A New Approach Towards the Golomb-Welch Conjecture

Peter Horak∗, Otokar Grošek†
∗University of Washington, Tacoma
†Slovak University of Technology, Slovakia

This paper is dedicated to the 75-th birthday of Tomaš Janovic

May 1, 2014

Abstract

The Golomb-Welch conjecture deals with the existence of perfect $e$-error correcting Lee codes of word length $n$, $PL(n,e)$ codes. Although there are many papers on the topic, the conjecture is still far from being solved. In this paper we initiate the study of an invariant connected to abelian groups that enables us to reformulate the conjecture, and then to prove the non-existence of linear $PL(n,2)$ codes for $n \leq 12$. Using this new approach we also construct the first quasi-perfect Lee codes for dimension $n = 3$, and show that, for fixed $n$, there are only finitely many such codes over $\mathbb{Z}$.

It turns out that the Lee metric is more suitable for some applications than the most frequently used Hamming metric. The Lee metric has been used for the first time in [16] and [24] when dealing with transmission of signals over noisy channels. Since then several types of codes in Lee metric have been studied. For example, the perfect error-correcting Lee codes introduced in [8], the negacyclic codes introduced by Berlekamp [4], see also [1], [2], [3], [7], and [13] for other types and results on Lee codes.

In this paper we focus on perfect and quasi-perfect error-correcting Lee codes. Except for practical applications, the Golomb-Welch conjecture [8] on the existence of perfect Lee codes has been the main motive power behind the
research in the area for more than 40 years. Although there are many papers on the topic, the conjecture is far from being solved. In these papers the authors use various methods when attacking the conjecture. However, each of these methods has its limitation and will not enable one to settle the conjecture completely. More detailed account on the methods used will be given in Section 3. Thus, in this paper we initiate the study of a new approach for tackling the conjecture. We have looked for a setting for the Golomb-Welch conjecture, also the G-W conjecture, in the area with a well developed theory containing many deep results. We have chosen an approach based on a new invariant related to homomorphisms of abelian groups. We will show how this invariant relates to linear $PL(n,e)$ codes. Using our approach we prove the non-existence of linear perfect 2-error correcting codes for $n = 7, ..., 11$. Proving the G-W conjecture for linear codes would constitute a big progress. To complete the proof of the Golomb-Welch conjecture it would be needed to show that if there is no linear perfect Lee code then there is no perfect Lee code. In other words, if there is no lattice tiling of $\mathbb{Z}^n$ by Lee spheres of radius $e$, then there is no tiling of $\mathbb{Z}^n$ by such Lee spheres. This means to answer in the affirmative a very special case of the second part of the Hilbert’s 18th problem. For more information on the problem we refer the reader to [15] and [19].

Although the G-W conjecture has not been solved yet it is widely believed that it is true. Therefore, instead of searching for perfect Lee codes, some codes that are ”close” to being perfect are considered; see e.g. [2], where quasi-perfect Lee codes have been introduced. We show, by means of our new approach, that these codes are a natural extension of perfect Lee codes. So far quasi-perfect Lee codes have been found only for $n = 2$. Using our new approach we construct first quasi-perfect Lee codes for $n > 2$. On the other hand we prove that, for each $n \geq 3$, there are at most finitely many values of $e$ for which there exists a quasi-perfect $e$-error-correcting Lee code in $\mathbb{Z}^n$.

1 Terminology and Basic Concepts

Throughout the paper we will use $\mathbb{Z}^n$ both for the $n$-fold Cartesian product of the set $\mathbb{Z}$ of integers and for the abelian(component-wise) additive group on $\mathbb{Z}^n$. It will always be clear from the context which of the two we have in mind. Because of the coding theory background the elements of $\mathbb{Z}^n$ will
be called words. The Lee distance (= the Manhattan distance) \( \rho_L(v, w) \) of two words \( v = (v_1, v_2, ..., v_n), w = (w_1, ..., w_n) \) in \( \mathbb{Z}^n \) is given by
\[
\rho_L(v, w) = \sum_{i=1}^{n} |v_i - w_i|.
\]
By \( S_{n,r} \) we denote the Lee sphere of radius \( r \) in \( \mathbb{Z}^n \) centered at the origin \( O \); that is, \( S_{n,r} = \{ w; \rho_L(O, w) \leq r \} \). The Lee sphere of radius \( r \) in \( \mathbb{R}^n \), denoted \( L_{n,r} \), is the union of unit cubes centered at words in \( S_{n,e} \). Further, \( e_i \) will stand for the word \( (0, ..., 0, 1, 0, ..., 0) \) with \( i \)-th coordinate equal to 1, and we will use \([a, b]\) as a shorthand for all integers \( k, a \leq k \leq b \); \([a, b]\) will also be called a segment or an interval in \( \mathbb{Z} \).

A code \( \mathcal{L} \) in \( \mathbb{Z}^n \) is a subset of \( \mathbb{Z}^n \). If a code \( \mathcal{L} \) is at the same time a lattice then \( \mathcal{L} \) is called a linear code. Linear codes play a special role as in this case there is a better chance for the existence of an efficient decoding algorithm. A code \( \mathcal{L} \) is called a perfect \( e \)-error-correcting Lee code in \( \mathbb{Z}^n \), denoted \( PL(n, e) \), if
(i) \( \rho_L(v, w) \geq 2e + 1 \) for every \( v, w \in \mathcal{L} \); and (ii) every word \( v \in \mathbb{Z}^n \) is at Lee distance at most \( e \) from a unique codeword in \( \mathcal{L} \). Another way how to introduce a \( PL(n, e) \) code is by means of a tiling. Let \( V \) be a subset of \( \mathbb{Z}^n \). By a copy of \( V \) we mean a translation \( V + x = \{ v + x, v \in V \} \) of \( V \), where \( x \in \mathbb{Z}^n \). A collection \( \mathcal{T} = \{ V + l; l \in \mathcal{L} \}, \mathcal{L} \subseteq \mathbb{Z}^n \), of copies of \( V \) constitutes a tiling of \( \mathbb{Z}^n \) by \( V \) if \( \mathcal{T} \) forms a partition of \( \mathbb{Z}^n \). \( \mathcal{T} \) is called periodic (lattice) tiling if \( \mathcal{L} \) is periodic (forms a lattice). Clearly, a set \( \mathcal{L} \) is a \( PL(n, e) \) code if and only if \( \{ S_{n,e} + l; l \in \mathcal{L} \} \) constitutes a tiling of \( \mathbb{Z}^n \) by Lee spheres \( S_{n,e} \).

