ABSTRACT

It is well known that the different stages of the Cartan-Whitehead decomposition of a 0-connected space can be obtained as the Adams cocompletion of the space with respect to suitable sets of morphisms. In this paper Cartan-Whitehead decomposition is obtained for a nilpotent space, in terms of Adams cocompletion, using the primary homotopy theory developed by Neisendorfer.

Indexing terms/Keywords

Category of fractions; Calculus of right fractions; Adams cocompletion; Primary homotopy theory; Nilpotent space; Cartan-Whitehead decomposition.

Academic Discipline And Sub-Disciplines

Mathematics, Category Theory, Algebraic Topology.

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TYPE (METHOD/APPROACH)

A new approach to Cartan-Whitehead decomposition of a nilpotent space.
1. INTRODUCTION

Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also studied the dual notion, namely the Adams cocompletion of an object in a category [9]. It is to be emphasized that many algebraic and geometrical constructions in Algebraic Topology, Differential Topology, Differentiable Manifolds, Algebra, Analysis, Topology, etc., can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms. Behera and Nanda [3] have shown that the different stages of the Cartan-Whitehead decomposition of a $0$-connected space are the Adams cocompletion of a space with respect to suitable sets of morphisms. In [12], Neisendonfer has studied the primary homotopic theory in an exhaustive manner. The central idea of this note is to study how Cartan-Whitehead decomposition of a $0$-connected nilpotent space is characterized in terms of its Adams cocompletions; it is done using the primary homotopy theory developed by Neisendonfer.

Let $C$ be an arbitrary category and $S$ a set of morphisms of $C$. Let $C[S^{-1}]$ denote the category of fractions of $C$ with respect to $S$ and $F : C \to C[S^{-1}]$ be the canonical functor. Let $S$ denote the category of sets and functions. Then for a given object $Y$ of $C$,

$$C[S^{-1}](Y, -) : C \to S$$

defines a covariant functor. If this functor is representable by an object $Y_S$ of $C$, that is, if $C[S^{-1}](Y, -) \cong C(Y_S, -)$ then $Y_S$ is called the (generalized) Adams cocompletion of $Y$ with respect to the set of morphisms $S$ or simply the $S$-cocompletion of $Y$. We shall often refer to $Y_S$ as the cocompletion of $Y$ [9].

Given a set $S$ of morphisms of $C$, the saturation of $S$, denoted as $\overline{S}$, is the set of all morphisms $u$ in $C$ such that $F(u)$ is an isomorphism in $C[S^{-1}]$. Furthermore, $\overline{S}$ is said to be saturated if $S = S[4,9]$.

Deleanu, Frei and Hilton have shown that if the set of morphisms $S$ is saturated then the Adams cocompletion of a space is characterized by a certain couniversal property ([9], dual of Theorem 1.2). In most of the applications, however, the set of morphisms $S$ is not saturated. There is a stronger version of Deleanu, Frei and Hilton’s characterization of Adams cocompletion in terms of couniversal property as described below.

Theorem 1.1. ([4], dual of Theorem 1.2) Let $S$ be a set of morphisms of $C$ admitting a calculus of right fractions. Then an object $Y_S$ of $C$ is the $S$-cocompletion of the object $Y$ with respect to $S$ if and only if there exists a morphism $e : Y_S \to Y$ in $\overline{S}$ which is couniversal with respect to the morphisms in $S$; given a morphism $s : Z \to Y$ in $\overline{S}$ there exists a unique morphism $t : Y_S \to Z$ in $\overline{S}$ such that $s = e$. In other words, the following diagram is commutative:

$$\begin{array}{ccc}
Y_S & \xrightarrow{e} & Y \\
\downarrow t & & \downarrow s \\
Z & \xrightarrow{s} & \overline{S}
\end{array}$$

Also the above theorem turns out to be essentially the dual of Theorem 1.2 [9] if we assume $S$ to be saturated; hence the proposition can be proved by recasting the dual of the proof of the Theorem 1.2 [9] with minor changes. The details are omitted.

The following Theorem (dual of Theorem 1.3, [4]) shows that under certain conditions the morphisms $e : Y_S \to Y$ always belongs to $S$.

