Protection of qubits by nonlinear resonances

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Abstract We show that quantized superconducting circuits are non-integrable at the classical level of description, adorned by nonlinear resonances amidst stochastic sea. The stable (elliptic) and unstable (hyperbolic) points occur in a way that by choosing the parameters of a system close to elliptic points, the dynamics is stable. Quantum mechanically, any disturbance has to tunnel the separatrix to reach the elliptic point. Thus, nonlinearity of the system provides protection. Based on these fundamental considerations from the Kolmogorov–Arnold–Moser theorem, we propose criteria for protection of qubits from any disturbance.

Circuit quantum electrodynamics studies the properties of quantum circuits using Josephson junctions coupled to the photon modes of microwave cavities \cite{1,2}. There is a tremendous interest in the study of these systems for their connections with fundamental problems in open quantum systems, quantum engineering of states, and decoherence \cite{3}. One of the important workhorses for realizing a qubit is a transmon—a Josephson tunnel junction shunted by a capacitance. This is a weakly nonlinear oscillator derived from a Cooper pair box \cite{4,5}. Transmons are playing the same role in a superconducting quantum computer as played by “harmonic oscillator” and “particle in a box (billiards)” at the advent of quantum mechanics. The major difference, of course, is that these are nonlinear systems at the classical level. Moreover, their simplicity is misleading as they are manifestations of collective effects of superconductors \cite{6}. Nevertheless, the transitions among the low-lying levels can be given a description in a manner akin to Rabi oscillations in atomic systems \cite{4,7}. Thus, we have a nonlinear quantum system with rich physics, termed in common parlance as an “artificial atom.”

The nonlinear artificial atomic systems are examples of non-integrable systems of a special kind, well-known in the literature as quasi-integrable systems or the KAM (Kolmogorov–Arnold–Moser) systems \cite{8}. We characterize these systems by studying the classical phase space as well as the spectral properties for several configurations in which transmons have been coupled for applications in quantum computing employing superconducting qubits. The fluctuations in their spectra bear properties akin to the systems described by random matrix theory (RMT) for systems with mixed phase space \cite{9–12}. To recall, quasi-integrable systems are obtained upon analytic perturbation of an integrable Hamiltonian. As the strength of the perturbation increases, the invariant tori in the phase space break. Eventually, when the last invariant torus is broken, the system becomes chaotic. In its transition to chaos, the phase space is typically comprised of regions termed as stable islands around elliptic points, surrounded by a stochastic sea. This is the scenario envisaged by the celebrated work of Kolmogorov, Arnold, and Moser (KAM theorem) \cite{8,13,14}. Some of the spectacular phenomena resulting due to stability provided by nonlinear resonances are rings of Saturn and asteroid belts in our Solar system \cite{15,16}. Our study presents a connection between quantum computing \cite{17,18} and quantum chaos \cite{19–21}. Some of the well-known computing platforms introduce disorder or detuning via nonlinear coupling with resonators to protect the qubits \cite{22}, thus bringing these systems close to classically chaotic regime. As stated above, our proposal is based on enhanced stability due to KAM theory whereby a system residing in islands of stability, close to elliptic points, is largely immune to external disturbance.

We shall concentrate on inductively coupled transmons \cite{1,23} and the much discussed, \(0 - \pi\) qubit \cite{24}. Pure capacitively coupled transmon system is classically non-integrable. However, in the vicinity of a nonlinear resonance, the dynamics is integrable \cite{25}. On the other hand, inductively coupled transmons possess a parameter regime where the system gets trapped in a nonlinear resonance. Trapping in the islands and scattering off the islands have been studied in great detail, classically \cite{26,27} and quantum mechanically \cite{28,29}. Outside this regime, the resonant region opens up, making the system susceptible to possible undesirable interaction with environment or noise, leading to decoherence. The configuration, termed a \(0 - \pi\) qubit, was proposed by Kitaev and developed further by Brooks, Kitaev, and Preskill \cite{24}; recently, it is experimentally realized \cite{30}. At the classical level, we show that it possesses a compact phase space and the system is trapped in two nonlinear resonances \cite{8,26}. On the basis of the behavior of dynamical systems possessing trapping regions created due to nonlinear resonances, and the associated quantum chaos \cite{11}, we propose criteria for protection of qubits. We believe that this proposal will help design new protected configurations for

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qubits by making sure that there exist deep trapping regions provided by primary nonlinear resonances. While classically forbidden, these trapping regions are accessible quantum mechanically via tunneling. The tunneling rate can be estimated semiclassically on the basis of the change in adiabatic invariant [27] when a system crosses a separatrix.