If the condition (ii) in the definition of the \( PL(n, e) \) code \( \mathcal{L} \) is relaxed to:
(iia) every word \( v \in \mathbb{Z}^n \) is at Lee distance at most \( e + 1 \) from at least one codeword in \( \mathcal{L} \), then \( \mathcal{L} \) is called a quasi-perfect \( e \)-error-correcting Lee code, a \( QPL(n, e) \) code. An efficient decoding algorithm for quasi-perfect codes has been given in [12].

## 2 Golomb - Welch Conjecture

In this section we present a short account of the state of the art in the Golomb-Welch conjecture, and describe various approaches how the conjecture has been tackled so far.

For \( n = 2 \) and all \( e \geq 1 \), and for \( e = 1 \) and all \( n \geq 1 \), \( PL(n, e) \) codes have been constructed by several authors, see [23]. Golomb and Welch [8] conjectured that:
Conjecture 1 There is no $PL(n,e)$ code for $n \geq 3$ and $e > 1$.

It is shown in [8] that for each $n$ there exists $e_n$, not specified in [8], so that for all $e > e_n$ there is no $PL(n,e)$ code. To prove this statement the authors use a clever geometric argument; we will describe it in detail in the proof of Theorem [7]. Unfortunately, the given type of argument cannot be used in the case when $e$ is relatively small to $n$.

Another type of a geometric argument, also making use of tiling by Lee spheres $L_{n,e}$, has been used in [9] to settle the G-W conjecture for $n = 3$ and all $e > 1$. It is an elegant ”picture says it all” proof that, unfortunately, cannot be extended to a higher dimension. Later Špacapan [21], whose proof is computer aided, showed the non-existence of a $PL(n,e)$ code for $n = 4$, and all $e > 1$. His method cannot be extended even to $n = 5$ as the number of cases needed to be checked in his approach grows too rapidly. It is proved in [11] that there is no $PL(n,e)$ code for $3 \leq n \leq 5$, and all $e > 1$. The only other value of parameters for which the Golomb-Welch conjecture is known to be true is $n = 6$ and $e = 2$, see [10]. The last two results are proved using an algebraic/counting argument showing that a $PL(n,e)$ code does not exist even in a ”local sense”. Unfortunately, this method is not suitable for bigger values of $n$.

In some papers the non-existence of special types of $PL(n,e)$ codes is proved. We mention here only two of them. Post [20] showed that there is no periodic $PL(n,e)$ code for $3 \leq n \leq 5$, $e \geq n - 2$, and for $n \geq 6$, and $e \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2} - 2)$. To prove it Post used generating functions. Also this method is unsuitable for small values of $e$. For $e \geq n \geq 3$, Post’s result has been improved in [22], where it is shown that there is no so called optimal Lee-type local structure for given parameters.

There are several reformulations of the Golomb-Welch conjecture. One is in terms of the perfect domination set in a graph isomorphic to Cartesian product of cycles, while a reformulation of the conjecture in terms of circulant graphs appears in [5]. So far these two reformulations have not been helpful in progressing with the Golomb-Welch conjecture.

At the end of this section we briefly describe two extensions of the Golomb-Welch conjecture. For a detailed account we refer the reader to [14]. Diameter-$d$ perfect codes have been introduced in the Hamming scheme by Ahlswede et al. in [1], while Etzion [7] extended the notion to Lee metric. Since, for $d$
odd, a diameter-\(d\) perfect Lee code is a \(PL(n, \frac{d+1}{2})\) code as well, these codes constitute a generalization of perfect error-correcting Lee codes. Therefore the conjecture stated by Etzion in [7] is an extension of the G-W conjecture. A further extension of Etzion’s conjecture to the perfect distance-dominating set in a graph \(G\) has been stated in [3]. Unfortunately, as with the mentioned reformulations of the G-W conjecture, the two extensions have not contributed yet to the solution of the conjecture. A tiling constructed by Minkowski [18] provides an exception to both extensions of the G-W conjecture. However, we do not believe that this indicates that there might be an exception to the G-W conjecture as well.

3 Embedding Abelian Groups

In this section we initiate the study of a new invariant of abelian groups. We show how this invariant is related to the G-W conjecture.

Let \(G\) be a finite abelian group and \(\phi : \mathbb{Z}^n \to G\) be a homomorphism. For \(g \in \phi(\mathbb{Z}^n)\) we set \(\pi(n, G, \phi, g) = \min \{ \rho_L(x, O) ; \phi(x) = g \}\), and say that \(g\) is embedded at the minimum distance \(\pi(n, G, \phi, g)\). If \(\phi\) is surjective, the embedding number of \(G\) into \(\mathbb{Z}^n\) with respect to \(\phi\) is defined to be the number \(\pi(n, G, \phi) = \sum_{g \in G} \pi(n, G, \phi, g)\), otherwise we put \(\pi(n, G, \phi) = \infty\). The embedding number \(\pi(n, G)\) of \(G\) in \(\mathbb{Z}^n\) is set to be \(\min_{\phi} \pi(n, G, \phi)\) where the minimum is taken over all homomorphisms \(\phi : \mathbb{Z}^n \to G\). Finally, for each \(k > 0\), we set \(\pi(n, k) = \min_{G} \pi(n, G)\), where the minimum runs over all abelian groups of order \(k\). We note that the value of \(\pi(n, G)\) is invariant under the group isomorphism; i.e., if \(G \cong H\) then \(\pi(n, G) = \pi(n, H)\).

In the following example we illustrate the definition of the embedding number by means of the cyclic group \(Z_{16}\).

Example. Consider a homomorphism \(\phi : \mathbb{Z}^2 \to Z_{16}\) given by \(\phi(e_1) = 1\) and \(\phi(e_2) = 5\). Then \(\pi(2, Z_{16}, \phi, g) = 0\) for \(g = 0\), \(\pi(2, Z_{16}, \phi, g) = 1\) for \(g = 1, 5, 11, 15\); \(\pi(2, Z_{16}, \phi, g) = 2\) for \(g = 2, 4, 6, 10, 12, 14\); \(\pi(2, Z_{16}, \phi, g) = 3\) for \(g = 3, 7, 9, 13\), and \(\pi(2, Z_{16}, \phi, g) = 4\) for \(g = 8\). Therefore, \(\pi(2, Z_{16}, \phi) = 0 \cdot 1 + 1 \cdot 4 + 2 \cdot 6 + 3 \cdot 4 + 4 \cdot 1 = 32\). The homomorphism \(\phi\) is illustrated in Fig.1. The Lee sphere \(S_{2,2}\) is bounded by a thick line, while \(S_{2,3}\) is bounded by a double line. The numbers given there are values of \(\phi((x, y)) \in Z_{16}\) at the given point of \(Z^2\). The elements in bold font and underlined are embeddings.
of elements of \( Z_{16} \) at the minimum distance from the origin. If there were more embeddings of an element at the minimum distance we have picked one of them at random; e.g., there are two embeddings of \( 10 \in Z_{16} \) at the minimum Lee distance 2, and of \( 13 \in Z_{16} \) at the minimum Lee distance 3.

