The Low Energy $\pi \pi$ Amplitude
To One and Two Loops

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Abstract

The low-energy $\pi \pi$ amplitude is computed explicitly to two-loop accuracy in the chiral expansion. It depends only on six independent (combinations of) low-energy constants which are not fixed by chiral symmetry. Four of these constants are determined via sum rules which are evaluated using $\pi \pi$ scattering data at higher energies. Dependence of the low-energy phase shifts and of the threshold parameters on the remaining two constants (called $\alpha$ and $\beta$) are discussed and compared to the existing data from $K_{l4}$ experiments. Using generalised $\chi$PT, the constants $\alpha$ and $\beta$ are related to fundamental QCD parameters such as the quark condensate $\langle 0 | \bar{q}q | 0 \rangle$ and the quark mass ratio $m_s/\hat{m}$. It is shown that forthcoming accurate low-energy $\pi \pi$ data can be used to provide, for the first time, experimental evidence in favour of or against the existence of a large quark-antiquark condensate in the QCD vacuum.

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1 Introduction

Current understanding of the mechanism of spontaneous breakdown of chiral symmetry (SB\(\chi\)S) in QCD is still lacking clear experimental support. The basic fact that in the limit \(m_u = m_d = m_s = 0\) the chiral symmetry of the lagrangian is spontaneously broken down to \(SU(3)_V\) is known to be a mathematical consequence of anomalous Ward identities \([1]\), quark confinement and the vector-like character of the theory \([2]\). It just means that there are eight massless Goldstone bosons coupled to eight conserved axial-vector currents. In the real world where \(m_u, m_d, m_s\) are nonzero but much smaller than the typical mass scale \(\Lambda_H \sim 1\,\text{GeV}\) of the first massive hadrons, the theoretical fact of SB\(\chi\)S and the Goldstone character of the eight lightest pseudoscalars are experimentally well founded. The chiral world, i.e. the first step in the expansion in powers of \(m_{\text{quark}}/\Lambda_H\), indeed strongly resembles the observed world.

That much is known and is sufficient to establish a low-energy effective theory in which the role of effective degrees of freedom is played by weakly interacting Goldstone bosons \([3]\). Equivalence of this effective theory with QCD is ensured step by step in a systematic expansion in powers of external momenta and quark masses, referred to as chiral perturbation theory \([4, 5]\) (\(\chi\)PT). The standard wisdom, however, involves much more than this established general framework of SB\(\chi\)S and of the corresponding low-energy effective theory since, in addition, it assumes a particular mechanism of symmetry breaking. What is commonly assumed is that in QCD, as in the Nambu-Jona-Lasinio model \([6]\), SB\(\chi\)S is triggered by strong condensation of quark-antiquark pairs in the vacuum. The quark condensate parameter

\[
B_0 = -\lim_{m_u, m_d, m_s \to 0} \frac{1}{F_\pi^2} \langle \Omega | \bar{q}q | \Omega \rangle
\]  

is then assumed to play a dominant role \([7]\) in the description of symmetry-breaking effects induced by quark masses. The assumption of a strong quark condensation and its consequences have not so far been tested experimentally. Moreover, in a vector-like theory like QCD, a natural alternative scenario is conceivable. Goldstone bosons with a non-vanishing coupling \(F_0 = \lim_{m_{\text{quark}} \to 0} F_\pi\) to conserved axial currents can be formed even if quark condensation is marginal or even absent. For all these reasons an experimental probe of the mechanism of SB\(\chi\)S in QCD becomes of fundamental importance. It has been pointed out that the most promising framework for testing the strength of quark condensation is low-energy \(\pi\pi\) scattering \([8, 9]\). In the present work the theoretical basis for such a test is elaborated in detail, in view of two experimental projects to improve considerably the accuracy of existing low-energy \(\pi\pi\) scattering data \([10, 11]\). The \(\pi\pi\) scattering amplitude is worked out explicitly to two-loop accuracy in the general low-energy expansion, independently of any prejudice concerning the size of the quark
Before describing the content of this article in detail, let us briefly comment on the theoretical possibility that in QCD, \( SB_S \chi \) need not necessarily imply a strong quark condensation. This is illustrated by an analogy with antiferromagnetic spin systems [12]. In this case, the spontaneous breakdown of rotation symmetry does not necessarily yield a large spontaneous magnetisation (as it would happen for a ferromagnet). The latter can even vanish to the extent that the structure of the ground state reaches the Néel-type magnetic order. From renormalised perturbation theory, we are used to consider as “unnatural” a situation where a quantity would vanish (or be kept unusually small) if it is not forced to do so for reasons of symmetry. The above example of antiferromagnetism suggests that this reasoning need not necessarily apply to quantities of genuine nonperturbative origin such as spontaneous magnetisation or the quark condensate. Indeed, in QCD (as in any local relativistic field theory) the necessary and sufficient criterion for \( SB_S \chi \) is the non-vanishing of the coupling \( F_0 \) of the Goldstone bosons to the Noether currents rather than a non-zero value of the quark condensate. It is instructive to formally express \( F_0^2 \) and the quark condensate by means of the euclidian functional integral of QCD in a finite box of size \( L \). Integrating over the fermions first (giving them a mass \( m \)), the double limit \( \lim_{m \to 0} \lim_{L \to \infty} \) of order parameters of \( SB_S \chi \) becomes sensitive to the small eigenvalues \( \lambda_n(A) \) of the Dirac operator \( iD / (A) \), averaged over all gluonic configurations. In particular, the crucial question is how dense does the spectrum become as \( L \to \infty \) [13, 2, 14]. If the average level spacing \( \Delta \lambda \) behaves as \( 1/L \), then chiral symmetry remains unbroken. Both \( F_0 \) and the quark condensate vanish. In order to have \( F_0 \neq 0 \), it is necessary and sufficient that \( \Delta \lambda \) behaves at least as \( 1/L^2 \) [15]. It is well known that quarks start to condense only provided \[14] \( \Delta \lambda \sim 1/L^4 |\langle \bar{q}q \rangle| \). The above discussion does not tell us what actually does happen in the QCD ground state. However, it demonstrates that the theoretical possibility of \( SB_S \chi \) (i.e., \( F_0 \neq 0 \)) without quark condensation might naturally arise within QCD. In fact, it is conceivable that small eigenvalues with \( \Delta \lambda \sim L^{-2} \) and \( \Delta \lambda \sim L^{-4} \) coexist and compete, giving rise to a marginal quark condensate, which then would be difficult to estimate in advance.

At this point, one should recall that the alternative scenario of a small or vanishing condensate, though not contradicted by a single experimental fact, does not seem to be favoured by existing lattice simulations when they are extrapolated to the chiral limit. However, lattice regularisation is known to mistreat chiral symmetry, in one way or another, especially in the quenched approximation, in which the most significant results are obtained. (For a recent review, see Ref. [16].) Lattice results constitute another reason to attempt a direct experimental test of theoretical prejudices regarding the value of \( \langle \bar{q}q \rangle \).

The quark condensate manifests itself exclusively through symmetry-breaking effects proportional to quark masses. The order parameter \( B_0 \) (1.1) is one of the constants of the low-energy effective lagrangian, but it enters into physical observables multiplied by a quark mass. An example of a meaningful question to ask concerns, for instance, the magnitude of the renormalisation group invariant product \((m_u + m_d)B_0\) measured in units of the pion mass.
The smallness of most symmetry-breaking effects is the main reason why experimental tests of quark condensation are difficult and have not been performed so far. Actually, the only symmetry-breaking effects which are easily accessible experimentally are the Goldstone boson masses. Their expansion in powers of quark masses can be schematically written as (for more details, see [17, 4])

\[ M^2_{\pi^+} = (m_u + m_d)B_0 + (m_u + m_d)^2A_0 + ... \]  
\[ M^2_{K^+} = (m_u + m_s)B_0 + (m_u + m_s)^2A_0 + ... \]

where \( F_0^2A_0 \) is a (suitably subtracted) massless QCD two-point function of scalar and pseudoscalar quark densities. \( A_0 \) is of the order of 1, typically \( A_0 = 1 \div 5 \). (A similar expression holds for the \( \eta \) mass, except that at the quadratic level, a new constant appears, reflecting the axial \( U(1) \) anomaly.) While the overall convergence of the expansion (1.2) is controlled by the small parameter \( m_{\text{quark}}/\Lambda_H \), the relative importance of the first two terms is determined by the relative sizes of the quark mass and of the mass scale

\[ m_0 = \frac{B_0}{2A_0} \]

set by the quark condensate. If \( m_{\text{quark}} \ll m_0 \), the condensate term dominates the expansion of Goldstone boson masses. On the contrary, if \( m_{\text{quark}} \sim m_0 \), the first and second order terms in Eq. (1.2) are of comparable size. This is what distinguishes the standard large-condensate wisdom from the alternative, low-condensate, scenario. The standard case \( (m_0 \sim \Lambda_H) \) is the basis of the standard \( \chi PT \), i.e. the expansion in powers of quark masses and external momenta in which one power of a quark mass is counted as two powers of the momenta:

\[ m_{\text{quark}} = O(p^2) \].

The consequences of this scheme are well known. At leading order, pseudoscalar masses satisfy the Gell-Mann–Okubo (GMO) formula, and the ratio of strange to non-strange quark masses gets determined to be [18]

\[ r = \frac{m_s}{\bar{m}} = 2\frac{M^2_{\bar{K}}}{M^2_{\bar{\pi}}} - 1 \simeq 26, \quad \bar{m} = \frac{1}{2}(m_u + m_d) \].

Historically, the emergence of the GMO formula at leading order was at the origin of the large \( B_0 \) scenario [7]. Today, the GMO formula can still be considered as an argument of plausibility, but not as an experimental test of the large-condensate hypothesis, for the following two reasons: i) The validity of the GMO relation does not imply the dominance of the condensate term in Eq. (1.3) and it does not rule out the small condensate alternative, and ii) the standard \( \chi PT \) is not yet able to predict the size of corrections to the leading order GMO formula; this would require an independent determination of the constant \( L_8 \) [3] from other symmetry-breaking effects.
On the other hand, if the quark condensate turned out to vanish, then the quark mass ratio (1.5) would become

\[ r = 2 \frac{M_K}{M_{\pi}} - 1 \simeq 6.3 . \tag{1.6} \]

Notice that even in this extreme case, the \( \eta \) mass would remain unrelated to \( M_K \) and \( M_{\pi} \), because of the anomaly contribution. However, the small-condensate alternative is more general than this: it covers the whole range \( m_0 \ll \Lambda_H \). For instance, if \( B_0 \sim F_0 \sim 90 \text{ MeV} \) (i.e. \( -\langle \bar{q}q \rangle \sim F_0^3 \)), then \( m_0 \) could easily be as small as \( \sim 20 \text{ MeV} \). In this case, \( m_{\text{quark}} \sim m_0 \) could happen even for \( u \) and \( d \) quarks, implying a value of the ratio \( (m_u + m_d)B_0/M_{\pi}^2 \) significantly below one. In the generic case \( m_0 \ll \Lambda_H \), the quark mass ratio \( r \) interpolates between the two extreme values (1.5) and (1.6). The standard \( \chi \)PT treats the terms \( m_{\text{quark}}/\Lambda_H \) and \( m_{\text{quark}}/m_0 \) in a similar way: they are both considered small and of order \( O(p^2) \). In the small-condensate alternative, the standard expansion of the (same) effective lagrangian has to be modified since, now, \( m_{\text{quark}}/m_0 \) is of order 1. This requires a modification of the chiral counting rule (1.4). \( B_0 \) now becomes an expansion parameter that counts as a small quantity of order \( O(p) \). The new counting rule reads

\[ m_{\text{quark}} = O(p) , \quad B_0 = O(p) . \tag{1.7} \]

The corresponding expansion of \( \mathcal{L}_{\text{eff}} \) defines the so-called “generalized chiral perturbation theory” (G\( \chi \)PT) [8, 19, 17]. Order by order, the G\( \chi \)PT contains the standard expansion as a special case, because at each order it includes additional terms, which the standard \( \chi \)PT relegates to higher orders. Notice that in both schemes the symbol \( O(p^n) \) has the same meaning: it represents a quantity with an order of magnitude \( (M_\pi/\Lambda_H)^n \). The symbol \( O(m_{\text{quark}}^n) \), in contrast, has to be interpreted according to (1.4) or (1.7), depending on the scheme. More details on the structure of the G\( \chi \)PT may be found in Refs. [8, 19, 17].

In order to test the strength of quark condensation, the constraints arising from the discussion of Goldstone boson masses have to be combined with experimental information on different symmetry-breaking effects. There are two particularly significant examples of such effects:

i) Comparing the observed deviation from the Goldberger-Treiman relation in \( \pi N, K\Lambda \) and \( K\Sigma \) channels one obtains a sum rule for the ratio \( r = m_s/\hat{m} \) valid up to higher order corrections [20]. The output of the sum rule is particularly sensitive to the precise value of the charged pion-nucleon coupling constant \( g_{\pi N} \), which is nowadays rather controversial. For the old value of Koch and Pietarinen [21], recently confirmed by the analysis of a new \( n - p \) charge-exchange scattering experiment [22], the value of \( r \) turns out to be significantly smaller than the standard value \( r \simeq 26 \), by at least a factor of 2 [21]. For lower values of \( g_{\pi N} \) the outcome is less conclusive. In any case, more accurate data as well as an estimate of higher order corrections are needed to transform this indication into a real measurement of \( m_s/\hat{m} \).
ii) The second relevant symmetry-breaking effect concerns the I=0 S-wave ππ scattering length. The standard χPT, based on the large-condensate hypothesis, predicts (at one-loop accuracy) \( a_0^0 = 0.20 \pm 0.01 \) \cite{23, 4}, while the current experimental value is \cite{24} \( a_0^0 = 0.26 \pm 0.05 \). One of the purposes of this work is to show that this one standard deviation effect finds a natural explanation within the low-condensate alternative. It has been known for some time \cite{8} that already at tree level Weinberg’s prediction \cite{25} \( a_0^0 = 0.16 \) (which implicitly uses the large-condensate hypothesis) can gradually increase towards a value \( a_0^0 = 0.27 \) provided the ratio \( (m_u + m_d)B_0/M_{\pi}^2 \) is allowed to decrease from 1 to 0. Here, this statement will be made more quantitative including the one-loop \cite{26} and the two-loop \( G\chi PT \) corrections. In view of forthcoming precise low energy ππ scattering data \cite{10, 11}, our analysis might imply a realistic possibility to disentangle the large- and low-condensate alternatives experimentally.

The paper is organized as follows. We first describe the relevant features of the effective action at order \( O(p^4) \) in the generalized case and obtain the corresponding expression for the ππ amplitude in terms of four independent combinations of the low-energy constants (Section 2). A preliminary version of this calculation has already been published \cite{26}. In Section 3, we show how this result can be rederived and easily extended to two-loop accuracy using a slightly different approach. At two-loop order, the ππ amplitude involves two additional constants which are not fixed by chiral symmetry. We determine four out of this total of six constants from sum rules and ππ production data at higher energies in Section 4, where the dependence of low-energy phase shifts and threshold parameters on the remaining two parameters is also discussed. The possibilities of exploiting future precise data on low-energy ππ scattering in order to distinguish between the small and large condensate alternatives are investigated in Section 5. A summary and concluding remarks are presented in Section 6, and miscellaneous formulae and auxiliary results have been collected in four Appendices.

Finally, it should be stressed that although the paper is motivated mainly by questions raised by the chiral structure of the QCD vacuum, this work contains new material concerning the phenomenology of low-energy ππ scattering which might be of interest for its own sake. The reader mostly interested in these aspects can directly read Sections 3 and 4 as well as appendices B, C and D, which form a self-contained core.

2 The ππ Amplitude to Order \( O(p^4) \) in \( G\chi PT \)

In this section, we briefly discuss the structure of the generating functional for the effective theory in the generalized framework up to order \( O(p^4) \). Then we derive the low-energy expansion, to this order, of the amplitude \( A(s|t, u) \) which describes ππ scattering in the absence of isospin breaking (\( i.e. \) we shall take \( m_u = m_d = \tilde{m} \) and we shall neglect electromagnetic corrections). We work within \( SU(3)_L \times SU(3)_R \), although ππ scattering is merely an \( SU(2)_L \times SU(2)_R \).
problem. The reason for this choice should be obvious from the preceding discussion on Goldstone boson masses. Working within the $SU(3)_L \times SU(3)_R$ chiral expansion offers the opportunity to make use of relationships between constants which characterise low-energy $\pi\pi$ scattering and QCD parameters (such as $m_s/\hat{m}$) or observables (such as $F_K/F_\pi$, $M_K/M_\pi$).

The strength of the quark-antiquark condensate in the QCD vacuum does not affect the structure of the effective lagrangian $\mathcal{L}_{\text{eff}}$. The latter is merely determined by chiral symmetry and the transformation properties of the chiral symmetry breaking mass term of QCD. Thus, $\mathcal{L}_{\text{eff}}$ consists of an infinite tower of invariants

$$\mathcal{L}_{\text{eff}} = \sum_{(k,l)} \mathcal{L}_{(k,l)},$$

(2.1)

where $\mathcal{L}_{(k,l)}$ contains $k$ powers of covariant derivatives of the Goldstone boson fields, and $l$ powers of scalar or pseudoscalar sources. In the chiral limit, $\mathcal{L}_{(k,l)}$ vanishes like the $k$-th power of external momenta $p$ and the $l$-th power of the quark masses,

$$\mathcal{L}_{(k,l)} \sim p^k m_{\text{quark}}^l.$$

(2.2)

For sufficiently small quark masses, such that both $m_{\text{quark}} \ll \Lambda_H$ and $m_{\text{quark}} \ll m_0 = B_0/2A_0$, one has $\mathcal{L} = O(p^{k+2l})$. In this case, the double expansion (2.1) can be reorganized as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \cdots,$$

(2.3)

where

$$\mathcal{L}^{(d)} = \sum_{k+2l=d} \mathcal{L}_{(k,l)}.$$

(2.4)

This expansion defines the standard $\chi$PT. On the other hand, if for the actual values of the quark masses one has $m_{\text{quark}} \sim m_0 \ll \Lambda_H$, then $\mathcal{L}_{(k,l)} = O(p^{k+l})$, and this new counting yields a different expansion of the same effective lagrangian (2.1):

$$\mathcal{L}_{\text{eff}} = \tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)} + \tilde{\mathcal{L}}^{(5)} + \tilde{\mathcal{L}}^{(6)} + \cdots,$$

(2.5)

with now

$$\tilde{\mathcal{L}}^{(d)} = \sum_{k+l+n=d} B^n_0 \mathcal{L}_{(k,l)}.$$

(2.6)

Although Eqs. (2.3) and (2.5) sum up the same effective lagrangian $\mathcal{L}_{\text{eff}}$ to all orders, their truncations at any finite order may differ.

The structure of the effective action $Z_{\text{eff}}$ in the generalized case has been discussed in [28] up to order $O(p^4)$ in the framework of the $SU(3)_L \times SU(3)_R$ chiral expansion. It is given by

$$Z_{\text{eff}} = \int d^4x \left\{ \tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)} \right\} + \tilde{Z}_{1\text{loop}}^{(4)}.$$

(2.7)
Whereas \( \tilde{\mathcal{L}}^{(n)} \), \( n = 2, 3, 4 \), gives the tree level contributions at order \( O(p^n) \), \( \tilde{\mathcal{L}}_{\text{1 loop}}^{(4)} \) contains the contributions from one-loop graphs with an arbitrary number of vertices from \( \tilde{\mathcal{L}}^{(2)} \) only. Thus, the leading order of the generalized expansion is described by \( \tilde{\mathcal{L}}^{(2)} \), which was first given in [8]:

\[
\tilde{\mathcal{L}}^{(2)} = \frac{1}{4} F_0^2 \left\{ \langle D_\mu U^+ D^\mu U \rangle + 2 B_0 \langle U^+ \chi + \chi^+ U \rangle + A_0 \langle (U^+ \chi)^2 + (\chi^+ U)^2 \rangle + Z_0^S \langle U^+ \chi + \chi^+ U \rangle^2 \\
+ Z_0^P \langle U^+ \chi - \chi^+ U \rangle^2 + H_0 \langle \chi^+ \chi \rangle \right\} .
\]

(2.8)

The notation is as in Refs. [1, 7], except for the consistent removal of the factor 2\( B_0 \) from \( \chi \), the parameter that collects the scalar and pseudoscalar sources,

\[
\chi = s + ip = \mathcal{M} + \cdots , \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s) .
\]

(2.9)

In G\( \chi \)PT, the next-to-leading-order corrections are of order \( O(p^3) \), and still occur at tree level only. They are embodied in \( \tilde{\mathcal{L}}^{(3)} = \mathcal{L}^{(2,1)} + \mathcal{L}^{(0,3)} \), which reads [3, 17]

\[
\tilde{\mathcal{L}}^{(3)} = \frac{1}{4} F_0^2 \left\{ \xi \langle D_\mu U^+ D^\mu U \rangle \langle \chi^+ U + U^+ \chi \rangle + \tilde{\xi} \langle D_\mu U^+ D^\mu U \rangle \langle \chi^+ U + U^+ \chi \rangle \\
+ \rho_1 \langle (\chi^+ U)^3 + (U^+ \chi)^3 \rangle + \rho_2 \langle (\chi^+ U + U^+ \chi) \chi^+ \chi \rangle \\
+ \rho_3 \langle \chi^+ U - U^+ \chi \rangle \langle (\chi^+ U)^2 - (U^+ \chi)^2 \rangle \\
+ \rho_4 \langle (\chi^+ U)^2 + (U^+ \chi)^2 \rangle \langle \chi^+ U + U^+ \chi \rangle \\
+ \rho_5 \langle \chi^+ U + U^+ \chi \rangle \\
+ \rho_6 \langle \chi^+ U - U^+ \chi \rangle \langle \chi^+ U + U^+ \chi \rangle + \rho_7 \langle \chi^+ U + U^+ \chi \rangle \langle \chi^+ U + U^+ \chi \rangle^3 \right\}.
\]

(2.10)

The tree-level contributions at order \( O(p^4) \) are contained in

\[
\tilde{\mathcal{L}}^{(4)} = \mathcal{L}^{(4,0)} + \mathcal{L}^{(2,2)} + \mathcal{L}^{(0,4)} + B_0^2 \mathcal{L}^{(0,2)} + B_0 \mathcal{L}^{(2,1)} + B_0 \mathcal{L}^{(0,3)} .
\]

(2.11)

The part without explicit chiral symmetry breaking, \( \mathcal{L}^{(4,0)} \), is described by the same low-energy constants \( L_1, L_2, L_3, L_9 \) and \( L_{10} \) as in the standard case [4]. The part with two powers of momenta and two powers of quark masses is given by:

\[
\mathcal{L}^{(2,2)} = \frac{1}{4} F_0^2 \left\{ A_1 \langle D_\mu U^+ D^\mu U \rangle \langle \chi^+ \chi + U^+ \chi^+ U \rangle \\
+ A_2 \langle D_\mu U^+ U \chi^+ D^\mu U U^+ \chi \rangle \\
+ A_3 \langle D_\mu U^+ U \langle \chi^+ D^\mu \chi^+ - D^\mu \chi \chi^+ \rangle \rangle + D_\mu U U^+ \langle \chi D^\mu \chi^+ - D^\mu \chi \chi^+ \rangle \rangle \\
+ A_4 \langle D_\mu U^+ D^\mu U \rangle \langle \chi^+ \chi \rangle \right\}.
\]

\[\text{(2.11)}\]

\[\text{At order } O(p^4), \text{ terms with odd intrinsic parity coming from the Wess-Zumino term are also present in the effective lagrangian, but since they start to contribute to } A(s|t, u) \text{ only at order } O(p^8), \text{ we do not mention them further here.}\]
Finally, the tree-level contributions which behave as $O(m_{\text{quark}}^4)$ in the chiral limit are contained in $\mathcal{L}_{(0,4)}$, which reads

$$\mathcal{L}_{(0,4)} = \frac{1}{4} F_0^2 \left\{ E_1 \langle (\chi^+ U)^4 + (U^+ \chi)^4 \rangle \\
+ E_2 \langle \chi^+ \chi (\chi^+ U^+ U + U^+ \chi U^+) \rangle \\
+ E_3 \langle \chi^+ \chi U^+ \chi^+ U \rangle \\
+ F_1 (\chi^+ U^+ U + U^+ \chi U^+)^2 \\
+ F_2 \langle (\chi^+ U)^3 + (U^+ \chi)^3 \rangle \langle \chi^+ U + U^+ \chi \rangle \\
+ F_3 \langle \chi^+ \chi (\chi^+ U + U^+) \rangle \langle \chi^+ U + U^+ \chi \rangle \\
+ F_4 \langle (\chi^+ U)^2 + (U^+ \chi)^2 \rangle \langle \chi^+ \chi \rangle \\
+ F_5 \langle (\chi^+ U)^2 + (U^+ \chi)^2 \rangle \langle \chi^+ U + U^+ \chi \rangle^2 \\
+ F_6 \langle \chi^+ \chi \rangle \langle \chi^+ U + U^+ \chi \rangle^2 + \cdots \right\} .$$

(2.12)

For $\mathcal{L}_{(2,2)}$ and $\mathcal{L}_{(0,4)}$, we have shown explicitly only those terms which will actually contribute to $A(s|t, u)$. For the complete list of counterterms which enter $\mathcal{L}_{(2,2)}$ and $\mathcal{L}_{(0,4)}$, we refer the reader to [28]. Notice also that in the standard framework all these contributions would count as order $O(p^6)$ and order $O(p^8)$, respectively.

