Logarithmic Singularities of Specific Heat and Related Properties of Liquid $^4$He Near $\lambda$-Point

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Abstract

The singularity of specific heat ($C_p$) and related properties (viz. thermal expansion coefficient, $\alpha_p$, compressibility, $\kappa_T$, and pressure coefficient, $\beta$) of liquid $^4$He at $\lambda$-point is studied and the accuracy of its logarithmic nature as concluded for the first time from a microscopic theory of a system of interacting bosons is examined. A very good agreement between the results of this theory and experiments concludes that the singularity is intrinsically logarithmic. However, as shown by other studies reporting carefully measured $C_p$ of liquid $^4$He around $T_\lambda$ for $|t| = |(T_\lambda - T)/T_\lambda|$ ranging between $10^{-1}$ to $10^{-9}$, in and out of earth’s gravitational field and in finite size samples, weak effects arising from earth’s gravity and small sample size round off and $C_p$ assumes asymptotic nature near $t \approx 0$.

1 Introduction:

The $\lambda$-transition of liquid $^4$He is characterized by logarithmic singularity of: (i) specific heat at constant pressure ($C_p$), (ii) expansion coefficient ($\alpha_p$), (iii) isothermal compressibility ($\kappa_T$) and (iv) pressure coefficient ($\beta$) [1,2] with varying strength. An excessively high $C_p$ at $\lambda$-point indicating its singular behavior was first observed in 1932 by Keesom and Clusius [3], while a series of more accurate measurements [4-9] later confirmed its logarithmic nature. Recently, Lipa and coworkers [10,11] have reported very carefully measured $C_p$ of liquid $^4$He around $T_\lambda$ for $|t| = |(T_\lambda - T)/T_\lambda|$ ranging between $10^{-1}$ to $10^{-9}$, in and out of earth’s gravitational field. To isolate the effect of gravity they made their measurements on STS-52 space craft, space shuttle Columbia. While the net dependence of their $C_p$ on $t = (T_\lambda - T)/T_\lambda$ is certainly asymptotic but their analysis indicates that once the observed $C_p$ is corrected for the factors which round off its values near $T_\lambda$, the corrected $C_p$ fits closely logarithmic variation. Similarly, careful investigation of the effects of finite size of the sample performed by Lipa et al [12-14] and Gasparini et al [15, 16] also reveals that $C_p$ variations are intrinsically logarithmic; they assume asymptotic dependence on $t$ when the sample size becomes smaller than coherence length.

One also finds that early measurements of density [17,18] showing peak at $\lambda$-point indicative of a divergence of $\alpha_p$ were repeated with improved accuracy by Atkins and Edwards [19], Kerr [20], Edwards [21], and Chase et. al. [22] for detailed analysis. These results are reviewed elegantly by Kerr and Taylor [23]. It is evident that $\alpha_p$ for its linear relation with $C_p$ should have logarithmic divergence at $\lambda$-point. Similarly, the divergence of $\beta$ investigated by Ahlers [9], Lounasmaa and Kaunisto [24], Lounasmaa [25], and Kiersted [26, 27] is ought to be logarithmic. We note that thermodynamic relations between various response functions, as discussed by Rice [28], Pippard[29, 30], and Buckingham and Fairbank [6], indicate that $K_T$ is an asymptotically linear function of $C_p$ and hence of $\alpha_p$. Evidently, $k_T$ is naturally
expected to show logarithmic singularity at \( \lambda \)–point. However, the identification of this behavior from experimental observation is obscured by large regular contribution.

Despite a long and rich history of experimental work and untiring efforts of nearly seven decades for developing a viable microscopic theory of liquid \( ^4\text{He} \), the exact nature and origin of the above stated singularities has been unknown. The importance of this is reflected by a comment from Feynman in his book [31]. Recently, Jain [32] used unconventional approach to develop long awaited microscopic theory of a system of interacting bosons such as liquid \( ^4\text{He} \) based on macro-orbital representation of a particle in a many body system and obtained a relation for the logarithmic singularity of \( C_p \). This paper examines his relation for its agreement with experiments and uses it to find similar relations for \( \alpha_p, K_T \) and \( \beta \).

