Unitary Kinematic Mixing of Electro-Weak Bosons

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Abstract
In the standard model, electro-weak bosons are developed as gauge-fields initially satisfying an SU(2) × U(1) local symmetry mediating interactions with a multi-component field. This symmetry gets broken when the component of the field associated with the (scalar) Higgs boson acquires a non-vanishing expectation value, generating masses for weak vector bosons (W⁺, W⁻, Z) as well as a massless photon (A) which couples to electric charge. Beyond an expression relating the masses of the W and Z bosons, there are few predictions concerning further relationships involving the Higgs mass or mixing angle. In this paper, kinematic relationships between unitarily mixed momentum-space spinors (supplemental to standard electro-weak modeling) will be developed. These relationships expressed in terms of kinematic invariants are shown to be consistent with current measured results.

Keywords Weak boson masses · Kinematic constraints on electroweak parameters

1 Introduction
Quantum fields representing fundamental particles are generally required to be eigenstates of the operator \( \hat{P}_\mu \eta^{\mu\nu} \hat{P}_\nu \), with eigenvalues given by \(- (mc)^2\). Field equations utilizing operator forms that are linear in the four-momentum have analytic behaviors that are directly manipulable for developing straightforward relationships using standard scattering theory, especially with regards to cluster decomposition properties. In a generalization of Dirac’s formulation [1], spinor fields that require the form \( \hat{\Gamma}^\mu \hat{P}_\mu \) to be a Lorentz scalar operation, while satisfying

\[
\Gamma^\beta \cdot \frac{\hbar}{i} \frac{\partial}{\partial x^\beta} \hat{\psi}^{(\Gamma)}(\vec{x}) = -\gamma mc \hat{\psi}^{(\Gamma)}(\vec{x}),
\] (1.1)

can be developed. For these fields, \( m \geq 0 \) always, \( \Gamma^\beta \) are the finite dimensional matrix representations of the operators \( \hat{\Gamma}^\beta \), and for massive states, \( \gamma \) are the eigenvalues of the hermitian operator \( \hat{\Gamma}^0 \). In (1.1), the particle mass \( m \) labels a unitary representation of the (extended) Poincare group of transformations on the particle state. The fundamental Dirac
representations are fermion fields with $\Gamma = \frac{1}{2}$, where the standard Dirac matrices $\gamma^\beta$ satisfy $\Gamma^\beta = \frac{i}{2} \gamma^\beta$.

Standard model electro-weak bosons originate from massless (configuration-space) gauge fields, with charged components ($W^\pm$) which become massive (while maintaining renormalizability) via interaction with a multi-component field that has a component acquire non-vanishing vacuum expectation value (plus the scalar Higgs boson) [2, 3]. The neutral components of the gauge fields furthermore mix to generate a neutral massive vector boson (Z), along with the photon (A). In this paper, electro-weak bosons that transform under the $\Gamma = 1$ representation of (1.1) are developed. The matrix representations $\Gamma^\beta$ are 10-dimensional, inherently combining scalars with vectors. For massive particles, the momentum-space forms of the $\Gamma = 1$ spinors depend on kinematic energy-momentum relationships (as in the Dirac representation), while the $m \to 0$ form of $z$-moving spinors are pure numbers. Kinematic unitary mixing of the degenerate $\gamma = 0$ momentum-space spinors will be explored, and invariant mixing angles will be developed. Furthermore, it will be demonstrated that an appropriate co-moving spinor with a mass consistent with that of the Higgs boson provides precisely the spinor forms needed for the mixing spinors to be eigenstates for massive representations of the group algebra. In what follows, natural units with $\hbar = 1, c = 1$ will be utilized, and Einstein’s summation convention over repeated superscripts will be assumed.

2 Review of Standard Electro-Weak Modeling

To begin, a brief description of relevant aspects of the standard electro-weak model [3] will be presented. Consider a non-interacting field $\phi$ whose dynamics are described using a Lagrangian density of the form $\mathcal{L}_o = -\frac{1}{4} (\partial_\mu \phi^\dagger \eta^{\mu \nu} \partial_\nu \phi + m^2 \phi^\dagger \phi)$. This field has a real mass $m$, with momentum-space components satisfying the usual relativistic energy-momentum dispersion relation.

