SPLIT EXTENSIONS OF GROUP WITH INFINITE CONJUGACY CLASSES

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ABSTRACT. We give a characterization of the group property of being with infinite conjugacy classes (or icc, i.e. $\neq 1$ and of which all conjugacy classes beside 1 are infinite) for split extensions of group.

INTRODUCTION

A group is said to be with infinite conjugacy classes (or icc) if it is non trivial, and if all its conjugacy classes beside $\{1\}$ are infinite. This property is motivated by the theory of Von Neumann algebra, since for any group $\Gamma$, a necessary and sufficient condition for its Von Neumann algebra $W^*_\lambda(\Gamma)$ to be a type $II - 1$ factor is that $\Gamma$ be icc (cf. [ROIV]).

The property of being icc has been characterized in several classes of groups : 3-manifolds and $PD(3)$ groups in [HP], groups acting on Bass-Serre trees in [Co], wreath products and finite extensions in [P1, P2]. We will focus here on groups defined by a split extension (also called semi-direct product).

Towards this direction particular results are already known. In [P2] has been proved the following :

Let $G$ be a finite extension of $K$ :

$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$

then $G$ is icc if and only if $K$ is icc and the natural homomorphism $\theta : Q \longrightarrow Out(K)$ is injective.

In particular, it applies when the finite extension splits :

Let $G = K \rtimes_\theta Q$, with $Q$ finite ; $G$ is icc if and only if $K$ is icc and the homomorphism $Q \longrightarrow Out(K)$ induced by $\theta$ is injective.

In [Co], has been proved, among other results, the following characterization of icc extensions by $\mathbb{Z}$ :

Let $G = D \rtimes_\theta \mathbb{Z}$ with $D \neq \{1\}$ ; then $G$ is not icc if and only if one of the following conditions is satisfied :

(i) $D$ contains a $\theta(Q)$-stable normal subgroup $N \neq \{1\}$ and either $N$ is finite or $D = \mathbb{Z}^n$ and if $\pi$ is the natural homomorphism from $G$ to $GL(n, \mathbb{Z})$ extending $\theta$, then $N$ has only finite $\pi(G)$-orbits,

(ii) the homomorphism $\mathbb{Z} \longrightarrow Out(D)$ induced by $\theta$ is non injective.

We give a generalization of these partial results by proposing a general characterization of split extensions with infinite conjugacy classes.

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1. Preliminaries

Let $G$ be a group, $H$ a non empty subset of $G$, and $u, g$ elements of $G$; then $Z_G(H)$ and $Z(G)$ denote respectively the centralizer of $H$ in $G$ and the center of $G$. The element $u^g$ of $G$ is defined as $u^g = g^{-1}ug$, while $u^H = \{u^g \mid g \in H\}$; in particular $u^G$ denotes the conjugacy class of $u$ in $G$. One immediately verifies that the cardinality of $u^G$ equals the index of $Z_G(u)$ in $G$ so that $u^G$ is finite if and only if $Z_G(u)$ has a finite index in $G$.

The set of elements having a finite conjugacy class in $G$ turns out to be a characteristic subgroup of $G$ that we denote by $FC(G)$; it is a so called $FC$-group, that is a group whom all conjugacy classes are finite. Obviously $G$ is icc if and only if $FC(G) = \{1\}$. The class of $FC$-groups has been extensively studied and it’s a well known fact that finitely generated $FC$-groups are precisely those groups defined by a central finite extension of a f.g. abelian group (c.f. [Ne]). In particular, in any f.g. $FC$-groups $K$, the subset of torsion elements $Tor(K)$ is a characteristic subgroup of $K$ and the quotient $K/Tor(K)$ is free abelian with a finite rank.

