Parity of the Partition Function $p(n, k)$

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Abstract

Let $p(n, k)$ denote the number of partitions of $n$ into parts less than or equal to $k$. We show several properties of this function modulo 2. First, we prove that for fixed positive integers $k$ and $m$, $p(n, k)$ is periodic modulo $m$. Using this, we are able to find lower and upper bounds for the number of odd values of the function for a fixed $k$.

1 Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum equals $n$. For instance, there are five partitions of 4: $4, 3 + 1, 2 + 2, 2 + 1 + 1$, and $1 + 1 + 1 + 1$. Let $p(n)$ denote the number of partitions of $n$ (For more details on this function, see [2]). It has been conjectured that the odd density of $p(n)$ is $\frac{1}{2}$. That is,

$$\lim_{N \to \infty} \frac{\# \{ n \leq N \mid p(n) \text{ is odd} \}}{N} = \frac{1}{2}.$$  

Although this problem remains open, several properties of the distribution of $p(n)$ modulo 2 have been discovered. In 1959, Newman [4] showed that $p(n)$ achieves both odd and even values infinitely often, and is not periodic modulo any integer. Nicolas, Ruzsa, and Sárközy [5] showed that the partition function satisfies

$$\sqrt{N} \ll \# \{ n \leq N \mid p(n) \text{ is even} \}.$$

Ahlgren [1] found similar bounds when taking $n$ in arithmetic progressions.

The strongest bounds known currently still do not preclude the possibility of the odd density of $p(n)$ being zero. However, the efforts made on the parity of $p(n)$ motivate a similar discussion for $p(n, k)$, the number of partitions of $n$ into parts less than or equal to $k$. It is not difficult to see that the generating function for $p(n, k)$ is given by

$$\sum_{n=0}^{\infty} p(n, k)q^n = \prod_{n=1}^{k} \frac{1}{1 - q^n}.$$  

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Since the generating function is a finite product rather than an infinite product, \( p(n, k) \) is in some ways more convenient to work with than the ordinary partition function. Kronholm [3] found that the function modulo odd primes \( k \) yields congruences of the form

\[
p(nk, k) - p(nk - \text{lcm}(1, 2, \cdots, k), k) \equiv 0 \pmod{k}
\]

for sufficiently large \( n \).

In this paper, we will find new properties of \( p(n, k) \). The most important result we will need, stated in the following theorem, reduces the odd density of \( p(n, k) \) to a finite problem.

**Theorem 1.1.** For any positive integers \( k \) and \( m \), the function \( p(n, k) \) modulo \( m \) is periodic on \( n \). That is, given \( k, m \in \mathbb{N} \), there exists \( L \in \mathbb{N} \) such that \( p(n, k) \equiv p(n + L, k) \pmod{m} \) for all values of \( n \).

Given this theorem, we can explicitly compute the odd density of \( p(n, k) \) for small values of \( k \). Additionally, it allows us to find an upper bound for the odd density in Section 3.

**Theorem 1.2.** There exist infinitely many values of \( k \) such that

\[
\lim_{N \to \infty} \frac{\# \{ n \leq N \mid p(n, k) \text{ is odd} \}}{N} \leq \frac{2}{3}.
\]

In the opposite direction, we can also look at the maximum consecutive number of even values of \( p(n, k) \). By observing how this number varies depending on \( k \), we find a weaker lower bound in Section 4.

**Theorem 1.3.** For all positive integers \( k \),

\[
\lim_{N \to \infty} \frac{\# \{ n \leq N \mid p(n, k) \text{ is odd} \}}{N} \geq \frac{2}{k(k + 1)}.
\]

2 **Periodicity of \( p(n, k) \) modulo \( m \)**

**Lemma 2.1.** For any integer \( n \) and any positive integer \( k > 1 \),

\[
p(n, k) = p(n - k, k) + p(n, k - 1),
\]

where \( p(0, k) = 1 \) and \( p(n, k) = 0 \) if \( n < 0 \).

**Proof.** For any \( k > 1 \), \( p(n - k, k) \) counts the number of partitions of \( n \) into parts of size at most \( k \) with at least one part of size \( k \). On the other hand, \( p(n, k - 1) \) counts the number of partitions of \( n \) into parts of size at most \( k \) with no parts of size \( k \). Summing both together, the identity follows. \( \square \)
2.1 Proof of Theorem 1.1

We prove Theorem 1.1 by induction on $k$.

For $k = 1$, the statement clearly holds since $p(n, 1) = 1 \equiv 1 \pmod{m}$. Hence $p(n, 1)$ has period 1.

