Representations of marked quivers

A.V. Roiter*

Institute of Mathematics of National Academy of Sciences of Ukraine,
3 Tereshchenkivska Street, 01601 Kyiv-4, Ukraine

We introduce a generalization of representations of quivers [1] that contains also representations of posets [2], vectorspace problems and other matrix problems. Many examples, some of which are given below, show that the language of marked quivers is rather convenient and that the notion of their representations can probably be considered as basic for a general theory of matrix problems. Some historical comments from my personal point of view are added in the Appendix. I am very grateful to the Referee for the numerous remarks and advices.

We compose maps on the right. Thus we denote the composition of \( \alpha : A \rightarrow B \) and \( \beta : B \rightarrow C \) by \( \alpha \beta \).

1. Let \( Q \) be a quiver, \( Q_v \) the set of vertices (points) of \( Q \), \( Q_a \) the set of arrows. We recall that a representation \( U \) of \( Q \) in a category \( K \) attaches to any \( a \in Q_v \) the object \( U(a) \in K \) and to any \( x \xrightarrow{\alpha} y \) the morphism \( U(\alpha) \in K(U(x), U(y)) \). Representations of \( Q \) in \( K \) form the category \( \text{Rep} Q = \text{Rep}(Q, K) \), and a morphism \( f : U \rightarrow W \) between the representations \( U \) and \( W \) consists of morphisms \( f(x) : U(x) \rightarrow W(x) \), \( x \in Q_v \) such that \( f(x)W(\alpha) = U(\alpha)f(y) \) for each arrow \( x \xrightarrow{\alpha} y \). Usually \( K \) is the category of modules over a field (or a commutative ring) \( k \) [1, 3].

We will denote by \( \Delta \) or \( \Delta_\alpha \) the quiver \( x \rightarrow y, (x \neq y), \Delta_v = \{x, y\}, \Delta_a = \{\alpha\} \). The category \( \text{Rep}(\Delta, K) \) is called in [4] the category of morphisms of a category \( K \).

We will say that a quiver \( Q \) is marked if to each \( x \in Q_v \) is attached a category \( K_x \), and to each arrow \( x \xrightarrow{\alpha} y \) is attached a functor \( \Phi_\alpha : K_x^\circ \times K_y \rightarrow \text{Sets} \), (i.e., a bifunctor contravariant in \( K_x \) and covariant in \( K_y \) and

*e-mail: roiter@imath.kiev.ua
taking values in the category of sets). Then \( M = \{ K_x, \Phi_{\alpha} \} \) is a marking of the quiver \( Q \).

A representation \( U \) of a marked quiver \( Q_M \) attaches to each \( a \in Q_v \) the object \( U(a) \in K_a \) and to each \( x \xrightarrow{\alpha} y \) the element \( U(\alpha) \in \Phi_{\alpha}(U(x), U(y)) \).

A morphism \( f : U \longrightarrow W \) consists of morphisms \( f(x) \in K_x(U(x), W(x)) \), \( x \in Q_v \) such that \( W(\alpha)\Phi_{\alpha} (f(x), 1_y) = U(\alpha)\Phi (1_x, f(y)) \) for each arrow \( x \xrightarrow{\alpha} y \).

Multiplication of morphisms (in both \( \text{Rep} Q \) and \( \text{Rep} Q_M \)) is defined in the natural way.

**Remark 1.** For any functor \( \Phi : K^\circ \times L \longrightarrow \text{Sets} \), \( t \in \Phi(A, B) \), \( \alpha \in K(A', A) \) and \( \beta \in L(B, B') \), where \( A, A' \in \text{Ob} K \) and \( B, B' \in \text{Ob} L \), we can agree to define the compositions \( \alpha t \in \Phi(A', B) \) and \( t\beta \in \Phi(A, B') \) by putting \( \alpha t = t\Phi(\alpha, 1_B) \) and \( t\beta = t\Phi(1_A, \beta) \). In fact by such agreement we attach (in the additive case) to the functor \( \Phi \) a bimodule \( K\Phi_L \) over the categories \( K \) and \( L \) (compare \([3], 2.2\)) with \( K\Phi_L \) and the second of these definitions repeats the first one word-for-word.

In order to obtain representations of the “usual” (unmarked) quiver \( Q \) in a category \( K \) we should of course put \( K_x = K \) for all \( x \in Q_v \), and \( \Phi_{\alpha} = \text{Hom}_K \) for all \( \alpha \in Q_a \).

We will say that a marking \( M \) is a point-marking (in a category \( K \)) if there exist a category \( K \) and functors \( \Phi_x : K_x \longrightarrow K(x \in Q_v) \) such that \( \Phi_{\alpha}(A, B) = K(\Phi_x(A), \Phi_y(B)) \) and \( t\Phi_{\alpha}(p, q) = \Phi_x(p) t \Phi_y(q) \in \Phi_{\alpha}(A', B') \) for any \( x \xrightarrow{\alpha} y \) (may be \( x = y \)) and \( t \in \Phi_{\alpha}(A, B) \). Here \( A, A' \in \text{Ob} K_x, B, B' \in \text{Ob} K_y, p \in K_x(A, A') \) and \( q \in K_y(B, B') \).

**2.** Let \( Q_M \) be a marked quiver and \( Q^i \) a subquiver of \( Q \). The marked quiver \( Q^i_M \) is defined naturally by restriction of the marking \( M \) to \( Q^i \). For \( x \in Q_v \), we define a functor \( Y^i_x : \text{Rep} Q_M \rightarrow K_x, U \mapsto U(x) \), and for \( x \in Q^i_v \) a functor \( Y^i_x : \text{Rep} Q^i_M \rightarrow K_x \).

Let \( \omega \) be a set, \( \{ Q^i| i \in \omega \} \) be a family of subquivers of \( Q \) such that \( Q_v = \bigsqcup_{i \in \omega} Q^i_v \), \( Q^a \cap Q^b = \emptyset \), if \( \{ i, j \} \subset \omega, i \neq j \). We introduce a quiver \( \Omega \) with vertex set \( \Omega_v = \omega \) and arrow set \( \Omega_a \) in one-one correspondence with \( Q^a \setminus \bigsqcup_{i \in \omega} Q^i_v \) and defined as follows. Corresponding to each arrow \( x \xrightarrow{\alpha} y \in Q^a \setminus \bigsqcup_{i \in \omega} Q^i_v \) with \( x \in Q^i_v \) and \( y \in Q^j_v \) there is an arrow \( i \xrightarrow{\alpha} j \in \Omega_a \); if \( i \) and \( j \) coincide, then \( i \) is a loop.

Let \( M \) be a marking of \( Q \). We define the marking \( \overline{M} \) of \( \Omega \), by putting
\[K_i = \text{Rep } Q_i^M, \quad \Phi_\alpha = (Y_i^x \times Y_j^y) \Phi_\alpha, \text{ where } x \xrightarrow{\alpha} y, \ x \in Q_i^v, \ y \in Q_i^v.\] From these definitions directly follows

**Proposition 2.** \(\text{Rep } Q_M \simeq \text{Rep } \Omega_M.\)

For \(z \in Q_v\), let \(\delta(z)\) denote the number of arrows incident on it.

Now consider the following example. Suppose that there is a vertex \(z \in Q_v\) with \(\delta(z) = 1\) and that \(\beta\) is the unique arrow incident on it. Let \(w \neq z\) be the second vertex of \(\beta\) (either \(z \xrightarrow{\beta} w\) or \(w \xrightarrow{\beta} z\)). We denote by \(Q'\) the subquiver of \(Q\) such that \(Q'_v = Q_v \setminus \{z\}\) and \(Q'_a = Q'_a \setminus \{\beta\}\).

