THE LATTICE OF VARIETIES OF IMPLICATION SEMIGROUPS

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Abstract. An implication semigroup is an algebra of type (2, 0) with a binary operation \( \rightarrow \) and a 0-ary operation 0 satisfying the identities

\[
(x \rightarrow y) \rightarrow z \approx (x \rightarrow (y \rightarrow z))' \quad \text{and} \quad 0'' \approx 0
\]

where \( u' \) means \( u \rightarrow 0 \). We completely describe the lattice of varieties of implication semigroups. It turns out that this lattice is non-modular and consists of 16 elements.

1. Introduction and summary

In the article [5], the second author introduced and examined a new type of algebras as a generalization of De Morgan algebras. These algebras are of type (2, 0) with a binary operation \( \rightarrow \) and a 0-ary operation 0 satisfying the identities

\[
(x \rightarrow y) \rightarrow z \approx (x \rightarrow (y \rightarrow z))' \quad \text{and} \quad 0'' \approx 0
\]

where \( u' \) means \( u \rightarrow 0 \). Such algebras are called implication semigroupoids. We refer an interested reader to [5] for detailed explanation of the background and motivations.

The class of all implication semigroupoids is a variety denoted by \( \text{IZ} \). It seems very natural to examine the lattice of its subvarieties. One of the important and interesting subvarieties of \( \text{IZ} \) is the class of all associative implication semigroupoids, that is algebras from \( \text{IZ} \) satisfying the identity

\[
(x \rightarrow y) \rightarrow z \approx x \rightarrow (y \rightarrow z).
\]

It is natural to call such algebras implication semigroups. The class \( \text{IS} \) of all implication semigroups forms a subvariety in \( \text{IZ} \). This subvariety was implicitly mentioned in [5, Lemma 8.21] and investigated more explicitly in the articles [1–3]. Incidentally, we should mention here that implication semigroupoids are referred to as “implicator groupoids” in [2].) But there, only the location of \( \text{IS} \) in the subvariety lattice of the variety \( \text{IZ} \) and “interaction” of \( \text{IS} \) with other varieties from this lattice were studied. The aim of this paper is to examine the lattice of subvarieties of the variety \( \text{IS} \). Our main result gives a complete description of this lattice.

For convenience of our considerations, we turn to the notation generally accepted in the semigroup theory. As usual, we denote the binary operation by a dot or (more often) by the absence of a symbol, rather than by \( \rightarrow \). Since this operation is associative, we agree not to use brackets in terms and put \( u^n := u \underbrace{u \cdots u}_{n \text{ times}} \). Besides that, the notation 0 for the 0-ary operation could be somewhat confusing in the context of semigroups, because it is associated with the operation of fixing the zero element in a semigroup with zero. For this reason, we will denote the 0-ary operation by the symbol \( \omega \) which does not have any predefined a priori meaning. In this

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notation, implication semigroups are defined by the associative law \((xy)z \approx x(yz)\) and the following two identities:

\begin{align}
(1.1) \quad xyz &\approx z\omega xyz\omega^2, \\
(1.2) \quad \omega^3 &\approx \omega.
\end{align}

To formulate the main result of the article, we need some notation. As usual, elements of the free implication semigroup over a countably infinite alphabet are called \textit{words}, while elements of this alphabet are called \textit{letters}. Words rather than letters are written in bold. We connect two sides of identities by the symbol \(\approx\), while the symbol \(=\) stands for the equality relation on the free implication semigroup. We denote by \(T\) the trivial variety of implication semigroups. The variety of implication semigroups defined by the set of identities \(\Sigma\), relative to IS, is denoted by \(\text{var} \Sigma\).

Let us fix notation for the following concrete varieties:

\begin{align*}
B &:= \text{var} \{x \approx x^2\}, \\
K &:= \text{var} \{xyz \approx x^2 \approx \omega, xy \approx yx\}, \\
L &:= \text{var} \{xyz \approx x^2 \approx \omega\}, \\
M &:= \text{var} \{xyz \approx \omega, xy \approx yx\}, \\
N &:= \text{var} \{xyz \approx \omega\}, \\
SL &:= \text{var} \{x \approx x^2, xy \approx yx\}, \\
ZM &:= \text{var} \{xy \approx \omega\}.
\end{align*}

It is appropriate to note here that the variety \(\text{IS}\) and several of its concrete sub-varieties have already appeared in the literature under other names (see [1–3, 5]). Moreover, the respective positions of these varieties are shown in the Hasse diagram of the poset of the (then) known varieties in [1].

