NONZERO POSITIVE SOLUTIONS OF FRACTIONAL LAPLACIAN SYSTEMS WITH FUNCTIONAL TERMS

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Abstract. We study the existence of non-zero positive solutions of a class of systems of differential equations driven by fractional powers of the Laplacian. Our approach is based on the notion of fixed point index, and allows us to deal with non-local functional weights and functional boundary conditions. We present two examples to shed light on the type of functionals and growth conditions that can be considered with our approach.

1. Introduction and preliminaries

The main aim of the present paper is to establish some existence and non-existence results for ‘functional’ Dirichlet problems driven by fractional powers of the classical Laplace operator. More precisely, if \( m \geq 1 \) is a fixed natural number, we shall be concerned with Dirichlet problems of the following form

\[
\begin{cases}
(-\Delta)^{s_i} u_i = \lambda_i f_i(x, u, P_i[u]) & \text{in } \Omega \\
u_i \equiv \eta_i \zeta_i(x) B_i[u], & \text{in } \mathbb{R}^n \setminus \Omega \\
u \geq 0 & \text{in } \mathbb{R}^n,
\end{cases}
\]

where \( \Omega \subseteq \mathbb{R}^n \) is a fixed open set, \( u = (u_1, \ldots, u_m) \) and, for \( i = 1, \ldots, m \),

- \( f_i \) is a real-valued function defined on \( \Omega \times \mathbb{R}^m \times \mathbb{R} \);
- \( \lambda_i, \eta_i \) are non-negative parameters;
- \( \zeta_i \) is a sufficiently regular, real-valued function defined on \( \mathbb{R}^n \);
- \( P_i, B_i \) are suitable functionals to be defined later.

Moreover, \( s_1, \ldots, s_m \in (0, 1) \) and \((-\Delta)^{s_i}\) denotes the standard fractional Laplace operator of order \( s_i \), which is the non-local operator defined as

\[
(-\Delta)^{s_i} v(x) = c_{n,s_i} \cdot \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+2s_i}} \, dy.
\]

Here, \( c_{n,s_i} > 0 \) is the ‘normalization’ constant defined as

\[
c_{n,s_i} := \left( \int_{\mathbb{R}^n} \frac{1 - \cos(y_1)}{|y|^{n+2s_i}} \, dy \right)^{-1}.
\]
Notice that, in addition to the fractional differential operators, in system (1.1) other non-local terms occur, both in the differential equations (having the role of *non-local functional weights*) and in the boundary conditions (BCs for short). In particular, we are interested in the existence/non-existence of *positive* solutions of (1.1), and our approach is based on the classical notion of fixed point index in cones. We work in the Banach space of the bounded continuous \( \mathbb{R}^m \)-valued functions defined in \( \mathbb{R}^n \), namely

\[
X := \left\{ u \in C(\mathbb{R}^n; \mathbb{R}^m) : \sup_{\mathbb{R}^n} |u_i| < \infty \text{ for all } i = 1, \ldots, m \right\},
\]

endowed with the supremum norm; accordingly, since we are interested in non-zero positive solutions, we look for solutions of (1.1) lying in the cone

\[
P := \{ u \in X : u_i \geq 0 \text{ on } \mathbb{R}^n \text{ for every } i = 1, \ldots, m \}.
\]

In view of these facts, it is natural to assume that the real-valued operators \( P_i, B_i \) are defined on \( X \) (for all \( i = 1, \ldots, m \)).

As it is by now well-known, equations involving the fractional Laplacian arise in several applications; because of this, they have been extensively studied in the last decades by many authors, we provide as a reference the comprehensive survey [13]. Among others, let us mention here the equations, driven by the fractional Laplacian, which have additional non-local terms and are often referred to as *Kirchhoff-type equations*. For instance, Kirchhoff-type equations on bounded domains have been recently studied in [8, 11, 16, 29], while *systems* on bounded domains are investigated, e.g., in [12]. Moreover, Kirchhoff-type equations on the whole of \( \mathbb{R}^n \) have been studied in [3, 6, 7, 26]. As regards concrete ‘real-word’ applications, the kind of problems that we are able to deal with seems to be of interest in, e.g., biological models: indeed, on one hand, equations with functional terms in the right-hand side commonly appear in models about cell-adhesion (see, e.g., [19, 23]); on the other hand, the fractional Laplacian is also used to model superdiffusive cells (see, e.g., [14, 24]).

In the above cited papers, variational methods are frequently used to prove the existence/multiplicity of solutions. To the best of our knowledge, not many papers have been devoted to equations driven by the fractional Laplacian from the point of view of topological methods. Let us mention here, for instance, the recent papers by Alves, de Lima and Nóbrega [11, 2]: in these papers, the authors obtained Rabinowitz-type global bifurcation results of positive solutions of a parametric fractional Laplacian equation in \( \mathbb{R}^n \) using the Leray-Schauder degree. On the other hand, due to the presence of the non-local functional weights, system (1.1) can be viewed as a *nonlinear fractional Kirchhoff-type problem*, even if in a slightly different direction than the one proposed in [12, 16].
As already pointed out, in this paper we adopt a topological approach based on the classical notion of fixed point index (see e.g. [18]) to prove our main existence result, namely Theorem 3.3 below; moreover, we prove a non-existence result via an elementary argument. In some sense, our existence result stems from a pioneering work by Amman [4, 5] and follows a line recently pursued by the authors in the study of elliptic PDEs [9, 20, 21, 22]. We point out that our approach permits to consider (possibly nonlinear) functional BCs: for example, in Section 4 we will discuss the solvability of the following problem:

\[
\begin{align*}
(\Delta)_{1/4}^4 u_1 &= \lambda_1 (1 - u_1) \int_{B_1} e^{u_2} \, dx \quad \text{in } B_1, \\
(\Delta)_{3/4}^4 u_2 &= \lambda_2 u_2 \cdot \text{osc}_{B_1}(u_1) \quad \text{in } B_1, \\
u_1 \big|_{\mathbb{R}^2 \setminus B_1} &= \eta_1 \cdot u_1(0)u_2(0), \\
u_2 \big|_{\mathbb{R}^2 \setminus B_1} &= \eta_2 \cdot \limsup_{|x| \to \infty} u_1(x),
\end{align*}
\]

in which by $B_1$ we denote the Euclidean ball in $\mathbb{R}^2$ centered at 0 with radius 1, and by $\text{osc}_{B_1}(\phi)$ we mean the oscillation of the function $\phi$ on $B_1$.

We now briefly describe the structure of our paper. In the first part, we perform a preliminary study of the fractional differential operators which occur in (1.1); in Section 2 we collect some properties and estimates of the solutions of the Dirichlet problem for $(-\Delta)^s$, $s \in (0, 1)$, which allow to define a Green operator, denoted by $G_s$, from $L^\infty(\Omega)$ to $C_0^{0,s}(\mathbb{R}^n)$. These properties are probably known to the experts in the field, nevertheless we include them for the sake of completeness. We refer the interested reader to the already quoted survey [13] for a detailed and self-contained introduction to the fractional Laplacian. We also discuss the positivity and compactness of the Green operator $G_s$, thought of as an operator from $L^\infty(\Omega)$ into itself, as well as spectral properties of $G_s$. Roughly speaking, these estimates yield the a priori bounds needed to compute the fixed point index in suitable cones of non-negative functions. We point out that a challenging feature of our investigation is the choice of the appropriate functional spaces to which the solutions belong. This is discussed in detail in Section 2. In Section 3 we prove our main results, while the last Section 4 contains a couple of examples illustrating both our existence and non-existence result.

2. Preliminaries and auxiliary results

In order to keep the paper as self-contained as possible, we collect in this section some definitions and results which shall be exploited in the sequel.
The \((-\Delta)^s\)-Green operator. Here we introduce the so-called \((-\Delta)^s\)-Green operator and we establish some of its basic properties. Throughout what follows, we take for fixed all the notation listed below.

