Continuum Soliton Chain Analog to Heisenberg Spin Chain System. Modulation Stability and Spectral Characteristics

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Received: 16 October 2021 / Accepted: 29 January 2022
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Abstract

The one-dimensional (1D) Heisenberg spin chain system (HSCS) allows to investigate anomalous features originating from strong quantum fluctuations, which become more significant than those in higher dimensions. A continuum model equation of the HSCS, based on the discrete model, was constructed in the literature. It is nonlinear Schrödinger equation (NLSE) with biquadratic dispersion and fifth degree nonlinearity. Rare research works were done in this area. Notably for deriving the exact solutions and investigate the physical phenomena produced. Our objective, here, is to obtain these solutions, which we think they are new. Further, an analog of the different geometric solutions structures to the known characteristics of HSCS is performed. The unified method is implemented to find the exact solutions of the continuum model equation. A variety of solutions are obtained where they are evaluated numerically and represented in graphs. In these graphs, it is remarked that the solutions exhibit soliton chain (or dense soliton chain). In an analog to spin chain. In the contour plots, they show different shapes of super lattices. Furthermore, complex chirped waves are observed. A significant result is that these solutions are bounded by $-1/4$ and $1/4$, which can be relevant to the spins $-1/2$ and $1/2$. The analysis of modulation stability is carried and it is found that there is a critical value for the dominant parameters, where below this value, modulation instability holds otherwise modulation stability occurs. For the spectral characteristics, it is shown that the wave number increases abruptly and decreases to an asymptotic state, while the frequency is monotonic increasing. The spectrum is periodic wave away from the origin, but near the origin it is soliton.

Keywords Heisenberg · Spin chain · Soliton chain · Super lattices · Unified method

1 Introduction

In the study of quantum phase and phase transition, and quantum computations are paradigmatic. In quantum spin systems, lower energy state for a nuclear spin in an external field is
spin (+1/2) while the higher energy corresponds to spin (−1/2). The quantum spin chains in a magnetic field was studied in [1], by using Bethe-ansatz solutions which arise from string solutions that continuously connect the mode of the lowest-energy excitation. In [2], It was reported that antiferromagnetic HSCs, in a 1D trap, are stabilized by strong repulsive interactions between the two spin components in the absence of an external potential. A scheme for conditional state transfer in a HSC, produced in a transistor, was proposed and analyzed in [3, 4]. Exact solutions of the generalized HSCS were found and the thermodynamic features were studied on the basis of the exact solutions [5]. Analytical and numerical studies of spin transport in a 1D Heisenberg model, in linear-response regime, were carried [6]. In [8], the integrability of 1D classical continuum inhomogeneous biquadratic HSC and the effect of nonlinear inhomogeneity on the soliton of completely integrable spin model are studied. The integrability aspects of a classical one-dimensional continuum isotropic biquadratic HSC in its continuum limit was studied. This was achieved via a differential geometric approach, the dynamical equation for the spin chain is expressed in the form of a higher-order NLSE [7]. The conserved quantities are expressed in terms of a sum over simple polynomials in spin variables. Very recently, It was shown that for a NLSE, whatever its formulation, is integrable (or completely integrable) when the real and imaginary parts are linearly dependent [9–12]. A direct construction of explicit expressions for all the quantum integrals of motion for the isotropic Heisenberg s=1/2 spin chain was presented [13]. Although the 1D Heisenberg ferromagnetic spin chain (HFSC) was rarely studied in the literature, the (2+1) dimensional HFSC was remarkably considered. This may be argued to that in the first case the model equation (ME) is taken a NLSE with biquadratic dispersion and fifth degree nonlinearity. While in the second case, the ME is taken a NLSE with quadratic dispersion and Kerr nonlinearity but in (2+1) dimensions. In [14], the (2+1)-dimensional HFSC that describes the nonlinear dynamics of magnet was studied. Two mathematical approaches for showing dark, bright, kink-type and singular soliton solutions to the HFSCS were presented. The NLSE in (2 + 1) dimensions, with beta derivative evolution, was considered to study nonlinear coherent structures for HFSC with magnetic exchanges [15]. In [16], the NLSE in (2+1)-dimensions for the HFSC, with anisotropic and bilinear interactions in the semi classical limit, where two integrating schemes were used, was studied. The (2+1)-dimensions HFSCS was considered for the objective of finding the exact solutions via a specific transformation and adopting a modified version of the Jacobi elliptic expansion method [17]. An ansatz method to solve The HFSC equation was used to get bright and dark 1-soliton solutions. Some conditions of integrability were given which guarantee the existence of solitons [18]. In [19], construction of further exact soliton solutions of the (2 + 1)-dimensional HFSCS and investigating the nonlinear dynamics of magnets and explains their ordering in ferromagnetic materials were carried. The collision dynamics of soliton in discrete classical ferromagnetic spin chain with Dzyaloshinskii-Moriya (DM) interaction in the classical limit are analyzed [20]. In [21], The conformable fractional derivative HFSC was considered via the complete discrimination system for polynomial method. The rational combined multi-wave solutions were obtained for HFSC by using the logarithmic transformation and symbolic computation with ansatz functions [22]. The NLSE that describes the spin dynamics of (2 + 1)-dimensional inhomogeneous IHFSC with bilinear and anisotropic interactions in the semi classical limit was investigated [23]. In [24], Hirota bilinear method with appropriate polynomial functions in bilinear forms, the one-order rogue waves solution and its existence condition were obtained. Different methods and techniques were used to solve nonlinear evolution equations; \( \text{Tanh} \) and \( \text{Exp-function} \) [25, 26], \( \frac{G'}{c} \) expansion [27], Darboux transformation [28], Kyrdashov
method, [29], Hirota-bilinear transformation [30], Lie symmetries of NLPDEs [31]. Very recently different methods and techniques were introduced, among them, the first integral method, the improved q- homotopy perturbation method and the unified algebraic and auxiliary equation expansion methods [32–41]. Here, the unified method (UM) [42–47] is used in this paper.

