QUANTUM GENERALIZED CLUSTER ALGEBRAS AND QUANTUM Dilogarithms OF HIGHER DEGREES

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Abstract. We extend the notion of the quantization of the coefficients of the ordinary cluster algebras to the generalized cluster algebras by Chekhov and Shapiro. In parallel to the ordinary case, it is tightly integrated with certain generalizations of the ordinary quantum dilogarithm, which we call the quantum dilogarithms of higher degrees. As an application, we derive the identities of these generalized quantum dilogarithms associated with any period of quantum Y-seeds.

1. Introduction

The generalized cluster algebras were introduced by Chekhov and Shapiro [CS14]. They naturally generalize the (ordinary) cluster algebras by Fomin and Zelevinsky [FZ03]. The main feature of the generalized cluster algebras is the appearance of polynomials in the exchange relations of cluster variables and coefficients, instead of binomials in the ordinary case. Generalized cluster algebras naturally appear so far in Poisson dynamics [GSV03], Teichmüller theory [CS14], representation theory [Gle14], exact WKB analysis [IN14], etc. It has been shown in [CS14, Nak14] that essentially all important properties of the ordinary cluster algebras are naturally extended to the generalized ones.

In this note we demonstrate that the notion of quantum cluster algebras is also extended to the generalized ones. To be more precise, there are two kinds of formulations of quantum cluster algebras, the one quantizing the cluster variables by [BZ05] and the one quantizing the coefficients by [FG09a, FG09b], and it is known that they are closely related to each other. Here, we concentrate on the latter one. As shown by [FG09a, FG09b], in the ordinary case, the quantization of the coefficients is tightly integrated with the quantum dilogarithm [FV93, FK94]. Similarly, in the generalized case, it is tightly integrated with certain generalizations of the quantum dilogarithm, which we call the quantum dilogarithms of higher degrees. As an application, we derive the identities of these generalized quantum dilogarithms associated with any period of quantum Y-seeds, which are also parallel to the ones in the ordinary case.

The main message of this note is that the fundamental (and perhaps all) features of the quantum cluster algebras are also extended to the generalized ones.

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2. **Quantum dilogarithms of higher degrees**

To begin with, let us recall some basic facts about the dilogarithm, the \(q\)-dilogarithm, and the quantum dilogarithm. The **dilogarithm** \(\text{Li}_2(x)\) is defined by

\[
\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.
\]

Let \(q\) be a formal variable. The **\(q\)-dilogarithm** is defined as follows.

\[
\mathcal{L}_{2,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(q^n - q^{-n})} = \frac{1}{q - q^{-1}} \sum_{n=1}^{\infty} \frac{x^n}{n[n]_q},
\]

where \([n]_q = (q^n - q^{-n})/(q - q^{-1})\) is the standard \(q\)-number. The power series (2.2) converges for \(|x| < 1\) and \(|q| < 1\), and the following asymptotic behavior holds when \(q \to 1^-\),

\[
\mathcal{L}_{2,q}(x) \sim \frac{\text{Li}_2(x)}{q^2 - 1} \sim \frac{\text{Li}_2(x)}{\log q^2}.
\]

This is clear from the second expression of \(\mathcal{L}_{2,q}(x)\) in (2.2) and the property \(\lim_{q \to 1} [n]_q = n\).

Following [FV93, FK94] (up to some convention), we introduce the **quantum dilogarithm** \(\Psi_q(x)\), which is a formal power series in \(x\) with coefficients in \(\mathbb{C}(q)\), as follows.

\[
\Psi_q(x) = \prod_{m=0}^{\infty} \left(1 + q^{2m+1}x\right)^{-1}.
\]

In particular, the quantum dilogarithm \(\Psi_q(x)\) should be distinguished from the \(q\)-dilogarithm \(\mathcal{L}_{2,q}(x)\). The formal power series \(\Psi_q(x)\) is characterized by the following recursion relation with initial condition,

\[
\Psi_q(0) = 1, \quad \Psi_q(q^{\pm 2}x) = (1 + qx)^{\pm 1} \Psi_q(x),
\]

where two relations in the latter equality are equivalent to each other. A little confusingly, the quantum dilogarithm is actually the exponential of the \(q\)-dilogarithm; namely,

\[
\Psi_q(x) = \exp \left(-\mathcal{L}_{2,q}(-x)\right).
\]

This is easily shown by using the recursion relation (2.5).

