A topological version of the Bergman property

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Abstract

A topological group $G$ is defined to have property (OB) if any $G$-action by isometries on a metric space, which is separately continuous, has bounded orbits. We study this topological analogue of the so-called Bergman property in the context of Polish groups, where we show it to have several interesting reformulations and consequences. We subsequently apply the results obtained in order to verify property (OB) for a number of groups of isometries and homeomorphism groups of compact metric spaces. We also give a proof that the isometry group of the rational Urysohn metric space of diameter 1 is Bergman.\footnote{MSC primary 03E15, secondary 22F05}

1 Introduction

We study in this paper a topological version of the Bergman property. This latter property has been the object of intense scrutiny by a number of people, since it was first discovered to hold for the infinite symmetric group, $S_\infty$, by George M. Bergman \cite{Bergman2003} sometime in 2003.

Definition 1.1 A group $G$ is said to have the Bergman property (or is Bergman) if whenever $W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n$, there are $n$ and $k$ such that $G = W_k^n$.

We now have a large number of interesting results concerning this property, but most surprising is perhaps the fact that many large permutation groups are indeed Bergman. One can pick some of these results from \cite{Bergman2003, DrosteHolland2011, Miller2012, KechrisRosendal2014, Shelah2015}. For example, it holds for automorphism groups of 2-transitive linear orders (Droste and Holland \cite{DrosteHolland2011}), the group of measure preserving automorphisms of the unit interval (Miller \cite{Miller2012}) and oligomorphic permutation groups with ample generics, e.g., automorphism groups of $\omega$-stable, $\omega$-categorical structures (Kechris and Rosendal \cite{KechrisRosendal2014}).

To see what the Bergman property is really worth, it is useful to consider some of its consequences and reformulations. First of all, it is clear that no Bergman group can be written as a union of a countable chain of proper subgroups, or, in other words, Bergman groups have uncountable cofinality. Similarly, if $1 \in E = E^{-1}$ is a generating set for a Bergman group, then there is some finite power $n$ such that every element of the group can be written as a word of length $n$ in $E$. We express this by saying that the group is Cayley bounded, since it corresponds to every Cayley graph being of bounded diameter with respect to the word metric. Both uncountable cofinality and Cayley boundedness have been studied in the literature, though apparently mostly independently of each other. Uncountable cofinality grew out of J.P. Serre’s work \cite{Serre1957} on property (FA), which is a fixed point property for actions on trees, in which it proved to be one of the three conditions in his equivalent formulation of property (FA) for uncountable groups. It was first proved to hold for certain profinite groups by S. Koppelberg and J. Tits \cite{KoppelbergTits1972} and has subsequently been verified for a large number of primarily subgroups of the infinite symmetric group. S. Shelah \cite{Shelah1984}, on
the other hand, has constructed a group of cardinality $\aleph_1$ having width 240 with respect to any generating set, so, in particular, the group is Cayley bounded, but moreover, has no uncountable proper subgroups. Thus, Shelah’s example also has uncountable cofinality. As a matter of fact, as was noticed by Droste and Holland \[11\], these two properties together are equivalent to being Bergman, so Shelah’s example is Bergman too.

However, perhaps more useful for the geometric theory of Bergman groups is the following basic characterisation, of which I learned the equivalence of (2) and (3) from B.D. Miller \[25\] and where the equivalence of (1) and (3) was independently noticed by Y. de Cornulier \[8\] and V. Pestov.

**Theorem 1.2** The following conditions are equivalent for a group $G$.

1. Whenever $G$ acts by isometries on a metric space $(X,d)$ every orbit is bounded.
2. Any left-invariant metric on $G$ is bounded.
3. $G$ is Bergman.
4. Whenever $G$ acts on a metric space $(X,d)$ by mappings, which are Lipschitz for large distances, every orbit is bounded.
5. Whenever $G$ acts by uniform homeomorphisms on a geodesic space $(X,d)$ every orbit is bounded.

Now, of course, (1),(4) and (5) are really properties one would tend to study in connection with topological groups modulo some continuity condition, and these are indeed the main objects of the present paper. We therefore propose the following definition.

**Definition 1.3** A topological group $G$ is said to have property (OB) if whenever $G$ acts by isometries on a metric space $(X,d)$, such that for every $x \in X$ the function $g \in G \mapsto gx \in X$ is continuous, then every orbit is bounded.

Similar properties have previously been considered by Jan Hejcman \[15\] in 1959 (see also his recent paper \[16\]) and later by Christopher Atkin \[1\] under the name of boundedness in the context of uniform spaces. However, let me point of the differences between their notions and property (OB). A topological group $G$ is bounded if for any non-empty open subset $U \subseteq G$ there is a finite set $A \subseteq G$ and a number $n$ such that $G = U^n A$. This, however, is easily seen to fail for our Bergman group par excellence, $S_\infty$. For if we choose $U$ to be an open subgroup of denumerable index, as for example the isotropy subgroup of $0 \in \mathbb{N}$, then clearly $U^n A = UA \neq S_\infty$ for all $n$ and finite $A$. Nevertheless, the two notions do turn out to be equivalent in the context of abelian groups, but most of the groups considered here are very much non-abelian.

One of the reasons for our interest in the property comes from the fact that it can be seen as an addition to a well-known spectrum of properties studied in geometric group theory, namely properties (FA), (FH), (T), amenability, etc. One easily sees that property (OB) implies property (FH) and actually, we shall see that it provides a fairly comprehensive class of new examples of non-locally compact groups with property (FH).

Much of the work on these properties has been restricted to the locally compact setting, where the strongest tools are available (e.g., Haar measure). But over the years, a number of very interesting results concerning the dynamics of non-locally compact Polish groups have surfaced, for example on the unitary group of $\ell_2$, where Gromov and Milman proved that it is extremely amenable \[14\] and Bekka proved that it has property (T) \[4\]. Moreover, in logic, where, e.g., automorphism groups of countable structures tend to be non-locally compact, there is a multitude of results on permutation groups, e.g., Truss \[35\] and Hodges, Hodkinson, Lascar, and Shelah \[18\], and also for more inclusive classes of Polish groups, e.g., Becker and Kechris \[9\] and Hjorth \[17\].

Thus for Polish groups it is natural to look for a similar characterisation of property (OB), and indeed we have the following result.
Theorem 1.4 The following are equivalent for a Polish group $G$:

(i) $G$ has property (OB).

(ii) Whenever $W_0 \subseteq W_1 \subseteq \ldots \subseteq G$ is an increasing exhaustive sequence of sets with the Baire property, there are $n$ and $k$ such that $G = W_n^k$.

(iii) Any compatible left-invariant metric on $G$ is bounded.

(iv) $G$ is finitely generated of bounded width over any non-empty open subset.

For example, a locally compact Polish group has property (OB) if and only if it is compact.

This gives an indication of how to think of these properties. Namely, one should think of the Bergman property as a strong generalisation of finiteness and of property (OB) as a strong generalisation of compactness. Surprisingly though, this "compactness" can, apart from compact groups, only be found in large, very much non-locally compact groups.

We study first the dynamics of property (OB) groups acting continuously by Hölder mappings, showing that in this case the closure of the orbits gives a decomposition of the phase space into pieces on which the group acts minimally (often denoted by semi-simplicity). And secondly we consider the closure properties of the class of property (OB) groups, for example, it is quite easily seen that it is closed under infinite products, group extensions over a property (OB) group, and behaves well with respect to short exact sequences. I.e., if $\pi : G \to H$ is a continuous homomorphism with dense image where $G$ has property (OB), then so does $H$. Most interesting in this connection is the fact that it passes to subgroups of finite index, for which we give a geometric proof. In his paper [3], Bergman originally asked whether his property was preserved between a group and its subgroups of finite index, and A. Khélif, in an announcement [21], stated that this is indeed the case. However, the mentioned geometric proof, which works also for the Bergman property, shows the usefulness of the reformulation of the Bergman property in terms of isometric actions, where one has the added advantage of geometric intuition.

The known examples of Bergman groups have mostly been groups of symmetries of various countable structures of spaces, and, maybe apart from the obvious counter-examples of profinite groups of which examples can be found in Saxl, Shelah and Thomas [28], Thomas [32] and de Cornulier [5], one has the feeling that this is the most likely place to discover these. Somehow there is an intuition that Bergman groups are not constructed from below beginning with the simplest groups and using algebraic constructions such as direct sums and group extensions. This feeling seems to be reinforced by the following result.

Theorem 1.5 A solvable group is Bergman if and only if it is finite.

However, a large chunk of the paper is concerned with the verification and construction of groups which are either Bergman or have property (OB). We consider a fair number of examples, beginning with groups connected with the unit circle. We subsequently turn Theorem 1.4 on its head and instead ask for when the isometry group of a bounded complete metric space has property (OB). We provide one sufficient condition that is also of independent interest and use this to show that the isometry group of the Urysohn metric space of diameter 1 has property (OB). However, in the case of the rational Urysohn metric space of diameter 1, one can take advantage of the recent deep results of S. Solecki from [31], and use this to show outright that its isometry group is Bergman.

Theorem 1.6 Let $\mathbb{U}_1$ be the Urysohn metric space of diameter 1 and $\Omega$ the rational Urysohn metric space of diameter 1. Then $\text{Iso}(\mathbb{U}_1)$ has property (OB) and $\text{Iso}(\Omega)$ the Bergman property.

We then study a model theoretic version of the unitary group of $\ell_2$ in some depth. This is a subgroup $U(V)$ that sits as a dense subgroup in $U(\ell_2)$, which we prove to have ample
generics, the main tool used in proving the Bergman property for automorphism groups of \(\omega\)-stable, \(\omega\)-categorical structures in [20] and previously introduced by Hodges, Hodkinson, Lascar, and Shelah [18]. From ample generics, we prove that also \(U(\mathbb{V})\) is Bergman and has a number of other properties, e.g., the small index property and satisfies automatic continuity of homomorphisms. We use the analysis of \(U(\mathbb{V})\) to further study the dynamics of actions by Hölder mappings \(U(\ell_2)\), obtaining the strange conclusion that these are to some extent determined by the action of any bilateral shift on \(\ell_2\).

Our final collection of examples comes from topology, where we prove property (OB) for homeomorphism groups of spheres and of the Hilbert cube.

**Theorem 1.7** \(\text{Hom}(S^m)\) and \(\text{Hom}(Q)\) have property (OB) and, by consequence, any Polish group is a subgroup of a Polish group with property (OB).

In the final section of the paper we consider property (FA), and provide a simple proof of a result of Dugald Macpherson and Simon Thomas stating that if a Polish group with a comeagre conjugacy class acts on a tree, there every element of the group fixes a vertex or an edge. Actually, we extend their theorem to all actions on \(\Lambda\)-trees, though of course, the case of simplicial trees it ultimately the most interesting, due to the structure theory of Serre for groups with property (FA) [29].

Though we shall from time to time use a little bit of descriptive set theory, the article should be comprehensible to the general analyst. Really, all one needs to know is the definition of a Polish space (a completely metrisable separable topological space), a Polish group (a topological group whose topology is Polish), Borel sets (the sets belonging to the \(\sigma\)-algebra generated by the open sets), analytic set (a subset in a Polish space which is the continuous image of a Borel set) and sets having the Baire property (i.e. sets \(A\) such that for some open set \(U\) and some meagre set \(M\), \(A = U \triangle M\)). A basic result of Lusin and Sierpiński says that analytic sets have the Baire property.

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2 The Bergman property

**Definition 2.1** A group \(G\) is said to have the Bergman property if whenever \(W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n\), there are \(n\) and \(k\) such that \(G = W_n^k\).

For the following, recall that a geodesic space is a metric space such that between any two points \(x\) and \(y\) there is a path of length \(d(x, y)\). For example, Banach spaces and \(\mathbb{R}\)-trees are geodesic spaces. Recall also that a mapping \(\phi : X \to X\), where \(X\) is a metric space, is called Lipschitz for large distances if there are constants \(c, K\) such that for all \(x, y \in X\), 
\[
d(\phi x, \phi y) \leq c \cdot d(x, y) + K.
\]

The following result states the basic equivalent formulations of the Bergman property. The equivalence of (2) and (3) I learned from B.D. Miller [25] and the equivalence of (1) and (3) was independently noticed by Y. de Cornulier [8] and V. Pestov.

**Theorem 2.2** The following conditions are equivalent for a group \(G\).

1. Whenever \(G\) acts by isometries on a metric space \((X, d)\) every orbit is bounded.
(2) Any left-invariant metric on \( G \) is bounded.
(3) \( G \) is Bergman.
(4) Whenever \( G \) acts on a metric space \( (X, d) \) by mappings which are Lipschitz for large distances, every orbit is bounded.
(5) Whenever \( G \) acts by uniform homeomorphisms on a geodesic space \( (X, d) \) every orbit is bounded.

Proof : Clearly, \( 1 \implies 2 \) is trivial.

