On existence of matter outside a static black hole

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(Dated: March 24, 2022)

It is expected that matter composed of a perfect fluid cannot be at rest outside of a black hole if the spacetime is asymptotically flat and static (non-rotating). However, there has not been a rigorous proof for this expectation without assuming spherical symmetry. In this paper, we provide a proof of non-existence of matter composed of a perfect fluid in static black hole spacetimes under certain conditions, which can be interpreted as a relation between the stellar mass and the black hole mass.

PACS numbers: 04.20.Cv 04.20.Ex 04.40Dg

I. INTRODUCTION

The issue of the final state of gravitational collapse is important from various points of view. Usually we expect the black holes will eventually form after the gravitational collapse. However, the resulting spacetime is expected to asymptotically converge to a stationary or static state. The perturbation analysis \(^{(1)}\) and numerical demonstrations support this picture. As a consequence, the limiting spacetime will be a stationary or static vacuum spacetime if all the matter is absorbed to the thus produced black hole. Nowadays we know that the uniqueness theorem for black holes holds in asymptotically flat and static (or) vacuum spacetimes and the resulting spacetimes are described by Kerr or Schwarzschild solutions \(^{(2,3,4)}\). Therefore we can have definite astrophysical predictions using these exact solutions.

While the uniqueness of the vacuum black hole has been established, it is interesting to ask if there is a uniqueness theorem for stationary or static spacetimes of a black hole plus matter. In the stationary case (i.e., time translation symmetry exists but their trajectories are not hypersurface orthogonal), clearly this is not the case because many spacetime solutions have been constructed. The perturbative solution of a slowly rotating black hole surrounded by an infinitesimal ring was constructed by Will \(^{(5)}\). There are numerical solutions of a black hole surrounded by an infinitely thin disk \(^{(6)}\) or by an differentially rotating ring \(^{(7)}\). Recently, the black hole with a uniformly rotating ring around it has been calculated with high accuracy \(^{(8)}\).

On the other hand, we expect that there should be some kind of uniqueness theorem for static spacetimes (i.e., trajectories of time-translation symmetry are hypersurface orthogonal) of a black hole with matter. This is because if we try to put the matter at rest outside a black hole, it is expected to fall into the black hole because of the gravitational attraction. However, we would like to point out that this intuitive picture does not have a rigorous reasoning. In fact, one can easily provide a counterexample as follows. If an infinitely thin disc composed of counter-rotating particles exists on the equatorial plane of the black hole spacetime, there should exist a static configuration after a fine tuning of system parameters. This example indicates the uniqueness of spacetimes of a black hole with matter depends not only on the energy condition but also the equation of state matter satisfies. As far as we know, there is Bekenstein’s work \(^{(9)}\) which proves the nonexistence of scalar fields outside a black hole in the static spacetime. However, without assuming the spherical symmetry, the question of nonexistence of matter composed of an ordinary perfect fluid outside a black hole in the static spacetime has not been addressed up to now.

As it turns out, proving the uniqueness in the setting above is rather a delicate problem. A similar situation is found for the proof of the spherical symmetry of a static isolated star composed of perfect fluid for a certain class of equations of state. This problem was solved relatively recently by Lindblom and Masood-ul-Alam \(^{(11)}\) after a long history \(^{(12,13)}\). In the work of \(^{(11)}\), the authors studied the condition that should hold inside of a star without assuming the spherical symmetry and proved the conformal flatness of the space that directly implies the spherical symmetry in turn.

Fortunately, many of the results in \(^{(11)}\) can be used for our situation because much of their analysis was done in quite a general setting. In this paper we would like to reformulate the analysis of \(^{(11)}\) for the goal of showing the uniqueness of the static spacetime of a black hole plus matter, taking care of the difference between the two setups. We will provide a partial proof for the nonexistence of matter outside a static black hole. The nonexistence we obtain here is conditional under an inequality between the black-hole mass and the stellar mass.

