COUPLING THE SHALLOW WATER EQUATION WITH A LONG TERM DYNAMICS OF SAND DUNES

MOUMADOU ALIOU M. T. BALDÉ AND DIARAF SECK

Université Cheikh Anta Diop de Dakar, BP 16889 Dakar Fann
Ecole Doctorale de Mathématiques et Informatique
Laboratoire de Mathématiques de la Décision et d’Analyse Numérique
(L.M.D.A.N) F.A.S.E.G, Senegal

ABSTRACT. In this paper we couple a long term dynamic equation of dunes of sand (LTDD) in [6] with a shallow water equation (SWE). And we study the evolution of sand dunes over long periods in the marine environment near the coast. We use works due to S. Klainerman & A. Majda [9] to show on the one hand existence and uniqueness results. On the other hand we give estimations of solutions for the dimensionless coupled system SWE-LTDD. And finally the coupled system is homogenized.

1. Introduction. Coastal erosion is a natural phenomenon existing since immemorial time. There are significant human and economic impacts of coastal erosion or encroachment of the sea on the changing world particularly for a country facing the coast. For example this phenomenon threatens the entire Senegalese coast in West Africa. The coastal erosion can be described as the transport of sand or sediment as a result of the motion of the sea.

In this work we intend to couple a long term dynamic equation of dune of sand (LTDD) in [6] with a shallow water equation (SWE). This coupled system models the motion of sand dunes over a long period under the effect of the motion of the sea near the coast.

Some authors have worked on the coupling of the Saint-Venant equation with dynamics equations of sediment and interesting results ensued [3], [5]. In their studies the authors of these two papers considered any type of sediments. But they focused their analysis on an numerical point of view. In [4] C. Berthon et al give an exact smooth solution for coupled shallow water equation with Exner equation in the steady state condition of flow for a given uniform discharge. And this solution is valid for large family of sedimentation laws which are widely used in erosion modeling such as the Grass model or those proposed by Meyer-Peter & Müller (see [4]). In [14], authors study the coupled system of a viscous version of shallow water and Exner diffusive equation in bounded domain with periodic boundary condition, that is two dimensional torus $T^2$. In [2], authors showed existence of solution of shallow water equation by using [8] and [9] and proved that the existence time of the solution does not depend on the small parameters $\epsilon$.

Our work is devoted to the case where we consider one type of sediment : the sand.

2010 Mathematics Subject Classification. 35L45, 58J15, 35K15, 58J35, 58J37.
Key words and phrases. Shallow water equations, singular PDE, hyperbolic, parabolic, asymptotics, sand dunes, homogenization.
But we aim to bring additional results. In fact in our study we couple a long term dynamic equation with the shallow water equation. And this combination implies interesting and more complicated situations to be studied.

Thus we shall study the evolution of sand dunes over long periods in the marine environment near the coast. Such a problem is known in the literature as morphodynamic problem, which consists on coupling an hydrodynamic model describing the water flow and an equation modeling the evolution of the topography.

To study the morphodynamic phenomenon coupling the shallow water equation (SWE) and long term dynamic equation of dunes of sand (LTDD), we use a mathematical model, where the unknown variables are combination of unknown variables coming from these two equations of hydrodynamic and topography.

We shall do the scaling of the system by using parameter size as in [6] to get dimensionless coupled system of SWE and LTDD. And it shall be followed by the stating of existence results.

As in [5] we shall give an asymptotic model by using two scale convergence, of the coupled system of SWE and long term dynamic of dune of sand equation.

The paper is organized as follows: In section 2, we shall present the model, do the scaling and give parameter size of the model, and finally we symmetrize the model to obtain the final form of the coupled system. The section 3 is devoted to the existence and uniqueness results for the coupled system. In section 4 we are going to prove the main theorem on the existence of solution. In section 5 we shall give the homogenized coupled system. And we are going to end this work by an appendix (section 5) in which we present useful details on the proofs of some estimates results.

2. The coupled system.

2.1. Model. In this work we consider the following system of partial differential equations:

$$\frac{\partial}{\partial t} (\xi - z) + \nabla \cdot [(\xi + H - z)u] = 0 \quad (1.1)$$

$$\frac{\partial}{\partial t} [(\xi + H - z)u] + \nabla \cdot [(\xi + H - z)u \otimes u] + g(\xi + H - z) \cdot \nabla (\xi + H - z) + g(\xi + H - z) \nabla z + f(\xi + H - z)u^\| + ku = 0 \quad (1.2)$$

$$\frac{\partial z}{\partial t} + \frac{\alpha}{1-p} \nabla \cdot \left[\chi(D_Gp \frac{|u|^2-w}{c^2})(\frac{m}{|u|} - \lambda \nabla z)\right] = 0 \quad (1.3)$$

where (see Figure 1):

- $t$ and $x = (x_1, x_2)$ are respectively the time variable and the 2-dimensional space variable.
- The function $\xi(t, x)$ stands for the free surface of the water.
- $H$ is constant that determine the mean height of the water.
- $z(t, x)$ is the height or the depth of the sand by considering the level $y = -H$.
- $h(t, x)$ is the water height from the free surface of the water to the surface of the dunes of sand. So $h = \xi + H - z \Rightarrow h = H = \xi - z$.
- We denote by $m = h - H = \xi - z$ to stands for the height variation.
- $u(t, x) = (u_1, u_2)$ is the velocity of the water and $u^\perp(t, x) = (-u_2, u_1)$. 


The tensor product
\[ u \otimes u = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_2 u_1 & u_2^2 \end{bmatrix} \]

The system of these two equations (1.1)-(1.2) is the shallow water equation with the gravity \( g \), the Coriolis term \( f \) and the friction coefficient \( k \).

The shallow water equations (SWE) are derived from the Navier-Stokes equations, which describes the motion of Newtonian fluids. The SWE models the evolution of the ocean and other incompressible fluids in the underlying assumption that the depth is small compared to the wave length of the fluid. This mean that the shallow water equations are the simplest form of the motion equations that can be used to describe the horizontal evolution of an incompressible fluid in response to gravitational and rotational accelerations.

The equation (1.3) is a transport equation based on [6], [13] and [7]. It model the dynamic of dunes of sand. \( D_G \) is the sand speck diameter, \( \rho \) is the water density, \( \alpha \) is a constant which order of magnitude is 100, \( p \in [0, 1) \) is the sand porosity, \( \lambda \) is the inverse value of maximum slope of the sediment surface when water velocity is 0, \( u_c \) is the threshold under which the water velocity does not make the sand move, \( C \) is a constant defined by \( C = \ln\left(\frac{12d}{3D_G}\right) \), \( d \) is the water height above the seabed (see [6]).

And \( \chi(\sigma) = \begin{cases} 0 & \text{if } \sigma < 0 \\ |\sigma^{3/2}| & \text{if } \sigma \geq 0 \end{cases} \) This equation is derived from equations of dynamic of sediments, which has been particularized for the transport of sand.

\[ \chi(\sigma) = \begin{cases} 0 & \text{if } \sigma < 0 \\ |\sigma^{3/2}| & \text{if } \sigma \geq 0 \end{cases} \]

\[ u(t,x) \]

\[ z(t,x) \]

\[ h \]

\[ \xi(t,x) \]

Figure 1. Water and dune of sand

2.2. Scaling. The characteristics values we use for the scaling are the same as in [6]. So we use and adapt them for the shallow water equation. And the subsection is organized as follows:
a) The characteristic dimensions: the interest is focused on:
- \( t' \) the characteristic time.
- \( \bar{L} \) the characteristic length.
- \( \xi \) characteristic of the free surface of the water.
- \( \bar{M} \) characteristic of the height variation.

b) The new variables and functions are calculated as follows:
- \( t' \cdot \bar{t} = t; \bar{L} \cdot \bar{x} = x \).
- \( \bar{\xi} \cdot \bar{\xi}'(t', \bar{x}') = \xi(\bar{t}', \bar{Lx}') \).
- \( \bar{z} \cdot \bar{z}'(t', \bar{x}') = \bar{z}(\bar{t}', \bar{Lx}') \).
- \( \bar{u} \cdot \bar{u}'(t', \bar{x}') = u(\bar{t}', \bar{Lx}') \).
- \( \bar{M} \cdot m'(t', \bar{x}') = \xi(\bar{t}', \bar{Lx}') - \bar{z}(\bar{t}', \bar{Lx}') \).

Let us point out that all above variables with the symbol prime ('') are dimensionless variables.

c) The derivatives are computed as follows:
- Space derivative: \( \nabla \cdot \bar{z}'(t', \bar{x}') = \bar{\xi} \nabla z(\bar{t}', \bar{Lx}') \).
- Time derivative: \( \frac{\partial}{\partial t'} \bar{z}'(t', \bar{x}') = \bar{\xi} \frac{\partial z}{\partial t} \).

d) Scaling of SWE. Let us consider the first equation in (1):
\[
\frac{\partial}{\partial t'} (\xi - z) + \nabla \cdot [((\xi + H - z)u] = 0.
\]

Then we have:
\[
\frac{1}{\bar{t}} \frac{\partial}{\partial t'} (\bar{\xi} \cdot \bar{\xi}' - \bar{z} \cdot \bar{z}') + \nabla \cdot \left[ (\bar{\xi} \cdot \bar{\xi}' + H - \bar{z} \cdot \bar{z}') \bar{u} \cdot \bar{u}' \right] = 0
\]
\[
\frac{\bar{z}}{\bar{t}} \frac{\partial}{\partial t'} (\frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' - \bar{z}') + \frac{\bar{z} \bar{u}}{\bar{L}} \nabla' \cdot \left[ (\frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' + H \bar{z} - \bar{z} \cdot \bar{z}) \cdot \bar{u}' \right] = 0
\]

Simplifying by \( \frac{\bar{z}}{\bar{t}} \) we have:
\[
\frac{\partial}{\partial t'} (\frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' - \bar{z}' + \frac{H}{\bar{z}} \bar{u}' \cdot \bar{u}') = 0 \quad (2)
\]

Let’s take now the second equation in (1):
\[
\frac{\partial}{\partial t'} [(\xi + H - z)u] + \nabla \cdot [(\xi + H - z)u \otimes u] + g(\xi + H - z) \nabla (\xi + H - z) + g(\xi + H - z) \nabla z + f(\xi + H - z)u' = -ku \quad (3)
\]

Using the same technique as for the first equation and simplifying by \( \frac{\bar{z} \bar{u}}{\bar{L}} \) we get:
\[
\frac{\partial}{\partial t'} \left[ \left( \frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' - \frac{H}{\bar{z}} \bar{u}' \cdot \bar{u}' \right) \right] + \frac{\bar{z} \bar{u}}{\bar{L}} \nabla' \cdot \left[ \left( \frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' + H \bar{z} - \bar{z} \cdot \bar{z} \right) \bar{u}' \right] + \frac{\bar{z}}{\bar{t}} \int' \left( \frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' - \frac{H}{\bar{z}} \bar{u}' \cdot \bar{u}' \right) = -\frac{\bar{t}k}{\bar{z}} k' \bar{u}' \quad (4)
\]

Thanks to the notations we have done previously, we get:
\[
m' = \frac{1}{M} m = \frac{1}{M} (h - H) = \frac{1}{M} (\xi - z) = \frac{1}{M} (\bar{\xi} \cdot \bar{\xi}' - \bar{z} \cdot \bar{z}') = \frac{\bar{z}}{M} \left( \frac{\bar{\xi}}{\bar{z}} \cdot \bar{\xi}' - \bar{z}' \right)
\]
So replacing in (2) and (4), simplifying by $\frac{M}{z}$ and dropping the ('') we get:

$$
\begin{align*}
\frac{\partial m}{\partial t} + \frac{\bar{u}}{L} \nabla \cdot [(m + \frac{H}{M})u] = 0 \\
\frac{\partial}{\partial t} \left[(m + \frac{H}{M})u\right] + \frac{\bar{u}}{L} \nabla \cdot [(m + \frac{H}{M})u \otimes u] + \\
g \frac{\bar{M}}{Lu} (m + \frac{H}{M}) \nabla (m + \frac{H}{M}) + \frac{\bar{e}}{Lu} (m + \frac{H}{M}) \nabla z + \\
f\tilde{f}(m + \frac{H}{M})u^+ = -\frac{\tilde{e}}{M} ku
\end{align*}
$$

(5)

e) Scaling of LTDD. Since we use the scaling in the paper [6], the equation for small sand specks is given by:

$$
\frac{\partial e}{\partial t} = \lambda \frac{\bar{u}}{1 - p} \frac{\tilde{e}}{3} (\rho D_G)^{3/2} \bar{L} \nabla \cdot [(1 - 3 \frac{\bar{M}}{H \ln(\frac{H}{D_G})})m] \chi(|u|^2 - \frac{u^2}{\bar{u}}) \nabla z
$$

$$
= \frac{1}{1 - p} \frac{\tilde{e}}{3} (\rho D_G)^{3/2} \bar{L} \nabla \cdot [(1 - 3 \frac{\bar{M}}{H \ln(\frac{H}{D_G})})m] \chi(|u|^2 - \frac{u^2}{\bar{u}}) \frac{u}{|u|},
$$

(6)

For more details on the computations, see Faye et al [6] and references therein.

Finally the dimensionless coupled system of SWE and LTDD is summed up by:

$$
\begin{align*}
\frac{\partial m}{\partial t} + \frac{\bar{u}}{L} \nabla \cdot [(m + \frac{H}{M})u] = 0 \\
\frac{\partial}{\partial t} \left[(m + \frac{H}{M})u\right] + \frac{\bar{u}}{L} \nabla \cdot [(m + \frac{H}{M})u \otimes u] + \\
g \frac{\bar{M}}{Lu} (m + \frac{H}{M}) \nabla (m + \frac{H}{M}) + \frac{\bar{e}}{Lu} (m + \frac{H}{M}) \nabla z + \\
f\tilde{f}(m + \frac{H}{M})u^+ = -\frac{\tilde{e}}{M} ku
\end{align*}
$$

$$
\begin{align*}
\frac{\partial e}{\partial t} - \lambda \frac{\bar{u}}{1 - p} \frac{\tilde{e}}{3} (\rho D_G)^{3/2} \bar{L} \nabla \cdot [(1 - 3 \frac{\bar{M}}{H \ln(\frac{H}{D_G})})m] \chi(|u|^2 - \frac{u^2}{\bar{u}}) \nabla z \\
- \alpha \frac{\bar{u}}{1 - p} (\rho D_G)^{3/2} \bar{L} \nabla \cdot [(1 - 3 \frac{\bar{M}}{H \ln(\frac{H}{D_G})})m] \chi(|u|^2 - \frac{u^2}{\bar{u}}) \frac{u}{|u|} = 0
\end{align*}
$$

(7)

2.3. Parameter size. In this subsection as for the scaling we use the same size of parameters for LTDD equation as in [6].

