THE STOKES PROBLEM WITH NAVIER SLIP BOUNDARY CONDITION: MINIMAL FRACTIONAL SOBOLEV REGULARITY OF THE DOMAIN

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Abstract. We prove well-posedness in reflexive Sobolev spaces of weak solutions to the stationary Stokes problem with Navier slip boundary condition over bounded domains Ω of class $W^{2-1/s}_s$, $s > n$. Since such domains are of class $C^{1,1-n/s}$, our result improves upon the recent one by Amrouche-Seloula, who assume Ω to be of class $C^{1,1}$. We deal with the slip boundary condition directly via a new localization technique, which features domain, space and operator decompositions. To flatten the boundary of Ω locally, we construct a novel $W^{2-1/s}_s$ diffeomorphism for Ω of class $W^{2-1/s}_s$. The fractional regularity gain, from $2-1/s$ to 2, guarantees that the Piola transform is of class $W^{1,s}_s$. This allows us to transform $W^{1,r}_r$ vector fields without changing their regularity, provided $r \leq s$, and preserve the unit normal which is Hölder. It is in this sense that the boundary regularity $W^{2-1/s}_s$ seems to be minimal.

Key words. Stokes problem, Navier slip boundary condition, reflexive Sobolev space, fractional Sobolev domain, localization approach, Piola transform.

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1. Introduction. A bounded connected domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ is said to be of fractional Sobolev class $W^{2-1/s}_s$, $(n < s < \infty)$ whenever its boundary $\partial \Omega$ is locally the graph of a function $\omega$ in $W^{2-1/s}_s$,loc($\mathbb{R}^{n-1}$). We refer to [17, 18] for an equivalent definition via the signed distance function. Our primary goal is to establish well posedness (in the sense of Hadamard) of the following Stokes problem

$$- \text{div} \sigma(u, p) = f, \quad \text{div} u = g \quad \text{in} \Omega,$$

(1.1a)

together with the Navier slip boundary condition,

$$u \cdot \nu = \phi, \quad \beta Tu + T^\top \sigma(u, p) \nu = \psi \quad \text{on} \partial\Omega,$$

(1.1b)

in the reflexive Sobolev spaces $W^1_r(\Omega) \times L^r(\Omega)$ with $s' = s/(s-1) \leq r \leq s$. Here $\sigma = \eta \varepsilon(u) - Ip$ is the stress tensor, $\eta$ is a constant viscosity parameter (Newtonian fluid), $\varepsilon(u) = (\nabla u + \nabla u^\top)$ is the strain tensor (or symmetric gradient), $\nu$ is the exterior unit normal to $\partial \Omega$, $\beta(x) \geq 0$ is the friction coefficient, and $T = I - \nu \otimes \nu$ is the projection operator onto the tangent plane of $\partial \Omega$. Notice that when $\phi = 0$, $\psi = 0$, and $\beta = 0$ in (1.1b) then the fluid slips along the boundary. The well-known no-slip condition $u = 0$ can be viewed as the limit of (1.1b) when $\phi = 0$, $\psi = 0$, and $\beta \to \infty$. The boundary condition (1.1b) is appropriate in dealing with free boundary problems; we refer to [31, 28, 41, 40, 39, 5, 4] and related references [3, 40].

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In contrast to (1.1b), the no-slip condition $u = 0$ has received a great deal of attention in the literature. It turns out that most of what is valid for the Poisson equation $-\Delta u = \text{div} \, f$ extends to the Stokes equation with no-slip condition. The a priori bound in $W^1_r(\Omega)$

$$\|u\|_{W^1_r(\Omega)} \leq C \|f\|_{L^r(\Omega)}$$

(1.2)

is valid for the Laplacian provided $\partial \Omega$ is Lipschitz and $s' \leq r \leq s$ for $s > n$; see [30, Theorem 1.1] for Dirichlet condition and [44, Theorem 1.6] for Neumann condition. Moreover, the range of $r$ becomes $1 < r < \infty$ provided $\partial \Omega$ is $C^1$; see [30, Theorem 1.1] for the Dirichlet problem. A similar bound with $1 < r < \infty$ is valid for the Stokes system with Dirichlet condition provided $\Omega$ is Lipschitz with a small constant, and in particular for $C^1$ domains [24]. This extends to Besov spaces on Lipschitz domains for the Stokes system with Dirichlet and Neumann conditions [37]; we refer to [16, 20] for earlier contributions. It is thus natural to wonder whether such estimates would extend to the Navier boundary condition (1.1b) with a domain regularity weaker than $C^1$.

We first mention some relevant literature and next argue that the domain regularity $W^{2-1/s}_s(\partial \Omega)$ seems to be minimal.

It is well-known that in case $\phi = 0, \psi = 0$, and $\beta = 0$ one can write (1.1b) equivalently as

$$u \cdot \nu = 0, \quad \nu \times \text{curl} \, u = 0 \quad \text{on} \, \partial \Omega. \quad (1.3)$$

However, such an equivalence is known to hold only when the boundary is of class $C^2$ [35, Section 2]. Starting from (1.3), Amrouche–Seloula [3] obtained recently well posedness of weak, ultra-weak, and strong solutions to (1.1a) in reflexive Sobolev spaces for $\Omega$ of class $C^{1,1}$. To our knowledge, this is the best result in the literature.

Our approach deals directly with (1.1b) which makes it amenable to free boundary problems. We elaborate below on this matter.

The pioneering work on the Stokes problem (1.1) in Hölder spaces is due to Solonnikov-Šcadilov [42]. This was extended by Beirão da Veiga [7], who showed existence of weak and strong solutions to a generalized Stokes system with $C^{1,1}$-domains in Hilbert space setting. We also refer to Mitrea-Monniaux [36] who proved existence of mild solutions in Lipschitz domains for the time-dependent Navier-Stokes (NS) equations with boundary condition (1.3). A survey for the stationary and the time-dependent Stokes and NS equations with slip boundary condition is given by Berselli [10]. Finally, well posedness for several variants of the time-dependent NS equations are shown by Málek and collaborators for $C^{1,1}$ domains [12, 13, 14, 15].

We now give three reasons why we find the regularity $W^{2-1/s}_s(\partial \Omega)$ of $\partial \Omega$ (nearly) minimal for (1.1), and stress that they all hinge on the critical role played by the unit normal $\nu$. We start with the boundary condition $u \cdot \nu = \phi$ in (1.1b). Given the scalar function $\phi \in W^{1-1/r}_r(\partial \Omega)$, in [7] we construct a vector-valued extension $\varphi \in W^1_r(\Omega)$ such that $\varphi = \phi \nu$ on $\partial \Omega$ in the sense of traces along with $\|\varphi\|_{W^1_r(\Omega)} \leq C\|\phi\|_{W^{1-1/r}_r(\partial \Omega)}$.

One possibility is to solve the following auxiliary problem subject to the compatibility condition $\int_\Omega g + \int_{\partial \Omega} \phi = 0$

$$- \Delta \xi = g \quad \text{in} \, \Omega \quad \partial_{\nu} \xi = \phi \quad \text{on} \, \partial \Omega, \quad (1.4)$$

and set $\varphi := \nabla \xi$. The requisite regularity $\xi \in W^2_r(\Omega)$, whence $\varphi \in W^1_r(\Omega)$, fails in general for $\partial \Omega$ Lipschitz or even $C^1$ [29]. If $\partial \Omega$ is of class $W^{2-1/s}_s$, instead, then we
can extend each component of $\mathbf{v} \in W^{2-1/s}_s$ and thus $\mathbf{v}$ to a function in $W^{1}_s(\Omega)$ (still denoted $\mathbf{v}$). If we also denote $\phi \in W^{1}_s(\Omega)$ an extension of $\phi$, then a simple calculation shows that $r \leq s$ and $s > n$ yields $\phi \mathbf{v} \in W^{1}_r(\Omega)$ and
\[
\|\mathbf{\phi}\|_{W^{2-1/s}_s(\Omega)} \leq C\|\phi\|_{W^{2-1/r}(\partial \Omega)}\|\mathbf{v}\|_{W^{1-1/s}_r(\partial \Omega)}.
\]  
(1.5)

Setting $\mathbf{v} := \mathbf{u} - \mathbf{\phi}$ we get a problem for $\mathbf{v}$ similar to (1.1) with modified data with the same regularity as $f$, $g$ and $\mathbf{\phi}$ but $\mathbf{v} \cdot \mathbf{v} = 0$. We study this problem in §3.6.

Our localization technique is the second instance for $\partial \Omega$ to be of class $W^{2-1/s}_s$. In §5 we construct a local $W^{2}_s$ diffeomorphism $\Psi$, such that $\Psi^{-1}$ flattens $\partial \Omega$, by suitably extending the function $\omega \in W^{2-1/s}_s$ describing $\partial \Omega$ locally. We exploit this small gain of regularity, from $2 - 1/s$ to 2, to define the Piola transform inverse $P^{-1} := \det(\nabla \Psi^{-1})\nabla \Psi^{-1} \in W^{1}_s(\Omega)$, which maps vector fields $\mathbf{v} \in W^{1}_s(\Omega)$ into vector fields $(P^{-1} \mathbf{v}) \circ \Psi \in W^{1}_s(\mathbb{R}^n_+)$ with the same divergence and normal trace in $\mathbb{R}^n_+$. This is instrumental to reduce (1.1) locally to a Stokes problem in $\mathbb{R}^n_+$ with variable coefficients and Navier condition (1.1b) on $\{x_n = 0\}$, and next make use of reflection arguments. We develop the localization framework in §6.

Our primary interest in studying (1.1) with minimal domain regularity is the Stokes problem defined in $\Omega$

$$\Omega := \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1 + \gamma(x)\}, \quad \Gamma := \{(x, 1 + \gamma(x)) : 0 < x < 1\}$$

with free boundary $\Gamma$ given by the graph of $1 + \gamma$ and $\gamma(0) = \gamma(1) = 0$. The free boundary condition on $\Gamma$ corresponds to surface tension effects, is overdetermined, and reads
\[
\mathbf{u} \cdot \mathbf{v} = 0, \quad \sigma(\mathbf{u}, p)\mathbf{v} = \chi H[\gamma] \mathbf{v},
\]  
(1.6)

with $H[\gamma] = \gamma \mathbf{x}$ the curvature of $\Gamma$, $Q[\gamma] := \sqrt{1 + |\mathbf{d}_x \gamma|^2}$, and $\chi > 0$ a surface tension coefficient, whereas a Dirichlet condition is imposed on the rest of $\partial \Omega$.

We realize that (1.6) includes (1.1b) besides the equation for the balance of forces $\nu^T \sigma(\mathbf{u}, p)\mathbf{v} = \chi H[\gamma] \mathbf{v}$ which determines the location of $\Gamma$. To formulate this problem variationally, we multiply the momentum equation in (1.1a) by $\mathbf{v}$, integrate by parts
\[
\int_{\Omega} \sigma(\mathbf{u}, p) : \nabla \mathbf{v} - f \cdot \mathbf{v} = \int_{\Gamma} \mathbf{v} \cdot \sigma(\mathbf{u}, p)\mathbf{v} = \int_{\Gamma} H[\gamma] |\mathbf{v} - \mathbf{v}| = \int_0^1 \mathbf{d}_x \gamma \mathbf{d}_x \mathbf{v} \cdot Q[\gamma]
\]
and use $\gamma(0) = \gamma(1) = 0$, where $\nu = Q[\gamma] \mathbf{v} \cdot \mathbf{v}$. We emphasize again the critical role that $\mathbf{v}$ plays: applying the Implicit Function Theorem enables us to prove that $\gamma \in W^{1-1/s}_s(0, 1) \subset W^{1}_s(0, 1)$, $s > 2$, whence $\mathbf{v} \in W^{1}_s(\Omega)$ implies $Q[\gamma] \mathbf{v} \cdot \mathbf{v} \in W^{1-1/s}_s(\Gamma)$ as already alluded to in (1.5). This would not be possible with mere Lipschitz regularity $\gamma \in W^{1}_s(0, 1)$. We refer to [4] for full details.

For the moment, we will make two simplifications in (1.1b). The first one is to treat the frictionless problem, i.e. $\beta = 0$; we will return to $\beta \neq 0$ in [8]. The second simplification concerns the non-trivial essential boundary condition $\phi$, which we address in [7] by the lifting argument already mentioned in (1.5). It is customary for the Stokes system (1.1) to let the pressure $p$ be defined up to a constant. Less apparent is that the velocity field kernel, namely $\nabla \mathbf{u} + \nabla \mathbf{u}^T = 0$, is non-trivial if and only if $\Omega$ is axisymmetric. More importantly, when this kernel is not empty, it is characterized by a small subspace of the rigid body motions,
\[
Z(\Omega) := \{\mathbf{z}(\mathbf{x}) = \mathbf{A} \mathbf{x} + \mathbf{b} : \mathbf{x} \in \Omega, \mathbf{A} = -\mathbf{A}^T \in \mathbb{R}^{n \times n}, \mathbf{b} \in \mathbb{R}^n, \mathbf{z} \cdot \mathbf{v} \big|_{\partial \Omega} = 0\}.
\]  
(1.7)
The standard variational formulation of (1.1) entails dealing with the Stokes bilinear form

$$S_\Omega(u, p)(v, q) := \int_\Omega \eta \varepsilon(u) : \varepsilon(v) - p \text{div} v + q \text{div} u.$$

(1.8)

which we obtain from (1.1a) upon formal multiplication by a test pair (v, q) and integration by parts. This also leads to the forcing term

$$F(v, q) := \int_\Omega f \cdot v + \int_{\partial\Omega} \psi \cdot \gamma_0 v + \int_\Omega g q,$$

upon invoking (1.11). To formulate (1.1) variationally we need two function spaces. The first space is that of trial functions $$X_r(\Omega),$$ which we define as

$$X_r(\Omega) := V_r(\Omega) \times L^r_0(\Omega) \quad s' \leq r \leq s,$$

(1.9a)

with $$V_r(\Omega) := \{ v \in W^1_r(\Omega) : \text{div} v = 0 \},$$ $$L^r_0(\Omega) := L^r(\Omega) / \mathbb{R}$$ and $$1/s + 1/s' = 1.$$ We have the following characterization of $$V_r(\Omega)$$ (see [19, p. 4]): a vector $$v \in V_r(\Omega)$$ if and only if

$$v \in \{ v \in W^1_r(\Omega) : v \cdot \nu = 0, \int_\Omega (\partial_{x_i} v^j - \partial_{x_j} v^i) \, dx = 0 \forall i, j = 1, \ldots, n \}.$$

It follows from its product definition that $$X_r(\Omega)$$ is complete under the norm

$$\| (v, p) \|_{X_r(\Omega)} := \| v \|_{W^1_r(\Omega)} + \| p \|_{L^r(\Omega)}.$$

(1.9b)

The second function space is that of prescribed data, which we take to be $$X'_{r^*}(\Omega),$$ the topological dual of $$X_r(\Omega)$$ where $$1/r + 1/r^* = 1.$$ Moreover, $$X_{r^*}(\Omega)^*$$ is complete under the operator norm

$$\| F \|_{X'_{r^*}(\Omega)} := \sup_{\| (v, q) \|_{X_{r^*}(\Omega)} = 1} | F(v, q) |.$$

(1.9c)

We note that $$\| F \|_{X'_{r^*}(\Omega)}$$ is finite provided $$g$$ is in $$L^r(\Omega),$$ $$f$$ belongs to $$V_{r^*}(\Omega)^*,$$ $$\psi$$ lies in the dual of the trace space $$\gamma_0(V_{r^*}(\Omega));$$ moreover all three functions must satisfy the compatibility conditions

$$\int_\Omega g = 0, \int_\Omega f \cdot z + \int_{\partial\Omega} \psi \cdot \gamma_0 z = 0 \forall z \in Z(\Omega).$$

(1.10)

The variational formulation of the strong equations (1.1) finally reads: solve

$$(u, p) \in X_r(\Omega) : S_\Omega(u, p)(v, q) = F(v, q) \forall (v, q) \in X_{r^*}(\Omega).$$

(1.11)

With the functional setting in place, we state our main result.

**Theorem 1.1** (well posedness of (1.1)). *Let $$\Omega$$ be a bounded domain of class $$W^{s-1}_s$$ with $$s > n,$$ and let $$s' \leq r \leq s.$$ For every $$F \in X_{r^*}(\Omega)^*$$ there exists a unique solution $$(u, p) \in X_r(\Omega)$$ of (1.1) such that

$$\| (u, p) \|_{X_r(\Omega)} \leq C_{\Omega, q, n, r} \| F \|_{X'_{r^*}(\Omega)}.$$

(1.12)
We say that the Stokes problem is \textit{well-posed} (in the sense of Hadamard) between the spaces $X_r(\Omega)$ and $X_{r'}(\Omega)^*$ whenever $[1.11]-[1.12]$ is satisfied.

The ideas explored in this paper can be summarized as follows. We develop a new localization technique which features \textit{domain}, \textit{space} and \textit{operator} decompositions. Instead of $[1.3]$, we rely solely on existence and uniqueness of solutions to the Stokes problem in the whole space $\mathbb{R}^n$ for compactly supported data. Finally, we develop an \textit{index-theoretical} framework to close the argument [34, Chapter 27]. Our approach is general and can be applied to a wide class of elliptic partial differential equations (PDEs). After a brief section about notation, we split the proof of Theorem 1.1 into six sections, which describe how the paper is organized:

\[ \text{[3]} \] gives a short proof for the Hilbert space case ($r = 2$). The importance of this result is the direct implication of uniqueness for solutions to $[1.11]$ when $r \geq 2$ and $\Omega$ is bounded.

\[ \text{[4]} \] presents fundamental results on well-posedness of the Stokes system in $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_+$. These two building blocks are instrumental in constructing a solution of $[1.11]$ for $s' \leq r \leq s$.

\[ \text{[5]} \] constructs a local $W^s_2$ diffeomorphism $\Psi$, whose inverse $\Psi^{-1}$ locally flattens $\partial \Omega$, and analyzes the Piola transform which preserves the essential boundary condition $u \cdot \nu = 0$.

\[ \text{[6]} \] develops a new localization procedure and uses index theory to prove the well-posedness of the Stokes system $[1.1]$ between the spaces $X_r(\Omega)$ and $X_{r'}(\Omega)^*$ for $s' \leq r \leq s$.

\[ \text{[7]} \] deals with the inhomogeneous essential boundary conditions.

\[ \text{[8]} \] extends the theory to the full Navier boundary condition, i.e. $\beta \neq 0$.

