Order Reduction of Optimal Control Systems

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Abstract

The paper presents necessary and sufficient conditions for the order reduction of optimal control systems. Exploring the corresponding Hamiltonian system allows to solve the order reduction problem in terms of dynamical systems, observability and invariant differential forms. The approach is applicable to non-degenerate optimal control systems with smooth integral cost function. The cost function is defined on the trajectories of a smooth dynamical control system with unconstrained controls and fixed boundary conditions. Such systems form a category of Lagrangian systems with morphisms defined as mappings preserving extremality of the trajectories. Order reduction is defined as a factorization in the category of Lagrangian systems.

Keywords: optimal control, order reduction, Lagrangian systems, Hamiltonian systems, factorization, decomposition, hierarchical control.

1 Introduction

The invariance in nonlinear control theory allows to approach and solve many important problems in an effective way. In particular, the invariance of dynamical control system with respect to the action of Lie group or Lie algebra allows order reduction by factoring out a system of smaller dimension. The review [16] and monographs [14], [12] present the current state of the art in this area.

The study of invariance in optimal control (like in classical works [15], [18] or in recent publications [11], [19]) can be built on the results of classical
mechanics [1], [2]. This places the geometric approach to optimal control into the rich context of Hamiltonian mechanics and symplectic geometry.

Different generalizations of the symplectic geometry allow to study wider range of symmetries of optimal control systems. In the recent work [11] an optimal control system is treated as a Hamiltonian system on the corresponding presymplectic manifold. The symmetries of optimal control system are symplectic (when considered on a special symplectic subspace) actions of Lie group that leave both dynamical system and Lagrange function invariant. It is shown that such symmetries allow to reduce the order of the optimal control system.

A more general point of view, based on theory of categories and theory of decomposition, allows to formulate the problem of order reduction in terms of general factor- and sub- objects (see [16] for review of theory of decomposition for control systems).

In [10] the order reduction of a smooth variational problem
\[
\int_0^T L(q, \dot{q})dt \rightarrow \text{extr}, \quad q \in \mathbb{R}^n, q(0) = q_0, q(T) = q_1
\]

was studied in categorial framework. Variational systems form a category with morphisms preserving extremals. The extremals are solutions of the corresponding Euler-Lagrange equations. These equations form a dynamical system, which can be studied and reduced using general geometric methods. Solving the inverse problem of variational calculus allows to rebuild the reduced variational system, when it exists, from the corresponding dynamical factorsystem. The work [10] presents necessary and sufficient conditions for order reduction (factorization) of variational systems.

One of the advantages of the categorial framework is that it allows to pose and solve the order reduction problem in the most general and complete way: the results of [10] cover all factorizations in the category of variational systems with morphisms preserving extremals.

Here we will define a more general category of Lagrangian systems that includes variational systems from [10] as a sub-category. The objects of the new category are optimal control systems with smooth integral criterion, smooth dynamical control system and unconstrained controls. Morphisms in the category of Lagrangian systems are also defined as mappings preserving extremals. Pontryagin’s maximum principle allows to convert Lagrangian system into corresponding Hamiltonian system. We will define a new category of Hamiltonian systems with morphisms that may not necessarily preserve
underlying symplectic structure. That will allow to study factorizations of Hamiltonian systems using methods for general dynamical systems. Finally we will show that factorizations of Hamiltonian systems and Lagrangian systems are corresponding to each other provided that observability condition for Hamiltonian factorsystem is satisfied.

The main results of this work were announced in [6] and [7].

2 Order Reduction as Factorization

The problem of order reduction could be approached for each particular type of mathematical objects individually. However we can study it in a more uniform way by placing it into the categorial framework where order reduction can be treated as factorization in the appropriate category.

For a brief informal illustration we will consider factorization of smooth dynamical systems. Smooth dynamical systems will form a category $DS$ if we define morphisms - mappings of the dynamical systems into each other. A natural morphism is a diffeomorphisms from one dynamical system into the other dynamical system that maps trajectories of the original dynamical system into the trajectories of the image system. Factorization is a special morphism that does not introduce anything additional to the factor object, which could not be derived from the original object. A morphism in category $DS$ defines a factorization if it is a surjective submersion (mapping onto of the full rank). A factor system in this case is the dynamical system on the image (factor) space. Any trajectory of the factor system has its inverse image - a set of trajectories of the original system. Also all trajectories of the original system map into some trajectories of the factor system. It is well known that such systems are described by $f$-related vector fields and provide the classical example of order reduction. Thus the order reduction of dynamical systems can be naturally described in very general terms of factorization.

A formal definitions for this approach could be derived from theory of structures [8] or, equivalently, from theory of categories [9]. For in depth discussion of theory of decomposition in application to control systems see [16].

In this paper we will apply categorial approach to the order reduction problem for optimal control systems.
3 The Category of Lagrangian Systems

We will consider optimal control system formed by a smooth control system and an integral cost function:

\[ \int_0^T L(q,u)dt \rightarrow \text{extr} \]  
\[ \dot{q} = f(q,u) \]  

where \( q \) is an \( n \)-dimensional vector of phase variables and \( u \) is an \( m \)-dimensional vector of controls. The vector of control \( u \) is not constrained, so \( u \in \mathbb{R}^m \). We will call such systems Lagrangian systems, because (under the appropriate conditions) the optimal control problem (1,2) with fixed boundary conditions is equivalent to Lagrange variational problem. The equivalent Lagrangian variational problem can be obtained by eliminating controls \( u \) from both (1) and (2), then transforming (2) to the implicit form \( F(q,\dot{q}) = 0 \).

Detailed definitions follow later.

We will assume that the optimal control system (1, 2) defines a field of extremal solutions. An extremal solution for (1, 2) is a curve \( \gamma(t) = (q(t),u(t)) \) such that it satisfies (2) and the functional (1) achieves on \( \gamma(t) \) an extremal value within the class of the curves with fixed boundary points. As usually, extremals are not necessarily optimal curves: extremality means that the curve has vanishing conditional variations of the functional in its vicinity.

We will be looking for an optimal control system of the same kind but of a lower order (a problem of order reduction or factorization):

\[ \int_0^T Q(y,v)dt \rightarrow \text{extr} \]  
\[ \dot{y} = F(y,v) \]  

such that there exists mapping \( y = y(q,u), v = v(q,u) \), which maps extremals of (1, 2) to the extremals of (3, 4).

In this paper we will derive the necessary and sufficient conditions for the factorization of optimal control systems (1, 2). We will use Pontryagin’s Maximum Principle to transform the optimal systems into Hamiltonian systems. Hamiltonian systems will allow for an intuitive geometric approach to the problem of factorization. Finally we will translate the results obtained
in the terms of Hamiltonian systems back into the domain of the original optimal control systems.

 Everywhere we assume smoothness and locality: all manifolds are open simply connected regions of $\mathbb{R}^n$, all functions are smooth (have as many derivatives as necessary), so we drop adjectives ”smooth” and ”local” in most cases. We use Einstein’s convention for summation: terms with repeating subscript and superscript index automatically sums.

4 Factorization of Hamiltonian systems

A Hamiltonian system is a dynamical system generated by the gradient flow of a Hamiltonian function defined on a symplectic manifold. More formally, a triplet $HS = (M, \omega^2, H)$ defines a Hamiltonian system on a symplectic manifold $M, \text{dim } M = 2n$, with the symplectic structure $\omega^2$ and Hamiltonian function $H : M \rightarrow \mathbb{R}$.

A Hamiltonian system $HS$ defines canonical equations in the form:

$$\dot{z} = \text{Id}H(z), \quad z \in M \quad (5)$$

where $I : TM \rightarrow T^*M$ is the isomorphism induced by the symplectic structure. We will also use $HS$ to denote the canonical equations (5).