The lattice of all varieties of implication semigroups is denoted by \(\mathbb{IS}\). The main result of the article is the following

\textbf{Theorem 1.1.} \textit{The lattice \(\mathbb{IS}\) has the form shown in Fig. 1.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The lattice \(\mathbb{IS}\)}
\end{figure}

In [5, Problem 5], the second author has raised the question as to whether the lattice of all varieties of implication zroupoids is distributive. The following assertion immediately follows from Fig. 1 and provides a negative answer to this question.
Corollary 1.2. The lattice \( \text{IS} \) is non-modular.

This article consists of three sections. Section 2 contains definitions, notation and auxiliary results. Section 3 is devoted to the proof of the main result.

2. Preliminaries

If \( u \) and \( v \) are words and \( \varepsilon \) is an identity then we will write \( u \varepsilon \approx v \) in the case when the identity \( u \approx v \) follows from \( \varepsilon \).

Lemma 2.1. The variety \( \text{IS} \) satisfies the following identities:

\[
\begin{align*}
(2.1) & \quad \omega^2 \approx \omega, \\
(2.2) & \quad \omega x \approx x \omega, \\
(2.3) & \quad xyz \approx xyz \omega, \\
(2.4) & \quad xyx \approx yx \omega, \\
(2.5) & \quad x^2 y \approx xy \omega.
\end{align*}
\]

Proof. The following five chains of identities provide deductions of the identities (2.1)–(2.5) from the identities that give the variety \( \text{IS} \) and the identities that were deducted before the currently deducted identity:

\[
\begin{align*}
(2.1) & \quad \omega^2 \approx \omega^{10} = \omega^9 \omega \omega^2 \omega^2 \omega^2 \approx \omega \omega^2 \omega^2 = \omega^5 \approx \omega, \\
(2.2) & \quad \omega x \approx \omega \omega x \approx x \omega x \omega^2 \approx (x \omega x \omega^2) \omega \approx (\omega x) \omega
\end{align*}
\]

\[
\begin{align*}
(2.3) & \quad xyz \approx z w x y \omega^2 \approx (z \omega x y \omega^2) \omega \approx (xy) \omega, \\
(2.4) & \quad xy x \approx x y x \omega \approx x \omega x \omega \approx x \omega \approx y x \omega, \\
(2.5) & \quad x^2 y \approx x^2 y \omega \approx x x y \approx \omega x \omega y \approx \omega^2 x y \approx \omega x y \approx xy \omega.
\end{align*}
\]

Lemma is proved. \( \square \)

A word that does not contain the symbol \( \omega \) is called a semigroup word. The last occurrence sequence of a word \( w \), denoted by \( \text{los}(w) \), is the semigroup word obtained from \( w \) by retaining only the last occurrence of each letter. If \( w \) does not contain letters (i.e., if \( w = \omega^n \) for some natural number \( n \)) then \( \text{los}(w) \) is the empty word. The length of a semigroup word \( w \) is denoted by \( \ell(w) \). If a word \( w \) contains the symbol \( \omega \) then we put \( \ell(w) = \infty \). We denote by \( \text{con}(w) \) the content of the word \( w \), i.e. the set of all letters occurring in \( w \).

Corollary 2.2. If \( w \) is a word of a length \( \geq 3 \) then the variety \( \text{IS} \) satisfies the identity

\[
(2.6) \quad w \approx \text{los}(w) \omega.
\]

Proof. Since \( \ell(w) \geq 3 \), the identity (2.3) allows us to assume that \( w \) is not a semigroup word. Further, in view of the identity (2.2), we can assume that \( w = w' \omega \) for some word \( w' \). The identities (2.4) and (2.5) permit to delete from \( w \) all but last occurrences of all letters. Therefore, we can assume that every letter from \( \text{con}(w) \) occurs in \( w \) at most one time. Finally, the identities (2.1) and (2.2) allow us to delete from \( w \) all but last occurrences of the symbol \( \omega \). Therefore, the identity (2.6) holds in \( \text{IS} \). \( \square \)

Note that the identities (2.3)–(2.5) are particular instances of the identity (2.6).
Corollary 2.3. If \( u \) and \( v \) are words such that \( \ell(u), \ell(v) \geq 3 \) and \( \text{los}(u) = \text{los}(v) \) then the variety \( \text{IS} \) satisfies the identity \( u \approx v \).

