A NOTE ON HANG-WANG’S HEMISPHERE RIGIDITY THEOREM

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Abstract. Let \((M, g)\) be a compact manifold with boundary and \(\text{Ric}_g \geq (n - 1)g\). Hang and Wang proved that \((M, g)\) is isometric to the standard hemisphere if \(\partial M\) is convex and isometric to \(S^{n-1}(1)\). We prove some rigidity theorems when \(\partial M\) is isometric to a product manifold where one factor is the standard sphere.

1. Introduction

Rigidity phenomenon of manifolds under various curvature conditions is a very interesting and important subject in differential geometry. A prominent example is the rigidity part of the positive mass theorem proved by Schoen-Yau [SY] and Witten [W], which generates enormous study on rigidity phenomena with assumptions on the scalar curvature. In particular, by Bartnik’s version of the positive mass theorem, any metric on \(\mathbb{R}^n\) with nonnegative scalar curvature, which agrees with the standard Euclidean metric outside a compact set, must be flat. For further developments, we refer the reader to the survey [B] and references therein.

In an attempt to tackle the Min-Oo conjecture, nevertheless remarkably disproved later by Brendle-Marques-Neves [BMN], Hang and Wang [HW] proved an interesting rigidity theorem for manifolds with boundary and positive Ricci curvature.

**Theorem A** (Hang-Wang). Let \((M, g)\) be a compact manifold with boundary, and suppose

- \(\text{Ric} \geq (n - 1)g\);
- \(\partial M\) is isometric to \(S^{n-1}(1)\);
- \(\partial M\) is convex, i.e., the second fundamental form \(h \geq 0\).

Then \((M, g)\) is isometric to the standard hemisphere \(S^n_+(1)\).

The above three curvature conditions are reminiscent of Serrin’s overdetermined problem [S]. Roughly speaking, the Ricci curvature can be viewed as the Laplacian of the metric. The boundary metric and the second fundamental form can be viewed as Dirichlet and Neumann boundary conditions for the metric respectively.

In this short note, we generalize Hang-Wang’s rigidity theorem in three settings depending on the sign of the Ricci curvature lower bound. We assume the boundary is isometric to a product manifold with the product metric, where one of the factors is isometric to a round sphere. The proofs mimic the proof of Theorem A with a new

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ingredient: the Obata type equation with Robin boundary condition, which has been carefully studied by Chen-Lai-Wang [CLW].

In our proofs, we first obtain functions on $M$ satisfying Obata type equations following the method of Hang-Wang. We then gain information on the second fundamental form of the inward equidistance hypersurface. The precise geometry of $M$ then follows by virtue of the boundary geometry and the Obata equation.

**Theorem 1.** Let $(M, g)$ be a compact manifold with boundary, and suppose

- $\text{Ric}_g \geq (n-1)g$;
- $\partial M$ is isometric to $S^{k-1}(\sin \theta) \times (N, g_N)$, where $\theta \in (0, \frac{\pi}{2})$ and $N$ is an $(n-k)$ dimensional closed manifold with a metric $g_N$;
- the second fundamental form $h$ on $\partial M$ satisfies $h(w, w) \geq \cot \theta |w|^2$, $\forall w$ tangent to $S^{k-1}$;
- $H \geq (k-1) \cot \theta - (n-k) \tan \theta \geq 0$.

Then $M$ is isometric to the doubly warped product $dr^2 + \sin^2(r)g_{S^{k-1}(1)} + \frac{\cos^2(r)}{\cos^2 \theta}g_N$, $r \in [0, \theta]$ and necessarily $\text{Ric}_{g_N} \geq \frac{(n-k-1)}{\cos^2 \theta}g_N$ in case $n-k \geq 2$.

