Hitting time and dimension in Axiom A systems and generic interval exchanges

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Abstract

In this note we prove that for equilibrium states of axiom A systems the time $\tau_B(x)$ needed for a typical point $x$ to enter for the first time in a typical ball $B$ with radius $r$ scales as $\tau_B(x) \sim r^d$ where $d$ is the local dimension of the invariant measure at the center of the ball. A similar relation is proved for a full measure set of interval exchanges. Some applications to Birkhoff averages of unbounded (and not $L^1$) functions are shown.1

1 Introduction

It is well known by classical recurrence results that a typical orbit of a dynamical systems comes back (in any reasonable neighborhood) near to its starting point. The quantitative study of recurrence quantifies the speed of this coming back, estimating, for example, how much time is needed to come back in any ball centered in the starting point (the reader can find and exposition of more and less recent developments about this kind of questions in the survey [CG]). It turns out that in many cases the scaling law of return times are related to the dimension of the invariant measure of the system. More precisely, let us consider a starting point $x$, a ball $B(x,r)$ and the time $\tau_{B(x,r)}(x)$ needed for the starting point $x$ to come back to $B(x,r)$. With these notations we have, for example, (S, STV) that in exponentially mixing systems or positive entropy (with some small tecnical assumptions) systems over the interval $\tau_{B(x,r)}(x) \sim r^{d(x)}$, where $d(x)$ is the local dimension at $x$. Moreover (Ba, BS) in general finite measure preserving systems the recurrence gives a lower bound to the dimension

$$\lim_{r \to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} \leq d(x).$$

A similar and strictly related (see e.g. LHV) problem is about the time needed for a typical point $x$ of an ergodic system to enter in some neighborhood

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of another point \( y \). This leads to the hitting time (also called waiting time) indicators. For example, let us denote by \( \tau_{B(y,r)}(x) \) the time needed for the point \( x \) to enter in the ball \( B(y,r) \) with center \( y \) and radius \( r \) (this in some sense generalizes the recurrence because we are allowed to consider an arriving point \( y \) different from the starting point \( x \)). Again we consider the scaling behavior of this hitting time for small \( r \). The waiting time indicator will have value \( R \) if \( \tau_{B(y,r)}(x) \sim r^{R} \) (precise definitions in section 2). Again, there are relations with the local dimension. Some general relations are proved in 

(see thm. \[ \text{[G]} \]) and show that the waiting time indicator gives an upper bound to the dimension. Moreover there is a class of systems where the waiting time is equal to dimension. This class of systems includes for example systems having exponential distribution of return times in small balls (this includes many more or less hyperbolic systems over the interval \( \text{[BSTV]} \)). We remark that exponential return times in balls is conjectured but yet not proved in general Axiom A systems, thus equality between hitting time and dimension for axiom A does not follow from such result.

We want to remark that there are also some relevant cases where the equality between recurrence or hitting time with dimension does not hold, hence this kind of questions are not trivial. Such cases includes rotations by Liouville numbers (see \[ \text{[BS, KS]} \]), and Maps having an indifferent fixed point and infinite invariant measure (\[ \text{[GKP]} \]).

A further motivation for this kind of studies is that the relations between recurrence (and similar) with dimension are used in the physical literature \[ \text{[HJ, GE, JKLPS]} \] to provide numerical methods for the study of the Hausdorff dimension of attractors. Since recurrence gives a lower bound to dimension and hitting time gives an upper bound, the combined use of these may produce efficient numerical estimators for the dimension of attractors.

The main result of this note is to show that in nontrivial nice examples such as Axiom A systems and typical interval exchanges, the hitting time indicator equals the local dimension \( d(y) \) of the considered measure. Hence, in such systems, for typical \( x \) and \( y \) we will have \( \tau_{B(y,r)}(x) \sim r^{d(y)} \). As an application of these results we give an estimation for the Birkhoff averages of functions (in ergodic systems) having some asymptote and no finite \( L^{1} \) norm. Here the Birkhoff average will increase to infinity as the number of iterations increases (this is trivially by ergodic theorem). The hitting time indicator will give an estimation about the speed of going to infinity of such average.