- \(\Omega \subseteq \mathbb{R}^n\) is a (non-void) open set with smooth boundary and \(s \in (0, 1)\);
- \(H^s(\mathbb{R}^n)\) is the usual fractional Sobolev space of order \(s\), i.e.,
  \[
  H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < \infty \right\}.
  \]
- If \(U \subseteq \mathbb{R}^n\) is any open set and \(\alpha \in (0, 1)\) is fixed, \(C^{0,\alpha}(U)\) is the set of the functions \(u : U \rightarrow \mathbb{R}\) which are H"older-continuous up to \(U\), i.e.,
  \[
  C^{0,\alpha}(U) := \left\{ u \in C(U) : [u]_{\alpha,U} := \sup_{x \neq y \in U} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}.
  \]

In particular, if \(u \in C^{0,\alpha}(\overline{U})\), we set
\[
\|u\|_{C^{0,\alpha}(\overline{U})} := \sup_{\overline{U}} |u| + [u]_{\alpha,U}.
\]
- \(C_b(\mathbb{R}^n)\) is the Banach space of the continuous functions on \(\mathbb{R}^n\) which are globally bounded on \(\mathbb{R}^n\), i.e., \(C_b(\mathbb{R}^n) = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\).

In order to introduce the \((-\Delta)^s\)-Green operator, our starting point is the following notable result due to Ros-Oton and Serra [25].

**Theorem 2.1.** [25 Prop. 1.1] Let \(f \in L^\infty(\Omega)\) be fixed. Then, there exists a unique (weak) solution \(u_f \in H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n)\) of the Dirichlet problem
\[
\begin{aligned}
(-\Delta)^s u &= f & \text{in } \Omega, \\
u &\equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

This means, precisely, that \(u_f \equiv 0\) pointwise in \(\mathbb{R}^n \setminus \Omega\) and
\[
\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_{\Omega} f \varphi \, dx \quad \forall \varphi \in C^0_0(\Omega).
\]

Furthermore, there exists a constant \(C = C(\Omega, s) > 0\) such that
\[
\|u_f\|_{C^{0,s}(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\Omega)}.
\]

Thanks to Theorem 2.1, the following definition is well-posed.

**Definition 2.2.** We define the \((-\Delta)^s\)-Green operator (relative to \(\Omega\)) as
\[
\mathcal{G}_s : L^\infty(\Omega) \rightarrow C^{0,s}(\mathbb{R}^n), \quad \mathcal{G}_s(f) := u_f,
\]
where \(u_f\) is the unique solution of \((2.1)\) (according to Theorem 2.1).

With Definition 2.2 at hand, we now proceed by proving some ‘topological’ properties of \(\mathcal{G}_s\). We begin with a couple of continuity/compactness results.

**Proposition 2.3.** The operator \(\mathcal{G}_s\) is continuous from \(L^\infty(\Omega)\) to \(C^{0,s}(\mathbb{R}^n)\).
Proof. On account of (2.2), for every $f \in L^\infty(\Omega)$ we have
\[
\|G_s(f)\|_{C^{0,s}(\mathbb{R}^n)} = \|uf\|_{C^{0,s}(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\Omega)},
\]
where $C > 0$ is a constant only depending on $\Omega$ and $s$. From this, since $G_s$ is obviously linear, we immediately infer that $G_s$ is continuous. \hfill \Box

Proposition 2.4. Let $\{f_k\}_{k=1}^\infty$ be a bounded sequence in $L^\infty(\Omega)$. Then, there exists $v_0 \in C(\mathbb{R}^n)$ such that $v_0 \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ and (up to a sub-sequence)
\[
(2.4) \quad \lim_{k \to \infty} G_s(f_k) = v_0 \quad \text{uniformly in } \mathbb{R}^n.
\]
In particular, $G_s$ is compact from $L^\infty(\Omega)$ into $L^\infty(\mathbb{R}^n)$.

Proof. First of all, since $\{f_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Omega)$, it follows from (2.2) that the family $\{G_s(f_k)\}_{k=1}^\infty \subseteq C^{0,s}(\mathbb{R}^n)$ is equi-continuous and equi-bounded; as a consequence, since $\Omega$ is compact, by Arzelà-Ascoli’s theorem there exists some function $g \in C(\Omega)$ such that (up to a sub-sequence)
\[
(2.5) \quad \lim_{k \to \infty} G_s(f_k) = g \quad \text{uniformly on } \Omega.
\]
In particular, since $G_s(f_k) \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ for every $k \in \mathbb{N}$, we have $g \equiv 0$ on $\partial \Omega$. Thus, introducing the function $v_0 : \mathbb{R}^n \to \mathbb{R}$ defined by
\[
v_0(x) := \begin{cases} g(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}
\]
from (2.5) (and since $G_s(f_k) \equiv v_0 \equiv 0$ on $\mathbb{R}^n \setminus \Omega$) we get
\[
\lim_{k \to \infty} \|G_s(f_k) - v_0\|_{L^\infty(\mathbb{R}^n)} = \lim_{k \to \infty} \|G_s(f_k) - v_0\|_{L^\infty(\Omega)} = 0,
\]
which is precisely the desired (2.4). This ends the proof. \hfill \Box

Then, we prove that $G_s$ is positive with respect to the cone
\[
C := \{ f \in L^\infty(\Omega) : f \geq 0 \text{ a.e. on } \Omega \}.
\]

Proposition 2.5. If $C$ is as in (2.6), then $G_s(C) \subseteq C$.

Proof. Let $f \in C$ be fixed, and let $u_f := G_s(f)$. By definition, $u_f$ is the unique solution of (2.1) in $H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n)$; thus, since $f \geq 0$ a.e. on $\Omega$ (as $f \in C$), from the Weak Maximum Principle and the continuity of $u_f$ we get
\[
u_f = G_s(f) \geq 0 \text{ point-wise in } \mathbb{R}^n.
\]
This ends the proof. \hfill \Box

We close this section by briefly studying the spectrum of $G_s$. To this end, we first recall a result on the eigenvalues of $(-\Delta)^s$, which easily follows by combining [27, Prop.9] and [25, Cor.1.6] (see also [28, Prop.4]).
Theorem 2.6. There exists a countable set \( \Lambda \subseteq (0, \infty) \) such that, if \( \lambda \in \Lambda \), there exists a solution \( e_\lambda \in H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n) \) of the eigenvalue problem

\[
\begin{cases}
(-\Delta)^s u = \lambda u & \text{in } \Omega, \\
u \equiv 0 & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

This means, precisely, that \( e_\lambda \equiv 0 \) in \( \mathbb{R}^n \setminus \Omega \) and

\[
\frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(e_\lambda(x) - e_\lambda(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy = \lambda \int_\Omega e_\lambda \varphi \, dx
\]

for every test function \( \varphi \in C^\infty_0(\Omega) \).

Thanks to Theorem 2.6, we can prove the following proposition.

Proposition 2.7. Let \( r(\mathcal{G}_s) \) denote the spectral radius of \( \mathcal{G}_s \), thought of as an operator from \( L^\infty(\Omega) \) into itself. Then, the following facts hold:

(i) \( r(\mathcal{G}_s) > 0 \);

(ii) there exists a function \( \phi \in C^{0,s}(\mathbb{R}^n) \setminus \{0\} \) such that

\[
\mathcal{G}_s(\phi|_\Omega) = r(\mathcal{G}_s) \phi \quad \text{and} \quad \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega.
\]

Proof. (i) On account of Theorem 2.6, we can find a real \( \lambda > 0 \) and a function \( e_\lambda \in C^{0,s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \), not identically vanishing, such that

\[
\begin{cases}
(-\Delta)^s e_\lambda = \lambda e_\lambda & \text{in } \Omega, \\
e_\lambda \equiv 0 & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

As a consequence, since \( e_\lambda|_\Omega \in L^\infty(\Omega) \) (as \( e_\lambda \in C^{0,s}(\mathbb{R}^n) \)), we have

\[
\mathcal{G}_s(e_\lambda|_\Omega) = \frac{1}{\lambda} e_\lambda,
\]

and this proves that \( r(\mathcal{G}_s) > 0 \), as desired.

(ii) We prove the assertion by using the well-known Krein-Rutman theorem. To this end we first observe that, if \( \mathcal{C} \) is the cone defined in (2.6), one has:

(a) \( \mathcal{C} - \mathcal{C} \) is dense in \( L^\infty(\Omega) \) (actually, \( \mathcal{C} - \mathcal{C} = L^\infty(\Omega) \));

(b) \( \mathcal{G}_s(\mathcal{C}) \subseteq \mathcal{C} \) (see Corollary 2.5).

