The absence of the $4\psi$ divergence in noncommutative chiral models

Maja Burić, Duško Latas and Voja Radovanović

Faculty of Physics, University of Belgrade,
P.O.Box 368, RS-11001 Belgrade, Serbia

Josip Trampetić
Rudjer Bošković Institute, Theoretical Physics Division,
P.O.Box 180 HR-10002 Zagreb, Croatia

(Dated: February 2, 2008)

In this paper we show that in the noncommutative chiral gauge theories the 4-fermion vertices are finite. The $4\psi$-vertices appear in linear order in quantization of the $\theta$-expanded noncommutative gauge theories; in all previously considered models, based on Dirac fermions, the $4\psi$-vertices were divergent and nonrenormalizable.

PACS numbers: 12.38.-t, 12.39.-x, 12.39.Dc, 14.20.-c

I. INTRODUCTION

Although the issue of regularization of quantized field theories was the original motivation to introduce the noncommutativity of coordinates in the fifties [1], the question of renormalizability of field theories on noncommutative Minkowski space is still far from being settled.

In a definition of ‘noncommutative theory’ there are several steps which are not straightforward and need to be specified. The first one is a definition of noncommutative space. By noncommutative Minkowski space one means the algebra of functions on commutative space. The second nonunique step is the very definition of a noncommutative theory and replace the commuting fields with the noncommuting ones and the ordinary products with the Moyal-Weyl ★-product. This product is associative but not commutative. The ★-product of functions integrated in the usual sense has a cyclic property, which is necessary to define the action and with it the noncommutative (NC) generalization of a classical field theory derived from the action principle.

The second nonunique step is the very definition of a field theory. This is because clearly, many theories can have the same commutative limit. In the flat NC space a most straightforward way is to start with a commutative theory and replace the commuting fields with the noncommuting ones and the ordinary products with the ★-products. A noncommutative scalar $\phi^4$ theory and the $U(N)$ gauge theory have been formulated in this manner initially [2, 3]. The most prominent result was that the short and the long distances were related: this was seen through the mixing of ultraviolet and infrared divergences. The UV/IR mixing also obstructs the renormalization. There are other variants of noncommutative scalar theories: as it has been shown recently [2], one can define a renormalizable noncommutative $\phi^4$ theory by modifying the original commutative action by a potential $x^2\phi^2$. There are similar proposals for gauge theory, too [2].

Generalization of the notion of gauge symmetry is also not unique. Initial proposals [3] contain UV/IR mixing as does the scalar field theory. In the most recent models, the symmetry in the ordinary sense is not changed; deformed is the coproduct in the Hopf algebra [4]. These models have not been tested for renormalizability yet.

We shall work in the framework of the $\theta$-expanded gauge theory [5]. The original idea [5] that, in addition to gauge symmetry, nonuniqueness of the Seiberg-Witten map can be used to establish renormalizability proved to be very useful in a couple of models. A rough summary of the obtained results [8, 9, 10, 11] is as follows. In general, divergencies related to the gauge fields are weaker than those for the fermions. When fermions are included an immediate obstacle to renormalizability is found – the so-called ‘$4\psi$’-divergence, which is of the form

$$D_{\text{div}} = \theta^{\mu
u} \epsilon_{\mu\rho\sigma} \langle \bar{\psi} \gamma^\rho \psi \rangle \langle \bar{\psi} \gamma^\sigma \gamma_5 \psi \rangle.$$ (1)

It appears independently of whether fermions are massive or massless. This divergent vertex of the form of Fermi interaction can not be regularized in any well-defined or systematic way.

Recently, some potentially interesting and encouraging results on renormalizability of the $\theta$-expanded theories have been obtained [12, 13]. It was shown that the pure $SU(N)$ noncommutative gauge theory is renormalizable; also, it was possible to define a generalization of the Standard Model (SM) which has the gauge sector free of divergencies. These results point out the importance of the choice of representation for the renormalizability properties. With this in mind and recalling that all previously considered models included only Dirac fermions, we decided to redo the calculation of divergencies for the chiral fermions. As an indicative and most important check we choose to do first the $4\psi$-divergence. The result was: for chiral fermions $4\psi$-divergence is absent! In this paper we present the calculations for the noncommutative $U(1)$ and $SU(2)$ gauge theories.
II. NONCOMMUTATIVE U(1) THEORY

A. Notation

We will work in the noncommutative Minkowski space, defined by the relation

\[ [x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \tag{2} \]

The commutator in (2) is a \( \star \)-commutator given by the Moyal-Weyl product,

\[ f(x) \star g(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu \bar{\theta}^\nu} f(x) g(y) \big|_{y \rightarrow x}. \tag{3} \]

