DERIVATIONS OF LIE ALGEBRAS OF DOMINANT UPPER TRIANGULAR LADDER MATRICES

PRAKASH GHIMIRE, HUAJUN HUANG

Abstract. We explicitly describe the Lie algebras $M_{\mathcal{L}}$ of ladder matrices in $M_n$ associate with dominant upper triangular ladders $\mathcal{L}$, and completely characterize the derivations of these $M_{\mathcal{L}}$ over a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$. We also completely characterize the derivations of Lie algebras $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ where $\mathcal{L}$ are strongly dominant upper triangular ladders and $\text{char}(\mathbb{F}) \neq 2, 3$.

Keywords: ladder matrix; upper triangular ladder; derivation algebra

1. Introduction

Ladder matrix is a natural extension of block upper triangular matrix. A ladder matrix is one that has zero entries outside of a ladder shape region. Let $[n] := \{1, 2, \ldots, n\}$. Given a field $\mathbb{F}$, let $M_{m n} \subseteq \mathbb{F}^{m \times n}$ be the set of $m \times n$ matrices over $\mathbb{F}$, and $M_n := M_{n n}$. We define a partial order on $\mathbb{Z}^+ \times \mathbb{Z}^+: (i_1, j_1)$ is said to dominate $(i_2, j_2)$, written as $(i_1, j_1) \succeq (i_2, j_2)$, whenever $i_1 \geq i_2$ and $j_1 \leq j_2$.

Definition 1.1. A subset $\mathcal{L} := \{(i_1, j_1), \cdots, (i_s, j_s)\}$ of the index set $[n] \times [n]$ of $M_n$ is called a ladder of step $s$ and size $n$, if

\[ i_1 < i_2 < \cdots < i_s \quad \text{and} \quad j_1 < j_2 < \cdots < j_s. \]

Each $(i_\ell, j_\ell) (\ell \in [s])$ is called a corner point of $\mathcal{L}$. The set $M_{\mathcal{L}}$ of $\mathcal{L}$-ladder matrices is a subset of $M_n$ defined by $M_{\emptyset} = \{0\}$ and

\[ M_{\mathcal{L}} := \sum_{(i,j) \in I(\mathcal{L})} \mathbb{F}E_{ij}, \]

where

\[ I(\mathcal{L}) := \{(i,j) \in [n] \times [n] : (i,j) \preceq (i_\ell, j_\ell) \text{ for some } \ell \in [s]\}, \]

and $E_{ij} \in M_n$ denotes the $(i,j)$ standard matrix that has 1 as the $(i,j)$ entry and 0 elsewhere.

In other words, $M_{\mathcal{L}}$ consists of matrices that have nonzero entries only in the upper right direction of some corner points $(i_\ell, j_\ell)$ of $\mathcal{L}$. In [2], Brice and Huang introduce the notion of ladder matrix and proved that $M_\mathcal{L} \cdot M_{\mathcal{L}'} = M_{\mathcal{L}''}$, where $\mathcal{L}$ and $\mathcal{L}'$ are two arbitrary ladders of size $n$, and $\mathcal{L}''$ is a ladder decided by $\mathcal{L}$ and $\mathcal{L}'$. In particular, if $\mathcal{L}$ is an upper triangular ladder (i.e., $i_\ell < j_{\ell + 1}$ for $\ell \in [s - 1]$, see Definition 2.2), then $M_{\mathcal{L}}$ is a matrix subalgebra of $M_n$. Naturally, $M_{\mathcal{L}}$ is a Lie subalgebra of $M_n$ (aka $\mathfrak{gl}(n, \mathbb{F})$) with respect to the standard Lie bracket $[X,Y] = XY - YX$.

Typical examples of Lie algebras $M_{\mathcal{L}}$ include those of block upper triangular matrices and of strictly block upper triangular matrices, $M_{pq}$ embedded in the upper right corner of $M_n$ (when $p \leq n$ and $q \leq n$), and $M_n$ itself. In 1957, Dixmier and Lister constructed a nilpotent
Lie algebra [4] to disprove the converse of a statement of Jacobson [6]: “a Lie algebra with a nonsingular derivation is nilpotent”; the corresponding derivation algebra is clearly embedded in a special nilpotent $M_L$.

A derivation of Lie algebra $\mathfrak{g}$ is a linear map $f \in \text{End}(\mathfrak{g})$ that satisfies

$$f([X, Y]) = [f(X), Y] + [X, f(Y)] \quad \text{for all } X, Y \in \mathfrak{g}.$$ 

The Lie derivations and generalized derivations of ladder shape matrix Lie algebras over a field or ring has drawn much attention in recent years. Here is a fairly incompletely list of literatures. Chen determines the structure of certain generalized derivations of a parabolic subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ over a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$ and $|\mathbb{F}| > n \geq 3$ [9]. Brice describes the derivations of parabolic subalgebra of a reductive Lie algebra over an algebraically closed and characteristics zero field, and proves the zero-product determined property of such derivation algebras [1]. Let $R$ be a communicative ring with identity. Cheung characterizes proper Lie derivations and gives sufficient conditions for any Lie derivation to be proper for triangular algebras over $R$ [3]. Du and Wang investigate the Lie derivations of $2 \times 2$ block generalized matrix algebras [5]. Wang, Ou, and Yu describe the derivations of intermediate Lie algebras between diagonal matrix algebra and upper triangular matrix algebra in $\mathfrak{gl}(n, R)$ [10]. Wang and Yu characterized all the derivations of parabolic subalgebras of $\mathfrak{gl}(n, R)$ [8]. Ou, Wang, and Yao describe the derivations of the Lie algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [7]. Ji, Yang, and Chen study the biderivations of the algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [12]. The Lie triple derivations are also extensively studied, for examples, on $\mathfrak{gl}(n, R)$ [13], on the algebra of upper triangular matrices of $\mathfrak{gl}(n, R)$ [14], and on the parabolic subalgebras of $\mathfrak{gl}(n, R)$ [11].

In this paper, we explicitly characterize the derivations of the Lie algebra $M_L$ associate with a dominant upper triangular (DUT) ladder $\mathcal{L}$ for $\text{char}(\mathbb{F}) \neq 2$ (Theorem 3.1), and the derivations of $[M_L, M_L]$ associate with a strongly dominant upper triangular (SDUT) ladder $\mathcal{L}$ for $\text{char}(\mathbb{F}) \neq 2, 3$ (Theorem 5.3). A ladder $\mathcal{L} = \{(i_1, j_1), \ldots, (i_s, j_s)\}$ is called DUT (resp. SDUT) if $j_\ell \leq i_\ell < j_{\ell+1}$ (resp. $j_\ell < i_\ell < j_{\ell+1}$) for $\ell \in [s-1]$. All $M_L$ associate with a DUT ladder $\mathcal{L}$ are completely characterized in Theorem 2.4.

- Theorem 2.4: $\mathcal{L}$ is DUT if and only if $M_L$ can be obtained by removing some non-consecutive diagonal blocks from the set of block upper triangular matrices corresponding to a partition of $[n]$.
- Theorem 3.1: When $\text{char}(\mathbb{F}) \neq 2$ and $\mathcal{L}$ is a DUT ladder, every derivation of $M_L$ is a sum of the adjoint action of a block upper triangular matrix and a linear map from $M_L/M_L$ to the center of $M_L$.
- Theorem 5.3: When $\text{char}(\mathbb{F}) \neq 2, 3$ and $\mathcal{L}$ is a SDUT ladder, every derivation of $[M_L, M_L]$ is the adjoint action of a block upper triangular matrix, so that it could be extended to a derivation of $M_L$.

In general, a derivation of a Lie algebra stabilizes each subalgebra appearing in the derived series. Moreover, the derived series of a non-solvable Lie algebra of upper triangular matrix algebras will terminate at $[M_L, M_L]$ for certain SDUT ladder $\mathcal{L}$. Therefore, knowledge on the derivation algebra of these $[M_L, M_L]$ would be useful to disclose the structure of derivations of Lie algebras of general upper triangular ladder matrices.

The paper is organized as follow: Section 2 provides some basic properties of ladder matrices; in particular, all DUT ladder matrix algebras are completely characterized (Theorem
2.4), and the counting of these algebras in \( M_n \) is done (Corollary 2.5). Section 3 characterizes the derivations of \( M_L \) for DUT ladders \( L \) and \( \text{char}(\mathbb{F}) \neq 2 \) (Theorem 3.1), and gives examples and applications, e.g. on the derivations of step 1 ladder matrix algebras (Theorem 3.5). Section 4 gives the proof of the main theorem in Section 3. Section 5 determines the derivations of \([M_L, M_L]\) for SDUT ladders \( L \) and \( \text{char}(\mathbb{F}) \neq 2, 3 \).

2. Preliminary

We develop some basic properties of ladders and ladder matrices in this section. Given a ladder \( L \subset [n] \times [n] \), the matrices in \( M_L \) could be viewed as block matrices with respect to suitable partitions. A partition of \([n]\) can be characterized by a subset

\[
\gamma = \{i_1, i_2, \ldots, i_s\} \subseteq [n - 1], \quad i_1 \leq i_2 \leq \cdots \leq i_s,
\]

where the corresponding partition in \( M_n \) is done right after the \( i_1, i_2, \ldots, i_s \) rows and columns. Every ladder \( L \) corresponds to one simplest compatible partition defined below.

**Definition 2.1.** Let \( L = \{(i_1, j_1), \ldots, (i_s, j_s)\} \subset [n] \times [n] \) be a ladder. The partition of \( L \) is a partition of \([n]\) characterized by

\[
\gamma_L := \{i_1, i_2, \ldots, i_s\} \cup \{j_1 - 1, j_2 - 1, \ldots, j_s - 1\} - \{0, n\}.
\]

The matrices in \( M_L \) could be viewed as block matrices with respect to the partition \( \gamma_L \). Denote by \([I(L)]\) the index set of nonzero blocks of \( M_L \) with respect to \( \gamma_L \). If we set \( t := |\gamma_L| + 1 \), then \([I(L)] \subseteq [t] \times [t]\).

The set of block upper triangular matrices corresponding to a partition \( \gamma_L = \{n_1, \ldots, n_t-1\} \) is exactly \( M_{L_B} \), where

\[
L_B = \{(n_1, 1), (n_2, n_1 + 1), \ldots, (n_{t-1}, n_{t-1} + 1), (n, n_{t-1} + 1)\}. \tag{2.1}
\]

We introduce some special ladders to be used in the paper.

**Definition 2.2.** A ladder \( L = \{(i_1, j_1), \ldots, (i_s, j_s)\} \) in \([n] \times [n]\) is called

- upper triangular: if \( i_\ell < j_{\ell+1} \) for \( \ell \in [s-1] \);
- strictly upper triangular: if \( i_\ell < j_\ell \) for \( \ell \in [s] \);
- dominant upper triangular (DUT): if \( j_\ell \leq i_\ell < j_{\ell+1} \) for \( \ell \in [s-1] \);
- strongly dominant upper triangular (SDUT): if \( j_\ell < i_\ell < j_{\ell+1} \) for \( \ell \in [s-1] \).

When \( L \) is upper triangular, a matrix in \( M_L \) is called an upper triangular (\( L \)-) ladder matrix. Similarly for the others.

The above different kinds of ladder \( L \) can be easily distinguished by the shape of \( M_L \). They can also be reinterpreted by the block form of \( M_L \) with respect to the partition \( \gamma_L \):

- \( L \) is upper triangular if \( M_L \subseteq M_{L_B} \) (resp. \( I(L) \subseteq I(L_B) \));
- \( L \) is strictly upper triangular if \( M_L \) is contained in the strictly block upper triangular part of \( M_{L_B} \);
- \( L \) is DUT if every block index \( (i, j) \in [I(L)] \) is dominated by a diagonal one \( (k, k) \in [I(L)] \);
- \( L \) is SDUT if \( L \) is DUT, and every nonzero diagonal block in \( M_L \) has size greater than 1.
Consider the ladder $\mathcal{L} = \{(1,1), (4,3), (5,5)\}$ of size 7. Then $\mathcal{L}$ is DUT but not SDUT. The matrix form of $M_\mathcal{L}$ is given in Figure 1(a). The index set $I(\mathcal{L})$ of $\mathcal{L}$ consists of $(i,j) \in [7] \times [7]$ dominated by at least one of $(1,1)$, $(4,3)$, and $(5,5)$:

$$I(\mathcal{L}) = \{(1,j) : 1 \leq j \leq 7\} \cup \{(i,j) : 2 \leq i \leq 4, \ 3 \leq j \leq 7\} \cup \{(5,j) : 5 \leq j \leq 7\}.$$ 

The partition of $\mathcal{L}$ is given by

$$\gamma_\mathcal{L} = \{1,4,5\} \cup \{1-1,3-1,5-1\} - \{0,7\} = \{1,2,4,5\}.$$ 

So matrices in $M_\mathcal{L}$ are partitioned after the 1, 2, 4, 5 rows and columns. Figure 1(b) indicates the block form of $M_\mathcal{L}$. The block index set $[I(\mathcal{L})]$ consists of $(i,j) \in [5] \times [5]$ dominated by at least one of $(1,1)$, $(3,3)$, and $(4,4)$:

$$[I(\mathcal{L})] = \{(1,j) : 1 \leq j \leq 5\} \cup \{(i,j) : 2 \leq i \leq 3, \ 3 \leq j \leq 5\} \cup \{(4,4),(4,5)\}.$$ 

(a) matrix form of $M_\mathcal{L}$

(b) block matrix form of $M_\mathcal{L}$

**Figure 1.** Ladder $\mathcal{L} = \{(1,1), (4,3), (5,5)\}$ of size 7

Now we can completely characterize DUT ladders and ladder matrices in terms of the associated partition.

**Theorem 2.4.**

1. A ladder $\mathcal{L}$ is DUT if and only if for each $i \in [t-1]$, at least one of $(i,i)$ and $(i+1,i+1)$ is in $[I(\mathcal{L})]$. In particular, $\mathcal{L}$ is DUT implies that $(i,j) \in [I(\mathcal{L})]$ for any $i,j \in [t]$ with $i < j$.

2. Equivalently, $M_\mathcal{L}$ is a set of DUT ladder matrices, if and only if it can be obtained by removing some non-consecutive diagonal blocks from the set of block upper triangular matrices corresponding to a partition of $[n]$.

**Proof.** It suffices to prove the first statement. Let $\mathcal{L} = \{(i_1,j_1), \ldots, (i_s,j_s)\}$ and $\gamma_\mathcal{L} = \{n_1, \ldots, n_{t-1}\}$, so that the set of block upper triangular matrices is $M_{\mathcal{L}_B}$ for the ladder $\mathcal{L}_B = \{(n_1,1), (n_2,n_1+1), \ldots, (n_{t-1},n_{t-2}+1), (n,n_{t-1}+1)\}$.

