Abstract

In 1969 Erdős proved that if \( r \geq 2 \) and \( n > n_0(r) \), every graph \( G \) of order \( n \) and \( e(G) > t_r(n) \) has an edge that is contained in at least \( n^{r-1}/(10r)^6r(r+1) \)-cliques. In this note we improve this bound to \( n^{r-1}/r+5 \). We also prove a corresponding stability result.

Keywords: extremal graph, clique, book, joint, jointsize

1 Introduction

Our notation and terminology are standard (see, e.g. [2]). Thus, \( G(n) \) is a graph of order \( n \) and \( G(n, m) \) is a graph of order \( n \) and size \( m \); for a graph \( G \) and a vertex \( u \in V(G) \) we write \( \Gamma(u) \) for the neighborhood of \( u \); \( d_G(u) = |\Gamma(u)| \) is the degree of \( u \); we write \( d(u) \) for \( d_G(u) \) when there is no danger of confusion.

We denote by \( k_r(G) \) the number of \( r \)-cliques of \( G \). We let \( T_r(n) \) be the Turán graph of order \( n \) with \( r \) classes and set \( t_r(n) = e(T_r(n)) \).

Erdős [3] proved that if \( r \geq 2 \) and \( n > n_0(r) \), every graph \( G = G(n, t_r(n) + 1) \) contains at least \( n^{r-1}/(10r)^6r \)-cliques of order \( r+1 \) sharing an edge. He used this result to estimate the minimum number of cliques in certain graphs.

In this note we strengthen and extend this result of Erdős. We start with a general definition. Let \( p,q,r \) be integers with \( p \geq r \), \( q > r \geq 1 \). We call the union of a \( p \)-clique \( H \) and \( t \) \( q \)-cliques, each one intersecting \( H \) in exactly \( r \) vertices, a \( (p,q,r) \)-joint of size \( t \) and denote it by \( J_{t}^{(p,q,r)} \). The maximum size of a \( (p,q,r) \)-joint in a graph \( G \) is called the \( (p,q,r) \)-jointsize of \( G \) and is denoted by \( js^{(p,q,r)}(G) \).

Observe that, in general, there may be many nonisomorphic \( (p,q,r) \)-joints with the same parameters \( p,q,r \).

In terms of joints the above assertion of Erdős can be stated as follows: for every integer \( r \geq 2 \) and \( n > n_0(r) \),

\[
js^{(2,r+1,2)}(G(n, t_r(n) + 1)) \geq \frac{n^{r-1}}{(10r)^6r}.
\]
In this note we shall show that, in fact, if $r \geq 2$ and $n > r^8$ then
\[ js^{(2,r+1,2)}(G(n, t_r(n) + 1)) > \frac{n^{r-1}}{r^{r+5}}. \]

Moreover, we shall show that if $r \geq 2$, $n > r^8$, and $0 < \alpha < 36r^8$ then, for every graph $G = G(n)$ with $e(G) > t_r(n) - \alpha n^2$, either
\[ js^{(2,r+1,2)}(G(n, t_r(n) + 1)) > \left(1 - \frac{1}{r^3}\right) \frac{n^{r-1}}{r^{r+5}} \]
or $G$ contains an induced $r$-chromatic subgraph of order at least $(1 - 2\sqrt{\alpha})n$.

2 Preliminary results

Recall that the following basic properties of the Turán graph $T_r(n)$
\[ \delta(T_r(n)) = \left\lfloor \frac{r-1}{r} n \right\rfloor \]
and
\[ t_r(n) = t_r(n-1) + \delta(T_r(n)). \]
Furthermore,
\[ t_r(n) = \frac{r-1}{2r} (n^2 - t^2) + \left(\frac{t}{2}\right), \]
where $t$ is the remainder of $n$ modulo $r$, and so
\[ \frac{r-1}{2r} n^2 - \frac{r}{8} \leq t_r(n) \leq \frac{r-1}{2r} n^2. \]

2.1 Bounds on $k_{r+1}(G)$ and $js^{(2,r+1,2)}(G)$

We start by establishing lower bounds for $k_{r+1}(G)$ and $js^{(2,r+1,2)}(G)$ in a graph $G$ of order $n$ with $e(G) > \frac{r+1}{2r}n^2$.