Proof. Indeed, we have \( u \overset{(2.6)}{\approx} \text{los}(u)\omega = \text{los}(v)\omega \overset{(2.6)}{\approx} v \). □

Recall that a semigroup is called a band if it satisfies the identity \( x^2 \approx x \). We call a variety of implication semigroups \( V \) a monoid variety if the identities \( x\omega \approx \omega x \approx x \) hold in \( V \). Obviously, this means that every algebra in \( V \) has a unit element and the operation \( \omega \) fixes just this element in each algebra in \( V \).

Proposition 2.4. A variety of implication semigroups is a monoid variety if and only if it is a variety of bands.

Proof. Necessity. Corollary 2.3 implies that the variety \( \text{IS} \) satisfies the identity \( x\omega \approx \omega x \approx x \). Hence any monoid variety satisfies the identities \( x \approx x\omega \approx x\omega x \approx x^2 \).

Sufficiency. It suffices to note that the identities \( \omega x \overset{(2.2)}{\approx} x \omega \overset{(2.3)}{\approx} x^3 \approx x \) hold in any variety of bands. □

We need a description of the identities of a few concrete varieties of implication semigroups. We say that a word \( w \) contains a square if \( w = ab^2c \) for some word \( b \) and some (possibly empty) words \( a \) and \( c \).

Lemma 2.5. A non-trivial identity \( u \approx v \) holds in the variety:

(i) \( \text{SL} \) if and only if \( \text{con}(u) = \text{con}(v) \);
(ii) \( \text{B} \) if and only if \( \text{los}(u) = \text{los}(v) \);
(iii) \( \text{ZM} \) if and only if \( \ell(u), \ell(v) \geq 2 \);
(iv) \( \text{K} \) if and only if either \( u \approx v \) is the commutative law or each of the words \( u \) and \( v \) either contains a square or has a length \( \geq 3 \);
(v) \( \text{L} \) if and only if each of the words \( u \) and \( v \) either contains a square or has a length \( \geq 3 \);
(vi) \( \text{M} \) if and only if either \( u \approx v \) is the commutative law or \( \ell(u), \ell(v) \geq 3 \);
(vii) \( \text{N} \) if and only if \( \ell(u), \ell(v) \geq 3 \).

Proof. (i) Proposition 2.4 implies that any identity that holds in one of the varieties \( \text{SL} \) or \( \text{B} \) is equivalent to a semigroup identity. It is well known and can be easily checked that a semigroup identity \( u \approx v \) holds in the variety of semilattices \( \text{SL} \) if and only if \( \text{con}(u) = \text{con}(v) \). This completes the proof of the assertion (i).

(ii) Proposition 2.4 shows that an arbitrary identity \( u \approx v \) is equivalent within \( \text{B} \) to the identity \( u\omega \approx \omega v \). This allows us to consider only identities, both sides of which have a length \( \geq 3 \). Corollary 2.2 allows us now to delete all but last occurrences of any letter in any word. Furthermore, Proposition 2.4 means that we can delete all occurrences of the symbol \( \omega \) in every word. This proves the claim (ii).

(iii)–(vii) These assertions are evident. □

Lemma 2.6. Let \( V \) be a variety of implication semigroups.

(i) If \( V \not\supseteq \text{B} \) then \( V \) satisfies the identity
\[
(2.7) \quad xy\omega \approx yx\omega.
\]

(ii) If \( V \not\supseteq \text{K} \) then \( V \) satisfies the identity
\[
(2.8) \quad xy \approx x\omega y.
\]

