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Neutrino magnetohydrodynamic instabilities in presence of two-flavor oscillations

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Abstract
The influence of neutrino flavor oscillations on the propagation of magnetohydrodynamic (MHD) waves and instabilities is studied in neutrino-beam driven magnetoplasmas. Using the neutrino MHD model, a general dispersion relation is derived which manifests the resonant interactions of MHD waves, not only with the neutrino beam, but also with the neutrino flavor oscillations. It is found that the latter contribute to the wave dispersion and enhance the magnitude of the instability of oblique magnetosonic waves. However, the shear-Alfvén wave remains unaffected by the neutrino beam and neutrino flavor oscillations. Such an enhancement of the magnitude of the instability of magnetosonic waves can be significant for relatively long-wavelength perturbations in the regimes of high neutrino number density and/or strong magnetic field, giving a convincing mechanism for type-II core-collapse supernova explosion.

1. Introduction
Neutrinos are generally produced due to very high explosions in the core of massive stars and can have significant impact on the cooling of white dwarfs and neutron stars [1, 2]. The most apparent source of neutrinos is the Sun, where they are produced due to the simplest nuclear fusion reaction in which two protons combine to form a deuterium nucleus with the emission of a positron and a neutrino. All other complex reaction processes that lead to heavier elements can also produce neutrinos which get away from the Sun at the speed of light in vacuum. Neutrinos are also produced by cosmic rays hitting up nuclei in the Earth’s atmosphere, similar to the reactions of terrestrial high-energy particle accelerators. Such nuclear reactions result not only in electron neutrinos as in the Sun, but also in two other flavors, namely, muon neutrinos and tau neutrinos—all of which were detected by super-Kamiokande detector [3]. On the other hand, the neutrino-producing fusion reactions in stars do not release energy in the form of light or heat that could provide pressure to stop gravitational collapse of the stellar core. So, the collapse occurs and it continues until the density in a nucleus is close to that in the core and suddenly a massive explosion occurs in producing all flavors of neutrinos. Such a sudden and higher optical gleam is known as a core-collapse supernova, e.g. SN1987A [4]. Although the interaction between neutrinos and matter is weak, in the gamma-ray bursts of a supernova explosion, the energy emitted from neutrinos can be very high (almost 99% of the gravitational binding energy of collapsing stars) and the intensity can be more than 10²⁸ W cm⁻². Furthermore, in the first few seconds of explosion, the neutrino burst that originates from the core of supernova is a source of free energy to drive collective oscillations and instabilities which may lead to the revival of a stalled supernova shock [5, 6]. Typically, neutrinos produced in the solar atmosphere or in the core-collapse of stars have energies ranging from 1 to 30 MeV. However, recent observations with IceCube data have indicated that neutrinos can have energies more or less 10¹⁵ eV [3, 7]. Such high-energy neutrinos are expected to be produced in astrophysical objects via the interactions of highly relativistic charged particles (Cosmic rays) with either target particles or photons [3].
Neutrinos produced from different sources can play significant roles in the formation of galaxies, galaxy clusters, and various coherent structures at large scales. Apart from their possible gravitational interactions, neutrinos interact weakly with matter and thus are very important in astrophysics. In regions where other particles get trapped or move through slow diffusion processes, neutrinos can still escape from them and thus connect those of matter without being detached from each other. In very hot or dense astrophysical objects, the emission of neutrinos can be an important energy-loss mechanism. The energy transfer rate can be faster and very efficient since neutrinos have almost zero mass and can travel at relativistic speeds. Furthermore, since neutrinos produced in the Sun can be detected at the Earth, they are useful to study nuclear reactions that can occur in the core of massive stars. Also, because neutrinos are electrically neutral like photons and hence uninfluenced by the strong magnetic fields, they tend to move back to the creation regions, and thus can provide useful information about the regions where particle creation and acceleration take place in the Universe. For more information about roles of neutrinos, see, e.g. [8].

In stellar environments, the collective plasma effects can remarkably modify the production rate of neutrinos, e.g. the decay of photons and plasmons into neutrino pairs, which is the dominant neutrino emission mechanism at high-density plasmas. The neutrino emission can also be possible in dense hot matter due to electron-positron annihilation [9], in ultra-relativistic plasmas due to positron and plasmino annihilation [10]. The neutrinos interacting with plasmas play key roles in many astrophysical situations including supernova explosions. Such interactions not only reform the neutrino flavor oscillations (in which neutrinos oscillate from one flavor state to another) and initiate resonant interactions of different flavors [11–13], but also produce an induced effective neutrino charge as well as induced electric and magnetic fields that can lead to collective plasma oscillations and an enhancement of collision cross sections. In this context, several authors have studied the neutrino-plasma interactions considering neutrino flavor oscillations, See, e.g. [14–17]. To discuss a few, in [16], it has been shown that the two-flavor neutrino-plasma oscillation equations admit an exact analytic solution for arbitrarily chosen electron neutrino populations. A hydrodynamic model has been introduced by Mendonça and Haas [17] to study the plasma and neutrino flavor oscillations in turbulent plasmas.

