ON BICONSERVATIVE SURFACES IN EUCLIDEAN SPACES

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ABSTRACT. In this paper, we study biconservative surfaces with parallel normalized mean curvature vector in $\mathbb{E}^4$. We obtain complete local classification in $\mathbb{E}^4$ for a biconservative PNMCV surface. We also give an example to show the existence of PNMCV biconservative surfaces in $\mathbb{E}^4$.

1. Introduction

Let $(M^m, g)$ and $(N^n, \tilde{g})$ be some Riemannian manifolds. Then, the bi-energy functional is defined by

$$E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 v_g$$

whenever $\psi : M \to N$ is a smooth mapping, where $\tau(\psi)$ denote the tension field of $\psi$.

A mapping $\psi : M \to N$ is said to be biharmonic if it is a critical point of $E_2$. In [12] it was proved that mapping $\psi$ is biharmonic if and only if it satisfies the Euler-Lagrange equation associated with this bi-energy functional given by

$$\tau_2(\psi) = 0,$$

where $\tau_2$ is the bitension field defined by $\tau_2(\psi) = \Delta \tau(\psi) - tr \tilde{R}(d\psi, \tau(\psi))d\psi$ (See also [13]).

In particular, if $\psi$ is an isometric immersion, then $M$ is said to be a biharmonic submanifold of $N$. In this case, by considering tangential and normal components of $\tau_2(\psi)$, one can obtain the following proposition.

Proposition 1.1. Let $x : M^m \to N^n$ be an isometric immersion between two Riemannian manifolds. Then, $x$ is biharmonic if and only if the equations

$$m \text{grad} \|H\|^2 + 4 \text{trace} A_{H} \cdot H = 0,$$

and

$$\text{trace} \alpha(\nabla H, \cdot) - \Delta^{-1} H + \nabla \tilde{R}(\cdot, H) \cdot ^\perp = 0,$$

are satisfied, where $A$, $H$ and $\alpha$ denote the shape operator, the mean curvature vector and second fundamental form of $\psi$, $\nabla ^\perp$ is the normal connection of $M$ and $\Delta ^\perp$ is the Laplacian associated with $\nabla ^\perp$.

From Proposition 1.1 one can see that an isometric immersion $x : M \to N$ is biharmonic if its mean curvature $H$ vanishes identically. In [1], Bang-Yen Chen conjectured that the converse of this statement is also true if the ambient space is Euclidean. Chen’s biharmonic conjecture has been verified in a lot of particular cases so far (see for example [2, 3, 4, 5, 9, 14]). However, the conjecture is still open.

On the other hand, a mapping $\psi : M \to N$ satisfying the condition

$$\langle \tau_2(\psi), d\psi \rangle = 0,$$

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that is weaker than (1.1), is said to be biconservative. In particular, if \( \psi = x \) is an isometric immersion, then (1.3) is equivalent to
\[
\tau_2(x)^T = 0,
\]
where \( \tau_2(x)^T \) denotes the tangential part of \( \tau_2(x) \). In this case, \( M \) is said to be a biconservative submanifold of \( N \). Before we proceed, we would like to note that one can conclude the following well-known proposition, by considering Proposition 1.1 (See for example [6]).

Proposition 1.2. Let \( x : M^m \to N^n \) be an isometric immersion between two Riemannian manifolds. Then, \( x \) is biconservative if and only if the equation (1.2) is satisfied.

In order to understand geometry of biharmonic submanifolds, biconservative immersions have been studied in many papers so far. For example, biconservative immersions into pseudo-Euclidean spaces were studied in [10, 11, 15, 18, 19]. On the other hand, in [16], Montaldo et al. study biconservative surfaces in four-dimensional space form with constant mean curvature. In [8], the complete classification of surfaces in product spaces \( S^n \times \mathbb{R} \) and \( H^n \times \mathbb{R} \) with parallel mean curvature vector was obtained by D. Fetcu, C. Oniciuc, and A. L. Pinheiro.

