PRIME PRINCIPAL RIGHT IDEAL RINGS

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ABSTRACT

Let R be a commutative ring with unity $1 \in R$. In this article, we introduce the concept of prime principal right ideal rings (PPRIR). A prime ideal P of R is said to be prime principal right ideal (PPRI) is given by $P = \{ar : r \in R\}$ for some element a. The ring R is said to be prime principal right ideal ring (PPRIR) if every prime ideal of R is a prime principal right ideal (PPRI). A prime principal right ideal ring R is called a prime principal right ideal domain (PPRID) if R is a domain. Several properties and characteristics of prime principal right ideal ring (PPRIR).

1. Introduction

Throughout this article, all rings are commutative ring with nonzero unity 1. An ideal P of the ring R is said to be prime ideal if for all $x, y \in R$ with $xy \in P$ then $x \in P$ or $y \in P$. Since prime ideals are so significant in commutative rings theory, this concept has been expanded and examined in a variety of ways. Some of these generalizations are as important as the prime ideals. Burton in [6] defined the maximal ideals of R, an ideal P of R is said to maximal ideal provided that $P \neq R$ and whenever I is an ideal of R with $P \subset I \subset R$, then $J = R$. A prime ideal P is said to be a prime ideal minimal over an ideal I if it is minimal among all prime ideals containing I.

Let P be a proper ideal of R. Then the radical of P is denoted by $\text{Rad}(P)$ and it is defined as written below:

$$\text{Rad}(P) = \{a \in R : a^n \in P \text{ for some positive integer } n\}$$

Note that $\text{Rad}(P)$ is actually an ideal of the ring R which contains P (see [6]).

Let P be an ideal of R, P is said to be principal right ideal is given by $P = \{ar : r \in R\}$ for some element a. Conrad defined Noetherian rings in [2]. A commutative ring R is called Noetherian if each ideal in R is finitely generated. In [4], R is said to be principal right ideal ring if every right ideal of R is a principal right ideal. We say that R is a principal right ideal domain if R is a domain.

A prime principal left ideal ring (PPLIR) and domain (PPLID) are defined symmetrically. A ring R is called a prime principal ideal ring (PPPIR) if R is both a prime principal right ideal ring (PPRIR) and a prime principal left ideal ring (PPLIR)

In Section Two, we introduce the concept of prime principal right ideal rings (PPRIR). We say that P is a prime principal right ideal (PPRI) of R is given by $P = \{ar : r \in R\}$ for some element a, when P is a prime ideal of R. A ring R we say that a prime principal right ideal ring (PPRIR) if every prime ideal is prime principal right ideal. A prime principal right ideal ring R is called a prime principal right ideal domain (PPRID) if R is a domain. We study rings with this property. Clearly, every principal right ideal rings (PRI) is prime principal right ideal rings (PPRIR). In Theorem 1, we show $R/P$ is a prime principal right ideal domain (PPRID) when $R$ is a prime principal right ideal ring (PPRIR) and P is a prime ideal of R. We investigate some basic properties of prime principal right ideal ring (PPRIR).
2. Prime Principal Right Ideal Rings

In this section, we introduce and study prime principal right ideal ring (PPRIR).

**Definition 1.** Let $R$ be a commutative ring with identity $1 \in R$.
1. A prime ideal $P$ of $R$ is prime principal right ideal (PPRI) is given by $P = \{ar : r \in R\}$ for some element $a$.
2. $R$ is a prime principal right ideal ring (PPRIR) if every prime ideal is prime principal right ideal. A prime principal right ideal ring $R$ is called a prime principal right ideal domain (PPRIPD) if $R$ is a domain.

Clearly, every principal right ideal rings (PRIR) is prime principal right ideal rings (PPRIR). We show examples of prime principal right ideal rings in the following examples. For a ring $R$ and $n \in \mathbb{Z}^+$, $M_n(R)$ means the ring of $n \times n$ matrices over $R$ and $I_n(R) = \{rI : r \in R\}$, where $I$ is the $n \times n$ identity matrix of $M_n(R)$.

**Example 1.** 1. Every field and division ring are prime principal right ideal ring.
2. Let $R$ be a principal right ideal ring and $J$ a subring of $M_n(R)$ containing $I_n(R)$. Then $J$ is a principal right ideal ring (see [3], Proposition 1.7), so $J$ is a prime principal right ideal ring. In particular, $M_n(R)$ is a prime principal right ideal ring.

