Universal energy distribution for interfaces in a random field environment

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We study the energy distribution function $\rho(E)$ for interfaces in a random field environment at zero temperature by summing the leading terms in the perturbation expansion of $\rho(E)$ in powers of the disorder strength, and by taking into account the non perturbational effects of the disorder using the functional renormalization group. We have found that the average and the variance of the energy for one-dimensional interface of length $L$ behave as, $(E)\sim L \ln L$, $\Delta E\sim L$, while the distribution function of the energy tends for large $L$ to the Gumbel distribution of the extreme value statistics.

It is well-known that the Langevin equation \ref{eq:1} can be reformulated in terms of the Fokker-Planck equation for the conditional probability density $P(z(x), t; z^0(x), t^0)$ to have the profile $z(x)$ at time $t$ by having the profile $z^0(x)$ at time $t^0$. This Fokker-Planck equation can be written as an integral equation, which, for an interface of a finite length $L$, reads

$$P(z, t; z^0, t^0) = P_0(z, t; z^0, t^0) - \mu \int_{t_0}^t dt \int Dz' P_0(z, t; z', t') \times \sum_{k'} \partial_{z'} g_{k'}(z') P(z', t'; z^0, t^0), \quad (2)$$

where $z_k = \int d^d x z(x) \exp(-i k x)$ and $g_{k}(z) = \int d^d x \exp(-i k x) g(x, z)$ ($k = (k_1, \ldots, k_d)$, $k_i = 2\pi j_i / L$, $j_i = 0, \pm 1, \ldots$) are the Fourier transforms of the interface height and the quenched force, respectively. $\int Dz$ in \ref{eq:2} stays for integrations over the modes $z \equiv \{z_k\}$. The bare conditional probability for non zero modes reads

$$P_0(z, t; z^0, t^0) = \prod_k \delta(z_k - z_k^0 \exp(-\gamma \mu k^2 (t - t^0))). \quad (3)$$

Analogously to the case of one Brownian particle \ref{eq:22}, the formal solution of Eq. \ref{eq:22} can be represented as a path integral.

The probability DF of the energy $E(z) = E_{el}(z) + E_{dis}(z) = \int d^d x (\frac{1}{2} \nabla z)^2 - \sum_{x} g(x, z)$ can be calculated using the conditional probability density $P(z(x), t; 0, 0)$ as follows

$$\rho(E(t)) = \int Dz(x) \delta(E - E(z)) P(z(x), t; 0, 0). \quad (4)$$

It is convenient instead of $\rho(E(t))$ to consider its Fourier transform $\hat{\rho}(s)$ which is obtained from Eq. \ref{eq:1} as

$$\hat{\rho}(s) = \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \sum_{m=0}^{n} C_n^m \langle E_{el}^m(t) E_{dis}^{n-m}(t) \rangle. \quad (5)$$

In this Letter we will restrict ourselves to the study of the energy DF in the steady state, i.e. for $t \to \infty$. In
this limit $\langle E_{\text{el}}^m E_{\text{dis}}^{n-m} \rangle$ is related to the static equilibrium correlation function

$$
\langle z(x_1)z(x_2) \rangle = \lim_{t \to \infty} \int Dz(x)z(x_1)z(x_2) \times P(z(x), t; 0, 0) = \int \frac{\Delta(0)}{(\gamma k^2)^n} e^{ik(x_1-x_2)},
$$

(6)

Let us for simplicity elucidate the computation of the $n$th moment of the elastic energy

$$
\langle E_{\text{el}}^n \rangle_c = \langle \gamma \rangle^n \int Dz \int_{k_1} k_1^2 |z_{k_1}|^2 \cdots \int_{k_n} k_n^2 |z_{k_n}|^2 P(z; t; 0, 0),
$$

(7)

where we expressed $E_{\text{el}}$ through the Fourier components of the interface height $z(x)$. For an interface of a finite size $L$ the integral $\int_{k}$ means $L^{-d} \sum_{k}$. To compute (7) to the lowest order in disorder strength we iterate Eq. 2, 2n times and insert it into (6). Expecting that the steady state does not depend on the initial interface configuration we have taken the latter in (6) and (7) to be flat at $t_0 = 0$. The average over the random forces, which is carried out by using the Wick theorem, yields connected and disconnected expressions. The connected expression contains only one integration over $k$, while the number of integrations over $k$ in a disconnected expression is equal to the number of connected parts in that expression. Let us consider the calculation of the connected part of $\langle E_{\text{el}}^n \rangle$. As a result of integrations by parts in (7) with $P(z; t; 0, 0)$ being iterated 2n times the 2n derivatives with respect to $z_{k_i}$ (see Eq. (2)) will act on $z_{k_i}$ in (7). This has the consequence that pairs of 2n momenta $k_1$, ..., $k_{2n}$, associated with the right-hand side of (2) (being iterated) become equal consecutively to one of $k_1$, ..., $k_n$ in (7). The number of such possibilities is $(2n)!$. The rid of 2n ordered time integrations in $P(z; t; 0, 0)$ gives the factor $1/(2^n)!$.

