Berry–Esseen Theorem for Sample Quantiles with Locally Dependent Data

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In this note, we derive a Gaussian Central Limit Theorem for the sample quantiles based on identically distributed but possibly dependent random variables with explicit convergence rate. Our approach is based on converting the problem to a sum of indicator random variables, applying Stein’s method for local dependence, and bounding the distance between two normal distributions. We also generalize this approach to the joint convergence of sample quantiles with an explicit convergence rate.

MSC2020 subject classifications: Primary: 60F05
Keywords: Sample median; Central Limit Theorem; Rate of convergence; Stein’s method; Multivariate normal approximation

1. Introduction

Central Limit Theorem (CLT) is one of the fundamental theorems in probability theory and statistics. In classical form, it states that the sum of i.i.d. finite variance random variables, appropriately centered and scaled, converges in distribution to the standard normal distribution. Since then, CLT has been strengthened and extended to various settings. The study of the asymptotic distribution of the sample median was first developed for the case of a continuous random variable by Sheppard in 1890 (according to [14]). Asymptotic properties of sample quantiles of independent random variables have been extensively studied for both continuous distributions (see e.g., [6] and references therein) and discrete ones (see e.g., [16]). Typically CLT for the median is derived either by converting the problem to a sum of indicators [19, Example 6.1] or by the Delta method [23, Example 20.1]. In this work, we build on the former method and extend it to the random variables with local dependencies. We derive an explicit convergence rate using the dependency graph approach for Stein’s method.

Stein’s method bounds the distance between the random variable of interest \(W\) and the standard normal variable \(Z\) in the following way

\[
d(W, Z) \leq \sup_{f \in D} \left| E(f'(W) - W f(W)) \right| \tag{1.1}
\]

for an adequately chosen class of functions \(D\) depending on the metric \(d(\cdot, \cdot)\). When \(W = \sum_{i=1}^{n} X_i\), it turns out that it is often easier to work with the right-hand side of (1.1), even if \(X_i\) have dependencies between each other. Depending on the structure of such dependencies, one would bound \(E(f'(W) - W f(W))\) in different ways. Hence Stein’s method can be used with variety of approaches such as exchangeable pairs [22], dependency graph or local dependencies [1, 5, 9, 10, 20], size-bias [13] and zero-bias couplings [11, 12], Stein coupling [4], and through Malliavin calculus [18] among others. In Section 1.3 we briefly discuss the basics of Stein’s method for local dependence (also known as the dependency graph approach) and refer to [3, 8, 21] for further reading on the topic.
One particular advantage of Stein’s method is that $E(f'(W) - W f(W))$ naturally preserves the additive structure of random variables. Hence, when $W = \sum_{i=1}^{n} X_i$, Stein’s method allows working with local interactions among $X_i$’s to derive global convergence. On the other hand, when one works with non-additive functions, it is unclear if Stein’s method is applicable. This work presents a way to apply Stein’s method to a non-additive function, namely sample median and, more generally, sample quantiles.

1.1. Setup

Let $X_1, X_2, \ldots, X_n$ be real valued random variables (not necessarily independent) with CDF’s $\{F_i(x)\}_{i \in [n]}$, respectively. Given $\alpha \in (0, 1)$, we define the $\alpha$th sample quantile as

$$Q_n^{(\alpha)} := \max \{x \in \mathbb{R} \mid |\{i \mid X_i \leq x\}| \leq \lfloor n\alpha \rfloor\}. \quad (1.2)$$

In particular, the sample median is defined as

$$M_n = Q_n^{(1/2)} := \max \{x \in \mathbb{R} \mid |\{i \mid X_i \leq x\}| \leq \lfloor n/2 \rfloor\}. \quad (1.3)$$

Fix an integer $\ell \geq 1$ and two increasing sequences of real numbers $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$ and $m_1 < m_2 < \cdots < m_\ell$. We assume the following.

**Assumption I.**

1. For all $k \in [\ell], i \in [n]$, the equations $F_i(x) = \alpha_k$ has a unique solution at $x = m_k$.
2. For all $k \in [\ell], i \in [n]$, the CDF $F_i$ is continuously twice differentiable at $m_k$.
3. There exists $\varepsilon, A > 0$ such that for all $|x| \leq \varepsilon$ we have

$$|F''_i(m_k + x)|, |F'_i(m_k + x)| \leq A \quad \text{for all } k \in [\ell], i \in [n]. \quad (1.4)$$

We define

$$\mu := (m_1, m_2, \ldots, m_\ell),$$

$$\theta_{k,n} := \frac{1}{n} \sum_{j=1}^{n} F'_j(m_k) \quad \text{and}$$

$$\Theta_n := \text{diag}(\theta_{1,n}, \theta_{2,n}, \ldots, \theta_{\ell,n}). \quad (1.5)$$

When $\alpha = 1/2$, for simplicity, we may assume that $m_{\alpha} = 0$, and thus

$$\theta := \frac{1}{n} \sum_{j=1}^{n} F'_j(0). \quad (1.6)$$

In this article, we consider the case when $(X_i)_{i \in [n]}$ is a locally dependent sequence of random variables. In particular, we will base our argument on the dependency graph approach.

**Assumption II.** The sequence of random variables $(X_1, X_2, \ldots, X_n)$ has dependency graph $G$ on the vertex set $[n]$, i.e.,

1. $\forall i \in [n], \exists N_i \subseteq [n]$ such that $X_i$ is independent of $(X_t)_{t \notin N_i}.$
2. \( \forall i \in [n] \) and \( j \in N_i \), \( \exists N_{ij} \subseteq [n] \) such that \((X_i, X_j)\) are independent of \((X_j)_{j \in N_i}\).

3. \( \forall i \in [n] \), \( j \in N_i \), and \( k \in N_{ij} \), \( \exists N_{ijk} \subseteq [n] \) such that \((X_i, X_j, X_k)\) are independent of \((X_s)_{s \in \mathbb{N}_{ij}^c}\).

We refer to the set \( N_i = \{ j \mid j \sim i \} \cup \{ i \} \) as the dependency neighborhood of \( X_i \) and let

\[
D_1 := \max_i |N_i|, \quad D_2 := \max_{i,j} |N_{ij}|, \quad \text{and} \quad D_3 := \max_{i,j,k} |N_{ijk}|.
\]

These quantities are the main parameters of a dependency graph and may depend on \( n \).

**Remark 1.1.** In literature, sometimes dependency graph is taken to be such that sets of vertices \( A \subseteq [n] \) and \( B \subseteq [n] \) share no edges if \((X_i)_{i \in A}\) and \((X_j)_{j \in B}\) are mutually independent. Notice that it is a stronger condition and implies that \( D_2 \leq 2D_1 \) and \( D_3 \leq 3D_1 \).

