Abstract. Based on energy considerations, we derive a class of dynamic outflow boundary conditions for the incompressible Navier-Stokes equations, containing the well-known convective boundary condition but incorporating also the stress at the outlet. As a key building block for the analysis of such problems, we consider the Stokes equations with such dynamic outflow boundary conditions in a halfspace and prove the existence of a strong solution in the appropriate Sobolev-Slobodeckij-setting with $L^p$ (in time and space) as the base space for the momentum balance. For non-vanishing stress contribution in the boundary condition, the problem is actually shown to have $L^p$-maximal regularity under the natural compatibility conditions. Aiming at an existence theory for problems in weakly singular domains, where different boundary conditions apply on different parts of the boundary such that these surfaces meet orthogonally, we also consider the prototype domain of a wedge with opening angle $\frac{\pi}{2}$ and different combinations of boundary conditions: Navier-Slip with Dirichlet and Navier-Slip with the dynamic outflow boundary condition. Again, maximal regularity of the problem is obtained in the appropriate functional analytic setting and with the natural compatibility conditions.

Introduction

In the numerical modeling of fluid flows from real world applications it is often not possible to model the complete flow domain up to physical boundaries. Instead, artificial boundaries usually need to be introduced into the problem description. In such cases the formulation of sensible boundary conditions, so-called artificial boundary conditions (ABCs, for short), is a non-trivial task since the flow can enter and, more problematic, leave the domain through open parts of the boundary. We speak of an “outflow boundary” if the mean flow points outwards, while locally a backflow – with fluid entering the domain – is allowed. One important class of ABCs at such outflow boundaries are “convective” boundary conditions like

$$\partial_t \phi + (a \cdot \nabla) \phi = 0$$

with a prescribed velocity $a$, where $\phi$ denotes a transported quantity, say a velocity component. Such dynamic ABCs are known since long in the area of hyperbolic problems, also called Sommerfeld radiation condition in this context. While $a$ usually denotes the phase velocity of the waves, which is hard to be known a priori, Orlanski used a local velocity $a$ in his numerical studies in [12]. In [6], using Fourier techniques and approximations in the transformed space similar to [5], the convective ABC above was derived as an approximation to the non-local exact boundary condition for a linear advection-diffusion equation. In [7], different approximations to the symbol of the exact boundary operator for the linearized incompressible Navier-Stokes equations have been derived, but these approximations often lead to non-local boundary conditions. One local condition given there for 2D flow is the combination of [1] for the normal velocity component with a homogeneous Neumann condition for the tangential part. The full incompressible Navier-Stokes equations are also treated in [9], where the resulting ABC is chosen to contain an additional (viscous) diffusion term acting in the tangential direction.

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Since the derivation of local ABCs of convective type are not strictly feasible for the incompressible Navier-Stokes or Stokes equations, we adopt a different approach based on energy considerations, somewhat in the spirit of [2]. These considerations also motivate the incorporation of additional stress terms and, moreover, lead to several variants of such dynamic outflow boundary conditions. Since it is very important also for the numerical applications that the chosen boundary conditions lead to wellposed initial-boundary-value problems, the main focus of the present work is the analysis of the resulting PDE system concerning the local-in-time wellposedness in appropriate Bessel potential and Sobolev-Slobodeckij spaces. To our knowledge, at least in the context of strong solutions such an analysis has not been done so far. But let us note that for other classes of ABCs, also employed at outflow boundaries, some analytical results are known; cf. [2] and the references given there.

Let us finally note that in the numerical description of real world flow problems, the computational domains usually contain edges at which different boundary conditions meet. Such mixed-type initial-boundary-value problems for the Navier-Stokes or Stokes equations in singular domains are very challenging concerning, e.g., their rigorous analysis. In some prototype cases, like a flow in a system of pipes, the flow domain can be chosen such that it is only weakly singular, meaning that if different boundary parts meet at a common edge, they locally form a right angle there. This is illustrated in Figure 1. There, the flow enters the domain via an inlet, while two outlets are available for the fluid to leave the domain. All in-/outlets are “connected” by an impermeable wall which forms the lateral boundary of the tube. Such a tube is a typical example of a weakly singular domain $\Omega \subseteq \mathbb{R}^n$, whose boundary may be decomposed into several smooth parts that meet each other orthogonally. For the example in Figure 1, the smooth parts of the boundary are the inlet $\Gamma^{\text{in}}$, the lateral boundary of the tube $\Gamma^{\text{wall}}$, and the outlets $\Gamma$.

For the right combinations of boundary conditions, such weakly singular domains can be treated for a variety of admissible boundary conditions as has been shown in [11]. The key model problem required to be treatable for such weakly domains are the corresponding PDE systems in a wedge of opening angle $\frac{\pi}{2}$. For this reason, the analysis for this prototype geometry is included in the present paper.

1. Dynamic Outflow Boundary Conditions

We aim at deriving physically meaningful boundary conditions at outflow boundaries which render the artificial boundary transparent in the sense that the boundary condition does not introduce unphysical dissipation into the system. While our motivation mainly stems from so-called non-reflecting boundary conditions developed for partial differential equations of hyperbolic character.
such as wave equations or compressible flows, their derivation requires a different approach because we aim at applications to flow problems for incompressible Newtonian fluids. The basic idea of our derivation is the preservation of kinetic energy in the following sense: if an outflow boundary $\Gamma^{\text{out}}$ is observed at arbitrary time $t = t_0$, the (infinitesimally thin) layer of fluid exiting the flow domain $\Omega$ at this time instant should not endure a change of its kinetic energy. In mathematical terms this means that

$$
\frac{d}{dt} \int_{\Gamma^{\text{out}}} \rho \frac{v(t, x(t; t_0, x_0))^2}{2} \, d\sigma(x_0) \big|_{t=t_0} = 0,
$$

where $\rho$ is the constant mass density and $x(\cdot) = x(\cdot; t_0, x_0)$ denotes the unique solution of

$$
\dot{x}(t) = v(t, x(t)), \quad x(t_0) = x_0.
$$

Let us note that the rate of change of kinetic energy given by the left-hand side of (2) is, in general, not the same as

$$
\frac{d}{dt} \int_{\Sigma(t)} \rho \frac{v(t, x)^2}{2} \, d\sigma(x) \big|_{t=t_0} = 0,
$$

where $\Sigma(t)$ is the surface composed of all fluid particles at time $t$ which exit through $\Gamma^{\text{out}}$ at $t_0$. The reason are the different surface measures in (2), resp. (4). To decide which expression is the physically correct one, notice first that the integral in (2) stands for a thin layer of fluid of a given constant thickness $\delta > 0$, say, since kinetic energy is stored in the mass of the fluid which requires a volume instead of an area to support for it. If this sheet of fluid is traced backwards along the flow trajectories, the thickness as well as the local surface area change. While the different surface measure in (2), corresponding to vanishing divergence of the velocity field.

Notice that (3) typically is an end value instead of an initial value problem, since the right-hand side in (3) is only defined for $t \leq t_0$ if the fluid trajectory is leaving the domain. But for a bounded and locally Lipschitz velocity field $v$, say, an extension of $v$ with the same regularity to a neighborhood of $\Omega$ is possible such that (3) then has unique solutions at least on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$. Then the derivative in (2) is also well-defined if the fluid locally enters the domain via $\Gamma^{\text{out}}$. Computing the derivative in (2) yields

$$
\int_{\Gamma^{\text{out}}} v(t_0, x_0) \cdot \rho (\partial_t v(t_0, x_0) + \nabla_x v(t_0, x_0) \cdot v(t_0, x_0)) \, d\sigma(x_0) = 0.
$$

Since this should hold for any time and any velocity field, the only appropriate local condition to assure (5) is the condition

$$
v \cdot (\partial_t v + (v \cdot \nabla)v) = 0 \quad \text{on } \Gamma^{\text{out}}.
$$

Evidently, the dynamic boundary condition

$$
(6a) \quad \partial_t v + (v \cdot \nabla)v = 0 \quad \text{on } \Gamma^{\text{out}}
$$

on the full velocity is sufficient for this to hold. In cases when it is reasonable to assume the outgoing flow to be perpendicular to the outflow boundary, the dynamic condition only needs to hold for the normal velocity component, i.e. the following variant is also sufficient:

$$
(6b) \quad P_T v = 0, \quad (\partial_t v + (v \cdot \nabla)v) \cdot \nu = 0 \quad \text{on } \Gamma^{\text{out}},
$$

where $\nu : \Gamma \rightarrow \mathbb{R}^n$ denotes the outer unit normal field at $\Gamma^{\text{out}}$ and $P_T := 1 - \nu \otimes \nu$ denotes the projection onto the tangent bundle. Another variant describes the normal (outgoing, say) velocity component and imposes only the tangential part of the dynamic condition, i.e.

$$
(6c) \quad v \cdot \nu = v^{\text{out}}(t, x) \cdot \nu, \quad P_T (\partial_t v + (v \cdot \nabla)v) = 0 \quad \text{on } \Gamma^{\text{out}}.
$$
with a given outflow velocity $v^\text{out}(t, x)$.

These dynamic ABCs are nonlinear boundary conditions which, in particular for numerical purpose, might be approximated by the linearized versions. For example, the linearized version of (6a) reads as

$$\partial_t v + (v^\text{out}(t, x) \cdot \nabla)v = 0 \quad \text{on } \Gamma^\text{out}$$

with a given outflow velocity $v^\text{out}(t, x)$. In practice, the latter velocity will also be unknown, but certain additional assumptions may be reasonable like perpendicular outflow velocity. Then (7a) becomes

$$\partial_t v + V(t, x)\partial_\nu v = 0 \quad \text{on } \Gamma^\text{out}$$

with a scalar function $V(t, x)$ which is assumed to be known. In case the mean flow across the outflow boundary is known, it is of special interest to consider (7b) with $V(t, x) \equiv V^\text{out}$, where $V^\text{out}$ is either constant or a known function of time.

At this point it is important to mention that the analysis below will also show that the Stokes problem in a half-space together with the dynamic ABC (7a), or even (7b), is not well-posed in the considered Sobolev-Slobodeckij-setting; cf. Remark 3.1. Therefore, an appropriate modification of this condition is required.

For this purpose, recall first that the kinetic energy

$$E_{\text{kin}} := \int_\Omega \rho \frac{v^2}{2} \, dx$$

contained in the full domain changes at the rate

$$\dot{E}_{\text{kin}} = -2\eta \int_\Omega D : \nabla v \, dx + \int_\Omega \rho b \cdot v \, dx + \int_\partial \Omega v \cdot S \nu \, d\sigma - \int_\partial \Omega \rho \frac{v^2}{2} v \cdot \nu \, d\sigma,$$

where $\eta$ is the dynamic viscosity of the fluid, $D = \frac{1}{2}(\nabla v + \nabla v^T)$ denotes the symmetric velocity gradient, $S = 2\eta D - p I$ denotes the stress tensor, $p$ is the pressure, and $b$ are the body force densities. We decompose the full boundary into disjoint parts according to $\partial \Omega = \Gamma^\text{in} \cup \Gamma^\text{wall} \cup \Gamma^\text{out}$, where we assume that $v \cdot S \nu = 0$ on $\Gamma^\text{wall}$. Hence, we obtain

$$\dot{E}_{\text{kin}} = -2\eta \int_\Omega D : \nabla v \, dx + \int_\Omega \rho b \cdot v \, dx$$

$$- \int_{\Gamma^\text{in}} \rho \frac{v^2}{2} v \cdot \nu \, d\sigma + \int_{\Gamma^\text{in}} v \cdot S \nu \, d\sigma - \int_{\Gamma^\text{out}} \rho \frac{v^2}{2} v \cdot \nu \, d\sigma + \int_{\Gamma^\text{out}} v \cdot S \nu \, d\sigma$$

as the rate of change of this energy functional. On the boundaries, the terms with $\rho v^2/2$ describe convective in- and output to the open domain $\Omega$, hence are not related to dissipation. Therefore, the condition for a non-dissipative outflow boundary becomes

$$v \cdot S \nu = 0 \quad \text{on } \Gamma^\text{out},$$

which is satisfied if, e.g., the homogeneous Neumann condition holds, i.e. $S \nu = 0$ on $\Gamma^\text{out}$. Other variants, analogous to the variants above, are $P_{\Gamma} v = 0$ and $S \nu \cdot \nu = 0$ or $v \cdot \nu = 0$ and $P_{\Gamma} S \nu = 0$.

These boundary conditions are natural conditions in the sense that they eliminate the corresponding boundary term in the variational formulation. In a Finite Element context, the omission of the boundary term is also referred to as the “do-nothing condition”; see [8]. Let us also note that well-posedness as well as $L^p$-maximal regularity are known for the Stokes and for the Navier-Stokes equations with Neumann boundary condition; see the remarks and references in [2, 11].

At this point, we have two different sets of artificial boundary conditions, which are all motivated from energy considerations. Somewhat similar to the Robin boundary condition as a linear combination of a Dirichlet and a Neumann condition, we consider the following types of dynamics
outflow boundary conditions, obtained by linear combination of a convective-type linearized dynamic condition and the corresponding variant of the Neumann-type condition: The fully dynamic condition
\[(8a)\quad \alpha (\partial_t v + (v^{out}(t, x) \cdot \nabla)v) + S \nu = 0 \quad \text{on } \Gamma^{out},\]
the normally dynamic variant
\[(8b)\quad P_T v = 0, \quad \alpha (\partial_t v + (v^{out}(t, x) \cdot \nabla)v) \cdot \nu + S \nu \cdot \nu = 0 \quad \text{on } \Gamma^{out},\]
and the tangentially dynamic variant
\[(8c)\quad v \cdot \nu = 0, \quad \alpha P_T (\partial_t v + (v^{out}(t, x) \cdot \nabla)v) + P_T S \nu = 0 \quad \text{on } \Gamma^{out}.\]
In all three ABCs above, $\alpha > 0$ is a model parameter. Let us note in passing that the new ABCs \[(8a)-(8c)\] could also be derived directly from a combined energy functional. In this case, also the nonlinear variants with $v$ instead of $v^{out}(t, x)$ would be reasonable choices.

**A Complete Model.** We now pass to the dimensionless form, writing $u$ for the non-dimensional velocity. Moreover, in order to economize the notation we write $\Gamma = \Gamma^{out}$ for an outflow boundary as in Figure 1. This yields
\[
(\text{NS})^{\text{f}}_{\text{Re}} \quad \partial_t u + (u \cdot \nabla u) - \frac{1}{\text{Re}} \Delta u + \nabla p = f \quad \text{in } J \times \Omega, \\
\quad \text{div } u = 0 \quad \text{in } J \times \Omega
\]
as the well-known dimensionless form of the Navier-Stokes equation inside the domain. Here $J := (0, a)$ with $a > 0$ denotes the time interval within which the flow is to be modeled, and $\text{Re} > 0$ is the Reynolds number. At the outflow boundary, we first record the full dynamic outflow boundary condition, i.e.
\[
\alpha (\partial_t u + (v^{out}(t, x) \cdot \nabla)u) + S \nu = 0 \quad \text{on } J \times \Gamma.
\]
Since the normal and the tangential parts are treated differently below, we also write the full dynamic outflow condition in the form
\[
(\text{FDO})^{v^{out}}_{\alpha, \text{Re}} \quad \alpha P_T (\partial_t u + (v^{out}(t, x) \cdot \nabla)u) + \frac{2}{\text{Re}} P_T D \nu = 0 \quad \text{on } J \times \Gamma, \\
\alpha (\partial_t u + (v^{out}(t, x) \cdot \nabla)u) \cdot \nu + \frac{2}{\text{Re}} D \nu \cdot \nu - p = 0 \quad \text{on } J \times \Gamma.
\]
A variant of this ABC imposes the dynamic condition on the normal part, only, and reads as
\[
(\text{NDO})^{v^{out}}_{\alpha, \text{Re}} \quad P_T u = 0 \quad \text{on } J \times \Gamma, \\
\alpha (\partial_t u + (v^{out}(t, x) \cdot \nabla)u) \cdot \nu + \frac{2}{\text{Re}} D \nu \cdot \nu - p = 0 \quad \text{on } J \times \Gamma.
\]
Finally, there is a third version which imposes the dynamic condition on the tangential component and reads as
\[
(\text{TDO})^{v^{out}}_{\alpha, \text{Re}} \quad \alpha P_T (\partial_t u + (v^{out}(t, x) \cdot \nabla)u) + \frac{2}{\text{Re}} P_T D \nu = 0 \quad \text{on } J \times \Gamma, \\
\quad u \cdot \nu = 0 \quad \text{on } J \times \Gamma.
\]
Note that the homogeneous version of the boundary condition above actually assumes an impermeable boundary, but the theorems to follow treat the nonhomogeneous case as well. For the last ABC, this means a prescribed outgoing normal velocity component. In all boundary conditions above, we assume the velocity $v^{out}$ to be a priori given and to be of the form
\[
(\text{CP}) \quad v^{out} = V \nu, \quad \text{where } V = V(t, x) \text{ satisfies } \alpha V + \frac{1}{\text{Re}} > 0.
\]
Let us note that in the main results to follow, we actually assume $V$ to be constant, since the considered prototype model problems result by a localization process.
Finally, in order to provide a full model for weakly singular domains like the tube in Figure 1, boundary conditions have to be prescribed for the other parts of the boundary as well. For an inlet like \( \Gamma^{\text{in}} \) it is reasonable to assume an inflow condition
\[
(\text{IF})^{\text{in}}\quad u = u^{\text{in}} \quad \text{on } J \times \Gamma^{\text{in}}
\]
with a prescribed velocity profile \( u^{\text{in}} \). On a lateral wall like \( \Gamma^{\text{wall}} \) a Navier type condition
\[
(W)_{\sigma, \text{Re}}\quad \sigma P_{\text{T}}u + \frac{2}{\text{Re}} P_{\text{T}} D\nu = 0 \quad \text{on } J \times \Gamma^{\text{wall}},
\]
\[
u \cdot u = 0 \quad \text{on } J \times \Gamma^{\text{wall}}
\]
with some friction/slip-length \( \sigma \geq 0 \) is suitable to describe the frictional flow along a wall.