Suppose for each $i \in [t-1]$, at least one of $(i,i)$ and $(i+1,i+1)$ is in $[I(\mathcal{L})]$. Then $\mathcal{L} \subseteq \mathcal{L}_B$. It is obvious that $j_\ell \leq i_\ell < j_{\ell+1}$ for $\ell \in [s-1]$. Hence $\mathcal{L}$ is DUT.

Now assume that $\mathcal{L}$ is DUT. Then every block index $(i,j) \in [I(\mathcal{L})]$ is dominated by a diagonal block index $(k,k) \in [I(\mathcal{L})]$, that is, $i \leq k \leq j$. If for some $i \in [t-1]$, neither $(i,i)$ nor $(i+1,i+1)$ is in $[I(\mathcal{L})]$, then $(i,i+1)$ is not in $[I(\mathcal{L})]$. Then $n_i \notin \gamma_\mathcal{L}$, which is a contradiction. Therefore, at least one of $(i,i)$ and $(i+1,i+1)$ is in $[I(\mathcal{L})]$.

A direct application of Theorem 2.4 is the counting of DUT ladder matrices.

**Corollary 2.5.** Let $\{F_t\}_{t=1}^\infty = \{1,1,2,3,5,\ldots\}$ be the Fibonacci sequence.
(1) The number of sets of DUT ladder matrices corresponding to a \( t \times t \) block form equals to \( b_t \), where

\[
\{b_t\}^\infty_{t=1} = \{F_{t+2}\}^\infty_{t=1} = \{2, 3, 5, 8, 13, \ldots\}. \tag{2.2}
\]

(2) The number of sets of DUT ladder matrices in \( M_n \) equals to \( a_n \), where

\[
\{a_n\}^\infty_{n=1} = \{F_{2n+1}\}^\infty_{n=1} = \{2, 5, 13, 34, 89, \ldots\}. \tag{2.3}
\]

Proof. (1) Clearly \( b_1 = 2 = F_3 \) and \( b_2 = 3 = F_4 \). (2.2) will be proved if \( \{b_t\} \) satisfies the same recursive formula as \( \{F_{t+2}\} \) does, that is,

\[
b_t = b_{t-1} + b_{t-2}. \tag{2.4}
\]

By Theorem 2.4, \( b_t \) equals to the number of ways to choose non-consecutive diagonal blocks in a given \( t \times t \) block form. If the first diagonal block is chosen, then the second one should be skipped, and there are \( b_{t-2} \) ways to choose the remaining diagonal blocks; if the first diagonal block is not chosen, then there are \( b_{t-1} \) ways to choose the remaining diagonal blocks. Therefore, (2.4) is true and (2.2) is proved.

(2) Given \( t \in [n] \), there are \( (n-1)! \) ways to partition matrices in \( M_n \) into a \( t \times t \) block form; each block form corresponds to \( b_t = F_{t+2} \) sets of DUT ladder matrices. Let \( r_1 := \frac{1+\sqrt{5}}{2} \) and \( r_2 := \frac{1-\sqrt{5}}{2} \) be the roots of \( x^2 - x - 1 = 0 \). The Binet’s Fibonacci number formula says that

\[
F_t = \frac{1}{\sqrt{5}}r_1^t - \frac{1}{\sqrt{5}}r_2^t.
\]

Therefore,

\[
a_n = \sum_{t=1}^{n} \binom{n-1}{t-1} F_{t+2} = \sum_{t=1}^{n} \binom{n-1}{t-1} \left( \frac{1}{\sqrt{5}}r_1^{t+2} - \frac{1}{\sqrt{5}}r_2^{t+2} \right) \\
= \frac{1}{\sqrt{5}} \left[ r_1^3(1 + r_1)^{n-1} - r_2^3(1 + r_2)^{n-1} \right] = \frac{1}{\sqrt{5}} \left[ r_1^3(1 + r_1)^{n-1} - r_2^3(1 + r_2)^{n-1} \right] = F_{2n+1}. \tag*{\Box}
\]

We give some notations that will be used in studying the partitioned matrices associated with \( M_L \).

**Definition 2.6.** Given an algebra \((M, +, \ast)\) and two subsets \( M', M'' \subseteq M \), define the subset

\[
M' \ast M'' := \left\{ \sum_{i=1}^{m} A_i \ast B_i \mid m \in \mathbb{N}, A_i \in M', B_i \in M'' \right\}.
\]

**Definition 2.7.** Consider the matrices in \( M_n \) with respect to a given partition \( \gamma_L \).

- Let \( \mathcal{M}_{ij} \) denote the set of all submatrices in the \((i, j)\) block of \( M_n \). Let \( E_{ij}^{[pq]} \) denote the \((p, q)\) standard matrix in \( \mathcal{M}_{ij} \).
- Let \( \widehat{\mathcal{M}}_{ij} \) denote the embedding of \( \mathcal{M}_{ij} \) in \( M_n \).
- For \( A \in M_n \), let \( A_{ij} \in \mathcal{M}_{ij} \) denote the \((i, j)\) block submatrix of \( A \).
- For \( B_{ij} \in \mathcal{M}_{ij} \), let \( \widehat{B}_{ij} \in \mathcal{M}_{ij} \) denote the matrix in \( M_n \) with \( B_{ij} \) in the \((i, j)\) block and zero elsewhere. Similarly for \( \widehat{B}_{ik} \) if \( B_{ik} \in \mathcal{M}_{ik} \) and \( B_{kj} \in \mathcal{M}_{kj} \).
- In \( \mathcal{M}_{kk} \), let \( I_{kk} \) denote the identity matrix, and \( \mathfrak{sl}_{kk} \) the set of traceless matrices, respectively.

A notation of double index, say \( \mathcal{M}_{ij} \), may be written as \( \mathcal{M}_{i,j} \) for clarity purpose.
The normalizer $N(M_L)$ and the centralizer $Z(M_L)$ of Lie subalgebra $M_L$ in $M_n$ are:

\[
N(M_L) = \{ A \in M_n : [A, B] \in M_L \text{ for all } B \in M_L \},
\]
\[
Z(M_L) = \{ A \in M_n : [A, B] = 0 \text{ for all } B \in M_L \}.
\]

They are explicitly described by the following two lemmas.

**Lemma 2.8.** If $L$ is a DUT ladder, then $N(M_L) = M_{L_B}$, the subalgebra of block upper triangular matrices with respect to the partition of $L$.

*Proof.* We first show that $N(M_L) \subseteq M_{L_B}$. Suppose on the contrary, there is $A \in N(M_L)$ such that the $(i, j)$ block $A_{ij} \neq 0$ for some $i > j$. There are two cases:

1. $i > j + 1$: We have $\widehat{M}_{j,j+1} \subseteq M_L$ by Theorem 2.4. So $[A, \widehat{M}_{j,j+1}] \subseteq M_L$. However, its $(i, j + 1)$ block is

\[
[A, \widehat{M}_{j,j+1}]_{i,j+1} = [A_{ij}, \widehat{M}_{j,j+1}] = A_{ij}\widehat{M}_{j,j+1} \neq \{0\},
\]

which contradicts the DUT assumption of $L$.

2. $i = j + 1$: By Theorem 2.4, either $\widehat{M}_{jj} \subseteq M_L$ or $\widehat{M}_{j+1,j+1} \subseteq M_L$. Without loss of generality, suppose $\widehat{M}_{jj} \subseteq M_L$. Then $[A, \widehat{M}_{jj}] \subseteq M_L$. However, its $(i, j)$ block is

\[
[A, \widehat{M}_{jj}]_{ij} = A_{ij}\widehat{M}_{jj} \neq \{0\},
\]

which contradicts the DUT assumption of $L$.

Therefore, $A \in M_{L_B}$ and thus $N(M_L) \subseteq M_{L_B}$.

For any $(i, j) \in [t] \times [t]$ with $i \leq j$, the possibly nonzero blocks of matrices in $[\widehat{M}_{ij}, M_L]$ are those $(i, q)$ blocks with $q \geq j$ and $(p, j)$ blocks with $p \leq i$, all of which belong to $M_L$. Hence $M_{L_B} \subseteq N(M_L)$.

\[\square\]

**Lemma 2.9.** Let $L$ be a DUT ladder and $t = |\gamma_L| + 1$.

1. If both the $(1, 1)$ and the $(t, t)$ blocks of $M_L$ are zero, then $Z(M_L) = \mathbb{F}I_n + \widehat{M}_{1t}$.

2. Otherwise, $Z(M_L) = \mathbb{F}I_n$.

*Proof.* Clearly $Z(M_L) \subseteq N(M_L)$. The possibly nonzero blocks of any $A \in Z(M_L)$ are $A_{ij}$ for some $1 \leq i \leq j \leq t$. If $A_{ij} \neq 0$ and $2 \leq i < j$, then $\widehat{M}_{i-1,i} \subseteq M_L$, and we can find $B_{i-1,i} \in \widehat{M}_{i-1,i}$ such that

\[
0 \neq B_{i-1,i}A_{i,j} = [B_{i-1,i}, A_{i,j}] = [B_{i-1,i}, A]_{i-1,j}.
\]

which contradicts the assumption $A \in Z(M_L)$. Thus $A_{ij} = 0$ for all $2 \leq i < j \leq t$. Similarly, $A_{ij} = 0$ for all $1 \leq i < j \leq t - 1$. So the only possibly nonzero blocks of $A \in Z(M_L)$ are $A_{1t}$ and $A_{it}$ for $i \in [t]$.

If $(1, 1) \in [I(L)]$, then $0 = [I_{11}, A]_{1t} = A_{1t}$. Similarly, $(t, t) \in [I(L)]$ implies that $A_{tt} = 0$. If neither $(1, 1)$ nor $(t, t)$ is in $[I(L)]$, then $\widehat{M}_{1t}$ is in $Z(M_L)$ by direct computation.

Now for any $i, j \in [t]$ with $i < j$ and $B_{ij} \in \widehat{M}_{ij} \subseteq M_L$,

\[
0 = [A, B_{ij}]_{ij} = A_{ii}B_{ij} - B_{ij}A_{jj}.
\]

Let $B_{ij}$ go through all standard matrices in $\widehat{M}_{ij}$ that have an entry one and zeros elsewhere. We can get $A_{ii} = \lambda I_{ii}$ and $A_{jj} = \lambda I_{jj}$ for a fixed $\lambda \in \mathbb{F}$.

In summary, $Z(M_L)$ is described by the statements (1) and (2). \[\square\]
3. The main theorem

In this section, we explicitly characterize the derivation algebra \( \text{Der}(M_L) \) for any DUT ladder \( \mathcal{L} \) over a field \( \mathbb{F} \) with \( \text{char}(\mathbb{F}) \neq 2 \), and provide some consequent results. Note that the adjoint representation \( \text{ad} : M_n \to \text{Der}(M_n) \) defined by \( \text{ad}A = [A, B] \) induces a Lie algebra homomorphism

\[
\text{ad}(\cdot)|_{M_L} : N(M_L)/Z(M_L) \to \text{Der}(M_L),
\]

which will be used in the following theorem.

**Theorem 3.1.** (Main theorem) Suppose \( \text{char}(\mathbb{F}) \neq 2 \). Let \( \mathcal{L} \) be a DUT ladder. Then the Lie algebra \( \text{Der}(M_L) \) can be decomposed as a direct sum of ideals:

\[
\text{Der}(M_L) = \text{ad}(N(M_L)/Z(M_L))|_{M_L} \oplus \mathcal{D}
\]

(3.1)

\[
= \left( \text{ad} \left( \frac{M_L}{Z(M_L) \cap M_L} \right) \times \bigoplus_{(k,k) \in [I(\mathcal{L}_B)]-[I(\mathcal{L})]} \text{ad} \left( \tilde{M}_{kk} \right) \right)_{M_L} \oplus \mathcal{D}
\]

(3.2)

where

- the normalizer \( N(M_L) \) and the centralizer \( Z(M_L) \) are described by Lemmas 2.8 and 2.9, respectively;
- the ideal \( \mathcal{D} \) is defined by

\[
\mathcal{D} := \{ \phi \in \text{End}(M_L) : \ker \phi \supseteq [M_L, M_L], \ \text{im} \phi \subseteq Z(M_L) \cap M_L \};
\]

(3.3)

in particular, \( \mathcal{D} \cong \text{Hom}_F(M_L/[M_L, M_L], Z(M_L) \cap M_L) \) as vector spaces.

Explicitly, we have the following cases with respect to the partition \( \gamma_L \) of \( \mathcal{L} \) (let \( t = |\gamma_L| + 1 \)):

1. If \( M_L \) is a set of block upper triangular matrices (i.e. \( \mathcal{L} = \mathcal{L}_B \)), then every \( f \in \text{Der}(M_L) \) corresponds to an \( X \in M_{L_B}/\mathbb{F}I_n \) and \( c_1, \ldots, c_t \in \mathbb{F} \), such that

\[
f(A) = \text{ad}X(A) + \left( \sum_{k \in [t]} c_k \text{tr}(A_{kk}) \right) I_n \quad \text{for} \quad A \in M_L.
\]

(3.4)

2. If \( M_L \) has some zero diagonal block(s), but at least one of its \((1,1)\) and \((t,t)\) blocks is nonzero, then every \( f \in \text{Der}(M_L) \) corresponds to an \( X \in M_{L_B}/\mathbb{F}I_n + \tilde{M}_{11t} \) and \( Y_{1t} \in M_{1t} \) for each \((k,k) \in [I(\mathcal{L})]\), such that

\[
f(A) = \text{ad}X(A) \quad \text{for} \quad A \in M_L.
\]

(3.5)

3. If both the \((1,1)\) and the \((t,t)\) blocks in \( M_L \) are zero, then every \( f \in \text{Der}(M_L) \) corresponds to an \( X \in M_{L_B}/(\mathbb{F}I_n + \tilde{M}_{11t}) \) and \( Y_{1t} \in M_{1t} \) for each \((k,k) \in [I(\mathcal{L})]\), such that

\[
f(A) = \text{ad}X(A) + \sum_{(k,k) \in [I(\mathcal{L})]} \text{tr}(A_{kk}) \tilde{Y}_{1tk} \quad \text{for} \quad A \in M_L.
\]

(3.6)

A detailed proof of Theorem 3.1 will be given in Section 4. The special case \( \mathcal{L} = \mathcal{L}_B \) (where \( M_L \) is a set of block upper triangular matrices) is included in a paper of Dengyin Wang and Qiu Yu [8, Theorem 4.1]. Moreover, Daniel Brice obtains a formula similar to (3.1) for the derivation algebra of the parabolic subalgebra of a reductive Lie algebra over a \( \mathbb{C} \)-like fields or over \( \mathbb{R} \) [1].
Example 3.2. Theorem 3.1 is not true when \( \text{char}(\mathbb{F}) = 2 \). Consider \( M_L = M_2 \) with the basis \( \mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\} \). Define \( f \in \text{End}(M_L) \) by \( f(E_{12}) = E_{21} \) and \( f(E_{ij}) = 0 \) for \((i, j) = (1, 1), (2, 1), (2, 2)\). It is straightforward to verify that
\[
f([E, E']) = [f(E), E'] + [E, f(E')]
\]
for any \( E, E' \in \mathcal{B} \), since there are only two cases that either side of (3.7) is nonzero: \( \{E, E'\} = \{E_{11}, E_{12}\} \) or \( \{E_{12}, E_{22}\} \). Therefore \( f \in \text{Der}(M_L) \). However, \( f \) is not an element of \( \text{ad}(N(M_L)/Z(M_L))|_{M_L} \oplus \mathcal{D} \) in (3.1).

