Estimation of Constrained Mean-Covariance of Normal Distributions

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Abstract: Estimation of the mean vector and covariance matrix is of central importance in the analysis of multivariate data. In the framework of generalized linear models, usually the variances are certain functions of the means with the normal distribution being an exception. We study some implications of functional relationships between covariance and the mean by focusing on the maximum likelihood and Bayesian estimation of the mean-covariance under the joint constraint \( \Sigma \mu = \mu \) for a multivariate normal distribution. A novel structured covariance is proposed through reparameterization of the spectral decomposition of \( \Sigma \) involving its eigenvalues and \( \mu \). This is designed to address the challenging issue of positive-definiteness and to reduce the number of covariance parameters from quadratic to linear function of the dimension. We propose a fast (noniterative) method for approximating the maximum likelihood estimator by maximizing a lower bound for the profile likelihood function, which is concave. We use normal and inverse gamma priors on the mean and eigenvalues, and approximate the maximum a-posteriori estimators by both MH within Gibbs sampling and a faster iterative method. A simulation study shows good performance of our estimators.

Abstract Keywords: Joint Mean-Covariance Estimation, Structured Covariance, Shrinkage Inverse Wishart, Normal-Inverse Gamma, Metropolis-Hastings within Gibbs Sampling.

1 Introduction

Mean and covariance estimation of multivariate normal distribution are essential in almost every area of classical multivariate statistics (Bibby et al., 1979). The range of modern applications includes astrophysics (Hamimeche and Lewis, 2009), economics (Ledoit and Wolf, 2004a), environmental sciences (Eguchi et al., 2010), climatology (Guillot et al., 2015) and genetics (Schäfer and Strimmer, 2005).

Estimation under joint constraints on the mean vector and covariance matrix of data from a \( N_p(\mu, \Sigma) \) distribution is relatively uncommon in multivariate statistics (Bibby et al., 1979). It is well-known that estimation of a covariance matrix alone is a daunting task...

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because of some standard constraints e.g. 1) positive definiteness and 2) the number of unknown covariance parameters growing quadratically with the dimension. Many strategies are developed to bypass the notorious positive definiteness e.g. spectral, Cholesky decomposition and factor models see Chiu et al. (1996); Pourahmadi (1999); Fan et al. (2008). Here due to the nature of the constraint on the mean and covariance, we rely on the spectral decomposition and introduce a new class of structured covariance matrices with several desirable properties. To the best of our knowledge this class of covariance matrices seems to be new and has not been studied before.

We consider the following joint constraint on the mean vector and covariance matrix of a multivariate normal distribution:

\[ \Sigma \mu = \mu, \quad |\Sigma| = 1, \]  

(1.1)

which appeared first in Paine et al. (2018) in the context of spherical data. Interestingly, the first constraint forces the mean vector to be an eigenvector of the covariance matrix corresponding to the eigenvalue one and is more consequential. The second which constraints the product of the remaining eigenvalues is less stringent and can be realized by rescaling, so that without loss of generality it is ignored from here on.

Nevertheless, the joint constraint on the mean and covariance matrix will definitely impact their estimators and the shape of the contours of the multivariate normal density function as gleaned from the spectral decomposition of the covariance matrix:

\[ \Sigma = PDP^\top = \sum_{i=1}^{p} \lambda_i P_i P_i^\top = \sum_{i=1}^{p-1} \lambda_i P_i P_i^\top + \mu \mu^\top \]  

(1.2)

where \( D = \text{diag}(1, \lambda_1, \lambda_2, \ldots, \lambda_{p-1}) \) is the diagonal matrix of ordered eigenvalues other than 1 and \( P = [\mu, P_1, P_2, \ldots, P_{p-1}] \) is an orthogonal matrix of eigenvectors.

Though these constraints arise in the context of directional data analysis Paine et al. (2018), they seem to resonate with some deep classical issues in statistical estimation theory. Note that the presence of the quadratic term \( \mu \mu^\top \) in (1.2) has resemblance with classical estimation in the \( N(\theta, \theta^2) \) distribution. It also encourages modeling the covariance matrix as a parsimonious quadratic function of the mean vector similar to Hoff and Niu (2012).

We propose a simple and novel structured covariance model based on the spectral decomposition (1.3) which overcomes several challenges in covariance estimation. Let \( u = \frac{\mu}{|\mu|} \) be the direction of the mean vector so that \( \mu = c_0 u \) where \( c_0 \in \mathbb{R}^+ \) and \( u \) lies on a unit sphere; \( c_0 \) can be interpreted as the radius of the sphere on which the mean vector lie. Our structured covariance model is

\[ \Sigma = \Sigma(u, \lambda_1, \ldots, \lambda_{p-1}) = P(u)DP^\top(u) \]  

(1.3)

where \( D = \text{diag}(1, \lambda_1, \ldots, \lambda_{p-1}) = \text{diag}(1, \lambda) \) is the matrix of eigenvalues, and for a given value of the mean direction \( u \), the orthogonal matrix of eigenvectors is \( P(u) = [u, V(u)] \) where \( V = V(u) \in \mathbb{R}^{p \times (p-1)} \) is a known matrix function so that orthogonality of \( P \) is ensured. A simple and prominent example of such a \( V \) is obtained by an application of the Gram-Schmidt procedure (Trefethen and Bau III, 1997) to the set \( \{u, e_1, e_2, \ldots, e_{p-1}\} \) where \( e_i, i = 1, 2, \ldots, p-1 \) denote the canonical basis of \( \mathbb{R}^p \) and \( u_0 \neq 0 \). With \( V \) assumed known in (1.3), the unknown parameters then are \( (\mu, \lambda) \), the vectors of mean and the \( (p-1) \) eigenvalues, so that the number of parameters drastically reduces from quadratic to linear i.e. \((2p-1)\) in the dimension. Moreover, the first constraint is automatically satisfied, since using \( \mu \perp V_i(u) = V_i \) (columns of the \( V \) matrix) for \( i = 1, 2, \ldots, (p-1) \), it follows that
\[
\Sigma \mu = P(u) \Lambda P^\top(u) \mu = \sum_{i=1}^{p-1} \lambda_i V_i V_i^\top \mu + uu^\top \mu = c_0 uu^\top u = \mu.
\] (1.4)

Our approximate MLE of the parameters in Section 2 provides positive estimates for the eigenvalues and hence guarantees the positive-definiteness of the estimated structured covariance matrix.

Next, we will provide two concrete examples to elucidate the idea behind the model and its components, particularly the eigenvectors as functions of the mean vector.