2. Main Results

The remaining part of the paper is devoted to the analysis of the Stokes equations \((S)_{\text{Re}, g, u_0}^{\text{g}, \text{uo}}\) subject to a dynamic outflow boundary condition \((\text{BDO})_{a, \text{Re}}^{\text{out}, h} \) with \( B \in \{ T, N, F \} \) in several prototype situations. Our approach is based on \( L_{p} \)-maximal regularity for suitable linearizations of the models. A generic approach to analyze the Stokes and Navier-Stokes equations subject to a large class of different boundary conditions in this setting has been developed in [2, 11]. In these sources the focus is set on so-called energy preserving boundary conditions which are of local and non-dynamic nature. However, this generic approach together with generic results on parabolic problems subject to dynamic boundary conditions as developed in [3] may be adapted to the Stokes equations subject to dynamic outflow boundary conditions \((\text{BDO})_{a, \text{Re}}^{\text{out}, h} \) with \( B \in \{ T, N, F \} \).

2.1. Prototype Models. Here we focus on two prototype models: Again, we set \( J := (0, a) \) with \( a > 0 \). We first study the fully inhomogeneous Stokes equations
\[
\partial_t u - \frac{1}{\text{Re}} \Delta u + \nabla p = f \quad \text{in } J \times \Omega,
\]
\[
\text{div} u = g \quad \text{in } J \times \Omega,
\]
\[
u(0) = u_0 \quad \text{in } \Omega
\]
in a halfspace \( \Omega = \mathbb{R}^n_+ := \{ (x, y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, \; y > 0 \} \), subject to a fully inhomogeneous linear dynamic outflow boundary condition on \( \Gamma = \partial \Omega \), i.e. we either consider the tangentially dynamic outflow boundary condition
\[
(\text{TDO})_{a, \text{Re}}^{\text{out}, h} \quad \alpha P_{\text{T}}(\partial_t u + (\nu^{\text{out}} \cdot \nabla)u) + \frac{2}{\text{Re}} P_{\text{T}} D\nu = P_{\text{T}} h \quad \text{on } J \times \Gamma,
\]
\[
u \cdot u = h \cdot \nu \quad \text{on } J \times \Gamma,
\]
or the normally dynamic outflow boundary condition
\[
(\text{NDO})_{a, \text{Re}}^{\text{out}, h} \quad P_{\text{T}} u = P_{\text{T}} h \quad \text{on } J \times \Gamma,
\]
\[
\alpha(\partial_t u + (\nu^{\text{out}} \cdot \nabla)u) \cdot \nu + \frac{2}{\text{Re}} D\nu \cdot \nu - p = h \cdot \nu \quad \text{on } J \times \Gamma,
\]
or the fully dynamic outflow boundary condition
\[
(\text{FDO})_{a, \text{Re}}^{\text{out}, h} \quad \alpha P_{\text{T}}(\partial_t u + (\nu^{\text{out}} \cdot \nabla)u) + \frac{2}{\text{Re}} P_{\text{T}} D\nu = P_{\text{T}} h \quad \text{on } J \times \Gamma,
\]
\[
\alpha(\partial_t u + (\nu^{\text{out}} \cdot \nabla)u) \cdot \nu + \frac{2}{\text{Re}} D\nu \cdot \nu - p = h \cdot \nu \quad \text{on } J \times \Gamma.
\]

Here, \( \nu : \Gamma \rightarrow \mathbb{R}^n \) again denotes the outer unit normal at the boundary while we denote by \( P_{\Gamma} := 1 - \nu \otimes \nu \) the projection onto the tangent bundle at the boundary. Based on our \( L_{p} \)-maximal regularity result Theorem 2.4, the localization procedure presented in [2, 11] leads to corresponding results for the fully inhomogeneous linear problem in bounded, smooth domains. However, the details of this localization procedure shall not be presented here.
As a second prototype problem we study the fully inhomogeneous Stokes equations \((S)^{f,g,u}_{\text{Re}}\) in a wedge \(\Omega = \mathbb{R}^n_+ := \{(x, y, z) \in \mathbb{R}^n : x \in \mathbb{R}^{n-2}, y > 0, z > 0\}\). This prototype domain has two smooth boundary parts which we denote by

\[
\partial_y \mathbb{R}^n_+ := \{(x, y, z) \in \mathbb{R}^n : x \in \mathbb{R}^{n-2}, y = 0, z > 0\},
\]
and \(\partial_z \mathbb{R}^n_+\), respectively. In order to be able to study domains like the tube in Figure 1, we consider the situation \(\Gamma^{\text{wall}} := \partial_y \mathbb{R}^n_+\) with a fully inhomogeneous Navier condition

\[
(W)^{\text{wall}}_{\sigma, \text{Re}} \quad \text{on} \quad (0, a) \times \Gamma^{\text{wall}},
\]

\[
\sigma \partial_t u + \frac{2}{\text{Re}} \partial_t D\nu = P_t^{\text{wall}} \quad \text{on} \quad (0, a) \times \Gamma^{\text{wall}},
\]

\[
u \cdot h^{\text{wall}} = 0 \quad \text{on} \quad (0, a) \times \Gamma^{\text{wall}}
\]
in combination with \(\Gamma^{\text{in}} := \partial_z \mathbb{R}^n_+\) with the inflow condition \((\text{IF})^{\text{in}}\). Moreover, we consider the situation \(\Gamma^{\text{wall}} := \partial_y \mathbb{R}^n_+\) with a fully inhomogeneous Navier condition \((W)^{\text{wall}}_{\sigma, \text{Re}}\) in combination with \(\Gamma := \partial_y \mathbb{R}^n_+\) with one of the fully inhomogeneous dynamic outflow boundary conditions \((\text{BDO})^{\text{out}, h}\) with \(B \in \{T, N, F\}\). Based on our \(L_p\)-maximal regularity results Theorems 2.4 and 2.6, the localization procedure presented in [11] Chapter 8 leads to corresponding results for the fully inhomogeneous linear problem in weakly singular domains like the one shown in Figure 1. However, the fully general notion of weakly singular domains is not needed in the present work. Moreover, due to space limitations, the details of the localization procedure are also not given here.

2.2. Necessary Regularity/Compatibility Conditions. Our approach leads to \(L_p\)-maximal regular solutions to \((S)^{f,g,u}_{\text{Re}}\), i.e., we assume \(f \in L_p(J \times \Omega)^n\) and obtain

\[
u \in H^1_p(J, L_p(\Omega)^n) \cap L_p(J, H^2_p(\Omega)^n), \quad \text{where} \quad J = (0, a), \quad \Omega \in \{\mathbb{R}^n_+, \mathbb{R}^n_+\}, \quad \text{and} \quad [H^2_p(J, \cdot), H^1_p(\Omega) : s \geq 0, 1 < p < \infty] \quad \text{denotes the scale of (vector-valued) Bessel-potential spaces.}
\]

Moreover, in order to handle the pressure we denote by \([H^2_p(\Omega) : s \geq 0, 1 < p < \infty] \) the scale of homogeneous Bessel-potential spaces. However, if the pressure does not appear in the boundary condition, then it is only unique up to an additive constant. Hence, in some situations we obtain a unique pressure \(p \in L_p(J, \dot{H}^1_p(\Omega))\) within the quotient space \(\dot{H}^1_p(\Omega) := H^1_p(\Omega)/\mathbb{R}\) for \(1 < p < \infty\). Now, standard trace theory leads to velocity traces at time \(t = 0\), and on smooth parts \(\Sigma \subseteq \partial \Omega\) of the boundary within the scale \([W^p_\sigma(J, \cdot), W^p_\sigma(\Omega), W^p_\sigma(\Sigma) : s \geq 0, 1 < p < \infty] \) of (vector-valued) Sobolev-Slobodeckij spaces. This implies regularity conditions for the initial velocity \(u_0\) and the right-hand side \(h\) of the boundary condition, while the mapping properties of the operator div imply a regularity condition for the right-hand side \(g\) of the divergence equation.

Besides the obvious compatibility conditions between \(g\) and \(u_0\) as well as between \(h\) and \(u_0\), there is a hidden compatibility condition between \(g\) and \(h\). To formulate this condition we argue as in [2] Section 2: For \(\Omega = \mathbb{R}^n_+\), and \(\Gamma = \partial \Omega\) we define a linear functional \(F(\psi, \eta)\) for \(\psi \in L_p(\Omega)\), and \(\eta \in L_p(\Gamma)\) as

\[
\langle \phi, F(\psi, \eta) \rangle := \int_{\Gamma} [\phi]_{\Gamma} \eta \, d\sigma - \int_{\Omega} \phi \psi \, dx, \quad \phi \in H^1_p(\Omega),
\]

where \(1 < p' < \infty\) with \(\frac{1}{p} + \frac{1}{p'} = 1\), and \([\cdot]_{\Gamma}\) denotes the trace of a quantity defined in \(\Omega\) on the boundary \(\Gamma\). Then we have

\[
\langle \phi, F(\text{div} u, [u]_{\Gamma} \cdot \nu) \rangle = \int_{\Omega} \nabla \phi \cdot u \, dx, \quad \phi \in H^1_p(\Omega),
\]

which implies

\[
|\langle \phi, \partial^m_p F(\text{div} u, [u]_{\Gamma} \cdot \nu) \rangle| \leq \|\partial^m_p u\|_{L_p(J \times \Omega)^n} \|\nabla \phi\|_{L_{p'}(\Omega)^n}, \quad \phi \in H^1_p(\Omega),
\]

for \(m = 0, 1\). Since a solution \(u\) to \((S)^{f,g,u}_{\text{Re}}\) satisfies \(\text{div} u = g\), this leads to a compatibility condition between \(g\) and \([u]_{\Gamma} \cdot \nu\), which may be a prescribed quantity depending on the boundary.
condition. To be precise, we have
\[
F(g, [u]_\Gamma \cdot \nu) \in H_p^1(J, \hat{H}^{-1}_p(\Omega))
\]
with \(\hat{H}^{-1}_p(\Omega) := (H_p^1(\Omega), \cdot | H_p^1(\Omega)')\) and \(|\cdot|_{H_p^1(\Omega)} = ||\nabla \cdot ||_{L_p(\Omega)^n}\). Analogously, for \(\Omega = \mathbb{R}_n^+\), \(\Sigma = \partial_y \mathbb{R}_n^+\), and \(\Gamma = \partial_z \mathbb{R}_n^+\) we define the linear functional \(F(\psi, \eta_{\Sigma}, \eta_{\Gamma})\) for \(\psi \in L_p(\Omega), \eta_{\Sigma} \in L_p(\Sigma)\), and \(\eta_{\Gamma} \in L_p(\Gamma)\) as
\[
\langle \phi, F(\psi, \eta_{\Sigma}, \eta_{\Gamma}) \rangle := \int_{\Sigma} [\phi \eta_{\Sigma} \nu \, d\sigma] + \int_{\Gamma} [\phi \eta_{\Gamma} \, d\sigma] - \int_{\Omega} \phi \psi \, dx, \quad \phi \in H_p^1(\Omega),
\]
and obtain
\[
||\langle \phi, \partial_t^m F(\div u, [u]_{\Sigma} \cdot \nu, [u]_{\Gamma} \cdot \nu) \rangle || \leq ||\partial_t^m u||_{L_p(J \times \Omega)^n} ||\nabla \phi||_{L_p(\Omega)^n}, \quad \phi \in H_p^1(\Omega),
\]
for \(m = 0, 1\). As above, this leads to a compatibility condition between \(g, [u]_{\Sigma} \cdot \nu\) and \([u]_{\Gamma} \cdot \nu\), which may be prescribed quantities depending on the boundary condition. In this case we have
\[
F(g, [u]_{\Sigma} \cdot \nu, [u]_{\Gamma} \cdot \nu) \in H_p^1(J, \hat{H}^{-1}_p(\Omega))
\]
with \(\hat{H}^{-1}_p(\Omega) := (H_p^1(\Omega), \cdot | H_p^1(\Omega)')\) as above.

Finally, for the model problems in \(\Omega = \mathbb{R}_n^+\) there are compatibility conditions on the edge \(\mathcal{E} = \partial_y \mathbb{R}_n^+ \cap \partial_z \mathbb{R}_n^+\) which have to be satisfied by the right-hand sides of the boundary conditions.

First, if we impose a Navier condition \((W)_{\sigma, Re}^{\text{wall}}\) on \(\Gamma_{\text{wall}} = \partial_y \mathbb{R}_n^+\) in combination with an inflow condition \((IF)^{\text{in}}\) on \(\Gamma^{\text{in}} = \partial_z \mathbb{R}_n^+\), then we necessarily have
\[
\begin{align*}
\sigma P_{\mathcal{E}} u^{\text{in}} + \frac{1}{\text{Re}} \partial_{t_{\text{wall}}} (P_{\mathcal{E}} u^{\text{in}}) + \frac{1}{\text{Re}} \nabla_{\mathcal{E}} (h_{\text{wall}} \cdot \nu_{\text{wall}}) &= P_{\mathcal{E}} h_{\text{wall}} & \text{on } J \times \mathcal{E}, \\
\sigma u^{\text{in}} \cdot \nu_{\Gamma} + \frac{1}{\text{Re}} \partial_{t_{\text{wall}}} (u^{\text{in}} \cdot \nu_{\Gamma}) + \frac{1}{\text{Re}} \partial_{t_{\Gamma}} (h_{\text{wall}} \cdot \nu_{\text{wall}}) &= h_{\text{wall}} \cdot \nu_{\Gamma} & \text{on } J \times \mathcal{E},
\end{align*}
\]
where we denote by \(P_{\mathcal{E}}\) the projection onto the tangent bundle of \(\mathcal{E}\), and by \(\nabla_{\mathcal{E}}\) the surface gradient. Note that this is a simplified form of the necessary compatibility conditions which is valid for the simple geometry of the wedge \(\mathbb{R}_n^+\). For a generic weakly singular domain additional curvature related terms appear in the first and last lines which stem from tangential derivatives of the normal fields \(\nu_{\Sigma}\) and \(\nu_{\Gamma}\).

Second, if we impose a Navier condition \((W)_{\sigma, Re}^{\text{wall}}\) on \(\Gamma_{\text{wall}} = \partial_y \mathbb{R}_n^+\) in combination with a dynamic outflow boundary condition \((\text{BDO})^{\text{out}, h}_{\alpha, Re}\) with \(B \in \{T, N, F\}\) on \(\Gamma = \partial_z \mathbb{R}_n^+\), then we necessarily have an analogous compatibility condition, where, however, the velocity profile on \(\Gamma\) is (in part) not prescribed. For \(B = T\) this leads to
\[
\begin{align*}
\sigma P_{\mathcal{E}} \xi + \frac{1}{\text{Re}} \partial_{t_{\text{wall}}} (P_{\mathcal{E}} \xi) + \frac{1}{\text{Re}} \nabla_{\mathcal{E}} (h_{\text{wall}} \cdot \nu_{\text{wall}}) &= P_{\mathcal{E}} h_{\text{wall}} & \text{on } J \times \mathcal{E}, \\
\xi \cdot \nu_{\text{wall}} &= h_{\text{wall}} \cdot \nu_{\text{wall}} & \text{on } J \times \mathcal{E},
\end{align*}
\]
for some function
\[
\xi \in W^{3/2-1/p,2}(J, L_p(\Gamma, TT)) \cap H_p^1(J, W_p^{1-1/p}(\Gamma, TT)) \cap L_p(J, W_p^{2-1/p}(\Gamma, TT))
\]
that is compatible with \(u_0\). For \(B = N\) we obtain
\[
\begin{align*}
\sigma P_{\mathcal{E}} h + \frac{1}{\text{Re}} \partial_{t_{\text{wall}}} (P_{\mathcal{E}} h) + \frac{1}{\text{Re}} \nabla_{\mathcal{E}} (h_{\text{wall}} \cdot \nu_{\text{wall}}) &= P_{\mathcal{E}} h_{\text{wall}} & \text{on } J \times \mathcal{E}, \\
 h \cdot \nu_{\text{wall}} &= h_{\text{wall}} \cdot \nu_{\text{wall}} & \text{on } J \times \mathcal{E},
\end{align*}
\]
\[
\begin{align*}
\sigma \eta + \frac{1}{\text{Re}} \partial_{t_{\text{wall}}} \eta + \frac{1}{\text{Re}} \partial_{t_{\Gamma}} (h_{\text{wall}} \cdot \nu_{\text{wall}}) &= h_{\text{wall}} \cdot \nu_{\Gamma} & \text{on } J \times \mathcal{E},
\end{align*}
\]
for some function
\[ \eta \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma)) \]
that is compatible with \( g \), and \( u_0 \). For \( B = F \) we have
\[ \sigma P_{\xi} \eta + \frac{1}{\text{Re}} \partial_{\nu_{\text{wall}}} (P_{\xi} \eta) + \frac{1}{\text{Re}} \nabla_\xi (h_{\text{wall}} \cdot \nu_{\text{wall}}) = P_\xi h_{\text{wall}} \] on \( J \times \mathcal{E} \),
\[ \xi \cdot \nu_{\text{wall}} = h_{\text{wall}} \cdot \nu_{\text{wall}} \] on \( J \times \mathcal{E} \),
\[ (\text{FDO/W})^{h,h_{\text{wall}},\xi,\eta} \]
\[ \sigma \eta + \frac{1}{\text{Re}} \partial_{\nu_{\text{wall}}} \eta + \frac{1}{\text{Re}} \partial_\eta (h_{\text{wall}} \cdot \nu_{\text{wall}}) = h_{\text{wall}} \cdot \nu_\Gamma \] on \( J \times \mathcal{E} \),
\[ \alpha (\partial_\xi (h_{\text{wall}} \cdot \nu_{\text{wall}}) + V \partial_{\xi} (h_{\text{wall}} \cdot \nu_{\text{wall}})) \]
\[ + h_{\text{wall}} \cdot \nu_\Gamma - \sigma \eta = h \cdot \nu_{\text{wall}} \] on \( J \times \mathcal{E} \)
for some functions \( \xi \), and \( \eta \) as above. Again these are simplified forms of the necessary compatibility conditions which are valid for the simple geometry of the wedge \( \mathbb{R}^n_+ \) and have to be modified for a generic weakly singular domain by additional curvature related terms.

2.3. Main Results. With the above preparations, we now formulate our main results, the proofs of which are carried out in Sections 3 and 4.

**Theorem 2.1.** Let \( a > 0 \), let \( J := (0,a) \) and let \( \Omega = \mathbb{R}^n_+ \) with \( \Gamma := \partial \Omega \). Let \( 1 < p < \infty \) with \( p \neq \frac{3}{2} \). Moreover, let \( B \in \{ T, N, F \} \), and let \( \alpha, \text{Re} > 0 \). Furthermore, let \( v_{\text{out}} = V \nu \) with \( V > -\frac{1}{\alpha \text{Re}} \), and let

- \( f \in L_p(J \times \Omega)^n \),
- \( g \in H^{1/2}_p(J, L_p(\Omega)) \cap L_p(J, H^1_p(\Omega)) \),
- \( h \in L_p(J, W^{1-1/p}_p(\Gamma))^n \),
- \( u_0 \in W^{2-2/p}_p(\Omega)^n \) with \( \text{div} u_0 = g(0) \) in \( \Omega \) for \( p \geq 2 \).

