ON T-CHARACTERIZED SUBGROUPS OF COMPACT ABELIAN GROUPS

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Abstract. We say that a subgroup $H$ of an infinite compact Abelian group $X$ is $T$-characterized if there is a $T$-sequence $u = \{u_n\}$ in the dual group of $X$ such that $H = \{x \in X : \langle u_n, x \rangle \to 1\}$. We show that a closed subgroup $H$ of $X$ is $T$-characterized if and only if $H$ is a $\mathbb{T}$-subgroup of $X$ and the annihilator of $H$ admits a Hausdorff minimally almost periodic group topology. All closed subgroups of an infinite compact Abelian group $X$ are $T$-characterized if and only if $X$ is metrizable and connected. We prove that every compact Abelian group $X$ of infinite exponent has a $T$-characterized subgroup which is not an $F_\sigma$-subgroup of $X$ that gives a negative answer to Problem 3.3 in [10].

1. Introduction

Notation and Preliminaries. Let $X$ be an Abelian topological group. We denoted by $\hat{X}$ the group of all continuous characters on $X$, $\hat{X}$ endowed with the compact-open topology is denoted by $X^\wedge$. The homomorphism $\alpha_X : X \to X^\wedge$, $x \mapsto (\chi \mapsto \langle \chi, x \rangle)$, is called the canonical homomorphism. Denote by $n(X) = \bigcap_{\chi \in X} \ker(\chi) = \ker(\alpha_X)$ the von Neumann radical of $X$. The group $X$ is called minimally almost periodic (MAP) if $n(X) = X$, and $X$ is called maximally almost periodic (MAP) if $n(X) = \{0\}$. Let $H$ be a subgroup of $X$. The annihilator of $H$ we denote by $H^\perp$, i.e., $H^\perp = \{\chi \in X^\wedge : \langle \chi, h \rangle = 1 \text{ for every } h \in H\}$.

Recall that an Abelian group $G$ is of finite exponent or bounded if there exists a positive integer $n$ such that $ng = 0$ for every $g \in G$. The minimal integer $n$ with this property is called the exponent of $G$ and is denoted by $\exp(G)$. When $G$ is not bounded, we write $\exp(G) = \infty$ and say that $G$ is of infinite exponent or unbounded. The direct sum of $\omega$ copies of an Abelian group $G$ we denote by $G^{(\omega)}$.

Let $u = \{u_n\}_{n \in \omega}$ be a sequence in an Abelian group $G$. In general no Hausdorff topology may exist in which $u$ converges to zero. A very important question whether there exists a Hausdorff group topology $\tau$ on $G$ such that $u_n \to 0$ in $(G, \tau)$, especially for the integers, has been studied by many authors, see Graev [22], Nieuwny [24], and others. Protasov and Zelenyuk [26] obtained a criterion that gives a complete answer to this question. Following [20], we say that a sequence $u = \{u_n\}$ in an Abelian group $G$ is a $T$-sequence if there is a Hausdorff group topology on $G$ in which $u_n$ converges to zero. The finest group topology with this property we denote by $\tau_u$.

The counterpart of the above question for precompact group topologies on $\mathbb{Z}$ is studied by Raczkowski [28]. Following [4] and motivated by [28], we say that a sequence $u = \{u_n\}$ is a $TB$-sequence in an Abelian group $G$ if there is a precompact Hausdorff group topology $\tau$ on $G$ in which $u_n$ converges to zero. For a $TB$-sequence $u$ we denote by $\tau_{\min}$ the finest precompact group topology on $G$ in which $u$ converges to zero. Clearly, every $TB$-sequence is a $T$-sequence, but in general, the converse assertion does not hold.

While it is quite hard to check whether a given sequence is a $T$-sequence (see, for example, [11] [12] [21] [26] [27]), the case of $TB$-sequences is much simpler. Let $X$ be an Abelian topological group and $u = \{u_n\}$ be a sequence in its dual group $X^\wedge$. Following [13], set

$$s_u(X) = \{x \in X : \langle u_n, x \rangle \to 1\}.$$  

In [4] the following simple criterion to be a $TB$-sequence was obtained:

Fact 1.1. [4] A sequence $u$ in a (discrete) Abelian group $G$ is a $TB$-sequence if and only if the subgroup $s_u(X)$ of the (compact) dual $X = G^\wedge$ is dense.

Motivated by Fact 1.1 Dikranjan et al. [13] introduced the following notion related to subgroups of the form $s_u(X)$ of a compact Abelian group $X$:

Definition 1.2. [13] Let $H$ be a subgroup of a compact Abelian group $X$ and $u = \{u_n\}$ be a sequence in $\hat{X}$. If $H = s_u(X)$ we say that $u$ characterizes $H$ and that $H$ is characterized (by $u$).

Note that for the torus $T$ this notion was already defined in [7]. Characterized subgroups has been studied by many authors, see, for example, [6] [7] [10] [12] [13] [16]. In particular, the main theorem of [12] (see also [4]) asserts
that every countable subgroup of a compact metrizable Abelian group is characterized. It is natural to ask whether a closed subgroup of a compact Abelian group is characterized. The following easy criterion is given in [10]:

**Fact 1.3.** [10] A closed subgroup $H$ of a compact Abelian group $X$ is characterized if and only if $H$ is a $G_\delta$-subgroup. In particular, $X/H$ is metrizable and the annihilator $H^\perp$ of $H$ is countable.

The next fact follows easily from Definition 1.2

**Fact 1.4.** ([9], see also [10]) Every characterized subgroup $H$ of a compact Abelian group $X$ is an $F_{\sigma \delta}$-subgroup of $X$, and hence $H$ is a Borel subset of $X$.

Facts 1.3 and 1.4 inspired in [10] the study of the Borel hierarchy of characterized subgroups of compact Abelian groups. For a compact Abelian group $X$ denote by $\text{Char}(X)$ the set of all characterized subgroups (respectively, $F_\sigma$-subgroups, $F_{\sigma \delta}$-subgroups and $G_\delta$-subgroups) of $X$. The next fact is Theorem E in [10]:

**Fact 1.5.** [10] For every infinite compact Abelian group $X$, the following inclusions hold:

$$\text{SG}_\delta(X) \subseteq \text{Char}(X) \subseteq \text{SF}_{\sigma \delta}(X) \quad \text{and} \quad \text{SF}_\sigma(X) \not\subseteq \text{Char}(X).$$

If in addition $X$ has finite exponent, then

$$\text{Char}(X) \not\subseteq \text{SF}_\sigma(X).$$

The inclusion 1.1 inspired the following question:

**Question 1.6.** [10] Problem 3.3] Does there exist a compact Abelian group $X$ of infinite exponent whose all characterized subgroups are $F_\sigma$-subsets of $X$?

**Main results.** It is important to emphasize that there is no any restriction on a sequence $u$ in Definition 1.2.

If a characterized subgroup $H$ of a compact Abelian group $X$ is dense, then, by Fact 1.1 a characterizing sequence is also a $TB$-sequence. But if $H$ is not dense, we can not expect in general that a characterizing sequence of $H$ is a $T$-sequence. Thus it is natural to ask:

**Question 1.7.** For which characterized subgroups of compact Abelian groups one can find characterizing sequences which are also $T$-sequences?

This question is of independent interest because every $T$-sequence $u$ naturally defines the group topology $\tau_u$ satisfying the following dual property:

**Fact 1.8.** [20] Let $H$ be a characterized subgroup of an infinite compact Abelian group $X$ by a $T$-sequence $u$. Then $(\hat{X}, \tau_u)^w = H(= s_u(X))$ and $u(\hat{X}, \tau_u) = H^\perp$ algebraically.

This motivates us to introduce the following notion:

**Definition 1.9.** Let $H$ be a subgroup of a compact Abelian group $X$. We say that $H$ is a $T$-characterized subgroup of $X$ if there exists a $T$-sequence $u = \{u_n\}_{n \in \omega}$ in $\hat{X}$ such that $H = s_u(X)$.

Denote by $\text{Char}_T(X)$ the set of all $T$-characterized subgroups of a compact Abelian group $X$. Clearly, $\text{Char}_T(X) \subseteq \text{Char}(X)$. Hence, if a $T$-characterized subgroup $H$ of $X$ is closed it is a $G_\delta$-subgroup of $X$ by Fact 1.3. Note also that $X$ is $T$-characterized by the zero sequence.