**Example 1:** Consider the case of equal means \( \mu = c1 \), then using Gram-Schmidt procedure (Trefethen and Bau III, 1997), it follows that \( P(u) \) is of the form:

\[
P(u) = \begin{bmatrix} 1_p & z^{(p)}_2 & z^{(p)}_{p-2} & \ldots & z^{(p)}_2 \\ u w_1 & w_2 & z^{(p)}_{p-2} & \ldots & z^{(p)}_2 \end{bmatrix}
\] (1.5)

where

\[
z^{(p)}_s = \begin{pmatrix} 0, \ldots, 0, z^{(s)} \end{pmatrix}^\top \quad \forall s \in \{2, 3, \ldots, p\}
\]

and \( 1_p = \begin{pmatrix} \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}}, \ldots, \frac{1}{\sqrt{p}} \end{pmatrix} \). The \( c \) drops due to normalization.

**Example 2:** As a generalization of Example 1, take

\[
\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_3)^\top
\]

where the first three entries of the mean vector are different and rest of them are the same as the third entry of the vector. Then, using the construction method described in the paragraph following (1.3), the \( P \) matrix takes the form:

\[
P(u) = \begin{bmatrix} u & w_1 & w_2 & z^{(p)}_{p-2} & \ldots & z^{(p)}_2 \end{bmatrix}
\] (1.7)

where \( z^{(p)}_s \)'s are same as in Example 1, and

\[
w_1 = (\mu_0^2, -\mu_1, \mu_2, -\mu_1, \mu_3, \ldots, -\mu_1, \mu_3)^\top / (\mu_0 ||u||) \\
w_2 = (0, (p-2) \mu_3, -\mu_2, \ldots, -\mu_2)^\top / ((p-2) \mu_0),
\] (1.8)

where \( \mu_0^2 = \mu_2^2 + (p-2) \mu_3^2 \). When \( p = 3 \), \( w_1 \) and \( w_2 \) reduce to the \( \xi_1 \) and \( \xi_2 \) in Paine et al. (2018, §2.3), so that this example is a generalization of their construction to the case of \( p > 3 \) or more general spherical data.

Spherical data arise in many scientific disciplines like shape analysis, geology, meteorology (e.g. Mardia and Jupp (2000)), text analysis (e.g. Hamsici and Martinez (2007)), etc. Paine et al. (2018) study spherical regression using their elliptically symmetric angular Gaussian distribution. Since our constraint is similar without the spherical nature of the distribution, we can always project them in the corresponding \( \mathbb{R}^p \) (Mardia and Jupp, 2000, §3.5.6) and fit a normal model with the constraint (1.1). This is helpful in developing multivariate techniques like discriminant analysis, clustering, etc. for spherical data. We have applied it on earth’s historic magnetic pole data by Schmidt (1976).
Our goal is to compute the maximum likelihood estimator and provide a Bayesian framework for estimation of the parameters in (1.3). It turns out that computing the maximum likelihood estimator is challenging due to the nonlinearity and intractability of $P(u)$. However, the MLE of the eigenvalues and $c_0$ have closed forms as a function of the mean direction, thus making it possible to compute its profile likelihood. At this stage, we approximate the MLE of the mean direction by first finding a lower bound for the concave profile log-likelihood function of the mean direction (for given $\lambda$) and then maximizing it.

In the Bayesian context, the maximum a posteriori (MAP) does not have a closed form, even though the posterior distribution obtained by using a Gaussian prior for the mean vector and inverse gamma on the eigenvalues does. These priors are suggested by extending the shrinkage inverse Wishart prior of Berger et al. (2020) to the case of nonzero means. We show in Section 3.4 that one can generate from the posterior distribution quite easily using Metropolis-Hastings within Gibbs sampling. But if we are only interested in point estimation (e.g. MAP), then we propose a simpler way of computing it. The idea is to follow the calculation of the approximate MLE by providing a lower bound for the posterior and maximize it using a modified version of Newton’s method.

The paper is organized as follows: Section 2 delves into the detailed computation of an approximate maximum likelihood estimator. Section 3 discusses the standard priors such as normal-inverse Wishart and normal-shrinkage inverse Wishart, motivates the normal-inverse gamma prior selection, provide details of the MH within Gibbs sampling process for generation from the posterior and imitates the approximate MLE calculation to propose a fast method to compute the Bayesian point estimate. Section 4 computes the risks of these methods of approximation through simulation and exhibits the performance of our method. Finally in section 5, we discuss the key take away for this paper.

## 2 MLE of $\mu$ and $\lambda$ in Model (1.3)

Let $x_1, x_2, \ldots, x_n$ be a sample of size $n$ from $N_p(\mu, \Sigma)$, where $\Sigma$ is parameterized as in (1.3) with $\mu = c_0u$ and $X$ is the $n \times p$ data matrix. Then, the log-likelihood can be written as

$$l(u, c_0, \lambda | X) \propto -\frac{n}{2} \sum_{i=1}^{p-1} \log(\lambda_i) - \frac{1}{2} \left[ \sum_{i=1}^{n} (c_0u - x_i)^\top \Sigma^{-1} (c_0u - x_i) \right] \quad (2.1)$$

$$\propto -\frac{n}{2} \sum_{i=1}^{p-1} \log(\lambda_i) - \frac{1}{2} \text{Tr} \left[ D^{-1} B(X, c_0, u) \right]$$

$$\propto -\frac{n}{2} \sum_{i=1}^{p-1} \log(\lambda_i) - \frac{1}{2} \left[ B(X, c_0, u)_{11} + \sum_{i=1}^{p-1} \frac{B(X, c_0, u)_{(i+1)(i+1)}}{\lambda_i} \right], \quad (2.2)$$

where $B(X, c_0, u) = P^\top(u)A(c_0u)P(u)$ and $A(\mu) = \sum_{i=1}^{n} (x_i - \mu)(x_i - \mu)^\top$.

Evidently, maximization of the log-likelihood function with respect to the mean direction is more challenging than those with respect to $c_0$ and $\lambda$. This suggests and we propose approximating the MLE of the parameters using the following three (non-iterative) steps:

1. Differentiate the log-likelihood function to obtain the MLE of $c_0$ and $\lambda_i$'s as a function of the mean direction vector $u$.

2. Compute the profile log-likelihood.
3. Find a lower bound for the profile log-likelihood and maximize it to obtain the approximate MLE of the mean direction \( \mathbf{u} \). Use the latter to compute the approximate MLE of \( c_0 \) and the eigenvalues.

Next, we provide further details about implementing these three steps.

