AN APPROXIMATE GLOBAL SOLUTION TO THE GRAVITATIONAL FIELD OF A PERFECT FLUID IN SLOW ROTATION *

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Abstract
Using the Post–Minkowskian formalism and considering rotation as a perturbation, we compute an approximate interior solution for a stationary perfect fluid with constant density and axial symmetry. A suitable change of coordinates allows this metric to be matched to the exterior metric to a particle with a pole-dipole-quadrupole structure, relating the parameters of both.

1. Introduction.
In the study of multipole particles in General Relativity, certain approximate solutions to the gravitational field outside such objects have been proposed. The information about their structure is collected in a series of parameters, which are interpreted as the multipole moments of the particle. Here, we use as a starting point the metric outside a particle with pole-dipole-quadrupole structure (PDQ particle), with stationary and axisymmetric moments [1]. Our aim is to find an approximate interior metric that will match it on a given surface. It is demanded that they should match in the sense of Lichnerowicz [2]: the metric and its first derivatives must be continuous on the matching surface. For simplicity’s sake we shall assume that the inside is made up of a homogeneous perfect fluid in rigid and slow rotation (this term will be precised below). The interior solution will be obtained by solving the Einstein equations using the Post–Minkowskian perturbation method [3]. From study of the motion equations of the material medium it is possible to extract precise information for obtaining the equation of the matching surface [4]. Finally, the matching conditions of both metrics will be translated into relationships between the parameters of the interior metric (density and angular speed of rotation), the mean radius of the surface, and the parameters of the exterior metric (multipole moments).

2. The exterior metric.
In a system of harmonic coordinates $X^\alpha$ such that $X^0 = t$ and $\phi$ are adapted to the Killings vectors ($R$, $\Theta$, and $\phi$ are the spherical coordinates associated to the harmonic

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ones), the exterior metric to an isolated PDQ particle with a stationary and axisymmetric structure is written as follows (see ref. 1):

\[
g_{00}^{\text{ext}} = -1 + 2 \frac{Gm}{Rc^2} \left( 1 - \frac{Gm}{Rc^2} + \frac{G^2m^2}{R^2c^4} - \frac{G^3m^3}{R^3c^6} \right) + 2 \frac{GB}{R^3c^2} P_2(\cos \Theta) \\
- \frac{G^2}{R^4c^4} \left[ 4mB P_2(\cos \Theta) - \frac{2S^2}{3c^2} \left[ 1 + 2P_2(\cos \Theta) \right] \right] + O(R^{-5})
\]

\[
g_{0i}^{\text{ext}} = 2 \frac{GS}{R^2c^3} \left( 1 - \frac{Gn}{Rc^2} + \frac{G^2m^2}{R^2c^4} \right) M_i + O(R^{-5}) \\
g_{ij}^{\text{ext}} = \left[ 1 + 2 \frac{Gm}{Rc^2} + \frac{G^2m^2}{R^2c^4} + 2 \frac{GB}{R^3c^2} P_2 \right. \\
- \frac{G^2}{R^4c^4} \left\{ \frac{mB}{3} \left[ 1 - 7P_2 \right] + \frac{S^2}{c^2} \left[ 1 - 3P_2 \right] \right\} \delta_{ij} \\
+ \left[ \frac{G^2m^2}{R^2c^4} \left[ 1 + 2 \frac{Gm}{Rc^2} + 2 \frac{G^2m^2}{R^2c^4} \right] \\
+ \frac{G^2}{R^2c^4} \left[ mB \left[ 1 + 5P_2 \right] + \frac{S^2}{c^2} \left[ 1 + 3P_2 \right] \right] \right\} N_i N_j \\
\left. + \frac{G^2}{R^4c^4} \left[ mB + 4 \frac{S^2}{c^2} \right] E_i E_j - 2 \frac{G^2}{R^4c^4} \left[ 3mB + 5 \frac{S^2}{c^2} \right] \cos \Theta N_i E_j \right)
\]

where

\[
E^i = \delta^i_3, \quad N^i = \frac{X^i}{R}, \quad M^i = \epsilon^{i}{}_{jk} N^j E^k
\]

\[P_l(\cos \Theta)\] is the Legendre polynomial of order \(l\) and the parameters \(m, S\) and \(B\), are respectively the mass, the angular momentum and the mass quadrupole moment of the particle.

