Polymer plats and multicomponent anyon gases

Franco Ferrari\textsuperscript{1}, Jaroslaw Paturej\textsuperscript{1,2}, Marcin Piątek\textsuperscript{1,3} and Yani Zhao\textsuperscript{1}

\textsuperscript{1} CASA* and Institute of Physics, University of Szczecin, Wielkopolska 15, 70451 Szczecin, Poland
\textsuperscript{2} Department of Chemistry, University of North Carolina, Chapel Hill, North Carolina 27599-3290, United States
\textsuperscript{3}Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980, Dubna, Russia

Anyon systems are studied in connection with several interesting applications including high $T_C$ superconductivity and topological quantum computing. In this work we show that these systems can be realized starting from directed polymers braided together to form a nontrivial link configuration belonging to the topological class of plats. The statistical sum of a such plat is related here to the partition function of a two-component anyon gas. The constraints that preserve the topological configuration of the plat are imposed on the polymer trajectories using the so-called Gauss linking number, a topological invariant that has already been well studied in polymer physics. Due to these constraints, short-range forces act on the monomers or, equivalently, on the anyon quasiparticles in a way that closely resembles the appearance of reaction forces in the constrained systems of classical mechanics. If the polymers are homogeneous, the anyon system reaches a self-dual point, in which these forces vanish exactly. A class of self-dual solutions that minimize the energy of the anyons is derived. The two anyon gas discussed here obeys an abelian statistics, while for quantum computing it is known that nonabelian anyons are necessary. However, this is a limitation due to the use of the Gauss linking invariant to impose the topological constraints, which is a poor topological invariant and is thus unable to capture the nonabelian characteristics of the braided polymer chains. A more refined treatment of the topological constraints would require more sophisticated topological invariants, but so far their application to the statistical mechanics of linked polymers is an open problem.
I. INTRODUCTION

Knots and links are a fascinating subject and are researched in connection with many concrete applications both in physics and biology [1–25]. In this paper we study the statistical mechanics of a system of two entangled polymer rings. Mathematically, two or more entangled polymers form what is called a link. Single polymer rings form instead knots. We will restrict ourselves to systems in the configurations of $2s$–plats. Roughly speaking, $2s$–plats are knots or links obtained by braiding together a set of $2s$ strings and connecting their ends pairwise [26]. A physical realization of $2s$–plats could be that of two rings topologically entangled together and with some of their points attached to two membranes or surfaces located at different heights. In nature $2s$–plats occur for example in the DNA of living organisms [11, 23, 24, 27]. Indeed, it is believed that most knots and links formed by DNA are in the class of $4$–plats [11]. These biological applications have inspired the research of Ref. [28], in which $4$–plats have been studied with the methods of statistical mechanics and field theory. In particular, in [28] it has been established an analogy between polymeric $4$–plats and anyons, showing in this way the tight relations between two component systems of quasiparticles and the theory of knots and links. After the publication of [28], interesting applications of analogous anyon systems to topological quantum computing has been proposed [29, 31]. These applications are corroborated by the results of experiments concerning the detection of anyons obeying a nonabelian statistics, see for example [32]. While these results have appeared in 2005 and are still under debate [31, 33], other systems in which non-abelian anyon statistics could be present have been discussed [34, 35]. In the present case, the topology of the original two-polymer link is distinguished by the Gauss linking invariant, which can be obtained from the amplitudes of an abelian BF model [36]. This implies that the statistics of the quasiparticles treated here is purely abelian. However, also abelian anyons may be exploited for quantum computations as it has been argued in Ref. [37].

Motivated by these recent advances, we study here the general case of $2s$–plats formed by two polymer rings. Among all knot and link configurations, the class of $2s$–plats is very special. For instance, it is possible to decompose the trajectory of a $2s$–plat into a set of $2s$ open subtrajectories that can be further interpreted as the trajectories of $2s$ polymer chains directed along a special direction. Without losing generality, we may suppose that
this direction coincides with the $z$–axis. When the system of directed polymers is mapped into a field theory, a model describing anyon quasiparticles is obtained. The $z$ coordinate can be related to "time", while the monomer densities of the $2s$ directed polymers become the quasiparticle densities of a multicomponent anyon model. A remarkable feature of polymers in $2s$–plat configurations is that they admit self-dual solutions in which the energy of the system is minimized \[28\]. Here we show that these solutions can be explicitly constructed by solving a sinh-Gordon equation. The conformations corresponding to such solutions should be particularly stable and thus observable, at least in principle. With the present technologies \[38\], in fact, it is possible to realize polymer $2s$–plats in the laboratory.

Another advantage of restricting ourselves to $2s$–plats configurations is that it is possible to distinguish their topological states in a more efficient way than what one could achieve in the general case of two linked polymer rings. Let us recall at this point that the trajectories of real polymers are impenetrable and thus, if no rupture occurs, they are bound to stay in the initial topological state while subjected to thermal fluctuations. However, in the Edwards' model used here polymers are "phantom" \[39\]. Without any control, their trajectories are allowed to cross themselves and thus the global topological configuration of the system may change. To find a powerful and reliable method in order to forbid such changes of topology is the most difficult problem of the statistical mechanics of polymer knots and links. Up to now there is no analytical model that is able to deal with the statistical mechanics of polymer knots. For this reason, in this work we assume that each polymer ring composing the link can be in any knot configuration. Only the topological configurations of the link formed by polymers belonging to the $2s$–plat will be distinguished. This goal is achieved by using the Gauss linking number in order to impose the necessary topological constraints. The Gauss linking number is a topological invariant given in the form of a double contour integral, where the contours coincide with the polymer trajectories. Unfortunately, it is a weak topological invariant, so that many nonequivalent topological configurations characterized by the same value of the Gauss linking number are allowed. However, once we restrict ourselves to a given $2s$–plat, we are implicitly imposing a much more stringent topological condition on the system. Indeed, its topological states are in this way not only limited by the value of the Gauss linking number, but are also forced to vary within the much smaller set of states that are compatible with the structure of the $2s$–plat.

Even if the Gauss linking number is one of the simplest topological invariants, its expres-
sion is very complicated. As a consequence, after imposing the topological constraints, the action of a system of topologically entangled polymers becomes both nonlocal and non-polynomial. The nonlocality is due to the double contour integral over the polymer trajectories. The nonpolynomiality arises from the fact that the integrand is a non-polynomial function of the components of the radius vectors determining the positions of the monomers in the space. The situation is somewhat reminiscent to that of holonomic constraints in the classical mechanics of particles. When these constraints are fixed by means of Lagrange multipliers, within the particle action new terms appear which are related to the reaction forces. The striking difference is that topological constraints are not holonomic and have "memory" in order to keep track of the global conformation of the chain. This last property causes the nonlocality of the action after fixing the constraints. The price to be paid to recover locality and to have a standard action is to introduce topological fields which interact with the monomers in such a way that the topological configuration of the link is preserved. To some extent, these interactions may be considered as the equivalents of the reaction forces in classical mechanics. The passage to the topological field theory description is not straightforward. In particular, it requires to find a topological field theory with an amplitude of metric independent and gauge invariant operators from which it is possible to isolate the particular topological invariant used to fix the topological constraints. If the topological invariant is the Gauss linking number and the two topologically linked rings are represented as continuous curves embedded in the space and parametrized by their arc-lengths, this task has been achieved in [40, 41]. In the present case, the two rings are constructed out of a set of $2s$ open subtrajectories parametrized by the $z$ coordinate and not by the arc-length. This parametrization is very peculiar because it identifies the parameter specifying the positions of the monomers with one coordinate of the space in which the monomers are fluctuating. For all the above reasons, the passage to topological field theories explained in [40, 41] cannot be straightforwardly applied to the present situation and has required a separate derivation. As a result of this derivation, we have been able to show that the path integral expressing the probability function of a $2s$–plat formed by two polymer rings entangled together is equivalent to the correlation function of a gas of $2s_1$ particles of type 1 and $2s_2$ particles of type 2, where $s_1 + s_2 = s$. The interactions between these particles are mediated by the vector fields of an abelian BF model. This is a topological gauge field theory that has been discussed in [36, 42]. The particles are also subjected to short-range interactions whose ori-
gin is the following. The 2s directed paths composing the two-polymer link are treated here 
like paths of directed polymers in random media \cite{43, 44}, which are subjected to quenched 
random potentials. After integrating over the random noise according to the prescriptions 
of Ref. \cite{43}, in the polymer action of the 2s directed polymers appear potentials describing 
short-range forces acting on the monomers.

The final passage to field theory is performed using the analog in statistical mechanics 
of the second quantization process. To this purpose, we generalized the method used by 
de Gennes and coworkers \cite{45} to achieve the field theory formulation in the case of a single 
polymer chain subjected to short-range interactions to the case of a set of 2s different 
polymers. In the “second quantized” version of the statistical mechanics of the 2s—plat, the 
scalar fields create and destroy monomers in different positions of the space. The square 
module of such fields may be related to the monomer density at a certain point. The BF 
fields take into account the interactions necessary to keep the system in its initial topological 
configuration. The abelian BF model has been quantized in the Coulomb gauge, because in 
this gauge the analogy with anyon field theories becomes particularly explicit. The obtained 
field theory is a multicomponent model of anyons such those described for instance in \cite{46}. 
This kind of theories exhibits the phenomenon of superconductivity. The only difference 
in our case is that the scalar fields containing the creation and annihilation operators for 
particles of type 1 and 2 are organized in replica multiplets, where at the end the limit of 
zero replicas should be taken.

This paper is organized as follows. First of all, to map the partition function of two 
linked polymer rings into that of anyons, it is necessary to split their trajectories into a 
set of 2s subtrajectories, which in the anyon model describe the evolution in time of the 
quasiparticles. The splitting procedure and the definition of a time variable that is able 
to parametrize the 2s subtrajectories is carefully described in Section III. A proof that it is 
possible to isolate from the amplitudes of the BF model the Gauss linking number also after 
splitting the trajectories and changing their parametrization, is presented in Section III. The 
fact that after quantizing the BF model in the Coulomb gauge it is still possible to recover 
the Gauss linking number from the amplitudes of the holonomies is shown in the particular 
case of a 4—plat in Appendix B. In Section IV the partition function of two linked polymers 
subjected to topological constraints imposed with the help of the Gauss linking invariant is 
transformed into a theory of 2s directed polymers interacting with the magnetic-like fields of
the BF model. Contrarily to Ref. [28], we treat the 2s subtrajectories as trajectories of real directed polymers. This requires the introduction of random potentials which complicates somewhat the passage to the anyon field theory performed in Section V. In the anyon formulation, the densities of monomers associated to the two original polymer rings can be regarded as the densities of anyon quasiparticles of type 1 and 2 interacting together. Thanks to a Bogomol’nyi transformation, the interactions may be split into a self-dual part and a part containing only short-range interactions. Remarkably, the latter interactions persists even if the short-range interactions coming from the random media are switched off. This is an effect of the presence of the topological constraint. In Section VI it is reviewed for completeness the case of a 4–plat studied in [28]. The static configurations of the anyon densities that minimize the Hamiltonian are computed. It is shown that the anyon model admits static self-dual points. The nature of the density configurations corresponding to these self-dual points is analyzed in Section VII. We prove that the solutions of the classical equations of motion that minimize the static Hamiltonian are self-dual configurations, whose exact form can be obtained after solving a sinh-Gordon equation. Finally, our conclusions are drawn in Section VIII.
II. POLYMERS AS 2s–PLATS

Let’s consider two closed loops $\Gamma_1$ and $\Gamma_2$ of lengths $L_1$ and $L_2$ respectively in a three dimensional space with coordinates $(r, z)$. The vectors $r = (x, y)$ span the two dimensional space $\mathbb{R}^2$. $z$ will play later on the role of time. The two loops will be labeled by using a indices the first letters of the latin alphabet: $a, b, \ldots = 1, 2$. We will assume that $\Gamma_1$ and $\Gamma_2$ form a 2s–plat. For convenience, we briefly review what is a 2s–plat. First of all, we recall that a single closed trajectory is from the mathematical point of view a knot, while a system of knots linked together forms a link. Knots and links may be represented after a projection onto a plane by diagrams like those of Fig. 1 and 3 in which the original three-dimensional structure is simulated by a system of crossings, see Fig. 2. Each crossing is composed by three arcs, one overpass and two underpasses. One may also realize that the treefoil diagram in Fig. 1 is characterized by two minima and two maxima. Two dimensional diagrams of this kind, deformed in such a way that the number 2s of minima and maxima is the smallest possible and the maxima and minima are aligned at the same heights $z_{\text{Max}}$ and $z_{\text{Min}}$ respectively, are called in knot theory 2s–plats. The height of a 2s–plat is measured here with respect to the $z$ axis. 2s–plats are used to classify knots and links by

1 Actually, to be rigorous one should still require that neither maxima nor minima occur at the crossing points.
dividing them into classes characterized by the same value of $s$. The concept of $2s$–plats arises naturally in biochemistry, see e. g. [11]. In the present case, with some abuse of language, we will call $2s$–plats also the two dimensional diagrams in which maxima and minima are not aligned, like for instance in Fig. 3. Let us denote with the symbols $\tau_{a,I_a}$, $I_a = 0, \ldots, 2s_a - 1$, the heights of the maxima and minima of each trajectory $\Gamma_a$, for $a = 1, 2$.