If \( B = T \), let
- \( P_{\xi} h \in W^{1/2-1/2p}_p(J, L_p(\Gamma))^n \), \( P_{\Gamma} [u_0]_\Gamma \in W^{2-2/p}_p(\Gamma)^n \),
- \( h \cdot \nu \in W^{1-1/2p}_p(J, L_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma)) \),
- \( F(g, h \cdot \nu) \in H^1_p(J, H^1_p(\Omega)) \),
- \( [u_0]_\Gamma \cdot \nu = h(0) \cdot \nu \) for \( p > \frac{3}{2} \);

if \( B = N \), let
- \( P_{\xi} h \in W^{1-1/2p}_p(J, L_p(\Gamma))^n \cap L_p(J, W^{2-1/p}_p(\Gamma))^n \),
- \( F(g, \eta) \in H^{1/2}_p(J, H^{-1/2}_p(\Omega)) \) for some \( \eta \in H^{1/2}_p(J, W^{1-1/p}_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma)) \) with \( [u_0]_\Gamma \cdot \nu = \eta(0) \) for \( p > \frac{3}{2} \),
- \( P_{\xi} [u_0]_\Gamma = P_{\eta}(h(0) \cdot \nu) \) for \( p > \frac{3}{2} \);

if \( B = F \), let
- \( P_{\xi} h \in W^{1/2-1/2p}_p(J, L_p(\Gamma))^n \), \( P_{\Gamma} [u_0]_\Gamma \in W^{2-2/p}_p(\Gamma)^n \),
- \( F(g, \eta) \in H^{1/2}_p(J, H^{-1/2}_p(\Omega)) \) for some \( \eta \in H^{1/2}_p(J, W^{1-1/p}_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma)) \) with \( [u_0]_\Gamma \cdot \nu = \eta(0) \) for \( p > \frac{3}{2} \).

Then the system \( \{S\}_\alpha \) admits a unique maximal regular solution
- \( u \in H^1_p(J, L_p(\Omega))^n \cap L_p(J, H^1_p(\Omega))^n \),
- \( p \in L_p(J, H^1_p(\Omega)) \) for \( B = T \), or
- \( p \in L_p(J, H^1_p(\Omega)) \) with \( [p] \in L_p(J, W^{1-1/p}_p(\Gamma)) \) for \( B \in \{ N, F \} \).

If \( B \in \{ T, F \} \), then we additionally have
- \( P_{\xi} [u]_\Gamma \in W^{3/2-1/2p}_p(J, L_p(\Gamma))^n \cap H^{1/2}_p(J, W^{1-1/p}_p(\Gamma))^n \cap L_p(J, W^{2-1/p}_p(\Gamma))^n \);

if \( B \in \{ N, F \} \), then we additionally have
- \( [u]_\Gamma \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma)) \).

The solutions depend continuously on the data in the corresponding spaces.
The proof of Theorem 2.1, which is based on a precise analysis of the corresponding boundary symbols, is carried out in Section 3. Here, however, some remarks seem to be in order.

Remark 2.2. There are some immediate corollaries of Theorem 2.1 which we want to mention without elaborate proofs.

(a) If \( B \in \{ N, F \} \), then the assumptions on the right-hand side of the boundary condition may be relaxed to \( h - \nu \in L_p(J, W^{1,p}_p(\Gamma)) \) to obtain a maximal regular solution as in Theorem 2.1 with \([p]\in L_p(J, W^{1,p}_p(\Gamma))\). Indeed, one first constructs an auxiliary pressure \( q \in L_p(J, H^1_p(\Omega)) \) as a weak solution to

\[
-\Delta q = 0 \quad \text{in} \ J \times \Omega,
q = -h \cdot \nu \quad \text{on} \ J \times \Gamma,
\]

and then solves \((S)^{\text{IF}}_{\text{Re}}, (BDO)^{\text{out}}_{\text{Re}}\) via Theorem 2.1 with the adjusted data \( f' = f - \nabla q, P_1 h' = P_1 h, \) and \( h' \cdot \nu = 0 \) to obtain a solution \((u', p')\) in the maximal regularity class. Then \( u = u', \ p = p' + q\) constitutes the unique maximal regular solution to the model problem with relaxed regularity assumptions. Conversely the relaxed version of Theorem 2.1 obviously implies Theorem 2.1, i.e. both formulations of the theorem are equivalent.

(b) One may assume \( v_{\text{out}} \) to be given based on

\[
V \in W^{1,1/2}_p(J, L_p(\Gamma)) \cap L_p(J, W^{2,1/2}_p(\Gamma))
\]

such that \((C1)\) is satisfied. Indeed, this problem may be reduced to Theorem 2.1 via a localization procedure.

Of course, Corollaries (a) and (b) are independent of each other and may be applied simultaneously.

Remark 2.3. Theorem 2.1 and its variants in Remark 2.2 are the cornerstones to obtain corresponding results for bounded, smooth domains \( \Omega \subseteq \mathbb{R}^n \) via well-known localization procedures as presented e.g. in 2. Based on well-known perturbation arguments, it is then also possible to obtain (local-in-time) strong solutions to the corresponding non-linear equations \((\text{NS})^{\text{Re}}\) with non-linear variants of the dynamic outflow boundary conditions.

Theorem 2.4. Let \( a > 0 \), let \( J := (0, a) \) and let \( \Omega = \mathbb{R}^n_+ \) with \( \Gamma_{\text{wall}} := \partial_p \mathbb{R}^n_+ \), and \( \Gamma_{\text{in}} := \partial_n \mathbb{R}^n_+ \). Let \( 1 < p < \infty \) with \( p \neq \frac{3}{2}, 3 \). Moreover, let \( \sigma \geq 0 \), and let

- \( f \in L_p(J \times \Omega)^n \),
- \( g \in H^{1/2}_p(J, L_p(\Omega)) \cap L_p(J, H^1_p(\Omega)) \),
- \( u^{\text{in}} \in W^{1,1/2}_p(J, L_p(\Gamma_{\text{in}}))^n \cap L_p(J, W^{2,1/2}_p(\Gamma_{\text{in}}))^n \),
- \( h^{\text{wall}} \in W^{1/2,1/2}_p(J, L_p(\Gamma_{\text{wall}}))^n \cap L_p(J, W^{2,1/2}_p(\Gamma_{\text{wall}}))^n \),
- \( u_0 \in W^{2,1/2}_p(\Omega)^n \) with \( \text{div} u_0 = g(0) \) in \( \Omega \) for \( p \geq 2 \), and \( [u_0]|_{\Gamma_{\text{in}}} = u^{\text{in}}(0) \) as well as \([u_0]|_{\Gamma_{\text{wall}}} \cdot \nu = h^{\text{wall}}(0) \cdot \nu \) for \( p > \frac{3}{2} \), and \( \sigma P_1[u_0]|_{\Gamma_{\text{wall}}} + \frac{2\sigma}{\text{Re}} P_1[D_0|_{\Gamma_{\text{wall}}} \nu = P_1 h^{\text{wall}}(0) \) for \( p > 3 \).

Furthermore, let the compatibility condition \([\text{IF}/W]^{u,h}_{\sigma,\text{Re}}\) be satisfied for \( p \geq 2 \). Then the system \((S)^{\text{IF}}_{\text{Re}}, (\text{IF})^{u,h}_{\text{Re}}\) admits a unique maximal regular solution

- \( u \in H^1_p(J, L_p(\Omega))^n \cap L_p(J, H^2_p(\Omega))^n \),
- \( p \in L_p(J, H^1_p(\Omega)) \).

The solutions depend continuously on the data in the corresponding spaces.

The proof of Theorem 2.4, which is based on a reflection technique and Theorem 2.1 is carried out in Section 4. Here, however, we have to compare it with known results.
Remark 2.5. Theorem 2.4 is contained as a special case in [11] Theorem 8.24. However, in order to keep this paper self-contained we give a short proof of Theorem 2.4 in Section 3 which is different (shorter and more descriptive) from that presented in [11], since we restrict our considerations to a special combination of boundary conditions.

Theorem 2.6. Let $a > 0$, let $J := (0, a)$ and let $\Omega = \mathbb{R}^n_+$ with $\Gamma^{wall} := \partial_y \mathbb{R}^n_+$, and $\Gamma := \partial_x \mathbb{R}^n_+$. Let $1 < p < \infty$ with $p \neq \frac{2}{n}$, 3. Moreover, let $\sigma \geq 0$, let $B \in \{ T, N, F \}$, and let $\alpha, \text{Re} > 0$. Furthermore, let $u^{\text{out}} = V \nu$ with $V > -\frac{1}{4 \text{Re}}$, and let

- $f \in L^p_p(J \times \Omega)^3$,
- $g \in H^{1/2}_p(J, L^p_p(\Omega)) \cap L^p_p(J, H^{1}_p(\Omega))$,
- $h \in L^p_p(J, W^{1-1/p}_p(\Gamma))$, \( h^{wall} \in W^{1-1/2p}_p(J, L^p_p(\Gamma^{wall})) \cap L^p_p(J, W^{1-1/p}_p(\Gamma^{wall})) \),
- $u_0 \in W^{2,2}_p(\Omega)$ with $u_0 = g(0)$ in $\Omega$ for $p \geq 2$, and $\nu \in W^{1-1/2p}_p(J, L^p_p(\Gamma^{wall})) \cap L^p_p(J, W^{2-1/p}_p(\Gamma^{wall}))$,
- $\eta_0 \in W^{2,2}_p(\Gamma^{wall})$ with $\partial \eta_0 \cdot \nu = h^{wall}(0) \cdot \nu$ for $p > \frac{3}{2}$, and $\sigma P_T[u_0]^{\text{wall}} + \frac{2}{\text{Re}} P_T[D_0]^{\text{wall}} \nu = P_T h^{\text{wall}}(0)$ for $p > 3$.

If $B = T$, let

- $P_T h \in W^{1-1/p}_p(J, L^p_p(\Gamma))$, $P_T[u_0] \in W^{2-2/p}_p(\Gamma)$,
- $h \cdot \nu \in W^{1-1/p}_p(J, L^p_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$,
- $F(g, h, \nu, \eta) \in H^1_p(J, H^{-1}_p(\Omega))$ for some $\eta \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$ with $[u_0] \cdot \nu = \eta(0)$ for $p > \frac{3}{2}$;
- the compatibility condition \([\text{DWO/W}]^{h,h^{wall}}\) be satisfied for $p \geq 2$ for some $\xi \in W^{2,1-1/2p}_p(J, L^p_p(\Gamma)) \cap H^1_p(J, W^{2-1/p}_p(\Gamma, \Gamma))$ with $P_T[u_0] = \xi(0)$ for $p > \frac{3}{2}$;

if $B = N$, let

- $P_T h \in W^{1-1/p}_p(J, L^p_p(\Gamma))$, $P_T[u_0] \in W^{2-2/p}_p(\Gamma)$,
- $F(g, h, \nu, \eta) \in H^1_p(J, H^{-1}_p(\Omega))$ for some $\eta \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$ with $[u_0] \cdot \nu = \eta(0)$ for $p > \frac{3}{2}$;
- the compatibility condition \([\text{DWO/W}]^{h,h^{wall}}\) be satisfied for $p \geq 2$;

if $B = F$, let

- $P_T h \in W^{1-1/2p}_p(J, L^p_p(\Gamma))$, $P_T[u_0] \in W^{2-2/p}_p(\Gamma)$,
- $F(g, h, \nu, \eta) \in H^1_p(J, H^{-1}_p(\Omega))$ for some $\eta \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$ with $[u_0] \cdot \nu = \eta(0)$ for $p > \frac{3}{2}$;
- the compatibility condition \([\text{DWO/W}]^{h,h^{wall}}\) \(\xi\) be satisfied for $p \geq 2$ for some $\xi \in W^{2,1-1/2p}_p(J, L^p_p(\Gamma)) \cap H^1_p(J, W^{2-1/p}_p(\Gamma, \Gamma))$ with $P_T[u_0] = \xi(0)$ for $p > \frac{3}{2}$;

Then the system \([S]^{f,g,u}_u, (\text{BDWO})^{h,h^{wall}}_{\alpha,\text{Re}}, (\text{W})^{h^{wall}}_{\text{Re}}\) admits a unique maximal regular solution

- $u \in H^1_p(J, L^p_p(\Omega)) \cap L^p_p(J, H^2_p(\Omega))$,
- $p \in L^p_p(J, H^1_p(\Omega))$ for $B = T$, or $p \in L^p_p(J, H^1_p(\Omega))$ with $[p] \in L^p_p(J, W^{1-1/p}_p(\Gamma))$ for $B \in \{ N, F \}$.

If $B \in \{ T, F \}$, then we additionally have

- $P_T[u] \in W^{1-1/2p}_p(J, L^p_p(\Gamma)) \cap H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$;

if $B \in \{ N, F \}$, then we additionally have

- $[u] \cdot \nu \in H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L^p_p(J, W^{2-1/p}_p(\Gamma))$.

The solutions depend continuously on the data in the corresponding spaces.
The proof of Theorem 2.4, which is based on a reflection technique and Theorems 2.1 and 2.3, is carried out in Section 2. Here, however, some remarks seem to be in order.

**Remark 2.7.** Again there are some immediate corollaries of Theorem 2.6, which we want to mention without elaborate proofs, cf. Remark 2.2.

(a) If \( B \in \{ N, F \} \), then the assumptions on the right-hand side of the boundary condition may be relaxed to \( h_v, v, w \in L_p(J, W^{1,1/p}(\Gamma)) \) to obtain a maximal regular solution as in Theorem 2.6 with \( [p]_\Gamma \in L_p(J, W^{1,1/p}(\Gamma)) \). The argument here is the same as used in Remark 2.2 (a) and both formulations of Theorem 2.6 are again equivalent.

(b) One may assume \( v^{\text{out}} \) to be given based on

\[
V \in W^{1,1/(2p)}_p(J, L_p(\Gamma)) \cap L_p(J, W^{2,1/p}_p(\Gamma))
\]

such that \([\text{CP}]\) is satisfied. Indeed, this problem may be reduced to Theorem 2.6 via a localization procedure.

Of course, Corollaries (a) and (b) are independent of each other and may be applied simultaneously.

**Remark 2.8.** Theorems 2.4 and 2.6 and the variants in Remark 2.7 are the cornerstones to handle realistic models in weakly singular domains \( \Omega \subseteq \mathbb{R}^n \) like the tube in Figure 1 via localization procedures as presented e.g. in [11, Chapter 8]. Based on well-known perturbation arguments, it is then also possible to obtain (local-in-time) strong solutions to the corresponding non-linear equations \([\text{NS}]_{\Gamma}^{\text{f,rec}}\) with non-linear variants of the dynamic outflow boundary conditions.

### 3. The Halfspace Case

This section is devoted to the first step of the proof of Theorem 2.1 where the halfspace \( \Omega := \mathbb{R}^n_+ \) is considered with \( \Gamma := \partial \Omega \). We assume \( a > 0 \), set \( J := (0, a) \), and assume \( 1 < p < \infty \) with \( p \neq \frac{2}{3}, 3 \). Furthermore, we assume \( \alpha, \text{Re} > 0 \) as well as \( v^{\text{out}} = V \nu \) with \( \sigma := \alpha V + \frac{\alpha}{\text{Re}}, \kappa := \alpha V + \frac{\alpha}{\text{Re}} > 0 \). We exploit the simple geometry of the halfspace and denote by \((x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}_+\) the generic point in \( \mathbb{R}^n_+ \), decomposed in its tangential part \( x \in \mathbb{R}^{n-1} \) and its normal part \( y > 0 \). Moreover, we employ the notation \( u = (v, w) \) to decompose the unknown velocity field into its tangential part \( v : J \times \mathbb{R}^n_+ \to \mathbb{R}^{n-1} \) and its normal part \( w : J \times \mathbb{R}^n_+ \to \mathbb{R} \). Finally, we denote by \([ \cdot ]_y : \mathbb{R}^n_+ \to \partial \mathbb{R}^n_+ \) the trace operator for the halfspace and frequently employ the identification \( \partial \mathbb{R}^n_+ \approx \mathbb{R}^{n-1} \), whenever this seems to be convenient. The right hand side of the boundary condition is decomposed as \( h = (h_v, h_w) \) into a tangential part \( h_v \) and a normal part \( h_w \). The same splitting is employed for the initial velocity, where we let \( u_0 = (v_0, w_0) \).

#### 3.1. The Condition TDO

We first consider the Stokes equations subject to a dynamic outflow boundary condition in tangential directions, i.e. the system \((\text{S})_{\Gamma}^{\text{f,rec}}(\text{TDO})_{\alpha, \text{Re}}^{\text{v,rec}}\) which reads

\[
\begin{align*}
\partial_t u - \frac{1}{\text{Re}} \Delta u + \nabla p &= f & \text{in } J \times \mathbb{R}^n_+, \\
\text{div} u &= g & \text{in } J \times \mathbb{R}^n_+, \\
\alpha \partial_t [v]_y - (\alpha V + \frac{1}{\text{Re}}) [\partial_y v]_y - \frac{1}{\text{Re}} \nabla_x [w]_y &= h_v & \text{on } J \times \mathbb{R}^{n-1}, \\
[w]_y &= h_w & \text{on } J \times \mathbb{R}^{n-1}, \\
u(0) &= u_0 & \text{in } \mathbb{R}^n_+.
\end{align*}
\]

Here, we require the data to satisfy the regularity and compatibility conditions as stated in Theorem 2.1, i.e. we have

- \( f \in L_p(J \times \mathbb{R}^n_+)^n \),
- \( g \in H^{1/2}_p(J, L_p(\mathbb{R}^n_+)) \cap L_p(J, H^1_p(\mathbb{R}^n_+)) \),
- \( h_v \in W^{1,2-1/(2p)}_p(J, L_p(R^{n-1}))^{n-1} \cap L_p(J, W^{1,1/p}_p(\mathbb{R}^{n-1}))^{n-1} \),
- \( h_w \in W^{1,2-1/(2p)}_p(J, L_p(R^{n-1})) \cap L_p(J, W^{2-1/p}_p(\mathbb{R}^{n-1})) \),
- \( u_0 \in W^{2-2/p}_p(\mathbb{R}^n_+) \) with \( \text{div} u_0 = g(0) \) in \( \mathbb{R}^n_+ \) for \( p \geq 2 \).
• $F(g, -h_w) \in H^1_p(J, \dot{H}^{1,-1}(\mathbb{R}^n_+))$,
• $[v_0]_y \in W^{2-2/p}_p(\mathbb{R}^{n-1})^{-1}$, and $[w_0]_y = h_w(0)$ for $p > \frac{3}{2}$.