In local canonical coordinates $(p, q)$ on $M$, we have:

$$\omega^2 = dp_i \wedge dq^i, \quad i = 1, n \quad (6)$$

and the system (6) has the form

$$\dot{p} = -\partial H/\partial q, \quad \dot{q} = \partial H/\partial p$$

Let $HS' = (N, \omega^2, G), \text{dim } N = 2n$ be another Hamiltonian system and let $\phi : M \rightarrow N$ be a smooth (not necessarily symplectic) mapping. If $(p, q)$ and $(x, y)$ are canonical coordinates on $M$ and $N$, then we can write $\phi$ in coordinates as $x = x(p, q), y = y(p, q)$.

Definition 1 Mapping $\phi : M \rightarrow N$ is called a morphism of Hamiltonian systems if for any solution $z(t), t \in [0, T]$ of the system $HS$ its image under the mapping $\phi$ is a solution of the system $HS'$ on the interval $t \in [0, T]$. 

Using categorial notation we will write $\phi : HS \to HS'$ for the morphism $\phi$ of Hamiltonian systems $HS$ and $HS'$.

The underlying symplectic structure does not participate in the definition of the morphism directly. So, if $\phi : HS \to HS'$ is a morphism, then by “erasing” symplectic structures $\omega^2$ and $\overline{\omega}^2$ from $HS$ and $HS'$ respectively we will obtain a morphism of general dynamical systems. This observation leads to:

**Proposition 1** Let $HS = (M,\omega^2, H)$ and $HS' = (N,\overline{\omega}^2, G)$ be Hamiltonian systems, and let $\phi : M \to N$ be a smooth mapping, then the following conditions are equivalent:

1. $\phi$ is morphism of Hamiltonian systems $HS$ and $HS'$

2. vector fields of the systems $HS$ and $HS'$ are $\phi$-related

If we denote by $I'$ the natural isomorphism $T^*N \to TN$ induced by $\overline{\omega}^2$, then the relation between vector fields of the Hamiltonian systems $HS$ and $HS'$ will have the form

$$I'^*dG = \phi_*IdH$$

Using Poisson bracket $(\cdot)_{p,q}$ on $M$ we can write the same relation in canonical coordinates on $M$ as

$$\partial G/\partial x_i = (H, y^i)_{p,q}, \quad -\partial G/\partial y^i = (H, x_i)_{p,q} \quad (7)$$

We will focus on the case when $\dim N < \dim M$ and the mapping $\phi$ is onto and of full rank, which corresponds to a factorization of Hamiltonian systems. Let’s find when $\phi : M \to N$ maps a Hamiltonian system $HS$ from $(M,\omega^2)$ into a Hamiltonian system on $(N,\overline{\omega}^2)$. The symplectic form $\overline{\omega}^2$ on the factor space $(N,\overline{\omega}^2)$ has to be an invariant of the factor-system by the definition of the Hamiltonian system. The form $\Omega^2 = \phi^*\overline{\omega}^2$ on $M$ is induced by $\phi$ from the form $\overline{\omega}^2$. If the canonical coordinates $(p, q)$ and $(x, y)$ on both spaces are fixed, then $\phi$ has the form $x = x(p, q), y = y(p, q)$, and we obtain a coordinate representation

$$\Omega^2 = dx_i(p, q) \wedge dy^i(p, q), \quad i = 1, m.$$
Proposition 2 Let $\phi : M \to N$ be a surjective submersion. The projection $\phi^* \text{Id}H$ of the field $\text{Id}H$ from $M$ to $N$ exists and is a Hamiltonian vector field on $(N, \o)$ iff 1-form $i_{\text{Id}H} \Omega^2$ is closed.

Here $i_{a}b$ denotes the internal product of a vector field $a$ and a form $b$.

Due to the locality, a closed form is automatically exact. That means it actually is a differential of some function on $M$ and, as we will show, on $N$ as well.

Note again that the fact that the vector field on $M$ is Hamiltonian is not used anywhere.

Proof. Necessity. By assumption $\phi^* \text{Id}H = I'dG$, where $I'dG$ is a Hamiltonian field on $(N, \o)$. Consider function $G = G \circ \phi$, which is defined on $M$.

The following is valid for $G$:

$$dG = d(G \circ \phi) = \phi^* dG = \phi^* i_{I'dG} \o = i_{\text{Id}H} \phi^* \o = i_{\text{Id}H} \Omega^2,$$

These equalities follow from the chain rule applied to $G \circ \phi$ and the equivalent transformation of a gradient 1-form into a vector field on a symplectic manifold.

To receive the same result in coordinates, let’s unfold $i_{\text{Id}H} \Omega^2$ in the coordinates $(p, q)$ on $M$:

$$i_{\text{Id}H} \Omega^2 \equiv i_{\text{Id}H} dx_i(p, q) \wedge dy^i(p, q)$$

Calculating the inner product in the right hand part we will get:

$$i_{\text{Id}H} dx_i(p, q) \wedge dy^i(p, q) = (y^i(p, q), H)_{p, q} dx_i(p, q) - (x_i(p, q), H)_{p, q} dy^i(p, q)$$

The right hand part here is the full differential of $G(x(p, q), y(p, q))$ because of $\phi$-relation of the vector fields expressed by (7).

Sufficiency. The equality $dG = i_{\text{Id}H} \Omega^2$ is equivalent to the fact that the gradient $dG$ can be linearly combined from the gradients of the independent mapping functions $x(p, q)$ and $y(p, q)$. In coordinates:

$$\frac{\partial G}{\partial p_k} = (y^i, H)_{(p, q)} \frac{\partial x_i}{\partial p_k} - (x_j, H)_{(p, q)} \frac{\partial y^j}{\partial p_k} \quad (8)$$

$$\frac{\partial G}{\partial q^k} = (y^i, H)_{(p, q)} \frac{\partial x_i}{\partial q^k} - (x_j, H)_{(p, q)} \frac{\partial y^j}{\partial q^k} \quad (9)$$
where \( k = 1, n \).

Because of the linear dependence of the gradients we have \( \overline{G} = G \circ \phi \) with some function \( G : N \to R \). Hence \( d\overline{G} = \phi^*dG \) and we have a new equality \( \phi^*dG = i_{1dH}\phi^*\omega^2 \). Since \( \phi \) has full rank the last equality implies that \( dG = i_{\phi^*1dH}\omega^2 \). From this follows that \( I'dG = \phi_\ast1dH \), i.e. the gradient vector fields are \( \phi \)-related.

The system (8), (9) can be viewed as a system of linear algebraic equations \( Ah = g \) with the matrix

\[
A = \begin{pmatrix}
\partial x/\partial p & \partial y/\partial p \\
\partial x/\partial q & \partial y/\partial q
\end{pmatrix}
\]

with the right hand part \( g = [\partial \overline{G}/\partial p, \partial \overline{G}/\partial q]^T \).

Both vectors \( [(y, H)_{p,q}, -(x, H)_{p,q}]^T \) and \( [\partial \overline{G}/\partial x, \partial \overline{G}/\partial y]^T \) satisfy the system: the first one is the solution by assumption and the second one as the result of the chain rule differentiation. The matrix \( A \) is of a full rank and the system is overdetermined. Hence the solution, if exists, is unique. This proves that (7) holds under our assumptions, which is equivalent to the \( \phi \)-relation of the corresponding vector fields. The proof is complete.

Using formula \( L_X = i_X \circ d + d \circ i_X \) for Lie derivative \( L_X \) along vector field \( X \), and the fact that \( d\Omega^2 = 0 \), we can get an equivalent proposition:

**Proposition 3** A vector field \( v \) on \( M \) maps onto a Hamiltonian field on \( (N, \omega^2) \) under surjective submersion \( \phi : M \to N \) iff 2-form \( \Omega^2 = \phi^*\omega^2 \) is an invariant of 1-parametric Lie group generated by the vector field \( v \).