When \( \mathcal{L} \) is an upper triangular ladder, an inner derivation \( (\text{ad } X)|_{M_L} (X \in M_L) \) satisfies that
\[
(\text{ad } X)|_{M_L}(\overline{M}_{ij}) \subseteq \overline{M}_{ij} + \sum_{k>j} \overline{M}_{ik} + \sum_{k<i} \overline{M}_{kj} \quad \text{for any} \quad \overline{M}_{ij} \subseteq M_L.
\]
So the inner derivation sends the \((i, j)\) block to a sum of blocks with the indices dominated by \((i, j)\). This dominance property also holds for all derivations of \( M_L \) when \( \mathcal{L} \) is a DUT ladder with some zero diagonal blocks (i.e. \( M_L \neq M_L\)).

Corollary 3.3. Let \( \mathcal{L} \) be a DUT ladder with some zero diagonal blocks. Then every \( f \in \text{Der}(M_L) \) maps any \((i, j)\) block of \( M_L \) to a sum of some blocks dominated by the \((i, j)\) block.

Proof. The corollary is a direct consequence of Theorem 3.1(2) and (3).

In general, Corollary 3.3 may not be true if \( \mathcal{L} \) is not a DUT ladder which can be seen via the following example.

Example 3.4. Suppose \( \mathbb{F} \) is an arbitrary field. Let \( n = 5 \) and \( \mathcal{L} = \{(1, 2), (3, 4)\} \). Then \( \mathcal{L} \) is not DUT, and \( M_L \) has the form:
\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} & a_{15} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad a_{ij} \in \mathbb{F}.
\]
So \( M_L \) has a basis \( \mathcal{B} = \{E_{12}, E_{13}, E_{14}, E_{15}, E_{24}, E_{25}, E_{34}, E_{35}\} \). Given \( a, b \in \mathbb{F} \), define \( f \in \text{End}(M_L) \) by
\[
f(E_{12}) := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad f(E_{13}) := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad f(E) = 0 \quad \text{for all other matrices} \ E \ \text{in the basis} \ \mathcal{B}.
\]
We prove that
\[
f([E, E']) = [f(E), E'] + [E, f(E')] \quad \text{for all} \quad E, E' \in \mathcal{B},
\]
so that \( f \) is a derivation of \( M_L \). On one hand, \([E, E'] \in \text{span}\{E_{14}, E_{15}\}\) and thus \( f([E, E']) = 0; \) on the other hand, in (3.8), \([f(E), E'] \neq 0 \) or \([E, f(E')] \neq 0 \) only when \( \{E, E'\} = \{E_{12}, E_{13}\} \), for which the equality (3.8) is easily verified. Therefore, \( f \in \text{Der}(M_L) \). However, \( f \) maps the block \( \overline{M}_{12} \) into \( \overline{M}_{23} \), where \((2, 3)\) is not dominated by \((1, 2)\).
An important family of ladders is that of 1-step ladders $\mathcal{L} = \{(i,j)\}$, where each $\mathcal{M}_L$ realizes $M_{pq}$ $(p,q \leq n)$ as a Lie subalgebra of $M_n$. Many 1-step ladders are DUT. The derivations of these $\mathcal{M}_L$ can be explicitly characterized here.

**Theorem 3.5.** Let $\mathcal{L} = \{(i,j)\} \subseteq [n] \times [n]$ be a 1-step ladder of size $n$.

(1) If $i < j$, then $\mathcal{M}_L$ is abelian and

$$\text{Der}(\mathcal{M}_L) = \text{End}(\mathcal{M}_L).$$

(2) If $i = n$ or $j = 1$, then

$$\text{Der}(\mathcal{M}_L) = \text{ad}(N(\mathcal{M}_L)/Z(\mathcal{M}_L))|_{\mathcal{M}_L}.$$ Explicitly, there are three subcases:

(a) If $i = n$ and $j = 1$, then $\mathcal{M}_L = M_n$, and $\text{Der}(\mathcal{M}_L) = \text{ad}(M_n/FI_n)$.

(b) If $i \neq n$ and $j = 1$, then

$$\mathcal{M}_L = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} : A_{11} \in M_i, A_{12} \in M_{i,n-i} \right\},$$

$$\text{Der}(\mathcal{M}_L) = \text{ad}\left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in M_i, X_{12} \in M_{i,n-i}, X_{22} \in M_{n-i} \right\}|_{\mathcal{M}_L}$$

$$= \text{ad} M_{\mathcal{M}_L} \times \text{ad}(\widehat{M_{22}})|_{\mathcal{M}_L}.$$ (c) If $i = n$ and $j \neq 1$, then

$$\mathcal{M}_L = \left\{ \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} : A_{12} \in M_{j-1,n-j+1}, A_{22} \in M_{n-j+1} \right\},$$

$$\text{Der}(\mathcal{M}_L) = \text{ad}\left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in M_{j-1}, X_{12} \in M_{j-1,n-j+1}, X_{22} \in M_{n-j+1} \right\}|_{\mathcal{M}_L}$$

$$= \text{ad} M_{\mathcal{M}_L} \times \text{ad}(\widehat{M_{11}})|_{\mathcal{M}_L}.$$ (3) If $n > i \geq j > 1$. Then

$$\text{Der}(\mathcal{M}_L) = \text{ad}(N(\mathcal{M}_L)/Z(\mathcal{M}_L))|_{\mathcal{M}_L} \oplus \mathcal{D}$$

where $\mathcal{D}$ is defined in (3.3). Explicitly, $\gamma_L = \{j-1,i\}$, and

$$\mathcal{M}_L = \left\{ A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\text{Der}(\mathcal{M}_L) = \text{ad}\left\{ \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \right\}|_{\mathcal{M}_L} \oplus \left\{ f_y : f_y(A) = \text{tr}(A_{22}) \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \left( \text{ad} M_{\mathcal{M}_L} \times \left( \text{ad}(\widehat{M_{11}}) \oplus \text{ad}(\widehat{M_{33}}) \right) \right) \oplus \left\{ f_y : f_y(A) = \text{tr}(A_{22}) \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$  

**Proof.** The cases (2) and (3) are done by Theorem 3.1. For case (1) where $\mathcal{M}_L$ is abelian, every $f \in \text{End}(\mathcal{M}_L)$ satisfies that

$$f([A,B]) = 0 = [f(A),B] + [A,f(B)], \quad A,B \in \mathcal{M}_L.$$ Therefore, $\text{Der}(\mathcal{M}_L) = \text{End}(\mathcal{M}_L).$ \qed
4. Proof of the main theorem

To prove Theorem 3.1, we give several auxiliary results here. The first two lemmas below connect the linear transformations within the four blocks of a $2 \times 2$ block matrix:

$$
\begin{bmatrix}
M_{mp} & M_{mq} \\
M_{np} & M_{nq}
\end{bmatrix}
$$

Let $E_{ij}^{(mn)}$ denote the $(i, j)$ standard matrix in $M_{mn}$.

**Lemma 4.1.** Suppose $\mathbb{F}$ is an arbitrary field. If linear transformations $\phi : M_{mp} \to M_{mq}$ and $\varphi : M_{np} \to M_{nq}$ satisfy that

$$
\phi(AB) = A\varphi(B) \quad \text{for all} \quad A \in M_{mn}, \ B \in M_{np},
$$

then there is $X \in M_{pq}$ such that $\phi(C) = CX$ for $C \in M_{mp}$ and $\varphi(D) = DX$ for $D \in M_{np}$.

**Proof.** For any $j \in [n]$ and any $B \in M_{np}$,

$$
\phi(E_{ij}^{(mn)} B) = E_{ij}^{(mn)} \varphi(B).
$$

All such $E_{ij}^{(mn)} B$ span the first row space of $M_{np}$. So $\phi$ maps the first row of $M_{mp}$ to the first row of $M_{mq}$. There exists a unique $X \in M_{pq}$ such that

$$
E_{ij}^{(mn)} \varphi(B) = \phi(E_{ij}^{(mn)} B) = E_{ij}^{(mn)} BX, \quad \text{for all} \quad j \in [n], \ B \in M_{np}.
$$

Therefore, $\varphi(B) = BX$. Then $\phi(AB) = A\varphi(B) = ABX$ for any $A \in M_{mn}$ and $B \in M_{np}$. Hence $\phi(C) = CX$ for all $C \in M_{mp}$. \hfill \Box

**Lemma 4.2.** Suppose $\mathbb{F}$ is an arbitrary field. If linear transformations $\phi : M_{mp} \to M_{np}$ and $\varphi : M_{mq} \to M_{nq}$ satisfy that

$$
\phi(BA) = \varphi(B)A \quad \text{for all} \quad A \in M_{ap}, \ B \in M_{nq},
$$

then there is $X \in M_{mn}$ such that $\phi(C) = XC$ for $C \in M_{mp}$ and $\varphi(D) = XD$ for $D \in M_{mq}$.

The proof (omitted) is similar to that of Lemma 4.1.

**Lemma 4.3.** Suppose $\mathbb{F}$ is an arbitrary field. If $X \in M_m$ and $Y \in M_n$ satisfy that $XA = AY$ for all $A \in M_{mn}$, then $X = \lambda I_m$ and $Y = \lambda I_n$ for certain $\lambda \in \mathbb{F}$.

**Proof.** For any $(i, j) \in [m] \times [n]$,

$$
XE_{ij} = E_{ij}Y.
$$

Comparing the $(i, j)$ entry, we get $x_{ii} = y_{jj}$. Comparing the $(p, j)$ entry for $p \neq i$, we get $x_{pi} = 0$. Comparing the $(i, q)$ entry for $q \neq j$, we get $0 = y_{jq}$. Therefore, $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$. \hfill \Box

In the remaining of this section, we assume that $\text{char}(\mathbb{F}) \neq 2$, $\mathcal{L}$ is a DUT ladder, and $t := |\gamma_{\mathcal{L}}| + 1$. Next we present several results on the image of a derivation of $M_{\mathcal{L}}$.

**Lemma 4.4.** For $f \in \text{Der} (M_{\mathcal{L}})$ and $(k, k) \in [I(\mathcal{L})]$, the $f$-images of the identity matrix and the standard matrices in the $(k, k)$ block satisfy that

$$
f(\tilde{I}_{kk}), \ f(\tilde{E}_{kk}) \in \sum_{i=1}^{k-1} \tilde{M}_{ik} + \sum_{j=k+1}^{t} \tilde{M}_{kj} + (Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}}) \quad \text{(4.3)}
$$

\[ \]
where (by Lemma 2.9)

\[
Z(M_\mathcal{L}) \cap M_\mathcal{L} = \begin{cases} 
\mathbb{F} I_n & \text{if } \mathcal{L} = \mathcal{L}_B; \\
\mathcal{M}_{1t} & \text{if } (1, 1) \notin [I(\mathcal{L})] \text{ and } (t, t) \notin [I(\mathcal{L})]; \\
0 & \text{otherwise.}
\end{cases}
\]  \tag{4.4}

**Proof.** We prove (4.3) for \( f(\widetilde{I}_{kk}) \) here, and the case of \( f(E_{kl}^{[kk]}) \) is similar.

(1) First we investigate \( f(\widetilde{I}_{kk})_{jj} \). When \( k < j \),

\[
f(\widetilde{A}_{kj})_{kj} = f(\widetilde{I}_{kk}, \widetilde{A}_{kj})_{kj} = [f(\widetilde{I}_{kk}), \widetilde{A}_{kj}]_{kj} + [\widetilde{I}_{kk}, f(\widetilde{A}_{kj})]_{kj}
\]

Therefore

\[
f(\widetilde{I}_{kk})_{kk} A_{kj} = A_{kj} f(\widetilde{I}_{kk})_{jj} \quad \text{for } A_{kj} \in \mathcal{M}_{kj}.
\]

Lemma 4.3 implies that \( f(\widetilde{I}_{kk})_{kk} = \lambda I_{kk} \) and \( f(\widetilde{I}_{kk})_{jj} = \lambda I_{jj} \) for a \( \lambda \in \mathbb{F} \). The same equation holds for \( k > j \). In the situation \( \mathcal{L} \neq \mathcal{L}_B \), there exists \( (p, p) \notin [I(\mathcal{L})] \), which forces \( f(\widetilde{I}_{kk})_{pp} = 0 \) and thus \( f(\widetilde{I}_{kk})_{jj} = 0 \) for all \( j \in [t] \).

(2) Next we prove that \( f(\widetilde{I}_{kk})_{ij} = 0 \) for \( i < j, i \neq k, j \neq k \), and \( (i, j) \neq (1, t) \). Either \( i > 1 \) or \( j < t \). Without loss of generality, suppose \( j < t \) (similarly for \( i > 1 \)). Then

\[
f(\widetilde{I}_{kk}, \widetilde{A}_{jt})_{it} = [f(\widetilde{I}_{kk}), \widetilde{A}_{jt}]_{it} + [\widetilde{I}_{kk}, f(\widetilde{A}_{jt})]_{it}.
\]  \tag{4.5}

(a) If \( k \neq t \), then (4.5) becomes \( 0 = f(\widetilde{I}_{kk})_{ij} A_{jt} \) for any \( A_{jt} \in \mathcal{M}_{jt} \). So \( f(\widetilde{I}_{kk})_{ij} = 0 \).

(b) If \( k = t \), then (4.5) becomes

\[
-f(\widetilde{A}_{jt})_{it} = f(\widetilde{I}_{kk})_{ij} A_{jt} - f(\widetilde{A}_{jt})_{it}.
\]

Again we get \( 0 = f(\widetilde{I}_{kk})_{ij} A_{jt} \) and thus \( f(\widetilde{I}_{kk})_{ij} = 0 \).

