On projections onto odometers of dynamical systems with the compact phase space. *†

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Abstract

We investigate projections to odometers (group rotations over adic groups) of topological invertible dynamical systems with discrete time and compact Hausdorff phase space.

For a dynamical system \((X, f)\) with a compact phase space we consider the category of its projections onto odometers. We examine the connected partial order relation on the class of all objects of a skeleton of this category. We claim that this partially ordered class always have maximal elements and characterize them. It is claimed also, that this class have a greatest element and is isomorphic to some characteristic for the dynamical system \((X, f)\) subset of the set \(\Sigma\) of ultranatural numbers if and only if the dynamical system \((X, f)\) is indecomposable (the space \(X\) could not be decomposed into two proper disjoint closed invariant subsets).

Introduction

An important role in analysis of convertible dynamical systems (d. s.) with discrete time (cascades) plays the information

— on what minimal dynamical systems there exists projection of the given dynamical system \((X, f)\);

— how these projections are arranged;

— how are dependent different projections of d. s. \((X, f)\) on minimal d. s.; in particular, whether for two projections \(h_1: (X, f) \to (Y_1, g_1)\) and \(h_2: (X, f) \to (Y_2, g_2)\) there exists a morphism \(\psi: (Y_1, g_1) \to (Y_2, g_2)\) of dynamical systems, such that \(h_2 = \psi \circ h_1\).

It is is highly nontrivial problem to receive the answers on these questions in general case. To approach to its solution, modern contributors consider projections of given d. s. not onto all minimal d. s., but on some ”convenient” classes of minimal d. s. (distal and equicontinuous minimal d. s., uniquely ergodic minimal d. s. etc.).

Suppose we consider some family \(\mathfrak{A}\) of minimal dynamical systems and explore properties of projections of d. s. \((X, f)\) onto elements of this family. In some cases elements of a class of all projections of d. s. \((X, f)\) onto d. s. from \(\mathfrak{A}\) can be put in order in the following sense.

*AMS classification 2000: 37B05, 37B20
†Keywords: compact, dynamical system, projection, odometer
Let \( h_1 : (X, f) \to (Y_1, g_1) \) and \( h_2 : (X, f) \to (Y_2, g_2) \) are projections. We shall say that \( h_1 \sim h_2 \), if there is an isomorphism of dynamical systems \( \psi : (Y_1, g_1) \to (Y_2, g_2) \), such that \( h_2 = \psi \circ h_1 \). Let us designate by \( \mathcal{B} \) a factor-set of all projections from \((X, f)\) onto elements of \( \mathcal{A} \) on this equivalence relation. We introduce binary relation \( \preceq \) on \( \mathcal{B} \). Let \( B_1, B_2 \in \mathcal{B} \). We say that \( B_1 \preceq B_2 \) if there exist representatives \( h_1 : (X, f) \to (Y_1, g_1) \) of the class \( B_1 \) and \( h_2 : (X, f) \to (Y_2, g_2) \) of the class \( B_2 \) and also morphism \( \psi : (Y_1, g_1) \to (Y_2, g_2) \) such that \( h_2 = \psi \circ h_1 \). The relation \( \preceq \) is easily checked to be defined correctly (i. e. it does not depend on the choice of representatives from \( B_1 \) and \( B_2 \)).

It is important to know, whether \( \preceq \) is the partial order relation on the class \( \mathcal{B} \). In the case of the positive answer to this question there appears a problem to describe properties of the class \( \mathcal{B} \) with the partial order \( \preceq \), in particular to determine classes of all its maximal and minimal elements and to find the greatest and least element (if they exist).

In what follows we consider the class \( \mathcal{A} \) of all odometers (group rotations over adic groups). This class is known to coincide with the class of all minimal distal dynamical systems with phase space homeomorphic to the Cantor set or some finite set. It is known also, that elements of the class \( \mathcal{A} \) are classified (up to topological conjugacy) by means of the lattice of so-called ultranatural numbers (\( \Sigma, \preceq \)).

We consider dynamical systems \((X, f)\) with Hausdorff compact phase space and their projections to elements of \( \mathcal{A} \) (mark that always there exists a trivial projection \((X, f) \to (\{pt\}, Id)\) on dynamical system, which phase space consists from one point).

It appears that an existence of nontrivial projections of d. s. \((X, f)\) on elements of \( \mathcal{A} \) is interconnected with an existence of so-called periodic partitions of the d. s. \((X, f)\) (finite closed partitions of \( X \) which elements are cyclically rearranged under the action of \( f : X \to X \)).

Let \( \mathcal{P}(X, f) \subseteq \mathbb{N} \) is a set of cardinalities of all periodic partitions of \((X, f)\). Then \( \mathcal{P}(X, f) \) is the topological invariant of \((X, f)\) and the existence of nontrivial projections of \((X, f)\) on elements of \( \mathcal{A} \) is equivalent to the inequality \( \mathcal{P}(X, f) \neq \{1\} \).

Designate by \( \mathcal{A}(X, f) \) a class of all elements of \( \mathcal{A} \) on which we can project dynamical system \((X, f)\). Let \( \Sigma(X, f) \) be a subset of the set of ultranatural numbers corresponding to \( \mathcal{A}(X, f) \). Let \( \mathcal{B}(X, f) \) be a set of all projections of \((X, f)\) on elements of \( \mathcal{A} \) and \( \mathcal{B}'(X, f) \) be a factor-class of \( \mathcal{B}(X, f) \) under the relation \( \sim \) (see above).

Among the main results obtained in the paper we can rank the following statements.

Let \((X, f)\) be a dynamical system with compact Hausdorff phase space and \( \mathcal{P}(X, f) \neq \{1\} \). Then

1. binary relation \( \preceq \) on \( \mathcal{B}'(X, f) \) is the relation of the partial order;
2. there exists a surjective map \( \Lambda_0 : (\mathcal{B}'(X, f), \preceq) \to (\Sigma(X, f), \preceq) \) which preserves order relation and such that a class of all maximal elements from \( (\mathcal{B}'(X, f), \preceq) \) coincides with a pre-image of the greatest element of \( (\Sigma(X, f), \preceq) \);
3. the ordered class \( (\mathcal{B}'(X, f), \preceq) \) is isomorphic to the ordered set \( (\Sigma(X, f), \preceq) \) if and only if d. s. \((X, f)\) is indecomposable (that is the space \( X \) can not be presented as a union of two disjoint proper closed invariant subsets).

To receive these results we explore in detail properties of periodic partitions, odometers and ultranatural numbers.

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It is easy to test, that if the relation \( \preceq \) is not partial order on \( \mathcal{B} \) then for some d. s. \((Y, g)\) from \( \mathcal{A} \) there is a morphism \( \alpha : (Y, g) \to (Y, g) \) such that the map \( \alpha : Y \to Y \) is not injective. The problem on existence of such minimal dynamical systems is interesting by itself.
In the last section we extract corollaries from the main results which relates to so-called almost one-to-one expansions of odometers, class of dynamical systems, which is intensively explored in last time (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]).

Finally, author would like to thank Igor Vlasenko, Sergey Kolyada, Vladimir Lubashenko, Sergey Maksimenko, Mark Pankov, Aleksander Prishlyak, Vladimir Sergeychuk, Vladimir Sharko, and Aleksander Sharkovskiy for discussion of results on seminars and series of valuable notes. The separate gratitude I want to express to Sergey Kolyada who has acquainted me with modern results on expansions of odometers.

**Preliminaries**

**Quotient spaces and factor-maps.**

Let $A$ be a certain set.

**Definition 0.1** Partition of set $A$ is a family $\{A_\alpha\}_{\alpha \in \Lambda}$ of nonempty subsets of $A$ which complies with the following requirements:

1) $A = \bigcup_{\alpha \in \Lambda} A_\alpha$;

2) $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha, \beta \in \Lambda, \alpha \neq \beta$.

**Definition 0.2** Partition $\{\tilde{A}_\gamma\}_{\gamma \in \Sigma}$ of $A$ is called the refinement of partition $\{A_\alpha\}_{\alpha \in \Lambda}$ if for every $\gamma \in \Sigma$ there exists $\alpha \in \Lambda$ such that $\tilde{A}_\gamma \subseteq A_\alpha$.

**Remark 0.1** Let partition $\{\tilde{A}_\gamma\}_{\gamma \in \Sigma}$ is the refinement of partition $\{A_\alpha\}_{\alpha \in \Lambda}$ of $A$. From property 2) of definition 0.1 it easily follows that for any $\alpha \in \Lambda$ and $\gamma \in \Sigma$ either $\tilde{A}_\gamma \subseteq A_\alpha$ or $\tilde{A}_\gamma \cap A_\alpha = \emptyset$.

**Remark 0.2** There exists a bijective correspondence between partitions of the set $A$ and equivalence relations on $A$.

1) With any partition $\{A_\alpha\}_{\alpha \in \Lambda}$ we can associate an equivalence relation $\rho$ with the help of relation

   $$(a_1 \rho a_2) \iff (\exists \alpha \in \Lambda : a_1, a_2 \in A_\alpha) ;$$

2) conversely, a partition on equivalence classes corresponds to any equivalence relation $\sigma$ on $A$.

Let $A$ is a set, $\mathfrak{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ is partition of $A$.

**Definition 0.3** Set $A/\mathfrak{A}$ which elements are the elements of partition $\mathfrak{A}$ is called the factor set of $A$ on the partition $\mathfrak{A}$.

The map $\text{pr} : A \to \mathfrak{A}$ which associate to every element $a \in A$ an element $A_\alpha \in A/\mathfrak{A}$ such that $a \in A_\alpha$ is called projection.
By analogy, it is possible to define a factor set \( A/\rho \) under the equivalence relation \( \rho \) (see remark 0.2).

Let \( X \) is a topological space, \( \mathcal{H} = \{ H_{\alpha} \}_{\alpha \in \Lambda} \) is a partition on \( X \).

Define topology on the set \( X/\mathcal{H} \) by the following rule: say that subset \( B \subseteq X/\mathcal{H} \) is open if and only if its pre-image \( pr^{-1}(B) \) is open in \( X \). This topology is named quotient topology and it is the weakest topology on \( X \) in which the map \( pr : X \to X/\mathcal{H} \) is continuous.

Let \( X \) and \( Y \) are topological spaces, \( \mathcal{H} \) is a partition on \( X \) and \( \mathcal{T} \) is a partition on \( Y \). Let \( f : X \to Y \) be a continuous map, which translates elements of the partition \( \mathcal{H} \) into elements of the partition \( \mathcal{T} \). Then it is defined a continuous factor-map \( \text{fact} f : X/\mathcal{H} \to Y/\mathcal{T} \) such that the following diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{pr_X} & & \downarrow_{pr_Y} \\
X/\mathcal{H} & \xrightarrow{\text{fact} f} & Y/\mathcal{T}
\end{array}
\]

Let again \( f : X \to Y \) is a continuous map. Designate by \( \text{zer} f \) a partition of \( X \) which elements are pre-images of points of \( Y \) under map \( f \). Let \( \mathcal{T} \) be a partition of \( Y \), which elements are points of \( Y \). It is clear that \( pr_Y = \text{Id} : Y \to Y/\mathcal{T} \) is identical map.

**Definition 0.4** Map \( \text{fact} f : X/\text{zer} f \to Y \) for which the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow_{pr_X} & & \downarrow_{pr_Y} \\
X/\text{zer} f & \xrightarrow{\text{fact} f} & Y/\mathcal{T}
\end{array}
\]

is commutative is called one-to-one factor of \( f \).

That the one-to-one factor is injective it is checked immediately.

**Definition 0.5** A continuous map \( f : X \to Y \) is referred as factorial, If \( f(X) = Y \) and one-to-one factor \( \text{fact} f : X/\text{zer} f \to Y \) is a homeomorphism.

**Proposition 0.1** (see. [11]) Suppose that the following requirements are fulfilled for a continuous map \( f : X \to Y \):

1. \( f(X) = Y \);
2. map \( f \) is open (is closed).

Then \( f \) is the factorial map.

In what follows we will need

**Lemma 0.1** Let \( X, Y_1, Y_2 \) are topological spaces, \( \varphi_1 : X \to Y_1 \) and \( \varphi_2 : X \to Y_2 \) are continuous maps.

If the map \( \varphi_1 \) is factorial then following conditions are equivalent:

1. partition \( \text{zer} \varphi_1 \) of \( X \) is the refinement of partition \( \text{zer} \varphi_2 \).
(2) there exists a continuous map \( \psi : Y_1 \to Y_2 \) such that \( \varphi_2 = \psi \circ \varphi_1 \).

**Proof.** 1. Let partition \( \text{zer} \varphi_1 \) is a refinement of partition \( \text{zer} \varphi_2 \). Then the map \( \varphi_2 \) translates elements of the partition \( \text{zer} \varphi_1 \) in points of space \( Y_2 \) and the factor-map \( \pi = \text{fact} \varphi_2 : X/ \text{zer} \varphi_1 \to Y_2 \) is well defined, for which the diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_2} & Y_2 \\
\downarrow \text{pr}_1 & & \downarrow \\
X/ \text{zer} \varphi_1 & \xrightarrow{\pi} & Y_2
\end{array}
\]

Let \( \chi = \text{fact} \varphi_1 : X/ \text{zer} \varphi_1 \to Y_1 \) be one-to-one factor of map \( \varphi_1 \), that is the diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_1} & Y_1 \\
\downarrow \text{pr}_1 & & \downarrow \\
X/ \text{zer} \varphi_1 & \xrightarrow{\chi} & Y_1
\end{array}
\]

Since the map \( \varphi_1 \) is factorial then \( \chi \) is homeomorphism of \( X/ \text{zer} \varphi_1 \) onto \( Y_1 \).

Consider the continuous map

\[ \psi = \pi \circ \chi^{-1} : Y_1 \to Y_2. \]

We have

\[ \psi \circ \varphi_1 = \pi \circ \chi^{-1} \circ \varphi_1 = \pi \circ \chi^{-1} \circ \chi \circ \text{pr}_1 = \pi \circ \text{pr}_1 = \varphi_2, \]

as it was required.

2. Suppose that there exists a continuous map \( \psi : Y_1 \to Y_2 \) to comply the equality \( \varphi_2 = \psi \circ \varphi_1 \).

We fix an element \( H_1 \) of partition \( \text{zer} \varphi_1 \). By definition there exists \( y_1 \in Y_1 \) such that \( H_1 = \varphi_1^{-1}(y_1) \). Let \( y_2 = \psi(y_1) \in Y_2 \). Then \( H_2 = \varphi_2^{-1}(y_2) = (\psi \circ \varphi_1)^{-1}(y_2) = \varphi_1^{-1}(\psi^{-1}(y_2)) \supseteq \varphi_1^{-1}(y_1) = H_1 \). Again by definition \( H_2 \) is the element of partition \( \text{zer} \varphi_2 \).

Due to arbitrariness in a choice of element \( H_1 \) of partition \( \text{zer} \varphi_1 \) we conclude that the partition \( \text{zer} \varphi_1 \) is refinement of the partition \( \text{zer} \varphi_2 \). □

**Categories and functors.**

**Definition 0.6** Category \( \mathcal{K} \) consists of a class of objects \( \text{Ob} \mathcal{K} \) and class of morphisms \( \text{Mor} \mathcal{K} \), which are linked by following conditions:

1) certain set \( H_\mathcal{K}(A, B) \) of morphisms of the category \( \mathcal{K} \) is associated to each ordered pair \( A, B \in \text{Ob} \mathcal{K} \);

2) each morphism of a category \( \mathcal{K} \) belongs to one and only one of sets \( H_\mathcal{K}(A, B) \);

3) in the class \( \text{Mor} \mathcal{K} \) the partial binary relation of multiplication is defined as follows: product \( \beta \circ \alpha \) of morphisms \( \alpha \in H_\mathcal{K}(A, B) \) and \( \beta \in H_\mathcal{K}(C, D) \) is defined if and only if \( B = C \) and in this case \( \beta \circ \alpha \in H_\mathcal{K}(A, D) \);

the partial multiplication is associative: \( \gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha \) for any \( \alpha \in H_\mathcal{K}(A, B) \), \( \beta \in H_\mathcal{K}(B, C) \) and \( \gamma \in H_\mathcal{K}(C, D) \);
4) for every \( A \in \text{Ob} \mathcal{K} \) the set \( \mathcal{H}_\mathcal{K}(A,A) \) contains a unit morphism \( 1_A \) such that \( \beta \circ 1_A = \beta \) and \( 1_A \circ \alpha = \alpha \) for any morphisms \( \alpha \in \mathcal{H}_\mathcal{K}(B,A) \) and \( \beta \in \mathcal{H}_\mathcal{K}(A,C) \).

**Definition 0.7** Category \( \mathcal{L} \) is called subcategory of category \( \mathcal{K} \) if

a) \( \text{Ob} \mathcal{L} \subseteq \text{Ob} \mathcal{K} \);

b) \( \text{Mor} \mathcal{L} \subseteq \text{Mor} \mathcal{K} \);

c) unit morphisms of \( \mathcal{L} \) are unit morphisms of \( \mathcal{K} \);

d) composition \( \beta \circ \alpha \) of morphisms \( \alpha, \beta \in \text{Mor} \mathcal{L} \) coincides with composition of these morphisms in \( \mathcal{K} \).

**Definition 0.8** Subcategory \( \mathcal{L} \) of category \( \mathcal{K} \) is named complete subcategory if \( \mathcal{H}_\mathcal{L}(A,B) = \mathcal{H}_\mathcal{K}(A,B) \) for every \( A, B \in \text{Ob} \mathcal{L} \).

**Definition 0.9** Morphism \( \sigma : A \to B \) is referred as monomorphism of category \( \mathcal{K} \) (\( \sigma \in \text{Mon} \mathcal{K} \)) if for any two morphisms \( \alpha, \beta : X \to A \) the equality \( \sigma \circ \alpha = \sigma \circ \beta \) implies \( \alpha = \beta \).

**Definition 0.10** Morphism \( \nu : A \to B \) is called epimorphism (\( \nu \in \text{Ep} \mathcal{K} \)) if for any \( \alpha, \beta : B \to Y \) from equality \( \alpha \circ \nu = \beta \circ \nu \) follows that \( \alpha = \beta \).

**Definition 0.11** Morphism \( \rho : A \to B \) is named bimorphism (\( \rho \in \text{Bim} \mathcal{K} \)) if \( \rho \in \text{Mon} \mathcal{K} \cap \text{Ep} \mathcal{K} \).

**Definition 0.12** Morphism \( \varphi : A \to B \) is called isomorphism (\( \varphi \in \text{Iso} \mathcal{K} \)) if there exists such morphism \( \psi : B \to A \) that \( \psi \circ \varphi = 1_A \) and \( \varphi \circ \psi = 1_B \).

**Definition 0.13** Two objects \( A, B \in \text{Ob} \mathcal{K} \) are referred as isomorphic if \( \mathcal{K}(A,B) \cap \text{Iso} \mathcal{K} \neq \emptyset \).

**Definition 0.14** Complete subcategory \( \mathcal{Z} \) of category \( \mathcal{K} \) which contains exactly one representative from each class of isomorphic objects of category \( \mathcal{K} \) is called skeleton of category \( \mathcal{K} \).

**Definition 0.15** Object \( O_\mathcal{K} \) of category \( \mathcal{K} \) is named right zero of category \( \mathcal{K} \) if for every \( A \in \text{Ob} \mathcal{K} \) there exists a unique morphism \( \alpha_A : A \to O_\mathcal{K} \).

**Definition 0.16** Object \( O_\mathcal{K} \) is called left zero of category \( \mathcal{K} \) if for every \( A \in \text{Ob} \mathcal{K} \) there exists a unique morphism \( \beta_A : 0_\mathcal{K} \to A \).

**Definition 0.17** By a (monadic) covariant functor from a category \( \mathcal{K} \) to a category \( \mathcal{L} \) we shall mean correspondence \( F : \mathcal{K} \to \mathcal{L} \) which satisfies the following requirements:

1) \( F(A) \in \text{Ob} \mathcal{L} \) for every \( A \in \text{Ob} \mathcal{K} \);

2) \( F(\alpha) \in \mathcal{H}_\mathcal{L}(F(A),F(B)) \) for every \( \alpha \in \mathcal{H}_\mathcal{K}(A,B) \);

3) \( F(1_A) = 1_{F(A)} \) for any unit morphism of category \( \mathcal{K} \);

4) if \( \alpha \in \mathcal{H}_\mathcal{K}(A,B), \beta \in \mathcal{H}_\mathcal{K}(B,C) \) then \( F(\beta \circ \alpha) = F(\beta) \circ F(\alpha) \).

**Definition 0.18** Monadic covariant functor which bijectively maps category \( \mathcal{K} \) on a category \( \mathcal{L} \) is called isomorphism of categories.
Dynamical systems.

Definition 0.19 Dynamical system with discrete time is a pair \((X, f)\), where \(X\) is a topological space and \(f : X \to X\) is a homeomorphism. Space \(X\) is called phase space of this dynamical system.

Consider a category \(K\) which objects are dynamical systems and morphisms of dynamical systems \((X, f)\) and \((Y, g)\) are continuous maps \(h : X \to Y\) of their phase spaces, for which the diagram is commutative

\[
\begin{array}{c}
X & \xrightarrow{f} & X \\
\downarrow h & & \downarrow h \\
Y & \xrightarrow{g} & Y
\end{array}
\]

Further we shall designate a morphism \(h\) of object \((X, f)\) in \((Y, g)\) as follows

\[ h : (X, f) \to (Y, g) . \]

Definition 0.20 Morphism \(h : (X, f) \to (Y, g)\) is called imbedding of dynamical system \((X, f)\) in \((Y, g)\) if the map \(h\) is injective. In this case \((X, f)\) is named subsystem of the dynamical system \((Y, g)\).

Definition 0.21 Morphism \(h : (X, f) \to (Y, g)\) is called projection if \(h(X) = Y\).

The dynamical system \((Y, g)\) is named factor-system of the dynamical system \((X, f)\).

The dynamical system \((X, f)\) is named expansion of the dynamical system \((Y, g)\).

Definition 0.22 Dynamical systems \((X, f)\) and \((Y, g)\) are topologically conjugate if there exists such morphism \(h : (X, f) \to (Y, g)\) that the map \(h : X \to Y\) is homeomorphism of the space \(X\) on \(Y\).

In all further considerations we shall restrict ourselves to the complete subcategory \(K_0\) of \(K\), which objects are dynamical systems with Hausdorff compact phase spaces. We shall name them dynamical systems or flows.

Definition 0.23 Let \((X, f)\) is a dynamical system (space \(X\) is Hausdorff and compact). Subset \(A \subseteq X\) is called invariant set of \((X, f)\) if \(f(A) = A\).

With each point \(x \in X\) of phase space of the dynamical system \((X, f)\) it is usual to associate following invariant sets:

- trajectory of the point \(x\)
  \[ \text{Orb}_f(x) = \bigcup_{n \in \mathbb{Z}} f^n(x) ; \]

- closure \(\overline{\text{Orb}_f(x)}\) of the trajectory of \(x\);

- \(\alpha\) and \(\omega\)-limit sets of the point \(x\)
  \[ \alpha(x) = \bigcap_{n<0} \bigcup_{k \leq n} f^k(x) , \quad \omega(x) = \bigcap_{n>0} \bigcup_{k \geq n} f^k(x) . \]
Definition 0.24 Point $x$ is called stable by Poisson in negative (positive) direction if $\alpha(x) = \text{Orb}_f(x)$ (if $\omega(x) = \text{Orb}_f(x)$).

Point $x$ is called stable by Poisson, if $\alpha(x) = \omega(x) = \text{Orb}_f(x)$.

Definition 0.25 Point $x$ is called recurrent, if for any neighborhood $U$ of $x$ there exists such $n(U) \in \mathbb{N}$ that for every $k \in \mathbb{Z}$ inequality is fulfilled

$$U \cap \bigcup_{i=k}^{k+n(U)-1} f^i(x) \neq \emptyset.$$ 

Definition 0.26 Point $x$ is named almost-periodic, if for any neighborhood $U$ of $x$ there exists such $n(U) \in \mathbb{N}$ that

$$\bigcup_{k \in \mathbb{Z}} f^{kn(U)}(x) \subseteq U.$$ 

Remark 0.3 Last definition is in no way conventional.

There is an other nomenclature (see [12, 21]) in which points stable by Poisson are called recurrent, and recurrent points are called almost-periodic.

Definition 0.27 Nonempty closed invariant set $A \subseteq X$ is called minimal set of dynamical system $(X, f)$ if $A$ does not contain any proper closed invariant subset of this dynamical system.

It is easy to see that for an object $(X, f)$ of category $K_0$ any minimal set $A$ is characterized by the following property: $\text{Orb}_f(x) = A$ for every $x \in A$.

Further we will need the following theorem (see [13, 14, 21])

Theorem 0.1 (Birkhoff) Each object $(X, f)$ of $K_0$ complies with the following statements:

- for every $x \in X$ the sets $\alpha(x)$ and $\omega(x)$ contain some minimal subsets of dynamical system $(X, f)$;

- for any recurrent point $x \in X$ the set $\text{Orb}_f(x)$ is minimal;

- each point $x \in A$ of an arbitrary minimal set $A$ is recurrent.

Definition 0.28 Dynamical system $(X, f)$ is called minimal, if its phase space $X$ is a minimal set.

In what follows we will take an advantage from the following

Lemma 0.2 Let $(X, f), (Y_1, g_1), (Y_2, g_2) \in \text{Ob} K_0$, $\varphi_1 : (X, f) \to (Y_1, g_1)$ and $\varphi_2 : (X, f) \to (Y_2, g_2)$ are morphisms.

Suppose the map $\varphi_1 : X \to Y_1$ is surjective. Then the following conditions are equivalent:

1. partition $\text{zer} \varphi_1$ of $X$ is refinement of the partition $\text{zer} \varphi_2$;

2. there exists a morphism $\psi : (Y_1, g_1) \to (Y_2, g_2)$ such that $\varphi_2 = \psi \circ \varphi_1$. 
Proof. 1. Suppose that partition $\varphi_1$ of space $X$ is the refinement of partition $\varphi_2$.

It is known, that any continuous mapping of a compact set into Hausdorff space is closed (see [11]). It is known also that continuous surjective closed map is factorial (see proposition 0.1).

Thus, the surjective map of compact sets $\varphi_1 : X \to Y_1$ is factorial and we are in the conditions of lemma 0.1.

Hence, there exists a continuous map $\psi : Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_1$.

Let us check commutability of the diagram

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{g_1} & Y_1 \\
\psi \downarrow & & \downarrow \psi \\
Y_2 & \xrightarrow{g_2} & Y_2
\end{array}
$$

Fix $y_1 \in Y_1$. Since the map $\varphi_1$ is surjective on the condition of Lemma then there exists $x \in \varphi_1^{-1}(y_1) \subseteq X$. Therefore $\psi \circ g_1(y_1) = \psi \circ g_2 \circ \varphi_1(x) = \varphi_2 \circ f(x) = \psi \circ \varphi_1 \circ f(x) = \psi \circ g_1 \circ \varphi_1(x) = \psi \circ g_1(y_1)$.

Due to arbitrariness in the choice of $y_1 \in Y_1$ we conclude that $\psi \circ g_1 = g_2 \circ \psi$ and $\psi \in \text{Mor}_0$.

2. Suppose there exists a morphism $\psi : (Y_1, g_1) \to (Y_2, g_2)$, such that $\varphi_2 = \psi \circ \varphi_1$.

Then $\varphi_2 = \psi \circ \varphi_1 : X \to Y_2$ and the further proof is an exact repetition of the second part of proof of Lemma 0.1. □

1 Periodic partitions.

1.1 Definition of periodic partition.

Suppose we have a compact Hausdorff space $X$ and a homeomorphism $f : X \to X$.

Definition 1.1 We call a finite family $W^{(m)} = \{W_i^{(m)}\}_{i=0}^{m-1}$ of subsets of space $X$ periodic partition of dynamical system $(X, f)$ of length $m$, if it satisfies to the following requirements:

(i) all $W_i^{(m)}$ are open-closed subsets of $X$;

(ii) $W_i^{(m)} = f(W_{i-1}^{(m)})$, $i = 1, \ldots, m - 1$ and $W_0^{(m)} = f(W_{m-1}^{(m)});$ 

(iii) $W_i^{(m)} \cap W_j^{(m)} = \emptyset$ when $i \neq j$;

(iv) $X = \bigcup_{i=0}^{m-1} W_i^{(m)}$.

Definition 1.2 Set of all lengths of all possible periodic partitions of dynamical system $(X, f)$ we call a set of periods of the dynamical system $(X, f)$ and designate it $\mathcal{P}(X, f)$.

Remark 1.1 For any dynamical system $(X, f)$ the set $\mathcal{P}(X, f)$ is not empty. Really, always exists trivial periodic partition $W^{(1)} = \{W_0^{(1)} = X\}$ of the dynamical system $(X, f)$ of length $1 \in \mathcal{P}(X, f)$. 


Remark 1.2 Let $W^{(m)} = \{W_i^{(m)}\}_{i=0}^{m-1}$ be a periodic partition of dynamical system $(X, f)$ of length $m$. From properties (ii) and (iii) of Definition 1.1 it immediately follows that for every $n \in \mathbb{Z}$ we have

$$f^n(W_0^{(m)}) = W_i^{(m)} \text{ when } n \equiv i \pmod{m} \text{ and }$$

$$f^n(W_0^{(m)}) \cap W_i^{(m)} = \emptyset \text{ when } n \not\equiv i \pmod{m}.$$ 

More commonly

$$f^n(W_i^{(m)}) = W_j^{(m)} \text{ when } n \equiv j - i \pmod{m} \text{ and }$$

$$f^n(W_i^{(m)}) \cap W_j^{(m)} = \emptyset \text{ when } n \not\equiv j - i \pmod{m}.$$ 

The base properties of the set $\mathcal{P}(X, f)$ are described by two following statements

**Proposition 1.1** Let $m \in \mathcal{P}(X, f)$ and $m$ is divided by $d \in \mathbb{N}$. Then $d \in \mathcal{P}(X, f)$.

