Resonant excitation of disk oscillations in deformed disks. VII.
Stability criterion in MHD systems

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Abstract

In a disk with an oscillatory deformation from an axisymmetric state with frequency \( \omega_D \) and azimuthal wavenumber \( m_D \), two normal mode oscillations with a set of frequency and azimuthal wavenumber being \((\omega_1, m_1)\) and \((\omega_2, m_2)\) are resonantly coupled to each other through the disk deformation when the resonant conditions \((\omega_1 + \omega_2 + \omega_D = 0 \text{ and } m_1 + m_2 + m_D = 0)\) are satisfied. In the case of hydrodynamical disks, the resonance amplifies the set of the oscillations if \((E_1/\omega_1)(E_2/\omega_2) > 0\) (Kato 2013b, PASJ, 65, 75), where \(E_1\) and \(E_2\) are wave energies of the two oscillations with \(\omega_1\) and \(\omega_2\), respectively. In this paper we show that this instability criterion is still valid even when the oscillations are ideal MHD ones in magnetized disks, if the displacements associated with the oscillations vanish on the boundary of the system.

Key words: accretion, accretion disks — instabilities — magnetic fields — resonance — waves

1 Introduction

Energy consideration is of primary significance in understanding various hydrodynamical and hydromagnetic instabilities. In a static system, energy associated with perturbations is always positive. In such a system stability can be examined by using the energy principle developed by Bernstein et al. (1958). In a stationary system with a shear flow, however, energy associated with perturbations is not always positive and can become negative. Well-known examples of negative wave-energy perturbations are waves in such a rotating system as galactic and accretion disks, where oscillations outside the radius of the corotation resonance have positive energy, while those inside have negative energy. It is known that in such systems two-wave coupling with opposite signs of wave energy can lead to instability. For example, spiral density waves in galactic disks are sustained by wave amplification (over-reflection) at the corotation resonance (Lin & Lau 1980). Papaloizou and Pringle instability (Papaloizou & Pringle 1984; Drury 1985) is also related to interaction between positive- and negative-energy waves through corotation radius. Amplification of p-mode oscillations at the corotation resonance in accretion disks (Lai & Tsang 2009, see also Fu & Lai 2011) also belongs to the instability of the same category.

A classical example which can be interpreted in terms of positive- and negative-wave coupling is Kelvin–Helmholtz instability (Cairns 1978). Especially, Khalzov, Smolyakov, and Ilgisonis (2008) suggest that coupling of waves with positive and negative energies is a universal mechanism for MHD (magnetohydrodynamic) instabilities of flowing media.

The importance of energy consideration is also recognized in a wave–wave resonant instability in deformed hydrodynamical disks (Kato et al. 2011; Kato 2013b).\(^1\)

\(^1\) This wave–wave resonant instability is essentially a generalization of Lubow’s tidal instability (1991) from the view point of emphasizing wave phenomena.
In these studies, however, we find that what is directly related to stability criterion is not the sign of wave energy, $E$, itself, but the sign of $E/\omega$, where $\omega$ is the frequency of oscillation. That is, let us consider a disk deviating from an axisymmetric state, where the deformation has frequency $\omega_0$ and azimuthal wavenumber $m_0$. On such deformed disks, two small-amplitude normal mode oscillations are superposed. The sets of frequency and azimuthal wavenumber, ($\omega_i$, $m_i$), of these two normal mode oscillations are ($\omega_1$, $m_1$) and ($\omega_2$, $m_2$). If

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad m_1 + m_2 + m_3 = 0,$$

are realized, the two oscillations with ($\omega_1$, $m_1$) and ($\omega_2$, $m_2$) are resonantly coupled to each other through the disk deformation. The instability condition is found to be (Kato 2013b)

$$E_1 E_2 \over \omega_1 \omega_2 > 0,$$

In particular cases where $\omega_0$ is sufficiently low, the resonant condition $\omega_1 + \omega_2 + \omega_3 = 0$ is realized for $\omega_1 \omega_2 < 0$. Then, the instability condition equation (2) is reduced to $E_1 E_2 < 0$. Except for such case, however, the instability condition is generally given by $(E_1/\omega_1)(E_2/\omega_2) > 0$.

The purpose of this paper is to show that the above instability condition, $(E_1/\omega_1)(E_2/\omega_2) > 0$, can be extended even to the cases of ideal MHD disks, under the condition that displacements associated with the oscillations vanish on the boundary of the system. Except that the disks are subject to magnetic fields, the procedures of analyses in this paper are quite parallel to those of Kato (2013b, hereafter referred to as Paper I). Hence, the parts almost parallel to those in Paper I will be described only briefly.

2 Linearized hydromagnetic equations and normal mode oscillations

The unperturbed disk is steady and axisymmetric. In the Lagrangian formulation, hydromagnetic perturbations superposed on the unperturbed disks can be described by extending the hydrodynamical formulation by Lynden-Bell and Ostriker (1967) to hydromagnetic cases as

$$D_0^{2}\xi = \Delta \left(-\nabla \psi - \frac{1}{\rho} \nabla \rho + \frac{1}{\rho} \text{curl}\,B \times B\right),$$

where $\xi(r, t)$ is a displacement vector associated with the perturbations, and $D_0/\Delta t$ is the time derivative along an unperturbed flow, $u_0(r)$, and is related to the Eulerian time derivative, $\partial / \partial t$, by

$$\frac{D_0}{\Delta t} = \frac{\partial}{\partial t} + u_0 \cdot \nabla.$$  

In equation (3) $\Delta (X)$ represents the Lagrangian variation of $X$, and $\psi$ is the gravitational potential. Other notations in equation (3) have their usual meanings.

In the case where the perturbations have small amplitude and are nondissipative, equation (3) is written as

$$\rho_0 \frac{\partial^2 \xi}{\Delta t^2} + 2 \rho_0 (u_0 \cdot \nabla) \frac{\partial \xi}{\Delta t} + L(\xi) = 0,$$

where $L(\xi)$ consists of hydrodynamic and hydromagnetic parts as

$$L(\xi) = L^G(\xi) + L^B(\xi).$$

Detailed expressions for $L^G(\xi)$ and $L^B(\xi)$ are unnecessary here. What we need here is that both of them are Hermitian in the following sense:

$$\int \eta \cdot L^G(\xi) dV = \int \overline{\xi} \cdot L^G(\eta) dV,$$

$$\int \eta \cdot L^B(\xi) dV = \int \overline{\xi} \cdot L^B(\eta) dV,$$

where $\eta$ and $\xi$ are any nonsingular functions of $r$ defined in the unperturbed volume of the disk and having continuous first and second derivatives everywhere. The integration is performed over the whole volume of the system, assuming that the surface integrals vanish. The Hermitian of the hydrodynamical part, $L^G(\xi)$, is shown by Lynden-Bell and Ostriker (1967) and that of the hydromagnetic part, $L^B(\xi)$, for example, by Bernstein et al. (1958) and Khalzov, Smolyakov, and Ilgisonis (2008). It is noted that $L(\xi)$ is Hermitian even when the perturbations are self-gravitating, but we consider hereafter only the cases of non-self-gravitating perturbations.