(iii) If \( V \not\supseteq \text{N} \) then \( V \) satisfies either the commutative law or the identity
\[
(2.9) \quad x\omega \approx x^2.
\]
Proof. Lemma 2.5(i) implies that the identities (2.7)–(2.9) hold in the variety $\text{SL}$. Whence, any of these identities is valid in a variety $V$ if and only if it is valid in $V \lor \text{SL}$. Further, if $V \lor \text{SL} \not\subseteq X$ where $X$ is one of the varieties $B$, $K$ or $N$ then $V \not\subseteq X$. These observations allow us to assume that $V \supseteq \text{SL}$. Then Lemma 2.5(i) applies and we conclude that if $V$ satisfies an identity $u \approx v$ then
\begin{equation}
\text{con}(u) = \text{con}(v).
\end{equation}

(i) By the hypothesis, there is an identity $u \approx v$ that is true in $V$ but fails in $B$. Lemma 2.5(ii) implies that los$(u) \neq \text{los}(v)$. If one of the words $u$ or $v$ has a length $\leq 2$ then we multiply the identity $u \approx v$ by $x^2$ for some $x \in \text{con}(u)$ from the left. Clearly, this does not change the words los$(u)$ and los$(v)$. Both the sides of the resulting identity have a length $\geq 3$. Thus we can assume without loss of generality that $\ell(u), \ell(v) \geq 3$. According to Corollary 2.2, we can assume also that $u = \text{los}(u)\omega$ and $v = \text{los}(v)\omega$. Then the equality (2.10) implies that $\text{con}(\text{los}(u)) = \text{con}(\text{los}(v))$. Whence there are letters $x$ and $y$ such that $x$ precedes $y$ in $u$ but $y$ precedes $x$ in $v$. Now we substitute $\omega$ for all letters from con$(u)$ except $x$ and $y$ in the identity $u \approx v$. In view of the identities (2.1) and (2.2), the obtained identity implies (2.7).

(ii) By the hypothesis, there is an identity $u \approx v$ that is true in $V$ but fails in $K$. Lemma 2.5(iv) implies that the identity $u \approx v$ is not the commutative law and one of the words $u$ and $v$, say $u$, has a length $\leq 2$ and is not a square. If $\ell(u) = 1$ then we can multiply the identity $u \approx v$ by some letter $y \notin \text{con}(u)$ from the right. Thus we can assume that $\ell(u) = 2$, whence $u = xy$. Further, $v \neq yx$ because the identity $u \approx v$ is not the commutative law. The equality (2.10) implies now that $\ell(v) \geq 3$. Hence $V$ satisfies the identities $xy = u \approx v \approx \text{los}(v)\omega$ and therefore, the identity
\begin{equation}
xy \approx \text{los}(v)\omega.
\end{equation}
If los$(v) = xy$ then we are done. Otherwise, los$(v) = yx$ and we have
\begin{align*}
xy & \approx yx\omega \approx xy\omega^2 \approx xy\omega.
\end{align*}

(iii) By the hypothesis, there is an identity $u \approx v$ that is true in $V$ but fails in $N$. Lemma 2.5(vii) implies that one of the words $u$ and $v$, say $u$, has a length $\leq 2$. We can assume without loss of generality that $\ell(u) \leq \ell(v)$. If $\ell(u) = 1$ then we can multiply the identity $u \approx v$ by some letter $y$ from the right. Thus we can assume that $\ell(u) = 2$, whence $u \in \{xy, x^2\}$. Recall that the equality (2.10) holds.

Suppose at first that $u = x^2$. Then $\ell(v) \geq 3$ and los$(v) = x$. Therefore, $V$ satisfies the identities $x^2 = u \approx v \approx \text{los}(v)\omega = x\omega$. Whence, the identity (2.9) holds in $V$.

Finally, let $u = xy$. If $v = yx$ then the variety $V$ is commutative, and we are done. Let now $v \neq yx$. Then $\ell(v) \geq 3$. Substituting $x$ for $y$ in $u \approx v$, we get the situation considered in the previous paragraph.

\section*{Lemma 2.7} Let $V$ be a variety of implication semigroups.

(i) If $V \not\subseteq \text{SL}$ then $V \subseteq N$.

(ii) If $V \not\subseteq \text{ZM}$ then $V \subseteq B$.