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2 Generalities and a criteria for hitting time and dimension

In the following we will consider a discrete time dynamical system \((X, T)\) were \(X\) is a separable metric space equipped with a Borel finite measure \(\mu\) and \(T : X \to X\) is a measurable map.

Let us consider the first entrance time of the orbit of \(x\) in the ball \(B(y, r)\) with center \(y\) and radius \(r\)
\[
\tau_r(x, y) = \min\{n \in \mathbb{N}, n > 0, T^n(x) \in B(y, r)\}.
\]

By considering the power law behavior of \(\tau_r(x, y)\) as \(r \to 0\) let us define the hitting time indicators as
\[
\overline{R}(x, y) = \limsup_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)}, \quad \underline{R}(x, y) = \liminf_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)}.
\]

If for some \(r\), \(\tau_r(x, y)\) is infinite then \(\overline{R}(x, y)\) and \(\underline{R}(x, y)\) are set to be equal to infinity. The indicators \(\overline{R}(x, y)\) and \(\underline{R}(x, y)\) of quantitative recurrence defined in \([BS]\) are obtained as a special case, \(\overline{R}(x, y) = \overline{R}(x, x), \underline{R}(x, y) = \underline{R}(x, x)\).

We recall some basic properties of \(\overline{R}(x, y)\):

**Proposition 2** \(\overline{R}(x, y)\) satisfies the following properties

- \(\overline{R}(x, y) = \overline{R}(T(x), y), \overline{R}(x, y) = \overline{R}(T(x), y)\).
- If \(T\) is \(\alpha - \text{Hoelder}\), then \(\overline{R}(x, y) \geq \alpha \overline{R}(x, T(y)), \overline{R}(x, y) \geq \alpha \overline{R}(x, T(y))\).
- If we consider \(T^n\) instead of \(T\) \(\overline{R}_T(x, y) \leq \overline{R}_{T^n}(x, y), \underline{R}_T(x, y) \leq \underline{R}_{T^n}(x, y)\).

**Proof.** The first two points comes from \([G]\) (and they comes directly from definitions). For the third one let us denote with \(\tau\) and \(\tau'\) the hitting time with resp. to \(T\) and \(T^n\). By definition \(\overline{R}_T(x, y) = \limsup_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)}\) but \(\tau_r(x, y) \leq n\tau'_r(x, y)\) and \(\frac{\log(\tau_r(x, y))}{-\log(r)} \leq \frac{\log(\tau'_r(x, y))}{-\log(r)}\), and taking the lim sup we are done.

The same can be done for the \(\liminf\). \(\square\)

In general systems the quantitative recurrence indicator gives only a lower bound on the dimension \([BS], [BE]\). The waiting time indicator instead give an upper bound \([C]\) to the local dimension of the measure at the point \(y\). This is summarized in the following

**Theorem 3** If \((X, T, \mu)\) is a dynamical system over a separable metric space, with an invariant measure \(\mu\). For each \(y\)
\[
\overline{R}(x, y) \geq d_\nu(y), \quad \overline{R}(x, y) \geq d_\mu(y)
\]
holds for \( \mu \) almost each \( x \). Where \( \underline{d}_\mu(y) \) and \( \bar{d}_\mu(y) \) are the lower and upper local dimension at \( y \). Moreover, if \( X \) is a closed subset of \( \mathbb{R}^n \), then for almost each \( x \in X \)

\[
\bar{R}(x, x) \leq \underline{d}_\mu(x) , \ R(x, x) \leq \bar{d}_\mu(x).
\]

A natural question which is important also from the numerical applications is whether equality can replace the above inequalities (see the results from \[5, 6, 8, 17\] already outlined in the introduction). The following is a general criteria that assures (together with theorem \[3\] for typical points, equality between waiting time and local dimension.