In the following the group $G$ stands for the split extension $G = K \rtimes_{\theta} Q$ (or semi-direct product) with normal factor $K$, retract factor $Q$ and associated homomorphism $\theta : Q \to Aut(K)$; with these notations, for any $k \in K$, $q \in Q$, $q^{-1}kq = \theta(q)(k)$. Let $\pi : G \to Aut(K)$ be the homomorphism defined by $\forall g \in G, k \in K, \pi(g)(k) = g^{-1}kg$; it extends on $G$ both $\theta$ and the natural homomorphism $\pi_K : K \to Inn(K)$, that is the diagram below commutes.

$$
\begin{array}{cccc}
1 & \to & K & \to & G & \to & Q & \to & 1 \\
\downarrow^{\pi_K} & \downarrow^{\pi} & \downarrow & \downarrow^{\theta} & \downarrow & \downarrow & 1 \\
1 & \to & Inn(K) & \to & \pi(G) & \to & \theta(Q) & \to & 1
\end{array}
$$

The subgroup $\pi(G)$ of $Aut(K)$ is an extension of $Inn(K)$ by $\theta(Q)$; in general the extension does not split, despite the above one does.

We shall write in the following $\theta_q$ and $\pi_q$ instead of $\theta(q)$ and $\pi(q)$. We will denote by $\Theta : FC(Q) \to Out(K)$ the homomorphism induced by $\theta : Q \to Aut(Q)$.

2. Statement of the main result

The first result we prove is the following characterization of semi-direct products with infinite conjugacy classes:

**Theorem 1.** Let $G = K \rtimes_{\theta} Q \neq 1$ be a split extension, and $\pi : G \to Aut(K)$, $\Theta : FC(Q) \to Out(K)$ be the homomorphisms defined as above. Then $G$ is not icc if and only if one of the following conditions is satisfied:

(i) $K$ contains a normal subgroup $N \neq 1$ preserved under the action of $\pi(G)$ and such that either $N$ is finite, or $N \approx \mathbb{Z}^n$ has only finite $\pi(G)$-orbits.

(ii) $\ker \Theta$ contains $q \neq 1$ with $\forall x \in K, \theta_q(x) = k^{-1}xk$, for some $k \in K$ with finite $\theta(Q)$-orbit.

**Remark 1.** Condition (ii) can be rephrased as:

(ii) either $\theta : FC(Q) \to Aut(K)$ is non injective or the subgroup $\pi_K^{-1}(\theta(FC(Q)))$ of $K$ contains $k \neq 1$ whose $\theta(Q)$-orbit is finite.

**Example.** Suppose $G = K \rtimes_{\theta} Q$; if $K$ satisfies any of the above assumptions, then $G$ is not icc:

- $K$ is a non trivial elementary group,
- $Z(K)$ contains a non trivial finite subgroup,
- $FC(K) \setminus 1$ contains a finite $\theta(Q)$-orbit,
– $\text{Tor}(FC(K))$ is a non trivial finite group.

(For each case condition (i) of theorem 1 is satisfied.)
– $\theta$ is non injective,

(For which case condition (ii) follows.)

The theorem 1 can be rephrased in several ways. The first rephrasing is by mean of the finite $\theta(Q)$-orbits in $K$.

**Theorem 2.** Let $O_{\theta}$ be the union of all finite $\theta(Q)$-orbits in $K$; $G$ is icc if and only if:

(a) $O_{\theta} \cap FC(K) = 1$, and

(b) $O_{\theta} \cap \pi^{-1}_{K}(\theta(FC(Q))) = 1$, and

(c) the restricted homomorphism $\theta : FC(Q) \rightarrow Aut(K)$ is injective.

Condition (a) of theorem 2 is equivalent to the negation of condition (i) of theorem 1.

Negation of condition (ii) of theorem 1 is equivalent to the conjunction of conditions (b) and (c) of theorem 2. So that theorem 2 can be seen as a way of reducing condition (ii) into the obvious condition : $\theta : FC(Q) \rightarrow Aut(K)$ non injective, and a residual one.

In this direction one can also reduce condition (ii) of theorem 1 into the condition that either $\theta : FC(Q) \rightarrow Aut(K)$ is non injective or $G$ contains –what we called– a twin $FC$-subfactor.

**Theorem 3.** In theorem 1, condition (ii) can be changed into :

(ii.a) $\theta : FC(Q) \rightarrow Aut(K)$ is non injective, or

(ii.b) $G$ contains a twin $FC$-subfactor $C \rtimes C$.

