NEW TYPE OF SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^N$

LIPENG DUAN AND MONICA MUSSO

Abstract. We construct a new family of entire solutions for the nonlinear Schrödinger equation
\[
\begin{cases}
-\Delta u + V(y)u = u^p, & u > 0, \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]
where $p \in (1, \frac{N+2}{N-2})$ and $N \geq 3$, and $V(y) = V(|y|)$ is a positive bounded radial potential satisfying
\[V(|y|) = V_0 + \frac{a}{|y|^m} + O\left(\frac{1}{|y|^{m+\sigma}}\right), \quad \text{as } |y| \to \infty,\]
for some fixed constants $V_0, a, \sigma > 0$, and $m > 1$. Our solutions are different from the ones obtained in [20] and have strong analogies with the doubling construction of entire finite energy sign-changing solution for the Yamabe equation in [13].

Keyword: Nonlinear Schrödinger equation, Infinitely many solutions, New solutions, Finite Lyapunov-Schmidt reduction

AMS Subject Classification: 35B34, 35J25.

1. Introduction

This paper is devoted to the construction of solutions to the following nonlinear elliptic problem
\[
\begin{cases}
-\Delta u + V(y)u = u^p, & u > 0, \quad \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]
(1.1)
where $1 < p < \frac{N+2}{N-2}$, $N \geq 3$, and $V$ is a bounded radially symmetric potential with $V(y) \geq V_0 > 0$. This problem arises when looking for standing waves solutions
\[
\psi(y, t) = e^{i\lambda t}u(y),
\]
to the time-dependent Schrödinger equations
\[
-i \frac{\partial \psi}{\partial t} = \Delta \psi - V_1(y)\psi + |\psi|^{p-1}\psi, \quad \text{for } (y, t) \in \mathbb{R}^N \times \mathbb{R},
\]
(1.2)
where $i$ is the imaginary unit and $V(y) = V_1(y) + \lambda$. This kind of equation arises in many applications, for instance in nonlinear optics, plasma physics, quantum mechanics, or condensed matter physics.

The energy functional associated to (1.1) is given by
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left\{|\nabla u|^2 + V(y)u^2\right\} - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]
(1.3)
This functional posses infinitely many critical points in the class of radially symmetric functions, as a direct consequence of the Ljusternik-Schnirelmann theory. Nonetheless, it is not clear that these critical points are solutions to (1.1) since they may not need to be positive. On the other hand, using the concentration compactness theorem one can prove that if the potential $V = V(y)$ satisfies

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together a large number of copies of building blocks given by the radial solutions to (1.1) with high energy. Roughly speaking, these solutions are obtained by gluing assumptions, Wei and Yan in [20] used a constructive method to produce infinitely many non-

possible to use the functions \( u \) the use of a refined version of the Pohozaev identity in the spirit of [12]. This property makes it operators around these solutions are invertible. This fact has been recently proven in [13] with degenerate in the class of functions sharing the same symmetry, in the sense that the linearized

degenerate, but not necessarily radially symmetric [10]. In fact, the solutions constructed in [20] happen to be non-
degenerate in the class of functions sharing the same symmetry, in the sense that the linearized operators around these solutions are invertible. This fact has been recently proven in [13] with the use of a refined version of the Pohozaev identity in the spirit of [12]. This property makes it possible to use the functions \( u_k^* \) as new building blocks to generate new constructions, see [13].

A natural question to ask is whether there are other type of building blocks for (1.1), which are not equal to the ones constructed in [20] nor cannot be obtained by gluing a number of them together. The purpose of this paper is to give an answer of this question with a new family of solutions to (1.1) which have a more complex concentration structure.

Let \( N \geq 3 \), \( k \) be an integer and introduce the points

\[
\begin{align*}
\bar{x}_j &= r \left( \sqrt{1 - h^2 \cos \frac{2(j-1)\pi}{k}}, \sqrt{1 - h^2 \sin \frac{2(j-1)\pi}{k}}, h, 0 \right), & j &= 1, \ldots, k, \\
\bar{y}_j &= r \left( \sqrt{1 - h^2 \cos \frac{2(j-1)\pi}{k}}, \sqrt{1 - h^2 \sin \frac{2(j-1)\pi}{k}}, -h, 0 \right), & j &= 1, \ldots, k,
\end{align*}
\]

Observe that a direct scaling argument enables us to just consider \( V_0 = 1 \). Under these same assumptions, Wei and Yan in [20] used a constructive method to produce infinitely many non-radial solutions to (1.1) with high energy. Roughly speaking, these solutions are obtained by gluing together a large number of copies of building blocks given by the bump \( U \) which is the only positive radially-symmetric solution to

\[
\begin{align*}
-\Delta u + u &= u^p, & \text{in } \mathbb{R}^N, \\
u(y) &\to 0, & \text{as } |y| \to \infty.
\end{align*}
\]

More precisely, for any large integer \( k \) the authors constructed a solution \( u_k^* \) looking like a sum of a \( k \) bumps \( U(x - x_j^*) \),

\[
u_k^*(y) \sim \sum_{j=1}^{k} U(y - x_j^*)
\]

where the location points \( x_j^* \) are distributed along the vertices of a regular \( k \)-polygon

\[
x_j^* = r \left( \cos \frac{2(j-1)\pi}{k}, \sin \frac{2(j-1)\pi}{k}, 0, \ldots, 0 \right), & \text{for } j = 1, \ldots, k
\]

with large radius \( r \sim k \log k \) as \( k \to \infty \). These solutions respect the polygonal symmetry in the \((x_1, x_2)\)-plane, and are radially symmetric in the other variables. This construction is stable in the sense that solutions with similar profile exist for other potential functions close to \( V \) but not necessarily radially symmetric [10]. In fact, the solutions constructed in [20] happen to be non-degenerate in the class of functions sharing the same symmetry, in the sense that the linearized operators around these solutions are invertible. This fact has been recently proven in [13] with the use of a refined version of the Pohozaev identity in the spirit of [12]. This property makes it possible to use the functions \( u_k^* \) as new building blocks to generate new constructions, see [13].

In this paper we assume that the potential \( V \) is radial and satisfies the following decay condition at infinity: there exist \( a, \sigma > 0 \), and \( m > 1 \) so that

\[
V(|y|) = V_0 + \frac{a}{|y|^m} + O_k \left( \frac{1}{|y|^{m+\sigma}} \right), & \text{as } |y| \to \infty. \quad \text{(H1)}
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A natural question to ask is whether there are other type of building blocks for (1.1), which are not equal to the ones constructed in [20] nor cannot be obtained by gluing a number of them together. The purpose of this paper is to give an answer of this question with a new family of solutions to (1.1) which have a more complex concentration structure.

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\bar{y}_j &= r \left( \sqrt{1 - h^2 \cos \frac{2(j-1)\pi}{k}}, \sqrt{1 - h^2 \sin \frac{2(j-1)\pi}{k}}, -h, 0 \right), & j &= 1, \ldots, k,
\end{align*}
\]
where \(0\) is the zero vector in \(\mathbb{R}^{N-3}\). The parameters \(h\) and \(r\) are positive numbers and are chosen in the range
\[ h \in [\alpha_0 \frac{1}{k}, \alpha_1 \frac{1}{k}], \quad r \in [\beta_0 k \ln k, \beta_1 k \ln k] \quad (1.7) \]
for \(\alpha_0, \alpha_1, \beta_0, \beta_1\) fixed positive constants, independent of \(k\). We define the approximate solution as
\[ W_{r,h}(y) = \sum_{j=1}^{k} U_{\hat{x}_j}(y) + \sum_{j=1}^{k} U_{\tilde{x}_j}(y), \quad (1.8) \]
where \(U_{\hat{x}_j}(y) = U(y - \hat{x}_j)\) and \(U_{\tilde{x}_j}(y) = U(y - \tilde{x}_j)\) and \(k\) would be sufficiently large. In this paper, we will prove that from any \(k\) large enough problem \((1.1)\) has a new type of solutions \(u_k\) which have the form
\[ u_k(y) \sim W_{r,h}(y), \quad (1.9) \]
as \(k \to \infty\). Our solutions will have polygonal symmetry in the \((x_1, x_2)\)-plane, will be even in the \(x_3\) direction and radially symmetric in the variables \(x_4, \ldots, x_N\). In particular, they do not belong to the same class of symmetry as the solutions built in \([20]\). Our solutions are thus different from the ones obtained in \([20]\) and have strong analogies with the doubling construction of entire finite energy sign-changing solution for the Yamabe equation in \([18]\).