**Lemma 4** Let \( x \in X \) and

\( SF^n_r(x) = X - B(x, r) \cap T^{-1}(X - B(x, r)) \cap \ldots \cap T^{-n}(X - B(x, r)) \)

be the set of points that after \( n \) steps never enters into \( B(x, r) \). If for each \( \epsilon > 0 \) we have \( \sum \mu(SF^n_r(B(x,2^{-n}))^{1-\epsilon}) < \infty \) then for almost each \( y \) it holds \( \bar{R}(y, x) \leq \underline{d}_\mu(x) \) and \( \bar{R}(y, x) \leq \bar{d}_\mu(x) \).

**Proof.** The proof follows by a Borel Cantelli argument. Let \( R_\epsilon = \{ y \in X, \bar{R}(y, x) \geq (1 + \epsilon)\bar{d}_\mu(x) \} \). If we prove that this set has measure zero for each \( \epsilon \) we are done. If we know that for some \( \epsilon \sum \mu(SF^n_r(B(x,2^{-n}))^{1-\epsilon}) < \infty \) this means that the set of points such that \( \tau_{2^{-n}}(y, x) = \mu(B(x,2^{-n}))^{1-\epsilon} \) for infinitely many \( n \) has zero measure. Taking logarithms and dividing by \( n \) we have \( \log(\tau_{2^{-n}}(x,y)) \leq (1 + \epsilon) \frac{\log(\mu(B(x,2^{-n})))}{n} \) eventually (as \( n \) increases) for a full measure set and then \( \bar{R}(y, x) = \limsup \frac{\log(\tau_{2^{-n}}(x,y))}{n} \leq (1 + \epsilon) \limsup \frac{\log(\mu(B(x,2^{-n})))}{n} = (1 + \epsilon)\underline{d}_\mu(x) \) on a full measure set. This is true for each \( \epsilon \) and we have the statement. The same can be done for the proof of \( \bar{R}(y, x) \leq \bar{d}_\mu(x) \).

### 3 Axiom A systems

In this section we will consider Axiom A systems, we will apply the properties of Gibbs measures to prove that they satisfy Lemma \[4\] at almost all points. We will prove the following

**Theorem 5** If \( X \) is a basic set of an axion A diffeomorphism, \( \mu \) is an equilibrium measure for an \( \text{Hoelder} \) potential defined on \( X \). Then \( (X, T, \mu) \) satisfies Lemma \[4\] at almost each \( x \in X \) and hence for almost each \( x \) it holds \( \bar{R}(y, x) = \underline{d}_\mu(x) , \bar{R}(y, x) = \bar{d}_\mu(x) \) for almost each \( y \).

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If \( X \) is a metric space and \( \mu \) is a measure on \( X \) the upper local dimension at \( x \) is defined as \( \underline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \limsup_{k \to 0} \frac{-\log(\mu(B(x,2^{-k})))}{k} \). The lower local dimension \( \bar{d}_\mu(x) \) is defined in an analogous way by replacing \( \limsup \) with \( \liminf \). If \( \underline{d}_\mu(x) = \bar{d}_\mu(x) = d \) almost everywhere the system is called exact dimensional. In this case many notions of dimension of a measure will coincide (see for example the book \[11\]).
First we need a general estimation on the behavior of a certain kind of sequences.

