The Sum Over Topological Sectors and $\theta$ in the 2+1-Dimensional $\mathbb{C}P^1$ $\sigma$-Model

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Abstract: We discuss the three spacetime dimensional $\mathbb{C}P^N$ model and specialize to the $\mathbb{C}P^1$ model. Because of the Hopf map $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$ one might try to couple the model to a periodic $\theta$ parameter. However, we argue that only the values $\theta = 0$ and $\theta = \pi$ are consistent. For these values the Skyrmions in the model are bosons and fermions respectively, rather than being anyons. We also extend the model by coupling it to a topological quantum field theory, such that the Skyrmions are anyons. We use techniques from geometry and topology to construct the $\theta = \pi$ theory on arbitrary 3-manifolds, and use recent results about invertible field theories to prove that no other values of $\theta$ satisfy the necessary locality.

1. Introduction

The functional integral definition of quantum field theory involves integrating over all possible configurations with a certain weight. It is often the case that the configuration space in the Euclidean functional integral breaks into topologically distinct sectors labeled by $\nu$. (These sectors and their characterization can depend on the Euclidean spacetime the theory is placed on.) Then, defining $Z_\nu$ as the sum over the configurations in the sector $\nu$, the total functional integral is given by a linear combination of $Z_\nu$

$$Z = \sum_\nu a_\nu Z_\nu.$$  \hfill (1.1)

The possible values of the coefficients $a_\nu$ are constrained by various consistency conditions like locality and unitarity. Different consistent choices of the $a_\nu$ correspond to distinct quantum field theories. An interesting problem is to find all possible consistent values of these coefficients, thus finding all possible theories constructed out of the building blocks $Z_\nu$.

A well known example is the quantum mechanical system of a single degree of freedom on a circle. Here, with Euclidean compact time the configuration space is the
space of maps $S^1 \to S^1$ and $\nu$ is the winding number. In this case the coefficients $a_\nu$ are constrained to be determined by a single periodic parameter $\theta$ as

$$a_\nu = e^{i\nu \theta}. \quad (1.2)$$

Another example is the 4d pure $SU(N)$ gauge theory, where $\nu$ is the instanton number and again we have (1.2). In these two cases we can express $\nu$ as an integral of a local gauge invariant density and we can interpret (1.2) as arising from a term in the fundamental Lagrangian. In many situations $\nu$ cannot be written as an integral over a local density, but still an expression like (1.2) exists. A typical example is the 1+1-dimensional $SO(3)$-gauge theory, where $\nu$ is defined modulo 2 as the second Stiefel-Whitney class of a principal $SO(3)$-bundle, and correspondingly the allowed values of $\theta$ in (1.2) are 0 and $\pi$.

Locality and unitarity do not require $a_\nu$ to be the exponential of the integral of a local density, but rather they must be the partition functions of an invertible field theory [1]. In physics terms, $\log a_\nu$ can be thought of as an action of a classical field theory, which is local, but not necessarily an integral of a local density. Recent progress in understanding the structure of invertible theories can be brought to bear on the problem of combining $Z_\nu$ into a well-defined theory.

One of the goals of this paper is to clarify this sum over sectors in the 2+1 dimensional nonlinear $\mathbb{C}P^1$ $\sigma$-model. Placing the theory on $S^3$ and using the Hopf invariant, which is associated with $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$, the label $\nu$ in (1.1) runs over the integers. It labels an instanton number. Then one might think that (1.2) is a consistent prescription for how to sum over these sectors and the theory is labeled by a continuous periodic parameter $\theta$. Explicitly, let $\tilde{n}^2 = 1$ be a coordinate on $\mathbb{C}P^1 \simeq S^2$. Define $\text{Hopf}(\tilde{n})$ to be a density such that $\int d^3x \text{Hopf}(\tilde{n}) \in \mathbb{Z}$ is the Hopf invariant. Then, we can modify the standard Euclidean Lagrangian for $\tilde{n}$ by adding a theta term (see e.g. [2,3] and many followup papers where this term was discussed) as follows

$$L = \frac{f}{2} (\partial \tilde{n})^2 + i \theta \text{Hopf}(\tilde{n}), \quad (1.3)$$

with a dimensionful parameter $f$. In this presentation it would seem that any $\theta$ is allowed and only $\theta \mod 2\pi$ matters. A hint that something might be wrong with this $\theta$ term comes from the fact that $\text{Hopf}(\tilde{n})$ does not have a local expression in terms of $\tilde{n}$. Furthermore, it is unclear how to define this theta term on other three-manifolds. Indeed, it has been known that $\theta = 0, \pi$ naturally arise in simple situations but not the other values of $\theta$. See e.g. [4] and references therein.

We will prove that, in fact, only $\theta = 0$ and $\pi$ are consistent. Furthermore, we will explicitly construct the corresponding mod 2 invariant on arbitrary spin three-manifolds. We will also present variants of the $\mathbb{C}P^1$ model, where the low-energy $\mathbb{C}P^1$ Goldstone bosons are coupled to a nontrivial TQFT leading to additional long range interactions such that $\theta$ behaves as if it has other values. These other values of $\theta$ are now allowed because we have modified the theory in the deep infrared. In condensed matter language, we could, for example, think about that as coupling the $\mathbb{C}P^1$ theory to a fractional quantum hall state.\(^1\)

\(^1\) The authors of [5] noted that certain microscopic 2+1 dimensional models of spins lead only to the values $\theta = 0$ and $\theta = \pi$. The same is true in 1+1 dimensions for such microscopic models. But unlike our claimed result in 2+1 dimensions, in 1+1 dimensions the $\mathbb{C}P^1$ model is well defined with arbitrary $\theta$ and not just at $\theta = 0, \pi$.

\(^2\) We thank P. Wiegmann for many useful discussions.
In Sect. 2 we will discuss a microscopic theory that flows at long distances to the $\mathbb{CP}^N$ model with a Wess–Zumino term. Here we will see that the $\mathbb{CP}^1$ model is special and this microscopic construction flows only to $\theta = 0$ or $\theta = \pi$. In Sect. 3 we will study the local operators in the theory and we will argue that the $\mathbb{CP}^1$ model makes sense only for these two values of $\theta$ and not for generic values. In Sect. 4 we will present modifications of the $\mathbb{CP}^1$ model, which behave as if they have other values of $\theta$. This is consistent with the arguments in Sect. 3, because the low energy theory is not simply the $\mathbb{CP}^1$ model, but it is coupled to a TQFT. In Sect. 5 we bring to bear results about invertible field theories. These arguments are based on extended locality and factorization and use an analysis of the theory on general Wick-rotated spacetimes. We freely employ techniques from global analysis and homotopy theory to construct the allowed terms at $\theta = 0$ and $\theta = \pi$ and prove that these are the only allowed values of $\theta$.

2. The 2 + 1 Dimensional $\mathbb{CP}^N$ Model from a Linear Model

We find it instructive to view the 3d $\mathbb{CP}^1$ model as an atypical special case of the 3d $\mathbb{CP}^N$ model. We study it on a closed (that is, compact and without a boundary) spin$^c$ (not necessarily spin) manifold $M_3$. In this section we will embed this model in a particular microscopic renormalizable field theory, which flows at long distances to the nonlinear model with particular Wess–Zumino terms.