Roughly speaking, a twin $FC$-subfactor is a transversal subgroup, either $\mathbb{Z}^{n} \times \mathbb{Z}^{n}$ or $C \rtimes C$ for $C$ a finite group, which is $\theta(Q)$-stable with only finite $\theta(Q)$-orbits, and such that $\theta^{-1} \circ \pi$ sends isomorphically the left factor on the right one. (cf. §6).

3. PROOF OF THEOREM 1

This section is entirely devoted to proving theorem 1.

**Proof of theorem 1**

We first prove the sufficient part of the assumption, that is, if either condition (i) or condition (ii) is satisfied, then $G$ is not icc.

**Fact 1.** Condition (i) implies that $G$ is not icc.

**Proof of the fact 1.** Suppose the condition (i) is satisfied. Since the conjugacy class in $G$ of an element of $K$ is its orbit under the action of $\pi(G)$, obviously each element of $N$ has a finite conjugacy class in $G$, and hence $G$ is not icc. □

**Fact 2.** Condition (ii) implies that $G$ is not icc.

**Proof of the fact 2.** Suppose the condition (ii) is satisfied; let $\omega = k^{-1}q \neq 1$, so that $Z_{G}(\omega) \supset K$. Let $\text{Stab}_{\theta}(k)$ denotes the stabilizer of $k$ in $\theta(Q)$; since it has a finite index in $\theta(Q)$, then $Q_{0} = \theta^{-1}(\text{Stab}_{\theta}(k))$ has a finite index in $Q$. Hence $Q_{1} = Z_{Q}(q) \cap Q_{0}$ also has a finite index in $Q$. Then for any $u \in Q_{1}$, $\omega^{u} = u^{-1}k^{-1}qu = \theta_{u}(k^{-1})q^{u} = k^{-1}q = \omega$; hence $Z_{G}(\omega) \supset Q_{1}$. It follows that $Z_{G}(\omega)$ contains $K \rtimes Q_{1}$ and hence has a finite index in $G$, so that $G$ is not icc. □

We now prove the necessary part of the assumption, that is, if $G$ is not icc then either condition (i) or condition (ii) is satisfied. Let $G$ be not icc: since $G \neq 1$, there exists $u \neq 1$ in $G$ such that $u^{G}$ is finite.

**Fact 3.** If $K$ contains $u \neq 1$ with $u^{G}$ finite, then condition (i) follows.
Proof of the fact 3. Let $N'$ be the subgroup of $K$ finitely generated by the set $u^G$. Then $N'$ is preserved under the action of $\pi(G)$, and in particular is normal in $K$. Since any element of $u^G$ has a finite orbit under $\pi(G)$, $N'$ contains only finite $\pi(G)$-orbits. In particular $N'$ is a finitely generated $FC$-group. It follows that $Tor(N')$ is a finite characteristic subgroup of $N'$ and $N'/Tor(N')$ is free abelian with finite rank (cf. \[\text{Ne}\]). Then one obtains a normal subgroup $N$ of $K$ satisfying condition (i) by : if $Tor(N') \neq 1$ then $N = Tor(N')$ and otherwise $N = N' = \mathbb{Z}^n$.

Fact 4. If $G \setminus K$ contains $u^G$ finite, then either condition (i) or (ii) is satisfied.

Proof of the fact 4. Let $u = k^{-1}q$ for some $k \in K$ and $q \neq 1$ lying in $Q$, such that $Z_G(u)$ has a finite index in $G$. Necessarily $q$ lies in $FC(Q)$, for $q^Q$ is the image of $u^G$ under the projection of $G$ onto $Q$.

Let $h \in K$ and $\omega = [u, h] \in K$ ; both $Z_G(u)$ and $Z_G(hu^{-1}h^{-1})$ have a finite index in $G$ and their intersection lies in $Z_G(\omega)$, so that $\omega$ is an element of $K$ having a finite conjugacy class in $G$. If $\omega \neq 1$, it follows from the fact 3 that condition (i) is satisfied. So we suppose in the following that for any $h \in K$, $[u, h] = 1$, so that $\pi_u$ is the identity on $K$. Hence $\theta_q$ is inner, for any $x \in K$, $\theta_q(x) = x^k$.

