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Deriving Fuzzy Weights of the Fuzzy Analytic Network Process via Fuzzy Inverse Matrix

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Abstract: The analytic hierarchical process/network process (AHP/ANP) is a popular multi-criteria decision making approach for determining the optimal alternative or weights of criteria. Many papers have extended the AHP/ANP to consider the fuzzy environment to reflect the subjective uncertainty of decision-makers. However, the fuzzy ANP (FANP) is not as popular as the fuzzy AHP (FAHP), because the calculation of the fuzzy supermatrix results in the divergence of the steady-state. In this paper, we provide a novel mathematical programming model to calculate the limiting distribution of the fuzzy supermatrix by considering a fuzzy inverse matrix rather than directly calculate the fuzzy supermatrix by limiting powers. In addition, we use a numerical example to illustrate the proposed method and compare the results with the previous method. The numerical results indicate the proposed method has the least spread of the fuzzy weights, thus justifying the usefulness of the proposed method.

Keywords: fuzzy analytic network process; fuzzy supermatrix; mathematical programming; fuzzy inverse matrix; fuzzy weights

1. Introduction

The analytic network process (ANP) [1] is a general form of the analytic hierarchy process (AHP) [2,3] to release the restriction of the hierarchical structure of problems; it has received much attention in decision sciences, operations research (OR), and multiple criteria decision-making (MCDM), e.g., by [4–7]. However, unlike the AHP, which already incorporates the concept of fuzzy numbers to reflect subjective uncertainty, the fuzzy ANP (FANP) has been less discussed by researchers than the fuzzy AHP (FAHP). Though Saaty explained why he believes that the fuzzy AHP/ANP should not be used [8], we think some realistic applications are needed to prove the degree of inaccuracy of the fuzzy method. The main problem of the FANP is the convergent problem of fuzzy numbers when calculating the limiting powers of the supermatrix. That is, the fuzzy numbers of the supermatrix range between $[0, \infty]$ by simply using the power operation.

Most papers have handled the problem of the FANP by first defuzzifying the fuzzy local weight vectors, which are calculated by the FAHP into the crisp vector, to form the supermatrix; they have then used the traditional Markov chain calculation, i.e., raising the supermatrix to limiting powers, to obtain the final weights of the criteria (e.g., [9–13]). Though this kind of method provides an easy way to deal with the ANP in the fuzzy environment, it loses the uncertainty information of weights which might be important for decision-makers.

Though [14] proposed a mathematical programming model to obtain the interval weights of the FANP, it is hard to apply in large and complicated systems due to the hard work of formulating mathematical programming equations and the need to process $2n$ runs for a problem with $n$ criteria.
Later, [15] provided an inverse matrix method to calculate the interval weights of the FANP. However, this method also needs to process $2n$ runs to obtain the interval weights and does not consider the issue of the minimum spread of a fuzzy matrix. In this paper, we handle the convergent problem of the FANP by proposing a novel mathematical programming with the minimum spread of the fuzzy inverse matrix. We transfer the issue of solving the FANP to finding the inverse of a specific fuzzy matrix. Hence, the problem of finding the limiting distribution of the fuzzy supermatrix can be overcome.

The rest of the paper is organized as follows. We first introduce the FANP in Section 2. The new formula for solving the ANP is proposed in Section 3. Section 4 presents the methods to derive the fuzzy inverse matrix, including the proposed method. A numerical example to illustrate the proposed method is listed in Section 5, and conclusions are in the last section.

2. Introduction of the FANP

The theory of the ANP relies on the concept of Markov chains [16] and the condition for a convergent result requires that the supermatrix is a stochastic matrix. This is not a problem when we deal with a crisp supermatrix. However, when we consider an interval/fuzzy supermatrix, the left and right values cannot satisfy the condition of the stochastic matrix and result in the problem of divergence by using traditional fuzzy operations.

To describe the method for solving the FANP, we consider the general form of the supermatrix as the following matrix for simplicity:

$$\Pi = \begin{bmatrix}
\pi_{11} & \pi_{12} & \cdots & \pi_{1m} \\
\pi_{21} & \pi_{22} & \cdots & \pi_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{m1} & \pi_{m2} & \cdots & \pi_{mm}
\end{bmatrix},$$

where $\sum_{i=1}^{m} \pi_{ij} = 1, \pi_{ij} \geq 0, \forall j = 1, \ldots, m$ to satisfy the condition of a stochastic matrix. Note that all elements in the supermatrix, called the local vectors, are derived from the AHP. In the original ANP, the influences are measured by cluster of alternatives. The weights of criteria obtained by the ANP method with and without the cluster of alternatives may be different [17]. To a certain degree, the differences question publications in which the ANP method is only used to obtain weights of criteria. Then, we can raise the supermatrix to limiting powers to obtain the global priority vectors as