We now consider three normal mode oscillations satisfying the linearized equation (5). The set of eigen-frequency and azimuthal wavenumber of these oscillations are denoted by ($\omega_1$, $m_1$), ($\omega_2$, $m_2$), and ($\omega_3$, $m_3$). The displacement vectors, $\xi_i(r, t)$, associated with these oscillations are expressed as

$$\xi_i(r, t) = \Re [\xi_i(r) \exp(i \omega_i t)]$$

$$= \Re [\xi_i \exp(i \omega_i t - m_i \varphi)] \quad (i = 1, 2, 3),$$

where $\Re$ denotes the real part, and $\varphi$ is the azimuthal coordinate of the cylindrical coordinates $(r, \varphi, z)$ whose center is at the disk center and the z-axis is the rotating axis of
the disk. We now assume that the following resonant conditions among the three oscillations are present:

\[ \omega_1 + \omega_2 + \omega_3 = \Delta \omega, \quad \text{and} \quad m_1 + m_2 + m_3 = 0, \]

(9)

where \( m_i \)'s \((i = 1, 2, 3)\) are integers. In order to include in our formulation the cases where three frequencies are slightly deviated from the exact resonant condition, \( \Delta \omega \) is introduced in the first relation of equation (9), where \(|\Delta \omega|\) is assumed to be much smaller than the absolute values of \( \omega_1 \) and \( \omega_2 \).\(^3\)

Our purpose in this paper is to examine how the resonant interactions among three oscillations change their amplitudes. Our main interest, however, is the case where among three oscillations, the \((\omega_3, m_3)\) oscillation has a particular position in the sense that it has a larger amplitude compared with the two others and its amplitude variation due to resonant interactions with other modes can be practically neglected or it is maintained at a fixed amplitude by an external force such as a tidal one. In other words, we consider resonant interactions between two oscillations with \((\omega_3, m_3)\) and \((\omega_2, m_2)\) in a deformed disk with \((\omega_3, m_3)\). In order to emphasize this situation, \( \xi_3(r, t) \) and \((\omega_3, m_3)\) are denoted hereafter \( \xi_D(r, t) \) and \((\omega_3, m_3)\), respectively.

Before examining the resonant interaction, we introduce the wave energy of oscillations, \( E_i \) \((i = 1, 2)\), defined by (e.g., Kato 2001)

\[
E_i = \frac{1}{2} \omega_i \left[ \rho_0 \left( \hat{\rho}_0 \hat{\xi}_i \right)^* - i \left( \hat{\rho}_0 \hat{\xi}_i \left( \mathbf{u}_0 \cdot \nabla \hat{\xi}_i \right) \right) \right]
\]

(10)

where the asterisk denotes the complex conjugate, and \( \langle X \rangle \) is the volume integration of \( X \). In the framework of linear theory, the wave energy (10) of each oscillation is conserved, i.e., \( \partial E_i / \partial t = 0 \), which is derived easily by using the facts that the operator \( L(\xi_i) \) is Hermitian and \( \xi_i \) follows the linear wave equation (5). It is instructive to note that the wave energy is expressed in the case of the oscillations in geometrically thin nonmagnetized disks as (e.g., see Kato 2001)

\[
E_i = \frac{\omega_i}{2} \left\{ \left( \omega_i - m_i \Omega \right) \rho_0 \left( \hat{\xi}_i \cdot \hat{\xi}_i \right) + \hat{\xi}_i \cdot \mathbf{L}(\hat{\xi}_i) \right\},
\]

(11)

where \( \Omega(r) \) is the angular velocity of disk rotation. It is noticed that the sign of wave energy depends on which side of the corotation radius \((\omega_i = m_i \Omega)\) the oscillation is.

\(^3\) The cases where \( \omega_3 \) (which is denoted later by \( \omega_D \)) can be included in our analyses.

### 3 Formulation of coupling processes and commutability of coupling terms

Now, we consider two normal modes of oscillations, \( \xi_1 \) and \( \xi_2 \), with \((\omega_1, m_1)\) and \((\omega_2, m_2)\), respectively. In the linear stage, the perturbation, \( \xi \), imposed on a deformed disk is simply the sum of these two oscillations:

\[
\xi(r, t) = A_1 \xi_1(r, t) + A_2 \xi_2(r, t)
\]

\[
= \Re \left\{ A_1 \hat{\xi}_1 \exp[i(\omega_1 t - m_1 \Omega)] \right\}
\]

+ \( A_2 \hat{\xi}_2 \exp[i(\omega_2 t - m_2 \Omega)] \}

(12)

where \( A_1 \) and \( A_2 \) are amplitudes and are arbitrary constants. In the quasi-nonlinear stage, the oscillations are coupled through disk deformation, \( \xi_D \), so that they satisfy a quasi-nonlinear wave equation. The non-linear wave equation is

\[
\rho_0 \frac{\partial^2 \hat{\xi}^*}{\partial t^2} + 2 \rho_0 (\mathbf{u}_0 \cdot \nabla) \frac{\partial \hat{\xi}}{\partial t} + L(\hat{\xi}) = C(\hat{\xi}, \xi_D).
\]

(13)

where \( C \) is the quasi-nonlinear coupling terms and consists of hydrodynamic and hydromagnetic terms:

\[
C(\hat{\xi}, \xi_D) = C^\xi(\hat{\xi}, \xi_D) + C^B(\hat{\xi}, \xi_D).
\]

(14)

A detailed expression for \( C^\xi(\hat{\xi}, \xi_D) \) is obtained by Kato (2004, 2008) and duplicated in Paper I. One of the major purposes of this paper is to derive \( C^B(\hat{\xi}, \xi_D) \). To obtain \( C^B \) in an explicit form of \( \xi \), we must express the Lagrangian variation of magnetic force, \( \Delta[(1/\rho)\mathbf{curl} \mathbf{B} \times \mathbf{B}] \) as far as the second order terms with respect to \( \xi \). To do so, we need the Lagrangian variation of \( \mathbf{B} \), i.e., \( \Delta \mathbf{B} \), expressed as far as the second order terms with respect to \( \xi \). The procedures of these derivations and the results are, however, given in appendices, since they are lengthy. The derivation of \( \Delta \mathbf{B} \) is in appendix 1, and that of \( \Delta[(1/\rho)\mathbf{curl} \mathbf{B} \times \mathbf{B}] \) is in appendix 2.

The disk oscillations, \( \xi(r, t) \), resulting from the quasi-nonlinear coupling through disk deformation, \( \xi_D(r, t) \), will be written generally in the form (Paper I)

\[
\xi(r, t) = \Re \sum_{i=1}^{2} A_i(t) \hat{\xi}_i(r) \exp(i\omega_0 t)
\]

+ \( \Re \sum_{i} \sum_{\omega \neq 1, 2} A_{\omega} \hat{\xi}_\omega(r) \exp(i\omega_0 t) \)

+ oscillating terms with other frequencies.

(15)

The two original oscillations, \( \xi_1 \) and \( \xi_2 \), resonantly interact with each other through the disk deformation. Hence,
their amplitudes secularly change with time, which is taken into account in equation (15) by taking the amplitudes, $A_1$, as slowly varying functions of time. The terms whose time-dependence is $\exp(i\omega_1 t)$ and whose spatial dependence is different from $\xi_1$ are expressed by a sum of a series of eigen-functions, $\hat{\xi}_\alpha (\alpha \neq 1$ and 2), assuming that they make a complete set. The terms whose time-dependences are different from $\exp(i\omega_1 t)$ are not written down explicitly in equation (15), since these terms disappear by taking long-time average when we are interested in phenomena of frequencies of $\omega_1$ ($i = 1$ and 2).