Proof. (i) Suppose that $\text{SL} \not\subseteq V$. In view of Lemma 2.5(i), $V$ satisfies an identity $u \approx v$ with con$(u) \neq \text{con}(v)$. We can assume without loss of generality that there is a letter $x \in \text{con}(u) \setminus \text{con}(v)$. One can substitute $\omega$ to all letters except $x$ in the identity $u \approx v$. The identities (2.1) and (2.2) imply that $V$ satisfies the identity $x^k\omega \approx \omega$ for some $k$. Corollary 2.3 implies that the variety $\text{IS}$ satisfies the identity $xyz \approx (xyz)^n\omega$ for any natural $n$. Then the identities $xyz \approx (xyz)^k\omega \approx \omega$ hold in $V$, whence $V \subseteq N$. 

(ii) Suppose that $\mathbf{ZM} \nsubseteq \mathbf{V}$. It suffices to verify that $\mathbf{V} \lor \mathbf{SL} \subseteq \mathbf{B}$. Whence, we can assume that $\mathbf{V} \supseteq \mathbf{SL}$. In view of Lemma 2.5(iii), $\mathbf{V}$ satisfies a non-trivial identity of the kind $x \approx v$. Lemma 2.5(i) implies that $\text{con}(v) = \{x\}$. Clearly, $\ell(v) > 1$. If $\ell(v) = 2$ then $v = x^2$, whence $\mathbf{V} \subseteq \mathbf{B}$. Let now $\ell(v) \geq 3$. Then Corollary 2.3 applies and we conclude that the identity $v^2 \approx v$ is true in the variety $\mathbf{IS}$. Therefore, $\mathbf{V}$ satisfies the identities $x^2 \approx v^2 \approx v \approx x$, whence $\mathbf{V} \subseteq \mathbf{B}$ again. 

3. PROOF OF THE MAIN RESULT

We denote by $L(X)$ the subvariety lattice of the variety $X$. We divide this section into six subsections.

3.1. The structure of the lattices $L(B)$ and $L(N)$. According to Proposition 2.4, $\mathbf{B}$ is a monoid variety. Therefore, it satisfies the identities $xyx \approx yxx \approx yx$. The lattice of varieties of band monoids is completely described in [6]. In view of [6, Proposition 4.7], the lattice $L(\mathbf{B})$ is the 3-element chain $\mathbf{T} \subseteq \mathbf{SL} \subseteq \mathbf{B}$.

The variety $\mathbf{N}$ satisfies the identities $\omega x \approx x \omega \approx x \omega^2 \approx \omega$. Hence every semigroup from $\mathbf{N}$ contains the zero element and the operation $\omega$ fixes just this element in each semigroup from $\mathbf{N}$. This means that $\mathbf{N}$ is nothing but the variety of all nilpotent of degree 3 semigroups. The subvariety lattice of this variety has the form shown in Fig. 1. This claim can be easily verified directly and is a part of a semigroup folklore. It is known at least from the beginning of 1970’s (see [4], for instance).

3.2. Identity bases for certain varieties. Here we prove the following equalities:

\begin{align}
(3.1) \quad & \mathbf{SL} \lor \mathbf{K} = \text{var}\{x\omega \approx x^2, xy \approx yx\}, \\
(3.2) \quad & \mathbf{SL} \lor \mathbf{L} = \text{var}\{x\omega \approx x^2, xy \omega \approx yx\omega\}, \\
(3.3) \quad & \mathbf{SL} \lor \mathbf{M} = \text{var}\{xy \approx yx\}, \\
(3.4) \quad & \mathbf{SL} \lor \mathbf{N} = \text{var}\{xy \omega \approx yx\omega\}, \\
(3.5) \quad & \mathbf{SL} \lor \mathbf{ZM} = \text{var}\{xy \approx yx\}.
\end{align}

Let $\varepsilon$ be one of these equalities. The fact that the left-hand side of $\varepsilon$ is contained in its right-hand side is evident. One can verify reverse inclusions.

We start with the equality (3.4). Let $u \approx v$ be an identity that holds in the variety $\mathbf{SL} \lor \mathbf{N}$. The items (i) and (vii) of Lemma 2.5 imply that $\text{con}(u) = \text{con}(v)$ and $\ell(u), \ell(v) \geq 3$. Put $\text{con}(u) = \text{con}(v) = \{x_1, x_2, \ldots, x_n\}$. Then Corollary 2.2 allows us to assume that our identity coincides with the identity

$$x_1x_2\cdots x_n\omega \approx x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}\omega$$

for some permutation $\pi$ on the set $\{1, 2, \ldots, n\}$. But an arbitrary identity of such a form follows from the identities (2.1), (2.2) and (2.7), whence $\text{var}\{x\omega \approx yx\} \subseteq \mathbf{SL} \lor \mathbf{N}$. This proves the equality (3.4).