Given a vector \( x \in \mathbb{R}^\ell \), we consider the centered random vector

\[
Y_{i,x} := \left( 1_{X_i \leq m_k + n^{-1/2}, x_k} - F_i(m_k + n^{-1/2}, x_k) \right)^\ell_{k=1}.
\]

Notice that the random vectors \( (Y_{i,x})_{i=1}^n \) inherit the dependency graph structure from \((X_i)_{i=1}^n\). Let

\[
\Sigma_x := \Sigma_{n,x} = \frac{1}{n} \text{Var}\left( \sum_{i=1}^n Y_{i,x} \right)
\]

be the variance-covariance matrix of the random vector \( \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_{i,x} \). In the univariate case, we denote it as \( \sigma^2_{n,x} \). Finally, to simplify notation we denote \( \sigma = \sigma_{n,0} \) and \( \Sigma = \Sigma_{n,0} \). We will assume that the matrix \( \Sigma \) is invertible.

**Remark 1.2.** In particular, if \( \alpha = 1/2 \) and \( m_\alpha = 0 \) and in addition to the Assumption I we know that \( \mathbb{P}(X_i \vee X_j \leq 0) = \frac{1}{4} \) for all \( i \neq j \), then \( 1_{X_i \leq 0} \) and \( 1_{X_j \leq 0} \) are uncorrelated for \( i \neq j \) and hence \( \sigma \equiv 1/2 \).

Similarly, when the random variables are independent, we have

\[
\Sigma_0 = (\langle \alpha_{i \wedge j} - \alpha_i \alpha_j \rangle)_{i,j=1}^\ell.
\]

### 1.2. Main results

To present the univariate result, we assume that \( \alpha = 1/2 \) and \( m_\alpha = 0 \), our argument can be easily extended to any other quantile that satisfies Assumption I.

**Theorem 1.3.** Let \( (X_i)_{i \in [n]} \) be random variables that satisfy Assumption I with \( \alpha = 1/2 \), \( m_\alpha = 0 \), and suppose that \( G = ([n], E) \) satisfies Assumption II. Let \( M_n \) be as in (1.3) and \( \theta \) be as in (1.6). Then

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\theta \sqrt{n} \cdot M_n \leq \sigma x) - \mathbb{P}(Z \leq x) \right| \lesssim \frac{A}{\theta^2} \cdot \left( 1 + \frac{D_1}{\sigma^2} \right) \cdot \frac{\log n}{\sqrt{n}} + \frac{D_1(D_2 + D_3)}{\sigma \sqrt{n}},
\]

where \( Z \sim N(0, 1) \).
Remark 1.4. One can replace the bounded second derivative assumption in (1.4) by Hölder continuity assumption for the $F'_i$. But this will give a slower rate of convergence in Theorem 1.3.

Now we present the multivariate version involving the joint distribution of sample quantiles. For two vectors $x, y \in \mathbb{R}^\ell$ we say that $x \preceq y$ if for all $i \in [\ell]$ we have that $x_i \leq y_i$.

Theorem 1.5. Let $(X_i)_{i \in [n]}$ be a sequence of random variables satisfying Assumption I and II. Let $\Theta, \mu$ be as in (1.5) and $Q_n := \left( Q_n^{(\alpha_1)}, Q_n^{(\alpha_2)}, \ldots, Q_n^{(\alpha_k)} \right)$, then

$$\sup_{x \in \mathbb{R}^\ell} \left| \mathbb{P} \left( \sqrt{n} \cdot \Theta Q_n - \mu \preceq x \right) - \mathbb{P} \left( \Sigma^{1/2} Z \preceq x \right) \right| \lesssim \frac{A}{\min_k \theta_k^2} \ell \left\| \Sigma^{-1} \right\|_{\text{op}} (D_1 \vee \sigma^2_{\max}) \frac{\log n}{\sqrt{n}} + \frac{\ell^{1/4}}{\sqrt{n}} \left\| \Sigma^{-1} \right\|_{\text{op}} D_1 \left( D_2 + D_3 \ell^{-1} \right),$$

where $\sigma^2_{\max} := \max_{i \in [\ell]} \Sigma_{ii}$ and $Z \sim \mathcal{N}(0, I_\ell)$.

While we present Theorem 1.3 under the assumption of local dependency, our approach can be applied to other dependency structures. The first step of our argument is applying the classical linearization technique that rewrites $\mathbb{P}(M_n \leq x)$ as the probability that the sum of Bernoulli random variables is less than some value. This step can be done under any dependence of $X_i$'s. However, the mean of these Bernoulli random variables $Y_i$ depends on $x$. While it is not an issue in the derivation of regular CLT, it poses significant complications in bounding the convergence rate when the value of $x$ is “far” from the true value of the median. Hence we consider two cases when $|x| \leq K_n$ and $|x| > K_n$ for some threshold function $K_n$. The next step is applying the appropriate approach of Stein’s method to treat the former case and establishing a concentration bound for Bernoulli random variables to treat the latter. In the end, we optimize over $K_n$ to derive the result. In conclusion, our argument can be adopted whenever the corresponding Bernoulli random variables $Y_i$ Stein’s method can be used to derive CLT, and one can derive a needed concentration inequality.

1.3. Local dependence

In this section, we state the results related to the dependency graph approach that will be used to prove the main results.

First, as mentioned above, we will use a CLT for a sum of indicator random variables with local dependencies. Various theorems cover this case in the univariate case (see [1, 5, 20] among others) and give the same order. We chose to present the multivariate version of the result.

Theorem 1.6 ([9, Theorem 2.1]). Suppose $X_1, X_2, \ldots$ are $d$-dimensional random vectors with $\mathbb{E} X_i = 0$, and variance-covariance matrix is given by $\Sigma$. Suppose $G$ is a dependency graph for $(X_i)_{i \in \mathbb{N}}$, and for all $i \in [n], j \in N_i, k \in N_{ij}$ we have that

$$|X_i| \leq \beta, \quad |N_i| \leq D_1, \quad |N_{ij}| \leq D_2, \quad \text{and} \quad |N_{ij}| \leq D_3,$$

then letting $W = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ we have that

$$\sup_{A \in \mathbb{C}} \left| \mathbb{P} (W \in A) - \mathbb{P} \left( \Sigma^{1/2} Z \in A \right) \right| \lesssim n \beta^{1/2} \cdot \left\| \Sigma^{-1/2} \right\|_{\text{op}} \cdot D_1 \left( D_2 + D_3 \ell \right),$$
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where $C$ is the collection of all convex sets in $\mathbb{R}^d$ and $\|\cdot\|_{\text{op}}$ denotes the operator norm.