Moreover, from (i) we know that \( r(\mathcal{G}_s) > 0 \). Thus, since Proposition 2.4 ensures that \( \mathcal{G}_s \) is compact from \( L^\infty(\Omega) \) into itself, we can invoke Krein-Rutman’s theorem, ensuring that \( \lambda = r(\mathcal{G}_s) \) is an eigenvalue of \( \mathcal{G}_s \). This means that there exists a function \( \phi \in L^\infty(\Omega) \), not identically vanishing, such that

\[
\mathcal{G}_s(\phi) = r(\mathcal{G}_s) \phi.
\]
On the other hand, since \( G_s(\phi) = u_\phi \in C^{0,s}(\mathbb{R}^n) \) and vanishes on \( \mathbb{R}^n \setminus \Omega \) (remind that \( u_\phi \) is the solution of (2.1)), from (2.8) we conclude that
\[
\phi = \frac{1}{r(G_s)} u_\phi \in C^{0,s}(\mathbb{R}^n) \quad \text{and} \quad \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega.
\]
This ends the proof.

2.2. The non-homogeneous Dirichlet problem for \((-\Delta)^s\). Due to its relevance in the sequel, we spend a few words about the non-homogeneous Dirichlet problem for \((-\Delta)^s\), that is,
\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega, \\
u \equiv \zeta & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
(2.9)

Avoiding to discuss the solvability of (2.9) for general \( f \) and \( \zeta \), here we limit to observe that, when \( \zeta \) is sufficiently regular, problem (2.9) can be reduced to (2.1) (with a different \( f \)). In fact, let us suppose that
\[
\zeta \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\overline{\Omega}),
\]
where \( \overline{\Omega} \subseteq \mathbb{R}^n \) is some open neighborhood of \( \overline{\Omega} \). Then, it is not difficult to see that \((-\Delta)^s \zeta\) can be computed pointwise in \(\Omega\), and
\[
(-\Delta)^s \zeta(x) = -\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{\zeta(x + z) + \zeta(x - z) - 2\zeta(x)}{|z|^{n+2s}} \, dz \quad (x \in \Omega).
\]
In particular, \((-\Delta)^s \zeta \in L^\infty(\Omega)\). Moreover, since \( \zeta \in H^s(\mathbb{R}^n) \), a standard ‘integration-by-parts’ argument gives, for every test function \( \varphi \in C_0^\infty(\Omega) \),
\[
\int_\Omega (-\Delta)^s \zeta \cdot \varphi \, dx = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} rac{(\zeta(x) - \zeta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy.
\]
(2.10)

On account of these facts, we easily derive the following result.

**Theorem 2.8.** Let \( f \in L^\infty(\Omega) \) and let \( \zeta \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\overline{\Omega}) \), where \( \overline{\Omega} \subseteq \mathbb{R}^n \) is an open neighborhood of \( \overline{\Omega} \). Then, the function
\[
u_{f, \zeta} := G_s(f - (-\Delta)^s \zeta) + \zeta \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)
\]
is the unique (weak) solution of (2.9). This means, precisely, that
\[
\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(v_{f, \zeta}(x) - v_{f, \zeta}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega),
\]
and \( v_{f, \zeta} \equiv \zeta \) pointwise on \( \mathbb{R}^n \setminus \Omega \).

**Remark 2.9.** We explicitly notice that, since \( f \) and \((-\Delta)^s \zeta\) are in \( L^\infty(\Omega) \), by the very definition of \( G_s \) one has \( v := G_s(f - (-\Delta)^s \zeta) \in C^{0,s}(\mathbb{R}^n) \); as a consequence, if \( \zeta \in C^{0,\alpha}(\mathbb{R}^n) \) for some \( \alpha \in (0, 1) \), we derive that
\[
v_{f, \zeta} = v + \zeta \in C^{0,\theta}(\mathbb{R}^n), \quad \text{where } \theta := \min\{s, \alpha\}.
\]
2.3. The fixed point index. For the sake of completeness, we collect in the following proposition some properties of the classical fixed point index that will be crucial in the proof of our existence result; for more details see, e.g., [4, 18].

In what follows the closure and the boundary of subsets of a cone $\hat{P}$ are understood to be relative to $\hat{P}$.

**Proposition 2.10.** Let $X$ be a real Banach space and let $\hat{P} \subset X$ be a cone. Let $D$ be an open bounded set of $X$ with $0 \in D \cap \hat{P}$ and $D \cap \hat{P} \neq \hat{P}$. Assume that $T : D \cap \hat{P} \to \hat{P}$ is a compact operator such that $x \neq T(x)$ for $x \in \partial(D \cap \hat{P})$.

Then the fixed point index $i_{\hat{P}}(T, D \cap \hat{P})$ has the following properties:

(i) If there exists $e \in \hat{P} \setminus \{0\}$ such that $x \neq T(x) + \sigma e$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 0$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 0$.

(ii) If $T(x) \neq \sigma x$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 1$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 1$.

(iii) Let $D^1$ be an open bounded subset of $X$ such that $(D^1 \cap \hat{P}) \subset (D \cap \hat{P})$. If $i_{\hat{P}}(T, D \cap \hat{P}) = 1$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 0$, then $T$ has a fixed point in $(D \cap \hat{P}) \setminus (D^1 \cap \hat{P})$.

The same holds if $i_{\hat{P}}(T, D \cap \hat{P}) = 0$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 1$.

3. Existence of positive solutions

In this section we state and prove our main existence result for positive solutions of (1.1), namely Theorem 3.3 below. Before doing this, we fix the relevant ‘structural’ assumptions which shall be tacitly understood in the sequel.

As already mentioned in the Introduction, our aim is to prove the existence of positive solutions for Dirichlet problems of the following form

\begin{equation}
\begin{cases}
(-\Delta)^{s_i} u_i = \lambda_i f_i(x, u, P_i[u]) & \text{in } \Omega \\
u_i(x) = \eta_i \zeta_i(x) B_i[u], & \text{for } x \in \mathbb{R}^n \setminus \Omega \quad (i = 1, \ldots, m).
\end{cases}
\end{equation}

Here, $m \geq 1$ is a fixed integer, $u = (u_1, \ldots, u_m)$, with $u_1, \ldots, u_m : \mathbb{R}^n \to \mathbb{R}$, $s_1, \ldots, s_m \in (0, 1)$ and, for every fixed $i \in \{1, \ldots, m\}$, we assume that

(H0) $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with smooth boundary;

(H1) $f_i$ is a real-valued function defined on $\overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}$;

(H2) $P_i, B_i$ are real-valued operators acting on the space

\begin{equation}
\mathbb{X} := \{u \in C(\mathbb{R}^n; \mathbb{R}^m) : \sup_{\mathbb{R}^n} |u_i| < \infty \text{ for all } i = 1, \ldots, m\};
\end{equation}

(H3) $\zeta_i \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\mathcal{O})$ and $\zeta_i \geq 0$ on $\mathbb{R}^n$, where

\begin{equation}
s = \min_{i = 1, \ldots, m} s_i;
\end{equation}

and $\mathcal{O} \subseteq \mathbb{R}^n$ is a suitable open neighborhood of $\overline{\Omega}$;
(H4) \( \lambda_i, \eta_i \) are non-negative parameters.

In dealing with vector-valued functions \( u = (u_1, \ldots, u_m) : \mathbb{R}^n \to \mathbb{R}^m \), it is more convenient to use on the spaces \( \mathbb{R}^p \) (for \( p \in \mathbb{N} \)) the maximum norm, that is,

\[
\|z\| := \max_{j=1,\ldots,p} |z_j| \quad \text{for all } z \in \mathbb{R}^p.
\]

Thus, if \( X \) is as in (3.2) and \( u \in X \), we define

\[
\|u\|_{\infty} := \sup_{x \in \mathbb{R}^n} \|u(x)\| = \max_{i=1,\ldots,m} \left( \sup_{\mathbb{R}^n} |u_i(x)| \right).
\]

Obviously, \((X, \| \cdot \|_{\infty})\) is a (real) Banach space.