So we set:

$$
\bar{u} = 1m/s, \ H = 50m, \ \bar{M} = 5m \text{ and } \lambda \frac{1}{1 - p} = 1 \text{ and } \frac{1}{1 - p} = 2
$$

$$
\tilde{e} \sim 16 \text{ years } \sim 5 \cdot 10^9 s, \ \frac{1}{\omega_c} \sim 1 \text{ month } \sim 2.6 \cdot 10^6 s
$$

$$
D_G = 7 \cdot 10^{-5}, \ \bar{z} = 1m, \ \bar{L} = 10m, \ u_c = 0 m/s
$$

Let’s define $\epsilon = \frac{1}{\omega_c} \sim \frac{1}{192}$: Then we are able to compute all the constants in the system SWE-LTDD:
Remark 1. 
• It is important to underline that the unknowns \( m, u, \) and \( z \) depend on \( \epsilon \).
• We shall use the methods introduced by T. Kato [8] and S. Klainerman and A. Majda [9] to study the above system. But it is important to point out that the matrices coming from this system are not symmetric. And then before adapting the methods quoted just above we need to do the symmetrization step in order to satisfy the Friedrich’s theory.
• As final remark before the next subsection, it is important to mention that \( m = m^\epsilon, u = u^\epsilon \) and \( z = z^\epsilon \).

2.4. Symmetrization. In this section we transform the dimensionless SWE in a matrix form and symmetrize it.

The system SWE-LTDD can be expressed in its more general form as follows:

\[
\begin{aligned}
\frac{\partial m^\epsilon}{\partial t} + \frac{a_1}{\epsilon_1^2} \nabla \cdot [(m^\epsilon + b^1)u^\epsilon] &= 0 \\
\frac{\partial}{\partial t} [(m^\epsilon + b^1)u^\epsilon] + \frac{a_1}{\epsilon_1^2} \nabla \cdot [(m^\epsilon + b^1)u^\epsilon \otimes u^\epsilon] &+ \frac{c_1}{\epsilon_1^4} (m^\epsilon + b^1) \nabla (m^\epsilon + b^1) + \frac{d_1}{\epsilon_1^4} (m^\epsilon + b^1) \nabla z^\epsilon \\
&+ \frac{f}{\epsilon} (m^\epsilon + b^1) u^\epsilon &= -\frac{k}{\epsilon} u^\epsilon \\
\frac{\partial z^\epsilon}{\partial t} - \frac{a}{\epsilon^2} \nabla \cdot [(1 - bcm^\epsilon)|u^\epsilon|^3 \nabla z^\epsilon] &+ \frac{\epsilon}{\epsilon^2} \nabla \cdot [(1 - bcm^\epsilon)|u^\epsilon|^2 u^\epsilon] &= 0,
\end{aligned}
\]

with \( m^\epsilon \), the one dimensional height variation, \( z^\epsilon \), the one dimensional dune height and \( u^\epsilon = (u^z_1, u^z_2) \), the 2-dimensional water velocity with \( u^\epsilon^\perp = (-u^z_2, u^z_1) \), and constants \( a_1, b^1, c^1, d^1, a, b, c > 0 \).
In the sequel we shall consider $t \in [0,T]$, $T > 0$ and the space variable $x = (x_1, x_2) \in \mathbb{T}^2$, the 2-dimensional torus i.e $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Let us set the following change of variables:

$$q^\varepsilon = (m^\varepsilon + b^1)u^\varepsilon = \left( \begin{array}{c} q_1^\varepsilon \\ q_2^\varepsilon \end{array} \right) = \left( \begin{array}{c} (m^\varepsilon + b^1)u_1^\varepsilon \\ (m^\varepsilon + b^1)u_2^\varepsilon \end{array} \right)$$

In the literature, the vector $q^\varepsilon$ is used to call water discharge.

So we can write

$$u_1^\varepsilon = \frac{q_1^\varepsilon}{m^\varepsilon + b^1} \text{ and } u_2^\varepsilon = \frac{q_2^\varepsilon}{m^\varepsilon + b^1}$$

$$|u^\varepsilon| = \sqrt{u_1^\varepsilon^2 + u_2^\varepsilon^2} = \sqrt{\frac{q_1^\varepsilon^2 + q_2^\varepsilon^2}{(m^\varepsilon + b^1)^2}} = \frac{|q^\varepsilon|}{(m^\varepsilon + b^1)}$$

$$|u^\varepsilon|^2 u^\varepsilon = \frac{|q^\varepsilon|^2 q^\varepsilon}{(m^\varepsilon + b^1)^3}$$

In this paper we need the realistic following hypothesis: the dunes are under the water. This means that the mean height of the water is greater than the height of any dune. It is translated by: $h > 0$ and $H > z(t,x) \forall t,x$. This can be written otherwise:

$$\exists \nu > 0 \text{ such that } m^\varepsilon + b^1 \geq \nu > 0$$

Therefore the system (9) becomes:

$$\frac{\partial m^\varepsilon}{\partial t} + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_1} [a^1 q_1^\varepsilon] + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_2} [a^1 q_2^\varepsilon] = 0$$

$$\frac{\partial q_1^\varepsilon}{\partial t} + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_1} \left[ \frac{a^1 q_1^\varepsilon^2}{m^\varepsilon + b^1} + \frac{c_1}{2}(m^\varepsilon + b^1)^2 \right] + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_2} \left[ \frac{a^1 q_1^\varepsilon q_2^\varepsilon}{m^\varepsilon + b^1} \right] - \frac{f}{\epsilon} q_2^\varepsilon + \frac{k}{\epsilon} q_1^\varepsilon + \frac{d^1}{\epsilon^4} \frac{b^1}{m^\varepsilon + b^1} \frac{\partial z^\varepsilon}{\partial x_1} = 0$$

$$\frac{\partial q_2^\varepsilon}{\partial t} + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_1} \left[ \frac{a^1 q_1^\varepsilon q_2^\varepsilon}{m^\varepsilon + b^1} \right] + \frac{1}{\epsilon^4} \frac{\partial}{\partial x_2} \left[ \frac{a^1 q_2^\varepsilon^2}{m^\varepsilon + b^1} + \frac{c_1}{2}(m^\varepsilon + b^1)^2 \right] + \frac{f}{\epsilon} q_1^\varepsilon + \frac{k}{\epsilon} q_2^\varepsilon + \frac{d^1}{\epsilon^4} \frac{b^1}{m^\varepsilon + b^1} \frac{\partial z^\varepsilon}{\partial x_2} = 0$$

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\epsilon^2} \frac{a(1 - b c^\varepsilon)}{|m^\varepsilon + b^1|^3} \frac{\partial^2 z^\varepsilon}{\partial x_1^2} - \frac{1}{\epsilon^2} \frac{a(1 - b c^\varepsilon)}{|m^\varepsilon + b^1|^3} \frac{\partial^2 z^\varepsilon}{\partial x_2^2} - \frac{1}{\epsilon^2} \frac{c(1 - b c^\varepsilon)}{|m^\varepsilon + b^1|^3} \frac{\partial^2 z^\varepsilon}{\partial x_1 \partial x_2} = 0$$

The above system can also be written in the following:

$$\frac{\partial \nu^\varepsilon}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^2 \frac{\partial}{\partial x_j} F_j(\nu^\varepsilon) = \frac{1}{\epsilon} \frac{\partial h(\nu^\varepsilon)}{\partial t} - \frac{1}{\epsilon^4} \frac{\partial P(\nu^\varepsilon, z^\varepsilon)}{\partial t}$$

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\epsilon^2} \nabla \cdot [A^\varepsilon \nabla z^\varepsilon] = \frac{1}{\epsilon^2} \nabla \cdot C^\varepsilon$$

\text{(10)}
where \(v^\varepsilon = \left( \frac{m^\varepsilon}{q_1^\varepsilon}, \frac{v^\perp}{q_2^\varepsilon} \right)\), \(v_{\varepsilon}^\perp = \left( -q_2^\varepsilon \right)\), \(h(v^\varepsilon) = -\frac{L}{\varepsilon}v^\perp + \frac{1}{\varepsilon}H(v^\varepsilon)\) and

\[
F(v^\varepsilon) = (F_1, F_2) = \left( \begin{array}{ccc} \frac{a_1 q_1^\varepsilon}{m^\varepsilon + b^\varepsilon} + \frac{c_1^2}{2} (m^\varepsilon + b^\varepsilon)^2 & \frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} & \frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} + \frac{c_1^2}{2} (m^\varepsilon + b^\varepsilon)^2 \\
\frac{a_1 q_1^\varepsilon}{q_2^\varepsilon} & \frac{a_1 q_2^\varepsilon}{q_1^\varepsilon} & \frac{a_1 q_2^\varepsilon}{q_1^\varepsilon} \\
\frac{a_1 q_2^\varepsilon}{q_1^\varepsilon} & \frac{a_1 q_2^\varepsilon}{q_1^\varepsilon} & \frac{a_1 q_2^\varepsilon}{q_1^\varepsilon} \\
\end{array} \right)
\]

with

\[
F_1(v^\varepsilon) = \left( \begin{array}{c} F_1^1 \\
F_1^2 \\
F_1^3 \end{array} \right) = \left( \begin{array}{c} \frac{a_1 q_1^\varepsilon}{m^\varepsilon + b^\varepsilon} + \frac{c_1^2}{2} (m^\varepsilon + b^\varepsilon)^2 \\
\frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} \\
\frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} + \frac{c_1^2}{2} (m^\varepsilon + b^\varepsilon)^2 \\
\end{array} \right)
\]

\[
F_2(v^\varepsilon) = \left( \begin{array}{c} F_2^1 \\
F_2^2 \\
F_2^3 \end{array} \right) = \left( \begin{array}{c} \frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} \\
\frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} + \frac{c_1^2}{2} (m^\varepsilon + b^\varepsilon)^2 \\
\frac{a_1 q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} \\
\end{array} \right)
\]

\[
H(v^\varepsilon) = \left( \begin{array}{c} -\frac{k q_1^\varepsilon}{m^\varepsilon + b^\varepsilon} \\
-\frac{k q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} \\
-\frac{k q_2^\varepsilon}{m^\varepsilon + b^\varepsilon} \end{array} \right)
\]

\[
P(v^\varepsilon, z^\varepsilon) = \left( \begin{array}{c} 0 \\
d^1 (m^\varepsilon + b^\varepsilon) \frac{\partial z^\varepsilon}{\partial x_1} \\
d^1 (m^\varepsilon + b^\varepsilon) \frac{\partial z^\varepsilon}{\partial x_2} \end{array} \right)
\]

\[
A^\varepsilon(v^\varepsilon) = \frac{a(1 - b e m^\varepsilon) |q^\varepsilon|^3}{m^\varepsilon + b^\varepsilon}^3 \quad \text{and} \quad C^\varepsilon(v^\varepsilon) = \frac{c(1 - b e m^\varepsilon) |q^\varepsilon|^2 |q^\varepsilon|}{(m^\varepsilon + b^\varepsilon)^3}.
\]

Let us remark that the equation

\[
\frac{\partial v^\varepsilon}{\partial t} + \frac{1}{\varepsilon^2} \sum_{j=1}^{2} \frac{\partial}{\partial x_j} F_j(v^\varepsilon) = \frac{1}{\varepsilon} h(v^\varepsilon) - \frac{1}{\varepsilon^2} P(v^\varepsilon, z^\varepsilon) \tag{11}
\]

is a hyperbolic system of first order with source term. It is usually called balance law equation. And the following one:

\[
\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2} \nabla \cdot [A^\varepsilon \nabla z^\varepsilon] = \frac{1}{\varepsilon^2} \nabla \cdot C^\varepsilon \tag{12}
\]

is a parabolic equation, which may become singular or degenerate for \(\varepsilon\) taking particular values: for example if \(\varepsilon = 0\) we are in front of a singular case, and if \(A^\varepsilon = 0\) for some values of \(\varepsilon\) the equation (12) is degenerated.

If we suppose that \(v^\varepsilon\) is smooth enough at least in the sense of distributions then:

\[
\frac{\partial}{\partial x_j} F_j(v^\varepsilon) = D_v F_j(v^\varepsilon) \cdot \frac{\partial v^\varepsilon}{\partial x_j}, \quad \forall \ j = 1, 2.
\]

In this case the system takes the following expression:

\[
\frac{\partial v^\varepsilon}{\partial t} + \frac{1}{\varepsilon^2} \sum_{j=1}^{2} D_v F_j(v^\varepsilon) \cdot \frac{\partial v^\varepsilon}{\partial x_j} = \frac{1}{\varepsilon} h(v^\varepsilon) - \frac{1}{\varepsilon^2} P(v^\varepsilon, z^\varepsilon) \tag{13}
\]
where \( D_v F_j(v^\epsilon) = \left( \frac{\partial}{\partial x} F_j^1, \frac{\partial}{\partial \epsilon} F_j^1, \frac{\partial}{\partial \epsilon} F_j^3, \frac{\partial}{\partial \epsilon} F_j^3 \right) \).

A simple calculation yields:

\[
D_v F_1(v^\epsilon) = \begin{pmatrix}
-a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & 0 & 2 a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} & 0 \\
0 & a^1 q_2^1 & 0 & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2}
\end{pmatrix}
\]

and

\[
D_v F_2(v^\epsilon) = \begin{pmatrix}
0 & 0 & a^1 & a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} \\
-a^1 q_2^1 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & a^1 q_2^1 & a^1 q_2^2 & 2 a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2}
\end{pmatrix}
\]

Setting \( A^j(v^\epsilon) = D_v F_j(v^\epsilon), \forall j = 1, 2 \), we have:

\[
\frac{\partial v^\epsilon}{\partial t} + \frac{1}{c^3} \sum_{j=1}^{2} A^j(v^\epsilon) \cdot \frac{\partial v^\epsilon}{\partial x_j} = \epsilon \partial h(v^\epsilon) - \frac{1}{c^2} P(v^\epsilon, z^\epsilon)
\]  

(14)

Let us consider the matrix \( B^0 \) defined by:

\[
B^0 = \begin{pmatrix}
1 & a^1 \left( \frac{q_1^2}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) \right) & -q_1^1 \frac{\epsilon}{(m^\epsilon + b_1)} & -q_2^1 \frac{\epsilon}{(m^\epsilon + b_1)} \\
-q_1^1 \frac{\epsilon}{(m^\epsilon + b_1)} & 1 & 0 & 0 \\
-q_2^1 \frac{\epsilon}{(m^\epsilon + b_1)} & 0 & 1 & 0
\end{pmatrix}
\]

Multiplying (14) by \( B^0 \), we have:

\[
B^0 \frac{\partial v^\epsilon}{\partial t} + \frac{1}{c^3} \sum_{j=1}^{2} B^j \cdot \frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{c^2} B^0 \cdot h(v^\epsilon) - \frac{1}{c^2} B^0 \cdot P(v^\epsilon, z^\epsilon)
\]  

(15)

where \( B^j = B^0 \cdot A^j, \forall i, j = 1, 2 \).

