Bottom quark mass and $|V_{cb}|$ matrix element

from $R(e^+e^- \rightarrow b\bar{b})$ and $\Gamma_{sl}(b \rightarrow c\ell\nu_\ell)$ in the
next-to-next-to-leading order.

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Abstract

We present a consistent analysis of \Upsilon sum rules and $B$-meson semileptonic width in
the next-to-next-to-leading order in the strong coupling constant. The analysis is based
on the analytical result for the heavy quark vector current correlator near threshold in
the second order in perturbative and relativistic expansion around the nonrelativistic
Coulomb approximation.

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1 Introduction.

The rich phenomenology of the heavy quark mesons has been recognized as a source of information on strong interaction dynamics and a clean place to determine fundamental parameters of the standard model such as the strong coupling constant, heavy quark masses and mixing angles. The necessity of the quantitative analysis of the strong interaction effects in the heavy flavor physics leads to developing new theoretical methods such as nonrelativistic QCD (NRQCD) \cite{1} for description of heavy quarkonium and heavy quark effective theory (HQET) \cite{2} for description of heavy-light mesons. One of the most promising applications of these techniques to the bottom quark physics is the analysis of the Υ sum rules \cite{3} and B-meson semileptonic decays (see \cite{4, 5} for a recent review). The strict estimate of the actual precision of theoretical predictions is now becoming important since the experimental precision is rather high and is quantitatively comparable with the last available terms of perturbative approximation. Recently a significant progress has been achieved in calculations of the high order corrections for the Υ sum rules \cite{6, 7, 8, 9, 10} and inclusive semileptonic width of B-mesons \cite{11, 12, 13}.

In this paper we present a simultaneous analysis of Υ sum rules and the B-meson inclusive semileptonic width Γ_{sl} in the next-to-next-to-leading order (NNLO). On the basis of this analysis we determine the mass parameter of the bottom quark in the second order of perturbation theory\footnote{Because quark states have not been observed as asymptotic free particle states the notion of their mass can not be unambiguously defined. There are different ways to reflect the fact the b-quark is heavy by introducing its mass \cite{14}. In the present paper we use the notion of the b-quark pole mass defined strictly within the finite order perturbation theory. Within this definition this mass parameter is equivalent (and can be reexpressed through a finite series in α_s) to any other perturbative definition of the quark mass parameter, in particularly, \overline{MS} mass in every order of the perturbation theory as a parameter of perturbative QCD.} and give a new accurate estimate of Cabibbo-Kobayashi-Maskawa (CKM) matrix element |V_{cb}|.
The results of the analysis of the Υ system in the NNLO approximation in the coupling constant and nonrelativistic expansion have been briefly reported in ref. [8]. Then two more papers with NNLO analysis of the Υ system appeared [9, 10]. Since there is some difference between the analyses of ref. [8] and refs. [9, 10] in some details of the treatment of the problem, in this paper we give more extensive discussion paying the main attention to the self-consistence of the approach, uncertainties of theoretical estimates and the reliability of results. We discuss also the general structure and the asymptotic character of perturbative expansion for the moments of the Υ system spectral density. By studying the behavior of the expansion around the Coulomb approximation for different moments we estimate the critical order where the asymptotic growth of the coefficients of the series in $\alpha_s$ for the moments starts.

The obtained moments of the Υ system spectral density are related then to the inclusive $B$-meson semileptonic width up to the NNLO order. In this way we avoid the strong dependence of $\Gamma_{sl}$ on $m_b$ and reduce the theoretical uncertainty in $|V_{cb}|$. In particular, the obtained perturbative expansion for $V_{cb}$ converges well (in heuristic sense) up to NNLO in $\alpha_s$.

The paper is organized as follows. In the next Section we describe the vacuum polarization function of heavy quarks near threshold in the NNLO. In Section 3 we discuss the Υ sum rules and numerical estimates for the $b$-quark pole and $\overline{\text{MS}}$ mass within perturbation theory. In Section 4 the inclusive $B$-meson semileptonic width and $|V_{cb}|$ matrix element are determined. The last Section contains our conclusions. In Appendix we give an explicit analytical formula for the polarization function of heavy quarks near threshold.
2 Next-to-next-to-leading order heavy quark vacuum polarization function near threshold.

We study the near threshold behavior of the vacuum polarization function $\Pi(s)$ of the $b$-quark vector current $j_\mu = \bar{b}\gamma_\mu b$

$$
(q_\mu q_\nu - g_\mu g_\nu q^2) \Pi(q^2) = i \int dx e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle.
$$

The finite order perturbation theory expansion in the strong coupling constant $\alpha_s$ is known to break down near the two-particle production threshold, $s \sim 4m_b^2$ where $m_b$ is the $b$-quark pole mass. However, near the threshold, i.e. for a small $b$-quark velocity $v = \sqrt{1 - 4m_b^2/s} \ll 1$, the nonrelativistic approximation becomes valid [1]. In the NNLO the nonrelativistic expansion for the polarization function $\Pi(s)$ has the form

$$
\Pi(s) = \frac{N_c}{2m_b^2} \left( C_h(\alpha_s) G(0, 0, k) + \frac{4}{3} \frac{k^2}{m_b^2} G_C(0, 0, k) \right)
$$

with $k = \sqrt{m_b^2 - s/4}$ being a natural energy variable near threshold. The quantity $C_h(\alpha_s)$ is a perturbative coefficient that matches correlators of relativistic and nonrelativistic vector currents and accounts for the hard part of the QCD corrections. It is now known to the second order in $\alpha_s$ expansion

$$
C_h(\alpha_s) = 1 - C^1_h C_F \frac{\alpha_s}{\pi} + C^2_h C_F \left( \frac{\alpha_s}{\pi} \right)^2
$$

with $C^1_h = 4$ [13] and

$$
C^2_h = \left( \frac{39}{4} - \zeta(3) + \frac{4\pi^2}{3} \ln 2 - \frac{35\pi^2}{18} \right) C_F - \left( \frac{151}{36} + \frac{13}{2} \zeta(3) + \frac{8\pi^2}{3} \ln 2 - \frac{179\pi^2}{72} \right) C_A
$$

$$
+ \left( \frac{44}{9} - \frac{4\pi^2}{9} + \frac{11}{9} n_f \right) T_F + 2\beta_0 \ln \left( \frac{m_b}{\mu} \right) + \pi^2 \left( \frac{2}{3} C_F + C_A \right) \ln \left( \frac{m_b}{\mu_f} \right)
$$

where $\alpha_s$ is defined in $\overline{\text{MS}}$ renormalization scheme with the scale parameter $\mu$. Another scale parameter $\mu_f$ is a factorization scale which separates contributions coming from the
hard and soft momentum regions and plays the role of an infrared cutoff in the diagrams contributing to the quantity $C_h(\alpha_s)$ [16, 17]. The color symmetry $SU(3)$ group invariants for QCD are $C_A = 3$, $C_F = 4/3$, $T_F = 1/2$, $n_f$ is the number of light fermion flavors, and $\beta_0 = 11C_A/3 - 4T_Fn_f/3$ is the first $\beta$-function coefficient. Here $\gamma_E = 0.577216\ldots$ is the Euler constant and $\zeta(z)$ is the Riemann $\zeta$-function. The quantity $G(x, y, k)$ is the nonrelativistic Green function. This Green function sums up the singular threshold corrections and satisfies the following Schrödinger equation

$$\left( -\frac{\Delta_x}{m_b} - \frac{\Delta_x^2}{4m_b^2} + V_C(x) + \frac{\alpha_s}{4\pi}\Delta_1 V(x) + \left(\frac{\alpha_s}{4\pi}\right)^2 \Delta_2 V(x) \right)$$

$$\left( + \Delta_{NA} V(x) + \Delta_{BF} V(x, \partial_x, S) + \frac{k^2}{m_b} \right) G(x, y, k) = \delta(x - y) \quad (3)$$

where $\Delta_x = \partial_x^2$, $V_C(x) = -C_F\alpha_s/x$ is the Coulomb potential, $x = |x|$, $\Delta_{NA} V(x) = -C_A C_F\alpha_s^2/(2m_b x^2)$ is the so called non-Abelian potential of quark-antiquark interaction [18], $\Delta_{BF} V(x, \partial_x, S)$ is the standard Breit-Fermi potential (up to the color factor $C_F$) containing the quark spin operator $S$, e.g. [19]. The terms $\Delta_i V (i = 1, 2)$ represent the first and second order perturbative QCD corrections to the Coulomb potential [20, 21, 22]²

$$\Delta_1 V(x) = \frac{\alpha_s}{4\pi} V_C(x) (C_0^1 + C_1^1 \ln(x\mu)), \quad \Delta_2 V(x) = \left(\frac{\alpha_s}{4\pi}\right)^2 V_C(x) (C_0^2 + C_1^2 \ln(x\mu) + C_2^2 \ln^2(x\mu)) \quad (4)$$

where

$$C_0^1 = a_1 + 2\beta_0 \gamma_E, \quad C_1^1 = 2\beta_0,$$

$$C_0^2 = \left(\frac{\pi^2}{3} + 4\gamma_E^2\right) \beta_0^2 + 2(\beta_1 + 2\beta_0 a_1) \gamma_E + a_2,$$