Alternatively, one may define the **dilogarithm** \(\text{Li}_2(x)\) by the integral

\[
\text{Li}_2(x) = -\int_0^x \log(1 - y) \frac{dy}{y} = -\int_0^{-x} \log(1 + y) \frac{dy}{y}.
\]

Then we have

\[
\log \Psi_q(x) = -\sum_{m=0}^{\infty} \log(1 + q^{2m+1}x)
\]

\[
= \frac{-1}{1 - q^2} \sum_{m=0}^{\infty} \log(1 + q^{2m+1}x) \frac{q^{2m+1}x - q^{2m+3}x}{q^{2m+1}x}
\]

\[
\sim \frac{-1}{1 - q^2} \int_0^x \log(1 + y) \frac{dy}{y} = \frac{1}{1 - q^2} \text{Li}_2(-x) \quad (q \to 1^-).
\]

This completely agrees with (2.3) and (2.6).
Now let us generalize the quantum dilogarithm Ψ_q(x) to the ones with higher degrees. For any field F, let F(q) be the field of the rational functions in the variable q.

**Definition 2.1.** Let F be a field, let d be a positive integer, and let z = (z_1, ..., z_{d-1}) be a d - 1-tuple of elements in F. When d = 1, z is regarded as the empty sequence (\( ). \) We set \( z_0 = z_d = 1 \). Then, we define a formal power series \( \Psi_{d,z,q}(x) \) in x with coefficients in \( F(q) \) as follows:

\[
\Psi_{d,z,q}(x) = \prod_{m=0}^{\infty} \left( \sum_{s=0}^{d} z_s q^{s(2m+1)} x^s \right)^{-1}.
\]

When \( d = 1 \), it is the usual quantum dilogarithm \( \Psi_{1,(),q}(x) = \Psi_q(x) \). We call \( \Psi_{d,z,q}(x) \) the quantum dilogarithm of degree \( d \) with coefficients z.

**Proposition 2.2.** The formal power series \( \Psi_{d,z,q}(x) \) is characterized by the following recursion relation with initial condition:

\[
\Psi_{d,z,q}(0) = 1,
\]

\[
\Psi_{d,z,q}(q^{\pm 2} x) = \left( \sum_{s=0}^{d} z_s q^{\pm s} x^s \right) \Psi_{d,z,q}(x),
\]

where two relations in (2.11) are equivalent to each other.

**Proof.** For example, we have

\[
\Psi_{d,z,q}(q^2 x) = \prod_{m=0}^{\infty} \left( \sum_{s=0}^{d} z_s q^{s(2m+1)} q^{2s} x^s \right)^{-1} = \prod_{m=1}^{\infty} \left( \sum_{s=0}^{d} z_s q^{s(2m+1)} x^s \right)^{-1} \left( \sum_{s=0}^{d} z_s q^s x^s \right) \Psi_{d,z,q}(x).
\]

The rest of the properties are easily shown. \( \square \)

For any integer a, let us introduce the sign function

\[
\text{sgn}(a) = \begin{cases} 
+ & a > 0 \\
0 & a = 0 \\
- & a < 0.
\end{cases}
\]

Here and below, we identify the signs \( \pm \) with numbers \( \pm 1 \).

The following formula will be useful later.