In this paper, we consider biconservative surfaces in Euclidean spaces. In Sect. 2, after we describe the notation that we will use, we give basic facts on biconservative submanifolds. In Sect. 3, we give complete classification of biconservative surfaces in Euclidean spaces with parallel normalized mean curvature vector field.

2. Biconservative submanifolds in Euclidean spaces

Let \( \mathbb{E}^n \) denote the Euclidean \( n \)-space with the canonical positive definite Euclidean metric tensor given by
\[
\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^{n} dx_i^2,
\]
where \( (x_1, x_2, \ldots, x_n) \) is a rectangular coordinate system in \( \mathbb{E}^n \) and \( \nabla \) stands for its Levi-Civita connection.

Let \( M \) be an \( m \)-dimensional submanifold of \( \mathbb{E}^n \) and \( \nabla^\perp \) denote its normal connection. A normal vector field \( \xi \) is called parallel if
\[
\nabla^\perp_X \xi = 0
\]
whenever \( X \) is tangent to \( M \). On the other hand, the mean curvature vector field \( H \) of \( M \) is defined by
\[
H = \frac{1}{2} \text{tr} h(\cdot, \cdot),
\]
where \( h \) is the second fundamental form of \( M \). Assume that the mean curvature (function) of \( M \) given by
\[
f = \langle H, H \rangle^{1/2}
\]
is a non-vanishing function. In this case, if the unit normal vector field along the mean curvature vector field \( H \) of \( M \) is parallel, then \( M \) is said to have parallel normalized mean curvature vector field and called a PNMCV submanifold of \( \mathbb{E}^n \). It is obvious that PNMCV submanifolds generalize non-minimal submanifolds with parallel mean curvature vector and a PNMCV submanifolds has parallel mean curvature vector if and only if it has constant mean curvature, i.e., \( f \) is constant. It is possible to find examples of PNMCV submanifolds with non-constant mean curvature (See for example [7, 17]).
Since the curvature tensor $\tilde{R}$ of $E^n$ vanishes identically, the following proposition is obtained immediately from (1.2).

**Proposition 2.1.** Let $M$ be an $m$-dimensional PNMCV submanifold of the Euclidean space $E^n$. Then, $M$ is biconservative if and only if

$$A_{m+1}(\text{grad} f) = -\frac{mf}{2}\text{grad} f,$$

where $e_{m+1}$ is the parallel mean curvature vector field and $A_{m+1}$ is the shape operator along $e_{m+1}$.

**Remark 2.2.** If the mean curvature vector of $M$ is parallel, then (2.2) is satisfied trivially. Therefore, after this point we will assume that $\text{grad} f$ does not vanish at any point of $M$.

### 2.1. Basic equations in the theory of surfaces of $E^n$.

Let $M$ be a surface in $E^n$, $\nabla$, $h$ and $A$ denote its the Levi-Civita connection, second fundamental form and shape operator, respectively. Note that $R$ and $\tilde{R}$ will stand for curvature tensor of $M$ and $E^n$, respectively.

For tangent vector fields $X, Y, Z$ on $M$ the Codazzi equation ($\tilde{R}(X,Y)Z) = 0$ and the Gauss equation ($\tilde{R}(X,Y)Z)^T = 0$ take the form

(2.3)  
$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$$

and

(2.4)  
$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$

respectively, where $(\nabla_X h)(Y, Z)$ is defined by

$$(\nabla_X h)(Y, Z) = \nabla^Y h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Ricci equation ($\tilde{R}(X,Y)\xi)^T = 0$ takes the form

(2.5)  
$$R^\xi(X,Y)\xi = h(X, A_\xi Y) - h(A_\xi X, Y)$$

whenever $X, Y$ are tangent and $\xi$ is normal to $M$.

### 3. Biconservative PNMCV Surfaces in $E^4$

In this section, we would like to obtain complete local classification of biconservative surfaces with parallel normalized mean curvature vector in the Euclidean 4-space $E^4$.

First we would like to obtain shape operator and Levi-Civita connection of a biconservative PNMCV surface in $E^4$.