For any \(x_i \in Q_v\) we define the subquiver \(Q_i^a \subset Q\) as follows. If \(x_i \neq w\), then \(Q_i^v = \{x_i\}, \ Q_i^a = \emptyset\); and if \(x_i = w\), then \(Q_i^v = \Delta_\beta\). We identify the quiver \(\Omega\) with \(Q'\). Let \(\overline{M}\) be the marking induced on \(Q'\), so that \(\overline{K}_x = K_x\) if \(x \neq w\), \(\overline{K}_w = \text{Rep } (\Delta_\beta)_M\) and the functors are defined naturally.

**Corollary 2.** \(\text{Rep } Q_M \simeq \text{Rep } Q'_M.\)

**Remark 2.** If \(M\) is a point-marking then \(\overline{M}\) in the corollary (but not in the proposition!) is also a point-marking.

3. Let \(k\) be a fixed algebraically closed field. By \(\text{mod } k\) is denoted the category of finite dimensional vector spaces over \(k\). Unless otherwise stated all categories will be assumed to be \(k\)-categories (i.e. the sets \(\text{Hom}(A, B)\) are endowed with a \(k\)-module structure such that the composition of maps is \(k\)-bilinear), all subcategories to be \(k\)-subcategories, and all functors to be \(k\)-functors [3]. In particular, the functors \(\Phi_\alpha\) of the definition of a marked quiver are \(k\)-functors \(K^x \times K_y \longrightarrow \text{mod } k\) (which, of course, may be also viewed as functors to \(\text{Sets}\)).

An additive \(k\)-category in which every idempotent has a kernel is an aggregate [3]. A category \(S\) is a spectroid if its objects are indecomposable (i.e., for any \(A \in \text{Ob } S\), \(S(A, A)\) contains exactly two idempotents \(1_A \neq O_A\) and pairwise non-isomorphic [3]. A subspectroid of \(\text{mod } k\) (i.e. a \(k\)-subcategory of \(\text{mod } k\), which is a spectroid) is a vectroid [5]. Spectroids are not additive, but any spectroid \(S\) generates in a natural way its “additive closure””, the aggregate \(\oplus S\) [3] (whose objects are the sequences \((X_1, \ldots, X_m),\ X_i \in \text{Ob } S\)). On the other hand, any aggregate \(\mathfrak{A}\) is defined by its spectroid \(S(\mathfrak{A})\), (such that \(\mathfrak{A} \simeq \oplus S(\mathfrak{A})\)).

If \(V\) is a vectroid, the aggregate \(\oplus V\) is naturally embedded in \(\text{mod } k\).
For a functor $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$, Ker $\Phi$ is the set (an ideal of $\mathfrak{A}$) of morphisms $\gamma$ of $\mathfrak{A}$ such that $\Phi(\gamma) = 0$; if $A \in \text{Ob} \mathfrak{A}$ then $A \in \text{Ker} \Phi$, means that $1_A \in \text{Ker} \Phi$. The functor $\Phi$ is faithful if Ker $\Phi = \{0\}$, then $\mathfrak{A} \simeq \text{Im} \Phi$.

We will say that $Q$ is $k$-marked if it is point-marked in mod $k$, all $K_x$ are aggregates, and the functors $\Phi_x (x \in Q_v)$ are faithful. So in this case we may consider $K_x (x \in Q_v)$ as subaggregates of mod $k$.

Choosing a representative in each indecomposable isoclass of $K_x (x \in Q_v)$, we obtain a vectroid $V_x$. So a $k$-marked quiver is defined by the collection of vectroids $V_x$, $x \in Q_v$ ($K_x = \oplus V_x$).

Unless otherwise stated we will consider only finite $k$-markings, i.e., we will assume, for a $k$-marked quiver $Q_M$, that $|Q_v| < \infty$, $0 < |Q_a| < \infty$, $|\text{Ob} V_x| < \infty$, ($x \in Q_v$). We will also consider only connected quivers.

For a category $C$, we denote by ind $C$ the set of isoclasses of indecomposables in $\text{Ob} C$; we abbreviate ind $\text{Rep} Q_M$ to ind $Q_M$. We say that $Q_M$ has finite type if $|\text{ind} Q_M| < \infty$.

For a fixed $x \in Q_v$, $T \in \text{ind} K_x$, we construct $U_T \in \text{ind} Q_M : U_T(x) = T$, $U_T(y) = 0$ for $x \neq y$, $U_T(\alpha) = 0$ for any $\alpha \in Q_a$. So we may consider $\bigsqcup_{x \in Q_v} \text{ind} K_x \subset \text{ind} Q_M$.

If $M$ is a point-marking of $Q$ in mod $k$ such that the $K_x$ are aggregates but the functors $\Phi_x$ are not faithful, we may define a $k$-marking $\tilde{M}$ (of $Q$) by putting $\tilde{K}_x \simeq K_x / \text{Ker} \Phi_x$, and taking $\tilde{\Phi}_x$ to be the embedding of $\tilde{K}_x = \text{Im} \Phi_x$ into mod $K$. The next statement is obvious.

**Lemma 3.** There is a natural bijection between $\text{ind} Q_M$ and $\text{ind} \tilde{Q}_M \sqcup \bigsqcup_{x \in Q_v} (\text{ind} K_x \cap \text{Ker} \Phi_x)$.

**4.** If $\mathfrak{A}$ and $\mathfrak{B}$ are two categories (in particular, vectroids) then $\mathfrak{A} \sqcup \mathfrak{B}$ is the category such that $\text{Ob} (\mathfrak{A} \sqcup \mathfrak{B}) = \text{Ob} \mathfrak{A} \sqcup \text{Ob} \mathfrak{B}$, $(\mathfrak{A} \sqcup \mathfrak{B})(A_1, A_2) = \mathfrak{A}(A_1, A_2)$, $(\mathfrak{A} \sqcup \mathfrak{B})(B_1, B_2) = \mathfrak{B}(B_1, B_2)$, $(\mathfrak{A} \sqcup \mathfrak{B})(A_1, B_1) = \{0\}$, $(\mathfrak{A} \sqcup \mathfrak{B})(B_1, A_1) = \{0\}$ for $A_1, A_2 \in \text{Ob} \mathfrak{A}$, $B_1, B_2 \in \mathfrak{B}$.

**Example 1.** Let $P$ be a poset considered as a category ($\text{Ob} P = P$, $|\text{Hom}(u, w)| = 1$, if $u \leq w$, $\text{Hom}(u, w) = \varnothing$ otherwise). Let $kP$ be its $k$-linearization [3], which may be viewed as a vectroid (to each $p \in P$ is associated a one-dimensional vector space $kp \in \text{mod} k$, $\dim \text{Hom}_{kP}(a, b) = 1$ if $a \leq b$, $\text{Hom}_{kP}(a, b) = \{0\}$ otherwise).

We define two vectroids, $k_n$ and $k^n$, as follows. The vectroid $k_n$, which we call linear, is the linearization of a linearly ordered set $P$ with $|P| = n$. On the other hand, the vectroid $k^n$ has one object $A$ and morphism space
Hom(A, A) = k(r) with \( r^n = 0 \). We take A to be an n-dimensional k-space with basis \( \{a_1, \ldots, a_n\} \) and define an action of r by \( a_i r = a_{i+1}, a_n r = 0 \). Observe that \( k_1 = k_1^1 = k \). We say that the vertex x is an unmarked point if \( V_x = k \) (so that \( K_x \simeq \text{mod} k \)).

If \( \Delta = x \xrightarrow{\alpha} y \), \( V_x = k \), \( V_y = kP \), then the category \( \text{Rep} \Delta_M \) coincides with the category of representations of the poset \( P \) in the sense of \([2]\) \( \bigoplus V_y \subset \text{mod} K \), \( \Phi_{\alpha} = \text{Hom}_k \), elements of \( \text{Hom}(X, Y) \), where \( X \in \text{mod} k \), \( Y \in \bigoplus V_y \), may be considered as matrices divided into vertical strips indexed by elements of \( P \).