2 Results

2.1 Specific heat \( C_p \)

For a small range of \( |t| < 0.1^\circ K \), Jain’s theory concludes

\[
C_p = -A \ln |t| + B
\]

with
In order to demonstrate the fact that Eqn. 1 obtained from his theory can correctly account for the logarithmic divergence of $C_p$, Jain [32] used $\nu = 0.55$ and $\delta \phi_\lambda(0) = \pi$ to conclude $A = 5.71$ (both for $T < T_\lambda$ and $T > T_\lambda$) matching closely with: (i) similar estimates based on Widom-Kadanoff scaling laws [30, 33-35] and (ii) $A = 5.1$ for $T < T_\lambda$ and 5.355 for $T > T_\lambda$ obtained from experimental $C_p$. While the close agreement between theoretical $A$ and experimental $A$ showed the accuracy of Eqn. 1, but the choice of parameters rendered $B = -10.35$ which, however, differs significantly from $B = -7.77$ (for $T > T_\lambda$) and 15.52 (for $T < T_\lambda$) estimated from experimental $C_p$ [2]. In this paper we examine this aspect more deeply to find that: (i) $\delta \phi_\lambda(0)$ should, in principle, be lower than $\pi$ because the shift in $\phi$ positions of all particles in the process of their order-disorder in $\phi$ space need not be equal to $\pi$ and (ii) $\delta \phi_\lambda(0)$ should not, necessarily, be equal for both sides of $T_\lambda$ because particles on $T_\lambda^+$ side nearly have random positions in $\phi$-space while the same on $T_\lambda^-$ side are largely locked with $\Delta \phi = 2n\pi$ which indicates that $\delta \phi_\lambda(0)$ on $+ve$ side should be slightly lower than $\pi$, while that on $-ve$ side should be fairly small. Guided by these points, we find that $\delta \phi_\lambda(0) = 0.75\pi$ for $T > T_\lambda$ and $\delta \phi = 0.084\pi$ for $T < T_\lambda$ with $\nu = 0.47$ render $A = 5.1148$ and $B = 16.9319$ for $T < T_\lambda$ and $A = 5.1148$ and $B = -6.8930$ for $T > T_\lambda$ which agree closely with their experimental values. This agreement can be better perceived from Figs. 1-3 where we plot our calculated $C_p$ vs. $T_\lambda - T$ along with the experimental values [8] for comparison. Further since $A$ depends on $\nu$ and our choice of its equal value for both sides of $T_\lambda$ renders $A^+ = A^-$ whose experimental values, however, have small difference [10, 36, 37]. Evidently, if $\nu$ is presumed to have slightly different values on two sides of $T_\lambda$ for the same reason, the difference in $A^+$ and $A^-$ is easily explained. Evidently, Jain’s theory accounts for the observed logarithmic singularity in its all details.

Fig. 2 : $C_p$ vs. log $|T - T_\lambda|$ curve for liquid $^4$He around $\lambda$-point on log scale.
Fig. 3: \( C_p \) vs. \( T_\lambda - T \) curve for liquid \(^4\)He around \( \lambda \)-point (the points in this figure are closer to \( \lambda \)-point in comparison to Fig. 1). Curves on two sides of \( \lambda \)-point join each other (as shown here) when weak perturbation on logarithmic singularity round it off.

### 2.2 Expansion Coefficient

PBF relation between \( C_p \) and \( \alpha_p \) [2] along an arbitrary thermodynamic path renders:

\[
C_p = T \left( \frac{\partial S}{\partial T} \right)_t + VT \left( \frac{\partial P}{\partial T} \right)_t \alpha_p
\]  

(4)

Close to the \( \lambda \) line, this relation yields

\[
\alpha_p = A_\alpha \log |T - T_\lambda| + B_\lambda
\]  

(5)

with

\[
A_\alpha = -2.3025 \frac{A}{T_\lambda V_\lambda (\partial P/\partial T)_\lambda}
\]  

(6)

and

\[
B_\alpha = \frac{B + A \ln T_\lambda - T_\lambda (\partial S/\partial T)_\lambda}{V_\lambda T_\lambda (\partial P/\partial T)_\lambda}
\]  

