NON-CHARACTERISTIC EXPANSIONS
OF LEGENDRIAN SINGULARITIES

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Abstract. This paper presents an algorithm to deform any Legendrian singularity to a
nearby Legendrian subvariety with singularities of a simple combinatorial nature. Furthermore,
the category of microlocal sheaves on the original Legendrian singularity is equivalent
to that on the nearby Legendrian subvariety. This yields a concrete combinatorial model for
microlocal sheaves, as well as an elementary method for calculating them.

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1. Introduction

This paper presents an algorithm to deform any Legendrian singularity to a nearby Legendrian subvariety with singularities of a simple combinatorial nature. Furthermore, the category of microlocal sheaves, as developed by Kashiwara-Schapira [13], on the original Legendrian singularity is equivalent to that on the nearby Legendrian subvariety. This yields a concrete combinatorial model for microlocal sheaves, in terms of modules over quivers, as well as an elementary method for calculating them.

Among other applications, it allows one to establish new homological mirror symmetry equivalences for Landau-Ginzburg models with singular thimbles [21]. It also provides a key tool in the foundations of microlocal sheaves, in particular in the recent development of its wrapped variant [23]. Furthermore, it underlies ongoing work on the structure of Weinstein manifolds [6] and their Fukaya categories [8].

In the rest of the introduction, we first state the main results of the paper, then sketch some of the arguments involved in their proof. Finally, we elaborate further on their place in the subject, in particular their original motivation in conjectures of Kontsevich [16] and a program of MacPherson and collaborators (see for example [4, 9]) devoted to combinatorial models of respectively Fukaya categories and microlocal sheaves.

1.1. Main results. A natural setting for the paper is local contact geometry. (See Sects. 2 and 3 below for detailed geometric preliminaries.)

Let $M$ be a smooth manifold with cotangent bundle $T^*M$ with its canonical exact symplectic structure. Introduce the cosphere bundle

$$
\pi : S^* M = (T^* M \setminus M) / \mathbb{R}_{>0} \longrightarrow M
$$

with its canonical cooriented contact structure. By the contact Darboux theorem, any contact manifold is locally equivalent to $S^* M$.

By a Legendrian subvariety $\Lambda \subset S^* M$, we will mean a closed Whitney stratified subspace of pure dimension $\dim \Lambda = \dim M - 1$ whose strata are isotropic for the contact structure. By a Legendrian singularity $\Lambda \subset S^* M$, we will mean the germ of a Legendrian subvariety at a point $z \in \Lambda$ which we refer to as its center.

We will assume that any Legendrian singularity $\Lambda \subset S^* M$ we encounter can be placed in generic position in the sense that the front projection $H = \pi(\Lambda) \subset M$ is the germ at the image of the center $x = \pi(z)$ of a Whitney stratified hypersurface, and the restriction

$$
\pi|_{\Lambda} : \Lambda \longrightarrow H
$$

is a finite map. (To simplify the exposition, we will also assume the Whitney stratification of $H$ satisfies some modest local connectivity which can always be arranged by refining the stratification, see Sect. 5.1 for details).

Fix a field $k$, and following Kashiwara-Schapira [13] (and reviewed in Sect. 6 below) given a Legendrian subvariety $\Lambda \subset S^* M$, introduce the dg category $\mathcal{M}Sh_{\Lambda}(S^* M)$ of constructible microlocal complexes of $k$-modules on $S^* M$ supported along $\Lambda$. (The results and arguments of the paper work equally well without the constructible hypothesis, but we include it to keep within a traditional setting.) In particular, for a Legendrian singularity $\Lambda \subset S^* M$ in generic
position, and a small open ball \( B \subset M \) around the image of the center \( x = \pi(z) \), there is a canonical quotient presentation

\[
\mu Sh_{\Lambda}(S^*M) \cong Sh_{\Lambda}(B)/\text{Loc}(B)
\]

in terms of the more concrete dg category \( Sh_{\Lambda}(B) \) of constructible complexes on \( B \) with singular support in \( \Lambda \), and its full dg subcategory \( \text{Loc}(B) \) of finite-rank derived local systems.

In the paper [20], we introduced a natural class of Legendrian singularities \( \Lambda_T \subset S^*\mathbb{R}^{|T|} \), called arboreal singularities, indexed by rooted trees \( T \) (finite connected acyclic graphs with a choice of root vertex), where \( |T| \) denotes the number of vertices of the underlying tree. Each is naturally stratified by strata \( \Lambda_T \), where

\[
\text{support in } \Lambda, \text{ and its full dg subcategory } \text{Loc}(B) \text{ of finite-rank derived local systems.}
\]

| \begin{align*}
|T| & \text{ denote the number of vertices of the underlying tree. Each is} \\
\text{support in } \Lambda, & \text{ and its full dg subcategory } \text{Loc}(B) \text{ of finite-rank derived local systems.}
\end{align*}

Passing to microlocal sheaves, we constructed in [20] a canonical equivalence

\[
\mu Sh_{\Lambda_T}(S^*\mathbb{R}^{|T|}) \cong \text{Perf}(T)
\]

where \( \text{Perf}(T) \) denotes the dg category of perfect complexes over the quiver associated to the rooted tree \( T \) where all edges are directed away from the root vertex. We also showed that for each stratum \( \Lambda_T(p) \subset \Lambda_T \), the corresponding microlocal restriction functor

\[
\mu Sh_{\Lambda_T}(S^*\mathbb{R}^{[T]}) \rightarrow \mu Sh_{\Lambda_R}(S^*\mathbb{R}^{[R]})
\]

is given by the integral transform

\[
\begin{array}{ccc}
\text{Perf}(T) & \xrightarrow{i^*} & \text{Perf}(S) \\
& \searrow & \downarrow q \\
& & \text{Perf}(R)
\end{array}
\]

where \( i^* \) kills the projective object \( P_{\alpha} \in \text{Perf}(T) \) attached to \( \alpha \in T \setminus S \), and and \( q \) identifies the projective objects \( P_{\alpha}, P_{\beta} \in \text{Perf}(S) \) attached to \( \alpha, \beta \in S \) such that \( q(\alpha) = q(\beta) \in R \).

In Sect. [2] below, we review the basic properties of arboreal singularities, and introduce natural generalizations \( \Lambda_{T^*} \subset S^*\mathbb{R}^{[T^*]} \) indexed by leafy rooted trees \( T^* \) (finite connected acyclic graphs with a choice of root vertex and a subset of leaf vertices), where \( |T^*| \) denotes the sum of the number of vertices of the underlying tree and the number of marked leaves. Similar statements to those recalled above hold for generalized arboreal singularities.

By an arboreal Legendrian subvariety \( \Lambda \subset S^*M \), we will mean a Legendrian subvariety such that its normal geometry along each of its strata is equivalent via Hamiltonian reduction to an arboreal singularity \( \Lambda_{T^*} \subset S^*\mathbb{R}^{[T^*]} \). Thus for an arboreal Legendrian subvariety \( \Lambda \subset S^*M \), the dg category of microlocal sheaves \( \mu Sh_{\Lambda}(S^*M) \) admits an elementary description: it is the global sections of a sheaf of dg categories over \( \Lambda \subset S^*M \) that assigns perfect modules over trees to small open sets, integral transforms for correspondences of trees to inclusions of small open sets, along with natural higher coherences.

Now to state the main result of this paper, we need one further key concept.

By a deformation of a Legendrian singularity \( \Lambda \subset S^*M \) to a Legendrian subvariety \( \Lambda' \subset S^*M \), we will mean a Legendrian subvariety \( \Lambda \subset S^*(M \times \mathbb{R}) \) supported over the parameters \( \mathbb{R}_{\geq 0} \) with special Hamiltonian reduction \( \Lambda_0 = \Lambda \subset S^*M \) the original Legendrian singularity, and generic Hamiltonian reductions \( \Lambda_t \simeq \Lambda' \subset S^*M \) all equivalent for \( t > 0 \).

It is not true in general that microlocal sheaves will be constant under such deformations, even if we insist on strong topological properties. (See Example [1,5] below.) By a non-characteristic
Theorem 1.1. Let $\Lambda \subset S^*M$ be a Legendrian singularity.

The expansion algorithm of Sect. [4] provides a non-characteristic deformation of $\Lambda \subset S^*M$ to an arboreal Legendrian subvariety $\Lambda_{arb} \subset S^*M$.

Our use of the term expansion reflects the idea that we perform a kind of “spherical real blowup” to expand complicated singularities into irreducible components which then interact in a combinatorial way. One could compare this with resolutions of singularities in algebraic geometry where complicated singularities become divisors with normal crossings. Proving the expansion algorithm is non-characteristic occupies Sect. [4] below and is the most technically involved part of the paper.

Remark 1.2. One can easily refine the expansion algorithm so that $\Lambda_{arb} \subset S^*M$ is as $C^0$-close to $\Lambda \subset S^*M$ as one wishes.

Corollary 1.3. Let $\Lambda \subset S^*M$ be a Legendrian singularity.

There is a canonical equivalence between the dg category $\mu Sh_{\Lambda}(S^*M)$ of microlocal sheaves and the global sections of a sheaf of dg categories over $\Lambda_{arb} \subset S^*M$ that assigns perfect modules over trees to small open sets, integral transforms for correspondences of trees to inclusions of small open sets, along with natural higher coherences.

One can view the theorem and corollary as providing analogies with basic results of Morse theory.

On the one hand, any germ of a smooth function $f : M \to \mathbb{R}$ can be deformed to a nearby function with Morse singularities. Moreover, Morse singularities are of a simple combinatorial form enumerated by their Morse index $0 \leq k \leq \dim M$. What results is a combinatorial description of the cohomology of $M$ in terms of the the Morse-Witten complex.

On the other hand, any Legendrian singularity $\Lambda \subset S^*M$ can be deformed to a Legendrian subvariety with arboreal singularities. Moreover, arboreal singularities are of a simple combinatorial form enumerated by leafy rooted trees. What results is a combinatorial description of microlocal sheaves along $\Lambda \subset S^*M$ in terms of a diagram of perfect modules over trees.

We have not attempted to formulate here the sense in which arboreal singularities are the “stable” Legendrian singularities, though we expect the analogy to extend in this direction.

Remark 1.4. We have used the term non-characteristic to mean that the dg category of microlocal sheaves is invariant under the deformation. We expect this to be a consequence of the following more basic geometric notion.

For any choice of Reeb vector field and resulting Reeb flow $\varphi_t$, a reasonable Legendrian singularity $\Lambda \subset S^*M$ will admit a small $\epsilon > 0$ so that the Reeb flow displaces $\Lambda \cap \varphi_t(\Lambda) = \emptyset$, for all $0 < t < \epsilon$.

In other words, there will be no positive Reeb trajectories from $\Lambda$ to itself of length less than $\epsilon$.

We expect a deformation $\tilde{\Lambda} \subset S^*(M \times \mathbb{R})$ will be non-characteristic if there is a small $\epsilon > 0$ so that the Reeb flow displaces $\tilde{\Lambda}_s \cap \varphi_t(\tilde{\Lambda}_s) = \emptyset$, for all $0 < t < \epsilon$, uniformly in $s$.

In other words, there will be no positive Reeb trajectories from $\tilde{\Lambda}_s$ to itself of length less than $\epsilon$ for all parameters $s \in \mathbb{R}_{\geq 0}$. 
Example 1.5. Take $M = \mathbb{R}^2$ with coordinates $x, y$. Introduce the hypersurface

$$H = \{y(y - x^2)(y + x^2) = 0\} \subset \mathbb{R}^2$$

as pictured in Fig. 1. It is the homeomorphic wavefront projection of a Legendrian subvariety $\Lambda \subset S^*\mathbb{R}^2$ given by the closure of the restriction of the $dy$ codirection to the conormal of the smooth locus of $H$. As a topological space, the curve $H$, and hence the Legendrian subvariety $\Lambda$, is the union of three real lines all glued to each other at zero to form a six-valent node.

![Figure 1. Front projection of initial Legendrian $\Lambda \subset S^*\mathbb{R}^2$.](image)

We will describe two deformations of $\Lambda \subset S^*M$ to nearby Legendrian subvarieties with simpler singularities, but only the first will be a non-characteristic deformation.

1) For $s \geq 0$, consider the family of hypersurfaces

$$H_s = \{y = 0\} \cup \{x \geq s, (y - (x - s)^2)(y + (x - s)^2) = 0\} \cup \{x \leq 0, (y - x^2)(y + x^2) = 0\} \subset \mathbb{R}^2$$

pictured in Fig. 2. It is the homeomorphic wavefront projection of a non-characteristic family of Legendrian subvarieties $\Lambda_s \subset S^*\mathbb{R}^2$ given by the closure of the restriction of the $dy$ codirection to the conormal of the smooth locus of $H_s$. When $s > 0$, as a topological space, the curve $H_s$, and hence the Legendrian subvariety $\Lambda_s$, has two singularities which are four-valent nodes.

![Figure 2. Front projection of non-characteristic deformation.](image)

2) For $s \geq 0$, consider the family of hypersurfaces

$$H_s = \{y = 0\} \cup \{(y - (x - s)^2) = 0\} \cup \{(y + (x + s)^2) = 0\} \subset \mathbb{R}^2$$

pictured in Fig. 3. It is the homeomorphic wavefront projection of a family of Legendrian subvarieties $\Lambda_s \subset S^*\mathbb{R}^2$ given by the closure of the restriction of the $dy$ codirection to the conormal of the smooth locus of $H_s$. When $s > 0$, as a topological space, the curve $H_s$, and
hence the Legendrian subvariety $\Lambda_s$, has two singularities which are four-valent nodes. But the family is not non-characteristic: for any small $\epsilon > 0$, there is a small $s > 0$ so that there is a geodesic in $\mathbb{R}^2$, positive with respect to $dy$, of length less than $\epsilon$, from a point of $\{y+(x+s)^2 = 0\}$ to a point of $\{y-(x-s)^2 = 0\}$ and orthogonal to each.

Figure 3. Front projection of characteristic deformation.

1.2. Sketch of arguments. We briefly sketch here the idea behind the expansion algorithm of Theorem 1.1 in the case of one-dimensional Legendrian singularities. As topological spaces, one-dimensional Legendrian subvarieties are nothing more than graphs, and it is not difficult to understand their deformations. But let us use this special case to give a hint about the proof of the general case which is a somewhat intricate inductive pattern built out of similar arguments.

First, it suffices to take $M = \mathbb{R}^2$ with coordinates $x, y$. Any Legendrian singularity $\Lambda \subset S^*\mathbb{R}^2$ in generic position has wavefront projection a singular plane curve $C = \pi(\Lambda) \subset \mathbb{R}^2$ as pictured in Fig. 4. We may assume $C$ passes through the origin $0 \in \mathbb{R}^2$, is smooth away from 0, so that $\Lambda$ defines a coorientation of $C \setminus \{0\} \subset \mathbb{R}^2$, and the fiber at the origin $\Lambda|_0 \subset S^*o^2$ is the single codirection $dy$. With this setup, the wavefront projection from the Legendrian $\Lambda \subset S^*\mathbb{R}^2$ to the curve $C \subset \mathbb{R}^2$ is a homeomorphism. (A significant complication in higher dimensions is the fact that it is only possible to arrange for the wavefront projection to be a finite map.)

Figure 4. Initial front projection $C \subset \mathbb{R}^2$.

Next, consider the circle $S(r) \subset \mathbb{R}^2$ of a small radius $r > 0$ around the origin $0 \in \mathbb{R}^2$. Let us assume $S(r')$ is transverse to $C$, for all radii $0 < r' \leq r$. For a very small constant $d > 0$, introduce the closed subarc of the circle

$$E = S(r) \setminus \{(x, y) \in S(r) \mid y < 0, |x| < d\} \subset \mathbb{R}^2$$
Consider the closed ball $B(r) \subset \mathbb{R}^2$ of radius $r$ around the origin $0 \in \mathbb{R}^2$, and form a new curve given by the union

$$C_{\text{pre}} = (C \setminus (C \cap B(r))) \cup E \subset \mathbb{R}^2$$

as pictured in Fig. 5.

**Figure 5.** Intermediate curve $C_{\text{pre}} \subset \mathbb{R}^2$.

Observe that $C_{\text{pre}}$ is smooth away from the finitely many points of the intersection $C \cap S(r)$. Moreover, away from these points, it has a canonical coorientation given by $\Lambda$ along $C \setminus (C \cap B(r))$, and by the outward radial differential $dr$ along $E \setminus (C \cap S(r))$. Working locally near each of the points of $C \cap S(r)$, we can smooth $C_{\text{pre}} \subset \mathbb{R}^2$ to a new homeomorphic curve $C_{\text{arb}} \subset \mathbb{R}^2$, as pictured in Fig. 6 that has an unambiguous coorientation defined everywhere. Thus there is a corresponding Legendrian $\Lambda_{\text{arb}} \subset S^*\mathbb{R}$ with homeomorphic wavefront projection to $C_{\text{arb}} \subset \mathbb{R}^2$.

**Figure 6.** Final curve $C_{\text{arb}} \subset \mathbb{R}^2$.

Finally, the singularities of $C_{\text{arb}}$, and hence those of $\Lambda_{\text{arb}}$, are of two combinatorial types. First, in a neighborhood of the points of the intersection $C \cap S(r)$, the singularities are trivalent nodes. With the exception of smooth points, these are the simplest example of arboreal singularities. In a neighborhood of the boundary ends of the subarc

$$\partial E = S(r) \setminus \{(x, y) \in S(r) \mid y < 0, |x| = d\} \subset C_{\text{arb}}$$

we find univalent nodes. These are the simplest example of the generalized arboreal singularities discussed in Sect. 4 below. One might hope to find only trivalent nodes and not univalent nodes, but if the original Legendrian $\Lambda \subset S^*\mathbb{R}^2$, and hence curve $C \subset \mathbb{R}^2$, itself had a univalent node, it would be awkward to try to deform away a univalent node rather than accept it as a reasonable singularity.
1.3. Motivations. We summarize here some of the prior and ongoing work motivating the results this paper.

To start, there is a well established expectation that the Fukaya category of an exact symplectic manifold, specifically a Weinstein manifold, admits a description in terms of microlocal sheaves supported on a Lagrangian skeleton. (Alternatively, to more closely approach the setting of this paper, one can formulate both theories on the contactification of the exact symplectic manifold along with the Legendrian lift of the skeleton.) There are two primary variants of this expectation: the infinitesimal Fukaya category \cite{7, 26} of compact branes running along the skeleton corresponds to constructible microlocal sheaves, and the wrapped Fukaya category \cite{11} of non-compact branes transverse to the skeleton corresponds to wrapped microlocal sheaves (see \cite{23} for further discussion). It is also possible to formulate a unified version involving non-compact skeleta and the partially wrapped Fukaya category \cite{2}. The expectation can be realized in many settings, notably for cotangent bundles \cite{18, 24}.

The Fukaya category and microlocal sheaves each have their advantages and challenges. On the one hand, the Fukaya category is evidently a symplectic invariant, and in particular independent of the choice of a skeleton. Its objects have clear geometric appeal, though its morphisms involves challenging global analysis. On the other hand, microlocal sheaves form a sheaf along the skeleton, and enjoy powerful functoriality developed by Kashiwara-Schapira \cite{13}. Its objects are often quite abstract, and in particular involve choices of local polarizations.

Without a comprehensive theory that the Fukaya category and microlocal sheaves coincide, one is naturally led to the following questions.

First, given a skeleton, is there a sheaf of dg categories whose global sections is the Fukaya category? This was conjectured by Kontsevich \cite{16} and establishing instances of the resulting local-to-global gluing is an active theme in the subject.

Second, is there an elementary way to define and calculate microlocal sheaves along a skeleton? This has been the focus of a longstanding program pioneered by MacPherson (notably for microlocal perverse sheaves in the holomorphic symplectic setting), and advanced for example by Gelfand-MacPherson-Vilonen \cite{9}.

The current paper provides an answer to this second question in terms of the linear algebra of modules over quivers. It can be applied to reduce any calculation about microlocal sheaves to a finite amount of linear algebra. This was implemented in \cite{21} to establish a central instance of mirror symmetry where the skeleton is a three-dimensional singular thimble. Namely, a simple application of the expansion algorithm provides the main step in the proof of the following.

**Theorem 1.6** \cite{21}. Set $M = \mathbb{C}^3$ with its conic exact symplectic structure, and $\Lambda \subset M$ the Lagrangian cone over the standard Legendrian torus $T^2 \subset S^5$.

There are are equivalences of dg categories

$$\mu \text{Sh}_\Lambda(M) \simeq \text{Coh}_{\text{tors}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \quad \mu \text{Sh}^w_\Lambda(M) \simeq \text{Coh}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$$

where $\mu \text{Sh}_\Lambda(M)$ and $\mu \text{Sh}^w_\Lambda(M)$ denote respectively traditional and wrapped microlocal sheaves on $M$ supported along $\Lambda$, and $\text{Coh}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and $\text{Coh}_{\text{tors}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ denote the dg category of respectively coherent complexes and coherent complexes with proper support on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

In another direction, the results of this paper were invoked in \cite{23} to establish that traditional microlocal sheaves and wrapped microlocal sheaves are dual in a sense parallel to the duality of perfect complexes and coherent complexes \cite{3}, or more basically, functions/cohomology and distributions/homology. More precisely, it provided the central tool in the proof of the following, reducing the assertion to a simple observation about modules over trees.
Theorem 1.7 ([23]). Let \( \Omega \subset T^*X \) be a conic open subset, and \( \Lambda \subset T^*X \) a closed conic Lagrangian subvariety.

The natural hom-pairing provides an equivalence

\[
\mu \text{Sh}_\Lambda(\Omega) \sim \text{Fun}^{ex}(\mu \text{Sh}^w_\Lambda(\Omega)^{op}, \text{Perf}_k)
\]

where \( \mu \text{Sh}_\Lambda(\Omega) \) and \( \mu \text{Sh}^w_\Lambda(\Omega) \) denote respectively traditional and wrapped microlocal sheaves on \( \Omega \) supported along \( \Lambda \), \( \text{Fun}^{ex} \) denotes the dg category of exact functors, and \( \text{Perf}_k \) denote the dg category of perfect \( k \)-modules.

At a more fundamental level, the current paper also provides the basis for answering the first question, about sheafifying the Fukaya category along a skeleton, via showing the Fukaya category and microlocal sheaves coincide. Namely, in ongoing work with Eliashberg and Starkston [6], its techniques are applied to show that any Weinstein manifold admits an arboreal skeleton. From here, ongoing work of Ganatra-Pardon-Shende [8] shows that given an arboreal skeleton, the Fukaya category is equivalent to microlocal sheaves along it. Going further, we expect that arboreal skeleta will provide a higher dimensional notion of ribbon graphs from which one can recover the ambient symplectic manifold itself. These developments rest upon the fact that the singularities of skeleton can be deformed to arboreal singularities.