**Example 2.** If \( Q = \Delta \), \( V_x = k \) and \( V_y \) is an arbitrary vectroid \( V \), then \( \text{Rep} \Delta_M \) coincides with the category \( \text{Rep} V \) for the vectroid \( V \) in the sense of \([5]\) (= categorical matrix problem \([6]\)= vectorspace problem \([7]\)), and we identify these two categories. So ind \( V \) \([5]\) is ind \( \Delta_M \), \( V \) is tame if \( \Delta_M \) is (see Section 5) and so on. If \( M \) is an arbitrary \( k \)-marking of \( \Delta \) then \( \Delta_M \) coincides with the category of representations of the pair \( (V_x, V_y) \) \([8]\).

If \( V_y = k \) and \( V_x = V \), then \( \text{Rep} \Delta_M \simeq \text{Rep} V^o \), where \( V^o \) is the vectroid opposite to \( V \) (i.e. there are bijections \( * \) between \( \text{Ob} V \) and \( \text{Ob} V^* \), between \( A \) and \( A^* \) for \( A \in V \), between \( \text{Hom}_V(A, B) \) and \( \text{Hom}_{V^o}(B^*, A^*) \) such that if \( a \varphi = b \), then \( b^* \varphi^* = a^* \) for \( a \in A, b \in B, \varphi \in \text{Hom}_V(A, B) \)).

**Example 3.** We recall \([3]\) that a right (finite dimensional) module \( M \) over a \( k \)-category \( \mathfrak{A} \) consists of a finite dimensional vector space \( M(X) \) (over \( k \)) for each object \( X \in \mathfrak{A} \), and of maps \( M(X) \times \mathfrak{A}(X, Y) \rightarrow M(Y), (m, f) \rightarrow mf \) that satisfy the usual axioms \( (m_1 x = m, m(fg) = (mf)g(m_1 + m_2)f = m_1 f + m_2 f \) and \( m(f_1 + f_2) = m f_1 + m f_2 \).

The notions of submodule, factormodule, and so on, are defined in the natural way. A functor \( \Phi_M : \mathfrak{A} \rightarrow \text{mod} k \) arises from the module \( M \). Then the category \( M^k \) defined in \([3]\) coincides with \( \text{Rep} \Delta_M \) where \( M \) is a point-marking in \( \text{mod} k \) (not a \( k \)-marking) of \( \Delta : x \) is unmarked, \( K_y = \mathfrak{A}, \Phi_y = \Phi_M \).

Lemma 3 holds.

**Example 4.** We call a family \( M_1, \ldots, M_p \) of modules over an aggregate \( \mathfrak{A} \) a bunch of modules. The category of representations of a bunch of modules is \( \text{Rep} Q_M \), where \( Q_e = \{x, y_1, \ldots, y_p\}, Q_a = \{\alpha_1, \ldots, \alpha_p\}, x \xrightarrow{\alpha_i} y_i, K_x = \oplus k, K_{y_i} = \mathfrak{A} (1 \leq i \leq p), \Phi_{\alpha_i} : (\oplus k)^o \times K_{y_i} \rightarrow \text{mod} k \) are induced by \( \Phi_M : \mathfrak{A} \rightarrow \text{mod} k (i = 1, \ldots, p) \) \([8]\). In this case \( M \) is not a point-marking.
Example 5. Let $Q$ be $c \leftarrow d \rightarrow b$. A representation of the (non-marked) $Q$ may be viewed as a matrix over $k$ divided into two vertical strips.

$$
\begin{array}{cc}
d & c & b \\
\end{array}
$$

Two representations are equivalent if one can be obtained from the other by means of any row transformation and any column transformations inside of each strip.

Let $M$ be the $k$-marking of $Q : V_d = k$, $V_c = k_2$, $V_b = k^2 \sqcup k$ (see Example 1).

A representation of $Q_M$ may be considered as a matrix over $k$ divided in the following way:

$$
\begin{array}{cccc}
d & c_1 & c_2 & x & x^* & y \\
\end{array}
$$

Here $c_1 > c_2$ correspond to the two objects of the partially ordered set underlying $k_2$, so $\text{Hom}(c_1, c_2) \neq 0$, and $x$ and $x^*$, with $xr = x^*$, correspond to the basis $\{x, x^*\}$ of the single object $X$ of $k^2$ with $\text{Hom}(X, X) = k[r]/r^2$. The strips $x$ and $x^*$ have the same numbers of columns, and when we perform a column transformation inside of $x$, we should perform the same column transformation in $x^*$. Moreover, we can perform any row transformation, any column transformation inside $c_1$, $c_2$, $y$, and add columns of $c_1$ to columns of $c_2$, and columns of $x$ to columns of $x^*$.

So we see that representations of $k$-marked quivers are obtained from representations of non-marked quivers by division of matrices, corresponding to the arrows, into vertical and (in the general case) horizontal strips and the restriction of admissible transformations. In the case of non-$k$-marking, these matrices may have fixed zero blocks.

Let us apply Corollary 2 to the arrow $d \xrightarrow{\beta} c$, $Q' = d \xrightarrow{\gamma} b$, $\overline{K}_b = K_b = \oplus V_b$, $\overline{K}_d = \text{Rep}(\Delta_\beta)_M$. It is easy to see that $\text{ind}(\Delta_\beta)_M$ consists of 5 representations $c_i^0$, $c_i^1$, $c_i^1$, $d^0$, where $c_i^0 \in k^{0 \times 1}$, $c_i^1 \in k^{1 \times 1}$, $d^0 \in k^{1 \times 0}$ ($i = 1, 2$). There arises a functor $\Phi : \text{Rep}(\Delta_\beta)_M \rightarrow \text{mod} k$ such that $\Phi(c_i^0) = \{0\}$, and $\text{Im} \Phi$ is $\oplus \overline{V}_d$, $\text{Ob} \overline{V}_d = \{c_1^1, c_2^1, d_0\}$, $\overline{V}_d \simeq k_3$.

Using Lemma 3 we conclude that there exists a bijection between $\text{ind}(Q_M)$ and $\text{ind}(Q'_M) \sqcup \{c_1^0, c_2^0\}$; here $M$ is a $k$-marking, $\overline{V}(d) = k_3$, $\overline{V}_b = V_b$. 

6
5. A bimodule $\mathfrak{M}_{\mathfrak{A}\mathfrak{B}}$ over categories $\mathfrak{A}$ and $\mathfrak{B}$ consists of $\mathfrak{M}(A, B) \in \text{mod } k$, $A \in \text{Ob } \mathfrak{A}$, $B \in \text{Ob } \mathfrak{B}$ with natural multiplication and axioms. By $\text{El } \mathfrak{M}$ we denote a category whose objects are the elements of the spaces $\mathfrak{M}(A, B)$, given $x \in \mathfrak{M}(A, B)$, $y \in \mathfrak{M}(C, D)$, $\text{El } \mathfrak{M}(x, y) = \{(\alpha, \beta) : \alpha \in \mathfrak{A}(A, C), \beta \in \mathfrak{B}(B, D), \alpha y = x \beta\}$. If $\mathfrak{A} = \mathfrak{B}$ then $\text{El } \mathfrak{M}$ is the following subcategory of $\text{El } \mathfrak{M}$: $x \in \mathfrak{M}(A, B)$ belongs to $\text{El } \mathfrak{M}$ if $A = B$; $(\alpha, \beta) \in \text{Mor } (\text{El } \mathfrak{M})$ if $\alpha = \beta$.

Let $M$ be a $k$-marking of $Q$. Denote by $D = D(Q_M)$ the Cartesian product of categories $K_x$, $x \in Q_v$. Let $Q_v = \{q_1, \ldots, q_n\}$; $X, Y \in \text{Ob } D$, $X = \{X_1, \ldots, X_n\}$, $Y = \{Y_1, \ldots, Y_n\}$, $X_i, Y_i \in \text{Ob } K_{q_i}$. If $q_i \xrightarrow{\alpha} q_j$, put $R(X, Y)_\alpha = \text{mod } k(X_i, Y_j)$. Let $R(X, Y)$ be the direct sum of vector spaces $R(X, Y)_\alpha$, $\alpha \in Q_a$.