$$\lim_{k \to \infty} \Pi^{(k)}$$

where $(k)$ denotes the power operation. When the situation of the ergodic Markov chains happens, the Cesaro sum is used to get the final priority weights. We should highlight that since the ANP stems directly from the AHP, it also inherits the theoretical weaknesses of the assumptions of the AHP which, above all, are the rank reversal problem, the priorities derivation method, and the comparison scale [18,19]. The rank reversal and priorities derivation method are closely related to each other. The rank reversal because of the formulation of the problem assumes that there is a ranking of alternatives determined with the use of the right eigenvector (the preference aggregation method). Solving a reversal problem and performing a preferences aggregation with the use of a left eigenvector method should, as a result, produce a reverse sequence of elements which are pairwise-compared in a matrix. However, this is not always true, in particular in the case of some inconsistencies in the pairwise comparison matrix [20]. Readers may refer to [21] for more the detailed properties of the ANP.
However, when fuzzy values are considered to reflect human uncertainty, we can formulate the following fuzzy supermatrix:

\[ \tilde{\Pi} = \begin{bmatrix}
\tilde{\pi}_{11} & \tilde{\pi}_{12} & \cdots & \tilde{\pi}_{1m} \\
\tilde{\pi}_{21} & \tilde{\pi}_{22} & \cdots & \tilde{\pi}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{\pi}_{m1} & \tilde{\pi}_{m2} & \cdots & \tilde{\pi}_{mm}
\end{bmatrix} \]

where \( \tilde{\pi}_{ij} \) is represented by an interval/fuzzy number. Note that a fuzzy number can be presented by an interval number by adopting the \( \alpha \)-cut operation. Clearly, this fuzzy supermatrix cannot obtain a convergent limiting distribution of the fuzzy supermatrix by simply using the traditional fuzzy power operation. Next, we introduce several FANP methods to derive global weights.

### 2.1. Defuzzy Local Weight Method

Ref. [11,22] extended Mikhailov’s fuzzy preference programming method [23,24] to consider the criteria weights by the FANP as the following three steps:

1. Construct the pairwise comparison matrices of all the criteria/sub-criteria in the system presented by fuzzy numbers and calculate local weights of criteria by using Mikhailov’s model [23,24]. The major advantage of Mikhailov’s method is that it is able to derive consistent values and crisp priority vectors of pairwise comparison matrices.

2. Form the supermatrix to show the interdependencies of the criteria and construct the inner dependence matrices for each criterion.

3. Calculate the global weights of the sub-criteria by multiplying the local weights of the sub-criteria and the interdependent weights of the related criteria.

Though the above method provides an easy way to derive the global weights by the FANP, it can only provide the crisp global weights of criteria, since Mikhailov’s method can derive the crisp weight vector. A similar concept was used to [25–27] to employee Chang’s extent analysis [28] to obtain the non-fuzzy local weights of all criteria/sub-criteria and then derive global weight of the ANP. However, these kinds of applications cannot meet the purpose of providing fuzzy global weights.

### 2.2. Huang’s Method

Ref. [14] provided the mathematical programming model to obtain the interval weights of the FANP as follows:

\[
\begin{align*}
\text{min} \quad & \text{max} \pi_{ij}, \quad \text{for } i = 1, \ldots, m, \\
\text{subject to} & \quad w_{i,n-1} = a_{i,n-1}w_{i,n}, \\
& \vdots \\
& \quad w_{i,1} = \frac{1}{\pi_{m-1}}(a_{i,12}w_{i,2} + a_{i,13}w_{i,3} + \cdots + a_{i,1n}w_{i,n}), \\
& \quad w_{i,1} + w_{i,2} + \cdots + w_{i,n} = 1, \text{for } i = 1, \ldots, k, \\
& \quad \pi_{1} = \pi_{11}\pi_{1} + \pi_{12}\pi_{2} + \cdots + \pi_{1m}\pi_{m}, \\
& \vdots \\
& \quad \pi_{(m-1)} = \pi_{(m-1)}\pi_{1} + \pi_{(m-1)}\pi_{2} + \cdots + \pi_{(m-1)}\pi_{m}, \\
& \quad \pi_{1} + \pi_{2} + \cdots + \pi_{m} = 1, \\
& \quad \pi_{i}(\alpha) = \min\{\pi_{11}\pi_{1} + \pi_{12}\pi_{2} + \cdots + \pi_{im}\pi_{m}\mid \pi_{11} \in [\pi_{11}[\alpha], \ldots, \pi_{im} \in [\pi_{im}[\alpha]], \\
& \quad \pi_{i}(\alpha) = \max\{\pi_{11}\pi_{1} + \pi_{12}\pi_{2} + \cdots + \pi_{im}\pi_{m}\mid \pi_{11} \in [\pi_{11}[\alpha], \ldots, \pi_{im} \in [\pi_{im}[\alpha]], \quad \forall i = 1, \ldots, m, \\
\end{align*}
\]

where \( w_{i,j} \) denotes the \( j \)th local weight of the \( i \)th cluster, \( n \) denotes the number of the criteria of the \( i \)th cluster and \( a_{i,j} \) denotes the subjective weight ratio of the \( i \)th cluster. Though the above model can obtain the interval weights by the FANP, it needs huge work to formulate the mathematical programming
model and hinder the possible application in large and complicated systems. In addition, we need to process $2m$ runs to obtain the interval weights of $m$ criteria.