1.4. Summary of sections. Here is a brief summary of the specific contents of the sections of the paper. Sect. 2 summarizes standard material from singularity theory, in particular Whitney stratifications and their control data which provide the language for our geometric constructions. Sect. 3 summarizes the basic structure of wavefront projections in the form of directed hypersurfaces and positive coray bundles. Sect. 4 reviews the notion of arboreal singularities from [20], then extends their exposition to generalized arboreal singularities. Sect. 5 contains our main geometric constructions: it presents the expansion algorithm that takes a Legendrian singularity to an arboreal Legendrian subvariety. Sect. 6 contains our main technical arguments: it proves that the expansion algorithm is non-characteristic in the sense that the dg category of microlocal sheaves is invariant under it. Finally, a brief appendix summarizes the data that goes into the expansion algorithm, in particular the hierarchy of the chosen constants.

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2. Preliminaries

This section collects standard material on stratification theory following Mather [17].

We write \( \mathbb{R} \) for the real numbers, \( \mathbb{R}_{>0} \) for the positive real numbers, and \( \mathbb{R}_{\geq 0} \) for the non-negative real numbers. All manifolds will be smooth and equidimensional and all maps will be smooth unless otherwise stated.

2.1. Whitney stratifications. Let \( M \) be an manifold and \( X \subset M \) a closed subspace. A Whitney stratification of \( X \) is a disjoint decomposition

\[
X = \bigcup_{\alpha \in A} X_\alpha
\]

into submanifolds \( X_\alpha \subset M \) satisfying:

(Axiom of the frontier) If \( X_\alpha \cap \overline{X}_\beta \neq \emptyset \), then \( X_\alpha \subset \overline{X}_\beta \). 

(Local finiteness) Each point $x \in M$ has an open neighborhood $U \subset M$ such that $U \cap X_\alpha = \emptyset$ for all but finitely many $\alpha \in A$.

(Whitney’s condition B) If sequences $a_k \in X_\alpha$ and $b_k \in X_\beta$ converge to some $a \in X_\alpha$, and the sequence of secant lines $[a_k b_k]$ (with respect to a local coordinate system) and tangent planes $T_{b_k}X_\beta$ both converge, then $\lim [a_k b_k] \subset \lim T_{X_\beta}$.

The index set $A$ is naturally a poset with $\alpha < \beta$ when $X_\alpha \neq X_\beta$ and $X_\alpha \subset \overline{X}_\beta$.

We will say that $X$ has dimension $k$ if it is the closure of its strata of dimension $k$.

Remark 2.1. Note that we can trivially extend any Whitney stratification of $X \subset M$ to a Whitney stratification of all of $M$ by including the open complement $M \setminus \emptyset$ itself as a stratum.

Given a Whitney stratification of $X \subset M$, by a small open ball $B \subset M$ around a point $x \in M$, we will always mean an open ball of radius $R > 0$ with respect to the Euclidean metric of some local coordinate system. Moreover, we will assume that for any $0 < r \leq R$, the sphere around $x$ of radius $r$ is transverse to the strata. Whitney’s condition $B$ guarantees this holds for any local coordinate system and small enough $R > 0$.

2.2. Control data. Let $M$ be an manifold.

A tubular neighborhood of a submanifold $Y \subset M$ consists of an inner product on the normal bundle $\nu : E \to Y$, and a smooth embedding

$$\varphi : E[\epsilon] = \{ v \in E \mid \langle v, v \rangle < \epsilon \} \to M$$

of the open ball bundle determined by some $\epsilon > 0$. The image $T = \varphi(E[\epsilon])$ is required to be an open neighborhood of $Y \subset M$, and the restriction of $\varphi$ to the zero section $Y \subset E$ is required to be the identity map to $Y \subset M$. By rescaling the inner product, we can assume that $\epsilon = 1$.

By transport of structure, the neighborhood $T$ comes equipped with the tubular distance function $\rho : T \to \mathbb{R}_{\geq 0}$ and tubular projection $\pi : T \to Y$ defined by

$$\rho(x) = \langle \varphi^{-1}(x), \varphi^{-1}(x) \rangle \quad \pi(x) = \rho(\varphi^{-1}(x))$$

We will write $(T, \rho, \pi)$ to denote the tubular neighborhood and remember that $\pi : T \to Y$ is the open unit ball bundle in a vector bundle with inner product inducing $\rho : T \to \mathbb{R}_{\geq 0}$.

Given small $\epsilon \geq 0$, we have the inclusions

$$j[\epsilon] : S[\epsilon] = \{ x \in T \mid \rho(x) = \epsilon \} \to T \quad j[\epsilon] : T[\epsilon] = \{ x \in T \mid \rho(x) < \epsilon \} \to T$$

and similarly with $<$ replaced by $\leq$, $>$, or $\geq$. Of course when $\epsilon = 0$, we have $T[\leq \epsilon] = \emptyset$, $T[\leq \epsilon] = S[\epsilon] = Y$, $T[\geq \epsilon] = T \setminus Y$, $T[\geq \epsilon] = T$.

Any Whitney stratified subspace $X \subset M$ admits a compatible system of control data consisting of a tubular neighborhood $(T_\alpha, \rho_\alpha, \pi_\alpha)$ of each stratum $X_\alpha \subset X$. Whenever $\alpha < \beta$, the tubular distance functions and tubular projections are required to satisfy

$$\pi_\alpha(\pi_\beta(x)) = \pi_\beta(x) \quad \rho_\alpha(\pi_\beta(x)) = \rho_\alpha(x)$$

on the common domain of points $x \in T_\alpha \cap T_\beta$ such that $\pi_\beta(x) \in T_\alpha$.

A key property of a system of control data is the fact that the product map

$$\rho_\alpha \times \pi_\alpha : T_\alpha \to \mathbb{R}_{\geq 0} \times X_\alpha$$

has surjective differential when restricted to any stratum $X_\beta \subset X$ with $\beta > \alpha$. 
2.3. Almost retraction. Let $M$ be a manifold.

Let $X \subset M$ be a closed subspace with Whitney stratification $\{X_\alpha\}_{\alpha \in A}$.

Suppose given a compatible system of control data $\{(T_\alpha, \rho_\alpha, \pi_\alpha)\}_{\alpha \in A}$.

Following Goresky [10, 11], we review some further fundamental constructions.

Fix once and for all a small $\epsilon > 0$.

Choose a family of lines subordinate to the system of control data. This consists of a retraction

$$r_\alpha : T_\alpha[\epsilon] \setminus X_\alpha \longrightarrow S_\alpha[2\epsilon]$$

for each $\alpha \in A$ satisfying the following. For $\alpha < \beta$, one requires $r_\alpha|_{X_\beta}$ is smooth and the compatibilities

$$r_\alpha r_\beta = r_\beta r_\alpha \quad \rho_\alpha r_\beta = \rho_\alpha \quad \rho_\beta r_\alpha = \rho_\beta \quad \pi_\alpha r_\alpha = \pi_\alpha \quad \pi_\alpha r_\beta = \pi_\alpha$$

on their natural domains. The retractions provide homeomorphisms

$$h_\alpha : T_\alpha[\epsilon] \setminus X_\alpha \xrightarrow{\sim} S_\alpha[2\epsilon] \times (0, 2\epsilon) \quad h_\alpha = r_\alpha \times \rho_\alpha$$

and for $B \subset A$, more general collaring homeomorphisms

$$h_B : \bigcap_{\alpha \in B}(T_\alpha[\epsilon] \setminus X_\alpha) \xrightarrow{\sim} \left(\bigcap_{\alpha \in B} S_\alpha[2\epsilon]\right) \times \prod_{\alpha \in B}(0, 2\epsilon)$$

$$h_B = (r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_k}) \times \prod_{\alpha \in B} \rho_\alpha$$

where $k = |B|$ and the indices $\alpha_i \in B$ can be arbitrarily ordered thanks to $r_{\alpha_i} r_{\alpha_j} = r_{\alpha_j} r_{\alpha_i}$.

Fix a smooth nondecreasing function $q : \mathbb{R} \to \mathbb{R}$ such that $q(t) = 0$, for $t \leq \epsilon$, and $q(t) = t$, for $t \geq 2\epsilon$. For each stratum $X_\alpha \subset X$, introduce the mapping

$$\Pi_\alpha : M \longrightarrow M \quad \Pi_\alpha(x) = \left\{ \begin{array}{ll} x & \text{when } x \notin T_\alpha[\epsilon] \\ h_\alpha^{-1}(r_\alpha(x), q(\rho_\alpha(x))) & \text{when } x \in T_\alpha[\epsilon] \end{array} \right.$$  

It is continuous, homotopic to the identity, and satisfies $\Pi_\alpha(x) = \pi_\alpha(x)$ when $x \in T_\alpha[\epsilon]$. Moreover, the mappings commute $\Pi_\alpha \Pi_\beta = \Pi_\beta \Pi_\alpha$, for $\alpha, \beta \in A$. To confirm this, if $x \notin T_\alpha[2\epsilon]$, then $\Pi_\alpha(x) = x$, and $\Pi_\beta(x) \notin T_\alpha[\epsilon]$ since $\rho_\alpha r_\beta(x) = r_\alpha(x)$, so $\Pi_\alpha \Pi_\beta(x) = \Pi_\beta(x) = \Pi_\beta \Pi_\alpha(x)$. If $x \in T_\alpha[\epsilon] \cap T_\beta[\epsilon]$, then

$$\Pi_\alpha \Pi_\beta(x) = h^{-1}_{(\alpha, \beta)}(r_\alpha r_\beta(x), q(\rho_\alpha(x)), q(\rho_\beta(x))) = \Pi_\beta \Pi_\alpha(x)$$

Now introduce the composition

$$r : M \longrightarrow M \quad r = \Pi_{\alpha_0} \Pi_{\alpha_1} \cdots \Pi_{\alpha_N}$$

where $N + 1 = |A|$ and the indices $\alpha_i \in A$ can be arbitrarily ordered thanks to $\Pi_\alpha \Pi_{\alpha_j} = \Pi_{\alpha_j} \Pi_\alpha$. It is continuous, homotopic to the identity, and satisfies

$$r(x) = \pi_\alpha(x) \quad \text{when } x \in T_\alpha[\epsilon] \setminus \bigcup_{\beta < \alpha} (T_\alpha[\leq \epsilon] \cap T_\beta[\leq 2\epsilon])$$

The restriction of $r$ to the open subspace

$$U[\epsilon] = \bigcup_{\alpha \in A} T_\alpha[\leq \epsilon] \subset M$$

is almost a retraction of $U[\epsilon]$ to $X$ in that it maps $U[\epsilon]$ to $X$ and the restriction of $r$ to $X$ is homotopic to the identity. In fact, the restriction of $r$ to $X$ is the identity on any $x \in X$ whenever $x \in X_\alpha \setminus \bigcup_{\beta < \alpha} (X_\alpha \cap T_\beta[\leq 2\epsilon])$ for some $\alpha \in A$. 

Remark 2.2. The above constructions are well suited to inductive arguments. Fix a closed stratum $X_0 \subset X$, and set $M' = M \setminus X_0$, $X' = X \setminus X_0$. The system of control data and family of lines for $X \subset M$ immediately provide the same for $X' \subset M'$ by deleting the data for $X_0 \subset X$. The resulting almost retraction $r' : M' \to M'$ satisfies the following evident compatibility with the almost retraction $r : M \to M$. Introduce the composition

$$
\hat{r}_0 : M \longrightarrow M \quad \hat{r}_0 = \Pi_{\alpha_1} \cdots \Pi_{\alpha_N}
$$

so that $r = \Pi_0 \hat{r}_0$. Then $\hat{r}_0|_{X_0} = \text{id}_{X_0}$ and $\hat{r}_0|_{M'} = r'$.

Remark 2.3. By convention, when $X = \emptyset$, we set $r = \text{id}_M : M \to M$ to be the identity. This would naturally result from invoking the above constructions with the complement $M \setminus X \subset M$ itself as a stratum. This is a trivial modification since the tubular neighborhood of such an open stratum is simply itself.

2.4. Multi-transversality. Let $M$ be a manifold.

We say that a finite set $\mathcal{F} = \{f_i\}_{i \in I}$ of functions $f_i : M \to \mathbb{R}$ is multi-transverse at a value $s_I = (s_i) \in \mathbb{R}^I$ if for any subset $J \subset I$, the product map

$$
F_J = \prod_{j \in J} f_j : M \longrightarrow \mathbb{R}^J
$$

is a submersion along $F_J^{-1}(s_J) \subset M$ where $s_J = (s_j) \in \mathbb{R}^J$ is the image of $s_I = (s_i) \in \mathbb{R}^I$ under the natural projection $\pi_J : \mathbb{R}^I \to \mathbb{R}^J$.

If $\mathcal{F} = \{f_i\}_{i \in I}$ is multi-transverse at $s_I = (s_i) \in \mathbb{R}^I$, then the level-sets

$$
H_i(s_i) = f_i^{-1}(s_i) \subset M
$$

are smooth hypersurfaces (if non-empty) and multi-transverse in the following sense. For any subset $J \subset I$, the intersection

$$
H_J(s_J) = \bigcap_{j \in J} H_j(s_j) \subset M
$$

is a smooth submanifold of codimension $|J|$ (if non-empty), and transverse to $H_i(s_i) \subset M$, for each $i \in I \setminus J$.

Suppose a finite set $\mathcal{F} = \{f_i\}_{i \in I}$ of functions $f_i : M \to \mathbb{R}$ is multi-transverse at a value $s_I = (s_i) \in \mathbb{R}^I$. Then given another function $f : M \to \mathbb{R}$ and a value $s \in \mathbb{R}$, we may find a nearby value $s' \in \mathbb{R}$ so that the extended set $\{f\} \coprod \{f_i\}_{i \in I}$ of functions is multi-transverse at the extended value $(s', s_i) \in \mathbb{R} \times \mathbb{R}^I$. This follows from Sard’s Theorem: there is a nearby regular value $s' \in \mathbb{R}$ for the function

$$
\coprod_{I \subset I} f|_{H_J(s_J)} : \coprod_{I \subset I} H_J(s_J) \longrightarrow \mathbb{R}
$$

Thus given any finite set $\mathcal{F} = \{f_i\}_{i \in I}$ of functions $f_i : M \to \mathbb{R}$ and a value $s = (s_i) \in \mathbb{R}^I$, by induction on any order of $I$, there is a nearby value $s' = (s'_i) \in \mathbb{R}^I$ such that $\mathcal{F} = \{f_i\}_{i \in I}$ is multi-transverse at $s' = (s'_i) \in \mathbb{R}^I$.

Example 2.4. (1) Let $M = \mathbb{R}$ with coordinate $x$. Set $f_1 = f_2 = x$. Then $\mathcal{F} = \{f_1, f_2\}$ is multi-transverse at $(s_1, s_2) \in \mathbb{R}^2$ if and only if $s_1 \neq s_2$.

(2) Let $M = \mathbb{R}^2$ with coordinates $x_1, x_2$. Set $f_1 = x_1$, $f_2 = x_2$, and $f_3 = x_1 + x_2$. Then $\mathcal{F} = \{f_1, f_2, f_3\}$ is multi-transverse at $(s_1, s_2, s_3) \in \mathbb{R}^3$ if and only if $s_3 \neq s_1 + s_2$. 
More generally, suppose given a set $\mathfrak{F} = \{f_i\}_{i \in I}$ of functions $f_i : U_i \to \mathbb{R}$ defined on a locally finite set $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets $U_i \subset M$. We will say that such a set $\mathfrak{F} = \{f_i\}_{i \in I}$ is multi-transverse at a value $s_I = (s_i) \in \mathbb{R}^I$ if for any finite subset $J \subset I$ the product map

$$F_J = \prod_{j \in J} f_i : \bigcap_{j \in J} U_j \longrightarrow \mathbb{R}^J$$

is a submersion along $F_J^{-1}(s_J) \subset M$. Note that if $I$ is finite, and $U_i = M$, for all $i \in I$, then we recover the previous notion.

For a key example of such a multi-transverse set of functions, consider a closed subspace $X \subset M$ with Whitney stratification $\{X_\alpha\}_{\alpha \in A}$. For any compatible system of control data $\{(T_\alpha, \rho_\alpha, \sigma_\alpha)\}_{\alpha \in A}$, the tubular distance functions $\mathfrak{F} = \{\rho_\alpha\}_{\alpha \in A}$ defined on the tubular neighborhoods $\mathcal{U} = \{T_\alpha\}_{\alpha \in A}$ are multi-transverse at any completely nonzero value $s_A = (s_\alpha) \in \mathbb{R}^A_0$. Moreover, for any stratum $X_\beta \subset X$, the restrictions $\mathfrak{F}_\beta = \{\rho_\alpha|_{X_\beta}\}_{\alpha < \beta}$ defined on the intersections $\mathcal{U}_\beta = \{T_\alpha \cap X_\beta\}_{\alpha < \beta}$ are multi-transverse at any completely nonzero value.

3. Directed hypersurfaces

3.1. Notation. Let $M$ be a manifold.

Let $T^*M$ denote its cotangent bundle, and $\theta \in \Omega^1(T^*M)$ the canonical one-form. We will identify $M$ with the zero-section of $T^*M$.

Introduce the spherical projectivization

$$S^*M = (T^*M \setminus M)/\mathbb{R}_{>0}$$

If we choose a Riemannian metric on $M$, we can canonically identify $S^*M$ with the unit cosphere bundle

$$U^*M = \{v \in T^*M \mid \|v\| = 1\}$$

The canonical one-form $\theta \in \Omega^1(T^*M)$ restricts to equip $U^*M$ and hence $S^*M$ with a contact form $\alpha \in \Omega^1(S^*M)$ (depending on the metric) and a canonical contact structure $\xi = \ker(\alpha) \subset TS^*M$ (independent of the choice of metric).

Introduce the projectivization

$$P^*M = (T^*M \setminus M)/\mathbb{R}^\times$$

We have the natural two-fold cover $S^*M \to P^*M$ which in particular equips $P^*M$ with a compatible canonical contact structure (though not a contact form).

Given a submanifold $Y \subset M$, we have its conormal bundle, its spherical projectivization, and its projectivization respectively

$$T_Y^*M \subset T^*M \quad S_Y^*M \subset S^*M \quad P_Y^*M \subset P^*M$$

The first is a conical Lagrangian submanifold and the latter two are Legendrian submanifolds.

3.2. Good position.

**Definition 3.1.** By a hypersurface $H \subset M$, we will mean a subspace admitting a Whitney stratification with $\dim H = \dim M - 1$.

Given a hypersurface $H \subset M$, and any open, dense smooth locus $H^{sm} \subset H$, we have a natural diagram of maps

$$S_{H^{sm}}M \longrightarrow P_{H^{sm}}M \longrightarrow H^{sm}$$

where the first is a two-fold cover and the second is a diffeomorphism.
**Definition 3.2.** A hypersurface \( H \subset M \) is said to be in *good position* if for some (or equivalently any) open, dense smooth locus \( H^{sm} \subset H \), the closure
\[
\mathcal{L} = P_{H^{sm}}^*M \subset P^*M,
\]
is finite over \( H \). If this holds, we refer to \( \mathcal{L} \) as the *coline bundle* of \( H \).

**Remark 3.3.** Equivalently, \( H \subset M \) is in good position if the closure
\[
\mathcal{R} = S_{H^{sm}}^*M \subset S^*M
\]
is finite over \( H \). If so, we refer to \( \mathcal{R} \) as the *coray bundle* of \( H \).

**Remark 3.4.** If \( H \subset M \) is in good position, we have a natural diagram of finite maps
\[
\mathcal{R} \longrightarrow \mathcal{L} \longrightarrow H
\]
where the first is a two-fold cover and the second is a diffeomorphism over \( H^{sm} \subset H \).

**Example 3.5.** (1) All real algebraic or subanalytic plane curves are in good position.
(2) Nondegenerate quadratic singularities (singular Morse level-sets) of dimension strictly greater than one are not in good position.

**Remark 3.6.** If \( H \subset M \) is in good position, then Whitney’s condition \( B \) (in fact Whitney’s condition \( A \)) implies its coline bundle \( \mathcal{L} \) and coray bundle \( \mathcal{R} \) are conormal to each stratum \( H_\alpha \subset H \) in the sense that
\[
\mathcal{L}|_{H_\alpha} \subset P_{H_\alpha}^*M \quad \mathcal{R}|_{H_\alpha} \subset S_{H_\alpha}^*M
\]

### 3.3. Coorientation.

**Definition 3.7.** By a *coorientation* of a hypersurface \( H \subset M \) in good position, we will mean a section
\[
\mathcal{R} \overset{\sigma}{\longrightarrow} \mathcal{L}
\]
of the natural two-fold cover from the coray to coline bundle.

**Definition 3.8.** (1) By a *directed hypersurface* inside of \( M \), we will mean a hypersurface \( H \subset M \) in good position equipped with a coorientation \( \sigma \).
(2) By the *positive coray bundle* of a directed hypersurface, we will mean the image of the coline bundle under the coorientation
\[
\Lambda = \sigma(\mathcal{L}) \subset S^*M
\]

### 4. Arboreal Singularities

We recall and expand upon the local notion of arboreal singularity from [20].

#### 4.1. Terminology.

We gather here for easy reference some language used below.

By a *graph* \( G \), we will mean a set of *vertices* \( V(G) \) and a set of *edges* \( E(G) \) satisfying the simplest convention that \( E(G) \) is a subset of the set of two-element subsets of \( V(G) \). Thus \( E(G) \) records whether pairs of distinct elements of \( V(G) \) are connected by an edge or not. We will write \( \{\alpha, \beta\} \in E(G) \) and say that \( \alpha, \beta \in V(T) \) are *adjacent* if an edge connects them.

By a *tree* \( T \), we will mean a nonempty, finite, connected, acyclic graph. Thus for any pair of vertices \( \alpha, \beta \in V(T) \), there is a unique minimal path (nonrepeating sequence of edges) connecting them. We call the number of edges in the sequence the *distance* between the vertices.

Given a graph \( G \), by a *subgraph* \( S \subset G \), we will mean a full subgraph (or vertex-induced subgraph) in the sense that its vertices are a subset \( V(S) \subset V(G) \) and its edges are the subset \( E(S) \subset E(G) \) such that \( \{\alpha, \beta\} \in E(S) \) if and only if \( \{\alpha, \beta\} \in E(G) \) and \( \alpha, \beta \in V(S) \). By
the complementary subgraph $G \setminus S \subset G$, we will mean the full subgraph on the complementary vertices $V(T \setminus S) = V(T) \setminus V(S)$.