Another useful result in this setting is a Hoeffding inequality for the dependency graph. We state a version in terms of the size of the largest dependency neighborhood $D_1$, which is a consequence of a stronger result [15, Theorem 2.1].

**Theorem 1.7** ([15]). Let $X_1, X_2, \ldots, X_n$ be a sequence of mean zero random variables taking values in $[a_i, b_i]$. Suppose $G$ is the dependency graph for $(X_i)_{i \in \mathbb{N}}$ as given in Assumption II. Then for any $t > 0$, we have

$$
P\left(\sum_{i=1}^{n} X_i \geq t \sqrt{n}\right) \leq \exp\left(-\frac{2nt^2}{D_1 \sum_{i=1}^{n} |b_i - a_i|^2}\right).
$$

1.4. Notations

Throughout this paper, we will use the following notations

- $Z$ always denotes a standard normal random variable, $\Phi, \phi$ denote the distribution and density function of $Z$, respectively.
- $Z = (Z_1, Z_2, \ldots, Z_\ell)$ denotes an $\ell$-dimensional standard Gaussian vector.
- $L(W)$ - the law of random variable $W$.
- $X := X - \mathbb{E}X$ denotes a centered version of a random variable $X$.
- $W_i$ represents its $i^{th}$ coordinate of a vector $W$.
- $d_{\text{KS}}(W, Z) := \sup_{x \in \mathbb{R}}|\mathbb{P}(W \leq x) - \mathbb{P}(Z \leq x)|$ - Kolmogorov-Smirnoff distance,
- $d_W(W, Z) := d_W(L(W), L(Z)) := \sup_{h:1-Lip.} |\mathbb{E}h(W) - \mathbb{E}h(Z)|$ - Wasserstein distance,
- $d_{\text{TV}}(W, Z) := d_{\text{TV}}(L(W), L(Z)) := \sup_{\text{Borel set } A} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)|$ - Total variation distance.
- $f \precsim g$ if $f = O(g)$, $f \preceq g$ if $f = o(g)$ and $f \preceq g$ if $c'g \preceq f \preceq cg$ for some universal constants $c, c' > 0$.
- $x \lor y := \max\{x, y\}$ and $x \land y := \min\{x, y\}$.

Recall that for a matrix $V = (v_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ the Frobenius norm of $V$ is defined as

$$
\|V\|_F := \sqrt{\sum_{i,j} |v_{ij}|^2}
$$

and the operator norm of $V$ is defined as

$$
\|V\|_{\text{op}} := \sup\{\|Vu\|_2 / \|u\|_2, u \neq 0\}.
$$

2. Application to the moving-average model

In this section, we present an application of Theorem 1.3 to the moving-average model. We refer to [2, Chapter 3] for an overview of moving-average models and their significance in time series analysis.

Suppose $\{\zeta_i\}_{i \geq 1}$ is a sequence of independent random variables that satisfies Assumption I with $\alpha = \frac{1}{2}$ and $m_\alpha = 0$. One can take $(\zeta_i)_{i \in \mathbb{N}}$ to be i.i.d. standard normal random variables. However, for our purposes, it is not needed, and moreover, $\zeta$ need not have a finite mean.
One can estimate the parameter $\mu$ for the following linear model using sample median based on the sample $(X_t)_{t \in [n]}$:

$$X_t = \mu + \sum_{i=1}^{q} c_i \zeta_{t-i} + \zeta_t$$

for all $t \in [n]$. This is also known as MA($q$) model with parameters $\mu, c_1, c_2, \ldots, c_q$.

Notice that $X_i$ is independent of $X_j$ whenever $|i-j| \geq q + 1$. Hence the parameter $D_1$ of the dependency graph is bounded by $q + 2$, $D_2 \leq 2D_1$, and $D_3 \leq 3D_1$. Therefore, Theorem 1.3 implies that for appropriate constants $\theta$ and $\sigma^2 = \frac{1}{4}$, the sample median obeys the central limit theorem with the following upper bound on the rate of convergence,

$$\sup_{x \in \mathbb{R}} \left| \left( \mathbb{P}(\theta \sqrt{n} \cdot M_n \leq x/2) - \Phi(x) \right) \right| \leq \frac{q^2 + q \log n}{\sqrt{n}},$$

where $Z \sim \mathcal{N}(0, 1)$.

3. Preliminary Results

3.1. Bound on the distance between two Gaussian vectors with the same mean

One of the terms in the upper bounds in each of our main theorems is of the form

$$\sup_{x \in \mathbb{R}} \left| \left( \mathbb{P}(\theta \sqrt{n} \cdot M_n \leq x/2) - \Phi(x) \right) \right| \leq \frac{q^2 + q \log n}{\sqrt{n}},$$

where $Z \sim \mathcal{N}(0, 1)$.

In the one-dimensional case, one can easily upper bound the Kolmogorov-Smirnov distance by the square root of the $L^1$-Wasserstein distance as follows. With $\rho \geq 1$, we have

$$d_{\text{KS}}(\mathcal{N}(0, \rho^2), \mathcal{N}(0, 1)) \leq \sqrt{4/\sqrt{2\pi}} \cdot d_{\text{TV}}(\mathcal{N}(0, \rho^2), \mathcal{N}(0, 1)) \leq \sqrt{4/\pi} \cdot |\rho^2 - 1|^{1/4}.$$

However, one can derive a better bound by applying Stein’s estimates [17, Proposition 3.6.1] or via direct computation as follows.

**Lemma 3.1.** For $\rho \geq 1$, we have

$$d_{\text{TV}}(\mathcal{N}(0, \rho^2), \mathcal{N}(0, 1)) \leq \frac{2}{\sqrt{\pi}} |\rho - 1| \sqrt{\log \rho \rho^2 - 1} \cdot \exp \left( -\frac{\log \rho}{\rho^2 - 1} \right) \leq \sqrt{\frac{2}{\pi e}} \cdot |\rho - 1|.$$

**Proof of Lemma 3.1.** Letting $\varphi$ be the density function of standard normal distribution, we get

$$d_{\text{TV}}(\mathcal{N}(0, \rho^2), \mathcal{N}(0, 1)) = \int_{0}^{\infty} \left| \rho^{-1} \varphi(x/\rho) - \varphi(x) \right| dx = 2 \int_{x^*}^{\infty} (\rho^{-1} \varphi(x/\rho) - \varphi(x)) dx,$$
where $x^* > 0$ satisfies the following equality $\rho^{-1} \exp(-x^2/2\rho^2) = \exp(-x^2/2)$, i.e.,

$$x^* = \sqrt{2(\log \rho)/(1 - \rho^{-2})}. \quad (3.2)$$

Letting $Z \sim N(0, 1)$, we get that

$$d_{TV}(N(0, \rho^2), N(0, 1)) = 2P \left( \frac{x^*}{\rho} \leq Z \leq x^* \right) \leq 2x^* \left( 1 - \rho^{-1} \right) \varphi(x^*/\rho) = \frac{2}{\sqrt{\pi}}(\rho - 1) \sqrt{\log \rho \rho^2 - 1} \cdot \exp \left( - \frac{\log \rho}{\rho^2 - 1} \right).$$

