A DYNAMICAL MODEL FOR THE EVOLUTION OF A PULSAR WIND NEBULA INSIDE A NONRADIATIVE SUPERNOVA REMNANT

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ABSTRACT

A pulsar wind nebula (PWN) inside a supernova remnant provides a unique insight into the properties of the central neutron star, the relativistic wind powered by its loss of rotational energy, its progenitor supernova, and the surrounding environment. In this paper, we present a new semianalytic model for the evolution of such a PWN throughout its lifetime. This model couples the dynamical and radiative evolution of the PWNe, and predicts both the dynamical (e.g., radius and expansion velocity) and radiative (radio to TeV γ-ray spectrum) properties of the PWN during this period. As a result, it is well suited for using the observed properties of a PWN to constrain the physical characteristics of the neutron star, pulsar wind, progenitor supernova, and surrounding environment. We also discuss the expected evolution for a particular set of these parameters, and show that it reproduces the large spectral break inferred from the radio and X-ray spectrum of many young PWNe, and the low break frequency, low radio luminosity, high TeV γ-ray luminosity, and high magnetization observed for several older PWNe.

The predicted spectrum of this PWN also contains spectral features which appear during different evolutionary phases detectable with new radio and γ-ray observing facilities such as the Extended Very Large Array and the Fermi Gamma-ray Space Telescope. Finally, this model has implications for determining if PWNe can inject a sufficient number of energetic electrons and positrons into their surroundings to explain the recent measurements of the cosmic-ray positron fraction by PAMELA and the cosmic-ray lepton spectrum by ATIC and HESS.

Key words: pulsars: general – supernova remnants

Online-only material: color figures

1 INTRODUCTION

The gravitational collapse of the core of a massive star into a neutron star (e.g., Baade & Zwicky 1934) releases enough energy to power a supernova explosion (e.g., Zwicky 1938). The neutron star is born spinning, but loses its rotational energy through a ultrarelativistic magnetic and particle outflow commonly referred to as a pulsar wind (e.g., Goldreich & Julian 1969; Kennel & Coroniti 1984a). The interaction between the pulsar wind and the neutron star’s environment creates a pulsar wind nebula (PWN). Initially, the neutron star and its PWN are inside the supernova remnant (SNR) created by the expansion of the supernova ejecta into the surrounding interstellar medium (ISM). Previous work (e.g., Pacini & Salvati 1973; Reynolds & Chevalier 1984) has demonstrated that, during this period, the properties of the PWN depend on the physical properties of the central neutron star (e.g., space velocity, initial period, surface magnetic field strength, spin-down timescale, and braking index), the pulsar wind (e.g., magnetization and energy spectrum of particles injected into the PWN), the progenitor supernova explosion (e.g., the mass and initial kinetic energy of the ejecta), and the surrounding ISM (e.g., density). Measuring these quantities is important for understanding the physical connection between core-collapse supernovae, the formation of neutron stars in these explosions, and properties of the progenitor star. Many of these quantities are extremely difficult to measure directly for most neutron stars or SNRs, but possible to infer indirectly using observations of PWNe. This requires understanding the evolution of a PWN inside an SNR.

As summarized in a recent review by Gaensler & Slane (2006), this is extremely complicated due to the rapid evolution of both the SNR and the central neutron star. Analytical (e.g., Ostriker & Gunn 1971; Pacini & Salvati 1973; Reynolds & Chevalier 1984) and numerical simulations (e.g., Bucciantini et al. 2007; Jun 1998; Kennel & Coroniti 1984a, 1984b; Ostriker & Gunn 1971; Pacini & Salvati 1973; Rees & Gunn 1974; Reynolds & Chevalier 1984; van der Swaluw et al. 2001, 2004; van der Swaluw 2003; Venter & de Jager 2006; Volpi et al. 2008, and references therein). While most reproduce the general...
evolutionary sequence outlined above, they differ significantly in their details, e.g., how they evolve the PWN’s magnetic field and how they inject particles into the PWN. These differences have significant consequences for the predicted evolution of the observable properties of a PWN (such as its size and broadband spectrum). Additionally, most of these models either predict the dynamical properties (e.g., size and expansion velocity; van der Swaluw & Wu 2001) or spectral properties (e.g., broadband spectrum; Volpi et al. 2008) of the PWN but not both, or apply solely to the initial expansion of the PWN inside the SNR (e.g., Chevalier 1977; Del Zanna et al. 2004).

In this paper, we present a new semianalytic model for the evolution of a PWN which predicts both the dynamical and radiative properties of a PWN inside a nonradiative SNR through the entire evolutionary sequence described above. This last feature is important because there are many systems of interest where the PWN likely has collided with the SNR reverse shock (e.g., G328.4+0.2, Gelfand et al. 2007; G327.1−1.1, Temim et al. 2009; Vela X, LaMassa et al. 2008). This model also self-consistently couples the dynamical and radiative evolution of the PWN, including the evolution of the PWN’s magnetic field, and calculates the broadband (radio–TeV γ-ray evolution) of the PWN. As a result, it is well suited for both examining the effect of different supernova, neutron star, pulsar wind, and ISM properties on the evolution of the resulting PWN, and for using the observed properties of a PWN to determine the physical properties of the progenitor supernova, central neutron star, and surrounding ISM.

This paper is structured as follows. In Section 2, we describe the physics underlying our model and its implementation. In Section 3, we present and discuss the predicted evolution of a PWN for a particular set of input parameters. Finally, in Section 4, we discuss the implication of these result and potential applications of this model.

2. MODEL DESCRIPTION AND IMPLEMENTATION

In this section, we describe the underlying physics of this model for the evolution of a PWN inside an SNR (Section 2.1) and its implementation (Section 2.2).

2.1. Model Description

This model assumes the PWN is surrounded by a thin shell of material with radius $R_{\text{pwn}}$ and mass $M_{\text{sw.pwn}}$ expanding with velocity $v_{\text{pwn}}$ (e.g., Ostriker & Gunn 1971; Gelfand et al. 2007). If $v_{\text{pwn}}$ is larger than the velocity of the material surrounding this PWN, $v_{\text{sh}}(R_{\text{pwn}})$, we assume this shell sweeps up the surrounding material. The difference in pressure between the PWN interior and the PWN, $P_{\text{pwn}}$, and the surrounding SNR, $P_{\text{snr}}(R_{\text{pwn}})$, applies a force, $F_{\Delta p}$, on this mass shell equal to (Gelfand et al. 2007)

$$F_{\Delta p} = 4\pi R_{\text{pwn}}^2 \left( P_{\text{pwn}} - P_{\text{snr}}(R_{\text{pwn}}) \right).$$  

(1)

The resultant change in momentum of this shell of material is simply

$$\frac{d}{dt} (M_{\text{sw.pwn}} v_{\text{pwn}}) = F_{\Delta p},$$  

(2)

and we use this equation to calculate the dynamical evolution (e.g., $R_{\text{pwn}}(t)$ and $v_{\text{pwn}}(t)$) of the PWN. This approach requires modeling the SNR’s density, velocity, and pressure profiles with time and the pressure inside the PWN, $P_{\text{pwn}}$. We evolve the properties of the SNR using the results of previously developed analytic models described in Appendix A. Since no such description exists for a radiative SNR, our model only applies to a PWN inside an SNR in its free-expansion or Sedov–Taylor evolutionary phase.

The evolution of $P_{\text{pwn}}$ depends on the rate at which the neutron star injects energy into the PWN, the content of the pulsar wind, and the evolution of the particle and magnetic components of the PWN. We assume that all of the spin-down luminosity of the neutron star $\dot{E}$ is injected into the PWN (e.g., Equation (5) in Gaensler & Slane 2006):

$$\dot{E}(t) = \dot{E}_0 \left( 1 + \frac{t}{\tau_{\text{sd}}} \right)^{-\frac{p}{4}},$$  

(3)

where $p$ is the braking index, $\dot{E}_0$ is the initial spin-down luminosity of the neutron star, and $\tau_{\text{sd}}$ is the spin-down timescale.

As mentioned in Section 1, it is believed that neutron star will eventually be stripped of its PWN. When this occurs, it no longer injects energy into the “relic” PWN but forms a new PWN inside the SNR (van der Swaluw et al. 2004). Based on the simulations of van der Swaluw et al. (2004), we assume the neutron star leaves the PWN only after the PWN has collided with the SNR reverse shock, and when the distance the neutron star has traveled since the supernova explosion, $r_{\text{psr}}$, satisfies $r_{\text{psr}} > R_{\text{pwn}}$.