Note that $N$ cliques $K_{r+1}$ of a graph $G$ cover some edge at least $N\binom{r+1}{2}/e(G)$ times, and so
\[ js^{(2,r+1,2)}(G) > k_{r+1}(G) \binom{r+1}{2} \binom{n}{2}^{-1}. \]

Lemma 1 For all $r \geq 3$, $c > 0$, if $G = G(n)$ and
\[ e(G) > \left(\frac{r-1}{2r} + c\right) n^2 \]
then
\[ k_{r+1}(G) > 2c \frac{n^{r+1}}{r+1} \]
and
\[ js^{(2,r+1,2)}(G) > 2c \left(\frac{n}{r}\right)^{r-1}. \]
Proof In [6] Moon and Moser stated the following assertion whose complete proof apparently appeared for the first time in [4] (see also [5], Problem 11.8).

If \( G = G(n) \) and \( k_s(G) > 0 \) then

\[
\frac{(s + 1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \geq \frac{sk_s(G)}{(s - 1)k_{s-1}(G)} - \frac{n}{s - 1}.
\]

Equivalently, if \( q \) is the clique number of \( G \) then, for \( q > s > t \geq 1 \), we have

\[
\frac{(s + 1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \geq \frac{(t + 1)k_{t+1}(G)}{tk_t(G)} - \frac{n}{t}.
\]  \hfill (10)

Since Turán’s theorem and (7) imply \( k_{r+1}(G) > 0 \), setting \( t = 1 \) in (10), we find that

\[
\frac{(s + 1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \geq \frac{2e(G)}{n} - n \geq \left( -\frac{1}{r} + 2c \right) n
\]
for every \( s = 2, \ldots, r \). Hence,

\[
\frac{(s + 1)k_{s+1}(G)}{sk_s(G)} > \left( \frac{1}{s} - \frac{1}{r} + 2c \right) n
\]
for every \( s = 1, \ldots, r \). Multiplying these inequalities for \( s = 1, \ldots, r \), we find that

\[
\frac{(r + 1)k_{r+1}(G)}{n} \geq n^r \prod_{s=1}^{r} \left( \frac{1}{s} - \frac{1}{r} + 2c \right) > 2cn^r \prod_{s=1}^{r-1} \left( \frac{1}{s} - \frac{1}{r} \right) = \frac{2c}{r} n^r,
\]
and hence (8) holds.

Taking into account (9), we find that

\[
js^{(2,r+1,2)}(G) \geq \left( \frac{r + 1}{2} \right) k_{r+1}(G) \left( \frac{n}{2} \right)^{-1} > \left( \frac{r + 1}{2} \right) \frac{2c}{(r + 1) r^r} n^{r+1} \left( \frac{n}{2} \right)^{-1},
\]
and (9) follows. \( \square \)

Since the inequality \( 2e(G) \geq \delta(G)v(G) \) holds for every graph \( G \), Lemma 4 implies the following corollary.

Corollary 2 For all \( r \geq 3, c > 0 \), if \( G = G(n) \) and

\[
\delta(G) > \left( \frac{r - 1}{r} + c \right) n
\]
then

\[
k_{r+1}(G) > \frac{c}{r} \left( \frac{n}{r} \right)^{r+1}
\]
and

\[
js^{(2,r+1,2)}(G) > c \left( \frac{n}{r} \right)^{r-2}.
\]
2.2 A Bonferroni-Zarankievicz type inequality

Suppose \( r \geq 3 \), \( X \) is a set of cardinality \( n \), and \( A_1, \ldots, A_r \) are subsets of \( X \). For every \( k \in [r] \), set

\[
S_k = \sum_{1 \leq i_1 < \ldots < i_k \leq r} \mu(A_{i_1} \cap \ldots \cap A_{i_k}).
\]

Then the following theorem holds.

