Lie algebras/Topology

Dirac families for loop groups as matrix factorizations

Familles d’opérateurs de Dirac pour les groupes de lacets et factorisations en matrices

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A B S T R A C T

We identify the category of integrable lowest-weight representations of the loop group \(LG\) of a compact Lie group \(G\) with the category of twisted, conjugation-equivariant curved Fredholm complexes on the group \(G\): namely, the twisted, equivariant matrix factorizations of a super-potential built from the loop rotation action on \(LG\). This lifts the isomorphism of \(K\)-groups of \([3–5]\) to an equivalence of categories. The construction uses families of Dirac operators.

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R É S U M É

On identifie la catégorie des représentations intégrables de plus bas poids du groupe de lacets \(LG\) d’un groupe de Lie compact \(G\) avec la catégorie des complexes de Fredholm tordus, courbés et équivariants pour conjugaison sur le groupe \(G\) : plus précisément, les factorisations en matrices d’un potentiel provenant de la rotation des lacets dans \(LG\). Cette construction relève l’isomorphisme de \(K\)-groupes de \([3–5]\) en une équivalence de catégories. La construction fait appel aux familles d’opérateurs de Dirac.

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1. Introduction and background

The group \(LG\) of smooth loops in a compact Lie group \(G\) has a remarkable class of linear representations whose structure parallels the theory for compact Lie groups [10]. The defining stipulation is the existence of a circle action on the representation, with finite-dimensional eigenspaces and spectrum bounded below, intertwining with the loop rotation action on \(LG\). We denote the rotation circle by \(T_r\); its infinitesimal generator \(L_0\) represents the energy in a conformal field theory.

Noteworthy is the projective nature of these representations, described (when \(G\) is semi-simple) by a level \(h \in H^2_G(G; \mathbb{Z})\) in the equivariant cohomology for the adjoint action of \(G\) on itself. The representation category \(\text{Rep}^h(LG)\) at a given level \(h\) is semi-simple, with finitely many simple isomorphism classes. Irreducibles are classified by their lowest weight (plus some supplementary data when \(G\) is not simply connected [5, Ch. IV]).
In a series of papers [3–5], the authors, jointly with Michael Hopkins, construct $K^0\mathfrak{g}ep^h(LG)$ in terms of a twisted, conjugation-equivariant topological $K$-theory group. To wit, when $G$ is connected, as we shall assume throughout this paper,\footnote{Twisted loop groups show up when $G$ is disconnected [5].} we have
\begin{equation}
K^0\mathfrak{g}ep^h(LG) \cong K_{C^+}^{\dim C}(G),
\end{equation}
with a twisting $\tau \in H^2(G; \mathbb{Z})$ related to $h$, as explained below.

**Remark 1.1.** One loop group novelty is a braided tensor structure\footnote{When $G$ is not simply connected, there is a constraint on $h$.} on $\mathfrak{g}ep^h(LG)$. The structure arises from the fusion product of representations, relevant to 2-dimensional conformal field theory. The $K$-group in (1.1) carries a Pontryagin product, and the multiplications match in (1.1).