Now let $Q_0 = Z_G(u) \cap Z_Q(q)$, $Q_0$ is obviously contained in $Z_Q(k)$, so that $\theta(Q_0)$ is contained in $Stab_\theta(k)$. Since $Q_0$ has a finite index in $Z_Q(q)$, it also has a finite index in $Q$, and then $Stab_\theta(k)$ has a finite index in $\theta(Q)$, so that $k$ has a finite $\theta(Q)$-orbit. Hence condition (ii) is satisfied.

4. FORMULATION BY MEAN OF FINITE $\theta(Q)$-ORBITS

One can formulate the theorem \[\Pi\] by mean of the finite $\theta(Q)$-orbits in $K$.

Theorem 2. Let $G = K \rtimes_\theta Q \neq 1$ and $O_\theta$ be the union of all finite $\theta(Q)$-orbits in $K$. Then $G$ is icc if and only if :

(a) $O_\theta \cap FC(K) = 1$, and
(b) $O_\theta \cap \pi_K^{-1}(\theta(FC(Q))) = 1$, and
(c) the restricted homomorphism $\theta : FC(Q) \rightarrow Aut(K)$ is injective.

Proof. Condition (i) obviously implies that $O_\theta \cap FC(K) \neq 1$. The converse is also true. For, since $FC(K)$ is a characteristic subgroup of $K$, $O_\theta \cap FC(K) \neq 1$ implies that $FC(K)$ contains a non trivial finite $\theta(Q)$-orbit $O \neq \{1\}$. The union of conjugates of $O$ in $K$ is finite and preserved under $\pi(G)$. So that for $k_0 \in O$, $k_0^G$ is finite, $k_0 \neq 1$, and condition (i) follows from the fact 3 in the proof of theorem \[\Pi\].

Conjunction of (b) and (c) is an immediate rephrasing of the negation of condition (ii). Conclusion follows from theorem \[\Pi\].

In particular, when $O_\theta = 1$ one obtains a very concise statement.

Corollary 1. Let $G = K \rtimes_\theta Q \neq 1$ such that all $\theta(Q)$-orbits in $K \setminus 1$ are infinite. Then $G$ is icc if and only if the restricted homomorphism $\theta : FC(Q) \rightarrow Aut(K)$ is injective.

5. ON WEAKENING CONDITION (ii)

As we just have seen, in specific cases, condition (ii) in theorem \[\Pi\] can be changed into the obvious : $\theta : FC(Q) \rightarrow Aut(K)$ is non injective. Further examples follow from :
Proposition 1. In the assumption of theorem [1] if one moreover suppose at least one of the following conditions:

- $K$ is abelian,
- $K \setminus 1$ contains only infinite $\theta(Q)$-orbits,
- the $\theta(Q)$-extension $\pi(G)$ of $\text{Inn}(K)$ splits, i.e. $\pi(G) = \text{Inn}(K) \rtimes \theta(Q)$,

then condition (ii) can be strengthened into:

- the restricted homomorphism $\theta : FC(Q) \rightarrow \text{Aut}(K)$ is non injective.

Proof. If either $K$ is abelian or $\pi(G) = \text{Inn}(K) \rtimes \theta(Q)$, then necessarily one has that $\theta(Q) \cap \text{Inn}(K) = 1$ so that $\pi^{-1}_K(\theta(\text{FC}(Q))) = 1$, and condition (ii) becomes equivalent with $\theta : FC(Q) \rightarrow \text{Aut}(K)$ is non injective.

In general one cannot strenghten condition (ii) so far. For example if $K$ is icc and $G = K \rtimes_\theta Z$ then $G$ is not icc each time $\Theta : Z \rightarrow \text{Out}(K)$ is non injective; which may happen while $\theta$ is injective.

One may expect to weaken condition (ii) into the condition that $\Theta : FC(Q) \rightarrow \text{Out}(K)$ is non injective; that is forgetting about hypothesis that $k$ has a finite $\theta(Q)$-orbit.