**Lemma 3.1.** Let $M$ be a surface in $E^4$ with non-vanishing mean curvature. Then, $M$ is a biconservative PNMCV surface if and only if there exists a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ such that

1. The Levi-Civita connection $\nabla$ and normal connection $\nabla^\perp$ of $M$ satisfy

   (3.1a)  
   $$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0,$$

   (3.1b)  
   $$\nabla_{e_2} e_1 = -3e_1(f)\frac{e_2}{4f}, \quad \nabla_{e_2} e_2 = 3e_1(f)\frac{e_1}{4f},$$

   (3.1c)  
   $$\nabla^\perp e_3 = \nabla^\perp e_4 = 0,$$

2. Shape operators along $e_3$ and $e_4$ have matrix representations given by

   (3.2)  
   $$A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} cf^{3/2} & 0 \\ 0 & -cf^{3/2} \end{pmatrix}$$

for some constant $c$ and a smooth non-vanishing function $f$ such that $e_2(f) = 0$ and $e_1(f) \neq 0$. 
Proof. Let the mean curvature vector of \( M \) be \( H \) and \( \langle H, H \rangle = f \). In order to prove the necessary condition, we assume that \( M \) is biconservative and \( H \) is parallel. We choose a local frame field \( \{e_3, e_4\} \) of the normal bundle of \( M \) as \( e_3 = \frac{H}{f} \). Note that (2.2) is satisfied for \( m = 2 \). On the other hand, by the assumption \( e_3 \) is parallel. \( e_4 \) is also parallel because the co-dimension of \( M \) is 2. Therefore, we have

\[
\nabla_X e_3 = \nabla_X e_4 = 0
\]

for any tangent vector field \( X \) which yields (3.1c). As \( H \) is proportional to \( e_3 \), we have

\[
(3.3) \quad \frac{1}{2} \text{tr} A_3 = f \quad \text{and} \quad \text{tr} A_4 = 0.
\]

We choose a local frame field \( \{e_1, e_2\} \) of the normal bundle of \( M \) so that \( A_3 = \text{diag}(h_{11}^3, h_{22}^3) \). Then, because of (2.2), we may assume \( e_1 = \nabla f / \|\nabla f\| \) and \( h_{11}^3 = -f \). We will prove that the frame field \( \{e_1, e_2; e_3, e_4\} \) satisfies the other conditions given in the lemma.

The first equation in (3.3) implies \( h_{22}^3 = 3f \). Since \( e_3 \) is parallel, the Ricci equation (2.5) for \( X = e_1, Y = e_2 \) and \( \xi = e_3 \) yields

\[
4fh(e_1, e_2) = 0.
\]

Therefore, we have \( \langle A_4(e_1), e_2 \rangle = 0 \). This equation and the second equation in (3.3) imply

\[
A_{e_3} = \begin{pmatrix} -f & 0 \\ 0 & 3f \end{pmatrix}, \quad \text{and} \quad A_{e_4} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}
\]

for a smooth function \( \lambda \).

By using Codazzi equation (2.3) for \( X = e_1, Y = Z = e_2 \), we obtain

\[
(3.4) \quad e_1(\lambda) = -2\lambda \omega_{12}(e_2) \quad \text{and} \quad e_1(f) = -\frac{4f}{3} \omega_{12}(e_2)
\]

which imply

\[
(3.5) \quad 2f e_1(\lambda) = 3\lambda e_1(f).
\]

The Codazzi equation (2.3) for \( X = e_2, Y = Z = e_1 \) gives

\[
(3.6) \quad \omega_{12}(e_1) = 0, \quad e_2(\lambda) = 0.
\]

By integrating equation (3.5) and considering (3.6), we have \( \lambda = cf^{3/2} \) for some constant \( c \). Moreover, since \( e_1 \) is proportional to \( \nabla f \), \( f \) satisfies \( \omega_2(f) = 0 \) and \( e_1(f) \neq 0 \). Therefore, we have the condition (1) of the lemma. On the other hand, the second equation in (3.4) and the first equation in (3.6) give the condition (2) of the lemma. Hence, we completed the proof of the necessary condition.

