TIME DECAY RATE OF GLOBAL STRONG SOLUTIONS TO NEMATIC LIQUID CRYSTAL FLOWS IN $\mathbb{R}^3_+$

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Abstract. In this paper, we obtain optimal time-decay rates in $L^r(\mathbb{R}^3_+)$ for $r \geq 1$ of global strong solutions to the nematic liquid crystal flows in $\mathbb{R}^3_+$, provided the initial data has small $L^3(\mathbb{R}^3_+)$-norm.

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1. Introduction and statement of main results

In this paper, we study a simplified nematic liquid crystal flow in the upper half three space $\mathbb{R}^3_+ = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0 \}$:

\[
\begin{aligned}
  u_t + u \cdot \nabla u + \nabla p &= \mu \Delta u - \lambda \nabla \cdot (\nabla d \odot \nabla d), \\
  \nabla \cdot u &= 0, \\
  d_t + u \cdot \nabla d &= \theta (\Delta d + |\nabla d|^2 d),
\end{aligned}
\]

(1.1)

where $u : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the fluid velocity field, $d : \mathbb{R}^3 \to S^2 \equiv \{ y \in \mathbb{R}^3 : |y| = 1 \}$ denotes the macroscopic orientation field of liquid crystal molecules, $p$ denotes the pressure function, $\nabla d \odot \nabla d = (\langle \nabla_i d, \nabla_j d \rangle)_{1 \leq i, j \leq 3}$, and $\mu, \lambda, \theta > 0$ represent the fluid viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relaxation time for the molecular orientation field respectively. The system (1.1) is equipped with the following initial
and boundary conditions:

\[
\begin{align*}
  u &= \frac{\partial d}{\partial x^3} = 0, \text{ on } \partial \mathbb{R}_+^3 \times (0, \infty), \\
  (u, d) &\to (0, e_3), \text{ as } |x| \to \infty, \\
  (u, d)|_{t=0} &= (u_0, d_0), \text{ in } \mathbb{R}_+^3,
\end{align*}
\]

where \(e_3 = (0, 0, 1) \in \mathbb{S}^2\).

The system \((1.1)\) couples the forced Navier-Stokes equation with the transported flow of harmonic maps to \(\mathbb{S}^2\), which has attracted considerable interests recently. The rigorous mathematical analysis of \((1.1)\) was first made by Lin-Liu \cite{23, 26}, in which they considered the Ginzburg-Landau approximation of \((1.1)\) by replacing \(|\nabla d|^2 \, d\) by \(\frac{1}{\epsilon^2}(1-|d|^2) \, d\) \((\epsilon > 0)\) and proved the existence of global weak solutions and their partial regularities. For the original system \((1.1)\), Lin-Lin-Wang \cite{23} have established the existence of a global weak solution that is smooth away from at most finitely many time in dimension two (see also \cite{15}, Hong-Xin \cite{16}, Huang-Lin-Wang \cite{18}, Li-Lei-Zhang \cite{21}, Wang-Wang \cite{38} for relevant results in dimension two). In dimension three, while the existence of global weak solutions of \((1.1)\) remains an open problem, there has been some interesting progress. For example, Ding-Wen \cite{39} have obtained the local existence and uniqueness of strong solutions in dimension three, Huang-Wang \cite{19} have proved the existence of global weak solutions and their partial regularities. For the original system \((1.1)\), Lin-Lin-Wang \cite{23} have established the existence of a global weak solution that is smooth away from at most finitely many time in dimension two (see also \cite{15}, Hong-Xin \cite{16}, Huang-Lin-Wang \cite{18}, Li-Lei-Zhang \cite{21}, Wang-Wang \cite{38} for relevant results in dimension two). In dimension three, while the existence of global weak solutions of \((1.1)\) remains an open problem, there has been some interesting progress. For example, Ding-Wen \cite{39} have obtained the local existence and uniqueness of strong solutions in dimension three, Huang-Wang \cite{19} have provided a blow-up criterion of strong solutions, and the well-posedness of \((1.1)\) for an initial data \((u_0, d_0)\) with small \(BMO^{-1} \times BMO\)-norm and with small \(L^3_{uloc}(\mathbb{R}^3)\)-norm has been shown by Wang \cite{37} and Hineman-Wang \cite{14} respectively. Most recently, Lin-Wang \cite{24} have shown the existence of global weak solutions in dimension three under the assumption that the initial director field \(d_0(\Omega) \subset \mathbb{S}^2_+\). Concerning the long time asymptotical behavior of global strong solutions to \((1.1)\) in \(\mathbb{R}^3\), Liu-Xu \cite{31} have established an optimal decay rate for \(\|(u, \nabla d)\|_{H^{m}(\mathbb{R}^3)}\) under the assumption that \((u_0, d_0) \in H^m(\mathbb{R}^3) \times H^{m+1}(\mathbb{R}^3, \mathbb{S}^2)\) \((m \geq 3)\) has sufficiently small \(\|(u_0, \nabla d_0)\|_{L^2(\mathbb{R}^3)}\)-norm; while Dai, and her coauthors, has obtained in \cite{4, 5} optimal decay rates in \(H^m(\mathbb{R}^3)\) provided \(\|u_0\|_{H^1(\mathbb{R}^3)} + \|d - e_3\|_{H^2(\mathbb{R}^3)}\) is sufficiently small.

A natural question is to ask for the large time asymptotical behavior of global solutions of \((1.1)\) on general domains. As a first step, we consider in this paper time decay rates in \(L^p(\mathbb{R}^3_+)\) of strong solutions of \((1.1) - (1.2)\) on the upper half space \(\mathbb{R}^3_+\). This consideration is also partly motivated by previous works on the corresponding Navier-Stokes equations on \(\mathbb{R}^3_+\), which has been relatively well understood. For example, the long time behavior of weak and strong solutions of \((1.1)\) in \(L^p(\mathbb{R}^3)\) has been investigated by Bae-Choe \cite{1}, Borchers-Miyakawa \cite{2}, Fujigaki-Miyakawa \cite{8}, Kozono \cite{20} in \(p \in (1, +\infty)\), and by Han \cite{10, 12} for the end point case \(p = 1\), which imposes difficulties due to the unboundedness of the Leray projection operator \(P : L^1(\mathbb{R}^3_+) \to L^1_0(\mathbb{R}^3_+)\). For the nematic liquid crystal flow \((1.1)\), the super-critical nonlinearity \(\nabla \cdot (\nabla d \odot \nabla d)\) in the momentum equation \((1.1)\) introduces new difficulties in establishing time decay estimates for solutions to \((1.1)\) in \(\mathbb{R}^3_+\). In particular,

- While the scaling of \(\nabla d\) is comparable with \(u\), the required estimates on \(\nabla \cdot (\nabla d \odot \nabla d)\) is more delicate than the convective term \(u \cdot \nabla u\), because \(\nabla d\) is not divergence free. In fact, third order derivatives of \(d\) emerge in the estimate of \(\|P (u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d))\|_{L^1(\mathbb{R}^3_+)\}.

which is equivalent to the estimate of \( \| \nabla d(t) \|_{H^2}^2 + \| \nabla^3 d(t) \|_{L^1} \). Therefore, higher order estimates of global solutions \((u, d)\) are needed. To achieve this, we utilize an iteration argument to derive the basic \( L^2 \)-decay estimate by first establishing \( \| \nabla d(t) \|_{L^2} \lesssim t^{-1} \) through a continuity argument, and then improving it to \( t^{-\frac34 + \epsilon} \) \((\epsilon > 0)\), and finally to \( t^{-\frac34} \).

We would also like to point out that

(i) in contrast with \([4, 5, 31]\) where they considered \((1.1)\) on \( \mathbb{R}^3 \), here we consider \((1.1)\) on \( \mathbb{R}^3_+ \) and hence we have to analyze the boundary contributions of global solutions, and

(ii) the time decay estimate in \( L^p(\mathbb{R}^3_+) \) in this paper holds for any initial data \((u_0, d_0) \in L^3(\mathbb{R}^3_+) \times W^{1,3}(\mathbb{R}^3_+, \mathbb{S}^2) \) that has small \( \| (u_0, \nabla d_0) \|_{L^3(\mathbb{R}^3_+)} \) norm, which improves the conditions on the initial data given by \([4, 5, 31]\).

In order to state the main results, we first recall some notations. Denote by \( C^{\infty}_{0, \sigma}(\mathbb{R}^3_+, \mathbb{R}^3) \) the space of smooth divergence-free vector fields with compact supports in \( \mathbb{R}^3_+ \), and \( L^r(\mathbb{R}^3_+, \mathbb{R}^3) \), \( r \in [1, \infty) \), the \( L^r \)-closure of \( C^{\infty}_{0, \sigma}(\mathbb{R}^3_+, \mathbb{R}^3) \) in \( L^r(\mathbb{R}^3_+, \mathbb{R}^3) \). For any nonnegative integer \( k \) and \( r \in [1, \infty) \), denote by \( W^{k, r}(\mathbb{R}^3_+) \) the \((k, r)\)-Sobolev space in \( \mathbb{R}^3_+ \), and \( W^{k, r}_0(\mathbb{R}^3_+) \) the \( W^{k, r} \)-closure of the set \( C^{\infty}_{0}(\mathbb{R}^3_+) \), and

\[
W^{k, r}(\mathbb{R}^3_+, \mathbb{S}^2) = \left\{ v \in W^{k, r}(\mathbb{R}^3_+, \mathbb{R}^3) : v(x) \in \mathbb{S}^2 \text{ for a.e. } x \in \mathbb{R}^3_+ \right\}.
\]

Set

\[
D^{k, r}(\mathbb{R}^3_+) = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^3_+) : \| \nabla^k v \|_{L^r(\mathbb{R}^3_+)} < \infty \right\},
\]

and \( D^{k}(\mathbb{R}^3_+) = D^{k, 2}(\mathbb{R}^3_+) \).

Our first theorem concerns the existence of a unique global strong solution of \((1.1)\) and its time-decay rate. More precisely, we have

**Theorem 1.1.** There exists an \( \varepsilon_0 > 0 \) such that if \( u_0 \in L^r(\mathbb{R}^3_+, \mathbb{R}^3) \) for \( r = 2, 3 \) and \( d_0 \in D^1(\mathbb{R}^3_+, \mathbb{S}^2) \) satisfies

\[
\| u_0 \|_{L^3(\mathbb{R}^3_+)} + \| \nabla d_0 \|_{L^3(\mathbb{R}^3_+)} \leq \varepsilon_0,
\]

then the system \((1.1) - (1.2)\) admits a unique global strong solution \((u, d)\) such that for any \( \tau > 0 \), the following properties hold:

\[
\begin{align*}
\begin{cases}
 u &\in C([0, \infty), L^2(\mathbb{R}^3_+) \cap L^2([0, \infty), D^1(\mathbb{R}^3_+)) ,
 u &\in C([0, \infty), D^2(\mathbb{R}^3_+) \cap L^2([0, \infty), D^3(\mathbb{R}^3_+)) ,
 u_t &\in C([0, \infty), L^2(\mathbb{R}^3_+) \cap L^2([0, \infty), W^{1,2}(\mathbb{R}^3_+)) ,
 u &\in L^\infty([0, \infty), L^2(\mathbb{R}^3_+)) , \nabla d &\in L^\infty([0, \infty), L^3(\mathbb{R}^3_+)) ,
 \| u \|_{L^2(0, \infty), \mathbb{R}^3_+} , \nabla d &\in L^2([0, \infty), D^1(\mathbb{R}^3_+)) ,
 \nabla d &\in C([0, \infty), L^2(\mathbb{R}^3_+) \cap L^2([0, \infty), D^1(\mathbb{R}^3_+)) ,
 \nabla d &\in C([0, \infty), D^2(\mathbb{R}^3_+) \cap L^2([0, \infty), D^3(\mathbb{R}^3_+)) ,
 d_t &\in L^2([0, \infty), L^2(\mathbb{R}^3_+)) ,
 d_t &\in C([0, \infty), W^{1,2}(\mathbb{R}^3_+) \cap L^2([0, \infty), D^2(\mathbb{R}^3_+)) .
\end{cases}
\end{align*}
\]
If, in addition, \( u_0, d_0 - e_3 \in L^1(\mathbb{R}^3_+) \), then we have the following decay estimates:

\[
\begin{align*}
\|u(\cdot, t)\|_{L^r(\mathbb{R}^3_+)} + \|(d - e_3)(\cdot, t)\|_{L^r(\mathbb{R}^3_+)} &\leq Ct^{-\frac{3}{2}(1 - \frac{1}{r})}, \\
\|\nabla d(\cdot, t)\|_{L^s(\mathbb{R}^3_+)} &\leq Ct^{-\frac{3}{2} - \frac{3}{2}(1 - \frac{1}{s})}, \\
\|\nabla u(\cdot, t)\|_{L^p(\mathbb{R}^3_+)} &\leq Ct^{-\frac{3}{2} - \frac{3}{2}(1 - \frac{1}{p})}, \\
\|\nabla^2 d(\cdot, t)\|_{L^q(\mathbb{R}^3_+)} &\leq Ct^{-1 - \frac{3}{2}(1 - \frac{1}{q})},
\end{align*}
\]

for any \( t > 0, r \in (1, \infty), s \in [1, \infty], p \in (1, 6], \) and \( q \in [1, 6]. \)

Now we state the second main result of this paper.

