New Approximations to the Fradkin representation for Green’s functions

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A new variant of the exact Fradkin representation of the Green’s function \(G_c(x, y|gU)\), defined for arbitrary external potential \(U\), is presented. Although this new approach is very similar in spirit to that previously derived by Fried and Gabellini, for certain calculations this specific variant, with its prescribed approximations, is more readily utilizable. Application of the simplest of these forms is made to the \(\lambda\Phi^4\) theory in four dimensions. As an independent check of these approximate forms, an improved version of the Schwinger-DeWitt asymptotic expansion of parametrix function is derived.

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I. Introduction

We here present a new approach for evaluating the (causal) Green’s function \(G_c(x, y|gU)\) and the related loop-generating functional \([gU]\), for a particle in the presence of an arbitrary external field \(U(x)\). For brevity’s sake, the simplest case when both the propagating particle and the external field are scalars is considered. The more interesting cases, when the particle has spinor structure and/or when it moves in a background gauge or gravitational field, will be presented in the next paper of this series.

The causal Green’s function for a scalar particle in the presence of a scalar background \(U(x)\) is a solution of the equation

\[
[\hat{P}_x^2 - m_c^2 + gU(x)]G_c(x, y|gU) = -\delta(x - y),
\]

and can be presented in the form

\[
G_c(x, y|gU) = i \int_0^\infty dse^{-ism^2}g(s|x, y|gU),
\]

where \(g(s|x, y|gU) := e^{i\hat{P}_s^2 + gU(x)}\delta(x - y)\) is the Parametrix function (with name taken from ref. [3]) of the problem. Since its relation to both Green’s function and loop-generating functional are very close, we will dedicate the major part of this paper to a derivation of different exact and approximate forms of the parametrix function, and results from the existing literature will be rephrased in terms of it. There exist two essentially different approaches for the construction of the parametrix: the Fradkin forms and the Schwinger-DeWitt asymptotic expansion. The first approach will be explained here, while the second is postponed until we reach Section IX.

The first approach is based on Fradkin’s [2] exact representation

\[
g(s|x, y|gU) = e^{-i \int_0^s dt\delta^2} e^{ig \int_0^s du (v + f_0^s u)} \delta (x - y - \int_0^s u) \bigg|_{v \to 0}
\]

where \(e^{-i \int_0^s dt\delta^2}\) denotes the linkage operator acting on the auxiliary variable \(v(t) (0 \leq t \leq s)\), and \(\delta^2 := \left(\frac{\delta}{\delta\epsilon(t)}\right)^2\). This form, on the one hand, leads directly to a path-integral formulation (Appendix B), a transition based on equivalence relations between the functional-linkage and functional integral forms (Appendix A). On the other hand, Fradkin’s expression (1.3) allows a derivation of an entire spectrum of different applications and approximations (1, 3). The majority of these are eikonal inspired, such as the Phase Averaged Approximation (\(< Ph >\)) of [3].

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1Notation: \(a \equiv b\) defines \(a\) in terms of \(b\); \(\Phi_p := \frac{\partial\Phi}{\partial x^\mu}\). \(\Phi\eta := \Pi_{\alpha=0}^\pm \frac{\partial\Phi(t)}{\partial \eta(\pm)}\). Metric has signature \((2 - D)\): \(\eta_{\alpha\nu} = \eta^{\alpha\nu} = \text{diag}(+1, -1, \ldots, -1)\). Also: \(z := x - y\); \(u(\xi) := y + \xi z\); \(m_+^2 := m^2 - ic\); \(\Phi_{\alpha}(x) := \partial_{\alpha}U(x)\); \(\mathcal{H}_{\mu\nu}(x) := \partial_{\mu}\partial_{\nu}U(x)\); We will use the hat symbol for operators that are of continuous nature (differential or integral) \((K)\), the hacheck symbol for matrices, \((\hat{H})\) and the bold face \((\mathbf{B})\) for objects that are both functional operators and matrices.
\[ g^{<\text{Ph}>}(s|xy|gU) = \int d\phi e^{ipz+isp^2} \int dq d\xi P e^{-i(q-y)(P+2p)} e^{iq \int_0^s dt U(q+tP)} \] (1.4)

It turns out that, based on the \(<\text{Ph}>=\) Approximation and a consideration of the approximate form \([1.3]\) in terms of Gaussian-weighted integrals over the set of auxiliary 4-vectors \(\{Q_n, P_n\}, n = 1, 3, 5, \ldots\), with the exact representation \([1.1, 1.2]\):

\[ g(s|xy|gU) = \int d\phi e^{ipz+isp^2} \int dq d\xi P e^{-i(P-q-y-t^2)2n} \int dQ_n dP_n e^{-i[Q^2_n + P^2_n]} \]

\[ e^{iq \int_0^s dt U(q+tP-2\xi^2 \sum_n \frac{1}{4} |P_n \cos(\frac{\pi}{4}) + Q_n \sin(\frac{\pi}{4})|)} \] (1.5)

where the prime (') denotes summation over odd natural numbers, \(\{Q_n, P_n\}\) are pairs of auxiliary 4-vectors. The steps needed to obtain this form are: Taylor-expand the exponential \(e^{iq \int_0^s dt U(y+tP)}\); use Fourier representations for each \(U\)-factor and for \(\delta x - y - \int_0^s v\); and evaluate the \(v\)-variable linkage operation. This generates exponentials with binary products \(k_i \cdot k_j\) of the Fourier momenta \(k_j\) for different \(U\)-factors, and these products are multiplied by functions of the difference of corresponding auxiliary (proper) time variables \(t_i - t_j\). One of these functions is \(\delta t_i - t_j\); and a suitable separation in terms of its Fourier modes has to be used to allow the factorization of expressions corresponding to different \(U\) factors. Once that separation is achieved, the reverse (inverse) Fourier transformation has to be performed, and the resulting series of factorized terms must be resumed. After some trivial rescalings, the resulting expression is \([1.3]\).

This form provides one with the possibility of approximating the full (exact) expression by taking into account only several pairs of variables \(\{Q_n, P_n\}\). The \(<\text{Ph}>=\) Approximation is obtained by neglecting all such \(\{Q_n, P_n\}\) variables in the argument \(U\) in \([1.3]\); also, the term \(-s P^2\) should be dropped \([1.3]\). Taking into account only the first pair of auxiliary variables \(\{Q_1, P_1\}\) in expression \([1.3]\) gives an improved \("<\text{Ph}\>|\) approximation, etc ... It is possible to quantify the quality of such approximations, in terms of the quality of corresponding approximations to the \(|t_i - t_j|\) function; and, according to \([1.3]\), the \(<\text{Ph}|1>\) may be expected to carry 81\% of contribution to the total expression, the \(<\text{Ph}|1|3>\) > 90\%, the \(<\text{Ph}|1|3|5>\) > 93\% etc ...

II. New representation for parametrix

In the course of searching for an alternate derivation of \([1.3]\) from Fradkin’s form \([1.3]\) (without resorting to a perturbative-like expansion and subsequent resummation), we have found a new, exact form, based on the mode expansion:

\[ v(t)^\mu = V^\mu + \sum_{n=1}^{\infty} [Q_n^\mu \cos(n\omega t) + P_n^\mu \sin(n\omega t)] \] (2.1)

where \(\omega = \frac{2\pi}{s}\). The main criteria in developing such an expansion is (the ad hoc) requirement that the delta-function restriction \(\int_0^s dt v(t) = \bar{z}\) constrains only the amplitude of the zero-frequency mode \(V\) (forcing it to be \(\bar{z}\)), thus leaving amplitudes of the oscillating modes unconstrained. The properly normalized basis on the interval \([0, s]\) is \(\{\phi_n\}\) = \(\{c_0(t) \equiv \sqrt{\frac{2}{s}}, \ c_n(t) \equiv \sqrt{\frac{2}{s}} \cos(n\omega t); \ s_n(t) \equiv \sqrt{\frac{2}{s}} \sin(n\omega t); \ (n \geq 1)\}\), where the scalar product is just the integral over that interval.

With such a basis the resolution of the identity on the same interval is provided:

\[ \delta(t-t') = \sum_{\lambda} \phi_\lambda(t)^* \phi_\lambda(t') = \frac{1}{s} + \frac{2}{s} \sum_{n=1}^{\infty} \cos(n\omega(t-t')); \] (2.2)

and, requiring that \(\frac{\delta v(t)^\mu}{\delta v(t)^n} = \delta(t-t') \delta_\mu^n\), the corresponding mode-decomposition of the functional derivative is given by \(\frac{\delta v(t)^\mu}{\delta v(t)^n} = \sum_{\lambda} \phi_\lambda(t)^* \frac{\delta}{\delta v(t)^n}\) \(\phi_\lambda\).

\(^2\) For the general decomposition \(v(t)^\mu = \sum_{\lambda} V_\lambda^\mu \phi(t)_\lambda\) where \(\{\phi(t)_\lambda\}\) is an ortho-normal basis; the corresponding functional derivative is given by \(\frac{\delta v(t)^\mu}{\delta v(t)^n} = \sum_{\lambda} \phi_\lambda(t)^* \frac{\delta}{\delta v(t)^n}\) \(\phi_\lambda\).
\[
\frac{\delta}{\delta v(t)^\mu} = \frac{1}{s} \frac{\partial}{\partial V^\mu} + \frac{2}{s} \sum_{n=1}^\infty \left\{ \cos(n \omega t) \frac{\partial}{\partial Q_n^\mu} + \sin(n \omega t) \frac{\partial}{\partial P_n^\mu} \right\}
\] (2.3)

Using (2.3), one can calculate some necessary expressions:

\[
\int_0^s dt \frac{\delta^2}{\delta v(t)^2} = \frac{1}{s} \frac{\partial^2}{\partial V^2} + \frac{2}{s} \sum_{n=1}^\infty \left\{ \frac{\partial^2}{\partial Q_n^2} + \frac{\partial^2}{\partial P_n^2} \right\},
\]

\[
\int_0^{t^\prime} dt' v(t') = Vt + \frac{s}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \{ Q_n \sin(n \omega t) + P_n [1 - \cos(n \omega t)] \},
\]

\[
\int_0^s dt v(t) = Vs;
\] (2.4)

and evaluate the V-linkage operation (using eq. (A2)), to obtain the new form for the parametrix (1.3):

\[
g(s|x, y|\, gU) = \frac{i}{(4\pi is)^2} e^{-\frac{m^2}{s^2}} \prod_{n=1}^\infty e^{-\frac{2}{ns^2}} \left\{ \frac{\alpha^2}{\pi^2 - n^2 \sigma^2} \right\}
\]

\[
e^{ig \int_0^s dt U(y + \frac{z}{s} + \frac{n}{s^2} \sum_{n=1}^\infty \frac{1}{n} \{ Q_n \sin(n \omega t) + P_n [1 - \cos(n \omega t)] \})}
\] (3.1)

This form is an exact representation of the Fradkin parametrix. Note that the characteristic “frequencies” for modes in auxiliary space [0, s] employed here are even multiples of \( \frac{s}{s} \), as opposed to the FG form (2.3), where the characteristic frequencies are odd multiples of that value. Also, here the variables \( V, \{ P_m, Q_m \} \)’s are just Fourier mode of the auxiliary variable \( v(t) \), while in form (1.3) no such simple connection between \( v(t) \) and their \( \{ P_m, Q_m \} \)’s is known.

In the following Section we will develop several approximate forms to (2.5), as well as one suitable reformulation of that exact result. That reformulation, in its turn, then will give rise to the some other approximate forms.