**Lemma 6** Let $0 < m < 1$, and $a_n$ be defined by \[ a_n = a_{n-1}m + s_n \quad \text{where} \quad a_0 = m^2 \]
\[ s_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \]
then for $n \geq 2$ it holds $a_n \leq \frac{m^2}{1-m} + \frac{4}{n^2}$.\]

**Proof.** We have
\[ a_n = m^{n+1} + m^{n-1}s_1 + m^{n-2}s_2 + m^{n-3}s_3 + \ldots + ms_{n-1} + s_n. \]
Since $s_i < 1$ and $m < 1$ then $a_n \leq \sum_{i=1}^{n} m^i + \sum_{i=1}^{n} s_i = m^{n+1} + \frac{1}{1-m} - \frac{1}{(n+1)^2}.$

**Proof of Theorem 5.** We already know that (thm. 3) $R(x,y) \geq d_\mu(y), R(x,y) \geq \overline{d}_\mu(y).$ For the opposite inequalities, first we remark that (see [Bow] pag. 72) $X = X_1 \cup \ldots \cup X_t$ where $T(X_i) = X_{i+1}$, $(T(X_i) = X_1)$ and $T^t|_{X_i}$ is topologically mixing.

By Lemma 2 we can suppose that $x, y$ belongs to the same $X_i$, and that $T$ is topologically mixing (replacing $T$ by $T^t$ we have a mixing transformation on $X_i$, moreover by the fourth point of Lemma 2 we see that if we have an upper bound for $R^t(x,y)$ and $\overline{R}^t(x,y)$, then this is also an upper bound for $R_T(x,y)$ and $\overline{R}_T(x,y)$. Since in this proof we are looking for an upper bound, by replacing $T$ with $T^t$ we can suppose that the map is topologically mixing).

To estimate the measure of the set $SF^n_T(x)$ let us consider a Markov partition $Z = \{Z_i\}$ of $X$. Let $Z^n_m = T^{-m}(Z) \cap \ldots \cap T^{-n}(Z)$. By uniform hyperbolicity there are constants $C, \lambda > 0$ such that $\text{diam}(Z^n_m) \leq Ce^{-\lambda n}$. By this we know that there is some
\[ n(r) \leq -\lambda^{-1}(\log r - \log C) \]
such that the partition is of size so small that there is one element $Z_0$ of the partition $Z = Z^n_m(r)$ which is included in $B(x, r)$.

Now $SF^n_T(x) \subset B^n_0 = X - Z_0 \cap T^{-1}(X - Z_0) \cap \ldots \cap T^{-n}(X - Z_0)$. We remark that $B^n_0$ is the union of many cylinders. The measure of $B^n_0$ decreases very fast by the weak Bernoulli property of the equilibrium measure $\mu$. Indeed by [Bow] pag. 90, we know that since the map $T$ can be supposed to be topologically mixing and then $\mu$ has the weak Bernoulli property: i.e. let us consider $t, s \geq 0$ and the partitions $P_s = Z \vee T^{-1}(Z) \vee \ldots \vee T^{-s}(Z)$ and $Q_t = T^{-t}(Z) \vee \ldots \vee T^{-t-k}(Z)$. For each $\epsilon$, if $t - s = N_2(\epsilon)$ is big enough, then $\sum_{P \subseteq Q, P \in Q_t} \mu(Q) = \mu(P)\mu(Q) < \epsilon.$ Moreover by [Bow], theorem 1.25 we can find an estimation for $N_2(\epsilon)$ as a function of $\epsilon$ (see [Bow] pag. 38): $N_2(\epsilon) = -c \log(\epsilon) + c'$, where $c, c'$ are constants depending on $\mu, T$ and $Z$.

The estimation for $\mu(B^n_0)$ follows from the fact that a non empty cylinder for the partition $Z = Z^n_m(r)$ is also a cylinder for the partition $Z$.

