathcal{F}}M$ between the total spaces which commute with the respective projections

$$\begin{array}{ccc}
\mathcal{F}M & \xrightarrow{\varphi} & \hat{\mathcal{F}}M \\
\pi & \downarrow & \hat{\pi} \\
M & & 
\end{array}$$

(1)

and thus map the fibers of $\mathcal{F}M$ to the fibers of $\hat{\mathcal{F}}M$ over the same point.

Intuitively, a non–linear or Ehresmann connection on a fiber bundle $\mathcal{F}M$ is a left hand side section $\nabla$ of the short exact sequence of vector bundles

$$0 \rightarrow \text{Vert } \mathcal{F}M \rightarrow \nabla \mathcal{F}M \rightarrow \pi^*TM \rightarrow 0$$

over $\mathcal{F}M$, the corresponding right hand side section is appropriately called the horizontal lift associated to $\nabla$. In order to studying connections we prefer the following version of this intuitive notion of a non–linear connection:

**Definition 2.1 (Non–linear Connections on Fiber Bundles).**

A general or a non–linear connection on a fiber bundle $\mathcal{F}M$ over a manifold $M$ is a field $\nabla \in \Gamma(\mathcal{F}M, \text{End } \mathcal{T}\mathcal{F}M)$ of projections $(\nabla^2)^2 = \nabla$ on the tangent bundle $\mathcal{T}\mathcal{F}M$ such that its image distribution equals the vertical foliation:

$$\text{im} \left( \nabla_{f} : T_{f}\mathcal{F}M \rightarrow T_{f}\mathcal{F}M \right) \overset{\dagger}{=} \text{Vert}_{f}\mathcal{F}M.$$ 

Every non–linear connection $\nabla$ on a fiber bundle $\mathcal{F}M$ allows us to define the directional derivative $D_{X}f \in \Gamma_{\text{loc}}(M, \text{Vert } \mathcal{F}M)$ of a given local section $f \in \Gamma_{\text{loc}}(M, \mathcal{F}M)$ in the direction of a vector field $X$ on $M$ by:

$$( D_{X}f )_{p} := \left( T_{p}M \xrightarrow{\nabla_{f}p} T_{f(p)}\mathcal{F}M \xrightarrow{p_{\nabla f(p)}} \text{Vert}_{f(p)}\mathcal{F}M \right) X_{p}.$$ 

These directional derivatives assemble into a first order differential operator

$$D_{\nabla} : \Gamma( M, TM ) \times \Gamma_{\text{loc}}( M, \mathcal{F}M ) \rightarrow \Gamma_{\text{loc}}( M, \text{Vert } \mathcal{F}M ),$$

which is the non–linear analogue of the classical definition of covariant derivatives on vector bundles. Somewhat annoyingly this covariant derivative $D_{X}f$ contains the redundant information $f = \pi_{\mathcal{F}M} \circ D_{X}f$, where $\pi_{\mathcal{F}M}$ denotes the vertical tangent bundle projection $\text{Vert } \mathcal{F}M \rightarrow \mathcal{F}M$. The simplicity of linear and principal connections stems from the fact that we can get rid of this redundancy altogether, the reduced covariant derivative $\nabla_{X}f$ captures only the partial derivatives of the section $f$.

The Nijenhuis or curvature tensor of a non–linear connection $\nabla$ on a fiber bundle $\mathcal{F}M$ over a manifold $M$ is the horizontal 2–form $R_{\nabla}$ on the total space $\mathcal{F}M$ of the fiber bundle with values in the vertical tangent bundle defined for two arbitrary vector fields $X, Y$ on
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\[ R^\nabla (X, Y) := - \nabla [ (\text{id} - \nabla) X, (\text{id} - \nabla) Y ] - (\text{id} - \nabla) [ \nabla X, \nabla Y ] \]

\[ = - \nabla [X, Y] + \nabla [X, \nabla Y] + \nabla [\nabla X, Y] - [\nabla X, \nabla Y]. \]

The strange sign is necessary to make this definition agree with the classical definition of the curvature of a linear connection, compare equation (8). By construction \( R^\nabla \in \Omega^2_{\text{hor}} (\mathcal{F}M, \text{Vert} \mathcal{F}M) \) is \( C^\infty (\mathcal{F}M) \)-bilinear and alternating. The integrability constraint for the vertical foliation \( \text{Vert} \mathcal{F}M \) tells us \((\text{id} - \nabla) [\nabla X, \nabla Y] = 0\), and the simplified expression

\[(4) \quad R^\nabla (X, Y) = - \nabla [ (\text{id} - \nabla) X, (\text{id} - \nabla) Y ],\]

evidently results in a vertical vector field for all vector fields \( X, Y \) on \( \mathcal{F}M \) and vanishes for a vertical vector field \( X \) or \( Y \) no matter what is the other. In particular the curvature \( R^\nabla \) measures exactly the failure of the horizontal distribution \( \ker \nabla \subseteq T\mathcal{F}M \) associated to \( \nabla \) to be integrable.

An interpretation of the curvature tensor along classical lines as a commutator of covariant derivatives requires a significant amount of extra work and so we will only sketch the basic idea. The iterated tangent bundle of a smooth manifold \( \mathcal{F} \) is a smooth manifold \( T(T\mathcal{F}) \) coming along with a canonical involutive diffeomorphism \( \Theta : T(T\mathcal{F}) \longrightarrow T(T\mathcal{F}) \) characterized by

\[(5) \quad \Theta \left( \left. \frac{d}{dt} \right|_0 \frac{d}{d\varepsilon} |_0 \gamma(t, \varepsilon) \right) = \left( \left. \frac{d}{dt} \right|_0 \frac{d}{d\varepsilon} |_0 \gamma(\varepsilon, t) \right)\]

for all \( \gamma : \mathbb{R}^2 \longrightarrow \mathcal{F} \). By construction, \( \Theta \) fits into the commutative diagram

\[
\begin{array}{ccc}
T(T\mathcal{F}) & \overset{\Theta}{\longrightarrow} & T(T\mathcal{F}) \\
\Pi \downarrow & & \downarrow \Pi \\
T\mathcal{F} \oplus T\mathcal{F} & \overset{\text{swap}}{\longrightarrow} & T\mathcal{F} \oplus T\mathcal{F}
\end{array}
\]

where swap interchanges the two summands and \( \Pi \) is the double projection:

\[ \Pi \left( \left. \frac{d}{dt} \right|_0 \frac{d}{d\varepsilon} |_0 \gamma(t, \varepsilon) \right) := \left. \frac{d}{d\varepsilon} |_0 \gamma(0, \varepsilon) \right|_0 \oplus \left. \frac{d}{dt} |_0 \gamma(t, 0) \right|_0. \]