The method used here is compared with the known methods, known in the literature.

Indeed, the UM [36] prevails the all known methods such as, the tanh, modified, and extended versions, the F-expansion, the exponential, the G'/G expansion method, the Kerdyashov methods, as it is of low time cost in symbolic computation. Further, it provides solutions which cannot be obtained by the other methods.

The outlines of the paper is in what follows. Section 2 is devoted to mathematical equations and outlines of the UM and GUM. The solutions in the polynomial form are presented in Section 3. While the solutions in the rational forms are carried in Section 4. Section 5 is devoted to modulation stability analysis. Conclusions are given in Section 6.

2 The Model equation and outlines of the UM

2.1 The Model equation

For the 1D classical HFSCS with biquadratic exchange and a bilinear exchange interaction, the Hamiltonian is,

\[ H = -J \sum_n (S_n \ldots S_{n+1}) - \alpha J \sum_n (S_n \ldots S_{n+1})^2, \quad S_n = (S_n^x, S_n^y, S_n^z), S_n^2 = 1, \quad (J > 0), \]

(1)

where \( S_n := S_n(t) \) and \( \alpha \) is the biquadratic exchange parameter, which is considered in a dominant parameter. By assuming that the lattice side, a, is small, in the continuum analog, we write \( x = na \) and \( S_{n+1} = S(x, t) + a S_x(x, t) + a^2 S_{xx}(x, t)/2! + \ldots \) Up to the order of \( a^4 \), the equation of motion takes the form,

\[ S_t = S \times [S_{xx} + v S_{xxx} + \beta ((S \ldots S_{xx}) S_x + \frac{2}{3} (S \ldots S_{xxx}) S_x)], \]

(2)

where \( v = a^2/12 \) and \( \beta = \alpha a^2/(1 + 2 \alpha) \). The continuum model equation, based on (2) was constructed [47]

\[ i w_t + w_{xx} + 2w \mid w \mid^2 + v w_{xxxx} - 4\delta w_{xx}^*, w^2 - 4 \mid w \mid^2 w_{xx} - 4\alpha \mid w \mid^2 w_{xx} - 4v_0 w^* (w x)^2 - 24\sigma \mid w \mid^4 w = 0, \]

(3)

where \( \alpha := 4\beta + 9v; \mu = 2\beta + 3v; \nu_0 = 2\beta + \frac{7\mu}{2}, \sigma = \frac{\beta}{2} + v; \delta = \beta + 2v, \text{ and } w := w(x, t). \) Equation (1) is a NLSE with the highest biquadratic dispersion and highest nonlinearity of fifth degree.

The spectral characteristics are, here, introduced. To this issue, we write,

\[ w(x, t) = | w(x, t) | e^{i(K x - \Omega t)}, \]

(4)

where \( K \) is the wavenumber and \( \Omega \) is the frequency;

\[ K = \frac{\int_0^\infty (\int_{-\infty}^\infty \mid w(x, t) \mid dx) dt}{\int_0^\infty (\int_{-\infty}^\infty \mid w(x, t) \mid dx) dt}, \quad \Omega = \frac{\int_0^\infty (\int_{-\infty}^\infty \mid w_t(x, t) \mid dx) dt}{\int_0^\infty (\int_{-\infty}^\infty \mid w(x, t) \mid dx) dt}, \]

(5)
and the spectrum content is,

$$W(k_0, t) = \int_{-\infty}^{\infty} e^{-ik_0x} w(x, t) dx. \quad (6)$$

For the objective of finding the solutions of (3), we introduce a transformation for \(w(x, t)\) with complex amplitude,

$$w(x, t) = (u(x, t) + iv(x, t))e^{i(kx - \omega t)}, \quad w(x, t)^* = (u(x, t) - iv(x, t))e^{-i(kx - \omega t)}. \quad (7)$$

This transformation allows to inspect the effect of soliton-periodic wave collision, which is elastic or inelastic depending on the waves solutions if they are smooth or not smooth. When inserting (7) into (3) gives rise, for the real and imaginary parts respectively, to,

$$-4\alpha k^2 u^3 + 4\delta k^2 u^3 + 4k^2 \mu u^3 + 4k^2 v^3 + k^4 v u - k^2 u - 24\sigma u^5 + 2u^3 + u \omega$$

$$-v_t - 4\alpha k^2 u'v^2 + 4\delta k^2 u'^2 + 4k^2 \mu v'^2 + 4k^2 v'u'^2 - 48\sigma u^3 v^2 + 24uv^2$$

$$-24\sigma uv^4 - 4k^3 vv_x + 8\alpha kuvv_x - 16\delta kuvu_x - 2kv_x - 4\alpha u (u_x)^2 - 4vou (u_x)^2$$

$$-8\alpha ku^2 v_x + 8\delta kuv^2 v_x + 8k\mu u^2 v_x + 8kouv^2 v_x - 8\delta kuv^2 v_x + 8\delta kuv^2 v_x$$

$$-6k^2 vuu_x + 8kouv^2 u_x - 4\delta u^2 u_{xx} - 4\mu \mu u_{xx} - 8vouuu_x v_x + u_{xx} - 4\alpha u (v_x)^2$$