**Proposition 2.3.** For any integer a, the following equality holds.

\[
\Psi_{d,z,q}(q^{2a} x) = \prod_{m=1}^{\lfloor a \rfloor} \left( \sum_{s=0}^{d} z_s q^{\text{sgn}(a)(2m-1)s} x^s \right)^{\text{sgn}(a)} \Psi_{d,z,q}(x).
\]

**Proof.** This is obtained from (2.11) by induction on a. \( \square \)

In some cases the quantum dilogarithms of higher degrees are factorized by the ordinary quantum dilogarithm.
Proposition 2.4. Factorization formula. Suppose that the following factorization
\[
\sum_{s=0}^{d} z_s x^s = \prod_{s=1}^{d} (1 - w_s x)
\]
occurs for some \(w_1, \ldots, w_d \in F\). Then, we have
\[
\Psi_{d, z, q}(x) = \prod_{s=1}^{d} \Psi_q(-w_s x).
\]

Proof. One can directly observe the factorization in (2.16) as
\[
\Psi_{d, z, q}(x) = \prod_{m=0}^{\infty} \left( \sum_{s=0}^{d} z_s q^{(2m+1)} x^s \right)^{-1} = \prod_{m=0}^{\infty} \prod_{s=1}^{d} (1 - w_s q^{2m+1} x)^{-1}.
\]
Alternatively, under the assumption (2.15), the right hand side of (2.16) satisfies (2.10) and (2.11). Thus, thanks to Proposition 2.3, we have (2.16). \(\square\)

Example 2.5. Let us consider the special case where \(F = \mathbb{C}\) and the coefficients \(z\) is trivial, i.e., \(z = 1 := (1, \ldots, 1)\). In this case we have the factorization
\[
\sum_{s=0}^{d} x^s = \prod_{s=1}^{d} (1 - \omega^s x),
\]
where
\[
\omega = \exp(2\pi i/(d + 1)).
\]
Thus, by Proposition 2.4 we have
\[
\Psi_{d, 1, q}(x) = \prod_{s=1}^{d} \Psi_q(-\omega^s x).
\]
On the other hand, there is another factorization formula,
\[
\Psi_{d, 1, q}(x) = \Psi_{q^{d+1}}(-x^{d+1}) \Psi_q(-x)^{-1}.
\]
This is due to the following alternative expression of \(\Psi_{d, 1, q}(x)\),
\[
\Psi_{d, 1, q}(x) = \prod_{m=0}^{\infty} \frac{1 - q^{2m+1} x}{1 - (q^{2m+1} x)^{d+1}}.
\]
Therefore, by (2.3) and (2.6), we have the following asymptotic behavior in the limit \(q \to 1^-\):
\[
\log \Psi_{d, 1, q}(x) \sim \frac{1}{1 - q^2} \sum_{s=1}^{d} \text{Li}_2(\omega^s x)
\]
\[
\sim \frac{1}{1 - q^2} \left( \frac{1}{d+1} \text{Li}_2(x^{d+1}) - \text{Li}_2(x) \right).
\]
In fact, these two expressions coincide due to the well-known identity for \(\text{Li}_2(x)\) called the factorization formula [Lew81, Eq. (1.14)],
\[
\frac{1}{d+1} \text{Li}_2(x^{d+1}) = \sum_{s=0}^{d} \text{Li}_2(\omega^s x).
\]
As a side remark, in view of (2.6), the expressions (2.20) and (2.21) imply the equality

\[ \mathcal{L}_{2,q^{d+1}}(x^{d+1}) = \sum_{s=0}^{d} \mathcal{L}_{2,q}(\omega^s x), \]

which is regarded as the \( q \)-analogue of (2.25). The equality (2.26) is also obtained directly from (2.2) and the equality

\[ \sum_{s=0}^{d} \omega^{sn} = \begin{cases} d + 1 & n \equiv 0 \mod d + 1 \\ 0 & n \not\equiv 0 \mod d + 1 \end{cases}. \]

(2.27)

**Example 2.6.** Let us consider the case where \( F = \mathbb{C} \) with arbitrary coefficients \( z \). Let us introduce the dilogarithm of degree \( d \) with coefficients \( z \) by the integral,

\[ \text{Li}_2; d, z(x) = -\int_{-x}^{x} \log \left( \sum_{s=0}^{d} z^s \frac{dy}{y} \right) \]