It is easy to check that for all $x \geq 0$ we have $\sqrt{x}e^{-x} \leq 1/\sqrt{2\pi}$. Simplifying, we get the result. ■

**Remark 3.2.** For general $\sigma_1 > \sigma_2$, letting $\rho = \sigma_1/\sigma_2 > 1$, by the scaling property of normal distribution, we have

$$d_{TV}(N(0, \sigma_1^2), N(0, \sigma_2^2)) = d_{TV}(N(0, \rho^2), N(0, 1)).$$

In the multivariate case, bounding (3.1) becomes more complex. We refer to [7] and references therein for the background on this question. While for our purposes, we need to bound the difference between two measures over specific sets, computing that does not seem feasible. Hence, similarly to the univariate case, we derive a bound in the total variation distance. One can bound this distance in terms of the eigenvalues of the $\Sigma^{-1/2} \Sigma_1^{-1/2}$ or, equivalently, $\Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2}$. We present that in Theorem 3.3, however for clarity of presentation we assume that $\Sigma_1 = I, \Sigma_2 = \Sigma$. Although our technique is simple, to our knowledge, it is not present in the literature, and we find it interesting on its own. For instance, it improves the constant in the upper bound of [7, Theorem 1.1].

**Theorem 3.3.** Let $Z_1$ and $Z_2$ be mean zero $d$-dimensional normal random vectors with variance-covariance matrices $I$ and $\Sigma$, respectively. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_d$ are the eigenvalues of $\Sigma$, then

$$d_{TV}(Z_1, Z_2)^2 \leq \sum_{i=1}^d \frac{(\sqrt{\lambda_i} - 1)^2}{\lambda_i + 1} \leq \|\Sigma - J\|^2_F.$$  

The following bound will be helpful when none of the variance-covariance matrices are identity matrices.

**Lemma 3.4.** Let $\Sigma_1$ be a $d \times d$ invertible symmetric matrix and $\Sigma_2$ be another $d \times d$ matrix. Then

$$\left\| \Sigma_1^{-1/2} \Sigma_2 \Sigma_1^{-1/2} \right\|_F \leq \left\| \Sigma_1^{-1} \right\|_{op} : \|\Sigma_2\|_F.$$

**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the eigenvalues of $\Sigma_1$, and $\{u_i \mid 1 \leq i \leq d\}$ be the corresponding orthonormal eigenbasis. Then $\Sigma_1^{-1}$ can be written as $\Sigma_1^{-1} = \sum_{i=1}^d \lambda_i^{-1} u_i u_i^T$. Using this, we get that

$$\left\| \Sigma_1^{-1/2} \Sigma_2 \Sigma_1^{-1/2} \right\|_F^2 = \sum_{i,j=1}^d (\lambda_i \lambda_j)^{-1} \cdot (u_i^T \Sigma_2 u_j)^2$$

...
\[ \| \Sigma_1^{-1} \|_{\text{op}}^2 \cdot \sum_{i,j=1}^{d} (u_j^T \Sigma_2 u_i)^2 = \| \Sigma_1^{-1} \|_{\text{op}}^2 \cdot \| \Sigma_2 \|_F^2. \]

This completes the proof.

It is well known that by Pinsker’s inequality bounds the square of the total variation distance is bounded by Kullback–Leibler divergence. This can be used to get a bound for the total variation distance between two multivariate Gaussian measures. We present a related but more intuitive way to bound the quantity. First, we show the following lemma.

**Lemma 3.5.** Let \( f, g \) be the density of the random variables \( X, Y \), respectively. Then

\[ d_{TV}(X, Y)^2 \leq 1 - \left( \int \sqrt{fg} \right)^2. \]

**Proof.** By definition of total variation distance, we have

\[ d_{TV}(X, Y) = \frac{1}{2} \int |f - g| = 1 - \int f \land g = \int f \lor g - 1. \]

Thus multiplying the last two expressions gives the square of the total variation distance,

\[ d_{TV}(X, Y)^2 = \left( 1 - \int f \land g \right) \left( \int f \lor g - 1 \right). \]

Notice that \( \int f \land g + \int f \lor g = 2 \). Thus

\[ d_{TV}(X, Y)^2 = 1 - \int f \land g \cdot \int f \lor g \leq 1 - \left( \int \sqrt{f \land g \cdot f \lor g} \right)^2 = 1 - \left( \int \sqrt{fg} \right)^2. \]

The inequality follows from the Cauchy-Schwarz inequality.

Hence to bound the total variation distance between two Gaussian vectors, it remains to bound \( \int \sqrt{fg} \) where \( f \) and \( g \) are corresponding probability density functions.

**Proof of Theorem 3.3.** By Lemma 3.5 it is enough to bound \( \int \sqrt{fg} \), where

\[ f(x) = (2\pi)^{-\ell/2} \cdot \exp \left( -x^T x / 2 \right) \text{ and } g(x) = (2\pi)^{-\ell/2} \cdot \det(\Sigma)^{-1/4} \cdot \exp \left( -x^T \Sigma^{-1} x / 2 \right). \]

Then

\[ \sqrt{f(x)g(x)} = (2\pi)^{-\ell/2} \cdot \det(\Sigma)^{-1/4} \cdot \exp \left( -x^T (I + \Sigma^{-1}) x / 4 \right) \]

\[ \quad = \frac{\det(\Sigma)^{1/4}}{\det((I + \Sigma)^{1/2})^{1/2}} \cdot (2\pi)^{-\ell/2} \cdot \frac{\det((I + \Sigma)/2)^{1/2}}{\det(\Sigma)^{1/2}} \cdot \exp \left( -x^T (I + \Sigma^{-1}) x / 4 \right). \]

Recognizing the density function of a normal distribution with mean zero and variance-covariance matrix \((I + \Sigma^{-1})/2\), we conclude that

\[ \left( \int \sqrt{fg} \right)^2 = \frac{\det(\Sigma)^{1/2}}{\det((I + \Sigma)/2)^{1/2}} = \prod_{i=1}^{\ell} \frac{2\sqrt{\lambda_i}}{\lambda_i + \lambda_i}. \]
Hence,
\[ d_{\text{TV}}(Z_1, Z_2)^2 \leq 1 - \prod_{i=1}^\ell \frac{2\sqrt{\lambda_i}}{1 + \lambda_i} \leq \sum_{i=1}^\ell \left( 1 - \frac{2\sqrt{X_i}}{1 + \lambda_i} \right) = \sum_{i=1}^\ell \left( \sqrt{\lambda_i} - 1 \right)^2. \]
The conclusion follows from the fact that \( \|\Sigma - I\|_F^2 = \sum_{i=1}^\ell (\lambda_i - 1)^2 \).