Figure 1: Homomorphism \( \phi : Z^2 \to Z_{16} \)

The (hypothetically) best value of \( \pi(2, Z_{16}) \) would be attained by a homomorphism, if any, \( \phi : Z^2 \to Z_{16} \), with the property that there are \( |S_{2,1}| - |S_{2,0}| = 5 - 1 = 4 \) elements \( g \) of \( Z_{16} \) with \( \pi(2, Z_{16}, \phi) = 1 \); \( |S_{2,2}| - |S_{2,1}| = 13 - 5 = 8 \) elements \( g \) of \( Z_{16} \) with \( \pi(2, Z_{16}, \phi) = 2 \); and finally \( |Z_{16}| - |S_{2,3}| = 16 - 13 = 3 \) elements of \( g \) with \( \pi(2, Z_{16}, \phi, g) = 3 \). Then, in total, \( \pi(2, Z_{16}, \phi) = 0 \cdot 1 + 1 \cdot 4 + 2 \cdot 8 + 3 \cdot 3 = 29 \). It will be shown in the next theorem that such an embedding for \( Z_{16} \) is attained by the homomorphism \( \phi \) given by \( \phi(e_1) = 2 \) and \( \phi(e_2) = 3 \). So, \( \pi(2, Z_{16}) = 29 \). As the lower bound applies to any abelian group of order 16, we also have \( \pi(2, 16) = 29 \).

To be able to show how the above introduced notion of group embeddings relates to the G-W conjecture we first present a lower bound on \( \pi(n, k) \) and then state a theorem proved in [14].

Let \( n, k \geq 1 \). Then there is a uniquely determined number \( r \) so that \( |S_{n,r}| \leq k < |S_{n,r+1}| \). To facilitate our discussion we set

\[
f(n, k) = \left[ \sum_{1 \leq i \leq r} i(|S_{n,i}| - |S_{n,i-1}|) \right] + (r + 1)(k - |S_{n,r}|).
\]
Theorem 2 Let \( k, n \geq 1 \). Then \( \pi(n, k) \geq f(n, k) \). Moreover, for \( |S_{n,r}| < k \) (for \( |S_{n,r}| = k \)), \( \pi(n, k) = f(n, k) \) if and only if there is an abelian group \( G \) of order \( k \) and a homomorphism \( \phi: \mathbb{Z}^n \rightarrow G \) such that the restriction of \( \phi \) to \( S_{n,r} \) is injective and the restriction of \( \phi \) to \( S_{n,r+1} \) is surjective (the restriction of \( \phi \) to \( S_{n,r} \) is a bijection).

We will say that a number \( k > 0 \) (an abelian group \( G \) of order \( k \)) has an optimal embedding in \( \mathbb{Z}^n \) if \( \pi(n, k) = f(n, k) \) (if \( \pi(n, G) = f(n, k) \)).

**Proof.** Denote by \( G_d \) the set \( \{ g \in G \text{ such that } \pi(n, G, \phi, g) = d \} \), and, for \( d \leq r \), \( \varepsilon_d = (|S_{n,d}| - |S_{n,d-1}|) - |G_d| \). Since there are in \( \mathbb{Z}^n \) exactly \( |S_{n,d}| - |S_{n,d-1}| \) words at distance \( d \) from the origin, we have \( |G_d| \leq |S_{n,d} - |S_{n,d-1}| \), and thus \( \varepsilon_d \geq 0 \). We get \( \pi(n, G, \phi) = \sum_{g \in G} \pi(n, G, \phi, g) = \sum_{d \geq 0} d |G_d| = \sum_{0 \leq d \leq r} d |G_d| + \sum_{d > r} d |G_d| = \left( \sum_{d \leq r} d (|S_{n,d}| - |S_{n,d-1}| - \varepsilon_d) \right) + (r + 1)(|G| - |S_{n,r}| + \sum_{d \leq r} \varepsilon_d) \). Therefore,

\[
\pi(n, G, \phi) = f(n, r) + \sum_{d \leq r} (r + 1 - d) \varepsilon_d + \sum_{d \geq r + 2} (d - r - 1) |G_d|. \tag{1}
\]

By (1), \( \pi(n, G, \phi) \geq f(n, r) \) for all homomorphisms \( \phi \). There is an equality in (1) iff \( \varepsilon_d = 0 \) for all \( d \leq r \), and \( |G_d| = 0 \) for all \( d > r + 1 \); i.e., iff the restriction of \( \phi \) to \( S_{n,r} \) is injective, and the restriction of \( \phi \) to \( S_{n,r+1} \) is surjective. For \( k = S_{n,r} \) this necessary and sufficient condition translates to \( \phi \) is a bijection on \( S_{n,r} \).

The following theorem has been stated in [14]

**Theorem 3** Let \( V \) be a subset of \( \mathbb{Z}^n \). Then there is a lattice tiling of \( \mathbb{Z}^n \) by \( V \) if and only if there is an abelian group \( G \) of order \( |V| \), and a homomorphism \( \phi: \mathbb{Z}^n \rightarrow G \) so that the restriction of \( \phi \) to \( V \) is a bijection.

Combining the above two theorems for \( V = S_{n,e} \) yields:

**Corollary 4** There exists a linear PL(n,e) code if and only if there is an optimal embedding of the number \( |S_{n,e}| \) in \( \mathbb{Z}^n \).

In turn, we get a reformulation of the G-W conjecture in the case of linear codes:
Conjecture 5 The number $|S_{n,e}|$ does not have an optimal embedding in $\mathbb{Z}^n$ for $n \geq 3$ and $e > 1$.

The following theorem constitutes another main results of the paper.

Theorem 6 Each $k \geq 1$ has an optimal embedding in $\mathbb{Z}^2$. In particular, for each $k \geq 1$, the cyclic group $\mathbb{Z}_k$ has an optimal embedding in $\mathbb{Z}^2$.