The involution \( \Theta \) and double projection \( \Pi \) generalize directly to iterated vertical tangent bundles, because we may identify the fibers of \( \text{Vert} \text{Vert} \mathcal{F}M \) considered as a fiber bundle over \( M \) in every point \( p \in M \) with the iterated tangent bundle \( T(T[\mathcal{F}pM]) = [\text{Vert} \text{Vert} \mathcal{F}M]_p \). The involutions and double projections thus defined on all the fibers of \( \text{Vert} \text{Vert} \mathcal{F}M \) assemble into an involutive fiber bundle automorphism \( \Theta \) and a double projection \( \Pi : \text{Vert} \mathcal{F}M \longrightarrow \text{Vert} \mathcal{F}M \oplus \text{Vert} \mathcal{F}M \). Making good use of the involution \( \Theta \) we extend a non–linear connection \( \nabla \) on \( \mathcal{F}M \) to a unique non–linear connection

\[ \mathcal{F}M \text{ by:} \]
on $\mathcal{F} M$ by stipulating the identity
\[
D_X^{\nabla_{\text{Vert}}} \left( \frac{d}{d\varepsilon} \bigg|_0 f_{\varepsilon} \right) = \Theta \left( \frac{d}{d\varepsilon} \bigg|_0 D_X f_{\varepsilon} \right)
\]
for every vector field $X \in \Gamma(M, TM)$ and every smooth one–parameter family $(f_{\varepsilon})_{\varepsilon \in \mathbb{R}}$ of local sections of $\mathcal{F} M$ with infinitesimal variation $\frac{d}{d\varepsilon} \bigg|_0 f_{\varepsilon} \in \Gamma_{\text{loc}}(M, \text{Vert } \mathcal{F} M)$. In this construction of the induced connection on $\text{Vert } \mathcal{F} M$, the involution $\Theta$ is proper remedy for the nuisance
\[
\Pi \left( D_X^{\nabla_{\text{Vert}}} \left( \frac{d}{d\varepsilon} \bigg|_0 f_{\varepsilon} \right) \right) = \left( \frac{d}{d\varepsilon} \bigg|_0 f_{\varepsilon} \right) \oplus \left( D_X f_{0} \right)
\]
\[
\Pi \left( \frac{d}{d\varepsilon} \bigg|_0 D_X f_{\varepsilon} \right) = \left( D_X f_{0} \right) \oplus \left( \frac{d}{d\varepsilon} \bigg|_0 f_{\varepsilon} \right),
\]
compare the commutative diagram (6). For essentially the same reason the involution $\Theta$ appears in the key identity linking the curvature $R^{\nabla}$ of a non–linear connection $P^{\nabla}$ on a fiber bundle $\mathcal{F} M$ over $M$ to the commutator
\[
\left( D_X^{\nabla} D_Y^{\nabla} f - \Theta(D_Y^{\nabla} D_X^{\nabla} f) \right) - D_{[X,Y]} f
\]
of covariant derivatives of a local section $f \in \Gamma_{\text{loc}}(M, \mathcal{F} M)$ in the direction of vector fields $X, Y \in \Gamma(M, TM)$, in which $R^{\nabla}_{X,Y} f$ is a simplified notation for the local section of $\text{Vert } \mathcal{F} M$ over $M$ defined in $p \in M$ by:
\[
\left( R^{\nabla}_{X,Y} f \right)_p := R^{\nabla}_{f(p)}( f_{*,p} X_p, f_{*,p} Y_p ) \in \text{Vert }_{f(p)} \mathcal{F} M.
\]
The basic tenet of symbolic calculus on jet bundles provides the correct interpretation for the differences in (8): The fibers of the double projection $\Pi : \text{Vert } \mathcal{F} M \rightarrow \text{Vert } \mathcal{F} M \oplus \text{Vert } \mathcal{F} M$ are affine spaces modelled on the fibers of $\text{Vert } \mathcal{F} M$ and thus come along with a difference map
\[
- : \text{Vert } \mathcal{F} M \times_{\text{Vert } \mathcal{F} M \oplus \text{Vert } \mathcal{F} M} \text{Vert } \mathcal{F} M \rightarrow \text{Vert } \mathcal{F} M,
\]
which determines the inner difference in the curvature identity (8), the outer difference is simply the difference in $\text{Vert } \mathcal{F} M$ considered as a vector bundle over $\mathcal{F} M$. Taking care with this delicate interpretation of differences we will prove the curvature identity (8) in local coordinates in Appendix A.

**Definition 2.2 (Parallel morphisms between Fiber Bundles).**

A parallel morphism between fiber bundles $\mathcal{F} M$ and $\hat{\mathcal{F} M}$ over the same manifold $M$ endowed with connections $P^{\nabla}$ and $\hat{P}^{\nabla}$ respectively is a morphism $\varphi : \mathcal{F} M \rightarrow \hat{\mathcal{F} M}$ of fiber bundles such that the following diagram commutes:
\[
\begin{array}{c}
T \mathcal{F} M \xrightarrow{\varphi^*} T \hat{\mathcal{F} M} \\
p^\nabla \downarrow \quad \downarrow p^\nabla \\
T \mathcal{F} M \xrightarrow{\varphi^*} T \hat{\mathcal{F} M}
\end{array}
\]
The constraint $\hat{\pi} \circ \varphi = \pi$ characterizing morphisms of fiber bundles readily implies
\[
\varphi_*(\text{Vert } \mathcal{F} M) \subset \text{Vert } \hat{\mathcal{F} M},
\]
hence a morphism \( \varphi \) of fiber bundles is parallel, if and only if \( \varphi_* \) maps the horizontal distribution of \( \mathcal{F}M \) to the horizontal distribution of \( \tilde{\mathcal{F}}M \):

\[
\varphi \text{ parallel } \iff \varphi_*(\ker P^\nabla) \subset \ker \tilde{P}^\nabla.
\]

3. Principal Connections

Having discussed general non–linear connections on fiber bundles in some detail we now want to specialize to principal connections on principal bundles in this section. Recall first of all that a principal bundle modelled on a Lie group \( G \) is a smooth fiber bundle \( GM \) with model fiber \( G \) endowed with a smooth right, fiber preserving action of \( G \) on its total space \( GM \)

\[
\rho: GM \times G \to GM, \quad (g, \gamma) \mapsto g\gamma,
\]

which is simply transitive on all fibers of \( GM \) over \( M \). Every local section \( \Gamma: U \to GM \) of the projection \( \pi: GM \to M \) defined over an open subset \( U \) extends to a \( G \)–equivariant local trivialization of \( GM \) over \( U \)

\[
(9) \quad \Phi: U \times G \xrightarrow{\times \text{id}} \pi^{-1}(U) \times G \xrightarrow{\rho} \pi^{-1}(U),
\]

satisfying \( \Phi(u, g\gamma) = \pi(u)g\gamma = \Phi(u, g)\gamma \) by construction. The existence of \( G \)–equivariant trivializations ensures that the unique set theoretic map

\[
(10) \quad \ \downarrow : GM \times_M GM \to G, \quad (g, \hat{g}) \mapsto g^{-1}\hat{g},
\]

satisfying \( g(g^{-1}\hat{g}) = \hat{g} \) for all \( g, \hat{g} \in GM \) in the same fiber is actually a smooth map, because it reads

\[
\Phi(u, g)^{-1}\Phi(u, \hat{g}) = g^{-1}\hat{g}
\]

in every such trivialization \( \Phi \).