In order to derive equations describing the time evolution of $A_1(t)$, we substitute equation (15) into the left-hand side of equation (13). Then, considering that $\hat{\xi}_1$'s and $\hat{\xi}_\alpha$'s are displacement vectors associated with eigen-functions of the linear wave equation (5), we find that the left-hand side of equation (13) becomes the real part of

$$
2\rho_0 \sum_{i=1}^{2} \frac{dA_i}{dt} \left[ i\omega_i + (u_0 \cdot \nabla) \right] \hat{\xi}_i \exp(i\omega_i t) + \rho_0 \sum_{i, \alpha \neq 1, 2} A_{i,\alpha} \left[ (\omega_i^2 - \omega_\alpha^2) - 2i(\omega_i - \omega_\alpha)(u_0 \cdot \nabla) \right] \times \hat{\xi}_\alpha \exp(i\omega_\alpha t) + \text{oscillating terms with other frequencies},
$$

where $d^2A_i/dt^2$ has been neglected, since $A_i(t)(i = 1$ and 2) are slowly varying functions of time. Now, the real part of equation (16) is integrated over the whole volume of disks after being multiplied by $\xi_1 \equiv \Re \xi_1 \exp(i\omega_1 t)$. Then, the term resulting from the second term of equation (16) vanishes and the results become

$$
\Re \left\{ \frac{dA_i}{dt} \right\} \left[ \rho_0 \hat{\xi}_1 (\omega_1 - i(u_0 \cdot \nabla)) \hat{\xi}_1 \right],
$$

which is further reduced to

$$
\Re \left\{ \frac{2E_1}{\omega_1} \frac{dA_1}{dt} \right\},
$$

where $E_1$ is the wave energy given by equation (10).

After the above preparations, the real part of equation (13) is multiplied by $\xi_1(r)$ and integrated over the whole volume to lead to

$$
\Re \left\{ \frac{2E_1}{\omega_1} \frac{dA_1}{dt} \right\} = \frac{1}{2} \Re \left[ A_1(t) A_D \left( \xi_1 \cdot C(\xi_2, \xi_D) \right) \exp(i\Delta\omega_1) \right].
$$

In deriving the right-hand side of equation (19), the time periodic terms with high frequencies (i.e., nonresonant terms) have been neglected due to time average.

Similarly, we multiply $\xi_2(r)$ by the real part of equation (13) and integrate over the whole volume to lead to

$$
\Re \left\{ \frac{2E_2}{\omega_2} \frac{dA_2}{dt} \right\} = \frac{1}{2} \Re \left[ A_2(t) A_D \left( \xi_2 \cdot C(\xi_1, \xi_D) \right) \exp(i\Delta\omega_2) \right].
$$

Equations (19) and (20) are formally the same as those in Paper I, but expressions for $C$'s in the present case are generalizations of $C$'s in Paper I to hydromagnetic cases [see equation (14)].

The most important characteristics of the coupling terms in equations (19) and (20) are that $\hat{\xi}_1$ and $\hat{\xi}_2$ in $\langle \hat{\xi}_1 \cdot C(\hat{\xi}_2, \hat{\xi}_D) \rangle$ and $\langle \hat{\xi}_2 \cdot C(\hat{\xi}_1, \hat{\xi}_D) \rangle$ are commutative. That is

$$
W \equiv \langle \hat{\xi}_1 \cdot C(\hat{\xi}_2, \hat{\xi}_D) \rangle = \langle \hat{\xi}_2 \cdot C(\hat{\xi}_1, \hat{\xi}_D) \rangle.
$$

A proof of this commutability is already given in Kato (2008) in the case of hydrodynamic oscillations. Even in the case of hydromagnetic oscillations the commutability is generally realized. We think there will be a simple way to the proof. However, we have not been able to construct a simple proof. Hence, we must be content with a clumsy analytical verification, which is very troublesome and thus given in appendix 3. A basic assumption involved in the proof is that when the volume integrations in equation (21) are performed by parts, the surface integrals vanish.

### 4 Conditions of resonant growth and discussions

Since the equations describing time evolution of resonant oscillations, equations (19)–(20), are formally the same as those in Paper I, we can derive the same conclusions as in Paper I. When the amplitude $A_D$ is assumed to be constant, what governed by equations (19) and (20) is the imaginary part of $A_1$ and $A_2$, i.e., $A_{1,\alpha}$ and $A_{2,\alpha}$. Their real parts are not related to the resonance, and we can neglect them in considering the exponential growth (or damping) of $A_{1,\alpha}$ and $A_{2,\alpha}$. Then, restricting our attention only to the case of $\Delta\omega = 0$, we have from equations (19) and (20)

$$
- \frac{2E_1}{\omega_1} \frac{dA_{1,\alpha}}{dt} = -\frac{1}{2} A_{2,\alpha} \Delta N \left( A_D W \right),
$$

$$
- \frac{2E_2}{\omega_2} \frac{dA_{2,\alpha}}{dt} = -\frac{1}{2} A_{1,\alpha} \Delta N \left( A_D W \right).
$$

The formula $\Re(A\Re(B)) = \frac{1}{2} \Re(AB + AB^*) = \frac{1}{2} \Re(AB + A^* B)$ is used, where $A$ and $B$ are complex variables and $B^*$ is the complex conjugate of $B$.

See an orthogonal relation given by equation (8) of Kato, Okazaki, and Oktarani (2011). The relation holds even in the present case of hydromagnetic perturbations, since what we use is only the fact that the operator $L$ is Hermitian.

$^4$ We suppose the presence of a simple way to the proof, since the commutability may be a general characteristic of conservative systems.
Eliminating $A_{2,i}$ from equations (22) and (23), we have
\[
\frac{d^2 A_{1,i}}{dt^2} = \frac{1}{16} \left( \frac{E_1 E_2}{\omega_1 \omega_2} \right)^{-1} |\Im(A_D W)|^2 A_{1,i}. \tag{24}
\]
The same equation is obtained for $A_{2,i}$ by eliminating $A_{1,i}$ instead of $A_{2,i}$. Equation (24) shows that if
\[
\frac{E_1 E_2}{\omega_1 \omega_2} > 0,
\tag{25}
\]
the oscillations grow with the growth rate given by
\[
\left( \frac{\omega_1 \omega_2}{16 E_1 E_2} \right)^{1/2} |\Im(A_D W)|. \tag{26}
\]
The above results show that even in the case of resonant coupling of ideal MHD oscillations, the formal criterion of instability obtained in Paper I for hydrodynamic oscillations is unchanged. That is, we have the following results.

(1) Two oscillations with $(\omega_1, m_1)$ and $(\omega_2, m_2)$ in a deformed disk with $(\omega_D, m_D)$ are ampliﬁed if
\[
\frac{E_1 E_2}{\omega_1 \omega_2} > 0, \tag{27}
\]
where $\omega_1 + \omega_2 + \omega_D = 0$ and $m_1 + m_2 + m_D = 0$.

(2) In the case where the frequency associated with the disk deformation, $\omega_D$, is low, the resonant condition, $\omega_1 + \omega_2 + \omega_D = 0$, is realized for $\omega_1 \omega_2 < 0$. Hence, in this case the amplification condition, equation (27), can be reduced to
\[
E_1 E_2 < 0. \tag{28}
\]
This might suggest that the resonant interaction between two oscillations with opposite signs of wave energy is the cause of amplification. However, in the case where $\omega_D$ is so high that the resonant condition $\omega_1 + \omega_2 = 0$ is realized for the same signs of $\omega_1$ and $\omega_2$, the amplification of $\omega_1$- and $\omega_2$-oscillations occurs when
\[
E_1 E_2 > 0. \tag{29}
\]
This is different from the case of $\omega_D$ being small. That is, the picture that the cause of resonant amplification is a direct energy exchange between two oscillations with positive and negative wave-energies is not always relevant.