The energy of the pulsar wind is distributed between the electrons and positrons ($\dot{E}_{\text{inj,}\text{e}}$), ions ($\dot{E}_{\text{inj,}\text{i}}$), and magnetic fields ($\dot{E}_{\text{inj,}\text{B}}$) that comprise this outflow, such that

$$\dot{E} = \dot{E}_{\text{inj,}\text{e}} + \dot{E}_{\text{inj,}\text{i}} + \dot{E}_{\text{inj,}\text{B}}.$$  

(4)

To parameterize the content of the pulsar wind, we define the following variables:

$$\eta_e(t) \equiv \frac{\dot{E}_{\text{inj,}\text{e}}(t)}{\dot{E}(t)},$$

$$\eta_i(t) \equiv \frac{\dot{E}_{\text{inj,}\text{i}}(t)}{\dot{E}(t)},$$

$$\eta_B(t) \equiv \frac{\dot{E}_{\text{inj,}\text{B}}(t)}{\dot{E}(t)},$$  

(5-7)

where $\eta_e + \eta_i + \eta_B \equiv 1$. It is important to emphasize that $\eta_i$ is not equivalent to the magnetization parameter, $\sigma$, defined as the ratio of magnetic-to-particle energy (Kennel & Coroniti 1984a). The pulsar wind is injected into the PWN at the “termination shock,” located where the ram pressure of the unshocked wind is equal to $P_{\text{pwn}}$ (e.g., Goldreich & Julian 1969). This occurs at a distance from the neutron star, $r_{\text{ts}}$, equal to (e.g., Slane et al. 2004; Gaensler & Slane 2006)

$$r_{\text{ts}} = \sqrt{\frac{\dot{E}}{4\pi \xi c P_{\text{pwn}}}},$$

(8)

where $\xi$ is the filling factor of the pulsar wind (an isotropic wind has $\xi = 1$). In this model, we only concern ourselves with the content of the pulsar wind just downstream of the termination shock ($r > r_{\text{ts}}$), which is likely very different than the content of the pulsar wind near the neutron star (e.g., Arons 2008).

\footnote{The braking index $p$ is defined as $\Omega_{\text{p}} = -k\Omega_{\text{p}}^0$, where $\Omega_{\text{p}}$ is the spin frequency of the neutron star ($\Omega_{\text{p}} \equiv 2\pi/P$, where $P$ is the rotational period of the neutron star; e.g., Shapiro & Teukolsky 1986).}
Since the energy of the pulsar wind is divided among electrons and positrons, ions, and magnetic fields, the same must be true for the energy inside the PWN, \( E_{\text{pwn}} \):

\[
E_{\text{pwn}} = E_{\text{pwn,e}} + E_{\text{pwn,i}} + E_{\text{pwn,B}},
\]

where \( E_{\text{pwn,e}} \) is the kinetic energy of electrons and positrons in the PWN, \( E_{\text{pwn,i}} \) is the kinetic energy of ions in the PWN, and \( E_{\text{pwn,B}} \) is the energy stored in the magnetic field of the PWN. Each component contributes separately to the total pressure inside the PWN, such that \( P_{\text{pwn}} \) is

\[
P_{\text{pwn}} = P_{\text{pwn,e}} + P_{\text{pwn,i}} + P_{\text{pwn,B}},
\]

where \( P_{\text{pwn,e}} \) is the pressure associated with electrons and positrons in the PWN, \( P_{\text{pwn,i}} \) is the pressure associated with ions in the PWN, and \( P_{\text{pwn,B}} \) is the pressure associated with the PWN’s magnetic field. The energy and associated pressure of these components evolve differently from each other, as explained below.

In this model, we assume the magnetic field inside the PWN is uniform and isotropic. As a result, \( E_{\text{pwn,B}} \) is

\[
E_{\text{pwn,B}} = \frac{B_{\text{pwn}}^2}{8\pi} V_{\text{pwn}},
\]

where \( B_{\text{pwn}} \) is the strength of the PWN’s magnetic field, and the pressure associated with this magnetic field \( P_{\text{pwn,B}} \) is

\[
P_{\text{pwn,B}} = \frac{B_{\text{pwn}}^2}{8\pi}.
\]

We also assume that the magnetic flux of the PWN is conserved as \( R_{\text{pwn}} \) changes. As a result, \( B_{\text{pwn}} \propto R_{\text{pwn}}^{-2} \) neglecting any input from the neutron star. In this case, \( E_{\text{pwn,B}} \propto R_{\text{pwn}}^{-4} \) from Equation (11) and \( P_{\text{pwn,B}} \propto R_{\text{pwn}}^{-4} \) from Equation (12).

We additionally assume that the electrons, positrons, and ions inside the PWN are relativistic. Since the density inside the PWN is very low, it is safe to assume these particles are essentially collisionless and therefore behave as an ideal gas with an adiabatic index \( \gamma = 4/3 \). As a result, the pressure associated with particles inside the PWN, \( P_{\text{pwn,p}} \equiv P_{\text{pwn,e}} + P_{\text{pwn,i}} \), is

\[
P_{\text{pwn,p}} = \frac{E_{\text{pwn,p}}}{3V_{\text{pwn}}},
\]

where \( E_{\text{pwn,p}} \) is the total particle energy of the PWN (\( E_{\text{pwn,p}} \equiv E_{\text{pwn,e}} + E_{\text{pwn,i}} \)) and \( V_{\text{pwn}} \) is the volume of the PWN (\( V_{\text{pwn}} \equiv \frac{4}{3} \pi R_{\text{pwn}}^3 \)). The evolution of \( E_{\text{pwn,p}} \) and \( P_{\text{pwn,p}} \) is sensitive to the adiabatic and radiative losses suffered by the particles inside the nebula. Since we are assuming that all the particles inside the PWN are relativistic (all three populations have \( \gamma = 4/3 \)), the adiabatic evolution of \( E_{\text{pwn,p}} \) is independent of both the distribution of particle energy between these species and their energy spectra. If the evolution of the PWN is purely adiabatic, then \( P_{\text{pwn,p}} \propto R_{\text{pwn}}^{4/3} \) is constant, implying that \( E_{\text{pwn,p}} \propto R_{\text{pwn}}^2 \) and \( P_{\text{pwn,p}} \propto R_{\text{pwn}}^{-4} \).

The total radiative losses of electrons/positrons and ions in the PWN depend greatly on their energy spectra. This, in turn, is highly sensitive to the injection spectrum of particles into the PWN. Observations of several PWNs (e.g., 3C58; Slane et al. 2008, PWN G0.9+0.1; Venter & de Jager 2006) suggest a broken power-law injection spectrum for electrons and positrons, and recent theoretical work (e.g., Spitkovsky 2008) argues that the injection spectrum of these particles is a Maxwellian with a nonthermal tail. For simplicity, we assume a simple power-law injection spectrum for the electrons, positrons, and ions, but our formalism can easily accommodate a more complex input spectrum.

Assuming the injection spectrum of electrons and positrons is a single power law, we have

\[
n_e = n_{0,e} \left( \frac{E}{E_0} \right)^{-\gamma_e} \text{ electrons s}^{-1} \text{ keV}^{-1},
\]

where \( n_e, \Delta E \Delta t \) is the number of electrons and positrons with energy between \( E \) and \( E + \Delta E \) injected into the PWN in time \( \Delta t \). For consistency with Equation (4), we require that

\[
\dot{E}_{\text{inj,e}} \equiv \int_{E_{\text{e,min}}}^{E_{\text{e,max}}} n_e E dE,
\]

where \( E_{\text{e,min}} \) and \( E_{\text{e,max}} \) are, respectively, the minimum and maximum energies of electrons and positrons injected into the PWN. Therefore, the rate at which electrons and ions are injected into the PWN, \( \dot{N}_{\text{inj,e}} \), is

\[
\dot{N}_{\text{inj,e}} = \begin{cases} 
\frac{2 - \gamma_e}{1 - \gamma_e} \left( \frac{E_{\text{e,max}}}{E_{\text{e,min}}} \right)^{1 - \gamma_e} - \frac{E_{\text{e,min}}^{1 - \gamma_e}}{E_{\text{e,max}}^{1 - \gamma_e}} & \gamma_e \neq 1, 2 \\
\frac{E_{\text{e,min}} - E_{\text{e,max}}}{\ln(E_{\text{e,max}}/E_{\text{e,min}})} & \gamma_e = 2 \\
\frac{E_{\text{e,min}} - E_{\text{e,max}}}{\ln(E_{\text{e,max}}/E_{\text{e,min}})} & \gamma_e = 1
\end{cases}
\]

Similarly, we assume the injection spectrum of ions is a single power law:

\[
n_i = n_{0,i} \left( \frac{E}{E_0} \right)^{-\gamma_i} \text{ ions s}^{-1} \text{ keV}^{-1},
\]

where \( n_i, \Delta E \Delta t \) is the number of ions with energy between \( E \) and \( E + \Delta E \) injected into the PWN in time \( \Delta t \). Again, for consistency with Equation (4) we require that

\[
\dot{E}_{\text{inj,i}} \equiv \int_{E_{\text{i,min}}}^{E_{\text{i,max}}} n_i E dE,
\]

where \( E_{\text{i,min}} \) and \( E_{\text{i,max}} \) are the minimum and maximum energies of ions injected into the PWN. Correspondingly, the injection rate of ions into the PWN, \( \dot{N}_{\text{inj,i}} \), has the same functional form as Equation (16).