Local gauge invariance: The dynamics of the field $\phi$ in $\mathcal{L}_o$ can be made invariant under unitary local gauge transformations $\phi(x) \to \tilde{\phi}(x) = U(\alpha)\phi(x) \equiv e^{ig \alpha^b(x) G^b(x)}$ (where the $G^b$ are hermitian generators of infinitesimal transformations using group parameters $\delta \alpha^b$) by introducing gauge potentials $B^{(a)}_\mu$. The gauge potentials are used to construct gauge-covariant derivatives which replace the standard gradients $\partial_\mu \to D_\mu \equiv \partial_\mu - ig B^{(a)}_\mu G^a$, and the potentials transform according to $B^{(a)}_\mu G^a \to \tilde{B}^{(a)}_\mu G^a = B^{(a)}_\mu U(\alpha) G^a U(\alpha)+\partial_\mu \alpha^a G^a$. In addition, a term of the form $\frac{1}{16\pi} F^{\mu \nu} F_{\mu \nu}$ (using Gaussian units) is included in the Lagrangian to generate the field equations fully describing how charges create and interact with the gauge potentials.

Spontaneous symmetry breaking: Suppose that an interaction potential $\mathcal{V}(\phi^\dagger \phi)$ is subtracted from the Lagrangian form $\mathcal{L}_o$. Such a potential maintains the symmetries of $\mathcal{L}_o$, and any linear term in the potential simply adjusts its mass term. Consider a potential of the form $\mathcal{V}(\phi^\dagger \phi) \equiv \frac{1}{2} (\phi^\dagger \phi)^2$ in the Lagrangian (with $\lambda > 0$), i.e. $\mathcal{L} = -\frac{1}{4} \{ \partial_\mu \phi^\dagger \eta^{\mu \nu} \partial_\nu \phi + m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \}$. The effective potential $\mathcal{V}_{eff} \equiv \frac{1}{2} [m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2]$ has a minimum at $<\phi> \equiv \langle \text{vac}|\phi|\text{vac} \rangle = 0$, indicating the vacuum as the lowest energy (ground) state of the system. However, if the mass becomes ‘tachyonic’ $m \to i\mu$, then $\mathcal{V}_{eff} \to \frac{1}{2} \left[ -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \right]$, which has a minimum when $<\phi> \geq \frac{\mu^2}{\sqrt{\lambda}}$, defining a new set of ground states for the complex field $\phi$. If the system settles into one of these ground states, the system itself loses a symmetry of the overall
Lagrangian, and gains a massless (Goldstone) mode for excitations along the un-chosen set of ground states. Thus, the system then manifests a broken symmetry.

2.0.1 The Glashow-Salam-Weinberg Model

Weak interactions maximally discriminate between left-handed and right-handed particles. To model this phenomenology, the field $\phi_H$ was chosen to initially manifest local gauge symmetry under the group of transformations $SU(2)_w \times U(1)_Y$, where the three $SU(2)$ (weak-isospin) generators will be denoted by $\hat{\tau}_a$, and the $U(1)$ (weak-hypercharge) generator will be denoted by $\hat{Y}$. The Lagrangian density that manifests this local invariance takes the form

$$L_{EW} = \left( (\partial_\mu - ig_{Ww} W_{\mu}^{(a)} \hat{\tau}_a - ig_{B} B_{\mu} \hat{Y}) \phi \right)^\dagger \eta^{\mu\nu} (\partial_\nu - ig_{Ww} W_{\nu}^{(b)} \hat{\tau}_b - ig_{B} B_{\nu} \hat{Y}) \phi - V(\phi^\dagger \phi)$$

(2.1)

where $V(\phi^\dagger \phi) \sim -\mu^2 \phi \phi^\dagger + \lambda(\phi^\dagger \phi)^2$. This potential form has a minimum when $|\phi^\dagger \phi| \sim \mu^2 \phi^2 \lambda$ rather than $|\phi^\dagger \phi| = 0$, allowing a component of the field $\phi$ to settle into a particular non-vanishing minimal value thereby breaking its symmetry from that of the Lagrangian.