Now, we have already anticipated that we aim to study the solvability of problem (3.1) by means of suitable fixed-point techniques; on the other hand, since the ‘non-local’ boundary conditions are prescribed on the complementary of \( \Omega \), one should work in the space \( C(\mathbb{R}^n; \mathbb{R}^m) \), which is not a Banach space.

To overcome this issue, we make the following key observation: if \( u : \mathbb{R}^n \to \mathbb{R}^m \) is a (continuous) function solving (3.1) point-wise in \( \mathbb{R}^n \), then

\[
u_i(x) = \eta_i \zeta_i(x) B_i [u] \quad \text{for all } x \text{ on } \mathbb{R}^n \setminus \Omega \quad (i = 1, \ldots, m);
\]

as a consequence, since \( u \) is continuous on \( \mathbb{R}^n \) and \( \zeta_i \in C_b(\mathbb{R}^n) \), we deduce that \( u \in X \), and \((X, \| \cdot \|_{\infty})\) is a Banach space. We then use the results in Section 2 to define a suitable functional \( T \), acting on the space \( X \), allowing us to rephrase problem (3.1) into the fixed-point equation

\[
T(u) = u \quad \text{in } X.
\]

To begin with, if \( i \in \{1, \ldots, m\} \) is fixed, for the sake of simplicity we denote \( G_i \) the \((-\Delta)^s_i\)-Green operator defined in (2.3). Let us recall that

\[
G_i : L^\infty(\Omega) \to C^{0,s_i}(\mathbb{R}^n)
\]

and, by definition, \( G_i(f) = u_i \) is the unique solution of \((P)_{f,0}\) in \( C^{0,s_i}(\mathbb{R}^n) \) (with \( f \in L^\infty(\Omega) \)). In particular, \( G_i(f) \equiv 0 \) on \( \mathbb{R}^n \setminus \Omega \). As a consequence, if \( s \in (0,1) \) is as in (3.3) we derive that

\[
G_i(L^\infty(\Omega)) \subseteq L^\infty(\mathbb{R}^n) \cap C^{0,s_i}(\mathbb{R}^n) \quad \text{for all } i = 1, \ldots, m.
\]

According to Proposition 2.7 we then let \( r_i = r(G_i) > 0 \) be the spectral radius of \( G_i \), thought of as an operator from \( L^\infty(\Omega) \) into itself, and we fix once and for all a function \( \phi_i \in C^{0,s_i}(\mathbb{R}^n) \setminus \{0\} \) such that (setting \( \mu_i := 1/r_i \))

\[
\phi_i = \mu_i G_i(\phi_i|_\Omega) \quad \text{and} \quad \phi_i \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega.
\]

To proceed further, we define the Nemytskii operator \( F_i \) as

\[
F_i(u) = f_i(\cdot, u(\cdot), P_i[u]) \quad (u \in X),
\]
and we assume for a moment that, for any \( u \in X \), we have \( F_i(u) \in L^\infty(\Omega) \); we will prove later that this holds under the assumptions of Theorem 3.3. In this case, taking into account (3.4), we can set

\[
I(u) := \left( \lambda_i G^i(F_i(u)) \right)_{i=1}^m \in X.
\]

Furthermore, if \( \gamma_i \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) is the unique solution of

\[
\begin{cases}
(-\Delta)^s u = 0 & \text{in } \Omega, \\
u \equiv \zeta_i & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
\]

(according to Theorem 2.8), given \( u \in X \) we define

\[
D(u) := \left( \eta_i \gamma_i(\cdot) B_i[u] \right)_{i=1}^m.
\]

We explicitly notice that, since \( \gamma_i \in C(\mathbb{R}^n) \), \( \zeta_i \in C_b(\mathbb{R}^n) \) and \( \gamma_i \equiv \zeta_i \) on \( \mathbb{R}^n \setminus \Omega \), we have \( \gamma_i \in X \); thus, since \( D(u) \) is a scalar multiple of \( \gamma_i \), one has \( D(u) \in X \).

Using the operators \( I \) and \( D \) just introduced, we can finally provide the precise definition of solution of problem (3.1).

**Definition 3.1.** We say that a function \( u = (u_1, \ldots, u_m) \in X \) is a solution of problem (3.1) if it satisfies the following properties:

(i) \( F_i(u) \in L^\infty(\Omega) \) for every \( i = 1, \ldots, m \);

(ii) \( u = I(u) + D(u) \), that is,

\[
u_i = \lambda_i G^i(F_i(u)) + \eta_i \gamma_i(\cdot) B_i[u].
\]

If, in addition, \( u_i \geq 0 \) for all \( i = 1, \ldots, m \) and there exists some \( i_0 \in \{1, \ldots, m\} \) such that \( u_{i_0} \neq 0 \), we say that \( u \) is a non-zero positive solution of (3.1).

**Remark 3.2.** On account of Remark 2.9, if \( \zeta_1, \ldots, \zeta_m \in C^0,\alpha(\mathbb{R}^n) \) for some \( \alpha \in (0, 1] \), we have that \( \gamma_i = u_{0,\zeta_i} \in C^0,\theta(\mathbb{R}^n) \) for all \( i = 1, \ldots, m \), where

\[
\theta = \min\{\alpha, s\}
\]

\( s \) being as in (3.3). As a consequence, we have \( D(X) \subseteq C^0,\theta(\mathbb{R}^n; \mathbb{R}^m) \) and, by (3.4), any solution of problem (3.1) actually belongs to \( C^0,\theta(\mathbb{R}^n; \mathbb{R}^m) \).

For our existence result, we make use of the classical fixed point index (see Section 2.3). We will work on the cone

\[
P := \left\{ u \in X : u_i \geq 0 \text{ on } \mathbb{R}^n \text{ for every } i = 1, \ldots, m \right\}.
\]

Given a finite sequence \( q = \{\rho_i\}_{i=1}^m \subseteq (0, +\infty) \), we define

\[
I(q) = \prod_{i=1}^m [0, \rho_i] \subseteq \mathbb{R}^m
\]
and set
\[
(3.11) \quad P(\varrho) := \left\{ u \in \mathbb{X} : u(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^n \right\}.
\]

**Theorem 3.3.** Let the assumptions (H0)-to-(H4) be in force. Moreover, let us suppose that there exists a finite sequence \( \varrho = \{\rho_i\}_{i=1}^m \subseteq (0, \infty) \) satisfying the following hypotheses:

(a) For every \( i = 1, \ldots, m \), one has that
\[
(a)_1 \quad P_i|_{P(\varrho)} \text{ is continuous, and there exist } \omega_{i,\varrho}, \overline{\omega}_{i,\varrho} \in \mathbb{R} \text{ such that } \\
\omega_{i,\varrho} \leq P_i[u] \leq \overline{\omega}_{i,\varrho}, \quad \text{for every } u \in P(\varrho);
\]

(b) \( f_i \) is continuous and non-negative on \( \overline{\Omega} \times I(\varrho) \times [\omega_{i,\varrho}, \overline{\omega}_{i,\varrho}] \);

(c) \( B_i|_{P(\varrho)} \) is continuous, non-negative, and bounded;

(b) There exist \( \delta \in (0, +\infty) \), \( i_0 \in \{1, 2, \ldots, m\} \) and \( \rho_0 \in (0, \min_i \rho_i) \) such that, if \( \varrho_0 \) denotes the finite sequence \( \varrho_0 := \{\rho_0\}_{i=1}^m \), we have
\[
(3.12) \quad f_{i_0}(x, z, \omega) \geq \delta z_{i_0} \quad \text{for every } (x, z, \omega) \in \Pi_0,
\]
where \( \Pi_0 := \overline{\Omega} \times I(\varrho_0) \times [\omega_0, \overline{\omega}_0] \) and
\[
\omega_0 := \inf_{u \in P(\varrho_0)} P_{i_0}[u], \quad \overline{\omega}_0 := \sup_{u \in P(\varrho_0)} P_{i_0}[u].
\]

(c) Setting, for every \( i = 1, \ldots, m \),
\[
M_i := \max \left\{ f_i(x, z, \omega) : (x, z, \omega) \in \overline{\Omega} \times I(\varrho) \times [\omega_{i,\varrho}, \overline{\omega}_{i,\varrho}] \right\} \quad \text{and}
\]
\[
B_i := \sup_{u \in P(\varrho)} B_i[u]
\]
the following inequalities are satisfied:
\[
(c)_1 \quad \mu_{i_0} \leq \delta \lambda_{i_0};
\]
\[
(c)_2 \quad \lambda_i M_i \|G^i(1)\|_{\infty} + \eta_i B_i \|\gamma_i\|_{\infty} \leq \rho_i.
\]