(3) Finally, if \( (1, 1) \in [I(\mathcal{L})] \) or \( (t, t) \in [I(\mathcal{L})] \), say \( (1, 1) \in [I(\mathcal{L})] \), then for any \( (k, k) \in [I(\mathcal{L})] \) and \( k \notin \{1, t\} \),

\[
0 = f(\widetilde{I}_{1k}, \widetilde{I}_{kk})_{it} = [f(\widetilde{I}_{11}), \widetilde{I}_{kk}]_{it} + [\widetilde{I}_{1k}, f(\widetilde{I}_{kk})]_{it} = f(\widetilde{I}_{kk})_{it}.
\]

Lemma 2.9 implies (4.4). Therefore, (4.3) is proved. \( \square \)

For \( (p, q) \in [I(\mathcal{L})] \), we have

\[
\widetilde{\mathcal{M}}_{pq} \cap [M_\mathcal{L}, M_\mathcal{L}] = \begin{cases} 
\mathfrak{sl}_{pp}, & \text{if } p = q; \\
\mathcal{M}_{pq}, & \text{if } p < q.
\end{cases}
\]

Next we investigate the image of derivations on each block in \([M_\mathcal{L}, M_\mathcal{L}]\).

**Lemma 4.5.** Suppose \( \text{char}(\mathbb{F}) \neq 2 \). For \( f \in \text{Der} (M_\mathcal{L}) \), \( (p, q) \in [I(\mathcal{L})] \), and \( \widetilde{A}_{pq} \in \widetilde{\mathcal{M}}_{pq} \cap [M_\mathcal{L}, M_\mathcal{L}] \),

\[
f(\widetilde{A}_{pq}) \in \widetilde{\mathcal{M}}_{pq} + \sum_{i=1}^{p-1} \widetilde{\mathcal{M}}_{iq} + \sum_{j=q+1}^{t} \widetilde{\mathcal{M}}_{pj}. \tag{4.6}
\]

**Proof.** There are two cases for \( (p, q) \in [I(\mathcal{L})] \):
(1) $p = q$: Then $\widetilde{M}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \widetilde{sl}_{pp} = [\widetilde{sl}_{pp}, \widetilde{sl}_{pp}]$. For $B_{pp}, C_{pp} \in \mathfrak{sl}_{pp}$,
\[ f([B_{pp}, C_{pp}]) = [f(B_{pp}), C_{pp}]) + [B_{pp}, f(C_{pp})]. \tag{4.7} \]
Since $f(B_{pp})$ and $f(C_{pp})$ are block upper triangular matrices with respect to $\gamma_{\mathcal{L}}$, the nonzero $(i, j)$ blocks of the right side of (4.7) satisfy that $p = i \leq j$ or $i \leq j = p$. Thus (4.6) holds in this case.

(2) $p < q$: Then $\widetilde{M}_{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \widetilde{M}_{pq}$. Let $q = p + k$ and we prove (4.6) by induction on $k$. For better display, we also use $\{\cdot\}_{ij}$ here to denote the embedding of $\mathcal{M}_{ij}$ to $\mathcal{M}_{ij} \subseteq M_n$.

(a) $k = 1$: By Theorem 2.4, at least one of $(p, p)$ and $(p + 1, p + 1)$ is in $[I(\mathcal{L})]$. Without loss of generality, suppose $(p, p) \in [I(\mathcal{L})]$. Then for $A_{p,p+1} \in \mathcal{M}_{p,p+1}$,
\[ f(A_{p,p+1}) = f([I_{pp}, A_{p,p+1}]) = [f(I_{pp}), A_{p,p+1}] + [I_{pp}, f(A_{p,p+1})] \]
\[ = \sum_{i=1}^{p-1} \left\{ f(I_{pp})_{ip} A_{p,p+1} \right\}_{i,p+1} + \sum_{j=p+1}^{t} \left\{ f(A_{p,p+1})_{pj} \right\}_{pj} - \sum_{i=1}^{p-1} \left\{ f(A_{p,p+1})_{ip} \right\}_{ip} \]
where the last equality is given by Lemma 4.4. Therefore,
\[ f(A_{p,p+1}) + \sum_{i=1}^{p-1} \left\{ f(A_{p,p+1})_{ip} \right\}_{ip} = \sum_{i=1}^{p-1} \left\{ f(I_{pp})_{ip} A_{p,p+1} \right\}_{i,p+1} + \sum_{j=p+1}^{t} \left\{ f(A_{p,p+1})_{pj} \right\}_{pj} \]

One one hand, as $\text{char}(\mathbb{F}) \neq 2$, the nonzero blocks on the left side of the above equality are those of $f(A_{p,p+1})$; on the other hand, the right side of this equality has nonzero $(i, j)$ blocks only for $1 \leq i \leq p - 1 < p + 1 = j$ or $i = p < p + 1 \leq j \leq t$. So $k = 1$ is done.

(b) $k = \ell$: Suppose the statement is true for all $k < \ell$ where $\ell \geq 2$. Now $\mathcal{M}_{p,p+\ell} = [M_{p,p+1}, M_{p+1,p+\ell}]$, and
\[ f([B_{p,p+1}, C_{p+1,p+\ell}]) = [f(B_{p,p+1}), C_{p+1,p+\ell}] + [B_{p,p+1}, f(C_{p+1,p+\ell})] \]
By induction hypothesis, $f(B_{p,p+1})$ has nonzero blocks only on the $p$ block row and the $(p + 1)$ block column, so that the nonzero blocks of $[f(B_{p,p+1}), C_{p+1,p+\ell}]$ only locate on the $p$ block row and the $(p+\ell)$ block row. Similarly for $[B_{p,p+1}, f(C_{p+1,p+\ell})]$. So (4.6) is true for $k = \ell$.

(c) Overall, (4.6) is verified for all the cases. \hfill \Box

Now we are ready to prove Theorem 3.1. The basic idea is to explore what remain in $\text{Der}(M_{\mathcal{L}})$ after factoring out $\text{ad}(N(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} = \text{ad}(M_{\mathcal{L}_B})|_{M_{\mathcal{L}}}$. Given $X \in M_{\mathcal{L}_B}$, $A \in M_{\mathcal{L}}$,
\[ \text{ad} X (A) = \sum_{1 \leq p \leq q \leq t} \sum_{(i,j) \in [I(\mathcal{L})]} [X_{pq}, A_{ij}]. \]
A summand $[X_{pq}, A_{ij}]$ is nonzero only if $i = q$ or $p = j$. In other words, $\text{ad} X_{pq}$ has nonzero action only on the $q$ block row or the $p$ block column of $A$. It motivates us to investigate the relationship of $f(\widetilde{A}_{ip})$ and $f(\widetilde{A}_{qj})$ for given $f \in \text{Der}(M_{\mathcal{L}})$ and $1 \leq p \leq q \leq t$. 

Proof of Theorem 3.1.

(1) If $M_{\mathcal{L}}$ is a set of block upper triangular matrices (i.e. $\mathcal{L} = \mathcal{L}_B$), by [8, Theorem 4.1] and the assumption $\text{char}(\mathbb{F}) \neq 2$, every $f \in \text{Der} (M_{\mathcal{L}})$ corresponds to $X \in M_{\mathcal{L}}$ and $\mu \in M_{\mathcal{L}}^*$ such that

$$f(A) = \text{ad} X(A) + \mu(A) I_n.$$  

Then $\mu([M_{\mathcal{L}}, M_{\mathcal{L}}]) = 0$ by derivation property. All $\mathcal{M}_{ij}$ with $i < j$ are in $[M_{\mathcal{L}}, M_{\mathcal{L}}]$. So $\mu(A) = \sum_{k \in [t]} \mu(A_{kk})$. Recall that the $(p, q)$ standard matrix in $\mathcal{M}_{ij}$ is denoted by $E_{pq}^{[ij]}$. Given $k \in [t]$, we have $\tilde{A}_{kk} - \text{tr}(A_{kk})E_{11}^{[kk]} \in [M_{\mathcal{L}}, M_{\mathcal{L}}]$ so that

$$\mu (\tilde{A}_{kk}) = \text{tr}(A_{kk}) \mu \left( E_{11}^{[kk]} \right).$$

Denote $c_k = \mu \left( E_{11}^{[kk]} \right)$. Then

$$f(A) = \text{ad} X(A) + \left( \sum_{k \in [t]} c_k \text{tr}(A_{kk}) \right) I_n.$$  

This is (3.4). The formulae (3.1) and (3.2) for $\mathcal{L} = \mathcal{L}_B$ immediately follow.

(2) In the remaining of the proof, we assume $\mathcal{L} \neq \mathcal{L}_B$, so that $M_{\mathcal{L}}$ has at least one zero diagonal block with respect to the partition $\gamma_{\mathcal{L}}$.

Suppose $(k, k) \in [I(\mathcal{L})]$. For any $A_{kk}, B_{kk} \in \mathcal{M}_{kk}$,

$$f(\tilde{A}_{kk}, B_{kk})_{kk} = [f(\tilde{A}_{kk})_{kk}, B_{kk}] + [A_{kk}, f(\tilde{B}_{kk})_{kk}].$$

So $f(\cdot)_{kk} : \mathcal{M}_{kk} \to \mathcal{M}_{kk}$ is a derivation of $\mathcal{M}_{kk}$. Since $\text{char}(\mathbb{F}) \neq 2$, according to [8, Corollary 5.1] \(^1\), there is $X_{kk} \in \mathcal{M}_{kk}$ and $\lambda_k \in \mathbb{F}$ such that

$$f(\tilde{A}_{kk})_{kk} = [X_{kk}, A_{kk}] + \lambda_k \text{tr}(A_{kk}) I_{kk} \quad \text{for } A_{kk} \in \mathcal{M}_{kk}.$$  

We prove that $\lambda_k = 0$ for all $k$. Recall that $E_{pq}^{[ij]}$ denotes the $(p, q)$ standard matrix in $\mathcal{M}_{ij}$. On one hand, the $(1, 1)$ entry of

$$f(E_{11}^{[kk]})_{kk} = [X_{kk}, E_{11}^{[kk]}] + \lambda_k I_{kk}$$

equals to $\lambda_k$. On the other hand, for any $\ell \in [t]$ with $\ell > k$,

$$f(E_{11}^{[\ell \ell]})_{k\ell} = f([E_{11}^{[\ell \ell]}, E_{11}^{[kk]}])_{k\ell} = [f(E_{11}^{[\ell \ell]}), E_{11}^{[kk]}]_{k\ell} + [E_{11}^{[kk]}, f(E_{11}^{[\ell \ell]})]_{k\ell}$$

$$= f(E_{11}^{[\ell \ell]})_{kk} E_{11}^{[kk]} - E_{11}^{[kk]} f(E_{11}^{[\ell \ell]}), E_{11}^{[kk]}]_{k\ell} + E_{11}^{[kk]} f(E_{11}^{[\ell \ell]})_{k\ell}.$$  

Therefore,

$$f(E_{11}^{[kk]})_{kk} E_{11}^{[kk]} = (I_{kk} - E_{11}^{[kk]} f(E_{11}^{[kk]})_{k\ell} + E_{11}^{[kk]} f(E_{11}^{[kk]})_{k\ell}.$$  

Comparing the $(1, 1)$ entry of both sides, we see that the $(1, 1)$ entries of $f(E_{11}^{[kk]})_{kk}$ and $f(E_{11}^{[kk]})_{k\ell}$ are equal. The same result holds for $\ell < k$. By assumption $\mathcal{L} \neq \mathcal{L}_B$.

\(^1\) Der (gl$(m, \mathbb{F})$) has additional elements when $\text{char}(\mathbb{F}) = 2$ and $m = 2$ [8, Corollary 5.1].
So there exists \((f, t) \notin [I(L)]\), where \(f(E_{kk})_{tt} = 0\). Hence \(\lambda_k = 0\). Overall, for any \((k, k) \in [I(L)]\), there exists \(X_{kk} \in M_{kk}\) such that
\[
f(A_{kk})_{kk} = [X_{kk}, A_{kk}] \quad \text{for all } A_{kk} \in M_{kk}.
\]

(3) Given \(p, q \in [I]\) and \(p < q\), we claim that there exists \(X_{pq} \in M_{pq}\) such that
\[
\begin{align*}
\widehat{f(A_{ip})}_{iq} &= \text{ad} \widetilde{X}_{pq}(A_{ip}), \quad \text{for any } (i, p) \in [I(L)], \quad \text{and} \\
\widehat{f(A_{jq})}_{pj} &= \text{ad} \widetilde{X}_{pq}(A_{jq}), \quad \text{for any } (q, j) \in [I(L)].
\end{align*}
\]

There are several situations:

(a) Suppose \((q, j) = (t, t) \in [I(L)]\). For any \(A_{tt}, B_{tt} \in M_{tt}\),
\[
f([A_{tt}, B_{tt})]_{tt} = [f(\tilde{A}_{tt}), \tilde{B}_{tt}]_{tt} + [A_{tt}, f(\tilde{B}_{tt})]_{tt} = f(\tilde{A}_{tt})_{tt}B_{tt} - f(\tilde{B}_{tt})_{tt}A_{tt}.
\]
Set \(B_{tt} = I_{tt}\). Then \(f(\tilde{A}_{tt})_{tt} = f(I_{tt})_{tt}A_{tt}\) for \(A_{tt} \in M_{tt}\). Denote \(X_{tt} := f(I_{tt})_{tt} \in M_{tt}\). We have \(f(\tilde{A}_{tt})_{tt} = X_{tt}A_{tt}\) and so
\[
f(\tilde{A}_{tt})_{tt} = \text{ad} X_{tt}(\tilde{A}_{tt}) \quad \text{for all } A_{tt} \in M_{tt}.
\]

(b) Suppose \((i, p) = (1, 1) \in [I(L)]\). Similarly, let \(Y_{1q} := -f(I_{11})_{1q} \in M_{1q}\) then
\[
f(\tilde{A}_{11})_{1q} = -A_{11}Y_{1q} = \text{ad} Y_{1q}(\tilde{A}_{11}) \quad \text{for all } A_{11} \in M_{11}.
\]