(2.28)

Then, by the same calculation as in (2.8), we have the following asymptotic behavior,

\[ \log \Psi_{d, z, q}(x) = \frac{-1}{1 - q^2} \sum_{m=0}^{\infty} \log \left( \sum_{s=0}^{d} z^s \left( q^{2m+1} x \right)^s \right) \frac{q^{2m+1} x - q^{2m+3} x}{q^{2m+1} x} \]

\[ \sim \frac{1}{1 - q^2} \text{Li}_2; d, z(-x) \quad (q \to 1^+). \]

(2.29)

3. **Generalized mutations of quantum \( Y \)-seeds**

In this section, following the idea of [FG09a, FG09b], we introduce the quantum version of the generalized mutation of generalized cluster algebras. Here, we use the formulation of generalized cluster algebras by [Nak14].

Let \( B = (b_{ij})_{n \times n} \) be a skew-symmetrizable integer matrix. Let \( d = (d_1, \ldots, d_n) \) be an \( n \)-tuple of positive integers. For given \( B \) and \( d \), we arbitrarily choose an \( n \)-tuple of positive integers \( r = (r_1, \ldots, r_n) \) such that

\[ r_i d_i b_{ij} = -r_j d_j b_{ji}. \]

(3.1)

Such an \( r \) exists (not uniquely) due to the skew-symmetric property of the matrix \( B \). Let \( q \) continue to be a formal variable, and let \( Y = (Y_i)_{i=1}^{n} \) be an \( n \)-tuple of noncommutative formal variables with commutation relation

\[ Y_i Y_j = q^{2r_i d_i b_{ij}} Y_j Y_i. \]

(3.2)

The relation (3.2) makes sense due to the skew-symmetric property in (3.1). We call such a pair \((B, Y)\) a quantum \( Y \)-seed.

We use the notation

\[ q_i := q^{r_i d_i}, \quad i = 1, \ldots, n. \]

(3.3)

Then, (3.2) is also written as

\[ Y_i Y_j = q_{i}^{2b_{ij}} Y_j Y_i = q_{i}^{-2b_{ij}} Y_j Y_i. \]

(3.4)

Let \( F \) be any field. For the above \( d = (d_1, \ldots, d_n) \) we arbitrarily choose a collection of elements in \( F \),

\[ z = (z_{i,s})_{i=1,\ldots,n; s=1,\ldots,d_i-1} \]

(3.5)
satisfying the \textit{reciprocity condition} in [Nak14]
\begin{equation}
  z_{i,s} = z_{i,d_i-s}.
\end{equation}
(The use of symbol $z$ here slightly conflicts with the one in Definition 2.1 but we find that it is convenient.) Let us set $z_{i,0} = z_{i,d_i} = 1$. We also introduce the notation
\begin{equation}
  z_i = (z_{i,s})_{s=1,...,d_i-1}, \quad i = 1, \ldots, n.
\end{equation}
Under these notations we have the associated quantum dilogarithm $\Psi_{d_i,z_i,q_i}(x)$ of degree $d_i$ for each $i = 1, \ldots, n$.

Below we assume that any element in $F$ commutes with variables $Y_i$.

\textbf{Definition 3.1.} For a quantum $Y$-seed $(B, Y)$, the $(d, z)$-\textit{mutation} (generalized mutation) $(B', Y') = \mu_k(B, Y)$ of $(B, Y)$ at $k$ is defined by
\begin{equation}
  b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + d_k([-b_{ik} + b_{kj} + b_{ik}[b_{kj}]]) & \text{if } i, j \neq k, \end{cases}
\end{equation}
\begin{equation}
  Y'_i = \begin{cases} Y = \left\{ \begin{array}{ll}
    Y_k^{d_k} & i = k \\
    \sum_{s=0}^{d_k} \sum_{m=1}^{d_k} z_{k,s} Y_k^{d_k} & i \neq k,
  \end{array} \right.
\end{cases}
\end{equation}
where $\varepsilon = \pm$, and
\begin{equation}
  [a]_+ = \begin{cases} a & a > 0 \\
    0 & a \leq 0.
  \end{cases}
\end{equation}
Actually, the right hand side of (3.9) does not depend on the choice of the sign $\varepsilon$ (see Lemma 3.2 (i)). We call $d$ and $z$ the \textit{mutation degrees} and the \textit{frozen coefficients}, respectively, in accordance with [Nak14].