**Proof.** Let $\{W_i^{(m)}\}_{i=0}^{m-1}$ be a periodic partition of length $m$. Let us present $m$ as $m = ad$, $a \in \mathbb{N}$. Consider a family of sets

$$V_j^{(d)} = \bigcup_{k=0}^{a-1} W_{j+kd}^{(m)} = \bigcup_{s \in \{0, \ldots, m-1\}, \ s \equiv j \pmod{d}} f^s(W_0^{(m)}), \quad j = 0, \ldots, d-1.$$ 

It is obvious that the family $\{V_j^{(d)}\}_{j=0}^{d-1}$ defined in this way complies with properties (i), (iii) and (iv) of definition 1.1. Let us check that it satisfies to property (ii) of this definition.

Since $m \equiv 0 \pmod{d}$ then congruences $s \equiv j \pmod{d}$ and $s \equiv j + tm \pmod{d}$ are equivalent for all $t \in \mathbb{Z}$. On the other hand, $f^{tm}(W_s^{(m)}) = W_s^{(m)}$, $t \in \mathbb{Z}$, $s = 0, \ldots, m-1$. Therefore

$$V_j^{(d)} = \bigcup_{s \in \{0, \ldots, m-1\}, \ s \equiv j \pmod{d}} f^s(W_0^{(m)}) = \bigcup_{t \in \mathbb{Z}, \ s \in \{0, \ldots, m-1\}, \ s \equiv j \pmod{d}} f^{tm+s}(W_0^{(m)}) =$$

$$= \bigcup_{t \in \mathbb{Z}, \ r \in \{tm, \ldots, (t+1)m-1\}, \ r \equiv j \pmod{d}} f^r(W_0^{(m)}) = \bigcup_{r \in \mathbb{Z}, \ r \equiv j \pmod{d}} f^r(W_0^{(m)}).$$

The validity of property (ii) of definition 1.1 is the obvious corollary of this sequence of equalities.

Proposition is proved. □

**Proposition 1.2** Let $m_1, m_2 \in \mathcal{P}(X, f)$ and $D$ is the least common multiple of $m_1$ and $m_2$. Then $D \in \mathcal{P}(X, f)$.

To prove Proposition 1.2 we need some additional inspection which will be done in the following subsection.
1.2 Main properties of periodic partitions.

It is clear that for any $m \in \mathcal{P}(X, f)$, $m > 1$, there exist more than one periodic partition of dynamical system $(X, f)$ of length $m$. Really, fix a partition $W^{(m)} = \{W^{(m)}_i\}_{i=0}^{m-1}$. With the help of cyclical permutation of indexes in the partition $W^{(m)}$ it is possible to construct periodic partition $W^{(m)}(k) = \{W^{(m)}_i(k)\}_{i=0}^{m-1}$,

$$W^{(m)}_j(k) = W^{(m)}_i \quad \text{when} \quad j \equiv i + k \pmod{m} \quad j = 0, \ldots, m - 1,$$

for arbitrary $k \in \{1, \ldots, m - 1\}$.

**Definition 1.3** Let $m \in \mathcal{P}(X, f)$. Two periodic partitions of dynamical system $(X, f)$ are called equivalent if one partition could be obtained from the other by means of cyclical permutation of indexes.

We ask a question: if $\mathcal{P}(X, f) \neq \{1\}$ then under what conditions on $(X, f)$ and $m \in \mathcal{P}(X, f)$ every two periodic partitions of length $m$ are equivalent?

**Definition 1.4** We say that dynamical system $(X, f)$ is indecomposable if it satisfies to the following property:

(A) If $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and $X_1, X_2$ are closed invariant subsets of $(X, f)$, then either $X_1 = \emptyset$ or $X_2 = \emptyset$.

**Remark 1.3** Assume $K$ is a closed invariant set of $(X, f)$ and $W^{(m)} = \{W^{(m)}_i\}_{i=0}^{m-1}$ is a periodic partition of length $m$.

For every $i = 0, \ldots, m - 1$ we have

$$f(W^{(m)}_i \cap K) = f(W^{(m)}_i) \cap f(K) = f(W^{(m)}_i) \cap K,$$

therefore, in particular $W^{(m)}_i \cap K \neq \emptyset$, $i = 0, \ldots, m - 1$, and if $K$ is open-closed in $X$ then the family $\{V^{(m)}_i = W^{(m)}_i \cap K\}_{i=0}^{m-1}$ satisfies to properties (i) – (iii) of Definition 1.1.

**Proposition 1.3** Let $m \in \mathcal{P}(X, f)$, $m > 1$. Dynamical system $(X, f)$ is indecomposable if and only if there exists unique up to the cyclical permutation of indexes periodic partition $W^{(m)}$ of length $m$.

**Proof.** 1. Assume that $W^{(m)}$ and $\overline{W}^{(m)}$ are two nonequivalent periodic partitions of dynamical system $(X, f)$ of length $m$.

From property (iv) of Definition 1.1 it follows that with the help of cyclical permutation of indexes in one of partitions we can achieve that $V^{(m)}_0 = W^{(m)}_0 \cap \overline{W}^{(m)}_0 \neq \emptyset$. Under our supposition $W^{(m)}_0 \neq \overline{W}^{(m)}_0$. Let, for instance, $K = W^{(m)}_0 \setminus \overline{W}^{(m)}_0 \neq \emptyset$.

Designate $V^{(m)}_i = f^i(V^{(m)}_0)$, $i = 1, \ldots, m - 1$. Remark, that $V^{(m)}_i \subset W^{(m)}_i$, therefore $V^{(m)}_i \cap W^{(m)}_0 = \emptyset$ when $i = 1, \ldots, m - 1$. This follows from the requirement (iii) of definition 1.1.

On the other hand,

$$f(V^{(m)}_{m-1}) = f^m(V^{(m)}_0) = f^m(W^{(m)}_0 \cap \overline{W}^{(m)}_0) = f^m(W^{(m)}_0) \cap f^m(\overline{W}^{(m)}_0) = W^{(m)}_0 \cap \overline{W}^{(m)}_0 = V^{(m)}_0.$$

The third equality is valid since $f$ is the homeomorphism, the penultimate equality follows from the requirement (ii) of definition 1.1.
Hence,

\[ X_1 = \bigcup_{i=0}^{m-1} V_i^{(m)} = \bigcup_{i=0}^{m-1} f^i(V_0^{(m)}) \]

is the invariant subset of \((X, f)\) (then also \(X_2 = X \setminus X_1\) is invariant). In this case \(X_1 \neq \emptyset\) on the construction and \(X_2 \neq \emptyset\) since \(X_1 \cap W_0^{(m)} = V_0^{(m)}\) and \(W_0^{(m)} \setminus V_0^{(m)} = K \subset X \setminus X_1\).

From the requirement (i) of Definition 1.1 it follows, that the set \(V_0^{(m)}\) is open-closed (and then all \(V_i^{(m)}\) are open-closed). Therefore, the sets \(X_1\) and \(X_2\) are open-closed in \(X\).

So, the dynamical system \((X, f)\) is not indecomposable.

The case \(\tilde{W}_0^{(m)} \setminus W_0^{(m)} \neq \emptyset\) is considered similarly.

2. Backwards, we shall assume that dynamical system \((X, f)\) is not indecomposable. We fix a partition \(X = X_1 \cup X_2\) of \(X\) on two proper disjoint invariant closed subsets. Mark, that the subsets \(X_1\) and \(X_2\) are open in \(X\) as well.

We fix periodic partition \(W^{(m)} = \{W_i^{(m)}\}_{i=0}^{m-1}\) of \((X, f)\) of length \(m\).

Nonempty families of sets \(\{V_i^{(m)}\}_{i=0}^{m-1} = W_i^{(m)} \cap X_1\) and \(\{V_i^{(m)}\}_{i=0}^{m-1} = W_i^{(m)} \cap X_2\) comply with properties (i) – (iii) of Definition 1.1 (see Remark 1.3).

It is easy to see that

\[ X_j = X_j \cap \bigcup_{i=0}^{m-1} W_i^{(m)} = \bigcup_{i=0}^{m-1} (W_i^{(m)} \cap X_j) = \bigcup_{i=0}^{m-1} V_i^{(m)}, \quad j = 1, 2. \]

Since \(X_1 \cap X_2 = \emptyset\), then \(V_r^{(m)} \cap V_s^{(m)} = \emptyset\) for every \(r, s \in \{0, \ldots, m - 1\}\).

We set

\[ \tilde{W}_i^{(m)} = V_i^{(m)} \cup V_{i-1}^{(m)} , \quad i = 1, \ldots, m - 1 , \]

\[ \tilde{W}_0^{(m)} = V_0^{(m)} \cup V_{m-1}^{(m)}. \]

The immediate check shows that the family \(\{\tilde{W}_i^{(m)}\}_{i=0}^{m-1}\) is the periodic partition of \((X, f)\) of length \(m\). In addition \(W_0^{(m)} \setminus \tilde{W}_0^{(m)} = V_0^{(m)} \neq \emptyset\) and \(W_0^{(m)} \setminus W_0^{(m)} = V_{m-1}^{(m)} \neq \emptyset\).

Therefore, the family \(\{\tilde{W}_i^{(m)}\}\) can not be obtained from \(\{W_i^{(m)}\}\) by cyclical permutation of indexes. □

Let \(m_1, m_2 \in \mathcal{P}(X, f)\). Suppose \(d\) and \(D\) are the greatest common divisor and respectively the least common multiple of numbers \(m_1\) and \(m_2\).

We consider periodic partitions \(\{W_i^{(m_1)}\}_{i=0}^{m_1-1}\) and \(\{W_j^{(m_2)}\}_{j=0}^{m_2-1}\) of \((X, f)\) of lengths \(m_1\) and \(m_2\).

**Proposition 1.4** Let for some \(k \in \{0, \ldots, m_1 - 1\}, l \in \{0, \ldots, m_2 - 1\}\) intersection \(W_k^{1} \cap W_l^{2}\) is not empty.

Then the family \(\{V_s^{(D)} = f^s(W_k^{(m_1)} \cap W_l^{(m_2)})\}_{s=0}^{D-1}\) complies with the requirements (i) — (iii) of Definition 1.1.

**Proof.** The set \(V_0^{(D)} = W_k^{(m_1)} \cap W_l^{(m_2)}\) is open-closed in \(X\) by definition 1.1. Since \(f\) is the homeomorphism, then all \(V_s^{(D)}\) are open-closed in \(X\) and the family \(\{V_s^{(D)}\}\) satisfies to the requirement (i) of Definition 1.1.

Again, taking into account the bijectivity of \(f\) we shall receive

\[ f^s(W_k^{(m_1)} \cap W_l^{(m_2)}) = f^s(W_k^{(m_1)}) \cap f^s(W_l^{(m_2)}), \quad s \in \mathbb{Z}, \quad (2) \]
In particular
\[ f(V_{D-1}^r) = f^D(V_0^r) = f^D(W_k^{r(m_1)} \cap W_l^{r(m_2)}) = f^D(W_k^{r(m_1)}) \cap f^D(W_l^{r(m_2)}) = W_k^{r(m_1)} \cap W_l^{r(m_2)} = V_0^{(D)}, \]

since by definition \( D \equiv 0 \pmod{m_r}, \ r = 1, 2. \) By this the fulfillment of property (ii) of Definition 1.1 is proved.

Taking into account property (ii) of Definition 1.1 and equality (2), for the proof of property (iii) it is enough to us now to show that
\[ V_0^{(D)} \cap V_s^{(D)} = \emptyset, \ s = 1, \ldots, D - 1. \]

Assume, that \( V_0^{(D)} \cap f^s(V_0^{(D)}) \neq \emptyset \) for certain \( s \in \mathbb{Z}. \) Then, in particular
\[ W_k^{r(m_1)} \cap f^s(W_k^{r(m_1)}) \neq \emptyset, \quad W_l^{r(m_2)} \cap f^s(W_l^{r(m_2)}) \neq \emptyset, \]

and it is possible by Remark 1.2 only if \( s \equiv 0 \pmod{m_1} \) and \( s \equiv 0 \pmod{m_2}, \) that is, only when \( s \) is the common multiple of numbers \( m_1 \) and \( m_2. \) Hence \( V_0^{(D)} \cap V_s^{(D)} = \emptyset \) for \( s = 1, \ldots, D - 1 \) and the family \( \{V_s^{(D)}\} \) complies with the condition (iii) of Definition 1.1. \( \square \)

We designate
\[ V_s^{(D)}(k,l) = f^s(W_k^{r(m_1)} \cap W_l^{r(m_2)}), \quad s = 0, \ldots, D - 1, \]
\[ A(k,l) = \bigcup_{s=0}^{D-1} V_s^{(D)}(k,l). \]

From Proposition 1.4 follows, that \( A(k,l) \) is open-closed invariant subset of \((X,f),\) and if \( V_0^{(D)}(k,l) \neq \emptyset, \) then the family \( \{V_s^{(D)}(k,l)\}_{s=0}^{D-1} \) is periodic partition of the dynamical system \((A(k,l), f|_{A(k,l)})\) of length \( D.\)

**Remark 1.4** If dynamical system \((X,f)\) is indecomposable, then either \( A(k,l) = \emptyset \) or \( A(k,l) = X,\) and in this case \( \{V_s^{(D)}(k,l)\}_{s=0}^{D-1} \) is periodic partition of the dynamical system \((X,f)\) of length \( D \) and \( D \in \mathcal{P}(X,f).\)

From property (iv) of Definition 1.1 it follows that there exist \( k,l, \) for which the set \( A(k,l) \) is not empty. According to what has been said we conclude that Proposition 1.2 is valid for indecomposable dynamical systems.

Generally speaking, if dynamical system \((X,f)\) is not indecomposable, it is not necessary that \( A(k,l) \in \{\emptyset\} \cup \{X\}.\)

**Definition 1.5** Let \( m_1, m_2 \in \mathcal{P}(X,f). \) Periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) of dynamical system \((X,f)\) are called compatible, if for any \( k \in \{0, \ldots, m_1 - 1\}, \ l \in \{0, \ldots, m_2 - 1\} \) either \( A(k,l) = \emptyset \) or \( A(k,l) = X.\)

**Remark 1.5** It easily follows from Proposition 1.4, that if \( m_2 = m_1, \) then compatibility of partitions \( \{W_i^{(m_1)}\}_{i=0}^{m_1-1} \) and \( \{W_j^{(m_2)}\}_{j=0}^{m_2-1} \) means, that these two partitions are equivalent.

**Remark 1.6** The immediate corollary of definitions of equivalence and compatibility of periodic partitions is the following statement.

Let periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible. If partition \( \widetilde{W}^{(m_2)} \) is equivalent to the partition \( W^{(m_2)} \), then periodic partitions \( W^{(m_1)} \) and \( \widetilde{W}^{(m_2)} \) are compatible.
If for some $m_1, m_2 \in \mathcal{P}(X, f)$ there exist compatible partitions of space $X$ of lengths $m_1$ and $m_2$, then iterating argument from Remark 1.4 we can claim, that the least common multiple $D$ of numbers $m_1$ and $m_2$ belongs to $\mathcal{P}(X, f)$. Hence, the proof of Proposition 1.2 is reduced to verification that for any pair $m_1, m_2 \in \mathcal{P}(X, f)$ there exist relevant compatible periodic partitions of the dynamical system $(X, f)$.

In the rest of subsection we shall prove a somewhat more common

**Proposition 1.5** Let $m_1, m_2 \in \mathcal{P}(X, f)$. For any periodic partition $W^{(m_2)}$ of dynamical system $(X, f)$ of length $m_1$ there exists a periodic partition $W^{(m_2)}$ of length $m_2$, which is compatible with the partition $W^{(m_1)}$.

**Corollary 1.1** Let dynamical system $(X, f)$ is indecomposable. Then any two periodic partitions of $(X, f)$ are compatible.

In order to prove Proposition 1.5 we shall study some properties of the constructions given above.

From Remark 1.3 it follows that the family $\{W^{(m_r)} \cap A(k, l)\}_{i=0}^{m_r}$, $r = 1, 2$, is periodic partition of length $m_r$ for a dynamical system $(A(k, l), f|_{A(k, l)})$.

Following statement gives the answer to a question about correlation of periodic partitions $\{V^{(D)}_s(k, l)\}_{s=0}^{D-1}$ and $\{W^{(m_r)}_i \cap A(k, l)\}_{i=0}^{m_r}$, $r = 1, 2$ of the dynamical system $(A(k, l), f|_{A(k, l)})$.

**Lemma 1.1** Let $V^{(D)}_0(k, l) \neq \emptyset$, $i \in \{0, \ldots, m_1 - 1\}$, $j \in \{0, \ldots, m_2 - 1\}$.

If $j - i \equiv l - k$ (mod $d$), then $W^{(m_1)}_i \cap W^{(m_2)}_j \subseteq A(k, l)$. Moreover $W^{(m_1)}_i \cap W^{(m_2)}_j = V^{(D)}_s(k, l)$, where $s \in \{0, \ldots, D - 1\}$ is a solution of the following system of congruences

$$\begin{cases} s \equiv i - k \text{ (mod $m_1$)}, \\ s \equiv j - l \text{ (mod $m_2$)}. \end{cases}$$

If $j - i \not\equiv l - k$ (mod $d$), then $W^{(m_1)}_i \cap W^{(m_2)}_j \cap A(k, l) = \emptyset$.

**Remark 1.7** We remind, that the system (3) is compatible if and only if $j - i \equiv l - k$ (mod $d$) (see [15]).

**Proof of Lemma 1.1.** 1. Let $j - i \equiv l - k$ (mod $d$). Then there exists unique (mod $D$) solution $s$ of system (3) (see [15]). Therefore,

$$A(k, l) \supseteq V^{(D)}_s(k, l) = f^s(W^{(m_1)}_i \cap W^{(m_2)}_j) = f^s(W^{(m_1)}_i) \cap f^s(W^{(m_2)}_j) = f^{i-k}(W^{(m_1)}_i) \cap f^{j-l}(W^{(m_2)}_j) = f^{i-k} \circ f^k(W^{(m_1)}_0) \cap f^{j-l} \circ f^l(W^{(m_2)}_0) = f^i(W^{(m_1)}_0) \cap f^j(W^{(m_2)}_0) = W^{(m_1)}_i \cap W^{(m_2)}_j.$$

Here we have taken advantage of that $f$ is the homeomorphism and, in particular, maps $f$ and $f^{-1}$ are one-to-one.

2. We shall assume now that $(W^{(m_1)}_i \cap W^{(m_2)}_j) \cap A(k, l) \neq \emptyset$. Then there exists $s \in \{0, \ldots, D - 1\}$, such that

$$\emptyset \neq (W^{(m_1)}_i \cap W^{(m_2)}_j) \cap V^{(D)}_s(k, l) = (W^{(m_1)}_i \cap W^{(m_2)}_j) \cap (f^s(W^{(m_1)}_i) \cap f^s(W^{(m_2)}_j)).$$

Hence, $W^{(m_1)}_i \cap f^s(W^{(m_1)}_i) \neq \emptyset$ and $W^{(m_2)}_j \cap f^s(W^{(m_2)}_j) \neq \emptyset$. From Remark 1.2 it follows now, that $s$ satisfies to system (3), and in particular $i - k \equiv j - l$ (mod $d$). □
Corollary 1.2 Let for some \( k_1, k_2 \in \{0, \ldots, m_1 - 1\} \), \( l_1, l_2 \in \{0, \ldots, m_2 - 1\} \) sets \( A(k_1, l_1) \), \( A(k_2, l_2) \) are not empty.

If \( k_1 - k_2 \equiv l_1 - l_2 \pmod{d} \), then \( A(k_1, l_1) = A(k_2, l_2) \). Otherwise \( A(k_1, l_1) \cap A(k_2, l_2) = \emptyset \).

Proof. We already know that \( A(k_1, l_1) \) and \( A(k_2, l_2) \) are closed invariant subsets of dynamical system \((X, f)\).

1. Suppose, \( k_1 - k_2 \equiv l_1 - l_2 \pmod{d} \). Then according to Lemma 1.1 we have \((W_{k_1}^{(m_1)} \cap W_{l_1}^{(m_2)}) \subseteq A(k_2, l_2)\). Hence,

\[
A(k_1, l_1) = \bigcup_{s=0}^{D-1} f^s(W_{k_1}^{(m_1)} \cap W_{l_1}^{(m_2)}) \subseteq \bigcup_{s=0}^{D-1} f^s(A(k_2, l_2)) = A(k_2, l_2).
\]

Changing roles of \( A(k_1, l_1) \) and \( A(k_2, l_2) \), we shall receive the inverse inclusion.

2. Suppose now that \( k_1 - k_2 \not\equiv l_1 - l_2 \pmod{d} \). Then

\[
\emptyset = \bigcup_{s=0}^{D-1} f^s(W_{k_1}^{(m_1)} \cap W_{l_1}^{(m_2)}) \cap f^s(A(k_2, l_2)) = \bigcup_{s=0}^{D-1} f^s(W_{k_1}^{(m_1)} \cap W_{l_1}^{(m_2)}) \cap A(k_2, l_2) = A(k_1, l_1) \cap A(k_2, l_2).
\]

Corollary is proved. \( \square \)

Corollary 1.3 If for some \( k \in \{0, \ldots, m_1 - 1\} \) and \( l \in \{0, \ldots, m_2 - 1\} \) the equality \( A(k, l) = X \) is valid, then periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible.

Corollary 1.4 If for some \( k \in \{0, \ldots, m_1\} \) and \( l \in \{0, \ldots, m_2\} \) there exists a set \( K \subseteq W_k^{(m_1)} \cap W_l^{(m_2)} \), such that

\[
\bigcup_{n \in \mathbb{Z}} f^n(K) = X,
\]

then periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible.

Proof. The statement of Corollary follows from the chain of equalities

\[
X = \bigcup_{n \in \mathbb{Z}} f^n(K) \subseteq \bigcup_{n \in \mathbb{Z}} f^n(W_k^{(m_1)} \cap W_l^{(m_2)}) = \bigcup_{m \in \mathbb{Z}} \bigcup_{r=0}^{D-1} f^{r+Dm}(V_0^{(D)}(k, l)) = \bigcup_{m \in \mathbb{Z}} f^{Dm}(A(k, l)) = A(k, l)
\]

and from Corollary 1.3 \( \square \)

Corollary 1.5 Let \( W_k^{(m_1)} \supseteq W_l^{(m_2)} \) for some \( k \in \{0, \ldots, m_1 - 1\} \) and \( l \in \{0, \ldots, m_2 - 1\} \).

Then

1) partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible;

2) \( m_1 \) divides \( m_2 \).
Obviously, assume that \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible and \( A(k,l) = X \).

2. From Lemma 1.1 follows, that \( W^{(m_1)}_i \cap W^{(m_2)}_l \neq \emptyset \) if and only if \( i \equiv k \pmod{d} \).

Assume, that \( d \neq m_1 \). Then there exists \( \tau \in \{0,\ldots,m_1 - 1\} \), \( \tau \neq k \), such that \( \tau \equiv k \pmod{d} \). Hence, \( W^{(m_1)}_\tau \cap W^{(m_2)}_l \neq \emptyset \). From the other side, we have \( W^{(m_1)}_l \subseteq W^{(m_1)}_k \) on condition of Corollary and \( W^{(m_1)}_k \cap W^{(m_1)}_\tau = \emptyset \) by Definition 1.1.

The obtained contradiction proves, that \( d = m_1 \) and \( m_1 \) divides \( m_2 \).

**Corollary 1.6** Let \( m_1, m_2 \in \mathcal{P}(X,f) \) and \( m_2 \) is divided by \( m_1 \).

Periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible if and only if the partition \( \{W^{(m_2)}_j\} \) of space \( X \) is refinement of the partition \( \{W^{(m_1)}_i\} \).

**Proof.** 1. Necessity. Suppose periodic partitions \( W^{(m_1)} \) and \( W^{(m_2)} \) are compatible.

We find \( k \in \{0,\ldots,m_1 - 1\} \) and \( l \in \{0,\ldots,m_2 - 1\} \), for which \( W^{(m_1)}_k \cap W^{(m_2)}_l \neq \emptyset \). Then \( A(k,l) = X \).

Since \( m_1 \) divides \( m_2 \), then \( d = m_1 \).

We fix \( j \in \{0,\ldots,m_2 - 1\} \). From Lemma 1.1 it follows, that \( W^{(m_1)}_i \cap W^{(m_2)}_j \neq \emptyset \) if and only if \( i \equiv j - l + k \pmod{m_1} \). There exists a unique \( \tau \in \{0,\ldots,m_1 - 1\} \), such that \( \tau \equiv j - l + k \pmod{m_1} \). Since

\[
X = \bigcup_{i=0}^{m_1-1} W^{(m_1)}_i
\]

and \( W^{(m_1)}_i \cap W^{(m_2)}_j = \emptyset \) when \( i \in \{0,\ldots,m_1 - 1\} \), \( i \neq \tau \), then \( W^{(m_2)}_j \subseteq W^{(m_1)}_\tau \).

By virtue of arbitrariness in the choice of \( j \in \{0,\ldots,m_2 - 1\} \) we conclude, that the partition \( \{W^{(m_2)}_j\} \) of space \( X \) is refinement of the partition \( \{W^{(m_1)}_i\} \).

2. Sufficiency follows from Corollary 1.5.

**Proof of Proposition 1.5.** Assume that \( W^{(m_1)} \) is periodic partition of length \( m_1 \). We fix some periodic partition \( W^{(m_2)} \) of length \( m_2 \).

Consider the following sets

\[
A(0,j) = \bigcup_{s=0}^{D-1} f^s \left( W^{(m_1)}_0 \cap W^{(m_2)}_j \right), \quad j = 0,\ldots,m_2 - 1.
\]

Obviously,

\[
W^{(m_1)}_0 = W^{(m_1)}_0 \cap \bigcup_{j=0}^{m_2-1} W^{(m_2)}_j = \bigcup_{j=0}^{m_2-1} \left( W^{(m_1)}_0 \cap W^{(m_2)}_j \right) \subseteq \bigcup_{j=0}^{m_2-1} A(0,j).
\]

Since \( A(0,j), j = 0,\ldots,m_2 - 1 \), are the invariant subsets of \( (X,f) \), then

\[
\bigcup_{j=0}^{m_2-1} A(0,j) = \bigcup_{j=0}^{m_2-1} \bigcup_{i=0}^{m_2-1} f^i \left( A(0,j) \right) = \bigcup_{i=0}^{m_1-1} \left( \bigcup_{j=0}^{m_2-1} A(0,j) \right) \supseteq \bigcup_{i=0}^{m_1-1} f^i(W^{(m_1)}_0) = X.
\]

We know from Corollary 1.2 that \( A(0,j) = A(0,k) \) when \( j \equiv k \pmod{d} \) and \( A(0,j) \cap A(0,k) = \emptyset \) if \( j \not\equiv k \pmod{d} \), therefore

\[
X = \bigcup_{j=0}^{d-1} A(0,j).
\]
In the last equality all sets \( A(0, j) \) are pairwise disjoint. Some of these sets can be empty. Let \( A_1 = A(0, k_1), \ldots, A_l = A(0, k_l) \) are all nonempty subsets from the family \( \{A(0, j)\}_{j=0}^{d-1} \).

Designate

\[
V_s^{(D)}(j) = V_s^{(D)}(0, k_j) = f^s \left( W_0^{(m_1)} \cap W_{k_j}^{(m_2)} \right), \quad j = 1, \ldots, l, \quad s = 0, \ldots, D - 1.
\]

Then

\[
X = \bigcup_{j=1}^{l} A_j = \bigcup_{j=1}^{l} \bigcup_{s=0}^{D-1} V_s^{(D)}(j) = \bigcup_{s=0}^{D-1} \left( \bigcup_{j=1}^{l} V_s^{(D)}(j) \right), \quad (4)
\]

Remark, that \( V_s^{(D)}(i) \cap V_r^{(D)}(j) = \emptyset \) if \( s \neq r \) or \( i \neq j \).

Really, \( V_s^{(D)}(i) \subset A_i, V_r^{(D)}(j) \subset A_j \), therefore on construction \( V_s^{(D)}(i) \cap V_r^{(D)}(j) \subset A_i \cap A_j = \emptyset \) when \( i \neq j \).

Let now \( i = j \). From Proposition 1.4 we know, that the family \( \{V_s^{(D)}(i)\}_{s=0}^{D-1} \) satisfies to properties (i)–(iii) of Definition 1.1. Hence, \( V_s^{(D)}(i) \cap V_r^{(D)}(i) = \emptyset \) when \( s \neq r \).

We set

\[
\tilde{W}_0^{(D)} = \bigcup_{j=1}^{l} V_0^{(D)}(j), \quad \tilde{W}_s^{(D)} = f^s \left( \tilde{W}_0^{(D)} \right) = \bigcup_{j=1}^{l} V_s^{(D)}(j), \quad s = 1, \ldots, D - 1.
\]

According to what has been said, we have \( \tilde{W}_s^{(D)} \cap \tilde{W}_r^{(D)} = \emptyset \) when \( s \neq r \), that is the family \( \{\tilde{W}_s^{(D)}\}_{s=0}^{D-1} \) complies with the requirement (iii) of Definition 1.1. From the formula (4) it follows, that this family satisfies to property (iv) of the indicated definition as well.