Arbitrarily, we choose $\tau_{1,0}$ and $\tau_{2,0}$ to be the heights of the lowest minima on the trajectories $\Gamma_1$ and $\Gamma_2$ respectively. Starting from $\tau_{a,0}$, we select the orientation of $\Gamma_a$ in such a way that, proceeding along the trajectory according to that orientation, we will encounter in the order the points $\tau_{a,1}, \tau_{a,2}, \ldots, \tau_{a,2s_a}$. Clearly, $\tau_{a,1}$ is a point of maximum, $\tau_{a,2}$ one of minimum and so on. Moreover, we should put for consistency:

$$\tau_{a,2s_a} \equiv \tau_{a,0}$$

The introduction of this double notation for the same height $\tau_{a,0}$ will be useful in the following in order to write formulas in a more compact form. In the following the $2s$–plats $\Gamma_1$ and $\Gamma_2$ will be decomposed into a set of directed trajectories $\Gamma_{a,I_a}$, $a = 1, 2$ and $I_a = 0, \ldots, 2s_a - 1$, whose ends are made to coincide in such a way that they form the topological configuration of two linked rings. An example when $s = 3$ is presented in Fig. 4. In the general case the set of points belonging to $\Gamma_{a,I_a}$ can be described by the formula:

$$\Gamma_{a,I_a} = \begin{cases}
\mathbf{r}_{a,I_a}(z_{a,I_a}) & a = 1, 2;
\tau_{a,I_a-1} \leq z_{a,I_a} \leq \tau_{a,I_a} & I_a \text{ odd}
\tau_{a,I_a} \leq z_{a,I_a} \leq \tau_{a,I_a-1} & I_a \text{ even}
\end{cases}$$
where the additional conditions:

\[
\begin{align*}
  \mathbf{r}_{a,I_a}(\tau_{a,I_a}) &= \mathbf{r}_{a,I_a+1}(\tau_{a,I_a}) & I_a = 1, \ldots, 2s_a - 1 \\
  \mathbf{r}_{a,1}(\tau_{a,0}) &= \mathbf{r}_{a,2s_a}(\tau_{a,0})
\end{align*}
\]

which connect together the subtrajectories $\Gamma_{a,I_a}$ so that the loop $\Gamma_a$ is reconstructed, are understood. In Eq. (2) $\mathbf{r}_{a,I_a}(z_{a,I_a})$ represents the projection of the trajectory $\Gamma_{a,I_a}$ onto the plane $x, y$ transverse to the longitudinal direction $z_{a,I_a}$. Let us note that we are using the same indexes $I_a$ to label the trajectories $\Gamma_{a,I_a}$ and the points $\tau_{a,I_a}$. However, in the first case $I_a = 1, \ldots, 2s_a$, while in the second case we have chosen $I_a = 0, \ldots, 2s_a - 1$. In the case of the variables $z_{a,I_a}$’s, the range of $I_a$ is the same as that of the $\Gamma_{a,I_a}$’s.

We notice that the variables $z_{a,I_a}$’s are always growing and do not take automatically into account the fact that the whole chain is continuous and has a given orientation. Better variables, both with respect to the continuity and orientation, are the following $t_{a,I_a}$’s:

\[
\begin{align*}
  t_{a,I_a} &= z_{a,I_a} & \text{when } I_a \text{ is odd} \\
  t_{a,I_a} &= -(z_{a,I_a} - \tau_{a,I_a}) + \tau_{a,I_a-1} & \text{when } I_a \text{ is even}
\end{align*}
\]

Assuming for instance that $I_a$ is odd, for two consecutive trajectories $\Gamma_{a,I_a}$ and $\Gamma_{a,I_a+1}$, we have that:

\[
\tau_{a,I_a-1} \leq t_{a,I_a+1} \leq \tau_{a,I_a}
\]
According to the above conventions, trajectories labeled by odd $I_a$'s are oriented from a point of minimum to a point of maximum, while trajectories with even values of $I_a$ go from a point of maximum to a point of minimum. In the new coordinate $t_{a,I_a}$, the trajectory $\Gamma_{a,I_a}$ becomes parametrized as follows:

$$
\Gamma_{a,I_a} = \left\{ \begin{array}{l l}
\{ r_{a,I_a}(t_{a,I_a}) \} & \quad a = 1, 2; \quad I_a = 1, \ldots, 2s_a \\
\tau_{a,I_a-1} \leq t_{a,I_a} \leq \tau_{a,I_a} & \quad I_a \text{ odd} \\
\tau_{a,I_a-1} \geq t_{a,I_a} \geq \tau_{a,I_a} & \quad I_a \text{ even}
\end{array} \right. \quad (9)
$$

where the boundary conditions (3) and (4) are understood.

### III. THE GAUSS LINKING NUMBER AND THE ABELIAN BF FIELD THEORY

To express the topological properties of the system of two linked loops $\Gamma_1$ and $\Gamma_2$, we use as a topological invariant the Gaussian linking number:

$$
\chi(\Gamma_1, \Gamma_2) = \frac{1}{4\pi} \varepsilon_{\mu\nu\rho} \oint_{\Gamma_1} dx_1^\mu(\sigma_1) \oint_{\Gamma_2} dx_2^\nu(\sigma_2) \frac{(\vec{x}_1(\sigma_1) - \vec{x}_2(\sigma_2))^\rho}{|\vec{x}_1(\sigma_1) - \vec{x}_2(\sigma_2)|^3} \quad (10)
$$

where the $\vec{x}_a^\mu(\sigma_a)$'s, $a = 1, 2$, are closed curves representing the loops $\Gamma_1$ and $\Gamma_2$ in the three dimensional space. The variables $\sigma_1$ and $\sigma_2$ used to parametrize $\Gamma_1$ and $\Gamma_2$ represent the respective arc-lengths of the two loops. They are defined in such a way that $0 \leq \sigma_a \leq L_a$. In the following, the trajectories of the two loops will be topologically constrained by the condition:

$$
m = \chi(\Gamma_1, \Gamma_2) \quad (11)
$$

$m$ being a given integer. The above constraints is imposed by inserting the Dirac delta function $\delta(m - \chi(\Gamma_1, \Gamma_2))$ in the partition function of the $2s$–plat, where the statistical sum over all conformations of $\Gamma_1$ and $\Gamma_2$ is performed. Of course, the analytical treatment of such a delta function in a path integral is difficult. Some simplification is obtained by passing to the Fourier representation:

$$
\delta(m - \chi(\Gamma_1, \Gamma_2)) = \int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{2\pi}} e^{-i\lambda(m - \chi(\Gamma_1, \Gamma_2))} \quad (12)
$$
However, even in the Fourier representation, the difficulty of having to deal with the Gauss linking number in the exponent appearing in the right hand side of Eq. (12) remains. Formally, this topological invariant introduces a term that resembles the potential of a two-body interaction. However, this potential is both nonlocal and nonpolynomial. It is for that reason that the treatment of the Gauss linking number in any microscopical model of topologically entangled polymers is usually very complicated. The best strategy do deal with it so far consists in rewriting the delta function

$$\delta(m - \chi(\Gamma_1, \Gamma_2)) = \int_{-\infty}^{+\infty} d\lambda e^{-i\lambda m} \tilde{Z}_{BF}(\lambda)$$

(13)

where

$$\tilde{Z}_{BF}(\lambda) = \int \mathcal{D} \tilde{B}_\mu(x) \mathcal{D} \tilde{C}_\mu(x) e^{-iS_{BF}[\tilde{B}, \tilde{C}]}$$

$$\times e^{-i\tilde{c}_1 \oint_{\Gamma_1} d\tilde{x}_1^\mu(\sigma_1) \tilde{B}_\mu(\tilde{x}_1(\sigma_1))} e^{-i\tilde{c}_2 \oint_{\Gamma_2} d\tilde{x}_2^\mu(\sigma_2) \tilde{C}_\mu(\tilde{x}_2(\sigma_2))}$$

(14)

In the above equation we have put \( x \equiv (x, t) \) to be dummy integration variables in the three dimensional space. Moreover, \( S_{BF}[\tilde{B}, \tilde{C}] \) denotes the action of the abelian BF-model:

$$S_{BF}[\tilde{B}, \tilde{C}] = \frac{\kappa}{4\pi} \int d^3 x \tilde{B}_\mu(x) \partial_\nu \tilde{C}_\mu(x) \epsilon^{\mu\nu\rho}$$

(15)

\( \epsilon^{\mu\nu\rho}, \mu, \nu, \rho = 1, 2, 3 \), being the completely antisymmetric \( \epsilon \)–tensor density defined by the condition \( \epsilon^{123} = 1 \). \( \kappa \) is the coupling constant of the BF-model. Finally, the constants \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are given by:

$$\tilde{c}_1 = \lambda \quad \tilde{c}_2 = \frac{\kappa}{8\pi^2}$$

(16)

While there is some freedom in choosing \( \tilde{c}_1 \) and \( \tilde{c}_2 \), one unavoidable requirement in order that Eq. (13) will be satisfied is that one of these parameters should be linearly dependent on \( \kappa \). In this way, it is easy to check that \( \kappa \) may be completely eliminated from Eq. (14) by performing a rescaling of one of the two fields \( \tilde{B}_\mu \) and \( \tilde{C}_\mu \). This is an expected result, because \( \kappa \) did not appear in the left hand side of Eq. (13), so that it cannot be a new parameter of the theory. By introducing the currents:

$$\tilde{J}_a^\mu(x) = \tilde{c}_a \oint_{\Gamma_a} d\tilde{x}_a^\mu(\sigma_a) \delta^{(3)}(x - \tilde{x}_a(\sigma_a)) \quad a = 1, 2$$

(17)

\( \tilde{Z}_{BF}(\lambda) \) may be rewritten in the more compact way:

$$\tilde{Z}_{BF}(\lambda) = \int \mathcal{D} \tilde{B}_\mu(x) \mathcal{D} \tilde{C}_\mu(x) e^{-iS_{BF}[\tilde{B}, \tilde{C}]} e^{-i \int d^3 x [\tilde{J}_1^\mu(x) \tilde{B}_\mu(x) + \tilde{J}_2^\mu(x) \tilde{C}_\mu(x) + \ldots]}$$

(18)
In all the above discussion, the two trajectories $\Gamma_1$ and $\Gamma_2$ have been parametrized with the help of the arc-lengths $\sigma_1$ and $\sigma_2$. However, in the case of a $2$-$s$–plat, the loops $\Gamma_a$ are realized as a set of open paths $\Gamma_{a,I_a}$ connected together by the conditions (3)–(4). The subtrajectories $\Gamma_{a,I_a}$'s are directed paths $r_{a,I_a}(t_{a,I_a}) = (x_{a,I_a}^1(t_{a,I_a}), x_{a,I_a}^2(t_{a,I_a}))$ parametrized by the new variables $t_{a,I_a}$, which are connected to the third spatial coordinates $z_{a,I_a} = x^3$ by the relations (3) and (4). Due to this difference of parametrization, the above method to express the Gauss linking number based on the BF-model, in particular Eq. (13), should be changed appropriately. Our starting point is the new partition function:

$$Z_{BF}(\lambda) = \int \mathcal{D}B_\mu(x)\mathcal{D}C_\mu(x)e^{-S_{BF}[B,C]}e^{-i\int d^3x[J_1^\mu(x)B_\mu(x)+J_2^\mu(x)C_\mu(x)]}$$

where $S_{BF}[B,C]$ coincides with the action (15) but with the fields $\bar{B}, \bar{C}$ renominated $B, C$ and

$$J_\mu(x) = \tilde{c}_a \sum_{I_a = 1}^{2s_a} \int_{\tau_{a,I_a}}^{\tau_{a,I_a+1}} dx_{a,I_a}(\tau_{a,I_a})\delta^{(3)}(x - x_{a,I_a}(t_{a,I_a}))$$

Let us note that the transformation $z_{a,I_a} \rightarrow t_{a,I_a}$ provided by Eqs. (5) and (6) leaves the BF action and the source terms unaffected, so that it does not change the form of the path integral $Z_{BF}(\lambda)$. For this reason, starting from Eq. (20), the directed paths $\Gamma_{a,I_a}$ are parametrized with the variables $t_{a,I_a}$ instead of $z_{a,I_a}$.

To show that the partition function $Z_{BF}(\lambda)$ of Eq. (19) coincides with the partition function $\tilde{Z}_{BF}(\lambda)$ of Eq. (18), we introduce new variables $\sigma_{a,I_a}$ as follows. Let $\varsigma_{a,I_a}$, $I_a = 0, \ldots, 2s_a - 1$ be the arc-length on $\Gamma_a$ of the point of maximum or minimum located at the height $\tau_{a,I_a}$. The $\sigma_{a,I_a}$'s are defined in such a way that they span the intervals:

$$\varsigma_{a,I_a} \leq \sigma_{a,I_a} \leq \varsigma_{a,I_a+1}$$

As a brief digression, even if it is not necessary for the present discussion, let us define the arc-length $\sigma'_{a,I_a}$ of each trajectory $\Gamma_{a,I_a}$. It is easy to show that $\sigma'_{a,I_a}$ is given by:

$$\sigma'_{a,I_a} = \sigma_{a,I_a} - \varsigma_{a,I_a}$$

As a matter of fact, $\sigma'_{a,I_a}$ ranges in the interval $[0, \varsigma_{a,I_a+1} - \varsigma_{a,I_a}]$ and $\varsigma_{a,I_a+1} - \varsigma_{a,I_a}$ is the total length of $\Gamma_{a,I_a}$ for $a = 1, 2$ and $I_a = , \ldots, 2s_a - 1$.

On each subtrajectory $\Gamma_{a,I_a}$ defined by Eq. (9), we can separately pass from the parameters $t_{a,I_a}$ to the arc-length of $\Gamma_{a,I_a}$ by a transformation of the kind $t_{a,I_a} = t_{a,I_a}(\sigma_{a,I_a})$. Putting

$$x_{a,I_a}^\mu(\sigma_{a,I_a}) = x_{a,I_a}^\mu(t_{a,I_a}(\sigma_{a,I_a}))$$
we may rewrite the currents $J^\mu_a(x)$ as follows:

$$J^\mu_a(x) = \sum_{I_a=1}^{2s_a} \tilde{c}_a \int_{\Gamma_a} d\tilde{x}_a^\mu(\sigma_a) \delta^{(3)}(x - \tilde{x}_a) \delta(3)(x - \tilde{x}_a(\sigma_a))$$  \hspace{1cm} (24)

It is easy to realize that the sum over all values of $I_a$ in Eq. (24) is equivalent to a contour integration over the whole loop $\Gamma_a$. As a consequence:

$$J^\mu_a(x) = \tilde{c}_a \oint_{\Gamma_a} d\tilde{x}_a^\mu(\sigma) \delta^{(3)}(x - \tilde{x}_a(\sigma_a)) \equiv \tilde{J}^\mu_a(x)$$ \hspace{1cm} (25)

i. e. the currents $J^\mu_a(x)$ coincide with the currents $\tilde{J}^\mu_a(x)$ defined in Eq. (17). Therefore, in the partition function $Z_{BF}(\lambda)$ of Eq. (19) the currents $J^\mu_a(x)$'s may be replaced by the $\tilde{J}^\mu_a(x)$'s:

$$Z_{BF}(\lambda) = \int DB\mu DC\mu e^{-iS_{BF}[B,C]} \times e^{-i\int d^3x [\tilde{J}^\mu_1(x)B_\mu(x) + \tilde{J}^\mu_2(x)C_\mu(x)$$  \hspace{1cm} (26)

Due to the fact that the BF fields $B_\mu$ and $C_\mu$ are just dummy field configurations over which a path integration is performed, it is possible to rename them $\tilde{B}_\mu$ and $\tilde{C}_\mu$ respectively. In conclusion, starting from the partition function $Z_{BF}(\lambda)$ of Eq. (19), the partition function $\tilde{Z}_{BF}(\lambda)$ appearing in Eq. (13) has been recovered. In other words, it has been shown that:

$$Z_{BF}(\lambda) = \tilde{Z}_{BF}(\lambda)$$ \hspace{1cm} (27)

and thus in the identity (13) we can replace $\tilde{Z}_{BF}(\lambda)$ by $Z_{BF}(\lambda)$:

$$\delta(m - \chi(\Gamma_1, \Gamma_2)) = \int_{-\infty}^{+\infty} d\lambda e^{-i\lambda m} Z_{BF}(\lambda)$$  \hspace{1cm} (28)

This is the desired final result. Thanks to Eq. (28), it will be possible to transform the path integral over all conformations of the $2s$–plat, which is complicated by the cumbersome presence of the dirac delta function containing the Gauss linking number, into a path integral over the trajectories of a system of $2s$ particles interacting with magnetic fields.