Conversely, let \( M \) be a surface with a local orthonormal frame field \( \{e_1, e_2; e_3, e_4\} \) satisfying the conditions given in the Lemma. Then, (3.2) implies \( H = fe_3 \) and \( \text{grad} f = e_1(f)e_1 + e_2(f)e_2 = e_1(f)e_1 \). Therefore, \( \text{grad} f \) is eigenvalue of the shape operator \( A_{e_3} \) and \( e_3 \) is the normalized mean curvature vector field of \( M \). Moreover, (3.1c) yields that \( e_3 \) is parallel. Therefore, \( M \) is a PNMCV surface. Moreover, Proposition 2.1 is satisfied which yields that \( M \) is biconservative. Hence, the proof of the sufficient condition is completed. \( \square \)

As an immediate consequence of Lemma 3.1 we would like to state the following corollary.

**Corollary 3.2.** Let \( M \) be a biconservative PNMCV surface in \( \mathbb{E}^4 \). Then, the Gaussian curvature \( K \) and the mean curvature \( f \) of \( M \) satisfy \( K = -3f^2 - c^2f^3 \) for a constant \( c \).

Let \( M \) be a PNMCV surface in \( \mathbb{E}^4 \), \( f \) its mean curvature and \( m \in M \) with \( f(m) \neq 0 \) and \( (\text{grad} f)(m) \neq 0 \). Next, by using Lemma 3.1 we would like to construct a local coordinate system on a PNMCV biconservative surface \( M \) in \( \mathbb{E}^4 \) on a neighborhood of \( m \in M \).
Lemma 3.3. Consider a local orthonormal frame field \( \{e_1, e_2, e_3, e_4\} \) on \( M \) given in Lemma 3.1. Then there exists local coordinate system \( \{s, t\} \) on a neighborhood of \( m \) such that
\[
f = f(s), \quad e_1 = \frac{\partial}{\partial s}, \quad \text{and} \quad e_2 = f(s)^{3/4} \frac{\partial}{\partial t}
\]

Proof. Let \( \{e_1, e_2, e_3, e_4\} \) be a a local orthonormal frame field given in Lemma 3.1. Because of \( \nabla e_1 e_2 = 0, \nabla e_2 e_1 = -\frac{3e_1(f)}{4f} e_2 \), we have \( [e_1, e_2] = \frac{3e_1(f)}{4f} e_2 \) which gives \( [e_1, E e_2] = 0 \) for any function \( E \) satisfying
\[
e_1(E) = \frac{-3e_1(f)}{4f} E.
\]

Thus, there exists a local coordinate system \( (s, t) \) such that \( e_1 = \frac{\partial}{\partial s} \) and \( E e_2 = \frac{\partial}{\partial t} \).

Moreover, \( e_2(f) = 0 \) which yields \( f = f(s) \) and because of (3.7) we can choose \( E \) as \( E = f(s)^{-3/4} \).

\[\square\]

Corollary 3.4. Let \( M \) be a biconservative surface with parallel normalized mean curvature vector in \( \mathbb{E}^4 \). Then, the mean curvature of \( M \) satisfies the following partial differential equation
\[
(3.8) \quad \frac{9f'(s)^2}{16f(s)^{7/2}} + c^2 f(s)^{3/2} + 9f(s)^{1/2} = c_2^2
\]

for a positive constant \( c_2 \), where \( s \) is the local coordinate given in Lemma 3.3.

Proof. Consider a local orthonormal frame field \( \{e_1, e_2, e_3, e_4\} \) given in Lemma 3.1 and local coordinate system \( (s, t) \) given in Lemma 3.3. Note that the Gauss equation (2.4) for \( X = Z = \partial_s \) and \( Y = \partial_t \) gives
\[
f(s)f''(s) - \frac{7}{4} f'(s)^2 + 4f(s)^4 + \frac{4}{3} c^2 f(s)^5 = 0.
\]

By multiplying this equation with \( \frac{9f'(s)}{8f(s)^{9/2}} \) and integrating the equation obtained, we get (3.8) for a constant \( c_2 \) which can be assumed to be positive.

\[\square\]

Proposition 3.5. Let \( M \) be a proper PNMVC biconservative surfaces in \( \mathbb{E}^4 \), where \( f \) is the mean curvature of \( M \) in \( \mathbb{E}^4 \) and \( e_1 = \frac{\nabla f}{|\nabla f|} \). Then,

(a) An integral curve of \( e_1 \) lies on a 3-dimensional hyperplane of \( \mathbb{E}^4 \).

