Nonlinear dynamics of the contact interaction of a three-layer plate-beam nanostructure in a white noise field

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Abstract. The mathematical model of the contact interaction of a multilayer nanostructure consisting of two nanoplates and nanobeams between them with small gaps was constructed for the first time. A modified moment theory is used to describe the size-dependent effects of a real nanostructure. The upper and lower layers are nanoplates, obeying Kirchhoff's kinematic hypothesis, and the middle layer is the Euler-Bernoulli nanobeam. Contact interaction is accounted for by the model B.Ya. Cantor. Nanoplates and nanobeam are isotropic, elastic, and they are connected through boundary conditions. The effect of the gap between the layers and the noise field is studied. To solve and analyze these constructively nonlinear problems, the methods of the qualitative theory of differential equations, wavelet analysis, and methods for analyzing the sign of the largest Lyapunov exponent are used. The differential equations system reduces to the Cauchy problem the Bubnov-Galerkin method in higher approximations and finite difference methods with approximation O(h²) and O(h⁴) with respect to the spatial coordinate. Next, the Cauchy problem is solved by the Runge-Kutta methods of the 4th, 6th, 8th accuracy order in time. The analysis showed that the gap size essentially depends on the interaction of the elements of the multilayer nanosystem and on the nature of their complex oscillations. Also, the presence of a noise field involves the contact interaction of elements that were at rest with the previous values of the remaining parameters.

1. Introduction
Multilayer beam-plate nanostructures are components of modern technologies of navigation systems in the aerospace industry, in oil and gas equipment, shipbuilding, gyroscopy and in other areas. For example, micro- and nanoscale beams and plates are widely used in micro- and nano electromechanical systems (MEMS, NEMS), such as vibration sensors [1], micro drives [2], micro switches [3]. The presence of small gaps between structural elements necessitates studying the influence of the contact interaction on the entire nanostructure behavior. Along with this, it is extremely important to study the dimensions of the dimension-dependent parameters of beam-plate structures on their oscillations nature. The classical mechanics of a rigid body cannot interpret and predict such a dimensionally dependent behavior. At present, there are several theories that allow modeling the scale effects in the continuum, such as the moment theory of elasticity [4, 5], the nonlocal theory of elasticity [6], the gradient theory of elasticity [7] and the surface theory of elasticity [8]. Thus, the nonlinear dynamics of a three-layer microplate is first studied in [9, 10]. On the basis of the theory of Kirchhoff plates and nonlinear deformations of von Karman, nonlinear size-dependent transverse and planar equations movement. In [11-13] mathematical models were constructed and the
contact interaction of several beams was studied. The purpose of the present study is to construct a mathematical model and to develop methods for studying the contact interaction of nanoplates and nanobeams depending on the size-dependent parameter.

2. Statement of the problem
A mathematical model of a three-layer nanosystem is constructed, which consists of two parallel nanoplates and nanobeams connected through an edge condition under the action of a transverse load. A drawing of the plate-beam structure is shown in Figure 1.

![Figure 1. The design scheme.](image)

The following hypotheses were used to construct this model: nanoplates and nanobeam are isotropic, elastic; for the description of nanoplates Kirchhoff’s kinematic hypothesis is applied, for the nanobeam — Euler-Bernoulli hypothesis. The contact interaction between the plates and the beam is taken into account by the model of B.Ya. Cantor [14]. The size-dependent effects are taken into account on the basis of the modified moment theory of elasticity [4, 5], which takes into account higher order moments \( m_{ijm} \) and additional independent material parameter of length \( l \), connected with the symmetric tensor of the rotation gradient. On the basis of the Hamilton-Ostrogradsky variation principle, differential equations, boundary and initial conditions of dimensional-dependent plate-beam systems are obtained:

\[
\begin{align*}
\ddot{w}_1 + \gamma_1 \dot{w}_1 &= q(t) - \left( \frac{p_1}{12 \gamma_1^2} + \frac{p_2}{2 \gamma_1^2} \right) \left( \frac{1}{\lambda_1^2} \frac{\partial^4 w_1}{\partial x^4} + \frac{1}{\lambda_1^2} \frac{\partial^4 w_1}{\partial y^4} + 2 \frac{\partial^4 w_1}{\partial x^2 \partial y^2} \right) + \nonumber \\
&+ k(w_1 - w_2 - h_k) \Psi_1, \\
\ddot{w}_2 + \gamma_2 \dot{w}_2 &= -\left( \frac{p_1}{12 \gamma_2^2} + \frac{p_2}{2 \gamma_2^2} \right) \left( \frac{1}{\lambda_2^2} \frac{\partial^4 w_2}{\partial x^4} - K(w_1 - w_2 - h_k) \Psi_1 + K(w_2 - w_3 - h_k) \Psi_2, \\
\ddot{w}_3 + \gamma_3 \dot{w}_3 &= -\left( \frac{p_1}{12 \gamma_3^2} + \frac{p_2}{2 \gamma_3^2} \right) \left( \frac{1}{\lambda_3^2} \frac{\partial^4 w_3}{\partial x^4} + \frac{1}{\lambda_3^2} \frac{\partial^4 w_3}{\partial y^4} + 2 \frac{\partial^4 w_3}{\partial x^2 \partial y^2} \right) - \\
&- K(w_2 - w_3 - h_k) \Psi_2, \\
\end{align*}
\]