\( 2 \implies 3 \) : Suppose that \( W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G \) is an exhaustive sequence of subsets of \( G \). Notice that then \( W_0 \cap W_0^{-1} \subseteq W_1 \cap W_1^{-1} \subseteq \ldots \subseteq G \) is also exhaustive. So we can suppose that the \( W_n \) are symmetric, and by renumbering the sequence, we can also suppose that \( W_0 = \{1\} \). Notice now that the following left-invariant metric on \( G \) is bounded if and only if \( G = W_n^k \) for some \( n \) and \( k \):

\[
d(f, g) = \min(k_1 + k_2 + \ldots + k_n \mid \exists h_i \in W_{k_i}, f h_1 \ldots h_n = g).
\]

\( 3 \implies 4 \) : Assume now that \( 3 \) holds and that \( G \) acts on a metric space \( (X, d) \) by mappings which are Lipschitz for large distances and find for each \( g \in G \) constants \( c_g \) and \( K_g \) witnessing this. Then, if \( g_1, \ldots, g_k \in G \) and \( c_{g_1}, \ldots, c_{g_k}, K_{g_1}, \ldots, K_{g_k} \leq M \),

\[
d(g_1 \ldots g_k \cdot x, g_1 \ldots g_k \cdot y) \leq M \cdot d(g_2 \ldots g_k \cdot x, g_2 \ldots g_k \cdot y) + M \leq M^2 \cdot d(g_3 \ldots g_k \cdot x, g_3 \ldots g_k \cdot y) + M^2 + M \leq \ldots \leq M^k \cdot d(x, y) + M^k + M^{k-1} + \ldots + M \leq M^k \cdot (d(x, y) + k)
\]

(1)

Now, fix an \( x_0 \in X \) and let for \( n \geq 1 \)

\[ W_n = \{ g \in G \mid c_g, K_g \leq n \& d(x_0, g \cdot x_0) \leq n \} \]

This is clearly an increasing exhaustive sequence of subsets of \( G \), so for some \( M \) and \( k \), \( G = W_M^k \). We claim that \( x_0 \)'s orbit is bounded in diameter by \( 2k^2M^k \). For if \( g = g_1 \ldots g_k \in G \), with \( g_1, \ldots, g_k \in W_M \),

\[
d(x_0, g \cdot x_0) = d(x_0, g_1 \ldots g_k \cdot x_0) \leq d(x_0, g_1 \cdot x_0) + d(g_1 \cdot x_0, g_1 g_2 \cdot x_0) + \ldots + d(g_1 \ldots g_{k-1} \cdot x_0, g_1 \ldots g_{k-1} g_k \cdot x_0) \leq d(x_0, g_1 \cdot x_0) + M(d(x_0, g_2 \cdot x_0) + 1) + M^2(d(x_0, g_3 \cdot x_0) + 2) + \ldots + M^{k-1}(d(x_0, g_k \cdot x_0) + k - 1) \leq M + M(M + 1) + M^2(M + 2) + \ldots + M^{k-1}(M + k - 1) \leq 2k^2M^k
\]

(2)

So if \( x \) is any other point of \( X \), then

\[
d(g \cdot x, x) \leq d(g \cdot x, g \cdot x_0) + d(g \cdot x_0, x_0) + d(x_0, x) \leq M^k(d(x, x_0) + k) + 2k^2M^k + d(x, x_0)
\]

(3)

Whence \( x \)'s orbit is bounded and thus showing 4.
orbits, then any action by isometries of \(G/H\) is bounded in \(X\),

\[d(x, y) \leq \epsilon \rightarrow d(\phi x, \phi y) \leq 1.\]

So if \(d(x, y) \leq N \cdot \epsilon\), there are, since \((X, d)\) is geodesic, \(x_0 = x, x_1, \ldots, x_N = y \in X\) with \(d(x_i, x_{i+1}) \leq \epsilon\), whence

\[d(\phi x, \phi y) \leq d(\phi x_0, \phi x_1) + d(\phi x_1, \phi x_2) + \ldots + d(\phi x_{N-1}, \phi x_N) \leq N.\]

In other words, for all \(x, y \in X\),

\[d(\phi x, \phi y) \leq \left\lceil \frac{d(x, y)}{\epsilon} \right\rceil < \frac{1}{\epsilon}d(x, y) + 1.\]

5 \(\implies\) 1: Suppose that \(G\) acts by isometries on a metric space \((X, d)\). Now, \((X, d)\) might not be geodesic, but can be extended to a geodesic space as follows:

For any \(x, y \in X\), let \(\alpha(x, y)\) be a distinct isometric copy of \([0, d(x, y)]\). We let \(\tilde{X}\) be the quotient space of \(X = \bigcup_{x,y \in X} \alpha(x, y)\) obtained by for all \(x, y, z \in X\) identifying the left endpoints of \(\alpha(x, y)\) and \(\alpha(x, z)\), identifying the right endpoints of \(\alpha(y, x)\) and \(\alpha(z, x)\), and identifying the right endpoint of \(\alpha(y, x)\) with the left endpoint of \(\alpha(x, z)\). We define a metric \(\tilde{d}\) on \(\tilde{X}\) as follows: For \(a \in \alpha(x, y)\) and \(b \in \alpha(z, u)\), put

\[
\tilde{d}(a, b) = \min \left\{ \left| a - 0 \right| + d(x, z) + \left| 0 - b \right|, \left| a - 0 \right| + d(x, u) + \left| d(z, u) - b \right|, \left| a - d(x, y) \right| + d(y, z) + \left| 0 - b \right|, \left| a - d(y, u) \right| + d(y, u) + \left| d(z, u) - b \right| \right\}
\]

Then \((\tilde{X}, \tilde{d})\) contains \((X, d)\) isometrically (sending \(x \in X\) to the equivalence class of the unique (end)point of \(\alpha(x, x)\)). Moreover, the isometric action of \(G\) on \(X\) extends to \(\tilde{X}\) by letting \(g \in G\) send \(\alpha(x, y)\) isometrically and order-preservingly to \(\alpha(gx, gy)\). Thus, \(G\) acts by isometries on the geodesic space \(\tilde{X}\) and hence, if 5 holds, then every orbit of \(\tilde{X}\) and hence every orbit of \(X\) is bounded.

This allows us to give the following nice proof that the Bergman property is preserved under short exact sequences: Clearly, if \(H \leq G\) and any action by isometries of \(G\) has bounded orbits, then any action by isometries of \(G/H\) has bounded orbits. Conversely, any action by isometries of \(G/H\) and of \(H\) has bounded orbits and that \(G\) acts by isometries on \((X, d)\). Let \(O\) be the closure of an \(H\) orbit in \(X\) and let \(A = \{g \cdot O \mid g \in G\}\) be equipped with the Hausdorff metric \(d_H\). Then \(G\) acts transitively by isometries on \((A, d_H)\) and the action factors through \(G/H\). Therefore, \(A\) is bounded and so any orbit is bounded in \(X\).

\section{Polish groups with property (OB)}

\textbf{Definition 3.1} A topological group \(G\) is said to have property (OB) if whenever \(G\) acts by isometries on a metric space \((X, d)\), such that for every \(x \in X\) the mapping \(g \in G \mapsto gx \in X\) is continuous, then every orbit is bounded.

The preceding section is exclusively concerned with discrete groups, but we shall see that in the case of Polish groups there are again nice equivalent formulations of property (OB). We first notice that property (OB) can be slightly reformulated for Polish groups.

\[\text{6}\]
Lemma 3.2 Let $G$ be a Polish group acting by homeomorphisms on a metrisable space $X$, such that the mapping $g \in G \mapsto gx \in X$ is continuous for every $x \in X$. Then the action is actually jointly continuous, i.e., $(g, x) \mapsto g \cdot x$ is continuous from $G \times X$ to $X$.

Proof: Assume that $g_n \to g$ and $x_n \to x$. Since the mapping $h \in G \mapsto hy \in X$ is continuous for every $y \in X$, we see that $G \cdot y$ is a continuous image of a separable space and thus separable for every $y$. Hence $Y = G \cdot x \cup \bigcup_n G \cdot x_n$ is an invariant separable subspace of $X$. Moreover, the action of $G$ on $Y$ is separately continuous, so, as $Y$ is metrisable, the action of $G$ on $Y$ is jointly continuous (Kechris [19] (9.16)). Therefore, $g_n x_n \to gx$ in $Y$ and thus also in $X$. 

In particular, a Polish group has property (OB) if and only if all of its continuous actions by isometries on separable metric spaces have bounded orbits.

Property (OB) and the Bergman property fit quite nicely into the well-known hierarchy of group theoretical fixed-point properties such as property (T), (FH), (FA) etc. As first sight they do not appear to be a fixed-point properties, but it all depends on the perspective, as, for example, the Bergman property is equivalent to a fixed point property for its induced actions on the hyperspace of bounded subsets of any metric space it acts upon.

Definition 3.3 A group $G$ is said to have property (FA) if whenever it acts by automorphisms on a combinatorial tree (i.e. a uniquely path connected graph) it either fixes a vertex or an edge.

A topological group $G$ has property (topFA) if whenever it acts by automorphisms on a combinatorial tree, such that the stabilisers of vertices are open, then it fixes either a vertex or an edge.

A group $G$ is said to have property (algFH) if whenever it acts by isometries on a real Hilbert space $\mathcal{H}$, then it fixes a vector.

A topological group $G$ is said to have property (FH) if whenever it acts by isometries on a real Hilbert space $\mathcal{H}$ such for all $\xi \in \mathcal{H}$ the mapping $g \in G \mapsto g \cdot \xi \in \mathcal{H}$ is continuous, then it fixes a vector.

Admittedly, the fixed point property on trees is mainly interesting in its algebraic version, property (FA). Indeed, it is the main object of Serre’s book [29] in which he shows that it is equivalent to the conjunction of (i) the group has no infinite cyclic quotients, (ii) the group is not a non-trivial free product with amalgamation and (iii) the group is not the union of a countable chain of proper subgroups. The fixed point property on Hilbert spaces has correspondingly mostly been studied for countable discrete groups (in which case properties (algFH) and (FH) coincide) and for locally compact groups, where one is interested in property (FH). It is also well-known that (algFH) is stronger than (FA) and similarly, (FH) is stronger than (topFA). Moreover, for a group of isometries of Hilbert space to fix a point it is enough that there should be a bounded orbit. This follows from the lemma of the centre (see Bekka, de la Harpe and Valette [5]). So the following proposition sums up the connections between our properties.

Proposition 3.4 The following diagram of implications holds for topological and abstract groups.

\[
\begin{array}{ccc}
\text{Bergman} & \implies & \text{Property (OB)} \\
\downarrow & & \downarrow \\
\text{Property (algFH)} & \implies & \text{Property (FH)} \\
\downarrow & & \downarrow \\
\text{Property (FA)} & \implies & \text{Property (topFA)}
\end{array}
\]
Now in turn, we will show the basic equivalences of the different formulations of property (OB) for Polish groups. The following extracts the basic properties of the usual proof of the Birkhoff-Kakutani metrisation theorem, see, e.g., Hjorth [17, Theorem 7.2).

**Lemma 3.5** Let $G$ be a topological group and $(V_n)_{n \in \mathbb{Z}}$ a neighbourhood basis at the identity consisting of open sets such that

(I) $V_n = V_n^{-1}$

(II) $G = \bigcup_{n \in \mathbb{Z}} V_n$

(III) $V_n^3 \subseteq V_{n+1}$

Let $\delta(g_1, g_2) = \inf(2^n \mid g_1^{-1} g_2 \in V_n)$ and put

$$d(g_1, g_2) = \inf(\sum_{i=0}^{k} \delta(h_i, h_{i+1}) \mid h_0 = g_1, h_k = g_1)$$

Then

$$\delta(g_1, g_2) \leq 2d(g_1, g_2) \leq 2\delta(g_1, g_2)$$

and $d$ is a left-invariant compatible metric on $G$.

**Definition 3.6** Let $G$ be a Polish group. We say that $G$ is topologically Bergman if whenever

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G$$

is an exhaustive sequence of subsets with the Baire property, then $G = B_n^k$ for some $n$ and $k$. If there is a $k$ which works for all sequences $(B_n)$, then we say that $G$ is topologically $k$-Bergman.

**Theorem 3.7** The following are equivalent for a Polish group $G$.

(i) $G$ has property (OB).

(ii) $G$ is topologically Bergman.

(iii) Any compatible left-invariant metric on $G$ is bounded.

(iv) $G$ is finitely generated of bounded width over any non-empty open subset.

**Proof** : (iii)$\implies$(ii): Assume that $G$ is not topologically Bergman as witnessed by some exhaustive sequence of subsets with the Baire property

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G.$$

By considering $B_0 \cap B_0^{-1} \subseteq B_1 \cap B_1^{-1} \subseteq \ldots$ we can assume that the $B_n$ are symmetric. Then, as $G$ is Polish, some $B_n$ must be non-meagre and contain $1_G$, whence $V = \text{int}(B_n^k) \neq \emptyset$ by Pettis’ Lemma. Thus

$$VB_n \subseteq VB_{n+1} \subseteq \ldots \subseteq G$$

is an exhaustive sequence of open sets and $(VB_m)^k \subseteq (B^3_m)^k \neq G$ for all $m \geq n$. Put now $V_m = (VB_{n+m})^{3m}$ and notice that $(V_m)_{m \in \mathbb{N}}$ is an increasing and exhaustive sequence of open neighbourhoods of the identity satisfying $V_m^3 \subseteq V_{m+1}$. Supplementing this sequence with suitable $V_m$ for $m < 0$ we get a neighbourhood basis $(V_m)_{m \in \mathbb{Z}}$ satisfying the conditions of Lemma 3.5. Moreover, as $V_m \neq G$ for all $G$, the resulting metric $d$ is left-invariant, compatible, but unbounded.

The proof of the implication (ii)$\implies$(i) can be done as in the proof of 3$\implies$4 in Theorem 2.2, and that (i) implies (iii) is trivial.

(ii)$\implies$(iv): If $V \subseteq G$ is non-empty open and $\{g_n\}_{n \in \mathbb{N}}$ is dense in $G$, then the sequence $B_n = g_0 V \cup \ldots \cup g_n V$ is increasing and exhaustive. So, if $G$ is topologically Bergman, then
$G = B_n^k$ for some $n$ and $k$. But then $G = (V \cup \{g_0, \ldots, g_m\})^{2k}$, showing that $G$ is finitely generated of bounded width over $V$.

(iv)$\Rightarrow$(ii): Suppose $G$ is finitely generated of bounded width over any non-empty open set and $B_0 \subseteq B_1 \subseteq \ldots \subseteq G$ is an increasing exhaustive sequence of sets with the Baire property. Then some $B_n$ is non-meagre and $V = \text{int} B_n^2 \neq \emptyset$. Find some $g_0, \ldots, g_m \in G$ and $k$ such that $G = (V \cup \{g_0, \ldots, g_m\})^k$. Then $G = (B_n^2 \cup \{g_0, \ldots, g_m\})^k \subseteq B_1^{2k}$ for $l \geq n$ large enough such that $1, g_0, \ldots, g_m \in B_1$.

It was shown in Droste and Holland [11] that a group $G$ has the Bergman property if and only if $G$ satisfies the conjunction of the following two properties:

- (Uncountable cofinality) Whenever $H_0 \leq H_1 \leq \ldots \leq G = \bigcup_n H_n$, then $G = H_n$ for some $n$.
- (Cayley boundedness) Whenever $1 \in E = E^{-1}$ generates $G$ then $G = E^n$ for some $n$.

In the same manner, we can define these concepts for Polish groups.

**Definition 3.8** Let $G$ be a Polish group. We say that $G$ has uncountable topological cofinality if $G$ is not the union of a chain of proper open subgroups (or equivalently, a countable chain of subgroups with the Baire property). $G$ is topologically Cayley bounded if it has finite width with respect to any analytic generating set.

**Proposition 3.9** A Polish group $G$ has property (OB) if and only if it has uncountable topological cofinality and is topologically Cayley bounded.

**Proof**: Suppose $G$ has property (OB), $H_0 \leq H_1 \leq \ldots \leq G = \bigcup_n H_n$ are open subgroups and $1 \in E = E^{-1}$ is an analytic set generating $G$. Then by property (OB) applied to the sequences $H_0 \subseteq H_1 \subseteq \ldots \subseteq G$ and $E \subseteq E^2 \subseteq \ldots \subseteq G$, we see that $G = H_n^k = H_n = (E^n)^k = E^{nk}$ for some $n$ and $k$. Thus, $G$ has both uncountable topological cofinality and is topologically Cayley bounded.

Conversely, suppose $G$ has uncountable topological cofinality, is topologically Cayley bounded and $W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n$ are sets with the Baire property. By considering instead a tail subsequence of the exhaustive sequence

$$W_0 \cap W_0^{-1} \subseteq W_1 \cap W_1^{-1} \subseteq \ldots \subseteq G$$

we can suppose each $W_n$ is non-meagre, symmetric and contains 1. Thus the sequence $(W_0) \leq (W_1) \leq \ldots \leq G$ consists of open subgroups and hence one of the $W_n$ generates $G$. As $W_n$ is symmetric and non-meagre, $\text{int} W_n^2$ is symmetric and non-empty, so $W_n \cdot \text{int} W_n^2 \cdot W_n \subseteq W_n^4$ is a symmetric generating open subset of $G$ containing 1. So $G = W_n^k$ for some $k$.

