On the Inverse Semigroup of Bimodules over a $C^*$-Algebra

V. M. Manuilov*

*Moscow Center for Fundamental and Applied Mathematics, Moscow State University, Leninskie Gory 1, 119991, Moscow, Russia
E-mail: manuilov@mech.math.msu.su

Abstract. It was noticed recently that, given a metric space $(X,d_X)$, the equivalence classes of metrics on the disjoint union of the two copies of $X$ coinciding with $d_X$ on each copy form an inverse semigroup $M(X)$ with respect to concatenation of metrics. Now put this inverse semigroup construction in a more general context, namely, we define, for a $C^*$-algebra $A$, an inverse semigroup $S(A)$ of Hilbert $C^*$-bimodules. When $A$ is the uniform Roe algebra $C_0^*(X)$ of a metric space $X$, we construct a mapping $M(X) \to S(C_0^*(X))$ and show that this mapping is injective, but not surjective in general. This allows to define an analog of the inverse semigroup $M(X)$ that does not depend on the choice of a metric on $X$ within its coarse equivalence class.

DOI 10.1134/S1061920822010071

1. INTRODUCTION

It was noticed in [5] that, given a metric space $(X,d_X)$, the equivalence classes of metrics on the disjoint union of the two copies of $X$ coinciding with $d_X$ on each copy form an inverse semigroup $M(X)$ with respect to concatenation of metrics. It was found that $M(X)$ depends on the choice of a metric on $X$ within the coarse equivalence class of $d_X$.

Now we are able to put this inverse semigroup construction in a more general context, namely, we define, for a $C^*$-algebra $A$, an inverse semigroup $S(A)$ of Hilbert $C^*$-bimodules. $C^*$-bimodules used to define this inverse semigroup are $A$-$A$-imprimitivity bimodules [8] without the requirement of being full.

In the case when $A$ is the uniform Roe algebra $C_0^*(X)$ of a metric space $X$, we construct a mapping $M(X) \to S(C_0^*(X))$ and show that this mapping is injective, but not surjective in general. This allows to define an analogue of the inverse semigroup $M(X)$ that does not depend on the choice of a metric on $X$ within its coarse equivalence class.

Our references for inverse semigroup theory are [2, 3]. Details on coarse geometry and Roe algebras can be found in [9].

2. ASSOCIATIVE HILBERT $C^*$-BIMODULES AS MORPHISMS. THE SEMIGROUP $S(A)$

Let $A$ and $B$ be $C^*$-algebras.

**Definition 1.** A Banach space $M$ is an **associative Hilbert $C^*$-bimodule (left over $A$ and right over $B$)** if

1. $M$ is a left $A$-module and a right $B$-module, and $(am)b = a(mb)$ for any $a \in A, b \in B, m \in M$, and both modules are nondegenerate, i.e., $AM$ and $MB$ are dense in $M$;

2. $M$ is a left Hilbert $C^*$-module over $A$ and a right Hilbert $C^*$-module over $B$ with respect to the inner products $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$, respectively;

3. $A$ (resp., $B$) acts on $M$ by bounded adjointable operators with respect to the Hilbert $C^*$-module structure over $B$ (resp., over $A$), i.e., $\langle am, bm' \rangle_B = \langle ma, bm' \rangle_B$ and $A\langle mb, m' \rangle_B = A \langle m, mb' \rangle_B$ for any $a \in A, b \in B, m, m' \in M$;

4. the norm on $M$ is equivalent to the norms $\|a\|_A = \|a\|_A$, $\|\cdot\|_B$, and $\|\cdot\|_B$;

5. $A(m,n)r = m\langle n, r \rangle_B$ for any $m, n, r \in M$. 


We denote the set of associative Hilbert $C^*$-bimodules for chosen $A$ and $B$ by $\mathcal{S}(A, B)$. When $B = A$, we write $\mathcal{S}(A)$. For the set of isomorphism classes of associative Hilbert $C^*$-bimodules we write $S(A, B)$ and $S(A)$ respectively.

