THE H-N FILTRATION OF BUNDLES AS FROBENIUS PULL-BACK

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Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ over an algebraic closed field $k$ of characteristic $p > 0$. Let $F : X \to X_1$ be the relative Frobenius morphism, and $E$ be a semistable torsion free sheaf on $X$. For a semistable vector bundle $E$, one may guess that the length of Harder-Narasimhan filtration is not more than $p$. In this paper, I give a negative answer to this by giving an explicit example.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraic closed field $k$ of characteristic $p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by $O_X \to O_X, f \mapsto f^p$. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$. One of the themes is to study its action on the geometric objects on $X$.

Recall that a vector bundle $E$ on a smooth projective curve is called semistable (resp. stable) if $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$) for any nontrivial proper subbundle $E' \subset E$, where $\mu(E)$ is the slope of $E$. Semistable bundles are basic constituents of vector bundle in the sense that any bundle $E$ admits a unique filtration

$$HN_i(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_i(E) = E'$$

which is the so called Harder-Narasimhan filtration, such that

1. $gr^H_N i (E) := HN_i(E)/HN_{i-1}(E) (1 \leq i \leq \ell)$ are semistable;

2. $\mu(gr^H_N 1 (E)) > \mu(gr^H_N 2 (E)) > \cdots > \mu(gr^H_N \ell (E))$. The integer $\ell$ is called the length of the Harder-Narasimhan filtration of bundle $E$. It measures how far is a vector bundle from being semistable in some sense, and it’s clear $E$ is semistable if and only if $\ell = 1$. It is known that $F_*$ preserves the stability of vector bundles ([6]), but $F^*$ does not preserve the semistability of vector bundle ([3] for example).

Date: April 20, 2012.

The author is supported by gucas.
Given a semistable vector bundle $E$ on $X$, then $F^*E$ may not be semistable, it’s natural to consider the length of the Harder-Narasimhan filtration of $F^*E$. If $E = F_*W$ where $W$ is a semistable bundle on $X$, the length of Harder-Narasimhan filtration of $F^*E$ is $p$ by Sun’s Theorem (cf. [6, Theorem 2.2]). In this case, the instability of $F^*E$ is $(p - 1)(2g - 2)$ which reaches the upper bound of $F^*E$. Actually, in [5], the authors prove the instability $I(F^*E) \leq (p - 1)(2g - 2)$ for a semistable bundle $E$ on $X$. In consideration of the above statement, one may guess that $\forall$ semistable bundle $E$ on $X$, the length of the Harder-Narasimhan filtration of $F^*E$ is not more than $p$.

In this note, we give a negative answer to the above guess by constructing an example. Briefly, give a polarized smooth projective surface $(Y, H)$ with $\mu(\Omega^1_Y) > 0$ is semistable, for a suitable semistable vector bundle $W$ on $Y$ (take $W$ to be line bundle for example), $F_*W$ is semistable and $F^*F_*W$ has the Harder-Narasimhan filtration whose length is bigger than $p$. Restricting $F^*F_*W$ and the Harder-Narasimhan filtration to a generic hyperplane $X$ of high degree, by some analysis, one can check it’s actually the Harder-Narasimhan filtration of $F^*(E|_X)$, where $E = F_*W$ and the stability of $E|_X$ would been proved in the text.

2. CONSTRUCTION OF THE EXAMPLE

Let $Y$ be a smooth projective variety of dimension $n$ over an algebraic closed field $k$ with $\text{char}(k) = p > 0$. Fix an ample divisor $H$ on $Y$, for a torsion free sheaf $E$, the slope of $E$ is defined as

$$\mu(E) = \frac{c_1(E) \cdot H^{n-1}}{rk(E)}$$

where $rk(E)$ denotes the rank of $E$. Then

**Definition 2.1.** A torsion free sheaf $E$ on $Y$ is called semistable (resp. stable) if for any subsheaf $E' \subset E$ with $E/E''$ torsion free, we have

$$\mu(E') \leq (\text{resp.} <) \mu(E).$$

**Theorem 2.2.** (Harder-Narasimhan filtration) For any torsion free sheaf $E$, there is a unique filtration

$$HN_\bullet(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E'$$

which is the so called Harder-Narasimhan filtration, such that

1. $gr^H_{i+1}(E) := HN_i(E)/HN_{i-1}(E)$ ($1 \leq i \leq \ell$) are semistable;
2. $\mu(gr^H_1(E)) > \mu(gr^H_2(E)) > \cdots > \mu(gr^H_\ell(E))$. 