In the case of locally compact groups, uncountable topological cofinality is clearly equivalent to compact generation. Moreover, if a locally compact, compactly generated group is also topologically Cayley bounded, then its compact generating set generates by a finite power and hence the group is compact. Conversely, compact groups trivially have property (OB). So property (OB) for locally compact Polish groups is just equivalent with compactness, just as the Bergman property for countable groups is equivalent with finiteness. However, we can actually provide a bit more information.

**Proposition 3.10** A compact Polish group is topologically 2-Bergman.

**Proof**: Assume that $G$ is compact and that

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq G$$
is an exhaustive sequence of subsets with the Baire property. Then there is some $B_{n_0}$ which is non-meagre and hence comeagre in some open set $Vf$, where $V$ is an open neighbourhood of the identity. Pick some symmetric open set $U \subseteq V$ such that $U^2 \subseteq V$ and $g_1, \ldots, g_m \in G$ such that $G = g_1U \cup \ldots \cup g_mU$. Then if $h_i \in g_iU$, we have $g_iU = h_i(h_i^{-1}g_i)U \subseteq h_i^2U^2 \subseteq h_iV$ and thus $G = h_1U^2 \cup \ldots \cup h_mU^2 = h_1V \cup \ldots \cup h_mV$. Considering now the sequences $(B_j \cap g_iU)_j$ for each $i = 1, \ldots, m$, we find $n_1 \geq n_0$ such that $B_{n_1}$ is non-meagre in each of the $g_iU$. Hence, we can find open sets $W_i \subseteq g_iU$ such that $B_{n_1}$ is comeagre in $W_i$ for each $i$. Pick now $h_i \in W_i \cap B_{n_1} \subseteq g_iU$ and notice that $G = Gf = h_iVf \cup \ldots \cup h_mVf$. But as $B_{n_1}$ is comeagre in both $W_i$ and $Vf$, we have by Pettis’ Lemma $h_iVf \subseteq W_iVf \subseteq B_{n_1}^2$. Thus, $G = B_{n_1}^2$, showing that $G$ is topologically $2$-Bergman. □

As a locally compact, non-compact Polish group cannot have property (OB), we know that it must have a compatible left-invariant unbounded metric. But actually we can see that this metric can be chosen to be Heine-Borel, i.e., such that any bounded closed set is compact.

**Proposition 3.11** (Folklore) Let $G$ be a locally compact Polish group. Then $G$ admits a left-invariant compatible Heine-Borel metric.

**Proof**: We start by fixing an open neighbourhood basis at the identity $(U_n)_{n \in \mathbb{N}}$ such that $U_{n+1}^3 \subseteq U_n$, $U_n = U_{n-1}^{-1}$ and $U_0$ being relatively compact. Since $G$ is $\sigma$-compact we can also find an increasing sequence of symmetric relatively compact sets $(V_n)_{n \in \mathbb{N}}$ such that $V_0 = U_0$, $V_{n+1}^3 \subseteq V_n$ and $G = \bigcup_{n \in \mathbb{N}} V_n$. Letting now $V_{-n} = U_n$, we see that the sequence $(V_n)_{n \in \mathbb{Z}}$ satisfies the conditions of Lemma 4. Let now $d$ be the metric given by the lemma, we claim that $d$ is Heine-Borel. For any $d$-bounded set $A$ is $\delta$-bounded and thus there is some $n$ such that for any $g, h \in A$, $g^{-1}h \in V_n$. In particular, $A$ is contained in some translate of $V_n$ and thus relatively compact. Hence if $A$ is closed it is compact, showing that $d$ is Heine-Borel. □

We should mention that locally compact Polish groups have complete left-invariant metrics and hence every left-invariant metric is complete, see Becker [2], section 3.

**Remark**: We should mention that there are examples of compact Polish groups not being Bergman. In fact, Koppelberg and Tits [22] prove that if $F$ is a finite non-trivial group, then $F^\mathbb{N}$ has uncountable cofinality if and only if $F$ is perfect. Thus as the Bergman property implies uncountable cofinality, we have compact profinite groups without the Bergman property. We shall see later that, in fact, no solvable infinite group can be Bergman.

We also see that the two properties (FH) and (OB) do not coincide. For there are plenty of examples of locally compact, non-compact Polish groups with property (FH), but of course without property (OB).

**Question 3.12** To what extent do homeomorphism groups of compact metric spaces have property (OB)? What about the isometry groups of bounded Polish metric spaces? (Some special cases will be verified in the following.)

**Definition 3.13** Recall that a mapping $f$ between metric spaces $X, d$ and $Y, \delta$ is called a Hölder($\alpha$) map for some $\alpha > 0$ if there is a constant $c \geq 1$ such that

$$\delta(f(x), f(y)) \leq c \cdot d(x, y)^\alpha$$

for all $x, y \in X$. Hölder(1) mappings are thus simply Lipschitz mappings.

**Proposition 3.14** Let $G$ be a Bergman group acting by Hölder maps on a metric space $(X, d)$. Then the action of $G$ is semi-simple, i.e., $\{g \cdot x\}_{x \in X}$ partitions $X$ into (bounded) invariant pieces each on which $G$ acts minimally. Moreover, there is an $N$ such that any $g \in G$ is Hölder($\alpha$) with constant $N$ for some $\alpha \in [1/N, N]$. 

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The same holds for Polish groups with property (OB) acting continuously and by Hölder maps on a Polish metric space $(X,d)$.

Proof : First assume that $G$ is Bergman. For each $g \in G$ let $\alpha_g \geq 0$ and $c_g \geq 1$ be such that $g$ is Hölder($\alpha_g$) with constant $c_g$. Thus

$$d(g_1 \cdot \ldots \cdot g_n x, g_1 \cdot \ldots \cdot g_n y) \leq c_g d(g_2 \cdot \ldots \cdot g_n x, g_2 \cdot \ldots \cdot g_n y)^{\alpha_g}$$

$$\leq c_g c_{g_2} \alpha_g \cdot \ldots \cdot \alpha_{g_n - 1} d(x,y)^{\alpha_{g_n}}$$

(5)

Put now $W_n = \{ g \in G \mid c_g \leq n \text{ & } \alpha_g \in [1/n,n] \}$ and notice that the sequence $W_n$ is increasing and exhaustive. So as $G$ is Bergman, $G = \bigcap W_n$ for some $n$ and $k$. By the inequality we see that there is a fixed $N$ such that any $g \in G$ is Hölder($\alpha$) with constant $N$ for some $\alpha \in [1/N,N]$. Thus we have

$$\forall \epsilon > 0 \exists \delta > 0 \forall x,y \forall g \quad (d(x,y) < \delta \rightarrow d(gx,gy) < \epsilon)$$

(6)

So suppose $x,y \in X$ and that $y \in G \cdot x$. Then for any $x' \in G \cdot x$ and $\epsilon > 0$, we can find $\delta > 0$ as above and $x'' \in G \cdot x$ with $d(x'',y) < \delta$. Now, if $x' = gx''$, then $d(x',gy) = d(gx'',gy) < \epsilon$, so $x' \in G \cdot y$, showing that $G \cdot x = G \cdot y$. Thus in $\overline{G \cdot x}$ every orbit is dense, and hence if $u \in \overline{G \cdot x} \cap \overline{G \cdot z}$ for any $u, z$, then $G \cdot x = G \cdot z = G \cdot u$. Therefore, $(G \cdot x)_{x \in X}$ partitions $X$ and $G$ acts minimally on each piece of the partition. The usual argument, as in the proof of Theorem 2.2 will also show that every orbit is bounded.

Now we only have to indicate the proof in the case that $G$ is a Polish group with property (OB). In this case we fix a countable dense set $\{x_m\}$ in $X$ and define $W_n$ by

$$W_n = \{ g \in G \mid \exists \alpha \in [1/n,n] \forall m,l d(gx_m,gx_l) \leq n \cdot d(x_m,x_l)^{\alpha} \}$$

Then $W_n$ is analytic (in fact closed) and hence has the Baire property. Notice also that if $g \in W_n$ then $g$ is indeed Hölder($\alpha$) with constant $n$ for some $\alpha \in [1/n,n]$. We can thus proceed as before using that $G$ is topologically Bergman.

In order to see that the result is not void, we can exhibit an action of $\mathbb{Z}$ by Lipschitz isomorphisms of $\mathbb{R}$ such that $\{\mathbb{Z} \cdot x\}_{x \in \mathbb{R}}$ does not partition $\mathbb{R}$. Namely, let $T(x) = 2x$, whence $T^n(x) = 2^n x$ for all $n \in \mathbb{Z}$. Then $T$ is a simple dilation of $\mathbb{R}$ and $0 \in \mathbb{Z} \cdot x$ for any $x \in \mathbb{R}$.

We thus see from statement that a Bergman group acting by Hölder maps actually acts equicontinuously. One might wonder if this also holds if the group acts by, e.g., uniform homeomorphisms, but this is false. For example, $S_\infty$, which is the Bergman group par excellence, acts continuously on $2^\mathbb{N}$, and thus by uniform homeomorphisms, but the action fails to be equicontinuous. Moreover, there is no decomposition of $2^\mathbb{N}$ into closed minimal pieces.

4 Closure properties

Proposition 4.1 Let $G$ be Polish and $H \leq G$ a closed subgroup. It both $G/H$ and $H$ have property (OB), so does $G$. Conversely, if $G$ has property (OB) and $\phi : G \rightarrow K$ is a continuous homomorphism into a Polish group $K$ with dense image, then $K$ has property (OB).
Proposition 4.2 Suppose \( \{G_n\}_n \) are Polish groups. Then \( G = \prod G_n \) has property (OB) if and only if each \( G_n \) has property (OB).

Proof: Suppose each \( G_n \) has property (OB) and assume \( W_0 \leq W_1 \leq \ldots \leq G \) is an exhaustive sequence of subsets with the Baire property. As in the proof of Proposition 3.9 we can suppose that each \( W_n \) is a symmetric open neighbourhood of the identity in \( G \).

Thus there is \( n \in \mathbb{N} \) such that

\[
\{1\} \times \ldots \times \{1\} \times \prod_{i>n} G_i \subseteq W_0
\]

and thus to prove that \( G = W_k^r \) for some \( n \) and \( k \), it is enough to prove that \( G_0 \times \ldots \times G_n \) is contained in some \( V_m^l \), where

\[
V_i = \{(g_0, \ldots, g_n) \in G_0 \times \ldots \times G_n \mid (g_0, \ldots, g_n, 1, 1, \ldots) \in W_i\}
\]

But \( V_0 \subseteq V_1 \subseteq \ldots \subseteq G_0 \times \ldots \times G_n \) is an increasing exhaustive sequence of open subsets, and as \( G_0 \times \ldots \times G_n \) has property (OB) by Proposition 4.1, the result follows. The other direction follows by Proposition 4.1. \( \square \)

In [0] Bergman poses the problem of whether (what we now subsequently call) the Bergman property passes from a group to a subgroup of finite index. In an announcement [21] A. Khelif states that this is indeed the case. We shall see that the concept of induced representations also leads to this result and, moreover, also solves the corresponding problem for Polish groups.

Proposition 4.3 Let \( G \) be a Polish group and \( H \leq G \) a finite index closed subgroup. The \( G \) has property (OB) if and only if \( H \) has.

Proof: First the easy direction. Assume \( H \) has property (OB). Then if \( G \) acts continuously by isometries on some space \((X, d)\), so does \( H \) and this latter has bounded orbits. Letting \( g_1, \ldots, g_n \) be representatives for the left cosets of \( H \) in \( G \), we see that \( G \cdot x = \bigcup_i g_i H \cdot x \), which is a finite union of bounded sets, and thus bounded.

For the other direction, consider first the abstract case of two groups \( G \) and \( H \) with \( H \) a finite index subgroup of \( G \). Fix a transversal \( 1 \in T \subseteq G \) for the left cosets of \( H \) in \( G \). Now assume that \( H \) acts by isometries on a metric space \((X, d)\). We define

\[
Y = \{ \xi : G \to X \mid \forall g \in G \forall h \in H \xi(gh) = h^{-1}\xi(g) \}
\]
For example, if $x_0 \in X$ is some fixed element, we can define $\xi_0 : G \to X$ by $\xi_0(ah) = h^{-1}x_0$ for all $h \in H$ and all $a \in T$. Then clearly $\xi_0 \in Y$. So $Y$ is non-empty.

We can now define the following metric $\partial$ on $Y$: $\partial(\xi, \zeta) = \sup_{g \in G} d(\xi(g), \zeta(g))$. If we can show that the supremum is finite, then this is clearly a metric. But

$$
\partial(\xi, \zeta) = \sup_{g \in G} d(\xi(g), \zeta(g))
= \sup_{a \in T, h \in H} d(\xi(ah), \zeta(ah))
= \sup_{a \in T, h \in H} d(h^{-1}\xi(a), h^{-1}\zeta(a))
= \sup_{a \in T, h \in H} d(\xi(a), \zeta(a))
= \sup_{a \in T} d(\xi(a), \zeta(a))
< \infty
$$

Where the last inequality holds since $T$ is finite. Now, let $G$ act on $Y$ by left translation

$$(g \cdot \xi)(f) = \xi(g^{-1}f)$$

This is an action by isometries.

$$
\partial(g \cdot \xi, \xi \cdot \zeta) = \sup_{f \in G} d((g \cdot \xi)(f), (g \cdot \zeta)(f))
= \sup_{f \in G} d(\xi(g^{-1}f), \zeta(g^{-1}f))
= \sup_{f' \in G} d(\xi(f'), \zeta(f'))
= \partial(\xi, \zeta)
$$

Now, if $G$ has the Bergman property, there is a bounded orbit $G \cdot \xi$ in $Y$. We now only need to see how this gives rise to a bounded orbit for $H$ in $X$. So let $x_0 = \xi(1)$ and notice that for $h \in H$

$$
d(x_0, h \cdot x_0) = d(\xi(1), h \cdot \xi(1))
= d(\xi(1), \xi(h^{-1}))
= d(\xi(1), (h \cdot \xi)(1))
\leq \sup_{g \in G} d(\xi(g), (h \cdot \xi)(g))
= \partial(\xi, h \cdot \xi)
\leq \text{diam}_G(G \cdot \xi)
$$

This shows that the Bergman property passes to subgroups of finite index.