We start with $N + 1$ scalar fields coupled to a $U(1)$ gauge field $b$. The Lagrangian is given by

$$\frac{1}{2e^2} (db)^2 + \sum_{I=1}^{N+1} |D_b \phi^I|^2 + \mu^2 \sum_{I=1}^{N+1} |\phi^I|^2 + (\sum_{I=1}^{N+1} |\phi^I|^2)^2. \tag{2.1}$$

If the scalar fields condense (e.g. when $\mu^2 < 0$), then the gauge field $b$ is Higgsed and the global $SU(N + 1)$ symmetry is spontaneously broken to $SU(N) \times U(1)$. Therefore we obtain in the deep infrared Goldstone bosons parameterising the coset $SU(N+1)/SU(N) \times U(1) = \mathbb{CP}^N$.

We could modify the model (2.1) and add to the action the Chern–Simons terms:

$$\int_{M_3} \left( \frac{K}{4\pi} bdb + \frac{1}{2\pi} dbA \right) \text{ for } K \in 2\mathbb{Z}$$

$$\int_{M_3} \left( \frac{K}{4\pi} bdb + \frac{1}{2\pi} dB \right) \text{ for } K \in 2\mathbb{Z} + 1, \tag{2.2}$$

where $A$ is a classical background spin$^c$ connection and $B$ is a classical background $U(1)$ gauge field (see [6, 7] for more details). For even $K$ each term in (2.2) is separately well defined (up to an additive $2\pi i \mathbb{Z}$, which does not affect the exponential of the action). The same is true for odd $K$ on a spin manifold. But on a general spin$^c$ manifold with odd $K$ only the sum of the two terms in (2.2) is well defined mod $2\pi i \mathbb{Z}$.

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3 Since all our three-manifolds are orientable, they admit a spin structure. When we say that they are not spin we mean that we do not pick a spin structure, and if we do, the answers are independent of that choice. When we later view the three-manifold as a boundary of a four-manifold, the latter is only assumed to be spin$^c$, not necessarily spin.

4 Here and in similar expressions below we are imprecise since the connection forms are not global forms on $M_3$. 
The terms (2.2) are made more precise by writing\(^5\) them as 4\(d\) integrals

\[
\begin{align*}
\int_{M_4} \left( \frac{K}{4\pi} dbdb + \frac{1}{2\pi} dbdA \right) & \quad \text{for } K \in 2\mathbb{Z} \\
\int_{M_4} \left( \frac{K}{4\pi} dbdB + \frac{1}{2\pi} dbdA \right) & \quad \text{for } K \in 2\mathbb{Z} + 1,
\end{align*}
\]

(2.3)

where the original spacetime \(M_3\) is the boundary of \(M_4\).

The first term in (2.2) (or, equivalently, (2.3)) is a Chern–Simons term. Due to it, the monopole operator acquires spin \(K/2\). The second term means that the monopole operator of the theory carries charge one under a global magnetic \(U(1)\) symmetry.\(^6\) Because of the Chern–Simons terms (2.2), this monopole operator carries charge \(K\). We can make the monopole gauge invariant by multiplying it by \(K\) charged scalars. For simplicity, let all of them be at the same point on the \(S^2\). This configuration is not invariant under the \(SU(2)\) isometry of the sphere. In order to have an \(SU(2)\) covariant description we replace the state with fixed position of the scalars with another wave function – we introduce collective coordinates for the action of the symmetry. Intuitively, they move the location of the scalars and hence they take values in \(S^2\). (If the scalars are at different positions, the collective coordinates are on the symmetric product \(\text{Sym}_K(S^2)\).) So the effective theory of the collective coordinates is a quantum mechanical system with an \(S^2\) target space. The Chern–Simons coupling (2.2) becomes a standard Wess–Zumino term in this quantum mechanical system. This is the familiar problem of a charge \(K\) particle in the background of a magnetic monopole. The result is that the system has spin \(K/2\) (and possible higher order excitations with higher spins, which are \(K/2 + \text{integer}\)). \(A\) or \(B\) are classical background fields for that symmetry. These monopole operators satisfy the spin/charge relation [6].

2.1. \(N > 1\). For \(N > 1\) the analysis of the linear model (2.1) with the Chern–Simons terms (2.2) is completely standard. With negative mass squared for the scalars \(\phi^I\) they obtain an expectation value. Then we integrate out \(b\) by using its equations of motion to find at low energies a nonlinear model on \(\mathbb{C}P^N\). The \(b\) equation of motion sets \(db = \omega + \cdots\), where the ellipses represent higher order terms in the inverse radius of the target space (\(\sqrt{f}\) in (1.3)) that we will ignore and \(\omega\) is the pullback of the Kähler form of the \(\mathbb{C}P^N\) target space. We normalize it such that its periods are integer multiples of \(2\pi\)

\[
\int_{M_2} \omega = 2\pi \mathbb{Z}.
\]

(2.4)

Then, (2.3) becomes

\[
\begin{align*}
\int_{M_4} \left( \frac{K}{4\pi} \omega \omega + \frac{1}{2\pi} \omega dB \right) & \quad \text{for } K \in 2\mathbb{Z} \\
\int_{M_4} \left( \frac{K}{4\pi} \omega \omega + \frac{1}{2\pi} \omega dA \right) & \quad \text{for } K \in 2\mathbb{Z} + 1.
\end{align*}
\]

(2.5)

\(^5\) One can prove that \(M_4\) and extensions of the gauge fields and spin\(^c\) structure exist.

\(^6\) A monopole operator is defined by removing a point from our spacetime and specifying boundary conditions on the \(S^2\) around it such that \(\int_{S^2} db = 2\pi\). (There are many such distinct operators.) There are several ways to see that such an operator carries \(SU(2)\) spin \(j \geq K/2\) with \(j - K/2 = 0 \mod 1\) (see [8], for example).
As we remarked after (2.2), on a spin manifold each term in (2.5) is separately meaningful as a 3d term. The same is true on a general spin$^c$ manifold with even $K$. But for odd $K$ on a general spin$^c$ manifold only the sum of the two terms in (2.5) is meaningful. This means that the first term, is not associated with $H^4$, but with a more subtle cohomology [9]; see Sect. 5.2.

The first term in (2.5) is a Wess–Zumino term of the nonlinear model. The second term has two complementary interpretations. The first interpretation involves the solitons of the model, which are known as Skyrmions. Viewing $M_2$ in (2.4) as our space, these are configurations with nonzero $\int_{M_2} \omega$. The second term in (2.5) means that they carry charge $\int_{M_2} \omega$ under the global $U(1)$ symmetry that $A$ or $B$ couple to. This is completely analogous to the situation in the 4d chiral Lagrangian [10,11], where the Skyrmions carry baryon number. The second interpretation is related to our discussion above of monopole operators of the microscopic theory. Similarly, we can discuss Skyrmion operators in the macroscopic $\mathbb{C}P^N$ theory. They are defined by removing a point from our spacetime $M_3$ and specifying boundary conditions on the small $S^2$ around the point such that with $S^2 = M_2$ in (2.4) we have $\int_{S^2} \omega = 2\pi$. These operators have spin $K/2$ and the second term in (2.5) means that they are charged under the global $U(1)$ symmetry.\footnote{The computation of their spin is very similar to the computation of the spin of the monopole operators in the previous footnote. This is not surprising because they are the macroscopic descendants of the microscopic monopole operators. More explicitly, the Skyrmion configuration breaks the $SU(2)$ isometry of the $S^2$ and the $SU(2)$ global symmetry of the target space. We introduce collective coordinates for their quantization. As above, the Wess–Zumino term in the 3d problem becomes a Wess–Zumino term in the 1d problem of the collective coordinates making the spin of the state $K/2$ or larger.}

Again, we see here the spin/charge relation [6].