2.\quad \textbf{Notation}. It will be convenient to distinguish the $n$-th dimension. A vector $x \in \mathbb{R}^n$, will be denoted by

$$x = (x^1, \ldots, x^{n-1}, x^n),$$

with $x^i \in \mathbb{R}$ for $i = 1, n$, $x^i \in \mathbb{R}^{n-1}$. We will make a distinction between the reference coordinate ($x$) and the physical coordinate ($\bar{x}$), such that $x = \Psi(\bar{x})$ where the properties of the map $\Psi$ are listed below. The symbols $B(x, \delta) \subset \mathbb{R}^n$ and $B(x, \delta) \subset \mathbb{R}^n$ will denote the balls of radius $\delta$ centered at $x$ and $\bar{x}$ respectively. Moreover, $\Omega \subset B(x, \delta)$ and $\Omega \supset B(x, \delta) \subset \mathbb{R}^n$ will be the disc and the lower half ball of radius $\delta$ centered at $x'$ and $\bar{x}$ respectively; see Figure 2.1 (right).

\textbf{Definition 2.1 ($W^{2-1/s}_s$-domain). An open and connected set $\Omega$ in $\mathbb{R}^n$ is called a $W^{2-1/s}_s$-domain, $s > n$, if at each point $x \in \partial \Omega$ there exists $\delta > 0$ and a function $\omega$ in $W^{2-1/s}_{s, \text{loc}}(\mathbb{R}^{n-1})$ such that, after a possible relabeling and reorientation of the coordinate axis

$$\Omega \cap B(x, \delta) = \{ y = (y', y^n) \in B(x, \delta) : y^n < \omega(y') \}; \quad (2.1)$$

see Figure 2.1. A $W^{2-1/s}_s$-domain where $\delta$ can be chosen independently of $x$ is said to be a uniform $W^{2-1/s}_s$-domain. It is easy to verify that every bounded $W^{2-1/s}_s$-domain is a uniform $W^{2-1/s}_s$-domain.

To study the problem $[1.1]$ near $x \in \partial \Omega$ we need to flatten $\partial \Omega$ locally. This is realized by a map $\Psi = (\Psi^1, \ldots, \Psi^n) : \mathbb{R}^n_{-} \rightarrow \mathbb{R}^n$ with the following properties (see Figure 2.1):
(P1) \( \Psi \) is a diffeomorphism of class \( W^2_\delta \) between \( \mathbb{R}^n_\text{c} \) and \( \Psi(\mathbb{R}^n_\text{c}) \subset \mathbb{R}^n \), \( y \in Q \subset \mathbb{R}^n \) denotes the reference coordinate and \( y \in \Psi(\mathbb{R}^n) \) the physical coordinate so that \( y = \Psi(y) \) and \( Q = \Psi(Q) \).

(P2) \( \Psi(D(x,\delta/2)) = \partial \Omega \cap B(x,\delta/2) \).

(P3) \( \Psi(\mathbb{R}^n_\text{c} \setminus B(x,\delta)) = I \) (identity).

We construct such a map \( \Psi \) in Section 5.1 but observe now the gain in regularity from \( W^2_\delta \) to \( W^2_s \). This regularity improvement is critical for our theory and is achieved by extending the function \( \omega \) suitably.

If \( X_p(\Omega) \) is a Banach space over \( \Omega \), we denote by \( \| \cdot \|_{X_p} \) its norm. By \( L^p(\Omega) \) with \( p \in [1, \infty] \) we denote the space of functions that are Lebesgue integrable with exponent \( p \). By \( W^k_p(\Omega) \) we denote the classical Sobolev space of functions whose distributional derivatives up to \( k \)-th order are in \( L^p(\Omega) \). We indicate with \( W^k_p(\Omega) \) the closure of \( C^\infty(\Omega) \) in \( W^k_p(\Omega) \). The Lebesgue conjugate to \( p \) will be denoted by \( p' \), i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \).

We denote by \( (\cdot, \cdot) \) the duality pairing and sometimes the \( L^2 \)-scalar product when it is clear from the context. The relation \( a \lesssim b \) indicates that \( a \leq Cb \), with the constant \( C \) that does not depend on \( a \) or \( b \). The value of \( C \) might change at each occurrence.

Given matrices \( P \in \mathbb{R}^{n \times n} \), \( M = (m^{i,k})_{i,k=1}^n \in \mathbb{R}^{n \times n} \), and vector \( w \in \mathbb{R}^n \), we define

\[
\Psi_P(M) = PMP^{-1},
\]

and note that

\[
\nabla (Mw) = \nabla M \odot w + M \nabla w,
\]

where the \((i,j)\) component of the \( n \times n \) matrix \( \nabla M \odot w \) is

\[
(\nabla M \odot w)^{i,j} = \sum_{k=1}^n \left( \partial_{x_i} m^{i,k} \right) w^k.
\]

3. The Hilbert Space Case. In this section we prove the well-posedness of the Stokes problem in \( X_2(\Omega) \times X_2(\Omega)^* \). Results in this direction are known for a generalized Stokes system on \( C^{1,1} \) domains \( \Omega \). We show that \( \Omega \) being Lipschitz is
sufficient for the homogeneous case $\phi = 0$. Our proof relies on Korn's inequality, Brezzi's inf-sup condition for saddle-point problems, and Nečas' estimate on the right inverse of the divergence operator. We collect these results in the sequel.

**Proposition 3.1 (Korn's inequality).** Let $1 < r < \infty$ and $D$ be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists constants $C_1$ and $C_2$ depending only on $D, n$ and $r$ such that for every $v \in W^{1,r}_0(D)$

$$\|v\|_{W^{1,r}_0(D)} \leq C_1 \left( \|v\|_{L^r(D)} + \|\varepsilon(v)\|_{L^r(D)} \right) \leq C_2 \|v\|_{W^{1,r}_0(D)}.$$  \hfill (3.1a)

Moreover, for every $v \in W^{1,r}_0(D)$ there exists a skew symmetric matrix $A$ in $\mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^n$ such that

$$\|v - (Ax + b)\|_{W^{1,r}_0(D)} \leq C_{D,n,r} \|\varepsilon(v)\|_{L^r(D)}.$$  \hfill (3.1b)

**Proof.** See [35] Theorem A.1 for $r = 2$, [19] Section 2] for $1 < r < \infty$ in bounded domains. \hfill \Box

**Lemma 3.2 (equivalence of norms).** Let $1 < r < \infty$ and $D$ be a bounded Lipschitz domain. For every $v \in V_r(D)$ the following holds

$$\|v\|_{W^{1,r}_0(D)} \leq C_{D,n,r} \|\varepsilon(v)\|_{L^r(D)}.$$  \hfill (3.2a)

**Proof.** This proceeds by contradiction. However, the argument is fairly standard, is based on [35] Theorem A.2 for $r = 2$, and is thus omitted. \hfill \Box

**Remark 3.3 (boundary condition $z \cdot \nu = 0$).** The difference between Lemma 3.2 and Proposition 3.1 is that the vector-fields $z$ in $Z(D)$ satisfy $z \cdot \nu = 0$ while the ones from Korn's inequality do not have this requirement. We further remark that Lemma 3.2 remains valid if the condition $v \cdot \nu = 0$ is imposed only on a subset of $\partial D$ with positive measure. We will use this result in Theorem 5.14, Lemma 6.10 and Lemma 6.21 below.

Next we state Brezzi's characterization of (1.11) as a saddle-point problem. We rewrite (1.11) as follows: find a unique $(u, p)$ in $V_r(\Omega) \times L^r_0(\Omega)$ such that

$$\eta \langle \varepsilon(u), \varepsilon(v) \rangle_{\Omega} - \langle p, \text{div } v \rangle_{\Omega} = F(v, 0) \quad \forall v \in V_r(\Omega),$$

$$\langle \text{div } u, q \rangle_{\Omega} = F(0, q) \quad \forall q \in L^r_0(\Omega).$$  \hfill (3.2)

**Lemma 3.4 (inf-sup conditions).** The saddle point problem (3.2) is well-posed in $(V_r(\Omega) \times L^r_0(\Omega)) \times (V_r(\Omega)^\ast \times L^r_0(\Omega)^\ast)$ if and only if there exist constants $\alpha, \beta > 0$ such that

$$\inf_{u \in V_r} \sup_{v \in V_r} \frac{\langle \varepsilon(u), \varepsilon(v) \rangle}{\|u\|_{V_r'} \|v\|_{V_r'}} = \inf_{v \in V_r} \sup_{u \in V_r} \frac{\langle \varepsilon(u), \varepsilon(v) \rangle}{\|u\|_{V_r'} \|v\|_{V_r'}} = \alpha > 0,$$  \hfill (3.3a)

$$\inf_{q \in L^r_0} \sup_{w \in V_r} \frac{\langle \text{div } w, q \rangle}{\|w\|_{V_r'} \|q\|_{L^r}} = \beta > 0,$$  \hfill (3.3b)

where $V_r := \left\{ w \in V_r(\Omega) : \langle \text{div } w, q \rangle = 0, \forall q \in L^r_0(\Omega) \right\}$. In addition, there exists $\gamma = \gamma(\alpha, \beta, \eta)$ such that the solution $(u, p)$ is bounded by

$$\|(u, p)\|_{X_r(\Omega)} \leq \gamma \|F\|_{X_r(\Omega)^\ast}.$$  \hfill (3.3c)
II.10.2 for the trace space results. The Stokes problem (1.11) is well-posed in $X_2(\Omega) \times X_2(\Omega)^*$. Proof. It suffices to check Brezzi’s conditions (3.3).

Remark 3.6 (boundary regularity). The Lipschitz regularity of $\partial \Omega$ is adequate only for $\phi = 0$ and $r = 2$, as alluded to in the introduction. In general, we need to work with domains of class $W^{2-1/s}_s$.

4. Stokes Problem on Unbounded Domains ($\mathbb{R}^n$ and $\mathbb{R}^n_-$). The purpose of this section is to prove the existence, uniqueness and local regularity of the Stokes problem (1.11) in the whole space $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_-$ for data with compact support. These two problems are the essential building blocks for the localization procedure in §4. This problem has been extensively studied under different functional frameworks; we refer to [2] Introduction for an overview.

Weighted Sobolev spaces are an extremely general framework for it provides a wealth of predictable behaviors at $\infty$ when considering different weight functions. A different framework is the one of Homogeneous Sobolev spaces, its main disadvantage being the lack of control on the $L^r$-norm of the function. Fortunately, these two frameworks are interchangeable as long as the data in question has compact support and one is not interested in the behavior at $\infty$ of the functions being analyzed [2, Proposition 4.8].

With this equivalence in hand, and the fact that our work was originally inspired by that of Galdi-Simader-Sohr [24], we choose to work with the Galdi-Simader’s characterization for homogeneous Sobolev spaces [23]. The rest of this section is split into three parts. In §4.1 we recall this essential characterization and define the equivalent $X_r(\mathcal{D})$ spaces for unbounded domains $\mathcal{D}$. In §4.2 we prove the well-posedness of the Stokes problem in its symmetric gradient form in $\mathbb{R}^n$. Finally, in §4.3 we extend the result to the half-space $\mathbb{R}^n_-$.  

4.1. Homogeneous Sobolev Spaces. The solution space $X_r(\Omega)$ is too small to prove an existence and uniqueness result for unbounded domains [24, Section 2]. In these cases we are led to consider the homogeneous Sobolev spaces

$$
G^1_r(\mathbb{R}^n) = \tilde{G}^1_r(\mathbb{R}^n) := \{C^\infty_c(\mathbb{R}^n)\}^{1|w^1_{r}(\mathbb{R}^n)},
$$

$$
G^1_r(\mathbb{R}^n_-) := \{C^\infty_c(\mathbb{R}^n_-)\}^{1|w^1_{r}(\mathbb{R}^n_-)},
$$

$$
\tilde{G}^1_r(\mathbb{R}^n_-) := \{C^\infty_c(\mathbb{R}^n_-)\}^{n-1 \times C^\infty(\mathbb{R}^n_-)}^{1|w^1_{r}(\mathbb{R}^n_-)},
$$

where $C^\infty_c(\mathcal{D})$ are $C^\infty$ functions with compact support in $\mathcal{D}$, the half-space $\mathbb{R}^n_-$ is given by $x = (x',x^n) \in \mathbb{R}^n$ with $x^n < 0$, and $\mathbb{R}^n = \mathbb{R}^n_+ \cup \partial \mathbb{R}^n$ with $x$ in $\partial \mathbb{R}^n$ if and only if $x^n = 0$. The statement $v = (v', v^n) \in (C^\infty_c(\mathbb{R}^n))^{n-1 \times C^\infty(\mathbb{R}^n_-)}$ implies $v|_{\partial \mathbb{R}^n} = (v'|_{\partial \mathbb{R}^n}, 0)$ for $v' \in (C^\infty_c(\mathbb{R}^n))^{n-1}$. For a detailed presentation of these spaces, their duals and trace spaces see [22, Chapter II], in particular [22, Theorem II.10.2] for the trace space results.

Next we recall a result by Galdi-Simader on the characterization of $G^1_r(\mathcal{D})$ and $\tilde{G}^1_r(\mathcal{D})$ with $\mathcal{D}$ equal to $\mathbb{R}^n$ or $\mathbb{R}^n_-$ [23, Lemma 2.2,8, Section 1], [22], and [24].

Proposition 4.1 (Galdi-Simader). Let $1 < r < \infty$ and $G^1_r(\mathcal{D})$ and $\tilde{G}^1_r(\mathcal{D})$ be
the spaces defined in \[4.1\]. The following characterization holds,

\[ G^1_r(\mathbb{R}^n) = \left\{ [v]_1 \in [L^r_{\text{loc}}(\mathbb{R}^n)]^n : \nabla v \in [L^r(\mathbb{R}^n)]^{n \times n} \right\}, \]

\[ \tilde{G}^1_r(\mathbb{R}^n) = \left\{ v = ([v']_1, v^0) \in [L^r_{\text{loc}}(\mathbb{R}^n)]^n : \nabla v \in L^r(\mathbb{R}^n)^{n \times n}, v^0|_{\partial \mathbb{R}^n} = 0 \right\}, \tag{4.2} \]

where \([v]_1\) is the equivalence class of functions in \([L^r_{\text{loc}}(\mathbb{D})]^n\) which differ by a constant vector. Furthermore, if \(1 < r < n\) then additionally

\[ G^1_r(\mathbb{D}) = \left\{ v \in [L^r_{\text{loc}}(\mathbb{D})]^n : \nabla v \in [L^r(\mathbb{D})]^{n \times n} \right\}, \quad \mathbb{D} = \mathbb{R}^n \text{ or } \mathbb{R}^n, \]

\[ \tilde{G}^1_r(\mathbb{R}^n) = \left\{ v \in ([L^r_{\text{loc}}(\mathbb{R}^n)]^{n-1} \times L^r(\mathbb{R}^n)) \cap G^1_r(\mathbb{R}^n) : v^0|_{\partial \mathbb{R}^n} = 0 \right\}, \tag{4.3} \]

where \(r^*\) is the Sobolev conjugate of \(r\) and is given by \(1/r^* = 1/r - 1/n\). Moreover, using Gagliardo-Nirenberg inequality we have for \(1 < r < n\)

\[ \|v\|_{L^{r^*}(\mathbb{D})} \lesssim \|\nabla v\|_{L^r(\mathbb{D})}. \tag{4.4} \]

We conclude by introducing the functional space \(X_r(\mathbb{D})\) when \(\mathbb{D}\) is \(\mathbb{R}^n\) or \(\mathbb{R}^n\). The distinction between this and \([1.9b]\) is that now we use the homogeneous Sobolev spaces defined above, and the pressure space is simply \(L^r(\mathbb{D})\), i.e.

\[ X_r(\mathbb{D}) := V_r(\mathbb{D}) \times L^r(\mathbb{D}) \quad 1 < r < \infty, \tag{4.5a} \]

with \(V_r(\mathbb{R}^n) = G^1_r(\mathbb{R}^n)\), and \(V_r(\mathbb{R}^n) = \tilde{G}^1_r(\mathbb{R}^n)\).

It follows from the product definition of \(X_r(\mathbb{D})\) that it is a complete space under the norm

\[ \|(v, p)\|_{X_r(\mathbb{D})} := |v|_{W^1_r(\mathbb{D})} + \|p\|_{L^r(\mathbb{D})}. \tag{4.5b} \]

The space for the prescribed data, is \(X_r(\mathbb{D})^*\), the topological dual of \(X_r(\mathbb{D})\), where \(1/r + 1/r^* = 1\). Moreover, \(X_r(\mathbb{D})^*\) is complete under the operator norm

\[ \|\mathcal{F}\|_{X_r(\mathbb{D})^*} = \sup_{\|(v, q)\|_{X_r(\mathbb{D})} = 1} |\mathcal{F}(v, q)|. \tag{4.5c} \]

**4.2. Stokes Problem in \(\mathbb{R}^n\).** In this section we investigate the well-posedness of the Stokes problem \([1.11]\) between the spaces \(X_r(\mathbb{R}^n)\) and \(X_r(\mathbb{R}^n)^*\). We begin by recalling the well-posedness of the usual Stokes problem without symmetric gradient \[22\] Section IV.2).

