Separation of Spin and Charge Quantum Numbers in Strongly Correlated Systems

Christopher Mudry and Eduardo Fradkin

Physics Department, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080

(September 10, 2021)

Abstract

In this paper we reexamine the problem of the separation of spin and charge degrees of freedom in two dimensional strongly correlated systems. We establish a set of sufficient conditions for the occurrence of spin and charge separation. Specifically, we discuss this issue in the context of the Heisenberg model for spin-1/2 on a square lattice with nearest ($J_1$) and next-nearest ($J_2$) neighbor antiferromagnetic couplings. Our formulation makes explicit the existence of a local SU(2) gauge symmetry once the spin-1/2 operators are replaced by bound states of spinons. The mean-field theory for the spinons is solved numerically as a function of the ratio $J_2/J_1$ for the so-called s-RVB Ansatz. A second order phase transition exists into a novel flux state for $J_2/J_1 > (J_2/J_1)_{cr}$. We identify the range $0 < J_2/J_1 < (J_2/J_1)_{cr}$ as the s-RVB phase. It is characterized by the existence of a finite gap to the elementary excitations (spinons) and the breakdown of all the continuous gauge symmetries. An effective continuum theory for the spinons and the gauge degrees of freedom is constructed just below the onset of the flux phase. We argue that this effective theory is consistent with the deconfinement of the spinons carrying the fundamental charge of the gauge group. We contrast this result with the study of the one dimensional quantum antiferromagnet
within the same approach. We show that in the one dimensional model, the spinons of the gauge picture are always confined and thus cannot be identified with the gapless spin-1/2 excitations of the quantum antiferromagnet Heisenberg model.

71.27.+a, 71.30.+h, 74.20.Mn, 75.10.Jm
I. INTRODUCTION

The issue of the possible breakdown of the Fermi-liquid picture is of considerable interest, both from a practical and conceptual point of view. The anomalous (with respect to a conventional Fermi-liquid scheme) properties of the metallic state of high $T_c$ materials, its proximity to an insulating magnetic state upon doping, might invalidate the use of Fermi-liquid theory. The microscopic foundation of Fermi-liquid theory itself has been a challenging theoretical problem for a long time. Fermi-liquid theory assumes that there is a one to one relationship between the low energy excitations of the non-interacting system and the low energy excitations of the interacting system which preserves quantum numbers like the spin, charge or particle number. In contrast, it has been proposed that for certain strongly interacting systems, the low lying excitations are of a radically different nature than in the non-interacting limit. For example, the spectrum just above the ground state would be made of “quasiparticles” carrying either the spin or the charge quantum numbers of the non-interacting excitations but not both simultaneously. This is the scenario of spin and charge separation which has been proposed by analogy to the Hubbard model in one dimension by some or to the fractional quantum Hall effect by others.

The purpose of this paper is to reexamine the conditions under which full separation of spin and charge occurs in two dimensional electronic systems. It is believed that a necessary condition for this phenomenon to take place is that the ground state in the spin sector is a spin liquid. A minimal requirement on a state describing a spin liquid is that it does not support any long range order breaking the continuous symmetry under global spin rotations, for example Néel ordering. However, under these conditions, many different types of spin liquids can still be constructed. Of crucial importance to applications to high $T_c$ superconductivity is whether the spin liquid is separated from the excited states by a gap or not and in the latter case the nature of the density of states.

The separation of spin and charge is assumed to occur for a certain range of electronic densities as a result of the competition between strong electronic correlations in the form
of antiferromagnetic (AF) interactions and the kinetic energy of the electrons. We will be interested in the Hubbard model [6] with moderate to strong $U$ or its generalization, the $t-J$ model [7], on a two dimensional lattice close to half-filling. At half-filling, the $t-J$ model and the $U/t \to \infty$ limit of the Hubbard model [8] reduce to the antiferromagnetic Heisenberg model for spin-1/2. The analytical treatment for the separation of spin and charge usually starts with the factorization of the creation operator $c_{i\alpha}^\dagger$ for the band electrons occurring in the Hubbard or $t-J$ model into a pair of operators satisfying fermionic and bosonic algebra, respectively [9,10]. At this point, the assignment for the spin and charge quantum numbers is arbitrary. We will choose the fermionic operator $s_{i\alpha}^\dagger$ to carry the spin quantum number ($\alpha = \uparrow, \downarrow$) but no charge and the state created by $s_{i\alpha}^\dagger$ is a bare spinon. The bosonic operator $h_i^\dagger$ must then carry the charge and the state it creates is similarly identified with a bare holon. Our choice is a matter of convenience. However, we believe it is well suited for the description of a spin liquid with the shortest possible AF correlations, i.e., of the order of the lattice constant. The alternative option consisting of fermionic holes and bosonic spins has been used to describe a spin liquid close to the onset of a state with long range Néel ordering [11].

The local phase of the bare spinon and holon operators is arbitrary. Accordingly, there exists a local gauge U(1) symmetry once the $c_{i\alpha}^\dagger = s_{i\alpha}^\dagger h_i$ representation is chosen. It thus follows that the bare spinon (holon) operators cannot create physical states. Indeed, these operators break the local symmetry and any theory based on a description in terms of spinons and holons must necessarily involve gauge fields. By construction, all of these theories are strongly coupled and, thus, the gauge fields vary rapidly. The likely scenario is then the confinement of all excitations transforming non-trivially under the gauge symmetry. Thus, it would appear that holons and spinons are permanently confined in bound states and that there may not be a separation of spin and charge in terms of spinons and holons carrying gauge quantum numbers. Nevertheless, several mean-field theories for the bare spinons and holons have been constructed at half-filling [12–18] and away from half-filling [19]. The challenge to any theory supposed to describe spin and charge separation is to construct
physical states associated with the spin and charge separately.

In this paper, we will concentrate on the pure spinon sector at zero-temperature by working at half-filling in which case the system is a *Mott insulator* \[20\]. A special feature of the Heisenberg limit is that the local U(1) symmetry is enlarged to SU(2) \[21,22\]. We will refer to this symmetry hereafter as the color symmetry to distinguish it from the symmetry under global spin rotations. We also do not have to deal with the complexities caused by the *dynamical interactions* between the spinons and the holons \[23\]. We will include *frustration* through a next-nearest-neighbor antiferromagnetic interaction in the Heisenberg limit. The issue of spin and charge separation then reduces to the existence of spin-1/2 excitations in the low energy sector of the theory.

Wen \[24\] has proposed a mean-field theory for the strongly interacting spinons which preserves the discrete symmetries under *time-reversal* and *parity* as well as the space group symmetries of the lattice. However, for a certain range of values for the ratio of the competing nearest \( (J_1) \) and next-nearest \( (J_2) \) interactions, his mean-field theory breaks completely the continuous part of the local color symmetry. *The fundamental assumption* is that an Anderson-Higgs mechanism \[25\], driven purely by the quartic spinon interaction, takes place and that its occurrence is sufficient to render the mean-field prediction of the existence of spinons in the physical spectrum robust to the strong gauge fluctuations caused by the constrained nature of the system. This BCS-like ground state is highly unconventional. Pairing takes place on the direct lattice and the “spontaneously broken” local gauge symmetry is not U(1) but SU(2). *It is thus argued that through the macroscopic condensation of spinon bound states (the typical size of which is the lattice spacing), the physical spectrum develops a gap to all excitations among which some carry the spinon quantum number.* Frustration is the trigger for this mechanism. A related mechanism has also been proposed by Read and Sachdev in their analysis of the Sp(N) \( J_1 - J_2 - J_3 \) antiferromagnetic Heisenberg model \[26\].

However, it is not at all clear to us that the spinon/holon picture is the appropriate one for the description of spin and charge separation. In fact, to our knowledge, it has never been demonstrated that the separation of spin and charge known to occur in many
one dimensional electronic models belonging to the Luttinger liquids universality class \[27\], becomes transparent in the spinon/holon representation. A first example is the Hubbard model on a linear chain. There, the separate spin-1/2 and charge sectors are created by soliton-like operators \[28\] which, when expressed in terms of the band electron operator \(c_{i\alpha}^\dagger\), are highly non-local and therefore cannot be described in any simple way by the spinon or holons. Another example is the antiferromagnetic quantum spin-1/2 Heisenberg chain with nearest-neighbor interactions for which the lowest excitations above the ground state are known to transform like the fundamental representation of SU(2) under spin rotations \[29\]. However, these excitations are again created by soliton-like operators which are related to the original spin-1/2 operators through a highly non-linear and non-local mapping, the Jordan-Wigner transformation \[30–32\]. Nevertheless, the spinon/holon picture offers one of the few analytical methods to study the issue of separation of spin and charge provided the mean-field predictions are not taken at face value. Moreover, aside from this issue, it offers a powerful technique to study non-Fermi liquid behaviour, once the local gauge invariance is properly accounted for \[33\].

In this paper, we will argue that the mechanism thought by Wen can but does not necessarily insure that the spinons remain in the physical spectrum. To this end, we have constructed an effective field theory for the low lying spinon modes in the presence of smooth fluctuations of the gauge modes. This effective field theory describes the low energy physics close to a second order phase transition from Wen’s s-RVB phase for \(0 < J_2/J_1 < (J_2/J_1)_{cr}\) into a gapless phase that, we found, is present at the mean-field level. The gapless state for \(J_2/J_1 > (J_2/J_1)_{cr}\) is qualitatively analogous to the flux state of Affleck and Marston \[34\]. However, in our problem, the gapless phase occurs at the mean-field level as a result of the competition between nearest \((J_1)\) and next-nearest \((J_2)\) neighbors. Hereafter, we refer to the gapless state as the flux state even though it is not the same state as in \[34\]. Close to and in the flux phase, the low energy spinon modes and large scale lattice gauge fluctuations are equivalent to a relativistic theory in 2+1 space-time of four Dirac fermions minimally coupled to non-Abelian gauge fields. The multiplication of the fermion species (flavors) is the result
of a degeneracy of the mean-field spectrum. Each flavor can be thought of as representing low lying fermionic excitations around one of the isolated but degenerate minima of the Brillouin Zone. The gauge fields are long wavelength fluctuations of the soft modes residing on the next-nearest-neighbor links of the lattice, i.e., the SU(2) color degrees of freedom. The nearest-neighbor exchange coupling $J_1$ causes the different fermionic flavors to interact through effective scalar fields. These scalar fields represent long wavelength fluctuations of the nearest-neighbor link degrees of freedom. It is important to stress that these bosonic degrees of freedom have no existence of their own independently of the spinons. They represent collective modes for the spinons. Just below the onset of the flux phase, the scalar gauge fields acquire expectation values which break all continuous gauge symmetries down to a discrete subgroup. Integration of the spinons then yields a local effective theory for the non-Abelian gauge fields and the scalar fields. We give a non-perturbative argument for the existence of the color quantum number of the spinon in the spectrum of the bosonized effective theory, provided both kinetic energy scales for the gauge fields and scalar fields (Higgs) are large enough. The calculation of these two scales will be published elsewhere. This mechanism is different from the one constructed from the chiral spin liquid since it does not rely on the breaking of parity and time reversal, which at the level of the effective continuum action implies the presence of a Chern-Simon term. Indeed, although, the four Dirac fermions acquire a mass it comes with an alternating sign for each flavor, and thus, upon fermionic integration, no net Chern-Simon term results.

In section II, we rapidly describe how to map the Heisenberg model into a SU(2) lattice gauge theory. Our strategy to study the issue of spin and charge separation is outlined in section III. In section IV, the numerical solutions of the saddle-point equations for the s-RVB Ansatz proposed by Wen are presented. A generalization of the s-RVB Ansatz to higher dimensions is also included. A field theory for the low lying excitations in the vicinity of the flux phase is derived in section V. We contrast our results with an application of our formalism to the one dimensional Heisenberg chain in section VI. Our conclusions are presented in section VII.
II. THE EQUIVALENCE OF THE HEISENBERG MODEL FOR SPIN-1/2 TO A SU(2) LATTICE GAUGE THEORY

The band electron operator \( c_{\uparrow(\downarrow)} \) of the Hubbard model can always be recast in the form

\[
c_{\uparrow(\downarrow)} = \begin{vmatrix} |0>><\downarrow| + (-)|\downarrow><\uparrow| \\
\end{vmatrix}
\]

\[
= h^\dagger s_{\uparrow(\downarrow)} + (-) s_{\downarrow(\uparrow)}^\dagger d ,
\]

provided the creation operators \( h^\dagger \), \( s^\dagger \) and \( d^\dagger \) for the holon, spinon and doublon states, respectively, satisfy the appropriate commutation relations. In the limit of strong on site repulsion, the term involving the doublon state is neglected [10]. We will choose the spinon operator to obey fermionic commutation relations.

At half-filling, the Hubbard model reduces to the Heisenberg model for spin-1/2 [8]:

\[
H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j
\] (2.2)

where \( <ij> \) is an ordered pair of sites on an arbitrary lattice \( \Lambda \), \( J_{ij} \) are antiferromagnetic coupling constants and

\[
\vec{S}_i = \frac{1}{2} s_{i\alpha} \vec{\sigma}_{\alpha\beta} s_{i\beta},
\] (2.3)

\( \vec{\sigma} \) being the Pauli matrices. The spinon representation, Eq. (2.3), for the spin-1/2 degrees of freedom must be supplemented with any of the three constraints

\[
\mathbb{I} = s_{i\alpha}^\dagger \delta_{\alpha\beta} s_{i\beta} \iff 0 = s_{i\uparrow}^\dagger s_{i\downarrow}^\dagger \iff 0 = s_{i\downarrow} s_{i\uparrow}
\] (2.4)

for all sites \( i \) of the lattice.

From the fully symmetric tensor \( \delta_{\alpha\beta} \) and the fully antisymmetric tensor \( \epsilon_{\alpha\beta} \) of SU(2), the two bilinear forms

\[
\chi_{ij} = s_{i\alpha}^\dagger \delta_{\alpha\beta} s_{j\beta}
\] (2.5)

and
\[ \eta_{ij}^\dagger = s_{i\alpha}^\dagger \epsilon_{\alpha\beta} s_{j\beta}^\dagger, \]  

(2.6)
can be used to describe a singlet pairing of the two spin-1/2 located on site \( i \) and \( j \), respectively [15,12]. Indeed, the identity

\[ \vec{S}_i \cdot \vec{S}_j = -\frac{1}{4} \eta_{ij}^\dagger \eta_{ij} - \frac{1}{4} \chi_{ij}^\dagger \chi_{ij} + \frac{1}{4} \mathbb{1} \]  

(2.7)
holds in the Hilbert space of one spinon per site. A spin liquid which, by definition, should not show any long range magnetic order, implies, in the spinon picture, the exponential decay with separation \( |i - j| \) of the vacuum expectation values \( \langle \eta_{ij}^\dagger \rangle \) or \( \langle \chi_{ij}^\dagger \rangle \).