Indeed the cylinder $Z_m = Z_{m+1}(Z_{m+2}) \cap \ldots \cap T^{-m-1}(Z_{m+2})$, $Z_i \in Z$ satisfies $Z_m = Z_{m+2n}$ where $z_{m+2n} = T^n(Z_{j_1}) \cap T^{n-1}(Z_{j_2}) \cap \ldots \cap T^{n-1}(Z_{j_{m+2n}})$ is
a cylinder of $Z$ and $Z_{ik} = T^{-k+n}Z_{jk} \cap \ldots \cap T^{-k-n}Z_{jk+2n}$. Since $\mu$ is preserved then $\mu(T^{-k+n}Z_{jk} \cap \ldots \cap T^{-k-n}Z_{jk+2n}) = \mu(T^{-k}Z_{jk} \cap \ldots \cap T^{-k-2n}Z_{jk+2n})$ hence we can apply the weak Bernoulli property to such cylinders obtaining that also $\overline{Z}$ satisfies such a property and

$$N_{\overline{Z}}(\epsilon) \leq c \log(\epsilon) + c' + 2n$$

(2)

(where $c, c'$ are the constants of $N_Z(\epsilon)$ as above). We recall that $n$ depends on $r$ and we can choose $n(r) \leq -\lambda^{-1}(\log r - \log C)$.

Finally let us apply the weak Bernoulli property of $Z$ to get an estimation for $\mu(B_0^n)$. Let us set $m = \mu(X - Z_0)$ and $\epsilon(i) = \frac{2i+1}{i(i+1)^2} = \frac{1}{i^2} - \frac{1}{(i+1)^2}$. We have (eq. 2) that setting $C'(r) = c' - 2\lambda^{-1}(\log r - \log C)$ then $N_Z(\epsilon(i)) \leq c \log(\epsilon(i)) + c' + 2n = c \log\left(\frac{2i+1}{i(i+1)^2}\right) + C'(r)$ and there is a $C$ s.t. $N_{\overline{Z}}(\epsilon) \leq C \log(i) + C'(r)$.

Let us set $n_i = \sum_{j \leq i} N_{\overline{Z}}(\epsilon(j))$. Thus for each $\delta$ there is a $K$ such that, if $i$ is big enough

$$n_i \leq -K i^{1+\delta} \log r.$$  (3)

The measure of $B_0^n$ can then be estimated applying $i$ times the Bernoulli property, with $\epsilon(i) = \frac{2i+1}{i(i+1)^2}$ as above, to subcylinders of increasing length $N_{\overline{Z}}(\epsilon(1)), N_{\overline{Z}}(\epsilon(1)) + N_{\overline{Z}}(\epsilon(2)), N_{\overline{Z}}(\epsilon(1)) + N_{\overline{Z}}(\epsilon(2)) + N_{\overline{Z}}(\epsilon(3))$... obtaining by the Bernoulli property of $\mu$

$$\mu(B_0^{N(\epsilon(1))}) \leq m^2 + \epsilon(1),$$
$$\mu(B_0^{N(\epsilon(1)) + N(\epsilon(2))}) \leq (m^2 + \epsilon(1))m + \epsilon(2),$$
$$\mu(B_0^{N(\epsilon(1)) + N(\epsilon(2)) + N(\epsilon(3))}) \leq ((m^2 + \epsilon(1))m + \epsilon(2))m + \epsilon(3), ...$$

Hence by Lemma 9 above

$$\mu(B_0^n) \leq \frac{m^{\frac{|Z|}}{1 - m} + \frac{4}{i^2}}.$$ 

We remarked that $SF_r^n \subset B_0^n$. If we consider another element $Z_1$ of $\overline{Z}$ with $Z_1 \subset B(x, r)$ and $B_0^n = X \cap T^{-1}(X - (Z_0 \cup Z_1)) \cap \ldots \cap T^{-n}(X - (Z_0 \cup Z_1))$, we have also $SF_r^n \subset B_1^n \subset B_0^n$. Now considering a sequence $Z_0, ..., Z_w$ of elements of $\overline{Z}$ with $Z_0, ..., Z_w \subset B(x, r)$ and $B^n_w = X \cap T^{-1}(X - (Z_0 \cup \ldots \cup Z_w)) \cap \ldots \cap T^{-n}(X - (Z_0 \cup \ldots \cup Z_w))$, we have also $SF_r^n \subset B^n_w$. The measure of $B^n_w$ can be estimated as above, obtaining $\mu(B^n_w) \leq \frac{m^{\frac{|Z_n|}}{1 - m}}{1 - \frac{4}{i^2}}$, where $m_w = \mu(X - (Z_0 \cup \ldots \cup Z_w))$.