**Lemma 4.2 (Well posedness in \(\mathbb{R}^n\)).** Let \(1 < r < \infty, n \geq 2\). For each \(f \in G^{-1}_r(\mathbb{R}^n) = G^1_r(\mathbb{R}^n)^*\) and \(g \in L^r(\mathbb{R}^n) = L^{r^*}(\mathbb{R}^n)^*\), there exists a unique pair \((u, p) \in G^1_r(\mathbb{R}^n) \times L^r(\mathbb{R}^n)\) satisfying

\[-\Delta u + \nabla p = f, \quad \text{div } u = g \quad \text{in } \mathbb{R}^n\]

in the sense of distributions, which depends continuously on the data, i.e.

\[ |u|_{W^1_r(\mathbb{R}^n)} + \|p\|_{L^r(\mathbb{R}^n)} \leq C_{n, r} \left( \|f\|_{G^{-1}_r(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)} \right). \]
Additionally, if $1 < t < \infty$, $f \in G_t^{-1}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $u \in G_t^1(\mathbb{R}^n) \cap G_t^1(\mathbb{R}^n)$ and $p \in L^t(\mathbb{R}^n) \cap \times L^t(\mathbb{R}^n)$.

This result, combined with the identity $\nabla u^\top = \nabla u$, yields the well-posedness of (1.11) in $\mathbb{R}^n$.

**Theorem 4.3** (well-posedness of $S_{\mathbb{R}^n}$). Let $1 < r < \infty$, $n \geq 2$. The Stokes problem $S_{\mathbb{R}^n}(u, p) = F$ is well-posed in the space pair $(X_r(\mathbb{R}^n), X_{r'}(\mathbb{R}^n)^*)$, namely (1.11) are satisfied. Additionally, if $1 < t < \infty$ and $F \in X_t(\mathbb{R}^n)^*$, then $(u, p) \in X_t(\mathbb{R}^n)$.

**4.3. Stokes Problem in $\mathbb{R}^n$.** In this section we show the well-posedness of the Stokes problem (1.11) in the space pair $(X_r(\mathbb{R}^n), X_{r'}(\mathbb{R}^n)^*)$. Although the reflection technique employed is well-known, the construction sets the stage for the localization section. A very general result in this direction is the work by Beirão da Veiga-Crispo-Grisanti [9].

**Theorem 4.4** (well-posedness of $S_{\mathbb{R}^n}$). Let $1 < r < \infty$, $n \geq 2$. The Stokes problem (1.11) is well-posed from $X_r(\mathbb{R}^n)$ to $X_{r'}(\mathbb{R}^n)^*$. Additionally, if $1 < t < \infty$ and $F \in X_t(\mathbb{R}^n)^*$, then $(u, p) \in X_t(\mathbb{R}^n)$.

**Proof.** In view of the uniqueness results in [21] Theorem 3.1, it suffices to construct a solution to the Stokes problem in the half-space which depends continuously on the data.

Define for each function $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ its upper and lower parts as

$$
\varphi_+(x) := \tilde{\varphi}(x', -x^n), \quad \varphi_-(x) := \tilde{\varphi}(x) \quad \text{for all } x \in \mathbb{R}^n.
$$

Take $(\hat{v}, \hat{q}) \in X_{r'}(\mathbb{R}^n)$ and define their pullbacks into $\mathbb{R}^n$ as follows:

$$
v := \frac{1}{2} (\hat{v}_' + \hat{v}_+ - \hat{v}_-^n)^\top, \quad q := \frac{1}{2} (\hat{q}_- + \hat{q}_+).
$$

It is simple to show that $(v, q) \in X_{r'}(\mathbb{R}^n)$ and

$$
[(v, q)]_{X_{r'}(\mathbb{R}^n)} \leq \left[|(\hat{v}, \hat{q})|\right]_{X_{r'}(\mathbb{R}^n)}.
$$

Let $F \in X_{r'}(\mathbb{R}^n)^*$ be fixed but arbitrary and define $\hat{F}(\hat{v}, \hat{q}) := F(v, q)$. It follows immediately from our current results that $\hat{F}$ is a linear functional on $X_{r'}(\mathbb{R}^n)$ and $\|\hat{F}\|_{X_{r'}(\mathbb{R}^n)^*} \leq \|F\|_{X_{r'}(\mathbb{R}^n)^*}$. Therefore, Theorem 4.3 for $\mathbb{R}^n$ asserts the existence and uniqueness of a solution $(\hat{w}, \hat{\pi})$ in $X_{r'}(\mathbb{R}^n)$ to (1.11) with the forcing function $\hat{F}$, i.e.

$$
S_{\mathbb{R}^n}(\hat{w}, \hat{\pi})(\hat{v}, \hat{q}) = \hat{F}(\hat{v}, \hat{q}) \quad \forall (\hat{v}, \hat{q}) \in X_{r'}(\mathbb{R}^n), \quad (4.6)
$$

and $\|((\hat{w}, \hat{\pi})\|_{X_{r'}(\mathbb{R}^n)} \leq C_{n, r}\|F\|_{X_{r'}(\mathbb{R}^n)^*}$.

Since the test functions $(\hat{v}, \hat{q})$ are arbitrary, we take $(v, q) \in X_{r'}(\mathbb{R}^n)$ and test (4.6) with $(\hat{v}, \hat{q})$ defined as even reflections for $q$ and $v'$, and an odd reflection for $v^n$, i.e.

$$
\hat{q}_+ = \hat{q}_- = q, \quad \hat{v}_+ = \hat{v}_- = v', \quad \hat{v}_n^+ = -\hat{v}_n^- = -v^n.
$$

We can immediately verify that $\hat{F}(\hat{v}, \hat{q}) = F(v, q)$ still holds, and after some technical computations,

$$
S_{\mathbb{R}^n}(\hat{w}, \hat{\pi})(\hat{v}, \hat{q}) = 2S_{\mathbb{R}^n}(u, p)(v, q),
$$
where \( p(x) = \frac{1}{2} (\tilde{\pi}(x) + \tilde{\pi}(x', -x^n)) \) and \( u = (u', u^n) \) is given by

\[
\begin{align*}
u'(x) &= \frac{1}{2} (\tilde{\nu}'(x) + \tilde{\nu}'(x', -x^n)), \\
u^n(x) &= \frac{1}{2} (\tilde{\nu}^n(x) - \tilde{\nu}^n(x', -x^n)).
\end{align*}
\]

Finally, \( u \cdot \nu(x', 0) = u^n(x', 0) = 0 \) and \( (u, p) \) also satisfies estimate (1.12). This concludes the proof. \( \square \)

5. Sobolev Domains and the Piola Transform.

5.1. Local Diffeomorphism on Sobolev Domains. We begin with a definition of the local-diffeomorphism.

Definition 5.1 (diffeomorphism). A map \( \Psi : U \rightarrow V \), where \( U, V \) are open subsets of \( \mathbb{R}^n \), is a \( W^2 \)-diffeomorphism if \( \Psi \) is of class \( W^2 \), is a bijection and \( \Psi^{-1} \) is of class \( W^2 \). A map \( \Psi \) is a local \( W^2 \)-diffeomorphism if for each point \( x \in U \) there exists an open set \( \tilde{U} \subset U \) containing \( x \), such that \( \tilde{U} \subset \mathbb{R}^n \) is open and \( \Psi : U \rightarrow \tilde{U} \) is a \( W^2 \)-diffeomorphism.

Given a \( W^{2-1/s}_s \)-domain \( \Omega \) it is essential to extend the local graph representation \( \omega \) in Definition 5.1 to a smooth, open, bounded subset of \( \mathbb{R}^n \) with the extension being \( W^2 \)-regular.

Definition 5.2 (bubble domain). An open set \( \Theta(x, \delta) \) is called a bubble domain of size \( \delta \) if its \( C^\infty \) boundary is obtained by smoothing the “corners” of the lower half-ball \( B_-(x, \frac{3}{2}\delta) = B_{\mathbb{R}^n}(x, \frac{3}{2}\delta) \) and still arrive at \( B_-(0, \frac{3}{2}\delta) \).

This is depicted in Figure 5.1.

Our strategy consists of the following four steps:

Step 1. In Lemma 5.3 we prove useful norm estimates for \( \omega \in W^{2-1/s}_s(D(x', \delta)) \), where \( D(x', \delta) \) is an open disc of radius \( \delta \) centered at \( x' \).

Step 2. It is not possible to invoke a standard extension of \( \omega \) to the bubble domain \( \Theta(x, \delta) \) and still arrive at \( W^2 \)-regularity for this extension. Therefore we introduce a smooth characteristic function \( \varphi \) in Definition 5.4 which permits us to define a compactly supported function \( \tilde{\varphi} \) on a disc \( D(x', \delta) \) in Lemma 5.5 and to prove norm estimates for \( \tilde{\varphi} \) in terms of \( \omega \).

Step 3. We define an harmonic extension \( \mathcal{E} \omega \) of \( \tilde{\varphi} \) to the bubble domain \( \Theta(x, \delta) \) and prove the \( W^2 \)-norm estimates for \( \mathcal{E} \omega \) in terms of \( \omega \) in Lemma 5.6.

Step 4. We define an extension \( \tilde{\mathcal{E}} \omega := \varphi \mathcal{E} \omega \) upon multiplying \( \mathcal{E} \omega \) by a cutoff function \( \varphi \) which is equal 1 in \( B(x, \delta/2) \) and vanishes outside \( B(x, \delta) \) (see 5.6), and prove its \( W^2 \)-regularity in Lemma 5.7.

This allows us to define \( \Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as follows and prove that \( \Psi \) is \( W^2 \)-regular in Corollary 5.8

\[
\begin{align*}
x' &= \Psi'(x) \\
x^n &= \Psi^n(x) + \tilde{\mathcal{E}} \omega(x) = \Psi^n(x),
\end{align*}
\]

where \( \tilde{\mathcal{E}} \omega \) is the extension of Step 4. We remark that the mapping \( x \mapsto \Psi^{-1}(x) = x \) “straightens out \( \partial \Omega \)”; see Figure 2.1. The construction of such a \( \Psi \) which is \( W^2 \)-regular, on a Sobolev domain which is only \( W^{2-1/s}_s \)-regular, is one of the key contributions of this paper. We point out that a standard diffeomorphism will only be \( W^{2-1/s}_s \)-regular, which is not sufficient for our purposes.
Given \( x \in \partial \Omega \), we rotate \( \Omega \) about \( x = x' \) so that \( \omega \) in Definition 2.1 satisfies additionally
\[
\nabla_{x'} \omega(x') = 0.
\]
We further translate \( \Omega \) in the \( e_n \) direction to enforce
\[
\int_{D(x', \delta)} \omega = 0.
\]

**Lemma 5.3 (properties of \( \omega \)).** Let \( \Omega \) be a domain of class \( W^{2-1/s}_s \), \( x \in \partial \Omega \), and \( \omega \) satisfy (5.2a), (5.3a). Then
\[
\|\omega\|_{L^\infty(D_\delta)} \lesssim \delta^{2-n/s} |\omega|_{W^{2-1/s}_s(D(x', \delta))},
\]
(5.3a)
\[
\|\nabla_{x'} \omega\|_{L^\infty(D_\delta)} \lesssim \delta^{1-n/s} |\omega|_{W^{2-1/s}_s(D(x', \delta))},
\]
(5.3b)
\[
|\omega|_{W^{1-1/s}_s(D_\delta)} \lesssim \delta^n \|\nabla_{x'} \omega\|_{L^\infty(D_\delta)}.
\]
(5.3c)

**Proof.** We proceed in three steps. For simplicity we set \( D_\delta := D(x', \delta) \) and \( \nabla' := \nabla_{x'} \).

1. Invoking (5.2b), in conjunction with Poincaré inequality we deduce
\[
\|\omega\|_{L^\infty(D_\delta)} \lesssim \delta \|\nabla' \omega\|_{L^\infty(D_\delta)}.
\]

2. Expression (5.2a) implies
\[
|\nabla' \omega(y')| = |\nabla' \omega(y') - \nabla' \omega(x')| \lesssim \delta^{1-n/s} |\omega|_{C^{1-1/s}_n(D_\delta)} \lesssim \delta^{1-n/s} |\omega|_{W^{2-1/s}_s(D_\delta)},
\]
because \( W^{2-1/s}_s(D_\delta) \subset C^{1-1/s}_n(D_\delta) \). We thus deduce (5.3b), and combining this with (5.2a) we obtain (5.3a).

3. To prove (5.3c), we use the definition of the \( W^{1-1/s}_s \) seminorm, i.e.
\[
|\omega|_{W^{1-1/s}_s(D_\delta)} = \int_{D_\delta} \int_{D_\delta} \frac{|\omega(x') - \omega(y')|^s}{|x' - y'|^{s+n-2}} \, dx' \, dy',
\]
because the exponent of the denominator is \((n-1) + s(1 - \frac{1}{s}) = s + n - 2\). A direct estimate further yields
\[
|\omega|_{W^{1-1/s}_s(D_\delta)} \leq \|\nabla' \omega\|_s \int_{D_\delta} \int_{D_\delta} \frac{1}{|x' - y'|^{n-2}} \, dx' \, dy',
\]
and the double integral is of order \( \delta^n \). This implies
\[
|\omega|_{W^{1-1/s}_s(D_\delta)} \lesssim \delta^n \|\nabla' \omega\|_s
\]
which is the asserted bound (5.3c). \( \Box \)

**Definition 5.4 (cutoff function).** Given \( x \in \mathbb{R}^n \), a function \( \varrho \) in \( C^\infty_c(\mathbb{R}^n) \) such that \( \varrho = 1 \) in \( B(x, \delta/2) \), \( 0 \leq \varrho \leq 1 \) in \( B(x, \delta) \setminus B(x, \delta/2) \) and \( \varrho = 0 \) in \( \mathbb{R}^n \setminus B(x, \delta) \) will be called a cutoff function of \( B(x, \delta/2) \).
LEMMA 5.5 (compactly supported graph). Let \( \Omega \) be a domain of class \( W^{2-1/s}_s \), \( x \in \partial \Omega \), and \( \omega \) satisfy (5.2a)–(5.2b). Then \( C\omega(y') := \rho(y', \omega(y'))\omega(y') \) for \( y' \in D(x', \delta) \) satisfies \( C\omega \in W^{2-1/s}_s(D(x', \delta)) \), \( \| C\omega \|_{W^{2-1/s}_s(D(x', \delta))} \leq \omega \), and

\[
\| C\omega \|_{W^{2-1/s}_s(D(x', \delta))} \leq \| \omega \|_{W^{2-1/s}_s(D(x', \delta))}.
\]

Proof. We proceed in four steps. For simplicity we again define \( D_\delta = D(x', \delta) \) and \( \nabla' = \nabla_{x'} \).

1. As \( \nabla' C\omega = (\nabla' \rho)\omega + \rho \nabla' \omega \), we deduce

\[
\| \nabla' C\omega \|_{L^\infty(D_\delta)} \leq \| \nabla' \rho \|_{L^\infty(D_\delta)} \| \omega \|_{L^\infty(D_\delta)} + \| \rho \|_{L^\infty(D_\delta)} \| \nabla' \omega \|_{L^\infty(D_\delta)}.
\]

Since \( \| \nabla' \rho \|_{L^\infty(D_\delta)} \leq \delta^{-1} \) and \( \| \omega \|_{L^\infty(D_\delta)} \leq \delta \| \nabla' \omega \|_{L^\infty(D_\delta)} \) (Poincaré inequality), we infer that \( \| \nabla' C\omega \|_{L^\infty(D_\delta)} \leq \| \omega \|_{W^{2-1/s}_s(D_\delta)} \).

2. Invoking the definition of the \( W^{2-1/s}_s \)-seminorm, we get

\[
\| \nabla' C\omega \|_{W^{2-1/s}_s(D_\delta)}^s = \int_{D_\delta} \left( \int_{D_\delta} \frac{|\nabla' C\omega(x') - \nabla' C\omega(y')|^s}{|x' - y'|^{s+n-2}} \right) dx' dy' \\
\leq \int_{D_\delta} \left( \int_{D_\delta} \frac{|\omega(x') \nabla' \rho(x') - \omega(y') \nabla' \rho(y')|^s}{|x' - y'|^{s+n-2}} \right) dx' dy' \\
+ \int_{D_\delta} \left( \int_{D_\delta} \frac{|\rho(x') \nabla' \omega(x') - \rho(y') \nabla' \omega(y')|^s}{|x' - y'|^{s+n-2}} \right) dx' dy' = I + II.
\]

We now proceed to tackle I and II separately.