(c) Suppose \((q, j) \in [I(L)] - \{(t, t)\}\). Either \(q < t\) or \(j < t\). Without loss of generality, suppose \(j < t\). Let \(j' := j + 1\). Then \((j, j'), (q, j'), (p, j), (p, j') \in [I(L)]\), and \(M_{qj} = M_{qj}M_{jj'} = [M_{qj}, M_{jj'}]\). For any \(A_{qj} \in M_{qj}, A_{jj'} \in M_{jj'}\),
\[
f(A_{qj}A_{jj'})_{pj} = f((A_{qj}, A_{jj'})_{pj})_{pj} = [f(\tilde{A}_{qj}), \tilde{A}_{jj'}]_{pj} + [A_{qj}, f(\tilde{A}_{jj'})_{pj}]_{pj} = f(\tilde{A}_{qj})_{pj}A_{jj'}
\]
Applying Lemma 4.2 to \(\phi : M_{qj} \rightarrow M_{pj}\) defined by \(\phi(C) := f(\tilde{C})_{pj}\) and \(\varphi : M_{qj} \rightarrow M_{pj}\) defined by \(\varphi(D) := f(\tilde{D})_{pj}\), we can find \(X_{pq} \in M_{pq}\) such that \(f(\tilde{A}_{qj})_{pj} = X_{pq}A_{qj}\) for all \(A_{qj} \in M_{qj}\), and \(f(\tilde{A}_{jj'})_{pj} = X_{pq}A_{jj'}\) for all \(A_{jj'} \in M_{jj'}\). In particular, \(X_{pq}\) is independent of \(j\). So
\[
f(\tilde{A}_{qj})_{pj} = \text{ad} X_{pq}(A_{qj}) \quad \text{for all } A_{qj} \in M_{qj}.
\]

(d) Suppose \((i, p) \in [I(L)] - \{(1, 1)\}\). Either \(i > 1\) or \(p > 1\). Without loss of generality, suppose \(i > 1\) (similarly for \(p > 1\)). Let \(i' := i - 1\). Then \((i', i), (i', p), (i, q), (i', q) \in [I(L)]\), and \(M_{i'p} = M_{i'p}M_{ip} = [M_{i'p}, M_{ip}]\). For \(A_{i'i} \in M_{i'i}\) and \(A_{ip} \in M_{ip}\),
\[
f(\tilde{A}_{i'i}A_{ip})_{i'q} = [f(\tilde{A}_{i'i}), \tilde{A}_{ip}]_{i'q} + [\tilde{A}_{i'i}, f(\tilde{A}_{ip})]_{i'q} = A_{i'i}f(\tilde{A}_{ip})_{iq}.
\]
Applying Lemma 4.1 to \(\phi : M_{i'p} \rightarrow M_{i'q}\) defined by \(\phi(C) := f(\tilde{C})_{i'q}\) and \(\varphi : M_{ip} \rightarrow M_{iq}\) defined by \(\varphi(D) := f(\tilde{D})_{iq}\), we can find \(-Y_{pq} \in M_{pq}\) such that \(f(\tilde{A}_{ip})_{iq} = -A_{ip}Y_{pq}\) for all \(A_{ip} \in M_{ip}\), and \(f(\tilde{A}_{ip})_{i'q} = -A_{i'p}Y_{pq}\) for all \(A_{i'p} \in M_{i'p}\). So \(-Y_{pq}\) is independent of \(i\) and
\[
f(\tilde{A}_{ip})_{iq} = \text{ad} Y_{pq}(A_{ip}) \quad \text{for all } A_{ip} \in M_{ip}.
\]
(e) Given any \((i, p), (q, j) \in [I(\mathcal{L})]\), we have \(\widetilde{A}_{ip}, \widetilde{A}_{jq} \) = 0, so that

\[
0 = f(\widetilde{A}_{ip}, \widetilde{A}_{jq})_{ij} = [f(\widetilde{A}_{ip}), \widetilde{A}_{jq}]_{ij} + [\widetilde{A}_{ip}, f(\widetilde{A}_{jq})]_{ij}
\]

\[
= f(\widetilde{A}_{ip})_{iq}A_{jq} + A_{ip}f(\widetilde{A}_{jq})_{pj} = -A_{ip}Y_{pq}A_{iq} + A_{ip}X_{pq}A_{jq}.
\]

Therefore, \(X_{pq} = Y_{pq}\).

Overall, we successfully find \(X_{pq}\) that satisfies (4.8) and (4.9).

(4) From (2) and (3), we can construct a matrix in \(\mathcal{M}\):

\[
X_0 := \sum_{(k, k) \in [I(\mathcal{L})]} \widetilde{X}_{kk} + \sum_{1 \leq p < q \leq t} \widetilde{X}_{pq}.
\]

Define the derivation

\[
f_1 := f - \text{ad} X_0.\]

Then for any \((k, k) \in [I(\mathcal{L})]\), \(1 \leq p < q \leq t\), and \((i, p), (q, j) \in [I(\mathcal{L})]\), we have

\[
f_1(\widetilde{M}_{kk})_{kk} = 0, \quad f_1(\widetilde{M}_{ip})_{iq} = 0, \quad f_1(\widetilde{M}_{jq})_{pj} = 0.
\]

By Lemmas 4.4 and 4.5, \(f_1\) belongs to the following set:

\[
D_0 := \{ g \in \text{Der}(\mathcal{M}) \mid g(\widetilde{M}_{kk}) \in Z(\mathcal{M}) \cap \mathcal{M} \text{ for } (k, k) \in [I(\mathcal{L})], \quad g(\widetilde{M}_{pq}) \subseteq \mathcal{M}_{pq} \text{ for } 1 \leq p < q \leq t \},
\]

(11.11)

It remains to describe the subalgebra \(D_0\) of \(\text{Der}(\mathcal{M})\).

(5) Given \(f' \in \text{Der}(\mathcal{M})\), \(p, q \in [t]\) with \(p < q\), and \(k \in [t]\) with \(p \leq k \leq q\), Lemmas 4.4 and 4.5 imply that

\[
f'(\widetilde{A}_{pk}A_{kq})_{pq} = f'(\widetilde{A}_{pk}, \widetilde{A}_{kq})_{pq} = [f'(\widetilde{A}_{pk}), \widetilde{A}_{kq}]_{pq} + [\widetilde{A}_{pk}, f'(\widetilde{A}_{kq})]_{pq}
\]

\[
= f'(\widetilde{A}_{pk})_{pk}A_{kq} + A_{pk}f'(\widetilde{A}_{kq})_{kq}.
\]

This formula will be frequently used in the following computations.

(6) We prove the following claim regarding \(f_1\) defined in (4.10): there exist \(Y_{ii} \in \mathcal{M}_{ii}\) for \(i \in [t]\), such that for each \(k \in [t]\), the derivation \(f_1^{(k)} := \left(f_1 - \sum_{i=1}^{k} \text{ad} Y_{ii}\right)|_{\mathcal{M}_i}\) has the images

\[
\begin{cases}
f_1^{(k)}(\widetilde{M}_{qq}) = f_1(\widetilde{M}_{qq}), & \text{for } (q, q) \in [I(\mathcal{L})], \quad q \leq k; \\
f_1^{(k)}(\widetilde{M}_{pq}) = 0, & \text{for } (p, q) \in [I(\mathcal{L})], \quad 1 \leq p < q \leq k.
\end{cases}
\]

(11.13)

Moreover, \(Y_{ii} \in \mathcal{F}I_{ii}\) whenever \((i, i) \in [I(\mathcal{L})]\).

The proof is proceeded by induction on \(k\):

(a) \(k = 1\) and \(2\): There are two subcases:

- If \((1, 1) \in [I(\mathcal{L})]\), we let \(Y_{11} = 0 \in \mathcal{M}_{11}\) so that \(f_1^{(1)} = f_1\). By (4.12),

\[
f_1^{(1)}(\widetilde{A}_{11})_{12} = f_1^{(1)}(\widetilde{A}_{11})_{11}A_{12} + A_{11}f_1^{(1)}(\widetilde{A}_{12})_{12} = A_{11}f_1^{(1)}(\widetilde{A}_{12})_{12}.
\]

By Lemma 4.1, there exists \(-Y_{22} \in \mathcal{M}_{22}\), such that \(f_1^{(1)}(\widetilde{A}_{12})_{12} = -A_{12}Y_{22}\).

Let \(f_1^{(2)} = f_1^{(1)} - \text{ad} \tilde{Y}_{22}\). Then \(f_1^{(2)}(\widetilde{A}_{12}) = 0\). If furthermore \((2, 2) \in [I(\mathcal{L})]\),

then by (4.12),

\[
0 = f_1^{(2)}(\widetilde{A}_{12})_{12} = f_1^{(2)}(\widetilde{A}_{12})_{12}A_{22} + A_{12}f_1^{(2)}(\widetilde{A}_{22})_{22} = A_{12}f_1^{(2)}(\widetilde{A}_{22})_{22}.
\]
Thus
\[ 0 = f_1^{(2)}(\widetilde{A}_{22})_{12} = f_1(\widetilde{A}_{22})_{12} - [Y_{22}, A_{22}] = -[Y_{22}, A_{22}]. \]
So \( Y_{22} \in \mathbb{F}I_{22} \) and \( f_1^{(2)}(A_{22}) = f_1(A_{22}). \) The claim holds for \( k = 1, 2. \)

- If \((1, 1) \notin [I(\mathcal{L})]\), then \((2, 2) \in [I(\mathcal{L})]\) by Theorem 2.4. By (4.12),
\[ f_1(A_{12}A_{22})_{12} = f_1(\widetilde{A}_{12})_{12}A_{22} + A_{12}f_1(\widetilde{A}_{22})_{22} = f_1(\widetilde{A}_{12})_{12}A_{22}. \]

By Lemma 4.2, there exists \( Y_{11} \in \mathcal{M}_{11} \) such that \( f_1(\widetilde{A}_{12})_{12} = Y_{11}A_{12}. \) Let \( Y_{22} = 0 \in \mathcal{M}_{22}, f_1^{(1)} = f_1 - \text{ad} \widetilde{Y}_{11}, \) and \( f_1^{(2)} = f_1^{(1)} - \text{ad} \widetilde{Y}_{22}. \) Then the claim holds for \( k = 1, 2. \)

(b) \( k = \ell > 2: \) Suppose the claim holds for \( k = \ell - 1 \geq 2. \) So there exist \( Y_{11} \in \mathcal{M}_{11}, \cdots, Y_{\ell - 1, \ell - 1} \in \mathcal{M}_{\ell - 1, \ell - 1} \), such that \( f_1^{(\ell - 1)} = f_1 - \sum_{i=1}^{\ell - 1} \text{ad} \widetilde{Y}_{ii} \) satisfies (4.13) for \( k = \ell - 1. \) Clearly \( f_1^{(\ell - 1)} \in D_0. \) For any \( p \in [\ell - 2], \) by (4.12),
\[ f_1^{(\ell - 1)}(A_p, \widetilde{A}_{\ell - 1, \ell})_{p \ell} = f_1^{(\ell - 1)}(A_p, \widetilde{A}_{\ell - 1, \ell})_{p \ell} - A_p f_1^{(\ell - 1)}(A_{\ell - 1, \ell})_{\ell - 1, \ell} = A_p f_1^{(\ell - 1)}(A_{\ell - 1, \ell})_{\ell - 1, \ell} + A_{\ell - 1, \ell} f_1^{(\ell - 1)}(A_p, \widetilde{A}_{p, \ell})_{\ell - 1, \ell}. \]

By Lemma 4.1, there exists \( -Y_{\ell \ell} \in \mathcal{M}_{\ell \ell}, \) such that
\[ f_1^{(\ell - 1)}(A_p, \widetilde{A}_{p \ell})_{p \ell} = -A_p Y_{\ell \ell} \quad \text{for all} \quad p \in [\ell - 1]. \]
Let
\[ f_1^{(\ell)} := f_1^{(\ell - 1)} - \text{ad} \widetilde{Y}_{\ell \ell}. \]
Then \( f_1^{(\ell)}(A_p, \widetilde{A}_{p \ell}) = 0 \) for \( p \in [\ell - 1]. \) In the case \((\ell, \ell) \in [I(\mathcal{L})], \) by (4.12),
\[ 0 = f_1^{(\ell)}(A_{\ell - 1, \ell}, \widetilde{A}_{\ell \ell})_{\ell - 1, \ell} = f_1^{(\ell)}(A_{\ell - 1, \ell}, \widetilde{A}_{\ell \ell})_{\ell - 1, \ell} - A_{\ell - 1, \ell} f_1^{(\ell)}(A_{\ell \ell})_{\ell - 1, \ell} = A_{\ell - 1, \ell} f_1^{(\ell)}(A_{\ell \ell})_{\ell - 1, \ell}. \]
So
\[ 0 = f_1^{(\ell)}(A_{\ell \ell})_{\ell \ell} = \left( f_1 - \sum_{i=1}^{\ell - 1} \text{ad} \widetilde{Y}_{ii} \right)(A_{\ell \ell})_{\ell \ell} = -[Y_{\ell \ell}, A_{\ell \ell}]. \]

Thus \( Y_{\ell \ell} \in \mathbb{F}I_{\ell \ell} \) and \( f_1^{(\ell)}(A_{\ell \ell}) = f_1(A_{\ell \ell}). \) The claim is proved for \( k = \ell. \)

(c) Overall, the claim holds for every \( k \in [t]. \)

(7) The derivation \( f_1^{(t)} = f_1 - \sum_{i=1}^{t} \text{ad} \widetilde{Y}_{ii} \) sends each \( \widetilde{M}_{kk} \) for \((k, k) \in [I(\mathcal{L})]\) to \( Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}}, \) and \( \widetilde{M}_{pq} \) for \( 1 \leq p < q \leq t \) to \( 0. \) For any \( A, B \in M_{\mathcal{L}}, \)
\[ f_1^{(t)}([A, B]) = [f_1^{(t)}(A), B] + [A, f_1^{(t)}(B)] = 0. \]
Therefore, \( f_1^{(t)} \in \mathcal{D} \) for \( \mathcal{D} \) defined in (3.3). Every \( \phi \in \mathcal{D} \) satisfies \( \phi([A, B]) = 0 = [\phi(A), B] + [A, \phi(B)] \) for \( A, B \in M_{\mathcal{L}}. \) Thus \( \mathcal{D} \subseteq \text{Der} M_{\mathcal{L}}. \) So far we have
\[ \text{Der} M_{\mathcal{L}} = (\text{ad } M_{\mathcal{L}^{(t)}})|_{M_{\mathcal{L}}} + \mathcal{D}. \]

If \((1, 1) \in [I(\mathcal{L})]\) or \((t, t) \in [I(\mathcal{L})], \) then \( Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = 0 \) implies that \( \mathcal{D} = 0. \) We get (3.5).

If neither \((1, 1)\) nor \((t, t)\) is in \([I(\mathcal{L})], \) then \( Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = \widetilde{M}_{11}. \) The set \( \{ E_{kk}^{(1)} \mid (k, k) \in [I(\mathcal{L})]\} \) spans a subalgebra complement to \([M_{\mathcal{L}}, M_{\mathcal{L}}]\) in \( M_{\mathcal{L}}. \) Then for any
\[\phi \in D \text{ and } A \in M_L, \]
\[\phi(A) = \sum_{(k,k) \in [I(L)]} \phi(A_{kk}) = \sum_{(k,k) \in [I(L)]} \text{tr}(A_{kk})\phi(E_{kk}).\]

Denote \(Y_{1k} := \phi(E_{1k}) \in \tilde{M}_k\) for \((k,k) \in [I(L)]\). We get (3.6).