Observe that, if we take \(h = 0\) in \((1.6)\), then \(\hat{x}_j = \hat{x}_j\) for any \(j\) and the two constructions \((1.5)\) and \((1.9)-(1.8)\) are the same. Under our assumptions on the range for the parameters \(h\) and \(r\) in \((1.7)\), the two constructions \((1.5)\) and \((1.9)-(1.8)\) are different. A main distinction between the present construction and the one in \([20]\) is that there are two parameters \(r\) and \(h\) to choose in the locations \(\hat{x}_j, \tilde{x}_j\) of the bumps in \((1.8)\).

We will discuss in details our result in the following subsection.

Throughout this paper, we employ \(C, C_j, \sigma, \sigma_j, \tau, \tau_j, j = 0, 1, 2, \cdots\) to denote certain constants. Furthermore, we also employ the common notation by writing \(O_k(f(r,h)), o_k(f(r,h))\) for the functions which satisfy
\[ \text{if } g(r,h) \in O_k(f(r,h)) \text{ then } \lim_{k \to +\infty} \left| \frac{g(r,h)}{f(r,h)} \right| \leq C < +\infty, \]
and
\[ \text{if } g(r,h) \in o_k(f(r,h)) \text{ then } \lim_{k \to +\infty} \frac{g(r,h)}{f(r,h)} = 0, \]
in the present paper.

1.1. Main result and scheme of the proof.

For \(j = 1, \ldots, k\), we divide \(\mathbb{R}^N\) into \(k\) parts:
\[ \Omega_j := \left\{ y = (y_1, y_2, y_3, y'') \in \mathbb{R}^3 \times \mathbb{R}^{N-3} \right\} \]
where \((\cdot, \cdot)_{\mathbb{R}^2}\) denote the dot product in \(\mathbb{R}^2\). For \(\Omega_j\), we divide it into two parts:
\[ \Omega_j^+ = \left\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 \geq 0 \right\}, \]
\[ \Omega_j^- = \left\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 < 0 \right\}. \]

We see that
\[ \mathbb{R}^N = \bigcup_{j=1}^{k} \Omega_j, \quad \Omega_j = \Omega_j^+ \cup \Omega_j^-. \]
and the interior of
\[ \Omega_j \cap \Omega_i, \quad \Omega_j^+ \cap \Omega_j^- \]
are empty sets for \( i \neq j \).

We now define the symmetric Sobolev space:
\[ H_{s} = \{ u : u \in H^1(\mathbb{R}^N), u \text{ is even in } y, \ell = 2, 4, 5, \ldots, N, \ u(\sqrt{y_1^2 + y_2^2} \cos \theta, \sqrt{y_1^2 + y_2^2} \sin \theta, \sqrt{y_3^2 + y_3''^2}) = u(\sqrt{y_1^2 + y_2^2} \cos (\theta + \frac{2j\pi}{k}), \sqrt{y_1^2 + y_2^2} \sin (\theta + \frac{2j\pi}{k}), y_3, y_3'') \}. \]

where \( \theta = \arctan \frac{y_2}{y_1} \).

In this paper, we always assume \((r, h) \in S_k =: \left[ (m - \beta)k \ln k, \left( m + \beta \right)k \ln k \right] \times \left[ \left( \frac{\pi (m + 2)}{m} - \alpha \right) \frac{1}{k}, \left( \frac{\pi (m + 2)}{m} + \alpha \right) \frac{1}{k} \right] \),
for some \( \alpha, \beta > 0 \) small, and independent of \( k \). We refer to Remark 2.6 for a discussion on the assumption (1.10) for \((r, h)\).

Our main result is the following

**Theorem 1.1.** Suppose that \( V(|y|) \) satisfies (H1) and the parameters \((r, h)\) satisfies (1.10). Then there is an integer \( k_0 \), such that for all integer \( k \geq k_0 \), (1.1) has a solution \( u_k \) of the form
\[ u_k = W_{r, h}(y) + \omega_k(y), \quad \text{(1.11)} \]
where \( \omega_k \in H_{s}, (r_k, s_k) \in S_k \) and \( \omega_k \) satisfies
\[ \int_{\mathbb{R}^N} (|\nabla \omega_k|^2 + V(y)|\omega_k|^2) \to 0, \quad \text{as } k \to \infty. \]

We will prove Theorem (1.1) by using the Lyapunov- Schmidt reduction technique adapted to our context as developed in [20]. Let us briefly sketch the scheme of the proof. A critical point \( u \) of the energy functional \( I \) defined in (1.3) corresponds to a solution for (1.1). Our solution \( u \) will have the form \( u = W_{r, h} + \phi \).

As we know, (1.4) has a unique positive solution \( U \), which is radially symmetric and
\[ \lim_{|y| \to +\infty} U(y)e^{|y|^2} = C < +\infty, \quad \text{and} \quad \lim_{|y| \to +\infty} \frac{U(y)}{U''(y)} = -1. \]
Moreover, \( U \) is non-degenerate, in the sense that the kernel of the linear operator \(-\Delta + I - pU^{p-1}\) in \( H^1(\mathbb{R}^N) \) is spanned by \( \left\{ \frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial y_N} \right\} \). For more details about (1.4), reader can refer to [14], [19].

We define
\[ J(\phi) = I(W_{r, h} + \phi), \quad \phi \in E. \]
The space \( E \) is given as follows. For \( j = 1, \ldots, k \), we define
\[ Z_{1j} = \frac{\partial U_{r_j}}{\partial r}, \quad Z_{1j} = \frac{\partial U_{s_j}}{\partial r}, \quad Z_{2j} = \frac{\partial U_{r_j}}{\partial h}, \quad Z_{2j} = \frac{\partial U_{s_j}}{\partial h}. \]
We define the constrained space
\[ E = \left\{ v : v \in H_s, \int_{\mathbb{R}^N} U_{Z_j}^{p-1} \nabla v = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \nabla v = 0, \quad j = 1, \ldots, k, \quad \ell = 1, 2 \right\}. \] (1.12)

The space \( E \) will be endowed with the norm
\[ \|v\|^2 = \langle v, v \rangle, \quad v \in E, \]
where
\[ \langle v_1, v_2 \rangle = \int_{\mathbb{R}^N} \left( \nabla v_1 \nabla v_2 + V(|y|) v_1 v_2 \right), \quad v_1, v_2 \in E. \]

Expand \( J(\phi) \) as follows:
\[ J(\phi) = J(0) + l(\phi) + \frac{1}{2} \langle L(\phi, \phi) - R(\phi), \phi \rangle, \quad \phi \in E, \]
where
\[ l(\phi) = \sum_{i=1}^{k} \int_{\mathbb{R}^N} \left( V(|y|) - 1 \right) \left( U_{\pi_i} + U_{\mu_i} \right) \phi + \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} \left( U_{\pi_i} + U_{\mu_i} \right)^p - W_{r,h}^p \right) \phi, \]
and
\[ R(\phi) = \frac{1}{p+1} \int_{\mathbb{R}^N} \left( |W_{r,h} + \phi|^{p+1} - |W_{r,h}|^{p+1} - (p+1)W_{r,h}^p - \frac{1}{2}(p+1)pW_{r,h}^{p-1}\phi^2 \right). \]

Furthermore \( L \) is a linear operator from \( E \) to \( E \), which satisfies
\[ \langle Lv_1, v_2 \rangle = \int_{\mathbb{R}^N} \left( \nabla v_1 \nabla v_2 + V(|y|) v_1 v_2 - pW_{r,h}^{p-1} v_1 v_2 \right), \quad \text{for all} \quad v_1, v_2 \in E. \]