This paper is organized as follows. In the next section, we provide our setup concerning the Einstein equation. In Sec.III, we summarize the results of Lindblom and Masood-ul-Alam \(^{(11)}\), paying particular attention to the part that are directly relevant to our argument. We prove our theorem in Sec.IV and summarize our paper with some discussion in Sec.V. We adopt the unit of \(c = G = 1\).
II. SETUP

We consider the static spacetime which has the metric

\[ ds^2 = -V^2(x)dt^2 + g_{ij}(x)dx^i dx^j, \]

where \( i = 1, 2, 3 \) and \( g_{ij}(x) \) is the induced metric of \( \{ t = \text{const.} \} \) hypersurfaces \( \Sigma \). In a static spacetime, its event horizon \( H \) is identified with Killing horizon \( \{ V = 0 \} \) and thus \( V(x) \) vanishes on the horizon. We assume that there is a perfect fluid with energy density \( \rho \) and pressure \( P \). The fluid is assumed to satisfy an equation of state \( P = P(\rho) \). We assume the surface of the star/fluid is a two-dimensional closed connected equipotential surface \( \{ x : V(x) = V_s > 0 \} \) for some positive constant \( V_s \).

The Einstein equation and equation for fluid are given by

\[ D^2V = 4\pi V(\rho + 3P), \]
\[ R_{ij} = \frac{1}{V}D_iD_j V + 4\pi (\rho - P)g_{ij}, \]
\[ D_i P = -\frac{1}{V} (\rho + P) D_i V, \]

where \( D_i \) and \( R_{ij} \) are the covariant derivative and Ricci tensor of the metric \( g_{ij}(x) \).

Eq. (2) indicates that the surface \( \{ \rho = \text{const.} \} \) is identical to the surface \( \{ V = \text{const.} \} \). Let us suppose for the moment the condition \( D_i V \neq 0 \) at the horizon. Except for the extremal charged black holes, this condition is satisfied in all known static black hole solutions. Then Eq. (2) says that the value of \( D_i P \) diverges at the horizon if matter exists at the horizon \( \{ V = 0 \} \), which is an unphysical situation. Hence, this would make the star surface disjoint from the horizon. We stress that the same conclusion can be obtained from Eq. (2) without any additional hypothesis. As Eq. (2) is elliptic, a standard boundary elliptic estimate (see, for example, Lemma 6.4 in Gilbarg-Trudinger [14]) says that near the horizon, the norm of the gradient of \( V \) is bounded by the sup norm of the right-hand side of Eq. (2) as well as that of \( V \), both of which are uniformly bounded in our case. As the horizon is the zero set of \( V \), while the surface of the star is the level set \( \{ V = V_s (> 0) \} \), the gradient bound for \( V \) provides a positive lower bound (dependent on \( V_s \)) for the distance between those two level sets, which in turn implies that the star is disjoint from the horizon.

At the moment pictures such as Fig.1 are possible configurations of our setup. Although the star surface shown in this figure is spherical, note that its topology is arbitrary as long as it is specified as a connected equipotential surface \( \{ V = V_s \} \). Thus our theorem will hold also for, e.g., a barotropic perfect fluid ring 1.

Here we stress that we will prove that such configurations as above do not occur under certain conditions. In doing so, we are also not assuming any symmetry of the spacelike slice.

Asymptotic flatness requires the following asymptotic behavior of \( V \) and the metric

\[ V = \left( 1 - \frac{M}{r} \right) + O(r^{-2}) \]

and

\[ g_{ij} = \left( 1 + \frac{2M}{r} \right) \delta_{ij} + O(r^{-2}), \]

where \( r = |\delta_{ij} x^i x^j|^{1/2} \) and \( M \) is the ADM mass.

Our strategy for the proof is as follows. We first show that the \( \{ t = \text{const.} \} \) hypersurface \( \Sigma \) is conformally flat. The main part of the proof is finding appropriate conformal transformations for \( \Sigma \) so that it becomes a hypersurface with zero ADM mass and non-negative Ricci scalar curvature. This wonderful idea was first introduced by Bunting and Masood-ul-Alam [14]. We then apply the positive energy theorem [15] to this surface to conclude that the surface is flat Euclidean space. In turn, the original hypersurface \( \Sigma \) is conformally flat and it will immediately follow that \( \rho = P = 0 \). In order to find the appropriate conformal transformation, we will use the Lindblom and Masood-ul-Alam’s theorems [14] that was used for proving the spherical symmetry of a static star. We review it in the next section.

