Multiple phase transitions in ER edge-coupled interdependent networks

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Abstract
Considering the real-world scenarios that there are interactions between edges in different networks and each network has different topological structure and size, we introduce a model of interdependent networks with arbitrary edge-coupling strength, in which $q_A$ and $q_B$ are used to represent the edge-coupling strength of network $A$ and network $B$ respectively. A mathematical framework using generating functions is developed based on self-consistent probabilities approach, which is verified by computer simulations. In particular, we carry out this mathematical framework on the Erdös–Rényi edge-coupled interdependent networks to calculate the values of phase transition thresholds and the critical coupling strengths which distinguish different types of transitions. Moreover, as contrast to the corresponding node-coupled interdependent networks, we find that for edge-coupled interdependent networks the critical coupling strengths are smaller, and the critical thresholds as well, which means the robustness of partially edge-coupled interdependent networks is better than that of partially node-coupled interdependent networks. Furthermore, we find that network $A$ will have hybrid percolation behaviors as long as the coupling strength $q_A$ belongs to a certain range, and the range does not affected by average degree of network $A$. Our findings may fill the gap of understanding the robustness of edge-coupled interdependent networks with arbitrary coupling strength, and have significant meaning for network security design and optimization.

1. Introduction

Many systems in the real world can be described as complex networks, such as social networks, power grids, transportation networks, financial networks, urban networks, and epidemic spreading networks [1–5]. Networks in real world usually are not isolated. They interact with each other and are modeled as interdependent networks. For example, the communication network needs the power support of the power grid, and the operation of the power grid requires the dispatch of the communication network. These interactions make the interdependent networks more fragile than single network because failures of nodes or edges in one network may cause nodes or edges in other network lose their functions [6]. To analyze the robustness of interdependent networks, researchers generally use percolation theory [7–9]. Randomly removing a fraction $1 - p$ of nodes or edges may trigger cascading failures due to the dependency between networks. When the removing fraction is greater than $1 - p_c$, the interdependent networks will collapse. The robustness of interdependent networks will be affected by the specific structure of the network, such as degree distribution [10, 11], degree correlation [12–14], coupling mode [15–17], spatial embedding [18, 19], cluster configuration [20]. Therefore, many related works have been published in recent years. In addition, the extension of different percolation problems, such as L-hop percolation, Bootstrap percolation, K-core percolation [21–24], and so on, also have abstracted lots of attention.

In recent years, researchers have proposed a variety of interdependent networks models [25–33]. Parshani [15] et al studied a partially interdependent networks model, in which only a fraction of nodes in...
one network depends on nodes in other network. They found that reducing the coupling strength will increase the robustness of the system. Yuan [25] et al studied a model of interdependent networks where a small fraction of nodes in this system can maintain their functions even if they are not in the giant component. It is found that reinforced a small fraction of nodes can eradicate catastrophic collapse. Kong [28] et al discussed a heterogeneous weakly interdependent networks model, in which the failure of a node will not cause the complete failure of the dependent nodes, but only the loss of part of the connection edges. These works mostly focus on the node-coupled model, where nodes in one layer are interdependent with nodes in other layers. However, in many real-world networks, interactions happen on the edges connecting the nodes rather than on the nodes. For instance, when a route in a transportation network is blocked, the regular supply between companies in a trade network is influenced, and the interruption of the supply relationship between companies may weak the stable demand and operation of the route in the transportation network. In order to describe such interdependent networks, Gao [34] et al proposed an edge-coupled interdependent networks model. This work only considered the special case of simplified one-to-one fully interdependent networks. However, this is a very realistic scenario considering that part of edges in real network is ‘coupled’ with part of edges in other real network because the two layers unnecessarily have identical sets of edges or nodes. For example, trade networks and transportation networks have different structures and sizes, and not all transactions need the support of transportation routes, and vice versa.