Now, refining again the partition $\overline{Z}$ if necessary (this is true because the diameter of each $Z_i$ is less or equal than $Ce^{-\lambda n}$), and this will only change the constants in $N(\epsilon)$ we can suppose that the diameter of each piece of the partition has diameter less than $r/4$. We then have that we can choose $Z_0, ..., Z_w$ such that $B(x, \frac{r}{4}) \subset Z_0 \cup \ldots \cup Z_w \subset B(x, r)$. Then $\mu(X - (Z_0 \cup \ldots \cup Z_w)) \leq \mu(X - B(x, \frac{r}{4}))$. This gives,

$$\mu(B^n_w) \leq \frac{(1 - \mu(B(x, \frac{r}{4})) \frac{r}{4}}{\mu(B(x, \frac{r}{4}))} + \frac{4}{i^2}. $$
By Proposition 2 we then have $i \geq \frac{n \delta^{n+1}}{(K \log \tau/4)^{\tau/2}}$, by this

$$\mu(SF_{2-n}^{(\mu(B(x, 2^{-n}))^{-1}} \leq \mu(B_{w}(B(x, 2^{-n-1}))) \leq \frac{(1 - \mu(B(x, 2^{-n-1})))((Kn + \log 4)^{\frac{1}{n+1}} \mu(B(x, 2^{-n-1})) \frac{1+\epsilon}{4} + \frac{(Kn + \log 4)^{\frac{1}{n+1}} \mu(B(x, 2^{-n-1}))}{\mu(B(x, 2^{-n-1}))} + \frac{1+\epsilon}{4})}{4}.$$ \hspace{1cm}

Since in our case $d_{u}(x) = d < \infty$ a.e., there is a constant $Q$, (which depends on the local dimension) such that $0 < Q < \frac{\mu(B(x, 2^{-n-1}))}{\mu(B(x, 2^{-n-1}))} < 1$ when $n$ is big, recalling that $\delta$ can be chosen as small as we want and hence smaller than $\epsilon$, then $\mu(SF_{2-n}^{(\mu(B(x, 2^{-n}))^{-1}}$ is less than about $\frac{(1 - \mu(B(x, 2^{-n-1})))((Kn + \log 4)^{\frac{1}{n+1}} \mu(B(x, 2^{-n-1})) \frac{1+\epsilon}{4} + \frac{(Kn + \log 4)^{\frac{1}{n+1}} \mu(B(x, 2^{-n-1}))}{\mu(B(x, 2^{-n-1}))}}{4} + 4(Kn + \log 4)^{\frac{1}{n+1}} \mu(B(x, 2^{-n-1})) \frac{1+\epsilon}{4} \mu(B(x, 2^{-n-1}))$ and we have $\sum_{n} \mu(SF_{2-n}^{(\mu(B(x, 2^{-n}))^{-1}} < \infty$. This is enough to apply the Lemma 4 and have the required statement. \[\square\]

4 Interval exchanges

An interval exchange is a piecewise isometry which preserves the Lesbegue measure. In this section we apply a result of Boshernitzan about a full measure class of uniquely ergodic interval exchanges maps to prove equality between hitting time and dimension at discontinuity points. We refer to [Bo2] for generalities on this important class of maps.

Theorem 7 For a typical interval exchange transformation $T$ (for a full measure set, in the space of interval exchanges) for each discontinuity point $x_{0}$ it holds $R(y, x_{0}) = 1$ for almost each $y \in [0, 1]$.