**Theorem 1.2.** Under the same assumptions of Theorem 1.1, if, in addition, \( u_0 \in D^1(\mathbb{R}^3, \mathbb{R}^3) \) and \( \nabla d_0 \in D^1(\mathbb{R}^3_+) \), then

\[
\|\nabla u(\cdot, t)\|_{L^1(\mathbb{R}^3_+)} \leq Ct^{-\frac{1}{2}}
\]

holds for any \( t > 0. \)

**Corollary 1.1.** Under the same assumptions of Theorem 1.1, if, in addition,

\[
\int_{\mathbb{R}^3_+} x_3 |u_0(x)| \, dx < \infty,
\]

then the estimates on \( u \) can be improved into

\[
\begin{align*}
\|u(\cdot, t)\|_{L^r(\mathbb{R}^3_+)} &\leq Ct^{-\frac{3}{2} - \frac{3}{2}(1 - \frac{1}{r})}, \\
\|\nabla u(\cdot, t)\|_{L^p(\mathbb{R}^3_+)} &\leq Ct^{-1 - \frac{3}{2}(1 - \frac{1}{p})},
\end{align*}
\]

for any \( t > 0, r \in (1, \infty], \) and \( q \in (1, 6]. \)

It remains to be an interesting question whether the director field \( d \) satisfies improved estimates on \( \nabla d, \) similar to \((1.7), \) provided \( \int_{\mathbb{R}^3_+} x_3 |\nabla d_0(x)| \, dx < \infty. \)

The strong solutions of \( (1.1) \) from Theorems 1.1 and 1.2 obey Duhamel’s formula:

\[
\begin{align*}
\begin{cases}
  u(t) = e^{-t\Delta} u_0 - \int_0^t e^{-(t-s)\Delta} \mathbb{P}(u(s) \cdot \nabla u(s) + \nabla \cdot (\nabla d(s) \circ \nabla d(s))) \, ds, \\
  (d - w_0)(t) = e^{t\Delta} (d_0 - w_0) - \int_0^t e^{(t-s)\Delta} (u(s) \cdot \nabla d(s) - |\nabla d(s)|^2 d(s)) \, ds,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
  u(t) = e^{-\frac{t}{2}\Delta} u\left(\frac{t}{2}\right) - \int_0^t e^{-(t-s)\frac{3}{2}\Delta} \mathbb{P}(u(s) \cdot \nabla u(s) + \nabla \cdot (\nabla d(s) \circ \nabla d(s))) \, ds, \\
  (d - w_0)(t) = e^{\frac{t}{2}\Delta} (d - w_0)(\frac{t}{2}) - \int_0^t e^{(t-s)\frac{3}{2}\Delta} (u(s) \cdot \nabla d(s) - |\nabla d(s)|^2 d(s)) \, ds,
\end{cases}
\end{align*}
\]

where \( \mathbb{A} = -\mathbb{P}\Delta \) is the Stokes operator.

Since the values of \( \mu, \lambda, \) and \( \theta \) do not play any role in this paper, we will henceforth assume \( \mu = \lambda = \theta = 1. \)
2. Preliminary estimates

In this section, we will provide a few basic estimates related to the Stokes operator \(\mathbb{A}\). We start with the \(L^p - L^q\) estimate for the Stokes semigroup, which can be found in [3].

**Lemma 2.1.** For \(n \geq 2\) and \(1 \leq q < \infty\), let \(a \in L^q_0(\mathbb{R}_+^n, \mathbb{R}^n)\), then for any non-negative integer \(k\), it holds

\[
\|\nabla e^{-tA}a\|_{L^p(\mathbb{R}_+^n)} \leq C_{k,p,q,n} t^{-\frac{k}{2} - \frac{n}{2}(1 - \frac{1}{p})} \|a\|_{L^q(\mathbb{R}_+^n)}, \quad \forall t > 0, \tag{2.1}
\]

where \(C_{k,p,q,n} > 0\) is independent of \(a\), provided either \(1 \leq q < p \leq \infty\) or \(1 < q \leq p < \infty\).

Furthermore,

\[
\|\nabla e^{-tA}\|_{L^1(\mathbb{R}_+^n)} \leq C_{1,n} t^{-\frac{1}{2}} \|a\|_{L^1(\mathbb{R}_+^n)}, \quad \forall t > 0, \tag{2.2}
\]

and (2.1) and (2.2) still hold, if we replace the operator \(e^{-tA}\) by \(e^{tA}\) with \(a \in L^q(\mathbb{R}_+^n, \mathbb{R}^n)\).

Recall that the Stokes operator is defined by

\[
\mathbb{A} = -\mathbb{P}\Delta : \mathcal{D}(\mathbb{A}) \mapsto L^2_\sigma(\mathbb{R}_+^3, \mathbb{R}^3),
\]

where \(D(\mathbb{A}) = H^2(\mathbb{R}_+^3, \mathbb{R}^3) \cap H^1_{0,\sigma}(\mathbb{R}_+^3, \mathbb{R}^3)\), and

\[
\mathbb{P} : L^r(\mathbb{R}_+^3, \mathbb{R}^3) \mapsto L^r(\mathbb{R}_+^3, \mathbb{R}^3)
\]

is the Leray projection operator, which is bounded for any \(1 < r < \infty\). It is well known that \(\mathbb{A}\) is a positive, self-adjoint operator on \(D(\mathbb{A}) \subseteq L^2_\sigma(\mathbb{R}_+^3, \mathbb{R}^3)\), and there exists a uniquely determined resolution \(\{E_\lambda : \lambda \geq 0\}\) of identity in \(L^2_\sigma(\mathbb{R}_+^3, \mathbb{R}^3)\) such that \(\mathbb{A}\) admits a spectral representation

\[
\mathbb{A} = \int_0^\infty \lambda dE_\lambda \quad \tag{2.3}
\]

so that

\[
\|\mathbb{A}u\|^2_{L^2} = \int_0^\infty \lambda^2 d\|E_\lambda u\|^2_{L^2}, \quad \text{for } u \in L^2_\sigma(\mathbb{R}_+^3, \mathbb{R}^3). \tag{2.4}
\]

For \(0 \leq \lambda_0 \leq \infty\), let \(E_{\lambda_0} = s - \lim_{\lambda \to \lambda_0} E_\lambda\) be the strong limit of operators. From [35], \(\{E_\lambda : \lambda \geq 0\}\) satisfies the following properties:

(i) \(E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda\) for \(0 \leq \lambda \leq \mu < \infty\);

(ii) \(E_\lambda = s - \lim_{\mu \to \lambda} E_\mu\) for \(0 < \lambda < \mu < \infty\);

(iii) \(E_0 = 0\), and \(s - \lim_{\mu \to \infty} E_\mu = I\), the identity operator.

For any \(\alpha \in (0,1)\), define the fractional order of Stokes operator \(\mathbb{A}^\alpha\) by

\[
\mathbb{A}^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda \tag{2.5}
\]

with

\[
\|\mathbb{A}^\alpha u\|^2_{L^2} = \int_0^\infty \lambda^{2\alpha} d\|E_\lambda u\|^2_{L^2}, \quad \text{for } u \in L^2_\sigma(\mathbb{R}_+^3, \mathbb{R}^3). \tag{2.6}
\]

Then we have that

\[
\|\nabla u\|^2_{L^2(\mathbb{R}_+^3)} = \|\mathbb{A}^\frac{1}{2} u\|^2_{L^2(\mathbb{R}_+^3)} = \int_0^{+\infty} \lambda d\|E_\lambda u\|^2_{L^2} \geq \rho \int_\rho^{+\infty} d\|E_\lambda u\|^2_{L^2}
\]

where \(\rho \leq 1\).
For any $f$ \eqref{2.7}
\begin{align}
(2.7) \quad \rho \int_0^\infty d\| E_\lambda u \|_{L^2}^2 - \rho \int_0^\mu d\| E_\lambda u \|_{L^2}^2 \\
= \rho(\| u \|_{L^2(\mathbb{R}^3)}^2 - \| E_\mu u \|_{L^2(\mathbb{R}^3)}^2). 
\end{align}

Motivated by \cite{10, 11} and \cite{12}, we perform a decomposition of $G$ in $L^1(\mathbb{R}^3, \mathbb{R}^3)$ as follows. For any $f : \mathbb{R}^3 \mapsto \mathbb{R}$, let $q : \mathbb{R}^3 \mapsto \mathbb{R}$ solve
\begin{align}
(2.8) \quad \begin{cases}
-\Delta q = f \text{ in } \mathbb{R}^3, \\
\frac{\partial q}{\partial x_3} = 0 \text{ on } \partial \mathbb{R}^3.
\end{cases}
\end{align}

Then $q$ can be represented by
\begin{align}
(2.9) \quad q = \mathcal{N}f, \quad \mathcal{N} = \int_0^\infty \mathcal{F}(\tau)d\tau,
\end{align}
where the operator $\mathcal{F}$ is defined by
\begin{align}
(2.10) \quad (\mathcal{F}(t)f)(x) = \int_{\mathbb{R}^3_+} (G_t((x' - y', x_3 - y_3), t) + G_t((x' - y', x_3 + y_3), t))f(y)dy,
\end{align}
and $G_t(x, t) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel in $\mathbb{R}^3$. Then we have
\begin{align}
(2.11) \quad \mathbb{P}(u \cdot \nabla u + \nabla \cdot (\nabla d \circ \nabla d)) \\
= (u \cdot \nabla u + \nabla \cdot (\nabla d \circ \nabla d) + \sum_{i,j=1}^3 \nabla \mathcal{N} \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle)).
\end{align}

We now need to show the following estimate: for any $0 < t < \infty$,
\begin{align}
(2.12) \quad \| \sum_{i,j=1}^3 \nabla \mathcal{N} \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle)(t) \|_{L^1(\mathbb{R}^3)} \\
\leq C(\| u(t) \|_{H^1(\mathbb{R}^3)}^2 + \| \nabla d(t) \|_{H^1(\mathbb{R}^3)}^2 + \| \nabla d(t) \|_{L^1(\mathbb{R}^3)}^2). 
\end{align}

In fact, for $1 \leq m \leq 3$,
\begin{align}
\| \sum_{i,j=1}^3 \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle)(t) \|_{L^1(\mathbb{R}^3)} \\
\leq \| \sum_{i,j=1}^3 \partial_m \int_0^\infty \mathcal{F}(\tau) \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle)(t)d\tau \|_{L^1(\mathbb{R}^3)} \\
\leq \| \sum_{i,j=1}^3 \partial_m \left( \int_0^1 + \int_1^\infty \right) G_\tau \ast \left( \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle) \right)(t)d\tau \|_{L^1(\mathbb{R}^3)} \\
\leq \| \sum_{i,j=1}^3 \partial_m \int_0^1 G_\tau \ast \left( \partial_i \partial_j (u_i u_j + \langle \partial_i d, \partial_j d \rangle) \right)(t)d\tau \|_{L^1(\mathbb{R}^3)} \\
+ \| \sum_{i,j=1}^3 \partial_m \partial_i \partial_j \int_1^\infty G_\tau \ast \left( u_i u_j + \langle \partial_i d, \partial_j d \rangle \right)(t)d\tau \|_{L^1(\mathbb{R}^3)}
\end{align}
This yields (2.14).

\[ \int_{t}^{\infty} \frac{1}{\tau} \left( -\ln(1 + \tau) \right) \| \mathcal{R}(\mathbf{u}) \|_{L^1(\mathbb{R}^3_+)} + \| \nabla \mathcal{G}(\mathbf{d}) \|_{L^1(\mathbb{R}^3_+)} + \| \nabla^2 \mathcal{G}(\mathbf{d}) \|_{L^1(\mathbb{R}^3_+)} \right) \]

(2.13) \leq C \left( \| \mathbf{u}(t) \|_{H^1(\mathbb{R}^3_+)}^2 + \| \nabla \mathbf{d}(t) \|_{L^1(\mathbb{R}^3_+)}^2 + \| \nabla^2 \mathbf{d}(t) \|_{L^1(\mathbb{R}^3_+)} \right),

where \( f \ast g \) represents the convolution of \( f \) and \( g \), and \( f_\alpha \) represents the even extension of function \( f \) with respect to \( x_3 \) from \( \mathbb{R}^3_+ \) to \( \mathbb{R}^3 \).

Next we need the following estimate (see also [3]).