Let us propose criteria which would help protection of qubits from decoherence or noise. The rest of the article constitutes arguments and calculations in support of these.

1. These systems belong to the class of quasi-integrable systems at the classical level of description, or, what are known as KAM systems. The perturbations to the classically integrable component may be smooth.

2. Due to this first point, there exists a hierarchy of islands of stability in the classical phase space, the primary resonance being the largest. The system parameters may be chosen so that the system sits close to the elliptic point of the primary island. This would provide a natural barrier to any external disturbance, which has to tunnel to reach the system.

3. If the available classical phase space is compact, then there will be even stronger protection.

Kolmogorov–Arnold–Moser (KAM) theorem provides the basis for the existence of invariants in nonlinear coupled systems. If an integrable system is perturbed smoothly, then the invariant curve is deformed. A nonlinear system under consideration is one whose frequencies are functions of action variables. When the ratio(s) of frequency(ies) is rational, the invariant tori break. For an integrable system, the deviation of the frequencies from quasi-rationality is limited to a neighborhood of stability which we exploit to protect the qubits.

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energy (in GHz) of the system, or tunnel in the protected region. Plotted against the separatrix passing through a hyperbolic point surrounded by two elliptic points. These elliptic points provide stability and protection from the external noise. To see this, we plot the square of the wavefunctions for $\beta_{12} = 10$ GHz in c of first three energy levels of the resonant Hamiltonian on top of the cosine potential (given by (6)) and are offset by their corresponding eigenenergies.

To quantify the protection, we show here a semiclassical estimate of the tunneling probability of a wavepacket across the separatix passing through a hyperbolic point. Plotted against the energy (in GHz) of the system, it shows the strength of the perturbation needed to tunnel out or tunnel in the protected region. Here, we have used the parameter values for $E_c$'s, $E_j$'s, and $\beta_{12}$ as in Fig. 1.

\begin{align}
+ E_{C_1}(4J^2I_2^2 - 8JI_2kR + 4k^2R^2) \\
+ \frac{\beta_{12}k}{2} \cos((l_1 - l_2)\phi)
\end{align}

for $m = n = k$; the general expression can be found but is quite cumbersome to be given here. We apply yet another canonical transformation $(R, \phi) \mapsto (P, \phi)$ employing the generating function, $W = (P + R_{\text{res}}(J))\phi$ with new variable $P = R - R_{\text{res}}$. We obtain the resonant Hamiltonian:

\begin{align}
H_{\text{res}}(P, \phi) &= 4k^2(E_{C_1} + E_{C_2})P^2 \\
&\quad + \frac{\beta_{12}k}{2} \cos((l_1 - l_2)\phi) + \Lambda(J),
\end{align}

where

\begin{align}
\Lambda(J) &= \frac{4E_{C_1}E_{C_2}(l_1 - l_2)^2}{E_{C_1} + E_{C_2}} J^2.
\end{align}

The values of the parameter that we have taken are: $E_{C_1} = 0.1$ GHz, $E_{C_2} = 0.3$ GHz, $E_{J_1} = 10$ GHz $E_{J_2} = 20$ GHz and $m/n = 1:1$ as primary resonance. The values of $E_c$'s and $E_j$'s are chosen for $\xi = \sqrt{\frac{2E_{C_1}}{E_{J_2}}} < 0.2$ [31], and accordingly, the value of $\beta$'s is chosen from [23]. Now, we are ready to exhibit the phase space surface corresponding to the resonance condition $\frac{E_{C_1}}{E_{C_2}} = E_{C_1}/E_{C_2} = 3$.