Proof. By a result of [Bo2] we have that if $T$ is a typical i.e.t. and $\delta(n)$ is the minimum distance between the discontinuity points of $T^{-n}$, then there is a constant $C$ and a sequence $n_{k}$ such that $\delta(n_{k}) \geq \frac{C}{n_{k}}$. If $x_{0}$ is a discontinuity point then for each $n_{k}$ it also hold that $\min_{i,j \leq n_{k}} d(T^{-i}(x_{0}), T^{-j}(x_{0})) \geq \frac{C}{n_{k}}$. Let us consider the set $J_{k} = \cup_{i \leq n_{k}} B(T^{-i}(x_{0}), \frac{C}{3n_{k}})$. Since it is a union of disjoint balls the measure of $J_{k}$ is $\frac{C}{3}$. This implies that in the interval $[0, 1]$ there is a positive measure set $J$ of points belonging to infinitely many $J_{k}$.

If a point $y$ is in $J$ then for a subsequence $n_{k}$, it holds $d(T^{n_{k}}(y), x_{0}) \leq \frac{C}{3n_{k}}$. This is true because by the Boshernitzan result there are no counter images of other discontinuity points in the interval $[y, T^{-n}(x_{0})]$ and then these two points cannot be separated during $n$ iterations of the map.

Since $\liminf_{n} d(T^{n}(y), x_{0}) \leq \frac{C}{3}$ then $R(y, x_{0}) \leq 1$ and we have the statement for $y$ varying in a positive measure set $J$. Since the system is ergodic, by Proposition 2 we have the statement. \[\square\]
Since in interval exchanges th only source of initial condition sensitivity is the discontinuity (the orbits of two points can be only separated by a discontinuity) we remark that an estimation of the approaching speed of typical orbits to the discontinuity is useful to estimate the kind of "weak" chaos that is present is such maps. The theorem above in some sense can give (using the construction done in [BGI]) an upper bound on the initial condition sensitivity of such maps. We will not go into details about this in this work however.

5 Hitting time and Birkhoff sums

Let us consider a function \( f : X - \{ x_0 \} \rightarrow \mathbb{R} \), \( f \geq 0 \) which is continuous and which satisfies \( \int_X f d\mu = \infty \) because it has an asymptote in \( x_0 \) where \( f(x) \sim d(x, x_0)^{-\alpha} \).

By the ergodic theorem we know that for almost each \( x \) the Birkhoff average \( S_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \) is such that \( S_n(x) \to \infty \). By the results of the previous sections, if we know the hitting time indicator at \( x_0 \) we can have an estimation for the speed of increasing of \( S_n(x) \).

**Theorem 8** Let us suppose that near \( x_0 \) we have \( 0 < \lim \frac{f(x)}{d(x, x_0)} < \infty \), for \( \alpha > 1 \) then for almost each \( x \)

\[
\frac{\alpha}{R(y, x_0)} \leq \limsup_{n \to \infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{R(y, x_0)} + 1
\]

**Proof.** By the definition of \( R(y, x_0) \) we obtain (see [3]) that for each \( \epsilon > 0 \)

\[
\liminf_{n \to \infty} \frac{1}{n^{\alpha - \epsilon}} d(T^n(y), x_0) = \infty.
\]

Then we have that if \( n \) is big enough \( d(T^n(y), x_0) \geq n^{-\frac{1}{\alpha}} \). Now we remark that since \( X \) is compact there are \( c_1, c_2 \) such that \( f(x) \leq \max(c_1, c_2 d(x, x_0)^{-\alpha}) \). Then if \( n \) is big enough

\[
\sum_{i=0}^{n} f(T^i(y)) \leq \sum_{i=0}^{n} \max(c_1, c_2 d(T^i(y) - x_0)^{-\alpha}) \leq \sum_{i=0}^{n} \max(c_1, c_2 n^{-\frac{1}{\alpha}} + \alpha \epsilon \leq nc_1 + c_2 n^{-\frac{1}{\alpha}} + \alpha \epsilon + 1
\]

and we have \( \limsup_{n \to \infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{R(y, x_0)} + 1 \). On the other hand, by the definition of \( R(y, x_0) \) we have that frequently \( d(T^n(y), x_0) \leq n^{\frac{1}{\alpha}} \), then frequently

\[
\sum_{i=0}^{n} f(T^i(y)) \geq cn^{\frac{\alpha}{R(y, x_0)} - \alpha} \epsilon.
\]

