Global asymptotic behaviour of positive solutions to a non-autonomous Nicholson’s blowflies model with delays

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This paper addresses the global existence and global asymptotic behaviour of positive solutions to a non-autonomous Nicholson’s blowflies model with delays. By using a novel approach, sufficient conditions are derived for the existence and global exponential convergence of positive solutions of the model without any restriction on uniform positiveness of the per capita dead rate. Numerical examples are provided to illustrate the effectiveness of the obtained results.

Keywords: Nicholson’s blowflies model; positive solutions; exponential convergence

AMS Subject Classification: 34C27; 34K14

1. Introduction

Nicholson [14] used the following delay differential equation:

\[ x'(t) = -\delta x(t) + Px(t - \tau) e^{-ax(t-\tau)}, \quad t \geq 0, \]  

where \( a, \delta, P \) and \( \tau \) are positive constants, to model laboratory population of the Australian sheep-blowfly. Biologically, \( x(t) \) is the size of the population at time \( t \), \( P \) is the maximum per capita daily egg production rate, \( 1/a \) is the size at which the population reproduces at its maximum rate, \( \delta \) is the per capita daily adult mortality rate and \( \tau \) is the generation time, or the time taken from birth to maturity. The dynamics of Equation (1) was later studied in [5,15], where this model was referred to as the Nicholson’s blowflies equation.

The theory of the Nicholson’s blowflies equation has made a remarkable progress in the past 40 years and attracted extensive attention from researchers (see, for example, [1] and the references therein). Many important results on the qualitative properties of the model such as existence of positive solutions, positive periodic/almost periodic solutions, persistence, permanence, oscillation and stability for the classical Nicholson’s model and its generalizations (in particular, to variable coefficients, time-varying delays and impulsive equations) have been established in the literature [2–4,7–13,16–22].

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However, it should be noted that, in most of the aforementioned works, the per capita daily adult mortality terms have been restricted to be uniformly positive in order to use the coincidence degree method \cite{3,18,21}, fixed point theorems \cite{7,12,13} or comparison principles \cite{9–11,17,19,20,22}. Furthermore, it is difficult to study the global asymptotic behaviour of the Nicholson’s blowflies model with variable coefficients and time-varying delays. So far, there has been no result in the literature considering the global existence and global exponential convergence to the zero equilibrium point of positive solutions of nonautonomous Nicholson’s blowflies model without the assumption on the uniform positiveness of the per capita daily mortality term.

Motivated by the above discussions, in this paper, we first consider the problem of global existence of positive solutions for a non-autonomous Nicholson’s blowflies model of the following form:

\[ x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x(t - \tau_j(t))e^{-\gamma_j(t)(t-\tau_j(t))}, \tag{2} \]

where \( m \) is a given positive integer, \( \alpha, \beta_j, \gamma_j, \tau_j, j \in m := \{1, 2, \ldots, m\} \), are continuous functions on \( \mathbb{R}^+ \), \( \alpha(t) > 0, \gamma_j(t) > 0 \) and \( \beta_j(t) \geq 0, \gamma_j(t) \geq 0 \) for all \( t \in \mathbb{R}^+, j \in m \). We then employ a novel proof to establish conditions for the global exponential convergence to the zero equilibrium point of model (2). It is worth noting that, the restriction on the uniform positiveness of \( \alpha(t) \) (that means, there is a positive constant \( \alpha^- \) such that \( \alpha(t) \geq \alpha^- \) for all \( t \geq 0 \)) as well as the upper and the lower bounds of \( \beta_j(t), \gamma_j(t), j \in m \), will be removed.

We assume that \( \tau_j(t) \leq \tau^+_j, t \geq 0 \), and let \( \tau = \max_{j \in m} \tau^+_j \). Throughout this paper, let \( C^+ = C^+([-\tau, 0], \mathbb{R}^+) \) be the set of nonnegative continuous functions with the usual supremum norm \( \| \cdot \| \) and \( C^+_0 = \{ \varphi \in C^+ : \varphi(0) > 0 \} \). In the biological interpretation of model (2), only positive solutions are meaningful and admissible. Thus we consider only the admissible initial conditions for Equation (2) as follows:

\[ x_{t_0} = \varphi \in C^+_0, \tag{3} \]

where \( x_{t_0} \) is defined as \( x_{t_0}(\theta) = x(t_0 + \theta) \) for all \( \theta \in [-\tau, 0] \).