The field is then expressed in the form of a weak iso-doublet $\hat{\phi} \equiv (\hat{\phi}^0 + i\hat{\phi}^2) \hat{\phi}^0 + i\hat{\phi}^3)$, where the scalar component acquires a non-vanishing vacuum expectation value $\langle \text{vac} | \hat{\phi}^0 | \text{vac} \rangle \equiv <\phi^0> = \mu \phi / \sqrt{2}$, spontaneously breaking its symmetry and defining a new ground state. Expanding about this new ground state $\hat{\phi}^0 = v / \sqrt{2} + \hat{h}$ defines a Higgs scalar field $\hat{h}(x)$ that has vanishing vacuum expectation value and real (time-like) mass $m_H = \sqrt{2} \mu \phi$. The remaining fields $\phi^1$, $\phi^2$, and $\phi^3$ continue to transform together under SU(2) maintaining gauge invariance, and thus can be eliminated by a suitable choice of gauge. These three degrees of freedom are ‘consumed’ in the generation of longitudinal components of what become massive $W^\pm$ and $Z$ bosons.

The presence of a symmetry-breaking vacuum expectation value in the Higgs sector has significant consequences for the gauge vector potentials $W_{\mu}^{(a)}$ and $B_{\mu}$ in (2.1). After substituting the form of $\hat{\phi}^0$ and examining quadratic terms for $W_{\mu}^{(\pm)}$ in (2.1), one discovers a mass term $m_{W^{(\pm)}} = 1 \mu [g_{W}]$ for this previously massless gauge boson. Thus the weak force mediated by these bosons becomes short ranged. The remaining gauge bosons $W_{\mu}^{(3)}$ and $B_{\mu}$ are then mixed to generate a massless photon that appropriately couples to electric charge, along with a massive charge-neutral $Z$ boson.

If one substitutes the form

$$W_{\mu}^{(3)} = Z_{\mu} \cos \vartheta_{ZW} - A_{\mu}^{(3)} \sin \vartheta_{ZW}, \quad B_{\mu} = Z_{\mu} \sin \vartheta_{ZW} + A_{\mu}^{(3)} \cos \vartheta_{ZW},$$

(2.2)

into (2.1), the proper coupling of electric charge $e$ to a massless photon field $A_\mu$ requires that $e = g_{W} \sin \vartheta_{ZW} = g_{B} \cos \vartheta_{ZW}$. Furthermore, the term quadratic in $Z_{\mu}$ defines its mass as $m_Z = \sqrt{\frac{1}{2} \mu [g_{W}^2 + g_{B}^2}]$. Combining these equations, one arrives at the standard electro-weak identifications:

$$e = g_{W} \sin \vartheta_{ZW} = g_{B} \cos \vartheta_{ZW}, \quad \cos \vartheta_{ZW} = \frac{m_{W}}{m_{Z}}, \quad v = \frac{2m_{W} \sin \vartheta_{ZW}}{|e|}. \quad (2.3)$$
3 Unitary Mixing of Degenerate Momentum-Space Bosonic Spinors

Momentum-space spinor fields described in (1.1) satisfy

$$\Gamma^\beta p_\beta \Phi_{\gamma,J_z}(\vec{p}) = -\gamma m \Phi_{\gamma,J_z}(\vec{p}),$$