When we formally set $q = 1$, the relation (3.9) reduces to
\begin{equation}
  Y'_i = \begin{cases} Y_k^{d_k} & i = k \\
    \sum_{s=0}^{d_k} \sum_{m=1}^{d_k} z_{k,s} Y_k^{d_k} \sum_{s=0}^{d_k} z_{k,s} Y_k^{d_k} & i \neq k,
  \end{cases}
\end{equation}
which is the \textit{generalized mutation} of coefficients ($y$-variables) in generalized cluster algebras formulated in [Nak14]. On the other hand, when we set $d_k = 1$, it reduces to the ordinary mutation of quantum $Y$-seeds by Fock and Goncharov [FG09a, FG09b].

The following properties are easily checked.

\textbf{Lemma 3.2.} (i) The right hand side of (3.9) does not depend on the choice of the sign $\varepsilon$.

(ii) For the matrix $B'$, the condition
\begin{equation}
  r_i d_i b'_{ij} = -r_j d_j b'_{ji},
\end{equation}
holds.

(iii) The $(d, z)$-mutation is involutive, i.e., $\mu_k(\mu_k(B, Y)) = (B, Y)$. 

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    \sum_{s=0}^{d_k} \sum_{m=1}^{d_k} z_{k,s} Y_k^{d_k} \sum_{s=0}^{d_k} z_{k,s} Y_k^{d_k} & i \neq k,
  \end{array} \right.
\end{cases}
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where $\varepsilon = \pm$, and
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Actually, the right hand side of (3.9) does not depend on the choice of the sign $\varepsilon$ (see Lemma 3.2 (i)). We call $d$ and $z$ the \textit{mutation degrees} and the \textit{frozen coefficients}, respectively, in accordance with [Nak14].

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\begin{equation}
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  \end{cases}
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\begin{equation}
  r_i d_i b'_{ij} = -r_j d_j b'_{ji},
\end{equation}
holds.

(iii) The $(d, z)$-mutation is involutive, i.e., $\mu_k(\mu_k(B, Y)) = (B, Y)$.
Lemma 3.3. The case $\varepsilon = 1$ it is called the “automorphism part” of (3.9) in [FG09a, FG09b].

Next let us consider the “monomial part” of (3.9). Let us set

$$Z_i := \begin{cases} Y_k^{-1} & i = k \\ b_i d_k [b_k]_+ Y_i Y_k d_k [b_k]_+ & i \neq k. \end{cases}$$

By Lemma 3.3 the $(d, z)$-mutation (3.9) is expressed as the composition

$$Y_i' = \text{Ad}(\Psi_{d_k, z_k, q_k}(Y_k^\varepsilon)) (Z_i^{(\varepsilon)}).$$

Lemma 3.4. The following commutation relation holds.

$$Z_i^{(\varepsilon)} Z_j^{(\varepsilon)} = q^{2r_j d_j b_j} Z_j^{(\varepsilon)} Z_i^{(\varepsilon)},$$

where $b_{ij}$ is given by (3.8).

