THE NUMBER OF CONNECTED COMPONENTS OF LINKS ASSOCIATED WITH THE THOMPSON GROUP

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Abstract. A few years ago Vaughan Jones devised a method to construct knots and links from elements of the Thompson groups. Given an element $g$ of $F_3$ we construct a permutation $P(g)$ which we call its Thompson permutation. By analogy with the braid group, we show that the number of the connected components of the link $L(g)$ is equal to the number of orbits of its Thompson permutation.

Introduction

The Thompson group $F$ was defined by Richard Thompson in the ’60s. About 20 years later the (Brown-)Thompson group $F_3$ was introduced by Kenneth Brown [11]. The latter group belongs to the family $F_k$, with $k \geq 2$, for which $F_2 = F$. All the $F_k$ can be described as groups of piecewise linear homeomorphisms of $[0,1]$.

Motivated by the aim of constructing a conformal field theory for every finite index subfactor, in [18] Vaughan Jones started a fascinating and multifaceted project involving the Thompson groups. One of the directions of this project has to do with knot theory. In particular, Jones defined a procedure to construct knots and links from elements of the Thompson group $F$ [18]. This procedure was very recently extended to the Thompson group $F_3$ in [20]. In passing, we mention that several representations of the Thompson groups were defined thanks to a general method introduced by Jones. Many of these representations are constructed from planar algebras [17] and are related to link and graph invariants [19, 5, 2, 6, 21] (see also [3, 4] for an elementary, but less powerful approach). More recently, other representations were introduced in [10], which are related to Cuntz algebras [12].

This connection between the Thompson groups and knots opened up the possibility of a full reboot of the theory of braids and links in the context of Thompson groups. In the framework of braid groups, it is well known that every link can be obtained as the closure of a braid. This striking result was proved by Alexander in 1923 [7]. In the new context of Thompson groups, results analogous to the Alexander theorem hold [18, 1]. In particular, all unoriented knots/links arise from elements of $F_3$, while for oriented knots/links the oriented subgroup $\bar{F}_3$ is needed (actually the binary oriented subgroup $\bar{F}_2$ is enough). We mention that the oriented subgroups were introduced by Jones in [18, 20] and studied in [14, 15, 23] (see also [22] for a study of the oriented subgroup $\bar{T} \leq T$). In the classical theory, the number of components of a link obtained as the closure of a braid is given by the number of orbits of the corresponding permutation. The aim of this paper is to give a similar
Figure 1. Pairs of opposing carets in $F_2$ and $F_3$.

\[ \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \]

Figure 2. The injection $\iota : F_2 \to F_3$ is induced by the following map between binary and ternary trees.

\[ \begin{array}{c}
\begin{array}{c}
\end{array}\end{array} \]

group theoretical interpretation for the Thompson group $F_3$ and, thus, providing an answer to Jones’s question [20, Question (4)]. Given an element $(T_+, T_-)$ of $F_3$, we introduce its Thompson permutation $\mathcal{P}(T_+, T_-)$, which is given by the composition of two permutations crafted from the two trees. Then, we prove that the number of components of $L(T_+, T_-)$ coincides with the number of orbits of its Thompson permutation $\mathcal{P}(T_+, T_-)$. There is a natural embedding $\iota : F_2 \to F_3$ and the construction of knots/links of $F_3$ restricted to $\iota(F_2)$ coincides with the original procedure from [18]. In particular, our result holds for $F$ as well.

This paper is divided in two sections. In Section 1, we recall the definitions of the Thompson groups and Jones’s construction of knots and links. In Section 2, we introduce the Thompson permutations. Then, we show that the number of orbits of $\mathcal{P}(T_+, T_-)$ coincides with the number of the connected components of $L(T_+, T_-)$.

1. Preliminaries on the Thompson groups and on Jones’s construction of knots.

There are several equivalent definitions of the Thompson groups $F = F_2$ and $F_3$. In this section we review the description of their elements that is most appropriate for our work in this paper, namely the one that uses tree diagrams. For further information we refer to [13] and [11]. An element of $F_2$ is given by a pair of rooted, planar, binary trees $(T_+, T_-)$ with the same number of leaves. As usual, we draw a pair of trees in the plane with one tree upside down on top of the other. Likewise, the elements of $F_3$ admit a description in terms of pairs of ternary trees. Two pairs of trees are equivalent if they differ by a pair of opposing carets, see Figure 1. Thanks to this equivalence relation, the following rule defines the multiplication in both $F_2$ and $F_3$: $(T_+, T) \cdot (T, T_-) := (T_+, T_-)$. The trivial element is represented
by any pair \((T, T)\) and the inverse of \((T_+, T_-)\) is just \((T_-, T_+)\). There is a natural injection \(\iota : F_2 \hookrightarrow F_3\). Given \((T_+, T_-) \in F_2\), firstly, add a new leaf to the middle of each vertex (thus turning every trivalent vertex into a quadrivalent one, see Figure 2). Then, join the new edges in the only planar way. This yields an element of \(F_3\). We provide an example in Figure 3.