\( B^1 \) and \( B^2 \) are matrices given respectively by the following expressions:

\[
\begin{pmatrix}
q_1^1 \frac{\epsilon}{(m^\epsilon + b_1)^2} - a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & -a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & a^1 q_1^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} \\
-a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} & 0 \\
-a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & 0 & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
q_2^1 \frac{\epsilon}{(m^\epsilon + b_1)^2} - a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & -a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} \\
-a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} & 0 \\
-a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2} + c^1 (m^\epsilon + b_1) & 0 & a^1 q_2^2 \frac{\epsilon}{(m^\epsilon + b_1)^2}
\end{pmatrix}
\]
And
\[ h(\nu^\epsilon) = H(\nu^\epsilon) - f \nu^\epsilon \perp = \begin{pmatrix}
0 \\
- \frac{kq_1^2}{m^\epsilon + b^1} + f q_2^\epsilon \\
- \frac{kq_2^2}{m^\epsilon + b^1} - f q_1^\epsilon 
\end{pmatrix} \]

If we set \( \tilde{h}(\nu^\epsilon) = B^0 : h(\nu^\epsilon) \), then
\[ \tilde{h}(\nu^\epsilon) = \begin{pmatrix}
\frac{kq^\epsilon}{(m^\epsilon + b^1)^2} \\
- \frac{kq_1^2}{m^\epsilon + b^1} + f q_2^\epsilon \\
- \frac{kq_2^2}{m^\epsilon + b^1} - f q_1^\epsilon 
\end{pmatrix}, \quad \tilde{p}(\nu^\epsilon, z^\epsilon) = B^0 : P(\nu^\epsilon, z^\epsilon). \]

Finally (15) becomes:
\[ B^0 \frac{\partial \nu^\epsilon}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^2 B^j \cdot \frac{\partial \nu^\epsilon}{\partial x_j} = \frac{1}{\epsilon} \tilde{h}(\nu^\epsilon) - \frac{1}{\epsilon^4} \tilde{p}(\nu^\epsilon, z^\epsilon). \] (16)

So having at hands the system of equations (12)-(16), we are in situation to use [9] to prove existence and uniqueness of solution with suitable initial conditions.

3. Existence, uniqueness and estimates. We shall propose in this section existence and uniqueness theorem for the coupled system SWE-LTDD(Shallow Water and Long Term Dynamic of Dunes of sand) and estimations results for the solution of SWE-LTDD. The estimates allow us to get the main theorem in the section devoted to homogenization. Before stating our results, let us say that in [4], C. Berthon et al give an exact smooth solution for coupled shallow water equation with Exner equation for a given uniform water discharge. In our case when we do the same hypotheses as in [4] the method of finding exact smooth solution remains valid for our system. But in this work, we emphasize on the fact that we do not impose a fixed value of the water discharge. And by using the methods developed in [9], we show existence, uniqueness and estimation results of solution of the global system.

For the coupled system since the small parameter \( \epsilon \) may go to 0 this yields singularities on coefficients of the system which become large and the interval of existence of classical solutions of (16) can shrink rapidly to zero as \( \epsilon \to 0 \). Hence we are in the conditions described in [9] to apply the method introduced by S. Klainerman and A. Majda for our system. So we need to introduce at first some definitions and notations before going on. All these definitions can be found in there general form in [9].

Let us define a parameter \( d(\epsilon) \) such that \( \lim_{\epsilon \to 0} d(\epsilon) = +\infty \), and the corresponding square norm of a splitting vector \( w := (w_1, w_2) \), \( w_1 \in \mathbb{R} \), \( w_2 \in \mathbb{R}^2 \):
\[ \|w\|_\epsilon^2 = \epsilon^2 |w_1|^2 + \|w_2\|^2 \]

Let’s \( D \) be matrix associated to this splitting of \( w \):
\[ D = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22} 
\end{pmatrix} \]

Then we can define the norms with \( D \) as follows:
\[ \|D\|_\epsilon = |D_{11}| + d(\epsilon) |D_{12}| + d(\epsilon)^{-1} |D_{21}| + |D_{22}| \]
for any splitting vector $w$.

It is not hard to see that there is a constant $C > 0$ such that

$$
\|Dw\|_e \leq C \|D\|_e \cdot \|w\|_e
$$

for any splitting vector $w$ defined as above.

Let us define too the associated Sobolev’s norm by:

$$
\|w\|^2_{s,e} = d(\epsilon)^2 \|w_1\|^2_s + \|w_2\|^2_s,
$$

for integers $s$ such that $s \geq s_0 + 1$ where $s_0 = 1$.

For the functional space $C([0,T], H^s(T^2, \mathbb{R}^3)) \cap C^1([0,T], H^{s-1}(T^2, \mathbb{R}^3))$, we consider the following norms:

$$
\|\|w\|\|_{s,T} = \sup_{t \in [0,T]} \|\|w(t)\|\|_e
$$

and

$$
\|\|w(t)\|\|_{s,e} = \|w(t)\|_{s,e} + \left| \frac{dw}{dt}(t) \right|_{s-1,e}.
$$

Since we have showed in the subsection 2.4 that the system (14) satisfies the Friedrich’s theory, we can introduce an important notion for our work: the structural conditions called the $\epsilon$-balanced property which we have to satisfy so that the method introduced in [9] should be valid.

**Definition 3.1.** The system (14) is said $\epsilon$-balanced around $v^0$, a given fixed vector, if the following structural conditions are satisfied:

There exists $\delta > 0$ independent of $\epsilon$ such that

$$
\theta_{1,s}(A^j, v^0, \delta), \theta_{1,s}(h, v^0, \delta), \theta_{1,s}(P, v^0, \delta) < \infty \text{ for } j = 1, 2 \text{ and } s \geq s_0 + 1, \quad \gamma_{1,1}(B^j, v^0, \delta) < \infty \text{ for } j = 0, 1, 2
$$

where

$$
\theta_{1,s}(\cdot, v^0, \delta) = \theta_{1,s}(\cdot)
$$

$$
\theta_{1,s}(\cdot, v^0, \delta) = \max_{\|p-v^0\|, \leq \delta} \sup_{1 \leq |s_1| + |s_2| \leq s} \left| \frac{D^{s_1} D^{s_2}}{D_{p_1} D_{p_2}} \cdot (p_1, p_2) \right|_e
$$

and

$$
\gamma_{1,s}(\cdot, v^0, \delta) = \gamma_{1,s}(\cdot)
$$

$$
\gamma_{1,s}(\cdot, v^0, \delta) = \max_{\|p-v^0\|, \leq \delta} \sup_{1 \leq |s_1| + |s_2| \leq s} \left| \frac{D^{s_1} D^{s_2}}{D_{p_1} D_{p_2}} \cdot (p_1, p_2) \right|_e
$$

Since $B^0(v^0, \epsilon)$ is symmetric definite positive, $(B^0(v^0, \epsilon)w, w)^1/2$ define a norm in $\mathbb{R}^3$ and by the equivalence of norm in finite dimension spaces, we get the relation:

$$
m \|w\|^2_e \leq (B^0(v^0, \epsilon)w, w) \leq M \|w\|^2_e,
$$

for a given $v^0$.

**Remark 2.** $B^0(v^0, \epsilon)$ is symmetric implies that $m$ and $M$ can be respectively its smallest and biggest eigenvalues and by its definite positivity, we get $m > 0$ and $M > 0$. 
The weighted matrix norm verifies:
\[\|Bw\|_\varepsilon \leq C \|B\|_\varepsilon \|w\|_\varepsilon,\]
(22)
\[|(Bw, w)| \leq C [B]_\varepsilon \|w\|_\varepsilon \|w\|_\varepsilon\]
(23)
where \((\ ,\ )\) is the usual inner product in \(\mathbb{R}^3\).

Before proceeding further we make the following assumptions as in [9].

We consider the initial value condition: \(v^\varepsilon(t = 0, x) = v_0(x) + \tilde{v}_0(x)\), with \(v_0 = (v_0^I, v_0^{II})\) such that:

\[
\begin{cases}
\ I. & v_0^I \text{ constant}, \|v_0^{II}\|_s + \left\| \sum_{j=1}^{2} A_j(v_0, \varepsilon) \frac{\partial v_0}{\partial x_j} \right\|_{s-1,\varepsilon} \leq K, \\
\ II. & \left\| \sum_{j=1}^{2} A_j(v_0, \varepsilon) \frac{\partial \tilde{v}_0}{\partial x_j} \right\|_{s-1,\varepsilon} \leq K, \\
\ III. & \|\tilde{v}_0\|_{s,\varepsilon} \leq \delta',
\end{cases}
\]

where \(\delta' > 0\) is considered small enough and to be chosen later.

The initial condition (CI2) implies that \(v_0\) satisfies the following:

\[
\begin{cases}
\ I. & v_0 \in H^s(T^2, \mathbb{R}^3), \ s \geq s_0 + 1 \\
\ II. & v_0^I \text{ constant}, \\
\ III. & \left\| \sum_{j=1}^{2} A_j(v_0, \varepsilon) \frac{\partial v_0}{\partial x_j} \right\|_{s-1,\varepsilon} \leq K.
\end{cases}
\]

**Remark 3.** \(\tilde{v}_0\) is a small perturbation of the initial data \(v_0\). And if the initial condition \(v_0\) is replaced by \(v_0 + \tilde{v}_0\), the main results in [9] work.

The first main result is the following:

**Theorem 3.2.** Let’s assume that \(v^\varepsilon(0, x)\) satisfies (CI2), \(z^\varepsilon(0, x) = z^0(x)\) such that \(z^0 \in H^1(T^2)\). Then there exists \(T > 0\) not depending on \(\varepsilon\) such that for any \(\varepsilon\) close to 0, the following Cauchy problem:

\[
\begin{cases}
\frac{\partial v^\varepsilon}{\partial t} + \frac{1}{\varepsilon} \frac{\partial}{\partial t} \sum_{j=1}^{2} A_j(v^\varepsilon, \varepsilon) \frac{\partial v^\varepsilon}{\partial x_j} = \frac{1}{\varepsilon} h(v^\varepsilon) - \frac{1}{\varepsilon^4} P(v^\varepsilon, z^\varepsilon), \\
\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2} \nabla \cdot \left[ A'(v^\varepsilon) \nabla z^\varepsilon \right] = \frac{1}{\varepsilon^2} \nabla \cdot C'(v^\varepsilon), \\
v^\varepsilon(0, x) = v_0(x) + \tilde{v}_0(x), \quad z^\varepsilon(0, x) = z^0(x)
\end{cases}
\]

(24)
has a unique solution \((\tilde{v}^\varepsilon, z^\varepsilon) \in C([0, T], H^3(T^2, \mathbb{R}^3)) \cap C^1([0, T], H^{s-1}(T^2, \mathbb{R}^3)) \times L^\infty([0, T), L^2(T^2))\).

And we have \(\sqrt{A'(v^\varepsilon)} \nabla z^\varepsilon \in L^2([0, T), L^2(T^2))\).

In addition the following estimates hold:

\[
\|z^\varepsilon\|_{L^\infty([0, T), L^2(T^2))} \leq \tilde{\gamma},
\]
\[
\left\| \sqrt{A'(v^\varepsilon)} \nabla z^\varepsilon \right\|_{L^2([0, T), L^2(T^2))}^2 \leq \tilde{\gamma}',
\]
\[
\|\tilde{v}^\varepsilon(t) - v_0\|_{s,\varepsilon} + \left\| \frac{d\tilde{v}^\varepsilon(t)}{dt} \right\|_{s-1,\varepsilon} \leq \Delta
\]
for any \( t \in [0, T] \), \( \Delta \) a constant independent of \( \epsilon \), and \( \gamma, \gamma' \) constants that depend on \( \epsilon \).

To prove of the theorem we use the method introduced in [9]. In fact we show that for a well chosen sequence \( v^{\epsilon,n-1} \) in a suitable set to be defined, the following system

\[
\begin{align*}
\frac{\partial v^{\epsilon,n}}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^{2} A^j(v^{\epsilon,n-1}) \frac{\partial v^{\epsilon,n}}{\partial x_j} &= \frac{1}{\epsilon} h(v^{\epsilon,n-1}) - \frac{1}{\epsilon^4} P(v^{\epsilon,n-1}, z^{\epsilon,n}) \\
\frac{\partial z^{\epsilon,n}}{\partial t} - \frac{1}{\epsilon^2} \nabla \cdot [A^e(v^{\epsilon,n-1}) \nabla z^{\epsilon,n}] &= \frac{1}{\epsilon^2} \nabla \cdot C^e(v^{\epsilon,n-1})
\end{align*}
\]

has a solution \( (v^{\epsilon,n}, z^{\epsilon,n}) \). Let us underline that we shall see that \( v^{\epsilon,n} \) and \( v^{\epsilon,n-1} \) are in the same set. And finally, thanks to a contraction map which we get by construction, we conclude that the solution converges to a solution \( (v^{\epsilon,\infty}, z^{\epsilon,\infty}) \) of our problem. That means:

\[
\begin{align*}
\frac{\partial v^{\epsilon,\infty}}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^{2} A^j(v^{\epsilon,\infty}) \frac{\partial v^{\epsilon,\infty}}{\partial x_j} &= \frac{1}{\epsilon} h(v^{\epsilon,\infty}) - \frac{1}{\epsilon^4} P(v^{\epsilon,\infty}, z^{\epsilon,\infty}) \\
\frac{\partial z^{\epsilon,\infty}}{\partial t} - \frac{1}{\epsilon^2} \nabla \cdot [A^e(v^{\epsilon,\infty}) \nabla z^{\epsilon,\infty}] &= \frac{1}{\epsilon^2} \nabla \cdot C^e(v^{\epsilon,\infty})
\end{align*}
\]

The proof of the theorem 3.2 is long and it shall be presented in the next section.

4. **Proof of theorem 3.2.** We start the proof of theorem 3.2 by stating and showing several results as propositions.

Let us consider the equation (14) and its symmetrized form (16), with initial condition \( v^e(t = 0, x) = v_0(x) + \tilde{v}_0(x) \).

Let's give now the definition of the functional framework in which we are going to look for solution. And in this functions spaces we shall apply a fixed point argument.

Let \( B^e_T(v_0) = B^e_T(v_0, \delta, \Delta) \) be a subset of \( C([0, T], H^s(T^2, \mathbb{R}^3)) \cap C^1([0, T], H^{s-1}(T^2, \mathbb{R}^3)) \) with \( s = s_0 + 1 = 2 \), such that

\[
\begin{align*}
\|v - v_0\|_s &\leq \delta \\
\|v - \tilde{v}_0\|_{s,T} &\leq \Delta, \quad \Delta \geq \delta > 0,
\end{align*}
\]

with \( \tilde{v}_0^l = v_0^l \) and \( \tilde{v}_0^{ll} = 0 \).

For \( v \in B^e_T(v_0, \delta, \Delta) \), let’s set \( \Phi(v) = w \) and consider the following linear Cauchy problem:

\[
\begin{align*}
\frac{\partial w}{\partial t} + \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial w}{\partial x_j} &= h(v, \epsilon) - P(v, z, \epsilon) \\
\frac{\partial z}{\partial t} - \nabla \cdot [A^e(v, \epsilon) \nabla z] &= \nabla \cdot C^e(v, \epsilon)
\end{align*}
\]

where \( A^j(v, \epsilon) = \frac{1}{\epsilon^4} A^j(v), h(v, \epsilon) = \frac{1}{\epsilon} h(v), P(v, z, \epsilon) = \frac{1}{\epsilon^4} P(v, z), A^e(v, \epsilon) = \frac{1}{\epsilon^2} A^e(v), C^e(v, \epsilon) = \frac{1}{\epsilon^2} C^e(v) \) and \( z_0 \in H^1(T^2) \).
We first use the second equation of (28) and give existence result and estimates.

**A) Existence of \( z^\epsilon \)**

By Ladyzenskaja, Solonnikov and Ural’ Ceva \([10]\) or Lions \([11]\) we have the existence and uniqueness of solution for the second equation of (28) on a time interval depending on \( \epsilon \). And in Faye et al \([6]\) authors have proved existence of solution on a time interval not depending of \( \epsilon \) for a type of solution of \( v^\epsilon \).