$R_\alpha(X, Y)$ may be naturally considered as a bimodule over $K_x$, $K_y$ (if $x \xrightarrow{\alpha} y$), so bimodule $D(R_D)$ arises.

It easy to see that the category $\text{El } D(R_D)$ in fact coincides with the category $\text{Rep } Q_M$. Indeed an element $U \in R(X, X)$ “is” the collection of $U(x) \in \text{Ob } K_x$ ($x \in Q_v$) and $U(\alpha) \in \text{mod } k(U(x_a), U(y_a))$, where $\alpha \in Q_a$, $x_a \xrightarrow{\alpha} y_a$, and morphisms in $\text{Rep } Q_M$ and $\text{El } D(R_D)$ are defined in fact in the same way.

So we may consider points (elements) of $R(X, X)$, $X \in \text{Ob } D$ as representations of $Q_M$ and talk about their indecomposibility and equivalence ($\simeq$).

A $k$-marked $Q_M$ is wild if for some $X$, $R(X, X)$ contains an affine plane $W_{X,X}$ consisting of indecomposable and pairwise non-isomorphic representations. A $k$-marked $Q_M$ is tame if $|\text{ind } Q_M| = \infty$ and for any $X$ there exists a finite set $T_{X,X}$ whose elements are affine subspaces of dimension 0 and 1 in $R(X, X)$ such that each indecomposable representation $U \in R(X, X)$ is equivalent to $w \in A \in T_{X,X}$ [9].

By aff $k$ we denote the following $k$-category: $\text{Ob } (\text{aff } k) = \text{Ob } (\text{mod } k)$, $\text{aff } k(X, Y) = \{(A, y) : A \in \text{mod } k(X, Y), y \in Y\}$, $(A, y)(B, z) = (AB, yB + z)$. Given $x \in X$, $(A, y) \in \text{aff } k(X, Y)$ put $x(A, y) = xA + y$.

A functor $\Phi : \text{El } \mathfrak{M}_{\mathfrak{A}\mathfrak{B}} \longrightarrow \text{El } \mathfrak{M}_{\mathfrak{A}\mathfrak{B}}$ is affine if there exists map $\overline{\Phi} : \text{Ob } \mathfrak{A} \longrightarrow \mathfrak{B}$ such that if $x \in \mathfrak{M}(\mathfrak{A}, \mathfrak{A})$ then $\Phi(x) \in \mathfrak{M}(\overline{\Phi}(\mathfrak{A}), \overline{\Phi}(\mathfrak{A}))$ and the induced map $\Phi_A : \mathfrak{M}(A, A) \longrightarrow \mathfrak{M}(\overline{\Phi}(A), \overline{\Phi}(A))$ is affine ($\in \text{Mor } \text{aff } k$) for each $A \in \text{Ob } \mathfrak{A}$.

From our definitions directly follows

**Lemma 5.** Let $D(R_D)$ and $D'(R_D')$ be bimodules attached respectively to $Q_M$ and $Q_M'$, $\Phi : \text{El } D(R_D) \longrightarrow \text{El } D(R_D)$ is an affine functor. The wildness of $Q_M'$
implies the wildness of $Q_M$, if conditions 1) and 2) hold:

1) $\Phi_X$ is embedding for each $X \in \text{Ob} \, D'$;

2) $\Phi_X(x)$ is indecomposable if $x$ is, $x \simeq y$ if $\Phi(x) \simeq \Phi(y)$ (for any $x, y \in \mathcal{R}'(X, X)$, $X \in \text{Ob} \, D'$).

The tameness of $Q'_M$ implies the tameness of $Q_M$, if conditions 3) and 4) hold:

3) for any $X \in \text{Ob} \, D$, there exists only a finite number of pairwise nonisomorphic objects $X'$ of $D'$, such that $\Phi(X') = X$;

4) if $X \in \text{Ob} \, D$ then for any indecomposable $x \in \mathcal{R}(X, X)$ except for a finite number there exists $x' \in \mathcal{R}'(X', X')$ such that $\Phi_X(x') \simeq x$, $\Phi(X') = X$.

Condition 2) clearly holds if, for each $X \in \text{Ob} \, D'$, $x_1, x_2 \in \mathcal{R}'(X, X)$ and $\varphi \in \widetilde{\text{El}} \, \mathcal{R}(\Phi_X(x_1), \Phi_X(x_2))$, there exists $\varphi' \in \widetilde{\text{El}} \, \mathcal{R}'(x_1, x_2)$, such that $\Phi(\varphi') = \varphi$.

**Remark 5.** An affine functor $\Phi : \widetilde{\text{El}} \, D' \to \widetilde{\text{El}} \, D$ naturally arises if $Q' = \Omega$, $M' = \widetilde{M}$ (see Proposition 2 and Lemma 3).

It is easy to see that $Q_M$ can not be both wild and tame. In [9] it is proved that any vectroid is either wild or tame or has finite type. In [10, 11] the same is proved for a wider class of matrix problems, but in a formally different sense.

The marked quiver $Q_M$ of Example 5 is wild. Let $D \in \text{Ob} \, K_d$, $C \in \text{Ob} \, K_c$ and $B \in \text{Ob} \, K_b$ all be 4-dimensional $k$-spaces. Then $\mathcal{R}(X, X) = \text{Hom}_k(D, C) \oplus \text{Hom}_k(D, B)$ contains the affine plane

$$
W_{X,X} = \begin{pmatrix}
  c & b \\
  0 & 0 & 1 & 0 & 0 & 0 & \mu \lambda \\
  0 & 0 & 0 & 1 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
$$

$$(\lambda, \mu \in k),$$

which consists of pairwise non-isomorphic indecomposable representations of $Q_M$.

**6.** Since a vectroid is a subcategory of mod $k$, its objects are vector spaces, and its morphisms are linear operators.

In [5], for any vectroid $V$, $\dim V = \sup_{A \in \text{Ob} \, V} \dim A$ is defined. It is well known [3, 5] that if $V$ has finite type then $\dim V \leq 3$. The rank of $V$, denoted
rank $V$, is by definition a supremum of the ranks of noninvertible additively indecomposable morphisms of $V$. Here a morphism $\varphi$ is called \textit{additively indecomposable} if $\varphi \neq \varphi_1 + \varphi_2$ where $\text{rank } \varphi_1 < \text{rank } \varphi$, $\text{rank } \varphi_2 < \text{rank } \varphi$. It is known \cite{5} that if $|\text{ind } V| < \infty$, then $\text{rank } V \leq 2$. It is easy to prove that if a vectroid $V$ is tame, then $\text{dim } V \leq 4$, $\text{rank } V \leq 3$.

We define the \textit{dimension}, $\text{dim } Q_M$, of a $k$-marked quiver $Q_M$ to be $\max_{x \in Q_v} \text{dim } V_x$.

It is clear that, if a $k$-marked quiver $Q_M$ has finite type, then $\text{dim } V_x \leq 3$ and $\text{rank } V_x \leq 2$ for all $x \in Q_v$ and that, if it has tame type, then $\text{dim } V_x \leq 4$ and $\text{rank } V_x \leq 3$ for all $x \in Q_v$.

It is easy to see that if $\text{dim } V = 1$, then $V$ is a linearization of some poset.

To an arbitrary vectroid $V$, we also associate a poset $S(V)$ \cite{5} in the following way. At first, consider the set $\overline{S}(V)$ consisting of all nonzero elements of all objects of $V$. Then we define on $\overline{S}(V)$ the relation $\prec_\prec x \prec_\prec y$ if there exists $\varphi \in V(X, Y)$ such that $\overline{x} \varphi = \overline{y}$ ($\overline{x} \in X$, $\overline{y} \in Y$; $X, Y \in \text{Ob } V$). This relation is not an ordering, because it can be that $x \prec_\prec y$ and $y \prec_\prec x$, but after factorization by the induced equivalence we obtain a poset $S(V)$.