**Proof.** Using infinitesimal criterion of the invariance we can conclude that \( L_v\Omega^2 = 0 \). Next, expanding this using the formula \( L_v = i_v \circ d + d \circ i_v \), we get: \( d \circ i_v\Omega^2 = 0 \) because \( d\Omega^2 = 0 \) by definition of symplectic form. This reduces Proposition 3 to Proposition 2.

**Corollary 1** If \( \phi : HS \to HS' \) is morphism of Hamiltonian systems then the function \( \overline{G} = G \circ \phi \), where \( G \) is a Hamiltonian of \( HS' \), is the first integral of \( HS \).

**Proof.** The corollary follows from the chain of equalities:

\[
(G, H)_{p,q} = 1dH(d\overline{G}) = i_{1dH}\phi^*dG = (i_{1dH})^2\phi^*\omega^2 = 0
\]
Here the first equality is by definition. The second one is an expansion of \( dG \). The third one is due to Proposition 3. The last equality holds because of the skew symmetry of 2-form \( \phi^2 \), which is symplectic on \( N \).

Remarks. In short this section says that we can reduce the order of a Hamiltonian system by projecting a general vector field and then converting the reduced dynamic system into the Hamiltonian form. The problem of recognizing a Hamiltonian system in a general dynamic system was studied in geometric mechanics (see [13] for linear quadratic case). The Hamiltonian form always exists locally whenever we can present \( 2m - 1 \) independent first integrals, which is always possible in the vicinity of a regular point [4]. We covered the subject in sufficient details mainly to establish the framework for the following sections. Also note that the propositions in this section can be easily generalized for the global case, but we keep it local for consistency with the later discussion.

5 Factorization of Optimal Control Systems

Here we introduce a category of Lagrangian systems to set up a framework for factorization of optimal control systems. Then we are going to establish a connection between factorizations in categories of Lagrangian and Hamiltonian systems.

**Definition 2** A Lagrangian system \( LS \) is a triplet \((M, CDS, L)\), where \( M \) is a manifold, \( \dim M = n \), \( CDS \) is a controllable dynamic system on \( M \), and \( L \) is a function \( M \times U \to R \).

By dynamical control system \( CDS \) in the definition 2 we understand a system:

\[
\dot{q}^i = f^i(q, u), \quad q \in M, i = 1, n
\]

where the vector of control \( u \in U = R^m \) is unconstrained.

A curve \( \gamma : [0, T] \to R \times U \) (or, in coordinates, \( \gamma(t) = (q(t), u(t)) \), \( t \in [0, T] \)) is called a solution for \( CDS \) if it satisfies the equation \( dq(t)/dt = f(q(t), u(t)) \) for \( \forall t \in [0, T] \). Also we will call such curve admissible.

The function \( L : M \times U \to R \) from the definition 2 defines a functional \( \mathcal{L}(\gamma) \) on the set of all admissible curves by the formula:
\[ \mathcal{L}(\gamma) = \int_0^T L(q(t), u(t)) dt \]

Solutions of a Lagrangian system are the extremals of \( \mathcal{L}(\gamma) \) in the class of curves with fixed boundaries.

**Definition 3** A solution \( \gamma(t) = (q(t), u(t)) \) of Lagrangian system \( LS = (M, CDS, L) \) is a curve providing a local extremum to the functional \( \mathcal{L} \) on the class of admissible curves with fixed boundary points.

Let \( LS' = (N, CDS', Q) \) be another Lagrangian system, such that: \( \dim N = \nu \), \( CDS' \) has the form \( \dot{y} = F(y, v) \), \( v \in V, V = R^\mu \) and \( Q = Q(y, v) \). Consider mapping \( \Psi : M \times U \to N \times V \), or in coordinates: \( y = y(q, u), v = v(q, u) \).

**Definition 4** A mapping \( \Psi : M \times U \to N \times V \) is called a morphism of Lagrangian systems from \( LS \) to \( LS' \) if it maps solutions of \( LS \) into solutions of \( LS' \).

In other words, if \( \gamma = \gamma(t) \) is a solution of \( LS \) then \( \gamma' = \Psi \circ \gamma \), \( \gamma'(t) = (y(q(t), u(t)), v(q(t), u(t))) \) is a solution of \( LS' \). We will denote a morphism of Lagrangian systems by the same mapping symbol \( \Psi : LS \to LS' \). We will be interested in morphisms that are onto and of full rank (factorizations) of Lagrangian systems.

Let \( p dq \) be the standard 1-form on \( T^*M \).

**Definition 5** A function \( H(p, q, u) = p_i f^i(q, u) - L(q, u) \) defined on \( (T^*M) \times U \) is called Pontryagin function of the Lagrangian system \( LS = (M, CDS, L) \).

We disregard singular systems, so \( p_0 \) in a more general Pontryagin function \( p_i f^i(q, u) - p_0 L(q, u) \) is never vanishing and we always have \( p_0 \equiv 1 \).

**Definition 6** A Lagrangian system \( LS \) is called non-degenerate if it satisfies the following conditions:

1. A system of nonlinear algebraic equations \( \partial H / \partial u^k = 0, k = 1, m \) can be resolved with respect to \( u \), the solution \( \hat{u} = \hat{h}(p, q) \) is unique and the mapping \( \hat{u} : T^*M \to U \) is smooth.
2. The matrix \( f_u \) has full rank:

\[
\text{rank}\left[ \frac{\partial f^i}{\partial u^k} \right] = m,
\]

where \( i = 1, n \) and \( k = 1, m \).

The mapping \( \dot{u} \) described above is the optimal synthesis for the optimal control system.

Note that \( m \leq n \) for a non-degenerate Lagrangian system (the dimension of control space does not exceed the dimension of the phase space).

From here we will consider only non-degenerate Lagrangian systems.

The correspondence between control variables \( u \) and dual variables \( p \) established by optimal synthesis is not one to one in case \( m < n \). Because of that we need the following definition of observability.

**Definition 7** A function \( S : T^*M \to R \) is called observable in \( LS \) if there exists a function \( \Psi : M \times U \to R \) such that the following diagram commutes

\[
\begin{array}{ccc}
T^*M & \xrightarrow{\Delta} & M \times U \\
\downarrow S & & \downarrow \Psi \\
R & & \end{array}
\]

where \( \Delta = \pi \times \dot{u} \) is morphism of fiber bundles \( \pi : T^*M \to M \) and \( \pi' : M \times U \to M \).

In other words, the definition requires that \( S(p,q) = \Psi(q,\dot{u}(p,q)) \) for observable in \( LS \) function \( S \). In the linear case this definition corresponds to the observability defined in [2].

The set of observable in \( LS \) functions will be denoted as \( \mathcal{F}_o(LS) \), or \( \mathcal{F}_o \) for brevity, when no confusion can happen.

We also need to define observability for morphisms of the Hamiltonian system derived from a Lagrangian system. Let \( HS' = (T^*N, \omega^2, G), \text{dim}N = \nu \) be a Hamiltonian system defined on \( T^*N \) with a natural symplectic form \( \omega^2 = dx_i \wedge dy^i \) where \( (x,y) \) are canonical coordinates on \( T^*N \). Let \( \phi : HS \to HS' \) be morphism. In coordinates \( x = x(p,q), y = y(p,q) \). Let \( \pi : T^*N \to N \) be a natural projection and \( L_h \) be Lie derivative along vector field \( h = \text{Id}H \) defined by \( HS = (T^*M, \omega^2, H) \).
Definition 8  Mapping \( \phi : T^*M \to T^*N \) is observable in \( LS \) if functions \( \pi \circ \phi \) and \( L_h \pi \circ \phi \) are observable in \( LS \).