III. THEOREMS OF STATIC STELLAR MODELS

In this section, we briefly review Lindblom and Masood-ul-Alam’s results [14] where the spherical symmetry of static stellar models was obtained under a certain condition on the equation of state. We are able to transcribe their argument mostly because the techniques used in treating the inside of stars are identical.

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1 Note, however, that our theorem is not applicable to a ring composed of counter-rotating particles around a black hole, as mentioned in Sec.I. This is because such matter has non-isotropic stress and its equation of state is not barotropic.
Because the \( \{V = \text{const.}\} \) surface is identical to \( \{\rho = \text{const.}\} \) surface due to Eq. 4 and the equation of state \( P = P(\rho) \), both \( \rho \) and \( P \) are regarded as functions of \( V \). The minimum value of \( V \) in the star is denoted by \( V = V_\text{s} \). Then quantities \( r_\mu(V) \), \( m_\mu(V) \) are defined as the solutions to the equations

\[
\frac{d r_\mu}{d V} = \frac{r_\mu(r_\mu - 2m_\mu)}{V(m_\mu + 4\pi r_\mu^2 P)},
\]

\[
\frac{d m_\mu}{d V} = \frac{4\pi r_\mu^3(r_\mu - 2m_\mu)}{V(m_\mu + 4\pi r_\mu^2 P)},
\]

where the boundary conditions for Eqs. 7 and 8 are

\[
r_\mu(V_\text{s}) = R_\mu := 2\mu/(1 - V_\text{s}^2),
\]

\[
m_\mu(V_\text{s}) = \mu.
\]

Here, \( \mu \) is some constant suitably chosen in the proof. As we see from Eqs. 7 and 8, \( m_\mu(V) \) holds and \( \mu \) could be interpreted as the local mass of the star. Now \( W_\mu(V) \) is defined by

\[
W_\mu := \begin{cases} 
(1 - V^2)^4 \frac{16\mu^2}{2 - 2m_\mu} \frac{2}{(1 + 2\mu)} & \text{outside of the star}, \\
0 & \text{inside of the star}.
\end{cases}
\]

Various lemmas on \( r_\mu(V) \), \( m_\mu(V) \), \( W_\mu(V) \) are introduced, such as the existence of the solution on \( (V_\mu, V_\text{s}) \) for some \( V_\mu \) (Lemma 1-3). In particular, assuming that the pressure is finite \( P(V) < \infty \), \( W_\mu \) can be positive for somewhat small \( \mu \) over the interval \( (V_\mu, V_\text{s}) \) (Lemma 8).

We require that the equation of state satisfies at least one of the following constraints.

\[
\frac{1}{5} \kappa^2 + 2\kappa + (\rho + P) \frac{d\kappa}{dP} \leq 0
\]

or

\[
\frac{5\rho^2}{6P(\rho + 3P)} \geq \kappa > \frac{10V_\text{s}^2}{c^2d(V_\text{s}) - V_\text{s}^2},
\]

where \( \kappa := (dp/dP)(\rho + P)/(\rho + 3P) \) and \( h(P) = \log(V_\text{s}/V) \). The lower bound on \( \kappa \) in the condition implies that the adiabatic index \( \gamma := (dp/d\rho)(\rho + P)/\rho \) must be less than or equal to \( (6/5)(1 + P/\rho)^2 \). Under this assumption, the following quantity

\[
\Sigma_\mu := \frac{dW_\mu}{dV} - \frac{8\pi}{3} V(\rho + 3P) + \frac{4W_\mu}{5V} \frac{\rho + P}{V + 3P} \frac{d\rho}{dP}
\]

is proved to be non-negative (Lemma 10).