Since \( W_{r,h}^p \) is bounded and has the symmetries of the space \( H_s \), we can easily prove \( L \) is a bounded linear operator from \( E \) to \( E \). We will show that \( l(\phi) \) is a bounded linear functional in \( E \). Thus there is an \( l_k \in E \), such that
\[ l(\phi) = \langle l_k, \phi \rangle. \]

Then a critical point of \( J(\phi) \) is also a solution of
\[ l_k + L(\phi) + R'(\phi) = 0. \] (1.13)

Thus a function \( u \) of the form \( u = W_{r,h} + \phi \) will be a solution to (1.11) if \( \phi \) is a solution of (1.13). The strategy now consists in first showing that, for any \( (r, h) \in S_k \), there exists a function \( \phi_{r,h} \)

solution of (1.13) in the space \( E \) (see Proposition 2.2). A second step will be to reduce the problem of finding a critical point \( u \) of \( I(u) \) to the problem of finding a stable critical point \( (r^*, h^*) \) of the function
\[ F(r, h) = I(W_{r,h} + \phi_{r,h}). \]

We will show that such a critical point exists in the set \( S_k \). The qualitative property of the solutions follows by their construction.

**Plan of the paper**

We organize the paper as follows. In section 2, we will give the proof of Theorem 1.1. In the Appendices, we will give some useful Lemmas, Propositions and the details for the energy of approximate solution expansion.
2. Proof of Theorem 1.1

We first give expansion for the energy of approximate solution.

**Proposition 2.1.** For all \((r, h) \in S_k\), there exist some small constant \(\sigma > 0\) such that

\[
I(W_{r,h}) = k\left(\frac{A_1}{m} + A_2 - 2B_1e^{-2\pi \sqrt{\frac{h^2}{r^m}}} - B_1e^{-2rh} + O_k\left(\frac{1}{r^{m+\sigma}}\right)\right)
+ kO_k(e^{-2(1+\sigma)rh}) + kO_k(e^{-2(1+\sigma)\sqrt{\frac{h^2}{r^m}}}),
\]

where

\[
A_1 = a \int_{\mathbb{R}^N} U^2, \quad A_2 = \left(1 - \frac{2}{p+1}\right) \int_{\mathbb{R}^N} U^{p+1}, \quad B_1 = \int_{\mathbb{R}^N} U^p e^{-yh}.
\]

**Proof of Proposition 2.1:** The proof of Proposition 2.1 is delayed to Appendices. \(\square\)

The next lemma gives the existence and boundness of inverse operator of \(L\) in \(E\).

**Lemma 2.2.** There is a constant \(\rho > 0\), independent of \(k\), such that for any \((r, h) \in S_k\)

\[
\|Lv\| \geq \rho \|v\|, \quad v \in E.
\]

**Proof of Lemma 2.2** We prove by contradiction. Suppose that when \(k \to +\infty\) there exist \(h_k, r_k \in S_k, v_k \in E\) satisfying

\[
\|Lv_k\| = o_k(1)\|v_k\|.
\]

Then easily

\[
\langle Lv_k, \varphi \rangle = o_k(1)\|v_k\|\|\varphi\|, \quad \forall \varphi \in E.
\]

Similar to [20], we assume \(\|v_k\|^2 = k\). Using the symmetric property of \(v_k, \varphi\), we can get

\[
\langle Lv_k, \varphi \rangle = \int_{\mathbb{R}^N} (\nabla v_k \nabla \varphi + V(|y|)v_k \varphi - pW_{r,h}^{p-1}v_k \varphi) = k \int_{\Omega_1} (\nabla v_k \nabla \varphi + V(|y|)v_k \varphi - pW_{r,h}^{p-1}v_k \varphi)
= o_k(1)\|v_k\|\|\varphi\| = o(\sqrt{k})\|\varphi\|,
\]

and

\[
\int_{\Omega_1} (|\nabla v_k|^2 + V(|y|)v_k^2) = 1.
\]

Inserting \(\varphi = v_k\) into (2.3), we can obtain immediately

\[
\int_{\Omega_1} (|\nabla v_k|^2 + V(|y|)v_k^2 - pW_{r,h}^{p-1}v_k^2) = o_k(1).
\]

We denote

\[
\overline{v}_k(y) = v_k(y + \overline{x}_1).
\]

For the sequence \(\overline{v}_k(y)\), we can prove that \(\overline{v}_k(y)\) is bounded in \(H^1_{\text{loc}}(\mathbb{R}^N)\). In fact, for any \(R > 0\), since \(|\overline{x}_2 - \overline{x}_1| = 2r^2 + \frac{2\pi}{k} \ln k\), we can choose \(k\) large enough such that \(B_R(\overline{x}_1) \subset \Omega_1\). As a result, we have

\[
\int_{B_R(0)} (|\nabla \overline{v}_k|^2 + V(|y|)\overline{v}_k^2) = \int_{B_R(\overline{x}_1)} (|\nabla v_k|^2 + V(|y - \overline{x}_1|)v_k^2)
\leq \int_{\Omega_1} (|\nabla v_k|^2 + V(|y - \overline{x}_1|)v_k^2) \leq 1.
\]

So we can conclude

\[
\overline{v}_k \to \overline{v} \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^N),
\]
and
\[ \overline{v}_k \to \overline{v} \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N). \] (2.7)
Since \( \overline{v}_k \) is even in \( y_d, d = 2, 4, \cdots, N \), then \( \overline{v} \) is even in \( y_d \).

From the orthogonal conditions for functions of \( E \)
\[ \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial r} v_k = 0 \]
and the identity
\[ \frac{\partial U_{\overline{x}_1}}{\partial r} = \sqrt{1 - h^2} \frac{\partial U_{\overline{x}_1}}{\partial y_1} + h \frac{\partial U_{\overline{x}_1}}{\partial y_3}, \]
we can get
\[ \sqrt{1 - h^2} \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial y_1} v_k + h \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial y_3} v_k = 0. \] (2.8)

Similarly, combining
\[ \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial h} v_k = 0, \]
and
\[ \frac{\partial U_{\overline{x}_1}}{\partial h} = - \frac{hr}{\sqrt{1 - h^2}} \frac{\partial U_{\overline{x}_1}}{\partial y_1} + \frac{r}{\partial y_3}, \]
we can get
\[ \frac{h}{\sqrt{1 - h^2}} \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial y_1} v_k - \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U_{\overline{x}_1}}{\partial y_3} v_k = 0. \] (2.9)

From (2.8), (2.9), we have
\[ \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_1} \overline{v}_k = \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_3} \overline{v}_k = 0. \]

Letting \( k \to +\infty \), we obtain
\[ \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_1} \overline{v} = \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_3} \overline{v} = 0. \] (2.10)

Next, we will show that \( \overline{v} \) is a solution of
\[ -\Delta \phi + \phi - pU^{-1} \phi = 0, \quad \text{in } \mathbb{R}^N. \] (2.11)

We define the constrained space as:
\[ \tilde{E}^+ = \left\{ \phi : \phi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_1} \phi = \int_{\mathbb{R}^N} U_{\overline{x}_1}^{-1} \frac{\partial U}{\partial y_3} \phi = 0 \right\}. \]

For the proof of (2.11), we first give a claim.

**Claim 1**: \( \overline{v} \) is a solution of
\[ -\Delta \phi + \phi - pU^{-1} \phi = 0, \quad \text{in } \tilde{E}^+. \]

Now we give the proof of the **Claim 1**.