Given a tree $T$, any subgraph $S \subset T$ is a disjoint union of trees. By a subtree $S \subset T$, we will mean a subgraph that is a tree. The complementary subgraph $T \setminus S \subset T$ is not necessarily a tree but in general a disjoint union of subtrees.

Given a tree $T$, by a quotient tree $T \twoheadrightarrow Q$, we will mean a tree $Q$ with a surjection $V(T) \twoheadrightarrow V(Q)$ such that each fiber comprises the vertices of a subtree of $T$. We will refer to such subtrees as the fibers of the quotient $T \twoheadrightarrow Q$.

By a partition of a tree $T$, we will mean a collection of subtrees $T_i \in T$, for $i \in I$, that are disjoint $V(T_i) \cap V(T_j) = \emptyset$, for $i \neq j$, and cover $V(T) = \bigsqcup_{i \in I} V(T_i)$. Note that the data of a quotient $T \twoheadrightarrow Q$ is equivalent to the partition of $T$ into the fibers.

By a rooted tree $T = (T, \rho)$, we will mean a tree $T$ equipped with a distinguished vertex $\rho \in V(T)$ called the root vertex. The vertices $V(T)$ of a rooted tree naturally form a poset with the root vertex $\rho \in V(T)$ the unique minimum and $\alpha < \beta \in V(T)$ if the former is nearer to $\rho$ than the latter. To each non-root vertex $\alpha \neq \rho \in V(T)$ there is a unique parent vertex $\hat{\alpha} \in V(T)$ such that $\hat{\alpha} < \alpha$ and there are no vertices strictly between them. The data of the root vertex $\rho$ and parent vertex relation $\alpha \mapsto \hat{\alpha}$ recover the poset structure and in turn the rooted tree.

By a forest $F$, we will mean a nonempty, finite, possibly disconnected graph with acyclic connected components. Thus $F = \bigsqcup_{i} T_i$ is a nonempty disjoint union of finitely many trees.

By a rooted forest $F$, we will mean a forest $F$ equipped with a distinguished root vertex in each of its connected components. Thus $F = \bigsqcup_{i} T_i = \bigsqcup_{i} (T_i, \rho_i)$ is a nonempty disjoint union of finitely many rooted trees. The vertices $V(F)$ of a rooted forest naturally form a poset with minima the root vertices and vertices in distinct connected components incomparable.

4.2. Arboreal singularities. To each tree $T$, there is associated a stratified space $L_T$ called an arboreal singularity (see [20] and in particular the characterization recalled in Thm. 4.1 below). It is of pure dimension $|T| - 1$ where we write $|T|$ for the number of vertices of $T$. It comes equipped with a compatible metric and contracting $\mathbb{R}_{>0}$-action with a single fixed point. We refer to the compact subspace $L_{T, \text{link}} \subset L_T$ of points unit distance from the fixed point as the arboreal link. The $\mathbb{R}_{>0}$-action provides a canonical identification

$$L_T \simeq \text{Cone}(L_{T, \text{link}})$$

so that one can regard the arboreal singularity $L_T$ and arboreal link $L_{T, \text{link}}$ as respective local models for a normal slice and normal link to a stratum in a stratified space. It follows easily from the constructions that the arboreal link $L_{T, \text{link}}$ is homotopy equivalent to a bouquet of $|T|$ spheres each of dimension $|T| - 1$.

As a stratified space, the arboreal link $L_{T, \text{link}}$, and hence the arboreal singularity $L_T$ as well, admits a simple combinatorial description. To each tree $T$, there is a natural finite poset $\mathcal{P}_T$ whose elements are correspondences of trees

$$p = (R \leftarrow^q S \xrightarrow{i} T)$$

where $i$ is the inclusion of a subtree and $q$ is a quotient of trees. Thus the tree $S$ is the full subgraph (or vertex-induced subgraph) on a subset of vertices of $T$; the tree $R$ results from contracting a subset of edges of $S$. Two such correspondences

$$p = (R \leftarrow^q S \xrightarrow{i} T) \quad p' = (R' \leftarrow^{q'} S' \xrightarrow{i'} T')$$
satisfy \( p \geq p' \) if there is another correspondence of the same form

\[
q = (R \leftarrow Q \rightarrow R')
\]
such that \( p = q \circ p' \). In particular, the poset \( \mathcal{P}_T \) contains a unique minimum representing the identity correspondence

\[
p_0 = (T \leftarrow T \rightarrow T)
\]

Recall that a **finite regular cell complex** is a Hausdorff space \( X \) with a finite collection of closed cells \( c_i \subset X \) whose interiors \( c_i^0 \subset c_i \) provide a partition of \( X \) and boundaries \( \partial c_i \subset X \) are unions of cells. A finite regular cell complex \( X \) has the **intersection property** if the intersection of any two cells \( c_i, c_j \subset X \) is either another cell or empty. The **face poset** of a finite regular cell complex \( X \) is the poset with elements the cells of \( X \) with relation \( c_i \leq c_j \) whenever \( c_i \subset c_j \). The **order complex** of a poset is the natural simplicial complex with simplices the finite totally-ordered chains of the poset.

**Theorem 4.1** (\[20\]). Let \( T \) be a tree.

The arboreal link \( L^{\text{link}}_T \) is a finite regular cell complex, with the intersection property, with face poset \( \mathcal{P}_T \setminus \{p_0\} \), and thus homeomorphic to the order complex of \( \mathcal{P}_T \setminus \{p_0\} \).

**Remark 4.2.** It follows that the normal slice to the stratum \( L_T(p) \subset L_T \) indexed by a partition

\[
p = (R \leftarrow S \rightarrow T)
\]
is homeomorphic to the arboreal singularity \( L_R \).

**Example 4.3.** Let us highlight the simplest class of trees.

When \( T \) consists of a single vertex, \( L_T \) is a single point.

When \( T \) consists of two vertices \( v_1, v_2 \) (necessarily connected by an edge), \( L_T \) is the local trivalent graph given by the cone over the three distinct points \( L^{\text{link}}_T \) representing the three correspondences

\[
(\{v_1\} \leftarrow \{v_1\} \rightarrow T) \quad (\{v_2\} \leftarrow \{v_2\} \rightarrow T) \quad (\{v\} \leftarrow T \rightarrow T)
\]

More generally, consider the class of \( A_n \)-trees \( T_n \) consisting of \( n \) vertices connected by \( n - 1 \) successive edges. The associated arboreal singularity \( L_{T_n} \) admits an identification with the cone of the \( (n - 2) \)-skeleton of the \( n \)-simplex

\[
L^{\text{link}}_{T_n} \simeq cone(sk_{n-2} \Delta^n)
\]
or in a dual realization, the \( (n - 1) \)-skeleton of the polar fan of the \( n \)-simplex.

4.3. **Arboreal hypersurfaces.** The basic notions and results about arboreal hypersurfaces from \[20\] generalize immediately from trees to forests. We will review this material in this generality and only comment where there is any slight deviation from the presentation of \[20\].

On the one hand, by convention, given a forest \( F = \coprod_i T_i \), we set the corresponding arboreal space to be the disjoint union of products of arboreal singularities with Euclidean spaces

\[
L_F = \coprod_i (L_{T_i} \times \mathbb{R}^{F \setminus T_i})
\]

where \( \mathbb{R}^{F \setminus T_i} \) denotes the Euclidean space of real tuples

\[
\{x_\gamma\}, \text{ with } \gamma \in V(F) \setminus V(T_i).
\]
or in other words, the Euclidean space of functions

\[
\{x_\gamma\} : V(F) \setminus V(T_i) \longrightarrow \mathbb{R}
\]
On the other hand, we can repeat the constructions of [20] for arboreal hypersurfaces starting from a rooted forest. Throughout the brief summary that follows, fix once and for all a rooted forest $\mathcal{F}$ which we can express as a disjoint union of rooted trees $\mathcal{F} = \coprod_i T_i = \coprod_i (T_i, \rho_i)$.

4.3.1. Rectilinear version. Let us write $\mathbb{R}^F$ for the Euclidean space of real tuples $\{x_\gamma\}$, with $\gamma \in V(\mathcal{F})$

or in other words, the Euclidean space of functions $\{x_\gamma\} : V(\mathcal{F}) \to \mathbb{R}$ so that we have the evident identity

$$\mathbb{R}^F = \prod_i \mathbb{R}^{T_i}.$$ 

**Definition 4.4.** Fix a vertex $\alpha \in V(\mathcal{F})$.

1. Define the quadrant $Q_\alpha \subset \mathbb{R}^F$ to be the closed subspace $Q_\alpha = \{x_\beta \geq 0 \text{ for all } \beta \leq \alpha\}$

2. Define the hypersurface $H_\alpha \subset \mathbb{R}^F$ to be the boundary $H_\alpha = \partial Q_\alpha = \{x_\beta \geq 0 \text{ for all } \beta \leq \alpha, \text{ and } x_\gamma = 0 \text{ for some } \gamma \leq \alpha\}$

**Remark 4.5.** Note that the hypersurface $H_\alpha \subset \mathbb{R}^F$ is homeomorphic (in a piecewise linear fashion) to a Euclidean space of dimension $|V(\mathcal{F})| - 1$.

**Definition 4.6.** The rectilinear arboreal hypersurface $H_\mathcal{F}$ associated to a rooted forest $\mathcal{F}$ is the union of hypersurfaces

$$H_\mathcal{F} = \bigcup_{\alpha \in V(\mathcal{F})} H_\alpha \subset \mathbb{R}^F$$

The rectilinear arboreal hypersurface admits the following less redundant presentations. Introduce the subspaces

$P_\alpha = \{x_\alpha = 0, x_\beta \geq 0 \text{ for all } \beta < \alpha\} \subset \mathbb{R}^F$ \quad $P_\alpha^0 = \{x_\alpha = 0, x_\beta > 0 \text{ for all } \beta < \alpha\} \subset \mathbb{R}^F$

**Lemma 4.7.**

$$H_\mathcal{F} = \bigcup_{\alpha \in V(\mathcal{F})} P_\alpha \subset \mathbb{R}^F \quad H_\mathcal{F} = \bigcup_{\alpha \in V(\mathcal{F})} P_\alpha^0 \subset \mathbb{R}^F$$

**Proof.** For the first identity, if $p \in P_\alpha$, then $x_\alpha(p) = 0$, $x_\beta(p) \geq 0$ for all $\beta < \alpha$, and hence $p \in H_\alpha$. Conversely, if $p \in H_\alpha$, then $x_\alpha(p) = 0$ for some $\gamma \leq \alpha$, and $x_\beta(p) \geq 0$ for all $\beta \leq \alpha$, in particular $x_\beta(p) \geq 0$ for all $\beta \leq \gamma$, and hence $p \in P_\beta$.

To see the second identity, clearly $P_\alpha^0 \subset P_\alpha$, and observe that if $p \in P_\alpha \setminus P_\alpha^0$, then $x_\beta(p) = 0$, for some $\beta < \alpha$, and if we take the minimum such $\beta$, then we have $p \in P_\beta^0$. \hfill \square

**Remark 4.8.** Introduce the inverse images under the natural projections

$$H_{F_i} = \pi_i^{-1}(H_{T_i}) \subset \mathbb{R}^F \quad \pi_i : \mathbb{R}^F = \coprod_i \mathbb{R}^{T_i} \to \mathbb{R}^{T_i}$$

Then we have the evident identities

$$H_{F_i} \simeq H_{T_i} \times \mathbb{R}^F \setminus T_i \quad H_\mathcal{F} = \bigcup_i H_{F_i}$$

Moreover, the inverse images $H_{F_i}$ are multi-transverse hypersurfaces being the inverse images of complementary projections.
4.3.2. Smoothed version. We recall here the smoothed version of arboreal hypersurfaces. We recall in the next section that the smoothed and rectilinear versions are homeomorphic as embedded hypersurfaces inside of Euclidean space.

Fix once and for all a small $\delta > 0$.

All of our constructions will depend on the choice of three functions denoted by

\[ b : \mathbb{R} \rightarrow \mathbb{R} \quad f : \mathbb{R}^2 \rightarrow \mathbb{R} \quad c : \mathbb{R} \rightarrow \mathbb{R} \]

the first two of which we will select now.

Choose a continuous function $b : \mathbb{R} \rightarrow \mathbb{R}$, smooth away from $0 \in \mathbb{R}$, with the properties:

1. $|b(t)| < \delta/4$, for all $t \in \mathbb{R}$.
2. $b(t) = 0$ outside of the interval $0 < t < \delta/4$.
3. $\lim_{t \to 0^+} b'(t) = -\infty$.

Choose a continuously differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with the properties:

1. $f$ is a submersion.
2. $\{ f(x_1, x_2) = 0 \} = \{ x_1 = 0, x_2 \geq 0 \} \cup \{ x_1 > 0, x_2 = b(x_1) \}$.
3. $f(x_1, x_2) = x_2$ over $\{ x_1 > 2\delta, |x_2| < \delta \}$.
4. $f(x_1, x_2) = x_1$ over $\{ |x_1| < \delta, x_2 > 2\delta \}$.
5. $f(x_1, x_2) < \delta$ implies $x_1 < \delta$ or $x_2 < \delta$.

**Remark 4.9.** If preferred, one can fix some $N \geq 1$, and arrange that $\lim_{t \to 0^+} b^{(k)}(t) = -\infty$, for all $1 \leq k \leq N$. Then one can choose $f$ to be correspondingly highly differentiable. One can also take $N = \infty$ and then choose $f$ to be smooth.

**Definition 4.10.** (1) For a root vertex $\rho \in V(F)$, set

\[ h_\rho = x_\rho : \mathbb{R}^F \rightarrow \mathbb{R} \]

(2) For a non-root vertex $\alpha \in V(F)$, inductively define

\[ h_\alpha : \mathbb{R}^F \rightarrow \mathbb{R} \quad h_\alpha = f(h_\hat{\alpha}, x_\alpha) \]

where $\hat{\alpha} \in V(F)$ is the parent vertex of $\alpha$.

**Remark 4.11.** For all $\alpha \in V(F)$, note that:

1. $h_\alpha$ is a submersion.
2. $h_\alpha$ depends only on the coordinates $x_\beta$, for $\beta \leq \alpha$.
3. $h_\alpha \geq 0$ implies $h_\beta \geq 0$, for $\beta \leq \alpha$.

**Definition 4.12.** Fix a vertex $\alpha \in V(F)$.

1. Define the halfspace $Q_\alpha \subset \mathbb{R}^F$ to be the closed subspace

\[ Q_\alpha = \{ h_\alpha \geq 0 \} \]

2. Define the hypersurface $H_\alpha \subset \mathbb{R}^F$ to be the zero-locus

\[ H_\alpha = \{ h_\alpha = 0 \} \]

**Definition 4.13.** The smoothed arboreal hypersurface $H_F$ associated to a rooted forest $F$ is the union of hypersurfaces

\[ H_F = \bigcup_{\alpha \in V(F)} H_\alpha \subset \mathbb{R}^F \]
Remark 4.14. Introduce the subspaces
\[ P_\alpha = \{ h_\alpha = 0, h_\hat{\alpha} \geq 0 \} \subset \mathbb{R}^F \quad P^\circ_\alpha = \{ h_\alpha = 0, h_\hat{\alpha} > 0 \} \subset \mathbb{R}^F \]
where \( \hat{\alpha} \in V(\mathcal{F}) \) is the parent vertex of \( \alpha \). Then the smoothed arboreal hypersurface admits the less redundant presentations
\[ H_\mathcal{F} = \bigcup_{\alpha \in V(\mathcal{F})} P_\alpha \subset \mathbb{R}^F \quad H^\circ_\mathcal{F} = \bigcup_{\alpha \in V(\mathcal{F})} P^\circ_\alpha \subset \mathbb{R}^F \]

Remark 4.15. Introduce the inverse images under the natural projections
\[ H_{\mathcal{F},i} = \pi_i^{-1}(H_{T_i}) \subset \mathbb{R}^F \quad \pi_i : \mathbb{R}^F = \prod_i \mathbb{R}^{T_i} \rightarrow \mathbb{R}^{T_i} \]
Then we have the evident identities
\[ H_{\mathcal{F},i} \simeq H_{T_i} \times \mathbb{R}^{F \setminus T_i} \quad H_\mathcal{F} = \bigcup_i H_{\mathcal{F},i} \]
Moreover, the inverse images \( H_{\mathcal{F},i} \) are multi-transverse hypersurfaces being the inverse images of complementary projections.

4.3.3. Comparison. We recall here that the rectilinear and smoothed arboreal hypersurfaces are homeomorphic as embedded hypersurfaces inside of Euclidean space.

Choose a smooth bump function \( c : \mathbb{R} \rightarrow [0, 1] \) with the properties:
\[
(1) \quad c(t) = 0 \text{ outside the interval } \{ |t| \leq \delta \}.
(2) \quad c(t) = 1 \text{ inside the interval } \{ |t| \leq \delta/2 \}.
\]
Using the functions \( b, c : \mathbb{R} \rightarrow \mathbb{R} \), introduce the vector field
\[ v = -b(x_1) c(x_2) \partial_{x_2} \in \text{Vect}(\mathbb{R}^2) \]
Observe that \( v \) is smooth except along the axis \( \{(0, x_2) \mid x_2 \in \mathbb{R} \} \subset \mathbb{R}^2 \) and satisfies:
\[
(1) \quad v = 0, \text{ outside the rectangle } \{ 0 \leq x_1 \leq \delta/4, |x_2| \leq \delta \}.
(2) \quad v = -b(x_1) \partial_{x_2}, \text{ inside the domain } \{ |x_2| \leq \delta/2 \}.
\]
Define the homeomorphism \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) to be the unit-time flow of the vector field \( v \).
Observe that \( \Phi \) is smooth except along the axis \( \{(0, x_2) \mid x_2 \in \mathbb{R} \} \subset \mathbb{R}^2 \) and satisfies:
\[
(1) \quad \Phi(x_1, x_2) = (x_1, x_2), \text{ outside the rectangle } \{ 0 \leq x_1 \leq \delta/4, |x_2| \leq \delta \}.
(2) \quad \Phi(x_1, x_2) = (x_1, x_2 - b(x_1)), \text{ inside the domain } \{ |x_2| \leq \delta/4 \}.
(3) \quad \text{For any fixed } a_1 \in \mathbb{R}, \text{ the restriction } \Phi|_{x_1 = a_1} : \mathbb{R} \rightarrow \mathbb{R}^2 \text{ is smooth.}
\]
The second property follows from the fact that \( |b(x_1)| < \delta/4 \), and \( c(x_2) = 1 \) when \( |x_2| \leq \delta/2 \), hence for less than or equal to unit-time, the flow of \( v = -b(x_1) c(x_2) \partial_{x_2} \) starting from inside the domain \( \{|x_2| \leq \delta/4\} \) stays inside the domain \( \{|x_2| \leq \delta/2\} \).
Introduce the continuous function \( \varphi = x_2 \circ \Phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by the second coordinate of \( \Phi \).
Observe that \( \varphi \) is smooth except along the axis \( \{(0, x_2) \mid x_2 \in \mathbb{R} \} \subset \mathbb{R}^2 \) and satisfies:
\[
(1) \quad \varphi(x_1, x_2) = x_2, \text{ outside the rectangle } \{ 0 \leq x_1 \leq \delta/4, |x_2| \leq \delta \}.
(2) \quad \varphi(x_1, x_2) = x_2 - b(x_1), \text{ inside the domain } \{ |x_2| \leq \delta/4 \}.
(3) \quad \text{For any fixed } a_1 \in \mathbb{R}, \text{ the restriction } \varphi|_{x_1 = a_1} : \mathbb{R} \rightarrow \mathbb{R} \text{ is a diffeomorphism.}
\]

Definition 4.16. (1) For a root vertex \( \rho \in V(\mathcal{F}) \), set
\[ F_\rho : \mathbb{R}^F \longrightarrow \mathbb{R} \quad F_\rho = x_\rho \]
(2) For a non-root vertex \( \alpha \in V(\mathcal{F}) \), set
\[ F_\alpha : \mathbb{R}^F \longrightarrow \mathbb{R} \quad F_\alpha = \varphi(h_\hat{\alpha}, x_\alpha) \]
where \( \hat{\alpha} \in V(\mathcal{F}) \) is the unique parent of \( \alpha \).

(3) Define the continuous map

\[
F_{\mathcal{F}} : \mathbb{R}^\mathcal{F} \longrightarrow \mathbb{R}^\mathcal{F} \quad F_{\mathcal{F}} = \{ F_\alpha \}
\]

In other words, the coordinates of \( F_{\mathcal{F}} \) are given by \( x_\alpha \circ F_{\mathcal{F}} = F_\alpha \).

**Remark 4.17.** Note that \( F_\alpha \) depends only on the coordinates \( x_\beta \), for \( \beta \leq \alpha \).

The map \( F_{\mathcal{F}} \) is evidently the product of maps

\[
F_{\mathcal{F}} = \prod_i F_{T_i} : \prod_i \mathbb{R}^{T_i} \longrightarrow \prod_i \mathbb{R}^{T_i}
\]

Consequently, the analogous result for trees from [20] immediately implies the following extension to forests.

**Theorem 4.18.** The map \( F_{\mathcal{F}} : \mathbb{R}^\mathcal{F} \longrightarrow \mathbb{R}^\mathcal{F} \) is a homeomorphism and satisfies \( F_{\mathcal{F}}(H_\mathcal{F}) = H_\mathcal{F} \), and in fact \( F_{\mathcal{F}}(Q_\alpha) = Q_\alpha \), \( F_{\mathcal{F}}(H_\alpha) = H_\alpha \), for all \( \alpha \in V(\mathcal{F}) \).

**Remark 4.19.** It follows that we also have \( F_{\mathcal{F}}(P_\alpha) = P_\alpha \), \( F_{\mathcal{F}}(P_\alpha^0) = P_\alpha^0 \), for all \( \alpha \in V(\mathcal{F}) \).

**Remark 4.20.** For \( \alpha \in V(\mathcal{F}) \), introduce the continuous map

\[
\tilde{F}_\alpha : \mathbb{R}^\mathcal{F} \longrightarrow \mathbb{R}^\mathcal{F}
\]

with coordinates given by

\[
x_\beta \circ \tilde{F}_\alpha = \begin{cases} 
F_\alpha, & \beta = \alpha \\
 x_\beta, & \beta \neq \alpha 
\end{cases}
\]

Fix a total order on \( V(\mathcal{F}) \) compatible with its natural partial order. Write \( \alpha_1, \ldots, \alpha_{n+1} \in V(\mathcal{F}) \) for the ordered vertices. Observe that \( F_{\mathcal{F}} \) factors as the composition

\[
F_{\mathcal{F}} = \tilde{F}_{\alpha_1} \circ \cdots \circ \tilde{F}_{\alpha_{n+1}}
\]

In particular, since \( F_{\mathcal{F}} \) is a homeomorphism, each \( \tilde{F}_\alpha \) is itself a homeomorphism.