Proof. The intersection $l_{r,m}$ of the sphere $S_{2,r}$ with the line $x + y = m$ is non-empty if and only if $-r \leq m \leq r$, and $l_{r,m}$ comprises points $(x,y)$ in $\mathbb{Z}^2$ with $\left[ \frac{m-x}{2} \right] \leq x \leq \left[ \frac{m+y}{2} \right]$, $\left[ \frac{m-y}{2} \right] \leq y \leq \left[ \frac{m+x}{2} \right]$, and $x + y = m$.

We split the sphere $S_{2,r}$ into the upper part $S'_{2,r} = \{(x,y): (x,y) \in S_{2,r}, x + y = m, 0 < m \leq r, m = 0 \text{ and } x < 0\}$, the lower part $S''_{2,r} = S_{2,r} - S'_{2,r} = \{(0,0)\}$, and the origin. As $(x,y) \in S'_{2,r}$ implies $-x,-y) \in S'_{2,r}$, we get $\phi(S'_{2,r}) = -\phi(S'_{2,r})$ for each homomorphism $\phi$ on $\mathbb{Z}^2$.

Let $\Phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ be a homomorphism given by $\Phi(e_1) = r$, and $\Phi(e_2) = r + 1$. Since $\Phi((x,y)) - \Phi((x+1,y+1)) = rx + (r+1)y - r(x+1) - (r+1)(y-1) = 1$, we get that $\Phi(l_{r,m})$ is a segment in $\mathbb{Z}$, and $\Phi$ is decreasing on $l_{r,m}$ in the $x$-coordinate. To prove that $\Phi(l_{r,m})$ and $\Phi(l_{r,m+1})$ constitute two consecutive segments it suffices to verify (we leave it to the reader) that for all $r,m$, the value of $\Phi$ at the point $(x,y)$ in $l_{r,m}$ with the smallest value of $x$, is by one smaller than the value of $\Phi$ at the point $(x,y)$ in $l_{r,m+1}$ with the largest value of $x$, i.e., $\Phi((m - \left[ \frac{m+1}{2} \right], \left[ \frac{m+1}{2} \right] + 1) = \Phi((m + 1 - \left[ \frac{m+1}{2} \right], \left[ \frac{m+1}{2} \right])$.

This in turn implies that $\Phi(S'_{2,r})$ is the union of consecutive segments and the restriction of $\Phi$ to $S'_{2,r}$ is an injection. Hence, $|\phi(S'_{2,r})| = |S'_{2,r}| = 2r^2 + 2r + 1$. In addition, we have $\Phi((0,0)) = 0$, and therefore $\Phi(S'_{2,r}) = [1, r(r+1)]$, and $\Phi(S''_{2,r}) = [-r(r+1), -1]$.

Now we are ready to prove the statement of the theorem. First, let

(i) $k = |S'_{2,r}| = 2r^2 + 2r + 1$ for some $r$. In this case the statement that $k$ has an optimal embedding in $\mathbb{Z}^2$ is equivalent to the statement that there is a tiling of $\mathbb{Z}^2$ by Lee spheres $L_{2,r}$; this has been shown by several authors, see e.g. [8]. It is not difficult to see that such a tiling is unique, up to a symmetry. Thus, in fact we show that the unique tiling is a lattice one, and moreover, the group associated with the lattice is the cyclic group. Consider the homomorphism $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}_k$ given by $\phi(e_1) = r$, $\phi(e_2) = r + 1$. Clearly, $\phi(u) = \Phi(u) \text{ (mod } k)$ for each $u \in \mathbb{Z}^2$, and thus $\phi((x,y)) = \Phi((x,y))$ for $(x,y) \in S''_{2,r}$ as $\Phi(S''_{2,r}) = [1, r(r+1)]$ and $r(r+1) < k$. In addition, $\phi$
is injective on $S_{2,r}$ because $\Phi$ is. As $\phi$ is a homomorphism, $\phi$ is injective on $S_{2,r}$ and $\phi(S_{2,r}) = \{1, r(r+1)\} \mod k = [r(r+1) + k, -1 + k]$ because $\Phi$ is. As $r > 2r^2 + 2r + 1$, we have $r(r+1) + k > r(r+1)$, and this implies $\phi(S_{2,r}) \supset \phi(S_{2,r}) = \emptyset$. Thus $\phi$ is an injection on $S_{2,r}$. To finish the proof we need to show that the restriction of $\phi$ to $S_{2,r+1}$ is a surjection. $\phi(X) = \Phi(X)\mod k$ yields $\phi(S_{2,r+1}) \supset \Phi(S_{2,r}) \cup \Phi((r+1)) = \{1, r(r+1)\} \cup \Phi((r+1), (0, r+1)) = [1, r(r+1)] \cup [r(r+1), (r+1)] = [1, (r+1)^2]$. Further, $\phi(S_{2,r}) = -\phi(S_{2,r}) \supset \phi(S_{2,r}) \supset \{r, r+1\} \mod k$, that is, $\phi(S_{2,r}) \supset \{r, r+1\} + k, -1 + k$. However, it is $(r+1)^2 > k - (r+1)^2$ because in this case $k < 2r^2 + 4r$. Therefore $\phi(S_{2,r+1}) = [0, k-1]$, i.e., $\phi$ is a surjection on $S_{2,r+1}$.

(ii) Let $r$ be so that $|S_{2,r}| < k < |S_{2,r+1}|$; that is, $2r^2 + 2r + 1 < k < 2r^2 + 6r + 5$. We split the proof into two cases.

(iia) $|S_{2,r}| = 2r^2 + 2r + 1 < k < 2r^2 + 4r$.
Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}_k$ be the same homomorphism as above. Then $\phi(u) = \Phi(u)\mod k$, and $\phi(S_{2,r}) = \{1, r(r+1)\}$. $\phi$ is injective on $S_{2,r}$, and $\phi(S_{2,r}) = -\phi(S_{2,r}) = \{1, r(r+1)\} \mod k = [1, r(r+1), 1, -1 + k]$. As $k > 2r^2 + 2r + 1$, we have $r(r+1) + k > r(r+1)$, and this implies $\phi(S_{2,r}) \cap \phi(S_{2,r}) = \emptyset$. Thus $\phi$ is an injection on $S_{2,r}$. To finish the proof we need to show that the restriction of $\phi$ to $S_{2,r+1}$ is a surjection. $\phi(X) = \Phi(X)\mod k$ yields $\phi(S_{2,r+1}) \supset \Phi(S_{2,r}) \cup \Phi((r+1)) = \{1, r(r+1)\} \cup \Phi((r+1), (0, r+1)) = [1, r(r+1)] \cup [r(r+1), (r+1)] = [1, (r+1)^2]$. Further, $\phi(S_{2,r}) = -\phi(S_{2,r}) \supset \phi(S_{2,r}) \supset \{r, r+1\} \mod k$, that is, $\phi(S_{2,r}) \supset \{r, r+1\} + k, -1 + k$. However, it is $(r+1)^2 > k - (r+1)^2$ because in this case $k < 2r^2 + 4r$. Therefore $\phi(S_{2,r+1}) = [0, k-1]$, i.e., $\phi$ is a surjection on $S_{2,r+1}$.