For any \( R > 0 \), let \( \phi \in C_0^\infty(B_R(0)) \cap \tilde{E}^+ \) which is even in \( y_d, d = 2, 4, \cdots, N \). Then denote
\[ \phi_k(y) =: \phi(y - \overline{x}_1) \in C_0^\infty(B_R(\overline{x}_1)). \]

Inserting \( \phi_k(y) = \varphi \) into (2.3) and combining (2.6), (2.7) and Lemma (3.1), we can get
\[ \int_{\mathbb{R}^N} \left( \nabla \overline{v} \nabla \phi + \overline{v} \phi - pU^{-1} \overline{v} \phi \right) = 0. \] (2.12)
On the other hand, since \( \varpi \) is even in \( y_d, d = 2, 4, \ldots, N \), then by using symmetric conditions we can conclude that (2.12) is valid for all functions \( \phi \in C_0^\infty \left( B_R(x_1) \cap \tilde{E}^+ \right) \) which is odd in \( y_d, d = 2, 4, \ldots, N \). Hence (2.12) is hold for all functions \( \phi \in C_0^\infty \left( B_R(x_1) \cap \tilde{E}^+ \right) \). Density argument implies that,

\[
\int_{\mathbb{R}^N} \left( \nabla \nabla \phi + \varpi \phi - pU^{p-1}\varpi \phi \right) = 0, \quad \text{for all } \phi \in \tilde{E}^+. \tag{2.13}
\]

The proof of Claim 1 is completed.

Combining Claim 1 and the fact that (2.11) is hold for \( \phi = \frac{\partial U}{\partial y_1} \) and \( \phi = \frac{\partial U}{\partial y_3} \), then we get

\[
\int_{\mathbb{R}^N} \left( \nabla \nabla \phi + \varpi \phi - pU^{p-1}\varpi \phi \right) = 0, \quad \text{for all } \phi \in H^1(\mathbb{R}^N), \tag{2.14}
\]

which is (2.11). By using the Non-degeneracy results for \( U \) and combining \( \varpi \) is even in \( y_d, d = 2, 4, \ldots, N \), we have

\[
\varpi = c_1 \frac{\partial U}{\partial y_1} + c_2 \frac{\partial U}{\partial y_3}, \tag{2.15}
\]

for some universal constants \( c_1, c_2 \). Combining (2.10), (2.15), we have

\[
 c_1 = c_2 = 0.
\]

Thus we have

\[
\varpi = 0. \tag{2.16}
\]

The direct result of (2.7) and (2.16) is that

\[
\int_{B_R(x_1)} v_k^2 = o_k(1).
\]

From Lemma (3.1), we have \( W_{r,h} \leq Ce^{-(1-\eta)|y-\varpi|} \). Then we can get, taking \( R \) large enough,

\[
o_k(1) = \int_{\Omega_1} \left( |\nabla v_k|^2 + V(|y|)v_k^2 - pW_{r,h}^{p-1}v_k^2 \right)
\]

\[
= \int_{\Omega_1} \left( |\nabla v_k|^2 + V(|y|)v_k^2 \right) - \int_{\Omega_1 \setminus B_R(x_1)} pW_{r,h}^{p-1}v_k^2 - \int_{B_R(x_1)} pW_{r,h}^{p-1}v_k^2
\]

\[
= \int_{\Omega_1} \left( |\nabla v_k|^2 + V(|y|)v_k^2 + o_k(1) + O_k(e^{-(1-\eta)R}) \right) \int_{\Omega_1} |v_k|^2
\]

\[
\geq \frac{1}{2} \int_{\Omega_1} \left( |\nabla v_k|^2 + V(|y|)v_k^2 + o_k(1) \right),
\]

which is a contradiction to (2.3). We complete the proof of Lemma (2.2). \( \square \)

Now we give the following Proposition which is crucial in the sequel.

**Proposition 2.3.** For any \( k \) sufficiently large enough, there exist a \( C^1 \) map \( \Phi : S_k \to \mathbb{E} \) such that \( \Phi(r, h) = \phi_{r,h}(y) \in \mathbb{E} \), and

\[
J'(\phi_{r,h}) = 0, \quad \text{in } \mathbb{E}. \tag{2.17}
\]

Moreover, there is a small \( \sigma > 0 \), such that

\[
\|\phi_{r,h}\| \leq \frac{C}{k^{\frac{m-1}{2}+\sigma}}.
\]
Proof of Proposition 2.2: The proof follows from a standard technique, which is based on the contraction mapping theorem. For the proof, it’s a slight modification of the proof of Proposition 2.2 in [20]. Here we omit for concise.

Next we will give the estimate for $l_k$. The estimate will play a role in carrying out the reduction argument in the proof of Theorem 1.1.

Lemma 2.4. For all $(r, h) \in S_k$, there is a small $\sigma > 0$, such that

$$
\|l_k\| \leq \frac{C}{k^{\frac{m-1}{2}+\sigma}}.
$$

Proof of Lemma 2.4. Recall that for any $\phi \in E$

$$
\langle l_k, \phi \rangle = \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(|y|) - 1)(U_{\tau_i} + U_{\varphi}) \phi + \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} (U_{\tau_i} + U_{\varphi})^p - W^p_{r,h} \right) \phi.
$$

Using the symmetric property for the functions, we can know

$$
\sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(|y|) - 1)(U_{\tau_i} + U_{\varphi}) \phi = 2k \int_{\mathbb{R}^N} (V(|y|) - 1)U_{\tau_1} \phi
$$

$$
= 2k \left( \int_{\mathbb{R}^N} (V(|y + \bar{\tau}_1|) - 1)^2 U^2 \right)^{\frac{1}{2}} \|\phi\|
$$

$$
= 2k \left( \int_{\mathbb{R}^N \setminus B_{\delta_0}(\bar{\tau}_1)(0)} (V(|y + \bar{\tau}_1|) - 1)^2 U^2 + \int_{B_{\delta_0}(\bar{\tau}_1)(0)} (V(|y + \bar{\tau}_1|) - 1)^2 U^2 \right)^{\frac{1}{2}} \|\phi\|
$$

$$
\leq kO_k \left( \frac{1}{r^m} \right) \|\phi\| \leq \frac{C}{k^{\frac{m-1}{2}+\sigma}} \|\phi\|.
$$

The last inequality hold as $m > 1$.

Note that explicitly

$$
U_{\tau_1} + U_{\varphi} \geq U_{\tau_1} + U_{\varphi} \quad \text{for} \quad y \in \Omega_1,
$$

$$
U_{\tau_1} \geq U_{\varphi} \quad \text{for} \quad y \in \Omega_1^+.
$$

For the second term in (2.19), we have

$$
\left| \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} (U_{\tau_i} + U_{\varphi})^p - W^p_{r,h} \right) \phi \right| = k \int_{\Omega_1} \left( \sum_{i=1}^{k} (U_{\tau_i} + U_{\varphi})^p - W^p_{r,h} \right) \phi
$$

$$
\leq k \int_{\Omega_1} (U_{\tau_1} + U_{\varphi})^{p-1} \sum_{i=2}^{k} (U_{\tau_i} + U_{\varphi}) \phi = 2k \int_{\Omega_1^+} (U_{\tau_1} + U_{\varphi})^{p-1} \sum_{i=2}^{k} (U_{\tau_i} + U_{\varphi}) \phi
$$

$$
\leq Ck \sum_{i=2}^{k} \int_{\Omega_1^+} U_{\tau_1}^{p-1} U_{\tau_i} \phi \leq Ck \sum_{i=2}^{k} e^{-\min\{p-1-\tau,1\}|\tau_1-\tau_i|} \left( \int_{\Omega_1^+} |\phi|^{p+1} \right)^{\frac{1}{p+1}}
$$

$$
\leq Ck^{\frac{p}{p+1}} \sum_{i=2}^{k} e^{-\min\{p-1-\tau,1\}|\tau_1-\tau_i|} \|\phi\|,
$$

where $\tau$ is the constant small enough.
Noting that for \((r, h) \in \mathbb{S}_k\), we can get
\[
\sum_{i=2}^{k} e^{-\min\{p-1-r,1\} |x_i-x_1|} \leq C e^{-\min\{p-1-r,1\}\frac{2\pi \sqrt{1-h^2}}{k}} \leq \frac{C}{k^{\min\{p-1-r,1\}(m-\beta)}}. 
\]

Combing (2.21) and the following fact
\[
\min\{p-1,1\}m - \frac{p}{p+1} > \frac{m-1}{2},
\]
we can obtain
\[
\left| \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} (U_{x_i} + U_{x_1})^p - W_{r,h}^p \right) \phi \right| \leq \frac{C}{k^{\frac{m-1+\sigma}{2}}} \|\phi\|. 
\]

The result of (2.18) follows from (2.20) and (2.22). \(\square\)

In order to prove the Theorem 1.1 we need the following proposition. Readers can refer to [15], [20] for more details about the proposition.

