Hawking Radiations of Kerr-Newman Black Hole in de Sitter Spacetime by Hamilton-Jacobi Method

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ABSTRACT

Incorporating Parikh and Wilczek's opinion Hawking radiations of Kerr-Newman-de Sitter (KNdS) black hole has been investigated by Hamilton-Jacobi method. We have assumed the space time background as dynamical, energy and angular momentum as conserved incorporating the self-gravitation effect of the emitted particles. We have shown that the massive particle tunneling rate is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum.

Keywords: Massive particle tunneling; KNdS black hole; black hole; thermal spectrum.

1. INTRODUCTION

Many researchers have attempted to provide various methods to correctly find out the Hawking radiations of different black holes because Hawking proved that black holes have emission of thermal radiation [1]. To describe Hawking radiation as a quantum tunneling process, a new

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This article is arranged as follows: In section 2 we describe the Kerr-Newman-de Sitter black hole spacetime with the position of event horizon and derive the new line element of KNdS black hole near the event horizon. In section 3 we describe the Hamilton-Jacobi method for the KNdS spacetime. Again, we consider the spacetime background as dynamical and self-gravitational interaction of the emitted particles, the non-thermal tunneling rate of KNdS black hole from massive particle tunneling process have been reviewed in section 4. In section 5 we develop the Hawking purely thermal rate from non-thermal rate. In section 6 we describe our result discussion. Finally, in section 7 we give our conclusion.

2. KERR-NEWMAN-DE SITTER BLACK HOLE

The most general black hole [30] solution can be expressed in the Boyer Lindquist coordinate as [31] an exact solution of the Einstein field equations with a positive cosmological constant \( \Lambda = 3/l^2 \) describes charged rotating black hole in asymptotically de Sitter space with cosmological radius \( l \), mass \( M \), charge \( q \), and angular momentum per unit mass \( a \) of the form

\[
ds^2 = - \frac{f(r) - f(\theta) a^2 \sin^2 \theta}{\rho^2} dt^2 + \frac{f(\theta)(r^2 + a^2)^2 - f(r)a^2 \sin^2 \theta}{\rho^2 \Sigma} d\theta^2 + \frac{\rho^2}{\Sigma} d\varphi^2 + \frac{\rho^2}{f(r)} dr^2 + \frac{\rho^2}{f(\theta)} d\theta^2
\]

Where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad f(\theta) = 1 + \frac{a^2 \cos^2 \theta}{l^2}, \quad \Sigma = 1 + \frac{a^2}{l^2},
\]

\[
f(r) = \left(1 - \frac{a^2}{l^2}\right) r^2 - 2Mr + a^2 \ - \frac{r^4}{l^2} + q^2.
\]

The de Sitter space are defined such that \(-\infty \leq t \leq \infty, \ r \geq 0, \ 0 \leq \theta \leq \pi, \ and \ 0 \leq \phi \leq 2\pi\). The metric (1) describes an interesting charged rotating AdS black hole called the Kerr-Newman-Anti-de Sitter (KNAdS) black hole if we replace \( l^2 \) by \(-l^2\). There are apparent singularities in the metric at the values of \( r \) for which

\[
f(r) = \left(1 - \frac{a^2}{l^2}\right) r^2 - 2Mr + a^2 \ - \frac{r^4}{l^2} + q^2 = 0
\]
The function \( f(r) = 0 \) with \( l^2 > a^2 \) has four distinct roots: \( r_h, r_-, r_c \) and \( r_{-\text{cauchy}} \). The real root \( r_h \) corresponds to the radius of the black hole’s outer event horizon, while the other real root \( r_+ \) represents the radius of the inner Cauchy horizon. Here we indicate \( r_c \) as the cosmological horizon and \( r_{-\text{cauchy}} \) the negative root of \( f(r) \) another cosmological horizon. Equation (3) can be written as

\[
 r^4 - (l^2 - a^2)r^2 + 2Ml^2r - l^2(a^2 + q^2) = 0
\]

Solving the above equation, the black hole event horizon and cosmological horizon can be written respectively of the form

\[
 r_h = \frac{l\beta}{\sqrt{\alpha}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3M \sqrt{3}}{M \alpha} \right) \right] \times \left( 1 + \sqrt{1 - \left( \frac{(a^2 + q^2)}{l^2 M} \right) \frac{2}{1+\delta} \cos \left[ \frac{1}{3} \sin^{-1} \left( \frac{3M \sqrt{3}}{M \alpha} \right) \right]} \right)
\]

