Information cloning of harmonic oscillator coherent states and its fidelity.

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We show that in the case of unknown harmonic oscillator coherent states it is possible to achieve what we call perfect information cloning. By this we mean that it is still possible to make arbitrary number of copies of a state which has exactly the same information content as the original unknown coherent state. By making use of this perfect information cloning it would be possible to estimate the original state through measurements and make arbitrary number of copies of the estimator. We define the notion of a Measurement Fidelity. We show that this information cloning gives rise, in the case of 1 \(\to\) \(N\), to a distribution of measurement fidelities whose average value is \(\frac{1}{2}\) irrespective of the number of copies originally made. Generalisations of this to the \(M \to MN\) case as well as the measurement fidelities for Gaussian cloners are also given.

I. INTRODUCTION

The no-cloning theorem expresses the inability to copy an unknown quantum state. The original version \(\frac{3}{3}\) of the theorem invoked the principle of linear superposition. The stronger restriction on cloning in fact comes from the unitarity of the cloning transformation itself. Accordingly, only mutually orthogonal states can be cloned universally. For class-dependent cloning the corresponding statement is that states belonging to the same class should still be mutually orthogonal but states belonging to distinct classes need not be so.

This precludes perfect cloning of even subclasses of harmonic oscillator coherent states. Cerf et al \(\frac{4}{4}\) have shown that there is an optimal fidelity to cloning coherent states. Many significant results regarding optimality of cloning of coherent states have appeared recently in the literature \(\frac{4–6}{4–6}\). Proposals for optical implementations have also been made \(\frac{7}{7}\). In this paper we wish to present an alternate route to the question of cloning coherent states. We show that it is possible to make arbitrary number of copies of coherent states with exactly the same information content as the original unknown state. Complete information about a coherent state is contained in the complex coherency parameter \(\alpha\) because of the theorem invoked the principle of linear superposition. Thus by information cloning what we mean is the ability to make arbitrary number of copies of coherent states whose coherency parameter is \(c(N)\alpha\) where \(\alpha\) is the coherency parameter of the unknown coherent state and \(c(N)\) is a known constant depending on the number of copies made.

We consider \(1 + N\) systems of harmonic oscillators whose creation and annihilation operators are the set \((a, a^\dagger), (b_k, b_k^\dagger)\) (where the index \(k\) takes on values \(1, \ldots, N\)) satisfying the commutation relations

\[
[a, a^\dagger] = 1; \quad [b_j, b_k^\dagger] = \delta_{jk}; \quad [a, b_k] = 0; \quad [a^\dagger, b_k] = 0
\]

Coherent states parametrised by a complex number are given by

\[
|\alpha > = D(\alpha) |0 >
\]

where \(|0 >\) is the ground state and the unitary operator \(D(\alpha)\) is given by

\[
D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}
\]

Let us consider a disentangled set of coherent states \(|\alpha > |\beta_1 >_1 |\beta_2 >_2 \ldots |\beta_N >_N\) and consider the action of the unitary transformation

\[
U = e^{i(\alpha^* \sum_j b_j a^\dagger - \alpha \sum_j a^\dagger b_j)}
\]

By an application of the Baker-Campbell-Hausdorff identity and the fact that \(U|0 > = |0 > \ldots |0 > = |0 >|0 >_1 \ldots |0 >_N\) it is easy to see that the resulting state is also a disentangled set of coherent states expressed by

\[
|\alpha' > |\beta_1' >_1 \ldots |\beta_N' >_N = U |\alpha > |\beta_1 >_1 \ldots |\beta_N >_N
\]

It is useful to transform the problem to the case where all \(\kappa_j\) are real through a redefinition of the creation and annihilation operators. Denoting \(\kappa_j = r_j e^{i\theta_j}\) we can introduce the tranformation

\[
\tilde{a} = a; \quad \tilde{b}_j = e^{i\theta_j} b_j; \quad \tilde{b}_j^\dagger = e^{-i\theta_j} b_j^\dagger
\]

This leaves the commutation relations of eqn (1) unchanged

\[
[\tilde{a}, \tilde{a}^\dagger] = 1; \quad [\tilde{b}_j, \tilde{b}_k^\dagger] = \delta_{jk}; \quad [\tilde{a}, \tilde{b}_k] = 0; \quad [\tilde{a}^\dagger, \tilde{b}_k] = 0
\]