In order to establish the analogy between polymers and anyons, which will be the subject of the next Section, it will be convenient to quantize the BF model in the Coulomb gauge:

$$\partial_i B^i = \partial_i C^i = 0$$  \hspace{1cm} (29)

2 A similar approach like that proposed here can be found in [48]. In Ref. [48] the plats are however static, they do not fluctuate, and the light-cone gauge has been used.
After the gauge choice (29), the action of the BF model (15) becomes:

\[ S_{BF,CG}[B, C] = \frac{\kappa}{4\pi} \int d^3x [B_3 \varepsilon^{ij} \partial_i C_j + C_3 \varepsilon^{ij} \partial_i B_j] \] (30)

with \( \varepsilon^{ij} = \varepsilon^{iij3} \) being the two-dimensional completely antisymmetric tensor. The gauge fixing term vanishes in the pure Coulomb gauge where the conditions (29) are strictly satisfied. Also the Faddeev-Popov term, which in principle should be present in Eq. (30), may be neglected because the ghosts decouple from all other fields. Moreover, the requirement of transversality of (29) in the spatial directions implies that the spatial components \( B_i \) and \( C_i \) of the BF fields may be expressed in terms of two scalar fields \( b \) and \( c \) via the Hodge decomposition:

\[ B_i = \varepsilon_{ij} \partial^j b \quad C_i = \varepsilon_{ij} \partial^j c \] (31)

After performing the above substitutions of fields in the BF action of Eq. (30), we obtain:

\[ S_{BF,CG} = \frac{\kappa}{4\pi} \int d^3x [B_3 \Delta c + C_3 \Delta b] \] (32)

Let’s compute now the propagator of the BF fields:

\[ G_{\mu\nu}(x, t; y, t') = \langle B_\mu(x, t), C_\nu(y, t') \rangle \] (33)

From Eq. (30) it turns out that only the following components of the propagator are different from zero:

\[ G_{3i}(x, t; y, t') = \frac{\delta(t - t')}{2\kappa} \varepsilon_{ij} \partial^j \log |x - y|^2 \] (34)

\[ G_{i3}(x, t; y, t') = -G_{3i}(x, t; y, t') \] (35)

The path integration over the scalar fields \( b \) and \( c \) in the partition function \( Z_{BF}(\lambda) \) is gaussian and could be in principle performed. A natural question that arise at this point is the interpretation of the topological constraint (11) in the Coulomb gauge? As a matter of fact, the BF propagator in the Coulomb gauge breaks explicitly the invariance of the BF model under general three dimensional transformation. It seems thus hard to recover the form (10) of the Gauss linking number in this gauge. Of course, an equivalent constraint should be obtained in the Coulomb gauge due to gauge invariance. In Appendix B it will be shown by a direct calculation in the case of a 4–plat that this is actually true. The computation of the expression of the equivalent of the Gauss linking number in the Coulomb gauge for a general 2s-plat is however technically complicated and will not be performed here.
IV. THE PARTITION FUNCTION OF A PLAT

In order to write the partition function of a $2s$-plat, we follow the strategy explained in the previous Section of dividing each trajectory $\Gamma_a$ into $2s_a$ open paths $\Gamma_{a,I_a}$, $I_a = 1, \ldots, 2s_a$. The statistical sum $Z(\lambda)$ of the system, that is performed over all possible configurations $r_{a,I_a}(t)$ of the subtrajectories $\Gamma_{a,I_a}$ using path integral methods, is defined as follows:

$$Z(m) = \prod_{a=1}^{2s} \prod_{I_a=1}^{2s_a} \int \text{boundary conditions} \mathcal{D}r_{a,I_a}(t_{a,I_a}) e^{-(S_{\text{free}} + S_{\text{EV}})} \delta (m - \chi(\Gamma_1, \Gamma_2))$$

(36)

In the above equation the boundary conditions on the trajectories enforce the constraints (3) and (4). The free part of the action $S_{\text{free}}$ is given by:

$$S_{\text{free}} = \sum_{a=1}^{2s} \sum_{I_a=1}^{2s_a} \int_{\tau_{a,I_a}}^{\tau_{a,I_a+1}} dt_{a,I_a} (-1)^{I_a-1} g_{a,I_a} \left| \frac{dr_{a,I_a}(t_{a,I_a})}{dt_{a,I_a}} \right|^2$$

(37)

The parameters $g_{a,I_a}$, with that $g_{a,I_a} > 0$, are proportional to the inverse of the Kuhn lengths of the trajectories $\Gamma_{a,I_a}$. They are also related to the total lengths of the trajectories $\Gamma_{a,I_a}$ as it is discussed in Appendix A. Let us note that $S_{\text{free}}$ is a positive definite functional despite the presence of the factors $(-1)^{I_a-1}$. This can be easily proved by performing inside $S_{\text{free}}$ the transformations of Eqs. (5) and (6) from the $t_{a,I_a}$’s to the $z_{a,I_a}$ variables:

$$S_{\text{free}} = \sum_{a=1}^{2s} \sum_{I_a=1}^{2s_a} \int_{\tau_{a,I_a}}^{\tau_{a,I_a+1}} dz_{a,I_a} g_{a,I_a} \left| \frac{dr_{a,I_a}(z_{a,I_a})}{dz_{a,I_a}} \right|^2$$

(38)

It is now evident that $S_{\text{free}}$ is either positive or, if the $r_{a,I_a}(z_{a,I_a})$’s are constants, equal to zero.

Since we wish to stress the analogy with directed paths moving in a random media, we have also to introduce a contribution with short-range interactions coming from the integration over the random noises [43]. This is the origin of the contribution $S_{\text{EV}}$ to the total action in Eq. (36). $S_{\text{EV}}$ is of the form:

$$S_{\text{EV}} = \sum_{l=1}^{s_1} \sum_{J=1}^{s_2} \int_{\tau_{1,l-1}}^{\tau_{1,l}} dt_{1,l} \int_{\tau_{2,J-1}}^{\tau_{2,J}} dt_{2,J} (-1)^{l+J-2} V(r_{1,l}(t_{1,l})) - r_{2,J}(t_{2,J})) \delta(t_{1,l} - t_{2,J})$$

$$+ \frac{1}{2} \sum_{a=1}^{2s} \sum_{I_a=1}^{2s_a} \sum_{J_a=1}^{2s_a} \int_{\tau_{a,I_a-1}}^{\tau_{a,I_a}} dt_{a,I_a} \int_{\tau_{a,J_a-1}}^{\tau_{a,J_a}} dt_{a,J_a} (-1)^{I_a+J_a-2} V(r_{a,I_a}(t_{a,I_a})) - r_{a,J_a}(t_{a,J_a}))$$

$$\times \delta(t_{a,I_a} - t_{a,J_a})$$

(39)
In the right hand side of the above equation, the first part describes the interactions between the monomers belonging to different loops, while the second part takes into account the interactions between the monomers of the same loop. If the random noises are gaussianly distributed, the two-body potential $V(r)$ is of the form:

$$V(r) \sim V_0 \delta(r)$$  \hspace{1cm} (40)

with $V_0$ being a positive constant. The factors $(-1)^{I+J-2}$ and $(-1)^{I_a+J_a-2}$ appearing in Eq. (39) are necessary to make the interactions repulsive. This can be easily proved by passing to the variables $z_{a,I_a}$ using the transformations of Eqs. (5) and (6).

As explained in the previous Section, the delta function $\delta(m - \chi(\Gamma_1, \Gamma_2))$ appearing in the partition function $Z(m)$ of Eq. (36) may be simplified by introducing the BF-fields $B_\mu, C_\mu$ with action

$$S_{BF} = \frac{\kappa}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} B_\mu \partial_\nu C_\rho$$  \hspace{1cm} (41)

After performing the Fourier transform of the delta function according to Eq. (12) and exploiting Eq. (28), the partition function $Z(m)$ becomes:

$$Z(m) = \int_{-\infty}^{+\infty} d\lambda e^{-i\lambda m} Z(\lambda)$$  \hspace{1cm} (42)

where

$$Z(\lambda) = \int DB_\mu DC_\mu e^{-iS_{BF}[B,C]} \prod_{a=1}^{2s_a} \prod_{I_a=1}^{2s_a} \int_{\text{boundary conditions}} D\mathbf{r}_{a,I_a}(t_{a,I_a}) e^{-S}$$  \hspace{1cm} (43)

The polymer action $S$ can be split as follows:

$$S = S_{\text{free}} + S_{\text{EV}} + S_{\text{top}}$$  \hspace{1cm} (44)

The expressions of $S_{\text{free}}$ and $S_{\text{EV}}$ are provided in Eqs. (37) and (39) respectively. Finally, using Eq. (19), it is possible to realize that the topological contribution $S_{\text{top}}$ to the polymer action turns out to be:

$$S_{\text{top}} = i \int d^3x [J^1_\mu(x) B_\mu(x) + J^2_\mu(x) C_\mu(x)]$$  \hspace{1cm} (45)

where the currents $J^a_\mu(x), a = 1, 2,$ are given in Eq. (20).
V. AN ANYON FIELD THEORY FORMULATION OF POLYMERIC $2s$–PLATS

The starting point in this Section is the polymer statistical sum $Z(\lambda)$ of Eq. (43). This is formally equivalent to the partition function of a multicomponent system of anyon particles. To write this partition function in terms of fields, we have to perform an integration over all polymer trajectories $r_{a,I_a}(t_{a,I_a})$, $a = 1, 2$ and $I_a = 1, \ldots, 2s_a$. The passage to the field theoretical formulation is not just a formal step, it allows to describe the short range and topological interactions by means a local and polynomial action. Before the introduction of fields, these interactions are both nonlocal and nonpolynomial. The standard procedure in polymer physics to pass from polymer trajectories to monomer densities and thus to a field theory consists in introducing auxiliary fields. In the case of the topological interactions, we have already seen that the auxiliary fields are the BF fields $B_\mu(x)$ and $C_\mu(x)$. The short range interactions in Eq. (39) require instead several scalar fields in order to be simplified. The minimal number of these fields is $2s_a + 2s_b + 2$. A couple of fields $\phi_1(x, t)$ and $\phi_2(x, t)$ is needed for the interaction between monomers belonging to different loops. The interactions between monomers belonging to the same loop will be taken into account by the fields $\varphi_{1,I_a}(x, t)$'s and $\varphi_{2,J_b}(x, t)$'s with $I_a = 1, \ldots, 2s_a$, $J_b = 1, \ldots, 2s_b$.

The passages that lead to the final field theory are well known in the polymer literature \[40, 41, 49, 50\]. After an integration over the auxiliary scalar fields $\phi_1(x, t)$, $\phi_2(x, t)$ and $\varphi_{1,I_a}(x, t)$, $\varphi_{2,J_b}(x, t)$, $I_a = 1, \ldots, 2s_a$, $J_b = 1, \ldots, 2s_b$, whose details are explained in Appendix C, the expression of the polymer partition function $Z(\lambda)$ of Eq. (43) becomes:

$$Z(\lambda) = \lim_{n_1 \to 0} \lim_{n_2 \to 0} \int DB_\mu DC_\mu \left[ \prod_{I=1}^{2s_1} \int D\bar{\Psi}_{1,I} D\Psi_{1,I} \psi_{1,I}^1(r_{1,I}(\tau_{1,I-1}), \tau_{1,I-1}) \psi_{1,I}^1(r_{1,I}(\tau_{1,I}), \tau_{1,I}) \right] \prod_{J=1}^{2s_2} \int D\bar{\Psi}_{2,J} D\Psi_{2,J} \psi_{2,J}^1(r_{2,J-1}(\tau_{2,J-1}), \tau_{2,J-1}) \psi_{2,J}^1(r_{2,J}(\tau_{2,J}), \tau_{2,J}) e^{-iS_{BF}} e^{-A} e^{-A_{EV}}$$

where the $\bar{\Psi}_{a,I_a}$ and $\bar{\Psi}_{a,I_a}$, $a = 1, 2$, $I_a = 1, \ldots, 2s_a$, are complex replica fields:

$$\bar{\Psi}_{a,I_a} = (\psi_{a,I_a}^1, \ldots, \psi_{a,I_a}^{n_a}) \quad \bar{\Psi}_{a,I_a}^* = (\psi_{a,I_a}^{1*}, \ldots, \psi_{a,I_a}^{n_a*})$$

