The Eigenfunctions of the $q$-Harmonic Oscillator on the Quantum Line

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Abstract
We construct a complete set of eigenfunctions of the $q$-deformed harmonic oscillator on the quantum line. In particular the eigenfunctions corresponding to the non-Fock part of the spectrum will be constructed.

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1 Introduction

In this paper we will construct a complete set of eigenfunctions for the \(q\)-deformed harmonic oscillator on the \(q\)-deformed line. We define the \(q\)-deformed oscillator [1, 2] as the unital \(*\)-algebra generated by the element \(a\) and its conjugate \(a^+\), subject to the relation

\[
aa^+ - q^{-2}a^+a = 1. \tag{1}\]

We take \(q > 1\), since this will be the case for the realization of these operators, which we consider below.

The spectrum of the Hamilton operator \(H \equiv a^+a\) consists of two parts [3]. A bounded part (Fock representation) and an unbounded part:

\[
\text{Spec}(H) = \left\{ \frac{1-q^{-2n}}{1-q^{-2}}, \quad n \in \mathbb{N} \right\}, \quad \left\{ \frac{1+q^{2(-\gamma-2m)}}{1-q^{-2}}, \quad m \in \mathbb{Z}, \quad \gamma \in \mathbb{R} \right\} \tag{2}\]

Both parts of the spectrum have an accumulation point at \(\frac{1}{1-q^{-2}}\). The Fock representation is a lowest weight representation with \(a^+\) acting as raising operator and \(a\) acting as lowering operator. For the second, i.e. the unbounded, part of the spectrum \(a\) is the raising operator.

It is known, that on the \(q\)-deformed line the \(q\)-deformed Hermite polynomials are related to the eigenfunctions corresponding to the Fock representation [4, 5]. However, these functions are not complete in the respective Hilbert space of square integrable functions. One has to consider the eigenfunctions related to the unbounded part of the spectrum as well. This we will do by using results of Ciccoli et.al. [6].

2 Representation on the \(q\)-deformed line

We will consider a realization of the \(q\)-oscillator on the \(q\)-deformed real line \(\mathbb{R}_q\), which is defined as being generated by the operators \(X, P, U\) with commutation relations

\[
q^{\frac{1}{2}}XP - q^{-\frac{1}{2}}PX = iU \\
UX = q^{-1}XU, \quad UP = qPU \tag{3}\]

and the following conjugation:

\[
X^+ = X, \quad P^+ = P, \quad U^+ = U^{-1} \tag{4}\]

A realization of the \(q\)-oscillator on \(\mathbb{R}_q\) is given by [4]:

\[
a = \alpha U^{-2} + \beta P \]
\[
a^+ = \bar{\alpha} U^2 + \bar{\beta} P \tag{5}\]
With \( \alpha, \beta \in \mathbb{C} \), such that
\[
\alpha \bar{\alpha} = \frac{1}{1 - q^{-2}} = \frac{q}{\lambda} \quad \text{and} \quad \alpha \bar{\beta} = \bar{\alpha} \beta.
\] (6)

The second relation implies \( \frac{\alpha}{\beta} = \frac{\bar{\alpha}}{\bar{\beta}} \in \mathbb{R} \); we define
\[
q^{-\gamma} \equiv \frac{\alpha}{\beta} = \frac{\bar{\alpha}}{\bar{\beta}}
\] (7)

The algebra of the \( q \)-deformed real line \( \mathbb{R}_q \) can be realized by operators acting on functions of one variable:
\[
X f(x) = xf(x); \quad Pf(x) = -iD_qf(x), \quad Uf(x) = q^{-\frac{1}{2}}f(q^{-1}x);
\] (8)

where a \( q \)-derivative in the following form has been used:
\[
D_q f(x) \equiv \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}.
\] (9)

The algebra acts on functions on a 'lattice' \( \xi_q^n, \xi, n \in \mathbb{Z} \).