The action for the left chiral fermion \( \varphi \) interacting with the U(1) gauge field \( A_\mu \) is, in commutative theory, given by

\[ S_C = \int d^4x \mathcal{L}_C = \int d^4x \left( i\bar{\varphi} \partial_\mu (\partial^\mu + iA_\mu) \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \tag{4} \]

The noncommutative U(1) symmetry can be realized by the U(1) gauge field \( A_\mu \), the NC field strength by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [\hat{A}_\mu, \hat{A}_\nu] \), and the NC Weyl spinor by \( \hat{\varphi} \). The noncommutative U(1) symmetry is of course nonabelian; it can however be related to the usual abelian U(1) symmetry by the Seiberg-Witten map \[14\], which gives the basic, the NC fields as expansions in their commutative approximations. The SW map to first order in \( \theta^{\mu\nu} \) reads \[15\]

\[ \hat{A}_\mu = A_\mu - \frac{1}{4} \theta^{\mu\nu} \{ A_\mu, \partial_\nu A_\rho + F_{\nu\rho} \} + \ldots, \tag{5} \]

\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2} \theta^{\mu\nu} \{ F_{\mu\nu}, F_{\rho\sigma} \} - \frac{1}{4} \theta^{\mu\nu} \{ A_\mu, \partial_\nu D_\sigma - D_\nu A_\sigma \} + \ldots, \tag{6} \]

\[ \hat{\varphi} = \varphi - \frac{1}{2} \theta^{\mu\nu} A_\mu \partial_\nu \varphi + \frac{i}{4} \theta^{\mu\nu} A_\mu A_\nu \varphi + \ldots. \tag{7} \]

The \( \{ , \} \) denotes the anticommutator and \( D_\mu \) is the commutative covariant derivative. The fields \( \varphi \) and strengths \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) in (5) carry representations of the commutative U(1) symmetry.

The action for noncommutative chiral electrodynamics is analogous to (6):

\[ S_{NC} = \int d^4x \mathcal{L}_{NC} = \int d^4x \left( i\bar{\varphi} \partial_\mu (\partial^\mu + i\hat{A}_\mu) \varphi - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right). \tag{8} \]

Expanding to first order in \( \theta^{\mu\nu} \) we obtain

\[ \mathcal{L}_{NC} = \mathcal{L}_0 + \mathcal{L}_{1,A} + \mathcal{L}_{1,\varphi}, \tag{9} \]

where

\[ \mathcal{L}_0 = \mathcal{L}_C = i\bar{\varphi} \sigma^\mu (\partial_\mu + iA_\mu) \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{10} \]

\[ \mathcal{L}_{1,A} = -\frac{1}{2} \theta^{\mu\nu} \left( F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \right), \tag{11} \]

\[ \mathcal{L}_{1,\varphi} = -\frac{1}{8} \theta^{\mu\nu} \Delta_{\mu\nu}^{\alpha\beta\gamma} F_{\alpha\beta} \bar{\varphi} \sigma^\rho (\partial_\gamma + iA_\gamma) \varphi. \tag{12} \]

The antisymmetric \( \Delta \) is defined by \( \Delta_{\mu\nu}^{\alpha\beta\gamma} = -\epsilon^{\alpha\beta\gamma\delta} \delta_{\mu\nu}\delta_{\rho\sigma} \). For nonabelian theories formulae (10) and (11) contain an additional trace in the group generators as we will see in the special case of SU(2) later.

Obviously the parameter \( \theta^{\mu\nu} \) of dimension \( (\text{length})^2 \) is small – of order of magnitude \( \lesssim (\text{TeV})^{-2} \) \[16\], and therefore the expansion (9) is useful to compute the almost-classical effects of noncommutativity. Its relevance in considerations of renormalizability is not quite clear since the divergent contributions can be nonperturbative in \( \theta^{\mu\nu} \); indeed this is what happens with the UV/IR mixing. Nevertheless we shall work with the truncated expression (9) for two reasons: first, an expansion like this might be a viable or a correct way to define a renormalizable theory. Second, we presume that an additional structure given by noncommutative Ward identities exists: it then relates the \( n \)-point functions of different orders in the \( \theta \)-expansion. Thus it is possible, in principle, to use NC Ward identities in order to lift renormalizability from \( \theta \)-linear to higher-\( \theta \) orders. In any case, if the theory is not renormalizable it will indeed show up in linear order, which is also a result of relevance.