²The value of the $a_2$ coefficient in eq. (4) obtained in ref. [21] exceeds the correct result obtained recently [22] by $2\pi^2 C_A^2$. Though the value of ref. [21] was used in the previous analysis of the $\Upsilon$ sum rules [8] and B-meson semileptonic width [13], its difference from the present one is numerically small and results in no significant change for the numerical estimates of $m_b$, $\alpha_s$ and $|V_{cb}|$ given in these papers. In the present paper the most recent value of ref. [22] is used.
\[ C_1^2 = 2(\beta_1 + 2\beta_0 a_1) + 8\beta_0^2 \gamma_E, \quad C_2^2 = 4\beta_0^2, \]

\[ a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_f, \]

\[ a_2 = \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) C_A^2 - \left( \frac{1798}{81} + \frac{56}{3} \zeta(3) \right) C_A T_F n_f \]

\[ - \left( \frac{55}{3} - 16\zeta(3) \right) C_F T_F n_f + \left( \frac{20}{9} T_F n_f \right)^2, \]

\[ \beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4C_F T_F n_f. \]

The second term in eq. (1) is generated by the operator of dimension five in the nonrelativistic expansion of the vector current (see, e.g. [23]). It contains the Green function of the pure Coulomb Schrödinger equation at the origin \( G_C(x, y, k) \rvert_{x,y=0} \) [24]. In the short distance limit \( x \to 0 \) the Coulomb Green function \( G_C(x, 0, k) \) has \( 1/x \) and \( \ln x \) divergent terms. These terms, however, have no imaginary part and do not contribute to the spectral density of the polarization function. Hence they can be subtracted. After the subtraction the (renormalized) Coulomb Green function takes the form

\[ G_C^r(0, 0, k) = -\frac{C_F \alpha_s m_b^2}{4\pi} \left( \frac{k}{C_F \alpha_s m_b} + \ln \left( \frac{k}{\mu_f} \right) + \gamma_E + \Psi_1 \left( 1 - \frac{C_F \alpha_s m_b}{2k} \right) \right) \]  \( (5) \)

where \( \Psi_n(z) = d^n \ln \Gamma(z)/dz^n \) and \( \Gamma(z) \) is the Euler \( \Gamma \)-function.

The solution of eq. (3) can be found within the standard nonrelativistic perturbation theory around the Coulomb Green function as a leading order approximation

\[ G(0, 0, k) = G_C(0, 0, k) + \Delta G(0, 0, k), \]

\[ \Delta G(0, 0, k) = -\int G_C(0, x, k) \left( -\frac{\Delta_2^2}{4m_b^2} + \frac{\alpha_s}{4\pi} \Delta_1 V(x) + \ldots \right) G_C(x, 0, k) dx + \ldots \]  \( (6) \)

\[ = \Delta \Delta_g, N_A, B F G + \Delta_1 G + \Delta_2^{(2)} G + \Delta_2^{(1)} G + \ldots \]

The corrections to the Coulomb Green function at the origin due to \( \Delta^2 \), \( V_{NA} \) and \( V_{BF} \) terms are known analytically [16, 17]. The nontrivial part of the calculation consists in a proper treatment of some integrals in eq. (6) that correspond to these corrections and
diverge, or become ill-defined at small $x$. This divergence is a consequence of the fact that the nonrelativistic approximation is not relevant for the description of the short distance effects. Moreover, in contrast to the next-to-leading order there exists a divergence in the imaginary part of the Green function in NNLO that contributes to the spectral density. The divergence can be regularized by introducing an ultraviolet cutoff $\mu_f$. Then, following the general line of the effective field theory approach one has to match the calculation of the Green function to the calculation of the coefficient $C_h(\alpha_s)$ i.e. to match the regularization procedures for the $G(0,0,k)$ and $C_h$. This can be done by comparing eq. (6) and the result of perturbative calculation of the spectral density \[25\] in the formal limit $\alpha_s \ll v \ll 1$ up to the order $\alpha_s^2$ \[16, 17\]. The result reads\[3\]

$$
\Delta_{\Delta^2, N_{A,BF}} G = \frac{C_F \alpha_s m_b^2}{4\pi} \left( \frac{5}{8} \frac{k^3}{C_F \alpha_s m_b^2} + \frac{2k^2}{m_b^2} \left( \ln \left( \frac{k}{\mu_f} \right) \right) + \frac{\gamma_E + \Psi_1}{2k} \left( 1 - \frac{C_F \alpha_s m_b}{2k} \right) - \frac{11}{16} \frac{C_F \alpha_s k}{m_b} \Psi_2 \left( 1 - \frac{C_F \alpha_s m_b}{2k} \right) + \frac{4\pi}{3} \frac{C_F \alpha_s}{m_b^2} \left( 1 + \frac{3}{2} \frac{C_A}{C_F} \right) G_C^r(0,0,k)^2 \right),$$

(7)

The next-to-leading (NLO) correction $\Delta_1 G$ in eq. (3) due to the first iteration of $\Delta_1 V$ term of the QCD potential and the NNLO correction $\Delta_2 G$ due to $\Delta_2 V$ part of the potential have been found in ref. [7]. The correction $\Delta_2^{(1)} G$ due to the second iteration of $\Delta_1 V$ term which has to be kept in NNLO approximation has been obtained in ref. [8]. We describe the details of this calculation below.

We use the following partial wave representation for the Coulomb Green function

$$
G_C(x, y, k) = \sum_{l=0}^{\infty} (2l + 1) G_l(x, y, k) P_l((xy)/xy)
$$

Calculating the NNLO corrections to the nonrelativistic Green function and the hard renormalization coefficient one encounters the divergences with the specific form depending on the regularization procedure. They are not included to eqs. \[6, 7\] because the divergent terms in the hard renormalization coefficient cancel the divergence in the corrections to the nonrelativistic Green function leaving the logarithmic dependence of eq. (6) on the cutoff $\mu_f$ which compensates corresponding dependence of eq. (5).
\[ G_l(x, y, k) = \frac{m_b k}{2\pi} (2kx)^l (2ky)^l e^{-k(x+y)} \sum_{m=0}^{\infty} \frac{L_l^{2l+1}(2kx)L_l^{2l+1}(2ky)m!}{(m + l + 1 - \alpha_s C_F m_b/(2k))(m + 2l + 1)!} \]  

(8)

where \( P_l(z) \) is a Legendre polynomial and \( L_l^\alpha(z) \) is a Laguerre polynomial

\[ L_l^\alpha(z) = e^z z^{-\alpha} \frac{(d/dz)^m}{m!} (e^{-z} z^{m+\alpha}). \]

Only \( l = 0 \) component of eq. (8) is necessary for the calculation of the corrections to the Green function at the origin up to NNLO approximation.

Let us consider first the NLO correction. It can be written as

\[ \Delta_1 G = \frac{\alpha_s}{4\pi} \left( \frac{m_b k}{2\pi} \right)^2 \sum_{m,n=0}^{\infty} H(m)H(n) \int e^{-2kx} L_m^1(2kx)L_n^1(2kx) \Delta_1 V(x) dx \]  

(9)

where

\[ H(m) = \left( m + 1 - \frac{C_F \alpha_s m_b}{2k} \right)^{-1} \]

and we use the equality

\[ L_m^\alpha(0) = \frac{\Gamma(m + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(m + 1)}. \]

The integrals in eq. (9) corresponding to \( \Delta_1 V \) term also diverge at small \( x \). The divergent part however is \( k \) independent and does not contribute to the spectral density. As a consequence no matching is necessary for calculation of these corrections. In the representation (9) the divergence of the integral at small \( x \) is transformed to the divergence of the sum. The divergent part of eq. (9) can be separated by the following method. Eq. (9) can be rewritten in the form

\[ \Delta_1 G = \frac{\alpha_s}{4\pi} \left( \frac{m_b k}{2\pi} \right)^2 \left( \sum_{m,n=0}^{\infty} F(m)F(n) \int e^{-2kx} L_m^1(2kx)L_n^1(2kx) \Delta_1 V(x) dx \right. \]

\[ + \left. 2 \sum_{m=0}^{\infty} F(m) \int e^{-2kx} L_m^1(2kx) \Delta_1 V(x) \right) \frac{dx}{2k} + \int \frac{e^{-2kx} \Delta_1 V(x)}{(2kx)^2} \]

(10)

where

\[ F(m) = \frac{C_F \alpha_s m_b}{(m + 1)2k} \left( m + 1 - \frac{C_F \alpha_s m_b}{2k} \right)^{-1} \]
and we used the property of the Laguerre polynomial
\[ \sum_{m=0}^{\infty} \frac{L_\alpha^m(z)}{m + \alpha} = z^{-\alpha} \Gamma(\alpha). \]

In eq. (10) all sums are convergent and the divergence is contained in the last term. Two divergent integrals in the last term of this equation after a regularization take the form
\[
\int \frac{e^{-2kx} \ln(\mu x)}{x} dx = -\gamma_E L(k) + \frac{1}{2} L(k)^2 + \ldots,
\]
\[
\int \frac{e^{-2kx}}{x} dx = L(k) + \ldots
\]
where \( L(k) = -\ln(2k/\mu) \) and ellipsis stands for inessential \( k \) independent divergent parts.