And

\[
 r_c = \frac{l\beta}{\sqrt{3}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3M \sqrt{3}}{M \alpha} \right) \right] \times \left( 1 + \delta \frac{3M \sqrt{l^2}}{\sqrt{3} \beta^2} \cos \left[ \frac{1}{3} \sin^{-1} \left( \frac{3M \sqrt{3}}{M \alpha} \right) \right] - 1 \right).
\]

Where

\[
 \delta = \sqrt{1 - \frac{4(a^2 + q^2)\beta^2}{3M^2}} \sin^2 \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}M}{l\alpha} \right) \right]
\]

\[
 \alpha = \sqrt{\left[ \frac{1}{l^2} + \frac{4q^2}{l^2} \right]^{2} + \frac{4q^2}{l^2}}, \quad \beta = \sqrt{1 - \frac{a^2}{l^2}}.
\]

And \( r_{-\text{cauchy}} = -(r_h + r_+ + r_c) \) is the another cosmological horizon. With \( \delta \approx 1 \) the black hole event horizon can be approximated as

\[
 r_h \approx \frac{l\beta}{\sqrt{\alpha}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3M \sqrt{3}}{M \alpha} \right) \right] \times \left( 1 + \sqrt{1 - \left( \frac{(a^2 + q^2)}{M^2} \right)} \right).
\]

Expanding \( r_h \) in terms of \( l, M, q \) and \( a \) with \( (a^2 + q^2)\alpha < M^2 \), we obtain

\[
 r_h = \frac{M}{\alpha} \left( 1 + \frac{4M^2}{l^2\beta^2\alpha} + \cdots \right) \left( 1 + \sqrt{1 - \left( \frac{(a^2 + q^2)}{M^2} \right)} \right),
\]

which can be written as

\[
 r_h = \frac{1}{\alpha} \left( 1 + \frac{4M^2}{l^2\beta^2\alpha} + \cdots \right) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha} \right).
\]
Obviously, the event horizon of the Kerr-Newman-de Sitter black hole is greater than the Kerr-Newman event horizon \( r_{\nu e} = M + \sqrt{M^2 - (a^2 + q^2)} \). It is interesting to note that it reduce to the Kerr-Newman black hole \([28-29]\) for \( l \to \infty \), Kerr-de Sitter black hole for \( q = 0 \), Kerr black hole \([18]\) for \( l \to \infty \), \( q = 0 \) and Schwarzschild-de Sitter black hole \([32]\) for \( a = 0 \) and \( q = 0 \). We perform the following effective transformation to obtain the Hawking radiation of the KNdS black hole.

\[
\frac{d\phi}{dt} = \frac{a\Sigma[f(\theta)(r^2 + a^2) - f(r)]}{f(\theta)(r^2 + a^2) - f(r)a^2 \sin^2 \theta}.
\]

Using (12) in the line element (1), the new line element of the Kerr-Newman-de Sitter black hole becomes

\[
ds^2 = -\frac{f(r)f(\theta)\rho^2}{f(\theta)(r^2 + a^2) - f(r)a^2 \sin^2 \theta} dt^2 + \frac{\rho^2}{f(r)} dr^2 + \frac{\rho^2}{f(\theta)} d\theta^2.
\]

The position of the event horizon is same as given in (11). Therefore, the new line element near the black hole horizon develops

\[
ds^2 = -\frac{f_{rr}(r_h)(r - r_h)\rho^2(r_h)}{(r^2 + a^2)^2} dt^2 + \frac{\rho^2(r_h)}{f_{rr}(r_h)(r - r_h)} dr^2 + \frac{\rho^2(r_h)}{f(\theta)} d\theta^2,
\]

where

\[
\rho^2(r_h) = r_h^2 + a^2 \cos^2 \theta \quad \text{and} \quad f_{rr}(r_h) = \frac{df}{dr}\bigg|_{r=r_h} = \frac{2}{\rho^2} \left( \beta^2 r_h - M - 2 \frac{r_h^3}{l^2} \right).
\]