where \( \gamma_i = \frac{a}{h_i}, \gamma_2 = \frac{l}{h}, p_1 = \frac{1-v}{(1+v)(1-2v)}, p_2 = \frac{1}{2(1+v)}, \) \( w_1, w_3, w_2 \) — the deflection functions of the plates and the beam, respectively, \( K \) — is the stiffness coefficient of the transverse compression of the structure in the contact zone, \( h_k \) — the gap between the plate and the beam (Figure 1); \( \Psi_1 = 1, \) if \( w_1 > w_2 + h_k \) — there is a contact between the top plate \( (w_1) \) and beam \( (w_2) \), otherwise \( \Psi_1 = 0, \) \( \Psi_1 = \frac{1}{2} \left[ 1 + \text{sign}(w_1 - h_k - w_2) \right]; \) \( \Psi_2 = 1, \) if \( w_2 > w_3 + h_k \) — there is a contact between the bottom plate \( (w_3) \)
and beam \( (w_2) \), otherwise \( \Psi_2 = 0 \), \( \psi_2 = \frac{1}{2} \left[ 1 + \text{sign}(w_2 - h_i - w_3) \right] \) [14]. The system of equations (1) is written in dimensionless form. The idea of reduction to a dimensionless form is presented in [15].

\[
q(x, y, t) = q_0 \sin(\omega_p t) + a_0 \left( \frac{2 \text{rand}()}{\text{rand \_max} + 1} - 1 \right) - \text{transverse load acting on the upper plate, where } q_0 - \text{amplitude, } \omega_p - \text{frequency.}
\]

The external noise field in the form of white noise affects only from the outside of the upper plate. White noise is a generalized stationary random process with a constant spectral density. The term "white" was assigned by analogy with white light, which in the visible part of the spectrum has the whole set of frequencies. In this work, white noise is added to the load, which is given by the formula \( a_0 \left( \frac{2 \text{rand}()}{\text{rand \_max} + 1} - 1 \right) \), where \( a_0 \) - is the noise impact intensity, \( \text{rand}() \) – is the standard function of the C++ language, which takes a random integer from 0 to RAND_MAX, RAND_MAX – is a constant equal to 65535. The expression \( \frac{2 \text{rand}()}{\text{rand \_max} + 1} - 1 \) takes arbitrary fractional values in the range \((-1;1)\). This mathematical model of white noise is offered by Perry R. Cook and Gary P. Scavone (Center for Computing Music and Acoustics (CCRMA) of Stanford University) [16].

Equations (1) should be supplemented by zero initial conditions, the conditions for the non-penetration of one system into the body of the other, and the boundary conditions — hinged support:

\[
w_{1,3} = 0; \quad w'_{1,3} = 0; \quad a_t = 0; \quad a_y = 0; \quad w_{1,3} = 0; \quad w'_{1,3} = 0; \quad a_t = 0; \quad a_y = 0;
\]

(2)

The resulting systems of constructively nonlinear partial differential equations (1) reduce to a system of ordinary differential equations of the second order by the Bubnov-Galerkin method in higher approximations with respect to spatial variables \( x \) and \( y \), and which, in turn, is solved by the fourth-order Runge-Kutta method. The functions \( w_1, w_3 \) and \( w_2 \), being solutions of the system (1), are approximated by an expression as a product of functions that depend on time and on the coordinates:

\[
w_{1,3} = \sum_{k=1}^{N} A_k(t) \sin(k\pi x) \sin(j\pi y), \quad w_2 = \sum_{k=1}^{N} A_k(t) \sin(k\pi y)
\]

(3)