3. We estimate the term I as follows: we add and subtract \( \omega(x') \nabla' \rho(y') \) to obtain

\[
I \leq \int_{D_\delta} \left( \int_{D_\delta} \frac{|\nabla' \rho(x') - \nabla' \rho(y')|^s}{|x' - y'|^{s+n-2}} \right) dx' dy' \\
+ \int_{D_\delta} \left( \int_{D_\delta} \frac{|\nabla' \rho(y')|^s}{|x' - y'|^{s+n-2}} \right) dx' dy' \\
\leq \| \omega \|_{L^\infty(D_\delta)} \| \nabla' \rho \|_{W^{2-1/s}_s(D_\delta)}^s \leq \| \nabla' \rho \|_{L^\infty(D_\delta)} \| \nabla' \omega \|_{W^{2-1/s}_s(D_\delta)} = III + IV.
\]

4. Introducing a co-ordinate transformation \( x' = \delta x \) implies

\[
|\rho|_{W^{2-1/s}_s(D_\delta)} = \delta^{2/n-1} |\rho|_{W^{2-1/s}_s(D_\delta)}, \\
|\nabla' \rho|_{W^{2-1/s}_s(D_\delta)} = \delta^{2/n-2} |\nabla' \rho|_{W^{2-1/s}_s(D_\delta)}, \\
|\nabla' \rho|_{L^\infty(D_\delta)} = \delta^{-1} |\nabla' \rho|_{L^\infty(D_\delta)},
\]

where \( D_\delta \) denotes the unit disc. We now show that III–VI are \( O(1) \) in \( \delta \). Using (5.3a)–(5.3c), we obtain

\[
III \leq \delta^{(2-\frac{n}{s})} \delta^{(\frac{2}{n}-2)} |\omega|_{W^{2-1/s}_s(D_\delta)} = |\omega|_{W^{2-1/s}_s(D_\delta)},
\]

\[
IV \leq \frac{1}{\delta^{2/n}} \delta^{n} \delta^{(1-\frac{2}{n})} |\omega|_{W^{2-1/s}_s(D_\delta)} = |\omega|_{W^{2-1/s}_s(D_\delta)}.
\]
as well as
\[ V \lesssim \delta^n(\frac{3}{2} - 1)\delta^{n(1 - \frac{3}{2})}|\omega|_{W^{2-1/s}_s(D_{\delta})} = |\omega|_{W^{2-1/s}_s(D_{\delta})}. \]

Since the estimate of \( V \) is immediate, we conclude the proof. \( \Box \)

Next we extend \( C\omega \) by zero to the boundary \( \partial \Theta(x, \delta) \) of the bubble domain \( \Theta(x, \delta) \) but still indicate it by \( C\omega \). We denote by \( \mathcal{E}\omega \) the harmonic extension of \( C\omega \) to \( \Theta(x, \delta) \) [25, Theorem 9.15], and point out that this is not the usual extension which retains the regularity properties of \( C\omega \). We collect the properties of \( \mathcal{E}\omega \) in the following lemma.

**Lemma 5.6** (harmonic extension). Let \( \Omega \) be a domain of class \( W^{2-1/s}_s, x \in \partial \Omega \), and \( \omega \) satisfy \((5.2a)-(5.2b)\). The harmonic extension \( \mathcal{E}\omega \) of \( C\omega \) to \( \Theta(x, \delta) \) satisfies
\[
\|\mathcal{E}\omega\|_{L^\infty(\Theta(x, \delta))} \lesssim \|\omega\|_{L^\infty(D(x', \delta))},
\]
\[
\|\mathcal{E}\omega\|_{W^{2}_s(\Theta(x, \delta))} \lesssim \|\omega\|_{W^{2-1/s}_s(D(x', \delta))}.
\]

**Proof.** As \( \partial \Theta(x, \delta) \) is smooth and \( C\omega \in W^{2-1/s}_s(\partial \Theta(x, \delta)) \), the harmonic extension \( \mathcal{E}\omega \) exists in \( W^{2}_s(\Theta(x, \delta)) \) and satisfies \((5.5b)\) [25, Lemma 9.17]. Expression \((5.5a)\) is due to the maximum principle. \( \Box \)

We remark that Lemma 5.6 is the first place where we have used the bubble domain and the smoothness of its boundary. Since the map \( \Psi \) in \((5.1)\) is defined over \( \mathbb{R}^n \), we introduce an extension \( \tilde{\mathcal{E}}\omega \) of \( \mathcal{E}\omega \) as follows
\[
\tilde{\mathcal{E}}\omega := \rho \mathcal{E}\omega,
\]
where \( \rho \) is the cutoff function in Definition 5.4. In view of \((5.6)\), we remark that \( \Psi \) in \((5.1)\) satisfies \((1.1)-(1.3)\) in \( \Omega \). It remains to show that \( \Psi \) is a local \( W^{2}_s \)-diffeomorphism, but that requires studying the following properties of \( \tilde{\mathcal{E}}\omega \).

**Lemma 5.7** (properties of \( \tilde{\mathcal{E}}\omega \)). Let \( \Omega \) be a domain of class \( W^{2-1/s}_s, x \in \partial \Omega \), and \( \omega \) satisfy \((5.2a)-(5.2b)\). Then
\[
\|\tilde{\mathcal{E}}\omega\|_{W^{2}_s(\Theta(x, \delta))} \lesssim \delta^{1-n/s}|\omega|_{W^{2-1/s}_s(D(x', \delta))},
\]
\[
\|\tilde{\mathcal{E}}\omega\|_{W^{2}_s(\Theta(x, \delta))} \lesssim \|\omega\|_{W^{2-1/s}_s(D(x', \delta))}.
\]

**Proof.** For simplicity we use the notation \( \Theta_\delta := \Theta(x, \delta), D_\delta := D(x', \delta), \) and \( \nabla = \nabla_x \).

1. The proof of \((5.7a)\) is tricky and we split it into two steps. Using the definition of \( \tilde{\mathcal{E}}\omega \) we obtain
\[
\|\nabla \tilde{\mathcal{E}}\omega\|_{L^\infty(\Theta_\delta)} \leq \|\nabla \rho\|_{L^\infty(\Theta_\delta)}\|\mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} + \|\rho\|_{L^\infty(\Theta_\delta)}\|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)}.
\]
As \( \|\nabla \rho\|_{L^\infty(\Theta_\delta)} \lesssim \frac{1}{\delta} \) and \( \|\mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} \lesssim \delta\|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} \) by Poincaré inequality, we deduce
\[
\|\nabla \tilde{\mathcal{E}}\omega\|_{L^\infty(\Theta_\delta)} \lesssim \|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)},
\]

2. Let \( B = B((x', \frac{3n}{4} - \frac{n}{4}, \frac{n}{4}), \frac{1}{4}) \) be the ball of center \((x', \frac{3n}{4} - \frac{n}{4})\) and radius \( \frac{n}{4} \) depicted in Figure 5.1. Adding and subtracting \( \nabla \mathcal{E}\omega := \int_B \nabla \mathcal{E}\omega \) to \( \|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} \), and applying triangle inequality we arrive at
\[
\|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} \leq \|\nabla \mathcal{E}\omega - \nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} + \|\nabla \mathcal{E}\omega\|_{L^\infty(\Theta_\delta)} = I + II.
We deal with terms I and II separately. To estimate I we use Poincaré inequality

$$I = \| \nabla \varepsilon \omega - \nabla \varepsilon \omega \|_{L^\infty(\Theta_d)} \lesssim \delta^{-\frac{7}{2}} \| \nabla \varepsilon \omega \|_{W^2_1(\Theta_d)} = \delta^{-\frac{7}{2}} | \varepsilon \omega |_{W^2_1(\Theta_d)}.$$

On the other hand, we estimate II as follows: $II \leq \int_B |\nabla \varepsilon \omega| \leq \| \nabla \varepsilon \omega \|_{L^\infty(B)}$.

Invoking the interior estimate for derivatives of a harmonic function [24, Theorem 2.10] we obtain

$$\| \nabla \varepsilon \omega \|_{L^\infty(B)} \leq n \delta^{-1} \| \varepsilon \omega \|_{L^\infty(\Theta_d)}.$$

Using (5.5a) followed by (5.3a) yields

$$II \lesssim \delta^{-1} \| \omega \|_{L^\infty(D_{\delta})} \lesssim \delta^{\frac{7}{2}} | \omega |_{W^{2-1/4}(D_{\delta})}.$$

The estimates for I and II yield $|\varepsilon \omega|_{W^2_1(\Theta_d)} \leq \delta^{\frac{7}{2}} | \omega |_{W^{2-1/4}(D_{\delta})}$, whence (5.7a) follows.

To prove (5.7b) we use the definition of $\tilde{\varepsilon} \omega$ to arrive at

$$\| \tilde{\varepsilon} \omega \|_{W^2_1(\Theta_d)} \lesssim \| D^2 \vartheta \|_{L^\infty(\Theta_d)} \| \varepsilon \omega \|_{L^\infty(\Theta_d)} + \| \nabla \vartheta \|_{L^2(\Theta_d)} \| \nabla \varepsilon \omega \|_{L^\infty(\Theta_d)} + \| \nabla \varepsilon \omega \|_{L^\infty(\Theta_d)} + \| \varepsilon \omega \|_{L^\infty(\Theta_d)} + \| \varepsilon \omega \|_{L^\infty(\Theta_d)}.$$

Since $\| D^2 \vartheta \|_{L^\infty(\Theta_d)} \lesssim \delta^{\frac{7}{2}}$ and $\| \varepsilon \omega \|_{L^\infty(\Theta_d)} \lesssim \delta \| \nabla \varepsilon \omega \|_{L^\infty(\Theta_d)}$ (Poincaré inequality), Step 2 shows that the first term is bounded by $|\omega |_{W^{2-1/4}(D_{\delta})}$. Similar arguments, in conjunction with (5.5b), shows that the remaining terms are also bounded by $|\omega |_{W^{2-1/4}(D_{\delta})}$, which is (5.7b).

**Corollary 5.8 (W^2-diffeomorphism).** Let $\Omega$ be a domain of class $W^{2-1/s}_s$, $x \in \partial \Omega$, and $\omega$ satisfy (5.2a)-(5.2b). If $\delta > 0$ is sufficiently small, then $\Psi$ defined in (5.1) is a local $W^2_2$-diffeomorphism and satisfies

$$\| 1 - \det \nabla_x \Psi \|_{L^\infty(\Theta(x, \delta))} \lesssim \delta^{1-n/s} | \omega |_{W^{2-1/4}(D(x', \delta))}.$$

**Proof.** In view of Definition 5.1 and the inverse function theorem, $\Psi$ is a local $W^2_2$-diffeomorphism, if and only if $\nabla \Psi(y)$ is an isomorphism for every $y \in \Theta(x, \delta)$. We first observe that the definition of $\Psi$ in (5.1) and the properties of $\tilde{\varepsilon} \omega$ in Lemma 5.7 yield $\Psi \in W^2_2(\mathbb{R}^n)$. Moreover, (5.1) implies

$$\det \nabla \Psi(y) = 1 - \partial_n \tilde{\varepsilon} \omega(y),$$
whence
\[ \| 1 - \det \nabla_y \Psi \|_{L^\infty(\Theta(x, \delta))} \leq \| \nabla_y \tilde{E} \omega \|_{L^\infty(\Theta(x, \delta))}. \]

Using (5.7a) we obtain (5.8). Finally, upon choosing \( \delta \) small enough in (5.8), for every \( y \in \Theta(x, \delta) \), we obtain \( | 1 - \det \nabla_y \Psi | < 1/2 \), whence \( | \det \nabla_y \Psi(y) | > 1/2 \). Thus, \( \nabla_y \Psi(y) \) is invertible for every \( y \in \Theta(y, \delta) \), which completes the proof.

5.2. Piola Transform. The purpose of this section is to analyze the Piola transform, a mapping which preserves the essential boundary condition \( u \cdot \nu = 0 \) after the boundary of \( \Omega \) has been flattened. We will restrict the presentation to a local \( W^2_s \)-diffeomorphism \( \Psi, s > n \), which maps the reference domain \( Q \) (bounded or unbounded) one-to-one and onto a physical domain \( Q \), i.e.,
\[ \Psi : Q \to Q, \quad x \mapsto \Psi(x), \]
with \( U = \text{supp}(I - \Psi) \) and \( U = \Psi(U) \). The construction of such a \( \Psi \) was the subject of Corollary 5.8, but we do not need in this section to use a particular form of \( \Psi \).

We remark that for the \( \Psi \) constructed in Corollary 5.8, we have \( U = \text{supp}(I - \Psi) \subset B_{-}(x, \delta) \).

Definition 5.9 (Piola transform). Let \( P : Q \to \mathbb{R}^{n \times n} \) and \( P^{-1} : Q \to \mathbb{R}^{n \times n} \) be the maps \( P := J_x^{-1} \nabla_x \Psi \), \( P^{-1} := J_x^{-1} \nabla_x \Psi^{-1} \) with \( J_x = \det \nabla_x \Psi \) and \( J_x = \det \nabla_x \Psi^{-1} \). We say two vector fields \( v : Q \to \mathbb{R}^n \) and \( \nu : Q \to \mathbb{R}^n \) are the Piola transforms of each other if and only if
\[ v \circ \Psi = P \nu, \quad \nu \circ \Psi^{-1} = P^{-1} v, \quad (5.9) \]
In view of the inverse function theorem we also have \( P^{-1} \circ \Psi = J_x((\nabla_x \Psi)^{-1} \circ \Psi) \).

The Piola transform is instrumental in dealing with vector-valued functions because it preserves the divergence and normals as the following identities illustrate.

Lemma 5.10 (Piola identities). If \( v \in [H^1(Q)]^n \) is the Piola transform of \( v \in [H^1(Q)]^n \) and \( q \in H^1(Q) \), then the following statements hold:
\[ \int_Q \nabla_x q \cdot v \, dx = \int_Q \nabla_x q \cdot \nu \, dx, \quad (5.10a) \]
\[ \int_Q q \, \text{div}_x v \, dx = \int_Q q \, \text{div}_x \nu \, dx, \quad (5.10b) \]
\[ \int_{\partial Q} q v \cdot \nu_x \, ds_x = \int_{\partial Q} q v \cdot \nu_x \, ds_x, \quad (5.10c) \]
where \( \nu_x \) and \( \nu_x \) are outward unit normals on \( \partial Q \) and \( \partial Q \) respectively. Moreover,
\[ v \cdot \nu_x \, ds_x = v \cdot \nu_x \, ds_x. \quad (5.11) \]

Proof. The Definition 5.9 is precisely what yields (5.10a). In fact,
\[ \int_Q v \cdot \nabla_x q \, dx = \int_Q (v \circ \Psi) \cdot ((\nabla_x \Psi)^{-T} \circ \Psi) \nabla_x q \, J_x \, dx, \]
and using $P^{-1} \circ \Psi = J_x((\nabla_x \Psi)^{-1} \circ \Psi)$ and Definition 5.9 we deduce

$$
\int_Q (v \circ \Psi) \cdot ((\nabla_x \Psi)^{-T} \circ \Psi) \nabla_x q \, dx = \int_Q ((P^{-1}v) \circ \Psi) \cdot \nabla_x q \, dx = \int_Q v \cdot \nabla_x q \, dx.
$$

Invoking the divergence theorem and (5.10a) we obtain

$$
\int_{\partial Q} qv \cdot \nu_x \, ds_x - \int_Q \text{div}_x v \, dx = \int_Q v \cdot \nabla_x q \, dx = \int_Q v \cdot \nabla_x q \, dx - \int_Q q \text{div}_x v \, dx + \int_{\partial Q} qv \cdot \nu_x \, ds_x
$$

Choosing $q \in H^1_r(Q)$, we obtain (5.10b), which in turn implies (5.10c) and (5.11). □

**Lemma 5.11** ($L^r$-norm equivalence). Let $1 < r < \infty$ and $\nu$ be the Piola transform of $v$. There exists constants $C'$ and $C''$ which depend only on the Lipschitz seminorms of $\Psi$ and $\Psi^{-1}$ such that

$$
C'\|v\|_{L^r(Q)} \leq \|v\|_{L^r(Q)} \leq C''\|v\|_{L^r(Q)}.
$$

**Proof.** This is a trivial consequence of Definition 5.9 because

$$
\|v\|_{L^r(Q)} = \left\|((P^{-1}v) \circ \Psi)\right\|_{L^r(Q)} \leq \left\|P^{-1} \circ \Psi\right\|_{L^\infty(Q)} \|J_x\|_{L^\infty(Q)} \|v\|_{L^r(Q)},
$$

and

$$
\|v\|_{L^r(Q)} = \left\|(Pv) \circ \Psi^{-1}\right\|_{L^r(Q)} \leq \left\|P \circ \Psi^{-1}\right\|_{L^\infty(Q)} \|J_x\|_{L^\infty(Q)} \|v\|_{L^r(Q)}.
$$

This concludes the proof. □

Lemma 5.10 discusses the transformation of the divergence operator from the reference domain $Q$ to the physical domain $Q$. In what follows, we need to transform the gradient operator as well.

**Lemma 5.12** (Piola gradient). Let $v$ and $\nu$ be Piola transforms of each other. The gradient operator admits the following decomposition

$$
\nabla_x v = J_x \Psi^{-1}(\nabla_x v \circ \Psi) + \nabla_x (P^{-1} \circ \Psi) \circ \nu \circ \Psi,
$$

(5.13a)

$$
\nabla_x \nu \circ \Psi = J_x^{-1} \Psi P(\nabla_x \nu) - J_x^{-1} \Psi P \left(\nabla_x (P^{-1} \circ \Psi) \circ P \nu\right),
$$

(5.13b)

where $\Psi$ is defined in (2.2). Moreover, for $s' \leq t^* \leq s$ and $1/t^* = 1/s + 1/t^o$

$$
|v|_{W^{s}_r(Q)} \leq C' \left(|v|_{W^{s}_r(Q)} + \|v\|_{L^{t^*}(U)}\right),
$$

|v|_{W^{s}_r(Q)} \leq C'' \left(|v|_{W^{s}_r(Q)} + \|v\|_{L^{t^*}(U)}\right),
$$

(5.13c)

where the constants $C'$ and $C''$ depend only on $n, r, s$, the Lipschitz and $W^2_s$ seminorms of $\Psi$ and $\Psi^{-1}$, and on the sets $U = \text{supp}(I - \Psi)$ and $U = \Psi(U)$.

**Proof.** To obtain (5.13a) it suffices to differentiate the Piola transform given in (5.9) and note that we use the chain rule to deal with the first term $\nabla_x v$ but not the second one $\nabla_x (P^{-1} \circ \Psi)$. 


To derive (5.13b) we multiply by $P$ on the left and $P^{-1}$ on the right of (5.13a) and then reorder terms
\[ P \left( \nabla_x v - \nabla_x (P^{-1} \circ \Psi) \circ v \circ \Psi \right) P^{-1} = J_x \nabla_x v \circ \Psi, \]
whence, upon using $v \circ \Psi = P v$ from (5.9), we achieve the desired expression.