The construction of a solution to (9) requires several steps.

**Step 1.** As a first step we show that we may w.l.o.g. assume $f = 0$, $g = 0$, $h_v = 0$ and $u_0 = 0$ in the following. Indeed, based on the compatibility condition between $g$ and $-h_w$, we may employ Proposition 3.6 to obtain $q \in L_p(J, \dot{H}^1_p(\mathbb{R}^+))$ such that $-\text{div} q = (\partial_t - \frac{1}{Re} \Delta)g$ in the sense of distributions. Then we solve the parabolic system with dynamic boundary conditions

$$
\partial_t u - \frac{1}{Re} \Delta u = W_p f - \nabla q \quad \text{in } J \times \mathbb{R}^n_+,
$$

$$
\alpha \partial_t[v]_y - (\alpha V + \frac{1}{Re})[\partial_y v]_y = h_v + \frac{1}{Re} \nabla_x [w]_y \quad \text{on } J \times \mathbb{R}^{n-1},
$$

$$
[\partial_y w]_y = [g]_y - \nabla_x [v]_y \quad \text{on } J \times \mathbb{R}^{n-1},
$$

$$
u(0) = u_0 \quad \text{in } \mathbb{R}^n_+.
$$

to obtain a unique solution $u$ in the desired regularity class via Proposition A.1. Here, we employ the Weyl projection $\mathcal{W} : L_p(\mathbb{R}^n_+) \rightarrow L_p(\mathbb{R}^n_+)$ that belongs to the topological decomposition $L_p(\mathbb{R}^n_+)^n = L_{p,s}(\mathbb{R}^n_+) \oplus \nabla_0 \dot{H}^1_p(\mathbb{R}^n_+)$ into

$$
L_{p,s}(\mathbb{R}^n_+) := \left\{ \phi \in L_p(\mathbb{R}^n_+) : \text{div } \phi = 0 \right\}, \quad \dot{H}^1_p(\mathbb{R}^n_+) := \left\{ \psi \in \dot{H}^1_p(\mathbb{R}^n_+) : [\psi]_y = 0 \right\},
$$

see e.g. Section 3]. If we then define $p \in L_p(J, \dot{H}^1_p(\mathbb{R}^n_+))$ via $\nabla p = \nabla q + (1 - W_p)f$, then

$$
\partial_t u - \frac{1}{Re} \Delta u + \nabla p = f \quad \text{in } J \times \mathbb{R}^n_+,
$$

$$
\alpha \partial_t[v]_y - (\alpha V + \frac{1}{Re})[\partial_y v]_y - \frac{1}{Re} \nabla_x [w]_y = h_v \quad \text{on } J \times \mathbb{R}^{n-1},
$$

$$
u(0) = u_0 \quad \text{in } \mathbb{R}^n_+.
$$

Moreover, we have by construction

$$
\partial_t \gamma - \frac{1}{Re} \Delta \gamma = 0 \quad \text{in } J \times \mathbb{R}^n_+,
$$

$$
[\gamma]_y = 0 \quad \text{on } J \times \mathbb{R}^{n-1},
$$

$$
\gamma(0) = 0 \quad \text{in } \mathbb{R}^n_+.
$$

for $\gamma = \text{div } u - g \in BC(J, W^{1-1/p}_p(\mathbb{R}^n_+)) \hookrightarrow BC(J, L_p(\mathbb{R}^n_+))$, which implies $\gamma = 0$ by uniqueness of weak solutions to the diffusion equation with Dirichlet boundary condition, see also the proof of Theorem 3.6. Thus, $\text{div } u = g$. Hence, we may assume $f = 0$, $g = 0$, $h_v = 0$ and $u_0 = 0$. Note that in this case the compatibility condition between the right-hand side of the divergence equation and the normal boundary condition implies

$$
h_w \in L^1_p(J, \dot{W}^{1-1/p}_p(\mathbb{R}^{n-1})) \cap W^{2-1/p}_p(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W^{2-1/p}_p(\mathbb{R}^{n-1})),
$$

which we will assume from now on.

**Step 2.** In order to solve the remaining problem, we will employ a Laplace transformation in time and a Fourier transformation in the tangential part of the spatial variables. Since this is only possible for an unbounded time interval, we will from now on consider the shifted problem

$$
\varepsilon u + \partial_t u - \frac{1}{Re} \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+,
$$

$$
div u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}_+,
$$

$$
(10) \quad \alpha \varepsilon[v]_y + \alpha \partial_t[v]_y - (\alpha V + \frac{1}{Re})[\partial_y v]_y - \frac{1}{Re} \nabla_x [w]_y = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}_+,
$$

$$
[w]_y = h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}_+,
$$

$$
u(0) = 0 \quad \text{in } \mathbb{R}^n_+.
$$

for an arbitrary $\varepsilon > 0$. Note that maximal regularity for this problem is equivalent to maximal regularity of the original problem (i.e. for $\varepsilon = 0$) on finite time intervals $J = (0, a)$. The strategy
to construct a solution to (10) is as follows: We compute the pressure derivative $-\partial_y p = \Pi h_w$ and show that it is given based on a bounded linear operator

$$\Pi : 0H^1_p(\mathbb{R}^+, \dot{W}^{-1/p}(\mathbb{R}^n)) \cap 0W^{-1/2p}(\mathbb{R}^+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}^+, W^{-1/p}(\mathbb{R}^n)) \rightarrow L_p(\mathbb{R}^+, W^{-1/p}(\mathbb{R}^n)).$$

(11)

Then we obtain the pressure $p \in L_p(\mathbb{R}^+, \dot{W}^1_p(\mathbb{R}^n))$ as a solution to the (weak) elliptic problem

$$-\Delta p = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n,$$

$$-\partial_y p = \Pi h_w \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n,$$

cf. [2, Proposition 3.3]. Finally, we obtain $u$ as a maximal regular solution to the parabolic problem

$$\varepsilon u + \partial_t u - \frac{1}{\mu} \Delta u = -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n,$$

$$\alpha \partial_y [v] + \alpha \partial_y [v] - (\alpha V + \frac{1}{\mu}) \partial_y [v] = \frac{1}{\mu} \nabla_x h_w \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n,$$

$$w_y = h_w \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n,$$

$$u(0) = 0 \quad \text{in } \mathbb{R}^n,$$

via Proposition A.1.

**Step 2.1.** We compute the Laplace-Fourier symbol of $\Pi$. The transformed equations (10) read:

$$\omega^2 \hat{v} - \frac{1}{\mu} \partial_y^2 \hat{v} + i\xi \hat{p} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0,$$

$$\omega^2 \hat{w} - \frac{1}{\mu} \partial_y^2 \hat{w} + \partial_y \hat{p} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0,$$

$$i\xi \cdot \hat{v} + \partial_y \hat{w} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0,$$

$$\alpha \lambda \hat{v} - \kappa \partial_y \hat{v} - \frac{1}{\mu} \partial_y \hat{w} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1},$$

$$\hat{w}_y = h_w \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1},$$

where $\hat{v}, \hat{w}, \hat{p}$ and $\hat{h}_w$ denote the transformed quantities, $\lambda \in \Sigma_{\pi - \theta} \subset \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \psi\}$ denotes the Laplace co-variable of $t$, where $\Sigma_{\psi} := \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \psi \}$ for $0 < \psi < \pi$, and $\xi \in \mathbb{R}^{n-1}$ denotes the Fourier co-variable of $x$. Moreover, we use the abbreviations

$$\lambda_{\varepsilon} := \varepsilon + \lambda, \quad \omega := \sqrt{\lambda_{\varepsilon} + |\xi|^2}, \quad \zeta := \frac{1}{\mu \sqrt{\xi}} \xi$$

The first three equations above are ordinary differential equations for $y > 0$, whose solutions admit a representation by linear combinations of fundamental solutions as

$$\begin{bmatrix}
\hat{v}(\lambda, \xi, y) \\
\hat{w}(\lambda, \xi, y) \\
\hat{p}(\lambda, \xi, y)
\end{bmatrix} =
\begin{bmatrix}
\omega & -i\xi \\
i\xi^T & |\xi| \\
0 & \frac{1}{\mu \sqrt{\xi}} \xi
\end{bmatrix}
\begin{bmatrix}
\hat{v}_\omega(\lambda, \xi) e^{-\sqrt{\mu \xi} \omega y} \\
\hat{w}_\omega(\lambda, \xi) e^{-\sqrt{\mu \xi} |\xi| y} \\
\hat{p}_\omega(\lambda, \xi)
\end{bmatrix}
$$

for a function $\tau = (\tau_v, \tau_w) : \mathbb{R}_+ \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$, which has to be determined based on the boundary conditions. Due to (12a), we have

$$\begin{bmatrix}
\hat{v}_y \\
\hat{w}_y \\
\hat{p}_y
\end{bmatrix} =
\begin{bmatrix}
\omega \hat{v}_\omega + i\xi \hat{w}_\omega \\
i\xi \omega^2 \hat{w}_\omega + \sqrt{\mu \xi} |\xi| i\xi \hat{w}_\omega \\
i\xi \omega^2 \hat{w}_\omega + \sqrt{\mu \xi} |\xi| i\xi \hat{w}_\omega
\end{bmatrix}
$$

and, thus, the boundary conditions read

$$\begin{bmatrix}
\alpha \sqrt{\mu \xi} \omega + \Re \kappa \omega^2 - (i\xi \otimes i\xi) \\
\alpha \sqrt{\mu \xi} \omega + \Re \kappa |\xi| + |\xi| i\xi
\end{bmatrix}
\begin{bmatrix}
\hat{v}_\omega \\
\hat{w}_\omega
\end{bmatrix} =
\begin{bmatrix}
0 \\
\hat{h}_w
\end{bmatrix}$$
and we obtain $\hat{\tau}_w = (|\zeta| + \beta i \zeta^T B^{-1} i \zeta)^{-1}\hat{h}_w$. Now,

$$(12c) \quad B^{-1}(\lambda, |\zeta|) = \frac{1}{\phi(\lambda, |\zeta|)} \left( 1 + \frac{i \zeta \otimes i \zeta}{\phi(\lambda, |\zeta|) + |\zeta|^2} \right), \quad \phi(\lambda, |\zeta|) = \alpha \sqrt{\Re \lambda_e \omega + \Re \kappa \omega^2},$$

which implies that

$$|\zeta| + \beta i \zeta^T B^{-1} i \zeta = |\zeta| + \frac{\beta |\zeta|^4}{\phi(\lambda, |\zeta|) + |\zeta|^2} = |\zeta| - \frac{\beta |\zeta|^2}{\phi(\lambda, |\zeta|) + |\zeta|^2} = |\zeta| \left( \frac{\phi + |\zeta|^2 - \beta |\zeta|}{\phi + |\zeta|^2} \right)$$

and, hence,

$$\hat{\Pi} h_w = \lambda_e |\zeta| \hat{\tau}_w = \left[ \frac{\alpha \sqrt{\Re \lambda_e + \Re \kappa (\omega + |\zeta|)}}{\alpha \sqrt{\Re \lambda_e + \Re \kappa (\omega + |\zeta|)}} + \frac{\Re \kappa |\zeta|}{\alpha \sqrt{\Re \lambda_e + \Re \kappa (\omega + |\zeta|)}} \right] \omega(\omega' + |\zeta|) \hat{h}_w.$$

Therefore, on a symbolic level we have

$$\Pi \sim (m_1(\lambda, |\zeta|) + m_2(\lambda, |\zeta|) + m_3(\lambda, |\zeta|) \mu(\lambda, |\zeta|)) \omega(\omega + |\zeta|) =: M(\lambda, |\zeta|),$$

which is the desired representation of $\Pi$.

**Step 2.2.** Based on the above considerations we have $\Pi = \text{Op}(M)$ and, thus, the mapping properties of $\Pi$ may be derived by studying its Fourier-Laplace symbol $M$. First note that

$$G := \text{Op}(\mu) = \partial_t \quad \text{and} \quad D := \text{Op}(|\zeta|) = \sqrt{-\frac{1}{\mu} \Delta_{\mathbb{R}^{n-1}}},$$

each admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\mathcal{R}\mathcal{H}^\infty$-angles $\phi_G^\infty = \frac{\pi}{2}$ and $\phi_D^\infty = 0$, respectively, within the scales $\mathcal{A}_r^s(\mathbb{R}^n)$ and $\mathcal{K}_r^s(\mathbb{R}^{n-1})$ for $\mathcal{A}, \mathcal{K} \in \{ H, W \}$, $r \geq 0$, and $s \in \mathbb{R}$ see e.g. [11 Corollary 2.10]. This combined with [10 Theorem 6.1] implies that the pair $(G, D)$ admits a joint $\mathcal{H}^\infty(\Sigma_{\mathbb{R}^{-\theta}} \times \Sigma_{\mathbb{R}^\theta})$-calculus for every $0 < \theta < \frac{\pi}{2}$. Now, it has been proved as part of [2 Theorem 2.3] that the operator $\text{Op}(\omega(\omega + |\zeta|))$ has the mapping properties ([11], cf. [2] Section 4, The Case $\alpha = 0$ and $\beta = 0$). Thus, it remains to prove that the functions

$$(\lambda, z) \mapsto m_j(\lambda, z), \mu(\lambda, z) : \Sigma_{\mathbb{R}^{-\theta}} \times \Sigma_{\mathbb{R}^\theta} \longrightarrow \mathbb{C}, \quad j = 1, 2, 3$$

are bounded and holomorphic for some $0 < \theta < \frac{\pi}{2}$; this implies the operators $\text{Op}(m_j) = m_j(G, D)$ for $j = 1, 2, 3$ and $\text{Op}(\mu) = \mu(G, D)$ to be bounded within the above scales of function spaces.

Since $\mu$ is clearly bounded and holomorphic in $\Sigma_{\mathbb{R}^{-\theta}} \times \Sigma_{\mathbb{R}^\theta}$, we restrict our considerations to the $m_j$ for $j = 1, 2, 3$. It suffices to prove that the reciprocals

$$m_1^{-1} = 1 + \frac{\Re \kappa (\omega(z) + z)}{\alpha \sqrt{\Re \lambda_e}}, \quad m_2^{-1} = 1 + \frac{\alpha \sqrt{\Re \lambda_e + \Re \kappa (\omega(z) + z)}}{\Re \kappa \omega(z)}, \quad m_3^{-1} = 1 + \frac{\alpha \sqrt{\Re \lambda_e + \Re \kappa (\omega(z) + z)}}{\Re \kappa \omega(z)},$$

with $\omega(z) = \sqrt{\lambda_e + z^2}$ are uniformly away from the origin for $(\lambda, z) \in \Sigma_{\mathbb{R}^{-\theta}} \times \Sigma_{\mathbb{R}^\theta}$. Here we exploit the following elementary but useful fact: if $|\arg z_1|, |\arg z_2|, |\arg z_1 - \arg z_2| < \pi$ then

$$\min\{\arg z_1, \arg z_2\} \leq \arg(z_1 + z_2) \leq \max\{\arg z_1, \arg z_2\},$$

Thus, for $(\lambda, z) \in \Sigma_{\mathbb{R}^{-\theta}} \times \Sigma_{\mathbb{R}^\theta}$ with $\arg \lambda \geq 0$ we have

$$-\theta < \arg(\lambda_e + z), \arg(\lambda_e + z^2) < \pi - \theta, \quad -\frac{\theta}{2} < \arg(\omega(z)), \arg(\omega(z) + z) < \frac{\pi}{2} - \frac{\theta}{2}.$$
which implies
\[-\pi + \frac{\theta}{2} < \arg \frac{\Re e(\omega(z)+z)}{\alpha \Re e \lambda_j} < \frac{\pi}{2} - \frac{\theta}{2}, \quad -\pi - \frac{\theta}{2} < \arg \frac{\alpha \Re e \lambda_j + \Re e \kappa z}{\Re e(\omega(z))} < \pi - \frac{\theta}{2},\]

i.e. \(m_j^{-1}\) is indeed uniformly away from the origin for \(j = 1, 2, 3\). By symmetry, we obtain the same behavior for \(\arg \lambda \leq 0\), which shows that \(\Pi\) has the mapping properties \([1]\). This finishes the proof of Theorem 2.1 for the boundary condition \([TDO]\).  

3.2. The Condition \(\text{NDO}\). Now we consider the Stokes equations subject to a dynamic outflow boundary condition in normal directions, i.e. the system \([S]_{\Re e, \alpha}([\text{NDO}]_{\max, \Re e})\) which reads

\[\begin{align*}
\partial_t u - \frac{1}{\Re e} \Delta u + \nabla p &= f \quad \text{in } J \times \mathbb{R}^n_+,\\
\text{div} u &= g \quad \text{in } J \times \mathbb{R}^n_+,\\n[u]_y &= h_v \quad \text{on } J \times \mathbb{R}^{n-1},\\n \alpha \partial_t [w]_y - (\alpha V + \frac{2}{\Re e}) [\partial_y w]_y + [p]_y &= h_w \quad \text{on } J \times \mathbb{R}^{n-1},\\n u(0) &= u_0 \quad \text{in } \mathbb{R}^n_+.
\end{align*}\]

We again require the regularity and compatibility conditions as stated in Theorem 2.1, i.e. we have

- \(f \in L_p(J \times \mathbb{R}^n_+)^n\),
- \(g \in H^{1/2}_p(J, L_p(\mathbb{R}^n_+)) \cap L_p(J, H^1_p(\mathbb{R}^n_+))\),
- \(h_v \in W^{1,1/2}_p(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W^{2,1}_p(\mathbb{R}^{n-1}))^{n-1}\),
- \(h_w \in L_p(J, W^{2,1}_p(\mathbb{R}^{n-1}))\),
- \(u_0 \in W^{2,1}_p(\mathbb{R}^n_+)^n\) with \(\text{div} u_0 = g(0)\) in \(\mathbb{R}^n_+\) for \(p \geq 2\),
- \(F(g, \eta) \in H^1_p(J, \tilde{H}^{-1}_p(\Omega))\) for some \(\eta \in H^1_p(J, W^{1,1/2}_p(\Gamma)) \cap L_p(J, W^{2,1}_p(\Gamma))\) with \([w_0]_y = \eta(0)\) for \(p > \frac{3}{2}\),
- \([\nabla u_0]_y = h_v(0)\) for \(p > \frac{3}{2}\).