Note that the terms containing the different moments have a specific dependence on the powers of \(R^{-1}\). More precisely, it may be seen that only the following factors appear:

\[
\frac{Gm}{Rc^2}, \quad \frac{GS}{R^2c^3}, \quad \frac{GB}{R^3c^2}
\]

and their powers and products.

Finally, it is worth mentioning that (2.1) coincides with the expansions, in certain harmonic coordinates, of Schwarzschild, Weyl and Kerr metrics, if the parameters \(m, S\) and \(B\) take the values:

\[
Gm = \alpha \quad GS = 0 \quad GB = 0 \quad \text{(Schwarzschild)}
\]

\[
Gm = -a_0 \quad GS = 0 \quad GB = \frac{1}{3}a_0^3 - a_2 \quad \text{(Weyl)}
\]

\[
Gm = \alpha \quad GS = -\alpha a \quad GB = -\alpha a^2 \quad \text{(Kerr)}
\]
3. Newtonian theory.

The Post–Minkowskian approximation allows one to obtain the interior metric in the form of an expansion in powers of G, which we shall match, at each iteration, with the terms of the same order of the exterior solution. Since the results obtained at first–order must necessarily coincide with those afforded by Newtonian mechanics, it makes sense to first study the problem posed within this framework.

It should be recalled that study of a self-gravitating fluid is already a non–linear problem in Newtonian mechanics. On one hand, the field equation:

\[ \triangle \psi_G = 4\pi G \rho \]  

allows one to obtain the gravitational potential, \( \psi_G \), once we know the mass distribution \( \rho \). On the other hand, Euler motion equations:

\[ \partial_t \rho + \partial_k (\rho v^k) = 0 \]
\[ \partial_t (\rho v^i) + \partial_k (\rho v^k v^i + \sigma^{k i}) = -\rho \partial_i \psi_G \]  

(where \( v^i \) is velocity and \( \sigma^{i k} \) is the stress tensor of the fluid) afford information about the evolution of the source of the gravitational field, if we have prior knowledge of the potential \( \psi_G \) inside it.

Let us assume that the fluid is perfect (\( \sigma^{i k} = p \delta^{i k} \)), that density is constant in time (\( \partial_t \rho = 0 \)) and that the fluid rotates in a rigid manner with a constant angular velocity \( \omega^i \) (\( v^i = \epsilon^i_{jk} \omega^j X^k \)). Under these conditions, equation (3.2) can be reduced to (we choose the \( X^3 \) axis in the direction of \( \omega^i \)):

\[ \partial_\phi \rho = 0 \quad , \quad \partial_i p = -\rho \partial_i \psi \]  

with

\[ \psi = \psi_G - \frac{1}{2} \omega^2 r^2 \sin^2 \Theta \]  

The second equation in (3.3) is a constraint on \( p, \rho \) and \( \psi \). If one considers this as an equation for \( p \), from its integrability condition

\[ \partial_i \rho \partial_j \psi = 0 \]  

it may be deduced that \( \rho \) must be a function of \( \psi \), \( \rho = \chi(\psi) \) (or the other way round); and its solution is:

\[ p = p_0 - \int_{\psi_0}^\psi \chi(\lambda) \, d\lambda \]  

The surface \( \Sigma \) of the fluid is determined by the equilibrium condition \( p \mid_\Sigma = 0 \), which leads to

\[ p = \int_{\psi}^{\psi_\Sigma} \chi(\lambda) \, d\lambda \]  

\[ \chi(\lambda) \, d\lambda \]
where $\psi_\Sigma$ is the constant value of $\psi$ on $\Sigma$ (hence, $\psi(R, \Theta) = \psi_\Sigma$ is the implicit equation of $\Sigma$). Moreover, since $p$ and $\rho$ are functions of $\psi$, they are constant on the surfaces $\psi(R, \Theta) = \text{constant}$.