Then the system \((3.1)\) has a non-zero positive solution \( u \in \mathbb{X} \) such that
\[
(3.14) \quad \|u\|_{\infty} \geq \rho_0 \quad \text{and} \quad \|u_i\|_{\infty} \leq \rho_i \text{ for every } i = 1, \ldots, m.
\]

**Proof.** As a preliminary fact, we explicitly observe that \( \mathcal{T} := I + D \) is a well-defined operator from \( P(\varrho) \) to \( \mathbb{X} \). For this purpose, we show that
\[
(3.15) \quad \mathcal{F}_i(P(\varrho)) \subseteq L^\infty(\Omega) \quad \text{(for every } i = 1, \ldots, m)\).
\]
In fact, given \( u \in P(\varrho) \), by assumption \( (a)_1 \) we have, for every \( i = 1, \ldots, m \),
\[
(x, u(x), P_i[u]) \in \overline{\Omega} \times I(\varrho) \times [\omega_{i,\varrho}, \overline{\omega}_{i,\varrho}] \quad \text{for all } x \in \overline{\Omega};
\]
as a consequence, from assumption \( (a)_2 \) we readily derive \((3.15)\). Notice that \( \mathcal{T} \) maps \( P(\varrho) \) into \( \mathbb{X} \) in view of \((3.4)\) and the very definition of \( D \).

We now show that the following assertions hold:
(i) $\mathcal{T}$ maps $P(\varrho)$ into $P \subseteq \mathbb{X}$;
(ii) $\mathcal{T} : P(\varrho) \to P$ is compact.

To prove (i), let $\mathbf{u} \in P(\varrho)$ and let $i \in \{1, \ldots, m\}$ be fixed. We first observe that, since $\mathbf{u} \in P(\varrho)$, by (a)$_3$ we have $\mathcal{B}_i[\mathbf{u}] \geq 0$; moreover, since $\zeta_i \geq 0$ on $\mathbb{R}^n$ (see assumption (H3)), from the Weak Maximum Principle we derive that $\gamma_i \geq 0$. Thus, the fact that $\eta_i$ is nonnegative implies that

$$D(\mathbf{u})_i = \eta_i \gamma_i(\cdot) \mathcal{B}_i[\mathbf{u}] \geq 0 \quad \text{on } \mathbb{R}^n.$$  

On the other hand, since $\mathbf{u} \in P(\varrho)$, by assumption (a)$_2$ we also have

$$\mathcal{F}_i(\mathbf{u}) = f_i(\cdot, \mathbf{u}(\cdot), \mathcal{P}_i[\mathbf{u}]) \geq 0;$$

as a consequence, from Corollary 2.5 we infer that $\mathcal{G}_i(\mathcal{F}_i(\mathbf{u})) \geq 0$ on $\mathbb{R}^n$, and thus, as $\lambda_i \geq 0$, we get

$$I(\mathbf{u})_i = \lambda_i \mathcal{G}_i(\mathcal{F}_i(\mathbf{u})) \geq 0 \quad \text{on } \mathbb{R}^n.$$  

Gathering together these facts, and bearing in mind the very definition of $\mathcal{T}$, we conclude that $\mathcal{T}(P(\varrho)) \subseteq P$, as claimed.

We now prove assertion (ii). To this end, we fix $i \in \{1, \ldots, m\}$ and we observe that, in view assumption (a)$_2$, we have that $\mathcal{F}_i(\mathbf{u}) : P(\varrho) \to L^\infty(\Omega)$ is continuous; moreover, by Proposition 2.4 one also has that $\mathcal{G}_i : L^\infty(\Omega) \to L^\infty(\mathbb{R}^n)$ is linear and compact. As a consequence, we deduce that

$$\mathcal{I}(\mathbf{u})_i = \lambda_i \mathcal{G}_i(\mathcal{F}_i(\mathbf{u})) \geq 0 \quad \text{on } \mathbb{R}^n.$$  

Gathering together these facts, and bearing in mind the very definition of $\mathcal{T}$, we conclude that $\mathcal{T}(P(\varrho)) \subseteq P$, as claimed.

To proceed further, we define

$$P_0 = \left\{ \mathbf{u} \in \mathbb{X} : \mathbf{u}(x) \in I(\varrho_0) \text{ for all } x \in \mathbb{R}^n \right\} \subseteq P(\varrho) \subseteq P,$$

where $\varrho_0$ is as in assumption (b). Moreover, we consider the open sets

$$D := \left\{ \mathbf{u} \in \mathbb{X} : \|\mathbf{u}_i\|_{\infty} < \rho_i \text{ for every } i = 1, \ldots, m \right\} \quad \text{and} \quad D^1 := \left\{ \mathbf{u} \in \mathbb{X} : \|\mathbf{u}_i\|_{\infty} < \rho_0 \text{ for every } i = 1, \ldots, m \right\}.$$

We explicitly observe that, if \( D, D^1 \) are as above, we have \( \partial(D \cap P) = \partial P(q) \) and \( \partial(D^1 \cap P) = \partial P_0 \), where both the boundaries are relative to \( P \).

Now, if the operator \( \mathcal{T} \) has a fixed point \( u_0 \in \partial P(q) \cup \partial P_0 \), then \( u_0 \) is a solution of problem (3.1) satisfying (3.14), and the theorem is proved. If, instead, \( \mathcal{T} \) is fixed-point free on \( \partial P(q) \cup \partial P_0 \), both the fixed-point indices
\[
i_P(\mathcal{T}, D \cap P) \quad \text{and} \quad i_P(\mathcal{T}, D^1 \cap P)
\]
are well-defined. Assuming this last possibility, we prove the following.

**Claim 1.** We claim that
\[
i_P(\mathcal{T}, D \cap P) = 1.
\]
According to Proposition 2.10-(ii), to prove (3.16) it suffices to show that
\[
A(u) \neq \sigma u \quad \text{for every} \quad u \in \partial P(q) \quad \text{and every} \quad \sigma > 1,
\]
To establish (3.17) we argue by contradiction, and we suppose that there exist a function \( u \in \partial P(q) \) and a real \( \sigma > 1 \) such that
\[
\sigma u = \mathcal{T}(u).
\]
Since \( u \in \partial P(q) \), there exists an index \( i \in \{1, \ldots, m\} \) such that \( \|u_i\|_\infty = \rho_i \). By assumption (a) and (3.13), we have
\[
\begin{align*}
0 & \leq F_i(u)(x) = f_i(x, u(x), \mathcal{P}_i[u]) \leq M_i \quad \text{for all} \quad x \in \overline{\Omega}; \\
\sigma u_i(x) & = \lambda_i \mathcal{G}^i(F_i(u))(x) + \eta_i \gamma_i(x) \mathcal{B}_i[u] \\
& \leq \lambda_i \mathcal{G}^i(M_i \hat{1})(x) + \eta_i \gamma_i(x) \mathcal{B}_i[u] \\
& \leq \|\lambda_i \mathcal{G}^i(M_i \hat{1})\|_\infty + \|\eta_i \mathcal{B}_i \gamma_i\|_\infty \\
& = \lambda_i M_i \mathcal{G}(\hat{1})\|_\infty + \eta_i \mathcal{B}_i \gamma_i \|_\infty \leq \rho_i
\end{align*}
\]
the last inequality following by assumption (c). As a consequence, by taking the supremum for \( x \in \overline{\Omega} \) in (3.19), as \( u \in \partial P(q) \subseteq P(q) \), we get
\[
\sup_{x \in \overline{\Omega}} |\sigma u_i(x)| \leq \sigma \rho_i \leq \rho_i,
\]
which is clearly a contradiction (since \( \sigma > 1 \)), and the claim is proved.