Let $J_A = A\langle M, M \rangle \subset A$ (resp., $J_B = \langle M, M \rangle_B \subset B$) be the closure of the linear span of $\langle m, n \rangle$ (resp., of $\langle m, n \rangle_B$), $m, n \in M$. These are ideals in $A$ and $B$, respectively (by ideals we mean closed two-sided ideals). The definition of $A$-$B$-imprimitivity bimodules [8] requires, besides the items 1-5 above, that $M$ is a full Hilbert $C^*$-bimodule, i.e., that $J_A = A$ and $J_B = B$. Then such bimodules are used to define strong Morita equivalence, but we don’t assume that $M$ is full. Note that we consider $M \in \mathcal{S}(A, B)$ as a left $J_A$-module and a right $J_B$-module. Then $M$ is a $J_A$-$J_B$-imprimitivity bimodule (and $J_A$ and $J_B$ are strongly Morita equivalent).

Associative Hilbert $C^*$-bimodules can be regarded as morphisms on the category of $C^*$-algebras. If $C$ is a $C^*$-algebra and $N \in \mathcal{J}(B, C)$ then $M \otimes_B N$ is an associative Hilbert $C^*$-bimodule, left over $A$ and right over $C$, with respect to the inner products determined by

$$A\langle m \otimes n, m' \otimes n' \rangle = A\langle m, m' \rangle_{B} \quad \text{and} \quad A\langle m \otimes n, m' \otimes n' \rangle_{C} = A\langle m, m' \rangle_{B} \otimes n \otimes n',$$

$m, m' \in M$, $n, n' \in N$. We write $M : N$ for $M \otimes_B N$. This makes $S(A)$ a semigroup.

For $M \in \mathcal{J}(A, B)$, the dual module $M^* \in \mathcal{J}(B, A)$ was defined in [8], Definition 6.17 (it was denoted there by $M$). Recall that $M^*$ as a set is the same as $M$, while the bimodule structure is given by

$$b\hat{m} = (mb)^*, \quad \hat{m}a = (a^*m)^*, \quad b\langle \hat{m}, \hat{n} \rangle = \langle m, n \rangle_B, \quad \langle \hat{m}, \hat{n} \rangle_A = A\langle m, n \rangle,$$

where we write $\hat{m}$ for $m \in M$ considered as an element of $M^*$. When $B = A$, we say that $M$ is self-adjoint if $M^*$ is isomorphic to $M$ as an associative Hilbert $C^*$-bimodule.

**Example 1.** Let $B = A$, $J \subset A$ an ideal. Then $J \in \mathcal{J}(A)$. In particular, $J = \{0\}$ (resp., $J = A$) represents the zero (resp., the unit) element in the semigroup $S(A)$.

**Lemma 1.** Let $M \in \mathcal{J}(A, B)$. Then $M \otimes A \cong J_B$, where $J_B = \langle M, M \rangle_B$.

**Proof.** It was shown in Lemma 6.22 of the article [8] that the mapping $R : M^* \otimes_J A \to J_B$ defined by $R(m \otimes n) = \langle m, n \rangle_B$ is an isomorphism. Since $M \otimes A = M^* \otimes JA$, it remains to check that $M^* \otimes JA = M^* \otimes A M$.

Let $m \in M$, $a \in A$, $\varepsilon > 0$, and let $(u_\lambda)_{\lambda \in A}$ be an approximate unit in the ideal $J_A$. Then

$$A\langle (a - u_\lambda a)m, (a - u_\lambda a)m \rangle = A\langle m, m \rangle(a - u_\lambda a)^* \leq \|a\| \cdot \|a - u_\lambda a\| = \|a\| \cdot \|a' - u_\lambda a'\|,$$

where $a' = aA(m, m) \in J_A$. Therefore, there exists a $j \in J_A$ of the form $j = u_\lambda a$ such that $\|am - jm\| < \varepsilon$. Then for any $m, n \in M$, any $a \in A$ and any $\varepsilon > 0$ there exists $j \in J$ such that

$$\|\hat{m}a \otimes n - \hat{m} \otimes an\| = 0 \text{ in } M \otimes_J A. M. \text{ As the kernel of the canonical quotient mapping } M^* \otimes_J A \to M^* \otimes A M \text{ is generated by differences } \hat{m}a \otimes n - \hat{m} \otimes an, a \in A, m, n \in M, \text{ this kernel is trivial.}$$