III. < 0 > Approximation

We are now able to develop the hierarchy of approximate forms (in the same spirit as those of the \( < Ph|1[3] \cdots > \)). The basic one is obtained by neglecting all oscillatory modes \( (Q_n \text{’s and } P_n \text{’s}) \) in the exponential argument of \( U \) in eq. (2.7):

\[
g^{<0>}(s|x, y|\, gU) = \frac{i}{(4\pi is)^2} e^{-\frac{m^2}{s^2}} + ig \int_0^s dt U(x(s))
\] (3.1)

where \( \sigma(\alpha) = y + \alpha z \). We will call this the \( < 0 > \)-Approximation. It has a simple meaning in coordinate space: propagation is replaced by motion along the straight line between points \( x \) and \( y \). In other words, the \( < 0 > \)-approximation ignores the ”structure” of the potential \( U(x) \), paying no attention to its ”valleys” and ”hills”. In terms of a propagating particle, this is the extreme high-energy (UV) approximation; in terms of a background potential, it is requirement that all its spatial derivatives are ”small” (i.e. \( |\partial_{\mu_1} \cdots \partial_{\mu_n} U| < \frac{|Q|}{L^2} \)), where the characteristic length \( L \) is \( \sim s < v^2 >^{1/2} \).

It is worth noticing that the corresponding Green’s function has the form of a free particle’s Green’s function with a path-dependent mass:

\[
G_{<0>}^{<0>}(x, y|\, gU) = - \int_0^\infty ds \frac{1}{(4\pi is)^2} e^{-isM(x,y)^2} \bigg|_{m^2 \rightarrow M(x,y)^2} = G_c(x, y|0)
\] (3.2)

where \( \mathcal{M}(x, y)^2 \equiv m^2 - g \int_0^1 d\alpha U(\sigma(\alpha)) \). From this form are evident the properties of symmetry, under \( x \leftrightarrow y \); and of the correct free-particle limit, \( U \to 0 \).

The \( < 0 > \)-approximation is not same as the \( < Ph > \). This can be seen by comparing a transformed (as closely as possible to the \( < Ph > \)-form; see Appendix B) expression for \( < 0 > \):
\[ g^{<0>}(x, y|gU) = \int dqdT e^{-i(q^{2}-m^{2}+gU_{x})} \int dq_{0} dqP e^{-iq(P^{2}-2P_{0}P)}  \]

with:

\[ g^{<Ph>}(x, y|gU) = \int dqdT e^{-i(q^{2}-m^{2}+gU_{x})} \int dq_{0} dqP e^{-iq(P^{2}-2P_{0}P)}  \]

The two forms are very similar; the only difference between them is in the argument of the potential \( U \). Instead of \([y - q - tP]\) (in \(<Ph>\)) we have \([y - \frac{1}{2}q - tP]\) (in \(<0>\)).

While the \(<0>\)-approximation has the free-particle-like coordinate form \((3.2)\) for the Green’s function, no such simplification can be made for the \(<Ph>\)-approximation. But, in momentum space picture, the \(<Ph>\)-approximation has a nice interpretation \([3]\), while the \(<0>\) remains nontransparent. Thus one may consider the \(<0>\)-approximation better suited for coordinate-space applications, while for diagrammatic applications, the \(<Ph>\) seems better adapted.

One more property (besides \( x \leftrightarrow y \) symmetry and the correct free-particle limit) that is shared by both the \(<0>\)- and \(<Ph>\)-approximation is their exactness for the plane-wave (laser-like) background \( U(x) = \phi(kx) \) (with \( k^{2} = 0 \)). The final form of the exact Green’s function of this field is the given by the same expression, \((3.2)\), which is only the \(<0>\)-approximation for other (non-laser) backgrounds. Also, it is worth noticing that in the family of approximations based on forms \((3.3)\) and \((3.4)\), where the argument of the potential \( U \) is replaced by \([y - qF_{1}(\alpha) - sPF_{2}(\alpha)]\), the laser-background case again simplifies to the explicit \(<0>\)-form \((3.2)\).

### IV. The loop generating functional \( L[gU] \)

Another important object (besides the Green’s functions) for functional field-theoretic calculations is the loop generating functional:

\[ L[gU] := -\frac{1}{2} \text{Tr} \left[ \frac{\hat{P}^{2} - m^{2} + gU_{x}}{\hat{P}^{2} - m^{2}} \right] = \frac{1}{2} \text{Tr det}(G_{c}[gU,G_{c}[0]^{-1}) \]

(4.1)

It can be expressed in the integral form \( L[gU] = \frac{1}{2} \int_{0}^{1} d\lambda \text{Tr}(G_{c}[\lambda gU]|gU) = \frac{1}{2} \int_{0}^{1} d\lambda \int dxG_{c}(x,x|\lambda gU)gU(x) \).

In the \(<0>\)-Approximation \((3.2)\) the Green’s function has a diagonal component of amount

\[ G_{c}^{<0>}(x, x|gU) = -\int_{0}^{\infty} ds \frac{1}{(4\pi is)^{2}} e^{-is(m^{2} - gU)} = \frac{-i}{(4\pi)^{2}}(m^{2} - gU)^{\frac{D}{2}} - 1 \Gamma \left( 1 - \frac{D}{2} \right) \]

(4.2)

and one finds

\[ L^{<0>}[gU] = \frac{1}{2} \int_{0}^{1} d\lambda \int dx \frac{i}{(4\pi)^{2}}(m^{2} - \lambda gU)^{\frac{D}{2}} - 1 \Gamma \left( 1 - \frac{D}{2} \right) \]

\[ = \frac{i}{2} \Gamma \left( -\frac{D}{2} \right) \int dx \left[ (m^{2} - gU)^{\frac{D}{2}} - m^{D} \right] \]

(4.3)

This is a divergent expression for even \( D \), with \( \Gamma \left( -\frac{D}{2} \right) \) expressing the UV-divergences. In what follows, we will assume that \( \Gamma \left( -\frac{D}{2} \right) \) is regularized in some implicit way\([\bar{1}]\) allowing explicit evaluation of \(<0>\)-approximate forms of some simple field theoretical models.

Note that for \( D = 2 \) and \( D = 4 \), expression \((4.3)\) simplifies further to

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\(^{3}\)One such regularization follows from the definition of the \( \Gamma\)-function, by replacing it’s UV-limit \( (\rho = 0) \) with the finite value \( M^{-2}\Lambda^{-2} \):

\[ \Gamma \left( -\frac{D}{2} \right) \sim \int_{M^{-2} \Lambda^{-2}}^{\infty} d\rho \rho^{-\frac{D}{2}} - 1 e^{-\rho} \sim \int_{M^{-2} \Lambda^{-2}}^{\infty} d\rho \rho^{-\frac{D}{2}} - 1 \sim \frac{2}{D} \left( \frac{\Lambda}{M} \right)^{D} \]

(4.4)

where \( M \) is some fixed mass scale. Therefore, \( \Gamma \left( -\frac{D}{2} \right) \sim \Lambda^{D} \)
\[
L^{<0>} [gU]_{D=2} = -i \frac{g \Gamma(-1)}{8\pi} \int dx U(x) \\
L^{<0>} [gU]_{D=4} = \Gamma(-2) \frac{g}{32\pi^2} \int dx [-2m^2 gU(x) + g^2 U(x)^2] 
\] (4.5)

V. Application to the $\lambda\phi^4$-model

To illustrate the $< 0 >$-approximate form, consider the $\lambda\phi^4$ model in Minkowski $1 + d$-dimensional space. Start from an action
\[
S = \int dx \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 + \phi^2 \right] 
\] (5.1)
which is (after evaluating the EOM of the field $\psi$) equivalent to the standard action with a potential term $V(\phi) = -\frac{1}{2} m^2 \phi^2 - \frac{1}{4} \phi^4$, where $\lambda = 2g^2/\mu^2$. This particular formulation is chosen in preparation for subsequent steps where it is important to have the action with all terms at most quadratic with respect to the any specific field (for example, the interaction term $g\phi^2\phi$ is quadratic with respect to the field $\phi$ and linear with respect to the field $\psi$). The field $\psi$ may be thought of as the classical form of a composite $\phi^2$.

The generating functional $Z[J, j] \equiv \int [D\phi D\psi] \exp \left\{ -iS[\phi, \psi] - i \int dx [\phi J + \psi j] \right\}$ can be transformed, following steps outlined in [4], into the form
\[
Z[J, j] = e^{L[2g\delta j]} e^{-\frac{i}{2} \int J \hat{G}_c[J, 2g\delta j] J} e^{\frac{i}{2\pi} \int j^2} 
\] (5.2)
where $\hat{G}_c[gU] := (\partial^2 + m_2^2 - gU)^{-1}$ is the causal Green’s function for the field $\phi$ in an external potential $U$, and $L[gU] := \frac{1}{2} \ln \det \left( \hat{G}_c[gU] \right)$ is the corresponding vacuum-loop generating functional.

In this section, the the 2n-point quantum correlation functions
\[
\Gamma_{2n}(x_1, \ldots, x_{2n}) := \langle \phi(x_1) \cdots \phi(x_{2n}) \rangle_+ \equiv \frac{1}{Z[0, 0]} \left. \frac{\delta^{2n} Z}{\delta J(x_1) \cdots \delta J(x_{2n})} \right|_{J,j=0} \\
= i^n \left[ G_c(x_1, x_2|2g\delta j) \cdots G_c(x_{2n-1}, x_{2n}|2g\delta j) + \text{permutations} \right] Z[j] \mid_{j=0} 
\] (5.3)
will be evaluated in the $< 0 >$-approximation. Here $Z[j] \equiv Z[0, j]/Z[0, 0] = Z[0, 0]^{-1} \exp \left\{ L[2g\delta j] \right\} \exp \left\{ \frac{\Gamma(-2)\lambda}{8\pi^2} \int j^2 \right\}$. Only the first of these terms will be observed, having all others related to this one by a permutations of coordinate arguments $\{x_1, \ldots, x_{2n}\}$.

For $D = 4$ use (4.3) and (A4) to obtain
\[
Z^{<0>} [j] = e^{\frac{\pi(-2)\lambda}{8\pi^2} \int dx \left( j^2 + \frac{2m^2}{\lambda} j \right)} 
\] (5.4)
where $a := \frac{\pi(-2)\lambda}{8\pi^2}$. From here, use [5.2]
\[
\Gamma_{2n}(x_1, \ldots, x_{2n}) = (-i)^n \prod_{j=1}^{n} \int_{0}^{\infty} ds_j \frac{1}{(4\pi is_j)^{d/2}} e^{-\frac{1}{2} \sum_j \left( \frac{s_j}{s_j} \right)^2 - \frac{1}{2} \sum_j s_j} \\
\int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dx_1 \cdots dx_{2n} \\
Z^{<0>} \left[ -2g \sum_j s_j \int_{0}^{1} da \delta(y - x_{2j} - x_{2j-1} - x_{2j}) \right] 
\] (5.5)
The $Z^{<0>}$ factor can be represented as a product of two factors $K_1 K_2$ where

\footnote{An alternate form is $Z[J, j] = e^{-\frac{\pi(-2)\lambda}{8\pi^2} \int dx \phi^2} e^{\frac{j}{2} \int J \hat{G}_c[j]} |_{\psi = \frac{j}{\lambda}}.$}
The exponential in $K_1$ has two sources of singularities: First, in "diagonal" terms $l = r$, the strongly UV-singular integrals $\int_0^1 d\beta \delta((\alpha - \beta)(x_{2r-1} - x_{2r}))$ is present. It can be transformed to the form $\int d\xi \left( (\xi_{2r-1} - \sin \xi_{2r-1})^2 \right)$, where $\xi_{2r-1} \equiv \frac{1}{2} k(x_{2r-1} - x_{2r})$, that it is well-behaved only in one direction (parallel to the four-vector $(x_{2r-1} - x_{2r})^\mu$). This gives an overall UV-behaviour $\sim \Lambda^4$, where $\Lambda$ is UV-cut-off parameter, the same as in (3.4).