The dynamics of these bilinear forms can be obtained from the vacuum persistence amplitude

\[ Z = \int \mathcal{D}[\vec{a}_{\vec{q}}] \int \mathcal{D}[s^\dagger] \mathcal{D}[s] e^{\pm i \int dt L'} \]  

(2.8)
where the lattice Lagrangian is

\[ L' = \sum_i s_{i\alpha}^\dagger i\partial_t s_{i\alpha} \]
\[ -\sum_i \left( \frac{1}{2} a_{0i}^- s_{i\alpha}^\dagger + \frac{1}{2} a_{0i}^+ s_{i\alpha} \right) \eta_{iit} + \frac{1}{2} a_{0i}^3 (\chi_{iit}^* - 1) \]
\[ + \sum_{<ij>} \frac{J_{ij}}{4} \left( \eta_{ij}^* \eta_{ijt} + \chi_{ij}^* \chi_{ijt} \right). \]  

(2.9)
The integration over the Lagrange multipliers

\[ \begin{pmatrix} a_{0it}^- \\ a_{0it}^+ \\ a_{0it}^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} (a_{0it}^1 - i a_{0it}^2) \\ \frac{1}{2} (a_{0it}^1 + i a_{0it}^2) \\ a_{0it}^3 \end{pmatrix} \]  

(2.10)
enforces the constraint of single occupancy on the spinon Hilbert space in a redundant way. But this redundancy allows for the mapping of Eq. (2.3) into the Lagrangian of a SU(2) lattice gauge theory [21], if a Hubbard-Stratanovich transformation with respect to the composite fields \( \eta \) and \( \chi \) is performed first.
With a particle-hole transformation of the spinons, our final Lagrangian will then take the form \[ 22 \]

\[
L = \sum_i \bar{\psi}_i \left( i \partial_t - A_{\delta it} \right) \psi_i \\
- \sum_{<ij>} \frac{J_{ij}}{4} \left( [\text{det } W_{ijt}] + (\psi_i^* W_{ijt} \psi_j + \psi_j^* W_{ijt}^\dagger \psi_i) \right).
\]

Here, the \( \psi' \)'s are \[ 37 \]

\[
\psi_i = \begin{pmatrix} s_{it}^\uparrow \\
               s_{it}^\downarrow \end{pmatrix}. \tag{2.12}
\]

The \( A_{\delta it} \)'s belong to the fundamental representation of the su(2) Lie algebra \[ 25 \]

\[
A_{\delta it} = \frac{1}{2} \vec{a}_{\delta it} \cdot \vec{\sigma}. \tag{2.13}
\]

Finally, the \( W' \)'s are \( 2 \times 2 \) matrices of the form \[ 24 \]

\[
W_{ijt} = \begin{pmatrix} -X_{ijt} & -E_{ijt} \\
                        -E_{ijt}^* & +X_{ijt}^* \end{pmatrix}, \tag{2.14}
\]

which satisfy \[ 23 \]

\[
W_{ijt} = W_{ji}^\dagger. \tag{2.15}
\]

The entries \( E \) and \( X \) of the \( W' \)'s are the Hubbard-Stratanovich degrees of freedom associated with the spinon bilinears \( \eta \) and \( \chi \), respectively.

The lattice Lagrangian in Eq. (2.11) is left unchanged by the local gauge transformations \[ 21 \]

\[
\begin{align*}
\psi_i \rightarrow \psi_i' &= U_{it} \psi_i, \\
A_{\delta it} \rightarrow A_{\delta it}' &= U_{it} A_{\delta it} U_{it}^\dagger + (i \partial_t U_{it}) U_{it}^\dagger, \\
W_{ijt} \rightarrow W_{ijt}' &= U_{it} W_{ijt} U_{jt}. \tag{2.16}
\end{align*}
\]

for all \( U_{it} \in SU(2) \). This local symmetry will be called a color symmetry. It is a different symmetry from the one generated by global spin rotations. Indeed, under the particle-hole transformation Eq. (2.12), the spin-1/2 operators of Eq. (2.3) are mapped into
\begin{align}
S_i^1 &= +\frac{1}{2} \left( \psi_{i_1}^\dagger \psi_{i_2}^\dagger + \psi_{i_2} \psi_{i_1} \right) \equiv +\frac{1}{2} \left( b_i^\dagger + b_i \right), \\
S_i^2 &= -\frac{i}{2} \left( \psi_{i_1}^\dagger \psi_{i_2}^\dagger - \psi_{i_2} \psi_{i_1} \right) \equiv -\frac{i}{2} \left( b_i^\dagger - b_i \right), \\
S_i^3 &= +\frac{1}{2} \left( \psi_i^\dagger \psi_i - 1 \right) \equiv +\frac{1}{2} \left( m_i - 1 \right). \tag{2.17}
\end{align}

The bilinears \(b\) and \(m\) defined above are left unchanged by the local color transformation Eq. (2.16) and thus the Heisenberg interaction explicitly transforms like a color singlet when expressed in terms of the \(\psi\)'s. Spin-spin correlations can be obtained from the generating functional

\[
Z[\vec{J}] = \int \mathcal{D}[W^\dagger] \mathcal{D}[W] \int \mathcal{D}[\vec{a}_0] \int \mathcal{D}[\psi^\dagger] \mathcal{D}[\psi] 
\times e^{i \int dt \left( L + \sum_i \vec{J}_i \cdot \vec{S}_i \right)} \tag{2.18}
\]

where the source term is written in terms of the bilinears \(b\) and \(m\) defined by Eq. (2.17).

It is important to realize that the gauge degrees of freedom in Eq. (2.11) are not independent from the fermionic degrees of freedom. For example, neither \(A_\delta\) nor the SU(2) factor of \(W_{ij}\) possess a lattice version of the kinetic energy. Their presence is simply a device to project the Fock space \(\mathcal{F}\) which is generated cyclically from the vacuum state \(|0\rangle_{\psi}\\) defined by

\[
\psi_{ia} |0\rangle_{\psi} = 0, \quad \forall i \in \Lambda, \quad a = 1, 2,
\]

onto the physical Hilbert space which is the tensorial product over all sites \(i\) of the vector spaces spanned by the color singlet states \(|0\rangle_{\psi}\) and \(b_i^\dagger |0\rangle_{\psi}\) (see [37]).

### III. SPIN LIQUID, DECONFINEMENT AND SPIN AND CHARGE SEPARATION.

In the previous section we have made use of the mapping of the spin-1/2 antiferromagnetic Heisenberg model into a SU(2) lattice gauge theory with the hope that the latter formulation is better suited for the description of a spin-liquid ground state and the possible existence of spin-1/2 excitations. The fact that the problem is, in effect, a gauge theory
indicates that in all likelyhood the spinons will necessarily be permanently confined inside SU(2) color singlet bound states. This is so since, below four space-time dimensions all (compact) gauge theories are generally in a confined phase. If this is the case, there is little hope that this mapping may be of great help in producing a mechanism for the separation of spin and charge quantum numbers in the physical spectrum of the system. Hence, the problem that needs to be solved is to find mechanisms that separate the spin and charge either in a manner consistent with the confinement of spinons or by dynamically driving the system into a deconfined state. In this section we will show that Wen’s s-RVB state effectively realizes the second scenario. In section [VI], we show that the first scenario is realized in the one dimensional version of Wen’s state.

In order to describe a spin liquid, an eigenstate of the Hamiltonian of Eq. (2.2) should show exponential decay with separation for all spin-spin correlations which break the global rotational spin symmetry. For example, to avoid Néel ordering we can imagine partitioning the lattice into distinct ordered pairs and choosing for each such pair the singlet state of the four dimensional Hilbert space associated with the two spin-1/2 degrees of freedom. The expectation value of \( \vec{S}_i \cdot \vec{S}_j \) in this state is only non-vanishing when \( i \) and \( j \) belong to the same ordered pair. However, such a state, the so-called valence bond state [38], breaks the lattice symmetry. If we require “featurelessness” from a spin liquid, then an easy remedy for a valence bond state with nearest-neighbor pairing, say on a square lattice, is to demand a uniform expectation value of the valence bond ordering operators for any spin and its four nearest-neighbors. This leads us naturally to the RVB overcomplete basis [2] for Hamiltonian Eq. (2.2). However, the RVB basis is extremely difficult to work with.

On the other hand, it appears to be very easy to implement featurelessness and exponential decay of the spin-spin correlations in the gauge picture, Eq. (2.11). Featurelessness amounts to selecting only those gauge equivalent classes of configurations \( C_{\{A_b, W\}} \) which are closed under the action of the lattice point group, i.e., gauge configurations on which the action of a point group transformation is equivalent to a gauge transformation. Exponential decay of the spin-spin correlations requires that within the class \( C_{\{A_b, W\}} \)
is satisfied, although this condition might not be sufficient (overcompleteness of the RVB basis).

This suggests the following mean-field approximation for a spin liquid:

(i) Choose a gauge field configuration consistent with the requirements of a spin liquid.
(ii) Solve the fermionic sector in the background of the gauge fields given in step (i).
(iii) Impose the self-consistency condition that the background gauge fields given in step (i) are the expectation value of the appropriate fermionic bilinear forms in the ground state found in step (ii).

The drawback of this approach is that the self-consistency condition only satisfies the local constraint of single occupancy for the spinons, Eq. (2.4), on average. In the worst case scenario, one could imagine that all the eigenstates of the mean-field Hamiltonian for the spinons belong to the unphysical Hilbert space, although on average the constraint of single occupancy is satisfied. For example, at half-filling, a state with all but two sites singly occupied by spinons is orthogonal to the physical subspace of the fermionic Hilbert space.

In the language of the $\psi$’s, single occupancy on any given site is the local constraint of Gauss’s law required by the local gauge invariance, namely the local physical states are the color singlets $|0 > \psi$ or $b_i^\dagger |0 > \psi$. It is then easy to convince oneself that the ground state and the single particle excitations of the mean-field theory have always a non-vanishing projection onto the unphysical Hilbert space. Indeed, the mean-field ground state is a Slater determinant for the $\psi$’s which can be obtained by a Fourier transformation of

$$\prod_{k \in \Omega_0} \psi_{k1}^\dagger \psi_{k2}^\dagger |0 > \psi,$$

(3.2)

where $\Omega_0$ denotes the set of all $|\Lambda|/2$ wave vectors entering in the Slater determinant of the mean-field ground state. The mean-field ground state and the excited states are thus convolutions in direct space whereas the Fourier transformation of the gauge invariant operator $b_i^\dagger = \psi_{i1}^\dagger \psi_{i2}^\dagger$ is a convolution in reciprocal space.
Since the problem of the quantum numbers of the physical states is non-perturbative by nature, it is important to take into account the quantum fluctuations of the gauge fields around the mean-field. The problem we end up with is that of fermions which are strongly coupled with gauge fields. If the mean-field theory predicts a gap in the fermionic spectrum a simpler picture can be drawn. In this case it is possible to integrate out the fermions to obtain a local theory. Although the resulting effective theory will be constructed from purely bosonic dynamical fields (which can be viewed as local fermionic composites), its spectrum can contain states with fermionic quantum numbers. We will show below that this is the case for the s-RVB Ansatz where the effective action involves gauge fields locally coupled to bosonic matter (scalars) which represent bound states of the spinons. We will also see that these bound states have simple transformation properties under the symmetries of the system. Of particular importance is that we will be able to find bound states that transform non-trivially under the local SU(2) invariance. Hence, at least qualitatively, our problem is closely related to that of finding the nature of the ground state of scalar fields coupled to SU(2) gauge fields. This is a problem that has been investigated quite extensively in the context of Lattice Gauge Theory. In what follows we will review those results, make connections with our problem and then draw conclusions. It will be apparent from our discussion that the key to the mechanism is to determine the transformation properties of the bound states.

Let us set the notation and define a set of scalar fields Φ_i on the sites i of a d + 1 dimensional space-time hypercubic lattice Λ and a set of SU(2) gauge fields V_{ij} on the ordered links < ij > of the same lattice. The gauge fields transform like group elements whereas the matter fields will transform under some representation of SU(2). In practice we will consider two cases: a) matter in the fundamental (or spinor) representation of SU(2) color and b) matter in the adjoint (or vector) representation of SU(2) color. A simple but generic action for this system is

\[ S = \beta \sum_{<ijkl>} \left( 1 - \frac{1}{4} \text{tr} \left[ V_{ij} V_{jk} V_{ik} V_{il}^\dagger + \text{H.C.} \right] \right) \]
The first term of Eq. (3.3), which is a sum over oriented plaquettes \(<ijkl>\), represents the gauge invariant kinetic energy for the SU(2) gauge fields while the rest of the action describes the dynamics of the matter and their coupling to the gauge fields. The scalar product of the matter fields \(\Phi\) in a given representation is denoted by \((\cdot, \\cdot)\). Finally, we have also allowed for a fluctuation of the norm of the matter field which, however, is irrelevant to the problem of confinement.

The properties of the phase diagram in the \(\beta - \kappa - \lambda\) space for Eq. (3.3) which are important to us depend only on the dimensionality of space-time and on the transformation properties of the matter fields \([39,40]\). We will distinguish two situations. In one case, the matter field \(\Phi\) is in the fundamental representation, in the other it is in the adjoint. Those two representations differ in one crucial aspect. The fundamental representation is one to one whereas the adjoint is two to one since it fails to distinguish between the two elements of the center \(C_{SU(2)}\) of the group SU(2) color \([41]\).

We first start our exploration of the phase diagram from the point \((\beta, \kappa, \lambda) = (0, 0, \infty)\). By letting \(\lambda \to \infty\), we have frozen the norm of the matter field to the value one. When we vary \(\beta\) from 0 to \(\infty\), we obtain a pure, non-Abelian lattice gauge theory. Along this line, the Wilson loop \(W_{f(a)}^\Gamma\) for sources in the fundamental (adjoint) representation

\[
W_{f(a)}^\Gamma = < tr \prod_{<ij>\in\Gamma} V_{ij} >
\]

where \(\Gamma\) is a closed path on the hypercubic lattice, can be used to detect a phase transition. For an appropriate choice of \(\Gamma\),

\[
\Gamma = \Box_R^T
\]

the potential
\[ V_{f(a)}(R) \equiv -\lim_{T \to \infty} \frac{1}{T} \ln W_{f(a)}(T, R) \]  

measures the energy of two static (infinitely heavy) sources in the fundamental (adjoint) representation which are separated by a distance \( R \).

For small values of \( \beta \), i.e., strong gauge coupling, the static potential is linear (accordingly the Wilson loop obeys an area law) and the static sources are confined. The lower critical space-time dimension is 4 for non-Abelian groups. We are interested in space-time of dimension 2+1 and therefore do not encounter a phase transition suggesting that for larger values of \( \beta \) deconfinement (the Wilson loop obeys a perimeter law) of the static sources occurs \([42]\).