**Claim 2.** We claim that
\[
i_P(\mathcal{T}, D^1 \cap P) = 0.
\]
According to Proposition 2.10-(i), to prove (3.20) it suffices to show that there exists a suitable function \( e \in P \setminus \{0\} \) satisfying the property
\[
\mathcal{T}(u) + \sigma e \neq u \quad \text{for every} \quad u \in \partial P_0 \quad \text{and every} \quad \sigma > 0.
\]
To establish (3.21), we let $e := (\phi_1, \ldots, \phi_m)$, where each component $\phi_1, \ldots, \phi_m$ is as in (3.3), and we argue by contradiction: we thus suppose that there exist $u \in \partial P_0$ and $\sigma > 0$ such that

$$u = T(u) + \sigma e.$$ 

Let $i_0$ be as in assumption (b). Since $T(u) \in P$, we have

$$u_{i_0} = T(u)_{i_0} + \sigma \phi_{i_0} \geq \sigma \phi_{i_0} \quad \text{on } \Omega.$$ 

Furthermore, again by assumption (b), for every $x \in \Omega$ we get

$$(3.22) \quad F_{i_0}(u)(x) = f_{i_0}(x, u(x), P_{i_0}[u]) \geq \delta u_{i_0}(x) \geq \delta \sigma \phi_{i_0}(x).$$ 

Gathering together all these facts, for every $x \in \Omega$ we have

$$u_{i_0}(x) = \lambda_{i_0} G_{i_0}(F_{i_0}(u))(x) + \eta_{i_0} \gamma_{i_0}(x) B_{i_0}[u] + \sigma \phi_{i_0}(x)$$

$$\geq \lambda_{i_0} G_{i_0}(\delta \sigma \phi_{i_0})(x) + \sigma \phi_{i_0}(x)$$

$$= \frac{\delta \lambda_{i_0}}{\mu_{i_0}} \cdot \sigma \phi_{i_0}(x) + \sigma \phi_{i_0}(x) \geq 2 \sigma \phi_{i_0}(x)$$

the last inequality following by assumption (c). 

By iterating the above argument, for every $x \in \Omega$ we get

$$u_{i_0}(x) \geq p \sigma \phi_{i_0}(x) \quad \text{for every } p \in \mathbb{N},$$

a contradiction since $u_{i_0}$ is bounded.

We are now ready to conclude the proof of the theorem: in fact, by combining Claims 1 and 2 and Proposition 2.10-(iii), we infer the existence of a fixed point $u_0 \in (D \cap P) \setminus P_0$ of $T$; thus, $u_0$ is a solution of (3.1) satisfying (3.14). $\square$

An elementary argument yields the following non-existence result.

**Theorem 3.4.** Let the assumptions (H0)-to-(H4) be in force. Moreover, let us suppose that there exists a finite sequence $\varrho = \{\rho_i\}_{i=1}^m \subseteq (0, \infty)$ such that, for every $i = 1, \ldots, m$, the following conditions hold:

(a) there exist $\omega_{i, \varrho}, \overline{\omega}_{i, \varrho} \in \mathbb{R}$ such that

$$(3.23) \quad \omega_{i, \varrho} \leq P_{i}[u] \leq \overline{\omega}_{i, \varrho} \quad \text{for every } u \in P(\varrho);$$

(b) there exist $\tau_i \in (0, +\infty)$ such that

$$f_i(x, z, \omega) \leq \tau_i z_i \quad \text{for every } (x, z, \omega) \in \Omega \times I(\varrho) \times [\omega_{i, \varrho}, \overline{\omega}_{i, \varrho}],$$

(c) there exist $\xi_i \in (0, +\infty)$ such that

$$|B_j[u]| \leq \xi_i \cdot \|u\|_{\infty}, \quad \text{for every } u \in P(\varrho),$$
(d) the following inequality holds:

\[ \lambda_i \tau_i \|G^i(\hat{1})\|_{\infty} + \eta_i \xi_i \|\gamma_i\|_{\infty} < 1. \]  

Then the system \([3.1]\) has at most the zero solution in \(P(\rho)\).

**Proof.** By contradiction, assume that \([3.1]\) has a solution \(u \in P(\rho) \setminus \{0\}\), that is, for every \(i = 1, \ldots, m\) we have (see Definition 3.1):

\[ F_i(u) \in L^\infty(\Omega) \quad \text{and} \quad u_i = \lambda_i G_i^i(F_i(u)) + \eta_i \gamma_i(B_i[u]). \]

Setting \(\rho := \|u\|_{\infty} > 0\), we let \(j \in \{1, 2, \ldots, m\}\) be such that \([3.25]\)

\[ \|u_j\|_{\infty} = \rho. \]

In view of assumptions (a)-(b), for every \(x \in \Omega\) we then have

\[ F_j(u)(x) = f_j(x, u(x), P_j[u]) \leq \tau_j u_j(x) \leq \tau_j \rho, \]

and thus (see Corollary 2.5 and recalling that \(F_j(u) \in L^\infty(\Omega)\))

\[ G^j(\tau_j \rho \cdot \hat{1} - F_j(u)) \geq 0 \iff G^j(F_j(u)) \leq \tau_j \rho \|G^j(\hat{1})\|_{\infty} \quad \text{on } \mathbb{R}^n. \]

As a consequence, we obtain

\[ u_j(x) = \lambda_j G^j(F_j(u))(x) + \eta_j \gamma_j(x) B_j[u] \]

\[ \leq \lambda_j \tau_j \rho \|G^j(\hat{1})\|_{\infty} + \eta_j \xi_j \|\gamma_j\|_{\infty} \]

\[ = (\lambda_j \tau_j \|G^j(\hat{1})\|_{\infty} + \eta_j \xi_j \|\gamma_j\|_{\infty}) \rho. \]

By taking the supremum in \([3.27]\) for \(x \in \Omega\), from \([3.24]\) and \([3.25]\) we get

\[ \rho = \sup_{x \in \Omega} u_j(x) \leq (\lambda_j \tau_j \|G^j(\hat{1})\|_{\infty} + \eta_j \xi_j \|\gamma_j\|_{\infty}) \rho < \rho, \]

and this is clearly a contradiction. Thus, we conclude that problem \([3.1]\) cannot have nonzero solutions in \(P(\rho)\), and the proof is complete. \(\square\)

4. **Examples**

In this last section we present a couple of concrete examples illustrating the applicability of Theorems 3.3 and 3.4. Before proceeding we remind the following result, which shall play a key role in our computations.

**Lemma 4.1.** Let \(r > 0\) be fixed, and let \(B_r \subseteq \mathbb{R}^n\) be the Euclidean ball centered at \(0\) with radius \(r\). Moreover, let \(s \in (0, 1)\). Then, the unique solution \(v_s\) of

\[
\begin{align*}
(-\Delta)^s v &= 1 \quad \text{in } B_r, \\
v &= 0 \quad \text{on } \mathbb{R}^n \setminus B_r
\end{align*}
\]
has the following explicit expression

\[ v_s(x) = \frac{2^{-2s} \Gamma(n/2)}{\Gamma(n^2/2)} (r^2 - \|x\|^2)^s, \]

where \(\| \cdot \|\) stands for the usual Euclidean norm and

\[ \Gamma(\alpha) = \int_{\mathbb{R}} x^{\alpha-1} e^{-x} \, dx \quad (\alpha > 0). \]

For a proof of Lemma 4.1 we refer, e.g., to [10, 17].