where the spin $J$ are integers satisfying $0 \leq J \leq \Gamma = 1$, and $-J \leq \gamma \leq +J$ represent an additional set of discrete quantum numbers. The $\Gamma^\beta$ are $10 \times 10$ matrices developed elsewhere [4–6]. The 10-spinors will be Hermitian normalized and labeled such that for a state at rest, all elements except one vanish, with ordering $\Phi_{\gamma,J_z}(\vec{p}) = \{ \Phi_{0,0}, \Phi_{1,1}, \Phi_{1,0}, \Phi_{0,1}, \Phi_{0,0}, \Phi_{0,-1}, \Phi_{-1,1}, \Phi_{-1,0}, \Phi_{-1,-1} \}$. It should also be noted that in the limit $m \to 0$, the forms $\mathcal{A}^\beta_{\gamma,J_z} \equiv \Gamma^\beta \Phi_{\gamma,J_z}(\vec{p})$ are momentum space representations of transverse vector fields that are degenerate with massive fields having $\gamma = 0$. This does not occur for Dirac representation fields $\Gamma = \frac{1}{2}$.

**General kinematic mixing:** It is of interest to examine (unitary) mixing of $\gamma = 0$ degenerate spinors satisfying (3.1), which does not occur amongst the $\gamma \neq 0$ spinors. Utilizing identifications consistent with the notations reviewed in Section 2, orthogonal spinors $W$ and $B$ can be mixed as long as one is ‘tachyonic’ (space-like), e.g. $m_B \to i\mu_B$, with

$$p_B \to \pm \frac{\mu_B}{m_W} \sqrt{m_W^2 + p_W^2}, \quad \sqrt{m_B^2 + p_B^2} \to \pm \frac{\mu_B}{m_W} p_W \geq 0,$$

where the signs insure positivity of $\epsilon_B$. A kinematic mix vertex is represented in Fig. 1.

![Fig. 1 Kinematic on-shell mixing requires $\vec{P}_W + \vec{P}_B = \vec{P}_{WB}$](#)
The degenerate $\gamma = 0$ spinors $W_0(\vec{p}_W)$ and $B(\vec{p}_B)$ can mix to generate new spinor forms $W_B$ and $B_W$ that continue to satisfy (3.1) according to

$$\begin{align*}
\cos \theta_{W_B} & W_{0, J_z}(\vec{p}_W) + \sin \theta_{W_B} B_{J_z}(\vec{p}_B) = W_{0, B}(\vec{p}_{WB}). \\
-\sin \theta_{W_B} & W_{0, J_z}(\vec{p}_W) + \cos \theta_{W_B} B_{J_z}(\vec{p}_B) = B_{0, J_z}(\vec{p}_{BW}).
\end{align*}$$

(3.3)

where for $z$-moving systems, the parameter $J_z$ labels the helicity of the particle. The index $\gamma = 0$ on a spinor should not be confused with any co-variant space-time index.

In what follows, only motions parallel to the $z$-axis will be examined, and general momentum eigenspinors can be developed through a spatial rotation. Labeling the momentum of boson $X$ by $p_X$, the normalized eigen-spinors are most elegantly expressed in terms of dimensionless kinematic angles $\zeta_X$ defined as

$$\zeta_X \equiv \sin^{-1}\left(\frac{p_X}{\sqrt{m_X^2 + 2p_X^2}}\right), \quad p_X = m_X \frac{\sin \zeta_X}{\sqrt{\cos 2\zeta_X}}, \quad \epsilon_X = m_X \frac{\cos \zeta_X}{\sqrt{\cos 2\zeta_X}}.$$  

(3.4)

The parameter $\zeta$ is directly related to the Lorentz boost $\beta \equiv \frac{\epsilon}{v} = \tan \zeta$, which implies that co-moving spinors have identical forms expressed in terms of $\zeta$. For massive particles, $-\frac{\pi}{4} < \zeta_m < +\frac{\pi}{4}$, while massless particles take the extremal values $\zeta_{massless} = \pm \frac{\pi}{4}$, and ‘tachyonic’ systems will satisfy $\frac{\pi}{4} < |\zeta_m| < \frac{\pi}{2}$.