**Definition 2.** An associative Hilbert $C^*$-bimodule $M \in \mathcal{J}(A)$ is idempotent if $M \cdot M \cong M$. A selfadjoint idempotent is a projection.

**Example 2.** Let $J \subset A$ be an ideal. Regard it as a Hilbert $C^*$-bimodule, $J \in \mathcal{J}(A)$. Then $J$ is a projection. Clearly, it is selfadjoint. Since $J_A = J$, by Lemma 1, $J^* \otimes A J \cong J$.

**Lemma 2.** If $M \in \mathcal{J}(A)$ is a projection, then there exists an ideal $J \subset A$ such that $M \cong J$.

**Proof.** If $M$ is a projection, then $M \cdot M \cong M^* \cdot M \cong M$. Let $(J, M)_A = J \subset A$. Then $M \cong M^* \cdot M \cong J$.

**Lemma 3.** The semigroup $S(A)$ is regular.

**Proof.** Recall that a semigroup $S$ is regular if any element $s \in S$ has a ‘pseudoinverse’ $t \in S$ such that $sts = s$ and $tst = t$. This follows from the isomorphism $M \otimes M^* \cdot M \cong M$ for any $M \in \mathcal{J}(A)$, so let us check this isomorphism. Let $(M, M)_A = J_A \subset A$. Then $M^* \cdot M \cong J_A$, hence $M^* \cdot M \cong M \otimes A J_A$. As in the proof of Lemma 1, using an approximate unit in $J_A$, it is easy to show that the canonical surjection $M = M \otimes_J J_A \to M \otimes A J_A$ is an isomorphism.
Lemma 4. Let $I, J \subset A$ be ideals. Then $I \cdot J = J \cdot I \cong I \cap J$.

Proof. The mapping $R_J : I \otimes_A J \to I$ is an isometry, hence it remains to show that the range of $R_J$ is $I \cap J$. This can be done by using that any element $a \in I \cap J$ can be written as a product $a = ij$, where $i \in I$, $j \in J$. Similarly, $I \otimes_A J \cong I \cap J$.

Theorem 1. The semigroup $S(A)$ is an inverse semigroup.

Proof. Recall that a regular semigroup $S$ is an inverse semigroup iff any two idempotents commute. By Lemma 4, any two projections commute. This suffices: if $M$ is idempotent then $M^*$ is idempotent as well, and

$$M^* = M^* \cdot M \cdot M^* = (M^* \cdot M) \cdot (M \cdot M^*) = (M \cdot M^*) \cdot (M^* \cdot M) = M \cdot M^* \cdot M = M,$$

hence $M$ is a projection.

Proposition 1. Let $A$ and $B$ be strongly Morita equivalent $C^*$-algebras. Then $S(A) \cong S(B)$.

Proof. Let $M \in \mathcal{S}(A, B)$ be an imprimitivity $A$-$B$-bimodule, i.e., a full associative Hilbert $C^*$-bimodule. Then the mappings $P \mapsto M \otimes_B P$ and $Q \mapsto M^* \otimes_A Q$, where $P \in \mathcal{S}(B)$, $Q \in \mathcal{S}(A)$, are inverse to each other.

Proposition 2. Let $J \subset A$ be an ideal. Then $S(J) \subset S(A)$.