**Theorem 3** If \( 1 \leq k \leq r \) then

\[
S_k \geq \left( \frac{\lfloor S_1/n \rfloor}{k-1} \right) \left( S_1 - \frac{k-1}{k} (\lfloor S_1/n \rfloor + 1) n \right) \tag{11}
\]

**Proof** Let \( H \) be a bipartite graph whose color classes are the sets \([r]\) and \( X \), and \( i \in [r] \) is joined to \( u \in X \) iff \( u \in A_i \). Clearly,

\[
S_1 = e(H) = \sum_{u \in X} d_H(u) \tag{12}
\]

and

\[
S_k = \sum_{u \in X} \left( \begin{array}{c} d_H(u) \end{array} \right)^k.
\]

The convexity of \( \binom{x}{k} \) implies that the minimum of \( S_k \), subject to \( \binom{\lfloor S_1/n \rfloor}{k-1} \), is attained when every vertex \( u \) has degree \( d_H(u) = \lfloor S_1/n \rfloor \) or \( d_H(u) = \lceil S_1/n \rceil \). Letting \( l \) be the number of those \( u \) with \( d_H(u) = \lfloor S_1/n \rfloor \) and setting \( x = l/n \), we see that

\[
(1 - x) \lfloor S_1/n \rfloor + x \lceil S_1/n \rceil = S_1/n,
\]

and so, \( x = S_1/n - \lfloor S_1/n \rfloor \). Since

\[
\binom{\lfloor S_1/n \rfloor}{k-1} = \binom{\lfloor S_1/n \rfloor + 1}{k} - \binom{\lceil S_1/n \rceil}{k}
\]

for \( x > 0 \), we have

\[
S_k \geq (1 - x) n \binom{\lfloor S_1/n \rfloor}{k} + x n \binom{\lceil S_1/n \rceil}{k}
\]

\[
= n \binom{\lfloor S_1/n \rfloor}{k} + x n \left( \binom{\lceil S_1/n \rceil + 1}{k} - \binom{\lfloor S_1/n \rfloor}{k} \right)
\]

\[
= n \binom{\lfloor S_1/n \rfloor}{k} + n (S_1/n - \lfloor S_1/n \rfloor) \binom{\lfloor S_1/n \rfloor}{k-1}
\]

\[
= \binom{\lfloor S_1/n \rfloor}{k-1} \left( S_1 - \frac{k-1}{k} (\lfloor S_1/n \rfloor + 1) n \right),
\]

as claimed. \( \Box \)
Lemma 4 Suppose \( r \geq 2, 0 < a < 1/r \), \( X \) is a set of cardinality \( n \), and \( A_1, \ldots, A_{r+1} \) are subsets of \( X \). If
\[
\sum_{i=1}^{r+1} |A_i| \geq \left( r - \frac{1}{r} - (r+1)a \right) n.
\]
Then some two members of \( \{A_1, \ldots, A_{r+1}\} \) have at least
\[
\left( \frac{r-2}{r} + \frac{2}{r^2(r+1)} - \frac{2(r-1)a}{r} \right) n
\]
elements in common.

Proof Applying Theorem \( \text{??} \) with \( k = 2 \) to the sets \( A_1, \ldots, A_{r+1} \), we find that
\[
S_2 \geq \left( \frac{r-1}{1} \right) \left( S_1 - \frac{r}{2}n \right) \geq (r-1) \left( r - \frac{1}{r} - (r+1)a - \frac{r}{2} \right) n
\]
\[
= \left( \frac{r(r-1)}{2} - \frac{r-1}{r} - (r^2-1)a \right) n.
\]
Since there are \( \binom{r+1}{2} \) pairwise intersections \( A_i \cap A_j \), for some \( 1 \leq k < l \leq r+1 \) we have
\[
|A_k \cap A_l| \geq S_2 \left( \frac{r+1}{2} \right)^{-1} \geq \left( \frac{r(r-1)}{2} - \frac{r-1}{r} - (r^2-1)a \right) \left( \frac{r+1}{2} \right)^{-1} n
\]
\[
= \left( \frac{r-1}{r+1} - \frac{2(r-1)}{r^2(r+1)} - \frac{2(r-1)a}{r} \right) n
\]
\[
= \left( \frac{r-2}{r} + \frac{2}{r^2(r+1)} - \frac{2(r-1)a}{r} \right) n.
\]
\( \square \)

The idea of the following lemma is due to Erdős; our proof techniques allow to improve his bound considerably.