Theorem 1.2. ([4], dual of Theorem 1.3) Let $S = S_1 \cap S_2$, be a set of morphisms in a category $C$ admitting a calculus of right fractions. Let $e : Y_S \to Y$ be the canonical morphism as defined in Theorem 1.1, where $Y_S$ is the $S$-cocompletion of $Y$. Assume further that $S_1$ and $S_2$ have the following properties:

(i) $S_1$ and $S_2$ are closed under composition.

(ii) If $f \in S_1$, implies that $f \in S_2$.

(iii) If $f \in S_2$, implies that $f \in S_2$.

Then $e \in S$.

2. The category $N_0$.

Let $S^m$ denote the $m$-dimensional sphere. Suppose $m \geq 2$, and let $k : S^m \to S^m$ denote a map of degree $k$. The space $S^{m-1} \cup_k e^{m}$ is denoted by $P^m_k(k)$ or $P^m_k(\mathbb{Z}/k\mathbb{Z})$. If $m \geq 2$, the $m$-th mod$_k$ homotopy group of $X$ is $[P^m(k); X]$, denoted by $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ [12]. If $m \geq 3$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is a group and $m \geq 4$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is an abelian group [12].
If \( f : X \to Y \) is a map, then there are induced maps \( f_* : \pi_n(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_n(Y; \mathbb{Z}/k\mathbb{Z}) \) defined by \( f_*[g] = [fg] \). If \( m \geq 3 \), \( f_* \) is a homomorphism and, if \( m = 2 \), \( f_* \) is compatible with the action of \( \pi_2 \).

For a group \( G \), the lower central series

\[
\cdots \subseteq \Gamma^{i+1}(G) \subseteq \Gamma^i(G) \cdots \subseteq G
\]

of \( G \) is defined by the setting

\[
\Gamma^1(G) = G, \quad \Gamma^{i+1}(G) = [G, \Gamma^i(G)], \quad i \geq 1.
\]

\( G \) is said to be nilpotent if \( \Gamma^j(G) = \{1\} \) for \( j \) sufficiently large [10].

A connected CW-complex \( X \) is said to nilpotent if \( \pi_1(X) \) is nilpotent and operates nilpotently on \( \pi_n(X) \) for every \( n \geq 2 \) [10].

Let \( \mathcal{N}_0 \) denote the category of 0-connected based nilpotent spaces and based maps and let \( \overline{\mathcal{N}}_0 \) be the corresponding homotopy category. We assume that the underlying sets of the elements of \( \overline{\mathcal{N}}_0 \) are the elements of \( \mathcal{U} \), where \( \mathcal{U} \) is a fixed Grothendieck universe. We now fix suitable sets of morphisms of \( \overline{\mathcal{N}}_0 \).

Let \( S_n \) denote the set of all maps \( \alpha \) in \( \overline{\mathcal{N}}_0 \) having the following property: \( \alpha : X \to Y \) is in \( S_n \) if and only if \( \alpha_* : \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y; \mathbb{Z}/k\mathbb{Z}) \) is an isomorphism for \( m > n \) and a monomorphism for \( m = n \).

**Proposition 2.1.** \( S_n \) admits a calculus of right fractions.

**Proof.** Clearly \( S_n \) is closed under composition. We shall verify conditions (i) and (ii) of Theorem 1.3* [9]. Only condition (ii) is in question. For proving this condition it is enough to prove that every diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\gamma} & D
\end{array}
\]

with \( \delta \in S_n \). Suppose \( \alpha = [f] \) and \( \gamma = [s] \). We replace \( f \) and \( s \) by fibrations \( f' \) and \( s' \) respectively: we have \( f' : r : \mathcal{U} \to \mathbb{Z}/k\mathbb{Z} \to B \) and \( s' : t : \mathcal{U} \to \mathbb{Z}/k\mathbb{Z} \to B \) where \( r \) and \( t \) are homotopy equivalences and \( P_r \) and \( P_t \) are mapping tracks of \( f \) and \( s \) respectively. Let \( f' \) and \( f \) be the homotopy inverses of \( r \) and \( t \) respectively. Let \( D \) be the usual pull-back of \( f' \) and \( s' \) and let \( D \to P_r \) be the respective projections. Let \( \delta = [FP] \) and \( \beta = [EQ] \). Thus \( a\delta = [FP][FP] = [FP][FP] = [FP] = [FP][FP] = \{s, t\} \) and \( \delta \gamma = [FP][FP] = [FP][FP] = [FP][FP] = \{s, t\} \gamma = \{s, t\} \gamma = \gamma \beta \).