As far as the $\pi\pi$ amplitude is concerned, the $O(p^4)$ loop corrections involve only graphs with a single or two vertices from $\tilde{\mathcal{L}}^{(2)}$:

$$\tilde{\mathcal{Z}}_{1\text{ loop}}^{(4)} = \tilde{\mathcal{Z}}_{\text{tadpole}}^{(4)} + \tilde{\mathcal{Z}}_{\text{unitarity}}^{(4)} + \cdots$$

(2.14)

The divergent parts of these one loop graphs have been subtracted at a scale $\mu$ in the same dimensional renormalisation scheme as described in [4, 5]. Accordingly, the low energy constants of $\mathcal{L}_{(4,0)}$, $\mathcal{L}_{(2,2)}$, and $\mathcal{L}_{(0,4)}$ stand for the renormalised quantities, with an explicit logarithmic scale dependence ($X(\mu)$ denotes generically any of these renormalised low-energy constants)

$$X(\mu) = X(\mu') + \frac{\Gamma_X}{(4\pi)^2} \ln(\mu'/\mu) .$$

(2.15)

At order $O(p^4)$, the low-energy constants of $\tilde{\mathcal{L}}^{(2)}$ and $\tilde{\mathcal{L}}^{(3)}$ also need to be renormalised. The corresponding counterterms, however, are of order $O(B_0^2)$ and $O(B_0)$, respectively, and they

\footnote{The standard $O(p^6)$ effective lagrangian $\mathcal{L}^{(6)} = \mathcal{L}_{(6,0)} + \mathcal{L}_{(4,1)} + \mathcal{L}_{(2,2)} + \mathcal{L}_{(0,3)}$ has been worked out in Ref. [24].}
are gathered in the three last terms of Eq. (2.11); in GχPT, renormalisation proceeds order by order in the expansion in powers of $B_0$. Alternatively, one may think of Eqs. (2.8) and (2.10) as standing for the combinations $\hat{L}^{(2)} + B_0^2 L'_{(2,2)}$ and $\hat{L}^{(3)} + B_0 L'_{(2,1)} + B_0 L'_{(3,3)}$, respectively, with the corresponding low-energy constants representing the renormalised quantities. The full list of β-function coefficients $\Gamma_X$ can be found in Ref. [28], where the explicit expressions of $\hat{Z}^{(4)}_{\text{tadpole}}$ and of $\hat{Z}^{(4)}_{\text{unitarity}}$ are also displayed.

Computing the $O(p^4)$ expression of $A(s|t, u)$ is then a straightforward exercise in field theory, and we shall merely quote the result, expressed in terms of the loop functions $J_{\pi\pi}$ and $M_{\pi\pi}$ defined as in [4, 5]:

$$A(s|t, u) = \frac{\beta^r}{F^2_\pi} \left( s - \frac{4}{3}M^2_\pi \right) + \alpha^r \frac{M^2_\pi}{3F^2_\pi}$$

$$+ \frac{4}{F^2_\pi} (2L^1_1 + L_3)(s - 2M^2_\pi)^2 + \frac{4}{F^4_\pi} L^2_2 \left( t - 2M^2_\pi \right)^2 \right]$$

$$+ \frac{1}{6F^2_\pi} J^r_{\pi\pi}(s) \left\{ 4s - \frac{M^2_\pi}{2} + 10\tilde{m}^2(A_0 + 2Z_0) \right\} - \left[ s - 2M^2_\pi - 8\tilde{m}^2(A_0 + 2Z_0) \right]$$

$$+ \frac{1}{4F^2_\pi} J^r_{\pi\pi}(t) \left[ t - 2M^2_\pi - 8\tilde{m}^2(A_0 + 2Z_0) \right]^2 + \frac{1}{4F^4_\pi} J^r_{\pi\pi}(u) \left[ u - 2M^2_\pi - 8\tilde{m}^2(A_0 + 2Z_0) \right]^2$$

$$+ \frac{1}{8F^2_\pi} J^r_{KK}(s) \left[ s + 8\tilde{m}^2(1 + r)(A_0 + 2Z_0) \right]^2$$

$$+ \frac{1}{18F^2_\pi} J^r_{\eta\eta}(s) \left[ M^2_\pi + 4\tilde{m}^2(1 + 2r)(A_0 + 2Z_0) + 8\tilde{m}^2(1 - r)(A_0 + 2Z_0) \right]^2$$

$$+ \frac{1}{2F^4_\pi} \left\{ (s - u)t[2M^r_{\pi\pi}(t) + M^r_{KK}(t)] + (s - t)u[2M^r_{\pi\pi}(u) + M^r_{KK}(u)] \right\}.$$ (2.16)

The coefficients $\alpha^r$ and $\beta^r$ denote two combinations of the (renormalised, scale dependent) low-energy constants of $\hat{L}^{(2)} + \hat{L}^{(3)} + \hat{L}^{(4)}$:

$$\beta^r = 1 + 2\tilde{m}\xi + 4\tilde{m}\tilde{\xi}$$

$$+ 2\tilde{m}^2 \left[ 3A_2 + 2A_3 + 4B_1 + 2B_2 + 8B_4 + 4C_1^{S} + 2(2 + r)D^S \right]$$

$$- 2\xi^2 - 4(2 + r)\xi^2 - 2(2 + r)(4 + r)\xi \tilde{\xi} \right] ,$$ (2.17)

and

$$\frac{F^2_\pi}{F_0^2} M^2_\pi \alpha^r = 2\tilde{m}B_0 + 16\tilde{m}^2 A_0 + 4(8 + r)\tilde{m}^2 Z^S_0 - 4\tilde{m}M^2_\pi(\xi + 2\tilde{\xi})$$

$$+ \tilde{m}^3 \left[ 81\eta_1 + \rho_2 + 2(82 + 16r + r^2)\rho_4 + (2 + r^2)\rho_5 + 12(2 + r)(14 + r)\rho_7 \right]$$

$$- 8\tilde{m}^2 M^2_\pi \left[ 2B_1 - 2B_2 + 4B_4 - A_3 - 4C_1^{S} + (4 + r)D^S \right]$$

$$+ 2\tilde{m}^4 \left[ 48A_3(A_0 + 2Z^S_0) + 128E_1 + 8E_2 \right.$$

$$+ 32(8 + r^2)F^1_0 + (272 + 81r + r^3)F^2_0 + (16 + r + r^3)F^3_0 + 8(2 + r^2)F^4_0$$

$$+ 4(144 + 82r + 16r^2 + r^3)F^5_0 + 2(16 + 2r + 8r^2 + r^3)F^6_0$$

$$+ (144 + 82r + 16r^2 + r^3)F^5_0 + 2(16 + 2r + 8r^2 + r^3)F^6_0.$$
\[
- \mu_\pi [3M_\pi^2 + 116\tilde{m}^2(A_0 + 2Z_0^S)] - 2\mu_K [M_\pi^2 + 4(7 + r)\tilde{m}^2(A_0 + 2Z_0^S)] \\
- \frac{1}{3}\mu_\eta [M_\pi^2 + 4(7 + 2r)\tilde{m}^2(A_0 + 2Z_0^S) + 8(4 - r)\tilde{m}^2(A_0 + 2Z_0^P)]] ,
\]

with \( \mu_P = M_P^2/(32\pi^2F_\pi^2) \ln(M_P^2/\mu^2) \), \( P = \pi, K, \eta \). The deviation of the pion decay constant \( F_\pi \) from its counterpart in the chiral limit \( F_0 \) is given, at order \( O(p^4) \), in Appendix A, together with the corresponding expansion of the pion mass.

As mentioned above, both \( \alpha^r \) and \( \beta^r \) depend of the same subtraction scale \( \mu \) in terms of which the renormalised loop functions \( J_{PQ}^r \) and \( M_{PQ}^r \) were defined. Using the results of Ref. \[28\], we find

\[
\alpha^r(\mu) = \alpha^r(\mu') + \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln(\mu'/\mu) \cdot \left\{ [1 + (1 + r)\tilde{c}] [1 + 3(1 + r)\tilde{c}] + \frac{1}{3} [1 + (1 + 2r)\tilde{c}] + 2\frac{\hat{\Delta}_{GMO}}{1 - r} \cdot [1 + 22\tilde{c} + 33\tilde{c}^2] \right\} ,
\]

and

\[
\beta^r(\mu) = \beta^r(\mu') + \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln(\mu'/\mu) \cdot [5 + (11 + r)\tilde{c}] ,
\]

where we have introduced

\[
\tilde{c} = \frac{4\tilde{m}^2}{M_\pi^2} (A_0 + 2Z_0^S) ,
\]

and

\[
\hat{\Delta}_{GMO} = \frac{4(\tilde{m} - m_s)^2}{M_\pi^2} (A_0 + 2Z_0^P) .
\]

From these formulae, one verifies that \( A(s|t, u) \) is indeed independent of the subtraction scale \( \mu \), thus providing us at the same time with a nontrivial check of our calculation.

Using the \( O(p^2) \) expressions for the pseudoscalar masses derived from \( \tilde{\mathcal{L}}^{(2)} \) \[20, 8, 9, 28\], one obtains

\[
\tilde{c} = 2 \frac{r_2 - r}{r_2 - 1} (1 + 2\zeta) [1 + O(m_{\text{quark}})] ,
\]

\[
\hat{\Delta}_{GMO} = (3M_0^2 - 4M_K^2 + M_\pi^2)/M_\pi^2 [1 + O(m_{\text{quark}})] ,
\]

with \( r_2 = 2M_K^2/M_\pi^2 - 1 \sim 26, \zeta = Z_0^S/A_0 \), which is expected to be small due to the Zweig rule, and \( r = m_s/\tilde{m} \). In \( G\chi\text{PT} \), the latter quark mass ratio remains a free parameter.

On the other hand, in the standard case, \( r \) is determined order by order in the chiral expansion. For instance, at leading order, \( r = r_2 \) (and \( \hat{\Delta}_{GMO} = 0 \)), while at order \( O(p^4) \), one

\footnote{Up to order \( O(p^3) \), \( \alpha^r \) and \( \beta^r \) are scale independent and coincide with the parameters \( \alpha_{\pi\pi} \) and \( \beta_{\pi\pi} \) which enter the \( O(p^5) \) expression of the \( \gamma\gamma \rightarrow \pi^0\pi^0 \) amplitude in generalised \( \chi\text{PT} \) \[19\].}
obtains

\[ r_{st} = r_2 - 2 \frac{M_K^2}{M_{\pi}^2} \Delta_M , \]  

(2.24)

\[ \Delta_M = -\mu_{\pi} + \mu_{\eta} + \frac{8}{F_{\pi}^2} (M_{K}^2 - M_{\pi}^2)(2L_8^r(\mu) - L_5^r(\mu)) . \]

Furthermore, the contributions from \( \mathcal{L}_{(0,3)}, \mathcal{L}_{(2,2)} \) and \( \mathcal{L}_{(4,0)} \) are relegated beyond order \( O(p^4) \). The remaining constants \( A_0, Z_0^S, Z_0^P, \xi \) and \( \bar{\xi} \) are related to the low-energy constants \( L_i \) of \( \mathcal{L}^{(4)} \), introduced by Gasser and Leutwyler [5], by (both sides of these equations refer to the renormalised quantities, defined at the same scale \( \mu \)):

\[
L_4^r = \frac{F_0^2}{8B_0} \bar{\xi} , \quad L_5^r = \frac{F_0^2}{8B_0} \xi , \\
L_6^r = \frac{F_0^2}{16B_0} Z_0^S , \quad L_7^r = \frac{F_0^2}{16B_0} Z_0^P , \quad L_8^r = \frac{F_0^2}{16B_0} A_0 .
\]  

(2.25)

Thus, upon replacing \( r \) by its standard expression \( r_{st} \) above, and upon keeping only those contributions that are of order \( O(p^4) \) in the standard counting, one recovers from Eqs. (2.16), (A.1), (A.4) and (2.25), as a special case, the expression of the one-loop \( \pi\pi \) amplitude in the \( SU(3)_L \times SU(3)_R \) framework of the standard chiral expansion [30]:

\[
A(s|t, u)_{st} = \frac{\beta_{st}^r}{F^{\pi}_r} \left( \frac{s}{3} - \frac{4}{3} M_{\pi}^2 \right) + \alpha_{st}^r \frac{M_{\pi}^2}{3 F^{\pi}_r} \left[ \frac{4}{F^{\pi}_r} (2L_4^r + L_3^r)(s - 2M_{\pi}^2)^2 + \frac{4}{F^{\pi}_r} L_2^r \left[ (t - 2M_{\pi}^2)^2 + (u - 2M_{\pi}^2)^2 \right] \right] + \frac{1}{2 F^{\pi}_r} \left[ s^2 - M_{\pi}^4 \right] J_{\pi\pi}^r(s) \]
\[
+ \frac{1}{4 F^{\pi}_r} \left[ (t - 2M_{\pi}^2)^2 J_{\pi\pi}^r(t) + (u - 2M_{\pi}^2)^2 J_{\pi\pi}^r(u) \right] \]
\[
+ \frac{1}{8 F^{\pi}_r} s^2 J_{KK}^r(s) + \frac{1}{18 F^{\pi}_r} M_{\pi}^4 J_{\eta\eta}^r(s) \]
\[
+ \frac{1}{2 F^{\pi}_r} \left\{ (s - u) t \left[ 2M_{\pi\pi}^r(t) + M_{KK}^r(t) \right] + (s - t) u \left[ 2M_{\pi\pi}^r(u) + M_{KK}^r(u) \right] \right\} ,
\]  

(2.26)

with

\[
\beta_{st}^r = 1 + \frac{8 M_{\pi}^2}{F^{\pi}_r} (2L_4^r + L_5^r) , \]

(2.27)

\[
\alpha_{st}^r = 1 - \frac{16 M_{\pi}^2}{F^{\pi}_r} (2L_4^r + L_5^r) + \frac{48 M_{\pi}^2}{F^{\pi}_r} (2L_6^r + L_8^r) .
\]  

(2.28)
The scale dependences of $\alpha_{\text{st}}^r$ and $\beta_{\text{st}}^r$ are readily obtained, either from Eqs. (2.19) and (2.20), upon taking $\hat{\epsilon}$ and $\hat{\Delta}_{GMO}$ equal to zero, or directly from the scale dependences of the $L_i$’s as given in Ref. [5]. Both ways lead to the same result:

$$\alpha_{\text{st}}^r(\mu) = \alpha_{\text{st}}^r(\mu') + \frac{7M_s^2}{3(4\pi F_\pi)^2} \ln(\mu'/\mu),$$

(2.29)

$$\beta_{\text{st}}^r(\mu) = \beta_{\text{st}}^r(\mu') + \frac{5M_s^2}{(4\pi F_\pi)^2} \ln(\mu'/\mu).$$

Starting from Eq. (2.26) above, it is then straightforward to further reduce this expression to the $SU(2)_L \times SU(2)_R$ case, and thus to recover the result, now expressed in terms of the constants $\bar{l}_i$, $i = 1, 2, 3, 4$, first obtained by Gasser and Leutwyler [23, 4]. We shall not pursue this matter for the moment.

3 The Use of Analyticity, Unitarity and Crossing Symmetry in $\chi$PT

In the previous section, we have obtained the $\pi\pi$ amplitude to order $O(p^4)$ in the generalised chiral expansion, and have compared it to its standard counterpart. At this stage, two features are worth being kept in mind: First, in both cases, the amplitude depends on only four independent combinations of low-energy constants of the respective $O(p^4)$ truncations of the (same) effective lagrangian. Secondly, the essential difference between the two cases arises from the fact that the quark mass ratio $r = m_s/\hat{m}$ remains a free parameter at each order of the generalised expansion. Upon fixing this ratio to its value predicted by the standard expansion, one recovers the $\pi\pi$ amplitude in the latter framework as a special case. These observations lead to the expectation that the form of the $\pi\pi$ amplitude might also be obtained beyond the $O(p^4)$ approximation independently of any prejudice concerning the size of the quark-antiquark condensate. In this section we shall establish the expression of the $\pi\pi$ amplitude up to corrections of order $O(p^8)$. More phenomenological aspects, such as how the actual value of $m_s/\hat{m}$ might actually be inferred from low-energy $\pi\pi$ data will be the subject of subsequent sections.

It has indeed been shown [3] that the Goldstone nature of the pion, combined with analyticity, crossing symmetry and unitarity of the S-matrix, determines the low-energy $\pi\pi$ scattering amplitude up to six arbitrary constants (not fixed by these general requirements) and up to corrections of the order $O([p/\Lambda_H]^8)$. (As usual, $\Lambda_H$ denotes the mass scale at which particles other than Goldstone bosons are created, $\Lambda_H \sim 1$GeV.) The G$\chi$PT one-loop $O(p^4)$ amplitude obtained in the preceding section has to be a particular case of this general six-parametrical formula. Moreover, since the general low energy representation of the amplitude extends up to and including order $O(p^6)$ (actually $O(p^7)$ in the generalised
counting), it must in fact contain the whole two-loop $\chi$PT amplitude both in the standard and in the generalised settings. In this section, we shall first rederive the $O(p^4)$ amplitude using dispersion relations, and subsequently these same dispersive methods will be used to work out the two-loop amplitude explicitly.

The starting point is the following Theorem, proved in Ref. [9]: For pion momenta $p \ll \Lambda_H$, the $\pi\pi$ scattering amplitude can be expressed as

$$
\frac{1}{32\pi} A(s|t, u) = \frac{1}{3} [W_0(s) - W_2(s)] + \frac{1}{2} [3(s - u)W_1(t) + W_2(t)] + \frac{1}{2} [3(s - t)W_1(u) + W_2(u)] + O \left( \frac{[p/\Lambda_H]^8}{p} \right).
$$

The three functions $W_a(s), a = 0, 1, 2$, are analytic except for a cut singularity at $s > 4M_{\pi}^2$. Their discontinuities across this cut are given by

$$
\text{Im} W_a(s) = (s - 4M_{\pi}^2)^{-\varepsilon_a} \text{Im} f_a(s) \theta(s - 4M_{\pi}^2),
$$

where

$$
\varepsilon_0 = \varepsilon_2 = 0, \quad \varepsilon_1 = 1,
$$

and with $f_a(s), a = 0, 1, 2$, denoting the three lowest partial-wave amplitudes $f^I_l(s)$ (the $I = 0$ and $I = 2$ S-waves, and the P-wave),

$$
f_0(s) \equiv f^0_0(s), \quad f_1(s) \equiv f^1_1(s), \quad f_2(s) \equiv f^2_0(s).
$$

(Notation, normalisation and other conventions are summarised in Appendix A of Ref. [9].)

The proof of this Theorem has been given in Ref. [9] with all details and it will not be reproduced here. Let us just recall that the main ingredient of this proof is the suppression of inelasticities and of higher partial waves in the chiral limit $p \to 0, M_\pi \to 0, p/M_\pi$ fixed: Below the threshold of production of non-Goldstone particles, one can write

$$
\text{Im} f_a(s) = \sqrt{\frac{s - 4M_{\pi}^2}{s}} |f_a(s)|^2 + O \left( \frac{[p/\Lambda_H]^8}{p} \right).
$$

Similarly, the absorptive parts of higher partial waves are suppressed as

$$
\text{Im} f^I_l(s) = O \left( \frac{[p/\Lambda_H]^8}{p} \right), \quad l \geq 2.
$$

As discussed in Ref. [9], the properties (3.3) and (3.6) are rigorous consequences of the fact that the whole amplitude $A(s|t, u)$ behaviors dominantly as $O(p^3)$ in the chiral limit, combined with analyticity, unitarity, crossing symmetry and chiral counting. At the same time, these two
equations explain why it is difficult to extend the representation (3.1) beyond two loops: At order \( O(p^8) \), inelasticities and higher partial waves set in.

Given \( \text{Im} \, f_a(s) \), then the four functions \( W_a(s) \), and consequently the whole amplitude \( A(s|t, u) \), are determined up to a polynomial. The maximum degree of this polynomial may be fixed according to the chiral order \( O(p^8) \) of the neglected contributions. Without loss of generality, the functions \( W_a(s) \) can thus be further specified by the asymptotic conditions

\[
\lim_{|s| \to \infty} s^{a-4} W_a(s) = 0.
\]

(3.7)

It then follows that for a given set of imaginary parts \( \text{Im} \, f_a(s) \) consistent with the bounds (3.7), the amplitude \( A(s|t, u) \) is determined up to six arbitrary constants.

It is worth stressing that in spite of Eq. (3.2), the functions \( W_a(s) \) do not coincide with the corresponding partial wave amplitudes \( f_a(s) \). The latter are obtained by the standard partial wave projections of Eq. (3.1),

\[
\begin{align*}
\hat{f}^0_l(s) &= \frac{1}{32\pi} \int_{4M_\pi^2}^{s} \frac{dt}{s - 4M_\pi^2} P_l \left(1 + \frac{2t}{s - 4M_\pi^2}\right) \{3A(s|t, u) + A(t|s, u) + A(u|s, t)\} \\
\hat{f}^1_l(s) &= \frac{1}{32\pi} \int_{4M_\pi^2}^{s} \frac{dt}{s - 4M_\pi^2} P_l \left(1 + \frac{2t}{s - 4M_\pi^2}\right) \{A(t|s, u) - A(u|s, t)\} \\
\hat{f}^2_l(s) &= \frac{1}{32\pi} \int_{4M_\pi^2}^{s} \frac{dt}{s - 4M_\pi^2} P_l \left(1 + \frac{2t}{s - 4M_\pi^2}\right) \{A(t|s, u) + A(u|s, t)\}
\end{align*}
\]

(3.8)

where the integrals are taken with \( s \) kept fixed and with \( u \) reexpressed as \( u = 4M_\pi^2 - s - t \). The functions \( W_a(s) \) have only a right-hand cut for \( s > 4M_\pi^2 \), whereas the partial wave amplitudes \( f_a(s) \) (3.8) do have both a right-hand cut (\( s > 4M_\pi^2 \)) and a left-hand cut (\( s < 0 \)), with the correct discontinuities, as dictated by unitarity and crossing symmetry. Actually, the partial waves with \( l \geq 2 \) obtained from (3.1) are real, in agreement with Eq. (3.6). The formula (3.1) may be viewed as the most general solution of the constraints imposed by analyticity, unitarity and crossing symmetry up to and including the chiral order \( O(p^6) \) \( (O(p^7) \) in the generalised case). In practice, the representation (3.1) should be a good approximation within a range of energies where the inelasticities in (3.3) and the contributions of higher partial waves in (3.6) remain small. Let us recall that the inelasticities, though in principle present above 540 MeV, are actually observed to be very small even up to 1 GeV. The same remark applies to the higher partial waves, \( \ell \geq 2 \).