If \( \phi : HS \to HS' \) is a factorization of Hamiltonian systems and \( \phi \) is observable in \( LS \) then \( HS' \) is called an observable in \( LS \) factorization of \( HS \).

Also we will need a rather technical definition of a regular point that would allow us to facilitate the proof of the main result later.

Definition 9  A point \((q_0, u_0) \in M \times U\) is called a regular point of \( LS \) if the set \( R = \pi^{-1}(q_0) \times \ker u_0 \Delta \) (in coordinates: \( R = \{(p, q) \in T^*M : q = q_0, u_0 = \hat{u}(p, q_0)\}\) does not contain singular points of \( HS \) and the rank of the set of the functions \( L_h q^i, i = 0, n, s = 0, 1, \ldots \) is constant in the vicinity of each point in \( R \).

This type of regularity will turn out to be quite natural, but we will see that only later in the discussion.

To show that the regular points do exist, consider Lagrangian system corresponding to a linear-quadratic optimization problem. One can show that regular points exist not only for linear-quadratic systems. Also it is possible to somewhat relax requirements for the regular points but for the price of much more technicalities that we would need to deal with. So we presented a simpler but more restricting version of regularity.

The final preparation before formulating the main result of the theory of factorization of Lagrangian systems is the following definition. A Lagrangian system \( LS' \) is a factor system for \( LS' \) iff there exists morphism \( \phi : LS \to LS' \) which is a surjective submersion.

Proposition 4  Let \( LS \) and \( LS' \) be Lagrangian systems. Let \( HS \) and \( HS' \) be corresponding Hamiltonian systems. Then the following two conditions are equivalent:

1. \( LS' \) is factorization of \( LS \)
2. \( HS' \) is observable in \( LS \) factorization of \( HS \)

Proof. \((2) \Rightarrow (1)\). Given a morphism of Hamiltonian systems \( \phi : HS \to HS' \) we will build the corresponding morphism \( \psi : LS \to LS' \), which, by definition, is a mapping \( \psi : M \times U \to N \times V \) that maps extremals of \( LS \) into extremals of \( LS' \).
The first half of the morphism mapping functions can be easily obtained from the observability assumption: \( y = y(p, q) = y(q, \hat{u}(p, q)) \), since \( y(p, q) \) is observable in \( LS \). Hence on the extremals of \( LS \) we have \( y = y(q, u) \).

In coordinates, if \( \gamma(t) = (p(t), q(t)) \) is a solution of \( HS \) then \( \phi(\gamma(t)) = (x(t), y(t)) \) is a solution of \( HS' \) which corresponds to an extremal \( \tilde{\gamma}(t) = (y(t), \tilde{v}(x(t), y(t)) \) in \( LS' \) where \( \tilde{v}(x, y) \) is the optimal synthesis in \( LS' \).

From observability of \( \phi \) we have \( L_h y^i = \tilde{F}^i(q, \hat{u}(p, q)) \) with some functions \( \tilde{F}^i, i = \overline{1, \nu} \). Here \( \hat{u} \) is optimal synthesis in \( LS \).

Since vector fields of \( HS \) and \( HS' \) are \( \phi \)-related, we have equalities: \( \tilde{F}^i(q, \hat{u}) = F^i(q(q, \hat{u}), \tilde{v}), i = \overline{1, \nu} \), which hold on the trajectories of \( HS \). The system \( LS' \) is non-degenerate, thus we can resolve these equations. Indeed, these equations are consistent on the trajectories of \( HS \) with respect to \( \tilde{v} \):

\[
\tilde{v} = v(q, u) \tag{10}
\]

It is easy to see that \( y = y(p, u) \) and \( v = v(q, u) \) are defining a morphism of the Lagrangian systems. If \( (q(t), u(t)) \) is an extremal of \( LS \), then there exists \( p(t) \) such that \( (p(t), q(t)) \) is a solution for \( HS \). Morphism of Hamiltonian systems maps this solution into a solution \( (x(t), y(t)) \) of \( HS' \). This solution defines an extremal \( (y(t), v(t)) \) with an optimal synthesis \( v(t) = \tilde{v}(x(t), y(t)) \). But on the trajectories of \( HS \) holds \( (10) \), so \( \tilde{v}(x(t), y(t)) = v(q(t), \hat{u}(p(t), q(t))) \) on the solutions of \( HS \). This means that the mapping \( v \) gives the same function of time as the optimal synthesis \( \tilde{v} \), so the extremal of \( LS \) was mapped into an extremal of \( LS' \).

(1) \( \Rightarrow \) (2) Given morphism \( \psi : LS \rightarrow LS' \) of Lagrangian systems we need to build morphism \( \phi : T^*M \rightarrow T^*N \) of Hamiltonian systems from \( HS \) into \( HS' \) and show that \( \phi \) is observable in \( LS \).

The first half of the morphism components is obvious: \( y(p, q) = y(q, \hat{u}(p, q)) \) where \( \hat{u} \) is the optimal synthesis and \( y(q, u) \) is the first part of the morphism \( \psi : M \times U \rightarrow N \times V \) of the Lagrangian systems. These functions are obviously observable.

If \( \gamma(t) = (q(t), u(t)) \) is an optimal trajectory in \( LS \) then its image \( \gamma' = (y(q(t), u(t)), v(q(t), u(t))) \) is an extremal, hence an admissible trajectory of \( LS' \). From this follows that on the trajectories of \( HS \) holds

\[
\frac{d}{dt} y(q, \hat{u}) = F(y(q, \hat{u}), v(q, \hat{u}))
\]
Thus the functions $L_h y^i$ are observable. The observability of the morphism is established and from now on we will write for brevity $y = y(p,q)$, $L_h y = \tilde{F}(p,q)$, collapsing the longer expression via $\tilde{u}(p,q)$.

Let $v = \dot{v}(x,y)$ be an optimal synthesis in $LS'$. By assumption the equality $F(y(p,q),\tilde{u}(p,q)) = \tilde{F}(p,q)$ holds whenever $(p(t),q(t))$ is a solution of $HS$. When this is the case, there exists a function $x(t)$ such that together with $y(p(t),q(t))$ it satisfies $HS'$. Under these conditions, our task is to find unknown components $x(p,q)$ of the mapping $\phi : T^*M \to T^*N$ while we know part of it $y(p,q)$ so that $\phi$ will be a morphism from $HS$ to $HS'$. The Lemma from the next section claims that such functions $x(p,q)$ exist. Proving the Lemma will finish the proof of the Proposition.

6 Existence of the Morphism

The previous section left us with a partial mapping of Hamiltonian systems that we need to extend to morphism. It turns out that the Hamiltonian structure is not important for that so we will consider general dynamical systems

$$\dot{x}^j = \xi^j(x), j = \overline{1,m}$$

(11)

and

$$\dot{y}^i = \eta^i(y,z), i = \overline{1,n_1}$$
$$\dot{z}^k = \zeta^k(y,z), k = \overline{1,n_2}$$

(12)
(13)

They define vector fields

$$X = \xi^j \frac{\partial}{\partial x^j}$$

(14)

$$Y = \eta^i(y,z) \frac{\partial}{\partial y^i} + \zeta^k(y,z) \frac{\partial}{\partial z^k}$$

(15)

where $j = \overline{1,m}, i = \overline{1,n_1}, k = \overline{1,n_2}$ and $m \geq n = n_1 + n_2$.
The fields $X$ and $Y$ are defined in $U_1 \in R^m$ and $U_2 \in R^n$ respectively.

We assume that there exists a mapping $y = y(x)$ such that for each trajectory $x(t)$ of the field $X$ there exists a trajectory $(y(t), z(t))$ of the field $Y$ such that $y$ maps $x(t)$ onto the corresponding components of the image. That means that if $x(t)$ is a solution of the system (11) and $y(t) = y(x(t))$ is provided by the mapping $y(x)$, then there exists $z(t)$ such that $(y(x(t)), z(t))$ satisfies the system (12, 13).