Proof. Let $M \in \mathcal{S}(J)$. Let us show that $M$ can be regarded as an $A$-$A$-bimodule. Let $m \in M$, $a \in A$, and let $\{u_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit in $J$. Then set $a \cdot m = \lim_\lambda (au_\lambda)m$. The existence of the limit follows from the estimate

$$\| (au_\lambda)m - (au_\mu)m \|^2 = \| (a(u_\lambda - u_\mu))m, (a(u_\lambda - u_\mu))m \|_A \leq \| a^*(u_\lambda - u_\mu)m, m \|_A (u_\lambda - u_\mu)a \|,$$

and from the convergence of $(u_\lambda - u_\mu)(m, m)_A$.

Recall that the isomorphism classes of imprimitivity bimodules over $A$ form a group, called the Picard group of $A$.

Proposition 3. If $A$ is simple then $S(A) = \{0\} \cup \text{Pic}(A)$.

Proof. Let $M \in \mathcal{S}(A)$, $M \neq 0$. As $M^* \cdot M$ and $M \cdot M^*$ are ideals in $A$, we have $M^* \cdot M \cong A \cong M \cdot M^*$, and these isomorphisms are given by $S : m \otimes n \mapsto (m, n)_A$ and $T : m \otimes n \mapsto (m, n)$, hence $M$ is a full Hilbert $C^*$-module, i.e., an imprimitivity bimodule.

Proposition 4. $S(\mathbb{C}^n)$ is the semigroup of partial bijections of the set $\{1, \ldots, n\}$.

Proof. Let $X = \{1, \ldots, n\}$, $\mathbb{C}^n = C(X) = A$, and let $M \in \mathcal{S}(X)$. Let $J_1 = A\langle M, M \rangle$, $J_2 = \langle M, M \rangle$. Then $J_1 = C(P)$, $J_2 = C(Q)$, where $P, Q \subset X$, and $M$ is $J_1$-$J_2$-imprimitivity bimodule. $C(P)$ and $C(Q)$ are strongly Morita equivalent only if $|P| = |Q|$. The Picard group of $C(P)$ is the group of permutations of $P$, so $M$ determines and is determined by the partial bijection $P \to Q$ (cf. [1]).

3. INVERSE SEMIGROUP FROM METRICS ON DOUBLES AND ROE BIMODULES

Let $X = (X, d_X)$ be a countable discrete metric space, and let $H_X = l^2(X)$ denote the Hilbert space of square-summable complex-valued functions on $X$, with the orthonormal basis consisting of delta functions $\delta_x$, $x \in X$, and with the inner product $\langle \cdot, \cdot \rangle$. An operator $T \in \mathcal{B}(H_X)$ is said to have propagation $\leq L$ if $T_{x,y} = 0$ whenever $d_X(x, y) > L$, where $T_{x,y} = (\delta_x, T\delta_y)$. The norm closure of the set of all operators of finite propagation is the uniform Roe algebra $C_u^*(X)$.

For a countable discrete metric space $X = (X, d_X)$, let $\mathcal{M}(X)$ denote the set of metrics on $X \times \{0,1\}$ such that

- for any $d \in \mathcal{M}(X)$, the restriction of $d$ to each copy of $X$ equals $d_X$;
- the distance between the two copies of $X$ is nonzero.
Let $M(X)$ be the set of coarse equivalence classes of metrics in $\mathcal{M}(X)$. It was shown in [5] that $M(X)$ is an inverse semigroup with respect to the concatenation of metrics.

Let $(X, d_X)$ and $(Y, d_Y)$ be two countable discrete metric spaces. Let $Z = X \sqcup Y$, and let $D_{X,Y}$ denote the set of all metrics $d$ on $Z$ such that $d|_X = d_X$ and $d|_Y = d_Y$. For each $d \in D_{X,Y}$, let $M_d[X,Y]$ denote the set of all bounded finite propagation operators $T : H_X \to H_Y$, and let $M_d(X,Y)$ be its norm closure in the bimodule $\mathbb{B}(H_X, H_Y)$ of all bounded operators from $H_X$ to $H_Y$.