$$+4vou (v_x)^2 - 4k v v_{xxx} + v u_{xxxx} + 4\delta v^2 u_{xx} - 4\mu \mu v_{xx} - 8\delta uvv_{xx} = 0,$$

$$-4\alpha k^2 u^2 v + 4\delta k^2 u'^2 + 4k^2 \mu v^2 + 4k^2 v'u'^2 + k^4 vu - 2^2 v + 2u^2 v + v \omega$$

$$2ku_x + u_t - 4\alpha k^2 v^3 + 4\delta k^2 v^3 + 4k^2 \mu v^3 + 4k^2 \mu v'^3 - 24\sigma u^4 v' - 24\sigma v^5$$

$$+ 2v^3 + [-4k^3 v + 8\delta ku^2 - 8k^2 u v^2 - 8\delta kuv^2 + 8\alpha k v^2 - 8kv^2 - 8kuv^2]u_x$$

$$-4\alpha v (u_x)^2 + 4vou (u_x)^2 - 8\alpha kuvv_x + 16\delta kuvv_x - 8vouuu_x v_x - 4\alpha v (v_x)^2$$

$$-4vou (v_x)^2 - 6k^2 v u_{xxx} + 4k v_{xxx} u (x, t)^2 - 44u^2 v_{xx} - 8\delta uvv_{xx}$$

$$-4\delta v^2 v_{xx} - 4\mu \mu v^2 v_{xx} + v v_{xxxx} + v_{xx} = 0. \quad (9)$$

We search for traveling waves solutions. To this end, we use the transformations \(u(x, t) = U(z), v(x, t) = V(z)\) and \(z = ax + bt\). Under these transformations (8) and (9) become, respectively,

$$-V'(8a^2v_0VU' - 4ak^3 v + 8akV^2(\delta - \mu - vo) + 2ak + b) + U^3$$

$$(4k^2(-\alpha + \delta + \mu + \nu) - 48\sigma V^2 + 2) + 4aU^2 (a(\delta + \mu)U'' + 2kV'(\alpha - \delta - \mu - vo))$$

$$-24\sigma U^5 + U (k^4v + V^2 (4k^2(-\alpha + \delta + \mu + vo) + 2) - k^2 + \omega$$

$$-4a^2(\alpha + vo) (U')^2 - 4a^2\alpha (V')^2 + 4a^2vo (V')^2 - 8aV (a(\delta V'' - k(\alpha - 2\delta)U')$$

$$-24\sigma V^4 + a^2 (aU^{(4)} - 4kV^{(3)}) + U'' (-6k^2 v + 4V^2(\delta - \mu + 1)) = 0, \quad (10)$$
\[ V^3 \left( 4k^2(-\alpha + \delta + \mu + \nu_0) - 48\sigma U^2 + 2 \right) - 24\sigma V^5 - k^2 - 24\sigma U^4 + \omega \\
+ U' \left( -4ak^3v + 2ak + b \right) + 8akU^2(\delta - \mu - \nu_0) - 8a^2\nu_0UV' + V \left( k^4v \right) + \\
+ U'^2 \left( 4k^2(-\alpha + \delta + \mu + \nu_0) + 2 \right) + 4a^2(\nu_0 - \alpha) \left( U' \right)^2 - 4a^2\alpha \left( V' \right)^2 - 4a^2\nu_0 \left( V' \right)^2 - 8aU \left( a\delta U'' + k(\alpha - 2\delta)V' \right) \]
\[ a^2 \left( av \left( aV^{(4)} + 4kU^{(3)} \right) + V'' \left( -6k^2v + 4U^2(\delta - \mu) + 1 \right) \right) - 4aV^2 \left( a(\delta + \mu)V'' + 2kU'(-\alpha + \delta + \mu + \nu_0) \right) = 0. \]  

Here, the solutions of (10) and (11) are found by using the UM and GUM. The Um asserts that the solutions of NLPDEs (NLODEs) are formulated in polynomial and rational forms, in auxiliary functions that satisfy appropriate auxiliary equations AE).

### 2.2 Outlines of the UM

#### 2.2.1 Polynomial forms

By considering (10) and (11), the polynomial forms are,

\[ U(z) = \sum_{j=0}^{m_1} a_j g(z)^j, \quad V(z) = \sum_{j=0}^{m_2} b_j g(z)^j, \quad (g'(z))^p = \sum_{j=0}^{r_p} c_j g(z)^j, \quad p = 1, 2. \]  

To determine \( m_1, i = 1, 2 \) and \( r \), we use the balance and compatibility conditions. First, we consider the case when \( p = 1 \). The balance condition is determined by balancing the highest order derivative and highest nonlinearity terms. In the present case, the balance condition reads \( m_1 = m_2 = r - 1 \). To determine the consistency condition, we require the following

(a) the number of equations that result from inserting (12) into (10) and (11) and by setting the coefficients of \( g(z)^i, i = 0, 1, 2, \ldots \) say \( h(k) \).

(b) the number of arbitrary parameters \( a_i, b_i, c_i \), say \( f(k) \). For integrable equations the condition is \( h(k) - f(k) \leq s \), where \( s \) is the highest order derivative (here \( s = 4 \)). In the present case the consistency condition reads \( 1 \leq r \leq 3 \).

The case when \( p = 2 \) can be analyzed by the same way. We mention that when \( p = 1 \), the solutions of the AE are elementary functions, while they are periodic or elliptic when \( p = 2 \).