Multiplying the second equation of (28) by \( z^\epsilon \) and integrating by parts over the torus \( T^2 \), we get:

\[
\frac{1}{2} \frac{d}{dt} \| z^\epsilon \|_{L^2(T^2)}^2 + \frac{1}{\epsilon^2} \int_{T^2} A^\epsilon(v)(\nabla z^\epsilon)^2 = \frac{1}{\epsilon^2} \int_{T^2} z^\epsilon \nabla \cdot C^\epsilon(v). \tag{29}
\]

For any \( \epsilon > 0 \) small enough, we have \( A^\epsilon(v^\epsilon) \geq 0 \) and then \( \frac{1}{\epsilon^2} \int_{T^2} A^\epsilon(v)(\nabla z^\epsilon)^2 \geq 0 \).

In addition, using Cauchy-Schwarz in the second member of (29) and the fact that \( \nabla \cdot C^\epsilon(v) \) is bounded (see proposition 5 in appendix for details), we get:

\[
\frac{1}{2} \frac{d}{dt} \| z^\epsilon \|_{L^2(T^2)}^2 \leq C(\epsilon) \| z^\epsilon \|_{L^2(T^2)} \tag{30}
\]

where \( C(\epsilon) \) is a constant depending on \( \epsilon \).

\[
\| z^\epsilon(s) \|_{L^2(T^2)}^2 - \| z_0 \|_{L^2(T^2)}^2 \leq 2C(\epsilon) \int_s^T \| z^\epsilon(t) \|_{L^2(T^2)} dt, \quad \forall s \in [0, T]
\]

\[
\max_{s \in [0, T]} \| z^\epsilon(s) \|_{L^2(T^2)} \leq 2C(\epsilon) T \max_{t \in [0, T]} \| z^\epsilon(t) \|_{L^2(T^2)} + \| z_0 \|_{L^2(T^2)}^2
\]

\[
\| z^\epsilon \|_{L^\infty([0, T], L^2(T^2))}^2 \leq 2C(\epsilon) T \| z_0 \|_{L^2(T^2)}^2 + \| z_0 \|_{L^2(T^2)}^2
\]

We get \( \| z^\epsilon \|_{L^\infty([0, T], L^2(T^2))} \leq C(\epsilon) T + (C(\epsilon)^2 T^2 + \| z_0 \|_{L^2(T^2)}^2)^{1/2} \) and we can set \( \gamma = C(\epsilon) T + (C(\epsilon)^2 T^2 + \| z_0 \|_{L^2(T^2)}^2)^{1/2} \) and \( \gamma \) depends on \( \epsilon \).

Using (29) and (30) we get:

\[
\int_{T^2} A^\epsilon(v)(\nabla z^\epsilon)^2 = \int_{T^2} z^\epsilon \nabla \cdot C^\epsilon(v) - \frac{\epsilon^2}{2} \frac{d}{dt} \| z^\epsilon \|_{L^2(T^2)}^2
\]

\[
\left\| \sqrt{A^\epsilon(v)} \| \nabla z^\epsilon \| \right\|_{L^2(T^2)}^2 \leq C \| z^\epsilon(t) \|_{L^2(T^2)} + C \| z^\epsilon(t) \|_{L^2(T^2)}
\]

\[
\left\| \sqrt{A^\epsilon(v)} \| \nabla z^\epsilon \| \right\|_{L^2([0, T], L^2(T^2))}^2 \leq CT \| z^\epsilon \|_{L^\infty([0, T], L^2(T^2))} \leq CT \gamma
\]

We have just shown that for any \( v \in B^\gamma_T(v_0) \), the second equation of the system (28) admits a solution \( z^\epsilon \) and even it is unique. And it remains to show that all the system (28) admits a solution. For this we are going to build a sufficient framework to get useful results and apply the classical fixed point theorem for the existence and uniqueness results stated in theorem 3.2. As already mentioned above, we endeavor to use the method introduced by S. Klainerman and A. Majda (see[9]). But since we do not get the same equation as in [9], it is impossible to apply directly the main existence result. In fact we need to verify the \( \epsilon \)-balanced property which is a crucial step for our proof. Before going on let’s point out that in the right hand the first side of equation of system (28) the term \( h(v^\epsilon, \epsilon) - P(v^\epsilon, z^\epsilon, \epsilon) \) does not appear in the case considered by S. Klainerman and A. Majda. And to end the proof it shall be important to show existence of appropriate constants \( T, \Delta \) and \( \delta \) not depending on \( \epsilon \) such that \( \Phi \) maps \( B^\gamma_T(v_0) \) into itself and moreover if \( \Phi \) is a contraction with the norm \( \| \cdot \|_{0, \epsilon} \).
Hence the proof of the theorem can be decomposed in three main steps:

- \( \epsilon \)-balanced
- construction of the set \( B^\epsilon_T(v^0, \delta, \Delta) \)
- \( \Phi \) is a contraction

B) \( \epsilon \)-balanced

We give at first the following result which is a crucial step of the proof of theorem 3.2:

**Proposition 1.** Let \( v^0 \) be the initial condition satisfying (CI1). Then the system (14) is \( \epsilon \)-balanced around \( v^0 \).

**Proof of the proposition 1.** At first we recall that \( A^j(v, \epsilon) = \frac{1}{\epsilon^4} A^j(v) \), \( \forall j = 1, 2 \), if \( B^j(v, \epsilon) = \frac{1}{\epsilon^4} B^j(v) \), \( \forall j = 1, 2 \), \( B^0(v, \epsilon) = B^0(v) \), \( h(v, \epsilon) = \frac{1}{\epsilon} h(v) \) and \( P(v, z, \epsilon) = \frac{1}{\epsilon^4} P(v, z) \).

Let’s set \( d(\epsilon) = \sqrt{\frac{c^T}{\epsilon^4 d^*(\epsilon)}} \) with \( d^*(\epsilon) \xrightarrow[\epsilon \to 0]{} 0 \), chosen conveniently such that \( \lim_{\epsilon \to 0} d^*(\epsilon) \| D^\alpha_\epsilon P(v^\epsilon, z^\epsilon) \| = 0 \), \( |\alpha| \geq 0 \).

Let’s begin with the matrix coefficient \( A^1 \), given by:

\[
A^1 = \begin{pmatrix}
0 & a^1 \\
-a^1 q_1^2 (m^\epsilon + b^1)^2 + c^1 (m^\epsilon + b^1) & 0
\end{pmatrix}
\]

For \( \alpha = (s_1, s_2) \), \( s_1 \in \mathbb{N} \) and \( s_2 \in \mathbb{N}^2 \), with \( 1 \leq |\alpha| \leq s \), we have:

\[
\| D^\alpha A^1(v, \epsilon) \|_\epsilon = \| D^\alpha A^1_{11} \| + d(\epsilon) \| D^\alpha A^1_{12} \| + d(\epsilon)^{-1} \| D^\alpha A^1_{21} \| + \| D^\alpha A^1_{22} \|.
\]

Due to the fact that all coefficients of the sub-matrix \( A^1_{11} \), \( A^1_{12} \) are constant or null, we have:

\[
\| D^\alpha A^1_{11} \| = 0, \quad \| D^\alpha A^1_{12} \| = 0
\]

and

\[
\| D^\alpha A^1(v, \epsilon) \|_\epsilon = d(\epsilon)^{-1} \| D^\alpha A^1_{21} \| + \| D^\alpha A^1_{22} \| \leq (d(\epsilon)^{-1} + 1) \| D^\alpha A^1(v, \epsilon) \|.
\]

Hence

\[
\| D^\alpha A^1(v, \epsilon) \|_\epsilon \leq (d(\epsilon)^{-1} + 1) \frac{1}{\epsilon^4} \| D^\alpha A^1(v) \|
\]

Now, taking the max and multiplying by \( d(\epsilon)^{-|s_1|} \) and replacing \( \alpha \) by \((s_1, s_2)\) we have:

\[
\theta_{1.s}(A^1, v^0, \delta) \leq \max_{\epsilon \to 0} \sum_{|s_1|+|s_2| \leq s} \frac{d(\epsilon)^{-|s_1|} d(\epsilon)^{-1} + 1}{\epsilon^4} \frac{1}{\epsilon^4} \max_{\|v-v^0\| \leq \delta} \| D^\alpha_\epsilon D^\beta_{\epsilon} A^1(v) \|
\]

By the proposition (5) which is at the appendix section, we have:

\[
\max_{\|v-v^0\| \leq \delta} \| D^\alpha_\epsilon D^\beta_{\epsilon} A^1(v) \| < \infty.
\]
Since $|s_1| \geq 1$ we have, \[\lim_{\epsilon \to 0} \frac{d(\epsilon)^{-(|s_1|+1)}}{\epsilon^4} + \frac{d(\epsilon)^{-|s_1|}}{\epsilon^4} < \infty.\]

Then $\theta_1,s(A^1, v^0, \delta) < \infty$.

The same method used for the matrix $A^1$ can be done for $A^2(v, \epsilon) = \frac{1}{\epsilon^3} A^2(v)$

\[
A^2(v) = \begin{pmatrix}
0 & 0 & a^2 \\
\frac{-a^1 q_1 q_2}{(m^* + b^1)^2} & \frac{a^1 q_2}{(m^* + b^1)} & \frac{a^1 q_1}{(m^* + b^1)} \\
\frac{-a^1 q_2^2}{(m^* + b^1)^2} + c^1 (m^* + b^1) & 0 & \frac{2a^1 q_2}{(m^* + b^1)}
\end{pmatrix} = \begin{pmatrix} A^2_{21} & A^2_{22} \end{pmatrix}.
\]

And for the same reason we conclude that $\theta_1,s(A^1, v^0, \delta) < \infty$.

To show that $\theta_1,s(h, v^0, \delta) < \infty$, let write at first:

\[
h(v) = \begin{pmatrix} 0 \\
\frac{-kq_1}{m^* + b^1} +fq_2 \\
\frac{-kq_2}{m^* + b^1} -fq_1
\end{pmatrix} = \begin{pmatrix} h^I \\
h^{II}
\end{pmatrix}
\]

$\forall \alpha = (s_1, s_2), s_1 \in \mathbb{N}$ and $s_2 \in \mathbb{N}^2$, with $1 \leq |\alpha| \leq s$. And it is easy to see :

\[
\|D^\alpha h(v, \epsilon)\|_\epsilon^2 = d(\epsilon)^2 \|D^\alpha h^I\|^2 + \|D^\alpha h^{II}\|^2.
\]

Since $h^I = 0$ we have :

\[
\|D^\alpha h(v, \epsilon)\|_\epsilon = \frac{1}{\epsilon} \|D^\alpha h^{II}(v)\|
\]

and

\[
\theta_1,s(h, v^0, \delta) = \max_{\epsilon \to 0} \sum_{1 \leq |s_1|+|s_2| \leq s} \frac{d(\epsilon)^{-|s_1|}}{\epsilon} \max_{\|\varepsilon-v^0\|, \leq \delta} \|D^\alpha_{\varepsilon, 1} D^\alpha_{\varepsilon, 2} h^{II}(v)\|)
\]

As previously, since $|s_1| \geq 1$ and $\max_{\|\varepsilon-v^0\|, \leq \delta} \|D^\alpha_{\varepsilon, 1} D^\alpha_{\varepsilon, 2} h^{II}(v)\| < \infty$, by the Proposition(5), we have $\theta_1,s(h, v^0, \delta) < \infty$.

The proof of $\theta_1,s(P, v^0, \delta) < \infty$ is similar than the previous ones. $\forall \alpha = (s_1, s_2), s_1 \in \mathbb{N}$ and $s_2 \in \mathbb{N}^2$, with $1 \leq |\alpha| \leq s$, we have :

\[
\|D^\alpha P(v, z, \varepsilon)\|_\epsilon^2 = d(\epsilon)^2 \|D^\alpha P^I\|^2 + \|D^\alpha P^{II}\|^2.
\]

Let’s remarks that $P^I = 0$ and

\[
\|D^\alpha P(v, z, \varepsilon)\|_\epsilon = \frac{1}{\epsilon^3} \|D^\alpha P^{II}(v)\|
\]

\[
\theta_1,s(P, v^0, \delta) = \max_{\epsilon \to 0} \sum_{1 \leq |s_1|+|s_2| \leq s} \frac{d(\epsilon)^{-|s_1|}}{\epsilon^4} \max_{\|\varepsilon-v^0\|, \leq \delta} \|D^\alpha_{\varepsilon, 1} D^\alpha_{\varepsilon, 2} P^{II}(v)\|
\]

And finally since $|s_1| \geq 1$ and $\lim_{\epsilon \to 0} d^*(\epsilon) \|D^\alpha_{\varepsilon, 1} D^\alpha_{\varepsilon, 2} P(v^*, z^*)\| = 0$, we have $\theta_1,s(P, v^0, \delta) < \infty$. 
Let’s prove that $\gamma_{1,s}(B^j, v^0, \delta) < \infty$ for $j = 0, 1, 2$, that is more general than $\gamma_{1,1}(B^j, v^0, \delta) < \infty$, $j = 0, 1, 2$.

\[ B^0 = \begin{pmatrix} \frac{1}{a^1} \frac{a^1 q^2}{(m^\epsilon + b^\epsilon)^2} + c^1 (m^\epsilon + b^\epsilon) & -\frac{q_1}{(m^\epsilon + b^\epsilon)} & -\frac{q_2}{(m^\epsilon + b^\epsilon)} \\ \frac{q_1}{(m^\epsilon + b^\epsilon)} & 1 & 0 \\ \frac{q_2}{(m^\epsilon + b^\epsilon)} & 0 & 1 \end{pmatrix} \]

$\gamma_{1,s}(B^0, v^0, \delta) = \max_{\varepsilon \to 0^+} \sum_{1 \leq |s_1| + |s_2| \leq s} d(\varepsilon)^{-|s_1|} \max_{\|v - v^0\| \leq \delta} [D^{s_1}_{u^1} D^{s_2}_{u^2} B^0(v)]$,

with

\[ [D^\alpha B^0(v, \varepsilon)]_s = d(\varepsilon)^{-2} |D^\alpha B^0_{11}| + d(\varepsilon)^{-1} (|D^\alpha B^0_{12}| + |D^\alpha B^0_{21}|) + |D^\alpha B^0_{22}|. \]

Then having in mind that $B^0(v, \varepsilon) = B^0(v)$, we have

$\gamma_{1,s}(B^0, v^0, \delta) \leq \max_{\varepsilon \to 0^+} \sum_{1 \leq |s_1| + |s_2| \leq s} d(\varepsilon)^{-|s_1|} (d(\varepsilon)^{-1} + d(\varepsilon)^{-1} + 1) \max_{\|v - v^0\| \leq \delta} [D^{s_1}_{u^1} D^{s_2}_{u^2} B^0(v)]$,

what is equivalent to

$\gamma_{1,s}(B^0, v^0, \delta) \leq \max_{\varepsilon \to 0^+} \sum_{1 \leq |s_1| + |s_2| \leq s} d(\varepsilon)^{-|s_1| + 2} + d(\varepsilon)^{-|s_1| + 1} + d(\varepsilon)^{-|s_1|} \max_{\|v - v^0\| \leq \delta} [D^{s_1}_{u^1} D^{s_2}_{u^2} B^0(v)]$.