If $T, S \in M_d[X,Y]$, then $T^* S$ and $T S^*$ are finite propagation operators in $l^2(X)$ and $l^2(Y)$, respectively, hence we can define uniform Roe algebra-valued inner products

$$\langle T, S \rangle_{C^*_u(X)} = T S^* \quad \text{and} \quad \langle T, S \rangle_{C^*_u(Y)} = T^* S.$$ 

Similarly, $M_d(X,Y)$ is a left $C^*_u(Y)$-module and a right $C^*_u(X)$-module. Clearly, this bimodule is associative. We write $M_d(X)$ when $Y = X$.

**Lemma 5.** Let $d_1, d_2 \in \mathcal{M}(X,Y)$, $M_j = M_{d_j}(X,Y)$, $j = 1, 2$, and let $M_1 \cong M_2$. Let $f : M_1 \to M_2$ be an isomorphism of associative Hilbert $C^*$-bimodules over $C^*_u(X)$ and $C^*_u(Y)$. Then $f(m) = \lambda m$ for any $m \in M_1$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$. In particular, $M_1 = M_2$.

**Proof.** Let $x \in X$, $y \in Y$, and let $e_{x,y}$ denote the elementary operator corresponding to these two points, i.e., $e_{x,y} \delta_x = \{ \delta_y \text{ if } z = x; \quad 0 \text{ if } z \neq x \}$, where $\delta_x \in H_X$ is the delta-function of the point $x$. Clearly, $e_{x,y} \in M_d(X,Y)$ for any $d \in \mathcal{M}(X,Y)$. The elementary operators $e_{x,x}$ and $e_{y,y}$ lie in $C^*_u(X)$ and in $C^*_u(Y)$, respectively. Then

$$f(e_{x,y}) = f(e_{x,y}e_{x,y}e_{x,y}) = e_{y,y}f(e_{x,y})e_{x,y} = \lambda_{x,y} e_{x,y}$$

for some $\lambda_{x,y} \in \mathbb{C}$, and as $f$ is an isometry, $|\lambda_{x,y}| = 1$.

Let $z \in X$. Then $e_{x,z} \in C^*_u(X)$ and $e_{x,y} = e_{x,y} e_{z,x}$, so

$$\lambda_{x,y} e_{z,y} = f(e_{z,y}) = f(e_{x,y}e_{z,x}) = f(e_{x,y})e_{z,x} = \lambda_{x,y} e_{z,y},$$

hence $\lambda_{x,y} = \lambda_{x,z}$ for any $x, z \in X$ and $y \in Y$. Similarly, $\lambda_{x,y} = \lambda_{x,u}$ for any $y, u \in Y$ and $x \in X$. Thus, $\lambda_{x,y} = \lambda$ for any $x \in X$, $y \in Y$.

Let $m \in M_1$, $m = \sum_{x \in X} \sum_{y \in Y} m_{x,y} e_{x,y}$. Then

$$f(m)_{x,y} = e_{x,y} f(m)e_{x,y} = f(e_{x,y} m e_{x,y}) = f(e_{x,y} m_{x,y} e_{x,y}) = m_{x,y} f(e_{x,y}) = \lambda m_{x,y} e_{x,y},$$

hence $f(m) = \lambda m$.

This gives a semigroup homomorphism

$$i : M(X) \to S(C^*_u(X)), \quad i([d]) = [M_d(X)].$$

(1)

**Theorem 2.** The mapping $i$ (1) is injective.

**Proof.** Let $d_1, d_2 \in \mathcal{M}(X)$, $M_j = M_{d_j}(X)$ $j = 1, 2$, and let $M_1 \cong M_2$. Assume the contrary, i.e., that $[d_1] \neq [d_2]$. Then there exist two sequences $(x_n, (y_n), n \in \mathbb{N}$, of points in $X$, and $C > 0$ such that $d_1((x_n, 0), (y_n, 1)) < C$ for any $n \in \mathbb{N}$, and $\lim_{n \to \infty} d_2((x_n, 0), (y_n, 1)) = \infty$. Set

$$m = \sum_{n \in \mathbb{N}} e_{(x_n,0),(y_n,1)} \in \mathbb{B}(H_X, H_Y)$$

(the sum is strongly convergent). Then $m \in M_1$, but $m \not\in M_2$, but this contradicts $M_1 \cong M_2$ by Lemma 5.