**Step 1:** For a given \( \mathbf{u} \), differentiating the log-likelihood in (2.1) with respect to \( c_0 \) (Bibby et al., 1979, §4.2.9) and in (2.2) with respect to \( \lambda_i \), we obtain

\[
\hat{c}_0 = \frac{\mathbf{u}^\top P(\mathbf{u})D^{-1}P^\top(\mathbf{u})\bar{x}}{\mathbf{u}^\top P(\mathbf{u})D^{-1}P^\top(\mathbf{u})\mathbf{u}}, \quad \text{and} \quad \hat{\lambda}_i = \frac{B(X, c_0, \mathbf{u})_{(i+1)(i+1)}}{n}. \tag{2.3}
\]

The expression for the \( \hat{c}_0 \) in (2.3) reduces to

\[
\hat{c}_0 = \frac{\mathbf{e}_1^\top D^{-1}P^\top(\mathbf{u})\bar{x}}{\mathbf{e}_1^\top D^{-1}\mathbf{e}_1} = \frac{1}{\hat{\lambda}_1} \frac{\mathbf{e}_1^\top P^\top(\mathbf{u})\bar{x}}{\mathbf{e}_1^\top P(\mathbf{u}) \mathbf{e}_1} = \mathbf{u}^\top \bar{x}, \tag{2.4}
\]

and note that the estimates of the eigenvalues are always positive.

**Step 2:** Substituting for \( \hat{c}_0 \) and \( \hat{\lambda}_i \) in the log-likelihood (2.2), the profile likelihood of the mean direction \( \mathbf{u} \) turns out to be

\[
l(\mathbf{u}, \hat{c}_0, \hat{\lambda} \mid \mathbf{X}) \propto -\frac{n}{2} \sum_{i=1}^{p-1} \log \left( \frac{B(X, \hat{c}_0, \mathbf{u})_{(i+1)(i+1)}}{n} \right) - \frac{1}{2} \left[ B(X, \hat{c}_0, \mathbf{u})_{11} + n(p - 1) \right]. \tag{2.5}
\]

Denoting the columns of \( \mathbf{V} \) by \( \mathbf{V}_i \) for \( i = 1, 2, \ldots, (p - 1) \), then the diagonal entries of the \( B(X, \hat{c}_0, \mathbf{u}) \) are

\[
\left\{ B(X, \hat{c}_0, \mathbf{u})_{11}, B(X, \hat{c}_0, \mathbf{u})_{22}, \ldots, B(X, \hat{c}_0, \mathbf{u})_{pp} \right\} = \left\{ \mathbf{u}^\top A(\hat{c}_0 \mathbf{u}) \mathbf{V}_1, \mathbf{V}_1^\top A(\hat{c}_0 \mathbf{u}) \mathbf{V}_1, \ldots, \mathbf{V}_{p-1}^\top A(\hat{c}_0 \mathbf{u}) \mathbf{V}_{p-1} \right\}.
\]

Since \( \mathbf{u} \perp \mathbf{V}_i \) for \( i = 1, 2, \ldots, (p - 1) \), one may further simplify the diagonal entries of \( B(X, \hat{c}_0, \mathbf{u}) \) to the following:

\[
\mathbf{V}_i^\top A(\hat{c}_0 \mathbf{u}) \mathbf{V}_i = \mathbf{V}_i^\top \left[ A(0) - n\hat{c}_0 \bar{x} u^\top - n\hat{c}_0 u \bar{x}^\top + n\hat{c}_0^2 u u^\top \right] \mathbf{V}_i = \mathbf{V}_i^\top A(0) \mathbf{V}_i
\]

\[
\mathbf{u}^\top A(\hat{c}_0 \mathbf{u}) \mathbf{u} = \mathbf{u}^\top A(\bar{x}) \mathbf{u}.
\tag{2.6}
\]

In spite of this simplification, the profile log-likelihood is hard to differentiate as a function of \( \mathbf{u} \), in general, when we are not assuming any specific form for the matrix \( \mathbf{V} \).

**Step 3:** We find a workable lower bound for (2.5) and maximize it with respect to the mean direction to obtain an approximate MLE for \( \mathbf{u} \). Alternatively, this amounts to assuming that the first sum in the profile likelihood (2.7) is constant. The expression in equation (2.6) is a quadratic form of the matrix \( A(0) = \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^\top \) and can be bounded by its largest eigenvalue (Rao, 1973, §1f.2.1). Consequently, using (2.6) and that columns of \( \mathbf{V} \) are orthonormal simplify the profile log-likelihood (2.5) and we arrive at the following lower bound:

\[
l(\mathbf{u}, \hat{c}_0, \hat{\lambda} \mid \mathbf{X}) \propto -\frac{n}{2} \sum_{i=1}^{p-1} \log \left( \frac{\mathbf{V}_i^\top A(0) \mathbf{V}_i}{n} \right) - \frac{1}{2} \left[ \mathbf{u}^\top A(\bar{x}) \mathbf{u} + n(p - 1) \right] \]
\[
\geq -\frac{n(p - 1)}{2} \log \left( \frac{\lambda_1 \{A(0)\}}{n} \right) - \frac{1}{2} \left[ \mathbf{u}^\top A(\bar{x}) \mathbf{u} + n(p - 1) \right] = h(\mathbf{u}) \tag{2.7}
\]
Evidently, the lower bound denoted by \( h(u) \) is maximized by minimizing the quadratic expression inside the second bracket. Since \( u \neq 0 \) and \( h \) is a quadratic function of \( u \), the maximum occurs when \( u \) is the eigenvector corresponding to the smallest eigenvalue of the matrix \( A(\bar{x}) \). As such the approximate MLE of \( u \) is unique only up to a sign.

Even though the computed mean direction \( \hat{\mu} \) is not the exact MLE, it is still a good approximation as confirmed by the simulation results in Table 2 of Section 4. The approximate maximum likelihood estimate of the eigenvalues and the constant \( c_0 \) are obtained by plugging in the approximate MLE of \( u \) in (2.3) and (2.4). Fortunately, this idea of approximating MLE can also be replicated in our posterior MAP estimator approximation developed in the next section, but the maximization of the lower bound is not as straightforward and requires employing a version of Newton’s iterative method.

3 Bayesian Estimation of \((\mu, \lambda)\) in (1.3)

In this section we develop a Bayesian methodology for estimating the parameters using two sets of priors reviewed in the next two subsections.

