A HIGHER ORDER WEIERSTRASS APPROXIMATION THEOREM – A NEW PROOF

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Abstract. The theorem studied is known before. Here is given a new proof. The proof has been part of a course material on Sobolev space theory with a special kind of outlook. The proof here is in accordance to this goal. At the same time several ideas of interest are shown that can be of general use.

Introduction

The definition of Sobolev space used here is for reasons to obvious later on that of the closure in the appropriate Sobolev space norm of the set of functions with $m$ times continuous derivatives on some open subset of $N$-dimensional Euclidean space.

The Weierstrass approximation theorem is stated below and concerns approximation of continuous functions by polynomials. The higher order Weierstrass approximation theorem concerns approximation of $m$ times continuously differentiable functions by polynomials in such a way that all derivatives up to order $m$ are approximated simultaneously. The theorem is stated below. The theorem is previously known. Here is given a very interesting proof that makes use of several ideas. These and their interplay displayed is the real goal! That a straightforward proof has less complexity is beside the point here.

The theorem holds for a cube as domain and with the open bounded domain denoted $\Omega$ in $\mathbb{R}^N$, then, if the domain allows the Sobolev functions considered to be extended to the full space, then the theorem follows for this domain as well. The argument is simply that take cube enclosing the domain and the do the extension of the function to full space and then restrict to the cube. Then use the approximation by polynom for the cube as in the theorem. Conclude by taking the restriction of the polynomial to the domain.

A very good extension theorem is that of P.W. Jones [1]. The domains allowing extension by his method are called uniform domains or also by the name $(\epsilon, \delta)$-domains. Domains with self-similar fractal boundary are simple examples of uniform domains. There is a point here that the Jones extension theorem also allows for $p = \infty$ as well.

A consequence is that polynomials are dense in integer order Sobolev spaces with $1 \leq p \leq \infty$ when the domain is bounded and is a so called uniform domain.
Theorem and Proof

Definition: Let $\Omega$ be open in $\mathbb{R}^N$. For $u$ a “nice” real function on $\Omega$ the Sobolev space norm is

$$||u||_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)} \sim \sum_{|\alpha| \leq m} (||D^\alpha u||^p_{L^p(\Omega)})^{\frac{1}{p}}.$$ 

Definition: Let $\Omega$ be open in $\mathbb{R}^N$. Define $C^m(\Omega)$ as the $m$ times continuously differentiable real-valued functions on $\Omega$.

Notation: We use $\nabla^k u$, the $k$-gradient, in integral expressions as follows

$$||\nabla^k u||_{L^p(\Omega)} = \sum_{|\alpha| = k} ||D^\alpha u||_{L^p(\Omega)}.$$ 

Notation: Let $\mathcal{P}$ be all polynomials in $\mathbb{R}^N$ and $\mathcal{P}_k = \{P \in \mathcal{P} : \nabla^{k+1} P = 0\}$.

Then some easy results without proofs.

Theorem: A Poincaré inequality, dim general, order one and general $p$.

Let $1 \leq p \leq \infty$ and $Q = [0, 1]^N$. Let $u \in C^0(Q)$ and $D_1 u \in C^0(Q)$. Furthermore $u|_{x_1 = 0} = 0$. Then

$$||u||_{L^p(Q)} \leq ||D_1 u||_{L^p(Q)}.$$ 

So, here $Q$ is a unit cube and $D_1$ is the derivative in the $x_1$-direction.

Corollary: The Standard Poincaré Inequality.

Let $u \in C^m_0(\Omega)$ for $\Omega$ bounded, open subset of $\mathbb{R}^N$ and $|\alpha| < m$. Let $A = A_{N,m,p,\Omega}$,

$$\sum_{|\alpha| < m} ||D^\alpha u||_{L^p(\Omega)} \leq A ||\nabla^m u||_{L^p(\Omega)}.$$ 

Corollary: A more detailed Poincaré inequality.