If an equation of state $p = f(\rho)$ is given as an input, (3.3) allows one to obtain a relation $\rho = \chi(\psi)$, which, substituted in (3.1), leads to a generally non-linear equation for the potential:

$$\Delta \psi_G = 4\pi \chi(\psi)$$

(3.8)

For example, for a polytropic-like state equation

$$p = a \rho^{1+\frac{1}{\gamma}} \quad (a = \text{constant} \, , \, \gamma > 0)$$

(3.9)

one obtains

$$\rho = C \left( \psi_\Sigma - \psi \right)^\gamma, \quad C = \left( \frac{\gamma - 1}{a \gamma} \right)^\gamma$$

(3.10)

and

$$\Delta \psi_G = 4\pi G C \left( \psi_\Sigma - \psi \right)^\gamma$$

(3.11)

which is not linear if $\gamma \neq 1$ [5].

The case of $\rho = \text{constant}$, which is not included in the previous example, is not even free of non-linearities. We have already mentioned that $\psi(R, \Theta) = \psi_\Sigma$ defines the surface $\Sigma$. However, to determine it we must first know the potential $\psi_G$, which in turn depends on the shape of the surface

$$\psi_G = -G \int_{(\text{interior of } \Sigma)} \frac{\rho}{|\vec{X} - \vec{X}'|} d^3\vec{X}'$$

(3.12)

Accordingly, $\psi = \psi_\Sigma$, rather than a definition of the surface, is a supplementary condition that must satisfy the solution.

Although different exact solutions to this problem are known [6], it can also be posed in terms of successive approximations. In the strict sense, this second procedure reproduces one of the known exact solutions (the McLaurin ellipsoids). Nevertheless, it is highly suitable for the problem we shall study in later sections.

As is well known, the expansion of (3.12) in spherical harmonics in the region exterior to $\Sigma$ is:

$$\psi_G^{\text{ext}} = -\frac{Gm}{R} - \sum_{l \geq 2} \frac{GQ_l}{R^{l+1}} P_l(\cos \Theta)$$

(3.13)

where

$$Q_l \equiv \int_{(\text{interior of } \Sigma)} R^l P_l(\cos \theta) \rho(\vec{R}) \, d^3\vec{R}$$

(3.14)

is the $2^l$-pole moment. We shall admit that there is a parameter $\Omega$ ($\Omega \ll 1$), related to the rotation, that measures the deformation of the fluid with respect to the spherical shape, such that $Q_l$ is at least of order $\Omega^l$. Furthermore, we shall assume that it is possible to expand the parametric function of the surface of the fluid, $R_\Sigma(\cos \Theta)$.
\( (\psi(R_\Sigma(\cos \Theta), \cos \Theta) = \psi_\Sigma) \), in powers of \( \Omega \). Direct, although tedious, calculation based on expressions (3.13) and (3.14) and the equation for \( \Sigma \) leads to the following results:

\[
R_\Sigma(\cos \theta) = R_0 \left[ 1 - \frac{5}{6} \Omega^2 P_2 + \Omega^4 \left( -\frac{5}{36} P_0 - \frac{50}{63} P_2 + \frac{15}{14} P_4 \right) + O(\Omega^6) \right]
\]

\[Q_2 = mR_0^2 \left[ -\frac{1}{2} \Omega^2 - \frac{5}{21} \Omega^4 + O(\Omega^6) \right]\]

\[Q_4 = mR_0^4 \left[ \frac{15}{28} \Omega^4 + O(\Omega^6) \right]\]

At the end of the process, it is possible to identify the parameter \( \Omega \), which proves to be:

\[
\Omega^2 = \frac{R_0^2 \omega^2}{Gm/R_0} = \frac{R_0 \omega^2}{Gm/R_0^2}
\]

which is the quotient between the rotational energy and the gravitational energy, or, between the mean values of the centrifugal force and the gravitational force on the surface.