Lemma 5 Suppose \( r \geq 3 \). If a graph \( G = G(n) \) contains a \( K_{r+1} \) and
\[
\delta(G) > \left( \frac{r-1}{r} - \frac{1}{r^2(r^2-1)} \right) n
\]
then
\[
js^{(2, r+1, 2)}(G) > \frac{n^{r-1}}{r^{r+3}}.
\]

Proof Indeed, let \( U \) be the vertex set of an \( (r+1) \)-clique in \( G \). Then
\[
\sum_{i \in U} |A_i| > (r+1) \left( \frac{r-1}{r} - \frac{1}{r^2(r^2-1)} \right) n = \left( \frac{r-1}{r} - \frac{r+1}{r^2(r^2-1)} \right) n.
\]
Hence, by Lemma 4, there are distinct \( u, v \in U \) such that \( M = |\Gamma(u) \cap \Gamma(v)| \) satisfies
\[
|M| \geq \left( \frac{r-2}{r} + \frac{2}{r^2(r+1)} - \frac{2(r-1)}{r} \frac{1}{r^2(r^2-1)} \right) n
= \left( \frac{r-2}{r} + \frac{2(r-1)}{r^3(r+1)} \right) n.
\]
(13)

For the graph \( G[M] \) induced by the set \( M \) we have
\[
\delta(G[M]) \geq \delta(G) - (n - |M|) > \left( \frac{r-1}{r} - \frac{1}{r^2(r^2-1)} \right) n - (n - |M|)
= |M| - \left( \frac{1}{r} + \frac{1}{r^2(r^2-1)} \right) n.
\]
(14)

By routine calculations we find that, for \( r \geq 3 \),
\[
\left( \frac{1}{r-2} - \frac{1}{r^2(r-1)^2} \right) \left( \frac{r-2}{r} + \frac{2(r-1)}{r^3(r+1)} \right) > \frac{1}{r} + \frac{1}{r^2(r^2-1)}.
\]
Recalling (13), this implies
\[
|M| \left( \frac{1}{r-2} - \frac{1}{r^2(r-1)^2} \right) > \left( \frac{1}{r} + \frac{1}{r^2(r^2-1)} \right) n,
\]
and furthermore,
\[
|M| - \left( \frac{1}{r} + \frac{1}{r^2(r^2-1)} \right) n > \left( \frac{r-3}{r-2} + \frac{1}{r^2(r^2-1)} \right) |M|.
\]

Hence, from inequality (14) we see that
\[
\delta(G[M]) \geq \left( \frac{r-3}{r-2} + \frac{1}{r^2(r-1)^2} \right) |M|.
\]

In view of (13), Corollary 5 implies
\[
k_{r-1}(G[M]) \geq \frac{r-2}{r^2(r-1)^2} \left( \frac{|M|}{r-2} \right)^{r-1} > \frac{r-2}{r^2(r-1)^3} n^{r-1} > \frac{n^{r-1}}{r^{r+3}}.
\]

To complete the proof observe that the number of \((r+1)\)-cliques of \( G \) containing the edge \( uv \) is exactly \( k_{r-1}(G[M]). \)
\[\square\]
3 Existence of large joints $J^{(2,r+1,2)}$

In this section we shall prove a Turán type result for large joints as stated in Theorem 7 below. We start with the following technical result.