By the proposition (5), \( \max_{\|v - v^0\| \leq \delta} \|D^{s_1}_{u^1} D^{s_2}_{u^2} B^0(v)\| < \infty \), and for \( |s_1| \geq 1 \), we have:

\[
\lim_{\varepsilon \to 0^+} \left( \sum_{1 \leq |s_1| + |s_2| \leq s} d(\varepsilon)^{-|s_1| + 2} + d(\varepsilon)^{-|s_1| + 1} + d(\varepsilon)^{-|s_1|} \right) < \infty.
\]

That means $\gamma_{1,s}(B^0, v^0, \delta) < \infty$.

The same techniques work for $B^1$ and $B^2$ which are similar to $B^0$. Hence we have $\gamma_{1,s}(B^1, v^0, \delta) < \infty$ and $\gamma_{1,s}(B^2, v^0, \delta) < \infty$. \( \square \)

**C) Construction of the set $B^*_s(v^0, \delta, \Delta)$**

Having at our disposal the proposition 1, we can deduce useful remarks. And after we show other preliminary results leading us in the way to built the functions space $B^*_s(v^0, \delta, \Delta)$.

**Remark 4.** Since the systems (14) and (16) are $\varepsilon$-balanced around $v^0$ we can say as in [9] that there exits a constant $\bar{K}$ such that for $\delta > 0$ and $s \geq s_0 + 1$,

\[
\left\{ \begin{array}{l}
\theta_{1,s}(A^1, v^0, \delta) \leq \bar{K}, \quad \theta_{1,s}(h, v^0, \delta) \leq \bar{K}, \quad \theta_{1,s}(P, v^0, \delta) \leq \bar{K}, \\
\gamma_{1,t}(B^j, v^0, \delta) \leq \bar{K} \end{array} \right.
\]

(31)

Let’s $\phi$ be a function defined by $\phi(t, x) = w(t, x) - \hat{v}_0(x) \Rightarrow w(t, x) = \phi(t, x) + \hat{v}_0(x)$.

Then, we have:

\[
\frac{\partial \phi}{\partial t} + \frac{\partial \hat{v}_0}{\partial t} + \sum_{j=1}^2 A^j(v, \varepsilon) \cdot \frac{\partial \phi}{\partial x_j} + \sum_{j=1}^2 A^j(v, \varepsilon) \cdot \frac{\partial \hat{v}_0}{\partial x_j} = h(v, \varepsilon) - P(v, z, \varepsilon)
\]
Then, let’s write:
\[
\frac{\partial \phi}{\partial t} + \frac{\partial \hat{v}_0}{\partial t} + \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial \phi}{\partial x_j} = h(v, \epsilon) - P(v, z, \epsilon) - \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial \hat{v}_0}{\partial x_j}
\]
And finally the function \( \phi \) satisfies the below Cauchy problem:
\[
\begin{cases}
\frac{\partial \phi}{\partial t} + \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial \phi}{\partial x_j} = h(v, \epsilon) - P(v, z, \epsilon) + F(v, \epsilon) \\
\phi(0, x) = v_0(x) - \hat{v}_0(x) + \tilde{v}_0(x)
\end{cases}
\]
with \( F(v, \epsilon) = - \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial \hat{v}_0}{\partial x_j} \).

Since \( \hat{v}_0 = v_0 \) constant and \( \hat{v}_0 = 0 \) then \( \frac{\partial \hat{v}_0}{\partial x_j} = 0 \)
\[\Rightarrow \frac{\partial \phi}{\partial t} + \sum_{j=1}^{2} A^j(v, \epsilon) \cdot \frac{\partial \phi}{\partial x_j} = h(v, \epsilon) - P(v, z, \epsilon)\]

Let:
\[\|\phi(t)\|_E^2 = \int_{\mathbb{T}^2} (B^0(v, \epsilon) \phi, \phi) \, dx \]
be an energy that is a norm.

Then multiplying the equation (32) by \( B^0 \) and by \( 2 \phi \) and integrating it with respect to \( x \in \mathbb{T}^2 \), we get:
\[
\frac{d}{dt} \|\phi\|_E^2 = \int_{\mathbb{T}^2} (\Gamma \phi, \phi) \, dx + 2 \int_{\mathbb{T}^2} (B^0 \cdot (h - P + F), \phi) \, dx
\]
where \( \Gamma = \frac{dB^0}{dt} + \sum_{j=1}^{2} \frac{dB^j}{dx_j} \).

Differentiating (32) with respect to \( (t, x) \) we have:
\[
D^\alpha \frac{\partial \phi}{\partial t} + \sum_{j=1}^{2} A^j D^\alpha \frac{\partial \phi}{\partial x_j} = H(\alpha)
\]
with \( H(\alpha) = D^\alpha h - D^\alpha P - G(\alpha) \), and \( G(\alpha) = \sum_{j=1}^{2} D^\alpha A^j \sum_{j=1}^{2} D^\alpha (A^j \frac{\partial \phi}{\partial x_j}) - A^j D^\alpha \frac{\partial \phi}{\partial x_j} \).

Here \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \) is multi-index with \( |\alpha| \leq N, \alpha_0 \leq 1 \). In what follows, we only consider \( N \) such that \( s_0 \leq N \leq s_0 + 1 \).

**Remark 5.** From (21), (31) we deduce that: \( \exists \delta > 0 \) small enough such that for any \( w \in \mathbb{R}^3 \):

- \( \frac{1}{2} m \|w\|_\epsilon^2 \leq (B^0(v, \epsilon)w, w) \leq 2M \|w\|_\epsilon^2 \)
- \( |B^0(v, \epsilon)|_\epsilon \leq CM \)

wherever \( v \) satisfies \( \|v - v^0\|_\epsilon \leq \delta \) (see[9] for more details).
An immediate corollary from this remark is: \( \forall \alpha = (\alpha_0, \alpha_1, \alpha_2), \ |\alpha| \leq s, \alpha_0 \leq 1 \) and \( t \in [0,T] \) we have:

\[
\frac{1}{C} m \|\phi(t)\|_\epsilon \leq \|D^\alpha \phi(t)\|_E \leq CM \|\phi(t)\|_\epsilon. \quad (35)
\]

Multiplying (32) by \( 2D^\alpha \phi \) and integrating it by parts, we have:

\[
\frac{d}{dt} \|D^\alpha \phi\|_E^2 \leq \sup_{x \in \Omega} \left[ \Gamma(v,\epsilon) \right] \cdot \|\phi(t)\|_\epsilon^2 + C \|\phi(t)\|_\epsilon \cdot \|H_\alpha(t)\|_{0,\epsilon} \quad (36)
\]

And in addition, the following estimation holds:

\[
\sup_{x \in \Omega} \left[ \Gamma(v,\epsilon) \right] \leq C\bar{K}\Delta, \ \forall t \in [0,T] \quad (37)
\]

By the lemmas (A.2), (A.4) (resp. p518, [9]) and corollary (3) (p519) in [9], we have the following estimation:

\[
\|G_\alpha(t)\|_{0,\epsilon} \leq C\bar{K}\Delta^s \|\phi(t)\|_\epsilon, \ \forall t \in [0,T] \quad (38)
\]

**Proposition 2.**

\[
\|D^\alpha h(v)\|_{0,\epsilon} \leq C\bar{K}\Delta^s \quad (39)
\]

\[
\|D^\alpha P(v, z, \epsilon)\|_{0,\epsilon} \leq C\bar{K}\Delta^s \quad (40)
\]

**Proof of proposition 2.** For \( |\alpha| \leq s \), we have:

\[
\|D^\alpha h(v)\|_{0,\epsilon}^2 = d(\epsilon)^2 \|D^\alpha h(v)\|_0^2 + \|D^\alpha h^I(v)\|_0^2.
\]

This implies

\[
\|D^\alpha h(v)\|_{0,\epsilon} \leq \|D^\alpha h(v)\|_{s-1,\epsilon}
\]

Applying the lemma(A.3) (p520, [9]) and corollary(4) (p522, [9]) to \( h(v,\epsilon) \) we have:

\[
\|D^\alpha h(v)\|_{0,\epsilon} \leq C\theta_1.s(h,v^0,\delta)\Delta^s.
\]

And since \( \theta_1.s(h,v^0,\delta) \leq \bar{K} \) we have:

\[
\|D^\alpha h(v)\|_{0,\epsilon} \leq C\bar{K}\Delta^s.
\]

The proof of the second estimation of \( \|D^\alpha P(v, z, \epsilon)\|_{0,\epsilon} \) is similar to the first one, since \( \theta_1.s(P,v^0,\delta) \leq \bar{K} \). \( \square \)

**Proposition 3.**

\[
\|H_\alpha(t)\|_{0,\epsilon} \leq C\bar{K}\Delta^s(1 + \|\phi(t)\|_\epsilon), \ \forall t \in [0,T]
\]

**Proof of proposition 3.** The proof is easy by using 38 and the proposition 2. Since \( \|H_\alpha(t)\|_{0,\epsilon} = \|D^\alpha h - D^\alpha P - G_\alpha\|_{0,\epsilon} \), we apply estimation of each term. \( \square \)

So, using estimation (37) and proposition(3) in (36) we have:

\[
\frac{d}{dt} \|D^\alpha \phi\|_E^2 \leq C\bar{K}\Delta^s(1 + \|\phi(t)\|_\epsilon) \|\phi(t)\|_\epsilon, \quad (\bar{C}T + \|\phi(0)\|_\epsilon) \exp\bar{C}T,
\]

Applying (35) with the norm \( \|\phi\|_E \) that is equivalent to \( \|\phi\|_\epsilon \), and the Gronwall’s inequality we have:

\[
\|\phi\|_\epsilon \leq (\bar{C}T + \|\phi(0)\|_\epsilon) \exp\bar{C}T. \quad (41)
\]
where \( \bar{C} = C\tilde{K}\Delta^s \). Hence
\[
\|\phi\|_{s,\epsilon} \leq (\bar{C}T + \|\phi(0)\|_{s,\epsilon} + \|\partial_t \phi(0)\|_{s-1,\epsilon}) \exp \bar{C}T. \tag{42}
\]
Knowing that \( \phi(0, x) = v^0(x) - \bar{v}^0(x) + \bar{v}^0(x) \) and \( \partial_t \phi(0, x) = -\sum_{j=1}^2 A^j(v^0) \frac{\partial}{\partial x_j} (v^0 + \bar{v}^0) \) and using (C12) we have:
\[
\|\phi(0)\|_{s,\epsilon} + \|\partial_t \phi(0)\|_{s-1,\epsilon} \leq \|v^0\|_{s,\epsilon} + \|\bar{v}^0\|_{s,\epsilon} + \left\| \sum_{j=1}^2 A^j(v^0) \frac{\partial}{\partial x_j} (v^0 + \bar{v}^0) \right\|_{s-1,\epsilon}.
\]
Therefore
\[
\|\phi(0)\|_{s,\epsilon} + \|\partial_t \phi(0)\|_{s-1,\epsilon} \leq CK.
\]
Now, replacing \( \phi \) by \( w - \bar{v}^0 \) we have:
\[
\|w - \bar{v}^0\|_{\epsilon} \leq (\bar{C}T + CK) \exp \bar{C}T.
\]
And if
\[
(\bar{C}T + CK) \exp \bar{C}T \leq \Delta,
\]
then
\[
\|w - \bar{v}^0\|_{\epsilon} \leq \Delta.
\]
Hence to have \( \|w - \bar{v}^0\|_{\epsilon} \leq \Delta \), it suffices to choose \( \Delta > \bar{C}T + CK \) and \( \bar{C}T \) small enough.

Now it remains to prove that \( \|w - \bar{v}^0\|_{\epsilon} \leq \delta \) to get \( w \in B^\delta_T(v^0) \). By Sobolev’s inequality it suffices to show that \( \|w(t) - v^0\|_{s_0,\epsilon} \leq \delta \) for \( t \in [0,T] \).

For this, let’s set \( \tilde{\phi} = w - v^0 \), then \( \tilde{\phi} \) satisfies:
\[
\frac{\partial \tilde{\phi}}{\partial t} + \sum_{j=1}^2 A^j(v, \epsilon) \cdot \frac{\partial \tilde{\phi}}{\partial x_j} = h(v, \epsilon) - P(v, z, \epsilon) + F(v, \epsilon) \tag{44}
\]
where \( F(v, \epsilon) = -\sum_{j=1}^2 A^j(v, \epsilon) \cdot \frac{\partial v^0}{\partial x_j} \) and \( \tilde{\phi}(0, x) = \bar{v}^0(x) \).

Differentiating (44) \( s_0 \)-times with respect to \( x \), we have:
\[
\frac{\partial D^\beta \tilde{\phi}}{\partial t} + \sum_{j=1}^2 A^j(v, \epsilon) \cdot \frac{\partial D^\beta \tilde{\phi}}{\partial x_j} = D^\beta h(v, \epsilon) - D^\beta P(v, z, \epsilon) - G(\beta) + D^\beta F(v, \epsilon), \tag{45}
\]
where \( \beta = (\beta_1, \beta_2), |\beta| \leq s_0 \).

\[
G(\beta) = \sum_{j=1}^2 D^\beta A^j \frac{\partial \tilde{\phi}}{\partial x_j} = \sum_{j=1}^2 D^\beta(A^j \frac{\partial \tilde{\phi}}{\partial x_j} - A^j D^\beta \frac{\partial \tilde{\phi}}{\partial x_j}) = [D^\beta, A^j] \frac{\partial \tilde{\phi}}{\partial x_j}.
\]

Thanks to the estimation (37) we get:
\[
\frac{d}{dt} \| D^\beta \tilde{\phi} \|_E^2 \leq C\tilde{K}\Delta \cdot \left( \| \tilde{\phi}(t) \|_{s_0,\epsilon}^2 + C \| \tilde{\phi}(t) \|_{s_0,\epsilon} (\|H_\beta(t)\|_{0,\epsilon} + \|D^\beta F(t)\|_{0,\epsilon}) \right) \tag{46}
\]
And from proposition (3) we have:
\[
\|H_\beta(t)\|_{0,\epsilon} \leq C\tilde{K}\Delta^s \cdot (1 + \|\tilde{\phi}(t)\|_{\epsilon}).
\]
Reminding that \( \tilde{\phi} = w - v^0 \), we have:
\[
\|\tilde{\phi}(t)\|_{\epsilon} \leq \|w - \bar{v}^0\|_{\epsilon,T} + \|v^0_{1,t}\| \leq \Delta + K,
\]
with \( \Delta \geq K \). That implies:
\[
\|H_\beta(t)\|_{0,\epsilon} \leq C\tilde{K}\Delta^s(1 + \Delta + K) \leq C\tilde{K}\Delta^{s+1} \tag{47}
\]

\[
\|D^\beta \tilde{\phi} \|_E \leq C\tilde{K}\Delta \cdot \left( \| \tilde{\phi}(t) \|_{s_0,\epsilon}^2 + C \| \tilde{\phi}(t) \|_{s_0,\epsilon} \|H_\beta(t)\|_{0,\epsilon} + \|D^\beta F(t)\|_{0,\epsilon} \right)
\]

\[
\|\tilde{\phi}(t)\|_{\epsilon} \leq \|w - \bar{v}^0\|_{\epsilon,T} + \|v^0_{1,t}\| \leq \Delta + K,
\]
with \( \Delta \geq K \). That implies:
\[
\|H_\beta(t)\|_{0,\epsilon} \leq C\tilde{K}\Delta^s(1 + \Delta + K) \leq C\tilde{K}\Delta^{s+1}.
\]
Let’s use the lemma (A.2)(p.518, in [9]), the initialization conditions (CI2) and the lemma(4.5)(p.507, in [9]), to get:

\[ ||D^\beta F(t)||_{0,\epsilon} \leq C\bar{K}\Delta^{s+2} \]  \hspace{1cm} (48)