(7)

where \( A \) and \( B \) are the same coefficients of Eqn.1 for \( C_p \). To estimate the coefficient \( A_\alpha \) and \( B_\alpha \) we use \( (\partial P/\partial T)_\lambda = -112.5 \text{ bar}/^oK \), \( (\partial S/\partial T)_\lambda = 102 \text{ cm}^3\text{/mole}/^oK^2 \) and \( V_\lambda = 27.38 \text{ cm}^3\text{/mole} \). These parameters give \( A_\alpha = 0.0166 \) and \( B_\alpha = 0.0032 \) for \( T < T_\lambda \) and \( A_\alpha = 0.0166 \) and \( B_\alpha = 0.0367 \) for \( T > T_\lambda \), while experiments reveal \( A_\alpha = 0.01680 \) and \( B_\alpha = 0.0025 \) for \( T < T_\lambda \) and \( A_\alpha = 0.0169 \) and \( B_\alpha = 0.0379 \) for \( T > T_\lambda \) [1]. We depict our results in Fig. 4 along with the experimental results of [19] and [9], respectively, marked as Exp.(1) and Exp(2).
2.3 Isothermal Compressibility \((\kappa_T)\):

In what follows from [2], the isothermal compressibility is related to \(\alpha_P\) and \(C_p\), respectively, through

\[
\alpha_P = V^{-1} \left( \frac{\partial V}{\partial T} \right)_t + \left( \frac{\partial P}{\partial T} \right)_t \kappa_T \tag{8}
\]

\[
\kappa_T = \left[ V T \left( \frac{\partial P}{\partial T} \right)_t \right]^{-1} C_p - \frac{1}{V} \left[ \left( \frac{\partial S}{\partial T} \right)_t \left( \frac{\partial P}{\partial T} \right)_t \right]^2 + \left( \frac{\partial V}{\partial T} \right)_t \left( \frac{\partial P}{\partial T} \right)_t \tag{9}
\]

This relation gives

\[
\kappa_T = A_\kappa \log |T - T_\lambda| + B_\kappa \tag{10}
\]

Using \((\partial V/\partial T)_\lambda = 43.82\text{cm}^3/\text{mole/K}\) and other parameters as used above we find \(A_\kappa = -0.0001\) and \(B_\kappa = 0.0141\) for \(T < T_\lambda\) and \(A_\kappa = -0.0001\) and \(B_\kappa = 0.0138\) for \(T > T_\lambda\). Our calculated \(\kappa_T\) is depicted in Fig 5 with the experimental values taken from [9].

2.4 Pressure Coefficient \((\beta)\)

From the thermodynamic relation

\[
\left( \frac{\partial P}{\partial T} \right)_V = - \left( \frac{\partial P}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P \tag{11}
\]
Fig. 5: $\kappa_T$ vs. $T_\lambda - T$ curve for liquid $^4He$ around $\lambda$–point.

Fig. 6: $\beta$ vs. $T_\lambda - T$ curve for liquid $^4He$ around $\lambda$–point.
one can obtain

$$\beta = \left( \frac{\partial P}{\partial T} \right)_V = \frac{\alpha P}{\kappa T}$$  \hspace{1cm} \text{(12)}$$

whose values at different $t$, obtained by using the calculated values of $\alpha P$ and $\kappa T$, are plotted in Fig 6. The experimental points obtained, respectively, from [9] and [19] are marked as Expt.(1) and Expt.(2). As expected, the divergence of $\beta$ at $\lambda-$point undoubtedly appears to be logarithmic.

3 Discussion

Several experimental studies of specific heat of liquid $^4\text{He}$, viz. in bulk investigated out of earth’s gravity [10,11] and confined to different geometries [12-16] have been performed near $\lambda-$point for various reasons discussed elegantly by these authors as well as by Bhattacharya and Bhattacherjee [38] and Schultka and Manousakis [39]. In particular, these studies aim at verifying several predictions of renormalization group theory. For the same reasons, extensive investigation of the $C_p$ of liquid $^4\text{He}$ mixtures near their $\lambda-$points have been also been made by Gasparini et al [40-41] and second sound propagation in liquid $^4\text{He}$ has been studied by Marek et al [42] and Swanson et al [43]. One, evidently, finds that the gravity of earth and the effects of finite size sample round off the $C_p$ of liquid $^4\text{He}$ at the $\lambda-$point and the position of its maximum is shifted to a temperature (say $T_m$) below $T\lambda$ by a small amount. However, in the absence of these effects $C_p$ divergence is logarithmic as concluded by Jain’s theory [32]. In this context it may be mentioned that Jain’s theory [32] does not incorporates weak effects arising from gravity of earth, finite size of the sample, etc. while concluding Eqn. 1. Since such effects would always be present, the experimental $C_p$ would always be rounded for one reason or the other as one tries to reach closer and closer to $T\lambda$.