(b) The curvature and torsion of an integral curve of \( e_1 \) are
\[
(3.9a) \quad \kappa(s) = f(s) \sqrt{1 + c^2 f(s)},
\]
\[
(3.9b) \quad \tau(s) = \frac{cf'(s)}{2 \sqrt{f(s)(1 + c^2 f(s))}}.
\]

(c) Any two integral curves of \( e_1 \) are congruent.

Proof. Let \( \{e_1, e_2, e_3, e_4\} \) be the local orthonormal frame on \( M \) given by Lemma 3.1 and we suppose that \( \gamma \) is an integral curve of \( e_1 \) and it is parametrized by \( \gamma(s) = x(s, t_0) \). Let \( T = \gamma' \) be tangent of \( M \) and consider the moving frame field span \( \{T(s), N(s), B(s), \tilde{B}(s)\} \).
of the curve $\gamma$ on $M$. We consider the following Frenet formulas

\[
\begin{align*}
\frac{dT}{ds} &= \kappa N, \\
\frac{dN}{ds} &= -\kappa T + \tau B, \\
\frac{dB}{ds} &= -\tau N + \tau_2 \tilde{B}, \\
\frac{d\tilde{B}}{ds} &= -\tau_2 B,
\end{align*}
\]

(3.10)

where $\kappa, \tau$ and $\tau_2$ are curvatures of $\gamma$.

We proceed to compute the curvature and torsion of an integral curve $\gamma$ of $e_1$. By combining (3.2) with Gauss formula and considering Weingarten formula we obtain

\[
\tilde{\nabla}_{T(s)} T(s) = -f(s)e_3(s) + cf(s)^{3/2}e_4(s),
\]

(3.11)

and

\[
\tilde{\nabla}_{e_1} e_3 = e_3'(s) = f(s)T
\]

and

\[
\tilde{\nabla}_{e_1} e_4 = e_4'(s) = -cf(s)^{3/2}T,
\]

(3.13)

where $e_3(s), e_4(s)$ are restrictions of $e_3$ and $e_4$ to $\gamma$.

By combining (3.10) with (3.11), we get (3.9a) and

\[
N(s) = \frac{dT}{ds} \frac{1}{\kappa} = \frac{-1}{\sqrt{1 + c^2 f(s)}} e_3(s) + c \sqrt{\frac{f(s)}{1 + c^2 f(s)}} e_4(s).
\]

(3.14)

By differentiation of (3.14) with respect to $e_1$ and using (3.12), (3.13), we obtain

\[
\frac{dN}{ds} = \frac{c^2 f'(s)}{2(1 + c^2 f(s))^{3/2}} e_3(s) + \frac{cf'(s)}{2 \sqrt{f(s)(1 + c^2 f(s))^{3/2}}} e_4(s) - \frac{f(s)(1 + c^2)}{\sqrt{1 + c^2 f(s)}} T.
\]

(3.15)

By combining equations (3.9a), (3.10) and (3.15), we obtain the torsion $\tau$ of $\gamma$ as given in (3.9b) and

\[
B(s) = \frac{c\sqrt{f(s)}}{\sqrt{1 + c^2 f(s)}} e_3(s) + \frac{1}{\sqrt{1 + c^2 f(s)}} e_4(s).
\]

(3.16)

By applying $e_1$ to the equation (3.16) and using (3.12), (3.13), we obtain

\[
\frac{dB}{ds} = \tau N(s),
\]

which yields that $\tau_2 = 0$. Hence, we have the part (a) and part (b) of the Lemma.