4. Relativistic Euler equations.

Within the framework of relativity, it is also possible to gain information about the matching surface from the evolution equations of the fluid. These equations are deduced from the energy–momentum tensor conservation condition, which, for a perfect fluid:

\[
T^{\alpha\beta} = \left( \rho + \frac{p}{c^2} \right) u^\alpha u^\beta + \frac{p}{c^2}g^{\alpha\beta} \quad (u^\alpha u_\alpha = -1)
\]

can be written

\[
u^\alpha \partial_\alpha \rho + \left( \rho + \frac{p}{c^2} \right) \partial_\alpha u^\alpha = 0
\]

\[
\left( \rho + \frac{p}{c^2} \right) u^\beta \partial_\beta u^\alpha + \frac{1}{c^2} (g^{\alpha\beta} + u^\alpha u^\beta) \partial_\beta \rho = 0
\]

If the metric is stationary and axysimmetric and, also, the velocity of the fluid is in the plane spanned by both Killing vectors, it is possible to find coordinates adapted to the Killing vectors in which

\[
u^\alpha = \psi \left( \delta^\alpha_t + \frac{\omega}{c} \delta^\alpha_\phi \right)
\]

and the metric is written in blocks [7]. From the condition \( u^\alpha u_\alpha = -1 \), one deduces that

\[
\psi = \left[ - \left( g_{tt} + 2\frac{\omega}{c} g_{t\phi} + \frac{\omega^2}{c^2} g_{\phi\phi} \right) \right]^{-\frac{1}{2}}
\]
Under the hypothesis that rotation is rigid ($\omega = \text{constant}$) and that density and pressure do not depend on either $t$ or $\phi$, the second set of Euler equations is reduced to

$$\partial_a p = (pc^2 + p)\partial_a \ln \psi \quad (a, b, \ldots = R, \Theta)$$

(4.5)

Considering this generalisation of the second equation of (3.3) as an equation for the pressure, the integrability condition leads us to $\rho = \chi(\psi)$, and the solution is:

$$p = \int_{\psi\Sigma}^{\psi} \frac{\chi(\lambda)}{\lambda^2} d\lambda$$

(4.6)

As in the Newtonian case, $\rho$ and $p$ are constant over the surfaces $\psi(R, \cos \Theta) = \text{constant}$, and $\Sigma$ (the surface on which pressure vanishes) has as its implicit equation $\psi(R, \cos \Theta) = \psi\Sigma$. If the fluid considered has a constant density, (4.6) leads to:

$$p = \rho \left( \frac{\psi}{\psi\Sigma} - 1 \right)$$

(4.7)

5. Parameters of the exterior metric.

As mentioned in the introduction, the exterior metric that we shall use as the starting point is a linear combination of products of factors (2.3) up to $R^{-4}$. Additionally, the interior metric will be calculated by the Post–Minkowskian approximation, considering at each order in $G$ a subexpansion in powers of $\Omega$. To match them to order $G$, it is of interest to know how the terms (2.3) are written as a function of these parameters. Instead of $G$, we shall use the adimensional parameter $g = Gm/R_0 c^2$ ($R_0$ is the mean radius of the ball of fluid) and we shall consider that the moments have an expansion in $\Omega$ similar to that obtained in Newtonian mechanics; that is:

$$Q_2 \equiv B = mR_0^2 \left[ \Omega^2 B_{(2)} + \Omega^4 B_{(4)} + O(\Omega^6) \right]$$

$$S \equiv mR_0 \omega \left[ S_{(0)} + \Omega^2 S_{(2)} + O(\Omega^4) \right]$$

(5.1)

One thus has:

$$\frac{Gm}{Rc^2} = g \frac{R_0}{R}$$

$$\frac{GS}{R^2 c^3} = g \frac{S}{mR_0 c} \left( \frac{R_0}{R} \right)^2 = g^3 \Omega \left[ S_{(0)} + O(\Omega^2) \right] \left( \frac{R_0}{R} \right)^2$$

$$\frac{GB}{R^3 c^2} = g \frac{B}{mR_0^2} \left( \frac{R_0}{R} \right)^3 = g \left[ \Omega^2 B_{(2)} + O(\Omega^4) \right] \left( \frac{R_0}{R} \right)^3$$