**Theorem 1.** Let $R$ be a prime principal right ideal ring (PPRIR). Then $P$ is a prime ideal of $R$ if and only if $R/P$ is a prime principal right ideal domain (PPRIPD).

**Proof.** First, take $P$ is a prime ideal of $R$ and $R$ is (PPRIR), so is the quotient ring $R/P$. The only thing left is to make sure $R/P$ doesn’t have any zero divisors. Assume the following:

$$(x + P)(y + P) = P$$

In other words, the zero element of the ring $R/P$ is the product of these two cosets. The preceding equation clearly equates to a requirement that $xy + P = P$, or $xy \in P$. But by assumption $P$ is a prime ideal, so $x \in P$ or $y \in P$. This is means that either the coset $x + P = P$ or $y + P = P$. Hence, $R/P$ is without zero divisors. Therefore $R/P$ is a (PPRIPD).

We just invert the argument to establish the converse. Assume that $R/P$ is (PPRIPD) and $xy \in P$. This indicates, in terms of cosets, that

$$(x + P)(y + P) = xy + P = P$$

By assumption $R/P$ has not zero divisors, so $x + P = P$ or $y + P = P$. In each case, one of $x$ or $y$ belongs to $p$, forcing $P$ to be a prime ideal of $R$.

Note that, let $R$ be a commutative ring with unity then every maximal ideal is a prime ideal (see [6]).

**Proposition 0.1.** If $R$ is a prime principal right ideal ring (PPRIR), then every maximal ideal is prime principal right ideal (PPRI).

**Proof.** Assume that $R$ is a prime principal right ideal ring (PPRIR), so $R$ is a commutative ring with unity. Then every maximal ideal is prime ideal, but every prime ideal of $R$ is a prime principal right ideal (PPRI). Hence the maximal ideal is a prime principal right ideal (PPRI).
The converse of Proposition 0.1 does not hold, the following example show that there exist prime ideal of the prime principal right ideal ring (PPRIR) which is not maximal ideal.

**Example 2.** Let \( R = \mathbb{Z} \times \mathbb{Z} \) be a prime principal right ideal ring and \( P = \mathbb{Z} \times \{0\} \) be a prime ideal of \( R \). Since \( \mathbb{Z} \times \{0\} \subset \mathbb{Z} \times \mathbb{Z}_e \subset R \) with \( \mathbb{Z} \times \mathbb{Z}_e \) an ideal of \( R \), then \( \mathbb{Z} \times \{0\} \) is not maximal ideal.

**Proposition 0.2.** Let \( R \) be a Boolean ring and (PPRIR). Then \( P \) is a prime principal right ideal (PPRI) of \( R \) if and only if \( P \) is a maximal ideal of \( R \).

**Proof.** It suffices to show that if \( P \) is the prime principal right ideal (PPRI), \( P \) is also the maximal ideal. Suppose that \( I \) is an ideal of \( R \) with \( P \subset I \subset R \), what we need to show is that \( I = R \). Let \( x \in I \) and \( x \notin P \), then \( x(1-x) = 0 \in P \). But \( P \) is a prime principal right ideal of \( R \) with \( x \notin P \), so \( 1-x \in P \subset I \). Then \( x \) and \( 1-x \) in \( I \), it follows that \( 1 = x + (1-x) \in I \). As a result, the ideal \( I \) contains the identity and, as a result, \( I = R \). Since there is no appropriate ideal between \( P \) and the entire ring \( R \), we come to the conclusion that \( P \) is a maximal ideal.

**Proposition 0.3.** Let \( R \) be a prime principal right ideal ring (PPRIR). If \( P \) is a prime principal right ideal (PPRI) of \( R \), then \( P \) is a semiprime ideal of \( R \).

**Proof.** Assume that \( P \) is a prime principal right (PPRI) of \( R \), so \( R/P \) is a prime principal right ideal domain (PPRID) by Theorem 1, and so \( R/P \) has no nonzero nilpotent elements. Hence \( P \) is a semiprime ideal of \( R \).

**Theorem 2.** A ring \( R \) is a prime principal right ideal ring (PPRIR) if and only if for each infinite increasing sequence of prime ideals \( P_1 \subset P_2 \subset P_3 \subset ... \) in \( R \), then \( P_n = P_{n+1} \) for all large \( n \).