**Proposition 2.5.** Suppose \(\Phi(r, h) = \phi_{r,h}(y)\) with \(\Phi(r, h)\) be the map defined in Proposition 2.5. Define
\[
F(r, h) = I(W_{r,h} + \phi_{r,h}(y), \quad \forall (r, h) \in \mathbb{S}_k.
\]
If \((r, h)\) is a critical point of \(F(r, h)\), then
\[
u = W_{r,h} + \phi_{r,h}(y)
\]
is a critical point \(I(u)\) in \(H^1(\mathbb{R}^N)\). \(\square\)

**Proof of Proposition 2.5.** The Proof is similar to the proof or Proposition 2.1. We omit it here.

Now, we give the proof of Theorem 1.1

**Proof of Theorem 1.1:** By Proposition 2.5 we need to show there is \((r_k, h_k) \in \mathbb{S}_k\) which is a critical point of \(F(r, h)\).

In fact, from Proposition 2.1 we can know
\[
F(r, h) = I(W_{r,h}) + I(\phi_{r,h}) + \frac{1}{2}(L\phi_{r,h}, \phi_{r,h}) - R(\phi_{r,h})
\]
\[
= I(W_{r,h}) + O_k(\|I_k\|\|\phi_{r,h}\| + \|\phi_{r,h}\|^2) = I(W_{r,h}) + O_k(\frac{1}{k^{m-1+\sigma}})
\]
\[
= k\left( \frac{A_1}{r^m} + A_2 - 2B_1 e^{-2\pi \sqrt{1-h^2}} - B_1 e^{-2r} 
\right.
\]
\[
+ O_k(e^{-2\pi(1+\sigma)\sqrt{1-h^2}}) + O_k(e^{-2(1+\sigma)r}) + O_k(\frac{1}{r^{m+\sigma}}) + O_k(\frac{1}{k^{m+\sigma}}) \right),
\]
where \(A_1, A_2, B_1\) are constants in (2.22). Define
\[
F_1(r, h) = \frac{A_1}{r^m} + A_2 - 2B_1 e^{-2\pi \sqrt{1-h^2}} - B_1 e^{-2r}
\]
Then we consider the system
\[
\begin{cases}
F_{1,r}(r, h) = -A_1 \frac{m}{r^{m+1}} + 4B_1 \pi \sqrt{1-h^2} e^{-2\pi \sqrt{1-h^2}} + 2B_1 h e^{-2r} = 0, \\
F_{1,h}(r, h) = -4B_1 \pi h \sqrt{1-h^2} e^{-2\pi \sqrt{1-h^2}} + 2B_1 r e^{-2r} = 0.
\end{cases}
\]
(2.23)
From (2.23), we can get

\[ -A_1 \frac{m}{r^{m+1}} + 4B_1 \pi \frac{e^{-2\pi \sqrt{1-h^2^r}}}{k} \left( \sqrt{1-h^2} + \frac{h^2}{\sqrt{1-h^2}} \right) = 0. \]

Define

\[ H(r, h) = e^{-2\pi \sqrt{1-h^2^r}}, \quad G(r, h) = e^{-2rh}. \]

Then (2.23) implies

\[
\begin{cases}
H(r, h) = \frac{A_1 k^{\frac{m}{r^{m+1}}}}{4B_1 \pi \left( \sqrt{1-h^2} + \frac{h^2}{\sqrt{1-h^2}} \right)}, \\
G(r, h) = 2\pi h e^{-2\pi \sqrt{1-h^2^r}}.
\end{cases}
\] (2.24)

We define the space \( \mathcal{S}_k \):

\[ \mathcal{S}_k = \left\{ (r_k, h_k) \mid r_k = \frac{(m + o_k(1))}{2} k \ln k, \text{ and } h_k = \left( \frac{\pi (m + 2)}{m} + o_k(1) \right) \frac{1}{k} \right\}, \]

and the mapping

\[ \mathbf{T} : \mathcal{S}_k \to \mathbb{R}^2 \text{ as } \mathbf{T}(r_k, h_k) = (H(r_k, h_k), G(r_k, h_k)). \]

The system (2.23) is equivalent to find a fixed point of

\[ (r, h) = \mathbf{T}^{-1} \left( \frac{A_1 k^{\frac{m}{r^{m+1}}}}{4B_1 \pi \left( \sqrt{1-h^2} + \frac{h^2}{\sqrt{1-h^2}} \right)} \right) + \frac{h}{2B_1 \sqrt{1-h^2} \left( \sqrt{1-h^2} + \frac{h^2}{\sqrt{1-h^2}} \right)} \]

\[ = \mathbf{A}(r, h) =: \left( \mathbf{a}_1(r, h), \mathbf{a}_2(r, h) \right). \] (2.25)

in \( \mathcal{S}_k \). By computing, for \( (r, h) \in \mathcal{S}_k \), we have

\[ \mathbf{A}(r, h) = \left( 1 + o_k(1) \right) \left( k \frac{(m + 1) \ln r - \ln k}{2\pi}, \frac{\ln h - \left( m + 1 \right) \ln r}{k \left( \ln k - \left( m + 1 \right) \ln r \right)} \right) \in \mathcal{S}_k \]

\[ = \left( \mathbf{a}_1(r, h), \mathbf{a}_2(r, h) \right). \] (2.26)

And it is easy to show that

\[ |\mathbf{a}_1(r_1, h_1) - \mathbf{a}_1(r_2, h_2)| + |\mathbf{a}_2(r_1, h_1) - \mathbf{a}_2(r_2, h_2)| \leq o_k(1) \left( |r_1 - r_2| + |h_1 - h_2| \right), \]

for all \( (r_1, h_1), (r_2, h_2) \in \mathcal{S}_k \). By using the Contraction Mapping principle we can prove that there exist a fixed point \( (\bar{r}_k, \bar{h}_k) \in \mathcal{S}_k \) for (2.25). That’s to say \( F_1(r, h) \) have a critical point \( (\bar{r}_k, \bar{h}_k) \in \mathcal{S}_k \).

Define

\[ \mathbf{M}_1(r, h) = \begin{pmatrix} F_{1,rr} & F_{1,rr} \\ F_{1,rr} & F_{1,rr} \end{pmatrix}. \]

By some simple calculations, we can know that

\[ F_{1,rr}(r, h) = (\bar{r}_k, \bar{h}_k) < 0, \quad F_{1,rr}(r, h) = (\bar{r}_k, \bar{h}_k) < 0 \]

and

\[ F_{1,rr} \times F_{1,rr}(r, h) = (\bar{r}_k, \bar{h}_k) - F_{1,rr}(r, h) > 0. \]

So we can know that \( (\bar{r}_k, \bar{h}_k) \) is a maximum point of \( F_1(r, h) \). Then the maximum of \( F_1(r, h) \) in \( \mathcal{S}_k \) can be achieved. Thus for the function \( F(r, h) \), we can find a maximum point \( (r_k, h_k) \) which is an interior point of \( \mathcal{S}_k \). So \( (r_k, h_k) \) is a critical point of \( F(r, h) \). Thus

\[ W_{r_k, h_k} + \phi_{r_k, h_k}(y) \]
is a critical point of $I(u)$. This complete the proof of Theorem 1.1 \hfill \Box

**Remark 2.6.** From (2.23) and the assumptions $h \in [\alpha_0 \frac{1}{1}, \alpha_1 \frac{1}{1}]$, $r \in [\beta_0 k \ln k, \beta_1 k \ln k]$, we can know that

$$e^{-2rh} = O_k \left( \frac{h}{k} \frac{1}{\sqrt{1 - h^2}} \right) e^{-2\pi \sqrt{1 - h^2} \frac{2}{k}} = O_k \left( \frac{1}{k^2} \right) e^{-2\pi \sqrt{1 - h^2} \frac{2}{k}}.$$

For the solvability system (2.23), we must have

$$e^{-2\pi \sqrt{1 - h^2} \frac{2}{k}} = O_k \left( \frac{1}{k^m} \right) \quad \text{and} \quad e^{-2rh} = O_k \left( \frac{1}{k^m + 2} \right).$$

That's the reason why we make the assumption (1.10) for $(r, h)$.