The most important technical result is Lemma 6 in [11]: Assuming that \( W_\mu > 0 \) and \( \Sigma_\mu \geq 0 \) on \( (V_\mu, V_\text{s}) \), then \( W_\mu \geq W := D_\alpha V^2 \) on \( (V_\mu, V_\text{s}) \) holds everywhere and \( \mu \leq M \). Because this lemma is relevant to our problem, we briefly review its proof. In [11], a differential inequality of elliptic type in the exterior vacuum region is derived:

\[
D_\alpha(V^{-1}d^2 \Delta_\mu^+) \geq 0,
\]

where

\[
\Delta_\mu^+ := (W - W_\mu)(1 - V_\mu^2)^3(1 + b - V_\text{s}^2)/(1 - V^2)^3(1 + b - V^2),
\]

\( b \) is some positive constant to be chosen later. The appropriate boundary condition at infinity is \( W - W_\mu = 0 \). From the maximum principle, \( \Delta_\mu^+ \) has the maximum value at the infinity or at the surface of the star \( \{V = V_\text{s}\} \). The possibility of maximum value of \( W - W_\mu \) occurring at \( \{V = V_\text{s}\} \) is then removed by choosing \( b \) appropriately. Thus \( \Delta_\mu^+ \leq 0 \) holds everywhere and \( W_\mu \geq W \). \( \mu \leq M \) is directly derived from the asymptotic expansion of \( W_\mu \geq W \). Note here that in our case where the black hole is additionally present, the possibility that \( \Delta_\mu^+ \) takes a maximal value at the event horizon cannot be removed, and we cannot use Lemma 6 in [11] without modifications. We will come back to this point in the next section.

They next introduce \( M^- \) and \( M^+ \) where \( M^- \) satisfies \( M^- < M^+ \) and \( M^- \) on \( [V_\mu, V_\text{s}] \) while \( M^+ \) satisfies \( M^+ > M \) with \( W_\mu = M^+(V_\mu) = 0 \) at a point \( V_\mu^+ \in (V_\mu, V_\text{s}) \). The existence of \( M^\pm \) is guaranteed by, e.g., Lemma 6 in [11]. Then, they defined \( \nu \) by \( \nu = \inf_{\mu \in [M^-, M^+]} \mu \), where \( S_{\text{c}} \) is a subset of \( [M^-, M^+] \) such that \( W_\mu > 0 \) on \( (V_\mu, V_\text{s}) \) and \( W_\mu(V_\text{s}) = 0 \).

In their Lemma 14, they proved that \( d^2 \psi(V)/dV^2 \) is positive, where \( \psi(V) \) is defined as the solution to the equation

\[
\frac{d \psi_\mu}{d V} = \frac{\psi_\mu}{2r_\mu} \frac{1 - 2m_\mu}{r_\mu},
\]

with \( \psi_\mu(V_\text{s}) = (1 + V_\text{s})/2 \) at the surface of the star \( \{V = V_\text{s}\} \). Finally they define a conformal metric \( g_{ij}^\pm = \Omega_\pm g_{ij} \), where

\[
\Omega_+ = \left( \frac{1 + V}{2} \right) \frac{\psi_\mu}{\psi_\mu(V_\text{s})} \quad \text{outside of the star},
\]

and found that the Ricci scalar has the expression

\[
R(g_{ij}^+) = (\tilde{W} - W_\mu) \frac{8}{\Omega_+^2} \frac{d^2 \Omega_+}{dV^2},
\]

where \( \tilde{W} = W_\mu \). By Lemma 6 and 14, this is positive, while the space has zero ADM mass. Here the case of

2 In Lemma 6 of Ref. [11], it is stated that \( W_\alpha \geq W \) holds on \( (V_\mu, V_\text{s}) \). However, we can easily see from the proof that it holds everywhere outside of the star.

3 In defining \( \tilde{W} \) precisely, we must take care of the differential structure at the center of stars. As the argument is rather technical, we refer this point to Lemma 11, 12, 13, 14 and the main theorem in Ref. [11].
equality in the positive energy theorem can be applied, and we conclude that the space is conformally flat, which then implies that an isolated star is spherically symmetric \[10\].

IV. PROOF OF NON-EXISTENCE OF MATTER OUTSIDE A STATICAL BLACK HOLE

Now we turn our attention to the proof of non-existence of matter outside a static black hole. The various functions inside of a star, \(r_\mu, m_\mu, \psi_\mu, W_\mu\) and \(\Delta_\mu\) can be introduced without modifications. We adopt the same condition on the equation of state. However, the modification for Lemma 6 is required as mentioned in the previous section.