The automorphism group bundle of a principal bundle \( GM \) over a manifold \( M \) is the Lie group bundle \( \text{Aut} GM \) over \( M \) defined by

\[
(11) \quad \text{Aut} GM := \{ (p, \psi) \mid \psi: G_pM \to G_pM \text{ is } G\text{–equivariant} \}
\]

with the bundle projection \( \pi_{\text{Aut} GM}: \text{Aut} GM \to M, (p, \psi) \mapsto p \). In mathematical physics, the Fréchet–Lie group \( \Gamma( M, \text{Aut} GM ) \) of all global sections of the automorphism bundle is called the gauge group of \( GM \).

The fiber of the Lie group bundle \( \text{Aut} GM \) over a point \( p \in M \) is a Lie group \( \text{Aut}_pGM \) isomorphic, although not canonically so, to the original group \( G \), in particular its Lie algebra \( \text{aut}_pGM \cong \mathfrak{g} \) is isomorphic to the Lie algebra of \( G \). All these Lie algebras assemble into a smooth Lie algebra bundle \( \text{aut} GM \), whose global sections \( \Gamma( M, \text{aut} GM ) \) form the Fréchet–Lie algebra of the gauge group \( \Gamma( M, \text{Aut} GM ) \) of the principal bundle \( GM \).

**Definition 3.1** (Principal Connections). A principal connection on a principal \( G \)–bundle \( GM \) over a manifold \( M \) is a non–linear connection \( \mathbb{P}^\nabla \) on the fiber bundle \( GM \), which is invariant under the right action of \( G \) on \( GM \) in the sense that the right translations \( R_\gamma: GM \to GM, g \mapsto g\gamma \), are parallel automorphisms for all \( \gamma \in G \).
In difference to general fiber bundles, the vertical tangent bundle of a principal bundle $GM$ is trivializable: The construction of the Maurer–Cartan form on a Lie group $G$ applies verbatim and results in the vertical trivialization isomorphism

$$\text{Vert } GM \xrightarrow{\cong} GM \times g , \quad \frac{d}{dt} \bigg|_0 g_t \longmapsto (g_0, \frac{d}{dt} \bigg|_0 g_0^{-1} g_t)$$

under the proviso that the curve $t \mapsto g_t$ chosen to represent the vertical tangent vector stays in the fiber so that the curve $t \mapsto g_0^{-1} g_t$ is defined by application (10). The composition of this vertical trivialization with the projection to $g$

$$v_{\text{triv}} : \text{Vert } GM \xrightarrow{\cong} GM \times g$$

induces isomorphisms $v_{\text{triv}}_g : \text{Vert}_g GM \xrightarrow{\cong} g$ at every $g \in GM$ and thus a bijection $\mathbb{P}^\text{V} \longleftrightarrow \omega$ between non–linear connections and $g$–valued 1–forms $\omega \in \Omega^1(GM, g)$, which agree with $v_{\text{triv}}$ on vertical tangent vectors:

$$\omega := v_{\text{triv}} \circ \mathbb{P}^\text{V} \iff \mathbb{P}^\text{V} := v_{\text{triv}}^{-1} \circ \omega .$$

This description of non–linear connections on $GM$ by $g$–valued 1–forms $\omega$ with $\omega|_{\text{Vert } GM} = v_{\text{triv}}$ is much more convenient and will be used for the rest of this article.

To characterize the principal connections specified in Definition 3.1 by their connection forms $\omega$ we recall that the right translation by $\gamma \in G$ is a parallel morphism

$$R_\gamma : GM \longrightarrow GM, \quad g \longmapsto g_\gamma,$$

if and only if its differential $R_\gamma^*$ maps the horizontal vectors $\ker \omega_g \subset T_g GM$ at a point $g \in GM$ to horizontal vectors $\ker \omega_{g_\gamma} \subset T_{g_\gamma} GM$. On the complementary vertical vectors the connection form behaves like $v_{\text{triv}} = \omega|_{\text{Vert } GM}$:

$$(v_{\text{triv}} \circ R_\gamma^*) \left( \frac{d}{dt} \bigg|_0 g_t \right) = \frac{d}{dt} \bigg|_0 (\gamma g_0)^{-1} (g_t \gamma) = \text{Ad}_{\gamma^{-1}} v_{\text{triv}} \left( \frac{d}{dt} \bigg|_0 g_t \right).$$

In consequence the condition $\omega \circ R_\gamma^* = \text{Ad}_{\gamma^{-1}} \circ \omega$ for all $\gamma \in G$ is necessary and sufficient for a $g$–valued 1–form $\omega$ agreeing on vertical vectors with $v_{\text{triv}} = \omega|_{\text{Vert } GM}$ to be the connection form of a principal connection $\mathbb{P}^\text{V}$. The chain rule in the guise of a Leibniz rule tells us for such a form

$$\omega_{g_0 \gamma_0} \left( \frac{d}{dt} \bigg|_0 g_t \gamma_t \right) = \omega_{g_0 \gamma_0} \left( \frac{d}{dt} \bigg|_0 g_t \gamma_0 + \frac{d}{dt} \bigg|_0 g_0 \gamma_t \right)$$

$$= \omega_{g_0 \gamma_0} \left( (R_{\gamma_0})^* \frac{d}{dt} \bigg|_0 g_t \right) + \frac{d}{dt} \bigg|_0 (g_0 \gamma_0)^{-1} (g_0 \gamma_t)$$

$$= \text{Ad}_{\gamma_0^{-1}} \omega_{g_0} \left( \frac{d}{dt} \bigg|_0 g_t \right) + \frac{d}{dt} \bigg|_0 \gamma_0^{-1} \gamma_t$$

for every curve $t \mapsto g_t$ in $GM$ and $t \mapsto \gamma_t$ in $G$. Put differently:
Lemma 3.2 (Principal Connection Axiom).