2.2. $N = 1$. Just as in the analogous 4d problem where the special case with two flavors is slightly different, the same is true in our case for $N = 1$. In these two situations there is no standard Wess–Zumino term, though there is a nonstandard one, defined for all $N$, which specializes for $N = 1$ to the mod 2 invariant at $\theta = \pi$; see Sect. 5.2.

Starting with the same microscopic linear model with a Chern–Simons term (2.3) we can follow the steps above to integrate out $b$. We again find $db = \omega + \cdots$. But here we cannot write (2.5). There are two related reasons for that. First, the first term, involving $\omega \omega$, clearly vanishes—there is no four-form on $\mathbb{C}P^1$. Second, in fact, we cannot always extend the fields on $M_3$ to $M_4$. Specifically, if $M_3 = S^3$ the $\mathbb{C}P^1$ configurations are labeled by an integer $N$ associated with the nontrivial Hopf invariant $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$ and configurations with odd $N$ cannot be extended to a 4d bulk (see Remark 5.8).

This is analogous to the similar situation in 4d, which is associated with $\pi_4(S^3) = \mathbb{Z}_2$. There the Wess–Zumino term is replaced by another term\footnote{A uniform picture using E-cohomology was presented for the 4d problem in [9], and will be discussed for the 3d problem in Sect. 5.} representing this $\mathbb{Z}_2$. It can be viewed as a discrete $\theta$ parameter term in the sigma model [10,11]. We can try to imitate it in our problem and to use $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$, which can be expressed as

$$N = \frac{1}{4\pi^2} \int_{M_3} b_0 db_0,$$

where $db_0 = \omega$. Then we can attempt to add to the action the $\theta$-term

$$\theta N = \frac{\theta}{4\pi^2} \int_{M_3} b_0 db_0.$$

(2.6)
Of course, the subtlety in this expression is in the fact that $b_0$ is a nonlocal expression in terms of the nonlinear model variables. This fact is at the root of our conclusion below that the theory is consistent only for $\theta = 0$ and $\theta = \pi$. Such expressions were studied in [2,3] and many followup papers.

Repeating the analysis for $N > 1$ we see that the microscopic Chern–Simons couplings (2.2) lead to $\theta = \pi K$; i.e. we find the $\mathbb{C}P^1$ theory with $\theta = 0$ or $\theta = \pi$. As for higher $N$, we can write

$$
\frac{1}{2\pi} \int_{M_3} \omega B \quad \text{for} \quad K \in 2\mathbb{Z}
$$

$$
\pi \mathcal{N} + \frac{1}{2\pi} \int_{M_3} \omega A \quad \text{for} \quad K \in 2\mathbb{Z} + 1.
$$

(2.8)

Therefore, for $\theta = 0$ the Skyrmions are bosons and for $\theta = \pi$ they are fermions [2,3].

Such constructions have recently appeared in [12].

3. Skyrmion Operators and Conditions on $\theta$

The previous discussion raises the following question. Starting with a microscopic theory we derived the $\mathbb{C}P^1$ model with $\theta = 0$ or $\theta = \pi$ (2.8). Are there microscopic models that lead to other values of $\theta$ as in (2.7)? In order to address this question we should study the $\mathbb{C}P^1$ model without relying on the details of its UV completion.

We are now going to argue that the $\mathbb{C}P^1$ model with generic $\theta$ is not a consistent quantum field theory, thus explaining why we cannot derive it from a microscopic model.

Consider the theory on $M_3 = S^2 \times \mathbb{R}$ and view $S^2$ as space and $\mathbb{R}$ as time. The theory has a topologically conserved current $\omega$, hence, the Hilbert space is decomposed into sectors with fixed soliton number $\int_{S^2} \omega$. Let us turn on (2.7) with a generic value of $\theta$.

It was argued in [2] that in this case the single Skyrmion has spin $\theta/2\pi$. This is a valid answer\(^9\) for a particle on a spatial $\mathbb{R}^2$, but it is not sensible for the system on $S^2$. The states in this case must be in representations of $SU(2)$ (i.e. the universal cover of the Euclidean isometry group, $SO(3)$). This is the case only for $\theta = 0$ or $\pi$.

An equivalent way to state it in $\mathbb{R}^3$ is the following. Particle states must be in representations of a multiple cover of the rotation group $SO(2)$; i.e. they can be in any representation of $\mathbb{R}$. As such, they can have arbitrary real spin. Local operators, on the other hand, must be in representations of the Lorentz group, which in Euclidean space is $SU(2)$. Now consider a Skyrmion operator. It is defined by removing a point from $\mathbb{R}^3$ and specifying boundary conditions that the integral over a small $S^2$ around it is $\int_{S^2} \omega = 1$.

In the presence of a $\theta$-term this operator has spin $\theta/2\pi$ and therefore $\theta$ should be 0 or $\pi$. The relation between this point about the local operators and the previous argument based on quantization on a spatial $S^2$ is standard and is clear using radial quantization.

These two equivalent perspectives can be stated also in the following way. Above we mentioned the effective quantum mechanics in a monopole or Skyrmion sector. When we quantize the system on a spatial $S^2$ in the sector with $\int_{S^2} \omega = \int_{S^2} db_0 = 2\pi$ the effective quantum mechanical problem includes a collective coordinate on $S^2$. The term (2.7) leads to a Wess–Zumino term in that quantum mechanical problem. It is $SU(2)$ invariant only for $\theta = 0$ or $\theta = \pi$. In more detail, the classical theory is well defined and is $SU(2)$ invariant for all $\theta$. In the quantum theory we have two options for generic

\(^9\) Particles in 2+1 dimensions are in representations of the universal cover of the little group $SO(2)$, i.e. they can have any real spin.
\( \theta \). We can have a well defined, but not \( SU(2) \) invariant expression (pick a point on the \( S^2 \), connect it to the position of the particle and the Lagrangian is the area swept by that line during the time \( (t, t + dt) \)) or we can have an \( SU(2) \) invariant expression by extending the map to a higher dimension, but then the answer depends on the choice of the extension.

We see that even though the partition function of the theory on \( S^3 \) allows arbitrary \( \theta \) the entire quantum field theory is consistent only for \( \theta = 0 \) or \( \theta = \pi \). In Sect. 5 we will find the same conclusion by imposing consistency of the theory on more complicated spacetimes \( \mathcal{M}_3 \). Here we argued it using \( \mathcal{M}_3 = \mathbb{R}^3 - \{ \text{point} \} \), or equivalently by studying the local operators in the theory.

This is reminiscent of the analysis of [13], where subtle choices in the coefficients \( a_\nu \) in (1.1) and corresponding subtle topological terms in the action were identified by studying a 4d theory on \( \mathbb{R}^4 \) minus some lines; i.e. by studying the consistency of line operators in the theory.