3.1 Normal-Inverse Wishart Priors:

For the parameters of a multivariate normal distribution the most popular prior is normal-inverse Wishart described in Gelman et al. (2013, Section 3.6) and Barnard et al. (2000) with the hyperparameters \((\mu_0, \kappa_0; \nu_0, \Lambda_0)\):

\[
\mu | \Sigma \sim N_p(\mu_0, \Sigma/\kappa_0), \quad \Sigma \sim \pi_{I\!W}(\nu_0, \Lambda_0^{-1}),
\]

where \( \nu_0 \) and \( \Lambda_0 \) are the degrees of freedom and the scale matrix. The suggested values for the hyperparameters in Gelman and Hill (2006) are \( \Lambda_0 = I \) and \( \nu_0 = p + 1 \). The remaining hyperparameters are the prior mean, \( \mu_0 \) and \( \kappa_0 \) on the \( \Sigma \) scale. Due to conjugacy, the posterior density has the following parameters:

\[
\mu_n = \frac{\kappa_0}{\kappa_0 + n} \mu_0 + \frac{n}{\kappa_0 + n} \bar{x},
\]

\[
\Lambda_n = \Lambda_0 + A + \frac{n\kappa_0}{\kappa_0 + n} (\bar{x} - \mu_0)(\bar{x} - \mu_0)^\top.
\]

As a point estimator posterior mode or the maximum a posteriori probability (MAP) estimator for \( \Sigma \) is generally the most popular choice (Murphy, 2012, §5.2.1). The posterior marginal distribution of the mean parameter \( \mu \) is multivariate t \( \nu_n-p+1 \left( \mu_n, \frac{\Lambda_n}{\nu_n-p+1} \right) \) with the MAP estimator \( \mu_n \), and the MAP estimator for \( \Sigma \) is given by

\[
\widehat{\Sigma}_{map} = \frac{\Lambda_n}{\nu_n + p + 2},
\]

where \( \nu_n = \nu_0 + n \) (proof is shown in appendix 1).
3.2 Shrinkage Inverse Wishart Priors

The most natural prior for mean and covariance i.e. normal-inverse Wishart reviewed above is known to overdisperse the eigenvalues of the posterior covariance estimator. In fact, Yang and Berger (1994) describes that the normal-inverse Wishart prior has the term \[ \prod_{i<j} (\lambda_i - \lambda_j) \] in the density where \( \lambda_i \)'s are the ordered eigenvalues of \( \Sigma \), thus forcing the eigenvalues apart and increasing the variability (Berger et al., 2020). This is one major motivation for using the normal-shrinkage inverse Wishart prior.

The shrinkage-inverse Wishart (SIW) prior has the same density as inverse Wishart prior save the extra term \( \prod_{i<j} (\lambda_i - \lambda_j) \) in the denominator:

\[
\pi_{SIW}(\Sigma \mid \nu_0, b, \Lambda_0^{-1}) \propto |\Sigma|^{-\nu_0+p+1/2} \exp\left[-\frac{1}{2} \text{Tr}(\Lambda_0 \Sigma^{-1})\right] \prod_{i<j} (\lambda_i - \lambda_j)^b, \tag{3.2}
\]

where \( \nu_0 \) is a real constant, \( b \in [0, 1] \) and \( \Lambda_0 \) is a positive semi-definite matrix. It is interesting to notice that \( b = 0 \) corresponds to some common priors like inverse Wishart, reference, Jefferey’s etc. which also contain the term \( \prod_{i<j} (\lambda_i - \lambda_j) \), see Berger et al. (2020). The posterior density for \((D, P)\),

using the one-to-one transformation from \( \Sigma \) to \( D = \text{diag}(\lambda_1, \ldots, \lambda_p) \) and the orthogonal eigenvector matrix \( P \), turns out to be

\[
\pi_{SIW}(D, P \mid \nu_0, b, \Lambda_0^{-1}) \propto |\Sigma|^{-\nu_0+p+1/2} \exp\left[-\frac{1}{2} \text{Tr}(\Lambda_0 PD^{-1}P^\top)\right] \prod_{i<j} (\lambda_i - \lambda_j)^{(b-1)} \lambda_1 \geq \lambda_2 \geq \ldots \lambda_p. \tag{3.3}
\]

Note that the posterior is zero whenever the eigenvalues are close together so that effectively it forces the eigenvalues apart. Moreover, when \( b = 1 \), the term in question goes away retaining the conjugacy property, even with the additional normal prior on the nonzero mean (Proof is shown in the Appendix 2) and the same posterior parameters as before. The next important consequence of choosing \( b = 1 \), from (3.3), is that the conditional distribution of the eigenvalues given the orthogonal matrix \( P \) is an ordered inverse gamma distribution (Berger et al., 2020, §3.2). This plays a central role in developing a computational scheme for computing the MAP.

Recall that the structured covariance model in (1.3) has its orthogonal matrix \( P \) determined by a non-zero mean vector. This requires working with nonzero mean vectors in contrast to the mean zero framework in Berger et al. (2020), while maintaining the conditional distribution of the eigenvalues given the mean vector as an inverse gamma distribution, see also Berger et al. (2020). This motivates us to select an inverse gamma prior for the eigenvalues as in (3.4) which, fortunately, in turn implies that the conditional distribution of eigenvalues is the inverse gamma (see section 3.4).

3.3 The Mean-Eigenvalue Priors

A convenient prior distribution on the mean vector is the multivariate normal and the inverse gamma on the ordered eigenvalues. The latter prior comes naturally from the shrinkage inverse Wishart prior of (Berger et al., 2020, §3.2) as the conditional distribution of eigenvalues given the matrix of eigenvectors (see section 3.2 for detailed discussion), see also Hoff (2009) (section 3.3). More specifically, our proposed prior is:
\( \mu | D \sim N_p(\mu_0, D/\kappa_0) \) and \( \lambda_i \sim \text{Inverse-gamma}(a-1, c_i/2) \), \( \text{(3.4)} \)

where \( a, c_i \)'s are the hyperparameters. Setting \( H_0 = \text{diag}(1, c_1, c_2, \ldots, c_{p-1}) \), then the posterior distribution has the form

\[
p(\mu, \lambda | X) \propto \left( \prod_{i=1}^{p-1} \lambda_i \right)^{-\frac{a+1+p}{2}} \exp\left[ -\frac{1}{2} Tr \{ D^{-1} H_N \} \right]
\]

(3.5)

with \( H_N = B(X, \mu) + \kappa_0(\mu - \mu_0)(\mu - \mu_0)^\top + H_0 \). Here \( B(X, \mu) \) has the same expression as \( B(X, \mu - c_0, u) \) which appeared in Section 2.

One needs to generate from this posterior distribution when computing the Bayesian point estimates, credible intervals and various other quantities. We show that Metropolis-Hastings within Gibbs sampling is possible here and suitable for our goals (Brooks et al., 2011, §1.12.10).

### 3.4 Gibbs Sampling

The Gibbs sampler (Gelfand and Smith, 1990) method is a numerical technique for sampling from the joint posterior distribution. Given an initial vector, the Gibbs sampling proceeds by sampling from the conditional posterior distribution. More generally, if the full conditional posterior distribution in any Gibbs step is of a non-standard form, using MH (Metropolis-Hastings) step is convenient (Brooks et al., 2011). The technique is known as Metropolis-Hastings within Gibbs Sampling or alternatively as single-component Metropolis-Hastings sampling as follows:

1. Since \( p(D | \mu, X) \propto p(\mu, \lambda | X) \) has an Inverse-Gamma distribution, the ordered eigenvalues are generated from independent Inverse Gamma \( \left( \frac{a+2-a}{2}, c_i^* / 2 \right) \) where \( c_i^* \) is the i-th diagonal entry of \( H_N \) matrix.