We recollect, that all systems of sets \( \{V_s^{(D)}(i)\}_{s=0}^{D-1}, i = 1, \ldots, l, \) satisfy to properties (i)–(iii) of Definition 1.1. From this in the first place it follows, that all sets of family \( \{\tilde{W}_s^{(D)}\} \) are open-closed in \( X \), secondly

\[
f \left( \tilde{W}_{D-1}^{(D)} \right) = \bigcup_{j=1}^{l} f(V_{D-1}^{(D)}(j)) = \bigcup_{j=1}^{l} V_0^{(D)}(j) = \tilde{W}_0^{(D)}.
\]

So, the system of sets \( \{\tilde{W}_s^{(D)}\}_{s=0}^{D-1} \) is periodic partition of \( (X, f) \) of length \( D \). By this we completely have proved Proposition 1.2.

Periodic partitions \( W^{(m_1)} \) and \( \tilde{W}^{(D)} \) are compatible since \( \tilde{W}_0^{(D)} \subseteq W_0^{(m_1)} \) on construction (see corollary 1.5).

We designate

\[
\tilde{W}_k^{(m_2)} = \bigcup_{s\in\{0, \ldots, D-1\}, \ s\equiv k \ (\text{mod} \ m_2)} \tilde{W}_s^{(D)}, \quad k = 0, \ldots, m_2 - 1.
\]

The easy immediate check shows, that \( \{\tilde{W}_k^{(m_2)}\}_{k=0}^{m_2-1} \) is the periodic partition of length \( m_2 \) (see proof of Proposition 1.1).

It follows from Corollary 1.4 that periodic partitions \( W^{(m_1)} \) and \( \tilde{W}^{(m_2)} \) are compatible since \( W_0^{(m_1)} \cap \tilde{W}_0^{(m_2)} \supseteq \tilde{W}_0^{(D)} \).

Proposition 1.5 is completely proved. \( \square \)
Remark 1.8 Generally speaking, if dynamical system \((X, f)\) is not indecomposable, then starting from an arbitrary fixed partition \(W^{(m_2)}\) it is possible to construct more than one periodic partition of length \(m_2\) compatible with a given partition \(W^{(m_1)}\) of length \(m_1\).

Namely, using notation introduced in the proof of Proposition 1.5 we assume

\[ \tilde{W}_0^{(D)}(t_1, \ldots, t_l) = \bigcup_{j=1}^{l} V_j^{(D)}(j), \]

where \(t_j \equiv 0 \pmod{m_1}\), \(t_j \in \{0, \ldots, D-1\}\), \(j = 1, \ldots, l\). Then, from Remark 1.2 we conclude that

\[ \tilde{W}_0^{(D)}(t_1, \ldots, t_l) \subseteq W_0^{(m_1)}. \]

Designate

\[ \tilde{W}_k^{(m_2)}(t_1, \ldots, t_l) = \bigcup_{s \in \{0, \ldots, D-1\}, \atop s \equiv k \pmod{m_2}} f^s(\tilde{W}_0^{(D)}(t_1, \ldots, t_l)), \quad k = 0, \ldots, m_2 - 1. \quad (5) \]

Now we can show, having applied the same argument as in the proof of Proposition 1.5, that the family of sets \(\{\tilde{W}_k^{(m_2)}(t_1, \ldots, t_l)\}_{k=0}^{m_2-1}\) is periodic partition of length \(m_2\), compatible with the partition \(W^{(m_1)}\).

Let \(\tilde{W}_k^{(m_2)}(t_1, \ldots, t_l)\) and \(\tilde{W}_r^{(m_2)}(\tau_1, \ldots, \tau_l)\) are two periodic partitions of form (5). Substituting them on equivalent, without breaking the compatibility relation it is possible to achieve, that \(t_1 = \tau_1 = 0\) (see Remark 1.5). We can easily see, that

\[ \bigcup_{s=0}^{m_2-1} \left( \tilde{W}_s^{(m_2)}(0, t_2, \ldots, t_l) \cap \tilde{W}_s^{(m_2)}(0, \tau_2, \ldots, \tau_l) \right) = A_1 \cup \bigcup_{t_j=\tau_j, \atop j \in \{2, \ldots, l\}} A_j. \]

Hence, periodic partitions \(\{\tilde{W}_s^{(m_2)}(0, t_2, \ldots, t_l)\}\) and \(\{\tilde{W}_s^{(m_2)}(0, \tau_2, \ldots, \tau_l)\}\) are compatible if and only if \(t_j = \tau_j\) for all \(j \in \{2, \ldots, l\}\).

Remark 1.9 Obviously, relation of the compatibility of two periodic partitions is reflexive and is symmetrical. However previous note shows, that generally speaking this relation is not transitive.

1.3 Sequences of periodic partitions.

In this subsection we shall prove a number of statements in order to analyze the transitivity problem of the compatibility relation of periodic partitions. The statements we are going to discuss will be used in further constructions.

Later on the following objects will be necessary for us.

Definition 1.6 Let a sequence of numbers \(\{n_i \in \mathcal{P}(X, f)\}_{i \in \mathbb{N}}\) is given.

We call a sequence \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) of periodic partitions of dynamical system \((X, f)\) regular, if it satisfies to the following conditions

1) \(n_k\) divides \(n_{k+1}\), \(k \in \mathbb{N}\);

2) partitions \(\{W_{s_k}^{(n_k)}\}\) and \(\{W_{s_{k+1}}^{(n_{k+1})}\}\) are compatible for all \(k \in \mathbb{N}\).
Remark 1.10 Using Corollary 1.6, it is easy to verify that every regular sequence \( \{W^{(n_k)}\}_{k \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\) complies with the requirement 3) periodic partitions \( \{W^{(m_k)}_{s_k}\} \) and \( \{W^{(n_l)}_{s_l}\} \) are compatible for all \( k, l \in \mathbb{N} \).

Really, let \( k, l \in \mathbb{N}, k < l \). In accord with Corollary 1.6, partition \( \{W^{(n_i)}_{s_i}\} \) of space \( X \) is the refinement of partition \( \{W^{(n_{i+1})}_{s_{i+1}}\} \) for every \( i \in \mathbb{N} \). Hence, partition \( \{W^{(m_i)}_{s_i}\} \) is the refinement of partition \( \{W^{(n_k)}_{s_k}\} \). Using again Corollary 1.6 we conclude that partitions \( \{W^{(n_i)}_{s_i}\} \) and \( \{W^{(m_k)}_{s_k}\} \) are compatible.

Proposition 1.6 Let a sequence of numbers \( \{n_k \in \mathcal{P}(X, f)\}_{k \in \mathbb{N}} \) is given, which complies with requirement 1) of Definition 1.6.

There exists a regular sequence \( \{W^{(n_k)}\}_{k \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\).

Proof. This statement is easily proved by the inductive application of Proposition 1.5. \( \square \)

Proposition 1.7 Let a regular sequence \( \{W^{(m_k)}\}_{k \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\) is given.

Let \( m_1, m_2 \in \mathcal{P}(X, f) \) and \( m_1 \) divides \( m_2 \).

Then

a) there exists periodic partition \( \{W^{(m_1)}_{i}\}_{i=0}^{m_1-1} \) of length \( m_1 \), which is compatible with each of partitions \( \{W^{(m_k)}_{s_k}\}, k \in \mathbb{N}\);

b) for any periodic partition \( \{W^{(m_1)}_{i}\} \) complying with item a) of the proposition there exists periodic partition \( \{W^{(m_2)}_{j}\}_{j=0}^{m_2-1} \) of length \( m_2 \), which is compatible with \( \{W^{(m_1)}_{i}\} \) and with each of partitions \( \{W^{(m_k)}_{s_k}\}, k \in \mathbb{N}\).

Proof of Proposition 1.7 is based on three lemmas.

Let \( m_1, m_2, n \in \mathcal{P}(X, f) \), and \( m_1 \) divides \( m_2 \).

Designate by \( d_i \) the greatest common divisor of numbers \( n \) and \( m_i \), \( i = 1, 2 \). Let also \( D_i \) be the least common multiple of numbers \( n \) and \( m_i \), \( i = 1, 2 \).

Lemma 1.2 Let \( \{W^{(m_1)}_{i}\}_{i=0}^{m_1-1} \), \( \{W^{(m_2)}_{j}\}_{j=0}^{m_2-1} \) and \( \{W^{(n)}_{k}\}_{k=0}^{n-1} \) are periodic partitions of dynamical system \((X, f)\) of lengths \( m_1, m_2 \) and \( n \), accordingly. Suppose, partitions \( \{W^{(n)}_{k}\} \) and \( \{W^{(m_2)}_{j}\} \) are compatible.

If the partitions \( \{W^{(m_1)}_{i}\} \) and \( \{W^{(m_2)}_{j}\} \) are compatible, then the partitions \( \{W^{(m_1)}_{i}\} \) and \( \{W^{(n)}_{k}\} \) are compatible too.

Lemma 1.3 Let \( d_1 = d_2 \).

Let \( \{W^{(m_1)}_{i}\}_{i=0}^{m_1-1} \) and \( \{W^{(n)}_{k}\}_{k=0}^{n-1} \) are compatible periodic partitions of dynamical system \((X, f)\) of lengths \( m_1 \) and \( n \), respectively.

Then any periodic partition \( \{W^{(m_2)}_{j}\}_{j=0}^{m_2-1} \) of length \( m_2 \), which is compatible with the partition \( \{W^{(m_1)}_{i}\} \), also is compatible with the partition \( \{W^{(n)}_{k}\} \).
Lemma 1.4 Let \( D_1 = D_2 \).

Let \( \{W_i^{(m_1)}\}_{i=0}^{m_1-1} \) and \( \{W_k^{(n)}\}_{k=0}^{n-1} \) are compatible periodic partitions of dynamical system \((X, f)\) of lengths \( m_1 \) and \( n \), respectively.

Then there exists periodic partition \( \{W_j^{(m_2)}\}_{j=0}^{m_2-1} \) of length \( m_2 \), which is compatible both with the partition \( \{W_i^{(m_1)}\} \) and with the partition \( \{W_k^{(n)}\} \).

Proof of Lemma 1.2. Substituting partition \( \{W_j^{(m_2)}\} \) on equivalent, we can regard that \( W_0^{(m_2)} \cap W_0^{(m_1)} \neq \emptyset \) (see Remark 1.6). Then, applying Corollary 1.6, we shall receive inclusion \( W_0^{(m_2)} \subseteq W_0^{(m_1)} \).

Similarly, substituting partition \( \{W_k^{(n)}\} \) on equivalent, we shall regard that \( K = W_0^{(n)} \cap W_0^{(m_2)} \neq \emptyset \). From Definition 1.5 it follows, that

\[
\bigcup_{t \in \mathbb{Z}} f^t(K) = \bigcup_{t=0}^{D_2-1} f^t \left( W_0^{(m_2)} \cap W_0^{(n)} \right) = X.
\]

On the other hand, \( K = W_0^{(m_2)} \cap W_0^{(n)} \subseteq W_0^{(m_1)} \cap W_0^{(n)} \), therefore from Corollary 1.4 it follows, that partitions \( W_i^{(m_1)} \) and \( W_k^{(n)} \) are compatible. \( \square \)

Proof of Lemma 1.3. As \( d_1 = d_2 \) and \( d_1 \) divides \( m_1 \), then \( d_2 \) divides \( m_1 \).

The replacement of a periodic partition on an equivalent does not affect the relation of compatibility (see note 1.6), therefore we can assume, that \( W_0^{(n)} \cap W_0^{(m_2)} \neq \emptyset \) and \( W_0^{(m_1)} \cap W_0^{(m_2)} \neq \emptyset \). Since partitions \( \{W_i^{(m_1)}\} \) and \( \{W_j^{(m_2)}\} \) are compatible, then \( W_0^{(m_2)} \subseteq W_0^{(m_1)} \) (see Corollary 1.6). Hence, \( W_0^{(n)} \cap W_0^{(m_1)} \neq \emptyset \).

We designate

\[
A = \bigcup_{s=0}^{D_2-1} f^s(W_0^{(m_2)} \cap W_0^{(n)}).
\]

Taking into account Corollary 1.3, to complete proof of lemma it is enough to check the equality \( A = X \).

Consider the pair of compatible partitions \( \{W_i^{(m_1)}\} \) and \( \{W_j^{(m_2)}\} \). From Definition 1.5, Lemma 1.1 and Corollary 1.6 we receive

\[
W_0^{(m_1)} = W_0^{(m_1)} \cap \bigcup_{s=0}^{m_2-1} f^s \left( W_0^{(m_1)} \cap W_0^{(m_2)} \right) = \bigcup_{s \in \{0, \ldots, m_2-1\}, \, s \equiv 0 \pmod{m_1}} W_s^{(m_2)}.
\]

Consider now the pair \( \{W_k^{(n)}\} \) and \( \{W_j^{(m_2)}\} \) of periodic partitions.

From Lemma 1.1 we get

\[
W_0^{(n)} \cap A = W_0^{(n)} \cap \bigcup_{r \in \{0, \ldots, m_2-1\}, \, r \equiv 0 \pmod{d_2}} W_r^{(m_2)}.
\]

As \( d_2 \) divides \( m_1 \), then congruence \( r \equiv 0 \pmod{d_2} \) is the consequence of the congruence \( r \equiv 0 \pmod{m_1} \) and

\[
W_0^{(m_1)} = \bigcup_{r \in \{0, \ldots, m_2-1\}, \, r \equiv 0 \pmod{m_1}} W_r^{(m_2)} \subseteq \bigcup_{r \in \{0, \ldots, m_2-1\}, \, r \equiv 0 \pmod{d_2}} W_r^{(m_2)}.
\]
Hence, $W_0^{(m_1)} \cap W_0^{(n)} \subseteq W_0^{(n)} \cap A \subset A$. But the set $A$ is $f$–invariant, therefore

$$A = \bigcup_{s=0}^{D_1-1} f^s(A) \supseteq \bigcup_{s=0}^{D_1-1} f^s\left(W_0^{(m_1)} \cap W_0^{(n)}\right) = X.$$ 

Last equality is valid, since periodic partitions $\{W_i^{(m_1)}\}$ and $\{W_k^{(n)}\}$ are compatible. □

**Proof of Lemma 1.4.** Taking into account Remark 1.6, we shall regard that $W_0^{(n)} \cap W_0^{(m_1)} \neq \emptyset$. Designate

$$V_s = f^s\left(W_0^{(n)} \cap W_0^{(m_1)}\right), \quad s = 0, \ldots, D_1 - 1.$$ 

Since periodic partitions $\{W_k^{(n)}\}$ and $\{W_i^{(m_1)}\}$ are compatible and $D_2 = D_1$ on a condition of Lemma, then

$$X = \bigcup_{s=0}^{D_1-1} V_s = \bigcup_{s=0}^{D_2-1} V_s$$

and $\{V_s\}_{s=0}^{D_2-1}$ is a periodic partition of dynamical system $(X, f)$ of length $D_2$.

We set

$$W_j^{(m_2)} = \bigcup_{s \in \{0, \ldots, D_2 - 1\}, \ s \equiv j \pmod{m_2}} V_s, \quad j = 0, \ldots, m_2 - 1.$$ 

We receive periodic partition $\{W_j^{(m_2)}\}_{j=0}^{m_2-1}$ of dynamical system $(X, f)$ of length $m_2$ (see proof of Proposition 1.1).

We shall prove now, that this periodic partition is compatible with each of partitions $\{W_i^{(m_1)}\}$ and $\{W_k^{(n)}\}$.

On construction we have $V_0 = W_0^{(m_1)} \cap W_0^{(n)} \subseteq W_0^{(m_2)}$, therefore $V_0 = W_0^{(m_1)} \cap W_0^{(n)} \subseteq W_0^{(m_2)} \cap W_0^{(n)}$. Now from property (iv) of Definition 1.1 and from Corollary 1.4 it follows, that periodic partitions $\{W_i^{(m_2)}\}$ and $\{W_k^{(n)}\}$ are compatible.

Similarly, on construction $V_0 = W_0^{(m_1)} \cap W_0^{(n)} \subseteq W_0^{(m_1)}$ and $V_0 \subseteq W_0^{(m_2)}$. Hence, $V_0 \subseteq W_0^{(m_1)} \cap W_0^{(m_2)}$ and from Corollary 1.4 we receive, that partitions $\{W_i^{(m_1)}\}$ and $\{W_j^{(m_2)}\}$ are compatible. □

**Proof of Proposition 1.7.**

a) Let $d_k^l$ is the greatest common divisor of numbers $n_k$ and $m_1$. From condition 1) of Proposition it follows, that $n_k + l$ is divided by $d_k^l$ for all $l \in \mathbb{N}$. Therefore

$$d_1^l \leq d_2^l \leq \ldots \leq d_k^l \leq \ldots.$$ 

On the other hand, $d_k^l \leq m_1$ for all $k \in \mathbb{N}$. Hence, there exists $l \in \mathbb{N}$, such that $d_k^l = d_k^l$ when $k \geq l$.

By using Proposition 1.5, we find periodic partition $\{W_i^{(m_1)}\}_{i=0}^{m_1-1}$ of length $m_1$, which is compatible with the partition $\{W_k^{(n)}\}$. Show, that for every $k \in \mathbb{N}$ this partition is compatible with the partition $\{W_k^{(n)}\}$.

Let $k > l$. Then $d_k^l = d_k^l$. From Remark 1.10 it follows, that partitions $\{W_i^{(m_1)}\}$ and $\{W_k^{(n)}\}$ are compatible. We apply Lemma 1.3 to periodic partitions $\{W_i^{(m_1)}\}$, $\{W_k^{(n)}\}$ and $\{W_k^{(n)}\}$, and conclude that partitions $\{W_i^{(m_1)}\}$ and $\{W_k^{(n)}\}$ are compatible.
Let now $k < l$. Again from Remark 1.10 we derive, that partitions $\{W_{s_l}^{(n_l)}\}$ and $\{W_{s_k}^{(n_k)}\}$ are compatible. We apply now Lemma 1.2 to periodic partitions $\{W_{s_k}^{(n_k)}\}, \{W_{s_l}^{(n_l)}\}$ and $\{W_{i}^{(m_i)}\}$ and conclude that partitions $\{W_{s_k}^{(m_k)}\}$ and $\{W_{s_k}^{(n_k)}\}$ are compatible.

b) Let $d_k^2$ is the greatest common divisor of numbers $n_k$ and $m_2$. Repeating argument from item a), we consequence that there exists $\tau \in \mathbb{N}$, such that $d_k^2 = d_2^2$ when $k \geq \tau$.

To complete the proof of item b) it is enough to us now to find periodic partition of length $m_2$, which is compatible both with $\{W_{s_{\tau}}^{(n_{\tau})}\}$ and $\{W_{i}^{(m_1)}\}$. Then by repetition of argument from item a) we shall prove, that it is compatible with every $\{W_{s_k}^{(n_k)}\}, k \in \mathbb{N}$.

Consider triple of numbers $m_1, m_2, n_{\tau} \in \mathcal{P}(X, f)$. Designate for convenience by $d_k$ and $D_k$ the greatest common divisor and the least common multiple of numbers $n_{\tau}$ and $m_k$, $k = 1, 2$.

As $m_1$ divides $m_2$ on the condition of Proposition, then $d_1$ divides $d_2$ and $D_1$ divides $D_2$. Set

$$m = \frac{m_1}{d_1} \cdot d_2.$$

It is clear, that $m_1$ divides $m$, since the number $d_2/d_1$ is integer. On the other hand,

$$\frac{m_2}{m} = \frac{m_2}{m_1} \cdot \frac{d_1}{d_2} = \left( \frac{m_2n_{\tau}}{d_2} \right) \left( \frac{m_1n_{\tau}}{d_1} \right)^{-1} = \frac{D_2}{D_1} \in \mathbb{N},$$

that is $m$ divides $m_2$.

Let $d$ and $D$ are the greatest common divisor and the least common multiple of numbers $m$ and $n_{\tau}$.

Obviously, $d_1$ divides $d$ and $d$ divides $d_2$. Since the number $m_1/d_1$ is integer, then $m$ is divided by $d_2$ and $d_2$ is the common divisor of numbers $m$ and $n_{\tau}$. Hence, $d_2 = d$.

On the other hand,

$$D = \frac{m n_{\tau}}{d} = \frac{m_1 d_2}{d_1} \cdot \frac{n_{\tau}}{d_2} = \frac{m_1 n_{\tau}}{d_1} = D_1.$$

Proposition 1.1 gives us the inclusion $m \in \mathcal{P}(X, f)$, since $m$ divides $m_2$ and $m_2 \in \mathcal{P}(X, f)$.

Applying Lemma 1.4 to numbers $m_1, m, n_{\tau} \in \mathcal{P}(X, f)$ and to compatible periodic partitions $\{W_{i}^{(m_1)}\}$ and $\{W_{s_{\tau}}^{(n_{\tau})}\}$, we shall find periodic partition $\{W_{k}^{(m)}\}_{k=0}^{m-1}$ of length $m$, which is compatible with each of partitions $\{W_{i}^{(m_1)}\}$ and $\{W_{s_{\tau}}^{(n_{\tau})}\}$.

Having used Proposition 1.5, we find periodic partition $\{W_{j}^{(m_2)}\}_{j=0}^{m_2-1}$ of length $m_2$, compatible with partition $\{W_{k}^{(m)}\}$. We apply Lemma 1.3 to numbers $m, m_2, n_{\tau}$ and periodic partitions $\{W_{k}^{(m)}\}, \{W_{j}^{(m_2)}\}, \{W_{s_{\tau}}^{(n_{\tau})}\}$, and conclude that partitions $\{W_{j}^{(m_2)}\}$ and $\{W_{s_{\tau}}^{(n_{\tau})}\}$ are compatible.

In accord with Corollary 1.6, partition $\{W_{j}^{(m_2)}\}$ of space $X$ is the refinement of the partition $\{W_{k}^{(m)}\}$. Consequently it all the more is the refinement of the partition $\{W_{i}^{(m_1)}\}$. Again applying Corollary 1.6 we conclude, that partitions $\{W_{i}^{(m_1)}\}$ and $\{W_{j}^{(m_2)}\}$ are compatible.

Proposition 1.7 is completely proved. □

Remark 1.11 Let a regular sequence $\{W_{s_i}^{(n_i)}\}_{s_i=0}^{n_i-1}$ of periodic partitions is given and

$$\bigcap_{i \in \mathbb{N}} W_{r_i}^{(n_i)} \neq \emptyset$$

for certain sequence $\{r_i\}_{i \in \mathbb{N}}$. Then from Corollary 1.6 we receive

$$W_{r_{i_1}}^{(n_{i_1})} \supseteq W_{r_{i_2}}^{(n_{i_2})} \supseteq \ldots \supseteq W_{r_{i_l}}^{(n_{i_l})} \supseteq \ldots.$$
Next, if \( K \subseteq X \) is the closed subset of a compactum \( X \), such that \( K \cap W_{r_i}^{(n_i)} \neq \emptyset \) for every \( i \in \mathbb{N} \), then a sequence of nested compact sets 
\[
(K \cap W_{r_i}^{(n_i)}) \supseteq \ldots \supseteq (K \cap W_{r_i}^{(n_i)}) \supseteq \ldots
\]
has nonempty intersection, in other words
\[
K \cap \left( \bigcap_{i \in \mathbb{N}} W_{r_i}^{(n_i)} \right) \neq \emptyset.
\]

**Definition 1.7** Two regular sequences \( \{W_{s_i}^{(n_i)}\}_{s_i=0}^{n_i-1}, i \in \mathbb{N} \), and \( \{W_{r_j}^{(m_j)}\}_{r_j=0}^{m_j-1}, j \in \mathbb{N} \), of periodic partitions of dynamical system \((X,f)\) are called compatible, if periodic partitions \( \{W_{s_i}^{(n_i)}\} \) and \( \{W_{r_j}^{(m_j)}\} \) are compatible for all \( i, j \in \mathbb{N} \).

Immediately from Proposition 1.7 we receive the following

**Proposition 1.8** Let \( \{W_{s_i}^{(n_i)}\}_{s_i=0}^{n_i-1}, i \in \mathbb{N} \), is regular sequence of periodic partitions of dynamical system \((X,f)\).

Let \( m_j \in \mathcal{P}(X,f), j \in \mathbb{N} \), and \( m_j \) divides \( m_{j+1} \) for every \( j \in \mathbb{N} \).

Then there exists a regular sequence \( \{W_{r_j}^{(m_j)}\}_{r_j=0}^{m_j-1}, j \in \mathbb{N} \), of periodic partitions of dynamical system \((X,f)\), which is compatible with sequence \( \{W_{s_i}^{(n_i)}\}, i \in \mathbb{N} \).

**Remark 1.12** It follows immediately from Corollary 1.1, that if dynamical system \((X,f)\) is indecomposable, then any two regular sequences of periodic partitions of this dynamical system are compatible.

**Remark 1.13** Let a sequence of numbers \( \{n_i \in \mathcal{P}(X,f)\}_{i \in \mathbb{N}} \) is given, which satisfies to condition 1) of Definition 1.6.

If dynamical system \((X,f)\) is not indecomposable, then there exist two incompatible periodic partitions \( W^{(n_1)} \) and \( \tilde{W}^{(n_1)} \) of dynamical system \((X,f)\) (see Proposition 1.3 and Remark 1.5).

Obviously, regular sequences \( \{W^{(n_i)}\}_{i \in \mathbb{N}} \) and \( \{\tilde{W}^{(n_i)}\}_{i \in \mathbb{N}} \) of periodic partitions, built from partitions \( W^{(n_1)} \) and \( \tilde{W}^{(n_1)} \) accordingly with the help of inductive application of Proposition 1.5, are not compatible.

### 1.4 Periodic partitions and returnability of trajectories of dynamical system.

Consider some dynamical system \((X,f)\).

**Lemma 1.5** Let for certain recurrent point \( x \in X \) there exists a closed neighborhood \( U \), which satisfies to the following property: there exists \( n \in \mathbb{N} \), such that
\[
\bigcup_{k \in \mathbb{Z}} f^{kn}(x) \subset U.
\]

Let
\[
m = \min\{n \in \mathbb{N} \mid f^{nk}(x) \in U \ \forall k \in \mathbb{Z}\}. \tag{6}
\]

Then dynamical system \((\text{Orb}_f(x),f)\) has periodic partition \( \{W_i\}_{i=0}^{m-1} \) of length \( m \), such that \( x \in W_0 \subseteq U \).
Proof. Under Birkhoff Theorem the set $\overline{\text{Orb}_f(x)}$ is minimal set of d. s. $(X, f)$. In particular, the space $\overline{\text{Orb}_f(x)}$ is compact.

We assume, that homeomorphism $f$ is given on space $\overline{\text{Orb}_f(x)}$ with the topology induced from $X$ and we shall consider in this topology all sets which will arise in the proof.

Designate

$$W = \bigcup_{k \in \mathbb{Z}} f^{mk}(x), \quad \tilde{W} = \text{Int}W.$$ 

Consider two families of sets

$$W_0 = W, \quad W_i = f^i(W) = f(W_{i-1}), \quad i = 1, \ldots, m - 1; \quad \tilde{W}_i = \text{Int}W_i, \quad i = 0, \ldots, m - 1.$$ 

It is clear, that for the family $\{W_i\}$ conditions (ii) and (iv) of Definition 1.1 are fulfilled. All sets $W_i$ are closed, therefore condition (i) is an immediate corollary of conditions (ii) and (iii).

So, in order to prove Lemma it is enough to verify condition (iii) of Definition 1.1.

At first we shall show, that $\tilde{W}_i \neq \emptyset$, $i = 0, 1, \ldots, m - 1$. Really, under Baire category Theorem it follows from (iv) that at least one of the sets $\{W_i\}$ is not the set of I-st category (and has nonempty interior in $\overline{\text{Orb}_f(x)}$). Since $f$ is the homeomorphism, then all $W_i$ have nonempty interior, that is $\tilde{W}_i \neq \emptyset$.

Now we shall check the relation $\tilde{W}_i \cap \tilde{W}_j = \emptyset$, $i \neq j$.

Assume, that it is not the case. Let $V = \tilde{W}_i \cap \tilde{W}_j \neq \emptyset$ and $i < j$ for a determinancy. Since the set $\bigcup_{k \in \mathbb{Z}} f^{km+i}(x)$ is dense in $\tilde{W}_i$ on construction, then $f^{ms+i}(x) \in V$ for some $s \in \mathbb{Z}$.

$$f^{ms+i}(x) \in V \subseteq \tilde{W}_j \subseteq W_j = \bigcup_{k \in \mathbb{Z}} f^{km+j}(x),$$

therefore there exists a sequence $\{l_r\}_{r \in \mathbb{N}}$ of integers, such that $f^{ml_r+j}(x) \to f^{ms+i}(x)$ when $r \to \infty$. This implies, that for every $k \in \mathbb{Z}$ we have

$$f^{ml_r-s+k+1-i}(x) \to f^{km}(x).$$

That is

$$\bigcup_{k \in \mathbb{Z}} f^{km}(x) \subseteq \bigcup_{k \in \mathbb{Z}} f^{km+i-j}(x) = W_{j-i}.$$ 

Passing to closures, we shall receive $W_0 \subseteq W_{j-i}$. Changing roles of $i$ and $j$, we shall receive inverse inclusion. Therefore $W_0 = W_{j-i}$.

Remark, that

$$f^{(j-i)k}(x) \in f^{(j-i)k}(W_0) = f^{(j-i)(k-1)} \circ f^{j-i}(W_0) = f^{(j-i)(k-1)}(W_0) = \cdots = W_0 \subseteq U$$

when $k \geq 0$ and

$$f^{-k(j-i)}(x) \in f^{-k(j-i)}(W_0) = f^{-k(j-i)}(W_{j-i}) = f^{-(k+1)(j-i) \circ f^{-(j-i)}(W_{j-i}) = f^{(j-i)k}(W_0) = \cdots = W_0 \subseteq U$$

for $k < 0$ (we remind, that the set $U$ is closed on condition of Lemma). That is $f^{k(j-i)}(x) \in U$ for all $k \in \mathbb{Z}$.