In this work, we introduce a model of partially edge-coupled interdependent networks under this realistic condition and develop a relatively simpler mathematical framework using generating functions based on self-consistent probabilities approach. For three different types of partially edge-coupled interdependent networks, numerical solutions and computer simulations results show excellent agreement, which verifies our mathematical framework. Similar work in reference [15] studied a partially node-coupled interdependent network model and found that reducing the coupling between the networks leads to a change from a first order percolation phase transition to a second order percolation transition at a critical point. As contrast to the results in reference [15], we find that the critical coupling strengths for edge-coupled interdependent networks are smaller than that for the corresponding node-coupled interdependent networks, and the critical thresholds as well, which means the robustness of partially edge-coupled interdependent networks is better than that of partially node-coupled interdependent networks. Furthermore, we find that network A will have hybrid percolation behavior as long as the coupling strength \(q_A\) belongs to a certain range, and the range does not change along with the average degree of network A.

The remainder of the paper is structured as follows. In section 2, we propose a partially edge-coupled interdependent networks model. In section 3, we give the mathematical framework of partially edge-coupled interdependent networks. In section 4, we discuss the phase transition of Erdős–Rényi (ER) partially edge-coupled interdependent networks. Conclusions are provided in section 5.

2. Model

The partially edge-coupled interdependent networks consist of two networks A and B. Their degree distributions are \(P_A(k)\) and \(P_B(k)\) respectively. Network A is composed of \(N_A\) nodes and \(E_A\) edges, among which \(E'_A\) edges have one-to-one interdependency relationship which edges in network B. Similarly, network B is composed of \(N_B\) nodes and \(E_B\) edges, among which \(E'_B\) edges depend on edges in network A following one to one corresponding pattern. We define \(q_A = \frac{E'_A}{E_A}\) and \(q_B = \frac{E'_B}{E_B}\) as the coupling strengths of network A and network B respectively. The functions of autonomous nodes and edges in the two-layer networks are maintained by their own networks. We observe the size of giant component in the system by randomly removing a fraction \(1 - p\) of edges in network A. Figure 1 shows the cascading failures process of partially edge-coupled interdependent networks. It can be seen that the phase transition behaviors of network A and network B will be different due to the existence of autonomous edges. This article defines that if one of the two-layer networks shows a first-order phase transition, the phase transition behavior of the system is first-order. If both layers of the network show a second-order phase transition, then the phase transition behavior of the system is second-order.

3. Mathematical framework

According to the theory of generating function, the generating function of degree distribution and the generating function of the underlying branching processes for a single network are \(G_{\text{net}}(x) = \sum_k P(k)x^k\) and \(G_{\text{net}}(x) = \sum_k P(k)kx^{k-1}/\langle k \rangle\) respectively, here \(P(k)\) is the degree distribution of the network, and \(\langle k \rangle\)
represents the average degree of the network. We define \( f \) as the probability of reaching the giant component in one direction along a randomly selected edge. The probability that a node with degree \( k \) is \( P(k) \), the probability that it belongs to the giant component is \( 1 - (1 - f)^k \). Therefore the probability that a randomly selected node with degree \( k \) belongs to the giant component is \( \sum_k P(k) [1 - (1 - f)^k] \). After randomly removing nodes in the network for a fraction of \( 1 - p \), the probability that a randomly selected node belongs to the giant component is [35]:

\[
u_\infty = p \cdot \sum_k P(k) [1 - (1 - f)^k] = p \cdot [1 - G_0(1 - f)].
\] (1)

In order to solve \( u_\infty \), we also need the equation of \( f \). According to the definition of \( f \), if a node with degree \( k \) reached by one edge in the network belongs to the giant component, at least one of the remaining \( k - 1 \) edges of this node leads to the giant component. This event happened with the probability \( 1 - (1 - f)^{k-1} \), and the probability that this node has degree \( k \) is \( \frac{\langle k \rangle}{k} \). After randomly removing a fraction \( 1 - p \) of nodes in the network, the following self-consistent equation is satisfied:

\[
f = p \cdot \sum_k \frac{P(k)}{\langle k \rangle} [1 - (1 - f)^{k-1}] = p \cdot [1 - G_i(1 - f)].
\] (2)

