On Einstein hypersurfaces of a remarkable class of Sasakian manifolds

Dario Di Pinto*¹ and Antonio Lotta**²

¹,²Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy.

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Abstract

We present a non existence result of complete, Einstein hypersurfaces tangent to the Reeb vector field of a regular Sasakian manifold which fibers onto a complex Stein manifold.

Key words: Einstein hypersurface · regular Sasakian manifold · Stein manifold.

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1 Introduction

In [4], I. Hasegawa established that a Sasakian space form with nonconstant sectional curvature admits no Einstein hypersurfaces. The aim of this note is to prove a new non existence result concerning Einstein hypersurfaces of a relevant class of regular Sasakian manifolds:

Theorem. If \((M, \varphi, \xi, \eta, g)\) is a regular Sasakian manifold which fibers onto a complex Stein manifold, then \(M\) does not admit any complete Einstein hypersurface tangent to \(\xi\).

* e-mail: dario.dipinto@uniba.it
** e-mail: antonio.lotta@uniba.it
We recall that a contact manifold \((M, \eta)\) is called regular provided the Reeb vector field \(\xi\) of the contact form \(\eta\) is, i.e. it determines a regular 1-dimensional foliation on \(M\), so that the space \(B = M/\xi\) of maximal integral curves of \(\xi\) is a manifold. When \(M\) carries a Sasakian metric \(g\) associated to \(\eta\), yielding a Sasakian structure \((\varphi, \xi, \eta, g)\) (we use the standard terminology and notation according to [2]), since \(L_\xi \varphi = 0\) and \(L_\xi g = 0\), \(g\) induces in a natural way a metric \(g'\) on \(M/\xi\) and \(\varphi\) also descends to an almost complex structure \(J\). Denoting by \(\pi : M \to B\) the canonical projection, it turns out by construction that \(\pi\) is a Riemannian submersion with \(\ker (d\pi)_x = \mathbb{R}\xi_x\) for every \(x \in N\), and

\[d\pi \circ \varphi = J \circ d\pi\]

and, moreover \((B, J, g')\) is a Kähler manifold (see for instance [9] and [8]). Hence our assumption on the Sasakian manifold is that \(B\), as a complex manifold, can be realized (up to a biholomorphism) as a closed complex submanifold of some Euclidean space \(\mathbb{C}^d\).

For instance, according to a result due to H. Wu ([3], Theorem 4.9), it is known that every simply connected, complete Kähler manifold with non-positive sectional curvature is a Stein manifold; in particular, the Hermitian symmetric spaces of non-compact type are Stein manifolds, hence Takahashi’s Sasakian globally \(\varphi\)-symmetric spaces of non-compact type (see [10]) provide a wide class of examples of Sasakian manifolds to which our result applies.

The proof of our result makes use of the natural CR structure of \(CR\) codimension 2 which is induced over any smooth hypersurface \(N \subset M\), under the assumption that \(N\) is everywhere tangent to the Reeb vector field \(\xi\) (see for instance [7] and section [2]). We establish a basic formula relating the Ricci tensor of \(N\) and the trace of a distinguished scalar Levi form of this \(CR\) structure (see [3.4]), implying that, in the Einstein case, \(\pi(N)\) is a weakly pseudoconcave real hypersurface of \(B\). Hence a non-compactness result by D. Hill and M. Nacinovich for weakly pseudoconcave \(CR\) submanifolds of Stein manifolds [5] is invoked to get the conclusion.

## 2 Preliminaries

Let’s start by recalling the definitions of \(CR\) manifolds, Levi-Tanaka forms and scalar Levi forms. In the following, given a vector bundle \(E\) over a smooth differential manifold \(M\), we will denote by \(\Gamma(E)\) the \(\mathcal{C}^\infty(M)\)-module
of global smooth sections of $E$.