The map from representations to topological $K$-classes is implemented by the following Dirac family. Calling $A \mathcal{g}$ the space of connections on the trivial $G$-bundle over $S^1$, the quotient stack $[G:G]$ under conjugation is equivalent to $[\mathcal{A}:LG]$ under the gauge action, via the holonomy map $A \to G$. Denote by $S^\pm$ the (lowest-weight) modules of spinors for the loop space $L_G$ of the Lie algebra and by $\psi(A) : S^\pm \to S^\mp$ the action of a Clifford generator $A$, for $d + Adt \in A$. A representation $H$ of $L_G$ leads to a family of Fredholm operators over $A$,
\begin{equation}
\mathfrak{p}_A : H \otimes S^+ \to H \otimes S^-, \quad \mathfrak{p}_A := \mathfrak{p}_0 + i\psi(A)
\end{equation}
where $\mathfrak{p}_0$ is built from a certain Dirac operator [7] on the loop group.\footnote{The normalized operator $-2^{-1/2}\mathfrak{p}_0$ is the square root $G_0$ of $L_0$ in the super-Virasoro algebra.} The family is projectively $LG$-equivariant; dividing out by the subgroup $\mathcal{O}G \subseteq LG$ of based loops leads to a projective, $G$-equivariant Fredholm complex on $G$, whose $K$-theory class $[\mathfrak{p}_A, H \otimes S^+] \in K_{C^+}^{\dim C}(G)$ is the image of $H$ in the isomorphism (1.1). When $\dim G$ is odd, $S^+ = S^-$ and skew-adjointness of $\mathfrak{p}_A$ leads instead to a class in $K^1$. The twisting $\tau$ is the level of $H \otimes S$ as an $LG$-representation, with a ($G$-dependent) shift from the level $h$ of $H$.

The shifts are best explained in the world of super-categories, with $Z/2$ gradings on morphisms and objects; odd simple objects have as endomorphisms the rank one Clifford algebra $\text{Cliff}(1)$, and in the semi-simple case, they contribute a free generator to $K^1$ instead of $K^0$. Consider the $\tau$-projective representations of $LG$ with compatible action of $\text{Cliff}(L_G)$, thinking of them as modules for the (not so well-defined) crossed product $LG \ltimes \text{Cliff}(L_G)$. They form a semi-simple super-category $\mathfrak{S}\mathfrak{g}ep^\tau$, and the isomorphism (1.1) becomes
\begin{equation}
K^\tau \mathfrak{S}\mathfrak{g}ep^\tau (LG \ltimes \text{Cliff}(L_G)) \cong K_{C^+}^{\tau + \dim C}(G)
\end{equation}
with the advantage of having no shift in degree or twisting. (For simply connected $G$, both sides live in degree $* = \dim G$, but both parities can be present for general $G$.) This isomorphism is induced by the Dirac families of (1.2): a super-representation $SH^\tau$ of $LG \ltimes \text{Cliff}(L_G)$ can be coupled to the Dirac operators $\mathfrak{p}_A$ without a choice of factorization as $H \otimes S^h$.

2. The main result

There is a curious mismatch in (1.3): the isomorphism is induced by a functor of underlying Abelian categories, from $Z/2$-graded representations to twisted Fredholm bundles over $G$, but this functor is far from an equivalence. The category $\mathfrak{S}\mathfrak{g}ep^\tau$ is semi-simple (in the graded sense discussed), but that of twisted Fredholm complexes is not so; we can even produce continua of non-isomorphic objects in any given $K$-class, by compact perturbation of a Fredholm family.

Here, we redress this problem by incorporating a super-potential, a celebrity in the algebraic geometry of 2-dimensional physics (the “$B$-model”). As explained by Orlov\footnote{Orlov discusses complex algebraic vector bundles; we found no exposition for equivariant Fredholm complexes in topology, and a discussion is planned for our follow-up paper.} [8], this deforms the category of complexes of vector bundles into that of matrix factorizations: the 2-periodic, curved complexes with curvature equal to the super-potential $W$. Our $W$ has Morse critical points, leading to a semi-simple super-category with one generator for each critical point; the generators are precisely the Dirac families of (1.2) on irreducible $LG$-representations. The artifice of introducing $W$ is redeemed by its natural topological origin in the loop rotation $\tau_i$-action on the stack $[G:G]$. The $\tau_i$-action is evident in the presentation $[\mathcal{A}:LG]$, but it rigidifies a $BGZ$-action on the stack. Furthermore, for twistings $\tau$ transgressed from $BG$, the $BGZ$-action lifts to the $G$-equivariant gerbe $G^\tau$ over $G$ which underlies the $K$-theory twisting. The logarithm of this lift is $2\pi iW$.