Let $Q = [0, 1]^N$ and let $\alpha$ and $\beta$ be multi-indices that are partially ordered by “$<$” in the natural way. Let $|\alpha| = t$, $|\beta| = m$ and $\alpha < \beta$. Assume that multi-indices $\{\gamma_k\}_1^m$ exist such that $\gamma_t = \alpha$ and $\gamma_m = \beta$ and also $\gamma_t < \gamma_{t+1} < \cdots < \gamma_{m-1} < \gamma_m$.

Let $u \in C^m(\overline{Q})$ and

$$(D^{\gamma_k} u|_{x_{\gamma_k-\gamma_k-1} = 0}) = 0$$

or

$$(D^{\gamma_k} u|_{x_{\gamma_k-\gamma_k-1} = 1}) = 0.$$ 

Then for $A = A_{N,m,t,p,Q}$,

$$||D^\alpha u||_{L^p(\Omega)} \leq A ||D^\beta u||_{L^p(\Omega)}.$$
Theorem: (Weierstrass Approximation Theorem.)

Let $\Omega \subset \mathbb{R}^N$ be connected, open and bounded. Then $\mathcal{P}$ is dense in $L^\infty$-norm in $C^0(\bar{\Omega})$.

This will be a tool for proving a higher order version.

The proof here is new.

Theorem: A Higher Order Weierstrass Approximation theorem.

Let $m > 0$, integer, and let $Q = [0, 1]^N$. Then $\mathcal{P}$ is dense in $W^{m,\infty}$-norm in $C^m(\bar{Q})$.

The difficulty with the use of Weierstrass approximation theorem here is that there are many partial derivatives. They shall all be approximated by the partial derivatives of only one polynomial. – A trick (=method) is called for.

First an observation.

Notation: Let $W^{m,p}(\bar{\Omega})$, where $\Omega$ is an open subset of $\mathbb{R}^N$, be the closure of $C^m(\bar{\Omega})$ in the $W^{m,p}(\Omega)$-norm.

Theorem: Let $Q = (0, 1)^N$ then $W^{m,p}(\bar{Q}) = W^{m,p}(Q)$.

Proof: Obviously $W^{m,p}(\bar{Q}) \subset W^{m,p}(Q)$. Hence it is enough to prove the other inclusion.

Let $u \in W^{m,p}(Q)$. Then scale=dilate the variables with a factor $1 - \epsilon$. Then $u$ is transformed into $\tilde{u}$ and $\tilde{u} \in W^{m,p}(Q_{1/1-\epsilon})$. Take restriction $\tilde{u}|_Q$. Clearly $\tilde{u}|_Q \in W^{m,p}(\bar{Q})$. Let $\epsilon$ take values $\{n^{-1}\}$, then $\{\tilde{u}_n\}$ a sequence which tends to $u$ in the norm of $W^{m,p}(Q)$.

End of Proof

Proof: (Wannebo) The higher order Weierstrass approximation theorem.

This proof can regarded as good training ground for the ideas discussed so far. It is not really meant for memorizing.

The key to get started with the proof is to change the setup and study another space.

We choose the space $u \in C^{Nm}(\bar{Q})$ instead of $C^m(\bar{Q})$.

Fix the special partial derivative $D^\beta = \prod_{i=1}^N D_i^m$. This way the multiindex $\beta$ is defined.

Define the set $S$ as the set of $2^{Nm}$ elements, each defined by how it acts on the monomial $x^\beta$. All transformations, which to each one–degree factor in $x^\beta$, say the factor $x_j$, substitute with either $x_j$ or $1 - x_j$ in this case. The set $S$ is all possible combinations of such transformations. We write $S = \{\sigma\}$

Identity:

$$1 = \sum \sigma(x^\beta)$$
– Check!