Proposition 2. In the assumption of theorem [1] if one moreover suppose at least one of the following conditions:

- $Z(K) = 1$,
- $Q$ is finite or cyclic,

then condition (ii) can be weakened into condition (ii'):

(ii') $\Theta : FC(Q) \rightarrow \text{Out}(K)$ is non injective.

Proof. Obviously condition (ii) implies condition (ii'). We prove the converse.

- $Z(K) = 1$. Condition (ii') implies that there exists $q \neq 1$ in $FC(Q)$ such that $\theta_q$ is inner, $\theta_q(x) = x^k$. Any element $p \in Z_Q(q)$ is such that $\theta_p(k) \in k.Z(K)$. For, $\forall x \in K$, $\theta_q(x) = k^{-1}xk = \theta_p \circ \theta_q \circ \theta^{-1}_p(x) = \theta_p(k^{-1})x \theta_p(k)$, implies that $\theta_p(k^{-1}) \in Z(K)$. So that with $Z(K) = 1$, necessarily $\theta(Z_Q(q))$ lies in $\text{Stab}_q(k)$. Since $Z_Q(q)$ has a finite index in $Q$, $\text{Stab}_q(k)$ has a finite index in $\text{Inn}(Q)$, so that condition (ii) is satisfied.

- $Q$ is finite or cyclic. Suppose there exists $q \neq 1$ in $FC(Q)$ such that $\theta_q$ is inner, $\theta_q(x) = x^k$. Since $<q> \subset Q$ has a finite index in $Q$ and fixes $k$, condition (ii) follows from condition (ii'). (Moreover, if $Q$ is finite, condition (i) is equivalent with $K$ not icc.)

We will see later several other particular cases for which the statement of theorem [1] becomes more concise. But in general, condition (ii) cannot be weakened into (ii') as noted in the following remark.

Remark 2. Condition (ii) of theorem [1] cannot in general be weakened in condition that $\Theta : FC(Q) \rightarrow \text{Out}(K)$ is non injective. For consider:

$$K=<a_1,a_2,k_1,k_2| [a_1,a_2], [a_1,k_1], i,j = 1,2 > \approx (\mathbb{Z} \oplus \mathbb{Z}) \times F(2)$$

$$A=<a_1,a_2>_K \approx \mathbb{Z} \oplus \mathbb{Z} \subset K$$

$$Q=<q_1,q_2|[q_1,q_2] > \approx \mathbb{Z} \oplus \mathbb{Z}$$

Let $\theta_1 \in \text{Inn}(K)$, s.t. $\forall x \in K, \theta_1(x) = x^{k_1}$ ; $\theta_1$ fixes $A$ pointwise. Let $\theta_2 \in \text{Aut}(K)$, s.t. $\theta_2$ is anosov on $A$, and $\theta_2(k_2) = k_2$, $\theta_2(k_1) = k_1\alpha$ for some $\alpha \neq 1$ lying in $A$. So defined, $\theta_1$ and $\theta_2$ commute, so that the map sending $q_1$ to $\theta_1$ and $q_2$ to $\theta_2$ extends to an homomorphism $\theta : Q \rightarrow \text{Aut}(K)$ ; moreover $\theta$ is injective.