Let $M$ be a smooth real manifold of dimension $n$, and let $m, k \in \mathbb{N}$ such that $2m + k = n$. If $HM$ is a real vector subbundle of rank $2m$ of the tangent bundle $TM$ and $J : HM \to HM$ is a bundle isomorphism such that $J^2 = -Id$, the couple $(HM, J)$ is called a CR structure on $M$ if the following properties hold for all $X, Y \in \Gamma(HM)$:

(i) $[JX, JY] - [X, Y] \in \Gamma(HM)$;

(ii) $N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$.

In this case $(M, HM, J)$ is called a CR manifold of type $(m,k)$ and $m, k$ are the CR dimension and the CR codimension of the CR structure, respectively.

**Remark 2.1.** Let $S$ be a real submanifold of a complex manifold $(M, J)$ and for any $p \in S$ set $H_pS := T_pS \cap J(T_pS)$. Because of the integrability of $J$, the couple $(HS, J|HS)$ canonically defines a CR structure on $S$ if the dimension of $H_pS$ is constant. In this case $S$ is termed a CR submanifold of $M$.

In particular, this condition is always satisfied when $S$ is a real hypersurface of $M$ and hence $S$ is a CR manifold of CR codimension 1.

**Definition 2.2.** Let $(M, HM, J)$ be a CR manifold of type $(m,k)$. Given a point $x \in M$, the Levi-Tanaka form of $M$ at $x$ is the bilinear map $L_x : H_x M \times H_x M \to T_x M / H_x M$ defined by

$$L_x(X, Y) := p_x([\tilde{X}, J\tilde{Y}]_x) \quad \forall X, Y \in H_x M,$$

where $\tilde{X}, \tilde{Y} \in \Gamma(HM)$ are two arbitrary extensions of $X, Y$ and $p : TM \to TM/HM$ is the canonical projection on the quotient bundle $TM/HM$.

It is known that $L_x$ is well defined, i.e. the value $p_x([\tilde{X}, J\tilde{Y}]_x)$ only depends on the values of $\tilde{X}, \tilde{Y}$ at $x$, that is on $X$ and $Y$.

Moreover, according to (i) above, $L_x$ turns to be a vector valued symmetric Hermitian form on the holomorphic tangent space $H_x M$ with respect to the complex structure $J := J_x$, that is

$$L_x(X, Y) = L_x(JX, JY), \quad L_x(X, Y) = L_x(Y, X)$$

(2.2)
for all $X, Y \in H_xM$.

Given a point $x$ on the CR manifold $(M, HM, J)$, we will denote by

$$H^0_x \overset{\tiny{\text{def}}}{=} \{ \omega \in T^*_xM \mid \omega(X) = 0 \quad \forall X \in H_xM \}$$

the annihilator of $H_xM \subset T_xM$. Then we recall the following definition.

**Definition 2.3.** Let $(M, HM, J)$ be a CR manifold, $x \in M$ and $\omega \in H^0_xM$. The Hermitian form

$$L_\omega : H_xM \times H_xM \to \mathbb{R} \quad \text{s.t.} \quad L_\omega(X, Y) := \omega L_x(X, Y) \quad (2.3)$$

is called the *scalar Levi form determined by* $\omega$ at $x$.

The next lemma represents a sort of naturality property of the Levi-Tanaka form with respect a particular class of maps between CR manifolds which preserve the CR structures.

**Definition 2.4.** Let $(M, HM, J)$ and $(N, HN, J')$ be two CR manifolds. A smooth map $\pi : M \to N$ is called CR map if $d\pi(HM) \subset HN$ and $d\pi \circ J = J' \circ d\pi$.