As in Definition 9 we will call a point $x_0 \in U_1$ a regular one if $x_0$ is a regular point for the field $X$ and the rank of the set of the functions $L^s_{X} \xi^i(x_0), s = 0, 1, ..., i = 1, m$ is constant in the vicinity of $x_0$.

**Lemma 1** Under the described assumptions in the vicinity of the regular points of $X$ and $Y$ there exists a mapping $z(x)$, not necessarily unique, such that the pair $(y(x), z(x))$ is a morphism of dynamical systems from (11) to (12, 13).

The key observation for proving the Lemma is that the existence of a complete mapping from (11) to (12, 13) is equivalent to the existence of a solution of a PDE system with identical principal part. This kind of systems was explored by V.I.Elkin in [12] and the proof of the Lemma relies on his results.

**Proof.** Since for each solution $x(t)$ of (11) there exists some solution $(y(t), z(t))$ of (12, 13), we can conclude that there exists a mapping $z = z(x)$ such that $F : x \rightarrow (y, z), y = y(x), z = z(x)$ maps initial conditions $x_0$ of the solutions of (11) into initial conditions $(y_0, z_0)$ of the solutions of (12, 13).

At this point we can’t claim yet that $F$ is the morphism we are looking for since we need to show that it will differentiate properly along the field $X$ to map $X$ into $Y$. To show that we can differentiate $y(x)$ part of the mapping along both fields. By the assumption the derivatives of $y$ have to be the same along both fields since $y = y(x)$ maps solutions into a partial solutions. That will result into a system of algebraic equations with respect to $y^i, z^k$:

\[ y^i = y^i(x) \]  \hspace{1cm} \text{(16)}

\[ L^s_x y^i = L^s_{X} y^i(x) \]  \hspace{1cm} \text{(17)}

where $i = 1, n_1, k = 1, n_2$ and $s = 1, 2, ...$. 
The graph of the mapping $F$ is a set $\{(x, y, z) \in U_1 \times U_2 : y = y(x), z = z(x)\}$ contained inside of the manifold $M \subset U_1 \times U_2$, which is defined by the equations (16, 17). The mapping $F$ is defined for all $x \in U_1$, so for each $x$ there exists some solution $(y, z)$ of (16, 17). Thus dependent equations in the system (16, 17) have to be identities with respect to $x$.

Next consider a regular point $(y_0, z_0) = (y(x_0), z(x_0))$ of the field $Y$. In the vicinity of the point $(x_0, y_0, z_0) \in U_1 \times U_2$ the rank $r$ of the set of the functions

$$L_s^i y^i, i = 1, n_1, s = 1, 2, ...$$

is constant.

If the set is of the maximum rank $r = n_1 + n_2$ then by the virtue of the implicit function theorem the equations (16, 17) define a function $z = z(x)$ that together with $y = y(x)$ defines the required morphism of the dynamical systems.

To finish the proof we have to consider the case $r < n_1 + n_2$. The fields $X$, $Y$ and $Z = X + Y$ have no singular points inside the area of consideration. Thus they define a set of the first integrals. For the field $Z$ we will have $n + m - 1$ first integrals which are independent functions in the area. The set of the integrals contain all the integrals $I^\nu(x), \nu = 1, m - 1$ of the field $X$ and some functions $J^\alpha(x, y, z), \alpha = 1, n$, such that:

$$\det \left[ \frac{\partial J^\alpha}{\partial y^i} \bigg| \frac{\partial J^\alpha}{\partial z^j} \right] \neq 0$$

where $i = 1, n_1, k = 1, n_2, n_1 + n_2 = n$.

We can add some of the functions $J^\alpha$ to the set (18) to make it of full rank $n$. Without loosing any generality we can assume that the functions used for that are $J^\beta(x, y, z), \beta = 1, n - r$. Let us add equations

$$J^\beta(x, y, z) = I^\beta(x), \quad \beta = 1, n - r$$

(19)

to the system (16, 17). The combined system (16, 17, 19) defines a manifold $M' \subset M$. If the manifold $M''$ is defined by the equations (19), then $M' = M \cap M''$. It is easy to see that the field $Z$ is tangent to each of the manifolds $M$ and $M''$, thus $Z$ is tangent to $M'$. It is known from [12] that such manifold
defines smooth functions \( y^i = y^i(x) \), \( z^k = z^k(x) \) satisfying a system of PDE with identical principal part:

\[
\begin{align*}
\xi^j \frac{\partial y^i}{\partial x^j} &= \eta^i(y, z) \tag{20} \\
\xi^j \frac{\partial z^k}{\partial x^j} &= \zeta^k(y, z) \tag{21}
\end{align*}
\]

where \( j = 1, m \), \( i = 1, n_1 \), \( k = 1, n_2 \). But this system is exactly equivalent to the condition that the functions \( y = y(x) \), \( z = z(x) \) define a morphism from (11) to (12, 13). This concludes the proof.

Remark. Although the set (18) contains infinite number of functions we can define the rank of this functions set with finite number of differentiations and calculations of determinant. That allows to reduce the system (16, 17) to a finite one that contains only independent equations.

7 Equations of Factorization. Building a Factorsystem.

Let’s denote \( \mathcal{F}_1(\text{LS}) \) (or, in a short form, \( \mathcal{F}_1 \)) the set of the functions on \( T^*M \) observable in \( \text{LS} \) together with its first derivative along \( \text{HS} \). Obviously \( \mathcal{F}_1 \subseteq \mathcal{F}_0 \). Both conditions \( f \in \mathcal{F}_0 \) and \( f \in \mathcal{F}_1 \) can be expressed in terms of differential equations for \( f \).

**Proposition 5** A Lagrangian system \( \text{LS} \) has non-trivial factorization iff there exist functions \( y^i(p, q) \in \mathcal{F}_1, \tilde{Q}(p, q) \in \mathcal{F}_0, x_i(p, q), i = 1, \nu, \nu < n \) such that the functions \( y^i(p, q), x_i(p, q), i = 1, \nu \) are independent and the following equation is satisfied:

\[
L_{\text{Id}H}(x_i dy^i) = d\tilde{Q} \tag{22}
\]

To prepare the proof we will use Cartan formula for Lie derivative to transform the equation (22) into its equivalent form:

\[
d(x_i(y^i, H)_{p,q} - \tilde{Q}) = i_{\text{Id}H}(dx_i \wedge dy^i) \tag{23}
\]
Necessity. Let there exist a factorsystem $LS'$

$$
\int_0^T Q(y, v) dt \rightarrow extr
$$

(24)

$$
\dot{y} = F(y, v)
$$

(25)

where $y \in N, dimN = \nu$ and $v = \hat{v}(x, y)$ is its optimal synthesis. Then the extremals of $LS'$ are described by the equations:

$$
\dot{y}^i = \frac{\partial T}{\partial x_i} \bigg|_{v = \hat{v}(x, y)} = F^i(y, \hat{v}(x, y))
$$

(26)

$$
\dot{x}_i = -\frac{\partial T}{\partial y^i} \bigg|_{v = \hat{v}(x, y)}
$$

(27)

Here $T = x_i F^i(y, v) - \tilde{Q}(y, v)$ is Pontryagin function of the Lagrangian system $LS'$. By assumption we have the equality $F^i(y, \hat{v}(x, y)) = (y^i, H)_{p,q}$. It follows from Proposition (1) that there exists a morphism $x = x(p, q), y = y(p, q)$ of the corresponding Hamiltonian systems. The Hamiltonian of the factor system in coordinates $(p, q)$ is:

$$
\bar{G}(p, q) = x_i(p, q)(y^i(p, q), H)_{p,q} - \tilde{Q}(p, q)
$$

From Proposition (2) we have equality $d\bar{G} = i_{idH}(dx_i \wedge dy^i)$. It immediately leads to the equation (28), if we set: $\bar{Q}(p, q) = Q(y(p, q), \hat{v}(x(p, q), y(p, q)))$.