To show (5.13c) we observe that $\Psi$ is the identity on $Q \setminus U$, whence $|v|_{W^s_r(Q \setminus U)} = |v|_{W^s_r(Q \setminus U)}$. To deal with the remaining part on $U$, we resort to (5.13a) and $\Psi \in W^s_r(U)$. Combining these results yields the first estimate of (5.13c). To obtain the second one, it suffices to follow the same steps above starting with (5.13b).

**Proposition 5.13** (Piola symmetric gradient). Let $v$ be the Piola transform of $v$. The symmetric gradient admits the following decomposition
\[ \varepsilon(v) \circ \Psi = J_x^{-1} \left( \varepsilon_P(v) - \theta_P(v) \right), \quad (5.14a) \]
with
\[ \varepsilon_P(v) := \Pi_P(\varepsilon(v)), \]
\[ \theta_P(v) := \frac{1}{2} \Pi_P \left( \nabla_x (P^{-1} \circ \Psi) \circ P v + \left( \nabla_x (P^{-1} \circ \Psi) \circ P v \right)^\top \right), \quad (5.14b) \]
Moreover, if $\Psi$ is a local $W^2_s$-diffeomorphism, $s > n$, and $t^0, t^\bullet$ satisfy $s' \leq t^0 \leq \infty$, $1 \leq t^\bullet \leq s$, and $1/t^\bullet = 1/s + 1/t^0$, then there holds
\[ \|\varepsilon_P(v)\|_{L^{s'}(Q)} \leq C' \|v\|_{W^s_r(Q)}, \]
\[ \|\theta_P(v)\|_{L^{t^\bullet}(Q)} \leq C'' \|v\|_{L^{t^0}(U)}, \quad (5.14c) \]
with constants $C'$ and $C''$ depending only on $n, r, s, t^0$, the Lipschitz and $W^s_r$ seminorms of $\Psi$ and $\Psi^{-1}$, and on the sets $U = \text{supp}(I - \Psi)$ and $U = \Psi(U)$.

**Proof.** The decomposition follows directly by the definition of the symmetric gradient and (5.13a). The bounds follow from the bounds in Lemma 5.12.

**Theorem 5.14** (space isomorphism). Let $s' \leq r \leq s$ and $\Psi$ be a local $W^s_r$-diffeomorphism between $Q$ and $Q$. The linear operator
\[ \mathcal{P} : X_r(Q) \rightarrow X_{s'}(Q), \quad (v, q) \rightarrow (v, q) = (P v, \Psi^{-1} \circ q) \quad (5.15) \]
is an isomorphism.

**Proof.** We only show that $\mathcal{P}$ is bounded because the same procedure applies to $\mathcal{P}^{-1}$. Consequently, given $(v, q) \in X_r(Q)$, we will show that $(v, q) \in X_{s'}(Q)$ and the norm
\[ \|(v, q)\|_{X_{s'}(Q)} = \|\mathcal{P}(v, q)\|_{X_r(Q)} = \|v\|_{V^r_r(Q)} + \|q\|_{L^{s'}(Q)} \]
is bounded. We first observe that $v \cdot \nu_x = 0$ implies $v \cdot \nu_x = 0$ because of (5.11). In view of Lemma 3.2 and (4.5b), $V^r_r(Q)$ is complete under the semi-norm $|.|_{V^r_r(Q)}$. Owing to (5.13c), for $r = t^\bullet$, we obtain
\[ |v|_{W^s_r(Q)} \lesssim |v|_{W^s_r(Q)} + \|v\|_{L^{t^0}(U)}, \]
and see that the first term is bounded because $v \in V_r(Q)$. To bound the second term, we use the Sobolev embedding theorem $W^{s,2}(U) \subset L^{r}(U)$ with $1/t^o = 1/r - 1/s$ to arrive at $\|v\|_{L^{r}(U)} \lesssim \|v\|_{W^{s,2}(U)}$.

If $Q$ is bounded then Lemma 3.2 implies $\|v\|_{W^{s,2}(U)} \lesssim \|\nabla v\|_{W^{s,2}(Q)} \lesssim \|v\|_{W^{s,2}(Q)}$. If $Q = \mathbb{R}^n$, then $P = I$ and there is nothing to prove. If $Q = \mathbb{R}^n$ then by Remark 3.3 we obtain $\|v\|_{W^{s,2}(U)} \leq \|v\|_{W^{s,2}(Q)}$. Altogether, we conclude that $v \in V_r(Q)$.

It remains to estimate $\|q\|_{L^r(Q)}$. Due to the change of variables and the fact that $\Psi$ is $W^s_2$ with $s > n$, we arrive at $\|q\|_{L^r(Q)} \lesssim \|J_x\|_{L^\infty(Q)} \|q\|_{L^r(Q)}$. This concludes the proof. □

**Remark 5.15 ($W^s_2$-regularity).** The proof of Theorem 5.14 reveals that it is absolutely necessary for $\Psi$ to have two derivatives, i.e. $\Psi \in W^s_2$, for the Piola transform to make sense as an isomorphism between $X_r(Q)$ and $X_r(Q)$. This differs from the canonical use of the Piola transform for $H(\text{div})$ spaces which hinges on (5.11).

**6. The Sobolev Space Case.** We start with a brief summary of index theory and related results following [34]. Let $X$, $Y$ and $Z$ be arbitrary Banach spaces with $X^*$, $Y^*$ and $Z^*$ being their duals.

A (bounded) linear operator $A : X \to Y$ is said to have finite index if it has the following properties:

(i) The nullspace $N_A$ of $A$ is a finite dimensional subspace of $X$.

(ii) The quotient space $Y/R_A$ is finite dimensional, with $R_A$ the range of $A$.

For such an operator we define the index as

$$\text{ind } A := \dim N_A - \dim Y/R_A.$$ 

Two bounded linear operators $A : X \to Y$ and $A^\dagger : Y \to X$ are called pseudoinverses of each other if

$$AA^\dagger = I_Y + K, \quad A^\dagger A = I_X + C,$$

where $K : Y \to Y$ and $C : X \to X$ are compact operators. Every bounded linear operator $A : X \to Y$ has a dual (operator) $A^* : Y^* \to X^*$ given by the relation,

$$\langle A^* y^*, x \rangle_{X^*, X} := \langle y^*, Ax \rangle_{Y^*, Y}, \quad x \in X, \quad y^* \in Y^*.$$

**Lemma 6.1** (index vs pseudoinverse [34] Chapter 27: Theorems 1,2]). A bounded linear operator $A : X \to Y$ has finite index if and only if $A$ has a pseudoinverse. Moreover,

$$\text{ind } A = - \text{ind } A^\dagger.$$

**Lemma 6.2** (compact perturbation [34] Chapter 21: Theorem 3]). Suppose that $A : X \to Y$ has finite index, and $K : X \to Y$ is a compact linear map. Then $A + K$ has finite index and

$$\text{ind}(A + K) = \text{ind } A.$$

**Lemma 6.3** (index of dual [34] Chapter 27: Theorem 4]). Let $A : X \to Y$ be a bounded linear operator. If $A$ has finite index, then so does its dual $A^*$. Moreover,

$$\text{ind } A^* = - \text{ind } A.$$
Corollary 6.4 (invertibility). Let $A : X \to Y$ be a bounded operator with a pseudoinverse. If $A$ and $A^*$ are injective then they are bijective.

Proof. From Lemma 6.1 we have that $A$ has finite index. Since $A$ and $A^*$ are injective, $\dim N_A = \dim N_{A^*} = 0$. According to Lemma 6.3 we have, $-\dim X^*/R_{A^*} = \dim Y/R_A$. Since the dimension of a space is not negative, we obtain

$$\dim X^*/R_{A^*} = \dim Y/R_A = 0,$$

i.e. $A$ and $A^*$ are surjective which concludes our proof. \[\square\]

Our strategy is to use Corollary 6.4 to infer the invertibility of the Stokes operator $\mathcal{S}_\Omega : X_r(\Omega) \to X_r(\Omega)^*$.  

- First, we will decompose $\mathcal{S}_\Omega$ into its interior and boundary parts (see §6.1).
- Second, we will use the boundedness of $\Omega$ to construct a pseudoinverse of $\mathcal{S}_\Omega$, hence showing that it has a finite index (see §6.2).
- Third, and last, we will show that $\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega^*$ are injective (see §6.3).

6.1. Localized Equations. The goal of this section is to localize the Stokes equations. The technique’s essence is to test the Stokes variational system with a cutoff version of a velocity-pressure pair $(v,q)$ defined over an unbounded domain. This exhibits the local behavior, in operator terms, of the Stokes linear map which splits (locally) into bounded operators including invertible $S_{\mathbb{R}^n}$ or $S_{\mathbb{R}^n}$ plus a compact part.

Definition 6.5 (localization operator). Let $\Omega$ be a $W^{2-1/s}$-domain. Let $x \in \Omega$ and $\zeta \in C_\infty(\overline{B(x,\delta)})$. Then for every $s' \leq r \leq s$,

$$\mathcal{R}_\zeta : X_r(Q) \to X_r(\Omega)$$

$$(v,q) \mapsto \zeta P(v,q)$$

$$\mathcal{E}_\zeta : X_r(\Omega) \to X_r(Q)$$

$$(v,q) \mapsto P^{-1}(\zeta v, \zeta q)$$

are localization operators if and only if

- when $x \in \Omega$, then $\delta = \text{dist}(x, \partial \Omega)$, and $\Psi = \mathcal{I}$, $P = \mathcal{I}$ and $Q = Q = \mathbb{R}^n$;
- when $x \in \partial \Omega$, then $\delta > 0$ is sufficiently small so that Corollary 5.3 holds, $U(x,\delta) = \supp(\mathcal{I} - \Psi) \subset B(x,\delta)$, $U(x,\delta) = \Psi(U)$ (see Figure 2.1), and $Q = \mathbb{R}^n$, $Q = \Psi(Q)$.

Lemma 6.6 (continuity). The operators $\mathcal{R}_\zeta$ and $\mathcal{E}_\zeta$ are continuous.

Proof. Since $\zeta$ is smooth, this follows from $P$ and $P^{-1}$ being continuous. This is due to Theorem 6.14 if $x \in \partial \Omega$ and to $P = \mathcal{I}$ if $x \in \Omega$. \[\square\]

Next we state the equations satisfied by the localized Stokes operator using $\mathcal{R}_\zeta$ and $\mathcal{E}_\zeta$. This process is applied both to the interior of $\Omega$ in Proposition 6.7 and to its boundary in Proposition 6.8.

Proposition 6.7 (interior localization). Let $x \in \Omega$. The Stokes (interior) operators $\mathcal{R}_\zeta^* S_{\mathcal{S}} : X_r(\Omega) \to X_r(\mathbb{R}^n)^*$ and $S_{\mathcal{S}} \mathcal{R}_\zeta : X_r(\mathbb{R}^n) \to X_r(\Omega)^*$ are linear continuous and can be written as

$$\mathcal{R}_\zeta^* S_{\mathcal{S}} = \mathcal{S} \mathcal{E}_\zeta + P^* K_\zeta,$$

$$S_{\mathcal{S}} \mathcal{R}_\zeta = \mathcal{E}_\zeta^* \mathcal{S} \mathcal{E}_\zeta + K_\zeta P,$$

where $\mathcal{S} = S_{\mathbb{R}^n}$, $P : X_r(\mathbb{R}^n) \to X_r(\mathbb{R}^n)$ is the identity, $K_\zeta : X_r(\mathbb{R}^n) \to X_r(\mathbb{R}^n)^*$ is given by

$$K_\zeta(u,p)(v,q) := -\langle p, \nabla_x \zeta \cdot v \rangle_{U(x,\delta)} - \langle \nabla_x \zeta \cdot u, q \rangle_{U(x,\delta)}$$

$$-\eta \langle \mathcal{E}_\zeta(u), \mathcal{E}(v) \rangle_{U(x,\delta)} + \eta \langle \mathcal{E}(u), \mathcal{E}(\zeta(v)) \rangle_{U(x,\delta)},$$

(6.1b)
and

\[ \epsilon_\zeta(w) := \frac{1}{2}(\nabla \zeta \otimes w + w \otimes \nabla \zeta), \]

\[ \|\epsilon_\zeta(w)\|_{L^t(U(\omega, \delta))} \lesssim \|w\|_{L^t(U(\omega, \delta))}, \]  
(6.1c)

where \( 1 \leq t \leq \infty \).

**Proof.** Let \((u, p) \in X_r(\Omega), (v, q) \in X_r(\mathbb{R}^n)\) be fixed. To localize \(S_\Omega\), we employ a test pair of the form \(\zeta(v, q)\) and switch the cut-off function \(\zeta\) as a multiplier of the solution pair \((u, p)\). To do so, we first realize that since \(R^*_\zeta S_\Omega\) and \(S_\Omega R^*_\zeta\) are compositions of linear and continuous operators, they are themselves linear and continuous.

To prove (6.1a) we recall Definition 6.5.

\[ R^*_\zeta(v, q) = \zeta P(v, q) = \zeta(v, q), \]

because \(P = I\). We multiply by \(S_\Omega(u, p)\) and rearrange terms to deduce

\[ \langle R^*_\zeta S_\Omega(u, p), (v, q) \rangle_{X_r(\mathbb{R}^n)^*, X_r(\mathbb{R}^n)} = S_\Omega(u, p)(v, \zeta q). \]

We now move the cutoff function \(\zeta\) from \((v, q)\) to \((u, p)\) to obtain

\[ S_\Omega(u, p)(v, \zeta q) = S_\Omega(u, \zeta p)(v, q) + K^*_\zeta(u, p)P(v, q). \]

Since \(P\) is the identity, we further have \((\zeta u, \zeta p) = E_\zeta(u, p)\) thereby getting the first expression of \((6.1a)\). The proof of the second expression is similar and thus omitted for brevity. Using Hölder’s inequality, and smoothness of \(\zeta\), we get the estimate \((6.1c)\). This concludes the proof. \(\square\)

For simplicity for the rest of this section we let \(\eta = 1\).

**Proposition 6.8 (boundary localization).** Let \(x \in \partial \Omega, Q = \Psi(\mathbb{R}^n), U(x, \delta) = \text{supp}(I - \Psi) \subset B_-(x, \delta),\) and \(U(x, \delta) = \Psi(U)\). The Stokes (boundary) operators \(R^*_\zeta S_\Omega : X_r(\Omega) \rightarrow X_r(\mathbb{R}^n)^* \) and \(S_\Omega R^*_\zeta : X_r(\mathbb{R}^n) \rightarrow X_r(\Omega)^*\) are continuous and can be written as

\[ R^*_\zeta S_\Omega = (\tilde{S} + C)E_\zeta + P^*K^*_\zeta, \quad S_\Omega R^*_\zeta = E_\zeta^*(\tilde{S} + C) + K^*_\zeta P, \]
(6.2a)

where \(\tilde{S} := S_Q + B\) with

\[ B(w, \pi)(v, q) := \langle \varepsilon_P(w), J^{-1}_x \varepsilon_P(v) \rangle_{\mathbb{R}^n} - \langle \varepsilon(w), \varepsilon(v) \rangle_{\mathbb{R}^n}, \]
(6.2b)

and

\[ C(w, \pi)(v, q) := \langle \partial_P(w), J^{-1}_x \partial_P(v) \rangle_{U(x, \delta)} \]
\[ - \langle \partial_P(w), J^{-1}_x \varepsilon_P(v) \rangle_{U(x, \delta)} - \langle \varepsilon_P(w), J^{-1}_x \partial_P(v) \rangle_{U(x, \delta)}. \]
(6.2c)

The operators \(\varepsilon_P\) and \(\partial_P\) are defined in \((5.14b)\) and \(K^*_\zeta\) is defined in \((6.1b)\).

**Proof.** Let \((u, p) \in X_r(\Omega)\) and \((v, q) \in X_r(\mathbb{R}^n)\) be fixed. As in Proposition 6.7 we again take \(\zeta(v, q)\) as a test function and switch the cut-off function \(\zeta\) as a multiplier for the solution \((u, p)\). We obtain

\[ S_\Omega(u, p)R^*_\zeta(v, q) = S_{U(\omega, \delta)}(\zeta u, \zeta p)P(v, q) + K^*_\zeta(u, p)P(v, q). \]
Lemma 3.2 again implies

\[
\int_{U(x,\delta)} \zeta_P \text{div}_x v \, dx = \int_{U(x,\delta)} \zeta_P \text{div}_x v \, dx.
\]

Moreover for the symmetric gradient we resort to Proposition 5.13 to write

\[
\int_{U(x,\delta)} \varepsilon(\zeta u) : \varepsilon(v) \, dx = \int_{U(x,\delta)} \big( \varepsilon_P(\zeta u) - \partial P(\zeta u) \big) : \big( \varepsilon_P(v) - \partial P(v) \big) J_x^{-1} \, dx.
\]

We add \( \langle \varepsilon(\zeta u), \varepsilon(v) \rangle \) to create the term \( S_{\mathbb{R}^n}(\zeta u, p)(v, q) \) and subtract it to compensate. The latter, together with the preceding terms give rise to \( B \) in \( (6.2b) \) and \( C \) in \( (6.2c) \). The expression for \( R_{\zeta}S_N \) follows analogously. \( \square \)

Lemma 6.9 \( (K_\zeta \text{ is compact}) \). Let \( s' \leq r \leq s, \ x \in \overline{\Omega} \) and \( Q = \Psi(Q) \), where \( Q = \mathbb{R}^n \) (if \( x \in \Omega \)), \( Q = \mathbb{R}_+^n \), (if \( x \in \partial\Omega \)). The operator \( K_\zeta : X_r(Q) \to X_{r'}(Q)^* \) is compact.