4. Changing the Quantization of the Coefficients

Next, following [14], we modify the theory such that the \( \mathbb{C}P^N \) model looks as if it can have Wess–Zumino terms with fractional coefficients and the \( \mathbb{C}P^1 \) model looks as if it has \( \theta \) that is a fractional multiple of \( \pi \). We do that by making the \( N + 1 \) scalars have charge \( q \) under the \( U(1) \) gauge field in the original microscopic model (2.1).

Again, with an appropriate potential the gauge symmetry is Higgsed and the low energy theory is a nonlinear model on \( \mathbb{C}P^N \). However, unlike the previous case, this is not the whole story. Now the microscopic \( U(1) \) gauge symmetry is Higgsed to \( \mathbb{Z}_q \). The coupling of this \( \mathbb{Z}_q \) gauge field to the \( \mathbb{C}P^N \) coordinates is obtained through the equation of motion \( q db = \omega + \cdots \). We give a global interpretation of this coupling in Sect. 5.4.

Following [15,16] it is convenient to represent the unbroken \( \mathbb{Z}_q \) gauge theory in terms of two \( U(1) \) gauge fields \( b \) and \( c \) as

\[
\begin{align*}
\int_{\mathcal{M}_3} \left( \frac{q}{2\pi} cdb + \frac{K}{4\pi} bdb + \frac{1}{2\pi} dbB - \frac{1}{2\pi} c\omega \right) & \quad \text{for } K \in 2\mathbb{Z} \\
\int_{\mathcal{M}_3} \left( \frac{q}{2\pi} cdb + \frac{K}{4\pi} bdb + \frac{1}{2\pi} dbA - \frac{1}{2\pi} c\omega \right) & \quad \text{for } K \in 2\mathbb{Z} + 1.
\end{align*}
\]

(4.1)

In the first term \( c \) is a Lagrange multiplier \( U(1) \) gauge field forcing \( b \) to be a \( \mathbb{Z}_q \) gauge field. It can be thought of as the dual of the overall phase of the fundamental scalars. As explained in [17], the terms proportional to \( K \) are Dijkgraaf–Witten terms [18] in this \( \mathbb{Z}_q \) gauge theory. For \( q = 1 \) the expressions (4.1) reduce to (2.5) since \( \frac{1}{2\pi} cdb \) is a trivial theory where we can use the equations of motion freely without missing global issues [19].

The third and fourth terms in (4.1) represent couplings of the \( \mathbb{Z}_q \) gauge theory to the background fields \( A \) or \( B \) and to the fields of the nonlinear model through the pull back of its Kähler form \( \omega \). The equation of motion of \( c \) sets \( qdb = \omega \) in agreement with the microscopic analysis. This determines \( db \) in terms of the nonlinear model fields, but leaves a \( \mathbb{Z}_q \) gauge field undetermined.

Consider the world line \( C \) of a small Skyrmion. We can approximate the nonlinear model configuration by \( \omega = 2\pi \delta^{(2)}(C) \). Therefore, the term with \( \omega \) in (4.1) can be

\[\text{For } \omega = 0 \text{ we indeed have a } \mathbb{Z}_q \text{ gauge theory, but for nonzero } \omega \text{ we have } qdb = \omega \text{ showing that } b \text{ is not a } \mathbb{Z}_q \text{ gauge field. We will discuss the geometric interpretation of this construction in Sect. 5.}\]
replaced by a Wilson line $e^{i \int_C c}$. A standard computation in the TQFT (4.1), which follows from the equations of motion, shows that this particle carries fractional $U(1)$ charge (under $A$ or $B$), which is $\frac{1}{q}$ and its spin is $\frac{K}{2q^2}$ mod 1.

We conclude that in this system (with $q \neq 1$) the Skyrmions become anyons. However, the total number of Skyrmions in a compact space

$$\frac{1}{2\pi} \int_{M_2} \omega = \frac{1}{2\pi} \int_{M_2} q dB \in q \mathbb{Z}$$

(4.2)

must be a multiple of $q$. In order to determine the quantum numbers of this configuration we deform it to $\omega = 2\pi q m \delta^{(2)}(C)$ with integer $m$. Substituting this in the TQFT (4.1) we see that all these anyons are combined to the line $e^{i q m \int_C c}$. This line carries integer charge and its total spin is $\frac{Km^2}{2}$ mod 1; i.e. it is either an integer or half-integer.

We can attempt to integrate out $b$ in (4.1). Such integration out is not legal because the field $b$ has long range interactions. If we do that anyway, using the equation of motion $q dB = \omega + \cdots$ and ignoring the fact that it does not determine $b$ (not even up to a gauge transformation), the expressions (2.5), (2.8) are modified.

For $N > 1$ (2.5) becomes

$$\int_{M_4} \left( \frac{K}{4\pi q^2} \omega \omega + \frac{1}{2\pi q} \omega dB \right) \quad \text{for} \quad K \in 2\mathbb{Z}$$

$$\int_{M_4} \left( \frac{K}{4\pi q^2} \omega \omega + \frac{1}{2\pi q} \omega dA \right) \quad \text{for} \quad K \in 2\mathbb{Z} + 1.$$ 

(4.3)

The coefficient of the Wess–Zumino term acquired a factor of $\frac{1}{q^2}$ and the coupling to the background field acquired a factor of $\frac{1}{q}$. This means that now the Skyrmions carry charge $\frac{1}{q}$, as we saw above. For $N = 1$ (2.8) becomes

$$\frac{\pi K}{q^2} N + \frac{1}{2\pi q} \int_{M_3} \omega B \quad \text{for} \quad K \in 2\mathbb{Z}$$

$$\frac{\pi K}{q^2} N + \frac{1}{2\pi q} \int_{M_3} \omega A \quad \text{for} \quad K \in 2\mathbb{Z} + 1.$$ 

(4.4)

As for higher $N$, the Skyrmions have fractional charge and the low energy $\theta$-parameter is a fractional multiple of $\pi$.

We should emphasize, however, that the expressions (4.3), (4.4) are useful in analyzing local properties, but they are imprecise and do not capture the global structure correctly. For that we need to keep the $\mathbb{Z}_q$ gauge field and not integrate it out.

We would like to relate this construction to the discussion in Sect. 3. In this system with generic $q$ we cannot place a single Skyrmion on a spatial $S^2$ because it is not invariant under the $\mathbb{Z}_q$ gauge symmetry. We can, however, place $q$ Skyrmions on $S^2$. Then their total spin is indeed in an $SU(2)$ representations. We can repeat this point using Skyrmion operators. These operators are not gauge invariant. They carry $\mathbb{Z}_q$ gauge charge. As such, they might not be in $SU(2)$ representations. We can construct a gauge invariant operator by fusing $q$ Skyrmion operators. This object has $\frac{1}{2\pi} \int_{S^2} \omega = q$ and it is in an $SU(2)$ representation.
To finish our discussion of effective fractional $\theta$ terms, we would like to present another construction, where the low energy theory consists of the $\mathbb{CP}^1$ model coupled to a TQFT. Let us start with the following Euclidean Lagrangian

$$\sum_{I=1,2} |D_b \phi^I|^2 + \mu^2 \sum_{I=1,2} |\phi^I|^2 + \sum_{I=1,2} (|\phi^I|^2)^2 + \frac{i}{2\pi} bdc \phi^I c d + \frac{iq}{4\pi} c d c + \begin{cases} \frac{i}{2\pi} c d B & \text{for } q \in 2\mathbb{Z} \\ \frac{i}{2\pi} c d A & \text{for } q \in 2\mathbb{Z} + 1, \end{cases}$$

(4.5)

where again $B$ is a background $U(1)$ gauge field and $A$ is a spin connection. Physically the gauge fields $b$ and $c$ could be interpreted as emergent gauge fields and $c$ represents a fractional quantum Hall effect. As above, we end up with $db = \omega + \cdots$ and the coupling to the $\mathbb{CP}^1$ degrees of freedom (2.8) becomes

$$\frac{1}{2\pi} \omega c + \frac{q}{4\pi} c d c + \begin{cases} \frac{1}{2\pi} c d B & \text{for } q \in 2\mathbb{Z} \\ \frac{1}{2\pi} c d A & \text{for } q \in 2\mathbb{Z} + 1. \end{cases}$$

(4.6)

We see here a $U(1)_q$ Chern–Simons theory of $c$ coupled to the nonlinear $\mathbb{CP}^1$ model and to a background field ($A$ or $B$).