2. Generate from \( p(\mu | D, X) \) using the MH algorithm with a proposal distribution \( q \) selected as

\[
q(. | \mu^{(i-1)}) = N_p\left( \mu^{(i-1)}, \frac{1}{n} P \left( \mu^{(i-1)} \right) D P^\top \left( \mu^{(i-1)} \right) \right).
\]

More concretely, the steps of MH within Gibbs algorithm in our context are as follows:

**Algorithm 1** MH within Gibbs for generating samples from the posterior distribution

1. Start with \( \mu^{(0)} = \bar{x} \).
2. **Repeat** \( s \) times: \( j - \text{th} \) step
   3. Generate \( \lambda^{(j)}_i \sim IG \left( \frac{a+2-a}{2}, c_i^*/2 \right) \) to form \( D^{(j)} \).
4. **Start with** \( \mu^{(j-1)} \), **Repeat MH Step** \( l \) times: \( k - \text{th} \) step
   5. Generate \( \mu^* \sim q( . | \mu^{(k-1)} ) \)
   6. Calculate

\[
r(\mu^*, \mu^{(k-1)}) = \min\left\{ 1, \frac{p(\mu^*, \lambda^{(j)} | X) q(\mu^{(k-1)} | \mu^*)}{q(\mu^* | \mu^{(k-1)}) p(\mu^{(k-1)} | \lambda^{(j)} | X)} \right\}
\]
An interesting observation is that the initial value of $\lambda$ is irrelevant. In the MCMC, the chain for the eigenvalues depend on the previous iteration only through the generation of mean vector. This is happening because the parameters for the conditional distribution of the eigenvalues are $(n + 2a - 1)/2$ and $c_i^*/2$ where $c_i^*$'s are the diagonal entries of the $H_N$ matrix and hence a function of the mean vector.

To illustrate the algorithm, we have generated 100 pairs of $(\mu, \lambda)$ from the posterior distribution by MH within Gibbs sampling. We find out the posterior maximizing $(\mu, \lambda)$ pair by exhaustive computation of posterior density at each pair. This is possible because the posterior density has a closed form barring the normalizing constants. We take the value of the mean vector from the posterior maximizing pair, set it as our MAP estimate for the mean vector discarding the corresponding $\lambda$ and make a final update on $\lambda$. Given the estimate of the mean vector, the posterior conditional distribution of $\lambda$ is inverse gamma and has a known mode. Using this, we can calculate the best eigenvalue vector $\lambda$ to be $\lambda_i^* = \frac{c_i^*}{n + 1 + 2a}$ for $i = 1, 2, \ldots, (p - 1)$. The newly calculated value $\lambda^*$, accompanied by the MAP estimate of $\mu$ constitute our MAP estimator $(\hat{\mu}, \hat{\lambda})$. The performance of such an estimator is assessed through a simulation study (see Section 4.2.1) and Table I of the simulation Section 4.

### 3.5 Approximation of MAP through a Lower Bound

The Gibbs sampling is computationally challenging and the time complexity increases exponentially with the dimensions. We follow the steps Section 2 to approximate the MAP estimator using a lower bound for the log posterior density. The main difference is in the maximization step of the lower bound as a function of $u$ where it is not as straightforward as the MLE case and does not have a closed-form. So, we resort to an iterative modified Newton-Raphson algorithm. Using $\mu = c_0 u$ where $c_0 \in \mathbb{R}$ as before we arrive at

\[
\log p(u, c_0, \lambda | X) \propto -t \sum_{i=1}^{p-1} \log \lambda_i - \frac{1}{2} \left[ f(c_0) + \text{Tr}(D^{-1}H_0) \right]
\]

where $t = \frac{n+1+2a}{2}$ and $f(c_0) = \sum_{i=1}^{n} (x_i - c_0 u)^\top \Sigma^{-1}(x_i - c_0 u) + \kappa_0(c_0 u - \mu_0)^\top D^{-1}(c_0 u - \mu_0)$. Following similar calculations as in (27.3), the estimates of $c_0$ and the eigenvalues in terms of the mean direction are as follows:

\[
\hat{c}_0 = \frac{nu^\top \bar{x} + \kappa_0 u^\top D^{-1} \mu_0}{n + \kappa_0 u^\top D^{-1} u}, \quad \hat{\lambda}_i = \frac{(H_N)_{i+1,i+1}}{2t}
\]

where $H_N = B(X, c_0, u) + \kappa_0(c_0 u - \mu_0)^\top D^{-1}(c_0 u - \mu_0) + H_0$. Here the expression of $\hat{c}_0$ and $\hat{\lambda}_i$ are intertwined. So we are doing the following calculation with the aim to propose an iterative algorithm for the MAP estimators. We will use equation (3.8) and (3.15) as our updating equation.
Using the expression of \( \hat{\lambda} \) from equation (3.8) in the log posterior likelihood function we obtain the following analogue of the profile likelihood in (2.5):

\[
\log p(u, c_0, \hat{\lambda} | X) \propto -t \sum_{i=1}^{p-1} \log \left( \frac{(H_N)_{i+1,i+1}}{2t} \right) - \frac{1}{2} \left[ (H_N)_{11} + (p-1)t \right]
\]  

(3.9)

where the diagonal entries of \( H_N \) are

\[
(H_N)_{i+1,i+1} = \begin{cases} 
    u^\top A(c_0 u) + \kappa_0 (c_0 u - \mu_0)^2 + (H_0)_{1,1} & \text{for } i = 0 \\
    V_i^\top A(0) V_i + \kappa_0 (c_0 u - \mu_0)^2 + (H_0)_{i+1,i+1} & \text{for } i \neq 0 \text{ and } \forall c_0 \end{cases}
\]

(3.10)

This follows from the expression of the matrix \( B(X, c_0, u) \) in equation (2.6) and the definition of \( H_N \). We observe the following

\[
(H_N)_{i+1,i+1} \leq \lambda_1 \{ A(0) \} + \kappa_0 \| c_0 u - \mu_0 \|^2 + (H_0)_{i+1,i+1}
\]

\[
= m_i + \kappa_0 \| c_0 u - \mu_0 \|^2 \quad \text{for } i \neq 0 \text{ and } \forall c_0 \in \mathbb{R}
\]

(3.11)