Consider $G = K \rtimes_\theta Q$ ; we show that $G$ is icc despite that $\Theta : FC(Q) \rightarrow \text{Out}(K)$ is non injective. For any non trivial $x \in K$, $x^G$ is infinite, so that in particular condition (i) of theorem [1] is not satisfied. If condition (ii) would be satisfied, it would follow that for some $n \geq 1$, $k_1^n$ would have a $\theta(Q)$-finite orbit. We show that this cannot arise.
Consider \( \theta_2 \in \theta(Q) \), \( \theta_2(k_1) = k_1 \alpha, \alpha \neq 1 \in A \), so that for any \( p \geq 1 \),
\[
\theta_2^p(k_1^n) = k_1^n \alpha^n \theta_2(\alpha^n) \theta_2^2(\alpha^n) \cdots \theta_2^{p-1}(\alpha^n)
\]
Let \( \phi_p : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \) be the map defined by \( \phi_p(x) = x \theta_2(x) \theta_2^2(x) \cdots \theta_2^{p-1}(x) \); \( \phi_p \) turns out to be an homomorphism. Let \( M_\theta \in SL(2, \mathbb{Z}) \) be the matrix associated with \( \theta_2 \); it has two distinct irrational eigen values \( \lambda_1, \lambda_2 \). Let \( M_p \) be the matrix associated with \( \phi_p \). Then \( M_p = \text{Id} + M_\theta + M_\theta^2 + \cdots + M_\theta^{p-1} \). \( M_p \) has two eigen values : \( l_i = 1 + \lambda_i + \lambda_i^2 + \cdots + \lambda_i^{p-1}, \)
\( i = 1, 2 \). They must be both non null because otherwise \( \lambda_i^p = 1 \) which contradicts that \( M_\theta \) is anosov. Hence, for any \( p \geq 1 \), \( \phi_p \) is injective. Since for any \( n \geq 1 \), \( \theta_2^p(k_1^n) = k_1^n \phi_p(\alpha^n) \), with \( \alpha^n \neq 1 \in A \), the \( \theta(Q) \)-orbit of \( k_1^n \) is infinite, so that condition (ii) is not satisfied. With theorem I \( G \) is icc, despite that the homomorphism \( \Theta : FC(Q) \to \text{Out}(K) \) is non injective.

6. FURTHER ON WEAKENING (ii) : THE TWIN FC-SUBFACTORS

We keep on refining condition (ii) by looking at what is in between the strengthed condition \( \theta : FC(Q) \to Aut(K) \) non injective, condition (ii) and the weakened condition (ii') : \( \Theta : FC(Q) \to \text{Out}(K) \) for theorem I.

**Definition.** Let \( G = K \rtimes_{\theta} Q \); We say that \( C \rtimes C' \) is a twin FC-subfactor of \( G \) when :
- \( C \) is a subgroup of \( K \), \( C \cap FC(K) = 1 \),
- \( C \) is \( \theta(Q) \)-stable with only finite \( \theta(Q) \)-orbits,
- \( C' \) is a normal subgroup of \( Q \), \( C' \subset FC(Q) \),
- \( \pi \) and \( \theta \) are injective respectively on \( C \) on \( C' \) and \( \pi(C) = \theta(C') \),
( so that \( \theta^{-1} \circ \pi | C : C \to C' \) is an isomorphism),
- \( C \neq 1 \) (and so \( C' \) is either finite or \( \mathbb{Z}^n \)).

A twin FC subfactor is either \( \mathbb{Z}^n \times \mathbb{Z}^n \) or \( C \rtimes_{\text{Inn}(C)} C' \), for some finite group \( C \). It is \( \theta(Q) \)-stable with only finite orbits, and \( \theta^{-1} \circ \pi \) sends isomorphically the normal factor on the retract one.

**Theorem 3.** Let \( G = K \rtimes_{\theta} Q \neq 1 \); Then \( G \) is not icc if and only if either condition (i) or at least one of the following conditions is satisfied :
- (ii.a) \( \theta : FC(Q) \to Aut(K) \) is non injective,
- (ii.b) \( G \) contains a twin FC-subfactor.

**Proof.** We first consider the sufficient part of the assumption. Condition (i) implies that \( G \) is not icc follows from theorem II; obviously condition (ii.a) also implies \( G \) not icc. If \( G \) contains a twin FC-subfactor \( C \rtimes_{\text{Inn}(C)} C' \), then condition (ii) of theorem II is satisfied with \( q \) being any non trivial element of \( C' \) and \( k = \theta \circ \pi^{-1}(q) \), so that \( G \) is not icc.

We now prove the necessary part of the assumption. We suppose in the following that \( G \) is not icc while it satisfies neither condition (i) nor condition (ii.a) and prove that condition (ii.b) must be satisfied.