Let $N$ be isometric to $S^{n-k}(\cos \theta)$, we immediately get

**Corollary 1.** Let $(M, g)$ be a compact manifold with boundary, and suppose

- $\text{Ric}_g \geq (n-1)g$;
- $\partial M$ is isometric to $S^{k-1}(\sin \theta) \times S^{n-k}(\cos \theta)$ for some $\theta \in (0, \frac{\pi}{2})$;
- the second fundamental form $h$ satisfies $h(w, w) \geq \cot \theta |w|^2$, $\forall w$ tangent to $S^{k-1}$;
- $H \geq (k-1) \cot \theta - (n-k) \tan \theta \geq 0$.

Then $(M, g)$ is isometric to $dr^2 + \sin^2 r g_{S^{k-1}} + \cos^2 r g_{S^{n-k}}$, $r \in [0, \theta]$. This is exactly the spherical domain bounded by a generalized clifford torus $S^{k-1}(\sin \theta) \times S^{n-k}(\cos \theta) \subset S^n$ whose boundary has nonnegative mean curvature with respect to the outward unit normal.

**Remark 1.** This corollary exhibits a new type of spherical region where Hang-Wang type rigidity holds. Its boundary is a generalized clifford torus. It is interesting to note that clifford tori are isoparametric hypersurfaces with two distinct principal curvatures in sphere. One might explore rigidity for other spherical regions bounded by isoparametric hypersurfaces. It may be also related to overdetermined problems in sphere. We also remark that Miao and Wang [MW] have obtained various interesting results on manifolds with boundary and Ricci curvature lower bound. Corollary 1 is a slight improvement of Theorem 1.5 in [MW], in view of the assumptions on the second fundamental form.

Analogously, we obtain

**Theorem 2.** Let $(M, g)$ be a compact manifold with boundary, and suppose

- $\text{Ric}_g \geq -(n-1)g$;
- $\partial M$ is isometric to $S^{k-1}(\sinh \theta) \times (N, g_N)$ ($\theta > 0$) and $N$ is an $(n-k)$ dimensional closed manifold with a metric $g_N$;

This corollary exhibits a new type of spherical region where Hang-Wang type rigidity holds. Its boundary is a generalized clifford torus. It is interesting to note that clifford tori are isoparametric hypersurfaces with two distinct principal curvatures in sphere. One might explore rigidity for other spherical regions bounded by isoparametric hypersurfaces. It may be also related to overdetermined problems in sphere. We also remark that Miao and Wang [MW] have obtained various interesting results on manifolds with boundary and Ricci curvature lower bound. Corollary 1 is a slight improvement of Theorem 1.5 in [MW], in view of the assumptions on the second fundamental form.
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The second fundamental form satisfies $h(w, w) \geq \coth \theta |w|^2$, $\forall w$ tangent to $S^{k-1}$.

$H \geq (k - 1) \coth \theta + (n - k) \tanh \theta$.

Then $M$ is isometric to the doubly warped product $dr^2 + \sinh^2(r) g_{S^{k-1}(1)} + \frac{\cosh^2(r)}{\cosh^2 \theta} g_N, r \in [0, \theta]$ and necessarily $\text{Ric}_{g_N} \geq -\frac{(n-k-1)}{\theta} \cosh^2 \theta g_N$, if $n - k \geq 2$.

**Theorem 3.** Let $(M, g)$ be a compact manifold with boundary, and suppose

- $\text{Ric}_g \geq 0$;
- $\partial M$ is isometric to $S^{k-1}(\theta) \times (N, g_N)$ ($\theta > 0$) and $N$ is an $(n-k)$ dimensional closed manifold with a metric $g_N$;
- the second fundamental form satisfies $h(w, w) \geq \frac{1}{\theta} |w|^2$, $\forall w$ tangent to $S^{k-1}$;
- $H \geq \frac{k-1}{\theta}$.

Then $M$ is isometric to $D^k(\theta) \times N$ with the product metric, where $D^k(\theta)$ is the Euclidean ball of radius $\theta$ and necessarily $\text{Ric}_{g_N} \geq 0$, if $n - k \geq 2$.