On every principal $G$–bundle $GM$ the association $\mathbb{P}^\nabla \leftrightarrow \omega$ characterized by $\omega := \text{vtriv} \circ \mathbb{P}^\nabla$ induces a bijection between principal connections in the sense of Definition 3.1 and $\mathfrak{g}$–valued 1–forms $\omega$ on $GM$ satisfying the axiom

$$\omega_{g_0 \gamma_0} \left( \frac{d}{dt} \right)_0 g_t \gamma_t = \text{Ad}_{g_0}^{-1} \omega_{g_0} \left( \frac{d}{dt} \right)_0 g_t + \frac{d}{dt} \left. \gamma_0^{-1} \gamma_t \right|_0$$

for all choices of smooth curves $t \mapsto g_t$ in $GM$ and curves $t \mapsto \gamma_t$ in $G$.

Cartan’s Second Structure Equation \cite{1} is a convenient description of the image of the composition of the curvature tensor $R^\nabla$ with the vertical trivialization $\text{vtriv}$ in terms of the exterior derivative of the connection form

$$\Omega := \text{vtriv} \circ R^\nabla \equiv d\omega + \frac{1}{2}[\omega \wedge \omega],$$

where $\frac{1}{2}[\omega \wedge \omega](X, Y) := [\omega(X), \omega(Y)]$; the usefulness of the factor $\frac{1}{2}$ in this definition is beyond question. Cartan’s Second Structure Equation can be easily verified directly. On the other hand it can be derived without too much additional effort from the curvature identity \cite{8} by considering the reduced covariant derivative associated to a principal connection $\nabla$ defined by $\nabla f := \text{vtriv}(D^\nabla f)$; in light of equation (13) this becomes

$$\left( \nabla f \right)_p := \text{vtriv}(\mathbb{P}^\nabla f_p(f_\ast p X_p)) \equiv (f^\ast \omega)_p(X)$$

for all $X \in \Gamma(M, TM)$. Iterating the vertical trivialization isomorphism $\text{Vert} GM \rightarrow GM \times \mathfrak{g}$ used to define $\text{vtriv}$ we obtain the isomorphism:

$$\Psi : \text{Vert} GM \xrightarrow{\cong} \text{Vert}(GM \times \mathfrak{g}) \xrightarrow{\cong} (GM \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}).$$

Its inverse can be written down explicitly by unravelling the definitions

$$\Psi^{-1}(g, X, Y, Z) = \left( \frac{d}{dt} \right)_0 \left( \frac{d}{d\xi} \right)_0 g e^{tX} e^{\xi(Y + tZ)} = \left( \frac{d}{dt} \right)_0 \left( \frac{d}{d\xi} \right)_0 g e^{tX + \xi Y + t\xi(Z + \frac{1}{2}[X, Y])},$$

the second equality is the Baker–Campbell–Hausdorff Formula \cite{2} in the Lie group $G$ ignoring as usual terms of order $O(t^2, \varepsilon^2)$. In particular the involution $\Theta$ defined in equation (5) simply by the interchange $t \leftrightarrow \varepsilon$ is not as innocent as it may appear, under the isomorphism $\Psi$ it becomes:

$$\Theta(g, X; Y, Z) \equiv \left( g, Y; X, Z + [X, Y] \right).$$

In accordance with \cite{3} the vertical trivialization isomorphism $\text{Vert} GM \rightarrow GM \times \mathfrak{g}$ is parallel with respect to the connections $\mathbb{P}^\nabla_{\text{Vert}}$ and $\mathbb{P}^\nabla \oplus \mathbb{P}^\text{triv}$ on $\text{Vert} GM$ and $GM \times \mathfrak{g}$ respectively, using the fiber bundle isomorphism $\Psi$ we thus obtain

$$\Psi(D^\nabla_{X} f) = (f, (f^\ast \omega)(X), (f^\ast \omega)(Y), X(f^\ast \omega)(Y))$$

$$\Psi(D^\nabla_{Y} f) = (f, (f^\ast \omega)(Y), (f^\ast \omega)(X), Y(f^\ast \omega)(X)).$$
for all local sections \( f \in \Gamma_{\text{loc}}(M, GM) \) and all \( X, Y \in \Gamma(M, TM) \). Evidently the double projections under \( \Pi \) of these two local sections of \( \text{Vert} \: GM \) do not agree unless we apply the involution \( \Theta \) first picking up the rather unexpected additional term of equation \((16)\) along the way:

\[
D_\mathbf{∇} \text{Vert} X D_\mathbf{∇} Y f - \Theta( D_\mathbf{∇} \text{Vert} Y D_\mathbf{∇} X f) = \left( f, X(f^*\omega)(Y) - Y(f^*\omega)(X) - [(f^*\omega)(Y), (f^*\omega)(X)] \right)
\]

Subtracting \( D_\mathbf{∇} [X,Y] f \) we obtain eventually

\[
R^\mathbf{∇}_{X,Y} f = (f, d(f^*\omega)(X,Y) + \frac{1}{2}[(f^*\omega) \wedge (f^*\omega)](X,Y))
\]

for all local sections \( f \in \Gamma_{\text{loc}}(M, GM) \) and all \( X, Y \in \Gamma(M, TM) \) by using the naturality \( d(f^*\omega) = f^*(d\omega) \) of the exterior derivative \( d \) and the definition of the 2–form \( \Omega \) in Cartan’s Second Structure Equation \((14)\).

### 4. Linear Connections

The strategy pursued for linear connections on vector bundles \( VM \) follows the model of principal connections closely. The tangent bundle of a vector space is canonically trivializable \( TV \cong V \times V \) by taking actual derivatives and this becomes via \( \text{Vert} \: VM \) the vertical trivialization

\[
\text{Vert} \: VM \cong VM \oplus VM, \quad \frac{d}{dt}\bigg|_0 v_t \longmapsto v_0 \oplus \lim_{t \to 0} \frac{1}{t}(v_t - v_0),
\]

again under the proviso that the representative curve \( t \longmapsto v_t \) is chosen to stay in the same vector space so that the quotient \( \frac{1}{t}(v_t - v_0) \in V_{\pi(v_0)} M \) is well–defined. Composing with the projection to the right factor we obtain

\[
v\text{triv} : \text{Vert} \: VM \xrightarrow{\cong} VM \oplus VM \xrightarrow{\text{pr}_R} VM
\]

which can be used to project out \( \nabla_X v := v\text{triv}(D_X^\mathbf{∇} v) \) the redundant information from the covariant derivative \( D_X^\mathbf{∇} \) of a section \( v \in \Gamma(M, VM) \):

**Definition 4.1** (Linear Connections on Vector Bundles).

A linear connection on a vector bundle \( VM \) on \( M \) is a non–linear connection \( \mathbb{P}^\mathbf{∇} \) on \( VM \) such that the reduced covariant derivative is \( \mathbb{R} \)-bilinear:

\[
\nabla : \Gamma(M, TM) \times \Gamma(M, VM) \longrightarrow \Gamma(M, VM)
\]