The parametrisation of the coherent states are correspondingly redefined as
\[ \hat{a} = \alpha \quad \hat{\beta}_j = e^{-i\delta_j} \beta_j \] (8)

The unitary operator \( U \) takes the form
\[ \hat{U} = e^{i(\hat{a}^\dagger \sum_j r_j \hat{b}_j - \hat{a} \sum_j r_j \hat{b}_j^\dagger)} \] (9)

The initial state is
\[ |I > = D(\hat{a}) \ D(\hat{\beta}_1) ... D(\hat{\beta}_N)_N \] |0 > |0 > |0 > |0 > (10)

Defining
\[ \hat{a}(t) = \hat{U} \hat{a} \hat{U}^\dagger \quad \hat{b}_j(t) = \hat{U} \hat{b}_j \hat{U}^\dagger \] (11)

one easily gets the differential equations
\[ \frac{d}{dt} \hat{a}(t) = - \sum_j r_j \hat{b}_j(t) \quad \frac{d}{dt} \hat{b}_j(t) = r_j \hat{a}(t) \] (12)

The solutions to these eqns are straightforward to find:
\[ \hat{a}(t) = \cos rt \hat{a} - \sum_j \frac{r_j}{r} \sin rt \hat{b}_j \]
\[ \hat{b}_j(t) = \frac{r_j}{r} \sin rt \hat{a} + \sum_k \hat{M}_{jk}(t) \hat{b}_k \] (13)

where \( r = \sqrt{\sum_j r_j^2} \) and
\[ \hat{M}_{jk} = \delta_{jk} - \frac{r_j r_k}{r^2} (1 - \cos rt) \] (14)

This transformation induces a transformation on the parameters \((\hat{a}, \hat{\beta}_j)\) which can be represented by the matrix \( \hat{U} \) i.e. \( \hat{a}_a(t) = \hat{U}_{ab} \hat{a}_b \). We have introduced the notation \( \hat{a}_a \) with \( a = 1, ..., N+1 \) such that \( \hat{a}_1 = \hat{a}, \hat{a}_k = \hat{\beta}_k-1(k \geq 2) \) and a similar notation for the \((\alpha, \beta_j)\) with \( U \) as the corresponding matrix. Then we have
\[ \hat{U}_{1a} = \left( \begin{array}{c} \cos rt \frac{r_a}{r} \sin rt \ldots \frac{r_N}{r} \sin rt \end{array} \right) \] (15)
\[ \hat{U}_{ab} = -\frac{r_{a-1}}{r} \sin rt \delta_{b1} + (1 - \delta_{b1}) \hat{M}_{a-1,b-1} \] (16)

where eqn (14) is defined for \( a \geq 2 \). Equivalently
\[ \hat{U} = \left( \begin{array}{cccc} \cos rt & \frac{r_1}{r} \sin rt & \ldots & \frac{r_N}{r} \sin rt \\ -\frac{r_1}{r} \sin rt & M_{11} & \ldots & M_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{r_N}{r} \sin rt & M_{N1} & \ldots & M_{NN} \end{array} \right) \] (17)

It is easy to show that the matrix \( \hat{U} \) is Orthogonal satisfying
\[ \sum_a \hat{U}_{ab} \hat{U}_{ac} = \delta_{bc} \] (18)

The orthogonality of this matrix can be demonstrated directly on physical grounds. It can easily be seen that \( U \) (respectively \( \hat{U} \)) commutes with \( a^\dagger a + \sum_k \hat{b}_k^\dagger \hat{b}_k \) (respectively \( \hat{a}^\dagger \hat{a} + \sum_k \hat{b}_k^\dagger \hat{b}_k \)). This implies that
\[ |\hat{a}(t)|^2 + \sum_k |\hat{\beta}_k(t)|^2 = |\hat{a}|^2 + \sum_k |\beta_k|^2 \]
\[ |\alpha(t)|^2 + \sum_k |\beta_k(t)|^2 = |\alpha|^2 + \sum_k |\beta_k|^2 \] (19)