With the theorem II there exists \( q \neq 1 \) in \( FC(Q) \) and \( k \neq 1 \) in \( K \), such that \( \theta_q(x) = x^k \) and \( \text{Stab}_q(k) \) has a finite index in \( Q \). Let \( C_Q \) be the subgroup of \( FC(Q) \) finitely generated by \( q^Q \); \( C_Q \) is a non trivial FC-group normal in \( Q \). Let \( Q_1 = \theta^{-1}(\text{Stab}_q(k)) \), it has a finite index in \( Q \). Clearly \( \text{Stab}_q(k) \) is included in \( Z_{\theta(q)}(\theta_q) \); if \( Q_1 \not\subset Z_{\theta(q)} \), there would exist \( p \in Q \) such that \( [p, q] \neq 1 \) and \( \theta([p, q]) = 1 \), which would contradict that \( \theta : FC(Q) \to Aut(K) \) is injective. Hence, \( Q_1 \subset Z_{\theta(q)} \), so that for \( q_0 = 1, q_1, \ldots, q_p \) a set of representatives of \( Q/Q_1, C_Q \) is generated by the finite family \( q, q^a, \ldots, q^{p^q} \).

Let \( k_i \) be such that \( k = \theta_{q_i}(k_i) \), then \( k_0 = k, k_1, \ldots, k_p \) is the \( \theta(Q) \)-orbit of \( k \) and moreover \( \theta_{q_i}^{-1} \circ \theta_q \circ \theta_{q_i}(x) = x^{k_i} \). Let \( C_K \) be the subgroup of \( K \) generated by \( k_0, k_1, \ldots, k_p \); \( C_K \) is preserved under \( \theta(Q) \) and contains only finite \( \theta(Q) \)-orbits. An element in \( C_K \cap
\( FC(K) \) has a finite conjugacy class in \( G \), so that \( C_K \cap FC(K) = 1 \) because otherwise as in fact 3 in the proof of theorem 1, condition (i) would follow.

By construction, \( \pi(C_K) = \theta(C_Q) \). Each element of \( Ker \pi|_{C_K} \) has a finite conjugacy class in \( G \) so that \( \pi \) must be injective on \( C_K \) because otherwise, as in fact 3 in the proof of theorem 1, condition (i) would follow. Moreover \( \theta \) is injective on \( C_Q \) because otherwise \( \theta : FC(Q) \to Aut(K) \) would be non injective. Hence \( \theta^{-1} \circ \pi|_{C_K} : C_K \to C_Q \) is an isomorphism. Now \( C_Q \neq 1 \) is a f.g. \( FC \)-group and hence with \( \text{Ne} \) either \( Tor(C_Q) \neq 1 \) is a finite normal subgroup in \( Q \), in which case let \( C' = Tor(C_Q) \), or \( C_Q \approx \mathbb{Z}^n \), in which case let \( C' = C_Q \). If \( C \) denotes \( \pi^{-1} \circ \theta(C') \), then \( C \rtimes_\theta C' \) is a twin \( FC \)-subfactor in \( G \).

In conclusion suppose that \( G = K \rtimes_\theta Q \) does not satisfy condition (i) of theorem 1. If \( \theta : FC(Q) \to Aut(K) \) is non injective then \( G \) is not icc. If \( G \) is not icc despite \( \theta \) is injective then \( G \) contains a twin \( FC \)-subfactor. It follows that \( \Theta : FC(Q) \to Out(K) \) is non injective. If \( \Theta \) is non injective, \( G \) may be icc as seen in remark 2; \( G \) is not icc whenever \( G \) contains a twin \( FC \)-subfactor.

**Example.** As in remark 2, consider:

\[
K = \langle a_1, a_2, k_1, k_2 | [a_1, a_2], [a_1, k_1], i, j = 1, 2 \rangle > \approx (\mathbb{Z} \oplus \mathbb{Z}) \times F(2)
\]

\[
A = \langle a_1, a_2 \rangle_{K} > \approx \mathbb{Z} \oplus \mathbb{Z} \subset K
\]

\[
Q = \langle q_1, q_2 | [q_1, q_2] > > \approx \mathbb{Z} \oplus \mathbb{Z}
\]