At the point \((\infty, 0, \infty)\), the pure gauge theory freezes into any of the classical gauge configurations with all plaquette variables set to unity. The simplest choice is to have all \( V_{ij} \) set to unity. We then start moving along the line \((\infty, \kappa, \infty)\). The action for the matter field in the fundamental (adjoint) representation and with frozen norm is equivalent to that of a classical \( O(4) \) (\( O(3) \)) Heisenberg model. For space-time dimension larger than 2 and irrespectively of the matter field representation, a second order phase transition will occur when \( \kappa \) is larger than \( \kappa_{cr}^{f(a)} \) \([43]\).

As of now, the choice for the representation of the matter field had no impact on the existence of a phase transition. However, along the line \((\beta, \infty, \infty)\), the physics will differ dramatically. At \((\infty, \infty, \infty)\), the matter field in the fundamental (adjoint) representation are frozen into a classical configuration, say \( \Phi_0 \), which minimizes the action and breaks the global \( O(4) \) (\( O(3) \)) symmetry, provided \( d + 1 > 2 \). By decreasing the value of \( \beta \), the gauge fields are unfrozen. However, the matter field configuration \( \Phi_0 \) acts like an external symmetry breaking magnetic field through the hopping term. Indeed, under a gauge transformation

\[(\Phi_0, V_{ij}\Phi_0) \to (\Phi_0, U_iV_{ij}U_j^{-1}\Phi_0).\]  

The adjoint representation differs from the fundamental representation in that if one chooses to multiply \( V_{ij} \) by the representative of \(-\sigma^0\) on the left and \(+\sigma^0\) on the right, then the hopping term remains unchanged in the adjoint while it changes sign in the fundamental.
In other words, there will always be a residual local $\mathbb{Z}_2$ symmetry along $(\beta, \infty, \infty)$ in the adjoint representation. Hence, if the matter field breaks all the continuous symmetry, no phase transition takes place in the fundamental representation, while a first order phase transition takes place at $\beta_c^a$ in the adjoint representation \cite{14}.

Finally, along the line $(0, \kappa, \infty)$, the action describes a theory of non-interacting links (see \cite{45}) which does not undergo a phase transition.

We can extract valuable informations from Eq. (3.3) in the limiting cases where either the matter field or the gauge degrees of freedom are passive bystanders to the dynamics. In the interior of the phase diagram of Fig. 1, gauge and matter degrees of freedom do not decouple from each other. Consequently, the Wilson loop for static charges carrying the same representation as the matter field always obeys the perimeter law \cite{16}. Indeed, pair creation of matter, which causes the screening of the static color charges \cite{47}, are induced by quantum fluctuations.

It also follows from this argument that for the matter in the fundamental representation, all the Wilson loops, irrespectively of the representation of the static sources, will obey the perimeter law in the interior of the phase diagram \cite{18}. Wilson loops cannot then be used anymore as order parameters. In fact, it was proven \cite{19,39} that there exists a strip of analyticity joining the small $\beta$ and small $\kappa$ region (Confinement region) to the large $\beta$ and large $\kappa$ region (Higgs region) of the phase diagram, in which all the gauge invariant Green functions are analytic. In short, there is only one phase, the so-called Confinement-Higgs phase, in the phase diagram $(\beta, \kappa, \infty)$ when the matter is in the fundamental representation. A line of first order phase transitions emerges from the point $(\infty, \kappa_c^f, \infty)$. But as in the Temperature-Pressure phase diagram for liquid and gas, it ends up at a critical point in the interior of the $\beta - \kappa$ plane (see Fig. 1(a)).

This analysis implies that if the semiclassical expansion around a mean-field Ansatz yields an effective bosonic action with the matter in the fundamental representation, there will not be any states in the spectrum which carry the spinon quantum numbers. Conversely, if the matter is in the adjoint representation and breaks all the continuous symmetries, then
a phase in which there are states that carry the fundamental color and spin-1/2 quantum numbers is possible. Indeed, the Wilson loop in the fundamental representation does not need to follow the perimeter law everywhere in the interior of the phase diagram. The dynamical matter field induced by quantum fluctuations carry the adjoint color charge and therefore cannot completely screen the fundamental charge of the static sources. Gauge invariant Green function are separately analytic in the confinement and Higgs region. On the upper boundary of the Higgs region, the Wilson loop obeys the perimeter law of the deconfining regime of a pure gauge $\mathbb{Z}_2$ theory. By continuity, the Wilson loop must obey the perimeter law in the Higgs region. The same argument applies to the confinement region where now the Wilson loop satisfies the area law. There must be a line of phase transition separating the two regimes and connecting the classical O(3) phase transition to the pure gauge $\mathbb{Z}_2$ phase transition [39] (see Fig. 1(b)).

When we allow the norm of the matter field to fluctuate, i.e., $\lambda < \infty$, changes occur in the $\beta - \kappa$ phase diagram. However, the deconfinement and Higgs regimes always remain analytically connected for matter fields in the fundamental representation. Conversely, the phase transition survives in the adjoint representation [50].

This is the important insight that the gauge picture provides us with. The problem we are now faced with is how to justify the use of action Eq. (3.3) where the matter is in the adjoint representations on the basis of Eq. (2.11). This we start now.

Let us reexamine Eq. (2.11) by parametrizing the $W$’s according to

$$W_{ijt} = +i \left( \sqrt{\det W_{ijt}} \right) V_{ijt}, \quad V_{ijt} \in \text{SU}(2).$$

The local gauge symmetry only affects the SU(2) factor of the link degrees of freedom. As it stands, Eq. (2.11) does not contain a kinetic term for the $A_0$’s and $V$’s. This corresponds to the infinitely strong gauge coupling limit, $\beta = 0$. This is to be expected from gauge fields implementing a local constraint.

So far, we have not made any approximations. However, in some mean-field states (in particular in Wen’s s-RVB) there are interesting operators which acquire expectation values.
For instance, a term which is absent from Eq. (2.11), but which would be allowed by the symmetry, is

$$\mathcal{M}_{ijt} = \frac{1}{2} \text{tr} \left( P_{it} W_{ijt} P_{jt}^{\dagger} W_{ijt}^{\dagger} \right),$$  

(3.9)

for any ordered pair $< i \, j >$ of lattice sites. The main point of Wen’s Ansatz is that the operator $P_{it}$, the path ordered product

$$P_{it} = \prod_{l=1}^{n} W_{i_{t-l}i_{t}}, \quad i_0 \equiv i, \quad i_n \equiv i,$$

(3.10)

(along the ordered pairs $< i_0 \, i_1 >, \ldots, < i_{t-1} \, i_t >, \ldots, < i_{n-1} \, i_n >$) does acquire an expectation value in the s-RVB ground state. Be aware that the $\mathcal{M}$’s do not depend on the determinant of the $W$’s. If we take $i$ and $j$ to be nearest-neighbor with $\epsilon$ being the lattice constant, say $j = i + \epsilon \hat{x}$, expand $P_{jt}$ and

$$W_{ijt} = +i \left( \sqrt{|\det W|} \right) e^{i\epsilon A_{\hat{x}it}}, \quad A_{\hat{x}it} \in \text{su}(2),$$

(3.11)

in powers of the lattice constant, then

$$\frac{1}{2} \left( \mathcal{M}_{ijt} + \mathcal{M}_{ijt}^{*} \right) = 1 - \frac{1}{4} \text{tr} \left( (D_{\hat{x}} P) (D_{\hat{x}} P)^{\dagger} \right) + O(\epsilon^3),$$

(3.12)

where $D_{\hat{x}}$ is the covariant derivative in the adjoint representation, i.e.,

$$D_{\hat{x}} = \partial_{\hat{x}} + i \left[ A_{\hat{x}}, \cdot \right].$$

(3.13)

The $P$’s can therefore be interpreted as matter fields in the adjoint representation of SU(2) color. The fluctuations around the s-RVB ground state should contain terms of the form of Eq. (3.12). Thus, we argue that the fluctuations around an s-RVB state should behave qualitatively like a system of matter fields in the adjoint (vector) representation of SU(2) (which break all continuous symmetries) coupled to SU(2) gauge fields. We are then under the general conditions of the theorem relating the confinement and the Anderson-Higgs
mechanisms of Fradkin and Shenker [39]. Notice, however, that the parameters of the effective Lagrangian need to be determined from a microscopic calculation. These parameters will, in turn, determine in which phase the symmetry is realized.

In summary, we have argued how deconfinement could occur for the fundamental charges (the spinons) in the phase diagram of a generic lattice gauge theory coupled to matter fields in the adjoint representation of the gauge group. Although Eq. (2.11) does not resemble a canonical SU(2) gauge-matter lattice theory, say for example Eq. (3.3), we have identified candidates for a matter field namely the $P$'s. Our goal in the next section will be to construct a mean-field theory for a spin liquid such that $P$’s are generated which break the gauge symmetry down to $Z_2$ [24].

IV. MEAN-FIELD THEORY.

A. The s-RVB Ansatz.

From now on, we will restrict ourself to a square lattice ($d=2$) with nearest ($J_1$) and next-nearest ($J_2$) neighbor antiferromagnetic interactions. The s-RVB Ansatz assumes that the true ground state of the system described by Eq. (2.11) is characterized by the configuration of the gauge fields [24]. The lattice spacing is taken to be one and $\hat{x} \equiv (1, 0)$, $\hat{y} \equiv (0, 1)$, $\hat{x}_+ \equiv (+1, +1)$ and $\hat{x}_- \equiv (+1, -1)$. The mean-field spectrum $\pm|\xi_k^4|$ for the s-RVB Ansatz is then given by

$$
\xi_k^1 = -\frac{a_1^3}{2} + J_2 \text{ Re } E \cos k_x \cos k_y,
$$

$$
\xi_k^2 = -\frac{a_2^3}{2} + J_2 \text{ Im } E \sin k_x \sin k_y,
$$

(4.2)
\[ \xi_k^3 = -\frac{a_0^3}{2} + J_1 X \left( \frac{1}{2} \cos k_x + \cos k_y \right). \]

Because of the particle-hole symmetry, the mean-field ground state is the state obtained by filling up all the one-particle eigenstates in the Brillouin Zone \( \Omega \) with negative energy eigenvalues:

\[ | \Psi_{s-RVB} > = \prod_{\vec{k}, |\vec{\xi}_k| \leq 0} \psi_{k1}^\dagger \psi_{k2}^\dagger | 0 >_\psi \] (4.3)

where \( | 0 >_\psi \) is the state annihilated by all the \( \psi \)'s.

As shown by Wen, all the discrete symmetries of the lattice as well as parity and time reversal are preserved by the s-RVB Ansatz when \( a_0^2 = a_3^2 = 0 \). But, most importantly to us is that the Ansatz is designed to break all the continuous color symmetries provided the complex mean-field parameter \( E \) is neither real nor purely imaginary and \( X \) is non-vanishing. Indeed, the counterclockwise (ccw) and clockwise (cw) products

\[ P_{it}^{ccw} \equiv W_{i(i+\hat{x})t} W_{(i+\hat{x})(i+\hat{x}+\hat{y})t} W_{(i+\hat{x}+\hat{y})it}, \]
\[ P_{it}^{cw} \equiv W_{i(i+\hat{x})t} W_{(i+\hat{x})(i+\hat{x}-\hat{y})t} W_{(i-\hat{x}+\hat{y})it}, \] (4.4)

reduce to

\[ \bar{P}_{ccw} = -X^2 \left( \text{Re} \ E \sigma^1 - \text{Im} \ E \sigma^2 \right), \]
\[ \bar{P}_{cw} = -X^2 \left( \text{Re} \ E \sigma^1 + \text{Im} \ E \sigma^2 \right), \] (4.5)

in the mean-field. Hence, \( \bar{P}_{ccw} \) and \( \bar{P}_{cw} \) do not commute if and only if \( X, \text{Re} \ E \) and \( \text{Im} \ E \) are all non-vanishing.

The s-RVB is the phase in which \( X, \text{Re} \ E \) and \( \text{Im} \ E \) are non-vanishing. In this phase, \( |\vec{\xi}_k| \) is also non-vanishing over the entire Brillouin Zone. The spinons have acquired a gap. The spin-spin correlation function can be calculated, within the mean-field approximation, following the methods of Marston and Affleck [34]. The spin-spin correlation function has an exponential decay (in imaginary time) as a function of separation since the s-RVB state has a gap. Also, as expected, in the BZA \((J_2/J_1 \rightarrow 0)\) regime the structure factor has a
peak at \((\pi, \pi)\) and, in the flux regime \((J_2/J_1 \geq (J_2/J_1)_{ct})\) it has peaks at \((\pi/2, \pi/2), (0, \pi)\) and \((\pi, 0)\).

B. The saddle-point equations and mean-field ground states.

The saddle-point equations in the infinite volume limit are

\[ 0 = F_i(\{\kappa\}), \quad i = 1, \cdots, 6, \]  

where \(\{\kappa\}\) is the set made of the mean-field parameters

\[ \kappa_1 = a_0^1, \quad \kappa_2 = a_0^2, \quad \kappa_3 = a_0^3, \]
\[ \kappa_4 = \text{Re} \ E, \quad \kappa_5 = \text{Im} \ E, \quad \kappa_6 = \mathcal{X}, \]  

and the functions \(F_i\) are defined implicitly by integrals over a square with the area \(4\pi^2\) (the Brillouin Zone \(\Omega\)):

\[ F_1 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_1}{|\xi|}, \quad F_4 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_1}{|\xi|} g_1 - \kappa_4, \]
\[ F_2 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_2}{|\xi|}, \quad F_5 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_2}{|\xi|} g_2 - \kappa_5, \]
\[ F_3 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_3}{|\xi|}, \quad F_6 = \frac{1}{4\pi^2} \int_{\Omega} \frac{\xi_3}{|\xi|} g_3 - \kappa_6. \]  

Here,

\[ g_1(\vec{k}) = \cos k_x \cos k_y, \]
\[ g_2(\vec{k}) = \sin k_x \sin k_y, \]
\[ g_3(\vec{k}) = \frac{1}{2} (\cos k_x + \cos k_y), \]  

each transform irreducibly under the point group of the square lattice. Hence, the integrals only need to be performed over the reduced Brillouin zone \(\Omega' = \{\vec{k} \in \Omega \mid 0 \leq |k_y| \leq \pi\}\).

It is also useful to consider the Jacobian

\[ F_{i,j} = \frac{\partial F_i}{\partial \kappa_j} \quad i, j = 1, \cdots, 6. \]  

22
There are two limiting cases for which the saddle-point equations can be solved analytically. The BZA limit is the limit when $J_2 = 0$. The mean-field parameters

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 0, \quad \kappa_6 = \frac{4}{\pi^2},$$

are (up to a sign) the unique solutions of the saddle-point equations. The single particle excitation spectrum is that of a tight-binding fermionic gas at half-filling. The edges of the square with vertices $(\pi, 0)$, $(0, \pi)$, $(-\pi, 0)$, $(0, -\pi)$ at which the spectrum is gapless is the Fermi surface. The Jacobian is formally given by the matrix

$$\begin{pmatrix}
F_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & F_{2,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_{4,4} & 0 & 0 \\
0 & 0 & 0 & 0 & F_{5,5} & 0 \\
0 & 0 & 0 & 0 & 0 & F_{6,6}
\end{pmatrix}_{\text{BZA}}.$$ (4.12)

It turns out that the non-vanishing entries are divergent integrals.