**Lemma 2.5.** Let $(M, HM, J)$ and $(N, HN, J')$ be two CR manifolds having the same CR dimension, let $\pi : M \to N$ be a CR map and assume that for every $x \in M$, $(d\pi)_x : H_xM \to H_{\pi(x)}N$ is an isomorphism. Then, given $x \in M$, the following diagram commutes:

$$
\begin{array}{ccc}
H_xM \times H_xM & \overset{L_x}{\longrightarrow} & T_xM/H_xM \\
\pi_* \times \pi_* & & \pi_* \\
H_yN \times H_yN & \overset{L'_y}{\longrightarrow} & T_yN/H_yN
\end{array}
$$

where $y = \pi(x)$, $\pi_* = (d\pi)_x$ and $L_x, L'_y$ are the Levi-Tanaka forms of $M$ and $N$ respectively.

**Proof.** Let us denote by $p_x : T_xM \to T_xM/H_xM$ and $q_y : T_yN \to T_yN/H_yN$ the canonical projections. As an immediate consequence of the definition of CR map, the differential $\pi_*$ descends to the quotient and, with abuse of notation, we still denote the quotient map by $\pi_* : T_xM/H_xM \to T_yN/H_yN$. Now consider $X, Y \in H_xM$: according to (2.1), $L'_y(\pi_*X, \pi_*Y) = q_y[Z, J'W]_y$, where

$$Z = p_xL_x(X) \quad \text{and} \quad W = p_xL_x(Y).$$
where $Z, W \in \Gamma(HN)$ are two extensions of $\pi_*X$ and $\pi_*Y$. Since for every $a \in M$, $(d\pi)_a : H_aM \to H_{\pi(a)}N$ is an isomorphism, we can define two extensions $\tilde{X}, \tilde{Y} \in \Gamma(HM)$ of $X$ and $Y$ respectively by putting

$$
\tilde{X}_a := (d\pi)^{-1}_a(Z_{\pi(a)}), \quad \tilde{Y}_a := (d\pi)^{-1}_a(W_{\pi(a)}).
$$

It turns out that $\tilde{X}$ and $\tilde{Y}$ are $\pi$-related to $Z$ and $W$ respectively, and hence $[\tilde{X}, J\tilde{Y}]$ is $\pi$-related to $[Z, J'W]$ too, since $d\pi$ commutes with the almost complex structures $J$ and $J'$. Finally, we have:

$$
\pi_* L_x(X, Y) = \pi_*(p_x[\tilde{X}, J\tilde{Y}]_x) = q_y(\pi_*[\tilde{X}, J\tilde{Y}]_y) = q_y[Z, J'W]_y = L'_y(\pi_*X, \pi_*Y).
$$

\[ \square \]

**Corollary 2.6.** In the same hypothesis and notation of the previous Lemma, for every $\psi \in H^0_0N$ one has that $\pi^*L'_y \psi = L_\pi^*\psi$.

We remark that the scalar Levi forms $L_\omega$ are symmetric and hence it makes sense to consider their index $i(L_\omega)$, defined as the minimum between the number of positive and negative eigenvalues of $L_\omega$.

More specifically, we recall the following terminology from CR geometry; see for instance [6].

**Definition 2.7.** Let $(M, HM, J)$ be a CR manifold of type $(m, k)$ and let $x \in M$.

- $M$ is called **pseudoconvex** at $x$ if $L_\omega$ is positive definite for some $\omega \in H^0_xM$.
- If there exists a global section $\omega \in \Gamma(H^0M)$ such that $L_\omega$ is positive definite at each point $x \in M$, $M$ is called **strongly pseudoconvex**.
- $M$ is said **pseudoconcave** at $x$ if $i(L_\omega) > 0$ for every $\omega \in H^0_xM$, $\omega \neq 0$.
- $M$ is said **weakly pseudoconcave** at $x$ if $L_\omega = 0$ or $i(L_\omega) > 0$ for every $\omega \in H^0_xM$.