More precisely, observe that for \( \rho \) a root vertex, \( \tilde{F}_\rho \) is the identity, and for \( \alpha \) not a root vertex, each \( \tilde{F}_\alpha \) is the unit-time flow of the vector field

\[
v_\alpha = -b(h_\hat{\alpha}) c(x_\alpha) \partial_{x_\alpha} \in \text{Vect}(\mathbb{R}^\mathcal{F})
\]

In particular, \( \tilde{F}_\alpha \) is the identity when \( h_\hat{\alpha} \leq 0 \) and is smooth when \( h_\hat{\alpha} > 0 \).

**Remark 4.21.** By scaling the original function \( b \) by a positive constant, one obtains a family of smoothed arboreal hypersurfaces all compatibly homeomorphic. Moreover, their limit as the scaling constant goes to zero is the rectilinear arboreal hypersurface. Thus one can view the smoothed arboreal hypersurface as a topologically trivial deformation of the rectilinear arboreal hypersurface.

**4.3.4. Microlocal geometry.** Finally, we recall the relation between arboreal singularities and smoothed arboreal hypersurfaces.

Recall that the smoothed arboreal hypersurface \( H_{\mathcal{F}} \) is the union

\[
H_{\mathcal{F}} = \bigcup_{\alpha \in V(\mathcal{F})} H_\alpha \subset \mathbb{R}^\mathcal{F}
\]

of hypersurfaces cut out by submersions

\[
H_\alpha = \{ h_\alpha = 0 \} \subset \mathbb{R}^\mathcal{F}
\]
Thus $H_F \subset \mathbb{R}^F$ is in good position, and moreover, each hypersurface $H_\alpha \subset \mathbb{R}^F$ comes equipped with a preferred coorientation $\sigma_\alpha$ given by the codirection pointing towards the halfspace

$$Q_\alpha = \{ h_\alpha \geq 0 \} \subset \mathbb{R}^F$$

Moreover, recall the inverse images under the natural projections

$$H_{F_\alpha} = \pi_i^{-1}(H_{T_i}) \subset \mathbb{R}^F \quad \pi_i : \mathbb{R}^F = \prod_i \mathbb{R}^{T_i} \rightarrow \mathbb{R}^{T_i}$$

and the evident identities

$$H_{F_\alpha} \simeq H_{T_i} \times \mathbb{R}^{F \setminus T_i} \quad H_F = \bigcup \pi_i H_{F_\alpha}$$

Note that the inverse images are multi-transverse hypersurfaces being the inverse images of complementary projections. By definition, we also have a parallel disjoint union identity

$$L_F = \prod_i (L_{T_i} \times \mathbb{R}^{F \setminus T_i})$$

Thus the analogous result for trees from [20] immediately implies the following extension to forests.

**Theorem 4.22.** Let $F$ be a rooted forest with arboreal singularity $L_F$ and smoothed arboreal hypersurface $H_F \subset \mathbb{R}^F$,

1. The smoothed arboreal hypersurface $H_F \subset \mathbb{R}^F$ is in good position with a natural coorientation $\sigma$ whose restriction to each $H_\alpha \subset H_F$ is the coorientation $\sigma_\alpha$.
2. The positive coray bundle $\Lambda_F \subset S^*\mathbb{R}^F$ of the directed hypersurface $H_F \subset \mathbb{R}^F$ with coorientation $\sigma$ is homeomorphic to $L_F$.

**4.4. Generalized arboreal singularities.** We introduce here a modest generalization of arboreal singularities akin to the generalization from manifolds to manifolds with boundary.

Let $F = \prod_i T_i = \prod_i (T_i, \rho_i)$ be a rooted forest.

By the **leaf vertices** $L = \prod_i L_i \subset V(F) = \prod_i V(T_i)$, we will mean the set of vertices that are maxima with respect to the natural partial order. (A root vertex is a maximum only if it is the sole vertex in its connected component; by the above definition such a vertex is also a leaf vertex.)

By a **leafy rooted forest** $F^* = (F, \ell) = \prod_i (T_i, \ell_i) = \prod_i (T_i, \rho_i, \ell_i)$, we will mean a rooted forest $F = \prod_i T_i = \prod_i (T_i, \rho_i)$ together with a subset $\ell = \prod_i \ell_i \subset L = \prod_i L_i$ of marked leaf vertices.

To any leafy rooted forest $F^* = (F, \ell)$, we associate a rooted forest $F^+$ by starting with $F$ with its natural partial order and adding a new maximum $\alpha^+ \in V(F^+)$ above each marked leaf vertex $\alpha \in \ell \subset V(F)$. We continue to denote by $\ell \subset V(F) \subset V(F^+)$ the originally marked vertices. We denote by $\ell^+ = V(F^+) \setminus V(F)$ the newly added vertices. Note that each $\alpha \in \ell^+ \subset V(F^+)$ has parent vertex $\hat{\alpha}^+ = \alpha \in \ell \subset V(F^+)$. Throughout what follows, let $F^* = (F, \ell)$ be a leafy rooted forest, and let $F^+$ be its associated rooted forest. Our constructions will devolve to previous ones when $\ell = \emptyset$ and hence $F^+ = F$.

**4.4.1. Rectilinear version.** For any directed forest and in particular $F^+$, recall the rectilinear arboreal hypersurface $H_{F^+} \subset \mathbb{R}^{F^+}$ admits the presentation as a union of closed subspaces

$$H_{F^+} = \bigcup_{\alpha \in V(F^+)} P_\alpha \subset \mathbb{R}^{F^+} \quad P_\alpha = \{ x_\alpha = 0, x_\beta \geq 0 \text{ for all } \beta < \alpha \} \subset \mathbb{R}^{F^+}$$

**Definition 4.23.** The **rectilinear arboreal hypersurface** $H_{F^*}$ associated to the leafy rooted forest $F^* = (F, \ell)$ is the union of closed subspaces

$$H_{F^*} = \bigcup_{\alpha \in V(F^+) \setminus \ell} P_\alpha \subset H_{F^+} \subset \mathbb{R}^{F^+}$$
Remark 4.24. If $\ell = 0$, so that $F^+ = F$, then $H_{F^*} = H_F$.

Example 4.25. If $\ell = F = \{\alpha\}$ consists of a single vertex, then $F^+ = \{\alpha, \alpha^+\}$ consists of two vertices satisfying $\alpha < \alpha^+$. The rectilinear arboreal singularity $H_{F^*}$ is the closed half-line
\[ H_{F^*} = P_{\alpha^+} = \{x_{\alpha^+} = 0, x_\alpha \geq 0\} \]

4.4.2. Smoothed version. For any directed forest and in particular $F^+$, recall the smoothed arboreal hypersurface $H_{F^*} \subset \mathbb{R}^{F^+}$ admits the presentation as a union of closed subspaces
\[ H_{F^*} = \bigcup_{\alpha \in V(F^+)} P_\alpha \subset \mathbb{R}^{F^+} \quad P_\alpha = \{h_\alpha = 0, h_\alpha \geq 0\} \subset \mathbb{R}^{F^+} \]

Definition 4.26. The smoothed arboreal hypersurface $H_{F^*}$ associated to the leafy rooted forest $F^* = (F, \ell)$ is the union of closed subspaces
\[ H_{F^*} = \bigcup_{\alpha \in V(F^+) \setminus \ell} P_\alpha \subset H_{F^*} \subset \mathbb{R}^{F^+} \]

Remark 4.27. If $\ell = 0$, so that $F^+ = F$, then $H_{F^*} = H_F$.

Remark 4.28. Recall the homeomorphism
\[ F_{F^+} : \mathbb{R}^{F^+} \overset{\sim}{\longrightarrow} \mathbb{R}^{F^+} \]

and that it satisfies $F_{F^+}(P_\alpha) = P_\alpha$

Alternatively, the smoothed arboreal hypersurface $H_{F^*}$ is the image of the rectilinear arboreal hypersurface $H_{F^*}$ under the inverse homeomorphism
\[ H_{F^*} = F_{F^+}^{-1}(H_{F^*}) \subset H_{F^*} \subset \mathbb{R}^{F^+} \]

4.4.3. Microlocal geometry. For any rooted forest and in particular $F^+$, recall that the smoothed arboreal hypersurface $H_{F^*} \subset \mathbb{R}^{F^+}$ is a directed hypersurface with a natural coorientation, and its positive coray bundle $\Lambda_{F^*} \subset S^*\mathbb{R}^{F^+}$ is homeomorphic to the arboreal singularity $L_{F^*}$.

By definition, the smoothed arboreal hypersurface $H_{F^*} \subset \mathbb{R}^{F^+}$ is a closed subspace of $H_{F^*} \subset \mathbb{R}^{F^+}$, and hence it is in good position and inherits a natural coorientation. Thus its positive coray bundle $\Lambda_{F^*} \subset S^*\mathbb{R}^{F^+}$ is a closed subspace of $\Lambda_{F^*} \subset S^*\mathbb{R}^{F^+}$, and hence homeomorphic to a closed subspace of the arboreal singularity $L_{F^*}$.

To identify this closed subspace, let us identify its open complement. Recall that $L_{F^*}$ is stratified by cells indexed by correspondences of the form
\[ p = (R \xleftarrow{q} S \xleftarrow{i} F^+) \]

where $i$ is the inclusion of a subtree and $q$ is a quotient of trees. (Strictly speaking, we have only stated this cell decomposition for trees, but it holds immediately for forests: by definition, the arboreal space of a forest is the disjoint union of the arboreal spaces of the connected components of the forest; and for correspondences of the above form, the inclusion $i$ must take its domain tree to a single connected component of its codomain forest.)

Given a marked leaf vertex $\alpha \in \ell \subset V(F^+)$, with added maximum vertex $\alpha^+ \in \ell^+ \subset V(F^+)$ so that $\hat{\alpha} = \alpha$, consider the two correspondences
\[ p_\alpha = (\{pt\} \xleftarrow{\alpha} \{\alpha\} \xleftarrow{F^+}) \]
\[ p_{\alpha^+, \alpha} = (\{pt\} \xleftarrow{\alpha^+, \alpha} \{\alpha^+, \alpha\} \xleftarrow{F^+}) \]

Since the correspondences begin with a singleton $\{pt\}$, they are maxima in the correspondence poset, and hence index open cells in $L_{F^*}$. 

Proposition 4.29. The positive coray bundle $\Lambda_{F^+} \subset S^*\mathbb{R}^{F^+}$ is homeomorphic to the closed subspace of the arboreal singularity $L_{F^+}$ given by deleting the open cells indexed by the correspondences $p_\alpha$, $p_{\alpha^+,\alpha}$, for all $\alpha \in \ell \subset V(F^+)$. 

Proof. For each $\alpha \in \ell \subset V(F^+)$, introduce the subspaces

$$P_\alpha^* = \{x_\alpha = 0, x_\beta > 0 \text{ for all } \beta < \alpha, x_{\alpha^+} \neq 0\} \subset H_{F^+} \subset \mathbb{R}^{F^+}$$

Observe that $P_\alpha^*$ is an open submanifold of $H_\alpha = \{h_\alpha = 0\}$ and hence comes with a preferred coorientation $\sigma_\alpha$ with associated positive coray bundle $\Lambda_\alpha^* \subset \Lambda_{F^+}$.

Lemma 4.30. $\Lambda_{F^+} = \Lambda_{F^*} \cup \bigcup_{\alpha \in \ell} \Lambda_{P_\alpha^*}$

Moreover, each $\Lambda_{P_\alpha^*}$ is disjoint from $\Lambda_{F^*}$ and each other.

Proof of Lemma 4.30. Observe that $H_{F^+} = H_{F^*} \cup \bigcup_{\alpha \in \ell} P_\alpha^*$

To see this, recall that $H_{F^+} = H_{F^*} \cup \bigcup_{\alpha \in \ell} P_\alpha$ and $H_{F^*} = \bigcup_{\alpha \in V(F^+) \setminus \ell} P_\alpha$

Suppose $p \in P_\alpha \setminus P_\alpha^*$. Then either $x_\beta(p) = 0$, for some $\beta < \alpha$, in which case $p \in P_\beta \subset H_{F^*}$, or $x_{\alpha^+}(p) = 0$, in which case $p \in P_{\alpha^+} \subset H_{F^*}$.

Thus applying $F_{F^+}^{-1}$, we also obtain $H_{F^+} = H_{F^*} \cup \bigcup_{\alpha \in \ell} P_\alpha^*$

Observe further that $P_\alpha^*$ is tangent to $P_\alpha$ along $h_\alpha = 0$, and tangent to $P_{\alpha^+}$ along $x_{\alpha^+} = 0$. Thus the boundary $\partial_{P_\alpha^*} \setminus \partial P_\alpha^*$ is contained in $H_{F^*}$, and hence we obtain the first assertion

$\Lambda_{F^+} = \Lambda_{F^*} \cup \bigcup_{\alpha \in \ell} \Lambda_{P_\alpha^*}$

Now let us turn to the second assertion.

First, $P_\alpha^* \cap P_{\alpha^+} = \emptyset$, since $p \in P_\alpha^*$ implies $x_{\alpha^+}(p) \neq 0$ so $p \notin P_{\alpha^+}$.

Similarly, if $\beta < \alpha$, then $P_\beta^* \cap P_{\beta^+} = \emptyset$, since $p \in P_\beta^*$ implies $x_{\beta}(p) > 0$ so $p \notin P_{\beta^+}$.

Finally, if $\gamma$ and $\alpha$ are incomparable, in particular if $\gamma$ also lies in $\ell$, then the following intersection is obviously transverse

$P_\alpha^* \cap P_{\gamma} = \{x_\alpha = 0, x_\beta > 0 \text{ for all } \beta < \alpha, x_{\alpha^+} \neq 0\} \cap \{x_\gamma = 0, x_\delta \geq 0 \text{ for all } \delta < \gamma\}$

We claim that the homeomorphism $F_{F^+}^{-1}$ preserves the transversality of the above intersection thus establishing the second assertion. To check this, fix a total order on $V(F)$ compatible with its natural partial order, write $\alpha_1, \ldots, \alpha_{n+1} \in V(F)$ for the ordered vertices, and recall the factorization $F_F = \hat{F}_{a_1} \circ \cdots \circ \hat{F}_{a_{n+1}}$.

Since each $\hat{F}_\beta$ preserves all coordinates except $x_\beta$, we are reduced to showing that $\hat{F}_{\alpha}^{-1}$ and $\hat{F}_{\beta}^{-1}$ preserve the transversality of the above intersection. But each only changes the corresponding coordinate as a function of the coordinates less than it in the partial order. Since $\alpha$ and $\gamma$ are incomparable by assumption, the asserted transversality follows. $\square$
Finally, to complete the proof of Prop. 4.29 by construction \cite{20}, the disjoint union of the open cells of $L_{F^+}$ indexed by the correspondences $p_\alpha$, $p_{\alpha^+}$ maps homeomorphically to $P^*_\alpha$ under the natural projection

$$S^*R^{F^+} \longrightarrow R^{F^+}.$$  

More precisely, the open cell indexed by $p_\alpha$ maps to the locus $P^*_\alpha \cap \{ x_{\alpha^+} < 0 \}$, and the open cell indexed by $p_{\alpha^+}$ maps to the locus $P^*_\alpha \cap \{ x_{\alpha^+} > 0 \}$. \hfill $\blacksquare$

**Remark 4.31.** If $\ell = 0$, so that $F^+ = F$, then $\Lambda_{F^+}$ is homeomorphic to $L_F$ itself.

**Example 4.32.** If $\ell = F = \{ \alpha \}$ consists of a single vertex, then $F^+ = \{ \alpha, \alpha^+ \}$ consists of two vertices satisfying $\alpha < \alpha^+$.

Recall that $L_{F^+}$ is the local trivalent graph given by the cone over three points indexed by the three correspondences

$$(\{ pt \} \xleftarrow{=} \{ \alpha \} \xrightarrow{=} F^+) \quad (\{ pt \} \xleftarrow{=} \{ \alpha^+ \} \xrightarrow{=} F^+) \quad (\{ pt \} \xleftarrow{=} F^+ \xrightarrow{=} F^+)$$

To obtain $\Lambda_{F^+}$, we start with $L_{F^+}$ and delete the two open cells indexed by the first and third of the above correspondences. What results is a closed half-line, the cone over the remaining point indexed by the middle correspondence. Note the agreement with Example 4.25.

### 5. Expansion Algorithm

#### 5.1. Setup.

Let $M$ be a manifold.

Let $H \subset M$ be a directed hypersurface with positive coray bundle $\Lambda \subset S^*M$.

Fix a Whitney stratification $\{ H_i \}_{i \in I}$ of the hypersurface $H \subset M$. (The reason for the presently superfluous underlining of the indices will become apparent soon below.) As usual, we will regard the index set $I$ of the stratification as a poset with partial order

$$i < j \, \, \text{ if and only if } \, \, H_i \subset H_j \neq H_j$$

To simplify the exposition, we will make the following first of several mild assumptions.

**Assumption 5.1.** We will assume there is a compactification $M \subset \overline{M}$ so that the stratification of $H \subset M$ is the restriction of a stratification of the closure $\overline{H} \subset \overline{M}$.

In particular, this implies the index set $I$ of the stratification is finite.

For each $i \in I$, introduce the restriction of the positive coray bundle

$$\Lambda_i = \Lambda \times_H H_i \subset \Lambda$$

Next, we will assume the following additional simplifying property of the stratification which can be achieved by refining the stratification if necessary, for example so that the strata are simply-connected.

**Assumption 5.2.** For each $i \in I$, we will assume the finite map $\Lambda_i \to H_i$ is a trivial bundle.

For each $i \in I$, fix once and for all a trivialization

$$\Lambda_i \cong H_i \times F_i$$

where $F_i$ is a finite set.

Introduce the set $I$ of pairs $i = (\hat{i}, f)$ where $\hat{i} \in \hat{I}$ and $f \in F_\hat{i}$, and the natural projection

$$I \longrightarrow \hat{I} \quad i = (\hat{i}, f) \quad \hat{i}$$

For each $\hat{i} \in \hat{I}$, we will regard $F_\hat{i}$ as a subset of $I$, and often write $i \in F_\hat{i}$ when $i \mapsto \hat{i}$ without specifying that $i = (\hat{i}, f)$. 

For each $i = (\hat{i}, f) \in F_{\hat{i}}$, we will write $\Lambda_i \subset \Lambda$ for the subspace
\[ \Lambda_i = H_{\hat{i}} \times \{ f \} \subset H_{\hat{i}} \times F_{\hat{i}} \simeq \Lambda_{\hat{i}} \subset \Lambda. \]
Note that projection provides a diffeomorphism
\[ \Lambda_i \sim H_{\hat{i}}. \]

We have a disjoint decomposition into submanifolds
\[ \Lambda = \bigsqcup_{i \in I} \Lambda_i. \]
The decomposition satisfies the axiom of the frontier but we will not worry about whether it is a Whitney stratification. We will regard the index set $I$ as a finite poset with partial order
\[ i < j \quad \text{if and only if} \quad \Lambda_i \subset \Lambda_j, \Lambda_i \neq \Lambda_j \]
The projection $I \to I$ respects the poset structures in the sense that $i < j$ implies $\hat{i} < \hat{j}$ (though not necessarily the converse).

Finally, to further simplify future notational demands, we will assume the following simplifying property of the stratification which can be achieved by further refining the stratification if necessary.

**Assumption 5.3.** For each $\hat{i} \in I$, we will assume the stratum $H_{\hat{i}} \subset H$ is locally connected.

The assumption has the following implication which will help simplify the exposition and notation around further constructions.

**Lemma 5.4.** Given $\hat{i} \in I$ and $j \in I$ with $\hat{i} < \hat{j}$, there exists a unique $i \in F_{\hat{i}}$ such that $i < j$.

**Proof.** First, note there exists $i \in F_{\hat{i}}$ with $i < j$ since the projection $\Lambda \to H$ is proper. Next, if there were two such $i, i' \in F_{\hat{i}}$, then $H_j \subset H$ would not be locally connected near $H_{\hat{i}} \subset H$. Namely, if we choose disjoint open neighborhoods $U_i \subset \Lambda$ of $\Lambda_i \subset \Lambda$, for all $i \in F_{\hat{i}}$, then near $H_{\hat{i}} \subset H$, we would have that $H_j \subset H$ is the disjoint union of the homeomorphic images of the open subsets $\Lambda_j \cap U_i \subset \Lambda_j$. \( \square \)

The above assertion immediately implies the following useful statements. Given a poset $I$, and an element $j \in I$, we will write $I_{\leq j} = \{ i \in I \mid i \leq j \}$ and $I_{\geq j} = \{ i \in I \mid i \geq j \}$ for the induced subposets. Given a subset $J \subset I$, we will write $I_{\leq J} = \cup_{j \in J} I_{\leq j}$ and $I_{\geq J} = \cup_{j \in J} I_{\geq j}$ for the induced subposets.

**Corollary 5.5.** (1) For each $j \in I$, the natural projection of subposets
\[ I_{\leq j} \longrightarrow I_{\leq \hat{j}} \]
is an isomorphism.

(2) Given $\hat{i} \in I$ with preimage $F_{\hat{i}} \subset I$, we have the decomposition
\[ I_{\geq F_{\hat{i}}} = \bigsqcup_{i \in F_{\hat{i}}} I_{\geq i} \]
into disjoint incomparable subposets.

The first assertion of the corollary implies for each $j \in I$, the natural projection of closed subspaces is a homeomorphism
\[ \bigcup_{i \leq j} \Lambda_i \sim \bigcup_{i \leq \hat{j}} H_{\hat{i}}. \]
The second assertion implies for each \( i \in \mathcal{I} \), there is a disjoint union decomposition of open subspaces
\[
\bigcup_{j \in I \supseteq i} \Lambda_j = \bigsqcup_{i \in F} \bigcup_{j \geq i} \Lambda_j
\]

5.2. **Expanded cylinder.** We continue with the setup of the preceding section.

Fix a compatible system of control data \( \{(T_i, \rho_i, \pi_i)\}_{i \in \mathcal{I}} \).

5.2.1. **Multi-transverse functions.** For each \( i \in \mathcal{I} \), choose a small positive radius \( r_i \in \mathbb{R}_{>0} \) so that \( r_i \neq r_i' \) whenever \( i = i' \).

**Definition 5.6.** For each \( i \in \mathcal{I} \), introduce the function
\[
f_i : T_i \rightarrow \mathbb{R} \quad f_i = \rho_i - r_i
\]

**Lemma 5.7.** The collection of functions \( \{f_i\}_{i \in \mathcal{I}} \) is multi-transverse at its total zero value.