Proof. This is easily verified by the case check. □

Proposition 3.5. Under the $(d, z)$-mutation in (3.9), the following commutation relation holds:

$$Y_i' Y_j' = q^{2r_j d_j b_j} Y_j' Y_i'.$$

Proof. By Lemma 3.3 and (3.17), we have

$$Y_i' Y_j' = \text{Ad}(\Psi_{d_k, z_k, q_k}(Y_k^\varepsilon)) (Z_i^{(\varepsilon)} Z_j^{(\varepsilon)})$$

$$= \text{Ad}(\Psi_{d_k, z_k, q_k}(Y_k^\varepsilon)) (q^{2r_j d_j b_j} Z_j^{(\varepsilon)} Z_i^{(\varepsilon)})$$

$$= q^{2r_j d_j b_j} Y_j' Y_i'.$$
4. Quantum dilogarithm identities of higher degrees

Let us give an application of generalized mutations of quantum Y-seeds to quantum dilogarithm identities of higher degrees. Since they are parallel to the one for ordinary quantum dilogarithm identities studied in [Kel11, KN11], we only give the minimal description here. We ask the reader to consult [KN11, Section 3] for more details.

Consider a sequence of \((d, z)\)-mutations of quantum Y-seeds,

\[(B(1), Y(1)) \leftrightarrow (B(2), Y(2)) \leftrightarrow \cdots \leftrightarrow (B(L), Y(L))\]

and suppose that it has the periodicity

\[(4.2) \quad b_{\sigma(i)\sigma(j)}(L + 1) = b_{ij}(1), \quad \sigma_{\sigma(i)}(L + 1) = \sigma_i(1)\]

for some permutation \(\sigma\) of \(1, \ldots, n\). Then, we have the associated sequence of \((d, z)\)-mutations of (nonquantum) Y-seeds of a (nonquantum) generalized cluster algebra,

\[(4.3) \quad (B(1), y(1)) \leftrightarrow (B(2), y(2)) \leftrightarrow \cdots \leftrightarrow (B(L), y(L + 1))\]

and it has the same periodicity

\[(4.4) \quad y_{\sigma(i)}(L + 1) = y_i(1).\]

Let us further assume the sign-coherence property of the sequence \([4.3]\) (see, e.g., [Nak14]). Let \(\varepsilon_t\) and \(c_t\) \((t = 1, \ldots, L)\) be the tropical sign and the \(c\)-vector of \(y_k(t)\) defined in [Nak14]. Let us denote the initial seed \((B(1), Y(1))\) as \((B, Y)\). Let \(\mathbb{T}(B)\) be the quantum torus generated by noncommutative variables \(Y^\alpha\) \((\alpha \in \mathbb{Z}^n)\) with the relations

\[(4.5) \quad q^{(\alpha, \beta)} Y^\alpha Y^\beta = Y^{\alpha + \beta}, \quad \langle \alpha, \beta \rangle = \sum_{i,j=1}^n \alpha_i d_{ij} b_{ij} \beta_j.\]

We identify \(Y_i = Y^{e_i}\), where \(e_i\) is the \(i\)th unit vector.

**Theorem 4.1.** Under the assumption of the periodicity \([4.2]\) and the sign-coherence property of the sequence \([4.3]\), we have the following identities of the quantum dilogarithms of higher degrees associated to the sequence \([4.1]\).

(i) Quantum dilogarithm identities in tropical form (cf. [KN11] Theorem 3.5]),

\[(4.6) \quad \Psi_{d_{k_1}} z_{\epsilon_{k_1}},q_{k_1} (Y^{\varepsilon_i e_i})^{\varepsilon_i} \cdots \Psi_{d_{k_L}} z_{\epsilon_{k_L}},q_{k_L} (Y^{\varepsilon_i e_i})^{\varepsilon_i} = 1,\]

where \(Y^{\varepsilon_i e_i} \in \mathbb{T}(B)\).

(ii) Quantum dilogarithm identities in universal form (cf. [KN11] Corollary 3.7]),

\[(4.7) \quad \Psi_{d_{k_1}} z_{\epsilon_{k_1}},q_{k_1} (Y_{k_1}(L))^{\varepsilon_i} \cdots \Psi_{d_{k_L}} z_{\epsilon_{k_L}},q_{k_L} (Y_{k_1}(1))^{\varepsilon_i} = 1.\]

We omit the proof, since it is completely parallel to the one for Theorem 3.5 and Corollary 3.7 of [KN11].