Jain’s approach [32] to the microscopic theory of liquid $^4\text{He}$ type systems differs from conventional approaches [44] for its use of a macro-orbital (a kind of pair waveform [45]) to represent the self superposition state of a particle in a many body system. One may find that each particle in such a system assumes self superposition state when its collision with other particle(s) forces its pre-collision and post-collision states (both represented, presumably, by plane waves of two different momenta) to have their superposition. This, naturally, organizes two particles in collision at phase positions with $\Delta \phi = 2n\pi$, particularly, when particles are of low energy (i.e., with wave length $\lambda$ equal to or larger than the inter-particle separation $d$) and this is achieved for nearly all particles in the system at low temperatures with thermal de Boroglie wave length $\lambda_T \approx 2d$ [32]. The fact, that Jain’s approach gives due importance to all consequences of this superposition which represents one of the simple truths of wave nature of a quantum particle, ensures a better agreement of its predictions with experiments and this fact is successfully demonstrated by this paper. In addition the merits of Jain’s approach also rest with the facts that: (i) it makes no assumption like conventional theories of a system of interacting bosons which assume that $p = 0$ condensate exists in superfluid state of liquid $^4\text{He}$ type systems or Cooper type bound pairs of $^3\text{He}$ atoms are the origin of superfluidity of liquid $^4\text{He}$ type systems, (ii) its all inferences are based on a systematic critical analysis of the solutions of $N-$particle Schrödinger equation, (iii) as seen from [32, 46], it proposes a single framework for all many body systems of interacting bosons (or fermions) like $^4\text{He}$ (or electrons in solids or $^3\text{He}$) atoms, (iv) its simple mathematical framework is easy to comprehend and
does not permit adjustable parameters like conventional theories where one finds widely different theoretical predictions depending on the values of such parameters (for example as discussed in [47], different papers using BCS picture predicted widely different superfluid $T_c$ for liquid $^3He$), and as shown in this paper it explains even those observations, viz. the logarithmic singularity of $C_p$ and related properties of liquid $^4He$, which found no basis in the framework of conventional theories, etc. In addition, the fact that the theoretical results, obtained by using Jain’s approach, for example for superfluid $T_c$ of liquid $^3He$ and its pressure dependence [47], excitation spectrum of liquid $^4He$ [48], and those reported here, match closely with experiments, clearly indicates that Jain’s theoretical approach has great potential to explain the physics of widely different many body systems. This has been demonstrated in his recent work related to the basic foundations of superconductivity [46] and ground state of $N$ hard core particles in 1-D box [49], unification of the physics of widely different many body systems [50] and wave mechanics of two hard core particles in 1-D system [45].

Recently, Fliessbach [51] reported a semi-phenomenological microscopic model of $\lambda-$transition of $He-II$, proposed to be known as “almost ideal Bose gas model” (AIBG). To this effect, he intuitively modifies the wave function $\Psi_{IBG} = \sum k \phi_k$ representing a state of ideal Bose gas (IBG) with $\phi_k = \exp (ik \cdot r)$ replaced by $\phi_k = \sin (qx_j + \theta_j)$ and introduces the concept of localized phase ordering by physical (or Dirichlet) boundary conditions at the walls of macroscopic volume $V$. While there is considerable resemblance of these aspects of Fliessbach’s single particle function $\phi_k = \sin (qx_j + \theta_j)$ with Jain’s macro-orbital $\phi_{mo} = \sin (q \cdot r) \exp (K \cdot R)$ function which identifies each atom like a particle of quantum size $\lambda/2 = \pi/q$ moving freely with momentum $K$, there is great deal of difference in the allowed values of $q$ permitted by his theory [51] (viz., $q \geq \pi/L$ with $L$ being the size of macroscopic $V$) and Jain’s theory [32] (viz., $q \geq \pi/d$ with $d = (V/N)^{1/3}$). It may be noted Jain’s theory uses purely microscopic considerations and the possibility of wave superposition to conclude a macro-orbital as the right wave function for a particle in a system like $He-II$ [32]. Interestingly, we note that Fliessbach finds that the energy, responsible for the logarithmic singularity of the specific heat of liquid $^4He$ at $\lambda-$ point, is related to phase ordering and it depends on $t$ as $t \ln |t|$ which agrees with Jain’s result forming the basis of Eqn.1 used in the present analysis, of course with different multiplying factors to $t \ln |t|$. It may be noted that Fliessbach does not use his relation to show the nature of agreement of his theoretical results with experiments, hence no comparative analysis of his results with our results could be possible. However, since Fliessbach’s model also uses phase ordering (although introduced intuitively) to explain the logarithmic nature of the said singularity, it may be argued that the phase ordering of particles, as concluded by Jain’s microscopic theory, has strong foundation and the fact that this ordering is an important characteristic of superfluid phase of liquid $^4He$ type systems, can not be ignored. In this context, it may be mentioned that studying the coupled low dimensional superfluids, Mathey et. al. [52] find that superfluids have strong tendency to phase lock which falls in line with an important inference of Jain’s theory that inter-particle phase locking is an inherent aspect of a superfluid.