A second source of singularities in $K_1$ are nondiagonal terms in case of intersecting linear trajectories; such trajectories occur whenever there exist $\alpha$ and $\beta$ such that $x_{2L} + \alpha(x_{2L-1} - x_{2L}) = x_{2r} + \beta(x_{2r-1} - x_{2r})$. Their contribution $\int_0^1 d\beta \int_0^1 d\delta \delta((\alpha - \beta)(x_{2r-1} - x_{2r}))$ can be transformed into the form $\int d\xi \left( (\xi_{2r-1} - \xi_{2L-1})^{-1} \sin \xi_{2r-1} \exp \left\{ \frac{1}{\Lambda} (x_{2L} + x_{2L-1} - x_{2r} - x_{2r-1}) \cdot k \right\} \right)$. From that expression one can infer that two $K$-space directions are convergent, and in one more the integration is made convergent by a fast-oscillating exponential factor. The remaining unregularized direction gives an overall $\Lambda^4$ UV-behaviour for such intersecting trajectories.

Then, since $a \sim \Gamma(-2) \sim \Lambda^4$, terms in the exponential of $K_1$ will display one of three UV-behaviours: $\Lambda^{3-4} = \Lambda^{-1}$ (diagonal singular terms), $\Lambda^{1-4} = \Lambda^{-3}$ (nondiagonal singular terms) or $\Lambda^{-4}$ (nondiagonal nonsingular terms, i.e. nonintersecting trajectories). In this way, in the $\Lambda \rightarrow \infty$ limit, one obtains $K_1 = 1$.

Notice that in the $\Lambda \rightarrow \infty$ limit, the value of bare-coupling constant $\lambda$ does not play any role, since it cancels: $\lambda = \frac{\lambda}{a-\alpha} \rightarrow -\frac{8\pi^2}{a(-2)} \sim \Lambda^{-4} \rightarrow 0$. Only way to make $\lambda$ nonzero (and finite) in the $\Lambda \rightarrow \infty$ limit is to accept peculiar behaviour of bare coupling $\lambda = \frac{1}{1 - \frac{\Lambda^{1-2}}{8\pi^2}} \sim \Lambda^{-4} \rightarrow 0$, which is completely opposite to the usual behaviour of bare couplings in the Abelian models.

Concerning the factor $K_2$, there are no other (than $a$) singularly behaving quantities in the exponential, and one can take the $\Lambda \rightarrow \infty$ limit right away. The result is $K_2 = e^{+im^2 \sum_{l=1}^n}$, $\tilde{G}_c$ in all subsequent expressions.

Then the $2n$-point correlator factorizes:

$$
\Gamma_{2n}(x_1, \cdots x_{2n}) \rightarrow \prod_{j=1}^n G_c(x_{2j-1}, x_{2j}|0) \bigg|_{m^2 \rightarrow M^2} \quad (5.7)
$$

into product of a free-particle Green functions.

Alternatively, one can evaluate the generating functional in naive $\Gamma(-2) \rightarrow \infty$ limit to get

$$
Z^{<0>[J, 0]} \rightarrow Z^{<0>[0, 0]|0} \frac{1}{J} \int G_c^{<0}>\left[ -a M^2 \right]^J = Z^{<0>[0, 0]|0} \frac{1}{J} \int G_c^{<0}|0, m^2 \rightarrow M^2 = Z[J, 0]|_{\lambda, m^2 \rightarrow M^2} \quad (5.8)
$$

i.e. such a limit produces a free theory of fields of mass $M$. Notice that it is not necessary to take the $< 0 >$-approximate form for $G_c^{<0}|0, m^2 \rightarrow M^2$ in (6.8): simplification in the $K_1$ factor in $Z^{<0>[J]$ suffices to validate the above expression with the exact $G_c^{<0}|0$'s.

The disappearance of the effective interaction in the UV-limit of $\varphi^4$ theory in four dimensions is the well-known phenomena of triviality, established or indicated in several different frameworks (Euclidean constructive field theory: [12]-[15]; Lattice calculations and "inspired" perturbative-renorm-group extrapolations: [17]-[22]; Functional approach: [16]), here, we have presented a simple analytic derivation of triviality. However, this method is not uniformly suitable for all number of dimensions: for example, in $D = 2$ the quadratic linkage operation is absent, leading to the absence of UV-singular quantities in denominators (that were of such importance in the $D = 4$ case), and thus to the non-triviality of the theory. In $D \neq 2, 4$, the linkage functional (4.4) is either (in case of even $D$) a polynomial of order higher than two, or a nonpolynomial (irrational) function (in the case of odd $D$) of $gU$; both circumstances generate linkage operators $e^{J[\varphi_0]$ that are neither shift-operators nor Gaussian linkages, preventing further evaluation in the absence of additional approximate steps.

The more detailed discussion, concerning more uniform regularization of sources UV-singularities, how this influence the derived trivial behaviour, and corrections due to higher approximations (using $< L >, < Q >$ and approximations obtained using the Schwinger-DeWitt's approach; all these are introduced in next few sections) will be presented elsewhere [23].
VI. Beyond the $< 0 >$-Approximation

How may one proceed beyond the $< 0 >$-approximation? The goal here is to obtain a sequentially better approximation to expression (2.3). Imitating the $< Ph | 1 | 3 \cdots >$-hierarchy does not help much, since the potential $U(x)$ is in general a nonlinear function of $x$: for example, retaining only $n = 1$ level variables $Q_1$ and $P_1$ (the $< 0 | 1 >$-Approximation) gives:

$$g^{< 1 >}(s|x,y|gU) = g(s|x,y|0)e^{-\frac{2i}{8\pi} \left( \frac{\partial}{\partial Q_1} + \frac{\partial}{\partial P_1} \right)} e^{ig \int_0^1 d\alpha U(\sigma(\alpha)) + \frac{\pi}{4} (Q_1 \sin(2n\pi\alpha) + P_1 [1 - \cos(2n\pi\alpha)])} \bigg|_{Q_1,P_1 \to 0}$$  

(6.1)

This expression can be evaluated analytically only in some special cases: e.g. if the potential $U(x)$ is a quadratic polynomial of its arguments, or if it represents a laser field (when it reduces to $< 0 >$-approximation). For all other cases one has to resort to some kind of further approximation, such as a Gaussian approximation. The same situation holds for $< 0 | 1 | 2 >$ and all higher approximations.

This indicates that it is perhaps not beneficial to insist on an application of the hierarchy $< 0 | 1 | 2 \cdots >$; but, instead, to proceed by an alternate route. Let us, in $< 0 >$, "remove" the variable $V$ from $v(t)$, while keeping all other components of $v(t)$ grouped into a new variable $u(t) \equiv v(t) - V$. The variational derivative $\delta_v$ then splits into $\delta_v^{(t)\mu} = \frac{1}{2} \partial_{v\nu} + \delta_{(t)\mu}$. Here the $\delta_{(t)\mu}$ is a constrained variational derivative that takes into account the constraint $\int_0^1 dt u(t)^\mu = 0$; its action on $u(t)^\nu$ is $\delta_{(t)\mu} = \eta_{\mu\nu} \Pi(t,t')$, where $\Pi(t,t') = [\delta(t-t') - \frac{1}{2}]$ is a projector on the $u$-subspace. In this way, one can write $\delta_v^{(t)\mu} = \Pi \delta_u^{(t)\mu}$ in terms of the unconstrained variational derivative $\delta_u$.

Then, the exact expression for the parametrix can be reformulated (Appendix [3]) as

$$g(s|x,y|gU) = \frac{i}{4\pi i s} e^{-\frac{i}{2} e^{-i \int_0^1 \delta u^2 \mu} e^{i \int_0^1 d\alpha W(\sigma(\alpha))}} \bigg|_{u \to 0}$$

(6.2)

While the $V$-integration (linkage operation) was easily performed in an explicit manner (using identity (A2)), the $u$-integration cannot be done in an exact way. Instead, one can assume that the $u$ variable is a "small" quantity, thus generating a Taylor expansion of the $U(\sigma(t/s) + \int_0^1 u)$ in the $u$ variable, and keeping only some of the lowest order terms. In what follows, derivation of the two such lowest order approximations is defined by retaining only up to $O(u^1)$ (giving the linear in $u$, $< L >$-Approximation) and $O(u^2)$ (quadratic in $u$, the $< Q >$-Approximation) terms.

VII. $< L >$-Approximation

The first of these, the $< L >$-approximate form for the parametrix is a product of the $< 0 >$-approximation parametrix and a linkage operation factor

$$e^{-i \int_0^1 \delta u^2} e^{i \int_0^1 a^2} \bigg|_{u \to 0} = e^{i \int_0^1 a^2 - \frac{1}{4} (\int_0^1 a^2)^2}$$

(7.1)

where $a_\mu(t) \equiv g s \int_t^1 d\alpha W_\mu(\sigma(\alpha))$ and $W_\mu \equiv \partial_\mu U$. Using formulas from Appendix [3], one can obtain:

$$\int_0^1 a^2 = 2 s^3 g^2 \int_0^1 d\alpha W_\mu(\sigma(\alpha)) W_\mu(\sigma(\beta))$$

(7.2)

$$\frac{1}{s} \left( \int_0^1 a^2 \right)^2 = s^3 g^2 \int_0^1 d\alpha W_\mu(\sigma(\alpha)) W_\mu(\sigma(\beta))$$

(7.3)

so that the final expression for the $< L >$-approximation is:

5 A similar approach is called the string inspired formalism by Bern-Kosower [10, 11]: the zero-mode variable is separated and the expansion over the oscillating modes is performed, allowing the efficient resummation of many parts of perturbative Feynman graphs.
\[ g^{<L>}(s|y|gU) = g^{<0>}(s|y|gU)e^{2is^3g^2\int_0^1 d\alpha(1-\alpha)W(\sigma(\alpha))\int_0^\infty d\beta \beta W'(\sigma(\beta))} \]  

At \( z = 0 \) it simplifies to:

\[ g^{<L>}(s|x,x|gU) = \frac{i}{(4\pi is)^2}e^{isgU(x) + i\frac{3}{4}W^2(x)} \]  

**VIII. \(<Q>-\text{Approximation}**

In the case of the \(<Q>-\text{Approximation} \) one retain up to quadratic terms (in the \( u \) variable) in the Taylor expansion of \( U \). The parametrix is again a product of the \(<0>\) form and a factor with the functional operations:

\[ e^{-i\int_0^1 s_u^2 e^{-i\int_0^{s_u} u + \frac{1}{2}\int_0^{s_u} u \cdot b_a} \int_{u=0}^1} = e^{-\frac{1}{2}\text{Tr}\ln(1-2\Pi \cdot b) + i\int a(1-2\Pi \cdot b)^{-1} \cdot \Pi \cdot a} \]

where \( b_{\mu\nu}(t,t') : = gs\int_{\max(t,t')}^1 d\alpha \mathcal{H}_{\mu\nu}(\sigma(\alpha)), \mathcal{H}(x) : = (\partial \otimes \partial U)(x) \), and eq (A4) has been used.

The determinantal term \( N[gU] : = -\frac{1}{2}\text{Tr}\ln(1-2\Pi \cdot b) = \sum_{n=1}^\infty \frac{\alpha^n}{n} \text{Tr}[(\Pi \cdot b)^n] \) can be evaluated term by term, but no general closed form can be obtained. The first two terms in that sum are (Appendix II):

\[ N[gU]_1 = +gs^2 \int_0^1 d\alpha(1-\alpha)\partial^2 U(\sigma(\alpha)) \]
\[ N[gU]_2 = 2gs^2 \int_0^1 d\alpha(1-\alpha)^2 \mathcal{H}_{\mu\nu}(\sigma(\alpha))\int_0^\infty d\beta \mathbf{b}^\mu \mathbf{b}^\nu(\sigma(\beta)) \]

The value of \( N[gU] \) at \( z = 0 \) can be obtained explicitly, using a different, more direct approach (Appendix III), with the result:

\[ N[gU]_{z=0} = -\frac{1}{2}\text{Tr} \ln \left( \frac{\sin \left( s\sqrt{2gH} \right)}{s\sqrt{2gH}} \right) = \frac{1}{4} \sum_{m=1}^\infty \frac{|B_{2m}|}{m(2m)!}(8gs^2)^m L_m(x) = \]
\[ = \frac{gs^2}{6}L_1(x) + \frac{g^2s^4}{90}L_2(x) + \frac{4g^3s^6}{2835}L_3(x) + \frac{g^4s^8}{4725}L_4(x) + \frac{16g^5s^{10}}{467779}L_5(x) + \cdots \]

where \( L_m(x) : = \text{Tr}_L(\mathcal{H}^m) \) and \( \text{Tr}_L \) is the trace over Lorentz indices.