The limiting case $J_1 = 0$ can also be solved analytically. The mean-field parameters

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_6 = 0,$$

$$\kappa_4 = \kappa_5 = \frac{1}{4\pi^2} \frac{1}{2} \int_{\Omega} \sqrt{g_1^2 + g_2^2} \approx 0.339,$$

are (up to signs) the unique solutions of the saddle-point equations. The single particle excitation spectrum has four discrete gapless points on the Brillouin Zone. They are located at

$$\vec{k}_{01} = (\frac{\pi}{2}, 0), \quad \vec{k}_{02} = (0, \frac{\pi}{2}),$$

$$\vec{k}_{03} = (-\frac{\pi}{2}, 0), \quad \vec{k}_{04} = (0, -\frac{\pi}{2}).$$ (4.14)

In the neighborhood of each of these nodes, the excitation spectrum takes the same form as that of a relativistic massless particle. It is shown in appendix that this gapless state is
characterized by an average mean-field flux of \( \pi \) through a cell bounded by the next-nearest-neighbor links and hence it will be called the flux state [51]. In this limit, the Jacobian is well defined and given by

\[
\begin{pmatrix}
F_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & F_{2,2} & 0 & 0 & 0 & 0 \\
0 & 0 & F_{3,3} & 0 & 0 & 0 \\
0 & 0 & 0 & F_{4,4} & F_{4,5} & 0 \\
0 & 0 & 0 & F_{5,4} & F_{5,5} & 0 \\
0 & 0 & 0 & 0 & 0 & F_{6,6}^{\text{flux}}
\end{pmatrix}
\]  

(4.15)

For arbitrary values of the ratio \( J_2/J_1 \), the saddle-point equations can only be solved numerically. The value of the solutions of the saddle-point equations as a function of \( J_2/J_1 \) are plotted in Fig. 2. One observes that \( \kappa_2 = \kappa_3 = 0 \). This results from the lattice symmetry. It also appears that there exists a critical value for the ratio \( J_2/J_1 \) above which the flux mean-field solution is always preferred. The numerics predict this critical value to be between 1.3 and 1.4.

This critical value can be found analytically by requiring the Jacobian determinant to satisfy

\[
\det \left( \frac{\partial F_i}{\partial \kappa_j} \right)^{\text{flux}} = 0.
\]

(4.16)

It is understood here that the Jacobian is calculated for the mean-field parameters in the flux limit but for arbitrary values of the ratio \( J_2/J_1 \). This condition is thus simply

\[
\frac{J_2}{J_1} = \frac{1}{4\pi^2} \frac{1}{\kappa_4} \int_{\Omega} \frac{g_3^2}{\sqrt{g_1^2 + g_2^2}} \approx 1.342
\]

(4.17)

if \( \kappa_4 \approx 0.339 \).

The numerics close to the BZA regime are too unreliable to predict the existence of a critical value of the ratio \( J_2/J_1 \) below which the BZA state prevails. The analytical argument used for the flux phase fails as well in this limit since the Jacobian is not of maximal rank and moreover is ill defined. Indeed, singularities associated with the Fermi surface renders all non-vanishing entries of the Jacobian infinite.
The numerical results clearly indicate that, between $J_2/J_1 = 0.6$ and $J_2/J_1 = 0.7$, the topography of the single particle energy crosses over from a regime with the absolute minima located on the diagonal of $\Omega'$ to a regime with the absolute minima located on the edges of the reduced Brillouin Zone (see Fig. 3). There should therefore exists a crossover value for the ratio $J_2/J_1$ for which there is an accidental degeneracy of the minima. The Lagrange multiplier $|a_0^1|$ appears to keep track of this crossover, the point at which it peaks. Our mean-field results are summarized in Fig. 4.

C. Higher dimensional generalization.

We will try in this section to generalize the s-RVB Ansatz to any spatial dimension $d > 2$. Let $\Lambda = \mathbb{Z}^d$ be a $d$ dimensional hypercubic lattice on which Eq. (2.11) is defined. It is helpful to introduce the $2d$ unit vectors

$$\hat{x}_{\sigma_\mu}^\mu \equiv (0, \cdots, 0, \sigma_\mu, 0, \cdots, 0),$$

$$\mu = 1, \cdots, d, \quad \sigma_\mu = \pm 1 \quad (4.18)$$

and the $2d(d-1)$ vectors of norm $\sqrt{2}$

$$\hat{x}_{\sigma_{\mu\nu}}^{\mu\nu} \equiv (0, \cdots, 0, \sigma_\mu, 0, \cdots, 0, \sigma_\nu, 0, \cdots, 0),$$

$$\mu, \nu = 1, \cdots, d, \quad \sigma_{\mu\nu} \equiv \sigma_\mu \cdot \sigma_\nu, \quad \mu < \nu, \quad (4.19)$$

which connect any given site with all its nearest-neighbors and next-nearest-neighbors, respectively. The straightforward generalization of Eq. (4.1) is

$$\bar{A}_{bit} = A_{\tilde{b}},$$

$$\bar{W}_{ijt} = \begin{cases} 
-X \sigma^3 & \text{if } j = i + \hat{x}_{\sigma_\mu}^\mu, \\
-\text{Re } E \sigma^1 + \text{Im } E \sigma^2 & \text{if } j = i + \hat{x}_{\sigma_{\mu\nu}}^{\mu\nu}, \\
-\text{Re } E \sigma^1 - \text{Im } E \sigma^2 & \text{if } j = i + \hat{x}_{\sigma_{\mu\nu}}^{\mu\nu}.
\end{cases} \quad (4.20)$$

The breaking of all the continuous color symmetries is automatically achieved once this is so for $d = 2$. However, we need to check that the Ansatz is, up to gauge transformations, invariant under all point group transformations of $\Lambda$ as well as under time reversal.
Time reversal is equivalent to complex conjugation. In turn, complex conjugation is
equivalent to $\text{Im } E \rightarrow -\text{Im } E$. But, this change can be undone by the gauge transformation

$$U_i \equiv \text{sgn}(i) \sigma^1,$$

provided $A_0 = \frac{1}{2} a_0^1 \sigma^1$. Here, sgn$(i)$ is the lattice function defined in Eq. (4.24).

We now turn to the point group transformations of $\Lambda$. They are generated cyclically
from the $d!$ permutations of the coordinates of $i \in \Lambda$:

$$(i_1, \cdots, i_d) \rightarrow (i_{n_1}, \cdots, i_{n_d}),$$

$$n_1, \cdots, n_d = 1, \cdots, d,$$

and from the $d$ transformations

$$\cdots, (i_1, \cdots, i_d) \rightarrow (-i_1, \cdots, i_d), \cdots$$

$$\cdots, (i_1, \cdots, i_d) \rightarrow (i_1, \cdots, -i_d).$$

When $d = 2$, the $d! = 2$ permutations are the identity and the reflection about the
diagonal $x = y$. They leave Eq. (4.20) unchanged. The remaining point group transfor-
mations are generated by reflections about the $x$ or $y$ axis. For $d = 2$, they are equivalent
to complex conjugation and Wen’s assertion that Eq. (4.20) describes a spin liquid follows.
But for higher dimensions, the equivalence between complex conjugation and any of the $d$
reflections about a $d - 1$ dimensional hyperplane, Eq. (4.23), is not true anymore.

A possible cure to this problem is to find a partitioning of all the next-nearest-neighbor
links into two disjoint families which are invariant under all point group transformations.
Notice first that the lattice function

$$\text{sgn} : \Lambda \rightarrow \mathbb{Z}_2, \ i \rightarrow \text{sgn}(i) = (-1)^{i_1 + \cdots + i_d}$$

is unchanged by any permutation of the coordinates of $i$. Furthermore, sgn$(i)$ is also left
unchanged when any one of the coordinates of $i$ has its sign changed. Consequently, the
partitioning of $\Lambda$ into the even (odd) sites $\Lambda_{e(o)}$ is not affected by the point group trans-
migrations.
We immediately see that the Ansatz

\[ \tilde{A}_{ijt} = A_{ij}, \quad (4.25) \]

\[ \tilde{W}_{ijt} = \begin{cases} 
-X \sigma^3 & \text{if } j = i + \hat{a}_{\sigma}^{\mu}, \\
- \text{Re } E \sigma^1 + \text{Im } E \sigma^2 & \text{if } j = i + \hat{a}_{\pm 1}^{\mu}, \ i \in \Lambda_e, \\
- \text{Re } E \sigma^1 - \text{Im } E \sigma^2 & \text{if } j = i + \hat{a}_{\pm 1}^{\mu}, \ i \in \Lambda_o, 
\end{cases} \]

respects all the point group symmetries. On the other hand, translational invariance and time reversal appear to be broken. However, they affect the Ansatz Eq. (4.25) in the same way as the gauge transformation Eq. (4.21) does. We conclude that this Ansatz describes a spin liquid which breaks all the continuous symmetries of color SU(2).

In two spatial dimensions, this new Ansatz reduces to the BZA Ansatz in both the \( J_2/J_1 \to 0 \) and \( J_1/J_2 \to 0 \) limits. Moreover, it is not possible to construct a mean-field plaquette term threaded by a flux of \( \pi \). We therefore expect that a gap to the mean-field spinon excitations is generated for all \( 0 < J_1/J_2 < \infty \).

V. GAUGE FIELDS FLUCTUATIONS CLOSE TO AND IN THE FLUX PHASE.

It is fortunate that the mean-field theory yields the flux phase since one can relatively easily identify a continuum effective theory for the long wavelength fluctuations of the soft modes. In contrast, it is much harder to do the same close to the BZA limit. This is so because in the flux phase the locus of points in the Brillouin Zone for which the mean-field single particle energy reaches a minima are isolated. From the location of the isolated minima, one can then extract a reciprocal vector around which the Fourier transform of the spinon and link degrees of freedom can be expanded, thus yielding the intermediate continuum limit. However, it is not feasible to carry out such an approach at the BZA point since there there are infinitely many degenerate minima along a square, the spinon Fermi surface, with vertices \((\pm \pi, 0)\) and \((0, \pm \pi)\) in the Brillouin Zone. Moreover, the difficulty is compounded by the existence of saddle points in the energy surface at the vertices of the spinon Fermi surface causing Van-Hove singularities in the mean-field density of states and
by the nesting property of the Fermi surface. One possible drastic approximation is to replace
the anisotropic Fermi surface altogether by a circle and then expand the nearest-neighbor
links degrees of freedom in powers of the gauge fields up to second order. The continuum
effective theory is that of gauge fields coupled to non-relativistic fermions. But, by doing so,
one neglects magnetic instabilities induced by the nesting property of the Fermi surface at
half-filling. Away from half-filling, such an approximation has been used by Lee and Nagaosa
in their study of the \( t - J \) model. It is then known that the fluctuations of the gauge
fields, when coupled to the “Fermi surface”, lead to severe infrared divergences \[52\] which
possibly signal an instability of the mean-field theory (and thus of the “Fermi surface”) to a
“strange metal” \[33\]. Given these difficulties, we will concentrate on the flux phase and will
not discuss the BZA regime any further. In any event, the mean-field theory predicts that
the BZA and flux regimes are smoothly connected. Thus, we expect that a detailed study of
the flux regime can throw light on the behavior near the BZA limit as well. Notice, however,
that the low energy fermionic states have different physical properties in both regimes given
the level crossing of the excited states shown in the previous section. Nevertheless, since the
mean-field ground state evolves smoothly (i.e., without level crossings) we will still assume
that the physics is qualitatively similar at both ends.

In the spirit of section \[11\], we need to identify the bosonic degrees of freedom of the
continuum limit. To this end we will perform a semiclassical expansion of the link degrees
of freedom around the mean-field Ansatz. In the language of critical phenomena, we will
use the fact that the mean-field theory of the previous section predicts a second order
phase transition at \( (J_2/J_1)_{cr} \). We will then think of the continuum limit as the tuning of
the coupling \( J_2/J_1 \) toward the point \( (J_2/J_1)_{cr} \) where a correlation length (the inverse of
the spinon mass) becomes infinite. In this scheme, the intermediate effective field theory
we obtain is the (infrared unstable) fixed point action of Eq. (2.11) in the sense of the
renormalization group (see Fig. 4). Notice that the semiclassical expansion is not fully
justified since the theory, as it stands, does not have a small parameter that will control
the fluctuations. Formally, we can regard our semiclassical expansion as a \( 1/N \) expansion
in which $N$ has been set to unity. While this approach seems tempting, we should be aware that at small values of $N$ the physics may be different from the semiclassical (mean-field) picture.

The intermediate effective theory so obtained will contain gauge fields and matter fields, which both transform like the adjoint (vector) representation of SU(2) color, and are coupled with (several species of) Dirac fermions. However, the bosons are still not truly dynamical since they do not possess a kinetic energy of their own. On the other hand, all the symmetry requirements that we are after are fulfilled, namely the bosonic fields break all the continuous gauge symmetries down to a discrete subgroup $\mathbb{Z}_2$. In the next subsections we derive the effective continuum theory in the vicinity of the flux phase. It turns out that Wen’s choice of gauge is not very natural in the flux regime. In appendix A we give a more convenient construction in a different (“flux”) gauge which makes simpler the description of the fluctuations around this state. The reader is referred to appendix A for the notation that we will use below.

### A. Continuum limit in the flux phase when $J_1 = 0$.

We will assume that $J_1$ has been turned off. The fermions on the even sublattice are then decoupled from those on the odd sublattice. We shall show how to take the continuum limit so as to obtain relativistic Dirac spinors in 2+1 dimensions which couple via the minimal coupling to two independent set of SU(2) gauge fields.

We will choose to parametrize small fluctuations of the $Q$’s (see Eq. (A12)) around the mean-field Ansatz by

\[
Q_{\hat{x} i}^{c_1} = +i |E| e^{-ik' A_{\hat{x} i}^{c_1}}, \quad Q_{\hat{x} i}^{c_2} = -i |E| e^{-ik' A_{\hat{x} i}^{c_2}}, \\
Q_{\hat{x} i}^{c_3} = +i |E| e^{-ik' A_{\hat{x} i}^{c_3}}, \quad Q_{\hat{x} i}^{c_4} = +i |E| e^{-ik' A_{\hat{x} i}^{c_4}}, \\
Q_{\hat{x} i}^{c_1} = -i |E| e^{-ik' A_{\hat{x} i}^{c_1}}, \quad Q_{\hat{x} i}^{c_2} = +i |E| e^{-ik' A_{\hat{x} i}^{c_2}}, \\
Q_{\hat{x} i}^{c_3} = +i |E| e^{-ik' A_{\hat{x} i}^{c_3}}, \quad Q_{\hat{x} i}^{c_4} = -i |E| e^{-ik' A_{\hat{x} i}^{c_4}},
\]
Here, \( \epsilon'(\bar{\epsilon}) \) is related to the lattice constant \( \epsilon \) by

\[
\epsilon' = \sqrt{2} \epsilon, \quad (\bar{\epsilon} = 2 \epsilon').
\]

(5.2)

Only SU(2) color fluctuations of the link degrees of freedom are considered since the determinant \( |E| \) is kept unchanged.