3. Some useful estimates and Lemmas

We first give some useful estimates and Lemmas.

**Lemma 3.1.** For $r, h$ be the parameters in (1.6) and any $\eta \in (0, 1]$, there is a constant $C > 0$, such that

$$\sum_{i=2}^{k} U^p_{\xi_i} (y) \leq Ce^{-\eta \sqrt{1 - h^2} \frac{2}{k}} e^{-(1-\eta)|y-\bar{z}_1|}, \quad \text{for all } y \in \Omega^1_1,$$

and

$$\sum_{i=2}^{k} U^p_{\xi_i} (y) \leq Ce^{-\eta \sqrt{1 - h^2} \frac{2}{k}} e^{-(1-\eta)|y-\bar{z}_1|}, \quad \text{for all } y \in \Omega^1_1,$$

$$U^p_{\xi_i} (y) \leq Ce^{-\eta hr} e^{-(1-\eta)|y-\bar{z}_1|}, \quad \text{for all } y \in \Omega^1_1.$$

**Proof of Lemma 3.1:** The proof of Lemma 3.1 is similar to Lemma A.1 in [20]. \hfill \Box

In this appendices, we assume $(r, h) \in S_k$, where $S_k$ is defined in (1.10).

**Lemma 3.2.** For $i = 2, \cdots, k$, there exist some small constant $\sigma > 0$ such that the following expansions hold

$$\int_{R^N} U^p_{\xi_i} U_{\xi_1} = B_1 e^{-|\xi_1 - \xi_i|} + O_k (e^{-(1+\sigma)|\xi_1 - \xi_i|}), \quad (3.1)$$

$$\int_{R^N} U^p_{\xi_i} U_{\xi_1} = B_1 e^{-|\xi_1 - \xi_i|} + O_k (e^{-(1+\sigma)|\xi_1 - \xi_i|}), \quad (3.2)$$

$$\int_{R^N} U^p_{\xi_i} U_{\xi_1} = B_1 e^{-2rh} + O_k (e^{-2(1+\sigma)rh}), \quad (3.3)$$

where $B_1$ is defined in (2.2).

**Proof of Lemma 3.2:** By calculation, we have

$$\int_{R^N} U^p_{\xi_1} U_{\xi_i} = \int_{R^N} U^p(z) U(z + \bar{\xi}_1 - \bar{\xi}_i)$$

$$= \int_{R^N} U^p(z) U(z + w_i) + \int_{R^N \setminus B_{|w_i|} (0)} U^p(z) U(z + w_i). \quad (3.4)$$
with $\delta > 0$ is small constant. Then we have
\[
\int_{R^N \setminus B_\delta |w_i(0)|} U^p(z)U(z + w_i) \leq C e^{-|w_i|} \int_{R^N \setminus B_\delta |w_i(0)|} e^{-(p-1)|z|} = O_k(e^{-(1+\sigma)|w_i|}) = O_k(e^{-(1+\sigma)|\mathbf{x}_1 - \mathbf{x}_i|}).
\] (3.5)
We have the expansion in $B_\delta |w_i(0)|$
\[
|z + |w_i|e_1| = |w_i|(1 + \frac{1}{|w_i|} e_1 z + O_k(\frac{|z|^2}{|w_i|^2})),
\]
where $e_1 = (1, \cdots, 0)$, then using the symmetry of $U$,
\[
\int_{B_\delta |w_i(0)|} U^p(z)U(z + w_i) = \int_{B_\delta |w_i(0)|} U^p(z)U(z + |w_i|e_1) = \int_{B_\delta |w_i(0)|} U^p(z)e^{-|z + |w_i|e_1|}
\]
\[
= \int_{B_\delta |w_i(0)|} U^p(z)e^{-|w_i|(1 + \frac{1}{|w_i|} e_1 z + O_k(\frac{|z|^2}{|w_i|^2}))}
\]
\[
= e^{-|\mathbf{x}_1 - \mathbf{x}_i|} \int_{R^N} U^p(z)e^{-z_1}(1 + o_k(1)).
\] (3.6)
Now combining (3.4), (3.5) and (3.6), we can get
\[
\int_{R^N} U^p_{\mathbf{x}_1} U_{\mathbf{x}_i} = B_1 e^{-|\mathbf{x}_1 - \mathbf{x}_i|} + O_k(e^{-(1+\sigma)|\mathbf{x}_1 - \mathbf{x}_i|}),
\]
with $B_1 = \int_{R^N} U^p(z)e^{-z_1}$. Similarity, we can easily obtain (3.2), (3.3). Thus we complete the proof of Lemma 3.2.

\[\square\]

**Lemma 3.3.** There exist some small constant $\sigma > 0$ such that the following expansions hold
\[
\sum_{i=2}^{k} \int_{R^N} U^p_{\mathbf{x}_1} U_{\mathbf{x}_i} = 2B_1 e^{-2\pi\sqrt{1-h^2} \frac{\pi}{k}} + O_k(e^{-2(1+\sigma)\pi\sqrt{1-h^2} \frac{\pi}{k}}),
\] (3.7)
\[
\sum_{i=1}^{k} \int_{R^N} U^p_{\mathbf{x}_i} U_{\mathbf{x}_1} = B_1 e^{-2\pi h} + O_k(e^{-2(1+\sigma)\pi h}) + O_k(e^{-2\pi(1+\sigma)\sqrt{1-h^2} \frac{\pi}{k}}).
\] (3.8)

**Proof of Lemma 3.3 :** Recalling the definitions $\mathbf{x}_j, \mathbf{r}_j$, we can know
\[
|\mathbf{r}_j - \mathbf{r}_1|^2 = 4r^2(1 - h^2)\sin^2 \frac{(j - 1)\pi}{k},
\]
\[
|\mathbf{x}_1 - \mathbf{r}_1|^2 = 4r^2 h^2,
\]
and
\[
|\mathbf{x}_j - \mathbf{x}_1|^2 = 4r^2 \left[ (1 - h^2)\sin^2 \frac{(j - 1)\pi}{k} + h^2 \right].
\]
From Lemma 3.2 we can get
\[
k \sum_{i=2}^{k} \int_{R^N} U^p_{\mathbf{x}_1} U_{\mathbf{x}_i} = kB_1 \sum_{i=2}^{k} e^{-|\mathbf{x}_1 - \mathbf{x}_i|} + k \sum_{i=2}^{k} O_k(e^{-(1+\sigma)|\mathbf{x}_1 - \mathbf{x}_i|}),
\] (3.9)
with $B_1 = \int_{\mathbb{R}^N} U^p(z)e^{-z_1}$. Without lose of generality, we can assume that $k$ is even. Then

$$
B_1 \sum_{j=2}^{k} e^{-|\mathbf{r}_1 - \mathbf{r}_j|} = B_1 \sum_{i=3}^{k/2} e^{-2r\sqrt{1-h^2}\sin \left(\frac{(j-1)\pi}{k}\right)}
$$

$$
+ B_1 \sum_{i=k/2+1}^{k-1} e^{-2r\sqrt{1-h^2}\sin \left(\frac{(j-1)\pi}{k}\right)} + 2B_1 e^{-2r\sqrt{1-h^2}\sin \frac{\pi}{k}}.
$$

(3.10)