**Proof.** Since the radii \( r_i \in \mathbb{R}_{>0} \) are distinct \( r_i \neq r_i' \) whenever \( i = i' \), the zero locus of a subcollection of functions is nonempty only if for each \( i \in \mathcal{I} \), the subcollection contains at most one function indexed by an \( i \in \mathcal{I} \) lying over \( i \). For such subcollections, the multi-transversality is the usual multi-transversality of the collection \( \{\rho_i\}_{i \in \mathcal{I}} \) of tubular distance functions at any collection of non-zero values.

5.2.2. **Truncated strata.**

**Definition 5.8.** For each \( i \in \mathcal{I} \), define the truncated stratum \( H_{tr}^i \subset H_i \) to be the closed subspace of \( x \in H_i \) cut out by the equations
\[
f_a(x) \geq 0, \text{ whenever } a < i \text{ and } x \in H_i \cap T_a
\]

**Lemma 5.9.**
1. The truncated stratum \( H_{tr}^i \subset H_i \) is a closed submanifold with corners.
2. The codimension \( k \) corners of \( H_{tr}^i \) are indexed by \( a_1, \ldots, a_k \in \mathcal{I} \) with \( a_1 < \cdots < a_k < i \).

**Proof.** Thanks to statement (1) of Corollary 5.5, the lemma reduces to the same assertion for the tubular distance functions of a system of control data which is a standard fact.

**Remark 5.10.** Of course if \( i \in \mathcal{I} \) is a minimum, so that \( H_i \subset H \) is a closed stratum, then we have \( H_{tr}^i = H_i \).

5.2.3. **Truncated cylinders.**

**Definition 5.11.** For each \( i \in \mathcal{I} \), define the truncated cylinder \( C_i \subset T_i \) to be the subspace of \( x \in T_i \) cut out by the equations
\[
f_i(x) = 0 \quad f_a(x) \geq 0, \text{ whenever } a < i \text{ and } x \in T_i \cap T_a
\]

**Remark 5.12.** Equivalently, by the axioms of a control system, the truncated cylinder \( C_i \subset T_i \) is the subspace of \( x \in T_i \) cut out by the equations
\[
f_i(x) = 0 \quad \pi_i(x) \in H_{tr}^i
\]

**Lemma 5.13.**
1. The truncated cylinder \( C_i \subset T_i \) is a closed submanifold with corners.
2. The projection \( \pi_i \) exhibits \( C_i \) as a \( (\text{codim}_M H_i - 1) \)-sphere bundle over \( H_{tr}^i \).

**Proof.** Immediate from Lemma 5.9.

**Remark 5.14.** Of course if \( i \in \mathcal{I} \) is a minimum, so that \( H_i \subset H \) is a closed stratum, then the truncated cylinder \( C_i \subset M \) is also closed and cut out simply by \( f_i(x) = 0 \).
5.2.4. Total cylinder.

Definition 5.15. Define the total cylinder \( C \subset M \) to be the union of truncated cylinders

\[
C = \bigcup_{i \in I} C_i
\]

Proposition 5.16. The singularities of the total cylinder \( C \subset M \) are rectilinear arboreal hypersurface singularities.

Proof. Fix a point \( p \in M \).

Let \( I_p \subset I \) comprise indices \( i \in I \) such that \( p \in C_i \subset T_i \), so in particular \( f_i : T_i \to \mathbb{R} \) vanishes at \( p \). We will regard \( I_p \subset I \) as a poset with the induced partial order: \( i, j \in I_p \) satisfy \( i < j \) inside of \( I_p \) if and only if \( i < j \) inside of \( I \).

By construction, it suffices to see that \( I_p \) is the poset of a rooted forest \( I_p \), and thus the singularity of \( C \) at the point \( p \) is the rectilinear arboreal hypersurface \( H_{I_p} \). More precisely, there will be an open ball \( U \subset \mathbb{R}^{2p} \times \mathbb{R}^{k_p} \), with \( k_p = \dim M - |I_p| \) and \( 0 \in U \), and a smooth open embedding

\[
\varphi : U \xrightarrow{\sim} \varphi(U) \subset M
\]

such that the following holds

\[
\varphi(0) = p \quad \varphi(U \cap (H_{I_p} \times \mathbb{R}^{k_p})) = \varphi(U) \cap C
\]

\[
x_i = f_i \circ \varphi : U \longrightarrow \mathbb{R} \quad \text{for all } i \in I_p
\]

Then for \( i \in I_p \), the constructions with the coordinates \( x_i \) immediately match those of Definitions 5.11 with the functions \( f_i \).

So let us check that \( I_p \) is the poset of a rooted forest \( I_p \). For this, it suffices to show that for any \( i \in I_p \) that is not a minimum, there is a unique parent \( i' \in I_p \) such that \( i < i' \) and no \( j \in I_p \) satisfies \( i < j < i' \). Recall that \( i \in I_p \) means \( p \in C_i \). By Lemmas 5.9 and 5.13 \( C_i^{tr} \subset M \) is a closed submanifold with codimension \( k \) corners indexed by (possibly empty) sequences \( a_1, \ldots, a_k \in I \) with \( a_1 < \cdots < a_k < i \) such that \( f_j(p) = 0 \) if and only if \( j = a_k \) for some \( \ell = 1, \ldots, k \). Now \( p \in C_i \) lies in some corner indexed by such a sequence. If the sequence is empty, then clearly \( i \in I_p \) is a minimum, else the unique parent of \( i \in I_p \) is clearly the maximum of the sequence \( i = a_k \in I_p \). \( \square \)

5.3. Smoothing into good position. The total cylinder \( C \subset M \) is a hypersurface with rectilinear arboreal hypersurface singularities. Our aim here is to amend its construction to produce a homeomorphic deformation of it to a directed hypersurface \( C \subset M \) with smoothed arboreal hypersurface singularities.

5.3.1. Good charts. Fix a point \( p \in M \).

Let \( I_p \subset I \) comprise indices \( i \in I \) such that \( p \in C_i \subset T_i \), so in particular \( f_i : T_i \to \mathbb{R} \) vanishes at \( p \). We will regard \( I_p \subset I \) as a poset with the induced partial order: \( i, j \in I_p \) satisfy \( i < j \) inside of \( I_p \) if and only if \( i < j \) inside of \( I \). Prop. 5.16 confirms that \( I_p \) is the poset of a rooted forest \( I_p \) and the arboreal singularity of \( C \) at the point \( p \) is that associated to \( I_p \).

By a good chart \( (U, \varphi) \) centered at \( p \in C \), we will mean an open ball \( U \subset \mathbb{R}^{2p} \times \mathbb{R}^{k_p} \), with \( k_p = \dim M - |I_p| \) and \( 0 \in U \), and a smooth open embedding

\[
\varphi : U \xrightarrow{\sim} \varphi(U) \subset M
\]

such that the following holds

\[
\varphi(0) = p \quad \varphi(U \cap (H_{I_p} \times \mathbb{R}^{k_p})) = \varphi(U) \cap C
\]
The proof of Prop. [5.10] confirms there is a good chart centered at any point.

**Remark 5.17.** A good chart \((U, \varphi)\) centered at \(p \notin H\) so that \(\mathcal{I}_p = \emptyset\) is simply a coordinate chart such that \(\varphi(U) \cap C = \emptyset\).

**Remark 5.18.** Suppose \((U_1, \varphi_1), (U_2, \varphi_2)\) are good charts centered at \(p \in H\). Introduce the open subsets

\[
U'_1 = \varphi_1^{-1}(\varphi_1(U_1) \cap \varphi_2(U_2)) \subset \mathbb{R}^x \times \mathbb{R}^k \\
U'_2 = \varphi_2^{-1}(\varphi_1(U_1) \cap \varphi_2(U_2)) \subset \mathbb{R}^x \times \mathbb{R}^k
\]

and the diffeomorphism

\[
\psi = \varphi_2^{-1} \circ \varphi_1 : U'_1 \longrightarrow U'_2
\]

By construction, \(\psi\) satisfies

\[
x_i = x_i \circ \psi : U'_1 \longrightarrow \mathbb{R} \quad \text{for all } i \in I_p
\]

Thus \(\psi\) is a shearing transformation in the sense that it takes the form

\[
\psi = \text{id}_{\mathbb{R}^x} \times \tilde{\psi} : U'_1 \sim \sim U'_2 \\
\tilde{\psi} : U'_1 \longrightarrow \mathbb{R}^k
\]

More generally, suppose \((U_1, \varphi_1), (U_2, \varphi_2)\) are good charts centered at \(p_1, p_2 \in H\) respectively. Then in the same notation as above, the diffeomorphism \(\psi\) satisfies

\[
x_i = x_i \circ \psi : U'_1 \longrightarrow \mathbb{R} \quad \text{for all } i \in I_{p_1} \cap I_{p_2}
\]

### 5.3.2. Global smoothing.

Choose an open covering of \(M\) by good charts \(\{(U_a, \varphi_a)\}_{a \in A}\) centered at points \(p_a \in M\).

Set \(I_a \subset I\) to contain indices \(i \in I\) such that \(p_a \in C_i \subset E_2\), so in particular \(f_i : T_2 \rightarrow \mathbb{R}\) vanishes at \(p_a\). Recall that \(I_a\) is the poset of a rooted forest \(\mathcal{I}_a\) and the arboreal singularity of \(C\) at the point \(p_a\) that is associated to \(\mathcal{I}_a\).

We will only be interested in a neighborhood of \(C \subset M\), so will throw out any \(a \in A\) such that \(\varphi_a(U_a) \cap C = \emptyset\). Since \(H\) is assumed to be compactifiable, \(C\) is also compactifiable, and hence we may assume \(A\) is finite.

By adjusting constants and refining the cover \(\{(U_a, \varphi_a)\}_{a \in A}\) if necessary, we can and will assume that they satisfy the following convenient conditions:

1. \(\varphi_a(U_a) \cap \varphi_b(U_b) \neq \emptyset\) implies \(I_a \subset I_b\) or \(I_b \subset I_a\).
2. \(\varphi_a(U_a) \cap \{f_i = 0\} = \emptyset\) implies \(\varphi_a(U_a) \cap \{f_i \leq 2\delta\} = \emptyset\).

For \(a \in I_a\), recall the function

\[
h_a : \mathbb{R}^{I_a} \longrightarrow \mathbb{R}
\]

appearing in the smoothing of Sect. [4.3.2]. Via the inclusion and projection

\[
U_a : \mathbb{R}^{I_a} \times \mathbb{R}^k \longrightarrow \mathbb{R}^{I_a}
\]

and diffeomorphism \(\varphi_a\), we can pull back and transfer \(h_a\) to a function

\[
h_{a, \alpha} : \varphi_a(U_a) \longrightarrow \mathbb{R}
\]

Recall that for any non-root vertex \(\alpha \in V(\mathcal{I}_a)\) there is a unique parent vertex which we will denote here by \(\hat{\alpha}_a \in V(\mathcal{I}_a)\) emphasizing its dependence on the poset \(\mathcal{I}_a\).
Lemma 5.19. For \( a, b \in A \), with \( \varphi_a(U_a) \cap \varphi_b(U_b) \neq \emptyset \), suppose \( I_b \subset I_a \). Then for any \( \alpha \in I_b \subset I_a \), we have the equality of functions
\[
h_{a,\alpha} = h_{b,\alpha}
\]
over the common domain \( \varphi_a(U_a) \cap \varphi_b(U_b) \).

Proof. It suffices to assume \( I_a = I_b \bigcup \{c\} \) for some \( c \in I \).

Recall for \( \alpha \in \mathcal{I}_a \), by definition for a root vertex \( \rho \in V(F) \), we have
\[
h_\rho = x_\rho : \mathbb{R}^F \longrightarrow \mathbb{R}
\]
and for a non-root vertex \( \alpha \in V(F) \), we inductively have
\[
h_{\alpha} : \mathbb{R}^F \longrightarrow \mathbb{R} \quad h_{\alpha} = f(h_{\hat{\alpha}}, x_\alpha)
\]
where \( \hat{\alpha} \in V(F) \) is the parent vertex of \( \alpha \).

Thus it suffices to suppose \( c = \hat{\alpha}_a \), or in other words, that \( c \) is the parent of \( \alpha \) inside of \( \mathcal{I}_a \).

Now we will consider two cases:

(i) \( c \) is a minimum in \( \mathcal{I}_a \). Then it suffices to show
\[
(5.1) \quad h_{a,\alpha} = h_{a,\hat{\alpha}_a} \quad \text{over } \varphi_a(U_a) \cap \varphi_b(U_b)
\]
Recall that \( \varphi_b(U_b) \cap \{f_c = 0\} = \emptyset \) implies \( \varphi_b(U_b) \cap \{f_c \leq 2\delta\} = \emptyset \). Thus by construction \( h_{a,\alpha} = f_c \) over \( \varphi_a(U_a) \cap \varphi_b(U_b) \) and so \([5.1]\) holds.

(ii) \( c \) is not a minimum in \( \mathcal{I}_a \). Then it suffices to show
\[
(5.2) \quad h_{a,\alpha} = h_{a,\hat{\alpha}_a} \quad \text{over } \varphi_a(U_a) \cap \varphi_b(U_b)
\]
Recall that \( \varphi_b(U_b) \cap \{f_c = 0\} = \emptyset \) implies \( \varphi_b(U_b) \cap \{f_c \leq 2\delta\} = \emptyset \). Thus by construction \([5.1]\) holds.

Next, for \( \alpha \in \mathcal{I}_a \), recall the vector field
\[
v_\alpha = -b(h_{\hat{\alpha}})c(x_\alpha)\partial_{x_\alpha} \in \text{Vect}(\mathbb{R}^{\mathcal{I}_a})
\]
appearing in the smoothing of Sect. 4.3.2. It naturally lifts to a vector field on the product \( \mathbb{R}^{\mathcal{I}_a} \times \mathbb{R}^k \), then via the inclusion
\[
U_a \hookrightarrow \mathbb{R}^{\mathcal{I}_a} \times \mathbb{R}^k
\]
and diffeomorphism \( \varphi_a \), we can restrict and transfer it to a vector field
\[
v_{a,\alpha} \in \text{Vect}(\varphi_a(U_a))
\]

Proposition 5.20. For \( a, b \in A \), with \( \varphi_a(U_a) \cap \varphi_b(U_b) \neq \emptyset \), suppose \( I_b \subset I_a \). Then for any \( \alpha \in I_b \), we have the equality of vector fields
\[
v_{a,\alpha} = v_{b,\alpha} + w
\]
over the common domain \( \varphi_a(U_a) \cap \varphi_b(U_b) \), where the vector field \( w \), transported via \( \varphi_b^{-1} \), points along the second factor of the product \( \mathbb{R}^{\mathcal{I}_b} \times \mathbb{R}^k \).

Proof. By Lemma 5.19 the ambiguity under change of good charts of the vector field \( v_\alpha = -b(h_{\hat{\alpha}})c(x_\alpha)\partial_{x_\alpha} \) is the ambiguity of the coordinate vector field \( \partial_{x_\alpha} \), and this is captured precisely by the shearing vector field \( w \).
Remark 5.21. Thanks to the axioms of a control system, we can additionally arrange so that the projection $\pi_a : T_a \to X_a$ is invariant with respect to $v_{a,\alpha}$ in the sense that $d\pi_a(v_{a,\alpha}) = 0$. This then in turn implies for $i \in I$ with $i \leq \alpha$, that the projection $\pi_i : T_i \to X_i$ is also invariant with respect to $v_{a,\alpha}$. We also have for $i \in I$ with $a$ and $i$ incomparable, that the vector field $v_{a,\alpha}$ vanishes near $H_i$.

Next fix a partition of unity $\{\delta_a\}_{a \in A}$ subordinate to the open cover $\{(U_a, \varphi_a)\}_{a \in A}$.

For any $a \in A$ and $i \in I$ with $i \not\leq \alpha$, set

$$v_{a,i} = 0 \in \text{Vect}(\varphi_a(U_a))$$

For each $i \in I$, introduce the global vector field

$$v_i = \sum_{a \in A} \delta_a v_{a,i} \in \text{Vect}(M)$$

For each $i \in I$, define the homeomorphism

$$\Phi_i : M \xrightarrow{\sim} M$$

to be the unit-time flow of the vector field $v_i$.

Remark 5.22. Note that we have arranged so that for $i \in I$, $j \in I$ with $i \not< j$, the projection $\pi_j : T_j \to X_j$ is invariant with respect to each $v_{a,i}$, thus also with respect to $v_i$, and thus finally with respect to $\Phi_i$.

Fix a total order on $I$ compatible with its natural partial order. Write $i_0, i_1, \ldots, i_N \in I$ for the ordered elements. Define the composite homeomorphism

$$\Phi = \Phi_{i_0} \circ \Phi_{i_1} \circ \cdots \circ \Phi_{i_N} : M \xrightarrow{\sim} M$$

Corollary 5.23. For any $a \in A$, under the good chart $\varphi_a$, the homeomorphism $\Phi$ takes the form $F_{I_a} \times \psi$.

Proof. Immediate from Prop. 5.20.

For each $i \in I$, introduce the inverse homeomorphism

$$\Psi_i = \Phi_i^{-1} : M \xrightarrow{\sim} M$$

Introduce the smoothing homeomorphism

$$\Psi = \Psi_{i_N} \circ \cdots \circ \Psi_{i_1} \circ \Psi_{i_0} : M \xrightarrow{\sim} M$$

Definition 5.24. Define the directed cylinder $C \subset M$ to be the image of the total cylinder

$$C = \Psi(C)$$

Theorem 5.25. The directed cylinder $C \subset M$ is a hypersurface in good position with a canonical coorientation and smoothed arboreal hypersurface singularities.

Proof. Immediate from Thm. 4.22, Prop. 5.16, and Cor. 5.23.

We will write $\Lambda_C \subset S^*M$ for the positive coray bundle of the directed cylinder $C \subset M$.

5.4. Expanded hypersurface. We continue with the constructions of the preceding sections, arriving in this section at our goal. Now taking into account the positive coray bundle $\Lambda \subset S^*M$, we cut out an expanded hypersurface $E \subset M$ inside the total cylinder $C \subset M$, and a directed expansion $E \subset M$ inside the directed cylinder $C \subset M$. 
5.4.1. Conormal sections. Recall for $i \in I$, and each $i \in I$ with $i \in F_i \subset I$, we have the subspace

$$\Lambda_i \subset S^*M|_{H_i}$$

The Whitney conditions imply the subspace lies in the spherically projectivized conormal bundle

$$\Lambda_i \subset S^*_H M$$

Furthermore, the projection $S^*M \to M$ restricts to a diffeomorphism

$$\Lambda_i \xrightarrow{\sim} H_i$$

and thus the subspace $\Lambda_i \subset S^*_H M$ is the image of a unique section

$$\lambda_i : H_i \to S^*_H M$$

Note that the conormal bundle $T^*_H M \to H_i$ is canonically isomorphic to the dual of the normal bundle $E_i \to H_i$. Hence for any inner product on the normal bundle $E_i \to H_i$, the section $\lambda_i$ naturally determines a unit-length section

$$\lambda_i : H_i \to E^*_i$$

Thus via the structures of the tubular neighborhood $(T_i, \rho_i, \pi_i)$, the section $\lambda_i$ naturally determines a fiber-wise linear function

$$\lambda_i : T_i \to \mathbb{R}$$

5.4.2. Expanded strata. Recall that to construct the total cylinder, we fixed a small positive radius $r_i \in \mathbb{R}_{>0}$, for each $i \in I$, so that $r_i \neq r_i'$ whenever $i \neq i'$. Now in addition, choose a small positive displacement $d_i \in \mathbb{R}_{>0}$, for each $i \in I$, and a small value $s_i \in \mathbb{R}$, for each $i \in I$.

**Definition 5.26.** For each $i \in I$, introduce the fiber-wise affine functions

$$g_i : T_\perp \to \mathbb{R}$$

$$g_i(x) = \lambda_i(x) + r_id_i - s_i$$

**Definition 5.27.** For each $i \in I$, define the expanded stratum $E_i \subset T_\perp$ to be the subspace of $x \in T_\perp$ cut out by the equations

$$f_i(x) = 0 \quad g_i(x) \geq 0 \quad f_a(x) \geq 0, \text{ whenever } a < i \text{ and } x \in T_\perp \cap T_a$$

**Remark 5.28.** Recall the truncated cylinder $C_i \subset T_\perp$ introduced in the previous section. Putting together the definitions, the expanded stratum $E_i \subset T_\perp$ is the subspace of $x \in C_i$ cut out by the equation $g_i(x) \geq 0$.

**Lemma 5.29.** Fix any $d_i, r_i \in (0, 1)$, and then sufficiently small $s_i \in \mathbb{R}$. 

1. The expanded stratum $E_i \subset T_\perp$ is a closed submanifold with corners.
2. The projection $\pi_\perp$ exhibits $E_i$ as a closed (codim $M H_\perp - 1$)-ball bundle over $H_\perp$.

**Proof.** For the moment, set $s_i = 0$, so that $g_i(x) = \lambda_i(x) + r_id_i$. Observe that for $d_i, r_i \in (0, 1)$, the pair $\{f_i, g_i\}$ of functions is multi-transverse at their total zero value $0 \in \mathbb{R}^2$ and the restriction $g_i|_{\{f_i = 0\}}$ takes both positive and negative values. Choosing small $s_i \in \mathbb{R}$, so that $g_i(x) = \lambda_i(x) + r_id_i - s_i$, the above facts continue to hold. Now the assertions follow from Lemmas 5.9 and 5.13. \qed
5.4.3. Total expansion. Recall that our constructions depend on constants \( d_i \in \mathbb{R}_{>0}, r_i \in \mathbb{R}_{>0}, s_i \in \mathbb{R}, \) for \( i \in I \). In what follows, we will always choose them in the following order. First, we will independently choose \( d_i \in (0, 1) \), for each \( i \in I \). Second, we will follow the poset structure on \( I \), working from the minima to the maxima, and choose small \( r_i \in \mathbb{R}_{>0}, \) for each \( i \in I \). Finally, we will again follow the poset structure on \( I \), working from the minima to the maxima, and choose small \( s_i \in \mathbb{R}, \) for each \( i \in I \). We will refer to such sufficiently small choices of constants as sequentially small.