It can be easily verified that $y^i \in F_1$ and $\tilde{Q} \in F_0$.

Sufficiency. We assume that functions $y^i(p, q) \in F_1, \tilde{Q}(p, q) \in F_0, x_i(p, q), i = 1, \nu, \nu < n$ satisfy equation (22) and that functions $y^i(p, q), x_i(p, q), i = 1, \nu$ are independent. Consider a function

$$
\bar{G} = x_i(p, q)(y^i(p, q), H)_{p,q} - \tilde{Q}(p, q)
$$

and denote $(y^i(p, q), H)_{p,q} = \bar{F}(p, q)$. We will show that $\bar{F} = F \circ \phi$ and $\tilde{Q} = Q \circ \phi$, where $\phi$ is the mapping defined by $(x(p, q), y(p, q))$.

From the condition (23) and Proposition (2) follows that $\bar{G} = G \circ \phi$. Then, according to Proposition (1) on fields $\phi$-relation, we have:
\[(y^i(p, q), H)_{p,q} = \frac{\partial G}{\partial x_i}(x(p, q), y(p, q)) = \mathcal{F}^i(x(p, q), y(p, q)) \quad (28)\]

Hence \(\tilde{F} = F \circ \phi\). Because of that we have \(x_i(p, q)\tilde{F}^i(p, q) = \Theta \circ \phi\), where \(\Theta = \Theta(x, y) = x_i\tilde{F}^i(x, y)\). Finally, from \(\tilde{Q} = x_i(p, q)(y^i(p, q), H)_{p,q} - G = \Theta \circ \phi - G \circ \phi\) we obtain that \(\tilde{Q} = \mathcal{Q} \circ \phi\).

Next we will consider independent functions \(v^k = \hat{v}^k(x, y), k = 1, \mu\) such that \(F^i = F^i(y, \hat{v}(x, y))\). The number \(\mu\) of such functions can be derived from (28):

\[
\text{rank} \left[ \frac{\partial^2 G}{\partial x_i \partial x_j} \right] = \text{rank} \left[ \frac{\partial F^i}{\partial v^k} \frac{\partial \hat{v}^k}{\partial x_j} \right]
\]

where \(i, j = 1, \nu, k = 1, \mu\). Since

\[
\text{rank} \left[ \frac{\partial \hat{v}^k}{\partial x_j} \right] = \mu
\]

then

\[
\text{rank} \left[ \frac{\partial F^i}{\partial v^k} \right] = \mu
\]

Having \(\tilde{F}^i = F^i(y, \hat{v})\) we can show that \(\mathcal{Q}\) depends on \(x\) via \(\hat{v}\): \(\mathcal{Q} = Q(y, \hat{v})\). From the definition of \(\mathcal{Q}\) we see that \(\mathcal{Q} = x_i\tilde{F}^i(y, \hat{v}) - G(x, y)\). Differentiation of \(\mathcal{Q}\) reveals linear dependence of the gradients:

\[
\frac{\partial \mathcal{Q}}{\partial x_j} = F^j(y, \hat{v}) + x_i \frac{\partial F^i}{\partial v^k} \frac{\partial \hat{v}^k}{\partial x_j} - \frac{\partial G}{\partial x_j} = \left( x_i \frac{\partial F^i}{\partial v^k} \right) \frac{\partial \hat{v}^k}{\partial x_j}
\]

Here the last equality follows from (28). From the linear dependence we conclude that \(\mathcal{Q}(x, y) = \mathcal{Q}(y, \hat{v}(x, y))\).

Now we can form a Lagrangian system:

\[
\int_0^T Q(y, v) dt \rightarrow \text{extr} \quad (29)
\]

\[
\dot{y}^i = F^i(y, v), \quad i = 1, \nu \quad (30)
\]
We will show that \( v = \hat{v}(x, y) \) is the optimal synthesis for this system. The Pontryagin function for this system is: 
\[
T(x, y, v) = x_iF^i(y, v) - Q(y, v).
\]
By construction \( T(x, y, \hat{v}(y, v)) = G(x, y) \), thus
\[
\frac{\partial G}{\partial x_j} = \frac{\partial T}{\partial x_j}(x, y, v) \bigg|_{v=\hat{v}(x,y)} + \frac{\partial T}{\partial v^k}(x, y, v) \bigg|_{v=\hat{v}(x,y)} \frac{\partial \hat{v}^k}{\partial x_j}
\]
On the other hand
\[
\frac{\partial T}{\partial v^k}(x, y, v) \bigg|_{v=\hat{v}(x,y)} = \frac{\partial G}{\partial x_j} = F^j(y, \hat{v}(x, y))
\]
Comparing last two equalities we see that:
\[
\frac{\partial T}{\partial v^k}(x, y, v) \bigg|_{v=\hat{v}(x,y)} = 0
\]
Since \( \hat{v}^k \) are independent functions of \( x \), we can satisfy the last equality only if
\[
\frac{\partial T}{\partial v^k}(x, y, v) \bigg|_{v=\hat{v}(x,y)} = 0
\]
But this is exactly the definition of the optimal synthesis as a stationary point of Pontryagin function.

From the observability of \( \tilde{F}(p, q) \) follows observability of the synthesis \( \hat{v}^k(x(p, q), y(p, q)) \). To proof this consider:
\[
\frac{\partial \tilde{F}^j}{\partial p_a} = A^j_{p}(p, q) \frac{\partial \hat{u}^a}{\partial p_a}
\]
with some functions \( A^j_{p}(p, q) \). Their existence follows from the observability of \( \tilde{F}(p, q) \). On the other hand
\[
\frac{\partial \tilde{F}^j}{\partial p_a} = \frac{\partial \tilde{F}^j}{\partial y^i} \frac{\partial y^i}{\partial p_a} + \frac{\partial \tilde{F}^j}{\partial \hat{v}^k} \frac{\partial \hat{v}^k}{\partial p_a} = \frac{\partial \tilde{F}^j}{\partial y^i} B^i_{a}(p, q) \frac{\partial \hat{u}^a}{\partial p_a} + \frac{\partial \tilde{F}^j}{\partial \hat{v}^k} \frac{\partial \hat{v}^k}{\partial p_a}
\]
where again the existence of the functions \( B^i_{a}(p, q) \) follows from the chain rule for differentiation of a compound function and from observability of \( y^i(p, q) \) and their derivatives. Combining all together we see that the gradient \( \frac{\partial \hat{v}^k}{\partial p} \) is linearly expressed via the gradients \( \frac{\partial \hat{u}^a}{\partial p} \) since the matrix \( [\partial F^j/\partial u^a] \) is of the maximum rank. This proves observability of \( \hat{v} \) in \( LS \).
Thus we built a factor system from the solution of (22) and also we built a morphism into the factor system. This concludes the proof.

We will call the equations from the Proposition (5) (and their equivalents (23)) “equations of factorization”.

Note that calculating the outer derivative $d$ on both sides of (23) we get $L_{tAH}(dx^i \wedge dy_i) = 0$ which is in agreement with Proposition 2. Also it is easy to verify that the identity morphism provides a solution for the equations of factorization(23): $x = p, y = q, \tilde{Q} = L$, which transforms these equations into the identity:

$$d(p_i \frac{\partial H}{\partial p_i} - L) = dH$$

Remark. A quite standard note is that systems with explicit dependence on time $t$ can be reduced to the investigated stationary case. We can add a new equation $\dot{t} = 1$ and a new pair of boundary conditions for the new variable $t(0) = 0$ and $t(T) = T$. The only difference of the new variable from the rest is that its boundary conditions are always the same. This does not affect our reasoning which was for the fixed ends case anyway. Finally, a field of extremals remains a local object in this case too.

