Note on the Fusion Map and Hopf Algebras

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Abstract

We discuss an inversion property of the fusion map associated to many semibialgebras. Please note that a characterisation of Hopf $k$-algebras has been added at the end of this version.

Let $C = (C, \otimes, I, c)$ be a symmetric (or just braided) monoidal category. A Von Neumann “core” in $C$ is firstly a semibialgebra in $C$, that is, an object $A$ in $C$ with an associative multiplication:

$$\mu : A \otimes A \rightarrow A$$

$$(\mu_3 = \mu(1 \otimes \mu) = \mu(\mu \otimes 1) : A \otimes A \otimes A \rightarrow A)$$

and a coassociative comultiplication:

$$\delta : A \rightarrow A \otimes A$$

$$(\delta_3 = (1 \otimes \delta)\delta = (\delta \otimes 1)\delta : A \rightarrow A \otimes A \otimes A)$$

such that:

$$\delta \mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) : A \otimes A \rightarrow A \otimes A$$

It is also equipped with an endomorphism

$$S : A \rightarrow A$$

in $C$ such that:

$$\mu_3(1 \otimes S \otimes 1)\delta_3 = 1 : A \rightarrow A$$

The name “Von Neumann core” stems partly from the notion of a Von Neumann regular semigroup, which is then precisely a VN-core in Set, while the free vector space on it is a particular type of VN-core in Vect, and partly from the properties of the paths which generate a (row-finite) graph algebra[5].

The fusion map

$$f = (1 \otimes \mu)(\delta \otimes 1) : A \otimes A \rightarrow A \otimes A$$

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then satisfies the fusion equation by the semibialgebra axiom of $A$ (see [6]), and if we set:
\[ g = (1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1) \]
as a tentative "inverse" to $f$, then we get the following (partial) results:

**Proposition 1.** [3] $fgf = f$ for any VN-core.

*Proof.* Define the (left) Fourier transform $l(\alpha)$ of a map $\alpha : A \rightarrow B$ to be the composite
\[ A \otimes B \xrightarrow{\delta \otimes 1} A \otimes A \otimes B \xrightarrow{1 \otimes \alpha \otimes 1} A \otimes B \otimes B \xrightarrow{1 \otimes \mu} A \otimes B \]
Then $l(\alpha * \beta) = l(\alpha)l(\beta)$, where $*$ is the convolution $\alpha * \beta = \mu(\alpha \otimes \beta)\delta$ of two maps $\alpha$ and $\beta$ from $A$ to $B$. Thus:
\[
fgf = l(1)l(S)l(1) \\
= l(1*S*1) \\
= l(1) \\
= f
\]
since $1*S*1 = 1$ by the definition of VN-core. \[\square\]

**Proposition 2.** $fgg = g$ if $S^2 = 1$ and $S$ is either an antipode or a coantipode.

The proof is straightforward.

Recall that a VN-core is called "unital" [1] if it satisfies the (stronger) axiom
\[ 1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_1 : A \rightarrow A \otimes A \]
where $A$ is assumed to have the unit $\eta : I \rightarrow A$. (A unital VN-core in $\mathcal{C} = \text{Set}$ is precisely a group).

**Proposition 3.** [1] $gf = 1$ for any unital VN-core.

Note that, in general, if for a map $f$ there exists a map $g$ with $fgf = f$, then we can always find a map $h$ with $fhf = f$ and $hfh = h$ provided idempotents split in $\mathcal{C}$.

A semialgebra is called a very weak bialgebra in [1] if it also has both a unit $\eta : I \rightarrow A$ ($\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1$) and a counit $\epsilon : A \rightarrow I$ ($((1 \otimes c)\delta = (c \otimes 1)\delta = 1$). A very weak bialgebra $A$ is then called a very weak Hopf algebra if it is equipped with a map $S : A \rightarrow A$ satisfying the axioms:
\[
\mu(S \otimes 1)\delta = t := (1 \otimes \varepsilon\mu)(c \otimes 1)(1 \otimes \delta\eta) \\
\mu(1 \otimes S)\delta = r := (\varepsilon\mu \otimes 1)(1 \otimes c^{-1})(\delta\eta \otimes 1) \\
\mu_3(S \otimes 1 \otimes S)\delta_3 = S
\]

Hence $S*1*S = S$ so that $fgg = g$ and, as a consequence of the semialgebra axiom, we have $1*t = 1$ (see [4]) whence $1*S*1 = 1$ so that $fgf = f$ (using $S*1 = t$ by the first axiom).
Remark 1. There should be some form of reconstruction theorem for VN-cores, involving bimonoidal functors \((U, r, r_0, i, i_0)\) for which \(ri = 1\) (cf. [1]).

**Example:** Suppose that \((A, \mu, \delta, \eta, \epsilon)\) is a bialgebra for which \(\delta\) is not known to be coassociative, and suppose that \(A\) is also equipped with a map \(S : A \rightarrow A\) (not necessarily an antipode, say), and invertible elements \(\alpha : I \rightarrow A\) and \(\beta : I \rightarrow A\) such that the standard Drinfel’d axioms hold, namely:

\[
\mu_3(S \otimes \alpha \otimes 1)\delta = \alpha \epsilon \\
\mu_3(1 \otimes \beta \otimes S)\delta = \beta \epsilon
\]

**Proposition 4.** This is a quasi-VN-core in the sense that both

\[
\mu_3(1 \otimes S \otimes 1)(\delta \otimes 1)\delta = 1
\]

and

\[
\mu_3(1 \otimes S \otimes 1)(1 \otimes \delta)\delta = 1
\]

Then at least

\[
(1 \ast S) \ast 1 = 1 \ast (S \ast 1)
\]

however here

\[
l(\alpha \ast 1) \neq l(\alpha)l(1)
\]

and

\[
l(1 \ast \beta) \neq l(1)l(\beta)
\]

in general.

Note that the two standard Drinfel’d conditions were still satisfied in the definition of a weak quasi-Hopf algebra (in the sense of Haring-Oldenburg et al. [2]).

Finally we note the following characterisation of Hopf \(k\)-algebras in terms of VN-cores, observing also that any VN-core \(A\) in \(\text{Vect}_k\) can be completed to the VN-bialgebra \(A \oplus k\).

**Proposition 5.** A VN-bialgebra in \(\text{Vect}_k\) with \(S\) an antihomomorphism of \(k\)-algebras and \(gf g = g\) is precisely a Hopf \(k\)-algebra.

The proof of this result uses the observation that the category of finite-dimensional representations of such a VN-bialgebra is left rigid with respect to the usual tensor product of left modules and duality. Thus Proposition 5 is related to the study of weak Hopf algebras, but only in that one can take the core of a given weak Hopf \(k\)-algebra \(A\) and complete it to a Hopf \(k\)-algebra structure \(A \oplus k\).

Enquiries (etc.) regarding this article can be made to the author through Micah McCurdy (Macquarie University), who kindly typed the manuscript.
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