SPECIAL SIMPLICES AND GORENSTEIN TORIC RINGS

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Abstract. Christos Athanasiadis [2] studies an effective technique to show that Gorenstein sequences coming from compressed polytopes are unimodal. In the present paper we will use such the technique to find a rich class of Gorenstein toric rings with unimodal h-vectors arising from finite graphs.

Introduction

Let \( P \subset \mathbb{R}^N \) be an integral convex polytope, i.e., a convex polytope each of whose vertices has integer coordinates. Let \( K[x, x^{-1}, t] = K[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}, t] \) denote the Laurent polynomial ring in \( (N + 1) \) variables over a field \( K \). The toric ring of \( P \) is the subalgebra \( K[P] \) of \( K[x, x^{-1}, t] \) generated by those Laurent polynomials \( x^a t \) such that \( a = (a_1, \ldots, a_N) \) is a vertex of \( P \). We will regard \( K[P] \) as a homogeneous algebra [3, p. 147] by setting each \( \deg x^a = 1 \) and write \( F(K[P], \lambda) \) for its Hilbert series. One has

\[
F(K[P], \lambda) = \frac{h_0 + h_1 \lambda + \cdots + h_s \lambda^s}{(1 - \lambda)^{d+1}}
\]

where each \( h_i \in \mathbb{Z} \) with \( h_s \neq 0 \) and where \( d \) is the dimension of \( P \). The sequence \((h_0, h_1, \ldots, h_s)\) is said to be the \( h \)-vector of \( K[P] \). If the toric ring \( K[P] \) is normal, then \( K[P] \) is Cohen–Macaulay. If \( K[P] \) is Cohen–Macaulay, then the \( h \)-vector of \( K[P] \) is nonnegative, i.e., each \( h_i \geq 0 \). Moreover, if \( K[P] \) is Gorenstein, then the \( h \)-vector of \( K[P] \) is symmetric, i.e., \( h_i = h_{s-i} \) for all \( i \).

An outstanding conjecture (which is still open) is that the \( h \)-vector of a Gorenstein toric ring is unimodal, i.e., \( h_0 \leq h_1 \leq \cdots \leq h_{s/2} \). One of the established techniques to show that the \( h \)-vector \((h_0, h_1, \ldots, h_s)\) of a Gorenstein toric ring \( K[P] \) is unimodal is to find a simplicial convex polytope [14] of dimension \( s-1 \) whose \( h \)-vector coincides with \((h_0, h_1, \ldots, h_s)\). On the other hand, however, given a Gorenstein toric ring \( K[P] \), it seems difficult to find such a simplicial convex polytope.

Christos Athanasiadis [2] introduces the concept of a special simplex of a convex polytope. Let \( P \subset \mathbb{R}^N \) be a convex polytope. A \((q-1)\)-simplex \( \Sigma \) each of whose vertices is a vertex of \( P \) is said to be a special simplex in \( P \) if each facet of \( P \) contains exactly \( q - 1 \) of the vertices of \( \Sigma \). Recall that an integral convex polytope \( P \subset \mathbb{R}^N \) is compressed [15, p. 337] (and [11]) if all “pulling triangulations” of \( P \) are unimodular. The toric ring \( K[P] \) of a compressed polytope \( P \) is normal. It turns out [2, Theorem 3.5] that if \( P \) is compressed and if there is a special simplex in \( P \), then the \( h \)-vector of \( K[P] \) is equal to the \( h \)-vector of a simplicial convex polytope.