B. Quantization

We start with the action (9) for noncommutative chiral electrodynamics and quantize it by using the path-integral method. We treat the \( \theta \)-dependent terms as interactions, the parameter \( \theta^{\mu\nu} \) as a coupling constant. The propagators for the spinors and for the gauge fields are the same as in the commutative theory. In order to compute the functional integral we need either to complexify the gauge potential or to work with the Majorana spinors; we choose the latter. Using

\[ \psi = \left( \begin{array} {c} \varphi \\ \bar{\varphi} \end{array} \right), \tag{13} \]

we can rewrite the commutative part of the Lagrangian (9) as

\[ \mathcal{L}_0 = \frac{1}{2} \bar{\psi} \gamma^\mu (\partial_\mu - i\gamma_5 A_\mu) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{14} \]
To obtain (13) from (10) we use the $\gamma$-matrices in the chiral representation; further details of the notation are given in the Appendix. Written in terms of the Majorana spinors the U(1) symmetry becomes axial; this is no surprise as chiral lagrangian is not invariant under parity. For the $\theta$-linear spinor part of the Lagrangian we obtain

$$\mathcal{L}_{1,\psi} = -\frac{1}{16} g_{\mu\nu} \Delta^{\alpha\beta\gamma}_{\mu\nu\rho} F_{\alpha\beta} \bar{\psi} \gamma^\rho (\partial_\gamma - i \gamma_5 A_\gamma) \psi,$$  \hspace{1cm} (15)$$

while the gauge self-interaction is given by (11).

In order to preserve gauge covariance we integrate using the background field method [10, 11]. Briefly: in the first step we expand fields around their classical configurations; we replace therefore in the classical action

$$S[A_\mu, \psi] = S[A_\mu, \psi] + \frac{1}{2} S\text{Tr} (\log B[A_\mu, \psi]).$$  \hspace{1cm} (17)$$

The first term $S[A_\mu, \psi]$ is the classical action and the second is the one-loop quantum correction $\Gamma_1$. The operator $B[A_\mu, \psi]$, a result of Gaussian integration, is the second functional derivative of $S[A_\mu, \psi]$; it can be obtained in our case by expanding $S[A_\mu + A_\mu, \psi + \Psi]$ to second order in $A_\mu$ and $\Psi$

$$S^{(2)} = \int d^4x \left( A_\mu \gamma_5 \Psi \right) B \left( A_\mu \gamma_5 \right).$$  \hspace{1cm} (18)$$

We can divide $S^{(2)}$ into its commutative part $B_0$ and a $\theta$-linear contribution $B_1$: $B = B_0 + B_1$. $B_0$, after the gauge fixing, is given by

$$B_0 = \frac{1}{2} \left( \frac{g^{\alpha\lambda}}{\gamma_5^\alpha \psi} i \theta + A_\gamma \right).$$  \hspace{1cm} (19)$$

It contains the kinetic part

$$B_{\text{kin}} = \frac{1}{2} \left( \begin{array}{cc} g^{\alpha\lambda} & 0 \\ 0 & 0 \end{array} \right),$$

and the interaction. In order to expand the logarithm in (17) around identity we have to multiply $B$ by $C$,

$$C = 2 \left( \begin{array}{cc} g^{\alpha\lambda} & 0 \\ 0 & -i \theta \end{array} \right).$$  \hspace{1cm} (20)$$

Then we can write

$$B C = \Box I + N_1 + T_1 + T_2,$$  \hspace{1cm} (21)$$

with

$$I = \left( \begin{array}{cc} g^{\alpha\lambda} & 0 \\ 0 & 0 \end{array} \right).$$  \hspace{1cm} (22)$$

The expression

$$\Gamma_1 = \frac{i}{2} \text{Str} (I + \Box^{-1} N_1 + \Box^{-1} T_1 + \Box^{-1} T_2)$$

$$= \frac{i}{2} \sum \left( \frac{(-1)^n+1}{n} \right) \text{Str} \left( \Box^{-1} N_1 + \Box^{-1} T_1 \right. + \Box^{-1} T_2)^n,$$

(23)

gives the perturbation expansion. $\Gamma_1$ can be identified with the one-loop effective action because

$$\text{Str} (\log B) = \text{Str} (\log \Box^{-1} B C) - \text{Str} (C \Box^{-1}).$$  \hspace{1cm} (24)$$

As the last term does not depend on the fields $A_\mu$ and $\psi$ it can be included in the (infinite) normalization.