As a result of the contact interaction between elements, the equations system is nonlinear, so it is not possible to solve it analytically. Therefore, to confirm the results reliability, the solution was carried out by the finite differences method with the approximation \( O(h^2) \) and \( O(h^4) \) and the Bubnov-Galerkin method in higher approximations, the convergence of the methods for different number of terms of the series (3) and the different number of partitions in the finite difference method \( O(h^2) \), \( O(h^4) \). Accuracy was established according to the Runge rule. Further investigation of the results was carried out by a qualitative method of the differential equations theory. For this purpose, signals, Poincaré sections, phase portraits, Fourier power spectra and wavelet analysis are analyzed. As the mother wavelet transform, different wavelets were chosen: Morlet, Gauss 8, 16, 32, Haar, in order to obtain reliable results. Preference is given to the Morlet wavelet, because it has better information at every time moment. On the basis of wavelet analysis, a method for studying the phase chaotic synchronization of mechanical dynamical systems was developed. To this end, the phase of the chaotic signal is introduced. Chaotic phase synchronization means that the phase of chaotic signals is captured, while the amplitudes of these signals remain unconnected to each other and appear to be chaotic. The phase capture leads to coincidence of signal frequencies. The frequency of a chaotic signal is defined as the average rate of phase change. In case of using wavelet transforms, the wavelet surface characterizes the system behavior at each time scale at any time moment.

A significant dependence on the initial conditions is a fundamental feature of chaos, according to the definition formulated by Gulik [17]. Also, the presence of chaos is possible when the function has a positive Lyapunov exponent at each domain point of its definition and therefore is not ultimately periodic. In the studies below, we follow the definition of chaos given by Gulik [17]. To this end, a
methodology has been proposed to identify true chaos for mechanical systems. The need to exclude the accumulation of numeric errors, which is easily taken for chaos, drew attention R. Lozi [18]. For this, in the present paper: 1) problems are considered as systems with an infinite number of freedom degrees, 2) systems of partial differential equations are reduced to a system of ordinary differential equations by several different methods (the finite difference method, the Bubnov-Galerkin method, etc.), 3) the Cauchy problem is solved by several different methods of Runge-Kutta type, 4) the convergence of these methods is investigated, 5) for the results reliability the signs of the largest Lyapunov exponent are investigated by the methods of Wolf [19], Kanz [20] and Rosenstein [21].

3. Numerical experiment

The text of your paper should be formatted as follows: In this paper, as an example, we considered the question of chaotic dynamics of a full-length (in system (1) \( \gamma_2 = 0 \)) three-layer mechanical structure, consisting of two parallel plates, between which there is a beam located at the plates center, and the gap between elements \( h_i = 0.1 \) (Figure 1). Let us study the character of the behavior of such a multilayer system under the action of an external distributed alternating load \( q(x, y, t) = q_0 \sin(\omega_0 t) \), applied to the upper plate, while taking into account the contact interaction of the layers. Here, the frequency of the external action \( \omega_0 = 5 \) is chosen to be close to the natural frequency of the plate, and \( \varepsilon = 1, \mu = 0.3 \).

We set the external load amplitude \( q_0 = 0.5 \), and in this case the upper plate performs harmonic oscillations at the frequency \( \omega_p = 5 \) and does not touch the reinforcing beam. The beam and the lower plate are at rest. At \( q_0 = 0.1 \) contact occurs between the upper plate and the beam and their oscillations are immediately chaotic in the triplicate period: \( \omega_p / 3 = 1.66, 2\omega_p / 3 = 3.33 \) and \( \omega_p = 5 \) (Figure 2a,c). Phase portraits are strange attractors. In the wavelet spectra of the upper plate and the beam, there are zones of frequency intermittency at different instants of time (Figs 2b, d), while the dominant oscillation frequency of the plate remains \( \omega_p = 5 \). Also at certain instants of time, a chaotic phase synchronization occurs between the oscillations of the upper plate and the beam in the vicinity of the frequency \( 2\omega_p / 3 \). Also in the course of the research, Lyapunov's largest indicator was calculated according to three methods. For the upper plate, the Lyapunov exponent by the Wolf method is 0.00374, according to the Rosenstein method it is 0.06997, according to the Kanz method it is 0.01855. All three methods give a positive sign of the higher Lyapunov exponent, which characterizes the chaotic state of the system according to the chaos definition according to Gulik [1]. The lower plate is at rest.

Figure 2. Fourier power spectra (a, c) and 2D wavelet spectra (b, d) for the upper plate and beam,
respectively at $q_0 = 1$.

With the load amplitude $q_0 = 1.5$ the contact interaction of all three elements of the multi-layered packet occurs. In this case, the oscillations character of the whole system is restructured. Both plates and the beam perform chaotic oscillations at the excitation frequency $\omega_p = 5$. The power spectra are solid bars and the phase portraits are strange attractors, a loop appears on the phase portraits for the plates. For the upper plate the largest Lyapunov exponent has positive sign: according to Wolf's method is 0.00241, according to the method of Rosenstein is 0.06776, according to the Kanz method is 0.02839. The phase difference graphs indicate that the chaotic phase synchronization of oscillations is increasing, now it is carried out in the frequency range $\omega \in [4;7]$ and at the frequency $\omega_p / 3 = 1.66$ over the entire time interval. Synchronization is mainly observed at the bottom plate and beam. A further increase in the load leads to a complete synchronization of the oscillations of the multilayer system.