In a similar way, for \( z \neq 0 \), one can obtain the \( n \)-th-order terms in Taylor expansion of the expression \( \chi : = i\int a \cdot (1 - 2\Pi \cdot b)^{-1} \cdot \Pi \cdot a \) The first two terms are (Appendix IV):

\[ \chi_0 : = i\int a \cdot \Pi \cdot a = i2s^3g^2 \int_0^1 d\alpha(1-\alpha)W(\sigma(\alpha))\int_0^\infty d\beta \mathbf{b} W'(\sigma(\beta)) \]
\[ \chi_1 : = i2s^3g^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma(\min(\alpha, \gamma) - \alpha \gamma)(\min(\beta, \gamma) - \beta \gamma) W(\sigma(\alpha)) W(\sigma(\beta)) \]

Notice that retaining only \( \chi_0 \) is giving the \(<L>-\text{approximation} \).

It is possible to obtain an closed exact form for \( \chi \) at \( z = 0 \) (Appendix IV):

\[ \chi|_{z=0} = \frac{isg}{2}W \cdot \frac{1}{\mathcal{H}} \cdot \left( \frac{\tan \left( s\sqrt{2\mathcal{H}} \right)}{s\sqrt{2\mathcal{H}}} - 1 \right) \cdot W \]

For example, several first members are:

\[ ^6\text{example: } \text{Tr}_L R : = R^{\alpha\alpha} \]
\[ \chi_0|_{z=0} = i\frac{s^3 g^2}{12} W^2(x) \]
\[ \chi_1|_{z=0} = i\frac{s^5 g^3}{60} (W \cdot \mathcal{H} \cdot W)(x) \]
\[ \chi_2|_{z=0} = i\frac{17}{5040} s^7 (W \cdot \mathcal{H}^2 \cdot W) \quad (8.6) \]

The closed exact form of \(<Q>\>-parametrix is then:

\[ g^{<Q>}(s|x, x|gU) = i \det \left( \frac{\sqrt{2g \mathcal{H}}}{4\pi i \sin (s\sqrt{2g \mathcal{H}})} \right)^{\frac{1}{2}} \exp \left\{ i\frac{s}{2} gW \cdot \frac{1}{\mathcal{H}} \left( \frac{\tan \left( \frac{s\sqrt{2g \mathcal{H}}}{2} \right)}{\frac{s\sqrt{2g \mathcal{H}}}{2}} - 1 \right) \cdot W \right\} \]
\[ = i \left( \frac{1}{4\pi is} \right)^{\frac{1}{2}} e^{isU + \frac{s^2}{12} \mathcal{L}_1 + \frac{s^4g^3}{720} W^2 + \frac{s^6g^5}{36864} \mathcal{L}_2 + \frac{s^8g^7}{5040} (W \cdot \mathcal{H} \cdot W) + \frac{s^{10}g^9}{87178291200} \mathcal{L}_3 + \cdots } \]
\[ \equiv \frac{1}{i(4\pi s)^{\frac{1}{2}}} e^{-iu \frac{s^2}{2}}. \]

(8.7)

\[ \text{IX. The Schwinger-DeWitt Asymptotic Expansion of the Parametrix} \]

The Schwinger-DeWitt Asymptotic Expansion method allows an arbitrarily precise approximation of the parametrix \( g(x, y|gU) \). The method is known under many different names (heat-kernel method, high-temperature asymptotic expansion, etc.) and was applied in various problems (spectral geometry of manifolds with boundaries, \( \mathcal{P} \), propagation of fields in general relativity, etc.). We are presenting here its simplest variant with the slight improvement allowed by the one-component nature of propagating field.

Starting from the Initial Value Problem (IVP) for the parametrix \( g(x, y|gU) \):

\[ -i\partial_x g(s|x, y|gU) = [-\partial_x^2 + gU(x)]g(s|x, y|gU) \]
\[ g(s = 0|x, y|gU) = \delta(x - y) \quad (9.1) \]

one may solve it in two stages: Firstly for the free case \( U \equiv 0 \), with the solution \( g_0(s|x, y) := g(s|x, y|0) = \frac{i}{(4\pi s)^{\frac{1}{2}}} e^{-\frac{s^2}{2u}}. \)

Secondly, for nonzero potential \( U \), introduce the ansatz \( g = g_0 \cdot h \). The IVP for the reduced Parametrix \( h \) is then:

\[ -i \left( \partial_s + \frac{1}{s} z \cdot \partial_x \right) h(s|x, y|gU) = [-\partial_x^2 + gU(x)]h(s|x, y|gU) \]
\[ h(s = 0|x, y|gU) = 1 \quad (9.2) \]

Although it is not possible to solve the first of these equations exactly, the method allows one to reach (in principle) any desired precision. Its idea is based on the notion that the differential operators on the LHS and RHS of IVP (9.2) are homogeneous of degrees \(-1\) and \(0\) respectively. Then, for the formal Taylor series \( h(s) = 1 + \sum_{n=1}^{\infty} (is)^n h_n \) (this is an asymptotic series), one will find a well-separated system (hierarchy) of recursive equations for the functions \( h_n \):

\[ (z \cdot \partial_s + 1) h_1 = gU \]
\[ (z \cdot \partial_s + 2) h_2 = -\partial_x^2 h_1 \]
\[ (z \cdot \partial_s + 3) h_3 = -\partial_x^2 h_2 + gU h_2 \]
\[ (z \cdot \partial_s + 4) h_4 = -\partial_x^2 h_3 + gU h_3 \]
\[ (z \cdot \partial_s + 5) h_5 = -\partial_x^2 h_4 + gU h_4 \]
\[ \cdots \]

\[ (9.3) \]

These can be solved in sequence, beginning with \( h_1 \), and continued to any finite order. Regularities will appear, suggesting that a better (in terms of computational costs) starting point is to use the exponential form \( h = \exp\{\sum_{n=1}^{\infty} (is)^n k_n\} \). One can always switch from one form to another, using algebraic relations which connect the two hierarchies of the coefficient functions \( \{h_n(x, y)\} \) and \( \{k_n(x, y)\} \). The hierarchy of equations for the coefficient functions \( k_n \) is:

\[ h_1 = k_1, h_2 = k_2 + \frac{1}{2} k_1^2, h_3 = k_3 + k_1 k_2 + \frac{1}{6} k_1^3, h_4 = k_4 + k_1 k_3 + \frac{1}{2} k_2^2 + \frac{1}{24} k_1^4 + \frac{1}{2} k_1^2 k_2 + \frac{1}{24} k_1^4, \cdots \]
One can resolve this hierarchy iteratively up to any desired level of accuracy $O(s^n)$. The first several terms are (Appendix K):

\begin{align*}
(z \cdot \partial_x + 1)k_1 &= gU \\
(z \cdot \partial_x + 2)k_2 &= -\partial^2_x k_1 \\
(z \cdot \partial_x + 3)k_3 &= -\partial^3_x k_2 - (\partial_x k_1)^2 \\
(z \cdot \partial_x + 4)k_4 &= -\partial^4_x k_3 - 2\partial_x k_1 \partial_x k_2 \\
(z \cdot \partial_x + 5)k_5 &= -\partial^5_x k_4 - (\partial_x k_2)^2 - 2\partial_x k_1 \partial_x k_3 \\
&\cdots
\end{align*}

(9.4)

Although straight-forward, the calculations become very tedious in higher orders.

The (exponential-)asymptotic expansion of the parametrix has the approximate form:

\begin{align*}
g(s|x, y|gU) &= \frac{i}{4\pi is} e^{i\frac{s^2}{2} e^s \int_0^1 d\alpha U(\sigma(\alpha))} (s^3/g^2)^n \sim s^3 + \cdots
\end{align*}

(9.6)

and one should note the agreement of the $U$-dependent exponential in that expansion (9.6) with terms appearing in previous approximate forms: for example, the $sg$ term comes from the $< 0 >$-approximation; the $s^3g^2$ term comes from the $< L >$-approximation; the $s^2g$ and $s^4g^2$ terms come from the $< Q >$-approximation ($N_2[gU_1]$ and $N_2[gU_2]$), as will all higher terms that depend on second derivatives matrix $\mathcal{H} := (\partial \otimes \partial U)$ only (they will be of order $(s^2g)^n \sim s^{2n}$); the $< Q >$ correction terms for $< L >$, of the type $W \cdot \mathcal{H}^n \cdot W \ (n > 0)$ are not visible here, since their order is $s^3g^2(s^2g)^n \sim s^{3+2n}$. The term $s^3g$ is proportional to the fourth derivative (hyper)matrix $(\partial^4 U)$, and cannot come from either $< L >$ or $< Q >$ approximations. Same is with $(\partial^6 g^2U)$ (with $s^2g$) and higher derivative terms.

The diagonal element (in coordinate space) of the parametrix (i.e. $z = 0, \sigma(\alpha) = x$) takes form:

\begin{align*}
g(s|x, x|gU) &= \frac{i}{4\pi is} e^{i\frac{s^2}{2} e^s \int_0^1 d\alpha U(\sigma(\alpha))} (s^3/g^2)^n \sim s^3 + \cdots
\end{align*}

(9.7)

This form can be useful for the evaluation of the functional determinant that represents vacuum loops in quantum field theory.

As a last point in this section, note that is possible to obtain the full expression for the collection $F_1(z|s)$ of the first order in $g$ terms in the exponential $k = \ln h$ (i.e. $h = e^{gF_1(z|s) + O(g^2)}$). The result is (Appendix [a])

10
\[ F_1(z|s) = is \int_0^1 d\alpha e^{-is(1-\alpha)\partial^2_y} U(y + \alpha z) = is \int_0^1 d\alpha \sum_{n=0}^{\infty} \left[ -is(1-\alpha)\partial^2_y \right]^n U(y + \alpha z) \] (9.8)

It’s diagonal \((z = 0)\) value is
\[ F_1(0|s) = is \sum_{n=0}^{\infty} \frac{1}{(2n+1)!!} \left( -\frac{is}{2} \partial^2_y \right)^n U(y) = isU + \frac{s^2}{6} \partial^2 U - \frac{is^3}{60} \partial^4 U - \frac{s^4}{840} \partial^6 U + \cdots \] (9.9)

which is in full agreement with previous derivations.

X. Conclusion

In this paper we have presented a new form of Fradkin’s parametrix, as well as a set of reasonable approximations. The first of them, the \(<0>-approximation,\) is eikonal in spirit, in the sense that it takes into account only the extreme high-energy behaviour of the propagating particle. It is exact in the case of a laser-like external field. That property will be shown (in the next paper) to be true in more complex cases (spinor particle, gauge and gravitational external fields). Since the \(<0>-form has a simple coordinate representation, we did employ it in calculation of full (nonperturbative) quantum correlation functions for \(\lambda\phi^4_{[D=4]}\) theory. The effective interaction vanishes, displaying the “triviality” of theory.