Two finite integrals in eq. (9) are explicitly given by the following expression
\[
\int e^{-z} L_\alpha^1(z) L_\alpha^1(z) \ln(z) z dz = \begin{cases}
(m + 1) \Psi_1(m + 2), & m = n, \\
-\frac{n + 1}{m - n}, & m > n,
\end{cases}
\]
\[
\int e^{-z} L_\alpha^1(z) \ln(z) dz = -2\gamma_E - \Psi_1(m + 1).
\] (11)

To compute the first integral we rewrite it in the form containing a derivative with respect to an auxiliary parameter \( \varepsilon \)
\[
\int e^{-z} L_\alpha^1(z) L_\alpha^1(z) \ln(z) z dz = \frac{d}{d\varepsilon} \left( \int e^{-z} L_\alpha^1(z) L_\alpha^1(z) z^{1+\varepsilon} dz \right) \bigg|_{\varepsilon=0}.
\] (12)

By using the relations
\[
L_\beta^\alpha(z) = \sum_{n=0}^{m} \frac{\Gamma(\beta - \alpha + n)}{\Gamma(\beta - \alpha) \Gamma(n + 1)} L_\alpha^n(z),
\]
\[
\int e^{-z} L_\alpha^\alpha(z) L_\alpha^\alpha(z) z^\alpha dz = \delta_{mn} \frac{\Gamma(m + \alpha + 1)}{\Gamma(m + 1)}
\]
for \( \beta = 1, \alpha = 1 + \varepsilon \) the integration in the right hand side of eq. (12) can be performed analytically. Then taking the derivative in \( \varepsilon \) at \( \varepsilon = 0 \) we get the first line of eq. (11). The second integral in eq. (11) can be computed using the same technique. Thus the final result for the correction is
\[
\Delta_1 G = \frac{\alpha_s}{4\pi} \frac{C_F \alpha_s m_b^2}{4\pi} \left( \sum_{m=0}^{\infty} F^2(m)(m + 1) \left( C_0^1 + (L(k) + \Psi_1(m + 2)) C_1^1 \right) \right)
\]
\[
-2 \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} F(m)F(n) \frac{n + 1}{m-n} C_1^1 + 2 \sum_{m=0}^{\infty} F(m) \left( C_0^1 + (L(k) - 2\gamma_E - \Psi_1(m + 1))C_1^1 \right) + L(k)C_0^1 + \left( -\gamma_E L(k) + \frac{1}{2}L(k)^2 \right) C_1^1 .
\]  

(13)

Note that the nontrivial part of the calculation is to find the correction to the Green function due to the logarithmic correction to the Coulomb potential \( (1) \). Computation of the correction due to the constant part of the correction to the Coulomb potential is trivial. It can be also found from the leading order result by changing the parameter of pure Coulomb solution \( \alpha_s \rightarrow \alpha_s(1 + C_0^1 \alpha_s/4\pi) \).

The \( \Delta_2^{(2)} G \) correction has been obtained in the same way. The \( \Delta_2^{(1)} G \) part is finite and requires no regularization. It can be computed directly using the representation \( (8) \) and the first integral of eq. \( (11) \). The results of the calculations are given in Appendix.

The Green function at the origin can be written in the form which includes only single poles in the energy variable. Such a form looks more natural for the Green function of a nonrelativistic Schrödinger equation

\[
G(0, 0, E) = \sum_{m=0}^{\infty} \frac{|\psi_m(0)|^2}{E_m - E} + \frac{1}{\pi} \int_0^{\infty} \frac{|\psi_{E'}(0)|^2}{E' - E} dE'
\]

(14)

where \( E = -k^2/m_b \), \( \psi_{m,E'}(0) \) is the wave function at the origin, the sum goes over bound states and the integral is over the state of continuous part of the spectrum. In this way the corrections to the Green function stemming from the discrete part of the spectrum reduce to corrections to the Coulomb bound state energy levels

\[
E_m = -\frac{C^2_F \alpha_s^2 m_b}{4(m + 1)^2} \left( 1 + \Delta_1 E_m + \Delta_2^{(2)}E_m + \Delta_{1A,BF}E_m + \Delta_{2A,BF} E_m + \Delta_{2(1)} E_m \right)
\]

(15)

and to the values of Coulomb bound state wave functions at the origin

\[
|\psi_m(0)|^2 = \frac{C^2_F \alpha_s^2 m_b^3}{8\pi (m + 1)^3} \left( 1 + \Delta_1 \psi_m^2 + \Delta_{d5} \psi_m^2 + \Delta_2^{(2)} \psi_m^2 + \Delta_{2A,BF} \psi_m^2 + \Delta_2^{(1)} \psi_m^2 \right)
\]

(16)

where \( \Delta_{d5} \psi_m^2 \) is the correction due to the second term in eq. \( (1) \).
In NLO an explicit analytical expression for the corrections to the bound state parameters has the form

$$\Delta_1 E_m = \frac{\alpha_s}{4\pi} \left( 2C_0^1 + 2(L(m) + \Psi_1(m + 2))C_1^1 \right), \quad (17)$$

$$\Delta_1 \psi_2^2 = \frac{\alpha_s}{4\pi} \left( 3C_0^1 + (3L(m) - 1) - 2\gamma_E + \frac{2}{m + 1} + \Psi_1(m + 2) - 2(m + 1)\Psi_2(m + 1)C_1^1 \right)$$

where

$$L(m) = \ln \left( \frac{(m + 1)\mu}{C_F\alpha_s m_b} \right).$$

The expressions of the NNLO corrections are rather cumbersome and given in Appendix.

The principal possibility of relating results of eqs. (15, 16) to the mass and leptonic width of corresponding $\Upsilon$ resonances with an account for nonperturbative corrections was discussed in [26].

Thus we have described some details of obtaining the complete analytical expressions for the vacuum polarization function of heavy quarks near the two-particle threshold in NNLO presented in [7, 8]. Note that our result does not determine the additive renormalization constant in the real part of the polarization function which has to be fixed according to the standard normalization condition $\Pi(0) = 0$. This constant, however, is inessential for physical applications.

### 3. $\Upsilon$ sum rules and $b$-quark mass

The result for the near threshold behavior of the vacuum polarization function of the heavy quark is now applied to the analysis of sum rules for the $\Upsilon$ system. Within the sum rules approach the moments $M_n^{th}$

$$M_n^{th} = \frac{12\pi^2}{n!} (4m_b^2)^n \left. \frac{d^n}{ds^n} \frac{\Pi(s)}{s} \right|_{s=0}^{s=\infty} = (4m_b^2)^n \int_0^\infty \frac{R(s)ds}{s^{n+1}}$$

4The NNLO vacuum polarization function of heavy quarks was also recently obtained in a different representation in ref. [10] where the “low scale” mass of $b$-quark was studied in the context of $\Upsilon$ sum rules.
of the spectral density $R(s) = 12\pi \text{Im}\Pi(s+i\epsilon)$ of the theoretical vacuum polarization function of the heavy quarks $\Pi(s)$ should be compared with the experimental ones

$$\mathcal{M}_n^{\text{exp}} = \left(\frac{4m_b^2}{Q^2_b}\right)^n \int_0^\infty \frac{R_b(s)ds}{s^{n+1}}$$

(18)

under the assumption of quark-hadron duality. Here $Q_b = -1/3$ is the $b$-quark electric charge and the theoretical parameter $m_b$ is included for convenience. The experimental moments $\mathcal{M}_n^{\text{exp}}$ are generated by the function $R_b(s)$ which is the normalized cross section of $e^+e^-$ annihilation $R_b(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ and, at least in principle, can be directly found from experiment for any $s$. In practice, however, the spectral density is well measured from experiments only for values of energy rather close to threshold where the well pronounced $\Upsilon$ resonances exist. Therefore numerical values for the experimental moments are obtained by saturating the integral in eq. (18) with the contribution of the first six $\Upsilon$ resonances while the tail of the spectral density at large energy is approximated theoretically

$$\mathcal{M}_n^{\text{exp}} = \left(\frac{4m_b^2}{Q^2_b}\right)^n \left(\frac{9\pi}{\alpha^2_{QED}(m_b)}\right) \sum_{k=1}^6 \frac{\Gamma_k}{M_k^{2n+1}} + \int_{s_0}^\infty \frac{ds}{s^{n+1}} R_b(s)$$

(19)

The leptonic widths $\Gamma_k$ and masses $M_k$ ($k = 1 \ldots 6$) of the resonances are known with good accuracy [28]. For example, for large $n$ the dominant contribution to the moments comes from the first $\Upsilon$ resonance for which $\Gamma_{\Upsilon(1S)} = 1.32 \pm 0.05$ keV and $M_{\Upsilon(1S)} = 19460.37 \pm 0.21$ MeV.