In (21) we have divided the interaction term in three parts in the following way. Operator $N_1$ contains the commutative 3-vertices, i.e. the terms with one classical and two quantum fields. Analogously, operator $T_1$ is a term linear in $g^{\mu\nu}$ containing one classical field, and $T_2$ is linear in $g^{\mu\nu}$ containing two classical fields. From (19) we see that $N_1$ equals to

$$N_1 = \left( \begin{array}{cc} 0 & -i \bar{\psi} \gamma_5 \psi \\ \gamma_5 \psi & \gamma_5 \psi \end{array} \right).$$  \hspace{1cm} (25)$$

The noncommutative vertices $T_1$ and $T_2$ require a bit more work. Using the Majorana spinor-identities we obtain

$$T_1 = \frac{1}{8} g^{\mu\nu} \Delta^{\alpha\beta\gamma}_{\mu\nu\rho} \left( V_{\alpha\beta\gamma \rho}^{\rho\kappa\lambda} - 2 \delta_{\rho}^{\alpha\beta\gamma} (\partial_\rho \bar{\psi}) \gamma_\rho \partial_\gamma \theta - 2 \delta_{\rho}^{\alpha\beta\gamma} (\partial_\rho \bar{\psi}) \gamma_\rho \partial_\gamma \theta - F_{\alpha\beta\gamma} \gamma_\rho \partial_\gamma \theta \right),$$  \hspace{1cm} (26)$$

and

$$T_2 = \frac{1}{8} g^{\mu\nu} \Delta^{\alpha\beta\gamma}_{\mu\nu\rho} \left( \delta_{\alpha}^{\alpha\beta\gamma} (2 \bar{\psi} \gamma_\rho \gamma_5 \psi \partial_\gamma + (\partial_\gamma \bar{\psi} \gamma_\rho \gamma_5 \psi)) - 2 \delta_{\rho}^{\alpha\beta\gamma} (\bar{\psi} \gamma_\rho \gamma_5 A_\gamma + \bar{\psi} \gamma_\rho \gamma_5 F_{\alpha\beta\gamma} - \bar{\psi} \gamma_\rho \gamma_5 A_\gamma \partial_\gamma \theta) \right),$$  \hspace{1cm} (27)$$

with the bosonic part $V_{\alpha\beta\gamma}^{\rho\kappa\lambda}$ given in [10]. As we are looking just the four-fermion divergence, $V_{\alpha\beta\gamma}^{\rho\kappa\lambda}$ will not
contribute to our calculations as we shall shortly explain.

C. Divergencies

We have noted, the full perturbation expansion is given in \[24\]. For example: the 4-point functions are contained in terms which have the sum of the operator-indices equal to 4. If in addition we look only for the contributions linear in \( \theta^{\mu \nu} \), the relevant expressions can have at most one of the \( T_1 \) or \( T_2 \). There are two such terms,

\[
D_1 = \text{STr} \left( (\Box^{-1} N_1)^3 (\Box^{-1} T_1) \right),
\]

and

\[
D_2 = \text{STr} \left( (\Box^{-1} N_1)^2 (\Box^{-1} T_2) \right).
\]

As here we restrict to the 4-fermion vertex, we can further simplify the calculation by putting \( A_{\mu} = 0 \). Under this condition we also have \( V_{\alpha \beta \gamma}^{\mu \nu} = 0 \).

In order to find the divergencies we write the traces in the momentum representation and afterwards perform the dimensional regularization. The result which we obtain is: the term \( D_1 \) is finite. For the divergent part of \( D_2 \) we get

\[ D_2|_{\text{div}} = \frac{1}{(4\pi)^2} \frac{3i}{8} \epsilon_{\mu \nu \rho \sigma} \theta^{\mu \nu} (\bar{\psi} \gamma^\rho \gamma^5 \psi) (\bar{\psi} \gamma^\sigma \gamma^5 \psi). \]

However the last expression vanishes too, due to the antisymmetry of the Levi-Civita symbol. In fact, in retrospect, it is easy to see that the 4-\( \psi \)-divergence has to be zero in the chiral case: because of antisymmetry of \( \epsilon^{\mu \nu \rho \sigma} \) the only possible expression is \( I \). On the other hand for the Majorana spinors \( \bar{\psi} \gamma^\mu \psi \equiv 0 \) and therefore \( I \) vanishes identically.

III. NONCOMMUTATIVE SU(2) THEORY

A. Lagrangian

We will now do an analogous analysis for the chiral fermions in the fundamental representation of SU(2). We start with a doublet of fermions and a vector potential:

\[
\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad A_\mu = A_{\mu}^a \frac{\sigma_a}{2},
\]

and the following commutative Lagrangian

\[
\mathcal{L}_0 = \mathcal{L}_{0,\psi} + \mathcal{L}_{0,A}
\]

\[
= i \bar{\varphi} \tilde{\sigma}^\mu (\partial_\mu + iA_\mu) \varphi - \frac{1}{4} \text{Tr} F_{\mu \nu} F^{\mu \nu},
\]

which we want to rewrite in terms of the Majorana spinors

\[
\psi_1 = \begin{pmatrix} \varphi_1 \\ \bar{\varphi}_1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \varphi_2 \\ \bar{\varphi}_2 \end{pmatrix}.
\]