The scalar product can be defined in terms of the Jackson integral:
\[
(f, g) \sim \sum_{n \in \mathbb{Z}} f(q^n)g(q^n)q^n
\] (10)

We will use the following notation:
\[
(a; q)_n \equiv \prod_{i=0}^{n-1} (1 - aq^i); \quad (a; q)_\infty \equiv \lim_{n \to \infty} (a; q)_n
\] (11)

and define the \( q \)-exponential function:
\[
e_q(x) \equiv \frac{1}{(x; q^{-2})_\infty}
\]
\[
D_q e_q(cx) = \frac{c}{\lambda} e_q(qcx), \quad c \in \mathbb{C}
\] (12)

Using the relations \( \ref{eq:6} \) and \( \ref{eq:8} \), it is easily seen, that the ground state of the Fock representation, i.e. the state satisfying \( a\psi_0(x) = 0 \), is given by:
\[
\psi_0(x) \equiv Ne_q(-i\frac{\alpha}{\beta} \lambda q^{-\frac{1}{2}} x),
\] (13)

where \( N \) is a normalization constant.

The Hamilton operator \( H \equiv a^+ a \) is in terms of \( D_q \) and \( U \):
\[
H = a^+ a = \alpha \bar{\alpha} - \beta \bar{\beta} D_q^2 - i\alpha \bar{\beta}(U + qU^{-1})D_q
\] (14)

With this, and the action \( \ref{eq:8} \), \( \ref{eq:9} \) of \( D_q \) and \( U \) on functions, the equation
\[
H f(x) = Ef(x)
\] (15)
for the eigenfunctions becomes a difference equation:

\[ Ex^2 \lambda^2 f(x) = f(x) \left\{ \alpha \lambda^2 + \beta \bar{\lambda} (q + q^{-1}) \right\} + f(q^2 x) \left\{ -q^{-1} \beta \bar{\lambda} - \mathcal{J} \alpha \bar{\lambda} q^2 x \lambda \right\} + f(q^{-2} x) \left\{ -q \beta \bar{\lambda} + i \mathcal{J} \alpha \bar{\lambda} q^2 x \lambda \right\} \] (16)

With the definition

\[ E = \frac{1 + \varepsilon}{1 - q^{-2}} \] (17)

the equation (16) for the eigenstates \( f(x) = \psi_0(x)g(x) \) with eigenvalue \( E \) becomes:

\[ 0 = g(x) \left\{ q + q^{-1} - \varepsilon q^{-2} \lambda^2 x^2 \right\} + q^{-1} g(q^2 x) - q g(q^{-2} x) \left\{ 1 + q^{-2} \lambda^2 x^2 \right\} \] (18)

Where we now use \( \gamma \), Eqn. (7), instead of \( \alpha \) and \( \beta \).

### 3 Orthonormal basis

To solve this equation, we use the basic hypergeometric series \( _1 \varphi_1 \) [7]. The function \( f(z) = _1 \varphi_1 (a; c; q, z) \) satisfies:

\[ (c - a) f(qz) + (-c + q + z) f(z) + q f(z/q) = 0 \] (19)

We define:

\[ \varphi_e(x) \equiv \varphi_1 (-\varepsilon^{-1}; q^{-2}; q^{-4}; \varepsilon q^{-2} \lambda^2 x^2) \]

\[ \varphi_o(x) \equiv \varphi_1 (-q^{-2} \varepsilon^{-1}; q^{-6}; q^{-4}; \varepsilon q^{-2} \lambda^2 x^2) \] (20)

Due to the relation (19) we find that the following functions solve the Eqn. (18):

\[ g(x) = \varphi_e(x), \quad g(x) = x \varphi_o(x) \] (21)

These two solutions correspond to parts of the spectrum (2), which are numbered by even and odd numbers respectively.

It is possible, to combine the two solutions (21) to a function, that yields the whole spectrum (3):

\[ (\varepsilon^k)^2 \varphi_1 \left( -\frac{1}{\varepsilon q^2}; -\frac{1}{\varepsilon}; 0; q^{-4}, -q^{4(k-1)} \right) \] (22)

\[ = C_e \varphi_1 \left( -\frac{1}{\varepsilon}; q^{-2}; q^{-4}, \varepsilon \right) + C_o q^{-2k} \varphi_1 \left( -\frac{1}{\varepsilon q^2}; q^{-6}; q^{-4}, \varepsilon \right) \]

with

\[ C_e = \frac{(\varepsilon^{-1} q^{-2}; q^{-4}, \varepsilon^{-1} q^{-4})_\infty}{(q^{-2}, -c, -c^{-1} q^{-4}; q^{-4})_\infty} \]