This model assumes the dominant radiative processes are synchrotron emission and inverse Compton scattering off background photons. For a particle with energy \( E \), mass \( m \), and charge \( q \), the rate at which it loses energy due to synchrotron emission, \( P_{\text{synch}} \), is (Equation (3.32) in Pacholczyk 1970)

\[
P_{\text{synch}} = \frac{2q^2}{3m^3c^2} B_{\text{pwn}}^2 \sin^2 \theta E^2,
\]

where \( \theta \) is the angle between the particle’s velocity and the magnetic field. We assume that the velocities of particles in the PWN are randomly oriented with respect to the magnetic field, such that \( \sin^2 \theta = 2/3 \) (Rybicki & Lightman 1979, p. 393). When the PWN is small, synchrotron self-absorption is important (e.g., Reynolds & Chevalier 1984). To determine its
\[ \nu = \text{that a particle emits all of its synchrotron radiation at a frequency} \]

\[ \text{effect on the dynamics and emission of the PWN, we first assume} \]

\[ \text{Ratio of the inverse Compton cross section of an electron scattering} \]

\[ \text{Figure 1.} \]

\[ \text{Table 1} \]

\[ \text{Model Input Parameters} \]

| Parameter | Units | Description |
|-----------|-------|-------------|
| \( E_{\text{kin,1}} \) | \( 10^{51} \text{ erg} \) | Initial kinetic energy of supernova ejecta |
| \( M_{\text{ej}} \) | \( M_{\odot} \) | Mass of supernova ejecta |
| \( \eta_{\text{num}} \) | \( \text{cm}^{-1} \) | Number density of the Surrounding ISM |
| \( \rho_{\text{rad}} \) | \( \text{...} \) | Characteristic spin-down timescale of the neutron star |
| \( E_{0,40} \) | \( 10^{40} \text{ erg s}^{-1} \) | Initial spin-down luminosity of the neutron star |
| \( v_{\text{par}} \) | \( \text{km s}^{-1} \) | Space velocity of the neutron star |
| \( \eta_{e} \) | \( \text{...} \) | Fraction of neutron's star spin-down luminosity injected as electrons |
| \( \eta_{i} \) | \( \text{...} \) | Fraction of neutron's star spin-down luminosity injected as ions |
| \( E_{e,\text{min}} \) | \text{Varies} | Minimum energy of electrons injected into the PWN |
| \( E_{e,\text{max}} \) | \text{Varies} | Maximum energy of electrons injected into the PWN |
| \( E_{i,\text{min}} \) | \text{Varies} | Minimum energy of ions injected into the PWN |
| \( E_{i,\text{max}} \) | \text{Varies} | Maximum energy of ions injected into the PWN |
| \( \gamma_{e} \) | \text{...} | Electron injection index |
| \( \gamma_{i} \) | \text{...} | Ion injection index |

where \( u_{\text{rad}} \) is the energy density of the background photon field, and \( f(E) \) is the inverse Compton scattering cross section relative to the Thomson cross section. \( f(E) \) is sensitive to both the particle energy and the spectrum of the background photon field. For most PWNe, the cosmic microwave background (CMB; \( \eta_{\text{rad}} \approx 4.17 \times 10^{-13} \text{ erg cm}^{-3} \)) is believed to be the dominant source of background photons, and we calculate \( f(E) \) for this field (Section 2.3 of Volpi et al. 2008; Figure 1). It is possible that other photon fields, e.g., starlight, not considered here are important for a given PWN—and a simple adjustment to the formalism presented here can account for these as well.

### 2.2. Model Implementation

To determine the evolution of a PWN inside an SNR, we use the equations presented in Section 2.1 to determine how the relevant properties of a PWN at a time \( t \) evolve to a time \( t + \Delta t \). We determine the initial conditions of the PWN inside an SNR using the equations given in Appendix B, and the free parameters of this model are listed in Table 1. We calculate the properties of the PWN at time \( t + \Delta t \) using the following procedure.

1. **Step 1.** Calculate the new radius of the PWN,

\[ R_{\text{pwn}}(t + \Delta t) = R_{\text{pwn}}(t) + v_{\text{par}}(t) \times \Delta t, \quad (23) \]

and the distance the neutron star has traveled inside the SNR since birth:

\[ r_{\text{psr}}(t + \Delta t) = v_{\text{psr}} \times (t + \Delta t), \quad (24) \]

where \( v_{\text{psr}} \) is the space velocity of the neutron star.

2. **Step 2.** Adjust the electron/positron, ion, and magnetic field energy of the PWN for the adiabatic work done by the PWN on its surroundings and, if the neutron star is still inside the PWN, the energy it injects between times \( t \) and \( t + \Delta t \):

\[ E_{\text{pwn,e}}(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E_{\text{pwn,e}}(t) + \eta_{e} \int_{t}^{t+\Delta t} \dot{E} \, dt, \quad (25) \]

\[ E_{\text{pwn,i}}(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E_{\text{pwn,i}}(t) + \eta_{i} \int_{t}^{t+\Delta t} \dot{E} \, dt, \quad (26) \]

\[ E_{\text{b}}(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E_{\text{b}}(t) + \eta_{B} \int_{t}^{t+\Delta t} \dot{E} \, dt, \quad (27) \]
**Table 2**
Trial Model Input Parameters

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $E_{nk,51}$ | 1     | $n_{e}$ | 0.999 |
| $M_{ej}$ | 10    | $n_{i}$ | 0     |
| $n_{gal}$ | 0.1   | $\eta_{n}$ | 0.001 |
| $p$ | 3.00  | $E_{e,\text{min}}$ | 511 keV |
| $\tau_{ad}$ | 500   | $E_{e,\text{max}}$ | 500 TeV |
| $E_{0,80}$ | 1     | $E_{i,\text{min}}$ | ... |
| $v_{psr}$ | 120   | $E_{i,\text{max}}$ | ... |
| $\eta_{r}$ | 1.6   | $\eta_{e}$ | ... |

$$E_{\text{pwn},i}(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E_{\text{pwn},i}(t) + \eta_{i} \int_{t}^{t + \Delta t} \dot{E} dt,$$

(26)

$$E_{\text{pwn},b}(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E_{\text{pwn},b}(t) + \eta_{B} \int_{t}^{t + \Delta t} \dot{E} dt,$$

(27)

where $\dot{E}$ is defined in Equation (3) and $\int \dot{E} dt$ is solved analytically. We then use $E_{\text{pwn},b}(t + \Delta t)$ to calculate $R_{\text{pwn}}(t + \Delta t)$ using Equation (11) and $P_{\text{pwn},b}(t + \Delta t)$ using Equation (12).

3. **Step 3.** Calculate the energy spectrum of electrons, positrons, and ions inside the PWN at time $t + \Delta t$ taking into account adiabatic, synchrotron, and inverse Compton losses. A particle with energy $E$ at time $t$ has an energy at time $t + \Delta t$ equal to

$$E(t + \Delta t) = \frac{R_{\text{pwn}}(t)}{R_{\text{pwn}}(t + \Delta t)} E - [P_{\text{synch}}(t) + P_{\text{IC}}(t)] \Delta t,$$

(28)

where $P_{\text{synch}}(t)$ (Equation (19)) is calculated using $B_{\text{pwn}}(t)$ and $P_{\text{IC}}(t)$ is defined in Equation (22), with the restriction that $E(t + \Delta t) \geq 0$. We then subtract from the value of $E_{\text{pwn},e}(t + \Delta t)$ and $E_{\text{pwn},i}(t + \Delta t)$ calculated in Step 3 the total synchrotron and inverse Compton losses of these particles. Even though the radiative losses of ions in the PWN are significantly smaller than that of electrons and positrons (both $P_{\text{synch}}$ and $P_{\text{IC}}$ are $\propto m^{-4}$), for completeness they are calculated. We then determine $P_{\text{pwn},i}(t + \Delta t)$ using Equation (13).

4. **Step 4.** Calculate the properties of the surrounding SNR [$\rho_{ej}(R_{\text{pwn}}, t + \Delta t)$, $v_{ej}(R_{\text{pwn}}, t + \Delta t)$, and $P_{\text{sm}}(R_{\text{sw}}, t + \Delta t)$] using the prescription described in Appendix A. If $v_{ej}(R_{\text{pwn}}, t + \Delta t) < v_{psr}(t)$, then

$$M_{\text{sw},\text{pwn}}(t + \Delta t) = M_{\text{sw},\text{pwn}}(t) + \frac{4}{3} \pi (R_{\text{pwn}}(t + \Delta t)^{3} - R_{\text{pwn}}(t)^{3}) \rho_{ej}(R_{\text{pwn}}, t + \Delta t),$$

(29)

where $\rho_{ej}(R_{\text{pwn}})$ is the density inside the SNR just outside the PWN. Otherwise, $M_{\text{sw},\text{pwn}}(t + \Delta t) = M_{\text{sw},\text{pwn}}(t)$.

5. **Step 5.** Using the properties of the PWN and SNR determined above, calculate $F_{\text{ap}}(t)$ (Equation (1)). The new velocity of the mass shell surrounding the PWN, $v_{\text{pwn}}(t + \Delta t)$ is

$$v_{\text{pwn}}(t + \Delta t) = \frac{1}{M_{\text{sw},\text{pwn}}(t + \Delta t)} \times [M_{\text{sw},\text{pwn}}(t)v_{\text{pwn}}(t) + \Delta M_{\text{sw},\text{pwn}}v_{ej}(R_{\text{pwn}}, t) + F_{\text{pwn}}(t) \times \Delta t],$$

(30)

from conservation of momentum arguments, where

$$\Delta M_{\text{sw},\text{pwn}} \equiv M_{\text{sw},\text{pwn}}(t + \Delta t) - M_{\text{sw},\text{pwn}}(t).$$

3. **MODEL PERFORMANCE**

In this section, we present the evolution of a PWN inside an SNR predicted by this model for the combination of the input variables listed in Table 2. It is important to emphasize that these results are specific to this set of parameters, and that different PWNe may have very different values for some or all of these quantities. The properties of the progenitor supernova ($E_{nk}$ and $M_{ej}$), central neutron star ($E_{i}$, $\tau_{ad}$, and $p$), and surrounding ISM ($\eta_{n}$) are the same as Model A in Blondin et al. (2001) — chosen by these authors to produce a PWN similar to that of the Crab Nebula. We also assume a purely electron/positron pulsar wind (i.e., $\eta_{e} = 0$), and that the properties of the pulsar wind ($\eta_{n}$, $E_{e,\text{min}}$, $E_{e,\text{max}}$, $\rho_{ej}$) remain constant with time. Lasty, we set $v_{psr} = 120$ km s$^{-1}$, the most recent measurement of the transverse velocity of the Crab pulsar (Kaplan et al. 2008). As shown in Figure 2, for this particular set of input parameters our model predicts four evolutionary stages.