Let \( u \in C^m(\bar{Q}) \). Then by the Weierstrass approximation theorem, given \( \epsilon > 0 \), there is a polynomial \( Q_\sigma \), such that

\[
||D^\beta[\sigma(x^\beta)u] - Q_\sigma||_{L^\infty(Q)} < \epsilon.
\]

Next step is to find a polynomial \( P_\sigma \) such that

\[
||D^\beta[\sigma(x^\beta)(u - P_\sigma)]||_{L^\infty(Q)} < \epsilon.
\]

In order to prove this observe that the equation

\[
D^\beta[\sigma(x^\beta)P_\sigma] - Q_\sigma = 0
\]

is solvable for \( P_\sigma \) for every \( Q_\sigma \). The equation is linear and it is enough to solve for \( P_\sigma \) when \( Q_\sigma = x^\gamma \) any \( \gamma \). Observe that

\[
D^\beta[\sigma(x^\beta)x^\gamma] = const \cdot x^\gamma + \text{lower order terms}.
\]

Hence it is possible by iteration to find \( P_\sigma \). Solve with higher orders monomials first subtract, iterate. Since the equation is linear the general \( Q_\sigma \) has a \( P_\sigma \) solution.

We are interested in the case \( |\alpha| \leq m \) and then it follows automatically that \( \alpha < \beta \). This is the reason for studying \( C^m(\bar{Q}) \) instead of \( C^m(\bar{Q}) \).

Collect the results so far and use the Identity. Then

\[
||D^\alpha[u - \sum_\sigma \sigma(x^\beta)P_\sigma]||_{L^\infty(Q)} = ||D^\alpha[\sum_\sigma \sigma(x^\beta)(u - P_\sigma)]||_{L^\infty(Q)} =
\]

\[
= ||\sum_\sigma D^\alpha[\sigma(x^\beta)(u - P_\sigma)]||_{L^\infty(Q)} \leq \sum_\sigma ||D^\alpha[\sigma(x^\beta)(u - P_\sigma)]||_{L^\infty(Q)} \leq
\]

Use the more detailed Poincaré inequality (the Corollary)

\[
\leq \sum_\sigma ||D^\beta[\sigma(x^\beta)(u - P_\sigma)]||_{L^\infty(Q)} \leq \text{const.} \cdot \epsilon.
\]

This proves that \( \mathcal{P} \) is dense in \( C^m(\bar{Q}) \) in the \( W^{m,\infty}(Q) \)-norm.

In order to finish the proof, it only remains to prove that \( C^m(\bar{Q}) \) is dense in \( C^m(\bar{Q}) \) in the \( W^{m,\infty}(Q) \)-norm.

Hence let \( u \in C^m(\bar{Q}) \), general. The space is complete in the \( W^{m,\infty}(Q) \)-norm. It is only needed now to find a Cauchy sequence

\[
\{v_n\}_1^\infty \in C^m(\bar{Q})
\]

which converges in \( W^{m,\infty}(Q) \)-norm to \( u \). As simplification, assume that \( Q \) has centre at the origin and side equal to 1.

Make the coordinate transformation

\[ x \to (1 - 1/(n + 1))x, \text{ with } n \text{ positive integer. We have the mapping } Q \to Q_n, \]

where \( Q_n \) has side \( 1 + 1/n \). There is some convolution kernel with radial symmetry, \( \Psi(r) \), which has support in a ball with radius 1 and with \( \Psi \in C^{m-N} \). Let

\[ \Psi_n(x) = \Lambda^N \Psi(\lambda x). \]
– a standard transformation. Let $\lambda_n \to \infty$ in fast enough. Now convolutions has such properties with respect to differentiation that

$$v_n = \Psi_n \ast u_n|_Q \in C^m(\bar{Q}).$$

But $C^m(\bar{Q})$ gives uniform continuity for the derivatives up to order $m$ so it follows (Check!) that $\{v_n\}$ is a Cauchy sequence with limit $u$.

End of Proof.

References

[1] P.W. Jones, *Quasiconformal mappings and and extendability of functions in Sobolev spaces*, Acta Math. 147 (1981), 71-88.