The items (vi) and (vii) of Lemma 2.5 imply that a unique identity that holds in $\mathbf{SL} \lor \mathbf{M}$ but fails in $\mathbf{SL} \lor \mathbf{N}$ is the commutative law. In view of what was said in the previous paragraph, this implies the equality (3.3).

Let now $u \approx v$ be an arbitrary identity that holds in $\mathbf{SL} \lor \mathbf{L}$. The items (i) and (v) of Lemma 2.5 imply that $\text{con}(u) = \text{con}(v)$ and each of the words $u$ and $v$ either has a length $\geq 3$ or contains a square. If one of the sides of the identity $u \approx v$ contains a square then we can apply the identity (2.9). Both the sides of the obtained identity will have a length $\geq 3$. Thus, we can assume that $\ell(u), \ell(v) \geq 3$. Then the items (i) and (vii) of Lemma 2.5 apply with the conclusion that the identity $u \approx v$ holds in $\mathbf{SL} \lor \mathbf{N}$. The equality (3.4) implies that $\text{var}\{x\omega \approx x^2, xy\omega \approx yx\omega\} \subseteq \mathbf{SL} \lor \mathbf{N}$. Therefore, the identity $u \approx v$ holds in $\text{var}\{x\omega \approx x^2, xy\omega \approx yx\omega\}$. Thus, the last variety is contained in $\mathbf{SL} \lor \mathbf{L}$. This proves the equality (3.2).
Let now \( u \approx v \) be an arbitrary identity that holds in \( SL \vee K \). Suppose that this identity differs from the commutative law. Comparison of the items (iv) and (v) of Lemma 2.5 shows that this identity holds in \( SL \vee L \). Besides that, the variety \( SL \vee K \) is commutative. The equality (3.2) proved in the previous paragraph implies now that the identity \( u \approx v \) holds in the variety

\[
\text{var}\{xw \approx x^2, \, xwy \approx yxw, \, xy \approx yx\} = \text{var}\{xw \approx x^2, \, xy \approx yx\}.
\]

Thus, the last variety is contained in \( SL \vee K \) in any case. This proves the equality (3.1).

Finally, let \( u \approx v \) be an identity that holds in \( SL \vee ZM \). The items (i) and (iii) of Lemma 2.5 imply that \( \text{con}(u) = \text{con}(v) \) and \( \ell(u), \ell(v) \geq 2 \). Further, the items (i) and (vii) of the same lemma imply that the identity

\[
uw \approx vw
\]

holds in \( SL \vee N \). The equality (3.4) implies that the last identity follows from the identity (2.7). In turn, the last identity follows from the identity

\[
xy \approx yxw.
\]

Indeed, if we multiply (3.7) on \( \omega \) from the right and use the identity (2.1), we obtain (2.7). Summarizing all we say, we get that the identity (3.6) holds in the variety \( \text{var}\{xy \approx yxw\} \). Besides that, it is clear that \( \text{var}\{xy \approx yxw\} \not\subseteq K \). Lemma 2.6(ii) implies that the variety \( \text{var}\{xy \approx yxw\} \) satisfies the identity (2.8). Hence this variety satisfies the identities \( u \approx u \omega \approx v \omega \approx v \). The equality (3.5) is proved.

### 3.3. The structure of the lattice \( L(SL \vee ZM) \)

Here we aim to verify that the variety \( SL \vee ZM \) contains only four subvarieties, namely \( T, SL, ZM \) and \( SL \vee ZM \). Let \( V \subseteq SL \vee ZM \). Results of Subsection 3.1 imply that \( SL \) and \( ZM \) are minimal non-trivial varieties of implication semigroups. Thus, it suffices to check that \( V \) is contained in one of the varieties \( SL \) or \( ZM \). Clearly, either \( SL \not\subseteq V \) or \( ZM \not\subseteq V \).

