ON THE CONNECTION PROBLEM FOR THE \( p \)-LAPLACIAN SYSTEM FOR POTENTIALS WITH SEVERAL GLOBAL MINIMA

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Abstract. We study the existence of solutions to systems of ordinary differential equations that involve the \( p \)-Laplacian for potentials with several global minima. We consider the connection problem for potentials with two minima in arbitrary dimensions and with three or more minima on the plane.

1. Introduction

We consider the problem of existence of solutions to systems of ordinary differential equations that involve the \( p \)-Laplacian operator, that is, systems of the form

\[
(|u_x|^p - 2 u_x)_x - \frac{1}{q} \nabla W(u) = 0
\]

for vector-valued functions \( u \) and potentials \( W \) that possess several global minima. The corresponding problem was considered in the papers Alikakos and Fusco [4] and Alikakos, Betelú, and Chen [3] for the standard Laplacian and here we provide extensions of these results for \( p > 1 \).

What is presented in the following is divided in two parts. In Section 2 we consider the problem in \( \mathbb{R}^N \) for potentials with two global minima and state without proof an existence theorem together with a variational characterization of the connecting solutions (for the full proofs we refer to [9]). The problem for potentials possessing three or more global minima (even for the case \( p = 2 \)) is significantly harder and essentially open. In Section 3, by restricting ourselves to \( N = 2 \), we are able to exhibit a class of potentials for which we have reasonably complete results. In particular, we establish a uniqueness theorem and also give some examples of potentials that exhibit non-existence and non-uniqueness properties.

2. The connection problem for potentials possessing two global minima

Let \( \Omega \) be an open and connected subset of \( \mathbb{R}^N \) and \( W : \Omega \to \mathbb{R} \) be a \( C^2 \) nonnegative potential function with two minima, that is, \( W > 0 \) in \( \mathbb{R}^N \setminus A \), with \( W = 0 \) on \( A = \{a^+, a^-\} \). In [4], Alikakos and Fusco analyze the existence of solutions to
the Hamiltonian system

\[ u_{xx} - \frac{1}{2} \nabla W(u) = 0, \quad \text{with} \quad \lim_{x \to \pm \infty} u(x) = a^\pm, \]

where \( u : \mathbb{R} \to \mathbb{R}^N \) is a vector-valued function. Such solutions are called \textit{heteroclinic connections}. The system (1) represents the motion of \( N \) material points of equal mass under the potential \(-W(u)\), with \( x \) standing for time and \( u \) for position. The approach in [4] is variational and is based on Hamilton’s principle of least action, that is, on the minimization of the action functional \( A : W^{1,2}(\mathbb{R}, \mathbb{R}^N) \to \mathbb{R} \), defined as

\[ A(u) = \frac{1}{2} \int_{\mathbb{R}} (|u_x|^2 + W(u)) \, dx. \]

The method depends on the introduction of a constraint leading to the existence of local minimizers. The constraint can later be removed and therefore provide a solution to (1).

In this section, we state without proof an extension of the results in [4] to the \( p \)-Laplacian operator. To this end, we consider the system

\[ (|u_x|^p - 2u_x)_{xx} - \frac{1}{q} \nabla W(u) = 0, \quad \text{with} \quad \lim_{x \to \pm \infty} u(x) = a^\pm, \]

where \( u : \mathbb{R} \to \mathbb{R}^N \) is again a vector-valued function and \( p, q > 1 \) are Hölder conjugates, that is, \( \frac{1}{p} + \frac{1}{q} = 1 \). Alternatively, by Hamilton’s principle of least action, the motion from one minimum of the potential to another is a critical point of the action functional \( A_p : W^{1,p}([t_1, t_2], \mathbb{R}^N) \to \mathbb{R} \), defined as

\[ A_p(u, (t_1, t_2)) := \int_{t_1}^{t_2} \left( \frac{|u_x|^p}{p} + \frac{W(u)}{q} \right) \, dx. \]

Therefore, the system (2) is the associated Euler–Lagrange equation with \( t_1 = -\infty \) and \( t_2 = +\infty \). To state our results, we assume that the following hypothesis holds.

