Symmetries and hamiltonians of Ince’s XXXVIII and XLIX equations

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Abstract. We discuss symmetries of Hamiltonians of I₃₈ and I₄₉ equations that appear on Ince’s list of fifty second-order differential equations with Painlevé property. This study is informed by structure of Weyl symmetries of Painlevé P₃.localized and mixed Painlevé P₃−V equations and provides insights into differences between the symmetries of Painlevé equations and symmetries of solvable equations on Ince’s list.

1. Introduction
Symmetry group analysis has been of crucial importance for studies of Painlevé equations and the singular behavior of solutions of second-order differential equations on the complex plane. In addition to the celebrated 6 Painlevé equations, there are also other 44 ordinary second-order differential equations with solutions that have no movable critical point other than poles. These equations presented in Ince’s book [4] are solvable, meaning that their solutions are expressible in terms of known functions. Comparing with literature on Painlevé equations, the Hamiltonian structure and symmetries of solvable equations on Ince’s list attracted much less attention with notable exceptions of few recent publications [5, 6, 7]. The present study fills this gap by studying symmetries of equations I₃₈ and I₄₉:

\[ I₃₈ : y_{xx} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y_x^2 + y(y-1) \left( A y(1) + B y - \frac{C}{y} + \frac{D}{(y-1)^2} \right) \]
\[ I₄₉ : y_{xx} = \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-A} \right) \frac{y_x^2}{2} + y(y-1)(y-A) \left( B + \frac{C}{y^2} + \frac{D}{(y-1)^2} + \frac{E}{(y-A)^2} \right) \]

from Ince’s list [4, 5, 6, 7]. For the full understanding of their symmetries it is instructive to first study how their structures emerge in the context of P₃−V model [3, 1]. Note that Ince’s equation I₃₈ (1) with \( D = 0 \) is equivalent to Ince’s equation I₄₉ (2) with \( A = 1 \).
These two incomplete forms of (1) (with $\mathcal{D} = 0$) or (2) (with $\mathcal{A} = 1$) equations emerge for two special values of the parameters:

1. $r_1 = 0, J = 1$
2. $r_0 = 0, J = -1$

of the $P_{III-V}$ model:

$$zq_z = q(q - r_1)(2p + r_0z) - (\alpha_1 + \alpha_3)q + \alpha_1r_1 + \epsilon_0r_0z^{-J}$$
$$zp_z = p(p + r_0z)(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \alpha_2r_0z - \epsilon_1r_1z^{J+1}.$$  

(4)

defined here in terms of the two first-order Hamiltonian equations. These equations depend on a number of parameters $J, \epsilon_0, r_0, r_1$ together with $\alpha_j, j = 0, 1, 2, 3$ (with $\sum_{j=0}^{3}\alpha_j = 1$) and can be obtained from the Hamiltonian:

$$zH = q(q - r_1)p(p + r_0z) - (\alpha_1 + \alpha_3)qp + (\alpha_1r_1 + \epsilon_0r_0z^{-J})p$$
$$+ (\alpha_2r_0z + \epsilon_1r_1z^{J+1})q.$$  

(5)

The above equations (1) are invariant under an automorphism $\pi$ such that

$$\pi(q) = -p/z, \pi(p) = (q - r_1)z, \pi(\alpha_i) = \alpha_{i+1}, \pi(J) = -J, \pi(\epsilon_j) = (-1)^i\epsilon_{i+1}, i = 0, 1,$$

(6)

together with $\pi(r_i) = r_{i+1}$. The automorphism $\pi$ satisfies $\pi^4 = 1$.

The $P_{III}$ Painlevé equation

$$P_{III} : y_{zz} = -\frac{1}{z}y_z + \frac{y^2}{y} + \frac{A}{z} + \frac{B}{y} + \frac{D}{z}$$  