3.1 First Iteration of the Unitarity Condition

The dispersion relation approach to \( \chiPT \) is based on the iteration of the unitarity condition. Here, the latter will be considered in the elastic form (3.5), i.e. below the \( KK \) and \( \eta\eta \) thresholds. This does not mean that we have to renounce the advantages of an \( SU(3)_L \times SU(3)_R \).
analysis of $\pi\pi$ scattering, but rather that the $K$ and $\eta$ loops will be expanded in powers of $s/(2M_K)^2$ and $s/(2M_\eta)^2$, respectively. Since in practice $2M_K$ and $2M_\eta$ are of the order of $\Lambda_H$, this should not affect the accuracy of our results at low energies $E \ll \Lambda_H$.

Chiral symmetry implies that the low-energy expansion of the $\pi\pi$ amplitude starts at order $O(p^2)$. Analyticity and crossing symmetry then restrict its form to

$$A(s|t, u) = \alpha \frac{M_\pi^2}{3F_\pi^2} + \beta \left( s - \frac{4}{3} M_\pi^2 \right) + O(p^4),$$

(3.9)

with $\alpha$ and $\beta$ two constants whose dominant behaviour in the chiral limit is of order $O(1)$. In $G\chi$PT, where odd chiral orders also occur, the constants $\alpha$ and $\beta$ in Eq. (3.9) consist of both $O(1)$ and $O(p)$ contributions, the latter being proportional to the first power of the quark masses. The Goldstone nature of the pions together with the general properties of their scattering amplitude, by themselves, do not fix the constants $\alpha$ and $\beta$. On the other hand, the values of these constants are actually at the center of our interest. They reflect the mechanism of spontaneous chiral symmetry breakdown in QCD and the corresponding chiral structure of the ground state: For instance, in the limit of a large quark condensate ($B_0 \sim \Lambda_H$, the special case of standard $\chi$PT), one has $\alpha = 1$, $\beta = 1$, whereas in the opposite limit of a vanishing condensate, $\alpha = 4$ and $\beta = 1$ at order $O(p^2)$. It is important that the dispersive approach to $\chi$PT and, in particular, the iteration of the unitarity condition, only assumes that $\alpha$ and $\beta$ are of the order $O(1)$ in the chiral counting, but otherwise proceeds independently of any particular values of these constants.

Equation (3.9) implies for the three lowest partial wave $f_a(s)$, $a = 0, 1, 2$,

$$\text{Re} \, f_a(s) = \varphi_a(s) + O(p^4), \quad \text{Im} \, f_a(s) = O(p^4),$$

(3.10)

where

$$\varphi_0(s) = \frac{1}{96\pi F_\pi^2} \left\{ 6\beta \left( s - \frac{4}{3} M_\pi^2 \right) + 5 \alpha M_\pi^2 \right\},$$

$$\varphi_1(s) = \frac{1}{96\pi F_\pi^2} \beta \left( s - 4 M_\pi^2 \right),$$

$$\varphi_2(s) = \frac{1}{96\pi F_\pi^2} \left\{ -3\beta \left( s - \frac{4}{3} M_\pi^2 \right) + 2 \alpha M_\pi^2 \right\}.$$  

(3.11)

Inserting this information into the unitarity condition (3.5), one obtains

$$\text{Im} \, f_a(s) = \sqrt{\frac{s - 4M_\pi^2}{s}} |\varphi_a(s)|^2 + O(p^6),$$

(3.12)

which are the discontinuities of the functions $W_a(s)$ up to and including the order $O(p^{4-2\varepsilon_a})$ ($O(p^{5-2\varepsilon_a})$ in $G\chi$PT). Due to the polynomial character of $\varphi_a(s)$, Eqs. (3.12) allow one to express the functions $W_a(s)$ as

$$W_a(s) = 16\pi (s - 4M_\pi^2)^{-\varepsilon_a} |\varphi_a(s)|^2 \bar{J}(s) + \text{polynomial} + O(p^{6-2\varepsilon_a}),$$

(3.13)

where \( \bar{J}(s) \) denotes the dispersion integral

\[
\bar{J}(s) = \frac{s}{16\pi^2} \int_0^\infty \frac{dx}{x} \frac{1}{x - s} \sqrt{\frac{x - 4M^2_\pi}{x}}.
\]  

(3.14)

The reader will easily recognise in the analytic function \( \bar{J}(s) \) the standard scalar-loop integral \( J^{\pi\pi}_{\pi\pi}(s) \) of the previous Section, subtracted at zero momentum transfer, \( J^{\pi\pi}_{\pi\pi}(s) = J^{\pi\pi}_{\pi\pi}(s) - J^{\pi\pi}_{\pi\pi}(0) \), (see Appendix C). The polynomial in Eq. (3.13) is at most of degree \( 2 - \varepsilon \) in \( s \). The coefficients of higher powers of \( s \) should be well behaved in the limit \( M_\pi \to 0 \), for \( s \) fixed, and consequently, such higher powers can be absorbed into the neglected \( O(p^{6-2\varepsilon a}) \) remainders. Notice that in the case of Eq. (3.13), the asymptotic bounds (3.7) are not only satisfied, but they are not even saturated (\( \bar{J}(s) \) grows as \( \ln s \)). 

Returning to the formula (3.1), the polynomial contributions in \( W_a(s) \) beyond the leading contributions (3.9) can be collected into

\[
\delta A(s|t, u) = \delta \alpha \frac{M^2_\pi}{3F^2_\pi} + \delta \beta \frac{3M^2_\pi}{4F^2_\pi} \left( s - \frac{4}{3}M^2_\pi \right) + \frac{\lambda_1}{5M^2_\pi} (s - 2M^2_\pi)^2 + \frac{\lambda_2}{5M^2_\pi} [(t - 2M^2_\pi)^2 + (u - 2M^2_\pi)^2].
\]  

(3.15)

Here, in the chiral limit, the dominant behaviour of \( \delta \alpha, \delta \beta, \lambda_1 \) and \( \lambda_2 \) is

\[
\delta \alpha = O(p^2), \quad \delta \beta = O(p^2), \quad \lambda_{1,2} = O(1).
\]  

(3.16)

It is convenient to absorb \( \delta \alpha \) and \( \delta \beta \) into the constants \( \alpha \) and \( \beta \) which characterise the leading order contribution (3.9). The constants \( \alpha \) and \( \beta \) so redefined by including higher-order contributions will keep their original names. Notice that in the expression (3.13) for \( W_a(s) \), the redefinitions \( \alpha \to \alpha - \delta \alpha, \beta \to \beta - \delta \beta \) induce terms of higher chiral orders, which can be reabsorbed into the neglected \( O(p^{6-2\varepsilon a}) \) remainders. Hence, the whole result of the first iteration of the unitarity condition can be expressed in terms of four parameters, \( \alpha, \beta, \lambda_1 \) and \( \lambda_2 \):

\[
A(s|t, u) = \frac{\beta}{F^2_\pi} \left( s - \frac{4}{3}M^2_\pi \right) + \alpha \frac{M^2_\pi}{3F^2_\pi} + \frac{\lambda_1}{5M^2_\pi} (s - 2M^2_\pi)^2 + \frac{\lambda_2}{5M^2_\pi} [(t - 2M^2_\pi)^2 + (u - 2M^2_\pi)^2]
\]  

(3.17)

where

\[
\bar{J}_{(\alpha,\beta)}(t, u) = \bar{J}(s)
\]  

(3.18)


3.2 Comparing Perturbative $O(p^4)$ and Dispersive $\chi$PT Formulae

The parameters $\alpha$, $\beta$, $\lambda_1$ and $\lambda_2$ in Eqs. (3.17) and (3.18) have their own expansions in powers of quark masses and they involve chiral logarithms. One has

$$
\alpha = \sum_{n=0}^{3} \alpha^{(n)}, \quad \beta = \sum_{n=0}^{3} \beta^{(n)}, \quad \lambda_{1,2} = \sum_{n=0}^{1} \lambda_{1,2}^{(n)},
$$

where $\alpha^{(n)}$ denotes the $O(p^n)$ component of $\alpha$ and likewise for the other constants. In the standard $\chi$PT, the odd orders are absent, $\alpha^{(0)} = \beta^{(0)} = 1$, and the formula (3.17) contains $O(p^2)$ and $O(p^4)$ parts only. In generalised $\chi$PT, the odd orders show up, $\beta^{(0)} = 1$, and $1 \leq \alpha^{(0)} \leq 4$, whereas the amplitude (3.17) involves $O(p^2)$, $O(p^3)$, $O(p^4)$ and $O(p^5)$ contributions before it reaches the (neglected) order $O(p^6)$. Detailed discussion of the values of the constants $\alpha$, $\beta$, $\lambda_{1,2}$ will be given in Sections 4 and 5.

It is now easy to show that the $O(p^4)$ result (2.16) obtained by a direct $G\chi$PT evaluation of one-loop graphs, starting from the Lagrangian $\tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \tilde{\mathcal{L}}^{(4)}$, is indeed of the form of Eqs. (3.17) and (3.18). To see this, one should first reexpress the scale dependent renormalised pion loop integrals $J_{\pi\pi}^r(s)$ and $M_{\pi\pi}^r(s)$ in terms of the scale independent function $\bar{J}(s)$:

$$
J_{\pi\pi}^r(s) = \bar{J}(s) - \frac{1}{16\pi^2} \left( \ln \frac{M_{\pi}^2}{\mu^2} + 1 \right),
$$

$$
M_{\pi\pi}^r(s) = \frac{1}{128} (s - 4M_{\pi}^2) \bar{J}(s) - \frac{1}{192\pi^2} \left( \ln \frac{M_{\pi}^2}{\mu^2} + \frac{1}{3} \right).
$$

Next, the $K$ and $\eta$ loop contributions $J_{KK}^r(s)$, $M_{KK}^r(s)$ and $J_{\eta\eta}^r(s)$ have to be expanded in powers of $s/(2M_K)^2$ and $s/(2M_\eta)^2$, respectively. Keeping only terms that contribute up to and including the order $O(p^4)$, one has

$$
J_{PP}^r(s) = -\frac{1}{16\pi^2} \left( \ln \frac{M_P^2}{\mu^2} + 1 \right) + O(s/4M_P^2), \quad P = K, \eta,
$$

$$
M_{KK}^r(s) = -\frac{1}{192\pi^2} \left( \ln \frac{M_K^2}{\mu^2} + 1 \right) + O(s/4M_K^2).
$$

Let us stress again at this point that it is perfectly possible to recover the full $K$ and $\eta$ loop contributions within the dispersive approach: for this, it is sufficient to include the $\bar{K}K$ and $\bar{\eta}\eta$ intermediate states into the unitarity condition. Here we do not present this extended analysis for reasons of simplicity, and because performing the expansions (3.21) in Eq. (2.16) should be a fairly good approximation at low energies.

Finally, in order to compare the two expressions, all contributions beyond the order $O(p^4)$ in Eqs. (3.17) and (3.18), i.e. all $O(p^5)$ contributions, should be dropped. In particular, only the
$O(p^4)$ part of the loop function $\bar{J}_{(\alpha,\beta)}(s|t, u)$ should be maintained. This amounts to replacing $\bar{J}_{(\alpha,\beta)}(s|t, u)$ by $\bar{J}_{(\alpha^{(0)},\beta^{(0)})}(s|t, u)$, where

$$\alpha^{(0)} = 1 + 3\hat{\epsilon}, \quad \beta^{(0)} = 1,$$  \hspace{1cm} (3.22)

are the leading parts of the constants $\alpha$ and $\beta$, as can be inferred from Eqs. (2.17), (2.18). Then one finds that Eqs. (2.16) and (3.17) are indeed the same, upon making the following identifications ($\hat{\epsilon}$ and $\tilde{\Delta}_{GMO}$ were defined in Eqs. (2.21) and (2.22))

$$\alpha = \alpha^r(\mu) - \frac{M^2_\pi}{32\pi^2 F^2_\pi} \left( \ln \frac{M^2_K}{\mu^2} + 1 \right) \left\{ 1 + (1 + r)\hat{\epsilon} \right\}^2,$$  \hspace{1cm} (3.23)

$$\beta = \beta^r(\mu) - \frac{M^2_\pi}{16\pi^2 F^2_\pi} \left( \ln \frac{M^2_\eta}{\mu^2} + 1 \right) \left\{ 1 + (1 + r)\hat{\epsilon} \right\}^2,$$  \hspace{1cm} (3.23)

and

$$\lambda_1 = \lambda_1^{(0)} = 4 (2L^r_1(\mu) + L_3) - \frac{1}{48\pi^2} \left\{ \ln \frac{M^2_\pi}{\mu^2} + \frac{1}{8} \ln \frac{M^2_K}{\mu^2} + \frac{35}{24} \right\},$$  \hspace{1cm} (3.25)

$$\lambda_2 = \lambda_2^{(0)} = 4 L^r_2(\mu) - \frac{1}{48\pi^2} \left\{ \ln \frac{M^2_\pi}{\mu^2} + \frac{1}{8} \ln \frac{M^2_K}{\mu^2} + \frac{23}{24} \right\}.$$  \hspace{1cm} (3.25)

Upon using Eqs. (2.13) and (2.20) and the scale dependences of the renormalised constants $L^r_1$ and $L^r_2$, it is straightforward to ascertain the scale independence of $\alpha, \beta, \lambda_1$ and $\lambda_2$.

At the end of Section 2 it was shown how the standard $SU(3)_L \times SU(3)_R$ $O(p^4)$ amplitude appears as a particular case of Eq. (2.16): It may be expressed in terms of the standard $O(p^4)$ constants $L_1,..., L_8$. The expressions (3.23) for $\lambda_1$ and for $\lambda_2$ remain unchanged, while one finds that $\alpha$ and $\beta$ become ($\alpha^r_{st} = \beta^r_{st} = 1$)

$$\alpha_{st} = 1 - 16 \frac{M^2_\pi}{F^2_\pi} (2L^r_4 + L^r_5) + 48 \frac{M^2_\pi}{F^2_\pi} (2L^r_6 + L^r_8)$$

$$- \frac{1}{32\pi^2 F^2_\pi} \left\{ \ln \frac{M^2_\pi}{\mu^2} + \ln \frac{M^2_K}{\mu^2} + \frac{1}{3} \ln \frac{M^2_\eta}{\mu^2} + \frac{7}{3} \right\},$$  \hspace{1cm} (3.26)

$$\beta_{st} = 1 + 8 \frac{M^2_\pi}{F^2_\pi} (2L^r_4 + L^r_5)$$

$$- \frac{1}{32\pi^2 F^2_\pi} \left\{ 4 \ln \frac{M^2_\pi}{\mu^2} + \ln \frac{M^2_K}{\mu^2} + 5 \right\}.$$  \hspace{1cm} (3.27)
If we use for the constants $L_i$ the values currently given in the literature \[31\] and take $F_\pi = 92.4$ MeV, $M_\pi = 139.57$ MeV, we obtain, for the corresponding scale independent constants $\alpha$, $\beta$, $\lambda_1$ and $\lambda_2$ the following values at order $O(p^4)$ in the standard case:

$$
\begin{align*}
\alpha_{\text{st}} &= 1.04 \pm 0.15 \\
\beta_{\text{st}} &= 1.08 \pm 0.03
\end{align*}
$$

(3.28)

$$
\begin{align*}
\lambda_{1,\text{st}} &= (-6.4 \pm 6.8) \times 10^{-3} \\
\lambda_{2,\text{st}} &= (10.8 \pm 1.2) \times 10^{-3} .
\end{align*}
$$

Similarly, the standard $SU(2)_L \times SU(2)_R$ $O(p^4)$ amplitude originally given by Gasser and Leutwyler \[23, 4\] in terms of the four scale independent parameters $\bar{l}_1, \bar{l}_2, \bar{l}_3$ and $\bar{l}_4$ can be put into the form (3.17) and (3.18), with the identifications

$$
\begin{align*}
\alpha_{\text{GL}} &= 1 + \frac{1}{32 \pi^2} \frac{M_\pi^2}{F_\pi^2} \left( 4 \bar{l}_4 - 3 \bar{l}_3 - 1 \right) , \\
\beta_{\text{GL}} &= 1 + \frac{1}{8 \pi^2} \frac{M_\pi^2}{F_\pi^2} \left( \bar{l}_4 - 1 \right) ,
\end{align*}
$$

(3.29)

and

$$
\begin{align*}
\lambda_{1,\text{GL}} &= \frac{1}{48 \pi^2} \left( \bar{l}_1 - \frac{4}{3} \right) , \\
\lambda_{2,\text{GL}} &= \frac{1}{48 \pi^2} \left( \bar{l}_2 - \frac{5}{6} \right) .
\end{align*}
$$

(3.30)

Using the numerical values recently updated by Gasser in \[32\]

$$
\begin{align*}
\bar{l}_1 &= -2.15 \pm 4.29 , \\
\bar{l}_2 &= 5.84 \pm 1.72 , \\
\bar{l}_3 &= 2.9 \pm 2.4 , \\
\bar{l}_4 &= 4.55 \pm .29 ,
\end{align*}
$$

(3.31)

one obtains

$$
\begin{align*}
\alpha_{\text{GL}} &= 1.06 \pm 0.06 \\
\beta_{\text{GL}} &= 1.103 \pm 0.008
\end{align*}
$$

(3.32)

$$
\begin{align*}
\lambda_{1,\text{GL}} &= (-7.35 \pm 9.06) \times 10^{-3} \\
\lambda_{2,\text{GL}} &= (10.57 \pm 3.63) \times 10^{-3} .
\end{align*}
$$

It thus appears clearly that the situation of a large quark condensate, \textit{i.e.} the standard version of $\chi$PT, is characterised by the values of the parameters $\alpha$ and $\beta$ remaining close to unity.

### 3.3 The Two-Loop $\pi \pi$ Amplitude

We now proceed to the second iteration of the unitarity condition (3.5). The starting point is the result of the previous step, \textit{i.e.} the one-loop amplitude (3.17). First, we use Eqs.
\((3.8)\) to calculate the S- and P-wave projections of the amplitude \((3.17)\). All the integrals are elementary and the result can be represented as

\[
\text{Re}\ f_a(s) = \varphi_a(s) + \psi_a(s) + O(p^6) ,
\]

(3.34)

where \(\varphi_a(s)\) are the polynomials \((3.11)\) and the new contributions \(\psi_a(s)\), which will be given shortly, are dominantly of order \(O(p^4)\). The unitarity condition \((3.5)\) now allows us to push the knowledge of \(\text{Im}\ f_a(s)\) one step further:

\[
\text{Im}\ f_a(s) = \sqrt{\frac{s - 4M^2}{s}} \left\{ |\varphi_a(s)|^2 + 2\varphi_a(s)\psi_a(s) \right\} + O(p^8) .
\]

(3.35)

It is convenient to express the functions \(\psi_a(s)\) arising from the partial wave projections in the following way \((s > 4M^2)\):

\[
\psi_a(s) = \frac{M^2}{F^4_\pi} \sqrt{\frac{s}{s - 4M^2}} \sum_{n=0}^{4} \xi_a^{(n)}(s) k_n(s) .
\]

(3.36)

Here, \(\xi_a^{(n)}(s)\) are polynomials in \(s\) of at most second degree, with coefficients given in terms of \(\alpha, \beta, \lambda_1\) and \(\lambda_2\). They are listed in Appendix B. \(k_n(s)\) represent the following set of elementary functions

\[
k_0(s) = \frac{1}{16\pi} \sqrt{\frac{s - 4M^2}{s}} , \quad k_1(s) = \frac{1}{8\pi} L(s) ,
\]

\[
k_2(s) = \frac{1}{8\pi} \left( 1 - \frac{4M^2}{s} \right) L(s) , \quad k_3(s) = \frac{3}{16\pi} \frac{M^2}{\sqrt{s(s - 4M^2)}} L^2(s) ,
\]

\[
k_4(s) = \frac{1}{16\pi} \frac{M^2}{\sqrt{s(s - 4M^2)}} \left\{ 1 + \sqrt{\frac{s}{s - 4M^2}} L(s) + \frac{M^2}{s - 4M^2} L^2(s) \right\} ,
\]

(3.37)

where

\[
L(s) = \ln \frac{1 - \sqrt{1 - \frac{4M^2}{s}}}{1 + \sqrt{1 - \frac{4M^2}{s}}} , \quad s > 4M^2 .
\]

(3.38)

Notice that in the chiral limit \((s \to 0, M^2 \to 0, s/M^2\) fixed) all functions \(k_n(s)\) are of order \(O(1)\). Furthermore, they are all asymptotically bounded by \(\ln s\). Eqs. \((3.34)\) and \((3.35)\) now lead to the following expressions for \(\text{Im}\ W_a(s)\)

\[
\text{Im}\ W_a(s) = \sum_{n=0}^{4} w_a^{(n)}(s) k_n(s) + O(p^{8-2\varepsilon_a}) ,
\]

(3.39)

where

\[
w_a^{(n)}(s) = (s - 4M^2)^{-\varepsilon_a} \left\{ 16\pi |\varphi_a(s)|^2 \delta_{n0} + \frac{2M^4}{F^4_\pi} \varphi_a(s)\xi_a^{(n)}(s) \right\}
\]

(3.40)
are polynomials of degree 3 − ε_a in s. (Remember that ϕ_1(s) is proportional to s − 4M_π^2.) Consequently,
\[ W_a(s) = \sum_{n=0}^{4} w_a^{(n)}(s) \tilde{K}_n(s) + \text{polynomial} + O(p^{8-2\varepsilon_a}) , \] (3.41)
where \( \tilde{K}_n(s) \) denote the dispersion integrals
\[ \tilde{K}_n(s) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dx}{x} k_n(x) \frac{k_n(x)}{x-s} . \] (3.42)
The formula (3.41) contains and extends the one-loop result (3.13): By definition,
\[ \tilde{K}_0(s) = \tilde{J}(s) , \] (3.43)
and the leading \( O(p^4) \) part in \( w_a^{(0)}(s) \) which contributes to Eq. (3.13) is manifest in the expression (3.40).

As for the polynomial part of Eq. (3.41), its degree in s can be at most 3 − ε_a: The coefficient of any power of s higher than 3 − ε_a should not blow up in the limit \( M_\pi \to 0, s \) fixed, and consequently, any such power can be absorbed into the neglected contribution \( O(p^{8-2\varepsilon_a}) \).