As in the previous construction (4.1), a small Skyrmion with a worldline $C$ is represented in the $U(1)_q$ theory by a line operator $e^{i/|c|c}$. It has spin $\frac{1}{2}\frac{1}{q}$ and charge $\frac{1}{q}$; i.e. it is an anyon. Also as in the previous example, we can incorrectly integrate out $c$ to find an effective $\mathbb{CP}^1$ theory with $\theta = -\pi/q$.

More generally, instead of considering the concrete examples leading to (4.1), (4.6), we could couple the gauge field $b$ to a general Chern–Simons TQFT with some matrix of Chern–Simons couplings $k_{ij}$.

5. Invertible Field Theories and Effective Actions

5.1. Generalities. The (effective) action of an $n$-dimensional field theory obeys strong locality constraints: it can be computed on a (Wick rotated) spacetime by assembling local contributions. For example, a typical kinetic term $\int |d\phi|^2/2$ is the integral of an expression computed locally from a field $\phi$. More interesting topological terms do not have such simple formulas yet obey many of the same locality properties. The strongest expression of that locality is encoded in the notion of an extended field theory [20–22]. Furthermore, the exponentiated action is the partition function of an invertible field theory: for example, if a closed $n$-manifold $M$ is cut along a codimension one submanifold $N$, then the vector space of “states” associated to $N$ is 1-dimensional. The notion of an invertible field theory systematizes the locality one demands in a classical action; the partition function is the exponential of what would be the classical action (which need only be well-defined up to shifts by $2\pi i$).

The structure of a field theory, in particular its locality, is captured by an Axiom System originally introduced by Segal [23] in the context of 2-dimensional conformal field theories and Atiyah [24] for topological field theories. It has since been expanded and used more generally; it is most developed for topological theories. In this framework an invertible field theory, after Wick rotation, becomes a map of spectra in the sense of stable homotopy theory. Recently an extended notion of unitarity, or rather its Wick rotated version—reflection positivity—was introduced in the invertible case [25]. Of course, we expect unitarity in any physical theory, so an invertible field theory used in the action should be reflection positive and in this paper we restrict to such field
theories. The theorems in [25] classify deformation classes of invertible theories as well as isomorphism classes of invertible topological theories, as we review shortly. Two exponentiated actions are in the same deformation class if they can be joined by a smooth path of exponentiated actions; in physics terms, they are related by adding a local term to the lagrangian. For example, \( t \mapsto \exp \int (1-t) |d\phi|^2/2, 0 \leq t \leq 1 \), is a path from the exponentiated kinetic action to the trivial action. Topologically nontrivial actions have nontrivial deformation classes, so are detected by the stable homotopy invariants introduced below.

The main result of [25] states that the abelian group of deformation classes of unitary invertible \( n \)-dimensional field theories is isomorphic to the abelian group of homotopy classes of maps \( B \to \Sigma^{n+1} I\mathbb{Z} \) from a Thom spectrum to the shifted Anderson dual to the sphere spectrum. We refer to [25, §5] and the references therein for exposition, and remark that the Anderson dual was introduced in this context in [26]. The torsion subgroup is the group of topological theories. We also consider unitary invertible \( n \)-dimensional theories with partition function an integer; they are classified up to isomorphism by a homotopy class of maps \( B \to \Sigma^n I\mathbb{Z} \). (For a theory of oriented or spin manifolds, ‘unitary’ here means that the partition function changes sign under orientation-reversal.) This classification statement is not proved in [25], nor do we give a full discussion here, but in any case it only enters peripherally in what follows.

The abelian groups computed here are generalized cohomology groups for the cohomology theory defined by the Anderson dual. They are not homotopy groups of a space, but rather generalized cohomology groups of a spectrum.

In the following subsections we treat the topological terms for the nonlinear \( \mathbb{C}P^N \) model on spin and spin\(^c\) manifolds. Then in Sect. 5.4 we comment briefly on the effective models in Sect. 4.

5.2. Spin manifolds. We begin by defining the Wess–Zumino term which appears in (2.5). We express it in terms similar to the WZ term in the effective action for pions [9] and the spin Chern–Simons action [27,28]. Namely, we use a generalized cohomology theory \( E \) which is a 2-stage Postnikov truncation of \( I\mathbb{Z} \): there is a map \( E \to I\mathbb{Z} \) which captures the top two nonzero homotopy groups of the co-connective spectrum \( I\mathbb{Z} \). We defer to [9, §1] for details about \( E \), which we freely use in the following. Two salient points are (i) \( E \) is spin-oriented, so \( E \)-cohomology classes can be evaluated on spin manifolds, and (ii) there is a long exact sequence

\[
\cdots \to H^q(X; \mathbb{Z}) \overset{i}{\to} E^q(X) \overset{j}{\to} H^{q-2}(X; \mathbb{Z}/2\mathbb{Z}) \overset{\beta \circ Sq^2}{\to} H^{q+1}(X; \mathbb{Z}) \to \cdots
\]

(5.1)

for any space \( X \), where \( \beta \) is the integer Bockstein map. This long exact sequence characterizes \( E \).

**Lemma 5.1.** For \( N \geq 2 \) there is an isomorphism \( E^4(\mathbb{C}P^N) \cong \mathbb{Z} \); the homomorphism \( H^4(\mathbb{C}P^N; \mathbb{Z}) \to E^4(\mathbb{C}P^N) \) maps a generator to twice a generator. Also, \( E^4(\mathbb{C}P^1) \cong \mathbb{Z} \).
\[ \mathbb{Z}/2\mathbb{Z} \text{ and a generator of } E^4(\mathbb{C}P^N) \text{ restricts to a generator of } E^4(\mathbb{C}P^1) \text{ under a linear inclusion } \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^N. \]

**Proof.** The first statement is part of the proof of [9, Proposition 1.9], where it is also shown that the generator is the characteristic class \( \lambda \in E^4(BO) \) of the real 2-plane bundle underlying \( \mathcal{O}(1) \to \mathbb{C}P^N \). Restricting that bundle under \( \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^N \) we obtain the second statement, after applying the long exact sequence (5.1) with \( q = 4 \) and \( X = \mathbb{C}P^1 \). \( \square \)

Fix \( N \in \mathbb{Z}_{\geq 1} \) and let \( \chi \) denote a generator of \( E^4(\mathbb{C}P^N) \). It has a unique lift \( \tilde{\chi} \in \tilde{E}^4(\mathbb{C}P^N) \) to the differential theory, as we see from [9, (1.8)]. Let \( M \) be a closed spin 3-manifold equipped with a smooth map \( \phi: M \to \mathbb{C}P^N \). The following is an exact analog of [9, Definition 4.1].