(5.2)

We thereby manage to substitute the expansion in powers of $R^{-1}$ (which lacks sense for the interior metric) by an expansion in $g$ and $\Omega$. This new approach leads to a different
assignation of orders to the terms of the expansion of the exterior metric. For example, according to the first criterion $G^2 mB/R^4 c^6$ and $G^2 S^2/R^4 c^6$ are $G^2/R^4$ terms. However, now, whereas the first is of order $g^2 \Omega^2$, the second proves to be of order $g^3 \Omega^2$. This is a direct consequence, in sum, of having chosen $\Omega$ instead of another parameter as an estimate of the effect of rotation. Nevertheless, this is the only way to recover the Newtonian results at order $g$.

Let us now see the meaning of making expansions in these parameters. The parameter $g$ is the quotient between the Schwarzschild radius of the body ($R_s = Gm/c^2$) and $R_0$. Therefore, $g$ will be small when $R_0 \gg R_s$. On the other hand, $\Omega$ has a strange relationship with $g$:

$$\Omega^2 = \frac{1}{g} \left( \frac{R_0 \omega}{c} \right)^2 \quad (5.3)$$

If the body is not very compact (small $g$), the typical velocity with which a point at the surface moves ($R_0 \Omega$) must be very small as compared with the speed of light, so that a small value of $\Omega$ can be attained and so that the approximation will conserve its validity. By contrast, when the body is compact, the surface velocity can be an important fraction of the speed of light. In the case of the Earth, the Sun and a typical pulsar ($m_{\text{pulsar}} \approx m_{\odot}$, $T \approx 1$ s, $R_0 \approx 10$ km), $g$ and $\Omega$ take the following values:

|        | Earth | Sun | pulsar |
|--------|-------|-----|--------|
| $g$    | $1.3 \times 10^{-9}$ | $2.1 \times 10^{-6}$ | $0.14$ |
| $\Omega^2$ | $1.8 \times 10^{-3}$ | $3.3 \times 10^{-5}$ | $2.9 \times 10^{-7}$ |

Finally, we can use the expansion in these parameters to determine the matching surface in an approximate way from the condition $\psi(R, \cos \Theta) = \psi_\Sigma$ as a function of the parameters of the exterior metric (we assume continuity of the metric on $\Sigma$). The final result is:

$$R_\Sigma(\cos \theta) = R_0 \left[ 1 + \Omega^2 \left( B_{(2)} - \frac{1}{3} + \frac{4}{3} g \left[ S_{(0)} - 1 \right] + O(g^2) \right) P_2 \right] + O(\Omega^4) \quad (5.5)$$

It should be noted that since the exterior metric does not contain the octupole moment metric (if we assume equatorial symmetry) nor the 16-pole moment, which would go as $\Omega^4$, there is little sense in considering expansions beyond $\Omega^2$.

### 6. The interior metric

The Post–Minkowskian perturbation method presupposes the existence of a formal expansion in the gravitational constant $G$ of $h_{\alpha\beta} \equiv \left( - \det(g_{\gamma\mu}) \right)^{1/2} g^{\alpha\beta} - \eta^{\alpha\beta}$ [8]. In our case, since in the exterior metric there are half–integer powers of $g$, we shall assume that the expansion is in half–integer powers of $g$ (an adimensional parameter).

$$h_{\alpha\beta}(x^\mu) = \sum_{n \geq 2} g^{n/2} h_{(n/2)}(x^\mu, \Omega) \quad (6.1)$$
Although later, as mentioned above, we subexpand in powers of $\Omega$ each of the terms of this expansion, the structure of (6.1) ensures that we can use the whole Post–Newtonian scheme, whose details, together with the explicit expressions for the development of the different magnitudes appearing in the Einstein equations, can be found in the papers cited in reference 7. Accordingly, they will not be referred to here.