### 3.2. Variance Control

We present our main results with the implicit variance \( \sigma = \sigma_{n,0} \). To compute \( \sigma_{n,0} \) explicitly, one would have to rely on particular properties of the model. For example, as we mentioned before, suppose in addition to Assumptions I we have that \( P(X_i \vee X_j \leq 0) = 1/4 \) for all \( i \neq j \), then \( \sigma \equiv 1/2 \).

**Remark 3.6.** In Lemma 3.7, \( D_1 \) can be replaced by the average degree in the dependency graph instead of the maximum one.

**Lemma 3.7.** In the setup of Theorem 1.5 for any \( x \in \mathbb{R}^\ell \) and \( A \) as in (1.4), we have that
\[ d_{\text{TV}}(N_{\ell+1}(0, \Sigma_x), N_{\ell+1}(0, \Sigma_0)) \leq \frac{3A\ell}{\min_k \theta_k} \left\| \Sigma_0^{-1/2} \Sigma_x \Sigma_0^{-1/2} - I \right\|_F \frac{D_1 \|x\|_\infty}{\sqrt{n}}. \]

**Proof.** Applying Theorem 3.3 and Lemma 3.4 gives us that
\[ d_{\text{TV}}(N_{\ell+1}(0, \Sigma_x), N_{\ell+1}(0, \Sigma_0)) \leq \left\| \Sigma_0^{-1/2} \Sigma_x \Sigma_0^{-1/2} - I \right\|_F \leq \left\| \Sigma_0^{-1} \right\|_{\text{op}} \cdot \left\| \Sigma_x - \Sigma_0 \right\|_F. \]

To compute \( \|\Sigma_x - \Sigma_0\|_F \), we bound the difference in each coordinate. Define the function
\[ f_k(x)(u) := \mathbb{1}_{0 < \ell \leq m_k + n^{-1/2} \cdot x_k} - \mathbb{1}_{0 < \ell \leq m_k + n^{-1/2} \cdot x_k / \theta_k, m_k} \]
for \( k \in [\ell] \). Then
\[ \mathbb{1}_{0 < \ell \leq m_k + n^{-1/2} \cdot x_k / \theta_k} = \mathbb{1}_{0 < \ell \leq m_k} + f_k(x)(u). \]
Thus for \( s \) and \( t \) in \([\ell]\), we get
\[
(\Sigma_x)_{st} - \Sigma_{st} = \frac{1}{n} \sum_{i, j \in N_i} \text{Cov} \left( \mathbb{1}_{X_i \leq m_s, f_t(x)(X_j)} \right) + \text{Cov} \left( f_s(x)(X_i), \mathbb{1}_{X_j \in I_{s, u}} \right) + \text{Cov} \left( f_s(x)(X_i), f_t(x)(X_j) \right)
\leq A \cdot \frac{D_1}{\sqrt{n}} \cdot \sqrt{\left( \frac{x_s}{\theta_s} + \frac{x_t}{\theta_t} \right)^2 + \sqrt{\frac{x_s x_t}{\theta_s \theta_t}}}
\leq 3A \cdot \frac{D_1}{\sqrt{n}} \cdot \left( \frac{x_s}{\theta_s} + \frac{x_t}{\theta_t} \right)
\]
Hence
\[ \|\Sigma_x - \Sigma_0\|_F \leq 3A \cdot \frac{D_1}{\sqrt{n}} \cdot \sqrt{\sum_{s, t} \left( \frac{x_s}{\theta_s} + \frac{x_t}{\theta_t} \right)^2} \leq 3\ell \cdot \frac{A}{\min_k \theta_k} \cdot \frac{D_1}{\sqrt{n}} \cdot \|x\|_\infty. \]
This completes the proof. ■

4. Proof of Main Results

4.1. Limiting distribution of the sample median

Proof of Theorem 1.3. Recall the definition of the sample median

\[ M_n = \max \{ x \in \mathbb{R} : |\{ i \mid X_i \leq x\}| \leq \lfloor n/2 \rfloor \}. \]

Fix \( x \in \mathbb{R} \). We have

\[
P(\theta \sqrt{n} M_n \leq x) = P(M_n \leq n^{-1/2} \cdot x/\theta) \\
\in \left[ P(\{|i \mid X_i \leq n^{-1/2} \cdot x/\theta\} \geq n/2), P(\{|i \mid X_i \leq n^{-1/2} \cdot x/\theta\} \geq n/2 - 1) \right] \\
= \left[ P\left( \sum_{i=1}^{n} 1_{X_i \leq n^{-1/2} \cdot x/\theta} \geq n/2 \right), P\left( \sum_{i=1}^{n} 1_{X_i \leq n^{-1/2} \cdot x/\theta} \geq n/2 - 1 \right) \right] (4.1)
\]

By subtracting \( \sum_{i=1}^{n} F_i \left( n^{-1/2} \cdot x/\theta \right) \) to each side of the inequality inside of the probability we get

\[
P\left( \sum_{i=1}^{n} 1_{X_i \leq n^{-1/2} \cdot x/\theta} \geq n/2 \right) = P\left( \sum_{i=1}^{n} \left( 1_{X_i \leq n^{-1/2} \cdot x/\theta} - F_i \left( n^{-1/2} \cdot x/\theta \right) \right) \geq -\sqrt{n} \cdot x_n \right) (4.2)
\]

where

\[ x_n = n^{-1/2} \cdot \left( \sum_{i=1}^{n} F_i \left( n^{-1/2} \cdot x/\theta \right) - n/2 \right). \]

By Assumption I, \( F_i \)'s are twice continuously differentiable at 0. Thus, using Taylor expansion for each \( F_i(x) \) at 0, \( x_n \) could be rewritten as

\[ x_n = \frac{1}{\sqrt{n}} (n/2 - n/2) + \frac{x}{\theta n} \sum_{i=1}^{n} F_i'(0) + R. \]

Thus,

\[ |x_n - x| = |R| \leq \frac{x^2}{2\sqrt{n}} \cdot \frac{A}{\theta^2}. \]