Note that, the function \( f : \mathbb{R}^+ \times C^+_0 \longrightarrow \mathbb{R} \) defined as

\[ f(t, \varphi) = -\alpha(t)\varphi(0) + \sum_{j=1}^{m} \beta_j(t)\varphi(-\tau_j(t))e^{-\gamma_j(t)\varphi(-\tau_j(t))} \]

is continuous and locally Lipschitz with respect to \( \varphi \in C^+_0 \). Thus, for each \( t_0 \in \mathbb{R}^+, \varphi \in C^+_0 \), there exists a unique locally solution \( x(t; t_0, \varphi) \) of Equations (2) and (3) (for more details, see \cite{6}). Let \([t_0, \eta(\varphi)]\) be the right maximal interval of existence of \( x(t; t_0, \varphi) \).

2. Main results

2.1. Global existence of positive solutions

In this section we will prove the global existence of positive solutions of Equation (2) for admissible initial conditions (3).

**Theorem 2.1** For any \( t_0 \in \mathbb{R}^+, \varphi \in C^+_0 \), the solution \( x(t; t_0, \varphi) \) of Equation (2) satisfies

\[ x(t; t_0, \varphi) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)) \]

and \( \eta(\varphi) = +\infty \).
Proof Let \( x(t; t_0, \varphi) \) be a solution of Equations (2) and (3). For convenience, let us denote \( x(t) = x(t; t_0, \varphi) \) if it does not make any confusion. We will show that

\[
x(t) > 0 \quad \forall t \in [t_0, \eta(\varphi)).
\]  

(4)

Suppose in contrary that Equation (4) does not hold. Then, there exists \( t_a \in (t_0, \eta(\varphi)) \) such that \( x(t_a) = 0 \) and \( x(t) > 0 \) for all \( t \in [t_0, t_a) \). Thus, \( x(t) \geq 0 \) for all \( t \in [t_0 - \tau, t_a] \). Observing that

\[
\beta_j(t)x(t - \tau_j(t)) \geq 0 \quad \text{for all} \quad t \in [t_0, t_a], \ j \in m,
\]

from Equation (2), we have

\[
x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x(t - \tau_j(t))e^{-\gamma_j(t)x(t - \tau_j(t))}
\]

\[
\geq -\alpha(t)x(t)
\]

which yields

\[
x(t) \geq x(t_0) \exp \left( -\int_{t_0}^{t} \alpha(s) \, ds \right), \quad t \in [t_0, t_a).
\]  

(5)

Let \( t \uparrow t_a \), it follows from Equation (5) that, \( x(t_0) \exp(-\int_{t_0}^{t_a} \alpha(s) \, ds) \leq 0 \), which contradicts with \( x(t_0) = \varphi(0) > 0 \). This shows that Equation (4) holds. Consequently, \( x(t, t_0, \varphi) \in C \) for all \( t \in [t_0, \eta(\varphi)) \).

Next, we will prove the global existence of \( x(t; t_0, \varphi) \), that means \( \eta(\varphi) = +\infty \). Note that

\[
x(t - \tau_j(t)) \geq 0 \quad \text{for all} \quad t \in [t_0, \eta(\varphi)). \]

Using the fact \( \sup_{t \geq 0} \alpha(t)x(t) = 1/\gamma e, \ \gamma > 0 \), we have

\[
\beta_j(t)x(t - \tau_j(t))e^{-\gamma_j(t)x(t - \tau_j(t))} \leq \frac{1}{e} \frac{\beta_j(t)}{\gamma_j(t)}, \quad j \in m, \ t \in [t_0, \eta(\varphi)).
\]

Therefore

\[
x'(t) \leq -\alpha(t)x(t) + \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)}, \quad t \in [t_0, \eta(\varphi)).
\]  

(6)

