SU(3) Latent Heat and Surface Tension from
Tree Level and Tadpole Improved Actions

B. Beinlich, F. Karsch and A. Peikert

Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany

ABSTRACT

We analyze the latent heat and surface tension at the SU(3) deconfinement phase transition with tree level and tadpole improved Symanzik actions on lattices with temporal extent $N_\tau = 3$ and 4 and spatial extent $N_\sigma/N_\tau = 4, 6$ and 8. In comparison to the standard Wilson action we do find a drastic reduction of cut-off effects already with tree level improved actions. On lattices with temporal extent $N_\tau = 4$ results for the surface tension and latent heat obtained with a tree level improved action agree well with those obtained with a tadpole improved action. A comparison with $N_\tau = 3$ calculations, however, shows that results obtained with tadpole action remain unaffected by cut-off effects even on this coarse lattice, while the tree level action becomes sensitive to the cut-off.

For the surface tension and latent heat we find $\sigma_1/T_c^3 = 0.0155 (16)$ and $\Delta\epsilon/T_c^4 = 1.40 (9)$, respectively.
1 Introduction

Tree level improved Symanzik actions [1] have been shown to yield a large reduction of finite cut-off effects on bulk thermodynamic observables even at temperatures close to $T_c$ [2, 3]. It has been found that the large cut-off dependence observed in calculations with the standard Wilson action [4] to a large extent does result from cut-off dependent corrections to the high temperature ideal gas limit. They, therefore, can be drastically reduced already with tree level improved Symanzik actions [2, 3] or fixed point actions [5], which are constructed in order to reduce or eliminate the cut-off dependence in the weak coupling / high temperature limit. Further cut-off dependent corrections of $O(a^2g^2)$ do seem to be small for thermodynamic observables in the high temperature plasma phase. This can be seen, for instance, from a calculation of the difference of energy density and three times the pressure. The leading ideal gas contributions cancel in this quantity. It thus is sensitive only to $O(g^2)$ and higher order corrections. Calculations of this quantity performed with the tree level improved Symanzik action on lattices with small temporal extent do agree with the continuum extrapolation obtained from simulations with the Wilson action [3]. Bulk thermodynamics of $SU(N)$ gauge theories can thus be studied with tree level improved actions showing little cut-off distortion already on lattices with temporal extent $N_\tau = 4$.

The success of improved actions for the calculation of bulk thermodynamics even at temperatures close to $T_c$ naturally leads to the question whether these actions also do lead to an improvement at $T_c$. This is, of course, a highly non-perturbative regime. However, observables like the latent heat and the surface tension, which characterize the discontinuities at the first order deconfinement phase transition in a $SU(3)$ gauge theory, do depend on properties of the low as well as the high temperature phase. As the latter is largely controlled by high momentum modes it may be expected that some improvement does result already from a perturbatively improved description of the high momentum modes in the deconfined phase. We will show here that this is indeed the case. In addition we will address the question to what extent a tadpole improved Symanzik action [6, 7] does lead to a further reduction of the cut-off dependence and would allow to perform calculations on even coarser lattices.

The latent heat and the surface tension at the $SU(3)$ deconfinement transition have been studied with the standard Wilson action on lattices up to temporal extent $N_\tau = 6$ [8, 9]. A strong cut-off dependence has been found when comparing calculations for $N_\tau = 4$ and 6 which is compatible in magnitude with the cut-off dependence visible in calculations of the energy density or pressure in the plasma phase. On the basis of these results an extrapolation to the continuum limit for these observables is not yet possible and calculations on larger temporal lattices do not seem to be feasible with the standard Wilson action given the present computa-
tional possibilities. One thus has to aim at a reduction of the cut-off dependence on lattices with small temporal extent in order to get reliable results for the continuum values of the latent heat and surface tension at $T_c$. We will present here results from calculations with a tree level as well as a tadpole improved Symanzik action on lattices of temporal extent $N_\tau = 3$ and 4 and spatial extent $N_\sigma/N_\tau$ ranging from 4 to 8.