For a given value of \( c_0 \), equation (3.9), provides a lower bound \( h(u) \), which we will maximize to approximate \( u \). This is equivalent to minimizing \( -h(u) \). We apply Newton-Raphson algorithm (Lange, 2013) to obtain an update for \( u \). The calculation of derivatives are shown below.

\[
\log p(u, c_0, \hat{\lambda} | X) \geq h(u),
\]

\[
-h(u) = t \sum_{i=1}^{p-1} \log \frac{m_i + \kappa_0 \| c_0 u - \mu_0 \|^2}{2t} + \frac{1}{2} \left[ u^\top A(c_0 u) u + \kappa_0 \| c_0 u - \mu_0 \|^2 + m_1 \right],
\]

(3.12)

\[
-\nabla h(u) = t \sum_{i=1}^{p-1} 4t \kappa_0 c_0 (c_0 u - \mu_0) + [A(0) u - c_0 n \hat{x} + \kappa_0 c_0 (c_0 u - \mu_0)],
\]

(3.13)

\[
-\nabla^2 h(u) = t \sum_{i=1}^{p-1} 4t \kappa_0 c_0 \left[ (m_i + \kappa_0 \| c_0 u - \mu_0 \|^2) I - 2(c_0 u - \mu_0) (c_0 u - \mu_0)^\top \right] + [A(0) + \kappa_0 c_0^2 (c_0 u - \mu_0) (c_0 u - \mu_0)^\top],
\]

(3.14)

Thus, the updating equation for the mean direction is

\[
u^{(k+1)} = \nu^{(k)} - [\nabla^2 h(u)]^{-1} \nabla h(u)
\]

(3.15)

Due to the concavity of the lower bound, Newton’s method is an appealing choice in our case. However there are two potential problems with Newton’s method (Lange, 2013, §10.3). First, it may be computationally expensive to invert the second derivative matrix in each step. Second, the Newton’s method is not really a ascent algorithm in the sense that \( h(u^{(k+1)}) \geq h(u^{(k)}) \) for a concave function. The second problem can be remedied by modifying the increment such that it is a partial step in the ascent direction. Let us denote:

\[
\nu = [\nabla^2 h(u)]^{-1} \nabla h(u)
\]

(3.16)

The idea is to take a sufficiently short increment in the direction of \( \nu \). If \( [u^{(k)} - \alpha \nu] \) shows increment in \( h(u) \) value, we update our mean vector in the iteration, otherwise we look at \( [u^{(k)} - \alpha^2 \nu] \) for \( j = 1, 2, \ldots \) until we observe an increment. Due to the good performance of the MLE, we use the MLE of \( c_0 \) and \( u \) as our initial values which make the convergence fast. Based on (3.8) and (3.15), an iterative algorithm for computing the approximate MAP is summarized in the following algorithm:
Algorithm 2 MAP Approximation

1: **Initialize**: Start with \( c_0^{(0)} = \text{MLE of } c_0, \ u^{(0)} = \text{MLE of } u \) and \( \lambda_i^{(0)} = \frac{H_N(d^{(0)}, u^{(0)})_{i+1,i+1}}{2t} \)

2: **For** \( k \to (k + 1) \)

3: Update \( c^{(k+1)} \) and \( \lambda_i^{(k+1)} \) from equation (3.8)

4: **For** \( j \to (j + 1) \)

5: \( u^{(j)} = u^{(k)} \) and \( v^{(j)} = \left[ \nabla^2 h(u^{(j)}) \right]^{-1} \nabla h(u^{(j)}) \)

6: **For** \( l \to (l + 1) \)

7: \( u^{(j+1)} = u^{(j)} - \alpha^l v^{(j)} \)

8: **If** \( h(u^{(j+1)}) > h(u^{(j)}) \)

9: Accept the update of \( u \)

10: **Break**

11: **Else** Change \( \alpha^l \to \alpha^{l+1} \)

12: **If** \( \|u^{(j+1)} - u^{(j)}\| < \epsilon \)

13: \( u^{(k+1)} = u^{(j+1)} \)

14: **Break**

15: **If** \( h(u^{(k+1)}) < h(u^{(k)}) \)

16: **Break**

17: Get the final value of \( c_0, \lambda \) and \( u \).

18: **End**

This algorithm produces the MAP approximation of the eigenvalues and the mean vector.

The continuity of the estimate of the covariance matrix, shown in Lemma 1, ensures that the covariance estimate will converge with the convergence of the mean direction vector \( u \).

**Lemma 1** If \( V(u) \), described after (1.3) is continuous, then the approximate MAP of the covariance matrix \( \Sigma \) is also a continuous function of \( u \).

**Proof** If \( V(u) \) is a continuous function of \( u \), then each column of \( V, \ V_j \) is also continuous. This implies diagonal entries of \( H_N \) and in turn the MAP estimate of the eigenvalues are also a continuous function of \( u \). The claim follows since the MAP of \( \Sigma \) given by

\[
\hat{\Sigma} = uu^T + \sum_{i=1}^{p-1} \hat{\lambda}_i V_i V_i^T = uu^T + \sum_{i=1}^{p-1} \frac{(H_N)_{i+1,i+1}}{n} V_i V_i^T
\]

(3.17)

is sum of continuous functions of \( u \).

4 Simulations

In this section, we perform several simulations to assess the performance of:

1. MLE approximation through lower bound maximization (see section 2) of the profile likelihood function,

2. MAP estimator with normal-inverse gamma prior (see section 3.3) approximated through Gibbs sampling in section 3.4, and

3. MAP approximation (see section 3.5) through posterior lower bound by computing their risks. Then the three estimators are compared with the MAP estimator with normal-inverse Wishart prior (see section 3.1). We have used RStudio 1.3.1093 and R 4.0.3 on a 64 bit 4 Core Windows 10 laptop for all our simulations.
4.1 The Simulation Set up:

We have taken the sample sizes and dimensions to be \( n = 50, 100, 300 \) and \( p = 3, 5, 10 \), respectively. In all cases the data generation mechanism and the risk function are kept the same. We have used Frobenius loss as our default loss function and risks are approximated by averaging the losses for 100 independent replications in each of the nine combinations of \((n,p)\).

The parameters of the Gaussian distributions used for data generation are selected in the following way:

- the entries of the mean vector \( \mu \) are taken to be independent standard Gaussian variables,
- the covariance matrix is generated from \( \Psi = LL^\top \) where
  \[
  L_{ij} \sim N(0, 1) \quad \text{for } i \neq j, \\
  L_{ij} \sim N(5, 1) \quad \text{for } i = j.
  \]

The larger diagonal entries of \( L \) ensure positive-definiteness of the modified covariance matrix \( \Sigma \).

- Since such \((\mu, \Psi)\) does not necessarily satisfy conditions (1.1), the covariance matrix \( \Sigma \) is constructed from applying Lemma 4 of Kundu and Pourahmadi (2020) on \((\mu, \Psi)\).