Beyond the \(<0>-approximation, two directions were explored: first, \(<L>-\text{and }<Q>-\text{approximation that are first two corrections to }<0>,\) derived in the spirit of a semiclassical approach; a difference from the semiclassical approach is that fluctuations are not taken around the classical trajectories of the particle, but around eikonal ones (straight-linear trajectories). Another approach is the Schwinger-DeWitt’s asymptotic expansion method (heat kernel method), which we have adapted and made explicit. The validity of its results is established by a comparison with \(<0/L/Q>-\text{results.}\) The majority of longer derivations have been placed into Appendices.

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APPENDIX A: Some useful linkage and functional identities

\[ e^{-i\alpha \partial^2_y} f(V) \bigg|_{V \to 0} = \int dV \frac{i}{(4\pi\alpha)^{3/2}} e^{-iv^2/4\alpha} f(V) \] (A1)

\[ e^{-i\alpha \partial^2_y} \delta_D(V - a)f(V) \bigg|_{V \to 0} = \frac{i}{(4\pi s)^{1/2}} e^{-i\alpha a^2} f(a) \] (A2)

\[ e^{-if_v dt \delta_z F(v)} \bigg|_{v \to 0} = \int \left[ Dv \right] N(s) e^{-i \int_0^s dt \partial u^2} F[v] \] (A3)

where \(N(s) \equiv \int D\eta e^{i\int_0^s \eta^2} = \left( \int Dv e^{i \int_0^s dt \partial u^2} \right)^{-1}.\)

\[ e^{\int \delta_u \cdot A_u \cdot} \delta_u e^{\int u \cdot B_u} \bigg|_{u \to 0} = \det(1 - 4AB)^{-i} e^{\int \alpha \cdot A^\alpha} \] (A4)
APPENDIX B: Functional form of Green’s function

It is possible (using the identity \( \Delta x = \int_0^1 d\epsilon f_0^s \delta v(t) \)) to write \( \Delta x \) as a functional integral over the auxiliary variable \( v(t) \):

\[
G(x, y|gU) = i \int_0^\infty ds e^{-ism^2} \int [Dv] N(s) e^{-\frac{i}{2} \int_0^s dt v(t) + i \int_0^s dt gU(y + \int_0^s v)} \delta (x - y - \int_0^s v) \tag{B1}
\]

where \( N(s) \) is a normalization constant fixed by the relation \( N(s)^{-1} = \int [Dv] e^{-\frac{i}{2} \int_0^s dt v(t)^2} \).

One can transform the functional-integral form \( \text{(B1)} \) further into a more-heuristically transparent form,

\[
G(x, y|gU) = i \int_0^\infty ds e^{-ism^2} \int DX \cdot e^{-i \int_0^s dt gU(X)} \delta (X(0) - y) \delta (X(s) - x) \tag{B2}
\]

by introducing an auxiliary variable \( X(t) \). Impose on it the constraint \( X(t) = y + \int_0^t \epsilon v \), which is equivalent to the IVP \( \dot{X}(t) = v(t), X(0) = y \). The idea is to solve that IVP for \( v \) (i.e. \( v(t) = \dot{X}(t) \)). Then one can perform a change of variables: \( v \rightarrow X \) by including the identity (see Appendix \( \mathcal{C} \))

\[
1 \equiv \int DX \delta (X(t) - y - \int_0^t \epsilon v) = \int DX \delta (\dot{X}(t) - v(t)) \delta (X(0) - y) \tag{B3}
\]

in \( \text{(B1)} \),

\[
G(x, y|gU) = i \int_0^\infty ds e^{-ism^2} \int [DX] N(s) \delta (X(0) - y) \delta (X(s) - x) e^{-i \int_0^s dt gU(X(t))} S \tag{B4}
\]

where the expression \( S \) is:

\[
S \equiv \int [Dv] e^{-\frac{i}{2} \int_0^s dt v(t)^2} \delta (\dot{X}(t) - v(t)) = \int [DPDv] e^{-\frac{i}{2} \int_0^s dt P \dot{X} + P \dot{v}} = \int [DP] e^{-i \int_0^s dt P X + i \int_0^s dt P^2} \tag{B5}
\]

Finally, one may assemble all terms to get \( \text{(B2)} \).

Its heuristic interpretation is very simple; that of a path-integral amplitude for a particle with Hamiltonian \( H(X, P) = P^2 + U(X) \), which propagates between initial position \( y \) and final position \( x \). Since the particle is relativistic, the reparametrisation symmetry of its world-line must be fixed, i.e. one should integrate over all possible choices of world-line length \( s \) (between these two points), with the "weight function" \( e^{-ism^2} \) determined by the mass of the particle.

APPENDIX C: Change of variables

The proof of the identity:

\[
\int DX \delta (X(t) - y - \int_0^t \epsilon v) = \int DX \delta (\dot{X}(t) - v(t)) \delta (X(0) - y) \tag{C1}
\]

may be given as follows. Start with the left hand side (LHS) of \( \text{(C1)} \), discretize it to

\[
LHS := \lim_{N \to \infty} \int dX_0 dX_1 \cdot \delta (X_0 - y) \delta (X_1 - y - \Delta t v_0) \delta (X_2 - y - \Delta t [v_0 + v_1]) \cdots \delta (X_N - y - \Delta t [v_0 + \cdots + v_{N-1}]) \tag{C2}
\]

where \( \Delta t = \epsilon = \Delta x \), and \( X_m := X((m-1)\Delta t) \) and \( v_m := v(m\Delta t) \). Note that the very first \( \delta \)-function does not have a \( v \)-variable in its argument. Then replace \( y \rightarrow X_0 \) in the second \( \delta \)-function (using first \( \delta \)-function), replace \( v_0 \rightarrow \frac{X_1 - X_0}{\Delta t} \) in the third \( \delta \)-function (now using the second \( \delta \)-function), etc ... As a result, one obtains

\[
\lim_{N \to \infty} (\Delta t)^{N-1} \int dX_0 dX_1 \cdot \delta (X_0 - y) \delta \left( \frac{X_1 - X_0}{\Delta t} - v_0 \right) \delta \left( \frac{X_2 - X_1}{\Delta t} - v_1 \right) \cdots \delta \left( \frac{X_N - X_{N-1}}{\Delta t} - v_{N-1} \right) \tag{C3}
\]

and then remove the regularization \( (N \to \infty) \). The final result is

\[
\int DX \delta (\dot{X}(t) - v(t)) \delta (X(0) - y) \tag{C4}
\]

where the \( \lim_{N \to \infty} (\Delta t)^{N-1} \) is included in the new measure \( DX \).
APPENDIX D: <0>-form versus <PH>-form

To transform the <0>-form (3.1) to (3.3), introduce the auxiliary variable $P := \frac{1}{s}z$ so that

$$g^{<0>}(s|x,y|gU) = \frac{i}{(4\pi is)^{\frac{D}{2}}} e^{-\frac{\mu^2}{4s} + ig\int_0^s dtU(\sigma(\frac{t}{s}))}$$

$$= i(-i\pi s)^{\frac{D}{2}} \int \mathcal{D}P \delta(z-sP) e^{-\frac{\mu^2}{4s} + ig\int_0^s dtU(y+itP)}$$

$$= i(-i\pi s)^{\frac{D}{2}} \int \mathcal{D}P \mathcal{D}q e^{ip\partial_x} \delta(z-sP) e^{-\frac{\mu^2}{4s} + ig\int_0^s dtU(y+itP)}.$$

(D1)

Then represent $i(-i\pi s)^{\frac{D}{2}}$ as $\int dq e^{ip^2}$,

$$g^{<0>}(s|x,y|gU) = \int \mathcal{D}P \mathcal{D}q e^{ip^2 - \frac{i\mu^2}{4s} + \frac{i}{2}q^2 + ig\int_0^s dtU(y+itP)}$$

$$= \int \mathcal{D}P \mathcal{D}q e^{ip^2 + ig\int_0^s dtU(y+itP)}$$

and shift $q \to q + \frac{1}{s}(P+2p)$ to obtain

$$g^{<0>}(s|x,y|gU) = \int \mathcal{D}P \mathcal{D}q e^{ip^2 + ig\int_0^s dtU(y+itP)}$$

$$= \int \mathcal{D}P \mathcal{D}q e^{ip^2 + ig\int_0^s dtU(y+itP)}$$

(D2)

where the last two steps were shifts $P \to P - \frac{1}{s}q$ and $q \to -q$.

That is expression (3.3), suitable for comparison with <PH>-form (3.4).

APPENDIX E: Proof of projected form

The easiest way to prove the form (3.3) is to notice that constrained variable $\Pi \cdot u = u - u_0$ ($u_0 := \frac{1}{s}\int_0^s u$) is given by the expansion (2.1) as $\Pi \cdot u(t) = \sum_{n=1}^\infty (Q_n \cos(n\omega t) + P_n \sin(n\omega t))$. Then $\int_0^s \delta^2 u = \frac{s}{2} \sum_{n=1}^\infty \left\{ \frac{\partial^2}{\partial Q_n^2} + \frac{\partial^2}{\partial P_n^2} \right\}$ and $\int_0^t u = tu_0 + \frac{s}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \{Q_n \sin(n\omega t) + P_n [1 - \cos(n\omega t)]\}$, giving:

$$g(s|x,y|gU) = \frac{i}{(4\pi is)^{\frac{D}{2}}} e^{-\frac{\mu^2}{4s} + \frac{2s}{s^2} \left\{ \frac{\partial^2}{\partial Q_n^2} + \frac{\partial^2}{\partial P_n^2} \right\}}$$

$$e^{ig\int_0^s dtU(\sigma(\frac{t}{s}) + tu_0 + \frac{t}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \{Q_n \sin(n\omega t) + P_n [1 - \cos(n\omega t)]\})} \rangle_{\{Q_n\}, \{P_n\}, u_0 \to 0}$$

(E1)

Since there is no linkage operation with respect to $u_0$, one can directly proceed to the $u_0 \to 0$ limit, reproducing the (2.7) form.

APPENDIX F: Evaluation of $N [gU]_{1,2}$

Start from (110):

$$(\Pi \cdot b)^{\mu \tau}_{\nu \nu} = g_s \int_0^1 \, d\alpha \mathcal{H}^{\mu \nu}(\sigma(\alpha)) \left[ \theta \left( \alpha - \max \left( \frac{t}{s}, \frac{\tau}{s} \right) \right) - \alpha \theta \left( \alpha - \frac{\tau}{s} \right) \right]$$

(F1)

where $\mathcal{H}^{\mu \nu} := (\partial^\mu \partial^\nu U)$. Then

$$N [gU]_1 = \text{Tr}(\Pi \cdot b) = g_s \int_0^s dt (\Pi \cdot b)^{\mu \nu}_{\nu \nu}$$

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\[= gs^2 \int_0^1 d\beta \int_0^1 d\alpha (\partial^2 U)(\sigma(\alpha))(1-\alpha)\theta(\alpha-\beta)\]
\[= gs^2 \int_0^1 d\beta \int_0^1 d\alpha (\partial^2 U)(\sigma(\alpha))(1-\alpha)\theta(\alpha-\beta)\]
\[= gs^2 \int_0^1 d\alpha (\partial^2 U)(\sigma(\alpha))\alpha(1-\alpha)\] (F2)

and

\[N[gU]_2 = \text{Tr}(\Pi \cdot b \cdot \Pi \cdot b)\]
\[= g^2 s^2 \int_0^\infty dt \int_0^1 d\tau \int_0^1 d\alpha \int_0^1 d\beta \mathcal{H}_{\mu \nu}(\sigma(\alpha))\mathcal{H}^{\mu \nu}(\sigma(\beta))\]
\[\left[\theta (\alpha - \max \left(\frac{t}{s}, \frac{\tau}{s}\right)) - \alpha \theta (\alpha - \frac{\tau}{s})\right] \cdot \left[\theta (\beta - \max \left(\frac{t}{s}, \frac{\tau}{s}\right)) - \beta \theta (\beta - \frac{\tau}{s})\right]\]
\[= g^2 s^4 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \int_0^1 d\delta \mathcal{H}_{\mu \nu}(\sigma(\alpha))\mathcal{H}^{\mu \nu}(\sigma(\beta))\]
\[\left[\theta (\alpha - \max (\gamma, \delta)) - \alpha \theta (\alpha - \delta)\right] \cdot \left[\theta (\beta - \max (\gamma, \delta)) - \beta \theta (\beta - \gamma)\right]\]
\[= g^2 s^4 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \int_0^1 d\delta \mathcal{H}_{\mu \nu}(\sigma(\alpha))\mathcal{H}^{\mu \nu}(\sigma(\beta))\]
\[\left[\min(\alpha, \beta)^2 - 2\alpha \min(\alpha, \beta) + \alpha^2 + \beta^2\right]\]
\[= 2g^2 s^4 \int_0^1 d\alpha (1-\alpha)^2 \mathcal{H}_{\mu \nu}(\sigma(\alpha)) \int_0^\infty d\beta (\beta - \alpha)^2\mathcal{H}^{\mu \nu}(\sigma(\beta))\] (F3)

where tools from Appendix J were utilized.