The construction of a solution to (13) requires several steps.

**Step 1.** As a first step we again show that we may w.l.o.g. assume \(f = 0, \ g = 0, \ h_v = 0\) and \(u_0 = 0\) in the following. Indeed, we may solve the Stokes equations with Dirichlet boundary conditions

\[\begin{align*}
\partial_t u - \frac{1}{\Re e} \Delta u + \nabla p &= f \quad \text{in } J \times \mathbb{R}^n_+,\\n\text{div} u &= g \quad \text{in } J \times \mathbb{R}^n_+,\\n[u]_y &= h_v \quad \text{on } J \times \mathbb{R}^{n-1},\\n [w]_y &= \eta \quad \text{on } J \times \mathbb{R}^{n-1},\\n u(0) &= u_0 \quad \text{in } \mathbb{R}^n_+.
\end{align*}\]

to obtain a unique solution in the desired regularity class using well-known results on maximal regularity of the Stokes equations, see e.g. [2] Theorem 2.3. This immediately leads to the desired reduction. Note, however, that we now have to assume \(h_w \in L_p(J, \tilde{H}^{-1}_p(\mathbb{R}^{n-1}))\) to obtain a pressure \(p \in L_p(J, \tilde{H}^1_p(\mathbb{R}^n_+))\) without additional regularity for \([p]_y\).

**Step 2.** In order to solve the remaining problem, we will again employ a Laplace transformation in time and a Fourier transformation in the tangential part of the spatial variables. This is again
due to (12b) and the divergence equation the boundary conditions read
\[ [v]_y = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \]
\[ \alpha \varepsilon [w]_y + \alpha \partial_t [w]_y - (\alpha V + \frac{2}{\Re \varepsilon}) [\partial_y w]_y + [p]_y = h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \]
\[ u(0) = 0 \quad \text{in } \mathbb{R}^n_+, \]
for an arbitrary \( \varepsilon > 0 \). Note that maximal regularity for this problem is again equivalent to maximal regularity of the original problem (i.e., for \( \varepsilon = 0 \)) on finite time intervals \( J = (0, a) \). The strategy to construct a solution to (14) is as follows: We compute the pressure trace \([p]_y = \Pi h_w\) as well as \( h_w - [p]_y = \Sigma h_w \) and show that these are given based on bounded linear operators
\[ \Sigma : L_p(\mathbb{R}_+, W^{1-1/p}_p(\mathbb{R}^{n-1})) \to L_p(\mathbb{R}_+, W^{1-1/p}_p(\mathbb{R}^{n-1})), \]
\[ \Pi : L_p(\mathbb{R}_+, W^{1-1/p}_p(\mathbb{R}^{n-1})) \to L_p(\mathbb{R}_+, W^{1-1/p}_p(\mathbb{R}^{n-1})). \]
Then we obtain the pressure \( p \in L_p(\mathbb{R}_+, \dot{H}^1_p(\mathbb{R}^n_+)) \) as a solution to the (weak) elliptic problem
\[ -\Delta p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \]
\[ [p]_y = \Pi h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \]
cf. [2, Proposition 3.1]. Finally, we obtain \( u \) as a maximal regular solution to the parabolic problem
\[ \varepsilon u + \partial_t u - \frac{1}{\Re \varepsilon} \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \]
\[ [v]_y = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \]
\[ \alpha \varepsilon [w]_y + \alpha \partial_t [w]_y - (\alpha V + \frac{2}{\Re \varepsilon}) [\partial_y w]_y + [p]_y = \Sigma h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \]
\[ u(0) = 0 \quad \text{in } \mathbb{R}^n_+, \]
via Proposition A.2.

Step 2.1. We compute the symbols of \( \Sigma \) and \( \Pi \). The transformed equations (14) read:
\[ \omega^2 \tilde{v} - \frac{1}{\Re \varepsilon} \partial_y^2 \tilde{v} + i \xi \tilde{p} = 0 \quad \lambda \in \Sigma_{\varepsilon \to 0}, \quad \xi \in \mathbb{R}^{n-1}, \quad y > 0, \]
\[ \omega^2 \tilde{w} - \frac{1}{\Re \varepsilon} \partial_y^2 \tilde{w} + \partial_y \tilde{p} = 0 \quad \lambda \in \Sigma_{\varepsilon \to 0}, \quad \xi \in \mathbb{R}^{n-1}, \quad y > 0, \]
\[ i \xi \cdot \tilde{v} + \partial_y \tilde{w} = 0 \quad \lambda \in \Sigma_{\varepsilon \to 0}, \quad \xi \in \mathbb{R}^{n-1}, \quad y > 0, \]
\[ [\tilde{v}]_y = 0 \quad \lambda \in \Sigma_{\varepsilon \to 0}, \quad \xi \in \mathbb{R}^{n-1}, \]
\[ \alpha \lambda \xi \tilde{v} - \sigma [\partial_y \tilde{v}]_y + [\tilde{p}]_y = \hat{h}_w \quad \lambda \in \Sigma_{\varepsilon \to 0}, \quad \xi \in \mathbb{R}^{n-1}, \]
where we used the same notations as in the Subsection 3.1. We again employ the ansatz (12a) and due to (12b) and the divergence equation the boundary conditions read
\[ \begin{bmatrix} \omega \\ \alpha \lambda \xi \tilde{v}^T \end{bmatrix} - \frac{i \tilde{\zeta}}{\sqrt{\Re \varepsilon}} \lambda \tilde{w} = \begin{bmatrix} \hat{\tau}_w \\ \hat{\rho}_w \end{bmatrix} \]
and we obtain
\[ \frac{1}{\sqrt{\Re \varepsilon}} \lambda \xi \hat{w} = \left( 1 + \sqrt{\Re \alpha} |\zeta| \left( 1 - \frac{|\zeta|^2}{\omega^2} \right) \right)^{-1} \hat{h}_w. \]
This implies
\[ \Sigma \hat{h}_w = \left( 1 + \sqrt{\Re \alpha} |\zeta| \left( 1 - \frac{|\zeta|^2}{\omega^2} \right) \right)^{-1} h_w, \]
\[ \Pi \hat{h}_w = \frac{1}{1 + \sqrt{\Re \alpha} |\zeta| \left( 1 - \frac{|\zeta|^2}{\omega^2} \right)} \hat{h}_w, \]
which are the desired representations of \( \Sigma \) and \( \Pi \).
Step 2.2. In order to derive the mapping properties \([15]\) based on the representations obtained above we employ the same techniques as in Step 2.2 of Subsection 3.1. By the very same arguments as used there we obtain that the symbol of \(\Pi\) is bounded and holomorphic in \(\Sigma_{\pi-\theta} \times \Sigma_{\theta/2}\) for some \(0 < \theta < \frac{\pi}{2}\). This yields the desired mapping properties of \(\Pi\). Moreover, based on its symbol, \(\Sigma\) has the same mapping properties as

\[
\text{Op}(\frac{|\cdot|}{1+|\cdot|}) : L_p(\mathbb{R}^+, \dot{W}^{1-1/p}(\mathbb{R}^{n-1})) \mapsto L_p(\mathbb{R}^+, W^{1-1/p}(\mathbb{R}^{n-1})),
\]

which yields the mapping properties \([15]\). This finishes the proof of Theorem 2.1 for the boundary condition \((\text{NDO})_{a, Re}^{\theta_{\text{out}}}\).

Remark 3.1. Note that we have

\[
\text{Sym}(\Sigma) \to 0, \quad \text{Sym}(\Pi) \to 1 \quad \text{as } a \to 0,
\]

which are the symbols of the corresponding operators for the boundary condition

\[
[v]_y = 0 \quad \text{on } J \times \mathbb{R}^{n-1},
\]

\[
-\frac{2}{Re} [\partial_y w]_y + [p]_y = h_w \quad \text{on } J \times \mathbb{R}^{n-1},
\]

which is one of the energy preserving boundary conditions considered in \([2]\). Now, one can either employ \([\partial_y w]_y = -\nabla_x \cdot [v]_y = 0\) or the fact that \(\Sigma = 0\) and \(\Pi = 1\) for this limit case to recognize that the absence of the pressure trace in \((16)\) leads to an ill-posed problem, since one boundary condition would be missing then. A similar defect applies to the dynamic outflow condition \((\text{NDO})_{a, Re}^{\theta_{\text{out}}}\) without the pressure trace, which would lead to an ill-posed problem.

3.3. The Condition FDO. Finally, we consider the Stokes equations subject to a fully dynamic outflow boundary condition, i.e. the system \((\text{S})_{Re}^{\theta_{\text{out}}}(\text{FDO})_{a, Re}^{\theta_{\text{out}}}\) which reads

\[
\begin{align*}
\partial_t u - \frac{1}{Re} \Delta u + \nabla p &= f \quad \text{in } J \times \mathbb{R}^n, \\
\text{div } u &= g \quad \text{in } J \times \mathbb{R}^n, \\
\alpha \partial_t [v]_y - (\alpha V + \frac{1}{Re}) [\partial_y v]_y - \frac{1}{Re} \nabla_x [w]_y &= h_v \quad \text{on } J \times \mathbb{R}^{n-1}, \\
\alpha \partial_t [w]_y - (\alpha V + \frac{2}{Re}) [\partial_y w]_y + [p]_y &= h_w \quad \text{on } J \times \mathbb{R}^{n-1}, \\
u(0) &= u_0 \quad \text{in } \mathbb{R}^n_+.
\end{align*}
\]

As in the previous steps we require the regularity and compatibility conditions as stated in Theorem 2.1, i.e. we assume that

- \(f \in L_p(J \times \mathbb{R}^n)^n,\)
- \(g \in H^{1/2}(J, L_p(\mathbb{R}^n)) \cap L_p(J, H^1_p(\mathbb{R}^{n-1})),\)
- \(h_v \in W^{1/2-1/2p}(J, L_p(\mathbb{R}^{n-1}))^{n-1} \cap L_p(J, W^{1-1/p}(\mathbb{R}^{n-1}))^{n-1},\)
- \(h_w \in L_p(J, W^{1-1/p}(\mathbb{R}^{n-1})),\)
- \(u_0 \in W^{2-2/p}(\mathbb{R}^n_+)^n\) with \(\text{div } u_0 = g(0)\) in \(\mathbb{R}^n_+\) for \(p \geq 2,\)
- \(F(g, \eta) \in H^{1/2}_p(J, H^1_p(\Omega))\) for some \(\eta \in H^1_p(J, W^{1-1/p}(\Gamma)) \cap L_p(J, W^{2-1/p}(\Gamma))\) with \([u_0]_y = \eta(0)\) for \(p > \frac{3}{2},\)
- \([v_0]_y \in W^{2-2/p}(\mathbb{R}^{n-1})^{n-1} .\)

The construction of a solution to \((17)\) requires several Steps.

Step 1. As a first step we again show that we may w.l.o.g. assume \(f = 0, g = 0, h_v = 0\) and \(u_0 = 0\) in the following. Indeed, since the proof Theorem 2.1 concerning the boundary condition \((\text{TD})_{a, Re}^{\theta_{\text{out}}}\) has already been given, we may now solve the Stokes equations subject to a tangential
dynamic outflow boundary condition

\[ \partial_t u - \frac{1}{\rho} \Delta u + \nabla p = f \quad \text{in } J \times \mathbb{R}_+^n, \]
\[ \text{div } u = g \quad \text{in } J \times \mathbb{R}_+^n, \]
\[ \alpha \partial_t [v]_y - (\alpha V + \frac{1}{\rho}) [\partial_y v]_y - \frac{1}{\rho} \nabla_x [w]_y = h_v \quad \text{on } J \times \mathbb{R}_+^{n-1}, \]
\[ [w]_y = \eta \quad \text{on } J \times \mathbb{R}_+^{n-1}, \]
\[ u(0) = u_0 \quad \text{in } \mathbb{R}_+^n. \]

to obtain a unique solution in the desired regularity class. This immediately leads to the desired reduction. Note, however, that we now have to assume \( h_w \in L_p(J, W^{1-1/p}_p(\mathbb{R}^{n-1})) \) to obtain a pressure \( p \in L_p(J, H^1_p(\mathbb{R}_+^n)) \) without additional regularity for \([p]_y\).

**Step 2.** In order to solve the remaining problem, we will again employ a Laplace transformation in time and a Fourier transformation in the tangential part of the spatial variables. This is again only possible for an unbounded time interval, i.e. we will from now on consider the shifted problem

\[ \varepsilon u + \partial_t u - \frac{1}{\rho} \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \]
\[ \text{div } u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \]
\[ \alpha \varepsilon [v]_y + \alpha \partial_t [v]_y - (\alpha V + \frac{1}{\rho}) [\partial_y v]_y - \frac{1}{\rho} \nabla_x [w]_y = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+^{n-1}, \]
\[ \alpha \varepsilon [w]_y + \alpha \partial_t [w]_y - (\alpha V + \frac{2}{\rho}) [\partial_y w]_y + [p]_y = h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+^{n-1}, \]
\[ u(0) = 0 \quad \text{in } \mathbb{R}_+^n \]

for an arbitrary \( \varepsilon > 0 \). Note that maximal regularity for this problem is again equivalent to maximal regularity of the original problem (i.e. for \( \varepsilon = 0 \)) on finite time intervals \( J = (0, a) \).

The strategy to construct a solution to \([18]\) is the same as for the problem \([14]\): We compute the pressure trace \([p]_y = \Pi h_w\) as well as \( h_w - [p]_y = \Sigma h_w \) and show that these operators are bounded and linear in the setting \([15]\). Then we obtain the pressure \( p \in L_p(\mathbb{R}_+, H^1_p(\mathbb{R}_+^n)) \) as a solution to the (weak) elliptic problem

\[ -\Delta p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \]
\[ [p]_y = \Pi h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+^{n-1}, \]

cf. \([2]\) Proposition 3.1]. Finally, we obtain \( u \) as a maximal regular solution to the parabolic problem

\[ \varepsilon u + \partial_t u - \frac{1}{\rho} \Delta u = -\nabla p \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \]
\[ \alpha \varepsilon [v]_y + \alpha \partial_t [v]_y - (\alpha V + \frac{1}{\rho}) [\partial_y v]_y = \frac{1}{\rho} \nabla_x [w]_y \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+^{n-1}, \]
\[ \alpha \varepsilon [w]_y + \alpha \partial_t [w]_y - (\alpha V + \frac{2}{\rho}) [\partial_y w]_y = \Sigma h_w \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+^{n-1}, \]
\[ u(0) = 0 \quad \text{in } \mathbb{R}_+^n \]

via Propositions \([A.1]\) and \([A.2]\).

**Step 2.1.** We compute the symbols of \( \Sigma \) and \( \Pi \). The transformed equations \([18]\) read:

\[ \omega^2 \ddot{v} - \frac{1}{\rho} \partial_y^2 \ddot{v} + i \xi \dot{v} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0, \]
\[ \omega^2 \ddot{w} - \frac{1}{\rho} \partial_y^2 \ddot{w} + \partial_y \dot{w} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0, \]
\[ i \xi \cdot \dot{v} + \partial_y \ddot{v} = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, y > 0, \]
\[ \alpha \lambda_c [v]_y - \kappa [\partial_y \dot{v}]_y - \frac{1}{\rho} \partial_y i \xi [\dot{w}]_y = 0 \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, \]
\[ \alpha \lambda_c [\dot{w}]_y - \sigma [\partial_y \dot{w}]_y + [p]_y = \dot{h}_w \quad \lambda \in \Sigma_{\pi - \theta}, \xi \in \mathbb{R}^{n-1}, \]
where we used the same notations as in the previous subsection. We again employ the ansatz (12a) and due to (12b) the boundary conditions read

\[
\begin{bmatrix}
\alpha \sqrt{\text{Re}} \lambda \omega + \text{Re} \kappa \omega^2 - (i\zeta \otimes i\zeta) \\
\beta(\lambda, |\zeta|)
\end{bmatrix} = B(\lambda, |\zeta|)
\]

\[
\begin{bmatrix}
(\alpha \sqrt{\text{Re}} \lambda) + \text{Re} \sigma \omega) i\zeta^T \\
\beta(\lambda, |\zeta|)
\end{bmatrix} = \begin{bmatrix}
\alpha \sqrt{\text{Re}} \lambda |\zeta| + \text{Re} \sigma |\zeta|^2 + \lambda \\
\beta(\lambda, |\zeta|)
\end{bmatrix}
\]

and we obtain \( \hat{p}_y = \frac{1}{\sqrt{\text{Re}}} \lambda \hat{\tau}_w = \lambda_e (\beta + \beta_v \beta_w i\zeta^T B^{-1} i\zeta)^{-1} \hat{h}_w \). Now, using (12c) we have

\[
\beta + \beta_v \beta_w i\zeta^T B^{-1} i\zeta = \beta + \frac{\beta_v \beta_w \zeta}{\phi + |\zeta|^2} = \beta - \frac{\beta_u \beta_w |\zeta|^2}{\phi + |\zeta|^2}
\]

and \( 1 + \text{Re} \kappa = \text{Re} \sigma \) together with

\[
\beta_v(\lambda, |\zeta|) = \alpha \sqrt{\text{Re}} \lambda_e + \text{Re} \sigma |\zeta|, \quad \beta(\lambda, |\zeta|) = \lambda_e + \beta_v(\lambda, |\zeta|) |\zeta|
\]

implies

\[
(\beta + \beta_v \beta_w i\zeta^T B^{-1} i\zeta)^{-1} = \frac{\phi + |\zeta|^2}{\lambda_e(\phi + |\zeta|^2) + \beta_u(\phi + |\zeta|^2) |\zeta| - \beta_u \beta_w |\zeta|^2}
\]