Consider

$$
c_3 \frac{(j-1)\pi}{k} \leq \sin \frac{(j-1)\pi}{k} \leq c_4 \frac{(j-1)\pi}{k}, \quad \text{for } j \in \{3, \ldots, k/2\},
$$

with $\frac{1}{2} < c_3 \leq c_4 \leq 1$. Then we have

$$
\sum_{j=3}^{k/2} e^{-2r\sqrt{1-h^2}\sin \left(\frac{(j-1)\pi}{k}\right)} \leq \sum_{j=3}^{k/2} e^{-2r\sqrt{1-h^2}c_3(j-1)\pi/k} = e^{-4r\sqrt{1-h^2}\frac{c_3\pi}{k}} - e^{-2r\sqrt{1-h^2}\frac{(j-3)\pi}{k}}
$$

$$
= \frac{e^{-4r\sqrt{1-h^2}\frac{c_3\pi}{k}} - e^{-2r\sqrt{1-h^2}\frac{(j-3)\pi}{k}}}{(1 - e^{-2r\sqrt{1-h^2}\frac{\pi}{k}})} = O_k(e^{-2(1+\sigma)\pi\sqrt{1-h^2}\frac{\pi}{k}}).
$$

(3.11)

Using symmetry of function $\sin x$, we can easily show

$$
B_1 \sum_{j=k/2+1}^{k-1} e^{-2r\sqrt{1-h^2}\sin \left(\frac{(j-1)\pi}{k}\right)} = O_k(e^{-2(1+\sigma)\pi\sqrt{1-h^2}\frac{\pi}{k}}),
$$

in a same manner as (3.11). Next, we have

$$
e^{-2r\sqrt{1-h^2}\sin \frac{\pi}{k}} = e^{-2r\sqrt{1-h^2}(\frac{\pi}{k} + O_k(\frac{3}{k}))}
$$

$$
= e^{-2r\sqrt{1-h^2}\frac{\pi}{k}} - 2r\sqrt{1-h^2}O_k(\frac{3}{k})
$$

$$
= e^{-2r\sqrt{1-h^2}\frac{\pi}{k}} + O_k(e^{-2(1+\sigma)\pi\sqrt{1-h^2}\frac{\pi}{k}}).
$$

(3.12)

Combining (3.9), (3.10), (3.11) and (3.12), we can get

$$
\sum_{i=2}^{k} \int_{\mathbb{R}^N} U^p_{\mathbf{r}_i} U_{\mathbf{r}_i} = 2B_1 e^{-2\pi\sqrt{1-h^2}\frac{\pi}{k}} + O_k(e^{-2(1+\sigma)\pi\sqrt{1-h^2}\frac{\pi}{k}}).
$$

(3.13)

Next, from (3.2) and (3.3), we have

$$
\sum_{j=1}^{k} \int_{\mathbb{R}^N} U^p_{\mathbf{z}_j} U_{\mathbf{z}_j}
$$

$$
= B_1 e^{-|\mathbf{r}_1 - \mathbf{z}_1|} + B_1 \sum_{j=2}^{k} e^{-|\mathbf{z}_j - \mathbf{r}_j|} + O_k(e^{-(1+\sigma)|\mathbf{r}_1 - \mathbf{z}_1|}) + O(\sum_{j=2}^{k} e^{-(1+\sigma)|\mathbf{z}_j - \mathbf{r}_1|})
$$

$$
= B_1 e^{-2rh} + B_1 \sum_{j=2}^{k} e^{-2r\left[\left(1-h^2\right)\sin^2 \left(\frac{(j-1)\pi}{k}\right) + h^2\right]}
$$

$$
= B_1 e^{-2rh} + B_1 \sum_{j=2}^{k} e^{-2r\left[\left(1-h^2\right)\sin^2 \left(\frac{(j-1)\pi}{k}\right) + h^2\right]}
$$
Recalling \((r, h) \in S_k\), we have
\[
-2r \left[ (1-h^2) \sin^2 \frac{\pi}{k} + h^2 \right]^{\frac{1}{2}} = e^{-2r(1-h^2) \frac{\pi}{k} \sin^{-1} \left( 1 \right) \frac{h^2}{1-h^2 \sin^2 \frac{\pi}{k}} }.
\]

Then together with (3.14) and (3.15), we can know easily
\[
\sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_j}^p U_{r,h} = B_1 e^{-2rh} + O_k(e^{-2(1+\sigma)r}h) + O_k(e^{-2(1+\sigma)\pi \sqrt{1-h^2} \frac{\pi}{k}}).
\]

We complete the proof of Lemma 3.3 \(\square\)

4. Proof of Proposition 2.1

In this section, we will turn to the proof of Proposition 2.1.

Proof of Proposition 2.1: Now we calculate \(I(W_{r,h})\):

\[
I(W_{r,h}) = \frac{1}{2} \int_{\mathbb{R}^N} \left\{ |\nabla W_{r,h}|^2 + V(|y|)W_{r,h}^2 \right\} - \frac{1}{p+1} \int_{\mathbb{R}^N} |W_{r,h}|^{p+1} \]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \left\{ |\nabla W_{r,h}|^2 + |W_{r,h}|^2 \right\} \]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} \left( V(|y|) - 1 \right) |W_{r,h}|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |W_{r,h}|^{p+1} \]
\[
= I_1 + I_2 + I_3.
\]