1. **Initial expansion.** This phase ends $t_{\text{col}} \sim 4500$ years after the supernova explosion when the PWN collides with the reverse shock.

2. **Reverse shock collision and first compression.** This phase ends $\sim 20,000$ years after the supernova explosion. During this phase, we expect the PWN to be stripped of its neutron star $\sim 17,000$ years after the supernova explosion.

3. **Re-expansion.** This phase ends $\sim 56,000$ years after the supernova. During this phase, the neutron star re-enters the “relic” PWN created during the first compression $\sim 30,000$ years after the supernova explosion.

4. **Second compression.** This phase continues until the SNR enters the radiative phase of its evolution. During this contraction, the PWN is stripped of its neutron star $\sim 70,000$ years after the supernova explosion.

The dynamical and radiative evolution of the PWN in these four phases is described in detail below.
The initial expansion of the PWN is the result of $P_{\text{pwn}} \gg P_{\text{sw}}(R_{\text{pwn}})$ (Figure 3). This is the case because $P_{\text{sw}}(R_{\text{pwn}}) \approx 0$ when $R_{\text{pwn}} < R_{\text{ss}}$, i.e., adiabatic expansion of the SNR makes the ejecta interior of the reverse shock cold (Appendix A). During this time, the PWN is expanding faster than the surrounding ejecta (Figure 4), and $M_{\text{sw,pwn}}$ increases (Figure 5). For most of this expansion, the energy input for the neutron star ($\dot{E}$) is larger than the adiabatic losses resulting from the PWN’s expansion and the radiative losses suffered by electrons inside the PWN (Figure 6), causing the internal energy of the PWN to rise (Figure 7). At the very earliest times ($t < 100$ years), the synchrotron luminosity of the PWN approaches $\dot{E}$ due to the PWN’s very strong magnetic field (Figure 8). This causes the magnetization parameter of the PWN, $\sigma$, to temporarily increase by more than an order of magnitude (Figure 9). For $t \lesssim 1000$ years, synchrotron losses dominate over adiabatic and inverse Compton losses (Figure 6), causing the PWN to expand slower than that predicted by Equation (B7). This difference is small at early times, so Equation (B7) does provide a reasonably accurate initial condition. The expansion of the PWN causes $\dot{E}$ to decrease rapidly (Figure 8), resulting in a steep decline in the PWN’s synchrotron luminosity. As a result, adiabatic losses dominate from $\sim 1000$ years after the supernova explosion until the PWN collides with the reverse shock. These adiabatic losses cause both $\dot{E}_{\text{pwn}}$ and $\dot{E}_{\text{pwn,B}}$ to decrease for $t > 1000$ years, while $E_{\text{pwn,B}}$ increases due to the continued injection of particles into the PWN. It is important to note that $\dot{E}_{\text{pwn}}$ does not follow $\dot{E}_{\text{pwn}} \propto 1/(1 + t/\tau)^\alpha$, as assumed in other work (e.g., Venter & de Jager 2006). We find that $\dot{E}_{\text{pwn}} \propto t^{-1.7}$ during the initial expansion, similar to the $t^{-1.3} - t^{-2}$ behavior derived by Reynolds & Chevalier (1984).

This behavior ends when the PWN collides with the SNR reverse shock. This collision shocks the swept-up ejecta surrounding the PWN, but not the PWN itself because the sound speed inside the PWN ($\sim c/\sqrt{3}$) is significantly higher than the velocity of the reverse shock. The collision marks the end of the PWN’s rapid expansion (Figure 2) because, at the time of this collision, $E_{\text{pwn}} \ll P_{\text{sw}}(R_{\text{pwn}})$ (Figure 3). This causes $\dot{E}_{\text{pwn}}$ to decrease significantly (Figure 4), though—due to the momentum...
of the mass shell surrounding the PWN—\(v_{\text{pwn}} > v_{\text{cs}}(R_{\text{pwn}})\) for the first \(\sim 100\) years after this collision (Figure 4). As a result, \(M_{\text{cw,pwn}}\) continues to rise, increasing from \(\sim 1 M_\odot\) to \(\sim 3 M_\odot\) (Figure 5) due to the high density behind the reverse shock. The sharp decrease in \(v_{\text{pwn}}\) also leads to a sharp decrease in adiabatic losses (Figure 6). Because the pulsar continues to inject energy into the PWN, both \(E_{\text{pwn,p}}\) and \(E_{\text{pwn,B}}\) initially rise rapidly after the PWN/reverse shock collision (Figure 7), though \(P_{\text{pwn}}\) remains \(P_{\text{pwn}} \ll P_{\text{sw}}(R_{\text{pwn}})\). As a result, the high-pressure ejecta downstream of the reverse shock will compress the PWN in the beginning \(\sim 6500\) years after the supernova explosion.

This compression begins a series of contractions and re-expansions that continue until the SNR enters the radiative phase of its evolution (Figure 2). Similar behavior was observed in previous work (e.g., Blondin et al. 2001; Bucciantini et al. 2004a; van der Swaluw et al. 2001, 2004). The adiabatic compression of the PWN causes a sharp rise in \(P_{\text{pwn}}\) (Figure 3) and both \(E_{\text{pwn,p}}\) and \(E_{\text{pwn,B}}\) (Figure 7). The compression of the PWN also causes a rapid rise in \(B_{\text{pwn}}\) (Figure 8), which in turn results in a rapid rise in the PWN’s synchrotron luminosity (Figure 6)—as previously noted by Reynolds & Chevalier (1984). Eventually, the synchrotron losses are larger than both the work done by the surrounding SNR on the PWN and \(\dot{E}\), causing \(E_{\text{pwn,p}}\) to decrease (this would have occurred during the second compression if we evolved the PWN further in time). This decrease in \(E_{\text{pwn,p}}\) leads to a decrease in the synchrotron luminosity of the PWN (Figure 6), even though \(B_{\text{pwn}}\) still increases (Figure 8). Since synchrotron losses do not diminish \(E_{\text{pwn,B}}\), \(\sigma\) increases significantly during the compression phases of the PWN (Figure 9).

Due to the space velocity of the neutron star, we expect the PWN to be stripped of its pulsar during both compressions. This result is strongly dependent on the space velocity of the pulsar. For a lower space velocity, it is possible that the PWN will not be stripped of its pulsar until the pulsar leaves the SNR, while for a higher velocity the PWN will not overtake the pulsar during its re-expansion. The departure of the pulsar causes a rapid increase of \(\sigma\) (Figure 9) and a rapid decrease in the PWN’s synchrotron luminosity since it removes the only source of high-energy particles that dominate this emission (Figure 6). The latter does not occur during the second compression due to the rapid rise in \(B_{\text{pwn}}\) when the pulsar exits the PWN. It is expected that the pulsar creates a new PWN inside the SNR after it leaves its original PWN (e.g., van der Swaluw et al. 2004) whose properties we do not calculate.

Eventually, during the compression stage \(P_{\text{pwn}}\) will exceed \(P_{\text{sw}}(R_{\text{pwn}})\), causing the PWN to re-expand. During the first contraction, the pressure inside the PWN begins to exceed that of the surrounding SNR \(\sim 5000\) years after the compression begins (Figure 3). At this time, the PWN is contracting at a very high velocity (\(>1000\) km s\(^{-1}\); Figure 4), and due to the considerable inertia of the swept-up material surrounding the PWN, it takes \(\sim 8000\) years for the PWN to re-expand (Figure 2). Therefore, \(P_{\text{pwn}} > P_{\text{sw}}(R_{\text{pwn}})\) when re-expansion begins (Figure 3). During this phase, the PWN sweeps up additional material though, due to the low density inside the SNR at this time, it only accumulates \(\sim 0.5 M_\odot\) of additional material (Figure 5). Initially during the re-expansion, \(E_{\text{pwn,p}}\), \(E_{\text{pwn,B}}\) (Figure 7), and \(B_{\text{pwn}}\) (Figure 8) rapidly decrease due to adiabatic losses. The decrease in \(E_{\text{pwn}}\) and increase in \(R_{\text{pwn}}\) result in a rapid decrease in \(P_{\text{pwn}}\) (Figure 3), and the decrease in both \(B_{\text{pwn}}\) and \(E_{\text{pwn,p}}\) results in a rapid decrease of the PWN’s synchrotron luminosity (Figure 6). During the re-expansion, \(\sigma\) initially continues to increase because the synchrotron and
The complicated dynamical evolution of a PWN described above causes the energy spectrum of electrons and positrons in the PWN, and the photons they radiate, to change considerably with time. As mentioned earlier, synchrotron losses play a very important role during the initial expansion of the PWN. For $t \ll \tau_{sd}$, the magnetic field of the PWN is so strong that the only high-energy ($E \gtrsim 1$ TeV) electrons and positrons in the PWN are those which were injected very recently (Figure 10). For $\tau_{sd} < t < \tau_{cal}$, the electron and positron spectrum of the PWN is dominated by previously injected particles at all energies, and the sharp high-energy cutoff observed at early times is replaced with a more gradual turnover whose sharpness increases with time (Figure 10). This is due to the increase peak energy of the electron and positron spectrum resulting from the decrease in $B_{pwn}$ (as first noted by Reynolds & Chevalier 1984).