On the other hand, the mapping $i$ (1) is far from being surjective. We show this by means of two examples, one very simple, and the other one much deeper.

**Example 3.** Let $X$ be a countable set with the metric $d_X$ defined by $d_X(x, y) = 1$ if $x \neq y$, $x, y \in X$. Then $C^*_u(X) = \mathbb{B}(H_X)$. Any two metrics in $\mathcal{M}(X)$ are coarsely equivalent, hence $M(X)$ consists of a single element. On the other hand, there are at least two different associative Hilbert $C^*$-bimodules over $\mathbb{B}(H_X)$: $\mathbb{B}(H_X)$ itself and the ideal $\mathbb{K}(H_X)$ of compact operators. It is easy to see that $i(M(X)) = [\mathbb{B}(H_X)]$. 

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 29 No. 1 2022
Before we turn to the deeper example, let us recall that a discrete metric space $X$ is of bounded geometry if the numbers of points in all balls of radius $R$ are uniformly bounded for any $R > 0$.

An element $a \in C^*_r(X)$ is a ghost element if $\lim_{r \to \infty} a_{x,z} = 0$, where $a_{x,z}$, $x, z \in X$, denote the matrix entries of $a$ with respect to the basis $\delta_x$, $x \in X$. The set of all ghost elements forms an ideal in $C^*_r(X)$ called the ghost ideal. This ideal contains the ideal $\mathbb{K}(H_X)$ of all compact operators, and it was shown in [10] that it equals $\mathbb{K}(H_X)$ if and only if $X$ has property A of Guoliang Yu.

**Proposition 5.** Let $X$ be a discrete metric space of bounded geometry without property A. Then the mapping $i$ is not surjective.

**Proof.** Let $I \subset C^*_r(X)$ denote the ghost ideal. Then it properly contains $\mathbb{K}(H_X)$. We claim that $[I] \in S(C^*_r(X))$ does not lie in the image of $i$. Assume the contrary: let $[d] \in \mathcal{M}(X)$, $i([d]) = [I]$. Since $[I]$ is an idempotent in $S(C^*_r(X))$, $[d]$ must be an idempotent in $M(X)$. The characterization of idempotents was obtained in [6]: any such $d$ is coarsely equivalent to the metric

$$d_d((x,0), (y,1)) = \inf \inf_{n \in \mathbb{N}} d_X(x,z) + n + d_X(z,y),$$

where $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of subsets in $X$ such that, for each $n \in \mathbb{N}$, $A_{n+1}$ contains the 1-neighborhood $N_1(A_n)$ of $A_n$, i.e., the set $N_1(A_n) = \{x \in X : d_X(x, A_n) \leq 1\}$.

The metric $d_d$ represents the zero element $0 \in M(X)$ if each $A_n$ is bounded. If $[d]=0$ then the bimodule $M_d(X)$ coincides with the set of compact operators, which differs from $[I]$, so $[d] \neq 0$. Then there exists an $n \in \mathbb{N}$ such that $A_n$ is not bounded, i.e., there is a sequence $\{x_k\}_{k \in \mathbb{N}}$ of points in $A_n$ such that $\lim_{k \to \infty} d_X(x_k, (x_1, \ldots, x_{k-1})) = \infty$. Then $d_d((x_k,0), (x_k,1)) \leq n$ for any $k \in \mathbb{N}$. As $d$ and $d_d$ are coarsely equivalent, there exists $C > 0$ such that $d((x_k,0), (x_k,1)) \leq C$ for any $k \in \mathbb{N}$. Set $m = \sum_{k \in \mathbb{N}} c_{(x_k,0), (x_k,1)}$ (the sum is strongly convergent). Then $m \in M_d(X)$, but $m \notin I$, hence $i([d]) = [M_d(X)] \neq [I]$ (as in the proof of Lemma 5, the isomorphism of associative Hilbert $C^*$-bimodules represented on $H_X \oplus H_X$ implies the equality of these bimodules).