**Lemma 2.2.** For any \( \alpha_1 \in (0, 1) \) and \( \alpha_2, \alpha_3 > 0 \), it holds

\[ \int_{t}^{t/2} (t - s)^{-\alpha_1 - \alpha_2} ds \leq C t^{1 - \alpha_1 - \alpha_2}, \quad \text{for } 0 < \alpha_2 < 1, \]

(2.14)

\[ \int_{t}^{t/2} (t - s)^{-\alpha_1 - \alpha_2} ds \leq C t^{1 - \alpha_1 - \alpha_2}, \quad \text{for } \alpha_2 > 0, \]

(2.15)

\[ \int_{0}^{t} (t - s)^{-\alpha_1} (1 + s)^{-\alpha_3} ds \leq \begin{cases} C t^{-\alpha_1}, & \text{for } \alpha_3 > 1, \\ C t^{1 - \alpha_1 - \alpha_3}, & \text{for } 0 < \alpha_3 < 1, \\ C t^{-\alpha_1} \ln(1 + t), & \text{for } \alpha_3 = 1, \end{cases} \]

and

(2.16)

\[ \int_{t/2}^{t} (t - s)^{-\alpha_1} (1 + s)^{-\alpha_3} ds \leq C t^{1 - \alpha_1 - \alpha_3} \text{ for } \alpha_3 > 0. \]

**Proof.** For convenience of readers, we sketch a proof. By direct calculations, we have that for \( \alpha_2 \in (0, 1) \),

\[
\int_{0}^{t} (t - s)^{-\alpha_1 - \alpha_2} ds = \int_{0}^{t/2} (t - s)^{-\alpha_1 - \alpha_2} ds + \int_{t/2}^{t} (t - s)^{-\alpha_1 - \alpha_2} ds \\
\leq \left( \frac{t}{2} \right)^{-\alpha_1} s^{1 - \alpha_2} \left| \frac{t}{2} \right|^{-\alpha_2} \left| \frac{t}{2} \right|^{1 - \alpha_1} t \\
\leq C t^{1 - \alpha_1 - \alpha_2}. 
\]

This yields (2.14).

Similarly, we have

\[
\int_{0}^{t} (t - s)^{-\alpha_1} (1 + s)^{-\alpha_3} ds = \int_{0}^{t/2} (t - s)^{-\alpha_1} (1 + s)^{-\alpha_3} ds + \int_{t/2}^{t} (t - s)^{-\alpha_1} (1 + s)^{-\alpha_3} ds 
\]
There exists □(3.1). equations over 0 then for any
In particular, there exists □(3.2)

This completes the proof. □

3. Proof of Theorem 1.1

This section will be devoted to the proof of Theorem 1.1. It is divided into several subsections and several Lemmas.

3.1. Global existence of strong solutions. The local existence of strong solutions as stated in Theorem 1.1 can be established by the same approach as [4], which is omitted. To show the time interval can be extended globally and to establish the optimal time decay rates, presented in subsection 3.3 below, we need to obtain a few a priori estimates.

Assume \((u, d, p)\) is a strong solution to \((1.1)-(1.2)\) in \(\mathbb{R}^3_+ \times [0, T]\) for some \(T > 0\), we have

Lemma 3.1. For any \(t \in [0, T]\), it holds that

\[
\frac{d}{dt} \left( \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla d(t)\|_{L^2(\mathbb{R}^3)}^2 \right) = -2 \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|(\Delta d + |\nabla d|^2 d(t))\|_{L^2(\mathbb{R}^3)}^2 \right).
\]

In particular, it holds that

\[
\mathbb{E}(t) = \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla d(t)\|_{L^2(\mathbb{R}^3)}^2 \leq \mathbb{E}(0) = \|u_0\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla d_0\|_{L^2(\mathbb{R}^3)}^2.
\]

Proof. Multiplying \((1.1)_1\) by \(u\) and \((1.1)_3\) by \((\Delta d + |\nabla d|^2 d)\), adding and integrating the resulting equations over \(\mathbb{R}^3_+\), applying \((1.1)_2\), the fact that \(|d| = 1\), and integration by parts, we can obtain \((3.1)\).

Lemma 3.2. There exists \(C > 0\), independent of \(T\), such that for any \(0 < t < T\), it holds

\[
\frac{d}{dt} \int_{\mathbb{R}^3_+} (|u(t)|^3 + |\nabla d(t)|^3) dx + \left( 1 - C \|u(t)\|_{L^3(\mathbb{R}^3)}^2 \right) \int_{\mathbb{R}^3_+} |u(t)| \|\nabla u(t)\|^2 dx \\
+ \left( 1 - C (\|u(t)\|_{L^3(\mathbb{R}^3)} + \|u(t)\|_{L^3(\mathbb{R}^3)} \|\nabla d(t)\|_{L^3(\mathbb{R}^3)} + \|\nabla d(t)\|_{L^3(\mathbb{R}^3)}^3) \right) \cdot \int_{\mathbb{R}^3_+} |\nabla d(t)| \|\nabla^2 d(t)\|^2 dx \leq 0.
\]

In particular, there exists \(\varepsilon_0 > 0\) such that if

\[
\|u_0\|_{L^3(\mathbb{R}^3)}^3 + \|\nabla d_0\|_{L^3(\mathbb{R}^3)}^3 \leq \varepsilon_0^3,
\]

then for any \(0 < t < T\),

\[
\|u(t)\|_{L^3(\mathbb{R}^3)}^3 + \|\nabla d(t)\|_{L^3(\mathbb{R}^3)}^3 \leq \|u_0\|_{L^3(\mathbb{R}^3)}^3 + \|\nabla d_0\|_{L^3(\mathbb{R}^3)}^3.
\]
Proof. We closely follow the proof of \cite{14} Lemma 2.1. In contrast with \cite{14} Lemma 2.1, we need to verify that boundary contributions are zero in the process of integration by parts. Take derivative of \eqref{3.1}, multiply the resulting equation by $|\nabla d|\nabla d$, and integrate over $\mathbb{R}^3_+$, we can check that, for example, there is no boundary contribution from the following term.

$$
\int_{\mathbb{R}^3_+} \nabla \Delta d : |\nabla d|\nabla d dx
$$

\begin{equation}
(3.3)
= - \int_{\partial \mathbb{R}^3_+} \left( \frac{\partial^2 d}{\partial x_j \partial x_3}, |\nabla d| \frac{\partial d}{\partial x_j} \right) d\sigma - \int_{\mathbb{R}^3_+ \cap \{ |\nabla d| > 0 \}} \nabla^2 d : \nabla (|\nabla d|\nabla d) dx.
\end{equation}

Since $\frac{\partial d}{\partial x_3} = 0$ on $\partial \mathbb{R}^3_+$, we have that $\frac{\partial^2 d}{\partial x_j \partial x_3} = 0$, for $j = 1, 2$, on $\partial \mathbb{R}^3_+$ and hence

$$
J = \sum_{j=1}^2 \int_{\partial \mathbb{R}^3_+} \left( \frac{\partial^2 d}{\partial x_j \partial x_3}, |\nabla d| \frac{\partial d}{\partial x_j} \right) d\sigma = 0.
$$

This yields that

$$
\int_{\mathbb{R}^3_+} \nabla \Delta d : |\nabla d|\nabla d dx = - \int_{\mathbb{R}^3_+ \cap \{ |\nabla d| > 0 \}} \left( |\nabla d| ||\nabla^2 d||^2 + \frac{|\nabla^2 d \cdot \nabla d|^2}{|\nabla d|} \right) dx
$$

\begin{equation}
(3.4)
\leq - \int_{\mathbb{R}^3_+ \cap \{ |\nabla d| > 0 \}} |\nabla d| ||\nabla^2 d||^2 dx.
\end{equation}

The remaining parts of proof follow \cite{14} Lemma 2.1 line by line, which is omitted. \qed

Lemma 3.3. There exists $C > 0$, independent of $T$, such that for any $t \in [0, T]$, it holds that

$$
\sigma(t) \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^3_+)}^2 + \|d_t(t)\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla^2 d(t)\|_{L^2(\mathbb{R}^3_+)}^2 \right) +
$$

\begin{equation}
(3.5)
\int_0^t \sigma(s) \left( \|u(s)\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla u(s)\|_{H^1(\mathbb{R}^3_+)}^2 + \|d_t(s)\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla^2 d(s)\|_{H^1(\mathbb{R}^3_+)}^2 \right) ds \leq C,
\end{equation}

where $\sigma(t) = \min\{1, t\}$.

Proof. Multiplying \eqref{1.1} by $u_t$ and integrating over $\mathbb{R}^3_+$, and applying the interpolation inequality and the Sobolev inequality, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3_+} \|\nabla u\|^2 dx + \int_{\mathbb{R}^3_+} |u_t|^2 \leq \int_{\mathbb{R}^3_+} \left( |u| \|\nabla u\| |u_t| + |\nabla d| \|\nabla^2 d\| |u_t| \right) dx
$$

\begin{equation}
(3.6)
\leq C \|\nabla u\|_{L^3(\mathbb{R}^3_+)} \|\nabla u\|_{L^6(\mathbb{R}^3_+)} \|u_t\|_{L^2(\mathbb{R}^3_+)} + C \|\nabla d\|_{L^1(\mathbb{R}^3_+)} \|\nabla^2 d\|_{L^6(\mathbb{R}^3_+)} \|u_t\|_{L^2(\mathbb{R}^3_+)}
\end{equation}

By the standard estimates of the Stokes equation on $\mathbb{R}^3_+$ (see \cite{9}), we obtain

$$
\|\nabla^2 u\|_{L^2(\mathbb{R}^3_+)} \leq C \left( \|u \cdot \nabla u\|_{L^2(\mathbb{R}^3_+)} + \|\nabla \cdot (\nabla d \odot \nabla d)\|_{L^2(\mathbb{R}^3_+)} + \|u_t\|_{L^2(\mathbb{R}^3_+)} \right)
$$

\begin{equation}
(3.7)
\leq C \left( \|u\|_{L^3(\mathbb{R}^3_+)} \|\nabla u\|_{H^1(\mathbb{R}^3_+)} + \|\nabla d\|_{L^3(\mathbb{R}^3_+)} \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)} + \|u_t\|_{L^2(\mathbb{R}^3_+)} \right).
\end{equation}

Combining the above two inequalities, we get

$$
\frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^3_+)} + \left( \|u_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla u\|_{H^1(\mathbb{R}^3_+)}^2 \right)
$$
\[
\leq C\left(\|u\|_{L^3(\mathbb{R}^3_+)} + \|u\|_{L^3(\mathbb{R}^3_+)}^2 + \|\nabla d\|_{L^3(\mathbb{R}^3_+)} + \|\nabla d\|_{L^3(\mathbb{R}^3_+)}^2\right)
\cdot (\|\nabla u\|_{L^3(\mathbb{R}^3_+)}^2 + \|u_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}^2) + C\|\nabla u\|_{L^2(\mathbb{R}^3_+)}^2.
\]

(3.8)

Next, taking \(\partial_t\) of (1.13), we have

\[
d_{tt} u_t + u_t \cdot \nabla d + u \cdot \nabla d_t = \Delta d_t + 2 \langle \nabla d, \nabla d_t \rangle d + |\nabla d|^2 d_t.
\]

(3.9)

Multiplying (3.9) by \(d_t\), integrating over \(\mathbb{R}^3_+\), applying \(\frac{\partial d}{\partial x_3} = 0\) on \(\partial \mathbb{R}^3_+\), (1.12), the fact \(|d| = 1\), and the interpolation inequality and the Sobolev inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3_+} |d_t|^2 dx + \int_{\mathbb{R}^3_+} |\nabla d_t|^2 dx
\leq C \int_{\mathbb{R}^3_+} (|u_t| \|\nabla d\|_{L^2(\mathbb{R}^3_+)} + |\nabla d|^2 |d_t|) dx
\leq C \|u_t\|_{L^2(\mathbb{R}^3_+)} \|\nabla d\|_{L^6(\mathbb{R}^3_+)} \|d_t\|_{L^6(\mathbb{R}^3_+)} + \|\nabla d\|_{L^2(\mathbb{R}^3_+)} \|d_t\|_{L^2(\mathbb{R}^3_+)}
\leq C \|u_t\|_{L^2(\mathbb{R}^3_+)} \|\nabla d\|_{L^3(\mathbb{R}^3_+)} \|d_t\|_{H^1(\mathbb{R}^3_+)} + \|\nabla d\|_{L^2(\mathbb{R}^3_+)} \|d_t\|_{H^1(\mathbb{R}^3_+)}
\]

(3.10)

By the elliptic estimate, we have

\[
\|\nabla^3 d\|_{L^2(\mathbb{R}^3_+)}
\leq C \left(\|\nabla d_t\|_{L^2(\mathbb{R}^3_+)} + \|\nabla (u \cdot \nabla d)\|_{L^2(\mathbb{R}^3_+)} + \|\nabla (|\nabla d|^2 d)\|_{L^2(\mathbb{R}^3_+)}\right)
\leq C \left(\|\nabla d_t\|_{L^2(\mathbb{R}^3_+)} + \|\nabla u\|_{L^6(\mathbb{R}^3_+)} \|\nabla d\|_{L^3(\mathbb{R}^3_+)}
+ \|u_t\|_{L^2(\mathbb{R}^3_+)} \|\nabla d\|_{L^3(\mathbb{R}^3_+)} \|\nabla^2 d\|_{L^6(\mathbb{R}^3_+)}\right)
\]