As the fundamental representation of SU(2) is not real, when we write the Lagrangian in the Majorana spinors we apparently break the SU(2) symmetry, i.e. we have to write each component of the vector potential separately. That is

\[
\mathcal{L}_{0,\psi} = i \bar{\psi}_1 \tilde{\sigma}^\mu (\partial_\mu + \frac{i}{2} A_{\mu}^a) \varphi_1 - \frac{1}{2} \bar{\psi}_1 \tilde{\sigma}^\mu A_{\mu}^a \varphi_2
\]

\[
- \frac{1}{2} \bar{\varphi}_2 \tilde{\sigma}^\mu A_{\mu}^a \varphi_1 + i \bar{\varphi}_2 \tilde{\sigma}^\mu (\partial_\mu - \frac{i}{2} A_{\mu}^a) \varphi_2
\]

\[ = \frac{1}{4} (\bar{\psi}_1 \bar{\psi}_2) \left( 2i \dot{\varphi} + A_{\mu}^3 \gamma_5 A_{\mu}^1 \gamma_5 + iA_{\mu}^2 \right) \psi_1 \psi_2.
\]

We have denoted \( A_{\mu}^\pm = A_{\mu}^1 \pm i A_{\mu}^2 \). Now of course \( \psi_1 \psi_2 \) is not a SU(2) doublet.

The \( \theta \)-linear bosonic part of the SU(2) Lagrangian

\[
\mathcal{L}_{1,A} = -\frac{1}{2} \theta^{\rho \sigma} \text{Tr} (F_{\mu \rho} F_{\nu \sigma} - \frac{1}{4} F_{\rho \sigma} F_{\mu \nu}) F^{\mu \nu}
\]

\[ = -\frac{1}{2} d^{abc} \theta^{\rho \sigma} (f_{\mu \rho} f_{\nu \sigma} - \frac{1}{4} f_{\rho \sigma} f_{\mu \nu}) f^{\mu \nu},
\]

vanishes, because it is proportional to the symmetric coefficients \( d^{abc} \)

\[ d^{abc} \sim \text{Tr} (\sigma^a [\sigma^b, \sigma^c]) = 0. \]

In fact, \( d^{abc} = 0 \) for all irreducible representations of SU(2). On the other hand, the fermionic linear part of the Lagrangian,

\[
\mathcal{L}_{1,\psi} = \frac{1}{32} \theta^{\mu \nu} \Delta^a_{\rho \sigma} \bar{\psi} \tilde{\sigma}^\rho \left( 2i F_{\alpha \beta}^a \gamma^\sigma \partial_\gamma + A_{\alpha}^a F_{\beta}^a - i d^{abc} A_{\alpha}^a F_{\beta}^b \gamma^c \right) \varphi,
\]

in the Majorana representation is rewritten as

\[
\mathcal{L}_{1,\psi} = \frac{1}{64} \theta^{\mu \nu} \Delta^a_{\rho \sigma} \bar{\psi} \tilde{\sigma}^\rho \left( 2i F_{\alpha \beta}^3 \gamma^\sigma \partial_\gamma + A_{\alpha}^3 F_{\beta}^3 - 2 F_{\alpha \beta}^3 \gamma^\rho \gamma_5 \gamma^\sigma \partial_\gamma \right) \psi_1 \psi_2.
\]
B. Quantization and divergencies

As we have written the Lagrangian in an appropriate form, the procedure of quantization is straightforward and follows closely that which was done for the U(1). It is interesting that the results, as we shall see shortly, are completely analogous, though the intermediate calculations are now considerably more involved. The part of the action which is of second order in quantum fields we write as

\[ S^{(2)} = \int \mathcal{A}^3 \left( A_\alpha^I, A_\eta^I, A_\xi^\beta, \bar{\Psi}_1, \bar{\Psi}_2 \right) B \left( A_\alpha^I, A_\eta^I, A_\xi^\beta, \bar{\Psi}_1, \bar{\Psi}_2 \right). \]  

(33)

The interactions of course differ: now we have the 4-boson vertex in the commutative part for example, etc. The one-loop effective action has the form

\[ \Gamma_1 = \frac{i}{2} \text{Tr} \log (I + \square^{-1}(N_1 + N_2 + T_1 + T_2 + T_3)). \]  

(34)