Correspondingly, the spectrum of photons radiated by the PWN during its initial expansion also evolves with time (Figure 11). As explained in Section 2, the only emission mechanisms we consider are synchrotron radiation and inverse Compton scattering off the CMB, and the photon spectrum is calculated using the procedure described in Section 2 of Volpi et al. (2008). At very early times ($t \lesssim 50$ years), the PWN is extremely luminous at GeV energies due to the large number of high-energy electrons and positrons recently injected by the neutron star (Figure 12). For $t < \tau_{sd}$, the PWN’s synchrotron...
emission peaks at photon energies $\gtrsim 100$ keV (Figure 11), and the PWN’s luminosity is highest in the hard X-ray regime (Figure 12). Since recently injected particles are energetically important, both the synchrotron and inverse Compton spectrum have two peaks—a lower-energy peak resulting from previously injected particles and a higher-energy peak resulting from recently injected ones. For $t_{\text{sd}} < t < t_{\text{col}}$, the synchrotron and inverse Compton spectrum have a single peak since recently injected particles no longer dominate. During the initial expansion, the energy peak of the synchrotron spectrum decreases from $\sim 1$ MeV at $t \ll t_{\text{sd}}$ to $\sim 1$–10 keV at $t \sim t_{\text{col}}$, causing a rapid decrease in the hard X-ray luminosity of the PWN. Slower decreases are predicted for the radio–soft X-ray luminosity (Figure 12) due to the gradual decline in the synchrotron luminosity of the PWN (Figure 6) resulting from the decreasing value of $B_{\text{pwn}}$ (Figure 8). Conversely, the energy peak of inverse Compton emission increases from $\sim 1$ TeV at $t \sim t_{\text{sd}}$ to $\sim 50$ TeV at $t \sim t_{\text{col}}$ due to the increase in the break energy of the electron spectrum discussed above, leading to a rapid rise in the GeV and TeV $\gamma$-ray luminosity of the PWN (Figure 12).

The energy spectrum of electrons and positrons inside the PWN changes significantly during the first contraction (Figure 10). Due to the strong magnetic field inside the PWN during this phase (Figure 8), the synchrotron lifetime of the highest energy ($E \gtrsim 10$ TeV) electrons and positrons in the PWN becomes significantly less than the age of the PWN. This results in a sharp cutoff in the energy spectrum where the synchrotron lifetime of electrons and positrons is the age of the PWN, above which recently injected particles dominate. The strengthening $B_{\text{pwn}}$ results in this energy decreasing with time. This causes the peak photon energy of the synchrotron emission to decrease significantly, from $\sim 10$ keV when the PWN collides with the reverse shock to $\sim 100$ eV when the neutron star exits the PWN. The spectrum of the inverse Compton emission radiated by the PWN also changes considerably during this time—with the energy peak decreasing from $\sim 10$ TeV at the time of the PWN/reverse shock collision to $\sim 100$ GeV when the neutron star leaves. The luminosity of the PWN in wave bands dominated by synchrotron emission (radio–soft X-rays; Figure 11) increases (Figure 12) due to the strengthening magnetic field. The decreasing energy of the inverse Compton peak causes the hard X-ray and GeV $\gamma$-ray luminosity of the PWN to increase and the TeV $\gamma$-ray luminosity to decrease (Figure 12), though the latter is mitigated by inverse Compton emission from the highest energy recently injected particles.

(A color version of this figure is available in the online journal.)
The departure of the neutron star from the PWN during the first contraction has a dramatic effect on the electron and positron energy spectrum in the PWN. When this occurs, the only high-energy ($E > 10$ TeV) electrons and positrons in the PWN are those recently injected by the neutron star. Therefore, the departure of the neutron star causes this plateau of high-energy electrons and positrons, inverse Compton emission from “old” electrons and positrons, synchrotron emission from “new” electrons and positrons, inverse Compton emission from “old” particles, and the radio luminosity of the PWN decreases (Figure 12) due to their decreasing energy. At all other wave bands, the emission is dominated by recently injected particles. Initially, the synchrotron emission from these electrons and positrons extends to $\sim 100$ keV, but due to adiabatic and synchrotron losses this decreases to $\sim 10$ keV by the PWN that begins its second contraction (Figure 11). As a result, after an initial increase which marks the neutron star’s re-entry, both the soft X-ray and hard X-ray luminosities of the PWN will decrease—though the decline in the soft X-ray luminosity is significantly slower than the hard X-ray luminosity (Figure 12). The inverse Compton emission from the recently injected electrons and positrons extends to TeV energies (Figure 11), and the $\gamma$-ray luminosity of the PWN increases between the re-entry of the pulsar and the second compression (Figure 12).

During the second contraction of the PWN, the energy spectrum of electrons and positrons inside the PWN maintains the two-component structure created during the PWN’s re-expansion (Figure 10). The peak energy of the “old” electron population increases slightly from $\sim 1$ GeV to $\sim 5$ GeV due to the work done on the PWN by the surrounding ejecta, but the energy peak of the “new” electron population decreases from $\sim 100$ TeV to $\sim 1$ TeV (Figure 10) due to the increasing strength of the PWN’s magnetic field and the second departure of the pulsar from the PWN $\sim 15,000$ years after the second contraction begins. During the second contraction, the photon spectrum of the PWN has four distinct regimes: from lowest to highest energy, they are synchrotron emission from “old” electrons and positrons, synchrotron emission from “new” electrons and positrons, inverse Compton emission from “old” particles, and inverse Compton emission from “new” particles (Figure 11). During the second compression, the energy peaks corresponding to emission from the old particles increase—the peak frequency of the synchrotron emission decreases from $\sim 100$ GeV to $\sim 10$ MeV (Figure 11), causing the PWN to be much more luminous at GeV energies than TeV energies (Figure 12).

As the PWN re-expands into the SNR, but before it overtakes the pulsar, the maximum energy of electrons and positrons in the PWN decreases from $\sim 10$ GeV to $\sim 1$ GeV (Figure 10). This fact, and the decrease in $B_{\text{sync}}$ (Figure 8), causes the peak frequency of the PWN’s synchrotron spectrum to decrease to $\sim 1$ GHz and the peak of the inverse Compton radiation to decrease to $\sim 100$ keV. As a result, when the PWN is re-expanding inside the SNR but before it overtakes the neutron star, almost all of its emission is in the radio, soft X-ray, and hard X-ray bands (Figure 12). When the neutron star re-enters the PWN, it resumes injecting high-energy electrons and positrons into the PWN—resulting in an electron spectrum with two distinct components: a lower-energy population composed of electrons and particle injected at earlier times, and a higher-energy population composed of recently injected particles (Figure 10). During the PWN’s re-expansion, the peak energy of the “old” particles continues to decrease due to adiabatic losses while the peak energy of the “new” particles decreases due to synchrotron losses (Figure 10). This dichotomy also extends to the photon spectrum (Figure 11). The radio emission from the PWN is dominated by synchrotron radiation from the old particles, and the radio luminosity of the PWN decreases (Figure 12) due to their decreasing energy. At all other wave bands, the emission is dominated by recently injected particles. Initially, the synchrotron emission from these electrons and positrons extends to $\sim 100$ keV, but due to adiabatic and synchrotron losses this decreases to $\sim 10$ keV by the PWN that begins its second contraction (Figure 11). As a result, after an initial increase which marks the neutron star’s re-entry, both the soft X-ray and hard X-ray luminosities of the PWN will decrease—though the decline in the soft X-ray luminosity is significantly slower than the hard X-ray luminosity (Figure 12). The inverse Compton emission from the recently injected electrons and positrons extends to TeV energies (Figure 11), and the $\gamma$-ray luminosity of the PWN increases between the re-entry of the pulsar and the second compression (Figure 12).

Figure 12. Photon luminosity of the PWN in the radio ($\nu = 10^7$–$10^{11}$ Hz, $L_{\text{radio}}$, red), mid-infrared ($\lambda = 3.6$–$160$ $\mu$m, orange), near-infrared/optical ($\lambda = 2.35$ $\mu$m–$354.3$ nm, yellow), soft X-ray ($h\nu = 0.5$–$10$ keV, dark green), hard X-ray ($h\nu = 15$ keV–$10$ MeV, blue), $\gamma$-ray ($h\nu = 10$ MeV–$100$ GeV, dark blue), and TeV $\gamma$-ray ($h\nu = 50$ GeV–$50$ TeV, purple) for $t \leq t_{\text{cut}}$ (top) and $t > t_{\text{cut}}$ (bottom). The definition of the wave bands was chosen to reflect the frequency/wavelength/energy coverage of current observing facilities, and is given in units usually associated with that portion of the electromagnetic spectrum. In the bottom plot, the vertical dotted lines demarcate the second, third, and fourth evolutionary phases discussed in Section 3. (A color version of this figure is available in the online journal.)
the synchrotron peak decreases from \( \sim 10 \) keV to \( \sim 1 \) eV, while the inverse Compton peak decreases from \( \sim 100 \) TeV to \( \sim 10 \) GeV, causing an increase in the optical and mid-IR luminosities and a decrease in the \( \gamma \)-ray luminosity of the PWN.