\footnote{the Baire category Theorem is applicable in spaces, complete on Cech, and it is known that any compact set is space, complete on Cech (see [11])}
On our supposition \(0 < j - i < m\), so we have received contradiction with the choice of \(m\) (see relation (6)). Hence, \(\overline{W}_i \cap \overline{W}_j = \emptyset\) when \(i \neq j\).

We shall check now equalities \(\overline{W}_i = W_i\), \(i = 0, 1, \ldots, m - 1\).

It is easy to see, that

\[
\overline{W}_i = \text{Int} W_i = \text{Int}(f(W_{i-1})) = f(\text{Int}(W_{i-1})) = f(\overline{W}_{i-1}) \quad i = 1, \ldots, m - 1,
\]

and \(\overline{W}_0 = f(\overline{W}_{m-1})\), since \(f\) is homeomorphism. Designate

\[
Q = \bigcup_{i=0}^{m-1} \overline{W}_i = \bigcup_{i=0}^{m-1} f^i(\overline{W}_0).
\]

The set \(Q\) is open invariant subset of dynamical system \((\overline{\text{Orb}}_f(x), f)\). Let \(K = \overline{\text{Orb}}_f(x) \setminus Q\). Obviously, \(K\) is the closed invariant subset of this dynamical system.

The set \(\overline{\text{Orb}}_f(x)\) is minimal, therefore either \(K = \emptyset\) or \(K = \overline{\text{Orb}}_f(x)\). We already have proved, that \(Q \neq \emptyset\), hence \(K = \emptyset\) and

\[
\overline{\text{Orb}}_f(x) = \bigcup_{i=0}^{m-1} \overline{W}_i.
\]

Sets from the family \(\{\overline{W}_i\}\) are pairwise disjoint, so all of them are open-closed and \(\overline{W}_i = W_i\), \(i = 0, 1, \ldots, m - 1\). On proved above from this immediately follows condition (iii) of definition 1.1.

Lemma is completely proved. \(\Box\)

## 2 Ultranatural numbers and subsets of natural numbers.

### 2.1 Ultranatural numbers.

**Definition 2.1** Let \(\mathbb{B} \subset \mathbb{N}\) is the set of all prime numbers ordered by increment. A sequence

\[
N = (N_2, N_3, \ldots, N_p, \ldots) , \quad N_p \in \mathbb{Z}_+ \cup \{\infty\} , \quad p \in \mathbb{B} ,
\]

is called ultranatural number.

The set of all ultranatural numbers we shall designate by \(\Sigma\).

We introduce the relation of partial order on \(\Sigma\). Say that

\[
M \leq N , \quad M, N \in \Sigma ,
\]

if \(M_p \leq N_p\) for every \(p \in \mathbb{B}\) (we shall regard that \(k \leq \infty\) for any \(k \in \mathbb{Z}_+\)). Elementary immediate verification shows the correctness of this definition.

Next, we introduce binary operation on \(\Sigma\). For \(M = (M_p)\) and \(N = (N_p)\) we set

\[
M \cdot N = K = (K_p) ;
\]

\[
K_p = \begin{cases} M_p + N_p , & \text{if } M_p \neq \infty \text{ and } N_p \neq \infty , \\ \infty , & \text{otherwise} . \end{cases} , \quad p \in \mathbb{B} .
\]

It is trivially checked, that \((\Sigma, \cdot)\) is a semigroup with unity \(E = (E_p = 0)\).
Remark 2.1 Easy immediate verification shows, that the equation \( M \cdot X = N \) has a solution in \((\Sigma, \cdot)\) only when \( M \leq N \).
However, this equation can have more than one solution.

Example 2.1 Let \( M = N = (N_p) \),

\[
N_p = \begin{cases} 
\infty, & \text{when } p = 2, \\
0, & \text{for } p \neq 2.
\end{cases}
\]

Then \( X^{(n)} = (X_p^{(n)}) \),

\[
X_p^{(n)} = \begin{cases} 
n, & \text{for } p = 2, \\
0, & \text{when } p \neq 2.
\end{cases}
\]
is a solution of the equation \( M \cdot X = N \) for every \( n \in \mathbb{Z} \cup \{\infty\} \).

In just the same way as in the semigroup \((\mathbb{N}, \cdot)\) we can well define the greatest common divisor and least common multiple for any two \( M, N \in \Sigma \). It is easy to see, that

\[
\gcd(M, N) = (d_p), \quad d_p = \min(M_p, N_p), \quad p \in \mathbb{P}, \tag{7}
\]

\[
\lcm(M, N) = (D_p), \quad D_p = \max(M_p, N_p), \quad p \in \mathbb{P}. \tag{8}
\]

Here we use the following agreements:

\[
\max(a, \infty) = \infty, \quad \min(a, \infty) = a, \quad a \in \mathbb{Z}^+ \cup \{\infty\}.
\]

We define monomorphism \( \Phi_0 : (\mathbb{N}, \cdot) \to (\Sigma, \cdot) \).
Let \( n \in \mathbb{N} \). Consider factorization

\[
n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}
\]
of the number \( n \) on prime factors (we regard that \( p_i \neq p_j \) when \( i \neq j \)). Set \( \Phi_0(n) = (\Phi_0(n)_p) \),

\[
\Phi_0(n)_p = \begin{cases} 
\alpha_i, & \text{when } p = p_i \in \{p_1, \ldots, p_k\}, \\
0, & \text{otherwise}.
\end{cases}
\]

Remark 2.2 Mark that for all \( m, n \in \mathbb{N} \) we have

\[
\Phi_0(\gcd(m, n)) = \gcd(\Phi_0(m), \Phi_0(n)),
\]

\[
\Phi_0(\lcm(m, n)) = \lcm(\Phi_0(m), \Phi_0(n)).
\]

2.2 Regular subsets of natural numbers.

Let \( A \subseteq \mathbb{N} \). For every \( p \in \mathbb{P} \) let

\[
\Phi(A)_p = \sup\{k \in \mathbb{Z}^+ \mid \exists a \in A : p^k \text{ divides } a\} = \sup_{a \in A} \Phi_0(a)_p.
\]

Designate by \( \Phi \) the map \( \Phi : A \mapsto \Phi(A) = (\Phi(A)_p) \) from class of all nonempty subsets of \( \mathbb{N} \) to the set \( \Sigma \) of ultranatural numbers.

Remark 2.3 It is easily checked that order relation on \( \Sigma \) defined above turns map \( \Phi \) into isotonic map, that is for any \( A, B \subseteq \mathbb{N} \) inclusion \( A \subseteq B \) implies \( \Phi(A) \leq \Phi(B) \).
Example 2.2 Let $A = \{a\}$ is a singleton. From definition it easily follows, that $\Phi(\{a\}) = \Phi_0(a)$.

Example 2.3 Let $A = \{a_1, \ldots, a_j\}$ is a finite subset of $\mathbb{N}$. Consider factorizations of numbers $a_1, \ldots, a_j$ on prime factors

$$a_i = \prod_{p \in \mathcal{P}} p^{n_p(i)}, \quad i = 1, \ldots, j$$

(here $n_p(i) \in \mathbb{Z}_+$, $p \in \mathcal{P}$, $i = 1, \ldots, j$).

By definition

$$\Phi(A)_p = \max\{n_p(1), \ldots, n_p(j)\}, \quad p \in \mathcal{P},$$

in other words $\Phi(\{a_1, \ldots, a_j\}) = \Phi(\{D\})$, where $D \in \mathbb{N}$ is the least common multiple of numbers $a_1, \ldots, a_j$.

Remark 2.4 Let $A \subseteq \mathbb{N}$. Immediately from definition follows, that $\Phi(\{a\}) \leq \Phi(A)$ for every $a \in A$.

Remark 2.5 Relations (7) and (8) imply, that for any nonempty $A, B \subseteq \mathbb{N}$

$$\Phi(A \cup B) = \text{lcm}(\Phi(A), \Phi(B)).$$

In addition if $A \cap B \neq \emptyset$, then

$$\Phi(A \cap B) \leq \text{gcd}(\Phi(A), \Phi(B)).$$

Definition 2.2 We call a nonempty subset $A \subseteq \mathbb{N}$ regular, if it satisfies to the following conditions:

(i) if $a \in A$ and $d \in \mathbb{N}$ divides $a$, then $d \in A$;

(ii) for any $a, b \in A$ their least common multiple $D$ also is contained in $A$.

We designate the family of all regular sets by $\mathcal{R}$.

Remark 2.6 It follows from Propositions 1.1 and 1.2, that for any dynamical system $(X, f)$ the set $\mathcal{P}(X, f)$ is regular.

Lemma 2.1 Let $A \in \mathcal{R}$. Then

$$A = \{a \in \mathbb{N} \mid \Phi(\{a\}) \leq \Phi(A)\} = \{a \in \mathbb{N} \mid \Phi_0(a) \leq \Phi(A)\}.$$  

Proof. Let $a \in \mathbb{N}$ and $\Phi(\{a\}) \leq \Phi(A)$. Suppose

$$a = p_1^{n_1} \cdots p_k^{n_k}$$

is the factorization of $a$ on prime factors. By definition of $\Phi$ there exist such $b_1, \ldots, b_k \in A$, that $p_i^{n_i}$ divides $b_i$, $i = 1, \ldots, k$. Assume $b$ is the least common multiple of numbers $b_1, \ldots, b_k$. Then $a$ divides $b$. But $b \in A$ by definition of regular set. Hence and $a \in A$. That is

$$A \supseteq \{a \in \mathbb{N} \mid \Phi(\{a\}) \leq \Phi(A)\}.$$  

Inverse inclusion

$$A \subseteq \{a \in \mathbb{N} \mid \Phi(\{a\}) \leq \Phi(A)\}.$$  

follows from Remark 2.4. □

An immediate corollary of Lemma 2.1 is following
Proposition 2.1  Mapping

\[ \Phi|_\mathcal{R}: \mathcal{R} \rightarrow \Sigma \]

is bijective.

Remark 2.7  Let \( A, B \in \mathcal{R} \). Obviously, \( 1 \in A \cap B \neq \emptyset \). From Lemma 2.1 the relation immediately follows

\[ \Phi(A \cap B) = \gcd(\Phi(A), \Phi(B)) . \]

Definition 2.3  Let \( A \subseteq \mathbb{N} \). Call a sequence \( \{a_i \in A\}_{i \in \mathbb{N}} \) regular, if \( a_i \) divides \( a_{i+1} \) for every \( i \in \mathbb{N} \).

Remark 2.8  It follows from Remark 2.3, that for any \( A \subseteq \mathbb{N} \) and any regular sequence \( \{a_i \in A\}_{i \in \mathbb{N}} \) we have the inequality

\[ \Phi(\{a_i | i \in \mathbb{N}\}) \leq \Phi(A) . \]

Proposition 2.2  Let \( A \in \mathcal{R} \). Then there exists a regular sequence \( \{b_i \in A\}_{i \in \mathbb{N}} \), such that

\[ \Phi(\{b_i | i \in \mathbb{N}\}) = \Phi(A) . \]

Proof.  Since \( A \subseteq \mathbb{N} \) is at most the enumerable set, we can enumerate all elements of \( A \) with the help of natural numbers, \( A = \{a_1, a_2, \ldots\} \). Let \( b_1 = a_1 \), \( b_i \) the least common multiple of numbers \( a_i \) and \( b_{i-1} \) for \( i > 1 \).

It is clear, that

\[ \Phi(\{b_i\}) \geq \Phi(\{a_1, \ldots, a_i\}), \quad i \in \mathbb{N} , \]

Therefore

\[ \Phi(\{b_i | i \in \mathbb{N}\}) \geq \Phi(\{a_i | i \in \mathbb{N}\}) = \Phi(A) . \]

On the other hand, \( b_i \in A, i \in \mathbb{N} \), since \( A \) is regular. Consequently, \( \{b_i\}_{i \in \mathbb{N}} \subseteq A \) and

\[ \Phi(\{b_i | i \in \mathbb{N}\}) \leq \Phi(A) . \]

In order to complete the proof it suffices to note, that on construction \( b_i \) divides \( b_{i+1} \), \( i \in \mathbb{N} \), hence the sequence \( \{b_i\}_{i \in \mathbb{N}} \) is regular. \( \square \)

Proposition 2.3  Let \( A \subseteq \mathbb{N} \). Suppose two regular sequences \( \{a_i \in A\}_{i \in \mathbb{N}} \) and \( \{b_j \in A\}_{j \in \mathbb{N}} \) are given.

The following conditions are equivalent:

1) \( \Phi(\{a_i | i \in \mathbb{N}\}) \leq \Phi(\{b_j \in \mathbb{N}\}) \);

2) for every \( i \in \mathbb{N} \) there exists \( j \in \mathbb{N} \), such that \( a_i \) divides \( b_j \).

Proof.  1) Let \( \Phi(\{a_i | i \in \mathbb{N}\}) \leq \Phi(\{b_j \in \mathbb{N}\}) \).

Consider the factorization

\[ a = a_i = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \]

of \( a_i \) on prime factors.

From Example 2.2 and Remark 2.4 it follows, that for \( m = 1, \ldots, k \)

\[ \alpha_m = \Phi(\{a\})_{p_m} \leq \Phi(\{a_i | i \in \mathbb{N}\})_{p_m} \leq \Phi(\{b_j \in \mathbb{N}\})_{p_m} . \]
Therefore, for every \( p_m, m = 1, \ldots, k \), there exists \( j_m \in \mathbb{N} \), such that \( p_m^{\alpha_m} \) divides \( b_{j_m} \).

Without loss of generality, we shall suppose that

\[
j_1 \leq j_2 \leq \ldots \leq j_k.
\]

Sequence \( \{b_j\} \) is regular, therefore \( b_{j_m} \) divides \( b_{j_k}, m = 1, \ldots, k \). Hence, \( p_m^{\alpha_m} \) divides \( b_{j_k}, m = 1, \ldots, k \).

Numbers \( p_1, \ldots, p_k \) are relatively prime on construction, therefore \( p_1^{\alpha_1} \ldots p_k^{\alpha_k} = a \) divides \( b_{j_k} \).

2) Let now assume that condition 2) of Proposition 2.3 is satisfied.

Suppose that for some \( p \in \mathbb{B} \) the inequality is fulfilled

\[
\Phi(\{a_i \mid i \in \mathbb{N}\})_p \geq \Phi(\{b_j \mid j \in \mathbb{N}\})_p.
\]

Then \( n = \Phi(\{b_j \mid j \in \mathbb{N}\})_p < \infty \) and on definition of \( \Phi \)

\[
b_j = p^{n_j} \tilde{b}_j, \quad n_j \leq n, \quad \gcd(p, \tilde{b}_j) = 1
\]

for every \( j \in \mathbb{N} \).

On the other hand, \( \Phi(\{a_i \mid i \in \mathbb{N}\})_p \geq n + 1 \). Hence there exists \( i_0 \in \mathbb{N} \), such that \( p^{n+1} \) divides \( a_{i_0} \). We take an advantage now of condition 2) of Proposition 2.3 and find \( j_0 \in \mathbb{N} \), such that \( a_{i_0} \) divides \( b_{j_0} \). All the more, \( p^{n+1} \) divides \( b_{j_0} \).

However, on construction \( \gcd(p^{n+1}, b_{j_0}) = p^{n_{j_0}} \leq p^n \).

The obtained contradiction proves that

\[
\Phi(\{a_i \mid i \in \mathbb{N}\})_p \leq \Phi(\{b_j \mid j \in \mathbb{N}\})_p \quad p \in \mathbb{B},
\]

That is condition 1) of Proposition 2.3 is valid. \( \Box \)

**Corollary 2.1** Let \( A \subseteq \mathbb{N} \). Assume \( \{a_i \in A \}_{i \in \mathbb{N}} \) is regular sequence.

For any subsequence \( \{b_j\}_{j \in \mathbb{N}} \) of \( \{a_i \}_{i \in \mathbb{N}} \) the equality is valid

\[
\Phi(\{a_i \mid i \in \mathbb{N}\}) = \Phi(\{b_j \mid j \in \mathbb{N}\}).
\]

### 3 Odometers and connected constructions.

#### 3.1 Definition of odometer.

We fix regular infinitely growing sequence \( \{n_i \in \mathbb{N}\}_{i \in \mathbb{N}} \).

Consider a sequence of finite cyclic groups \( \mathbb{Z}_{n_i} = \mathbb{Z}/n_i\mathbb{Z} \) and group homomorphisms

\[
\varphi_i : \mathbb{Z}_{n_{i+1}} \rightarrow \mathbb{Z}_{n_i},
\]

\[
\varphi_i : 1 \mapsto 1.
\]

Take the inverse limit \( A = \lim_{i \rightarrow \infty} \mathbb{Z}_{n_i} \) of this sequence of groups and homomorphisms. We shall obtain an Abelian group \( (A, +) \).

We endow each set \( \mathbb{Z}_{n_i} = \{0, 1, \ldots, n_i - 1\} \) with the discrete topology. Each of maps \( \varphi_i \) is continuous in this topology. Space \( A \) with the topology \( \mathcal{T} \) of inverse limit is homeomorphic to Cantor set \( \Gamma \).

It is easy to see, that in the group \( (A, +) \) operation of addition and passage to opposite element are continuous in the topology \( \mathcal{T} \), thus \( A \) turns to be the continuous group.
Remark 3.1 We remind, that the inverse limit $A = \lim_{i \to \infty} \text{inv } \mathbb{Z}_{n_i}$ could be imagined as a subset

$$A = \{ \vec{a} = (a_i \in \mathbb{Z}_{n_i}) \mid \varphi_i(a_{i+1}) = a_i, \ i \in \mathbb{N} \} \quad (9)$$

of the direct product

$$\prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i}. \quad \ (10)$$

In this notation the operation of addition in $A$ is defined component-wise, that is $\vec{a} + \vec{b} = (a_i + b_i)$ for any $\vec{a} = (a_i), \vec{b} = (b_i) \in A$.

It is known, that the topology of direct product (10) is set with the help of basis, which consists of so-called cylindrical sets

$$U(x_{i_1}, \ldots, x_{i_k}) = \{(a_i) \mid a_{i_s} = x_{i_s}, \ s = 1, \ldots, k \}; \ x_{i_s} \in \mathbb{Z}_{n_{i_s}}, \ i_1 < \ldots < i_k, \ k \in \mathbb{N}.$$

From definition of $A$ (see relation (9)) it is easy to see, that

$$U(x_{i_1}, \ldots, x_{i_k}) \cap A = U(x_{i_k}) \cap A$$

for any $k \in \mathbb{N}, i_1 < \ldots < i_k$ and $x_{i_s} \in \mathbb{Z}_{n_{i_s}}$. So, the family of sets

$$V_{x_j} = U(x_j) \cap A = \{(a_i) \in A \mid a_j = x_j\} = \{(a_i) \in A \mid a_j = x_j, \ a_k = \varphi_k \circ \ldots \circ \varphi_{j-1}(x_j) \text{ for } k < j\}; \ j \in \mathbb{N}, \ x_j \in \mathbb{Z}_{n_j} \quad (11)$$

appears to be basis of topology of space $A$.

The natural metric $d : A \times A \to \mathbb{R}_+$ on $A$, associated with the sequence $\{n_i\}$, is defined by the following correlation

$$d(\vec{x}, \vec{y}) = \frac{1}{n_m}, \quad m = \min \{i \in \mathbb{N} \mid x_k = y_k \text{ when } k < i \text{ and } x_i \neq y_i\}.$$

The correctness of this definition is verified immediately.

Consider an element $\vec{e} = (1) = (1, \ldots, 1, \ldots) \in A$. It is called generator of the group $A$ and the cyclical subgroup $\langle \vec{e} \rangle$, generated by this element is dense in $A$ in the topology $\mathcal{T}$.

The translation map

$$g : A \to A,$$

$$g : \vec{x} \mapsto \vec{x} + \vec{e}$$

obviously is homeomorphism.

**Definition 3.1** Dynamical system $(A, g)$ is called odometer.

**Remark 3.2** From that fact the subgroup $\langle \vec{e} \rangle$ is dense in $A$ it immediately follows, that each trajectory of dynamical system $(A, g)$ is dense in $A$. In other words odometer is always minimal dynamical system.

**Lemma 3.1** For any $k \in \mathbb{N}$ and $x_k \in \mathbb{Z}_{n_k}$ a family of sets $\{W_{j}^{(nk)} = V_{x_k+j} \}_{j=0,\ldots,n_i-1}$ is periodic partition of dynamical system $(A, g)$ of length $n_k$. 
Proof. Obviously,
\[ A = \bigcup_{s \in Z_{n_k}} V_s = \bigcup_{j \in Z_{n_k}} V_{x_k+j}. \]

Hence, for the family \( \{ W_j^{(n_k)} \} \) condition (iv) of Definition 1.1 is fulfilled.

Since all sets \( V_{x_k+j}, j \in Z_{n_k}, \) are open by definition and pairwise disjoint, the collection \( \{ W_j^{(n_k)} \} \) satisfies also to conditions (i) and (iii) of definition of periodic partition.

To complete the proof it remains to verify that \( g(V_{a_k}) = V_{a_k+1} (1 \in Z_{n_k}) \) for every \( a_k \in Z_{n_k}. \)

Let \( \vec{b} = (b_i) \in V_{a_k}. \) Then \( b_k = a_k \) and \( g(\vec{b}) = \vec{b} + \vec{e} = (b_i + 1) \in V_{a_k+1}. \) Therefore, \( g(V_{a_k}) \subseteq V_{a_k+1}. \)

Conversely, let \( \vec{c} = (c_i) \in V_{a_k+1}. \) Then \( c_k = a_k + 1 \) and \( g^{-1}(\vec{c}) = \vec{c} - \vec{e} = (c_i - 1) \in V_{a_k}. \)

Hence, \( g(V_{a_k}) \supseteq V_{a_k+1}. \) □

Remark 3.3 It immediately follows from relation (11), that \( d(\vec{x}, \vec{y}) = d(g(\vec{x}), g(\vec{y})) \)

for all \( \vec{x}, \vec{y} \in A. \)

3.2 Regular sequences of periodic partitions and associated partitions on a phase space of dynamical system.

Let \( (X, f) \) be a dynamical system with compact phase space, \( \{ n_i \in P(X, f) \}_{i \in \mathbb{N}} \) is an unlimited regular sequence. Let \( \{ W_{i}^{(n_i)} \} \) be a regular sequence of periodic partitions of dynamical system \( (X, f). \)

Let \( x \in X. \) Remark, that in the strength of properties of periodic partitions for every \( i \in \mathbb{N} \) there exists unique \( \alpha_i(x) \in Z_{n_i}, \) such that \( x \in W_{\alpha_i(x)}^{(n_i)}. \) In other words the map is correctly defined

\[ F : X \to \prod_{i \in \mathbb{N}} Z_{n_i}, \]

\[ F : x \mapsto (\alpha_i(x)) \ , \ x \in X. \]

We associate with every \( x \in X \) a subset

\[ H(x) = \bigcap_{i \in \mathbb{N}} W_{\alpha_i(x)}^{(n_i)} \ni x \]

of the space \( X. \) It follows from Definitions 1.1, 1.6 and Corollary 1.6, that

1) all \( H(x) \) are nonempty closed sets;

2) \( H(x) = H(y) \) if \( F(x) = F(y) \) and \( H(x) \cap H(y) = \emptyset \) if \( F(x) \neq F(y); \)

3) \( F(f^{\pm 1}(x)) = F(x) \pm \vec{e} \) for all \( x \in X \) (remind, that \( \vec{e} = (1, 1, \ldots, 1, \ldots) \in A). \)

For every \( \vec{a} = (a_i) \in F(X) \) we fix \( x \in F^{-1}(\vec{a}) \) and designate \( H_{\vec{a}} = H(x). \) It follows from 2), that the set \( H_{\vec{a}} \) does not depend on a choice of \( x \in F^{-1}(\vec{a}). \)
From 1) and 2) it immediately follows, that the family of sets $\mathcal{H} = \{H_\vec{a} : \vec{a} \in F(X) = \text{zer}(F)\}$ is partition of the space $X$, elements of which are pre-images of points of space $F(X)$. Also diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{F} & F(X) \\
pr \downarrow & & \\
X/\text{zer}(F) & \xrightarrow{\text{fact}} & X/\mathcal{H} \xrightarrow{\text{fact}} F(X)
\end{array}
$$

**Proposition 3.1** $F$ is the continuous map.

**Proof.** Consider a subbasis of topology

$$U_{x_j} = \{\vec{a} = (a_i) \in \prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i} \mid a_j = x_j\}, \quad x_j \in \mathbb{Z}_{n_j}, \quad j \in \mathbb{N}$$

of the space $\prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i}$.

The easy immediate verification shows, that

$$F^{-1}(U_{x_j}) = W^{(n_j)}_{x_j}, \quad x_j \in \mathbb{Z}_{n_j}, \quad j \in \mathbb{N}.$$ 

To complete the proof it suffices to recollect, that all sets $W^{(n_j)}_{x_j}$ are open in $X$ by definition. □

$X$ is the compact set, fact $F$ is continuous bijective map of $X/\mathcal{H}$ on $F(X)$ and space $F(X)$ is Hausdorff, therefore fact $F$ is the homeomorphism (see [11]).

For every $\vec{a} \in F(X)$ from 2) and 3) the equality is easily received $f(F^{-1}(\vec{a})) = F^{-1}(\vec{a} + \vec{e})$. Thus, if we designate

$$g : F(X) \to F(X), \quad g : \vec{a} \mapsto \vec{a} + \vec{e};$$

$$\bar{f} = \text{fact } f : X/\mathcal{H} \to X/\mathcal{H}, \quad \bar{f} : H_\vec{a} \mapsto H_{\vec{a} + \vec{e}};$$

then we receive the commutative diagram

$$
\begin{array}{ccc}
(X, f) & \xrightarrow{pr} & (F(X), g) \\
\downarrow & & \\
(X/\mathcal{H}, \bar{f}) & \xrightarrow{\text{fact}} & (F(X), g)
\end{array}
$$

We now ask the question: what is the set $F(X)$?

We fix $x \in X$ and consider the set

$$F(\text{Orb}_f(x)) = \{F(x) + n\vec{e} \mid n \in \mathbb{Z}\} = F(x) + \langle \vec{e} \rangle.$$ 

It is clear, that $F(x) + \langle \vec{e} \rangle \subseteq F(\text{Orb}_f(x)) \subseteq F(x) + \langle \vec{e} \rangle = F(x) + \langle \vec{e} \rangle = F(x) + A$ ($A$ is adic group constructed on the sequence $\{n_i\}$, see above).

Since $\text{Orb}_f(x)$ is compact set, then $F(\text{Orb}_f(x))$ is closed in $\prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i}$. The set $F(x) + \langle \vec{e} \rangle$ is dense in $F(x) + A$, therefore $F(\text{Orb}_f(x)) = F(x) + A$.

Let now $y$ is another point of the space $X$. $\text{Orb}_f(x)$ is the closed invariant subset of dynamical system $(X, f)$. Hence $W^{(n_i)}_{s_i} \cap \text{Orb}_f(x) \neq \emptyset$ for all $i \in \mathbb{N}, s_i \in \mathbb{Z}_{n_i}$ (see Remark 1.3).
Let $F(y) = (\beta_i)$. It follows from Remark 1.11 that $H(\beta_i) \cap \text{Orb}_f(x) \neq \emptyset$ and $F(y) = F(H(\beta_i)) = F(\text{Orb}_f(x)) = F(x) + A$.

As a result we receive

$$F(X) = F(x) + A,$$

In other words $F(X)$ is the coset of group $\prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i}$ on the subgroup $A$.

**Remark 3.4** Obviously, $F(X) = A$ if and only if

$$\bigcap_{i \in \mathbb{N}} W_0^{(n_i)} \neq \emptyset. \quad (13)$$

**Definition 3.2** A regular sequence $\{W^{(n_i)}\}_{i \in \mathbb{N}}$ of periodic partitions of dynamical system $(X, f)$ is called coherent, if it satisfies to the relation (13).

From Proposition 1.6, Remark 1.5 and construction, given above, we get

**Proposition 3.2** Let $(X, f)$ is a dynamical system with Hausdorff compact phase space.

For any unlimited regular sequence $\{n_i \in \mathcal{P}(X, f)\}_{i \in \mathbb{N}}$ there exists a projection $\pi : (X, f) \to (A, g)$ onto odometer $(A, g)$, constructed on the sequence $\{n_i\}_{i \in \mathbb{N}}$.

Let now $\{n_i \in \mathcal{P}(X, f)\}_{i \in \mathbb{N}}$ is a regular sequence and let $\{W^{(n_i)}\}$ and $\{\tilde{W}^{(n_i)}\}$ are two compatible regular sequences of periodic partitions of dynamical system $(X, f)$. Then (see Remark 1.5 and Definition 1.3) periodic partitions $W^{(n_i)}$ and $\tilde{W}^{(n_i)}$ are equivalent for every $i \in \mathbb{N}$. From this we immediately conclude, that

$$\mathfrak{F} = \mathfrak{F}(\{W^{(n_i)}\}_{i \in \mathbb{N}}) = \mathfrak{F}(\{\tilde{W}^{(n_i)}\}_{i \in \mathbb{N}}) = \tilde{\mathfrak{F}}.$$

**Proposition 3.3** Let $\{W^{(n_i)}\}_{i \in \mathbb{N}}$ is a regular sequence of periodic partitions of a dynamical system $(X, f)$.