Proof. Let \( \{ (u_\ell, p_\ell) \}_{\ell \in \mathbb{N}} \subset X_r(Q) \) be a bounded sequence. Since \( X_r(Q) \) is reflexive, there exists a subsequence \( \{ (u_\ell, p_\ell) \}_{\ell \in \mathbb{N}} \) (not relabeled) such that

\( (u_\ell, p_\ell) \to (u, p) \) in \( X_r(\Omega) \),

and due to Rellich-Kondrachov theorem (cf. [II Theorem 6.2])

\( \nabla u_\ell \to \nabla u \) in \( W_r^1(\Omega)^* \), \( p_\ell \to p \) in \( W_r^1(\Omega)^* \).

For simplicity we denote \( U = U(x, \delta) \). We need to prove

\[
K_\zeta(u_\ell, p_\ell) \to K_\zeta(u, p) \text{ in } X_{r'}(Q)^*.
\]

We will proceed in several steps. We write \( (6.1b) \) as follows

\[
K_\zeta(u_\ell, p_\ell)(v, q) = I + II + III + IV.
\]

In view of Lemma 3.2 we estimate I as

\[
|I| \lesssim \|p_\ell\|_{W_r^1(U)} \cdot \|\nabla_X \zeta \cdot v\|_{W_r^1(U)} \\
\lesssim \|p_\ell\|_{W_r^1(U)} \cdot \|v\|_{W_r^1(U)} \lesssim \|p_\ell\|_{W_r^1(U)} \cdot |v|_{W_r^1(U)} \lesssim \|p_\ell\|_{W_r^1(U)} \cdot |v|_{W_r^1(U)}.
\]

For II and III we use the definition \( (6.1c) \) of \( \varepsilon_\zeta \) to obtain

\[
|II|, |III| \lesssim \|u_\ell\|_{L_r(\Omega)} \left( \|q\|_{L_r'(\Omega)} + \|v\|_{W_r^1(\Omega)} \right) \lesssim \|u_\ell\|_{L_r(\Omega)} \left( \|q\|_{L_r'(Q)} + \|v\|_{W_r^1(Q)} \right).
\]

Lemma 3.2 again implies

\[
|IV| \lesssim \|\varepsilon(u_\ell)\|_{W_r^1(U)} \cdot \|v\|_{W_r^1(U)} \\
\lesssim \|\varepsilon(u_\ell)\|_{W_r^1(\Omega)} \cdot \|v\|_{W_r^1(\Omega)} \lesssim \|\varepsilon(u_\ell)\|_{W_r^1(\Omega)} \cdot |v|_{W_r^1(Q)}.
\]
Collecting estimates, we obtain for all \((v, q) \in X_r(Q)\)

\[
\sup_{(v, q) \in X_r(Q)} \left| \mathcal{K}_\mathcal{C}(u_\ell, p_\ell)(v, q) - \mathcal{K}_\mathcal{C}(u, p)(v, q) \right| \\
\lesssim \left\| p_\ell - p \right\|_{W_r^s(\Omega)} + \left\| u_\ell - u \right\|_{L^r(\Omega)} + \left\| \varepsilon(u_\ell) - \varepsilon(u) \right\|_{W_r^s(\Omega)}.
\]

which tends to 0 as \(\ell \to \infty\) and implies (6.3). This concludes the proof. \(\square\)

**Lemma 6.10** \((\mathcal{C} \text{ is compact})\). Let \(s' \leq r \leq s\) and \(x \in \partial \Omega\). The operator \(\mathcal{C} : X_r(\mathbb{R}^n) \to X_r(\mathbb{R}^n)^*\) defined in (6.2) is compact.

**Proof.** The starting point of the proof is the same as in Lemma 6.9. For simplicity we set \(U = U(x, \delta)\) and \(U = U(x, \delta)\). Let \(\{(u_\ell, p_\ell)\}_{\ell \in \mathbb{N}} \subset X_r(\mathbb{R}^n)\) be a bounded sequence. Since \(X_r\) is reflexive, there exists a subsequence \(\{(u_\ell, p_\ell)\}_{\ell \in \mathbb{N}}\) (not relabeled), such that

\[
u_\ell \to u \quad \text{in} \quad V_r(\mathbb{R}^n), \quad p_\ell \to p \quad \text{in} \quad L^r(\mathbb{R}^n).
\]

Setting

\[
\frac{1}{r} = \frac{1}{s} + \frac{1}{r'}, \quad \frac{1}{r'} = \frac{1}{s} + \frac{1}{(r')^s};
\]

the following embeddings are compact

\[
W_r^1(U) \subset L^{r'}(U), \quad W_r^1(U) \subset L^{(r')^s}(U).
\]

We rewrite \(\mathcal{C}(u_\ell, p_\ell)(v, q) = I + II + III\) with

\[
I = \langle \partial_p(u_\ell), J_x^{-1}\partial_p(v) \rangle_u, \quad II = -\langle \partial_p(u_\ell), J_x^{-1}\varepsilon_P(v) \rangle_u, \quad III = -\langle \varepsilon_P(u_\ell), J_x^{-1}\partial_p(v) \rangle_u,
\]

and estimate each term separately. The estimate for \(I\) reads

\[
\left| I \right| \lesssim \left\| \partial_p(u_\ell) \right\|_{L^r(U)} \left\| \partial_p(v) \right\|_{L^{r'}(U)} \lesssim \left\| u_\ell \right\|_{L^{r}(U)} \left\| v \right\|_{L^{(r')^s}(U)} \lesssim \left\| u_\ell \right\|_{L^{r}(U)} \left\| v \right\|_{W_r^2(\mathbb{R}^n)};
\]

in view of (5.14c) as well as \(\left\| v \right\|_{L^{(r')^s}(U)} \lesssim \left\| v \right\|_{W_r^1(U)} \lesssim \left\| v \right\|_{W_r^2(U)}\), the latter being a consequence of Remark 3.3.

Applying Hölder's inequality and using \(\Psi \in W_r^2(\mathbb{R}^n)\) in conjunction with (5.14c), we obtain

\[
\left| II \right| \lesssim \left\| \partial_p(u_\ell) \right\|_{L^r(U)} \left\| \varepsilon_P(v) \right\|_{L^{r'}(U)} \lesssim \left\| u_\ell \right\|_{L^{r}(U)} \left\| v \right\|_{W_r^2(\mathbb{R}^n)}.
\]

Instead of directly estimating \(III\), for every \((v, q) \in X_r(\mathbb{R}^n)\), we consider

\[
\left| \langle \varepsilon_P(u_\ell - u), J_x^{-1}\partial_p(v) \rangle_u \right|,
\]

where we have used the linearity of \(\varepsilon_P(\cdot)\). As \(\varepsilon_P(\cdot) : W_r^1(U) \to L^r(U)\) is continuous and \(J_x^{-1}\partial_p(v) \in L^{r'}(U)\), we deduce that

\[
\lim_{\ell \to \infty} \left| \langle \varepsilon_P(u_\ell - u), J_x^{-1}\partial_p(v) \rangle_u \right| = 0.
\]
Combining the estimates for I, II, III, we obtain
\[
\lim_{\ell \to \infty} |C(u_\ell, p_\ell)(v, q) - C(u, p)(v, q)| = 0,
\]
for every \((v, q) \in X_r(\mathbb{R}^n)\). This completes the proof. \(\square\)

Now that we have obtained a local decomposition of \(S_\Omega\) it is important to show that the perturbed Stokes operator \(\tilde{S}\) in Propositions 6.7 and 6.8 is invertible and enjoys the same smoothing property as \(S_{\mathbb{R}^n}\) and \(S_{\mathbb{R}^n}\). The strategy is to use Neumann perturbation theorem [32] Chapter 4: Theorem 1.16, [24] Lemma 3.1 which we restate in a form that suits our needs.

**Lemma 6.11** (perturbation of identity). Consider two Banach spaces \(X\) and \(Y\), and two bounded linear operators \(A\) and \(B\) from \(X\) to \(Y\). Suppose \(A\) has a bounded inverse from \(Y\) to \(X\) and that
\[
\|Bx\|_Y \leq C\|Ax\|_Y \quad \forall x \in X,
\]
with a constant \(0 < C < 1\). Then \(A + B : X \to Y\) is bijective with a bounded inverse.

**Theorem 6.12** (well-posedness of \(\tilde{S}\)). Let \(s' \leq r \leq s\) and \(x \in \partial \Omega\). There exists a constant \(C = C(n, r, s, \partial \Omega)\) such that if \(\delta \leq C\), then the (perturbed) Stokes problem
\[
\tilde{S}(w, \pi) = F
\]
is well-posed from \(X_r(Q)\) to \(X_r(Q)^*\), where \(Q = \mathbb{R}^n\) if \(x \in \Omega\) or \(Q = \mathbb{R}^n_+\) if \(x \in \partial \Omega\).

Additionally, if \(r < t \leq s\) and \(F \in X_t(Q)^*\), then \((w, \pi) \in X_t(Q)\). We recall that \(\delta \leq C\) is defined in (5.14b) and assume for the remainder of this proof that \(\|\cdot\|_{W^{1,r}(\mathbb{R}^n)} = 1\). We can readily estimate I using (5.14c) and (5.8)
\[
|I| \leq C\|1 - \nabla x \Psi\|_{L^\infty(U)}\|w\|_{W^{1,r}(U)} \leq C\delta^{1-n/s}\|w\|_{W^{1,r}(U)}.
\]

The constant \(C\) depends on \(n, r, s, U, \|\Psi\|_{W^{2}(\mathbb{R}^n)}\) but is independent of \(\delta\). Next note that \(\Psi_P(M) = PMP^{-1}\) for any matrix \(M\), according to (2.2), whence
\[
\Psi_P(M) - M = \nabla x \Psi M (\nabla x \Psi)^{-1} - M
= (\nabla x \Psi - I) M (\nabla x \Psi)^{-1} + M (\nabla x \Psi)^{-1} (I - \nabla x \Psi).
\]
We recall that \( \tilde{\omega} = \Psi - \mathcal{I} \), whence \( \| \mathcal{I} - \nabla_s \Psi \|_{L^\infty(\Omega)} = \| \tilde{\omega} \|_{H^{-1}(\Omega)} \), and apply the preceding expression in conjunction with \( (5.7a) \) to bound the remaining terms II and III as follows:

\[
\| \Pi_1 \|, \| \Pi_3 \| \leq C \delta^{1-n/s} |w|_{W^{1\gamma}(\Omega)},
\]

Collecting the estimates for I, II and III, and using the invertibility of \( S_{\mathbb{R}^n} \), we obtain

\[
\begin{aligned}
\| B(w, \pi) \|_{X_s(\mathbb{R}^n)^*} &\leq C \delta^{1-n/s} \| (w, \pi) \|_{X_s(\mathbb{R}^n)} = C \delta^{1-n/s} \left\| S_{\mathbb{R}^n}^{-1} S_{\mathbb{R}^n} (w, \pi) \right\|_{X_s(\mathbb{R}^n)} \\
&\leq C \delta^{1-n/s} \| S_{\mathbb{R}^n}^{-1} \|_{L(X_s(\mathbb{R}^n)^*, X_s(\mathbb{R}^n))} \| S_{\mathbb{R}^n} (w, \pi) \|_{X_s(\mathbb{R}^n)},
\end{aligned}
\]

where \( C \) depends on \( n, r, s, U \) and \( \| \Psi \|_{W^{1\gamma}(\mathbb{R}^n)} \). By choosing \( \delta \) small enough we satisfy the assumption of Lemma \( 6.11 \) and conclude the first part of our theorem.

To prove the second part involving further regularity in \( X_t(Q) \) for \( t > r \) we simply follow Galdi-Simader-Sohr [24, p. 159]. This concludes the proof. \( \Box \)

6.2. \( S_\Omega \) has finite index. In this section we prove that \( S_\Omega \) has finite index or equivalently, according to Lemma \( 6.1 \), that \( S_\Omega \) has a pseudo-inverse.

**Lemma 6.13** (domain decomposition). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain of class \( W^{2,1/2}_x \). There exists a finite open covering of \( \Omega \subset \bigcup_{i=1}^k B(x_i, \delta_i/2) \), such that

(i) if \( x_i \in \Omega \) then \( B(x_i, \delta_i/2) \cap \partial \Omega = \emptyset \).

(ii) if \( x_i \in \partial \Omega \) then the associated local \( W^{2,1}_x \)-diffeomorphism \( \Psi_i \) is a bijection between \( U(x_i, \delta) = \text{supp}(I - \Psi_i) \subset B(x_i, \delta) \) and \( U(x_i, \delta) = \Psi_i(U(x_i, \delta)) \) with the disc \( D(x'_i, \delta_i/2) \) being mapped to \( \partial \Omega \cap B(x_i, \delta_i/2) \) (see Figure 2.7), i.e. it flattens the boundary of \( \Omega \) near \( x_i \).

(iii) the (perturbed) Stokes operator \( \tilde{S}_i \) is invertible.

**Proof.** Since \( \Omega \) is compact, the trivial covering generated by the \( \{ B(x_i, \delta_i/2) \} \), with \( x_i \in \Omega \) and \( \delta_i \) computed in Theorem \( 6.12 \) has a finite sub-covering. Results (i)-(iii) follow immediately. \( \Box \)

We subordinate to the finite covering of Lemma \( 6.13 \) the following set of functions:

(a) a smooth partition of unity \( \{ \phi_i \}_{i=1}^k \) of \( \Omega \), i.e. \( \phi_i \in C_0^\infty(\Omega) \) with \( 0 \leq \phi_i \leq 1 \), \( \sum_{i=1}^k \phi_i(x) = 1 \) for every \( x \in \Omega \).

(b) smooth characteristic functions \( \{ \rho_i \}_{i=1}^k \) of \( B(x_i, \delta_i/2) \) with support on \( B(x_i, \delta_i) \), i.e. \( \rho_i \in C_0^\infty(\Omega) \) with \( \rho_i = 1 \) on \( B(x_i, \delta_i) \).

**Lemma 6.14** (space decomposition of \( X_r(\Omega) \)). Let \( s' \leq r \leq s \). The following identities hold

\[
\mathcal{I}_{X_r(\Omega)} = \sum_{i=1}^k \mathcal{R}_{\phi_i} \mathcal{E}_{\rho_i}, \quad \mathcal{I}_{X_r(\Omega)^*} = \sum_{i=1}^k \mathcal{E}_{\phi_i}^* \mathcal{R}_{\phi_i}^*.
\]

As a result we may decompose \( X_r(\Omega) \) and \( X_r(\Omega)^* \) as follows:

\[
X_r(\Omega) = \sum_{i=1}^k \mathcal{R}_{\phi_i} \mathcal{X}_r(Q_i), \quad X_r(\Omega)^* = \sum_{i=1}^k \mathcal{E}_{\phi_i}^* \mathcal{X}_r(Q_i)^*.
\]

where \( Q_i = \mathbb{R}^n \) or \( \mathbb{R}^n \).

**Proof.** We begin by proving the relation for \( \mathcal{I}_{X_r(\Omega)} \) and simultaneously show the decomposition of \( X_r(\Omega) \). Let \( (u, p) \) in \( X_r(\Omega) \) be fixed but arbitrary. Using that
$\varphi_i \varrho_i = \varphi_i$ and $\sum_{i=1}^k \varphi_i = 1$, Definition 6.5 implies

$$(u, p) = \sum_{i=1}^k \varphi_i \varrho_i (u, p) = \sum_{i=1}^k \varphi_i \mathcal{P}_i \mathcal{P}_i^{-1} (\varrho_i u, \varrho_i p) = \sum_{i=1}^k \mathcal{R}_{\varphi_i} \mathcal{E}_{\varphi_i} (u, p).$$

This proves the identity relation for $\mathcal{I}_{X_r(\Omega)}$.

Since the operators $\mathcal{E}_{\varrho_i} : X_r(\Omega) \to X_r(\Omega_i)$ are continuous, due to Lemma 6.6, it follows that the “vector” $(\mathcal{E}_{\varrho_i} (u, p))_{i=1}^k \in \prod_{i=1}^k X_r(\Omega_i)$. Conversely, if $(u_i, p_i)_{i=1}^k \in X_r(\Omega_i)$, then the continuity of $\mathcal{R}_{\varphi_i} : X_r(\Omega_i) \to X_r(\Omega)$, implies $\mathcal{R}_{\varphi_i} (u_i, p_i) \in X_r(\Omega)$ and $\sum_{i=1}^k \mathcal{R}_{\varphi_i} (u_i, p_i) \in X_r(\Omega)$ because $k$ is finite. This implies the decomposition of $X_r(\Omega)$.

The remaining decompositions of $\mathcal{I}_{X_r(\Omega)}$ and $X_r(\Omega)^*$ can be proven similarly. \(\Box\)

**Theorem 6.15** (pseudoinverse of $S_{\Omega}$). Let $s' \leq r \leq s$, $S_{\Omega} : X_r(\Omega) \to X_r(\Omega)^*$ be the Stokes operator defined in (1.11), and $\tilde{S}_{i} : X_r(\Omega_i) \to X_r(\Omega_i)^*$ be the perturbed Stokes operator defined in Propositions 6.7 and 6.8. The operator $S_{\Omega}^\dagger : X_r(\Omega)^* \to X_r(\Omega)$

$$S_{\Omega}^\dagger := \sum_{i=1}^k \mathcal{R}_{\varrho_i} \tilde{S}_i^{-1} \mathcal{R}_{\varphi_i}^*$$

is a pseudoinverse of $S_{\Omega}$.