### 3. THE H-J METHOD FOR KNdS SPACETIME

In this section, we have applied the standard Hamilton-Jacobi method \([9-12,33]\) developed by Angheben et al. \([13-14,26-27]\) which is an extension of complex path analysis proposed by Padmanabhan et al. \([9-12,33]\) to calculate the imaginary part of the true action for the process of s-wave emission across the horizon. Using the WKB approximation \([34]\), the emission rate satisfies the following relation

\[
\Gamma \sim \exp(-2 \Im I),
\]

where \( I \) is the action of the outgoing particle. In the Hamilton-Jacobi method we avoid the exploration of the equation of motion in the Painlevé’s coordinates systems. The classical action of the radiation particle tunnels across the event horizon satisfies the relativistic Hamilton-Jacobi equation

\[
g^{ij} \left( \frac{\partial I}{\partial x^i} \right) \left( \frac{\partial I}{\partial x^j} \right) + u^2 = 0,
\]

where \( u \) is the mass of the particle and \( g^{ij} \) are the inverse metric tensors derived from the metric (14), namely as follows

\[
g^{11} = -\frac{(r_h^2 + a^2)^2}{f_{rr}(r_h)(r - r_h)\rho^2(r_h)}, \quad g^{22} = \frac{f_{rr}(r_h)(r - r_h)}{\rho^2(r_h)}, \quad g^{33} = -\frac{f(\theta)}{\rho^2(r_h)},
\]

and others are null. Substituting them in the (17), we get

\[
g^{11} \left( \frac{\partial I}{\partial t} \right)^2 + g^{22} \left( \frac{\partial I}{\partial r} \right)^2 + g^{33} \left( \frac{\partial I}{\partial \theta} \right)^2 + u^2 = 0.
\]
For the HJ equation, to find the solution we use the separation of variables method for the action \( I(t, r, \theta, \phi) \) as follows

\[
I = -\omega t + R(r) + H(\theta) + j\phi,
\]

where \( \omega \) is the energy of the emitted particle, \( j \) is the angular momentum with respect to \( \phi \), \( R(r) \) and \( H(\theta) \) are the generalized momentums. The angular velocity of the particle at the horizon is

\[
\Omega_h = \left. \frac{d\phi}{dt} \right|_{r=r_h} = \frac{a\Sigma}{r_h^2 + a^2}.
\]

Inserting (20) into (19) and solving we obtain

\[
R(r) = \pm \frac{r_h^2 + a^2}{f_{rr}(r_h)} \int \frac{dr}{r-r_h} \sqrt{(\omega - j\Omega_h)^2 - \frac{f_{rr}(r_h)(r-r_h)\rho^2(r_h)}{(r_h^2 + a^2)^2}} \left[ \frac{\bar{g}^{33}}{\frac{dH(\theta)}{d\theta}} + u^2 \right]
\]

4. NON-THERMAL TUNNELING RATE

For the convenience of research, let’s the emitted particle as an ellipsoid shell of energy \( \omega \) to tunnel across the event horizon. The quadratic form of (19) is the reason of \( \pm \) signatures that summarized in the above equation. Solution of (22) with “+” signature corresponds to outgoing particles and the other solution i.e., the solution with “-” signature refers to the ingoing particles. The solution given by (22) is singular at \( r = r_h \) which corresponds to the event horizon. Finishing the above integral by using the Cauchy’s integral formula, we obtain

\[
R(r) = \pm \frac{2\pi i(r_h^2 + a^2)}{f_{rr}(r_h)} (\omega - j\Omega_h).
\]

Substituting the above result in (22), the imaginary part of the action \( I \) corresponding to the outgoing particle is obtained by \( \pi \) times the residue of the integrand

\[
R(r) = \frac{2\pi (r_h^2 + a^2)}{f_{rr}(r_h)} (\omega - j\Omega_h)
\]

\[
= \frac{\pi\Sigma^2(r_h^2 + a^2)}{\beta^2 r_h - M - 2\gamma \frac{r_h}{I}} (\omega - j\Omega_h). \tag{24}
\]