The functions \( W_a(s) \) then satisfy the bounds (3.7), since the functions \( \tilde{K}_n(s) \) grow at most as (ln s)^2. This means that Eq. (3.41) determines the amplitude \( A(s|t, u) \) up to a general crossing symmetric polynomial \( \delta A(s|t, u) \) of at most cubic order. The most general polynomial of this type contains six arbitrary parameters and it can be written as
\[ \delta A(s|t, u) = \delta \alpha \frac{M_\pi^2}{3F^2_\pi} + \delta \beta \frac{1}{F^2_\pi} \left( s - \frac{4}{3}M_\pi^2 \right) \]
\[ + \frac{\delta \lambda_1}{F^4_\pi} (s - 2M_\pi^2)^2 + \frac{\delta \lambda_2}{F^4_\pi} [(t - 2M_\pi^2)^2 + (u - 2M_\pi^2)^2] \]
\[ + \frac{\lambda_3}{F^6_\pi} (s - 2M_\pi^2)^3 + \frac{\lambda_4}{F^6_\pi} [(t - 2M_\pi^2)^3 + (u - 2M_\pi^2)^3] . \] (3.44)
This formula represents the polynomial contribution to \( A(s|t, u) \) beyond the one-loop order (3.17). In the chiral limit, the six constants involved in Eq. (3.44) dominantly behave as
\[ \delta \alpha, \delta \beta = O(p^4) , \delta \lambda_{1,2} = O(p^2) , \lambda_{3,4} = O(1) . \] (3.45)
Furthermore \( \delta \alpha, \delta \beta, \delta \lambda_1 \) and \( \delta \lambda_2 \) can be absorbed into the parameters \( \alpha, \beta, \lambda_1 \) and \( \lambda_2 \), respectively, which characterise the one loop result (3.17). This redefinition simply extends the expansion (3.13) in powers of quark masses and consequently the parameters \( \alpha, \beta \) and \( \lambda_{1,2} \) so redefined will be referred to by their original names. Notice that in the expressions (3.41) for the functions \( W_a(s) \), the redefinitions \( \alpha \to \alpha - \delta \alpha, \beta \to \beta - \delta \beta \) and \( \lambda_{1,2} \to \lambda_{1,2} - \delta \lambda_{1,2} \) induce terms of higher chiral order which can be reabsorbed into the neglected \( O(p^{8-2\varepsilon_a}) \) remainders.
This implies that the whole $\pi\pi$ scattering amplitude up to and including two loops can be expressed in terms of only six parameters, and the result reads:

\[
A(s|t, u) = \frac{\beta}{F_\pi^2} \left( s - \frac{4}{3} M_\pi^2 \right) + \alpha \frac{M_\pi^2}{3F_\pi^2} \\
+ \frac{\lambda_1}{F_\pi^4} (s - 2M_\pi^2)^2 + \frac{\lambda_2}{F_\pi^4} \left[ (t - 2M_\pi^2)^2 + (u - 2M_\pi^2)^2 \right] \\
+ \frac{\lambda_3}{F_\pi^6} (s - 2M_\pi^2)^2 + \frac{\lambda_4}{F_\pi^6} \left[ (t - 2M_\pi^2)^3 + (u - 2M_\pi^2)^3 \right] \\
+ \bar{K}(s|t, u) + O(p^8/\Lambda_H^8),
\] (3.46)

where

\[
\bar{K}(s|t, u) = 32\pi \sum_{n=0}^4 \left\{ \frac{1}{3} \left[ w_0^{(n)}(s) - w_2^{(n)}(s) \right] \bar{K}_n(s) \\
+ \frac{1}{2} \left[ w_2^{(n)}(t) + 3(s-u)w_1^{(n)}(t) \right] \bar{K}_n(t) \\
+ \frac{1}{2} \left[ w_2^{(n)}(u) + 3(s-t)w_1^{(n)}(u) \right] \bar{K}_n(u) \right\}.
\] (3.47)

The one-loop contribution $\bar{J}_{(\alpha, \beta)}(s|t, u)$, cf. Eq. (3.18), is hidden in the $n = 0$ part of the sum (3.47): It arises from the first terms, proportional to $F_\pi^{-4}$, in (3.40). The terms proportional to $F_\pi^{-6}$ in the polynomials $w_0^{(n)}(s)$ represent the $O(p^6)$ (and $O(p^7)$) contributions: The parts linear in $\lambda_1$ and $\lambda_2$ correspond to single loop graphs with one $O(p^4)$ vertex, while the remaining parts, which are cubic in $\alpha$ and $\beta$, give rise to the genuine two-loop contributions.

If one wishes to identify a given chiral order in the general formula (3.46), the parameters $\alpha, \beta, \lambda_i, i = 1, 2, 3, 4,$ have to be expanded in powers of quark masses. The only difference with the previous expansion (3.19), is that now more terms have to be included, until one reaches the first neglected order $O(p^8)$. Hence

\[
\alpha = \sum_{n=0}^5 \alpha^{(n)}, \quad \beta = \sum_{n=0}^5 \beta^{(n)}, \quad \lambda_{1,2} = \sum_{n=0}^3 \lambda_{1,2}^{(n)}, \quad \lambda_{3,4} = \sum_{n=0}^1 \lambda_{3,4}^{(n)},
\] (3.48)

where $\alpha^{(n)} = O(p^n)$ and likewise for the other constants. The particular case of the standard $\chi$PT is again characterised by the absence of odd chiral orders.

Finally, we turn to the dispersion integrals $\bar{K}_n(s)$ (3.42) which occur in the two loop formula (3.46), (3.47). It is remarkable that all these integrals are elementary and that they can be expressed in terms of powers of the standard one-loop function $\bar{J}(s)$ (3.14):

\[
\]
\[ \bar{K}_0(s) = J(s) \]
\[ \bar{K}_1(s) = \frac{1}{16\pi^2} \frac{s}{s - 4M^2_\pi} \left[ 16\pi^2 J(s) - 2 \right]^2 \]
\[ \bar{K}_2(s) = \frac{s - 4M^2_\pi}{s} \bar{K}_1(s) - \frac{1}{4\pi^2} \]
\[ \bar{K}_3(s) = \frac{1}{16\pi^2} \frac{M^2_\pi}{s - 4M^2_\pi} \left\{ \frac{s}{s - 4M^2_\pi} \left[ 16\pi^2 J(s) - 2 \right]^3 + \pi^2 \left[ 16\pi^2 J(s) - 2 \right] \right\} - \frac{1}{32} \]
\[ \bar{K}_4(s) = \frac{1}{16\pi^2} \frac{M^2_\pi}{s - 4M^2_\pi} \left\{ 16\pi^2 J(s) - 2 + 8\pi^2 \bar{K}_1(s) + \frac{16\pi^2}{3} \bar{K}_3(s) + \frac{\pi^2}{3} \right\} - \frac{1}{32\pi^2} + \frac{1}{192} \]

The proof of these formulae and a discussion of some properties of the functions \( \bar{K}(s) \) can be found in Appendix C.

4 Phenomenological Aspects

The preceding discussion has led us to the expression (3.46) for the \( \pi\pi \) amplitude at twoloop order, which depends on the six parameters \( \alpha, \beta \) and \( \lambda_i, i = 1, 2, 3, 4 \). A priori, neither the one-loop formula (3.17) nor the two-loop expression (3.46), by itself, contains information about these parameters. On the other hand, Eqs. (3.23), (3.24) or (3.25) relate them to the low-energy constants, such as \( \hat{m}_B_0, \hat{m}^2 A_0, \hat{m}_\xi, L_1, L_2, \ldots \), which appear in the chiral expansion of the effective action \( Z_{\text{eff}} \). To lowest chiral orders, these same low-energy constants are also present in the expansion of other observables, such as the masses of the pseudoscalars, \( F_\pi/F_K \), the scalar form-factor of the pion, the \( K_{44} \) form-factors, etc... This kind of information, which reflects the Ward identities obeyed by the QCD correlation functions, is rather crucial and has always played a central role in applications of chiral perturbation theory. In particular, it allows one to pin down approximate values of the parameters \( \beta, \lambda_1 \) and \( \lambda_2 \) from independent experiments, and it provides a clear interpretation of the leading order value of \( \alpha \) in terms of the mechanism of spontaneous breakdown of chiral symmetry in QCD, by relating this value to the quark mass ratio \( r = m_s/\hat{m} \), cf. Eq. (3.22). It is also clear, however, that the higher orders in the expansions of \( \alpha, \beta \) and the \( \lambda_i \)'s in powers of quark masses involve a rapidly increasing number of new low-energy constants from \( Z_{\text{eff}} \). The task of estimating them from independent measurements of different observables, and of re-evaluating the values of the previous low-energy constants taking into account the higher-order corrections, seems rather out of reach at the moment.

Fortunately, it turns out to be possible to pin down four of the parameters (\( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \)), which enter the two-loop expression (3.46) of the \( \pi\pi \) scattering amplitude, using sum rules and available \( \pi\pi \) data at medium energies. The derivation of these sum rules and their evaluations are sketched in the following subsection (a more detailed account will be postponed to a forthcoming publication). Next, we consider a few applications. In particular, we discuss
how the low-energy data constrain the remaining two parameters $\alpha$ and $\beta$. We remind the reader that $\alpha$ is the key parameter for deciding whether the chiral symmetry is broken according to the standard scenario, with a large $\langle \bar{q}q \rangle$ condensate, or not.

4.1 Sum Rule Evaluation of the $\lambda_i$’s

Consider the $s$-channel isospin $I=0,1,2$ amplitudes $F^I(s,t)$ defined as ($u = 4M_\pi^2 - t - s$)

\[
F^0(s,t) = \frac{1}{32\pi} \{3A(s|t, u) + A(t|s, u) + A(u|s, t)\}
\]
\[
F^1(s,t) = \frac{1}{32\pi} \{A(t|s, u) - A(u|s, t)\}
\]
\[
F^2(s,t) = \frac{1}{32\pi} \{A(t|s, u) + A(u|s, t)\}.
\]

Because of the Froissart bound, they satisfy twice-subtracted, fixed-$t$, dispersion relations. These form the starting point for the derivation of the sum rules for the constants $\lambda_i$. The next, and somewhat tedious, step consists in transforming this set of fixed-$t$ dispersion relations into a form which allows direct comparison with the two-loop chiral expression. This is performed by first identifying and rejecting the contributions which are of chiral order $O(p^8)$ or more. In order to do so, one has to split the integration ranges of the dispersive integrals into two regions, a low-energy region $[4M_\pi^2, E^2]$ and a high energy region $[E^2, \infty]$. In the first region, one may drop the contributions from the partial waves with $l \geq 2$, while in the latter region one may expand in powers of $s, t, u$ divided by $E^2$ and stop at the third power. The borderline energy $E$ corresponds, roughly speaking, to the point where the chiral expansion starts to break down, i.e. $E$ is in the range 500-600 MeV. Secondly, one has to impose crossing symmetry. In practice, one proceeds in close analogy with subsection III-C of Ref. [9]. There, starting from thrice-subtracted dispersion relations, it was shown how, up to corrections of chiral order $O(p^8)$, crossing symmetry entirely determines the form of the amplitude except for six arbitrary parameters. If, instead, one uses twice-subtracted dispersion relations one finds that two parameters remain undetermined ($\alpha$ and $\beta$), while the four others ($\lambda_1, \ldots, \lambda_4$) obey well-convergent sum rules. Details will be provided elsewhere. Here, we shall only give the final result:

\[
\frac{\lambda_1}{F_4^0} = \frac{1}{3} \left( T_0^{(2)} - T_2^{(2)} \right) + 2M_\pi^2 \left( T_0^{(3)} - T_2^{(3)} \right) - 3T_1^{(1)} - 6M_\pi^2 T_1^{(2)} + \frac{16\pi}{3} (h_0 - 4h_2)
\]
\[
\frac{\lambda_2}{F_4^0} = \frac{1}{2} T_2^{(2)} + 3M_\pi^2 T_2^{(3)} + \frac{3}{2} T_1^{(1)} + 3M_\pi^2 T_1^{(2)} - \frac{16\pi}{3} (h_0 - h_2)
\]
\[
\frac{\lambda_3}{F_4^0} = \frac{1}{3} \left( T_0^{(3)} - T_2^{(3)} \right) + T_1^{(2)} + \frac{32\pi}{9} (-9h_1 + h_0' - h_2')
\]

As a matter of fact, if one assumes conventional Regge asymptotics for the $t$-channel $I=1$ amplitude, one can derive a slowly convergent sum rule for $\beta$ as well, which is equivalent to the one derived long ago by Olsson [33].
\[
\frac{\lambda_4}{F_\pi^2} = \frac{1}{2} I_1^{(3)} - \frac{1}{2} I_1^{(2)} + \frac{32\pi}{9} (h'_0 - h'_2) .
\] (4.2)

In these formulae, the quantities \( I_a^{(k)} \) are given in terms of integrals over the low-energy range \([4M^2, E^2]\), while \( h_n \) and \( h'_n \) involve integrals from \( E^2 \) to infinity (detailed formulae will be given below). Since experimental data on \( \pi\pi \) phase shifts exist down to roughly 500 MeV, we can calculate the latter integrals essentially using experimental input. In the low-energy integrals we shall use the two-loop chiral expression for the phase shifts. Stability of the result with respect to variations of the energy \( E \) within the range indicated above measures the extent to which the \( \chiPT \) phases do match to the experimental ones at this energy. Note that since the \( \chiPT \) phases themselves depend on the parameters \( \lambda_i \), the set of relations (4.2) must actually be solved in a self-consistent way.

Let us now give the explicit expressions for the various entries in Eqs. (4.2). First, we define the \( I_a^{(k)} \)'s as

\[
I_a^{(k)} = \frac{32\pi}{k!} \frac{d}{ds^k} \left\{ \frac{s^{2-z_a}}{\pi} \int_{4M^2}^{E^2} \frac{dx}{x^{2-z_a}} \frac{\text{Im} W_a}{x - s} - \sum_{n=0}^{4} w_a^{(n)}(s) \tilde{K}_n(s) \right\}_{s=0},
\] (4.3)

where the polynomials \( w_a^{(n)}(s) \) and the functions \( \tilde{K}_n(s) \) are those which occur in the expression of the two-loop amplitude of the preceding section. The quantities \( h_a \) and \( h'_a \) are defined in terms of the following high-energy integrals

\[
h_0(t) = \frac{-1}{\pi} \int_{E^2}^{\infty} dx \frac{2x + t - 4M^2}{x^2(x + t - 4M^2)^2} \text{Im} \left[ \frac{1}{3} F^0(x, t) + F^1(x, t) + \frac{5}{3} F^2(x, t) \right] ,
\] (4.4)

and

\[
h_2(t) = \frac{-1}{\pi} \int_{E^2}^{\infty} dx \frac{2x + t - 4M^2}{x^2(x + t - 4M^2)^2} \text{Im} \left[ \frac{1}{3} F^0(x, t) - \frac{1}{2} F^1(x, t) + \frac{1}{6} F^2(x, t) \right] .
\] (4.5)

Then, \( h_a \) and \( h'_a \) are given simply by

\[
h_a \equiv h_a(0) , \quad h'_a \equiv h'_a(0) , \quad a = 0, 2 .
\] (4.6)

Finally, \( h_1 \) is given by

\[
h_1 = \frac{-1}{\pi} \int_{E^2}^{\infty} \frac{dx}{x(x - 4M^2)} \text{Im} \left[ \frac{1}{3} F^0(x, 0) + \frac{1}{2} F^1(x, 0) - \frac{5}{6} F^2(x, 0) \right] .
\] (4.7)

Notice that the three integrals (4.4), (4.3) and (4.7) involve the \( t\)-channel isospin \( I_t = 0, 2 \) and \( 1 \) combinations, respectively. In addition to the constraints it imposes on the low-energy parts, crossing symmetry also implies the following relation between these high-energy integrals:

\[
9h_1 = 2h'_0 - 5h'_2 .
\] (4.8)
In practice, the most reliable source of information on the $\pi\pi$ phase shifts remains the old high-statistics CERN-Munich experiment [34] (see e.g. the reviews by Ochs [35] and by Morgan and Pennington [36] for critical discussions of more recent experiments). In order to estimate the errors in the evaluation of the high-energy integrals, we have varied the experimental data within their error bars and we have also compared the results obtained using different analyses of the production data of Ref. [34]. The errors in the low-energy integrals, which reflect the presence of $O(p^8)$ contributions, were estimated from the sensitivity to variations in the value of the matching energy $E$ and by trying different ansätze for $\text{Im} f_a$ (which all differ only by $O(p^8)$ contributions) in equations (4.3). Furthermore, a small dependence of the result upon the remaining two parameters $\alpha$ and $\beta$ was absorbed in the error. More details of this analysis will be given elsewhere. We end up with the following estimates for the values of $\lambda_1$ and $\lambda_2$:

\begin{align}
\lambda_1 &= (-5.3 \pm 2.5) \times 10^{-3} \\
\lambda_2 &= (9.7 \pm 1.0) \times 10^{-3} .
\end{align} \tag{4.9}

These numbers are in agreement with the result of the analysis of Bijnens et al. [31], which incorporates constraints from sum rules for D-wave threshold parameters as well as from $K_{l4}$ decays. An independent estimate was recently made by Pennington and Portolès [37], who derived simple, but approximate, formulae by setting the isospin $I=2$ contributions in both direct and crossed channels exactly equal to zero. From their analysis it follows that $\lambda_1 = (-4.3 \pm 2.3) \times 10^{-3}$ and $\lambda_2 = (8.8 \pm 1.1) \times 10^{-3}$ for $F_\pi = 92.4$ MeV. Our result for $\lambda_2$ lies between those of these two references. Finally, for the remaining two parameters, we obtain

\begin{align}
\lambda_3 &= (2.9 \pm 0.9) \times 10^{-4} \\
\lambda_4 &= (-1.4 \pm 0.2) \times 10^{-4} .
\end{align} \tag{4.10}

### 4.2 Low-energy Phase Shifts

It is well known that the chiral expansions of the $\pi\pi$ partial-wave amplitudes do not satisfy unitarity exactly, but rather in a perturbative sense. The deviations from unitarity provide an estimate for the energy range where the chiral expansion, at a given order, can be trusted. We show in Fig. 1 the Argand diagram for the amplitude $f^0_0$ computed at one-loop and at two-loop order. The figure shows that there is a considerable improvement with respect to unitarity at low energies in going from the one-loop result to the two-loop result. On the other hand, it also suggests that one should expect sizeable $O(p^8)$ corrections starting at 450 MeV.

In spite of the fact that the unitarity relation is not exactly satisfied, it is possible to define a chiral expansion for the phase shifts (see, for instance, the discussion in [38]). At lowest order, and corresponding to the amplitude (3.9), the partial waves for $l < 2$ are defined as ($a = 0, 1, 2$)

\begin{align}
\delta_a &= \sqrt{s - 4M^2} \frac{\varphi_a}{s} + O(p^4) .
\end{align} \tag{4.11}
The one-loop form (3.17) of the amplitude extends the above definition of the phases to
\[
\delta_a = \sqrt{s - 4M^2} \left[ \varphi_a + \psi_a \right] + O(p^6) ,
\]
whereas the two-loop expression (3.46) leads to
\[
\delta_a = \sqrt{s - 4M^2} \left[ \text{Re} f_a + \frac{2}{3} (\varphi_a)^3 \right] + O(p^8) ,
\]
where in this last formula \( \text{Re} f_a \) must be evaluated from the full two-loop amplitude (3.46). The higher partial waves receive contributions starting only at the one-loop approximation, otherwise the formulae are exactly the same. The result for the phases as computed from the successive approximations (1.11), (1.12) and (1.13) are shown in Figs. 2a and 2b. For simplicity, we have used the same values for \( \alpha \) and \( \beta \) (\( \alpha = 2, \beta = 1.08 \)) in all three cases (strictly speaking, of course, one should use the expansions (3.48) of \( \alpha \) and \( \beta \) at the given order). Surprisingly, for \( I = l = 0 \) (Fig. 2a) the one-loop and two-loop phases remain fairly close at energies where one would expect (say from the Argand diagram) that \( O(p^8) \) terms should become important (approximately above 500 MeV). This is somewhat fortuitous, and does not happen for the isospin I=2 wave or the P-wave. In the latter case, it is interesting to observe how the \( \rho \) resonance is gradually built up as the order of the perturbation increases. Finally, Fig. 4a shows the behaviour, at increasing orders of the chiral expansion, of the difference \( \delta_0^0 - \delta_1^1 \).

We investigate next the sensitivity of the phase shifts on the values of the parameters \( \alpha \) and \( \beta \). Results are shown in Figure 3, representing \( \delta_0^0, \delta_2^0 \) and \( \delta_1^1 \) respectively, and Fig. 4b which shows the difference \( \delta_0^0 - \delta_1^1 \) in the energy range accessible in \( K_{l4} \) decays. We have taken values of \( \alpha \) ranging from 1 to 3. Figure 4b suggests that the data of Rosselet et al. [39] are approximately compatible with such a range. The figures show that the sensitivity upon \( \alpha \) is most significant at very low energies. An important issue for us is to assess the extent to which the availability of a new generation of data on \( K_{l4} \) decays from e.g. the \( \text{DAΦNE} \) facility will allow a really precise determination of these parameters \( \alpha \) and \( \beta \). For this purpose we show in Fig. 5 how the errors on \( \lambda_1, \ldots, \lambda_4 \) affect the prediction for \( \delta_0^0 - \delta_1^1 \). The figure shows that the situation where \( \alpha = 1 \) and the one where \( \alpha = 2 \) can be distinguished, provided sufficiently precise data are available in the energy range below \( E = 340 \) MeV. Let us now examine in a more quantitative way how the presently available data constrain the values of \( \alpha \) and \( \beta \), using the two-loop form of the amplitude and our estimates of the values of the parameters \( \lambda_i \). Since there are only five data points provided in Ref. [39], it is useful to look for an additional constraint between \( \alpha \) and \( \beta \). Such a relation can indeed be obtained from the “Morgan-Shaw Universal Curve” [40]. The latter represents a correlation between the \( I = 0 \) and \( I = 2 \) S-wave scattering lengths, obtained by fitting the data in the region between 600 MeV and 1 GeV (i.e. outside of the range of applicability of the two-loop chiral expansion) using the Roy equations (see [11], [12]). We reproduce here the form quoted by Petersen in Ref. [24]:
\[
2a_0^0 - 5a_0^2 = 0.692 \pm 0.027 + 0.9 (a_0^0 - 0.3) + 1.2 (a_0^0 - 0.30)^2 .
\]
(4.14)
Upon using the explicit expressions for the scattering lengths (see Appendix D), this equation transcribes into the desired relation between $\alpha$ and $\beta$, which is plotted in Fig. 6. We have performed a chi-square minimisation using this relation in addition to the data of Rosselet et al. [39] and to the results (4.9), (4.10). We find

$$\alpha = 2.16 \pm 0.86, \quad \beta = 1.074 \pm 0.053,$$

(4.15)

with a $\chi^2$ equal to 2.01 for four degrees of freedom. The threshold parameters will be discussed in detail below. Let us give here the corresponding value for $a_0^0$,

$$a_0^0 = 0.263 \pm 0.052.$$

(4.16)

Interestingly enough, this result is in agreement with the one quoted in Ref. [24], which is based on a fit using the Roy equations.