For the case of Polish groups $G$ and $H$, with $H$ being a finite index closed and thus clopen subgroup, we of course restrict our attention to continuous $\xi$. Again we see that $\xi_0 \in Y \neq \emptyset$. We claim that the action of $G$ on $Y$ is separately continuous. In the second variable this is trivial, as $G$ acts by isometries. On the other hand, if we fix some $\xi \in Y$ and suppose that $g_n \to g$ in $G$, then by Equation (9)

$$
\partial(g_n \cdot \xi, g \cdot \xi) = \sup_{a \in T} d((g_n \cdot \xi)(a), (g \cdot \xi)(a)) = \sup_{a \in T} d(\xi(g_n^{-1}a), \xi(g^{-1}a)) \to 0 \quad n \to \infty
$$

Thus, by Lemma 3.2, the action of $G$ on $Y$ is continuous and we can finish the proof as in the discrete case. □

These calculations will also help us prove
**Theorem 4.4** A solvable Bergman group is finite.

This result reinforces our feeling that Bergman groups are not something that can be constructed from below using only simple construction methods, but are rather groups that have to be given in one single step.

That infinite abelian groups might not be Bergman was suggested to me by Pandelis Dodos. Actually, we shall see that there is a very specific reason for this to be true.

**Definition 4.5** Let $H \leq K$ be abelian groups. $H$ is said to be pure in $K$ if for every $n \geq 0$, $nK \cap H = nH$, i.e., if some $x \in H$ is divisible by $n$ in $K$ then $x$ is divisible by $n$ in $H$.

If, moreover, $K$ is a torsion group, we say that $H$ is a basic subgroup of $K$ if (i) $H$ is pure in $K$, (ii) $H$ is a direct sum of cyclic subgroups and (iii) $K/H$ is divisible.

A theorem of Kulikov (confer Robinson (4.3.4.) \(^{27}\)) states that any abelian torsion group contains a basic subgroup.

**Lemma 4.6** An infinite abelian group has a countably infinite quotient group.

**Proof:** Let $G$ be an infinite abelian group and $T$ its torsion subgroup. Then $G/T$ is torsion free and hence embeds into a direct sum of copies of $\mathbb{Q}$, $G/T \subseteq \oplus_{i \in I} \mathbb{Q}$. If $G$ is not itself a torsion group, then the projection of $G/T$ onto one of these summands must be non-trivial and thus there is some non-trivial quotient of $G/T$, and hence of $G$, which is isomorphic to a subgroup of $\mathbb{Q}$. Clearly, this quotient is countably infinite.

So we can assume that $G = T$, i.e., that $G$ is a torsion group. Therefore, by the theorem of Kulikov, let $H$ be a basic subgroup of $G$. Assume first that $|G/H| > 1$. Then as $G/H$ is divisible, by the structure theorem of divisible abelian groups, it is the direct sum of isomorphic copies of quasicyclic groups. Since each of these is countably infinite, by projecting $G/H$ surjectively onto one of these we thus produce a countably infinite quotient of $G$. Assume now that $G = H$. Then $G$ is a direct sum of cyclic subgroups and, as $G$ is a torsion group, each of the summands is finite. So let $G = \oplus_{i \in J} F_i$, where $F_i \neq \{0\}$ is a finite group. As $G$ is infinite, so is $J$. Picking a countably infinite subset $J_0 \subseteq J$ and projecting $G$ onto $\oplus_{i \in J_0} F_i$, we again end up with the countably infinite quotient.

Since no countably infinite group is Bergman, no infinite abelian group is Bergman.

**Proof of Theorem 4.4** Assume that $G$ is solvable and that $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ is an abelian series for $G$, i.e., that each quotient $G_{i+1}/G_i$ is abelian. Suppose that $G = G_n$ is Bergman and assume by induction that $G_{i+1}$ is Bergman. Then also $G_{i+1}/G_i$ is Bergman and, being abelian, it is also finite. Thus $G_i$ is a finite index subgroup of a Bergman group and hence Bergman. So this shows inductively that each of $G_i$ is Bergman, and so all the quotients $G_{i+1}/G_i$ are finite. Therefore also $G$ is finite.

## 5 Circle groups

We shall first consider the homeomorphism group of the unit circle $S^1$ and its model-theoretic counterpart, the automorphism group of the countable dense circular order, $\text{Aut}(\mathbb{C})$.

Let first $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ and let $d$ be the metric on $S^1$ induced by the metric on $\mathbb{R}$. I.e., $d(x, y) = \text{dist}(\pi^{-1}(x), \pi^{-1}(y))$. So $d$ takes values in $[0, 1/2]$.

Let $\mathbb{C} \subseteq S^1$ be a countable dense set, for concreteness we can take $\mathbb{C} = \pi[\mathbb{Q}]$, and $\text{Aut}(\mathbb{C})$ the set of all permutations of $\mathbb{C}$ that preserve the relation $B \subseteq \mathbb{C}^3$ defined as follows:

For $x, y, z \in \mathbb{C}$ let $B(x, y, z)$ if and only if
• $x, y$ and $z$ are distinct.

• Going clockwise along the unit circle $S^1$ from $x$ to $z$ one passes through $y$.

In this case, we say that $y$ is between $x$ and $z$.

**Notation 5.1** For $x_1, \ldots, x_n \in \mathfrak{C}$ write $\odot x_1 x_2 \ldots x_n$ if for all $i < j < l$, $B(x_i, x_j, x_l)$.

**Lemma 5.2** Suppose $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathfrak{C}$, $\odot x_1 \ldots x_n y_1 \ldots y_m$. Then

$$\text{Aut}(\mathfrak{C}) = \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n) \cdot \text{Aut}(\mathfrak{C}, y_1, \ldots, y_m) \cdot \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n)$$

A similar statement holds for $\text{Hom}_+(S^1)$, which is the group of orientation preserving homeomorphisms of $S^1$.

**Proof**: Let $g \in \text{Aut}(\mathfrak{C})$ be given and let $I \subseteq \{1, \ldots, n\}$ be the set of $i$ such that $B(y_n, g(x_i), y_1)$. Notice that $I$ is an interval, $I = \{i_0, i_0 + 1, \ldots, i_1\}$.

Pick some $f \in \text{Aut}(\mathfrak{C}, y_1, \ldots, y_m)$ such that $f(x_i) = g(x_i)$ for $i \in I$,

$$B(y_n, f(x_i), x_1), \quad B(y_n, f(x_i), f(x_{i_0}))$$

for $i = 1, \ldots, i_0 - 1$, and

$$B(x_n, f(x_i), y_1), \quad B(f(x_{i_1}), f(x_i), y_1)$$

for $i = i_1 + 1, \ldots, n$.

Pick now

$$h \in \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n, f(x_{i_0}), \ldots, f(x_{i_1}))$$

such that $hf(x_i) = g(x_i)$ for $i \notin I$. Then

$$hf \upharpoonright \{x_1, \ldots, x_n\} = g \upharpoonright \{x_1, \ldots, x_n\}$$

whence $(hf)^{-1}g \in \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n)$ and

$$g = hf \cdot (hf)^{-1}g \in \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n) \cdot \text{Aut}(\mathfrak{C}, y_1, \ldots, y_m) \cdot \text{Aut}(\mathfrak{C}, x_1, \ldots, x_n)$$

The statement and the proof for $\text{Hom}_+(S^1)$ is similar. $\square$

**Definition 5.3** If $X$ is a set and $\text{Sym}(X)$ the group of all permutations of $X$, the permutation group topology on $\text{Sym}(X)$ has as open neighbourhood basis at the identity the sets $\text{Sym}(X, x_1, \ldots, x_n) = \{g \in \text{Sym}(X) \mid g(x_1) = x_1, \ldots, g(x_n) = x_n\}$, where $x_1, \ldots, x_n$ are any elements of $X$.

Recall that on $\text{Hom}_+(S^1)$ the two topologies of pointwise and uniform convergence and the compact-open topology coincide. Moreover, $\text{Hom}_+(S^1)$ is a Polish group in this topology.

As $\text{Aut}(\mathfrak{C})$ is a closed subgroup of $\text{Sym}(\mathbb{N}) = \mathcal{S}_\infty$ in the permutation group topology, $\text{Aut}(\mathfrak{C})$ is also Polish.

**Theorem 5.4** $\text{Aut}(\mathfrak{C})$ and $\text{Hom}(S^1)$ are Bergman.
Proof: Let us first notice that for any \( x \in \mathcal{C} \), the groups \( \text{Aut}(\mathcal{C}, x) \) and \( \text{Aut}(\mathcal{Q}, <) \) are naturally isomorphic. Thus if

\[
W_0 \subseteq W_1 \subseteq \ldots \subseteq \text{Aut}(\mathcal{C}) = \bigcup_n W_n
\]

then there are \( n \) and \( k \) such that \( \text{Aut}(\mathcal{C}, x) \subseteq W_n^k \). This follows from the result of Droste and Göbel [10] that \( \text{Aut}(\mathcal{Q}, <) \) is Bergman. Taking \( g \in \text{Aut}(\mathcal{C}) \) such that \( x \neq g(x) \), we see by Lemma 5.2 that

\[
\begin{align*}
\text{Aut}(\mathcal{C}) &= \text{Aut}(\mathcal{C}, x) \cdot \text{Aut}(\mathcal{C}, g(x)) \cdot \text{Aut}(\mathcal{C}, x) \\
&= \text{Aut}(\mathcal{C}, x) \cdot g \text{Aut}(\mathcal{C}, x) g^{-1} \cdot \text{Aut}(\mathcal{C}, x) \\
&= W_n^k g W_n^k g^{-1} W_n^k \\
&= W_m^k
\end{align*}
\]

for some sufficiently large \( m \).

The same argument applies to \( \text{Hom}^+(S^1) \), using that for any \( x \in S^1 \) the groups

\[
\text{Hom}^+(S^1, x) = \{ g \in \text{Hom}^+(S^1) \mid g(x) = x \}
\]

and \( \text{Hom}^+(\mathbb{R}) \) are isomorphic. Again \( \text{Hom}^+(\mathbb{R}) \) is Bergman by the results of Droste and Göbel, so by Lemma 5.2 also \( \text{Hom}^+(S^1) \) is Bergman. Now the Bergman property for \( \text{Hom}(S^1) \) follows, as \( \text{Hom}^+(S^1) \) is a subgroup of index 2 in \( \text{Hom}(S^1) \).

\(\Box\)

6 Groups of isometries

**Definition 6.1** Let \((X, d)\) be a metric space and \(G \leq \text{Iso}(X, d)\). We let \(d_\infty\) denote the supremum metric on the spaces \(X^m\) induced by \(d\). The group \(G\) is said to be approximately oligomorphic if for any \(n \geq 1\) and \(\epsilon > 0\) there is a finite set \(A \subseteq X^n\) such that \(G \cdot A\) is \(\epsilon\)-dense in \(X^n\) with respect to \(d_\infty\).

**Theorem 6.2** Let \((X, d)\) be a Polish metric space and \(G\) a closed subgroup of \(\text{Iso}(X, d)\) with the topology of pointwise convergence. If \(G\) is approximately oligomorphic, then \(G\) has property (OB).

**Proof:** We need to show that \(G\) is finitely generated of bounded width over any non-empty open set \(V \subseteq G\). So find \(\overline{\pi} = (x_1, \ldots, x_n) \in X^n\) and \(\epsilon > 0\) such that

\[
U = \{ g \in G \mid \forall i \leq n \ d(x_i, gx_i) < \epsilon \} \subseteq V V^{-1}
\]

We claim that there is a finite set \(B \subseteq X^n\) such that \(U \cdot B\) is \(\frac{\epsilon}{2}\)-dense in \(X^n\). To see this, let \(A \subseteq X^n \times X^n\) be a finite set such that \(G \cdot A\) is \(\frac{\epsilon}{2}\)-dense in \(X^n \times X^n\). Define \(A' \subseteq A\) to be the set of \(\overline{\pi} = (\overline{x_1}, \overline{x_2}) \in A\) such that for some \(g_{\overline{x}} \in G\), \(d_\infty(\overline{x}, g_{\overline{x}}\overline{x}) < \frac{\epsilon}{2}\). Finally, put \(B = \{ g_{\overline{x_2}} \mid \overline{x} \in A' \}\).

Then, if \(\overline{x} \in X^n\), there are \(\overline{a} = (\overline{a}_1, \overline{a}_2) \in A\) and \(g \in G\) such that

\[
d_\infty((\overline{x}, \overline{a}), (g_{\overline{x}}\overline{a}, g_{\overline{x}}\overline{a})) < \frac{\epsilon}{2}
\]

In particular, \(\overline{a} \in A'\) and thus

\[
d_\infty(g_{\overline{x}}^{-1}\overline{x}, \overline{a}) = d_\infty(g_{\overline{x}}^{-1}\overline{x}, g^{-1}\overline{a}) \\
\leq d_\infty(g_{\overline{x}}^{-1}\overline{x}, \overline{a}) + d_\infty(\overline{a}, g^{-1}\overline{a}) \\
= d_\infty(\overline{x}, g_{\overline{x}}\overline{a}) + d_\infty(g_{\overline{x}}\overline{a}, \overline{a}) < \epsilon
\]

(14)
Thus, $U \cdot B$ is $\frac{\varepsilon}{2}$-dense in $X^n$. Let now $B' \subseteq B$ be the set of $\bar{b} \in B$ such that

$$d_{\infty}(U \cdot \bar{b}, G \cdot \bar{c}) < \frac{\varepsilon}{2}$$

So for some $h_{\bar{b}} \in G$ and $g_{\bar{c}} \in U$,

$$d_{\infty}(g_{\bar{c} \bar{b}}, h_{\bar{b}}) < \frac{\varepsilon}{2}$$

Then, if $f \in G$, we can find $\bar{b} \in B$ and $g \in U$ such that $d_{\infty}(f \bar{b}, g \bar{c}) < \frac{\varepsilon}{2}$. Thus, $\bar{b} \in B'$ and

$$d_{\infty}(\bar{c}, f^{-1}g_{\bar{c} \bar{b}}) = d_{\infty}(g^{-1}f \bar{c}, g^{-1}h_{\bar{b} \bar{b}})$$

$$\leq d_{\infty}(g^{-1}f \bar{c}, \bar{b}) + d_{\infty}(\bar{b}, g^{-1}h_{\bar{c} \bar{c}})$$

$$= d_{\infty}(f \bar{c}, g \bar{b}) + d_{\infty}(g_{\bar{c} \bar{c}}, h_{\bar{b} \bar{b}})$$

$$< \varepsilon$$

So $f^{-1}g_{\bar{c} \bar{c}}h_{\bar{c} \bar{c}} \in U$ and if $H = \{h_{\bar{b}} \mid \bar{b} \in B'\}$, we see that $G = UU^{-1}HU^{-1}$. □

**Corollary 6.3** Let $G$ be an oligomorphic closed subgroup of $S_{\infty}$. Then $G$ has property (OB).

*Proof:* Notice that if we let $\mathbb{N}$ have the metric in which all points have distance 1, then $G$ is approximately oligomorphic as a group of isometries exactly when it is oligomorphic. □

The *Urysohn metric space* $\mathbb{U}$ is the unique separable complete metric space containing each finite metric space and such that any isometry between finite subsets extends to a full isometry of the space. This space, constructed by Urysohn [36] is also characterised by being separable, complete and satisfying the following extension property:

($\ast$): If $\phi : X \to \mathbb{U}$ is an isometric embedding of a finite metric space $X$ into $\mathbb{U}$ and $Y = X \cap \{y\}$ is a one point metric extension of $X$, then $\phi$ extends to an isometric embedding of $Y$.