The main goal of the article is to obtain a complete description of closed $T$-characterized subgroups (see Theorem 1.10) and to study the Borel hierarchy of $T$-characterized subgroups (see Theorem 1.13) of compact Abelian groups. In particular, we obtain a complete answer to Question 1.7 for closed characterized subgroups and give a negative answer to Question 1.6.

Note that, if a compact Abelian group $X$ is finite, then every $T$-sequence $u$ in $\hat{X}$ is eventually equal to zero. Hence $s_u(X) = X$. Thus $X$ is the unique $T$-characterized subgroup of $X$. So in what follows we shall consider only infinite compact groups.

The following theorem describes all closed subgroups of compact Abelian groups which are $T$-characterized.

**Theorem 1.10.** Let $H$ be a proper closed subgroup of an infinite compact Abelian group $X$. Then the following assertions are equivalent:

1. $H$ is a $T$-characterized subgroup of $X$;
2. $H$ is a $G_\delta$-subgroup of $X$ and the countable group $H^\perp$ admits a Hausdorff MinAP group topology;
(3) $H$ is a $G_\delta$-subgroup of $X$ and one of the following holds:
(a) $H^\perp$ has infinite exponent;
(b) $H^\perp$ has finite exponent and contains a subgroup which is isomorphic to $\mathbb{Z}(\exp(H^\perp))^{(\omega)}$.

**Corollary 1.11.** Let $X$ be an infinite compact metrizable Abelian group. Then the trivial subgroup $H = \{0\}$ is $T$-characterized if and only if $\hat{X}$ admits a Hausdorff MinAP group topology.

As an immediate corollary of Fact 1.3 and Theorem 1.10 we obtain a complete answer to Question 1.7 for closed characterized subgroups.

**Corollary 1.12.** A proper closed characterized subgroup $H$ of an infinite compact Abelian group $X$ is $T$-characterized if and only if $H^\perp$ admits a Hausdorff MinAP group topology.

If $H$ is an open proper subgroup of $X$, then $H^\perp$ is non-trivial and finite. Thus every Hausdorff group topology on $H^\perp$ is discrete. Taking into account Fact 1.3 we obtain:

**Corollary 1.13.** Every open proper subgroup $H$ of an infinite compact Abelian group $X$ is a characterized non-$T$-characterized subgroup of $X$.

Nevertheless (see Example 2.11 below) there is a compact metrizable Abelian group $X$ with a countable $T$-characterized subgroup $H$ such that its closure $\hat{H}$ is open. Thus it may happened that the closure of a $T$-characterized subgroup is not $T$-characterized.

It is natural to ask for which compact Abelian groups all their closed $G_\delta$-subgroups are $T$-characterized. The next theorem gives a complete answer to this question.

**Theorem 1.14.** Let $X$ be an infinite compact Abelian group. The following assertions are equivalent:

1. All closed $G_\delta$-subgroups of $X$ are $T$-characterized;
2. $X$ is connected.

By Corollary 2.8 of [10], the trivial subgroup $H = \{0\}$ of a compact Abelian group $X$ is a $G_\delta$-subgroup if and only if $X$ is metrizable. So we obtain:

**Corollary 1.15.** All closed subgroups of an infinite compact Abelian group $X$ are $T$-characterized if and only if $X$ is metrizable and connected.

Theorems 1.10 and 1.14 are proved in Section 2.

In the next theorem we give a negative answer to Question 1.6.

**Theorem 1.16.** Every compact Abelian group of infinite exponent has a dense $T$-characterized subgroup which is not an $F_\sigma$-subgroup.

As a corollary of the inclusion (1.1) and Theorem 1.16 we obtain:

**Corollary 1.17.** For an infinite compact Abelian group $X$ the following assertions are equivalent:

1. $X$ has finite exponent;
2. every characterized subgroup of $X$ is an $F_\sigma$-subgroup;
3. every $T$-characterized subgroup of $X$ is an $F_\sigma$-subgroup.

Therefore, $\text{Char}(X) \subseteq \text{SF}_\sigma(X)$ if and only if $X$ has finite exponent.

In the next theorem we summarize the obtained results about the Borel hierarchy of $T$-characterized subgroups of compact Abelian groups.

**Theorem 1.18.** Let $X$ be an infinite compact Abelian group $X$. Then:

1. $\text{Char}_T(X) \subseteq \text{SF}_\sigma(X)$;
2. $\text{SG}_\delta(X) \cap \text{Char}_T(X) \subseteq \text{Char}_T(X)$;
3. $\text{SG}_\delta(X) \subseteq \text{Char}_T(X)$ if and only if $X$ is connected;
4. $\text{Char}_T(X) \cap \text{SF}_\sigma(X) \subseteq \text{SF}_\sigma(X)$;
5. $\text{Char}_T(X) \subseteq \text{SF}_\sigma(X)$ if and only if $X$ has finite exponent.

We prove Theorems 1.16 and 1.18 in Section 3.

The notions of $g$-closed and $g$-dense subgroups of a compact Abelian group $X$ were defined in [13]. In the last section of the paper, in analogy to these notions, we define $g_T$-closed and $g_T$-dense subgroups of $X$. In particular, we show that every $g_T$-dense subgroup of a compact Abelian group $X$ is dense if and only if $X$ is connected (see Theorem 4.22).
2. The Proofs of Theorems 1.10 and 1.14

The subgroup of a group $G$ generated by a subset $A$ we denote by $\langle A \rangle$.

Recall that a subgroup $H$ of an Abelian topological group $X$ is called dually closed in $X$ if for every $x \in X \setminus H$ there exists a character $\chi \in H^*$ such that $(\chi, x) \neq 1$. $H$ is called dually embedded in $X$ if every character of $H$ can be extended to a character of $X$. Every open subgroup of $X$ is dually closed and dually embedded in $X$ by Lemma 3 of [25].

The next notion generalizes the notion of the maximal extension in the class of all compact Abelian groups introduced in [11].

**Definition 2.1.** Let $G$ be an arbitrary class of topological groups. Let $(G, \tau) \in G$ and $H$ be a subgroup of $G$. The group $(G, \tau)$ is called a maximal extension of $(H, \tau|_H)$ in the class $G$ if $\sigma \leq \tau$ for every group topology on $G$ such that $\sigma|_H = \tau|_H$ and $(G, \sigma) \in G$.

Clearly, the maximal extension is unique if it exists. Note that in Definition 2.1 we do not assume that $(H, \tau|_H)$ belongs to the class $G$.

If $H$ is a subgroup of an Abelian group $G$ and $\mathbf{u}$ is a $T$-sequence (respectively, a $TB$-sequence) in $H$, we denote by $\tau_\mathbf{u}(H)$ (respectively, $\tau_{\mathbf{u}}(H)$) the finest (respectively, precompact) group topology on $H$ generated by $\mathbf{u}$. We use the following easy corollary of the definition of $T$-sequences.

**Lemma 2.2.** For a sequence $\mathbf{u}$ in an Abelian group $G$ the following assertions are equivalent:

1. $\mathbf{u}$ is a $T$-sequence in $G$;
2. $\mathbf{u}$ is a $T$-sequence in every subgroup of $G$ containing $\langle \mathbf{u} \rangle$;
3. $\mathbf{u}$ is a $T$-sequence in $\langle \mathbf{u} \rangle$.

In this case, $\langle \mathbf{u} \rangle$ is open in $\tau_\mathbf{u}$ (and hence $\langle \mathbf{u} \rangle$ is dually closed and dually embedded in $(G, \tau_\mathbf{u})$), and $(G, \tau_\mathbf{u})$ is the maximal extension of $\langle \mathbf{u}, \tau_\mathbf{u}(\langle \mathbf{u} \rangle) \rangle$ in the class $G$.

**Proof.** Evidently, (1) implies (2) and (2) implies (3). Let $\mathbf{u}$ be a $T$-sequence in $\langle \mathbf{u} \rangle$. Let $\tau$ be the topology on $G$ whose base is all translates of $\tau_\mathbf{u}(\langle \mathbf{u} \rangle)$-open sets. Clearly, $\mathbf{u}$ converges to zero in $\tau$. Thus $\mathbf{u}$ is a $T$-sequence in $G$. So (3) implies (1).