(3.11)

\[
\leq C \left[\|\nabla d_t\|_{L^2(\mathbb{R}^3_+)} + \|u_t\|_{L^2(\mathbb{R}^3_+)} \|\nabla d\|_{L^3(\mathbb{R}^3_+)} \|\nabla d\|_{H^1(\mathbb{R}^3_+)} \right] \left(\|\nabla u\|_{H^1(\mathbb{R}^3_+)} + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}\right),
\]

where we have used the fact that \(|\nabla d|^2 = -d \cdot \Delta d \leq |\Delta d|\) in the second inequality. Combining these two inequalities and using the Cauchy inequality, we obtain

\[
\frac{d}{dt} \|d_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|d_t\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}^2
\leq C \left(\|u\|_{L^3(\mathbb{R}^3_+)}^2 + \|\nabla d\|_{L^3(\mathbb{R}^3_+)} + \|\nabla d\|_{L^3(\mathbb{R}^3_+)}^2\right)
\cdot (\|u_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|u_t\|_{H^1(\mathbb{R}^3_+)}^2 + \|d_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|d_t\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}^2)
\]

(3.12)

Combining (3.8) and (3.12), we have

\[
\frac{d}{dt} \left(\|u\|_{L^3(\mathbb{R}^3_+)}^2 + \|d_t\|_{L^2(\mathbb{R}^3_+)}^2\right)
+ (1 - C \mathcal{G}(t)) \left(\|u\|_{L^3(\mathbb{R}^3_+)}^2 + \|\nabla u\|_{H^1(\mathbb{R}^3_+)}^2 + \|d_t\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}^2\right)
\leq C \left(\|u\|_{L^3(\mathbb{R}^3_+)}^2 + \|d_t\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{L^2(\mathbb{R}^3_+)}^2\right).
\]

(3.13)

where

\[
\mathcal{G}(t) = \left(\|u(t)\|_{L^3(\mathbb{R}^3_+)}^2 + \|u(t)\|_{L^3(\mathbb{R}^3_+)}^2 + \|\nabla d(t)\|_{L^3(\mathbb{R}^3_+)} + \|\nabla d(t)\|_{L^3(\mathbb{R}^3_+)}^2\right).
\]
On the other hand, let \( \sigma(t) = \min\{1, t\} \). Then we have
\[
\frac{d}{dt} \left[ \sigma(t) \int_{\mathbb{R}^3_+} (|\nabla u|^2 + |d_t|^2) \, dx \right]
\leq \int_{\mathbb{R}^3_+} (|\nabla u|^2 + |d_t|^2) \, dx + \sigma(t) \frac{d}{dt} \int_{\mathbb{R}^3_+} (|\nabla u|^2 + |d_t|^2) \, dx.
\]
(3.14)

Multiplying (3.13) by \( \sigma(t) \) and substituting it into (3.14), we have
\[
\frac{d}{dt} \left[ \sigma(t) (|\nabla u|_{L^2(\mathbb{R}^3_+)}^2 + |d_t|_{L^2(\mathbb{R}^3_+)}^2) \right] + (1 - C\mathcal{G}(t)) \sigma(t) (|u_t|_{L^2(\mathbb{R}^3_+)}^2 + |\nabla u|_{H^1(\mathbb{R}^3_+)}^2 + |d_t|_{H^1(\mathbb{R}^3_+)}^2 + |\nabla^2 d|_{H^1(\mathbb{R}^3_+)}^2)
\leq C (|\nabla u|_{L^2(\mathbb{R}^3_+)}^2 + |d_t|_{L^2(\mathbb{R}^3_+)}^2 + |\nabla^2 d|_{L^2(\mathbb{R}^3_+)}^2).
\]
(3.15)

Observe that
\[
\int_0^t |d_t(s)|_{L^2(\mathbb{R}^3_+)}^2 \, ds \leq C \int_0^t (\|d_t + u \cdot \nabla d\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)}^2 \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)}^2) \, ds
\leq C.
\]
(3.16)

This, together with \( \int_0^t |\nabla u(s)|_{L^2(\mathbb{R}^3_+)}^2 \, ds \leq C \), implies that there exists \( t_k \to 0 \) such that
\[
t_k (|\nabla u(t_k)|_{L^2(\mathbb{R}^3_+)}^2 + |d_t(t_k)|_{L^2(\mathbb{R}^3_+)}^2) \to 0.
\]
(3.17)

By the elliptic estimates, we have
\[
|\nabla^2 d|_{L^2(\mathbb{R}^3_+)} \leq C |d_t + u \cdot \nabla d|_{L^2(\mathbb{R}^3_+)} + C |\nabla d|_{L^4(\mathbb{R}^3_+)}^2
\leq C |d_t + u \cdot \nabla d|_{L^2(\mathbb{R}^3_+)} + C |\nabla d|_{L^3(\mathbb{R}^3_+)} \|\nabla d\|_{L^6(\mathbb{R}^3_+)}
\leq C |d_t + u \cdot \nabla d|_{L^2(\mathbb{R}^3_+)} + C \varepsilon_0 |\nabla^2 d|_{L^2(\mathbb{R}^3_+)}
\]
(3.18)

where we have used the Sobolev inequality\(^1\)
\[
|\nabla d|_{L^6(\mathbb{R}^3_+)} \leq C |\nabla^2 d|_{L^2(\mathbb{R}^3_+)}.
\]

Thus if we choose a sufficiently small \( \varepsilon_0 > 0 \), then we obtain that
\[
\int_0^t |\nabla^2 d(s)|_{L^2(\mathbb{R}^3_+)}^2 \, ds \leq C.
\]
(3.19)

By integrating (3.15) over \([t_k, t]\) and sending \( k \) to \( \infty \), we finally obtain
\[
\sigma(t) (|\nabla u(t)|_{L^2(\mathbb{R}^3_+)}^2 + |d_t|_{L^2(\mathbb{R}^3_+)}^2) +
\int_0^t \sigma(s) (|u_t(s)|_{L^2(\mathbb{R}^3_+)}^2 + |\nabla u(s)|_{H^1(\mathbb{R}^3_+)}^2 + |d_t(s)|_{H^1(\mathbb{R}^3_+)}^2 + |\nabla^2 d(s)|_{H^1(\mathbb{R}^3_+)}^2) \, ds \leq C.
\]
(3.20)

By the elliptic estimate (3.11), we have
\[
|\nabla^2 d|_{L^2(\mathbb{R}^3_+)} \leq C |d_t|_{L^2(\mathbb{R}^3_+)} + C \varepsilon_0 |\nabla^2 d|_{L^2(\mathbb{R}^3_+)}.
\]

\(^1\)In fact, since \( \frac{\partial d}{\partial x_3} = 0 \) on \( \partial \mathbb{R}^3_+ \), this follows from an even extension of d from \( \mathbb{R}^3_+ \) to \( \mathbb{R}^3 \). See also [2]
By choosing a sufficiently small $\varepsilon_0 > 0$, this yields
\begin{equation}
\|\nabla^2 d\|_{L^2(\mathbb{R}^3_+)} \leq C \|d_t\|_{L^2(\mathbb{R}^3_+)}.
\end{equation}

Hence we obtain
\begin{equation}
\sigma(t)\left(\|\nabla u(t)\|^2_{L^2(\mathbb{R}^3_+)} + \|d_t(t)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^2 d(t)\|^2_{L^2(\mathbb{R}^3_+)}\right) + \int_0^t \sigma(s)\left(\|u_t(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla u(s)\|^2_{H^1(\mathbb{R}^3_+)} + \|d_t(s)\|^2_{H^1(\mathbb{R}^3_+)} + \|\nabla^2 d(s)\|^2_{H^1(\mathbb{R}^3_+)}\right) ds \leq C.
\end{equation}

This completes the proof. \hfill \Box

It is clear that \[(3.22)\] implies that for any small $\tau > 0$, there exists a positive constant $C_\tau$ such that for $\tau < t < T$,
\begin{equation}
\left(\|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^2 d(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|d_t(\tau)\|^2_{L^2(\mathbb{R}^3_+)}\right) + \int_\tau^t \left(\|u_t(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla u(s)\|^2_{H^1(\mathbb{R}^3_+)} + \|d_t(s)\|^2_{H^1(\mathbb{R}^3_+)} + \|\nabla^2 d(s)\|^2_{H^1(\mathbb{R}^3_+)}\right) ds \leq C_\tau.
\end{equation}

Theorem 1.1 follows from higher order elliptic estimates. More precisely, we have the following Lemma.

**Lemma 3.4.** For any $0 < \tau < t < T$, there exists a positive constant $C_\tau$ such that
\begin{equation}
\left(\|\nabla^2 u(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_t(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^3 d(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|d_t(\tau)\|^2_{L^2(\mathbb{R}^3_+)}\right) + \int_\tau^t \left(\|\nabla u_t(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^2 u(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^2 d_t(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla^4 d(s)\|^2_{L^2(\mathbb{R}^3_+)}\right) ds \leq C_\tau.
\end{equation}

**Proof.** First, taking $\partial_t$ of (1.1), multiplying the resulting equations by $u_t$, and integrating over $\mathbb{R}^3_+$, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla u_t\|^2_{L^2(\mathbb{R}^3_+)} \leq \int_{\mathbb{R}^3_+} |u_t| |u| |\nabla u_t| + |\nabla d_t| |\nabla d| |\nabla u_t| \, dx,
\end{equation}
where we have used
\[
\int_{\mathbb{R}^3_+} (u_t \cdot \nabla) u \cdot u_t \, dx = - \int_{\mathbb{R}^3_+} (u_t \cdot \nabla) u_t \cdot u.
\]
Using the Hölder inequality, the Cauchy inequality and the Sobolev inequality, we get
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla u_t\|^2_{L^2(\mathbb{R}^3_+)} \leq C \|u\|^2_{L^5(\mathbb{R}^3_+)} \|u_t\|^2_{L^5(\mathbb{R}^3_+)} + C \|\nabla d\|^2_{L^3(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^3(\mathbb{R}^3_+)}
\end{equation}
\begin{equation}
\leq C \|u\|^2_{L^5(\mathbb{R}^3_+)} \|\nabla u_t\|^2_{L^2(\mathbb{R}^3_+)} + C \|\nabla d\|^2_{L^3(\mathbb{R}^3_+)} \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)}
\end{equation}
where we have used the fact that
\begin{equation}
\|\nabla^2 d_t\|_{L^2(\mathbb{R}^3_+)} = \|\Delta d_t\|_{L^2(\mathbb{R}^3_+)}.
\end{equation}

In fact, by integration by parts and (1.2), one has
\[
\int_{\mathbb{R}^3_+} |\Delta d_t|^2 \, dx = - \int_{\mathbb{R}^3_+} \langle \partial_j \partial_i d_t, \partial_j d_t \rangle \, dx + \int_{\partial \mathbb{R}^3_+} \langle \partial_i \partial_i d_t, \partial_3 d_t \rangle \, d\sigma
\]
\[
= - \int_{\mathbb{R}^3_+} \langle \partial_j \partial_i d_t, \partial_j d_t \rangle \, dx.
\]
\[ \int_{\mathbb{R}^3_+} |\nabla^2 d_t|^2 \, dx = \int_{\partial \mathbb{R}^3_+} \left( \sum_{j=1}^2 \langle \partial_j \partial_3 d_t, \partial_j d_t \rangle + \langle \partial_3^2 d_t, \partial_3 d_t \rangle \right) d\sigma \]
\[ = \int_{\mathbb{R}^3_+} |\nabla^2 d_t|^2 \, dx. \]

Next, taking \( \partial_t \) of (3.13) and multiplying the resulting equations by \( \Delta d_t \), integrating over \( \mathbb{R}^3_+ \), using integration by parts, (3.12) and the Cauchy inequality, we have

\[ \frac{1}{2} \frac{d}{dt} \| \nabla d_t \|^2_{L^2(\mathbb{R}^3_+)} + \| \Delta d_t \|^2_{L^2(\mathbb{R}^3_+)} \]
\[ \leq C \int_{\mathbb{R}^3_+} (|u_t|^2 |\nabla d_t|^2 + |u_t|^2 |\nabla d_t|^2 + |\nabla d_t|^2 |\nabla d_t|^2) \, dx - \int_{\mathbb{R}^3_+} |\nabla d_t|^2 (d_t \cdot \Delta d_t) \, dx \]
(3.28) \[ = I_1 + I_2. \]

By Hölder’s inequality, we can estimate \( I_1 \) by

\[ |I_1| \leq C \|u_t\|^2_{L^6(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)} + C \|u_t\|^2_{L^6(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)} + C \|\nabla d_t\|^2_{L^6(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^6(\mathbb{R}^3_+)} \]
(3.29) \[ \leq C (\|u_t\|^2_{L^{3,\infty}(\mathbb{R}^3_+)} + \|\nabla d_t\|^2_{L^{3,\infty}(\mathbb{R}^3_+)})(\|\nabla u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)}). \]

While \( I_2 \) can be estimated by

\[ |I_2| \leq \int_{\mathbb{R}^3_+} |\nabla d_t|^2 |\nabla d_t|^2 \, dx + 2 \int_{\mathbb{R}^3_+} (\nabla d_t, \nabla^2 d_t) \, dx \]
\[ \leq C \|\nabla d_t\|^2_{L^6} \|\nabla d_t\|_{L^2} + C \|\nabla d_t\|_{L^6} \|\nabla^2 d_t\|_{L^2} \|\Delta d_t\|_{L^2} \|\nabla d_t\|_{L^6} \]
(3.30) \[ \leq \left( C \|\nabla d_t\|^2_{L^6} + \frac{1}{2} \right) \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)} + C \|\nabla d_t\|^4_{H^1(\mathbb{R}^3_+)} \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)}. \]