In the same manner, there is a Urysohn metric space of diameter 1, designated by $\mathbb{U}_1$, which is the unique complete separable metric space whose diameter is at most 1 and satisfying the extension property ($\ast$), when $Y$ varies over metric spaces of diameter at most 1.

Similarly, one can construct variants of the Urysohn metric space, where the metric takes values only in $\mathbb{Q} \cap [0, 1]$. Thus, the rational Urysohn metric space of diameter 1, denoted by $\Omega$, is the unique countable metric space whose metric takes values in $\mathbb{Q} \cap [0, 1]$ and satisfying the extension property for $Y$, whose metric also takes values in $\mathbb{Q} \cap [0, 1]$.

**Theorem 6.4** Let $\mathbb{U}_1$ be the Urysohn metric space of diameter 1. Iso($\mathbb{U}_1$) is approximately oligomorphic and hence has property (OB).

For this proof we need some notions of metric theory. Let $D_n$ be the set of $n \times n$ matrices $[a_{ij}]$ with entries in $[0, 1]$ such that $d(i,j) = a_{ij}$ defines a pre-metric on $\{1, \ldots, n\}$. Consider $D_n$ as a subset of $[0, 1]^{n^2}$ with the supremum metric $d_{\infty}$. Clearly, the triangle inequality is a closed condition, so $D_n$ is compact.

We define also the following distance $d_1$ on $D_n$ à la Gromov and Hausdorff (see Chapter 3, Gromov [13]):

$$d_1(A, B) = \min(\text{trace}(E) \mid \begin{bmatrix} A & E \\ E^t & B \end{bmatrix} \in D_{2n})$$

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Notice that the infimum is indeed attained, as we are minimising over a compact space. So if \( A, B \in D_n \) are representing pre-metrics \( a \) and \( b \) on \( \{1, \ldots, n\} \) and \( \{1', \ldots, n'\} \) respectively (thus of diameter at most 1), \( d_1 \) is the minimum of \( \sum_i c(i, i') \), where \( c \) varies over all pre-metrics on \( \{1, \ldots, n, 1', \ldots, n'\} \) of diameter at most 1 agreeing with \( a \) on \( \{1, \ldots, n\} \) and with \( b \) on \( \{1', \ldots, n'\} \). Therefore, \( d_1 \) measures how far the spaces have to be from each other, when they are both embedded into a metric space of diameter at most 1.

**Lemma 6.5** \( 2d_1 \leq nd_{\infty} \leq nd_1 \).

**Proof:** Let \( A, B \in D_n \) and let \( a \) and \( b \) be the corresponding pre-metrics on \( \{1, \ldots, n\} \). Assume that
\[
\delta = d_{\infty}(A, B) = \sup_{i,j} |a(i, j) - b(i, j)|
\]
and let \( c \) be defined on \( \{1, \ldots, n, 1', \ldots, n'\} \) by
\[
\begin{align*}
c(i, j) &= a(i, j) \\
c(i', j') &= b(i, j) \\
c(i, j') &= a(i, l) + \min_l (a(i, l) + \delta/2 + b(l, j))
\end{align*}
\]
We claim that \( c \) is a pre-metric and that \( i \mapsto i \) and \( i \mapsto i' \) are isometric embeddings of the spaces given by \( a \) and \( b \) respectively. Clearly, the triangle inequality is satisfied separately on \( \{1, \ldots, n\} \) and on \( \{1', \ldots, n'\} \), and
\[
c(i, k') + c(k', j) = \min_l (a(i, l) + \delta/2 + b(l, k)) + \min_p (b(k, p) + \delta/2 + a(p, j))
\]
\[
= \delta + \min_l (a(i, l) + b(l, k)) + \min_p (b(k, p) + a(p, j))
\]
\[
\geq \delta + \min_{i,p} (a(i, l) + a(l, p) + a(p, j))
\]
\[
\geq a(i, j)
\]
\[
= c(i, j)
\]
Similarly, \( c(i', j') \leq c(i', l) + c(l, j') \). And
\[
c(i, j') \leq \min_k (a(i, k) + \delta/2 + b(k, j))
\]
\[
\leq a(i, l) + \min_k (a(l, k) + \delta/2 + b(k, j))
\]
\[
= c(i, l) + c(l, j')
\]
Similarly, \( c(i, j') \leq c(i, l') + c(l', j') \), so the triangle inequality is verified. Unfortunately, \( c \) does not necessarily have diameter bounded by 1, but this can be remedied by letting \( c'(x, y) = \min \{c(x, y), 1\} \). Clearly, this does not affect the distances on \( \{1, \ldots, n\} \) and \( \{1', \ldots, n'\} \) separately, and only decreases other distances. So \( c'(i, i') = \frac{\delta}{2} \). Let now
\[
C' = \begin{bmatrix} A & E \\ E^t & B \end{bmatrix} \in D_{2n}
\]
be the matrix corresponding to \( c' \), and notice that \( d_1(A, B) \leq \text{trace}(E) = n\frac{\delta}{2} = \frac{n}{2}d_{\infty}(A, B) \). Thus, \( d_1 \leq \frac{n}{2}d_{\infty} \).
On the other hand, if the two sets \( \{1, \ldots, n\} \) and \( \{1', \ldots, n'\} \) are very close to each other, pointwise, in some common metric space, then the distance between \( i \) and \( j \) cannot differ very much from the distance between \( i' \) and \( j' \). And in fact, \( d_{\infty} \leq d_1 \).

**Proof of Theorem 6.4.** Fix some \( n \geq 1 \) and \( \epsilon > 0 \) and let \( \partial \) be the metric on \( U_1 \). As \( D_n \) is compact, we can find some finite \( A \subseteq D_n \), which is \( \epsilon \)-dense in the metric \( d_1 \). By the universality property of the Urysohn metric space \( U_1 \), this means that for any \( \mathfrak{r} = (x_1, \ldots, x_n) \in U^n_1 \), there is \( \mathfrak{y} = (y_1, \ldots, y_n) \in U^n_1 \) with distance matrix \( A \in A \), such that \( \partial_{\infty}(\mathfrak{r}, \mathfrak{y}) \leq \sum_i \partial(x_i, y_i) \leq \epsilon \). So pick for each \( A \in A \) some \( \mathfrak{r} \in U^n_1 \) with distance matrix \( A \), and let \( A \) be the set of these. Then, if \( \mathfrak{r} = (x_1, \ldots, x_n) \in U^n_1 \), there is \( \mathfrak{y} = (y_1, \ldots, y_n) \) as above, and hence some \( \mathfrak{z} = (z_1, \ldots, z_n) \in A \) isometric to \( \mathfrak{y} \). But then as \( U_1 \) is ultrahomogeneous, we see that \( \mathfrak{y} \) and \( \mathfrak{z} \) are in the same orbit of \( \text{Iso}(U_1) \), showing that \( \text{Iso}(U_1) \) is approximately oligomorphic.

We will now show that if we consider the isometry group of the rational Urysohn metric space of diameter 1, \( \Omega \), then we actually get the Bergman property outright. The results here were finally clear after a late night discussion at Caltech with Stevo Todorčević.

Two tuples \( \mathfrak{r} \) and \( \mathfrak{y} \) in \( \Omega \) are said to be *uniformly of distance 1 from each other* if \( d(x_i, y_j) = 1 \) for all \( i, j \).

**Lemma 6.6** If \( \mathfrak{r} \) and \( \mathfrak{y} \) in \( \Omega \) are uniformly of distance 1 and some \( \mathfrak{z} \) in \( \Omega \) is given, then there are \( \mathfrak{r}' \) and \( \mathfrak{z}' \) such that \( (\mathfrak{r}, \mathfrak{z}, \mathfrak{y}) \) and \( (\mathfrak{r}', \mathfrak{z}', \mathfrak{y}) \) are isometric and \( \mathfrak{r} \) is uniformly of distance 1 from both \( \mathfrak{r}' \) and \( \mathfrak{z}' \).

**Proof:** Notice that the distances between \( \mathfrak{r}, \mathfrak{y}, \mathfrak{z}, \mathfrak{r}' \) are completely specified by the lemma, so we need only specify the distances between \( \mathfrak{r} \) and \( (\mathfrak{r}', \mathfrak{z}') \). We let \( \mathfrak{r} \) be uniformly of distance 1 from \( \mathfrak{r}' \) and put

\[
d(z_i, z_j') = \min\{1, \inf_{y_i} d(z_i, y_i) + d(y_i, z_j)\}
\]

The triangle inequality holds, which can be checked by hand, so let us just give a few representative cases.

- Clearly, \( d(x_i', y_j) \leq d(x_i', v) + d(v, y_j) \) for all \( v \in \mathfrak{r}, \mathfrak{z}, \mathfrak{y} \), since \( \mathfrak{r}' \) is uniformly of distance 1 from all of \( \mathfrak{r}, \mathfrak{z}, \mathfrak{y} \). Moreover, it also holds for \( v \in \mathfrak{r}', \mathfrak{z}' \), since \( (\mathfrak{r}', \mathfrak{z}', \mathfrak{y}) \) is isometric to \( (\mathfrak{r}, \mathfrak{z}, \mathfrak{y}) \) and thus is a metric space.

- Clearly, for all \( v \in \mathfrak{r}, \mathfrak{y}, \mathfrak{r}', \mathfrak{z} \), \( d(x_i, z_j') \leq d(x_i, v) + d(v, z_j') \), since in this case one of the distances on the right hand side must equal 1. So for \( v = z_j \) we have

\[
d(x_i, z_k) + d(z_k, z_j') \geq \min\{1, \inf_{y_i} d(x_i, z_k) + d(z_k, y_i) + d(y_i, z_j)\}
\]

\[
\geq \min\{1, \inf_{y_i} d(x_i, y_i) + d(y_i, z_j)\}
\]

\[
\geq 1
\]

\[
= d(x_i, z_j') \quad (18)
\]

- Clearly, for all \( v \in \mathfrak{r}, \mathfrak{y}, \mathfrak{z}, \mathfrak{z} \), \( d(z_i, y_j) \leq d(z_i, v) + d(v, y_j) \), since in the first two cases one of the distances on the right hand side must equal 1 and in the last two cases it reduces to the triangle inequality on \( (\mathfrak{r}, \mathfrak{y}) \). And for \( v = z_j' \) we have

\[
d(z_i, z_k') + d(z_k', y_j) \geq \min\{1, \inf_{y_i} d(z_i, y_i) + d(y_i, z_k) + d(z_k, y_j)\}
\]

\[
\geq \min\{1, d(z_i, y_j)\}
\]

\[
\geq d(z_i, y_j) \quad (19)
\]

\[
\square
\]
Lemma 6.7 Assume that \( \varpi \) is a tuple in \( \Omega \). Then there is some \( l \in \text{Iso}(\Omega) \) such that

\[
\text{Iso}(\Omega) = (l \cdot \text{Iso}(\Omega, \varpi))^4
\]

Proof: Find some \( \varpi \), isometric to \( \varpi \) and uniformly of distance 1 from it. Let \( l(\varpi) = \varpi \) and \( l(\varpi) = \varpi \). Then \( l \cdot \text{Iso}(\Omega, \varpi) \cdot l = \text{Iso}(\Omega, \varpi) \). Let \( g \in \text{Iso}(\Omega) \) be any element and put \( \varpi = g(\varpi) \).

By Lemma 6.6 we can find \( \varpi, \varpi' \) such that \( (\varpi, \varpi, \varpi) \) and \( (\varpi, \varpi, \varpi) \) are isometric and \( \varpi \) is uniformly of distance 1 from both \( \varpi' \) and \( \varpi' \). Thus there is some \( h \in \text{Iso}(\Omega, \varpi) \) such that \( h(\varpi) = \varpi' \) and \( h(\varpi) = \varpi' \). Now, since \( (\varpi, \varpi') \) and \( (\varpi, \varpi') \) are isometric, there is some \( f \in \text{Iso}(\Omega, \varpi) \) such that \( f(\varpi') = \varpi' \). And finally, as \( (\varpi, \varpi') \) and \( (\varpi, \varpi') \) are isometric, we can find \( k \in \text{Iso}(\Omega, \varpi) \) such that \( k(\varpi') = \varpi' \).

Therefore, \( kfhg(\varpi) = kfh(\varpi) = kfh(\varpi') = k(\varpi') = \varpi \) and \( kfhg \in \text{Iso}(\Omega, \varpi) \). So

\[
g = h^{-1}f^{-1}k^{-1}(kfhg) \in (\text{Iso}(\Omega, \varpi) \cdot \text{Iso}(\Omega, \varpi))^2 = (l \cdot \text{Iso}(\Omega, \varpi) \cdot l \cdot \text{Iso}(\Omega, \varpi))^2.
\]

\( \square \)

Theorem 6.8 The isometry group of the rational Urysohn metric space of diameter 1, \( \text{Iso}(\Omega) \), is Bergman.

Proof: The proof relies on the deep result of S. Solecki [31], also independently announced by A.M. Vershik, that for any finite rational metric space \( X \) there is another finite rational metric space \( Y \) containing \( X \) and such that any partial isometry of \( X \) extends to a full isometry of \( Y \).

First of all, we notice that this also implies the corresponding result for rational metric spaces of bounded diameter 1. For if \( X \) is of bounded diameter 1, then we find first some \( Y' \) (not necessarily of bounded diameter 1) extending \( X \) such that every partial isometry of \( X \) extends to a full isometry of \( Y' \). Now, if \( d' \) is the metric on \( Y' \), let \( d \) be the metric given by \( d(y_0, y_1) = \min\{1, d'(y_0, y_1)\} \) and let \( Y \) be the metric space obtained. Then we see that the distances between points in \( X \) are preserved and thus \( X \) is still a subspace of \( Y \), and if \( f \) is an isometry of \( Y' \) it is also an isometry of \( Y \). Thus, every partial isometry of \( X \) extends to a full isometry of \( Y \).

We now need the following concept, which will also be used in a later section.

Definition 6.9 Suppose \( G \) is a Polish group and consider for each finite \( m \geq 1 \) the diagonal conjugacy action of \( G \) on \( G^m \) given by

\[
g \cdot (h_1, \ldots, h_m) = (gh_1g^{-1}, \ldots, gh_mg^{-1})
\]

\( G \) is said to have ample generics if for each \( m \geq 1 \) there is a comeagre orbit in \( G^m \) for this action.