#### 2.2.2 Rational forms

In the UM the rational solutions are written,

\[ U(z) = \frac{a_1g(z) + a_0}{s_1g(z) + s_0}, \quad V(z) = \frac{b_1g(z) + b_0}{s_1g(z) + s_0}, \quad (g'(z))^p = \sum_{j=0}^{r_p} c_j g(z)^j, \quad p = 1, 2. \]

### 3 Polynomial solutions of (10) and (11)

Here, we consider the following cases.
3.1 When \( p = 1 \) and \( r = 2 \)

In this case (12) reduces to

\[
U(z) = a_1 g(z) + a_0, \quad V(z) = b_1 g(z) + b_0, \quad g'(z) = c_2 g(z)^2 + c_1 g(z) + c_0. \tag{14}
\]

When inserting (14) into (10) and (11), and by setting the coefficients of \( g(z)^i = 0, \ i = 0, 1, \ldots \), we have, (for linearity dependent solutions, \( b_0 = a_0 b_1/a_1 \)),

\[
v = \frac{(a_1^2 + b_1^2) (a^2 c_2^2 (\alpha + 2b + 2\mu + \nu_0) + 6c (a_1^2 + b_1^2))}{6a^2 c_2^2}, \quad c_2 = -r^2, \\
v_0 = \frac{6c (a_1^2 + b_1^2)}{a^2 c_2^2} + \alpha + 4\delta, \quad \delta := \frac{1}{12s a_1^2 c_2^2 (a_1^2 + b_1^2)} \left( 4a_0^2 b_1^2 c_2 \left( 12b_1^2 \sigma - a^2 c_2^2 (\alpha - 2\mu) \right) \right)
\]

\[
-48a_0^2 c_2 \sigma + a_1^4 \left( 4a^2 c_2 c_2^2 (\alpha - 2\mu) + c_2 \left( 48a_0^2 \sigma + 6\mu k^2 - 3 \right) - 96c_0 b_1^2 \sigma \right)
\]

\[
+ a_1^2 \left( b_1^2 c_2 \left( 4a^2 c_2 c_0 (\alpha - 2\mu) + 6a_0^2 \sigma + 6\mu k^2 - 3 \right) \right) + c_1 (a^2 + b_1^2) (4a_0^2 c_2 (\alpha - 2\mu) - 48b_1^2 \cos) \]

\[
c_1 = \frac{2a_0 c_2}{a_1}, \quad b = \frac{\mu}{\sigma}, \quad P = 2ak \left( \alpha (12k^2 \nu - 1) + 2\mu - 3\nu \right)
\]

\[
+ 3a^4 r^8 (4k^2 \nu + 1) a^2 r^4 (a_1^2 + b_1^2), \quad H = (a_1^2 + b_1^2) (12\sigma (a_1^2 + b_1^2) - a^2 r^4 (\alpha - 2\mu)),
\]

\[
c_0 = \frac{H_1}{H_2}. \quad H_1 = -a^4 r^8 \left( a_1^2 (4a_0^2 (\alpha - 2\mu) + 12k^2 \nu + 3) + 4a_0^2 b_1^2 (\alpha - 2\mu) \right)
\]

\[
+ 24a_1^2 b_1^2 (a_1^2 + b_1^2)^2 + a^2 r^4 (a_1^2 + b_1^2)^2, \quad H_2 = 4a_1^2 a^2 r^4 (a_1^2 + b_1^2) (a^2 r^4 (\alpha - 2\mu) - 12\sigma (a_1^2 + b_1^2)) .
\]

\[
\omega = \frac{H_3}{H_2}. \quad H_3 = 90a_1^{10} k^2 \nu^2 (\alpha - 2\mu) (4k^2 \nu + 1) - 432a^2 k^2 r^4 (a_1^2 + b_1^2)^4 (2k^2 (\alpha + 2\mu) - 1)
\]

\[
- 1728k^4 \delta^3 (a_1^2 + b_1^2)^5 - 3a_0^2 a_1^2 + b_1^2 r^6 \left( -8k^2 \mu^2 \left( 13k^2 \nu + 2 \right) \right)
\]

\[
+ 2a^2 k^2 (17k^2 \nu + 4) + \alpha (232k^4 \mu k^2 (8\mu + 42\nu) + 3)
\]

\[
+ \mu (84k^2 \nu + 6) \left( 9 \left( 11k^2 \nu^2 + 16k^2 \nu - 3 \right) - \alpha (232k^4 \mu k + 2 (8\mu + 42\nu) + 3) \right)
\]

\[
+ 9a^4 r^8 \sigma (a_1^2 + b_1^2)^3 (4k^4 (5\alpha^2 - 28\alpha \mu - 12\mu^2 + 168\nu \sigma)
\]

\[
+ 4k^2(\alpha + 10\mu + 12\sigma) - 3) \sigma (a_1^2 + b_1^2) \left( -144k^4 (\mu^2 - 14\nu \sigma) + 12k^2 (8\mu + 33\sigma) + 9) \right),
\]

\[
H_4 = 6a^4 r^8 (a_1^2 + b_1^2) (a^2 r^4 (\alpha - 2\mu) - 12\sigma (a_1^2 + b_1^2))^2 .
\]

(15)

By solving the AE in (14), the solutions of (8) and (9) are,

\[
u(x, t) = a_1 \frac{K \tanh \left( \frac{x + A_0 K}{k_0} \right)}{K_0},
\]

\[
K = \sqrt{3a^4 r^8 (4k^2 \nu + 1) - (24k^2 \sigma (a_1^2 + b_1^2) + M)}.
\]
\[ M = a^2 r^4 \left( 2k^2(5\alpha + 2\mu) - 3 \right) \left( a_1^2 + b_1^2 \right) \]
\[ K_0 = 2ar^2 \sqrt{a_1^2 + b_1^2} / 12\sigma \left( a_1^2 + b_1^2 \right) - a^2 r^4 (\alpha - 2\mu). \]

(16)

\[ v(x, t) = \frac{b_1}{a_1} u(x, t), \ z = ax + bt. \]

The solutions in (16) are evaluated numerical and the results are used to display \( Re w \) in Fig. 1(i)-(v).