(7)

emerges from $P_{III-V}$ model for either $r_1 = 0$ and $J \neq 1$ or $r_0 = 0$ and $J \neq -1$ and is invariant under the extended affine Weyl group $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ in the former case and by $W[s_1, s_3, \pi_1, \pi_3, \pi^2]$ in the latter case. The extended affine Weyl group $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ is generated by transformations

$$\pi_0(q) = -\frac{\epsilon_0}{q}, \pi_0(p) = \frac{1}{\epsilon_0}(q^2p + \alpha_2q)$$
$$\pi_0(\alpha_1 + \alpha_3) = -2\alpha_2 - \alpha_1 - \alpha_3, \pi_0(\alpha_2) = \alpha_2, \pi_0(\alpha_0) = 2 - \alpha_0$$
$$\pi_2(q) = \frac{\epsilon_0}{q}, \pi_2(p) = -\frac{1}{\epsilon_0}(q^2(p + r_0z) + (1 - \alpha_2 - \alpha_1 - \alpha_3)q) - r_0z$$
$$\pi_2(\alpha_1 + \alpha_3) = -2 + 2\alpha_2 + \alpha_1 + \alpha_3, \pi_2(\alpha_2) = 2 - \alpha_2, \pi_2(\alpha_0) = \alpha_0$$

(9)

$$s_2(q) = q + \frac{\alpha_2}{p}, \quad s_2(p) = p,$$
$$s_2(\alpha_1 + \alpha_3) = 2\alpha_2 + \alpha_1 + \alpha_3, \quad s_2(\alpha_2) = -\alpha_2,$$
$$s_0(q) = q + \frac{1}{p + r_0z}, \quad s_0(p) = p$$

(10)

$$s_0(\alpha_1 + \alpha_3) = 2 - 2\alpha_2 - \alpha_1 - \alpha_3, \quad s_0(\alpha_2) = \alpha_2,$$

(11)

that satisfy relations:

$$s_i^2 = 1 = p, \quad \pi^2_0\pi^2_1\pi^2_2 = \pi_{i+2}, \quad \pi^2_0\pi^2_i = s_{i+2}, \quad i = 0, 2,$$

(12)

for

$$\pi^2(q) = -q, \quad \pi^2(p) = -p - r_0z, \quad \pi^2(\alpha_i) = \alpha_{i+2}, \quad \pi^2(\epsilon_0) = -\epsilon_0.$$
as well as commutativity rules:

\[ s_is_{i+2} = s_{i+2}s_i, \quad \pi_i\pi_{i+2} = \pi_{i+2}\pi_i, \quad \pi_is_{i+2} = \pi_{i+2}s_i, \quad i = 0, 2. \]  

(13)

Relations (12) and (13) amount to the following Coxeter group relations:

\[
(s_0s_2)^2 = 1, \quad (\pi_0\pi_2)^2 = 1, \quad (\pi_2s_0)^2 = 1.
\]

(14)  

(15)

In the setting of P_{III} model it is possible to realize \( W[s_0, s_2, \pi_0, \pi_2, \pi^2] \) symmetry as an extended affine Weyl group \( C_2^{(1)} \) [3].

Ince’s equation \( I_{12} \): 

\[
I_{12} : y_{xx} = \frac{y^2}{y} + Ay^3 + By^2 + C + \frac{D}{y}
\]

(16)

as well as the incomplete \( I_{38} \) (with \( D = 0 \)) and the incomplete \( I_{49} \) (with \( A = 1 \)) are obtained from \( P_{III-V} \) for either either of two values of parameters given in (3). For these models the symmetry is still given by \( W_0[s_0, s_2, \pi_0, \pi_2, \pi^2] \) (or \( W[s_1, s_3, \pi_1, \pi_3, \pi^2] \)). However for parameters in (3) actions of \( \alpha_j \) become identical to those of \( s_i \), and connection with the affine extended Weyl group \( C_2^{(1)} \) realization can no longer be established [2].

We will illustrate the above comments by providing a brief derivation of relevant Ince’s equations for a first choice \( r_1 = 0, J = 1 \) of the parameters among two listed in (3). Defining

\[
x = \ln z, \quad w = qz, \quad f = p/z
\]

we obtain from (11) equations:

\[
w_x = w^2 (2f + r_0) - (\alpha_1 + \alpha_3 - 1)w + \epsilon_0r_0, \\
f_x = f(f + r_0)(-2w) + (\alpha_1 + \alpha_3 - 1)f - \alpha_2r_0,
\]

(17)

that also can be reproduced as Hamilton equations following from a Hamiltonian:

\[ H_{12} = f^2w^2 + w^2fr_0 + (\alpha_1 + \alpha_3 - 1)fw + \epsilon_0r_0f + \alpha_2r_0w. \]