Recall that the set \( \{f_i\}_{i \in I} \) of functions is multi-transverse at its total zero value \( 0 \in \mathbb{R}^I \). Recall the role of the constants \( d_i \in \mathbb{R}_{>0}, r_i \in \mathbb{R}_{>0}, s_i \in \mathbb{R}, \) for \( i \in I \), in the definition of the functions \( g_i(x) = \lambda_i(x) + r_id_i - s_i \). In particular, since we select the values \( s_i \in \mathbb{R}, \) for \( i \in I \), after the others, we may select sequentially small constants such that the extended set \( \{f_i\}_{i \in I} \coprod \{g_i\}_{i \in I} \) of functions is multi-transverse at its total zero value \( (0, 0) \in \mathbb{R}^I \times \mathbb{R}^I \).

**Definition 5.30.** Define the total expansion \( E \subset M \) to be the union of expanded strata

\[
E = \bigcup_{i \in I} E_i
\]

**Proposition 5.31.** There exist sequentially small constants \( d_i \in \mathbb{R}_{>0}, r_i \in \mathbb{R}_{>0}, s_i \in \mathbb{R}, \) for \( i \in I \), such that the singularities of the total expansion \( E \subset M \) are generalized rectilinear arboreal hypersurface singularities.

**Proof.** Let us first appeal to Prop. 5.16.

Fix a point \( p \in M \).

Let \( I_p \subset I \) comprise indices \( i \in I \) such that \( p \in C_i \subset T_i^2 \), so in particular \( f_i(p) = 0 \). We will regard \( I_p \subset I \) as a poset with the induced partial order: \( i, j \in I_p \) satisfy \( i < j \) inside of \( I_p \) if and only if \( i < j \) inside of \( I \).

Recall that Prop. 5.16 established that \( I_p \) is the poset of a rooted forest \( T_p \), and thus the singularity of the total cylinder \( C \subset M \) at the point \( p \) is the rectilinear arboreal hypersurface \( H_{T_p} \). More precisely, there is an open ball \( U \subset \mathbb{R}^{T_p} \times \mathbb{R}^{k_p} \), with \( k_p = \dim M - |I_p| \) and \( 0 \in U \), and a smooth open embedding

\[
\varphi : U \xrightarrow{\sim} \varphi(U) \subset M
\]

such that the following holds

\[
\varphi(0) = p, \quad \varphi(U \cap (H_{T_p} \times \mathbb{R}^{k_p})) = \varphi(U) \cap C
\]

\[
x_i = f_i \circ \varphi : U \longrightarrow \mathbb{R} \quad \text{for all } i \in I_p
\]

Now let \( J_p \subset I_p \) comprise indices \( i \in I_p \) such that \( p \in E_i \subset C_i \), so additionally \( g_i(p) \geq 0 \). We will regard \( J_p \subset I_p \) as a poset with the induced partial order: \( i, j \in J_p \) satisfy \( i < j \) inside of \( J_p \) if and only if \( i < j \) inside of \( I_p \). It will follow from the discussion below that at most \( J_p \) results from deleting from \( I_p \) some of its leaf vertices.

Let \( \ell_p \subset J_p \) comprise indices \( i \in J_p \) so that \( g_i(p) = 0 \). It will follow from the discussion below that \( \ell_p \) is a subset of the leaf vertices of \( J_p \).

To see the poset \( J_p \) (if nonempty), together with the marked vertices \( \ell_p \), arise from a leafy rooted forest \( J^*_p = (J_p, \ell_p) \), it suffices to establish the claim: for sequentially small constants, if \( g_i(p) \leq 0 \), for some \( i \in I_p \), then \( i \) is a leaf vertex of \( I_p \). If the claim holds, then the above embedding \( \varphi \) will identify the singularity of the total expansion \( E \subset M \) at the point \( p \) with the rectilinear arboreal hypersurface \( H_{J^*_p} \).

To prove the claim, we will appeal to the following.
Lemma 5.32. For any \( d_i \in (0, 1) \), sufficiently small \( r_i \in \mathbb{R}_{>0} \), further sufficiently small \( s_i \in \mathbb{R} \), and any \( a \in I \) with \( a > i \), the restriction of \( g_i : T_1 \rightarrow \mathbb{R} \) to the intersection \( H_a \cap C_i \subset T_1 \) is strictly positive.

Proof. Recall that \( g_i(x) = \lambda_i(x) + r_i d_i - s_i \). Thus it suffices to prove the assertion with \( s_i = 0 \).

Fix \( d_i \in (0, 1) \). Suppose there is a sequence of radii \( r_i(n) \in \mathbb{R}_{>0} \), with \( r_i(n) \rightarrow 0 \), with corresponding truncated cylinder \( C_i(n) \subset T_1 \), and points \( x(n) \in H_a(n) \cap C_i(n) \), with \( x(n) \rightarrow x \in H_a \), such that \( g_i(x(n)) \leq 0 \). Then it is a simple calculation to check with respect to any local coordinates that a subsequence of the secant lines \([x(n), x]\) converges to a line not contained in \( \ker(\lambda_i) \subset T_2 M \). But this contradicts Whitney’s condition \( B \) for the pair of strata \( H_1 \subset H_2 \).

Returning to the claim, for any \( i \in I \), we can invoke the lemma to choose a small radius \( r_i \in \mathbb{R}_{>0} \) to be sure that the restriction of \( g_i : T_1 \rightarrow \mathbb{R} \) to the intersection \( H_a \cap C_i \subset T_1 \) is strictly positive, for all \( a \in I \) with \( a > i \). Then later in our sequence of choices of constants, for each \( a \in I \) with \( a > i \), we can choose a small radius \( r_a \in \mathbb{R}_{>0} \), so that \( C_a \subset T_1 \) is as close as we like to \( H_a \), hence ensuring that the restriction of \( g_i : T_1 \rightarrow \mathbb{R} \) to the intersection \( C_a \cap C_i \subset T_1 \) is strictly positive. Thus if \( i \in I_p \) is not a leaf vertex, so there is \( a \in I_p \) with \( a > i \), we must have \( g_i(p) > 0 \).

Thus the claim holds and this completes the proof of the proposition. \( \square \)

5.4.4. Smoothed total expansion. Recall the smoothing homeomorphism

\[ \Psi : M \xrightarrow{\sim} M \]

Definition 5.33. Define the directed expansion \( E \subset M \) to be the image of the total expansion \( E = \Psi(E) \).

Theorem 5.34. The directed expansion \( E \subset M \) is a hypersurface in good position with a canonical coorientation and generalized smooth arboreal hypersurface singularities.

Proof. Immediate from Thm. 4.22, Cor. 5.23, and Prop. 5.31. \( \square \)

We will write \( \Lambda_E \subset S^* M \) for the positive coray bundle of the directed expansion \( E \subset M \).

6. Invariance of sheaves

Fix once and for all a field \( k \) of characteristic zero.

Let \( M \) be a manifold with spherically projective cotangent bundle \( \pi : S^* M \rightarrow M \).

6.1. Singular support. For the material reviewed here, the standard reference is \[13\].

6.1.1. Basic notions. Let \( Sh(M) \) denote the dg category of complexes of sheaves of \( k \)-vector spaces on \( M \) such that each object is constructible with respect to some Whitney stratification. (This choice of definition has the pitfall that finite collections of Whitney stratifications do not necessarily admit a common refinement, but we will always work with specific Whitney stratifications and never come near this danger.) We will abuse terminology and refer to objects of \( Sh(M) \) as sheaves on \( M \).

To any object \( F \in Sh(M) \), one can associate its singular support \( ss(F) \subset S^* M \). This is a closed Legendrian recording those codirections in which the propagation of sections of \( F \) is obstructed. Its behavior under standard functors is well understood including its behavior under Verdier duality \( ss(D_M(F)) = -ss(F) \). One has the vanishing \( ss(F) = \emptyset \) if and only if the cohomology sheaves of \( F \) are locally constant. We will abuse terminology and refer to such objects of \( Sh(M) \) as local systems on \( M \).
Example 6.1. To fix conventions, suppose \( i : U \to M \) is the inclusion of an open submanifold whose closure is a submanifold with boundary modeled on a Euclidean halfspace. Then the singular support \( \Lambda_U = ss(i_! k_U) \subset S^* M \) of the extension by zero \( i_! k_U \in Sh(M) \) consists of the spherical projectivization of the outward conormal codirection along the boundary \( \partial U \subset M \). If near a point \( p \in \partial U \), we have \( U = \{ x < 0 \} \), for a local coordinate \( x \), then \( \Lambda_U \mid_p = ss(i_! k_U) \mid_p \) is the spherical projectivization of the ray \( \mathbb{R}_{\geq 0}(dx) \).

More generally, suppose \( i : U \to M \) is the inclusion of an open submanifold whose closure is a submanifold with corners modeled on a Euclidean quadrant. Then the singular support \( \Lambda_U = ss(i_! k_U) \subset S^* M \) consists of the spherical projectivization of the outward conormal cone along the boundary \( \partial U \subset M \). If near a point \( p \in \partial U \), we have \( U = \{ x_1, \ldots, x_k < 0 \} \), for local coordinates \( x_1, \ldots, x_k \), then \( \Lambda_U \mid_p = ss(i_! k_U) \mid_p \) is the spherical projectivization of the cone \( \mathbb{R}_{\geq 0}(dx_1, \ldots, dx_k) \).

Fix a closed Legendrian \( \Lambda \subset S^* M \). For example, given \( S = \{ X_\alpha \}_{\alpha \in \Lambda} \) a Whitney stratification of \( M \), one could take the union of the spherically projectivized conormals to the strata

\[
\Lambda_S = \bigcup_{\alpha \in \Lambda} S_{X_\alpha} M \subset S^* M
\]

In general, given any closed Legendrian \( \Lambda \subset S^* M \), we will always assume \( M \) admits a Whitney stratification \( S \) such that \( \Lambda \subset \Lambda_S \).

Let \( Sh_\Lambda(M) \subset Sh(M) \) denote the full dg subcategory of objects with singular support lying in \( \Lambda \subset S^* M \). For example, for \( S \) a Whitney stratification, \( Sh_{\Lambda_S}(M) \subset Sh(M) \) consists precisely of \( S \)-constructible sheaves. In general, if \( \Lambda \subset \Lambda_S \), then objects of \( Sh_\Lambda(M) \subset Sh(M) \) are in particular \( S \)-constructible, while possibly satisfying further constraints.

6.1.2. Non-characteristic isotopies. Let us recall a key property of singular support. Suppose \( \Lambda_1, \Lambda_2 \subset S^* M \) are closed Legendrians, and \( \psi_t : M \to M \) is an isotopy such that \( \psi_t(\Lambda_1) \cap \Lambda_2 = \emptyset \), for all \( t \). Then for any \( \mathcal{F}_1 \in Sh_{\Lambda_1}(M), \mathcal{F}_2 \in Sh_{\Lambda_2}(M) \), the complex \( \text{Hom}_{Sh(M)}(\psi_t(\mathcal{F}_1), \mathcal{F}_2) \) is locally independent of \( t \) in the sense that it forms a local system on the space of parameters \( t \).

For a basic example of this, recall that given an open subset \( i : U \to M \), there is a functorial identification

\[
\Gamma(U, \mathcal{F}) \simeq \text{Hom}_{Sh(M)}(i_! k_U, \mathcal{F})
\]

Suppose \( \psi_t : M \to M \) is an isotopy, and \( i_t : U_t \to M \) is family of open submanifolds with boundary given by the isotopy \( U_t = \psi_t(U_0) \). Let \( \Lambda \subset S^* M \) be a closed Legendrian disjoint from the outward conormal direction \( \Lambda_{U_t} \subset S^* M \) along the boundary \( \partial U_t \subset M \), for all \( t \). Then for any \( \mathcal{F} \in Sh_\Lambda(M) \), the sections

\[
\Gamma(U_t, \mathcal{F}) \simeq \text{Hom}_{Sh(M)}(i_t k_{U_t}, \mathcal{F})
\]

are locally independent of \( t \). Similarly, for the closed complement \( j_t : Y_t = M \setminus U_t \to M \), the sections

\[
\Gamma(Y_t, \mathcal{F}) \simeq \Gamma(Y_t, j_t^! \mathcal{F}) \simeq \text{Cone}(\text{Hom}(\Gamma(M, \mathcal{F}) \to \Gamma(U_t, \mathcal{F}))[−1])
\]

are locally independent of \( t \).

For a specific instance of this, suppose \( \Lambda \subset S^* M \) is a closed Legendrian, and \( f : M \to N \) is a proper fibration that is \( \Lambda \)-non-characteristic in the sense that the spherical projectivization of \( \text{Im}(df^*) \subset T^* M \) is disjoint from \( \Lambda \). Then for any \( \mathcal{F} \in Sh_{\Lambda}(M) \), the pushforward \( f_* \mathcal{F} \in Sh(N) \) is a local system. This can be put into the above setup by recalling for \( U \subset N \) an open subset with inverse image \( i : f^{-1}(U) \to M \), the functorial identifications

\[
\Gamma(U, f_* \mathcal{F}) \simeq \Gamma(f^{-1}(U), \mathcal{F}) \simeq \text{Hom}_{Sh(M)}(i_! k_{f^{-1}(U)}, \mathcal{F})
\]
6.2. **Projections and orthogonality.** Let $H \subset M$ be a directed hypersurface with positive coray bundle $\Lambda \subset S^*M$. Fix a Whitney stratification of $H \subset M$ satisfying the setup of Sect. 5.1 and fix a compatible system of control data.

In this section, we will focus on a single closed stratum and its tubular neighborhood, and thus break from our usual notational conventions to reduce clutter.

6.2.1. **Microlocal projections.** Let $i_Y : Y \to H$ be the inclusion of a closed stratum with tubular neighborhood $T \subset M$, tubular distance function $\rho : T \to \mathbb{R}$ and tubular projection $\pi : T \to Y$. Let $j_Y : T' = T \setminus Y \to T$ be the inclusion of the open complement. In what follows, we can take $M = T$.

Recall there are finitely many codirections $\lambda_i : Y \to S_Y^* M$, for $i = 1, \ldots, k$, as well as disjoint union decompositions

$$\Lambda|_Y = \prod_{i=1}^k \lambda_i(Y) \quad \Lambda|_T = \prod_{i=1}^k \Lambda_i$$

such that $\Lambda_i|_Y = \lambda_i(Y)$. The front projection of $\Lambda_i \subset S^*T$ is itself a directed hypersurface $H_i \subset T$ with positive coray bundle $\Lambda_i \subset S^*T$.

We have the evidently fully faithful inclusions $Sh_{\Lambda_i}(T) \subset Sh_{\Lambda}(T)$. In the other direction, microlocal cut-offs provide canonical functors

$$\Psi_i : Sh_{\Lambda}(T) \longrightarrow Sh_{\Lambda_i}(T)$$

equipped with natural transformations

$$p_i : F \longrightarrow \Psi_i(F) \quad F \in Sh_{\Lambda}(T)$$

Taking the direct sum, we obtain a natural transformation

$$\bigoplus_{i=1}^k p_i : F \longrightarrow \bigoplus_{i=1}^k \Psi_i(F) \quad F \in Sh_{\Lambda}(T)$$

The cone $L = Cone(\bigoplus_{i=1}^k p_i)$ has no singular support so is a local system. We have a functorial presentation of $F \in Sh_{\Lambda}(B)$ itself as a cone

$$F \simeq Cone(\bigoplus_{i=1}^k \Psi_i(F) \longrightarrow L)$$

6.2.2. **Single codirection.** Now suppose further that $\Lambda|_Y = \lambda(Y)$ for a single codirection $\lambda : Y \to S_Y^* M$. Then for $F \in Sh_{\Lambda}(T)$, we have two canonical morphisms

$$\gamma : \pi^* \pi_! F \longrightarrow F \quad \gamma_c : F \longrightarrow \pi^! \pi_! F$$

Observe that $\pi_* F$, $\pi_! F$ are local systems, so $\pi^* \pi_!, \pi^! \pi_!$ are local systems, since $\pi$ is non-characteristic with respect to $S$, and hence with respect to $\Lambda$ since $\Lambda \subset \Lambda_S$.

Introduce the full subcategories

$$Sh_{\Lambda}(T)^0 \subset Sh_{\Lambda}(T) \quad Sh_{\Lambda}(T)_{!}^0 \subset Sh_{\Lambda}(T)$$

de of $F \in Sh_{\Lambda}(T)$ with $\pi_* F \simeq 0$ respectively $\pi_! F \simeq 0$. Observe that $F \in Sh_{\Lambda}(T)^0$ respectively $F \in Sh_{\Lambda}(T)_{!}^0$ if and only if $i_Y^* F \simeq 0$ respectively $i_Y^! F \simeq 0$, or in turn, if and only if the canonical map $j_Y j_Y^! F \to F$ respectively $F \to j_Y j_Y^* F$ is an isomorphism. Verdier duality restricts to an equivalence

$$D_B : (Sh_{\Lambda}(T)^0)_{\text{op}} \longrightarrow Sh_{-\Lambda}(T)_{!}^0$$

The cones $F^0_* \subset Sh_{\Lambda}(T)^0$ respectively $F^0_{!} \subset Sh_{\Lambda}(T)_{!}^0$ satisfy the vanishing $\pi_* F^0_* \simeq 0$, $\pi_! F^0_{!} \simeq 0$ or in other words lie in the full subcategories

$$F^0_* \in Sh_{\Lambda}(T)^0 \quad F^0_{!} \in Sh_{\Lambda}(T)_{!}^0$$
There are functorial presentations of \( \mathcal{F} \in \text{Sh}_A(T) \) itself as a cone

\[
\mathcal{F} \simeq \text{Cone}(\mathcal{F}_0^0[-1] \longrightarrow \pi^*\pi_*\mathcal{F}) \quad \quad \quad \mathcal{F} \simeq \text{Cone}(\pi^*\pi_*\mathcal{F} \longrightarrow \mathcal{F}_0^0)[-1]
\]

Continuing with \( \Lambda|_Y = \lambda(Y) \) for a single codirection \( \lambda : Y \to S^*_Y M \), choose any smooth path \( \ell : \mathbb{R} \to T \) so that \( \ell(0) \in Y \) is the only intersection of \( \ell(\mathbb{R}) \) with \( H \), and also \( \lambda(\ell'(0)) > 0 \). Then for any \( \mathcal{F} \in \text{Sh}_A(T) \), the pullbacks \( \ell^*(\mathcal{F}), \ell^!(\mathcal{F}) \in \text{Sh}(\mathbb{R}) \) are constructible with respect to \( \{0\}, \mathbb{R} \setminus \{0\} \). Furthermore, the singular support conditions imply the following are local systems

\[
\ell^!(\mathcal{F})|_{\mathbb{R} < 0} \in \text{Loc}(\mathbb{R} < 0) \quad \quad \quad \ell^*(\mathcal{F})|_{\mathbb{R} > 0} \in \text{Loc}(\mathbb{R} > 0)
\]

Thus in particular for \( \mathcal{F}_0^0 \in \text{Sh}_A(T)_0^0, \mathcal{F}_1^0 \in \text{Sh}_A(T)_1^0 \), the vanishings \( i^*_Y(\mathcal{F}_0^0) \simeq 0, i^*_Y(\mathcal{F}_1^0) \simeq 0 \) respectively imply the vanishings

\[
(6.1) \quad \ell^!(\mathcal{F}_0^0)|_{\mathbb{R} < 0} \simeq 0 \quad \quad \ell^*(\mathcal{F}_1^0)|_{\mathbb{R} > 0} \simeq 0
\]

Informally speaking, if we think of \( \lambda \) as pointing “up” along \( Y \), then \( \mathcal{F}_0^0 \) vanishes “above” \( H \), and \( \mathcal{F}_1^0 \) vanishes “below” \( H \).

6.2.3. Orthogonality of codirections. Now let us return to the possibility that \( \Lambda|_Y \) has more than one codirection and focus on the interaction of two distinct codirections \( \lambda_1, \lambda_2 : Y \to \Lambda|_Y \) with \( \Lambda_1 = \lambda_1(Y), \Lambda_2 = \lambda_2(Y) \).

**Lemma 6.2.** For any \( \mathcal{F}_1 \in \text{Sh}_A(T)_1^0, \mathcal{F}_2 \in \text{Sh}_A(T)_2^0 \), we have \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, \mathcal{F}_2) \simeq 0 \).

For any \( \mathcal{F}_1 \in \text{Sh}_A(T)_1^0, \mathcal{F}_2 \in \text{Sh}_A(T)_2^0 \), we have \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, \mathcal{F}_2) \simeq 0 \).

**Proof.** The second statement follows from the first by duality.

To prove the first, we will move \( \mathcal{F}_2 \) through a non-characteristic isotopy to a position where it is evident that \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, \mathcal{F}_2) \simeq 0 \).

Note that it suffices to prove the assertion locally in \( Y \). Thus we may fix a smooth identification \( T \simeq \mathbb{R}^{k+\ell+1}, Y \simeq \mathbb{R}^k \times \{0\} \) such that \( \pi : T \to Y \) is the standard projection \( \mathbb{R}^{k+\ell+1} \to \mathbb{R}^k \).

Moreover, for each \( i = 1, 2 \), we can arrange that \( \Lambda|_{\mathcal{Y}} \simeq \mathbb{R}^k \times \{\lambda_i\} \subset S^*_Y T \simeq \mathbb{R}^k \times S^i \), and that \( \Lambda_i \subset S^i T \simeq \mathbb{R}^{k+i+1} \times \{\lambda_i\} \).

*(Step 1)* If \( \lambda_2 = -\lambda_1 \), then proceed to *(Step 2)* below. Else \( \lambda_1, \lambda_2 \) are linearly independent so span a two-dimensional plane \( P \subset \mathbb{R}^{k+\ell+1} \subset \mathbb{R}^{k+\ell+1} \). For \( \theta \in [0, 1] \), let \( R_{\theta} : \mathbb{R}^{k+\ell+1} \to \mathbb{R}^{k+\ell+1} \) be the orthogonal rotation of \( P \) fixing \( P^\perp \), such that \( R_0 = \text{id}, R_1(\lambda_2) = -\lambda_1 \), and \( R_0(\lambda_2) \), for \( \theta \in [0, 1] \), traverses the short arc of directions in \( P \) from \( \lambda_2 \) to \( -\lambda_1 \) (so not passing through \( \lambda_1 \)).

Viewing \( R_{\theta} : \mathbb{R}^{k+\ell+1} \to \mathbb{R}^{k+\ell+1} \) as an isotopy, observe that it satisfies \( R_{\theta}(\Lambda_2) \cap \Lambda_1 = \emptyset \), for \( \theta \in [0, 1] \). Thus \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, R_{\theta}(\mathcal{F}_2)) \) is independent of \( \theta \in [0, 1] \).