In the Heisenberg representation, the equations of motion satisfied by the fermions are, to lowest order in \( \bar{\epsilon} \),

\[
iD_o^{e1} f_i^{e1} = v_f \left[ \left( i\partial_{\hat{x}_-} + \frac{1}{2}(A^{e1}_{\hat{x}_-} + A^{e2}_{\hat{x}_-}) \right) f_i^{e2} \right.
\]

\[
\quad - \left( i\partial_{\hat{x}_+} + \frac{1}{2}(A^{e1}_{\hat{x}_+} + A^{e4}_{\hat{x}_+}) \right) f_i^{e4} \right],
\]

\[
iD_o^{e2} f_i^{e2} = v_f \left[ \left( i\partial_{\hat{x}_-} + \frac{1}{2}(A^{e2}_{\hat{x}_-} + A^{e1}_{\hat{x}_-}) \right) f_i^{e1} \right.
\]

\[
\quad + \left( i\partial_{\hat{x}_+} + \frac{1}{2}(A^{e2}_{\hat{x}_+} + A^{e3}_{\hat{x}_+}) \right) f_i^{e3} \right],
\]

\[
iD_o^{e3} f_i^{e3} = v_f \left[ \left( i\partial_{\hat{x}_-} + \frac{1}{2}(A^{e3}_{\hat{x}_-} + A^{e4}_{\hat{x}_-}) \right) f_i^{e4} \right.
\]

\[
\quad + \left( i\partial_{\hat{x}_+} + \frac{1}{2}(A^{e3}_{\hat{x}_+} + A^{e1}_{\hat{x}_+}) \right) f_i^{e2} \right],
\]

\[
iD_o^{e4} f_i^{e4} = v_f \left[ \left( i\partial_{\hat{x}_-} + \frac{1}{2}(A^{e4}_{\hat{x}_-} + A^{e3}_{\hat{x}_-}) \right) f_i^{e3} \right.
\]

\[
\quad - \left( i\partial_{\hat{x}_+} + \frac{1}{2}(A^{e4}_{\hat{x}_+} + A^{e1}_{\hat{x}_+}) \right) f_i^{e1} \right].
\]

(5.3)

Here, the flux velocity \( v_f \) is

\[
v_f = \frac{J_2 |E|}{4},
\]

(5.4)

and the covariant time derivative is

\[
D_o^{ea} = \partial_t + iA_{0i}^{ea}, \quad a = 1, 2, 3, 4.
\]

(5.5)

The same equations hold on \( \Lambda_0^{(1)} \).

In the continuum limit, we make the following identifications:
\[
\lim_{\bar{\epsilon} \to 0} \frac{f_i^{e_1} + f_i^{e_2}}{\bar{\epsilon}} \equiv u^1(\bar{r}t), \quad \lim_{\bar{\epsilon} \to 0} \frac{f_i^{e_3} + f_i^{e_4}}{\bar{\epsilon}} \equiv v^1(\bar{r}t),
\]
\[
\lim_{\bar{\epsilon} \to 0} \frac{f_i^{e_3} - f_i^{e_4}}{\bar{\epsilon}} \equiv u^2(\bar{r}t), \quad \lim_{\bar{\epsilon} \to 0} \frac{f_i^{e_2} - f_i^{e_1}}{\bar{\epsilon}} \equiv v^2(\bar{r}t),
\]
\[
\lim_{\bar{\epsilon} \to 0} A_{0i}^{e_3} \equiv A_1^e(\bar{r}t), \quad \lim_{\bar{\epsilon} \to 0} A_{0i}^{e_4} \equiv A_2^e(\bar{r}t),
\]
\[
\lim_{\bar{\epsilon} \to 0} \bar{\epsilon} \sum_{i \in \Lambda_0^{(1)}} e_2 \equiv \int d^2r,
\]

on \(\Lambda_0^{(1)}\). The counterparts to the fields \(u, v\) and \(A_\mu^e\) on sublattice \(\Lambda_0^{(1)}\) are \(w, z\) and \(A_\mu^o\), respectively.

The equations of motion, when rewritten in terms of
\[
\begin{align*}
    u &= \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \\
    v &= \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \\
    w &= \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \\
    z &= \begin{pmatrix} z^1 \\ z^2 \end{pmatrix},
\end{align*}
\]

take the same form as those for four independent massless Dirac spinors in 2+1 space-time dimensions, the speed of light being here the flux velocity. To describe the relativistic structure, we need the three gamma matrix
\[
\gamma^\mu \equiv \left(\tau^2 \tau^0, -\tau^2 \tau^3, -\tau^2 \tau^1\right) = \left(\tau^2, -i\tau^1, +i\tau^3\right),
\]
\[
D_\mu^{e(o)} \equiv \gamma^\mu D_\mu^{e(o)} \equiv \gamma^\mu \left(\partial_\mu + iA_\mu^{e(o)}\right),
\]

where the \(\tau\) are a new set of Pauli matrices. The long wavelength, low energy limit of Eq. (2.11) when \(J_1 = 0\) and for gauge fields close to the flux phase configuration is
\[
L = \int d^2r \, \mathcal{L},
\]
\[
\mathcal{L} = v_f \left[ \bar{u} i D_\mu^{e} u + \bar{w} i D_\mu^{o} w + \bar{v} i D_\mu^{e} v + \bar{z} i D_\mu^{o} z \right].
\]

The dynamics due to fluctuations in the determinant of the W’s has been neglected

There are thus four flavors of Dirac spinors which interact with two independent set of SU(2) color gauge fields through the minimal coupling. In the continuum limit, the original \(16 = 2 \times (4 + 4)\) fermionic degrees of freedom per unit cell of \(\Lambda_0^{(1)} \cup \Lambda_0^{(1)}\) have become \(16 = 4 \times 2 \times 2\) fields carrying flavor, color and spinor indices, respectively. On the other
hand, the original $16 = 2 \times 4 + 2 \times 4$ degrees of freedom residing on the oriented links of the unit cell have been projected onto the smaller manifold of $4 = 2 + 2 \text{su}(2)$ spatial gauge fields. What we have done could have been achieved by expanding the equations of motion in reciprocal space around the minima of the mean-field spinon spectrum and keeping only the lowest terms in powers of the reciprocal lattice. The gauge fields that are neglected vary rapidly compared to the one kept. Choosing a staggered gauge transformation is equivalent to choosing the origin in reciprocal space to be the location of one of the degenerate minima. The different flavors of the continuum spinons is caused by the degeneracy of the mean-field spectrum.

The local gauge symmetry of Eq. (5.9) is that of the semi-simple gauge group $\text{SU}(2) \times \text{SU}(2)$. Indeed, if we arrange the fermions and gauge fields into the $(j_1, j_2) = (1/2, 1/2)$ reducible representation of $\text{SU}(2) \times \text{SU}(2)$ \cite{54}

$$\Psi \equiv \begin{pmatrix} u \\ w \end{pmatrix}, \quad \Theta \equiv \begin{pmatrix} v \\ z \end{pmatrix}, \quad A_\mu \equiv \begin{pmatrix} A^e_\mu & 0 \\ 0 & A^o_\mu \end{pmatrix},$$

then

$$\mathcal{L} = v_f \left[ \bar{\Psi} i \not{D} \Psi + \bar{\Theta} i \not{D} \Theta \right]$$

(5.11)

is invariant under the local gauge transformation

$$\Psi(\Theta) \rightarrow U \Psi(\Theta), \quad A_\mu \rightarrow U A_\mu U^\dagger + (i \partial_\mu U) U^\dagger,$$

$$\forall U \equiv \begin{pmatrix} U^e & 0 \\ 0 & U^o \end{pmatrix} \in \text{SU}(2) \times \text{SU}(2).$$

(5.12)

**B. Continuum limit in the flux phase when $J_1 \neq 0$.**

The fermions on the even and odd sublattices are coupled by pure fluctuations of the degrees of freedom $M$, Eq. (A13), on the nearest-neighbor links. These fluctuations are of order $O(\epsilon)$ and therefore it is meaningless to separate the contributions from the $\text{SU}(2)$ color and determinant factors \cite{54}. Thus, the fluctuating $M$ form a four dimensional real
vector space. By repeating the same steps as in the previous subsection, one finds the long
correlation length, low energy continuum limit of Eq. (2.11) to be given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2. \quad (5.13)$$

The first contribution $\mathcal{L}_0$ is the one given by Eq. (5.11) in the limit $J_1 = 0$. The second
and third contribution account for the fluctuations of the $M$ on the nearest-neighbor links.
There are 16 such degrees of freedom.

Four of them, $N_{1e}^{eo}, \ldots, N_{4e}^{eo}$, transform like scalars under Lorentz transformations. They
are linearly independent combinations of the original lattice degrees of freedom connecting
even to odd sites and have been rescaled so has to carry the dimensions of inverse length.

With the help of the four dimensional representation

$$\mathcal{N}_1 \equiv \begin{pmatrix} 0 & iN_{1e}^{eo} \\ -iN_{1e}^{eo} & 0 \end{pmatrix}, \quad \mathcal{N}_2 \equiv \begin{pmatrix} 0 & iN_{2e}^{eo} \\ -iN_{2e}^{eo} & 0 \end{pmatrix},$$

$$\mathcal{N}_3 \equiv \begin{pmatrix} 0 & iN_{3e}^{eo} \\ -iN_{3e}^{eo} & 0 \end{pmatrix}, \quad \mathcal{N}_4 \equiv \begin{pmatrix} 0 & iN_{4e}^{eo} \\ -iN_{4e}^{eo} & 0 \end{pmatrix}, \quad (5.14)$$

the second contribution to Eq. (5.13) is

$$\mathcal{L}_1 = -\frac{J_1}{16} \sum_{n=1}^{4} \text{tr} \left[ \mathcal{N}_n \mathcal{N}_n^\dagger \right]$$

$$+ \frac{J_1 v_f}{\sqrt{2J_2|E|}} \left[ \bar{\Psi} \mathcal{N}_1 \Psi + \bar{\Psi} \mathcal{N}_2 \Theta + \bar{\Theta} \mathcal{N}_3 \Psi + \bar{\Theta} \mathcal{N}_4 \Theta \right]. \quad (5.15)$$

Notice the mixing of the four Dirac species (or flavors). Under a local $\text{SU}(2) \times \text{SU}(2)$
color gauge transformation $U$, the $\mathcal{N}$’s transform like the adjoint representation:

$$\mathcal{N}_n \rightarrow U \mathcal{N}_n U^\dagger, \quad n = 1, \ldots, 4. \quad (5.16)$$

As it turns out, the $\mathcal{N}$’s pick up mean-field expectation values below the flux phase which
break the $\text{SU}(2) \times \text{SU}(2)$ symmetry down to $\mathbb{Z}_2$, the center of $\text{SU}(2)$ color.

The description of the remaining twelve degrees of freedom follows the same line as that
for the $\mathcal{N}$’s except that they can be grouped into four families, $A_{\mu}^{eo}, B_{\mu}^{eo}, C_{\mu}^{eo}, D_{\mu}^{eo}$, each
transforming like a three-vector under Lorentz transformation. With the help of the four
dimensional representation

\[
A_\mu \equiv \begin{pmatrix} 0 & A^{\mu \text{eo}}_\mu \\ A^{\mu \text{eo}}_\mu & 0 \end{pmatrix}, \quad B_\mu \equiv \begin{pmatrix} 0 & B^{\mu \text{eo}}_\mu \\ C^{\mu \text{eo}}_\mu & 0 \end{pmatrix}, \\
C_\mu \equiv \begin{pmatrix} 0 & C^{\mu \text{eo}}_\mu \\ B^{\mu \text{eo}}_\mu & 0 \end{pmatrix}, \quad D_\mu \equiv \begin{pmatrix} 0 & D^{\mu \text{eo}}_\mu \\ D^{\mu \text{eo}}_\mu & 0 \end{pmatrix},
\]

(5.17)

the third contribution to Eq. (5.13) is

\[
\mathcal{L}_2 = -\frac{J_1}{16} \text{tr} \left[ A^{\mu} A^{\mu \dagger} + B^{\mu} B^{\mu \dagger} + C^{\mu} C^{\mu \dagger} + D^{\mu} D^{\mu \dagger} \right] + \frac{J_1 v_f}{\sqrt{2} J_2 |E|} \left[ \bar{\Psi} A \Psi + \bar{\Psi} B \Theta + \bar{\Theta} C \Psi + \bar{\Theta} D \Theta \right].
\]

(5.18)

The structure of \( \mathcal{L}_2 \) is the same as that for \( \mathcal{L}_1 \) except for the relativistic transformation properties of the dynamical fields. Any mean-field expectation value of the \( A_\mu, \ldots, D_\mu \)'s below the flux phase would imply a loss of relativistic covariance. But, as it turns out, these fields do not pick up a mean-field expectation value just below the onset of the flux phase. We do not expect that they play a relevant role to the issue of the confinement of the spinons.

C. Continuum limit just below \( (J_2/J_1)_{\text{cr}} \).

Below the onset of the flux phase, the mean-field solutions of the saddle-point equations take the form

\[
a_1^0 \neq 0, \quad a_2^2 = 0, \quad a_3^3 = 0,
\]

(5.19)

\[
\text{Re} \ E \neq 0, \quad \text{Im} \ E \neq 0, \quad X \neq 0.
\]

The six mean-field parameters evolve continuously into their flux phase values as \( J_2/J_1 \) approaches from below its critical value. What continuum limit can we expect close to the
onset of the flux phase? Strictly speaking, the locations of the minima for the mean-field excitation spectrum are functions of the parameters $\tilde{a}_0$, $E$ and $X$, i.e., of $J_2/J_1$ and it would appear that any analytical approach is hopeless.