The electromagnetic coupling constant is renormalized to the energy of order $m_b$ with the result $\alpha^2_{QED}(m_b) = 1.07\alpha^2$. The rest of the spectrum beyond the resonance region for energies larger than $s_0 \approx (11.2 \text{ GeV})^2$ (continuum contribution) is approximated by the theoretical spectral density multiplied by the parameter $0.5 < t < 1.5$ which accounts for the uncertainty in the experimental data in this energy region. The allowed range for the parameter $t$ is chosen to be sufficiently large that gives a very conservative estimate of uncertainty caused by all other contributions but the resonance one. For instance, the numerical change of the experimental moments produced by a manifest account for the physical continuous spectrum due to the open $BB$ production at $s > (10.6 \text{ GeV})^2$ is well within the error bars introduced
by variation of the $t$ parameter.

The ordinary perturbative expansion in the coupling constant taken up to a finite order is not accurate enough for sufficiently high moments since the coefficients of the expansion in low orders of perturbation theory grow fast with the order of the moment. Therefore a kind of resummation of the terms giving a dominant contribution is required. These terms are identified as those connected with a strong Coulomb interaction in the final state of quark-antiquark production. The importance of the Coulomb resummation for large $n$ can by directly inferred from the explicit expression for the moments in a pure Coulomb approximation

$$M_n^C = 3\pi N_c \left( \frac{\Gamma\left(n - \frac{1}{2}\right)}{4\Gamma\left(\frac{1}{2}\right)\Gamma\left(n + 1\right)} + \frac{C_F\alpha_s}{n} + \sum_{m=2}^{\infty} \left(\frac{C_F\alpha_s}{2}\right)^m \frac{\zeta(m)\Gamma\left(n + \frac{m-1}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(n + 1\right)} \right).$$

(20)

The terms of the series in eq. (20) first increase numerically and then decrease. The number $\tilde{m}$ of the term with a maximal numerical magnitude is determined from the relation

$$\Psi_1\left(n + \frac{\tilde{m} - 1}{2}\right) - \Psi_1\left(\frac{\tilde{m} - 1}{2}\right) = 2 \ln\left(\frac{2}{C_F\alpha_s}\right).$$

(21)

In eq. (21) we neglect a weak $m$ dependence of $\zeta(m)$ for large $m$. For several first terms with $m < n$ and large $n$ the coefficients of the series increase by $\sqrt{n}$ in subsequent orders of the expansion that makes the Coulomb resummation necessary. Note that for large $m > n$ and fixed $n$ the character of asymptotic behavior in $m$ changes and terms become proportional to

$$\left(\frac{C_F\alpha_s}{2}\right)^m \zeta(m) \left(\frac{m - 1}{2}\right)^n$$

and the series (20) is convergent in $\alpha_s$ at $C_F\alpha_s/2 < 1$.

On the other hand, no resummation is necessary for small $n$ and finite order perturbation theory is sufficient for the phenomenological analysis giving a reasonable precision. Moreover, the nonrelativistic approximation is not relevant for the analysis of low moments. The ordinary perturbation theory expressions for several first moments of the spectral density
are now available with $\alpha_s^2$ accuracy [27]. These low moments, however, cannot be used in theoretical formulas for sum rules directly because they get a sizable contribution from the large momentum region far from threshold. In this case, the experimental moment necessary for comparison can not be found with sufficiently high precision because the spectrum is not well known experimentally.

For numerical estimates it is important to fix the allowed range for the normalization point which is present in the explicit formula of the polarization function. Following the general line of the renormalization group approach the normalization point has to be chosen to minimize the higher order corrections. In fact, the normalization points of $\alpha_s$ entering the coefficient $C_h$ and the nonrelativistic Green function can be different when NNLO corrections are considered. The difference between the normalization points of the hard and soft corrections can be noticed only in higher orders of perturbative expansion. At first sight this gives an additional possibility to improve the convergence of the perturbation theory for the moments. The typical hard scale of the problem is the heavy quark mass $m_b$. Indeed, one can see that for $\mu \sim m_b$ the NNLO correction to $C_h$ is small in comparison with the NLO one. However a naive estimate of the soft “physical scale” as a characteristic scale of the Coulomb problem $\mu \sim m_b\alpha_s$ [8, 9] is not acceptable since the direct calculation of the NNLO corrections shows that the perturbation theory series for the moments blow up for this value of the soft normalization point in $\overline{\text{MS}}$ renormalization scheme. This phenomenon can be clearly seen from Fig. 1 and Fig. 2 where the relative weights of the NLO and NNLO corrections to the parameters of the nonrelativistic Green function are plotted. Though the NLO corrections reach their minimal magnitude at $\mu \sim m_b\alpha_s$ the NNLO corrections to the Green function and the moments are completely out of control at this point. At first glance this seems to contradict our physical intuition. However since the normalization scale is defined in the rather artificial $\overline{\text{MS}}$ scheme the connection of which with $b\bar{b}$ physics is not straightforward there is no reason for coincidence of $\mu$ parameter with any physical scale of
the process. The relative weight of the NNLO corrections is stabilized at \( \mu \sim m_b \). Moreover, here the \( \mu \) dependence of the moments is minimal so in our opinion there is no reason to split the hard and soft normalization points and use the smaller numerical value for the soft normalization scale. The dependence of the moments on \( \mu_f \) is rather weak and we put \( \mu_f = \mu \).

The range of \( n \) which can be used for reliable estimates is also quite restricted. Indeed, the low moments cannot be used in sum rules because of the large uncertainty on the experimental side due to the poor knowledge of the spectral density at large energies as has been already pointed out. From Table 1 one sees that for \( n \geq 8 \) the experimental moments are rather sensitive to the form of the continuous spectrum beyond the resonance region. Note that we assume rather large uncertainty of the continuum to be on the safe side. However, its contribution is essentially suppressed in comparison with the resonance one and the resulting error of the whole quantity in eq. (19) is of the same order of magnitude as the uncertainties introduced by the resonance contribution.

| \( t \) | \( M_{5}^{exp} \) | \( M_{10}^{exp} \) | \( M_{15}^{exp} \) |
|---|---|---|---|
| 0.5 | 1.185 | 1.015 | 1.057 |
| 1.0 | 1.315 | 1.028 | 1.059 |
| 1.5 | 1.446 | 1.041 | 1.061 |

Table 1: Sensitivity of the experimental moments to the continuum contribution above \( s_0 \) for \( s_0 = (11.2 \text{ GeV})^2 \).

On the other hand, the leading nonperturbative power correction to the polarization function due to gluonic condensate which can introduce a large uncertainty on the theoretical side of sum rules is known to be important for \( n > 20 \). The smallness of the nonperturbative contribution happened to be the only practical restriction on maximal allowed \( n \) in NLO
analysis \cite{3}. In NNLO analysis, however, the upper limit on $n$ is stronger and is connected with the behavior of the perturbative expansion for the moments as well. From Tables \cite{2, 3} one sees that for $n \gtrsim 12$ the perturbative series for the moments is not well convergent and the $\mu$ dependence of the moments becomes strong that can be considered as an indication that the higher order corrections are large here. Perhaps this can mean that the Coulomb solution is not a good leading order approximation for these $n$. A possible explanation of this phenomenon can be obtained from the analysis of an asymptotic character of the series in $\alpha_s$ for the moments. Indeed, the Coulomb resummation (20) extracts from the coefficients of perturbation theory for the moments the part that forms a convergent series. This means that resummation cannot change the analytic structure of the moment as a function of the coupling constant. On the other hand the Coulomb approximation for the $n$th moment is saturated by the large terms of the order $m \sim \tilde{m}$ where $\tilde{m}$ is the solution of eq. (21). Now suppose that the full perturbative expansion for the moments is an asymptotic series with $m_{opt}$ being the critical order where the series starts to diverge. In this case if the number of the moment is large enough i.e. if $\tilde{m}$ is close to $m_{opt}$ the asymptotic growth of the high order coefficients of the full series becomes more important than the resummed Coulomb part of the series. For such $n$ the Coulomb approximation does not saturate the full series coefficients in the orders which are dominant in the Coulomb approximation itself and large corrections naturally appear. The condition for applicability of the Coulomb solution as a leading order approximation now is $\tilde{m} < m_{opt}$. Taking into account that the perturbative expansion around the Coulomb solution is not well convergent for $n \gtrsim 12$ from the above condition and eq. (21) for the physical value of $\alpha_s$ in $\overline{\text{MS}}$ scheme one finds $m_{opt} \sim 3 - 4$ which is a reasonable value for a QCD series.

Here a remark concerning a specific form of Green function is in order. The difference between two representations (8) and (14) for the Green function appears only in higher orders in the coupling constant. We do not attempt to compute the spectral density at positive energies
Table 2: The 0th, 1st and 2nd order theoretical moments for $\alpha_s(M_z) = 0.118$, $m_b = 4.8$ GeV and $\mu = m_b$. The continuum contribution above $s_0$ is subtracted. Here the star (*) stands for the moments obtained by using the representation (14) for Green function.

in the threshold region because the nonperturbative effects are large and prevent point-wise evaluation of this quantity. We are interested in the polarization function in Euclidean region where some high order derivatives can be obtained. The nonrelativistic Green function has no immediate meaning for us as a source for the spectral density at positive energies in threshold region and only serves as a tool for resummation of a special contributions of perturbation theory expansion. Therefore these two representations are equivalent in NNLO.