Figures 4 and 5. In (iv), the contour plots is displayed, while in (v) the variation of \( Re w \) against \( x \) for different values of \( t \) is done.

Figure 1(i) shows “continuum” soliton chain with trapping, while Fig. 1(ii) and (iii) show complex soliton chain. Fig. 1(iv) shows super lattices and Fig. 1(v) shows “continuum” soliton chain.

The Spectral characteristics which are given in (5) and (6) are shown in Fig. 2(i)-(iii), for the wave number, the frequency and spectrum respectively.

Figure 2(i) shows the there a critical value of \( \nu \) where it increases and decreases abruptly. Figure 2(ii) shows that the frequency increases with \( \nu \). Figure 2(iii) shows soliton chain with small amplitude apart near \( x = 0 \).

3.2 When \( p = 2 \) and \( r = 2 \)

In this case consider the solution in (14), but the AE is,

\[ g'(z) = g(z) \sqrt{c_2 g(z)^2 + c_1 g(z) + c_0}. \]

(17)

Fig. 1 The 3D plot is displayed for \( Re w \) against \( x \) and \( t \) by varying the values of \( \alpha \) and \( \nu \) when \( r := 1.2, k = 1.5, a := 1.2, a_1 = 0.7, b_1 = 0.5, \sigma = 0.6, \mu = 0.5, A_0 = -5 \). In (i) \( \alpha = 1.5, \nu = 0.3 \), (ii) \( \alpha = 1.5, \nu = 1.3 \), (iii) \( \alpha = 2.5, \nu = 0.3 \).
By substituting from (14) and (17) into (10) and (11), by the same way as in Section 3.1, we get,

\[ a_1 = \frac{\sqrt{3a^2c_2v-b_1^2(-\delta+\mu+\nu)}}{\sqrt{-\delta+\mu+\nu}}, \quad a_0 = \frac{c_1}{4c_2}a_1, \quad c_0 = \frac{3a^3c_1^2k^2v+ac_1k(8k^2v-4)-2bc_2}{8a^3c_2v}, \]

\[ \sigma = \frac{(\alpha+4\delta-\nu)(\delta-\mu-\nu)}{18v}, \quad b = \frac{2ak(3a(4k^2v-1)+\delta(4k^2v-7)-4k^2\mu v-4k^2\nu v+\mu-3v+\nu)}{3a+8\delta-2(\mu+\nu)}, \]

\[ c_2 = \frac{a^2(3a+8\delta-2(\mu+\nu))}{4(-6ak^2v+\delta(12k^2v-1)+\mu+3v+\nu)}, \]

\[ \omega = \frac{3(\alpha+4\delta-\nu)(\delta-6ak^2v+12k^2v+\mu+3v+\nu)^2}{4(\delta-\mu-\nu)(3a+8\delta-2(\mu+\nu))^2} + k^2(-\nu) + k^2 \]

\[ -\frac{3(2k^2(-\alpha+\delta+\mu+\nu)+1)(-6ak^2v+\delta(12k^2v-1)+\mu+3v+\nu)}{2(\delta-\mu-\nu)(3a+8\delta-2(\mu+\nu))}. \]

Finally, the solutions of (8) and (9) are,

\[ u(x, t) = \frac{K_1}{K_2}, \quad K_1 = -6ak^2v + \delta(12k^2v - 1) + \mu + 3v + \nu \exp:\endgroup (A_0+z) (3k^2(4k^2+1)+\mu+\nu) K) + a^2c_1 \exp:\endgroup (A_0+z) (\delta+6ak^2v) K) v(3a+8\delta-2(\mu+\nu)) \]

\[ \sqrt{-\frac{3a^2c_1^2v^2(3a+8\delta-2(\mu+\nu))}{4(-6ak^2v+\delta(12k^2v-1)+\mu+3v+\nu))} + b_1^2(\delta-\mu-\nu), \]

\[ \begin{aligned} K_2 &= a^2c_1 v\sqrt{-\delta+\mu+\nu}(3a+8\delta-2(\mu+\nu)) \\
&\quad + a^2c_1 v(3a+8\delta-2(\mu+\nu)) \exp((A_0+z)(\delta+6ak^2v) K)). \end{aligned} \]

\[ K = \sqrt{-\frac{a^3v(2(-4\delta+\mu+\nu)-3a)(-6ak^2v+\delta(12k^2v-1)+\mu+3v+\nu))}{4a^3v(2(-4\delta+\mu+\nu)-3a)(-6ak^2v+\delta(12k^2v-1)+\mu+3v+\nu))}, \]

\[ v(x, t) = \frac{b_1}{a_1}u(x, t), \quad z = ax + \frac{2ak(3a(4k^2v-1)+\delta(4k^2v-7)-4k^2\mu v-4k^2\nu v+\mu-3v+\nu)}{3a+8\delta-2(\mu+\nu)} t. \]

The solutions in (20) are used to display Rew in Fig. 3 (i)-(iii)