In all the cases, the equations (3.5) and (3.6) as well as (3.4) derived in (1) imply (3.1) and (3.2) by direct verification. So Theorem 3.1 is completely proved. \(\square\)

5. Derivations of \([M_L, M_L]\) for SDUT ladder \(L\)

In this section, we will give an explicit description of the derivation algebra of \([M_L, M_L]\) for a SDUT ladder \(L\) when \(\text{char}(\mathbb{F}) \neq 2, 3\). The Lie subalgebra \([M_L, M_L]\) consists of matrices in \(M_L\) that have zero trace on every diagonal block of \(M_L\). To see the motivations of studying \(\text{Der}([M_L, M_L])\), we make the following notation.

**Definition 5.1.** Given an upper triangular ladder \(L\), let \(M_L^0\) denote the Lie subalgebra of \(M_L\) consisting of matrices with zero trace on every diagonal block of \(M_L\) with respect to the partition of \(L\).

Any derivation of a Lie algebra \(g\) preserves the lower central series, upper central series, and derived series of \(g\). Given an upper triangular ladder \(L\), the derived series of \(M_L\) is:

\[M_L = M_L^{(0)} \supseteq M_L^{(1)} \supseteq M_L^{(2)} \supseteq \cdots, \quad M_L^{(k)} := [M_L^{(k-1)}, M_L^{(k-1)}].\]

The following observations are straightforward in the view point of partitioned matrices:

1. When \(k \geq 1\), each \(M_L^{(k)} = M_{L_k}^0\) for some upper triangular ladder \(L_k\) contained in \(L\), that is,

\[I(L) \supseteq I(L_1) \supseteq I(L_2) \supseteq \cdots\]

2. The Lie algebra \(M_L\) is non-solvable if and only if its derived series terminates at a nonzero \(M_L^0\), where \(L_s\) is the maximal SDUT ladder contained in \(L\). Precisely,

\[L_s = \{(i_1, j_1) \in L : i_1 > j_1\}.\]

If \(L_s\) given above is an empty set, then \(M_L\) is solvable, and the derived series of \(M_L\) terminates at 0.

3. Every \(f \in \text{Der}(M_L)\) stabilizes \(M_{L_s}^0\), and induces a derivation \(f|_{M_{L_s}^0} \in \text{Der}(M_{L_s}^0)\). The restriction map \(\pi : \text{Der}(M_L) \to \text{Der}(M_{L_s}^0)\) is a Lie algebra homomorphism.

**Example 5.2.** In \(M_8\), the forms of \(M_L\), \(M_{L_s}\), and \(M_{L_s}^0\), associate with an upper triangular \(L\) are illustrated below. In particular, we see \(L_s = \{(2, 1), (7, 6)\}\) is SDUT.

| \(M_L\) | \(M_{L_s}\) | \(M_{L_s}^0\) |
|---|---|---|
| \[
\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
0 & * & * & * & * \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
0 & * & * & * & * \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
a & * & * & * & * & * \\
-\frac{a}{b} & * & * & * & * & * \\
0 & * & * & * & * & * \\
0 & 0 & * & * & * & * \\
b & * & * & * & * & * \\
0 & \frac{b}{a} & * & * & * & * \\
\end{bmatrix}
\] |
The above observations indicate that the structure of \( \text{Der} \left( M_2^0 \right) \) for SDUT ladders \( \mathcal{L} \) (where \( \mathcal{L} = \mathcal{L}_\ast \)) will be useful in studying the structure of \( \text{Der} \left( M_{\mathcal{L}} \right) \) for non-solvable upper triangular ladders \( \mathcal{L}' \). In the rest of this section, we assume that \( \mathcal{L} \) is a SDUT ladder, unless otherwise specified. Let \( t := |\gamma_{\mathcal{L}}| + 1 \) as before.

**Theorem 5.3.** Suppose \( \text{char}(\mathbb{F}) \neq 2, 3 \). Let \( \mathcal{L} \) be a SDUT ladder of size \( n \). Then every derivation \( f \in \text{Der} \left( M_2^0 \right) \) can be extended to a derivation \( f^+ \in \text{Der} \left( M_{\mathcal{L}} \right) \) such that \( f^+|_{M_2^0} = f \). In particular, there exists a block upper triangular matrix \( X \in M_{\mathcal{L}B} \) such that

\[
 f(B) = \text{ad} \ X(B) = [X,B], \quad \text{for all } B \in M_2^0.
\]

We can write

\[
 \text{Der} \left( M_2^0 \right) = \text{ad}(N(M_{\mathcal{L}})/Z(M_{\mathcal{L}}))|_{M_2^0}.
\]

The proof of Theorem 5.3 will be deferred to the end of this section.

**Corollary 5.4.** When \( \text{char}(\mathbb{F}) \neq 2, 3 \), and \( \mathcal{L} \) is a SDUT ladder, we have the split exact sequence:

\[
 0 \rightarrow \mathcal{D} \rightarrow \text{Der} \left( M_{\mathcal{L}} \right) \xrightarrow{\pi} \text{Der} \left( M_2^0 \right) \rightarrow 0,
\]

where \( \mathcal{D} \) is defined in (3.3).

**Proof.** Theorem 5.3 shows that the restriction map \( \pi : \text{Der} \left( M_{\mathcal{L}} \right) \rightarrow \text{Der} \left( M_2^0 \right) \) is surjective. Theorem 3.1 shows that \( \text{Der} \left( M_{\mathcal{L}} \right) = \text{ad}(N(M_{\mathcal{L}})/Z(M_{\mathcal{L}}))|_{\mathcal{L}B} \oplus \mathcal{D} \). It is easy to check that \( Z(M_{\mathcal{L}}) = Z(M_2^0) \) and \( \text{Ker} \ \pi = \mathcal{D} \). Therefore, we get the split exact sequence (5.3). \( \square \)

**Example 5.5.** When \( \text{char}(\mathbb{F}) = 2 \) or 3, we show by counterexamples that \( \text{Der} \left( M_2^0 \right) \) is not in the form of (5.2).

- \( \text{char}(\mathbb{F}) = 2 \): Let \( M_{\mathcal{L}} = M_2 \), so that \( M_2^0 = s\mathfrak{l}_2 \). Let \( f \) be the derivation of \( M_2 \) given in Example 3.2, that is, \( f(E_{12}) = E_{21} \), and \( f(E_{ij}) = 0 \) for \( (i,j) \in \{(1,1),(2,2),(2,1)\} \). Then \( f|_{s\mathfrak{l}_2} \) is a derivation of \( s\mathfrak{l}_2 \). However, there is no \( X \in M_{\mathcal{L}B} = M_2 \) such that \( f|_{s\mathfrak{l}_2}(E_{12}) = [X,E_{12}] \).
- \( \text{char}(\mathbb{F}) = 3 \): Let \( n = 4 \), \( \mathcal{L} = \{(2,1)\} \). Then \( M_{\mathcal{L}}^2 \) consists of matrices in \( M_4 \) that takes the following forms:

\[
 \begin{bmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} \\
 a_{21} & -a_{11} & a_{23} & a_{24} \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad a_{ij} \in \mathbb{F}.
\]

So \( M_2^0 \) has a basis \( \mathcal{B} = \{E_{11} - E_{22}, E_{12}, E_{13}, E_{14}, E_{21}, E_{23}, E_{24}\} \). Define \( f \in \text{End} \left( M_2^0 \right) \) by \( f(E_{12}) := E_{24} \), and \( f(E) = 0 \) for all other matrices \( E \) in the basis \( \mathcal{B} \). We prove that

\[
 f([E,E']) = [f(E),E'] + [E,f(E')]
\]

for any distinct \( E, E' \in \mathcal{B} \), so that \( f \in \text{Der} \left( M_2^0 \right) \). The only case that the left side or the right side of (5.4) is nonzero is \( \{E, E'\} = \{E_{11} - E_{22}, E_{12}\} \), in which

\[
 f([E,E']) = 2f(E_{12}) = 2E_{24}, \quad [f(E),E'] + [E,f(E')] = -E_{24}.
\]

Since \( \text{char}(\mathbb{F}) = 3 \), the equality (5.4) holds for this case. Therefore, (5.4) holds for all \( \{E, E'\} \subseteq \mathcal{B} \), and \( f \in \text{Der} \left( M_2^0 \right) \). However, there is no matrix \( X \in M_4 \), such that \( f(E_{12}) = [X,E_{12}] \).

In order to prove Theorem 5.3, we first give two lemmas similar to Lemmas 4.1 and 4.2.
Lemma 5.6. Suppose $n \geq 2$. If linear transformations $\phi : M_{mn} \to M_{mq}$ and $\varphi : \mathfrak{sl}_n \to M_{nq}$ satisfy that
\[ \phi(AB) = A\varphi(B) \quad \text{for all} \quad A \in M_{mn}, \quad B \in \mathfrak{sl}_n, \] then there is $X \in M_{nq}$ such that $\phi(C) = CX$ for $C \in M_{mn}$ and $\varphi(D) = DX$ for $D \in \mathfrak{sl}_n$.

Lemma 5.6 is very similar to a special case ($p = n$) of Lemma 4.1, except that the domain of $\varphi$ is $\mathfrak{sl}_n$ instead of $M_{nn} = M_n$. The proof of Lemma 5.6 (omitted) is totally parallel to that of Lemma 4.1, using the key fact that $\{E_{ij}^{mn}B \mid j \in [n], B \in \mathfrak{sl}_n\}$ still spans the first row space of $M_{mn}$. Similarly, we have the following lemma.

Lemma 5.7. Suppose $n \geq 2$. If linear transformations $\phi : M_{nq} \to M_{nq}$ and $\varphi : \mathfrak{sl}_n \to M_{mn}$ satisfy that
\[ \phi(BA) = \varphi(B)A \quad \text{for all} \quad A \in M_{nq}, \quad B \in \mathfrak{sl}_n, \] then there is $X \in M_{mn}$ such that $\phi(C) = XC$ for $C \in M_{nq}$ and $\varphi(D) = XD$ for $D \in \mathfrak{sl}_n$.

Next we give two lemmas related to the bracket operation.

Lemma 5.8. Suppose $\text{char}(\mathbb{F}) \neq 2, 3$. If a linear transformation $\phi : \mathfrak{sl}_n \to M_{nn}$ satisfies that
\[ \phi(AB - BA) = A\phi(B) - B\phi(A), \quad \text{for all} \quad A, B \in \mathfrak{sl}_n, \] then there is $X \in M_{nn}$ such that $\phi(C) = CX$ for $C \in \mathfrak{sl}_n$.

Proof. The case $n = 1$ is obviously true. We now assume that $n \geq 2$. Let $\{E_{ij} \mid i, j \in [n]\}$ be the standard basis of $M_n$. Then $\mathfrak{sl}_n$ has the standard basis $\{E_{ij} \mid i, j \in [n], i \neq j\} \cup \{H_i \mid i \in [n - 1]\}$, where $H_i := E_{ii} - E_{i+1,i+1}$. We have $M_n = \mathfrak{sl}_n \oplus \mathbb{F}E_{11}$.

First we prove that the only possibly nonzero row of $\phi(E_{ij})$ ($i \neq j$) is the $i$-th row, and the only possibly nonzero rows of $\phi(H_i) = \phi(E_{ii} - E_{i+1,i+1})$ ($i \in [n - 1]$) are the $i$-th and the $(i + 1)$-th rows.

Suppose $i, j \in [n]$ with $i < j$. Denote $E := E_{ij}$, $F := E_{ji}$, and $H := E_{ii} - E_{jj}$. Then $\{H, E, F\} \in \mathfrak{sl}_n$ is the standard triple of a $\mathfrak{sl}_2$ subalgebra. We have
\[ 2\phi(E) = \phi([H, E]) = H\phi(E) - E\phi(H) \quad \Rightarrow \quad (2I_n - H)\phi(E) = -E\phi(H). \]
When $\text{char}(\mathbb{F}) \neq 2, 3$, the matrix $2I_n - H = \text{diag}(2, 2, \cdots, 1, \cdots, 3, \cdots, 2)$ is invertible and diagonal. The matrix $(2I_n - H)^{-1}$ is again diagonal with 1 as the $i$-th diagonal entry. So we have
\[ \phi(E) = -(2I_n - H)^{-1}E_{ij}\phi(H) = -E_{ij}\phi(H). \]
In particular, $\phi(E_{ij}) = \phi(E)$ has zeros outside of the $i$-th row. Similar argument works for $E_{ji}$.

For $H_i = E_{ii} - E_{i+1,i+1}$, we have
\[ \phi(H_i) = \phi([E_{i,i+1}, E_{i+1,i}]) = E_{i,i+1}\phi(E_{i+1,i}) - E_{i+1,i}\phi(E_{i,i+1}). \]
Therefore, $\phi(H_i)$ has zeros outside of the $i$-th row and the $(i + 1)$-th row.

Next we extend the map $\phi$ from the domain $\mathfrak{sl}_n$ to the domain $M_n$ such that property (5.7) still hold in $M_n$. Define the linear transformation $\phi^+ : M_n \to M_{nn}$ as follow:
\[
\begin{align*}
\phi^+(A) &= \phi(A), \quad \text{for} \quad A \in \mathfrak{sl}_n; \\
\phi^+(E_{11}) &= E_{12}\phi(E_{21}).
\end{align*}
\]
Then \( \phi^+ \) is an extension of \( \phi \) from \( \mathfrak{sl}_n \) to \( M_n \). To verify (5.7)-like property for \( \phi^+ \) in \( M_n \), it suffices to prove the following equality for all \( A \) in the standard basis of \( \mathfrak{sl}_n \):

\[
\phi^+(E_{1i}A - AE_{1i}) = E_{1i}\phi^+(A) - A\phi^+(E_{1i}) = E_{1i}\phi(A) - AE_{12}\phi(E_{21}). 
\]  

(5.8)

(1) \( A = E_{ij}, \ 1 \neq j \in [n] \): the left side of (5.8) is \( \phi^+(E_{ij}) = \phi(E_{ij}) \). The right side of (5.8) is \( E_{1i}\phi(E_{ij}) \). Both sides are clearly equal since \( \phi(E_{ij}) \) has zero entries outside of the first row.

(2) \( A = E_{ii}, \ 1 \neq i \in [n] \): the proof is similar.

(3) \( A = E_{ij}, \ i, j \in [n] - \{1\}, i \neq j \): both sides of (5.8) are zero.