Degenerate spinors moving parallel to the $z$-axis take a general form similar to

$$\Phi^{(1,J=1)}_{0, J_z=1}(\vec{p}_X) \equiv X_{0, +1}(\zeta_X) = \begin{pmatrix} 0 \\ \sin \zeta_X \\ 0 \\ \cos \zeta_X \\ 0 \\ \sin \zeta_X \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

(3.5)

(with $J_z = 0$ components mixing scalar and longitudinal vector elements 1,3,9, while $J = 1$, $J_z = 0$, has only a single non-zero component of 1 as the 6th element). Notice that the $z$-moving spinors with $m_X \rightarrow 0$ have components that are pure numbers (though not rotationally invariant). Assuming only orthogonality $[B(\zeta_B)]^\dagger W_0(\zeta_W) = 0$ and $\theta_{WB} > 0$ in (3.3), one can show that if $\text{Sign}(\zeta_B) = -\text{Sign}(\zeta_W)$, then $\zeta_B = \zeta_W - \text{Sign}(\zeta_W)\frac{\pi}{2}$ and $\text{Sign}(\zeta_W)(\zeta_W - \zeta_{WB}) = \theta_{WB}$.

The form of the parameter $\zeta_W$ can be expressed in terms of Lorentz invariants by defining $(p_W^\mu + p_B^\mu) \eta_{\mu \nu} (p_W^\nu + p_B^\nu) \equiv -M^2$ while using (3.2) to obtain

$$\zeta_W^M \equiv \pm \sin^{-1}\left(\frac{-2m_W \mu_B + \sqrt{M^4 + 2M^2(\mu_B^2 - m_W^2) + (\mu_B^2 + m_W^2)^2}}{\sqrt{2}(M^2 - m_W^2)^2 + 2(M^2 + m_W^2)\mu_B^2 + \mu_B^4}\right).$$

(3.6)

where the sign matches that of the momentum of the $W_0$ spinor. The analogous formula for $\zeta_B^M_{\mu_W}$ has a sign opposite $\zeta_W^M_{\mu_B}$ with the labels $W$ and $B$ interchanged. The angle $\zeta_W^M_{\mu_B}$
vanishes when \( M \rightarrow \sqrt{m_W^2 - \mu_B^2} < m_W \). The kinematic angle of the spinor \( W B \) resulting from this mixing satisfies (assuming \( p_W \geq 0 \))

\[
\zeta W B^{M}_{\mu_B} = \sin^{-1}\left(\frac{p_W - \frac{\mu_B}{m_W} \epsilon_W}{\sqrt{M^2 + 2 \left(p_W - \frac{\mu_B}{m_W} \epsilon_W\right)^2}}\right) = \zeta W^{M}_{\mu_B} - \theta_{WB}, \quad (3.7)
\]

where \( \epsilon_W \equiv \sqrt{m_W^2 + p_W^2} \).

**Mixing to a massless A**: The resultant kinematic angle for mixing to a massless spinor \( M \rightarrow 0 \) must be \( \pm \frac{\pi}{4} \). The kinematics then constrains \( \tilde{m}_B \leftrightarrow i \mu_B \) such that

\[
\zeta B_0^{m_W} |\tilde{m}_B \rightarrow \tilde{\theta}_{WB} - \frac{\pi}{4}, \quad \zeta W_0^{m_Z} = \frac{\pi}{4} - \tilde{\theta}_{WB} \Rightarrow \tilde{\theta}_{WB} = m_W \tan \tilde{\theta}_{WB}. \quad (3.8)
\]

This analytically fixes a mixing angle independent of the kinematic parameter \( M \), a necessary characteristic for quantum field mixing defined in configuration space. With this identification, the form (3.7) satisfies \( \zeta W B^{M}_{\mu_B} \Rightarrow M \rightarrow 0 \) \( \pm \frac{\pi}{4} \).