2 Critical Couplings

We use the (1,2) Symanzik action which is defined by adding a planar (1,2) Wilson loop to the standard Wilson action defined in terms of a (1,1) Wilson loop (plaquette),

$$S(\beta) = \sum_{x,\nu > \mu} \left( \frac{5}{3} W^{1,1}_{\mu,\nu}(x) - \frac{1}{6u_0(\beta)} W^{1,2}_{\mu,\nu}(x) \right).$$

Here $W^{k,l}_{\mu,\nu}(x)$ denotes a symmetrized combination of $k \times l$ Wilson loops in the $(\mu, \nu)$-plane of the lattice,

$$W^{k,l}_{\mu,\nu}(x) = 1 - \frac{1}{2N} \left( \text{Re} \text{ Tr} V^{(k)}_{x,\mu} V^{(l)\dagger}_{x+k\mu,\nu} V^{(k)+\dagger}_{x+l\nu,\mu} V^{(l)+\dagger}_{x,\nu} + (k \leftrightarrow l) \right),$$

with $V^{(k)}_{x,\mu} = \prod_{j=0}^{k-1} U_{x+j\mu,\mu}$ and $x = (n_1, n_2, n_3, n_4)$ denoting the sites on an asymmetric lattice of size $N_3 N_\tau$. We consider the tree level improved action, corresponding to $u_0 \equiv 1$, as well as a tadpole improved action \[6\] where

$$u_0^4 = \frac{1}{6N_3^3 N_\tau} \left\{ \sum_{x,\nu > \mu} (1 - W^{1,1}_{\mu,\nu}(x)) \right\}$$

is the self-consistently determined plaquette expectation value. We have determined this tadpole improvement factor at several values of the gauge coupling $\beta$ and then used a spline interpolation in order to define $u_0$ at intermediate values of $\beta$. This allows to give an unambiguous definition of derivatives of the tadpole term with respect to the gauge coupling which is needed in thermodynamic observables. Moreover, it allows us to use the Ferrenberg-Swendsen reweighting technique in the vicinity of $T_c$. We note that the tadpole improved action does depend on the gauge coupling $\beta$ through $u_0$. This does lead to some modifications of the relations usually explored for thermodynamic observables, which are defined through derivatives of the partition function, $Z = \int \mathcal{D}U \exp(-\beta S(\beta))$, with respect to $\beta$. For the purpose
of the calculations presented here this is of relevance for the analysis of the latent heat.

Most of our calculations have been performed close to the critical coupling on lattices with temporal extent $N_\tau = 3$ and 4. On the smaller lattices we have performed simulations at several nearby $\beta$-values. Typically 20,000-40,000 configurations have been generated at each $\beta$-value using an overrelaxed heat bath algorithm (1 iteration = 4 overrelaxation + 1 heat bath step). The critical couplings have then been determined from the location of peaks in the Polyakov-loop susceptibilities,

$$\chi = \langle |L|^2 \rangle - \langle |L| \rangle^2,$$

(2.4)

where $L$ denotes the Polyakov-loop

$$L = \frac{1}{N_\sigma^3} \sum_{\vec{x}} \text{Tr} \prod_{x_4=1}^{N_\tau} U(\vec{x},x_4),$$

(2.5)

We have used an interpolation based on the Ferrenberg-Swendsen reweighting method to determine the location of the maximum in $\chi$. In Table 1 we summarize our results for the critical couplings. In the case of the Symanzik improved tree level action our results do agree well with those of earlier calculations [10]. For $N_\tau = 4$ we have used results from different spatial lattice sizes to perform an infinite volume extrapolation of the critical couplings using the ansatz

$$\beta_c(N_\tau) = \beta_c(\infty) - h(N_\tau/N_\sigma)^3,$$

(2.6)

which is valid for first order phase transitions. The coefficient $h$ characterizing the infrared sensitivity of the thermodynamics at $T_c$ has been found to be similar for both improved actions. We find $h = 0.101 (34)$ and $0.068 (45)$ for the tree level and tadpole improved actions, respectively. This also agrees well with the volume dependence observed in the case of the standard Wilson action [8, 11].