(3) The amplification of $\omega_1$- and $\omega_2$-oscillations will be generally interpreted as a result of energy exchange between the $\omega_D$-oscillation and the set of $\omega_1$- and $\omega_2$-oscillations. This is based on the following consideration. If the $\omega_D$-oscillation is not maintained externally, unlike the treatment in this paper, equations (22) and (23) and a similar equation for $A_{D,i}$ [equation (38) in Paper I] give
\[
\frac{E_1}{\omega_1} \frac{dA_{2,i}}{dt} = \frac{E_2}{\omega_2} \frac{dA_{1,i}}{dt} = \frac{E_D}{\omega_D} \frac{dA_{D,i}}{dt}. \tag{30}
\]
Multiplying $E_1/\omega_1$ by these equations, we see that when the instability condition $(E_1/\omega_1)(E_2/\omega_2) > 0$ is satisfied and thus $A_{2,i}$ and $A_{D,i}$ grow with time, we have
\[
\frac{E_1 E_D}{\omega_1 \omega_D} \frac{dA_{D,i}}{dt} > 0. \tag{31}
\]
This inequality means
\[
\frac{dA_{D,i}}{dt} < 0, \tag{32}
\]
since $(E_1/\omega_1)(E_D/\omega_D)$ is usually negative in the case of $(E_1/\omega_1)(E_2/\omega_2) > 0$ [see inequalities (41) and (42) in Paper I]. The same conclusion can be derived by multiplying $E_2/\omega_2$ by equation (30). For more discussions, see equations (43)–(45) in Paper I and the arguments following them.

In summary, wave energy, $E$, is an important concept in considering resonant amplification of oscillations, but generally speaking, the more important quantity which is directly related to amplification is not $E$ but $E/\omega$.

It is important to notice here that simplicity of the instability condition, $(E_1/\omega_1)(E_2/\omega_2) > 0$, comes from the commutability relation of the coupling term $W$. The commutability is proved in this paper by integrating the volume integrals in $W$ by parts, neglecting the surface integrals (appendix 3). This neglect of the surface integrals will be acceptable, if the boundaries of the system are taken far outside and the displacement vectors associated with perturbations can be taken to be small there. Such assumptions may not be always acceptable in realistic cases, and careful consideration may be necessary. We think, however, that the commutability relation is a general characteristic in conservative systems. Hence, there will be a simple direct proof of the commutability, which will give more physically perspective conditions of the commutability.

Deformed disks will appear in various astrophysical objects, e.g., in disks of binary systems and in disks where large
scale instabilities are present. In previous work, we have applied the present wave–wave resonant amplification process to describe two astrophysical phenomena. The first one is the positive and negative superhumps in dwarf novae (Kato 2013a, 2013b, 2014). This is one of the examples where $|\omega_0|$ is comparable with $|\omega_1|$ or $|\omega_2|$. We have been also trying, since Kato (2004), to apply the present wave–wave resonant process to the high frequency quasi-periodic oscillations (HFQPOs) observed in low-mass X-ray binaries (LMXBs). This is an example of application of the wave–wave resonant process to the case where $|\omega_0|$ is so small that $\omega_1 + \omega_2 + \omega_3 = 0$ leads to $\omega_1 \omega_2 < 0$ and the instability condition is reduced to $E_1 E_2 < 0$.

When the present wave–wave resonant process is applied to describe the oscillatory phenomena in disks, however, magnetic fields are not always of importance. In dwarf novae, for example, magnetic fields are too weak to have major effects on disk oscillations. In the case of LMXBs, however, this is not the case. In some microquasars, twin HFQPOs are observed with frequency ratio close to 3 : 2. Furthermore, their central sources (black holes) have extremely high spins. Any discoseismological models in which effects of magnetic fields are not taken into account are not successful so far in describing simultaneously both the frequency ratio close to 3 : 2 and the high spin of the central source. That is, magnetic fields will be important in describing the HFQPOs in microquasars by discoseismological models.

The importance of magnetic fields in considering discoseismology of black-hole sources partially comes from the fact that the decrease of epicyclic frequency in the radial direction toward the central source by general relativistic effects (Okazaki et al. 1987) is strongly modified by the presence of poloidal magnetic fields, even if the fields do not have small plasma $\beta$-values (Fu & Lai 2009). Using this fact and assuming in advance that the instability condition $(E_1/\omega_1)(E_2/\omega_2) > 0$ of wave–wave resonant process is unchanged even in the case of magnetized disks, Kato (2012) tried to describe the HFQPOs by the present wave–wave resonant process, considering the resonant interaction between a set of a two-armed vertical p-mode oscillation and an axisymmetric g-mode oscillation in a disk with two-armed deformation. The results seem to be able to simultaneously describe both the frequency ratio and the high spin in conceivable ranges of strength of magnetic fields. This suggests that further examinations of HFQPOs of black-hole sources on this line is worthwhile.

Appendix 1. Lagrangian and Eulerian variations of magnetic fields as far as second-order terms with respect to displacement vector $\xi$

Our purpose here is to express the frozen-in condition of magnetic fields, as far as the second-order terms with respect to displacement vector $\xi$. Let us consider a small rectangular solid in a fluid (see the left-hand panel of figure 1). Among eight vertices the cartesian coordinates of four, $A_0$, $B_0$, $C_0$, and $D_0$ are, respectively, $A_0(r_1, r_2, r_3)$, $B_0(r_1 + \Delta r_1, r_2, r_3)$, and $C_0(r_1, r_2 + \Delta r_2, r_3)$, and $D_0(r_1, r_2, r_3 + \Delta r_3)$, where $\Delta r_1, \Delta r_2, \Delta r_3$ are assumed to be sufficiently small. In the unperturbed state the flux of magnetic fields passing through a piece of the rectangular plane specified by two vectors $A_0 \vec{B}_0$ and $A_0 \vec{C}_0$ is $B_{03} \Delta r_1 \Delta r_2$

Similarly, the flux passing through the plane specified by $A_0 \vec{D}_0$ and $A_0 \vec{B}_0$ is $B_{02} \Delta r_1 \Delta r_3$, and the flux passing through the plane specified by $A_0 \vec{C}_0$ and $A_0 \vec{D}_0$ is $B_{03} \Delta r_2 \Delta r_3$. Here, the first subscript 0 attached to magnetic field $B$ shows the unperturbed value and the second shows the spatial component, and is not to be confused with the symbol of point $B_0$.

Now, we assume that a fluid element at point $A_0$ moves to point $A_1(a_1, a_2, a_3)$ (see the right-hand panel of figure 1) by a perturbation. Here, $a = (a_1, a_2, a_3)$ is related to $r = (r_1, r_2, r_3)$ through the displacement vector $\xi = (\xi_1, \xi_2, \xi_3)$ as

$$a(r) = r + \xi(r).$$

(A1)

Corresponding to this displacement, a fluid element at point $B_0$ moves to point $B_1$, which is

$$B_1 \left( a_1 + \frac{\partial a_1}{\partial r_1} \Delta r_1, a_2 + \frac{\partial a_2}{\partial r_1} \Delta r_1, a_3 + \frac{\partial a_3}{\partial r_1} \Delta r_1 \right).$$

(A2)

Similarly, fluid elements at $C_0$ and $D_0$ move, respectively, to points $C_1$ and $D_1$, which are

$$C_1 \left( a_1 + \frac{\partial a_1}{\partial r_2} \Delta r_2, a_2 + \frac{\partial a_2}{\partial r_2} \Delta r_2, a_3 + \frac{\partial a_3}{\partial r_2} \Delta r_2 \right).$$

(A3)

$$D_1 \left( a_1 + \frac{\partial a_1}{\partial r_3} \Delta r_3, a_2 + \frac{\partial a_2}{\partial r_3} \Delta r_3, a_3 + \frac{\partial a_3}{\partial r_3} \Delta r_3 \right).$$

(A4)

Fig. 1. A small rectangular solid element in fluid (left-hand panel). Its deformation after a perturbation (right-hand panel).