The family of sets $\{pr(W^{(n_i)}_{s_i}) \mid i \in \mathbb{N} \ s_i \in \mathbb{Z}_{n_i}\}$ satisfies to the following properties:

1) it is the regular sequence of periodic partitions of the dynamical system $(X/\mathfrak{F}, \tilde{f})$;

2) it is the basis of topology on space $X/\mathfrak{F}$.

**Proof.** It immediately follows from what has been said that the sequence $\{W^{(n_i)}\}$ could be considered as coherent.

Now proposition follows from the relations

$$W^{(n_i)}_{s_i} = F^{-1}(V_{s_i}), \quad s_i \in \mathbb{Z}_{n_i}, \ i \in \mathbb{N}$$

(see formula (11), Lemma 3.1 and Corollary 1.6) and from commutative diagram (12), the lower arrow in which is homeomorphism. □

**Proposition 3.4** Let $\{n_i \in \mathcal{P}(X, f)\}_{i \in \mathbb{N}}$ and $\{m_j \in \mathcal{P}(X, f)\}_{j \in \mathbb{N}}$ are two regular sequences, such that $\Phi(\{n_i \mid i \in \mathbb{N}\}) \leq \Phi(\{m_j \mid j \in \mathbb{N}\})$.

Let regular sequences $\{W^{(n_i)}\}_{i \in \mathbb{N}}$ and $\{\tilde{W}^{(m_j)}\}_{j \in \mathbb{N}}$ of periodic partitions of dynamical system $(X, f)$ are compatible.

Then the partition $\tilde{\mathfrak{F}}$ of space $X$, which is induced by sequence $\{\tilde{W}^{(m_j)}\}$, is the refinement of the partition $\mathfrak{F}$, induced by the sequence $\{W^{(n_i)}\}$.
Proof. We fix \( x \in X \). There exist such \( (\alpha_i) \in \prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i} \) and \( (\beta_j) \in \prod_{j \in \mathbb{N}} \mathbb{Z}_{m_j} \), that
\[
x \in \bigcap_{i \in \mathbb{N}} W_{\alpha_i}^{(n_i)} \cap \bigcap_{j \in \mathbb{N}} \tilde{W}_{\beta_j}^{(m_j)}.\]

According to Proposition 2.3, for every \( i \in \mathbb{N} \) there exists \( k(i) \in \mathbb{N} \), such that \( n_i \) divides \( m_{k(i)} \). Since periodic partitions \( W^{(n_i)} \) and \( \tilde{W}^{(m_{k(i)})} \) are compatible, then by Corollary 1.6 second of them is the refinement of first one.
Thus, \( \tilde{W}_{\beta_{k(i)}}^{(m_{k(i)})} \subseteq W_{\alpha_i}^{(n_i)} \) for every \( i \in \mathbb{N} \). Therefore, we have
\[
H_{(\alpha_i)} = \bigcap_{i \in \mathbb{N}} W_{\alpha_i}^{(n_i)} \supseteq \bigcap_{i \in \mathbb{N}} \tilde{W}_{\beta_{k(i)}}^{(m_{k(i)})} \supseteq \bigcap_{j \in \mathbb{N}} \tilde{W}_{\beta_j}^{(m_j)} = \tilde{H}_{(\beta_j)}.\]

By virtue of arbitrariness in a choice of \( x \in X \) we conclude from what was said above, that for an arbitrary \( (\alpha_i) \in F(X) \) and \( (\beta_j) \in \tilde{F}(X) \) either \( H_{(\alpha_i)} \cap \tilde{H}_{(\beta_j)} = \emptyset \) or \( H_{(\alpha_i)} \supseteq \tilde{H}_{(\beta_j)} \). \( \square \)

Corollary 3.1 If in the conditions of Proposition 3.3 the equality takes place \( \Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\{m_j \mid j \in \mathbb{N}\}) \), then partitions \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) of the space \( X \) coincide.

Proposition 3.5 Let \( \{n_i \in \mathcal{P}(X,f)\}_{i \in \mathbb{N}} \) and \( \{m_j \in \mathcal{P}(X,f)\}_{j \in \mathbb{N}} \) are two regular sequences.
Let \( \{W_{\alpha_i}^{(n_i)}\}_{i \in \mathbb{N}} \) and \( \{\tilde{W}_{\beta_j}^{(m_j)}\}_{j \in \mathbb{N}} \) are regular sequences of periodic partitions of dynamical system \( (X,f) \). Assume that \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) are partitions of the space \( X \), induced by these sequences.
Let sequences \( \{W_{\alpha_i}^{(n_i)}\} \) and \( \{\tilde{W}_{\beta_j}^{(m_j)}\} \) are not compatible.
Then \( H(x) \setminus \tilde{H}(x) \neq \emptyset \) and \( \tilde{H}(x) \setminus H(x) \neq \emptyset \) for every \( x \in X \).

Proof. Let \( x \in X \). There exist such \( (\alpha_i) \in \prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i} \) and \( (\beta_j) \in \prod_{j \in \mathbb{N}} \mathbb{Z}_{m_j} \), that
\[
H(x) = H_{(\alpha_i)} = \bigcap_{i \in \mathbb{N}} W_{\alpha_i}^{(n_i)}, \quad \tilde{H}(x) = \tilde{H}_{(\beta_j)} = \bigcap_{j \in \mathbb{N}} \tilde{W}_{\beta_j}^{(m_j)}.\]

On definition \( H(x) \cap \tilde{H}(x) \neq \emptyset \).

According to conditions of Proposition there exist such \( k, l \in \mathbb{N} \), that periodic partitions \( W^{(n_k)} \) and \( \tilde{W}^{(m_l)} \) are not compatible. That is
\[
x \in A_{k,l} = \bigcup_{t \in \mathbb{Z}} f^t \left(W_{\alpha_k}^{(n_k)} \cap \tilde{W}_{\beta_l}^{(m_l)}\right) \neq X.\]

So, the space \( X \) falls into the union of two disjoint closed invariant sets \( A_{k,l} \) and \( B_{k,l} = X \setminus A_{k,l} \) of dynamical system \( (X,f) \) (see Proposition 1.4).
Obviously, \( H(x) \cap \tilde{H}(x) \subseteq W_{\alpha_k}^{(n_k)} \cap \tilde{W}_{\beta_l}^{(m_l)} \subseteq A_{k,l} \).

However,
\[
H(x) \setminus \tilde{H}(x) \supseteq H(x) \cap B_{k,l} = \bigcap_{i \in \mathbb{N}} \left(W_{\alpha_i}^{(n_i)} \cap B_{k,l}\right) \neq \emptyset
\]
(see Remarks 1.3 and 1.11). Similarly, \( \tilde{H}(x) \setminus H(x) \neq \emptyset \). \( \square \)
Proposition 3.6 Let \( \{n_i \in \mathcal{P}(X,f) \}_{i \in \mathbb{N}} \) and \( \{m_j \in \mathcal{P}(X,f) \}_{j \in \mathbb{N}} \) are two regular sequences.

Let \( \{W^{(n_i)} \}_{i \in \mathbb{N}} \) and \( \{\widetilde{W}^{(m_j)} \}_{j \in \mathbb{N}} \) are regular sequences of periodic partitions of dynamical system \((X,f)\). Assume \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) are the partitions of space \( X \), induced by these sequences.

If there exists \( x \in X \), such that
\[
H(x) = \bigcap_{i \in \mathbb{N}} W^{(n_i)}_{\alpha_i(x)} \supseteq \bigcap_{j \in \mathbb{N}} \widetilde{W}^{(m_j)}_{\alpha_j(x)} = \tilde{H}(x),
\]
then the partition \( \tilde{\mathcal{H}} \) is refinement of the partition \( \mathcal{H} \), sequences \( \{W^{(n_i)}\} \) and \( \{\widetilde{W}^{(m_j)}\} \) are compatible and \( \Phi(\{n_i \mid i \in \mathbb{N}\}) \leq \Phi(\{m_j \mid j \in \mathbb{N}\}) \).

Corollary 3.2 Let \( \{W^{(n_i)} \}_{i \in \mathbb{N}} \) and \( \{\widetilde{W}^{(m_j)} \}_{j \in \mathbb{N}} \) are regular sequences of periodic partitions of dynamical system \((X,f)\). Assume \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) are the partitions of space \( X \), induced by these sequences.

The following statements are equivalent:

1) the partition \( \tilde{\mathcal{H}} \) is refinement of the partition \( \mathcal{H} \) (respectively, \( \mathcal{H} = \tilde{\mathcal{H}} \));

2) sequences \( \{W^{(n_i)}\} \) and \( \{\widetilde{W}^{(m_j)}\} \) are compatible and \( \Phi(\{n_i \mid i \in \mathbb{N}\}) \leq \Phi(\{m_j \mid j \in \mathbb{N}\}) \) (respectively, \( \Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\{m_j \mid j \in \mathbb{N}\}) \)).

In order to prove Proposition 3.6 we will need following almost obvious

Lemma 3.2 Let \( X \) be a Hausdorff space,
\[
K_1 \supseteq K_2 \supseteq \ldots \supseteq K_i \supseteq \ldots
\]
is a sequence of nonempty compact subsets of \( X \).

For any open neighborhood \( U \) of the set
\[
K = \bigcap_{i \in \mathbb{N}} K_i
\]
there exists \( n \in \mathbb{N} \), such that \( K_i \subseteq U \) for \( i \geq n \).

Proof. Assume that there exist a neighborhood \( U \supseteq K \) and a sequence \( \{x_i \in K_i \setminus U\}_{i \in \mathbb{N}} \).

Since \( x_i \in K_1 \), \( i \in \mathbb{N} \), on the construction and \( K_1 \) is the compact set, then this sequence has at least one limit point \( x \in K_1 \setminus U \subseteq X \setminus U \).

It follows from condition of Lemma that \( x_i \in K_i \subseteq K_m \) for every \( m \in \mathbb{N} \) and \( i \geq m \). Hence, \( x \in K_m \), \( m \in \mathbb{N} \) and \( x \in K \setminus U \).

The obtained contradiction proves Lemma. \( \square \)

Proof of Proposition 3.6. 1. We shall prove, that for every \( i \in \mathbb{N} \) there exists \( j(i) \in \mathbb{N} \), which comply with the following requirements:

- \( n_i \) divides \( m_{j(i)} \);

- periodic partition \( \widetilde{W}^{(m_{j(i)})} \) is the refinement of partition \( W^{(n_i)} \).
We fix \( i \in \mathbb{N} \). Obviously, the open neighborhood \( W^{(n)}_{\alpha_i(x)} \) of the set \( \hat{H}(x) \subseteq H(x) \subseteq W^{(n)}_{\alpha_i(x)} \) and the sequence of closed sets
\[
\hat{W}^{(m_1)}_{\tilde{\alpha}_1(x)} \supseteq \hat{W}^{(m_2)}_{\tilde{\alpha}_2(x)} \supseteq \ldots \supseteq \hat{W}^{(m_j)}_{\tilde{\alpha}_j(x)} \supseteq \ldots
\]
satisfy to condition of Lemma 3.2. Hence, there exists \( j(i) \in \mathbb{N} \), for which \( \hat{W}^{(m_j(i))}_{\tilde{\alpha}_j(i)(x)} \subseteq W^{(n)}_{\alpha_i(x)} \).

From Corollary 1.5 we conclude, that periodic partitions \( \hat{W}^{(m_j(i))} \) and \( W^{(n)} \) are compatible and \( n_i \) divides \( m_j(i) \). Now it follows from Corollary 1.6 that the partition \( \hat{W}^{(m_j(i))} \) is refinement of the partition \( W^{(n)} \).

2. Verify, that \( H(y) \supseteq \hat{H}(y) \) for every \( y \in X \), in other words the partition \( \tilde{\mathcal{S}} \) is refinement of the partition \( \mathcal{S} \).

Really, it follows from what we said above, that
\[
\hat{H}(y) = \bigcap_{j \in \mathbb{N}} \hat{W}^{(m_j)}_{\tilde{\alpha}_j(y)} \subseteq \bigcap_{i \in \mathbb{N}} \hat{W}^{(m_j(i))}_{\tilde{\alpha}_j(i)(y)} \subseteq \bigcap_{i \in \mathbb{N}} W^{(n_i)}_{\alpha_i(y)} = H(y)
\]
for every \( y \in X \).

3. The previous item and Proposition 3.5 implies, that the sequences of periodic partitions \( \{W^{(n)}\}_{i \in \mathbb{N}} \) and \( \{W^{(m)}\}_{j \in \mathbb{N}} \) are compatible.

4. We conclude from item 1 and Proposition 2.3, that \( \Phi(\{n_i \mid i \in \mathbb{N}\}) \leq \Phi(\{m_j \mid j \in \mathbb{N}\}) \).

\( \square \)

### 3.3 Main properties of odometers

We show now on the example of odometers how the proved above statements could be applied.

**Proposition 3.7** Let \((X, f)\) and \((Y, g)\) are dynamical systems, \( p : (X, f) \to (Y, g) \) is a projection. If \( n \in \mathcal{P}(Y, h) \) and \( W^{(n)} = \{W_i^{(n)}\}_{i \in \mathbb{Z}_n} \) is a periodic partition of dynamical system \((Y, g)\), then \( n \in \mathcal{P}(X, f) \) and \( \tilde{W}^{(n)} = \{W_i^{(n)} = p^{-1}(W_i^{(n)})\}_{i \in \mathbb{Z}_n} \) is the periodic partition of dynamical system \((X, f)\).

**Proof.** is the simple immediate verification. \( \square \)

**Corollary 3.3** Let \((Y, h)\) is a factor–system of a dynamical system \((X, f)\). Then \( \mathcal{P}(Y, h) \subseteq \mathcal{P}(X, f) \).

We take an advantage of Remark 2.3 and obtain

**Corollary 3.4** In conditions of Corollary 3.3 the inequality \( \Phi(\mathcal{P}(Y, h)) \leq \Phi(\mathcal{P}(X, f)) \) is fulfilled.

**Remark 3.5** Thus, if dynamical systems \((X, f)\) and \((Y, g)\) are topologically conjugate, then \( \Phi(\mathcal{P}(X, f)) = \Phi(\mathcal{P}(Y, g)) \). Hence, \( \Phi(\mathcal{P}(X, f)) \in \Sigma \) is topological invariant of the dynamical system \((X, f) \in \mathcal{K}_0 \).
Proposition 3.8 Let \((A, g)\) is an odometer constructed on a regular sequence \(\{n_i\}_{i \in \mathbb{N}}\).

Then \(\Phi(\{\mathcal{P}(A, g)\}) = \Phi(\{n_i \mid i \in \mathbb{N}\})\).

Let \(\{W^{(m_j)}\}\) is a regular sequence of periodic partitions of dynamical system \((A, g)\).

The family of sets \(\{W^{(m_j)} \mid r_j \in \mathbb{Z}_{m_j}, j \in \mathbb{N}\}\) is basis of topology of the space \(A\) if and only if \(\Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(\{\mathcal{P}(A, g)\})\).

Proof. It follows from Lemma 3.1 and relation (11) that the family

\[
W^{(n_i)} = \{W^{(n_i)} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\},
\]

is the regular sequence of periodic partitions of dynamical system \((A, g)\). We construct on this sequence the associated partition \(\mathcal{H}\) of space \(A\). Since the family (11) is basis of the topology of space \(A\), then \(H_a = \{\tilde{a}\}\) for every \(\tilde{a} \in A\).

1. The set \(\mathcal{P}(A, g)\) is admissible (see Remark 2.6), therefore there exists regular sequence \(\{m_j \in \mathcal{P}(A, g)\}_{j \in \mathbb{N}}\), such that

\[
\Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g)) \geq \Phi(\{n_i \mid i \in \mathbb{N}\})
\]

(see Proposition 2.2 and Remark 2.8).

Construct on this sequence regular sequence of periodic partitions \(\{W^{(m_j)}\}_{j \in \mathbb{N}}\), which is compatible with the sequence \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) (see Proposition 1.8).

Let \(\tilde{\mathcal{H}}\) is the partition of space \(A\), induced by the sequence \(\{\tilde{W}^{(m_j)}\}\). Then it follows from Proposition 3.4 that the partition \(\tilde{\mathcal{H}}\) is refinement of the partition \(\mathcal{H}\). And it is possible only if \(\tilde{\mathcal{H}} = \mathcal{H}\). Now from Proposition 3.6 we receive

\[
\Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(\{n_i \mid i \in \mathbb{N}\})
\]

2. Let now \(\{\tilde{W}^{(m_j)}\}_{j \in \mathbb{N}}\) is a certain regular sequence of periodic partitions of the dynamical system \((A, g)\). Assume \(\mathcal{H}\) is the partition of space \(A\), induced by the sequence \(\{\tilde{W}^{(m_j)}\}\).

We conclude from the first part of Proposition 3.8 and Remark 2.8 that \(\Phi(\{n_i \mid i \in \mathbb{N}\}) \geq \Phi(\{m_j \mid j \in \mathbb{N}\})\).

Odometer \((A, g)\) is the minimal dynamical system (see. Remark 3.2), so this dynamical system is indecomposable and regular sequences \(\{W^{(n_i)}\}\) and \(\{\tilde{W}^{(m_j)}\}\) are compatible (see Remark 1.12).

Applying Corollary 3.3 we conclude, that the partition \(\mathcal{H}\) of space \(A\) is refinement of the partition \(\tilde{\mathcal{H}}\), with \(\tilde{\mathcal{H}} = \mathcal{H}\) if and only if

\[
\Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g)).
\]

Remind, that a family \(\{U_\alpha\}_{\alpha \in \Lambda}\) of open subsets of topological space \(X\) is its basis of topology, when the following conditions (see [16]) are fulfilled:

(a) if \(U_\alpha \cap U_\beta \neq \emptyset\) for certain \(\alpha, \beta \in \Lambda\), then there exists \(\gamma \in \Lambda\), such that \(U_\gamma \subseteq U_\alpha \cap U_\beta\);

(b) for every \(x \in X\) and any open neighborhood \(U\) of \(x\) there exists \(\alpha \in \Lambda\), such that \(x \in U_\alpha \subseteq U\).
Remark, that for any regular sequence of periodic partitions condition (a) is always fulfilled (it follows immediately from Definition 1.6 and Remark 1.10).

Let \( \Phi(\{m_j \mid j \in \mathbb{N}\}) \not\subseteq \Phi(\{n_i \mid i \in \mathbb{N}\}) \). Then partitions \( \mathcal{H} \) and \( \mathcal{\tilde{H}} \) do not coincide (see Corollary 3.2) and there exist two points \( x_1, x_2 \in A, x_1 \neq x_2 \), contained in the same element of the partition \( \mathcal{\tilde{H}} \). Hence, for every \( \mathcal{\tilde{W}}_{s_j}(m_j) \), \( s_j \in \mathbb{Z}, j \in \mathbb{N} \), either \( \mathcal{\tilde{W}}_{s_j}(m_j) \cap \{x_1, x_2\} = \emptyset \) or \( \{x_1, x_2\} \subseteq \mathcal{\tilde{W}}_{s_j}(m_j) \), and the condition (b) is not fulfilled.

Let now \( \Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(\{n_i \mid i \in \mathbb{N}\}) \). In this case partitions \( \mathcal{H} \) and \( \mathcal{\tilde{H}} \) coincide.

Let \( x \in X \) and \( U \) is an open neighborhood of \( x \). On definition of the partition \( \mathcal{\tilde{H}} \) there exists the unique sequence \( \{\alpha_j \in \mathbb{Z}_{m_j}\}_{j \in \mathbb{N}} \), such that
\[
\{x\} = H(x) = \mathcal{\tilde{H}}(x) = \bigcap_{i \in \mathbb{N}} \mathcal{\tilde{W}}_{\alpha_j}(m_j).
\]

In this case all sets from intersection in the right-hand part of the equality are compact (being closed subsets of the compact space \( A \)) and \( \mathcal{\tilde{W}}_{\alpha_{j+1}}(m_{j+1}) \subseteq \mathcal{\tilde{W}}_{\alpha_j}(m_j) \), \( j \in \mathbb{N} \) (see Definition 1.6). We apply Lemma 3.2 and conclude that there exists \( k \in \mathbb{N} \), for which \( x \in \mathcal{\tilde{W}}_{\alpha_{j+k}}(m_j) \subseteq U \). \( \square \)

From Proposition 3.2, Corollary 3.4, Proposition 3.8, Remark 2.6 and Proposition 2.2 we receive the following statement.

**Theorem 3.1** Let \((X, f)\) is a dynamical system with Hausdorff compact phase space, \((A, g)\) is an odometer.

The following statements are equivalent:

(i) there exists a projection \( \pi: (X, f) \to (A, g) \);

(ii) the inequality \( \Phi(P(A, g)) \leq \Phi(P(X, f)) \) is fulfilled.

In order to formulate the following statement, we require two definitions.

Let \((X, f)\) is a dynamical system with compact metric phase space \((X, \rho)\).

**Definition 3.3** Points \( x, y \in X, x \neq y \), are distal, if there exists \( \delta > 0 \), such that \( \rho(f^n(x), f^n(y)) > \delta \) for every \( n \in \mathbb{Z} \).

The dynamical system \((X, f)\) is called distal, if any pair of points \( x, y \in X, x \neq y \), is distal.

**Definition 3.4** The dynamical system \((X, f)\) is called equicontinuous, if the family of maps \( \{f^n\}_{n \in \mathbb{Z}} \) is equicontinuous under the metrics \( \rho \), that is if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \), such that if \( \rho(x, y) < \delta \) for some \( x, y \in X \), then \( \rho(f^n(x), f^n(y)) < \varepsilon \) for every \( n \in \mathbb{Z} \).

**Remark 3.6** It is easy to see, that distality and equicontinuity of dynamical system \((X, f)\) do not depend (by virtue of compactness of \( X \)) on a choice of metric function, which generates the given topology on \( X \). In other words, distality and equicontinuity are topological properties of dynamical system \((X, f)\) with a metrizable compact phase space \( X \).

**Theorem 3.2 (see. [17])** Let \((\Gamma, f)\) is a minimal dynamical system on Cantor set \( \Gamma \).

Then the following conditions are equivalent:

1. d. s. \((\Gamma, f)\) is topologically conjugate with an odometer;
2. d. s. $(\Gamma, f)$ is distal;

3. d. s. $(\Gamma, f)$ is equicontinuous.

**Proof.** Equivalence of conditions 2. and 3. for dynamical systems with zero-dimensional compact phase space immediately follows from results, obtained in paper [18].

Verify implication 1. $\Rightarrow$ 3.

Let d. s. $(\Gamma, f)$ is topologically conjugate with the help of a homeomorphism $h : \Gamma \to A$ with an odometer $(A, g)$, which is generated by admissible sequence $\{n_i\}_{i \in \mathbb{N}}$. We transfer the natural metric $d$ from the space $A$ on $\Gamma$ with the help of the relations

$$\rho(x, y) = d(h(x), h(y)), \quad x, y \in \Gamma.$$ 

Remark 3.3 consequences, that the map $f$ is isometric under the metric $\rho$. Thus, d. s. $(\Gamma, f)$ is equicontinuous.

Prove implication 3. $\Rightarrow$ 1.

We fix the metrics $\rho : \Gamma \times \Gamma \to \mathbb{R}_+$. Let the dynamical system $(\Gamma, f)$ is equicontinuous under the metrics $\rho$.

Let $x \in \Gamma$ and $V$ is an open-closed neighborhood of $x$. Since the closed sets $V$ and $\Gamma \setminus V$ are disjoint and $\Gamma$ is compact, then

$$\rho(V, \Gamma \setminus V) = \varepsilon > 0.$$ 

There exists $\delta > 0$, such that for any $y_1, y_2 \in \Gamma$ the implication is valid

$$(\rho(y_1, y_2) < 2\delta) \Rightarrow (\rho(f^n(y_1), f^n(y_2)) < \varepsilon, \quad n \in \mathbb{Z}).$$

Let $U = U_\delta(x)$. Then $\text{Diam} U < 2\delta$ and $\text{Diam} f^n(U) < \varepsilon$ for all $n \in \mathbb{Z}$. Hence, for every $n \in \mathbb{Z}$ either $V \cap f^n(U) = \emptyset$ or $f^n(U) \subseteq V$.

Since the dynamical system $(\Gamma, f)$ is minimal, then there exists $k \in \mathbb{N}$, such that $f^k(x) \in U$.

Then $f^k(U) \cap U \neq \emptyset$. It is clear that since $f$ is the homeomorphism, we have

$$f^{k+n}(U) \cap f^n(U) \neq \emptyset, \quad n \in \mathbb{Z}.$$ 

Check, that $f^{kn}(U) \subseteq V$, $n \in \mathbb{Z}$.

We shall carry out verification on an induction for negative $n$.

**Basis of induction.** Since $\emptyset \neq f^{-k}(U) \cap U \subseteq f^{-k+1}(U) \cap V$, then $f^{-k}(U) \subseteq V$.

**Step of induction.** Let $f^{-ki}(U) \subseteq V$ for some $i \in \mathbb{N}$. Then $\emptyset \neq f^{-k(i+1)}(U) \cap f^{-ki}(U) \subseteq f^{-k(i+1)}(U) \cap V$ and $f^{-k(i+1)}(U) \subseteq V$.

On the induction we conclude, that $f^{kn}(U) \subseteq V$ for all $n < 0$.

The proof of this inclusion for positive $n$ is done similarly.

So,

$$\bigcup_{n \in \mathbb{Z}} f^{kn}(x) \subseteq \bigcup_{n \in \mathbb{Z}} f^{kn}(U) \subseteq V.$$ 

We conclude from Lemma 1.5 that there exist $m \in \mathcal{P}(\Gamma, f)$ and periodic partition $W^{(m)}$ of dynamical system $(\Gamma, f)$ of length $m$, such that $x \in W^{(m)} \subseteq V$.

We fix $x \in \Gamma$ and sequence $\{\varepsilon_i \geq 0\}_{i \in \mathbb{N}}$, such that $\varepsilon_i \to 0$ when $i \to \infty$. For every $\varepsilon_i$ we can find $\delta_i > 0$, such that for all $y_1, y_2 \in \Gamma$

$$(\rho(y_1, y_2) < \delta_i) \Rightarrow (\rho(f^n(y_1), f^n(y_2)) < \varepsilon_i, \quad n \in \mathbb{Z}).$$
Now we shall construct on an induction a coherent regular sequence of periodic partitions \( \{W^{(n_i)}\}_{i \in N} \) of dynamical system \((\Gamma, f)\), such that for every \( i \in N \)
\[
\operatorname{Diam}W_{s_i}^{(n_i)} < \varepsilon_i, \quad s_i \in \mathbb{Z}_{n_i}.
\]
(14)

The space \( \Gamma \) is zero-dimensional, therefore there exists a sequence \( \{V_i \ni x\}_{i \in N} \) of open-closed sets, such that \( \operatorname{Diam}V_i < \delta_i, \ i \in N \).

**Basis of induction.** Find periodic partition \( W^{(n_1)} \) of dynamical system \((\Gamma, f)\), such that \( x \in W_0^{(n_1)} \subseteq V_1 \). Then \( \operatorname{Diam}W_0^{(n_1)} < \delta_1 \) and \( \operatorname{Diam}W_{s_1}^{(n_1)} = \operatorname{Diam}f^{s_1}(W_0^{(n_1)}) < \varepsilon_1, \ s_1 \in \mathbb{Z}_{n_1} \), according to the choice of \( \delta_1 \).

**Step of induction.** Assume the family \( \{W^{(n_i)}\}_{i=1}^k \) of periodic partitions of dynamical system \((\Gamma, f)\) is already constructed, such that \( n_i \) divides \( n_{i+1} \), \( i = 1, \ldots, k-1 \),
\[
W_0^{(n_i)} \supseteq \ldots \supseteq W_0^{(n_k)} \ni x
\]
(we conclude from Corollary 1.5 that every two periodic partitions from this family are compatible), and which complies with the relations (14).

Designate \( \widetilde{V}_{k+1} = V_{k+1} \cap W_0^{(n_k)} \ni x \). Obviously, \( \operatorname{Diam}\widetilde{V}_{k+1} < \delta_{k+1} \).

Find periodic partition \( W^{(n_{k+1})} \) of the dynamical system \((\Gamma, f)\), such that \( x \in W_0^{(n_{k+1})} \subseteq \widetilde{V}_{k+1} \subseteq W_0^{(n_k)} \).

On one hand \( \operatorname{Diam}W_0^{(n_{k+1})} < \operatorname{Diam}\widetilde{V}_{k+1} < \delta_{k+1} \), hence \( \operatorname{Diam}W_{s_1}^{(n_{k+1})} = \operatorname{Diam}f^{s_1}(W_0^{(n_{k+1})}) < \varepsilon_{k+1}, \ s_1 \in \mathbb{Z}_{n_{k+1}} \).

On the other hand, \( n_k \) divides \( n_{k+1} \) and periodic partitions \( W^{(n_{k+1})} \) and \( W^{(n_k)} \) are compatible by Corollary 1.5.

On an induction we receive the coherent sequence of periodic partitions \( \{W^{(n_i)}\}_{i \in N} \) of dynamical system \((\Gamma, f)\), all elements of which satisfy to relation (14).

Construct the partition \( \mathcal{S} \) of space \( \Gamma \) on the sequence \( \{W^{(n_i)}\} \) and the projection \( F : (\Gamma, f) \to (A, g) \).