However, let us expand formally the mean-field energy excitations in powers of the lattice spacing $\epsilon$ by inserting

$$\tilde{k}_l = k_{0l} + \epsilon \tilde{p}, \quad l = 1, 2, 3, 4,$$

(5.20)

into the mean-field dispersion relation. Here, the lattice spacing $\epsilon$ merely plays the role of a bookkeeping device. We now assume that

$$\exists \delta, C > 0,$$

$$\left\{ \begin{array}{l}
\lim_{\epsilon \to 0} \frac{1}{\epsilon^{1+\delta}} |a_0^1| < C, \\
\lim_{\epsilon \to 0} \frac{1}{\epsilon^{\delta}} |\text{Re} \ E - \text{Re} \ E_f| < C, \\
\lim_{\epsilon \to 0} \frac{1}{\epsilon^{\delta}} |\text{Im} \ E - \text{Im} \ E_f| < C, \\
\lim_{\epsilon \to 0} \frac{1}{\epsilon^{\delta}} |X| < C,
\end{array} \right.$$  

(5.21)

so that the formal expansion in powers of $\epsilon$ of the mean-field excitation energies around any of the four points within the Brillouin Zone at which the flux spectrum is gapless, yields the relativistic spectra

$$|\tilde{\xi}_{k_l}| = v_f \sqrt{\tilde{p}^2 + m^2 v_{f}^2} + O(\epsilon^2).$$

(5.22)

The mean-field parameters of Eq. (5.22) are taken in the flux phase and

$$v_f = J_2 |\text{Re} \ E| \epsilon, \quad m \equiv \frac{J_1 |X|}{2v_{f}^2}.$$  

(5.23)

This simplification suggests that the problem becomes tractable if we can argue that the deviations of $a_0^1$ and $E$ from their flux phase values are negligible compared to the deviations of the mean-field parameter $X$ from its flux phase value. By inspection of the saddle-point solutions, one extracts that on approaching the flux phase from below (see Fig. 3)
\[ |a_0^1| \propto \left( 1 - \frac{(J_2/J_1)}{(J_2/J_1)_{cr}} \right)^{\beta_1}, \quad \beta_1 = 1.52, \]
\[ |\text{Re} E - \text{Re} E_1| \propto \left( 1 - \frac{(J_2/J_1)}{(J_2/J_1)_{cr}} \right)^{\beta_2}, \quad \beta_2 = 1.79, \]
\[ |\text{Im} E - \text{Im} E_1| \propto \left( 1 - \frac{(J_2/J_1)}{(J_2/J_1)_{cr}} \right)^{\beta_3}, \quad \beta_3 = 1.77, \]
\[ |X| \propto \left( 1 - \frac{(J_2/J_1)}{(J_2/J_1)_{cr}} \right)^{\beta_4}, \quad \beta_4 = 0.84. \] (5.24)

The important insight that follows from Eq. (5.24) is that \( \beta_4 = \min\{\beta_1, \ldots, \beta_4\} \). Thus, the deviation of \( X \) from its flux value is more important than the deviations of \( a_0^1 \), \( \text{Re} E \) and \( \text{Im} E \) from their flux values. We will now show how to recover the massive relativistic dispersion relation of Eq. (5.22) from the relativistic theory that was constructed in the previous sections when \( a_0^1 \) and \( J_2E \) are assumed to take their flux phase values whereas \( J_1X \) is non-vanishing.

In the flux gauge of appendix A the sixteen degrees of freedom of Eq. (5.14) and Eq. (5.17) have the mean-field values
\[
\bar{A}_\mu^{eo} = \bar{B}_\mu^{eo} = \bar{C}_\mu^{eo} = \bar{D}_\mu^{eo} = 0, \quad \mu = 0, 1, 2, \\
\bar{N}_n^{eo} = \frac{2X}{\bar{\epsilon}} V_n, \quad n = 1, \ldots, 4, \] (5.25)
where
\[
V_1 \equiv \frac{1}{\sqrt{2}} (i\sigma^0 - W_-) = +ie^{i\pi_4 W_-}, \\
V_2 \equiv \frac{1}{\sqrt{2}} (W_+ - W_3) = +ie^{-i\pi_4 2\sqrt{2}(W_+ - W_3)}, \\
V_3 \equiv \frac{1}{\sqrt{2}} (W_+ + W_3) = +ie^{-i\pi_4 \sqrt{2}(W_+ + W_3)}, \\
V_4 \equiv \frac{1}{\sqrt{2}} (i\sigma^0 + W_-) = +ie^{-i\pi_4 W_-}. \] (5.26)

Consequently, at mean-field Eq. (5.13) reduces to
\[
\bar{\mathcal{L}} = \bar{\mathcal{L}}_0 + \bar{\mathcal{L}}_1 + \bar{\mathcal{L}}_2. \] (5.27)

Here, if
\[ q \equiv \begin{pmatrix} \Psi \\ \Theta \end{pmatrix} = \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix}, \] (5.28)

then

\[ \mathcal{L}_0 = v_f \bar{q} i \partial_\mu q, \] (5.29)

\[ \mathcal{L}_1 = -\frac{J_1}{16} \sum_{n=1}^{4} \text{tr} \left[ \bar{N}_n N_n^\dagger \right] + v_f \bar{q} m v_f M q, \] (5.30)

\[ \mathcal{L}_2 = 0. \] (5.31)

The matrix \( m \bar{M} \) plays the role of a mass matrix for the \( q \)’s. It is given in a 4×4 dimensional representation by

\[ \bar{M} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & +V_1 & 0 & +V_2 \\ -V_1^\dagger & 0 & -V_3^\dagger & 0 \\ 0 & +V_3 & 0 & +V_4 \\ -V_2^\dagger & 0 & -V_4^\dagger & 0 \end{pmatrix}. \] (5.32)

It can be checked that the matrix \( \bar{M} \), which, in its full glory, is a 16 by 16 matrix, has the alternating eigenvalues ±1. Hence, the dispersion relation Eq. (5.22) is reproduced by Eq. (5.27). The global SU(2)×SU(2) color symmetry is broken by Eq. (5.27) all the way down to \( Z_2 \). Again, the generation of the masses for the four Dirac species \( u, w, v \) and \( z \) is equivalent to the breaking of all the continuous color symmetries.

Finally, by using a linear parametrization of the scalar gauge fields fluctuations, the non-linear parametrization of the gauge fluctuations on the next-nearest-neighbor links of Eq. (5.1) and a linear parametrization of the nearest-neighbor links fluctuations, we find from Eq. (5.13) and Eq. (5.30) the intermediate effective theory

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_1 + \mathcal{L}_2. \] (5.33)
D. Symmetries of the intermediate effective theory in the flux phase.

In the past subsections we showed that it is possible to construct an effective continuum theory for the slow modes of the system near the flux regime. The effective theory that we constructed has all the right ingredients for the scenario of section III to work. In particular, below the flux phase, all the continuous gauge symmetries are broken down to a discrete $\mathbb{Z}_2$ and the fermions are all massive. A loop expansion in the gauge coupling constant is well defined order by order and yields a local action for the gauge fields. The only low energy degrees of freedom left are the adjoint bosonic matter fields (which have acquired dynamics through the fermions) and the gauge fields. Hence, the effective action for these degrees of freedom will have the form indicated in section III. We will present elsewhere the calculation of this effective bosonic action [35]. By construction, it will have the two crucial features that all continuous symmetries are broken and that the only remaining symmetry is discrete namely $\mathbb{Z}_2$. In view of the arguments given at the beginning of section III, the spinon quantum numbers are present in the spectrum of the theory provided the effective scales for the kinetic energies of the gauge and matter fields are large enough. By varying the ratio $J_2/J_1$, the parameters $\beta$ and $\kappa$ in Eq. (3.3) of the effective theory change. However, the mean-field results alone do not guarantee that it is possible to move from the confining regime to the Anderson-Higgs phases of the effective theory. If this can be done, then we have a mechanism which does not break parity and time-reversal and allows for the presence of the color spinon quantum number in the spectrum of the theory. The only other mechanism that we are aware of has its origin in the chiral spin liquid for which the effective continuum limit contains a Chern-Simons term [17]. The presence of the Chern-Simons term in the bosonized action removes the interactions responsible otherwise for the confinement of the fundamental color charge [56].

Let us take the limit $J_2/J_1 \to \infty$ for which the Heisenberg model reduces to two independent, nearest-neighbor antiferromagnetic Heisenberg models defined on the even and odd sublattices, respectively. Although the exact ground state of the nearest-neighbor, spin-1/2,
antiferromagnetic Heisenberg model is not known, there are strong evidences that it carries Néel order \[57\]. We therefore expect to find degenerate ground states carrying independent Néel order on the even and odd sublattices and our mean-field theory must be unstable \[58\] to the Néel ordering. The natural question to ask is whether Eq. (5.9) can account for this instability or if we need to include fluctuations with higher energy and/or shorter wavelength. Since Néel ordering requires the breaking of a global symmetry, let us see what symmetries beside the local gauge symmetry Eq. (5.9) possesses.

When $J_2/J_1 \to \infty$, there exists an additional global $U(4)$ (flavor) symmetry under the transformation

$$
\begin{pmatrix}
  u \\
  w \\
  v \\
  z
\end{pmatrix}
\to V
\begin{pmatrix}
  u \\
  w \\
  v \\
  z
\end{pmatrix},
\forall V \in U(4) = U(1) \times SU(4).
$$

(5.34)

A possible manifestation of an instability of the flux phase is the dynamical breaking of the global flavor symmetry as a result of the gauge fluctuations. We have listed in table II of appendix B all 15 lattice Hamiltonians $H_a$ expressed in terms of the spinon operators defined by Eq. (2.3) which yield all 15 generators $\bar{q} T_a q$ breaking the $SU(4)$ flavor symmetry. One sees that the 9 generators $T_1, T_3, T_4, T_6, T_7, T_9, T_{11}, T_{13}$ and $T_{15}$ are all related to different types of valence bond ordering which do not break the spin symmetry \[59\]. The remaining 6 generators are related to spin densities and spin currents. For example, $\bar{q} T_2 q$, measures the Néel ordering on the unit cell of the even sites whereas $\bar{q} T_4 q$ measures the Néel ordering on the odd sites. Finally, the generator of $U(1)$ is also related to valence bond ordering. We thus conclude that Eq. (5.9) has enough symmetries to account for the instability towards Néel ordering. In fact our effective action could be unstable to many other types of ordering including Peierls ordering.

It is an open problem to show from Eq. (5.9) that the flavor symmetry is broken dynamically \[60\]. The difficulty comes from the non-perturbative nature of such a mechanism. For example, one can formally integrate the fermions in Eq. (5.9). To one loop, the resulting
action for the gauge fields is finite but non-local \[61\]. However, higher loop contributions are infrared divergent due to the masslessness of the fermions. The infrared problem can be bypassed if one chooses a dimensionless expansion parameter such as \(1/N_F\) with \(N_F\) the flavor number, instead of the dimensionfull gauge coupling. Within this approximation, the lowest order in \(1/N_F\) already predicts spontaneous symmetry breaking of SU(4) \[62\] although this result has to be taken with caution \[60\]. When \((J_2/J_1)_{cr} < J_2/J_1 < \infty\), the effective theory Eq. (5.13) has only a U(1) global flavor symmetry. For a small number of flavors (such as \(4 = 2 \times 2\) in our case), it is likely that the instability of the flux phase with respect to the quantum fluctuations at \((J_2/J_1)_{cr}\) is preempted by a first order transition from a state carrying long range spin-spin correlations to a spin liquid. In any case, Eq. (5.13) seems rich enough to describe the phase diagram of the \(J_1 - J_2\) spin-1/2 antiferromagnetic Heisenberg model as inferred from a bosonic representation of a Sp(N) generalization of the model \[26\].

VI. THE ONE DIMENSIONAL CASE.

In the previous section we have discussed a scenario for separation of spin and charge (or the occurrence of unusual quantum numbers for excitations) to take place in two dimensional systems. In this section we want to see what picture this scenario yields about the one dimensional Heisenberg chain. Our purpose is twofold. Firstly, the one dimensional Heisenberg chain is known to exhibit separation of spin and charge in the sense that it has in its excitation spectrum gapless excitations with spin-1/2. These excitations are nowadays referred to as the spinons of the one dimensional chain. Secondly, we want to make a comparison with the two dimensional problem. In particular, we want to see what is the relation, if any, between the gapless spinons of the one dimensional chain and the spinons of the gauge picture.

The one dimensional Heisenberg chain has been extensively studied by many methods and much is known about the physics of this system. The Bethe Ansatz yields the ground state and the excitations above it. Conformal field theory gives the critical exponents of the
correlation functions. The excitation spectrum thus found consists of gapless integer spin excitations ("spin waves") and spin-1/2 gapless excitations ("spinons").

The integer spin excitations are the expected spin flips. The gapless spin-1/2 excitations are the ones we are interested in. The simplest way to see what the "spinons" are is by means of the Jordan-Wigner [30] transformation of the spin-1/2 operators:

\[
c_i^\dagger = e^{i\pi \sum_{j<i} S_j^+ S_j^-} S_i^+ e^{-i\pi \sum_{j<i} S_j^+ S_j^-},
\]

\[
c_i = e^{i\pi \sum_{j<i} S_j^+ S_j^-} S_i^+, \tag{6.1}
\]

From Eq. (2.17), we recognize in the raising and lowering spin-1/2 operators \( S_i^\pm \) the color singlet "baryon" operators \( b_i^\dagger \) and \( b_i \), respectively:

\[
S_i^+ \equiv S_i^1 + iS_i^2 = b_i^\dagger,
\]

\[
S_i^- \equiv S_i^1 - iS_i^2 = b_i, \tag{6.2}
\]

whereas the "mesonic" operator \( m_i^\dagger \) is related to

\[
S_i^+ S_i^- = \frac{1}{2} + S_i^3 = \frac{1}{2} m_i. \tag{6.3}
\]

The fermion operators \( c_i \) defined by the Jordan-Wigner transformation can be viewed as spinless fermions in the sense that they do not carry an explicit spin index. Naturally, the expression of Eq. (6.1) depends on the choice of the spin orientation. In this sense the \( c \)'s do transform under the global SU(2) of spin. But, and this is what matters for a comparison with the gauge picture, they are singlets under the local SU(2) (color) and consequently gauge invariant. Thus, these are allowed excitations which are compatible with confinement of the color degrees of freedom. Another important feature of the "spinon" operators of Eq. (6.1) is the fact that they disturb the boundary conditions and, hence, are topological excitations. These operators are non-local and gauge invariant. In contrast, the spinons of the gauge picture are local but not invariant under local SU(2) (color) transformations. In addition, the spinons of the gauge picture, are absent from the physical spectrum if the system is realized in a confining state. Clearly, the Jordan-Wigner spinons are not.
To sum up, the fermion $c_i$, when expressed in terms of the baryons $b_i$ and mesons $m_i$, give a non-local and non-linear realization of an operator transforming like the fundamental representation of SU(2) under spin rotations and transforming like a color singlet under gauge transformations. This is very suggestive of how confinement precludes any simple description of the spin-1/2 low energy sector of the one dimensional Heisenberg model in terms of color carrying spinons.

Let us see what the SU(2) gauge approach tells us about the one dimensional case. In particular we would like to see how the known physics of this system is recovered in the gauge field approach. It is straightforward to apply the methods of the previous sections to the one dimensional case. In a separate publication we do so [35]. Here, we will discuss several qualitative differences that arise when this picture is applied to the one dimensional case.