**Example 4.2.** Let us consider the simplest nontrivial example of a generalized cluster algebra with

\[(4.8) \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad d = (2, 1), \quad z = (z_1, 1).\]
Among them, the calculation of \( c \) and the periodicity of the sequence \((4.11)\) with \( \sigma = \text{id} \) is given by
\[
Y_1 Y_2 = q^4 Y_2 Y_1.
\]
Let us set \((B(1), Y(1)) := (B, Y)\) and consider the following sequence of mutations
\[
(B(1), Y(1)) \xrightarrow{t_1} (B(2), Y(2)) \xrightarrow{t_2} (B(3), Y(3)) \xrightarrow{t_1} (B(4), Y(4)) \xrightarrow{t_2} (B(5), Y(5)) \xrightarrow{t_1} (B(6), Y(6)) \xrightarrow{t_2} (B(7), Y(7)).
\]
Then, we have
\[
B(t) = (-1)^{t+1} B,
\]
and the quantum \( Y \)-variables mutate as follows, where we set \( z = z_{1,1} \) for simplicity. (If we set \( q = 1 \), we recover the result in [Nak14, Table 1] for the nonquantum case.)

\[
\begin{align*}
Y_1(1) &= Y_1, \\
Y_2(1) &= Y_2, \\
Y_1(2) &= Y_1^{-1}, \\
Y_2(2) &= Y_2(1 + z q^2 Y_1 + q^4 Y_2^2), \\
Y_1(3) &= Y_1^{-1}(1 + q^2 Y_2 + z Y_1 Y_2 + q^{-2} Y_1^2 Y_2), \\
Y_2(3) &= Y_2^{-1}(1 + z q^{-2} Y_1 + q^{-4} Y_2^2)^{-1}, \\
Y_1(4) &= Y_1(1 + q^{-2} Y_2 + z q^{-4} Y_1 Y_2 + q^{-6} Y_1^2 Y_2)^{-1}, \\
Y_2(4) &= q^{-4} Y_1^{-2} Y_2^{-1}(1 + q^2 Y_2 + q^6 Y_2^2 + q^8 Y_2^2 + + z Y_1 Y_2 + z q^2 Y_1 Y_2^2 + q^{-4} Y_1^2 Y_2^2), \\
Y_1(5) &= q^{-2} Y_1^{-1} Y_2^{-1}(1 + q^2 Y_2), \\
Y_2(5) &= q^{-4} Y_1^2 Y_2(1 + q^6 Y_2 + q^{-2} Y_2 + q^{-8} Y_2^2 + + z q^{-4} Y_1 Y_2 + z q^{-10} Y_1^2 Y_2^2 + q^{-10} Y_1^2 Y_2^2)^{-1}, \\
Y_1(6) &= q^{-2} Y_1 Y_2(1 + q^{-2} Y_2), \\
Y_2(6) &= Y_2^{-1}, \\
Y_1(7) &= Y_1, \\
Y_2(7) &= Y_2.
\end{align*}
\]