We will also consider on $S(V)$ the equivalence relation $\sim$: $x \sim y$ if $\overline{x} \in X$, $\overline{y} \in Y$ and $X = Y$, where $\overline{x}$, $\overline{y}$ are the elements of $\overline{S}(V)$ corresponding to $x$, $y$; $X, Y \in \text{Ob } V$. We say that $x \in S(V)$ is \textit{big}, if there exists $y \sim x$, $y \neq x$.

We will say that the vectroid $V$ is \textit{halflinear} if it is not linear and

1) $\text{dim } V \leq 2$
2) $\text{rank } V = 1$
3) if $a$ is big then $a$ is comparable with each point of $S(V)$
4) if $a$ is small then $a$ can be incomparable with only one point of $S(V)$.

It is easy to see that a halflinear (or linear) vectoid $V$ is determined uniquely by $(S(V), \leq, \sim)$.

A vertex $x \in Q_v$ is said to be \textit{linearly marked} if $V_x$ is linear and to be \textit{halflinearly marked} if $V_x$ is halflinear. The marked quiver $Q_M$ is said to be \textit{linear} if each $x \in Q_v$ is linearly marked, and to be half linear if each $x$ is either halflinearly or linearly marked.

By $G(Q)$ we denote the non-oriented graph that corresponds to a quiver $Q$; we will view a $k$-marking of $Q$ also as a marking of $G(Q)$ – associating to each $x \in G(Q)$ a vectoid $V_x$. If all $V_x$ are linear or halflinear, we will write over each vertex $x$ of $G(Q)$ the number $n$ if $V_x = k_n$, the symbol $\infty$ if $V_x$ is halflinear, and nothing if $V_x = k$.

A $k$-marked quiver $Q_M$ is \textit{Gelfand} if and only if either $Q = \Delta$, and both...
vectroids $V_x, V_y$ are halflinear, or $G(Q)_M = \bullet \xrightarrow{\infty} \bullet$. Representations of the Gelfand $k$-marked quivers were treated in [12], where in fact their tameness was proved. (If $Q = z \leftarrow x \rightarrow y$, then $\text{Rep} Q_M = \text{Rep} \Delta M$, where $\nabla x = V_x, \nabla y = k \sqcup k$. If $Q = z \rightarrow x \rightarrow y$, then Corollary 2 implies $\text{Rep} Q_M = \text{Rep} (z \leftarrow x \rightarrow y) M$ where $\nabla x = V_x^\circ, (\nabla y = \nabla z = k)$.

For a halflinear vectroid $V$, we denote by $V^-$ the vectroid obtained from $V$ by excluding those one-dimensional objects which correspond to points of $S(V)$ comparable to all other points of $S(V)$. We say that two halflinear vectroids $V_1$ and $V_2$ are almost equivalent if $V_1^- \simeq V_2^-$. Lemma 6. $Q_M$ is wild if it is a $k$-marked quiver for which $G(Q)_M$ is one of the following types:

1) $\bullet \xrightarrow{x} \bullet \xrightarrow{y}$, where $V_x$ is not linear, $V_y \neq k$.
2) $\bullet \xrightarrow{x} \bullet \xrightarrow{y}$, where $V_x$ is not linear and not halflinear.
3) $\bullet \xrightarrow{x} \bullet \xrightarrow{y}$, where both $V_x, V_y$ are not linear, and $V_x$ is not halflinear.
4) a cycle $\tilde{A}_n$ containing a vertex $a$ with $V_a \neq k$.

The proof is straightforward.

Remark 6. Let $Q_M$ be a $k$-marked quiver with an arrow $x \xrightarrow{\alpha} y, V_x = V_y$ and let $L_N$ be the $k$-marked quiver obtained from $Q_M$ by excluding the arrow $\alpha$ and uniting the vertices $x$ and $y$ in one vertex. Then $\text{Rep} Q_M$ contains a full subcategory $R' \simeq \text{Rep} L_N$. Indeed, consider the full subcategory $R' \subset \text{Rep} Q_M$, defined by $U \subseteq \text{Ob} R'$ if $U(x) = U(y), U(\alpha) = 1_{U_x}$. So if $L_N$ is wild, then $Q_M$ also is.

7. Let $\beta$ be the arrow of the $k$-marked $Q_M$ considered in Section 2, that is, $Q$ contains a vertex $z$ with $\delta(z) = 1$ and either $z \xrightarrow{\beta} w$ or $w \xrightarrow{\beta} z$, with $w \neq z$ and $|Q_a| > 1$. We will say that $\beta$ is reducible in one of the following cases:

1) $V_w = k, V_z = k_m$
2) $V_w = k, V_z$ is halflinear
3) $V_z = k, V_w = k_2$.

Lemma 7. If $\beta$ is reducible, then there exists a natural bijection between $\text{ind} Q_M \setminus \text{Ob} V_z$ and $\text{ind} Q'_M$. If $Q'_M$ is tame, then so is $Q_M$; if it is wild then $Q_M$ is wild.
Here $Q'$ is as defined in Section 2. Thus $V'(x) = V(x)$ if $x \neq w$. Furthermore $V'(w) = k_{m+1}$ in case 1); $V'_w$ is halflinear and $V'_w$ is almost equivalent to $V_w$ in case 2); $V_w$ is halflinear and $S(V_w) = kP$, where $P$ is the poset \{p_1, p_2, p_3, p_4 | p_1 < p_2 < p_4, p_1 < p_3 < p_4\} in case 3).

The proof follows from Corollary 2, Lemma 3, Lemma 5, Remark 5 and the calculation of $\text{Rep}(\Delta_{3})_M$ in cases 1.2.3.

**Example 6.** Let $Q$ be a quiver containing subquiver $Q^0 = \Delta = w \rightarrow z$, such that $\delta(z) = 1$, and $M$ be such marking of $Q$ that we have the situation of the Lemma, case 2). Namely $K_w = \text{mod} \ k$, $V_z$ is halflinear, $\text{Ob} \ V_z = \{A, B, C, D\}$, $\dim A = 2$, $\dim B = \dim C = \dim D = 1$, $S(V_z) = \{a, a^*, b, c, d\}$, $a < b < a^* < c$, $a^* < d$. Let $Y^0_w : \text{Rep} \Delta \rightarrow K_w = \text{mod} \ k$ be the functor determined in Section 2. We may consider $\text{Ob} \ V_z \subset \text{ind} \Delta_M$. Let $T \in \text{ind} \Delta_M$, then $Y^0_w(T) = 0$ if and only if $T \in \text{Ob} \ V_z$. Here $\text{ind} \Delta \setminus \text{Ob} \ V_z = \{a_1, a_1^* \times, (c, d), b_1, c_1, d_1, w_0\}$. 

$$a_1 = (1|0), a_1^* = (0|1),$$

$$\overline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

(c, d) = (1|1), $t_1 = (1)$ for $t \in \{b, c, d\}$.

After application of Corollary 2 and Lemma 3 we get $Q'_M$, where $\text{Ob} \ V'_w = Y^0_w(\text{ind} \Delta \setminus \text{Ob} \ V_z) = \{w_0, b_1, c_1, d_1, a_1, a_1^*, \overline{A}, (c, d)\}$ where $\dim \overline{A} = 2$, (the dimensions of the others are equal to 1). Furthermore the poset $S(V'_w) = \{w_0, b_1, c_1, d_1, a_1, a_1^*, \overline{A}, \overline{A}, (c, d)\}, (\overline{A} < \overline{A}^*), a_1 < \overline{A} < b_1 < a_1^* < \overline{A}^* < (c, d) < d_1 < w_0, c, d < c_1, w_0$. The vectroid $V'_w$ is clearly almost equivalent to $V_w$. It is easy to see that the conditions of Lemma 5 (in this and the general cases) hold.