(47)
The action $\mathcal{A}$ in Eq. (46) contains the free part and the topological interactions:

$$\mathcal{A} = \sum_{I=1}^{2s_1} \int_{\tau_{I-1}}^{\tau_{I}} dt (-1)^{I-1} \int d^2x \bar{\Psi}_{1,I}^*(x, t) \cdot \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{1,I}} \left( \nabla - i\lambda(-1)^{I-1}B(x, t) \right)^2 + i\lambda(-1)^{I-1}B_3(x, t) \right] \bar{\Psi}_{1,I}(x, t)$$

$$+ \sum_{J=1}^{2s_2} \int_{\tau_{J-1}}^{\tau_{J}} dt (-1)^{J-1} \int d^2x \bar{\Psi}_{2,J}^*(x, t) \cdot \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{2,J}} \left( \nabla - \frac{i\kappa}{2\pi}(-1)^{J-1}C(x, t) \right)^2 + \frac{i\kappa}{2\pi}(-1)^{J-1}C_3(x, t) \right] \bar{\Psi}_{2,J}(x, t)$$

\( (48) \)

The action $\mathcal{A}_{EV}$, given by

$$\mathcal{A}_{EV} = \sum_{I,J,I',J'=1}^{2s_1} \frac{V_0}{2} \int_{\tau_{I-1}}^{\tau_{I}} dt \int_{\tau_{I'-1}}^{\tau_{I'}} dt' \delta(t - t') \int d^2x (-1)^{I+I'-2} \left| \bar{\Psi}_{1,I}(x, t) \right|^2 \left| \bar{\Psi}_{1,I'}(x, t') \right|^2$$

$$+ \sum_{J,J',J''=1}^{2s_2} \frac{V_0}{2} \int_{\tau_{J-1}}^{\tau_{J}} dt \int_{\tau_{J'-1}}^{\tau_{J'}} dt' \delta(t - t') \int d^2x (-1)^{J+J'-2} \left| \bar{\Psi}_{2,J}(x, t) \right|^2 \left| \bar{\Psi}_{2,J'}(x, t') \right|^2$$

$$+ V_0 \sum_{I=1}^{2s_1} \sum_{J=1}^{2s_2} \int_{\tau_{I-1}}^{\tau_{I}} dt \int_{\tau_{J-1}}^{\tau_{J}} dt' \int d^2x (-1)^{I+J-2} \left| \bar{\Psi}_{1,I}(x, t) \right|^2 \left| \bar{\Psi}_{2,J}(x, t) \right|^2 \delta(t - t')$$

\( (49) \)

is the analog of the action $S_{EV}$ written in the language of second quantized fields and describes the short-range interactions. Looking at Eqs. (46)-(49), we see that we have succeeded in our task, i.e. the original polymer partition function \[43\] has been transformed in an anyon field theory. The action $\mathcal{A}$ is formally equivalent to the action of a multicomponent system of anyons subjected to the Coulomb interactions described by $\mathcal{A}_{EV}$. Similar systems have been discussed in connection with the fractional quantum Hall effect and high $T_C$ superconductivity \[46\]. The only differences are the boundaries of the integrations over the time, which in the present case depend on the heights of the points of maxima and minima of the two trajectories $\Gamma_1, \Gamma_2$ and the fact that here the quasiparticles are bosons of spin $n_1$ or $n_2$ considered in the limit $n_a \to 0, a = 1, 2$.

VI. SELF-DUALITY OF THE TWO-POLYMER PROBLEM

In this section we restrict ourselves for simplicity to 4–plats. The partition function of a 4–plat formed by two linked polymers is obtained by putting $s_1 = s_2 = 1$ in the general
partition function of a $2s$-plat given in Eq. (46). Accordingly, the action $\mathcal{A}$ of Eq. (48) becomes in this particular case:

$$
\mathcal{A} = \int_{\tau_1,0}^{\tau_1,1} dt \int d^2x \left\{ \bar{\Psi}^*_{1,1} \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{1,1}} \nabla^2 (\lambda, \mathbf{B}) + i\lambda \mathbf{B}_3 \right] \bar{\Psi}_{1,1} + \bar{\Psi}^*_{1,2} \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{1,2}} \nabla^2 (\lambda, \mathbf{B}) - i\lambda \mathbf{B}_3 \right] \bar{\Psi}_{1,2} \right\} + \int_{\tau_2,0}^{\tau_2,1} dt \int d^2x \bar{\Psi}^*_{2,1} \left\{ \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{2,1}} \nabla^2 \left( \frac{\kappa}{2\pi}, \mathbf{C} \right) + \frac{i\kappa}{2\pi} \mathbf{C}_3 \right] \bar{\Psi}_{2,1} \right\} + \bar{\Psi}^*_{2,2} \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{2,2}} \nabla^2 \left( \frac{\kappa}{2\pi}, \mathbf{C} \right) - \frac{i\kappa}{2\pi} \mathbf{C}_3 \right] \bar{\Psi}_{2,2} \right\} \tag{50}
$$

In writing the above equation, we have used the notations:

$$
D(\pm \lambda, \mathbf{B}) = \nabla \pm i\lambda \mathbf{B} \quad \quad D \left( \pm \frac{\kappa}{2\pi}, \mathbf{C} \right) = \nabla \pm i\frac{\kappa}{2\pi} \mathbf{C} \tag{51}
$$

The short-range interaction term $\mathcal{A}_{EV}$ of Eq. (49) simplifies in the case of a 4-plat as follows:

$$
\mathcal{A}_{EV} = \sum_{I,J=1}^{2} V_0 \int_{\tau_{1,0}}^{\tau_{1,1}} dt \int d^2x |\bar{\Psi}_{1,I}(x, t)|^2 |\bar{\Psi}_{2,J}(x, t)|^2 + \sum_{I \neq I'}^{2} \frac{V_0}{2} \int_{\tau_{1,0}}^{\tau_{1,1}} dt \int d^2x |\bar{\Psi}_{1,I}(x, t)|^2 |\bar{\Psi}_{1,I'}(x, t)|^2 + \sum_{J \neq J'}^{2} \frac{V_0}{2} \int_{\tau_{2,0}}^{\tau_{2,1}} dt \int d^2x |\bar{\Psi}_{2,J}(x, t)|^2 |\bar{\Psi}_{2,J'}(x, t)|^2 \tag{52}
$$

Finally, the BF contribution $iS_{BF}$ defined in Eq. (41) remains unchanged.

The next goal is to find the classical field configurations which minimize the the energy $\mathcal{F}$ of the two-polymer system. A sketchy derivation of these configurations can be found in Ref. [28]. In the following, we will provide the details that were missing in [28]. The energy $\mathcal{F}$ is given by:

$$
\mathcal{F} = iS_{BF} + \mathcal{A} + \mathcal{A}_{EV} \tag{53}
$$

where the expressions of $S_{BF}$, $\mathcal{A}$ and $\mathcal{A}_{EV}$ are defined in Eqs. (41), (50) and (52) respectively. To simplify the task of its minimization, the short-range interactions will be neglected putting $V_0 = 0$ in Eq. (52), so that $\mathcal{A}_{EV} = 0$. This approximation is valid for instance for polymer solutions which are at the theta point. To proceed, we notice that the third components $B_3$ and $C_3$ of the BF fields play the role of pure Lagrange multipliers. Thus, they can
be integrated out from the partition function (46) giving as a result the following constraints:

\[ B = 2(|\bar{\Psi}_{2,1}|^2 - |\bar{\Psi}_{2,2}|^2)\theta(\tau_{2,1} - t)\theta(t - \tau_{2,0}) \tag{54} \]

\[ C = \frac{4\pi\lambda}{\kappa}(|\bar{\Psi}_{1,1}|^2 - |\bar{\Psi}_{1,2}|^2)\theta(\tau_{1,1} - t)\theta(t - \tau_{1,0}) \tag{55} \]

where \( B \) and \( C \) are the magnetic fields associated to the vector potentials \( B_i \) and \( C_i \) respectively:

\[ B = \partial_1 B_2 - \partial_2 B_1 = \varepsilon^{ij}\partial_i B_j \tag{56} \]

\[ C = \partial_1 C_2 - \partial_2 C_1 = \varepsilon^{ij}\partial_i C_j \tag{57} \]

In Eqs. (54) and (55), \( \theta(t) \) denotes the Heaviside function \( \theta(t) = 0 \) if \( t < 0 \) and \( \theta(t) = 1 \) if \( t \geq 0 \). We will look here only for static field configurations, i.e. those which satisfy the relations:

\[ \frac{\partial}{\partial t}\psi^{\sigma_a}_{a,I_a} = \frac{\partial}{\partial t}\bar{\psi}^{\sigma_a}_{a,I_a} = 0 \tag{58} \]

for all values of \( a = 1, 2 \ I_a = 1, 2 \) and \( \sigma_a = 1, \ldots, n_a \), where the \( n_a \)’s denote the numbers of replicas. To avoid problems with the presence of the Heaviside functions in the expression of the magnetic fields, we will assume that

\[ \tau_{1,0} = \tau_{2,0} \equiv \tau_0 \quad \text{and} \quad \tau_{1,1} = \tau_{2,1} = \tau_1 \tag{59} \]

In this way, the parameter \( t \), whose range is changing depending on which subtrajectory is parametrized, is always be defined in the interval \([\tau_0, \tau_1]\) as a real time. At this point, the static energy \( F_{st} \) may be written as follows:

\[
F_{st} = (\tau_1 - \tau_0) \int d^2x \left[ \frac{1}{4g_{1,1}} |D(-\lambda, B)\bar{\Psi}_{1,1}|^2 + \frac{1}{4g_{1,2}} |D(\lambda, B)\bar{\Psi}_{1,2}|^2 \right] \\
+ (\tau_1 - \tau_0) \int d^2x \left[ \frac{1}{4g_{2,1}} |D\left(-\frac{\kappa}{2\pi}, C\right)\bar{\Psi}_{2,1}|^2 + \frac{1}{4g_{2,2}} |D\left(\frac{\kappa}{2\pi}, C\right)\bar{\Psi}_{2,2}|^2 \right] \tag{60}
\]

The vector potentials \( B \) and \( C \) in the above equations are determined by the relations (54–57). The analogy with the anyon problem suggests the application of the Bogomol’nyi identities [51]. For a single theory of complex scalar fields \( \psi^*, \psi \) minimally coupled to an abelian gauge field \( a \), these identities look as follows:

\[ |D(\gamma, a)\psi|^2 = |D_+(\gamma, a)\psi|^2 \pm \gamma b|\psi|^2 \pm \varepsilon^{ijk}\partial_i j_k \tag{61} \]

where \( D(\gamma, a) = \nabla - i\gamma a \) is the covariant derivative and

\[ D_\pm(\gamma, a) = D_1(\gamma, a) \pm iD_2(\gamma, a) \tag{62} \]
Here \( D_i(\gamma, a), i = 1, 2 \), denotes the components of \( \mathbf{D}(\gamma, a) \), while

\[
\mathbf{b} = \partial_1 a_2 - \partial_2 a_1
\]

is the magnetic field. Finally

\[
j_k = \frac{1}{2i} \left[ \psi^* D_k(\gamma, a) \psi - \psi D_k(\gamma, a) \psi^* \right]
\]

is the current related to the abelian gauge group of symmetry. Let us notice that the term in Eq. (61) containing \( j_k \) is a total derivative, so that it can be omitted in our case, in which the space has no boundaries.

Coming back to the problem of minimizing the static free energy of Eq. (60), we can now apply the Bogomol’nyi identities for all \( 2n_1 + 2n_2 \) replica fields. Actually, Eq. (61) defines two different identities, depending on the choice of sign. This fact may be used to simplify the calculations. In particular, we will choose the + sign when the scalar fields are coupled to \( B \) and the – sign when the scalar fields are coupled to \( C \). As a result we obtain:

\[
\mathcal{F}_{st} = \mathcal{F}_1 + \mathcal{F}_2
\]

with

\[
\mathcal{F}_1 = (\tau_1 - \tau_0) \int d^2x \left\{ \frac{1}{4g_{1,1}} \left| D_+(-\lambda, B) \bar{\Psi}_{1,1} \right|^2 + \frac{1}{4g_{1,2}} \left| D_+(\lambda, B) \bar{\Psi}_{1,2} \right|^2 \right. \\
\left. + \frac{1}{4g_{2,1}} \left| D_-(\frac{\kappa}{2\pi} C) \bar{\Psi}_{2,1} \right|^2 + \frac{1}{4g_{2,2}} \left| D_-(\frac{\kappa}{2\pi} C) \bar{\Psi}_{2,2} \right|^2 \right\}
\]

\[
\mathcal{F}_2 = (\tau_1 - \tau_0) \int d^2x \left\{ \frac{\lambda}{4g_{1,1}} B \left| \bar{\Psi}_{1,1} \right|^2 - \frac{\lambda}{4g_{1,2}} B \left| \bar{\Psi}_{1,2} \right|^2 - \frac{\kappa}{8\pi g_{2,1}} C \left| \bar{\Psi}_{2,1} \right|^2 \right. \\
\left. + \frac{\kappa}{8\pi g_{2,2}} C \left| \bar{\Psi}_{2,2} \right|^2 \right\}
\]

If we substitute in Eq. (67) the expressions of the magnetic fields \( B \) and \( C \) in terms of the replica fields given by Eqs. (54)-(55), it is easy to realize that \( \mathcal{F}_2 \) becomes of the form:

\[
\mathcal{F}_2 = \frac{\lambda}{2}(\tau_1 - \tau_0) \int d^2x \left[ \frac{1}{g_{1,1}} \left( \left| \bar{\Psi}_{2,1} \right|^2 - \left| \bar{\Psi}_{2,2} \right|^2 \right) \left| \bar{\Psi}_{1,1} \right|^2 - \frac{1}{g_{1,2}} \left( \left| \bar{\Psi}_{2,1} \right|^2 - \left| \bar{\Psi}_{2,2} \right|^2 \right) \left| \bar{\Psi}_{1,2} \right|^2 \right. \\
\left. - \frac{1}{g_{2,1}} \left( \left| \bar{\Psi}_{1,1} \right|^2 - \left| \bar{\Psi}_{1,2} \right|^2 \right) \left| \bar{\Psi}_{2,1} \right|^2 + \frac{1}{g_{2,2}} \left( \left| \bar{\Psi}_{1,1} \right|^2 - \left| \bar{\Psi}_{1,2} \right|^2 \right) \left| \bar{\Psi}_{2,2} \right|^2 \right]
\]
It turns out from the above equation that the presence of the topological constraints induces changes in the energy of two linked polymers which consists in the appearance of short-range interactions with coupling constants proportional to

\[
\frac{\lambda}{g_{a,I_a}} (\tau_1 - \tau_0)
\]  

These interactions clearly interfere with the short-range interactions given in Eq. (52), which have potentials of the same structure, characterized by fourth-order powers of the fields, but have different coupling constants. In particular, in Eq. (52) the coupling constant \( V_0 \) is always positive, while the coupling constants in Eq. (69) can be either positive or negative. This shows that the topological constraints have nontrivial effects on the short-term interactions acting on the monomers. These effects have been already observed in experiments, see for example Ref. [14]. Analytically, the influence of the topological constraints has been quantitatively described using various approximations [52–54]. Thanks to the analogy between anyons and 2s–plats established here, we have been able to derive Eq. (68), which represents a direct confirmation at a nonperturbative level of the appearance of interactions associated to topological constraints.