Notice that since \( \Omega \) is countable, \( \text{Iso}(\Omega) \) is a Polish group in the permutation group topology. In section 5 (A) of Kechris and Rosendal [20] it is shown how the above extension property for finite rational metric spaces of bounded diameter 1 implies that \( \text{Iso}(\Omega) \) has ample generics and the results of section 5 (F) of [20] implies that a Polish group with ample generics has the Bergman property if and only if it has property (OB). Thus it is enough to show that \( \text{Iso}(\Omega) \) is finitely generated of bounded width over any non-empty open subset. But this follows from Lemma 6.7. \( \square \)
7 A dense Bergman subgroup of the unitary group

In the following $\ell_2$ will be the complex Hilbert space on the countable orthonormal basis $(e_i)_{i \in \mathbb{N}}$ and with usual inner product

$$\langle \sum a_i e_i \mid \sum b_i e_i \rangle = \sum a_i \bar{b}_i$$

We will also fix a countable algebraically closed field $\mathbb{Q} \subseteq \mathbb{Q} \subseteq \mathbb{C}$ closed under complex conjugation. In fact, it will only be essential that $\mathbb{Q}$ is closed under square root, and we could therefore work in some subfield of $\mathbb{R}$ too. This would give similar results for the orthogonal group of the real separable Hilbert space, but we shall be content with the above setting. Notice first that $\mathbb{Q}$ is dense in $\mathbb{C}$.

We let $V$ be the $\mathbb{Q}$-vector space with basis $(e_i)$ and notice that $V$ is a dense subset of $\ell_2$. The inner product restricts to an inner product on $V$ taking values in $\mathbb{Q}$, as $\mathbb{Q}$ is a field closed under complex conjugation. Since $\mathbb{Q}$ is algebraically closed, the norm of an element of $V$ also belongs to $\mathbb{Q}$. This will give us enough space to perform the usual tasks of Gram-Schmidt orthonormalisation etc.

Our first result is the following extension property.

**Lemma 7.1** Let $T$ be a $\mathbb{Q}$-linear isometry of $V$. Then $T$ extends to a unique unitary operator on $\ell_2$.

**Proof:** Since $V$ is dense in $\ell_2$ and $\ell_2$ is complete, any isometry of $V$ extends to a unique isometry of $\ell_2$. Hence $T$ extends to an isometry of $\ell_2$ preserving the origin. A simple argument shows that the extension is $\mathbb{C}$-linear. \(\square\)

So the group $U(V)$ of $\mathbb{Q}$-linear isometries of $V$ can be seen as a subgroup of $U(\ell_2)$. It will be useful represent unitary operators as infinite matrices with respect to the canonical basis $(e_i)_{i \in \mathbb{N}}$. Since we are only considering finite $\mathbb{Q}$-linear combinations, this means that any row and any column is eventually zero. The following operators in $U(V)$ are of particular interest.

**Definition 7.2** An operator $T \in U(V)$ is **finitary** if it is the identity outside of a finite-dimensional subspace of $V$.

So the finitary operators are those that are supported on a finite-dimensional subspace and hence can be represented as

$$\begin{bmatrix}
A & 0 \\
0 & I
\end{bmatrix}$$

where $I$ is the infinite identity matrix and $A$ some finite unitary matrix.

Clearly, the finitary operators form a subgroup of $U(V)$.

The unitary group $U(\ell_2)$ naturally comes with the **strong topology**, which makes it a Polish group. The strong topology is the weakest topology that makes all the maps

$$T \mapsto T(x)$$

continuous, where $x$ varies over the elements of $\ell_2$.

Similarly, as $V$ is countable, $U(V)$ is naturally isomorphic to a subgroup if the group $\text{Sym}(V)$ of all permutations of $V$, with the Polish permutation group topology. Moreover, $U(V)$ is easily seen to be closed in $\text{Sym}(V)$ and hence is a Polish group itself. Notice that this topology is stronger than the topology induced by $U(\ell_2)$.

We need that the Gram-Schmidt orthonormalisation procedure can be done in $V$. 

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Lemma 7.3 If \( W \) is any subspace of \( V \) and \( w_1, \ldots, w_n \) is an orthonormal set of vectors in \( W \), then there is an orthonormal basis of \( W \) extending \( \{w_1, \ldots, w_n\} \).

Proof: Let \( \{x_1, x_2, x_3, \ldots\} \) be a \( Q \)-vector space basis of \( W \) such that \( x_1 = w_1, \ldots, x_n = w_n \). Now, define inductively \( y_m, w_m \) by

\[
y_{m+1} = x_m - \sum_{i=1}^{m} \langle x_{m+1} \mid w_i \rangle w_i
\]

and notice that as \( \langle x_{m+1} \mid w_i \rangle \in Q \) also \( y_{m+1} \in V \). Now, put

\[
w_{m+1} = \frac{y_{m+1}}{\|y_{m+1}\|}
\]

and again as \( \|y_{m+1}\| \in Q \) (as \( Q \) is algebraically closed), \( w_{m+1} \in V \). So as usual, \( \{w_1, w_2, \ldots\} \) is an orthonormal basis of \( W \).

Lemma 7.4 Suppose \( S \) is a linear isometry between finite-dimensional spaces \( W_0 \) and \( W_1 \). Then \( S \) extends to a finitary operator \( \tilde{S} \) in \( U(V) \).

Proof: This is clear from Lemma 7.3. For choose an orthonormal basis \( v_1, \ldots, v_n \) for \( W_0 \) and find some sufficiently big \( i \) such that \( W_0, W_1 \subseteq [e_1, \ldots, e_i] \). Then we can extend \( v_1, \ldots, v_n \) and \( S(v_1), \ldots, S(v_n) \) respectively to orthonormal bases \( u_1, \ldots, u_i \) and \( w_1, \ldots, w_i \) of \( [e_1, \ldots, e_i] \).

Letting \( \tilde{S}(u_j) = w_j \) for \( j \leq i \) and \( \tilde{S}(e_j) = e_j \) for \( j > i \), we have the result. \( \Box \)

In particular, if \( \{v_1, \ldots, v_n\} \) and \( \{u_1, \ldots, u_n\} \) are orthonormal sets in \( V \), then there is a finitary operator \( F \) sending the ordered basis \( \{v_1, \ldots, v_n\} \) to the ordered basis \( \{u_1, \ldots, u_n\} \).

We recall the following fact (see, e.g., Proposition 2.2. in Kechris and Rosendal [20]):

Proposition 7.5 Let \( G \) be a Polish acting continuously on a Polish space \( X \). Then the following are equivalent for any \( x \in X \):

(i) The orbit \( G \cdot x \) is non-meager.
(ii) For every open neighbourhood \( V \subset G \) of the identity, \( V \cdot x \) is somewhere dense.

Proposition 7.6 \( U(V) \) has ample generics.

Proof: By abstract methods (see Truss [35] and Kechris and Rosendal [20]) it is enough to show that certain amalgamation properties are satisfied, but in our case an outright description of the comeagre orbits is not much longer, so we give this instead.

So fix an \( m \geq 1 \). We need to construct \( K_1, \ldots, K_m \in U(V) \) such that the conjugacy class of the \( m \)-tuple \( (K_1, \ldots, K_m) \) is comeagre in \( U(V) \).

By Lemma 7.4, we see that for any \( P_1, \ldots, P_m \in U(V) \) and \( v_1, \ldots, v_k \in V \) there are finitary operators \( H_1, \ldots, H_m \in U(V) \) such that for all \( t \leq m, s \leq k \), \( P_t(v_s) = H_t(v_s) \).

So list all \( m \)-tuples \( K = (K_1, \ldots, K_m) \) of unitary operators on some common finite-dimensional subspace \( \mathbb{V}_i = [e_1, \ldots, e_i] \) as

\[
K_1 = (K_1^1, \ldots, K_1^m), K_2 = (K_2^1, \ldots, K_2^m), \ldots
\]

and let \( a_n \) be the dimension of the space \( [e_1, \ldots, e_i] \) on which the \( K_1^n, \ldots, K_m^n \) act. Let also \( b_m = \sum_{j=1}^{a_n} a_j \). We suppose furthermore that each \( K_i \) is repeated infinitely often.
We can now paste these operators together as

$$M_t = \begin{bmatrix} K_t^1 & & \\ & K_t^2 & \\ & & K_t^3 \\ & & & \ddots & \\ & & & & K_t \\ \\ & & & & \end{bmatrix}$$

In other words, $M_1, \ldots, M_m$ are disjoint sums of unitary operators on finite-dimensional spaces of the form $[e_1, \ldots, e_k]$ such that each conjugacy type of $m$-tuples appears infinitely often.

To see that the conjugacy type of $(M_1, \ldots, M_m)$ is comeagre in $\mathcal{U}(\mathbb{V})^m$, we show first that it is dense and non-meagre. Thus, by Proposition 7.5, it is enough to show that it is dense and that for every $l \in \mathbb{N}$ the set

$$\mathcal{A} = \{(T^{-1}M_1T, \ldots, T^{-1}M_mT) \mid T = \begin{bmatrix} I_l & 0 \\ 0 & A \end{bmatrix} \text{ for some } A\}$$

is somewhere dense in $\mathcal{U}(\mathbb{V})^m$, where $I_l$ is the $l \times l$ identity matrix. To see this latter, find first some $b_l \geq l$. We claim that $\mathcal{A}$ is dense in the open set of $(P_1, \ldots, P_m)$ such that for every $t \leq m$,

$$P_t = \begin{bmatrix} K_t^1 & & \\ & \ddots & \\ & & K_t^i \\ & & & \ddots & \\ & & & & K_t \\ & & & & \end{bmatrix} = \begin{bmatrix} M_t \upharpoonright_{V_{b_l}} & A_t \end{bmatrix}$$

for some $A_t$. For if $(P_1, \ldots, P_m)$ is above, then the tuple can be approximated arbitrarily well by a tuple of finitary operators. So we can suppose that $P_1, \ldots, P_m$ are finitary themselves. Assume that we want to approximate $P_1, \ldots, P_m$ on $V_k = [e_1, \ldots, e_k]$, where $k > b_l$ is such that $P_t(e_p) = e_p$ for all $t \leq m$ and $p \geq k$. Then we can find $j > i$ such that $k = b_l + a_j$ and

$$P_t = \begin{bmatrix} M_t \upharpoonright_{V_{b_l}} & K_t^j \\ & I \end{bmatrix}$$

Find a unitary operator $T$ such that $T \upharpoonright [e_1, \ldots, e_{b_l}] = I_{b_l}$ and $T$ sends the ordered basis $\{e_{b_l+1}, \ldots, e_{b_l+a_j}\}$ to $\{e_{b_l+1}, \ldots, e_{b_l+a_j}\}$. Then

$$TM_tT^{-1} = T \begin{bmatrix} K_t^1 & & \\ & K_t^2 & \\ & & \ddots \\ & & & \ddots & \\ & & & & K_t \\ & & & & \end{bmatrix} T^{-1} = \begin{bmatrix} M_t \upharpoonright_{V_{b_l}} & K_t^j \\ & B_t \end{bmatrix}$$

for some $B_t$. Thus $TM_tT^{-1}$ agrees with $P_t$ on $V_k = [e_1, \ldots, e_{b_l+a_j}]$ for every $t = 1, \ldots, m$.

A similar argument shows that the conjugacy class of $(K_1, \ldots, K_m)$ is dense. But in fact, this also follows from the next proposition. Thus, as there is a dense orbit, the diagonal conjugacy action of $\mathcal{U}(\mathbb{V})$ on $\mathcal{U}(\mathbb{V})^m$ is generically ergodic, i.e. any invariant Borel set is either meagre or comeagre. So, as the conjugacy class of $(K_1, \ldots, K_m)$ is non-meagre, it must be comeagre.

**Definition 7.7** A Polish group $G$ is said to have a cyclically dense conjugacy class if there are elements $g, h \in G$ such that $\{g^n h g^{-n}\}_{n \in \mathbb{Z}}$ is dense in $G$. We say that $G$ has an ample cyclically dense conjugacy class if there is a $g \in G$ and some infinite sequence $(h_k)_{k \in \mathbb{N}}$ such that the set $\{(g^n h_k g^{-n})_k \}_{n \in \mathbb{Z}}$ is dense in $G^\mathbb{N}$. 

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Proposition 7.8 $\mathcal{U}(\mathcal{V})$ has an ample cyclically dense conjugacy class.

Proof: Notice first that if $G$ is a Polish group and for some $g \in G$ there is some $(h_1, \ldots, h_m) \in G^m$ such that the set
$$\{(g^n h_1 g^{-n}, \ldots, g^n h_m g^{-n})\}_{n \in \mathbb{Z}}$$
is dense in $G^m$, then set of such $(h_1, \ldots, h_m)$ is certainly dense in $G^m$. Moreover, since it is also $G_\delta$, it is comeagre. Thus, if for each $m \in \mathbb{N}$ there is such $(h_1, \ldots, h_m) \in G^m$, then there is an infinite sequence $(h_k)_k \in G^\mathbb{N}$ such that $\{(g^n h_k g^{-n})\}_{n \in \mathbb{Z}}$ is dense in $G^\mathbb{N}$.

Therefore, we only need to find some unitary operator $S$ that fills the rôle of $g$. For this we will consider instead a biinfinite orthonormal basis $(e_i)_{i \in \mathbb{Z}}$ of $\mathcal{V}$ and let $S$ be the bilateral shift on this basis. Fix also some dimension $m$.

We can now take $H_t = I + M_t$, where we let the identity $I$ act on $\mathcal{V}_- = [e_{-2}, e_{-1}, e_0]$ and let $M_t$ be as in the proof of Proposition 7.8 defined on $\mathcal{V}_+ = [e_1, e_2, e_3, \ldots]$. One easily sees that $(S^{-n} H_1 S^n, \ldots, S^{-n} H_m S^n)_{n \in \mathbb{N}}$ is dense in $\mathcal{U}(\mathcal{V})^m$. For suppose we wish to approximate some $(P_1, \ldots, P_m)$, which we can suppose are finitary, on some space $\mathcal{W} = [e_{-n}, \ldots, e_n]$. Since each $P_t$ is finitary, we can find $k > n$ such that each $P_t$ is on the form
$$P_t = \begin{pmatrix} I & A_t \\ A_t^* & I \end{pmatrix}$$
where $A_t$ is a $(2k + 1) \times (2k + 1)$ matrix acting on $[e_{-k}, \ldots, e_k]$. Now find some $j$ such that $K_t^j = A_t$ for each $t \leq m$. Then we see that $H_t [e_{b_{j-1} + 1}, \ldots, e_{b_j}]$ and thus
$$S^{-b_{j-1} - 1 - k} H_t S^{b_{j-1} + 1 + k} [e_{-k}, \ldots, e_k] = A_t$$
Thus
$$(S^{-b_{j-1} - 1 - k} H_1 S^{b_{j-1} + 1 + k}, \ldots, S^{-b_{j-1} - 1 - k} H_m S^{b_{j-1} + 1 + k})$$
agrees with $(P_1, \ldots, P_m)$ on $[e_{-k}, \ldots, e_k] \supseteq \mathcal{W}$.