**Proof.** \((\Rightarrow)\) If \( P_1 \subset P_2 \subset P_3 \subset ... \) is an increasing sequence of prime ideals. Let \( P = \bigcup_{n \geq 1} P_n \) be a prime ideal of \( R \). Since \( R \) is a prime principal right ideal ring (PPRIR), so \( P \) is a Prime principal right ideal (PPRI). Now, by increasing condition for each finite subset of \( P \) lies in a common \( P_n \), so a finite set of \( P \) is in some \( P_m \). Thus \( P \subset P_m \) as well as that \( P_m \subset P \), so \( P = P_m \). Then for all \( n \geq m \), \( P_m \subset P_n \subset P = P_m \), so \( P_n = P_m \).

\((\Leftarrow)\) Suppose that \( R \) is a not prime principal right ideal ring, so \( R \) has a prime ideal \( P \) that is not prime principal right ideal. Take \( p_1 \in P \). Since \( P \) is not prime principal right ideal, so \( P \neq \{pr : r \in R\} = (p_1) \). Consequently, there is a \( p_2 \in P - (p_1) \) but \( P \neq (p_1,p_2) \). Follow the same steps to take \( p_n \) in \( P \) for all \( n \geq 1 \) by making \( p_n \in P - (p_1,p_2,...,p_{n-1}) \) for all \( n \geq 2 \). Then we have an increasing sequence of prime ideals \( (p_1) \subset (p_1,p_2) \subset ... \subset (p_1,p_2,...,p_{n-1}) \subset ... \) in \( R \) where each prime ideal is completely contained within the one after it, so this is contradiction. Hence \( R \) is a prime principal right ideal ring (PPRIR).

**Theorem 3.** Let \( R \) be a prime principal right ideal domain (PPRID). If \( \phi \) from \( R \) to \( R \) is a surjective ring homomorphism, then \( \phi \) is an injective ring homomorphism and thus
is an isomorphic ring homomorphism.

**Proof.** Let $\phi : R \to R$ be a surjective ring homomorphism. The $n$th iteration $\phi^n$, let $P_n = \ker(\phi^n)$ be a prime ideal of $R$ and these prime ideals form an increasing chain $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots$ since $x \in P_n$ then $\phi^n(x) = 0$ and then $\phi^{n+1}(x) = \phi(n^i(x)) = \phi(0) = 0$, so $x \in P_{n+1}$. Since $R$ is a prime principal right ideal ring (PPRIR) by Theorem 2 $P_n = P_{n+1}$ for some $n$. Take $x \in \ker(\phi)$, so $\phi(x) = 0$. The map $\phi^n$ is surjective map, so $x = \phi^n(y)$ for some $y \in R$. Thus $0 = \phi(x) = \phi(n^i(y)) = \phi^{n+1}(y)$. Therefore $x \in \ker(\phi^{n+1}) = \ker(\phi^n)$, so $x = \phi^n(y) = 0$. Thus $\ker(\phi) = \{0\}$, so $\phi$ is an injective ring homomorphism.

The converse of Theorem 3 is not true, the following example show this.

**Example 3.** Let $R[X]$ be a prime principal right ideal domain (PPRID) since it is a principal right ideal domain (PRID) and let $\phi : R[X] \to R[X]$ by $\phi(x) = x^2$. The homomorphism ring $\phi$ is an injective but not surjective.

**Theorem 4.** If $R$ is a prime principal right ideal ring (PPRIR) then $R[X]$ is a prime principal right ideal ring (PPRIR).

**Proof.** It is clearly if $R = \{0\}$, so suppose that $R \neq \{0\}$. To prove the theorem must prove every prime ideal of $R[X]$ is a prime principal right ideal (PPRI). By contradiction, assume that $P$ is a prime ideal of $R[X]$ but not prime principal right ideal. We have $P \neq \{0\}$. Defined a sequence of polynomials $f_1, f_2, f_3, \ldots$ in $P$ as the following:
1. Take $f_1 \in P - \{0\}$ with minimal degree.
2. Since $P \neq \{f_1\}$, take $f_2 \in P - \{f_1\}$ with minimal degree. Note that $\deg(f_1) \leq \deg(f_2)$.
3. For $n \geq 2$, if we have defined $f_1, f_2, \ldots, f_{n-1}$ in $P$ then $P \neq \{f_1, f_2, \ldots, f_n\}$, so we may take $f_{n+1} \in P - \{f_1, f_2, \ldots, f_n\}$ with minimal degree.

We have $\deg(f_n) \leq \deg(f_{n+1})$ for all $n$. The case $n = 1$ was already examined and for $n \geq 2$, $f_n$ and $f_{n+1}$ are in $P - f_1, f_2, \ldots, f_{n-1}$ so $\deg(f_n) \leq \deg(f_{n+1})$.