A physically interesting quantity which is related to the $\pi\pi$ phase shifts is the phase of the CP violation parameter $\epsilon'$

$$\phi(\epsilon') = 90 + \delta_0^2 - \delta_0^0 \quad \text{(degrees)},$$

(4.17)

where the phases are to be evaluated at the energy $E = M_{K^0} = 497.7$ MeV. The preceding discussion shows that it makes sense to use the two-loop approximation at this energy and, indeed, Gasser and Meißner have discussed $\phi(\epsilon')$ at $O(p^4)$ in the standard chiral expansion and find $\phi = (45 \pm 6)^\circ$. Using the values of $\alpha$ and $\beta$ (4.15) derived from the fit discussed above, we obtain

$$\phi = (43.5 \pm 2 \pm 6)^\circ,$$

(4.18)

where we have shown separately the error coming from the uncertainties in the fit values (4.15) of $\alpha$ and $\beta$ ($\pm 2^\circ$), and the error due to the uncertainties in our determinations (4.9) and (4.10) of the $\lambda_i$’s ($\pm 6^\circ$). At this energy, a significant gain in precision is achieved by going through a numerical solution of the Roy equations, rather than by using the chiral expansion. This has been done by Ochs [35], who found a result very similar to ours but with a much smaller error: $\phi = (46 \pm 3)^\circ$.

### 4.3 Threshold Parameters

From the two-loop amplitude (3.46), it is straightforward to obtain explicit expressions for the scattering lengths $a_l^1$ and the slope parameters $b_l^1$. For the reader’s convenience, we have gathered, in Appendix D, a selection of formulae relevant for the discussion that follows.

Table 1 shows the numerical values of the threshold parameters at two-loop precision for different values of $\alpha$ and $\beta$, with the $\lambda_i$’s fixed at their central values as given by Eqs. (4.9) and (4.10) above. The second line of the table gives the two-loop threshold parameters for
Table 1: Threshold parameters for $F_\pi = 92.4$ MeV, in units of $M_{\pi^+} = 139.57$ MeV for different values of $\alpha$ and of $\beta$. 

| $\alpha$ | $\beta$ | $a_0^0$ | $b_0^0$ | $-10a_0^0$ | $-10b_0^0$ |
|----------|---------|---------|---------|------------|------------|
| 1.04     | 1.08    | 0.210   | 0.26    | 0.43       | 0.78       |
| 2        | 1.127   | 0.264   | 0.27    | 0.32       | 0.83       |
|          | 1.071   | 0.255   | 0.25    | 0.29       | 0.78       |
|          | 1.016   | 0.246   | 0.24    | 0.26       | 0.74       |
| 2.16     | 1.074   | 0.263   | 0.25    | 0.27       | 0.79       |
| 2.5      | 1.085   | 0.283   | 0.25    | 0.22       | 0.80       |
| 3.0      | 1.102   | 0.311   | 0.25    | 0.16       | 0.82       |
| 3.5      | 1.122   | 0.342   | 0.25    | 0.10       | 0.85       |
| Error bars | ±0.006 | ±0.02   | ±0.02   | ±0.06      |            |

| Experiment [24] | 0.26±0.05 | 0.25±0.03 | 0.28±0.12 | 0.82±0.08 |

| $\alpha$ | $\beta$ | $10a_1^1$ | $10^2b_1^1$ | $10^2a_2^0$ | $10^3a_2^2$ | $10^4a_3^1$ |
|----------|---------|-----------|------------|------------|------------|-----------|
| 1.04     | 1.08    | 0.37      | 0.60       | 0.17       | 0.08       | 0.56      |
| 2        | 1.127   | 0.39      | 0.55       | 0.17       | 0.13       | 0.65      |
|          | 1.071   | 0.37      | 0.55       | 0.17       | 0.14       | 0.63      |
|          | 1.016   | 0.35      | 0.55       | 0.18       | 0.14       | 0.62      |
| 2.16     | 1.074   | 0.37      | 0.54       | 0.17       | 0.15       | 0.65      |
| 2.5      | 1.085   | 0.37      | 0.53       | 0.17       | 0.17       | 0.68      |
| 3        | 1.102   | 0.38      | 0.50       | 0.17       | 0.20       | 0.74      |
| 3.5      | 1.122   | 0.38      | 0.48       | 0.17       | 0.23       | 0.80      |
| Error bars | ±0.01   | ±0.15     | ±0.05      | ±0.28      | ±0.18      |           |

| Experiment [24] | 0.38±0.02 | 0.17±0.03 | 0.13±0.30 | 0.6±0.2 |

Table 1: Threshold parameters for $F_\pi = 92.4$ MeV, in units of $M_{\pi^+} = 139.57$ MeV for different values of $\alpha$ and of $\beta$. 

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the values of $\alpha$ and $\beta$ corresponding to the $O(p^4)$ predictions \((3.28)\) of the standard case \(^5\) within the $SU(3)_L \times SU(3)_R$ analysis, while the fourth line corresponds to the central values \((4.13)\), obtained from the fit of the two-loop phases to the Rosselet data cum Universal Curve constraint. For the remaining entries, we have, for each value of $\alpha$, chosen for $\beta$ the central value given by the Universal Curve \((4.14)\), except for $\alpha = 2$, where we have also displayed the results corresponding to the maximal and minimal values of $\beta$ as allowed by \((4.14)\). This variation of $\beta$ at fixed $\alpha$ only affects the values of the S- and P-wave scattering lengths and the S-wave slopes, which are the only threshold parameters to depend on $\beta$ already at leading order. The effect is strongest in $a_0^2$ and is of the order of about $\pm 10\%$ in average. The S-wave scattering lengths $a_0^0$ and $a_0^2$ are clearly the most sensitive to variations in the value of $\alpha$. The penultimate line of Table 1 gives the error bars (which are independent on both $\alpha$ and $\beta$ at the level of numerical accuracy we consider here) on the threshold parameters. They only include the effects of varying the values of the $\lambda_i$’s within their ranges as given by Eqs. \((4.9)\) and \((4.10)\). The last line gives the values, as quoted in \([24]\), obtained from Roy-equation analyses of the $K_{l4}$ and production data.

Figure 7 shows the S-wave scattering lengths as functions of $\alpha$. The shaded area gives the uncertainties coming from the errors on the $\lambda_i$’s, as given by Eqs. \((4.9)\) and \((4.10)\), whereas the band delimited by the solid lines includes the error on the $\alpha$-dependence of $\beta$ inferred from the Universal Curve \((4.14)\). The present experimental value $a_0^0 = 0.26 \pm 0.05$ allows $\alpha$ to lie between 1 and 3.4. In this whole range of $\alpha$, corrections to $a_0^0$ represent a 25% increase with respect to tree level, and the two-loop corrections are about 5% of the one-loop result. (see also the discussion at the end of Appendix D). The observed convergence rate leads one to expect that higher order effects should be within the error bars quoted in Table 1. In addition, we show (Fig. 7) the combination $|a_0^0 - a_0^2|$ which can be extracted from the measurement of the $\pi^+\pi^-$-atom lifetime planned at CERN \([10]\).

5 Testing the Strength of Quark Condensation

In Section 4.2 it has been shown how the two-loop expression for the scattering amplitude can be exploited to extract values of the parameters $\alpha$ and $\beta$ from available low-energy data. The lack of precision in the result, viz. $\alpha = 2.16 \pm 0.86$, merely reflects the large error bars in currently existing data, rather than the uncertainty attached to the parameters $\lambda_1, \ldots, \lambda_4$. More accurate data are awaited, and with them a rather precise measurement of $\alpha$ and $\beta$ should become possible. We have already stressed the key role played by the parameter $\alpha$ in the measurement of the amount of quark condensation in the QCD vacuum: In the standard, large...
condensate alternative of symmetry breaking, \( \alpha \) should remain close to 1 (cf. Eqs. (3.28) and (3.33)), whereas the case of a marginal quark condensation is characterised by a considerably higher value, \( \alpha \sim 3 - 4 \). In both cases, \( \beta \) should stay close to 1.

In this section, we attempt a more quantitative interpretation of the low-energy constants \( \alpha \) and \( \beta \), relating them to fundamental QCD parameters such as the quark mass ratio \( r = m_s/\hat{m} \) and the condensate parameter \( 2\hat{m}B_0 \). To leading order, this relation has been given before [8, 9]. The complexity of this relationship rapidly increases with increasing chiral order to which \( \alpha \) and \( \beta \) are expanded. Here, we restrict our discussion to one-loop order \( O(p^4) \), using the corresponding perturbative expressions (2.17) and (2.18) for \( \beta \) and \( \alpha \). Let us stress that this is a perfectly consistent procedure: It is conceivable that two-loop accuracy in the expansion of the amplitude is important for a precise measurement of \( \alpha \) and \( \beta \) using, in addition to other input, energy-dependent phase shift data above threshold. On the other hand, to make use of the results of such a measurement, one-loop accuracy in the expansion of \( \alpha \) and \( \beta \) in powers of quark masses could be sufficient to distinguish a value of \( r \sim 25 \) from, say, \( r \sim 10 \), or the case \( 2\hat{m}B_0/M^2 \sim 1 \) from \( 2\hat{m}B_0/M^2 \sim 0 \).

### 5.1 Dimensional Analysis and Order of Magnitude Estimates

Before proceeding to a more refined analysis, it might be useful to recall some crude and simple order of magnitude estimates of various contributions to \( \alpha \) and \( \beta \) that are exhibited in Eqs. (2.17) and (2.18). A constant of \( \mathcal{L}^{\text{eff}} \) multiplying an invariant that is made up from \( n \) covariant derivatives and \( m \) powers of the quark mass is, for \( n + m \geq 2 \), expected to be of order \( F_0^2\Lambda^2_H^{-n-m} \), unless it is suppressed by the Zweig rule. Indeed, this is the expected order of magnitude of a (massless) QCD correlation function that consists of \( n \) vector and/or axial-vector and of \( m \) scalar and/or pseudoscalar current densities, at low external momenta, with the singularities arising from exchanges of Goldstone bosons subtracted. This estimate is obtained by saturating correlation functions that are smooth enough at short distances by the lowest massive bound states \( (\rho, a_1, f_0, \ldots) \), which have masses of the order of \( \Lambda_H \sim 1 \text{ GeV} \). \( F_0 \) is a scale characterizing the coupling of mesons \( M \) to currents \( J: \left\langle 0 | J | M \right\rangle \sim F_0 \Lambda_H \). This argument is a special case of the “naive dimensional analysis” of Ref. [43], except that we leave open the question as to whether it may be extended to the “one-point function” \( (n = 0, m = 1) \) representing the quark condensate. The reason for this restriction is quark confinement, which prevents us from analysing the quark condensate by the same dispersion techniques as in the case of \( (n + m) \)-point functions for \( n + m \geq 2 \).

Factoring out \( F_0^2 \) from all terms of \( \mathcal{L}^{\text{eff}} \), cf. Eqs. (2.8), (2.10), (2.12) and (2.13), the \( \hat{\mathcal{L}}^{(2)} \) constants \( A_0 \) and \( Z_0^p \) are two-point functions divided by \( F_0^2 \), and consequently they are expected to be of the order of 1. (A more detailed estimate, using QCD sum rules, gives \( A_0 \sim 3 \pm 2 \).) Similarly, the constants \( \xi, \rho_1, \rho_2 \) are expected to be of the order of \( 1/\Lambda_H \), whereas the \( \hat{\mathcal{L}}^{(4)} \) constants \( A_1, A_2, A_3, B_1, B_2, E_1, E_2, E_3 \) should be of the order \( 1/\Lambda_H^2 \). The constants \( Z_0^g, \tilde{\xi}, \rho_4, \)
$\rho_5$ and $\rho_7$ violate the Zweig rule, and they are therefore expected to be suppressed compared to the corresponding dimensional estimates.

It should be stressed that for $n + m \geq 2$ these estimates are independent of any chiral counting or of the importance of the quark condensate. On the other hand, attempts to estimate the value of the $u$ and $d$ running quark masses especially $\hat{m} = (m_u + m_d)/2$, depend in one way or another on the presumed mechanisms of chiral symmetry breakdown in QCD. A model-independent evaluation of corresponding QCD sum rules is at present problematic, due to the complete absence of experimental information on the size of the spectral function of the divergence of the axial current beyond the one-pion contribution. The existing evaluations of $\hat{m}$ usually normalise [44, 45] the unknown spectral function by its low-energy behaviour as predicted by the standard $\chi$PT. However, in generalised $\chi$PT the latter prediction is modified by an enhancement factor which can be as large as 13.5, depending on the actual importance of the quark condensate [46, 17]. The issue can and should be settled experimentally, by measuring the magnitude of the divergence of the axial current in hadronic $\tau$ decays [46]. Fortunately, the existing determinations of the quark mass difference $m_s - m_u$ (see e.g. the recent analyses in [17, 48, 49], from which earlier references can be traced back) are on less speculative grounds. They are based on sum rules which involve the two-point function of the divergence of the vector current $\bar{s}\gamma_\mu u$; the corresponding spectral function can be normalised using experimental data in a rather model-independent way. For definiteness, we shall adopt the corresponding result of Ref. [17], viz. $m_s - m_u = (184 \pm 32)$ MeV, which is compatible, within the error bars, with the values obtained by the authors of Refs. [48] and [49]. For the sake of our order of magnitude estimates we shall thus use for $\hat{m}$ a value which depends on the ratio $r = m_s/\hat{m}$ in the following way

$$\hat{m} = \frac{(184 \pm 32)}{r - 1} \text{ MeV} , \quad (5.1)$$

keeping in mind that in generalised $\chi$PT, $r$ is not fixed from the outset. Even for $r$ as small as $r \sim 8$, the corresponding value of $\hat{m}$ would be

$$\hat{m} \sim (26 \pm 4.6) \text{ MeV} , \quad (5.2)$$

which would keep the expansion parameter $\hat{m}/\Lambda_H$ reasonably small, although somewhat larger than in the standard $\chi$PT. Given the above value for $m_s - m_u$, Eq. (5.2) represents the upper bound for $\hat{m}$ in $G\chi$PT.

We are now in a position to estimate orders of magnitude of various terms contributing to $\alpha$, dividing Eq. (2.18) by $M_\pi^2$. For definiteness, we set $\Lambda_H \simeq M_\rho$ and we first consider the low $B_0$ alternative, taking, e.g., $r \simeq 10$. In this case, both terms $2\hat{m}B_0/M_\pi^2$ and $16\hat{m}^2A_0/M_\pi^2$ should

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6 All QCD running quantities will be understood to be normalised at the scale 1 Gev $\sim \Lambda_H$. The combinations of constants times powers of quark masses appearing in the expansions of $\alpha$ and $\beta$ are QCD renormalisation group invariant.
be of order unity. The $\tilde{L}^{(3)}$ contributions $4\hat{m}\xi$ and $81\hat{m}^3\rho_1/M^2_\pi$ should both reach the ten-percent level, whereas the $L_{(2,2)}$ terms $\hat{m}^2B_1$, $\hat{m}^2A_3$ and the $L_{(0,4)}$ terms, including their large coefficients – viz., $256\hat{m}^4E_1/M^2_\pi$ – should hardly reach the one-percent level. Chiral logarithms and the Zweig rule violating, (explicitly) $r$-dependent, terms will be discussed shortly.

In the standard large $B_0$ alternative ($r \sim 26$), the whole expression (2.18) is dominated by the Gell-Mann–Oakes–Renner (GOR) ratio $2\hat{m}B_0/M^2_\pi$, which remains close to 1. All remaining terms should contribute at most at the one-percent level, reflecting the smaller value of $\hat{m} \sim 6$ MeV.

5.2 Constraints from Pseudoscalar Meson Masses and Decay Constants

More information on the low-energy constants contained in $\tilde{L}^{(2)}$ and $\tilde{L}^{(3)}$ can be obtained from the expansion of the Goldstone boson masses and decay constants $F_\pi$ and $F_K$. These expansions, up to and including order $O(p^4)$ contributions in generalised $\chi$PT, are collected in Appendix A. To leading, $O(p^2)$ order, $\hat{m}^2A_0$ and $\hat{m}^2Z^P_0$ can be expressed as functions of $r = m_s/\hat{m}$ and of pseudoscalar meson masses:

$$\frac{4\hat{m}^2A_0}{M^2_\pi} = 2 \frac{r_2 - r}{r^2 - 1} \equiv \epsilon(r)$$

$$\frac{8\hat{m}^2Z^P_0}{M^2_\pi} = -\epsilon(r) + \frac{\Delta_{GMO}}{(r-1)^2},$$

where (using $M_\pi = 135$ MeV, $M_K = 495.7$ MeV, $M_\eta = 547.5$ MeV)

$$r_2 = 2 \frac{M^2_K}{M^2_\pi} - 1 = 26$$

$$\Delta_{GMO} = (3M^2_\eta + M^2_\pi - 4M^2_K)/M^2_\pi = -3.6.$$  

The ratio $2\hat{m}B_0/M^2_\pi$ depends, in addition, on the Zweig rule violating constant $Z^S_0$:

$$\frac{2\hat{m}B_0}{M^2_\pi} = 1 - \epsilon(r)[1 + (r + 2)\zeta],$$

where

$$\zeta \equiv \frac{Z^S_0}{A_0} = \frac{L_6}{L_8}$$

is expected to be small. For $r = r_2 + O(m_{\text{quark}}/\Lambda_H)$, the ratio (5.5) is close to 1, whereas the constants (5.3) are relegated to higher orders, provided, of course, that $\Delta_{GMO} = O(m_{\text{quark}}/\Lambda_H)$.  

33
This is the standard scenario, in which the quark condensate is large enough to dominate the expansion of Goldstone boson masses as well as the expansion (2.18) of the parameter $\alpha$. As $r$ decreases, the ratio (5.3) becomes smaller than 1 and (neglecting $\zeta$) it vanishes for

$$r = r_1 = 2 \frac{M_K}{M_\pi} - 1 = 6.3.$$ (5.7)

Notice that $r$ cannot be smaller than this critical value, since vacuum stability does not allow the ratio (5.7) to be negative. For $r < r_2$, the ratio $\epsilon(r)$ (5.3) gradually rises towards 1, in agreement with the dimensional analysis mentioned above: Assuming the quark mass $\hat{m}$ to be given by Eq. (5.1), and taking, for instance, $r = 10$, Eq. (5.3) implies $\epsilon \simeq 0.3$ and $A_0 \simeq 3.4$. In generalised $\chi$PT, the leading order contribution to $\alpha$ is obtained by summing up the first three terms in the expression (2.18):

$$\alpha^{(0)} = 1 + 6 \frac{r_2 - r}{r^2 - 1} (1 + 2\zeta),$$ (5.8)

whereas

$$\beta^{(0)} = 1.$$ (5.9)

Hence, the quark mass ratio $r$, instead of being determined by pseudoscalar meson masses, may be used as a measure of the amount of quark condensation. It determines the leading order value of the parameter $\alpha$, which lies in the range $1 \lesssim \alpha^{(0)} \lesssim 4$.

Let us now consider the constants of $\mathcal{L}^{(3)}$ characterizing the next to the leading order $O(p^3)$. To that order, the constant $\hat{m}\xi$ can be inferred from the splitting of the decay constants $F_K$ and $F_\pi$, cf. Appendix A (we take the determination $F_K/F_\pi = 1.22 \pm 0.01$ from Ref. [50]),

$$\hat{m}\xi = \frac{1}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) = \frac{0.488 \pm 0.024}{r - 1}.$$ (5.10)

This value agrees with the dimensional analysis estimate within a factor of 2. The next to the leading order contribution to $\beta$ is then

$$\beta = 1 + \frac{2}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) \left( 1 + 2\hat{\xi}/\xi \right) + \ldots$$ (5.11)

The $O(p^3)$ corrections affect the leading order formulae (5.3), (5.3) and (5.8). The most important effect arises precisely from the splitting $F_K \neq F_\pi$. It amounts to replacing $\epsilon(r)$ everywhere by $\epsilon^*(r)$, where

$$\epsilon^*(r) = 2 \frac{r_2^* - r}{r^2 - 1}, \quad r_2^* = 2 \left( \frac{F_K M_K}{F_\pi M_\pi} \right)^2 - 1.$$ (5.12)

The relative importance of this correction is greatest for $r \sim r_2$ ($r_2 = 26$, whereas $r_2^* = 39$). In the low $B_0$ alternative, this $\mathcal{L}_{(2,1)}$ correction shows up mainly as a slight increase in the critical
value of \( r \) for which the quark condensate vanishes: This critical values moves from \( r = r_1 \simeq 6.3 \) towards \( r = r_1^* \), with

\[
 r_1^* = 2 \frac{F_K M_K}{F_\pi M_\pi} - 1 \simeq 8 .
\] (5.13)

Finally, we discuss the \( \mathcal{L}_{(0,3)} \) parameters \( \rho_i \), especially \( \rho_1 \) and \( \rho_2 \) which are not suppressed by the Zweig rule. The magnitude of one of them – the parameter \( \rho_2 \) – can be inferred from isospin breaking effects induced by \( m_d \neq m_u \). It has been pointed out in [8] that (in the absence of electromagnetism) there exists a particular linear combination of the \( K^+, K^0 \) and \( \pi^+ \) masses, namely

\[
\Delta(r, R) = M_K^2 - M_\pi^2 - R \Delta M_K^2 + (M_K^2 - \frac{r + 1}{2} M_\pi^2) \frac{r - 1}{r + 1} ,
\] (5.14)

where

\[
R = \frac{m_s - m}{m_d - m_u} , \quad \Delta M_K^2 = (M_{K^0}^2 - M_{K^+}^2)_{QCD} ,
\] (5.15)

which has remarkable properties: Neglecting terms quadratic in \( m_d - m_u \), all contributions to \( \Delta(r, R) \) of the type \( m_{\text{quark}} B_0 \), \( m_{\text{quark}}^2 \) and even \( m_{\text{quark}} \ln m_{\text{quark}} \) cancel out. Consequently, in the standard \( \chi \)PT, \( \Delta(r, R) \) vanishes at leading order; however, it receives \( O(p^3) \) contributions from \( \tilde{\mathcal{L}}^{(3)} \). In both cases,

\[
\Delta(r, R) \big|_{m_d = m_u} = \left( \frac{F_K^2}{F_\pi^2} - 1 \right) \left( M_K^2 - \frac{r + 1}{2} M_\pi^2 \right) + (r - 1)^2 (r + 1) \tilde{m}^3 \rho_2 + O(m_{\text{quark}}^3 B_0, m_{\text{quark}}^4) .
\] (5.16)

Let us write, as usual,

\[
\Delta M_K^2 = (M_{K^0}^2 - M_{K^+}^2)_{\text{exp}} + \gamma (M_{\pi^0}^2 - M_{\pi^+}^2) ,
\] (5.17)

where the coefficient \( \gamma - 1 \) measures the violation of Dashen’s theorem. We shall take

\[
\gamma = 1.8 \pm 0.4 ,
\] (5.18)

based on the recent estimates [51]. Taking for the ratio \( R \) the value [52]

\[
R = 43.7 \pm 2.7
\] (5.19)

inferred from baryon mass splittings and from \( \omega - \rho \) interference, one obtains

\[
\frac{R \Delta M_K^2}{M_\pi^2} = 14.4 \pm 2.1 .
\] (5.20)

Upon inserting this value into Eq. (5.16), one finds that the central value of \( \rho_2 \) is largely dominated by the uncertainty: For \( r < 15 \), \( \rho_2 \) is compatible with zero; more precisely,

\[
\frac{\tilde{m}^3 \rho_2}{M_\pi^2} \simeq -0.6 \pm 2.1 \quad \frac{1}{(r - 1)^2 (r + 1)}.
\] (5.21)
In comparison, for \( r = 25 \), the above numerator would be \(-2.9 \pm 2.1\). It is interesting to notice that the dimensional analysis of Section 5.1 gives (the error bars come from the uncertainty on \( m_s \))

\[
\left| \frac{\tilde{m}^3 \rho_i}{M^2_{\pi}} \right|_{DA} = 0.4 \pm 0.2 \left( r - 1 \right)^3, \quad i = 1, 2.
\] (5.22)

The constants \( \rho_1 \) and \( \rho_2 \) also describe the leading \( SU(3)_\chi \)-breaking effects in two-point functions of the scalar and pseudoscalar densities, which might lead to additional information. We have however not performed a complete analysis along these directions so far. In the sequel, we shall use the estimate (5.22), since it is compatible with the more direct determination from \( \Delta(r, R) \) in the case of \( \rho_2 \).