We find, however, that when the representation (14) is used the perturbative series for the moments are more divergent especially for large $n$ (see Table 2) and the dependence of the results on $n$ and $\mu$ is stronger. Therefore we consider the representation (6) as a preferable one. The fact that resummation of the corrections keeping the Coulomb form of the nonrelativistic Green function spoils the properties of the perturbative expansion can be considered as another indication that for large $n$ the Coulomb resummation is not enough and a more adequate leading order approximation is necessary with a proper large $n$ behavior.

Now we turn to a discussion of numerical values of parameters extracted from the analysis of sum rules. The sum rules for the $\Upsilon$ system are not very sensitive to $\alpha_s$ so NLO [7] and
Table 3: The scale dependence of the theoretical moments in the NNLO approximation for $\alpha_s(M_Z) = 0.118$ and $m_b = 4.8$ GeV (the continuum contribution above $s_0$ is subtracted).

NNLO \cite{8} analyses give a rather rough estimate

$$\alpha_s(M_Z) = 0.118 \pm 0.006$$

which is in agreement with other available data \cite{28}. This result is obtained by simultaneous fit for $\alpha_s$ and $m_b$. The central values are found by the standard least $\chi^2$ method \cite{6}. The uncertainty of the fit is estimated by fixing $m_b$ to its central value and varying $\mu$ and $n$ within allowed range. On the other hand, the sum rules are much more sensitive to the $b$-quark mass so it is instructive to fix $\alpha_s$ to the “world average” value $\alpha_s(M_Z) = 0.118$ \cite{28} and then extract $m_b$. The final estimate of the bottom quark pole mass is

$$m_b = 4.80 \pm 0.06 \text{ GeV}.$$  \hspace{1cm} (24)

The uncertainty corresponds to the interval $3.5 \text{ GeV} \leq \mu \leq 6.5 \text{ GeV}$ for $8 \leq n \leq 12$. In fact the central value changes slightly in a wider interval $5 < n < 20$ but we restrict the range of $n$ to minimize the uncertainty related to $\mu$ and $t$ dependence. Note that the scale dependence of the moments is mainly due to the scale dependence of $\alpha_s(\mu)$ while the explicit dependence on $\mu$ is rather weak.

\footnote{In ref. \cite{8} the interval $4.74 \text{ GeV} \leq m_b \leq 4.82 \text{ GeV}$ has been obtained for $b$-quark pole mass. In the present paper we use the different range of $n$ and $\mu$ for the analysis of the sum rules and give more conservative estimate of the uncertainty.}
The perturbation theory relation between the perturbative pole mass and the mass $m_b(\mu)$ defined in the $\overline{\text{MS}}$ renormalization scheme is known up to two-loop level \cite{29}. It is therefore straightforward to find the numerical value for the $\overline{\text{MS}}$ mass of $b$-quark using result of eq. (24)

$$m_b(m_b) = 4.21 \pm 0.11 \text{ GeV}$$

where we assume the interval \cite{23} for the strong coupling constant. An interesting fact is that the convergence of the perturbative series for $\overline{\text{MS}}$ mass obtained order by order from the $\Upsilon$ sum rules

$$\overline{m}_b(\overline{m}_b) = (\overline{m}_b(\overline{m}_b))^{LO}(1 - 0.085 - 0.021)$$

is much better than for the pole mass

$$m_b = m_b^{LO}(1 - 0.001 + 0.021)$$

though the second order terms in the above expansions are rather close numerically.

Let us emphasize that the convergence of the perturbation theory for the vacuum polarization function of heavy quark near the threshold and to the moments is not fast in the $\overline{\text{MS}}$ renormalization scheme. We have found the NNLO corrections to exceed the NLO ones. Furthermore, in the case of $b$-quark the corrections due to the perturbative modification of the Coulomb instantaneous potential (i.e. related to $\Delta G_1$ and $\Delta G_2^{(i)}$ terms) dominate the total correction in the NLO and NNLO. Inclusion of these corrections is quite important for consistent analysis of sum rules for the $\Upsilon$ system. A conjecture that this fact is a consequence of the asymptotic character of the series which leads to the intrinsic ambiguity in the heavy quark pole mass was studied in detail in the literature \cite{10, 30, 31, 32}. It mainly based on consideration of the renormalon contribution \cite{30, 32}. Within this picture the high order contributions to the moments are saturated by the corrections that are associated with $(\alpha_s \beta_0 \ln(\mu r))^m$ terms in perturbative series for the heavy quark potential. These corrections are generated by terms stemming from the running of the coupling constant that can be found from renormalization group analysis. However, the explicit calculation shows
that in NNLO the corresponding contribution (the one proportional to the coefficient $C_2^2$) provides only $\sim 10\%$ of the total NNLO correction \textit{i.e.} the NNLO correction turns out to be essentially larger than the renormalon picture predicts. Thus the asymptotic series for the pole mass seems to reach its critical order and starts to diverge. If this is a case then eq. (24) gives a numerical estimate of a finite order sum of the asymptotic series. This value extracted from the sum rules can be used as an auxiliary parameter in the expressions for the physical quantities order by order of QCD perturbation theory. This, however, is sufficient for physical applications because the pole mass is not an observable.

An independent NNLO analysis of $\Upsilon$ sum rules has been done in ref. [9] where the Laplace transform of the polarization function was studied in the spirit of ref. [6] while the explicit NNLO expression of the polarization function near the threshold has not been obtained. We found that for a given set of the parameters the numerical values of the theoretical moments obtained on the basis of our result for the polarization function is in a good agreement with the results of ref. [9]. However, in ref. [9] an essentially lower range of the soft normalization scale $1.5 \text{ GeV} < \mu < 3.5 \text{ GeV}$ was used for the phenomenological analysis. This does not affect strongly the result for the strong coupling constant $0.96 < \alpha_s(M_Z) < 0.124$ obtained in ref. [6] but leads to larger value of the $b$-quark pole mass $4.78 \text{ GeV} < m_b < 4.98 \text{ GeV}$ (the result of the constrained fit for the fixed $\alpha_s$ of ref. [9]) than the number given by eq. (24). From Fig. 2 we, however, see that such a low normalization scale is unsuitable for a reliable estimate because the perturbation theory around the Coulomb solution diverges in this case.

### 4 Inclusive $B$-meson semileptonic width and $|V_{cb}|$ matrix element.

The theoretical expression for the inclusive semileptonic width of $B$-meson up to the second order in the strong coupling constant and up to the first order of heavy quark expansion
reads
\[
\Gamma_{\text{sl}} = \frac{G_F^2 m_b^5}{192 \pi^3} |V_{cb}|^2 \left( F_1 \left( \frac{m_c^2}{m_b^2} \right) C_\Gamma(\alpha_s) \left( 1 - \frac{\mu^2 - \mu_G^2}{2 m_b^2} \right) - 2 F_2 \left( \frac{m_c^2}{m_b^2} \right) \frac{\mu_G^2}{m_b^2} \right) - 2 F_2 \left( \frac{m_c^2}{m_b^2} \right) \frac{\mu_G^2}{m_b^2} \]  
(25)
\]
where \( F_1(x) = 1 - 8x - 12x^2 \ln x + 8x^3 - x^4 \) is the phase space factor, \( F_2(x) = (1 - x)^4 \), \( \mu_\pi \) and \( \mu_G \) are the HQET parameters \[33, 34\]. The hadronic matrix element of the gluon dipole operator \( \mu_G \) of the heavy quark is directly related to the masses of \( B \)-mesons with different spin structure
\[\mu_G^2 (1 + O(1/m_b)) = \frac{4}{3} (M_{B^*}^2 - M_B^2) = 0.36 \text{ GeV}^2.\]
The hadronic matrix element of the heavy quark kinetic operator \( \mu_\pi \) suffers from larger uncertainty. For numerical estimates we use the result
\[\mu_\pi^2 = (0.5 \pm 0.15) \text{ GeV}^2\]

obtained within QCD sum rules framework \[35\]. The perturbative coefficient \( C_\Gamma(\alpha_s) \) up to the second order in \( \alpha_s \) \[12, 36\] is
\[C_\Gamma = 1 - 1.67 \frac{\alpha_s(\mu)}{\pi} - (8.4 \pm 0.4) \left( \frac{\alpha_s}{\pi} \right)^2\]  
(26)
for the normalization point \( \mu = \sqrt{m_b m_c} \).

Along with ordinary power corrections coming from heavy quark expansion an account for truncation of perturbative series for semileptonic width can generate further contributions to eq. (25) which within the renormalon picture scale as \( \Lambda_{QCD}/m_b \). We discuss some details related to this issue later.