Using (11) and (21) into (24), we get the imaginary part of the action as

\[
\text{Im} = \frac{\pi\Sigma^2}{\alpha}(1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right)^2 - \frac{\pi\Sigma^2}{\alpha}(1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right)^3 - M - A
\]

\[
\frac{\pi\Sigma^2}{\alpha} \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right) - M - A
\]

\[
\text{Im} = \frac{\pi\Sigma^2}{\alpha}(1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right)^2 - \frac{\pi\Sigma^2}{\alpha}(1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right)^3 - M - A
\]

\[
\frac{\pi\Sigma^2}{\alpha} \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right) - M - A
\]

Where

\[
A = \frac{2}{\beta^2\alpha^3}(1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2} \right) = \frac{2}{\beta^2\alpha^3} k_1^3 k_2^3.
\]

For more convenient, we have set

\[
k_1 \equiv (1 + \frac{4M^2}{\beta^2(\omega/\alpha)3}) \text{ and } k_2 \equiv ( M + \sqrt{M^2 - (a^2 + q^2)\alpha^2}).
\]
Now (25) becomes
\[ \text{Im} = \frac{\pi \Sigma^2 k_2^2}{\beta^2 \alpha k_1 k_2 - B^2} \omega + \frac{\pi a^2 \Sigma^2}{\beta^2 \left( k_1 k_2 - \frac{Ma}{\beta^2} - \frac{aA}{\beta^2} \right)} \omega - \frac{\pi a \Sigma^2}{\beta^2 \left( k_1 k_2 - \frac{Ma}{\beta^2} - \frac{aA}{\beta^2} \right)} j. \] (27)

where \( B = \frac{Ma}{\beta^2} \left( 1 + \frac{aM^2}{\beta^2} + \cdots \right) + \frac{2}{\alpha^2 \beta^2} k_1 k_2^2. \)

To get the maximum value of the integration, neglecting above second order terms of black hole parameter ‘mass’ from the denominator, we then get
\[ \text{Im} = \frac{\pi \Sigma^2}{\beta^2 \alpha} \cdot \frac{k_2^2}{\left( k_2^2 - \frac{Ma}{\beta^2} \right)} \omega + \frac{\pi a^2 \Sigma^2 \alpha}{\beta^2 \left( k_2^2 - \frac{Ma}{\beta^2} \right)} \omega - \frac{\pi a \Sigma^3 \alpha}{\beta^2 \left( k_2^2 - \frac{Ma}{\beta^2} \right)} j. \] (28)

Let us now focus on a semi-classical treatment of the associated radiation and adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out - no classical escape route exists - where it materializes a real particle while the negative energy particle is absorbed by the black hole, resulting in a decrease in the mass and angular momentum of the black hole.

If the particle’s self-gravitational interaction is taken into account, equations (1) to (28) should be changed. Therefore the imaginary part of the true action can be calculated from (28) in the following integral
\[ \text{Im} = \frac{\pi \Sigma^2}{\beta^2 \alpha} \cdot \int_0^\omega \frac{k_2^2}{k_2^2 + (M - \frac{Ma}{\beta^2})} d\omega' + \frac{\pi a^2 \Sigma^2 \alpha}{\beta^2} \cdot \int_0^\omega \frac{1}{k_2^2 + (M - \frac{Ma}{\beta^2})} d\omega' \]
\[ - \frac{\pi a \Sigma^3 \alpha}{\beta^2} \cdot \int_0^j \frac{1}{k_2^2 + (M - \frac{Ma}{\beta^2})} d\omega'. \] (29)

For the maximum value of integration, neglecting \( \left( 1 - \frac{\alpha}{\beta^2} \right) M. \) Now (29) becomes
\[ \text{Im} = \frac{\pi \Sigma^2}{\beta^2 \alpha} \cdot \int_0^\omega \frac{k_2^2}{\sqrt{M^2 - (a^2 + q^2) \alpha}} d\omega' + \frac{\pi a^2 \Sigma^2 \alpha}{\beta^2} \cdot \int_0^\omega \frac{1}{\sqrt{M^2 - (a^2 + q^2) \alpha}} d\omega' \]
\[ - \frac{\pi a \Sigma^3 \alpha}{\beta^2} \cdot \int_0^j \frac{1}{\sqrt{M^2 - (a^2 + q^2) \alpha}} d\omega'. \] (30)