This means that \( \hat{U}, U \) are Unitary matrices. Since \( \hat{U} \) is real it is Orthogonal. The orthogonality of \( \hat{U} \) leads to another very important quadratic invariant in addition to the ones stated in eqn(19), namely,
\[ \hat{a}(t)^2 + \sum_k \hat{\beta}_k(t)^2 = \hat{a}^2 + \sum_k \beta_k^2 \] (20)

In fact for two independent sets of coherent state parameters \((\alpha, \beta), (\eta, \xi)\) we have the invariants
\[ \hat{a}(t) \ \eta(t) + \sum_k \hat{\beta}_k(t) \ \xi(t) = \alpha \ \eta + \sum_k \beta_k \ \xi_k \]
\[ \hat{a}^*(t) \ \eta(t) + \sum_k \hat{\beta}_k^*(t) \ \xi(t) = \alpha^* \ \eta + \sum_k \beta_k^* \ \xi \] (21)

Inverting the transformations in eqn (8) it is straightforward to obtain
\[ \hat{U} = \left( \begin{array}{cccc} \cos rt & \frac{r_1}{r} e^{-i\delta_1} \sin rt & \ldots & \frac{r_N}{r} e^{-i\delta_N} \sin rt \\ -\frac{r_1}{r} e^{i\delta_1} \sin rt & M_{11} & \ldots & M_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{r_N}{r} e^{i\delta_N} \sin rt & M_{N1} & \ldots & M_{NN} \end{array} \right) \] (22)

where
\[ M_{jk} = \delta_{jk} - e^{i\delta_j - i\delta_k} \frac{r_j r_k}{r^2} (1 - \cos rt) \] (23)

The corresponding quadratic invariants are
\[ \alpha(t) \ \eta(t) + \sum_k e^{-2i\delta_k} \beta_k(t) \ \xi(t) = \alpha \ \eta + \sum_k e^{-2i\delta_k} \beta_k \ \xi \]
\[ \alpha^*(t) \ \eta(t) + \sum_k \beta_k^*(t) \ \xi(t) = \alpha^* \ \eta + \sum_k \beta_k^* \ \xi \] (24)

With these results we first illustrate the hurdles to be overcome in cloning the harmonic oscillator coherent states. For this purpose let us look at the case \( N = 1 \). In this case the state obtained after applying the unitary transformation \( U \) is
\[ \alpha(t) = \cos rt \ \alpha + e^{-i\delta} \sin rt \ \beta \]
\[ \beta(t) = \cos rt \ \beta - e^{i\delta} \sin rt \ \alpha \] (25)
Here the best that can be achieved is to set \( \sin rt = 1 \) and remove known phases through the unitary operator \( e^{i r_1 a_1} \) and one sees that it only amounts to swapping but not cloning. Next, in the general \( N \)-case, the coefficients of \( \alpha \) in all \( \beta_k(t) \) must be made to have the same magnitude implying \( r_1 = r_2 = \ldots = r_n \). With the choice \( \beta_1 = \beta_2 = \ldots = \beta_N \) one gets

\[
\beta_k(t) = -e^{i \alpha} \frac{\sin rt}{\sqrt{N}} \alpha
\]  

(26)

With the optimal choice of \( \sin rt = -1 \) and using appropriate unitary transformations to remove known phases one gets \( N \) copies of the state \( \frac{\alpha}{\sqrt{N}} \).

A. Information cloning

Thus we are able to produce \( N \)-copies not of the original state \( |\alpha\rangle \) but of a state of the form \( \frac{\alpha}{\sqrt{N}} \) which has the same information content as \( |\alpha\rangle \) in the sense that a complete determination of the latter is equivalent to a complete determination of the former. This is what we would like to call cloning of information in contrast to cloning of the quantum state itself. It is quite plausible that in many circumstances of interest cloning in this more restricted sense may suffice.

Superficially this may appear to be a triviality in the sense that one can always apply known unitary transformations on unknown quantum states to produce states with the same information content in the sense used above. But what is nontrivial in our construction is that arbitrary number of copies of such information-equivalent states can be produced.