Along the lines of section IV C, we try the mean-field Ansatz

$$\bar{A}_{bit} = A_\sigma,$$  \hspace{1cm} (6.4)

$$\bar{W}_{ijt} = \begin{cases} 
-X\sigma^3 & \text{if } j = i + 1, \\
-Re\ E\sigma^1 + Im\ E\sigma^2 & \text{if } j = i + 2, i \text{ even}, \\
-Re\ E\sigma^1 - Im\ E\sigma^2 & \text{if } j = i + 2, i \text{ odd}, \\
0 & \text{otherwise.}
\end{cases}$$

It describes a spin liquid if and only if

$$A_\sigma = \frac{1}{2} a_{\sigma}^1 a_1^1.$$

(6.5)

This Ansatz satisfies all of the requirements: a) it is translationally invariant (up to gauge transformations), b) it respects all point group symmetries and time reversal invariance (up to gauge transformations) and c) it achieves a complete breaking of the local continuous gauge symmetry through the operators

$$P_{i}^e = W_{(i+1)(i+2)} W_{(i+1)(i+2)}^\dagger;$$

$$P_{i+1}^o = W_{(i+2)(i+3)} W_{(i+2)(i+3)}^\dagger, \quad \forall i \text{ even.}$$

(6.6)
In particular, just as in the two dimensional case, it leaves a discrete $\mathbb{Z}_2$ symmetry unbroken.

However, discrete gauge symmetries play a very different role in one and two space dimensions. This is so since the lower (space-time) critical dimension for gauge theories with a discrete symmetry group (such as $\mathbb{Z}_2$ in the case of interest for our problem) is three. Hence, in the case of the one dimensional quantum antiferromagnet, which lives in two space-time dimensions, the effective $\mathbb{Z}_2$ gauge theory along the topmost line of the phase diagram in Fig. 1 is below the lower critical dimension and thus is always in a confining phase. No phase transition is encountered along the line $(\beta, \kappa, \lambda) = (\beta, \infty, \infty)$ and there can only be one phase in the phase diagram, the confinement-Higgs phase. In this phase, the spinon carrying the fundamental color charge is confined and is not part of the spectrum of finite energy states. Clearly, the spin-1/2 excitations of the one dimensional quantum Heisenberg antiferromagnet are not the spinons of the gauge picture, as has been claimed rather loosely in the literature of the subject. As we stressed above, the fermion states described by the Jordan-Wigner operators are manifestly gauge invariant under the local SU(2) symmetry and are unaffected by confinement.

VII. CONCLUSION.

We have studied the issue of spin and charge separation for strongly correlated electronic systems in two spatial dimensions by studying the frustrated Heisenberg model for spin-1/2 in a representation of the spin degrees of freedom in terms of fermions (spinons) and gauge fields. Our treatment preserves explicitly the existing SU(2) local gauge symmetry which allows us to treat on the same footing the Affleck-Marston and the Anderson order parameters. We have given a set of sufficient conditions for the system to have excitations in which the spinons carry the separated spin quantum numbers, i.e., spin-1/2 degrees of freedom, inspite of the existence of strong gauge fluctuations.

We solved the saddle-point equations for an isotropic mean-field Ansatz first proposed by Wen and depending on four real parameters. There exists a second order phase transition
to a new flux phase which is triggered by the competition between the nearest $J_1$ and next-nearest-neighbor $J_2$ antiferromagnetic couplings of the model. The flux phase is chosen for strong enough $J_2$. Below the flux phase, a gap for the spinons opens and reaches a maxima for $J_2/J_1 \approx 0.7$. Although no level crossing occurs for the mean-field ground states below the flux phase, there is a level crossing for the excited states when $J_2/J_1 \approx 0.7$. On approaching the pure nearest-neighbor limit, the features of a tight-binding Fermi surface at half-filling emerge. The phase $0 < J_2/J_1 < (J_2/J_1)_{cr}$, the s-RVB phase, is characterized by an energy gap and the breaking of all continuous gauge symmetries.

We derived a continuum theory for the soft degrees of freedom in the vicinity of the flux phase which describes bosonic gauge and matter fields interacting with the spinons. In the flux phase, the continuum theory is likely to be unstable towards formation of long range antiferromagnetic order or dimer ordering. A more detailed calculation is needed to decide which ordering is chosen by the system but, in any case, deconfinement of the spinons is not expected to take place. We showed that below the flux phase, the only important remaining soft modes are those of a $Z_2$ gauge theory in three space-time dimensions due to the breaking of all continuous gauge symmetries by the s-RVB Ansatz. Deconfinement of the gauge spinons is then allowed in the phase diagram of a generic theory with the same symmetry attributes but in which the gauge and matter fields have independent kinetic energy scales. In our case, the calculation of the ratio of the effective energy scales for the gauge and matter fields through the integration of the spinons is needed to decide whether the system is in the confining or deconfining regime. *Consequently, although deconfinement of the spinons is allowed it is not guaranteed* since it depends on the values of microscopic parameters. This result stands in sharp contrast to the analysis of the one dimensional quantum antiferromagnet which predicts that the gauge spinons are always confined, irrespectively of the value of the effective energy scales for the gauge and matter fields, although the system still exhibits separation of spin and charge. The mechanism for separation of spin and charge in one dimension is topological and it is unrelated to the issue of confinement.
ACKNOWLEDGMENTS

We have benefitted from useful discussions with J.-F. Lagaë. This work was supported in part by NSF grants No. DMR91-22385 at the Department of Physics of the University of Illinois at Urbana-Champaign, DMR89-20538 at the Materials Research Laboratory of the University of Illinois, an IBM Graduate Student Fellowship and by a grant of the Research Board of the University of Illinois at Urbana-Champaign. We have made extensive use of the facilities of the Materials Research Laboratory Center for Computations.

APPENDIX A: THE FLUX PHASE

We have found in section IV that above a critical value of the coupling constant $J_2/J_1$, the s-RVB phase collapses to the case

$$\vec{a}_0 = 0, \quad \text{Re} \, E = \text{Im} \, E, \quad X = 0,$$

i.e., the spectrum takes the simple form

$$|\vec{\xi}_k| = J_2 |\text{Re} \, E| \sqrt{\cos^2 k_x \cos^2 k_y + \sin^2 k_x \sin^2 k_y}.$$

The spectrum is thus gapless and linearization around the four minima of Eq. (4.14) yields a massless relativistic dispersion relation.

Another characterization of this phase can be made. We introduce the orthogonal and normalized basis

$$W_- = -\frac{\sigma^1 + \sigma^2}{\sqrt{2}}, \quad W_+ = -\frac{\sigma^1 - \sigma^2}{\sqrt{2}}, \quad W_3 = -\sigma^3,$$

of the Lie Algebra $su(2)$. This basis satisfies the $su(2)$ Algebra with

$$W_- W_+ = iW_3, \quad W_+ W_3 = iW_-, \quad W_3 W_- = iW_+.$$
and can be used to rewrite
\[
\tilde{W}_{ijt} = \begin{cases} 
0 & \text{along the links } \hat{x} \text{ and } \hat{y}, \\
|E|W_+ & \text{along the link } \hat{x}_+, \\
|E|W_- & \text{along the link } \hat{x}_-, 
\end{cases}
\] (A5)

whenever \( J_2 \) is larger than its critical value. This regime is characterized uniquely by the most elementary path ordered operator along a closed path:
\[
\bar{P}_o \equiv W_-W_+W_+^\dagger W_+^\dagger = -\sigma^0. 
\] (A6)

Here, \( \sigma^0 \) is the neutral element of SU(2). Eq. (A6) tells us that there is a mean-field flux of \( \pi \) per elementary plaquette, a situation commonly referred to as the flux phase \([14,15]\).

As of now, the mean-field Ansatz had an explicit translational invariance. In order to describe small fluctuations of the link and time-like \( su(2) \) degrees of freedom around their mean-field value in the flux phase, it turns out to be convenient to choose a gauge in which the color structure on the next-nearest-neighbor links becomes trivial. The price to be payed is a reduction of the translational symmetry. First we need some notation.

We introduce eight sublattices (see Fig. 3) by partitioning first the square lattice \( \Lambda \), whose sites \( i \in \mathbb{Z}^2 \) are labelled by pairs of integer, into two interpenetrating sublattices \( \Lambda_e \) and \( \Lambda_o \):
\[
\Lambda_e = \{ i \in \Lambda | i_1 + i_2 = 0 \mod 2 \}, \\
\Lambda_o = \{ i \in \Lambda | i_1 + i_2 = 1 \mod 2 \}. 
\] (A7)

The even and odd sublattices are then partitioned into four sublattices \( \Lambda_e^{(a)} \), \( \Lambda_o^{(a)} \), \( a = 1, 2, 3, 4 \) by choosing
\[
\Lambda_e^{(1)} = \left\{ i \in \Lambda_e | \exists j \in \mathbb{Z}^2, i = j_1 2\hat{x}_- + j_2 2\hat{x}_+ \right\}, \\
\Lambda_o^{(1)} = \left\{ i \in \Lambda_o | \exists j \in \Lambda_e^{(1)}, i = j + \hat{x} \right\}. 
\] (A8)

and using translations to define

46
\[ \Lambda_e^{(2)} = \left\{ i \in \Lambda_e \mid \exists j \in \Lambda_e^{(1)}, i = j + \hat{x}_- \right\}, \]
\[ \Lambda_e^{(3)} = \left\{ i \in \Lambda_e \mid \exists j \in \Lambda_e^{(2)}, i = j + \hat{x}_+ \right\}, \]
\[ \Lambda_e^{(4)} = \left\{ i \in \Lambda_e \mid \exists j \in \Lambda_e^{(3)}, i = j - \hat{x}_- \right\}, \] (A9)

and similarly for the odd sublattices \( \Lambda_o^{(2)} \), \( \Lambda_o^{(3)} \) and \( \Lambda_o^{(4)} \). From now on, \( i \) will always be taken to belong to \( \Lambda^{(1)}_e \).

The fermionic doublets will be renamed

\[ \psi_i \rightarrow f_{i}^{e1}, \quad \psi_{i+\hat{x}} \rightarrow f_{i}^{o1}, \]
\[ \psi_{i+\hat{x}_-} \rightarrow f_{i}^{e2}, \quad \psi_{i+\hat{x}+\hat{x}_-} \rightarrow f_{i}^{o2}, \]
\[ \psi_{i+\hat{x}_-+\hat{x}_+} \rightarrow f_{i}^{e3}, \quad \psi_{i+\hat{x}_-+\hat{x}_+-\hat{x}_+} \rightarrow f_{i}^{o3}, \]
\[ \psi_{i+\hat{x}_+} \rightarrow f_{i}^{e4}, \quad \psi_{i+\hat{x}+\hat{x}_+} \rightarrow f_{i}^{o4}. \] (A10)

In the same way, there will be \( 2 \times 4 = 8 \) independent time-like gauge fields:

\[ A_{0i} \rightarrow A_{0i}^{e1}, \ldots, A_{0(i+\hat{x}+\hat{x}_+)} \rightarrow A_{0i}^{o4}, \] (A11)

\( 2 \times 8 = 16 \) next-nearest-neighbor link degrees of freedom:

\[ W_{i(i+\hat{x}_-)} \rightarrow Q_{\hat{x}^-i}^{e1}, \ldots \]
\[ \ldots, W_{i(i+\hat{x}_-+\hat{x}_+)(i+\hat{x}_-+\hat{x}_+)} \rightarrow Q_{\hat{x}^-i}^{o4}, \] (A12)

... \( 2 \times 8 = 16 \) nearest-neighbor link degrees of freedom:

\[ W_{i(i+\hat{x})} \rightarrow M_{\hat{x}i}^{e1}, \ldots \]
\[ \ldots, W_{i(i+\hat{x}+\hat{x}_+)(i+\hat{x}_++\hat{y})} \rightarrow M_{\hat{y}i}^{o4}. \] (A13)

We are now ready to perform the staggered gauge transformation

\[ f_{i}^{e1} \rightarrow \text{sgn}(i) (+\sigma^0) f_{i}^{e1}, \quad f_{i}^{o1} \rightarrow \text{sgn}(i) (+\sigma^0) f_{i}^{o1}, \]
\[ f_{i}^{e2} \rightarrow \text{sgn}(i) (-iW_-) f_{i}^{e2}, \quad f_{i}^{o2} \rightarrow \text{sgn}(i) (-iW_-) f_{i}^{o2}, \]
\[ f_{i}^{e3} \rightarrow \text{sgn}(i) (-iW_3) f_{i}^{e3}, \quad f_{i}^{o3} \rightarrow \text{sgn}(i) (-iW_3) f_{i}^{o3}, \]
\[ f_{i}^{e4} \rightarrow \text{sgn}(i) (+iW_+) f_{i}^{e4}, \quad f_{i}^{o4} \rightarrow \text{sgn}(i) (+iW_+) f_{i}^{o4}, \] (A14)
for all $i \in \Lambda_0^{(1)}$. Here, $\text{sgn}(i)$ is the sign function which distinguishes between even and odd sites of sublattice $\Lambda_0^{(1)}$. Under this gauge transformation, the mean-field Ansatz for the nearest-neighbor links takes the simple form

$$
\bar{Q}_{x,i}^{e_1} = +i|E| \sigma^0, \quad \bar{Q}_{x,i}^{e_2} = -i|E| \sigma^0,
$$

$$
\bar{Q}_{x,i}^{e_3} = +i|E| \sigma^0, \quad \bar{Q}_{x,i}^{e_4} = +i|E| \sigma^0,
$$

and similarly on $\Lambda_0^{(1)}$. One easily verifies that Eq. (A6) still holds.

**APPENDIX B: CONTINUUM LIMIT FOR SOME LATTICE HAMILTONIANS.**

We have collected in this appendix all the 15 lattice Hamiltonians which yield the 15 generators of the global SU(4) symmetry of the Lagrangian density Eq. (5.9). They are displayed in table I below.
REFERENCES

[1] *High Temperature Superconductivity, Proceedings of the 1989 Los Alamos Symposium*, eds. K. S. Bedell, et al., Addison-Wesley, Redwood City, CA, 1990.

[2] P. W. Anderson, Science **235**, 1196 (1987).

[3] P. W. Anderson, Phys. Rev. Lett. **64**, 1839 (1990).

[4] R. B. Laughlin, Science **242**, 525 (1988).

[5] G. S. Canright and M. D. Johnson, Comments Cond. Mat. Phys. **15**, 77 (1990).

[6] J. Hubbard, Proc. Roy. Soc. London A **281**, 401 (1964).

[7] J. E. Hirsch, Phys. Rev. Lett. **54**, 1317 (1985); C. Gros, R. Joynt and T. M. Rice, Phys Rev. B **36**, 381 (1987); F. C. Zhang and T. M. Rice, Phys. Rev. B **37**, 3759 (1988).

[8] V. J. Emery, Phys. Rev. B **14**, 2989 (1976).

[9] S. E. Barnes, J. Phys. F **6**, 1375 (1976), and **7**, 2637 (1977); P. Coleman, Phys. Rev. B **29**, 3035 (1984); G. Kotliar and A. E. Ruckenstein, Phys. Rev. Lett. **57**, 1362 (1986).