We can see that there are islands of stability in Fig. 1b for an inductively coupled transmon system. However, the wavefunction of the system can tunnel out, or, noise can tunnel into these islands across the separatix. The probability of tunneling in or the wavefunction of the system tunneling out depends on the energy of the system or on the intensity of the noise. In Fig. 2, we show the tunneling probability for the wavepacket to cross the separatix. This is calculated by using the WKB approximation, integrating the imaginary part of the action over the turning points of the potential barrier. From Fig. 1c, we see that the potential barrier is 5 GHz; therefore, the probability increases with energy of the system, approaching one as the energy tends to the barrier strength.

The (0 – π) qubit consists of identical pairs of small Josephson junctions, large shunting capacitors and superinductors, organized in a small closed-loop geometry with four nodes (Fig. 3a). It can be shown that this circuit has four degrees of freedom, denoted by (say) $\theta$, $\phi$, $\xi$, $\Sigma$ modes [30], corresponding to linear combinations of phase difference between the superconducting order parameter across the elements in the circuit. The $\phi$ and $\theta$ modes describe qubit degrees of freedom of the circuit with the two-mode Hamiltonian [30].

![Image](392x634 to 541x735)
Fig. 3  a The circuit diagram of the $0 - \pi$ qubit. There are four nodes, large capacitors and superinductors, b and c normal modes considered here with signs at the nodes representing their amplitudes.

Fig. 4  In a & c, we have the phase space plots of the Hamiltonian near 1:1 resonance, respectively, for $\Phi_{\text{ext}} = 0$ and $\Phi_{\text{ext}} = \pi$. Here, values of $\Phi_{\text{ext}}$ and other variables are taken from a recently reported experiment [30]. In b and d, ten lowest energy eigenfunctions of the resonant Hamiltonian are plotted along with the potential. Wavefunctions are shown along with corresponding eigenenergies. As expected from a & c, probability is higher in bounded regions.

$$H_{0-\pi} = 4E_C^\theta (n_\theta - n_\theta^g)^2 + 4E_C^\phi n_\phi^2 - 2E_J \cos \theta \cos (\phi - \pi \Phi_{\text{ext}}/\Phi_0) + E_L \phi^2,$$

(8)

where $\Phi_0 = h/2e$, $\Phi_{\text{ext}}$ is the external magnetic flux and $n_\theta^g$ is the offset-charge bias due to electrostatic environment.

To unravel the phase space structure of the classical Hamiltonian (8), we follow the same procedure as above and state the resonance condition between $\theta$- and $\phi$-modes:

$$m\omega_1 - n\omega_2 = m \frac{d\theta}{dt} - n \frac{d\phi}{dt} = m \frac{\partial H}{\partial n_\theta} - n \frac{\partial H}{\partial n_\phi} = mE_C^\theta (n_\theta - n_\theta^g) - nE_C^\phi n_\phi = 0.$$

(9)

Canonically transforming the Hamiltonian $(n_\theta, n_\phi, \theta, \phi) \mapsto (R, J, \Phi, \psi)$ using the generating function $W = (m\theta - n\phi)R - (l_1\theta - l_2\phi)J$, where $l_1, l_2$ are integers such that $ml_2 - nl_1 = 1$,

$$n_\theta = \frac{\partial W}{\partial R} = mR - l_1J, \quad n_\phi = \frac{\partial W}{\partial J} = -nR + l_2J,$$

$$\Phi = \frac{\partial W}{\partial R} = m\theta - n\phi, \quad \psi = \frac{\partial W}{\partial J} = -l_1\theta + l_2\phi.$$  