Let us prove the last assertion. By the definition of $\tau_\mathbf{u}$ we have also $\tau \leq \tau_\mathbf{u}$, and hence $\tau|_{\langle \mathbf{u} \rangle} = \tau_\mathbf{u}(\langle \mathbf{u} \rangle) \leq \tau|_{\langle \mathbf{u} \rangle}$. Thus $\langle \mathbf{u} \rangle$ is open in $\tau_\mathbf{u}$, and hence it is dually closed and dually embedded in $(G, \tau_\mathbf{u})$ by [25]Lemma 3.3. On the other hand, $\tau_\mathbf{u}|_{\langle \mathbf{u} \rangle} \leq \tau_\mathbf{u}(\langle \mathbf{u} \rangle) = \tau|_{\langle \mathbf{u} \rangle}$ by the definition of $\tau_\mathbf{u}(\langle \mathbf{u} \rangle)$. So $\tau_\mathbf{u}$ is an extension of $\tau_\mathbf{u}(\langle \mathbf{u} \rangle)$. Now clearly, $\tau = \tau_\mathbf{u}$ and $(G, \tau_\mathbf{u})$ is the maximal extension of $\langle \mathbf{u}, \tau_\mathbf{u}(\langle \mathbf{u} \rangle) \rangle$ in the class $G$.

For $TB$-sequences we have the following:

**Lemma 2.3.** For a sequence $\mathbf{u}$ in an Abelian group $G$ the following assertions are equivalent

1. $\mathbf{u}$ is a $TB$-sequence in $G$;
2. $\mathbf{u}$ is a $TB$-sequence in every subgroup of $G$ containing $\langle \mathbf{u} \rangle$;
3. $\mathbf{u}$ is a $TB$-sequence in $\langle \mathbf{u} \rangle$.

In this case, the subgroup $\langle \mathbf{u} \rangle$ is dually closed and dually embedded in $(G, \tau_{\mathbf{u}})$, and $(G, \tau_{\mathbf{u}})$ is the maximal extension of $\langle \mathbf{u}, \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle) \rangle$ in the class of all precompact Abelian groups.

**Proof.** Evidently, (1) implies (2) and (2) implies (3). Let $\mathbf{u}$ be a $TB$-sequence in $\langle \mathbf{u} \rangle$. Then $\langle \mathbf{u}, \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle) \rangle$ separates the points of $\langle \mathbf{u} \rangle$. Let $\tau$ be the topology on $G$ whose base is all translates of $\tau_{\mathbf{u}}(\langle \mathbf{u} \rangle)$-open sets. Then $\langle \mathbf{u}, \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle) \rangle$ is an open subgroup of $(G, \tau)$. It is easy to see that $(G, \tau)^+$ separates the points of $G$. Since $\mathbf{u}$ converges to zero in $\tau$, it is also converges to zero in $\tau^+$, where $\tau^+$ is the Bohr topology of $(G, \tau)$. Thus $\mathbf{u}$ is a $TB$-sequence in $G$. So (3) implies (1).

The last assertion follows from Proposition 1.8 and Lemma 3.6 in [11].

For a sequence $\mathbf{u} = \{u_n\}_{n \in \omega}$ of characters of a compact Abelian group $X$ set

$$K_{\mathbf{u}} = \bigcap_{n \in \omega} \ker(u_n).$$

The following assertions is proved in [10]:

**Fact 2.4.** [10] Lemma 2.2(i)] For every sequence $\mathbf{u} = \{u_n\}_{n \in \omega}$ of characters of a compact Abelian group $X$, the subgroup $K_{\mathbf{u}}$ is a closed $G_\delta$-subgroup of $X$ and $K_{\mathbf{u}} = \langle \mathbf{u} \rangle^\perp$. 
The next two lemmas are natural analogues of Lemmas 2.2(ii) and 2.6 of [10].

Lemma 2.5. Let $X$ be a compact Abelian group and $u = \{u_n\}_{n \in \omega}$ be a $T$-sequence in $\tilde{X}$. Then $s_u(X)/K_u$ is a $T$-characterized subgroup of $X/K_u$.

Proof. Set $H := s_u(X)$ and $K := K_u$. Let $q : X \to X/K$ be the quotient map. Then the adjoint homomorphism $q^\wedge$ is an isomorphism from $(X/K)^\wedge$ onto $K^\perp$ in $X^\wedge$. For every $n \in \omega$, define the character $\tilde{u}_n$ of $X/K$ as follows:

$$(\tilde{u}_n, q(x)) = (u_n, x)$$

$$(\tilde{u}_n \text{ is well-defined since } K \subseteq \ker(u_n)).$$

Then $\tilde{u} = \{\tilde{u}_n\}_{n \in \omega}$ is a sequence of characters of $X/K$ such that $q^\wedge(\tilde{u}_n) = u_n$. Since $u \subseteq K^\perp$, $u$ is a $T$-sequence in $K^\perp$ by Lemma 2.5. Hence $\tilde{u}$ is a $T$-sequence in $(X/K)^\wedge$ because $q^\wedge$ is an isomorphism.

We claim that $H/K = s_u(X/K)$. Indeed, for every $h+K \in H/K$, by definition, we have $(\tilde{u}_n, h+K) = (u_n, h) \to 1$. Thus $H/K \subseteq s_u(X/K)$. If $x + K \in s_u(X/K)$, then $(\tilde{u}_n, x + K) = (u_n, x) \to 1$. This yields $x \in H$. Thus $x + K \in H/K$.

Let $u = \{u_n\}_{n \in \omega}$ be a $T$-sequence in an Abelian group $G$. For every natural number $m$ set $u_m = \{u_n\}_{n \geq m}$. Clearly, $u_m$ is a $T$-sequence in $G$, $\tau_u = \tau_{u_m}$ and $s_u(X) = s_{u_m}(X)$ for every natural number $m$.

Lemma 2.6. Let $K$ be a closed subgroup of a compact Abelian group $X$ and $q : X \to X/K$ be the quotient map. Then $H$ is a $T$-characterized subgroup of $X/K$ if and only if $q^{-1}(H)$ is a $T$-characterized subgroup of $X$.

Proof. Let $\tilde{H}$ be a $T$-characterized subgroup of $X/K$ and let a $T$-sequence $\tilde{u} = \{\tilde{u}_n\}_{n \in \omega}$ characterize $\tilde{H}$. Set $H := q^{-1}(\tilde{H})$. We have to show that $H$ is a $T$-characterized subgroup of $X$.

Note that the adjoint homomorphism $q^\wedge$ is an isomorphism from $(X/K)^\wedge$ onto $K^\perp$ in $X^\wedge$. Set $u = \{u_n\}_{n \in \omega}$, where $u_n = q^\wedge(\tilde{u}_n)$. Since $q^\wedge$ is injective, $u$ is a $T$-sequence in $K^\perp$. By Lemma 2.6, $u$ is a $T$-sequence in $\tilde{X}$. So it is enough to show that $H = s_u(X)$. This follows from the following chain of equivalences. By definition, $x \in s_u(X)$ if and only if

$$(u_n, x) \to 1 \iff (\tilde{u}_n, q(x)) \to 1 \iff q(x) \in H = H/K \iff x \in H.$$

The last equivalence is due to the inclusion $K \subseteq H$.

Conversely, let $H := q^{-1}(\tilde{H})$ be a $T$-characterized subgroup of $X$ and a $T$-sequence $u = \{u_n\}_{n \in \omega}$ characterize $H$. Proposition 2.5 of [10] implies that we can find $m \in \mathbb{N}$ such that $K \subseteq K_{u_m}$. So, taking into account that $H = s_u(X) = s_{u_m}(X)$ for every natural number $m$, without loss of generality we can assume that $K \subseteq K_u$. By Lemma 2.6, $H/K_u$ is a $T$-characterized subgroup of $X/K_u$. Denote by $q_u$ the quotient homomorphism from $X$ onto $X/K_u$. Then $\tilde{H} = q^{-1}(H/K_u)$ is $T$-characterized in $X/K_u$ by the previous paragraph of the proof.

The next theorem is an analogue of Theorem B of [10], and it reduces the study of $T$-characterized subgroups of compact Abelian groups to the study of $T$-characterized subgroups of compact Abelian metrizable groups:

Theorem 2.7. A subgroup $H$ of a compact Abelian group $X$ is $T$-characterized if and only if $H$ contains a closed $G_\delta$-subgroup $K$ of $X$ such that $H/K$ is a $T$-characterized subgroup of the compact metrizable group $X/K$.

Proof. Let $H$ be a $T$-characterized subgroup of $X$ by a $T$-sequence $u = \{u_n\}_{n \in \omega}$ in $\tilde{X}$. Set $K := K_u$. Since $K$ is a closed $G_\delta$-subgroup of $X$ by Fact 2.2, $X/K$ is metrizable. By Lemma 2.4, $H/K$ is a $T$-characterized subgroup of $X/K$.