Different from traditional node-coupled interdependent networks, what we want to study is the partially edge-coupled interdependent networks model. Before discussing this model, we first discuss the fully edge-coupled interdependent network. Gao [34] et al derived the percolation equations of the fully edge-coupled interdependent networks based on self-consistent probabilities method. Following this method, we also define \( u_i \) as the size of the giant component of network \( i \), and use \( f_i \) to represent the probability of reaching the giant component in one direction along a randomly selected edge. After randomly removing a fraction \( 1 - p \) of edges in the network \( A \), the percolation equations of fully edge-coupled interdependent networks can be expressed as follows by the generating function [34]:

\[
\begin{aligned}
u_\infty^A &= 1 - G_{iA}(1 - f_i) \\
u_\infty^B &= 1 - G_{iB}(1 - f_i) \\
f_i &= p \cdot [1 - G_{iA}(1 - f_i)] \cdot [1 - G_{iB}(1 - f_i)]^2 \\
f_i &= p \cdot [1 - G_{iB}(1 - f_i)] \cdot [1 - G_{iA}(1 - f_i)]^2,
\end{aligned}
\] (3)

where \( 1 - G_{iA}(1 - f_i) \) represents the survival probability of a randomly selected edge \( e_A \) in network \( A \), and \( 1 - G_{iB}(1 - f_i) \) represents the survival probability of the corresponding dependent edge of \( e_A \) in network \( B \).

Based on the above framework, we will discuss our partially edge-coupled interdependent networks, where \( q_A \) edges in network \( A \) depends on \( q_B \) edges of network \( B \) and \( q_B \) edges in network \( B \) depends on \( q_A \).
the giant component is randomly selected in network \(G\). Numerical results obtained by solving equations (4)–(7) for different networks, which verified our above following condition:

\[
\sum_{k=\text{min}}^{\text{max}} \frac{\partial f_B}{\partial f_A} = 1
\]

The next important step is to derive the self-consistent equations of \(f_A\) and \(f_B\). As we known, \(f_A\) needs to meet the conditions of intralayer and interlayer. Since network \(A\) has autonomous edges with a fraction of \(1 - q_A\), the probability that an edge is randomly selected to depend on network \(B\) is \(q_A\). When edge \(e_A\) is randomly selected in network \(A\), there are two cases. First case is only considering the intralayer conditions for autonomous edge, the probability that this edge has not been initially deleted and leads to the giant component is \(p \cdot (1 - G_{A1}(1 - f_A))(1 - q_A)\). Second, if edge \(e_A^{'}\) is non-autonomous edge which has dependent edge \(e_B^{'}\) in network \(B\), the conditions in the intralayer are the same as the autonomous edge, and the interlayer condition is that \(e_B^{'}\) in network \(B\) also survive, so the survival probability of a non-autonomous edge is 

\[
\sum_{k=\text{min}}^{\text{max}} \frac{\partial f_B}{\partial f_A} = 1
\]

Similarly, for network \(B\), there are autonomous edges with a fraction of \(1 - q_B\) in network \(B\). The probability that an edge of network \(B\) is randomly selected to depend on network \(A\) is \(q_B\). The conditions intralayer and interlayer must also be met. The probability that a randomly selected autonomous edge belongs to the giant component is \(1 - G_{B1}(1 - f_B)\). The survival probability of a randomly selected the non-autonomous edge is \(G_{B1}(1 - f_B)\). So that \(f_B\) can be expressed as:

\[
\sum_{k=\text{min}}^{\text{max}} \frac{\partial f_A}{\partial f_B} = 1
\]

The generating function of the degree distribution for random regular (RR) networks is \(G_0(x) = x^{\langle k\rangle}\), analogously, the generating function of the underlying branching processes is \(G_1(x) = x^{\langle k\rangle - 1}\). Substituting them into equations (4)–(7), we can plot the giant component \(u_{\infty}\) of RR interdependent networks as a function of \(p\), as shown in figure 2. For ER networks, \(G_0(x) = G_1(x) = e^{-\langle k\rangle (1-x)}\).