The number of possibilities to get a connected loop diagram shown in Fig. 1 with $n$ continuous lines is $2^{n-1}(n-1)!$. Integrations over $x_1$, ..., $x_{2n-1}$ arising from the above expression of $g_{z}(x)$ provides that the momenta of the modes being connected by a dashed line, which is associated with the disorder correlator, become equal. The integration over $x_{2n}$ gives the factor $L^d$. The intermediate $z_{k_i}'$ are zero for flat initial interface configuration due to delta functions in (3). As a result the arguments of disorder correlators $\Delta(z)$ become zero. Collecting all combinatorial factors and taking the limit $t \to \infty$ we find the following expression of the connected part of $\langle E_{\text{el}}^n \rangle$

$$
\frac{1}{n!} \langle E_{\text{el}}^n \rangle_c = \frac{1}{2^n} \Delta(0)^n \gamma^{-n} L^d \int_k \frac{\Delta(0)}{(k^2)^n}.
$$

(8)

The computation of $\langle E_{\text{el}}^m E_{\text{dis}}^{n-m} \rangle$ to the same order is similar and gives $\langle E_{\text{el}}^m E_{\text{dis}}^{n-m} \rangle = \frac{1}{2^n} (-2)^{n-m}(n-1)! \Delta(0)^n \gamma^{-n} L^d \int_1 \frac{1}{(k^2)^n}$. The use of the latter yields $\frac{1}{m!} \langle E^n \rangle_c$, which is obtained from (8) multiplied by the factor $(-1)^n$. The expression $\frac{1}{n!} \langle E^n \rangle_c$ is associated with the loop diagram consisting of $n$ continuous lines (see Fig. 1). The factor $2n$ is the symmetry number of the diagram. The straightforward analysis gives that the expansion (8) can be represented as a series of loop diagrams. The use of the connectivity theorem enables us to write the Fourier transform of the energy distribution DF as an exponential of the series of connected loop diagrams shown in Fig. 1

$$
\dot{\rho}(s) = \exp\left(-\frac{1}{2} L^d \int_k \ln \left(1 - \frac{i s \Delta(0)}{\gamma k^2}\right)\right).
$$

(9)

Notice that $\dot{\rho}(s)$ given by the diagram series in Fig. 1 is closely related to the loop expansion of the effective potential in Quantum Field Theory studied in Ref. [23].

Replacing the integral in (10) by the sum according to $L \int f(k) \to \sum_{j=-\infty}^{\infty} f(2\pi j/L)$ we find in $d = 1$

$$
\dot{\rho}(s) = \prod_{j=1}^{\infty} (1 + i s E_{0}/j^2)^{-1} = \frac{\pi \sqrt{i s E_{0}}}{\sinh(\pi \sqrt{i s E_{0}})},
$$

(10)

where $E_{0} = -\Delta(0) L^2/(4\pi^2 \gamma)$ is the characteristic energy for an interface with the perturbational roughness $w \propto L^{3/2}$, which follows from $w \propto L^{(d-4)/2}$ for $d = 1$.

Eq. (10) has only simple poles $s = i j^2 / E_{0}$ in the lower half-plane, so that the inverse Fourier transformation of (10) can be easily performed as a sum over all poles by using the Jordan’s lemma. As a result we obtain the DF as $\rho(E) = \frac{1}{|E_{0}|} f(E/E_{0})$, $E < 0$, where

$$
f(x) = 2 \sum_{j=1}^{\infty} (-1)^{(j+1)} j^2 e^{-xj^2}.
$$

(11)

The derivation of $\rho(E)$ shows that (11) describes also the DF of the elastic energy ($x = E/E_{0} > 0$) and the disorder energy ($x = E/2E_{0} > 0$). Eq. (11) coincides with the dimensionless width DF for the one-dimensional random-walk interface studied in Ref. [24]. Using the method of stationary phase it was shown in [24] that (11) can be well approximated for small $x$ by $f(x) \approx \sqrt{\pi/\sqrt{2 \pi}(x^2 - x)} e^{-x^2/4x}$.