*(Step 2)* By *(Step 1)*, we may assume \( \Lambda_2 = -\lambda_1 \). Without loss of generality, we may further assume \( \lambda_1 = dy_0 \) so \( \lambda_2 = -dy_0 \). For \( t \in \mathbb{R} \), let \( T_t : \mathbb{R}^{k+\ell+1} \to \mathbb{R}^{k+\ell+1} \) be the translation \( T_t(x_1, \ldots, x_k, y_0, y_1, \ldots, y_\ell) = (x_1, \ldots, x_k, y_0 + t, y_1, \ldots, y_\ell) \). Viewing \( T_t : \mathbb{R}^{k+\ell+1} \to \mathbb{R}^{k+\ell+1} \) as an isotopy, observe that it satisfies \( T_t(\Lambda_2) \cap \Lambda_1 = \emptyset \), for \( t \in \mathbb{R} \). Thus \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, T_t(\mathcal{F}_2)) \) is independent of \( t \in \mathbb{R} \).

Finally, for \( t > 0 \), the vanishing \( (6.1) \) implies the supports of \( \mathcal{F}_1, T_t(\mathcal{F}_2) \) are disjoint. Hence \( \text{Hom}_{\text{Sh}_A(T)}(\mathcal{F}_1, T_t(\mathcal{F}_2)) \simeq 0 \) and we are done.

6.3. Specialization of sheaves. Let \( X \subset M \) be a closed subspace with Whitney stratification \( S = \{S_\alpha\}_{\alpha \in A} \). Fix a compatible system of control data \( \{(T_\alpha, \rho_\alpha, \pi_\alpha)\}_{\alpha \in A} \).

Fix a small \( \epsilon > 0 \). For each \( \alpha \in A \), recall the mapping \( \Pi_\alpha : M \to M \) and the almost retraction

\[
r : M \longrightarrow M \quad \quad r = \Pi_{\alpha_0}\Pi_{\alpha_1} \cdots \Pi_{\alpha_N}
\]
where $N + 1 = |A|$ and the indices $\alpha_i \in A$ can be arbitrarily ordered.

We will record some of its simple properties; we leave the details of the proofs to the reader.

**Lemma 6.3.** For each $\alpha \in A$, pushforward along $\Pi_\alpha : M \to M$ is canonically equivalent to the identity when restricted to local systems

$$\Pi_\alpha \simeq \text{id} : \text{Loc}(M) \xrightarrow{\sim} \text{Loc}(M)$$

More generally, it is canonically equivalent to the identity when restricted to $S$-constructible sheaves

$$\Pi_\alpha \simeq \text{id} : \text{Sh}(S(M)) \xrightarrow{\sim} \text{Sh}(S(M))$$

The same assertions hold for pushforward along $r : M \to M$.

**Proof.** We leave the assertions for $\Pi_\alpha$ to the reader. Since $r = \Pi_{\alpha_0} \Pi_{\alpha_1} \cdots \Pi_{\alpha_N}$, the assertions for $\Pi_\alpha$ imply them for $r$. \hfill $\square$

**Lemma 6.4.** Let $X_0 \subset X$ be a closed stratum with tubular neighborhood $T_0 \subset M$.

Restriction of $S$-constructible sheaves is an equivalence

$$\text{Sh}(S(M \setminus X_0)) \xrightarrow{\sim} \text{Sh}(S(M \setminus T_0[\leq \epsilon]))$$

with an inverse provided by the pushforward

$$\Pi_0 : \text{Sh}(S(M \setminus T_0[\leq \epsilon])) \xrightarrow{\sim} \text{Sh}(S(M \setminus X_0))$$

Suppose in addition $X$ is a directed hypersurface with positive coray bundle $\Lambda$. Then restriction of sheaves is an equivalence

$$\text{Sh}(S(M \setminus X_0)) \xrightarrow{\sim} \text{Sh}(S(M \setminus T_0[\leq \epsilon]))$$

with an inverse provided by the pushforward

$$\Pi_0 : \text{Sh}(S(M \setminus T_0[\leq \epsilon])) \xrightarrow{\sim} \text{Sh}(S(M \setminus X_0))$$

**Proof.** For the first assertion, the mapping $\Pi_0 : M \setminus T_0[\leq \epsilon] \to M \setminus X_0$ is a stratum-preserving homeomorphism and the identity on $M \setminus T_0[\leq 2\epsilon]$.

For the second assertion, thanks to the first, it suffices to show $\Pi_0$ does not introduce any spurious singular support outside of $\Lambda$. More generally, it suffices to show the following. Let $p \in X$ be a point in a closed stratum $X_0 \subset X$, and $B(p) \subset M$ a small open ball around $p$. Let $q \in B(p)$ be another point in the same stratum $X_0 \subset X$, and $B(q) \subset B(p)$ a small open ball around $q$. Then it suffices to show for any $\mathcal{F} \in \text{Sh}(S(M))$, if $ss(\mathcal{F})|_{B(q)} \subset \Lambda$, then $ss(\mathcal{F})|_{B(p)} \subset \Lambda$.

The assertion is local and we may assume $M = T_0 = \mathbb{R}^{k+\ell+1}$, $X_0 = \mathbb{R}^k$, and the projection $\pi_0 : T_0 \to X_0$ is the standard projection $\mathbb{R}^{k+\ell+1} \to \mathbb{R}^k$.

Suppose some $\xi \in T_p^* \mathbb{R}^{k+\ell+1} \simeq \mathbb{R}^{k+\ell+1}$ represents a point of $ss(\mathcal{F})$ but not a point of $\Lambda$. Since $\mathcal{F}$ is $S$-constructible, we have $\xi \in T_p^* \mathbb{R}^{k+\ell+1}|_{p} \simeq \mathbb{R}^{\ell+1}$. Consider the corresponding linear function $\xi : \mathbb{R}^{k+\ell+1} \to \mathbb{R}$. Fix a small $\epsilon < 0$, and consider the inclusion $i : \{\xi \leq \epsilon\} \to \mathbb{R}^{k+\ell+1}$. Then it suffices to see that $\pi_0 i^* \mathcal{F}$ is locally constant on $\mathbb{R}^k$, since then its vanishing at some $q$ will imply its vanishing at $p$. But since $X$ is in good position, and $\xi$ does not represent a point of $\Lambda$, the map $\pi_0 \times \xi$ is non-characteristic near the value $\xi = \epsilon$, and the assertion follows. \hfill $\square$

**Lemma 6.5.** Let $X_0 \subset X$ be a closed stratum with tubular neighborhood $T_0 \subset M$.

Introduce the mapping

$$r' : M \setminus X_0 \longrightarrow M \setminus X_0 \quad r' = \Pi_{\alpha_1} \Pi_{\alpha_2} \cdots \Pi_{\alpha_N}$$

where $\Pi_0$ is omitted from the composition.
For $\mathcal{F} \in \text{Sh}(M)$, suppose $r'_i(\mathcal{F}|_{M \setminus X_0}) \in \text{Sh}(M \setminus X_0)$ is $S$-constructible. Then there is a functorial equivalence

$$r_*(\mathcal{F})|_{M \setminus X_0} \simeq r'_i(\mathcal{F}|_{M \setminus X_0})$$

Proof. By Lemma 6.4 we have

$$r'_i(\mathcal{F}|_{M \setminus X_0}) \simeq \Pi_0*(r'_i(\mathcal{F}|_{M \setminus X_0})|_{M \cap T_0(\leq \epsilon)})$$

By construction, we also have

$$\Pi_0*(r'_i(\mathcal{F}|_{M \setminus X_0})|_{M \cap T_0(\leq \epsilon)}) \simeq \Pi_0*(r'_i(\mathcal{F}|_{M \setminus X_0}))|_{M \setminus X_0} \simeq r_*(\mathcal{F})|_{M \setminus X_0}$$

$\square$

6.4. Singular support of specialization. Let $H \subset M$ be a directed hypersurface with positive coray bundle $\Lambda \subset S^*M$. Fix a Whitney stratification $S = \{H_i\}_{i \in I}$ satisfying the setup of Sect. 5.1 and fix a compatible system of control data $\{(T_i, \rho_i, \pi_i)\}_{i \in I}$.

Fix a small $\epsilon > 0$. For each $i \in I$, recall the mapping $\Pi_i : M \rightarrow M$ and the almost retraction

$$r : M \longrightarrow M \quad r = \Pi_{i_0} \Pi_{i_1} \cdots \Pi_{i_N}$$

where $N + 1 = |I|$ and the indices can be arbitrarily ordered.

Fix sequentially small parameters $d_i > 0, r_i > 0, s_i > 0$, for $i \in I$. Fix a small smoothing constant $\delta > 0$. Recall the directed cylinder $C \subset M$ with positive coray bundle $\Lambda_C \subset S^*M$, and the directed expansion $E \subset M$ with positive coray bundle $\Lambda_E \subset S^*M$.

The main goal of this section is Theorem 6.7 below that states that pushforward along the almost retraction $r : M \rightarrow M$ induces a functor

$$r_* : \text{Sh}_{\Lambda_E}(M) \longrightarrow \text{Sh}_{\Lambda}(M)$$

In other words, pushforward takes sheaves with singular support in $\Lambda_E$ to sheaves with singular support in $\Lambda$.

6.4.1. Interaction with directed cylinder. We will arrive at our main goal after the following coarser estimate.

Proposition 6.6. Pushforward along the almost retraction $r : M \rightarrow M$ induces a functor

$$r_* : \text{Sh}_{\Lambda_C}(M) \longrightarrow \text{Sh}_{\Lambda_S}(M)$$

In other words, pushforward takes sheaves with singular support in $\Lambda_C$ to sheaves with singular support in $\Lambda_S$, or in other words, to $S$-constructible sheaves.

Proof. By induction on the number of strata of $H \subset M$.

The base case $H = \emptyset$ is immediate: $r$ is the identity map of $M$.

Suppose given a closed stratum $t_0 : H_0 \rightarrow M$.

Set $M(> \epsilon) = M \setminus T_0(\leq \epsilon)$, $H(> \epsilon) = H \cap M(> \epsilon)$. The Whitney stratification, system of control data and family of lines for $H \subset M$ immediately provide the same for $H(> \epsilon) \subset M(> \epsilon)$.

Denote the induced Whitney stratification by $S(> \epsilon) = S(> \epsilon)$, and the resulting almost retraction by $r(> \epsilon) = r|_{M(> \epsilon)}$. Furthermore, starting with these data, the expansion algorithm yields a directed cylinder $C(> \epsilon) \subset M(> \epsilon)$ with positive coray bundle $\Lambda_{C(> \epsilon)} \subset S^*M(> \epsilon)$ such that $C(> \epsilon) = C(> \epsilon)$, $\Lambda_{C(> \epsilon)} = \Lambda_{C(> \epsilon)}$.

By induction, since $H(> \epsilon)$ has fewer strata than $H$, pushforward induces a functor

$$r(> \epsilon)_* : \text{Sh}_{\Lambda_{C(> \epsilon)}}(M(> \epsilon)) \longrightarrow \text{Sh}_{\Lambda_{S(> \epsilon)}}(M(> \epsilon))$$
For any $F \in Sh_{\Lambda}(M)$, by construction, we have
\[ r^*_s(F)|_{M \setminus H_0} \simeq \Pi_0 r^*[>\epsilon](F|_{M[>\epsilon]}) \]
Thus Lemma 6.5 implies that $r^*_s(F)|_{M \setminus H_0}$ is $S$-constructible.

Now it suffices to show $i_0^* r^*_s F$ is a local system. By base change, we have
\[ i_0^* r^*_s F \simeq r^*_s j^{[\leq \epsilon]} F \]
where $j^{[\leq \epsilon]} : T_0[\leq \epsilon] \to T_0 \to M$ is the inclusion.

Recall that the almost retraction $r$ is independent of the ordering of the indices so that in particular $r = \Pi_{i_1} \cdots \Pi_{i_N} \Pi_0$. Recall that $\Pi_0|_{T_0[\leq \epsilon]} = \pi_0$, and that $\Pi_{i}|_{H_0} = id$ for all $i \neq 0$. Thus we have
\[ r^*_s j^{[\leq \epsilon]} F \simeq \Pi_{i_1^*} \cdots \Pi_{i_N^*} \Pi_{0^*} j^{[\leq \epsilon]} F \simeq \Pi_{i_1^*} \cdots \Pi_{i_N^*} \pi_0 j^{[\leq \epsilon]} F \simeq \pi_0 j^{[\leq \epsilon]} F \]
Recall the smoothing homeomorphism
\[ \Psi = \Psi_{i_N} \circ \cdots \circ \Psi_{i_1} \circ \Psi_0 : M \sim M \]
where $N + 1 = |I|$, and the elements $0, i_1, \ldots, i_N \in I$ are ordered compatibly with the partial order on $I$. Recall that for $j \in I$ with image $j \in L$, the restriction $\Psi_j|_{T_0}$ satisfies $\Psi_j|_{T_0} = id$, when $i$ is incomparable to $0$, and $\pi_0 \Psi_j|_{T_0} = \pi_0$, when $i$ is greater than or equal to $0$. Thus altogether $\pi_0 \Psi|_{T_0} = \pi_0$.

For any $i \in I$, the projection $\pi_0$ is non-characteristic with respect to the truncated cylinder $C_i \subset M$, and hence to the total cylinder $C \subset M$. Thanks to the identity $\pi_0 \Psi|_{T_0} = \pi_0$, the projection $\pi_0$ is also non-characteristic with respect to the smoothing $C_i = \Psi(C_i)$, and hence to the directed cylinder $C = \Psi(C)$, and in particular to its positive coray bundle $\Lambda_C$. Finally, since $\pi_0$ is a proper fibration and non-characteristic with respect to $\Lambda_C$, and hence with respect to $ss(F) \subset \Lambda_C$, we conclude that $\pi_0 j^{[\leq \epsilon]} F$ is a local system on $H_0$. \qed

6.4.2. Interaction with directed expansion. Now we will prove the main assertion of this section.

**Theorem 6.7.** Pushforward along the almost retraction $r : M \to M$ induces a functor
\[ r^*_s : Sh_{\Lambda}(M) \longrightarrow Sh_{\Lambda}(M) \]
In other words, pushforward takes sheaves with singular support in $\Lambda_E$ to sheaves with singular support in $\Lambda$.

**Proof.** By induction on the number of strata of $H \subset M$.

The base case $H = \emptyset$ is immediate: $r$ is the identity map of $M$.

Suppose given a closed stratum $i_0 : H_0 \to M$.

Set $M[>\epsilon] = M \setminus T_0[\leq \epsilon]$, $H[>\epsilon] = H \cap M[>\epsilon]$, $A[>\epsilon] = A|_{M[>\epsilon]}$. The Whitney stratification, system of control data and family of lines for $H \subset M$ immediately provide the same for $H[>\epsilon] \subset M[>\epsilon]$. Denote the resulting almost retraction by $r[>\epsilon] : M[>\epsilon] \to M[>\epsilon]$ and note that $r[>\epsilon] = r|M[>\epsilon]$. Additionally, starting with these data, our constructions give a directed expansion $E[>\epsilon] \subset M[>\epsilon]$ with positive coray bundle $\Lambda_{E[>\epsilon]} \subset S^* M[>\epsilon]$ such that $E[>\epsilon] = E \cap M[>\epsilon]$, $\Lambda_{E[>\epsilon]} = \Lambda_{E|M[>\epsilon]}$.

By induction, since $H[>\epsilon]$ has fewer strata than $H$, pushforward induces a functor
\[ r[>\epsilon] : Sh_{\Lambda_{E[>\epsilon]}}(M[>\epsilon]) \longrightarrow Sh_{\Lambda[>\epsilon]}(M[>\epsilon]) \]
By the second assertion of Lemma 6.4 for $F \in Sh_{\Lambda}(M)$, we then have
\[ (r_s F)|_{M \setminus H_0} \in Sh_{\Lambda}(M \setminus H_0) \]
Therefore, for $\mathcal{F} \in \text{Sh}_{\Lambda_E}(M)$, it only remains to show

$$ss(r_*\mathcal{F})|_{H_0} \subset \Lambda|_{H_0}$$

By construction, we have an inclusion of directed hypersurfaces $\mathcal{E} \subset \mathcal{C}$ and positive coray bundles $\Lambda_\mathcal{E} \subset \Lambda_\mathcal{C}$. Thus we have $\text{Sh}_{\Lambda_\mathcal{E}}(M) \subset \text{Sh}_{\Lambda_\mathcal{C}}(M)$, hence thanks to Prop. 6.6 we have $r_*\mathcal{F} \in \text{Sh}_{\mathcal{G}}(M)$, so in particular

$$ss(r_*\mathcal{F})|_{H_0} \subset S^{\mathcal{E}}_{H_0}M$$

Hence for each $x \in H_0$, we may restrict to the normal slice

$$\pi_0^{-1}(x) \subset T_0$$

Without loss of generality, we may assume $\pi_0^{-1}(x) = \mathbb{R}^n$, $x = 0$, and $\rho_0|_{\pi_0^{-1}(x)}$ is the standard Euclidean inner product. The positive corays $\Lambda|_x = \{\lambda_1, \ldots, \lambda_k\}$ are represented by pairing with nonzero vectors $v_1, \ldots, v_k \in \mathbb{R}^n$.

For $\xi \notin \Lambda|_x$, we seek a small $\delta > 0$ so that

$$\Gamma_{\xi \geq 0}(B(\delta), r_*\mathcal{F}) \simeq 0$$

where $B(\delta) \subset \mathbb{R}^n$ is the open ball of radius $\delta > 0$ around $0$.

For any $t \in [0, \epsilon]$, introduce the subspace $B(t, 2\epsilon)^- \subset B(2\epsilon)$, where $t \leq \rho \leq 2\epsilon$, $\xi < 0$, so in particular $B(0, 2\epsilon)^- = B(2\epsilon)^-$. Unpacking the constructions, we seek that

$$H^*(B(2\epsilon), B(\epsilon, 2\epsilon)^-; \mathcal{F}) \simeq 0$$

where $\epsilon > 0$ is the original constant selected once and for all.

We will proceed by induction on the finite set $\Lambda|_x$.

The arguments in the base case, when $\Lambda|_x = \{\xi_1\}$ is a single codirection, and in the general inductive step will be similar. We will show by a series of non-characteristic moves that we can change the subspace $B(\epsilon, 2\epsilon)^-$ of the pair to be the entire ambient space $B(2\epsilon)$.

(Step 1) The natural map is an isomorphism

$$H^*(B(2\epsilon), B(\epsilon, 2\epsilon)^-; \mathcal{F}) \xrightarrow{\sim} H^*(B(2\epsilon), B(2\epsilon)^-; \mathcal{F})$$

since the isotopy of pairs

$$(B(2\epsilon), B(t, 2\epsilon)^-) \quad t \in [0, \epsilon]$$

is non-characteristic with respect to $\Lambda_\mathcal{E}$.

(Step 2) Suppose $\Lambda|_x = \{\lambda_1, \ldots, \lambda_k\}$ with corresponding radii constants $r_1 < \cdots < r_k < \epsilon$.

For any $t \in [0, 2\epsilon]$, introduce the subspace $U(t, 2\epsilon)^- \subset B(2\epsilon)$, where either $\rho \leq t$ or $t \leq \rho \leq 2\epsilon$, $\xi < 0$, so the union

$$U(t, 2\epsilon)^- = B(t) \cup B(t, 2\epsilon)^-$$

In particular, we have $U(0, 2\epsilon)^- = B(2\epsilon)^-$ and $U(2\epsilon, 2\epsilon)^- = B(2\epsilon)$.

Fix $r$ such that $r_{k-1} < r < r_k$ (when $k = 1$, fix $r$ such that $0 < r < r_k$). We claim the natural map is an isomorphism

$$H^*(B(2\epsilon), B(\epsilon, 2\epsilon)^-; \mathcal{F}) \xrightarrow{\sim} H^*(B(2\epsilon), U(r, 2\epsilon)^-; \mathcal{F})$$

If $k = 1$, then the assertion is clear since the intersection of the isotopy of pairs

$$(B(2\epsilon), U(t, 2\epsilon)^-) \quad t \in [0, r]$$

is non-characteristic with respect to $\Lambda_\mathcal{E}$, since in fact it has constant intersection with $\mathcal{E}$. 
If $k > 1$, then the claim follows by induction: in the locus $r < \rho \leq 2\epsilon$, the pairs are precisely the same, and in the locus $\rho \leq r$, the pairs and $\Lambda_\mathcal{E}$ are precisely what one encounters for $\Lambda|_x = \{\xi_1, \ldots, \xi_{k-1}\}$.

(Step 3) Continuing with the notation of (Step 2), recall that $\xi_0 \neq \lambda_k$. For $\theta \in [0, 1]$, let $\xi_\theta$ be the short arc (not passing through $\lambda_k$) of the great circle of codirections passing through $\xi_0 = \xi$, $\xi_1 = -\lambda_k$.

For any $\theta \in [0, 1]$, introduce the subspace $U(t, 2\epsilon)^-_\theta \subset B(2\epsilon)$, where either $\rho < t$ or $t < \rho \leq 2\epsilon$.

We claim that the isotopy of pairs $$(B(2\epsilon), U(r, 2\epsilon)^-_\theta) \quad \theta \in [0, 1]$$
is non-characteristic with respect to $\Lambda_\mathcal{E}$.

First, note that we can excise the locus $\rho < r$. The only remaining issue is when $\xi_\theta$ passes through or near to some $\lambda_i$, for $i < k$. But here we have two transverse functions $\rho$ and $\theta$ and $\Lambda_\mathcal{E}$ is the disjoint union of something $\rho$-characteristic and something $\theta$-characteristic. In general, such a situation is non-characteristic.

(Step 4) Finally, observe that any linear function lifting $\xi_1 = -\lambda_k$ is non-characteristic with respect to $\Lambda_\mathcal{E}$ on the pair $$(B(2\epsilon), U(r, 2\epsilon)_1^-) \quad \theta \in [0, 1]$$
Thus the relative cohomology vanishes
$$H^*(B(2\epsilon), U(r, 2\epsilon)_1^-; \mathcal{F}) \simeq 0$$
This completes the proof of the theorem. \hfill \Box

6.5. Inverse functor. Our aim here is to show that the functor
$$r_* : Sh_{\Lambda_\mathcal{E}}(M) \longrightarrow Sh_{\Lambda}(M)$$
of Theorem 6.4 is in fact an equivalence.

We will first establish an inductive version of the assertion. Suppose given a closed stratum $i_0 : H_0 \to M$. Suppose $\Lambda|_{H_0} = \lambda(H_0)$ for a single codirection $\lambda : H_0 \to S^*_{H_0}M$.

Recall the constant $r_0 > 0$, indexed by the minimum $0 \in I$, appearing in the construction of the directed expansion $\mathcal{E} \subset M$ with positive coray bundle $\Lambda_\mathcal{E} \subset S^*M$.

