Pauli-Lubanski scalar in the Polygon Approach to 2+1-Dimensional Gravity

M. Welling and M. Bijlsma

Institute for Theoretical Physics
University of Utrecht
Princetonplein 5
P.O. Box 80006
3508 TA Utrecht
The Netherlands

Abstract

In this paper we derive an expression for the conserved Pauli-Lubanski scalar in ’t Hooft’s polygon approach to 2+1-dimensional gravity coupled to point particles. We find that it is represented by an extra spatial shift $\Delta$ in addition to the usual identification rule (being a rotation over the cut). For two particles this invariant is expressed in terms of ’t Hooft’s phase-space variables and we check its classical limit.

1E-mail: welling@fys.ruu.nl
1 Introduction

In 1992 't Hooft introduced the polygon approach to 2+1-D gravity \[2\]. By using the fact that gravity in 2+1 dimensions has only matter degrees of freedom he managed to write down an exact solution to 2+1-D gravity coupled to point particles. In this formulation one tessellates a time-slice with polygons (see figure 1) and defines a Lorentz frame on each polygon. The velocity of the boundary between two adjacent polygons (generally with different Lorentz frames) is determined by the equation \( t_1 = t_2 \) i.e. the time on both polygons must be equal. This ensures that no time jumps take place and the surface is a proper Cauchy surface. One can show that in this case the boundary can only move perpendicular to itself with a constant velocity. The lengths of the boundaries will be taken as the configuration variables of phase space, and are denoted by \( L_i \).

As a momentum variable conjugate to \( L_i \), we must take \( 2\eta_i \), where \( \eta_i \) is the rapidity of the edge \( L_i \) (\( v_{\text{edge}} = \tanh \eta \)). Thus we have: \( \{L_i, 2\eta_j\} = \delta_{ij} \). The line segments \( L_i \) end either in a vertex, where three edges meet (three-vertex), or in a particle (one-vertex), as depicted in figure 1. At the three-vertex the three dimensional curvature must vanish as there is no matter present at this location. Exploiting this fact one can derive a set of equations which can be used e.g. to express the three angles \( \alpha_i \) in terms of the adjacent rapidities \( \eta_j \). If there is a particle at the end of a line \( L_i \) we pick up a nontrivial holonomy if we move around the particle. The length of an edge \( L_i \) may shrink or grow, and this can be described by a Hamiltonian formulation. If we take the Hamiltonian as the sum of the angle deficits at the particle-sites and at the three-vertices we generate the correct dynamical equations:

\[
\frac{d}{dt} L_i = \{H, L_i\} \quad (1)
\]

\[
\frac{d}{dt} \eta_i = \{H, \eta_i\} \quad (= 0) \quad (2)
\]
As the system evolves, one of the lengths $L_i$ may shrink to zero and a transition to a new set of polygons takes place. The possible transitions are listed in [2]. Another aspect of the Hamiltonian formulation is the fact that there are constraints on the variables. The constraints among the variables result from the fact that the polygons must close. These constraints are first class and generate time translations and Lorentz transformations of the polygon. We also refer to the literature for a detailed description of these constraints [2]. In this paper we will actually concentrate only on one (open) polygon (figure 3).

The polygon formulation of 2+1-D gravity was recently revisited by Franzosi and Guadagnini who stressed the importance of the braid group [4]. Using holonomy loops they obtained a formula expressing the conservation of energy-momentum during particle interactions. In this paper we will derive a similar kind of conservation law for the Pauli-Lubanski scalar $J$. It is long known that this conservation law exists [3]. In this paper we will investigate this conservation law in the polygon approach where we cannot allow for time jumps. We find that it is represented by an extra spatial shift $\Delta$ in the identification over the cut.

2 Observables in 2+1-Dimensional Gravity

The usual way to describe point particles in 2+1-D gravity is by cutting out wedges from space-time and identifying the edges according to certain rules. For instance in the case of two moving particles we have the following identification rule [1]:

$$\tilde{x} = Lx + q$$

$$= (B_1R_1B_1^{-1}B_2R_2B_2^{-1})x$$

$$+ (-B_1R_1B_1^{-1}B_2R_2B_2^{-1}a_2 + B_1R_1B_1^{-1}(a_2 - a_1) + a_1)$$

Here $a_i$ are the locations of the particles, $B_i$ are boost matrices in $SO(2,1)$, $R_i$ are rotation matrices over an angle $m_i$ (equal to the mass of particle $i$), $L$ is a Lorenz transformation of the form: $L = BRB^{-1}$ and $q$ is a translation vector. We have chosen units such that $8\pi G = 1$. In the above formulas we omitted the indices. In [1] it was found that the angular momentum is given by $q^0$, which is a quantity that transforms under Lorentz transformations and translations. The ”observables” in this theory must be given by the invariants of the Poincaré group. There are two such invariants [3], namely:

1. $$\text{Tr}L = 1 + 2 \cos M$$
   
   where $M$ is the total mass of the system, and

2. $$J = \frac{-1}{2 \sin M} \varepsilon^{abc}L_{ab}q_c = [B^{-1}_{com}q]^0$$
The matrix $B_{\text{com}}$ is the boost-matrix for the effective center of mass particle. The second expression can be derived using $L = B_{\text{com}} R(M)_{\text{com}} B^{-1}_{\text{com}}$ and $L J^a L^{-1} = J^b L_a^b$ with $(J^a)_b^c = (\varepsilon^a)_b^c$. From ii) we see that the second invariant is really proportional to the angular momentum in the center of mass frame. It is in fact the Pauli-Lubanski scalar, which is proportional to the spin of a particle in its rest frame.