3 Surface Tension

The $SU(3)$ deconfinement transition is known to be of first order. At the critical temperature the deconfined and confined phases can coexist in a mixed phase. Both coexisting phases are separated by an interface. The additional energy needed to build up such interfaces does, however, make these configurations more unlikely than pure states of one or the other phase. This is reflected in the double-peaked
Table 1: Critical couplings for the tree level and tadpole improved actions. Errors are obtained from a jack-knife analysis. Columns four and seven give the combined statistics from different $\beta$-values entering the Ferrenberg-Swendsen analysis of the susceptibilities, surface tension and latent heat. The number of $\beta$-values entering this analysis are given in columns three and six.

| $N_g^3 \times N_T$ | tree level | tadpole |
|---------------------|------------|---------|
|                     | $\beta_c$  | $\# \beta$ | $\#$ iterations | $\beta_c$ | $\# \beta$ | $\#$ iterations |
| $12^2 \times 3$     | 3.9079 (6) | 5        | 80000           | 4.1868 (4)| 6         | 85500           |
| $16^3 \times 4$     | 4.0715 (4) | 8        | 157500          | 4.3512 (4)| 7         | 133400          |
| $24^3 \times 4$     | 4.0722 (4) | 3        | 75200           | 4.3519 (5)| 1         | 38500           |
| $32^3 \times 4$     | 4.0729 (3) | 1        | 46800           | 4.3522 (4)| 1         | 59000           |
| $(\infty)^3 \times 4$ | 4.0730 (3) | -       | -               | 4.3523 (4)| -         | -               |

To be specific we consider the probability distribution of the absolute value of the Polyakov loop, $|L|$, and follow the analysis presented in Ref. [9] to extract the surface tension. The probability distribution at the minimum is proportional to

$$P(|L|) \sim \exp\left(-\left[f_1 V_1 + f_2 V_2 + 2\sigma_I A\right]/T\right)$$

(3.1)

where $f_i$ denotes the free energy in the phase $i$, and $V_i$ is the volume occupied by that phase. The last term denotes the extra free energy needed to create a surface to separate both phases. Here $\sigma_I$ denotes the surface free energy and $A$ is the area of the interface. An additional factor of two does appear here because we consider finite volumes with periodic boundary conditions and one always has to create two interfaces in this case. At $T_c$ the free energy density in both phases is identical, $f_1 = f_2$. In our simulations we have observed frequent flips between the confined and deconfined phase on the $12^3$, $16^3$ and $24^3$ lattices which suggests that we have properly sampled both phases. On the $32^3$ lattice the data samples include, however, only 4 - 5 flips at each $\beta$-value.

In order to determine the maxima and the minimum of $P(|L|)$ the region in the vicinity of the extrema has been fitted with third order polynomials [9]. A Ferrenberg-Swendsen reweighting has been used to shift the $\beta$-value to a point where the two maxima, $P_{\text{max},1}$ and $P_{\text{max},2}$, in the distribution function of the Polyakov-loop
have equal height. A simple estimate of the surface tension is then given by

\[ \frac{\sigma_L}{T_c^3} = \frac{1}{2} \left( \frac{N_{\sigma}}{N_{\tau}} \right)^2 \ln \left( \frac{P_{\min}}{P_{\max}} \right) , \]  