The performance of the estimators is assessed using the Frobenius (scaled \( L_2 \)) risk as in Ledoit and Wolf (2004b, §3.1):

\[
R(\mu, \hat{\mu}^\ast) = \mathbb{E} \left[ \frac{1}{p} \| \hat{\mu}^\ast - \mu \|_2^2 \right] , \quad R(\Sigma, \hat{\Sigma}^\ast) = \mathbb{E} \left[ \frac{1}{p} \| \hat{\Sigma}^\ast - \Sigma \|_2^2 \right],
\]

where \( \hat{\mu}^\ast \) and \( \hat{\Sigma}^\ast \) are the final estimators in each of the cases.

4.2 Simulation Results:

Here we provide the details of the simulation studies for the MLE approximation, MAP estimator computed through Gibbs sampling and using a lower bound of the posterior density. We have used normal-inverse Wishart as a yardstick to compare the risk of our estimators (MLE approximation and the two MAP’s).

4.2.1 Gibbs Sampling

The risk of the MAP estimator is computed through Gibbs sampling as described in Section 3.4. Let us denote the initial values to be \((\mu^{(0)}, \lambda^{(0)})\). We have chosen \( \mu^{(0)} = \mu_0 = \bar{x}, H_0 = I, \kappa_0 = 1.5 \) and \( a = (p + 1) \). The first iteration takes in \( \mu^{(0)} \) and calculates the value of \( H_N \) as described after equation (3.5). Using this, we can generate the value of \( \lambda^{(1)} \) from the distribution \( p(D \mid \mu, X) \) (see step 1 of Gibbs sampling in section 3.4) which by construction makes sure that each eigenvalue is positive. We can easily see that the initial value of \( \lambda \) i.e. \( \lambda^{(0)} \) does not affect the Gibbs sampling.

We have generated 100 samples from the posterior distribution using the MH within Gibbs algorithm described in section 3.4 We repeat this experiment 100 times and approximate the risk. The performance of the Gibbs sampling is displayed in the following table.
We can observe that the performance is improving as the number of observations \((n)\) increase. For low dimensional cases the risk of the MAP estimator of the covariance matrix calculated from Gibbs sampling is equivalent and better as compared to its normal-inverse Wishart counterpart.

A well-known problem with Gibbs sampling is that as the dimension increases the time complexity of the MAP approximation increases significantly. The acceptance rate of the MH step within Gibbs sampling is reasonable and decreases with increment of dimension.

Table 1: Risk ratio of MAP approximation (from MH within Gibbs sampling) relative to the MAP of normal-inverse Wishart

| n  | p  | Normal-Inverse Gamma Risk | Acc Rate (in MH) | Time (Sec.) |
|----|----|---------------------------|------------------|-------------|
| 50 | 3  | 1.1173 0.8928             | 0.4135           | 980.52      |
| 50 | 5  | 1.0824 1.0963             | 0.2822           | 2029.15     |
| 50 | 10 | 1.2423 1.2798             | 0.1218           | 13574.59    |
| 100| 3  | 1.0542 0.9872             | 0.4352           | 910.92      |
| 100| 5  | 1.1677 1.3662             | 0.2908           | 1901.41     |
| 100| 10 | 1.1963 1.6229             | 0.1245           | 12183.56    |
| 300| 3  | 1.1661 1.2546             | 0.4287           | 1256.81     |
| 300| 5  | 1.5729 2.0163             | 0.2895           | 3215.97     |
| 300| 10 | 1.5533 2.4053             | 0.1294           | 9385        |

4.2.2 MLE and MAP Approximation using a Lower Bound

The structured covariance matrix defined in equation (1.3) does not allow us to calculate the maximum likelihood estimate and the MAP estimator directly due to the intractability of the likelihood function. Hence, we approximate them through a lower bound described in Sections 2 and 3.5, respectively. The approximation of MLE performs well.

Table 2: Risk ratio of MLE and MAP approximation (through a lower bound) relative to the MAP of normal-inverse Wishart

| n  | p  | MLE Approx. Risk | MAP Approx. Risk | Time (Sec) |
|----|----|------------------|------------------|------------|
| 50 | 3  | 0.4253 1.1331    | 0.4253 1.1331    | 3.25       |
| 50 | 5  | 0.6625 1.1528    | 0.6625 1.1528    | 2.59       |
| 50 | 10 | 1.5009 1.195     | 1.5009 1.195     | 5.22       |
| 100| 3  | 0.3481 1.4383    | 0.3481 1.4383    | 2.52       |
| 100| 5  | 0.5342 1.5065    | 0.5342 1.5065    | 2.69       |
| 100| 10 | 1.401 1.63       | 1.401 1.63       | 2.84       |
| 300| 3  | 0.3159 2.3753    | 0.3159 2.3753    | 2.7        |
| 300| 5  | 0.5797 2.6429    | 0.5797 2.6429    | 2.95       |
| 300| 10 | 1.303 2.6493     | 1.303 2.6493     | 3.41       |

A modified version of the Newton-Raphson method (described in the paragraph fol-
lowing equation (3.15)) converges relatively fast i.e. two to five iterations for MAP when initialized with the approximation of MLE of \( c_0 \) and \( u \).

The performance of these approximations (both MLE and MAP), assessed by the risk, are equivalent in dimension three and five to the Gibbs sampling technique discussed in the last section with the added benefit of being significantly faster (see the last column of table 2). For \( p = 10 \), the performance is worse than Gibbs sampling (see Table 2). The usual observations like the decline of risk as \( n \) increases and \( p \) decreases are generally true. One interesting observation is that the performance of MLE approximation is not significantly different from the MAP approximation.

4.2.3 An Example: Estimates of the Historic Position of Earth’s Magnetic Pole

The dataset collected by Schmidt (1976) contains the site mean direction estimates of the Earth’s historic magnetic pole collected from 33 different sites in Tasmania. The longitude and latitudes from the data set is transformed to \( X_1,X_2,\ldots,X_{33} \) on a three dimensional unit sphere (Preston and Paine, 2017). The angular gaussian distribution family is the marginal directional component of a multivariate normal distribution with ESAG distribution as a sub-family. Paine et al (2018) provided strong evidence in favor of ESAG distribution which satisfy the constraint over isotropic angular Gaussian distribution while analyzing this dataset. This inspire us to make normality assumption under the constraint similar to ESAG distribution disregarding the spherical nature of the transformed dataset. The constrained maximum likelihood estimate under the structured covariance model is:

\[
\hat{\mathbf{u}}^\top = \begin{bmatrix} -0.44 & 0.32 & 0.75 \end{bmatrix}, \quad \hat{\Sigma} = \begin{bmatrix} 1.63 & -0.91 & -2.13 \\ -0.91 & 1.09 & 1.53 \\ -2.13 & 1.53 & 4.02 \end{bmatrix}
\]

which are comparable to the maximum likelihood estimate calculated using elliptically symmetric angular Gaussian distribution with the parametrization in three dimension (Paine et al, 2018) i.e. \( \hat{\mathbf{u}}^\top = (-0.56, 0.24, 0.79) \).