In the case of scale free (SF) networks, \(G_0(x) = \sum_{k=\text{min}}^{\text{max}} \frac{\langle k\rangle!}{(\langle k\rangle + 1)!} \cdot x^k\), where, \(G_1(x) = \sum_{k=\text{min}}^{\text{max}} k \cdot \left(\frac{\langle k\rangle}{k+1}\right)^{\langle k\rangle - 1} \cdot x^k / k\).

Figure 2 shows the excellent agreement between computer simulations of the cascade failures and the numerical results obtained by solving equations (4)–(7) for different networks, which verified our above mathematical framework.

Theoretically, equation (7) can be expressed as \(f_A = F_1(p, f_B, q_B)\), and equation (6) can be expressed as \(f_B = F_2(p, f_A, q_A)\). If the system has first-order phase transition, the phase transition point satisfies the following condition:

\[
\frac{\partial F_1(p, f_B, q_B)}{\partial f_B} \cdot \frac{\partial F_2(p, f_A, q_A)}{\partial f_A} = 1
\]
where $p^V_i$ represents the first-order phase transition threshold and can be obtained by solving the equation (8). When a continuous phase transition occurs in the system, second-order phase transition threshold $p^V_{\bar{c}}$ can be solved by satisfying equations (6) and (7) and the condition $f_A = 0$ or $f_B = 0$.

As well known, reducing the coupling strength $q_A$ or $q_B$ leads to a change from a first-order percolation phase transition to a second-order percolation transition at a critical tipping point corresponding to critical coupling strength $q_{A_c}$ and $q_{B_c}$ respectively. Next, taking ER networks as an example, we will analyse the critical coupling strengths and percolation thresholds.

4. Phase transition in ER edge-coupled interdependent networks

For ER partially edge-coupled interdependent networks, the average degrees of network $A$ and network $B$ are $\langle k_A \rangle$ and $\langle k_B \rangle$ respectively. The generating function of degree distribution and the generating function of the underlying branching processes can be expressed as:

$$G_{A0}(f_A) = G_{A1}(f_A) = e^{-\langle k_A \rangle(1-f_A)}$$

$$G_{B0}(f_B) = G_{B1}(f_B) = e^{-\langle k_B \rangle(1-f_B)}.$$  

Substituting equations (9) and (10) into equations (4)–(7), we can obtain:

$$u^A_c = 1 - e^{-\langle k_A \rangle f_A}$$

$$u^B_c = 1 - e^{-\langle k_B \rangle f_B}$$

$$f_A = p \cdot (1 - e^{-\langle k_A \rangle f_A}) \cdot [(1 - q_A) + q_A \cdot (1 - e^{-2\langle k_B \rangle f_B})]$$

$$f_B = (1 - e^{-\langle k_B \rangle f_B}) \cdot [(1 - q_B) + p \cdot q_B \cdot (1 - e^{-2\langle k_A \rangle f_A})].$$

Equation (13) can be reexpressed as follows:

$$f_B = F_3(p, f_A, q_A) = \frac{1}{2\langle k_B \rangle} \log \left[ \frac{1}{q_A} - \frac{f_A}{pq_A(1 - e^{-\langle k_A \rangle f_A})} \right]$$

Similarly, equation (14) can be reexpressed as:

$$f_B = F_4(p, f_B, q_B) = \frac{1}{2\langle k_A \rangle} \log \left[ 1 + \frac{1 - q_B}{pq_B(1 - e^{-\langle k_B \rangle f_B})} \right].$$

When the system shows first-order phase transition, the following conditions need to be satisfied:

$$\frac{\partial F_3(p, f_A, q_A)}{\partial f_A} \cdot \frac{\partial F_4(p, f_B, q_B)}{\partial f_B} = 1.$$  

When the system shows second-order phase transition, $p^V_{\bar{c}}$ can be solved by satisfying equations (13) and (14) and the condition $f_A = 0$ or $f_B = 0$.