**Definition 5.2.** The WZ factor in the \( \sigma \)-model exponentiated action on spin manifolds is

\[
W_M(\phi) = \exp \left( 2\pi i \, \pi^M_\ast \phi^\ast \tilde{\chi} \right).
\]

The projection \( \pi^M: M \to \text{pt} \) induces the pushforward \( \pi^M_\ast: \tilde{E}^4(M) \to \tilde{E}^1(\text{pt}) \cong \mathbb{R}/\mathbb{Z} \). We emphasize that Definition 5.2 works for \( N = 1 \) as well as \( N \geq 2 \).

**Remark 5.3.** Formula (5.2) corresponds to \( K = 1 \) in Sect. 2; the formula for arbitrary \( K \) multiplies the quantity in parentheses by \( K \). An application of Stokes’ theorem for differential \( E \)-theory analogous to [9, (4.3)] reproduces the WZ term in (2.5)—the first term in the formula with \( K = 1 \)—assuming that \( M \) bounds a compact spin 4-manifold \( W \) and \( \phi \) extends to a map \( \Phi: W \to \mathbb{C}P^N \). (See Remark 5.8 below.)

**Remark 5.4.** As in [9, (4.10)] the \( \sigma \)-model with WZ factor encodes the statistics of skyrmions.

**Remark 5.5.** The hyperplane bundle \( \mathcal{O}(1) \to \mathbb{C}P^N \) has a natural \( SU(2) \)-invariant metric and connection. The WZ factor (5.2) is the lowest level spin Chern–Simons invariant of its pullback via \( \phi \).

The WZ factor varies smoothly with \( \phi \) if \( N \geq 2 \), but is a topological invariant if \( N = 1 \). Next, we give a topological description for \( N = 1 \) which does not use \( E \)-cohomology. Let \( M \) be a closed spin 3-manifold and \( \phi: M \to \mathbb{C}P^1 \). Fix a regular value \( p \in \mathbb{C}P^1 \) and a basis \( e_1, e_2 \) of \( T_p \mathbb{C}P^1 \). This produces a normal framing of the 1-manifold \( S := \phi^{-1}(p) \), which after applying Gram-Schmidt we can assume is orthonormal. (The contractible choice of a Riemannian metric on \( M \) does not affect the mod 2 invariant we are defining.) At each point of \( S \) there is a unique completion \( e_1, e_2, e_3 \) to an oriented orthonormal basis, and so two lifts to the Spin\(_3\)-bundle of frames of \( M \). The resulting double cover of \( S \) may be identified with the spin bundle of frames of a spin structure on \( S \).

**Lemma 5.6.** Set \( N = 1 \). Then

1. \( W_M(\phi) = (-1)^{[S]} \), where \( [S] \in \Omega_{\text{Spin}}^1 \cong \mathbb{Z}/2\mathbb{Z} \) is the spin bordism class of \( S \).
2. \( W_{S^3}(\phi) \) is the mod 2 Hopf invariant of \( \phi: S^3 \to \mathbb{C}P^1 \).
$S$ is a finite union of spin circles, each of which is bounding (Neveu-Schwarz) or nonbounding (Ramond). The invariant in (i) is $±1$ depending on the parity of the number of nonbounding components. Alternatively, the normal framing of $S$ determines an element of framed bordism, which is isomorphic to spin bordism in dimension one. Assertion (ii) shows that $W_M(\phi)$ extends the $\theta = \pi$ term for the Hopf invariant.

**Proof.** The inclusion $\iota : \{p\} \hookrightarrow \mathbb{C}P^1$ with normal framing $e_1, e_2$ induces a pushforward $\iota_* : E^2(\{p\}) \longrightarrow E^4(\mathbb{C}P^1)$. Let $\alpha \in E^2(\{p\}) \cong \mathbb{Z}/2\mathbb{Z}$ be the generator and $\tilde{\alpha} \in \tilde{E}^2(\{p\})$ the unique lift to the differential group. The commutativity of the diagram

$$
\begin{array}{ccc}
E^2(\{p\}) & \longrightarrow & E^4(\mathbb{C}P^1) \\
\downarrow j & & \downarrow j \\
H^0(\{p\}; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^2(\mathbb{C}P^1; \mathbb{Z}/2\mathbb{Z})
\end{array}
$$

(5.3)

implies that $\iota_*\alpha = \chi$. It follows that $\pi^M_\ast \phi^\ast \tilde{\chi} = (\pi^S_\ast(\pi^S)^\ast)(\tilde{\alpha})$. The right hand side contains $\pi^S_\ast: \tilde{E}^2(S) \rightarrow \tilde{E}^1(\text{pt})$, which equals the topological pushforward

$$
\pi^S_\ast : E^1(S; \mathbb{R}/\mathbb{Z}) \longrightarrow E^0(\text{pt}; \mathbb{R}/\mathbb{Z})
$$

(5.4)
on the flat part of the differential $E$-theory groups, after identifying $\tilde{\alpha} \in E^1(\text{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Rewrite (5.4) as $\pi^S_\ast : ko^{-3}(S; \mathbb{R}/\mathbb{Z}) \rightarrow ko^{-4}(\text{pt}; \mathbb{R}/\mathbb{Z})$, identify $\tilde{\alpha}$ with the generator $a \in ko^{-3}(\text{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and write $\pi^S_\ast((\pi^S)^\ast(a)) = \pi^S_\ast(1) a$. Finally, (i) follows from $\pi^S_\ast(1) = [S] \in ko^{-1}(\text{pt})$ and [29, (B.10)].

For (ii) we observe that both $\phi \mapsto \pi^M_\ast \phi^\ast \tilde{\chi}$ and the mod 2 Hopf invariant define homomorphisms $\pi_3(\mathbb{C}P^1) \rightarrow \mathbb{Z}/2\mathbb{Z}$, so it suffices to verify that the equality on the Hopf map $S^3 \rightarrow \mathbb{C}P^1$, which amounts to verifying that the induced spin structure on a fiber of the Hopf map is nonbounding, or equivalently the normal framing is nontrivial. That follows since the Hopf map represents the nontrivial element of $\Omega^1_{\text{framed}}$ via the Pontrjagin–Thom construction. \(\square\)

We now turn to the classification of possible topological terms, so to bordism computations of invertible field theories of spin 3-manifolds equipped with a map to $\mathbb{C}P^1$. If $B$ is any spectrum then there is a short exact sequence [26, (B.3)]

$$
0 \longrightarrow \text{Ext}^1(\pi_{q-1}B, \mathbb{Z}) \longrightarrow [B, \Sigma^q I\mathbb{Z}] \longrightarrow \text{Hom}(\pi_q B, \mathbb{Z}) \longrightarrow 0
$$

(5.5)

where $[X, Y]$ denotes the group of homotopy classes of spectrum maps $X \rightarrow Y$. The Thom bordism spectrum of spin manifolds equipped with a map to $\mathbb{C}P^1$ is $^{15}$

$$
M\text{Spin} \wedge \mathbb{C}P^1 \cong M\text{Spin} \wedge \mathbb{C}P^1 \vee M\text{Spin} \cong \Sigma^2 M\text{Spin} \vee M\text{Spin}.
$$

(5.6)

$^{14}$ which is a spectrum $E_{\mathbb{R}/\mathbb{Z}}$ with $\pi_0E_{\mathbb{R}/\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$ and $\pi_{-1}E_{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$ connected by a nontrivial $k$-invariant.