Later, [15] proposed another concise mathematical model to derive the FANP as follows. Let $\Pi$ be a supermatrix of the ANP and $\Pi^{(\kappa)}$ as the ever entry positive (i.e., $\Pi$ is regular), there is a unique column matrix $\pi$ satisfying $\Pi \pi = \pi$, and the entries of $\pi$ are positive and sum to 1, where $\pi$ is the global weight vector of the ANP. The above matrix identity can be represented as:

$$
\begin{align*}
\pi_1 &= \pi_{11}\pi_1 + \pi_{12}\pi_2 + \ldots + \pi_{1m}\pi_m, \\
\pi_2 &= \pi_{21}\pi_1 + \pi_{22}\pi_2 + \ldots + \pi_{2m}\pi_m, \\
&\vdots \\
\pi_m &= \pi_{m1}\pi_1 + \pi_{m2}\pi_2 + \ldots + \pi_{mm}\pi_m.
\end{align*}
$$

(3)

By moving the right-side of Equation (3) to the left-side, we can obtain:

$$
\begin{align*}
(1 - \pi_{11})\pi_1 - \pi_{12}\pi_2 - \ldots - \pi_{1m}\pi_m &= 0, \\
-\pi_{21}\pi_1 + (1 - \pi_{22})\pi_2 - \ldots - \pi_{2m}\pi_m &= 0, \\
&\vdots \\
-\pi_{m1}\pi_1 - \pi_{m2}\pi_2 - \ldots + (1 - \pi_{mm})\pi_m &= 0.
\end{align*}
$$

(4)

Since the last equation of Equation (4) is superfluous, we replace it with the constraint $\pi'\mathbf{1} = 1$, where $\mathbf{1}' = (1, 1, \ldots, 1)$. As such, Equation (4) can be represented as Equation (5) as the following matrix form:

$$
C\pi = \mathbf{e},
$$

(5)

where

$$
C = \begin{bmatrix}
1 - \pi_{11} & -\pi_{12} & \cdots & -\pi_{1m} \\
-\pi_{21} & 1 - \pi_{22} & \cdots & -\pi_{2m} \\
& \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}, \quad \pi = \begin{bmatrix}
\pi_1 \\
\pi_2 \\
\vdots \\
\pi_m
\end{bmatrix}, \text{ and } \mathbf{e} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}.
$$

Finally, if $C$ is the nonsingular matrix, the global weight vector can be derived as:

$$
\pi = C^{-1}\mathbf{e},
$$

(6)

where $C^{-1} = [c'_{ij}]$ denotes the inverse of $C$ and $CC^{-1} = I$.

If we consider Equation (6) under the fuzzy environment, it can be modified as:

$$
\tilde{C}\tilde{\pi} = \mathbf{e},
$$

(7)

and the fuzzy global weights can be given as:

$$
\tilde{\pi} = \tilde{C}^{-1}\mathbf{e}.
$$

(8)

That is, the fuzzy global weights of the FANP can be given as the last column of $\tilde{C}^{-1}$. Huang (2012) gave the following mathematical programming to calculate the interval number of $\tilde{C}^{-1}$ with respect to a specific $\alpha$-cut by consider the constraint $CC^{-1} = I$ as:
where
\[ c_{11}c'_{11} + c_{12}c'_{21} + \cdots + c_{1n}c'_{n1} = 1, \]
\[ c_{11}c'_{12} + c_{12}c'_{22} + \cdots + c_{1n}c'_{n2} = 1, \]
\[ \vdots \]
\[ c_{n1}c'_{1n} + c_{n2}c'_{2n} + \cdots + c_{mn}c'_{mn} = 1, \]
where \( c'_{ij}[\alpha] = [\min c_{ij}, \max c_{ij}] \) denotes the interval number of the \( i \)th row, and the \( j \)th column element of \( \tilde{C}^{-1} \) and \( [\alpha] \) is the \( \alpha \)-cut operation.

However, [15] only derived the fuzzy weights based on the Zadeh’s extension principle without further considering the minimum spread of fuzzy weights. Though the final result of the model might still provide rational interval weights, we can provide a new model to obtain fuzzy weights with less spread. In addition, Huang’s method needs to process \( 2n \) runs for \( n \) criteria. Hence, in this paper, we propose a new mathematical programming model to derive the fuzzy weights of the FANP by only processing one time.