In the present paper we will use [2, Theorem 3.5] to study the \( h \)-vector of the toric ring of the edge polytope of a finite graph satisfying the odd cycle condition as well as that of the stable polytope of a perfect graph.
1. Two polytopes arising from finite graphs

Let $G$ be a finite graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ having no loops and no multiple edges, and $E(G)$ the edge set of $G$. We associate each subset $W \subseteq [n]$ with the $(0,1)$-vector $\rho(W) = \sum_{j \in W} e_j \in \mathbb{R}^n$. Here $e_j$ is the $j$-th unit coordinate vector in $\mathbb{R}^n$. Thus in particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^n$. A subset $W \subseteq [n]$ is called stable (resp. a clique) if $\{i,j\} \not\in E(G)$ (resp. $\{i,j\} \in E(G)$) for all $i,j \in W$ with $i \neq j$. Note that the empty set as well as each single-element subset of $[n]$ is both stable and a clique. Let $S(G)$ denote the set of stable sets of $G$.

We now introduce two convex polytopes arising from a finite graph $G$ on $[n]$. First, the edge polytope $[8]$ of $G$ is the $(0,1)$-polytope $\mathcal{P}_G \subseteq \mathbb{R}^n$ which is the convex hull of $\{\rho(e) : e \in E(G)\}$. Second, the stable polytope $[4]$ of $G$ is the $(0,1)$-polytope $\mathcal{Q}_G \subseteq \mathbb{R}^n$ which is the convex hull of $\{\rho(W) : W \in S(G)\}$.

**Example 1.1.** Let $P$ be a finite poset on $[n]$ and $\text{com}(P)$ its comparability graph. Thus $\text{com}(P)$ is the finite graph on $[n]$ such that $\{i,j\}$ with $i \neq j$ is an edge of $\text{com}(P)$ if and only if $i$ and $j$ are comparable in $P$. Then the stable polytope of $\text{com}(P)$ coincides with the chain polytope $[16]$ of $P$.

The problem when the toric ring $K[\mathcal{P}_G]$ is normal and the problem when the edge polytope $\mathcal{P}_G$ possesses a unimodular covering $[8, p. \ 420]$ are studied in $[8]$ (and $[13]$).

**Theorem 1.2** ([8]). Given a finite connected graph $G$, the following conditions are equivalent:

(i) The toric ring $K[\mathcal{P}_G]$ is normal;
(ii) The edge polytope $\mathcal{P}_G$ possesses a unimodular covering;
(iii) $G$ satisfies the odd cycle condition, i.e., if each of $C'$ and $C''$ is an odd cycle (a cycle of odd length) of $G$ and if $C'$ and $C''$ possess no common vertex, then there exists an edge $\{i,j\}$ of $G$ such that $i$ is a vertex of $C'$ and $j$ is a vertex of $C''$.

Thus in particular the edge polytope of a finite connected bipartite graph possesses a unimodular covering and its toric ring is normal.

A chromatic number of a finite graph $G$ on $[n]$ is the smallest integer $\ell > 0$ for which there is a map $\varphi : [n] \to [\ell]$ with the property that $\varphi(i) \neq \varphi(j)$ if $\{i,j\} \in E(G)$. A finite graph $G$ is called perfect if, for all induced subgraphs $H$ of $G$ including $G$ itself, the chromatic number of $H$ is equal to the maximal cardinality of cliques contained in $H$. The comparability graph of a finite partially ordered set is perfect ([1]).

The facets of the edge polytope $\mathcal{P}_G$ of a finite connected graph $G$ is completely determined ([8, Theorem 1.7]). On the other hand, the facets of the stable polytope $\mathcal{Q}_G$ is completely determined when $G$ is a perfect graph ([4, Theorem 3.1]).