The observable properties discussed above are calculated using the predictions of the model. Unfortunately, for most PWNe the only observational data available are a flux density measurement at two or three radio frequencies (typically, 1.4, 4.8, and 8.5 GHz) and a measurement of the X-ray spectrum between \( E_{\gamma} \sim 0.5 \) and 10 keV—though due to interstellar absorption, for several PWNe it is only possible to measure the spectrum between \( E_{\gamma} = 2 \) and 10 keV (e.g., G328.4+0.2; Gelfand et al. 2007). This information is then used to estimate the properties of the PWN, for example, its energetics and strength of its magnetic field. The radio and unabsorbed X-ray spectrum are usually fit to a power law, \( L_{\nu} \propto \nu^\alpha \), where \( \alpha \) is the spectral index. (The X-ray spectrum is often fit using \( N_\nu \propto \nu^{-1} \), where \( N_\nu \) is the number density of observed photons and \( \Gamma \) is the photon index, where \( \Gamma = 1 - \alpha \).) We derive the spectral index of the PWN in the radio, 0.5–10 keV, and 2–10 keV bands throughout its evolution. The radio spectral index is \( \sim -0.3 \) at all times except for two periods where \( \alpha \ll -1.6 \) (Figure 13)—during

\[ v_b = 10^{21} \left( \frac{B_{\text{pwn, break}}}{10^{-6} \text{ G}} \right)^{-3} \left( \frac{t}{10^3 \text{ years}} \right)^{-2} \text{ Hz,} \]  

where \( B_{\text{pwn, break}} \) is the magnetic field strength derived for a given age \( t \) and \( v_b \) inferred from the radio and X-ray spectrum. This method does a poor job of reproducing \( B_{\text{pwn}} \), either underpredicting or overpredicting \( B_{\text{pwn}} \) by several orders of magnitude at most times and does not reproduce the evolution of \( B_{\text{pwn}} \) discussed above (Figure 16). Another, more complicated method uses the observed radio and X-ray spectral index of the PWN, \( v_b \), the inferred luminosity density of the PWN at this break frequency \( (L_{\nu_b}) \), and assumes that \( E_{\text{pwn, B}} = (3/4)E_{\text{pwn, p}} \) (minimum energy estimate; Chevalier 2005). This method does a significantly better job of reproducing \( B_{\text{pwn}} \) than Equation (31), though it overpredict \( B_{\text{pwn}} \) by factors of a few at most times (Figure 16). This method also underpredicts \( E_{\text{pwn}} \) by a factor of \( \sim 5-10 \) (Figure 17), largely because it assumes a value of \( \sigma (\sigma = 3/7) \) considerably higher than the actual value of \( \sigma \). (Figure 9).

The above discussion assumes the PWN is not disrupted by any hydrodynamical instabilities as it evolves inside the SNR. However, this is not necessarily the case. The shell of swept-up material surrounding the PWN is unstable to Rayleigh–Taylor
instabilities when \( P_{\text{pwn}} > P_{\text{sh}}(R_{\text{pwn}}) \) because the low-density pulsar wind \((\rho_{\text{pwn}})\) is accelerating this much higher-density shell \((\rho_{\text{sw}})\) (Chandrasekhar 1961). Unfortunately, since our model is inherently one dimensional, it cannot directly determine the growth rate of these instabilities—but does calculate the properties of the PWN necessary for an analytical estimate. The growth rate of these instabilities depends strongly on the magnetic field strength inside the PWN parallel to the boundary between the pulsar wind and swept-up supernova ejecta, \( B_{\text{pwn},l} \). Numerical simulations of the magnetic field inside the PWN suggest that, at early times, the PWN’s magnetic field is largely toroidal (e.g., van der Swaluw 2003), in which case \( B_{\text{pwn},l} \approx B_{\text{pwn}} \)—though polarized radio observations of older PWNe (e.g., Vela X; Milne 1980, Dodson et al. 2003) suggest a strong radial component to their outer magnetic field. Assuming \( B_{\text{pwn},l} = B_{\text{pwn}} \), the growth rate \( \omega_{\text{p}}(k) \) of a Rayleigh–Taylor (Kruskal–Schwarzschild; Kruskal & Schwarzschild 1954) instability with wavenumber \( k \equiv 2\pi/\lambda \) is (Chandrasekhar 1961; Bucciantini et al. 2004a)

\[
\omega_{\text{rt}}^2(k) = \frac{a_{\text{pwn}} k (\rho_{\text{sw}} - \rho_{\text{pwn}})}{\rho_{\text{sw}} + \rho_{\text{pwn}}} - \frac{B_{\text{pwn}}^2 k^2}{2\pi (\rho_{\text{sw}} + \rho_{\text{pwn}})},
\]

(32)

where \( a_{\text{pwn}} \) is the acceleration of the shell of swept-up material \((a_{\text{pwn}} \equiv dv_{\text{pwn}}/dt)\). Hydrodynamic simulations of the expansion of the PWN inside an SNR suggest that this shell of swept-up material that surrounds the PWN has thickness \( \approx R_{\text{pwn}}/24 \) (van der Swaluw et al. 2001), and we assume this is true at all times when calculating \( \rho_{\text{sw},\text{pwn}} \). As a result, the maximum wavenumber \( k_{\text{crit}} \) of a Rayleigh–Taylor instability which can grow is (Bucciantini et al. 2004a)

\[
k_{\text{crit}} = \frac{2\pi a_{\text{pwn}}}{B_{\text{pwn}}^2} (\rho_{\text{sw}} - \rho_{\text{pwn}}),
\]

(33)

and the wavenumber of the Rayleigh–Taylor instability with the highest growth rate \( k_{\text{max}} = k_{\text{crit}}/2 \) (Chandrasekhar 1961; Stone & Gardiner 2007). Rayleigh–Taylor instabilities between the pulsar wind and the swept-up ejecta result in “bubbles” of swept-up ejecta entering the PWN. Three-dimensional numerical simulations of the growth of Rayleigh–Taylor instabilities in a PWN-like scenario (e.g., a strong magnetic field in the light fluid parallel to the interface with the heavy fluid) suggest that the penetration of bubbles is relatively unimpeded by the presence of a magnetic field, though mixing between these two fluids is suppressed (Stone & Gardiner 2007). To estimate the penetration of swept-up ejecta bubbles into the PWN, we use the results of relevant laboratory experiments, which suggest the penetration depth of these bubbles, \( h_{\text{rt}} \), is (Dimonte & Schneider 1996; Dimonte et al. 2007)

\[
h_{\text{rt}} = \alpha_{b} A \left[ \int \sqrt{a_{\text{pwn}}(t)} dt \right]^2,
\]

(34)

where \( \alpha_{b} \) is an experimental derived constant \((\alpha_{b} \approx 0.061; \text{Dimonte & Schneider 1996})\), and \( A \) is the Atwood number of this system:

\[
A \equiv \frac{\rho_{\text{sw}} - \rho_{\text{pwn}}}{\rho_{\text{sw}} + \rho_{\text{pwn}}}
\]

(35)

Since \( \rho_{\text{sw}} \gg \rho_{\text{pwn}} \), \( A \approx 1 \).

We find that the PWN is unstable to Rayleigh–Taylor instabilities during the initial expansion \((t < t_{\text{crit}})\), and parts of the first contraction, re-expansion \((t \sim 15,000–30,000 \text{ years})\), and second contraction \((t \gtrsim 85,000 \text{ years})\). During the initial expansion, we expect that only Rayleigh–Taylor instabilities on the smallest angular scales \((\lesssim 1')\) are suppressed by the nebular magnetic field (Figure 18), while during the first and second contractions Rayleigh–Taylor instabilities at significantly larger angular scale are suppressed (Figure 18) due to the PWN’s strong magnetic field (Figure 8). The growth rate of these instabilities depends significantly on their angular scale, and during the initial expansion instabilities with an angular size of \( \sim 6' \) grow the fastest (Figures 18 and 19). Initially, during the contractions, the Rayleigh–Taylor instabilities with the highest growth rates have an angular scale \( \sim 5' \) (Figures 18 and 19). However, as the PWN re-expands, the growth rate of instabilities on these angular scales decreases and eventually suppressed—for example, Rayleigh–Taylor instabilities with an angular size of \( \sim 6' \) can grow for only \( \sim 5000 \text{ years} \) (Figure 19). As a result, during this period Rayleigh–Taylor instabilities with large angular scales \((\gtrsim 30')\) likely experience the most growth (Figure 19).
that the depth of this mixing layer decreases significantly, with surrounding the Crab Nebula (Hester et al. 1996). After the length scale is similar to the size of the optical filaments contained inside this penetration layer. It is interesting to note that this mixing layer as do the hydrodynamical simulations cited above. Recent simulations of the growth of Rayleigh–Taylor instabilities inside a PWN suggest the growth of \( h_{\text{rt}} \) is highly suppressed for high \( \sigma \) (Bucciantini et al. 2004a). Since this is the case during the contraction of the PWN (Figure 9), this approach likely underpredicts the lifetime of a PWN inside an SNR.

4. DISCUSSION AND CONCLUSIONS

In this paper, we present a general model for the evolution of a PWN inside an SNR (Section 2) and the specific evolution predicted by this model for a particular set of neutron star, pulsar wind, supernova explosion, and ISM properties (Section 3). This model—based largely on previous analytical work on this subject (e.g., Reynolds & Chevalier 1984)—accounts for the continued injection of energy into the PWN by the pulsar, the role of the PWN’s magnetic field on its evolution, couples the dynamical and radiative evolution of the PWN to each other, and is able to reproduce the evolutionary sequence identified by more complicated numerical simulations of such objects (e.g., van der Swaluw et al. 2001; Bucciantini et al. 2003).