3. Solving the ANP by Inverse Matrix

Let \( \{X_n : n=0,1,2,\ldots\} \) be a discrete-time stochastic process with a finite state space \( S = \{1,2,\ldots,m\} \). The transition matrix is defined by \( P = [p_{ij}] \) where \( p_{ij} = P[X_n = j|X_{n-1} = i] \) for all \( i, j \in S, n \geq 1 \). The stationary distribution \( \{\pi_i, i \in S\} \) exists iff the Markov chain is irreducible, i.e., every state can be reached from every state. For a finite irreducible Markov chain, the steady state vector \( \pi^T = (\pi_1, \pi_2, \ldots, \pi_m) \) satisfies Equation (10):

\[ \pi^T(I - P) = 0^T \text{ with } \pi^T e = 1 \]

where \( e^T = (1, 1, \ldots, 1) \). Then, according to the proof by [29], the matrix \( (I - P + tu^T)^{-1} \) is nonsingular iff \( \pi^T t \neq 0 \) and \( u^T t \neq 0 \). That is, if \( \pi^T t \neq 0 \) and \( u^T t \neq 0 \), then \( (I - P + tu^T)^{-1} \) is invertible. Hence, we have the property shown in Equations (11) and (12):

\[ (I - P + tu^T)(I - P + tu^T)^{-1} = I \]

and

\[ (I - P)(I - P + tu^T)^{-1} = I - tu^T(I - P + tu^T)^{-1}. \]

If we consider Equation (13):

\[ \pi^T(I - P + tu^T) = \pi^T(I - P) + \pi^T tu^T = \pi^T tu^T \]

We can obtain:

\[ \pi^T = (\pi^T t)u^T(I - P + tu^T)^{-1} \Rightarrow \pi^T u^T = u^T(I - P + tu^T)^{-1} \]

Then, we can multiple \( e \) to Equation (14) and obtain Equation (15) as:

\[ u^T(I - P + tu^T)^{-1} e = \frac{\pi^T e}{u^T t} \Rightarrow \pi^T = \frac{1}{u^T(I - P + tu^T)^{-1} e} \times \frac{1}{t} \]

where

\[ \frac{1}{t} = u^T(I - P + tu^T)^{-1} \]
Hence, let $G^{-1} = (I - P + tu^T)^{-1}$, where $u$ and $t$ are any vectors such that $\pi^T t \neq 0$ and $u^T e \neq 0$; then, we can derive the formulation of $\pi^T$ as shown in Equation (16):

$$\pi^T = \frac{u^T G^{-1}}{u^T G^{-1} e}$$

(16)

If we set $t = e$, we can derive $\pi^T t = 1$, i.e., weight normalization, and $u^T Ge = 1$ based on Equation (16). Then, if $G^{-1} = (I - P + eu^T)^{-1}$ where $u^T e \neq 0$, we can obtain Equation (17) as:

$$\pi^T = u^T G^{-1}$$

(17)

Furthermore, if we claim $u^T = e^T$, then $G^{-1} = (I - P + ee^T)^{-1}$, where $ee^T$ is an all-one matrix. If we let $G^{-1} = [g'_{ij}]$, we can obtain:

$$\pi_j = \sum_{k=1}^{m} g'_{kj}$$

(18)

That is, the $j$th element of the steady state of a Markov chain is the $j$th column sum of $G^{-1}$. Since the transition matrix $P$ in a Markov chain is the transpose of the supermatrix $\Pi$ in the ANP, we can simply obtain the result of the ANP based on the Equation (18).

For example, consider a supermatrix represented as:

$$\Pi = \begin{bmatrix}
0.1290 & 0.6223 & 0.5171 & 0.0657 \\
0.6066 & 0.0000 & 0.1243 & 0.2146 \\
0.1984 & 0.1307 & 0.0000 & 0.1869 \\
0.0660 & 0.2470 & 0.3586 & 0.5327 \\
\end{bmatrix}$$

Then, we set $P = \Pi^T$ and obtain:

$$G^{-1} = (I - P + ee^T)^{-1} = \begin{bmatrix}
0.8021 & 0.1098 & -0.1687 & -0.4932 \\
0.0676 & 0.6986 & -0.2037 & -0.3125 \\
-0.0582 & -0.2311 & 0.6816 & -0.1423 \\
-0.5147 & -0.3158 & -0.1613 & 1.2417 \\
\end{bmatrix}$$

Finally, the global weights of the ANP can be calculated by the sum of the column of $G^{-1}$ as:

$$\pi^T = [0.2968 \ 0.2615 \ 0.1480 \ 0.2937]$$

The purpose of this paper is to propose a method to extend the above equations under the fuzzy environment. Specifically, we propose a rational way to calculate $\tilde{G}^{-1}$. Since the problem involves the calculation of a fuzzy inverse matrix, we should first consider the literature review of the fuzzy inverse matrix as follows.

4. Deriving Fuzzy Inverse Matrix

From the above description, it can be seen that the key to derive the FANP is to calculate the inverse of $G^{-1}$. Next, we introduce the methods of solving the fuzzy inverse matrix from the past research as follows.
4.1. Zadeh’s Extension Principle

The inverse of a fuzzy matrix can be easily derived by Zadeh’s extension principle [30,31] and described as follows. If we let \( \tilde{A} \) be a square regular fuzzy matrix, we can define the set as:

\[
\Omega(\alpha) = \left\{ A^{-1} | AA^{-1} = I, A = [a_{ij}], a_{ij} \in [\tilde{a}_{ij}]_{\alpha} \right\}
\]

for \( \alpha \in [0, 1] \).