As the formulae are cumbersome we shall here restrict immediately to the subset of fermionic diagrams defined by the condition \( A^\mu_\eta = 0 \), which of course simplifies the calculations. With this restriction we also have \( N_2 = 0 \) and \( T_3 = 0 \). The remaining interaction vertex is

\[
B_{\text{kin}} = \frac{1}{2} \begin{pmatrix}
\begin{bmatrix}
\square \delta \xi' \\
\square \delta \eta' \\
\square \delta \epsilon' \\
\end{bmatrix} & 0 \\
0 & i \phi \\
\end{pmatrix} = \frac{1}{2} \text{diag} \left( \square \delta \xi', \square \delta \eta', \square \delta \epsilon', i \phi, i \phi \right),
\]

\[
C = 2 \begin{pmatrix}
\delta \xi' \xi' & \delta \eta' \eta' & 0 \\
0 & -i \phi & -i \phi \\
\end{pmatrix} = 2 \text{diag} \left( \delta \xi' \xi', \delta \eta' \eta', -i \phi, -i \phi \right).
\]

In \( \theta \)-linear order we have the following \( T_1 \) matrix:

\[
T_1 = -\frac{1}{8} \theta^{\mu \nu} \Delta^{\alpha \beta \gamma}_{\mu \nu \rho} \begin{pmatrix}
0 & -i \bar{\psi}_2 \gamma^5 \gamma_5 \phi \theta & -i \psi_1 \gamma^5 \gamma_5 \phi \theta \\
-\bar{\psi}_2 \gamma^5 \gamma_5 \phi \theta & \psi_1 \gamma^5 \gamma_5 \phi \theta & -i \psi_1 \gamma^5 \gamma_5 \phi \theta \\
0 & -i \bar{\psi}_2 \gamma^5 \gamma_5 \phi \theta & \psi_1 \gamma^5 \gamma_5 \phi \theta \\
\end{pmatrix}
\]

(35)

while, \( T_2 \) matrix is given by

\[
T_2 = \frac{1}{32} \theta^{\mu \nu} \Delta^{\alpha \beta \gamma}_{\mu \nu \rho} \begin{pmatrix}
2 \delta \xi^\alpha_{\rho \gamma} E^\rho & \delta \eta^\eta_{\rho \gamma} F^\rho & -\delta \xi^\xi_{\rho \gamma} G^\rho \\
-\delta \xi^\alpha_{\rho \gamma} F^\rho & 2 \delta \eta^\eta_{\rho \gamma} E^\rho & \delta \xi^\xi_{\rho \gamma} H^\rho \\
-\delta \xi^\alpha_{\rho \gamma} G^\rho & \delta \eta^\eta_{\rho \gamma} H^\rho & 0 \\
\end{pmatrix},
\]

(36)

where

\[
E^\rho = \bar{\psi}_1 \gamma^\rho \gamma_5 \psi_1 + \bar{\psi}_2 \gamma^\rho \gamma_5 \psi_2,
\]

(37)

\[
F^\rho_\gamma = -i (\partial_\gamma \bar{\psi}_1) \gamma^\rho \psi_1 + i \bar{\psi}_1 \gamma^\rho (\partial_\gamma \psi_1) + i (\partial_\gamma \bar{\psi}_2) \gamma^\rho \psi_2 - i \bar{\psi}_2 \gamma^\rho (\partial_\gamma \psi_2),
\]

(38)
\[ G_\gamma^\mu = (\partial_\gamma \overline{\psi}_1) \gamma^\rho \gamma_5 \overline{\psi}_2 - \overline{\psi}_1 \gamma^\rho \gamma_5 (\partial_\gamma \overline{\psi}_2) \]
\[ - (\partial_\gamma \overline{\psi}_2) \gamma^\rho \gamma_5 \overline{\psi}_1 + \overline{\psi}_2 \gamma^\rho \gamma_5 (\partial_\gamma \overline{\psi}_1), \quad (39) \]

\[ H_\gamma^\mu = -i(\partial_\gamma \overline{\psi}_1) \gamma^\rho \psi_2 + i\overline{\psi}_1 \gamma^\rho (\partial_\gamma \psi_2) \]
\[ -i(\partial_\gamma \overline{\psi}_2) \gamma^\rho \psi_1 + i\overline{\psi}_2 \gamma^\rho (\partial_\gamma \psi_1). \quad (40) \]

As before, the divergent contributions in principle come from \( D_1 \) and \( D_2 \). However, \( D_1 \) is finite, while for the divergent part of \( D_2 \) we obtain
\[ D_2|_{\text{div}} = -\frac{1}{(4\pi)^2} \frac{9i}{64} \theta^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} (\overline{\psi}_1 \gamma^\rho \gamma_5 \psi_1 + \overline{\psi}_2 \gamma^\rho \gamma_5 \psi_2) \]
\[ \times (\overline{\psi}_1 \gamma^\sigma \gamma_5 \psi_1 + \overline{\psi}_2 \gamma^\sigma \gamma_5 \psi_2), \quad (41) \]

which identically vanishes, too.