In this regard we recall that a Sasakian manifold $(M, \phi, \xi, \eta, g)$, as defined in [2], is a particular kind of strongly pseudoconvex CR manifold of hypersurface type, i.e. of CR codimension 1. We shall refer to [2] for the notation and basic facts concerning Sasakian geometry. We only remark that in this case the CR structure is given by the contact distribution $D = \ker \eta = \langle \xi \rangle ^\perp$ and the almost complex structure is $J = \phi|_D$. Therefore, for any $x \in M$,
$H^0_x M$ is spanned by $\eta_x$ and, up to scaling, we have only one scalar Levi form $\mathcal{L}_{\eta_x}$. Moreover, since $M$ is a contact metric manifold, the identity
\[ d\eta(X,Y) = g(X,\varphi Y) \]
yields that
\[ \mathcal{L}_{\eta_x} = 2g_x|_{H_x M \times H_x M}. \]

We end this section by recalling the definition of Stein manifold (for more information, see for instance [3]) and a theorem due to Hill and Nacinovich [5, 6], which provides a basic restriction to the topology of $CR$ weakly pseudoconcave submanifolds of a Stein manifold.

**Definition 2.8.** A *Stein manifold* is a closed complex submanifold of $\mathbb{C}^d$, for some $d \geq 1$.

**Theorem 2.9.** Every weakly pseudoconcave $CR$ submanifold of a Stein manifold cannot be compact.

## 3 Main result

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N$ be a hypersurface of $M$, tangent to the Reeb vector field $\xi$. At each point $x \in N$, let us consider the linear subspace of $T_x N$ defined by
\[ H_x N := \{ X \in T_x N \mid X \perp \xi_x \text{ and } \varphi X \in T_x N \}. \]
Observe that, if $\nu \in T_x N^\perp$ is a unit normal vector at $x$, then we have the following orthogonal decomposition:
\[ T_x N = \langle \xi_x \rangle \oplus \langle \varphi \nu \rangle \oplus H_x N. \]
It follows that $HN$ is a subbundle of $TN$ with constant rank and in [7] M. Munteanu proved that the couple $(HN, \varphi|_{HN})$ defines a $CR$ structure of $CR$ codimension 2 on $N$. We remark that he assumes the orientability of $N$, but this is unnecessary for our aim and the result holds true even if $N$ is not orientable.

The $CR$ structure $(HN, \varphi|_{HN})$ on the hypersurface $N$ allows us to consider, for every unit normal vector $\nu$, the scalar Levi form $\mathcal{L}_\omega$ attached to the covector
\[ \omega(X) = g_x(X, \varphi \nu) \quad \forall X \in T_x N. \quad (3.1) \]
We shall denote this scalar Levi form with the symbol $\mathcal{L}_\nu$, and in the following proposition we establish the relationship between $\mathcal{L}_\nu$ and the second fundamental form of the hypersurface $N$.

**Proposition 3.1.** Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N \subset M$ be a hypersurface, tangent to $\xi$, with second fundamental form $\alpha$. Let $\nu$ be a unit normal vector at some point $x \in N$. Then one has:

$$\mathcal{L}_\nu(X, X) = g_x(\alpha(X, X) + \alpha(\varphi X, \varphi X), \nu) \tag{3.2}$$

for every $X \in H_x N$.

**Proof.** First we recall that Sasakian manifolds are characterized by means of the following identity, involving the covariant derivatives of $\varphi$ with respect to the Levi-Civita connection (see [2]):

$$(\nabla_X \varphi) Y = g(X, Y)\xi - \eta(Y)X. \tag{3.3}$$

Now, fix $x \in N$, $X \in H_x N$ and consider a smooth section in $\Gamma(HN)$ which extends $X$ and a local normal vector field extending $\nu$. Then $\varphi X$ is again tangent to $N$. Using the fact that $X$, $\varphi X$ and $\varphi \nu$ are all orthogonal to $\xi$ and identity (3.3), we get:

$$\mathcal{L}_\nu(X, X) =$$

$$= g_x([X, \varphi X], \varphi \nu) =$$

$$= g_x(\nabla_X \varphi X, \varphi \nu) - g_x(\nabla_{\varphi X} X, \varphi \nu) =$$

$$= g_x(\varphi \nabla_X X, \varphi \nu) + g_x(\varphi \nabla_{\varphi X} X, \nu) =$$

$$= g_x(\nabla_X X, \nu) + g_x(\nabla_{\varphi X} \varphi X, \nu) =$$

$$= g_x(\alpha(X, X) + \alpha(\varphi X, \varphi X), \nu).$$

We shall use this formula to establish an identity relating the trace (with respect to $g$) of $\mathcal{L}_\nu$ and the Ricci tensor field of $N$. Hereinafter we will denote with an overline the relevant geometric entities of the hypersurface $N$ (Levi-Civita connection, curvature, etc.).