Set $M[> r_0] = M \setminus T_0[\leq r_0]$, $H[> r_0] = H \cap M[> r_0]$, $\Lambda[> r_0] = \Lambda_{M[> r_0]}$. The Whitney stratification, system of control data and family of lines for $H \subset M$ immediately provide the same for $H[> r_0] \subset M[> r_0]$. Denote the resulting almost retraction by $r[> r_0] : M[> r_0] \to M[> r_0]$. The expansion data for $H \subset M$ immediately restricts to expansion data for $H[> r_0] \subset M[> r_0]$. Denote the resulting directed expansion by $\mathcal{E}[> r_0] = \mathcal{E} \cap M[> r_0]$ with positive coray bundle $\Lambda[> r_0] = \Lambda_{M[> r_0]}$. Note as well that $E \cap T_0[\leq r_0] = \mathcal{E} \cap S_0[r_0] = E_0$

where $E_0 \subset T_0$ is the expanded stratum of $H_0$.

**Proposition 6.8.** Let $H_0 \subset H$ be a closed stratum.
Suppose $\Lambda|_{H_0} = \lambda(H_0)$ for a single codirection $\lambda : H_0 \to S^*_{H_0}M$.

Suppose the functor $$r[> r_0]_* : Sh_{\Lambda_{E[> r_0]}}(M[> r_0]) \longrightarrow Sh_{\Lambda[> r_0]}(M[> r_0])$$
is an equivalence.
Then the functor
\[
r_* : Sh_{\Lambda E}(M) \xrightarrow{\sim} Sh_{\Lambda}(M)
\]
is also an equivalence.

Proof. We will construct an explicit inverse functor denoted by
\[
s : Sh_{\Lambda}(M) \xrightarrow{} Sh_{\Lambda E}(M)
\]
By our hypotheses, there exists an inverse
\[
s[> r_0] : Sh_{\Lambda[> r_0]}(M[> r_0]) \xrightarrow{\sim} Sh_{\Lambda[> r_0]}(M[> r_0])
\]
Observe that it suffices to prove the assertion for \(M = T_0\).

Recall \(Sh_{\Lambda}(T_0)^0 \subset Sh_{\Lambda}(T_0)\) denotes the full subcategory of \(\mathcal{F} \in Sh_{\Lambda}(T_0)\) with \(\pi_0(\mathcal{F}) \simeq 0\), or equivalently \(i_0^* \mathcal{F} \simeq 0\), or again equivalently, the canonical map \(\mathcal{F} \to j_0^* j_0^* \mathcal{F}\) is an isomorphism.

More generally, recall for \(\mathcal{F} \in Sh_{\Lambda}(T_0)\), the functorial presentation
\[
\mathcal{F} \simeq Cone(\pi_0^* \pi_0 \mathcal{F} \xrightarrow{} \mathcal{F}^0)\][−1]

where \(\mathcal{F}^0 \subset Sh_{\Lambda}(T_0)^0\) is the cone of of the canonical morphism \(\mathcal{F} \to \pi_0^* \pi_0 \mathcal{F}\). Note as well that \(\pi_0^* \pi_0 \mathcal{F} \in \text{Loc}(T_0)\).

By Lemma 6.3 on the full subcategory \(\text{Loc}(T_0) \subset Sh_{\Lambda}(T_0)\), we may set the inverse \(s\) to be the identity. Now we will construct the inverse on the full subcategory \(Sh_{\Lambda}(T_0)^0 \subset Sh_{\Lambda}(T_0)\) as a composition of several functors. To do so, let us walk back through some steps in the construction of the directed hypersurface \(E\) with positive coray bundle \(\Lambda_E\).

Introduce the open inclusion
\[
j[> r_0] : T_0[> r_0] \xrightarrow{} T_0
\]
For \(\mathcal{F} \in Sh_{\Lambda}(T_0)^0\), define the candidate inverse to be the functorial composition
\[
s(\mathcal{F}) = j[> r_0] \circ s[> r_0](\mathcal{F}|_{T_0[> r_0]}) \in Sh(T_0)
\]

Claim 6.9. For \(\mathcal{F} \in Sh_{\Lambda}(T_0)^0\), we have \(ss(s(\mathcal{F})) \in \Lambda_E\).

Proof. Over \(T_0[> r_0]\), the assertion is evident by construction.

Over \(T_0[< r_0]\), we have \(s(\mathcal{F}) = 0\) by definition.

Along \(S_0[r_0]\), we have
\[
ss(s(\mathcal{F})) \subset \mathbb{R}_{\geq 0}(d\rho_0)
\]
thanks to the fact that
\[
\Lambda_E|_{S_0(r)} \subset \mathbb{R}_{\geq 0}(d\rho_0)
\]
and the behavior of singular support under \(*\)-pushforwards [25].

Thus it remains to see that
\[
ss(s(\mathcal{F}))|_{S[r_0] \backslash E_0} = \emptyset
\]
In fact, we have that
\[
s(\mathcal{F})|_{S[r_0] \backslash E_0} = 0
\]
To see this, recall that \(\mathcal{F} \in Sh_{\Lambda}(T_0)^0\) implies by equation [6.1] that \(\mathcal{F}\) vanishes on the \((\lambda < 0)\)-component of the complement \(T_0 \backslash H\). Thus \(s[> r_0](\mathcal{F}|_{T_0[> r_0]})\) vanishes on the \((\lambda < 0)\)-component of the complement \(T_0[> r_0] \backslash E[> r_0]\). Since \(S[r_0] \backslash E_0\) is in the closure of this component, we obtain the asserted vanishing.

The claim confirms we have a well-defined functor \(s : Sh_{\Lambda}(T_0)^0 \to Sh_{\Lambda E}(T_0)\).
Claim 6.10. $r_* \circ s \simeq \text{id}$.

Proof. Recall that $\epsilon > r_0$, and that $r^{-1}(T_0[>0]) = T_0[>\epsilon]$.

Thus by induction, we have a canonical isomorphism

$$(r_*s(F))[T[>0]] \simeq F[T[>0]]$$

Recall that $F \in Sh_{\Lambda}(T_0)_0$ implies the canonical map $F \to j_0j_! F$ is an isomorphism. Therefore it suffices to show that $r_*s(F) \in Sh_{\Lambda}(T_0)_0$, or in other words that $i_0^1r_*s(F) \simeq 0$.

Working locally in $H_0$, by base change, it suffices to show that

$$\Gamma_{T_0[\leq s]}(T_0, s(F)) \simeq 0$$

Unwinding the definitions, we seek

$$\Gamma_{T_0[>r_0]\cap T_0[\leq s]}(T_0, s[>r_0](F|_{T_0[>r_0]})) \simeq 0$$

We have seen that $ss(s[r_0 > 0](F|_{T_0[>r_0]})) \subset \Lambda_{E[>r_0]}$. Since $d\rho_0$ is disjoint from $\Lambda_{E[>r_0]}$, the above relative cohomology vanishes. □

Claim 6.11. For $L \in \text{Loc}(T_0)$, $F \in Sh_{\Lambda}(T_0)_0$, we have canonically

$$\text{Hom}_{Sh(T_0)}(s(F), L) \simeq 0 \quad \text{Hom}_{Sh(T_0)}(L, F) \simeq \text{Hom}_{Sh(T_0)}(L, s(F))$$

Proof. We may work locally in $H_0$, so in particular may assume $L$ is constant.

For the first assertion, by duality, it suffices to show

$$\Gamma(T_0, D(s(F))) \simeq 0$$

Unwinding the definitions, we seek

$$\Gamma(T_0, j[>r_0] D(s[>r_0](F|_{T_0[>r_0]}))) \simeq 0$$

We have seen that $ss(s[>r_0](F|_{T_0[>r_0]})) \subset \Lambda_{E[>r_0]}$. Since $d\rho_0$ is disjoint from $-\Lambda_{E[>r_0]}$, the above relative cohomology vanishes.

For the second assertion, it suffices to show

$$\Gamma(T_0, F) \simeq \Gamma(T_0, s(F))$$

But by the previous claim, we have

$$\Gamma(T_0, F) \simeq \Gamma(T_0, r_*s(F)) \simeq \Gamma(T_0, s(F))$$

□

The claim confirms the functor extends $s : Sh_{\Lambda}(T_0) \to Sh_{\Lambda_0}(T_0)$.

Claim 6.12. $s \circ r_* \simeq \text{id}$.

Proof. Thanks to what we have seen, it suffices to check the assertion on the full subcategory $Sh_{\Lambda_0}(T_0)_0 \subset Sh_{\Lambda_0}(T_0)$ given by objects $F \in Sh_{\Lambda_0}(T_0)$ with $\pi_0 F \simeq 0$.

Since $d\rho_0$ is disjoint from $\Lambda_{E[>r_0]}$, we have $\pi_0 F \simeq 0$ if and only if the canonical map $F \to j[>r_0] j[>r_0]^* F$ is an isomorphism. But we have seen that then $r_* F \in Sh_{\Lambda}(T_0)_0$. By induction, we then have that

$$F|_{T_0[>r_0]} \simeq s[>r_0](r_* F|_{T_0[>r_0]})$$

and so by the definition of $s$, we obtain the assertion. □
Now we will use the previous proposition to establish our main goal.

**Theorem 6.13.** Pushforward along the almost retraction induces an equivalence

\[ r_* : \text{Sh}_{\Lambda}^e(M) \xrightarrow{\sim} \text{Sh}_{\Lambda}(M) \]

**Proof.** By induction on the number of strata of \( H \).

The base case \( H = \emptyset \) is immediate: \( r \) is the identity map of \( M \).

It suffices to focus on a closed stratum \( H_0 \subset M \) with tubular neighborhood \( T_0 \subset M \) and in fact to assume \( M = T_0 \).

Recall the disjoint union decompositions

\[ \Lambda|_{H_0} = \bigoplus_{i=1}^k \Lambda_i(H_0) \quad \Lambda|_{T_0} = \bigoplus_{i=1}^k \Lambda_i \]

such that \( \Lambda|_{H_0} = \lambda_i(H_0) \). The front projection of \( \Lambda_i \subset S^*T_0 \) is itself a directed hypersurface \( H_i \subset T_0 \) with positive coray bundle \( \Lambda_i \subset S^*T_0 \).

Recall the functorial presentation of any \( F \in \text{Sh}_{\Lambda}(T_0) \) as a cone

\[ F \cong \text{Cone}(\oplus_{i=1}^k F_i \longrightarrow L) \]

where \( F_i \in \text{Sh}_{\Lambda_i}(T_0) \), and \( L \in \text{Loc}(T_0) \).

By Proposition 6.8, the restriction of \( r_* \) to each full subcategory \( \text{Sh}_{\Lambda_{\ell_i}}(T_0) \subset \text{Sh}_{\Lambda}(T_0) \) is an equivalence. We also have the full inclusions \( \text{Loc}(T_0) \subset \text{Sh}_{\Lambda_{\ell_i}}(T_0) \). Thus in particular \( r_* \) is essentially surjective.

It remains to check that for distinct codirections \( \lambda_1, \lambda_2 : H_0 \to \Lambda|_{H_0} \) with \( \Lambda_1 = \lambda_1(H_0), \Lambda_2 = \lambda_2(H_0) \), and \( F_1 \in \text{Sh}_{\Lambda_{\ell_j}}(T_0) \), \( F_2 \in \text{Sh}_{\Lambda_{\ell_2}}(T_0) \), the functorial map

\[ r_* : \text{Hom}_{\text{Sh}(T_0)}(F_1, F_2) \longrightarrow \text{Hom}_{\text{Sh}(T_0)}(r_* F_1, r_* F_2) \]

is an isomorphism.

Since the assertion is clear when one sheaf is a local system, it suffices to check it when \( \pi_0 F_1 \cong 0, \pi_0 F_2 \cong 0 \). Note that \( \pi_0 r_* = \pi_0 \), so that then \( \pi_0 r_* F_1 \cong 0, \pi_0 r_* F_2 \cong 0 \) as well. Thus by Lemma 6.2 we have

\[ \text{Hom}_{\text{Sh}(T_0)}(r_* F_1, r_* F_2) \cong 0 \]

so we are left to show

\[ \text{Hom}_{\text{Sh}(T_0)}(F_1, F_2) \cong 0 \]

We will now appeal to the proof of Lemma 6.2. It should be possible to directly apply the proof to \( F_1, F_2 \), except the isotopies involved are less clearly non-characteristic. To take care of this, let us note the following inductive simplification. Recall that \( r = \Pi_0 \hat{r}_0 \) where \( \hat{r}_0 = \Pi_{A_1} \cdots \Pi_{A_N} \). Thus by induction, it suffices to show

\[ \text{Hom}_{\text{Sh}(T_0)}(\hat{r}_0 F_1, \hat{r}_0 F_2) \cong 0 \]

Now for the sheaves \( \hat{r}_0 F_1, \hat{r}_0 F_2 \), we can simply repeat the proof of Lemma 6.2 to move \( \hat{r}_0 F_2 \) through a non-characteristic isotopy to a position where the vanishing is evident. This concludes the proof of the theorem. \( \square \)
6.6. Microlocal sheaves. Let us apply the preceding constructions to microlocal sheaves.

Let $H \subset M$ be a direct hypersurface with positive coray bundle $\Lambda \subset S^* M$. Let $\mu Sh_\Lambda$ denote the dg category of microlocal sheaves supported along $\Lambda$. It is the global sections of a sheaf of dg categories on $\Lambda$.

To understand $\mu Sh_\Lambda$ concretely, let $p \in \Lambda$ be a point, and let $\mu Sh_\Lambda|_p$ be the stalk of $\mu Sh_\Lambda$. Let $x \in H$ be the image of $p \in \Lambda$, assume $\Lambda|_x \subset \Lambda$ consists only of $p$, and that $M$ is itself a small ball around $x$. Then there is the concrete realization as a quotient category

$$\mu Sh_\Lambda|_p \simeq Sh_\Lambda(M)/\text{Loc}(M)$$

There is also the concrete realization as the full subcategory

$$\mu Sh_\Lambda|_p \simeq Sh_\Lambda(M)_1^0 \subset Sh_\Lambda(M)$$

of objects with $\Gamma_c(M, F) \simeq 0$.

Now let $E \subset M$ be the directed expansion of $H \subset M$ with positive coray bundle $\Lambda_E$. Let $\mu Sh_{\Lambda_E}$ be the dg category of microlocal sheaves along $\Lambda_E$.

**Proposition 6.14.** Let $p \in \Lambda$ be a point with image $x \in H$. Assume $\Lambda|_x \subset \Lambda$ consists only of $p$, and that $M$ is itself a small ball around $x$.

Then the natural functor is an equivalence

$$Sh_{\Lambda_E}(M)/\text{Loc}(M) \xrightarrow{\sim} \mu Sh_{\Lambda_E}$$

**Proof.** Regard $x \in H$ as a closed stratum. Let $E_0 \subset E$ be its expanded stratum.

Choose an open cover $\{B_\kappa\}_{\kappa \in K}$ of a neighborhood of $E_0 \subset M$ by a finite collection of small balls $B_\kappa \subset M$ centered at points of $E_0$. Arrange so that their intersections $B_J = \cap_{\kappa \in J} B_\kappa$, for $J \subset K$, are also small balls or empty. Since $E$ deformation retracts to $E_0$, we have the identification

$$\mu Sh_{\Lambda_E} \simeq \lim_{J \subset K} Sh_{\Lambda_E}(B_J)/\text{Loc}(B_J)$$

Any object $F \in Sh_{\Lambda_E(B_J)}/\text{Loc}(B_J)$ admits a canonical representative $F^0[-1] \in Sh_{\Lambda_E(B_J)}^0$ defined by the triangle

$$F \xrightarrow{k_{B_J} \otimes \Gamma_c(B_J, F)} F^0$$

Observe that $F^0[-1] \in Sh_{\Lambda_E(B_J)}^0$ admits the alternative characterization as the canonical representative vanishing below $E_0$. Such representatives are compatible and yield an identification

$$\mu Sh_{\Lambda_E} \simeq \lim_{J \subset K} Sh_{\Lambda_E}(B_J)^0$$

Similarly, we have parallel equivalences

$$Sh_{\Lambda_E}(M)/\text{Loc}(M) \simeq Sh_{\Lambda_E}(M)_1^0 \simeq \lim_{J \subset K} Sh_{\Lambda_E}(B_J)_1^0$$

□

**Corollary 6.15.** Pushforward along the almost retraction induces an equivalence

$$r_* : \mu Sh_{\Lambda_E} \xrightarrow{\sim} \mu Sh_\Lambda$$

**Proof.** Let $p \in \Lambda$ be a point with image $x \in H$. It suffices to prove the assertion when $\Lambda|_x \subset \Lambda$ consists only of $p$, and $M$ is itself a small ball around $x$. Then we have a commutative diagram
of equivalences

\[
\begin{align*}
Sh_{\Lambda}(M)/\text{Loc}(M) & \xrightarrow{\sim} \mu Sh_{\Lambda} \\
\downarrow \sim \downarrow \sim & \\
Sh_{\Lambda}(M)/\text{Loc}(M) & \xrightarrow{\sim} \mu Sh_{\Lambda}
\end{align*}
\]

where the bottom horizontal arrow is the usual quotient presentation. The top horizontal arrow is an equivalence by Prop. 6.14. The left vertical arrow is an equivalence by Thm. 6.13. Thus the right vertical arrow is an equivalence. □

7. Appendix: expansion data

We collect here for convenient reference the hierarchy of constructions and sequentially small constants that enter into the expansion algorithm of Sect. 5.

Let \( H \subset M \) be a directed hypersurface with positive coray bundle \( \Lambda \subset S^*M \). Fix a Whitney stratification \( \{ H_i \}_{i \in I} \) of the hypersurface \( H \subset M \) satisfying the setup of Sect. 5.1. One obtains a compatible decomposition \( \{ \Lambda_i \}_{i \in I} \) of the positive coray bundle \( \Lambda \subset S^*M \) over the map \( I \to I \).

Fix a compatible system of control data \( \{ (T_i, \rho_i, \pi_i) \}_{i \in I} \).

Choose a small \( \epsilon > 0 \). Fix a compatible family of lines, and construct the almost retraction \( r : M \to M \).

Choose a small displacement \( d_i > 0 \), for each \( i \in I \), without concern for the poset structure of \( I \). These will not be used until the construction of the expanded strata, but should be chosen before the radii chosen immediately below.

Choose a small radius \( r_i > 0 \), for each \( i \in I \), following the poset structure on \( I \) from minima to maxima. Construct the truncated cylinders \( C_i \subset M \), for \( i \in I \), and total cylinder \( C \subset M \).

Choose a small value \( s_i \), for each \( i \in I \), following the poset structure on \( I \) from minima to maxima. Construct the expanded strata \( E_i \subset M \), for \( i \in I \), and total expansion \( E \subset M \).

Choose a smoothing constant \( \delta > 0 \). Construct the smoothing homeomorphism \( \Psi : M \to M \), the directed cylinder \( C = \Psi(C) \subset M \) with positive coray bundle \( \Lambda_C \subset S^*X \), and the directed expansion \( E = \Psi(E) \subset M \) with positive coray bundle \( \Lambda_E \subset S^*X \).

References

[1] M. Abouzaid and P. Seidel, An open string analogue of Viterbo functoriality, Geom. Topol. 14 (2010), 627–718.
[2] D. Auroux, Fukaya categories of symmetric products and bordered Heegaard-Floer homology, J. Gökova Geom. Topol. 4 (2010), 1–54.
[3] D. Ben-Zvi, D. Nadler, and A. Preygel, Integral transforms for coherent sheaves, arXiv:1312.7164.
[4] T. Braden and M. Grinberg, Perverse sheaves on rank stratifications, Duke Math. J. Volume 96, Number 2 (1999), 317–362.
[5] T. Dyckerhoff and M. Kapranov, Triangulated surfaces in triangulated categories. arXiv:1306.2545.
[6] Y. Eliashberg, D. Nadler, and L. Starkston, Arboreal Weinstein Skeleta, in preparation.
[7] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian intersection Floer theory: Anomaly and Obstruction, AMS/IP Studies in Advanced Math. 46, International Press/Amer. Math. Soc., 2009.
[8] S. Ganatra, J. Pardon, and V. Shende, Localizing the Fukaya category of a Weinstein manifold, in preparation.
[9] S. Gelfand, R. MacPherson, and K. Vilonen, Microlocal Perverse Sheaves, arXiv:math/0509440.
[10] M. Goresky, Triangulation of stratified objects. Proc. Amer. Math. Soc. 72 (1978), 193–200.
[11] M. Goresky, Whitney stratified chains and cochains. Trans. Amer. Math. Soc. 267 (1981), 175–196.
[12] M. Kashiwara and T. Kawai, On the holonomic systems of microdifferential equations III, Publ. RIMS, Kyoto Univ. 17 (1981), 813–979.
[13] M. Kashiwara and P. Schapira, Sheaves on manifolds. Grundlehren der Mathematischen Wissenschaften 292, Springer-Verlag (1994).
[14] B. Keller, On differential graded categories, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich (2006).
[15] J. Lurie, Higher Algebra.
[16] M. Kontsevich, Symplectic Geometry of Homological Algebra, lecture notes: http://www.ihes.fr/~maxim/TEXTS/Symplectic_AT2009.pdf
[17] J. Mather, Notes on Topological Stability, Bull. of the AMS, Vol. 49, No. 4, 2012, 475–506.
[18] D. Nadler, Microlocal branes are constructible sheaves, Selecta Math. (N.S.) 15 (2009), 563–619.
[19] D. Nadler, Cyclic symmetries of $A_n$-quiver representations, Advances in Math. 269 (2015), 346–363.
[20] D. Nadler, Arboreal singularities, to appear in Geom. Topol., arXiv:1309.4122
[21] D. Nadler, A combinatorial calculation of the Landau-Ginzburg model $M = \mathbb{C}^3, W = z_1z_2z_3$, to appear in Selecta Math., arXiv:1507.08735
[22] D. Nadler, Mirror symmetry for the Landau-Ginzburg A-model $M = \mathbb{C}^n, W = z_1 \cdots z_n$, arXiv:1601.02977
[23] D. Nadler, Wrapped microlocal sheaves on pairs of pants, arXiv:1604.00114
[24] D. Nadler and E. Zaslow, Constructible sheaves and the Fukaya category, J. Amer. Math. Soc. 22 (2009), 233–286.
[25] W. Schmid and K. Vilonen, Characteristic cycles of constructible sheaves, Inventiones Math. 124 (1996), 451–502.
[26] P. Seidel, Fukaya Categories and Picard-Lefschetz Theory, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
[27] D. Treumann, Exit paths and constructible stacks, Compositio Math. 145 (2009), 1504–1532.
[28] I. Waschkies, Microlocal Riemann-Hilbert Correspondence, Publ. RIMS, Kyoto Univ. 41 (2005), 37–72.