As mentioned in Section 1, the ultimate goal of this model is to reproduce the observed dynamical and radiative properties of a PWN in order to study the central neutron star, progenitor supernova explosion, and pulsar wind. This requires that our model accurately reproduces the observed properties of a well-studied and constrained PWN. The best test case is the Crab Nebula, the brightest radio and X-ray PWN in the Milky Way, and the only PWN whose age and neutron star properties \( (E_{0}, \rho, \tau_{\text{sd}}) \) are well known. As mentioned earlier, the input parameters of the simulation discussed in Section 3 were based on previous analyses of this source. As shown in Table 3, for these parameters our model reproduces the size of the Crab Nebula, the radius of the termination shock, the expansion velocity of the PWN, and the spectral index of both the radio and 0.5–10 keV emission from the PWN. However, for the

**Figure 18.** Minimum angular scale unstable to Rayleigh–Taylor instabilities (\( \theta_{\text{rt, crit}} \equiv \lambda_{\text{rt, crit}} / R_{\text{pwn}} \); dashed) and the angular scale maximally unstable to Rayleigh–Taylor instabilities (\( \theta_{\text{rt, max}} \equiv \lambda_{\text{rt, max}} / R_{\text{pwn}} \); solid). The vertical dotted lines demarcate the evolutionary phases discussed in Section 3.

**Figure 19.** Growth rate \( (\omega_{\text{rt}}, \text{Equation (32)}) \) of Rayleigh–Taylor instabilities with different angular scales. The vertical dotted lines demarcate the evolutionary phases discussed in Section 3.

(A color version of this figure is available in the online journal.)

Additionally, the growth rate of Rayleigh–Taylor instabilities at all angular scales is much lower after the PWN has collided with the reverse shock than during the initial expansion due to the stronger nebular magnetic field and significantly higher density of the mass shell surrounding the PWN (Figure 19). Since the density of these perturbations grows as \( \rho \propto \rho_{0} e^{\tau_{\text{sd}}} \) (Chandrasekhar 1961), during the initial expansion it is likely that a significant fraction of the mass swept up by the PWN will be in these filaments.

Rayleigh–Taylor instabilities will cause “bubbles” of swept-up material to penetrate the PWN. During the initial expansion of the PWN, the depth of these bubbles is \( h_{\text{rt}} = (0.01–0.1)R_{\text{pwn}} \) (Figure 20), so \( \sim 5\%–20\% \) of the volume of the PWN is contained inside this penetration layer. It is interesting to note that this length scale is similar to the size of the optical filaments surrounding the Crab Nebula (Hester et al. 1996). After the PWN collides with the reverse shock, Equation (34) predicts that the depth of this mixing layer decreases significantly, with \( h_{\text{rt}} \approx 0 \) when the PWN begins to contract (Figure 20). This mixing layer will grow again when \( P_{\text{pwn}} > P_{\text{snr}}(R_{\text{pwn}}) \). When the PWN re-expands into the SNR, \( h_{\text{rt}} \approx R_{\text{pwn}} \) (Figure 20).
young PWNe, though not their low break frequency (Woltjer breaks inferred from radio and X-ray observations of several does have some interesting implications for existing questions in and not just the Crab Nebula.

A thorough examination of the possible parameter space, which can identify what regimes produce a low break frequency and neutron star, pulsar wind, supernova, and ISM properties, we must deposit energy spectrum flatter than $E^{-2}$ which extends up to an energy of $\sim 1$ TeV (Malyshev et al. 2009). For the set of parameters modeled in Section 3, these conditions are met for only a short period of time during the evolution of this PWN. This condition is satisfied when $h_{rt} \approx R_{pwn}$ (Section 3), when a PWN is likely to be disrupted and injects its particles into the surrounding ISM. Using this model, it is possible to determine what sets of the neutron star, pulsar wind, supernova, and ISM parameters are required for a PWN to satisfy these criteria, and evaluate different models for particle escape from the PWN and their effect on the PWN’s evolution—particularly if these particles escape gradually or suddenly from the PWN. While similar analyses exist (e.g., Büsching et al. 2008), with this model we can more realistically calculate the evolution of the particle spectrum inside a PWN and how these particles escape into the ISM than previous work. In fact, such an analysis is likely to be the only of determining if past PWNe are the origin this anomaly in the cosmic-ray electron spectrum and positron fraction.

One major limitation of this model is that, since it assumes a constant pressure and magnetic field strength inside the PWN, it is inherently one dimensional. Therefore, it is insensitive to any pressure and magnetic field variations inside the PWN. Recent Chandra observations of PWNe have revealed the existence of internal structures (e.g., the torus and jets in the Crab Nebula; Weisskopf et al. 2000) and spectral changes (Weisskopf et al. 2000; Mori et al. 2004) indicative of a nonuniform pressure and/or magnetic field inside the PWN. We are also insensitive to the possibly significant effects asymmetries in the PWN/reverse shock collision resulting from either the space velocity of the neutron star or inhomogeneities in the progenitor supernova and/or surrounding ISM that can have on the PWN’s evolution (e.g., van der Swaluw et al. 2004). Additionally, while we estimate the growth rate of Rayleigh–Taylor instabilities in the shell of swept-up material surrounding the PWN, we cannot determine the role they might play in enabling particles to escape from the PWN—critical in determining if such PWNe are responsible for the PAMELA, ATIC, and HESS results discussed above.

To summarize, we present a one-dimensional model for the evolution of a PWN inside an SNR. This model self-consistently evolves the magnetic, dynamical, and radiative properties of the PWN throughout its evolution, and therefore represents a significant improvement over other currently existing models. The model described here provides a framework for investigating the dependence of a PWN’s evolution on the properties of the central neutron star, pulsar wind, progenitor supernova, and surrounding ISM (similar to previous studies, e.g., Reynolds & Chevalier 1984; Bucciantini et al. 2004b), the effect of more complicated descriptions of pulsar winds, e.g., the presence of

| Observed Property | Value | Reference | Model Prediction |
|-------------------|-------|-----------|-----------------|
| $R_{pwn}$         | 1.5–2 pc | Green (2006) | 1.7 pc |
| $r_{IS}$          | 0.07–0.14 pc | Weisskopf et al. (2000) | 0.24 pc |
| $v_{pwn}$         | $\sim 1270$ km s$^{-1}$ | Temim et al. (2006) | 2000 km s$^{-1}$ |
| Radio luminosity  | $1.8 \times 10^{37}$ erg s$^{-1}$ | Frail & Scharringhausen (1997) | $2.2 \times 10^{38}$ erg s$^{-1}$ |
| $\alpha_{\nu_{\gamma}}$ | $-0.3$ | Green (2006) | $-0.3$ |
| $L_{\nu_{\gamma},5–10\text{keV}}$ | $1.3 \times 10^{37}$ erg s$^{-1}$ | Mori et al. (2004) | $\sim 3 \times 10^{38}$ erg s$^{-1}$ |
| $\Gamma_{5–10\text{keV}}$ | 1.9–3.0 | Mori et al. (2004) | 1.8 |
| $L_{50\text{GeV}–50\text{TeV}}$ | $\sim 10^{34}–10^{35}$ erg s$^{-1}$ | Aharonian et al. (2004) | $1.6 \times 10^{36}$ erg s$^{-1}$ |
ions in the pulsar wind, on the dynamical and radiative evolution of the PWN, as well as determining if there exists any unique observational signatures of such processes. Additionally, it is well suited for using the observed properties of a PWN to constrain the properties of the central neutron star, its pulsar wind, progenitor supernova, and surrounding ISM. Its applicability to a large number of known, well-studied PWNe makes it an extremely powerful tool for studying these intriguing systems.

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APPENDIX A

EVOLUTION OF A NONRADIATIVE SUPERNOVA REMNANT

In this model, we assume that the progenitor core-collapse supernova injected material with mass $M_{ej}$ and initial kinetic energy $E_{sn}$ into a constant density $\rho_{ISM}$ ISM. We assume the supernova ejecta initially is composed of a constant density inner core surrounded by a $\rho \propto r^{-9}$ outer envelope—the standard assumption for core-collapse supernovae (e.g., Blondin et al. 2001)—where the transition velocity $v_t$ between these two components is (Equation (3) in Blondin et al. 2001)

$$v_t = \left( \frac{40 E_{sn}}{18 M_{ej}} \right)^{1/2}. \quad (A1)$$

Under this assumption, the density of material inside the SNR evolves as (Equations (1) and (2) in Blondin et al. 2001)

$$\rho_{ej}(r, t) = \begin{cases} \frac{10}{9\pi} E_{sn} v_t^{-5} t^{-3} & \text{for } r < v_t t \\ \frac{10}{9\pi} E_{sn} v_t^{-5} \left( \frac{r}{v_t t} \right)^{-9} t^{-3} & \text{for } r > v_t t \end{cases}. \quad (A2)$$