(iiib) $2r^2 + 4r + 1 \leq k < 2r^2 + 6r + 5$.
Consider a homomorphism $\Phi' : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\Phi'(e_1) = r+1, \Phi'(e_2) = r+2$. Translating the results obtained above for the homomorphism $\Phi$ into the language of $\Phi'$ we get that the restriction of $\Phi'$ to $S_{2,r+1}$ is injective, and $\Phi'(S_{2,r+1}) = [1, (r+1)(r+2)], \Phi'(S_{2,r+1}) = [-r(r+1)+1, k-1]$. Further, we have $\max \Phi'(S_{2,r}) = \Phi'(0, r) = r+2$.

Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}_k$ be a homomorphism given by $\phi(e_1) = r+1, \phi(e_2) = r+2$. Hence, $\phi(u) = \Phi'(u)\mod k$, which in turn implies $\phi(S_{2,r+1}) = \{1, r(r+1)(r+2)\} \supset \{1, \frac{k-1}{2}\}$. In addition, $\phi(S_{2,r+1}) = -\phi(S_{2,r+1}) \supset [-\frac{k-1}{2}, -1] \mod k = [-\frac{k-1}{2} + k, -1 + k] = [\frac{k+1}{2}, k-1]$. In aggregate, $\phi(S_{2,r+1}) = \phi(S_{2,r+1}) \cup \phi(S_{2,r+1}) \cup \{0\} = [0, k-1]$. Thus, the restriction of $\phi$ to $S_{2,r+1}$ is surjective. Now we prove that the restriction of $\phi$ to $S_{2,r}$ is injective. We recall that the restriction of $\Phi'$ to $S_{2,r}$ is injective and $\max \Phi'(S_{2,r}) = \Phi'(0, r) = r+2$. Thus $\Phi'(S_{2,r}) \subset [1, (r+1)(r+2)]$. Therefore $\phi$ is injective on $S_{2,r}$, $\phi(S_{2,r}) \subset [1, \frac{k-1}{2}]$. Further, $\phi$ is injective on $S_{2,r}$, and
\( \phi(S_{2,r}) = -\phi(S_{2,r}) \subset [-\frac{k+1}{2}, -1] (\text{mod } k) = \left[ \frac{k-1}{2}, k, -1 + k \right] = \left[ \frac{k+1}{2}, k - 1 \right]. \)

Hence, \( \phi(S_{2,r}) \cap \phi(S_{2,r}) \) is empty. \( \blacksquare \)

Now we prove that also in this case the results for \( n \geq 3 \) dramatically differ from those for \( n = 2 \).

**Theorem 7** For each \( n \geq 3 \), there is a \( k_n \) so that no \( k \geq k_n \) has an optimal embedding in \( \mathbb{Z}^n \).

**Proof.** This proof uses ideas developed in [8]. Let \( P_{n,r} \) be the smallest convex polytope containing the \( 2n \) points \( \pm (r + \frac{1}{2})e_i, i = 1, ..., n \). In other words, \( P_{n,r} \) is the smallest convex polytope containing \( 2n \) center points of \( (n-1) \)-dimensional extremal hyperfaces of the Lee sphere \( L_{n,r} \). Thus, \( P_{2,r} \) is a square while \( P_{3,r} \) is a regular octahedron. For the \( n \)-dimensional hypervolume of the regular polytope \( P_{n,r} \) we have: \( V(P_{n,r}) = \frac{(2r+1)^n}{n!} \).

If there was a tiling \( T \) of \( \mathbb{R}^n \) by Lee spheres \( L_{n,r} \), then this tiling would induce a packing of \( \mathbb{R}^n \) by regular polytopes \( P_{n,r} \).

Assume now that an integer \( k \) has an optimal embedding in \( \mathbb{Z}^n \). Then, by Theorem 2 there is a homomorphism \( \phi : \mathbb{Z}^n \rightarrow G \), an abelian group of order \( k \), so that the restriction of \( \phi \) to \( S_{n,r} \) is injective and the restriction of \( \phi \) to \( S_{n,r+1} \) is surjective. Therefore we are able to choose a set \( K \) of \( k \) points in \( \mathbb{Z}^n \) so that \( S_{n,r} \subseteq K \subseteq S_{n,r+1} \) and the restriction of \( \phi \) on \( K \) is a bijection.

By Theorem 3 \( \phi \) induces a lattice tiling of \( \mathbb{Z}^n \) by copies of \( K \). This in turn implies that there is a lattice tiling \( T \) of \( \mathbb{R}^n \) by the tile \( T_K \) comprising unit cubes centered at points in \( K \). Since \( S_{n,r} \subseteq K \), the Lee sphere \( L_{n,r} \) is a subset of \( T_K \). Therefore, tiling \( T \) induces a packing of \( \mathbb{R}^n \) by Lee spheres \( L_{n,r} \), which in turn induces a packing of \( \mathbb{R}^n \) by polytopes \( P_{n,r} \).

It is known, see [6], that regular polytopes \( P_{n,k} \) do not tile \( \mathbb{R}^n \). Further, cf. [8], if a polytope does not tile \( \mathbb{R}^n \), then the packing efficiency \( \alpha \) of \( \mathbb{R}^n \) by this polytope is strictly less than 1. As the packing of \( \mathbb{R}^n \) by copies of \( P_{n,r} \) has been induced by tiling \( T \) of \( \mathbb{R}^n \) by the cluster of unit cubes \( T_K \), \( V(P_{n,r}) \) by \( \alpha \).