**Remark 2.** Notice that when $N$ is a one-dimensional closed manifold, namely a circle, then $M$ in Theorem 2 is in fact a hyperbolic manifold as the sectional curvature of $dr^2 + \sinh^2(r) g_{S^{n-2}(1)} + \frac{\cosh^2(r)}{\cosh^2 \theta} g_N, r \in [0, \theta]$ is $\equiv -1$. $M$ in Theorem 3 is a flat manifold $D^{n-1}(\theta) \times S^1$.

We present the detailed proof of Theorem 1 in the next section, and only indicate necessary changes in proofs of Theorem 2 and Theorem 3.

2. PROOF OF THE MAIN THEOREM

*Proof of Theorem 1.* The proof follows closely with the proof in [HW]. First let us recall Reilly’s result [R]. Let $(M, g)$ be a compact manifold with boundary and $\text{Ric}_g \geq (n - 1)g$. Suppose $\partial M$ is mean convex, i.e., the mean curvature $H \geq 0$, then the first eigenvalue $\lambda_1$ of the Laplacian operator with the Dirichlet boundary condition satisfies $\lambda_1 \geq n$. Moreover the equality holds if and only if $(M, g)$ is isometric to the standard hemisphere.

In view of assumptions on the boundary, we have $\lambda_1 > n$. Thus there exists a unique solution to

\[
\begin{cases}
\Delta u + nu = 0, & \text{in } M; \\
u = f, & \text{on } \partial M,
\end{cases}
\]

for any $f \in C^\infty(\partial M)$.

Denote by $\tilde{g}$ the induced metric of $M$ on $\partial M$, and denote by $\tilde{\nabla}, \tilde{\Delta}$ the induced operators with respect to $\tilde{g}$ on $\partial M$.

Since $\partial M$ is isometric to $S^{k-1}(\sin \theta) \times N$, we could choose $f$ to be one of the coordinate functions on $S^{k-1}$ that does not depend on the second factor. By direct computations, we get

\[
\tilde{\Delta} f + (k - 1) \csc^2 \theta f = 0, \quad \text{on } \partial M,
\]

\[
\sin^2 \theta |\tilde{\nabla} f|^2 + f^2 = \sin^2 \theta \quad \text{on } \partial M.
\]
Set \( \frac{\partial u}{\partial \nu} =: \varphi \) and \( u^2 + |\nabla u|^2 =: \phi \). By the Bochner formula, \( \text{Ric} \geq (n - 1)g \) and (1), we have
\[
\Delta \phi = \Delta (u^2 + |\nabla u|^2) = 2\{u\Delta u + |\nabla u|^2 + |\nabla^2 u| + \nabla\Delta u \cdot \nabla + \text{Ric}(\nabla u, \nabla u)\} \geq 2\{-nu^2 + |\nabla u|^2 + \frac{(\Delta u)^2}{n} - n|\nabla u|^2 + (n - 1)|\nabla u|^2\} = 0.
\]

Our goal is to show \( \phi \) is constant. Suppose not, then there exists \( p \in \partial M \) such that
\[
\phi(p) = \max_{x \in M} \phi(x) \quad \text{and} \quad \frac{\partial \phi}{\partial \nu}(p) > 0.
\]

The latter is due to the Hopf lemma.

We compute \( \frac{\partial \phi}{\partial \nu} \) as follows:
\[
\frac{1}{2} \frac{\partial \phi}{\partial \nu} = \frac{\partial u}{\partial \nu} + D^2 u(\nu, Du) = \varphi(f + D^2 u(\nu, \nu)) + D^2 u(\nu, \nabla f) = \varphi(f + D^2 u(\nu, \nu)) + \nabla \varphi \cdot \nabla u \cdot \nu = \varphi(f + D^2 u(\nu, \nu)) + \nabla f \cdot \nabla \varphi - h(\nabla f, \nabla f).
\]

Set \( A = \varphi(f + D^2 u(\nu, \nu)) \) and \( B = \nabla f \cdot \nabla \varphi - h(\nabla f, \nabla f) \).