Notice that by Assumption I.3, the quantity \( A/\theta^2 \) is bounded. On the other hand the left hand side of the inequality in side of probability in (4.2) is a scaled sum of centered Bernoulli random variables

\[ Y_{i,x} := 1_{X_i \leq n^{-1/2} \cdot x/\theta} - F_i \left( n^{-1/2} \cdot x/\theta \right), \quad i \in [n]. \]
We treat the right bound of interval (4.1) similarly. This allows us to rewrite the original quantity of interest in the following way.

\[
P\left(\theta\sqrt{n}M_n \leq x\right) \in \left[ P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right), P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n - 1\right) \right]. \tag{4.3}
\]

Moreover, for each \(x \in \mathbb{R}\) random variables \(\{Y_{i,x}\}\) have the same dependency graph as \(\{X_i\}\). This allows us to consider two cases when \(|x| > K_n\) in which we will apply a concentration inequality and \(|x| \leq K_n\) where we would use Stein’s method for normal approximation.

Recall that \(\sigma^2_{n,x} = \frac{1}{n}\) \text{Var}\left(\sum_{i=1}^n Y_{i,x}\right)\). Hence the Kolmogorov–Smirnov distance can be approximated in terms of \(\{Y_{i,x}\}_{i \leq n}\) as

\[
\sup_{x \in \mathbb{R}} |P\left(\theta\sqrt{n}M_n \leq x\right) - \Phi(x/\sigma)| = \sup_{x \in \mathbb{R}} |P\left(\theta\sqrt{n}M_n \leq x\right) - \Phi(x/\sigma)| \leq \sup_{x \in \mathbb{R}} \left\{ P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) - \Phi(x/\sigma) \right\} \right\} - \Phi(x/\sigma) \right\}. \tag{4.4}
\]

Focusing on the first term inside of the maximum in (4.4) we rewrite it as

\[
\sup_{x \in \mathbb{R}} P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) - \Phi(x/\sigma) \leq \text{Err}_1 + \text{Err}_2,
\]

where

\[
\text{Err}_1 = \sup_{|x| > K_n} P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) - \Phi(x/\sigma), \tag{4.5}
\]

and

\[
\text{Err}_2 = \sup_{|x| < K_n} P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) - \Phi(x/\sigma). \tag{4.6}
\]

**Bound on** \(\text{Err}_1\): We bound the first term by considering two cases: when \(x_n\) is negative and when it is positive. When \(x_n < 0\) both quantities in (4.5) are negligible. By Theorem 1.7 we have the following concentration bound

\[
P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) \leq \exp\left(-\frac{2K_n^2}{D_1}\right)
\]

and since \(\max\{\Phi_{\sigma^2}(-K_n), 1 - \Phi_{\sigma^2}(K_n)\} \leq \exp\left(-\frac{K_n^2}{2\sigma^2}\right)\) we get that

\[
\sup_{x < -K_n} P\left(\sum_{i=1}^n Y_{i,x} \geq -\sqrt{n}\cdot x_n\right) - \Phi(x/\sigma) \leq 2 \exp\left(-\frac{2K_n^2}{(D_1 \vee 4\sigma^2)}\right).
\]
On the other hand, when $x_n > 0$, both terms are close to 1,

$$
\sup_{x > K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \geq -\sqrt{n} \cdot x_n \right) - \Phi(x/\sigma) \right| = \sup_{x > K_n} P \left( \sum_{i=1}^{n} Y_{i,x} \leq -\sqrt{n} \cdot x_n \right) + \Phi(-x/\sigma)
$$

$$
\leq 2 \exp \left( -2K_n^2/(D_1 \lor 4\sigma^2) \right).
$$

**Bound on $\text{Err}_2$.** We consider the rest of the terms $|x| \leq K_n$.

$$
\text{Err}_2 \leq \sup_{|x| \leq K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \leq -\sqrt{n} \cdot y \right) - \Phi(y/\sigma_{n,x}) \right| - \Phi(x_n/\sigma_{n,x}) - \Phi(x/\sigma)
$$

$$
+ \sup_{|x| \leq K_n} |\Phi(x_n/\sigma_{n,x}) - \Phi(x_n/\sigma)| + \sup_{|x| \leq K_n} |\Phi(x_n/\sigma) - \Phi(x/\sigma)|.
$$

(4.7)

Since all random variables $|Y_i| \leq 1$ by Theorem 1.6 we have a quantitative version of CLT for $\{Y_i\}_{i \geq 1}$. Since $Y_{i,x}$ are centered we we can rewrite (4.7) as

$$
\sup_{y \in \mathbb{R}} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \leq \sqrt{n} \cdot y \right) - \Phi(y/\sigma_{n,x}) \right| \leq \frac{D_1}{\sigma_{n,x} \sqrt{n}} (D_2 + D_3).
$$

Lemma 3.1 and Lemma 3.7 imply that for $|x| \leq K_n$

$$
(4.8) \leq \left| 1 - \sigma_{n,x}^2/\sigma^2 \right| \leq \frac{3A}{\theta} \cdot \frac{D_1 |x|}{\sigma^2 \sqrt{n}} \leq \frac{A}{\theta^2} \cdot \frac{D_1 K_n}{\sigma^2 \sqrt{n}}.
$$

Finally, term (4.9) is bounded as follows

$$
(4.9) \leq \frac{|x_n - x|}{\sigma^2} = \frac{x_n^2}{\sigma^2 \sqrt{n}} \cdot \frac{A}{\theta^2} \leq \frac{A}{\theta^2} \cdot \frac{K_n^2}{\sigma^2 \sqrt{n}}.
$$

(4.10)

Putting these bounds together, we conclude that

$$
\sup_{x \in \mathbb{R}} \left| P \left( \sum_{i=1}^{n} Y_i \geq -\sqrt{n} \cdot x_n \right) - \Phi(x/\sigma) \right|
$$

$$
\leq \exp \left( -\frac{2K_n^2}{D_1 \lor 4\sigma^2} \right) + \frac{A}{\theta^2} \cdot \frac{D_1 K_n}{\sigma^2 \sqrt{n}} + \frac{A}{\theta^2} \cdot \frac{K_n^2}{\sigma^2 \sqrt{n}} + \frac{D_1}{\sigma_{n,x} \sqrt{n}} (D_2 + D_3)
$$

$$
\leq \frac{A}{\theta^2} \cdot \left( 1 \lor \frac{D_1}{\sigma^2} \right) \frac{\log n}{\sqrt{n}} + \frac{D_1 (D_2 + D_3)}{\sigma \sqrt{n}}.
$$

(4.11)