(18)

Eliminating \( f \) one obtains the second order equation for \( w \):

\[
w_{xx} = \frac{w_x^2}{w} + w^3r_0^2 + w^2r_0(\alpha_2 - \alpha_0) - \epsilon_0r_0(\alpha_0 + \alpha_2) - \frac{\epsilon_0r_0^2}{w},
\]

in which we recognize equation \( I_{12} \) of Ince [16]. The second order equation for \( f \) written in terms of \( y \) such that

\[
f = -\frac{r_0y}{y - 1}
\]

leads to Ince’s equation \( I_{38} \) (1) with \( A = \alpha_0^2/2, B = -\alpha_3^2/2, C = -2\epsilon_0r_0^2 \) and \( D = 0 \) and thus the equation obtained in this limit is only an incomplete version of \( I_{38} \) equation (1).

We now turn our attention to the remaining case listed in (3): \( r_0 = 0 \) and \( J = -1 \). Inserting these values directly in (1) with \( r_0 = 0 \) yields (for \( x = \ln z \))

\[
q_x = q(q - r_1)2p - (\alpha_1 + \alpha_3)q + \alpha_1r_1
\]

\[
p_x = p^2(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \epsilon_1r_1
\]
Let us set \( q = w, p = f \) and note that the Hamiltonian that reproduces the above equations is:

\[
H_{12} = f^2 w^2 - w f^2 r_1 - (\alpha_1 + \alpha_3) f w + \epsilon_1 r_1 w + \alpha_1 r_1 f,
\]

(note that the major difference from \( H_{12} \) in (18) is the term \( w f^2 \) instead for \( w^2 f \).)

For the quantity \( f = p \) we find from the above equations a second order equation:

\[
f_{xx} = \frac{f^2}{f} + f^3 r_1^2 + f^2 r_1 (-\alpha_1 + \alpha_3) + \epsilon_1 r_1 (\alpha_1 + \alpha_3) - \frac{\epsilon_1^2 r_1^2}{f},
\]

in which we again recognize the XII-th equation of Ince (16). Furthermore we derive:

\[
w_{xx} = \frac{w^2}{2} \left( \frac{1}{w} + \frac{1}{w - r_1} \right) - 2 r_1 \epsilon_1 w^2 + \alpha_1^2 r_1
\]

\[
+ \frac{2 r_1^3 \epsilon_1 w^2}{w - r_1} - w r_1 \frac{\alpha_1^2 + \alpha_3^2 + 4 \beta \epsilon_1}{2(w - r_1)} + \frac{r_1^3 \alpha_1^2}{2(w - r_1)}.
\]

Defining \( y \) in terms of \( w \) as

\[
y = \frac{w}{w - r_1} \quad \text{or} \quad w = \frac{r_1 y}{y - 1},
\]

we obtain a special incomplete case of Ince’s equation XLIX (2) with the parameters \( A = 1, B = \alpha_3^2/2, C = -\alpha_1^2/2 \) and \( D + E = 2 r_1^2 \epsilon_1 \).

2. Completing Hamiltonian (18) to obtain Ince’s eq. 38

To obtain a Hamiltonian that would reproduce a complete equation I38 we symmetrize Hamiltonian structures \( H_{12} \) from (18) and \( H_{12} \) from (19) by adding a term \( w f^2 \) to the Hamiltonian \( H_{12} \):

\[
H_{38} = f^2 w^2 + \alpha w^2 f + \kappa w f^2 + \beta f w + \gamma f + \delta w,
\]

where we also replaced \( r_0 \) by \( \alpha, -(\alpha_1 + \alpha_3 - 1) \) by \( \beta, \epsilon_0 r_0 \) by \( \gamma \) and \( \alpha_2 r_0 \) by \( \delta \) to have a more general expression. The corresponding Hamilton equations are

\[
w_x = 2 w^2 f + \alpha w^2 + \beta w + 2 \kappa w f + \gamma,
\]

\[
f_x = -2 w f^2 - 2 \alpha w f - \beta f - \delta - \kappa f^2.
\]