Now, suppose that $p(n, k)$ is periodic with period $L$. We claim that $p(n, k+1)$ has period $m(k+1)L$. To see this, note that for all positive integers $n$,

$$ p(n + (k+1)L, k+1) - p(n, k+1) $$

$$ = \sum_{i=1}^{L} p(n + (k+1)i, k+1) - p(n + (k+1)(i-1), k+1). $$

By (2.1), we can rewrite the above equation as

$$ p(n + (k+1)L, k+1) - p(n, k+1) = \sum_{i=1}^{L} p(n + (k+1)i, k). \quad (2.2) $$

Since (2.2) holds for all positive integers $n$, for any non-negative integer $a$, we can substitute $n + a(k+1)L$ for $n$:

$$ p(n + (a+1)(k+1)L, k+1) - p(n + a(k+1)L, k+1) $$

$$ = \sum_{i=1}^{L} p(n + a(k+1)L + (k+1)i, k). $$

But since $p(n, k)$ modulo $m$ has period $L$, we can simplify this to

$$ p(n + (a+1)(k+1)L, k+1) - p(n + a(k+1)L, k+1) $$

$$ \equiv \sum_{i=1}^{L} p(n + (k+1)i, k) \pmod{m}, $$

which we observe does not depend on $a$. Summing over $a$, we obtain

$$ p(n + m(k+1)L, k+1) - p(n, k+1) $$

$$ = \sum_{a=0}^{m-1} p(n + (a+1)(k+1)L, k+1) - p(n + a(k+1)L, k+1) $$

$$ \equiv \sum_{a=0}^{m-1} \sum_{i=1}^{L} p(n + (k+1)i, k) \pmod{m} $$

$$ \equiv 0 \pmod{m}. $$

So, $p(n, k+1)$ is periodic with period $m(k+1)L$, as desired. The result follows by induction. More directly, we can take the period of $p(n, k)$ to be $m^{k-1}k!$. 

Remark 2.2. We can use the periodicity of $p(n, k)$ modulo 2 to determine the odd density of the function, simply by checking the parity of $p(n, k)$ for $2^{k-1} k!$ consecutive values of $n$. With the use of a computer, this approach gives us the densities for the first 10 values of $k$, which are shown in the following table.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| Odd Density of $p(n, k)$ | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{2}$ | $\frac{27}{56}$ | $\frac{27}{56}$ | $\frac{5}{2}$ | $\frac{5}{56}$ | $\frac{5}{56}$ | $\frac{1}{2}$ |

3 Upper bound for Odd Density of $p(n, k)$

Let $k$ be fixed. Because $p(n, k)$ is periodic modulo 2, its odd density exists and is determined by the number of odd values of $p(n, k)$ over a finite range of $n$. We can use this fact to easily relate the number of odd values of $p(n, k - 1)$ to the number of odd values of $p(n, k)$ over $n$.

Lemma 3.1. For all integers $k > 1$, $p(n, k)$ satisfies

$$
\lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd}, p(n, k - 1) \text{ is odd} \}}{N} = \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is even}, p(n, k - 1) \text{ is odd} \}}{N}.
$$

Proof. Let $k \in \mathbb{N}$ be fixed. Consider

$$
\lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd} \}}{N} = \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd}, p(n, k - 1) \text{ is even} \}}{N} + \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd}, p(n, k - 1) \text{ is odd} \}}{N}.
$$

By (2.1), we can simplify this to

$$
\lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd} \}}{N} = \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n + k, k) - p(n + k, k - 1) \text{ is odd} \}}{N}.
$$

Since we are taking $N \to \infty$, we can replace instances of $n + k$ with $n$.

$$
= \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) - p(n, k - 1) \text{ is odd} \}}{N} = \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is odd}, p(n, k - 1) \text{ is even} \}}{N} + \lim_{N \to \infty} \frac{\# \{ n \leq N | p(n, k) \text{ is even}, p(n, k - 1) \text{ is odd} \}}{N}.
$$
By comparing this to the original expression and cancelling terms, the desired result follows.

Let \( k \) be a fixed positive integer. Lemma 3.1 tells us that for every two odd values of \( p(n, k - 1) \), there is one even value of \( p(n, k) \). For instance, if \( p(n, k - 1) \) is odd for all \( n \), then \( p(n, k) \) will be odd for half of the values of \( n \). In this sense, odd values are “better behaved” than even values. We can predict the odd density of \( p(n, k) \) using the number of odd values of \( p(n, k - 1) \), while even values give no information. Using this property, we can set a bound for the odd density of \( p(n, k) \) for a fixed \( k \).

### 3.1 Proof of Theorem 1.2

We will prove the result by showing that if the odd density of \( p(n, k) \) is greater than \( \frac{2}{3} \) for some \( k \), the odd density of \( p(n, k + 1) \) must be less than \( \frac{2}{3} \). Hence the odd density would have to be at most \( \frac{2}{3} \) for infinitely many \( k \).