[10] Z. Zou and P. W. Anderson, Phys. Rev. B **37**, 627 (1988).

[11] D. P. Arovas and A. Auerbach, Phys. Rev. B **38**, 316 (1988); R. Shankar, Phys. Rev. Lett. **63**, 203 (1989); N. Read and S. Sachdev, Phys. Rev. B **42**, 4568 (1990); N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991); N. Dorey and N. E. Mavromatos Phys. Rev. B **44**, 5286 (1991).

[12] G. Baskaran, Z. Zou and P. W. Anderson, Solid State Commun. **63**, 973 (1987).

[13] A. E. Ruckenstein, P. Hirschfeld, J. Appel, Phys. Rev. B **36**, 857 (1987).

[14] G. Kotliar, Phys. Rev. B **37**, 3664 (1988).

[15] I. K. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).
[16] T. Dombre and G. Kotliar, Phys. Rev. B 39, 855 (1989).

[17] X. G. Wen, F. Wilcecek and A. Zee, Phys. Rev. B 39, 11413 (1989).

[18] N. Read and S. Sachdev, Nucl. Phys. B 316, 609 (1989).

[19] M. Grilli and G. Kotliar, Phys. Rev. Lett. 64, 1170 (1990); S. Sachdev, Phys. Rev. B 41, 4502 (1990); J. P. Rodriguez and B. Douçot, Europhys. Lett. 11, 451 (1990); M. U. Ubbens and P. A. Lee, Phys. Rev. B 46, 8434 (1992). For a variational point of view see P. W. Anderson, B. S. Shastry and D. Hristopoulos, Phys. Rev. B 40, 8939 (1989); P. Lederer, D. Poilblanc and T. M. Rice, Phys. Rev. Lett. 63, 1519 (1989).

[20] We understand under Mott insulator an insulator with an odd number of electron per unit cell.

[21] I. K. Affleck, Z. Zou, T. Hsu and P. W. Anderson, Phys. Rev. B 38, 745 (1988).

[22] E. Dagotto, E. Fradkin and A. Moreo, Phys. Rev. B 38, 2926 (1988).

[23] L. B. Ioffe and A. I. Larkin, Phys. Rev. B 39, 8988 (1989); R. Shankar, Nucl. Phys. B330, 433 (1990); N. Nagaosa and P. A. Lee, Phys. Rev. Lett. 64, 2450 (1990).

[24] X. G. Wen, Phys. Rev. B 44, 2664 (1991).

[25] P. W. Anderson, Phys. Rev. 130, 439 (1963); F. Englert and R. Brout, Phys. Rev. Lett. 13, 321 (1964); P. W. Higgs, Phys. Rev. 145, 1156 (1966); T. Kibble, Phys. Rev. 155, 1554 (1967).

[26] S. Sachdev and N. Read, Int. J. Mod. Phys. B 1&2, 219 (1991).

[27] F. D. H. Haldane, J. Phys. C 14, 2585 (1981).

[28] For a review, see V. J. Emery in Highly Conducting One-dimensional Solids, eds. J. T. Devreese, R. P. Evrard and V. E. Van Doren, Plenum, 1979.

[29] L. D. Faddeev and L. A. Takhtajan, Phys. Lett. A 85, 375 (1981).
[30] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).

[31] A. Luther and I. Peshel, Phys. Rev. B **12**, 3908 (1975).

[32] F. D. M. Haldane in *Electron correlation and magnetism in narrow-band systems*, ed. T. Moriya, Springer-Verlag, Berlin 1981.

[33] P. A. Lee and N. Nagaosa, Phys. Rev. B **46**, 5621 (1992).

[34] J. B. Marston and I. K. Affleck, Phys. Rev. B **39**, 11538 (1989).

[35] C. Mudry and E. Fradkin (unpublished).

[36] J. Hubbard, Proc. Roy. Soc. London A **285**, 542 (1965).

[37] The identities $|0 >_{s_i} = \psi^\dagger_2 |0 >_{\psi_i}$, $s^\dagger_1 |0 >_{s_i} = \psi^\dagger_1 \psi^\dagger_2 |0 >_{\psi_i}$, $s^\dagger_2 |0 >_{s_i} = |0 >_{\psi_i}$, and $s^\dagger_1 s^\dagger_2 |0 >_{s_i} = \psi^\dagger_1 |0 >_{\psi_i}$, where $|0 >_{s_i}$ is the spinon vacuum at site $i$, hold.

[38] S. Kivelson, D. S. Rokhsar and J. P. Sethna, Phys, Rev. B **35**, 8865 (1987).

[39] E. Fradkin and S. Shenker, Phys. Rev. D **19**, 3682 (1979).

[40] For a review see H. G. Evertz and M. Marcu in, TeV Physics, Proceedings of the Johns Hopkins Workshop on Current Problems in Particle Theory (12th: 1988, Baltimore, Md), eds. G. Domokos and S. Kovesi-Domokos, World Scientific, Singapore, 1988.

[41] The center $C_{SU(2)}$ of the group SU(2) is the set of all matrices of SU(2) which commute with all other elements of SU(2). It is made of two elements, the unit matrix $+\sigma^0$ and $-\sigma^0$. As a group, the center is isomorphic to $\mathbb{Z}_2$: $C_{SU(2)} = \{-\sigma^0, +\sigma^0\} \cong \{-1, +1\} = \mathbb{Z}_2$.

The adjoint representation maps the two elements of the center on the 3 by 3 unit matrix whereas the fundamental representation of SU(2) is SU(2) itself.

[42] For a review, see M. J. Creutz, *Quarks, Gluons and Lattices*, Cambridge University Press, 1983.

[43] For a review, see N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization*
Any finite dimensional representation of SU(2) can be constructed from the tensor product of the fundamental representation. Physically, any source (color charged particle) carrying the color charge of a higher dimensional representation than the fundamental, can be thought as a composite of the fundamental sources (e.g., hadrons are composite of quarks).

The Affleck-Marston flux state has the same flux \( \pi \) threading a different cell, namely the unit cell of the square lattice \( \Lambda \). Correspondingly, it has four minima located at \( (\pm \pi/2, \pm \pi/2) \) in the BZ. The Affleck-Marston flux state and ours then differ in the location in reciprocal space of the maxima for their instantaneous spin-spin correlations \( < \vec{S}_{\vec{Q},t=0} \cdot \vec{S}_{-\vec{Q},t=0} > \) due to the different momentum transfers \( \vec{Q} \) in scattering processes mixing the degenerate minima.

The usual justification for this approximation is that the fluctuations of the determinant cost an energy of order \( J_{2(1)}^2 \), which is not the lowest scale of energy due the softness of
the fluctuations of the SU(2) degrees of freedom. This approximation does not affect the issue of confinement as shown in the discussion based on the \((\beta, \kappa, \infty)\) phase diagram of section \[\text{III}\]. However, this approximation does not allow the effective continuum theory to be unstable towards dimerization, an issue which is of particular importance for the one dimensional antiferromagnet with competing nearest and next-nearest-neighbor interactions.

[54] \(j_{1(2)}\) labels all \((2j_{1(2)} + 1)\) dimensional irreducible representations of SU(2).

[55] Since we are including determinant fluctuations of the nearest-neighbor links \(M\), our effective theory can, in principle, detect an unstability of the mean-field theory towards dimerization on nearest-neighbor links.

[56] E. Fradkin and F. A. Schaposnik, Phys. Rev. Lett. 66, 276 (1991).

[57] E. Manousakis, Rev. of Mod. Phys. 63, 1 (1991).

[58] The same argument applies to the BZA limit \(J_2 = 0\) in which we expect a ground state with Néel order, but there we do not have an effective theory to study how the quantum fluctuations affect the mean-field Ansatz.

[59] One might wonder why the valence bond (dimer) ordering results from the breaking of a continuous symmetry in the continuum theory when on the lattice it is only associated with the breaking of discrete symmetries. This is because we have neglected (dangerous) irrelevant operators in our gradient expansion which modify the symmetries of the continuum theory.

[60] M. C. Diamantini, P. Sodano and G. W. Semenoff, Phys. Rev. Lett. 70, 3848 (1993).

[61] R. Jackiw and S. Templeton, Phys. Rev. D 15, 2291 (1981).

[62] T. Appelquist and D. Nash, Phys. Rev. Lett. 64, 721 (1990).
FIGURES

FIG. 1. Phase diagram for a gauge-Higgs lattice theory in 2+1 dimensional space-time. \( \beta \) is the energy scale for the gauge fields, \( \kappa \) that for the matter (Higgs). The amplitude of the Higgs is frozen \( (\lambda = \infty) \). (a) shows the case of the matter (Higgs) in the fundamental representation. (b) shows the case of the matter (Higgs) in the adjoint representation.

FIG. 2. The solutions \( a_0^1 \), Re \( E \), Im \( E \) and \( X \) as a function of the dimensionless ratio \( J_2/J_1 \) of the saddle-point equations for Wen’s Ansatz.

FIG. 3. The upper branch \(+|\vec{\xi}_k|\) of the MF single-particle excitations (spinons) in the first quadrant of the Brillouin Zone for: (a) \( J_2/J_1 = 1.4 \), (b) \( J_2/J_1 = 0.9 \), (c) \( J_2/J_1 = 0.5 \), (d) \( J_2/J_1 = 0 \). The minima move from the edges (a) and (b) to the diagonal (c) and (d).

FIG. 4. Phase diagram of Wen’s mean-field Ansatz. There exists a second order phase transition into a phase, the flux phase, without any gap to the mean-field excitations. Below the threshold \( (J_2/J_1)_{cr} \approx 1.342 \), the mean-field excitations have a gap, except at the BZA point \( J_2/J_1 = 0 \). The critical point \( (J_2/J_1)_{cr} \) is infrared unstable as indicated by the RG flow of \( J_2/J_1 \). The location of the minima in the s-RVB phase, \( 0 < J_2/J_1 < (J_2/J_1)_{cr} \), jumps discontinuously for \( J_2/J_1 \approx 0.7 \), signaling a level crossing of the excitations. The values of the gap are plotted in units of \( J_1 \).

FIG. 5. The scaling exponents \( \beta_1 \), \( \beta_2 \), \( \beta_3 \) and \( \beta_4 \) of the s-RVB parameters \( a_0^1 \), Re \( E \), Im \( E \) and \( X \) as a function of \( J_2/J_1 \) can be read from a log-log plot. We have used \( (J_2/J_1)_{cr} = 1.342 \) and \( \text{Re } E_{f} = \text{Im } E_{f} = 0.339 \).

FIG. 6. Partitioning of the square lattice \( \Lambda \) into eight sublattices. A sublattice is labelled by the letter \( e \) or \( o \) for even or odd respectively and by an integer between 1 and 4.
TABLE I. The 15 bilinears $\int d^2 x \, q \, T_a \, q$ are given together with their lattice counterparts. Here, $T_a$, $a = 1, \cdots, 15$ are the generators of SU(4) flavor. The spinors $s$ were introduced in section II and are relabelled according to the lattice partitioning of appendix A. The $u$, $w$, $v$ and $z$ are defined in section V.

| $a$ | Lattice Hamiltonian $H_a$ | $\int d^2 x \, q \, T_a \, q$ |
|-----|--------------------------|-----------------------------|
| 1   | $-\sum_{i,l} \left[ s_i^{+1} \sigma_0 s_i^{e2} + s_i^{+4} \sigma_0 s_i^{e3} - \text{H.C.} \right]$ + $\int d^2 x [\bar{w} + \bar{v}]$ | |
| 2   | $\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ - $\int d^2 x [\bar{w} - \bar{v}]$ | |
| 3   | $-\sum_{i,l} \left[ -s_i^{+1} \sigma_0 s_i^{e4} + s_i^{+2} \sigma_0 s_i^{e3} - \text{H.C.} \right]$ + $\int d^2 x [\bar{u} - \bar{v}]$ | |
| 4   | $-\sum_{i,l} \left[ s_i^{+1} \sigma_0 s_i^{e2} + s_i^{+4} \sigma_0 s_i^{e3} - \text{H.C.} \right]$ + $\int d^2 x [\bar{u} - \bar{v}]$ | |
| 5   | $\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ - $\int d^2 x [\bar{u} - \bar{v}]$ | |
| 6   | $-\sum_{i,l} \left[ -s_i^{+1} \sigma_0 s_i^{e4} + s_i^{+2} \sigma_0 s_i^{e3} - \text{H.C.} \right]$ + $\int d^2 x [\bar{u} - \bar{v}]$ | |
| 7   | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ + $\int d^2 x [\bar{u} + \bar{v}]$ | |
| 8   | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ - $\int d^2 x [\bar{u} + \bar{v}]$ | |
| 9   | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ + $\int d^2 x [\bar{u} + \bar{v}]$ | |
| 10  | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ - $\int d^2 x [\bar{u} + \bar{v}]$ | |
| 11  | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ + $\int d^2 x [\bar{w} + \bar{v}]$ | |
| 12  | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ - $\int d^2 x [\bar{w} + \bar{v}]$ | |
| 13  | $-\sum_{i,l} \left[ s_i^{+1} \sigma_3 s_i^{e1} - s_i^{+2} \sigma_3 s_i^{e2} + s_i^{+3} \sigma_3 s_i^{e3} - s_i^{+4} \sigma_3 s_i^{e4} + \text{H.C.} \right]$ | |
\[ -s_i^{21} \sigma^0 s_i^{3} s_{1-2\hat{z}+} - s_i^{21} \sigma^0 s_i^{1} s_{1-2\hat{z}+} + s_i^{11} \sigma^0 s_i^{3} s_{1-2\hat{z}+} - s_i^{11} \sigma^0 s_i^{1} s_{1-2\hat{z}+} - \text{H.C.} \] + \int d^2 x [\bar{v} \tilde{z} + \tilde{z} \bar{v}]

14 \[ -\sum_{i \in \Lambda^1_i} \left[ s_i^{e3} \sigma^3 s_i^{1} s_{1-2\hat{z}+} - s_i^{e3} \sigma^3 s_i^{1} s_{1-2\hat{z}+} + s_i^{e4} \sigma^4 s_i^{1} s_{1-2\hat{z}+} - s_i^{e4} \sigma^4 s_i^{1} s_{1-2\hat{z}+} + s_i^{e11} \sigma^3 s_i^{1} s_{1-2\hat{z}+} - s_i^{e11} \sigma^3 s_i^{1} s_{1-2\hat{z}+} + \text{H.C.} \right] - i \int d^2 x [\bar{v} \tilde{z} - \tilde{z} \bar{v}]

15 \[ -\frac{1}{2} \sum_{i \in \Lambda^1_i} \left[ s_i^{e11} \sigma^0 s_i^{e3} + s_i^{e11} \sigma^0 s_i^{e4} - s_i^{e21} \sigma^0 s_i^{e3} - s_i^{e21} \sigma^0 s_i^{e4} - (e \leftrightarrow o) - \text{H.C.} \right] + \int d^2 x [\bar{v} \tilde{v} - \tilde{v} \bar{v}]

56