With a view to simplifying the problem as much as possible, we shall assume that the perfect fluid is homogeneous ($\rho = \text{constant}$), in which case pressure will be given by (4.7). Since $\psi \simeq 1 + O(g)$ pressure is order $g$, which means that the energy-momentum tensor at zero–order will be expressed only in terms of $\rho$ (Newtonian theory).

**Order $g$.** The solution of the Einstein equations at this order, which is regular at the origin, is

$$h^{00}_{(1)} = 2 \frac{\rho'}{m} \left( \frac{r}{R_0} \right)^2 + \sum_{n \geq 0} H^{(1)}_n \left( \frac{r}{R_0} \right)^n P_n$$

$$h^{ij}_{(1)} = h^{ij}_{(1)} = \text{solution of the homogeneous equation}$$

with $\rho' \equiv (4\pi/3)\rho R_0^3$. The fact that the parameter of the exterior metric $m$ appears in (6.2) arises from having chosen $g$ as the constant in which we expand the solution, and has no further transcendence.

Our aim is to match this metric with (2.1) in the sense of Lichnerowicz (see ref. 2): the metric and its first derivatives should be continuous on the surface determined by (5.5). For the exterior metric (2.1), $h^{0i}_{\text{ext}(1)} = h^{ij}_{\text{ext}(1)} = 0$; we thus choose the solution of the homogeneous equation mentioned in (6.2) equal to zero. Also, the fact that in the exterior metric only terms in $P_0$ and $P_2$ appear implies that, by continuity, $H^{(1)}_n = 0$ if $n > 2$. Since there are four constants, $m$, $B$, $H^{(1)}_0$ and $H^{(1)}_2$, and since four conditions are imposed by the matching (terms in $P_0$ and $P_2$ in $h^{00}_{(1)}$ and their normal derivative to $\Sigma$), the matching of the two metrics is possible by identifying the exterior $X^\alpha$ and interior $x^\beta$ coordinates. Thus, $X^\alpha = x^\alpha + O(g^{3/2})$ and the metrics match if:

$$m = \rho' + O(g)$$

$$B = -\frac{1}{2} \rho' R_0^2 \Omega^2 \left[ 1 + O(g) \right]$$

$$R_\Sigma = R_0 \left[ 1 - \frac{5}{6} \Omega^2 P_2 + O(g\Omega^2) + O(\Omega^4) \right]$$

(6.3)

Finally, one has that

$$h^{00}_{(1)} = 2 \left( \frac{r}{R_0} \right)^2 - 6 + 2 \Omega^2 \left( \frac{r}{R_0} \right)^2 P_2 + O(\Omega^4)$$

(6.4)

**Order $g^{3/2}$.** Calculation of the solution at this order passes through the same steps as the previous order and in a completely similar way, since only the $h^{0i}_{(3/2)}$ components are
different from zero and, in fact, depend on a single function. Neither is it necessary here
to modify the interior or exterior coordinates; that is $X^\alpha = x^\alpha + O(g^2)$. The matching
conditions demand that

$$S = \frac{2}{5} \rho' R_0^2 \omega [1 + O(\Omega^2) + O(g)]$$

Finally,

$$h_{(3/2)}^{0i} = \left[ 2 \frac{r}{R_0} - \frac{6}{5} \left( \frac{r}{R_0} \right)^3 + O(\Omega^2) \right] \Omega m^i$$

$$h_{(3/2)}^{00} = h_{(3/2)}^{ij} = 0$$

Order $g^2$. At this order, the solution obtained

$$h_{(2)}^{00} = \left( \frac{\rho'}{m} \right)^2 \left[ 18 \left( \frac{r}{R_0} \right)^2 - \frac{5}{2} \left( \frac{r}{R_0} \right)^4 \right] + \Omega^2 \frac{\rho'}{m} \left( \frac{2}{5} - \frac{37}{7} P_2 \right) \left( \frac{r}{R_0} \right)^4$$