It follows from Equation (6)

\[
x(t) \leq \exp \left( -\int_{t_0}^{t} \alpha(s)ds \right) \left[ x(t_0) + \frac{1}{e} \sum_{j=1}^{m} \int_{t_0}^{t} \frac{\beta_j(s)}{\gamma_j(s)} \exp \left( \int_{t_0}^{s} \alpha(u) \, du \right) ds \right]
\]

\[
\leq x(t_0) \exp \left( -\int_{t_0}^{t} \alpha(s)ds \right) + \frac{1}{e} \sum_{j=1}^{m} \int_{t_0}^{t} \frac{\beta_j(s)}{\gamma_j(s)} \exp \left( -\int_{t}^{\infty} \alpha(u) \, du \right) ds
\]

\[
\leq x(t_0) + \frac{1}{e} \sum_{j=1}^{m} \int_{t_0}^{t} \frac{\beta_j(s)}{\gamma_j(s)} \, ds
\]

(7)

for all \( t \in [t_0, \eta(\varphi)) \).
Suppose in contrary that \( \eta(\varphi) < +\infty \), then
\[
\limsup_{t \to \eta(\varphi)} x(t) = +\infty.
\] (8)

On the other hand, from Equation (7), we have
\[
\limsup_{t \to \eta(\varphi)} x(t) \leq x(t_0) + \frac{1}{\alpha} \sum_{j=1}^{m} \int_{t_0}^{\eta(\varphi)} \frac{\beta_j(s)}{\gamma_j(s)} \, ds < +\infty
\]
which yields a contradiction with Equation (8). Therefore, \( \eta(\varphi) = +\infty \). The proof is completed.  

\[ \blacksquare \]

\textbf{Remark 1}  
In the proof of Theorem 2.1, we do not require the upper and the lower boundedness of \( \alpha(t), \beta_j(t), \gamma_j(t), j \in m \), by positive constants as considered in many other results (see, e.g., [3,4,7,9–13,17–22]). Furthermore, if \( \alpha(t), \beta_j(t), \gamma_j(t), j \in m \) are assumed to be upper and lower bounded by positive constants then the boundedness of any solution \( x(t; t_0, \varphi) \) on \([t_0, \eta(\varphi))\) can be proved easily as follows.

For a given bounded continuous function \( g(t) \) on \( \mathbb{R}^+ \), let us denote
\[
g^+ = \sup_{t \in \mathbb{R}^+} g(t), \quad g^- = \inf_{t \in \mathbb{R}^+} g(t).
\]
From Equation (7), we obtain
\[
x(t) \leq \| \varphi \| + \frac{1}{\alpha} \sum_{j=1}^{m} \beta_j^+ \int_{t_0}^{\eta(\varphi)} \frac{\gamma_j^-(s)}{\gamma_j(s)} \, ds \forall t \in [t_0 - \tau, \eta(\varphi)).
\]
This shows that \( x(t; t_0, \varphi) \) is bounded on \([t_0, \eta(\varphi))\) and thus \( \eta(\varphi) = +\infty \) as concluded in Theorem 2.1. As shown in the following example, if \( \alpha(t) \) is not uniformly positive then positive solutions of Equation (2) may not be bounded.

\textbf{Example 2.2}  
Consider the following equation:
\[
x'(t) = -e^{-t}x(t) + \frac{\sqrt{t}}{1 + \sqrt{t}}x(t - |\sin t|)e^{-1/(1+t)|x(t-|\sin t|)|}, \quad t \geq 0.
\] (9)
It can be seen that \( \alpha(t) = e^{-t}, \beta(t) = \sqrt{t}/(1 + \sqrt{t}) \) and \( \gamma(t) = 1/(1 + t) \) are continuous functions, \( \alpha(t) > 0, \gamma(t) > 0 \) and \( \beta(t) \geq 0 \) for all \( t \in \mathbb{R}^+ \) but \( \alpha(t) \) and \( \gamma(t) \) are not uniformly positive. For any \( t_0 \in \mathbb{R}^+, \varphi \in C^+([-1, 0], \mathbb{R}^+), \varphi(0) > 0 \), by Theorem 2.1, Equation (9) has a unique positive solution \( x(t; t_0, \varphi) \) on \([t_0, +\infty)\). For illustrative purpose, in the following numerical simulation, we take \( \varphi(\theta) = 20e^{-2 \sin(10\theta)} \) for \( \theta \in [-1, 0] \). To visualize the effect of initial condition, we simulate the state trajectory from \( t = -1 \). It is shown in Figure 1 that the corresponding solution \( x(t; 0, \varphi) \) of Equation (9) is unbounded.