(10)
The resonance condition in the new coordinates is
\[
mE_C^0(mR - l_1J - n_g) - nE_C^\phi(-nR + l_2J) = 0,
\]
\[
R_{\text{res}} = \frac{(ml_1E_C^0 + nl_2E_C^\phi)J + mn_g}{m^2E_C^0 + n^2E_C^\phi}
\]
being the resonant Hamiltonian. Near resonance, integrating over the fast variable (\(\psi\)):
\[
H_1(R, J, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(R, J, \phi, \psi)d\psi
\]
\[
= 4E_C^0(J + n_g - R)^2 + 4E_C^\phi(R - 2J)^2
\]
\[
- E_J \cos(\Phi + \Phi_{\text{ext}}) + E_L \Phi^2 + \frac{\pi^2E_L}{3}.
\]
(12)
The expression for general \(m, n\) is found to be too cumbersome to be presented; moreover, for our purpose, the primary resonance (\(m/n = 1:1\)) is relevant. After another canonical transformation, \((R, J, \phi) \mapsto (P, \Phi)\) using the generating function \(W' = (P + R_{\text{res}}(J))\Phi\) where new variable \(P = R - R_{\text{res}}\), we get the Hamiltonian near resonance:
\[
H(P, \Phi) = 4(E_C^0 + E_C^\phi)P^2
\]
\[
+ \frac{(E_L(\pi^2 + 3\Phi^2) - 3E_J \cos(\Phi_{\text{ext}} + \Phi))}{3} + \Lambda(J),
\]
(13)
where \(\Lambda(J) = \frac{4E_C^0E_C^\phi(n_g-J)^2}{E_C^0 + E_C^\phi}\). We have taken the parameters [30] where \(E_C^0 = 92\ \text{MHz}, E_C^\phi = 1.14\ \text{GHz}, E_J = 6\ \text{GHz}, E_L = 0.38\ \text{GHz}, n_g = 0\). The resonance condition for \(1:1\) resonance is \(\frac{2\pi}{n_\phi} = \frac{E_C^\phi}{E_C^0} \simeq 12.39\). In Fig. 4a, c, we observe that the classical phase space surrounding the resonance condition is compact. The corresponding potentials and the states thereof are shown in Fig. 4b, d. These results satisfy our Criterion 3, where we would like the system to be quasi-integrable.

Quantum circuits can be designed according to details of the desired trapping region with the choice of suitable parameters. The primary resonance condition suggests a relation among the system parameters like the values of capacitances, inductances and so on. As illustrated by the examples considered above, we have quantitatively shown that a wavepacket will have a negligible tunneling probability to interact with the transmon system situated deep inside the primary island (Fig. 2, for instance). The quantum circuits can be manipulated nevertheless, or tuned, by introducing time-dependent, well-controlled external probes. For instance, a time-dependent flux can tune a 0 \(\rightarrow\) \(π\) qubit in a way that we desire. A systematic gauge-invariant Hamiltonian for such systems coupled to resonators provides an interesting possibility for design of architecture in two and three dimensions [32].

The cases considered in this article were lower dimensional. We would like to comment on the validity of the criteria to a chain of qubits where the dimensionality of phase space of the corresponding classical system would be high. Let us recall that the fundamental Poincaré–Birkhoff theorem, forming the basis of our arguments is valid in any dimension. Also, the idea of nonlinear resonances is universal. Due to higher dimensionality, there would be several resonance conditions to be simultaneously satisfied. This would constrain the values of the parameters quite strongly. In some of the cases, these constraints could guide consideration of new designs. Also, quantum tunneling in the wake of phase space diffusion (like Arnold diffusion) in higher-dimensional systems could make protection difficult. Thus, as stated in the criteria, it would be in best interests to design qubit architecture so that the underlying classical system is quasi-integrable. We imagine that this suggests designing near-symmetric architectures, so that certain invariants persist. To give an example, in the design of current-mirror qubit, the coupling capacitances (which are large) could all be kept at the same value.

To conclude, we have presented criteria for protection of qubits by exploiting nonlinearity, drawing inspiration from well-known theory of stability of classical nonlinear systems and its quantum analogues.

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Two capacitively, nonlinearly coupled transmons have two degrees of freedom, possessing two frequencies. The frequencies are functions of action variables, their rational ratios present nonlinear resonances. Presence of such a resonance modifies the phase space in an interesting way, with a chain of stable (elliptic) and unstable (hyperbolic) fixed points, owing to the celebrated Poincaré-Birkhoff theorem [14].

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