3.1. About the influence of external noise and the size of the gap on the character of the structure oscillations under the action of a transverse alternating load on the upper plate

Let us study the effect of an external noise field on the contact interaction of the system. For this purpose, we place the top plate in the field of white noise, that is, we add noise to the transverse load applied to the upper plate, with the same gap value $h_k$. Let us also set $q_0 = 1$ and the noise intensity $a_0 = 1*10^{-4}$. In this case, all three elements of the structure come into contact: the beam contacts the upper plate and the lower plate (Figure 3a). The Fourier power spectra for each element are a solid bars (Figure 3b). For the upper plate, the largest Lyapunov exponent has a positive sign: according to Wolff's method it is 0.00230, Rosenstein's method is 0.06762, according to the Kanz method it is 0.03269.

![Figure 3. Joint vibrations and the Fourier power spectrum at $q_0 = 1$ and the noise intensity $a_0 = 1*10^{-4}$.](image)

We increase the gap between the layers of the beam-plate structure up to $h_k = 0.15$. In this case, in the absence of noise ($a_0 = 0$) and under the action of a transverse load with amplitude $q_0 = 1$ only the upper plate and the beam contact, the lower plate is at rest. The Fourier power spectra for the upper plate and beams are solid bars, the phase portraits are strange attractors. For the upper plate, the largest Lyapunov exponent has a positive sign: according to Wolff's method it is 0.00216, according to the Rosenstein method it is 0.06255, according to the Kanz method it is 0.03550. Further gradually, we will increase the noise effect on the upper plate with the former remaining parameters, and with noise intensity $a_0 = 20$ all three elements of the structure come into contact, the upper plate comes into contact with the beam, and the lower plate touches beam. The Fourier power spectra for each element are a solid pedestal, the phase portraits are a solid spot. The nature of the oscillations of both plates and the beam is chaotic. For the upper plate, the largest Lyapunov exponent has a positive sign: according to Wolff's method it is 0.00167, according to the Rosenstein method it is 0.07187, according to the Kanz method it is 0.02752.
Increase the gap between the layers of the beam-plate structure \( h_k = 0.2 \). Since the individual equations of the system are linear, then the maximum deflection is 0.2, so the maximum gap for such systems is \( h_k = 0.2 \). In this case, in the absence of noise \( (a_0 = 0) \) and under the transverse load action with amplitude \( q_0 = 1 \) only the upper plate and the beam contact, the lower plate is at rest. The oscillations nature is chaotic on a tripling period. For the upper plate, the largest Lyapunov exponent is 0.00220 according to the Wolf method, and according to the Rosenstein method it is 0.05861, according to the Kanz method it is 0.01987. Lyapunov's indices for all three methods are positive, which confirms the chaotic system state. The contact interaction of all three elements begins with the action of noise with intensity \( a_0 = 30 \), while their oscillations nature is chaotic. For the upper plate, the largest Lyapunov exponent has a positive sign: according to Wolf's method it is 0.00179, according to the method of Rosenstein it is 0.07420, according to the Kanz method it is 0.02833.

Substantially reduce the gap \( h_k = 0.05 \). In this case, under the influence of a transverse load with amplitude \( q_0 = 1 \) all three elements of the structure come into contact, the upper plate comes into contact with the beam, and the lower plate touches the beam. Hopf bifurcations occur and the oscillations are chaotic. For the upper plate, the largest Lyapunov exponent has a positive sign: according to Wolf's method it is 0.00153, according to the Rosenstein method it is 0.06341, according to the Kanz method it is 0.02845.

4. Conclusions
The gap size between the elements of the structure significantly affects the oscillations nature. With the gap size \( 0.05 < h_k < 0.2 \) chaos arises on tripling the excitation frequency, intermittency zones appear. With a gap value \( h_k < 0.05 \) chaos occurs at the first Hopf bifurcation. The transition from harmonic oscillations to chaotic ones occurs according to the Pomo-Manneville scenario.

In the work at each stage of the study, the solution and analysis were carried out by several alternative methods, their convergence was investigated: the Bubnov-Galerkin method and the finite difference method for reduction to the Cauchy problem; methods of Runge-Kutta type for solving the Cauchy problem; different mother wavelets and Fourier analysis to determine the oscillations nature; Wolf, Rosenstein and Kanz methods to find the largest Lyapunov exponent. This confirms the reliability of the results.

The presence of a noise field has a significant effect, since it involves elements in the contact interaction that were at rest with the same unchanging control parameters of the system.

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