In other contexts, the neutrino-plasma coupling in magnetized plasmas can lead to different types of hydrodynamic instabilities which may influence the neutrino beam transport by improving the properties of the background medium. Several studies have focused on the physics of collective neutrino-plasma interactions in different astrophysical situations [6, 18, 19]. Also, the parametric instabilities in intense neutrino flux and collective plasma oscillations have been studied by Bingham et al [5]. Furthermore, the generation of neutrino-beam driven wavefields [20], neutrino streaming instability [21–23], and neutrino Landau damping [22] have been studied in different contexts. The latter effect can be implemented to the cooling process of strongly turbulent plasmas. Furthermore, it has been shown that neutrinos can contribute to the generation of both the inhomogeneities and magnetic fields in the early universe [24, 25].

Recently, Haas et al [26] proposed a neutrino MHD (NMHD) model in magnetoplasmas by considering the neutrino-plasma interactions as well as the coupling between MHD waves and neutrino fluids. This model was studied for the propagation of magnetosonic waves in a specific geometry, i.e. when the propagation direction is perpendicular to the external magnetic field. However, the theory was later advanced with an arbitrary direction of propagation [27]. Motivated by these works, the influence of intense neutrino beams on the hydrodynamic Jeans instability has been studied by Prajapati in a magnetized quantum plasma [28]. It turns out that the NMHD model has become very useful to establish connections between various astrophysical phenomena and neutrino-plasma coupling processes in magnetized media.

In this work, we aim to advance the previous theory of NMHD waves [27] by considering (in addition to the neutrino beam effects) the influence of two neutrino favor (electron- and muon-neutrinos) oscillations on the neutrino-beam driven MHD waves and instabilities. We show that the two-flavor oscillations, not only resonantly interact with the oblique magnetosonic wave, but can have a significant contribution to the growth rate of instability.

The paper is organized as follows: in section 2, we describe the NMHD model, which is coupled to the dynamics of two neutrino flavors, namely, the electron-neutrino and muon-neutrino. Using the perturbation analysis, a general linear dispersion relation is derived in section 3 to show the coupling of MHD waves with the resonant neutrino beam and the resonant neutrino flavor oscillations. The instability growth rates for both the fast and slow magnetosonic waves are obtained in section 4, and analyzed numerically in section 5. Finally, section 6 is left for concluding remarks.

2. Physical model

We consider a homogeneous magnetized system composed of electrons and ions, as well as the neutrino beams of electron neutrinos and muon neutrinos. We also assume that the fluid descriptions for both the plasma
electrons and ions, and the neutrino beams are valid for the length scale of the order of electron skin depth and the time scale of the order of ion gyroperiod. In the NMHD description, the continuity and momentum equations for the MHD fluids read [27]

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{U}) = 0, \quad (1)
\]

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \rho_m + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho_m} + \frac{\mathbf{F}_e}{m_i}, \quad (2)
\]

where \(\rho_m = m_e n_e + m_i n_i \approx n_m\), \((n_e = n_i = n)\) is the mass density,

\(\mathbf{U} = (m_n \mathbf{u}_e + m_i \mathbf{u}_i)/(m_e n_e + m_i n_i) \approx (m_e \mathbf{u}_e + m_i \mathbf{u}_i)/m_i\) is the plasma velocity, \(\mu_0\) is the permeability of free space, \(V_i = \sqrt{k_B T_e/m_i}\) is the ion-acoustic velocity, and \(F_e\) is the neutrino-plasma (electroweak) interaction force. Here, \(m_{e(i)}\) denotes the electron (ion) mass, \(n_{e(i)}\) the electron (ion) number density, \(u_{e(i)}\) the electron (ion) fluid velocity, \(\mathbf{B}\) is the magnetic field, \(T_e\) is the electron temperature, and \(k_B\) the Boltzmann constant. In addition, the equation for the magnetic flux modified by the electroweak force is given by

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{U} \times \mathbf{B} - \frac{\mathbf{F}_e}{e} \right), \quad (3)
\]

where \(e\) is the elementary charge, \(F_e = \sqrt{2} G_F (E_e + \mathbf{U} \times \mathbf{B}_e)\) with \(G_F\) denoting the Fermi coupling constant and \(E_e, B_e\) the effective electric (magnetic) field induced by the weak interactions of neutrinos with plasmas, given by,

\[
E_e = -\nabla N_e - \frac{1}{c^2} \frac{\partial}{\partial t} (N_e \mathbf{v}_e), \quad (4)
\]

\[
B_e = \frac{1}{c^2} \nabla \times (N_e \mathbf{v}_e). \quad (5)
\]