2. Gorenstein toric rings

When $K[\mathcal{P}_G]$ (resp. $K[\mathcal{Q}_G]$) is normal, it follows easily that $K[\mathcal{P}_G]$ (resp. $K[\mathcal{Q}_G]$) coincides with the Ehrhart ring [6, p. 97] of $\mathcal{P}_G$ (resp. $\mathcal{Q}_G$). On the other hand, since the equations of the facets of $\mathcal{P}_G$ (resp. $\mathcal{Q}_G$) are known, when $K[\mathcal{P}_G]$ (resp.
Let $G$ be a finite connected graph on $[n]$. Given a subset $V \neq \emptyset$ of $[n]$, write $G_V$ for the induced subgraph of $G$ on $V$. We say that $G$ is 2-connected if $G$ together with $G_{[n]\setminus \{i\}}$ for all $i \in [n]$ is connected. If $i \in [n]$, then $N(G; i)$ stands for the set of vertices $j$ with $\{i, j\} \in E(G)$. If $T \subset [n]$, then $N(G; T) = \bigcup_{i \in T} N(G; i)$. The bipartite graph induced by a stable set $T \neq \emptyset$ of $G$ is the bipartite graph on the vertex set $T \cup N(G; T)$ consisting of those edges $\{i, j\}$ of $G$ with $i \in T$ and $j \in N(G; T)$.

Recall that a matching of $G$ is a set of edges $\{e_1, \ldots, e_m\}$ such that $e_i \cap e_j = \emptyset$ for all $i \neq j$. A matching $\{e_1, \ldots, e_m\}$ of $G$ is called perfect if $\bigcup_{i=1}^m e_i = [n]$. In particular, $n$ is even and $m = n/2$ if $G$ possesses a perfect matching $\{e_1, \ldots, e_m\}$. It follows that $G$ possesses a perfect matching if and only if the monomial $x_1x_2 \cdots x_n$ belongs to the toric ring $K[\mathcal{P}_G]$.

**Theorem 2.1.** (a) Let $G$ be a finite connected graph on $[n]$ satisfying the odd cycle condition and suppose that every connected component of $G_{[n] \setminus \{i\}}$ possesses at least one odd cycle for all $i \in [n]$. Then the toric ring $K[\mathcal{P}_G]$ of the edge polytope $\mathcal{P}_G$ of $G$ is Gorenstein if and only if (i) $G$ possesses a perfect matching, (ii) one has $|N(G; T)| = |T| + 1$ for each stable set $T$ of $G$ such that the bipartite graph induced by $T$ is connected with $T \cup N(G; T) \neq [n]$ and that every connected component of $G_{[n] \setminus (T \cup N(G; T))}$ has at least one odd cycle and (iii) one has $|T| = n/2 - 1$ for each stable set $T$ of $G$ such that the bipartite graph induced by $T$ is connected with $T \cup N(G; T) = [n]$.

(a') Let $G$ be a bipartite graph on $[n] = V_1 \cup V_2$ and suppose that $G$ is 2-connected. Then the toric ring $K[\mathcal{P}_G]$ of the edge polytope $\mathcal{P}_G$ of $G$ is Gorenstein if and only if (i) $G$ possesses a perfect matching and (ii) one has $|N(G; T)| = |T| + 1$ for every subset $T \subset V_1$ such that $G_{T \cup N(G; T)}$ is connected and that $G_{[n] \setminus (T \cup N(G; T))}$ is a connected graph with at least one edge.

(b) The toric ring $K[\mathcal{Q}_G]$ of a stable polytope $\mathcal{Q}_G$ of a perfect graph $G$ is Gorenstein if and only if all maximal cliques have the same cardinality.

**Proof.** (a) The edge polytope $\mathcal{P}_G \subset \mathbb{R}^n$ lies on the hyperplane $\mathcal{H}$ defined by the equation $z_1 + \cdots + z_n = 2$. Let $\pi : \mathbb{R}^{n-1} \to \mathcal{H}$ denote the affine map defined by setting $\pi(z_1, \ldots, z_{n-1}) = (z_1, \ldots, z_{n-1}, 2 - (z_1 + \cdots + z_{n-1}))$. Then $\pi$ is an affine isomorphism with $\pi(\mathbb{Z}^{n-1}) = \mathcal{H} \cap \mathbb{Z}^n$. Hence $\pi^{-1}(\mathcal{P}_G) \subset \mathbb{R}^{n-1}$ is an integral convex polytope with $\dim \pi^{-1}(\mathcal{P}_G) = n - 1$ and the toric ring $K[\pi^{-1}(\mathcal{P}_G)]$ is isomorphic to $K[\mathcal{P}_G]$ as homogeneous algebras over $K$.