**Proposition 7.** Let $Q_M$ be $k$-marked, $Q = \Delta_a, x \overset{\omega}{\rightarrow} y$ and $V_x = k_m, m > 1$. Then there exists a natural bijection between $\text{ind} \ Q_M$ and $\text{ind} \ V \setminus \text{Ob} \ k_{m-1}$, where $V = V_y \cup k_{m-1}$. If $Q_M$ is tame, $V$ is tame; if $Q_M$ is wild, $V$ is wild. If $V$ is tame then $Q_M$ also is.

**Proof.** We consider $\overline{Q} = y \leftarrow x \rightarrow z \ \overline{\nu}_x = k, \ \overline{\nu}_y = V_y, \overline{\nu}_z = k_{m-1}$. Then the first and the second statements follow from the application of the Lemma to the arrow $x \rightarrow z$. 

11
In order to prove that tameness of $V$ implies tameness of $Q_M$ we use a matrix construction, although it probably should follow from some general consideration. In the picture below we will for simplicity assume $m = 3$. The representation $U$ of $V_y \sqcup k_2$ has the form

$$
\begin{array}{c|c|c}
U' & & \\
\hline
\end{array}
$$

where $U'$ is a representation of $V_y$ and the empty columns are for a representation of $k_2$ which we reduce to a standard form in the next diagram.

We say that a representation $U$ has preliminary form if

$$
U = \begin{bmatrix}
U_1 & U_{11} & U_{12} \\
U_2 & 0 & U_{22} \\
U_3 & 0 & 0 \\
\end{bmatrix}
$$

where $U_{11}$ and $U_{22}$ are matrices with linearly independent rows. Let $r_t(U)$ be the number of rows of $U_t$ ($t = 1, 2, 3$). Of course any representation may be reduced to preliminary form.

Let $P(V)$ be a full subcategory of $\text{Rep} V$ consisting of representations in preliminary form.

Consider a map $F : \text{Ob} P(V) \to \text{Ob} \text{Rep} Q_M$:

$$
F(U) = \begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
\end{bmatrix}
$$

It is clear that for each $Z' \in \text{Ob} D(Q_M)$ (see Section 5) there exists $Z \in \text{Ob} D(V)$ such that $F(R(Z, Z) \cap P(V)) = R(Z', Z')$. (If $Q = \Delta$, $Z' = (X, Y) \in \text{Ob} D(Q_M)$, $X \in K_x$, $Y \in K_y$ then $R(Z', Z') \simeq \text{Hom}_k(X, Y)$).

It is easy to see (using Corollary 2 or directly) that $F$ can be considered as a functor.

Let $A$ be an affine subspace of $R(Z, Z)$, $Z \in \text{Ob} D(V)$, $\dim A = 1$. We will write $A \subset P(V)$ if $A \supset \overline{A}$, $|A \backslash \overline{A}| < \infty$, $\overline{A} \subset P(V)$ and $r_t(a_i) = r_t(a_j)$ for $a_i, a_j \in \overline{A}$ ($t = 1, 2, 3$). It is clear that if $A \subset P(V)$ then there exists an affine subspace $F(A)$ of $R(Z', Z')$ such that $\dim F(A) \leq 1$, $|F(A) \backslash \overline{F(A)}| < \infty$.

It is easy to show that row-transformations may be used to reduce any $A$ to (an affine subspace) $A \subset P(V)$ (i.e. an invertible matrix $M$ exists such
that $MA = A')$. Now we may consider that if $V$ is tame, then for each $Z \in \text{Ob } D(V)$ there exist $A_1, \ldots, A_n \subset \mathcal{P}(V)$ ($\dim A_i = 1$) which generate almost all indecomposable points of $R(Z, Z)$. Consequently $F(A_1), \ldots, F(A_n)$ generate almost indecomposable points of all those $R(Z', Z')$ for which $F(R(Z, Z) \cap P(V)) = R(Z', Z')$. So $Q_M$ is tame with $V$.

**Remark 7.** A similar method may be used to prove that, in the situation of the Lemma, case 1, the tameness of $Q_M$ implies the tameness of $Q_{M'}$.

For a halflinear $k$-marked $Q_M$ we construct a (non-oriented) graph $G(Q_M)$ in the following way. For each vertex $x$ such that $V_x = k_a$, we add to the graph $G(Q)$ $(n - 1)$ vertices $a_1^n, x - a_2^n - \cdots - a^n_n$, for each vertex $y$ such that $V_y$ is halflinear we add to $G(Q)$ two vertices $b_1^n, b_2^n - y - b_3^n$.

It is easy to see that under the conditions of the Lemma in cases 1 and 3, $G(Q_M) \simeq G(Q_{M'})$. In case 2, if $G(Q_M)$ is a Dynkin or extended Dynkin diagram, then $G(Q_M)$ is $D_n$ or $\tilde{D}_n$, with $n > 3$, respectively, and $G(Q_{M'})$ is $D_{n-1}$ or $\tilde{D}_{n-1}$, respectively. Conversely, if $G(Q_{M'})$ is a Dynkin or extended Dynkin diagram, then $G(Q_{M'})$ is $D_n$ or $\tilde{D}_n$, with $n > 3$, and $G(Q_M)$ is $D_{n+1}$ or $\tilde{D}_{n+1}$, respectively.

**Theorem 7.** If $Q_M$ is halflinear, then it is tame, or has finite type, if and only if $G(Q_M)$ is an extended Dynkin diagram, or a Dynkin diagram, respectively.

Suppose there is a vertex $a \in Q_M$ with $V_a$ neither a halflinear nor a linear vectroid. Then $Q_M$ is tame, or has finite type, if and only if $G(Q) = A_n$ with $a = a_1$, $V_a = k$ for $1 < i < n$ and $V_n = k_m$ for some $m \geq 1$, and the vectroid $V \sqcup k_{m+n-3}$ is tame, or has finite type, respectively; here $V = V_a$ or $V_a^\circ$ according as $a$ is the end or beginning of an arrow of $Q_M$.

**Proof.** We will assume that for unmarked quivers the statement is known. Although usually tameness is defined in a different way, it is easy to see that the proofs are valid for tameness in our sense.

So we may consider that $G(Q)$ is a Dynkin diagram or an extended Dynkin diagram.

We will also assume that $G(Q)$ (and so $G(Q_M)$) is acyclic because an unmarked cycle is tame and a marked cycle $Q_M$ is wild by Lemma 6 (case 4) (and $G(Q_M)$ is neither a Dynkin nor an extended Dynkin diagram).

a) Let $Q_M$ be tame, or of finite type $a \in Q_v$, $V_a$ be neither linear nor halflinear. Then $\delta(a) = 1$ by Lemma 6 (2).
a) $Q_v \ni x \neq a$, $V_x$ is not linear. Using Remark 6 we have a contradiction to Lemma 6 (3).

b) $Q_v \ni x \neq a$, $V_x \neq k$, $\delta(x) \neq 1$. We show that $Q_M$ is wild (that contradicts which our assumption).

Without losing of the generality we can (using Remark 6) assume that $G(Q) = \bullet \rightarrow \infty \leftarrow \bullet$, $V_x = k_2$, $V_y = k$.

Using the Lemma (case 3) we get $Q'_M = \bullet \rightarrow \infty$, where $V'_x$ is halflinear ($V'_a = V_a$), $Q'_M$ is wild by Lemma 6 (3) and so also is $Q_M$ (the Lemma).

c) $Q_v \ni x$, $\delta(x) > 2$. This case is reduced to a2) by several applications of the Lemma (case 1) and Remark 6.

d) $Q = A_n$, $a = a_1$, $V_{a_2} = \cdots = V_{a_{n-1}} = k$, $V_{a_n} = k_m$. By $n - 2$ applications of the Lemma (case 1) and (at the end) of the proposition reduce Rep $Q_M$ to Rep $(V_a \sqcup k_{n+m-3})$ or Rep $(V''_a \sqcup k_{n+m-3})$.