Let us now go back to the expression of the static energy \( F_{st} \) of Eq. (65). Looking at the form of its components \( F_1 \) and \( F_2 \) of Eqs. (66) and (68), it is possible to conclude that \( F_{st} \) is formally equivalent to the Hamiltonian of a set of complex scalar fields coupled to the BF fields \( B_\mu \) and \( C_\mu \). This kind of theory is known to have self-dual solutions [46, 51]. In general, the search of self-dual solutions is not a simple task, because of the non-linear character of the classical equations of motion. Up to now, this problem has been solved in general only using numerical methods. Despite these difficulties, however, it is still possible to investigate the self-dual point analytically by restricting ourselves to the region of the space of physical parameters in which the attractive and repulsive forces appearing in \( F_2 \) counterbalance themselves. In the present context, the self-duality is achieved when the following conditions are satisfied:

\[
g_{1,1} = g_{1,2} = g_{2,1} = g_{2,2} = g
\]  

If the equalities in Eq. (70) are valid, in fact, the potentials in the right hand side of Eq. (68) vanish identically. As a consequence, the energy (65) becomes self-dual, i.e. it can be written
as a sum of self-dual contribution:

\[
\mathcal{F}_{st} = \frac{(\tau_1 - \tau_0)}{4g} \int d^2x \left[ \left| D_+(-\lambda, B)\Psi_{1,1} \right|^2 + \left| D_+(\lambda, B)\Psi_{1,2} \right|^2 \right] + \frac{(\tau_1 - \tau_0)}{4g} \int d^2x \left[ \left| D_-(\frac{-\kappa}{2\pi}, C)\Psi_{2,1} \right|^2 + \left| D_-(\frac{\kappa}{2\pi}, C)\Psi_{2,2} \right|^2 \right]
\]

(71)

In anyon field theories the self-duality condition (70) has a very physical meaning, see for example [51], but its interpretation in the case of the 2s-plat is much more difficult. Certainly the self-duality condition (70) is related both to the length and rigidity of the polymer trajectories. Indeed, the parameters \(g_{a,Ia} \) can be identified with the inverse of the Kuhn lengths of the subtrajectories \( \Gamma_{a,Ia} \) and thus determine their rigidity. Moreover, in Appendix A it is shown how the lengths of the subtrajectories depend on the \( g_{a,Ia} \)'s, see Eq. (A13). Therefore, it is clear that the relations (70) are also imposing conditions on the lengths of the trajectories \( \Gamma_{1,1}, \Gamma_{1,2}, \Gamma_{2,1} \) and \( \Gamma_{2,2} \), which must have in the average the same lengths in order to attain the self-dual point.

In the next Section we will derive some explicit self-dual configurations which minimize the free energy \( \mathcal{F}_{st} \) of Eq. (71).

\section*{VII. SELF-DUAL SOLUTIONS OF THE TWO-POLYMER PROBLEM}

The task of this Section is to find classical solutions of the equations of motion which minimize the energy \( \mathcal{F}_{st} \) of Eq. (71). The classical equations of motion read as follows:

\[
D_+(-\lambda, B)\psi_{1,1}^{\sigma_1} = 0
\]

(72)

\[
D_+(\lambda, B)\psi_{1,2}^{\sigma_1} = 0
\]

(73)

\[
D_\left(\frac{-\kappa}{2\pi}, C\right)\psi_{2,1}^{\sigma_2} = 0
\]

(74)

\[
D_\left(\frac{-\kappa}{2\pi}, C\right)\psi_{2,2}^{\sigma_2} = 0
\]

(75)

\(\sigma_1, \sigma_2\) being replica indexes. To Eqs. (72–75) one should add the constraints (54) and (55):

\[
\epsilon^{ij}\partial_i \partial_j = 2 \left(\left| \Psi_{2,1} \right|^2 - \left| \Psi_{2,2} \right|^2 \right) \theta(\tau_1 - t) \theta(t - \tau_0)
\]

(76)

\[
\epsilon^{ij}\partial_i \partial_j = 4\pi\lambda \kappa \left(\left| \Psi_{1,1} \right|^2 - \left| \Psi_{1,2} \right|^2 \right) \theta(\tau_1 - t) \theta(t - \tau_0)
\]

(77)

To avoid analytical complications due to the presence of the Heaviside theta functions, we have assumed as in Eq. (59) that \( \tau_{1,0} = \tau_{2,0} = \tau_0 \) and \( \tau_{1,1} = \tau_{2,1} = \tau_1 \). Moreover, in the
following we will restrict ourselves to the replica symmetric solutions of Eqs. (72–75) and (76–77) by putting:

\[
\psi_{1,I}^{\sigma_1} = \psi_{1,I} \quad \text{for} \quad 1 \leq \sigma_1 \leq n_1 \quad I = 1, 2
\]

\[
\psi_{2,J}^{\sigma_2} = \psi_{2,J} \quad \text{for} \quad 1 \leq \sigma_2 \leq n_2 \quad J = 1, 2
\]

(78)

In this way, the explicit form of the equations of motion (72–75) and of the constraints (76–77) looks as follows:

\[
[\partial_1 - i\lambda B_1 + i (\partial_2 - i\lambda B_2)] \psi_{1,1} = 0
\]

(79)

\[
[\partial_1 + i\lambda B_1 + i (\partial_2 + i\lambda B_2)] \psi_{1,2} = 0
\]

(80)

\[
\left[ \partial_1 - \frac{i\kappa}{2\pi} C_1 - i \left( \partial_2 - \frac{i\kappa}{2\pi} C_2 \right) \right] \psi_{2,1} = 0
\]

(81)

\[
\left[ \partial_1 + \frac{i\kappa}{2\pi} C_1 - i \left( \partial_2 + \frac{i\kappa}{2\pi} C_2 \right) \right] \psi_{2,2} = 0
\]

(82)

\[
\epsilon^{ij} \partial_i B_j = 2n_1 \left( |\psi_{1,1}|^2 - |\psi_{2,2}|^2 \right)
\]

(83)

\[
\epsilon^{ij} \partial_i C_j = \frac{4n_2 \pi \lambda}{\kappa} \left( |\psi_{1,1}|^2 - |\psi_{1,2}|^2 \right)
\]

(84)

At this point we pass to polar coordinates by performing the transformations:

\[
\psi_{a,I} = e^{i\omega_{a,I} a} \rho_{a,I}^{1/2}
\]

(85)

After the above change of variables in Eqs. (79–84), we obtain by separating the real and imaginary parts:

\[
\partial_1 \omega_{1,1} - \lambda B_1 + \frac{1}{2} \partial_2 \log \rho_{1,1} = 0
\]

(86)

\[
-\partial_2 \omega_{1,1} + \lambda B_2 + \frac{1}{2} \partial_1 \log \rho_{1,1} = 0
\]

(87)

\[
\partial_1 \omega_{1,2} + \lambda B_1 + \frac{1}{2} \partial_2 \log \rho_{1,2} = 0
\]

(88)

\[
-\partial_2 \omega_{1,2} - \lambda B_2 + \frac{1}{2} \partial_1 \log \rho_{1,2} = 0
\]

(89)

\[
\partial_1 \omega_{2,1} - \frac{\kappa}{2\pi} C_1 - \frac{1}{2} \partial_2 \log \rho_{2,1} = 0
\]

(90)

\[
\partial_2 \omega_{2,1} - \frac{\kappa}{2\pi} C_2 + \frac{1}{2} \partial_1 \log \rho_{2,1} = 0
\]

(91)

\[
\partial_1 \omega_{2,2} + \frac{\kappa}{2\pi} C_1 - \frac{1}{2} \partial_2 \log \rho_{2,2} = 0
\]

(92)

\[
\partial_2 \omega_{2,2} + \frac{\kappa}{2\pi} C_2 + \frac{1}{2} \partial_1 \log \rho_{2,2} = 0
\]

(93)
\[ \epsilon^{ij} \partial_i B_j = 2n_1 (\rho_{2,1} - \rho_{2,2}) \]  
\[ \epsilon^{ij} \partial_i C_j = \frac{4n_2 \pi \lambda}{\kappa} (\rho_{1,1} - \rho_{1,2}) \]  

To solve equations (86–93) with respects to the unknowns \( \omega_{a,I} \) and \( \rho_{a,I} \), we proceed as follows. First of all, we isolate from Eq. (86) and Eq. (88) the same quantity \( \lambda B_1 \). By requiring that the expressions of \( \lambda B_1 \) provided by Eqs. (86) and (88) are equal, we obtain the consistency condition:

\[ \partial_1 \omega_{1,1} + \frac{1}{2} \partial_2 \log \rho_{1,1} = -\partial_1 \omega_{1,2} - \frac{1}{2} \partial_2 \log \rho_{1,2} \]  

A possible solution of Eq. (96) is:

\[ \omega_{1,1} = -\omega_{1,2} \quad \text{and} \quad \rho_{1,1} = \frac{A_1}{\rho_{1,2}} \]  

where \( A_1 \) is a constant factor. As well, we could require that the two different expressions of the quantity \( \lambda B_2 \) obtained from Eqs. (87) and (89) are equal. However, in this way one obtains once again the condition (96), which can be solved by applying the ansatz (97). In a similar way, it is possible to extract from equations (90–93) the conditions:

\[ \omega_{2,1} = -\omega_{2,2} \quad \text{and} \quad \rho_{2,1} = \frac{A_2}{\rho_{2,2}} \]  

with \( A_2 \) being a constant.

Thanks to (97) and (98), the number of unknowns to be computed is reduced. For instance, if we know the expressions of \( \omega_{1,1}, \omega_{2,1}, \rho_{1,1} \) and \( \rho_{2,1} \), the classical field configurations \( \omega_{1,2}, \omega_{2,2}, \rho_{1,2} \) and \( \rho_{2,2} \) can be derived using Eqs. (97) and (98). As a consequence, the system of equations (86–95) reduces to:

\[ \lambda B_1 = \partial_1 \omega_{1,1} + \frac{1}{2} \partial_2 \log \rho_{1,1} \]  
\[ \lambda B_2 = \partial_2 \omega_{1,1} - \frac{1}{2} \partial_1 \log \rho_{1,1} \]  
\[ \frac{\kappa}{2\pi} C_1 = \partial_1 \omega_{2,1} - \frac{1}{2} \partial_2 \log \rho_{2,1} \]  
\[ \frac{\kappa}{2\pi} C_2 = \partial_2 \omega_{2,1} + \frac{1}{2} \partial_1 \log \rho_{2,1} \]  
\[ \partial_1 B_2 - \partial_2 B_1 = 2n_1 \left( \rho_{2,1} \frac{A_2}{\rho_{2,1}} \right) \]  
\[ \partial_1 C_2 - \partial_2 C_1 = \frac{4n_2 \pi \lambda}{\kappa} \left( \rho_{2,1} \frac{A_2}{\rho_{2,1}} \right) \]
where we have used the fact that $\epsilon^{ij}\partial_i B_j = \partial_i B_2 - \partial_2 B_1$ and $\epsilon^{ij}\partial_i C_j = \partial_i C_2 - \partial_2 C_1$. Eqs. (104) contain only the unknowns $\omega_{1,1}, \omega_{2,1}, \rho_{1,1}$ and $\rho_{2,1}$ that have still to be determined.

By subtracting term by term the two equations resulting from the derivation of Eqs. (99) and (100) with respect to the variables $x^2$ and $x^1$ respectively, we obtain as an upshot the relation:

$$\lambda (\partial_1 B_2 - \partial_2 B_1) = \partial_1 \partial_2 \omega_{1,1} - \partial_2 \partial_1 \omega_{1,1} - \frac{1}{2} \Delta \log \rho_{1,1}$$

(105)

with $\Delta = \partial_1^2 + \partial_2^2$ being the two-dimensional Laplacian.