In order to characterize linear connections by a condition easier to verify we consider a vector bundle \( VM \) with projection \( \pi : VM \longrightarrow M \) and a local basis of sections \( v_1, \ldots, v_n \in \Gamma_{\text{loc}}(M, VM) \) defined over the open domain \( U \subseteq M \) of a local coordinate chart \( x : U \longrightarrow \mathbb{R}^n \).
\(x(U) \subseteq \mathbb{R}^m\). In this situation we can construct linear local coordinates for the total space \(VM\) of the vector bundle \((x, v) : \pi^{-1}(U) \rightarrow x(U) \times \mathbb{R}^n\) characterized by:

\[
(x^1, \ldots, x^m; v^1, \ldots, v^n) \leftrightarrow v^1 v_1 + \ldots + v^n v_n \in V_{(x^1, \ldots, x^m)} M.
\]

In these linear local coordinates a general non-linear connection \(\mathbb{P}^\nabla\) reads

\[
(\mathbb{P}^\nabla_{(x,v)} \left( \frac{\partial}{\partial v^\alpha} \right) = \frac{\partial}{\partial v^\alpha} + \nabla^\mu \Gamma^\alpha_\mu(x,v) \frac{\partial}{\partial v^\alpha}
\]

with local coefficient functions \(\Gamma^\alpha_\mu(x,v)\) generalizing the classical Christoffel symbols, because the coordinate vector fields \(\frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\) span the vertical tangent bundle in every point of \(\pi^{-1}(U) \subseteq VM\).

With respect to the local trivialization of the vector bundle \(VM\) provided by the local basis \(v_1, \ldots, v_n\) every section \(v \in \Gamma(M, VM)\) can be expanded as a sum \(v = v^1 v_1 + \ldots + v^n v_n\) with local coefficient functions \(v^1, \ldots, v^n\) in \(C^\infty(U)\), which of course reappear in the local coordinate description of the section \(v \in \Gamma(M, VM)\) considered as a smooth map:

\[
(x^1, \ldots, x^m) \leftrightarrow (x^1, \ldots, x^m; v^1(x^1, \ldots, x^m), \ldots, v^n(x^1, \ldots, x^m)).
\]

Calculating the differential of \(v\) in these coordinates is straightforward:

\[
\mathbb{P}^\nabla \left( v_\alpha \left( \frac{\partial}{\partial x^\mu} \right) \right) = \mathbb{P}^\nabla \left( \frac{\partial}{\partial x^\mu} + \nabla^\mu \Gamma^\nu_\mu \left( x \right) \frac{\partial}{\partial v^\alpha} \right) = \sum_{\alpha=1}^n \left( \frac{\partial v^\alpha}{\partial x^\mu} \right) \frac{\partial}{\partial v^\alpha}.
\]

Pushing this expression to a section of \(VM\) via the vertical trivialization \(v_{\text{triv}}\) in diagram (17) converts the coordinate vector field \(\frac{\partial}{\partial v^\alpha}\) back to the corresponding local section \(v^\alpha\), hence it reduces the covariant derivative to:

\[
\nabla^\alpha = \sum_{\alpha=1}^n \left( \frac{\partial v^\alpha}{\partial x^\mu} \left( x \right) + \Gamma^\alpha_\mu \left( x, v(x) \right) \right) \frac{\partial}{\partial v^\alpha}.
\]

Directly from this expansion we conclude that a connection \(\mathbb{P}^\nabla\) on \(VM\) will not be a linear connection in the sense of Definition 4.1 unless its Christoffel symbols \(\Gamma^\beta_\mu(x,v)\) are linear in the coefficient coordinates with an expansion

\[
\Gamma^\alpha_\mu(x^1, \ldots, x^m; v^1, \ldots, v^n) = \sum_{\omega=1}^n \Gamma^\alpha_{\mu\omega}(x^1, \ldots, x^m) v^\omega
\]

and suitable local coefficient functions \(\Gamma^\alpha_{\mu\omega}(x)\). In the chosen linear coordinates the scalar multiplication \(\Lambda_\lambda : VM \rightarrow VM, v \mapsto \lambda v\), with \(\lambda \in \mathbb{R}\) reads \(x^1, \ldots, x^m; v^1, \ldots, v^n \mapsto (x^1, \ldots, x^m; \lambda v^1, \ldots, \lambda v^n)\). It is thus a parallel endomorphism of \(VM\) with respect to the connection \(\mathbb{P}^\nabla\), if and only if we have for all \(\mu\) equality between the two tangent vector
expressions

\[
(\Lambda_\lambda)_*(x,v)\left(\frac{\partial}{\partial x^\mu} - \sum_{\alpha=1}^n \Gamma^\alpha_{\mu}(x,v) \frac{\partial}{\partial v^\alpha}\right)
\]

\[
= \frac{\partial}{\partial x^\mu} - \sum_{\alpha=1}^n \lambda \Gamma^\alpha_{\mu}(x,v) \frac{\partial}{\partial v^\alpha} \equiv \frac{\partial}{\partial x^\mu} - \sum_{\alpha=1}^n \Gamma^\alpha_{\mu}(x,\lambda v) \frac{\partial}{\partial v^\alpha}
\]
due to \((\Lambda_\lambda)_*\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\) and \((\Lambda_\lambda)_*\frac{\partial}{\partial v^\alpha} = \lambda \frac{\partial}{\partial v^\alpha}\). Note that the tangent vectors on the left and right hand side of this equation are horizontal in the points \((x,v)\) and \((x,\lambda v)\) of \(VM\) respectively.

In consequence the scalar multiplications \(\Lambda_\lambda\) are parallel endomorphisms of \(VM\) for all scalars \(\lambda \in \mathbb{R}\), if and only if all the generalized Christoffel symbols \(\Gamma^\alpha_{\mu}(x,v)\) are homogeneous functions of degree one in the coefficient coordinates \(v^1, \ldots, v^n\). On the other hand every smooth homogeneous function of degree one is actually a linear function, hence we have proved:

**Lemma 4.2 (Characterization of Linear Connections).**

A non-linear connection \(\mathbb{P}^\nabla\) on a vector bundle gives rise to an \(\mathbb{R}\)-bilinear covariant derivative \(\nabla : \Gamma(M, TM) \times \Gamma(M, VM) \longrightarrow \Gamma(M, VM)\), if and only if the multiplication by every \(\lambda \in \mathbb{R}\) is a parallel endomorphism:

\[
\Lambda_\lambda : VM \longrightarrow VM, \quad v \mapsto \lambda v.
\]

**Appendix A. Explicit Calculations in Local Coordinates**

In order to illustrate the rather abstract and functorial discussion of non-linear connections and their curvature in this article we will repeat parts of the arguments of Section 2 in this appendix by explicit calculations in local coordinates. In particular we will use these calculations in local coordinates to prove the general curvature identity \([8]\) relating the Nijenhuis tensor definition \([4]\) of the curvature \(R^\nabla\) of a non-linear connection on a fiber bundle \(\mathcal{F}M\) to the commutator of iterated covariant derivatives of its sections.