**Remark 2.1.** The conceptual description of a super-potential as logarithm of a $BGZ$-action on a category of sheaves is worked out in [9]; the matrix factorization category is the Tate fixed-point category for the $BGZ$-action. For varieties, $W$ is a function and $\exp(2\pi iW)$ generates a $BGZ$-action on sheaves; on a stack, a geometric underlying action can also be present, as in this case. With respect to [9], our $W_\tau$ below should be re-scaled to take integer values at all critical points; we will omit this detail in order to better connect with the formulas in [4,5].
To spell this out, recall that a stack is an instance of a category, and a \( B\mathbb{Z} \)-action thereon is described by its generator, an automorphism of the identity functor. This is a section over the space of objects, valued in automorphisms, which is central for the groupoid multiplication. For \( [G : G] \), the relevant section is the identity map \( G \to G \), from objects to morphisms. Intrinsically, \( [G : G] \) is the mapping stack from \( B\mathbb{Z} \) to \( BG \), and the \( B\mathbb{Z} \)-action in question is the self-translation action of \( B\mathbb{Z} \). This rigidifies the geometric \( T_\tau \)-action on the homotopy equivalent spaces \( LBG \sim BLG \sim A/IG \).

A class \( \tilde{t} \in H^4(BG; \mathbb{Z}) \) transgresses to a \( t \in H^2(G, \mathbb{Z}) \), with the latter having a natural \( T_\tau \)-equivariant refinement. This can also be rigidified, as follows. The exponential sequence lifts \( \tilde{t} \) uniquely to \( H^3(BG, \mathbb{T}) \), the group cohomology with smooth circle coefficients. That defines a Lie 2-group \( G^\mathbb{T} \), a multiplicative \( \mathbb{T} \)-gerbe over \( G \). (Multiplicativity encodes the original \( \tilde{t} \).) The mapping stack from \( B\mathbb{Z} \) to \( B\mathbb{T} \) is the quotient \( [G^\mathbb{T} : G] \) under conjugation, and carries the \( B\mathbb{Z} \)-action from the self-translations of the latter. Because \( B\mathbb{T} \to G \) is strictly central, the self-conjugation action of \( G^\mathbb{T} \) factors through \( G \), and the quotient stack \( [G^\mathbb{T} : G] \) is our \( B\mathbb{Z} \)-equivariant gerbe over \( G \) with band \( \mathbb{T} \). We denote this central circle by \( T_\mathbb{C} \), to distinguish it from \( T_\tau \).

The \( B\mathbb{Z} \)-action gives an automorphism \( \exp(2\pi i W_\tau) \) of the identity of \([G^\mathbb{T} : G]\), lifting the geometric one on \([G : G]\). Concretely, \([G^\mathbb{T} : G]\) defines a \( T_\mathbb{C} \)-central extension of the stabilizer of \([G : G]\), and \( \exp(2\pi i W_\tau) \) is a trivialization of its fiber over the automorphism \( g \) at the point \( g \in G \) (see Section 3 below). The logarithm \( W_\tau \) is multi-valued and only locally well-defined; nevertheless, the category \( MF^\mathbb{T}_G(G; W_\tau) \) of twisted matrix factorizations is well-defined, and its objects are represented by \( \tau \)-twisted \( G \)-equivariant Fredholm complexes over \( G \) curved by \( W_\tau + \mathbb{Z} \cdot 1d \).

**Theorem 2.2.** The following defines an equivalence of categories from \( \mathcal{E}Rep^\mathbb{T} \) to \( MF^\mathbb{T}_G(G; -2W_\tau) \): a graded representation \( SH^\pm \) goes to the twisted and curved Fredholm family \( (\mathcal{D}_\tau, SH^\pm) \) whose value at the connection \( d + A dt \in A \) is the \( \tau \)-projective \( LG \)-equivariant curved Fredholm complex

\[ \mathcal{D}_\tau = \mathcal{D}_0 + i\psi(A) : SH^+ \cong SH^- \]

**Remark 2.3.**

(i) The factor \((-2)\), stemming from our conventions [5], can be absorbed by scaling the operators.