If \( ZM \not\subseteq V \) then Lemma 2.7(ii) implies that \( V \subseteq B \). Since the variety \( SL \vee ZM \) and therefore, \( V \) is commutative, we have that \( V \subseteq SL \) in this case. Suppose now that \( SL \not\subseteq V \). Lemma 2.7(i) implies that \( V \subseteq N \). Suppose that \( K \subseteq V \). Then \( K \subseteq SL \vee ZM \). The equality (3.5) implies then that the identity (3.7) is true in \( K \). But this identity implies within \( N \) the identities \( xy \approx yxw \approx \omega \) (see the second paragraph in Subsection 3.1). The identity \( xy \approx \omega \) evidently fails in \( K \), a contradiction. We prove that \( V \subseteq N \) but \( K \not\subseteq V \).

The description of the lattice \( L(N) \) given in Subsection 3.1 implies now that \( V \subseteq ZM \) (see Fig. 1).

### 3.4. The structure of the lattice \( L(SL \vee N) \)

Here we are going to prove that the lattice \( L(SL \vee N) \) has the form shown in Fig. 1. Let \( V \subseteq SL \vee N \). By Lemma 2.7(i), the lattice \( L(SL \vee N) \) is the set-theoretical union of the lattice \( L(N) \) and the interval \( [SL, SL \vee N] \). The lattice \( L(N) \) is described in Subsection 3.1. It remains to verify that the interval \( [SL, SL \vee N] \) has the form shown in Fig. 1. In other words, we need to check that if \( SL \subseteq V \subseteq SL \vee N \) then \( V \) coincides with one of the varieties \( SL, SL \vee ZM, SL \vee K, SL \vee L \) or \( SL \vee M \). Since \( N \not\subseteq V \), Lemma 2.6(iii) and the equalities (3.2) and (3.3) imply that \( V \) is contained in one of the varieties \( SL \vee L \) or \( SL \vee M \). If \( V \) coincides with one of these varieties then we are done. Let now \( V \neq SL \vee L \) and \( V \neq SL \vee M \).

Suppose that \( SL \vee K \subseteq V \). If \( V \subseteq SL \vee L \) then the variety \( V \) is commutative by the items (iv) and (v) of Lemma 2.5. Then the equalities (3.1) and (3.2) show that \( V = SL \vee K \). Suppose now that \( V \subseteq SL \vee M \). Then the items (iv) and (vi) of Lemma 2.5 imply that \( V \) satisfies a non-trivial identity of the kind \( u \approx v \) where the word \( u \) contains a square and \( \ell(u) < 3 \). Clearly, \( u = x^2 \). Thus, \( V \) satisfies a non-trivial identity of the kind \( x^2 \approx v \). Lemma 2.5(i) implies that \( \text{con}(v) = \{x\} \).

It is clear that \( \ell(v) \neq 2 \). If \( \ell(v) = 1 \) then our identity implies \( x^3 \approx x^2 \). Finally,
let $\ell(v) \geq 3$. Then $V$ satisfies the identities $x^2 \approx v \overset{(2.6)}{=} \text{los}(v)w = xw$. Besides that, $V$ is commutative because $V \subseteq SL \lor M$. The equality (3.1) implies now that $V = SL \lor K$.

It remains to consider the case when $SL \lor K \not\subseteq V$. Lemma 2.6(ii) and the equality (3.4) imply that the identities $xy \overset{(2.8)}{=} xy\omega \overset{(2.7)}{=} yx\omega$ hold in $V$. Therefore, $V$ satisfies the identity (3.7). This fact together with the equality (3.5) imply that $V \subseteq SL \lor ZM$. The result of Subsection 3.3 implies now that either $V = SL \lor ZM$ or $V = SL$ (see Fig. 1).

3.5. The structure of the lattice $L(B \lor ZM)$. Here we aim to check that the lattice $L(B \lor ZM)$ has the form shown in Fig. 1. First of all we prove that

$$B \lor ZM = \text{var}\{xy \approx xy\omega\}.$$  

(3.8)

The inclusion $B \lor ZM \subseteq \text{var}\{xy \approx xy\omega\}$ follows from Proposition 2.4 and the definition of the variety $ZM$. One can verify the inverse inclusion. Let $u \approx v$ be an arbitrary identity that holds in $B \lor ZM$. The items (ii) and (iii) of Lemma 2.5 imply that $\text{los}(u) = \text{los}(v)$ and $\ell(u), \ell(v) \geq 2$. Clearly, $\text{los}(u\omega) = \text{los}(v\omega)$.