Conversely, let $H$ contain a closed $G_\delta$-subgroup $K$ of $X$ such that $H/K$ is a $T$-characterized subgroup of the compact metrizable group $X/K$. Then $H$ is a $T$-characterized subgroup of $X$ by Lemma 2.6.

As it was noticed in [13] before Definition 2.33, for every $T$-sequence $u$ in an infinite Abelian group $G$ the subgroup $\langle u \rangle$ is open in $(G, \tau_u)$ (see also Lemma 2.2), and hence, by Lemmas 1.4 and 2.2 of [3], the following sequences are exact:

$$0 \to \langle (u), \tau_u \rangle \to (G, \tau_u) \to G/\langle u \rangle \to 0,$$

$$0 \to (G/\langle u \rangle)^\wedge \to (G, \tau_u)^\wedge \to \langle (u), \tau_u \rangle^\wedge \to 0,$$

where $(G/\langle u \rangle)^\wedge \cong \langle u \rangle^\perp$ is a compact subgroup of $(G, \tau_u)^\wedge$ and $(\langle u \rangle, \tau_u)^\wedge \cong (G, \tau_u)^\wedge/\langle u \rangle^\perp$.

Let $u = \{u_n\}_{n \in \omega}$ be a $T$-sequence in an Abelian group $G$. It is known [27] that $\tau_u$ is sequential, and hence $(G, \tau_u)$ is a $k$-space. So the natural homomorphism $\alpha := \alpha_{(G, \tau_u)} : (G, \tau_u) \to (G, \tau_u)^\wedge$ is continuous by [2] 5.12. Let us recall that $(G, \tau_u)$ is MinAP if and only if $(G, \tau_u) = \ker(\alpha)$.

To prove Theorem 2.10, we need the following:

Fact 2.8. [10] For each $T$-sequence $u$ in a countably infinite Abelian group $G$ the group $(G, \tau_u)^\wedge$ is Polish.
Now we are in position to prove Theorem 1.10.

Proof of Theorem 1.10. (1) ⇒ (2) Let \( H \) be a proper closed \( T \)-characterized subgroup of \( X \) and a \( T \)-sequence \( u = \{ u_n \}_{n \in \omega} \) characterize \( H \). Since \( H \) is also characterized it is a \( G_\delta \)-subgroup of \( X \) by Fact 1.3. We have to show that \( H^\perp \) admits a MinAP group topology.

Our idea of the proof is the following. Set \( G := \hat{X} \). By Fact 1.8 \( H^\perp \) is the von Neumann radical of \( (G, \tau_\alpha) \). Now assume that we found another \( T \)-sequence \( v \) which characterizes \( H \) and such that \( (v) = H^\perp \) (maybe \( v = u \)). By Fact 1.8, we have \( n(G, \tau_\alpha) = H^\perp = (v) \). Lemma 2.2 implies that the subgroup \( (v, \tau_v|_\langle v \rangle) \) of \( (G, \tau_\alpha) \) is open, and hence it is dually closed and dually embedded in \( (G, \tau_\alpha) \). Hence \( n((v, \tau_v|_\langle v \rangle)) = n(G, \tau_\alpha)(= (v)) \) by Lemma 4 of 1.10. So \( (v, \tau_v|_\langle v \rangle) \) is MinAP. Thus \( H^\perp = (v) \) admits a MinAP group topology, as desired.

We find such a \( T \)-sequence \( v \) in 4 steps (in fact we show that \( v \) has the form \( u_m \) for some \( m \in \mathbb{N} \)).

Step 1. Let \( q : X \to X/K_u \) be the quotient map. For every \( n \in \omega \), define the character \( \bar{u}_n \) of \( X/K_u \) by the equality \( u_n = \bar{u}_n \circ q \) (this is possible since \( K_u \subseteq \ker(u_n) \)). As it was shown in the proof of Lemma 2.5, the sequence \( \bar{u} = \{ \bar{u}_n \}_{n \in \omega} \) is a \( T \)-sequence which characterizes \( H/K_u \) in \( X/K_u \). Set \( \hat{X} := X/K_u \) and \( \hat{H} := H/K_u \). So that \( \hat{H} = s_u(\hat{X}) \). By 19 5.34 and 24.11 and since \( K_u \subseteq H \), we have

\[
(2.2) \quad H^\perp \cong (X/H)^\wedge \cong \left( \hat{X}/\hat{H} \right)^\wedge \cong \hat{H}^\perp.
\]

By Fact 1.3, \( \hat{X} \) is metrizable. Hence \( \hat{H} \) is also compact and metrizable, and \( \hat{G} := \hat{X} \) is a countable Abelian group by 19 24.15. Since \( H \) is a proper closed subgroup of \( X \), (2.2) implies that \( \hat{G} \) is non-zero.

We claim that \( \hat{G} \) is countably infinite. Indeed, suppose for a contradiction that \( \hat{G} \) is finite. Then \( X/K_u = \hat{X} \) is also finite. Now Fact 2.4 implies that \( (u) \) is a finite subgroup of \( G \). Since \( u \) is a \( T \)-sequence, \( u \) must be eventually equal to zero. Hence \( H = s_u(X) = X \) is not a proper subgroup of \( X \), a contradiction.

Step 2. We claim that there is a natural number \( m \) such that the group \( \langle (\bar{u}_m), \tau_\alpha|_\langle \bar{u}_m \rangle \rangle = \langle (\bar{u}_m), \tau_\alpha|_\langle \bar{u}_m \rangle \rangle \) is MinAP.

Indeed, since \( \hat{G} \) is countably infinite, we can apply Fact 1.8. So \( \hat{H} = (\hat{G}, \tau_\alpha)^\wedge \) algebraically. Since \( \hat{H} \) and \( (\hat{G}, \tau_\alpha)^\wedge \) are Polish groups (see Fact 2.8), \( H \) and \( (\hat{G}, \tau_\alpha)^\wedge \) are topologically isomorphic by the uniqueness of the Polish group topology. Hence \( (\hat{G}, \tau_\alpha)^\wedge = \hat{H}^\perp \) is discrete. As it was noticed before the proof, the natural homomorphism \( \bar{\alpha} : (\hat{G}, \tau_\alpha) \to (\hat{G}, \tau_\alpha)^\wedge \) is continuous. Since \( (\hat{G}, \tau_\alpha)^\wedge \) is discrete we obtain that the von Neumann radical \( \ker(\bar{\alpha}) \) of \( (\hat{G}, \tau_\alpha) \) is open in \( \tau_\alpha \). So there exists a natural number \( m \) such that \( \bar{u}_n \in \ker(\bar{\alpha}) \) for every \( n \geq m \). Hence \( \langle \bar{u}_m \rangle \subseteq \ker(\bar{\alpha}) \). Lemma 2.4 implies that the subgroup \( \langle \bar{u}_m \rangle \) is open in \( (\hat{G}, \tau_\alpha) \), and hence it is dually closed and dually embedded in \( (\hat{G}, \tau_\alpha) \). Now Lemma 4 of 1.10 yields \( \langle \bar{u}_m \rangle = \ker(\bar{\alpha}) \) and \( \langle \bar{u}_m \rangle \) is MinAP.

Step 3. Set \( v = \{ v_n \}_{n \in \omega} \), where \( v_n = u_n + m \) for every \( n \in \omega \). Clearly, \( v \) is a \( T \)-sequence in \( G \) characterizing \( H \), \( \tau_v = \tau_\alpha \) and \( K_u \subseteq K_v \). Let \( t : X \to X/K_v \) and \( r : X/K_u \to X/K_v \) be the quotient maps. Analogously to Step 1 and the proof of Lemma 2.5, the sequence \( \bar{v} = \{ v_n \}_{n \in \omega} \) is a \( T \)-sequence in \( \hat{X}/\hat{K}_v \) which characterizes \( H/K_v \) in \( X/K_v \), where \( v_n = \bar{v}_n \circ t \). Since \( t = r \circ q \) we have

\[
v_n = \bar{v}_n \circ t = t^\wedge(\bar{v}_n) = q^\wedge(r^\wedge(\bar{u}_n)),
\]

where \( t^\wedge \), \( r^\wedge \) and \( q^\wedge \) are the adjoint homomorphisms to \( t \), \( r \) and \( q \) respectively.