(ii) Matrix factorizations obtained from irreducible representations are supported on single conjugacy classes, the so-called Verlinde conjugacy classes in \( G \), for the twisting \( \tau \). These are the supports of the co-kernels of the Dirac families [12], [5, §12].

(iii) There is a braided tensor structure on \( \mathcal{E}Rep^\mathbb{T}(LG \rtimes \text{Cliff}(LG)) \) (without \( T_\mathbb{C} \)-action). A corresponding structure on \( MF^\mathbb{T}_G(G; W_\tau) \) should come from the Pontryagin product. We do not know how to spell out this structure, partly because the \( T_\mathbb{C} \)-action is already built into the construction of \( MF^\mathbb{T} \), and the Pontryagin product is not equivariant thereunder.

(iv) The values of the automorphism \( \exp(2\pi i W_\tau) \) at the Verlinde conjugacy classes determine the ribbon element in \( \mathcal{E}Rep^B(LG) \); see [2] for the discussion when \( G \) is a torus.

**Theorem 2.2** has a \( \tilde{t} \to \infty \) scaling limit, which is needed in the proof. In this limit, the representation category of \( LG \) becomes that of \( G \). On the topological side, noting that each \( \tilde{t} \) defines an inner product on \( g \), we magnify a neighborhood of \( 1 \in G \) to fix the scale. The \( \tau \)-central extensions of stabilizers near 1 have natural splittings, and \( W_\tau \) converges to a super-potential \( W \), a central element of the crossed product algebra \( G \rtimes \text{Sym}(g^\mathbb{T}) \). In a basis \( \xi_a \) of \( g \) with dual basis \( \xi^a \) of \( g^\mathbb{T} \), we will find in Section 3 that

\[ W = -i \cdot \xi_a(\delta_1) \otimes \xi^a + \frac{1}{2} \sum_a \| \xi^a \|^2 \]

with \( \xi_a(\delta_1) \) denoting the \( \xi_a \)-derivative of the delta-function at \( 1 \in G \). This leads to a \( G \)-equivariant matrix factorization category \( MF_G(g, W) \) on the Lie algebra.

To describe this limiting case, recall from [5, §4] the \( G \)-analogue of the Dirac family [12]. Kostant’s cubic Dirac operator [6] on \( G \) is left-invariant, and the Peter–Weyl decomposition gives an operator \( \mathcal{D}_0 : V \otimes S^\pm \to V \otimes S^\mp \) for any irreducible representation \( V \) of \( G \), coupled to the spinors \( S^\mp \) on \( g \). As before, let us work with graded modules \( SV \) for the super-algebra \( G \rtimes \text{Cliff}(g) \).

**Theorem 2.4.** Sending \( SV^\pm \) to \( (\mathcal{D}_\tau, SV^\pm) \), the curved complex over \( g \) given by

\[ g \ni \mu \mapsto \mathcal{D}_\mu = \mathcal{D}_0 + i\psi(\mu) : SV^+ \cong SV^- \]

provides an equivalence of super-categories from graded \( G \rtimes \text{Cliff}(g) \)-modules \( SV^\pm \) to \( G \)-equivariant, \((-2W)\)-matrix factorizations over \( g \).

With \( \lambda \) denoting the lowest weight of \( V \) and \( T(\mu) \) the \( \mu \)-action on \( SV \), we have [5, Cor. 4.8]

\[ \mathcal{D}_\mu^2 = -\| \lambda_V + \rho \|^2 + 2i \cdot T(\mu) - \| \mu \|^2 \in (-2W) + \mathbb{Z} \]
3. Outline of the proof