$$+ \sum_{n \geq 0} H_n^{(2)} \left( \frac{r}{R_0} \right)^n P_n + O(\Omega^4)$$

$$h_{(2)}^{0i} = \text{solution of the homogeneous equation}$$

$$h_{(2)}^{ij} = \left[ \left( \frac{\rho'}{m} \right)^2 \left[ \frac{6}{5} \left( \frac{r}{R_0} \right)^2 - \frac{3}{7} \left( \frac{r}{R_0} \right)^4 \right] + \Omega^2 \frac{\rho'}{m} \left[ \frac{2}{5} \left( \frac{r}{R_0} \right)^2 + \frac{1}{126} (89 - 203 P_2) \left( \frac{r}{R_0} \right)^4 \right] \right] \delta^{ij} + \Omega^2 \frac{\rho'}{m} \left[ \frac{2}{5} \left( \frac{r}{R_0} \right)^2 + \left( \frac{67}{63} + \frac{5}{18} P_2 \right) \left( \frac{r}{R_0} \right)^4 \right] n^i \bar{n}^j$$

$$+ \left[ \left( \frac{\rho'}{m} \right)^2 \left[ - \frac{3}{5} \left( \frac{r}{R_0} \right)^2 + \frac{2}{7} \left( \frac{r}{R_0} \right)^4 \right] + \Omega^2 \frac{\rho'}{m} \left[ \frac{2}{5} \left( \frac{r}{R_0} \right)^2 + \left( \frac{67}{63} + \frac{5}{18} P_2 \right) \left( \frac{r}{R_0} \right)^4 \right] \right] n^{i} \bar{n}^{j}$$

$$- \frac{5}{6} \Omega^2 \frac{\rho'}{m} \left( \frac{r}{R_0} \right)^4 e^i e^j + \frac{7}{3} \Omega^2 \frac{\rho'}{m} \left( \frac{r}{R_0} \right)^4 \cos \theta e^{(i} n^{j)}$$

$$+ \text{divergence free solution of the homogeneous equation} + O(\Omega^4)$$

(where $e^i = \delta^i_3$, $n^i = x^i/r$ y $m^{i} = e^{i} j k n^{j} e^{k}$) does not match the part corresponding to
the exterior metric if one does not make a change of coordinates; this even occurs if $\Omega = 0$
(Schwarzschild).

The coordinate change

$$X^\alpha = x^\alpha - g^2 f^\alpha(X^\mu) + O(g^5/2)$$

(6.8)
induces the change
\[ h'^{\alpha\beta} = h^{\alpha\beta} + g^2 \left[ \epsilon^{\alpha\beta} \partial_\mu f^\mu - 2\delta^{(\alpha} f^{\beta)} \right] + O(g^{5/2}) \tag{6.9} \]
in the metric density. In order to preserve the structure and explicit symmetry of the metric, one must take:
\[ f^0 = 0 \]
\[ f^i = \sum_{l \geq 1} f^i_l = \sum_{l \geq 1} \left[ \chi_l(R) \frac{P^i_l}{\sin \theta} (N^i - \cos \theta E^i) + \Gamma_l(R) P^l E^i \right] \tag{6.10} \]
\((P^i_l \text{ are the associated Legendre functions of order } 1)\) where \(\chi_l\) and \(\Gamma_l(R)\) are arbitrary functions. The new coordinates \(X^\alpha\) are still harmonic, for the interior metric, if \(\Delta f = 0\).

It is possible to check that the change in the spatial components of the metric density that induces the most general solution of this equation, which is regular at the origin of coordinates, has the same structure as the solution of the same type and divergence free of the homogeneous equation that must be satisfied by the spatial components of \(h^{ij}_{(2)}\). Accordingly, we can take this term out of (6.7) since it is included in the change of coordinates.