Since \( q^\wedge \) and \( r^\wedge \) are embeddings, we have \( r^\wedge(\bar{u}_n) = \bar{u}_{n+m} \). In particular, \( \langle v \rangle \cong \langle \bar{v} \rangle \cong \langle \bar{u}_m \rangle \) and

\[
\langle \bar{u}_m \rangle, \tau_\alpha|_\langle \bar{u}_m \rangle \rangle = \langle \bar{u}_m \rangle, \tau_\alpha|_\langle \bar{u}_m \rangle \rangle \cong \langle \bar{v} \rangle, \tau_v|_\langle \bar{v} \rangle \rangle \cong \langle v \rangle, \tau_v|_\langle v \rangle \rangle.
\]

By Step 2 \( \langle \bar{u}_m \rangle \) is MinAP. Hence \( \langle v \rangle, \tau_v|_\langle v \rangle \rangle \) is MinAP as well.

Step 4. By the second exact sequence in 2.1, applying to \( v \), Fact 1.8 and since \( \langle v \rangle, \tau_v|_\langle v \rangle \rangle \) is MinAP (by Step 3), we have \( H = s_v(X) = (\hat{G}, \tau_v)^\wedge = (G/\langle v \rangle)^\wedge = (v)^\perp \) algebraically. Thus \( H^\perp = (v) \), and hence \( H^\perp \) admits a MinAP group topology generated by the \( T \)-sequence \( v \).

(2) ⇒ (1): Since \( H \) is a \( G_\delta \)-subgroup of \( X \), \( H \) is closed by Proposition 2.4 and \( X/H \) is metrizable (due to the well known fact that a compact group of countable pseudocharacter is metrizable). Hence \( H^\perp = (X/H)^\wedge \) is countable. Since \( H^\perp \) admits a MinAP group topology, \( H^\perp \) must be countably infinite. By Theorem 3.8 of 21, \( H^\perp \) admits a MinAP group topology generated by a \( T \)-sequence \( u = \{ u_n \}_{n \in \omega} \). By Fact 1.8, this means that \( s_u(X/H) = \{ \langle u_n \rangle \} \). Let \( q : X \to X/H \) be the quotient map. Set \( u_n = \bar{u}_n \circ q \). Since \( q^\wedge \) is injective, \( u \) is a \( T \)-sequence in \( \hat{X} \) by Lemma 2.2. We have to show that \( H = s_u(X) \). By definition, \( x \in s_u(X) \) if and only if \( (u_n, x) = (\bar{u}_n, q(x)) \to 1 \Leftrightarrow q(x) \in s_u(X/H) \Leftrightarrow q(x) = 0 \Leftrightarrow x \in H \).

(2)⇔(3) follows from Theorem 3.8 of 21. The theorem is proved. □
Proof of Theorem 1.14. (1) ⇒ (2): Suppose for a contradiction that \( X \) is not connected. Then, by [24, 24.25], the dual group \( G = X^\wedge \) has a non-zero element \( g \) of finite order. Then the subgroup \( H := (g) \perp \) of \( X \) has finite index. Hence \( H \) is an open subgroup of \( X \). Thus \( H \) is not \( T \)-characterized by Corollary 1.13. This contradiction shows that \( X \) must be connected.

(2) ⇒ (1): Let \( H \) be a proper \( G_\delta \)-subgroup of \( X \). Then \( H \) is closed by [10, Proposition 2.4], and \( X/H \) is connected and non-zero. Hence \( H^\perp \cong (X/H)^\wedge \) is countably infinite and torsion free by [23, 24.25]. Thus \( H^\perp \) has infinite exponent. Therefore, by Theorem 1.10 \( H \) is \( T \)-characterized. □

The next proposition is a simple corollary of Theorem B in [10].

Proposition 2.9. The closure \( \hat{H} \) of a characterized (in particular, \( T \)-characterized) subgroup \( H \) of a compact Abelian group \( X \) is a characterized subgroup of \( X \).

Proof. By Theorem B of [10], \( H \) contains a compact \( G_\delta \)-subgroup \( K \) of \( X \). Then \( \hat{H} \) is also a \( G_\delta \)-subgroup of \( X \). Thus \( \hat{H} \) is a characterized subgroup of \( X \) by Theorem B of [10].

In general we cannot assert that the closure \( \hat{H} \) of a \( T \)-characterized subgroup \( H \) of a compact Abelian group \( X \) is also \( T \)-characterized as the next example shows.

Example 2.10. Let \( X = \mathbb{Z}(2) \times \mathbb{T} \) and \( G = \hat{X} = \mathbb{Z}(2) \times \mathbb{Z} \). It is known (see the end of (1) in [13]) that there is a \( T \)-sequence \( u \) in \( G \) such that the von Neumann radical \( u(G, \tau_u) \) of \( (G, \tau_u) \) is \( \mathbb{Z}(2) \times \{0\} \), the subgroup \( H := s_u(X) \) is countable and \( \hat{H} = \{0\} \times \mathbb{T} \). So the closure \( \hat{H} \) of the countable \( T \)-characterized subgroup \( H \) of \( X \) is open. Thus \( \hat{H} \) is not \( T \)-characterized by Corollary 1.13.

We do not know answers to the following questions:

Problem 2.11. Let \( H \) be a characterized subgroup of a compact Abelian group \( X \) such that its closure \( \hat{H} \) is \( T \)-characterized. Is \( H \) a \( T \)-characterized subgroup of \( X \)?

Problem 2.12. Does there exists a metrizable Abelian compact group which has a countable non-\( T \)-characterized subgroup?

3. The Proofs of Theorems 1.10 and 1.18

Recall that a Borel subgroup \( H \) of a Polish group \( X \) is called polishable if there exists a Polish group topology \( \tau \) on \( H \) such that the inclusion map \( i : (H, \tau) \to X \) is continuous. Let \( H \) be a \( T \)-characterized subgroup of a compact metrizable Abelian group \( X \) by a \( T \)-sequence \( u = \{u_n\}_{n \in \omega} \). Then, by [16] Theorem 1, \( H \) is polishable by the metric

\[
\rho(x, y) = d(x, y) + \sup\{|(u_n, x) - (u_n, y)|, n \in \omega\},
\]

where \( d \) is the initial metric on \( X \). Clearly, the topology generated by the metric \( \rho \) on \( H \) is finer than the induced one from \( X \).

To prove Theorem 1.16 we need the following three lemmas.

For a real number \( x \) we write \([x]\) for the integral part of \( x \) and \( \|x\| \) for the distance from \( x \) to the nearest integer. We also use the following inequality proved in [15]

\[
|\pi|\varphi| \leq |1 - e^{2\pi i \varphi}| \leq 2|\varphi|, \quad \varphi \in \left[-\frac{1}{2}, \frac{1}{2}\right).
\]

Lemma 3.1. Let \( \{a_n\}_{n \in \omega} \subset \mathbb{N} \) be such that \( a_n \to \infty \) and \( a_n \geq 2, n \in \omega \). Set \( u_n = \prod_{k \leq n} a_n \) for every \( n \in \omega \). Then \( u = \{u_n\}_{n \in \omega} \) is a \( T \)-sequence in \( X = \mathbb{T} \), and the \( T \)-characterized subgroup \( H = s_u(T) \) of \( T \) is a dense non-\( F_\sigma \)-subset of \( T \).

Proof. We consider the circle group \( T \) as \( \mathbb{R}/\mathbb{Z} \) and write it additively. So that \( d(0, x) = \|x\| \) for every \( x \in \mathbb{T} \). Recall (see, for example, the proof of Lemma 1 in [13]) that every \( x \in \mathbb{T} \) has the unique representation in the form

\[
x = \sum_{n=0}^{\infty} \frac{c_n}{u_n},
\]

where \( 0 \leq c_n < a_n \) and \( c_n \neq a_n - 1 \) for infinitely many indices \( n \).

It is known [1] (see also (12) in the proof of Lemma 1 of [15]) that \( x \) with representation (3.3) belongs to \( H \) if and only if

\[
\lim_{n \to \infty} \frac{c_n}{a_n} \equiv 0 \quad (\text{mod } 1).
\]
Hence $H$ is a dense subgroup of $\mathbb{T}$. Thus $\mathfrak{u}$ is even a $TB$-sequence in $\mathbb{Z}$ by Fact 11.