5.3 \( \alpha \) and \( \beta \) as Functions of the Quark Mass Ratio \( r = m_s/\hat{m} \)

Up to now, nothing has been said about chiral logarithms. In generalised \( \chi \)PT, they start to contribute at the order \( O(p^4) \), i.e. in the expansions of \( \alpha \) and of \( \beta \), they first appear in the next-to-next-to leading order corrections \( \alpha^{(2)} \) and \( \beta^{(2)} \). As already pointed out, one particular feature of generalised \( \chi \)PT is that \( O(p^4) \) loops renormalise the \( O(p^2) \) constants \( A_0 \) and \( Z^S_0 \) by higher order terms proportional to \( B^2_0 \) (remember that \( B_0 \) counts as \( O(p) \)). Similarly, the constants \( \xi, \tilde{\xi} \) and \( \rho_i \) of \( \tilde{L}^{(3)} \) are also subject to \( O(p^4) \) renormalisations, proportional to \( B_0 \). The constants of \( \tilde{L}^{(2)} \) and of \( \tilde{L}^{(3)} \), which appear in the expressions (2.18) and (2.17) for \( \alpha^r \) and for \( \beta^r \) (or in the formulae of Appendix A), are the renormalised constants at scale \( \mu \), with the \( O(p^4) \) \( B_0 \)-dependent counterterms exhibited in Eq. (2.11) included. For a quantitative discussion of \( O(p^4) \) effects, it is useful to explicitly split off these \( \mu \)-dependent parts from the constants that are dominantly of order \( O(p^2) \) or \( O(p^3) \). This separation is needed in particular when dealing with the Zweig rule violating constants \( Z^S_0, \tilde{\xi}, \rho_4, \rho_5 \) and \( \rho_7 \): While it makes perfect sense to declare these constants to be small at their respective \( O(p^2) \) and \( O(p^3) \) tree levels, they will, however, be generated at the \( O(p^4) \) accuracy by Zweig rule violating chiral loops. In what follows, we shall assume that there exists a scale \( \Lambda \) (not related to the renormalisation scale \( \mu \)), at which these constants vanish. Upon using the coefficients \( \Gamma_X \), Eq. (2.15), of the corresponding beta functions \(^7\), this assumption amounts to writing

\[
\frac{4\tilde{m}^2 Z^S_0}{M^2_{\pi}} = -\frac{11 M^2}{288 \pi^2 F^2_{\pi}} x^2 \ln \frac{\mu^2}{\Lambda^2}.
\]

\[
\frac{2\tilde{m}\tilde{\xi}}{M^2_{\pi}} = -\frac{M^2}{32 \pi^2 F^2_{\pi}} x \ln \frac{\mu^2}{\Lambda^2}.
\]

\[
\frac{\tilde{m}^3 \rho_1}{M^2_{\pi}} = -\frac{M^2}{192 \pi^2 F^2_{\pi}} x \left( \frac{11y}{3} - 2z \right) \ln \frac{\mu^2}{\Lambda^2}.
\]

\(^7\) For \( Z^S_0 \) and for \( \tilde{\xi} \), they can be inferred from the renormalisations of the constants \( L_6 \) and \( L_4 \), respectively, as given in Ref. [3], whereas for the \( \rho_i \)’s, see Ref. [28].
\[ \frac{\hat{m}^3 \rho_5}{M^2_\pi} = -\frac{M^2_\pi}{64\pi^2 F^2_\pi} x \left( y - \frac{4z}{3} \right) \ln \frac{\mu^2}{\Lambda^2} \tag{5.23} \]
\[ \frac{\hat{m}^3 \rho_7}{M^2_\pi} = -\frac{M^2_\pi}{576\pi^2 F^2_\pi} x z \ln \frac{\mu^2}{\Lambda^2}, \]
where we have introduced the notation
\[ x = \frac{2\hat{m}^2 F_0^2 B_0}{F^2_\pi M^2_\pi}, \quad y = \frac{4\hat{m}^2 F_0^2 A_0(\Lambda)}{F^2_\pi M^2_\pi}, \quad z = \frac{4\hat{m}^2 F_0^2 Z_0^\prime}{F^2_\pi M^2_\pi}. \tag{5.24} \]

The vanishing at a common scale \( \Lambda \) of all the \( \hat{\mathcal{L}}^{(2)} \) and \( \hat{\mathcal{L}}^{(3)} \) Zweig-rule violating constants is by no means obvious. Here it is assumed mainly for reasons of simplicity. An experimental determination of the sizes of \( Z_0^\delta(\Lambda) \) and of \( \xi(\Lambda) \) is in principle possible from a simultaneous analysis of \( K_{\mu4} \) form factors and of the low-energy \( \gamma\gamma \to \pi^0\pi^0 \) cross-section, when more precise data on these quantities become available [1].

It is now rather natural to use the scale \( \Lambda \) in order to separate the induced \( O(p^4) \) components of the constants \( A_0(\mu) \) and \( \xi(\mu) \). In terms of the notation introduced in Eq. (5.24), one has
\[ \frac{4\hat{m}^2 F_0^2 A_0(\mu)}{F^2_\pi M^2_\pi} = y - \frac{5M^2_\pi}{96\pi^2 62 F^2_\pi} x^2 \ln \frac{\mu^2}{\Lambda^2} \tag{5.25} \]
\[ 2\hat{m} \xi = 2\hat{m} \xi(\Lambda) - \frac{3M^2_\pi}{32\pi^2 F^2_\pi} x \ln \frac{\mu^2}{\Lambda^2}. \]

Similarly, the constants \( \rho_1(\mu) \) and \( \rho_2(\mu) \) of \( \hat{\mathcal{L}}^{(3)} \) which are not suppressed by the Zweig rule may be decomposed as
\[ \frac{\hat{m}^3 F_0^2 \rho_1}{F^2_\pi M^2_\pi} = \hat{\rho}_1 - \frac{M^2_\pi}{64\pi^2 F^2_\pi} \left( \frac{xy}{6} + xz \right) \ln \frac{\mu^2}{\Lambda^2} \tag{5.26} \]
\[ \frac{\hat{m}^3 F_0^2 \rho_2}{F^2_\pi M^2_\pi} = \hat{\rho}_2 - \frac{M^2_\pi}{64\pi^2 F^2_\pi} \left( \frac{xy}{6} - 3xz \right) \ln \frac{\mu^2}{\Lambda^2}, \]
where
\[ \hat{\rho}_i = \frac{\hat{m}^3 F_0^2 \rho_i(\Lambda)}{F^2_\pi M^2_\pi}, \quad i = 1, 2. \tag{5.27} \]

We are now in a position to reexpress various contributions to \( \alpha \) and \( \beta \) in terms of observables, including \( O(p^4) \) accuracy. From the expressions of \( F^2_\pi/F_0^2 \) and \( F^2_\pi/F_0^2 \) given in Appendix A, one obtains
\[ 2\hat{m} \xi(\Lambda) = \frac{2}{r-1} \left( \frac{F^2_K}{F^2_\pi} - 1 \right) + \frac{4}{(r-1)^2} \left( \frac{F^2_K/F^2_\pi - 1}{r-1} \right)^2 - \frac{4\hat{m}^2}{r-1} \left[ \delta^{(2,2)}_{F,K} - \delta^{(2,2)}_{F,\pi} \right] \]
\[ + \frac{3M^2_\pi}{32\pi^2 F^2_\pi} x \ln(\mu^2/\Lambda^2) + \frac{1}{r-1} (-5\mu_\pi + 2\mu_K + 3\mu_\eta) + O(p^3), \tag{5.28} \]

\[ \textsuperscript{8} \text{An attempt to estimate the possible effects of non-vanishing } Z_0^\delta \text{ and } \hat{\xi} \text{ has been made in a preliminary version of this work [26]. See in particular Fig. 1 in this reference.} \]
where $\delta^{(2)}_{F,K} - \delta^{(2)}_{F,\pi}$ collects all $O(p^4)$ tree contributions arising from $L_{(2,2)}$. Of course, the entire expression (5.28) is $\mu$-independent. All results collected so far in this Section can now be used to reexpress the full parameter $\alpha$, i.e. $\alpha^r(\mu)$ as given by Eq. (2.18), plus the logarithmic unitarity corrections displayed in Eq. (3.23), as follows:

$$
\alpha = x + 4y - \frac{4}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) + 81 \hat{\rho} + \hat{\rho}_2 + \Delta_\alpha(\mu) + \omega_\alpha(x, y, r) + O(p^3).
$$

(5.29)

Here, $\Delta_\alpha(\mu)$, which is of order $O(p^2)$, collects all the tree contributions arising from $L_{(2,2)}$ and $L_{(0,4)}$. It can be read off from Eqs. (2.18), (5.28) and the formulae collected in Appendix A; however, its explicit expression would be of little use here. $\omega_\alpha(x, y, r)$ involves all the logarithms arising both from tadpoles and from the unitarity corrections:

$$
\omega_\alpha(x, y, r) = - \frac{M^2}{192\pi^2 F_\pi^2} \left[ \frac{1}{3} (1945 + 352r + 31r^2) xy 
+ (10 - 4r^2) xz + 2 \left( 148 + 11r \right) x^2 - 24x \right] \ln(\mu^2/\Lambda^2)
+ \mu_\pi \left[ \frac{10}{r - 1} - 3 - 29y \right] - 2\mu_K \left[ \frac{2}{r - 1} + 1 + (r + 7)y \right]
- \frac{1}{3}\mu_\eta \left[ \frac{18}{r - 1} + 1 + 15y - 4(r - 4)z \right]
- \frac{M^2}{32\pi^2 F_\pi^2} (1 + 22y + 33y^2) \ln(M^2_\pi/\mu^2) + 1
- \frac{M^2}{32\pi^2 F_\pi^2} [1 + (r + 1)y] [1 + 3(r + 1)y] \ln(M^2_K/\mu^2) + 1
- \frac{M^2}{32\pi^2 F_\pi^2} \left[ 1 + (2r + 1)y - \frac{2}{r - 1} \hat{\Delta}_{GMO} \right]^2 \ln(M^2_\eta/\mu^2) + 1 \right] .
$$

(5.30)

Similarly, the full parameter $\beta$, i.e. $\beta^r$ of Eq. (2.17), plus unitarity corrections given in Eq. (3.24), reads

$$
\beta = 1 + \frac{2}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) + \beta^{(2)}_{\text{tree}}(\mu) + \beta^{(2)}_{\text{loop}}(\mu) + O(p^3) .
$$

(5.31)

Here again, $\beta^{(2)}_{\text{tree}}(\mu)$, which is of order $O(p^2)$, collects all the tree contributions from $L_{(2,2)}$, whereas $\beta^{(2)}_{\text{loop}}(\mu)$ contains the logarithms:

$$
\beta^{(2)}_{\text{loop}}(\mu) = - \frac{M^2}{16\pi^2 F_\pi^2} x \ln(\mu^2/\Lambda^2)
+ \frac{1}{r - 1} (-5\mu_\pi + 2\mu_K + 3\mu_\eta)
- \frac{M^2}{16\pi^2 F_\pi^2} (2 + 5y) \ln(M^2_\pi/\mu^2) + 1
- \frac{M^2}{32\pi^2 F_\pi^2} [1 + (1 + r)y] \ln(M^2_K/\mu^2) + 1 .
$$

(5.32)
The parameters $x$ and $y$ \([5.24]\) satisfy the following two equations, which may be obtained by multiplying the expansion formulae for $F_\pi^2 M_\pi^2$ and $F_K^2 M_K^2$ (see App. A) by the factor $F_\pi^2 / F_\pi^2 M_\pi^2$:

$$x + y = 1 - 9\bar{\rho}_1 - \bar{\rho}_2 + \Delta_\pi(\mu) + \omega_\pi(x, y, m r),$$

\[x + \frac{r + 1}{2} y = \frac{2}{r + 1} \left( \frac{F_K M_K}{F_\pi M_\pi} \right)^2 - 3(r^2 + r + 1)\bar{\rho}_1 - (r^2 - r + 1)\bar{\rho}_2 + \Delta_K(\mu) + \omega_K(x, y, r). \tag{5.33}\]

As before, the $\mu$-dependence of the tree contributions $\Delta_{\pi,K}(\mu)$, which are $O(p^2)$, is compensated by the collections $\omega_{\pi,K}$ of all chiral logarithms, which read

$$\omega_\pi(x, y, r) = \frac{M_\pi^2}{576\pi^2 F_\pi^2} \times \left[ (74 + 22r)x^2 + (253 + 88r + 31r^2)xy - (42 + 12r^2)xz \right] \ln(\mu^2/\Lambda^2),$$

$$+ \mu_\pi(3x + 8y) + 2\mu_K[x + (r + 2)y] + \frac{1}{3}\mu_\eta[x + 4y + 4(r - 1)z]$$

$$\omega_K(x, y, r) = \frac{M_\pi^2}{192\pi^2 F_\pi^2} \times \left[ \frac{1}{3}(59 + 37r)x^2 + (52 + 45r + 27r^2)xy - (16 - 10r + 12r^2)xz \right] \ln(\mu^2/\Lambda^2)$$

$$+ \frac{3}{4}\mu_\pi[2x + (5 + r)y] + \frac{3}{2}\mu_K[2x + 3(r + 1)y]$$

$$+ \frac{1}{12}\mu_\eta[10x + (5 + 17r)y + 8(r - 1)z]. \tag{5.34}\]

In order to get the full expressions of the parameters $\alpha$ and $\beta$ as functions of $r$ up to and including the $O(\hat{m}^2)$ terms, it is sufficient to solve the system of equations \([5.33]\) for $x$ and $y$. This can be done perturbatively: $x, y$ and $z$ in the functions $\omega_\pi, \omega_K$ can be replaced by their leading order values $x_0, y_0, z_0$:

$$x_0 = 1 - y_0, \quad y_0 = \epsilon(r), \quad z_0 = -\frac{\epsilon(r)}{2} + \frac{\Delta_{\text{GMO}}}{2(r - 1)^2}, \tag{5.35}\$$

since they multiply terms of order $O(p^2)$. The final expression for $\alpha$ as a function of $r$ then reads

$$\alpha(r) = 1 + 3\epsilon^*(r) - \frac{4}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) + 18(2 - r)\bar{\rho}_1 - 6r\bar{\rho}_2 + \alpha^{(2)}(r) + O(p^3), \tag{5.36}\$$

where $\epsilon^*(r)$ is defined in Eq. \([5.12]\) and the $O(p^2)$ part $\alpha^{(2)}$ is given by $\alpha^{(2)} = \alpha^{(2)}_{\text{tree}} + \alpha^{(2)}_{\text{loop}}$, with

$$\alpha^{(2)}_{\text{tree}} = \frac{6}{r - 1}(\Delta_K - \Delta_\pi) + \Delta_\alpha + \Delta_\pi, \tag{5.37}\$$

$$\alpha^{(2)}_{\text{loop}} = \frac{6}{r - 1}[\omega_K(x_0, y_0, r) - \omega_\pi(x_0, y_0, r)] + \omega_\alpha(x_0, y_0, r) + \omega_\pi(x_0, y_0, r). \tag{5.38}\$$
The various contributions to $\alpha$ and $\beta$, displayed in Eqs. (5.36) and (5.31), are shown in Figures 8 and 9, respectively. The leading contribution $\alpha^{(0)}$ is given by Eq. (5.28), $(\beta^{(0)} = 1)$. The $O(p)$ correction $\alpha^{(1)}$ consists of terms proportional to $F_K/F_\pi - 1$ and of the $\hat{\rho}_1$ and $\hat{\rho}_2$ terms, which are enhanced by factors of $r$. Using the estimate (5.22), the uncertainty in $\alpha^{(0)} + \alpha^{(1)}$ induced by the latter terms is shown as the doubly-hatched area in Fig. 8a. Notice that $\beta^{(0)} + \beta^{(1)}$ (Fig. 9a) is free from such an uncertainty – cf. the first two terms in Eq. (5.31). Considering the $O(p^2)$ components $\alpha^{(2)}$ and $\beta^{(2)}$, two facts should be kept in mind. First, there is at present no direct way to determine the constants of $\mathcal{L}_{\alpha(2,2)}$ and $\mathcal{L}_{\alpha(0,4)}$ entering the corresponding tree contributions $\alpha^{(2)}_{\text{tree}}(\mu)$ and $\beta^{(2)}_{\text{tree}}(\mu)$. Next, the splitting of $\alpha^{(2)}$ and $\beta^{(2)}$ into the ($\mu$-dependent) tree and loop parts is a matter of convention. At present, the only available information on $\alpha^{(2)}$ and $\beta^{(2)}$ that is unambiguous concerns the variation of the loop contributions $\alpha^{(2)}_{\text{loop}}(\mu)$ and $\beta^{(2)}_{\text{loop}}(\mu)$ with the renormalisation scale $\mu$ (see Figs. 8b and 9b). We use this variation between $\mu = M_\eta$ and $\mu = 1$ GeV as an estimate of the size of the contributions of $\alpha^{(2)}$ and $\beta^{(2)}$. The hatched region of Figs 8a and 9a then shows the total $\alpha$ and $\beta$ respectively, with uncertainties arising from the $\rho$’s and from the $O(p^2)$ contributions added. (It should be noted that the final $\alpha(r)$ and $\beta(r)$ displayed in Figs. 1a and 1b of Ref. [26] have been obtained in a slightly different manner: the contributions $\alpha^{(2)}_{\text{loop}}(\mu)$ and $\beta^{(2)}_{\text{loop}}(\mu)$ have been simply added to $\alpha^{(0)} + \alpha^{(1)}$ and $\beta^{(0)} + \beta^{(1)}$ respectively. While in the case of $\alpha$ the two procedures practically coincide, the curve $\beta(r)$ shown in Fig. 1 of Ref. [26] should be considered as slightly overestimated.) The final uncertainties in $\alpha(r)$ and $\beta(r)$ could be reduced when an independent estimate of the constants from $\mathcal{L}_{\alpha(2,2)}$ and $\mathcal{L}_{\alpha(0,4)}$ becomes available.

5.4 The $SU(2)_L \times SU(2)_R$ GOR Ratio

The above expression for the parameters $\alpha$ and $\beta$ as functions of the quark mass ratio $r = m_s/\hat{m}$ is characteristic of the $SU(3)_L \times SU(3)_R$ $\chi$PT and it is subject to the corresponding uncertainty. One may ask whether there is a direct relation between $\alpha$ and $\beta$ and the condensate defined in the $SU(2)_L \times SU(2)_R$ symmetry limit, which would not be based on an expansion in $m_s$ of a quantity such as $M_K^2$. The existence of such a relationship can be indeed be inferred already from the standard $SU(2)_L \times SU(2)_R$ one-loop analysis of Gasser and Leutwyler [23]. In this case, the parameters $\alpha$ and $\beta$ are given by Eqs. (3.29) and (3.30) in terms of the constants $\bar{\ell}_3$ and $\bar{\ell}_4$. It turns out that $\bar{\ell}_3$ also measures the deviation from the $SU(2)_L \times SU(2)_R$ GOR relation [4]:

$$\frac{2\hat{m}\tilde{B}}{M_\pi^2} = 1 + \frac{M_\pi^2}{32\pi^2 F_\pi^2} \bar{\ell}_3 + O(M_\pi^4),$$  \hspace{1cm} (5.39)

where

$$\tilde{B} = - \lim_{m_u, m_d \to 0} \frac{\langle \Omega | \bar{u}u | \Omega \rangle}{F_\pi^2}. \hspace{1cm} (5.40)$$
Eliminating the constants $\bar{\ell}_3$ and $\bar{\ell}_4$, one obtains

$$\frac{2\bar{m}\bar{B}}{M^2_\pi} = 1 - \frac{\alpha - \beta}{3} + \frac{M^2_\pi}{32\pi^2 F^2_\pi} + O(M^4_\pi).$$  \hfill (5.41)$$

This formula clearly shows that the ratio on the left hand side should be indeed considerably less than 1, provided $\alpha$ is as large as indicated by the central value of existing data. However, Eq. (5.41) is based on the standard $\chi$PT and there is no need for it to hold if either $\alpha$ or $\beta$ differ from 1 by more than an amount of order $O(M^2_\pi/\Lambda^2_H)$. It is convenient to rewrite Eq. (5.41) in terms of the GOR ratio

$$x_{GOR} = -\frac{2\bar{m}}{F^2_\pi M^2_\pi} \lim_{m_{u,d} \to 0 \atop m_s \text{ fixed}} \langle \Omega | \bar{u}u | \Omega \rangle = \frac{2\bar{m}\bar{B}F^2_\pi}{F^2_\pi M^2_\pi},$$  \hfill (5.42)$$

where $\bar{F} = \lim_{m_{u,d} \to 0} F_\pi$. Using the standard $\chi$PT relation

$$\frac{F^2_\pi}{F^2} = \beta + \frac{M^2_\pi}{8\pi^2 F^2_\pi},$$  \hfill (5.43)$$

Eq. (5.41) can be reexpressed as

$$x_{GOR} = 2 - \frac{\alpha + 2\beta}{3} - \frac{3M^2_\pi}{32\pi^2 F^2_\pi}. \hfill (5.44)$$

Within the standard expansion, Eqs. (5.41) and (5.44) are strictly equivalent. It turns out, however, that Eq. (5.44) is of a more general validity: It remains true at the leading order of $G\chi$PT, i.e. irrespective of how much $\alpha$ does deviate from 1. Here, we are going to discuss the $O(p)$ and $O(p^2)$ corrections to Eq. (5.44).

The $SU(2)_L \times SU(2)_R$ condensate (5.40) can be obtained from Appendix A as follows

$$\frac{\bar{F}^2}{F^2_0} \bar{B} = \lim_{m_{u,d} \to 0 \atop m_s \text{ fixed}} \frac{F^2_\pi M^2_\pi}{2\bar{m}F^2_0}.$$  \hfill (5.45)$$

The result is given in the form of an expansion in powers of $m_s$,

$$\frac{\bar{F}^2}{F^2_0} \bar{B} = B_0 + 2m_s Z^S_0 + m^2_s \left( \rho_4 + \frac{1}{2} \rho_5 + 6 \rho_7 \right) + m^3_s \left( F^S_2 + F^S_3 + 4F^{SS}_5 + 2F^{SSS}_6 \right) - 2\bar{\mu}_K \left( B_0 + 2m_s A_0 + 6m_s Z^S_0 \right) - \frac{1}{3} \bar{\mu}_\eta \left( B_0 + 10m_s Z^S_0 - 8m_s Z^P_0 \right),$$  \hfill (5.46)$$

where $\bar{M}_K$ and $\bar{M}_\eta$ denote the kaon and eta masses in the $SU(2)_L \times SU(2)_R$ chiral limit, and

$$\bar{\mu}_{K,\eta} = \lim_{m_{u,d} \to 0 \atop m_s \text{ fixed}} \mu_{K,\eta} = \frac{\bar{M}_{K,\eta}}{32\pi^2 F^2} \ln \left( \frac{\bar{M}^2_{K,\eta}}{\mu^2} \right). \hfill (5.47)$$
The formula \((\ref{5.46})\) collects Zweig-rule violating contributions to the \(SU(2)_L \times SU(2)_R\) condensate \(\bar{B}\) which are induced by the non-vanishing strange quark mass. It includes, in particular, \(K\) and \(\eta\) loops, which, being proportional to a power of \(m_s\) (or of \(r\)), may be rather large and sensitive to the subtraction scale \(\mu\). This does not mean, however, that the difference \(F^2 \bar{B} - F^2_0 B_0\) of the two-massless flavour and three-massless flavour condensates should be expected to be large, but rather that the chiral logarithms alone can hardly be used here in order to estimate the size of this difference.