Here, \(N_e (\mathbf{v}_e)\) denotes the number density (velocity) of electron neutrinos. For a coherent neutrino beam with an energy \(E_0\), the continuity equations for electron and muon neutrinos (with number density \(N_{\nu_e}, \nu_{\nu_e}\)), respectively, are [15, 29]

\[
\frac{\partial N_{\nu_e}}{\partial t} + \nabla \cdot (N_e \mathbf{v}_e) = \frac{1}{2} N \Omega_0 P_2, \quad (6)
\]

\[
\frac{\partial N_{\nu_\mu}}{\partial t} + \nabla \cdot (N_\mu \mathbf{v}_\mu) = -\frac{1}{2} N \Omega_0 P_2, \quad (7)
\]

where \(P_2\) corresponds to the neutrino coherence in the flavor polarization vector \(\mathbf{P} = (P_1, P_2, P_3)\), \(N = N_e + N_\mu\) is the total neutrino fluid density, and \(\Omega_0 = \omega_0 \sin(2\theta_h)\). Here, \(\omega_0 = \beta m c^2/2h \mathcal{E}_n\) with \(\beta m^2\) denoting the squared neutrino mass difference, \(c\) the speed of light in vacuum, \(h\) the reduced Planck’s constant, and \(\theta_h\) the neutrino oscillation mixing angle. While the left-hand sides of equations (6) and (7) involve the convective terms due to the flows of neutrinos into plasmas, the terms on the right-hand sides appear due to the neutrino flavor oscillations along with the rates of changes of the electron- and muon-neutrino fluid densities. We also require the global neutrino fluid densities to be conserved, i.e.

\[
\frac{d}{dt} \int (N_e + N_\mu) d^3r = -\int \nabla \cdot (N_e \mathbf{v}_e + N_\mu \mathbf{v}_\mu) d^3r = 0. \quad (8)
\]

Next, the electron neutrino and muon neutrino equations of motion are

\[
\frac{\partial \mathbf{p}_e}{\partial t} + \mathbf{v}_e \cdot \nabla \mathbf{p}_e = -\frac{\sqrt{2} G_F}{m_i} \nabla \rho_m, \quad (9)
\]

\[
\frac{\partial \mathbf{p}_\mu}{\partial t} + \mathbf{v}_\mu \cdot \nabla \mathbf{p}_\mu = 0, \quad (10)
\]

where \(\mathbf{p}_e = \mathcal{E}_e \mathbf{v}_e/c^2\) and \(\mathbf{p}_\mu = \mathcal{E}_\mu \mathbf{v}_\mu/c^2\) are the momenta of electron and muon neutrinos with \(\mathcal{E}_{e,\mu} = (E_{e,\mu}^2 + m_{e,\mu}^2 c^4)^{1/2}\) denoting the electron- and muon-neutrino energies, \(m_{e,\mu}\) the electron (muon) neutrino mass, and \(\nu_{e(\mu)}\) the electron (muon) neutrino velocity.

To complete the description of neutrino-plasma interactions, we require the time evolution equations of the components of the flavor polarization vector \(\mathbf{P} = (P_1, P_2, P_3)\) as [15]

\[
\frac{dP_1}{dt} = -\Omega(n_e) P_2, \quad (11)
\]

\[
\frac{dP_2}{dt} = \Omega(n_\mu) P_1 - \Omega_0 P_3, \quad (12)
\]
\[
\frac{dP_3}{dt} = \Omega_0 P_2,
\]
where \(\Omega(n_e) = \omega_0 [\cos(2\theta_0) - \sqrt{2} G_F n_\nu / (m_\nu \omega_0)]\). The total time derivatives appearing in equations (11)-(13) should, in general, be different. However, for a mono-energetic neutrino beam, the velocity of each neutrino flavor can be assumed to be identical so that \(v_e = v_\mu = v\). One can then consider the total time derivative as \(d/dt \equiv \partial_t + v \cdot \nabla\). Since we are interested in the linear regime, the convective parts will be less important and can thus be disregarded in the analysis in section 3.