Now we assume that $Q_M$ is halflinear.

b) Let $Q_v \ni b$, $V_b$ be halflinear.

b1) $\delta(b) \neq 1$. If $G(Q) = \bullet \rightarrow \infty \leftarrow \bullet$ ($Q_M$ is Gelfand, $G(Q_M) = \tilde{D}_4$), then it is tame. (If $Q$ is $\bullet \rightarrow \bullet \leftarrow \bullet$ or $\bullet \leftarrow \bullet \rightarrow \bullet$ then Rep $Q_M \simeq$ Rep $\Delta_M$ where $\overline{V}_x = V_b$, $\overline{V}_y = k \sqcup k$ or $\overline{V}_x = k \sqcup k$, $\overline{V}_y = V_b$, so $\Delta_M$ is tame by the results of [12]. If $Q = \bullet \rightarrow \bullet \leftarrow \bullet$, then Rep $Q_M \simeq$ Rep $Q'_M$ where $Q^* = \bullet \rightarrow \bullet \leftarrow \bullet$, $V_x = V_y = k$, $V^* = V''_b$).

Suppose $Q_M$ is not Gelfand. Then it is easy to see that $G(Q_M)$ is not an extended Dynkin (and not Dynkin) diagram. If $Q_v \ni x$, $V_x \neq k$ then $Q_M$ is wild by Lemma 6 (1) and Remark 6. If all vertices except for $b$ are unmarked but $|Q_v| > 3$, we reduce this case to the case above using the Lemma (case 1).

b2) $\delta(b) = 1$. In this case we will prove the theorem by induction on $|Q_v|$.

If $|Q_v| = 2$, $Q_M$ is Gelfand and tame (see [12]) or of finite type, then $G(Q_M)$ is $\tilde{D}_5$ or $D_n$ respectively. So let $|Q_v| > 2$ and suppose that

$G(Q_M) \ni \bullet \rightarrow \bullet \leftarrow \bullet$.

If $V_x = k$ we apply the Lemma (case 2). Now $G(Q'_M)$ is an extended Dynkin (Dynkin) diagram if and only if $G(Q_M)$ is. Since $|Q'_v| < |Q_v|$ our statement follows from the inductive assumption.

Let $V_x = k_m$, $m > 1$ ($x$ can not be halflinear by b1). If $m \geq 3$ then $G(Q_M)$ is neither an extended Dynkin nor a Dynkin diagram and we show that $Q_M$ is not tame. We may assume $Q_M = \bullet \rightarrow \bullet \leftarrow \bullet$.
Consider $L_N = \overset{\infty}{b} \overset{2}{\overset{\frac{w}{z}}{\overset{x}{}}}$ Then $Q_M = L'_N$ (the Lemma, case 1).

On the other hand we apply the Lemma, case 2 to the vertex $b$ of $L_N$ and get $\overset{2}{\overset{\infty}{\overset{\cdot}{\cdot}}}$ $\cdot$. The last $k$-marked quiver is wild by Lemma 6 (1), so also is $L_N$ (by the Lemma, case 2). Now $Q_M$ is not tame because otherwise $L_N$ should be tame by the Lemma (case 1). Let $m = 2$. The cases $V_y \neq k$ and $|Q_v| > 3$ are considered on the analogy of the cases $m \geq 3$ and $b_1$.

If $G(Q_M) = \overset{\infty}{\overset{2}{\cdot}} \cdot \cdot$ we get a tame $Q_M$ by the Lemma, case 3 and [12] $G(Q_M) = \tilde{D}$.

c) $Q_M$ is linear. We consider an unmarked quiver $\overline{Q}$ such that $G(\overline{Q}) = G(Q_M)$ and by the several applications of the Lemma case 1 to $\overline{Q}$ we reduce it to $Q_M$. The remark implies that if $\overline{Q}$ is tame or of finite type, then $Q_M$ is of the same type. The converse statement follows from the Lemma.

The theorem gives a criterion for wildeness and finiteness of type for halflinear (in particular linear) $Q_M$, and for a $k$-marked quiver of dimension 1 (together with [13, 14]), as well as a criterion for the finiteness of type if $\dim Q_M = 2$ (together with [15, 8], see also [3, 16–20]).

If $\dim Q_M \geq 3$ and $|\text{ind} Q_M| < \infty$, then $\dim Q_M = 3$ (see Section 6), and it is easy to see that in this case $Q = \Delta$, and if $Q_v = \{x, y\}$ and $\dim V_y = 3$, then $V_x = k$, so we have representations of a vectroid $V$, $\dim V = 3$. For this case a criterion for the finiteness of type is formulated in [21], but only the necessity of it has been proven.

From our considerations in fact it follows that the tameness in our sense coincides with the tameness in sense of [10, 11]. In fact Lemma 5, Lemma 7 and Proposition 7 hold for tameness and wildness in the sense of [10, 11] (the last statement in the Proposition 7 follows from the second by Drozd’s theorem).

**Appendix**

Here I attempt to present my personal point of view on the trends in investigations connected with the subject of this article.

In the first half of the XXth century it was the opinion of a majority of mathematicians that after the classical results of Jordan, Weierstrass and Kronecker, linear algebra as a branch of pure mathematics was completely finished. As an exception, one should mention the remarkable work of Szekerez [22].
Concerning the classical results mentioned above, it seemed that in spite of their importance for applications they were far from abstract algebra, and in particular from representation theory. I remember that 40 years ago all Soviet algebraists were surprised and even in some sense disappointed when V.A. Bashev (a student of I.R. Shafarevich) had solved [23] the problem of classification of representations of the Klein 4-group over a field of characteristic 2 (which was considered then as very difficult) by a trivial reduction of this problem to the problem of Kronecker.

Between 1960 and 1970 several different problems were solved by their reduction to some classification problems of linear algebra [24–26].

The concept of matrix problems (or combinatorial problems of linear algebra) as a special branch of mathematics arose about 1970 from at least three sources.

In [27, 28] (see also [29]) it was conjectured in particular that the category of Harish-Chandra modules for any semisimple group is equivalent to a certain category of diagrams in the category of finite dimensional vector spaces, and a boom in linear algebra was predicted on the basis of the new categorical and homological methods.

In [30, 31, 2] representations of posets were introduced, and it was claimed that many other matrix problems can and should be reduced to them.

At last, but of course not least, it was clear that the subject of [1] was very wide and deep. I hope that this article of mine underlines once more the importance of representations of quivers.

In the coming years the theories of representations of quivers and posets were developed successfully. It was clear from the beginning that these two theories were very close. Representations of a majority of quivers can be reduced to representations of posets (see for example [32]). Note that “on the way” from representations of quivers to representations of posets, in fact, representations of marked quivers arose. Many important generalizations of representations of posets also were introduced. In particular, representations of vectroids played a big role in many questions. P. Gabriel showed that representations of arbitrary finite dimensional algebra can be reduced to them [9].

However the general theory of matrix problems was not developed so well. It may be partially explained by the absence of a natural basic definition. The widest class of matrix problems are the representations of $DQC$ or bocses, which were introduced by M. Kleiner and me [33]. But this class is not only wide enough, but may be in some sense too wide. It includes
the representations of posets, vectroids, and “Gelfand problems”, but these most important problems are not picked out naturally in the very wide area of bocses. The general definition of bocses is more or less clear (though is not so natural as quivers and posets), but additional conditions which are necessary to get a majority of really important classes of matrix problems are rather complicated. So the theory of bocses became convenient to prove general theorems [10, 11], but not to develop a systematic theory of different classes of matrix problems.

Conversely, in the terms of this article the most important classes of matrix problems are picked out very naturally, and it seems that the general definition can stimulate investigations of many other natural and useful classes of matrix problems. So it seems that the notion of representations of marked quivers should be a better basic notion for classification problems of linear algebra.