Fixing the ADM mass, charge and angular momentum of the total spacetime and allow these of the black hole to vary. Then we should replace \( M \) by \( M - \omega \) and \( j \) by \( j - j \), and putting the value of \( k_2 \), we have
\[ \text{Im} = \frac{\pi \Sigma^2}{\beta^2 \alpha} \cdot \int_{M - \omega}^{M - \omega} \frac{(M - \omega + \sqrt{(M - \omega)^2 - (a^2 + q^2) \alpha})^2}{\sqrt{(M - \omega)^2 - (a^2 + q^2) \alpha}} d(M - \omega) \]
+ \frac{\pi a^2 \Sigma^2 \alpha}{\beta^2}, \int_M^{M-\omega} \frac{1}{\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha}} d(M - \omega)

- \frac{\pi a \Sigma^3 \alpha}{\beta^2}, \int_j^{j-j} \frac{1}{\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha}} d(j-j),

where \( j-j = \frac{(M-\omega)\alpha}{\beta^2} \).

Using (32) into (31) and finishing the integral, the imaginary part of the action finally yields

\[ \text{Im} I = -\frac{\pi \Sigma^2}{\beta^2 \alpha} \int_M^{M-\omega} \frac{2(M-\omega)^2 + 2(M-\omega)\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha}}{\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha}} d(M - \omega) \]

\[ + \frac{\pi \Sigma^2}{\beta^2 \alpha} \int_M^{M-\omega} \frac{(a^2 + q^2)\alpha}{\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha}} d(M - \omega) \]

Doing the \( \omega \) integral, finally we get

\[ \text{Im} I = -\frac{\pi \Sigma^2}{\beta^2 \alpha} \left\{ (M-\omega)\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha} + (M-\omega)^2 - M\sqrt{M^2 - (a^2 + q^2)\alpha} \right\} \]

\[ = -\frac{\pi \Sigma^2}{\beta^2 \alpha} \left\{ 2(M-\omega)\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha} + 2(M-\omega)^2 - 2M\sqrt{M^2 - (a^2 + q^2)\alpha} - 2M^2 \right\} \]

\[ = -\frac{\pi \Sigma^2}{\beta^2 \alpha} \left\{ 2(M-\omega)\sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha} \right\} - \left( M + \sqrt{M^2 - (a^2 + q^2)\alpha} \right)^2 \]

\[ = -\frac{1}{2} \exp[\pi(r_f^2 - r_i^2)] \]

\[ = -\frac{1}{2} \exp(\Delta S_{BH}). \]

Here \( r_i = \frac{\Sigma}{\sqrt{\alpha}} (M + \sqrt{M^2 - (a^2 + q^2)\alpha}) \) and \( r_f = \frac{\Sigma}{\sqrt{\alpha}} (M-\omega) + \sqrt{(M-\omega)^2 - (a^2 + q^2)\alpha} \) are the locations of the KNdS event horizon before and after the particle emission respectively, and \( \Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M) \) is the difference of Bekenstein-Hawking entropy.

Utilizing (16), the relationship between the tunneling rate and the imaginary part of the action of the radiative particle for the KNdS black hole is given by

\[ \Gamma \sim \exp(-2\text{Im} J) = \exp(\Delta S_{BH}) \]

(35)

5. PURELY THERMAL RADIATION

The non-thermal emission rate described by (35) is related to the change of Bekenstein-Hawking entropy, and is consistent with an underlying unitary theory and the radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of KNdS black hole. The pure thermal radiation spectrum can be derived from (35) by expanding the tunneling rate in power of \( \omega \) up to second order as follows
\( \Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{ -\omega \frac{\partial S_{BH}(M)}{\partial M} + \frac{\omega^2}{2} \frac{\partial^2 S_{BH}(M)}{\partial M^2} \right\} \). \hspace{1cm} (36)

From (34), we can write

\[ S_{BH}(M - \omega) = \frac{\pi \Sigma^2}{\beta^2 \alpha} \left[ (M - \omega) + \sqrt{(M - \omega)^2 - (a^2 + q^2)\alpha} \right]^2. \hspace{1cm} (37) \]

At \( \omega = 0 \),

\[ \frac{\partial S_{BH}(M)}{\partial M} = \frac{2\Sigma^2}{\beta^2 \alpha} \left[ 2M + \sqrt{M^2 - (a^2 + q^2)\alpha} + \frac{M^2}{\sqrt{M^2 - (a^2 + q^2)\alpha}} \right] \hspace{1cm} (38) \]