Let $\delta$ denote the smallest integer for which the interior of $\delta(\pi^{-1}(\mathcal{P}_G))$ contains at least one integer point $(a_1, \ldots, a_{n-1})$. Since every connected component of $G_{[n] \setminus \{i\}}$ possesses at least one odd cycle, it follows from [8, Theorem 1.7 (a)] that the hyperplane defined by the equation $z_i = 0$ is a supporting hyperplane which defines a facet of $\mathcal{P}_G$ for each $1 \leq i \leq n$. Thus the hyperplane defined by the equation $z_i = 0$ is a supporting hyperplane which defines a facet of $\pi^{-1}(\mathcal{P}_G)$ for each $1 \leq i < n$. Thus by using [5, Corollary (1.2)] one has each $a_i = 1$ if $K[\pi^{-1}(\mathcal{P}_G)]$ is Gorenstein. If $e_1 + \cdots + e_{n-1} + qe_n$ belongs to the interior of $\delta \mathcal{P}_G$ for some integer $q > 0$, then $e_1 + \cdots + e_{n-1} + qe_n$ belongs to the interior of $\delta \mathcal{P}_G$ for some integer $q > 0$. [3]
Since $K[P_G]$ coincides with the Ehrhart ring of $P_G$, it follows that there are edges $\epsilon_1, \ldots, \epsilon_m$ of $G$ with $\epsilon_1 + \cdots + \epsilon_{n-1} + q\epsilon_n = \rho(\epsilon_1) + \cdots + \rho(\epsilon_m)$. Hence $q = 1$ and $\epsilon_1 + \cdots + \epsilon_n = \rho(\epsilon_1) + \cdots + \rho(\epsilon_m)$. Thus $G$ possesses a perfect matching with $\delta = n/2$ if $K[\pi^{-1}(P_G)]$ is Gorenstein.

For a while, suppose that $G$ possesses a perfect matching with $\delta = n/2$. Let $P^\circ \subset \mathbb{R}^{n-1}$ denote the integral convex polytope $\delta(\pi^{-1}(P_G)) - (\epsilon_1 + \cdots + \epsilon_{n-1})$. Then $P^\circ$ is of standard type, i.e., $\text{dim } P = n - 1$ and the origin of $\mathbb{R}^{n-1}$ belongs to the interior of $P$. Then [5, Corollary (1.2)] guarantees that the toric ring $K[\pi^{-1}(P_G)]$ is Gorenstein if and only if the dual polytope [5, p. 631] of $P^\circ$ is integral.

Now, by using [8, Theorem 1.7 (a)] again, it turns out that the equations of the supporting hyperplanes which defines the facets of $P^\circ$ are the followings:

- $z_i = -1$ for each $1 \leq i < n$;
- $\sum_{i \in [n] \setminus (T \cup N(G; T))} z_i + 2 \sum_{j \in N(G; T)} z_j = |T| - |N(G; T)|$ if $n \in T$;
- $2 \sum_{i \in T} z_i + \sum_{j \in [n] \setminus (T \cup N(G; T))} z_j = |N(G; T)| - |T|$ if $n \in N(G; T)$;
- $\sum_{i \in T} z_i - \sum_{j \in N(G; T)} z_j = |N(G; T)| - |T|$ if $n \notin T \cup N(G; T)$,

where $T \neq \emptyset$ is a stable set of $G$ for which the bipartite graph induced by $T$ is connected and for which either $T \cup N(G; T) = [n]$ or every connected component of the induced subgraph $G_{[n] \setminus (T \cup N(G; T))}$ has at least one odd cycle.