Taking a second derivative of the first equation in (21) and eliminating \( f \) one obtains a second-order differential equation for \( w \) which agrees with I38 in (1) with the parameters:

\[
A = \frac{1}{\kappa^2} (\alpha^2 \kappa^4 + \beta^2 \kappa^2 - 2 \alpha \kappa^3 \beta + 2 \alpha \kappa^2 \gamma + \gamma^2 - 2 \beta \kappa \gamma), \quad B = - \frac{\gamma^2}{2 \kappa^2},
\]

\[
C = \frac{\kappa}{2} (-4 \delta + 2 \alpha \beta - 3 \alpha^2 \kappa), \quad D = - \kappa^2 \alpha^2,
\]

when using the variable \( y \):

\[
y = \frac{w}{w + \kappa}.
\]

Similarly obtaining a second-order differential equation for \( f \) from (21) and defining \( y \) through \( f = -\alpha y/(y - 1) \) will also yield equation (1). Due to the fact that addition of \( \kappa w f^2 \) term rendered the system (21) explicitly symmetric in \( w, f \) the Hamilton equations (21) are invariant under \( w, f \) rotation here referred to as \( R \) operation:

\[
R(w) = -f, \quad R(f) = -w, \quad R(\kappa) = -\alpha, \quad R(\alpha) = -\kappa
\]

\[
R(\gamma) = -\delta, \quad R(\delta) = -\gamma, \quad R(\beta) = \beta, \quad R(x) = -x.
\]
Obviously $R^2 = 1$. Also the Hamilton equations (21) are invariant under

$$\pi_2(w) = -w, \quad \pi_2(f) = -f - \alpha, \quad \pi_2(\alpha) = \alpha, \quad \pi_2(\beta) = \beta - 2\kappa \alpha,$$

$$\pi_2(\gamma) = -\gamma, \quad \pi_2(\kappa) = -\kappa, \quad \pi_2(\delta) = -\delta + \alpha \beta - \kappa \alpha^2,$$

which also squares to 1. In addition the model is also invariant under modified transformations (11) and (10):

$$s_2(w) = w + \frac{\delta}{\alpha f}, \quad s_2(f) = f, \quad s_2(\alpha) = \alpha, \quad s_2(\kappa) = \kappa$$

$$s_2(\beta) = \beta - 2\frac{\delta}{\alpha}, \quad s_2(\gamma) = \gamma - \frac{\kappa \delta}{\alpha}, \quad s_2(\delta) = -\delta$$

and

$$s_0(w) = w + \frac{\beta - \delta/\alpha - \kappa \alpha}{f + \alpha}, \quad s_0(f) = f, \quad s_0(\alpha) = \alpha, \quad s_0(\kappa) = \kappa$$

$$s_0(\beta) = -\beta + 2\frac{\delta}{\alpha} + 2\kappa \alpha, \quad s_0(\gamma) = \gamma - \kappa(\beta - \delta \alpha - \kappa), \quad s_0(\delta) = \delta$$

connected to each other via $s_0 = \pi_2 s_2 \pi_2$ and both being a symmetries of the Hamilton equations (21).

We can also define transformations

$$S_2 = Rs_2 R, \quad S_0 = Rs_0 R,$$

with an explicit action for $S_0$ and $S_2$ being:

$$S_2(w) = w, \quad S_2(f) = f + \frac{\gamma}{w \kappa}, \quad S_2(\beta) = \beta - 2\gamma / \kappa,$$

$$S_2(\delta) = \delta - \gamma \alpha / \kappa, \quad S_2(\gamma) = -\gamma, \quad S_2(\alpha) = \alpha,$$

$$S_0(w) = w, \quad S_0(f) = f + \frac{\beta - 2}{w + \kappa}, \quad S_0(\beta) = -\beta + 2\gamma / \kappa + 2\kappa \alpha, \quad S_0(\kappa) = \kappa$$

$$S_0(\delta) = \delta - \alpha(\beta - \gamma / \kappa - \kappa), \quad S_0(\gamma) = \gamma, \quad S_0(\alpha) = \alpha,$$

with both transformations keeping the Hamilton equations invariant. We note that $w$ remains invariant under actions of $S_0, S_2$ while $f$ remains invariant under actions of $s_0, s_2$.

In conclusion addition of an additional cubic term to Hamilton structures (18) or (19) associated with $I_{12}$ yields a Hamilton structure (20) of $I_{38}$ with a symmetry group no longer involving the automorphisms $\pi_i, i = 0, 2$. The manifest symmetry between $f$ and $w$ variables gives raise to a new $w - f$ rotation $R$.