\[ C_o = \frac{(\varepsilon^{-1} q^{-6}; q^{-4})_\infty}{(q^{-6}, -c, -c^{-1} q^{-4}; q^{-4})_\infty} \] (23)
We will discuss some properties of these functions and consider later the relation to the lattice and Hilbert space coming from representations of the algebra Eq. (8). First we consider the Fock representation \( \varepsilon = -q^{-2p}, p \in \mathbb{N} \). For the lattice points \( x = \xi q^{2n} \) one has, using the results of [5] (Theorem 4.1) in the case \( \varepsilon = -q^{-4p} \):

\[
\varphi^R_\varepsilon(n) = \varphi_1(q^{4p}; q^{-2}; q^{-4}, -q^{-4p}q^{-2\gamma-3}\xi^2q^{4n})
\]

(24)

with \( c \equiv q^{-2\gamma-1}\xi^2\lambda^2 \) one obtains:

\[
\sum_{n=-\infty}^{+\infty} q^{2n} = \delta_{rs} q^{4r} \frac{(q^{-4}; q^{-4})_\infty (q^{-4}_r)_\infty}{(q^{-2}; q^{-4})_\infty (q^{-2}_r; q^{-4})_\infty}
\]

(25)

Notice, that a \( q \)-exponential function turns up as measure under the Jackson integral, since \( (x; q)_\infty (-x; q)_\infty = (x^2; q)_\infty \). This is similar to the undeformed case, where \( e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}x^2} = e^{-x^2} \) leads to the orthogonality measure for the Hermite polynomials.

For \( \varepsilon = -q^{-4p-2} \) one obtains

\[
\varphi^L_\varepsilon(n) = q^{2n} \varphi_1(q^{4p}; q^{-6}; q^{-4}, -q^{-4p-2}q^{-2\gamma-5}\xi^2q^{4n})
\]

(26)

and

\[
\sum_{n=-\infty}^{+\infty} q^{2n} = \delta_{rs} q^{4r} \frac{(q^{-4}; q^{-4})_\infty (q^{-4}_r)_\infty}{(q^{-6}; q^{-4})_\infty (q^{-6}_r; q^{-4})_\infty}
\]

(27)

Now we use (22) to combine these two parts. For \( \varepsilon = -q^{-4n} \) one finds:

\[
C_c = (-c)^{-n} q^{4n^2-2n}(q^{-2}; q^{-4})_n, \quad C_o = 0
\]

(28)

Furthermore Eqn. (22) becomes

\[
(\pm \sqrt{cq^{-2k}})^{2n} \varphi_1 \left( q^{4n-2}; q^{4n}; 0; q^{-4} \right) \frac{q^{-4}}{(\pm \sqrt{cq^{-2k}})^2}
\]

(29)

\[
= (-)^n q^{4n^2-2n}(q^{-2}; q^{-4})_n \varphi_1 \left( q^{4n}; q^{-2}; q^{-4}, -q^{-4n-2}(\pm \sqrt{cq^{-2k}})^2 \right)
\]

For \( \varepsilon = -q^{-4n-2} \) we find

\[
C_c = 0, \quad C_o = (-c)^{-n} q^{4n^2+2n}(q^{-6}; q^{-4})_n
\]

(30)

and

\[
(\pm \sqrt{cq^{-2k}})^{2n+1} 2 \varphi_1 \left( q^{4n}; q^{4n+2}; 0; q^{-4} \right) \frac{q^{-4}}{(\pm \sqrt{cq^{-2k}})^2}
\]

(31)

\[
= (-)^{n+1} \sqrt{cq^{-2n-2k}}(q^{-6}; q^{-4})_n \varphi_1 \left( q^{4n}; q^{-6}; q^{-4}, -q^{-4n-6}(\pm \sqrt{cq^{-2k}})^2 \right)
\]
Using $m = 2n$ in the first case $\varepsilon = -q^{-4n}$ and $m = 2n + 1$ for $\varepsilon = -q^{-4n-2}$ the left hand sides of the Eqns. (24) and (31) are identical:

$$\tilde{h}_m(x) = x^m \tilde{\varphi}_1 \left( q^{2m-2}, q^{2m}; 0; q^{-4}, -q^{-4} x^2 \right)$$

(32)