One sees from formula (25) that the semileptonic width depends rather strongly on \( m_b \). Therefore if \( m_b \) is taken from a theoretical expression for some other experimental quantity it should be determined with a great accuracy for obtaining a reasonable precision of the \( |V_{cb}| \) determination. In the previous Section we have described the moments of the vacuum polarization function and have extracted the numerical value of the \( b \)-quark pole mass entering the theoretical perturbation theory expressions for the moments. Being analyzed
independently, the perturbative series in $\alpha_s$ for the moments and for the width expressed in terms of the pole mass $m_b$ seem not to enjoy a fast apparent convergence that can lead to a large uncertainty due to higher order contributions. In principle the apparent convergence of the finite order perturbation theory series can be changed by redefinition of the mass [30, 37, 38] or the coupling constant [7, 39]. However $m_b$ is not an observable and has no immediate physical meaning. Therefore it can be safely removed from relations between physical observables. The idea of trading the unphysical parameters in favor of direct relations between observables constitutes now the most general trend in high precision phenomenological analyses, its application to the particular case of the $B$-meson semileptonic decay width was discussed earlier (see e.g. [3, 38] and references therein). In this Section we establish the direct relation between inclusive $B$-meson semileptonic width and one of the moments obtained in the previous section. This relation is independent of any redefinition of the quark mass and of using different couplings so the accuracy of the relation reveals the actual precision of the approximation used for comparison of two physical observables. It happens that the convergence of the perturbation theory for this relation is rather fast and the subsequent approximations converge very well in a heuristic sense that the higher order approximation is close to the previous one.

Our analysis consists in direct relating the factor $m_b^5$ in eq. (25) to the $n$th moment of the $\Upsilon$ sum rules

$$m_b^5 = \left( \frac{\mathcal{M}_n^{th}}{\mathcal{M}_n^{exp}} \right)^{\frac{2}{3n}}.$$  \hspace{0.5cm} (27)

where (dimensionful) moments $\tilde{\mathcal{M}}_n^{exp} = \mathcal{M}_n^{exp}/m_b^{2n}$ is a purely experimental quantity. The theoretical moment $\mathcal{M}_n^{th}$ is a dimensionless quantity which depends on $m_b$ only logarithmically in a finite order in $\alpha_s$. Thus, the substitution of relation (27) to eq. (25) substantially reduces its dependence on $m_b$ though it also introduces an explicit uncertainty due to $\mathcal{M}_n^{exp}$.

We use eqs. (25, 27) to find for the mixing angle $|V_{cb}|$

$$|V_{cb}| = (192\pi^3)^{1/2} K_{th} \frac{\Gamma_{{\text{sl}}}}{G_F} \frac{(\tilde{\mathcal{M}}_n^{exp})^{5/4n}}{G_F}$$  \hspace{0.5cm} (28)
where the functions

\[ K_{th} = (M_n^{th})^{-5/4n} \left( F_1 \left( \frac{m_c^2}{m_b^2} \right) C_\Gamma(\alpha_s) \left( 1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \right) - 2F_2 \left( \frac{m_c^2}{m_b^2} \right) \frac{\mu_G^2}{m_b^2} \right)^{-\frac{1}{2}} \]  

(29)

accumulate theoretical information depending on \( m_b, m_c, \alpha_s, \) etc.

The function \( F_1(m_c^2/m_b^2) \) gives rather large theoretical uncertainty if masses of \( b \) - and \( c \) -quarks are considered as independent variables. However there is almost model independent constraint of the form

\[ m_b - m_c - \mu_\pi^2 \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right) + O(1/m_{b,c}^2) = \bar{M}_B - \bar{M}_D = 3.34 \text{ GeV} \]

(30)

where \( \bar{M}_B = 5.31 \text{ GeV}, \bar{M}_D = 1.97 \text{ GeV} \) denote the spin-average meson masses, e.g. \( \bar{M}_B = \frac{1}{4}(M_B + 3M_{B^*}). \) With this constraint the function \( F_1(m_c^2/m_b^2) \) becomes a function of a single variable \( \bar{F}_1(m_b) \). Note that in such a setting the \( m_b \) dependence of the function \( \bar{F}_1(m_b) \) partly cancels the large \( m_b^5 \) dependence of the width. Furthermore if the relation (27) is used to express the \( b \)-quark pole mass in the argument of the function \( \bar{F}_1(m_b) \) in terms of the moments of the spectral density then only logarithmic dependence on \( m_b \) appears in the right hand side of eq. (28).

Now we analyze the theoretical factor \( K_{th} \) numerically order by order in \( \alpha_s \). The result reads

\[ K_{th}|_{n=10} = 1.366(1 + 0.088 + 0.028). \]

(31)

where the value of \( b \)-quark pole mass in the fixed order in \( \alpha_s \) with Coulomb resummation is found from eq. (27). For comparison, the perturbative series for \( m_b \) that follows from eq. (27) and the series for \( C_\Gamma \) are

\[ m_b|_{n=10} = 4.71(1 - 0.001 + 0.021), \]

(32)

\[ C_\Gamma = 1 - 0.146 - 0.064. \]

(33)

Thus we find that in the expansions of the theoretical moment (as well as \( m_b \) itself) and width expressed in terms of \( m_b \) the NNLO corrections are of the order of the NLO ones while
the perturbative series for the mixing angle, or the theoretical coefficient eq. (31), converges much better.

Let us discuss the problem of convergence in more details. Though there is no rigorous result on the asymptotic structure of the expansion (26) it is widely believed (mainly due to consideration of the renormalon contribution) that this asymptotic series (as well as the series for the moments) starts to diverge already in some low orders of perturbation theory. The truncation of the asymptotic series at the optimal order results in the inherent uncertainty in these quantities which parametrically as large as $\Lambda_{QCD}/m_b$ within the renormalon picture. An argument in favor of this conjecture is the relatively large value of the NNLO corrections both to the moments and the semileptonic width. Since the detailed structure of the above asymptotic expansions remains unknown we cannot conclude that the divergence of the series for the moments and the series for the width cancels each other exactly. But because the terms in the perturbative expansion (31) decrease rapidly we can reasonably hope that the critical order where the series (31) starts to diverge is higher than ones of eq. (32) and eq. (33). Moreover, a partial cancellation of the divergences has been found within the renormalon picture [10, 32].

As for numerics, we use the following central values for our experimental inputs (see [28] for more detail):

$$BR(B \rightarrow X_c \ell \nu_\ell) = 10.5\%, \quad \tau_B = 1.55 \text{ ps},$$

$$\left( M_{10}^{exp} \right)^{1/4} = 3.95 \times 10^{-4} \text{ GeV}^{-5}, \quad \alpha_s(M_Z) = 0.118.$$  

With these numbers we obtain the value of the matrix element $|V_{cb}|$

$$|V_{cb}| = 0.0423 \left( \frac{BR(B \rightarrow X_c \ell \nu_\ell)}{0.105} \right)^{\frac{1}{2}} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{\frac{1}{2}} \times \left( 1 - 0.01 \frac{\alpha_s(M_Z) - 0.118}{0.006} \right) (1 \pm \Delta_{npt}) (1 \pm \Delta_{tr})$$

where $\Delta_{npt} \sim 0.02$ is the uncertainty in the nonperturbative contribution [3] induced mainly by the uncertainty of $\mu_\pi$ in eq. (30). We have introduced also the uncertainty due to trun-
cation of the perturbative series for the $K_{th}$ parameter $\Delta_{tr} \sim 0.01$ which is taken as a half of the last term in eq. (31).

The typical scale of uncertainty of key parameters is also indicated. Another important source of the uncertainty is the scale dependence of the theoretical moment and the experimental errors in the value of $\alpha_s(M_Z)$ because of rather high sensitivity of the theoretical moment to $\alpha_s$. In fact these uncertainties are closely related since as it has already been pointed out the scale dependence of $M^{th}_{10}(\alpha_s(\mu), \mu)$ is mainly due to the scale dependence of $\alpha_s(\mu)$. The pointed error bars roughly correspond to the interval $3.5 \text{ GeV} \leq \mu \leq 6.5 \text{ GeV}$ at fixed $\alpha_s(M_Z) = 0.118$. The central value in eq. (34) does not change in the interval $5 \leq n \leq 15$ and we have chosen $n = 10$ for our final estimate by the reasons discussed in the previous section.

The main part of the experimental uncertainty is related to the uncertainty in the experimentally measured inclusive semileptonic width. The experimental situation changes rather quickly and data are improving fast that means that the experimental uncertainties will be smaller (see e.g. [40]). The uncertainty in $M^{exp}_{10}$ comes mainly from the uncertainties in leptonic widths $\Gamma_k$ of $\Upsilon$ resonances. It is about 5% and leads only to 0.3% uncertainty in $|V_{cb}|$ so we do not include it to the error bars in eq. (34).

Now the advantage of our approach becomes clear – large errors due to the uncertainty in $m_b$ is now partly shifted to more direct experimental data. The expression for $|V_{cb}|$ matrix element has a very weak dependence on $m_b$ so a possible uncertainty in $b$-quark pole mass does not lead to any uncertainty in $|V_{cb}|$. Furthermore the terms in the perturbative expansion (31) decrease rapidly which is an indication that the higher orders corrections to the obtained result are small enough.

Our result is in a good agreement with the previous estimate $|V_{cb}| = 0.0419$ [3]. Our value, however, is somewhat larger than the estimate $|V_{cb}| = 0.039$ of ref. [4]. There is no much hope to reduce the uncertainty in the nonperturbative contribution. Thus, in our
opinion, the model independent result presented in the paper provides one with the most reliable and accurate estimate of the CKM matrix element $|V_{cb}|$ from the inclusive $B$-meson semileptonic width.