(3.2)

where \( P_{\max} \equiv P_{\max,1} \equiv P_{\max,2} \). In order to obtain a reliable error estimate we have performed a jack-knife analysis at this \( \beta \)-value. As the relative maxima are then no longer of equal height in the course of the analysis, we define the surface tension through the jack-knife average of

\[ \frac{\sigma_L}{T_c^3} = \frac{1}{2} \left( \frac{N_{\sigma}}{N_{\tau}} \right)^2 \ln \left( \frac{P_{\min}}{P_{\max,1} P_{\max,2}} \right) . \]  

(3.3)
where $\gamma_1$ and $\gamma_2$ characterize the relative weight of the confined and deconfined phases at $\beta_c$ and are fixed by demanding that the Polyakov loop expectation value is given by the weighted sum of the values $L_1$ and $L_2$ at $P_{max,1}$ and $P_{max,2}$, respectively, i.e.

$$\langle |L| \rangle = \gamma_1 L_1 + \gamma_2 L_2 \quad \text{with} \quad \gamma_1 + \gamma_2 = 1 . \quad (3.4)$$

Of course, we do find that the surface tension defined this way agrees within errors with the global value extracted from Eq. 3.2. For the relative weight $\gamma_1$ we find in all cases values close to 0.4 which also is similar to the case of the standard Wilson action\(^1\).

The results for $\sigma_1/T^3_c$ are given in Table 2 for the improved actions analyzed by us on lattices with temporal extent $N_\tau = 3$ and 4. Also given there are the results from [9] obtained with the standard Wilson action on $N_\tau = 4$ and $N_\tau = 6$ lattices.

| $N_\tau$ | volume | tree level | tadpole | Wilson |
|----------|---------|------------|---------|--------|
| 3        | $12^3$  | 0.0234 (24)| 0.0158 (11) |
| 4        | $12^2 \times 24$ | | |
| 4        | $24^2 \times 36$ | | 0.0241 (27) |
| 4        | $16^3$  | 0.0148 (16)| 0.0147 (14) |
| 4        | $24^3$  | 0.0136 (25)| 0.0119 (21) |
| 4        | $32^3$  | 0.0116 (23)| 0.0125 (17) |
| 4        | $\infty$ | 0.0152 (26)| 0.0152 (20)| 0.0295 (21) |
| 6        | $20^3$  | | 0.0123 (28) |
| 6        | $24^3$  | | 0.0143 (22) |
| 6        | $36^2 \times 48$ | | 0.0164 (26) |
| 6        | $\infty$ | | | 0.0218 (33) |

\(^1\)There is a misprint in the labeling of the last column of Table I in Ref. [9]. This column does give $\gamma_2$ rather than $\gamma_1$.

Table 2: Surface tension for the improved actions on several finite spatial lattices with temporal extent $N_\tau = 4$ and the Wilson action for $N_\tau = 4$ and 6. The latter are taken from Ref. [9]. The second column gives the spatial lattice size.

In order to extract the infinite volume result for the surface tension one has to take into account finite volume corrections, which result from the finite width of
the (gaussian) peaks in the pure phases, the contribution of zero modes resulting from the translational invariance as well as from contributions of fluctuations of the interface. Taking these into account, the interface tension should be extrapolated to infinite volume using the ansatz given in Ref. [9],

\[
\sigma_{IT} \left( \frac{T^3}{c} \right) = \left( \frac{\sigma_{IT}}{T^3} \right) - \left( \frac{N_T}{N_\sigma} \right)^2 \left[ c + \frac{1}{4} \ln N_\sigma \right].
\]

(3.5)

The result of such an extrapolation is also given in Table 2 for the \( N_T = 4 \) calculations. Even without performing the infinite volume extrapolations, it is, however, obvious already from Table 2 that the surface tension is reduced in the improved action calculations relative to that of the Wilson action with the same temporal extent.

We note that the tree level improved action still shows some cut-off dependence when comparing the \( N_T = 3 \) and 4 results while the results for the tadpole improved action coincide within errors. Combining both results for the tadpole improved action we find as an estimate for the surface tension

\[
\frac{\sigma_{IT}}{T^3} = 0.0155 \pm 0.0016.
\]

(3.6)

It is interesting to note that this agrees well with an extrapolation in \( 1/N_T^2 = (aT_c)^2 \) of the results obtained with the Wilson action for \( N_T = 4 \) and 6.