$^{15}$ The ‘+’ denotes a disjoint basepoint, which occurs since the Thom spectrum of $W \oplus 0 \rightarrow B\text{Spin} \times \mathbb{C}P^1$ is the smash product of the Thom spectra of the universal bundle $W \rightarrow B\text{Spin}$ and $0 \rightarrow \mathbb{C}P^1$; the latter is $\mathbb{C}P^1_+$. 
We first ask if there is an integer-valued invertible field theory\(^{16}\) whose partition function extends the Hopf invariant, so a map \(\text{M}^{\text{Spin}} \wedge \mathbb{C}P^1_+ \rightarrow \Sigma^3 I\mathbb{Z}\). (We use the unproved assertion towards the end of Sect. 5.1.) It follows from (5.6) and low dimension spin bordism that \(\left[ \text{M}^{\text{Spin}} \wedge \mathbb{C}P^1_+, \Sigma^3 I\mathbb{Z} \right] \cong \mathbb{Z}/2\mathbb{Z}\), and so the partition function of any integer-valued theory vanishes. In particular, there is no possibility to write \(e^{iv\theta}\) in an exponentiated action if \(v\) specializes on \(S^3\) to the Hopf invariant.

More directly, we can compute the group of deformation classes of \textit{topological} field theories.

\textbf{Theorem 5.7.} The group of deformation classes of unitary invertible topological field theories of spin 3-manifolds equipped with a map to \(\mathbb{C}P^1\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\); it is also the group of isomorphism classes of theories of this type. The generator has partition function (5.2).

There are non-topological invertible field theories whose partition function is an exponentiated \(\eta\)-invariant. They depend on a metric (but not on the map \(\phi\) to \(\mathbb{C}P^1\)). Theorem 5.7 rules out values of \(\theta\) other than \(\theta = 0\) and \(\theta = \pi\), as deduced in Sect. 3 by a different argument.

\textit{Proof.} According to the main theorem in [25] the group in the theorem is the torsion subgroup of \(\left[ \text{M}^{\text{Spin}} \wedge \mathbb{C}P^1_+, \Sigma^4 I\mathbb{Z} \right]\), which is the Ext\(^1\) group in (5.5). Using (5.6) we deduce

\[\pi_3\left( \text{M}^{\text{Spin}} \wedge \mathbb{C}P^1_+ \right) \cong \pi_1 \text{M}^{\text{Spin}} \cong \mathbb{Z}/2\mathbb{Z}\] 

(5.7) since Ext\(^1\)(\(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\)) \(\cong \mathbb{Z}/2\mathbb{Z}\). The isomorphism \(\pi_3\left( \text{M}^{\text{Spin}} \wedge \mathbb{C}P^1_+ \right) \cong \pi_1 \text{M}^{\text{Spin}}\) maps a spin 3-manifold \(M\) equipped with a map \(\phi: M \rightarrow \mathbb{C}P^1\) to the inverse image of a regular value, so the identification of the partition function follows from the construction before Lemma 5.6. \(\Box\)

\textit{Remark 5.8.} The nontriviality of the bordism group (5.7) shows that the WZ factor (5.2) cannot always be computed by extending over a bounding 4-manifold, for example for \(M = S^3\) equipped with the Hopf map \(\phi: S^3 \rightarrow \mathbb{C}P^1\), as was done in (2.5).

5.3. \textit{Spin}^c \textit{manifolds.} We repeat the analysis for \textit{spin}^c manifolds. The cohomology theory \(E\) does not have a \textit{spin}^c orientation—\(E\)-cohomology classes cannot be integrated over \textit{spin}^c manifolds—so differential \(E\)-cohomology does not enter our analysis. On the other hand, complex \(K\)-theory is \textit{spin}^c-oriented, and so we use differential \(K\)-theory, but only implicitly as we express the integral of a differential \(K\)-theory class as an exponentiated \(\eta\)-invariant [30,31].

Recall that the group \(\text{Spin}^c_n\) is a central extension \(1 \rightarrow \mathbb{T} \rightarrow \text{Spin}^c_n \rightarrow SO_n \rightarrow 1\) of the special orthogonal group \(SO_n\) by the circle group \(\mathbb{T}\) of phases. A \textit{spin}^c manifold \(M\) is an oriented Riemannian manifold equipped with a principal \(\text{Spin}^c_n\)-bundle \(P \rightarrow M\) lifting its oriented orthonormal bundle of frames and a connection—the \textit{spin}^c \textit{connection}—on \(P \rightarrow M\) compatible with the Levi-Civita connection. There is a homomorphism \(\text{Spin}^c_n \rightarrow \mathbb{T}\) and so an associated circle bundle with connection over \(M\), called the \textit{characteristic bundle}. A \textit{spin}^c manifold has a canonical Dirac operator. A

\(^{16}\) If \(v\) in (1.2) is an integer invariant of spin manifolds which extends the Hopf invariant, then to use it in the action of a quantum field theory it should be fully local, hence the partition function of an invertible integer-valued topological field theory.
spin structure on a spin\(^c\) manifold is equivalent to a flat trivialization of its characteristic bundle.

The analog of Definition 5.2 on a spin\(^c\) manifold depends on the spin\(^c\) connection and the map \(\phi : M \to \mathbb{CP}^N\), but not on the Riemannian metric. It uses the \(\eta\)-invariant of Atiyah–Patodi–Singer [32]. Recall that these authors define a more refined invariant \(\xi = (\eta + \dim \ker) / 2\) and that the exponentiated \(\eta\)-invariant \(\exp(2\pi i \xi)\) varies smoothly with parameters. Let \(M\) be a closed spin\(^c\) 3-manifold equipped with a smooth map \(\phi : M \to \mathbb{CP}^N\). Let \(O(1) \to \mathbb{CP}^N\) be the hyperplane line bundle with its standard covariant derivative.

**Definition 5.9.** The WZ factor in the \(\sigma\)-model exponentiated action on spin\(^c\) manifolds is the exponentiated \(\eta\)-invariant of the Dirac operator coupled to the virtual bundle \(\phi^*O(1) - 1\).

This is the ratio of the exponentiated \(\eta\)-invariant of the spin\(^c\) Dirac operator coupled to \(\phi^*O(1)\) and the exponentiated \(\eta\)-invariant of the uncoupled spin\(^c\) Dirac operator.

We claim that this reproduces (2.5) for \(K = 1\) in case \(M\) bounds a spin\(^c\) 4-manifold \(W\) equipped with a map \(\Phi : W \to \mathbb{CP}^N\) which extends \(\phi\). In that case the main theorem in [32] computes the WZ factor as the exponential of the integral over \(W\) of the Chern–Weil differential forms which represent the degree 4 term in

\[
\hat{A}(W)e^{c/2}(c^x - 1).
\]  

(5.8)

In this expression \(c, x \in H^2(W; \mathbb{Z})\) are the Chern classes of the characteristic line bundle and \(\Phi^*O(1)\), respectively; each has degree 2. The degree 4 term is \((x^2 + xc) / 2\), which matches the differential form expression (2.5).