It follows from the results of Kechris and Rosendal (see Proposition 5.16 [20]) that a Polish group with ample generics has property (OB) if and only if it is Bergman.

Proposition 7.9 $\mathcal{U}(\mathcal{V})$ has the Bergman property.

Proof: Since $\mathcal{U}(\mathcal{V})$ has ample generics, it is enough to show that it has property (OB). We show that $\mathcal{U}(\mathcal{V})$ is finitely generated of bounded width over any open neighbourhood $U$ of the identity. So suppose $k$ is given such that $U$ contains all operators fixing $\mathcal{W}_0 = [e_1, \ldots, e_k]$ pointwise. Find an operator $M$ such that $M[\mathcal{W}_0] = \mathcal{W}_1 = [e_{k+1}, \ldots, e_{2k}]$ and notice that $MUM^{-1}$ contains all operators fixing $\mathcal{W}$ pointwise. Let now $T \in \mathcal{U}(\mathcal{V})$ and find a finite dimensional space $\mathbb{H}_0 \subseteq (\mathcal{W}_0 \oplus \mathcal{W}_1)^\perp$ such that $T[\mathcal{W}_0] \subseteq \mathcal{W}_0 \oplus \mathcal{W}_1 + \mathbb{H}_0$. Let $R_0 \in U$ send $\mathcal{W}_1$ into $(\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathbb{H}_0)^\perp$ and fix $\mathcal{W}_0 \oplus \mathbb{H}_0$ pointwise. Thus
$$R_0 T[\mathcal{W}_0] \subseteq R_0 [\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathbb{H}_0] \subseteq \mathcal{W}_0 \oplus R_0 [\mathcal{W}_1] \oplus \mathbb{H}_0 \subseteq \mathcal{W}_1$$
We can therefore find some $R_1 \in MUM^{-1}$ such that $R_1 R_0 T$ is the identity on $\mathcal{W}_0$ and $R_1$ fixes $\mathcal{W}_1$ pointwise. Hence, $R_1 R_0 T \in U$ and $T \in R_0^{-1} R_1^{-1} U \subseteq U^{-1} M U^{-1} M^{-1} U$. Thus, $\mathcal{U}(\mathcal{V}) = U^{-1} M U^{-1} M^{-1} U$. \qed

Lemma 7.10 $\mathcal{U}(\mathcal{V})$ is dense in $\mathcal{U}(\ell_2)$.
are normalised vectors such that \( \|v_i\| = 1 \).
By the continuity of the inner product, we can find \( v \in V \) such that \( \langle v, u_i \rangle = \delta \) for every \( i \), then if \( v \) and \( u \) are the orthonormal bases obtained by applying the Gram-Schmidt orthonormalisation process to \( v \) and \( u \), respectively, we still have \( \|v_i - v\| < \epsilon/2 \) and \( \|T(v_i) - Tu_i\| < \epsilon/2 \) for every \( i \). Thus choose \( v, u \in V \) as above and pick some \( R \in \mathcal{U}(\mathcal{V}) \) sending the ordered basis \( v, \ldots, v_n \) to the ordered basis \( u, \ldots, u_n \). Then

\[
\|T(v_i) - R(x_i)\| \leq \|T(v_i) - R(v_i)\| + \|R(v_i) - R(x_i)\| \leq \epsilon
\]

and hence approximating \( T \) on \( x_1, \ldots, x_n \).

So let us sum up the results so far.

**Theorem 7.11** \( \mathcal{U}(\mathcal{V}) \) has ample generics, an ample cyclically dense conjugacy class and the Bergman property. Thus, \( \mathcal{U}(\ell_2) \) has property (OB) and an ample cyclically dense conjugacy class.

Christopher Atkin [11] has actually proved something a lot stronger for the full unitary group, \( \mathcal{U}(\ell_2) \), namely that it has property (OB) in the norm topology.

We should mention that the existence of ample generics in a Polish group has quite remarkable consequences for the structure of the group, for example, it implies that any homomorphism from it into a separable group is automatically continuous, and the group cannot be covered by countably many non-open cosets (see Hodges, Hodkinson, Lascar and Shelah [18], Kechris and Rosendal [20]).

**Theorem 7.12** Let \( \mathcal{U}(\ell_2) \) act continuously by Hölder maps on a complete metric space \((X,d)\). If the bilateral shift \( S \) induces a relatively compact orbit on \( X \), then \( \mathcal{U}(\ell_2) \) fixes a point of \( X \).

**Proof**: First, we can evidently suppose that \( X \) is in fact separable and thus Polish. So \( \mathcal{U}(\ell_2) \) acts continuously on \( K(X) \). Moreover, if \( g \in G \) is Hölder (\( \alpha \)) with constant \( c \) on \((X,d)\), then \( g \) is Hölder (\( \alpha \)) with constant \( c \) on \((K(X),d_H)\), where \( d_H \) is the Hausdorff metric. For

\[
d_H(K, L) = \max \left( \sup_{x \in K} d(x, L), \sup_{x \in L} d(K, x) \right)
\]

But

\[
\sup_{x \in K, y \in L} d(x, gy) = \sup_{x \in K, y \in L} c \cdot d(x, y) = c \cdot \left( \sup_{x \in K, y \in L} d(x, y) \right)^\alpha
\]

and thus

\[
d_H(gK, gL) = \max \left( \sup_{x \in K} d(x, gL), \sup_{y \in L} d(gK, y) \right)
\]

\[
\leq \max \left( c \cdot \left( \sup_{x \in K} d(x, y) \right)^\alpha, c \cdot \left( \sup_{y \in L} d(x, y) \right)^\alpha \right)
\]

\[
= c \cdot \max \left( \sup_{x \in K} d(x, y), \sup_{y \in L} d(x, y) \right)^\alpha
\]

Let now \( \mathcal{O} = \{S^n \cdot x\}_{n \in \mathbb{Z}} \) be compact and find by the proof of Proposition [18] some \( T \in \mathcal{U}(\ell_2) \) such that \( \{S^n TS^{-n} \}_{n \in \mathbb{Z}} \) is dense in \( \mathcal{U}(\ell_2) \). Then

\[
d_H(T \cdot \mathcal{O}, \mathcal{O}) = d_H(TS^{-n} \mathcal{O}, S^{-n} \mathcal{O})
\]

\[
\leq c_n d_H(S^nTS^{-n} \mathcal{O}, S^n S^{-n} \mathcal{O})^{\alpha_n}
\]

\[
\leq c_n d_H(S^n TS^{-n} \mathcal{O}, \mathcal{O})^{\alpha_n}
\]

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where $S^n$ is Hölder($\alpha_n$) with constant $c_n$. Using now that $U(\ell_2)$ has property (OB), we find a universal $N$ such that we can choose all the $\alpha_n \in [1/N, N]$ and $c_n \leq N$. Picking a subsequence $n_i$ such that $S^{n_i}$TS$^{-n_i} \rightarrow I$, we see that
\[ c_{n_i} d_H(S^{n_i}TS^{-n_i}\mathcal{O}, \mathcal{O})^{n_i} \rightarrow 0 \]
and thus $d_H(T\cdot \mathcal{O}, \mathcal{O}) = 0$. Hence, $\mathcal{O}$ is both $S$ and $T$ invariant and thus also $U(\ell_2)$-invariant. Moreover, as $\mathcal{O}$ is compact and $U(\ell_2)$ is extremely amenable (this is a result of Gromov and Milman [14]), $U(\ell_2)$ fixes a point of $\mathcal{O}$. \hfill \Box

8 Groups of homeomorphisms

Spheres

**Theorem 8.1** Let $\text{Hom}(S^m)$ be the group of homeomorphisms of the $m$-dimensional sphere with the topology of uniform convergence. Then $\text{Hom}(S^m)$ has property (OB).

**Proof:** Let $d$ be the euclidean metric on $\mathbb{R}^{m+1}$ and $d_\infty$ the supremum metric on $\text{Hom}(S^m)$, $d_\infty(g, f) = \sup_{x \in S^m} d(gx, fx)$. We show that $\text{Hom}(S^m)$ is finitely generated of bounded width over any non-empty open subset $U$. So pick an $\epsilon_0 > 0$ such that
\[ V = \{ g \in \text{Hom}(S^m) \mid d_\infty(g, id) < 3\epsilon_0 \} \subseteq UU^{-1} \]
Let also $x_0 = (1, 0, 0, \ldots, 0) \in S^m$. Then for any $\epsilon_0 > \delta > 0$ there is some homeomorphism $\phi_\delta$ of $S^m$ such that $\phi_\delta(B_{\epsilon_0}(x_0)) = B_\delta(x_0)$ and $d_\infty(\phi_\delta, id) < \epsilon_0$. Moreover, there is an involution homeomorphism $\psi$ of $S^m$ fixing $\partial B_{\epsilon_0}(x_0)$ pointwise, while switching $int B_{\epsilon_0}(x_0)$ with $ext B_{\epsilon_0}(x_0) = S^m \setminus B_{\epsilon_0}(x_0)$. Finally, let $\iota$ be the orientation inverting involution
\[ \iota(x_0, x_1, x_2, \ldots, x_m) = (x_0, -x_1, x_2, \ldots, x_m) \]
We notice that $SO(m + 1)$ is a compact subgroup of $\text{Hom}(S^m)$, so $\bigcup_n (SO(m + 1) \cap VV^{-1})^n$ is an open subgroup of $SO(m + 1)$. But this latter is connected and compact, so $SO(m + 1) \subseteq (VV^{-1})^k$ for some $k$.

**Claim 8.2** $\text{Hom}(S^m) \subseteq (VV^{-1})^k \{ \iota, id \} V^2 \psi V \psi V^{-1}$

To see this, let $g \in \text{Hom}(S^m)$ and find $f \in SO(m + 1)$ such that $fg(x_0) = x_0$. Then put
\[ \hat{f} = \begin{cases} f & \text{if } fg \text{ is orientation preserving} \\ \iota f & \text{if } fg \text{ is orientation preserving} \end{cases} \]
It follows that $\hat{f}g$ preserves the orientation and fixes $x_0$. Therefore, by Lemma 3.1. of Glasner and Weiss [12], which itself relies on the proof of the annulus conjecture, there is some $\epsilon_0 > \delta > 0$ and a homeomorphism $h$ of $S^m$ such that
\[ \begin{align*}
\forall x \in S^m \quad & d(x, x_0) > \epsilon_0 \rightarrow hx = \hat{f}gx \\
\forall x \in S^m \quad & d(x, x_0) < \delta \rightarrow hx = x
\end{align*} \tag{23} \]
In particular,
\[ d_\infty(h, \hat{f}g) = \sup_{x \in S^m} d(hx, \hat{f}gx) = \sup_{x \in B_{\epsilon_0}(x_0)} d(hx, \hat{f}gx) < 3\epsilon_0 \]
So $fgh^{-1} \in V$ and $g \in \hat{f}^{-1}Vh$. Thus,

$$\phi_{\delta}^{-1}h\phi_{\delta} | B_{x_0}(x_0) = id$$

and

$$\psi\phi_{\delta}^{-1}h\phi_{\delta}\psi | \text{ext}B_{x_0}(x_0) = id$$

In particular, $d_{\infty}(\psi\phi_{\delta}^{-1}h\phi_{\delta}\psi, id) < 3\epsilon_0$, so $\psi\phi_{\delta}^{-1}h\phi_{\delta}\psi \in V$. Therefore, $h \in \phi_{\delta}V\psi\phi_{\delta}^{-1}$ and

$$g \in \hat{f}^{-1}Vh$$

$$\subseteq \hat{f}^{-1}V\phi_{\delta}V\psi\phi_{\delta}^{-1}$$

$$\subseteq (VV^{-1})^k \cdot \{id, \iota\} \cdot V^2\psi V^{-1}$$

(24)

The Hilbert cube

Consider now the Hilbert cube $Q = [0,1]^\mathbb{N}$ and its homeomorphism group $\text{Hom}(Q)$ equipped with the topology of uniform convergence. We let $d$ be the metric on $Q$ given by

$$d((x_n), (y_n)) = \sum_{n \in \mathbb{N}} \frac{|x_n - y_n|}{2^{n+1}}$$

and $d_{\infty}$ the supremum metric on $\text{Hom}(Q)$ given by

$$d_{\infty}(f, g) = \sup_{\bar{x} \in Q} d(f(\bar{x}), g(\bar{x}))$$

which is right invariant.

**Theorem 8.3** $\text{Hom}(Q)$, with the topology of uniform convergence, has property (OB).

**Proof** : Fix some open neighbourhood $V$ of the identity in $\text{Hom}(Q)$, which we can suppose is of the form

$$V = \{g \in \text{Hom}(Q) \mid d_{\infty}(g, id) < 2\epsilon\}$$

for some $\epsilon > 0$. Thus, if $n$ is sufficiently large such that $2^{-n} < \epsilon$, then for any $f \in \text{Hom}(Q)$ the does not change the first $n$ coordinates of any $\bar{x} \in Q$, i.e.,

$$f((x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots)) = (x_0, x_1, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)$$

for all $\bar{x} \in Q$, we have $f \in V$.

**Claim 8.4** If $\frac{1}{k} < \epsilon$, then there is a finite set $F \subseteq \text{Hom}(Q)$ such that for every $\bar{x} \in Q$, $\bar{0} = (0, 0, 0, \ldots) \in FV^{k+1} \cdot \bar{x}$.

**Proof of claim** : Let $\frac{1}{k} < \epsilon$. For each $s \in \{0, \frac{1}{k}, 1\}^n$ let $\bar{z}'_s = (s_0, s_1, \ldots, s_{n-1}, 0, 0, \ldots)$. As $Q$ is homogeneous, $\text{Hom}(Q)$ acts transitively on $Q$ (see van Mill, Theorem 6.1.6. [24]), and we can therefore find some $h_s \in \text{Hom}(Q)$ such that $h_s(\bar{z}'_s) = \bar{0}$ for each $s$. Let $F = \{h_s \mid s \in \{0, \frac{1}{k}, 1\}^n\}$. So it is enough to show that $\exists s \bar{z}'_s \in V^{k+1} \cdot \bar{x}$. So first use the homogeneity of $Q$ to adjust the tail $(x_n, x_{n+1}, \ldots)$ of $\bar{x}$ by some element of $V$ to become $(0, 0, \ldots)$. This can be done since a homeomorphism leaving the first $n$ coordinates invariant belongs to $V$. Now we can subsequently adjust each of the first $n$ coordinates (leaving the tail invariant) to be equal.
to either 0, $\frac{1}{2}$ or 1. For this operation it is enough to use a product of at most $k$ elements of $V$.

Now, it follows from Brouwer’s fixed point Theorem that any homeomorphism of $Q$ fixes a point. Thus, up to a conjugate by an element of the set $FV^{k+1}$ from Claim 8.4, any homeomorphism of $Q$ fixes $\bar{0}$. As we wish to show that $\text{Hom}(Q)$ is finitely generated of bounded width over $V$, we can suppose that any homeomorphism fixes $\bar{0}$.