Remark 2.3. In [4, Theorem 1.3.4], the proof of existence of the filtration is given in terms of Gieseker stability. In particular, \( gr_i^{HN}(E) \) are Gieseker semistable, thus they are \( \mu \) semistable torsion free sheaves. We call the integer \( \ell \) the length of the Harder-Narasimhan filtration.

Definition 2.4. Let \( E \) be a semistable sheaf. A Jordan-Hölder filtration of \( E \) is a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E
\]

such that the factors \( gr_i(E) = E_i/E_{i-1} \) are stable.

Remark 2.5. Jordan-Hölder filtration always exists. As in Remark 2.3, we can get the factors \( gr_i(E) \) are torsion free.

Let \( F : Y \to Y_1 := Y \times_k k \) denote the relative Frobenius morphism over \( k \).

Lemma 2.6. Let \( X \) be a closed subvariety of \( Y \), then the induced morphism \( F|_C : C \to C_1 \) is the relative Frobenius morphism.

Proof. For the case of absolute Frobenius morphism, it’s trivial. In the relative case, it’s just a translation of the absolute case to the relative case. \( \square \)

Now, let’s restrict \( Y \) to be a smooth projective variety of dimension \( n \) with \( \Omega^1_Y \) is semistable and \( \mu(\Omega^1_Y) > 0 \) (For example, \( Y \) can been chosen to be \( C \times C \) with \( C \) is a smooth projective curve with genus \( g \geq 2 \)). Let \( W \) be a torsion free sheaf on \( Y \), define \( V_0 := V = F^*(F_*W) \), \( V_1 = \ker(F^*(F_*W)) \to W \),

\[
V_{\ell+1} := \ker((V_\ell) \to V \otimes_{\mathcal{O}_Y} \Omega_Y^1 \to (V/V_\ell) \otimes_{\mathcal{O}_Y} \Omega_Y^1)
\]

where \( \nabla : V \to V \otimes_{\mathcal{O}_Y} \Omega_Y^1 \) is the canonical connection. Actually, for the above filtration we have

Theorem 2.7. ([6, Theorem 3.7]) The filtration defined above is

\[
0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)
\]

which has the following properties

1. \( \nabla(V_{\ell+1}) \subset V_\ell \otimes \Omega_Y^1 \) for \( \ell \geq 1 \), and \( V_0/V_\ell \cong W \).
2. \( V_\ell/V_{\ell+1} \to (V_{\ell-1}/V_\ell) \otimes \Omega_Y^1 \) are injective for \( 1 \leq \ell n(p-1) \), which induced isomorphisms

\[
\nabla^\ell : V_\ell/V_{\ell+1} \cong W \otimes_{\mathcal{O}_Y} T^\ell(\Omega_Y^1), \quad 0 \leq \ell \leq n(p-1)
\]
The vector bundle $T^\ell(\Omega^1_Y)$ is suited in the exact sequence

$$0 \to \text{Sym}^{\ell-(p)}(\Omega^1_Y) \otimes F^*\Omega^q_Y \xrightarrow{\phi} \text{Sym}^{\ell-(q(p)-1)}(\Omega^1_Y) \otimes F^*\Omega^p_Y \xrightarrow{\phi} \text{Sym}^{\ell-(q(p)-1)}(\Omega^1_Y) \otimes F^*\Omega^q_Y \to 0$$

where $\ell(p) > n$ is the integer such that $\ell - \ell(p) \cdot p < p$.

**Lemma 2.8.** If $\Omega^1_Y$ is strongly semistable with $\mu(\Omega^1_Y) > 0$ and $W$ is strongly semistable, then the filtration defined in Theorem 2.7

$$0 = V_n(p-1) + V_n(p-1) \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)$$

is the Harder-Narasimhan filtration of $F^*(F_*W)$.

**Proof.** From the Lemma 4.5 of [7], one can get

$$\mu(V_\ell/V_{\ell+1}) = \frac{l}{n}K_Y \cdot H^{n-1} + \mu(W).$$

So the rest is enough to show that $W \otimes_{\mathcal{O}_Y} T^\ell(\Omega^1_Y)$ is semistable. By $W$ is strongly semistable, it's enough to show $T^\ell(\Omega^1_Y)$ is strongly semistable. In the long exact sequence of Theorem 2.7, every terms have slope $\ell \cdot \mu(\Omega^1_Y)$ by direct computations and strongly semistable by easily checking. Now, the lemma comes from the following fact: For a short exact sequence of torsion free sheaves with the same slope:

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

then one of them is strongly semistable if the other two is strongly semistable.