We now review Jones’s construction of knots from elements of \(F_3\) by giving an explicit example. Consider the element of \(F_3\)

\[
X = \frac{T_+}{T_-} = \begin{array}{c}
\text{Diagram of } X
\end{array}
\]
Now join the two roots by an edge. Wolog we may suppose that the new edge passes through the point \((0,0)\).

At this stage all the vertices are quadrivalent, change them according to the rule displayed in Figure 4 to obtain a knot diagram.

**Figure 4.** The rules needed to turn 4-valent vertices into crossings.
Therefore, in our example we get the knot $\mathcal{L}(T_+, T_-)$

\[
\mathcal{L}(T_+, T_-) = \begin{array}{c}
\end{array}
\]

2. From trees to permutations and the main result.

In this section we want to show that, given an element $(T_+, T_-)$ of the Thompson group, there exists a permutation $\mathcal{P}(T_+, T_-)$, herein called the Thompson permutation of $(T_+, T_-)$, whose number of orbits coincides with the number of components of $\mathcal{L}(T_+, T_-)$.

It is appropriate to fix the notation for the permutations, see e.g. [16] Chapter 3. Given $k$ distinct integers $i_1, \ldots, i_k$ in $\{0, \ldots, n\}$, the symbol $(i_1, \ldots, i_k)$ represents the permutation $p : \{0, \ldots, n\} \to \{0, \ldots, n\}$, where $p(i_j) = i_{j+1}$ for $j < k$, $p(i_k) = i_1$, and $p(s) = s$ for all $s \in \{0, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. When $k = 2$, the permutation is called a transposition. A permutation of the form $(i_1, \ldots, i_k)$ is called a $k$-cycle. Two cycles are said to be disjoint if they have no integers in common. Every permutation is the product of disjoint cycles.

Given a rooted ternary tree, say with $2n + 1$ leaves, we show how to construct a permutation on the set $\{0, \ldots, 2n + 1\}$ associated with the tree. We illustrate it with a couple of examples. Consider the pair of trees

\[
T_+ = \begin{array}{c}
\end{array} \quad T_- = \begin{array}{c}
\end{array}
\]

where we numbered the leaves of each tree from left to right (starting from 1). We start with the tree $T_+$. We consider each leaf and take a path according to the rules
displayed in Figure 5. Each path ends when we meet another leaf or the root (the paths for $T_+$ are highlighted in red in the figure below).

{\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (root) at (0,0) {$\ldots$};
    \node (1) at (-1,-1) {1};
    \node (2) at (0,-1) {2};
    \node (3) at (1,-1) {3};
    \node (4) at (-1,-2) {4};
    \node (5) at (0,-2) {5};
    \draw (root) -- (1);
    \draw (root) -- (2);
    \draw (root) -- (3);
    \draw (1) -- (4);
    \draw (2) -- (5);
    \draw (3) -- (4);
    \draw (3) -- (5);
\end{tikzpicture}
\caption{Rules for calculating the permutation.}
\end{figure}

We note that in every tree there exists exactly one path from a leaf, say $f$, to the root. For this path we consider the permutation $(0, f)$. For example, in our case we have $(1, 5), (2, 4), (0, 3), (4, 2), (5, 1)$. Since all the transpositions (but the one corresponding to the root) occur exactly twice, we set aside only one of each. Now we define the permutation $\pi(T_+) : \{0, 1, \ldots, 2n + 1\} \to \{0, 1, \ldots, 2n + 1\}$ to be the product of all these transpositions. We call $\pi(T_+)$ the tangled permutation associated with $T_+$. In this example, we get $\pi(T_+) = (1, 5)(2, 4)(0, 3)$.

For the second tree we follow the same procedure. For $T_-$ the paths are

{\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (root) at (0,0) {$\ldots$};
    \node (1) at (-1,-1) {1};
    \node (2) at (0,-1) {2};
    \node (3) at (1,-1) {3};
    \node (4) at (-1,-2) {4};
    \node (5) at (0,-2) {5};
    \draw (root) -- (1);
    \draw (root) -- (2);
    \draw (root) -- (3);
    \draw (1) -- (4);
    \draw (2) -- (5);
    \draw (3) -- (4);
    \draw (3) -- (5);
\end{tikzpicture}
\end{figure}

and the transpositions are $(1, 4), (0, 2), (3, 5), (4, 1), (5, 3)$. Therefore, the permutation associated to $T_-$ is $\pi(T_-) = (0, 2)(1, 4)(3, 5)$. Now, the permutation associated with $(T_+, T_-)$ is defined as $\mathcal{P}(T_+, T_-) := \pi(T_+) \circ \pi(T_-)$. We call $\mathcal{P}(T_+, T_-)$ the Thompson permutation of $(T_+, T_-)$. In our example we have