4 Conclusion

This study establishes that $C_p$ of liquid $^4He$, when observed in absence of the effects of earth’s gravity or finite size of the sample, etc., exhibits logarithmic singularity at $T_\lambda$ and this agrees closely with Jain’s
microscopic theory [32]. As shown in [32], this singularity is a consequence of order-disorder of particles in phase space forced by the wave nature of particles or quantum correlation between a pair of bosons. We find that Jain’s theory [32] is capable of accounting for the difference in $C_p$ at $T < T_\lambda$ and $T > T_\lambda$ for the same value of $|T_\lambda - T|$. It is noted that $\alpha_p$, $\kappa_T$ and $\beta$ also exhibit logarithmic divergence for their relation with $C_p$ which implies that this effect also originates from the ordering of the particles in phase space. The agreement of our theoretical values obtained from Jain’s theory [32] with experimental results undoubtedly confirms the accuracy of Eqn. 1. Evidently it may be concluded that Jain’s macro-orbital theory is equipped to explain the origin of logarithmic divergence of thermodynamic response functions and it is more advantageous for its accuracy, simplicity and clarity.

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References

1. J. Wilks, The Properties of Liquid and Solid Helium, Clarendon Press, Oxford (1967).
2. G. Ahlers, in “The Physics of Solid and Liquid Helium”, Part I [K.H. Bennemann and J.B. Katterson, eds.], pp 85-206, John Wiley and Sons, New York (1976).
3. W.H. Keessom and K.Clausius, Leiden Comm. 219e; Proc. Sect. Sci. K. ned. Akad. Wet. 35, 307 (1932).
4. W.M. Fairbank, M.J. Buckingham, and C.F. Kellers, in “Proceedings of the Fifth International Conference on Low Temperature Physics and Chemistry”, [J.R. Dillinger, ed.], p. 50, University of Wisconsin Press, Madison, Wisc (1958).
5. C.F. Kellers, Ph.D. Thesis, Duke University, Durham, N.C. (1960).
6. M.J. Buckingham and W.M. Fairbank, in “Progress in Low Temperature Physics”, [C.J. Gorter, ed.], Vol III, p. 80, North-Holland, Amsterdam (1961).
7. W.M. Fairbank and C.F. Kellers, in “Critical Phenomena, Proceedings of a Conference”, [M.S. Green and J.V. Sengers, Natl. Bur.Std. Misc. Pub. No. 273, p. 71, U.S. GPO, Washington, D.C. (1966).
8. G. Ahlers, Phys. Rev. A, 3, 696 (1971).
9. G.Ahlers, Phys.Rev. A, 8, 530 (1973).
10. J. A. Lipa, D.R. Swanson, J.A. Nissen, T.C.P. Chui, U.E. Israelsson, Phys. Rev. Lett., 76, 944 (1996).
11. J.A.Lipa, J.A. Nissen, D.A. Stricker, D.R. Swanson, and T.C.P. Chui, Phys. Rev. B, 68, 174518-1 (2003).
12. J.A.Lipa and T.C.P. Chui, Phys. Rev. Lett., 51, 2291 (1983).
13. M. Coleman and J.A.Lipa, Phys. Rev. Lett., 74, 286 (1995).
14. J.A. Lipa, D.R. Swanson, J.A. Nissen, Z.K. Geng, P.R. Williamson, D.A. Stricker, T.C.P. Chui, U.E. Israe1sson, and M. Larson, Phys. Rev. Lett., 84, 4894 (2000).