On \( \partial M \), we have
\[
0 = \Delta u + nu = \bar{\Delta} f + H \varphi + D^2 u(\nu, \nu) + nf.
\]

Plugging (2) in the above, it follows
\[
A = \varphi(-H \varphi + (k - 1) \cot^2 \theta f - (n - k)f) \leq ((k - 1) \cot \theta - (n - k) \tan \theta) \varphi[\cot \theta f - \varphi],
\]
where we have used the fact \( H \geq (k - 1) \cot \theta - (n - k) \tan \theta \geq 0 \).

By (3), \( \phi|_{\partial M} = 1 + \varphi^2 - \cot^2 \theta f^2 \). Since \( p \) is maximal for \( \phi \) and \( f \) takes value 0 somewhere, it follows that \( \varphi^2(p) \geq \cot^2 \theta f^2(p)^2 \) and
\[
0 = \nabla \phi(p) = 2\varphi(p) \nabla \varphi(p) - 2 \cot^2 \theta f(p) \nabla f(p).
\]

There are two cases:

- Case 1: \( \varphi(p) \neq 0 \). Since changing \( f \) to \( -f \) results a change \( \varphi \) to \( -\varphi \), but leaves \( \phi \) unchanged, thus we can without loss of generality assume that \( \varphi(p) > 0 \). Hence \( -\varphi(p) \leq \cot \theta f(p) \leq \varphi(p) \), from which we obtain \( A(p) \leq 0 \). For \( B \), we infer by (6) that
\[
B(p) = \cot^2 \theta \frac{f(p)}{\varphi(p)} |\nabla f|^2 - h(\nabla f, \nabla f) \leq \cot \theta |\nabla f|^2 (\cot \theta \frac{f(p)}{\varphi(p)} - 1) \leq 0.
\]
Here we use the fact $\nabla f$ is tangent to $S^{k-1}$, where $h \geq \cot \theta$. This contradicts with $\frac{\partial \phi}{\partial \nu}(p) > 0$.

- Case 2: $\varphi(p) = 0$. In this case, $f(p) = 0$ as well and $A(p) = 0$. For $B$, we use the fact $p$ is the maximum point of $\phi$, thus for any $X \in T(\partial M)$, we have
  \[ 0 \geq \nabla_{X,X}^2 \varphi(p) = \nabla_{X,X}^2(1 + \varphi^2 - \cot^2 \theta f^2)(p) = 2(X \cdot \nabla \varphi)^2 - 2 \cot^2 \theta(X \cdot \nabla f)^2. \]

Here we use the fact $p$ is minimum for both $\varphi^2$ and $f^2$. Letting $X = \tilde{\nabla} f$, we get $\cot^2 \theta|\nabla f|^2 \geq \tilde{\nabla} f \cdot \nabla \varphi$, which implies $B(p) \leq 0$. We again get a contradiction.

Therefore $\phi$ is constant on $M$, which implies $D^2u + ug = 0$ in $M$. Moreover $\varphi^2 - \cot^2 \theta f^2$ is also a constant on $\partial M$. Since
\[ 0 = \frac{\partial \phi}{\partial \nu} = A + B, \]
combining the estimate (5) and (7), we have $\varphi = \cot \theta f$. Plugging it back in (4), we have
\[ 0 \equiv A + B = \varphi(f + D^2u(\nu, \nu)) + \nabla f \cdot \nabla \varphi - h(\nabla f, \nabla f) = \varphi^2(-H + (k - 1) \cot \theta - (n - k) \tan \theta) + \cot \theta |\nabla f|^2 - h(\nabla f, \nabla f). \]

In view of assumptions that $H \geq (k - 1) \cot \theta - (n - k) \tan \theta$ and $h(\nabla f, \nabla f) \geq \cot \theta |\nabla f|^2$, we infer that $H \equiv (k - 1) \cot \theta - (n - k) \tan \theta$, and $h(\nabla f, \nabla f) = \cot \theta |\nabla f|^2$. Since there exist coordinate functions $f_1, \cdots f_k$ on $S^{k-1}$ such that $\{\nabla f_i\}_{i=1}^k$ span the tangent subspace along $S^{k-1}$, we get
\[ h(\nu, \nu) \equiv \cot \theta, \quad \forall \text{ unit vector tangent } \nu \text{ along } S^{k-1}. \]