By the above result and the previous ones it easily follows:
1. (by thm. 3) In a general system, if the local dimension at \( x_0 \) is \( d_\mu(x_0) \). Then for almost each \( x \)

\[
\limsup_{n \to \infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{d_\mu(x_0)} + 1
\]

2. (by thm. 4) If \( T \) is an IET and \( x_0 \) is a discontinuity point then

\[
\alpha \leq \limsup_{n \to \infty} \frac{\log(S_n(x))}{\log(n)} \leq \alpha + 1
\]

3. (by thm. 5) If \( (X, T) \) is axiom A (with an equilibrium measure, as in thm. 5), \( x_0, x \) are typical and \( d \) is the dimension of the measure then

\[
\frac{\alpha}{d} \leq \limsup_{n \to \infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{d} + 1.
\]

References

[BS] Barreira L, Saussol B, Hausdorff dimension of measures via Poincaré recurrence, Commun. Math. Phys., 219 (2001), 443-463.

[BGI] Bonanno C, Isola S, Galatolo S Recurrence and algorithmic information Nonlinearity 17 (2004), no. 3, 1057–1074.

[Bow] Bowen R. Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics 470, Springer-Verlag, 1975.

[Bo] Boshernitzan M D, Quantitative recurrence results, Invent. Math. 113 (1993), 617-631

[Bo2] Boshernitzan M D, A condition for minimal interval exchange maps to be uniquely ergodic. Duke Math. J. 52 (1985), no. 3, 723–752

[BSTV] Bruin H., Saussol B., Troubetzkoy S., Vaienti, S. Return time statistics via inducing, Ergodic Theory Dynam. Systems 23 (2003), no. 4, 991–1013.

[ChG] Chazottes J. R., Galatolo S. Using Hitting & Return times to study strange attractors (work in preparation).

[CG] Carletti T., Galatolo S., Numerical Estimates of dimension by recurrence and waiting time (work in preparation).

[KS] Kim D.H., Seo B.K., The waiting time for irrational rotations, Nonlinearity 16 (2003), no. 5, 1861–1868.

[G] Galatolo, S. Dimension via waiting time and recurrence, Math. Res. Lett. 12, no 3, May 2005, 377-386
Galatolo S., Kim D.H., Koh Park K. *work in preparation*

Gratrix S., Elgin J.N., *Pointwise Dimensions of the Lorenz Attractor*, Phys. Rev. Lett. 92, 014101 (2004).

Halsey T.C., Jensen M.H., *Hurricanes and butterflies*, Nature 428, 11 March 2004, 127–128.

Jensen M.H., Kadanoff L.P., Libchaber A., Procaccia I., Stavans J., *Global universality at the onset of chaos: results of a forced Rayleigh-Bénard experiment*, Phys. Rev. Lett. 55, no. 25, 2798–2801.

Lacroix Y., Haydn N., Vaienti S., *Hitting and return times in ergodic dynamical systems*, to appear in Ann. Probab.

Pesin Y *Dimension theory in dynamical systems* Chicago lectures in Mathematics (1997).

Saussol B., Troubetzkoy S., Vaienti S., *Recurrence, dimensions and Lyapunov exponents*, J. Stat. Phys. 106 (2002), 623-634.

Saussol B., *Recurrence rate in rapidly mixing dynamical systems*, preprint.