Moreover, if \( \alpha = [f] \), let \( \mu : \mathcal{U} \to \mathcal{U} \). We assume that the map \( \alpha : A \to B \) is a fibration with fibre \( Q \). We note that \( Q \) is also the fibre of \( \beta : D \to \mathcal{U} \). We have the following commutative diagrams:

\[
\begin{array}{cccc}
\cdots & \to & \pi_{m+1}(C; \mathbb{Z}/k\mathbb{Z}) & \to \\
\cdots & \to & \pi_m(Q; \mathbb{Z}/k\mathbb{Z}) & \to \\
\downarrow & & \downarrow & \\
\gamma & \quad & \delta & \quad \\
\cdots & \to & \pi_{m-1}(Q; \mathbb{Z}/k\mathbb{Z}) & \to
\end{array}
\]

By Five Lemma \( \delta \) is an isomorphism for \( m > n \) and a monomorphism for \( m = n \). This completes the proof of Proposition 2.1.

In fact, the set \( S_n \) admits a strong calculus of right fractions. A set \( S \) of morphisms of a small \( \mathcal{V} \)-category \( \mathcal{C} \) is a Grothendieck universe, admits a strong calculus of right fractions [14] if

(i) \( S \) admits a calculus of right fractions.
(ii) for any set \( \{ S_i ; B_i \to A , i \in I \} \) \( A \in \mathcal{V} \)-set\}, there exists a commutative completion \( \{ f_i ; C \to B_i , i \in I \} \) such that \( s_i f_i \in S \) for every \( i \in I \).

**Proposition 2.2.** \( S_n \) admits a strong calculus of right fractions.

**Proof.** Let \( \{ S_i ; Y_i \to X_i , i \in I \} \) be a given set of morphisms \( \mathcal{N}_0 \) with every \( S_i \in S_n \) and \( i \in \mathcal{U} \). We have a map from \( X \to P^nX \), where \( P^nX \) denotes the Postnikov decomposition of \( X \) ([10], proof of Proposition 1.1). Convert this into a fiberation \( X \to \mathcal{N}_0 \to X \to P^nX \), being its fibre. Considering the homotopy exact sequence of this fibration we get \( \pi_m(X_i; \mathbb{Z}/k\mathbb{Z}) = 0 \) for \( m \leq n \) and \( \pi_m(X_i; \mathbb{Z}/k\mathbb{Z}) \cong \pi_m(X; \mathbb{Z}/k\mathbb{Z}) \) for \( m > n \). Thus \( u_S \in S_n \). Since \( \pi_1(X_i; \mathbb{Z}/k\mathbb{Z}) = 0 \) we have a map (lifting) \( f_i : X_\mathcal{N} \to Y_i \) such that \( s_i f_i = u_S \) and the proposition is proved.

**Remark 2.3.** We note that the map \( \pi \to X \) is independent of the index.

**Proposition 2.4.** For a given object \( X \) of the category \( \mathcal{N}_0 \), there exists a subset \( S_X \) of the set \( \{ s : X' \to X \mid s \in S_n \} \) such that \( S_X \) is an element of the universe \( \mathcal{U} \), and for each \( s : X' \to X \), \( s \in S_n \), there exists a map \( s' : X' \to X \) in \( S_X \) and a morphism \( u : X' \to X \) in \( S_X \) such that \( s' u = s' \).