We claim that
\[ h = \cot \theta \tilde{g}|_{S^{k-1}} \oplus (-\tan \theta) \tilde{g}|_N. \]

To this end, we summarize the known facts to find that $u$ satisfies the Obata equation with the Robin boundary condition:
\[ \begin{cases} 
D^2u + ug = 0, & \text{in } M \\
\frac{\partial u}{\partial \nu} - \cot \theta u = 0, & \text{on } \partial M. 
\end{cases} \tag{8} \]

The restriction of this equation on the boundary implies that there are at most two distinct principal curvatures: $\cot \theta$ and $-\tan \theta$. (see Lemma 2.1 of [CLW] for detailed computation) Moreover each is of constant multiplicity. In view of $H \equiv (k - 1) \cot \theta - (n - k) \tan \theta$, it follows the tangent subspace along $S^{k-1}$ is the eigenspace with respect to the principal curvature $\cot \theta$ and the tangent subspace along $N$ is the eigenspace with respect to the principal curvature $-\tan \theta$. Thus the claim follows.

To determine the metric structure of $M$, let $\partial M_t$ be the inward $t$-equidistance hypersurface of $\partial M$, where $M_t := \{x \in M\mid \text{dist}(x, \partial M) \geq t\}$. Thus for small $t$, say $t \in (0, t_0)$, $\partial M_t$ is diffeomorphic to $\partial M$. The diffeomorphism $\Pi_t : \partial M \to \partial M_t$ is explicitly given by the inward normal exponential map, i.e. $\Pi_t(x) = \exp_x(-t\nu(x))$, for $x \in \partial M$. 

Along a normal geodesic $\gamma_x(t)$ with $\gamma_x(0) = x$ and $\gamma_x'(0) = -\nu(x)$, $u$ satisfies
\[ u'' \circ (\gamma_x(t)) + u \circ (\gamma_x(t)) = 0. \]
Note that $u' \circ (\gamma_x(t)) = -\frac{\partial u}{\partial \nu} = -\cot \theta u(0)$ by (8), it follows that
\[ u \circ (\gamma_x(t)) = u(0) \cos t - \cot \theta u(0) \sin t. \]
Then
\[ u' \circ (\gamma_x(t)) = -u(0) \sin t - \cot \theta u(0) \cos t. \]
Thus on $\partial M_t$,
\[ \frac{\partial u}{\partial t} - \sin t + \cot \theta \cos t = 0. \]
For simplicity, let us denote $\frac{\sin t + \cot \theta \cos t}{\cos t - \cot \theta \sin t}$ by $a(t)$.
Hence the Obata equation (8) holds on $\mathcal{M}_t := \{x \in M | dist(x, \partial M) \geq t\}$, $\forall t \in (0, t_0)$, i.e.,
\[ u \in \mathcal{H}_{\mathcal{M}_t}, \quad \left\{ \begin{array}{l} D^2 u + u g = 0, \quad \text{in } \mathcal{M}_t \\ \frac{\partial u}{\partial \nu} - a(t) u = 0, \quad \text{on } \partial \mathcal{M}_t. \end{array} \right. \tag{10} \]
Restricting $\nabla^2 u + u g = 0$ on $\partial \mathcal{M}_t$, we infer that there are at most two distinct principal curvatures of $\partial \mathcal{M}_t$: $a(t)$ and $-\frac{1}{a(t)}$. Moreover the multiplicity is necessarily constant by continuity, and thus the principal curvatures of $\partial \mathcal{M}_t$ are
\[ a(t) \quad \text{of multiplicity } k - 1, \quad -\frac{1}{a(t)} \quad \text{of multiplicity } n - k. \]
We claim that eigenspace of $a(t)$ is the tangent subspace along $S^{k-1}$ factor of $\partial \mathcal{M}_t$.
To this end, first denote by $\bar{g}(t)$ the restriction of $g$ on $\partial \mathcal{M}_t$, let $e_1, \ldots, e_n$ be an orthonormal frame with $e_n = \nu$, the unit normal vector field along $\partial \mathcal{M}_t$.
Using (10), for $i \neq n$, we have
\[ 0 = \nabla^2_{e_i, e_n} u = \nabla_{e_i} \nabla_{e_n} u - \nabla_{e_i} e_n u = a(t) u_i - h(t) u_j. \]
Hence $\nabla_t u$ is an eigenvector of $h(t)$ with respect to the eigenvalue $a(t)$.
We can choose the boundary value $f$ to be coordinate functions $f^{(1)}, \ldots, f^{(k)}$ such that $\{\nabla f^{(i)}\}_{i=1}^k$ span the tangent subspace along $S^{k-1}$. Let $u^{(i)}$, $i = 1, \ldots, k$ be the corresponding solutions, thus they all satisfy (8) and (10). For any $p \in S^{k-1}$, by (9) it follows that the restriction of $u^{(i)}$ on $\Pi_t(\{p\} \times N)$ is constant, thus $\nabla_t u^{(i)}$ is perpendicular to the tangent subspace along $\Pi_t(\{p\} \times N)$. Clearly $\nabla_t u^{(i)}$ span the tangent subspace along the $S^{k-1}$ factor of $\partial \mathcal{M}_t$, the claim thus follows. Consequently, the eigenspace of $-\frac{1}{a(t)}$ is the tangent subspace along $N$ factor.
The restriction of $\nabla^2 u + u g = 0$ on $\partial \mathcal{M}_t$ reads as
\[ \nabla_t^2 u + \frac{\partial u}{\partial \nu_t} h(t) + u \bar{g}(t) = 0. \]
In view of the eigenvalue description of $h(t)$, it follows that $\nabla_t^2 u$ has two eigenvalues
\[ a^2(t) + 1, \quad \text{whose eigenspace is the tangent subspace along } S^{k-1}; \]
\[ 0, \quad \text{whose eigenspace is the tangent subspace along } N. \]

Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal frame, such that $\{e_1, \ldots, e_{k-1}\}$ forms an orthonormal frame for the tangent subspace along $S^{k-1}$. Let $\tilde{\nabla}_t$ denote the induced Levi-Civita connection of $\tilde{g}(t)$ on its submanifolds of the form
\[ S^{k-1} \times \{q\}, \quad q \in N \quad \text{or} \quad \{p\} \times N, \quad p \in S^{k-1}. \]

We have
\[ \tilde{\nabla}_{te_i,e_j}^2 u = \tilde{\nabla}_{te_i,e_j}^2 u - \tilde{\nabla}_t(\nabla_{te_i,e_j})^1 u, \quad 1 \leq i, j \leq k - 1. \]
Since $(\nabla_{te_i,e_j})^1$ is along $N$ direction, the last term vanishes. Hence
\[ \tilde{\nabla}_{te_i,e_j}^2 u + (1 + a(t)^2)u\tilde{g} = 0, \]
holds on $S^{k-1} \times \{q\}$, for any $q \in N$. By Obata theorem [O], it follows that $\tilde{g}$ on $S^{k-1} \times \{q\}$ is isometric to the round sphere of radius $\frac{1}{\sqrt{1+a(t)^2}}$.

On the other hand, we have
\[ \tilde{\nabla}_{te_i,e_j}^2 u = \tilde{\nabla}_{te_i,e_j}^2 u - \tilde{\nabla}_t(\nabla_{te_i,e_j})^1 u, \quad k \leq i, j \leq n - 1. \]
Since $u$'s restriction on $\{p\} \times N$ is constant, $\tilde{\nabla}_{te_i,e_j}^2 u$ vanishes. Therefore $\tilde{\nabla}_t(\nabla_{te_i,e_j})^1 u = 0$, which implies that $\{p\} \times N$ is totally geodesic for any $p \in S^{k-1}$.