Assuming that $\omega_{1,1}$ is a regular function satisfying the relation

$$\partial_1 \partial_2 \omega_{1,1} - \partial_2 \partial_1 \omega_{1,1} = 0$$

(106)

Eq. (105) becomes:

$$\lambda (\partial_1 B_2 - \partial_2 B_1) = -\frac{1}{2} \Delta \log \rho_{1,1}$$

(107)

An analogous identity can be derived starting from Eqs. (101) and (102):

$$\kappa \pi (\partial_1 C_2 - \partial_2 C_1) = \Delta \log \rho_{2,1}$$

(108)

The compatibility of (107) and (108) with the constraints (103) and (104) respectively leads to the following conditions between $\rho_{1,1}$ and $\rho_{2,1}$:

$$\Delta \log \rho_{1,1} = 4\lambda n_1 \left( \frac{A_2}{\rho_{2,1}} - \rho_{2,1} \right)$$

(109)

$$\Delta \log \rho_{2,1} = 4\lambda n_2 \left( \rho_{1,1} - \frac{A_1}{\rho_{1,1}} \right)$$

(110)

The fact that $\rho_{1,1}$ and $\rho_{2,1}$ appear in a symmetric way in Eqs. (109) and (110), suggests the following ansatz:

$$\rho_{2,1} = \frac{A_3}{\rho_{1,1}}$$

(111)

$A_3$ being a constant. It is easy to check that with this ansatz Eqs. (109) and (110) remain compatible provided:

$$\frac{A_2}{A_3} = -\frac{n_2}{n_1} \text{ and } \frac{A_3}{A_1} = -\frac{n_2}{n_1}$$

(112)

We choose $A_1$ to be the independent constant, while $A_2$ and $A_3$ are constrained by Eq. (112) to be dependent on $A_1$:

$$A_2 = \left( \frac{n_2}{n_1} \right)^2 A_1 \quad A_3 = -\frac{n_2}{n_1} A_1$$

(113)
We are now left only with the task of computing the explicit expression of \( \rho_{1,1} \). This may be obtained by solving the equation:

\[
\Delta \log \rho_{1,1} = 4\lambda n_2 \left( \frac{A_1}{\rho_{1,1}} - \rho_{1,1} \right)
\]  

(114)

The other quantities \( \rho_{2,1}, \rho_{1,2} \) and \( \rho_{2,2} \) can be derived using the relations (111), (97) and (98) respectively. Eq. (114) may be cast in a more familiar form by putting: \( \eta = \ln \left( \frac{\rho_{1,1}}{\sqrt{A_1}} \right) \). After this substitution, Eq. (114) becomes the Euclidean sinh–Gordon equation with respect to \( \eta \):

\[
\Delta \eta = 8\lambda n_2 \sqrt{A_1} \sinh \eta
\]  

(115)

Next, it is possible to determine the magnetic fields \( B \) and \( C \) from Eqs. (103) and (104). In the Coulomb gauge, in fact, the two dimensional vector potentials \( B \) and \( C \) can be represented using two scalar fields \( b \) and \( c \) as follows (see also Eq. (31)):

\[
B = (-\partial_2 b, \partial_1 b) \quad C = (-\partial_2 c, \partial_1 c)
\]  

(116)

Performing the above substitutions in Eqs. (103) and (104), it turns out that \( b \) and \( c \) satisfy the relations:

\[
\Delta b = 2n_2 (\rho_{1,1} - \frac{A_1}{\rho_{1,1}})
\]  

(117)

\[
\Delta c = \frac{4n_2 \pi \lambda}{\kappa} (\rho_{1,1} - \frac{A_1}{\rho_{1,1}})
\]  

(118)

The solution of Eqs. (117) and (118) can be easily derived with the help of the method of the Green functions once the expression of \( \rho_{1,1} \) is known. Finally, the phases \( \omega_{1,1}, \omega_{1,2}, \omega_{2,1} \) and \( \omega_{2,2} \) are computed using Eqs. (99)–(102). In fact, remembering that we assumed that \( \omega_{1,1} = -\omega_{1,2} \) and \( \omega_{2,1} = \omega_{2,2} \) in (97) and (98) respectively, we have only to determine \( \omega_{1,1} \) and \( \omega_{2,1} \). By deriving Eq. (99) with respect to \( x^1 \) and Eq. (100) with respect to \( x^2 \), we obtain:

\[
\lambda \partial_1 B_1 = \partial_1^2 \omega_{1,1} + \frac{1}{2} \partial_1 \partial_2 \log \rho_{1,1}
\]

\[
\lambda \partial_2 B_2 = \partial_2^2 \omega_{1,1} - \frac{1}{2} \partial_2 \partial_1 \log \rho_{1,1}
\]  

(119)

On the other side, by adding term by term the above two equations and using the fact that in the Coulomb gauge the magnetic field \( B \) is completely transverse, it is possible to show that:

\[
\Delta \omega_{1,1} = 0
\]  

(120)
Proceeding in a similar way with Eq. (101) and (102) it is possible to derive also the relation satisfied by $\omega_{2,1}$:

$$\Delta \omega_{2,1} = 0 \quad (121)$$

VIII. CONCLUSIONS

In this work a system of two polymers forming a nontrivial link has been considered. The topological properties of the link have been described by using the Gauss linking invariant. This is a weak topological invariant, but when applied to a 2$s$–plat configuration, which cannot be destroyed because the 2$s$ points of maxima and minima are kept fixed, its capabilities to distinguish the changes of topology are greatly enhanced. The reason is the synergy between the constraint imposed by the Gauss linking number and those imposed by the fact that the polymer system cannot escape the set of conformations allowed in a 2$s$–plat. We have also seen in Appendix B what is the meaning of these constraints from the point of view of the 2$s$–plat. Basically, the sum of the winding numbers of all pairs of the subtrajectories $\Gamma_{a,I}$ are constrained to be equal to some integer multiple of $2\pi$. Moreover, since the endpoints of the trajectories are fixed, also the winding number between two different trajectories is fixed up to multiples of $2\pi$. Allowed are only the topology changes such that an amount of the winding angle of two subtrajectories is transferred in units of $2\pi$ to the winding angle of another couple of subtrajectories. This result paves the way to a treatment of polymer knots or links constructed from tangles. Polymers of this kind are relevant in biochemistry because nontrivial knot configurations appearing as a major pattern in DNA rings are mostly in the form of tangles [11].

A crucial point of the connection shown in this paper between 2$s$–plats and anyons is the possibility of eliminating the cumbersome topological constraint (11) from the partition function $Z(\lambda)$ of a 2$s$–plat by introducing BF fields. Indeed, the delta function fixing the constraint (11) can be represented using the Fourier transform of the amplitude of gauge invariant and metric independent observables of an abelian BF model. This has been proved in Eq. (28). The proof of this relation is not trivial because the 2$s$ subtrajectories in which the original 2$s$–plat has been split are open and parametrized by a special variable, the parameter $t$, which is proportional to one of the spatial components of the subtrajectories themselves.
Thanks to the identity \((28)\) it has been possible to interpret the problem of the statistical mechanics of a \(2s-\)plat as that of a two-component anyon gas with \(2s_1\) particles of kind 1 and \(2s_2\) particles of kind 2 interacting via short range potentials, see Eqs.\((42-45)\). The trajectories of the quasi-particles correspond in the polymer analog to the \(2s\) directed trajectories \(\Gamma_{a,I_a}\), \(a = 1, 2\), \(I_a = 1, \ldots, 2s_a\), which are traversed by the fictitious currents \((20)\). The Gauss linking number can be interpreted as the circulation of the magnetic field generated by the current traversing the loop \(\Gamma_1\) with respect to the closed contour formed by the loop \(\Gamma_2\) \([50, 55]\).

The system of quasiparticles with partition function Eq. \((43)\) has been further mapped into a two-component anyon field theory, whose final form is displayed in Eqs. \((46)\) and \((48-49)\). Similar field theories have been proposed in the past to explain the superconductivity of high temperature superconductors without breaking the \(P\) and \(T\) invariance, see \([31]\). The analogy between directed polymers and vortex lines has been studied in connections with high \(T_C\) superconductors in \([56]\). As in superconductors of type II, also in the present case attractive and repulsive forces appear, which vanish at some self-dual point of the theory. What is remarkable here, is that these interactions do not need the introduction of any potential and are purely related to the topological constraints imposed on the trajectories of the original polymer system. Indeed, they remain even if the short-range interactions are switched off as shown in Eq. \((68)\). From the condition \((70)\), which determines the existence of the self-dual point or not, it is possible to predict that \(2s-\)plats consisting of homogeneous polymers should have a profoundly different behavior than their counterparts built out of block copolymers. As a matter of fact, we have seen that the physical characteristics of the \(2s\) directed polymers into which the \(2s-\)plat has been split are described by the constants \(g_{a,I_a}\). In particular, the rigidity of the trajectories may be specified by choosing the \(g'_{a,I_a}\)s appropriately. Clearly, from Eq. \((70)\) it turns out that the self-dual point is attained only if these constants are all equal, implying that either the polymer rings are homopolymers or their subtrajectories \(\Gamma_{a,I_a}\) contain monomers of different types but, after averaging over the distance of many monomer sizes, they have identical physical properties. We have also derived the equations of motion that minimize the action the anyon field theory in the case of a \(4-\)plat. These equation describe the self-dual point of the two-component anyon gas. After many simplifications, the relevant degrees of freedom can be derived by solving the sinh-Gordon equation \((115)\) and the Laplace equations \((120)\) \([121]\).
From the polymer point of view, the physical meaning of the self-duality is unclear, because here only static conformations have been considered. In principle, these static solutions could become physically relevant in the case of a very long $4^-$plat in which the monomer concentration does not depend on the height $z$. What is however more important, is that Eq. (115), which determines the static density of monomers of type 1, is a sinh-Gordon equation identical to that obtained in [57] for the static vortices of a relativistic abelian Higgs model on a special type of Riemann surfaces. This analogy between field theories on Riemann surfaces and polymers, together with the connections between linked polymers and multicomponent anyon systems, that are related both to topological quantum computing and to high-$T_C$ superconductors, should be further explored. It is true that for topological computing nonabelian anyon systems are necessary, while our discussion has been limited to the abelian case. However, this limitation is only apparent. In principle, instead of the Gauss linking invariant, we could have used much more refined topological invariants that would have led to nonabelian anyon field theories [58, 59]. Up to now, however, nobody has succeeded to formulate completely a nonabelian theory of topological entanglement for polymer systems based on such topological invariants, apart from a few exceptions [49, 60]. Also the possibility of studying the statistical mechanics of knots constructed from tangles should be investigated, because up to now there is no analytical model which is able to describe the statistical properties of knots.

IX. ACKNOWLEDGMENTS

F. Ferrari would like to thank E. Szuszkiewicz for pointing out Ref. [61] and inspiring the present work. We wish to thank heartily also M. Pyrka, V. G. Rostiashvili and T. A. Vilgis for fruitful discussions. The support of the Polish National Center of Science, scientific project No. N N202 326240, is gratefully acknowledged. The simulations reported in this work were performed in part using the HPC cluster HAL9000 of the Computing Centre of the Faculty of Mathematics and Physics at the University of Szczecin.
Appendix A: The length $L$ of a directed polymer as a function of the height

In this Appendix we consider the partition function
\[ Z = \int D\mathbf{r}(z)e^{-S} \quad (A1) \]
where $S$ is the action of the free open polymer, whose trajectory $\Gamma$ is parametrized by means of the height $z$ defined in some interval $[\tau_0, \tau_1]$:
\[ S = g \int_{\tau_0}^{\tau_1} dz \left| \frac{d\mathbf{r}}{dz} \right|^2 \quad (A2) \]
We want now to determine how the total length of the curve $\Gamma$ depends on the constant parameter $g$. To understand what we mean by that, let us consider the standard case of an ideal chain whose trajectory is parametrized with the help of the arc-length $\sigma$. We denote with $a$ the average statistical length (Kuhn length) of the $N$ segments composing the polymer. In the limit of large $N$ and small $a$ such that the product $Na$ is constant, the total length $L$ of the polymer satisfies the relation
\[ L = Na \quad (A3) \]
We wish to obtain a similar identity connecting $L$ with $N$ and $g$ in the present situation, which is somewhat different. To this purpose, we first discretize the interval of integration $[\tau_0, \tau_1]$ splitting it into $N$ small segments of length:
\[ \Delta z = \frac{\tau_1 - \tau_0}{N} \quad (A4) \]
As a consequence, we may approximate the action as follows:
\[ S \sim g \sum_{w=1}^{N} \frac{\Delta r_w}{\Delta z}^2 \Delta z \quad (A5) \]
where the symbol $\Delta r_w$ means
\[ \delta r_w = r_{w+1} - r_w \quad (A6) \]
and
\[ r_w = r(\tau_0 + w\Delta z) \quad (A7) \]
The discretized partition function becomes thus the partition function of a random chain composed by $N$ segments:
\[ Z_{disc} = \int \prod_{w=1}^{N} d\mathbf{r}_w e^{- \sum_{w=1}^{N} g \frac{\Delta r_w^2}{\Delta z}} \quad (A8) \]
Using simple trigonometric arguments it is easy to realize that the length of each segment is:

\[ \Delta L = \sqrt{|\Delta r_w|^2 + \Delta z^2} \]  

This is of course an average length, dictated by the fact that, from Eq. (A8), the values of \(|\Delta r_w|\) should be gaussianly distributed around the point:

\[ |\Delta r_w|^2 = \frac{\Delta z}{g} \]  

In the limit \( \Delta z \to 0 \), the distribution of length of \( \Delta r_w \) becomes the Dirac \( \delta \)-function:

\[ \lim_{\Delta z \to 0} \frac{1}{2\sqrt{\Delta z}} e^{-g|\Delta r_w|^2/\Delta z} \sim \delta \left( |\Delta r_w| - \sqrt{\frac{\Delta z}{g}} \right) \]  

If \( N \) is large enough, we can therefore conclude that the total length of the chain \( \Gamma \) is:

\[ L \sim N\Delta L = N\sqrt{\frac{\Delta z}{g} + \Delta z^2} \]  

Since \( N\Delta z = \tau_1 - \tau_0 \), we get:

\[ L^2 = |\tau_1 - \tau_0|^2 + \frac{N(\tau_1 - \tau_0)}{g} \]  

In the limit \( N \to \infty \), while keeping the ratio \( \frac{N}{g} \) finite, Eq. (A13) becomes the desired relation between the length of \( \Gamma \) and \( g \) which replaces Eq. (A3).