**Hypothesis 1.** The potential \( W \) is such that \( \liminf_{|u| \to \infty} W(u) > 0 \) and also there exists \( R > 0 \) such that the map \( r \mapsto W(a^+ + r\xi) \) has a strictly positive derivative for every \( r \in (0, R) \) and for every \( \xi \in S^{N-1} := \{ u \in \mathbb{R}^N : |u| = 1 \} \), with \( R < |a^+ - a^-| \).

Then, under the above hypothesis, we have the following theorems.

**Theorem 1.** Let \( W : \mathbb{R}^N \to \mathbb{R} \) be a non-negative \( C^2 \) potential function and let \( a^- \neq a^+ \in \mathbb{R}^N \) be such that \( W(a^\pm) = 0 \). Also assume that Hypothesis 1 holds. Then, there exists a connection \( U \) between \( a^- \) and \( a^+ \).

**Theorem 2.** Let \( U \) be the minimizer provided by Theorem 1 above and let \( R \) be as defined in Hypothesis 1. Also let \( \mathcal{A} \) be the set that consists of all functions \( u \in W^{1,p}(\mathbb{R}; \mathbb{R}^N) \) for which there exist \( x^-_u < x^+_u \) (depending on \( u \)) such that

\[ \begin{cases} |u(x) - a^-| \leq R/2, & \text{for all} \ x \leq x^-_u, \\ |u(x) - a^+| \leq R/2, & \text{for all} \ x \geq x^+_u. \end{cases} \]

Then,

\[ A_p(U) = \min_{u \in \mathcal{A}} A_p(u). \]
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The above theorems yield the existence of the desired heteroclinic connection and also a variational characterization. Our approach (for full details, see [9]) follows the lines of the method established in [3] and therefore it is based on the minimization of the action functional $A_p$ over the whole real line. The proof of existence of a connection is generally straightforward, since the test functions constructed in Lemmas 3.1 and 3.2 in [4] succeed in the pointwise reduction of both the gradient and the potential terms.

3. The connection problem on the complex plane for potentials possessing several global minima

In this section we extend to the $p$-Laplacian operator the existence and uniqueness results in Alikakos, Betelú, and Chen [3]. Here, we consider potentials possessing three or more global minima and restrict ourselves to the planar case $N = 2$ for which we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

We tackle the problem by utilizing Jacobi’s principle, which deals with curves and detects geodesics. Specifically, one considers the length functional

$$L_p(u) = \int_{t_1}^{t_2} \sqrt{W(u)|u_x|} \, dx,$$

which is independent of parametrizations and hence is more properly denoted by

$$L_p(\Gamma) = \int_{\Gamma} \sqrt{W(\Gamma)} \, d\Gamma.$$

Notice that the functional $A_p$ is defined on functions while the functional $L_p$ is defined on curves. The relationship between the two is that critical points of $L_p$ parametrized under the equipartition parametrization (that is, a parameter $t$ is such that $|u_t|^p = W(u)$, for all $t$ in an interval $(a, b)$) render critical points of $A_p$. In this case, we study the system of ordinary differential equations

$$\left(|u_x|^{p-2}u_x \right)_x - \frac{1}{q} \frac{\partial W(u^1, u^2)}{\partial u^i} = 0, \text{ for } i = 1, 2,$$

where $u = (u^1, u^2) : \mathbb{R} \to \mathbb{R}^2$. We identify $u = (u^1, u^2)$ with the complex number $z = u^1 + iu^2$, and similarly we write $W(u)$ as $W(z)$. Since $W(\cdot)$ is non-negative, we can write $W(z) = |f(z)|^q$ for some analytic function $f$. We can also verify that the equations in (2) are equivalent to

$$\left(|z_x|^{p-2}z_x \right)_x = \left(\overline{\bar{f}'} \right)^{\frac{2}{p-2}} \int_{\Gamma} \overline{\bar{f}}^{rac{2}{p-2}} \, d\Gamma,$$

where the bar represents complex conjugation.