**Proposition 3.2.** Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N \subset M$ be a hypersurface tangent to $\xi$. Let $x \in N$ and let $\nu \in T_x N^\perp$ be a unit normal vector. Then one has:

$$\overline{\text{Ric}}(\xi, \varphi \nu) = \frac{1}{2} \text{tr}(\mathcal{L}_\nu). \tag{3.4}$$
Proof. By a well known property of Sasakian manifolds (see [2]), for every $X \in \mathfrak{X}(M)$,

$$R(\xi, X)X = g(X, X)\xi - \eta(X)X.$$  \hfill (3.5)

Then, given $X \in \Gamma(HN)$, since $\xi$ and $X$ are normal to $\varphi\nu$, it follows that

$$R(\xi, X, \varphi\nu, X) = g(R(\xi, X)X, \varphi\nu) = 0.$$  \hfill (3.6)

Since $\varphi X = -\nabla_X \xi$ is still tangent to $N$, we also deduce that the normal component of $\nabla_X \xi$ vanishes, i.e. $\alpha(X, \xi) = 0$. Moreover,

$$\alpha(X, \varphi\nu) = g(\nabla_X \varphi\xi, \nu) = g(-\varphi^2 \nu, \nu) = \nu.$$  \hfill (3.7)

Therefore, by using the Gauss formula, for every $X \in \Gamma(HN)$ we have that

$$\overline{R}(\xi, X, \varphi\nu, X) = g(\alpha(X, X), \nu).$$  \hfill (3.8)

Thus, fixed a local orthonormal frame of $TN$ of type $\{\xi, \varphi\nu, E_i, \varphi E_i\}_{i=1,...,n-1}$, with $E_i, \varphi E_i \in \Gamma(HN)$, from (3.8) and (3.2) we get:

$$\overline{\text{Ric}}(\xi, \varphi\nu) = \sum_{i=1}^{n-1} \left[ R(\xi, E_i, \varphi\nu, E_i) + \overline{R}(\xi, \varphi E_i, \varphi\nu, \varphi E_i) \right]$$

$$= \sum_{i=1}^{n-1} g(\alpha(E_i, E_i) + \alpha(\varphi E_i, \varphi E_i), \nu)$$

$$= \sum_{i=1}^{n-1} \mathcal{L}_\nu(E_i, E_i) = \frac{1}{2} \text{tr}(\mathcal{L}_\nu),$$

where the last equality follows from the fact that $\mathcal{L}_\nu$ is Hermitian and symmetric. \qed

Now we come to the proof of our main result.\hspace{1em} \textbf{Theorem 3.3.} If $(M, \varphi, \xi, \eta, g)$ is a regular Sasakian manifold which fibers onto a complex Stein manifold, then $M$ does not admit any complete Einstein hypersurface tangent to $\xi$.

\textit{Proof.} Assume by contradiction that $M$ admits a complete Einstein hypersurface $N$ tangent to $\xi$, with Einstein constant $c$.