4 The Latent Heat

The latent heat is calculated from the discontinuity in the energy density, \( \epsilon \), or more conveniently directly from the discontinuity in \( (\epsilon - 3P) \). The latter is obtained from the discontinuity in the various Wilson loops entering the definition of the improved actions,

\[
\frac{\Delta \epsilon}{T^4} = \Delta \left( \frac{\epsilon - 3P}{T^4} \right) = \frac{1}{6} \left( \frac{N_T}{N_\sigma} \right)^3 \left( a \frac{d\beta}{da} \right) \left( \langle \tilde{S} \rangle_+ - \langle \tilde{S} \rangle_- \right),
\]

(4.1)

with \( \tilde{S} \equiv S - dS/d\beta \) and \( ad\beta/da \) denoting the SU(3) \( \beta \)-function.

The difference of the expectation values at \( \beta_c \) is obtained by calculating expectation values separately in the two coexisting phases at the critical coupling \( \beta_c \). This does require large spatial lattices in order to separate well the two coexisting phases. We therefore have performed the analysis of the latent heat only for the
$32^3 \times 4$ lattices. Following the approach used in Ref. [8] we have analyzed the time histories of the Polyakov loop values and introduced a cut to classify configurations belonging to either of the two phases. We then have performed averages in the two phases separately and left out configurations belonging to the transition region. We have checked that the expectation values calculated in both phases are within errors insensitive to the precise choice of the cut and the width of transition regions. In

![Figure 2: Probability distributions of $\tilde{S}$ at the critical couplings.](image)

Figure 2 we show the distribution of $\tilde{S}$ for the improved actions as well as the Wilson action on lattices with temporal extent $N_\tau = 4$. It is obvious from this Figure that the discontinuity is drastically reduced for the improved actions relative to the Wilson action.

In order to extract the latent heat we still need the $\beta$-function entering the definition of $\Delta\epsilon/T_c^4$ in Eq. (4.1). We have determined this through the calculation of the string tension in units of the cut-off following the approach discussed in Ref. [4]. These calculations have been performed on lattices of size $16^4$ [3]. We also have reanalyzed the results for the latent heat presented in [8] using the newly determined $\beta$-function of [4]. Further details of our determination of the temperature scale will be presented in Ref. [3].

The resulting values for the latent heat are summarized in Table 3. There we also give results obtained with the perturbative $\beta$-function, $ad\beta/da = 33/4\pi^2$. The ratio of the fourth and fifth column in Table 3 thus reflects the deviations of the non-perturbative determination of the $\beta$-functions from the perturbative result. We note that these ratios are, in fact, quite similar, which shows that the improved actions do not lead to an improved asymptotic scaling behaviour. The results summarized in Table 3 do, however, show that the improved actions do reduce drastically the cut-off dependence of the latent heat. Results obtained with the tree level and tadpole improved actions for $N_\tau = 4$ agree with each other. They are about 30% smaller.
Table 3: Latent heat for the three improved actions and the Wilson action. Results for the Wilson action are taken from [8] using the non-perturbative $\beta$-function calculated in Ref. [4].

\[
\Delta \varepsilon / T_c^4 = 1.40 \pm 0.09 .
\] (4.2)

5 Conclusion

We have analyzed the surface tension and the latent heat at the first order deconfinement phase transition of the SU(3) gauge theory using tree level and tadpole improved actions. We do find that these actions drastically reduce the cut-off dependence observed previously in calculations of these quantities with the standard Wilson action. Results obtained with a Symanzik improved tree level action agree within errors with those obtained with a tadpole improved action for $N_\tau = 4$ lattices. The results obtained with the tree level action do, however, show a clear cut-off dependence when comparing calculations on the $N_\tau = 3$ and 4 lattices while results obtained with the tadpole improved action remain unaffected by the change in lattice cut-off. We thus expect that our results obtained on the $N_\tau = 4$ lattice with the tadpole improved action do provide a good estimate for the continuum limit result of discontinuities in thermodynamic observables at $T_c$.