\section*{IV. CONCLUSIONS}

When one thinks about the quantization of the chiral models, the first question which naturally arises is the one about anomalies. The issue of chiral anomalies for the \( \theta \)-expanded models has been analyzed in details in [17], and the result was that, for the compact gauge groups, anomalies are the same as in the commutative theory. This, for example, means that the noncommutative chiral electrodynamics which we analyzed in Section 2 cannot be quantized consistently. But on the other hand, it also means that we can build the particle physics models as in the ordinary theory, for example the noncommutative chiral \( U(1) \times SU(2) \) gauge theory is consistent if lepton and quark multiplets are the same as in the Standard Model. Our present result asserts that in addition such model has no four-fermion divergencies.

Construction of a consistent noncommutative standard model (NCMS is the main motivation of our investigation. We have previously proposed a model which is renormalizable in the gauge sector [13, 16] and the result opens a possibility to extend it. Of course the Higgs sector should also be investigated [15]. There is a number of phenomenological predictions of the NCMS models [16, 21, 22]; they would become more robust if one could prove the one-loop renormalizability.

Obviously, there is a long way to go to show the full renormalizability of the NC chiral gauge models: what we have done here is just an initial step. As from the one-loop renormalizability no immediate conclusions can be made about the all-loop properties, likewise from the renormalizability in \( \theta \)-linear order nothing automatically follows for the full SW expansion. There are many steps to be done: some, as extension from linear to higher orders in \( \theta^{\mu\nu} \), we just see as viable possibilities. Other, like the analysis of all one-loop divergent vertices in the \( \theta \) linear order, are straightforward and require additional work, and this is what we plan to do in our following work. One should remember that in the \( \theta \)-expanded theories one has an additional tool for renormalizability, the SW field redefinition. Note also that the renormalizability principle could help to minimize or even cancel most of the ambiguities of the higher order SW maps [23].

If indeed the \( \theta \)-linear order of the chiral gauge models proves to be renormalizable, then it will really be important to analyze the noncommutative Ward identities and their implications to renormalizability more systematically.

\section*{Acknowledgments}

The work of M. B., V. R. and D. L. is done within the project 141036 of the Serbian Ministry of Science. The work of J. T. is supported by the project 098-0982930-2900 of the Croatian Ministry of Science Education and Sports. Our collaboration was partly supported by the UNESCO project 875.834.6 through the SEENET-MTP and by ESF in the framework of the Research Networking Programme on 'Quantum Geometry and Quantum Gravity'.

\section*{APPENDIX A: CONVENTIONS}

The notation and the rules of chiral-spinor algebra follow basically [24]. We use the following chiral representation of the \( \gamma \)-matrices
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (A1) \]
with
\[ \sigma^\mu = (1, \bar{\sigma}), \quad \bar{\sigma}^\mu = (1, -\sigma). \quad (A2) \]
This means in particular
\[ \bar{\sigma}^{\mu\alpha} = \epsilon^{\alpha\beta} \epsilon^{\sigma\delta} \sigma^\mu_{\beta\delta}. \]

The chiral \( \psi, \chi \) spinors multiply as
\[ \varphi \chi = \chi \varphi, \quad \varphi \bar{\chi} = \bar{\chi} \bar{\varphi}, \quad (A3) \]
\[ \bar{\varphi} \bar{\sigma}^\mu \chi = -\chi \sigma^\mu \bar{\varphi}, \quad (\chi \sigma^\mu \bar{\varphi})^\dagger = \varphi \sigma^\mu \bar{\chi}. \]

Those relations, as can be seen easily, give the usual identities for the Majorana spinors \( \phi, \psi \) which we use
\[ \bar{\phi} \psi = \bar{\psi} \phi, \quad \bar{\sigma}_5 \bar{\gamma}_5 \psi = \bar{\gamma}_5 \bar{\psi} \phi, \]
\[ \bar{\phi} \gamma^\mu \psi = -\bar{\psi} \gamma^\mu \phi, \quad \bar{\sigma}^\mu \gamma^\mu \psi = \bar{\psi} \gamma^\mu \gamma_5 \phi. \]