**Proof.** For a given object \( X \) in \( \mathcal{N}_0 \), let \( S_X \) denote the set of morphisms \( S_X = \{ s : Y \to X \mid s \in S_n , Y \} \) is an object of \( \mathcal{N}_0 \). We assert that \( S_X \) is an element of \( \mathcal{U} \). For any object \( Y \) of \( \mathcal{N}_0 \), let \( S_{Y,X} = \{ s : Y \to X , s \in S_n \} \). It is clear that \( S_X = U_Y S_{Y,X} \) and \( S_{Y,X} = S_n \cap Mor_{\mathcal{N}_0}(Y,X) \). Since \( \mathcal{N}_0 \) is a small \( \mathcal{U} \)-category, \( Mor_{\mathcal{N}_0}(Y,X) \) belongs to \( \mathcal{N}_0 \) and so \( S_{Y,X} \) being a subset of \( Mor_{\mathcal{N}_0}(Y,X) \). Therefore, the set \( S_X \), being a union of sets all belonging to \( \mathcal{U} \) and indexed by the objects \( Y \) of \( \mathcal{N}_0 \) (which is a subset of \( \mathcal{U} \)) is itself in \( \mathcal{U} \). In view of Proposition 2.2 and Remark 2.3, there exists a lifting \( f_i : X_\mathcal{N} \to Y_i \) of \( u \) such that \( s_i f_i = u_S \) where \( s \in S_X \) is arbitrary, \( u_S \) is the map as constructed in Proposition 2.2. This completes the proof of the Proposition 2.4.

**Corollary 2.5.** \( u_S \in S_n \) and with respect to any \( s' \in S_X \), \( u_S \) has couniversal property.

3. **Existence of Adams cocompletion in \( \mathcal{N}_0 \).**

Since the category \( \mathcal{N}_0 \) as stated above is neither complete nor small, the dual of Theorem 2.6 [7] cannot be used to show the existence of Adams cocompletion of an object in the category \( \mathcal{N}_0 \) with respect to the set of morphisms \( S_n \). The following theorem shows that under certain conditions the Adams cocompletion of an object in the category \( \mathcal{N}_0 \) always exists; the theorem is essentially the dual of Theorem 4.7 [1] and dual of Theorem 3.8 [2] (it is also a generalization of the dual of the Theorem in [7]).

**Theorem 3.1.** Let \( \mathcal{U} \) be a fixed Grothendieck universe. Let \( \mathcal{C} \) be the category defined as follows: the objects of \( \mathcal{C} \) are connected based nilpotent spaces whose underlying sets are elements of \( \mathcal{U} \); the morphisms of \( \mathcal{C} \) are based homotopy classes of based-point preserving maps between such based nilpotent spaces. Let \( S \) be a family of morphisms of \( \mathcal{C} \) admitting a calculus of right fractions and satisfying the following axioms of compatibility with products:

(P) \( \prod_{i \in I} S_i : X_i \to Y_i \) lies in \( S \) for each \( i \in I \), where the index set \( I \) is an element of \( \mathcal{U} \), then \( \prod_{i \in I} S_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \) lies in \( S \).

Assume that the family \( S \) and the object \( X \) of \( \mathcal{C} \) satisfy the condition:

(\*) There exists a subset \( S_X \) of the set \( \{ s : X' \to X \mid s \in S \} \) such that \( S_X \) is an element of the universe \( \mathcal{U} \) and for each \( s : X' \to X , s \in S_X \), there exist an \( s' : X' \to X \) in \( S_X \) and a morphism \( u : X' \to X \) of \( \mathcal{C} \) rendering the following diagram is commutative:

\[
\begin{array}{ccc}
X' & \xrightarrow{s'} & X \\
\uparrow u & \quad & \quad \\
X & & \\
\end{array}
\]

Then the Adams cocompletion \( X_S \) of \( X \) does exist.

As remarked by Adams of page 34 of [2] this result remains valid if \( \mathcal{C} \) is the homotopy category of 0-connected based nilpotent spaces (whose underlying sets belong to \( \mathcal{U} \)). It is to be emphasized that condition (\*) is essential in order to be able to apply E.H. Brown’s representability theorem to prove this result.

From the Propositions 2.1, 2.2 and 2.4 and Remark 2.3, we note that the conditions of Theorem 3.1 are satisfied and so by Theorem 1.2, we obtain the following theorem.

**Theorem 3.2.** Every object \( X \) of the category \( \mathcal{N}_0 \) has an Adams cocompletion \( X_S \) with respect to the set of morphisms \( S_X \) and there exists a morphism \( e_n : X_{S_n} \to X \) in \( S_X \) which is couniversal with respect to morphisms in \( S_X \).
Proposition 3.3. The morphism $e_n: X_{n+1} \to X$ as constructed in Theorem 3.2 is in $S_n$.