**Theorem 6** If $r \geq 2$ and $n > r^8$, every graph $G = G(n)$ with

$$
\begin{equation}
\label{eq15}
e(G) > t_r(n)
\end{equation}
$$

has an induced subgraph $G' = G(n')$ with $n' > (1 - 1/r^2) n$ such that either

$$
\begin{equation}
\label{eq16}
K_{r+1} \subseteq G', \quad \text{and} \quad \delta(G') > \left( \frac{r-1}{r} - \frac{1}{r^2 (r^2-1)} \right) n',
\end{equation}
$$

or

$$
\begin{equation}
\label{eq17}
e(G') > \left( \frac{r-1}{2r} + \frac{1}{r^4 (r^2-1)} \right) (n')^2.
\end{equation}
$$

**Proof** Let the sequence $u_1, ..., u_n$ be an enumeration of the vertices such that $d(u_1) = \delta(G)$ and

$$
\begin{equation}
\label{eq18}
d(u_i) = \delta(G - u_1 - ... - u_{i-1}) \quad \text{for } 1 < i \leq n.
\end{equation}
$$

Set $G_0 = G$, and set $G_i = G - u_1 - ... - u_i$, $i = 1, ..., n-1$, so that

$$
\begin{equation}
\label{eq19}
e(G_i) - e(G_{i+1}) = \delta(G_i)
\end{equation}
$$

for every $i \in [n-1]$.

Set $\beta = \frac{1}{r(r^2-1)}$ and let $k - 1$ be the largest integer such that $1 \leq k \leq n$ and

$$
\begin{equation}
\delta(G_{k-1}) \leq \left( \frac{r-1}{r} - \beta \right) (n - k + 1).
\end{equation}
$$

From (18), for every $s \in [k]$, we have

$$
\begin{equation}
\label{eq20}
e(G) - e(G_s) = \sum_{i=0}^{s-1} \delta(G_i) \leq \left( \frac{r-1}{r} - \beta \right) \sum_{i=0}^{s-1} (n - i)
\end{equation}
$$

$$
\leq \left( \frac{r-1}{r} - \beta \right) \left( \frac{n+1}{2} - \frac{n - s + 1}{2} \right)
\leq \left( \frac{r-1}{r} - \beta \right) \left( \frac{n^2}{2} + \frac{(n-s)^2}{2} + \frac{s}{2} \right).
$$

From (15) and (18) we have

$$
\begin{equation}
\label{eq21}e(G) > \frac{r-1}{2r} n^2 - \frac{r}{8}.
\end{equation}
$$
Hence, for every $s \in [k]$, we deduce
\[
e(G_s) > e(G) - \left(\frac{r-1}{r} - \beta\right) \left(\frac{n^2}{2} - \frac{(n-s)^2}{2} + \frac{s}{2}\right) > \frac{r-1}{2r} n^2 - \frac{r}{8} \left(\frac{r-1}{2r} - \beta\right) \left(n^2 - (n-s)^2 + s\right) = \beta \frac{n^2}{2} + \left(\frac{r-1}{2r} - \frac{\beta}{2}\right) (n-s)^2 - \left(\frac{r-1}{2r} - \frac{\beta}{2}\right) s - \frac{r}{8} > \frac{r-1}{2r} (n-s)^2 + \frac{\beta}{2} \left(n^2 - (n-s)^2\right) - \frac{r}{8} - \frac{s}{2}.
\]
(19)

In the rest of the proof we shall consider two cases - (a) $k > n/r^2$ and (b) $k \leq n/r^2$.

(a) Let $n > r^8$, assume that $k > n/r^2$, and set $l = \lfloor n/r^2 \rfloor$. Then we have
\[
n - l \leq \left(1 - \frac{1}{r^2}\right) (n+1),
\]
(20)
implying
\[
(n-l)^2 \leq (n+1)^2 \left(1 - \frac{1}{r^2}\right)^2 \leq \left(n^2 - \frac{n}{\beta}\right) \left(1 + \frac{2}{r^2}\right)^{-1}.
\]
Hence, from (19), it follows
\[
e(G_l) > \frac{r-1}{2r} (n-l)^2 + \frac{\beta}{2} \left(n^2 - (n-l)^2\right) - \frac{r}{8} - \frac{n}{2r^2} > \frac{r-1}{2r} (n-l)^2 + \frac{\beta}{2} \left(n^2 - \frac{n}{\beta} - (n-l)^2\right) > \left(\frac{r-1}{2r} + \frac{\beta}{r^2}\right) (n-l)^2.
\]
This, together with (20), implies (14) with $G' = G_l$.