### 3. Linear waves: general dispersion relation

In order to obtain a general dispersion relation for NMHD waves, we Fourier analyze the system of equations (1)-(13) about the following equilibrium state:

\[
U = 0, \quad N_e = N_{e0}, \quad N_\nu = N_{\nu0},
\]
\[
v_e = v_\mu = v_0, \quad N_0 = N_{e0} + N_{\nu0},
\]
\[
P_1 = \frac{\Omega_e}{\Omega_0}, \quad P_2 = 0, \quad P_3 = \frac{\Omega(n_\nu)}{\Omega_0} = \frac{N_{\nu0} - N_{e0}}{N_0},
\]
where \(\delta_f\) denotes the perturbation of a physical quantity \(f\). Also, from equation (9), we have

\[
(\omega - k \cdot v_0) \delta P_e = \left[ \delta v_e + \left( 1 - \frac{v_0^2}{c^2} \right) \frac{v_0 \cdot \delta v_e}{c^2} v_0 \right] = \sqrt{2} G_F \frac{k}{m_i} \rho_{m},
\]

So, for nonrelativistic fluid flow with \(v_0 \ll c\), we get

\[
\delta v_e = \sqrt{2} G_F \frac{c^2 k \rho_{m}}{E_0(\omega - k \cdot v_0)} \left[ \frac{c^2 k \rho_{m}}{m_i} - \left( \frac{k \cdot v_0 \rho_{m1}}{m_i} \right) - \frac{n_{e0} \omega}{c^2 v_0} \delta U \right].
\]

Using the perturbed form of equation (1), we can rewrite equation (18) as

\[
\delta v_e = \sqrt{2} G_F \frac{c^2 \rho_{m0}}{E_0(\omega - k \cdot v_0)} (k \cdot \delta U) k.
\]

Next, from equations (11)-(13) one obtains

\[
\delta P_2 = -i \sqrt{2} \Omega_{e0} \omega G_F \frac{c^2 \rho_{m0}}{(\omega^2 - \Omega_{e0}) m_i \omega} \delta P_m.
\]
and using this expression of $\delta P_2$, we obtain from equation (6) the following equation for the perturbed density.

$$
\delta N_e = N_0 \Theta^2 \frac{\sqrt{2} m_0 c^2 \rho_{m0}}{E_0(\omega - k \cdot v_0)^2} k^2 (k \cdot \delta U) + \frac{\sqrt{2} \rho_{m0} N_0 \rho_{m0}}{2 m_1 \hbar \Omega_0 (\omega - k \cdot v_0)(\omega^2 - \Omega_0^2)} (k \cdot \delta U).
$$

(21)

The expressions for the perturbed density and velocity of electron neutrinos (equations (19) and (21)) together with the continuity equation (6) can then be used in the expression for the perturbed neutrino fluid force to show that its magnitude is enhanced for $\omega \approx k \cdot v_0$ and/or $\omega \approx \Omega_0$. It follows that the MHD waves can have resonant-like interactions with the streaming neutrino beam and the neutrino flavor oscillations for which the energy exchange can take place leading to the MHD instability. Although the resonant contribution of the neutrino beam is known in [27], we will, however, consider both the resonances in order to study the relative influence of the neutrino flavor oscillations on the MHD instability. Finally, from equations (16), (19), and (21), we obtain the following dispersion relation.

$$
\omega^2 \delta U = \left( V_A^2 + V_N^2 + V_N^2 \frac{c^2 k^2 - \omega^2}{(\omega - k \cdot v_0)^2} \right) (k \cdot \delta U) k + \left( (k \cdot V_A) \{ (k \cdot V_A) \delta U - (\delta U \cdot V_A) k - (k \cdot \delta U) V_A \} + V_{osc}^2 \frac{\Omega_0^2 \epsilon^2 \omega_0^2 c^2 k^2 - \omega(k \cdot v_0)}{2 \epsilon^2 \omega_0 \Omega_0 (\omega - k \cdot v_0)(\omega^2 - \Omega_0^2)} \right) k (k \cdot \delta U).
$$

(22)

Here, $V_A = B_0 / (\mu_0 \rho_{m0})^{1/2}$ is the Alfvén velocity associated with the MHD wave, $V_N = [2G_0 \rho_{m0} N_0 / (m_1^2 \epsilon^2 \Omega_0)]^{1/2}$ is the velocity associated with the electron-neutrino beam (which involves the densities of both the MHD and neutrino fluids, highlighting the mutual coupling between them), and $V_{osc} = [2G_0 \rho_{m0} N_0 / (m_1^2 \epsilon^2 \Omega_0)]^{1/2}$ that due to the coupling of MHD waves with both the electron- and muon-neutrino flavor oscillations. Thus, the terms proportional to $V_N^2$ and $V_{osc}$ in equation (22) appear due to the neutrino beam effect (electron-neutrino) and two-flavor (both the electron-neutrino and muon-neutrino) oscillations. As noted before and is clear from equation (22) that, in addition to the phase velocity resonance (at the neutrino beam velocity, i.e. $\omega \approx k \cdot v_0$), there also can occur the resonance (at the frequency of two-flavor oscillations, i.e. $\omega \approx \Omega_0$) due to the coupling between MHD waves and two neutrino flavor oscillations. Furthermore, disregarding the contribution of the neutrino flavor oscillations from equation (22), one can recover the same dispersion relation as in [27]. Thus, the dispersion equation (22) generalizes the previous theory with the effects of neutrino flavor oscillations.