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3 See Osaki (1985) and Lubow (1991) on the origin of the positive superhumps, and Lubow (1991) and references in Kato (2014) on possible origins of the negative superhumps.
Hence, the vectors defined by $A_1 \vec{B}_1 (\equiv \vec{B})$, $A_1 \vec{C}_1 (\equiv \vec{C})$, and $A_1 D_1 (\equiv \vec{D})$ are expressed, respectively, in the forms:

\begin{align}
\vec{B} &= \left( \frac{\partial a_1}{\partial r_1}, \frac{\partial a_2}{\partial r_1}, \frac{\partial a_3}{\partial r_1} \right) \Delta r_1, \\
\vec{C} &= \left( \frac{\partial a_1}{\partial r_2}, \frac{\partial a_2}{\partial r_2}, \frac{\partial a_3}{\partial r_2} \right) \Delta r_2, \\
\vec{D} &= \left( \frac{\partial a_1}{\partial r_3}, \frac{\partial a_2}{\partial r_3}, \frac{\partial a_3}{\partial r_3} \right) \Delta r_3.
\end{align}  

(A5)

(A6)

(A7)

Then, the scalar product of $\vec{B} \times \vec{C}$ and $\vec{B}(r + \xi)$ [where $B(r + \xi)$ is the magnetic field at the displaced position, i.e., $B(r + \xi) = B_0(r) + \Delta B$ and $\Delta B$ is the Lagrangian variation of magnetic field $B$] is magnetic flux through the surface defined by vectors $\vec{B}$ and $\vec{C}$. This flux must be equal to $B_0_3 \Delta r_1 \Delta r_2$, since the magnetic fields are assumed to be frozen in the fluid. This frozen-in condition is written as

\[
\frac{1}{\Delta r_1 \Delta r_2} (\vec{B} \times \vec{C}) \cdot B(a) = \left[ 1 + a \xi_1/\partial r_1 \quad a \xi_2/\partial r_1 \quad a \xi_3/\partial r_1 \right]_{\Delta r_1} \Delta r_2.
\]

(A8)

Similarly, we have

\[
\frac{1}{\Delta r_2 \Delta r_3} (\vec{C} \times \vec{D}) \cdot B(a) = \left[ 1 + a \xi_1/\partial r_2 \quad a \xi_2/\partial r_2 \quad a \xi_3/\partial r_2 \right]_{\Delta r_2} \Delta r_3.
\]

(A9)

Equations (A8)–(A10) are a set of simultaneous algebraic equations with respect to $\Delta B_1, \Delta B_2,$ and $\Delta B_3$. Their expressions in terms of displacement vector $\xi$ can be derived easily after lengthy but straightforward calculations, since what we need here is their successive forms expressed in terms of $\xi$, i.e., their first- and second-order expressions with respect to $\xi$.

After deriving the first-order expressions for $\Delta B_1, \Delta B_2,$ and $\Delta B_3$, we can easily summarize their expressions in a single vector form, which is

\[
(\Delta B_1) = B_0 (\delta \xi_i/\partial r_j - B_0 \text{div} \xi),
\]

(A11)

where $(\Delta B_1)$ denotes the first-order quantity of $\Delta B_i$. Of course, we can easily derive this expression for $(\Delta B_2)$ from the induction equation by integrating it with respect to time.\(^{10}\) By substituting equation $(\Delta B_1)$ given above into equations (A8)–(A10), we obtain the second-order expression for $\Delta B_i$, i.e., $(\Delta B_i)_{2}$, which is

\[
(\Delta B_2) = B_0 (\delta \xi_i/\partial r_j + 1/2 B_0 \left[ \delta \xi_i \delta \xi_i/\partial r_k \partial r_k + (\text{div} \xi)^2 \right]).
\]

(A12)

Next, let us summarize the second-order expression of the Eulerian variation of $B$. The Eulerian variation of $B(r)$, i.e., $\delta B$, is defined by $\delta B = B(r) - B_0(r)$. In order to explicitly demonstrate the order with respect to $\xi$, $\delta B_i(r)$ is written as $(\delta B_1)_i + (\delta B_2)_i + \cdots$, where the subscripts 1 and 2 denote, respectively, the first- and second-order quantities with respect to $\xi$, i.e.,

\[
\delta B_i(r) = B_i(r) - B_0(r) = (\delta B_1)_i + (\delta B_2)_i + \cdots.
\]

(A13)

Since $\Delta B_i = B_i(r + \xi(r)) - B_0(r)$, we have, after Taylor-expanding $B_i(r + \xi)$ around $B_i(r)$,

\[
(\Delta B_1)_i = \delta B_i_1 + \xi_j \frac{\partial}{\partial r_j} B_0
\]

(A14)

in the first-order with respect to $\xi$, and

\[
(\Delta B_2)_i = \delta B_i_2 + \xi_j \frac{\partial}{\partial r_j} \delta B_i_1 + 1/2 \xi_j \xi_l \frac{\partial^2}{\partial r_j \partial r_l} B_0
\]

(A15)

in the second-order with respect to $\xi$. By using expressions for $(\Delta B_1)_i$ and $(\Delta B_2)_i$ given, respectively, by equations (A11) and (A12), we have

\[
(\delta B_1)_i = \frac{\partial}{\partial r_j} (B_0 \xi_l - B_0 \xi_l).
\]

(A16)

\(^{8}\) Here and hereafter, the Lagrangian variation of $B$ is denoted by $\Delta B$, and $\delta B$ is reserved for the Eulerian variation of $B$, like Lynden-Bell and Ostriker (1967).

\(^{9}\) Here and hereafter, we use Einstein’s convention on indices; that is, we take summation if a term has the same index variable twice.

\(^{10}\) The induction equation, $\partial B/\partial t = \text{curl}(u \times B)$, is rewritten as $\partial B/\partial t = \text{curl}(u \times B) + (u \cdot \nabla) B = (B \cdot \nabla) u - B \text{div} u$. By integrating this equation with respect to time, we can obtain equation (A11) as the first-order expression for $\Delta B$. 

Writing down explicitly the second order terms of $\Delta F_{Bi}$ with respect to $\xi$, i.e., $(\Delta F_{Bi})_2$, we have, from equation (A19),
\[4\pi\rho_0(\Delta F_{Bi})_2\]
\[= -\frac{\partial}{\partial r_i} \left[ B_{0j}(\delta B_j)_2 + \frac{1}{2}(\delta B_j)_1^2 \right] + \frac{\partial}{\partial r_j} \left[ B_{0j}(\delta B_j)_2 + B_{0i}(\delta B_i)_1 + (\delta B_i)_1 (\delta B_i)_1 \right] + \xi_k \frac{\partial}{\partial r_i} \xi_k \frac{\partial^2}{\partial r_i \partial r_k} (B_{0k} B_{0i}) + \frac{1}{2} \left( \xi_j \frac{\partial^2}{\partial r_i \partial r_k} (B_{0k} B_{0i}) \right) + \frac{1}{2} \left( \xi_j \frac{\partial^2}{\partial r_i \partial r_k} (B_{0k} B_{0i}) \right) + \frac{1}{4\pi\rho_0} (\Delta \rho)_2 \left[ \frac{1}{2} \left( \xi_j \frac{\partial^2}{\partial r_i \partial r_k} (B_{0k} B_{0i}) \right) \right], \tag{A21}
\]
where $(\Delta \rho)_2 = \frac{1}{2} \left( \xi_j \frac{\partial^2}{\partial r_i \partial r_k} (B_{0k} B_{0i}) \right)$ has been used (Kato 2004).

If Eulerian quantities, $(\delta B_j)_1$ and $(\delta B_j)_2$, in equation (A21) are expressed in terms of $\xi$ by using equations (A16) and (A17), we have an expression for $(\Delta F_{Bi})_2$ explicitly expressed in terms of $\xi$. The detailed expression for $(\Delta F_{Bi})_2$ is, however, unnecessary here. What we need is commutative relations of the coupling terms resulting from $(\Delta F_{Bi})_2$, which are discussed in the next appendix.