Then, the membership function of \( \tilde{B} \in \mathcal{F}(\mathbb{R})^{n \times n} \) by:

\[
\mu_{\tilde{B}}(B) = \sup \{ \alpha | B \in \Omega(\alpha) \}
\]

for \( B \in \Omega(0) \) and define \( \mu_{\tilde{B}}(B) = 0 \) when \( B \notin \Omega(0) \). Hence, the inverse of \( \tilde{A} \) is \( \tilde{B} \). However, \( \tilde{B} \) is not necessarily a fuzzy matrix, unless all of its entries are fuzzy numbers. Though the extension principle provides a method to derive a fuzzy inverse matrix, it does not necessarily result a fuzzy matrix [32].

4.2. Rohn’s Scheme

The idea of Rohn’s method [33,34] is to assume an interval regular matrix \( \hat{A} \) is represented by an interval \([A, \tilde{A}]\). The narrowest interval matrix containing the set \( \left\{ A^{-1} | A \in \hat{A} \right\} \) is the inverse matrix of \( \hat{A} \), i.e., \( \tilde{B} = [B, \bar{B}] \). Then, the bounds of \( \tilde{B} \) are defined by

\[
b_{ij} = \min \left\{ (a^{-1})_{ij} | A \in \hat{A} \right\}
\]

and

\[
\bar{b}_{ij} = \max \left\{ (a^{-1})_{ij} | A \in \hat{A} \right\}
\]

Let \( Y = \left\{ y \in \mathcal{R}^n | y_j = 1, \forall j = 1, \ldots, n \right\} \), and for each vector \( q \in Y_n \), the \( n \times n \) diagonal is matrix defined by \((T_q)_{ii} = q_i\) and \((T_q)_{ij} = 0\) for \( i \neq j, \forall i, j = 1, \ldots, n\). Then, we can define \( A_{yz} \) by

\[
A'_{yz} = A'_c - T_y \Delta' T_z, \forall y, z \in Y_n,
\]

where \( T_y \) and \( T_z \) are the diagonal matrices with diagonal vectors \( y \) and \( z \), respectively, and \( A'_c \) and \( \Delta' \) denote the center and radius matrices of \( \hat{A} \) which can be defined as:

\[
A'_c = \frac{\max[\hat{A}]_0 + \min[\hat{A}]_0}{2}
\]

\[
\Delta' = \frac{\max[\hat{A}]_0 - \min[\hat{A}]_0}{2}
\]

where \( [\hat{A}]_0 \) is the 0-cut of fuzzy matrix \( \hat{A} \).

Then, the solutions of \( \underline{B} \) and \( \overline{B} \) are given by [33,34] as the following formulas:

\[
\underline{B} = \min \left\{ A'_{yz}^{-1} | y, z \in Y_n \right\}
\]

and

\[
\overline{B} = \max \left\{ A'_{yz}^{-1} | y, z \in Y_n \right\}
\]

where \( i, j = 1, \ldots, n \). When we consider a fuzzy matrix \( \hat{A} \), we can use the concept of \( \alpha \)-cut \( [\hat{A}]_\alpha \) to transform a fuzzy matrix into an interval matrix and follow the concept of Rohn’s method to derive the fuzzy inverse matrix.
4.3. $\varepsilon$-Inverse Method

The $\varepsilon$-inverse method is extended from Dubois and Prade’s fuzzy operators [31] to avoid the invalid situation of a fuzzy inverse matrix. Let an LR fuzzy matrix be $A = (A, M, N)$; we say $(\bar{A}, \bar{M}, \bar{N})$ is the left $\varepsilon$-inverse of the fuzzy matrix $A$ if

$$ (\bar{A}, \bar{M}, \bar{N}) \otimes (A, M, N) = (I, \varepsilon I, \varepsilon I) $$

and $(\bar{A}, \bar{M}, \bar{N})$ is the right $\varepsilon$-inverse of the fuzzy matrix $A$ if

$$ (A, M, N) \otimes (\bar{A}, \bar{M}, \bar{N}) = (I, \varepsilon I, \varepsilon I) $$

Then, if we consider the nonnegative fuzzy matrix $\bar{A} = (A, M, N)$ and suppose that $n_{ij} = 0$ if $a_{ij}$ is zero:

$$ (A^{-1}, A^{-1}(\varepsilon I - MA^{-1}), A^{-1}(\varepsilon I - NA^{-1})) $$

is the left $\varepsilon$-inverse of the fuzzy matrix $\bar{A}$, where $\varepsilon > 0$ is chosen in the following interval:

$$ \{ \max[\max\{\frac{n_{ij}}{a_{ij}}|a_{ij} \neq 0\}, \max\{\frac{m_{ij}}{a_{ij}}|a_{ij} \neq 0\}], 1 + \max\{\frac{m_{ij}}{a_{ij}}|a_{ij} \neq 0\} \} $$

On the other hand, let $\bar{A} = (A, M, N)$. The nonnegative matrix $(A^{-1}, M^{-1}, N^{-1})$ is the right tolerance $\varepsilon$-inverse of $\bar{A}$ if $\varepsilon > 0$ which fulfills

$$ \max_{ij}\left\{ \frac{a_{ij} + \sum_{h} \sum_{p} n_{hk} A^{-1}_{kp} n_{pj}}{n_{ij}} \right\} \leq \varepsilon \leq \max_{ij}\left\{ \frac{a_{ij} + \sum_{h} \sum_{p} m_{hk} A^{-1}_{kp} m_{pj}}{m_{ij}} \right\} $$

where $A^{-1}_{kp}$ denotes the $(k, p)$th element of $A^{-1}$. However, this method only considers the nonnegative fuzzy inverse matrix, which is not the case in this paper.