5 Discussion

This article demonstrates how simple ideas can be used to solve a challenging joint estimation problem under constraint (1.1). The structured covariance model takes into account of the constraint, reduces dimensions significantly and do not suffer from the usual issue of positive definiteness. The intractability of the model is resolved through a lower bound of the profile likelihood to obtain MLE, MAP and sampling from posterior density is performed through Metropolis within Gibbs. We have proposed a simple prior for our structured model i.e. normal on mean vector and inverse gamma on eigenvalues inspired by Berger et al (2020). Although this model works nicely here for comparatively smaller dimensions, it is possible to replicate the idea in higher dimensions with some modifications. Other priors can also be explored under such a model in future.
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Appendices

A Proofs of Results:

1. In case of normal-inverse Wishart prior the posterior density with parameter \((\mu_n, \kappa_n, \nu_n, \Lambda_n)\) is the following:

\[
p(\mu, \Sigma \mid \mu_n, \kappa_n, \nu_n, \Lambda_n) \propto \kappa_n^{-\frac{p}{2}} \mid \Sigma \mid ^{-\left(\frac{p+\nu_n}{2}\right)} \exp \left[ -\frac{1}{2} \{ \operatorname{Tr}(\Sigma^{-1} \Lambda_n) + \kappa_n (\mu - \mu_n)^\top \Sigma^{-1} (\mu - \mu_n) \} \right]
\]

After taking logarithm and differentiating we get:

\[
\frac{\partial \log p(\mu, \Sigma \mid \mu_n, \kappa_n, \nu_n, \Lambda_n)}{\partial \Sigma^{-1}} = \left( \frac{\nu_n + p}{2} + 1 \right) \Sigma - \frac{1}{2} \Lambda_n - \frac{\kappa_n}{2} (\mu - \mu_n)(\mu - \mu_n)^\top = 0
\]

\[
\hat{\Sigma}_{map} = \frac{\Lambda_n + \kappa_n (\mu - \mu_n)(\mu - \mu_n)^\top}{\nu_n + p + 2}
\]

By applying Theorem 4.2.1 of [Bibby et al. (1979)] we can say that this is indeed the MAP estimator.

2. Let \(x_1, x_2, \ldots, x_n \sim N_p(\mu, \Sigma)\) and the prior on \((\mu, \Sigma)\) is normal-shrinkage inverse Wishart prior i.e. \(\mu \mid \Sigma \sim N_p(\mu_0, \Sigma/\kappa_0)\) and \(\Sigma \sim \pi_{SIW}(\nu_0, b, \Lambda^{-1}_0)\). Here we are interested with \(b = 1\) as stated earlier. The posterior distribution is calculated as follows:

\[
\pi(\mu, \Sigma \mid x_1, \ldots, x_n, \mu_0, \kappa_0, \nu_0, \Lambda_0) \propto \prod_{i=1}^n N_p(x_i; \mu, \Sigma) N_p(\mu; \mu_0, \Sigma/\kappa_0) \pi_{SIW}(\Sigma; \nu_0, 1, \Lambda^{-1}_0)
\]

\[
\propto \prod_{i=1}^n N_p(x_i; \mu, \Sigma) N_p(\mu; \mu_0, \Sigma/\kappa_0) \frac{\pi_{IW}(\Sigma; \nu_0, \Lambda^{-1}_0)}{\prod_{i<j} (\lambda_i - \lambda_j)}
\]

This is because shrinkage-inverse-Wishart can be obtained from inverse-Wishart density by dividing it with \(\prod_{i<j} (\lambda_i - \lambda_j)\). We know that the normal-inverse Wishart prior is conjugate with posterior parameters described in (3.1), \(\kappa_n = \kappa_0 + n\) and \(\nu_n = \nu_0 + n\) respectively. The numerator is the posterior distribution corresponding to the normal-inverse Wishart prior and gives us the following:

\[
\pi(\mu, \Sigma \mid x_1, \ldots, x_n, \mu_0, \kappa_0, \nu_0, \Lambda_0) \propto \frac{\pi_{NIW}(\mu, \Sigma \mid \mu_n, \kappa_n, \nu_n, \Lambda^{-1}_n)}{\prod_{i<j} (\lambda_i - \lambda_j)}
\]

\[
\propto N_p(\mu \mid \Sigma, \mu_n, \kappa_n) \pi_{IW}(\Sigma \mid \nu_n, \Lambda^{-1}_n) \prod_{i<j} (\lambda_i - \lambda_j)
\]

\[
\propto N_p(\mu \mid \Sigma, \mu_n, \kappa_n) \pi_{SIW}(\Sigma \mid \nu_n, 1, \Lambda^{-1}_n)
\]

From the above calculation we can see that the posterior is normal-shrinkage inverse Wishart with the same parameters as in normal-inverse Wishart proving our hypothesis.
3. Calculation of the Posterior Distribution:

\[
p(\mu, \lambda \mid X) \propto \left( \prod_{i=1}^{p-1} \lambda_i \right)^{-\frac{n}{2}} \exp \left[ -\frac{1}{2} \text{Tr} \left\{ \Lambda^{-1} B(X, \mu) \right\} \right] \left( \prod_{i=1}^{p-1} \lambda_i \right)^{-\frac{a}{2}} \exp \left[ -\frac{1}{2} \text{Tr} \left\{ \Lambda^{-1} \left( \mu - \mu_0 \right) \left( \mu - \mu_0 \right)^\top \right\} \right] \exp \left[ -\frac{1}{2} \text{Tr} \left\{ \Lambda^{-1} H_0 \right\} \right] \
\]

\[
= \left( \prod_{i=1}^{p-1} \lambda_i \right)^{-\frac{n+2(a+1)}{2}} \exp \left[ -\frac{1}{2} \text{Tr} \left\{ \Lambda^{-1} \left( B(X, \mu) + (\mu - \mu_0)(\mu - \mu_0)^\top + H_0 \right) \right\} \right] \
\]

\[
= \left( \prod_{i=1}^{p-1} \lambda_i \right)^{-\frac{n+2(a+1)}{2}} \exp \left[ -\frac{1}{2} \text{Tr} \left\{ \Lambda^{-1} H_N \right\} \right] \
\]

where \( B(X, \mu) = P(\mu)^\top A(\mu) P(\mu) \).