We note that the adiabatic sound speed is also modified by the effects of neutrino flavor oscillations. Thus, defining $\tilde{V}_s$ by

$$
\tilde{V}_s^2(\omega, k) = V_A^2 + V_N^2 \frac{c^2 k^2 - \omega^2}{(\omega - k \cdot v_0)^2} + V_{osc}^2 \frac{\Omega_0^2 \epsilon^2 \omega_0^2 c^2 k^2 - \omega(k \cdot v_0)}{2 \epsilon^2 \omega_0 \Omega_0 (\omega - k \cdot v_0)(\omega^2 - \Omega_0^2)}.
$$

(23)

Equation (22) can be recast as

$$
\omega^2 \delta U = (V_A^2 + \tilde{V}_s^2)(k \cdot \delta U) k + (k \cdot V_A) \{ (k \cdot V_A) \delta U - (\delta U \cdot V_A) k - (k \cdot \delta U) V_A \}.
$$

(24)

The form of equation (24) is the same as the well-known dispersion equation for the propagation of linear waves in a compressible, nonviscous, perfectly conducting magnetofluid. Consequently, the usual method of recovering various modes applies, which we discuss as follows.

We consider the wave propagation at an arbitrary angle $\theta$ with respect to the constant magnetic field $B_0 = B_0 \hat{z}$ and assume, without loss of generality, that the wave vector $k$ lies in the $xz$-plane. Thus, equating the coefficient determinant of the homogeneous system (24) for the components of $\delta U$ to zero, we obtain the following linear dispersion relation for the coupling of MHD waves with the neutrino beam and the neutrino flavor oscillations.

$$
(\omega^2 - k^2 \tilde{V}_s^2 \cos^2 \theta) \omega^4 - k^2 (V_A^2 + \tilde{V}_s^2) \omega^2 + k^4 \tilde{V}_s^4 \tilde{V}_s^2 \cos^2 \theta \tilde{\Omega} = 0.
$$

(25)

From equation (25), it is evident that the first factor, when equated to zero, gives the dispersion relation for oblique Alfvén or shear-Alfvén waves, i.e. $\omega = k \tilde{V}_s \cos \theta$. Such waves are neither influenced by the neutrino beam nor by the neutrino flavor oscillations. This is expected as shear-Alfvén waves are characterized by both $\delta U \cdot B_0 = 0$ and $k \cdot \delta U = 0$. The latter, however, eliminates the contributions of neutrinos in equation (22). So,
we are interested in the second factor to obtain the following dispersion relation.

\[ \omega^4 - k^2(V_A^2 + V_S^2)\omega^2 + k^4V_A^2 V_S^2 \cos^2 \theta = 0. \]  

(26)
equation (26) reveals the coupling of the oblique magnetosonic waves with the neutrino beam and the neutrino two-flavor oscillations. Note that for wave propagation perpendicular to the magnetic field (\( \theta = \pi/2 \)), typical magnetosonic mode is recovered [26], which is, however, modified by the influence of neutrino flavor oscillations mediated through the term proportional to \( V_{\text{osc}}^2 \). Furthermore, due to smallness of the Fermi constant \( \frac{\Gamma}{4\pi} \), and hence \( V_N^2 \) and \( V_{\text{osc}}^2 \), the contributions from the neutrino beam and neutrino flavor oscillations are typically small. So, they can be considered as small perturbations to the squared acoustic speed \( V_s^2 \). Physically, these perturbations, as they develop in the resonant interactions of MHD waves with the streaming neutrino beam and the neutrino flavor oscillations, may lead to instabilities due to energy gain from neutrinos that can be radiated due to core collapse of massive stars in supernova explosions. In section 4, we will investigate the qualitative features of these instabilities in details.