4.4. Basaran’s Method

[35] considered solving a fuzzy linear equation system to derive the inverse of a fuzzy matrix. First, Basaran re-defined the fuzzy one and zero numbers as follows. Let the center value of a fuzzy number be 1 and the left and right spread values be $\delta$ and $\lambda$, where $0 < \delta$ and $\lambda < 1$; this fuzzy number is called fuzzy one number and is denoted by $\bar{1} = (1, \delta, \lambda)$. If the center value of a fuzzy number is 0 and the left and right spread values are $\alpha$ and $\beta$ where $0 < \alpha, \beta < 1$, this fuzzy number is called fuzzy zero number and is denoted by $\bar{0} = (1, \alpha, \beta)$. Then, let an LR fuzzy matrix be $\bar{A} = (A, M, N)$, and the fuzzy equation $\bar{A} \otimes \bar{x} \approx \bar{1}$ can be represented as:

$$ (\bar{a}_{11} \otimes \bar{x}_{11}) \oplus (\bar{a}_{12} \otimes \bar{x}_{21}) \oplus \cdots \oplus (\bar{a}_{1n} \otimes \bar{x}_{n1}) = \bar{1} $$

$$ (\bar{a}_{21} \otimes \bar{x}_{11}) \oplus (\bar{a}_{22} \otimes \bar{x}_{22}) \oplus \cdots \oplus (\bar{a}_{2n} \otimes \bar{x}_{n2}) = \bar{0} $$

$$ \vdots $$

$$ (\bar{a}_{n1} \otimes \bar{x}_{1n}) \oplus (\bar{a}_{n2} \otimes \bar{x}_{2n}) \oplus \cdots \oplus (\bar{a}_{nn} \otimes \bar{x}_{nn}) = \bar{1} $$

We can solve the above equation to derive the center and spread parameters to derive $\bar{x}$ by setting specific $\alpha$ and $\delta$ values. However, Basaran’s method simply assumes all the elements of the fuzzy inverse are non-negative fuzzy numbers, while some of the elements of a fuzzy inverse matrix may be negative fuzzy numbers [36]. Here, we compare the proposed method with Rohn’s methods, because his methods can obtain a rational result and do not need to assume the non-negative fuzzy numbers of a fuzzy inverse matrix.
4.5. The Proposed Method

Here, we focus on the problem of deriving the fuzzy weights of the FANP given a known fuzzy supermatrix, since the previous steps before forming the fuzzy supermatrix can be handled by the fuzzy AHP. The proposed method to calculate the fuzzy weights of the FANP is presented as follows.

Let \( \widetilde{G} = I - \widetilde{P} + ee^T \) and \( \widetilde{G}^{-1} = \left[ \widetilde{s}_{ij} \right] = (I - \widetilde{P} + ee^T)^{-1} \), where \( \widetilde{s}_{ij} = [s_{ij}^l, s_{ij}^c, s_{ij}^r] \) are defined as triangular fuzzy numbers. From the previous description, we know that \( \widetilde{G}^{-1} \) is the key role in determining the fuzzy weights of the FANP. Hence, if we hope to minimize the spread of the fuzzy weights, we should minimize the spread of \( \left[ \widetilde{s}_{ij} \right] \) and result in the objective as:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} \|s_{ij}^c - s_{ij}^l\| + \sum_{i=1}^{n} \sum_{j=1}^{n} \|s_{ij}^c - s_{ij}^r\|
\]

(34)

where \( G^c = [s_{ij}^c] = (I - P^c + ee^T)^{-1} \) is known and \( s_{ij}^l \leq s_{ij}^c \leq s_{ij}^r \).

Then, we can consider the condition of \( \widetilde{G} \widetilde{G}^{-1} = I \) as:

\[
(2 - p_{11})s_{11}^r + (1 - p_{12})s_{21}^r + \cdots + (1 - p_{1n})s_{n1}^r > 1, \\
(1 - p_{21})s_{11}^l + (2 - p_{22})s_{21}^l + \cdots + (1 - p_{2n})s_{n1}^l > 0, \\
\vdots
\]

(35)

and

\[
(2 - p_{11})s_{11}^l + (1 - p_{12})s_{21}^l + \cdots + (1 - p_{1n})s_{n1}^l < 1, \\
(1 - p_{21})s_{11}^c + (2 - p_{22})s_{21}^c + \cdots + (1 - p_{2n})s_{n1}^c < 0, \\
\vdots
\]

where \( p_{ij} \in \overline{p}_{ij}[\alpha], \alpha \in [0, 1] \).