5 Conclusion.

In this paper we have presented the determination of the $b$-quark pole and $\overline{\text{MS}}$ mass and Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$ from $\Upsilon$ sum rules and $B$-meson semileptonic width in the NNLO in the strong coupling constant. In our opinion it seems to be a final result for these problems within the framework of analytical treatment: the next order approximation is too complicated to deal with analytically within QCD or NRQCD. In particular, the nonrelativistic approximation in next order in $\alpha_s$ should be supplemented by the real gluon radiation that takes the problem to the completely new level of complexity and makes it practically unsolvable within the framework presented here.

The self-consistence of the $\Upsilon$ sum rules has been checked and the intervals for the relevant normalization scale and the moment numbers which can be used for a reliable estimates have been found. An asymptotic character of the perturbative expansion for the moments of the $\Upsilon$ system spectral density was discussed and a conjecture on the critical order where the series in $\alpha_s$ for the moments starts to diverge has been made. We have also presented a new representation of the heavy quark vector current correlator near threshold based on the explicit formulas for the NNLO correction to the $^3S_1$ heavy quark bound state parameters$^6$.

We have constructed the direct relation between the moments of the $\Upsilon$ system spectral density and the inclusive $B$-meson semileptonic width up to the NNLO order. We have found that when the unphysical variable (the $b$-quark pole mass) is removed the residual perturbation theory in the coupling constant for $\Gamma_{sl}$ (or $|V_{cb}|$) works well and demonstrates $^6$When this paper was in preparation a paper [10] appeared where the similar result was obtained.
nice convergence.

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References

[1] W.E.Caswell and G.E.Lepage, Phys.Lett. B167(1986)437;  
    G.E.Lepage et al., Phys.Rev. D46(1992)4052.

[2] E.Eichten, B.Hill, Phys.Lett. B234(1990)511;  
    H.Georgi, Phys.Lett. B240(1990)447.

[3] V.A.Novikov et al., Phys.Rev.Lett. 38(1977)626;  
    V.A.Novikov et al., Phys.Rep. C41(1978)1;  
    M.B.Voloshin, Yad.Fiz. 36(1982)247;  
    M.B.Voloshin and Yu.M.Zaitsev, Usp.Fiz.Nauk 152(1987)361.

[4] M.Neubert, “B Decays And the Heavy Quark Expansion”,  
    to appear in the Second Edition of: Heavy Flavours, edited by A.J. Buras and M. Lindner (World Scientific, Singapore), hep-ph/9702375.

[5] N.Uraltsev, “Heavy Quark Expansion in Beauty and its Decays”, Lectures given at the International School of Physics Enrico Fermi “Heavy Flavour Physics: a Probe of Nature’s Grand Design”, hep-ph/9804275.

[6] M.Voloshin, Int.J.Mod.Phys. A10(1995)2865.

[7] J.H.Kühn, A.A.Penin and A.A.Pivovarov, Nucl.Phys. B534(1998)356.

[8] A.A.Penin and A.A.Pivovarov, Phys.Lett. B435(1998)413.

[9] A.H.Hoang, Preprint UCSD/PTH 98-02, hep-ph/9803454.

[10] K.Melnikov and A.Yelkhovsky, Preprint TTP-98-17, hep-ph/9805270.

[11] M.Luke, M.J.Savage and M.B.Wise, Phys.Lett. B345(1995)301;  
    P.Ball, M.Beneke and V.Braun, Phys.Rev. D51(1995)1125.
[12] A.Czarnecki, Phys.Rev.Lett. 76(1996)4124;
    A.Czarnecki and K.Melnikov, Nucl.Phys. B505(1997)65; Phys.Rev.Lett. 78(1997)3630;
    TTP-98-14, hep-ph/9804213.

[13] A.A.Penin and A.A.Pivovarov, Preprint INR-98-979, hep-ph/9805344.

[14] R.Tarrach, Nucl.Phys. B183(1981)384.

[15] G.Källen and A.Sarby, K.Dan.Vidensk.Selsk.Mat.-Fis.Medd. 29(1955), N17, 1.

[16] A.H.Hoang and T.Teubner, Phys.Rev. D58(1998)114023.

[17] K.Melnikov and A.Yelkhovsky, Nucl.Phys. B528(1998)59.

[18] S.N.Gupta and S.F.Radford, Phys.Rev. D24(1981)2309; Phys.Rev. D25(1982)3430 (Erratum);
    S.N.Gupta, S.F.Radford and W.W.Repko, Phys.Rev. D26(1982)3305.

[19] L.D.Landau and E.M.Lifshitz, Relativistic Quantum Theory, Part 1 (Pergamon, Oxford, 1974).

[20] W.Fisher, Nucl.Phys B129(1977)157;
    A.Billoire, Phys.Lett. B92(1980)343.

[21] M.Peter, Phys.Rev.Lett. 78(1997)602; Nucl.Phys B501(1997)471.

[22] Y.Schröder, Preprint DESY 98-191, hep-ph/9812205.

[23] G.T.Bodwin, E.Braaten and G.P.Lepage, Phys.Rev. D51(1995)1125.

[24] J.Schwinger, J.Math.Phys. 5(1964)1606.

[25] A.H.Hoang, Phys.Rev. D56(1997)7276;
    A.H.Hoang, J.H.Kühn and T.Teubner, Nucl.Phys. B452(1995)173;
A.Czarnecky and K.Melnikov, Phys.Rev.Lett. 80(1998)2531;
M.Beneke, A.Signer and V.A.Smirnov, Phys.Rev.Lett. 80(1998)2535.

[26] M.B.Voloshin, Nucl.Phys. B154(1979)365;
H.Leutwyler, Phys.Lett. B98(1981)447.

[27] K.G.Chetyrkin, J.H.Kühn and M.Steinhauser, Phys.Lett. B371(1996)93;
Nucl.Phys. B482(1996)213.

[28] Particle Data Group, Phys.Rev. D54(1996)1.

[29] N.Gray, D.J.Broadhurst, W.Grafe and K.Schilcher, Z.Phys. C48(1990)673.

[30] I.Bigi, M.Shifman, N.Uraltsev and A.Vainstein, Phys.Rev. D50(1994)2234;
M.Beneke and V.Braun, Nucl.Phys. B426(1994)301.

[31] M.Jezabek, M.Peter and Y.Sumino, Phys.Lett. B428(1998)352.

[32] A.H.Hoang et al., Preprint UCSD/PTH 98-13, hep-ph/9804227;
M.Beneke, Phys.Lett. B434(1998)115.

[33] I.Bigi, N.Uraltsev and A.Vainstein, Phys.Lett. B293(1992)430;
B.Blok and M.Shifman, Nucl.Phys. B399(1993)441, 459;
I.Bigi, M.Shifman, N.Uraltsev and A.Vainstein, Phys.Rev.Lett. 71(1993)496.

[34] A.Manohar and M.B.Wise, Phys.Rev. D49(1994)1310.

[35] E.Bagan, P.Ball, V.Braun and G.Gosdzinsky, Phys.Lett. B342(1995)362.

[36] Y.Nir, Phys.Lett. B221(1989)184.

[37] I.Bigi, M.Shifman, N.Uraltsev and A.Vainstein, Phys.Rev. D56(1997)4017.

[38] I.Bigi, M.Shifman and N.Uraltsev, Ann.Rev.Nucl.Part.Sci. 47(1997)591.
[39] N.Uraltsev, Int.J.Mod.Phys. A11(1996)515.

[40] S.Stone, “Prospects For b Physics in the Next Decade”, HEPSY-96-01, hep-ph/9610305

[41] A.Pineda and F.J.Yndurain, Phys.Rev. D58(1998)094022.
Appendix.

A. The correction $\Delta_2^{(2)} G$ due to the $\Delta_2 V$ part of the potential is (see eq. (4))

$$\Delta_2^{(2)} G = \left( \frac{\alpha_s}{4\pi} \right)^2 \frac{C_F \alpha_s m^2}{4\pi} \left( \sum_{m=0}^{\infty} F^2(m) \left( (m+1) \left( C_0^2 + L(k)C_1^2 + L^2(k)C_2^2 \right) \right. \right.$$ 

$$+ (m+1)\Psi_1(m+2) \left( C_1^2 + 2L(k)C_2^2 \right) + I(m)C_2^2 \right)$$

$$+ 2 \sum_{m=1}^{m-1} \sum_{n=0}^{n-1} F(m) F(n) \left( - \frac{n+1}{m-n} \left( C_1^2 + 2L(k)C_2^2 \right) + J(m,n)C_2^2 \right)$$

$$+ 2 \sum_{m=0}^{\infty} F(m) \left( C_0^2 + L(k)C_1^2 + (L^2(k) + K(m))C_2^2 - (2\gamma_E + \Psi_1(m+1)) \left( C_1^2 + 2L(k)C_2^2 \right) \right)$$

$$+ \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F^2(m) \left( m^2 + 1 \right) \left( \Psi_1(m+1) + \frac{\pi^2}{3} - \frac{2}{(m+1)^2} \right)$$