$\mathcal{P}(T_+, T_-) = (1, 5)(2, 4)(0, 3)(0, 2)(1, 4)(3, 5) = (1, 4, 2, 0, 3, 5)$
Before stating the main result of this paper, we provide a characterisation of tangled permutations. We recall that a permutation on \( \{0, \ldots, n\} \) determines a partition consisting of its orbits. We say that a 4-tuple \( a < b < c < d \) is a crossing if \( \{a, c\} \) and \( \{b, d\} \) belong to two different orbits. In general, if we say that \( \{e, f, g, h\} \) is a crossing, we mean that if we arrange them in increasing order, the first and third elements belong to an orbit, the second and fourth to another orbit. See e.g. [9] for a study of crossing partitions.

**Proposition 1.** Every tangled permutation on the set \( \{0, 1, \ldots, 2n + 1\} \), \( n \geq 1 \), satisfies the following properties

1. it consists of \( n + 1 \) disjoint 2-cycles;
2. for every orbit \( \{a_1, a_3 = p(a_1)\} \) there exists at least another orbit \( \{a_2, a_4 = p(a_2)\} \) such that \( \{a_1, a_2, a_3, a_4\} \) is a crossing;
3. there are no \( a_1 < a_2 < a_3 < a_4 < a_5 < a_6 \) such that \( p(a_1) = a_4, p(a_3) = a_6, p(a_2) = a_5 \);
4. there are no \( a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8 \) such that \( p(a_1) = a_4, p(a_5) = a_8, p(a_2) = a_7, p(a_3) = a_6 \);
5. the number of crossings is \( n \).

Conversely, every permutation satisfying these properties is a tangled permutation. Moreover, the map \( T \mapsto \pi(T) \) between the set of ternary rooted planar trees with \( 2n + 1 \) leaves and tangled permutations on the set \( \{0, 1, \ldots, 2n + 1\} \) is bijective.

**Proof.** For the first part of the claim follows from the definition of tangled permutations.

For the second part, the proof is done by induction on \( n \). If \( n = 1 \), there exists just one permutation, i.e. \( (0, 2)(1, 3) \), and one ternary tree with 3 leaves, so the claim is clear. Suppose that the claim is true for \( n \) and consider a permutation \( p \) on \( \{0, \ldots, 2n + 3\} \). It is easy to see that there exists an index \( i \in \{1, \ldots, 2n + 3\} \) with \( p(i) = i + 2 \).

The idea is to consider now the permutation \( p \circ (i, p(i)) \), which acts trivially on \( \{i, p(i)\} \), re-number the indices \( \{0, \ldots, 2n + 1\} \setminus \{i, p(i)\} \) and construct a new permutation \( \hat{p} \) on \( \{0, \ldots, 2n + 1\} \) on which we may apply the induction hypothesis. Define
the permutation \( \hat{p} \) on \( \{0, \ldots, 2n + 1\} \) as

\[
\hat{p}(j) := \begin{cases} 
  i & \text{if } p(j) = i + 1 \\
  p(j + 1) & \text{if } j = i \\
  p(j) & \text{if } j \leq i - 1 \text{ and } p(j) \leq i - 1 \\
  p(j) - 2 & \text{if } j \leq i - 1 \text{ and } p(j) \geq i + 3 \\
  p(j + 2) & \text{if } j \geq i + 1 \text{ and } p(j + 2) \leq i - 1 \\
  p(j + 2) - 2 & \text{if } j \geq i + 1 \text{ and } p(j + 2) \geq i + 3 
\end{cases}
\]

By construction, \( \hat{p} \) satisfies the above properties and by induction is of the form \( \pi(T') \) for some tree \( T' \) with \( 2n + 1 \) leaves. By adding one caret below the \( i \)-th leaf we get a tree \( T \) such that \( \pi(T) = p \). Therefore, all the permutations satisfying (1), (2), (3) arise from ternary rooted planar trees.

\textbf{Remark 1.} The previous result provides a description of the elements of the Thompson group \( F_3 \) in terms of pairs of tangled permutations with the same number of disjoint cycles.

We are now in a position to state the main result of this paper.

\textbf{Theorem.} Let \((T_+, T_-)\) be a tree diagram in \( F_3 \) with \( 2n + 1 \) leaves. Then, the number of components of \( \mathcal{L}(T_+, T_-) \) is equal to the number of orbits of its Thompson permutation \( \mathcal{P}(T_+, T_-) \).