In the last inequality, we take $K_n^2 = c \cdot (D_1 \lor \sigma^2) \cdot \log n$ for some large constant $c > 0$. The second term in the maximum in (4.4) can be treated similarly.
4.2. Limiting joint distribution of sample quantiles

Proof of Theorem 1.5. Define

$$\overline{Q}_n := Q_n - \mu.$$ 

Since each centered empirical quantile can be rewritten as

$$\overline{Q}_n^{(\alpha_k)} := \max \{ x \in \mathbb{R} \mid \{ i : X_i \leq x \} \leq \lfloor n\alpha_k \rfloor \} - m_k,$$

for any \(x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell\) we rewrite the joint distribution of quantiles in the following way

$$P(\Theta \sqrt{n} \cdot (Q_n - \mu) \preceq x)$$

$$= P \left( \theta_1 \sqrt{n} \overline{Q}_n^{(\alpha_1)} \preceq x_1, \ldots, \theta_\ell \sqrt{n} \overline{Q}_n^{(\alpha_\ell)} \preceq x_\ell \right)$$

$$\in \left[ P \left( \sum_{i=1}^{n} \mathbb{1}_{X_i - m_k \leq (\theta_k \sqrt{n})^{-1} x_k} \geq \lfloor n\alpha_k \rfloor, \text{ for all } k \in [\ell] \right), \right.$$

$$\left. P \left( \sum_{i=1}^{n} \mathbb{1}_{X_i - m_k \leq (\theta_k \sqrt{n})^{-1} x_k} \geq \lfloor n\alpha_k \rfloor - 1, \text{ for all } k \in [\ell] \right) \right].$$

(4.13)

Following the same idea as in the proof of Theorem 1.3 we treat each of the probabilities from (4.13) similarly but separately. In particular, by subtraction corresponding values of CDF functions \(F_i\) on both sides of the inequalities inside of the probabilities and then using Taylor expansion, we get an analogous equation to (4.3), namely

$$P(\Theta (Q_n - \mu) \preceq x) \in \left[ P \left( \sum_{i=1}^{n} Y_{i,x} \succeq -\sqrt{n} \cdot x_n \right), P \left( \sum_{i=1}^{n} Y_{i,x} \succeq -\sqrt{n} \cdot x_n - 1 \right) \right],$$

(4.14)

where \(1 := (1, 1, \ldots, 1)\) and

$$x_n := \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} F_i \left( m_k + n^{-1/2} \cdot x_k / \theta_k \right) - \lfloor n\alpha_k \rfloor \right)_{k=1}^{\ell},$$

satisfies

$$\|x_n - x\|_2 \leq \frac{\|x\|_2^2}{2\sqrt{n}}, \quad \frac{A}{\min_k \theta_k^2} \leq \frac{\ell \|x\|_\infty^2}{2\sqrt{n}}, \quad \frac{A}{\min_k \theta_k^2}.$$

(4.15)

For some function \(K_n\), each of the terms on the right-hand side of (4.14) can be bounded similarly as follows:

$$\sup_x \left| P \left( \sum_{i=1}^{n} Y_{i,x} \succeq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \succeq -x \right) \right| \leq \widehat{Err}_1 + \widehat{Err}_2,$$

where

$$\widehat{Err}_1 = \sup_{\|x\|_\infty \geq K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \succeq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \succeq -x \right) \right|$$

(4.16)
and \( \hat{\text{Err}}_2 = \sup_{\|x\|_\infty \leq K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \geq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \geq -x \right) \right| \) \hspace{1cm} (4.17)

**Bound on \( \hat{\text{Err}}_1 \).** In the first term, we bound similarly to the way we bounded (4.5) in the one-dimensional case. If \( x_{\min} := \min_k x_k \leq -K_n \) then using a union bound and Theorem 1.7,

\[
\sup_{x_{\min} \leq -K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \geq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \geq -x \right) \right| \leq 2\ell \exp \left( -2K_n^2 / D_1 \vee 4\sigma_{\max}^2 \right).
\]

On the other hand if \( x_{\min} > -K_n \) and \( x_{\max} := \max_k x_k \geq K_n \), then divide coordinates into two sets \( A := \{ k : x_k \in (-K_n, K_n) \} \) \( B := \{ k : x_k \leq -K_n \} \). For simplicity let

\[
E_k := \left\{ \sum_{i=1}^{n} (Y_{i,x})_k \geq -\sqrt{n} \cdot (x_n)_k \right\} \quad \text{and} \quad F_k := \left\{ \Sigma^{1/2} Z_k \geq -(x)_k \right\}.
\]

For all \( x \) such that \( x_{\min} > -K_n \) and \( x_{\max} \geq K_n \) we have

\[
\left| P \left( \sum_{i=1}^{n} Y_{i,x} \geq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \geq -x \right) \right| \\
= \left| P \left( \bigcap_{k \in A} E_k \bigcap_{k' \in B} E_{k'} \right) - P \left( \bigcap_{k \in A} F_k \bigcap_{k' \in B} F_{k'} \right) \right| \\
\leq \left| P \left( \bigcap_{k=1}^{\ell} E_k \right) - P \left( \bigcap_{k=1}^{\ell} F_k \right) \right| \\
+ \left| P \left( \bigcap_{k \in A} E_k \left( \bigcap_{k' \in B} E_{k'} \right)^c \right) - P \left( \bigcap_{k \in A} F_k \left( \bigcap_{k' \in B} F_{k'} \right)^c \right) \right|, \hspace{1cm} (4.18)
\]

where in (4.19) we switched to the compliment events enabling the application a concentration inequality.

\[
(4.19) \leq \left| P \left( \bigcup_{k \in B} \left\{ \sum_{i=1}^{n} Y_{i,x} < -\sqrt{n} \cdot (x_n)_k \right\} \right) - P \left( \bigcup_{k \in B} \left\{ \Sigma^{1/2} Z_k < -(x)_k \right\} \right) \right| \\
\leq 2\ell \exp \left( -2K_n^2 / (D_1 \vee 4\sigma_{\max}^2) \right)
\]

where \( \sigma_{\max}^2 := \max_{i \in [n]} \Sigma_{ii} \). On the other hand (4.18) can be bounded similarly to the term \( \hat{\text{Err}}_2 \).