Proof. The proof follows from the dual of Theorem 1.3 [4]. We take $S^1_n = \{ \alpha: X \to Y \text{ in } \mathcal{N}_0 \} \cap \mathcal{N}_n$, and $S^2_n = \{ \alpha: X \to Y \text{ in } \mathcal{N}_0 \} \cap \mathcal{N}_n$, where $\mathcal{N}_0$ is a monomorphism for $m \geq n$ and $S^2_n = \{ \alpha: X \to Y \text{ in } \mathcal{N}_0 \} \cap \mathcal{N}_n$ is an epimorphism for $m \geq n + 1$. Clearly (a) $S_n = S^1_n \cap S^2_n$ and (b) $S^1_n$ and $S^2_n$ satisfy all conditions of Theorem 1.2; hence $e \in S_n$. This completes the proof of the Proposition 3.3.

4. A primary decomposition of a 0-connected based nilpotent space.

Now we obtain the primary decomposition of a 0-connected based nilpotent space with the help of the set of morphisms $S_n$ as described below. In fact the different stages of the Cartan-Whitehead decomposition of a 0-connected nilpotent space are the Adams cocompletions of the space with respect to the sets of morphisms $S_n$. In the process, starting from a 0-connected based nilpotent space $X$ we get a tower of spaces,

$$\cdots \to X_{n+1} \xrightarrow{\theta_{n+1}} X_n \to \cdots \to X_2 \xrightarrow{\theta_2} X_1 \xrightarrow{\theta_1} X_0$$

and the direct limit of this tower gives us a space which in some sense is the Cartan-Whitehead decomposition of $X$. First we prove the following proposition.

Proposition 4.1. $X_n$, as constructed in the proof of Proposition 2.2, is homotopically equivalent to $X_{n+1}$, as constructed in Theorem 3.2.

Proof. By the couniversal property of $u_n: X_n \to X$ we have a map $s: X_n \to X_{n+1}$ such that $e_ns = u_n$. By the couniversal property of $e_n: X_{n+1} \to X$ we have a map $t: X_{n+1} \to X_n$ such that $u_nt = e_n$. Thus $e_n = u_nt$ implies that $st = 1_{X_n}$ and $e_n = u_n\theta_n$ implies that $st = 1_{X_{n+1}}$. and the required homeomorphism between $X_n$ and $X_{n+1}$ is obtained. This completes the proof of Proposition 4.1.

Theorem 4.2. Let $X$ be a 0-connected based nilpotent space. Then for $n \geq 3$, there exist 0-connected based nilpotent spaces $X_n$, maps $s_n: X_n \to X$ and fibrations $\theta_{n+1}: X_{n+1} \to X_n$ such that

(a) $e_n: \pi_m(X_n; \mathbb{Z} / k\mathbb{Z}) \to \pi_m(X; \mathbb{Z} / k\mathbb{Z})$ is an isomorphism for $m > n$ and $\pi_m(X_n; \mathbb{Z} / k\mathbb{Z}) = 0$ for $m \leq n$,

(b) $e_{n+1} = e_n \circ \theta_{n+1}$.

Proof. For each integer $n \geq 3$, let $X_n$ be the $S_n$-completion of $X$ and $e_n: X_n \to X$ be the canonical map as stated in Theorem 3.1. Since $e_n \in S_n \subset S_{n+1}$, it follows from the couniversal property of $e_{n+1}$ that there exists a unique morphism $\theta_{n+1}: X_{n+1} \to X_n$ such that $e_{n+1} = e_n \circ \theta_{n+1}$, i.e., the following diagram is commutative:

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{e_{n+1}} & X \\
\downarrow \theta_{n+1} & & \searrow e_n \\
X_n & & \\
\end{array}
$$

The maps $\theta_n$ can of course be replaced by fibrations in the usual manner. Since $e_n \in S_n$, $e_n: \pi_m(X_n; \mathbb{Z} / k\mathbb{Z}) \to \pi_m(X; \mathbb{Z} / k\mathbb{Z})$ is an isomorphism for $m > n$; it is already proved in Proposition 2.2 that $\pi_m(X_n; \mathbb{Z} / k\mathbb{Z}) = 0$ for $m \leq n$.

This completes the proof of the Theorem 4.2.

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