In order to obtain a suitable representation of the symbols of \( \Sigma \) and \( \Pi \) we first observe that

\[
\phi + |\zeta|^2 = \alpha \sqrt{\text{Re}} \lambda_e \omega + \text{Re} \kappa \omega^2 + |\zeta|^2 = \alpha \sqrt{\text{Re}} \lambda_e \omega + \text{Re} \kappa \lambda_e + \text{Re} \sigma |\zeta|^2
\]

while

\[
\text{Re} \sigma \lambda_e |\zeta|^2 + \beta_v(\text{Re} \kappa \lambda_e |\zeta| + \text{Re} \sigma |\zeta|^3 - \beta_w |\zeta|^2)
\]

\[
= \text{Re} \sigma \lambda_e |\zeta|^2 + \beta_v(\text{Re} \kappa \lambda_e |\zeta| + \text{Re} \sigma (|\zeta| - \omega)|\zeta| + \alpha \sqrt{\text{Re}} \lambda_e |\zeta|^2)
\]

\[
= \text{Re} \sigma \lambda_e |\zeta|^2 + \text{Re} \kappa \text{Re} \sigma \lambda_e |\zeta|^2 + \text{Re} \sigma (|\zeta| - \omega) \text{Re} \sigma |\zeta|^3
\]

\[
+ \alpha \sqrt{\text{Re}} \lambda_e \text{Re} \kappa \lambda_e + \text{Re} \sigma (|\zeta| - \omega) |\zeta| - \beta_u |\zeta| |\zeta|
\]

\[
= \text{Re} \sigma \lambda_e \text{Re} \sigma (|\omega| + |\zeta|) |\zeta|^2 - \text{Re} \sigma \lambda_e \text{Re} \sigma |\zeta|^3
\]

\[
\omega + |\zeta|
\]

\[
+ \alpha \sqrt{\text{Re}} \lambda_e (\text{Re} \kappa \lambda_e + \text{Re} \sigma (|\zeta| - \omega) |\zeta| - \beta_u |\zeta| |\zeta|
\]

\[
\omega + |\zeta|
\]

\[
\text{Re} \sigma \zeta_e \omega + |\zeta| \text{Re} \sigma |\zeta|^2 + \alpha \sqrt{\text{Re}} \lambda_e (\text{Re} \kappa \lambda_e + \text{Re} \sigma (|\zeta| - \omega) |\zeta| - \beta_u |\zeta| |\zeta|
\]

\[
\alpha \sqrt{\text{Re}} \lambda_e (\beta_v(\omega - |\zeta|) + \text{Re} \kappa \lambda_e + \text{Re} \sigma (|\zeta| - \omega) |\zeta| - \beta_u |\zeta| |\zeta|
\]

\[
= \alpha \sqrt{\text{Re}} \lambda_e \left[ \frac{\beta_v \omega - \text{Re} \sigma \lambda_e |\zeta|}{\omega + |\zeta|} + \text{Re} \kappa \lambda_e \right] |\zeta|
\]

\[
= \alpha \sqrt{\text{Re}} \lambda_e \left[ \frac{\alpha \sqrt{\text{Re}} \lambda_e^2 \omega + |\zeta|}{\omega + |\zeta|} + \text{Re} \kappa \lambda_e \right] |\zeta|,
\]

which implies

\[
\Pi \hat{h}_w = \frac{\alpha \sqrt{\text{Re}} \lambda_e \omega + \text{Re} \kappa \lambda_e + \text{Re} \sigma |\zeta|^2}{\alpha \sqrt{\text{Re}} \lambda_e \omega + \text{Re} \kappa \lambda_e + \alpha \sqrt{\text{Re}} \lambda_e \left( \alpha \sqrt{\text{Re}} \lambda_e + \text{Re} \kappa (\omega + |\zeta|) \right) \frac{|\omega + |\zeta|}{|\omega + |\zeta|} + \text{Re} \sigma \omega + \frac{\omega + |\zeta|}{\omega + |\zeta|} \text{Re} \sigma |\zeta|^2}
\]

Based on this representation of \( \Pi \) we also obtain

\[
\Sigma \hat{h}_w = \frac{\alpha \sqrt{\text{Re}} \lambda_e \left( \alpha \sqrt{\text{Re}} \lambda_e + \text{Re} \kappa (\omega + |\zeta|) \right) \frac{|\omega + |\zeta|}{|\omega + |\zeta|} + \text{Re} \kappa (\omega - |\zeta|) \text{Re} \sigma |\zeta|^2}{\alpha \sqrt{\text{Re}} \lambda_e \omega + \text{Re} \kappa \lambda_e + \alpha \sqrt{\text{Re}} \lambda_e \left( \alpha \sqrt{\text{Re}} \lambda_e + \text{Re} \kappa (\omega + |\zeta|) \right) \frac{|\omega + |\zeta|}{|\omega + |\zeta|} + \text{Re} \sigma \omega + \frac{\omega + |\zeta|}{\omega + |\zeta|} \text{Re} \sigma |\zeta|^2}
\]
These are the desired representations of $\Sigma$ and $\Pi$.

**Step 2.2.** In order to derive the mapping properties \([\text{15}]\) based on the representations obtained above we employ the same techniques as in Step 2.2 of Subsection \([\text{3.1}]\). By the very same arguments as used there we obtain that the symbol of $\Pi$ is bounded and holomorphic in $\Sigma_{\pi-\theta} \times \Sigma_{\theta/2}$ for some $0 < \theta < \frac{\pi}{2}$. This yields the desired mapping properties of $\Pi$. Moreover, based on its symbol, $\Sigma$ has the same mapping properties as

\[ \text{Op}(\frac{|k|}{w(z)}): L_p(\mathbb{R}^+, \dot{W}^{1-1/p}(\mathbb{R}^{n-1})) \to L_p(\mathbb{R}^+, W^{1-1/p}(\mathbb{R}^{n-1})), \]

which yields the mapping properties \([\text{15}]\). This finishes the proof of Theorem \([\text{2.1}]\) for the boundary condition \([\text{FDO}]\).

4. The Wedge Case

This section is devoted to the proofs of Theorems \([\text{2.4}]\) and \([\text{2.6}]\). Here, we first note that we can always assume $a = 0$, since the corresponding term is of lower order and may be added using a standard perturbation argument. Moreover, we assume $a > 0$, set $J := (0, a)$, and assume $1 < p < \infty$ with $p \neq \frac{3}{2}, 3$. Furthermore, we assume $\alpha, \text{Re} > 0$. Since we solve the Stokes equations in the wedge $\Omega := \mathbb{R}_+^n$, it is convenient to denote the velocity field as $(u, v, w): J \times \Omega \to \mathbb{R}^n$, i.e. we employ a decomposition into a purely tangential part $u: J \times \Omega \to \mathbb{R}^{n-2}$, and two normal parts $v, w: J \times \Omega \to \mathbb{R}$. The spatial coordinates are denoted by $(x, y, z) \in \mathbb{R}^n$, with $x \in \mathbb{R}^{n-2}$, and $y, z > 0$. Finally, $E = \partial_y \mathbb{R}^n_+ \cap \partial_z \mathbb{R}^n_+$ and $[\cdot]_y$ and $[\cdot]_z$ denote the trace of a quantity defined in $\mathbb{R}^n_+$ on the boundaries $\partial_y \mathbb{R}^n_+$ and $\partial_z \mathbb{R}^n_+$, respectively.

4.1. Combination of Inflow/Navier Conditions. In order to prove Theorem \([\text{2.4}]\) we have to study the model problem

\begin{equation}
\begin{aligned}
\partial_t u - \frac{1}{\text{Re}} \Delta u + \nabla_x p &= f_u \quad \text{in } J \times \mathbb{R}^n_+, \\
\partial_t v - \frac{1}{\text{Re}} \Delta v + \partial_y p &= f_v \quad \text{in } J \times \mathbb{R}^n_+, \\
\partial_t w - \frac{1}{\text{Re}} \Delta w + \partial_z p &= f_w \quad \text{in } J \times \mathbb{R}^n_+, \\
\text{div}_x u + \partial_y v + \partial_z w &= g \quad \text{in } J \times \mathbb{R}^n_+, \\
-\frac{1}{\text{Re}} [\partial_y u]_y - \frac{1}{\text{Re}} \nabla_x [v]_y &= h^{\text{wall}}_u \quad \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[v]_y &= h^{\text{wall}}_v \quad \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[u]_z &= u^{\text{in}}_w \quad \text{on } J \times \partial_z \mathbb{R}^n_+, \\
[v]_z &= w^{\text{in}}_w \quad \text{on } J \times \partial_z \mathbb{R}^n_+, \\
[w]_z &= w^{\text{in}}_w \quad \text{on } J \times \partial_z \mathbb{R}^n_+, \\
u(0) &= u_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{in } \mathbb{R}^n_+,
\end{aligned}
\end{equation}

where the data $f = (f_u, f_v, f_w), g, u^{\text{in}} = (u^{\text{in}}_w, u^{\text{in}}_w, u^{\text{in}}_w), h^{\text{wall}} = (h^{\text{wall}}_w, h^{\text{wall}}_w, h^{\text{wall}}_w)$, and the initial data $(u_0, v_0, w_0)$ are subject to the regularity/compatibility conditions stated in Theorem \([\text{2.4}]\), i.e.

- $f \in L_p(J \times \mathbb{R}^n_+),$
- $g \in H^{1/2}_p(J, L_p(\mathbb{R}^n_+) ) \cap L_p(J, H^{1/2}_p(\mathbb{R}^n_+) )$,
- $u^{\text{in}} \in W^{1-1/2p}(J, L_p(\partial_z \mathbb{R}^n_+) ) \cap L_p(J, W^{2-1/p}(\partial_z \mathbb{R}^n_+) )$,
- $(h^{\text{wall}}_w, h^{\text{wall}}_w) \in W^{1-2/2p}(J, L_p(\partial_z \mathbb{R}^n_+) ) \cap L_p(J, W^{2-1/p}(\partial_z \mathbb{R}^n_+) )$,
- $h^{\text{wall}}_w \in W^{1-1/2p}(J, L_p(\partial_y \mathbb{R}^n_+) ) \cap L_p(J, W^{2-1/p}(\partial_y \mathbb{R}^n_+) )$,
- $F(g, -h^{\text{wall}}_w, -u^{\text{in}}_w) \in H^1(J, H^{-1}(\mathbb{R}^n_+))$,
- $(u_0, v_0, w_0) \in W^{2-3/p}(J \times \mathbb{R}^n_+)$.
with

\[(20a)\]
\[\text{div}_x u_0 + \partial_y v_0 + \partial_z w_0 = g(0) \quad \text{in } \mathbb{R}^n_+, \quad \text{if } p \geq 2\]
as well as

\[(20b)\]
\[-\frac{1}{\Re} [\partial_y u_0]_y - \frac{1}{\Re} \nabla_x [v_0]_y = h_w^{\text{wall}}(0) \quad \text{on } \partial_y \mathbb{R}^n_+, \quad \text{if } p > 3,\]

\[(20c)\]
\[-\frac{1}{\Re} [\partial_y w_0]_y - \frac{1}{\Re} \nabla_z [v_0]_y = h_w^{\text{wall}}(0) \quad \text{on } \partial_y \mathbb{R}^n_+, \quad \text{if } p > 3,\]
together with

\[(0) = 0, \quad \text{if } p > \frac{3}{2},\]

\[(20a)\]
\[u_0|_z = u_0^{in}(0) \quad \text{on } \partial_z \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2},\]

\[(20b)\]
\[v_0|_z = v_0^{in}(0) \quad \text{on } \partial_z \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2},\]

\[(20c)\]
\[w_0|_z = w_0^{in}(0) \quad \text{on } \partial_z \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2},\]

and, due to (IF/W)$_x^{u^{in}, h^{\text{wall}}}$ with

\[(20d)\]
\[-\frac{1}{\Re} [\partial_y u_0^{in}]_y - \frac{1}{\Re} \nabla_x [h_v^{\text{wall}}]_z = [h_y^{\text{wall}}]_z \quad \text{on } J \times \mathcal{E},\]

for $p \geq 2$. The construction of a solution to [19] requires several steps.

**Step 1.** We first show that we can w.l.o.g. assume $f = 0$, as well as $g(0) = 0$, if $p \geq 2$, and $h_v^{\text{wall}}(0) = 0$, if $p > \frac{3}{2}$, and $h_u^{\text{wall}}(0) = h_w^{\text{wall}}(0) = 0$, if $p > 3$, as well as $u^{in}(0) = 0$, if $p > \frac{3}{2}$, together with $u_0 = v_0 = w_0 = 0$. Indeed, we may choose

\[f \in L_p(J \times \mathbb{R}^n), \quad u_0 \in W^{2-2/p}_p(\mathbb{R}^n)^{n-2}, \quad v_0, w_0 \in W^{2-2/p}_p(\mathbb{R}^n)^n\]
as extensions of $f$, $u_0$, $v_0$, and $w_0$, respectively. Note that such extensions may be constructed using a linear extension operator as provided e.g. by [11] Theorem 4.32. Then the problems

\[\partial_t \hat{u} - \frac{1}{\Re} \Delta \hat{u} = \hat{f}_u \quad \text{in } J \times \mathbb{R}^n,\]
\[\partial_t \hat{v} - \frac{1}{\Re} \Delta \hat{v} = \hat{f}_v \quad \text{in } J \times \mathbb{R}^n,\]
\[\partial_t \hat{w} - \frac{1}{\Re} \Delta \hat{w} = \hat{f}_w \quad \text{in } J \times \mathbb{R}^n,\]

\[\hat{u}(0) = \hat{u}_0, \quad \hat{v}(0) = \hat{v}_0, \quad \hat{w}(0) = \hat{w}_0 \quad \text{in } \mathbb{R}^n\]

admit unique solutions

\[\hat{u} \in H^1(J, \mathbb{R}^n)^{n-2} \cap L_p(J, H^2_p(\mathbb{R}^n))^{n-2},\]
\[\hat{v}, \hat{w} \in H^1(J, \mathbb{R}^n) \cap L_p(J, H^2_p(\mathbb{R}^n)).\]

Now, if we define $u$, $v$, and $w$ to be the restrictions of $\hat{u}$, $\hat{v}$, and $\hat{w}$ to $\mathbb{R}^n_+$, then $(u, v, w)$ together with $p := 0$ belong to the desired regularity class and solve

\[\partial_t u - \frac{1}{\Re} \Delta u + \nabla_x p = f_u \quad \text{in } J \times \mathbb{R}^n_+,\]
\[\partial_t v - \frac{1}{\Re} \Delta v + \partial_y p = f_v \quad \text{in } J \times \mathbb{R}^n_+,\]
\[\partial_t w - \frac{1}{\Re} \Delta w + \partial_z p = f_w \quad \text{in } J \times \mathbb{R}^n_+,\]

\[u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{in } \mathbb{R}^n_+.\]

Hence, we may assume $f = u_0 = v_0 = w_0 = 0$ together with the assumptions on $g(0)$, $h_v^{\text{wall}}(0)$, and $u^{in}(0)$ stated above. Note that this reduction of the problem does not affect the regularity and compatibility assumptions of Theorem [2.4] i.e. we may still assume

- $g \in H^{1/2}_p(J, L_p(\mathbb{R}^n_+)) \cap L_p(J, H^1_p(\mathbb{R}^n_+))$,
- $u^{in} \in W^{1-1/2p}_p(J, L_p(\partial_n \mathbb{R}^n_+))^n \cap L_p(J, W^{2-1/p}(\partial_2 \mathbb{R}^n_+))^n$,
Step 2. We show that we may w. l. o. g. assume as well as compatibility condition (20d) in the remaining part of the proof. Hence, we may assume of Theorem 2.4, nor the simplifications obtained in Step 1, i.e. we may still assume Note that this reduction of the problem neither affects the regularity and compatibility assumptions notation admit unique solutions as extensions of \( h^\text{wall}_u \), \( h^\text{wall}_v \), and \( u^\text{in}_w \), respectively, where we denote by \( \mathbb{R}^n_{[y>0]} := \{ (x, y, z) \in \mathbb{R}^n : x \in \mathbb{R}^{n-2}, y > 0, z \in \mathbb{R} \} \), and \( \mathbb{R}^n_{[z>0]} \), which is defined analogously, the two halfspaces, whose intersection is given by \( \mathbb{R}^n_+ \). Note that such extensions may be constructed using a linear extension operator as provided e.g. by \( \text{[1] Theorem 4.26} \). Then the problems

\[
\begin{align*}
\partial_t \hat{u} - \frac{1}{\rho \epsilon} \Delta \hat{u} = 0 & \quad \text{in } J \times \mathbb{R}^n_{[y>0]}, \\
\partial_t \hat{v} - \frac{1}{\rho \epsilon} \Delta \hat{v} = 0 & \quad \text{in } J \times \mathbb{R}^n_{[y>0]}, \\
\partial_t \hat{w} - \frac{1}{\rho \epsilon} \Delta \hat{w} = 0 & \quad \text{in } J \times \mathbb{R}^n_{[z>0]}, \\
- \frac{1}{\rho \epsilon} [\hat{p} \hat{u}]_y = \hat{h}^\text{wall}_u + \frac{1}{\rho \epsilon} \nabla_x \hat{h}^\text{wall}_v & \quad \text{on } J \times \partial \mathbb{R}^n_{[y>0]}, \\
[\hat{v}]_y = \hat{h}^\text{wall}_v & \quad \text{on } J \times \partial \mathbb{R}^n_{[y>0]}, \\
[\hat{w}]_z = \hat{u}^\text{in}_w & \quad \text{on } J \times \partial \mathbb{R}^n_{[z>0]}, \\
\hat{u}(0) = 0, & \quad \hat{v}(0) = 0 \quad \text{in } \mathbb{R}^n_{[y>0]}, \\
\hat{w}(0) = 0 & \quad \text{in } \mathbb{R}^n_{[z>0]}
\end{align*}
\]

admit unique solutions

\[
\begin{align*}
\hat{u} & \in H^1_p(J, L_p(\mathbb{R}^n_{[y>0]}))^{n-2} \cap L_p(J, H^2_p(\mathbb{R}^n_{[y>0]}))^{n-2}, \\
\hat{v} & \in H^1_p(J, L_p(\mathbb{R}^n_{[y>0]})) \cap L_p(J, H^2_p(\mathbb{R}^n_{[y>0]})), \\
\hat{w} & \in H^1_p(J, L_p(\mathbb{R}^n_{[z>0]})) \cap L_p(J, H^2_p(\mathbb{R}^n_{[z>0]})).
\end{align*}
\]

Now, if we define \( u, v, \) and \( w \) to be the restrictions of \( \hat{u}, \hat{v}, \) and \( \hat{w} \) to \( \mathbb{R}^n_+ \), then \( (u, v, w) \) together with \( p := 0 \) belong to the desired regularity class and solve

\[
\begin{align*}
\partial_t u - \frac{1}{\rho \epsilon} \Delta u + \nabla_x p = 0 & \quad \text{in } J \times \mathbb{R}^n_+, \\
\partial_t v - \frac{1}{\rho \epsilon} \Delta v + \partial_y p = 0 & \quad \text{in } J \times \mathbb{R}^n_+, \\
\partial_t w - \frac{1}{\rho \epsilon} \Delta w + \partial_z p = 0 & \quad \text{in } J \times \mathbb{R}^n_+, \\
- \frac{1}{\rho \epsilon} [\partial_y u]_y - \frac{1}{\rho \epsilon} \nabla_x [v]_y = h^\text{wall}_u & \quad \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[v]_y = h^\text{wall}_v & \quad \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[u]_z = u^\text{in}_w & \quad \text{on } J \times \partial_z \mathbb{R}^n_+, \\
u(0) = u_0, & \quad v(0) = v_0, \quad w(0) = 0 \quad \text{in } \mathbb{R}^n_+.
\end{align*}
\]

Hence, we may assume \( h^\text{wall}_u = h^\text{wall}_v = u^\text{in}_w = 0 \) together with the assumption on \( g \) stated above. Note that this reduction of the problem neither affects the regularity and compatibility assumptions of Theorem 2.4 nor the simplifications obtained in Step 1, i.e. we may still assume
Hence, we may assume \( h \) as well as the compatibility condition

\[
\frac{1}{\text{Re}} [\partial_y u_w^\text{in}]_y = 0 \quad \text{on } J \times \mathcal{E},
\]

\[
[u_w^\text{in}]_y = 0 \quad \text{on } J \times \mathcal{E},
\]

\[
[h_w^\text{wall}]_z = 0 \quad \text{on } J \times \mathcal{E},
\]

which stems from [20d], in the remaining part of the proof.