15. T. Chen and F.M. Gasparini, Phys. Rev. Lett., 40, 331 (1978).

16. S. Mehta and F.M. Gasparini, Phys. Rev. Lett., 78, 2596 (1997).

17. H.K. Onnes and J.D.A. Boks, Leiden Comm. 170b (1924).

18. E. Mathias, C.A. Crommelin, H.K. Onnes and J.C. Swallow, Leiden Comm. 172 (1925).

19. K.R. Atkins and M.H. Edwards, Phys. Rev., 97, 1429 (1955).

20. E.C. Kerr., J. Chem. Phys., 26, 292 (1957).

21. M.H. Edwards, Can. J. Phys., 36, 884 (1058).

22. C.E. Chase, E. Maxwell and W.E. Millett, Physica, 27, 1129 (1961).

23. E.C. Kerr and R.D. Taylor, Ann. Phys. 26, 292 (1964).

24. O. Lounasmaa and L. Kaunisto, Ann. Acad. Sci. Fenn. Sev. A VI (Finland) No. 59 (1960).

25. O.V. Lounasmaa, Phys. Rev., 130, 847 (1963).

26. H.A. Kiersted, Phys. Rev., 138, A1594 (1965).

27. H.A. Kiersted, Phys. Rev., 153, 258 (1967).

28. O.K. Rice, J. Chem. Phys., 22, 1535 (1954).

29. A.B. Pippard, Phil. Mag. 1, 473 (1956).

30. A.B. Pippard, The Elements of Classical Thermodynamics, Chap. IX, Cambridge Univ. Press (1957).

31. R.P. Feynman, Statistical Mechanics, W.A. Benjamin, Inc., Reading (1972), p.34.

32. Y.S. Jain, J. Sci. Expl., 16, 77 (2002); more detailed version of this paper is available for ready reference as a paper entitled, “Macro-orbitals and microscopic theory of a system of interacting bosons” www.arxiv.org/cond-mat/0606571.

33. M.E. Fisher, Reports on the Progress of Physics, 30, 615 (1967).

34. B. Widom, J. Chem. Phys., 43, 3892 (1965); 43, 3898 (1965).

35. R.B. Griffiths, Phys. Rev., 158, 176 (1967).

36. L.P. Kadanoff, W. Gotze, D. Hamblen, R. Hecht, E.A. S. Lewis, V.V. Palciauskas, M. Rayl, and J. Swift, Rev. Mod. Phys., 39, 395 (1967).

37. G. Ahlers, Phys. Rev. Lett., 23, 464, 739 (1969).

38. S. Bhattacharya and J. K. Bhattacharjee, Phys. Rev. B 59, 3341 (1991).
39. N. Schultka and E. Manousakis, Phys. Rev. Lett. 75, 2710 (1995).

40. F.M. Gasparini and M.R. Moldover, Phys. Rev. B, 12, 93 (1975).

41. F.M. Gasparini and A.A. Gaeta, Phys. Rev. B, 17, 1466 (1978).

42. D. Marek, J.A. Lipa and D. Phillips, Phys. Rev. B, 38, 4465 (1988).

43. D.R. Swanson, T.C.P. Chui and J.A. Lipa, Phys. Rev. B, 46, 9041 (1992).

44. See Ref.[32] for a brief discussion on the two basic conventional approaches used to understand a system like liquid $^4$He.

45. Y.S. Jain, Central Europ. J. Phys. 2, 709 (2004).

46. Y.S. Jain, Basic foundations of microscopic theory of superconductivity www.arxiv.org/cond-mat/0603784

47. Y.S. Jain, Superfluid $T_c$ of liquid helium-3 and its pressure dependence, www.arxiv.org/cond-mat/0611298.

48. Y.S. Jain, A study of elementary excitations of liquid helium-4 using macro-orbital microscopic theory www.arxiv.org/cond-mat/0609418.

49. Y.S. Jain, Ground state of a system of N hard core quantum particles in 1-D box www.arxiv.org/cond-mat/0606409.

50. Y.S. Jain, J. Sc. Explor. 16, 117 (2002).

51. T. Fliessbach, A model for $\lambda$ transition of helium, www.arxiv.org/cond-mat/0203353.

52. L. Mathey, A. Polkovnikov and A.H. Castro Neto, Phase locking transition of coupled low dimensional superfluids, www.arxiv.org/cond-mat/0612174.