\subsection*{2.2. \textit{Global exponential convergence to the zero equilibrium point}}

In this section, we will establish conditions for the global exponential convergence to the zero equilibrium point of positive solutions of model (2).

Let us consider the following assumptions:

(A1) There exists \( m[\alpha] > 0 \) such that \( \sup_{t-r_j(t) \geq 0} \int_{t-r_j(t)}^{t} \alpha(s) \, ds \leq m[\alpha], j \in m \).
Figure 1. Unbounded state trajectory of (9) with ϕ(θ) = 20e^{-2\sin^2(100)}.

(A2) \lim_{t \to +\infty} \int_{0}^{t} \alpha(s) \, ds = +\infty.

(A3) \limsup_{t \to +\infty} \sum_{j=1}^{m} (\beta_j(t)/\alpha(t)) < 1.

Remark 2 If α(t) is assumed to be upper- and lower-bounded by positive constants then assumptions (A1) and (A2) are obviously removed.

Proposition 2.3 Let assumption (A3) holds. Then there exists a positive constant δ such that any solution x(t; t_0, ϕ) of Equation (2) satisfies

\[ x(t; t_0, ϕ) \leq \|ϕ\| + δ \quad \forall t \in [t_0, +\infty). \]  

Proof By (A3), there exists T > 0 such that

\[ \sum_{j=1}^{m} \frac{\beta_j(t)}{\alpha(t)} \leq 1 \quad \text{for all } t \geq T. \]  

Let us define

\[ δ = \frac{1}{e} \sum_{j=1}^{m} \int_{0}^{T} \frac{\beta_j(t)}{γ_j(t)} \, dt \]

then δ is a positive constant. Suppose x(t; t_0, ϕ) be a solution of Equation (2). Without loss of generality, we assume that t_0 ≤ T. From Equation (7), we have

\[ x(t) \leq \|ϕ\| + δ \quad \forall t \in [t_0, T]. \]
We will prove that Equation (12) holds for all $t \in [t_0, +\infty)$. For given $\epsilon > 0$, assume that there exists $\bar{t} > T$ such that
\[ x(\bar{t}) = \|\varphi\| + \delta + \epsilon, \quad x(t) < \|\varphi\| + \delta + \epsilon \quad \forall t \in [t_0 - \tau, \bar{t}). \]

Then, for $t \in [T, \bar{t})$, from Equation (2) we have
\[
x'(t) \leq -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t) \sup_{t-\tau \leq s \leq s} x(s) \leq -\alpha(t)x(t) + \alpha(t)(\|\varphi\| + \delta + \epsilon).
\]

Therefore,
\[
x(t) \leq \exp\left(-\int_{T}^{t} \alpha(s) ds\right) \left[ x(T) + (\|\varphi\| + \delta + \epsilon) \int_{T}^{t} \alpha(s) \exp\left(\int_{T}^{s} \alpha(u) du\right) ds \right] \leq x(T) \exp\left(-\int_{T}^{t} \alpha(s) ds\right) + (\|\varphi\| + \delta + \epsilon) \left(1 - \exp\left(-\int_{T}^{t} \alpha(s) ds\right)\right) \tag{13}
\]

Let $t \to \bar{t}$, from Equation (13) we obtain
\[
\epsilon \exp\left(-\int_{T}^{\bar{t}} \alpha(s) ds\right) \leq 0
\]
which yields a contradiction. Thus,
\[ x(t) < \|\varphi\| + \delta + \epsilon \quad \forall t \in [t_0, +\infty). \]

Let $\epsilon \to 0^+$ we finally obtain
\[
x(t) \leq \|\varphi\| + \frac{1}{e} \sum_{j=1}^{m} \int_{0}^{T} \beta_j(t) \frac{\gamma_j(t)}{\gamma_j(t)} dt \quad \forall t \geq t_0.
\]

This completes the proof. \[ \square \]

**Remark 3** It can be seen from the proof of Proposition 2.3 that, if Equation (11) holds for all $t \in \mathbb{R}^+$ then every solution $x(t; t_0, \varphi)$ of Equation (2) satisfies $x(t) \leq \|\varphi\|$ for all $t \geq t_0$.

We now prove the global exponential convergence to the zero equilibrium point of positive solutions of Equation (2) as given in the following theorem.