The function $\psi_0(x) \tilde{h}(x)$ is the eigenfunction corresponding to the eigenvalue $\varepsilon = -q^{-2m}$. In both cases we obtain from the Eqns. (25), (27)

$$\sum_{k=\infty}^{\infty} \tilde{h}_m(\sqrt{c} q^{-2k}) \tilde{h}_m(\sqrt{c} q^{-2k}) q^{-2k} = N_c \left( q^{-2}; q^{-2} \right)_m q^{-2m^2}$$

(33)

where $N_c = (q^{-2}; q^{-2})_\infty$. The functions $\tilde{h}_m$ for even $m$ are not orthogonal to functions $\tilde{h}_m$ with odd $m$. Since according to the definition (32) the functions $\tilde{h}_m(x)$ are even and odd for even $m$ and odd $m$ respectively, we extend the sum to negative $x$-values. Then we have:

$$\sum_{k=-\infty}^{\infty} \tilde{h}_m(\sigma \sqrt{c} q^{-2k}) \tilde{h}_m(\sigma \sqrt{c} q^{-2k}) q^{-2k} = 2 N_c \left( q^{-2}; q^{-2} \right)_m \delta_{mn},$$

(34)

which is the well known orthogonality relation for the $q$-Hermite II polynomials, that are known to be related to the $q$-oscillator [5, 8].

Including negative eigenvalues of $x$, i.e. taking the direct sum of two irreducible representations of the algebra generated by $X$, $P$ and $U$, is also a possibility to obtain a Hilbert space, on which $X$ and $P$ are represented by self-adjoint operators [9].

We now turn to the unbounded part of the spectrum, i.e. $\varepsilon = q^{2\gamma-2m}, m \in \mathbb{Z}$. To connect our solutions with the results of [6] (Eqn (4.2)), it is necessary, to set $\xi^2 \lambda^2 = q$ or, equivalently:

$$c = q^{-2\gamma}.$$ 

(35)

For the functions $\tilde{\varphi}_1(-cq^{-4\nu+2p}; q^{-4\nu-4}, q^{-4}, -q^{-4} xq^{-2p-4})$, with $\nu = \pm \frac{1}{2}$, the square of the norm is according to [6]:

$$\delta_{pr} c q^{4p+2} \frac{(-c^{-1} q^{-4p-4}, -c^{-1} q^{-2} q^{-4})_\infty}{(-c^{-1} q^{-4p-6}, -cq^{-2}, -c, -c^{-1} q^{-4}; q^{-4})_\infty} \left( \frac{(q^{-4}, -cq^{-2}; q^{-4})_\infty}{(q^{-4}; q^{-4})_\infty} \right)^2$$

(36)

and

$$\delta_{pr} c q^{-4p-2} \frac{(-c^{-1} q^{-4p-4}, -c^{-1} q^{-2}; q^{-4})_\infty}{(-c^{-1} q^{-4p-2}, -cq^{-6}, -c, -c^{-1} q^{-4}; q^{-4})_\infty} \left( \frac{(q^{-4}, -cq^{-6}; q^{-4})_\infty}{(q^{-6}; q^{-4})_\infty} \right)^2$$

(37)

If we take [6] into account we find for the constants $C_e, C_o$ in [22]: In the case $\varepsilon = q^{2\gamma-4p-2} = c^{-1} q^{-4p-2}$

$$C_e = (-c)^p q^{4p^2+2p} \frac{(q^{-2}; q^{-4})_\infty}{(-c^{-1} q^{-4p-4}; q^{-4})_\infty}, \quad C_o = 0$$

(38)
and for $\varepsilon = q^{2\gamma-4p} = c^{-1}q^{-4p}$

$$C_e = 0, \quad C_o = -(-c)^p q^{4p^2-2p} \left(\frac{q^{-6}, q^{-4}}{(-c^{-1}q^{-4p^{-1}}, q^{-4})}\right)$$

(39)

Now, take $m = 2p + 1$ in the first case and $m = 2p$ in the second case. Such that $\varepsilon = q^{2\gamma-2m}$ with $m \in \mathbb{Z}$. Putting together these results, the Norm of the combined solution (22) becomes for both cases

$$\delta_{pr} c^m q^{2m^2} \frac{M_c}{(-c^{-1}q^{-2m-2}, q^{-2})},$$

(40)

with $M_c = \frac{(q^{-4}, q^{-4}, -c^{-1}q^{-2}, -cq^{-4})}{(-c, -c, c^{-1}q^{-1}, q^{-1})}$. We define

$$\tilde{k}_m(x) = (-x)^{m-\gamma}\sqrt{c^{-m}q^{-m}}\frac{1}{\varphi_1} \left(-q^{2m-2\gamma-2}, -q^{2m-2\gamma}; 0; q^{-4}, -\frac{q^{-4}}{x^2}\right);$$