(b) Assume that $k \leq n/r^2$. The way the graphs $G_1, ..., G_k$ are constructed, together with (3) and (15), implies
\[
e(G_k) > t_r (n-k),
\]
and by Turán's theorem $K_{r+1} \subset G$. Since $(n-k) \geq (1 - 1/r^2) n$, and
\[
\delta(G_k) > \left(\frac{r-1}{r} - \frac{1}{r^2 (r^2 - 1)}\right) (n-k),
\]
condition (16) holds with $G' = G_k$. The proof is completed. \[\square\]

After this proposition we are ready to prove our main theorem, strengthening inequality (1).
Theorem 7 For \( r \geq 2 \) and \( n > r^8 \), every graph \( G = G(n) \) with \( e(G) \geq t_r(n) \) satisfies

\[
j_s^{(2, r+1, 2)}(G) > \frac{n^{r-1}}{r^{r+5}}
\]

unless \( G = T_r(n) \).

Proof Assume first that \( e(G) > t_r(n) \). By Theorem 6 \( G \) contains an induced subgraph \( G' = G(n') \) with \( n' > (1 - 1/r^2) n \) and such that either 16 or 17 holds. If 16 is true, applying Lemma 5 to the graph \( G' \), we see that

\[
j_s^{(2, r+1, 2)}(G') > \frac{(n')^{r-1}}{r^{r+3}} > \left(1 - \frac{1}{r^2}\right)^{r-1} \frac{n^{r-1}}{r^{r+3}} > \left(1 - \frac{1}{r^2}\right) \frac{n^{r-1}}{r^{r+3}},
\]

and the assertion follows.

If 17 holds then, by Lemma 4 we see that

\[
j_s^{(2, r+1, 2)}(G') > \frac{2}{r^4(r^2 - 1)} \left(\frac{n'}{r}\right)^{r-1} > \frac{2}{r^4(r^2 - 1)} \left(1 - \frac{1}{r^2}\right)^{r-1} \left(\frac{n}{r}\right)^{r-1}
\]

\[
> \frac{2}{r^4(r^2 - 1)} \left(1 - \frac{1}{r^2}\right) \left(\frac{n}{r}\right)^{r-1} > \frac{n^{r-1}}{r^{r+5}},
\]

and the assertion follows.

Assume now that \( e(G) = t_r(n) \). If \( G \) has a vertex \( u \) with \( d(u) < \delta(T_r(n)) \) then

\[
e(G - u) > t_r(n - 1),
\]

and therefore, the graph \( G - u \) contains an induced subgraph \( G' = G(n') \) with \( n' > (1 - 1/r^2)(n - 1) \) and such that either 16 or 17 holds. Using the arguments from the first part of our proof we see that either

\[
j_s^{(2, r+1, 2)}(G') > \frac{(n')^{r-1}}{r^{r+3}} > \left(1 - \frac{1}{r^2}\right)^{r-1} \frac{n^{r-1}}{r^{r+3}}
\]

or

\[
j_s^{(2, r+1, 2)}(G') > \frac{2}{r^4(r^2 - 1)} \left(\frac{n'}{r}\right)^{r-1} > \frac{2}{r^4(r^2 - 1)} \left(1 - \frac{1}{r^2}\right)^{r-1} \left(\frac{n}{r}\right)^{r-1}
\]

\[
> \frac{2}{r^4(r^2 - 1)} \left(1 - \frac{1}{r^2}\right) \left(\frac{n}{r}\right)^{r-1} > \frac{n^{r-1}}{r^{r+5}},
\]

completing the proof in this case.

It remains the case when \( \delta(G) = \delta(T_r(n)) \). Hence, in view of \( n > r^8 \), we find that

\[
\delta(G) = \left\lfloor \frac{r - 1}{r} n \right\rfloor \geq \left(\frac{r - 1}{r} - \frac{1}{r^2(r^2 - 1)}\right) n.
\]

(22)
If \( G \neq T_r(n) \), Turán’s theorem implies that \( G \) contains a \( K_{r+1} \); thus, in view of Lemma 5 and (22), the proof is completed.