For actual calculations the most convenient definition of the tangent bundle of a smooth manifold \(\mathcal{F}\) is as a set \(T\mathcal{F} := \text{Jet}_0^1(\mathbb{R}, \mathcal{F})\) of equivalence classes of curves \(\gamma : \mathbb{R} \longrightarrow \mathcal{F}\) under first order contact in \(0 \in \mathbb{R}\). The second iterated tangent bundle can be defined analogously as the set \(T(T\mathcal{F})\) of equivalence classes of maps \(\gamma : \mathbb{R}^2 \longrightarrow \mathcal{F}, (t,\varepsilon) \mapsto \gamma(t,\varepsilon)\), under a modified equivalence relation of second order contact in (0,0): Two maps \(\gamma : \mathbb{R}^2 \longrightarrow \mathcal{F}\) and \(\hat{\gamma} : \mathbb{R}^2 \longrightarrow \mathcal{F}\) determine the same equivalence class

\[
\left.\frac{d}{dt}\right|_0 \left.\frac{d}{d\varepsilon}\right|_0 \gamma(t, \varepsilon) = \left.\frac{d}{dt}\right|_0 \left.\frac{d}{d\varepsilon}\right|_0 \hat{\gamma}(t, \varepsilon) \quad \in \ T(T\mathcal{F}),
\]
if and only if for some and hence every local coordinate chart \((f^1, \ldots, f^n)\) of \(\mathcal{F}\) defined in a neighborhood of \(\gamma(0,0) = \hat{\gamma}(0,0)\) the 4n real numbers

\[
\begin{align*}
f^\alpha \left( \frac{d}{dt} \bigg|_0 \frac{d}{d\varepsilon} \bigg|_0 \gamma(t, \varepsilon) \right) &= (f^\alpha \circ \gamma)(0,0) \\
\dot{f}^\alpha \left( \frac{d}{dt} \bigg|_0 \frac{d}{d\varepsilon} \bigg|_0 \gamma(t, \varepsilon) \right) &= \frac{\partial}{\partial t} \bigg|_{(0,0)} (f^\alpha \circ \gamma) \\
\delta f^\alpha \left( \frac{d}{dt} \bigg|_0 \frac{d}{d\varepsilon} \bigg|_0 \gamma(t, \varepsilon) \right) &= \frac{\partial}{\partial \varepsilon} \bigg|_{(0,0)} (f^\alpha \circ \gamma) \\
\delta \dot{f}^\alpha \left( \frac{d}{dt} \bigg|_0 \frac{d}{d\varepsilon} \bigg|_0 \gamma(t, \varepsilon) \right) &= \frac{\partial^2}{\partial t \partial \varepsilon} \bigg|_{(0,0)} (f^\alpha \circ \gamma)
\end{align*}
\]

(22)

for \(\gamma\) agree with the corresponding numbers for \(\hat{\gamma}\). In this case these 4n numbers are of course the coordinates of the common equivalence class \((21)\) in the induced coordinate chart \((f^\alpha; \dot{f}^\alpha; \delta f^\alpha; \delta \dot{f}^\alpha)\) of \(T(T,\mathcal{F})\). In contrast to \(T,\mathcal{F}\) the manifold \(T(T,\mathcal{F})\) is not a true jet space, the modification \((22)\) of the second order contact relation depends on the existence of tautological coordinates \((t, \varepsilon)\) on \(\mathbb{R}^2\). From the definition \((22)\) of the equivalence relation we see directly that the involution \(\Theta : T(T,\mathcal{F}) \longrightarrow T(T,\mathcal{F})\) defined in equation \((5)\) by precomposing \(\gamma : \mathbb{R}^2 \longrightarrow \mathcal{F}\) with the swap \((t, \varepsilon) \mapsto (\varepsilon, t)\) is well–defined on equivalence classes and takes the simple form

\[
\Theta ( f^\alpha; \dot{f}^\alpha; \delta f^\alpha; \delta \dot{f}^\alpha ) = ( f^\alpha; \delta f^\alpha; \dot{f}^\alpha; \delta \dot{f}^\alpha )
\]

(23)

in local coordinates, compare equation \((16)\). Similarly the double projection \(\Pi : T(T,\mathcal{F}) \longrightarrow T,\mathcal{F} \oplus T,\mathcal{F}\) becomes in the chosen local coordinates

\[
\Pi( f^\alpha; \dot{f}^\alpha; \delta f^\alpha; \delta \dot{f}^\alpha ) = ( f^\alpha; \delta f^\alpha ) \oplus ( f^\alpha; \dot{f}^\alpha ) = ( f^\alpha; \delta f^\alpha; \dot{f}^\alpha ),
\]

because \(\oplus\) for vector bundles over \(\mathcal{F}\) is essentially the same as \(\times\) and effectively removes the redundant copy of the coordinates \(f^\alpha\). For an alternative coordinate chart \((h^1, \ldots, h^n)\) of the manifold \(\mathcal{F}\) the chain rule applied to the composition \(h^\omega \circ \gamma = (h^\omega \circ f^{-1}) \circ (f \circ \gamma)\) tells us

\[
\bar{\partial} h^\omega(f; \dot{f}; \delta f; \delta \dot{f}) = \sum_{\alpha, \beta = 1}^n \frac{\partial^2 h^\omega}{\partial f^\alpha \partial f^\beta}(f) \delta f^\alpha \dot{f}^\beta + \sum_{\alpha = 1}^n \frac{\partial h^\omega}{\partial f^\alpha}(f) \delta \dot{f}^\alpha
\]

for the induced change of coordinates on \(T(T,\mathcal{F})\) with similar formulas for \(\hat{h}^\omega\) and \(\bar{\partial} h^\omega\). It is the symmetry of \(\frac{\partial^2 h^\omega}{\partial f^\alpha \partial f^\beta}(f)\) under the interchange \(\alpha \leftrightarrow \beta\), which allows \(\Theta\) to read \((23)\) in all local coordinates on \(T(T,\mathcal{F})\) induced by coordinates on \(\mathcal{F}\)! In taking differences the quadratic term drops out, hence the fibers of \(\Pi\) are affine spaces modelled on \(T,\mathcal{F}\) under the difference map

\[
(f^\alpha; \dot{f}^\alpha; \delta f^\alpha; \hat{F}^\alpha) - (f^\alpha; \dot{f}^\alpha; \delta f^\alpha; \hat{F}^\alpha) := (f^\alpha; \hat{F}^\alpha - \hat{F}^\alpha)
\]

(24)

in all local coordinates on \(T(T,\mathcal{F})\) induced by local coordinates on \(\mathcal{F}\).