Putting these two estimates together, we obtain

\[ \frac{d}{dt} (\|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)} + (1 - CG(t)) (\|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)})) \]
(3.31) \[ \leq C \|\nabla d_t\|^4_{H^1(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)}. \]

Multiplying (3.31) by \( \sigma^3(t) \) and choosing a sufficiently small \( \varepsilon_0 \), we obtain

\[ \frac{d}{dt} \left[ \sigma^3(t) (\|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)}) \right] + \sigma^3(t) (\|\nabla u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\Delta d_t\|^2_{L^2(\mathbb{R}^3_+)})) \]
(3.32) \[ \leq 3\sigma^2(t) \sigma'(t) (\|u_t\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)} + C \sigma^3(t) \|\nabla d_t\|^4_{H^1(\mathbb{R}^3_+)} \|\nabla d_t\|^2_{L^2(\mathbb{R}^3_+)}). \]

Integrating the above inequality in \([0, t]\) and using (3.1) and (3.22), we get

\[ (\|u_t(\tau)\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d_t(\tau)\|^2_{L^2(\mathbb{R}^3_+)})) + \int^t_\tau (\|\nabla u_t(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\Delta d_t(s)\|^2_{L^2(\mathbb{R}^3_+)}) \, ds \]
(3.33) \[ \leq C_\tau. \]

This, combined with the standard elliptic estimates, implies (3.24). □
3.2. **Uniqueness.** To show the uniqueness of global strong solutions obtained in Theorem 1.1, let \((u_i, d_i, p_i)\) be two strong solutions to (1.1)–(1.2) as in Theorem 1.1. Set

\[ \tilde{u} = u_1 - u_2, \quad \tilde{p} = p_1 - p_2, \quad \tilde{d} = d_1 - d_2. \]

Then we have

\[
\begin{cases}
\tilde{u}_t + \tilde{u} \cdot \nabla u_1 + u_2 \cdot \nabla \tilde{u} + \nabla \tilde{p} = \Delta \tilde{u} - \nabla \cdot (\nabla \tilde{d} \odot \nabla d_1) - \nabla \cdot (\nabla d_2 \odot \nabla \tilde{d}), \\
\nabla \cdot \tilde{u} = 0, \\
\tilde{d}_t + \tilde{u} \cdot \nabla d_1 + u_2 \cdot \nabla \tilde{d} = \Delta \tilde{d} + (\nabla \tilde{d}, (\nabla d_1 + \nabla d_2))d_1 + |\nabla d_2|^2 \tilde{d},
\end{cases}
\tag{3.34}
\]

along with the initial and boundary conditions

\[
\begin{cases}
\tilde{u} = \frac{\partial \tilde{u}}{\partial x_3} = 0, \text{ on } \partial \mathbb{R}_+^3 \times (0, \infty), \\
(\tilde{u}, \tilde{d}) \to 0, \text{ as } |x| \to \infty, \\
(\tilde{u}, \tilde{d}) \big|_{t=0} = 0, \text{ in } \mathbb{R}_+^3.
\end{cases}
\tag{3.35}
\]

Multiplying (3.34) by \(\tilde{u}\) and (3.35) by \(\Delta \tilde{d}\), and applying integration by parts, (3.34), (3.35), the Hölder inequality, the Sobolev inequality and the Cauchy inequality, we obtain

\[
\frac{d}{dt} \left( ||\tilde{u}||^2_{L^2(\mathbb{R}_+^3)} + ||\nabla \tilde{d}||^2_{L^2(\mathbb{R}_+^3)} \right) + (1 - \tilde{G}(t)) (||\nabla \tilde{u}||^2_{L^2(\mathbb{R}_+^3)} + ||\nabla^2 \tilde{d}||^2_{L^2(\mathbb{R}_+^3)})
\leq - \int_{\mathbb{R}_+^3} \nabla d_2 \cdot \Delta \tilde{d} dx
\leq \int_{\mathbb{R}_+^3} |\nabla d_2|^2 |\nabla \tilde{d}|^2 dx + 2 \int_{\mathbb{R}_+^3} |\nabla d_2| ||\nabla^2 \tilde{d}|| |\nabla \tilde{d}| dx
\leq ||\nabla d_2||^2_{L^4(\mathbb{R}_+^3)} ||\nabla \tilde{d}||^2_{L^6(\mathbb{R}_+^3)} + 2 \int_{\mathbb{R}_+^3} |\nabla d_2| ||\nabla^2 \tilde{d}|| |\nabla \tilde{d}| dx
\leq ||\nabla d_2||^2_{L^4(\mathbb{R}_+^3)} ||\nabla \tilde{d}||^2_{L^6(\mathbb{R}_+^3)} + \frac{1}{2} ||\nabla^2 d_2||^2_{L^2(\mathbb{R}_+^3)} ||\nabla \tilde{d}||^2_{L^6(\mathbb{R}_+^3)} + C ||\nabla d_2||^2 ||\nabla^2 \tilde{d}||^2_{L^2}
\leq C ||\nabla d_2||^2 ||\nabla^2 \tilde{d}||^2_{L^2(\mathbb{R}_+^3)},
\tag{3.36}
\]

where

\[
\tilde{G}(t) = \sum_{i=1}^2 \left( ||u_i||^2_{L^3(\mathbb{R}_+^3)} + ||\nabla d_i||^2_{L^3(\mathbb{R}_+^3)} + ||\nabla d_i||^2_{L^3(\mathbb{R}_+^3)} \right).
\]

By choosing a sufficiently small \(\varepsilon_0\) and integrating over \([0, t]\), and applying Lemma 3.2, we can conclude that

\[
||\tilde{u}||^2_{L^2(\mathbb{R}_+^3)}(t) + ||\nabla \tilde{d}||^2_{L^2(\mathbb{R}_+^3)}(t) \leq ||\tilde{u}_0||^2_{L^2(\mathbb{R}_+^3)} + ||\nabla \tilde{d}_0||^2_{L^2(\mathbb{R}_+^3)} = 0
\]

for any \(t > 0\). This implies that \((u_1, d_1) \equiv (u_2, d_2)\) and completes the proof of uniqueness.
3.3. Time-decay estimates. In this subsection, we will apply the continuity argument to derive the time decay rates stated as in Theorem 1.1.

Lemma 3.5. There exists $\tilde{C} > 0$ such that if

$$\|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)} \leq 2\tilde{C}(1 + t)^{-1},$$

for any $t \in [0, T]$, then

$$\|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)} \leq \frac{3}{2}\tilde{C}(1 + t)^{-1},$$

for any $t \in [0, T]$. Moreover, there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(1 + t)^{-\frac{4}{5}}$$

for any $t \in [0, T]$.

Proof. Without loss of generality, we assume that $t \geq 1$. From (3.3), we have

$$\frac{d}{dt}(\|u\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d\|^2_{L^2(\mathbb{R}^3_+)}) + \|\nabla u\|^2_{L^2(\mathbb{R}^3_+)} \leq 0.$$

This, combined with (2.7), implies that

$$\frac{d}{dt}(\|u\|^2_{L^2(\mathbb{R}^3_+)} + \|\nabla d\|^2_{L^2(\mathbb{R}^3_+)}) + \rho\|u\|^2_{L^2(\mathbb{R}^3_+)} + \rho\|\nabla d\|^2_{L^2(\mathbb{R}^3_+)} \leq C\rho\|E_\rho u\|^2_{L^2(\mathbb{R}^3_+)} + \rho\|\nabla d\|^2_{L^2(\mathbb{R}^3_+)}.$$

It follows from (1.9) that

$$E_\rho u(t) = E_\rho e^{-\frac{4}{5}E_\rho u(t)} - E_\rho \int_0^t e^{-\frac{4}{5}(t-s)}\mathbb{P}(u \cdot \nabla u)(s)ds$$

$$= -E_\rho \int_0^t e^{-(t-s)}\mathbb{P} \cdot \nabla \cdot (\nabla d \circ \nabla d)(s)ds$$

$$= I_1 - I_2 - I_3.$$

By (2.3) and calculations similar to (2) (page 150), we have

$$I_2 = \int_0^t \left[ \int_0^\rho e^{-\lambda(t-s)}d(E_\lambda(\mathbb{P}(u \cdot \nabla u))(s)) \right]ds$$

$$= \int_0^t e^{-\rho(t-s)}E_\rho(\mathbb{P}(u \cdot \nabla u))(s)ds$$

$$+ \int_0^t (t-s)\left[ \int_0^\rho e^{-\lambda(t-s)}E_\lambda(\mathbb{P}(u \cdot \nabla u))(s)d\lambda \right]ds.$$
By the Minkowski inequality and the Hölder inequality, Lemma [2.1] for \( p = 2 \) and \( q = r \in (1, 2) \), the boundedness of
\[
P : L^r(\mathbb{R}^3_+, \mathbb{R}^3) \rightarrow L^r_\sigma(\mathbb{R}^3_+, \mathbb{R}^3), \quad \forall \ r \in (1, \infty),
\]
the fact
\[
\| E_0 f \|_{L^2(\mathbb{R}^3_+)} \leq \| f \|_{L^2(\mathbb{R}^3_+)} , \quad f \in L^2_\sigma(\mathbb{R}^3_+, \mathbb{R}^3),
\]
and (3.20), we can estimate \( I_3 \) as follows.
\[
\| - I_3 \|_{L^2(\mathbb{R}^3_+)} = \| E_0 \int_0^t e^{-(t-s)A} \mathbb{P} \cdot (\nabla d \circ \nabla d)(s) \, ds \|_{L^2(\mathbb{R}^3_+)}
\leq \| \int_0^t e^{-(t-s)A} \mathbb{P} \cdot (\nabla d \circ \nabla d)(s) \, ds \|_{L^2(\mathbb{R}^3_+)}
\leq \int_0^t \| e^{-(t-s)A} \mathbb{P} \cdot (\nabla d \circ \nabla d)(s) \|_{L^2(\mathbb{R}^3_+)} \, ds
\leq C \int_0^t (t-s)^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \| \mathbb{P} \cdot (\nabla d \circ \nabla d)(s) \|_{L^2(\mathbb{R}^3_+)} \, ds
\leq C \int_0^t (t-s)^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \| \nabla d(s) \|_{L^{\frac{2r}{r-2}(\mathbb{R}^3_+)}} \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)} \, ds
\leq C \int_0^t (t-s)^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{2} \right)} \| \nabla d(s) \|_{L^2(\mathbb{R}^3_+)} \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)}^{1 + \frac{2r-3}{2r}} \, ds.
\]
This, combined with the Hölder inequality, (3.37), and Lemma [2.2] yields
\[
\| I_3 \|_{L^2(\mathbb{R}^3_+)}
\leq C \left( \int_0^t (t-s)^{-2 \left( \frac{1}{r} - \frac{1}{2} \right)} \| \nabla d(s) \|_{L^2(\mathbb{R}^3_+)}^{\frac{4(3-2r)}{2r}} \, ds \right)^{\frac{1}{2}} \left( \int_0^t \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)}^{\frac{4(1+\alpha-2r)}{2r}} \, ds \right)^{\frac{1}{2}}
\leq C \left( \int_0^t (t-s)^{-2 \left( \frac{1}{r} - \frac{1}{2} \right)} \left( \sup_{s \in \left[ \frac{t}{2}, t \right]} \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)} \right)^{\frac{7}{2} - \frac{3}{r}} \left( \int_0^t \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)}^{\frac{2}{2}} \, ds \right)^{\frac{1}{2}}
\leq C \left( \int_0^t (t-s)^{-2 \left( \frac{1}{r} - \frac{1}{2} \right)} \left( \sup_{s \in \left[ \frac{t}{2}, t \right]} \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)} \right)^{\frac{7}{2} - \frac{3}{r}} \left( \int_0^t \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)}^{\frac{2}{2}} \, ds \right)^{\frac{1}{2}}
\leq C \left( \int_0^t (t-s)^{-2 \left( \frac{1}{r} - \frac{1}{2} \right)} \left( \sup_{s \in \left[ \frac{t}{2}, t \right]} \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)} \right)^{\frac{7}{2} - \frac{3}{r}} \left( \int_0^t \| \nabla^2 d(s) \|_{L^2(\mathbb{R}^3_+)}^{\frac{2}{2}} \, ds \right)^{\frac{1}{2}}
\leq C (1 + t)^{-\frac{3}{2} \left( 1 - 2 \left( \frac{1}{r} - \frac{1}{2} \right) - \frac{4(3-2r)}{2r} \right)} \leq C (1 + t)^{-\frac{3}{2}},
\]
for some \( r \in (1, \frac{18}{17}) \).