4.1. Analysis and discussion of percolation threshold and critical tipping point

In this part, we will show how to get the critical coupling strength $q_{A_c}$ and percolation threshold $p_c$ for a given $q_B$.

At the critical tipping point where the system changes from first-order phase transition to second-order phase transition, equations (15) and (16) have three different types of solutions shown in below. In order to simplify the calculation, we assume that $\langle k_A \rangle = \langle k_B \rangle = \langle k \rangle$.

(i) The giant components of network $A$ and network $B$ are zero ($f_A = 0$ and $f_B = 0$), as showed in figure 3(a). The phase transition critical point parameters can be obtained from the condition (17) and $f_A = 0, f_B = 0$ as follows:

$$q_{B_c} = 1 - \frac{1}{\langle k \rangle}$$

$$q_{A_c} = 1 + \frac{4\sqrt{q_{B_c}}(1 - q_{B_c})}{(q_{B_c} - 1)\sqrt{4q_{B_c} + 1} + \sqrt{\sqrt{4q_{B_c} - 1} - 2}}$$
Figure 3. Illustrations of the different graphical solutions of equations (15)–(17).

Figure 4. (a) Plot of coupling strength critical points $q_{Ac}$ as functions of coupling strength $q_{Bc}$ for different values of $\langle k \rangle$. (b) Plot of percolation threshold points $p_c$ as functions of coupling strength $q_{Bc}$ for different values of $\langle k \rangle$. $p_c = -\frac{(q_{Bc} - 1)\sqrt{3q_{Bc}} + 1 + \sqrt{(2q_{Bc} - 2)}}{4\sqrt{q_{Bc}}}$. (20)

For this case, critical coupling strength $q_{Ac}$ and percolation $p_c$ as function of $q_{Bc}$ for different $\langle k \rangle$ are plotted by the dotted lines in figures 4(a) and (b) respectively.

(ii) The giant component of network $A$ is zero, and the giant component of network $B$ is nonzero ($f_A = 0$ and $f_B > 0$), as showed in figure 3(b). Given $q_{Bc} < 1 - \frac{1}{\langle k \rangle}$, the relationship $1 - q_{Bc} = \frac{f_B}{1 - e^{-q_{Bc}}}$ can be obtained through equation (16). Solving the numerical solution of $f_B$ and substitute it into equation (17), we can find the values of $q_{Ac}$ and $p_c$. For this case, $q_{Ac}$ and $p_c$ as functions of $q_{Bc}$ for different $\langle k \rangle$ are plotted by the red lines in figures 4(a) and (b) respectively.

(iii) The giant component of network $A$ is nonzero, and the giant component of network $B$ is zero ($f_A > 0$ and $f_B = 0$), as showed in figure 3(c). Given $q_{Bc} > 1 - \frac{1}{\langle k \rangle}$, $q_{Ac}$ and $p_c$ can be obtained from equations (15)–(17). For this case, the relationship diagrams of $q_{Ac}$ and $p_c$ with respect to $q_{Bc}$ for different $\langle k \rangle$ are plotted by the black lines in figures 4(a) and (b) respectively.

As can be seen from figure 4(a), when $q_{Bc} = 1$, networks with different average degrees have the same critical coupling strength $q_{Ac} = 0.1074$, which is smaller than that ($q_{Ac} = 0.20794$ found in reference [15]) of the partially node-coupled interdependent networks. In addition, figure 4(b) shows the relationship between $p_c$ and $q_{Bc}$ of the system under different average degrees. Referring to figure 4 in [15], it can be seen that the phase transition thresholds of the partially edge-coupled interdependent networks is smaller than that of the partially node-coupled interdependent networks, which indicate that the robustness of the partially edge-coupled interdependent networks is better than that of corresponding partially node-coupled interdependent networks.
Figure 5. Simulation results for ER interdependent networks under the conditions of \( N_A = 200,000, \langle k \rangle = 4, q_B = 1, q_A = 0.25 \). The solid line represents numerical results, and the hollow marker represents computer simulations.