Turning to fiber bundles over a manifold \(M\) we choose local coordinates on the total space of a fiber bundle by lifting local coordinates \((x^1, \ldots, x^m)\) of \(M\) to functions on \(\mathcal{F}M\) still denoted by \(x^1, \ldots, x^m\). Complemented by suitable functions \(f^1, \ldots, f^n \in C^\infty(\mathcal{F}M)\) we obtain a system of local coordinates \((x^1, \ldots, x^m; f^1, \ldots, f^n)\) with the property that the
differentials $dx^1, \ldots, dx^m$ of the coordinates constant along the fibers of $\mathcal{F} M$ generate the horizontal forms so that the vertical tangent bundle is spanned by:

$$\text{Vert}_{(x,f)} \mathcal{F} M := \text{span}_\mathbb{R} \left\{ \frac{\partial}{\partial f^1}, \ldots, \frac{\partial}{\partial f^n} \right\} \subseteq T_{(x,f)} \mathcal{F} M$$

All the constructions discussed in the first part of this appendix for the second iterated tangent bundle $T(T \mathcal{F})$ extend without further ado to the second iterated vertical tangent bundle $\text{Vert} \text{Vert} \mathcal{F} M$ by throwing in the additional coordinates $(x^\mu; f^\alpha; \delta f^\alpha; \delta^\alpha)$ of the base manifold $M$ as passive coordinates. In this specific local coordinate setup for a fiber bundle an arbitrary non–linear connection $\mathbb{P}^V$ is determined by

$$\mathbb{P}^V_{(x,f)} \left( \frac{\partial}{\partial f^\alpha} \right) = \frac{\partial}{\partial f^\alpha} \quad \mathbb{P}^V_{(x,f)} \left( \frac{\partial}{\partial x^\mu} \right) = \sum_{\alpha=1}^n \Gamma^\alpha_{\mu}(x,f) \frac{\partial}{\partial f^\alpha}$$

with suitable local coefficient functions $\Gamma^\alpha_{\mu}(x,f)$ generalizing the standard Christoffel symbols. Needless to say the curvature $R^V$ of the non–linear connection $\mathbb{P}^V$ is determined by these generalized Christoffel symbols, due to horizontality, it suffices to calculate expression \[4\] for the vector fields:

$$R^V \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)$$

$$= - \mathbb{P}^V \left[ \frac{\partial}{\partial x^\mu} - \sum_{\alpha=1}^n \Gamma^\alpha_{\mu}(x,f) \frac{\partial}{\partial f^\alpha} - \sum_{\beta=1}^n \Gamma^\beta_{\nu}(x,f) \frac{\partial}{\partial f^\beta} \right]$$

$$= \sum_{\beta=1}^n \frac{\partial \Gamma^\beta_{\nu}}{\partial x^\mu}(x,f) \frac{\partial}{\partial f^\beta} - \sum_{\alpha, \beta=1}^n \Gamma^\alpha_{\mu}(x,f) \frac{\partial \Gamma^\beta_{\nu}}{\partial f^\alpha}(x,f) \frac{\partial}{\partial f^\beta} - \sum_{\alpha=1}^n \frac{\partial \Gamma^\alpha_{\nu}}{\partial x^\mu}(x,f) \frac{\partial}{\partial f^\alpha} + \sum_{\beta, \alpha=1}^n \Gamma^\beta_{\nu}(x,f) \frac{\partial \Gamma^\alpha_{\mu}}{\partial f^\beta}(x,f) \frac{\partial}{\partial f^\alpha}.$$ 

Interchanging the dummy indices $\alpha \leftrightarrow \beta$ if necessary, this results in a sum of the form

$$R^V \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right) = \sum R^\alpha_{\mu\nu}(x,f) \frac{\partial}{\partial f^\alpha}$$

with coefficients given by:

$$(25) \quad R^\alpha_{\mu\nu} := \frac{\partial \Gamma^\alpha_{\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_{\mu}}{\partial x^\nu} + \sum_{\beta=1}^n \left( \Gamma^\beta_{\nu} \frac{\partial \Gamma^\alpha_{\mu}}{\partial f^\beta} - \Gamma^\beta_{\mu} \frac{\partial \Gamma^\alpha_{\nu}}{\partial f^\beta} \right)$$

In case the fiber bundle is actually a vector bundle we have a distinguished class of local coordinates $(x^1, \ldots, x^m; v^1, \ldots, v^n)$ on the total space $VM$ associated to a basis $v_1, \ldots, v_n \in \Gamma_{\text{loc}}(M, VM)$ of local sections:

$$(x^1, \ldots, x^m; v^1, \ldots, v^n) \longleftrightarrow v^1 v_1 + \ldots + v^n v_n \in V_{(x^1, \ldots, x^m)} M.$$ 

In these linear local coordinates on $VM$ the generalized Christoffel symbols $\Gamma^\alpha_{\mu}(x,v)$ of a linear connection are linear functions with an expansion

$$\Gamma^\alpha_{\mu}(x^1, \ldots, x^m; v^1, \ldots, v^n) = \sum_{\omega=1}^n \Gamma^\alpha_{\mu\omega}(x^1, \ldots, x^m) v^\omega$$
with local coefficient functions $\Gamma^\alpha_{\mu \omega}(x)$. In particular the expansion \(^{(25)}\) of the curvature of a linear connection simplifies in linear coordinates on $VM$

$$R^{\nabla}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \sum_{\alpha, \omega = 1}^{n} R^\alpha_{\mu \nu \omega}(x^1, \ldots, x^m) \psi^\omega \frac{\partial}{\partial \psi^\alpha}$$

with coefficients $R^\alpha_{\mu \nu \omega}(x)$ given by the classical formula:

\(^{(26)}\)

$$R^\alpha_{\mu \nu \omega} = \frac{\partial \Gamma^\alpha_{\nu \omega}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_{\mu \omega}}{\partial x^\nu} + \sum_{\beta = 1}^{n} \left( \Gamma^\alpha_{\mu \beta} \Gamma^\beta_{\nu \omega} - \Gamma^\alpha_{\nu \beta} \Gamma^\beta_{\mu \omega} \right).$$

Coming back to fiber bundles we want to discuss the covariant derivative of a local section $f \in \Gamma_{loc}(M, \mathcal{F}M)$ of $\mathcal{F}M$, which in the chosen local coordinates on $M$ and $\mathcal{F}M$ becomes an $n$–tuple of smooth functions:

$$f(x^1, \ldots, x^m) = (x^1, \ldots, x^m; f^1(x^1, \ldots, x^m), \ldots, f^n(x^1, \ldots, x^m)).$$