Recall that two metrics, $d_X$ and $b_X$, on $X$, are coarsely equivalent if there exists a homeomorphism $\varphi$ of $[0, \infty)$ such that $\varphi^{-1}(b_X(x, y)) < d_X(x, y) < \varphi(b_X(x, y))$ for any $x, y \in X$. It was shown in [7] that the inverse semigroup $M(X)$ is not coarsely invariant, e.g. there exists a space $X$ with two coarsely equivalent metrics, $d_X$ and $d'_X$ such that $M(X, d_X)$ is commutative, while $M(X, d'_X)$ is not. Now we can define a new semigroup $M_c(X)$ for a metric space $(X, d_X)$ by $M_c(X) = \cap_{d \in [d_X]} i(M(X, d))$.

**Proposition 6.** $M_c(X)$ is an inverse semigroup, invariant under coarsely equivalent metrics on $X$.

**Example 4.** Let $x_n = n^2$, $X = \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ equipped with the metric $d_X$ induced from the standard metric on $\mathbb{R}$. Let $Y = \{n^3 : n \in \mathbb{N}\}$, and let $f : X \to Y$ be the mapping defined by $f(x_n) = \begin{cases} (2n)^3, & \text{if } n \geq 0; \\ -(2n+1)^3, & \text{if } n < 0. \end{cases}$ Let $b_X$ be the metric on $X$ defined by $b_X(x_n, x_m) = |f(x_n) - f(x_m)|$, $n, m \in \mathbb{Z}$. It is easy to see that $X$ and $Y$ are coarsely equivalent, hence the metrics $d_X$ and $b_X$ are coarsely equivalent. While $M(X, d_X)$ is not commutative, $M(S, b_X)$ is commutative by Proposition 7.1 of [5]. Thus, $M_c(X)$ must be commutative.

**FUNDING**

The research was supported by RSF (project No. 21-11-00080).

**REFERENCES**

[1] L. G. Brown, P. P. Green, and M. A. Rieffel, “Stable Isomorphism and Strong Morita Equivalence of $C^*$-Algebras”, Pacific J. Math, 71 (1977), 349-363.

[2] J. M. Howie, Fundamentals of Semigroup Theory, Oxford Univ. Press, 1995.

[3] M. V. Lawson, Inverse Semigroups: The Theory of Partial Symmetries, World Scientific, 1998.

[4] V. Manuilov, “Roe Bimodules as Morphisms of Discrete Metric Spaces”, Russian J. Math. Phys., 26 (2019), 470–478.

[5] V. Manuilov, “Metrics on Doubles as an Inverse Semigroup”, J. Geom. Anal, 31 (2021), 5721–5739.

[6] V. Manuilov, “Metrics on Doubles as an Inverse Semigroup II”, J. Math. Anal. Appl, 496 (2021), 124821.

[7] V. Manuilov, “Metrics on Doubles as an Inverse Semigroup III. Commutativity and (In) Finiteness of Idempotents”, arXiv.org 2101.01013.

[8] M. A. Rieffel, “Induced Representations of $C^*$-Algebras”, Adv. Math, 13 (1974), 176–257.

[9] J. Roe, “Lectures on Coarse Geometry”, Amer. Math. Soc, 31 (2003).

[10] J. Roe and R. Willett, “Ghostbusting and Property A”, J. Funct. Anal, 266 (2014), 1674–1684.