### Appendix 3. Commutability of magnetic coupling terms

So far, we have attached subscripts 1, 2, and 3 (or D) to $\xi$, like $\xi_1$, $\xi_2$, and $\xi_3$ (or $\xi_D$), in order to represent the set of three oscillation modes. In this appendix, however, the superscripts (1), (2), and (3) (or (D)) are attached to represent them, respectively, like $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$ (or $\xi^{(D)}$), in order to save confusion with other subscripts.

Let us consider three modes of oscillations, $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$, which are resonantly coupled to each other. For the sake of simplicity, $\xi^{(3)}$ is regarded as a mode of disk deformation, i.e., $\xi^{(D)}$, and the oscillation modes of $\xi^{(1)}$ and $\xi^{(2)}$ are resonantly coupled to each other through $\xi^{(D)}$.

The wave equation describing $\xi^{(1)}$ has quasi-nonlinear terms on the right-hand side of the equation [cf., equation
(13)], which consist of terms of two groups. The first are coupling terms due to hydrodynamical processes, and the second are those resulting from hydromagnetic couplings. The former are already examined in Paper I, and expressed there as $C^i(\xi_2, \xi_1)$, including the case where the deformation is maintained by external tidal force. This term has been expressed as $C^i(\xi_2, \xi_1)$ in section 3 of this paper, and is $C^i(\xi_2, \xi_1)$, if the notation of this appendix is used. The coupling term of the latter group is $\rho_0 \Delta F_B(\xi_2, \xi_1)$, where the arguments of $\Delta F_B$ are written explicitly, since we are now interested in the coupling between $\xi_2$ and $\xi_1$, which feeds back to $\xi_1$. Hereafter, we denote $\rho_0 \Delta F_B(\xi_2, \xi_1)$ by $C^0(\xi_2, \xi_1)$, i.e.,

$$\rho_0 \Delta F_B(\xi_2, \xi_1) = C^0(\xi_2, \xi_1),$$

(A23)

which is nothing but $C^i(\xi_2, \xi_1)$ in section 3 of this paper. The sum of $C^0(\xi_2, \xi_1)$ and $C^i(\xi_2, \xi_1)$ is the total coupling terms [see equation (14) in section 3], i.e., in the present notation we have

$$C(\xi_2, \xi_1) = C^0(\xi_2, \xi_1) + C^i(\xi_2, \xi_1).$$

(A24)

It is noted that $\xi_2$ and $\xi_1$ are commutative in $C^i$, i.e., $C^i(\xi_2, \xi_1) = C^i(\xi_1, \xi_2)$.

The purpose of this appendix is to show that in the volume integration of

$$\int \xi(1) \cdot C(\xi_2, \xi_1) dV,$$

(A25)

$\xi(1)$ and $\xi(2)$ are exchangeable as

$$\int \xi(1) \cdot C(\xi_2, \xi_1) dV = \int \xi(2) \cdot C(\xi_1, \xi_1) dV,$$

(A26)

where the integration is performed over the whole volume of the system. A basic assumption involved in this derivation of commutability is that the surface integrals which appear by performing the integration by parts can be neglected. The commutability of the hydrodynamical terms has been shown in Paper I. Hence, in this appendix we show only the commutability of hydromagnetic coupling terms; i.e.,

$$\int \xi(1) \cdot C^0(\xi_2, \xi_1) dV = \int \xi(2) \cdot C^0(\xi_1, \xi_1) dV.$$  

(A27)

Let us multiply $\xi(1)$ by equation (A21) and integrate by parts, assuming that the surface integrals can be neglected. Then, some terms resulting from the third and fifth lines of equations (A21) are cancelled out, and we have

$$4\pi \int \xi(1) \cdot C^0(\xi_2, \xi_1) dV = \int C^0(\xi_1, \xi_2) dV,$$

(A28)

where

$$G^\delta = \text{div} \xi(1) \left[ B_{0j}(\delta B_j)_2 + \frac{1}{2}(\delta B_i)_1 \right]$$

- $\frac{\partial \xi(1)}{\partial r_j} \left[ B_{0j}(\delta B_i)_1 + B_{0j}(\delta B_j)_1 \right]$

+ $\frac{\partial \xi(1)}{\partial r_j} \left( \frac{\partial B_0}{\partial r_j} [ B_{0k}(\delta B_k)_1 + B_{0k}(\delta B_k)_1 ] \right)$

+ $\frac{1}{2} \xi(1) \frac{\partial}{\partial r_j} \left[ \frac{\partial B_0}{\partial r_j} + \frac{\partial B_0}{\partial r_j} \right]$

+ $\frac{1}{2} \xi(1) \left[ \text{div} \xi(2) - \frac{\partial \xi(1)}{\partial r_j} \delta B_j \frac{\partial}{\partial r_k} \right]$ \times $\left[ \frac{1}{2} \frac{\partial B_0^2}{\partial r_j} + \frac{\partial}{\partial r_j} (B_{0i} B_{0i}) \right].$  

(A29)

In the above expression for $G^\delta$, $\xi(1)$ is written explicitly, but $\xi(2)$ and $\xi(3)$ are not. Each term of equation (A29) has the form of a component of $\xi(1)$ (or its spatial derivative) times a quadratic term consisting of a component of $\xi(2)$ (or its spatial derivative) and a component of $\xi(3)$ (or its derivative), although some terms are not explicitly expressed in such forms. Each quadratic term mentioned above is the sum of two terms where the part of $\xi(2)$ and $\xi(3)$ are changed. With this convention we have neglected to attach the superscript (2) or (3) in terms related to $\xi$ in equation (A29), in order to unravel the complexity of notation. This convention is used hereafter.

The terms $(\delta B_1)_2$ and $(\delta B_2)_2$ in equation (A29) (there are three terms in all) can be expressed as $(\delta B_1)_2 = -\xi \partial(\delta B_1)_1/\partial r_1 + \ldots$ and $(\delta B_1)_2 = -\xi \partial(\delta B_1)_1/\partial r_1 + \ldots$ as shown in equation (A17). These terms with $-\xi \partial(\delta B_1)_1/\partial r_1$ or $-\xi \partial(\delta B_1)_1/\partial r_1$ in equation (A29) are integrated by parts in equation (A28), neglecting the surface integrals. Then, contributions of the terms in $G^\delta$ become as

$$(\delta B_1)_1 \frac{\partial}{\partial r_k} \left( B_{0j} \xi(1) \delta B_j \frac{\partial \xi(1)}{\partial r_j} \right) - (\delta B_1)_1 \frac{\partial}{\partial r_k} \left( B_{0j} \delta \xi(1) \frac{\partial}{\partial r_j} \right).$$

(A30)

Furthermore, all the terms of the third line of equation (A29) are integrated by parts in order to erase the terms with spatial derivatives of $(\delta B_1)_1$ and $(\delta B_1)_1$. The result is

$$- \left( \xi \frac{\partial}{\partial r_j} \text{div} \xi(1) + \frac{\partial \xi(1)}{\partial r_j} \right) [ B_{0k} (\delta B_k)_1 ]$$

+ $\frac{\partial \xi(1)}{\partial r_j} \frac{\partial \xi(1)}{\partial r_k} [ B_{0k} (\delta B_k)_1 + B_{0k} (\delta B_k)_1 ].$  