Figure 6. Plot of \( p_c \) as a function of \( q_A \) for different \( \langle k \rangle \) in the case of \( q_B = 1 \). The vertical dotted lines represent the boundaries of different phase transition types.

4.2. Discussion of hybrid phase transitions

When observing the phase transition behaviors of the system, under certain conditions, although it is a first-order phase transition for the entire system, one may observe a hybrid phase transition for network A. For example, in the case of \( \langle k \rangle = \langle k_A \rangle = \langle k_B \rangle = 4, N_A = 200,000, q_B = 1, q_A = 0.25 \), we plot giant components \( u_A^\infty \) and \( u_B^\infty \) as functions of \( p \) respectively in figure 5. The figure shows a phenomenon of hybrid phase transition. The giant component of the network A abruptly changes to a relatively small value due to an iterative process of cascading failures, and then continuously approaches zero.

To discuss the hybrid phase transition more specifically, for the case \( q_B = 1 \) we plot \( p_c \) as function of \( q_A \) for different \( \langle k \rangle \) in figure 6. The vertical dotted lines represent the boundaries of different phase transition types. With the decrease of \( q_A \) from 1 to 0, the phase transition behaviors of network A are divided into three different regions. When \( q_A > q_{A_{\text{cl}}} = 0.37 \), the network A presents a first-order phase transition, when \( 0.1074 = q_{A_{\text{cl}}} < q_A < q_{A_{\text{cl}}} = 0.37 \), it shows the characteristic of hybrid phase transition, and when \( q_A < q_{A_{\text{cl}}} = 0.1074 \), the network A presents a second-order phase transition. In addition, it can be seen from figure 6 that \( q_{A_{\text{cl}}} \) and \( q_{A_{\text{cl}}} \) do not affected by average degree \( \langle k \rangle \).

Further analysis shows that network A will have hybrid phase transitions when \( q_B > 1 - \frac{1}{\langle k \rangle} \) and \( q_{A_{\text{cl}}} > q_A > q_{A_{\text{cl}}} \), in which \( q_{A_{\text{cl}}} \) is the critical parameter \( q_A \) for the above phase transition type (iii). Substituting \( q_{A_{\text{cl}}} \) and \( \langle k \rangle \) to equations (13)–(15) and condition \( q_A = 1 - \frac{1}{\langle k \rangle p} \), we can solve \( q_{A_{\text{cl}}} \).

5. Conclusion

In real scenarios, interactions happen on the edges connecting the nodes rather than on the nodes. We therefore introduce an edge-coupled interdependent networks model with arbitrary coupling strength that is applicable to many real networks. In this model, the coupling strengths of network A and network B are expressed by \( q_A \) and \( q_B \) respectively. To analyze the phase transition behaviors of the system, we develop a
mathematical framework for the partially edge-coupled interdependent networks using generation functions based on self-consistent probability approach. Especially for the ER partially edge-coupled interdependent networks, we have carried out computer simulations and numerical solutions to verify our mathematical framework. Moreover, we analyze the tipping points and get the corresponding critical coupling strengths. Compared with the partially node-coupled interdependent networks in the same coupling strength case, it is found that the partially edge-coupled interdependent networks is more robust, and the critical coupling strength is smaller than that of the partially node-coupled interdependent networks. In addition, we find that network \( A \) will have hybrid percolation behaviors as long as the coupling strength \( q_A \) belongs to a certain range, and the range does not change along with the average degree of network \( A \). This paper only discusses the edge-coupled interdependent networks with arbitrary coupling strengths. Actually, the edge-based coupling modes have different types, such as multiple-dependence relation, correlated coupling, and so on. In addition, in many real settings displaying edge interdependency, the networks are likely to be spatially embedded. Would spatial embeddedness be expected to further increase or decrease the robustness of an edge-based coupling system? Future works need to be done to extend the understanding of the edge-coupled interdependent networks.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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