We have to show that $H$ is not an $F_{\sigma}$-subset of $\mathbb{T}$. Suppose for a contradiction that $H$ is an $F_{\sigma}$-subset of $\mathbb{T}$. Then $H = \bigcup_{n \in \mathbb{N}} F_n$, where $F_n$ is a compact subset of $\mathbb{T}$ for every $n \in \mathbb{N}$. Since $H$ is a subgroup of $\mathbb{T}$, without loss of generality we can assume that $F_n - F_n \subseteq F_{n+1}$. Since all $F_n$ are closed in $(H, \rho)$ as well, the Baire theorem implies that there are $0 < \varepsilon < 0.1$ and $m \in \mathbb{N}$ such that $F_m \supseteq \{ x : \rho(0, x) \leq \varepsilon \}$.

Fix arbitrarily $l > 0$ such that $\frac{2}{u_{l-1}} < \frac{\varepsilon}{20}$. For every natural number $k > l$, set

$$x_k := \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[ \frac{(a_n - 1)\varepsilon}{20} \right].$$

Then, for every $k > l$, we have

$$x_k = \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[ \frac{(a_n - 1)\varepsilon}{20} \right] < \sum_{n=l}^{k} \frac{1}{u_{n-1}} \cdot \frac{\varepsilon}{20} < \frac{1}{u_{l-1}} \sum_{n=0}^{k-l} \frac{1}{2^n} < \frac{2}{u_{l-1}} < \frac{\varepsilon}{2} < \frac{1}{2}.$$ 

This inequality and (3.2) imply that

$$(3.5) \quad d(0, x_k) = \| x_k \| = x_k < \frac{\varepsilon}{20}, \quad \text{for every } k > l.$$ 

For every $s \in \omega$ and every natural number $k > l$, we estimate $|1 - (u_s, x_k)|$ as follows.

Case 1. Let $s < k$. Set $q = \max\{s + 1, l\}$. By the definition of $x_k$, we have

$$2\pi \left[ (u_s \cdot x_k) \mod 1 \right] = 2\pi \left[ \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[ \frac{(a_n - 1)\varepsilon}{20} \right] \mod 1 \right] < 2\pi \sum_{n=q}^{k} \frac{u_s}{u_n} \cdot \frac{(a_n - 1)\varepsilon}{20}$$

$$< \frac{\pi \varepsilon}{10} \left( 1 + \frac{1}{a_{s+1}} + \frac{1}{a_{s+1}a_{s+2}} + \frac{1}{a_{s+1}a_{s+2}a_{s+3}} + \ldots \right)$$

$$< \frac{\pi \varepsilon}{10} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \right) = \frac{\pi \varepsilon}{10} \cdot 2 < \frac{2\varepsilon}{3} < \frac{1}{2}. $$

This inequality and (3.2) imply that

$$(3.6) \quad |1 - (u_s, x_k)| = |1 - \exp \left( 2\pi i \cdot \left[ (u_s \cdot x_k) \mod 1 \right] \right)| < \frac{2\varepsilon}{3}. $$

Case 2. Let $s \geq k$. By the definition of $x_k$, we have

$$(3.7) \quad |1 - (u_s, x_k)| = 0.$$ 

In particular, (3.7) implies that $x_k \in H$ for every $k > l$.

Now, for every $k > l$, (3.1) and (3.5)-(3.7) imply

$$\rho(0, x_k) < \frac{\varepsilon}{20} + \frac{2\varepsilon}{3} < \varepsilon.$$ 

Thus $x_k \in F_m$ for every natural number $k > l$. Clearly,

$$x_k \to x := \sum_{n=l}^{\infty} \frac{1}{u_n} \cdot \left[ \frac{(a_n - 1)\varepsilon}{20} \right] \quad \text{in } \mathbb{T}.$$ 

Since $F_m$ is a compact subset of $\mathbb{T}$, we have $x \in F_m$. Hence $x \in H$. On the other hand, we have

$$\lim_{n \to \infty} \frac{1}{a_n} \cdot \left[ \frac{(a_n - 1)\varepsilon}{20} \right] \mod 1 = \frac{\varepsilon}{20} \neq 0.$$ 

So (3.1) implies that $x \not\in H$. This contradiction shows that $H = s_{\mathfrak{u}}(\mathbb{T})$ is not an $F_{\sigma}$-subset of $\mathbb{T}$. Q.E.D.

For a prime number $p$, the group $\mathbb{Z}(p^\infty)$ is regarded as the collection of fractions $m/p^n \in [0, 1)$. Let $\Delta_p$ be the compact group of $p$-adic integers. It is well known that $\hat{\Delta}_p = \mathbb{Z}(p^\infty)$.

**Lemma 3.2.** Let $X = \Delta_p$. For an increasing sequence of natural numbers $0 < n_0 < n_1 < \ldots$ such that $n_{k+1} - n_k \to \infty$, set

$$u_k = \frac{1}{p^{n_k + 1}} \in \mathbb{Z}(p^\infty).$$
Then the sequence $\mathbf{u} = \{u_k\}_{k \in \omega}$ is a $T$-sequence in $\mathbb{Z}(p^\infty)$, and the $T$-characterized subgroup $H = s_\mathbf{u}(\Delta_p)$ is a dense non-$F_\sigma$-subset of $\Delta_p$.

Proof. Let $\omega = (a_n)_{n \in \omega} \in \Delta_p$, where $0 \leq a_n < p$ for every $n \in \omega$. Recall that, for every $k \in \omega$, \cite[25.2]{23} implies

$$
(3.8) \quad (u_k, \omega) = \exp \left\{ \frac{2\pi i}{p^{n_k+1}} (a_0 + pa_1 + \cdots + p^{n_k}a_{n_k}) \right\}.
$$

Further, by \cite{23}[10.4], if $\omega \neq 0$, then $d(0, \omega) = 2^{-n}$, where $n$ is the minimal index such that $a_n \neq 0$.

Following \cite[2.2]{17}, for every $\omega = (a_n) \in \Delta_p$ and every natural number $k > 1$, set

$$
m_k = m_k(\omega) = \max\{j_k, n_k-1\},
$$

where

$$
j_k = n_k \text{ if } 0 < a_{n_k} < p-1,
$$

and otherwise

$$
j_k = \min\{j : \text{ either } a_s = 0 \text{ for } j < s \leq n_k, \text{ or } a_s = p-1 \text{ for } j < s \leq n_k\}.
$$

In \cite[2.2]{17} it is shown that

$$
(3.9) \quad \omega \in s_\mathbf{u}(\Delta_p) \text{ if and only if } n_k - m_k \to \infty.
$$

So $H := s_\mathbf{u}(\Delta_p)$ contains the identity $1 = (1, 0, 0, \ldots)$ of $\Delta_p$. By \cite[Remark 10.6]{23}, $(1)$ is dense in $\Delta_p$. Hence $H$ is dense in $\Delta_p$ as well. Now Fact \cite{14} implies that $u$ is a $T$-sequence in $\mathbb{Z}(p^\infty)$.

We have to show that $H$ is not an $F_\sigma$-subset of $\Delta_p$. Suppose for a contradiction that $H = \cap_{n \in \mathbb{N}} F_n$ is an $F_\sigma$-subset of $\Delta_p$, where $F_n$ is a compact subset of $\Delta_p$ for every $n \in \mathbb{N}$. Since $H$ is a subgroup of $\Delta_p$, without loss of generality we can assume that $F_n - F_n \subseteq F_{n+1}$. Since all $F_n$ are closed in $(H, p)$ as well, the Baire theorem implies that there are $0 < \varepsilon < 0.1$ and $m \in \mathbb{N}$ such that $F_m \supseteq \{x : \rho(0, x) \leq \varepsilon\}$.

Fix a natural number $s$ such that $\frac{1}{2^s} < \frac{\varepsilon}{20}$. Choose a natural number $l > s$ such that, for every natural number $n \geq l$, we have

$$
n_{w+1} - n_w > s.
$$

For every $r \in \mathbb{N}$, set

$$
\omega_r := (a^*_n), \text{ where } a^*_n = \begin{cases} 
1, & \text{if } n = n_{l+i} - s \text{ for some } 1 \leq i \leq r, \\
0, & \text{otherwise.}
\end{cases}
$$

Then, for every $r \in \mathbb{N}$, \eqref{3.10} implies that $\omega_r$ is well-defined and

$$
(3.11) \quad d(0, \omega_r) = \frac{1}{2^m} < \frac{1}{2^i} < \frac{1}{2} < \frac{1}{2^s} < \frac{\varepsilon}{20}.
$$

Note that

$$
(3.12) \quad 1 + p + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1} < p^{k+1}.
$$

For every $k \in \omega$ and every $r \in \mathbb{N}$, we estimate $|1 - (u_k, \omega_r)|$ as follows.