Hence the dual polytope of $P^\circ$ is integral if and only if (a) one has $|N(G; T)| = |T| + 1$ for each nonempty stable set $T$ of $G$ such that the bipartite graph induced by $T$ is connected with $T \cup N(G; T) \neq [n]$ and that every connected component of $G_{[n] \setminus (T \cup N(G; T))}$ has at least one odd cycle and (b) one has $|T| = n/2 - 1$ for each stable set $T$ of $G$ such that the bipartite graph induced by $T$ is connected with $T \cup N(G; T) = [n]$.

Consequently, when the toric ring $K[P_G]$ is Gorenstein, the conditions (i), (ii) and (iii) are satisfied. Conversely, suppose that the conditions (i), (ii) and (iii) are satisfied. Since the hyperplane defined by the equation $z_i = 0$ is a supporting hyperplane which defines a facet of $P_G$ for each $1 \leq i \leq n$, if $\gamma P_G$, where $\gamma > 0$, contains at least one integer point $(a_1, \ldots, a_n)$, then each $a_i > 0$ and $\gamma \geq [(n+1)/2]$.

It follows from (i), (ii) and (iii) that $n$ is even and $e_1 + \cdots + e_n$ belongs to the interior of $(n/2)P_G$. Thus the smallest number $\delta > 0$ for which $\delta P_G$ contains at least one integer point is $\delta = n/2$ and $e_1 + \cdots + e_n$ belongs to the interior of $\delta P_G$. Our discussion done already in the preceding paragraph guarantees that the toric ring $K[P_G]$ is Gorenstein, as desired.

(a') In imitation of the preceding proof of (a) by using [8, Theorem 1.7 (b)] instead of [8, Theorem 1.7 (a)], one can easily give a proof of (a').

(b) The facets of the stable polytope $Q_G$ is completely determined when $G$ is a perfect graph ([4, Theorem 3.1]). In fact, when $G$ is perfect, the equations of the supporting hyperplanes which defines the facets of $Q_G$ are either $z_i = 0$ for $1 \leq i \leq n$ or $\sum_{W \subseteq [n]} z_i = 1$, where $W$ is a maximal cliques of $G$. Let $\delta$ denote the smallest integer $\delta > 0$ for which the interior of $Q_G$ contains at least one integer point. Then $\delta - 1$ coincides with the maximal cardinality of cliques of $G$ and $e_1 + \cdots + e_n$ belongs to the interior of $\delta Q_G$. It follows that the dual polytope of the integral polytope
\[ \delta Q_G - (e_1 + \cdots + e_n) \subseteq \mathbb{R}^n \] of standard type is integral if and only if all maximal cliques of \( G \) have the cardinality \( \delta - 1 \). Hence [5, Corollary (1.2)] guarantees that the toric ring \( K[Q_G] \) is Gorenstein if and only if all maximal cliques have the same cardinality. \( \square \)

**Example 2.2.** The toric ring of the edge polytope of each of the finite connected graphs \( G_1 \) and \( G_2 \) drawn below is normal and Gorenstein.

![Graphs G1 and G2](#)

3. **Unimodal Gorenstein sequences**

Let \( \mathcal{P} \subset \mathbb{R}^N \) be a convex polytope. Recall that a \((q - 1)\)-simplex \( \Sigma \) each of whose vertices is a vertex of \( \mathcal{P} \) is said to be a *special simplex* [2] in \( \mathcal{P} \) if each facet of \( \mathcal{P} \) contains exactly \( q - 1 \) of the vertices of \( \Sigma \).

**Theorem 3.1.** (a) Let \( G \) be a finite connected graph as in Theorem 2.1 (a) or (a') and suppose that the toric ring \( K[\mathcal{P}_G] \) of the edge polytope \( \mathcal{P}_G \) of \( G \) is Gorenstein. Then there is a special simplex in \( \mathcal{P}_G \).