(A31)
The sum of equations (A30) and (A31) becomes, after some manipulations,
\[
\text{div} \xi^{(1)}(\partial B_j) \frac{\partial}{\partial r_k} (\xi_k B_{0j}) \\
+ \frac{\partial \xi^{(1)}}{\partial r_j} \left[ -\frac{\partial}{\partial r_k} (\xi_k B_{0j}) + (\partial B_j) (\partial B_k) + (\partial B_k) (\partial B_j) \right] \\
+ (\delta B_k) \left[ -\frac{\partial}{\partial r_j} B_{0k} + B_{0j} \frac{\partial}{\partial r_k} \right]. \tag{A32}
\]

The remaining terms in the first and second lines of equation (A29) are arranged as, using equation (A15),
\[
\text{div} \xi^{(1)} \left\{ B_{0j} \left[ (\Delta B_j) + \left( \frac{1}{2} \xi_j \xi_k \frac{\partial^2 B_{0j}}{\partial r_i \partial r_k} \right) + \frac{1}{2} (\partial B_j)^2 \right] \\
- \frac{\partial \xi^{(1)}}{\partial r_j} \left[ B_{0i}(\Delta B_j) + B_{0j}(\Delta B_i) + (\partial B_i)(\partial B_j) \right] \\
- \frac{1}{2} \xi_k \xi_k \left[ B_{0i} B_{0j} + B_{0j} B_{0i} + \frac{\partial^2 B_{0i}}{\partial r_i \partial r_k} \right] \right\}. \tag{A33}
\]

Summing equations (A32), (A33), and the remaining terms in equation (A29), we finally obtain, after some manipulations, an expression for \( G^B \) expressed explicitly in quadratic forms with respect to \( \xi \):
\[
G^B = \text{div} \xi^{(1)} \left\{ -B_{0j} B_{0i} \frac{\partial \xi_i}{\partial r_j} \text{div} \xi + \frac{1}{2} B_{0i} \left[ \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} + (\text{div} \xi)^2 \right] \\
+ \frac{1}{2} \xi_i \xi_j \frac{\partial B_{0j}}{\partial r_i} B_{0k} - \frac{1}{2} \left( B_{0j} \text{div} \xi + \xi_i \frac{\partial B_{0j}}{\partial r_i} \right)^2 \\
- \frac{1}{2} \xi_k \xi_k \frac{\partial^2 B_{0i}}{\partial r_i \partial r_k} \right\} \\
+ \frac{\partial \xi^{(1)}}{\partial r_j} \left[ -\frac{\partial \xi_i}{\partial r_j} \frac{\partial B_{0i}}{\partial r_k} + \frac{\partial}{\partial r_k} \left( B_{0i} B_{0j} \right) \\
- \frac{\partial \xi_i}{\partial r_j} B_{0j} B_{0k} - \frac{\partial^2 B_{0i}}{\partial r_i \partial r_k} \left( B_{0k} B_{0j} \right) \right] \\
+ \frac{1}{2} \xi_i \xi_k \frac{\partial^2 B_{0i}}{\partial r_i \partial r_k} \left[ \xi_j \frac{\partial B_{0j}}{\partial r_k} + \frac{\partial}{\partial r_k} \left( B_{0i} B_{0j} \right) \\
- \frac{\partial \xi_j}{\partial r_k} B_{0j} B_{0i} \right] \right\}. \tag{A34}
\]

Now, we introduce the following scalar quantities:
\[
A(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = B_{0j} \left( \frac{1}{2} \text{div} \xi^{(1)} \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} + \text{div} \xi \frac{\partial \xi^{(1)}}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} \right), \tag{A35}
\]
\[
B_1(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = \frac{1}{2} \frac{\partial B_{0j}^2}{\partial r_j} \left( \frac{1}{2} \xi_j \xi_j \text{div} \xi^{(1)} + \xi_j \text{div} \xi \frac{\partial \xi^{(1)}}{\partial r_i} \right), \tag{A36}
\]
\[
B_2(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = \frac{1}{2} \frac{\partial B_{0j}^2}{\partial r_j} \left( \frac{1}{2} \xi_j \xi_j \text{div} \xi^{(1)} + \xi_j \text{div} \xi \frac{\partial \xi^{(1)}}{\partial r_i} \right), \tag{A37}
\]
\[
C(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = -\frac{1}{4} \frac{\partial B_{0j}}{\partial r_j} \xi_i \xi_i \left( \xi_j \frac{\partial \xi^{(1)}}{\partial r_i} \right), \tag{A38}
\]
\[
D(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = \frac{1}{4} \frac{\partial B_{0j}}{\partial r_j} \xi_i \xi_i \left( \xi_j \frac{\partial \xi^{(1)}}{\partial r_i} \right). \tag{A39}
\]

The right-hand sides of the above equations are written by use of the convention mentioned just below equation (A29). That is, each term on the right-hand side is quadratic in such a sense that one of a component of \( \xi \) (or its derivative) is that of \( \xi^{(2)} \) (or its derivative) and the other is a component of \( \xi^{(D)} \) (or its derivative). The inverse case is also involved. That is, the term is involved where the former component of \( \xi \) is that of \( \xi^{(D)} \) (or its derivative) and the latter is that of \( \xi^{(2)} \) (or its derivative). An important characteristic of the quantities defined by equations (A35)–(A38) is that they are commutative to an exchange of \( \xi^{(1)}, \xi^{(2)}, \) and \( \xi^{(3)} \). For example, we have
\[
A(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) = A(\xi^{(2)}, \xi^{(1)}, \xi^{(D)}) = A(\xi^{(D)}, \xi^{(1)}, \xi^{(2)}) = \ldots \tag{A40}
\]

Then, the terms which are proportional to \( B_{0j}^2 \) or to its spatial derivatives on the right-hand side of equation (A34) are all summarized by using \( A_1, B_1, B_2, C, \) and \( D \). And then \( G^B \) is reduced to
\[
G^B = A + B_1 + B_2 + C + D \\
+ \text{div} \xi^{(1)} \left( -B_{0j} B_{0i} \frac{\partial \xi_i}{\partial r_j} \text{div} \xi + \frac{1}{2} B_{0j} B_{0k} \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_i} \right) \\
+ \frac{\partial \xi^{(1)}}{\partial r_j} \left[ B_{0j} B_{0i} \frac{\partial \xi_i}{\partial r_j} \frac{\partial \xi_j}{\partial r_k} - \frac{\partial \xi_i}{\partial r_j} B_{0j} B_{0k} \frac{\partial \xi_j}{\partial r_k} \right] \\
+ \frac{1}{2} \xi_j \xi_j \frac{\partial^2 B_{0i}}{\partial r_i \partial r_k} \left[ \frac{\partial \xi_j}{\partial r_k} - \frac{\partial \xi_k}{\partial r_j} \right] \right\} \right\}. \tag{A34}
\]
\begin{align*}
+ \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \left[ -\xi^i B_{0j} \frac{\partial B_{0k}}{\partial r^\ell} + B_{0k} \frac{\partial}{\partial r^\ell} (B_{0i} \xi_k - B_{0k} \xi_i) \right] \\
+ \frac{1}{2} \xi^i B_{0j} \xi^j \frac{\partial^2}{\partial r_j \partial r_k \partial r^\ell} (B_{0i} B_{0k}) \\
+ \xi^i \xi^j \xi^k \frac{\partial^2}{\partial r^\ell \partial r_j \partial r_k} (B_{0i} B_{0j}) \\
+ \frac{1}{2} \xi^i \left[ (\text{div}\xi)^2 - \frac{\partial \xi^i}{\partial r_k} \frac{\partial \xi^j}{\partial r^\ell} \right] \frac{\partial}{\partial r_j} (B_{0i} B_{0j}). \quad (A41)
\end{align*}