(4) \( A = H_1 = E_{11} - E_{22} \): the left side of (5.8) is zero. The right side of (5.8) is

\[
E_{11}\phi(H_1) - H_1E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}\phi(E_{21}). 
\]

We have

\[-2\phi(E_{21}) = \phi([H_1, E_{21}]) = H_1\phi(E_{21}) - E_{21}\phi(H_1) = -\phi(E_{21}) - E_{21}\phi(H_1),
\]

where the last equality holds since \( \phi(E_{21}) \) has zeros outside of the second row. Therefore, \( \phi(E_{21}) = E_{21}\phi(H_1) \), and the right side of (5.8) is

\[
E_{11}\phi(H_1) - E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}E_{21}\phi(H_1) = 0.
\]

So both sides are equal.

(5) \( A = H_i, \ i \in [n - 1] - \{1\} \): Both sides of (5.8) are clearly zero.

Overall, (5.8) is proved. We have

\[
\phi^+(AB - BA) = A\phi^+(B) - B\phi^+(A), \quad \text{for all } A, B \in M_n. 
\]  

(5.9)

Finally, let \( B = I_n \) in (5.9), then

\[
0 = A\phi^+(I_n) - I_n\phi^+(A) \quad \Rightarrow \quad \phi^+(A) = A\phi^+(I_n).
\]

Setting \( X := \phi^+(I_n) \), we get \( \phi(A) = AX \) for all \( A \in \mathfrak{sl}_n \). \( \square \)

Similarly, we have the following result.

**Lemma 5.9.** Suppose \( \text{char}(\mathbb{F}) \neq 2, 3 \). If a linear transformation \( \phi : \mathfrak{sl}_n \to M_{mn} \) satisfies that

\[
\phi(AB - BA) = \phi(A)B - \phi(B)A, \quad \text{for all } A, B \in \mathfrak{sl}_n,
\]

then there is \( X \in M_{mn} \) such that \( \phi(C) = XC \) for \( C \in \mathfrak{sl}_n \).

The statements of Lemmas 5.8 and 5.9 also hold when \( \text{char}(\mathbb{F}) = 2 \), but the proofs should be adjusted slightly. We will not need the case \( \text{char}(\mathbb{F}) = 2 \) here. The following counterexample shows that Lemma 5.8 is not true when \( \text{char}(\mathbb{F}) = 3 \). Likewise for Lemma 5.9.

**Example 5.10.** Suppose \( \text{char}(\mathbb{F}) = 3 \). In \( M_2 \), let \( H := E_{11} - E_{22} \), and \( \phi : \mathfrak{sl}_2 \to M_2 \) the linear map given by

\[
\phi(E_{12}) := E_{21}, \quad \phi(E_{21}) := 0, \quad \phi(H) := 0.
\]

Then \( \phi \) satisfies (5.7) since

\[
\phi([H, E_{12}]) = 2\phi(E_{12}) = 2E_{21} = -E_{21} = H\phi(E_{12}) - E_{12}\phi(H),
\]

\[
\phi([H, E_{21}]) = -2\phi(E_{21}) = 0 = H\phi(E_{21}) - E_{21}\phi(H),
\]

\[
\phi([E_{12}, E_{21}]) = \phi(H) = 0 = E_{12}\phi(E_{21}) - E_{21}\phi(E_{12}).
\]

However, there is no \( X \in M_2 \) such that \( \phi(E_{12}) = E_{21} = E_{12}X \).
Lemma 5.11. Suppose \( \text{char}(\mathbb{F}) \neq 2 \). Then for any \( f \in \text{Der}(M^0_\mathcal{L}) \):

\[
f(\tilde{sl}_{kk}) \subseteq \tilde{sl}_{kk} + \sum_{i=1}^{k-1} \tilde{M}_{ik} + \sum_{j=k+1}^{t} \tilde{M}_{kj}, \quad \text{for} \ (k, k) \in [I(\mathcal{L})]; \\
f(\tilde{M}_{pq}) \subseteq \tilde{M}_{pq} + \sum_{i=1}^{p-1} \tilde{M}_{iq} + \sum_{j=q+1}^{t} \tilde{M}_{pj}, \quad \text{for} \ 1 \leq p < q \leq t.
\]

The proof is similar to that of Lemma 4.5, with some slight adjustments.

Proof. Given \((k, k) \in [I(\mathcal{L})]\), we have \([\tilde{sl}_{kk}, \tilde{sl}_{kk}] = \tilde{sl}_{kk}\) in \(M^0_\mathcal{L}\). For \(A_{kk}, B_{kk} \in sl_{kk}\),

\[
f([A_{kk}, B_{kk}]) = [f(A_{kk}), B_{kk}] + [A_{kk}, f(B_{kk})] \in \tilde{sl}_{kk} + \sum_{i=1}^{k-1} \tilde{M}_{ik} + \sum_{j=k+1}^{t} \tilde{M}_{kj}.
\]

So (5.11) is done.

Given \(1 \leq p < q \leq t\), we prove (5.12) by induction on \(\ell := q - p\):

1. \(\ell = 1\): Here \((p, q) = (p, p+1) \in [I(\mathcal{L})]\). By Theorem 2.4, at least one of \((p, p)\) and \((p + 1, p + 1)\) is in \([I(\mathcal{L})]\). Without loss of generality, suppose \((p, p) \in [I(\mathcal{L})]\). Since \(\mathcal{L}\) is SDUT, the matrices in \(sl_{pp}\) have the size \(m \geq 2\). Therefore \([sl_{pp}, M_{p,p+1}] = M_{p,p+1}\) in \(M^0_\mathcal{L}\). Let \(\{\cdot\}_{ij}\) also denote the embedding of \(M_{ij}\) to \(M_{ij} \in \mathbb{M}_n\). For \(A_{pp} \in sl_{pp}, A_{p,p+1} \in M_{p,p+1}\),

\[
f(A_{pp}A_{p,p+1}) = f([A_{pp}, A_{p,p+1}]) = [f(A_{pp}), A_{p,p+1}] + [A_{pp}, f(A_{p,p+1})] \\
\in \tilde{M}_{p,p+1} + \sum_{i=1}^{p-1} M_{i,p+1} + \sum_{j=p+2}^{t} \tilde{M}_{pj} \\
- \sum_{i=1}^{p-1} \left\{ f(A_{p,p+1})_{ip} A_{pp} \right\}_{ip} + \left\{ A_{pp}, f(A_{p,p+1})_{pp} \right\}_{pp}.
\]

To get (5.12) for \(q - p = 1\), it remains to prove that \(f(E^{[p,p+1]}_{k_j})_{ip} = 0\) for any given standard matrix \(E^{[p,p+1]}_{k_j}\) in \(M_{p,p+1}\) and \(i \in [p]\). There are two cases:

- \(i \in [p - 1]\): (5.13) shows that for \(A_{pp} \in sl_{pp}\) and \(A_{p,p+1} \in M_{p,p+1}\),

\[
f(A_{pp}A_{p,p+1})_{ip} = -f(A_{p,p+1})_{ip} A_{pp}.
\]

Since the size \(m\) of \(sl_{pp}\) is no less than 2, we can choose \(s \in [m] - \{k\}\). Then

\[
f(E^{[p,p+1]}_{k_j})_{ip} = f(E^{[p,p+1]}_{k_s} E^{[p,p+1]}_{s_j})_{ip} = -f(E^{[p,p+1]}_{s_j})_{ip} E^{[p,p+1]}_{k_s}.
\]
However, we also have
\[
0 = f(\tilde{E}_{sj}^{[p,p+1]}, E_{sj}^{[p,p+1]}))_{i,p+1} \\
= [f(E_{sj}^{[p,p+1]}), E_{sj}^{[p,p+1]}]_{i,p+1} + [E_{sj}^{[p,p+1]}, f(E_{sj}^{[p,p+1]}))]_{i,p+1} \\
= f(E_{sj}^{[p,p+1]}))_{ip} E_{sj}^{[p,p+1]} - f(E_{sj}^{[p,p+1]}))_{ip} E_{sj}^{[p,p+1]} \\
= -f(E_{sj}^{[p,p+1]}))_{ip} E_{ks}^{[p,p+1]} - f(E_{sj}^{[p,p+1]}))_{ip} E_{kj}^{[p,p+1]} \\
= -2f(E_{sj}^{[p,p+1]}))_{ip} E_{kj}^{[p,p+1]}.
\]

Since \(\text{char}(F) \neq 2\), the \(k\)-th column of \(f(E_{sj}^{[p,p+1]}))_{ip}\) must be zero. Then (5.15) shows that \(f(E_{sj}^{[p,p+1]}))_{ip} = 0\).

- \(i = p\) \((5.13)\) shows that for \(A_{pp} \in \mathfrak{sl}_{pp}\) and \(A_{p,p+1} \in \mathcal{M}_{p,p+1}\),

\[
f(A_{pp}E_{p,p+1})_{pp} = [A_{pp}, f(A_{p,p+1})_{pp}] = A_{pp}f(A_{p,p+1})_{pp} - f(A_{p,p+1})_{pp}A_{pp}.
\]

In particular, for \(r \in [m] - \{k\}\), we have \(E_{kr}^{[pp]} \in \mathfrak{sl}_{pp}\) and

\[
f(E_{kr}^{[p,p+1]})_{pp} = f(E_{kr}^{[p,p+1]} E_{kr}^{[p,p+1]})_{pp} = E_{kr}^{[pp]} f(E_{kr}^{[p,p+1]})_{pp} - f(E_{kr}^{[p,p+1]})_{pp} E_{kr}^{[pp]}. \quad (5.16)
\]

Denote

\[
A = [a_{ij}]_{m \times m} := f(E_{kj}^{[p,p+1]})_{pp}.
\]

(5.16) implies that all nonzero entries of \(A\) are located in the \(k\)-th row and the \(r\)-th column. If \(m \geq 3\), we can replace \(r\) by any \(s \in [m] - \{k, r\}\) in (5.16) to show that all nonzero entries of \(A\) are located in the \(k\)-th row.

In both \(m = 2\) and \(m \geq 3\) cases, we have

\[
A = E_{kk}^{[pp]} A + a_{rr} E_{rr}^{[pp]}. \quad (5.17)
\]

Applying (5.16) twice, we get

\[
A = \left[ E_{kr}^{[pp]}, f(E_{kr}^{[p,p+1]}))_{pp} \right] = \left[ E_{kr}^{[pp]}, E_{kr}^{[pp]}, f(E_{kj}^{[p,p+1]})_{pp} \right] \\
= E_{kk}^{[pp]} A - E_{kr}^{[pp]} AE_{kr}^{[pp]} - E_{rk}^{[pp]} AE_{kr}^{[pp]} + AE_{rr}^{[pp]} \\
= (A - a_{rr} E_{rr}^{[pp]} - E_{kk}^{[pp]} (A + a_{rr} E_{rr}^{[pp]})) - E_{rk}^{[pp]} AE_{kr}^{[pp]} + AE_{rr}^{[pp]} \\
= A - a_{rr} (E_{rr}^{[pp]} + E_{kk}^{[pp]}) - E_{rk}^{[pp]} AE_{kr}^{[pp]} + AE_{rr}^{[pp]}.
\]

Therefore,

\[
a_{rr} (E_{rr}^{[pp]} + E_{kk}^{[pp]}) + E_{rk}^{[pp]} AE_{kr}^{[pp]} = AE_{rr}^{[pp]}.
\]

Comparing the \((k, k)\) (resp. \((r, r), (k, r)\)) entry, we get \(a_{rr} = 0\) (resp. \(a_{kk} = 0\), \(a_{kr} = 0\)). Since \(r \in [m] - \{k\}\) is arbitrary, we have \(f(E_{kj}^{[p,p+1]})_{pp} = 0\).

We finish the proof for \(\ell = 1\).
(2) Suppose (5.12) is true for all $\ell < k$. Now for any $(p, p + k) \in [I(\mathcal{L})]$, we have

\[
\widetilde{[M_{p,p+1}, M_{p+1,p+k}]} = \widetilde{M}_{p,p+k} \text{ in } M^0_{\mathcal{L}},
\]

and by induction hypothesis,

\[
f(A_{p,p+1}A_{p+1,p+k}) = f([\widetilde{A}_{p,p+1}, A_{p+1,p+k}]) = [f(\widetilde{A}_{p,p+1}), A_{p+1,p+k}] + [\widetilde{A}_{p,p+1}, f(A_{p+1,p+k})]
\]

\[
\in \widetilde{M}_{p,p+k} + \sum_{i=1}^{p-1} \widetilde{M}_{i,p+k} + \sum_{j=p+k+1}^{t} \widetilde{M}_{pj}.
\]

Therefore, (5.12) is true for $\ell = k$.

(3) Overall, (5.12) is proved for all $(p, q) \in [I(\mathcal{L})]$ with $p < q$. □

Lemma 5.12. Suppose char($\mathbb{F}$) $\neq 2, 3$. Let $f \in \text{Der}(M^0_{\mathcal{L}})$. Then for any $1 \leq p < q \leq t$, there exists $X_{pq} \in M_{pq}$ such that

\[
f(A_{ip})_{iq} = -A_{ip}X_{pq}, \quad \text{for all } (i, p) \in [I(\mathcal{L})] \text{ and } \tilde{A}_{ip} \in \tilde{M}_{ip} \cap M^0_{\mathcal{L}}, \quad (5.18)
\]

\[
f(A_{jq})_{pj} = X_{pq}A_{qj}, \quad \text{for all } (q, j) \in [I(\mathcal{L})] \text{ and } \tilde{A}_{qj} \in \tilde{M}_{qj} \cap M^0_{\mathcal{L}}. \quad (5.19)
\]

The proof is similar to part (3) of the proof of Theorem 3.1 in Section 4.

Proof. Given $p < q$ in $[t]$, we consider the following four situations:

(1) Suppose $(q, j) = (t, t) \in [I(\mathcal{L})]$. For any $A_{tt}, B_{tt} \in \mathfrak{s}l_{tt}$,

\[
f([\tilde{A}_{tt}, B_{tt}])_{pt} = [f(\tilde{A}_{tt}), B_{tt}]_{pt} + [\tilde{A}_{tt}, f(B_{tt})]_{pt} = f(\tilde{A}_{tt})_{pt}B_{tt} - f(B_{tt})_{pt}A_{tt}.
\]

Applying Lemma 5.9 to the map $\phi : \mathfrak{s}l_{tt} \to M_{pt}$ defined by $\phi(C) = f(\tilde{C})_{pt}$, we can find $X_{pt} \in M_{pt}$ such that $f(\tilde{A}_{tt})_{pt} = X_{pt}A_{tt}$ for $A_{tt} \in \mathfrak{s}l_{tt}$.

(2) Similarly, when $(i, p) = (1, 1)$, there exists $Y_{1q} \in M_{1q}$ such that $f(\tilde{A}_{11})_{1q} = -A_{11}Y_{1q}$ for all $A_{11} \in \mathfrak{s}l_{11}$.