Case 1. Let $k \leq l$. By \eqref{3.8}, \eqref{3.10} and the definition of $\omega_r$ we have

$$
(3.13) \quad |1 - (u_k, \omega_r)| = 0.
$$

Case 2. Let $l < k \leq l + r$. Then \eqref{3.12} yields

$$
\frac{2\pi}{p^{n_k+1}} |p^{n_k+s} + \cdots + p^{n_k-s}| < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_k-s+1} = \frac{2\pi}{p^s} \leq \frac{2\pi}{2^s} < \frac{\varepsilon}{2}.
$$

This inequality and the inequalities \eqref{3.12} and \eqref{3.8} imply

$$
(3.14) \quad |1 - (u_k, \omega_r)| = \left| 1 - \exp \left\{ \frac{2\pi i}{p^{n_k+1}} (p^{n_k+s} + \cdots + p^{n_k-s}) \right\} \right| < \frac{\varepsilon}{2}.
$$

Case 3. Let $l + r < k$. By \eqref{3.12} we have

$$
\frac{2\pi}{p^{n_k+1}} |p^{n_k+s} + \cdots + p^{n_k-r-s}| < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_k-r-s+1} < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_k-s+1} = \frac{2\pi}{p^s} \leq \frac{2\pi}{2^s} < \frac{\varepsilon}{2}.
$$
These inequalities, \(3.12\) and \(3.13\), immediately yield
\[
|1 - (u_k, \omega_r)| = \left| 1 - \exp \left\{ \frac{2\pi i}{p^{n_k+1}} (p^{n_{k+1}} - s + \cdots + p^{n-r-s}) \right\} \right| \leq \frac{\varepsilon}{2},
\]
and
\[
|1 - (u_k, \omega_r)| < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_{k+1} - s + 1} \to 0, \quad \text{as } k \to \infty.
\]
So, \(3.10\) implies that \(\omega_r \in H\) for every \(r \in \mathbb{N}\).

For every \(r \in \mathbb{N}\), by \(3.1, 3.11\) and \(3.13 - 3.15\) we have
\[
\rho(0, \omega_r) = d(0, \omega_r) + \sup \{ |1 - (u_k, \omega_r)|, \, k \in \omega \} < \frac{\varepsilon}{20} + \frac{\varepsilon}{2} < \varepsilon.
\]
Thus \(\omega_r \in F_m\) for every \(r \in \mathbb{N}\). Evidently,
\[
\omega_r \to \bar{\omega} = (\bar{a}_n) \in \Delta_p, \quad \text{where } \bar{a}_n = \begin{cases} 1, & \text{if } n = n_{l+i} - s \text{ for some } i \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}
\]
Since \(F_m\) is a compact subset of \(\Delta_p\), we have \(\bar{\omega} \in F_m\). Hence \(\bar{\omega} \in H\). On the other hand, it is clear that \(m_k(\bar{\omega}) = n_k - s\) for every \(k \geq l + 1\). Thus for every \(k \geq l + 1\), \(n_k - m_k(\bar{\omega}) = s \neq \infty\). Now \(3.10\) implies that \(\bar{\omega} \notin H\).

This contradiction shows that \(H\) is not an \(F_\sigma\)-subset of \(\Delta_p\).

**Lemma 3.3.** Let \(X = \prod_{n \in \omega} Z(b_n), \) where \(1 < b_0 < b_1 < \ldots\) \(G := \hat{\prod} = \bigoplus_{n \in \omega} Z(b_n)\). Set \(u = \{u_n\}_{n \in \omega}, \) where \(u_n = 1 \in Z(b_n)^\omega \subset G\) for every \(n \in \omega\). Then \(u\) is a T-sequence in \(G\), and the T-characterized subgroup \(H = s_u(X)\) is a dense non-\(F_\sigma\)-subset of \(X\).

**Proof.** Set \(H := s_u(X)\). In \(17\) 2.3 it is shown that
\[
\omega = (a_n) \in s_u(X) \text{ if and only if } \left\| \frac{a_n}{b_n} \right\| \to 0.
\]
So \(\bigoplus_{n \in \omega} Z(b_n) \subseteq H\). Thus \(H\) is dense in \(X\). Now Fact \(14\) implies that \(u\) is a T-sequence in \(G\).

We have to show that \(H\) is not an \(F_\sigma\)-subset of \(X\). Suppose for a contradiction that \(H = \bigcup_{n \in \mathbb{N}} F_n\) is an \(F_\sigma\)-subset of \(X\), where \(F_n\) is a compact subset of \(X\) for every \(n \in \mathbb{N}\). Since \(H\) is a subgroup of \(X\), without loss of generality we can assume that \(F_n - F_n \subseteq F_{n+1}\). Since all \(F_n\) are closed in \((H, \rho)\) as well, the Baire theorem yields that there are \(0 < \varepsilon < 0.1\) and \(m \in \mathbb{N}\) such that \(F_m \supseteq \{ \omega \in X : \rho(0, \omega) \leq \varepsilon \}\).

Note that \(d(0, \omega) = 2^{-l}\), where \(0 \neq \omega = (a_n)_{n \in \omega} \in X\) and \(l\) is the minimal index such that \(a_l \neq 0\). Choose \(l\) such that \(2^{-l} < \varepsilon/3\). For every natural number \(k > l\), set
\[
\omega_k := (a_n^k), \quad \text{where } a_n^k = \begin{cases} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor, & \text{for every } n \text{ such that } l \leq n \leq k, \\ 0, & \text{otherwise.} \end{cases}
\]
Since \((u_n, \omega_k) = 1\) for every \(n > k\), we obtain that \(\omega_k \in H\) for every \(k > l\). For every \(n \in \omega\) we have
\[
2\pi \cdot \frac{1}{b_n} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor < \frac{2\pi \varepsilon}{20} < \varepsilon < \frac{1}{2}.
\]
This inequality and the inequalities \(3.1, 3.2\) imply
\[
\rho(0, \omega_k) = d(0, \omega_k) + \sup \{ |1 - (u_n, \omega_k)|, \, n \in \omega \}
\]
\[
\leq \frac{1}{2} + \max \left\{ 1 - \exp \left\{ 2\pi i \frac{1}{b_n} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor \right\}, \, l \leq n \leq k \right\}
\]
\[
\leq \frac{\varepsilon}{3} + 2\pi \cdot \max \left\{ \frac{1}{b_n} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor, \, l \leq n \leq k \right\} < \frac{\varepsilon}{3} + \frac{2\pi \varepsilon}{20} < \varepsilon.
\]
Thus \(\omega_k \in F_m\) for every natural number \(k > l\). Evidently,
\[
\omega_k \to \bar{\omega} = (\bar{a}_n)_{n \in \omega} \text{ in } X, \quad \text{where } \bar{a}_n = \begin{cases} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor, & \text{if } l \leq n. \\ 0, & \text{if } 0 \leq n < l, \end{cases}
\]
Since $F_m$ is a compact subset of $X$, we have $\bar{w} \in F_m$. Hence $\bar{w} \in H$. On the other hand, since $b_n \to \infty$ we have
\[
\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = \lim_{n \to \infty} \frac{1}{b_n} \left\lfloor \frac{\varepsilon b_n}{20} \right\rfloor = \frac{\varepsilon}{20} \neq 0.
\]
Thus $\bar{w} \notin H$ by (3.17). This contradiction shows that $H$ is not an $F_\sigma$-subset of $X$. □

Now we are in position to prove Theorems 1.16 and 1.18.

Proof of Theorem 1.16. Let $X$ be a compact Abelian group of infinite exponent. Then $G := \hat{X}$ also has infinite exponent. It is well-known that $G$ contains a countably infinite subgroup $S$ of one of the following forms:

- (a) $S \cong \mathbb{Z}$;
- (b) $S \cong \mathbb{Z}(p^\infty)$;
- (c) $S \cong \bigoplus_{n \in \omega} (b_n)$, where $1 < b_0 < b_1 < \ldots$.