Therefore, there is no tiling \( T \) of \( \mathbb{R}^n \) by a tile \( T_k \) for

\[
\frac{V(P_{n,r})}{V(T_K)} > \alpha.
\]  

However, for the volume \( V(T_k) \) we have \( V(L_{n,r}) \leq V(T_k) < V(L_{n,r+1}) \). Therefore, for \( n \) fixed,
Thus, there is a \( k_n \) so that, for all \( k > k_n \), we have \( \frac{V(P_{n,r})}{V(T_k)} > \alpha \). Hence, by (2), for \( k > k_n \), there is no tiling of \( \mathbb{R}^n \) by \( T_k \), that is, there is no tiling of \( \mathbb{Z}^n \) by the set \( K \) in this case. This in turn implies that \( k = |K| \) does not have an optimal embedding in \( \mathbb{Z}^n \). ■

At the end of this section we prove the non-existence of a linear \( PL(n,e) \) code for some new values of parameters.

**Theorem 8** There is no linear \( PL(n,2) \) code for \( 7 \leq n \leq 12 \).

**Proof.** By Corollary 4 it suffices to prove that, for \( n = 7, \ldots, 12 \), the number \( k_n = |S_{n,2}| = 2n^2 + 2n + 1 \) does not have an optimal embedding in \( \mathbb{Z}^n \). For \( n = 7, \ldots, 12, k_n \) is a square free number. Thus each abelian group of the order \( k_n \) is isomorphic to the cyclic group \( \mathbb{Z}_{k_n} \). We need to show that there is no homomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}_{k_n} \) such that the restriction of \( \phi \) to \( S_{n,2} \) is bijective. Each homomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}_{k_n} \) is determined by the values of \( \phi(e_i), i = 1, \ldots, n \), and \( |\phi(S_{n,2})| = |\{ \pm \phi(e_i), \pm \phi(e_i) \pm \phi(e_j); 1 \leq i \leq j \leq n \}| < |S_{n,2}| \) if \( \phi \) is not a bijection on \( S_{n,2} \). Hence, it is sufficient to show that for each \( n \)-tuple \( (g_1, \ldots, g_n) \) of elements in \( \mathbb{Z}_{k_n} \)

\[
|\{ \pm g_i, \pm g_i \pm g_j; 1 \leq i \leq j \leq n \}| < |S_{n,2}|. \tag{3}
\]

This can be proved by a brute force computer test, where all \( (k_n)^n \approx (2n^2)^n \) \( n \)-tuples of elements in \( \mathbb{Z}_{k_n} \) are shown to satisfy (3). In what follows we exhibit a way how to substantially reduce the computational complexity of the test.

Assume that there exists a homomorphism \( \phi : \mathbb{Z}^n \to \mathbb{Z}_{k_n} \) such that the restriction of \( \phi \) to \( S_{n,2} \) is a bijection. Then there would have to be such a homomorphism \( \phi' \) with \( \phi'(e_j) \leq \frac{k_n}{2} \) (if \( \phi(e_i) > \frac{k_n}{2} \) we set \( \phi'(e_i) = -\phi(e_i) \) ) for all \( 1 \leq i \leq n \), and also \( \phi'(i) < \phi'(j) \) for all \( 1 \leq i < j \leq n \). Therefore, we need to show that (3) is satisfied by any of \( \binom{k_n/2}{n} \) \( n \)-tuples \( (g_1, \ldots, g_n) \), where \( 1 \leq g(i) \leq \frac{k_n}{2} \), and \( g(i) < g(j) \) for all \( 1 \leq i < j \leq n \). Using the Stirling formula we have \( \binom{k_n/2}{n} \approx \frac{(en)^n}{\sqrt{2\pi n}} \). To reduce the computational complexity even
further we used a backtracking algorithm that enables to check (3) only for a portion of $\binom{k_n/2}{n}$ of $n$-tuples. The algorithm is based on the following two simple observations: First, let $(g_1, ..., g_n)$ be a $n$-tuple such that for some $m < n$

$$|\{\pm g_i, \pm g_i \pm g_j; 1 \leq i \leq j \leq m\}| < |S_{m,2}|,$$ \hspace{1cm} (4)

then $(g_1, ..., g_n)$ satisfies (3) as well. Second, it suffices to choose $g_m > g_{m-1}$ and $g_m \in T_m = [1, ..., \frac{k_n-1}{2}] - \{\pm g_i, \pm g_i \pm g_j; 1 \leq i \leq j \leq m-1\}, 1 \leq m \leq n$, as otherwise (4) is trivially satisfied. The algorithm comprises $n$ nested cycles. In the $m$-th cycle we choose the $m$-th element of the tuple $(g_1, ..., g_n)$. It suffices to choose $g_m > g_{m-1}$ and $g_m \in T_m$, see above. We test $(g_1, ..., g_m)$ for (4). If (4) is satisfied, we replace $g_m$ by the next element from $T_m$, if none is available, we backtrack to the previous cycle that chooses $g_{m-1}$. Otherwise, if there is an equality in (4), in the next nested cycle we choose $g_{m+1} > g_m$ from the set $T_{m+1}$.

Thus, in aggregate, the test (4) is performed at most

$$\frac{1}{n!} \prod_{m=1}^{n} |T_m| = \frac{1}{n!} \prod_{m=1}^{n} \left( \frac{|S_{n,2}| - 1}{2} - \frac{|S_{m-1,2}| - 1}{2} \right) = \frac{1}{n!} \prod_{m=0}^{n-1} [(n^2+n) - (m^2+m)] =

\frac{1}{n!} \prod_{m=0}^{n-1} (n + m + 1)(n - m) = \frac{(2n)!}{n!} \approx \sqrt{2} \left( \frac{4n}{e} \right)^n$$

times. We note that the factor $\frac{1}{n!}$ stands for the fact that from all $n$-tuples $(g_1, ..., g_n)$ that differ only by the order of elements we test only the one with $g(i) < g(j)$ for all $1 \leq i < j \leq n$. Clearly, this is a crude upper bound because if $(g_1, ..., g_m)$ does not satisfy (4), then we skip testing all $t$-tuples $(g_1, ..., g_m, ..., g_t)$ for each $t > m$. The backtracking algorithm described above was used to prove the statement. \hfill

**Remark 9** We point out that the backtracking algorithm described above can be used any time when $k_n = |S_{n,2}| = 2n^2+2n+1$ is a square free number, and sufficient computing power is available. E.g., for all $n \leq 19$, the number $k_n$ is square free. However, we have verified the statement of the theorem only for $n \leq 12$, as for the bigger values of $n$ the computation has not been feasible for our computer lab. We note that the computation can easily be distributed over several machines as verifying (4) for distinct $n$-tuples is independent on each other. In fact we used this distributed approach for all $n \geq 9$
We end this section by a conjecture related to group embeddings. If true, it would be much simpler to determine the value of $\phi(n, k)$.