**Mixing to a massive Z**: The particular case in which the invariant mass of the \( WB \) mixture \( M = m_Z \equiv \frac{m_W}{\cos \tilde{\theta}_{WB}} \) analytically generates a unique kinematic solution for which

\[
\zeta W^{m_Z}_{\mu_B} = \frac{\tilde{\theta}_{WB}}{2} = -\zeta W B^{m_Z}_{\mu_B}. \quad (3.9)
\]

Furthermore, a massive \( H \) spinor with the momentum \( p_H = p_W (\zeta W^{m_Z}_{m_H}) \) for space-like mixing with a \( W \) to produce a \( Z \) is co-moving with the \( W_0 \) involved in the unitary mixing (as well as anti-moving with the \( Z \)), \( \zeta_H (p_W (\zeta W^{m_Z}_{m_H})) = \zeta W^{m_Z}_{\mu_B} \). The vector components of this \( H \) spinor then contribute the momentum-space spinor (and phase-space factors) to be ‘consumed’ by now massive non-mixing \( W^\pm \) states, as well as the \( W_0 \) mix with \( B \) to the massive \( Z \) state. This co-moving concurrence only occurs for \( M = \frac{m_W}{\cos \tilde{\theta}_{WB}} \). Thus, the relation \( \zeta_H (p_W (\zeta W^{m_Z}_{m_H})) = \frac{\tilde{\theta}_{WB}}{2} \) yields an additional kinematic condition given by

\[
\frac{m_W}{m_Z} = \frac{2m_H^3}{2m_H^3 - 2m_H m_W^2 + m_W \sqrt{(m_H^2 + m_W^2)^2 + 2(m_H^2 - m_W^2)m_H^2 + m_W^4}}, \quad (3.10)
\]

which determines another mass ratio. It should be noted that the mixing angle defined by \( W + H \rightarrow Z \) is significantly different from \( \tilde{\theta}_{WB} \).

Substitution of mass values from the Particle Data Group (PDG) \([7]\) results in a calculation of the mass of the Higgs boson well within the experimental uncertainty, \( \frac{\Delta m_H}{m_H} \sim 6.4 \times 10^{-5} \) vs. \( 1.1 \times 10^{-3} \). Alternatively, the calculated mixing angle deviates from the quoted values by \( \frac{\Delta \cos \tilde{\theta}_{WB}}{m_W/m_Z} \sim 6.3 \times 10^{-6} \), also well within the experimental uncertainty of \( 1.5 \times 10^{-4} \).

### 4 Additional Kinematic Correspondences

As asserted in the previous section, the Higgs scalar is the only component of the \( H \)-spinor not consumed to produce massive vector bosons. These components have momenta satisfying \( p_H (\zeta W^{m_Z}_{m_H}) = p_W (\zeta W^{m_Z}_{m_H}) \). It is of interest to next explore analogous relationships involving energies.
At a fundamental level, one expects the $Z$ spinor to have the potential to directly mix into other degenerate states. In particular, examine the mixing of a $Z$ with an orthogonal $\phi$ (whose symmetry was spontaneously broken) having 'tachyonic' mass $\mu_\phi = \frac{m_H}{\sqrt{2}}$. The $Z$ should be able to self-generate $Z + \phi \to Z$, with kinematic angle given by $\zeta Z^M_{\mu_\phi}$, calculated from (3.6). Furthermore, using (3.8) this can define a particular $Z + \phi$ mixing angle into a massless state as

$$\tilde{\theta}_{Z\phi} \equiv \tan^{-1} \frac{\mu_\phi}{m_Z} = \tan^{-1} \frac{\mu_H}{\sqrt{2} m_Z} \Rightarrow \zeta Z^M_{m_Z \tan \tilde{\theta}_{Z\phi}} = \zeta Z^M_{\mu_\phi}. \quad (4.1)$$

More generally, $\theta_{Z\phi}$ represents the Lorentz invariant mixing angle of $Z + \Phi \to X_M$ for all invariant energies $M$. Relationships of kinematic angles involving known particles ($W$, $Z$, $H$) that are co-moving with $\zeta Z^M_{\mu_\phi}$ will be examined for further insights.