Now Corollary 2.3 implies that the variety $IS$ satisfies the identity $u\omega \approx v\omega$. Then the variety $\text{var}\{xy \approx xy\omega\}$ satisfies the identities $u \approx u\omega \approx \omega \overset{(2.8)}{=} v$, whence $u \approx v$ holds in $\text{var}\{xy \approx xy\omega\}$. The equality (3.8) is proved.

Let $V \subseteq B \lor ZM$. In view of the results of Subsections 3.1 and 3.3, it remains to verify that either $V \subseteq B$ or $V \subseteq SL \lor ZM$. Clearly, either $ZM \not\subseteq V$ or $B \not\subseteq V$.

If $ZM \not\subseteq V$ then Lemma 2.7(ii) implies that $V \subseteq B$. Suppose now that $B \not\subseteq V$. Then Lemma 2.6(i) implies that $V$ satisfies the identity (2.7). Besides that, the equality (3.8) implies that the identity (2.8) holds in $V$. Therefore, the identities $xy \overset{(2.8)}{=} xy\omega \overset{(2.7)}{=} yx\omega$ are satisfied by $V$. We see that $V$ satisfies the identity (3.7). This fact and the equality (3.5) implies that $V \subseteq SL \lor ZM$.

3.6. Completion of the proof. First of all, we verify that $IS = B \lor N$. Let $u \approx v$ be an arbitrary identity that holds in $B \lor N$. The items (ii) and (vii) of Lemma 2.5 imply that $\text{los}(u) = \text{los}(v)$ and $\ell(u), \ell(v) \geq 3$. Then Corollary 2.3 implies that the variety $IS$ satisfies the identity $u \approx v$, and we are done.

Now we verify that

$$B \lor K = \text{var}\{x\omega \approx x^2\}.$$  

(3.9)

Proposition 2.4 and the definition of the variety $B$ imply that this variety satisfies the identity (2.9). The variety $K$ also satisfies this identity by Lemma 2.5(iv). Hence $B \lor K \subseteq \text{var}\{x\omega \approx x^2\}$. One can verify the inverse inclusion. Let $u \approx v$ be an arbitrary identity that holds in $B \lor K$. The items (ii) and (iv) of Lemma 2.5 and the fact that the variety $B$ is non-commutative imply that $\text{los}(u) = \text{los}(v)$ and each of the words $u$ and $v$ either has a length $\geq 3$ or contains a square. If a word contains a square then we can apply the identity (2.9) and obtain a word of a length $\geq 3$. Thus, we can assume that $\ell(u), \ell(v) \geq 3$. Then Corollary 2.3 implies that the identity $u \approx v$ holds in the variety $IS$ and therefore, in $\text{var}\{x\omega \approx x^2\}$. The equality (3.9) is proved.

Now we are well prepared to quickly complete the proof of Theorem 1.1. Let $V \subseteq IS$. Since $IS = B \lor N$, either $B \not\subseteq V$ or $N \not\subseteq V$. In the former case, Lemma 2.6(i) and the equality (3.4) imply that $V \subseteq SL \lor N$, while in the latter case $V \subseteq B \lor K$ by Lemma 2.5(iv) and the equality (3.9). As we have seen in Subsection 3.4, the lattice $L(SL \lor N)$ has the form shown in Fig. 1. It remains to consider the lattice $L(B \lor K)$.

Let $V \subseteq B \lor K$. Then either $B \not\subseteq V$ or $K \not\subseteq V$. As we have seen in the previous paragraph, $V \subseteq SL \lor N$ in the former case. Then $V \subseteq (SL \lor N) \land (B \lor K)$. Now
the equalities (3.2), (3.4) and (3.9) apply with the conclusion that $V \subseteq SL \vee L$. According to results of Subsection 3.4, the lattice $L(SL \vee L)$ has the form shown in Fig. 1. Finally, let $K \nsubseteq V$. Then Lemma 2.6(ii) and the equality (3.8) imply that $V \subseteq B \vee ZM$. It remains to refer to results of Subsection 3.5.

Theorem 1.1 is proved.

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