We begin by sketching the method for the standard triple-well potential $W(z) = |z^3 - 1|^q$, the minima of which are taken at the points $1, e^{\frac{2\pi i}{3}},$ and $e^{-\frac{2\pi i}{3}}$. We will construct the connection between $a = 1$ and $b = e^{\frac{2\pi i}{3}}$, by considering the variational problem

$$\min L_p(u)$$

along embeddings on the plane connecting $a$ to $b$. Since the functional $L_p$ is independent of parametrizations, we choose $u : (0, 1) \to \mathbb{R}^2$ with $u(0) = a, u(1) = b$, and set $z(t) = u^1(t) + u^2(t)$. Then,

$$L_p(u) = \int_0^1 |z'(\tau)||z^3(\tau) - 1| \, d\tau = \int_0^1 \left| \frac{d}{d\tau} g(z(\tau)) \right| \, d\tau = \int_0^1 |w'(\tau)| \, d\tau,$$
where \( w = g(z) = z - z^4/4 \). It is clear that minimizing \( L_p \) over the set of curves connecting \( a \) to \( b \) is reduced to the simple problem of minimizing the length functional on the \( w \)-plane for curves connecting \( g(a) = 3/4 \) to \( g(b) = 3e^{2\pi i}/4 \), which of course is minimized by the line segment connecting these image points. Now, by choosing the following parametrization for the line segment

\[
 g(z(\tau)) = \tau g(a) + (1 - \tau)g(b) = \frac{3}{4}(\tau + (1 - \tau)e^{2\pi i}), \quad 0 \leq \tau \leq 1,
\]

we can show that the curve \( z(\tau) = r(\tau)e^{i\theta(\tau)} \) satisfies the parameter-free equation

\[
 4r \cos(\theta - \frac{\pi}{3}) = r^4 \cos(4\theta - \frac{\pi}{3}) + 3\cos\frac{\pi}{3}, \quad 0 \leq \theta \leq \frac{\pi}{3} \quad \text{and} \quad 0 < r < 1,
\]

which is exactly the same equation presented in [3] for \( p = q = 2 \). The dependence on \( p \) is through the parametrization

\[
 \frac{dt}{dx} = \frac{\sqrt{W(z(t))}}{\left|z'(t)\right|}, \quad \text{with} \quad t(0) = \frac{1}{2},
\]

which leads to the connection \( u(x) = z(t(x)) \). So, naturally, one would expect that the theory presented in [3] could be extended for any \( p > 1 \), and although this is true, it is not without some extra technical effort.

**Theorem 3.** Let \( W(u_1, u_2) = |f(z)|^q \), where \( f = g' \) is holomorphic in an open subset \( D \) of \( \mathbb{R}^2 \), and let the point \( (u_1, u_2) \) be identified with the complex number \( z(x) = u_1(x) + iu_2(x) \). Additionally, let \( \gamma = \{u(x) : x \in (a,b)\} \) be a smooth curve in \( D \) and \( x \) an equipartition parameter, that is, \( |u_x|^p = W(u) \). Also, set \( \alpha = u(a) \) and \( \beta = u(b) \). Then, \( u \) is a solution to

\[
 |u_x|^{p-2}u_x - \frac{1}{q}\nabla W(u) = 0, \quad \text{in} \quad (a,b)
\]

if and only if

\[
 \text{Im} \left( \frac{g(z) - g(\alpha)}{g(\beta) - g(\alpha)} \right) = 0, \quad \text{for all} \quad z \in \gamma.
\]

**Proof.** Let \( u = (u_1, u_2) : (a, b) \to D \) be a solution to (3) with \( |u_x|^p = W(u) \). Also, let \( L \) be the total arclength and \( l \) the arclength parameter defined by

\[
 L = \int_a^b |u_x|\sqrt{W(u)} \, dx \quad \text{and} \quad l = \int_a^x |u_x(y)|\sqrt{W(u(y))} \, dy.
\]