Now, we want to show part (c) of the Lemma. Let $m_1, m_2 \in M$ lie on the same integral curve of $e_2$. Consider the local coordinate system $(s, t)$ given in Lemma 3.3. Note that $e_2(f) = 0$ and $\nabla_{e_1} e_2 = 0$ yields $e_2(e_1(f)) = 0$ which implies $e_1(f) = f'(s)$. Therefore, because of (3.9), any integral curves $\gamma_1$ and $\gamma_2$ of $e_1$ have the same curvature and torsion functions. Hence, they are congruent to each other. □

Now, we are ready to get main classification theorem

**Theorem 3.6.** Let $M$ be a PNMVC surface in the 4-dimensional Euclidean space $\mathbb{E}^4$ with a point $m \in M$ at which $f(m) > 0$, $(\text{grad } f)(m) \neq 0$, where $f$ is the mean curvature of $M$. 
If $M$ is biconservative, then there exists a neighborhood of $m$ on which $M$ is congruent to the simple rotational surface
\begin{equation}
  x(s, t) = (\alpha_1(s) \cos t, \alpha_1(s) \sin t, \alpha_2(s))
\end{equation}
with arc-length parametrized smooth profile curve $\alpha(s) = (\alpha_1(s), \alpha_2(s))$.

whose curvature and torsion are given by (3.9).

Proof. We consider a local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given in Lemma 3.1 with
\[ e_1 = \frac{\partial}{\partial s} \quad \text{and} \quad e_2 = f(s)^{3/4} \frac{\partial}{\partial t}, \]
where $(s, t)$ is local coordinate system given in Lemma 3.3. Note that we also have $f = f(s)$ which satisfies (3.8) for a constant $c_2$ because of Corollary 3.4. Moreover, the induced metric of $M$ is
\begin{equation}
  g = ds \otimes ds + \frac{1}{f(s)^{3/2}} dt \otimes dt.
\end{equation}

By combining (3.1b) with (3.2), we have
\begin{align}
  \tilde{\nabla}_{\partial_t} \partial_s &= \frac{3f'}{4f} \partial_t, \\
  \tilde{\nabla}_{e_2} e_2 &= \frac{3f'}{4f} \partial_s + 3f e_3 - cf^{3/2} e_4, \\
  \tilde{\nabla}_{\partial_s} e_3 &= -3f \partial_t, \\
  \tilde{\nabla}_{\partial_s} e_4 &= cf^{3/2} \partial_t.
\end{align}

Let $x : M \to \mathbb{E}^4$ be an isometric immersion. Then, (3.19a) becomes
\begin{equation}
  x_{ts} = \frac{-3f'}{4f} x_t.
\end{equation}

By solving this equation, we get
\begin{equation}
  x(s, t) = f^{-3/4} \Theta(t) + \Gamma(s)
\end{equation}
for some $\mathbb{E}^4$-valued smooth functions $\Theta, \Gamma$.

By combining (3.20) with (3.19b), we obtain
\begin{equation}
  f(s)^{3/4} \Theta''(t) + \frac{9f'(s)^2}{16f(s)^{11/4}} \Theta(t) - \frac{3f'(s)}{4f(s)} \Gamma'(s) - 3f(s)e_3 + cf(s)^{3/2}e_4 = 0.
\end{equation}

By applying $\partial_t$ to this equation and using (3.19a), (3.19c) and (3.19d) we obtain
\begin{equation}
  \Theta'''(t) + \left( \frac{9f'(s)^2}{16f(s)^{7/2}} + c^2 f(s)^{3/2} + 9f(s)^{1/2} \right) \Theta'(t) = 0.
\end{equation}

By combining this equation with (3.8), we get
\begin{equation}
  \Theta''''(t) + c^2 \Theta'(t) = 0.
\end{equation}