**APPENDIX G: Evaluation of** \(N[gU]_{z=0}\)

In this appendix the "loop-generator" \(N[gU] := -\frac{1}{2} \text{Tr} \ln(A)\) for matrix \(A := 1 - 2\Pi \cdot b\) for the case \(z = 0\) is evaluated. Matrix elements of \(A\) are given by

\[A^{\mu \nu}_{\mu \tau} = \eta^{\mu \nu} \delta (t - \tau) - 2(\Pi \cdot b)^{\mu \nu}_{\mu \tau} \quad (J1)\]
\[\equiv \frac{1}{s} \left\{ \eta^{\mu \nu} \delta \left( \frac{t}{s} - \frac{\tau}{s} \right) - gs^2 \mathcal{H}^\mu \nu(x) K \left( \frac{t}{s}, \frac{\tau}{s} \right) \right\}\]

where \(K(\alpha, \beta) := 1 - 2 \max(\alpha, \beta) + \beta^2\). We will exploit fact that \(N[gU]\) can be represented in terms of dimensionless variables \((\alpha = t/s, \text{ etc}...)\) in the form \(N[gU] = -\frac{1}{2} \text{Tr} \ln(\overline{A})\), where \(\overline{A}^{\mu \nu}_{\mu \tau} = s A^{\mu \nu}_{\mu \tau} \quad (J1)\) and \(\overline{A}^{\mu \nu}_{\mu \tau} = s \eta^{\mu \nu} \delta (t - \beta) - gs^2 \mathcal{H}^\mu \nu(x) K(\alpha, \beta)\), where \(\overline{A} = s \int_0^\infty d\alpha (..., (\alpha, \alpha)\).

The operator \(\hat{K}\) whose kernel is \(K(\alpha, \beta)\) is not completely diagonalisable, since it is not a **normal operator**, i.e. it does not satisfy condition of normality \(\hat{K}^\dagger \hat{K} = \hat{K} \hat{K}^\dagger\). To see this, evaluate matrix elements of both sides:

\[(\hat{K}^\dagger \hat{K})(\alpha, \gamma) = \int_0^1 d\beta K(\beta, \alpha) K(\beta, \gamma)\]
\[(\hat{K} \hat{K}^\dagger)(\alpha, \gamma) = \int_0^1 d\beta K(\beta, \alpha) K(\beta, \gamma)\]

where \(\gamma \geq \alpha\) is assumed, and (114), (117) were used.

We will here first resolve the decomposition problem of \(\hat{K}\), and only then we will return to evaluation of spectral problem for operator \(\overline{A}\).

**1. Structure of** \(\hat{K}\)

Since \(\hat{K}\) is not normal, it cannot be completely diagonalized. Even worst, its left- and right-hand-side eigenvectors are not the same, and those sets of eigenvectors are not complete basis sets.
Here, we will find the right-hand-side-eigenvectors of \( \hat{K} \). To find them, start with the eigen-problem

\[
\int_0^1 d\beta K(\alpha, \beta) R(\beta) = \lambda R(\alpha)
\]

and its two differential consequences in explicit forms:

\[
\int_0^1 d\beta R(\beta) - 2\alpha \int_0^\alpha d\beta R(\beta) - 2 \int_0^1 d\beta R(\beta) + \int_0^1 d\beta \beta^2 R(\beta) = \lambda R(\alpha) \tag{G4}
\]

\[-2 \int_0^\alpha d\beta R(\beta) = \lambda R'(\alpha) \tag{G5}\]

\[-2 R(\alpha) = \lambda R''(\alpha) \tag{G6}\]

Substituting the general solution \( R(\alpha) = A e^{i\nu\alpha} + B e^{-i\nu\alpha} \) of that last equation (where \( \lambda = \frac{2}{\nu^2} \)) into (G3) one gets \( B = A \), so \( R(\alpha) = 2A \cos(\nu\alpha) \). Equation (G4) further constrains the solution to frequencies \( \nu \) such that \( \sin(\nu\alpha) = 0 \). This gives possible eigen-frequencies \( \nu_n \equiv n\pi \) (\( n \geq 1 \)), corresponding to the eigenvalues \( \lambda_n = \frac{2}{\pi^2n^2} \) and normalized (in terms of scalar product \( <a|b> = \int_0^1 da da(b)\)) eigenvectors \( R_n(\alpha) = \sqrt{2} \cos(n\pi\alpha) \). Notice that the \( R_n \) are not the left-hand-side-eigenvectors of \( \hat{K} \).

Also, although the vectors \( R_n \) are mutually orthogonal, they are not a complete set: the supposed decomposition of unity gives the projection operator \( \sum_{n=1}^\infty R_n(\alpha)R_n(\beta) = \delta(\alpha - \beta) - 1 \equiv \pi(\alpha, \beta) \) with defect one.

A similar procedure for left-hand-side-eigenvectors does not allow solutions in terms of elementary functions, and will not be presented here. The only left-hand-side-eigenvector that is easy to obtain is \( L_0(\alpha) = 1 \), with corresponding eigenvalue 0. It is not the right-hand-side-eigenvector of \( \hat{K} \).

In terms of ortho-normal basis \( \{ L_0 = 1, R_n = \sqrt{2} \cos(n\pi\alpha): n \geq 1 \} \) one can write the resolution \( \hat{K} = \hat{K}_D + \hat{K}_J \) on \( \hat{K}_D \), the diagonalisable part, and \( \hat{K}_J \), the non-diagonalisable (Jordan type) part:

\[
\hat{K}_D = \sum_{n=1}^\infty \lambda_n |R_n><R_n|
\]

\[
\hat{K}_J = -\sqrt{2} \sum_{n=1}^\infty (-)^n \lambda_n |R_n><L_0| \equiv |KL_0><L_0| \tag{G7}
\]

where \( \lambda_n = \frac{2}{\pi^2n^2} \), and \( |KL_0><L_0| = -\sqrt{2} \sum_{n=1}^\infty (-)^n \lambda_n |R_n| \) (note that \( |L_0><KL_0| \equiv 0 \)).

In the above basis, the matrix form of full operator \( \hat{K} \) is

\[
\hat{K} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\sqrt{2}\lambda_1 & \lambda_1 & 0 & 0 & 0 & \cdots \\
-\sqrt{2}\lambda_2 & 0 & \lambda_2 & 0 & 0 & \cdots \\
\sqrt{2}\lambda_3 & 0 & 0 & \lambda_3 & 0 & \cdots \\
-\sqrt{2}\lambda_4 & 0 & 0 & 0 & \lambda_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\tag{G8}
\]

In coordinate form, kernels of the \( \hat{K}_{D,J} \) operators are

\[
K_J(\alpha, \beta) = \int_0^1 d\gamma K(\alpha, \gamma) = \frac{1}{3} - \alpha^2
\]

\[
K_D(\alpha, \beta) = \int_0^1 d\gamma K(\alpha, \gamma)\pi(\gamma, \beta) = \frac{2}{3} - 2\max(\alpha, \beta) + \alpha^2 + \beta^2 = \sum_{n=1}^\infty \frac{4}{\pi^2n^2} \cos(n\pi\alpha) \cos(n\pi\beta) \tag{G9}
\]

Also: \( KL_0(\alpha) = \frac{1}{3} - \alpha^2 \).
2. Spectral structure of $\mathcal{A}$: Evaluation of $N[gU]|_{z=0}$

Using the known decomposition of the operator $\hat{K}$, one can obtain the corresponding representation of the operator $\mathcal{A}$:

$$
\mathcal{A} = 1 \otimes \hat{1} - gs^2 \hat{K} \otimes \hat{\mathcal{H}} = 
\left(\left[|L_0\rangle\langle L_0| + \sum_{n=1}^{\infty} |R_n\rangle\langle R_n|\right) \otimes \hat{1} - gs^2 \left(\sum_{n=1}^{\infty} \lambda_n |R_n\rangle \{<R_n| - \sqrt{2}(-)^n < L_0|\}\right) \otimes \hat{\mathcal{H}} = 
\begin{pmatrix}
1 & 0 & 0 & \ldots \\
-gs^2\sqrt{2}\lambda_1 \hat{\mathcal{H}} & 1 - gs^2\lambda_1 \hat{\mathcal{H}} & 0 & \ldots \\
gs^2\sqrt{2}\lambda_2 \hat{\mathcal{H}} & 0 & 1 - gs^2\lambda_2 \hat{\mathcal{H}} & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix} 
$$

From that representation one can read its eigenvalues:

$$
\lambda_{a,n}(x) \equiv 1 - gs^2\Lambda_a(x)\lambda_n = 1 - \frac{2gs^2\Lambda_a(x)}{\pi^2n^2} ; \quad n \geq 1
$$

where $\Lambda_a(x)$ ($a = 1, D$) are the eigenvalues of the matrix $\hat{\mathcal{H}}(x)$. The $n = 0^{th}$ eigenvalue is equal 1.

In this way, $N[gU]|_{z=0}$ is given as

$$
N[gU]|_{z=0} = -\frac{1}{2} \sum_{a=1}^{D} \sum_{n=1}^{\infty} \ln \left(1 - \frac{2gs^2\Lambda_a(x)}{\pi^2n^2}\right) = -\frac{1}{2} \text{Tr}_L \ln \left(\frac{\sin \left(s\sqrt{2g\mathcal{H}}\right)}{s\sqrt{2g\mathcal{H}}}\right) 
$$

To obtain the expansion in $s$, Taylor expand the logarithm in the first form. The first term is:

$$
N[gU]_1|_{z=0} = -\frac{1}{2} \sum_{a=1}^{D} \sum_{n=1}^{\infty} \left(\frac{2gs^2\Lambda_a(x)}{\pi^2n^2}\right) = \frac{gs^2}{\pi^2} \left(\sum_{a=1}^{D} \Lambda_a(x)\right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \frac{gs^2}{6}(\partial^2 U)(x) 
$$

where $S_2 = \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{1 - \frac{1}{n^2}} = \frac{\pi^2}{6}$ and $L_1(x) := \sum_{a=1}^{D} \Lambda_a(x) = \text{Tr}_L \hat{\mathcal{H}}(x) = (\partial^2 U)(x)$.