**3. Hamiltonian for Ince’s $I_{49}$ and its symmetries**

We will here derive Ince’s equation $I_{49}$ (2) from the Hamilton function:

$$H_{49} = kf^3 w^2 + f^2 w^2 + \alpha f w^2 + \kappa w f^2 + \beta f w + \gamma f + \delta w,$$

where we allowed for the first time a term of the 5-th power in $f, w : kf^3 w^2$ in addition to terms already present in $H_{38}$ (20). A term of this dimension appears in the Hamiltonian of the Painlevé VI equation. The corresponding Hamilton equations are

$$w_x = +3kf^2 w^2 + 2f(2f + \alpha) + 2kw f + \beta w + \gamma,$$

$$f_x = -2kf^3 w + f(f + \alpha)(-2w) - \kappa f^2 - \beta f - \delta,$$
leading to Ince’s equation I₄⁹ (2) when coefficients k and α are related through the condition:

\[ k = - (\alpha + 1), \quad \alpha \neq -1 \]  

(30)
or \( \alpha = -(k + 1) \). The coefficients \( A, B, C, E, D \) of equation (2) are given in terms of coefficients \( \kappa, \alpha, \beta, \gamma, \delta \) from (28) as follows:

\[
A = -\alpha / (\alpha + 1), \quad C = \frac{1}{2} \delta^2 (\alpha + 1) / \alpha, \quad B = 2 \gamma (\alpha + 1) + \kappa^2 / 2 \\
D = \frac{\alpha + 1}{2(2\alpha + 1)} (\delta + \beta + \kappa)^2, \quad E = -\frac{((1 + \alpha)(\delta + \alpha(-\beta + \delta)) + \alpha^2 \kappa)^2}{2\alpha(1 + \alpha)^2(1 + 2\alpha)}.
\]

(31)
The parameter \( \alpha \) needs to have values different from \( = -1, -1/2, 0 \), to avoid that \( A = 0 \) or \( A = 1 \) or \( D \) or \( E \) becoming infinite.

As we will see below the condition (30) required so that the Hamilton equations derived from (28) would reproduce equation I₄⁹ in (2) also enables several symmetries of (28) system. In the discussion below we will assume that the condition (30) holds and consider those symmetries that maintain this relation.

One symmetry transformation that keeps equations (29) invariant is

\[
s_2(w) = w + \frac{\delta}{\alpha f}, \quad s_2(f) = f \\
s_2(\alpha) = \alpha, \quad s_2(\delta) = -\delta, \quad s_2(\beta) = \beta - 2 \frac{\delta}{\alpha} \\
s_2(\gamma) = \gamma - \frac{\kappa \delta}{\alpha} - (\alpha + 1) \frac{\delta^2}{\alpha^2}, \quad s_2(\kappa) = \kappa + 2(\alpha + 1) \frac{\delta}{\alpha},
\]

(32)

with

\[ s_2^2 = 1. \]

For equations (29) with condition \( k = -\alpha - 1 \) satisfied we can also define additional symmetry \( \pi_0 \) made possible by presence of term \( kf^3 w^2 \) in addition to \( \alpha f w^2 \) in \( H_{49} \). This symmetry is defined as follows

\[
\pi_0(f) = -\frac{\alpha}{(\alpha + 1)f}, \quad \pi_0(w) = \left(\frac{\alpha + 1}{\alpha} f^2 w + \frac{X}{\alpha} f\right) \\
\pi_0(\beta) = \beta - 2 \frac{X}{\alpha + 1}, \quad \pi_0(\kappa) = \frac{1}{\alpha} (-2X\alpha + \delta(\alpha + 1)) \\
\pi_0(\delta) = \frac{\alpha}{\alpha + 1} (\kappa + 2X), \quad \pi_0(k) = k, \quad \pi_0(\alpha) = \alpha \\
\pi_0(\gamma) = -\gamma + \frac{X(\delta(\alpha + 1) + \kappa \alpha)}{\alpha(\alpha + 1)},
\]

(33)
in terms of \( X \) being a root of a quadratic equation:

\[-\gamma(\alpha + 1) + \kappa X + X^2 = 0.\]