One may now proceed as follows. First, one subtracts from \(F^2_\pi M^2_\pi \alpha_r\) given by Eq. \((\ref{2.18})\) the expansion of \(4F^2_\pi M^2_\pi\) displayed in Appendix A. This eliminates \(\hat{m}^2 A_0\). Then, one replaces \(F^2_0 B_0\) by \(\bar{F}^2 \bar{B}\), using Eq. \((\ref{5.41})\). One finds that the dangerous Zweig-rule violating terms enhanced by higher powers of \(r\) cancel. Next, one eliminates \(\hat{m}(\xi + 2 \tilde{\xi})\) in favour of \(\beta r\), using Eq. \((\ref{2.17})\). Finally, one replaces \(\alpha r\) and \(\beta r\) by \(\alpha\) and \(\beta\), introducing the unitarity logarithms displayed in Eqs. \((3.23)\) and \((3.24)\). As usual (cf. the discussion in Section 11 of Ref. \([5]\)), the remainders of \(K\) and \(\eta\) loops are absorbed into the \(O(p^4)\) tree-level constants. In this way, one arrives at the desired relation

\[
x_{GOR} = 2 - \frac{\alpha + 2\beta}{3} + 15\hat{\rho}_1 - \hat{\rho}_2 + \Delta x(\mu) + x_{GOR}^{\text{loop}}(\mu). \tag{5.48}
\]

Here, \(\Delta x(\mu) = O(\hat{m}^2)\) collects all (redefined) tree contributions arising from \(\mathcal{L}_{(2,2)}\) and \(\mathcal{L}_{(0,4)}\). The \(O(p^2)\) pion-loop contribution reads

\[
x_{GOR}^{\text{loop}} = -\frac{M^2_\pi}{192\pi^2 F^2_\pi} \left(\frac{311y}{3} + 22z\right) \ln \frac{\mu^2}{\Lambda^2}
- \mu_\pi \left(17y + 11y^2\right) - \frac{M^2_\pi}{32\pi^2 F^2_\pi} \left(3 + 14y + 11y^2\right), \tag{5.49}
\]

where the parameter \(y\) may be replaced by its leading order expression in the \(SU(2)_L \times SU(2)_R\) expansion:

\[
y = -2z = 1 - x_{GOR}\text{[lead]} = \frac{\alpha + 2\beta}{3} - 1. \tag{5.50}
\]

Eq. \((\ref{5.48})\) gives the \(SU(2)_L \times SU(2)_R\) GOR ratio \((\ref{5.42})\) as a function of \((\alpha + \beta)/3\). It is seen that for \(\alpha\) and \(\beta\) close to 1, one recovers the standard \(\chi PT\) formula \((\ref{5.44})\) up to and including \(O(p^2)\) accuracy. Notice that, using the estimate \((\ref{5.22})\), the \(O(p)\) contribution \(15\hat{\rho}_1 - \hat{\rho}_2\) to \(x_{GOR}\) becomes negligible in the whole range of \(r\). Fig. 10a shows the GOR ratio \((\ref{5.48})\) with the \(O(p^2)\) uncertainty obtained from \(x_{GOR}^{\text{loop}}(\mu)\) as described at the end of Subsection 5.3.

6 Summary and Conclusions

In the present article, we have obtained the explicit expression of the \(\pi \pi\) scattering amplitude \(A(s|t, u)\) to two-loop accuracy in the chiral expansion. This constitutes our main result,
and it is contained in Eqs. (3.46), (3.47) and (3.49). At this order, the $\pi\pi$ amplitude is entirely determined by the general requirements of analyticity, crossing symmetry, unitarity and the Goldstone-boson nature of the pion, up to the six independent parameters $\alpha, \beta, \lambda_i, i = 1, 2, 3, 4$, which are not fixed by chiral symmetry.

For the four constants $\lambda_i$, we have established a set of sum rules which are evaluated using available information on $\pi\pi$ interaction at medium and high energies ($E \gtrsim 600$ MeV). Details about the derivation of these sum rules and the corresponding numerical analysis will be published separately. The resulting values of the $\lambda_i$'s are given in Eqs. (4.9) and (4.10). For $\lambda_1$ and $\lambda_2$, they are compatible with previous determinations, but with a much smaller error bar in the case of $\lambda_1$. The constants $\lambda_3$ and $\lambda_4$ are genuine $O(p^6)$ contributions. To the best of our knowledge, they have not been discussed before.

The most important improvement when going from one loop to two loops concerns unitarity: The latter is only satisfied in the perturbative sense by the amplitude $A(s|t, u)$. A study of the Argand plots for the lowest partial waves (cf. the partial wave $f_0^0$ in Figure 1) shows that violations of the unitarity bounds become important only at energies greater than 450 MeV. From the size of the two-loop corrections, we also expect that higher orders are under control. Typically, for $e.g.$ the S-wave scattering lengths, the one-loop corrections represent a 25% effect. The two-loop contributions, however, do not modify the one-loop result by more than 5%, independently of the value of $\alpha$. (At this level of accuracy, isospin-breaking effects due to $m_d \neq m_u$ and to electromagnetism could start to play a role. We intend to discuss these effects elsewhere.)

As far as the two remaining parameters, $\alpha$ and $\beta$, are concerned, let us stress once more the particular importance attached to the first one: $\alpha$ measures the amount of explicit chiral symmetry breaking in the $\pi\pi$ amplitude due to the non-vanishing of the light quark masses. As such, it is intimately correlated to the ratio $x = -2\hat{m}\langle 0|\bar{q}q|0\rangle/F_\pi^2M_\pi^2$ or equivalently, to the quark mass ratio $r = m_u/\hat{m}$. The commonly accepted picture that spontaneous breakdown of chiral symmetry results from a strong condensation of quark-antiquark pairs in the QCD vacuum requires that $x \sim 1$ (or $r \sim 25$), and it is only compatible with values of $\alpha$ and $\beta$ close to unity, as predicted by standard $\chi$PT. A sizeable deviation of $\alpha$ from 1, say $e.g.$ $\alpha \gtrsim 2$, even if not “a major earthquake” [53], would nevertheless shake our confidence in the validity of the standard lore. These connections between $\alpha$ and the ratios $x$ or $r$ are well established at leading, $O(p^2)$, and at next to leading, $O(p^3)$, orders, cf. Eqs. (5.48), (5.36) and (5.31) and Figs. 8, 9, and 10. Our estimates of $O(p^4)$ corrections suggest that these correlations do not suffer major changes even at this order. In the particular case of the standard $\chi$PT ($r \sim 25$), higher order corrections to $\alpha$ are under even tighter control: The large-condensate hypothesis can hardly be compatible with $\alpha > 1.2$. Thus, an accurate determination of $\alpha$ and $\beta$ would confirm or contradict the standard picture of the chiral structure of the QCD vacuum. Unfortunately, the result of a fit to the presently available $K_{l4}$ data [33] remains inconclusive in this respect. We
obtain
\[ \alpha = 2.16 \pm 0.86, \quad \beta = 1.074 \pm 0.053. \]

Clearly, additional information, which would make such fits more accurate is needed. This may be provided by new high statistics \( K_{l4} \) experiments, that could be done e.g., at DAΦNE, or by a precise measurement of the lifetime of \( \pi^+\pi^- \)-atoms at CERN.

It is, however, interesting to stress that the above fit result leads to values for the threshold parameters which are in perfect agreement with the results obtained some time ago from analyses of \( K_{l4} \) data based on the Roy equations. For instance, the fitted value of the I=0 S-wave scattering length, \( a_0^0 = 0.263 \pm 0.052 \), reproduces the result quoted in \cite{24}. Similar conclusions hold for the other threshold parameters, see Table 1. It seems thus quite reasonable to conclude that in the energy range accessible in \( K_{l4} \) decays, our two-loop chiral result contains all the relevant information on the \( \pi\pi \) interaction which is already encoded in the Roy equation (for instance, the constraints coming from the production data at higher energies are satisfactorily reproduced by our determination of the \( \lambda_i \)'s).

The fact that the two-loop expression of the \( \pi\pi \) scattering amplitude was obtained, in Section 3, in a rather direct and easy way from general S-matrix properties, and without any reference to the effective lagrangian \( \mathcal{L}_{\text{eff}} \) or to Feynman diagrams, should not be surprising. After all, the effective lagrangian is merely a technical device which collects the general properties of transition amplitudes among Goldstone bosons. All the results obtained with the help of the effective lagrangian are, in principle, reproducible using directly Ward Identities, analyticity, crossing symmetry and unitarity within a systematic low-energy expansion of QCD correlation functions. It just happens that in the case of the \( \pi\pi \) amplitude, it is technically simpler to use the direct S-matrix method than to evaluate all two-loop graphs generated by the effective lagrangian. In any case, the results have to be identical in both approaches. We have shown that this is indeed the case at the \( O(p^4) \) level (see Section 3.2), and since we understand that work towards an effective lagrangian computation of \( A(s|t,u) \) to order \( O(p^6) \) in the standard framework is in progress \cite{54,55}, we hope that it will soon become possible to check this statement to two loops by a direct comparison at least in this particular case. Undertaking a similar enterprise in the generalised case would be considerably more difficult, due to the proliferation of low-energy constants in \( \mathcal{L}_{\text{eff}} \) to that order. For the same reason, it is not clear that it would be particularly useful.

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Appendix A  Expansions of Masses and of Decay Constants to Order $O(p^4)$

From the expression of the effective action at order $O(p^4)$ one computes

$$
\frac{F_0^2}{F_0^2} = 1 + 2\tilde{m}[(2 + r)\xi] + 2\tilde{m}^2\delta_{F,\pi}^{(2,2)} - 4\mu - 2\mu_K,
$$

(A.1)

and

$$
\frac{F_K^2}{F_0^2} = 1 + \tilde{m}[(r + 1)\xi + 2(2 + r)\xi] + 2\tilde{m}^2\delta_{F,K}^{(2,2)} - \frac{3}{2}\mu - 3\mu_K - \frac{3}{2}\mu_\eta,
$$

(A.2)

where $\delta_{F,\pi}^{(2,2)}$ and $\delta_{F,K}^{(2,2)}$ contain the contributions from $\mathcal{L}_{(2,2)}$, cf. Eq. (2.12),

$$
\delta_{F,\pi}^{(2,2)} = A_1 + \frac{1}{2}A_2 + 2A_3 + \frac{1}{2}(2 + r^2)(A_4 + 2B_4) + B_1 - B_2 + 2(2 + r)D^S,
$$

(A.3)

$$
\delta_{F,K}^{(2,2)} = \frac{1}{2}(1 + r^2)(A_2 + A_3 + B_1) + \frac{r}{2}(A_2 + 2A_3 - 2B_2)
$$

$$
+ \frac{1}{2}(2 + r^2)(A_4 + 2B_4) + (r + 1)(2 + r)D^S.
$$

Similarly, the expansion of the pion and kaon masses read

$$
\frac{F_0^2}{F_0^2}M_\pi^2 = 2\tilde{m}B_0 + 4\tilde{m}^2A_0 + 4(2 + r)\tilde{m}^2Z_0^S
$$

$$
+ 2\tilde{m}^3\delta_{M,\pi}^{(0,3)} + 2\tilde{m}^4\delta_{M,\pi}^{(0,4)} + 4\tilde{m}^2M_\pi^2A_3
$$

$$
- \mu_\pi \left[3M_\pi^2 + 20\tilde{m}^2(A_0 + 2Z_0^S)\right]
$$

$$
- 2\mu_K \left[M_\pi^2 + 4(1 + r)\tilde{m}^2(A_0 + 2Z_0^S)\right]
$$

$$
- \frac{1}{3}\mu_\eta \left[M_\pi^2 + 4(1 + 2r)\tilde{m}^2(A_0 + 2Z_0^S) + 8(1 - r)\tilde{m}^2(A_0 + 2Z_0^P)\right],
$$

(A.4)

$$
\frac{F_K^2}{F_0^2}M_K^2 = (r + 1)\tilde{m}B_0 + (r + 1)^2\tilde{m}^2A_0 + 2(r + 1)(2 + r)\tilde{m}^2Z_0^S
$$

$$
+ (r + 1)\tilde{m}^3\delta_{M,K}^{(0,3)} + (r + 1)\tilde{m}^4\delta_{M,K}^{(0,4)} + (r + 1)^2\tilde{m}^2M_K^2A_3
$$

$$
- \frac{3}{2}\mu_\pi \left[2M_K^2 + 4(r + 1)\tilde{m}^2(A_0 + 2Z_0^S)\right]
$$

$$
- 3\mu_K \left[M_K^2 + 2(1 + r)^2\tilde{m}^2(A_0 + 2Z_0^S)\right]
$$

$$
- \frac{1}{6}\mu_\eta \left[5M_K^2 + 4(r + 1)(1 + 2r)\tilde{m}^2(A_0 + 2Z_0^S) - 4(1 - r^2)\tilde{m}^2(A_0 + 2Z_0^P)\right].
$$

(A.5)
The $O(p^3)$ tree-level contributions $\delta^{(0,3)}_{M,P}$, $P = \pi, K$ are given as follows

\begin{align*}
\delta^{(0,3)}_{M,\pi} &= \frac{9}{2}\rho_1 + \frac{1}{2}\rho_2 + (10 + 4r + r^2)\rho_4 + \frac{1}{2}(2 + r^2)\rho_5 + 6(2 + r)^2\rho_7, \\
\delta^{(0,3)}_{M,K} &= \frac{3}{2}(1 + r + r^2)\rho_1 + \frac{1}{2}(1 - r + r^2)\rho_2 + 3(2 + 2r + r^2)\rho_4 + \frac{1}{2}(2 + r^2)\rho_5 + 6(2 + r)^2\rho_7.
\end{align*}

The $O(p^4)$ tree level contributions from $\bar{\mathcal{L}}^{(0,4)}$ contained in $\delta^{(0,4)}_{M,\pi}$ and in $\delta^{(0,4)}_{M,K}$ are not displayed explicitly.
Appendix B  The Polynomials $\xi^{(n)}_a(s)$

We give here the list of polynomials $\xi^{(n)}_a(s)$ which define the one-loop partial-wave projections in Eqs. (3.34) and (3.36). The center-of-mass momentum $q$ of the pions is given (in units of $M_\pi$) by

$$q = \sqrt{\frac{s - 4M_\pi^2}{4M_\pi^2}}.$$  \hfill (B.1)

\begin{align*}
\xi^{(0)}_0(s) &= \frac{1}{144\pi^2} \left[ 35\alpha^2 + 80\alpha\beta + 134\beta^2 \right] + 10 (\lambda_1 + 2\lambda_2) \\
&\quad + \left\{ \frac{1}{72\pi^2} (60\alpha + 209\beta)\beta + 16(2\lambda_1 + 3\lambda_2) \right\} q^2 \\
&\quad + \left\{ \frac{311}{108\pi^2} \beta^2 + \frac{8}{3} (11\lambda_1 + 14\lambda_2) \right\} q^4 \\
\xi^{(1)}_0(s) &= \frac{1}{192\pi^2} \left[ 5\alpha^2 + 4\beta^2 \right] + \frac{1}{9\pi^2} \beta^2 q^2 + \frac{7}{36\pi^2} \beta^2 q^4 \\
\xi^{(2)}_0(s) &= \frac{1}{1152\pi^2} \left[ 5\alpha + 16\beta + 24\beta q^2 \right]^2 \\
\xi^{(3)}_0(s) &= \frac{1}{288\pi^2} \left[ -5\alpha^2 + 4\beta^2 \right] + \frac{1}{12\pi^2} \beta^2 q^2 \\
\xi^{(4)}_0(s) &= 0 \\
\xi^{(0)}_2(s) &= \frac{1}{288\pi^2} \left[ 31\alpha^2 - 122\alpha\beta + 220\beta^2 \right] + 4 (\lambda_1 + 2\lambda_2) \\
&\quad + \left\{ \frac{1}{144\pi^2} (-69\alpha + 268\beta)\beta + 8(\lambda_1 + 3\lambda_2) \right\} q^2 \\
&\quad + \left\{ \frac{265}{216\pi^2} \beta^2 + \frac{16}{3} (\lambda_1 + 4\lambda_2) \right\} q^4 \\
\xi^{(1)}_2(s) &= \frac{1}{576\pi^2} \left[ 9\alpha^2 - 42\alpha\beta + 60\beta^2 \right] \\
&\quad + \frac{1}{144\pi^2} (-9\alpha + 37\beta) \beta q^2 \\
&\quad + \frac{1}{72\pi^2} \beta^2 q^4 \hfill (B.3)
\end{align*}
\[ \xi_2^{(2)}(s) = \frac{1}{288\pi^2} \left[ \alpha - 4\beta - 6\beta^2 \right]^2 \]

\[ \xi_2^{(3)}(s) = \frac{1}{288\pi^2} \left[ -3\alpha^2 + 2\alpha\beta - 12\beta^2 \right] - \frac{1}{24\pi^2} \beta^2 q^2 \]

\[ \xi_2^{(4)}(s) = 0 \]

\[ \xi_1^{(0)}(s) = \frac{1}{576\pi^2} \left\{ 5\alpha^2 - 80\alpha\beta + 10\beta^2 \right\} \]
\[ + \left\{ \frac{1}{432\pi^2} [55\alpha - 68\beta] \beta - \frac{8}{3} (\lambda_1 - \lambda_2) \right\} q^2 \]
\[ - \left\{ \frac{\beta^2}{108\pi^2} + \frac{8}{3} (\lambda_1 - \lambda_2) \right\} q^4 \]

\[ \xi_1^{(1)}(s) = \frac{1}{288\pi^2} (-5\alpha + 7\beta) \beta + \frac{1}{144\pi^2} (5\alpha - 3\beta) \beta q^2 - \frac{1}{72\pi^2} \beta^2 q^4 \]

\[ \xi_1^{(2)}(s) = \frac{1}{72\pi^2} \beta^2 q^4 \]

\[ \xi_1^{(3)}(s) = \frac{1}{864\pi^2} \left[ -5\alpha^2 + 10\alpha\beta + 28\beta^2 \right] + \frac{1}{24\pi^2} \beta^2 q^2 \]

\[ \xi_1^{(4)}(s) = - \frac{5}{144\pi^2} \left[ \alpha^2 + 4\alpha\beta - 2\beta^2 \right] \]
Appendix C  The Loop Integrals $\bar{K}_n(s)$

In this Appendix, we discuss some properties of the functions $\bar{K}_n(s)$, cf. Eqs. \((3.42)\), \((3.37)\) and \((3.49)\). Let us start with the function $\bar{J}(s)$ which denotes the usual $d$-dimensional one-loop integral

\[
J(p^2) = -i \int \frac{d^dq}{(2\pi)^d} \frac{1}{(M_\pi^2 - q^2)(M_\pi^2 - (q - p)^2)}
\]  

(C.1)

subtracted at $p^2 = 0$. $\bar{J}(s)$ is finite for $d = 4$ and may be expressed in the standard parametrical form \([4, 5]\)

\[
\bar{J}(s) = -\frac{1}{16\pi^2} \int_0^1 d\lambda \ln \left[ 1 - \frac{s}{M_\pi^2} \lambda (1 - \lambda) \right].
\]  

(C.2)

For $s < 4M_\pi^2$, the latter formula can be integrated by parts and, after introducing a new integration variable

\[
x = \frac{M_\pi^2}{\lambda (1 - \lambda)},
\]  

(C.3)

it can be transformed into the dispersive representation \((3.14)\). For real $s$, one has

\[
16\pi^2 \bar{J}(s) = \begin{cases} 
2 + \sigma (\ln \frac{1 - \sigma}{1 + \sigma} + i\pi), & \text{if } s \geq 4M_\pi^2 \\
2 - 2 \left( \frac{4M_\pi^2}{s} \right)^{1/2} \arctan \left( \frac{s}{4M_\pi^2 - s} \right)^{1/2}, & \text{if } 0 \leq s \leq 4M_\pi^2 \\
2 + \sigma \ln \sigma^{-1}, & \text{if } s \leq 0,
\end{cases}
\]  

(C.4)

where

\[
\sigma = \left( 1 - \frac{4M_\pi^2}{s} \right)^{1/2}.
\]  

(C.5)

The function $J(s)$ was known at times as the Chew-Mandelstam function \([56]\).