Note that (21) is tight up to a factor of order at most \( r^{-6} \), as seen by taking the graph \( T_r(n) \) and adding an edge to its largest chromatic class.

4 A stability theorem about large joints \( J^{(2, r+1, 2)} \)

Theorem 7 may be used to prove a stability result about large joints \( J^{(2, r+1, 2)} \) as stated in the theorem below. In the course of our proof we shall need the following result of Andrásfai, Erdős and Sós [1]: if \( G \) is a \( K_{r+1} \)-free graph of order \( n \) with minimal degree

\[
\delta(G) > \left(1 - \frac{3}{3r - 1}\right)n
\]

then \( G \) is \( r \)-chromatic.

Theorem 8 Let \( r \geq 2 \), \( n > r^8 \), and \( 0 < \alpha < r^{-8}/36 \). If a graph \( G = G(n) \) satisfies

\[
e(G) > \left(\frac{r - 1}{2r} - \alpha\right)n^2,
\]

then either

\[
J^{(2, r+1, 2)}(G) > \left(1 - \frac{1}{r^3}\right)n^{r-1}\frac{r^r}{r^r+5}, \tag{23}
\]

or \( G \) contains an induced \( r \)-chromatic subgraph \( G_0 \) of order at least \((1 - 2\sqrt{\alpha})n\) with minimum degree

\[
\delta(G_0) > \left(1 - \frac{1}{r} - 6\sqrt{\alpha}\right)n. \tag{24}
\]

Proof We may assume that \( \alpha n^2 \geq 1 \), since otherwise we have \( e(G) \geq t_r(n) \) and the assertion follows from Theorem 7. Set

\[
\varepsilon = 2\sqrt{\alpha} < \frac{1}{3r^4}, \tag{25}
\]

and define \( M_\varepsilon \subset V \) as

\[
M_\varepsilon = \left\{ u \in V(G) : d(u) \leq \left(\frac{r - 1}{r} - \varepsilon\right)n \right\}.
\]

Assume that (23) does not hold. Our aim is to show that (a) \(|M_\varepsilon| < 2\varepsilon n\), and (b) the subgraph \( G_0 \) of \( G \) induced by \( V(G) \setminus M_\varepsilon \) has the properties required in the theorem.
\[
\frac{(s + 1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} > \frac{3k_3(G)}{2m} - \frac{n}{2} \geq \frac{1}{2m} \sum_{u \in V(G)} d^2(u) - n
\]

\[
= \frac{2m}{n} + \frac{1}{2m} \sum_{u \in V(G)} \left( d(u) - \frac{2m}{n} \right)^2 - n
\]

\[
\geq \frac{2m}{n} - n + \frac{1}{2m} \sum_{u \in M} \varepsilon^2 > \frac{2m}{n} - n + \varepsilon^2 |M|.
\]

(a) Assume, for a contradiction, that \(|M| \geq 2\varepsilon n\) and let \(M' \subset M\) satisfy

\[
\left(1 - \sqrt{1/2}\right) \varepsilon n < |M'| < \left(1 + \sqrt{1/2}\right) \varepsilon n.
\]

Such a set \(M'\) exists since \(\sqrt{2\varepsilon n} > 2\sqrt{2\varepsilon n} > 2\sqrt{2}\). Let \(G'\) be the subgraph of \(G\) induced by \(V \setminus M'\). Then

\[
e(G) = e(G') + e(M', V \setminus M') + e(M') \leq e(G') + \sum_{u \in M'} d(u)
\]

\[
\leq e(G') + |M'| \left( \frac{r - 1}{r} - \varepsilon \right) n.
\]

Observe that the second inequality of (26) implies that

\[
n - |M'| > n - 2\varepsilon n.
\]

Hence, if

\[
e(G') > \frac{r - 1}{2r} (n - |M'|)^2
\]

then, by Theorem 7 and (26),

\[
js^{(2, r+1, 2)}(G) \geq js^{(2, r+1, 2)}(G') > \frac{(n - |M'|)^{r-1}}{r^{r+5}} > \left(1 - 2\varepsilon\right)^{r-1} \frac{n^{r-1}}{r^{r+5}}
\]

\[
> (1 - 2(r - 1)\varepsilon)^r \frac{n^{r-1}}{r^{r+5}} \left(1 - \frac{1}{r^5}\right) \frac{n^{r-1}}{r^{r+5}}.
\]

Thus (28) holds, contradicting our assumption.