The second term in that expansion gives

$$
N[gU]_2|_{z=0} = -\frac{1}{2} \sum_{a=1}^{D} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right) \left(\frac{2gs^2\Lambda_a(x)}{\pi^2n^2}\right)^2 = \frac{gs^4}{\pi^4} L_2(x) S_4 = \frac{gs^4}{90} \mathcal{H}^2(x) 
$$

where $L_2(x) := \sum_{a=1}^{D} \Lambda_a(x)^2 = \text{Tr}_L [\hat{\mathcal{H}}(x)^2] = \mathcal{H}_{\mu\nu} \mathcal{H}^{\mu\nu}$ and $S_4 := \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Similarly, one can evaluate higher terms in $N[gU]|_{z=0}$, to obtain

$$
N[gU]|_{z=0} = \frac{1}{4} \sum_{m=1}^{\infty} \frac{|B_{2m}|}{m(2m)!} (8gs^2)^m L_m(x) 
$$

$$
= \frac{gs^2}{6} L_1(x) + \frac{gs^4}{90} L_2(x) + \frac{4gs^{3}S_6}{2835} L_3(x) + \frac{4g^2S_8}{4725} L_4(x) + \frac{16g^5S_{10}}{467775} L_5(x) + \cdots 
$$

where $L_m(x) := \text{Tr}_L [\hat{\mathcal{H}}(x)^m]$. 

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APPENDIX H: On evaluation of $\chi[gU]$

Starting from

$$a_\mu(t) = gs \int_{t/s}^1 d\alpha W_\mu(\sigma(\alpha))$$
$$b_{\mu\nu}(t_1, t_2) = gs \int_{\max(t_1, t_2)}^1 d\alpha H_{\mu\nu}(\sigma(\alpha))$$

(H1)

and $\Pi(t_1, t_2) = \delta(t_1 - t_2) - \frac{1}{s}$, we obtain:

$$(a_\perp)_{\mu}(t) : (a \cdot \Pi)_{\mu}(t) = gs \int_0^1 d\alpha W_\mu(\sigma(\alpha)) \left[ \theta \left( \alpha - \frac{t}{s} \right) - \alpha \right]$$
$$(b_{\perp \perp})_{\mu\nu}(t_1, t_2) : (\Pi \cdot b \cdot \Pi)_{\mu\nu}(t_1, t_2) = gs \int_0^1 d\alpha H_{\mu\nu}(\sigma(\alpha)) \left[ \theta \left( \alpha - \frac{t_1}{s} \right) - \alpha \right] \left[ \theta \left( \alpha - \frac{t_2}{s} \right) - \alpha \right]$$

(H2)

Then

$$\chi_0 = i \int a_\perp \cdot a_\perp = i \int_0^s dt g^2 s^2 \int_0^1 d\alpha \int_0^1 d\beta W_\mu(\sigma(\alpha)) W^\mu(\sigma(\beta)) \left[ \theta \left( \alpha - \frac{t_1}{s} \right) - \alpha \right] \left[ \theta \left( \beta - \frac{t_2}{s} \right) - \beta \right]$$

$$= i g^2 s^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma W_\mu(\sigma(\alpha)) W^\mu(\sigma(\beta)) \left[ \theta \left( \alpha - \gamma \right) - \alpha \right] \left[ \theta \left( \beta - \gamma \right) - \beta \right]$$

$$= i g^2 s^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma W_\mu(\sigma(\alpha)) W^\mu(\sigma(\beta)) \left[ \theta \left( \alpha - \gamma \right) - \alpha \right] \left[ \theta \left( \beta - \gamma \right) - \beta \right]$$

(H3)

On the same way:

$$\chi_1 = 2i \int a_\perp \cdot b_{\perp \perp} \cdot a_\perp =$$

$$(J18) \quad i g s^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma W_\mu(\sigma(\alpha)) W^\mu(\sigma(\beta)) \left[ \theta \left( \alpha - \gamma \right) - \alpha \right] \left[ \theta \left( \beta - \gamma \right) - \beta \right]$$

$$= i g^2 s^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma W_\mu(\sigma(\alpha)) W^\mu(\sigma(\beta)) \left[ \theta \left( \alpha - \gamma \right) - \alpha \right] \left[ \theta \left( \beta - \gamma \right) - \beta \right]$$

(H4)

At $z = 0$ the $\alpha, \beta$ and $\gamma$ integrals can be performed explicitely, using formulas from Appendix I, to obtain same results as in Appendix H. However, derivation of $\chi_m$’s (at $z = 0$) given there is superior to this one (much less tedious).

APPENDIX I: Evaluation of $\chi[gU]|_{z=0}$

From (G10) is easy to obtain the inverse of operator $\overline{A}$ as:

$$\overline{A}^{-1} = |L_0 > < L_0| \otimes 1 + \sum_{n=1}^\infty |R_n > < R_n| \otimes (1 - g s^2 \lambda_n \hat{H})^{-1} +$$

$$- \sum_{n=1}^\infty \sqrt{2} (-)^n g s^2 \lambda_n |R_n > < L_0| \otimes (1 - g s^2 \lambda_n \hat{H})^{-1} \hat{H}$$

(I1)

To evaluate $\chi$, only projection of that inverse is necessary:

$$\overline{A}^{-1} \cdot \pi = \sum_{n=1}^\infty |R_n > < R_n| \otimes (1 - g s^2 \lambda_n \hat{H})^{-1}$$

(I2)
where \( \pi = \sum_{n=1}^{\infty} |R_n| < R_n \otimes \mathbf{1} \). To obtain \( A^{-1} \cdot \Pi \) multiply above result with \( s \). Then
\[
\chi|_{z=0} = i \int \int a \cdot A^{-1} \cdot \Pi \cdot a = is \int_0^1 d\alpha \int_0^1 d\beta \sum_{n=1}^{\infty} R_n(\alpha)R_n(\beta)a^\mu(s\alpha) \left( \frac{1}{1 - gs^2\lambda_n \mathcal{H}(x)} \right) \cdot a^\nu(s\beta)
\]
where \( a^\mu(s\alpha) = gs(1 - \alpha)W^\mu(x), W^\mu(x) \equiv \partial^\mu U(x) \). Then
\[
\chi|_{z=0} = 2is^3g^2 \sum_{n=1}^{\infty} \left[ \int_0^1 d\alpha (1 - \alpha) \cos(n\pi\alpha) \right]^2 W \cdot (1 - gs^2\lambda_n \mathcal{H}(x))^{-1} \cdot W = \frac{2is^3g^2}{\pi^4} \sum_{n=1}^{\infty} \frac{1 - (-)^n|n^2|}{n^4} W \cdot (1 - gs^2\lambda_n \mathcal{H}(x))^{-1} \cdot W
\]

From this point one can proceed in two alternate routes. First is to Taylor-expand matrix factor in the above form and to obtain an explicit expressions for \( \chi_m \) (\( \chi = \sum_{m=0}^{\infty} \chi_m \)): 
\[
\chi_m|_{z=0} = \frac{i2^{m+1}s^{2m+3}g^{m+2}}{\pi^{2m+4}} (W \cdot \mathcal{H}^m \cdot W) \tilde{S}_{2m+4}
\]
where
\[
\tilde{S}_{2m+4} \equiv \sum_{n=1}^{\infty} \frac{1 - (-)^n|n^2|}{n^{2m+4}} = 4(1 - 2^{-2m-4}) \zeta(2m + 4) = \frac{2(2^{2m+4} - 1)\pi^{2m+4}}{(2m + 4)!}|B_{2m+4}|
\]
giving
\[
\chi_m|_{z=0} = \frac{i2^{m+2}(2^{2m+4} - 1)s^{2m+3}g^{m+2}B_{2m+4}}{(2m + 4)!} (W \cdot \mathcal{H}^m \cdot W)
\]
Several first members of that series are:
\[
\begin{align*}
\chi_0|_{z=0} &= \frac{i s^3 g^2 W^2}{12} \\
\chi_1|_{z=0} &= \frac{i s^3 g^3 W \cdot \mathcal{H} \cdot W}{60} \\
\chi_2|_{z=0} &= \frac{i 17 s^7 g^4 W \cdot \mathcal{H}^2 \cdot W}{5040} \\
\chi_3|_{z=0} &= \frac{i 31 s^9 g^5 W \cdot \mathcal{H}^3 \cdot W}{37800}
\end{align*}
\]
The second possible route is to regroup terms in (I4):
\[
\chi|_{z=0} = \frac{2is^3g^2}{\pi^4} W \cdot \sum_{n=1}^{\infty} \frac{1 - (-)^n|n^2|}{n^2 - a^2} \cdot W
\]
where \( a^2 \equiv \frac{2gs^2}{\pi^4} \mathcal{H} \), and to evaluate given sum
\[
\sum_{n=1}^{\infty} \frac{1 - (-)^n|n^2|}{n^2 - a^2} \cdot \frac{1}{n^2 - a^2} = \frac{\pi^2}{2a^2} \left( \tan \left( \frac{\pi a}{2} \right) - 1 \right)
\]
Then
\[
\chi|_{z=0} = \frac{is^3g^2}{\pi^2} W \cdot \frac{1}{a^2} \left( \tan \left( \frac{\pi a}{2} \right) - 1 \right) \cdot W = \frac{isg}{2} W \cdot \frac{1}{\mathcal{H}} \left( \tan \left( \frac{s \sqrt{\frac{2\mathcal{H}}{a^2}}}{2} \right) - 1 \right) \cdot W
\]
APPENDIX J: Tools for parametrix calculations

\[
(z \partial_z + n)k_n(x, y) = j_n(x, y) \Rightarrow k_n(x, y) = \int_0^1 d\alpha^{n-1} j_n(\alpha, y) \tag{J1}
\]

\[
\frac{\partial\alpha}{\partial x} f(\alpha) = \alpha^{n-1}(\partial f)(\alpha) \tag{J2}
\]

\[
\int_0^1 d\alpha^{n-1} f(\alpha) = \beta^{n-1} \int_0^\beta d\alpha f(\alpha) \tag{J3}
\]

\[
\int_0^1 d\beta \int_0^1 d\alpha f(\alpha) = \int_0^1 d\alpha (\alpha^m - \alpha^l) f(\alpha) \tag{J4}
\]

\[
\int_0^1 d\beta f(\alpha) = \frac{1}{n} \int_0^1 d\alpha (1 - \alpha^n) f(\alpha) \tag{J5}
\]

\[
\int_0^1 d\gamma \int_0^1 d\alpha f(\alpha) g(\alpha) = \frac{1}{l - m} \int_0^1 d\alpha (\alpha^m - \alpha^l) f(\alpha) \int_0^\alpha d\beta g(\beta) + \int_0^\alpha d\beta (1 - \beta^{l-m-n}) g(\beta) \tag{J6}
\]

\[
\int_0^1 d\gamma \int_0^1 d\alpha f(\alpha) g(\beta) = \frac{1}{l - m} \int_0^1 d\alpha (\alpha^m - \alpha^l) f(\alpha) \int_0^\alpha d\beta g(\beta) \tag{J7}
\]

\[
\int_0^1 d\alpha f(\alpha) \int_0^\alpha d\beta g(\beta) = \int_0^1 d\alpha g(\alpha) \int_0^\alpha d\beta f(\beta) \tag{J8}
\]

\[
\int_0^1 d\beta \Theta(\alpha - \max(\beta, \gamma)) = \alpha \Theta(\alpha - \gamma) \tag{J9}
\]

\[
(\Pi \cdot b)_{\mu \nu}(t_1, t_2) = gs \left[ \int_{\max(t_1, t_2)/s}^1 d\alpha (\partial_\alpha \partial_\mu U)(\sigma(\alpha)) - \int_{t_2/s}^1 d\alpha (\partial_\alpha \partial_\mu U)(\sigma(\alpha)) \right] \tag{J10}
\]

\[
(\Pi \cdot b)_{\mu \nu}(t_1, t_2)_{t=0} = gs \left[ \frac{1}{2} \max \left(\frac{t_1}{s}, \frac{t_2}{s}\right) + \frac{1}{2} \left(\frac{t_2}{s}\right)^2 \right] H_{\mu \nu}(x) = \frac{1}{2} g s H_{\mu \nu}(x) K \left(\frac{t_1}{s}, \frac{t_2}{s}\right) \tag{J11}
\]