Thus, $\Theta$ has the form $\Theta(t) = \cos(c_2 t)A_1 + \sin(c_2 t)A_2 + A_3$ for some constant vectors $A_1, A_2, A_3$. Therefore, (3.20) becomes
\begin{equation}
  x(s, t) = f(s)^{-3/4} \cos(c_2 t)A_1 + f(s)^{-3/4} \sin(c_2 t)A_2 + f(s)^{-3/4} A_3 + \Gamma(s).
\end{equation}
By considering (3.18), we obtain
\[ \langle A_1, A_1 \rangle = a_1^2, \quad \langle A_2, A_2 \rangle = \langle A_3, A_3 \rangle = \frac{1}{c_2^2}, \]
\[ \langle A_i, A_j \rangle = \langle \Gamma'(s), A_2 \rangle = \langle \Gamma'(s), A_3 \rangle = 0 \quad \text{if } i \neq j \]
and
\[ \langle \Gamma'(s), \Gamma'(s) \rangle = -\frac{3f''(s)}{2f(s)^{7/4}} \langle A_1, \Gamma'(s) \rangle + \frac{9f'(s)^2}{16f(s)^{7/2}} \left( a_1^2 + \frac{1}{c_2^2} \right) = 1 \]
for a non-zero constant \( a_1 \). Therefore, up to a suitable isometry of \( \mathbb{E}^4 \), we may assume
\[ A_1 = (0, 0, a_1, 0), \]
\[ A_2 = \left( \frac{1}{c_2}, 0, 0, 0 \right), \]
\[ A_3 = \left( 0, \frac{1}{c_2}, 0, 0 \right), \]
(3.22)
\[ \Gamma(s) = \left( 0, 0, \alpha_2(s) - \frac{a_1}{f(s)^{3/4}} \alpha_3(s) \right) \]
for some smooth functions \( \alpha_2, \alpha_3 \). Hence, (3.21) gives (3.17) after re-defining \( t \) properly. Since \( t = \text{const.} \) is an integral curve of \( e_1 \), Proposition 3.5 implies that the curvature and torsion of the curve \( \alpha(s) = \left( \frac{1}{c_2 f(s)^{3/4}}, \alpha_2(s), \alpha_3(s) \right) \) are the functions \( \kappa \) and \( \tau \) given in (3.9).

Next, we obtain the converse of the above theorem.

**Theorem 3.7.** Let \( M \) be the simple rotational surface in \( \mathbb{E}^4 \) given by (3.17) with arc-length parametrized smooth profile curve \( \alpha(s) = \left( \frac{1}{c_2 f(s)^{3/4}}, \alpha_2(s), \alpha_3(s) \right) \) whose curvature \( \kappa \) is given by (3.9a), where \( f : (a, b) \rightarrow \mathbb{E}^4 \) is a positive function satisfying (3.8) for a constant \( c_2 \). Then, \( M \) is a PNMCV biconservative surface. Furthermore, its mean curvature is \( f \).

**Proof.** Let \( f \) be a positive function satisfying (3.8) which is equivalent to
\[ f'(s) = \frac{4}{3} \sqrt{c^2 f(s)^5 + c_2^2 f(s)^{7/2} - 9f(s)^4} \]
for a constant \( \varepsilon \in \{-1, 1\} \). Consider the curve \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) with \( \alpha_1 = \frac{1}{c_2 f(s)^{3/4}} \), curvature \( \kappa \) given by (3.9a) and torsion \( \tau \). Let \( t = (t_1, t_2, t_3), n = (n_1, n_2, n_3), b = (b_1, b_2, b_3) \) be the unit tangent, unit normal and unit binormal of \( \alpha \). By a simple computation considering (3.23), we obtain
\[ t_1 = \alpha'_1 = -\frac{\varepsilon \sqrt{c^2 f(s)^5 + c_2^2 f(s)^{7/2} - 9f(s)^4}}{c_2 f(s)^{3/4}}, \]
\[ n_1 = \frac{t'_1}{\kappa} = \frac{f(s)^{1/4} (c_2 f(s) + 3)}{c_2 \sqrt{c^2 f(s) + 1}}, \]
(3.24)
Furthermore, \( t_1^2 + n_1^2 + b_1^2 = 1 \) and \( \tau b = -n_1' - \kappa t \) give
\[ b_1 = -\frac{2c f(s)^{3/4}}{c_2 \sqrt{c^2 f(s) + 1}}, \]
\[ \tau = \frac{2c \varepsilon \sqrt{c_2^2 f(s)^{9/2} - f(s)^3 (c^2 f(s) + 9)}}{3c^2 f(s) + 3}. \]
By considering (3.23), one can check that (3.24d) is equivalent to (3.9b).