Consequently, we may assume that

\[
e(G') \leq \frac{r - 1}{2r} (n - |M'|)^2.
\]

Since

\[
e(G') \geq e(G) - \sum_{u \in M} d(u)
\]

\[
\geq \left( \frac{r - 1}{2r} - \alpha \right) n^2 - |M'| \left( \frac{r - 1}{r} - \varepsilon \right) n,
\]

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it follows that
\[
\frac{r-1}{2r} (n - |M'|)^2 \geq \left( \frac{r-1}{2r} - \alpha \right) n^2 - |M'| \left( \frac{r-1}{r} - \varepsilon \right)n.
\]
Setting \( x = |M'| / n \) we find that
\[
\frac{r-1}{2r} (1-x)^2 + x \left( \frac{r-1}{r} - \varepsilon \right) - \left( \frac{r-1}{2r} - \alpha \right) \geq 0.
\]
and so,
\[
x^2 - 2\varepsilon x + 2\alpha \geq 0.
\]
Hence, either
\[
|M'| \leq \left( \varepsilon - \sqrt{\varepsilon^2 - 2\alpha} \right)n = \varepsilon \left( 1 - \sqrt{1/2} \right)n
\]
or
\[
|M'| \geq \left( \varepsilon + \sqrt{\varepsilon^2 - 2\alpha} \right)n = \varepsilon \left( 1 + \sqrt{1/2} \right)n,
\]
contradicting (26). Therefore, \( |M_\varepsilon| < 2\varepsilon n \).

(b) Note first that \( G_0 \) has \( n - |M_\varepsilon| > (1 - 2\varepsilon) n \) vertices. By our choice of \( M_\varepsilon \), for \( u \in V \setminus M_\varepsilon \), we have
\[
d_G(u) > \left( \frac{r-1}{r} - \varepsilon \right)n,
\]
so
\[
d_{G_0}(u) > \left( \frac{r-1}{r} - \varepsilon \right)n - |M_\varepsilon| > \left( \frac{r-1}{r} - 3\varepsilon \right)n = \left( \frac{r-1}{r} - 6\sqrt{\alpha} \right)n, \tag{28}
\]
and (24) holds.

All that remains to prove is that \( G_0 \) is \( r \)-chromatic. From (28) we have
\[
\delta(G_0) > \left( \frac{r-1}{r} - 6\sqrt{\alpha} \right)n \geq \left( \frac{r-1}{r} - 6\sqrt{\alpha} \right)v(G_0)
\]
\[
> \left( \frac{r-1}{r} - \frac{1}{r^4} \right)v(G_0) > \left( 1 - \frac{3}{3r-1} \right)v(G_0) \tag{29}
\]
If \( G_0 \) contains a \( K_{r+1} \), by Lemma 5 we have
\[
js^{(2,r+1,2)}(G) \geq js^{(2,r+1,2)}(G_0) > \frac{(n - |M'|)^{r-1}}{r^{r+5}} > (1 - 2\varepsilon)^{r-1} \frac{n^{r-1}}{r^{r+5}}
\]
\[
> (1 - 2(r-1)\varepsilon) \frac{n^{r-1}}{r^{r+5}} > \left( 1 - \frac{1}{r^3} \right) \frac{n^{r-1}}{r^{r+5}}.
\]
Therefore, (28) holds, contradicting our assumption.

We may assume that \( G_0 \) is \( K_{r+1} \)-free. In view of (24), the theorem of Andrásfai, Erdős and Sós implies that \( G_0 \) is \( r \)-chromatic, completing our proof.
\[\square\]
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