**Proof.** To simplify the notation take $C_i = 0$ whenever $x_i \in \Omega$. In view of Propositions 6.7 and 6.8, we can write

$$S_{\Omega}^\dagger S_{\Omega} = \sum_{i=1}^k \mathcal{R}_{\varrho_i} \tilde{S}_i^{-1} (\tilde{S}_i \mathcal{E}_{\varphi_i} + C_i \mathcal{E}_{\varphi_i} + \mathcal{P}_i^* \mathcal{K}_{\varphi_i})$$

$$= \sum_{i=1}^k \mathcal{R}_{\varrho_i} \mathcal{E}_{\varphi_i} + \sum_{i=1}^k \mathcal{R}_{\varrho_i} \tilde{S}_i^{-1} (C_i \mathcal{E}_{\varphi_i} + \mathcal{P}_i^* \mathcal{K}_{\varphi_i})$$

$$= \mathcal{I}_{X_r(\Omega)} + \sum_{i=1}^k \mathcal{R}_{\varrho_i} \tilde{S}_i^{-1} (C_i \mathcal{E}_{\varphi_i} + \mathcal{P}_i^* \mathcal{K}_{\varphi_i}).$$

Since $C_i$ and $\mathcal{K}_{\varphi_i}$ are compact, according to Lemmas 6.9 and 6.10 and so is $S_{\Omega}^\dagger S_{\Omega} - \mathcal{I}_{X_r(\Omega)}$, Chapter 21: Theorem 1]. The proof that $S_{\Omega} S_{\Omega}^\dagger - \mathcal{I}_{X_r(\Omega)^*}$ is compact follows along the same lines. \(\Box\)

**6.3. $S_{\Omega}$ and $S_{\Omega}^\dagger$ are injective.** In view of our strategy to use Corollary 6.4 to infer the invertibility of the Stokes operator $S_{\Omega}$, it is essential to prove the injectivity of $S_{\Omega}$ and $S_{\Omega}^\dagger$. This is precisely the aim of this section. We proceed by making an immediate observation.

**Proposition 6.16.** Let $2 \leq r \leq s$. The Stokes operator $S_{\Omega} : X_r(\Omega) \to X_r(\Omega)^*$ is injective.

**Proof.** Owing to Theorem 3.5, if $(u, p) \in X_2(\Omega)$ solves $S_{\Omega}(u, p) = 0$, then $(u, p) = (0, 0)$. To prove the assertion for $r > 2$ we use that $\Omega$ is bounded so that $X_r(\Omega) \hookrightarrow X_2(\Omega)$, and as a consequence any solution $(u, p) \in X_r(\Omega)$ of $S_{\Omega}(u, p) = 0$ necessarily belongs to $X_2(\Omega)$. We conclude that $S_{\Omega}$ is injective for $2 \leq r \leq s$. \(\Box\)

To prove that $S_{\Omega}$ is injective for $s' \leq r < 2$ we will develop an induction argument; for a somewhat related result see [24] pg. 159. It seems that the induction argument is rooted in the work introduced by Moser in the context of elliptic PDEs.
Section 8.5]. Leveraging the boundedness of \( \Omega \) it is sufficient to prove this argument for \( r = s' \). We will show in a finite number of steps that a homogeneous solution in \( X_{s'}(\Omega) \) is in fact in \( X_t(\Omega) \) for some \( t \geq 2 \).

**Definition 6.17** (smoothing sequence). Let \( s > n \) and \( t_{-1} = r = s' \). We introduce a smoothing sequence

\[
\frac{1}{t_0} := 1 - \frac{2}{s + n}, \quad \frac{1}{t_m} := \frac{1}{t_{m-1}} + \frac{1}{s} - \frac{1}{n} \quad \text{for } m = 1, \ldots, M
\]

where \( M = \left( \left( \frac{1}{n} - \frac{1}{s} \right)^{-1} \left( \frac{1}{2} - \frac{2}{s + n} \right) \right) \) guarantees that \( t_M \geq 2 \).

**Remark 6.18** (properties of \( t_m \)). We observe that \( t_m \) is monotone increasing because \( s > n \). Moreover, \( t_{m-1} < 2 \leq n \) implies \( t_m' < n \) for \( m \geq 1 \). In fact, the final conclusion is due to the definitions of \( t_m \) and \( t_m' \), namely

\[
\frac{1}{t_m'} = 1 - \frac{1}{t_m} = \frac{s'}{t_m} = \frac{1}{t_{m-1}} + \frac{1}{n},
\]

and that \( t_{m-1} \) is monotone increasing with \( t_{m-1} > s' = t_{-1} \).

**Lemma 6.19** (Sobolev embedding). Let \( \mathcal{D} \subset \mathbb{R}^n \) be a bounded Lipschitz domain. The following holds for \( m \geq 0 \) and \( t_{m-1} < 2 \leq n \)

\[
W_{t_{m-1}}^1(\mathcal{D}) \hookrightarrow L_t^m(\mathcal{D}), \quad W_{t_m'}^1(\mathcal{D}) \hookrightarrow L_{t_{m-1}}^r(\mathcal{D}).
\]

Equivalently, for every \( u \) in \( W_{t_{m-1}}^1(\mathcal{D}) \) and \( v \) in \( W_{t_m'}^1(\mathcal{D}) \),

\[
\|u\|_{L_t^m(\mathcal{D})} \leq C_{m,n,s,\mathcal{D}} \|u\|_{W_{t_{m-1}}^1(\mathcal{D})}, \quad \|v\|_{L_{t_{m-1}}^r(\mathcal{D})} \leq C_{m,n,s,\mathcal{D}} \|v\|_{W_{t_m'}^1(\mathcal{D})}.
\] (6.4)

**Proof.** We split the proof into two cases. We recall the Sobolev number \( t^* \) associated with \( t < n \):

\[
\frac{1}{t^*} = \frac{1}{t} - \frac{1}{n}.
\]

1. Case \( m = 0 \): Since \( t_{-1} = s' < 2 \leq n \), \( W_{t_{-1}}^1(\mathcal{D}) \hookrightarrow L_t^0(\mathcal{D}) \) if \( t_0 \leq t^*_{-1} \). This is true because

\[
\frac{1}{t_0} - \frac{1}{t^*_{-1}} = 1 - \frac{2}{s + n} - \left( \frac{1}{t_{-1}} - \frac{1}{n} \right) = \frac{1}{s} + \frac{1}{n} - \frac{2}{s + n} = \frac{n^2 + s^2}{sn(s + n)} > 0.
\]

For the second embedding we note \( t'_{-1} = s \), and \( t'_0 = (s + n)/2 > n \), thus we have \( W_{t'_0}^1(\mathcal{D}) \hookrightarrow L_{t'_{-1}}^r(\mathcal{D}) \).

2. Case \( m \geq 1 \): Since \( t_{m-1} < 2 \leq n \) and \( W_{t_{m-1}}^1(\mathcal{D}) \hookrightarrow L_{t_{m-1}}^{t^*_m}(\mathcal{D}) \), to get the first embedding, it suffices to verify \( t_m \leq t^*_m \):

\[
\frac{1}{t_m} - \frac{1}{t^*_m} = \frac{1}{t_{m-1}} + \frac{1}{s} - \frac{1}{n} - \left( \frac{1}{t_{m-1}} - \frac{1}{n} \right) = \frac{1}{s} \geq 0.
\]
Using Definition 6.17, we deduce \( t'_{m-1} \leq (t'_m)\) whence \( W^1_{t'_m}(D) \hookrightarrow L^{t'_{m-1}}(D) \). This completes the proof.

**Lemma 6.20** (interior regularity of homogeneous solution). Let \( \mathbf{x} \in \Omega \). If \((\mathbf{u}, p) \in X_{t_m-1}(\Omega)\) satisfies \( S_\Omega(\mathbf{u}, p) = 0 \), then \( E_\zeta (\mathbf{u}, p) \in X_{t_m}(\mathbb{R}^n) \).

**Proof.** Since \( S_\Omega(\mathbf{u}, p) = 0 \) we have from (6.1a) that
\[
\tilde{S}E_\zeta (\mathbf{u}, p) = -\mathcal{P}^*K_\zeta (\mathbf{u}, p).
\]

As \( \mathcal{P} \) is an isomorphism (cf. Theorem 5.14), the strategy is to show that \( K_\zeta (\mathbf{u}, p) \in X_{t'_m}(\mathbb{R}^n)^* \) provided \((\mathbf{u}, p) \in X_{t_m-1}(\mathbb{R}^n)\) and use Theorem 6.12 to conclude that \( E_\zeta (\mathbf{u}, p) \in X_{t_m}(\mathbb{R}^n) \). For simplicity we use the notation \( \tilde{U} = U(\mathbf{x}, \delta) \).

Let \((\mathbf{v}, q) \in X_{t'_m}(\mathbb{R}^n)\). In view of the definition (6.1b), we split \( K_\zeta (\mathbf{u}, p) \) into four terms
\[
I = -\langle p, \nabla_x \zeta \cdot \mathbf{v} \rangle_{U}, \quad II = -\langle \nabla_x \zeta \cdot \mathbf{u}, q \rangle_{U},
\]
\[
III = -\langle (\zeta(\mathbf{u}), (\mathbf{v})) \rangle_{U}, \quad IV = \langle (\zeta(\mathbf{u}), (\mathbf{v})) \rangle_{U},
\]
and estimate them separately. Invoking Hölder’s inequality and (6.4), we obtain
\[
|I| \lesssim \|p\|_{L^{t_m-1}(U)} \|\mathbf{v}\|_{L^{t'_m}(U)} \lesssim \|p\|_{L^{t_m-1}(U)} \|\mathbf{v}\|_{W^{1}_{t'_m}(U)},
\]
\[
|II| \lesssim \|\mathbf{u}\|_{L^{t_m}(U)} \|q\|_{L^{t'_m}(U)} \lesssim \|\mathbf{u}\|_{W^{1}_{t_m}(U)} \|q\|_{L^{t'_m}(U)}.
\]

Using Hölder’s inequality again, this time in conjunction with (6.1c) and (6.4), we see that
\[
|III| \lesssim \|\mathbf{u}\|_{L^{t_m}(U)} \|\mathbf{v}\|_{L^{t'_m}(U)} \lesssim \|\mathbf{u}\|_{W^{1}_{t_m}(U)} \|\mathbf{v}\|_{W^{1}_{t'_m}(U)}.
\]
\[
|IV| \lesssim \|\mathbf{u}\|_{L^{t_m}(U)} \|\mathbf{v}\|_{L^{t'_m}(U)} \lesssim \|\mathbf{u}\|_{L^{t_m}(U)} \|\mathbf{v}\|_{W^{1}_{t'_m}(U)}.
\]

Since \( \|\mathbf{v}\|_{W^{1}_{t'_m}(U)} \lesssim \|\mathbf{v}\|_{W^{1}_{t'_m}(\Omega)} \lesssim \|\mathbf{v}\|_{W^{1}_{t'_m}(\mathbb{R}^n)} \), the latter being the norm of \( V_{t'_m}(\mathbb{R}^n) \) according to (4.1) and (4.5a), we deduce
\[
|K_\zeta (\mathbf{u}, p)(\mathbf{v}, q)| \lesssim \left( \|\mathbf{u}\|_{W^{1}_{t_m}(\Omega)} + \|p\|_{L^{t_m-1}(\Omega)} \right) \|\mathbf{v}\|_{X_{t'_m}(\mathbb{R}^n)}. \]

Finally, using Lemma 3.2 (Korn’s inequality) namely \( \|\mathbf{u}\|_{L^{t_m}(U)} \lesssim \|\zeta(\mathbf{u})\|_{L^{t_m-1}(\Omega)} \lesssim \|\mathbf{u}\|_{W^{1}_{t_m}(\Omega)} \), we obtain
\[
\|K_\zeta (\mathbf{u}, p)\|_{X_{t'_m}(\mathbb{R}^n)} \leq \|\mathbf{u}\|_{X_{t_m-1}(\Omega)}. \]

This completes the proof.

**Lemma 6.21** (boundary regularity of homogeneous solution). Let \( \mathbf{x} \in \partial \Omega \). If \( (\mathbf{u}, p) \in X_{t_m-1}(\Omega) \) satisfies \( S_\Omega(\mathbf{u}, p) = 0 \), then \( E_\zeta (\mathbf{u}, p) \in X_{t_m}(\mathbb{R}^n) \).

**Proof.** Since \( S_\Omega(\mathbf{u}, p) = 0 \) we have from (6.2a)
\[
\tilde{S}E_\zeta (\mathbf{u}, p) = -\mathcal{P}^*K_\zeta (\mathbf{u}, p).
\]

The strategy is the same as in Lemma 6.20, we show that the right-hand-side is in \( X_{t'_m}(\mathbb{R}^n)^* \) provided \((\mathbf{u}, p) \in X_{t_m-1}(\mathbb{R}^n)\), and use Theorem 6.12 to conclude that \( E_\zeta (\mathbf{u}, p) \in X_{t_m}(\mathbb{R}^n) \). In particular, the regularity for \( K_\zeta (\mathbf{u}, p) \) follows from the exact
We only need to prove the additional regularity for \( C_\varepsilon(u, p) \).

Let \((v, q) \in X_{t_m}(\mathbb{R}^n)\) be fixed but arbitrary and set \((u, p) = C_\varepsilon(u, p)\). In view of definition \(6.2c\), we split \( C(u, p)(v, q) \) into three terms

\[
I = \langle \vartheta_p(u), J_{n-1}\vartheta_p(v) \rangle_U, \quad II = -\langle \vartheta_p(u), J_{n-1}\varepsilon_p(v) \rangle_U, \\
III = -\langle \varepsilon_p(u), J_{n-1}\vartheta_p(v) \rangle_U,
\]

where \( U = U(x, \delta) = \Psi(U) \) is the physical domain and \( U \) is the bubble domain (see Figure 2.1). We also recall the Sobolev number \( t^* \) associated with \( t < n \):

\[
\frac{1}{t^*} = \frac{1}{t} - \frac{1}{n}.
\]

We split the proof into two parts depending on whether \( m \geq 1 \) or \( m = 0 \).

**Case** \( m \geq 1 \): Since \( t_{m-1} < 2 \leq n \) we have \( u \in L^{t_m}(U) \), and applying \(6.14c\) with

\[
\frac{1}{t^*_{m-1}} = \frac{1}{s} + \frac{1}{t^*_{m-1}} = \frac{1}{t_{m-1}} + \frac{1}{s} - \frac{1}{n} = \frac{1}{t_m}
\]

we deduce \( \vartheta_p(u) \in L^{t_m}(U) \) with

\[
\| \vartheta_p(u) \|_{L^{t_m}(U)} \lesssim \| u \|_{W^{1}_{t_{m-1}}(U)},
\]

as a consequence of \(6.4\). On the other hand, combining Remark \(6.18\) with \(4.4\) yields \( v \in L^{(t_m)^*}(U) \), whence invoking \(5.14c\) with

\[
\frac{1}{(t^*_m)^*} = \frac{1}{(t^*_m)^*} + \frac{1}{s} = \frac{1}{t^*_{m-1}},
\]

noticing that \( t^*_m < t^*_{m-1} \) and using Gagliardo-Nirenberg inequality, we obtain

\[
\| \vartheta_p(v) \|_{L^{t^*_m}(U)} \lesssim \| v \|_{L^{(t^*_m)^*}(U)} \lesssim \| v \|_{W^{1}_{t^*_m}(U)}.
\]

This implies

\[
\| I \| \lesssim \| \vartheta_p(u) \|_{L^{t_m}(U)} \| \vartheta_p(v) \|_{L^{t^*_m}(U)} \lesssim \| u \|_{W^{1}_{t_{m-1}}(U)} \| v \|_{W^{1}_{t^*_m}(U)}.
\]

Using Lemma \(3.2\) (Korn’s inequality), namely \( \| u \|_{W^{1}_{t_{m-1}}(U)} \lesssim \| \varepsilon(u) \|_{L^{t_{m-1}}(\Omega)} \lesssim \| u \|_{W^{1}_{t_{m-1}}(\Omega)} \), we arrive at

\[
\| I \| \lesssim \| u \|_{W^{1}_{t_{m-1}}(\Omega)} \| v \|_{W^{1}_{t^*_m}(U)}.
\]

To estimate II we resort to \(6.14c\), \(6.5\), and Lemma \(3.2\) to deduce

\[
\| II \| \lesssim \| \vartheta_p(u) \|_{L^{t_m}(U)} \| \varepsilon_p(v) \|_{L^{t^*_m}(U)} \lesssim \| u \|_{W^{1}_{t_{m-1}}(U)} \| v \|_{W^{1}_{t^*_m}(U)} \lesssim \| u \|_{W^{1}_{t_{m-1}}(\Omega)} \| v \|_{W^{1}_{t^*_m}(U)}.
\]

We next deal with III. Since

\[
\frac{1}{t^*_{m-1}} - \frac{1}{s} = \frac{1}{t^*_m} - \frac{1}{n} = \frac{1}{(t^*_m)^*} > 0,
\]

...
applying (5.14) yields \( \|\vartheta_P(v)\|_{L^{t'_{-1}}(U)} \lesssim \|v\|_{L^{t'_{-1}}(U)} \lesssim \|v\|_{W^1_U(U)} \), where the last inequality is due to Gagliardo-Nirenberg inequality. Lemma 3.2 (Korn’s inequality) further leads to

\[
\|\|\varepsilon_P(u)\|_{L^{t'_{-1}}(U)}\|\vartheta_P(v)\|_{L^{t'_{-1}}(U)} \lesssim |u|_{W^1_U(U)}|v|_{W^1_U(U)}
\]

Combining the estimates for I, II, and III gives

\[
|\mathcal{C}(u,p)(v,q)| \lesssim \|(u,p)\|_{X_{t_{-1}}(\Omega)}(v,q)\|_{X^{t'_{-1}}(\mathbb{R}^n)}
\]

thereby leading to \( \mathcal{C}(u,p) \in X_{t_{-1}}(\mathbb{R}^n)^* \) as desired.