Example 1. An optimal control system

$$\int_0^T (q_1 u_1 u_2 + q_1 q_2) dt \rightarrow extr$$

$$\dot{q}_1 = u_1, \quad \dot{q}_2 = u_2$$

has a factorization. This was shown by A.N.Chernoplekov in [10]. The system (31, 32) is especially well suited to show that the factorization theory developed here is quite natural and generalizes factorization of variational problems in [10].

The Hamiltonian equations for the optimal control system (31, 32) are:

$$\dot{p}_1 = \frac{p_1 p_2}{q_1^2} + q_2, \quad \dot{q}_1 = \frac{p_2}{q_1}$$

$$\dot{p}_2 = q_1, \quad \dot{q}_2 = \frac{p_1}{q_1}$$

The mapping $(x = 2p_2, y = q_1^2)$ defines a morphism into the Hamiltonian factor system
The Lagrangian system that corresponds to this Hamiltonian system is

\[ \int_{0}^{T} \left( \frac{1}{2} v^2 + \frac{4}{3} y^{3/2} \right) dt \rightarrow \text{extr} \tag{35} \]

\[ \dot{y} = v \tag{36} \]

and the optimal synthesis is \( \dot{v} = x \). The conditions of observability in this case are satisfied automatically. The morphism of the Lagrangian systems in this case is

\[ y = q_1^2, \quad v = 2q_1u_1 \]

Also let’s write down the equations of the factorization and check that they are satisfied:

\[ (2p_2 d(q_1^2)) = d\left( \frac{1}{2}(2p_2)^2 + \frac{4}{3}q_1^3 \right) \]

Here (\( \cdot \)) denotes differentiation \( L_{IdH} \) along the vector field of the original Hamiltonian system \( H \).

The morphism of the Hamiltonian systems in this example is essentially non-symplectic, meaning that there is no coordinate change that will make it symplectic. It is easily follows from the observation that for any two functions \( f_1(q_1,p_2), f_2(q_1,p_2) \) we have \( (f_1, f_2)_{p,q} = 0 \).

It is also possible to verify that the Lagrangian system from this example does not allow any symmetries as defined in [11]. To show that we will search for a vector filed in the form (see proof of Theorem 1 in [11]):

\[ \tilde{\xi} = \xi_1(q_1,q_2) \frac{\partial}{\partial q_1} + \xi_2(q_1,q_2) \frac{\partial}{\partial q_2} + \zeta_1(q_1,q_2,u_1,u_2) \frac{\partial}{\partial u_1} + \zeta_2(q_1,q_2,u_1,u_2) \frac{\partial}{\partial u_2} \]

If \( X \) is vector field defined by dynamical system \( (32) \), then invariance of Lagrangian system with respect to a field \( \tilde{\xi} \) is given in [11] by conditions \( L_{\tilde{\xi}}X = 0 \) (invariance of the vector field \( X \)) and \( L_{\tilde{\xi}}L = 0 \) (invariance of the Lagrangian \( L(q,u) \)).

Expanding these conditions into the system of PDE we obtain:
\[ \begin{align*}
\zeta_1 &= u_1 \frac{\partial \xi_1}{\partial q_1} + u_2 \frac{\partial \xi_1}{\partial q_2} \quad (37) \\
\zeta_2 &= u_1 \frac{\partial \xi_2}{\partial q_1} + u_2 \frac{\partial \xi_2}{\partial q_2} \quad (38) \\
\xi_1(u_1u_2 + q_2) + \xi_2q_1 + q_1u_2\zeta_1 + q_1u_1\zeta_2 &= 0 \quad (39)
\end{align*} \]

We can substitute \( \zeta_i \) into the last equation. Then we can break it into a system of equations by powers of \( u_{1,2} \) since the solution \( \xi \) does not depend on \( u \). The resulting system

\[ \begin{align*}
\xi_1q_2 + \xi_2q_1 &= 0 \\
q_1 \frac{\partial \xi_2}{\partial q_1} &= 0, \quad q_1 \frac{\partial \xi_1}{\partial q_2} = 0 \quad (41) \\
\xi_1 + q_1 \frac{\partial \xi_1}{\partial q_1} + q_1 \frac{\partial \xi_2}{\partial q_2} &= 0 \quad (42)
\end{align*} \]

has only trivial solution.

Thus the Lagrangian system in this example has no symmetries in terms of \( u \) yet it allows order reduction within the introduced category of Lagrangian systems.

**Example 2.** A system

\[ \begin{align*}
\int_0^T (q_1q_2 + \frac{1}{2}u_1^2)dt \to extr \\
\dot{q}_1 &= q_2 - u_1, \quad \dot{q}_2 = q_1 + u_1 \quad (44)
\end{align*} \]

offers another example of order reduction via factorization in the category of Lagrangian systems.

One of the possible strategies of order reduction for this system is to sum up equations of the control system and eliminate \( u_1 \). Then we can integrate the resulting ODE, but that will result in introducing time \( t \) in the right part.

However solving the equations of factorization will allow for more elegant order reduction. The solution that leads to a simpler system is

\[ \begin{align*}
x &= q_1 - q_2 + p_2 - p_1, \quad y = p_2 - p_1, \quad \tilde{Q} = \frac{1}{2}(q_1 - q_2 + p_2 - p_1)^2 - (p_2 - p_1)^2 \quad (45)
\end{align*} \]
The corresponding factor-system is:

\[ \int_0^T \frac{1}{2} (v^2 - y^2) dt \rightarrow extr \]  \hspace{1cm} (45)

\[ \dot{y} = v \]  \hspace{1cm} (46)

with the morphism

\[ y = u_1 \quad v = q_1 - q_2 + u_1 \]

into the factorsystem.

**Example 3.** The so called "horizontal decomposition" (see [16]) can be achieved for the system:

\[ \int_0^T \frac{1}{2} u_2^2 - u_1 u_3 - \frac{1}{4} q_1^2 - \frac{1}{4} q_3^2 dt \rightarrow extr \]  \hspace{1cm} (47)

\[ \dot{q}_1 = u_1, \quad \dot{q}_2 = q_1 + u_2 + u_3, \quad \dot{q}_3 = q_2 - u_2 + u_3 \]  \hspace{1cm} (48)

This system is equivalent to the pair of two independent factor systems:

\[ \int_0^T (y_1 y_2 + \frac{1}{2} v_1^2) dt \rightarrow extr \]  \hspace{1cm} (49)

\[ \dot{y}_1 = y_2, \quad \dot{y}_2 = v_1 \]  \hspace{1cm} (50)

and

\[ \int_0^T \frac{1}{2} (v_3^2 - y_3^2) dt \rightarrow extr \]  \hspace{1cm} (51)

\[ \dot{y}_3 = v_3 \]  \hspace{1cm} (52)

The corresponding morphisms of the Lagrangian systems are:

\[ y_1 = q_1, \quad y_2 = u_1, \quad v_1 = \frac{1}{2} (q_2 + q_3) \]  \hspace{1cm} (53)

\[ y_3 = -u_2 \quad v_3 = \frac{1}{2} (q_2 - q_3) \]  \hspace{1cm} (54)

And the morphisms of their Hamiltonian systems are:
\[ x_1 = p_1, \quad x_2 = \frac{1}{2}(q_2 + q_3), \quad y_1 = q_1, \quad y_2 = -p_2 - p_3 \quad (55) \]
\[ x_3 = \frac{1}{2}(q_2 - q_3), \quad y_3 = p_3 - p_2 \quad (56) \]

8 Boundary Conditions, Constrained Control and Factorization

In this section we will discuss how boundary conditions transform under factorization of Lagrangian systems.