3.1. Executive summary

The category $\text{MF}^r_c(G; W_\tau)$ sheafifies over the conjugacy classes of $G$. Near a $g \in G$ with centralizer $Z$, the stack $[G : G]$ is modeled on a neighborhood of 0 in the adjoint quotient $[3 : Z]$ of the Lie algebra $\mathfrak{g}$, via $3 \ni \xi \mapsto g \cdot \exp(2\pi \xi)$. The equivariant gerbe $[G : G]$ is locally trivialized (possibly on a finite cover of $Z$) uniquely up to discrete choices, differing by $Z$-characters. We will compute $W_\tau$ locally in those terms in $Z \times \mathbb{C}^\infty(3)$, recovering (2.1), up to a $(g$-dependent) central translation in $3$. We then show that $\text{MF}^r$ vanishes near singular elements $g$. Assuming for brevity that $\pi_1(G)$ is torsion-free, we are then left with the case when $Z$ is the maximal torus $T \subseteq G$, where the super-potential $W_\tau$ turns out to have Morse critical points, located precisely at the Verlinde conjugacy classes. The local category is freely generated by the respective Atiyah–Bott–Schapiro Thom complex; the latter is quasi-isomorphic to our Dirac family for a specific irreducible representation, associated with the Verlinde class $[3, 12]$.

3.2. Crossed module description

We will describe $G^i$ as a Whitehead crossed module $[11]$. This is an exact sequence of groups

$$\mathbb{T}_c \hookrightarrow K \xrightarrow{\varphi} G,$$

equipped with an action $\alpha : H \to \text{Aut}(K)$ which lifts the self-conjugation of $H$ and factors the self-conjugation of $K$. Call $h$ an $H$-lift of $g \in G$ and $C$ the pre-image of $Z$ in $H$. Define the central extension $\tilde{Z}$ by means of a $\mathbb{T}_c$-central extension $\tilde{C}$ of $C$ trivialized over $\varphi(K)$.

The commutator $c \mapsto hch^{-1}c^{-1}$ gives a crossed homomorphism $\chi : C \to \varphi(K)$ with respect to the conjugation action of $C$ on $\varphi(K)$. The lift $\alpha$ lets $\chi$ pull back the central extension $\tilde{K} \to \varphi(K)$ to one $\tilde{C} \to C$; further, $\tilde{C}$ is trivialized over $\varphi(K)$, since $\alpha(h)$ identifies the fibers of $K$ over $c$ and $hch^{-1}$, when $c \in \varphi(K)$. Finally, noticing that $h^2h^{-1}h^{-1} = 1$ trivializes the fiber of $\tilde{C}$ over $c = h$ and gives our $\exp(2\pi i W_\tau)$ at $g \in Z$.

3.3. Local computation of $W_\tau$

Following [1], take $K = \Omega^2 \mathcal{G}$, the $\tau$-central extension of the group of smooth maps $[0, 2\pi] \to G$ sending $[0, 2\pi]$ to 1, and $H = \mathcal{P}_1 G$, the group of smooth paths starting at 1 in $G$ but free at the end. With the $\tilde{\tau}$-inner product $(\cdot | \cdot)$, the crossed module action of $\gamma \in H$ on the Lie algebra $\mathfrak{i} \mathfrak{r} \oplus \Omega^\mathcal{G}$ of $K$ is

$$\gamma.(x \oplus \omega) = \left(x - \frac{i}{2\pi} \int_0^{2\pi} \langle \gamma^{-1} \frac{d\gamma}{d\tau} | \omega \rangle \right) \oplus \text{Ad}_\gamma(\omega) \quad (3.1)$$

extending the $\text{Ad}$-action of $\Omega^2 \mathcal{G}$ $[10, \text{Prop. 4.3.2}]$, and exponentiating to an $H$-action on $K$.$^5$