(34)

Acting with \( \pi_0 \) directly on \( X = (-\kappa \pm \sqrt{\kappa^2 + 4\gamma(\alpha + 1)}) / 2 \) yields

\[
\pi_0(X) = X - \delta \frac{\alpha + 1}{2\alpha} \pm \frac{1}{2} \sqrt{\delta^2(\alpha + 1)^2 / \alpha^2}.
\]
Thus there are two possible values for $\pi_0(X)$:

$$\pi_0(X) = \begin{cases} X = \pi_0^+(X) \\ X - \delta \frac{\delta}{\alpha} = \pi_0^-(X) \end{cases}, \quad (35)$$

where we associated two different transformations $\pi_0^+$ and $\pi_0^-$ to two possible actions of $\pi_0$ on $X$. Both $\pi_0^+$ and $\pi_0^-$ act on other quantities in accordance with (33).

Both transformations $\pi_0^+$ and $\pi_0^-$ keep equations (29) invariant and preserve the condition $k = -(\alpha + 1)$. Furthermore it holds that:

$$\pi_0^2 = 1, \quad \pi_0^4 = 1, \quad (36)$$

with few intermediate explicit formulas being:

\begin{align*}
\pi_0^- (w) &= s_2(w), \quad \pi_0^- (f) = s_2(f), \quad \pi_0^- (\alpha) = s_2(\alpha), \quad \pi_0^- (\beta) = s_2(\beta) \\
\pi_0^- (\delta) &= s_2(\delta), \quad \pi_0^- (\kappa) = s_2(\kappa), \quad \pi_0^- (\gamma) = s_2(\gamma) \\
\pi_0^- (X) &= -X - \frac{\delta(\alpha + 1)}{\alpha} - \kappa. \quad (37)
\end{align*}

From relations (37) and $s_2(X) = X - \frac{\delta}{\alpha} + \frac{1}{\alpha}$, (38)

and $s_2\pi_0^- (X) = X$ it follows that

$$(s_2\pi_0^\pm)^4 = 1. \quad (39)$$

Thus both $\pi_0^+$ and $\pi_0^-$ satisfy the relation $(s_2\pi_0)^4 = 1$ although only $\pi_0^+$ squares to one! In addition $\pi_0^+$ and $\pi_0^-$ transformations satisfy the relation

$$(\pi_0^-\pi_0^+)^2 = 1, \quad (40)$$

that can be rewritten equivalently as

$$\pi_0^-\pi_0^+ = \pi_0^+, \quad \pi_0^2\pi_0^2 = \pi_0^+, \quad (41)$$

using relation (36). The last identity can also be written as

$$(\pi_0^+\pi_0^-)^2 = 1. \quad (42)$$

Equations (29) with the condition $k = -(\alpha + 1)$ are also invariant under transformations of $\pi^2$ defined as

\begin{align*}
\pi^2 (w) &= (2 + 3\alpha)w, \quad \pi^2 (f) = -f + 1 \\
\pi^2 (k) &= -\frac{\alpha + 1}{(2 + 3\alpha)}, \quad \pi^2 (\alpha) = -\frac{2\alpha + 1}{2 + 3\alpha}, \quad \pi^2 (\beta) = \beta + 2\kappa \\
\pi^2 (\kappa) &= -\kappa, \quad \pi^2 (\delta) = -\delta - \kappa - \beta, \quad \pi^2 (\gamma) = (2 + 3\alpha)\gamma \quad (43)
\end{align*}

From

$$(2 + 3\alpha)(2 + 3\pi^2 (\alpha)) = (2 + 3\alpha)(2 - \frac{2\alpha + 1}{2 + 3\alpha}) = 1$$

it follows that

$$(\pi^2)^2 = 1.$$
and since \( \pi^2(k) = -\pi^2(\alpha) - 1 \) then \( \pi^2 \) defined in (13) is an automorphism that squares to 1 and leaves (29) with the condition \( k = -(\alpha + 1) \) invariant!