4. Growth rate of instability

To study the instabilities of oblique magnetosonic waves, we rewrite the dispersion equation (26) as

\[ \omega^4 - k^2(V_A^2 + V_S^2)\omega^2 + k^4V_A^2 V_S^2 \cos^2 \theta \]

\[ = V_N^2 k^2(c^2k^2 - (k \cdot v_0)^2)(\omega^2 - k^2V_A^2 \cos^2 \theta) \]

\[ + V_{\text{osc}}^2 \Omega_0^2 \omega E_c(c^2k^2 - \omega(k \cdot v_0))(\omega^2 - k^2V_A^2 \cos^2 \theta) \]

\[ = \frac{2c^2k^2/\Omega_0(\omega - k \cdot v_0) (\omega^2 - k^2V_A^2 \cos^2 \theta)}{2c^2k^2/\Omega_0(\omega - k \cdot v_0) (\omega^2 - \Omega_0^2)}. \]  

(27)

Since the influences of the neutrino streaming beam and the flavor oscillations on the instability growth rates are of our prime interest, we assume

\[ \omega = \tilde{\omega} + \delta \omega, \quad \text{with } |\delta \omega| \ll \tilde{\omega}, \]  

(28)

together with the double resonance condition

\[ \omega = \Omega_\nu \approx \tilde{\omega} = k \cdot v_0, \]  

(29)

where \( \tilde{\omega} \) is a solution of the following dispersion equation (in absence of the effects of neutrinos)

\[ \omega^4 - k^2(V_A^2 + V_S^2)\omega^2 + k^4V_A^2 V_S^2 \cos^2 \theta = 0. \]  

(30)

From equation (30), the frequencies of the fast (with the suffix +) and slow (with the suffix −) classical magnetosonic modes can be obtained as

\[ \omega = \Omega_{\pm} = kV_\pm, \]  

(31)

where \( V_{\pm} \) are the corresponding phase velocities, given by,

\[ V_\pm = \left[ \frac{1}{2}(V_A^2 + V_S^2 \pm \sqrt{(V_A^2 - V_S^2)^2 + 4V_A^2 V_S^2 \sin^2 \theta}) \right]^{1/2}. \]  

(32)

Thus, from equations (27) to (29) and using the fact that \( V_{\pm} \ll c^2 \) for non-relativistic fluid flow, we obtain

\[ (\delta \omega)^4 \approx \pm \left[ \frac{V_N^2 c^2k^2(V_A^2 - V_A^2 \cos^2 \theta)}{2V_\pm \sqrt{(V_A^2 - V_S^2)^2 + 4V_A^2 V_S^2 \sin^2 \theta}} + \frac{G^2 \rho_{\text{neu}} N_0 \Omega_0^2(V_{\pm}^2 - V_A^2 \cos^2 \theta)}{4V_\pm^2 \Omega_0^2 \sqrt{(V_A^2 - V_S^2)^2 + 4V_A^2 V_S^2 \sin^2 \theta}} \right]. \]  

(33)

The instability growth rate \( \gamma = \mathcal{J}(\delta \omega) > 0 \) is then obtained as

\[ \gamma \equiv \gamma_{\pm} = [I(\gamma_{\pm}^+) + I(\gamma_{\pm}^-)]^{1/3}, \]  

(34)

where we have defined the dimensionless parameter \( \Delta = V_N^2/c^2 \) and the expressions for \( \gamma_{\pm}^+ \) and \( \gamma_{\pm}^- \), respectively, are
Here, we again note that while the quantity $\gamma_\nu$ is associated with the interactions of MHD waves with the streaming (with velocity $v_\nu$) neutrino beam, the quantity $\gamma_{\text{osc}}$ appears due to coupling of MHD waves with neutrino two-flavor oscillations (with frequency $\Omega_{\nu}$). In absence of the latter, one can recover exactly the same result as in [27]. Furthermore, the conditions for the weak perturbations due to the neutrino beam and neutrino flavor oscillations can be validated so that $\gamma_\nu/\Omega_{\nu} \ll 1$ and $\gamma_{\text{osc}}/\Omega_{\nu} \ll 1$ since the terms in the square brackets in equation (35) can be made less than unity after an appropriate normalization.

The relative influence of the neutrino flavor oscillations on the growth rate of instabilities can be noted and it is given by

$$\frac{\gamma_{\text{osc}}}{\gamma_\nu} = \frac{1}{2^{1/3}k} \left[ \frac{(\delta m^2 c^2)^2 \sin^2(2\theta_{12})}{\hbar^2 E_0 V_\nu} \right]^{1/3} \left( \frac{N_{\odot}}{N_{\odot}} \right)^{1/3}. \quad (36)$$

From equation (36), it is to be noted that while the quantity $\gamma_\nu$ explicitly depends on the wave number $k$, $\gamma_{\text{osc}}$ is independent of $k$, which means the growth rate ratio is inversely proportional to $k$. Thus, it follows that in the regimes of sufficiently large wave numbers (provided that the wavelength is not too small) of magnetosonic perturbations, the neutrino beam contribution to the MHD instability can be larger than that of neutrino flavor oscillations. In contrast, the two-flavor oscillations can dominate over the neutrino beam-plasma interactions if initially the muon-neutrino beam density ($N_{\mu\nu}$) is much higher than that of electron neutrinos ($N_{e\nu}$) or the streaming neutrino spinor energy is relatively low. Furthermore, depending on the angle of propagation $\theta$, a relatively low magnetic field strength and/or low thermal energies of MHD fluids can enhance the neutrino flavor oscillation correction.