(b) Let \( G \) be a perfect graph and suppose that the toric ring \( K[Q_G] \) of the stable polytope \( Q_G \) of \( G \) is Gorenstein. Then there is a special simplex in \( Q_G \).

**Proof.** (a) Let \([n]\) be the vertex set of \( G \). Since \( G \) possesses a perfect matching, it follows that \( n = 2m \) is even and there exist \( m \) edges \( e_1, \ldots, e_m \) of \( G \) with \( \rho(e_1) + \ldots + \rho(e_m) = e_1 + \cdots + e_n \). Let \( \Sigma \) denote the \((m - 1)\)-simplex whose vertices are \( \rho(e_1), \ldots, \rho(e_m) \). We claim that \( \Sigma \) is special in \( \mathcal{P}_G \).

Theorem 2.1 together with [8, Theorem 1.7] give the complete information about the equations of the supporting hyperplanes which define the facets of the edge polytope \( \mathcal{P}_G \). Let \( \mathcal{H}_i \) denote the hyperplane defined by the equation \( z_i = 0 \). Then \( \rho(e_j) \in \mathcal{H}_i \) if and only if \( i \notin e_j \). Let \( \mathcal{H}_T \) denote the hyperplane defined by the equation \( \sum_{i \in T} z_i = \sum_{j \in N(G; T)} z_j \), where \( T \) is a stable set of \( G \), which is the supporting hyperplane of a facet of \( \mathcal{P}_G \). Let \( C \) denote the set of those \( 1 \leq i \leq m \) with \( \rho(e_i) \cap T \neq \emptyset \) and \( D \) the set of those \( 1 \leq i \leq m \) with \( \rho(e_i) \cap N(G; T) \neq \emptyset \). In either
the case of \(|N(G; T)| = |T| + 1\) or the case of \(|N(G; T)| = |T| + 1 = m\), one has \(C \subset D\) with \(|C| = |D| - 1\). Let \(i_0 \in D \setminus C\). Then \(\varphi(e_j) \in \mathcal{H}_T\) if and only if \(j \neq i_0\). Thus \(\Sigma\) is special in \(\mathcal{P}_G\) as desired.

(b) Let \(G\) be a perfect graph on \([n]\) and suppose that all maximal cliques have the cardinality \(q\). Since \(G\) is perfect, the chromatic number of \(G\) is equal to \(q\). Thus there is a map \(\varphi : [n] \to [q]\) with the property that \(\varphi(i) \neq \varphi(j)\) if \(\{i, j\} \in E(G)\). Let \(W'_q\) denote the stable set \(\{i \in [n] : \varphi(i) = \ell\}\) for each \(1 \leq \ell \leq q\). We assume that \(Q_G\) is not a simplex. Thus one of the stable sets \(W'_1, \ldots, W'_q\) contains at least two vertices. Let, say, \(W'_1\) contain at least two vertices and fix \(i_0 \in W'_1\). Let \(W_0 = \{i_0\}, W_1 = W'_1 \setminus \{i_0\}\) and \(W_\ell = W'_\ell\) for \(2 \leq \ell \leq q\). Each of \(W_0, W_1, \ldots, W_q\) is a stable set of \(G\) and \(\sum_{\ell=0}^{q} \varphi(W_\ell) = e_1 + \cdots + e_n\). Let \(\Sigma\) denote the \(q\)-simplex with \(q + 1\) vertices \(\rho(W_0), \rho(W_1), \ldots, \rho(W_q)\). We claim that \(\Sigma\) is special in \(Q_G\).