Then using (47) and (48) in (46), we have:

\[ \frac{d}{dt} ||D^\beta \phi||_E \leq C\bar{K}\Delta^{s+2} \cdot ||\hat{\phi}(t)||_{s-1,\epsilon} \cdot (1 + ||\hat{\phi}(t)||_{s-1,\epsilon}) \]

Thanks to (35) and the Gronwall’s inequality, we have:

\[ ||\hat{\phi}(t)||_{s-1,\epsilon} \leq (\hat{\cal C}T + ||\hat{\phi}(0)||_{s-1,\epsilon}) \exp \hat{\cal C}T, \]  \hspace{1cm} (49)

where \( \hat{\cal C} = C\bar{K}\Delta^{s+2} \cdot ||\hat{\phi}(0)||_{s-1,\epsilon} = ||\hat{\varphi}_\epsilon^0||_{s-1,\epsilon} \leq ||\varphi_\epsilon^0||_{s,\epsilon} \leq \delta' \Rightarrow ||\hat{\phi}(0)||_{s-1,\epsilon} \leq \delta'. \)

And with \( \hat{\phi}(t,x) = w(t,x) - v^0 \), we have:

\[ ||w(t) - v^0||_{s-1,\epsilon} \leq (\hat{\cal C}T + \delta') \exp \hat{\cal C}T, \]  \hspace{1cm} (50)

By Sobolev’s inequality, there is \( L > 0 \) such that

\[ ||w(t,x) - v^0||_\epsilon \leq L ||w(t) - v^0||_{s-1,\epsilon}. \]

and using (50) we have:

\[ ||w(t,x) - v^0||_\epsilon \leq L(\hat{\cal C}T + \delta') \exp \hat{\cal C}T. \]

if we choose \( T \) sufficiently small and \( \delta' \leq \delta/L \) we can get

\[ (\hat{\cal C}T + \delta') \exp \hat{\cal C}T \leq \delta/L, \]  \hspace{1cm} (51)

hence \( L(\hat{\cal C}T + \delta') \exp \hat{\cal C}T \leq \delta. \) Therefore, the following inequality is obtained

\[ ||w(t,x) - v^0||_\epsilon \leq \delta. \]

So we can conclude that for \( T \) chosen sufficiently small, depending only on \( K, \bar{K}, m \) and \( M \), such that (43) and (51) are satisfied with \( \Delta > K \) and the functional \( \Phi \) maps the set \( B^T_T(v^0, \delta, \Delta) \) into itself.

**D) \( \Phi \) is a contraction :**

Now, we aim to prove that \( \Phi \) is a contraction in \( B^T_T(v^0, \delta, \Delta) \) with respect to the norm \( ||.||_{0,\epsilon} \).

**Proposition 4.** By choosing \( T \) sufficiently small, depending only on \( K, \bar{K}, m \), and \( M \), \( \Phi \) maps the set \( B^T_T(v^0, \delta, \Delta) \) into itself and

\[ ||\Phi(v(t)) - \Phi(\bar{v}(t))||_{0,\epsilon} \leq \mu ||v(t) - \bar{v}(t)||_{0,\epsilon}, \]

\( 0 < \mu < 1 \), for every \( t \in [0, T] \) and every \( v, \bar{v} \in B^T_T(v^0). \)

**Proof.** Let’s set \( \psi = w - \bar{w} \) with \( w = \Phi(v) \) and \( \bar{w} = \Phi(\bar{v}) \). By definition of \( \psi \) we have

\[ \frac{\partial \psi}{\partial t} + \sum_{j=1}^2 A_j^i(v, \epsilon) \cdot \frac{\partial \psi}{\partial x_j} = H - \mathcal{P} + F \]  \hspace{1cm} (52)

where \( H = h(v, \epsilon) - h(\bar{v}, \epsilon), \mathcal{P} = -P(v, z, \epsilon) + P(\bar{v}, z, \epsilon) \) and \( F = \sum_{j=1}^2 (A_j^i(v, \epsilon) - A_j^i(\bar{v}, \epsilon)) \frac{\partial \bar{w}}{\partial x_j}. \)
Let’s quote that $\psi(0, x) = 0$.

From (52), proceeding as before in the derivation of (36), we get:

$$\frac{d}{dt} \|\psi\|^2_E \leq C \tilde{K} \Delta \cdot \|\psi(t)\|_{0, \epsilon}^2 + C \|\psi(t)\|_{0, \epsilon} \cdot (\|H(t)\|_{0, \epsilon} + \|P\|_{0, \epsilon} + \|F(t)\|_{0, \epsilon}).$$

(53)

By the lemma(4.5)(p.507, in[9]) we have

$$\|H(t)\|_{0, \epsilon} \leq C \theta_{1, 1}(h, v^0, \delta) \|v(t) - \bar{v}(t)\|_{0, \epsilon} \leq C \tilde{K} \|v(t) - \bar{v}(t)\|_{0, \epsilon}$$

(54)

And for the estimation of $\|F(t)\|_{0, \epsilon}$, we use again lemma(4.5)(p.507, in[9]) and Sobolev’s inequality to obtain:

$$\|F(t)\|_{0, \epsilon} \leq C \|A^j(v, \epsilon) - A^j(\bar{v}, \epsilon)\|_{0, \epsilon} \left\|\frac{\partial \bar{v}}{\partial x_j}\right\|_{s_0, \epsilon}$$

$$\leq C \tilde{K} \|v(t) - \bar{v}(t)\|_{0, \epsilon} \left\{\|v(t) - \bar{v}^0\|\epsilon + \|\bar{v}(t) - \bar{v}^0\|\epsilon\right\},$$

and then

$$\|F(t)\|_{0, \epsilon} \leq C \tilde{K} \Delta \|v(t) - \bar{v}(t)\|_{0, \epsilon}.$$  

(55)

Since $\|P\|_{0, \epsilon} = \|P(v, z, \epsilon) - P(\bar{v}, \bar{z}, \epsilon)\|_{0, \epsilon}$ we have:

$$\|P\|_{0, \epsilon} \leq \|P(v, z, \epsilon) - P(\bar{v}, \bar{z}, \epsilon)\|_{0, \epsilon} + \|P(\bar{v}, \bar{z}, \epsilon) - P(\bar{v}, \bar{z}, \epsilon)\|_{0, \epsilon}.$$ 

By the lemma(4.5)(p.507, in[9]) we get:

$$\|P\|_{0, \epsilon} \leq C \theta_{1, 1}(P, v^0, \delta) \|v - \bar{v}\|_{0, \epsilon} + C$$

$$\leq C \tilde{K} \|v - \bar{v}\|_{0, \epsilon} + C$$

Since $\|v - \bar{v}\|_{0, \epsilon}$ is bounded, there exists $\tilde{C} > 0$ such that:

$$\tilde{C} \|v - \bar{v}\|_{0, \epsilon} \leq C \tilde{K} \|v - \bar{v}\|_{0, \epsilon} + C$$

And if $\|v - \bar{v}\|_{0, \epsilon} = 0$, by uniqueness of the solution of the second equation of (28), we have $z = \bar{z}$. That implies $\|P\|_{0, \epsilon} = 0$. Therefore the following estimation holds:

$$\|P\|_{0, \epsilon} \leq \tilde{C} \|v - \bar{v}\|_{0, \epsilon}.$$  

(56)

Now using (54), (55) and (56) in (53) and applying the Gronwall’s inequality we have

$$\|\psi(t)\|_{0, \epsilon} \leq (\|\psi(0)\|_{0, \epsilon} + \tilde{C} \|v(t) - \bar{v}(t)\|_{0, \epsilon}) \cdot T \exp \tilde{C} T,$$

(57)

where $\tilde{C} = C \tilde{K}^2 \Delta^2$. This last step allows us to get:

$$\|\Phi(v(t)) - \Phi(\bar{v}(t))\|_{0, \epsilon} \leq \tilde{C} T \cdot \|v(t) - \bar{v}(t)\|_{0, \epsilon} \exp \tilde{C} T.$$

\[\square\]

**D) End of the proof.** Finally, one can quote that the above method builds a sequence of solutions $(v^{\epsilon, n}, z^{\epsilon, n})_{n \in \mathbb{N}}$, of the system (28), defined as follows:

$$\begin{cases}
  v^{\epsilon, 0} \in B^\epsilon_r(v^0) \\
  v^{\epsilon, n} = \Phi(v^{\epsilon, n-1}) \\
  z^{\epsilon, n} = \Phi(v^{\epsilon, n-1}) 
\end{cases} \quad \forall n \in \mathbb{N},$$

(58)
and satisfying (25), where $\Phi$ maps $B_T^\varepsilon(v^0)$ into itself and is a contraction for the norm $\|\cdot\|_{0,\varepsilon}$. That means there is $\mu, 0 < \mu < 1$ such that

$$
\|v^{\varepsilon,n} - v^{\varepsilon,n-1}\|_{0,\varepsilon} \leq \mu \|v^{\varepsilon,n-1} - v^{\varepsilon,n-2}\|_{0,\varepsilon}
$$

$$
\leq \mu^n \|v^{\varepsilon,1} - v^{\varepsilon,0}\|_{0,\varepsilon}
$$

$$
\Rightarrow \|v^{\varepsilon,n} - v^{\varepsilon,n-1}\|_{0,\varepsilon} \to 0, \text{ as } n \to \infty.
$$

Then, we have $\|v^{\varepsilon,n} - v^{\infty}\|_{0,\varepsilon} \to 0$ uniformly in $t$ and $v^{\infty}$ is a fixed point of $\Phi$ translated by $\Phi(v^{\infty}) = v^{\infty}$.

In addition, since $A'(v)$, $C'(v)$ are respectively $C^1_b(\mathbb{R}^3)$ and $C^1(\mathbb{R}^3, \mathbb{R}^2)$ (see proposition 5 in Appendix section) by the continuity argument, we have $(v^{\varepsilon,n}, \varepsilon,n)_{n \in \mathbb{N}}$ converges to $(v^{\varepsilon,\infty}, \varepsilon,\infty)$, solution of the system (28).

In addition

$$
v^{\varepsilon,\infty} \in C([0, T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap C^1([0, T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3)) \text{ and }
$$

$$
z^{\varepsilon,\infty} \in L^\infty([0, T], L^2(\mathbb{T}^2))
$$

**Remark 6.** By Sobolev’s lemma (see [9]), we have

$$
C([0, T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap C^1([0, T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3)) \subseteq C^1([0, T] \times \mathbb{T}^2)
$$

and then $v^{\varepsilon,\infty}$ is regular and it follows that $v^{\varepsilon,\infty} \in L^\infty([0, T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap Lip([0, T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3))$ (see [9]).

That ends of the proof of theorem 3.2

**Remark 7.** If there is $C > 0$ such that for any $\epsilon > 0 \|P(v^{\varepsilon}, z^{\varepsilon})\| \leq C$, then $\|\nabla z^{\varepsilon}\| \leq C$; because $m^\varepsilon + b^\varepsilon$ is bounded. Then we have $z^{\varepsilon}(t) \in C_b(\mathbb{T}^2)$ and more precisely $z^{\varepsilon} \in L^\infty([0, T], C_b(\mathbb{T}^2))$ and $\nabla z^{\varepsilon} \in L^\infty([0, T], L^\infty(\mathbb{T}^2))$.

**Remark 8. A particular case**

In this remark we study the coupled system (10) in the case where $m^\varepsilon(t, x) = 0$ and $Q_0$ not depending on $t$ and $x$. This particular case appears in the paper of Faye et al [6].

That means $v^\varepsilon = (0, q^\varepsilon(\theta))$ and $\nabla F(v^\varepsilon) = 0$, where $\theta \in \mathbb{R}$ is an other time scale not depending on $t, x$. Hence (10) becomes

$$
\begin{align*}
\frac{\partial q^\varepsilon(\theta)}{\partial t} &= \frac{1}{\varepsilon} h(v^\varepsilon) - \frac{1}{\varepsilon^2} P(v^\varepsilon, z^\varepsilon) \\
\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2} \nabla \cdot [A^\varepsilon \nabla z^\varepsilon] &= \frac{1}{\varepsilon^2} \nabla \cdot C^\varepsilon
\end{align*}
$$

It follows

$$
\begin{align*}
\frac{1}{\varepsilon} \left( \frac{k}{b^1} q_1^\varepsilon(\theta) + f q_2^\varepsilon(\theta) \right) - \frac{1}{\varepsilon^2} d^1 b^1 \frac{\partial z^\varepsilon}{\partial x_1} &= 0 \\
\frac{1}{\varepsilon} \left( \frac{k}{b^2} q_2^\varepsilon(\theta) - f q_1^\varepsilon(\theta) \right) - \frac{1}{\varepsilon^2} d^1 b^1 \frac{\partial z^\varepsilon}{\partial x_2} &= 0 \\
\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2} \nabla \cdot \left[ \frac{a}{(b^1)^3} ||q^\varepsilon(\theta)||^3 \nabla z^\varepsilon \right] &= 0.
\end{align*}
$$
For the sake of simplifying, we shall note this limit by : 

\[
\begin{align*}
\varepsilon^{2} & = \frac{1}{\epsilon} \frac{d^{2} b}{d \epsilon d x_{1}} \cdot \frac{\partial^{2} z^{\varepsilon}}{\partial t \partial x_{1}} = 0 \\
\varepsilon^{2} & = \frac{1}{\epsilon} \frac{d^{2} b}{d \epsilon d x_{2}} \cdot \frac{\partial^{2} z^{\varepsilon}}{\partial t \partial x_{2}} = 0 \\
\nabla \left( \frac{\partial z^{\varepsilon}}{\partial t} \right) - \frac{1}{\epsilon^{2}} \nabla \cdot \left[ \frac{a}{(b^{1})^{3}} \| q^{\prime} (\theta) \|^{3} \nabla z^{\varepsilon} \right] & = 0
\end{align*}
\]

Hence, we get \( \nabla \left( \frac{\partial z^{\varepsilon}}{\partial t} \right) = \frac{\partial \nabla z^{\varepsilon}}{\partial t} = 0 \) and finally :

\[
\nabla \left( \frac{\partial z^{\varepsilon}}{\partial t} \right) = \frac{\partial \nabla z^{\varepsilon}}{\partial t} = 0 \Rightarrow \nabla \cdot \left[ a \left( \frac{(b^{1})^{3}}{\| q^{\prime} (\theta) \|^{3} \nabla z^{\varepsilon} \} \right] = C
\]

with \( C > 0 \). In other words :

\[
\frac{a}{(b^{1})^{3}} \| q^{\prime} (\theta) \|^{3} \Delta z^{\varepsilon} = C.
\]

And then, if \( \| q^{\prime} (\theta) \| \neq 0 \), a solution can be obtained as follows :

\[
z^{\varepsilon}(\theta, x) = \frac{C}{2} \left( \frac{a}{(b^{1})^{3}} \| q^{\prime} (\theta) \|^{3} \right)^{-1} \| x \|^{2} + (C_{1}, x) + C_{2}
\]

with \( C_{1} \in \mathbb{R}^{2}_{+}, C_{2} > 0 \) and \((\cdot, \cdot)\) is the inner product in \( \mathbb{R}^{2} \).