I want to underline that in spite of the close relation between matrix problems and representations of algebras, it does not seems correct to consider the first theory only as part of the second. Many matrix problems arose from applications to other branches of mathematics, and we hope that the number of such applications will increase in the future.

The first aim in the theory of matrix problems is to finish a description of vectroids of finite type and their representations. As follows from this article, this will imply that such a theory also exists for representations of $k$-marked quivers. For the point-marked and non point-marked (see Example 4) quivers such theories should be more varied but also solid.

In the theory of representations of tame vectroids (and tame $k$-marked quivers) the first important class is locally semisimple vectroids (i.e. $\text{Hom}(A, A)$ is a semisimple algebra for any $A$). A criterion of tameness for such vectroids was announced by L.A. Nasarova in 1985 (see [35]).

At the end I want to note that although a majority of matrix problems are wild, their investigations may make sense. For any matrix problem (representations of quiver, poset, marked quiver, boc) all representations of fixed dimension form a vector-space. Any affine subspace of this space generates some sets of representations of this and bigger dimensions.

**Conjecture.** For any matrix problem there exists an $n$ such that all indecomposable representations of fixed dimension are generated by a finite number of affine subspaces of dimensions at most $n$, whose elements are indecomposable.

---

1 Such categories were already considered by J.M. Gelfand and G.E. Shilow in 1963 [34].
and pairwise inequivalent.

For $n = 0$ it would imply the first Brauer-Thrall conjecture [36], and for $n = 1$, Drozd’s theorem [10, 11, 9].

Of course this formulation may be too strong, but it seems that some its modifications will be typical for the investigations in the new millenium.
References

1. Gabriel P.: Unzerlegbare Darstellungen I, *Manuscr. Math.* 6 (1972), 71–103.

2. Nazarova L.A., Roiter A.V.: Representations of partially ordered sets, *Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova* 28 (1972), 5–31, English transl.: *J. Sov. Math.* 3 (1975), 585–606.

3. Gabriel P., Roiter A.V.: Representations of Finite-Dimensional Algebras: Springer-Verlag, Algebra VIII, 1992.

4. Bass H.: Algebraic K-theory. W.A. Benjamin, INC. New York, 1969.

5. Belousov K.I., Nazarova L.A, Roiter A.V., Sergeichuk V.V.: Elementary and multielementary representations of vectroids, *Ukr. Math. J.* 47 (1995), 1451–1477.

6. Nazarova L.A., Roiter A.V.: Kategorielle Matrizen-Problems und Brauer-Thrall - Vermutung, Mitt. Aus dem. Math. Sem. Giessen, 1975.

7. Ringel C.M.: Tame algebras and integral quadratic form, *Lecture Notes Math.* 1099 (1984).

8. Nazarova L.A., Roiter A.V.: Finitely represented dyadic sets, *Ukr. Math. J.* 52 (2000), 1363–1396.

9. Gabriel P., Nazarova L.A., Roiter A.V., Sergeichuk V.V., Vossieck V.: Tame and wild subspace problems, *Ukr. Math. J.* 45 (1993), 313–352.

10. Drozd Ju.A.: Tame and wild matrix problems (Ottava, 1979), *Lect. Notes Math.* 832 (1980), 242–258.

11. Crawley - Boevey W.W.: On tame algebras and bocses, *Proc. Lond. Math. Soc. III Ser.* 56 (1988), 451–483.

12. Nazarova L.A., Roiter A.V.: On a problem of I.M. Gelfand, *Funk. Anal. and Appl.*, 7 (1973), 54–69.

13. Nazarova L.A.: Partially ordered sets of infinite type, *Izv. A.C. USSR, 39* (1975), 963–991.

14. Kleiner M.M.: Partially ordered sets of finite type, *Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova*, 281 (1972), 32–41.
15. Roiter A.V., Belousov K.I., Nazarova L.A.: Representations of finitely represented dyadic sets, *Algebras and modules II, ICRA 1996 CMS Conf. Proc.* 24 (1996), 61–77.

16. Nazarova L.A., Roiter A.V.: Representations of biinvolutive posets I, Preprint 91.34, Kiev, Math. Inst., 1991.

17. Nazarova L.A., Roiter A.V.: Representations of biinvolutiv posets II, Preprint, Kiev, Math Inst., 1994.

18. Guidon T.: Representations of dyadic sets 2, Diss. Uni. Zurich, 1996, 1–47.

19. Hassler U: Representations of dyadic sets 1, Diss. Uni. Zurich, 1996, 1–62.

20. Guidon T., Hassler U., Nazarova L.A., Roiter A.V.: Dyadic sets S: a dichotomy for indecomposable S-matrices, Comptes - Rendus Acad. Sc. Paris, 324, Series I (1997), 1205–1210.

21. Belousov K.I., Nazarova L.A., Roiter A.V.: Finitely represented tryadic sets, *Alg. and Analis* 9(4) (1997), 3–27.

22. Szekeres G.: Determination of certain family of finite metabelian groups, *Trans. Amer. Math. Soc.* 66 (1949), 11–43.

23. Bashev V.A.: Representations of group $Z_2 \times Z_2$ in a field of characteristic 2, *DAN SSSR* 141:5 (1961), 1015–1018.

24. Nazarova L.A.: Integral representations of Klein’s four group, *Dokl. Akad. Nauk SSSR* 140 (1961), 1011–1014; Representations of the local ring of a curve with 4 branches, *Izv. Akad. Nauk SSSR. Ser. Math.* 31 (1967), 1361–1378.

25. Drozd Yu.A., Roiter A.V.: Commutative rings with a finite number of integral indecomposable representations, *Izv. Akad. Nauk SSSR* 31 (1967), 783–798.

26. Gelfand I.M., Ponomarev V.A.: Indecomposable representations of Lorentz group, *Usp. Math. Nauk* 32 (1968), 3–60.

27. Gelfand I.M.: The cohomology of infinite dimensional Lie algebras, some questions of integral geometry, *Actes I.C. Math.* 1/10 (1970) /NICE/ France 95–111.

28. Gelfand I.M., Ponomarev V.A.: Problems of linear algebras and classification of quadruples of subspaces in finite-dimensional vector space,
Colloq. math. Soc. Janos Bolyai 5 (Tihany 1970) (1972) 163–237; 
DAN SSSR 197:4 (1971), 762–765.

29. Bernstein I.N., Gelfand I.M., Ponomarev V.A.: Coxeter functors and 
Gabriel’s theorem, Izv. Mat. Nauk 28 (1973), 19–33; English transl. 
Russ. Math. Surv. 28 (1973), 17–32.

30. Nazarova L.A., Roiter A.V.: Matrix Questions and the Brauer-Thrall 
Conjectures on Algebras with an Infinite Number of Indecomposable 
representations, Proc. Amer. Math. Soc. (1971), 111–115.

31. Roiter A.V.: Matrix problems and the representations of bisystems, 
Zap. Nauchn. Sem. LOMI 29 (1972), 130–139.

32. Nazarova L.A.: Representations of quivers of infinite type, Izv. AN 
SSSR 37:4 (1973), 752–791.

33. Kleiner M.M., Roiter A.V.: Representations of D.G.C., in Matrix prob-
lems, Kiev, Inst. Math. AN USSR, 1977, 5–71.

34. Gelfand I.M., Shilov G.E.: Categories of finite-dimensional spaces, Usp. 
Math. Nauk Ser. 1 (1963), 27–49.

35. Roiter A.V.: Representations of posets and tame matrix problems, in 
Proc. Durham Simp. 1985, Cambridge, 1986, 91–109.

36. Roiter A.V.: Unboundness of the dimension of the indecomposable 
representations of algebras, that have an infinite number of indecom-
posable representations, Izv. AN SSSR 32 (1968), 1275–1282.