**Remark 2.9.** Under the above condition, one can deduce $F_*W$ is semistable, denote it by $E$. Actually, it is a direct corollary of the following Theorem.

**Theorem 2.10.** ([7] Theorem 4.2] When $K_Y \cdot H^{n-1} \geq 0$, we have, for any $E \subset F_*W$,

$$\mu(F_*W) - \mu(E) \geq -\frac{1(W, X)}{p}.$$

In particular, if $W \otimes T^\ell(\Omega^1_Y)$, $0 \leq \ell \leq n(p-1)$, are semistable, then $F_*W$ is semistable. Moreover, if $K_Y \cdot H^{n-1} > 0$, the stability of the bundles $W \otimes T^\ell(\Omega^1_Y)$, $0 \leq \ell \leq n(p-1)$, implies the stability of $F_*W$.

**Lemma 2.11.** ([2] Corollary 5.4] Let $E$ be a torsion-free sheaf of rank $r > 1$ on $Y$. Assume that $E$ is $\mu$ semistable with respect to $(D_1, \cdots, D_{n-1})$ and let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ be the corresponding Jordan-Hölder filtration of $E$, set $F_i = E_i/E_{i-1}$, $r_i = \text{rk}F_i$. 


Let $D \in |kD|$ be a normal divisor such that all the sheaves $F_i|_D$ have no torsion. If

$$k > \left[ \frac{r-1}{r} \Delta(E)D_2 \cdots D_{n-1} + \frac{1}{dr(r-1)} + \frac{(r-1)\beta_r}{dr} \right]$$

then $E|_D$ is $\mu$ semistable with $(D_2|_D, \cdots, D_{n-1}|_D)$.

**Remark 2.12.** For the notation, one can see [2] for detail. Note that if $E$ is torsion free then the restriction $E|_D$ is also torsion free for a general divisor $D$ in a base point-free system (see [4, Corollary 1.1.14] for a precise statement). Take $k \gg 0$, using the Bertini’s Theorem (cf. [1, Theorem 8.8]), we can choose $D \in |kD|$ to be smooth projective variety and $E|_D$ is a semistable torsion free sheaf with respect to $(D_2|_D, \cdots, D_{n-1}|_D)$. Repeating the above process, we can get a smooth projective curve $X$ which is a subvariety of $Y$, such that $E|_C$ is a semistable vector bundle. From our choice, the curve $C$ has large genus.

Now, we can construct the example using the above preliminaries.

**Construction 2.13.** Let $Y$ to be a smooth projective variety of dimension $n$ with $\Omega^1_Y$ is strongly semistable and $\mu(\Omega^1_Y) > 0$, for example we can take $Y = C \times \cdots \times C$ with $C$ is a smooth curve of genus $G \geq 2$. Let $W$ to be a strongly semistable bundle on $Y$, for example we can take $W$ to be the copies of line bundle. Consider the following diagram:

$$
\begin{array}{ccc}
Y & \overset{F}{\rightarrow} & Y \\
\uparrow & & \uparrow \\
X & \overset{F}{\rightarrow} & X
\end{array}
$$

where $X$ is smooth projective curve which is chosen as in Remark 2.13, and the commutative diagram comes from Lemma 2.6. Denote $F_*W$ by $E$, then $E$ is semistable from Remark 2.9 and $E|_X$ is semistable from Remark 2.12. Denote the Harder-Narasimhan filtration of $F^*E$

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)$$

by $\text{HN}$, we have the length of $\text{HN}$ is $n(p-1)+1$. Consider the restriction of the filtration $HN$ to $X$ which denoted by $\text{HN}|_X$, $(V_i/V_{i+1})|_X = (V_i|_X)/(V_{i+1}|_X)$ is semistable by Remark 2.12 and $\mu((V_i|_X)/(V_{i+1}|_X))$ is strictly increasing. Meanwhile, $(F^*E)|_X = F^*(E|_X)$ by the commutative diagram and $\text{HN}|_X$ is the Harder-Narasimhan filtration of $F^*(E|_X)$. Above all, we construct a semistable bundle $E|_X$ over a smooth curve $X$ with genus $g > 2$, such that the length of the Harder-Narasimhan filtration is $n(p-1)+1$. So there is no bound for the
length of the Harder-Narasimhan filtration of a bundle as Frobenius pull-back of a semistable bundle.

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