5. **Asymptotic behavior.** In this section, we shall prove that the solution of the dimensionless Shallow Water and Long Term Dynamic of Dunes system 2-scale converges to a profile which is solution of a homogenized SWE-LTDD system.

In the sequel, in order to simplify the writings of the spaces, \( L^{p}_{\text{per}}(\mathbb{R}, X), p \in [1, \infty] \) and \( C^{s}_{\text{per}}(\mathbb{R}, X) \) shall mean respectively the spaces of \( L^{p}, C^{s} \) 1-periodic and valued functions in \( X \).

Let’s begin with what we understand by 2-scale convergence according to the definitions of Allaire[1] and N’getse[n][12].

**Definition 5.1.** Let \((v^{\varepsilon})\) in \( L^{\infty}([0, T), L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3})) \) be a sequence of functions.

It two-scale converges to \( V \in L^{\infty}([0, T), L^{\infty}_{\text{per}}(\mathbb{R}, L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3}))) \) if for every \( \psi \in C([0, T], C_{\text{per}}(\mathbb{R}, L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3}))), \) we have :

\[
\lim_{\epsilon \to 0} \int_{\mathbb{T}^{2}} \int_{0}^{T} v^{\varepsilon}(t, x) \psi(t, \frac{t}{\epsilon}, x) dtdx = \int_{\mathbb{T}^{2}} \int_{0}^{T} V(t, \theta, x) \psi(t, \theta, x) d\theta dtdx
\]

For the sake of simplifying, we shall note this limit by : \( v^{\varepsilon} \Rightarrow_{2} V \).

**Definition 5.2.** A sequence of functions \((v^{\varepsilon})\) in \( L^{\infty}([0, T), L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3})) \) is said to strongly two-scale converge to \( V \in L^{\infty}([0, T), L^{\infty}_{\text{per}}(\mathbb{R}, L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3}))) \) if

\[
v^{\varepsilon} \Rightarrow_{2} V, \quad \text{and} \quad \lim_{\epsilon \to 0} \left\| v^{\varepsilon} \right\|_{L^{\infty}([0, T), L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3}))} = \left\| V \right\|_{L^{\infty}([0, T), L^{\infty}_{\text{per}}(\mathbb{R}, L^{2}(\mathbb{T}^{2}, \mathbb{R}^{3})))}
\]

Before going on, let’s recall the following theorem which we plan to use if the hypothesis is satisfied.
Theorem 5.3. If a sequence of functions \((v^\epsilon)\) is bounded in \(L^\infty([0, T), L^2(\mathbb{T}^2, \mathbb{R}^3))\), there exists a sub-sequence still denoted \((v^\epsilon)\) and a function \(V \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))\) such that
\[
v^\epsilon \to^\ast V.
\]

From the proof of theorem 3.2 we can see that : \((\tilde{v}^\epsilon, \tilde{z}^\epsilon) \in L^\infty([0, T), H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap Lip([0, T), H^{s-1}(\mathbb{T}^2, \mathbb{R}^3)) \times L^\infty([0, T), L^2(\mathbb{T}^2)),\) and the solution satisfies the estimates condition :
\[
\|\tilde{v}^\epsilon(t) - v^0\|_{s, \epsilon} + \left\| \frac{\partial \tilde{v}^\epsilon}{\partial t} \right\|_{s-1, \epsilon} \leq \Delta,
\]
for all \(t \in [0, T]\) and \(\Delta\) not depending on \(\epsilon\).

Having this estimates at hands, we can deduce estimates with the classical Sobolev norm in \(H^s\) and \(H^{s-1}\). Then, we have :
\[
\|\tilde{v}^\epsilon(t) - v^0\|_s + \left\| \frac{\partial \tilde{v}^\epsilon}{\partial t} \right\|_{s-1} \leq \Delta,
\]
This implies that
\[
\|\tilde{v}^\epsilon(t) - v^0\|_s + \left\| \frac{\partial \tilde{v}^\epsilon}{\partial t} \right\|_{s-1} \leq C,
\]
with \(C\) not depending on \(\epsilon\).

For mathematical reasons, we can also take \(\tilde{v}^\epsilon\) in the space \(L^\infty([0, T), L^2(\mathbb{T}^2, \mathbb{R}^3))\) with the following estimates
\[
\|\tilde{v}^\epsilon(t) - v^0\|_0 + \left\| \frac{\partial \tilde{v}^\epsilon}{\partial t} \right\|_0 \leq C.
\]

One can see that the two equations in the system (28) do not have the same order with respect to the power of \(\epsilon\). And one needs to do a careful analysis.

Before stating the homogenization theorem, let’s give useful remarks and hypotheses for our framework of study.

\[
\begin{cases}
\text{For any } \epsilon > 0, \text{ the source term of the dimensionless LTDD can’t become infinite. That means : } \exists C > 0 \forall \epsilon \in (0, 1] \text{ small } \quad (H) \\
\text{such that } |\nabla \cdot C'(v^\epsilon)| \leq \epsilon^2 C.
\end{cases}
\]

Remark 9. • To get an estimation of \(z^\epsilon\) independent of \(\epsilon\), we need \((H)\). Such situations may occur, see for instance [6], where it is showed that \(|\nabla \cdot C'| \leq C\epsilon\), with \(C\) a constant not depending on \(\epsilon\).
• Under the assumption \((H)\), \(\bar{\gamma}\) and \(\bar{\gamma}'\) introduced in theorem 3.2, section 4-A do not depend on \(\epsilon\).

Now, we can state the homogenization theorem for the system.

Theorem 5.4. Under all the assumptions as in theorem 3.2, we have
\[
\begin{cases}
\tilde{v}^\epsilon \to^\ast \tilde{v} \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3))) \\
\tilde{z}^\epsilon \to^\ast \tilde{z} \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\mathbb{T}^2)))
\end{cases}
\]
And the limit functions satisfy :
\[
\begin{cases}
\bar{B}^1 \frac{\partial \tilde{v}}{\partial x_1} + \bar{B}^2 \frac{\partial \tilde{v}}{\partial x_2} = -\bar{p}(\tilde{v}, \tilde{z}) \quad \text{on } [0, T) \times \mathbb{R} \times \mathbb{T}^2 \\
-\nabla \cdot (\bar{A} \nabla \tilde{z}) = \nabla \cdot \tilde{C} \quad \text{on } [0, T) \times \mathbb{R} \times \mathbb{T}^2,
\end{cases}
\]
with \( B^1(t, \frac{t}{\epsilon}, x) \) \( 2^{-x} \bar{B}^1(t, \theta, x) \), \( B^2(t, \frac{t}{\epsilon}, x) \) \( 2^{-x} \bar{B}^2(t, \theta, x) \), \( p(v^\epsilon, z^\epsilon) \) \( 2^{-x} \bar{p}(\bar{v}, \bar{z}) \), \( A'(v^\epsilon) = A(t, \frac{t}{\epsilon}, x) \) \( 2^{-x} \bar{A}(t, \theta, x) \), and \( C'(v^\epsilon) = C(t, \frac{t}{\epsilon}, x) \) \( 2^{-x} \bar{C}(t, \theta, x) \).

**Proof.** Let’s recall the equation (16) to ease the reading of the proof

\[
 B^0 \frac{\partial}{\partial t}[v^\epsilon] + \frac{1}{\epsilon^4} \sum_{j=1}^{2} B^j \cdot \frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{\epsilon} \tilde{h}(v^\epsilon) - \frac{1}{\epsilon^4} \tilde{b}(v^\epsilon, z^\epsilon)
\]

Considering \( \psi^\epsilon(t, x) = \psi(t, \frac{t}{\epsilon}, x) \) regular enough and \( \mathbb{R}^3 \) valued test functions with \( supp(\psi^\epsilon) \subset [0, T) \times T^2 \) and \( \theta \mapsto \psi(t, \theta, x) \) is 1-periodic in \( \theta \).

Multiplying (16) by \( \psi^\epsilon \) and integrating it yields:

\[
 \int_0^T \int_{T^2} B^0 \frac{\partial v^\epsilon}{\partial t} \psi dtdx + \frac{1}{\epsilon^4} \sum_{j=1}^{2} \int_0^T \int_{T^2} B^j \frac{\partial v^\epsilon}{\partial x_j} \psi dtdx + \frac{1}{\epsilon^4} \sum_{j=1}^{2} \int_0^T \int_{T^2} B^2 \frac{\partial v^\epsilon}{\partial x_j} \psi dtdx \\
= \frac{1}{\epsilon} \int_0^T \int_{T^2} \tilde{h} \psi dtdx - \frac{1}{\epsilon^4} \int_0^T \int_{T^2} \tilde{b} \psi dtdx
\]

\[
- \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^0 \psi^\epsilon}{\partial t} \right) dtdx - \frac{1}{\epsilon^4} \sum_{j=1}^{2} \int_0^T \psi^\epsilon \left( \frac{\partial B^j \psi^\epsilon}{\partial x_j} \right) dtdx = 0
\]

Note that \( \frac{\partial (B^0 \psi^\epsilon)}{\partial t} (t, \frac{t}{\epsilon}, x) = \frac{\partial (B^0 \psi^\epsilon)}{\partial t} (t, \theta, x) \) and integrating it yields:

Then, we get:

\[
- \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^0 \psi^\epsilon}{\partial t} \right) dtdx - \frac{1}{\epsilon^4} \sum_{j=1}^{2} \int_0^T \psi^\epsilon \left( \frac{\partial B^j \psi^\epsilon}{\partial x_j} \right) dtdx
\]

Multiplying by \( \epsilon^4 \) we get:

\[
- \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^0 \psi^\epsilon}{\partial t} \right) dtdx - \epsilon^4 \int_{T^2} \psi^\epsilon \left( \frac{\partial B^0 \psi^\epsilon}{\partial t} \right) dtdx + \epsilon^4 \int_{T^2} B^0 \psi^\epsilon \psi^\epsilon (0, 0, x) dx
\]

and passing now to the limit it yields:

\[
- \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^0 \psi^\epsilon}{\partial t} \right) dtdx - \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^1 \psi^\epsilon}{\partial t} \right) dtdx + \int_{T^2} \int_0^T \psi^\epsilon \left( \frac{\partial B^2 \psi^\epsilon}{\partial t} \right) dtdx
\]
Remark 10. As in [6], if we consider following assumptions on the solutions $v^\epsilon$:
\begin{align*}
m^\epsilon &= M_1(\theta, x) + c^2 M_2(\theta, x) \\
q^\epsilon(t, x) &= Q_0(\theta) + Q_1(\theta, x) + Q_2(\theta, x) + \epsilon Q_3(\theta, x) \\
&\quad + c^2 Q_4(\theta, x) + c^4 Q_5(\theta, x),
\end{align*}
then we can see that
\begin{align*}
q^\epsilon(t, x) &= q_0(t, \frac{t}{\epsilon}, x) 2^{-s} Q_0(\theta) \quad \text{and} \quad m^\epsilon(t, x) = m_0(t, \frac{t}{\epsilon}, x) 2^{-s} M_1(\theta, x)
\end{align*}
Hence
\begin{align*}
B^1(t, \frac{t}{\epsilon}, x) 2^{-s} B^1(t, \theta, x), \quad B^2(t, \frac{t}{\epsilon}, x) 2^{-s} B^2(t, \theta, x)
\end{align*}
where $B^1, B^2$ are respectively:
\[
\begin{pmatrix}
\frac{Q_{01}}{M_1 + b^1} [a^1 Q_0, c^1 M_1] - [a^1 Q_{01}, c^1 M_1] - \frac{a^1 Q_0 Q_{02}}{(M_1 + b^1)^2} & 0 \\
-a^1 Q_{02} & \frac{Q_{02}}{M_1 + b^1} [a^1 Q_0, c^1 M_1] - \frac{a^1 Q_{02}}{(M_1 + b^1)^2} [a^1 Q_0, c^1 M_1] \\
\frac{a^1 Q_{01} Q_{01} Q_{02}}{(M_1 + b^1)^2} & \frac{a^1 Q_{01} Q_{02}}{(M_1 + b^1)^2} [a^1 Q_0, c^1 M_1] - \frac{a^1 Q_{01} Q_{02}}{(M_1 + b^1)^2} [a^1 Q_0, c^1 M_1]
\end{pmatrix}
\]
where $[a^1 Q_0, c^1 M_1] = \frac{a^1 Q_0^2}{(M_1 + b^1)^2} - c^1(M_1 + b^1)$, and $[a^1 Q_{01}, c^1 M_1] = \frac{a^1 Q_{01}^2}{(M_1 + b^1)^2} - c^1(M_1 + b^1)$, $i = 1, 2$.

Now let’s recall the equation of long term dynamic of dunes of sand (12):
\[
\begin{align*}
\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon^2} \nabla \cdot [A^\epsilon(\tilde{v}) \nabla z^\epsilon] = \frac{1}{\epsilon^2} \nabla \cdot C^\epsilon(\tilde{v}^\epsilon) \\
z^\epsilon(t = 0, x, y) = z_0 \in H^1(\mathbb{T}^2)
\end{align*}
\]
We have by the theorem (3.2), the existence of solution $z^\epsilon$ of equation (12) belonging to $L^\infty([0, T], L^2(\mathbb{T}^2))$ and bounded independently of $\epsilon$. Then multiplying (12) by test functions $\psi^\epsilon(t, x) = \psi(t, \frac{t}{\epsilon}, x)$ regular enough with compact support in $[0, T] \times \mathbb{T}^2$ and $\theta \mapsto \psi(t, \theta, x)$ 1-periodic in $\theta$, integrating over $[0, T] \times \mathbb{T}^2$, multiplying again by $\epsilon^2$ and finally passing to the limit it yields:
\[
- \int_0^T \int_{\mathbb{T}^2} \int_0^1 \tilde{z} \nabla \cdot (\tilde{A} \nabla \psi) d\theta dx dt = \int_0^T \int_{\mathbb{T}^2} \int_0^1 \nabla \cdot \tilde{C} \psi d\theta dx dt.
\]
Then we get:
\[
\nabla \cdot (\mathcal{A} \nabla \tilde{z}) = \nabla \cdot \tilde{\mathcal{C}} \quad \text{on} \quad [0, T) \times \mathbb{R} \times \mathbb{T}^2
\]
with
\[
\mathcal{A}(t, \frac{t}{\epsilon}, x)^{2-\epsilon} \tilde{\mathcal{A}}(t, \theta, x) \in L^\infty([0, T), L^\infty_{per}(\mathbb{R}, L^2(\mathbb{T}^2)))
\]
\[
\mathcal{C}(t, \frac{t}{\epsilon}, x)^{2-\epsilon} \tilde{\mathcal{C}}(t, \theta, x) \in L^\infty([0, T), L^\infty_{per}(\mathbb{R}, L^2(\mathbb{T}^2)))
\]

**Remark 11.** With the same assumptions as in the remark 10 we deduce:

- \(\tilde{\mathcal{A}}(t, \theta, x)\) and \(\tilde{\mathcal{C}}(t, \theta, x)\) are expressed as follows:
  \[
  \tilde{\mathcal{A}}(t, \theta, x) = a_{\mathcal{A}} \left( \frac{|Q_0(\theta)|}{b^1} \right) Q_0(\theta)
  \]
  \[
  \tilde{\mathcal{C}}(t, \theta, x) = a_{\mathcal{C}} \left( \frac{|Q_0(\theta)|}{b^1} \right) \frac{Q_0(\theta)}{|Q_0(\theta)|}
  \]
- \(\tilde{\mathcal{C}}\) does not depend on \(x\) that means \(\nabla \cdot \tilde{\mathcal{C}} = 0\).