When \( v_0 = 3.2, k := 1.5, a := 1.2, a_1 = 0.7, b_1 = 0.1, \sigma = \sqrt{0.6}, \alpha = 0.5, \mu := 2.5, v = 1.3, A_0 = -5, \delta = 0.1, c_1 = 0.6. \) Theses figures show “continuum” soliton chain.
Here, we consider the AE,

\[ g'(z) = \sqrt{c_4g(z)^4 + c_2g(z)^2 + c_0}. \] (20)

By inserting (14) and (20) into (10) and (11), we have,

\[ a_1 = \frac{\sqrt{3a^2c_2v+b_1^2(-\delta+\mu+v_0)}}{\sqrt{-\delta+\mu+v_0}}, \quad a_0 = 0, \quad b = -2ak(2a^2c_2v - 2k^2v + 1), \]

\[ \sigma = \frac{(\alpha+4\delta-v_0)(\delta-\mu-v_0)}{18v}, \quad k = \frac{\sqrt{-6aa^2c_2v-\delta(16a^2c_2v+1)}+4a^2c\mu v+4a^2c_0v+\mu+3v+v_0}{\sqrt{6v(\alpha-28)}}, \]

\[ \omega = a^4v \left( c_2 + \frac{12c_4co(a+\delta-\mu)}{\delta-\mu-v_0} \right) + \frac{(-6aa^2c_2v-16a^2c\delta v+4a^2c\mu v+4a^2c_0v+\mu+3v+v_0)^2}{36v(\alpha-28)^2} + \frac{a^2c_2(-\alpha(6a^2c_2v+1)+16a^2c_\delta v+4a^2c_\mu v+4a^2c_0v+\delta+\mu+3v+v_0)}{\alpha-28} + \frac{-6aa^2c_2v-\delta(16a^2c_2v+1)+4a^2c\mu v+4a^2c_0v+\mu+3v+v_0}{6v(\alpha-28)}, \]

\[ c_4 = m^2, \quad c_0 = r^2, \quad c_2 = -n^2. \] (21)

The solutions of (8) and (9) are,

\[ u(x, t) = \frac{\sqrt{3a^2m^2v-b_1^2(-\delta+\mu+v_0)}}{\sqrt{-\delta+\mu+v_0}} \]

\[ \sqrt{2\left(n^2-\sqrt{n^4-4m^2r^2+2m^2r^2-n^4}\right)s} \left( \sqrt{2r} \sqrt{n^2-\sqrt{n^4-4m^2r^2}}(\alpha+\gamma), \frac{m^2}{n^2+\sqrt{n^4-4m^2r^2}} \right) \] (22)

\[ v(x, t) = \frac{b_1}{a_1}u(x, t), \quad z = ax - 2ak(2a^2c_2v - 2k^2v + 1) t. \]
The solutions in (22) are used to display $|w|$ in Fig. 4 (i)-(iii).

When $v_0 = 3.2; k := 1.5; a = 1.2; a_1 = 0.7; b_1 = 0.1; \sigma = 0.6; \alpha = 0.5; \mu = 2.5; \nu = 1.3; A_0 = -5; \delta = 0.3; m := 0.6; r = 1.2; n = 3.$

Figure 4(i) shows complex chirped waves, while Fig. 4(ii) shows super lattices. Figure 4(iii) shows dense “continuum” soliton chain.

4 Rational solutions of (10) and (11)

We consider the solution in (13), together with AE,

$$g'(z) = c_1g(z) + c_0. \quad (23)$$

By using (13) and (23) into (10) and (11), we get,

$$\omega = \frac{1}{s_1}(2b_1^2s_1^2(2k^2(\alpha - \delta - \mu - v_0) - 1) + k^2s_1^4(1 - k^2\nu) + 24a_1^4\sigma + 24b_1^4\sigma + a_1^2(s_1^2(4k^2(\alpha - \delta - \mu - v_0) - 2)) + 48b_1^2\sigma),$$

$$b = \frac{1}{a_1c_1s_1^2}(a^2b_1c_1^2s_1^4(a^2c_1^2\nu - 6k^2\nu + 1) - 4b_1^3s_1^2(a^2c_1^2(\delta + \mu) + 2k^2(\alpha - \delta - \mu - v_0) - 1)$$

$$- 2aa_1c_1k_3s_1^2(s_1^2(2a^2c_1^2\nu - 2k^2\nu + 1) + 4b_1^2(\delta - \mu - v_0)) + 8aa_1^3c_1k_3s_1^2(-\delta + \mu + v_0) - 96a_1^4b_1\sigma - 96b_1^5\sigma$$

$$- 4a_1^2(b_1^2s_1^2(a^2c_1^2(\delta + \mu) + 2k^2(\alpha - \delta - \mu - v_0) - 1) + 48b_1^3\sigma),$$

$$c_0 = -\frac{3c_1s_0}{2s_1}, \mu = \frac{P_1}{Q}, \quad P_1 = \frac{5a^5b_1^2s_1^2 + 242a^4a_1b_1c_1^4k_1s_1^2 + 4a_1b_1k(a_1^2(11 - 38ak^2) + b_1^2(11 - 38ak^2) + 3k_1^2s_1^2(6k_1^2 - 1) + a_1^3c_1^2s_1^2(45a^2k_1^2s_1^2 + b_1^2(9a^2k_1^2 + 1))) + 2a_1b_1c_1^2k(-4\alpha(a_1^2 + b_1^2)) + s_1^2(114k_1^2s_1^2 + 13) + a_1c_1(b_1^4(4 - 16ak^2) + 3b_1^2k_1^2s_1^2(1 - 6k_1^2) - 4a_1^2(b_1^2(4ak_1^2 - 1) + 12ak_1^2s_1^2)), \quad Q = 16a_1b_1k(a_1^2 + b_1^2)(3k_1^2 - a_1^2c_1^2),$$