**Theorem 2.4** Under assumptions (A1)–(A3), all positive solutions of Equation (2) converge exponentially to the zero equilibrium point of Equation (2). More precisely, there exist positive constants $\kappa, \lambda, \delta$ such that every solution $x(t; t_0, \varphi)$ of Equations (2) and (3), with $\varphi \in C^1_0$, satisfies
\[
x(t; t_0, \varphi) \leq \kappa(\|\varphi\| + \delta)e^{-\lambda t_0} \int_{0}^{t} \alpha(s) ds, \quad t \geq t_0. \tag{14}
\]

**Remark 4** It is worth noting that, estimation of (14) and (A2) guarantee the global exponential convergence to zero of all positive solutions of (2) which we will refer to generalized exponential convergence.
Proof. Let \( x(t) = x(t; t_0, \varphi) \) be a solution of Equations (2) and (3). By Proposition 2.3, there exists a constant \( \delta > 0 \) such that
\[
x(t) \leq \| \varphi \| + \delta \quad \forall t \geq t_0.
\]
Also, by (A3),
\[
\sigma := \limsup_{t \to +\infty} \frac{\sum_{j=1}^{m} \beta_j(t)}{\alpha(t)} < 1,
\]
and hence, \( \frac{1-\sigma}{2} > 0 \). Therefore, there exists \( T_* \geq T \) (defined in Equation (11)) such that
\[
\sum_{j=1}^{m} \frac{\beta_j(t)}{\alpha(t)} \leq \sigma + \frac{1-\sigma}{2} = \frac{1+\sigma}{2} \quad \forall t \geq T_*. \tag{15}
\]
Furthermore, we can assume that \( t - \tau_j(t) \geq t_0 \) for all \( t \geq T_*, j \in m \). Then, by (A1), we have
\[
\int_{t-\tau_j(t)}^{t} \alpha(s) \, ds \leq m[\alpha] \quad \forall t \geq T_*, \ j \in m.
\]
Now, we consider the following scalar equation:
\[
H(\lambda) = \lambda + \frac{1+\sigma}{2} e^{\lambda m[\alpha]} - 1 = 0, \quad \lambda \in [0, +\infty). \tag{16}
\]
Note that, \( H(\lambda) \) is a continuous function, \( H(0) = -(1-\sigma)/2 < 0, \ H(\lambda) \to +\infty \) as \( \lambda \) tends to infinity and its derivative \( H'(\lambda) = 1 + m[\alpha](1+\sigma)/2 e^{\lambda m[\alpha]} > 0 \) for all \( \lambda \geq 0 \). Therefore, Equation (16) has a unique positive solution \( \lambda_* \). Moreover, \( H(\lambda) \leq 0 \) for all \( \lambda \in (0, \lambda_*) \). Then, it can be verified that
\[
\lambda_* \alpha(t) + \sum_{j=1}^{m} \frac{\beta_j(t)}{\alpha(t)} e^{\lambda_* \int_{t-\tau_j(t)}^{t} \alpha(s) \, ds} \leq H(\lambda) + 1
\]
for all \( t \geq T_*, \lambda \in (0, \lambda_*) \), from which we obtain
\[
\lambda_* \alpha(t) + \sum_{j=1}^{m} \beta_j(t) e^{\lambda_* \int_{t-\tau_j(t)}^{t} \alpha(s) \, ds} \leq \alpha(t) \quad \forall t \geq T_*. \tag{17}
\]
Let us consider the following function:
\[
v(t) = \kappa (\| \varphi \| + \delta) e^{-\lambda_* \int_{t_0}^{t} \alpha(s) \, ds}, \quad t \geq t_0,
\]
where \( \kappa = e^{\lambda_* \int_{t_0}^{t_*} \alpha(t) \, dt} \). Note that
\[
v(t - \tau_j(t)) = e^{\lambda_* \int_{t-\tau_j(t)}^{t} \alpha(s) \, ds} v(t),
\]
and thus, by Equation (17),
\[
v'(t) \geq -\alpha(t) v(t) + \sum_{j=1}^{m} \beta_j(t) v(t - \tau_j(t)) \quad \forall t \geq T_* \tag{18}
\]
We will show that
\[
x(t) \leq v(t) = \kappa (\| \varphi \| + \delta) e^{-\lambda_* \int_{t_0}^{t} \alpha(s) \, ds} \quad \forall t \geq t_0. \tag{19}
\]
For given \( \epsilon > 0 \), note that \( x(t) \leq v(t) \) for all \( t \in [t_0, T_*] \), we have \( x(T_*) < v(T_*) + \epsilon \). Assume that there exists a \( \tilde{t} > T_* \) satisfying \( x(\tilde{t}) = \epsilon + v(\tilde{t}) \), \( x(t) < \epsilon + v(t) \) for all \( t \in [T_*, \tilde{t}] \). Then, \( x(t) \leq
Taking integral on both sides of the above inequality we obtain
\[ 2.5 \]