Define now

\[
s_0 = \pi^2 s_2 \pi^2,
\]

which obviously squares to one. Then

\[
\begin{align*}
s_0(w) &= w - \frac{\beta + \delta + \kappa}{(1 + 2\alpha)(f - 1)}, \\
s_0(f) &= f, \\
s_0(\alpha) &= \alpha, \\
s_0(\beta) &= \frac{\beta - 2\alpha(\delta + \kappa)}{1 + 2\alpha}, \\
s_0(\gamma) &= \frac{\gamma - (1 + \alpha)\left(-4\alpha \gamma + (\delta + \beta)^2\right) - (\delta + \beta)\kappa + \alpha \kappa^2}{(1 + 2\alpha)^2}, \\
s_0(\delta) &= \delta, \\
s_0(\alpha) &= \frac{2(1 + \alpha)(\beta + \delta) + \kappa}{1 + 2\alpha}.
\end{align*}
\] (44)

It follows then that

\[
(s_2 s_0)^2 = 1.
\]

For the transformation

\[
\pi_2 \pm = \pi^2 \pi_0 \pm \pi^2,
\]

we are able to find based on (33) and (43) the transformations rules:

\[
\begin{align*}
\pi_2(f) &= \frac{\alpha + (\alpha + 1)f}{(\alpha + 1)(f - 1)}, \\
\pi_2(w) &= -\frac{1}{2\alpha + 1} \left( (\alpha + 1)(f - 1)^2 w + \frac{X}{\alpha}(f - 1) \right), \\
\pi_2(\beta) &= \frac{1}{2\alpha + 1} \left( \beta - 2\alpha \kappa + 2\delta(\alpha + 1) \right) - \frac{2X\alpha}{\alpha + 1}, \\
\pi_2(\kappa) &= -(\delta + \kappa + \beta) \frac{\alpha + 1}{2\alpha + 1} - 2X, \\
\pi_2(\delta) &= -\delta + \frac{\alpha}{2\alpha + 1}(\delta + \beta) - \kappa \frac{\alpha^2}{(2\alpha + 1)(\alpha + 1)}, \\
\pi_2(\alpha) &= \alpha, \\
\pi_2(\gamma) &= -\gamma + X \left( \frac{\delta + \beta}{2\alpha + 1} + \kappa \left( \frac{1}{2\alpha + 1} - \frac{1}{\alpha + 1} \right) \right).
\end{align*}
\] (45)

In addition it holds

\[
\pi_2(X) = \begin{cases} 
X = \pi_2 + (X) \\
X + (\delta + \kappa + \beta) \frac{\alpha + 1}{2\alpha + 1} = \pi_2 - (X).
\end{cases}
\] (46)

Furthermore it follows that

\[
\pi_2^2 = 1, \\
\pi_2^{3+} = 1
\]

and also

\[
(s_0 \pi_2)^4 = 1.
\]

We also find from relations (40) and (42) that:

\[
\begin{align*}
\pi_2^{2-} &= 1, \\
\pi_2^{2+} &= 1
\end{align*}
\]

(47)

In conclusion adding a 5-th power term to Hamilton structure (20) of \( I_{38} \) yielded Hamilton structure (29) of \( I_{49} \) and restored \( \pi_0 \) symmetry that was absent in \( H_{38} \). Remarkably, the underlying symmetry group contains transformations \( s_0, s_2, \pi_0, \pi_2 \) and \( \pi^2 \) of \( W[s_0, s_2, \pi_0, \pi_2, \pi^2] \). However the Coxeter group relations are in some cases (e.g (39) and (47)) of higher order as compared with simple Coxeter group relations (14) and (15) of symmetry transformations of P\(_{III}\) or I\(_{12}\) models.
4. Conclusions

We presented here a study of Hamiltonian structures of I_{12}, I_{38} and I_{49} and their symmetries. The mixed P_{III−V} equations taken for various special values of the underlying parameters provided a useful starting point for this analysis.

The Hamiltonian structure of I_{12} shared its symmetry generators and underlying Coxeter relations with P_{III} equation although its symmetry generators acted differently on a set parameters hindering its interpretation as an extended affine Weyl group as described in details in [2]. To obtain I_{38} and I_{49} hamiltonian structures extra terms needed to be added to the Hamiltonian for I_{12}. For I_{38} that resulted in additional symmetry rotation operation and a totally different content of the underlying symmetry group. For I_{49} model the underlying symmetry structure contains the same generators as those of P_{III} or I_{12} but the added higher dimensional term in the Hamiltonian resulted in different higher order Coxeter relations among symmetry generators.

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