Using (11) we have computed the average energy, $\langle E \rangle = \pi^2 E_{0}/6$, and the variance $\Delta E = (\langle E^2 \rangle - \langle E \rangle^2)^{1/2} = \pi^2 E_{0}^{3/2}/(3 \sqrt{10})$. Eq. (11) is the exact perturbational result generalizing the result established by Efetov and Larkin [25] for...
the height-height correlation functions [1], which can be readily proved using supersymmetry [20]. Nevertheless, both [8] and [14] are wrong due to the fact that [8] gives the value $(4 - d)/2$ for the roughness exponent instead of the correct Imry-Ma [27] value $\zeta = (4 - d)/3$.

We now will take into account the effect of the renormalization on the energy DF using the results of the FRG [10]. According to [10], the renormalized disorder correlator $\Delta R(0)$ acquires the scale dependence $R^{2 - \varepsilon}$. Taking into account the latter in [10] gives the correct value of the roughness exponent $\zeta$. To enable a crossover to the perturbative regime, we obtain the Fourier transform of the renormalized distribution on the energy DF using the results of the FRG [10]. According to [10] the renormalized disorder correlator $\Delta R(0)$ is asso-

cuted with the Larkin length $L_c \sim (\gamma^2 a^2/\Delta(0))^{1/\varepsilon}$. The ansatz [12] describes the scale dependence of $\Delta R(0)$ at the cusped fixed-point solution of the disorder correlator, $\Delta R(0) \sim (\Delta(0)/k_c)^{2 - \varepsilon}$ for $k \ll k_c$, and describes the crossover to the perturbative regime, $\Delta R(0) \sim \Delta(0)$ for $k \gg k_c$. Using the renormalized $\Delta R(0)$ in Eq. [10] we obtain the Fourier transform of the renormalized distribution of the energy in $d = 1$ as

$$\hat{\rho}_R(s) = \prod_{j=1}^{\infty} \left(1 + \frac{i s \hat{E}_0}{j(1 + j/\eta)}\right)^{-1},$$

where $\eta = L/L_c$ and $\hat{E}_0 = -\Delta(0)L_c/L/(4\pi^2\gamma)$. Carrying out the inverse Fourier transformation of [13] we obtain $\rho_R(E) = [\hat{E}_0]^{-1} f_R(E/\hat{E}_0; \eta)$, where the function $f_R(x; \eta)$ is given by

$$f_R(x; \eta) = \sum_{j=1}^{\infty} (-1)^j \frac{\Gamma(j + \eta + 1)}{\Gamma(j + 1)} \frac{(1 + 2j/\eta)}{\eta} e^{-j(1 + j/\eta)x}. \quad (14)$$

For string lengths much shorter than the Larkin length, $\eta \ll 1$, the DF [12] pass over to the perturbational result [11]. Similar to the height-height correlation function at equilibrium we expect that [14], which is the result of the renormalization of [11] to order $\varepsilon$, is exact. Eq. [14] applies to order $\varepsilon$ at the depinning transition too with the difference that in this case there are corrections to [14] of order $\varepsilon^2$. However, we expect that the latter will be small as it is the case for corrections of order $\varepsilon^2$ to the interface width distribution at the depinning transition [28]. The average energy $\langle E \rangle_R$ derived from Eq. [13] is

$$\langle E \rangle_R = \hat{E}_0 \sum_{j=1}^{\infty} \frac{1}{j(1 + j/\eta)} = \left[\Psi(\eta + 1) + C\right] \hat{E}_0 \approx \ln \eta + C \hat{E}_0 + O(1/\eta) \propto L \ln L, \quad (15)$$

where $\Psi(x)$ is the digamma function and $C = 0.5772...$ is the Euler’s constant. The energy fluctuation $\Delta E_R$ ob-

tained from [13] reads

$$\Delta E_R = |\hat{E}_0| \left[\frac{\pi^2}{6} + \Psi'(\eta + 1) - 2(C + \Psi(\eta + 1))/\eta\right]^{1/2} \approx \frac{\pi}{\sqrt{6}} |\hat{E}_0| + O(\ln \eta/\eta) \propto L. \quad (16)$$

The result, $\Delta E_R \propto L$, agrees with the estimate of the energy by using dimensionality arguments with correct roughness exponent $\zeta$. Notice that due to the logarithmic term in [10], $\langle E \rangle_R$ and $\Delta E_R$ scale in different way, and the relative fluctuation, $\Delta E_R/\langle E \rangle_R$, disappears as $1/\ln L$ for large $L$, which is in contrast to $1/\sqrt{L}$ behavior for a Gaussian distribution. The latter reflects the relevance of fluctuations over all length scales. The higher moments of the energy distribution [14] scale as $\langle (E - \langle E \rangle)^n \rangle \propto \Delta E_R^n$.