Fidelity of cloning was interpreted by \( \| \) as \( \langle \alpha | \rho_1 | \alpha \rangle \) where \( |\alpha\rangle \) is the unknown coherent that was cloned and \( \rho_1 \) is the one-particle reduced density matrix of the output. In the Gaussian-cloners of the type considered in \( \| \) there are \( N \)-copies of \( \rho_1 \) which are all mixed states. In contrast our information cloning produces \( N \)-copies which are pure states. We call the fidelity introduced by \( \| \) Overlap Fidelity. This overlap fidelity for our information cloning is

\[
F_{\text{info}} = e^{-|\alpha|^2(1-\frac{1}{\sqrt{N}})^2}
\]  

(27)

Not only can this be very small it is also not universal.

We introduce another notion of fidelity which we call Measurement Fidelity by which we mean the best reconstruction of the original unknown state that can be achieved through actual measurements performed in some optimal way. Before proceeding we give the formula for the optimal overlap fidelity of \( N \rightarrow M \) cloning of coherent states \( \| \)

\[
F_{\text{overlap}}^{N,M} = \frac{MN}{MN+M-N}
\]  

(28)

We now propose using the copies of the information-equivalent states to estimate the parameter \( \alpha \). Normally when the available number of copies of a state is very large, one can estimate the state quite accurately and use that to create arbitrary number of clones of the original coherent state. However, in our proposal for information cloning even though the number of copies \( N \) can be arbitrarily large, the coherency parameter given by \( \frac{\alpha}{\sqrt{N}} \) becomes arbitrarily small while the variances in \( \alpha \) remain the same as in the original state. This raises the question as to how best the original state can be reconstructed and about the statistical significance of our information cloning procedure.

On introducing momentum and position operators \( \hat{p}, \hat{x} \) through

\[
\hat{x} = \frac{(a + a^\dagger)}{\sqrt{2}} \quad \hat{p} = \frac{(a - a^\dagger)}{\sqrt{2i}}
\]  

(29)

the probability distributions in position and momentum representations are given by

\[
|\psi_{\text{clone}}(x)|^2 = \frac{1}{\sqrt{\pi}} e^{-\left(x - \sqrt{\frac{\pi}{2}} \alpha_R\right)^2}
\]

\[
|\psi_{\text{clone}}(p)|^2 = \frac{1}{\sqrt{\pi}} e^{-\left(p - \sqrt{\frac{\pi}{2}} \alpha_I\right)^2}
\]  

(30)

Let us distribute our \( N \)-copies into two groups of \( N/2 \) each and use one to estimate \( \alpha_R \) through position measurements and the other to estimate \( \alpha_I \) through momentum measurements. Let \( y_N \) denote the average value of the position obtained in \( N/2 \) measurements and let \( z_N \) denote the average value of momentum also obtained in \( N/2 \) measurements. The central limit theorem states that the probability distributions for \( y_N, z_N \) are given by

\[
f_x(y_N) = \sqrt{\frac{N}{2\pi}} e^{-\frac{1}{2}(y_N - \sqrt{\pi} \alpha_R)^2}
\]

\[
f_p(z_N) = \sqrt{\frac{N}{2\pi}} e^{-\frac{1}{2}(z_N - \sqrt{\pi} \alpha_I)^2}
\]  

(31)

The estimated value of \( \alpha \) is

\[
\alpha_{\text{est}} = \frac{y_N + iz_N}{\sqrt{2}} \sqrt{\frac{N}{\pi}}
\]  

(32)

The measurement fidelity \( F_{\text{meas}} \) can be understood as the quantity \( |\langle \alpha | \alpha_{\text{est}} \rangle|^2 \):

\[
F_{\text{meas}} = e^{-|\alpha - \alpha_{\text{est}}|^2}
\]  

(33)

The probability distribution for \( F \) is given by

\[
p(F) dF = \int dz_N dy_N \delta(z_N^2 + y_N^2 - \frac{2}{N} |\alpha_{\text{est}}|^2) f_x(y_N) f_p(z_N)
\]  

(34)

It is straightforward to show that
\[ p(F)dF = dF \]  

Consequently the average value of \( F_{\text{meas}} \) is  
\[ F_{1,N}^{\text{meas}} = 1/2 \]  

Now we generalise our results to the \( M' \rightarrow N' \) case. We start with \( M \) copies and let each copy be information cloned to \( N \) copies so we have \( MN \) copies finally.  