In particular, from (3.4), the kinematic angle for a mass $m$ with energy $\epsilon$ generally satisfies

$$\zeta_m^\epsilon(\epsilon) = \sin^{-1} \sqrt{\frac{\epsilon^2 - m^2}{2\epsilon^2 - m^2}}. \quad (4.2)$$

Consider a particle with mass $m_*$ having threshold rest-energy for particle $m_{\text{th}}$, which has a spinor with kinematic angle

$$\zeta_{m_*}^\epsilon(m_{\text{th}}) = \sin^{-1} \sqrt{\frac{m_{\text{th}}^2 - m_*^2}{2m_{\text{th}}^2 - m_*^2}}, \quad (4.3)$$

establishing a minimal kinematic relationship between the masses. For the general mixing $m + X \to M_{mX}$, the value of $M_{mX}$ in $\zeta m_{\text{MAX}}^M_{m \tan \tilde{\theta}_{mX}}$ for which it co-moves with $\zeta_{m_*}^\epsilon(m_{\text{th}})$ in (4.3) is obtained using (3.6), yielding

$$M_{mX}(m_*, m_{\text{th}}; \tilde{\theta}_{mX}) = \sqrt{\frac{m^2 m_*^2 \cos(2\tilde{\theta}_{mX}) + 2 m_{\text{th}}^2 \sqrt{m_{\text{th}}^4 - m_*^2 m_*^2} \sin(2\tilde{\theta}_{mX})}{m_* \cos \tilde{\theta}_{mX}}}, \quad (4.4)$$

where $\tilde{\theta}_{mX}$ is the previously defined mixing angle of $m + X$ that also generates a massless partner.

As an example of $Z + \phi$ mixing, consider a $Z$ spinor that is co-moving with a $W$ spinor at the threshold energy of a $Z$. One can analytically show that the following kinematic parameters are equal:

$$\zeta Z^W_{\mu_\phi} = \zeta W^e(m_Z) = \zeta Z^e \left( \frac{m_Z^2}{m_W} \right) = \zeta H \left( \frac{m_H m_Z}{m_W} \right) \quad (4.5)$$

where $M^W_{Z\phi} \equiv M_{Z\phi}(m_W, m_Z; \tilde{\theta}_{Z\phi})$. Thus, (4.5) generates no new kinematic condition.
Next, consider the relationship involving a $W$ at $H$ threshold, $\xi^e_H(m_H) = \xi^e_H(m_H^2/m_W)$, which is analytically true. For $Z + \phi$ mixing, a $Z$ satisfying the energy analog of $\xi_H(p_W(\xi W m_Z^2)) = \xi W m_Z^2$ in $W + B$ mixing has

$$\xi^e_H \left( \frac{m_H^2}{m_W} \right) \equiv \xi^e_Z \mu_\phi \Rightarrow \xi^e_Z M^e = \xi^e_W(m_H), \quad (4.6)$$

providing an invariant $M_\phi$ that can be connected to the previous $W + B$ kinematic mixing. Using symmetry, the kinematic angle for the $W + B$ mixing satisfies

$$\xi^e_Z \left( \frac{m_H^2}{m_W} \right) = \xi^e_W \left( \frac{m_H^2}{m_Z} \right) \equiv \xi W m_Z^2 \tan \tilde{\theta}_{WB}, \quad (4.7)$$

which determines $M_\phi$. Reversing the previous step using the analytic relationship $\xi^e_H(\epsilon_W = m_H^2/m_Z) = \epsilon^e_Z(m_H)$ defines the invariant $M_\phi$ as

$$M_\phi = M_{WB}(m_Z, m_H; \tilde{\theta}_{WB}) \sim 245.7\, GeV. \quad (4.8)$$

However, unlike the first equality in (4.5), the relationship $\xi^e_Z M^e \approx \epsilon^e_W(m_H)$ is not an analytic identity, differing in the 6th significant digit from PDG [7] values, thus establishing an additional kinematic condition.