### 5. Results and discussion

In order to examine the qualitative features, we numerically study the growth rates of instability for both the fast and slow magnetosonic waves, as well as the relative influence of the flavor oscillations. To this end, we consider the parameters that are relevant for type-II core-collapse supernova SN1987A [27]. In such scenarios, we can expect a fluid flow of $10^{58}$ neutrinos of all flavors and the streaming energy $E_0 \approx 10 - 15$ MeV. There may also exist a strong magnetic field $B_0 \sim 10^6 - 10^8$ T and high neutrino beam densities, $N_{\nu} \sim 10^{14} - 10^{17}$ m$^{-3}$. We, however, consider $n_0 = 10^{24}$ m$^{-3}$, $N_{\nu} = 10^{27}$ m$^{-3}$, and two different values of each of $N_{\nu}$ and $B_0$, namely $N_0 = 5 \times 10^{27}$ m$^{-3}$, $10^{28}$ m$^{-3}$ and $B_0 = 5 \times 10^6$ T, $2 \times 10^7$ T. Furthermore, $\tau = 0.1$ MeV, $k = 10^6$ m$^{-1}$ $\Delta m^2 c^4 = 3 \times 10^{-6}$ (eV)$^2$, $\sin(2\theta_{12}) = 10^{-1}$, $E_0 = 10$ MeV, and $G_{\nu} = 1.45 \times 10^{-27}$ J M$^{-1}$. With these parameters, the non-relativistic conditions $V_{\nu}/c \ll 1$ and $V_{\nu}/c \ll 1$, as well as the simplifying assumption [26]:

$$\frac{ck}{\omega_{pe}} \ll \frac{1}{\omega_{pe}}, \quad \frac{1}{\omega_{ce}} \ll \frac{1}{\omega_{ce}}$$

where $\omega_{pe}$ and $\omega_{ce}$ are, respectively, the electron plasma and electron cyclotron frequencies, for the present model are also satisfied. Figure 1 displays the growth rates of both the fast and slow magnetosonic modes when only the influence of the neutrino beam is present and when both the neutrino beam and the neutrino flavor oscillations are present. These growth rates are also plotted with two different magnetic field strengths. It is found that in the regimes of relatively low magnetic field [Subplots (a) and (b)], the contribution of the two-flavor oscillation to the growth rate becomes significant. It enhances the growth rates of both the fast and slow magnetosonic modes. In this case, the growth rate for the fast mode exhibits an inverted bell-shaped curve, while that for the slow mode displays a symmetric double-hump even in absence of the flavor oscillations. Consequently, the growth rate of the fast mode reaches its minimum at $\theta = \pi/2$ and maximum at $\theta = 0$ and $\theta = \pi$, whereas that for the slow mode has two cut-offs at $\theta = 0$ and $\pi$, and the maximum at $\theta = \pi/4$ and $3\pi/4$.

From figure 1, the growth rate can be estimated as $\gamma_{\nu}/\Omega_{\nu} \sim 10^{-3}$ for $\Omega_{\nu} \sim 10^4$ rad/s. Since $\Omega_{\nu} \sim 10^4$ rad/s for a small value of $k \sim 10^5$, we have $\gamma_{\nu}/\Omega_{\nu} \sim 10^{-7} \ll 1$, i.e. the weak beam and weak flavor oscillations assumptions hold for the fast mode in the entire regime of $\theta$. However, for the slow mode, the similar estimation applies except at $\theta = \pi/2$ where $\Omega_{\nu} = 0$ and $\gamma_{\nu}$ is not defined. Thus, in difference to the fast magnetosonic mode, which is likely to be more unstable in the parallel and anti-parallel propagation, the slow mode becomes unstable for propagation at angles $\theta = \pi/4$ and $3\pi/4$. Such distinctive features of the instability growth rates of magnetosonic modes have not been reported in the earlier work [27].