**Appendix B: The expression of the Gauss linking invariant in the Coulomb gauge.**

To fix the ideas, we will study here the particular case of a 4–plat. In the partition function (43) we isolate only the terms in which the BF fields appear, because the other contributions are not connected to topological constraints and thus are not relevant. As a consequence, we have just to compute the following partition function:

\[ Z_{CS,CG}(\lambda) = \int DB_\mu DC_\mu e^{-iS_{BF,CG} - S_{tot}} \]  

where the BF action in the Coulomb gauge \( S_{BF,CG} \) has been already defined in Eq. (30) and \( S_{top} \) has been given in Eq. (45). In the case of a 4–plat, \( S_{top} \) becomes:

\[ S_{top} = i\lambda \int_{\tau_{1,0}}^{\tau_{1,1}} dt \left[ \frac{dx_{1,1}^\mu(t)}{dt} B_\mu(r_{1,1}(t), t) - \frac{dx_{1,2}^\mu(t)}{dt} B_\mu(r_{1,2}(t), t) \right] + \frac{i\kappa}{2\pi} \int_{\tau_{2,0}}^{\tau_{2,1}} dt \left[ \frac{dx_{2,1}^\mu(t)}{dt} C_\mu(r_{2,1}(t), t) - \frac{dx_{2,2}^\mu(t)}{dt} C_\mu(r_{2,2}(t), t) \right] \]  

(B2)
where we recall that $x_{a,I}^\mu(t) = (r_{a,I}(t), t)$, $a = 1, 2$, $I = 1, 2$. Using the Chern-Simons propagator of Eqs. (34)-(35), it is easy to evaluate the path integral over the gauge fields in Eq. (B1). The result, after two simple Gaussian integrations, is:

$$Z_{BF,CG}(\lambda) = \exp \left\{ \frac{i\lambda}{2\pi} \sum_{I,J=1}^{2} (-1)^{I+J-2} \varepsilon_{ij} \int_{\tau_0}^{\tau_1} d(t) \left( \frac{r_I(t)}{r_{1,I}(t) - r_{2,J}(t)} \right)^2 \right\}$$  (B3)

In the above equation we have put for simplicity:

$$\tau_0 = \max[\tau_{1,0}, \tau_{2,0}]$$
$$\tau_1 = \min[\tau_{1,1}, \tau_{2,1}]$$  (B4)

For instance, if the polymer configurations are as in Fig. 5, we have that $\tau_0 = \tau_{1,0}$ and $\tau_1 = \tau_{2,1}$. Moreover, we remember that in our notation $r_{a,I}(t) = (x_{a,1}(t), x_{a,2}(t))$. Apparently,

![FIG. 5. Example of configuration of a 4-plat.](image)

the elements of the trajectories $\Gamma_1$ and $\Gamma_2$ which lie below $\tau_0$ and above $\tau_1$ do not take the part in the topological interactions. Thus is due to the presence of the Dirac $\delta$-function $\delta(t - t')$ in the components of the Chern-Simons propagator (34)-(35). However, we will see later that also the contributions of these missing parts are present in the expression of $Z_{CS}(\lambda)$. In order to proceed, we notice that the exponent of the right hand side of Eq. (B3) consists in a sum of integrals over the time $t$ of the kind:

$$D_{1,2,J}(\tau_1) - D_{1,2,J}(\tau_0) = \varepsilon_{ij} \int_{\tau_0}^{\tau_1} d(t) \left( \frac{x_{1,I}(t) - x_{2,J}(t)}{r_{1,I}(t) - r_{2,J}(t)} \right)^2$$  (B5)
The above integrals can be computed exactly. It is in fact well known that the function $D_{1,1,2,1}(t)$ is the winding angle of the vector $r_{1,1}(t) - r_{2,1}(t)$ at time $t$:

$$D_{1,1,2,1}(t) = \arctan \left( \frac{x_{1,1}^1(t) - x_{2,1}^1(t)}{x_{1,1}^2(t) - x_{2,1}^2(t)} \right)$$  \hspace{1cm} (B6)

Thus, the quantity $D_{1,1,2,1}(\tau_1) - D_{1,1,2,1}(\tau_0)$ is a difference of winding angles which measures how many times the trajectory $\Gamma_{1,1}$ turns around the trajectory $\Gamma_{2,1}$ in the slice of time $\tau_0 \leq t \leq \tau_1$. At this point, without any loss of generality, we suppose that the configurations of the curves $\Gamma_1$ and $\Gamma_2$ is such that the maxima and minima $\tau_{a,I}$ are ordered as follows:

$$\tau_{2,0} < \tau_{1,0} < \tau_{2,1} < \tau_{1,1}$$  \hspace{1cm} (B7)

As example of loop configurations that respect this ordering is given in Fig. 5. As a consequence, we have:

$$\tau_0 = \tau_{1,0} \quad \text{and} \quad \tau_1 = \tau_{2,1}$$  \hspace{1cm} (B8)

Now we notice that the logarithm of the gauge partition function $Z_{BF,CG}(\lambda)$ in Eq. (B3) contains a sum of differences of the winding angles defined in Eq. (B6):

$$\frac{2\pi \log Z_{BF,CG}(\lambda)}{i\lambda} = \left[ D_{1,1,2,1}(\tau_{2,1}) - D_{1,1,2,1}(\tau_{1,0}) + D_{1,2,2,2}(\tau_{2,1}) - D_{1,1,2,2}(\tau_{2,1}) 
+ D_{1,2,2,1}(\tau_{1,0}) - D_{1,2,2,1}(\tau_{2,1}) + D_{1,1,2,2}(\tau_{1,0}) - D_{1,2,2,2}(\tau_{1,0}) \right]$$  \hspace{1cm} (B9)

Further, assuming that the curves $\Gamma_1$ and $\Gamma_2$ are oriented as in Fig. 5, if we start from the minimum point at $\tau_0 = \tau_{1,0}$, we can isolate in the right hand side of Eq. (B9) the following four contributions:

1. In the time slice $\tau_{1,0} \leq t \leq \tau_{2,1}$ the angle which measures the winding of the trajectory $\Gamma_{1,1}$ around the trajectory $\Gamma_{2,1}$ is given by the difference $D_{1,1,2,1}(\tau_{2,1}) - D_{1,1,2,1}(\tau_{1,0})$.

2. In the region $\tau_{2,1} \leq t \leq \tau_{1,1}$ only the trajectory $\Gamma_1$ continues to evolve, going first upwards with the subtrajectory $\Gamma_{1,1}$ and then downwards with $\Gamma_{1,2}$. After this evolution, the winding angle between the two trajectories $\Gamma_1$ and $\Gamma_2$ has changed by the quantity $D_{1,2,2,2}(\tau_{2,1}) - D_{1,1,2,2}(\tau_{2,1})$.

3. Next, in the region $\tau_{2,1} \geq t \geq \tau_{1,0}$, the winding angle which measures how many times the subtrajectory $\Gamma_{1,2}$ winds up around $\Gamma_{2,2}$ is given by the difference $D_{1,2,2,1}(\tau_{1,0}) - D_{1,2,2,1}(\tau_{2,1})$. 
4. Finally, in the region $\tau_{1,0} \geq t \geq \tau_{2,0}$ only the second trajectory $\Gamma_2$ continues to evolve, going first downwards with the curve $\Gamma_{2,2}$ and then upwards with $\Gamma_{2,1}$. The net effect of this evolution is that the winding angle between $\Gamma_1$ and $\Gamma_2$ changes by the quantity $D_{1,1;2,2}(\tau_{1,0}) - D_{1,2;2,2}(\tau_{1,0})$.

It is thus clear that the right hand side of Eq. (B9), apart from a proportionality factor $i\lambda$, counts how many times the trajectory $\Gamma_1$ winds around the trajectory $\Gamma_2$. If we wish to identify the quantity in the right hand side of Eq. (B9) with the Gauss linking number $\chi(\Gamma_1, \Gamma_2)$, we should check for consistency that it takes only integer values as the Gauss linking number does. Indeed, it is easy to see that, modulo $2\pi$, the following identities are holding:

\begin{align}
D_{1,1;2,1}(\tau_{2,1}) &= D_{1,1;2,2}(\tau_{2,1}) \\
D_{1,1;2,2}(\tau_{1,0}) &= D_{1,2;2,2}(\tau_{1,0}) \\
D_{1,2;2,2}(\tau_{2,1}) &= D_{1,2;2,1}(\tau_{2,1}) \\
D_{1,1;2,1}(\tau_{1,0}) &= D_{1,2;2,1}(\tau_{1,0})
\end{align} 

(B10)

For example, the first of the above equalities states that the angle formed by the vector $r_{1,1} - r_{2,1}$ connecting the subtrajectories $\Gamma_{1,1}$ and $\Gamma_{2,1}$ at the height $\tau_{2,1}$ is equal to the angle formed by the vector $r_{1,1} - r_{2,2}$ connecting the subtrajectories $\Gamma_{1,1}$ and $\Gamma_{2,2}$ at the same height. The reason of this identity is trivial: At that height, the subtrajectories $\Gamma_{2,1}$ and $\Gamma_{2,2}$ are connected together at the same point. Applying the above relations to Eq. (B9), one may prove that:

$$\frac{2\pi \log Z_{BF,CG}(\lambda)}{i\lambda} = 0 \quad \text{mod } 2\pi$$

(B11)

As a consequence, we can write:

$$Z_{BF,CG}(\lambda) = e^{i\lambda \chi(\Gamma_1, \Gamma_2)}$$

(B12)

where $\chi(\Gamma_1, \Gamma_2)$ is the Gauss linking number. Concluding, the above analysis shows that also in the Coulomb gauge the BF fields in the polymer partition function fix the topological constraints correctly, in full consistency with the results obtained in the covariant gauges. Of course this consistency was expected due to gauge invariance. Yet, it is interesting that, using the Coulomb gauge, one may express the Gauss linking number invariant in a way that is quite different from the usual form given in Eq. (10).
Appendix C: From polymers to anyon field theories

In this Appendix the passage from the polymer partition function (43) to the field theoretical formulation of Eq. (46) is performed. To this purpose, we have to integrate over all polymer trajectories \( r_{a,I_a}(t_{a,I_a}) \), \( a = 1, 2 \) and \( I_a = 1, \ldots, 2s_a \). The standard procedure to pass to field theory in polymer physics consists in introducing auxiliary fields. This procedure works of course also in the present case, but it is complicated by the splitting of the trajectories \( \Gamma_a \) into \( 2s_a \) subtrajectories. First of all, we have to introduce external sources for each subtrajectory as follows:

\[
J_{a,I_a}(x,t) = \int_{\tau_{a,I_a-1}}^{\tau_{a,I_a}} dt_{a,I_a} \delta(x-r_{a,I_a}(t_{a,I_a})) \delta(t-t_{a,I_a})(-1)^{I_a-1}
\]  

(C1)

Here the coordinates \((x,t)\) are allowed to span the whole \( \mathbb{R}^3 \) space. Now it is possible to write the following identity:

\[
\exp \left[ -\int_{\tau_{a,I_a-1}}^{\tau_{a,I_a}} dt_{a,I_a} \int_{\tau_{b,J_b-1}}^{\tau_{b,J_b}} dt_{b,J_b} (-1)^{I_a+J_b-2} V(r_{a,I_a}(t_{a,I_a}) - r_{b,J_b}(t_{b,J_b})) \right] = \exp \left[ -\int d^2x d^2y dt dt' J_{a,I_a}(x,t)V(x,y)\delta(t-t')J_{b,J_b}(y,t') \right] 
\]  

(C2)

where \( a, b = 1, 2 \), \( I_a = 1, \ldots, 2s_a \), \( J_b = 1, \ldots, 2s_b \). Clearly, the right hand side of the above equation can be interpreted as the generating functional of a free scalar field theory with propagator \( G(x,y; t, t') = V(x,y)\delta(t-t') \). At this point we notice that the weight \( e^{-S_{EV}} \) that takes into account the short-term interactions in the partition function (43) is a product of exponents of the kind given in Eq. (C2). Thus, \( e^{-S_{EV}} \) coincides formally with the generating functional of a multi-component scalar field theory. The minimum number of scalar fields that is necessary to express \( e^{-S_{EV}} \) as a generating functional is \( 2s_a + 2s_b + 2 \). Let’s call these fields \( \varphi_{1,I_a}(x,t), \varphi_{2,J_b}(x,t), I_a = 1, \ldots, 2s_a \), \( J_b = 1, \ldots, 2s_b \) and \( \phi_1(x,t), \phi_2(x,t) \). \( \phi_1(x,t) \) and \( \phi_2(x,t) \) will be responsible for the interaction between monomers belonging to different loops, while the \( \varphi_{1,I_a}(x,t) \)’s and \( \varphi_{2,J_b}(x,t) \)’s will take into account the interactions of monomers belonging to the same loop.

Remembering that in the present case \( V(x,y) = V_0\delta(x - y) \), it is possible to verify the
validity of the following identity:

\[ e^{-S_{EV}} = \int \mathcal{D} \phi_1 \mathcal{D} \phi_2 \exp \left( \frac{-1}{V_0} \int d^2 x dt \phi_1(x, t) \phi_2(x, t) \right) \times \prod_{I=1}^{2s_1} \int \mathcal{D} \varphi_{1,I}(x, t) \prod_{J=1}^{2s_2} \int \mathcal{D} \varphi_{2,J}(x, t) \times \exp \left[ -\frac{1}{2V_0} \int d^2 x dt \left( \sum_{I_1, I'_1 = 1}^{2s_1} \varphi_{1,I_1}(x, t) \varphi_{1,I'_1}(x, s) \alpha_{I_1 I'_1}^{-1} \right) \right] \times \exp \left[ -\frac{1}{2V_0} \int d^2 x dt \left( \sum_{J_2, J'_2 = 1}^{2s_2} \varphi_{2,J_2}(x, t) \varphi_{2,J'_2}(x, s) \alpha_{J_2 J'_2}^{-1} \right) \right] \times \exp \left[ -i \sum_{I=1}^{2s_1} \int d^2 x dt J_{1,I}(x, t) (\phi_2(x, t) + \varphi_{1,I}(x, t)) \right] \times \exp \left[ -i \sum_{J=1}^{2s_2} \int d^2 x dt J_{2,J}(x, t) (\phi_1(x, t) + \varphi_{2,J}(x, t)) \right] \tag{C3} \]

where \( \alpha_{I_a I'_a}, I_a, I'_a = 1, \ldots, 2s_a \), is the off-diagonal matrix

\[ \alpha_{I_a I'_a} = \begin{cases} 0 & \text{if } I_a = I'_a \\ 1 & \text{if } I_a \neq I'_a \end{cases} \tag{C4} \]

and \( \alpha_{I_a I'_a}^{-1} \) represents its inverse. Using equation (C3), the partition function (43) becomes:

\[ Z(\lambda) = \int \mathcal{D} B_{\mu} \mathcal{D} C_{\mu} e^{-i S_{BF}} \int \mathcal{D} \phi_1 \mathcal{D} \phi_2 e^{-\frac{1}{V_0} \int d^2 x dt \phi_1(x, t) \phi_2(x, t)} \times \prod_{I=1}^{2s_1} \int \mathcal{D} \varphi_{1,I}(x, t) \prod_{J=1}^{2s_2} \int \mathcal{D} \varphi_{2,J}(x, t) \times \exp \left[ -\frac{1}{2V_0} \int d^2 x dt \left( \sum_{I_1, I'_1 = 1}^{2s_1} \varphi_{1,I_1}(x, t) \varphi_{1,I'_1}(x, s) \alpha_{I_1 I'_1}^{-1} \right) + \sum_{J_2, J'_2 = 1}^{2s_2} \varphi_{2,J_2}(x, t) \varphi_{2,J'_2}(x, s) \alpha_{J_2 J'_2}^{-1} \right] \times \prod_{I=1}^{2s_1-1} \int \mathcal{D} r_{1,I}(t_1, t) \prod_{I=1}^{2s_1} \int \mathcal{D} r_{1,2s_1}(t_1, t) \prod_{I=1}^{2s_2-1} \int \mathcal{D} r_{2,J}(t_2, t) \prod_{I=1}^{2s_2} \int \mathcal{D} r_{2,2s_2}(t_2, t) e^{-S_{eff}} \tag{C5} \]
with