Over the small range of couplings we have explored in this study the tadpole improvement factor varies only slightly, $0.86 < u_0(\beta) < 0.88$. Although we do take into account this variation in our simulations we note that we do effectively work with an over-improved Symanzik action where the weight of the contribution of $(1 \times 2)$-loops relative to the $(1 \times 1)$-plaquette term is increased from $1/10$ by about 30%. Testing the role of the $\beta$-dependence of the tadpole improvement factor does require an analysis of the thermodynamics in a larger temperature interval. This will be presented elsewhere [3]. It also would be interesting to investigate the
sensitivity of the analysis presented here to even larger changes in the weight of the 
$(1 \times 2)$-loop contribution. In the case of the renormalization group improved action 
[12, 13], for instance, the relative weight is as large as 0.1815.

Acknowledgements:
The work has been supported by the DFG under grant Pe 340/3-3. Numerical 
calculations have been performed on the QUADRICS parallel computers funded by 
the DFG under contract no. Pe 340/6-1.

References

[1] K. Symanzik, Nucl. Phys. B226 (1983) 187 and Nucl. Phys. B226 (1983) 205.
[2] B. Beinlich, F. Karsch and E. Laermann, Nucl. Phys. B462 (1996) 415.
[3] B. Beinlich, F. Karsch, A. Peikert and E. Laermann, String Tension and Ther-
mosdynamics with a Tree Level and Tadpole Improved Symanzik Action, Bielefeld 
preprint, in preparation.
[4] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lütgemeier and 
B. Petersson, Phys. Rev. Lett. 75 (1995) 4169 and Nucl. Phys. B469 (1996) 419.
[5] T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer, Nucl. Phys. 
B454 (1995) 615; 
A. Papa, SU(3) Thermodynamics on Small Lattices, BUTP-96/13 ([hep-
lat/9605002]), May 1996.
[6] G.P. Lepage and P.B Mackenzie, Phys. Rev. D48 (1993) 2250 
M. Alford, W. Dimm, G.P. Lepage, G. Hockney and P.B. Mackenzie, Nucl. 
Phys. B (Proc. Suppl.) 42 (1995) 787.
[7] D.W. Bliss, K. Hornbostel and G.P. Lepage, The Deconfinement Transition on 
Coarse Lattices, SMUHEP-96-05 ([hep-lat/9605041]), May 1996.
[8] Y. Iwasaki, K. Kanaya, T. Yoshié, T. Hoshino, T. Shirakawa, Y. Oyanagi, S. 
Ichii and T. Kawai, Phys. Rev. D 46 (1992) 4657.
[9] Y. Iwasaki, K. Kanaya, L. Kärkkäinen, K. Rummukainen and T. Yoshié, Phys. 
Rev. D49 (1994) 3540.
[10] G. Cella, G. Curci, R. Tripiccione and A. Viceré, Phys. Rev. D49 (1994) 511; 
G. Cella, G. Curci, A. Viceré and B. Vigna, Phys. Lett. B333 (1994) 457.
[11] M. Fukugita, M. Okawa and A. Ukawa, Nucl. Phys. B337 (1990) 181.
[12] S. Itoh, Y. Iwasaki and T. Yoshié, Phys. Rev. D33 (1986) 1806 and references 
therein.
[13] Y. Iwasaki, K. Kanaya, T. Kaneko and T. Yoshié, Scaling of the critical tem-
perature and the quark potential with a renormalization group improved SU(3) 
gauge action, UTHEP-341 ([hep-lat/9608090]), August 1996.