We will show that \( g(\gamma) \) is the line segment \( [g(\alpha), g(\beta)] \). We begin by modifying equation (3). Here we note that the equipartition relationship gives

\[
 |z_x|^p = W(u) = (f\overline{f})^{\frac{q}{p}}
\]

and that since

\[
 \frac{\partial W}{\partial u_1} = (|f(z)|^q)_{u_1} = (f\overline{f})^{\frac{q}{p}}(f\overline{f} + f\overline{f}),
\]

\[
 \frac{\partial W}{\partial u_2} = (|f(z)|^q)_{u_2} = \frac{q}{2}(f\overline{f})^{\frac{q}{p}}(f\overline{f} - f\overline{f}),
\]

we have

\[
 \frac{\partial W}{\partial u_1} + i\frac{\partial W}{\partial u_2} = \frac{q}{2}(f\overline{f})^{\frac{q}{p}}(f\overline{f} + f\overline{f} - f\overline{f} + f\overline{f}) = q(f\overline{f})^{\frac{q}{p}}f\overline{f}.
\]
Hence, based on (4), equation (3) can be written as

\[
0 = (|u_x|^{p-2}u_x)_x - \frac{1}{q} \nabla W(u) = \frac{p}{2} |z_x|^{p-2} z_{xx} + (\frac{p}{2} - 1)|z_x|^{p-4} z_x^2 f_{xx} - \left( \int f f' \right)^{\frac{p-2}{2}} f f'.
\]

Now, by multiplying the above equation by \(|z_x|^{4-p}\) and by further simplifying, we see that equation (3) is equivalent to

\[
p|z_x|^2 z_{xx} + (p-2) z_x^2 z_{xx} - 2|z_x|^{(3-p)} f f' = 0.
\]

In addition, by differentiating the equipartition relationship \(|z_x|^p = (ff')^{\frac{p}{2}}\), we obtain the equation

\[
p|z_x|^{p-2}(z_{xx} z_x^2 + z_{xx} z_{x}^2) = q(ff')^{\frac{p}{2}} (f f' z_x + f' f f_{xx})
\]

which simplified, is equivalent to

\[
(p - 1)|z_x|^{2(p-2)} (z_{xx} z_x^2 + z_{xx} z_{x}^2) = f' f f_{xx} + f f' z_x.
\]

Finally, differentiating the function \(g\), we have

\[
\frac{dg(z)}{dl} = \frac{g'(z) z_x}{(ff')^{\frac{p}{2}}} = \frac{z_x}{(ff')^{\frac{p}{2} - 1}} = \frac{z_x}{(ff')^{\frac{p}{2}}}.
\]

and

\[
\frac{d^2 g(z)}{dl^2} = \frac{ff' z_{xx} + \frac{2-q}{2} ff' z_x^2 - \frac{q}{2} z_x^2 f f'}{(ff')^{\frac{p}{2}+1}} = \frac{2(p-1)ff' z_{xx} + (p-2)ff' z_x^2 - p|z_x|^2 f f'}{(ff')^{\frac{p}{2}+1}} = \frac{2(p-1)ff' z_{xx} + z_x((p-2)ff' z_x - p|z_x|^2 f f')}{(ff')^{\frac{p}{2}+1}} = \frac{2(p-1)ff' z_{xx} + z_x((p-2)ff' z_x - p|z_x|^2 f f') - 2(p-1)z_x f f'}{(ff')^{\frac{p}{2}+1}}.
\]

Now utilizing (8), equation (9) becomes

\[
\frac{d^2 g(z)}{dl^2} = \frac{2ff' z_{xx} + (p-2)|z_x|^2|z_x|^{2p-4} z_{xx} + (p - 2)|z_x|^{2(p-2)} z_x^2 z_{xx} - 2|z_x|^2 f f'}{2(ff')^{\frac{p}{2}+1}}
\]