And

\[ \frac{\partial^2 S_{BH}(M)}{\partial M^2} = \frac{2\Sigma^2}{\beta^2 \alpha} \left[ 2 + \frac{M}{\sqrt{M^2 - (a^2 + q^2)\alpha}} + \frac{M^3}{(M^2 - (a^2 + q^2)\alpha)^{3/2}} \right] \hspace{1cm} (39) \]

With the help of (38) and (39), the pure thermal emission rate is of the form

\[ \Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{ \pi \left( -\omega \eta + \frac{\omega^2}{2} \lambda \right) \right\}, \hspace{1cm} (40) \]

where \( \eta = \frac{2\Sigma^2}{\beta^2 \alpha} \left[ 2M + \sqrt{M^2 - (a^2 + q^2)\alpha} + \frac{M^2}{\sqrt{M^2 - (a^2 + q^2)\alpha}} \right] \) and \( \lambda = \frac{2\Sigma^2}{\beta^2 \alpha} \left[ 2 + \frac{M}{\sqrt{M^2 - (a^2 + q^2)\alpha}} + \frac{M^3}{(M^2 - (a^2 + q^2)\alpha)^{3/2}} \right] \).

6. RESULTS AND DISCUSSION

We now like to point out that some of the previous results which can be enclosed as special cases. In particular, when cosmological constant vanishes, then \( \Sigma = \beta = \alpha = 1 \) and hence the pure thermal spectrum can be reduced for the Kerr-Newman black hole [24]. The position of the event horizon before and after the emission of the particle with energy \( \omega \) are \( r_i = M + \sqrt{M^2 - (a^2 + q^2)} \) and \( r_f = (M - \omega) + \sqrt{(M - \omega)^2 - (a^2 + q^2)} \) respectively. From (34), the non-thermal tunneling rate for the Kerr-Newman black hole can be written as

\[ \Gamma \sim \exp(-2\text{Im}l) = \exp\left\{ \pi \left( (M - \omega) + \sqrt{(M - \omega)^2 - (a^2 + q^2)} \right)^2 \right\} \times \exp\left\{ \left( M + \sqrt{M^2 - (a^2 + q^2)} \right)^2 \right\} \]

\[ = -\frac{1}{2} \exp\left\{ \pi (r_f^2 - r_i^2) \right\} \]

\[ = -\frac{1}{2} \exp(\Delta S_{BH}), \hspace{1cm} (41) \]

and the purely thermal rate of the Kerr-Newman black hole can be written as

\[ \Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{ -8\pi \omega \left( \eta - \frac{\omega}{2} \lambda \right) \right\}, \hspace{1cm} (42) \]

where \( \eta = \frac{1}{4} \left[ 2M + \sqrt{M^2 - (a^2 + q^2)} + \frac{M^2}{\sqrt{M^2 - (a^2 + q^2)}} \right] \) and \( \lambda = \frac{1}{4} \left[ 2 + \frac{3M}{\sqrt{M^2 - (a^2 + q^2)}} + \frac{M^3}{(M^2 - (a^2 + q^2))^{3/2}} \right] \).
It is interesting that for \( q = 0 \), it reduces to the result of Kerr-de Sitter black hole \[35\] and for \( q = 0, a = 0 \), it becomes to the result of SdS black hole \[32\]. Finally, if one sets \( \omega \rightarrow \infty, a = 0 \), and \( q = 0 \) gives the result for the Schwarzschild black hole \[6\].

7. CONCLUSION

We have developed the non-thermal and purely thermal Hawking radiations as massive particles tunneling process from KNdS black hole \[35\] by taking into account the self-gravitational interaction, the background spacetime as dynamical and the energy as conservation. We have explored that the tunneling rate at the event horizon of KNdS black hole is related to the change of Bekenstein-Hawking entropy. The results are in accordance with Parikh and Wilczek's opinion \[6, 36-37\] from spherically symmetric black hole. We also conclude that the actual radiation spectrum of KNdS black hole is not precisely thermal, which provides an interesting correction to the Hawking pure thermal spectrum.

DISCLAIMER

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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