$$\delta = \frac{P_2}{Q}, \quad P_2 = -102a^4a_1b_1c_1^4k_1s_1^2 + 5a^5b_1^2c_1^5s_1^2$$

$$+ 12a_1b_1k(a_1^2 + b_1^2)(2ak^2 - 1)$$

$$+ ab_1c_1^2(4a_1^2(4ak_1^2 - 1) + 4b_1^2(4ak_1^2 - 1) + 3k_1^2s_1^2(6k_1^2 - 1))$$

$$+ 2a_1b_1c_1^2k(4\alpha(a_1^2 + b_1^2) - 3s_1^2s_1^2(8k_1^2 - 1))$$

$$- a_1^3c_1^2s_1^2(120a_1^2k_1^2s_1^2 + b_1^2(99k_1^2 + 1)) \quad a = \frac{k}{\sqrt{2c_1}, \quad k = \frac{1}{2\sqrt{\alpha}},$$

$$b_1 = \frac{1}{4}\sqrt[5]{\frac{\sqrt{2}c_1^2\sqrt{\sigma(\alpha - 16v)}}{a_1\sigma}} - 16a_1^2, \quad a_0 = -\frac{4a_1s_0}{s_1}. \quad (24)$$
The solutions of (8) and (9) are,

\[
\begin{align*}
    u(x, t) &= \frac{2A_0a_1s_1e^{i1^2}-5a_1s_0}{2A_0s_1^2e^{i1^2}+5a_1s_0} \quad v(x, t) = \frac{1}{4} \sqrt{s_1^2 \frac{2}{\pi} \sqrt{\sigma(8\alpha-17\nu)}} - 16a_1^2 u(x, t), \\
    z &= \frac{H_3}{H_0}, \quad H_3 = -3\sigma s_1^4 t \left( 448\alpha^2 - 8384\alpha\nu + 15793\nu^2 \right) \\
    &+ 24 a_1 \alpha \sigma s_1^2 \sqrt{\sigma(8\alpha-17\nu)} \left( \sqrt{30} a_1 t (56\alpha - 449\nu) \\
    &+ 4 (-6\alpha t + 99\nu t + 10a^3/2) \sqrt{\frac{s_1^2 \sqrt{30} \sqrt{\sigma(8\alpha-17\nu)-240\alpha a_1^2 \sigma}}{\alpha \sigma}} \right) \\
    H_4 &= 1920 \sqrt{2} a_1^3 a_1 \sigma s_1^2 \sqrt{\sigma(8\alpha-17\nu)} \sqrt{\frac{s_1^2 \sqrt{30} \sqrt{\sigma(8\alpha-17\nu)-240\alpha a_1^2 \sigma}}{\alpha \sigma}}.
\end{align*}
\]

The results in (25) are used to display \( Rew \) in Fig. 5 (i)-(iii).

In Fig. 5(i) and (ii0, the 3D and contour plots of \( Rew \) are displayed, while the variation of \( Rew \) against \( x \) for different values of \( t \) is displayed.

Figure 5(i) shows “—continuum” soliton chain with trap, while jii) shows mixed lattice-solitons.

Fig. 5 When \( a_1 = 0.5, v_0 = 3.2, k = 1.5, \alpha := 1.2, \sigma=0.6, \alpha=1.5, \mu:=2.5, v=0.5, A_0 = 5, s_1 = 3, s_0 = 1.5, c_1 = 2.5 \)
5 Modulation stability analysis

To study the modulation instability of a system, it should exhibit a normal mode. That is a periodic standing waves. Here, (3) has a solution of the form.

\[ w_m(x, t) = Q e^{i(rx - qt)} \]

By inserting (24) into (3), we get

\[ q = 4\alpha Q^2 r^2 - 4\delta Q^2 r^2 - 4\mu Q^2 r^2 - 4\nu Q^2 r^2 + 24 Q^4 \sigma - 2 Q^2 - vr^4 + r^2. \]  

(27)

We write the solution expansion near \( w_m \),

\[ w(x, t) = Q e^{i(rx - qt)} \left( 1 + \epsilon_1 e^{\lambda t} (U(x) + iV(x)) + O(\epsilon^2) \right) \]

\[ w(x, t) = Q e^{-i(rx - qt)} \left( 1 + \epsilon_2 e^{\lambda t} (U(x) - iV(x)) + O(\epsilon^2) \right), \]

in (3) and Calculations give rise to,

\[ H \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0, \quad H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \]

\[ h_{11} = U \left( g + Q^2 (8r^2 - (\alpha + \delta + \mu + \nu) + 4) - 72 Q^4 \sigma + r^2 (vr^2 - 1) \right) \]

\[ + 2r V' \left( Q^2 (4(\mu + \nu) - 2\alpha) + 2vr^2 - 1 \right) - 4\mu Q^2 U'' - 6vr^2 U'' \]

\[ - 4v r V(3) + \nu U(4) + U''', \]

\[ h_{21} = V \left( g + Q^2 (8r^2 - (\alpha + \delta + \mu + \nu) + 4) - 72 Q^4 \sigma + r^2 (vr^2 - 1) \right) \]

\[ + 4r Q^2 r U - 8\mu Q^2 r U' - 8\nu r^2 Q^2 r U' - 4\mu Q^2 V'' - 4r^3 U' \]

\[ - 6vr^2 V'' + 4vr U(3) + 2r U' + \lambda U + \nu V(4) + V''', \]

\[ h_{12} = 2Q^2 U \left( -24 Q^2 \sigma + 2r^2 (\alpha - \delta + \mu + \nu) + 1 \right) \]

\[ - 4Q^2 \left( r(\alpha - 2\delta) V' + \delta U'' \right), \]

\[ h_{22} = 2Q^2 \left( V (24 Q^2 \sigma + 2r^2 (\alpha - \delta - \mu - \nu) - 1) - 2r(\alpha - 2\delta) U' + 2\delta V'' \right). \]