Finally, the fuzzy weights with normalization \( \overline{w}_i = [w_i^l, w_i^r] \) can be formulated by interval numbers as:

\[
\sum_{i=1}^{n} s_{ij}^r = w_i^r, \\
\sum_{i=1}^{n} s_{ij}^l = w_i^l
\]

(36)

We should highlight that the proposed model derives the minimum spread of \( \widetilde{G}^{-1} \) and hence the minimum spread of fuzzy weights, because the addition of fuzzy numbers does not increase their spread. Next, we give two numerical examples to demonstrate the proposed method.

5. Two Numerical Example

In this section, we consider two numerical example to justify the proposed method and compare the result with other approaches. The first example is used to compare Rohn and Huang’s methods, and the second is used to compare Buckley’s fuzzy Markov chain [37].

Example 1. Here, we give an example to illustrate the proposed method and compare the results with Huang’s and Rohn’s methods. We revised the previous crisp case as a fuzzy case and let the fuzzy supermatrix be presented by the interval elements as:

\[
\overline{\Pi} = \begin{bmatrix}
0.1090, 0.1490 & 0.6023, 0.6423 & 0.4971, 0.5371 & 0.0457, 0.0857 \\
0.5866, 0.6266 & 0.0000, 0.0000 & 0.1043, 0.1443 & 0.1946, 0.2346 \\
0.1784, 0.2184 & 0.1107, 0.1507 & 0.0000, 0.0000 & 0.1669, 0.2069 \\
0.0460, 0.0860 & 0.2270, 0.2670 & 0.3386, 0.3786 & 0.5127, 0.5527
\end{bmatrix}
\]
Then, we can calculate the center matrix, $G^c$, of $(I - P^c + ee^T)$ as:

$$
G^c = \begin{bmatrix}
1.8710 & 0.3777 & 0.4829 & 0.9343 \\
0.3934 & 2.0000 & 0.8757 & 0.7854 \\
0.8016 & 0.8693 & 2.0000 & 0.8131 \\
0.9340 & 0.7530 & 0.6414 & 1.4673
\end{bmatrix}
$$

Furthermore, we can use the proposed mathematical programming model to derive $\tilde{G}^{-1}$ as follows:

$$
\tilde{G}^{-1} = \begin{bmatrix}
[0.7993, 0.8056] & [0.1040, 0.1161] & [-0.1734, -0.1664] & [-0.4943, -0.4908] \\
[0.0667, 0.0765] & [0.6975, 0.6989] & [-0.2064, -0.2031] & [-0.3195, -0.3112] \\
[-0.0668, -0.0572] & [-0.2329, -0.2281] & [0.6755, 0.6876] & [-0.1431, -0.1363] \\
[-0.5157, -0.5089] & [-0.3191, -0.3058] & [-0.1431, -0.1363] & [1.2431, 1.2462]
\end{bmatrix}
$$

Finally, we can sum each column to obtain the fuzzy weights of each criterion in the FANP. The results of the fuzzy weights are compared with Huang’s and Rohn’s methods with different $\alpha$-cuts, as shown in Table 1.

| Methods       | $\tilde{w}_1$       | $\tilde{w}_2$       | $\tilde{w}_3$       | $\tilde{w}_4$       |
|---------------|---------------------|---------------------|---------------------|---------------------|
| Rohn’s ($\alpha$-cut = 0) | [0.1229, 0.4768]   | [0.1197, 0.4089]   | [0.0190, 0.2765]   | [0.0324, 0.5549]   |
| Huang’s ($\alpha$-cut = 0)  | [0.2686, 0.3249]   | [0.2388, 0.2833]   | [0.1325, 0.1628]   | [0.2565, 0.3309]   |
| Proposed ($\alpha$-cut = 0) | [0.2835, 0.3159]   | [0.2478, 0.2810]   | [0.1306, 0.1580]   | [0.2838, 0.3078]   |
| Rohn’s ($\alpha$-cut = 0.2) | [0.1575, 0.4400]   | [0.1478, 0.3787]   | [0.0448, 0.2508]   | [0.0851, 0.5022]   |
| Huang’s ($\alpha$-cut = 0.2) | [0.2764, 0.3198]   | [0.2442, 0.2791]   | [0.1357, 0.1599]   | [0.2624, 0.3226]   |
| Proposed ($\alpha$-cut = 0.2) | [0.2897, 0.3081]   | [0.2532, 0.2729]   | [0.1334, 0.1534]   | [0.2839, 0.3041]   |
| Rohn’s ($\alpha$-cut = 0.4) | [0.1922, 0.4036]   | [0.1761, 0.3489]   | [0.0706, 0.2252]   | [0.1375, 0.4499]   |
| Huang’s ($\alpha$-cut = 0.4) | [0.2815, 0.3147]   | [0.2486, 0.2748]   | [0.1387, 0.1569]   | [0.2682, 0.3154]   |
| Proposed ($\alpha$-cut = 0.4) | [0.2914, 0.3052]   | [0.2552, 0.2700]   | [0.1370, 0.1520]   | [0.2863, 0.3015]   |
| Rohn’s ($\alpha$-cut = 0.6) | [0.2269, 0.3677]   | [0.2044, 0.3195]   | [0.0964, 0.1995]   | [0.1897, 0.3977]   |
| Huang’s ($\alpha$-cut = 0.6) | [0.2865, 0.3096]   | [0.2529, 0.2706]   | [0.1418, 0.1539]   | [0.2739, 0.3081]   |
| Proposed ($\alpha$-cut = 0.6) | [0.2932, 0.3024]   | [0.2573, 0.2672]   | [0.1407, 0.1507]   | [0.2888, 0.2989]   |
Based on the results of Table 1, it can be seen that the proposed method provides rational fuzzy weights which range between [0,1] and have a minimum spread of fuzzy numbers compared to other methods. In contrast, Rohn’s method has the maximum spread of the fuzzy numbers, except for the \( \alpha \)-cut = 1, and it is hard for a decision-maker to make a concrete decision because the information is too vague.