Let \( y \in \Gamma \), \( (\alpha_i) = F(y) \in A \). Remark that for every \( k \in N \) we have
\[
\operatorname{Diam}H(y) \leq \operatorname{Diam}\left( \bigcap_{i=1}^k W^{(n_i)}_{\alpha_i} \right) = \operatorname{Diam}W^{(n_k)}_{\alpha_k} \leq \varepsilon_k \to 0.
\]
Hence, \( H(y) = \{y\} \) for every \( y \in \Gamma \) and the projection map \( pr : \Gamma \to \Gamma/\mathcal{S} \) is bijective. Therefore the map \( F = (\text{fact } F) \circ pr : \Gamma \to A \) is also bijective. Since \( \Gamma \) is compact set, then \( F \) is the homeomorphism, which conjugates dynamical systems \((\Gamma, f)\) and \((A, g)\) (see commutative diagram (12)). □

**Remark 3.7** When we defined odometer constructed on a regular sequence \( \{n_i\}_{i \in N} \), we required, that this sequence should be unlimited. Actually this requirement can be written in the following way
\[
\Phi(\{n_i \mid i \in N\}) \in \Sigma \setminus \Phi_0(N).
\]
Let us look, what will change, if \( \Phi(\{n_i \mid i \in N\}) = \Phi_0(m) \in \Phi_0(N) \). In this case \( A = \lim_{i \to \infty} \operatorname{inv} \mathbb{Z}_{n_i} = \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \), \( \bar{e} = 1 \in \mathbb{Z}_m \), and the dynamical system \((A, g)\) consists from unique periodic trajectory of length \( m \).

We expand definition of odometers, including in it the case \( \Phi(\{n_i \mid i \in N\}) \in \Phi_0(N) \).

It is trivially checked, that everything we said in subsections 3.2 and 3.3, except for theorem 3.2, is valid for our new definition.
Theorem 3.3 (see [19, 5, 10]) 1. For every $N \in \Sigma$ there exists an odometer $(A, g)$, such that $\Phi(\mathcal{P}(A, g)) = N$.

2. Odometers $(A_1, g_1)$ and $(A_2, g_2)$ are topologically conjugate if and only if $\Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2))$.

Proof. 1. Let $N \in \Sigma$. We can find admissible set $\overline{A} \subseteq \mathcal{R}$, such that $\Phi(\overline{A}) = N$ (see Proposition 2.1). Fix also regular sequence $\{n_i \in \overline{A}\}_{i \in \mathbb{N}}$, for which $\Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\overline{A})$ (see Proposition 2.2). Construct on the sequence $\{n_i\}$ odometer $(A, g)$. From Proposition 3.8 we conclude, that $\Phi(\mathcal{P}(A, g)) = \Phi(\overline{A}) = N$.

2. (a) Assume that odometers $(A_1, g_1)$ and $(A_2, g_2)$ are topologically conjugate with the help of a homeomorphism $h : A_1 \to A_2$.

We have two projections $h : (A_1, g_1) \to (A_2, g_2)$ and $h^{-1} : (A_2, g_2) \to (A_1, g_1)$. From Corollary 3.4 we conclude that $\Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2))$.

(b) Let now $\Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2)) = N \in \Sigma$.

We find regular sequence $\{n_i\}_{i \in \mathbb{N}}$, for which $\Phi(\{n_i \mid i \in \mathbb{N}\}) = N$ (see above) and construct on it odometer $(A, g)$.

Since $\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2))$, then we receive from Proposition 2.1 and Remark 2.6 the relation $\mathcal{P}(A, g) = \mathcal{P}(A_1, g_1) = \mathcal{P}(A_2, g_2)$. It follows from Lemma 3.1 that $n_i \in \mathcal{P}(A, g)$, $i \in \mathbb{N}$. Consequently, $n_i \in \mathcal{P}(A_k, g_k)$, $k = 1, 2, i \in \mathbb{N}$.

We fix coherent sequence $\{W^{(n_i)}\}_{i \in \mathbb{N}}$ of periodic partitions of dynamical system $(A_1, g_1)$ and construct on it the partition $\overline{\mathcal{F}}$ of space $A_1$.

Similarly, we take coherent sequence $\{W^{(n_i)}\}_{i \in \mathbb{N}}$ of periodic partitions of dynamical system $(A_2, g_2)$ and partition $\overline{\mathcal{F}}$ of the space $A_2$, induced by this sequence.

Consider the commutative diagram

$$
\begin{array}{ccc}
(A_1, g_1) & \xrightarrow{F} & (A, g) & \xleftarrow{\overline{F}} & (A_2, g_2) \\
pr \downarrow & & \uparrow & & \tilde{pr} \\
(A_1/\mathcal{F}, \text{fact } g_1) & \xrightarrow{\text{fact } F} & (A, g) & \xleftarrow{\text{fact } \overline{F}} & (A_2/\mathcal{F}, \text{fact } g_2)
\end{array}
$$

We know already, that all maps in the lower line of this diagram are isomorphisms in category $\mathcal{K}_0$.

Maps $pr$ and $\tilde{pr}$ are one-to-one on Proposition 3.8. Since spaces $A_1$ and $A_2$ are compact, then $pr$ and $\tilde{pr}$ are isomorphisms in category $\mathcal{K}_0$.

From what has been said it follows, that the morphism

$$
\overline{F}^{-1} \circ F = \tilde{pr}^{-1} \circ (\text{fact } \overline{F})^{-1} \circ (\text{fact } F) \circ pr
$$

is isomorphism and dynamical systems $(A_1, g_1)$ and $(A_2, g_2)$ are topologically conjugate. $\square$

Remark 3.8 Let $(X, f), (Y, g) \in \mathcal{K}_0$, $h : (X, f) \to (Y, g)$ is a morphism. If dynamical system $(Y, g)$ is minimal, then $h$ is the projection.

Really, we fix $y \in Y$. Under the Birkhoff theorem we have $\overline{\text{Orb}_y(h(y))} = Y$. On the other hand, since $X$ is compact set, then $h(X)$ is the closed subset of space $Y$ and, obviously, $h(X) \supseteq \overline{\text{Orb}_y(h(y))} = h(\text{Orb}_f(y))$. 
Proposition 3.9 Let \((A, g)\) is an odometer, \(h : (A, g) \rightarrow (A, g)\) is a morphism. Then \(h\) is isomorphism.

Proof. We fix regular sequence \(\{n_i \in \mathcal{P}(A, g)\}_{i \in \mathbb{N}}\), such that \(\Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g))\) (see Remark 2.6 and Proposition 2.2).

Fix regular sequence of periodic partitions \(\{W_{(n_i)}\}_{i \in \mathbb{N}}\).

\(h\) is epimorphism according to Remark 3.8, therefore from Proposition 3.7 follows, that for every \(i \in \mathbb{N}\) the family of sets \(\widehat{W}_{(s_i)} = \{W_{s_i}^{(n_i)} = h^{-1}(W_{s_i}^{(n_i)})\}_{i \in \mathbb{N}}\) is periodic partition of the dynamical system \((A, g)\) of length \(n_i\).

Odometer \((A, g)\) is the minimal dynamical system, therefore \((A, g)\) is indecomposable. We apply Corollary 1.1 and conclude that \(\{\widehat{W}_{(n_i)}\}_{i \in \mathbb{N}}\) is regular sequence of periodic partitions of \((A, g)\).

From Proposition 3.8 it follows, that each of the families of sets \(\{W_{s_i}^{(n_i)} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) and \(\{\widehat{W}_{s_i}^{(n_i)} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) is basis of the topology of space \(A\). Therefore \(h\) is continuous one-to-one map. Since \(A\) is compact set, then \(h\) is homeomorphism. \(\Box\)

Corollary 3.5 Let \((Y_1, h_1)\) and \((Y_2, h_2)\) are two dynamical systems, which are topologically conjugate with some odometers.

If the objects \((Y_1, h_1), (Y_2, h_2) \in \text{Ob} \mathcal{K}_0\) are isomorphic, then any morphism \(\alpha : (Y_1, h_1) \rightarrow (Y_2, h_2)\) is isomorphism.

Proof. 1. Let dynamical system \((Y_1, h_1)\) is topologically conjugate with an odometer \((A, g)\). Let \(\rho : (Y_1, h_1) \rightarrow (Y_1, h_1)\) is a morphism. Then \(\rho\) is isomorphism.

Really, we fix isomorphism \(\phi : (Y_1, h_1) \rightarrow (A, g)\) and consider morphism \(\phi \circ \rho \circ \phi^{-1} : (A, g) \rightarrow (A, g)\). According to Proposition 3.9 \(\phi \circ \rho \circ \phi^{-1}\) is isomorphism. Then also \(\rho = \phi^{-1} \circ (\phi \circ \rho \circ \phi^{-1}) \circ \phi\) is isomorphism.

2. Assume that there exists an isomorphism \(\psi : (Y_1, h_1) \rightarrow (Y_2, h_2)\).

Morphism \(\chi = \alpha \circ \psi^{-1} : (Y_2, h_2) \rightarrow (Y_2, h_2)\) is isomorphism (see above). Hence also \(\chi \circ \psi = \alpha \circ \psi^{-1} \circ \psi = \alpha\) is isomorphism. \(\Box\)

Let \((A, +)\) is an adic group. Let \(g : A \rightarrow A, g : \tilde{a} \mapsto \tilde{a} + \tilde{c} \quad \tilde{a}, \tilde{c} \in A\).

Let \((A, g)\) is the relevant odometer.

We fix \(\tilde{a}, \tilde{b} \in A\). Consider the map

\[
h_{\tilde{a}, \tilde{b}} : A \rightarrow A, \\
h_{\tilde{a}, \tilde{b}} : \tilde{c} \mapsto \tilde{c} + (\tilde{b} - \tilde{a}), \quad \tilde{c} \in A.
\]

Proposition 3.10 \(h_{\tilde{a}, \tilde{b}} \circ g = g \circ h_{\tilde{a}, \tilde{b}}\).

Proof. This is the obvious corollary of commutability of group \((A, g)\). \(\Box\)

Remark 3.9 Let dynamical system \((X, f)\) is minimal, \(h_1, h_2 : (X, f) \rightarrow (Y, g)\) are two morphisms, such that \(h_1(x) = h_2(x)\) for some \(x \in X\).

Then \(h_1 = h_2\).
Realistically, for every \( y = f^n(x) \in \text{Orb}_f(x) \) we have \( h_1(y) = h_1 \circ f^n(x) = g^n \circ h_1(x) = g^n \circ h_2(x) = h_2 \circ f^n(x) = h_2(y) \). Therefore \( h_1|_{\text{Orb}_f(x)} = h_2|_{\text{Orb}_f(x)} \). Since \( X = \text{Orb}_f(x) \) under Birkhoff theorem, then \( h_1 = h_2 \).

Thus, combining Propositions 3.9, 3.10 and Remark 3.9, we receive

**Corollary 3.6** Let \((A, g)\) is a dynamical system, topologically conjugate with an odometer. For any pair of points \( x, y \in A \) there exists unique morphism \( h_{x,y} : (A, g) \to (A, g) \), such that \( h_{x,y}(x) = y \), and this morphism is isomorphism.

### 3.4 Deviation. One categorial construction.

Let \( \mathcal{L} \) is a category.

**Definition 3.5** We say that \( \mathcal{L} \) has property **LU** (Lifting Upstairs), if for every objects \( A, B \in \text{Ob} \mathcal{L} \) and for arbitrary morphisms \( \alpha \in H_\mathcal{L}(A, B) \) and \( e_B \in H_\mathcal{L}(B, B) \cap \text{Iso} \mathcal{L} \) there exists morphism \( e_A \in H_\mathcal{L}(A, A) \cap \text{Iso} \mathcal{L} \), such that

\[
e_B \circ \alpha = \alpha \circ e_A .
\]

**Definition 3.6** We say that \( \mathcal{L} \) has property **LD** (Lifting Downstairs), if for any objects \( A, B \in \text{Ob} \mathcal{L} \) and for arbitrary morphisms \( \alpha \in H_\mathcal{L}(A, B) \) and \( f_A \in H_\mathcal{L}(A, A) \cap \text{Iso} \mathcal{L} \) there exists morphism \( f_B \in H_\mathcal{L}(B, B) \cap \text{Iso} \mathcal{L} \), such that

\[
\alpha \circ f_A = f_B \circ \alpha .
\]

Let \( \mathcal{L} \) is a category. For each pair of objects \( A, B \in \text{Ob} \mathcal{L} \) we shall define binary relation \( \sim \) on the set \( H_\mathcal{L}(A, B) \). Say that \( \alpha \sim \beta, \alpha, \beta \in H_\mathcal{L}(A, B) \), if there exist such \( e_A \in H_\mathcal{L}(A, A) \cap \text{Iso} \mathcal{L} \) and \( e_B \in H_\mathcal{L}(B, B) \cap \text{Iso} \mathcal{L} \), that

\[
\alpha \circ e_A = e_B \circ \beta .
\]

It is easy to see, that \( \sim \) is the equivalence relation. An equivalence class of a morphism \( \alpha \) shall be designated by \([\alpha]\).

**Proposition 3.11** Assume that category \( \mathcal{L} \) has one of properties **LU** or **LD**.

Then the category \( \overline{\mathcal{L}} \) is correctly defined, for which the objects are same with the objects of category \( \mathcal{L} \) and for any pair of objects \( A, B \in \overline{\mathcal{L}} \) a set of morphisms \( H_{\overline{\mathcal{L}}}(A, B) \) is the set of equivalence classes of morphisms from \( H_\mathcal{L}(A, B) \).

**Proof.** Assume, that the category \( \mathcal{L} \) has property **LU**.

It is trivially checked, that \( \overline{\mathcal{L}} \) satisfies to properties 1) and 2) of category.

In order to define correctly composition of morphisms in \( \overline{\mathcal{L}} \), we shall prove that for any triple of objects \( A, B, C \in \mathcal{L} \) and for any morphisms \( \alpha \in H_\mathcal{L}(A, B), \beta \in H_\mathcal{L}(B, C) \) the equality is fulfilled

\[
[\beta \circ \alpha] = [\beta] \circ [\alpha] = \{ \beta' \circ \alpha' \mid \alpha' \in [\alpha], \beta' \in [\beta] \} .
\]  

(15)

Let \( \alpha' \in [\alpha], \beta' \in [\beta] \). Then there exist such isomorphisms \( e_A \in H_\mathcal{L}(A, A) \cap \text{Iso} \mathcal{L}, e_B, f_B \in H_\mathcal{L}(B, B) \cap \text{Iso} \mathcal{L} \) and \( f_C \in H_\mathcal{L}(C, C) \cap \text{Iso} \mathcal{L} \), that

\[
\beta' \circ \alpha' = (f_C \circ \beta \circ f_B^{-1}) \circ (e_B \circ \alpha \circ e_A^{-1}) = f_C \circ \beta \circ (f_B^{-1} \circ e_B) \circ \alpha \circ e_A .
\]
Obviously, $f_B^{-1} \circ e_B \in H_\Sigma(B, B) \cap \text{Iso } \Sigma$. From property $LU$ we conclude, that there exists $g_A \in H_\Sigma(A, A) \cap \text{Iso } \Sigma$, for which $(f_B^{-1} \circ e_B) \circ \alpha = \alpha \circ g_A$. Hence,

$$
\beta' \circ \alpha' = f_C \circ \beta \circ (f_B^{-1} \circ e_B) \circ \alpha \circ e_A^{-1} = f_C \circ \beta \circ \alpha \circ (g_A \circ e_A^{-1}),
$$

$\beta' \circ \alpha' \in [\beta \circ \alpha]$ and $[\beta \circ \alpha] \supseteq \{\beta' \circ \alpha' \mid \alpha' \in [\alpha], \beta' \in [\beta]\}$.

Inversely, let $\gamma \in [\beta \circ \alpha]$. It means, that for some $e_A \in H_\Sigma(A, A) \cap \text{Iso } \Sigma$ and $e_C \in H_\Sigma(C, C) \cap \text{Iso } \Sigma$ the relation takes place

$$
\gamma = e_C \circ (\beta \circ \alpha) \circ e_A^{-1} = (e_C \circ \beta) \circ (\alpha \circ e_A^{-1}).
$$

Obviously, $\alpha \circ e_A^{-1} = 1_B \circ \alpha \circ e_A^{-1} \in [\alpha]$ and $e_C \circ \beta = e_C \circ \beta \circ 1_B^{-1} \in [\beta]$. Hence, $[\beta \circ \alpha] \subseteq \{\beta' \circ \alpha' \mid \alpha' \in [\alpha], \beta' \in [\beta]\}$.

So, we have established, that partial multiplication of equivalence classes of morphisms does not depend on a choice of representatives, hence it is defined correctly.

Associativity of multiplication of morphisms in $\Sigma$ follows from associativity of multiplication of morphisms in $\Sigma$.

To complete the proof it suffices to note, that for any $A \in \text{Ob } \Sigma$ the unit morphism of object $A$ in $\Sigma$ is $[1_A]$. If the category $\Sigma$ has property $LD$, proof is conducted similarly. $\square$

**Remark 3.10** The described above construction is a special case of so-called factor–category (see [20]).

### 3.5 Main properties of odometers (continuation).

Let $(A_1, g_1)$, $(A_2, g_2)$ are dynamical systems, topologically conjugate with odometers, $\pi_1, \pi_2 : (A_1, g_1) \to (A_2, g_2)$ and $h : (A_2, g_2) \to (A_2, g_2)$ are morphisms.

Designate by $F$ the set of all morphisms $f : (A_1, g_1) \to (A_1, g_1)$, such that the diagram is commutative

$$
\begin{array}{ccc}
(A_1, g_1) & \xrightarrow{f} & (A_1, g_1) \\
\downarrow_{\pi_1} & & \downarrow_{\pi_2} \\
(A_2, g_2) & \xrightarrow{h} & (A_2, g_2)
\end{array}
$$

It follows from Proposition 3.9 that $h \in \text{Iso } K_0$ and $F \subseteq \text{Iso } K_0$.

**Proposition 3.12** The set $F$ is not empty.

For any $y \in A_2$ and $x_1 \in \pi_1^{-1}(y)$ the equality is valid

$$
F = \{h_{x_1, x_2} \mid x_2 \in \pi_2^{-1}(h(y))\}.
$$

Proof of Proposition 3.12 is based on the following Lemmas.

**Lemma 3.3** Let $(X, f)$ is an indecomposable dynamical system, $(A, g)$ is dynamical system, topologically conjugate with an odometer, $\pi_1, \pi_2 : (X, f) \to (A, g)$ are projections.

Then the partitions $\zer \pi_1$ and $\zer \pi_2$ of space $X$ coincide.
Lemma 3.4 Let \((X, f), (A, g), \pi_1, \pi_2 : (X, f) \to (A, g)\) are the same, as in Lemma 3.3.
For any morphism \(h : (X, f) \to (X, f)\) the continuous map fact \(\pi_2 \circ h : A \to A\) is well
defined to comply with the relation \(g \circ (\text{fact} \pi_2 \circ h) = \text{fact} \pi_2 \circ h \circ g\).
Therefore, the commutative diagram is valid
\[
\begin{array}{ccc}
(X, f) & \xrightarrow{h} & (X, f) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
(A, g) & \xrightarrow{\text{fact} \pi_2 \circ h} & (A, g)
\end{array}
\]

Proof of Lemma 3.3. We fix a regular sequence \(\{n_i \in \Phi(\mathcal{P}(A, g))\}_{i \in \mathbb{N}}\), such that \(\Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g))\) (see Remark 2.6 and Proposition 2.2).
We construct regular sequence \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) of periodic partitions of dynamical system \((A, g)\).
According to Proposition 3.8 the family of sets \(\{W^{(n_i)}_{s_i} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) is basis of the topology
of space \(A\).
It follows from Proposition 3.7 that for every \(i \in \mathbb{N}\) the systems of sets \(\tilde{W}^{(n_i)} = \{\tilde{W}^{(n_i)}_{s_i} = \pi_1^{-1}(W^{(n_i)}_{s_i}) \mid s_i \in \mathbb{Z}_{n_i}\}\) and \(\hat{W}^{(n_i)} = \{\hat{W}^{(n_i)}_{s_i} = \pi_2^{-1}(W^{(n_i)}_{s_i}) \mid s_i \in \mathbb{Z}_{n_i}\}\) are periodic partitions of
of dynamical system \((X, f)\) of length \(n_i\).
Dynamical system \((X, f)\) is indecomposable, therefore we conclude from Corollary 1.1 and
Remark 1.12 that \(\{\tilde{W}^{(n_i)}\}_{i \in \mathbb{N}}\) and \(\{\hat{W}^{(n_i)}\}_{i \in \mathbb{N}}\) are compatible regular sequences of periodic
partitions of \((X, f)\).
Let \(\tilde{\mathcal{H}}\) and \(\hat{\mathcal{H}}\) are the partitions of space \(X\), induced respectively by sequences \(\{\tilde{W}^{(n_i)}\}_{i \in \mathbb{N}}\) and \(\{\hat{W}^{(n_i)}\}_{i \in \mathbb{N}}\). Then partitions \(\tilde{\mathcal{H}}\) and \(\hat{\mathcal{H}}\) coincide by Corollary 3.2.
The family of sets \(\{\tilde{W}^{(n_i)}_{s_i} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) is pre-image of the basis of topology
\(\{W^{(n_i)}_{s_i} \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) under action of the projection \(\pi_1\), therefore partitions \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\) coincide.
Similarly, the partitions \(\mathcal{H}\) and \(\hat{\mathcal{H}}\) coincide. \(\square\)

Proof of Lemma 3.4. Let \(h : (X, f) \to (X, f)\) is a morphism.
Since dynamical system \((A, g)\) is minimal, then morphism \(\pi_2 \circ h : (X, f) \to (A, g)\) is the projection.
From Lemma 3.3 it follows, that \(\mathcal{H} = \text{zer} \pi_1\) and \(\mathcal{H} = \text{zer} \pi_2 \circ h\). To complete the proof it
remains to apply Lemma 0.2 to morphisms \(\varphi_1 = \pi_1\) and \(\varphi_2 = \pi_2 \circ h\). \(\square\)

Proof of Proposition 3.12. Dynamical system \((A_1, g_1)\) is minimal, therefore it is indecomposable.
We fix \(y \in A_2\) and \(x_1 \in \pi_1^{-1}(y)\). Let \(x_2 \in \pi_2^{-1}(h(y))\). Consider morphism \(h_{x_1, x_2} : (A_1, g_1) \to (A_2, g_2)\).
We conclude from Lemma 3.4 that morphism fact \(\pi_2 \circ h_{x_1, x_2} : (A_2, g_2) \to (A_2, g_2)\) is correctly defined and the sequence of equalities is valid
\[
\text{fact} \pi_2 \circ h_{x_1, x_2}(y) = (\text{fact} \pi_2 \circ h_{x_1, x_2}) \circ \pi_1(x_1) = \pi_2 \circ h_{x_1, x_2}(x_1) = \pi_2(x_2) = h(y).
\]
Therefore, it follows from Corollary 3.6 that fact \(\pi_2 \circ h_{x_1, x_2} = h\) and \(\mathcal{F} \supseteq \{h_{x_1, x_2} \mid x_2 \in \pi_2^{-1}(h(y))\}\).
On the other hand, for any \(f \in \mathcal{F}\) the relation \(f(x_1) \in f(\pi_1^{-1}(y)) = \pi_2^{-1}(h(y))\) should be
fulfilled (see Lemma 3.3). Corollary 3.6 consequences that \(f = h_{x_1, f(x_1)}\) and \(\mathcal{F} \subseteq \{h_{x_1, x_2} \mid x_2 \in \pi_2^{-1}(h(y))\}\). \(\square\)

Consider the complete subcategory \(\mathcal{A}\) of \(\mathcal{K}_0\), objects of which are all dynamical systems, topologically conjugate with odometers.
Corollary 3.7 The category $\mathcal{A}$ has properties LU and LD.

Proof. 1. Condition LU follows from Proposition 3.12, with $\pi_1 = \pi_2 = \alpha$, and from Corollary 3.6.

2. Condition LD is the consequence of Lemma 3.4, where $(X,f)$ and $(A,g)$ are topologically conjugate with odometers and $\pi_1 = \pi_2 = \alpha$, and from Corollary 3.6. □

We fix a skeleton $\mathcal{A}_0$ of category $\mathcal{A}$.

Remark 3.11 From Theorem 3.3 and Remark 3.5 we conclude that for every $N \in \Sigma$ category $\mathcal{A}_0$ contains exactly one object $(A,g)$, such that $\Phi(P(A,g)) = N$.

Remark 3.12 Theorem 3.1 consequences that for any two objects $(A_1,g_1)$ and $(A_2,g_2)$ of category $\mathcal{A}_0$ the following statements are equivalent:

(i) $\Phi(P(A_1,g_1)) \geq \Phi(P(A_2,g_2))$;

(ii) $H_{\mathcal{A}_0}((A_1,g_1),(A_2,g_2)) \neq \emptyset$.

We take advantage of Proposition 3.11 and construct category $\overline{\mathcal{A}}_0$ by the category $\mathcal{A}_0$. From Lemma 3.4 and Corollary 3.6 we obtain

Corollary 3.8 Let $(A_1,g_1), (A_2,g_2) \in \text{Ob} \mathcal{A}_0$, $\pi_1, \pi_2 \in H_{\mathcal{A}_0}((A_1,g_1),(A_2,g_2))$. Then $[\pi_1] = [\pi_2]$.

Now from Remark 3.12 we have

Corollary 3.9 Let $(A_1,g_1), (A_2,g_2) \in \text{Ob} \overline{\mathcal{A}}_0$. Then

(i) if $\Phi(P(A_1,g_1)) \geq \Phi(P(A_2,g_2))$, then the set $H_{\overline{\mathcal{A}}_0}((A_1,g_1),(A_2,g_2))$ contains exactly one element;

(ii) $H_{\overline{\mathcal{A}}_0}((A_1,g_1),(A_2,g_2)) = \emptyset$, otherwise.

By partially ordered set $\Sigma$ we construct category $\mathfrak{L}(\Sigma)$, which objects are the elements of the set $\Sigma$, and morphisms are all possible pairs of elements $(M,N)$, such that $M \geq N$. For any two elements $M, N \in \Sigma$ the set $H_{\mathfrak{L}(\Sigma)}(M,N)$ consists of one morphism $(M,N)$ if $M \geq N$, otherwise this set is empty.

Remark 3.11 and corollary 3.9 give us the following

Theorem 3.4 Correspondence $\Psi_0 : \text{Ob} \overline{\mathcal{A}}_0 \rightarrow \text{Ob} \mathfrak{L}(\Sigma)$,

$\Psi_0 : (A,g) \mapsto \Phi(P(A,g))$, $(A,g) \in \text{Ob} \overline{\mathcal{A}}_0$,

is uniquely extended to functor $\Psi : \overline{\mathcal{A}}_0 \rightarrow \mathfrak{L}(\Sigma)$.

The functor $\Psi$ sets the isomorphism of categories $\overline{\mathcal{A}}_0$ and $\mathfrak{L}(\Sigma)$. 

4 Expansions of odometers.

4.1 General case.

Let \((X, f)\) is a dynamical system with Hausdorff compact phase space.

Remind, that for any dynamical system \((A, g)\) \(\in\) \(\text{Ob} \mathcal{A}\) category \(\mathcal{K}_0\) contains a morphism

\[ h : (X, f) \rightarrow (A, g) \] if and only if \(\Phi(\mathcal{P}(A, g)) \leq \Phi(\mathcal{P}(X, f))\) (see Theorem 3.1).

Consider a complete subcategory \(\mathcal{A}(X, f)\) of \(\mathcal{A}\), objects of which are dynamical systems \((A, g)\) \(\in\) \(\text{Ob} \mathcal{A}\), such that \(\Phi(\mathcal{P}(A, g)) \leq \Phi(\mathcal{P}(X, f))\).

It follows from Remark 3.11 that the category \(\mathcal{A}(X, f)\) has properties \(\text{LU}\) and \(\text{LD}\).

We fix a skeleton \(\mathcal{A}_0(X, f)\) of \(\mathcal{A}(X, f)\) and construct category \(\mathcal{A}_0(X, f)\), taking an advantage of Proposition 3.11.

In the same way as we did it for Corollary 3.9, we prove

Proposition 4.1 Let \((A_1, g_1), (A_2, g_2) \in \text{Ob} \mathcal{A}_0(X, f)\).

Then

(i) if \(\Phi(\mathcal{P}(A_1, g_1)) \geq \Phi(\mathcal{P}(A_2, g_2))\), then the set \(\mathcal{H}_{\mathcal{A}_0(X,f)}((A_1, g_1), (A_2, g_2))\) contains exactly one element;

(ii) \(\mathcal{H}_{\mathcal{A}_0(X,f)}((A_1, g_1), (A_2, g_2)) = \emptyset\), otherwise.

Consider the subset \(\Sigma(X, f) = \{N \in \Sigma \mid N \leq \Phi(\mathcal{P}(X, f))\}\) of \((\Sigma, \leq)\) and construct category \(\mathcal{L}(\Sigma(X, f))\) by this partially ordered set.

Similarly to Theorem 3.4 we prove

Theorem 4.1 Correspondence \(\Psi_0(X, f) : \text{Ob} \mathcal{A}_0(X, f) \rightarrow \text{Ob} \mathcal{L}(\Sigma(X, f))\),

\[ \Psi_0(X, f) : (A, g) \mapsto \Phi(\mathcal{P}(A, g)), \quad (A, g) \in \text{Ob} \mathcal{A}_0(X, f), \]

is uniquely extended to functor \(\Psi(X, f) : \mathcal{A}_0(X, f) \rightarrow \mathcal{L}(\Sigma(X, f))\).

The functor \(\Psi(X, f)\) sets the isomorphism of categories \(\mathcal{A}_0(X, f)\) and \(\mathcal{L}(\Sigma(X, f))\).