\[
\int_0^1 d\gamma \int_0^1 d\delta \Theta(\alpha - \max(\gamma, \delta)) \Theta(\beta - \max(\gamma, \delta)) = \min(\alpha, \beta)^2 \tag{J12}
\]

\[
\int_0^1 d\gamma \int_0^1 d\delta \Theta(\alpha - \max(\gamma, \delta)) \Theta(\beta - \gamma) = \alpha \min(\alpha, \beta) \tag{J13}
\]

\[
\int_0^1 d\beta \max(\beta, \alpha) = \frac{1}{n+2} \left(1 + \frac{1}{n+1} \alpha^{n+2} \right) \tag{J14}
\]

\[
\int_0^1 d\beta \max(\beta, \alpha) = \frac{1}{2} (1 + \alpha^2) \tag{J15}
\]

\[
\int_0^1 d\beta \max(\beta, \alpha) = \frac{1}{4} \left(1 + \frac{1}{3} \alpha^4 \right) \tag{J16}
\]

\[
\int_0^1 d\beta \max(\beta, \alpha) = \frac{1}{3} + \frac{1}{2} \alpha \min(\gamma, \alpha) + \frac{1}{6} \max(\gamma, \alpha)^3 \tag{J17}
\]

\[
\int_0^1 d\gamma \left[\theta(\alpha - \gamma) - \alpha\right] \left[\theta(\beta - \gamma) - \beta\right] = \min(\alpha, \beta) - \alpha \beta \tag{J18}
\]

\[
\theta(\gamma - \max(\mu, \nu)) \theta(\gamma - \nu) = \theta(\gamma - \mu) \theta(\gamma - \nu) \tag{J19}
\]

\[
\int_0^1 d\beta (\min(\alpha, \beta) - \alpha \beta) = \frac{1}{2} \alpha (1 - \alpha) \tag{J20}
\]

APPENDIX K: Schwinger-DeWitt calculations

Rewrite the system (9.5) in the form

\[
(z \cdot \partial_z + n)k_n[y, z] = j_n[y, z] \tag{K1}
\]

where \( n \geq 1, k_n[y, z] = k_n(x, y) \) and: \( j_1 = gU, j_2 = -\partial_z k_1, j_3 = -\partial_z^2 k_2 - (\partial_z k_1)^2, j_4 = -\partial_z^2 k_3 - 2(\partial_z k_1)(\partial_z k_2), \ldots \). Then, for every given (known) \( j_n[y, z] \), the corresponding \( k_n[y, z] \) is given by the expression (Appendix \( \square \zeta \), Eq \( \square \zeta \))
Then, with known $k_n[y,z]$, one can evaluate the next source function $j_{n+1}[y,z]$, etc. During the calculations several characteristic procedures are taking place. As an illustration, let us calculate the first several $k$s.

Since $j_1[y,z] = gU(y + z)$, it follows that $k_1[y,z] = g\int_0^1 daU(y + az)$.

The second source function takes the form $j_2[y,z] = -g\partial_z^2 \int_0^1 daU(y + az)$. Then, each $\partial_z$ converts into $\alpha \partial_y$ upon action on $U(y + az)$. So $j_2[y,z] = -g\int_0^1 da \alpha \partial^2 U(y + az)$. That was first characteristic procedure: $\partial_z \rightarrow \alpha \partial_y$.

Then $k_2[y,z] = \int_0^1 da \alpha j_2[y,az] = -g\int_0^1 da \int_0^1 d\beta \beta^2(\partial^2 U)(y + az)$. Rescale $\beta \rightarrow \beta/\alpha$ to make argument of $U$ free of $\alpha$: $k_2[y,z] = -g\int_0^1 da \int_0^\alpha d\beta \beta^2(\partial^2 U)(y + \beta z) = -g\int_0^1 da \alpha \partial^2 U(y + \beta z)$ (The second procedure: $\beta \rightarrow \beta/\alpha$). Extend the range of integration of $\beta$ back to $[0,1]$, introducing the step function $\theta(\alpha - \beta)$, switch the order of integration (first to perform the $\alpha$ integral, and only then the $\beta$ integral), and limit the range of integration of $\alpha$ to $[\beta, 1]$ (according to the argument of the $\theta$ function): $k_2[y,z] = -g\int_0^1 d\beta \beta^2(\partial^2 U)(y + \beta z) \int_0^\alpha d\alpha \alpha^{-2}$ (Third procedure: switch the order of integrals). The integral over $\alpha$ can be performed, giving $\int_0^\alpha d\alpha \alpha^{-2} = \beta^{-1}(1 - \beta)$ and $k_2[y,z] = -g\int_0^1 d\beta \beta(1 - \beta)(\partial^2 U)(y + \beta z)$.

The third source function becomes $j_3[y,z] = -\partial_z^2 k_2 - (\partial_z k_2)^2 = g\int_0^1 d\beta \beta^2(1 - \beta)(\partial^2 U)(y + \beta z) - g^2\int_0^1 d\beta \beta(\partial^2 U)(y + \beta z) \int_0^\alpha d\gamma (\partial U)(y + \gamma z)$. Replaying the same order of procedures as above, one can obtain (in seven lines of derivation) $k_3[y,z] = \frac{g^2}{2} \int_0^1 d\alpha \beta^2 (1 - \alpha)^2 (\partial^2 U)(\sigma(\alpha)) - 2g^2\int_0^1 d\alpha(1 - \alpha)(\partial U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(\partial U)(\sigma(\beta))$.

In the same way, the fourth $k$ function can be obtained as:

$$
k_4[y,z] = -\frac{1}{6}g\int_0^1 da \alpha^3 (1 - \alpha)^2 (\partial^2 U)(\sigma(\alpha)) +
+g^2\int_0^1 da \alpha (1 - \alpha)^2 (\partial^2 U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(\partial U)(\sigma(\beta)) +
+2g^2\int_0^1 da \alpha (1 - \alpha)^2 (\partial_\alpha \partial_z U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(\partial^2 U)(\sigma(\beta)) +
+g^2\int_0^1 da \alpha (1 - \alpha)^2 (\partial U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(\partial^2 U)(\sigma(\beta)) +
+g^2\int_0^1 da \alpha (1 - \alpha)(\partial U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(\partial^2 U)(\sigma(\beta)) +
+g^2\int_0^1 da \alpha(\partial U)(\sigma(\alpha)) \int_0^\alpha d\beta \beta(1 - \beta)(\partial^2 U)(\sigma(\beta))
$$

(3)

This takes about 30 lines (one and one-half pages of calculation), that is a factor of four longer than the evaluation of $k_3$. One can expect that the length of computation for higher order functions will grow at least as a geometric progression. It is interesting that all steps (procedures) are straightforward, and one could be tempted to program some symbolic mathematical tool to perform them.

Some of the time-saving formulas obtained during calculations are placed in Appendix I.

It is interesting that if one asks different kind of questions, then certain exact (i.e., in all orders in $s$) partial information can be obtained. For example, in the Appendix I the first order in $g$ contribution to $k$ is derived.

APPENDIX L: Derivation of $F_1(z|s)$

From (2.2) follows the IVP for $F_1(z|s)$:

$$
[-i(s\partial_x + z \cdot \partial_z) + s\partial_z^2] F_1(z|s) = sU(x)
F_1(z|0) = 0
$$

(1.1)

Then, using Schwinger’s parametrization one can write

$$
F_1(z|s) = i \int_0^\infty dt e^{-it[-i(s\partial_x + z \cdot \partial_z) + s\partial_z^2]} sU(x)
$$

(2.2)
For a moment switch to an algebraic structure by introducing differential generators \( \hat{a} := -s \partial_s \), \( \hat{b} := -z \cdot \partial_z \) and \( \hat{c} := -is \partial_y^2 \) with the commutational relations \([\hat{a}, \hat{b}] = 0, [\hat{a}, \hat{c}] = -\hat{c}, [\hat{a}, \hat{b}] = 2\hat{c} \). Then we are interested in \( \hat{r} := e^{(\hat{a} + \hat{b} + \hat{c})} \). Assume that it can be factorized in form \( \hat{r} = e^{\hat{A}(t)\hat{a}}e^{\hat{B}(t)\hat{b}}e^{\hat{C}(t)\hat{c}} \). Then the \( [\hat{A}] \) gives equation

\[
\hat{a} + \hat{b} + \hat{c} = \hat{A}\hat{a} + \hat{B}\hat{b} + \hat{C}\hat{c}.
\]

Since \( e^{\hat{A}\hat{a}}(\hat{b}) = \hat{b}, e^{\hat{B}\hat{b}}(\hat{c}) = e^{2\hat{B}\hat{c}} \) and \( e^{\hat{A}\hat{a}}(\hat{c}) = e^{-\hat{A}\hat{c}} \), the IVPs for \( A, B \) and \( C \) follow: \( \hat{A} = \hat{B} = 1, \hat{C} = e^{\hat{A} - 2\hat{B}}, A(0) = B(0) = C(0) = 0 \). Their solution is \( A(t) = B(t) = t, C(t) = 1 - e^{-t} \) giving: \( \hat{r} = e^{\hat{a}t}e^{\hat{b}t}e^{(1-e^{-t})\hat{c}} \).

So:

\[
F_1(z|s) = i \int_0^\infty dt e^{-ts\partial_s} e^{-tz\partial_z} e^{-is(1-e^{-t})\partial_y^2} sU(x)
\]

Use now the Fourier representation for \( U(x) \) \( (x = y + z) \) to obtain:

\[
F_1(z|s) = \int dp e^{ipz} i \int_0^\infty dt e^{-ts\partial_s} e^{-tz\partial_z} e^{+ips} e^{+is(1-e^{-t})p^2} s\tilde{U}(p)
\]

\[
= \int dp e^{ipz} i \int_0^\infty dt e^{-ts\partial_s} e^{ipze^{-t}} e^{-is(1-e^{-t})p^2} s\tilde{U}(p)
\]

\[
= i s \int_0^\infty dt e^{-t} \int dp e^{ipz} e^{-is(1-e^{-t})p^2} \tilde{U}(p)
\]

i.e.

\[
F_1(z|s) = is \int_0^\infty dt e^{-t} \sum_{n=0}^\infty \frac{1}{n!} (-is(1-e^{-t})\partial_y^2)^n U(y + ze^{-t})
\]

Introduce \( \alpha = e^{-t} \) and the final expression is established:

\[
F_1(z|s) = is \int_0^1 d\alpha \sum_{n=0}^\infty \frac{1}{n!} (-i\alpha(1-\alpha)\partial_y^2)^n U(y + z\alpha)
\]

\[
= is \int_0^1 d\alpha U(\sigma(\alpha)) + s^2 \int_0^1 d\alpha(1-\alpha)\partial^2 U(\sigma(\alpha)) +
\]

\[
- \frac{i s^3}{2} \int_0^1 d\alpha \alpha^2(1-\alpha)^2 \partial^4 U(\sigma(\alpha)) - \frac{s^4}{6} \int_0^1 d\alpha(1-\alpha)^3 \partial^6 U(\sigma(\alpha)) + \cdots
\]

It is easy to check that first four terms indeed do agree with expressions obtained through more tedious calculations in previous Appendix.

At \( z = 0 \) the \( \alpha \) dependence disappears from argument of \( U \) and integrals over \( \alpha \) can be performed \((\int_0^1 d\alpha \alpha^n(1-\alpha)^n = B(n + 1, n + 1))\), giving

\[
F_1(z|0) = is \sum_{n=0}^\infty \frac{1}{(2n+1)!!} (-i s/2 \partial_y^2)^n U(y)
\]

\[
= isU(y) + \frac{s^2}{6}(\partial^2 U)(y) + \frac{i s^3}{60}(\partial^4 U)(y) - \frac{s^4}{840}(\partial^6 U)(y) + \cdots
\]

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