Claim 8.5 (Glasner and Weiss [12]) If $f \in \text{Hom}(Q)$ fixes $\bar{0}$, then there is some $\delta > 0$ and $g \in \text{Hom}(Q)$ such that $d_{\infty}(g, f) < \epsilon$ and $g |_{B_{1}(\bar{0})} = id$.

**Proof of claim:** Pick $\delta > 0$ sufficiently small such that $\inf_{x \in B_{1}(\bar{0})} d(f(\bar{x}), \bar{x}) < \epsilon$. As both $\partial B_{1}(\bar{0})$ and $\partial f^{-1} B_{1}(\bar{0})$ are $Z$-sets, we can extend the homeomorphism $f^{-1} : \partial f^{-1} B_{1}(\bar{0}) \rightarrow \partial B_{1}(\bar{0})$ to a homeomorphism $h \in \text{Hom}(Q)$ satisfying $d_{\infty}(h, id) < \epsilon$ (see van Mill, Theorem 6.4.6. [22]). Thus, $d_{\infty}(h, f) = d_{\infty}(h, id) < \epsilon$ and we can let

$$g(\bar{y}) = \begin{cases} \bar{y} & \text{if } \bar{y} \in B_{1}(\bar{0}) \\ h\bar{f}(\bar{y}) & \text{otherwise} \end{cases}$$

Claim 8.6 For any $g \in \text{Hom}(Q)$ and $0 < \delta < \epsilon$ such that $g |_{B_{1}(\bar{0})} = id$, there is $h \in V^{2}$ such that $h^{-1}gh |_{[0, \epsilon]^{n+1} \times [0, 1]^{n}} = id$.

**Proof of claim:** Notice that $[0, \epsilon]^{n+1+l} \times [0, 1]^{n} \subseteq B_{1}(\bar{0})$ for some $l > 0$. Moreover, it is not hard to see that $[0, \epsilon]^{l+1}$ is homeomorphic to $[0, \delta]^{l+1}$ by some function $a$, which is a homeomorphism of $[0, 1]^{l+1}$. Thus,

$$h_{0} = \text{id}_{[0, 1]^{n}} \otimes a \otimes \text{id}_{[0, 1]^{n}} : Q \rightarrow Q$$

belongs to $V$ and sends

$$[0, 1]^{n} \times ([0, \epsilon] \times [0, 1]) \times [0, 1]^{n}$$

to

$$[0, 1]^{n} \times [0, \delta]^{l+1} \times [0, 1]^{n}.$$  

Now, let $h_{1} : Q \rightarrow Q$ be a homeomorphism that moves the set $[0, \epsilon]^{n} \times [0, 1]^{n} \subseteq B_{1}(\bar{0})$ to $[0, \delta]^{n} \times [0, 1]^{n}$, preserves all coordinates $\geq n$ and $d_{\infty}(h_{1}, id) < \epsilon$. Then $h_{0}, h_{1} \in V$ and $h = h_{1}h_{0}$ moves $[0, \epsilon]^{n+1} \times [0, 1]^{n}$ to $[0, \delta]^{n+1+l} \times [0, 1]^{n}$.

Now, let $i \in \text{Hom}([0, 1])$ be an involution homeomorphism that fixes $\epsilon$ and switches 0 and 1. Define $i \in \text{Hom}(Q)$ by

$$i(x_{0}, x_{1}, \ldots, x_{n}, x_{n+1}, \ldots) = (i(x_{0}), i(x_{1}), \ldots, i(x_{n}), x_{n+1}, x_{n+2}, \ldots).$$

Then $i$ interchanges $[0, \epsilon]^{n+1} \times [0, 1]^{n}$ and $[\epsilon, 1]^{n+1} \times [0, 1]^{n}$.

We can now conclude our result. For suppose $f \in \text{Hom}(Q)$. Then, up to a conjugate by an element of $FV^{k+1}$ we can suppose that $f$ fixes $\bar{0}$. By Claim 8.6, we can find $g \in Vf$ and $0 < \delta < \epsilon$ such that $g |_{B_{1}(\bar{0})} = id$. So pick by Claim 8.6 some $h \in V^{2}$ such that $h^{-1}gh |_{[0, \epsilon]^{n+1} \times [0, 1]^{n}} = id$. But then $ih^{-1}ghi |_{[\epsilon, 1]^{n+1} \times [0, 1]^{n}} = id$ and hence $ih^{-1}ghi \in V$. All in all, this shows that $f \in (FV^{k+1})^{-1}V^{-1}V^{2}V^{2}V^{-1}FV^{k+1}$.

Since any Polish group is a closed subgroup of $\text{Hom}(Q)$ (see Uspenski [37] or the exposition in Kechris [19]), we have

Corollary 8.7 Any Polish group is topologically isomorphic to a closed subgroup of a Polish group with property (OB).
9 Actions on trees

Comeagre conjugacy classes

We give first a simple proof of a result of Dugald Macpherson and Simon Thomas. We will actually prove a result stronger than theirs, for which we need some basic computations by M. Culler and J.W. Morgan \[9\]. Note first that if a group $G$ acts by isometries on an $\mathbb{R}$-tree $T$, then each $g \in G$ has associated a characteristic non-empty subtree $T_g$ of $T$, which either is the set of points fixed by $g$ (in which case $g$ is called \textit{elliptic}) or a line on which $g$ acts by translation (in which case $g$ is called \textit{hyperbolic}). We let $\|g\| = \inf (r \in \mathbb{R}_+ \mid \exists x \in T \ d(x, g \cdot x) = r)$. This infimum is in fact attained as shown in \[9\]. Thus, $g$ is elliptic if and only if $\|g\| = 0$.

The interested reader should consult the very readable article by Culler and Morgan for more information on the general theory of group actions on $\mathbb{R}$-trees.

Lemma 9.1 \[9\] Suppose $g$ and $h$ are isometries of an $\mathbb{R}$-tree $T$. If $T_g \cap T_h$ is empty, then

$$\|gh\| = \|g\| + \|h\| + 2 \text{dist}(T_g, T_h)$$

Lemma 9.2 \[9\] Let $g$ and $h$ be hyperbolic isometries of an $\mathbb{R}$-tree $T$ such that $T_g \cap T_h \neq \emptyset$. Then

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

From Lemma 9.1 follows the following important special case.

Theorem 9.3 (Serre’s Lemma) Suppose $g$, $h$ and $gh$ are elliptic isometries of an $\mathbb{R}$-tree $T$. Then $T_g \cap T_h \neq \emptyset$.

Theorem 9.4 (D. Macpherson and S. Thomas for combinatorial trees \[22\].) Suppose $G$ is a Polish group with a comeagre conjugacy class $C$ acting by isometries on an $\mathbb{R}$-tree $T$. Then every element of $G$ is elliptic.

\textit{Proof}: We claim that $\|\cdot\|$ is constantly 0 on $C$. Assume towards a contradiction that this is not the case. Notice first that $\|\cdot\|$ is conjugacy invariant, so constant on $C$. Pick $g, h \in C$ such that also $gh, gh^{-1} \in C$. By Lemma 9.1 if $T_g \cap T_h = \emptyset$ then

$$\|gh\| = \|g\| + \|h\| + 2 \text{dist}(T_g, T_h) > \|g\|$$

contradicting that $\|\cdot\|$ is constant on $C$. So $T_g \cap T_h \neq \emptyset$, whence by Lemma 9.2

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\| > \|g\|$$

again contradicting that $\|\cdot\|$ is constant on $C$ and thus proving the claim.

Assume now that $f$ is an arbitrary element of $G$ and pick $g, h \in C$ such that $f = hg$. Then

$$C^{-1} \cap C \cap hC^{-1} \cap gC^{-1} \cap fC^{-1} \neq \emptyset$$

so we can find $k_0, k_1, k_2, k_3 \in C$ with $k_0 = h k_1^{-1} = g k_2^{-1} = f k_3^{-1}$, $k_0^{-1}, k_2, k_2^{-1} \in C$, i.e., $k_0 k_1 = h$, $k_0 k_2 = g$ and $k_0 k_3 = f = k_0 k_1 k_0 k_2$.

Notice that $k_0, k_1, k_0 k_1 \in C$, $k_1, k_0 k_2, k_1 k_0 k_2 = k_3 \in C$ and $k_0^{-1}, k_0 k_2, k_0^{-1} k_0 k_2 = k_2 \in C$, so applying Serre’s Lemma to each of these three situations, we have $T_{k_0} \cap T_{k_1} \neq \emptyset$, $T_{k_1} \cap T_{k_0 k_2} \neq \emptyset$ and $T_{k_0} \cap T_{k_0 k_3} \neq \emptyset$. The three trees $T_{k_0}$, $T_{k_1}$ and $T_{k_0 k_2}$ therefore intersect pairwise, and thus there is some $x$ in their common intersection. But then clearly $f \cdot x = k_0 k_1 k_0 k_2 \cdot x = x$, whence $f$ is elliptic. \qed

\number{29}
Notice that if a group $G$ acts by automorphisms on a tree $T$, then the action extends to the tree $T'$ obtained from $T$ by adding a midpoint on every edge. Moreover, the action on $T'$ is without inversion, i.e., there are no vertices $a \neq b$ in $T'$ such that $\{a, b\}$ is an edge and $g \cdot a = b, g \cdot b = a$ for some $g \in G$. We also see that $G$ fixes a vertex if $T'$ if and only if $G$ fixes either a vertex or an edge of $T$. Thus, to see that a group has property (FA) it is enough to show that any action without inversion on a tree has a fixed vertex.

The proof of Theorem 9.9 translates word for word into the corresponding proof for $\Lambda$-trees (a generalisation of $\mathbb{R}$-trees with a metric taking values in an arbitrary ordered abelian group). The only thing that has to be checked is that the appropriate lemmas are true also in this setting. Well, here they are. In the following, $\Lambda$ is a fixed ordered abelian group and $(X, \rho)$ a given $\Lambda$-tree. We define the norm of elements of $G$ in the same manner as for actions on $\mathbb{R}$-trees.

**Lemma 9.5** (Lemma 2.1.11 in [7]) Suppose $X_1, \ldots, X_n$ are subtrees of $X$ such that $X_i \cap X_j \neq \emptyset$ for all $i, j$. Then $X_1 \cap \ldots \cap X_n \neq \emptyset$.

**Lemma 9.6** (Lemma 3.2.2 in [7]) Suppose $g$ and $h$ are isometries of $(X, \rho)$, which are not inversions, such that $T_g \cap T_h = \emptyset$. Then

$$\|gh\| = \|g\| + \|h\| + 2\text{dist}(T_g, T_h)$$

**Lemma 9.7** (Lemmas 3.2.3 and 3.3.1 in [7]) Suppose $g$ and $h$ are hyperbolic isometries of $(X, \rho)$ such that $T_g \cap T_h \neq \emptyset$. Then

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

From Lemma 9.6 we have again a version of Serre’s Lemma.

**Lemma 9.8** Suppose $g$, $h$ and $gh$ are elliptic isometries of a $\Lambda$-tree $(X, \rho)$. Then $T_g \cap T_h \neq \emptyset$.

**Theorem 9.9** Suppose $G$ is a Polish group with a comeagre conjugacy class $C$ acting by isometries and without inversion on a $\Lambda$-tree $(X, \rho)$. Then every element of $G$ is elliptic.

**Dense conjugacy classes**

**Lemma 9.10** Suppose a topological group $G$ acts continuously and without inversion on a tree $T$, i.e., such that the stabilisers of vertices in $T$ are open in $G$. Then $\|\cdot\| : G \to \mathbb{N}$ is continuous.

**Proof:** Suppose first that $\|g\| = 0$. Then for some $a \in T$, $g \cdot a = a$, i.e. $g \in G_a$ and $G_a$ is an open neighbourhood of $g$ on which $\|\cdot\|$ is constantly 0.

Now suppose $\|g\| = n > 0$. Then by a theorem of Tits (Proposition 24 in [29]) there is a line $\ell_g = (a_i \mid i \in \mathbb{Z})$ in $T$ such that $g \cdot a_i = a_{i+n}$ for all $i$. Now, if $h \in G$ is elliptic, then for any $a \in T$, $h$ fixes the midpoint of the geodesic from $a$ to $h \cdot a$. So if $h \cdot a_0 = g \cdot a_0 = a_n$ then $n = 2m, m > 0$ and $h \cdot a_m = a_m \neq g \cdot a_m$, by uniqueness of the geodesic. Hence if

$$U = \{f \in G \mid f \cdot a_0 = g \cdot a_0, \ldots, f \cdot a_n = g \cdot a_n\}$$

then $U$ is an open neighbourhood of $g$ containing only hyperbolic points of norm $\leq n$.

Moreover, if $h$ is hyperbolic of norm $k < n$, then $\ell_h$ would contain exactly the $k + 1$ midpoints of the arc $a_0, a_1, \ldots, a_n$ from $a_0$ to $h \cdot a_0 = a_n$. So for some $0 < i < n$,

$$\text{dist}_T(a_i, h \cdot a_i) = k \neq n$$

which is a contradiction. So $U$ only contains hyperbolic points of norm $n$. \qed
Proposition 9.11 Suppose $G$ is a Polish group with a dense conjugacy class, which is not the union of a countable sequence of proper open subgroups. Then whenever $G$ acts continuously and without inversion on a tree $T$, it fixes a vertex of $T$. In other words, $G$ has property (topFA).

Proof: Notice first that $||·||$ is conjugacy invariant and continuous, so must be constantly 0 on $G$. I.e. every element of $G$ is elliptic. So if $G$ does not fix a vertex, it fixes an end $\alpha = (a_0, a_1, \ldots) \subseteq T$ (Tits, Exercise 2, page 66 [29]). But then $G = \bigcup_n G_{(a_n, a_{n+1}, \ldots)}$, where $G_{(a_n, a_{n+1}, \ldots)}$ is the pointwise stabiliser of the set $\{a_n, a_{n+1}, \ldots\}$. Since these subgroups are closed, almost all of them must be open, as $G$ satisfies Baire’s category theorem. And as $G$ is not the union of a countable chain of proper open subgroups, $G = G_{(a_n, a_{n+1}, \ldots)}$ for some $N$, contradicting that $G$ did not fix a vertex. $\blacksquare$

S. Solecki [31] has shown that the isometry group of the rational Urysohn metric space, $\text{Iso}(\mathbb{U}_\mathbb{Q})$, with the permutation group topology, has ample generics and a cyclically dense conjugacy class. Moreover, in Kechris and Rosendal [20] it is shown that Polish groups with ample generics and a cyclically dense conjugacy class cannot be written as the union of a countable chain of proper subgroups. So this means that $\text{Iso}(\mathbb{U}_\mathbb{Q})$ has property (FA). Moreover, V.G. Pestov [26] shows that $\text{Iso}(\mathbb{U})$ has no non-trivial continuous representations by isometries in a reflexive Banach space, so in particular it has property (FH). However, this does not solve the corresponding problem for $\text{Iso}(\mathbb{U}_\mathbb{Q})$.

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