Let $d_n = \deg(f_n)$ for $n \geq 1$ and $c_n$ be the leading coefficient of $f_n$, so $d_n = d_{n+1}$ and $f_n(X) = C_nX^{d_n} + \text{terms of a lower degree}$.

Now choose the prime ideal $(c_1, c_2, \ldots)$ in $R$ is a prime principal right ideal (PPRI) since $R$ is a prime principal right ideal ring (PPRIR). This prime ideal has elements all linear combination of finitely many $c_n$, so $(c_1, c_2, \ldots) = (c_1, c_2, \ldots, c_m)$ for some $m$.

Since $c_{m+1} \in (c_1, c_2, \ldots, c_m)$, we have

$$c_{m+1} = \sum_{n=1}^{m} x_n c_n$$

for some $x_n \in R$. From $d_n \leq d_{m+1}$ for $n \leq m$, the leading term in $f_n(X) = c_nX^{d_n} + \ldots$ can be transformed into a degree $d_{m+1}$ by using $f_n(X)X^{d_{m+1}-d_n} = c_nX^{d_{m+1}} + \ldots$, and this is in $P$ since $f_n(X) \in P$ and $P$ is a prime ideal of $R[X]$. By the previous linear combination $c_{m+1}$ the linear combination

$$\sum_{n=1}^{m} x_n f_n(X)X^{d_{m+1}-d_n}$$

is in the ideal $(f_1, f_2, \ldots, f_m)$ and its coefficient of $X^{d_{m+1}}$ is $\sum_{n=1}^{m} x_n c_n$ which equals the
leading coefficient $c_{m+1}$ of $f_{m+1}(X)$ in degree $d_{m+1}$. The difference $$f_{m+1}(X) - \sum_{n=1}^{m} x_n f_n(X) X^{d_{m+1} - d_n}$$ is in $P$ and not 0 since $f_{m+1} \in P - (f_1, f_2, ..., f_m)$ with degree less than $d_{m+1}$. But $f_{m+1}$ has a minimal degree in $P - (f_1, f_2, ..., f_m)$ and the previous difference is in $P - (f_1, f_2, ..., f_m)$ with lower degree than $d_{m+1}$. This is the contradiction. Thus $P$ is a prime principal right ideal ($PPRIR$) of $R[X]$.

In a single sentence, we would want to summarize this proof, ”use a prime ideal of leading coefficients”.

In the proof of Theorem 4, the prime principal right ideal ring ($PPRIR$) property of $R$ where we mentioned is used $(c_1, c_2, c_3, ...) = (c_1, c_2, ..., c_m)$ for some $m$. To get the contradiction in the evidence, all we need to do is $c_{m+1} \in (c_1, c_2, ..., c_m)$ for some $m$. Since $(c_1) \subset (c_1, c_2) \subset (c_1, c_2, c_3) \subset ...$, the following property is what we require: for each infinite increasing sequence of prime ideals $P_1 \subset P_2 \subset P_3 \subset ...$ in $R$, $P_m = P_{m+1}$ for some $m$. Of course this is implied by prime principal right ideal ring ($PPRIR$) property, but it also implies the prime principal right ideal ring ($PPRIR$) property since a non-prime principal right ideal ring has an infinite increasing sequence of ideals with strict containment at each step see the proof of Theorem 2.

Where did we apply the assumption that $R$ is a prime principal right ideal ring ($PPRIR$) in the proof of the Theorem 4? It is how we know the prime ideals $(c_1, c_2, ..., c_n)$ for $n \geq 1$ stabilize for large $n$, so $c_{m+1} \in (c_1, c_2, ..., c_m)$ for some $m$. The contradiction we obtain from that really shows $c_{m+1} \notin (c_1, c_2, ..., c_m)$ for all $m$, so the proof of Theorem 4 could be construed as proving the converse: if $R[X]$ is not prime principal right ideal ring then $R$ is not prime principal right ideal ring.

**Remark.** The converse of Theorem 4 is true, if the ring $R[X]$ is a prime principal right ideal domain ($PPRID$) then $R$ is a prime principal right ideal domain ($PPRID$), since $R \cong R[x]/(x)$.

**Corollary 4.1.** If $R$ is a prime principal right ideal ring then $R[X_1, ..., X_n]$ is a prime principal right ideal ring.

**Proof.** The case of $n=1$ is in Theorem 4. For $n \geq 2$, write $R[X_1, ..., X_n]$ as $R[X_1, ..., X_{n-1}][X_n]$, with $R[X_1, ..., X_{n-1}]$ being a prime principal right ideal ring by the inductive hypothesis. As a result, we have been limited to the most basic case.