and by virtue of (4), equation (10) becomes

\[
\frac{d^2 g(z)}{dl^2} = \frac{2|z_x|^{2(p-1)} z_{xx} + (p-2)|z_x|^{2(p-1)} z_{xx} + (p - 2)|z_x|^{2(p-2)} z_x^2 z_{xx} - 2|z_x|^2 f f'}{2(ff')^{\frac{p}{2}+1}}.
\]
which after simplification can be written as
\[
\frac{d^2 g(z)}{dl^2} = p|z_x|^2z_{xx} + (p - 2)|z_x|^2z_{x}^2z_{xx} - 2|z_x|^2ff'
\]
\[
= p|z_x|^2z_{xx} + (p - 2)|z_x|^2z_{x}^2z_{xx} - 2|z_x|^2(3-p)ff'
\]
\[
= 0,
\]
as a result of (7). Thus \( dg(z)/dl = C \) is constant. Integrating this equation and evaluating it at \( l = L \) gives respectively \( g(z) = g(\alpha) + Cl \) and \( Cl = g(\beta) - g(\alpha) \). In addition, we have
\[
\left| \frac{dg(z)}{dl} \right| = \left| \frac{z_x}{f^{\frac{q}{2}}} \right| = \frac{|z_x||z_x|^q}{|z_x|^p} = \frac{|z_x||z_x|^{p-1}}{|z_x|^p} = 1 = |C|,
\]
hence \( L = |g(\beta) - g(\alpha)| \) and \( C = \frac{g(\beta) - g(\alpha)}{|g(\beta) - g(\alpha)|} \), which means that
\[
g(z(l)) = \frac{L-l}{L}g(\alpha) + \frac{l}{L}g(\beta).
\]

Thus, \( g \) is the desired line segment.

For the converse, assume that \( \gamma = u((a, b)) \) satisfies (4) and the parameter \( x \) is an equipartition parameter for \( u \). Then, equation (11) can be written as
\[
g(z) - g(\alpha) = s(x)(g(\beta) - g(\alpha)),
\]
where \( s(x) \) is a real-valued function. Upon differentiation, we obtain
\[
s_x(g(\beta) - g(\alpha)) = g'(z)z_x = f(z)z_x.
\]
This equation implies that
\[
|s_x| = \left| \frac{f(z)}{|g(\beta) - g(\alpha)|} \right| = \left| \frac{f(z)^q}{|g(\beta) - g(\alpha)|} \right| > 0
\]
and since \( s(x) \in \mathbb{R} \), with \( s(a) = 0 \) and \( s(b) = 1 \), we must have \( s_x(x) > 0 \). Hence,
\[
s_x(x) = \frac{|f(z)|^q}{|g(\beta) - g(\alpha)|}.
\]
Consequently,
\[
(11)
\]
\[
z_x = \frac{s_x(g(\beta) - g(\alpha))}{f(z)} = C\left(\frac{ff'}{f}\right)^{\frac{q}{2}},
\]
where \( C = \frac{g(\beta) - g(\alpha)}{|g(\beta) - g(\alpha)|} \).

Lastly, utilizing (11), we construct the differential equation (7) as follows. First, we have
\[
|z_x|^2 = (ff')^{q-1},
\]
\[
z_{xx} = \frac{q-2}{2}C^2f^{q-3}ff' + \frac{q}{2}f^{q-1}f^{q-2}f',
\]
\[
z_x^2 = C^2f^{q-2}f',
\]
\[
\overline{z}_{xx} = \frac{q-2}{2}C^2f^{q-3}ff' + \frac{q}{2}f^{q-1}f^{q-2}f',
\]
so from the above relations it follows that
\[ (12) \quad p|z_x|^2 z_{xx} = \frac{pq - 2p}{2} C^2 f^{2q - 4} f' + \frac{pq}{2} f^{2q - 2} f'' \]
\[ (13) \quad (p - 2) z_{xx}^2 = \frac{pq - 2p - 2q + 4}{2} f^{2q - 2} f'' + \frac{pq - 2q}{2} C^2 f^{2q - 4} f' \]
Adding (12) and (13) gives
\[ (14) \quad p|z_x|^2 z_{xx} + (p - 2) z_{xx}^2 = (pq - p - q) C^2 f^{2q - 4} f' + (pq - p - q + 2) f^{2q - 2} f'' \]
and utilizing the fact that \(|z_x|^{2-p} = (f f')^{2q - 3}\), equation (14) becomes
\[ p|z_x|^2 z_{xx} + (p - 2) z_{xx}^2 = 2(f f')^{2q - 3} f f'' = 2|z_x|^{2(3-p)} f f'' \]
This equation is equivalent to \(u\) being a solution to \(q(|u_x|^{p-2}u_x)_x = W_u(u)\) and the proof is complete.