**FIG. 2:** The renormalized distribution of the energy for a line in a random field environment. Dashed line: $L/L_c = 10^2$; solid line: the Gumbel distribution.

We now will consider the asymptotic behavior of [14] in the limit of long lines, $L \gg L_c$. Changing $x$ in favor of $x - \ln \eta \equiv y$ and taking the limit $\eta \to \infty$ we calculate the sum over $j$ in [14] and arrive at

$$f_R(y) = P(y) \equiv \exp(-y - \exp(-y)), \quad (17)$$

which is nothing but the Gumbel distribution of the extreme value statistics [29]. The universality of $f_R(y)$ is due to the universal character of fluctuations on large scales, which are described by the fixed-point solution of the FRG [10]. Notice that the expectation value of $y$ calculated with [17] gives the Euler’s constant $C$ which is in consistence with Eq. [19] of the average energy. We have checked that the limit of the distribution $f_R(x; \eta)$ for $\eta \to \infty$ is insensitive to the details of the renormalization at scales smaller than the Larkin scale.

The Gumbel distribution is one of the three possible limit distributions in the extreme value statistics [29], which concern the distribution of the maximum $M_n = \max\{\xi_1, ..., \xi_n\}$ of the set of identically distributed random variables $\xi_i$ ($i = 1, 2, ..., n$). The asymptotic distribution $P_n(x)$ for $M_n$ in limit $n \to \infty$ does not depend on
detailed of the distribution of $\xi_i$ and under fulfilling some conditions \[24\] has the form $P_n(x) \simeq \mathcal{P}(x - \ln n)$ where $\mathcal{P}(y)$ is given by Eq. \[17\]. The combination $y = x - \ln n$, where $n$ is the number of random variables guarantees that the distribution remains invariant for $n \to \infty$.

Vinokur et al. \[16\] have used the Gumbel distribution to describe in a phenomenological way the energy barriers distribution for a flux line in a random environment. The creep motion of the flux line in the limit of small driving force $F$ and low temperature is controlled by thermally activated jumps. Thermally activated advance of the flux segment of length $L$ is controlled by the global barrier $U = \max\{U_1, \ldots, U_n\}$, where $U_i$ is the barrier for the subsegment $i$ of length $L_c$ with the number of subsegments $n = L/L_c$. It was suggested in \[10\] that the probability distribution for a given segment $L$ is $\mathcal{P}(U/L_c - \ln(L/L_c))$, where $U_c \simeq \gamma a^2 L_c^{-2}$ is the minimum average barrier between neighboring metastable positions of a pinned segment $L_c$, so that the typical barrier of a segment of length $L$ scales then as $U \propto U_c \ln(L/L_c)$.

Bouchaud and Mézard \[17\] showed that the Gumbel distribution describes the energy distribution in a class of random energy models possessing the one-step RSB. The Gumbel and related distributions were used in \[25\] to describe universal fluctuations in correlated systems. It was shown in \[31\] that the Gumbel distribution appears in systems with $1/f$ power spectra.

The relation of the energy distribution with the extreme value statistics can be illustrated as follows. It is known from the treatment of the problem in the framework of the replica variational approach \[11\] that the system under consideration demonstrates RSB \[32\]. The RSB is related to the existence of many metastable states and to the loss of the ergodicity of the system. On the contrary, the perturbational result \[11\] corresponds to the average over states which belong to different nonergodic subsystems. The metastable states, which correspond to the minima of the total energy of the interface, are rare states among other states of the energy landscape, and has to reflect the statistics of these extreme states \[16\]. It was shown in \[21\] that energy minima behave in the same way as barriers, which as suggested in \[16\] are distributed in accordance with the Gumbel distribution.

In conclusion, we have studied the distribution function of the elastic, disorder, and the total energies of an interface in a random field environment by summing the leading terms of the perturbation expansion in powers of the disorder. The nonperturbational effects of the disorder are taken into account using the FRG method. We have found that the average and the fluctuation of the energy for one-dimensional interfaces behaves as, $\langle E \rangle_R \propto L \ln L$, $\Delta E_R \propto L$, while the energy DF tends for large $L$ to a universal function which coincides with the Gumbel distribution.

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\[29\] The RSB solution \[11\] gives the same form of the height-height correlation function and the same roughness exponent $\gamma$ as that predicted by the FRG, and consequently the same energy distribution function.