We will say that the Lagrangian system defines an optimal control problem if we specify and fix some boundary conditions allowing to pose a fixed ends boundary conditions problem for the corresponding Hamiltonian system. As an example we will consider conditions \( q(0) = q_0, q(T) = q_1 \). Under the factorization mapping they transform into boundary conditions in factor spaces \( T^*N \) and \( N \times V \). The transformed boundary conditions define some manifolds where the trajectory of the factor system has to start and to end. Depending on the dimension of these manifolds we can end up with over-, well- or under-determined boundary problem for factorsystem.

The over- and well-determined cases are directly useful. If the boundary problem for the original system has a solution it is obvious that the factor problem is also solvable, even if it may appear overdetermined. Such factorization allows for classical hierarchial control when we can solve the factor problem and then extend its solution to the solution of the original problem. The discussion of such well-behaving factorization naturally falls into the framework developed in Y.N.Pavlovskii, V.I.Elkin [16] and will be essentially the same as in the paper [10] by A.N.Chernoplekov for the case of variational systems.

The case of under-determined boundary factor problem is less obvious. Example 4. Consider Lagrangian system

\[ \int_0^T (u_1 u_2 + q_2) dt \to extr \quad (57) \]
\[ \dot{q}_1 = u_1, \quad \dot{q}_2 = u_2 \quad (58) \]

It has factor system
\[ \int_0^T \frac{1}{2} v^2 dt \rightarrow extr \] (59)

\[ \dot{y} = v \] (60)

with the morphism \( y = u_1 u_2, v = u_2 \) into it. The fixed boundary conditions \( q_1(0), q_2(0), q_1(T), q_2(T) \) define a unique trajectory in the original Lagrangian system. But, when mapped into the factorsystem, they don’t provide enough information to build the appropriate boundary conditions for the factor system. In fact they don’t impose any restrictions on the trajectory ends at all since the boundary manifolds coincide with the entire space.

The case of an under-determined factor system does not allow to immediately benefit from knowing the factorsystem. We still can build a hierarchy out of factor- and quotient- systems. Its functioning may rely on a differential game with two players: the center (factorsystem) and subordinate (quotient) system. The goals of functioning of both players are the same but the natural information structure does not allow to achieve the optimum without using some additional interaction between the players. A differential games resulting from such factorization may resemble situations when the center operates in terms that are not directly related to the reality in which the subordinate system has to function, even though they share the same goal. Thus the practical value of such factorization is less obvious.

Note that the discussed here factorization of Lagrangian systems can be easily generalized to an optimal control problem with free end(s). That follows from the locality of the field of extremals. Instead of the boundary conditions \( q(0) = q_0, q(T) = q_1 \) we will have \( q(0) = q_0, p(T) = 0. \) That change of boundary conditions does not affect any of our constructions.

Finally we will briefly touch the case of constrained controls. Since the approach we used here is based on smooth objects, any direct generalization to the constrained case might me problematic. However in many practical cases we can approximate the original constrained optimal control problem with a smooth unconstrained one by introducing smooth penalty functions to represent constraints. Apparently it could be done in many different ways potentially leading to different factorizations or no factorizations at all.
9 Conclusion

In this paper we interpreted order reduction of optimal control systems as factorization in the category of Lagrangian systems. We established sufficient and necessary conditions of factorization for Lagrangian systems. Factorization can be described in terms of the corresponding Hamiltonian systems that appear from Pontryagin’s maximum principle.

Morphisms of Hamiltonian systems that we use in the paper differ from the classical. Our definition does not require the mapping into factor system to be symplectic. That means that we are not necessarily able to extend it to a canonical change of coordinates. Because of that Hamiltonian factor systems does not preserve the original symplectic form on the base space. Instead the symplectic form in the factor space has to be invariant under the flow of the original Hamiltonian system. This invariance of the symplectic form gives sufficient and necessary condition of the factorization of Hamiltonian systems. Naturally, Hamiltonian of the factor system turns out to be first integral of the original Hamiltonian system. Finally, factorization of Hamiltonian system allows to build Lagrangian factor system iff the morphism is an observable mapping. Observability here means that the mapping and its Lie derivative along the original Hamiltonian field depends on the dual variables only via optimal synthesis functions.

We discussed factorization of boundary conditions for Lagrangian systems. The mapping of the fixed boundary conditions under the factorization does not always allow to obtain a well defined boundary condition problem for the factor systems. Also we discussed some of the possible interpretations of that situation from the point of view of differential games.

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References

[1] R. Abraham, J.E. Marsden *Foundations of Mechanics*, Addison-Wesley 1997, 806 pp.

[2] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt *Mathematical Aspects of Classical and Celestial Mechanics* Springer 1989, 291 pp.

[3] V.I. Arnold *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics Vol. 60, Springer Verlag 1989, 516 pp.

[4] G.D. Birkhoff *Dynamical Systems*. AMS Colloq. Publ. v.9, 1927.

[5] G. Blankenstein and A.J. van der Schaft, Optimal control and implicit Hamiltonian systems. In: Nonlinear Control in the Year 2000, vol.1, pp.185-206, (A. Isidori, F. Lamnabhi-Laggarique and W. Respondek, Editors), Springer-Verlag, 2000.

[6] I.A. Borovikov *Factorizatsia v odnom klasse optimalnyh processov*, in: Metody matematicheskogo modelirovania i obrabotki informatsii, MIPT, 1987, Moscow, pp. 121-126.

[7] I.A. Borovikov *Vybor dekompozitsionnoy struktury optimalnyh protsessov*, Ph.D. Thesis, MIPT, 1990, 140 pp.

[8] N. Bourbaki *Theorie Des Ensembles*, Hermann, 1960, 455 pp.

[9] I. Bucur, A. Deleanu, P. J. Hilton *Introduction to the Theory of Categories and Functors*, 1968, 224 pp.

[10] A.N. Chernoplekov *Factorizatsia variatsionnyh system*, Kibernetika i vichislitel'naia tekhnika, Kiev, Naukova Dumka, 1982, No 55, p. 45-51.

[11] A. Echeverría-Enríquez, J. Marín-Solano, M. C. Muñoz-Lecanda, N. Román-Roy *Geometric reduction in optimal control theory with symmetries*, http://arxiv.org/pdf/math-ph/0206036, 2003, 24 pages.

[12] V.I. Elkin *Reduction of Nonlinear Control Systems, A Differential Geometric Approach*, Mathematics and its applications, Vol. 472, Kluwer Academic Press 1999, 248 pp.
[13] V.V.Kozlov *Linear-Quadratic Systems*. Applied mathematics and mechanics, VINITI series, Moscow 1992, vol.56 No. 6, pp. 900-906.

[14] H. Nijmeijer, A.J. van der Schaft, *Nonlinear dynamical control systems*, Springer-Verlag, New-York 1990.

[15] A.J. van der Shaft, *Symmetries in optimal control*, SIAM J. Control and Optimization 25(2) 1987, pp. 245-259.

[16] Y. N. Pavlovskii, V. I. Elkin *Decomposition of models of control processes*, Journal of Mathematical Sciences, Vol. 88, No 5, 1998, pp.723-761

[17] L.S.Pontryagin, V.G.Boltyaniskii, R.V.Gamkrelidze, E.F.Mishchenko, *The Mathematical Theory of Optimal Processes*, Wiley, New York 1962.

[18] H.J. Sussman, *Symmetries and integrals of motion in optimal control*. In *Geometry in nonlinear control and differential inclusions* (Warsaw, 1993), pp. 379-393, Polish Acad. Sci., Warsaw, 1995.

[19] D.F.M. Torres. *Conservation laws in optimal control*. In *Dynamics, Bifurcations and Control*, vol 273 of Lecture Notes in Control and Information Sciences, pp. 287-296, Springer-Verlag, Berlin, Heidelberg, 2002.