**Conjecture 10** For each $n \geq 2$, and $k > 0$, the value $\pi(n, k)$ is attained by the cyclic group $\mathbb{Z}_k$.

## 4 Quasi-Perfect Lee codes

We start with a theorem that shows how the quasi-perfect codes relate to optimal embeddings introduced in the previous section. In fact it turns out that quasi-perfect Lee codes are a natural extension of perfect Lee codes.

**Theorem 11** A linear $QPL(n, e)$ code exists if and only if there is a number $k$, $|S_{n,e}| \leq k < |S_{n,e+1}|$, having an optimal embedding in $\mathbb{Z}^n$.

**Proof.** Let $\phi : \mathbb{Z}^n \rightarrow G$, an abelian group $G$ of order $k$, be a homomorphism so that the restriction of $\phi$ to $S_{n,e}$ is an injection and the restriction of $\phi$ to $S_{n,e+1}$ is a surjection. Choose a set $K$, $|K| = k$, of words in $\mathbb{Z}^n$ so that the restriction of $\phi$ to $K$ is a bijection and $S_{n,e} \subseteq K \subseteq S_{n,e+1}$. By Theorem 3, $\phi$ induces a lattice tiling $T$ of $\mathbb{Z}^n$. Consider the lattice $L = \ker(\phi)$. For any two words $u, v \in L$ we have $\rho_L(u, v) \geq 2e + 1$ as $S_{n,e} \subseteq K$. Since $T$ is a tiling, to each word $w \in \mathbb{Z}^n$ there exists a copy $K_w$ of $K$ so that $w \in K_w$, and $K \subseteq S_{n,e+1}$ guarantees that $w$ is at distance $\leq e + 1$ from at least one word in $L$. Thus, $L$ constitutes a linear $QPL(n, e)$ code.

Now, assume that $L \subseteq \mathbb{Z}^n$ is a linear $QPL(n, e)$ code. We will prove that there is a number $k$, $|S_{n,e}| \leq k < |S_{n,e+1}|$, so that $k$ has an optimal embedding in $\mathbb{Z}^n$. It is well known that $\mathbb{Z}^n / L \simeq G$, where $G$ is an abelian group. We show that $G$ has an optimal embedding, and $|S_{n,e}| \leq |G| < |S_{n,e+1}|$. Consider the natural homomorphism $\phi : \mathbb{Z}^n \rightarrow G$. Then $\ker(\phi) = L$. Assume that there are two words $u, v \in S_{n,e}$, $u \neq v$, such that $\phi(u) = \phi(v)$. Set $w = u - v$. Then $\phi(w) = \phi(u - w) = \phi(u) - \phi(v) = 0$. Thus, $w \in L$ is a codeword. However, this is a contradiction as $\rho_L(w, O) = \rho_L(u, v) < 2e + 1$ which contradicts the condition (i) in the definition of $QPL(n, e)$ code. Hence we proved that $\phi$ is an injection on $S_{n,e}$, which at the same time implies that $|G| \geq |S_{n,e}|$.

To prove that the restriction of $\phi$ to $S_{n,e+1}$ is surjective, consider an element $g \in G$. Let $u$ be a word in $\mathbb{Z}^n$ with $\phi(u) = g$. As $L$ is a $QPL(n, e)$ code, for each word $u \in \mathbb{Z}^n$, there is a codeword $w \in L$ so that $\rho_L(u, w) \leq e + 1$,
and for \( u - w \) we have \( \rho_L(O, u - w) \leq e + 1 \), i.e., \( u - w \in S_{n, e+1} \). As \( \phi \) is a homomorphism, \( \phi(u - w) = \phi(u) - \phi(w) = g - 0 = g. \)

In practical applications we deal with finite perfect Lee codes over the alphabet \( \mathbb{Z}_p^n \). These codes are usually denoted as \( PL(n, 2, p) \) codes (\( QPL(n, e, p) \) codes). As a corollary of Theorem 11 we get:

**Corollary 12** There is a linear \( QPL(2, e, k) \) code for each \( |S_{2,e}| \leq k < |S_{2,e+1}| \).

**Proof.** By Theorem 6, each \( k \geq 1 \) has an optimal embedding in \( \mathbb{Z}^2 \), and by Theorem 11 there is a linear \( QPL(n, e) \) code \( L \), where \( \mathbb{Z}^n / L \cong G \) is an abelian group of order \( |G| = k \). Denote by \( \phi \) the natural homomorphism \( \phi: \mathbb{Z}^n \rightarrow G \). Further, for the smallest period \( p \) of \( L \) we have \( p = \text{l.c.m.} \{ \text{ord}(\phi(e_i)), i = 1, \ldots, n \} \), where \( \text{ord}(g) \) stands for the order of the element \( g \) in the group \( G \). Thus \( p \) divides \( |G| \), hence \( L \) is a linear \( QPL(n, e) \) code that is \( k \)-periodic, and thus induces a linear \( QPL(n, e, k) \) code. \( \square \)

By Theorem 7 and Theorem 11 we immediately get:

**Corollary 13** For each \( n > 2 \), there are at most finitely many values of \( e \) for which there exists a linear \( QPL(n, e) \) code.

Now we concentrate on the case of \( n = 3 \) that is most likely to be used in a real-life application. It follows from the result of Gravier et al. [9], that there is no \( PL(3, e) \) code for \( e > 1 \), so there is no optimal embedding for \( k = |S_{3,e}| \) in \( \mathbb{Z}^3 \).

Set \( K = [1, 21] \cup [27, 50] \cup \{55\} \cup [70, 102] \cup \{117, 145\} \cup [147, 151] \cup [153, 156] \cup [158, 165] \cup [167, 172] \cup [174, 177] \cup [182, 183, 190, 260, 261, 263, 264, 266, 267, 268, 270] \cup [272, 276] \cup \{279, 282, 286, 288, 292, 300, 421, 422, 426, 438, 455\} \).

**Theorem 14** If \( k \in K \), then \( k \) has an optimal embedding in \( \mathbb{Z}^3 \). In particular, there is a linear \( QPL(3, e) \) code for each \( e, 1 \leq e \leq 6 \).