**Step 3.** We show that we may w.l.o.g. assume \( h_w^\text{wall} = 0 \). To accomplish this we define

\[
h_w^\text{wall} := E^- h_w^\text{wall} \in W^{1,2-1/p}(J, L_p(\partial\mathcal{R}_{n>0})) \cap L_p(J, W^{2-1/p}(\partial\mathcal{R}_{n>0})),
\]

where we denote by \( E^- \) the odd extension operator w.r.t. \( z \). Note that \( [h_w^\text{wall}]_z = 0 \) thanks to the compatibility condition (21). This ensures that the odd extension of \( h_w^\text{wall} \) w.r.t. \( z \) has the desired spatial regularity. Now, the problem

\[
\partial_t \hat{w} - \frac{1}{\text{Re}} \Delta \hat{w} = 0 \quad \text{in } J \times \mathbb{R}_{n>0}^n,
\]

\[
-\frac{1}{\text{Re}} [\partial_y \hat{w}]_y = h_w^\text{wall} \quad \text{on } J \times \partial\mathbb{R}_{n>0}^n,
\]

\[
\hat{w}(0) = 0 \quad \text{in } \mathbb{R}_{n>0}^n,
\]

admits a unique solution

\[
\hat{w} \in H^1_p(J, L_p(\mathbb{R}_{n>0}^n)) \cap L_p(J, H_0^2(\mathbb{R}_{n>0}^n)),
\]

which is odd w.r.t. \( z \) by construction. Hence, if we set \( u = v = 0 \) and define \( w \) to be the restriction of \( \hat{w} \) to \( \mathbb{R}_+^n \), then \( (u, v, w) \) together with \( p := 0 \) belong to the desired regularity class and solve

\[
\partial_t u - \frac{1}{\text{Re}} \Delta u + \nabla_x p = 0 \quad \text{in } J \times \mathbb{R}_+^n,
\]

\[
\partial_t v - \frac{1}{\text{Re}} \Delta v + \partial_y p = 0 \quad \text{in } J \times \mathbb{R}_+^n,
\]

\[
\partial_t w - \frac{1}{\text{Re}} \Delta w + \partial_z p = 0 \quad \text{in } J \times \mathbb{R}_+^n,
\]

\[
-\frac{1}{\text{Re}} [\partial_y u]_y - \frac{1}{\text{Re}} \nabla_x [v]_y = 0 \quad \text{on } J \times \partial\mathbb{R}_+^n,
\]

\[
[v]_y = 0 \quad \text{on } J \times \partial\mathbb{R}_+^n,
\]

\[
-\frac{1}{\text{Re}} [\partial_y w]_y - \frac{1}{\text{Re}} \partial_z [v]_y = h_w^\text{wall} \quad \text{on } J \times \partial\mathbb{R}_+^n,
\]

\[
[w]_z = 0 \quad \text{on } J \times \partial\mathbb{R}_+^n,
\]

\[
u(0) = u_0, \quad v(0) = v_0, \quad w(0) = 0 \quad \text{in } \mathbb{R}_+^n.
\]

Hence, we may assume \( h_w^\text{wall} = 0 \). Note that this reduction of the problem neither affects the regularity and compatibility assumptions of Theorem 2.4 nor the simplifications obtained in Steps 1, and 2, i.e. we may still assume

- \( g \in H^{1/2}_p(J, L_p(\mathbb{R}_+^n)) \cap L_p(J, H^1_p(\mathbb{R}_+^n)) \),

- \( (u_w^\text{in}, u_w^\text{in}) \in W^{1-1/p}_p(J, L_p(\partial\mathcal{R}_{n>0}))^{n-1} \cap L_p(J, W^{2-1/p}(\partial\mathcal{R}_{n>0}))^{n-1} \),

as well as the compatibility condition

\[
-\frac{1}{\text{Re}} [\partial_y u_w^\text{in}]_y = 0 \quad \text{on } J \times \mathcal{E},
\]

\[
[u_w^\text{in}]_y = 0 \quad \text{on } J \times \mathcal{E},
\]

which stems from [21], in the remaining part of the proof.
Step 4. Finally, we solve the reduced problem as obtained by Steps 1, 2, and 3. To accomplish this we define
\[
\hat{g} := E_y^+ g \in H^1_p(J, 0) H^{-1/2}(\mathbb{R}_y^{n+1}) \cap H^1_p(J, L^2(\mathbb{R}_y^{n+1})) \cap L_p(J, H^2_p(\mathbb{R}_y^{n+1}))
\]
as well as
\[
\hat{u}_u := E_y^+ u_u \in W^{1,2}_{pc} J, L^2(\mathbb{R}_y^{n+1})) \cap L_p(J, W^{-1,2}_{pc} (\mathbb{R}_y^{n+1})) \cap L_p(J, W^{-1,2}_{pc} (\mathbb{R}_y^{n+1})),
\]
where we denote by \( E_y^+ \) the even, and odd extension operator w.r.t. \( y \), respectively. Note that
\[
\langle \phi, F(\hat{g}, 0) \rangle = \int \phi \hat{g} \, dV = \int R^+_y(1 + R^+_y) \phi g \, dV = \langle (1 + R^+_y) \hat{\phi} | x_+^2 \rangle, F(g, 0, 0) \]
for \( \phi \in H^1_p(\mathbb{R}_y^{n+1}), \) where \( R^+_y \) denotes the even reflection operator w.r.t. \( y \). This implies that \( \hat{g} \) has the desired temporal and spatial regularity. Also note that \( [\partial_y u_u y] = [\hat{u}_u y] = 0 \) thanks to the compatibility condition \( (22) \). This ensures that the even extension of \( u_u \), and the odd extension of \( u_v \) w.r.t. \( y \) have the desired spatial regularity. Now, the Stokes equations in the halfspace
\[
\begin{align*}
\partial_t \hat{u} - \frac{1}{Re} \Delta \hat{u} + \nabla_x \hat{p} &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\partial_t \hat{v} - \frac{1}{Re} \Delta \hat{v} + \partial_y \hat{p} &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\partial_t \hat{w} - \frac{1}{Re} \Delta \hat{w} + \partial_z \hat{p} &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\text{div}_x \hat{u} + \partial_y \hat{v} + \partial_z \hat{w} &= \hat{g} \quad \text{in } J \times \mathbb{R}_y^{n+1},
\end{align*}
\]
are even while \( \hat{v} \) is odd w.r.t. \( y \) by construction. Hence, the restrictions \( u, v, w, \) and \( p \) of \( \hat{u}, \hat{v}, \hat{w}, \) and \( \hat{p} \) to \( \mathbb{R}_y^{n+1} \) belong to the desired regularity class and solve
\[
\begin{align*}
\partial_t u - \frac{1}{Re} \Delta u + \nabla_x p &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\partial_t v - \frac{1}{Re} \Delta v + \partial_y p &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\partial_t w - \frac{1}{Re} \Delta w + \partial_z p &= 0 \quad \text{in } J \times \mathbb{R}_y^{n+1}, \\
\text{div}_x u + \partial_y v + \partial_z w &= g \quad \text{in } J \times \mathbb{R}_y^{n+1},
\end{align*}
\]
thanks to \( (23) \), where \( \hat{u}, \hat{w}, \) and \( \hat{p} \) are even while \( \hat{v} \) is odd w.r.t. \( y \) by construction.
which is the reduced form of problem (19) after Steps 1, 2, and 3. This completes the proof of Theorem 2.4

4.2. Combination of Dynamic Outflow/Navier Conditions. In order to prove Theorem 2.6 in addition to the assumptions and the notation introduced at the beginning of this section, we assume \( v^{\text{out}} = V \nu \) with \( \alpha V + \frac{2}{Re} > \alpha V + \frac{1}{Re} > 0 \). We have to study the model problem

\[
\begin{align*}
\partial_t u - \frac{1}{Re} \Delta u + \nabla_x p &= f_u \quad \text{in } J \times \mathbb{R}^n, \\
\partial_t v - \frac{1}{Re} \Delta v + \partial_y p &= f_v \quad \text{in } J \times \mathbb{R}^n, \\
\partial_t w - \frac{1}{Re} \Delta w + \partial_z p &= f_w \quad \text{in } J \times \mathbb{R}^n, \\
\text{div}_x u + \partial_y v + \partial_z w &= g \quad \text{in } J \times \mathbb{R}^n, \\
-\frac{1}{Re} \partial_y u \big|_y - \frac{1}{Re} \nabla_x [v] \big|_y &= h_w^{\text{wall}} \quad \text{on } J \times \partial_y \mathbb{R}^n, \\
[v]_y &= h_v^{\text{wall}} \quad \text{on } J \times \partial_y \mathbb{R}^n, \\
-\frac{1}{Re} \partial_y w \big|_y - \frac{1}{Re} \partial_z v \big|_y &= h_w^{\text{wall}} \quad \text{on } J \times \partial_y \mathbb{R}^n, \\
u(0) &= u_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{in } \mathbb{R}^n, 
\end{align*}
\]

(23)

together with a dynamic outflow boundary condition in tangential directions

\[
\begin{align*}
\alpha \partial_t [u] - (\alpha V + \frac{1}{Re}) [\partial_z u] - \frac{1}{Re} \nabla_x [w]_z &= h_u \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
\alpha \partial_t [v] - (\alpha V + \frac{1}{Re}) [\partial_z v] - \frac{1}{Re} \partial_y [w]_z &= h_v \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
[w]_z &= h_w \quad \text{on } J \times \partial_z \mathbb{R}^n, 
\end{align*}
\]

(24a)
or a dynamic outflow boundary condition in normal direction

\[
\begin{align*}
[u]_z &= h_u \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
[v]_z &= h_v \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
\alpha \partial_t [w] - (\alpha V + \frac{2}{Re}) [\partial_z w]_z + [p]_z &= h_w \quad \text{on } J \times \partial_z \mathbb{R}^n, 
\end{align*}
\]

(24b)
or a full dynamic outflow boundary condition

\[
\begin{align*}
\alpha \partial_t [u] - (\alpha V + \frac{1}{Re}) [\partial_z u] - \frac{1}{Re} \nabla_x [w]_z &= h_u \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
\alpha \partial_t [v] - (\alpha V + \frac{1}{Re}) [\partial_z v] - \frac{1}{Re} \partial_y [w]_z &= h_v \quad \text{on } J \times \partial_z \mathbb{R}^n, \\
\alpha \partial_t [w] - (\alpha V + \frac{2}{Re}) [\partial_z w]_z + [p]_z &= h_w \quad \text{on } J \times \partial_z \mathbb{R}^n, 
\end{align*}
\]

(24c)

where the data \( f = (f_u, f_v, f_w), g, h = (h_u, h_v, h_w), h^{\text{wall}} = (h_w^{\text{wall}}, h_v^{\text{wall}}, h_u^{\text{wall}}), \) and \( (u_0, v_0, w_0) \) are subject to the regularity/compatibility conditions stated in Theorem 2.6 i.e.

- \( f \in L_p(J \times \mathbb{R}^n)^n \)
- \( g \in H_0^{1/2}(J, L_p(\mathbb{R}^n)) \cap L_p(J, H_0^1(\mathbb{R}^n)) \)
- \( (h_u, h_v) \in W_p^{3/2-2/p}(J, L_p(\mathbb{R}^n))^{n-1} \cap L_p(J, W_p^{\frac{\alpha}{p}-1/p}(\partial_2 \mathbb{R}^n))^n \)
- \( h_w \in L_p(J, W_p^{\frac{1}{p}-1/p}(\partial_2 \mathbb{R}^n)) \)
- \( (h_u^{\text{wall}}, h_v^{\text{wall}}) \in W_p^{1/2-1/p}(J, L_p(\partial_y \mathbb{R}^n))^{n-1} \cap L_p(J, W_p^{1-1/p}(\partial_y \mathbb{R}^n))^n \)
- \( h_w^{\text{wall}} \in W_p^{1-1/p}(J, L_p(\partial_y \mathbb{R}^n))^{n-1} \cap L_p(J, W_p^{2-1/p}(\partial_y \mathbb{R}^n))^n \)
- \( F(g, -h_w^{\text{wall}}, -\eta) \in H_p^1(J, H_p^{-1}(\mathbb{R}^n)) \)
- \( (u_0, v_0, w_0) \in W_p^{2-2/p}(\mathbb{R}^n)^n \)
- \( ([u_0]_z, [v_0]_z) \in W_p^{2-(\kappa+1)/p}(\partial_2 \mathbb{R}^n)^n \)

with \( \kappa = 1 \) for conditions (24a), and (24c), while \( \kappa = 2 \) for condition (24b). Furthermore, \( \eta = h_w \in W_p^{-1/p}(J, L_p(\partial_2 \mathbb{R}^n)) \cap L_p(J, W_p^{-1/p}(\partial_2 \mathbb{R}^n)) \) for condition (24a), while otherwise \( \eta \in H_p^1(J, W_p^{1-1/p}(\partial_2 \mathbb{R}^n)) \cap L_p(J, W_p^{2-1/p}(\partial_2 \mathbb{R}^n)) \) is given by assumption. In all cases the compatibility conditions (20a), (20b) are satisfied, and, due to \( \text{(TDO/W)}^{h,w^{\text{wall}}}, \text{(NDO/W)}^{h,w^{\text{wall}}} \)
or \((\text{FDO/W})^{h,h_{\text{wall}},\xi,\eta}\) we have

\[
\begin{align*}
-\frac{1}{\text{Re}} [\partial_y \xi_u]_y - \frac{1}{\text{Re}} \nabla_x [h_{\text{wall}}]_z &= [h_{\text{u}}]_z & \text{on } J \times \mathcal{E}, \\
\xi_v &= [h_{\text{v}}]_z & \text{on } J \times \mathcal{E}, \\
-\frac{1}{\text{Re}} [\partial_y \eta]_y - \frac{1}{\text{Re}} [\partial_z h_{\text{wall}}]_z &= [h_{\text{w}}]_z & \text{on } J \times \mathcal{E}
\end{align*}
\] (25a)

for \(p \geq 2\) with \((\xi_u, \xi_v) = (h_u, h_v)\) for condition \((\text{24a})\) while otherwise

\[
(\xi_u, \xi_v) \in W_p^{3/2-1/2p}(J, L_p(\partial_2 \mathbb{R}^n_+))^{n-1} \cap H_p^{1}(J, W_p^{1-1/p}(\partial_2 \mathbb{R}^n_+))^{n-1} \cap L_p(J, W_p^{2-2/p}(\partial_2 \mathbb{R}^n_+))^{n-1}
\]

is given by assumption such that we always have

\[
[u_0]_z = \xi_u(0) \quad \text{on } \partial_2 \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2},
\]

\[
[v_0]_z = \xi_v(0) \quad \text{on } \partial_2 \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2},
\]

\[
[w_0]_z = \eta(0) \quad \text{on } \partial_2 \mathbb{R}^n_+, \quad \text{if } p > \frac{3}{2}.
\]

In case of the boundary conditions \((\text{24a})\) and \((\text{24c})\) the compatibility conditions \((\text{TDO/W})^{h,h_{\text{wall}},\xi}\) and \((\text{FDO/W})^{h,h_{\text{wall}},\xi,\eta}\) imply

\[
\alpha \partial_t [h_{\text{v}}]_z - \alpha V[\partial_t h_{\text{wall}}]_z + [h_{\text{w}}]_z = [h_{\text{v}}]_y \quad \text{on } J \times \mathcal{E}.
\] (25c)

The construction of solutions to this problem requires two steps.