The proof implies that (3) is equivalent to the first-order ordinary differential equation
\[ (15) \quad z_x = C \frac{(f f')^\frac{q}{2}}{f}, \quad \text{for } C \in \mathbb{C} \text{ with } |C| = 1. \]
Multiplying (15) by \(\frac{f(z)}{C}\) gives
\[ \frac{d}{dx} \frac{g(z)}{C} = |f(z)|^q = W > 0 \]
and integrating this equation gives
\[ \text{Im} \left( \frac{g(z) - g(\alpha)}{C} \right) = 0 \]
and
\[ \frac{g(z) - g(\alpha)}{C} = \int_a^x W(z(t))dt = l. \]
This in particular implies that the map \(x \mapsto g(z(x))\) is a one-to-one map. Also, in the case of \(u\) being a solution, the theorem states that the set \(g(\gamma) = \{g(z) : z \in \gamma\}\) is a line segment with end points \(g(\alpha)\) and \(g(\beta)\) and that the partial transition energy is given by
\[ \int_a^y \left( \frac{|u_x|^p}{p} + \frac{W(u)}{q} \right) dx = \int_a^y \left| u_x \sqrt{W(u)} \right| dx \]
\[ = \int_a^y \left| \frac{d}{dx} g(u) \right| dx \]
\[ = |g(u(y)) - g(\alpha)|, \quad \text{for all } y \in (a, b]. \]

**Theorem 4.** There exists at most one trajectory connecting any two minima of a holomorphic potential, that is, if \(W(z) = |f(z)|^q\), where \(f\) is holomorphic on \(\mathbb{C}\), then there exists at most one solution of (2) that connects any two roots of \(W(z) = 0\).
Proof. Let $g$ be an antiderivative of $f$ and suppose that $\gamma_1$ and $\gamma_2$ are two trajectories to (2) with the same end points $\alpha, \beta$. Since the energy $|g(\beta) - g(\alpha)|$ is positive, it follows that $g(\beta) \neq g(\alpha)$ and we can define the function

$$\hat{g} = \frac{|g(\beta) - g(\alpha)|}{g(\beta) - g(\alpha)} (g(z) - g(\alpha)),$$

for all $z \in \mathbb{C}$.

Then, $\hat{g}$ is real on $\gamma_1 \cup \gamma_2$. If $\gamma_1 \neq \gamma_2$, then $\gamma_1$ and $\gamma_2$ will enclose an open domain $D$ in $\mathbb{C}$. As the imaginary part of $\hat{g}$ on $\partial D = \gamma_1 \cup \gamma_2 \cup \{\alpha, \beta\}$ is zero, it has to be identically zero in $D$. This implies that $\hat{g}$ is a constant function in $\mathbb{C}$, which is impossible. Thus, $\gamma_1 = \gamma_2$. □

Finally, we present some specific examples of non-existence and non-uniqueness of connections between the minima of various potentials. Specifically, it can be proved that for both the potentials

$$W(z) = |z^n - 1|^q,$$

where $n \geq 2$ is an integer, and

$$W(z) = |(1 - z^2)(z^2 + \varepsilon^2)|^q,$$

where $0 < \varepsilon < \infty$, there always exists a unique connection between each pair of their minima. We also refer to a non-existence and non-uniqueness phenomenon for the potentials

$$W(z) = |(1 - z^2)(z - i\varepsilon)|^q,$$

where $0 \leq \varepsilon < \infty$, and

$$W(z) = |(z - 1)(z + a)/z|^q,$$

where $0 < a < 1$, respectively. In the first case it can be proved that there exists a connection between $-1$ and $1$ if and only if $|\varepsilon| > \sqrt{2\sqrt{3} - 3}$, while in the second case there exist exactly two connections between $-a$ and $1$, one in the upper half-plane and one in the lower half-plane. We conclude by stating that all examples given in [3] have exact analogs since the only modification needed for the transformation to the $p$-case is the change from potentials of the form $W = |f|^2$ (cf. [3]) to potentials of the form $W = |f|^q$.

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