**Proof.** It suffices to prove that the cyclic group \( \mathbb{Z}_k \) has an optimal embedding in \( \mathbb{Z}^3 \) for each \( k \in K \). A required homomorphism \( \phi \), uniquely determined by the values of \( \phi(e_i), i = 1, 2, 3 \), has been found by a computer search. For example, for \( k = 7, \ldots, 13 \), it suffices to choose \( \phi(e_i) = i, i = 1, 2, 3 \). The values \( \phi(e_i), i = 2, 3 \), for the other \( k \in K \) are given in Appendix, while \( \phi(e_1) = 1 \) except for \( k = 438 \) where \( \phi(e_1) = 2 \).
For $n = 3$, it is $|S_{3,e}| = \frac{4}{3}e^3 + 2e^2 + \frac{8}{3}e + 1$. Thus, for $e = 1, 2, 3, 4, 5, 6, 7$, we get that $|S_{3,e}| = 7, 25, 63, 129, 231, 377, 575$, respectively. To prove the second part of the statement it suffices to notice that for each $e, 1 \leq e \leq 6$, there is a $k \in K$ with $|S_{3,e}| \leq k < |S_{3,e+1}|$. ■

The last theorem asserts that $QP L(3, e)$ codes exist only for finitely many values of $e$. We note that the statement of the theorem could be proved with the condition linear dropped.

**Theorem 15** There is no linear $QP L(3, e)$ code for $e \geq 55$.

**Proof.** Suppose that there is a linear $QP L(3, e)$ code. By Theorem 11 there is a $k, |S_{3,e}| \leq k < |S_{3,e+1}|$, so that $k$ has an optimal embedding in $\mathbb{Z}^3$. Using the language of the proof of Theorem 7 this implies that there is a lattice tiling of $\mathbb{R}^n$ by a cluster $T_k$ of unit cubes, $L_{3,e} \subseteq T_k \subset L_{n,e+1}$, with its volume $V(T_k) = k$. Also, by proof of Theorem 7 there is no tiling $\mathcal{T}$ of $\mathbb{R}^n$ by a cluster of unit cubes $T_k$ with

$$\frac{V(P_{3,e})}{V(T_k)} > \alpha,$$

where $\alpha$ is the packing efficiency of the regular polytope $P_{3,e}$. The packing efficiency $\alpha = \frac{18}{19}$ of the regular octahedron has been determined by Minkowski in [17]. To prove the non-existence of a linear $QP L(n, e)$ code we need to show that (5) is satisfied by all $k, |S_{3,e}| \leq k < |S_{3,e+1}|$. Clearly, it suffices to show that (5) is satisfied by $k = |S_{3,e+1}| - 1$ as $\frac{V(P_{3,e})}{V(T_k+1)} < \frac{V(P_{3,e})}{V(T_k)}$. Solving (5) for $V(P_{3,e}) = \frac{(2e+1)^3}{3}$ and $V(T_k) = |S_{3,e+1}| - 1 = \frac{4}{3}(e+1)^3 + 2(e+1)^2 + \frac{8}{3}(e+1)$ we get that there is no linear $QP L(3, e)$ code for $e \geq 55$. ■

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## 5 Appendix- Optimal Embeddings in $\mathbb{Z}^3$

| k | $\phi(e_2)$ | $\phi(e_3)$ | k | $\phi(e_2)$ | $\phi(e_3)$ | k | $\phi(e_2)$ | $\phi(e_3)$ | k | $\phi(e_2)$ | $\phi(e_3)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 5 | 48 | 7 | 18 | 95 | 6 | 37 | 169 | 10 | 72 | 300 | 14 | 132 |
| 15 | 2 | 49 | 7 | 11 | 96 | 6 | 37 | 170 | 9 | 64 | 421 | 16 | 182 |
| 16 | 2 | 60 | 8 | 12 | 97 | 7 | 36 | 171 | 12 | 70 | 422 | 72 | 112 |
| 17 | 2 | 55 | 5 | 21 | 98 | 7 | 36 | 172 | 11 | 52 | 426 | 36 | 50 |
| 18 | 2 | 70 | 16 | 25 | 99 | 7 | 37 | 174 | 14 | 34 | 438 | 45 | 122 |
| 19 | 2 | 71 | 7 | 30 | 100 | 6 | 22 | 175 | 10 | 53 | 455 | 16 | 199 |
| 20 | 5 | 72 | 8 | 30 | 101 | 11 | 27 | 176 | 16 | 41 |
| 21 | 2 | 73 | 6 | 21 | 102 | 10 | 43 | 177 | 17 | 28 |
| 27 | 5 | 74 | 8 | 20 | 117 | 16 | 22 | 182 | 35 | 64 |
| 28 | 5 | 75 | 6 | 22 | 145 | 9 | 61 | 183 | 21 | 29 |
| 29 | 5 | 76 | 7 | 18 | 147 | 9 | 62 | 190 | 22 | 30 |
| 30 | 5 | 77 | 7 | 18 | 148 | 32 | 46 | 260 | 40 | 94 |
| 31 | 5 | 78 | 7 | 30 | 149 | 12 | 52 | 261 | 36 | 61 |
| 32 | 6 | 79 | 6 | 32 | 150 | 16 | 26 | 263 | 11 | 97 |
| 33 | 5 | 80 | 6 | 21 | 151 | 10 | 63 | 264 | 16 | 55 |
| 34 | 5 | 81 | 8 | 21 | 153 | 17 | 41 | 266 | 40 | 127 |
| 35 | 5 | 82 | 7 | 26 | 154 | 8 | 58 | 267 | 12 | 99 |
| 36 | 5 | 83 | 6 | 31 | 155 | 9 | 66 | 268 | 40 | 98 |
| 37 | 5 | 84 | 6 | 31 | 156 | 10 | 47 | 270 | 14 | 117 |
| 38 | 6 | 85 | 7 | 25 | 158 | 9 | 48 | 272 | 14 | 118 |
| 39 | 6 | 86 | 6 | 32 | 159 | 10 | 67 | 273 | 12 | 81 |
| 40 | 4 | 87 | 6 | 32 | 160 | 14 | 34 | 274 | 102 | 128 |
| 41 | 4 | 88 | 6 | 26 | 161 | 10 | 68 | 275 | 44 | 60 |
| 42 | 6 | 89 | 6 | 37 | 162 | 34 | 75 | 276 | 104 | 117 |
| 43 | 6 | 90 | 6 | 37 | 163 | 11 | 68 | 279 | 54 | 89 |
| 44 | 6 | 91 | 7 | 24 | 164 | 10 | 69 | 282 | 74 | 100 |
| 45 | 6 | 92 | 10 | 38 | 165 | 9 | 71 | 286 | 14 | 88 |
| 46 | 6 | 93 | 6 | 26 | 167 | 15 | 39 | 288 | 84 | 106 |
| 47 | 6 | 94 | 6 | 26 | 168 | 12 | 69 | 292 | 40 | 102 |