(41)

if we extend the lattice to negative values as above, these functions are orthogonal:

$$\sum_{k=-\infty}^{+\infty} \tilde{k}_m(\sigma \sqrt{c} q^{-2k}) \tilde{k}_n(\sigma \sqrt{c} q^{-2k}) q^{-2k} = 2M_c \frac{c^m q^{2m^2}}{(-c^{-1}q^{-2m-2}, q^{-2})} \delta_{mn},$$

(42)

Also the scalar product of two functions $\tilde{h}_n$ and $\tilde{k}_m$ vanishes. The results of (40) imply that the set of functions $\tilde{h}_n$, $n \in \mathbb{N}$ together with the functions $\tilde{k}_m$, $m \in \mathbb{Z}$ form a basis of the Hilbert space.

The $q$-Heisenberg algebra (3) is represented on the space of square integrable functions on the set

$$\Lambda = \{\sigma \xi_o q^n | \sigma = \pm 1, n \in \mathbb{Z}\}, \xi_o \in \mathbb{R}.$$

(43)

With the scalar product given by a Jackson integral:

$$(f, g) = \xi_o(q - q^{-1}) \sum_{\sigma, n} f(\sigma \xi_o q^n) g(\sigma \xi_o q^n) q^n.$$

(44)

The operators $X$, $P$, $U$ act according to (8). $X$ and $P$ are essentially self-adjoint, $U$ is unitary. A representation is characterized by $\xi_o \in [1, q]$. The $q$-difference equation (16) for the eigenvalues of the Hamilton operator $H = a^+ a$ only connects even and odd lattice points among them self. That means, there is a twofold degeneracy in the spectrum.

Depending on the lattice that we consider (even or odd), the parameter $c$ has to be related to $\xi_o$ in different ways. For even lattice points, $\sigma \xi_o q^{2n}$, we have $\sqrt{c} = \xi_o$. For odd lattice points, $\sigma \xi_o q^{2n+1}$, we find $q \sqrt{c} = \xi_o$ and a shift in $\gamma$ occurs, which however does not change the spectrum.
4 Summary

It is well known, that in general spectra of Hamilton operators related to
deformed oscillator algebras consist of several different parts \(10\). The example
discussed in this paper shows, that in order to get self-adjoint representations,
one has to take into account all parts. It would be very interesting to see,
whether these parts play a role in a quantum field theory, that is constructed
with the aid of a deformed oscillator algebra.

In our specific case it turned out, that a nice way to get a self-adjoint rep-
resentation is to consider not only positive lattice points, which already form
an irreducible representation of the algebra \(9\), but also negative. This has
also been done in \(9\) in order to get a representation, such that \(X\) and \(P\) are
both essentially self-adjoint. It also was shown, that in the unbounded part
\[\frac{1+q^{2\gamma-2m}}{1-q^{2}}\] \(m \in \mathbb{Z}\), \(\gamma \in \mathbb{R}\) of the spectrum one \(\gamma\) is singled out by the chosen
lattice, cf. \(35\).

As explained in \(6\), these results also show, that the momentum proble-
m associated to the weight function \[\frac{1}{(-x^2\,q^{-1})_\infty}\], which appears in the orthogonality
relations, e.g. Eq. \(34\), is indetermined. Each of the functions \(\hat{k}_s\) is bounded:
\[|\hat{k}_s(\pm \sqrt[q]{q^{-2n}})| < C\], for all \(n\). Since the functions \(\hat{k}_s\) are orthogonal to the
Hermite polynomials \(\hat{h}_m\) and therefore to all polynomials, the moments will not
change if one uses for example \[\frac{1+C^{-1}\hat{k}_s(x)}{(-x^2\,q^{-1})_\infty}\] as weight function. From another
point of view, see e.g. \(11\), this happens, because the operator \(X\) is not self-
adjoint in the space spanned by the \(q\)-Hermite polynomials together with the
scalar product, that is given by the moments. In some sense one may interpret
specifying \(\gamma\) as choosing a self-adjoint extension.

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