(3) Suppose $(q, j) \in [I(\mathcal{L})], (q, j) \neq (t, t)$. Then $q < t$. Given any $j < j'$ in $[t]$, we have $(j, j'), (q, j'), (p, j), (p, j') \in [I(\mathcal{L})]$, and $\tilde{M}_{qj} = \tilde{M}_{qj'} = \tilde{M}_{qj}, \tilde{M}_{jj'}.$

- If $q = j$, then for $A_{qj} \in \mathfrak{s}l_{qq}$ and $A_{jj'} \in \mathcal{M}_{jj'},$

\[
f(A_{qj}A_{jj'})_{pj'} = f([\tilde{A}_{qj}, A_{jj'}])_{pj'} = [f(\tilde{A}_{qj}), A_{jj'}]_{pj'} + [\tilde{A}_{qj}, f(A_{jj'})]_{pj'} = f(A_{qj})_{pj}A_{jj'}.
\]

Applying Lemma 5.7 to the map $\phi : M_{qj} \to M_{pj'}$ defined by $\phi(C) = f(\tilde{C})_{pj'}$, and $\varphi : \mathfrak{s}l_{qq} \to \tilde{M}_{pq}$ defined by $\varphi(D) = f(\tilde{D})_{pj}$, there exists $X_{pq} \in M_{pq}$ such that $f(A_{qj})_{pj} = X_{pq}A_{qj}$ for $A_{qj} \in \mathfrak{s}l_{qq}$, and $f(A_{qj'})_{pj'} = X_{pq}A_{qj'}$ for any $j' > j$ in $[t]$ and any $A_{qj'} \in \mathcal{M}_{qd}$.

- If $q < j$, then for $A_{qj} \in \mathcal{M}_{qj}$ and $A_{jj'} \in \mathcal{M}_{jj'}$, we still have

\[
f(A_{qj}A_{jj'})_{pj'} = f([\tilde{A}_{qj}, A_{jj'}])_{pj'} = [f(\tilde{A}_{qj}), A_{jj'}]_{pj'} + [\tilde{A}_{qj}, f(A_{jj'})]_{pj'} = f(A_{qj})_{pj}A_{jj'}.
\]

Applying Lemma 4.2, there exists a (unique) $X_{pq} \in M_{pq}$ such that $f(A_{qj})_{pj} = X_{pq}A_{qj}$ for all $j > q$ in $[t]$.

(4) Suppose $(i, p) \in [I(\mathcal{L})]$ and $(i, p) \neq (1, 1).$ Similar to the preceding argument, there exists $-Y_{pq} \in M_{pq}$ such that $f(A_{ip})_{pq} = -A_{ip}Y_{pq}$ for $(i, p) \in [I(\mathcal{L})]$ and $A_{ip} \in \mathfrak{M}_{ip}$.

(5) For any $(i, p), (q, j) \in [I(\mathcal{L})]$, we have $[\tilde{A}_{ip}, \tilde{A}_{qj}] = 0$. So

\[
0 = f([\tilde{A}_{ip}, \tilde{A}_{qj}])_{ij} = [f(\tilde{A}_{ip}), \tilde{A}_{qj}]_{ij} + [\tilde{A}_{ip}, f(\tilde{A}_{qj})]_{ij}
\]

\[
= f(\tilde{A}_{ip})_{qj}A_{qj} + A_{ip}f(\tilde{A}_{qj})_{pj} = -A_{ip}Y_{pq}A_{qj} + A_{ip}X_{pq}A_{qj}.
\]
Therefore, \( X_{pq} = Y_{pq}. \)

Now we are ready to prove Theorem 5.3.

**Proof of Theorem 5.3.** We have the Lie subalgebra decomposition
\[
M_\mathcal{L} = \text{span}\{E_{11}^{[kk]} \mid (k, k) \in [I(\mathcal{L})]\} \ltimes M_\mathcal{L}^0.
\]
Given \( f \in \text{Der} (M_\mathcal{L}^0), \) we define \( f^+(A) := f(A) \) for \( A \in M_\mathcal{L}^0. \) The next step is to define \( f^+(E_{11}^{[kk]}) \) for each \( (k, k) \in [I(\mathcal{L})] \) appropriately so that \( f^+ \in \text{Der} (M_\mathcal{L}). \) We will let
\[
f^+(E_{11}^{[kk]}) = sl_{kk} + \sum_{i=1}^{k-1} \tilde{M}_{ik} + \sum_{j=k+1}^{t} \tilde{M}_{kj}
\]
and define the nonzero blocks of \( f^+(E_{11}^{[kk]}) \) as follow.

1. The \((k, k)\) block: it is easy to see that \( f(\cdot)_{kk} : sl_{kk} \to sl_{kk}, \ A_{kk} \mapsto f(\tilde{A}_{kk})_{kk}, \) is a derivation of \( sl_{kk}. \) Since \( \text{char}(\mathbb{F}) \neq 2, \) there exists \( X_{kk} \in sl_{kk} \) such that \( f(\tilde{A}_{kk})_{kk} = [X_{kk}, A_{kk}] \) for \( A_{kk} \in sl_{kk}. \) Define
\[
f^+(E_{11}^{[kk]})_{kk} := [X_{kk}, E_{11}^{[kk]}].
\] (5.20)

2. The \((i, k)\) block, \( i < k: \) by Lemma 5.12, there exists \( X_{ik} \in \mathcal{M}_{ik} \) such that \( f(\tilde{A}_{kj})_{ij} = X_{ik}A_{kj} \) for any \( (k, j) \in [I(\mathcal{L})]. \) Define
\[
f^+(E_{11}^{[kk]})_{ik} := X_{ik}E_{11}^{[kk]} \text{ for all } i \in [k - 1].
\] (5.21)

3. The \((k, j)\) block, \( k < j: \) by Lemma 5.12, there exists \( X_{kj} \in \mathcal{M}_{kj} \) such that for all \( (i, k) \in [I(\mathcal{L})] \) we have \( f(\tilde{A}_{ik})_{ij} = -A_{ik}X_{kj}. \) Define
\[
f^+(E_{11}^{[kk]})_{kj} := -E_{11}^{[kk]}X_{kj} \text{ for all } k < j \leq t.
\] (5.22)

The above process uniquely defines a linear map \( f^+ \in \text{End}(M_\mathcal{L}) \) such that \( f^+|_{M_\mathcal{L}^0} = f. \) Next we verify that \( f^+ \in \text{Der}(M_\mathcal{L}). \) It suffices to prove that for every \( (i, j) \in [I(\mathcal{L})], \)
\[
f^+([E_{11}^{[kk]}, \tilde{A}_{ij}]) = [f^+(E_{11}^{[kk]}), \tilde{A}_{ij}] + [E_{11}^{[kk]}, f^+(\tilde{A}_{ij})] \text{ for all } \tilde{A}_{ij} \in \tilde{M}_{ij} \cap M_\mathcal{L}^0.
\] (5.23)
Denote
\[
X_k := \tilde{X}_{kk} + \sum_{i=1}^{k-1} \tilde{X}_{ik} + \sum_{j=k+1}^{t} \tilde{X}_{kj}.
\] (5.24)

Then (5.20), (5.21), and (5.22) imply that \( f^+(E_{11}^{[kk]}) = [X_k, E_{11}^{[kk]}]. \) So (5.23) is equivalent to
\[
f([E_{11}^{[kk]}, \tilde{A}_{ij}]) = [[X_k, E_{11}^{[kk]}], \tilde{A}_{ij}] + [E_{11}^{[kk]}, f(\tilde{A}_{ij})] \text{ for all } \tilde{A}_{ij} \in \tilde{M}_{ij} \cap M_\mathcal{L}^0.
\] (5.25)

We will prove (5.25) for each block \((i, j) \in [I(\mathcal{L})]:\)

(1) \((k, k) \in [I(\mathcal{L})]: \) the matrices \( X_{kk}, X_{ik} (i < k), \) and \( X_{kj} (k < j) \) satisfy that
\[
f(\tilde{A}_{kk}) = [X_k, \tilde{A}_{kk}] \text{ for all } \tilde{A}_{kk} \in sl_{kk},
\]
where \( X_k \) is given by (5.24). Therefore, (5.25) is true for \((i, j) = (k, k) \in [I(\mathcal{L})].\)
(2) \((k, j), k < j \leq t\): when \((i, j) = (k, j)\), we have
\[
[E_{ij}^{[kk]}, \tilde{A}_{kj}] = E_{ij}^{[kk]} \tilde{A}_{kj} = E_{ij}^{[kk]} E_{21}^{[kk]} \tilde{A}_{kj} = [E_{ij}^{[kk]}, [E_{21}^{[kk]}, \tilde{A}_{kj}]].
\]
So (5.25) is equivalent to the following equalities:
\[
f([E_{ij}^{[kk]}, [E_{21}^{[kk]}, \tilde{A}_{kj}]]) = [[X_{k}, E_{11}^{[kk]}], \tilde{A}_{kj}] + [E_{11}^{[kk]}, f(\tilde{A}_{kj})]
\]
\[
\iff f(E_{ij}^{[kk]} E_{21}^{[kk]} \tilde{A}_{kj} + E_{ij}^{[kk]} f(E_{21}^{[kk]} \tilde{A}_{kj}) + E_{ij}^{[kk]} E_{21}^{[kk]} f(\tilde{A}_{kj})) = [X_{k}, E_{11}^{[kk]}] \tilde{A}_{kj} + E_{11}^{[kk]} f(\tilde{A}_{kj})
\]
\[
\iff f(E_{ij}^{[kk]} E_{21}^{[kk]} \tilde{A}_{kj} + E_{ij}^{[kk]} f(E_{21}^{[kk]} \tilde{A}_{kj}) = [X_{k}, E_{11}^{[kk]}] \tilde{A}_{kj} \text{ (for all } A_{kj} \in \mathcal{M}_{kj})
\]
\[
\iff f(E_{ij}^{[kk]} E_{21}^{[kk]} + E_{ij}^{[kk]} f(E_{21}^{[kk]})) = [X_{k}, E_{11}^{[kk]}]
\]
\[
\iff [X_{k}, E_{ij}^{[kk]} E_{21}^{[kk]} + E_{ij}^{[kk]} [X_{k}, E_{11}^{[kk]}]] = [X_{k}, E_{11}^{[kk]}].
\]

The last equality is obviously true.

(3) \((i, k), 1 \leq i < k\): similarly, we can prove (5.25) for the case \((i, j) = (i, k)\).

(4) \((i, j) \in [I(\mathcal{L})], i \neq k, j \neq k\): the left side of (5.25) is zero. We investigate the right side of (5.25) in three cases:

(a) \(i \leq j < k\): the only possibly nonzero block in the right side of (5.25) is the \((i, k)\) block, which is
\[
[[X_{k}, E_{11}^{[kk]}], \tilde{A}_{ij}]_{ik} + [E_{11}^{[kk]}, f(\tilde{A}_{ij})]_{ik} = -A_{ij}[X_{k}, E_{11}^{[kk]}]_{ik} - f(\tilde{A}_{ij})_{ik} E_{11}^{[kk]}
\]
\[
= -A_{ij}[X_{k}, E_{11}^{[kk]}]_{ik} + A_{ij} X_{jk} E_{11}^{[kk]} \text{ (by Lemma 5.12)}
\]
\[
= -A_{ij} X_{jk} E_{11}^{[kk]} + A_{ij} X_{jk} E_{11}^{[kk]} \text{ (by (5.24))}
\]
\[
= 0.
\]

So (5.25) is done for this case.

(b) \(k < i \leq j\): similarly, we can prove (5.25) for this case.

(c) \(i < k < j\): the right side of (5.25) is
\[
[[X_{k}, E_{11}^{[kk]}], \tilde{A}_{ij}] + [E_{11}^{[kk]}, f(\tilde{A}_{ij})] = 0 + 0 = 0.
\]

So (5.25) holds.

Overall, we have proved (5.25). Therefore, \(f^{+} \in \text{Der}(M_{\mathcal{L}})\) and \(f^{+}|_{M_{\mathcal{L}}^0} = f\). By Theorem 3.1, there is \(X \in M_{\mathcal{L}B}\) such that \(f(B) = [X, B]\) for all \(B \in M_{\mathcal{L}}^0\).

\[\square\]

References

[1] D. Brice, *On derivations of parabolic Lie algebras*, submitted, arXiv:1504.08286 [math.RA].
[2] D. Brice, H. Huang (2014), *On zero product determined algebra*, Linear and Multilinear Algebra, 63 (2015) 326-342 DOI:10.1080/03081080.2013.866668.
[3] Wai-Shun Cheung (2003) Lie Derivations of Triangular Algebras, Linear and Multilinear Algebra, 51:3, 299-310, DOI: 10.1080/0308108031000096993
[4] J. Dixmier, W.G. Lister, *Derivations of Nilpotent Lie algebras*, Proceedings of the American Mathematical Society, Vol. 8, No 1 (1957) 155–158.
[5] Y. Du, Y. Wang *Lie derivations of generalized matrix algebras*, Linear Algebra and Its Applications, Vol. 437 (2012) 2719–2726.
[6] N. Jacobson, *A Note on automorphisms and derivations of Lie algebras*, Proceedings of the American Mathematical Society, Vol. 6 (1955) 281–283.
[7] S. Ou, D. Wang, R. Yao, Derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring, Linear Algebra and Its Applications 424 (2007) 378–383.

[8] D. Wang, Q. Yu, Derivation of the parabolic subalgebras of the general linear Lie algebra over commutative ring, Linear Algebra and Its Applications 418 (2006) 763–774.

[9] Z. X. Chen, Generalized derivations on parabolic subalgebras of general linear Lie algebras, Acta Math. Sci. Ser. B Engl. Ed. 34 (2014), no. 3, 814–828.

[10] D. Wang, S. Ou, Q. Yu, Derivations of the intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices over a commutative ring, Linear Multilinear Algebra 54 (2006), no. 5, 369–377.

[11] J. Li, Y. Cao, Z. Li, Triple and generalized triple derivations of the parabolic subalgebras of the general linear Lie algebra over a commutative ring, Linear Multilinear Algebra 61 (2013), no. 3, 337–353.

[12] P. S. Ji, X. L. Yang, J. H. Chen, Biderivations of the algebra of strictly upper triangular matrices over a commutative ring, J. Math. Res. Exposition 31 (2011), no. 6, 965–976.

[13] X. Y. Kong, L. L. Zhou, N. N. Li, Lie triple derivations of general linear Lie algebras over a commutative ring, J. Math. (Wuhan) 32 (2012), no. 4, 663–668.

[14] D. Benkovič, Lie triple derivations on triangular matrices, Algebra Colloq. 18 (2011), Special Issue No.1, 819–826.

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

E-mail address: pzg0011@auburn.edu, huanghu@auburn.edu