\textbf{Proof.} This result follows from the very definition of Thompson permutations. Indeed, given an element \((T_+, T_-) \in F_3\), one can number (from 0 to \( 2n + 1 \)) the intersection points between the \( x \)-axis and the knot diagram \( \mathcal{L}(T_+, T_-) \). This is precisely the numbering used for the trees \( T_\pm \). So we can move along this knot/link starting from the point \((1, 0)\) in the part of the knot/link contained in the lower-half plane. We will eventually come across the \( x \)-axis again, on the point prescribed by the permutation \( \pi(T_-) \) because the rules in Figure\ref{fig:thompson_permutation} describe exactly how the strands of the link \( \mathcal{L}(T_+, T_-) \) are connected. Now we continue moving along the strand, but in the upper-half plane. Also in this case, we will eventually come across the \( x \)-axis again. The point is prescribed by the permutation \( \pi(T_+) \). We can continue travelling along the strands of the tree diagram and we will eventually come back to the point \((1, 0)\) after visiting all the points of the \( x \)-axis with \( x \)-coordinate in the orbit of 1 for the permutation \( \mathcal{P}(T_+, T_-) \). This journey is completely described by \( \mathcal{P}(T_+, T_-) \). If \( \mathcal{P}(T_+, T_-) \) consists of just one cycle, this means that we have come across all the point \((0,0), \ldots, (2n + 1, 0)\) and \( \mathcal{L}(T_+, T_-) \) is a knot, that is \( \mathcal{L}(T_+, T_-) \) has only one connected component. Otherwise, we can consider one of the points that we have not met before (so another orbit of \( \mathcal{P}(T_+, T_-) \)) and continue our journey in \( \mathcal{L}(T_+, T_-) \). \hfill \Box

\textbf{Remark 2.} In [18] Section 5.3 an alternative procedure for the construction of knots and links was described. This alternative construction is associated with real values of \( A \) for the Kauffman bracket and produces only alternating knots (all the crossings are positive in this case). The previous result holds in this setting as well.
Indeed, changing the sign of the crossings does not affect the number of connected components.

**Example** (The $4_1$ knot). Consider the element of $F_3$ represented by the following two trees.

For the tree $T_+$ we get the following transpositions: $(1, 3), (2, 8), (3, 1), (4, 12), (5, 7), (6, 9), (7, 5), (8, 2), (9, 6), (10, 18), (11, 14), (12, 4), (13, 15), (14, 11), (15, 13), (16, 0), (17, 19), (18, 10), (19, 17)$. Therefore, the corresponding permutation is

$$\pi(T_+) = (1, 3)(2, 8)(4, 12)(5, 7)(6, 9)(10, 18)(11, 14)(13, 15)(16, 0)(17, 19)$$

Similarly, for $T_-$ we have the transpositions $(1, 4), (2, 0), (3, 6), (4, 1), (5, 8), (6, 3), (7, 18), (8, 5), (9, 11), (10, 14), (11, 9), (12, 19), (13, 16), (14, 10), (15, 17), (16, 13), (17, 15), (18, 7), (19, 12)$. The corresponding permutation is

$$\pi(T_-) = (1, 4)(2, 0)(3, 6)(5, 8)(7, 18)(9, 11)(10, 14)(12, 19)(13, 16)(15, 17)$$

Therefore, the permutation associated with $(T_+, T_-)$ is

$$P(T_+, T_-) = (1, 4, 12, 19, 17, 15, 13, 16, 0, 2, 8, 5, 7, 18, 10, 14, 11, 9, 6, 3)$$

We have only 1 orbit, therefore $\mathcal{L}(T_+, T_-)$ is a knot. Actually, after applying some Reidemeister moves (to be precise a sequence consisting of five Reidemeister moves
of type II and four of type I), one sees that $L(T_+, T_-)$ is the $4_1$ knot.

By analogy with the braid index, Jones defined the $F$-index of a knot/link \cite{18}. This is the smallest number of leaves required by an element of $F$ to give that link (or the number of 4-valent vertices plus 1 in $T_+$). Since $(T_+, T_-) = \iota(T'_+, T'_-)$, with $T'_\pm$ having 10 leaves, it means that the $F$-index of the $4_1$ knot is at most 10.

We end this paper posing the following question.

**Question.** How many Thompson permutations exist for every $n$? Is it possible to classify these permutations in terms of their cycles?
We point out that classifying these permutations in terms of the number of cycles might also be useful for a question posed in [14, Problems 6.17], where the authors propose to consider a random walk on $F$ and study the properties of the corresponding links.

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