To estimate \( I_1 \), we need to estimate the upper bound of \( \| u(t) \|_{L^a(\mathbb{R}^3_+)} \) for \( a \in (1, \frac{3}{2}) \). From [1.31], Lemma [2.1] and Lemma [3.1], we have that
\[
\| u(t) \|_{L^a(\mathbb{R}^3_+)} \leq C t^{-\frac{3}{2} \left( 1 - \frac{1}{2} \right)} \| u_0 \|_{L^1(\mathbb{R}^3_+)} + \| \int_0^t e^{-(t-s)A} \mathbb{P} \left( u \cdot \nabla u - \nabla \cdot (\nabla d \circ \nabla d) \right) \, ds \|_{L^a(\mathbb{R}^3_+)}
\leq C t^{-\frac{3}{2} \left( 1 - \frac{1}{2} \right)} \| u_0 \|_{L^1(\mathbb{R}^3_+)} + \int_0^t (t-s)^{-\frac{3}{2} \left( 1 - \frac{1}{2} \right)} \, ds \leq C t^{-\frac{3}{2} \left( 1 - \frac{1}{2} \right)},
\]
(3.48)
where we have used the following estimate: for any \( a \in (1, \frac{3}{2}) \) and \( t > 0 \),
\[
\left\| \int_0^t e^{-(t-s)A} \mathbb{P} (u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d)) \, ds \right\|_{L^a(\mathbb{R}^3_+)} \\
= \sup_{\varphi \in C_0^\infty(\mathbb{R}^3_+, \mathbb{R}^3), \| \varphi \|_{L^\infty(\mathbb{R}^3_+)} \leq 1} \left| \langle \int_0^t e^{-(t-s)A} \mathbb{P} (u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d)) \, ds, \varphi \rangle \right| \\
= \sup_{\varphi \in C_0^\infty(\mathbb{R}^3_+, \mathbb{R}^3), \| \varphi \|_{L^\infty(\mathbb{R}^3_+)} \leq 1} \left| \langle \int_0^t e^{-(t-s)A} \nabla \cdot (u \otimes u - \nabla d \odot \nabla d) \, ds, \varphi \rangle \right| \\
= \sup_{\varphi \in C_0^\infty(\mathbb{R}^3_+, \mathbb{R}^3), \| \varphi \|_{L^\infty(\mathbb{R}^3_+)} \leq 1} \left| \langle \int_0^t (u \otimes u - \nabla d \odot \nabla d) \, ds, \nabla e^{-(t-s)A} \varphi \rangle \right| \\
\leq \sup_{\varphi \in C_0^\infty(\mathbb{R}^3_+, \mathbb{R}^3), \| \varphi \|_{L^\infty(\mathbb{R}^3_+)} \leq 1} \int_0^t \| \nabla e^{-(t-s)A} \varphi \|_{L^\infty(\mathbb{R}^3_+)} \| u \otimes u - \nabla d \odot \nabla d \|_{L^1(\mathbb{R}^3_+)} \, ds.
\]

(3.49) \( \leq \int_0^t (t - s)^{-\frac{1}{2} - \frac{3}{2}(1 - \frac{1}{a})} (\| u(s) \|^2_{L^2(\mathbb{R}^3_+)} + \| \nabla d(s) \|^2_{L^2(\mathbb{R}^3_+)}) \, ds. \)

Now we can estimate \( \| I_1 \|_{L^2(\mathbb{R}^3_+)} \) by
\[
\| I_1 \|_{L^2(\mathbb{R}^3_+)} \leq C \| e^{-\frac{t}{2}A} u(t) \|^2_{L^2(\mathbb{R}^3_+)} \\
\leq C t^{-\frac{3}{2}(\frac{1}{a} - \frac{1}{2})} \| u(t) \|^2_{L^a(\mathbb{R}^3_+)} \\
\leq C t^{-\frac{3}{2}(\frac{1}{a} - \frac{1}{2})} t^{1 - \frac{3}{2}(1 - \frac{1}{a})} \\
\leq C (1 + t)^{-\frac{1}{2}}
\]

(3.50)

Putting (3.44), (3.45), (3.47), (3.50), together with (3.37) yields that
\[
\frac{dg}{dt} + \rho g \leq C \rho \left[ (1 + t)^{-\frac{1}{2}} + \rho^{\frac{3}{2}} \left( \int_0^t \| u(s) \|^2_{L^2(\mathbb{R}^3_+)} \, ds \right)^2 + (1 + t)^{-\frac{a}{2}} \right],
\]
where
\[
g(t) = \| u(t) \|^2_{L^2(\mathbb{R}^3_+)} + \| \nabla d(t) \|^2_{L^2(\mathbb{R}^3_+)}.
\]

For \( k \) sufficiently large, let \( \rho = \frac{k}{1+t} \) and multiply (3.51) by \( (1 + t)^k \). Then we have
\[
\left( (1 + t)^k g \right)' \leq C (1 + t)^{-1+k} \left( (1 + t)^{-\frac{1}{2}} + (1 + t)^{-\frac{a}{2}} \right) \\
\leq C (1 + t)^{-\frac{1}{2} + k}.
\]

(3.52)

This, after integrating over \([1, t]\) and applying Lemma 3.1, implies that
\[
\| u(t) \|^2_{L^2(\mathbb{R}^3_+)} + \| \nabla d(t) \|^2_{L^2(\mathbb{R}^3_+)} \leq C (1 + t)^{-\frac{1}{2}}.
\]

(3.53)

Inserting (3.53) into (3.49) first and then (3.48), we obtain that
\[
\| u(t) \|^2_{L^a(\mathbb{R}^3_+)} \leq C t^{-\frac{a}{2}(1 - \frac{1}{a})},
\]

(3.54)

This can be used to improve estimate of \( I_1 \) to
\[
\| I_1(t) \|_{L^2(\mathbb{R}^3_+)} \leq C (1 + t)^{-\frac{1}{2}},
\]

(3.55)
which can then be used to improve (3.52) to

\[(1 + t^k g)' \leq C(1 + t)^{-1 + k}(1 + t)^{-\frac{3}{2}}.\]

Thus we obtain

\[(3.56)\quad \|u(t)\|_{L^2(\mathbb{R}^4_+)} \leq C(1 + t)^{-\frac{3}{4}}\]

so that (3.39) holds.

To show (3.38), first observe that by (1.8), (3.37), (3.56) and Lemma 2.1 we have that for small $\delta > 0$,

\[
\|d - e_3\|_{L^6(\mathbb{R}^4_+)} \leq C_0 \delta \quad \text{and} \quad \|\nabla d\|_{L^\infty(\mathbb{R}^4_+)} \leq C_1 \delta.
\]

provided $\varepsilon_0 \leq \bar{C}^{-1}$. From here to the end of this section, $C$ denotes a positive constant depending on $\bar{C}$, and $\tilde{C}$ denotes a positive constant that is independent of $\bar{C}$.

It follows from (1.8) that

\[(3.57)\quad \nabla d(t) = \nabla e^{\frac{3}{2} \Delta} (d - u_0)(\frac{t}{2}) - \int_0^t \nabla e^{\Delta(t - s)} \big( u(s) \cdot \nabla d(s) - |\nabla d(s)|^2 d(s) \big) \, ds.
\]

From Lemma 2.1 and (3.57), we obtain that for any small $\delta > 0$,

\[
\|\nabla d(t)\|_{L^2(\mathbb{R}^4_+)} \leq C_{1,2,3,4} \delta t^{-\frac{3}{2}} \|d - u_0\|_{L^6(\mathbb{R}^4_+)} + C \int_0^t (t - s)^{-\frac{3}{2}} \big( \|u\|_{L^6(\mathbb{R}^4_+)} \|\nabla d(s)\|_{L^2(\mathbb{R}^4_+)} + \|\nabla d(s)\|_{L^2(\mathbb{R}^4_+)} \|\nabla d(s)\|_{L^2(\mathbb{R}^4_+)} \big) \, ds
\]

where we choose $q \in (\frac{6}{5}, 2)$ so that $\frac{3}{2} - \frac{3}{2q} > -1$ and $\lim_{q \to \frac{6}{5}} \left( \frac{3}{4} - \frac{3}{2q} \right) = -1$.

It is readily seen that if we choose the constant $\tilde{C} = 2C$ in (3.37), then it follows from (3.37) and (3.59) that

\[(3.59)\quad \|\nabla d(t)\|_{L^2(\mathbb{R}^4_+)} \leq \tilde{C}(1 + t)^{-1}
\]

\[
(3.60)\quad + C \int_0^t (t - s)^{-\frac{3}{2}} \|u(s)\|_{L^6(\mathbb{R}^4_+)} \|\nabla d(s)\|_{L^2(\mathbb{R}^4_+)} + \|\nabla d(s)\|_{L^6(\mathbb{R}^4_+)}^2 \, ds.
\]
where $q \in \left(\frac{6}{7}, \frac{3}{2}\right)$. 

Applying (3.61), (3.62), (3.63), (3.64), (3.37), (3.56) and Lemma 3.3, we can obtain

\[
\|u(s)\|_{L^2(R^3_+)} \|\nabla d(s)\|_{L^{2q}(R^3)} \\
= \|u(s)\|_{L^2(R^3_+)} \|\nabla d(s)\|_{L^{\frac{2q}{3}}(R^3_+)} \|\nabla d(s)\|_{L^{\frac{2q}{3}}(R^3_+)} \\
\leq C \|u(s)\|_{L^2(R^3_+)} \|\nabla d(s)\|_{L^{\frac{2q}{3}}(R^3_+)} \|\nabla^2 d(s)\|_{L^{\frac{10}{7}}(R^3)} \|\nabla d(s)\|_{L^3(R^3_+)} \\
\leq C \varepsilon_0^{\frac{1}{5}-\frac{2}{3q}} (1 + s)^{\frac{11}{12} - \frac{5}{9q}},
\]

and

\[
\|\nabla d(s)\|_{L^{2q}(R^3_+)}^2 = \|\nabla d(s)\|_{L^{2q}(R^3_+)}^3 \|\nabla d(s)\|_{L^{2q}(R^3_+)}^\frac{1}{2} \\
\leq C \|\nabla d(s)\|_{L^2(R^3_+)} \|\nabla^2 d(s)\|_{L^\frac{2q}{3}(R^3_+)} \|\nabla d(s)\|_{L^3(R^3_+)} \\
\leq C \varepsilon_0^{\frac{3}{4} - \frac{3}{2q}} (1 + s)^{\frac{7}{4} - \frac{3}{4q}} ds
\]

for $s \in \left[\frac{t}{2}, t\right]$. 

Substituting (3.65) and (3.66) into (3.60) and applying Lemma 2.2 yields that

\[
\|\nabla d(t)\|_{L^2(R^3_+)} \\
\leq \widetilde{C}(1 + t)^{-1} + C \varepsilon_0^{\frac{1}{2} - \frac{3}{4}} \int_{\frac{t}{2}}^t (t - s)^{\frac{1}{2} - \frac{1}{2q}} (1 + s)^{\frac{7}{4} - \frac{5}{4q}} ds \\
+ C \varepsilon_0^{\frac{3}{4} - \frac{3}{2q}} \int_{\frac{t}{2}}^t (t - s)^{\frac{1}{4} - \frac{1}{2q}} (1 + s)^{\frac{7}{4} - \frac{15}{4q}} ds \\
\leq \widetilde{C}(1 + t)^{-1} + C \varepsilon_0^{\frac{1}{2} - \frac{3}{4}} (1 + t)^{\frac{13}{8} - \frac{4}{7}} + C \varepsilon_0^{\frac{3}{4} - \frac{3}{2q}} (1 + t)^{3 - \frac{4q}{3}} \\
\leq \frac{3\widetilde{C}}{2} (1 + t)^{-1}
\]

provided that we choose $q \in \left(\frac{6}{7}, \frac{24}{11}\right)$ and $\varepsilon_0$ so small that

\[
C \varepsilon_0^{\frac{1}{2} - \frac{3}{4}} + C \varepsilon_0^{\frac{3}{4} - \frac{3}{2q}} \leq \frac{\widetilde{C}}{2}.
\]
This implies (3.38) and completes the proof. □

By the standard continuity argument, we can then complete the proof of Lemma 3.5.

**Corollary 3.1.** Under the same assumptions of Theorem 1.1, if \((u, d)\) is the global strong solution obtained by Theorem 1.1, then
\[
\|u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(1 + t)^{-\frac{3}{4}}, \quad \|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(1 + t)^{-1}, \quad \forall \ t > 0.
\]

With some further calculations, we also have

**Corollary 3.2.** Under the same assumptions of Theorem 1.1, if \((u, d)\) is the global strong solution obtained by Theorem 1.1, then it holds that
\[
\|d - e_3(t)\|_{L^r(\mathbb{R}^3_+)} \leq C t^{-\frac{3}{4}(1 - \frac{1}{r})}, \quad \forall \ t > 0, \ \forall r \in (1, 2],
\]
\[
\|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(1 + t)^{-\frac{3}{4}}, \quad \forall \ t > 0.
\]

**Proof.** By (1.8), Lemma 2.1, (3.71), (3.61), and (3.62), we then have
\[
\|d - e_3(t)\|_{L^r(\mathbb{R}^3_+)} \leq C t^{-\frac{3}{4}(1 - \frac{1}{r})} \|d_0 - e_3\|_{L^1(\mathbb{R}^3_+)}
\]
\[
+ C \int_0^t (t - s)^{-\frac{3}{4}(1 - \frac{1}{r})} \left(\|u(s)\|_{L^2(\mathbb{R}^3_+)} \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)} + \|\nabla d(s)\|_{H^1(\mathbb{R}^3_+)}^2\right) ds
\]
\[
\leq C t^{-\frac{3}{4}(1 - \frac{1}{r})} + C \int_0^t (t - s)^{-\frac{3}{4}(1 - \frac{1}{r})} \left[ 1 + s \right]^{-\frac{7}{4} + (1 + s)^{-2}} ds
\]
\[
\leq C t^{-\frac{3}{4}(1 - \frac{1}{r})} + \int_0^t (t - s)^{-\frac{3}{4} + \frac{3q}{4} - \frac{3}{q}} ds
\]
\[
\leq C(1 + t)^{-\frac{3}{4}} + C(1 + t)^{\frac{5}{4} - \frac{3q}{4}}
\]
for any \(q \in \left(\frac{6}{5}, \frac{3}{2}\right)\).