Example 2. In this example, we consider Buckley’s example [37]. Let a fuzzy transition matrix be

\[
\tilde{\Pi} = \begin{bmatrix}
[0.4, 0.5, 0.6] & 0 & [0.4, 0.5, 0.6] \\
[0.2, 0.25, 0.3] & [0.7, 0.75, 0.8] & 0 \\
0 & [0.5, 0.6, 0.7] & [0.3, 0.4, 0.5]
\end{bmatrix}
\]

Then, the center matrix, \( G^c \), of \( (I - \tilde{P} + ee^T) \) can be calculated as:

\[
G^c = \begin{bmatrix}
1.50 & 1 & 0.50 \\
0.75 & 1.25 & 1 \\
1 & 0.40 & 1.60
\end{bmatrix}
\]

Finally, we can derive the steady-state of the fuzzy Markov chain and compare it with Buckley’s method, as shown in Table 2. Note that we let \( \alpha \)-cuts = 0 and 1, respectively.

Table 2. The results between Buckley’s and the proposed methods.

| Methods           | \( \tilde{w}_1 \)         | \( \tilde{w}_2 \)         | \( \tilde{w}_3 \)         |
|-------------------|---------------------------|---------------------------|---------------------------|
| Buckley’s (\( \alpha \)-cut = 0) | [0.1923, 0.3443] | [0.4255, 0.6176] | [0.1600, 0.2857] |
| Proposed (\( \alpha \)-cut = 0)   | [0.2455, 0.2762] | [0.5071, 0.5362] | [0.2012, 0.2335] |
| Buckley (\( \alpha \)-cut = 1)    | [0.2609, 0.2609] | [0.5217, 0.5217] | [0.2174, 0.2174] |
| Proposed (\( \alpha \)-cut = 1)   | [0.2609, 0.2609] | [0.5217, 0.5217] | [0.2174, 0.2174] |

From the results of Table 2, it can be seen that the proposed method significantly reduces the spread of the fuzzy interval of the weights and justifies the usefulness for the fuzzy Markov chain problem.

6. Conclusions

The ANP is a useful approach to determine the weights of the interdependency and feedback effects between criteria. However, when involving subjective uncertainty, the ANP is hard to extend to the FANP because of the problem of calculating the limiting powers of a fuzzy supermatrix. Though several papers have used the defuzzified method to obtain a crisp supermatrix, they can only drive the
crisp weights of the FANP, which might lose important information for decision-makers to make a sound decision.

In this paper, we proposed a novel mathematical programming model to derive the fuzzy weights of the FANP by using the fuzzy inverse matrix based on the criterion of the minimum spread of fuzzy numbers. In addition, we proposed another formula, compared with Huang’s method, to derive the weights of the FANP. We used two numerical examples to illustrate the proposed method with different $\alpha$-cuts and compared the results with Huang’s and Rohn’s methods in the first example and Buckley’s method in the second method. The first empirical results showed that Rohn’s method has the maximum spread of the fuzzy weights. It could be realized that Rohn’s method simply inverses a fuzzy matrix by the interval operations without considering further constraints. In contrast, Huang’s method uses the Zadeh’s extension principle and considers that the constraints of the sum of the fuzzy weights is equal to one. Hence, the fuzzy weights are more rational and have less spread compared to the Rohn’s method. The second example also indicated that, compared to Buckley’s method, the proposed method can obtain a lesser spread of fuzzy numbers in the problem of the fuzzy Markov chain.

In addition, we can considered the further applications of the proposed method. For example, forecasts are interesting due to the fact that they can counteract adverse actions and simulate decisions. Therefore, for forecasts, it is necessary to use methods that cope well with uncertainty and inaccuracy [38]. Furthermore, risk evaluation [39] should be another important research issue because it considers different types of uncertainty.

In this paper, we proposed a new mathematical programming model to derive the fuzzy weights of the FANP by simultaneously minimizing the spread of fuzzy numbers and constraining the sum of all the fuzzy weights to equal to one. In addition, we also used the center matrix of $G$, which is a useful information to derive the minimum spread of the fuzzy weights of the FANP. Hence, the proposed method has the minimum spread of the fuzzy weights of the three methods and increases the possible applications of the FANP.

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