On the other hand, when the magnetic field strength is relatively high, the similar features as in [27] (but in contrast to the previous case with a low magnetic field) are noticed. The growth rate for the fast mode is found to be slightly increased by the influence of the flavor oscillations. This increase is, however, pronounced except for
parallel and anti-parallel propagation. The weak perturbation assumptions, as said before, still hold for this mode and it displays a stronger instability for propagation nearly perpendicular to the magnetic field. The effects of the flavor oscillations on the instability growth rates are rather significant for the slow magnetosonic mode over the entire domain of the propagation angle except at $\theta = \pi/2$ where $\Omega_{\nu} = 0$ and $\gamma_\pm$ is not defined. However, close to this value of $\theta$, the growth rate for the slow mode tends to reach its minimum value, implying the stability of the slow magnetosonic mode therein. The instabilities are rather stronger for the parallel and anti-parallel propagation. Furthermore, the growth rates for the fast waves decrease with increasing values of the magnetic field, whereas the opposite trend occurs for the slow waves. From figure 1, it is also noted that the growth rates for both the fast and slow magnetosonic modes remain finite at $\theta \to 0$ and $\theta \to \pi$ as the values of $\gamma_{\nu}^+ \pm$ and $\gamma_{\nu}^\pm$ assume nonzero values therein and that $\gamma_{\pm} \sim 10^{-1}$ (i.e. $1/\gamma_{\pm} \sim 10$), and it becomes higher (or $1/\gamma_{\pm}$ becomes lower) with the influence of flavor oscillations. This means that the instability of the magnetosonic wave occurs faster in presence of the flavor oscillations than that due to its absence. Also, this typical time of instability is relatively shorter than the characteristic time scale of supernova explosion $\sim 100$ s. Thus, the neutrino flavor oscillations have a remarkable impact on the instability of magnetosonic waves in neutrino-beam driven magnetoplasmas.

The relative influence of the neutrino flavor oscillations can be qualitatively analyzed in two different density regimes and with the variations of the wave number. It is evident from figure 2 that as one approaches towards a domain of higher wave numbers, the relative influence of flavor oscillations on the instability growth rate...
becomes less important. However, it can be significant for both the slow and fast magnetosonic modes at some higher density regimes, provided one limits the neutrino density below a critical value, since at higher density regimes the relativistic degeneracy of electrons should come into the picture [29] which is not considered in the present model.

6. Conclusion

We have studied the influences of the neutrino two-flavor oscillations on the propagation of MHD waves and instabilities that are driven by the streaming neutrino beams. Special emphasis is given to analyzing the characteristics of fast and slow magnetosonic waves propagating in an arbitrary direction to the static magnetic field. In this way, the previous theory of neutrino MHD waves [27] is generalized with the inclusion of neutrino flavor oscillations in the wave dynamics. Using the neutrino MHD equations and assuming the weak neutrino beam-plasma interactions, as well as the weak coupling between MHD waves and two-flavor oscillations, a general dispersion relation for MHD waves is derived which accounts for the contributions from the resonant-like interactions of MHD waves with both the streaming neutrino beam and flavor oscillations. It is found that while the latter modify the wave dispersion and intensify the magnitude of the instability of oblique magnetosonic waves, the shear-Alfvén wave remains unaffected by the neutrino beam and neutrino flavor oscillations. The growth rates of instabilities, so generated due to the energy exchange of neutrinos with the wave, are obtained and analyzed with the parameters relevant to type-II supernova explosion due to the core collapse of massive stars. It is found that the relative influence of the flavor oscillations can be significant in the regimes of high neutrino beam densities and/or relatively low magnetic fields provided the wavelength is moderate or high. Here, one must restrict the neutrino beam density to some limit. Otherwise, in high density regimes, where the relativistic degeneracy effects will come into the picture, one must deal with the relativistic NMHD model which is, however, a project for our future work. Furthermore, in the regimes of two different magnetic fields, the growth rate profiles exhibit almost opposite characters, implying that the instabilities can be strong enough not only for the parallel and perpendicular propagation of waves but also for other directions of propagation with $\theta = \pi/4$ and $\theta = 3\pi/4$.

To conclude, since the growth rate of instability becomes higher (or its inverse becomes lower $\lesssim 10^{-1}$ s) due to the effects of the neutrino flavor oscillations, the instabilities of magnetosonic waves can occur within a shorter time than that in absence of the flavor oscillations. Consequently, this instability should appear to be fast enough to provoke the neutrino radiation and two-flavor neutrino mixing in core-collapse supernovae [30], since the typical time scale for supernova explosion is $\sim 1 - 100$ s. Although the present model restricts to two neutrino flavor oscillations, the extension to three flavor states is also seemingly important, however, left for future studies.

Figure 2. The relative influences of the neutrino two-flavor oscillations ($\gamma_{\text{osc}}^+$) and the neutrino streaming beam ($\gamma_{\text{str}}^+$) on the growth rates of instabilities of oblique magnetosonic waves are shown against the wave number for two different values of the total neutrino density $N_0$ as in the legend. The fixed parameter values are $B_0 = 5 \times 10^6$ T, $\theta = \pi/4$ and others as in the text.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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