**Example 4.2.** In Euclidean space \(\mathbb{R}^2\), let us consider the following BVP

\[
\begin{cases}
(\Delta)^{\frac{1}{2}} u_1 = \lambda_1 (1 - u_1) \int_{B_1} e^{u_2} \, dx & \text{in } B_1, \\
(\Delta)^{\frac{1}{2}} u_2 = \lambda_2 u_2 \cdot \text{osc}_{B_1}(u_1) & \text{in } B_1,
\end{cases}
\]

where \(B_1\) is the Euclidean ball centered at 0 with radius 1, and

\[ \text{osc}_{B_1}(\phi) := \sup_{B_1} \phi - \inf_{B_1} \phi \quad (\text{for all } \phi \in X). \]

Clearly, problem (4.2) is of the form (3.1), with

1. \(\Omega = B_1, m = 2, s_1 = 1/4, s_2 = 3/4;\)
2. \(f_1 : \overline{B_1} \times \mathbb{R}^2 \times \mathbb{R}, f_1(x, z, w) := (1 - z_1) w;\)
3. \(f_2 : \overline{B_1} \times \mathbb{R}^2 \times \mathbb{R}, f_2(x, z, w) := z_2 w;\)
4. \(P_1 : X \to \mathbb{R}, P_1[u] := \int_{B_1} e^{u_2} \, dx;\)
5. \(P_2 : X \to \mathbb{R}, P_2[u] := \text{osc}_{B_1}(u_1);\)
6. \(B_1 : X \to \mathbb{R}, B_1[u] := u_1(0) u_2(0);\)
7. \(B_2 : X \to \mathbb{R}, B_1[u] := \limsup_{|x| \to \infty} u_1(x);\)
8. \(\zeta_1 \equiv \zeta_2 \equiv 1.\)

Moreover, it is straightforward to recognize that all the ‘structural’ assumptions (H0)-to-(H4) listed at the beginning of Section 3 are fulfilled. We now turn to prove that, in this case, also assumptions (a)-to-(c) of Theorem 3.3 are satisfied for a suitable choice of the nonnegative parameters \(\lambda_1, \lambda_2, \eta_1, \eta_2.\)

To this end, we consider the (finite) sequence \(\varrho\) defined as follows:

\[ \varrho = \{\rho_1, \rho_2\}, \quad \text{where } \rho_1 := \frac{1}{2} \text{ and } \rho_2 := 1. \]

According to this choice of \(\varrho\), we have (see (3.10)–(3.11))

\[ I(\varrho) = [0, 1/2] \times [0, 1], \quad P(\varrho) = \{ u \in X : u(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^2 \}. \]
Assumption (a). First of all, it is easy to see that $\mathcal{P}_1, \mathcal{P}_2$ are continuous when restricted to the set $P(\varrho) \subseteq \mathbb{X}$; moreover, for every $u = (u_1, u_2) \in \mathbb{X}$ we have
\[
\pi \leq \mathcal{P}_1[u] = \int_{\Omega_1} e^{u_2} \, dx \leq \pi \cdot e \quad \text{and} \quad 0 \leq \mathcal{P}_2[u] = \operatorname{osc}(u_1) \leq \frac{e}{2},
\]
so that assumption (a)$_1$ is satisfied with the choices
\[
(4.5) \quad \omega_{1, e} := \pi, \quad \omega_{1, e} := \pi \cdot e, \quad \omega_{2, e} := 0, \quad \omega_{2, e} := 1/2.
\]
As regards assumption (a)$_2$, we first notice that $f_1, f_2 \in C(\overline{\mathcal{B}_1} \times \mathbb{R}^2 \times \mathbb{R})$; moreover, by taking into account (4.4) and (4.5), we get
\[
f_1(x, z, w) = (1 - z_1) w \geq \frac{\pi}{2} > 0 \quad \text{on} \quad \overline{\mathcal{B}_1} \times I(\varrho) \times [\pi, \pi \cdot e] \quad \text{and}
\]
\[
f_2(x, z, w) = z_2 w \geq 0 \quad \text{on} \quad \overline{\mathcal{B}_1} \times I(\varrho) \times [0, 1/2],
\]
so that (a)$_2$ is fulfilled. Finally, as regards assumption (a)$_3$, it is not difficult to check that $\mathcal{B}_1, \mathcal{B}_2$ are continuous and non-negative when restricted to $P(\varrho)$; moreover, since for every $u = (u_1, u_2) \in P(\varrho)$ we have
\[
|\mathcal{B}_1[u]| = |u_1(0)u_2(0)| \leq \frac{1}{2} \quad \text{and} \quad |\mathcal{B}_2[u]| = |\limsup_{|x| \to \infty} u_1(x)| \leq \frac{1}{2}
\]
we conclude that $\mathcal{B}_1, \mathcal{B}_2$ are bounded on $P(\varrho)$.

Assumption (b). First of all, if $\rho_0 \in (0, 1/2)$ is arbitrarily fixed, we have
\[
\omega_0 := \inf_{u \in P(\varrho_0)} \mathcal{P}_1[u] = \pi,
\]
where $\varrho_0 := \{\rho_0, \rho_0\}$; moreover, for every $(x, z) \in \overline{\mathcal{B}_1} \times I(\varrho_0)$ and every $w \geq \pi$, one has
\[
f_1(x, z, w) = (1 - z_1) w \geq \frac{\pi}{2}.
\]
Gathering together these facts, we easily conclude that (3.12) holds for every choice of $\delta > 0$. In fact, given any such $\delta$, we define
\[
(4.6) \quad \rho_0 = \rho_0(\delta) := \min \left\{ \frac{1}{4}, \frac{\pi}{2\delta} \right\} \in (0, 1/2);
\]
then, for every $(x, z) \in \overline{\mathcal{B}_1} \times I(\varrho_0)$ and every $w \geq \pi = \omega_0$, we get
\[
f_1(x, z, w) \geq \frac{\pi}{2\delta} \cdot \delta \geq \delta z_1,
\]
and thus assumption (b) is satisfied with $i_0 = 1$ (and for every $\delta > 0$).

Assumption (c). We start by computing the constants appearing in (3.13). First of all, using (4.4), (4.5) and the definition of $f_1, f_2$ we get
\[
M_1 = \sup \{ f_1(x, z, w) : (x, z, w) \in \overline{\mathcal{B}_1} \times I(\varrho) \times [\omega_{1, e}, \omega_{1, e}] \} = \pi \cdot e \quad \text{and} \quad
\]
\[
(4.7) \quad M_2 = \sup \{ f_2(x, z, w) : (x, z, w) \in \overline{\mathcal{B}_1} \times I(\varrho) \times [\omega_{2, e}, \omega_{2, e}] \} = \frac{1}{2},
\]
Moreover, again by (4.4), (4.5) and the definition of $B_1, B_2$ we get
\begin{equation}
B_1 = \sup_{u \in P(\mathcal{Q})} B_1[u] = \frac{1}{2} \quad \text{and} \quad B_2 = \sup_{u \in P(\mathcal{Q})} B_2[u] = \frac{1}{2}
\end{equation}
We then turn our attention to the functions $G^i(\hat{1}) = G_{s_i}(\hat{1})$ and $\gamma_i$ (for $i = 1, 2$). To begin with, according to the very definition of $(-\Delta)^{s_i}$-Green operator, we know that $G_i(\hat{1})$ is the unique solution in $C^{0,s_i}(\mathbb{R}^2)$ of
\begin{equation}
\begin{cases}
(-\Delta)^{s_i} v = 1 & \text{in } B_1, \\
v \equiv 0 & \text{on } \mathbb{R}^2 \setminus B_1.
\end{cases}
\end{equation}
On the other hand, thanks to Lemma 4.1 we can write the explicit expression of $G_i(\hat{1})$: in fact, we have (remind that $n = 2, r = 1, s_1 = 1/4$ and $s_2 = 3/4$)
\begin{align*}
G_1(\hat{1}) &= G_{1/4}(\hat{1}) = \left( \frac{2^{-1/4}}{\Gamma(5/4)} \right)^2 (1 - \|x\|^2)^{1/4}_+ \\
G_2(\hat{1}) &= G_{3/4}(\hat{1}) = \left( \frac{2^{-3/4}}{\Gamma(7/4)} \right)^2 (1 - \|x\|^2)^{3/4}_+.
\end{align*}
As a consequence, we obtain
\begin{equation}
\|G_1^i(\hat{1})\|_\infty = \|G_2^i(\hat{1})\|_\infty = \frac{1}{\sqrt{2} \Gamma^2(5/4)} \approx 0.860682 \quad \text{and}
\end{equation}
\begin{equation}
\|G_2^i(\hat{1})\|_\infty = \frac{1}{\sqrt{8} \Gamma^2(7/4)} \approx 0.418567.
\end{equation}
As for the functions $\gamma_i$, the computations are much more easier: first of all, since $\zeta_1 \equiv \zeta_2 \equiv 1$, we know from (3.7) that $\gamma_i$ is the unique solution of
\begin{equation}
\begin{cases}
(-\Delta)^{s_i} u = 0 & \text{in } B_1, \\
u \equiv 1 & \text{on } \mathbb{R}^2 \setminus B_1.
\end{cases}
\end{equation}
On the other hand, since the above problem is solved by the constant function $\hat{\gamma} \equiv 1$ (independently of $s_i$), we get $\gamma_1 \equiv \gamma_2 \equiv 1$. Hence, we have
\begin{equation}
\|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1.
\end{equation}
Gathering together all the facts established so far, we are finally in a position to apply Theorem 3.3, taking into account (4.4), (4.7), (4.8), (4.9) and (4.10), for every choice of parameters $\lambda_1, \lambda_2, \eta_1, \eta_2 \geq 0$ satisfying
\begin{equation}
\begin{align*}
\lambda_1 \frac{\pi \cdot e}{\sqrt{2} \Gamma^2(5/4)} + \frac{\eta_1}{2} & \leq \frac{1}{2} \quad \text{(see assumption (c)_2 with } i = 1) \\
\lambda_2 \frac{1}{\sqrt{8} \Gamma^2(7/4)} + \frac{\eta_2}{2} & \leq 1 \quad \text{(see assumption (c)_2 with } i = 2) \\
\lambda_1 & > 0 \quad \text{(see assumption (c)_1 with } i_0 = 1)
\end{align*}
\end{equation}
there exists a solution $u_0 \in C^{0,1/4}(\mathbb{R}^2)$ of (4.2), further satisfying
\[
\|u\|_\infty \geq \rho_0(\delta) \quad \text{and} \quad \|u_1\|_\infty \leq \frac{1}{2}, \quad \|u_2\|_\infty \leq 1.
\]
Here, $\rho(\delta) \in (0,1/2)$ is as in (4.6) and $\delta > 0$ is chosen in such a way that
\[
\delta \lambda_1 \geq \mu_1,
\]
where $\mu_1$ is the inverse of the spectral radius of $(-\Delta)^{1/4}$. More explicitly, given $\lambda_1, \lambda_2, \eta_1, \eta_2$ satisfying (4.11), one first chooses $\delta > 0$ in such a way that $\delta \lambda_1 \geq \mu_1$ (see assumption (c.1)); then, one lets $\rho_0 = \rho_0(\delta)$ be as in (4.6).