Lemma 6 in \[11\] says that \(\tilde{W} - W \geq 0\) holds everywhere. In the case of \(\tilde{W} = W\), this inequality was proved by showing \(\Delta^+_{\mu} < 0\) at \(V = V_s\) and by applying the maximum principle for the inequality \(D_{\mu}(V^{-1}D^\alpha \Delta^+_{\mu}) \geq 0\) which indicates that \(\Delta^+_{\mu} = 0\) at infinity is the maximum value. However, presently we have an additional boundary, which is the horizon \(\{V = 0\}\), and a possibility of \(\Delta^+_{\mu} > 0\) there remains. We thus have to assume \(\Delta^+_{\mu} < 0\) at the horizon or equivalently

\[
(W - W_\mu)_{V=0} \leq 0. \tag{20}
\]

We will discuss the physical meaning of this hypothesis in the last section. Under this assumption, \(W_\mu - W \geq 0\) is guaranteed everywhere, and the results in \[11\] that was proven using Lemma 6 become available. In particular, the existence of \(\nu\) is guaranteed.

Now we show the conformal flatness of this space. Consider the two conformal transformations defined by

\[
g^+_{ij} = \Omega^4_{\pm} g_{ij}, \tag{21}
\]

where \(\Omega_+\) is the same as Eq. \(18\) and

\[
\Omega_- = \frac{1}{2}(1 - V). \tag{22}
\]

Now we have two manifolds \((\Sigma^\pm, g^+_{ij})\). As in Bunting and Masood-ul-Alam \[14\], we can make a smooth manifold out of \(\Sigma^+ \cup \Sigma^-\) by gluing along \(\{V = 0\}\).

The asymptotic behavior of each metric is:

\[
g^+_{ij} = \delta_{ij} + O(r^{-2}), \tag{23}
\]

and

\[
g^-_{ij} dx^i dx^j = (M/2r)^4 (dr^2 + r^2 d\Omega_2^2) + O(r^{-5}). \tag{24}
\]

In the manifold \((\Sigma^-, g^-_{ij})\), \(r = \infty\) corresponds to a regular point in a 3-dimensional surface. Indeed, introducing a new coordinate \(\tilde{R} = M^2/4r\), the metric near \(r = \infty\) can be written as

\[
g^-_{ij} dx^i dx^j = d\tilde{R}^2 + R^2 d\Omega_2^2. \tag{25}
\]

If the Ricci scalar of this space is non-negative, positive energy theorem tells us that \(\Sigma^+ \cup \Sigma^-\) is the flat Euclidean space and thus \(\Sigma\) is conformally flat, because this manifold has the zero ADM mass and non-negative Ricci scalar. Hence we want to show the non-negativity of the Ricci scalars \(R(g^+_{ij})\).

The three-dimensional Ricci scalar becomes the same as \(19\) for \(\Sigma^+\) and

\[
R(g^+_{ij}) = 8\pi \Omega^-_\mu \left(1 + V \right) \rho + 6VP \geq 0, \tag{26}\]

for \(\Sigma^-\). Since \(R(g^-_{ij}) \geq 0\) holds as above, it remains to show the non-negativity of \(R(g^+_{ij})\). \(d^2 \Omega_+ /dV^2 \geq 0\) is guaranteed by Lemma 14 in \[11\] and \((W - W) \geq 0\) is guaranteed by the hypothesis \(20\). Then the conformally transformed space \(\Sigma^+ \cup \Sigma^-\) has non-negative Ricci scalar (See Eq. \(19\)) and zero ADM mass, which implies that \(\Sigma^+ \cup \Sigma^-\) is flat.

Now we have proven under the assumption \(20\) that the original space \(\Sigma\) is conformally flat. Going back to Eqs. \(19\) and \(23\), we find that \(R(g^+_{ij}) = R(g^-_{ij}) = 0\) holds and they in turn imply \(\tilde{W} = W\) and \(\rho = P = 0\), respectively. The latter implies that the spacetime should be vacuum. Our result excludes any static configurations of a black hole with a star whose surface is given by \(\{V = V_s\}\) as was suggested in Fig.1.