Let \( \theta_1 \in Inn(K) \), s.t. \( \forall x \in K, \theta_1(x) = x^{\theta_1} \). Let \( \theta_2 \in Aut(K) \), s.t. \( \theta_2 \) is anosov on \( A \), and \( \theta_2 \) is the identity on \( < k_1, k_2 >_{K} \). Hence \( \theta_1 \) and \( \theta_2 \) commute so that the map sending \( q_1 \) to \( \theta_1 \) and \( q_2 \) to \( \theta_2 \) extends to an injective homomorphism \( \theta : Q \to Aut(K) \). So defined, \( \theta(Q) \) fixes \( k_1 \) so that condition (ii) of theorem 1 is satisfied and \( G \) is not icc; condition (i) is not satisfied since \( FC(K) = A \) contains only infinite \( \theta(Q) \)-orbits. Let \( C \subset K, C' \subset Q \) be generated respectively by \( k_1 \) and \( q_1 \), then \( C \rtimes_\theta C' = \mathbb{Z} \times \mathbb{Z} \) is a twin \( FC \)-subfactor in \( G \).

**Corollary 2.** Let \( G = K \rtimes_\theta Q \), s.t. \( K \) does not contain any \( \theta(Q) \)-invariant subgroup \( H \neq 1 \), either finite or \( \mathbb{Z}^n \) with only finite \( \theta(Q) \)-orbits. Then \( G \) is icc if and only if the restricted homomorphism \( \theta : FC(Q) \to Aut(K) \) is injective.

**Proof.** Under these hypothesis, one cannot verifies condition (i), and \( G \) cannot contain any twin \( FC \)-subfactor. \( \square \)

7. Split extension of icc groups

We now consider the special case where at least one of the factors is icc.

**Theorem 4.** Let \( G = K \rtimes_\theta Q \), with \( K \) icc. Then \( G \) is icc if and only if \( \Theta : FC(Q) \to Out(K) \) is injective.

**Proof.** Since \( K \) is icc, on the one hand condition (i) cannot arise and on the other \( Z(K) = 1 \) so that the conclusion follows from proposition 2 and theorem 1. \( \square \)

The following follows directly from theorem 2.

**Theorem 5.** Let \( G = K \rtimes_\theta Q \) with \( Q \) icc. Then \( G \) is icc if and only if the action of \( \theta(Q) \) does not have any finite orbit in \( FC(K) \setminus 1 \).

**Corollary 3.** The icc property is stable under split extension.
Note that a non elementary word hyperbolic group $K$ is icc if and only if it does not contain any non trivial finite normal subgroup. For, in a word hyperbolic group the centralizer of any $\mathbb{Z}$ subgroup is virtually $\mathbb{Z}$. So that $FC(K)$ is a torsion $FC$-group. Since in a word hyperbolic group there is only finitely many conjugacy classes of torsion elements, $FC(K)$ is a finite normal subgroup of $K$.

**Theorem 6.** Let $G = K \rtimes Q$ with $K$ a non trivial word hyperbolic group. Then $G$ is icc if and only if $K$ is icc and $\Theta : FC(Q) \rightarrow Out(K)$ is injective.

**Proof.** Note that, as already stated, $G = K \rtimes Q$ is not icc whenever $K \neq 1$ is elementary. So we suppose in the following that $G$ is non elementary. It follows that $Z(K)$ is finite (cf. [Gr]). If $Z(K) \neq 1$ then $K$ contains a non trivial finite normal subgroup so that both $K$ and $G$ are not icc. If $Z(K) = 1$, it follows from proposition 2 that $G$ is not icc if and only if it satisfies either condition (i) or condition (ii'). In a word hyperbolic group the normalizer of a $\mathbb{Z}$ subgroup is virtually $\mathbb{Z}$; in particular, a non elementary hyperbolic group cannot contain any $\mathbb{Z} \oplus \mathbb{Z}$ nor normal $\mathbb{Z}$ subgroup. Hence condition (i) becomes that $K$ contains a non trivial finite normal subgroup, that is, that $K$ is not icc. \[\square\]

**Corollary 4.** Let $G = K \rtimes Q$ with $K$ a non cyclic torsion free word hyperbolic group. Then $G$ is icc if and only if $\Theta : FC(Q) \rightarrow Out(K)$ is injective.

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