Lift $g$ to $h = \exp(\tau \mu) \in \mathcal{P}_1 G$, $\mu \in \frac{1}{2\pi} \log g$, and assume first that $Z$ centralizes $\mu$. Instead of the entire group $C$ of Section 3.2, consider the subgroup $\mathcal{P}_c Z$ of paths in $Z$. This centralizes $h$, trivializing $\tilde{C}$ over $\mathcal{P}_c Z$. In this 'lucky' trivialization, $W_\tau = 0$. However, over $\Omega Z = \varphi(K) \cap \mathcal{P}_c Z$, the trivialization of Section 3.2 differs from the lucky one by adding the (exponentiated) character

$$\omega \mapsto -\frac{i}{2\pi} \int_0^{2\pi} (\mu | \omega) dt,$$

as per formula (3.1). We can trivialize $\tilde{Z}$ locally by extending this to a character of $\mathcal{P}_c Z$, accomplished by exponentiating the same integral. Now, $2\pi i W_\tau(g) = \pi i | \mu | ^2 \oplus 2\pi \mu \in i \mathbb{R} \oplus \mathfrak{g}$.

Even when $Z$ does not centralize $\mu$, $W_\tau$ is trivialized (for $\pi_1(G)$ torsion-free) by restriction to a maximal torus. Continuity also pins it down: the assumption on $\mu$ can be satisfied for generic $g$.

3.4. Vanishing of singular contributions

Take for simplicity $g = 1$, $Z = G$, $W$ on $\mathfrak{g}$ as in (2.1), plus possibly a central linear term $\mu$. Koszul duality equates the localized category $\text{MF}^r_c(G; W)$ with the super-category of modules over the differential super-algebra

$$(G \ltimes \text{Cliff}(\mathfrak{g}), [\mathfrak{p}_\mu, \cdot]), \quad \text{with} \quad \mathfrak{p}_\mu = \mathfrak{p}_0 + i \psi(\mu)$$

$^5$ The trivialization will be normalized by $C$-conjugation, thus descending the central extension to $Z$.

$^6$ Acting on other components of $\Omega^2 \mathcal{G}$ requires more topological information from $\tilde{\tau}$. 
of Theorem 2.4. Ignoring $\bar{p}_\mu$, the algebra is semi-simple, with simple modules the $V \otimes S^\pm$. Now, $\bar{p}_\mu^2 = -\|\lambda V + \mu + \rho\|^2$ cannot vanish for any $V$ for non-abelian $\lambda$, so $[\bar{p}_\mu, \bar{p}_\mu]$ provides a homotopy between $0$ and the central unit $\bar{p}_\mu^2$. This makes the super-category of graded modules quasi-equivalent to 0.

3.5. Globalization for the torus

We describe the stack $[T^\xi : T]$ and potential $W_\xi$ in the presentation $T = [t : \Pi]$ of the torus as a quotient of its Lie algebra by $\Pi \cong \pi_1(T)$. Lifted to $t$, the gerbe of stabilizers $\tilde{T}$ is trivial with band $\mathbb{T}_c \times T$. The descent datum under translation by $p \in \Pi$ is the shearing automorphism of $\mathbb{T}_c \times T$ given by the $\mathbb{T}_c$-valued character $\exp(p \log t)$, $t \in T$. In the same trivialization over $t$, the super-potential is

$$2\pi i W_\xi(\mu) = \pi \|\mu\|^2 \oplus 2\pi \mu \in i\mathbb{R} \oplus t.$$

With $\Lambda$ denoting the character lattice of $T$, the crossed product algebra of the stack $[T^\xi : T]$ can be identified with the functions on $(\coprod_{\lambda \in \Lambda} \mathbb{T}_c t_\lambda) / \Pi$, with the action of $\Pi$ by simultaneous translation on $\Lambda$ and $t$. On the sheet $\lambda \in \Lambda$, $W_\xi = -\langle \lambda | \mu \rangle + \|\mu\|^2/2$ has a single Morse critical point at $\mu = \lambda$.

It follows that the super-category $\text{MF}_\xi^\xi(T; W_\xi)$ is semi-simple, with one generator of parity $\dim t$ at each point in the kernel of the isogeny $T \to T^*$ derived from the quadratic form $\tilde{e} \in H^4(BT; \mathbb{Z})$. The kernel comprises precisely the Verlinde points in $T$ [2], concluding the proof of our main result.

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