**Bound on \( \hat{\text{Err}}_2 \).** We rewrite \( \hat{\text{Err}}_2 \) in the following way

\[
\hat{\text{Err}}_2 = \sup_{\|x\|_\infty \leq K_n} \left| P \left( \sum_{i=1}^{n} Y_{i,x} \geq -\sqrt{n} \cdot x_n \right) - P \left( \Sigma^{1/2} Z \geq -x_n \right) \right| \hspace{1cm} (4.20)
\]
Applying multivariate CLT as in Theorem 1.6

\[
\begin{align*}
&\sup_{\|x\|\leq K_n}\|\Sigma^{-1}\|_{\text{op}}\|\Sigma_{\bar{x}} - \Sigma\|_F \\
&\leq \sup_{\|x\|\leq K_n}\frac{3A}{\min_k \theta_k} \cdot \ell \cdot \|\Sigma^{-1}\|_{\text{op}} \cdot \frac{D_1 \|x\|_{\infty}}{\sqrt{n}} \quad (4.23)
\end{align*}
\]

Finally the term (4.22) we bound coordinate-wise is in one-dimensional case (see (4.11)).

\[
(4.22) \leq \|\Sigma^{-1}\|_{\text{op}} \frac{\ell K_n^2}{2\sqrt{n}} \frac{A}{\min_k \theta_k^2} \leq \frac{\ell A}{\min_k \theta_k^2} \|\Sigma^{-1}\|_{\text{op}} \frac{K_n^2}{\sqrt{n}} \quad (4.24)
\]

Putting this all together and optimizing over \(K_n\) yields

\[
\sup_{\bar{x}} \left| \mathbb{P}\left( \sum_{i=1}^{n} Y_{i,\bar{x}} \geq -\sqrt{n} \cdot x_n \right) - \mathbb{P}\left( \Sigma^{1/2} Z \geq -x_n \right) \right| \\
\leq \ell \cdot \exp \left( \frac{K_n^2}{D_1 \sqrt{4\sigma_{\text{max}}^2}} \right) \frac{\ell^{1/4}}{\sqrt{n}} \|\Sigma^{-1/2}\|_{\text{op}} D_1 \left( D_2 + D_3 \ell^{-1} \right) \\
+ \frac{A}{\min_k \theta_k^2} \cdot \ell \|\Sigma^{-1}\|_{\text{op}} \frac{D_1 K_n}{\sqrt{n}} + \frac{A}{\min_k \theta_k^2} \cdot \ell \|\Sigma^{-1}\|_{\text{op}} \frac{K_n^2}{\sqrt{n}} \\
\leq \frac{A}{\min_k \theta_k^2} \ell \|\Sigma^{-1}\|_{\text{op}} \left( D_1 \sqrt{\sigma_{\text{max}}^2} \right) \frac{\log n}{\sqrt{n}} + \frac{\ell^{1/4}}{\sqrt{n}} \frac{\ell^{1/2}}{\text{op}} D_1 \left( D_2 + D_3 \ell^{-1} \right),
\]

where in the last inequality, we take \(K_n^2 = c \cdot (D_1 \sqrt{\sigma_{\text{max}}^2}) \cdot \log n\) for some large constant \(c > 0\). Similarly, treating the second term in (4.14) gives the desired result. ■
5. Optimal rate of convergence and examples

When $D_1, D_2, \text{ and } D_3$ are constants, Theorem 1.3 gives the bound of order $\frac{\log n}{\sqrt{n}}$. In this section, we analyze the i.i.d. case to motivate the following conjecture of the optimal rate.

Conjecture 5.1. In the situation of Theorem 1.3, if $D_i = O(1)$ for $i \in \{1, 2, 3\}$, then

$$\sup_{x \in \mathbb{R}} |\Phi(x) - \phi(x)| = \frac{1}{\sqrt{8\pi}} |F''(0)|.$$  

Lemma 5.2. Let $n = 2m + 1$ for some integer $m$. Suppose $(X_i)_{i \in [n]}$ are i.i.d. continuous random variables with CDF $F(x)$ that satisfy Assumption I. 1-2 with $\alpha = 1/2$ and $m_\alpha = 0$. Assume that $F''(0)$ exists. Then

$$n^{1/2} \sup_{x \in \mathbb{R}} |\Phi(x) - \phi(x)| \rightarrow \frac{1}{\sqrt{8\pi}} |F''(0)|.$$  

Recall that when $X_i$ are uncorrelated we have that $\sigma = 1/2$ and $\theta = F'(0)$.

Proof. By definition of the sample median and independence of $X_i$’s, we have

$$\mathbb{P} \left( \sqrt{2m} \cdot M_n \leq x \right) = (2m + 1) \binom{2m}{m} \int_{-\infty}^{x/(\sqrt{2m})} (1 - F(t))^m F(t)^m dt \quad (5.1)$$

where $x_m = 2\sqrt{2m} (F(x/(\sqrt{2m})) - F(0))$. Here in the last equality, we use the change of variable $t = \sqrt{2m} \cdot (F(t) - 1/2)$. Define

$$a_m = \sqrt{2\pi} \cdot \frac{2m + 1}{\sqrt{2m} \cdot 2^{m+1}} \binom{2m}{m} = 1 + \frac{3}{8m} + O(m^{-2}),$$

where the last equality follows by Stirling’s approximation. Thus, we have

$$\int_{-\sqrt{2m}}^{x_m} \left( g(t/2m) \right)^m \phi(t) dt (5.1) = a_m \int_{-\sqrt{2m}}^{x_m} \left( g(t/2m) \right)^m \phi(t) dt$$

where $g(s) := (1 - s) \exp(s)$. Notice that for all $|s| \leq 1$, we have

$$1 - s^2 \leq g(s) \leq 1 - s^2/4.$$  

Thus we get,

$$\mathbb{P} \left( \sqrt{2m} \cdot M_n \leq x \right) - \Phi(x) = \int_{-\sqrt{2m}}^{x_m} \left( g(t^2/2m)^m - 1 \right) \phi(t) dt + \int_{x_m}^{x} \phi(t) dt + O(1/m)$$
Berry–Esseen Theorem for Sample Quantiles

\[
\int_{-\sqrt{2m}x_m}^{x_m} \frac{t^4}{2m} \phi(t) \, dt + \phi(x)(x_m - x) + O(1/m) \\
= O(1/m) + \phi(x)(x_m - x).
\]

Recall that \(x_m = 2\sqrt{2m} (F(x/(2\theta\sqrt{2m})) - F(0))\). Hence using Taylor expansion, we have,

\[
x_m - x = 2\sqrt{2m} \left( \frac{x}{2\theta\sqrt{2m}} \cdot F'(0) + \frac{x^2}{16\theta^2m} F''(0) + R \right) - x = \frac{F''(0)}{4F'(0)^2} \cdot \frac{x^2}{\sqrt{n}} (1 + o(1)),
\]

where \(R = O(1/n)\). This completes the proof.

Acknowledgments. We would like to thank Sabyasachi Chatterjee for many insightful conversations.

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