In order to simplify the discussion of the two-loop integrals, it is convenient to introduce the notation

\[
F(s) \equiv 16\pi^2 \bar{J}(s) - 2.
\]  

(C.6)

The function $F(s)$ vanishes at threshold; more precisely, for $s - 4M_\pi^2 \to 0^+$, one has the expansion

\[
F(s) = i\pi\sigma - 2\sigma^2 - \frac{2}{3}\sigma^4 - \frac{2}{5}\sigma^6 + \ldots,
\]  

(C.7)

whereas for $s \to 0$,

\[
F(s) = -2 + \frac{1}{6} \frac{s}{M_\pi^2} + \ldots.
\]  

(C.8)

The dispersion integrals \((3.42)\) can now be directly determined by algebraic manipulations with the functions $F(s)$, without calculating a single integral. It is sufficient to construct functions
that are analytic except for a branch-cut singularity on the positive real half-axis starting at $s = 4 M^2_\pi$, with the discontinuities given by the functions $k_n(s)$, Eqs. (3.37):

$$\text{Im} K_n(s) \equiv \frac{1}{2i} [\tilde K_n(s + i\epsilon) - \tilde K_n(s - i\epsilon)] = k_n(s) \theta(s - 4 M^2_\pi) .$$

(C.9)

Such analytic functions are, of course, only defined up to a polynomial. The latter, however, can be unambiguously fixed, using the boundary conditions

$$\lim_{|s| \to \infty} s^{-1} \tilde K_n(s) = 0 , \quad \tilde K_n(0) = 0 ,$$

(C.10)

which follow from Eqs. (3.37) and (3.42). The main ingredients of this construction are the analytic functions $F(s)$, $F^2(s)$, and $F^3(s)$, with the discontinuities

$$\text{Im} F(s) = \pi \sqrt{s - 4 M^2_\pi} \theta(s - 4 M^2_\pi)$$

(C.11)

$$\text{Im} F^2(s) = 2\pi \left( \frac{s - 4 M^2_\pi}{s} \right) L(s) \theta(s - 4 M^2_\pi)$$

(C.12)

$$\text{Im} F^3(s) = \left( \frac{s - 4 M^2_\pi}{s} \right)^{3/2} (3\pi L^2(s) - \pi^3) \theta(s - 4 M^2_\pi) ,$$

(C.13)

where $L(s)$ is the logarithmic function (3.38). Since $F(4 M^2_\pi) = 0$, new analytic functions with the desired discontinuities can be obtained upon dividing by $(s - 4 M^2_\pi)$. This leads to

$$\text{Im} \frac{F(s)}{s - 4 M^2_\pi} = \frac{\pi}{\sqrt{s(s - 4 M^2_\pi)}} \theta(s - 4 M^2_\pi) ,$$

(C.14)

$$\text{Im} \frac{s F^2(s)}{s - 4 M^2_\pi} = 2\pi L(s) \theta(s - 4 M^2_\pi) .$$

(C.15)

This procedure can be extended further: Subtracting from the function $s F^2(s)/(s - 4 M^2_\pi)$ its threshold value

$$\lim_{s \to 4 M^2_\pi} \frac{s F^2(s)}{s - 4 M^2_\pi} = -\pi^2 ,$$

(C.16)

one obtains from Eq. (C.15)

$$\text{Im} \frac{1}{s - M^2_\pi} \left[ \frac{s F^2(s)}{s - M^2_\pi} + \pi^2 \right] = \frac{2\pi L(s)}{s - M^2_\pi} \theta(s - 4 M^2_\pi) .$$

(C.17)

The functions $k_0(s)$, $k_1(s)$ and $k_2(s)$ can be recognised on the right-hand sides of Eqs. (C.11), (C.15) and (C.12) respectively. Similarly, Eqs. (C.14) and (C.17) reproduce the first two terms in the expression of $k_4(s)$. The remaining terms, quadratic in $L(s)$, can be inferred from
Eq. (C.13): The function $F^3(s) + \pi^2(s - 4M^2_\pi)F(s)/s$ exhibits a double zero at $s = 4M^2_\pi$. Consequently, one can write
\[
\text{Im} \frac{s}{(s - 4M^2_\pi)^2} \left[ F^3(s) + \frac{\pi^2}{s}(s - 4M^2_\pi)F(s) \right] = \frac{3\pi L^2(s)}{\sqrt{s(s - 4M^2_\pi)}} \theta(s - 4M^2_\pi). \tag{C.18}
\]

Finally, subtracting from the analytic function on the left-hand side of the last equation its threshold value, one obtains
\[
\text{Im} \frac{1}{s - 4M^2_\pi} \left\{ \frac{M^2_\pi}{(s - 4M^2_\pi)^2} \left[ sF^3(s) + \pi^2(s - 4M^2_\pi)F(s) \right] - \pi^2 \right\} = \frac{3\pi M^2_\pi}{s - 4M^2_\pi} \cdot \frac{L^2(s)}{\sqrt{s(s - 4M^2_\pi)}} \theta(s - 4M^2_\pi). \tag{C.19}
\]

All the elements needed to write down the functions $K_n(s)$ are now collected. Eqs. (C.11), (C.15), (C.12), (C.18) imply, in succession,
\[
\begin{align*}
K_0(s) &= \frac{1}{16\pi^2} F(s) + \kappa_0(s) \\
K_1(s) &= \frac{1}{16\pi^2} \frac{s}{s - 4M^2_\pi} F^2(s) + \kappa_1(s) \\
K_2(s) &= \frac{1}{16\pi^2} F^2(s) + \kappa_2(s) \\
K_3(s) &= \frac{1}{16\pi^2} \frac{sM^2_\pi}{(s - 4M^2_\pi)^2} \left\{ F^3(s) + \frac{\pi^2}{s}s - 4M^2_\pi F(s) \right\} + \kappa_3(s),
\end{align*}
\tag{C.20}
\]

and from Eqs. (C.14), (C.17) and (C.19) one reconstructs $K_4(s)$:
\[
\begin{align*}
K_4(s) &= \frac{M^2_\pi}{16\pi^2} \frac{F(s)}{s - 4M^2_\pi} \\
&\quad + \frac{M^2_\pi}{32\pi^2} \frac{1}{s - 4M^2_\pi} \left[ \frac{s}{s - 4M^2_\pi} F^2(s) + \pi^2 \right] \\
&\quad + \frac{M^2_\pi}{48\pi^2} \frac{1}{s - 4M^2_\pi} \left\{ \frac{M^2_\pi}{(s - 4M^2_\pi)^2} \left[ F^3(s) + \frac{\pi^2}{s}s - 4M^2_\pi F(s) \right] - \pi^2 \right\} + \kappa_4(s).
\end{align*}
\tag{C.21}
\]

In Eqs. (C.20) and (C.21), $\kappa_n(s)$ denote sofar arbitrary polynomials. They may be fixed using the boundary conditions (C.10). Since $F(s)$ is bounded by $\ln s$ for $|s| \to \infty$, the asymptotic condition (C.10) implies that all $\kappa_n(s)$ must be constant. Their values are then determined by the conditions $K_n(0) = 0$:
\[
\kappa_0 = \frac{1}{8\pi^2}, \quad \kappa_1 = 0, \quad \kappa_2 = -\frac{1}{4\pi^2}, \quad \kappa_3 = -\frac{1}{32}, \quad \kappa_4 = \frac{1}{132} - \frac{1}{32\pi^2}. \tag{C.22}
\]

Eqs. (C.20) and (C.21) now coincide with the formula (3.49) given in Section 3 without proof. It is rather interesting that all two-loop contributions to the $\pi\pi$ scattering amplitude can be obtained by algebraic manipulations with the Chew-Mandelstam function, without having to evaluate a single integral.
Appendix D  Threshold Parameters

In this Appendix, we display the explicit two-loop expressions of the threshold parameters \( a_l^i \) and \( b_l^i \). They are defined from the low-\( q^2 \) expansions of the real parts of the corresponding partial waves \( f_l^i(s) \)

\[
\text{Re} f_l^i(s) = q^{2l} \left\{ a_l^i + b_l^i q^2 + \cdots \right\}, \tag{D.1}
\]

with \( q^2 = (s - 4M_r^2)/4M_r^2 \). The partial waves are defined by taking the corresponding projections (3.8) of the two-loop amplitude (3.40), (3.47).

\[
a_0^0 = \frac{1}{96 \pi} \frac{M_r^2}{F_\pi^2} (5 \alpha + 16 \beta) + \frac{5}{8 \pi} \frac{M_r^4}{F_\pi^4} (\lambda_1 + 2 \lambda_2) + \frac{1}{4608 \pi^3} \frac{M_r^4}{F_\pi^4} (5 \alpha + 16 \beta)^2
\]
\[
+ \frac{1}{4 \pi} \frac{M_r^6}{F_\pi^6} (3 \lambda_3 - 6 \lambda_4) + \frac{5}{192 \pi^3} \frac{M_r^6}{F_\pi^6} (\lambda_1 + 2 \lambda_2) (5 \alpha + 16 \beta)
\]
\[
+ \frac{25}{221184 \pi^5} \frac{M_r^6}{F_\pi^6} (23 - 2 \pi^2) \alpha^3 + \frac{5}{13824 \pi^5} \frac{M_r^6}{F_\pi^6} (33 - 2 \pi^2) \alpha^2 \beta
\]
\[
+ \frac{5}{55296 \pi^5} \frac{M_r^6}{F_\pi^6} (198 - \pi^2) \alpha \beta^2 + \frac{1}{3456 \pi^5} \frac{M_r^6}{F_\pi^6} (70 - \pi^2) \beta^3 \tag{D.2}
\]

\[
b_0^0 = \frac{1}{4 \pi} \frac{\beta}{F_\pi^2} + \frac{1}{\pi} \frac{M_r^2}{F_\pi^4} (2 \lambda_1 + 3 \lambda_2) + \frac{1}{3 \pi^3} \frac{M_r^4}{F_\pi^4} (\beta^2 \alpha \gamma) + \frac{5}{96} \alpha \beta - \frac{5}{256} \alpha^2
\]
\[
+ \frac{3}{16 \pi^3} \frac{M_r^4}{F_\pi^4} (3 \lambda_3 - \lambda_4) + \frac{5}{216 \pi^3} \frac{M_r^4}{F_\pi^4} (\lambda_1 (47 \alpha + 67 \beta) + \frac{5}{27 \pi} \frac{M_r^4}{F_\pi^4} \lambda_2 (\frac{29}{16} \alpha + 13 \beta)
\]
\[
- \frac{5}{331776 \pi^5} \frac{M_r^4}{F_\pi^4} (161 + \frac{61}{24} \pi^2) \alpha^3 + \frac{5}{9216 \pi^5} \frac{M_r^4}{F_\pi^4} \left( \frac{13}{2} - \frac{37}{9} \pi^2 \right) \alpha \beta^2
\]
\[
+ \frac{5}{20736 \pi^5} \frac{M_r^4}{F_\pi^4} (89 - \frac{85}{4} \pi^2) \alpha \beta^2 + \frac{1}{41472 \pi^5} \frac{M_r^4}{F_\pi^4} \left( \frac{4793}{2} - \frac{823}{3} \pi^2 \right) \beta^3 \tag{D.3}
\]

\[
a_2^0 = \frac{1}{30 \pi} \frac{1}{3 \pi^3} \frac{M_r^2}{F_\pi^4} (\lambda_1 + 4 \lambda_2) + \frac{1}{34560 \pi^3} \frac{M_r^4}{F_\pi^4} (\alpha^2 - 48 \beta)
\]
\[
- \frac{1}{5 \pi} \frac{M_r^2}{F_\pi^4} (3 \lambda_3 + 4 \lambda_4) - \frac{1}{36 \pi^3} \frac{M_r^4}{F_\pi^4} \lambda_1 (\frac{7}{22} \alpha - \frac{53}{225} \beta) - \frac{1}{36 \pi^3} \frac{M_r^4}{F_\pi^4} \lambda_2 (\frac{7}{36} \alpha + \frac{53}{225} \beta)
\]
\[
+ \frac{11}{7464960 \pi^5} \frac{M_r^4}{F_\pi^4} (\frac{41}{4} - \pi^2) \alpha^3 - \frac{1}{138240 \pi^5} \frac{M_r^4}{F_\pi^4} (\frac{121}{18} - \pi^2) \alpha \beta^2
\]
\[
- \frac{1}{248832 \pi^5} \frac{M_r^4}{F_\pi^4} (89 - \frac{46}{5} \pi^2) \alpha \beta^2 + \frac{1}{9331200 \pi^5} \frac{M_r^4}{F_\pi^4} (\frac{1679}{2} - 329 \pi^2) \beta^3 \tag{D.4}
\]
\[ a_1^1 = \frac{1}{24\pi} \frac{M_\pi^2}{F_\pi^2} \beta - \frac{1}{6\pi} \frac{M_\pi^2}{F_\pi^2} (\lambda_1 - \lambda_2) + \frac{1}{41472\pi^3} \frac{M_\pi^2}{F_\pi^4} (5\alpha^2 - 40\alpha\beta - 16\beta^2) \\
+ \frac{1}{2\pi} \frac{M_\pi^2}{F_\pi^6} (\lambda_3 - \lambda_4) + \frac{1}{144\pi^3} \frac{M_\pi^4}{F_\pi^6} \lambda_1 (\frac{5}{12}\alpha - \beta) + \frac{1}{81\pi^3} \frac{M_\pi^4}{F_\pi^6} \lambda_2 (\frac{5}{32}\alpha - \beta) \\
+ \frac{5}{497664\pi^5} \frac{M_\pi^2}{F_\pi^6} (1 - \frac{\pi^2}{6})\alpha^3 + \frac{5}{331776\pi^5} \frac{M_\pi^4}{F_\pi^6} (5 - \frac{\pi^2}{3})\alpha^2\beta \\
- \frac{5}{497664\pi^5} \frac{M_\pi^2}{F_\pi^6} \left( \frac{374}{3} - 13\pi^2 \right)\alpha\beta^2 - \frac{5}{746496\pi^5} \frac{M_\pi^4}{F_\pi^6} \left( \frac{73}{2} + 5\pi^2 \right)\beta^3 \quad (D.5) \]

\[ b_1^1 = -\frac{1}{6\pi} \frac{M_\pi^2}{F_\pi^2} (\lambda_1 - \lambda_2) + \frac{1}{4320\pi^3} \frac{1}{F_\pi^4} (\frac{47}{3}\beta^2 - \frac{65}{6}\alpha\beta - \frac{5}{24}\alpha^2) \\
+ \frac{1}{35\pi F_\pi^6} (\lambda_3 - \lambda_4) + \frac{1}{294\pi^3 F_\pi^6} \lambda_1 (\frac{23}{240}\alpha - \beta) + \frac{1}{3528\pi^3 F_\pi^6} \lambda_2 (\alpha - \frac{211}{10}\beta) \\
+ \frac{1}{20321280\pi^5 F_\pi^6} \left( \frac{163}{12} - \pi^2 \right)\alpha^3 + \frac{1}{8128512\pi^5 F_\pi^6} \left( \frac{311}{10} - \pi^2 \right)\alpha^2\beta \\
- \frac{1}{2903040\pi^5 F_\pi^6} \left( \frac{2537}{14} - 17\pi^2 \right)\alpha\beta^2 - \frac{1}{3386880\pi^5 F_\pi^6} \left( \frac{8011}{36} - \frac{13}{5}\pi^2 \right)\beta^3 \quad (D.6) \]

\[ a_3^1 = \frac{1}{13230\pi^3 M_\pi^2 F_\pi^4} \left( \frac{\alpha^2}{64} + \frac{17}{32}\alpha\beta + \beta^2 \right) \\
+ \frac{1}{35\pi F_\pi^6} (\lambda_3 - \lambda_4) + \frac{1}{294\pi^3 F_\pi^6} \lambda_1 (\frac{23}{240}\alpha - \beta) + \frac{1}{3528\pi^3 F_\pi^6} \lambda_2 (\alpha - \frac{211}{10}\beta) \\
+ \frac{1}{20321280\pi^5 F_\pi^6} \left( \frac{163}{12} - \pi^2 \right)\alpha^3 + \frac{1}{8128512\pi^5 F_\pi^6} \left( \frac{311}{10} - \pi^2 \right)\alpha^2\beta \\
- \frac{1}{2903040\pi^5 F_\pi^6} \left( \frac{2537}{14} - 17\pi^2 \right)\alpha\beta^2 - \frac{1}{3386880\pi^5 F_\pi^6} \left( \frac{8011}{36} - \frac{13}{5}\pi^2 \right)\beta^3 \quad (D.7) \]

\[ a_0^2 = \frac{1}{48\pi} \frac{M_\pi^2}{F_\pi^2} (\alpha - 4\beta) + \frac{1}{4\pi} \frac{M_\pi^4}{F_\pi^4} (\lambda_1 + 2\lambda_2) + \frac{1}{1152\pi^3} \frac{M_\pi^4}{F_\pi^4} (\alpha - 4\beta)^2 \\
- \frac{1}{2\pi} \frac{M_\pi^6}{F_\pi^6} \lambda_3 + \frac{1}{48\pi^3} \frac{M_\pi^6}{F_\pi^6} (\lambda_1 + 2\lambda_2)(\alpha - 4\beta) \\
+ \frac{1}{18432\pi^5} \frac{M_\pi^6}{F_\pi^6} \left( \frac{29}{3} - \pi^2 \right)\alpha^3 + \frac{1}{9216\pi^5} \frac{M_\pi^6}{F_\pi^6} (37 - \frac{23}{6}\pi^2)\alpha^2\beta \\
+ \frac{1}{1536\pi^5} \frac{M_\pi^6}{F_\pi^6} (17 - \frac{31}{18}\pi^2)\alpha\beta^2 - \frac{1}{384\pi^5} \frac{M_\pi^6}{F_\pi^6} (47 - \frac{\pi^2}{2})\beta^3 \quad (D.8) \]
\[
\begin{align*}
t_0^2 &= \frac{1}{8\pi} \frac{\beta}{F^2_\pi} + \frac{1}{2\pi} \frac{M^2_\pi}{F^4_\pi} (\lambda_1 + 3\lambda_2) + \frac{1}{4608\pi^3} \frac{M^2_\pi}{F^4_\pi} (112\beta^2 - 8\alpha\beta - 7\alpha^2) \\
&- \frac{3}{2\pi} \frac{M^4_\pi}{F^6_\pi} (\lambda_3 - \lambda_4) + \frac{1}{432\pi^3} \frac{M^4_\pi}{F^6_\pi} \lambda_1 (\frac{17}{4}\alpha - 83\beta) + \frac{1}{108\pi^3} \frac{M^4_\pi}{F^6_\pi} \lambda_2 (\frac{59}{8}\alpha - 61\beta) \\
&- \frac{1}{165888\pi^5} \frac{M^4_\pi}{F^6_\pi} (18 - \frac{23}{\pi^2})\alpha^3 - \frac{1}{36864\pi^5} \frac{M^4_\pi}{F^6_\pi} (91 - \frac{169}{9}\pi^2)\alpha^2\beta \\
&+ \frac{1}{82944\pi^6} \frac{M^4_\pi}{F^6_\pi} (71 + \frac{445}{2}\pi^2)\alpha^2 - \frac{1}{182944\pi^6} \frac{M^4_\pi}{F^6_\pi} (6745 - \frac{1237}{3}\pi^2)\beta^3
\end{align*}
\]

\[
\begin{align*}
a_2^2 &= \frac{1}{30\pi} \frac{1}{F^2_\pi} (\lambda_1 + \lambda_2) + \frac{1}{7200\pi^3} \frac{1}{F^4_\pi} (\frac{1}{8} \alpha^2 + \frac{13}{6} \alpha\beta - \frac{13}{3} \beta^2) \\
&- \frac{1}{5\pi} \frac{M^2_\pi}{F^6_\pi} (\lambda_3 + \lambda_4) - \frac{1}{32400\pi^3} \frac{M^2_\pi}{F^6_\pi} \lambda_1 (\frac{151}{2}\alpha - 191\beta) - \frac{1}{8100\pi^3} \frac{M^2_\pi}{F^6_\pi} \lambda_2 (\frac{41}{2}\alpha - 61\beta) \\
&+ \frac{1}{9331200\pi^5} \frac{M^2_\pi}{F^6_\pi} (1223 - 7\pi^2)\alpha^3 + \frac{7}{1382400\pi^5} \frac{M^2_\pi}{F^6_\pi} (\frac{113}{18} - \pi^2)\alpha^2\beta \\
&+ \frac{1}{518400\pi^5} \frac{M^2_\pi}{F^6_\pi} (119 - \frac{257}{24}\pi^2)\alpha^2 + \frac{1}{18662400\pi^5} \frac{M^2_\pi}{F^6_\pi} (\frac{3289}{2} - 271\pi^2)\beta^3
\end{align*}
\]

The contributions coming from various orders of the chiral expansion are easy to distinguish. The lowest order contributions, corresponding to the amplitude (3.9), give terms linear in $\alpha$ and $\beta$, proportional to $F^{-2}_\pi$. At this order, only the S- and P-wave scattering lengths and the S-wave slope parameters are non-vanishing. The one-loop precision brings in two types of contributions, both proportional to $F^{-4}_\pi$: Those which are linear in $\lambda_1$ and $\lambda_2$ come from $O(p^4)$ and $O(p^5)$ tree graphs, and those which are quadratic in $\alpha, \beta$ correspond to genuine one-loop graphs. Finally, the contributions at order two-loop, proportional to $F^{-6}_\pi$, come from $O(p^6)$ and $O(p^7)$ tree-graphs (terms linear in $\lambda_3$ and $\lambda_4$), from one-loop graphs with a $\lambda_1$ or $\lambda_2$ vertex, whereas the genuine two-loop graphs generate the contributions which are cubic in $\alpha, \beta$. 

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Figure Captions

• 1- Argand diagram of the amplitude \( f_0^0 \) (i.e. \( \sqrt{1 - 4M_\pi^2/s} \text{Im} f_0^0 \) versus \( \sqrt{1 - 4M_\pi^2/s} \text{Re} f_0^0 \)). The result at one loop is represented by the dashed line and the result at two loops by the solid line. Both cases correspond to the same parameter choice \( \alpha = 2, \beta = 1.08 \) and \( \lambda_i \) as in (4.9) and (4.10).

• 2- Phase shifts computed at increasing chiral order (see formulas (4.11)-(4.13) and the discussion following them). The tree, one-loop and two-loop orders are represented by the dotted, dashed and dash-dotted lines respectively. Shown for comparison are the data of Ref. \[39\] and \[57\] for \( \delta_0^0 \) and Ref. \[58\] for \( \delta_0^2 \) and \( \delta_2^2 \). For \( \delta_1^1 \) and \( \delta_0^0 \) the Breit-Wigner contributions of the \( \rho \) and the \( f_2 \) resonances are shown as crosses.

• 3- Illustration of the sensitivity of the phase shifts to the values of the parameters \( \alpha \) and \( \beta \). The dotted line is the result of the standard \( \chi \)PT (\( \alpha = 1.04, \beta = 1.08 \)), the dashed line corresponds to \( \alpha = 2, \beta = 1.08 \) and the solid line to \( \alpha = 3, \beta = 1.12 \).

• 4- The phase shift difference \( \delta_0^0 - \delta_1^1 \) in the range of energies accessible in the \( K_{l4} \) decays is shown: a) for increasing chiral orders as in Fig. 2 and, b) for several values of \( \alpha \) and \( \beta \), as in Fig. 3.

• 5- Influence of the uncertainties affecting the evaluation of \( \lambda_1, ..., \lambda_4 \) on the prediction for \( \delta_0^0 - \delta_1^1 \). The band delimited by solid lines corresponds to varying the \( \lambda_i \)'s inside the ranges indicated in (4.9) and (4.10) with \( \alpha, \beta \) fixed to the standard \( \chi \)PT values, whereas the band delimited by dashed lines corresponds to \( \alpha = 2, \beta = 1.08 \).

• 6- Plot of the correlation between \( \alpha \) and \( \beta \) as inferred from the Morgan-Shaw universal band given by Eq. (4.14). The hatched region covers the set of points consistent with Eqs. (5.36) and (5.31). If the \( O(p^2) \) terms are ignored, this set shrinks to the doubly-hatched sub-region shown. See the discussion in Section 5.

• 7- Predictions for the scattering lengths as functions of \( \alpha \). The inner bands are obtained by varying the \( \lambda_i \)'s inside their error bars with \( \beta \) fixed at the center of the Morgan-Shaw band. Including the variations in \( \beta \) allowed by the latter gives the outer bands.

• 8- In a), we show the parameter \( \alpha(r) \) at increasing chiral order (see Eqs. (5.36) and the following discussion). The doubly-hatched area represents uncertainties in the contributions from \( \tilde{\rho}_1, \tilde{\rho}_2 \), while the hatched band includes the estimated uncertainty of \( \alpha^{(2)} \), as described in the text. In b) we show the contribution \( \alpha_{\text{loop}}^{(2)}(r) \) for different values of \( \mu \). Each band corresponds to variations of the scale \( \Lambda \) in the range \( M_\eta \leq \Lambda \leq 1 \text{ GeV} \).
9- In a), we show the parameter $\beta(r)$ at increasing chiral order (see Eq. (5.31) and the discussion following Eq. (5.36)). The hatched area represents the estimated uncertainty in contribution from $\beta^{(2)}$. In b) we show the contribution $\beta_{\text{loop}}^{(2)}(r)$ for different values of $\mu$. Each band corresponds to variations of the scale $\Lambda$ in the range $M_\eta \leq \Lambda \leq 1$ GeV.

10- In a) we show the Gell-Mann–Oakes–Renner ratio $x_{\text{GOR}}$ as a function of the linear combination $(\alpha + 2\beta)/3$ (see Eq. (5.48)). The hatched area represents the expected range of $O(p^2)$ contributions. The vertical dotted lines show the range in $(\alpha + 2\beta)/3$ corresponding to the fit values Eq. (4.15). In b) we show the contribution $x_{\text{GOR}}^\text{loop}$ for different values of $\mu$. Each band corresponds to variations of the scale $\Lambda$ in the range $M_\eta \leq \Lambda \leq 1$ GeV.
Figure 2b
Three values of $\alpha$

![Graph showing three curves for $\delta_0^0$, $\delta_0^2$, and $\delta_1^1$ as functions of energy (E in MeV).](image)

**Figure 3**
Morgan–Shaw universal curve
Figure 8
Figure 9