Since \(\lim_{q \to \frac{6}{5}} \left(\frac{5}{2} - \frac{9}{2q}\right) = -\frac{5}{4}\), it follows from (3.72) that for any \(\epsilon > 0\), there exists \(C_\epsilon > 0\) such that
\[
\|\nabla d(t)\|_{H^1(\mathbb{R}^3_+)} \leq C_\epsilon (1 + t)^{-\frac{3}{4} + \epsilon}.
\]

Substituting (3.73) into (3.72) and running the same argument as in (3.72), we would obtain the sharp estimate
\[
\|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(1 + t)^{-\frac{3}{4}}.
\]

This completes the proof. □
Corollary 3.3. Under the same assumptions of Theorem 1.1, if \((u, d)\) is the global strong solution obtained by Theorem 1.1, then the following estimates hold

\[
\begin{align*}
\|u(t)\|_{L^r(\mathbb{R}^3_+)} & \leq Ct^{-\frac{3}{2}(1-\frac{1}{r})}, \\
\|\nabla d(t)\|_{L^s(\mathbb{R}^3_+)} & \leq Ct^{-\frac{3}{2}(1-\frac{1}{s})}
\end{align*}
\]

for any \(r \in (1, 2]\) and \(s \in [1, 2]\).

**Proof.** By (3.49), Corollary 3.1 and Corollary 3.2, we have that for any \(t \in \left(1, \frac{3}{2}\right)\), it holds

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq C t^{-\frac{3}{2}(1-\frac{1}{r})}\|u_0\|_{L^1(\mathbb{R}^3_+)} + C \int_0^t (t-s)^{-\frac{3}{2}(1-\frac{1}{r})}(1+s)^{-\frac{3}{2}} ds
\]

\[
\leq C t^{-\frac{3}{2}(1-\frac{1}{r})}.
\]

Then for any \(r \in \left[\frac{3}{2}, 2\right)\), using the interpolation inequality, we have

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq \|u(t)\|_{L^2(\mathbb{R}^3_+)}^{\frac{\alpha}{2}}\|u(t)\|_{L^2(\mathbb{R}^3_+)}^{1-\frac{\alpha}{2}} \leq C t^{-\frac{3}{2}(1-\frac{1}{r})}\|u_0\|_{L^1(\mathbb{R}^3_+)}^{\frac{\alpha}{2}}\|u_0\|_{L^1(\mathbb{R}^3_+)}^{1-\frac{\alpha}{2}} \leq Ct^{-\frac{3}{2}(1-\frac{1}{r})},
\]

where \(\alpha \in (0, 1)\) satisfies \(\frac{2}{r} + \frac{1-\alpha}{2} = \frac{1}{r}\).

While, by (1.8), (2.2), Lemma 2.2, Corollary 3.1 and Corollary 3.2, we have that

\[
\|\nabla d(t)\|_{L^1(\mathbb{R}^3_+)} \leq C t^{-\frac{3}{2}}d_0 - e_3\|\nabla d\|_{L^1(\mathbb{R}^3_+)}
\]

\[
+ C \int_0^t (t-s)^{-\frac{3}{2}}\left(\|u(s)\|_{L^2(\mathbb{R}^3_+)}\|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)} + \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)}^2\right) ds
\]

\[
\leq C t^{-\frac{3}{2}} + C \int_0^t (t-s)^{-\frac{3}{2}} \left[ (1+s)^{-2} + (1+s)^{-\frac{5}{2}} \right] ds
\]

\[
\leq C t^{-\frac{3}{2}}.
\]

This, combined with (3.74) and the interpolation inequality, implies that for any \(1 \leq s \leq 2\),

\[
\|\nabla d(t)\|_{L^s(\mathbb{R}^3_+)} \leq \left(\|\nabla d(t)\|_{L^1(\mathbb{R}^3_+)}^{\frac{s-1}{s}}\|\nabla d(t)\|_{L^2(\mathbb{R}^3_+)}^{\frac{2-s}{s}}\right) \leq C t^{-\frac{3}{2} - \frac{3}{2}(1-\frac{1}{s})}.
\]

This completes the proof. \(\square\)

**Lemma 3.6.** Assume there exist \(C_1, C_2 > 0\) such that if the global strong solution \((u, d)\) given by Theorem 1.1 satisfies

\[
\begin{align*}
\|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} & \leq 2C_1 t^{-\frac{7}{4}}, \\
\|\nabla^2 d(t)\|_{L^6(\mathbb{R}^3_+)} & \leq 2C_2 t^{-\frac{9}{4}}
\end{align*}
\]

for any \(t \in (0, T]\). Then the following estimates

\[
\begin{align*}
\|u(t)\|_{L^r(\mathbb{R}^3_+)} & \leq Ct^{-\frac{3}{2}(1-\frac{1}{r})}, \\
\|(d - e_3) (t)\|_{L^r(\mathbb{R}^3_+)} & \leq Ct^{-\frac{3}{2}(1-\frac{1}{r})}, \\
\|\nabla d(t)\|_{L^r(\mathbb{R}^3_+)} & \leq Ct^{-\frac{3}{2} - \frac{3}{2}(1-\frac{1}{r})}, \\
\|\nabla u(t)\|_{L^s(\mathbb{R}^3_+)} & \leq \frac{3}{2} C_1 t^{-\frac{7}{4}},
\end{align*}
\]

(3.87) \[ \|\nabla^2 d(t)\|_{L^6(\mathbb{R}^3_+)} \leq \frac{3}{2} C_2 t^{-\frac{3}{4}} \]

hold for any \( t \in (0, T] \). Here \( r \in [2, \infty] \).

Proof. Without loss of generality, we may assume that \( t \geq 1 \). We divide the proof into two steps:

**Step I.** By Gagliardo-Nirenberg inequality \(^2\) and Corollary 3.3 we know that for any \( 0 < t \leq T \),

\[
\|u(t)\|_{L^\infty(\mathbb{R}^3_+)} \leq \|u(t)\|_{L^2(\mathbb{R}^3_+)}^{\frac{1}{7}} \|\nabla u(t)\|_{L^3(\mathbb{R}^3_+)}^{\frac{3}{7}}
\]

(3.88)

Then the case for \( r \in (2, \infty) \) directly follows from the interpolation inequality. This yields (3.88). Similar arguments also yield first (3.85) and then (3.84).

**Step II.** We want to show the estimate of \( \|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} \). To see this, first observe that by (1.9), Lemma 2.1, and the interpolation inequality, we have

\[
\|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} \leq \|u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C \|u(t)\|_{L^2(\mathbb{R}^3_+)} \|\nabla u(t)\|_{L^3(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{4}} \|
\]

(3.89)

Then the case for \( r \in (2, \infty) \) directly follows from the interpolation inequality. This yields (3.88). Similar arguments also yield first (3.85) and then (3.84).

\[
\|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} \leq \|u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C \|u(t)\|_{L^2(\mathbb{R}^3_+)} \|\nabla u(t)\|_{L^3(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{4}} \|
\]

(3.89)

To bound the second term in the right hand of (3.89), we first estimate by using interpolation inequality, Lemma 3.2 (3.81) and (3.88) that

\[
\int_{\frac{t}{2}}^{t} (t - s)^{-\frac{1}{7}} \|u(s)\|_{L^6(\mathbb{R}^3_+)} \|\nabla u(s)\|_{L^6(\mathbb{R}^3_+)} \| \leq C \int_{\frac{t}{2}}^{t} (t - s)^{-\frac{1}{7}} \|u(s)\|_{L^6(\mathbb{R}^3_+)} \|\nabla u(s)\|_{L^6(\mathbb{R}^3_+)} \| ds
\]

(3.90)

Similarly, we can estimate

\[
\int_{\frac{t}{2}}^{t} (t - s)^{-\frac{1}{7}} \|\nabla d(s)\|_{L^4(\mathbb{R}^3_+)} \|\nabla^2 d(s)\|_{L^4(\mathbb{R}^3_+)} ds \leq C \varepsilon_0 t^{-\frac{7}{4}}.
\]

(3.91)

Substituting the estimates (3.90) and (3.91) into (3.89) and choosing a sufficiently small \( \varepsilon_0 \), we conclude that

\[
\|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} \leq \frac{3}{2} C_1 t^{-\frac{7}{4}}.
\]

---

\(^2\)Indeed, to see this, one has only to extend \( u \) from \( \mathbb{R}^3_+ \) to \( \mathbb{R}^3 \) by odd or even extension and then apply the \( \mathbb{R}^3 \)-version.
Finally, we want to estimate \( \| \nabla^2 d(t) \|_{L^6(\mathbb{R}^3_+)} \). By taking \( \nabla \) of \((1.4)_3\), we have that
\[
(\nabla d)_t - \Delta (\nabla d) = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d)
\]
so that by Duhamel’s formula, we have
\[
\nabla d(t) = e^{\frac{t}{2}} \Delta \nabla d(t) + \int_{\frac{t}{2}}^t e^{(t-s)\Delta} \left( -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d) \right)(s) \, ds,
\]
and
\[
\nabla^2 d(t) = e^{\frac{t}{2}} \Delta \nabla d(t) + \int_{\frac{t}{2}}^t \nabla e^{(t-s)\Delta} \left( -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d) \right)(s) \, ds.
\]
This, combined with Lemma 2.1 and a similar argument as (3.90), yields that
\[
\begin{align*}
\| \nabla^2 d(t) \|_{L^6(\mathbb{R}^3_+)} & \leq C_{1,6.2,3} t^{-1} \| \nabla d(t) \|_{L^2(\mathbb{R}^3_+)} + \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla (u \cdot \nabla d) - \nabla (|\nabla d|^2 d) \|_{L^3(\mathbb{R}^3_+)} \, ds \\
& \leq C_2 t^{-\frac{9}{8}} + \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla u \| \| \nabla d \| + |u| \| \nabla^2 d \| + |\nabla d| \| \nabla d \| + |\nabla d|^3 \|_{L^3(\mathbb{R}^3_+)} \, ds \\
& \leq C_2 t^{-\frac{9}{8}} + K_1 + K_2 + K_3 + K_4,
\end{align*}
\]
where
\[
K_1 = \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla u \| \| \nabla d \|_{L^3(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla u(s) \|_{L^6(\mathbb{R}^3_+)} \| \nabla d(s) \|_{L^3(\mathbb{R}^3_+)} \| \nabla d(s) \|_{L^1(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \varepsilon_0 t^{-\frac{11}{4}},
\]
\[
K_2 = \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| u \| \| \nabla^2 d \|_{L^3(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| u(s) \|_{L^6(\mathbb{R}^3_+)} ^{\frac{1}{2}} \| u(s) \|_{L^{12}(\mathbb{R}^3_+)} \| \nabla^2 d(s) \|_{L^6(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \varepsilon_0 t^{-\frac{9}{4}},
\]
\[
K_3 = \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla d \| \| \nabla^2 d \|_{L^3(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla d(s) \|_{L^3(\mathbb{R}^3_+)} \| \nabla d(s) \|_{L^{12}(\mathbb{R}^3_+)} \| \nabla^2 d(s) \|_{L^6(\mathbb{R}^3_+)} \, ds
\]
\[
\leq C \varepsilon_0 t^{-\frac{9}{4}},
\]
\[
K_4 = \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \| \nabla d(s) \|_{L^3(\mathbb{R}^3_+)} \, ds
\]
Corollary 3.5. Under the same assumptions as Corollary 3.4, it holds that for any $r \in \varepsilon \leq C\varepsilon^{\frac{1}{2}} - \frac{2}{3}$. Thus (3.87) follows by choosing $\varepsilon_0$ sufficiently small. This completes the proof. □

Now we can apply the continuity argument to show the following Corollary.

**Corollary 3.4.** Under the same assumptions of Theorem 1.1, if $(u, d)$ is the global strong solution obtained by Theorem 1.1, then the following estimates

\begin{align*}
&\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{2}{3}(1-\frac{1}{r})}, \quad (3.97) \\
&\|(d - e_3)(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{2}{3}(1-\frac{1}{r})}, \quad (3.98) \\
&\|\nabla d(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2} - \frac{2}{3}(1-\frac{1}{r})}, \quad (3.99) \\
&\|\nabla u(t)\|_{L^6(\mathbb{R}^3_+)} \leq Ct^{-\frac{7}{4}}, \quad (3.100) \\
&\|\nabla^2 d(t)\|_{L^6(\mathbb{R}^3_+)} \leq Ct^{-\frac{9}{4}}, \quad (3.101)
\end{align*}

hold for any $t > 0$, where $r \in [2, \infty]$.