Fix such a subgroup $S$. Set $K = S^\perp$ and $Y = X/K \cong S_d^\perp$, where $S_d$ denotes the group $S$ endowed with the discrete topology. Since $S$ is countable, $Y$ is metrizable. Hence $\{0\}$ is a $G_\delta$-subgroup of $Y$. Thus $K$ is a $G_\delta$-subgroup of $X$. Let $q : X \to Y$ be the quotient map. By Lemmas 3.1 and 3.3, the compact group $Y$ has a dense $T$-characterized subgroup $\tilde{H}$ which is not an $F_\sigma$-subset of $Y$. Lemma 2.6 implies that $H := q^{-1}(H)$ is a dense $T$-characterized subgroup of $X$. Since the continuous image of an $F_\sigma$-subset of a compact group is an $F_\sigma$-subset as well, we obtain that $H$ is not an $F_\sigma$-subset of $X$. Thus the subgroup $H$ of $X$ is $T$-characterized but it is not an $F_\sigma$-subset of $X$. The theorem is proved. □

Proof of Theorem 1.18. (1) follows from Fact 1.2.

(2) By Lemma 3.6 in [10], every infinite compact Abelian group $X$ contains a dense characterized subgroup $H$. By Fact 1.1, $H$ is $T$-characterized. Since every $G_\delta$-subgroup of $X$ is closed in $X$ by Proposition 2.4 of [10], $H$ is not a $G_\delta$-subgroup of $X$.

(3) follows from Theorem 1.14 and the aforementioned Proposition 2.4 of [10].

(4) follows from Fact 1.5.

(5) follows from Corollary 1.14. □

It is trivial that $\text{Char}_T(X) \subseteq \text{Char}(X)$ for every compact Abelian group $X$. For the circle group $\mathbb{T}$ we have.

Proposition 3.4. $\text{Char}_T(\mathbb{T}) = \text{Char}(\mathbb{T})$.

Proof. We have to show only that $\text{Char}(\mathbb{T}) \subseteq \text{Char}_T(\mathbb{T})$. Let $H = s_u(\mathbb{T}) \in \text{Char}(\mathbb{T})$ for some sequence $u$ in $\mathbb{Z}$.

If $H$ is infinite, then $H$ is dense in $\mathbb{T}$. So $u$ is a $T$-sequence in $\mathbb{Z}$ by Fact 1.1. Thus $H \in \text{Char}_T(\mathbb{T})$.

If $H$ is finite, then $H$ is closed in $\mathbb{T}$. Clearly, $\bar{H}$ has infinite exponent. Thus $H \in \text{Char}_T(\mathbb{T})$ by Theorem 1.10. □

Note that, if a compact Abelian group $X$ satisfies the equality $\text{Char}_T(X) = \text{Char}(X)$, then $X$ is connected by Fact 1.3 and Theorem 1.14. This fact and Proposition 3.4 justify the next problem:

Problem 3.5. Does there exists a connected compact Abelian group $X$ such that $\text{Char}_T(X) \neq \text{Char}(X)$? Is it true that $\text{Char}_T(X) = \text{Char}(X)$ if and only if $X$ is connected?

For a compact Abelian group $X$, the set of all subgroups of $X$ which are both $F_{\sigma\delta}$- and $G_{\delta\sigma}$-subsets of $X$ we denote by $\text{S}\Delta_0^\delta(X)$. To complete the study of the Borel hierarchy of $(T)$-characterized subgroups of $X$ we have to answer to the next question.

Problem 3.6. Describe compact Abelian groups $X$ of infinite exponent for which $\text{Char}(X) \subseteq \text{S}\Delta_0^\delta(X)$. For which compact Abelian groups $X$ of infinite exponent there exists a $T$-characterized subgroup $H$ that does not belong to $\text{S}\Delta_0^\delta(X)$?

4. $g_T$-closed and $g_T$-dense subgroups of compact Abelian groups

The following closure operator $g$ of the category of Abelian topological groups is defined in [13]. Let $X$ be an Abelian topological group and $H$ its arbitrary subgroup. The closure operator $g = g_X$ is defined as follows
\[
g_X(H) := \bigcap_{u \in \hat{X}} \{ s_u(X) \mid H \leq s_u(X) \},
\]
and we say that $H$ is $g$-closed if $H = g(H)$, and $H$ is $g$-dense if $g(H) = X$. 

The set of all $T$-sequences in the dual group $\hat{X}$ of a compact Abelian group $X$ we denote by $T_s(\hat{X})$. Clearly, $T_s(\hat{X}) \subseteq \hat{X}$. Let $H$ be a subgroup of $X$. In analogy to the closure operator $g$, $g$-closure and $g$-density, the operator $g_T$ is defined as follows

$$g_T(H) := \bigcap_{u \in T_s(\hat{X})} \{ s_u(X) : H \leq s_u(X) \},$$

and we say that $H$ is $g_T$-closed if $H = g_T(H)$, and $H$ is $g_T$-dense if $g_T(H) = X$.

In this section we study some properties of $g_T$-closed and $g_T$-dense subgroups of a compact Abelian group $X$. Note that every $g$-dense subgroup of $X$ is dense by Lemma 2.12 of \cite{13}, but for $g_T$-dense subgroups the situation changes:

**Proposition 4.1.** Let $X$ be a compact Abelian group.

1. If $H$ is a $g_T$-dense subgroup of $X$, then the closure $\bar{H}$ of $H$ is an open subgroup of $X$.
2. Every open subgroup of a compact Abelian group $X$ is $g_T$-dense.

**Proof.** (1) Suppose for a contradiction that $\bar{H}$ is not open in $X$. Then $X/\bar{H}$ is an infinite compact group. By Lemma 3.6 of \cite{10}, $X/\bar{H}$ has a proper dense characterized subgroup $S$. Fact \cite{13} implies that $S$ is a $T$-characterized subgroup of $X/\bar{H}$. Let $\bar{q} : X \to X/\bar{H}$ be the quotient map. Then Lemma 2.6 yields that $q^{-1}(S)$ is a $T$-characterized dense subgroup of $X$ containing $H$. Since $q^{-1}(S) \neq X$, we obtain that $H$ is not $g_T$-dense in $X$, a contradiction.

(2) Let $H$ be an open subgroup of $X$. If $H = X$ the assertion is trivial. Assume that $H$ is a proper subgroup (so $X$ is disconnected). Let $u$ be an arbitrary $T$-sequence such that $H \subseteq s_u(X)$. Since $H$ is open, $s_u(X)$ is open as well. Now Corollary \cite{13} implies that $s_u(X) = X$. Thus $H$ is $g_T$-dense in $X$. □

Proposition \cite{11}(1) shows that $g_T$-density may essentially differ from the usual $g$-density. In the next theorem we characterize all compact Abelian groups for which all $g_T$-dense subgroups are also dense.

**Theorem 4.2.** All $g_T$-dense subgroups of a compact Abelian group $X$ are dense if and only if $X$ is connected.

**Proof.** Assume that all $g_T$-dense subgroup of $X$ are dense. Proposition \cite{11}(2) implies that $X$ has no open proper subgroups. Thus $X$ is connected by \cite{23} 7.9.

Conversely, let $X$ be connected and $H$ be a $g_T$-dense subgroup of $X$. Proposition \cite{11}(1) implies that the closure $\bar{H}$ of $H$ is an open subgroup of $X$. Since $X$ is connected we obtain that $\bar{H} = X$. Thus $H$ is dense in $X$. □

For $g_T$-closed subgroups we have:

**Proposition 4.3.** Let $X$ be a compact Abelian group.

1. Every proper open subgroup $H$ of $X$ is a $g$-closed non-$g_T$-closed subgroup.
2. If every $g$-closed subgroup of $X$ is $g_T$-closed, then $X$ is connected.

**Proof.** (1) The subgroup $H$ is $g_T$-dense in $X$ by Proposition \cite{11}. Therefore $H$ is not $g_T$-closed. On the other hand, $H$ is $g$-closed in $X$ by Theorem A of \cite{10}.

(2) Item (1) implies that $X$ has no open subgroups. Thus $X$ is connected by \cite{23} 7.9. □

We do not know whether the converse in Proposition \cite{13}(2) holds true:

**Problem 4.4.** Let a compact Abelian group $X$ be connected. Is it true that every $g$-closed subgroup of $X$ is also $g_T$-closed?

**Historical Note.** This paper (with $a_n = n$ in Lemma 3.1) was sent for possible publications to the journal “Topology Proceedings” at 25 November 2012. However, the author till now did not received even a report from the referee. Since the paper is cited in \cite{10,21} and other articles which have already been published, the author decided to put it in ArXiv.

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