Hence $\tilde{g}(t)$ is indeed a product metric. Writing $\tilde{g}(t) = \alpha(t)g_{S^{k-1}}(t) \oplus \beta(t)N$, where $\alpha(t)$ and $\beta(t)$ are two families of metrics on $S^{k-1}$ and $N$ respectively. We thus have
\[ (11) \quad \begin{pmatrix} \frac{1}{2}a'(t) & 0 \\ 0 & \frac{1}{2}\beta'(t) \end{pmatrix} \begin{pmatrix} \alpha(t) & 0 \\ 0 & \alpha(t) \end{pmatrix} = \begin{pmatrix} -a(t)\alpha(t) & 0 \\ 0 & \alpha(t)\beta(t) \end{pmatrix}. \]
The boundary condition means
\[ \alpha(0) = \sin^2 \theta g_{S^{k-1}(1)}, \quad \beta(0) = g_N. \]
Solving (11), we obtain
\[ \alpha(t) = \sin^2(\theta - t)g_{S^{k-1}(1)}, \quad \beta(t) = \frac{\cos^2(\theta - t)}{\cos^2 \theta}dg_N. \]

So far we only solve the metric for small $t \in (0, t_0)$. By continuity of the metric, $\partial M_{t_0}$ is isometric to $S^{k-1}(\sin \theta - t_0) \times N$ with a product metric. Therefore, we could continue the same argument as long as $a(t)$ is well defined, i.e., for $t \in [0, \theta)$.

Hence $g = dt^2 + \sin^2(\theta - t)g_{S^{k-1}(1)} + \frac{\cos^2(\theta - t)}{\cos^2 \theta}dg_N$ for $t \in [0, \theta]$. After metric completion, which corresponds to a submanifold diffeomorphic to $N$ for $t = \theta$, it yields a compact manifold with boundary, thus it is necessarily the whole $M$. Letting $r = \theta - t$, then $(M, g)$ is isometric to the doubly warped product
\[ dr^2 + \sin^2(r)g_{S^{k-1}(1)} + \frac{\cos^2(r)}{\cos^2 \theta}g_N, \quad r \in [0, \theta]. \]
By direction computation of the Ricci curvature for this doubly warped product, we have
\[
\text{Ric}(\partial_r) = (n-1)\partial_r, \\
\text{Ric}(X) = (n-1)X, \quad \forall X \text{ tangent to the } S^{k-1} \text{ factor} \\
\text{Ric}(Y) = \text{Ric}_{g_N}(Y) - ((n-k-1) \frac{\sin^2 r}{\cos^2 r} - k)Y, \quad \forall Y \text{ tangent to the } N \text{ factor}.
\]

The assumption $\text{Ric}_g \geq (n-1)g$ implies that $\text{Ric}_{g_N} \geq (\frac{n-k-1}{\cos^2 \theta})g_N$. □

Using the same method, we can prove Theorem 2 and Theorem 3.

Proof of Theorem 2. Let $u$ be a solution of
\[
\begin{cases}
\Delta u - nu = 0, & \text{in } M; \\
u = f, & \text{on } \partial M,
\end{cases}
\]
for any $f \in C^\infty(\partial M)$. Note it is always solvable. Take $f$ to be a coordinate function on $S^{k-1}$, which does not depend on the $N$ factor. This time we consider $\phi := \frac{1}{2}|\nabla u|^2 - u^2$. A similar argument shows that $\phi = \text{const}$ and consequently $u$ satisfies
\[
\begin{cases}
D^2 u - ug = 0, & \text{in } M \\
\frac{\partial u}{\partial \nu} - \coth \theta u = 0, & \text{on } \partial M.
\end{cases}
\]

Along a normal geodesic $\gamma_x(t)$ with $\gamma_x(0) = x \in \partial M$ and $\gamma'_x(0) = -\nu(x)$, $u$ satisfies
\[
u''(\gamma_x(t)) - u(\gamma_x(t)) = 0.
\]