Based on the estimate (3.97), we can deduce

**Corollary 3.5.** Under the same assumptions as Corollary 3.4, it holds that for any $t > 0$ and $r \in (1, 6]$,

$$\|\nabla u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{1}{2} - \frac{3}{4}(1-\frac{1}{r})}. \quad (3.102)$$

**Proof.** In fact, it follows from (3.97) and Lemma 2.1 that, for any $r \in (1, 6)$,

\begin{align*}
\|\nabla u(t)\|_{L^r(\mathbb{R}^3_+)} &\leq C(1 + t)^{-\frac{3}{2}}\|u(t)\|_{L^r(\mathbb{R}^3_+)} + C \int_{\frac{t}{2}}^{t}(t - s)^{-\frac{1}{2}}\|u\|\nabla u\| + \|\nabla d\|\nabla^2 d\|_{L^r(\mathbb{R}^3_+)} ds \\
&\leq Ct^{-\frac{1}{2} - \frac{3}{4}(1-\frac{1}{r})} + C \int_{\frac{t}{2}}^{t}(t - s)^{-\frac{1}{2}}(\|u(s)\|_{L^{\frac{6r}{r-6}}(\mathbb{R}^3_+)}\|\nabla u(s)\|_{L^6(\mathbb{R}^3_+)} + \|\nabla d(s)\|_{L^{\frac{6r}{r-6}}(\mathbb{R}^3_+)}\|\nabla^2 d(s)\|_{L^6(\mathbb{R}^3_+)} ds \\
&\leq Ct^{-\frac{1}{2} - \frac{3}{4}(1-\frac{1}{r})}.
\end{align*}

This, combined with (3.100), completes the proof. □

We also enlarge the range for the estimate of $\nabla^2 d(t)$. More precisely,

**Corollary 3.6.** Under the same assumptions as Corollary 3.4, it holds that for any $t > 0$ and $s \in [1, 6]$,

$$\|\nabla^2 d(t)\|_{L^s(\mathbb{R}^3_+)} \leq Ct^{-\frac{4}{3}(1-\frac{1}{s})}. \quad (3.103)$$
Proof. From \([3.95]\), Lemma 2.1 (3.79), and Corollary 3.4 we obtain that
\[
\|\nabla^2 d(t)\|_{L^1(\mathbb{R}^3_+)}
\leq C_{1,1,3} t^{-\frac{3}{2}} \|\nabla d(t)\|_{L^1(\mathbb{R}^3_+)}
\]
\[+
\int_{\frac{3}{2}}^t (t-s)^{-\frac{3}{2}} \|\nabla(u \cdot \nabla d)\|_{L^1(\mathbb{R}^3_+)}
\leq C t^{-1} + \int_{\frac{3}{2}}^t (t-s)^{-\frac{3}{2}} \|\nabla u(s)\|_{L^2(\mathbb{R}^3_+)} \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)} + \|u(s)\|_{L^2(\mathbb{R}^3_+)} \|\nabla^2 d(s)\|_{L^2(\mathbb{R}^3_+)}
\]
\[+
\int_{\frac{3}{2}}^t (t-s)^{-\frac{3}{2}} \|\nabla^2 d(s)\|_{L^2(\mathbb{R}^3_+)} + \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)} + \|\nabla d(s)\|_{L^3(\mathbb{R}^3_+)}(s)ds
\leq C t^{-1}.
\]

The conclusion now follows from (3.101) and the interpolation inequality.

Combining all conclusions in this section, we prove the time decay estimates in Theorem 1.1

4. PROOF OF THEOREM 1.2

In this section, we will give a proof of Theorem 1.2, which follows from Lemma 4.1 below.

Under the assumptions in Theorem 1.2, by integrating \((3.13)\) over \([0,t]\), and applying \((3.1), (3.16), (3.21)\) and the elliptic estimates, we have that for any \(t > 0\),
\[
(\|\nabla u(t)\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla^2 d(t)\|_{L^2(\mathbb{R}^3_+)}^2 + \|d_t(t)\|_{L^2(\mathbb{R}^3_+)}^2)
\]
\[+
\int_0^t \left(\|u_{tt}\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla u\|_{H^1(\mathbb{R}^3_+)}^2 + \|d_t\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla^2 d\|_{H^1(\mathbb{R}^3_+)}^2\right) ds \leq C.
\]

Lemma 4.1. Under the same assumptions as in Theorem 1.2, it holds that for any \(t > 0\),
\[
\|\nabla u(t)\|_{L^1(\mathbb{R}^3_+)} \leq C t^{-\frac{1}{2}}.
\]

Proof. Without loss of generality, we may assume that \(t \geq 1\). By \((1.8)_{1, 2, 11}\), \((2.11)\), \((2.12)\), the Hölder inequality, \((4.1)\) and Lemma 3.4 we obtain
\[
\|\nabla u\|_{L^1(\mathbb{R}^3_+)}
\leq C t^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}^3_+)} + \int_0^t \|\nabla e^{-(t-s)\Delta} \mathbb{P} (u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d) ) (s)\|_{L^1(\mathbb{R}^3_+)} ds
\leq C t^{-\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R}^3_+)}
\[+
C \int_0^t (t-s)^{-\frac{1}{2}} \left(\|u\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla d\|_{H^1(\mathbb{R}^3_+)}^2\right)
\]
\[+
\|\nabla d\|_{L^1(\mathbb{R}^3_+)} + \|\nabla^2 d\|_{L^1(\mathbb{R}^3_+)}(s)ds
\leq C t^{-\frac{1}{2}} + C \int_0^t (t-s)^{-\frac{1}{2}} \left(\|u(s)\|_{H^1(\mathbb{R}^3_+)}^2 + \|\nabla d(s)\|_{H^1(\mathbb{R}^3_+)}^2\right)
\]
\[+
\|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla^2 d(s)\|_{L^2(\mathbb{R}^3_+)}^2 + \|\nabla d(s)\|_{L^2(\mathbb{R}^3_+)} \|\nabla^3 d(s)\|_{L^2(\mathbb{R}^3_+)} ds
\leq C t^{-\frac{1}{2}} + C \int_0^t (t-s)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{4}} ds.
\]
Lemma 5.1. Under the same assumptions of Theorem 1.1, if \( u, d \) is the global strong solution of (1.1) obtained by Theorem 1.1, then, for any \( r \in (1, 3/2) \), it holds that

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})}.
\]  

Proof. By using (1.39), (3.49), (5.1) and Theorem 1.1, we have,

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \int_{\mathbb{R}^3_+} (1 + x_3) |u_0(x)| dx \\
+ \int_0^t \int_{\mathbb{R}^3_+} (t-s)^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \left( \|u(s)\|_{L^6(\mathbb{R}^3_+)}^2 + \|\nabla d(s)\|_{L^6(\mathbb{R}^3_+)}^2 \right) ds \\
\leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} + \int_0^t (t-s)^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} (1 + s)^{-\frac{3}{2}} ds \\
\leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})}.
\]

This completes the proof of this Lemma. \( \square \)

5. Proof of Corollary 1.1

In this section, we will prove Corollary 1.1. In fact, the conclusions of Corollary 1.1 follow from Lemma 5.1 and Lemma 5.2 below.

We first recall an revised estimates with weighted condition (1.6). For any \( p \in (1, +\infty) \), it holds

\[
\|\nabla^k e^{-t\Delta} u_0\|_{L^p(\mathbb{R}^3_+)} \leq Ct^{-\frac{k+1}{2} - \frac{3}{2}(1-\frac{1}{p})} \int_{\mathbb{R}^3_+} (1 + x_3) |u_0(x)| dx,
\]

for \( k = 0, 1 \).

Lemma 5.1. Under the same assumptions of Theorem 1.1 if \( u, d \) is the global strong solution of (1.1) obtained by Theorem 1.1 then, for any \( r \in (1, 3/2) \), it holds that

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})}.
\]

Proof. By using (1.39), (3.49), (5.1) and Theorem 1.1, we have,

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \int_{\mathbb{R}^3_+} (1 + x_3) |u_0(x)| dx \\
+ \int_0^t (t-s)^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} \left( \|u(s)\|_{L^6(\mathbb{R}^3_+)}^2 + \|\nabla d(s)\|_{L^6(\mathbb{R}^3_+)}^2 \right) ds \\
\leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} + \int_0^t (t-s)^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})} (1 + s)^{-\frac{3}{2}} ds \\
\leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})}.
\]

This completes the proof. \( \square \)

Lemma 5.2. Under the same assumptions of Theorem 1.1 if \( u, d \) is the global strong solution of (1.1) obtained by Theorem 1.1 then, for any \( r \in \left[\frac{3}{4}, \infty\right) \) and \( p \in (1, 6] \), it holds that

\[
\|u(t)\|_{L^r(\mathbb{R}^3_+)} \leq Ct^{-\frac{3}{2}-\frac{3}{2}(1-\frac{1}{r})},
\]

\[
\|\nabla u(t)\|_{L^p(\mathbb{R}^3_+)} \leq Ct^{-1-\frac{3}{2}(1-\frac{1}{p})}.
\]
Proof. Without loss of generality, assume that \( t \geq 1 \). On one hand, for \( k = 0, 1 \), from Lemma 2.1 and Lemma 3.1 it holds, for any \( r \in \left[ \frac{3}{2}, \infty \right) \),
\[
\| \nabla^k e^{-\frac{k}{2} \Delta} u(t) \|_{L^r(\mathbb{R}^3)} \leq C t^{-\frac{k}{2} - \frac{3}{2}(\frac{1}{r} - \frac{1}{2})} \| u(t) \|_{L^r(\mathbb{R}^3)} \\
\leq C t^{-\frac{k+1}{2} - \frac{3}{2}(\frac{1}{r} - \frac{1}{2})} t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{r})}
\]
(5.6)

Here we choose some \( \tilde{r} \in (1, \frac{3}{2}) \).

On the other hand, by using Lemma 2.1 Lemma 2.2 and Theorem 1.1 we have, for any \( r \in \left[ \frac{3}{2}, \infty \right) \), it holds that,

\[
\left\| \int_{\frac{1}{2}}^{t} e^{-(t-s)\Delta^2} \left( u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d) \right) (s) ds \right\|_{L^r(\mathbb{R}^3)} \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})} \left( \| u(s) \cdot \nabla u(s) \|_{L^3(\mathbb{R}^3)} + \| \nabla \cdot (\nabla d \odot \nabla d)(s) \|_{L^3(\mathbb{R}^3)} \right) ds \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})} \left( \| u(s) \|_{L^8(\mathbb{R}^3)} \| \nabla u(s) \|_{L^6(\mathbb{R}^3)} + \| \nabla d(s) \|_{L^8(\mathbb{R}^3)} \| \nabla^2 d(s) \|_{L^6(\mathbb{R}^3)} \right) ds
\]
(5.7) \leq C t^{-\frac{3}{2} - \frac{3}{2}(1 - \frac{1}{r})},

and

\[
\left\| \int_{\frac{1}{2}}^{t} e^{-(t-s)\Delta^2} \left( u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d) \right) (s) ds \right\|_{L^r(\mathbb{R}^3)} \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{3}{2}} \left( \| u(s) \cdot \nabla u(s) \|_{L^3(\mathbb{R}^3)} + \| \nabla \cdot (\nabla d \odot \nabla d)(s) \|_{L^3(\mathbb{R}^3)} \right) ds \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{3}{2}} \left( \| u(s) \|_{L^6(\mathbb{R}^3)} \| \nabla u(s) \|_{L^6(\mathbb{R}^3)} + \| \nabla d(s) \|_{L^6(\mathbb{R}^3)} \| \nabla^2 d(s) \|_{L^6(\mathbb{R}^3)} \right) ds
\]
(5.8) \leq C t^{-\frac{3}{4} + \frac{3}{4} - \frac{1}{4}} ds \leq C t^{-2},

Then (5.4) follows from (1.9), (5.6), (5.7) and (5.8).

Furthermore, for any \( p \in (1, 6] \) and for some \( \tilde{p} \) such that \( \frac{3p}{3+p} < \tilde{p} < p \), one obtains

\[
\left\| \int_{\frac{1}{2}}^{t} \nabla e^{-(t-s)\Delta^2} \left( u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d) \right) (s) ds \right\|_{L^p(\mathbb{R}^3)} \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{1}{2}} \left( \| u(s) \cdot \nabla u(s) \|_{L^p(\mathbb{R}^3)} + \| \nabla \cdot (\nabla d \odot \nabla d)(s) \|_{L^p(\mathbb{R}^3)} \right) ds \\
\leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{1}{2}} \left( \| u(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \| \nabla u(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \right) + \| \nabla d(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \| \nabla^2 d(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \right) ds
\]
(5.9) \leq \int_{\frac{1}{2}}^{t} \left( t-s \right)^{-\frac{1}{2}} \left( \| u(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \| \nabla u(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \right) + \| \nabla d(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \| \nabla^2 d(s) \|_{L^{6\tilde{p}}(\mathbb{R}^3)} \right) ds \leq C t^{-1 - \frac{3}{4}(1 - \frac{1}{p})}.
Then (5.5) follows from (5.6) with $k = 1$ and (5.9).

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