**Remark 5.10.** Even for \(N = 1\) the WZ factor on a spin\(^c\) manifold depends on the spin\(^c\) connection and the map \(\phi\) (not just up to homotopy), so is not a topological invariant.

**Remark 5.11.** The exponentiated \(\eta\)-invariant is the partition function of an extended invertible unitary field theory, so a valid factor in an exponentiated action. As a nonextended field theory of 2- and 3-manifolds this follows from the theorems in [33]. To construct the extended theories we can use differential \(K\)-theory, following the ideas in [26].

**Proposition 5.12.** The spin\(^c\) WZ factor in Definition 5.9 reduces to the spin WZ factor (5.2) on a spin manifold.

In particular, by Lemma 5.6(ii), it extends the mod 2 Hopf invariant.

**Proof.** Suppose first that the spin 3-manifold \(M\) bounds a spin 4-manifold \(W\) over which \(\phi\) extends. A spin structure on a spin\(^c\) manifold trivializes the characteristic class \(c\) of the characteristic bundle, so the degree 4 term in (5.8) reduces to \(x^2 / 2\). The corresponding statement about differential forms follows since the curvature of the characteristic bundle vanishes. Then the integral over \(W\) which computes the spin\(^c\) WZ factor reduces to the one for the spin WZ factor alluded to in Remark 5.3. This proves the proposition in the bounding case.

If \(N = 1\) then (5.7) computes the relevant bordism group to be cyclic of order 2 with generator the Hopf map \(\phi : S^3 \to \mathbb{CP}^1\) (see Remark 5.8), so we cannot directly apply the argument in the previous paragraph. Observe, however, that if \(\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N\) is a linear embedding then the bundle \(O(1) \to \mathbb{CP}^N\) with its covariant derivative restricts
to the bundle \( \mathcal{O}(1) \to \mathbb{CP}^1 \) with its covariant derivative.\(^\dagger\) Since \( \tilde{\chi} \) in Definition 5.2 is a differential characteristic class of \( \mathcal{O}(1) \to \mathbb{CP}^N \), it follows that both WZ factors are unchanged by composing \( \phi: M \to \mathbb{CP}^1 \) with the embedding \( \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N \). So it suffices to prove that any spin 3-manifold \( M \) equipped with a map \( \phi: M \to \mathbb{CP}^N \) bounds for any \( N \geq 2 \). The map \( \phi \) can be homotoped into the 4-skeleton \( \mathbb{CP}^2 \), so it suffices to take \( N = 2 \).

First, arguing as in (5.6) we are reduced to showing \( A := \pi_3(M \text{Spin} \wedge \mathbb{CP}^2) = 0 \). The cofibration sequence \( s^3 \xrightarrow{\eta} s^2 \to \mathbb{CP}^2 \) gives rise to the exact sequence

\[
\pi_3(M \text{Spin} \wedge s^3) \xrightarrow{\eta} \pi_3(M \text{Spin} \wedge s^2) \to \pi_3(M \text{Spin} \wedge \mathbb{CP}^2) \to \pi_2(M \text{Spin} \wedge s^3),
\]

which simplifies to \( \pi_0M \text{Spin} \to \pi_1M \text{Spin} \to A \to 0 \). The Hopf map \( \eta \) induces a surjective map \( \pi_0M \text{Spin} \to \pi_1M \text{Spin} \) (as stated above it represents the generator of stable framed bordism), from which we conclude \( A = 0 \). \( \square \)

Next, we observe that there is no unitary integer-valued invertible field theory of spin\(^c\) manifolds equipped with a map to \( \mathbb{CP}^1 \) whose partition function specializes to the Hopf invariant. For if there were, it would restrict to a theory of spin manifolds with that property and that was ruled out after (5.6). Similarly, it follows from Theorem 5.7 that there is no invertible \( \mathbb{CP}^1 \)-valued spin\(^c\) theory which specializes to a theory whose partition function for \( \phi: S^1 \to \mathbb{CP}^1 \) is \( e^{i\nu \theta} \), where \( \nu \) is the Hopf invariant and \( \theta \neq 0, \pi \); any such would restrict to a spin theory with those properties.

Finally, we justify that (2.5) can be used in the spin\(^c\) case by computing that every 3-dimensional spin\(^c\) manifold \( M \) equipped with a map \( \phi: M \to \mathbb{CP}^N \) bounds, i.e., \( B(\mathcal{N}) := \pi_3(M \text{Spin}^c \wedge \mathbb{CP}^N) = 0 \). First, as in previous arguments we reduce the case \( N \geq 2 \) to \( N = 2 \), and we can omit the disjoint basepoint ‘+’ since \( \pi_3M \text{Spin}^c = 0 \). For \( N = 2 \) we use the exact sequence (5.9) with \( M \text{Spin}^c \) replacing \( M \text{Spin} \), and since \( \pi_1M \text{Spin}^c = 0 \) we deduce \( B(\mathcal{N}) = 0 \) for \( N \geq 2 \). For \( N = 1 \) we also see \( B(1) \cong \pi_1M \text{Spin}^c = 0 \).

5.4. The variation in Sect. 4. The long distance theory derived in Sect. 4 has fields (i) a map \( \phi: M \to \mathbb{CP}^N \), (ii) a line bundle with connection \( L \to M \), and (iii) an isomorphism

\[
\theta: \phi^*\mathcal{O}(1) \cong \bigotimes L^q
\]

of line bundles with connection. The set of isomorphism classes of pairs \( (L, \theta) \) is a torsor over the set of isomorphism classes of principal \( \mathbb{Z}/q\mathbb{Z} \)-bundles: more precisely,\(^\dagger\) given two pairs \( (L, \theta), (L', \theta') \) there is a canonical \( \mathbb{Z}/q\mathbb{Z} \)-bundle \( Q \to M \) such that \( (L', \theta') \cong (L, \theta) \otimes Q \). The spin\(^c\) WZ factor has an easy generalization in this case.

Definition 5.13. The \( q \)-WZ factor in the \( \sigma \)-model exponentiated action on spin\(^c\) manifolds is the exponentiated \( \eta \)-invariant of the Dirac operator coupled to the virtual bundle \( L - 1 \).

\(^\dagger\) The hermitian metric and holomorphic structure restrict, so too does the resultant Chern covariant derivative.

\(^\dagger\) The following analog may be helpful: a spin structure on a Riemann surface \( \Sigma \) is a pair \( (L, \theta) \) where \( L \to \Sigma \) is a holomorphic line bundle and \( \theta: K_{\Sigma} \cong L^2 \) an isomorphism of the canonical bundle of \( \Sigma \) with the square of \( L \). Any two spin structures are related by a double cover.
Assuming the 3-manifold $M$ and all its geometric data extend over a compact 4-manifold with boundary $M$, then (5.8) is modified by substituting $x \rightarrow x/q$, the consequence of (5.10) on Chern classes. The degree 4 term is then $(x^2/q^2 + xc/q)/2$, which matches (4.4).

As in Proposition 5.12 we can restrict this definition to spin manifolds, or alternatively take the direct image of the differential $E$-characteristic class $\lambda$ of the line bundle with connection $L \rightarrow M$, analogous to Definition 5.2. (See the proof of Lemma 5.1.)

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