The key point in this argument is that, since (3.12) holds for every $\delta > 0$ (by accordingly choosing $\rho_0$), one is free to choose $\delta > 0$ in such a way that (4.12) holds (provided that $\lambda_1 > 0$), without the need of an explicit knowledge of $\mu_1$.

**Example 4.3.** In Euclidean space $\mathbb{R}^2$, we consider the following BVP
\[
\begin{aligned}
(-\Delta)^{1/4} u_1 &= \lambda_1 u_1^2 (1 - u_1) \int_{B_1} e^{u_2} \, dx \quad \text{in } B_1, \\
(-\Delta)^{1/4} u_2 &= \lambda_2 u_2 \cdot \text{osc}_{B_1}(u_1) \quad \text{in } B_1, \\
 u_1|_{\mathbb{R}^2 \setminus B_1} &= \eta_1 \cdot u_1(0) u_2(0), \\
 u_2|_{\mathbb{R}^2 \setminus B_1} &= \eta_2 \cdot \limsup_{|x| \to \infty} u_1(x),
\end{aligned}
\]
where $B_1$ is the Euclidean unit ball, and $\text{osc}_{B_1} (\cdot)$ is as in (4.3). Clearly, problem (4.13) is of the form (3.1), with

1. $\Omega = B_1, \ m = 2, \ s_1 = 1/4$ and $s_2 = 3/4$;
2. $f_1 : \overline{B_1} \times \mathbb{R}^2 \times \mathbb{R}, \ f_1(x, z, w) := z_1^2 (1 - z_1) w$;
3. $f_2 : \overline{B_1} \times \mathbb{R}^2 \times \mathbb{R}, \ f_2(x, z, w) := z_2 w$;
4. $\mathcal{P}_1 : X \to \mathbb{R}, \ \mathcal{P}_1[u] := \int_{B_1} e^{u_2} \, dx$;
5. $\mathcal{P}_2 : X \to \mathbb{R}, \ \mathcal{P}_2[u] := \text{osc}_{B_1}(u_1)$;
6. $\mathcal{B}_1 : X \to \mathbb{R}, \ \mathcal{B}_1[u] := u_1(0) u_2(0)$;
7. $\mathcal{B}_2 : X \to \mathbb{R}, \ \mathcal{B}_1[u] := \limsup_{|x| \to \infty} u_1(x)$;
8. $\zeta_1 \equiv \zeta_2 \equiv 1$.

Moreover, it is straightforward to recognize that all the ‘structural’ assumptions (H0)-to-(H4) listed at the beginning of Section 3 are fulfilled. We now aim to show that, despite the similarity between problems (4.13) and (4.2), in this case it is possible to choose the parameters $\lambda_1, \lambda_2, \eta_1, \eta_2$ in such a way that assumptions (a)-to-(d) of the *non-existence* Theorem 3.4 are satisfied.

To this end, we consider the (finite) sequence $\varrho$ defined as
\[
\varrho := \{\rho_1, \rho_2\}, \quad \text{where } \rho_1 = \rho_2 = 1;
\]
According to this choice of $\varrho$, we have (see (3.10)-(3.11))
\[
I(\varrho) = [0,1] \times [0,1], \quad P(\varrho) = \{u \in X : u(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^2\}.
\]
Assumption (a). Given any $u = (u_1, u_2) \in P(\rho)$, we have
\[ \pi \leq P_1[u] = \int_{B_1} e^{u_2} \, dx \leq \pi \cdot e \quad \text{and} \quad 0 \leq P_2[u] = \text{osc}(u_1) \leq 1; \]
hence, assumption (a) is fulfilled with the choices
\[ (4.15) \quad \omega_{1,\rho} := \pi, \quad \omega_{2,\rho} := \pi \cdot e, \quad \omega_{2} := 0, \quad \omega_{2,\rho} := 1. \]

Assumption (b). We first observe that, clearly, both $f_1$ and $f_2$ are continuous on the whole of $\overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}$; moreover, we have (see (4.14) and (4.15))
\[ 0 \leq f_1(x, z, w) = z_1^2 (1 - z_1)w \leq (\pi \cdot e) z_1 \quad \text{on} \quad \overline{B}_1 \times I(\rho) \times [\pi, \pi \cdot e] \]
and
\[ 0 \leq f_2(x, z, w) = z_2 w \leq z_2 \quad \text{on} \quad \overline{B}_1 \times I(\rho) \times [0, 1]. \]

Thus, assumption (b) is satisfied with the choices
\[ (4.16) \quad \tau_1 = \pi \cdot e \quad \text{and} \quad \tau_2 = 1. \]

Assumption (c). Given any $u = (u_1, u_2) \in P(\rho)$, we see that
\[ 0 \leq B_1[u] = u_1(0)u_2(0) \leq u_1(0) \leq \|u\|_\infty \quad \text{and} \quad 0 \leq B_2[u] = \limsup_{|x| \to \infty} u_1(x) \leq \|u\|_\infty. \]

Hence, assumption (c) is satisfied with the choices
\[ (4.17) \quad \xi_1 = \xi_2 = 1. \]

Assumption (d). First of all, by exploiting all the computations carried out in Example 4.2 (see, respectively, (4.9) and (4.10)), we know that
\[ (a) \quad \|G_1(\hat{1})\|_\infty = \frac{1}{\sqrt{2\Gamma^2(5/4)}} \quad \text{and} \quad \|G_2(\hat{1})\|_\infty = \frac{1}{\sqrt{8\Gamma^2(7/4)}}; \]
\[ (b) \quad \|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1; \]
As a consequence, by gathering together (4.16), (4.17) and the above (a)-(b), we can apply Theorem 3.4 for every choice of $\lambda_1, \lambda_2, \eta_1, \eta_2 \geq 0$ satisfying
\[ \lambda_1 \frac{\pi \cdot e}{\sqrt{2\Gamma^2(5/4)}} + \eta_1 < 1 \quad \text{and} \quad \lambda_2 \frac{\sqrt{8\Gamma^2(7/4)}}{\sqrt{8\Gamma^2(7/4)}} + \eta_2 < 1, \]
the BVP (4.13) possesses only the zero solution in $P(\rho)$ (notice that the constant function $u \equiv 0$ is indeed a solution of problem (4.13)).

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