Now, let $M$ be the simple rotational surface in $\mathbb{E}^4$ given by (3.17) with profile curve $\alpha$. Consider the local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on $M$ given by

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{\alpha(s)} \frac{\partial}{\partial t},$$

$$e_3 = (n_1(s) \cos t, n_1(s) \sin t, n_2(s), n_3(s)), \quad e_4 = (b_1(s) \cos t, b_1(s) \sin t, b_2(s), b_3(s)).$$

By a direct computation considering (3.24a)-(3.24c) and (3.9b), we can obtain (3.1a), (3.1b) and

$$A_{\hat{e}_3} = \begin{pmatrix} f(s)\sqrt{c^2 f(s) + 1} + 1 & 0 \\ 0 & -\frac{f(s)(c^2 f(s) + 3)}{\sqrt{c^2 f(s) + 1}} \end{pmatrix}, \quad A_{\hat{e}_4} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2cf(s)^{3/2}}{\sqrt{c^2 f(s) + 1}} \end{pmatrix}.$$  

Furthermore, the normal connection of $M$ satisfies

$\nabla^\perp_{e_2} \hat{e}_3 = 0$,

$$\nabla^\perp_{e_1} \hat{e}_3 = \frac{2c \sqrt{c^2 f(s)^{5/2} - f(s)^3(c^2 f(s) + 9)}}{3c^2 f(s) + 3} \hat{e}_4.$$  

Note that the mean curvature vector of $M$ is

$$H = -\frac{f(s)}{\sqrt{c^2 f(s) + 1}} \hat{e}_3 + \frac{cf(s)^{3/2}}{\sqrt{c^2 f(s) + 1}} \hat{e}_4$$

which yields that the mean curvature of $M$ is $H = f$. Thus, the normalized mean curvature vector of $M$ is

$$\hat{e}_3 = -\frac{1}{\sqrt{c^2 f(s) + 1}} \hat{e}_3 + \frac{cf(s)^{1/2}}{\sqrt{c^2 f(s) + 1}} \hat{e}_4$$

and we put

$$\hat{e}_4 = \frac{cf(s)^{1/2}}{\sqrt{c^2 f(s) + 1}} \hat{e}_3 + \frac{1}{\sqrt{c^2 f(s) + 1}} \hat{e}_4$$

(3.25), (3.27) and (3.28) give (3.2). Furthermore, (3.1c) follows from a direct computation considering (3.23), (3.26) and (3.27). Thus, $M$ is a PNMCV surface and the orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ satisfies conditions of Lemma 3.1 which yields that $M$ is biconservative.

In order to show the existence of PNMCV biconservative surfaces in $\mathbb{E}^4$, we would like to give the following example.

**Example 3.8.** Let $f$ be a positive function satisfying (3.8) and assume that $f'$ does not vanish. Then, the PNMCV simple rotational surface $M$ given by

$$x(s, t) = \left(\frac{1}{c_2 f(s)^{3/4}} \cos t, \frac{1}{c_2 f(s)^{3/4}} \sin t, \frac{1}{c_2} \int_{s_0}^s \cos(\theta(\xi)) f(\xi)^{1/4} \sqrt{c^2 f(\xi) + 9} d\xi, \frac{1}{c_2} \int_{s_0}^s \sin(\theta(\xi)) f(\xi)^{1/4} \sqrt{c^2 f(\xi) + 9} d\xi \right),$$

(3.29) for a function $\theta$ satisfying

$$\theta'(s) = \frac{2c_2 f(s)^{5/4}}{c_2^2 f(s) + 9}$$

is biconservative where $s_0, c \neq 0$ are some constants and $c_2$ is positive constant.
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