We define category \(\mathfrak{B}(X, f)\) as follows:

— objects of \(\mathfrak{B}(X, f)\) are the pairs \((h, (A, g))\), where \(h \in \text{Mor}_{\mathcal{K}_0}((X, f), (A, g))\) and \((A, g) \in \text{Ob} \mathcal{A}(X, f)\);

— for any two \((h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathfrak{B}(X, f)\) the set \(\mathcal{H}_{\mathfrak{B}(X,f)}((h_1, (A_1, g_1)), (h_2, (A_2, g_2)))\) consists of all triples \(((h_1, (A_1, g_1)), \pi, (h_2, (A_2, g_2)))\), such that \(\pi \in \mathcal{H}_{\mathcal{A}(X,f)}((A_1, g_1), (A_2, g_2))\) and \(h_2 = \pi \circ h_1\).

The correctness of this definition is checked immediately.

Lemma 4.1 Let \((h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathfrak{B}(X, f)\).

Then

(i) the set \(\mathcal{H}_{\mathfrak{B}(X,f)}((h_1, (A_1, g_1)), (h_2, (A_2, g_2)))\) contains no more than one element;

(ii) if \(\mathcal{H}_{\mathfrak{B}(X,f)}((h_1, (A_1, g_1)), (h_2, (A_2, g_2))) \neq \emptyset\), then \(\Phi(\mathcal{P}(A_1, g_1)) \geq \Phi(\mathcal{P}(A_2, g_2))\).
Proof. (i) Let \( \pi_1, \pi_2 : (A_1, g_1) \to (A_2, g_2) \) are two morphisms, such that \( h_2 = \pi_1 \circ h_1 = \pi_2 \circ h_1 \). Let \( x \in X \). Then \( h_2(x) = \pi_1(h_1(x)) = \pi_2(h_1(x)) \). Since dynamical system \((A_1, g_1)\) is minimal, then it follows from Remark 3.9 that \( \pi_1 = \pi_2 \).

(ii) This statement immediately follows from Theorem 3.1. \( \square \)

Lemma 4.2 Let \((h, (A, g)) \in \text{Ob} \mathfrak{B}(X, f)\), \(N \in \Sigma(X, f)\) and \( \Phi(\mathcal{P}(A, g)) \leq N \).

Then there exists \((h_1, (A_1, g_1)) \in \text{Ob} \mathfrak{B}(X, f)\), such that

(i) \( \Phi(\mathcal{P}(A_1, g_1)) = N \);

(ii) \( H_{\mathfrak{B}(X, f)}((h_1, (A_1, g_1)), (h, (A, g))) \neq \emptyset \).

Proof. On the condition of Lemma and by definition of the set \( \Sigma(X, f) \) we have following inequalities \( \Phi(\mathcal{P}(A, g)) \leq N \leq \Phi(\mathcal{P}(X, f)) \).

Proposition 2.1 consequences that there exists unique regular set \( Q \in \mathbb{N} \), such that \( \Phi(Q) = N \). Since the sets \( \mathcal{P}(A, g) \) and \( \mathcal{P}(X, f) \) are regular (see Remark 2.6), then from Lemma 2.1 we receive the inclusions \( \mathcal{P}(A, g) \subseteq Q \subseteq \mathcal{P}(X, f) \).

We use Proposition 2.2 and find regular sequences \( \{n_i \in \mathcal{P}(A, g)\}_{i \in \mathbb{N}} \) and \( \{m_j \in Q\}_{j \in \mathbb{N}} \), such that \( \Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g)) \) and \( \Phi(\{m_j \mid j \in \mathbb{N}\}) = \Phi(Q) = N \).

Fix regular sequence \( \{V^{(n_i)}\}_{i \in \mathbb{N}} \) of periodic partitions of dynamical system \((A, g)\). Consider pre-images \( W^{(n_i)} = \{W^{(n_i)}_n = h^{-1}(V^{(n_i)}_n)\}_{s \in \mathbb{Z}_n}, i \in \mathbb{N} \), of periodic partitions of the sequence \( \{V^{(n_i)}\}_{i \in \mathbb{N}} \). We conclude from Proposition 3.7, Definition 1.6 and Corollaries 1.5 and 1.6 that \( \{W^{(n_i)}\}_{i \in \mathbb{N}} \) is the regular sequence of periodic partitions of dynamical system \((X, f)\).

Let \( \mathfrak{H} \) is the partition of space \( X \), induced by the sequence \( \{W^{(n_i)}\} \). In accord with Proposition 3.8 the family of sets \( \{V^{(n_i)}_s \mid s_i \in \mathbb{Z}_n, i \in \mathbb{N}\} \) is basis of the topology of space \( A \). Therefore, \( \mathfrak{H} \) coincides with the partition zero \( h \).

We take advantage of Proposition 1.8 and Remark 1.6 and construct coherent sequence \( \{\tilde{W}^{(n_i)}\}_{j \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\), which is compatible with the sequence \( \{W^{(n_i)}\} \).

Let \( \tilde{\mathfrak{H}} \) is the partition of space \( X \), induced by the sequence \( \{\tilde{W}^{(n_i)}\} \). Taking into account Remark 3.4, we have the commutative diagram (see relation (12))

\[
\begin{array}{ccc}
(X, f) & \xrightarrow{F} & (A_1, g_1) \\
pr \downarrow & & \downarrow id \\
(X/\mathfrak{H}, \tilde{f}) & \xrightarrow{\text{fact}_F} & (A_1, g_1)
\end{array}
\]

(16)

In this diagram \((A_1, g_1)\) is an odometer, \( F \) is a projection, \( \text{fact} F \) is homeomorphism and \( \Phi(\mathcal{P}(X/\tilde{\mathfrak{H}}, \tilde{f})) = \Phi(\mathcal{P}(A_1, g_1)) = \Phi(\{m_j \mid j \in \mathbb{N}\}) = N \).

From Corollary 3.2 we conclude that the partition \( \tilde{\mathfrak{H}} = \text{zer} F \) is refinement of the partition \( \mathfrak{H} = \text{zer} h \) of space \( X \). Designate \( h_1 = F \). Applying now Lemma 0.2 to projections \( \varphi_1 = h_1 \) and \( \varphi_2 = h \) we consequence that there exists \( \psi \in H_K((A_1, g_1), (A, g)) \), such that \( h = \psi \circ h_1 \). \( \square \)

We define the “forgetful” functor \( \Theta : \mathfrak{B}(X, f) \to \mathcal{A}(X, f) \) with the help of the relations

\[
\Theta : (h, (A, g)) \mapsto (A, g), \quad (h, (A, g)) \in \text{Ob} \mathfrak{B}(X, f);
\]

\[
\Theta : ((h_1, (A_1, g_1)), \pi, (h_2, (A_2, g_2))) \mapsto \pi, \quad ((h_1, (A_1, g_1)), \pi, (h_2, (A_2, g_2))) \in \text{Mor}(\mathfrak{B}(X, f)).
\]

Consider pre-image \( \mathfrak{B}'(X, f) \) of the skeleton \( \mathcal{A}_0(X, f) \) under action of \( \Theta \). The easy immediate verification shows that \( \mathfrak{B}'(X, f) \) is complete subcategory of \( \mathfrak{B}(X, f) \).
Remark 4.1 Let $\mathcal{L}'$ is a complete subcategory of a category $\mathcal{L}$. Let $A, B \in \text{Ob}\,\mathcal{L}'$. By definition we have $H_{\mathcal{L}'}(A, B) = H_{\mathcal{L}}(A, B)$.

Hence, the objects $A$ and $B$ are isomorphic in $\mathcal{L}'$ if and only if they are isomorphic in $\mathcal{L}$.

Let $\mathfrak{B}'(X, f)$ is a skeleton of $\mathfrak{B}'(X, f)$. Obviously, $\mathfrak{B}_0'(X, f)$ is the complete subcategory of $\mathfrak{B}(X, f)$.

Proposition 4.2 Category $\mathfrak{B}_0'(X, f)$ is the skeleton of the category $\mathfrak{B}(X, f)$.

Proof. It follows from definition of the category $\mathfrak{B}_0'(X, f)$ and from Remark 4.1 that this subcategory contains no more than one representative from every class of isomorphic objects of the category $\mathfrak{B}(X, f)$.

We shall prove that for every $(h, (A, g)) \in \text{Ob}\,\mathfrak{B}(X, f)$ there exists an object contained in $\mathfrak{B}_0'(X, f)$, which is isomorphic to $(h, (A, g))$.

Consider $(A, g) = \Theta(h, (A, g)) \in \text{Ob}\,\mathcal{A}(X, f)$. Subcategory $\mathcal{A}_0(X, f)$ is the skeleton of $\mathcal{A}(X, f)$, therefore there exists exactly one object $(A', g') \in \text{Ob}\,\mathcal{A}_0(X, f)$, which is isomorphic to $(A, g)$. Let $\rho : (A, g) \rightarrow (A', g')$ is an isomorphism.

Designate $h' = \rho \circ h : (X, f) \rightarrow (A', g')$. Consider the object $(h', (A', g'))$ of category $\mathfrak{B}(X, f)$. Obviously, $(h', (A', g')) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$ and $((h, (A, g)), \rho, (h', (A', g'))) \in \text{Iso}\,\mathfrak{B}(X, f)$. □

We shall introduce now binary relation $\preceq$ on the class $\text{Ob}\,\mathfrak{B}_0'(X, f)$. We say that $(h_1, (A_1, g_1)) \preceq (h_2, (A_2, g_2))$ if $H_{\mathfrak{B}(X,f)}((h_2, (A_2, g_2)), (h_1, (A_1, g_1))) \neq \emptyset$.

Proposition 4.3 Relation $\preceq$ is the partial order and the following statements hold true:

(i) each element $(h, (A, g)) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$ is majorized by certain element $(h', (A', g')) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$, such that $\Phi(P(A', g')) = \Phi(P(X, f))$;

(ii) an element $(h, (A, g)) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$ is maximal under order relation $\preceq$ if and only if $\Phi(P(A, g)) = \Phi(P(X, f))$.

Before we proceed with the proof of Proposition 4.3, let us prove

Lemma 4.3 Let $(h, (A, g)) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$, $N \in \Sigma(X, f)$ and $\Phi(P(A, g)) \leq N$.

There exists such object $(h', (A', g')) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$, that

(i) $\Phi(P(A', g')) = N$;

(ii) $H_{\mathfrak{B}(X,f)}((h', (A', g')); (h, (A, g))) \neq \emptyset$.

Proof. According to Lemma 4.2 there exists $(h_1, (A_1, g_1)) \in \text{Ob}\,\mathfrak{B}(X, f)$, such that $\Phi(P(A_1, g_1)) = N$ and $H_{\mathfrak{B}(X,f)}((h_1, (A_1, g_1)), (h, (A, g))) \neq \emptyset$.

We conclude from Proposition 4.2 that there exists an object $(h', (A', g')) \in \text{Ob}\,\mathfrak{B}_0'(X, f)$, which is isomorphic to $(h_1, (A_1, g_1))$. Lemma 4.1 consequences that $\Phi(P(A', g')) = \Phi(P(A_1, g_1)) = N$. In addition, we have $H_{\mathfrak{B}(X,f)}((h', (A', g')); (h, (A, g))) \neq \emptyset$ on the construction. □

Proof of Proposition 4.3. For every object $B \in \text{Ob}\,\mathfrak{B}_0'(X, f)$ there exists unit morphism $1_B$ on definition of category, therefore relation $\preceq$ is reflexive.

Composition of any morphisms $\alpha \in H_{\mathfrak{B}(X,f)}(B, B')$ and $\beta \in H_{\mathfrak{B}(X,f)}(B', B'')$ is element of the set $H_{\mathfrak{B}(X,f)}(B, B'')$, hence relation $\preceq$ is transitive.
Let \( \alpha \in H_{\mathfrak{B}(X,f)}(B, B') \), \( \beta \in H_{\mathfrak{B}(X,f)}(B', B) \) for certain \( B, B' \in \text{Ob} \mathfrak{B}'_0(X, f) \). According to Lemma 4.1 we have \( H_{\mathfrak{B}(X,f)}(B, B) = \{ 1_B \} \), \( H_{\mathfrak{B}(X,f)}(B', B') = \{ 1_B' \} \). Hence, \( \beta \circ \alpha = 1_B \), \( \alpha \circ \beta = 1_{B'} \) and the objects \( B \) and \( B' \) are isomorphic in category \( \mathfrak{B}(X, f) \). Since \( \mathfrak{B}'_0(X, f) \) is the complete subcategory of \( \mathfrak{B}(X, f) \), then \( B \) and \( B' \) are isomorphic in \( \mathfrak{B}'_0(X, f) \) (see Remark 4.1).

From what has been said we conclude, that the relation \( \preceq \) is antisymmetric.

So, \( \preceq \) is the partial order relation.

Statement (i) of Proposition 4.3 follows from Lemma 4.3 and Theorem 3.1.

We shall prove now statement (ii).

Let \( (h, (A, g)), (h', (A', g')) \in \text{Ob} \mathfrak{B}'_0(X, f) \), \( \Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f)) \) and \( (h, (A, g)) \preceq (h', (A', g')) \). From Lemma 4.1 and Theorem 3.1 we conclude that \( \Phi(\mathcal{P}(A, g)) \leq \Phi(\mathcal{P}(A', g')) \leq \Phi(\mathcal{P}(X, f)) \). Hence, \( \Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f)) = \Phi(\mathcal{P}(A, g)) \) and any morphism \( \rho : (A', g') \to (A, g) \) is isomorphism (see Theorem 3.3 and corollary 3.5).

Since the set \( H_{\mathfrak{B}(X,f)}((h', (A', g')),(h, (A, g))) \) is not empty on our supposition, then the objects \( (h', (A', g')) \), \( (h, (A, g)) \) in \( \mathfrak{B}'_0(X, f) \) are isomorphic. Hence, \( (h', (A', g')) \preceq (h, (A, g)) \) (see Proposition 4.2) and \( (h, (A, g)) \) is the maximal element under relation \( \preceq \).

Let now \( (h, (A, g)) \) is a maximal element under \( \preceq \). The equality \( \Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f)) \) follows from the statement (i) of Proposition 4.3 and Theorem 3.1. \( \square \)

On definition we have \( \Theta(\text{Ob} \mathfrak{B}'_0(X, f)) = \text{Ob} \mathfrak{A}_0(X, f) = \text{Ob} \overline{\mathfrak{A}_0(X, f)} \), therefore the map is correctly defined

\[
\Lambda_0 = \Psi_0 \circ \Theta : \text{Ob} \mathfrak{B}'_0(X, f) \to \text{Ob} \mathfrak{L}(\Sigma(X, f)) = (\Sigma(X, f), \preceq)
\]

From Lemma 4.1 and Proposition 4.3 we obtain

**Corollary 4.1** The map \( \Lambda_0 \) preserves the order relation.

Pre-image \( \Lambda_0^{-1}(\Phi(\mathcal{P}(X, f))) \) of the greatest element of the set \( (\Sigma(X, f), \preceq) \) coincides with the class of all maximal elements from \( (\text{Ob} \mathfrak{B}'_0(X, f), \preceq) \).

From definition of the category \( \mathfrak{L}(\Sigma(X, f)) \), Lemma 4.1 and Corollary 4.1 we get

**Corollary 4.2** Map \( \Lambda_0 : \text{Ob} \mathfrak{B}'_0(X, f) \to \text{Ob} \mathfrak{L}(\Sigma(X, f)) \) is uniquely extended to the functor

\[
\Lambda : \mathfrak{B}'_0(X, f) \to \mathfrak{L}(\Sigma(X, f)).
\]

For any two objects \( B, B' \in \text{Ob} \mathfrak{B}'_0(X, f) \) the map

\[
\Lambda_{B, B'} : H_{\mathfrak{B}_0(X,f)}(B, B') \to H_{\mathfrak{L}(\Sigma(X,f))}(\Lambda(B), \Lambda(B'))
\]

is injective.

**Remark 4.2** 1) It follows from Lemma 4.1, that the equality \( \Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2)) \) is the necessary condition for two objects \( (h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathfrak{B}(X, f) \) to be isomorphic.

2) From Corollary 3.5 we obtain the statement: if \( \Phi(\mathcal{P}(A_1, g_1)) = \Phi(\mathcal{P}(A_2, g_2)) \), then objects \( (h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathfrak{B}(X, f) \) are isomorphic if and only if at least one of the sets

\[
H_{\mathfrak{B}(X,f)}((h_1, (A_1, g_1)), (h_2, (A_2, g_2))) \quad \text{and} \quad H_{\mathfrak{B}(X,f)}((h_2, (A_2, g_2)), (h_1, (A_1, g_1)))
\]

is not empty.
4.2 Indecomposable dynamical systems.

We have natural desire to “compare” somehow categories $\mathfrak{B}(X, f)$ and $L(\Sigma(X, f))$.

In what follows we shall see, that if dynamical system $(X, f)$ is indecomposable, then the category $L(\Sigma(X, f))$ is isomorphic to a skeleton of the category $\mathfrak{B}(X, f)$ (and the isomorphism is set by the functor $\Lambda$).

In the case, when dynamical system $(X, f)$ is not indecomposable, generally speaking it is not clear how to “compare” categories $\mathfrak{B}(X, f)$ and $L(\Sigma(X, f))$, as shows following

**Lemma 4.4** Let $(h, (A, g)) \in \text{Ob} \mathfrak{B}'_0(X, f)$, $N \in \Sigma(X, f)$ and $\Phi(P(A, g)) \not\subseteq N$.

Object $(h', (A', g')) \in \text{Ob} \mathfrak{B}'_0(X, f)$, which satisfies to Lemma 4.3, is defined uniquely if and only if dynamical system $(X, f)$ is indecomposable.

**Proof.** 1) Assume, that dynamical system $(X, f)$ is indecomposable.

Let $(h_1, (A_1, g_1))$, $(h_2, (A_2, g_2)) \in \text{Ob} \mathfrak{B}'_0(X, f)$, $\Phi(P(A_1, g_1)) = \Phi(P(A_2, g_2)) = N$ and $(h_1, (A, g)) \leq (h_i, (A_i, g_i)), i = 1, 2$.

Since $\Phi(P(A_1, g_1)) = \Phi(P(A_2, g_2))$, then dynamical systems $(A_1, g_1)$ and $(A_2, g_2)$ are topologically conjugate. We fix isomorphism $\rho : (A_1, g_1) \rightarrow (A_2, g_2)$.

We conclude from Remark 3.8 and Lemma 3.4 that the following commutative diagram holds true

$$
\begin{array}{ccc}
(X, f) & \xleftarrow{h_1} & (X, f) \\
\rho^{-1} \circ h_2 & & \circ \rho \\
\downarrow & & \downarrow \\
(A_1, g_1) & \xrightarrow{\text{fact}(\rho^{-1} \circ h_2)} & (A_1, g_1) \\
& & \rho \circ (A_2, g_2)
\end{array}
$$

In accord with Corollary 3.5 the map $\rho \circ \text{fact}(\rho^{-1} \circ h_2) : (A_1, g_1) \rightarrow (A_2, g_2)$ is isomorphism. $\mathfrak{B}'_0(X, f)$ is the skeleton of $\mathfrak{B}(X, f)$, therefore $(h_1, (A_1, g_1)) = (h_2, (A_2, g_2))$.

2) Let now dynamical system $(X, f)$ is not indecomposable.

Designate $\Phi(P(A, g)) = M \in \Sigma(X, f)$. Since $M \not\subseteq N$ on condition of Lemma, then there exists prime $p \in \mathbb{P}$, such that $M_p \not\subseteq N_p$. This implies, that $M_p \neq \infty$. Let $M_p = k$. Then $N_p \geq k + 1$.

We fix regular sequence $\{n_i\}_{i \in \mathbb{N}}$, such that $\Phi(\{n_i \mid i \in \mathbb{N}\}) = M$ (see. Proposition 2.2). It follows from definition of function $\Phi$ that

- $n_i = p^{k_i}a_i, k_i \leq k$, $\gcd(a_i, p) = 1$, $i \in \mathbb{N}$;
- there exists such $i_0 \in \mathbb{N}$, that $k_{i_0} = k$.

Sequence $\{n_i\}_{i \in \mathbb{N}}$ is regular, so $k_i = k$ for all $i \geq i_0$. Without loss of generality (see Corollary 2.1), we can consider that

$$
n_i = p^k a_i, \quad \gcd(a_i, p) = 1, \quad i \in \mathbb{N}.
$$

Once again we take advantage of Corollary 2.1 and we shall consider, that $n_1 = p^k$.

Fix regular sequence $\{m_j\}_{j \in \mathbb{N}}$, such that $\Phi(\{m_j \mid j \in \mathbb{N}\}) = N$(see the beginning of proof of Lemma 4.2). Sequence $\{m_j\}_{j \in \mathbb{N}}$ is regular and $N_p \geq k + 1$, therefore there exists $j_0 \in \mathbb{N}$, such that $m_j$ is divided by $p^{k+1}$ for every $j \geq j_0$.

Again using Corollary 2.1 we assume, that $m_j$ is divided by $p^{k+1}$ for all $j \in \mathbb{N}$ and $m_1 = p^{k+1}$. 
We fix regular sequence \( \{V^{(n_i)}\}_{i \in \mathbb{N}} \) of periodic partitions of dynamical system \((A, g)\). Consider pre-images \( W^{(n_i)} = \{W_{s_i}^{(n_i)} = h^{-1}(V_{s_i}^{(n_i)})\}_{s_i \in \mathbb{Z}_{n_i}}, \ i \in \mathbb{N} \), of periodic partitions of the sequence \( \{V^{(n_i)}\}_{i \in \mathbb{N}} \).

Repeating argument from the proof of Lemma 4.2 we conclude, that \( \{W^{(n_i)}\}_{i \in \mathbb{N}} \) is the regular sequence of periodic partitions of dynamical system \((X, f)\) and the partition \( \mathcal{U} \) of space \( X \), which is induced by this sequence, coincides with the partition \( \text{zer} \ h \).

Since \( N \leq \Phi(P(A, g)) \) on condition of Lemma, then there exists a coherent regular sequence \( \{U^{(m_j)}\}_{j \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\), which is compatible with the sequence \( \{W^{(n_i)}\}_{i \in \mathbb{N}} \) (see Proposition 1.8). Therefore the family of sets
\[
\mathcal{U}^{(m_j)} = P_{s_i}^{(m_j)} \cup f^{m_j}(Q_{s_i}^{(m_j)}) = P_{s_i}^{(m_j)} \cup f^{m_j}(Q_{s_i}^{(m_j)}) , \quad s_i \in \mathbb{Z}_{m_j},
\]
is the periodic partition of dynamical system \((X, f)\) of length \( m_1 = p^{k+1} \) (see the proof of Proposition 1.3).

Periodic partitions \( U^{(m_j)} \) and \( W^{(n_i)} \) are compatible and \( n_i \) divides \( m_1 \), hence there exists \( \tau \in \mathbb{Z}_{m_1} \), such that \( U^{(m_j)} = P^{(m_j)}_0 \cup Q^{(m_j)}_0 \subseteq \mathcal{W}^{(m_j)}_\tau \) (see Corollary 1.6).

Remark, that since \( f : X \to X \) is homeomorphism, then
\[
f^{m_j}(Q^{(m_j)}_0) = f^{m_j}(Q^{(m_j)}_0 \cap \mathcal{W}^{(m_j)}_\tau) = f^{m_j}(Q^{(m_j)}_0) \cap f^{m_j}(\mathcal{W}^{(m_j)}_\tau) = f^{m_j}(Q^{(m_j)}_0) \cap \mathcal{W}^{(m_j)}_\tau.
\]
Hence \( \mathcal{U}^{(m_j)} \subseteq \mathcal{W}^{(m_j)}_\tau \). From Corollary 1.5 we conclude, that periodic partitions \( \mathcal{U}^{(m_j)} \) and \( \mathcal{W}^{(n_i)} \) are compatible.

The inductive application of Proposition 1.7 and Remark 1.6 gives us a coherent sequence \( \{\mathcal{U}^{(m_j)}\}_{j \in \mathbb{N}} \) of periodic partitions of dynamical system \((X, f)\), which is compatible with the sequence \( \{\mathcal{W}^{(n_i)}\}_{i \in \mathbb{N}} \).

Let \( \mathcal{F} \) and \( \mathcal{W} \) are the partitions of space \( X \), induced by sequences \( \{U^{(m_j)}\}_{m_j \in \mathbb{N}} \) and \( \{\mathcal{U}^{(m_j)}\}_{m_j \in \mathbb{N}} \), respectively.

Iterating argument from the proof of Lemma 4.2, we shall find \((h_1, (A'_1, g'_1)), (h_2, (A'_2, g'_2)) \in \text{Ob } \mathfrak{W}(X, f)\), such that \( \mathcal{F} = \text{zer} \ h_1 \), \( \mathcal{W} = \text{zer} \ h_2 \) and
\[
h_{\mathfrak{W}(X, f)}((h_1, (A'_1, g'_1)), (h, (A, g))) \neq \emptyset , \quad i = 1, 2 .
\]

Find \((\pi_1, (A_1, g_1)), (\pi_2, (A_2, g_2)) \in \text{Ob } \mathfrak{W}'(X, f)\), which are isomorphic to objects \((h_1, (A'_1, g'_1))\) and \((h_2, (A'_2, g'_2))\) respectively. Obviously,
\[
h_{\mathfrak{W}(X, f)}((\pi_1, (A_1, g_1)), (h, (A, g))) \neq \emptyset , \quad i = 1, 2 .
\]

In order to verify the inequality \((\pi_1, (A_1, g_1)) \neq (\pi_2, (A_2, g_2))\) it suffices to show, that objects \((h_1, (A'_1, g'_1))\) and \((h_2, (A'_2, g'_2))\) are not isomorphic. For this purpose we shall take an advantage of Remark 4.2 (remind, that \( \Phi(P(A'_1, g'_1)) = \Phi(P(A'_2, g'_2)) = N \) on the construction, hence dynamical systems \((A'_1, g'_1)\) and \((A'_2, g'_2)\) are topologically conjugate).

Let us check the equality
\[
H_{\mathfrak{W}(X, f)}((h_1, (A'_1, g'_1)), (h_2, (A'_2, g'_2))) = \emptyset . \quad (17)
\]
Proposition 4.4

The category relation \( \pi \)

Proof. Obviously, there exists exactly one projection \( \pi \) and relations are fulfilled hence \( \mathcal{S} = \ker h_2 = \ker (\tilde{\alpha} \circ h_1) = \ker h_1 = \mathcal{S} \).

On the other hand, sequences \( \{U^{(m_i)}\}_{j \in \mathbb{N}} \) and \( \{\tilde{U}^{(m_i)}\}_{j \in \mathbb{N}} \) are not compatible. Really, on the construction we have \( \emptyset \neq U_0^{(m_1)} \cap \tilde{U}_0^{(m_1)} = P_0^{(m_1)} \subseteq X_1 \), Hence

\[
\bigcup_{n \in \mathbb{Z}} f^n(U_0^{(m_1)} \cap \tilde{U}_0^{(m_1)}) \subseteq X_1
\]

and periodic partitions \( U^{(m_1)} \) and \( \tilde{U}^{(m_1)} \) are not compatible. Therefore Corollary 3.2 consequences that \( \mathcal{S} \neq \mathcal{S} \).

The obtained contradiction proves the equality (17).

The equality

\[
H_{B(X,f)}((h_2, (A'_2, g'_2)), (h_1, (A'_1, g'_1))) = \emptyset
\]

is proved similarly. \( \Box \)

Remark 4.3

Obviously, the set \( (\Sigma, \leq) \) has the least element \( E = (E_r = 0)^{r \in \mathbb{B}} = \Phi_0(1) \).

Hence, for any dynamical system \((X,f)\) the element \( E \in \text{Ob } \mathfrak{L}(\Sigma(X,f)) \) is the right zero of category \( \mathfrak{L}(\Sigma(X,f)) \).

In the category \( \mathcal{A}_0(X,f) \) dynamical system \((\{pt\}, Id)\) with a phase space consisting of one point corresponds to the element \( E \) in the sense that \( \Phi(\{pt\}, Id) = E \).

Proposition 4.4

The category \( \mathfrak{B}_0'(X,f) \) has the right zero \( 0_R \in \text{Ob } \mathfrak{B}_0'(X,f) \).

Proof. Obviously, there exists exactly one projection \( \pi_0 : X \to \{pt\} \) and it satisfies to the relation \( \pi_0 \circ f = Id \circ \pi_0 \). Designate \( 0_R = (\pi_0, (\{pt\}, Id)) \).

Let \( (h, (A,g)) \in \text{Ob } \mathfrak{B}_0'(X,f) \). Obviously, the projection \( \pi : A \to \{pt\} \) is uniquely defined and relations are fulfilled \( \pi \circ g = Id \circ \pi \) and \( \pi \circ h = \pi_0 : (X, f) \to (\{pt\}, Id) \). \( \Box \)

Remark 4.4

Since any two different objects of category \( \mathfrak{B}_0'(X,f) \) are not isomorphic (on definition of skeleton of a category), then the right zero is defined uniquely.

Now we can extract from Lemma 4.4 the following statements.

Corollary 4.3

Let \( \Phi(\mathcal{P}(X,f)) \neq E, N \in \Sigma(X,f) \) and \( N \neq E \).

There exists \( (h, (A,g)) \in \text{Ob } \mathfrak{B}_0'(X,f) \), such that \( \Phi(\mathcal{P}(A,g)) = N \) and the following statements are equivalent:

(i) object \( (h, (A,g)) \in \text{Ob } \mathfrak{B}_0'(X,f) \), such that \( \Phi(\mathcal{P}(A,g)) = N \), is defined uniquely;

(ii) dynamical system \((X,f)\) is indecomposable.