The position and momentum distributions are still given by eqn (30) but now \( MN/2 \) measurements are carried out for position and momentum. Consequently  
\[ f_x(y_{MN}) = \sqrt{\frac{MN}{2\pi}} e^{-\frac{MN}{2N}(y_{MN}^2 - \frac{\sqrt{\alpha}}{\sqrt{\alpha}})^2} \]  
\[ f_p(z_{MN}) = \sqrt{\frac{MN}{2\pi}} e^{-\frac{MN}{2N}(z_{MN}^2 - \frac{\sqrt{\alpha}}{\sqrt{\alpha}})^2} \]  

The estimated value of \( \alpha \) is still given by eqn (12). One finally obtains  
\[ p_{M,MN}(F)dF = MF^{M-1}dF \]  

The average measurement fidelity in this case is given by  
\[ F_{M,MN}^{\text{meas}} = \frac{M}{M+1} \]  

This approaches 1 as \( M \rightarrow \infty \).  

These fidelities should not be directly compared with eqns (28). As emphasised by Massar and Popescu [8] there can be many notions of fidelities and two schemes should be compared only with the same criterion for fidelity. So we compute the measurement fidelity for Gaussian cloners. Each copy is the Gaussian mixture  
\[ \rho = \int d^2\alpha \frac{A_{M,MN}}{\pi} e^{-A_{M,MN}|\alpha|^2} [\alpha_0 < \alpha < \alpha_0 + \alpha] \]  

Where \( A_{M,MN} \) given by  
\[ A_{M,MN} = \frac{MN}{N-1} \]  

reproduces eqn (28). The position and momentum distributions in the Gaussian mixture are given by  
\[ p_{\text{Gauss}}(x) = \frac{1}{\sqrt{\pi}} \frac{A_{M,MN}}{A_{M,MN} + \frac{2}{2}} e^{-\frac{(x-x_0)^2}{2\alpha}} \]  
\[ p_{\text{Gauss}}(p) = \frac{1}{\sqrt{\pi}} \frac{A_{M,MN}}{A_{M,MN} + \frac{2}{2}} e^{-\frac{(p-p_0)^2}{2\alpha}} \]  

The analogues of eqns (30) are given by  
\[ f_x(y_{MN}) = \sqrt{\frac{MN}{2\pi(A_{M,MN} + \frac{2}{2})}} e^{-\frac{MN}{2N}(y_{MN}^2 - \frac{\sqrt{\alpha}}{\sqrt{\alpha}})^2} \]  
\[ f_p(z_{MN}) = \sqrt{\frac{MN}{2\pi(A_{M,MN} + \frac{2}{2})}} e^{-\frac{MN}{2N}(z_{MN}^2 - \frac{\sqrt{\alpha}}{\sqrt{\alpha}})^2} \]  

The estimate for the coherency parameter is  
\[ \alpha_{\text{est}} = \frac{y_{MN} + i\sqrt{2N}}{\sqrt{2}} \]  

The resulting measurement fidelity distribution is  
\[ p_{M,MN}^{\text{Gauss}}(F)dF = F_{M,MN}^{MN \times \frac{MN}{2}} - 1 \]  

while the average measurement fidelity is  
\[ F_{M,MN}^{\text{Gauss}} = \frac{M^2N^2}{M^2N^2 + 2MN + 4N - 4} \]  

For \( M = 1, N = 2 \) the measurement fidelities for Gaussian and Information cloning are 1/3 and 1/2 respectively. For \( M = 1, N = 4 \) these become 4/9 and 1/2. For \( M = 2, N = 2 \) these are 4/7 and 2/3 while for \( M = 2, N = 4 \) they become 16/23 and 2/3 respectively.  

B. Conclusion  

In this paper we have demonstrated the concept of information cloning for harmonic oscillator coherent states. The principal difference with the Gaussian cloning of [8] is that in our case the outputs are pure and disentangled states. The coherency parameter for the output states is reduced by the factor \( \frac{N}{N^2} \) where \( N \) is the number of copies. The variances are unchanged. We have also introduced the notion of measurement fidelity which is different from the notion of fidelity introduced in [4]. For purposes of comparison we have calculated the measurement fidelities for Gaussian cloners also. In the case of \( d \)-level quantum states a formula is available giving the fidelity that can be achieved given \( N \) copies [8]. Our formula eqn (13) is such a relation for coherent states.  

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