Let $\nabla$ be the Levi-Civita connection of $N$. Since $\nabla_\xi \xi = 0$, from the Gauss
equation we deduce that $\nabla_\xi \xi = 0$. Moreover, since $\xi$ is a Killing vector field on $N$, the operator $A_\xi := -\nabla_\xi$ is skew-symmetric and hence

$$\overline{\text{Ric}}(\xi, \xi) = -\text{div}(A_\xi \xi) - \text{tr}(A_\xi^2) = -\text{tr}(A_\xi^2) \geq 0.$$ 

It follows that

$$c = cg(\xi, \xi) = \overline{\text{Ric}}(\xi, \xi) \geq 0.$$ 

If $c = 0$, then $A_\xi = 0$, i.e. $\xi$ is $\nabla$-parallel and this leads to a contradiction. Indeed, if we consider $X \in \Gamma(HN)$, with $X \neq 0$, from $\nabla_X \xi = 0$ and the Gauss equation we would get

$$-\varphi X = \nabla_X \xi = \alpha(X, \xi),$$

where $-\varphi X \in \Gamma(HN)$ is non zero and tangent to $N$, while $\alpha(X, \xi)$ is normal. Therefore $c > 0$ and, because of completeness of $N$, Myers’ theorem ensures that $N$ is compact.

Moreover, from Proposition 3.2 we have:

$$\text{tr}(L_\nu) = 2\overline{\text{Ric}}(\xi, \varphi \nu) = 2cg(\xi, \varphi \nu) = 0. \quad (3.9)$$

Now, let $\pi : M \to M/\xi$ be the canonical projection, where $(M/\xi, J, g')$ is a Stein manifold; $\pi$ is a Riemannian submersion whose fibers are 1-dimensional submanifolds of $M$ tangent to $\xi$ and

$$d\pi \circ \varphi = J \circ d\pi. \quad (3.10)$$

Since at every $x \in M$, $\ker(d\pi)_x = \mathbb{R}\xi_x$, we have that $\pi|_N : N \to M/\xi$ has constant rank. Hence, according to Theorem 3.5.18 in [1], $S := \pi(N)$ is a smooth hypersurface of $M/\xi$ and it carries a $CR$ structure (defined as in Remark 2.2), having the same $CR$ dimension of $N$. Moreover, (3.10) implies that $\pi : N \to S$ is a $CR$ map, such that at every point $x \in N$ the differential $(d\pi)_x : H^0_x N \to H^0_{\pi(x)} S$ is an isomorphism.

Fix a point $y = \pi(x) \in S$, with $x \in N$; if $\psi \in H^0_y S$, then $\pi^* \psi$ belongs to the vector space $H^0_x N$, which is spanned by $\omega$ and $\eta$, with $\omega$ as in (3.1). Actually, if $\pi^* \psi = \alpha \omega + \beta \eta$, for some numbers $\alpha, \beta$, evaluating at $\xi$ we obtain $\beta = 0$ and hence $\pi^* \psi = \alpha \omega$. Using Corollary 2.6 we get

$$\pi^* \mathcal{L}_\psi = \mathcal{L}_{\pi^* \psi} = \alpha \mathcal{L}_\nu$$

and by (3.9) we conclude that $\text{tr}(\mathcal{L}_\psi) = 0$, so that $\mathcal{L}_\psi = 0$ or $i(\mathcal{L}_\psi) > 0$. Therefore $S$ is a compact weakly pseudoconcave $CR$ hypersurface of the complex Stein manifold $M/\xi$, thus contradicting Theorem 2.9. □
With just a small change in the previous proof, we also get the following result.

**Theorem 3.4.** If $(M, \varphi, \xi, \eta, g)$ is a regular Sasakian manifold which fibers on a complex Stein manifold, then $M$ cannot admit any compact hypersurface $N$, tangent to $\xi$ and such that at any point of $N$ $\xi$ is an eigenvector of the Ricci operator $Q$ of $N$.

**Proof.** It suffices to note that if $Q\xi = \alpha \xi$ along $N$, with $\alpha \in C^\infty(N)$, then one has
\[
\text{tr}(\mathcal{L}_\nu) = 2\text{Ric}(\xi, \varphi \nu) = 2g(Q\xi, \varphi \nu) = 0.
\]
Hence the proof ends with the same argument of the previous one. \qed

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