Majorana Lagrangians are obtained from the corresponding chiral ones using the identities [A3] and the fact that Lagrangians are real.
[1] H. S. Snyder, Phys. Rev. 71 (1947) 38; H. S. Snyder, Phys. Rev. 72 (1947) 68.
[2] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002 (2000) 020 [arXiv:hep-th/9912072]; I. Chepelev and R. Roiban, JHEP 0005 (2000) 037 [arXiv:hep-th/9911098].
[3] A. Mátéusis, L. Susskind and N. Toumbas, JHEP 0002 (2000) 020 [arXiv:hep-th/9912072]; M. Hayakawa, Phys. Lett. B 478 (2000) 394 [arXiv:hep-th/9912094]; M. Van Raamsdonk, JHEP 0111 (2001) 006 [arXiv:hep-th/0110093]; C. P. Martin and D. Sanchez-Ruiz, Phys. Rev. Lett. 83 (1999) 476 [arXiv:hep-th/9903077].
[4] H. Grosse and R. Wulkenhaar, Commun. Math. Phys. 256 (2005) 305 [arXiv:hep-th/0401128]; H. Grosse and H. Steinacker, Nucl. Phys. B 746, 202 (2006) [arXiv:hep-th/0512029].
[5] H. Grosse and M. Wohlgenannt, Eur. Phys. J. C 52, 435 (2007) [arXiv:hep-th/0703169].
[6] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml and J. Wess, Lett. Math. Phys. 78, 61 (2006) [arXiv:hep-th/0603024].
[7] J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 16 (2000) 161 [hep-th/0001203]; B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 21 (2001) 383 [hep-th/0104153].
[8] R. Wulkenhaar, JHEP 0203 (2002) 024 [arXiv:hep-th/0112248].
[9] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, JHEP 06 (2001) 013 [hep-th/0104097].
[10] M. Buric and V. Radovanovic, JHEP 0210 (2002) 074 [arXiv:hep-th/0208204].
[11] M. Buric and V. Radovanovic, JHEP 0402 (2004) 040 [arXiv:hep-th/040103]; M. Buric and V. Radovanovic, Class. Quant. Grav. 22 (2005) 525 [arXiv:hep-th/0410085].
[12] M. Buric, D. Latas and V. Radovanovic, JHEP 0602 (2006) 046 [arXiv:hep-th/0510133]; D. Latas, V. Radovanovic and J. Trampetic, Phys. Rev. D 76 (2007) 085006.
[13] M. Buric, V. Radovanovic and J. Trampetic, JHEP 0703, 030 (2007) [arXiv:hep-th/0609073].
[14] N. Seiberg and E. Witten, JHEP 09 (1999) 032 [hep-th/9908142].
[15] P. Schupp, J. Trampetic, J. Wess and G. Raffelt, Eur. Phys. J. C 36 (2004) 405 [hep-ph/0212292].
[16] M. Buric, D. Latas, V. Radovanovic and J. Trampetic, Phys. Rev. D 75 (2007) 097701.
[17] C. P. Martin, Nucl. Phys. B 652, 72 (2003) [arXiv:hep-th/0211614]; F. Brandt, C.P. Martin and F. Ruiz Ruiz, JHEP 07 (2003) 068 [hep-th/0307292].
[18] C. P. Martin, D. Sanchez-Ruiz and C. Tamarit, JHEP 0702 (2007) 065; C. P. Martin and C. Tamarit, arXiv:0706.4052 [hep-th].
[19] W. Behr, N.G. Deshpande, G. Duplančić, P. Schupp, J. Trampetić and J. Wess, Eur. Phys. J. C 29 (2003) 441 [hep-ph/0202121]; G. Duplančić, P. Schupp and J. Trampetić, Eur. Phys. J. C 32 (2003) 141 [hep-ph/0309138].
[20] B. Melić, K. Pasek-Kumericki, J. Trampetić, P. Schupp and M. Wohlgenannt, Eur. Phys. J. C 42 (2005) 483, [arXiv:hep-ph/0502249]; ibid 499, [arXiv:hep-ph/0503064].
[21] J. Trampetić, Acta Phys. Polon. B 33 (2002) 4317 [hep-ph/0212309]; P. Minkowski, P. Schupp and J. Trampetić, Eur. Phys. J. C 37 (2004) 123; B. Melić, K. Pasek-Kumericki and J. Trampetić, Phys. Rev. D 72 (2005) 054004; ibid 057502.
[22] T. Ohl and J. Reuter, Phys. Rev. D70 (2004); A. Alboteanu, T. Ohl and R. Rückl, PoS HEP2005 (2006) 322 [arXiv:hep-ph/0511188]; A. Alboteanu, T. Ohl and R. Rückl, Phys. Rev. D 74, 096004 (2006);
[23] L. Möller, JHEP 10 (2004) 063; A. Alboteanu, T. Ohl and R. Rückl, 0707.3505[hep-th]; Josip Trampetić and Michael Wohlgenannt 0710.2182[hep-th].
[24] M. E. Peskin and D. V. Schroeder, “An Introduction To Quantum Field Theory,” Reading, USA: Addison-Wesley (1995) ; J. Wess and J. Bagger, “Supersymmetry and supergravity,” Princeton, USA: Univ. Pr. (1992)