Note that $u'(\gamma_x(0)) = -\frac{\partial u}{\partial \nu} = -\coth \theta u(0)$ by (12), it follows that
\[u(\gamma_x(t)) = u(0) \cosh t - \coth \theta u(0) \sinh t.
\]

Thus on the $t$-equidistance hypersurface $\partial M_t$,
\[
\frac{\partial u}{\partial t} - \coth \theta \cosh t - \sinh t \cosh t - \coth \theta \sinh t u = 0.
\]

For simplicity, let $b(t) = \frac{\coth \theta \cosh \frac{t}{2} \sinh \frac{t}{2}}{\cosh \frac{t}{2} - \coth \theta \sinh \frac{t}{2}}$. A similar computation shows that there are two distinct principal curvatures: $b(t)$ whose eigenspace is the tangent subspace along $S^{k-1}$ and $\frac{1}{b(t)}$ whose eigenspace is the tangent subspace along $N$. Similarly $\bar{g}(t) = \alpha(t)_{S^{k-1}} \oplus \beta(t)_{N}$ is a product metric on the $t$-equidistance hypersurface $\partial M_t$, and we get the metric ODE
\[
\begin{pmatrix}
\frac{1}{2} \alpha'(t) & 0 \\
0 & \frac{1}{2} \beta'(t)
\end{pmatrix} = 
\begin{pmatrix}
-b(t) \alpha(t) & 0 \\
0 & -\frac{1}{b(t)} \beta(t)
\end{pmatrix}.
\]

Solving it using the boundary condition, we get
\[\alpha(t) = \sinh^2(\theta - t)g_{S^{k-1}(1)}, \quad \beta(t) = \cosh^2(\theta - t)g_N.
\]

The desired result thus follows. □
Proof of Theorem 3. Let \( u \) be a solution of
\[
\begin{cases}
\Delta u = 0, & \text{in } M; \\
u = f, & \text{on } \partial M,
\end{cases}
\]
for any \( f \in C^\infty(\partial M) \). Take \( f \) to be a coordinate function of \( S^{k-1} \), which does not depend on the \( N \) factor. We consider \( \phi := \frac{1}{2}|\nabla u|^2 \). A similar argument shows that \( \phi = \text{const} \) and thus we have \( u \) satisfies
\[
\begin{cases}
D^2 u = 0, & \text{in } M \\
\frac{\partial u}{\partial \nu} - \frac{1}{\theta} u = 0, & \text{on } \partial M.
\end{cases}
\] (13)

Thus along a normal geodesic \( \gamma_x(t) \) with \( \gamma_x(0) = x \in \partial M \) and \( \gamma_x'(0) = -\nu(x) \), \( u \) satisfies
\[
uu \circ (\gamma_x(t)) = 0.
\]
Note that \( uu \circ (\gamma_x(0)) = -\frac{\partial u}{\partial \nu} = -\frac{1}{\theta} u(0) \) by (13), it follows that
\[
u \circ (\gamma_x(t)) = u(0) - \frac{u(0)}{\theta} t.
\]
Thus on the \( t \)-equidistance hypersurface \( \partial M_t \),
\[
\frac{\partial u}{\partial t} - \frac{1}{\theta - t} u = 0.
\]
For simplicity, let \( c(t) = \frac{1}{\theta - t} \).

Again, there are two distinct principal curvatures \( \frac{1}{c(t)} \) and 0, corresponding to the tangent subspace along \( S^{k-1} \) and \( N \), respectively. The metric \( \tilde{g}(t) \) is of a product metric, and we get the corresponding metric ODE
\[
\left( \begin{array}{cc}
\frac{1}{2}\alpha'(t) & 0 \\
0 & \frac{1}{2}\beta'(t)
\end{array} \right) = \left( \begin{array}{cc}
-\frac{1}{c(t)} & \alpha(t) \\
0 & 0
\end{array} \right).
\]

In view of the boundary condition, we get
\[
\alpha(t) = (\theta - t)^2 g_{S^{k-1}(1)}, \quad \beta(t) = g_N,
\]
which leads to the desired conclusion. \( \square \)

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