3 Pauli-Lubanski Scalar in the Polygon Approach

In this section we will see how $\mathcal{J}$ is represented in the polygon approach to 2+1-D gravity. As mentioned in the introduction the polygon approach is a Cauchy formulation and no closed timelike curves are allowed by construction. It also implies that we must choose the cuts in such a way that there are no time jumps anywhere in the plane.

The total energy of the system is given by the total angle deficit of an "effective" center of mass particle and can be expressed in terms of the rest masses of the particles and the momenta $\eta_i$ across the edges. Similarly we expect that $\mathcal{J}$ will be given by the spin of the center of mass particle and that it can be expressed in terms of the $L_i$ and $\eta_i$ of the constituent particles. Although we know that close to the (spinning) center of mass particle (where closed timelike curves are possible) the Cauchy construction of ’t Hooft is not possible, we still expect that far away from the particles the identification rule for a single center of mass particle is valid as an effective description for the system of particles.

The general identification rule for a moving and spinning particle situated at the origin is:

$$\tilde{x} = B R B^{-1} x + B s \quad s = (S, 0, 0)$$

We want to choose the cut in such a way that (at $t = 0$) there is no time jump across this cut:

$$\tilde{x}^0 = [B R B^{-1} x + B s]^0 = 0$$

Next we choose this particle to move in the positive $x$-direction. The condition (7) then gives the following line of points (parametrized by $\ell$) that will not experience a time jump under the transformation (6):

$$x = \ell$$

$$y = \ell \tan \Sigma - \frac{S}{v \sin M}$$

with $\tan \Sigma = \cosh \xi \tan \frac{M}{2}$ and $v = \tanh \xi$. This line is mapped by (6) to:

$$\tilde{x} = \ell - \frac{S}{\sinh \xi}$$

$$\tilde{y} = -\ell \tan \Sigma - \frac{\cot M S}{v}$$
Figure 2: Wedge cut out of space-time for a moving spinning particle ($\sigma = \sinh \xi$).

This mapping is pictured in figure 2. First we note that the point of intersection of the two lines in figure 2 is not at the position of the particle itself. Secondly, the mapping is not only a rotation over the angle $2\Sigma$ but also contains a shift over a distance $\Delta = S/(\sinh \xi \cos \Sigma)$. We see that close to the particle the construction becomes pathological as anticipated. This, however, presents no problem as this part of space will be replaced by the space of the moving particles without spin. Far away from the particles however, $J$ is still given by:

$$J = \Delta \cos \Sigma \sinh \xi$$

We can, in the case of two particles, express this quantity $J$ in terms of the polygon variables $L_i$ and $\eta_i$ (figure 3) as follows;

$$J = -2 \sinh \xi \{L_1 \sin \beta_1 \sin(\beta_2 + \frac{1}{2}(\alpha_2 - \alpha_1 - \alpha_3)) - L_2 \sin \beta_2 \sin(\beta_1 + \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3))\}$$

The angles $\alpha_i$ and $\beta_i$ can be expressed in terms of the rapidities defined across the edges (see (2)). Furthermore, $\sinh \xi = \sinh \eta / \sin \frac{1}{2}M$, where $\eta$ is the rapidity.

Figure 3: Two particles with angular momentum.
over the edge $L_\infty$ and $M$ is the total mass of the system, which can also be expressed in terms of the rest masses and rapidities of the constituent particles.
In order to check the classical limit we have to take:

\[
\sinh \xi \to v_{\text{com}} \quad (12)
\]

\[
2 \sin \beta_i \to m_i \quad (13)
\]

and find (figure 4):

\[
\mathcal{J} = - \sum_{i=1}^{2} m_i \vec{r}_i \times \vec{v}_{\text{com}} \quad (14)
\]

If we choose the origin as in figure 4, such that the angular momentum vanishes in this (boosted) frame, this quantity indeed represents the total angular momentum of the two particles in their center of mass frame, written in terms of the boosted variables:

\[
\mathcal{J} = \sum_i m_i \vec{r}_i \times \vec{w}_i = \sum_i m_i \vec{r}_i \times (\vec{v}_i - \vec{v}_{\text{com}}) = - \sum_i m_i \vec{r}_i \times \vec{v}_{\text{com}} \quad (15)
\]

where \( \vec{w}_i \) is the velocity in the c.o.m. frame and \( \vec{v}_i \) is the velocity in the boosted frame.

4 Discussion

The first step towards quantizing 2+1-D gravity in the polygon approach has been made recently by 't Hooft [5]. In this paper the quantization of one particle in its own gravitational field is considered. Surprisingly, he finds that space-time is discretized in this case. The next logical step would be to quantize the two particle problem. As it is very convenient to have a Hamiltonian formulation as
a starting point for quantization, the polygon approach seems to be the natural choice for describing the system. The two particle system is expected to have two observables: energy and angular momentum of the center of mass frame (or Pauli-Lubanski scalar). These conserved quantities must be "measurable" at spatial infinity. In the polygon approach energy was identified with the total angle deficit of space but no geometrical quantity was identified as the Pauli-Lubanski scalar. We found in this paper that it is given by a spatial shift in the identification rule at spatial infinity.

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