Suppose that

\[
\lim_{N \to \infty} \frac{\#\{n \leq N | p(n, k) \text{ is odd}\}}{N} > \frac{2}{3}.
\]

Then we claim that the odd density in \( p(n, k + 1) \) is less than or equal to \( \frac{2}{3} \). To see this, we apply Lemma 3.1 to rewrite the inequality as

\[
2 \lim_{N \to \infty} \frac{\#\{n \leq N | p(n, k) \text{ is odd}, p(n, k + 1) \text{ is even}\}}{N} > \frac{2}{3},
\]

\[
\lim_{N \to \infty} \frac{\#\{n \leq N | p(n, k) \text{ is odd}, p(n, k + 1) \text{ is even}\}}{N} > \frac{1}{3}.
\]

So certainly,

\[
\lim_{N \to \infty} \frac{\#\{n \leq N | p(n, k + 1) \text{ is even}\}}{N} > \frac{1}{3}.
\]

Therefore, there exist infinitely many values of \( k \) for which the odd density is less than or equal to \( \frac{2}{3} \).

### 4 Lower Bound for Odd Density of \( p(n, k) \)

Let \( k \) be a fixed positive integer. By Theorem 1.1, we know that \( p(n, k) \) is periodic modulo 2, so we can write its generating function as

\[
\sum_{n=0}^{\infty} p(n, k)q^n = \prod_{n=1}^{k} \frac{1}{1 - q^n} = \frac{a(q)}{1 - q^L} \pmod{2},
\]

where \( L \) is the period of \( p(n, k) \) and \( a(q) \) is a polynomial with degree less than \( L \). Rearranging to eliminate the denominators, we obtain

\[
a(q) \prod_{n=1}^{k} (1 - q^n) \equiv 1 - q^L \pmod{2},
\]
from which we see that
\[ \deg a(q) = L - \frac{k(k + 1)}{2} \]

But this means that the last \( \frac{k(k+1)}{2} - 1 \) terms of a period of \( p(n, k) \) must be zero modulo 2, since the last nonzero coefficient of \( a(q) \) is that of \( a^{L - \frac{k(k+1)}{2}} \). So for any fixed \( k \), there exist \( \frac{k(k+1)}{2} - 1 \) consecutive values of \( n \) such that \( p(n, k) \) is divisible by 2. We shall show that this is the longest consecutive string of zeroes.

**Theorem 4.1.** For any positive integer \( k \), there exist at most \( \frac{k(k+1)}{2} - 1 \) consecutive values of \( n \) such that \( p(n, k) \) is even.

**Proof.** We prove by induction on \( k \). If \( k = 1 \), then \( p(n, 1) = 1 \) for all positive integers \( n \), so the statement is true. This establishes the base case. Now, suppose that the statement is true for \( k = j - 1 \). Then, if
\[
p(i, j), p(i + 1, j), \ldots, p(i + j(j + 1)/2 - 2, j)
\]
are all even, then
\[
p(i + j, j) - p(i, j), p(i + 1 + j, j) - p(i + 1, j), \ldots,
p(i + j(j + 1)/2 - 2, j) - p(i + j(j + 1)/2 - 2 - j, j)
\]
are all even. Stated differently, this implies that
\[
p(i + j, j - 1), p(i + j + 1, j - 1), \ldots, p(i + j(j + 1)/2 - 2, j - 1)
\]
are all even. This is a sequence of \( \frac{(j-1)(j)}{2} - 1 \) consecutive even numbers. So, by the inductive hypothesis, \( p(i + j(j + 1)/2 - 1, j - 1) \) must be odd. But we already know that \( p(i + j(j + 1)/2 - 1 - j, j) \) is even, so
\[
p(i + j(j + 1)/2 - 1, j) = p(i + j(j + 1)/2 - 1, j - 1) + p(i + j(j + 1)/2 - 1 - j, j)
\]
is odd. The result follows by induction. \( \square \)

The preceding theorem immediately yields a lower bound for the odd density of \( p(n, k) \).

### 4.1 Proof of Theorem 1.3

Fix a positive integer \( k \) and consider the sequence of terms of \( p(n, k) \). By Theorem 4.1, there can be at most \( \frac{k(k+1)}{2} - 1 \) even terms in a row. Therefore, among every \( \frac{k(k+1)}{2} \) consecutive terms of \( p(n, k) \), there must be at least one odd term. Thus,
\[
\lim_{N \to \infty} \frac{\# \{ n \leq N \mid p(n, k) \text{ is odd} \}}{N} \geq \frac{2}{k(k+1)}.
\]
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