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Reflective anisotropic hyperbolic lattices of rank 4

N. V. Bogachev

1. Introduction and formulation of the result. By definition, a hyperbolic lattice is a free Abelian group with a non-degenerate integral symmetric bilinear form (called an inner product) of signature \((n, 1)\). A lattice \(L\) is said to be isotropic if the corresponding quadratic form represents zero, otherwise it is said to be anisotropic.

Let \(L\) be a hyperbolic lattice. We shall assume that it is embedded in the Minkowski space \(E^{n,1} = L \otimes \mathbb{R}\), and we shall take one of the connected components of the hyperboloid
\[
\{x \in E^{n,1} : (x, x) = -1\}
\] (1)
as a model of the \(n\)-dimensional Lobachevsky space \(\mathbb{L}^n\).

A primitive vector \(e\) in \(L\) is called a root or, more precisely, a \(k\)-root if \((e, e) = k > 0\) and \(2(e, x) \in k\mathbb{Z}\) for all \(x \in L\). (For \(k = 1, 2\) the last condition is fulfilled automatically.) Every root \(e\) defines an orthogonal reflection \(R_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e\) (called a \(k\)-reflection if \((e, e) = k\)) in \(E^{n,1}\) which preserves the lattice \(L\) and defines a reflection with respect to the hyperplane
\[
H_e = \{x \in \mathbb{L}^n : (x, e) = 0\}
\]
in the space \(\mathbb{L}^n\).

It is known that the group \(O'(L)\) of automorphisms of \(L\) leaving invariant each connected component of the hyperboloid (1) acts discretely on the Lobachevsky space, and its fundamental polyhedron has finite volume. In addition, the fundamental polyhedron of \(O'(L)\) is bounded if and only if the lattice \(L\) is anisotropic.

We denote by \(O_r(L), \ O_r^{(2)}(L), \text{ and } \ O_r^{(1,2)}(L)\) the subgroups of \(O'(L)\) generated by all the reflections, all the 2-reflections, and all the 1- and 2-reflections in \(O'(L)\), respectively. The lattice \(L\) is said to be reflective, 2-reflective, or 1.2-reflective if the subgroup \(O_r(L), \ O_r^{(2)}(L), \text{ or } \ O_r^{(1,2)}(L)\), respectively, has finite index in \(O'(L)\). It is obvious that every finite extension of a 2-reflective (or 1.2-reflective) hyperbolic lattice is also a 2-reflective (respectively, 1.2-reflective) lattice.

Nikulin (see [1] and [2]) classified all 2-reflective hyperbolic lattices of rank distinct from 4, and Vinberg [3] classified those of rank 4. Then in [4] Nikulin found all reflective hyperbolic lattices of rank 3 with square-free discriminants. Subsequently, Allcock [5] classified all reflective lattices of rank 3. Scharlau and Walhorn [6]–[8] found all maximal groups of the form \(O_r(L)\), where \(L\) is a reflective isotropic hyperbolic lattice of rank 4 or 5.

We denote by \([C]\) the quadratic lattice with inner product given by a symmetric matrix \(C\) in some basis, and we denote by \(L \oplus M\) the orthogonal sum of two lattices \(L\) and \(M\).

The main result of this paper is the following theorem.

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Theorem 1. Every maximal 1.2-reflective anisotropic hyperbolic lattice of rank 4 is isomorphic either to the lattice $[-7] \oplus [1] \oplus [1] \oplus [1]$ or to the lattice $[-15] \oplus [1] \oplus [1] \oplus [1]$.

These lattices are in fact 2-reflective [3].

2. Main ideas of the proof. The $(n - 1)$-dimensional faces of a polyhedron will be called just faces. The Gram matrix of a system of vectors $v_1, \ldots, v_k$ will be denoted by $G(v_1, \ldots, v_k)$.

In [9] Nikulin proved the following assertion.

Theorem 2. Let $M$ be an acute-angled convex polyhedron in $\mathbb{L}^n$ and let $e_0$ be a fixed interior point of $M$. If $F$ is a face of $M$ that is outermost from $e_0$, then $\cosh \rho(F_1, F_2) \leq 14$ for any faces $F_1$ and $F_2$ of $M$ adjacent to $F$, where $\rho(\cdot, \cdot)$ denotes the distance in the Lobachevsky space.

Let $P$ be the fundamental polyhedron of the group $O_{r(1,2)}^+(L)$ for a maximal anisotropic hyperbolic lattice $L$ of rank 4. The lattice $L$ is 1.2-reflective if and only if $P$ is bounded. In this case each vertex belongs to exactly three faces. Let $e_0$ be a fixed point in the interior of $P$, and let $E$ be an edge that is outermost from this point.

Let $e_1, e_2$ (respectively, $e_3, e_4$) be 1- or 2-roots of the lattice $L$ that are orthogonal to the faces containing $E$ (respectively, to the faces containing a vertex of $E$, but not $E$ itself) and are outward normals to these faces.

Using Theorem 2, we obtain an explicit upper bound for the absolute values of elements of the Gram matrix $G(e_1, e_2, e_3, e_4)$. This leaves only a finite number of possible Gram matrices $G(e_1, e_2, e_3, e_4)$. Further, we choose from these matrices the ones that define an anisotropic quadratic form.

The vectors $e_1, e_2, e_3, e_4$ generate some sublattice $L'$ of finite index in the lattice $L$. Using the fact that $[L : L']^2$ divides the discriminant of $L$, we find for each such lattice $L'$ all its maximal extensions. Then using Vinberg’s algorithm (see [10]) and some additional considerations, we pick the 1.2-reflective extensions among them.

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