**Appendix.** In this section we give the proof of some estimates we have already used in the previous steps of this work. In fact, we are going to prove that \(B^j(v'(t, x))\), \(j = 1, 2, 3\), \(\tilde{h}(v'(t, x))\), \(\tilde{p}(v'(t, x))\), \(\mathcal{A}'(v'(t, x))\), and \(\mathcal{C}'(v'(t, x))\) are \(C^\alpha\)-differentiable with respect to \(v'(t, x) \in \mathbb{R}^3\) and are valued functions in \(B(\mathbb{R}^3)\) or \(\mathbb{R}^3\) for \(v'\) belonging to a set included in \(H^s([0, T) \times \mathbb{T}^2, \mathbb{R}^3)\).

For an integer \(k \geq 0\), \(C^k_b(\mathbb{R}^3, \mathbb{R}^3)\) is the space of all \(\mathbb{R}^3\)-valued functions on \(\mathbb{R}^3\) that have continuous and bounded derivatives of order at most \(k\). Also we define \(C^k_b(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))\) as the space of all \(\mathcal{B}(\mathbb{R}^3)\)-valued functions on \(\mathbb{R}^3\) having continuous and bounded derivatives of at most \(k\), with \(\mathcal{B}(\mathbb{R}^3)\), the spaces of bounded operators from \(\mathbb{R}^3\) into \(\mathbb{R}^3\).

Let \(s\) be a positive integer and \(H^s(\mathbb{T}^2, \mathbb{R}^3)\) be the classical Sobolev space of \(\mathbb{R}^3\)-valued functions on the torus \(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2\). Let’s define the classical Sobolev norm in the Sobolev space \(H^s(\mathbb{T}^2, \mathbb{R}^3)\) by:

\[
\|\cdot\|_s = \left\| \sum_{|\alpha| \leq s} |D^\alpha \cdot |_0 \right\|^{1/2},
\]
where \(\alpha = (\alpha_1, \alpha_2)\) is a multi-index, \(\alpha_1, \alpha_2 \in \mathbb{N}\), \(|\alpha| = \alpha_1 + \alpha_2\) and

\[
\|u\|_0^2 = \sum_{1 \leq i \leq 3} \|u_i\|^2_{L^2}.
\]

We can define other norms and let’s quote them:

- in \(L^\infty([0, T], H^s(\mathbb{T}^2, \mathbb{R}^3))\) by
  \[
  \|v\|_{s,T} = \max_{t \in [0,T]} \|v(t)\|_s
  \]
- in the space \(C^s_b(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))\):
  \[
  \|\|B\|\|_{C^s_b} = \sum_{|\alpha| \leq s} \sup_{p \in \mathbb{R}^3} \left\| D^\alpha_p B(p) \right\|
  \]
Proposition 5.

In the sequel, we are going to consider the following notation: \( \| \cdot \| \)

or we can use the subordinated norm:

\[
\| B(u) \| = \| B^j(u) \|_\infty = \max_{1 \leq k \leq 3} ( \sum_{1 \leq k \leq 3} |B^j_{kl}(u)|) = \sup_{p \neq 0} \frac{\| B^j(u) \cdot p \|_\infty}{\| p \|_\infty}
\]

with \( \| p \|_\infty = \max |p_i|, p = (p_1, p_2, p_3)^t \)

- in \( C_b^s(\mathbb{R}^3, \mathbb{R}^3) \):
  \[
  \| h \|_{C^s_b} = \sum_{|\alpha| \leq s} \sup_{p \in \mathbb{R}^3} \| D^\alpha_p h(p) \|_{\mathbb{R}^3}
  \]

In the sequel, we are going to consider the following notation: \( \| \cdot \| \) defines the norm in the \( \mathbb{R}^n \) space \( n \geq 2 \).

**Proposition 5.** Let \( \omega_0 \) in \( H^s(T^2, \mathbb{R}^3) \) and \( r > 0 \). We suppose that \( s > 2 \) (or \( s \geq 3 \)) and consider the set \( \Pi = \{ v \in L^\infty([0, T], H^s(T^2, \mathbb{R}^3))/ ||v - \omega_0||_{s,T} < r \} \). Then we have:

1. \( B^j \in C^s_b(\mathbb{R}^3, B(\mathbb{R}^3)), 0 \leq j \leq 2 \)
2. \( \dot{h} \in C^s_b(\mathbb{R}^3, \mathbb{R}^3) \)
3. \( \mathcal{A}^c() \) and \( \mathcal{C}^c() \) are respectively \( C^s_b(\mathbb{R}^3) \) and \( C^s_b(\mathbb{R}^3, \mathbb{R}^2) \).

**Proof of Proposition 5.**

1. At first, let us prove that for \( 1 \leq j \leq 2 \), \( B^j(u), u \in \mathbb{R}^3 \)

   is a bounded operator. For this we show that each coefficient of the matrix \( B^j, j = 0, 1, 2 \) is bounded with the matrix norm defined below.

   a) Let us start with a first coefficient:

   \[
   \left| \frac{q_j^1}{m^e + b^1} - \frac{a^1 q_j^1 q_j^2}{(m^e + b^1)^2} - c^1(m^e + b^1) \right| = \left| \frac{a^1 q_j^1 q_j^2}{(m^e + b^1)^3} - c^1 q_j^1 \right|
   \]

   \[
   \left| \frac{a^1 q_j^1 q_j^2}{(m^e + b^1)^3} - c^1 q_j^1 \right| = \frac{a^1 q_j^1 q_j^2}{(m^e + b^1)^3} - c^1 q_j^1 \leq \frac{a^1}{(v^2)^3} |q_j^1 q_j^2 | + c^1 |q_j^1 | \leq \max\left( \frac{a^1}{(v^2)^3}, c^1 \right) |q_j^1 | (|q_j^2|^2 + 1)
   \]

   Then we have:

   \[
   \left| \frac{q_j^1}{m^e + b^1} - \frac{a^1 q_j^1 q_j^2}{(m^e + b^1)^2} - c^1(m^e + b^1) \right| \leq C ||v||_s (||v||_s^2 + 1)
   \]

   b) Now look at an other coefficient:

   \[
   \left| \frac{-a^1 q_j^2}{(m^e + b^1)^2} - c^1(m^e + b^1) \right| \leq \max\left( \frac{a^1}{(v^2)^3}, c^1, c^b b^1 \right) (||q_j^2||_s^2 + ||m^e||_s + 1)
   \]

   \[
   \left| \frac{-a^1 q_j^2}{(m^e + b^1)^2} + c^1(m^e + b^1) \right| \leq C (||v||_s^2 + ||v||_s + 1)
   \]

   c) For the coefficient \( \left| \frac{-a^1 q_j^1 q_j^2}{(m^e + b^1)^2} \right| \), we have:

   \[
   \left| \frac{-a^1 q_j^1 q_j^2}{(m^e + b^1)^2} \right| \leq \frac{a^1}{(v^2)^2} ||q_j^1||_s ||q_j^2||_s \leq \frac{a^1}{(v^2)^2} ||v||_s^2
   \]

   d) For the last coefficient for the matrix \( B^j \), we have:

   \[
   \left| \frac{a^1 q_j^1}{m^e + b^1} \right| \leq \frac{a^1}{v} ||q_j^1||_s \leq \frac{a^1}{v} ||v||_s
   \]
e) An other coefficient :
\[
\left| \frac{1}{a^1} \left( \frac{a^1 q^2}{m^1 + b^1} \right)^2 + c^3 (m^1 + b^1) \right| \leq \frac{1}{(v^1)^2} |q^2|^2 + \frac{c^1}{a^1} |m^1| + \frac{c^1 b^1}{a^1}
\]
For \( i = 1, 2 \) we have :
\[
\left| \frac{\alpha^2}{m^1 + b^1} \right| \leq \frac{1}{(v^1)^2} |q^2| \leq \frac{1}{(v^1)^2} |v^1| \leq \frac{1}{(v^1)^2} |v^1|_s
\]
So we conclude that \( \forall k, l = 1, 2, 3 \) there is \( C > 0 \) such that
\[
|B^0_j(v^r)| \leq C(|v^r|_{2, s}^2 + |v^r|_{s, t, 1} + 1) < \infty
\]
Hence for \( j = 0, 1, 2 \)
\[
|B^j(v^r)| \leq C(|v^r|_{2, s}^2 + |v^r|_{s, t, 1}) < \infty
\]
Since each coefficient of the matrices \( B^j(\xi), \ j = 1, 2, 3 \) are \( s \)-continuously differentiable with respect to \( \xi \in \mathbb{R}^3 \), we deduce that
\[
\forall \xi \in \mathbb{R}^3 \quad |||D^s_\xi B^j(\xi)||| < \infty \quad \text{with} \quad D^s_\xi B^j(\xi) \in C^s_{-\infty}(\mathbb{R}^3, \mathbb{B}(\mathbb{R}^3))
\]
So \( B^j \in C^s_{b}(\mathbb{R}^3, \mathbb{B}(\mathbb{R}^3)) \) and is bounded \( \Rightarrow B^j \in C^s_{b}(\mathbb{R}^3, \mathbb{B}(\mathbb{R}^3)) \). Indeed
\[
\forall \xi \in \mathbb{R}^3 \quad |||D^s_\xi B^j(\xi)||| < \infty \Rightarrow \sup_{\xi \in \mathbb{R}^3} |||D^s_\xi B^j(\xi)||| < \infty \iff |B^j|_{C^s_{b}} < \infty
\]
So \( B^j \in C^s_{b}(\mathbb{R}^3, \mathbb{B}(\mathbb{R}^3)) \), \( \forall j = 0, 1, 2 \).
2. Now let’s prove that \( \tilde{h} \in C^s_{b}(\mathbb{R}^3, \mathbb{R}^3) \).
\[
\tilde{h}(v^r) = \begin{pmatrix}
\frac{kq^2}{(m^1 + b^1)^2} \\
\frac{kq_1}{m^1 + b^1} + f^2q^2 \\
\frac{kq_2}{m^1 + b^1} - f^2q_1
\end{pmatrix}
\]
It is easy to see that if \( \forall i = 1, 2, 3 \), \( \tilde{h}_i \in C^s(\mathbb{R}^3) \) then \( \tilde{h} \in C^s(\mathbb{R}^3, \mathbb{R}^3) \). For each component of \( \tilde{h} \) we have the following estimations :
\[
|\frac{kq^2}{(m^1 + b^1)^2}| \leq k \frac{1}{v^2} |v^r|_s^2
\]
\[
|\frac{kq_1}{m^1 + b^1} + f^2q^2| \leq k \frac{1}{v^1} |q^1| + f |q^2| \leq C |v^r|_s
\]
\[
|\frac{kq_2}{m^1 + b^1} - f^2q_1| \leq C |v^r|_s
\]
Hence \( \forall i = 1, 2, 3 \), \( |\tilde{h}_i| \leq C |v^r|_{s, t}(|v^r|_{s, t, 1}) \).
For the same reason as for \( B^j, \ j = 0, 1, 2 \), we have \( |||D^s_\xi \tilde{h}(p)|||_{\mathbb{R}^3} < \infty \). That implies \( |||\tilde{h}||_{C^s_{b}} = \sum_{|a| \leq s} (\sup_{p \in \mathbb{R}^3} |||D^s_\xi \tilde{h}(p)|||_{\mathbb{R}^3}) < \infty \) therefore \( \tilde{h} \in C^s_{b}(\mathbb{R}^3, \mathbb{R}^3) \).
3. Now for \( A^r(v^r) \) and \( C^r(v^r) \) we have :
\[
|A^r(v^r)| = \left| \frac{a(1 - b m^1)}{m^1 + b^1} \right| |q^r|^3 \leq \frac{a}{v^1} |q^r|^3 \leq \frac{a}{v^1} |v^r|_s^3,
\]
\[
|C^r(v^r)| = \left| \frac{c(1 - b m^1)}{m^1 + b^1} \right| |q^r|^2 |q^r|_s \leq \frac{c}{v^1} |q^r|^2 \leq \frac{c}{v^1} |v^r|_s^3.
\]
And as above we can prove that $\mathcal{A}(\cdot)$ and $C^s(\cdot)$ are respectively $C^s_b(\mathbb{R}^3)$ and $C^s_b(\mathbb{R}^3,\mathbb{R}^2)$.

Acknowledgments. We would like to thank Emmanuel Frénod for fruitful discussions and the referee for the opportunity given to us to improve this paper.

REFERENCES

[1] G. Allaire, Homogenization and Two-Scale convergence, SIAM J. Math. Anal., 23 (1992), 1482–1518.
[2] P. Alliot, E. Frénod and V. Monbet, Modelling the coastal ocean over a time period of several weeks, Journal of Differential Equations, 248 (2010), 639–659.
[3] E. Audusse, F. Benkhadlou, J. Sainte-Marie and M. Seaid, Multilayer Saint-Venant equations over movable beds, Discrete and Continuous Dynamical Systems - B, 15 (2011), 917–934.
[4] C. Berthon, S. Cordier, O. Delestre and M. H. Le, An analytical solution of shallow water system coupled to Exner equation, Comptes Rendus Mathématique, Elsevier, 350 (2012), 183–186.
[5] S. Cordier, C. Lucas and J. D. D. Zabsonré, A two time-scale model for tidal bed-load transport, Communications in Mathematical Sciences, 10 (2012), 875–888.
[6] I. Faye, E. Frénod and D. Seck, Long term behavior of singularity perturbed parabolic degenerated equation, To appear in journal of non linear analysis and application.
[7] D. Idier, D. Astruc and S. J. M. H. Hulcher, Influence of bed roughness on dune and megaripple generation, Geophysical Research Letters, 31 (2004), 1–5.
[8] T. Kato, The Cauchy problem for quasi-linear symmetric system, Arch. Ration. Mech. Anal., 58 (1975), 181–205.
[9] S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Commun. Appl. Math., 34 (1981), 481–524.
[10] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’ceva, Linear and Quasi-linear Equations of Parabolic Type, AMS Translation of Mathematical Monographs, 1968.
[11] J. L. Lions, Remarques sur les équations différentielles ordinaires, Osaka Math. J., 15 (1963), 131–142.
[12] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., 20 (1989), 608–623.
[13] L. C. Van Rijn, Handbook on Sediment Transport by Current and Waves, Tech. Report H461:12.1-12.27, Delft Hydraulics, 1989.
[14] J. de D. Zabsonré, C. Lucas and E. Fernandez-Nieto, An energetically consistent viscous sedimentation model, Math. Model. Meth. Appl. Sci., 19 (2009), 477–499.

Received February 2015; Final revision April 2016.

E-mail address: mouhamadouamt.balde@ucad.edu.sn
E-mail address: diaraf.seck@ucad.edu.sn