Initially, the evolution of the SNR is self-similar (e.g., Chevalier 1982), and therefore described in terms of characteristic length ($R_{ch}$), time ($t_{ch}$), and mass ($M_{ch}$) scales. For an SNR expanding into a constant density ISM, these scales are (Equations (1)–(3) in Truelove & McKee 1999)

$$R_{ch} \equiv M_{ej}^{1/3} \rho_{ISM}^{-1/3}, \quad (A3)$$

$$t_{ch} \equiv E_{sn}^{1/2} M_{ej}^{5/6} \rho_{ISM}^{-1/3}, \quad (A4)$$

$$M_{ch} \equiv M_{ej}. \quad (A5)$$

The expanding supernova ejecta drives a shock wave into the surrounding ISM (called the “forward shock”) which marks the outer boundary of the SNR ($R_{snr}$). Initially, the dynamics of the SNR is dominated by expanding supernova ejecta because the ejecta mass is much greater than the mass of the ISM material swept up and shocked by the SNR ($M_{sw,snr} \equiv \frac{4}{3\pi} R_{snr}^3 \rho_{ISM}$). During this period, $R_{snr}(t)$ is (Equation (75) in Truelove & McKee 1999)

$$R_{snr}(t) = 1.12 R_{ch} \left( \frac{t}{t_{ch}} \right)^{2/3}, \quad (A6)$$

where 1.12 is specific for a $\rho \propto r^{-9}$ outer ejecta envelope, though it varies little for different values of the power-law exponent (Truelove & McKee 1999). Therefore, the expansion velocity $v_{snr} (v_{snr}(t) \equiv dR_{snr}/dt)$ of the SNR is (Equation (76) in Truelove & McKee 1999)

$$v_{snr} = 0.75 R_{ch} \left( \frac{t}{t_{ch}} \right)^{-1/3}. \quad (A7)$$

As the SNR grows, $M_{sw,snr}$ increases and eventually will reach the point where $M_{sw,snr} \approx M_{ej}$. At this point, the swept-up ISM material will begin to dominate the dynamics of the SNR, and the SNR is said to enter the Sedov–Taylor phase of its evolution (Sedov 1959; Taylor 1950). For an SNR with the ejecta profile given in Equation (A2), this transition occurs at time $t_{ST} \approx 0.52t_{ch}$ (Truelove & McKee 1999). During the Sedov–Taylor phase, $R_{snr}$ is (Equation (56) in Truelove & McKee 1999)

$$R_{snr}(t) = \left[ R_{snr,ST}^{5/2} + \left( \frac{2.026 E_{sn}}{\rho_{ISM}} \right)^{1/2} \left( t - t_{ST} \right) \right]^{2/5}, \quad (A8)$$

where $R_{snr,ST} \equiv R_{snr}(t_{ST})$. The Sedov–Taylor phase ends when the gas recently shocked by the expanding SNR cools radiatively, which occurs approximately at (Equation (3) in Blondin et al. 1998)

$$t_{rad} = 2.9 \left( \frac{E_{sn}}{10^{51} \text{erg}} \right)^{4/17} \left( \frac{\rho_{ISM}}{m_p \text{cm}^{-3}} \right)^{-9/17} \times 10^4 \text{years} \quad (A9)$$

after the supernova, where $m_p$ is the mass of a proton.

The pressure $P$ and density $\rho$ profiles of an SNR change significantly over time. Initially, the expansion velocity of the SNR is significantly larger than the sound speed of the surrounding ISM. As a result, the swept-up material is shocked by the surrounding ejecta. Assuming that energy losses due to cosmic-ray acceleration is negligible, the pressure of the recently shocked ISM material [$P_{snr}(R_{snr})$] is

$$P_{snr}(R_{snr}, t) = \frac{4}{3} \rho_{ISM} v_{snr}(t)^2, \quad (A10)$$

assuming an adiabatic index $\gamma = 5/3$ (nonrelativistic gas). The pressure of the shocked ISM is significantly higher than that of the ejecta inside the SNR, which has cooled significantly since the explosion due to adiabatic expansion. As a result, the layer of the shocked ISM expands inside the SNR, driving a shock wave, referred to as the reverse shock, into the supernova ejecta. When the reverse shock is in the outer envelope of the supernova ejecta, its radius $R_{rs}$ is (Chevalier 1982; Truelove & McKee 1999)

$$R_{rs}(t) = \frac{1}{1.19} R_{snr}(t), \quad (A11)$$

and therefore, its velocity relative to the surrounding ISM $v_{rs}$ is

$$v_{rs}(t) = \frac{1}{1.19} v_{snr}(t). \quad (A12)$$

The properties of ejecta recently shocked by the reverse shock depend on the velocity of the reverse shock relative to the unshocked ejecta, $v_{ej}$. The standard assumption is that the
unshocked ejecta is expanding ballistically \[v_{ej}(r, t) \equiv r/t\]. Therefore, the relative velocity of the reverse shock, \(\bar{v}_{rs} \equiv v_{ej}(R_{rs}, t) - v_{rs}(t)\), is (Trueup & McKee 1999)

\[
\bar{v}_{rs}(t) = \frac{1}{2.38} v_{snr}(t).
\] (A13)

Eventually, the reverse shock will enter the constant density core at the center of the SNR. For an SNR with an initial ejecta density profile defined in Equation (A2), this occurs at a time \(t_{core} \approx 0.25t_{ch}\) (Equation (79) in Trueup & McKee 1999), where \(t_{ch}\) is defined above in Equation (A4). After this time, the radius of the reverse shock evolves as (Equation (83) in Trueup & McKee 1999)

\[
R_{rs}(t) = \left[1.49 - 0.16 \frac{t - t_{core}}{t_{ch}} - 0.46 \ln \left(\frac{t}{t_{core}}\right)\right] \frac{R_{ch}}{t_{ch}}.
\] (A14)

During this stage, \(\bar{v}_{rs}\) is equal to (Equation (84) in Trueup & McKee 1999)

\[
\bar{v}_{rs} = \left[0.50 + 0.16 \frac{(t - t_{core})}{t_{ch}}\right] \frac{R_{ch}}{t_{ch}}.
\] (A15)

Since the velocity of the reverse shock is much greater than the sound speed in the unshocked ejecta, the reverse shock—like the forward shock—is a strong shock. Therefore, the density of the ejecta recently shocked by the reverse shock \(P_{rs}\) is (Trueup & McKee 1999)

\[
P_{rs}(t) \equiv \frac{\rho_{ej}(R_{rs}, t)\bar{v}_{rs}(t)^2}{\rho_{ISM}v_{snr}(t)^2} P_{snr}(t).
\] (A16)

When the reverse shock is in the outer envelope of the supernova ejecta, the density, velocity, and pressure profiles of the SNR between the reverse shock and forward shock \((R_{snr} < r < R_{sur})\) are calculated using the equations derived by Chevalier (1982). These equations are no longer valid once the reverse shock enters the constant density ejecta core. At this point, we model the density, velocity, and pressure structure of the SNR using the solution for a Sedov-Taylor SNR presented in Appendix A of Bandiera (1984). This is valid until the SNR goes radiative \((t = t_{rad})\). Unfortunately, the internal structure of a radiative SNR is poorly understood, and therefore we do not attempt to model the evolution of a PWN in this environment. However, previous work suggests that the radius of a PWN \(R_{pwn}\) inside a radiative SNR evolves as (Blondin et al. 2001; van der Swaluw & Wu 2001)

\[
\frac{R_{pwn}}{R_{snr}} \propto t^{0.075}.
\] (A17)

**APPENDIX B**

**INITIAL PROPERTIES OF A PWN INSIDE AN SNR**

In order to model the evolution of a PWN inside an SNR, it is necessary to estimate the initial conditions. To estimate the initial energy of the PWN, we assume that adiabatic losses dominate and the PWN is expanding with a constant velocity. In this case,

\[
\frac{dE_{pwn}}{dt} = \dot{E}_0 \left(1 + \frac{t}{\tau_{sd}}\right)^{-\frac{\eta e}{1 + \eta e}} - \frac{E_{pwn}}{t}.
\] (B1)

Defining \(y = (p + 1)/(p - 1)\), \(\epsilon \equiv E_{pwn}/(\dot{E}_0\tau_{sd})\), and \(x \equiv t/\tau_{sd}\), one derives that

\[
\frac{d\epsilon}{dx} = -\frac{\epsilon}{x} + (1 + x)^{-y}.
\] (B2)

Using the boundary conditions that \(\epsilon(x = 0) = 0\) (initially, the PWN has zero energy), one derives that, for \(y = 2\) \((p = 3)\),

\[
\epsilon = \frac{\ln (1 + x)}{x} - \frac{1}{x + 1}
\] (B3)

and, for \(y \neq 2\) \((p \neq 3)\),

\[
\epsilon = \frac{(1 + x)^{-y} - (1 + x)^{2-y}}{1 - y} + \frac{1}{x(1-y)(2-y)^2}.
\] (B4)

We also assume that initially the fraction of the PWN’s energy in magnetic fields is \(\eta_B\), the fraction of the PWN’s energy in electrons and positrons is \(\eta_e\), and the fraction of the PWN’s energy in ions is \(\eta_i\). Additionally, we assume that the initial spectrum of particles inside the PWN has the same shape as the injection spectrum.

If adiabatic losses dominate, the equation of motion of the PWN for \(P_{snr}(R_{pwn}) \equiv 0\) is (Ostriker & Gunn 1971; Chevalier & Fransson 1992)

\[
M_{snr} \frac{d\bar{v}_{pwn}}{dt} = 4\pi R_{pwn}^2 [P_{pwn} - \rho_{ej}(R_{pwn})] \times (v_{pwn} - v_{ej}(R_{pwn}))^2,
\] (B5)

where

\[
\frac{dE_{pwn}}{dt} = \dot{E} - 4\pi R_{pwn}^2 v_{pwn}.
\] (B6)

At early times \((t \ll \tau)\), \(\dot{E} \approx \dot{E}_0\), and it is possible to solve Equations (B5) and (B6) analytically (e.g., Chevalier & Fransson 1992). For the initial supernova ejecta density given in Equation (A2), \(R_{pwn}\) can be expressed as (e.g., Chevalier 1977, Equation (2.6) in Chevalier & Fransson 1992, Equation (5) in Blondin et al. 2001)

\[
R_{pwn}(t) = 1.44 \left(\frac{E_{pwn}^3}{M_{ej}^2}\right)^{1/10} t^{6/5}.
\] (B7)

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