In order to not introduce into the metric polynomials in \(\cos \Theta\) of a higher degree than necessary, only \(f^i_1\) and \(f^j_3\) must be taken into account, with the restriction
\[ 3\chi_3 + \Gamma_3 = 0 \tag{6.11} \]

The \(h^{0i}_{(2)}\) component, which is not modified by the coordinate change, must be zero in order to match the exterior. Also, the matching of \(h^{00}_{(2)}\) implies four conditions and that of \(h^{ij}_{(2)}\) twelve (the term \(\delta^{ij}\) provides four, \(n^i n^j\), four, \(n^i e^j\), two and \(e^i e^j\) two more). As free parameters, one has \(H^0_{(2)}, H^2_{(2)}, m\) and \(B\) [by the latter two we understand the corrections of order \(g\) to the Newtonian values (6.3)], which allow us to match \(h^{00}_{(2)}\) with and without a coordinate change (see order \(g\)), and also the parameters provided by the coordinate change.

The change (6.10) with the restrictions indicated introduces three independent arbitrary functions \(\chi_1, \Gamma_1\) and \(\chi_3\). If \(\Omega = 0\) (spherical symmetry), (6.9) reduces to \(f^i = f(R) N^i\); that is, a single arbitrary function that is a particular combination of \(\chi_1\) and \(\Gamma_1\). Therefore, all the functions, with the exception of this combination, are at least of order \(\Omega^2\). The constants introduced by the coordinate change are the values of these functions and their first and second derivatives on the surface \(\Sigma\). At the order of approximation required, this is equivalent, for functions of order \(\Omega^2\), to taking the value of the function (and respectively its derivatives) in \(R_0\); for example:
\[ \chi_3 |_{\Sigma} = \chi_3(R_0) + O(\Omega^4) \tag{6.12} \]
However, for the combination that we have called $f$, we must conserve the first two terms of its expansion in powers of $\Omega$

\[ f \big|_\Sigma = f(R_0) - \frac{5}{6} \Omega^2 f'(R_0) P_2 + O(\Omega^4) \]  

(6.13)

and the same for its derivatives. To conclude, we have available $4 + 3 + 3 = 10$ arbitrary constants (actually, nine because under matching conditions two of them always appear in the same combination) to satisfy twelve conditions. As a result, three of them must be linear combinations of the rest. This is the case here.

Having achieved the continuity of the $h_{ij}^{(2)}$ components and their derivatives, and therefore having determined the constants associated with the coordinate change, we look at the $h_{00}^{(2)}$ component. From the matching we obtain:

\[
m = \rho' \left[ 1 + g \left[ 3 + \frac{2}{5} \Omega^2 + O(\Omega^4) \right] + O(g^2) \right]
\]

\[
B = \rho' R_0^2 \left[ \Omega^2 \left[ -\frac{1}{2} + \left( \frac{1682}{105} g + O(g^2) \right) \right] + O(\Omega^4) \right]
\]

(6.14)

Finally, it should be mentioned that by changing the coordinates not only of the interior but also of the exterior metric one can find a global harmonic coordinate system; i.e., in these coordinates the metric and its first derivatives are continuous on $\Sigma$.

[1] J. Martín and E. Ruiz, Phys. Rev. D32, 2550 (1985); Y. Gürsel, Gen. Rel. Grav. 15, 737 (1983).
[2] A. Lichnerowicz, Théories Relativistes de la Gravitation et de l’Électromagnétisme (Mason, Paris, 1955).
[3] Referring to the method we use as Post–Minkowskian, this is perhaps misleading. In fact, a true Post–Minkowskian approximation should exhibit explicit Minkowskian covariance. This is not the case in our approach, since we use definite systems of coordinates all along the calculations (“mass–centered” coordinate systems).
[4] R. H. Boyer, Proc. Camb. Phil. Soc. 61, 527 (1965).
[5] It is possible to obtain for $\rho$ an equation like (3.11). See for instance, S. Chandrasekar, Stellar Structure (Dover, New York, 1939).
[6] J. P. Luminet, Ann. Phys. Fr. 10, 101 (1985); C. Y. Wong, Astrophys. J. 190, 675 (1974).
[7] D. Kramer, H. Stephani, M. MacCallum and H. Herlt, Exact Solutions of Einstein’s Field Equation (Cambridge University Press, Cambridge, 1980).
[8] L. Bel, T. Damour, N. Deruelle, J. Ibañez and J. Martín, Gen. Rel. Grav. 13, 963 (1981); the second paper cited in ref. 1.