Among them, the calculation of \( Y_2(4) \) is the most tedious one. Now we observe the periodicity of the sequence \((4.11)\) with \( \sigma = \text{id} \). We also have the following data of the tropical signs and the \( c \)-vectors in [Nak14, Section 3.4]
\[
\begin{align*}
\varepsilon_1 &= \varepsilon_2 = +, & \varepsilon_3 &= \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = -, \\
c_1 &= (1, 0), & c_2 &= (0, 1), & c_3 &= (-1, 0), \\
c_4 &= (-2, -1), & c_5 &= (-1, -1), & c_6 &= (0, -1),
\end{align*}
\]
which can be also read off from the above result by setting $q = 1$. Now, by substituting these data, the identity \((4.6)\) reads
\[
\Psi_{2,(z),q^2}(Y_1)\Psi_{q^2}(Y_2)\Psi_{2,(z),q^2}(Y_1)^{-1}
\times \Psi_{q^2}(q^{-4}Y_1^2Y_2)^{-1}\Psi_{2,(z),q^2}(q^{-2}Y_1Y_2)^{-1}\Psi_{q^2}(Y_2)^{-1} = 1,
\]
while the identity \((4.7)\) reads
\[
\Psi_{q^2}(Y_2)^{-1}\Psi_{2,(z),q^2}((1 + q^2Y_2)^{-1}q^2Y_2Y_1)^{-1}
\times \Psi_{q^2}((1 + q^2Y_2 + q^6Y_2 + q^8Y_2^2 + zY_1Y_2 + zq^2Y_1Y_2^2 + q^{-4}Y_1^2Y_2^2)^{-1}q^4Y_2Y_1^{-1})^{-1}
\times \Psi_{2,(z),q^2}((1 + q^2Y_2 + zY_1Y_2 + q^{-2}Y_1Y_2^{-1})^{-1})^{-1}
\times \Psi_{q^2}(Y_2((1 + zq^2Y_1 + q^4Y_2^2))\Psi_{2,(z),q^2}(Y_1) = 1.
\]

In general, we conjecture that if the underlying nonquantum sequence \((4.3)\) has a periodicity \((4.4)\), then the the corresponding quantum sequence \((4.1)\) also has the same periodicity \((4.2)\). (The converse is trivial as already stated.) This was proved for the ordinary cluster algebras in \([Nak14, Proposition 3.4]\) when $B = B(1)$ is skew-symmetric.

References

[BZ05] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, Adv. in Math. 195 (2005), 405–455; arXiv:math.QA/0404446.

[CS14] L. Chekhov and M. Shapiro, Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables, Int. Math. Res. Notices 2014 (2014), 2746–2772; arXiv:1111.3963 [math-ph].

[FG09a] V. V. Fock and A. B. Goncharov, Cluster ensembles, quantization and the dilogarithm, Annales Sci. de l’Ecole Norm. Sup. 42 (2009), 865–930; arXiv:math090311245 [math.AG].

[FG09b] V. V. Fock and A. B. Goncharov, Cluster ensembles, quantization and the dilogarithm II: Finite type classification, Invent. Math. 172 (2008), 223–286; arXiv:math0702397 [math.QA].

[FK94] L. D. Faddeev and R. M. Kashaev, Quantum dilogarithm, Mod. Phys. Lett. A9 (94), 427–434; arXiv:hep-th/93010070.

[FV03] L. D. Faddeev and A. Yu. Volkov, Abelian current algebra and the Virasoro algebra on the lattice, Phys. Lett. 315 (1993), 311–318; arXiv:hep-th/9307048.

[FZ03] S. Fomin and A. Zelevinsky, Cluster algebras II. Finite type classification, Invent. Math. 154 (2003), 63–121; arXiv:math0208229 [math.RA].

[Gle14] A. Gleitz, Quantum affine algebras at roots of unity and generalized cluster algebras, 2014, in preparation.

[GSV03] M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Poisson geometry, Moskov Math. J. 3 (2003), 899–934; arXiv:math0308033 [math.QA].

[IN14] K. Iwaki and T. Nakanishi, Exact WKB analysis and cluster algebras II: simple poles, orbifold points, and generalized cluster algebras, 2014, arXiv:1409.4641 [math.CA].

[Kel11] B. Keller, On cluster theory and quantum dilogarithm identities, Representations of algebras and related topics (A. Skowroński and K. Yamagata, eds.), EMS Series of Congress Reports, European Mathematical Society, 2011, pp. 85–116; arXiv:1102.4148 [math.RT].

[KN11] R. M. Kashaev and T. Nakanishi, Classical and quantum dilogarithm identities, SIGMA 7 (2011), 102, 29 pages, arXiv:1104.4630 [math.QA].

[Lew81] L. Lewin, Polylogarithms and associated functions, North-Holland, Amsterdam, 1981.

[Nak14] T. Nakanishi, Structure of seeds in generalized cluster algebras, 2014, arXiv:1409.5967 [math.RA].

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