The solution of (26) is \( det H = 0 \), which yields,

\[ (V (24 Q^2 \sigma + 2r^2 (\alpha - \delta - \mu - \nu) - 1) - 2r(\alpha - 2\delta) U' + 2\delta V'') \]

\[ (U (g + Q^2 (8r^2 - (\alpha + \delta + \mu + \nu) + 4) - 72 Q^4 \sigma + r^2 (vr^2 - 1) + 1) + 2r V' \]

\[ (Q^2 (4(\mu + \nu) - 2\alpha) + 2vr^2 - 1) - 4\mu Q^2 U'' - 6vr^2 U'' - 4v r V(3) + \nu U(4) + U'' \]

\[ - (U (24 Q^2 \sigma + 2r^2 (\alpha + \delta + \mu + \nu) + 1) - 2r(\alpha - 2\delta) V' + \delta U'')) \]

\[ (V (g + Q^2 (8r^2 - (\alpha + \delta + \mu + \nu) + 4) - 72 Q^4 \sigma + r^2 (vr^2 - 1)) + \lambda U \]

\[ + 4\alpha Q^2 r U - 8\mu Q^2 r U' - 8\nu r^2 Q^2 r U' - 4\mu Q^2 V'' \]

\[ - 4r^3 U' - 6vr^2 V'' + 4vr U(3) + 2r U' + \nu V(4) + V''') = 0. \]

(30)

We solve the eigenvalue problem in (27) subjected to the boundary conditions \( U(\pm \infty) \leq U_0 \) and \( V(\pm \infty) \leq V_0 \). Thus the eigenfunctions take the form.

\[ U(x) = U_0 e^{i(hx)}, \quad V(x) = V_0 e^{i(hx)}. \]

(31)

By substituting from (28) into (27), we have,

\[ \lambda = \frac{1}{Q} 2V_0 \left( 2 \delta h^6 v + 2 Q^2 (24 Q^2 \sigma + 2r^2 (\alpha - \delta - \mu - \nu) - 1) \right) \]

\[ + h^4 (4Q^2 (\delta + \mu) + 1) - 4v r^4 (4\alpha - 7\delta - 3(\mu + \nu)) \]

\[ (24 Q^2 \sigma - 1) + 2r^2 (-\mu + 3v + \nu + (4\alpha^2 + 4\delta^2 + 4\mu^2 + 4\mu \nu) Q^2 \]

\[ + \alpha (3 - 4 Q^2 (3\delta + 3\mu + 2\nu)) + \delta (4 Q^2 (6\mu + 5\nu) - 5) - 72 v Q^2 \sigma) \right) \],

(32)
When $r=1.2$, $k=1.5$, $\delta = 0.2$, $h=0.5$, $\mu=0.5$, $U_0 = 5$, $V_0 = 3$. In (i) $\alpha=2.5$, $\nu = 2.3$, (ii) $\alpha=2.5$, $\nu = 1.3$

$$Q := \frac{\sqrt{U_0^2 - V_0^2} \sqrt{2h^2 \nu + 2\nu r^2 - 1}}{2\sqrt{U_0^2 (\alpha - \delta - \mu - \nu_0) + V_0^2 (-\delta + \mu + \nu_0)}},$$

(33)

together with a lengthy equation for $\sigma$, which will not be produced here. The eigenvalue $\lambda$ given in (29) is displayed against the dominant parameters $\nu$, $\alpha$ and $\nu_0$ in Fig. 6 (i)-(iii) and (iii); $\alpha = 2.5$, $\nu = 1.7$, $\nu_0 = 2.3$.

Figure 6(i) shows that modulation stability holds against $\nu$. While, in Fig. 6(ii) and (iii) there are the critical values $\nu_{ocr} = 1.25$ and $\alpha_{cr} = 1.4$, where below these values instability holds otherwise stability occurs.

### 6 Conclusions

The 1D Heisenberg spin chain system is considered. In continuum analog a equation was derived in the literature. Which is a nonlinear Schrodinger equation with bi-quadratic dispersion and fifth degree nonlinearity. Here, this equation is studied for the objectives of finding the exact solutions and investigating the relevant phenomena vis-a-vis spin chain. The exact solutions are obtained by using the unified method in polynomial and rational forms. A transformation that enables us to inspect the effects of soliton-periodic wave collision is proposed. The collision can be elastic or inelastic according to the waves solutions are smooth or not smooth. Numerical evaluation of the solutions are carried and displayed in figures. It is found that the solutions exhibit “continuum” soliton chain, while the contour plots show super lattices or lattices with trapping. It is remarked the solutions are bounded by $-1/4$ and $1/4$ which may be relevant with the spin $-1/2$ and $1/2$. The modulation stability analysis is carried and it is shown that there is a critical value of the dominant parameters that separates stability and instability. It is worth noticing that the waves found here are smooth, so waves collision is elastic.

**Funding** Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

**Declarations**

**Conflict of Interests** The author declares that there is no conflict of interest.

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