We summarize what we have obtained as the following theorem:

**Theorem:** In asymptotically flat static black hole spacetimes, the star, which is composed of a perfect fluid satisfying the dominant energy condition and has the surface of level surface set \(\{V = V_s(>0)\}\), cannot exist if (i) the equations of state \(P = P(\rho)\) satisfies the condition \[12\] or \[13\], and (ii) for \(W_\mu\) defined by Eq. \(11\) with Eqs. \(7\) and \(8\), the inequality \(W - W_\mu \leq 0\) holds on the event horizon \((V = 0)\).

As a result, the ordinary uniqueness theorem \[2\] of vacuum spacetimes can be applied and then the spacetime in our setup is reduced to the Schwarzschild spacetime.

V. DISCUSSION

We have proven the non-existence of matter composed of a perfect fluid under the hypothesis \((W - W_\mu)_{V=0} \leq 0\). Because the physical meaning of this hypothesis is still unclear, we examine it in this section. On the event horizon, the value of \(\sqrt{W}\) coincides with the surface gravity \(\kappa_H\) of the black hole. We introduce the Komar integral on the event horizon \(M_{BH} = \kappa_H A_H/4\pi\) that indicates the local mass of the black hole, where \(A_H\) is the area of the horizon. Then, \((W - W_\mu)_{V=0} \leq 0\) corresponds to

\[
\mu \leq \frac{1}{4\kappa_H} = \frac{A_H}{16\pi M_{BH}}. \tag{27}
\]

The first law of the black hole thermodynamics for the static spacetime \(\delta M_{BH} = \kappa_H \delta A_H / 8\pi\) implies \(A_H \propto
$M^2_{\text{BH}}$. Hence $A_H \sim 16\pi M^2_{\text{BH}}$ and the right hand side of the inequality (27) is $O(M_{\text{BH}})$. In order to find some upper bound on $\mu$ in terms of the quantities of the star, recall that $(W - W_{\mu})_{V=V_c} \leq 0$ holds on the surface of the star as appeared in the proof of Lemma 6 of [11]. This is rewritten as $\sqrt{W_s} \leq (1 - V_s^2)/2\mu$. Integrating over the surface of the star, we find

$$\mu \leq \frac{(1-V_s)^2}{16\pi M_*} A_s, \quad \text{(28)}$$

where $M_*$ is the Komar mass of the star and $A_s$ is the area of the surface of the star. Hence, $(W - W_{\mu})_{V=0} \leq 0$ holds if

$$(1-V_s^2)\frac{A_s}{16\pi M_*} \leq \frac{A_H}{16\pi M_{\text{BH}}} \sim M_{\text{BH}}. \quad \text{(29)}$$

In order to simplify this inequality, let us consider the situation where a ball-shaped star exists outside a black hole and the distance between them is sufficiently large. In this case $V_s^2$ is approximated as $V_s^2 \simeq 1 - 2M_*/r_s$ and $A_s \simeq 4\pi r_s^2$, where $r_s$ is the radius of the star. Then the inequality becomes

$$M_* \lesssim M_{\text{BH}}. \quad \text{(30)}$$

Therefore the main theorem states that a star with smaller mass than a black hole mass cannot exist outside of the black hole in static spacetimes.

Intuitively, heavy stars are also not permitted to exist outside of black holes in static spacetimes. We hope to have a different argument for proving the statement because our current approach of adapting Lindblom and Masood-ul-Alam’s results [11] is not expected to produce optimal statements.

Lastly we note that the star surface is assumed to be a surface of one connected component. In order to exclude the configuration of a star whose surface has two or more components such as a shell surrounding the black hole, further considerations are needed.

**Acknowledgements**

The work of TS was supported by Grant-in-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture of Japan(No.13135208, No.14102004, No. 17740136 and No. 17340075), the Japan-U.K. and Japan-France Research Cooperative Program. The work of SY is supported by Grant-in-Aid for Scientific Research (No.17740030,) as well as Exploratory Research Program for Young Scientists from Tohoku University. The work of HY was partially supported by a Grant for The 21st Century COE Program (Holistic Research and Education Center for Physics Self-Organization Systems) at Waseda University.

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