\[ S_{\text{eff}} = \sum_{l=1}^{2s_1} \int_{\tau_l, l-1}^{\tau_{l, l}} dt_{l, l} \left[ (-1)^{l-1}g_{l, l} \left| \frac{d\mathbf{r}_{l, l}}{dt_{l, l}} \right|^2 + (-1)^{l-1}i(\phi_2(\mathbf{r}_{l, l}(t_{l, l})), t_{l, l}) + \varphi_1(\mathbf{r}_{l, l}(t_{l, l})), t_{l, l}) \right] \]

\[ + i\lambda B_3(\mathbf{r}_{l, l}(t_{l, l})), t_{l, l}) + i\lambda \frac{d\mathbf{r}_{l, l}}{dt_{l, l}} \cdot \mathbf{B}(\mathbf{r}_{l, l}(t_{l, l})), t_{l, l}) \]

\[ + \sum_{j=1}^{2s_2} \int_{\tau_{2, j-1}}^{\tau_{2, j}} dt_{2, j} \left[ (-1)^{j-1}g_{2, j} \left| \frac{d\mathbf{r}_{2, j}}{dt_{2, j}} \right|^2 + (-1)^{j-1}i(\phi_2(\mathbf{r}_{2, j}(t_{2, j})), t_{2, j}) + \varphi_2(\mathbf{r}_{2, j}(t_{2, j})), t_{2, j}) \right] \]

\[ + i\kappa C_3(\mathbf{r}_{2, j}(t_{2, j})), t_{2, j}) + \frac{i\kappa}{2\pi} \frac{d\mathbf{r}_{2, j}}{dt_{2, j}} \cdot \mathbf{C}(\mathbf{r}_{2, j}(t_{2, j})), t_{2, j}) \]  

(C6)

Of course, the ends of the trajectories \( \Gamma_{1, l} \) and \( \Gamma_{2, j} \) appearing in the limits of path integration over \( \mathbf{r}_{1, l}(t_{1, l}) \) and \( \mathbf{r}_{2, j}(t_{2, j}) \) in Eq. (C5) are not all independent, because they are subjected to the constraints (3) and (4). We will get rid of these constraints later when passing to the field theoretical representation.

Let us use at this point the so-called complex replica fields defined in Eq. (47). Then, the path integrals over the trajectories \( \mathbf{r}_{a, l} \) may be rewritten as follows [50]:

\[ \lim_{n_1 \to 0} \left[ \prod_{l=1}^{2s_1} \int_{\tau_{l, l-1}}^{\tau_{l, l}} d\mathbf{r}_{l, l} \frac{d\mathbf{r}_{l, l}}{dt_{l, l}}^{\psi_1^{1*}}(\mathbf{r}_{l, l}(t_{l, l}), t_{l, l}) \right] \]

\[ \times \psi_1^{1*}(\mathbf{r}_{1, 2s_1+1}(\tau_{1, 2s_1+1}), \tau_{1, 2s_1+1}) \psi_1^{1*}(\mathbf{r}_{1, 1}(\tau_{1, 1}), \tau_{1, 1}) \]

\[ \times \prod_{l=1}^{2s_1} \exp \left\{ - \int_{\tau_{l, l-1}}^{\tau_{l, l}} dt_{l} (-1)^{l-1} \int d^2 \mathbf{x} \bar{\psi}_{1, l}^{1*}(\mathbf{x}, t) \cdot \left[ \frac{\partial}{\partial t} - \frac{1}{4g_{1, l}} (\nabla - i\lambda (-1)^{l-1}\mathbf{B}(\mathbf{x}, t))^2 \right] \bar{\psi}_{1, l}^{1*}(\mathbf{x}, t) \right\} \]

\[ + i\lambda (-1)^{l-1} B_3(\mathbf{x}, t) + i(\phi_2(\mathbf{x}, t) + \varphi_1(\mathbf{x}, t)) \]  

(C7)
Let us note that with the above choice of arguments of the fields $\psi_{1,I}^*, \psi_{1,I}$ and $\psi_{2,J}^*, \psi_{2,J}$, $I = 1, \ldots, 2s_1$, $J = 1, \ldots, 2s_2$, the constraints (3) and (4) are already taken into account. Inserting this result in Eq. (C5) and integrating out the auxiliary fields $\varphi_{1,I}, \varphi_{2,J}, \phi_1$ and $\phi_2$, we obtain the final expression of the polymer partition function given by Eqs. (46), (48) and (49).

[1] A. Yu. Grosberg Phys.-Usp. 40, 12 (1997).
[2] W. R. Taylor, Nature (London) 406, 916 (2000).
[3] V. Katritch, J. Bednar, D. Michoud, R. G. Scharein, J. Dubochet, A. Stasiak, Nature 384, 142 (1996).
[4] V. Katritch, W. K. Olson, P. Pieranski, J. Dubochet and A. Stasiak, Nature 388, 148 (1997).
[5] M. A. Krasnow, A. Stasiak, S. J. Spengler, F. Dean, T. Koller and N. R. Cozzarelli, Nature 304, 559 (1983).
[6] B. Laurie, V. Katritch, J. Dubochet and A. Stasiak, Biophys. Jour. 74, 2815 (1998).
[7] J. I. Sułkowksa, P. Sułkowksa, P. Szymczak and M. Cieplak, Phys. Rev. Lett. 100, 058106 (2008).
[8] J. F. Marko, Phys. Rev. E 79 (2009), 051905.
[9] Z. Liu, E. L. Zechiedrich, and H. S. Chan, Biophys. J. 90, 2344 (2006).
[10] S. A. Wasserman and N. R. Cozzarelli, Science 232, 951 (1986).
[11] D. W. Sumners, aÅJKnot theory and DNA,aÅI in New Scientific Applications of Geometry and Topology, edited by D. W. Sumners, Proceedings of Symposia in Applied Mathematics, Vol. 45, ÌŚAmerican Mathematical Society, Providence, RI, 1992, 39.
[12] A. V. Vologodski, ÎĘA. V. Lukashin, M. D. Frank-Kamenetski ÎĘ and V. V. Anshelevich, Zh. Eksp. Teor. Fiz. 66, 2153 (1974); Sov. Phys. JETP 39, 1059 (1975); M. D. Frank-Kamenetskii, A. V. Lukashin and A. V. Vologodskii, Nature (London) 258, 398 (1975).
[13] E. Orlandini, S. G. Whittington, Rev. Mod. Phys. 79, 611 (2007); C. Micheletti, D. Marenduzzo, and E. Orlandini, Phys. Reports 504, 1 (2011).
[14] S. D. Levene, C. Donahue, T. C. Boles and N. R. Cozzarelli, Biophys. J. 69 (1995) 1036.
[15] T. Vettorel, A. Yu. Grosberg and K. Kremer, Phys. Biol. 6, 025013 (2009).
[16] P. Virnau, Y. Kantor and M. Kardar, J. Am. Chem. Soc. 127 (43), 15102 (2005).
[17] P. Pierański, S. Przybył and A. Stasiak, EPJ E 6 (2), 123 (2001).
[18] J. Yan, M. O. Magnasco and J. F. Marko, Nature 401 (1999), 932.
[19] J. Arsuaga, M. Vazquez, S. Trigueros, D. W. Sumners and J. Roca, PNAS 99 (2002), 5373.
[20] J. Arsuaga, M. Vazquez, P. McGuirk, S. Trigueros, D. W. Sumners and J. Roca, PNAS 102 (2005), 9165.
[21] R. Metzler, A. Hanke, P. G. Dommersnes, Y. Kantor and M. Kardar, Phys. Rev. Lett. 88, 188101 (2002).
[22] P. Pieranski, S. Clausen, G. Helgesen and A. T. Skjeltorp, Phys. Rev. Lett. 77, 1620 (1996).
[23] Y. Diao, C. Ernst and E. J. Janse van Rensburg in Ideal Knots, A. Stasiak, V. Katritch and L. H. Kauffman (Eds), (World Scientific, Singapore, 1998) p.52.
[24] D. W. Sumners, Notices of the Am. Math. Soc. 42 (5) (1995), 528.
[25] L. Faddeev and A. J. Niemi, Nature 387 (1997), 58.
[26] J. S. Birman, Braids, links, and mapping class groups, (Princeton University Press 1974).
[27] I. K. Darch and R. G. Scharein, Bioinformatics, 22 (14) (2006), 1790.
[28] F. Ferrari, Phys. Lett. A323, (2004), 351, [cond-mat/0401104]
[29] Das Sarma, S., M. Freedman, and C. Nayak, Topological quantum computation, Phys. Today 59 I7(7I8) (2006), 32.
[30] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, Rev. mod. Phys. 80 (2008), 1083.
[31] F. Wilczek, New kinds of quantum statistics, article published in The Spin, Progress in Mathematical Physics 55 (2009), 61.
[32] V. Goldman, J. Liu and A. Zaslavsky, Phys. Rev. B71 (2005), 153303; F. Camino, F. Zhou and V. Goldman, Phys. Rev. Lett. 98 (2007), 076805.
[33] G. Ben-Shach, C. R. Laumann, I. Neder, A. Yacoby, and B. I. Halperin, Phys. Rev. Lett. 110 (2013), 106805.
[34] V. Gurarie, L. Radzihovsky, and A. V. Andreev, Phys. Rev. Lett. 94 (2005), 230403.
[35] M. Dolev, M. Heiblum, V. Umansky, A. Stern, and D. Mahalu, Nature 452 (2008), 829; I. Radu, J. Miller, C. Marcus, M. Kastner, L. Pfeiffer, and K. West, Science 320 (2008), 899.
[36] M. Blau and G. Thompson, Annals Phys. 205 (1991), 130.
[37] D. F. Milne, N. V. Korolkova and P. van Loock, Phys. Rev. A. 85 (5) (2012), 052325.
[38] See e. g. G. Decher, E. Kuchinka, H. Ringsdorf, J. Venzmer, D. Bitter-Suermann and C. Weisgerber, *Angew. Makromol. Chem.*, 71 (1989), 166; J. Simon, M. Kühner, H. Ringsdorf and E. Sackmann, *Chem. Phys. Lipids.* 76 (2) (1995), 241; G. Blume and G. Cevc, *Biochim. Biophys. Acta* 1029 (1990), 91; D. D. Lasic, F. J. Martin, A. Gabizon, S. K. Huang, D. Papahadjopoulos; *Biochim. Biophys. Acta* 1070 (1991), 187.

[39] S. F. Edwards, *Proc. Phys. Soc.* 91 (1967), 513; *Proc. Phys. Soc.* 92 (1967), 9.

[40] F. Ferrari and I. Lazzizzera, *Phys. Lett.* B444 (1998), 167.

[41] F. Ferrari and I. Lazzizzera, *Jour. Phys. A: Math. Gen.* 32 (1999), 1347, hep-th/9803008.

[42] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* 209 (1991), 129.

[43] M. Kardar, *J. Appl. Phys.* 61 (1987), 3601; M. Kardar, G. Parisi, Y.-C. Zhang, *Phys. Rev. Lett.* 56 (1986), 889; M. Kardar, Y.-C. Zhang, *Phys. Rev. Lett.* 58 (1987), 2087.

[44] R. D. Kamien, P. Le Doussal and D. Nelson, *Phys. Rev.* A 45 (1992), 8727.

[45] P. G. de Gennes, *Phys. Lett.* A38 (1972), 339; J. des Cloiseaux, *Phys. Rev. A* 10 (1974), 1665; V. J. Emery, *Phys. Rev.* B11 (1975), 239.

[46] F. Wilczek, *Phys. Rev. Lett.* 69 (1992), 132.

[47] M. Blau and G. Thompson, Ann. Phys. 205 (1991), 130.

[48] J. Froehlich and C. King, *Comm. Math. Phys.* 126 (1) (1989), 167.

[49] F. Ferrari, *Topological field theories with non-semisimple gauge group of symmetry and engineering of topological invariants*, chapter published in *Trends in Field Theory Research*, O. Kovras (Editor), Nova Science Publishers (2005), ISBN:1-59454-123-X. See also the reprint of this article in *Current Topics in Quantum Field Theory Research*, O. Kovras (Editor), Nova Science Publishers (2006), ISBN: 1-60021-283-2.

[50] F. Ferrari, *Annalen der Physik* (Leipzig) 11 (2002) 4, 255–290.

[51] G. Dunne, *Self-Dual Chern-Simons Theories*, Lecture Notes in Physics, New Series M: Monographs, Vol. 36, (Springer Verlag, 1995).

[52] F. Ferrari and I. Lazzizzera, *Nucl. Phys.* B559 (3) (1999), 673.

[53] M. Otto and T. A. Vilgis, *Phys. Rev. Lett.* 80 (1998), 881.

[54] M. Otto, *J. Phys. A: Math. Gen.* 34 (12) (2001), 2539.

[55] M. G. Brereton, *Jour. Mol. Struct.* (Theochem), 336 (1995), 191.

[56] D. R. Nelson and H. S. Seung, *Phys. Rev.* B39 (1989), 9153.
[57] M. Dunajski, *Abelian vortices from Sinh–Gordon and Tzitzeica equations*, *Phys. Lett* **B710** (2012), 236, [arXiv:1201.0105v2 [hep-th]].

[58] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, (World Scientific Publishing, 3rd Ed., Singapore, 2003).

[59] A.L. Kholodenko and T.A. Vilgis, *Phys. Rep.* **298** (1998), 251.

[60] F. Ferrari, *Jour. Math. Phys.* **44** (1) (2003), 138, [hep-th/0210100].

[61] G. P. Collins, *Scientific American* **294** (4) (2006), 56.