\[ \text{Corollary} \]

many other works in the literature (e.g., [3,11,12,17,20,22]) then we obtain the following corollary.

\[ \alpha \]

\[ \epsilon \]

\[ 142 \]

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where
\[ \kappa > 0, \quad \text{and} \quad \sum_{j=1}^{m} \beta_j(t) \]

\[ \beta_j(t) \]

\[ \alpha(\tilde{t}) \]

\[ \sin 2t \]

\[ 2 \]

\[ 2.3. \quad \text{An example} \]

Example 2.6

Consider the following Nicholson’s blowflies model with time-varying delay
\[ x'(t) = -\alpha(t)x(t) + \beta(t)x(t - \tau(t))e^{-\gamma(t)x(t-\tau(t))}, \quad t \geq 0, \quad (21) \]

where
\[ \alpha(t) = \frac{1}{\sqrt{t+1}}, \quad \beta(t) = \frac{\sqrt{t}}{2t+1}, \quad \gamma(t) = 1 + \frac{|\cos t|}{t+1}, \quad \tau(t) = |\sin 2t|. \]

It should be noted that, for this model, the obtained results in the literature cannot be applied to conclude the convergence of positive solutions of Equation (21). In this case we have
\[ m[\alpha] = 1, \quad \int_{0}^{t} \alpha(s) = 2(\sqrt{1+t-1}) \to +\infty, \quad t \to +\infty, \]

\[ \beta(t) < \alpha(t), \quad \forall t \geq 0, \quad \sigma = \lim_{t \to +\infty} \frac{\beta(t)}{\alpha(t)} = \frac{1}{2}. \]
Therefore, assumptions (A1)–(A3) hold. The scalar equation $\lambda + \frac{3}{4}e^\lambda - 1 = 0$ has a unique positive solution $\lambda_* = 0.1385$. By Remark 3 and Theorem 2.4, every solution $x(t, \varphi)$ of Equation (21) with $\varphi \in C^r([-1, 0], \mathbb{R}^+), \varphi(0) > 0$, satisfies

$$x(t, \varphi) \leq \|\varphi\|e^{-0.277(\sqrt{t+1} - 1)}, \quad t \geq 0,$$

where $\|\varphi\| = \sup_{-1 \leq \theta \leq 0} \varphi(\theta)$. Furthermore, it can be seen that, there is no positive constant $\eta$ such that $2\sqrt{t+1} \geq \eta t + \nu, \nu \in \mathbb{R}$, for all $t \geq 0$. And thus, a classical exponential estimation, that is, $x(t, \varphi) \leq M\|\varphi\|e^{-\eta t}, \eta > 0, M > 0$, does not exist. Therefore, the exponential estimation proposed in this paper is less conservative and is expected to relax conditions for the exponential convergence of the model. In the following simulation, we take initial function $\varphi(\theta) = e^{-0.2 \sin^2(8\theta)}, \theta \in [-1, 0]$. As shown in Figure 2, the corresponding state trajectory of Equation (21) satisfies a generalized exponential estimation $x(t, \varphi) \leq e^{-0.277(\sqrt{t+1} - 1)}, t \geq 0$. Furthermore, a classical exponential estimation does not exist. For illustrative purpose, we take $\eta = 0.2$ then it can be seen in Figure 2 that $x(t, \varphi) \geq e^{-0.2t}$ for all $t \geq 20$.

3. Conclusion

This paper has dealt with the global existence and global asymptotic behaviour of positive solutions to a non-autonomous Nicholson’s blowflies model with time-varying delays. By using a new approach, we have derived sufficient conditions for the global generalized exponential convergence of positive solutions of the model without any restriction on the uniform positiveness of the per capita dead rate term. Numerical examples have been provided to illustrate the effectiveness of the obtained results.
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