Numerically solving this new relationship

$$\xi^e_Z M^e = \epsilon^e_W(m_H) \quad (4.9)$$

using (4.3), (4.4), and (4.8), along with $\cos \tilde{\theta}_{WB} = m_W/m_Z$ and (3.10), determines all mass ratios, leaving only a single undetermined mass scale. For convenience, these relationships can be reduced to two dimensionless equations. Defining the dimensionless mass ratio $m_r/m_s \equiv \rho_{rs},$ (3.10) can be re-expressed as

$$\rho_{WZ} = \frac{2\rho_{HZ}^3}{2\rho_{HZ}^3 - 2\rho_{HZ}\rho_{WZ}^2 + \rho_{WZ}\sqrt{\rho_{HZ}^4 + (1 - \rho_{WZ}^2)^2 + 2\rho_{HZ}^2(1 + \rho_{WZ}^2)}} \quad (4.10)$$

Furthermore, defining substitutions $\Delta_{WHZ} \equiv \sqrt{(1 - \rho_{WZ}^2)(\rho_{HZ}^4 - \rho_{WZ}^2)}$ and $\delta_{WZ} \equiv 1 - \rho_{WZ}^2$, the remaining equations can be expressed in the dimensionless form

$$\left(64\Delta_{WHZ}\rho_{HZ}^8 + 16\delta_{WZ}\rho_{WZ}^2 + \rho_{HZ}\rho_{WZ}\left(16\Delta_{WHZ}(\mu_{HZ}^2 - 4\delta_{WZ}) - 63\rho_{HZ}^2\rho_{WZ} + 8(1 + 8\rho_{HZ}^2)\rho_{WZ}^3\right)\right)\rho_{WZ}^2 = 8(2\rho_{HZ}^3 - \rho_{HZ}\rho_{WZ}^2)^2. \quad (4.11)$$

The resulting mass ratios

$$\frac{m_w}{m_Z} = 0.88146747846957...$$

$$\frac{m_H}{m_Z} = 1.37196507659589...$$

$$\tilde{\theta}_{WB} = 0.49183562809461... \quad (4.12)$$

all fall well within the uncertainties in measured values: $\frac{\Delta_{WZ} \cos \hat{\theta}_{WB}}{m_{WZ}^2/m_Z^2} \approx 1.2 \times 10^{-6}$ vs. experimental uncertainty $1.7 \times 10^{-4}$, and $m_H \approx 125.106\, GeV$ vs. $125.1 \pm 0.14\, GeV$. It should be noted that the previously reported value [6] for the mixing angle of a (different) self-generating system $\theta_{WB}$ differs from $\tilde{\theta}_{WB}$ at the 6th significant figure, making them inconsistent. It is felt that the value presented here is more fundamental and robust.
5 Discussion and Conclusions

Kinematic constraints relating electro-weak boson masses in terms of a single mass scale have been demonstrated. These constraints follow from unitarily mixing momentum-space spinors that transform as the first boson states beyond the fundamental fermion (Dirac) spinors, with the scalar component of an $H$ spinor being associated with the Higgs scalar. The vector components of the $H$ spinor are ‘consumed’ to generate massive spinor forms for the $W^\pm$ and $Z$. The presented constraints fall well within the reported experimental uncertainties.

The developed relationships are consistent with the standard mixing of space-time dependent fields in Lagrangian dynamics, which can depend only on kinematic invariants. However, these results do not imply that dynamic particle couplings are independent of four-momentum transfers. Both non-perturbative unitary (vertex) structure functions, as well as renormalized perturbative effective couplings are typically not kinematically invariant. It should be emphasized that this formulation does not contradict the Standard Model, but rather provides supplementary insight into the Higgs sector of that model.

Furthermore, any bosons of mass $M$ resulting from $\zeta Z^M_{\mu\phi}$ mixing do not couple to standard electromagnetic charge. Although the kinematic invariant energy $M_\ast \sim 245.7 GeV$ was developed as a convenience for connecting $Z + \phi$ mixing to $W + B$ mixing through this shared value, any physical massive or massless particle resulting from $Z + \phi$ mixing would provide suggestive and intriguing inferences of electromagnetically-dark sector dynamics. Promising expressions relating particle-gauge field couplings to weak boson kinematics are actively being examined as on-going research.

Declarations

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