$$- 2(\gamma_1(m+1) + \gamma_E),$$

$$J(m,n) = 2 \frac{n+1}{m-n} \left( \Psi_1(m-n) - \frac{1}{n+1} + 2\gamma_E \right)$$

$$+ 2 \frac{m+1}{m-n} (\Psi_1(m-n+1) - \Psi_1(m+1)),$$

$$K(m) = 2(\psi_1(m+1) + \gamma_E)^2 + \Psi_2(m+1) - \psi_2(m+1) + 2\gamma_E^2, $$

$$N(k) = \left( \gamma_1 + \frac{\pi^2}{6} \right) L(k) - \gamma_1 L^2(k) + \frac{1}{3} L^3(k).$$

where dots stand for inessential $k$ independent divergent part and

$$I(m) = (m+1) \left( \Psi_1(m+2) - \Psi_2(m+2) + \frac{\pi^2}{3} - \frac{2}{(m+1)^2} \right)$$

$$- 2(\psi_1(m+1) + \gamma_E).$$
The correction $\Delta_2^{(1)} G$ due to the second iteration of $\Delta_1 V$ term

$$\Delta_2^{(1)} G = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{C_F \alpha_s}{4\pi} \right)^2 \frac{m_b^3}{2k} \left( \sum_{m=0}^{\infty} H^3(m)(m+1) \left( C_1^1 + (\Psi(m+2) + L(k)) C_1^1 \right) \right)^2$$

$$-2 \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \frac{n+1}{m-n} C_1^1 \left( H^2(m)H(n) \left( C_0^1 + \left( \Psi(m+2) + L(k) - \frac{1}{2(m-n)(m+1)} \right) C_1^1 \right) \right)$$

$$+ H(m)H^2(n) \left( C_0^1 + \left( \Psi(n+2) + L(k) - \frac{1}{2(m-n)(m+1)} \right) C_1^1 \right)$$

$$+ 2(C_1^1)^2 \left( \sum_{m=2}^{\infty} \sum_{l=1}^{m-1} \sum_{n=0}^{l-1} H(m)H(n)H(l) \frac{n+1}{(l-n)(m-n)} \right)$$

$$+ \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \sum_{l=0}^{n-1} H(m)H(n)H(l) \frac{l+1}{(n-l)(m-n)}$$

$$+ \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \sum_{l=0}^{m-1} H(m)H(n)H(l) \frac{(l+1)(m+1)}{(n+1)(n-l)(m-n)}$$

\[\text{B. The NNLO corrections to the square of the Coulomb } {}^3S_1 \text{ heavy quark bound state wave function at the origin (eq. (15))} \]

$$\Delta_{d5} \psi_m^2 = \frac{1}{3} \frac{C_F^2 \alpha_s^2}{(m+1)^2},$$

$$\Delta_{\Delta^2, NA, BF} \psi_m^2 = -C_F^2 \alpha_s^2 \left( \frac{15}{8} \frac{1}{(m+1)^2} + \frac{2}{3} + \frac{C_A}{C_F} \right) \left( -\ln \left( \frac{2\mu_f(m+1)}{C_F \alpha_s m_b} \right) \right)$$

$$+ \gamma_E + \Psi_1(m+1) - \frac{1}{(m+1)} \right) \right),$$

$$\Delta_2^{(2)} \psi_m^2 = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 3(C_0^2 + L(m)C_1^2 + L^2(m)C_2^2) + (-1 - 2\gamma_E + \frac{2}{m+1} + \Psi_1(m+2)$$

$$-2(m+1)\Psi_2(m+1))(C_1^2 + 2L(m)C_2^2) + \left( \frac{I(m)}{m+1} + 2K(m) - 2\Psi_1(m+2)$$

$$+ 2 \sum_{n=0}^{m-1} \frac{m+1}{(n-m)(n+1)} J(m,n) + 2 \sum_{n=m+1}^{\infty} \frac{m+1}{(n-m)(n+1)} J(n,m) \right) C_2^2),$$

$$\Delta_2^{(1)} \psi_m^2 = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 3(C_0^1 + (L(m) + \Psi_1(m+2))C_1^1) \right)^2$$
\[ +2C_1 \left( \sum_{n=0}^{m-1} \frac{(n+1)(m+1)}{(n-m)^3} \left( C_0^1 + \left( (L(m) + \Psi_1(n+2) + \frac{1}{2} \frac{n+1}{(n-m)(m+1)} \right) C_1^1 \right) \right. \]

\[- \sum_{n=m+1}^{\infty} \frac{(m+1)^2}{(n-m)^3} \left( C_0^1 + \left( (L(m) + \Psi_1(n+2) - \frac{1}{2} \frac{1}{n-m} \right) C_1^1 \right) \right) \]

\[ + 2C_1 \left( C_0^1 + (L(m) + \Psi_1(m+2)) C_1^1 \right) \left( \frac{5}{2} + \sum_{n=0}^{m-1} \frac{n+1}{(n-m)^2} U(m,n) \right) \]

\[- \sum_{n=m+1}^{\infty} \frac{m+1}{(n-m)^2} U(m,n) + 2(C_1^1)^2 \left( \frac{1}{2} - \sum_{n=0}^{m-1} \frac{n+1}{(n-m)^2} + \sum_{n=m+1}^{\infty} \frac{m+1}{(n-m)^2} \right) \]

\[+ \frac{1}{2} \sum_{n=0}^{m-1} \frac{n+1}{(n-m)^3} U(m,n) + \frac{1}{2} \sum_{n=m+1}^{\infty} \frac{(m+1)^2}{(n-m)^3(n+1)} U(m,n) \]

\[+ \sum_{n=1}^{m-1} \sum_{l=0}^{n-1} \left( \frac{(l+1)(m+1)}{(n-m)^2(l-m)^2} - \frac{(l+1)(m+1)}{(n-m)^2(l-m)(n-l)} - \frac{(l+1)(m+1)}{(l-m)^2(n-m)(n-l)} \right) \]

\[+ \sum_{n=m+1}^{\infty} \sum_{l=0}^{m-1} \left( \frac{(l+1)(m+1)}{(n-m)^2(l-m)^2(n-l)} + \frac{(l+1)(m+1)^2}{(n-m)^2(l-m)^2} \right) \]

\[+ \sum_{n=2l=m+1}^{\infty} \sum_{l=m+1}^{\infty} \left( \frac{(m+1)^2}{(n-m)^2(l-m)^2} + \frac{(l+1)(m+1)^2}{(n-m)^2(l-m)(n-l)} \right) \]

where

\[ U(m,n) = 3 + \frac{n+1}{m+n+2} - 2 \frac{(n+1)^2}{(n-m)(n+m+2)} \]

**C. The NNLO corrections to the Coulomb $^3S_1$ heavy quark bound state energy levels**

(eq. (10))

\[ \Delta \Delta^{(2)}_{NABF} E_m = \frac{C_F^2 \alpha_s^2}{(m+1)} \left( \frac{C_A}{C_F} + \frac{2}{3} - \frac{11}{16} \frac{1}{(m+1)} \right) \]

\[ \Delta^{(2)}_m = 2 \left( \frac{\alpha_s}{4\pi} \right)^2 \left( C_0^2 + L(m)C_1^2 + L^2(m)C_2^2 + \Psi_1(m+2)(C_1^2 + 2L(m)C_2^2) \right. \]

\[+ \frac{I(m)}{(m+1)C_2^2} \]

\[ \Delta_1^{(2)} E_m = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \left( C_0^1 + (L(m) - 2 + \Psi_1(m+2))C_1^1 \right) \left( C_0^1 + (L(m) + \Psi_1(m+2))C_1^1 \right) \right) \]

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The results for the heavy quark bound state parameters are obtained on the basis of corrections found in [8]. When this work was in preparation ref. [10] appeared where the correction to the 1S3 heavy quark bound state wave function at the origin were computed up to NNLO. This calculation is consistent with our result for \( \Delta_1 \psi^2_m \), \( \Delta_{dS} \psi^2_m \), \( \Delta_{N_{A,BF}} \psi^2_m \) and \( \Delta^{(2)}_2 \psi^2_m \). The expression for \( \Delta^{(1)}_2 \psi^2_m \) was obtained in ref. [10] in a different representation so direct comparison with our formula is difficult. At the same time numerically the results are in a good agreement. Our result for the NNLO correction to the heavy quark 1S3 bound state energy levels coincides with the result of ref. [11].

**Figure captions**

**Fig. 1.** The relative weight of the NLO corrections to the ground state energy \( E_{0}^{NLO}/E_{0}^{LO} - 1 \) (curve a) and to the square of the ground state wave function at the origin \( (1/|\psi(0)|^2)^{NLO}/(1/|\psi(0)|^2)^{LO} - 1 \) (curve b) as a function of the normalization point \( \mu \).

**Fig. 2.** The relative weight of the NNLO corrections to the ground state energy \( E_{0}^{NNLO}/E_{0}^{NLO} - 1 \) (curve a) and to the square of the ground state wave function at the origin \( (1/|\psi(0)|^2)^{NNLO}/(1/|\psi(0)|^2)^{NLO} - 1 \) (curve b) as a function of the normalization point \( \mu \).
Fig. 1.

Fig. 2.