**Step 1.** We first show that we can w.l.o.g. assume \(f = g = h_{\text{wall}} = \xi = \eta = 0\) together with \(u_0 = v_0 = w_0 = 0\) as well as \(h_u = h_v = 0\) in case condition \((\text{24b})\) is applied, and \(h_w = 0\) in case condition \((\text{24a})\) is applied. Indeed, thanks to \((\text{20a}), (\text{20b}), (\text{25a})\) and \((\text{25b})\) all necessary regularity and compatibility conditions are satisfied in order to apply Theorem \(2.4\) to solve

\[
\begin{align*}
\partial_t u - \frac{1}{\text{Re}} \Delta u + \nabla_x p &= f_u & \text{in } J \times \mathbb{R}^n_+, \\
\partial_t v - \frac{1}{\text{Re}} \Delta v + \partial_y p &= f_v & \text{in } J \times \mathbb{R}^n_+, \\
\partial_t w - \frac{1}{\text{Re}} \Delta w + \partial_z p &= f_w & \text{in } J \times \mathbb{R}^n_+, \\
\partial_x u + \partial_y v + \partial_z w &= g & \text{in } J \times \mathbb{R}^n_+, \\
-\frac{1}{\text{Re}} [\partial_y u]_y - \frac{1}{\text{Re}} \nabla_x [v]_y &= h_{\text{u}} & \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[v]_y &= h_{\text{v}} & \text{on } J \times \partial_y \mathbb{R}^n_+, \\
-\frac{1}{\text{Re}} [\partial_y w]_y - \frac{1}{\text{Re}} \partial_z [v]_y &= h_{\text{w}} & \text{on } J \times \partial_y \mathbb{R}^n_+, \\
[u]_z &= \xi_u & \text{on } J \times \partial_z \mathbb{R}^n_+, \\
[v]_z &= \xi_v & \text{on } J \times \partial_z \mathbb{R}^n_+, \\
[w]_z &= \eta & \text{on } J \times \partial_z \mathbb{R}^n_+, \\
u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0 & \text{in } \mathbb{R}^n_+.
\end{align*}
\]

Then \((u, v, w)\), and \(p\) belong to the desired regularity class, except that the pressure trace only satisfies \([p]_z \in L_p(J, W_p^1(\partial_2 \mathbb{R}^n_+))\). This shows that we can assume \(f = g = h_{\text{wall}} = \xi = \eta = 0\) together with \(u_0 = v_0 = w_0 = 0\) as well as \(h_u = h_v = 0\) in case condition \((\text{24b})\) is applied, and \(h_w = 0\) in case condition \((\text{24a})\) is applied. Note that this reduction of the problem does not affect the regularity and compatibility assumptions of Theorem \(2.6\) except for a potential lower regularity of \(h_w\) that stems from the potential lower regularity of \([p]_z\), i.e. we may now assume

- \((h_u, h_v) \in W_p^{1/2-1/2p}(J, L_p(\partial_2 \mathbb{R}^n_+))^{n-1} \cap L_p(J, W_p^{1-1/p}(\partial_2 \mathbb{R}^n_+))^{n-1},
- h_w \in L_p(J, W_p^{1-1/p}(\partial_2 \mathbb{R}^n_+))
\]
as well as the compatibility condition

\[
[h_{\text{v}}]_y = 0 \quad \text{on } J \times \mathcal{E},
\] (26)
which stems from (25a), with \( u_h = h_v = 0 \) in case of condition (24a), and \( h_w = 0 \) in case of condition (24a).

**Step 2.** Finally, we solve the reduced problem as obtained by Step 1. To accomplish this we define

\[
\begin{align*}
\hat{h}_u &:= E_y^+ h_u \in W_p^{1/2-1/2p}(J, L_p(\partial \mathbb{R}^n_{y,z}^+) )^{n-2} \cap L_p(J, W_p^{1-1/p}(\partial \mathbb{R}^n_{y,z}^+) )^{n-2}, \\
\hat{h}_v &:= E_y^+ h_v \in W_p^{1/2-1/2p}(J, L_p(\partial \mathbb{R}^n_{y,z}^+) ) \cap L_p(J, W_p^{1-1/p}(\partial \mathbb{R}^n_{y,z}^+) ),
\end{align*}
\]

in case condition (24a) or (24c) is applied as well as

\[
\hat{h}_w := E_y^+ h_w \in L_p(J, W_p^{1-1/p}(\partial \mathbb{R}^n_{y,z}^+) )
\]

in case condition (24b) or (24c) is applied. Note that \([h_v]_y = 0\) thanks to the compatibility condition (26). This ensures that the odd extension of \( h_v \) w. r. t. \( y \) have the desired spatial regularity. Now, the Stokes equations in the halfspace

\[
\partial_t \hat{u} - \frac{1}{\text{Re}} \Delta \hat{u} + \nabla_x \hat{p} = 0 \quad \text{in} \ J \times \mathbb{R}^n_{y,z}^+, \\
\partial_t \hat{v} - \frac{1}{\text{Re}} \Delta \hat{v} + \partial_y \hat{p} = 0 \quad \text{in} \ J \times \mathbb{R}^n_{y,z}^+, \\
\partial_t \hat{w} - \frac{1}{\text{Re}} \Delta \hat{w} + \partial_z \hat{p} = 0 \quad \text{in} \ J \times \mathbb{R}^n_{y,z}^+, \\
\text{div}_x \hat{u} + \partial_y \hat{v} + \partial_z \hat{w} = 0 \quad \text{in} \ J \times \mathbb{R}^n_{y,z}^+, \\
\hat{u}(0) = 0, \quad \hat{v}(0) = 0, \quad \hat{w}(0) = 0 \quad \text{in} \ \mathbb{R}^n_{y,z}^+,
\]

together with the dynamic outflow boundary condition in tangential directions

\[
\begin{align*}
\alpha \partial_t [\hat{u}]_z - (\alpha V + \frac{1}{\text{Re}}) [\partial_x \hat{u}]_z - \frac{1}{\text{Re}} \nabla_x [\hat{w}]_z &= \hat{h}_u \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
\alpha \partial_t [\hat{v}]_z - (\alpha V + \frac{1}{\text{Re}}) [\partial_x \hat{v}]_z - \frac{1}{\text{Re}} \partial_y [\hat{w}]_z &= \hat{h}_v \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
[\hat{w}]_z &= 0 \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, 
\end{align*}
\]

the dynamic outflow boundary condition in normal direction

\[
\begin{align*}
[\hat{u}]_z &= 0 \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
[\hat{v}]_z &= 0 \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
\alpha \partial_t [\hat{w}]_z - (\alpha V + \frac{2}{\text{Re}}) [\partial_x \hat{w}]_z + [\hat{p}]_z &= \hat{h}_w \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, 
\end{align*}
\]

the full dynamic outflow boundary condition

\[
\begin{align*}
\alpha \partial_t [\hat{u}]_z - (\alpha V + \frac{1}{\text{Re}}) [\partial_x \hat{u}]_z - \frac{1}{\text{Re}} \nabla_x [\hat{w}]_z &= \hat{h}_u \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
\alpha \partial_t [\hat{v}]_z - (\alpha V + \frac{1}{\text{Re}}) [\partial_x \hat{v}]_z - \frac{1}{\text{Re}} \partial_y [\hat{w}]_z &= \hat{h}_v \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, \\
\alpha \partial_t [\hat{w}]_z - (\alpha V + \frac{2}{\text{Re}}) [\partial_x \hat{w}]_z + [\hat{p}]_z &= \hat{h}_w \quad \text{on} \ J \times \partial \mathbb{R}^n_{y,z}^+, 
\end{align*}
\]

respectively, admit a unique solution

\[
(\hat{u}, \hat{v}, \hat{w}) \in H^1_p(J, L_p(\mathbb{R}^n_{y,z}^+)) \cap L_p(J, H^{2}_p(\mathbb{R}^n_{y,z}^+)), \\
\hat{p} \in L_p(J, H^{1}_p(\mathbb{R}^n_{y,z}^+))
\]

with increased regularity of \([\hat{u}]_z, [\hat{v}]_z, \) and \([\hat{w}]_z, \) according to the dynamic boundary condition thanks to Theorem 2.1 and Remark 2.2 (a), where \( \hat{u}, \hat{w}, \) and \( \hat{p} \) are even while \( \hat{v} \) is odd w. r. t. \( y \) by construction. Hence, the restrictions \( u, v, w, \) and \( p \) of \( \hat{u}, \hat{v}, \hat{w}, \) and \( \hat{p} \) to \( \mathbb{R}^n_{y,z}^+ \) belong to the desired
regularity class and solve
\[
\begin{align*}
\partial_t u - \frac{1}{\Re} \Delta u + \nabla_x p &= 0 \quad \text{in} \; J \times \mathbb{R}^n_+, \\
\partial_t v - \frac{1}{\Re} \Delta v + \partial_y p &= 0 \quad \text{in} \; J \times \mathbb{R}^n_+, \\
\partial_t w - \frac{1}{\Re} \Delta w + \partial_z p &= 0 \quad \text{in} \; J \times \mathbb{R}^n_+, \\
\text{div}_x u + \partial_y v + \partial_z w &= 0 \quad \text{in} \; J \times \mathbb{R}^n_+, \\
-\frac{1}{\Re}[\partial_y u]_y - \frac{1}{\Re} \nabla_x [v]_y &= 0 \quad \text{on} \; J \times \partial_y \mathbb{R}^n_+, \\
[v]_y &= 0 \quad \text{on} \; J \times \partial_y \mathbb{R}^n_+, \\
-\frac{1}{\Re}[\partial_y w]_y - \frac{1}{\Re} \partial_z [v]_y &= 0 \quad \text{on} \; J \times \partial_y \mathbb{R}^n_+, \\
u(0) = 0, \; u(0) = 0, \; w(0) = 0 \quad \text{in} \; \mathbb{R}^n_+
\end{align*}
\]
together with the dynamic outflow boundary condition in tangential directions
\[
\alpha \partial_t [u]_z - (\alpha V + \frac{1}{\Re})[\partial_z u]_z - \frac{1}{\Re} \nabla_x [w]_z = h_u \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
\[
\alpha \partial_t [v]_z - (\alpha V + \frac{1}{\Re})[\partial_z v]_z - \frac{1}{\Re} \partial_y [w]_z = h_v \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
\[
[w]_z = 0 \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
the dynamic outflow boundary condition in normal direction
\[
[u]_z = 0 \quad \text{on} \; J \times \partial_x \mathbb{R}^n_+,
\]
\[
[v]_z = 0 \quad \text{on} \; J \times \partial_x \mathbb{R}^n_+,
\]
\[
\alpha \partial_t [w]_z - (\alpha V + \frac{2}{\Re})[\partial_z w]_z + [\alpha [w]_z = h_w \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
the full dynamic outflow boundary condition
\[
\alpha \partial_t [u]_z - (\alpha V + \frac{1}{\Re})[\partial_z u]_z - \frac{1}{\Re} \nabla_x [w]_z = h_u \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
\[
\alpha \partial_t [v]_z - (\alpha V + \frac{1}{\Re})[\partial_z v]_z - \frac{1}{\Re} \partial_y [w]_z = h_v \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
\[
\alpha \partial_t [w]_z - (\alpha V + \frac{2}{\Re})[\partial_z w]_z + [\alpha [w]_z = h_w \quad \text{on} \; J \times \partial_z \mathbb{R}^n_+,
\]
respectively, which is the reduced form of problem \((23)\) together with \((24a), \; (24b), \; (24c)\), respectively, after Step 1. This completes the proof of Theorem 2.6.

Appendix A
Parabolic Equations subject to Dynamic Boundary Conditions

In this appendix we collect some useful results on parabolic equations subject to dynamic boundary conditions. The first result is essentially contained in [3].

**Proposition A.1.** Let \(0 < a \leq \infty\), let \(J := (0, a)\), let \(\varepsilon \geq 0\) with \(\varepsilon > 0\), if \(a = \infty\). Let \(\Omega := \mathbb{R}^n_+\) with \(\Gamma = \partial \Omega\). Let \(1 < p < \infty\) with \(p \neq \frac{3}{2}, \; 3\) and let \(\alpha, \beta, \mu > 0\). Then for every
\[
f \in L_p(J \times \Omega), \quad h \in W^{3/2-1/p}_p(J, L_p(\Gamma)) \cap L_p(J, W^{1-1/p}_p(\Omega)), \quad u_0 \in W^{2-2/p}_p(\Omega)
\]
with \([u_0]_\Gamma \in W^{2-2/p}_p(\Gamma)\) the parabolic problem
\[
\begin{align*}
\varepsilon u + \partial_t u - \mu \Delta u &= f \quad \text{in} \; J \times \Omega, \\
\alpha \varepsilon u + \alpha \partial_t u + \beta \partial_t u &= h \quad \text{on} \; J \times \Gamma, \\
u(0) &= u_0 \quad \text{in} \; \Omega
\end{align*}
\]
admits a unique maximal regular solution
\[
u \in H^1_p(J, L_p(\Omega)) \cap L_p(J, H^{2}_p(\Omega)),
\]
\[
[u]_\Gamma \in W^{3/2-1/2p}_p(J, L_p(\Omega)) \cap H^1_p(J, W^{1-1/p}_p(\Gamma)) \cap L_p(J, W^{2-1/p}_p(\Gamma))
\]
The solutions depend continuously on the data.
Concerning the proof, first note that the problem fits into the framework of [3] Theorem 2.1; cf. also [3] Example 3.1. Strictly speaking, [3] Theorem 2.1 is formulated for the case that \( \Gamma \) is a sufficiently smooth, compact manifold, \( a < \infty \), and \( \varepsilon = 0 \). However, the proof given in [3] employs a localization procedure, where the problem in the halfspace \( \Omega = \mathbb{R}_+^n \) with \( a = \infty \), and \( \varepsilon > 0 \) is a model problem, which is dealt with in [3] Section 4.

Now, in order to obtain maximal regular solutions for the Stokes equations subject to a dynamic boundary condition involving the pressure it is necessary to have a result at hand that requires a lower regularity for the right-hand side of the boundary condition.

**Proposition A.2.** Let \( 0 < a \leq \infty \), let \( J := (0, a) \), let \( \varepsilon \geq 0 \) with \( \varepsilon > 0 \), if \( a = \infty \). Let \( \Omega := \mathbb{R}_+^n \) with \( \Gamma = \partial \Omega \). Let \( 1 < p < \infty \) with \( p \neq \frac{3}{2}, 3 \) and let \( \alpha, \beta, \mu > 0 \). Then for every

\[
    f \in L_p(J \times \Omega), \quad h \in L_p(J, W^{1-1/p}(\Gamma)), \quad u_0 \in W^{2-2/p}(\Omega)
\]

with \([u_0]_\Gamma \in W^{2-2/p}(\Gamma)\) the parabolic problem

\[
    \varepsilon u + \partial_t u - \mu \Delta u = f \quad \text{in} \ J \times \Omega,
\]

\[
    \alpha \varepsilon u + \alpha \partial_t u + \beta \partial_u u = h \quad \text{on} \ J \times \Gamma,
\]

\[
    u(0) = u_0 \quad \text{in} \ \Omega
\]

admits a unique maximal regular solution

\[
    u \in H^1_p(J, L_p(\Omega)) \cap L_p(J, H^2_p(\Omega)),
\]

\[
    [u]_\Gamma \in H^1_p(J, W^{1-1/p}(\Gamma)) \cap L_p(J, W^{2-1/p}(\Gamma)).
\]

The solutions depend continuously on the data.

**Proof.** It is sufficient to consider the case \( a = \infty \), and \( \varepsilon > 0 \). Moreover, using Proposition A.1 we may assume \( f = u_0 = 0 \) in the following. A Laplace transformation w.r.t. time and a Fourier transformation w.r.t. the tangential spatial variables leads to

\[
    \omega^2 \hat{u} - \mu \partial^2 \hat{u} = 0, \quad \lambda \in \Sigma_{\varepsilon-\theta}, \ \xi \in \mathbb{R}^{n-1}, \ y > 0,
\]

\[
    \alpha \lambda [\hat{u}]_y - \beta \partial \hat{u}]_y = \hat{h}, \quad \lambda \in \Sigma_{\varepsilon-\theta}, \ \xi \in \mathbb{R}^{n-1},
\]

where we employ the notation from Subsection 3.1 with \( \omega = \sqrt{\lambda_\varepsilon + \mu |\xi|^2} \). We immediately obtain

\[
    \hat{u}(\lambda, \xi, y) = \hat{\tau} e^{-\omega(\varepsilon/\sqrt{\mu})y}
\]

for an unkown boundary value \( \hat{\tau} : \mathbb{R}_+ \times \mathbb{R}^{n-1} \rightarrow \mathbb{R} \), which has to be determined based on the boundary condition

\[
    \alpha \lambda [\hat{u}]_y - \beta \partial \hat{u} : \hat{h} = \left( \alpha \lambda + \beta \frac{\omega}{\sqrt{\mu}} \right) \hat{\tau} \hat{h}.
\]

This implies that

\[
    [\hat{u}]_y = \hat{\tau} = \frac{\sqrt{\mu}}{\alpha \sqrt{\mu} \lambda + \beta \omega} \hat{h}
\]

and since the symbols

\[
    (\lambda, z) \mapsto \frac{\sqrt{\mu} \lambda}{\alpha \sqrt{\mu} \lambda + \beta \omega}, \quad \frac{\sqrt{\mu} \lambda}{\alpha \sqrt{\mu} \lambda + \beta \omega} : \Sigma_{\varepsilon-\theta} \times \Sigma_{\theta/2} \rightarrow \mathbb{C}
\]

are bounded and holomorphic for \( 0 < \theta < \frac{\pi}{2} \), we obtain the desired regularity of \([u]_y\) by the bounded \( H^\infty \)-calculus of the operators \( \partial_t \) and \( \sqrt{-\Delta_\Gamma} \), cf. Subsection 3.1. \( \square \)

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References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Academic Press, 2 edition, 2003.

[2] D. Bothe, M. Köhne, and J. Prüss. On a Class of Energy Preserving Boundary Conditions for Incompressible Newtonian Flows. *SIAM J. Math. Anal.*, 45(6):3768–3822, 2013.

[3] R. Denk, J. Prüss, and R. Zacher. Maximal $L_p$-Regularity of Parabolic Problems with Boundary Dynamics of Relaxation Type. *J. Funct. Anal.*, 255:3149–3187, 2008.

[4] R. Denk, J. Saal, and J. Seiler. Inhomogeneous Symbols, the Newton Polygon, and Maximal $L_p$-Regularity. *Russian J. Math. Phys.*, 15(2):171–192, 2008.

[5] B. Engquist and A. Majda. Absorbing Boundary Conditions for the Numerical Simulation of Waves. *Math. Comp.*, 31:629–651, 1977.

[6] L. Halpern. Artificial Boundary Conditions for the Linear Advection Diffusion Equation. *Math. Comp.*, 46:425–439, 1986.

[7] L. Halpern and M. Schatzman. Artificial Boundary Conditions for Incompressible Viscous Flows. *SIAM J. Math. Anal.*, 20:308–353, 1989.

[8] J. G. Heywood, R. Rannacher, and S. Turek. Artificial Boundaries and Flux and Pressure Conditions for Incompressible Navier-Stokes Equations. *Int. J. Numer. Meth. Fluids*, 22:325–352, 1992.

[9] G. Jin and M. Braza. A Nonreflecting Outlet Boundary Condition for Incompressible Unsteady Navier-Stokes Calculations. *J. Comp. Phys.*, 107:239–253, 1993.

[10] N. J. Kalton and L. Weis. The $H^\infty$-Calculus and Sums of Closed Operators. *Math. Ann.*, 321:319–345, 2001.

[11] M. Köhne. $L_p$-Theory for Incompressible Newtonian Flows. Energy Preserving Boundary Conditions, Weakly Singular Domains. Springer Spektrum, Wiesbaden, 2013.

[12] I. Orlanski. A Simple Boundary Condition for Unbounded Hyperbolic Flows. *J. Comp. Phys.*, 21:251–269, 1976.

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