**Proposition 0.4.** Let $R$ be a prime principal right ideal ring ($PPRIR$). If $P$ is a primary ideal of $R$ then $\text{Rad}(P)$ is a prime principal right ideal ($PPRI$) of $R$.

**Proof.** Suppose that $P$ is a primary ideal of $R$, by proposition 4.1 in [1] the $\text{Rad}(P)$ is a prime ideal of $R$. But $R$ is a prime principal right ideal ring ($PPRIR$), so $\text{Rad}(P)$ is a prime principal right ideal of $R$.

**Proposition 0.5.** Let $R$ be a prime principal right ideal ring ($PPRIR$). If $P$ is a prime ideal of $R$ then $\text{Rad}(P)$ is a prime principal right ideal ($PPRI$) of $R$.

**Proof.** Suppose that $P$ is a prime ideal of $R$, so $\text{Rad}(P) = P$. But $R$ is a prime principal right ideal ring ($PPRIR$), then every prime ideal of $R$ is a prime principal right ideal ($PPRI$), and then $\text{Rad}(P)$ is prime principal right ideal ($PPRI$).
Theorem 5. A ring $R$ is a prime principal right ideal ring ($\text{PPRIR}$) if and only if every nonempty collection $T$ of a prime ideals of $R$ contains the maximum element in terms of inclusion there is a prime ideal in $T$ not strictly contained in another prime ideal in $T$.

Proof. ($\Rightarrow$) Suppose that there is a nonempty collection $T$ of a prime ideals in $R$ which there is no maximum member in terms of inclusion. Thus suppose we begin with a prime ideal $P_1$ in $R$, we can find prime ideals recursively $P_2, P_3, \ldots$ such that $P_{n-1} \subset P_n$ for all $n \geq 2$. $P_{n-1}$ would be a maximum element of $T$ if there was no prime ideal in $T$ strictly containing $P_{n-1}$, which does not exist. This is contradiction of Theorem 2 .

($\Leftarrow$) Let $P$ be a prime ideal in $R$ and $T$ be the set of all a prime principal right ideal ($\text{PPRI}$) contained in $P$. Suppose that $I \in T$ no other element of $T$ contains this, so $I$ is a prime principal right ideal ($\text{PPRI}$) in $P$ and no other prime principal right ideal ($\text{PPRI}$) of $R$ contains $I$. we will prove $P = I$ by contradiction, which would prove $P$ is a prime principal right ideal ($\text{PPRI}$). If $P \neq I$, take $x \in P - I$, since $I$ is a prime principal right ideal ($\text{PPRI}$), also $I + (x)$ is a prime principal right ideal ($\text{PPRI}$) where $(x)$ is a prime ideal in $R$. So $I + (x) \in T$ and so $I \in I + (x)$ this is a contradiction of maximality of $I$ as a member of $T$. Hence $P = I$.

Theorem 6. Let $R$ be a prime principal right ideal ring ($\text{PPRIR}$) and $I$ be a proper ideal of $R$. If every prime ideal minimal over $I$ is prime principal right ideal ($\text{PPRI}$), then there are only finitely many prime ideals minimal over $I$.

Proof. Let $T = \{P_1, \ldots P_n : \text{each } P_i \text{ is a prime ideal minimal over } I\}$. If $H \in T$ we have $H \subseteq I$, then any prime ideal $P$ minimal over $I$ contains some $P_i$, so $\{P_1, \ldots, P_n\}$ is the set of minimal prime ideals of $I$. Now assume that $H \nsubseteq P$ for some $H \in T$. Let the set $S = \{J : J \text{ is a prime ideal of } R \text{ with } I \subseteq J \text{ and } H \nsubseteq J \text{ for some } H \in T\}$. Since every element in $T$ is a prime principal right ideal ($\text{PPRI}$), $S$ is inductive and hence by Zorn’s Lemma has a maximal element $Q$. But $I \subseteq Q$, $Q$ contains a prime ideal $P$ minimal over $I$ (see [5], Theorem 10). Thus $P \in T$, this is a contradiction.

3. Conclusions

In this study, we introduced the concept of prime principal right ideal rings which is a generalization of principal right ideal rings. We investigated some basic properties of prime principal right ideal rigs. As a proposal to further the work on the topic, we are going to study the concepts of $S$-prime principal right ideal rings and graded $S$-prime principal right ideal rings.

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