By using the symmetry and the equation for \(U\), we can get
\[
I_1 = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{j=1}^{k} \left( - \Delta U_{x_j} + U_{x_j} - \Delta U_{x_j} + U_{x_j} \right) \cdot \sum_{i=1}^{k} (U_{x_i} + U_{x_i})
\]
\[
= \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (U_{x_j}^p + U_{x_j}^p) \cdot (U_{x_i} + U_{x_i})
\]
\[
= \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (U_{x_j}^p U_{x_i} + U_{x_j}^p U_{x_i} + U_{x_j}^p U_{x_i} + U_{x_j}^p U_{x_i})
\]
\[
= k \int_{\mathbb{R}^N} U^{p+1} + k \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x_i}^p U_{x_i} + k \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_j}^p U_{x_i}.
\]
Next we calculate $I_2$, using symmetry and Lemma 3.1
\[
I_2 = \frac{1}{2} \int_{\mathbb{R}^N} (V(|y|) - 1)|W_{r,h}|^2
\]
\[
= k \int_{\Omega_1^+} (V(|y|) - 1)(U_{\mathfrak{z}_1} + U_{\mathfrak{T}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})^2
\]
\[
= k \int_{\Omega_1^+} (V(|y|) - 1)\left(U_{\mathfrak{T}_1} + O_k\left(e^{-\frac{1}{2}hr}e^{-\frac{1}{2}|y-\mathfrak{T}_1|} + e^{-\frac{1}{2}hr}e^{-\frac{1}{2}|y-\mathfrak{T}_1|}\right)^2
\]
\[
= k \int_{\Omega_1^+} (V(|y|) - 1)U_{\mathfrak{T}_1}^2 + kO_k\left(\int_{\Omega_1^+} (V(|y|) - 1)e^{-\frac{1}{2}|y-\mathfrak{T}_1|} U_{\mathfrak{T}_1}\right)
\]
\[
= k \left\{\frac{A_1}{p^m} + O_k\left(\frac{1}{p^{m+\sigma}}\right)\right\}, \quad (4.3)
\]
where $A_1$ is defined in (2.2) and the last equality holds due to the asymptotic expression of $V(y)$. We consider $I_3$. Suppose $p \leq 3$, then for $y \in \Omega_1^+$,
\[
|W_{r,h}|^{p+1} = \left(U_{\mathfrak{z}_1} + U_{\mathfrak{T}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j}\right)^{p+1}
\]
\[
= U_{\mathfrak{T}_1}^{p+1} + (p+1)U_{\mathfrak{T}_1}^p(U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})
\]
\[
+ O_k\left(U_{\mathfrak{T}_1}^p(U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})^{p+1}\right).
\]
Using the Lemma 3.1, we can get for $y \in \Omega_1^+$,
\[
U_{\mathfrak{T}_1}^{p+1}(U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})^{p+1}
\]
\[
= U_{\mathfrak{T}_1}^{p+1} (U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})(U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j})^{p+1}
\]
\[
\leq CU_{\mathfrak{T}_1}^{(1-\eta)p}\left(e^{-hr}e^{-(1-\eta)|y-\mathfrak{T}_1|} + e^{-\eta\sqrt{1-h^2}+\frac{\eta}{k}Ce^{-\frac{(1-\eta)|y-\mathfrak{T}_1|}}\right)^{p+1}
\]
\[
\cdot (U_{\mathfrak{z}_1} + \sum_{j=2}^{k} U_{\mathfrak{T}_j} + \sum_{j=2}^{k} U_{\mathfrak{z}_j}).
\]
For any $(r, h) \in S_k$, and $y \in \Omega_1^+$ we have
\[
U_{\mathfrak{T}_1}U_{\mathfrak{z}_1} \leq Ce^{-|\mathfrak{T}_1-\mathfrak{z}_1|} = Ce^{-2hr},
\]
\[
U_{\mathfrak{T}_1} \sum_{j=2}^{k} U_{\mathfrak{T}_j} \leq C \sum_{j=2}^{k} e^{-|\mathfrak{T}_1-\mathfrak{T}_j|} \leq Ce^{-2\pi\sqrt{1-h^2}\frac{k}{\pi} + Ce^{-2(1+\sigma)\pi\sqrt{1-h^2}\frac{k}{\pi}},
\]
and
\[
U_\mathcal{T} \sum_{j=2}^{k} U_{\mathcal{Z}_j} \leq C \sum_{j=2}^{k} e^{-|x_1 - z_j|} \leq C e^{-2\pi \sqrt{1-h^2}r_\mathcal{T}}.
\]
So, we obtain that for \( p \in (1, 3] \),
\[
\mathbb{I}_3 = -\frac{1}{p + 1} \int_{\mathbb{R}^N} |W_{r,h}|^{p+1}
\]
\[
= -\frac{2k}{p + 1} \int_{\Omega_1^+} \left( U_{\mathcal{X}_1} + U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j} \right)^{p+1}
\]
\[
= -\frac{2k}{p + 1} \int_{\Omega_1^+} \left( U_{\mathcal{X}_1}^{p+1} + (p + 1)U_{\mathcal{X}_1}^p (U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j}) \right)
\]
\[
- \frac{2k}{p + 1} \int_{\Omega_1^+} O_k \left( U_{\mathcal{X}_1}^{p+1} (U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j}) \right)^{\frac{p+1}{2}}
\]
\[
+ k \left( e^{-hr} + e^{-\eta(1-h^2)r_\mathcal{T}} + e^{-\eta_1(1-h^2)r_\mathcal{T}} \right)^{\frac{p+1}{2}}
\]
\[
\cdot O_k(e^{-|x_1 - z_j|} + \sum_{j=2}^{k} e^{-|x_1 - x_j|} + \sum_{j=2}^{k} e^{-|x_1 - z_j|})
\]
\[
= -\frac{2k}{p + 1} \int_{\mathbb{R}^N} \left( U_{\mathcal{X}_1}^{p+1} + (p + 1)U_{\mathcal{X}_1}^p (U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j}) \right)
\]
\[
+ kO_k(e^{-2(1+\sigma)\sqrt{1-h^2}r_\mathcal{T}} + e^{-2(1+\sigma)r_\mathcal{T}}).
\]
Now we consider \( p > 3 \). Then for any \( y \in \Omega_1^+ \),
\[
|W_{r,h}|^{p+1} = \left( U_{\mathcal{X}_1} + U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j} \right)^{p+1}
\]
\[
= U_{\mathcal{X}_1}^{p+1} + (p + 1)U_{\mathcal{X}_1}^p (U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j})
\]
\[
+ O_k \left( U_{\mathcal{X}_1}^{p+1} (U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j})^2 \right).
\]
Since \( p - 1 > 2 \), similar to the proof of (4.4), we have
\[
\mathbb{I}_3 = -\frac{1}{p + 1} \int_{\mathbb{R}^N} |W_{r,h}|^{p+1}
\]
\[
= -\frac{2k}{p + 1} \int_{\Omega_1^+} \left( U_{\mathcal{X}_1} + U_{\mathcal{X}_1} + \sum_{j=2}^{k} U_{\mathcal{X}_j} + \sum_{j=2}^{k} U_{\mathcal{Z}_j} \right)^{p+1}
\]
Combining (4.1), (4.2), (4.3), (4.4), (4.5), we can get

\[ I = \int_{\mathbb{R}^N} \left( U_{x_1}^p + (p + 1) U_{x_1}^p U_{x_2} + \sum_{j=2}^k U_{x_j} + \sum_{j=2}^k U_{x_j} \right) + k O_k \left( \int_{\Omega_k^{+}} U_{x_1}^p \right) \]

\[ = -\frac{2k}{p + 1} \int_{\mathbb{R}^N} \left( U_{x_1}^p + (p + 1) U_{x_1}^p U_{x_2} + \sum_{j=2}^k U_{x_j} + \sum_{j=2}^k U_{x_j} \right) \]

\[ + k O_k \left( \int_{\Omega_k^{+}} U_{x_1}^p \right) \]

\[ = -\frac{2k}{p + 1} \int_{\mathbb{R}^N} \left( U_{x_1}^p + (p + 1) U_{x_1}^p U_{x_2} + \sum_{j=2}^k U_{x_j} + \sum_{j=2}^k U_{x_j} \right) \]

\[ + k \left( e^{-hr} + e^{-\eta \sqrt{1 - \eta^2} \frac{x^2}{h^2}} + e^{-\eta \sqrt{1 - \eta^2} \frac{x^2}{h^2}} \right) \cdot O_k \left( e^{-2hr} + e^{-2\pi \sqrt{1 - \eta^2} \frac{x^2}{h^2}} \right) \]

\[ = -\frac{2k}{p + 1} \int_{\mathbb{R}^N} \left( U_{x_1}^p + (p + 1) U_{x_1}^p U_{x_2} + \sum_{j=2}^k U_{x_j} + \sum_{j=2}^k U_{x_j} \right) \]

\[ + k O_k \left( e^{-2(1 + \sigma) \pi \sqrt{1 - \eta^2} \frac{x^2}{h^2}} + e^{-2(1 + \sigma) rh} \right). \]  (4.5)

Combining (4.4), (4.2), (4.3), (4.4), (4.5), we can get

\[ I(W, r, h) = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 \]

\[ = k \int_{\mathbb{R}^N} U_{x_1}^p + \sum_{i=2}^k \int_{\mathbb{R}^N} U_{x_1}^p U_{x_i} + k \int_{\mathbb{R}^N} U_{x_1}^p U_{x_1} \]

\[ - \frac{2k}{p + 1} \int_{\mathbb{R}^N} \left( U_{x_1}^p + (p + 1) U_{x_1}^p U_{x_2} + \sum_{j=2}^k U_{x_j} + \sum_{j=2}^k U_{x_j} \right) \]

\[ + k \left( A \frac{1}{\eta m} + O_k \frac{1}{\eta m + \sigma} \right) \]

\[ = k \left( A \frac{1}{\eta m} + A_2 + O_k \frac{1}{\eta m + \sigma} \right) - k \sum_{i=2}^k \int_{\mathbb{R}^N} U_{x_1}^p U_{x_i} + k \int_{\mathbb{R}^N} U_{x_1}^p U_{x_1} \]

\[ + k O_k \left( e^{-2(1 + \sigma) \pi \sqrt{1 - \eta^2} \frac{x^2}{h^2}} + e^{-2(1 + \sigma) rh} \right). \]  (4.6)

Combining Lemma 3.3 and (4.6), we have

\[ I(W, r, h) = k \left( A \frac{1}{\eta m} + A_2 - B_1 e^{-2 \pi \sqrt{1 - \eta^2} \frac{x^2}{h^2}} - B_1 k e^{-2 rh} \right) \]

\[ + O_k \left( \frac{1}{\eta m + \sigma} \right) + O_k \left( e^{-2 \pi (1 + \sigma) \sqrt{1 - \eta^2} \frac{x^2}{h^2}} + O_k \left( e^{-2(1 + \sigma) rh} \right) \right), \]  (4.7)

where \( A_1, A_2, B_1 \) are defined in (2.2).

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