2 Case \( m = 0 \): Since \( t_{-1} = s' < n \) we have \( u \in L^{t'_{-1}}(U) \), whence \( \vartheta_P(u) \in L^{t'_{-1}}(U) \) with

\[
\frac{1}{t'_{-1}} = \frac{1}{s} + \frac{1}{t'_{-1}} = \frac{1}{s} + \frac{1}{s'} - 1 = 1 - \frac{1}{n} = \frac{1}{n'}
\]

and

\[
\|\vartheta_P(u)\|_{L^{s'}(U)} \lesssim \|u\|_{W^1_U(U)}
\]

because of (5.14c). On the other hand, the fact that

\[
\frac{1}{t'_{0}} = 1 - \frac{1}{t'_{0}} = \frac{2}{s + n} < \frac{1}{n}
\]

implies

\[
\|\vartheta_P(v)\|_{L^{s'}(U)} \lesssim \|v\|_{L^{\infty}(U)} \lesssim \|v\|_{W^1_{t'_{0}}(U)}
\]

Consequently

\[
|\mathcal{C}(u,p)(v,q)| \lesssim \|(u,p)\|_{X_{t_{-1}}(\Omega)}(v,q)\|_{X^{t'_{-1}}(\mathbb{R}^n)}
\]

Since \( v \in W^1_{t'_{0}}(U) \), we see that

\[
|\mathcal{C}(u,p)(v,q)| \lesssim \|(u,p)\|_{X_{t_{-1}}(\Omega)}(v,q)\|_{X^{t'_{-1}}(\mathbb{R}^n)}
\]

Applying Lemma 3.2 (Korn’s inequality), we deduce \( |u|_{W^1_{t'_{-1}}(U)} \lesssim |u|_{W^1_{t'_{-1}}(U)} \), and further using Remark 3.3 we obtain \( \|v\|_{W^1_{t'_{-1}}(U)} \lesssim \|v\|_{W^1_{t'_{-1}}(U)} \). This implies that

\[
|\mathcal{C}(u,p)(v,q)| \lesssim \|(u,p)\|_{X_{t_{-1}}(\Omega)}(v,q)\|_{X^{t'_{-1}}(\mathbb{R}^n)}
\]

and as a consequence that \( \mathcal{C}(u,p) \in X_{t_{-1}}(\mathbb{R}^n)^* \). This concludes the proof. □

Corollary 6.22 (global regularity). If \((u,p)\) in \( X_{t_{-1}}(\Omega) \) satisfies \( S(u,p) = 0 \), then \((u,p)\) in \( X_{t_{-1}}(\Omega) \).
Proof. For a given \((u, p) \in X_{t_m-1}(\Omega), \text{Lemma 6.14}\) (space decomposition of \(X_r(\Omega)\)) with \(r = t_m-1\) implies
\[
(u, p) = \sum_{i=1}^k \mathcal{R}_{\varphi_i} E_{\psi_i}(u, p).
\]
For \(i = 1, \ldots, k\), Lemmas 6.20 and 6.21 yield \(E_{\psi_i}(u, p) \in X_{t_m}(Q_i)\) with \(Q_i = \mathbb{R}^n\) or \(\mathbb{R}^n\). Finally, using Lemma 6.6 (continuity of \(\mathcal{R}_{\varphi_i}\)) leads to the asserted result. □

**Proposition 6.23** (injectivity of \(S_\Omega\)). Let \(s' \leq r < 2\). The Stokes operator \(S_\Omega : X_r(\Omega) \to X_r(\Omega)^*\) is injective.

*Proof*. As \(\Omega\) is bounded, it suffices to prove the assertion for \(r = s'\). Let \((u, p) \in X_{s'}(\Omega)\) solve \(S_\Omega(u, p) = 0\). Applying Corollary 6.22 \(M\) times, where \(M\) is given in Definition 6.17, we obtain
\[
(u, p) \in X_{t_{m-1}-s'}(\Omega) \cap \ldots \cap X_{t_m}(\Omega),
\]
with \(t_M \geq 2\). Proposition 6.16 further implies \((u, p) = (0, 0)\), which concludes the proof. □

**Proposition 6.24** (injectivity of \(S^*_\Omega\)). Let \(s' \leq r \leq s\). The dual Stokes operator \(S^*_\Omega\) is injective.

*Proof*. We use the subscript \(r\) on the operator \(S_\Omega\) to indicate that \(S_\Omega\) is defined on \(X_s(\Omega)\), i.e., \(S_{\Omega, r} : X_s(\Omega) \to X_r(\Omega)^*\). Since \(X_r(\Omega)\) is reflexive, we further deduce \(S^*_{\Omega, r} : X_r(\Omega) \to X_r(\Omega)^*\). Let \((v, q) \in X_r(\Omega)\) satisfy
\[
S^*_{\Omega, r}(v, q)(u, p) = 0, \quad \forall (u, p) \in X_r(\Omega).
\]
Using the definition of adjoint operator
\[
S_{\Omega, r}(u, p)(v, q) = 0, \quad \forall (u, p) \in X_r(\Omega).
\]
Owing to the definition of \(S_{\Omega, r}\) we have
\[
S_{\Omega, r}(v, -q)(u, -p) = 0, \quad \forall (u, p) \in X_r(\Omega).
\]
As \(S_{\Omega, r}\) is injective, thus \((v, q) = (0, 0)\), which completes the proof. □

**Corollary 6.25** (invertibility of \(S_\Omega\)). The Stokes operator \(S_\Omega : X_s(\Omega) \to X_{s'}(\Omega)^*\) is invertible for \(s' \leq r \leq s\).

*Proof*. The index of \(S_\Omega\) is finite because \(S_\Omega\) has a pseudoinverse (Lemma 6.1 and Theorem 6.15). In addition, \(S_\Omega\) is injective (Proposition 6.16 and 6.23) and \(S_\Omega\) is also injective (Proposition 6.24). Apply Corollary 6.4 to conclude the assertion. □

### 7. The non-homogeneous case \(u \cdot \nu \neq 0\)

We present a framework to treat the nonhomogeneous essential boundary condition \((1.1b)\). It relies on the standard practice of lifting the data inside the domain. By the principle of superposition it suffices to study the case when \(\phi\) is the only non-trivial data. Note that the compatibility condition \((1.1b)\) becomes \(\int_{\partial \Omega} \phi = 0\).

Since the domain is of class \(W^{2-1/s}_s\), for \(s > n\), the unit normal satisfies \(\nu \in W^{1-1/s}_s(\partial \Omega)\). We extend each component of \(\nu\) and thus \(\nu\) to a function in \(W^{1}_s(\Omega)\) (still denoted \(\nu\)). Given a scalar function \(\phi \in W^{1-1/r}_{s'}(\partial \Omega)\), we still denote \(\phi \in W^1_s(\Omega)\) its extension to \(\Omega\). These extensions are possible because \(\partial \Omega\) is Lipschitz. We define \(\varphi := \nu \phi\) and note that, for \(r \leq s\) and \(s > n\), a simple calculation yields
\[
\|\varphi\|_{W^{2}_s(\Omega)} \leq C_{\Omega, n, r, s} \|\phi\|_{W^{1-1/r}_s(\partial \Omega)} \|\nu\|_{W^{1-1/r}_s(\partial \Omega)}.
\] (7.1)
In fact, \( \nu \nabla \phi \in L^r(\Omega) \) is obviously valid. Therefore, the only problematic term is \( \phi \nabla \nu \) when \( r \leq n \), for otherwise \( \phi \) is bounded and \( \phi \nabla \nu \in L^s(\Omega) \). In such a case, \( \phi \in L^r(\Omega) \) with \( \frac{1}{r} > \frac{1}{s} - \frac{1}{n} \), whence \( \frac{1}{s} = \frac{1}{r} - \frac{1}{n} < \frac{1}{n} \). Since \( s > n \) we can choose \( t \) so that \( n < t^* \leq s \) and Hölder’s inequality gives \( \phi \nabla \nu \in W_1^1(\Omega) \). Similar estimates are reported in [13] Lemma 13.3 and [26] Corollary 1.1 [1] Theorem 7.39.

**Corollary 7.1.** The pair \( (u, p) \in X_r(\Omega) \) is a solution to (1.1) with only \( \phi \) non-trivial if and only if \( (w, p) = (u - \varphi, p) \in X_r(\Omega) \) is a weak solution to (1.1) with
\[
    f = \eta \text{div} \varepsilon(\varphi), \quad g = - \text{div} \varphi, \quad \psi = - \eta T^\top \varepsilon(\varphi) \nu, \quad w \cdot \nu = 0.
\]

In particular, \( (w, p) \) satisfies
\[
    S_{\Omega}(w, p)(v, q) = \eta \langle \varepsilon(\varphi), \varepsilon(v) \rangle_{\Omega} + \langle p, \text{div} v \rangle + \langle \psi, \gamma_0 v \rangle_{\partial \Omega} - \langle \text{div} \varphi, q \rangle_{\Omega}
\]
for all \( (v, q) \in X_r(\Omega) \), and its norm is controlled by the data \( \phi \), namely
\[
    \| (w, p) \|_{X_r(\Omega)} \leq C_{\Omega, n, r, \eta} \| \phi \|_{W_1^{-1/r}(\partial \Omega)}.
\]

**Proof.** The expressions for \( f, g, \psi \), and \( w \cdot \nu \) are straightforward to obtain, so we skip their derivation. The variational form follows after integrating by parts and recalling that the test functions \( v \) are tangential on the boundary. Finally, the continuity estimate is a direct application of the results when \( w \cdot \nu = 0 \) as well as (7.1). \( \Box \)

**Remark 7.2 (alternative lift).** Given data \( \phi \) in \( W_1^{1-r}(\partial \Omega) \), consider solving the Neumann problem
\[
    -\Delta \varphi = 0 \quad \text{in} \quad \Omega \quad \partial_{\nu} \varphi = \phi \quad \text{on} \quad \partial \Omega \quad (7.2)
\]
in \( W_2^2(\Omega) \). Then, the pair \( (u, p) \) is a solution of (1.1a), (1.1b), with only \( \phi \) non-trivial, if and only if \( (w, p) = (u - \nabla \varphi, p) \) is a solution of (1.1a), (1.1b), with
\[
    f = \eta \text{div} \varepsilon(\nabla \varphi), \quad g = - \text{div} \nabla \varphi, \quad \psi = - \eta T^\top \varepsilon(\nabla \varphi) \nu, \quad w \cdot \nu = 0.
\]

We point out that existence of a strong solution to the inhomogeneous Neumann problem (7.2) for Lipschitz or even \( C^1 \) domains may fail in general; see [29]. On the other hand, it is well-known that for \( C^{1,1} \) domains a strong solution always exists [27]. In this respect, our fractional Sobolev domain regularity appears to be (nearly) optimal.

**8. The Navier boundary condition.** The goal of this section is to consider the Stokes problem (1.1a) with the Navier boundary condition, i.e.
\[
    u \cdot \nu = 0, \quad \beta T u + T^\top \sigma(u, p) \nu = 0 \quad \text{on} \quad \partial \Omega \quad (8.1)
\]
with \( \beta > 0 \). The strategy is to notice that the term \( T u \) is a compact perturbation of the pure-slip problem, and in view of Lemma 6.2 we have that the index of this new problem is zero. Therefore, its well-posedness is governed only by its finite dimensional null-space. We structure the rest of this section as follows: first, we state mild integrability assumptions on the parameter \( \beta \) which still guarantee compactness of the added term; second, we show that the perturbed problem is injective by constructing
a smoothing sequence as was done in [6.3] finally, we state the main result as another consequence of Corollary 6.4.

**Lemma 8.1** ($T_\Omega$ is compact). Let $T_\Omega(u)(v) := \int_{\partial\Omega} \beta Tu \cdot Tv$, $s' \leq r \leq s$, and $\beta \in L^2(\partial\Omega)$ with

$$q \geq \frac{r}{n'} \text{ if } r > n, \quad q > n - 1 \text{ if } n' \leq r \leq n, \quad q > \frac{r'}{n'} \text{ if } r < n',$$

and $n' = n/(n-1)$. Then the operator $T_\Omega : W^s_r(\Omega) \rightarrow W^s_{r'}(\Omega)$ is compact.

**Proof.** We first observe that the projection operator $T = I - \nu \otimes \nu$ is in $L^\infty(\partial\Omega)$. We employ Sobolev embeddings and the trace theorems. We only consider the case $n' \leq r \leq s$ because the other two are similar. We have the following embeddings

$$W^s_r(\Omega) \hookrightarrow L^p(\partial\Omega) \quad \frac{1}{p} > \frac{n'}{r} - \frac{1}{n-1},$$

$$W^s_{r'}(\Omega) \hookrightarrow L^q(\partial\Omega) \quad \frac{1}{q} = \frac{n'}{r'} - \frac{1}{n-1},$$

the former being compact. Since

$$\left(\frac{n'}{r} - \frac{1}{n-1}\right) + \left(\frac{n'}{r'} - \frac{1}{n-1}\right) = 1 - \frac{1}{n-1},$$

for any $q > n-1$ there exists a $p$ satisfying the above inequality as well as $\frac{1}{r} + \frac{1}{q} + \frac{1}{s} = 1$. Therefore

$$|T_\Omega(u)(v)| \leq C\|\beta\|_{L^s(\partial\Omega)} \|u\|_{W^s_r(\Omega)} \|v\|_{W^s_{r'}(\Omega)}$$

and $T_\Omega$ is compact. □

We remark that a sufficient condition on $q$ to ensure the conclusion of Lemma 8.1 is $q > n - 1$. For simplicity from hereon we will work under this assumption on $q$.

**Definition 8.2** (smoothing sequence). Let $s > n$, $q > n - 1$ and $t_0 = r = s'$. The smoothing sequence conformal to $S_\Omega + T_\Omega$ is given by

$$\frac{1}{t_m} := \frac{1}{t_{m-1}} - \frac{1}{n} \left(1 - (n-1)\frac{1}{q}\right) \quad \text{for } m = 1, \ldots, M$$

where $M \geq n \left(1 - \frac{n-1}{q}\right)^{-1} \left[\frac{1}{t_0} - \frac{1}{2}\right]$ guarantees that $t_M \geq 2$.

**Lemma 8.3** ($S_\Omega + T_\Omega$ is injective). Let $s' \leq r \leq s$ and suppose $\beta$ is strictly positive in a set $\Gamma \subset \partial\Omega$ of positive measure. If $(u, p) \in X_r(\Omega)$ is a homogeneous solution to the Stokes problem with Navier slip boundary conditions (6.1), then $(u, p) = (0, 0)$.

**Proof.** Since $\Omega$ is bounded, the case $2 < r \leq s$ follows from the embedding $X_r(\Omega) \hookrightarrow X_2(\Omega)$. The Hilbert space case $r = 2$ follows from the coercivity estimate

$$0 = S(u, p)(u, p) + T_\Omega(u)(u)$$

$$= \eta\|\varepsilon(u)\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \beta|Tu|^2 + \|u\|^2_{L^2(\partial\Omega)} + \beta_0\|Tu\|^2_{L^2(\Gamma)}$$

where $\beta_0 > 0$. Since $\|\varepsilon(u)\|_{L^2(\Omega)} = 0$ we recall Proposition 3.1 to conclude that $u$ is an element of $Z(\Omega)$, i.e. it is an affine vector field of the form $u(x) = Ax + b$ with $u \cdot \nu = 0$ on $\partial\Omega$. By using that $Tu = 0$ a.e. on $\Gamma$, we conclude that $u = 0$. The uniqueness of $p$, up to a constant, follows as in [3].
To obtain injectivity for $s' \leq r < 2$ we suppose that $(u, p) \in X_{t_0}(\Omega)$ is a homogeneous solution to $S_\Omega + T_\Omega$ with $t_0 = r = s'$. We then use the smoothing property of the Stokes operator, and the same induction argument as in Proposition 6.23, to obtain in $M$ steps that 

$$(u, p) \in X_{t_0}(\Omega) \cap \ldots \cap X_{t_M}(\Omega) \subset X_2(\Omega).$$

where $t_m$ is the sequence from Definition 8.2. This implies $(u, p) = (0, 0)$ as desired.

**Remark 8.4 (uniqueness).** If the set $\Gamma = \partial \Omega$ in Lemma 8.3, then we may take the set $Z = \emptyset$, i.e. the velocity field $u$ is unique and the pressure is unique up to a constant. 

**Theorem 8.5 (slip with friction).** Let $\Omega$ be a bounded domain of class $W^{2-1/s}_s$ and $\beta$ satisfy the assumptions of Lemmas 8.1 and 8.3. For every $F \in X_{r'}(\Omega)^*$ there exists a unique $(u, p) \in X_r(\Omega)$ such that 

$$S_\Omega(u, p)(v, q) + T_\Omega(u)(v) = F(v, q) \quad \forall (v, q) \in X_{r'}(\Omega),$$

and 

$$\|(u, p)\|_{X_r(\Omega)} \leq C_{\Omega, n, \eta, r} \|F\|_{X_{r'}(\Omega)^*},$$

where $s' \leq r \leq s$.

**Proof.** The proof relies on the boundedness of $\Omega$ and the compactness of $T_\Omega$. We start by noting that $S^{-1}_\Omega$ is a pseudo-inverse of $S_\Omega + T_\Omega$, i.e.

$$S^{-1}_\Omega(S_\Omega + T_\Omega) = I_{X_r} + S^{-1}_\Omega T_\Omega, \quad (S_\Omega + T_\Omega) S^{-1}_\Omega = I_{X_{r'}},$$

because $S^{-1}_\Omega T_\Omega$ and $T_\Omega S^{-1}_\Omega$ being the product of a bounded operator and a compact one are compact. Moreover, in view of Lemma 8.1 we have that 

$$\text{ind}(S_\Omega + T_\Omega) = - \text{ind} S^{-1}_\Omega = 0.$$ 

Using Lemma 8.3 and the definition of the index we have that $	ext{codim} R_{S_\Omega + T_\Omega} = \dim N_{S_\Omega + T_\Omega} = 0$, i.e. $S_\Omega + T_\Omega$ is bijective. The Open Mapping Theorem guarantees the asserted estimate. 

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