In turn its covariant derivative with respect to the connection $\nabla$ reads:

\(^{(27)}\)

$$\left(\frac{\partial}{\partial x^\mu} f\right)(x) = \mathbb{P}(x, f(x)) \left( \frac{\partial}{\partial x^\mu} + \sum_{\alpha = 1}^{n} \frac{\partial f^\alpha}{\partial x^\mu}(x) \frac{\partial}{\partial f^\alpha} \right)$$

$$= \sum_{\alpha = 1}^{n} \left( \frac{\partial f^\alpha}{\partial x^\mu}(x) + \Gamma^\alpha_{\mu \beta}(x, f(x)) \right) \frac{\partial}{\partial f^\alpha},$$

In this formula we have considered the result as a vertical vector field, alternatively we may write the result as a local section of $\text{Vert} \mathcal{F}M$

$$\left(\frac{\partial}{\partial x^\mu} f\right)(x) = \left( x; f^\alpha(x); \frac{\partial f^\alpha}{\partial x^\mu}(x) + \Gamma^\alpha_{\mu \beta}(x, f(x)) \right).$$

For a smooth family $(f_\epsilon)$ of local sections of $\mathcal{F}M$ we obtain the variation

$$\left(\frac{\partial}{\partial \epsilon} f_\epsilon\right)(x) = \left( x; f^\alpha(x); \frac{\partial f^\alpha}{\partial x^\mu}(x) + \Gamma^\alpha_{\mu \beta}(x, f(x)) \right);$$

$$\delta f^\alpha(x); \frac{\partial \delta f^\alpha}{\partial x^\mu}(x) + \sum_{\beta = 1}^{n} \frac{\partial \Gamma^\alpha_{\mu \beta}}{\partial f^\beta(x)}(x, f(x)) \delta f^\beta(x) \right)$$

with $f^\alpha := f^\alpha_0$ and $\delta f^\alpha := \frac{\partial}{\partial \epsilon} f^\alpha_0$. Evidently we need to apply the fiberwise involution $\Theta$ to interpret the result as the covariant derivative of the variation $\frac{\partial}{\partial \epsilon} f_\epsilon \in \Gamma_{loc}(M, \text{Vert} \mathcal{F}M)$ with respect to the non–linear connection $\mathbb{P}_\text{Vert}^{\nabla}$ induced by equation \(^{(7)}\) on the vertical tangent bundle:

$$\left(\frac{\partial}{\partial \epsilon}^{\nabla}_{\text{Vert}} f_\epsilon\right)(x) = \left( x; f^\alpha(x); \frac{\partial f^\alpha}{\partial x^\mu}(x) + \Gamma^\alpha_{\mu \beta}(x, f(x)) \right);$$

$$\delta f^\alpha(x); \frac{\partial \delta f^\alpha}{\partial x^\mu}(x) + \sum_{\beta = 1}^{n} \frac{\partial \Gamma^\alpha_{\mu \beta}}{\partial f^\beta(x)}(x, f(x)) \delta f^\beta(x) \right).$$
Comparing the resulting expression with the formula (27) for a general non-linear connection we conclude that the connection $P^\nabla_{\text{Vert}}$ is given by

$$P^\nabla_{\text{Vert}}(x, f, \delta f)(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial f^\alpha}) = \sum_{\alpha, \beta = 1}^n \frac{\partial \Gamma^\alpha_\mu(x, f)}{\partial f^\beta} \frac{\partial}{\partial f^\alpha} \frac{\partial}{\partial \delta f^\beta},$$

note that the vertical tangent vectors $\frac{\partial}{\partial f^\alpha}$ and $\frac{\partial}{\partial \delta f^\beta}$ are fixed by every non-linear connection on $\text{Vert} \mathcal{F} M$. Substituting the variation $\delta f^\alpha(x)$ by the corresponding expression $\frac{\partial f^\alpha}{\partial x^nu}(x) + \Gamma^\alpha_\mu(x, f(x)) \frac{\partial}{\partial x^\mu} f(x)$ in the covariant derivative $D^\nabla_{\text{Vert}} f$ of the local section $f \in \Gamma_{\text{loc}}(M, \mathcal{F} M)$ we obtain directly:

$$\left( D^\nabla_{\text{Vert}} \frac{\partial}{\partial x^\mu} f \right)(x) = \left( x; f^\alpha(x), \frac{\partial f^\alpha}{\partial x^\nu}(x) + \Gamma^\alpha_\mu(x, f(x)), \frac{\partial f^\alpha}{\partial x^\mu}(x) + \Gamma^\alpha_\mu(x, f(x)) \right)$$

$$+ \left( \frac{\partial^2 f^\alpha}{\partial x^\mu \partial x^\nu}(x) + \frac{\partial \Gamma^\alpha_\nu}{\partial x^\mu}(x) + \sum_{\beta = 1}^n \frac{\partial \Gamma^\alpha_\mu}{\partial f^\beta}(x, f(x)) \Gamma^\beta_\nu(x, f(x)) \right)$$

$$+ \left( \sum_{\beta = 1}^n \frac{\partial \Gamma^\alpha_\nu}{\partial f^\beta}(x, f(x)) \frac{\partial f^\beta}{\partial x^\mu}(x) + \sum_{\beta = 1}^n \frac{\partial \Gamma^\alpha_\mu}{\partial f^\beta}(x, f(x)) \frac{\partial f^\beta}{\partial x^\nu}(x) \right).$$

Evidently it is necessary to apply the involution $\Theta$ first to form the difference

$$\left( D^\nabla_{\text{Vert}} \frac{\partial}{\partial x^\mu} f - \Theta \left( D^\nabla_{\text{Vert}} \frac{\partial}{\partial x^\nu} f \right) \right)(x) = \left( x; f^\alpha(x), \frac{\partial \Gamma^\alpha_\nu}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_\mu}{\partial x^\nu} + \sum_{\beta = 1}^n \left( \Gamma^\beta_\nu \frac{\partial \Gamma^\alpha_\mu}{\partial f^\beta} - \Gamma^\beta_\mu \frac{\partial \Gamma^\alpha_\nu}{\partial f^\beta} \right) \right)$$

$$= \left( x; f^\alpha(x), R^\alpha_\mu(x, f(x)) \right) = \left( R^\nabla_{\text{Vert}} \frac{\partial}{\partial x^\mu} f \right)(x),$$

where the coefficients $R^\alpha_\mu(x, f)$ of the curvature of the connection $P^\nabla$ have been calculated in equation (25). Due to $[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}] = 0$ the latter identity is exactly the local coordinates formulation of the curvature identity (8).

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