Proof. We apply Lemmas 4.3 and 4.4 to object \( 0_R \in \text{Ob } \mathfrak{B}_0'(X,f) \) and number \( N \in \Sigma(X,f) \). \( \Box \)

Remark 4.5

In other words Corollary 4.3 can be formulated as follows:

— map \( \Lambda_0 : \text{Ob } \mathfrak{B}_0'(X,f) \to \text{Ob } \mathfrak{L}(\Sigma(X,f)) \) is surjective;
— if $\Phi(\mathcal{P}(X, f)) \neq E$, then injectivity of $\Lambda_0$ is equivalent to that the dynamical system $(X, f)$ is indecomposable.

**Corollary 4.4** If dynamical system $(X, f)$ is indecomposable, then for any two objects $(h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathcal{B}_0'(X, f)$ the inequality

$$H_{(X,f)}((h_2, (A_2, g_2)), (h_1, (A_1, g_1))) \neq \emptyset$$

is fulfilled if and only if $\Phi(\mathcal{P}(A_1, g_1)) \leq \Phi(\mathcal{P}(A_2, g_2))$.

**Proof.** Let $(h_1, (A_1, g_1)), (h_2, (A_2, g_2)) \in \text{Ob} \mathcal{B}_0'(X, f)$.

If $H_{(X,f)}((h_2, (A_2, g_2)), (h_1, (A_1, g_1))) \neq \emptyset$, then Lemma 4.1 consequences that $\Phi(\mathcal{P}(A_1, g_1)) \leq \Phi(\mathcal{P}(A_2, g_2))$.

Let now $\Phi(\mathcal{P}(A_1, g_1)) \leq \Phi(\mathcal{P}(A_2, g_2))$. We take advantage of Lemma 4.3 and find $(h_2', (A_2', g_2')) \in \text{Ob} \mathcal{B}_0'(X, f)$, such that $\Phi(\mathcal{P}(A_2', g_2')) = \Phi(\mathcal{P}(A_2, g_2))$ and

$$H_{(X,f)}((h_2', (A_2', g_2')), (h_1, (A_1, g_1))) \neq \emptyset.$$

From Corollary 4.3 we conclude, that $(h_2', (A_2', g_2')) = (h_2, (A_2, g_2))$. □

**Lemma 4.5** Let $\Phi(\mathcal{P}(X, f)) \neq E$. Then the following statements are equivalent:

(i) category $\mathcal{B}_0'(X, f)$ has the left zero $0_L \in \text{Ob} \mathcal{B}_0'(X, f)$;

(ii) dynamical system $(X, f)$ is indecomposable.

**Proof.** 1) Let dynamical system $(X, f)$ is indecomposable.

From Corollary 4.3 it follows, that there exists a unique object $(h, (A, g)) \in \text{Ob} \mathcal{B}_0'(X, f)$, such that $\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))$.

Theorem 3.1 guarantees that for any $(h', (A', g')) \in \text{Ob} \mathcal{B}_0'(X, f)$ the inequality $\Phi(\mathcal{P}(A', g')) \leq \Phi(\mathcal{P}(A, g))$ is fulfilled. Now Corollary 4.4 and Lemma 4.1 show, that for every $(h', (A', g')) \in \text{Ob} \mathcal{B}_0'(X, f)$ the set

$$H_{(X,f)}((h, (A, g)), (h', (A', g')))$$

contains exactly one element, hence $(h, (A, g))$ is the left zero of $\mathcal{B}_0'(X, f)$.

2) Let dynamical system $(X, f)$ is not indecomposable.

Assume, that there exists a left zero $(h, (A, g))$ of category $\mathcal{B}_0'(X, f)$. From Theorem 3.1 and Lemma 4.1 we conclude, that $\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))$.

Corollary 4.3 implies, that there exists $(h', (A', g')) \in \text{Ob} \mathcal{B}_0'(X, f)$, such that $\Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f))$ and $(h', (A', g')) \neq (h, (A, g))$.

Since $\mathcal{B}_0'(X, f)$ is the skeleton of $\mathcal{B}(X, f)$ (see Proposition 4.2), then objects $(h', (A', g'))$ and $(h, (A, g))$ are not isomorphic in $\mathcal{B}(X, f)$. From Remark 4.2 we conclude, that

$$H_{(X,f)}((h, (A, g)), (h', (A', g'))) = \emptyset$$

and the object $(h, (A, g))$ can not be left zero of $\mathcal{B}_0'(X, f)$.

The obtained contradiction finishes the proof. □

**Theorem 4.2** Let $\Phi(\mathcal{P}(X, f)) \neq E$, $\mathcal{B}_0'(X, f)$ is a skeleton of the category $\mathcal{B}(X, f)$.

The following statements are equivalent:
(i) dynamical system \((X, f)\) is indecomposable;

(ii) categories \(\mathfrak{B}_0(X, f)\) and \(\mathfrak{L}((\Sigma(X, f))\) are isomorphic.

Proof. On definition \(\mathfrak{L}((\Sigma(X, f))\) has left zero \(\Phi((\mathcal{P}(X, f)) \in \text{Ob} \mathfrak{L}((\Sigma(X, f))\). Hence, the existence of left zero in the category \(\mathfrak{B}_0(X, f)\) is the necessary condition for categories \(\mathfrak{B}_0(X, f)\) and \(\mathfrak{L}((\Sigma(X, f))\) to be isomorphic.

Subcategory \(\mathfrak{B}_0'(X, f)\) is the skeleton of \(\mathfrak{B}_0(X, f)\) (see Proposition 4.2), therefore it is isomorphic to the skeleton \(\mathfrak{B}_0(X, f)\). If dynamical system \((X, f)\) is not indecomposable, then we conclude from Lemma 4.5, that there is no left zero in the category \(\mathfrak{B}_0(X, f)\) and it can not be isomorphic to \(\mathfrak{L}((\Sigma(X, f))\).

Assume now that \((X, f)\) is an indecomposable dynamical system. We shall prove, that the functor \(\Lambda : \mathfrak{B}_0'(X, f) \rightarrow \mathfrak{L}((\Sigma(X, f))\) (see Corollary 4.2) sets the isomorphism of categories \(\mathfrak{B}_0'(X, f)\) and \(\mathfrak{L}((\Sigma(X, f))\).

From Remark 4.5 the map \(\Lambda_0 : \text{Ob} \mathfrak{B}_0'(X, f) \rightarrow \text{Ob} \mathfrak{L}((\Sigma(X, f))\) appears to be bijective.

Let \(M, N \in \text{Ob} \mathfrak{L}((\Sigma(X, f))\). Corollary 4.4 consequences that inequalities

\[ H_{\mathfrak{L}((\Sigma(X, f))}(M, N) \neq \emptyset \quad \text{and} \quad H_{\mathfrak{B}_0'(X, f)}(\Lambda_0^{-1}(M), \Lambda_0^{-1}(N)) \neq \emptyset \]

are equivalent. Since each of the sets \(H_{\mathfrak{L}((\Sigma(X, f))}(M, N)\) and \(H_{\mathfrak{B}_0'(X, f)}(\Lambda_0^{-1}(M), \Lambda_0^{-1}(N))\) contain no more than one element (see definition of category \(\mathfrak{L}((\Sigma(X, f))\) and Lemma 4.1), then

\[ \Lambda_{\Lambda_0^{-1}(M), \Lambda_0^{-1}(N)} : H_{\mathfrak{B}_0'(X, f)}(\Lambda_0^{-1}(M), \Lambda_0^{-1}(N)) \rightarrow H_{\mathfrak{L}((\Sigma(X, f))}(M, N) \]

is bijective map for any pair \(M, N \in \text{Ob} \mathfrak{L}((\Sigma(X, f))\).

Thus, \(\Lambda : \mathfrak{B}_0'(X, f) \rightarrow \mathfrak{L}((\Sigma(X, f))\) sets the isomorphism of categories \(\mathfrak{B}_0'(X, f)\) and \(\mathfrak{L}((\Sigma(X, f))\).

Therefore, categories \(\mathfrak{B}_0(X, f)\) and \(\mathfrak{L}((\Sigma(X, f))\) are also isomorphic. \(\square\)

4.3 Expansions of odometers and almost periodic points

Proposition 4.5 Let \((A, g) \in \text{Ob} \mathcal{A}(X, f)\).

Suppose there exists projection \(\pi : (X, f) \rightarrow (A, g)\), such that for a certain \(x \in X\) the equality \(\pi^{-1}(\pi(x)) = \{x\}\) is fulfilled.

Then the following statements hold true:

(i) \(x \in X\) is the almost periodic point of dynamical system \((X, f)\);

(ii) for any \(y \in X\) the inclusion is valid \(\text{Orb}_f(x) \subseteq \alpha(y) \cap \omega(y)\) (hence, \(\text{Orb}_f(x)\) is the unique minimal set of dynamical system \((X, f)\), in particular \((X, f)\) is indecomposable);

(iii) \(\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))\);

(iv) for any projection \(\pi' : (X, f) \rightarrow (A', g')\), \((A', g') \in \text{Ob} \mathcal{A}\), the following conditions are equivalent:

a) \((\pi')^{-1}(\pi'(x)) = \{x\}\);

b) \(\Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f))\);

(v) if dynamical system \((X, f)\) is minimal, then for every almost periodic point \(y \in X\) of dynamical system \((X, f)\) the equality \(\pi^{-1}(\pi(y)) = \{y\}\) is fulfilled.
Before we proceed to prove Proposition 4.5 and extract corollaries from it, we shall prove three lemmas.

**Lemma 4.6** Let \((A_1, g_1), (A_2, g_2)\) \(\in\) \(\text{Ob} \mathcal{A}\), \(h : (A_1, g_1) \to (A_2, g_2)\) is a morphism.

If there exists \(x \in A_1\), such that \(h^{-1}(h(x)) = \{x\}\), then \(h \in \text{Iso} \mathcal{A}\).

**Proof.** Since \((A_2, g_2)\) is minimal dynamical system (see Remark 3.2), then \(h\) is projection (see Remark 3.8).

Let \(y \in A_1\). Designate by \(H(y)\) the element of partition \(\text{zer} \ h\) of space \(A_1\), which contains \(y\).

We know from Corollary 3.6 that there exists unique isomorphism \(h_{y, x} : (A_1, g_1) \to (A_1, g_1)\), such that \(h_{y, x}(y) = x\). Consider a projection \(\tilde{h} = h \circ h_{y, x} : (A_1, g_1) \to (A_2, g_2)\). Designate by \(\tilde{H}(y)\) the element of partition \(\text{zer} \ \tilde{h}\) of space \(A_1\), which contains \(y\).

Let \(z = h(x) \in A_2\). It is clear, that

\[
\tilde{h}^{-1}(z) = h_{y, x}^{-1}(h^{-1}(z)) = h_{y, x}^{-1}(x) = \{y\} = \tilde{H}(y).
\]

Dynamical system \((A_2, g_2)\) is minimal, hence it is indecomposable. We conclude from Lemma 3.3 that \(\text{zer} \ h = \text{zer} \ \tilde{h}\). Hence, \(h^{-1}(h(y)) = H(y) = \tilde{H}(y) = \{y\}\). \(\square\)

**Lemma 4.7** Let dynamical system \((X, f)\) is minimal and \(x \in X\) is almost periodic point of this dynamical system.

Then there exist dynamical system \((A, g) \in \mathcal{A}(X, f)\) and projection \(\pi : (X, f) \to (A, g)\), such that \(\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))\) and \(\pi^{-1}(\pi(x)) = \{x\}\).

**Proof.** We fix regular sequence \(\{n_i\}_{i \in \mathbb{N}}\), such that \(\Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(X, f))\), and build regular sequence \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) of periodic partitions of dynamical system \((X, f)\). Without loss of generality we can suppose that \(x \in W^{(n_i)}_0, i \in \mathbb{N}\) (see Remark 1.6).

Let \(\mathcal{H}\) is the partition of space \(X\), induced by the sequence \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) and let

\[
F : (X, f) \to (X/\mathcal{H}, \overline{\mathcal{H}}) = (A, g)
\]

is the projection to dynamical system \((A, g) \in \mathcal{A}(X, f)\) (see relation (12), Remark 3.4 and Proposition 3.3). Then \(\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))\).

Since \(\text{zer} \ F = \mathcal{H}\), then in order to complete the proof it is enough to verify the equality

\[
\{x\} = \bigcap_{i \in \mathbb{N}} W^{(n_i)}_0.
\]

Assume, that this equality is invalid and there exists \(y \neq x\), such that

\[
y \in \bigcap_{i \in \mathbb{N}} W^{(n_i)}_0.
\]

The space \(X\) is Hausdorff, therefore there exists a closed neighborhood \(U \subseteq X \setminus \{y\}\) of \(x\). Since \(x\) is the almost periodic point, then we conclude from Lemma 1.5 that there exist \(m \in \mathcal{P}(X, f)\) and periodic partition \(\overline{W}^{(m)}\) of dynamical system \((X, f)\), such that \(x \in \overline{W}^{(m)}_0 \subseteq U\).

Consider the stationary regular sequence \(\{m_j = m\}_{j \in \mathbb{N}}\). From Lemma 2.1 it follows, that \(\Phi_0(m) = \Phi(\{m_j \mid j \in \mathbb{N}\}) \leq \Phi(\mathcal{P}(X, f)) = \Phi(\{n_i \mid i \in \mathbb{N}\})\). Therefore Proposition 2.3 consequences that there exists \(k \in \mathbb{N}\), such that \(m_1 = m\) divides \(n_k\).
The dynamical system \((X, f)\) is minimal, hence it is indecomposable. Corollary 1.1 implies, that periodic partitions \(\widetilde{W}^{(m)}\) and \(W^{(nk)}\) are compatible. Therefore (see Corollary 1.6) the inclusions hold true \(x \in W^{(nk)}_0 \subseteq \widetilde{W}^{(m)}_0 \subseteq U \subseteq X \setminus \{y\}\) and

\[
y \notin \bigcap_{i \in \mathbb{N}} W^{(n_i)}_0.
\]

The obtained contradiction completes the proof. \(\square\)

**Lemma 4.8** Let \(x \in Y_1\) is an almost periodic point of a dynamical system \((Y_1, h_1)\). Let \(\pi : (Y_1, h_1) \to (Y_2, h_2)\) is a projection.

Then \(y = \pi(x) \in Y_2\) is an almost periodic point of the dynamical system \((Y_2, h_2)\).

**Proof.** Let \(U \subseteq Y_2\) is an open neighborhood of \(y\). Since \(\pi : Y_1 \to Y_2\) is continuous map, then \(V = \pi^{-1}(U)\) is the open neighborhood of the almost periodic point \(x\). Hence, there exists \(n(V) \in \mathbb{N}\), such that

\[
\bigcup_{k \in \mathbb{Z}} h_1^{k n(V)}(x) \subseteq V.
\]

Then

\[
\bigcup_{k \in \mathbb{Z}} h_2^{k n(V)}(y) = \bigcup_{k \in \mathbb{Z}} h_2^{k n(V)} \circ \pi(x) = \bigcup_{k \in \mathbb{Z}} \pi \circ h_1^{k n(V)}(x) \subseteq \pi(V) = U.
\]

From arbitrariness in the choice of a neighborhood \(U\) it follows, that \(y = \pi(x)\) is almost periodic point of dynamical system \((Y_2, h_2)\). \(\square\)

**Proof of Proposition 4.5.** (i) We fix regular sequence \(\{n_i\}_{i \in \mathbb{N}},\) such that \(\Phi(\{n_i \mid i \in \mathbb{N}\}) = \Phi(\mathcal{P}(A, g))\). Construct regular sequence \(\{V^{(n_i)}\}_{i \in \mathbb{N}}\) of periodic partitions of dynamical system \((A, g)\). Without loss of generality we can assume, that \(z = \pi(x) \in V^{(n_i)}_0, i \in \mathbb{N}\) (see Remark 1.6).

According to Proposition 3.8 the family of sets \(\{V^{(n_i)}_s \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) is basis of the topology of space \(A\). Therefore,

\[
\{z\} = \bigcap_{i \in \mathbb{N}} V^{(n_i)}_0.
\]

In addition since the sequence \(\{V^{(n_i)}\}_{i \in \mathbb{N}}\) is regular, then \(V^{(n_{i+1})}_0 \subseteq V^{(n_i)}_0, i \in \mathbb{N}\) (see Corollary 1.6).

Consider pre-images \(\{W^{(n_i)}_{s_i} = \pi^{-1}(V^{(n_i)}_s) \mid s_i \in \mathbb{Z}_{n_i}, i \in \mathbb{N}\}\) of periodic partitions \(\{V^{(n_i)}_s \mid s_i \in \mathbb{Z}_{n_i}\}\).

It is clear, that \(W^{(n_{i+1})}_0 \subseteq W^{(n_i)}_0, i \in \mathbb{N}\), and

\[
\{x\} = \pi^{-1}(z) = \bigcap_{i \in \mathbb{N}} W^{(n_i)}_0.
\]

We consequence from Proposition 3.7 and Corollary 1.5, that \(\{W^{(n_i)}\}_{i \in \mathbb{N}}\) is the regular sequence of periodic partitions of dynamical system \((X, f)\).

Let \(U \subseteq X\) is an open neighborhood of \(x\). Since \(X\) is compact, then all sets \(W^{(n_i)}_0, i \in \mathbb{N}\), are compact. Applying Lemma 3.2, we find \(k \in \mathbb{N}\), such that \(x \in W^{(nk)}_0 \subseteq U\).
On definition of periodic partition we have
\[ \bigcup_{m \in \mathbb{Z}} f^{mN}(x) \subseteq \bigcup_{m \in \mathbb{Z}} f^{mN}(W^{\prime}(n)) = W^{\prime}(n) \subseteq U. \]

By virtue of arbitrariness in the choice of a neighborhood \( U \) the point \( x \) is almost periodic.

(ii) Let \( y \in X, t = \pi(y) \in A \). Dynamical system \((A, g)\) is minimal (see Remark 3.2), consequently \( \alpha(t) = \omega(t) = \text{Orb}_g(t) = A \). Hence there exists monotonic unlimited sequence of numbers \( \{n_i \in \mathbb{Z}\}_{i \in \mathbb{N}} \), such that \( z = \pi(x) = \lim_{i \to \infty} g^n(t) \).

Consider the sequence \( \{f^{n_i}(y) \in X\}_{i \in \mathbb{N}} \). Space \( X \) is compact, therefore this sequence has at least one limit point \( x' \in X \). Passing to a subsequence we can assume, that \( x' = \lim_{i \to \infty} f^{n_i}(y) \).

Hence, \( x' \in \omega(y) \).

On the other hand, \( \pi \circ f^{n_i}(y) = g^{n_i} \circ \pi(y) = g^{n_i}(t) \), therefore
\[ \pi(x') = \lim_{i \to \infty} \pi \circ f^{n_i}(x) = \lim_{i \to \infty} g^{n_i}(t) = \pi(x) \]
and \( x = x' \in \omega(y) \). Since \( \omega(y) \) is closed invariant set of dynamical system \((X, f)\), then \( \text{Orb}_f(x) \subseteq \omega(y) \).

The relation \( \text{Orb}_f(x) \subseteq \alpha(y) \) is proved similarly.

(iii) We consider \((\pi, (A, g)) \in \text{Ob} \mathcal{B}(X, f)\). It follows from Theorem 3.1, that \( \Phi(\mathcal{P}(A, g)) \leq \Phi(\mathcal{P}(X, f)) \). Apply Lemma 4.2 and find \((\pi', (A', g')) \in \text{Ob} \mathcal{B}(X, f)\), such that \( \Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f)) \) and exists \( h \in H_{\mathcal{B}(X, f)}((\pi', (A', g'))), (\pi, (A, g))). \)

In other words, there exist \((A', g') \in \mathcal{A}(X, f)\) and \( h : (A', g') \to (A, g) \), such that \( \Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f)) \) and \( \pi = h \circ \pi' \).

We conclude from Remark 3.8, that map \( \pi' : X \to A' \) is surjective. Hence for any subset \( B \subseteq A' \) the equality \( \pi'((\pi')^{-1}(B)) = B \) is fulfilled.

Let \( z' = \pi'(x) \in A' \). Then \( h(z') = h \circ \pi'(x) = z \). The equalities hold true \((\pi')^{-1}(h^{-1}(z)) = \pi^{-1}(z) = \{x\}, \) hence
\[ h^{-1}(h(z')) = h^{-1}(z) = \pi'(\pi'^{-1}(h^{-1}(z))) = \{\pi'(x)\} = \{z'\}. \]

Applying Lemma 4.6 we conclude, that \( h : (A', g') \to (A, g) \) is isomorphism. Hence, \( \Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f)). \)

(iv) We have already proved in (iii) the implication \( a) \Rightarrow b) \).

Let \((A', g') \in \mathcal{A}(X, f)\), \( \Phi(\mathcal{P}(A', g')) = \Phi(\mathcal{P}(X, f)) \) and \( \pi' : (X, f) \to (A', g') \) is a projection.

Dynamical system \((X, f)\) is indecomposable (see. (ii)), therefore from Lemma 4.4 it follows (we shall remind, that \( \mathcal{B}_0(X, f) \) is the skeleton of \( \mathcal{B}(X, f) \)), that objects \((\pi', (A', g'))\) and \((\pi, (A, g))\) of the category \( \mathcal{B}(X, f) \) are isomorphic. Hence there exists isomorphism \( \rho : (A, g) \to (A', g') \), such that \( \pi' = \rho \circ \pi \). In addition it is obvious that \( \pi = \rho^{-1} \circ \pi' \). Applying Lemma 0.2 we conclude, that partitions \( \text{zer} \pi \) and \( \text{zer} \pi' \) coincide.

The sets \( \pi^{-1}(\pi(x)) \) and \( \pi'^{-1}(\pi'(x)) \) are elements of the partitions \( \text{zer} \pi \) and \( \text{zer} \pi' \) respectively, and contain the point \( x \). Hence, \( \pi'^{-1}(\pi'(x)) = \pi^{-1}(\pi(x)) = \{x\} \).

(v) Let dynamical system \((X, f)\) is minimal and \( y \in X \) is almost periodic point of this dynamical system.

Applying Lemma 4.7, we find projection \( \pi' : (X, f) \to (A', g'), (A', g') \in \mathcal{A}(X, f) \), such that \( (\pi')^{-1}(\pi')(y) = \{y\} \). Then changing roles of projections \( \pi \) and \( \pi' \), we shall receive from (iv) that \( (\pi)^{-1}(\pi)(y) = \{y\} \). \( \Box \)
Definition 4.1. Dynamical system \((Y_1, h_1)\) is called almost one-to-one expansion of a dynamical system \((Y_2, h_2)\), if there exist projection \(\pi : (Y_1, h_1) \to (Y_2, h_2)\) and a dense subset \(Q \subseteq Y_1\), such that \(\pi^{-1}(\pi(y)) = \{y\}\) for any \(y \in Q\).

Corollary 4.5 (see [10]). Dynamical system \((X, f)\) is almost one-to-one expansion of an odometer if and only if \(X = \text{Orb}_f(x)\) for an almost periodic point \(x \in X\).

Proof. Let \(\pi : (X, f) \to (Y, h)\) is a projection and \(\pi^{-1}(\pi(x)) = \{x\}\) for a certain \(x \in X\).

Remark, that since \(f : X \to X\) and \(h : Y \to Y\) are the homeomorphisms, then for every \(n \in \mathbb{Z}\) equalities hold true

\[
\{x\} = \pi^{-1}(\pi(x)) = \pi^{-1}(h^{-n} \circ \pi \circ f^n(x)) = (h^n \circ \pi)^{-1}(\pi \circ f^n(x)) = \\
= (\pi \circ f^n)^{-1}(\pi \circ f^n(x)) = f^{-n}(\pi^{-1}(\pi(f^n(x)))).
\]

Hence

\[
\{f^n(x)\} = f^n \circ f^{-n}(\pi^{-1}(\pi(f^n(x)))) = \pi^{-1}(\pi(f^n(x))) .
\]

In other words, for every \(y \in \text{Orb}_f(x)\) equality \(\pi^{-1}(\pi(y)) = \{y\}\) is fulfilled.

If the dynamical system \((X, f)\) is minimal, then \(X = \text{Orb}_f(x)\) and \((X, f)\) is almost one-to-one expansion of the dynamical system \((Y, h)\).

1. Let \(X = \text{Orb}_f(x)\) for some almost-periodic point \(x \in X\). Then dynamical system \((X, f)\) is minimal by Birkhoff Theorem and applying Lemma 4.7 and argument mentioned above, we conclude that \((X, f)\) is almost one-to-one expansion of an odometer.

2. Let dynamical system \((X, f)\) is almost one-to-one expansion of an odometer \((A, g)\).

We fix a projection \(\pi : (X, f) \to (A, g)\), such that for each point \(y \in \text{Orb}_f(y)\) from some everywhere dense set \(Q \subseteq X\) the equality \(\pi^{-1}(\pi(y)) = \{y\}\) is fulfilled.

From Proposition 4.5, item (i), we conclude, that any point \(y \in Q\) is an almost-periodic point of \((X, f)\). Hence, for every \(y \in Q\) the set \(\text{Orb}_f(y)\) is minimal set of dynamical system \((X, f)\).

From here we consequence that either \(\text{Orb}_f(y_1) = \text{Orb}_f(y_2)\) or \(\text{Orb}_f(y_1) \cap \text{Orb}_f(y_2) = \emptyset\) for arbitrary \(y_1, y_2 \in Q\).

We fix \(x \in Q\). For any \(y \in Q\) we have inclusions (see. Proposition 4.5, item (iii)) \(\text{Orb}_f(x) \subseteq \alpha(y) \cap \omega(y) \subseteq \text{Orb}_f(y)\). Hence, \(\text{Orb}_f(x) = \text{Orb}_f(y)\), in particular \(y \in \text{Orb}_f(x)\). Since \(Q\) is dense in \(X\), then \(X = Q = \text{Orb}_f(x)\). \(\square\)

Corollary 4.6. Let \((Y_1, h_1)\) is almost one-to-one expansion of an odometer, \(\pi : (Y_1, h_1) \to (Y_2, h_2)\) is a projection.

Then \((Y_2, h_2)\) is almost one-to-one expansion of an odometer.

Proof. Since map \(\pi : Y_1 \to Y_2\) is surjective and continuous, then for any dense subset \(Q\) of space \(Y_1\) its image \(\pi(Q)\) is dense in \(Y_2\).

We conclude from Corollary 4.5 that there exists almost periodic point \(x \in Y_1\), such that \(Y_1 = \text{Orb}_{h_1}(x)\). From Lemma 4.8 it follows, that \(\pi(x)\) is an almost-periodic point of the dynamical system \((Y_2, h_2)\), and \(\text{Orb}_{h_2}(\pi(x)) = Y_2\) (see above).

Again applying Corollary 4.5, we come to a conclusion, that \((Y_2, h_2)\) is almost one-to-one expansion of odometer. \(\square\)
Corollary 4.7 Let \((X, f)\) is almost one-to-one expansion of certain odometer, \((A, g) \in \mathcal{A}(X, f)\) and \(\pi : (X, f) \to (A, g)\) is a projection.

Let \(Q = \{y \in X \mid \pi^{-1}(\pi(y)) = \{y\}\}\). If \(\Phi(\mathcal{P}(A, g)) = \Phi(\mathcal{P}(X, f))\), then \(Q\) coincides with the set of all almost periodic points of the dynamical system \((X, f)\). If \(\Phi(\mathcal{P}(A, g)) \neq \Phi(\mathcal{P}(X, f))\), then \(Q = \emptyset\).

**Proof.** From Corollary 4.5 and Birkhoff theorem we conclude, that the dynamical system \((X, f)\) is minimal.

The first part of the current statement follows from Lemma 4.7 and Proposition 4.5, items (i), (iv) and (v).

The second part follows from Proposition 4.5, item (iii). \(\Box\)

**Corollary 4.8** (see [10]) Let \((X, f)\) is a transitive dynamical system.

The dynamical system \((X, f)\) is topologically conjugate with an odometer if and only if each point of space \(X\) is an almost periodic point of dynamical system \((X, f)\).

**Proof.** 1. Let each point of space \(X\) is an almost-periodic point of \((X, f)\). Since dynamical system \((X, f)\) is transitive, then on definition there exists \(x \in X\), such that \(X = \text{Orb}_f(x)\).

Applying Corollary 4.5 we conclude, that \((X, f)\) is almost one-to-one expansion of an odometer (in particular, dynamical system \((X, f)\) is minimal under Birkhoff theorem). We take advantage of Lemma 4.7 and find \((A, g) \in \mathcal{A}(X, f)\) and projection \(\pi : (X, f) \to (A, g)\), such that \(\pi^{-1}(\pi(x)) = \{x\}\).

From Corollary 4.7 we consequence, that map \(\pi : X \to A\) is bijective. Space \(X\) is compact, hence \(\pi\) is the homeomorphism and \(\pi : (X, f) \to (A, g)\) is isomorphism in the category \(\mathcal{K}_0\).

2. Let \((X, f) \in \mathcal{A}\). Consider the unit morphism \(Id : (X, f) \to (X, f)\). Since map \(Id : X \to X\) is bijective, then we conclude from Proposition 4.5, item (i), that every point of space \(X\) is an almost-periodic point of dynamical system \((X, f)\). \(\Box\)

**Corollary 4.9** (see [10]) Let dynamical system \((A, g)\) is topologically conjugate with an odometer, \(\pi : (A, g) \to (X, f)\) is a projection.

Then dynamical system \((X, f)\) is topologically conjugate with an odometer.

**Proof.** The dynamical system \((A, g)\) is minimal (see. Remark 3.2), therefore it is transitive. Hence the dynamical system \((X, f)\) is transitive too (see proof of Corollary 4.6).

From Corollary 4.8 and Lemma 4.8 we conclude, that each point of the space \(X\) is an almost-periodic point of dynamical system \((X, f)\).

In order to complete the proof it suffices to apply Corollary 4.8 once again. \(\Box\)

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