Recall that the equation of the supporting hyperplanes which defines the facets of the stable polytope of \(G\) are either (i) \(x_i = 0\) for \(1 \leq i \leq n\) or (ii) \(\sum_{i \in W} x_i = 1\), where \(W\) is a maximal clique of \(G\). If \(F_i\) is the facet defined by \(x_i = 0\), then \(\rho(W_\ell) \in F_i\) if and only if \(i \not\in W_\ell\). Since \([n]\) is the disjoint union \(W_0 \cup W_1 \cup \cdots \cup W_q\), it follows that \(F_i\) contains exactly \(q\) of the vertices of \(\Sigma\). Let \(F'_W\) denote the facet defined by \(\sum_{i \in W} x_i = 1\), where \(W\) is a maximal clique of \(G\). Since each of the subsets \(W \cap (W_0 \cup W_1), W \cap W_2, \ldots, W \cap W_q\) of \([n]\) consists of one element, it follows that each of the vertices \(\rho(W_2), \ldots, \rho(W_q)\) belongs to \(F'_W\) and that \(\rho(W_0) \not\in F'_W\) (resp. \(\rho(W_1) \in F'_W\)) if and only if \(i_0 \in W\) (resp. \(i_0 \not\in W\)). Hence \(F'_W\) contains exactly \(q\) of the vertices of \(\Sigma\).

By using Example 1.1 together with [16, Theorem 3.2], it turns out that the above Theorem 3.1 (b) generalize Reiner–Welker [12, Corollary 3.8].

Now, by virtue of [2, Theorem 3.5], one has a rich class of unimodal Gorenstein sequences [17, p. 66]. It is known [11, Example 1.3 (c)] the stable polytope of a perfect graph is compressed.

The edge polytope of a finite connected graph is unimodular, i.e., all of its triangulations are unimodular, if and only if any two odd cycles of \(G\) possess at least one common vertex. In particular the edge polytope of a finite connected bipartite graph is unimodular. The edge polytope of \(G_1\) of Example 2.2 is compressed ([10]) but not unimodular, and that of \(G_2\) is unimodular. A combinatorial characterization of finite graphs \(G\) for which the edge polytope \(\mathcal{P}_G\) is compressed is given in [7, Theorem 4.1].

**Corollary 3.2.** (a) Let \(G\) be a finite connected graph as in Theorem 2.1 (a) and suppose that the edge polytope \(\mathcal{P}_G\) is compressed and that the toric ring \(K[\mathcal{P}_G]\) is Gorenstein. Then the h-vector of \(K[\mathcal{P}_G]\) is unimodal.

(a') Let \(G\) be a finite 2-connected bipartite graph and suppose that the toric ring \(K[\mathcal{P}_G]\) is Gorenstein. Then the h-vector of \(K[\mathcal{P}_G]\) is unimodal.

(b) Let \(G\) be a perfect graph and suppose that the toric ring \(K[Q_G]\) of the stable polytope \(Q_G\) of \(G\) is Gorenstein. Then the h-vector of \(K[Q_G]\) is unimodal.

We conclude the present paper with
Example 3.3. Let $n \geq 3$ and $G$ the finite connected graph on $[2n]$ drawn below. Let $n$ be odd. (If $n$ is even, then $K[P_G]$ is not Gorenstein by Theorem 2.1 (a’).) By virtue of [7, Theorem 4.1] it turns out that the edge polytope $P_G$ of $G$ is compressed. By using Theorem 2.1 (a) it follows that the toric ring $K[P_G]$ is (normal and) Gorenstein. Moreover, we can compute the $h$-vector explicitly. Since the graph $G$ satisfies the condition in [9, Theorem 1.2], the “toric ideal” $I_G$ of $G$ is generated by quadratic binomials which correspond to even cycles of $G$ of length 4. There exists a reverse lexicographic order such that the initial monomials of the quadratic binomials are relatively prime. Since the set of quadratic binomials is a Gröbner basis of $I_G$ with respect to $<_\text{rev}$, it follows that the initial ideal is generated by $n$ monomials which are squarefree, quadratic and relatively prime. Thus the $h$-vector of $K[P_G]$ is $(1,n,\binom{n}{2}, \cdots, \binom{n}{n-2}, n, 1)$.

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