Next, let us consider the terms proportional to \(B_{0i} B_{0j}\) on the right-hand side of equation (A41). The sum of the second term of the second line of equation (A41) and the first term of the third line is summarized as \(A_{C1}\):

\begin{align*}
A_{C1}(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) \\
= B_{0i} B_{0j} \left( \frac{1}{2} \text{div} \xi^i \frac{\partial \xi^j}{\partial r_j} - \frac{\partial \xi^i}{\partial r_i} \frac{\partial \xi^j}{\partial r_j} \right), \quad (A42)
\end{align*}

where the right-hand side of equation (A42) is written by using the convention mentioned before [see sentences below equation (A29)]. (This conventional expression is used hereafter for some scalar quantities without notice.) It is important to note that \(A_{C1}(\xi^{(1)}, \xi^{(2)}, \xi^{(D)})\) is commutative with respect to \(\xi^{(1)}, \xi^{(2)}\), and \(\xi^{(D)}\). In equation (A41) the remaining terms which have \(B_{0i} B_{0j}\) with no spatial derivatives are

\begin{align*}
B_{0i} B_{0j} \left( -\text{div} \xi^i \frac{\partial \xi^j}{\partial r_j} \xi^k - \frac{\partial \xi^i}{\partial r_i} \frac{\partial \xi^j}{\partial r_j} \frac{\partial \xi^k}{\partial r_k} - \frac{\partial \xi^i}{\partial r_i} \frac{\partial \xi^j}{\partial r_j} \right) \\
+ \frac{\partial \xi^i}{\partial r_i} \frac{\partial \xi^j}{\partial r_j} \frac{\partial \xi^k}{\partial r_k} \text{div}\xi. \quad (A43)
\end{align*}

The final term of equation (A43) can be reformulated by integrating by parts, neglecting the surface integral, as

\begin{align*}
B_{0i} B_{0j} \left[ \frac{1}{2} \xi^i \frac{\partial}{\partial r_j} (\text{div}\xi)^2 + \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \text{div}\xi \right] \\
+ \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \xi^j \text{div}\xi. \quad (A44)
\end{align*}

The first term of equation (A44) is further integrated by parts, neglecting the surface integrals. Then, equation (A44) is reduced to

\begin{align*}
B_{0i} B_{0j} \left[ -\frac{1}{2} \frac{\partial \xi^i}{\partial r_j} (\text{div}\xi)^2 + \xi^i \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} \text{div}\xi \right] \\
- \frac{1}{2} \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \xi^j (\text{div}\xi)^2 + \frac{\partial}{\partial r_k} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_j} \text{div}\xi. \quad (A45)
\end{align*}

Here, we introduce quantities defined by

\begin{align*}
A_{C2}(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) \\
= -B_{0i} B_{0j} \left[ \frac{1}{2} \frac{\partial \xi^i}{\partial r_j} (\text{div}\xi)^2 + \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} \text{div}\xi \right]. \quad (A46)
\end{align*}

\begin{align*}
A_{C3}(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) \\
= -B_{0i} B_{0j} \left( \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \frac{\partial \xi^k}{\partial r^\ell} \frac{\partial}{\partial r^\ell} + 2 \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r^\ell} \frac{\partial}{\partial r^\ell} \text{div}\xi \right). \quad (A47)
\end{align*}

Then, equation (A43) can be arranged in the form of

\begin{align*}
A_{C2} + A_{C3} \\
+ B_{0i} B_{0j} \left( \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \frac{\partial \xi^k}{\partial r^\ell} \frac{\partial}{\partial r^\ell} + \frac{\partial \xi^i}{\partial r_j} \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r^\ell} \frac{\partial}{\partial r^\ell} \text{div}\xi \right) \\
- \frac{1}{2} \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \xi^j (\text{div}\xi)^2 + \frac{\partial}{\partial r_k} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_j} \text{div}\xi. \quad (A48)
\end{align*}

The term of \(B_{0i} B_{0j}(\partial \xi^i / \partial r_j)(\partial \xi^j / \partial r_i)(\partial \xi^k / \partial r_k)\) in equation (A48) is integrated by parts to lead to \(-\xi^i (\partial / \partial r_j) [B_{0i} B_{0j}(\partial \xi^j / \partial r_i)(\partial \xi^k / \partial r_k)]\). Furthermore, \(B_{0i} B_{0j}(\partial \xi^i / \partial r_j)(\partial \xi^j / \partial r_i)(\partial \xi^k / \partial r_k)\) is also integrated by parts to lead to \(-\xi^i (\partial / \partial r_i) [B_{0i} B_{0j}(\partial \xi^i / \partial r_j)(\partial \xi^k / \partial r_k)]\). Then, by introducing \(A_{C4}\) defined by

\begin{align*}
A_{C4}(\xi^{(1)}, \xi^{(2)}, \xi^{(D)}) \\
= -(B_{0i} B_{0j}) \left( \xi^i \frac{\partial^2 \xi^j}{\partial r_i \partial r_j} + \xi^i \frac{\partial \xi^j}{\partial r_i} \frac{\partial^2 \xi^k}{\partial r_i \partial r_k} \right) \\
+ \xi^i \frac{\partial \xi^j}{\partial r_i} \frac{\partial^2 \xi^k}{\partial r_i \partial r_k} \right), \quad (A49)
\end{align*}

the sum of the three terms in the second line of equation (A48) is summarized as

\begin{align*}
A_{C4} = \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_i} \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r^\ell} \frac{\partial}{\partial r^\ell} \text{div}\xi. \quad (A50)
\end{align*}

Hence, equation (A43) is reduced to

\begin{align*}
A_{C2} + A_{C3} + A_{C4} \\
- \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_k} \frac{\partial}{\partial r_k} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_i} \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r^\ell} \frac{\partial}{\partial r^\ell} \text{div}\xi \\
- \frac{1}{2} \frac{\partial}{\partial r_j} (B_{0i} B_{0j}) \xi^i \xi^j (\text{div}\xi)^2 + \frac{\partial}{\partial r_k} (B_{0i} B_{0j}) \xi^i \frac{\partial \xi^j}{\partial r_j} \text{div}\xi. \quad (A51)
\end{align*}
It should be noticed that in $A_{C2}$, $A_{C3}$, and $A_{C4}$ defined above, $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$ are all commutative with each other.

Summing up the results obtained so far, we can write down $G^B$ given by equation (A34) in the form:

$$G^B \Rightarrow A + A_{C1} + A_{C2} + A_{C3} + A_{C4} + B_1 + B_2 + C + D +$$

$$- \frac{\partial}{\partial r_j} (B_0 B_0) \hat{\xi}_i^{(1)} \hat{\xi}_k^{(1)} \frac{\partial \hat{\xi}_k}{\partial r_k} - \frac{\partial}{\partial r_j} (B_0 B_0) \hat{\xi}_i^{(1)} \hat{\xi}_k^{(1)} \frac{\partial \hat{\xi}_k}{\partial r_k} \frac{\partial}{\partial r_j}$$

$$- \frac{1}{2} \frac{\partial}{\partial r_j} (B_0 B_0) \frac{\partial}{\partial r_j} (\text{div}\xi)^2 + \frac{\partial}{\partial r_k} (B_0 B_0) \frac{\partial}{\partial r_k} (\text{div}\xi)$$

$$+ \frac{\partial}{\partial r_j} \left[ \xi_\ell \text{div}\xi \frac{\partial}{\partial r_k} (B_0 B_0) \right]$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

$$+ \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} (B_0 B_0)$$

The first term on the second line of equation (A52) and a part of the last line of equation (A52) are summarized in the form of $B_{C1}$ which is

$$B_{C1}(\xi^{(1)},\xi^{(2)},\xi^{(3)})$$

$$= - \frac{\partial}{\partial r_j} (B_0 B_0) \left( \frac{1}{2} \xi_i^{(1)} \xi_j^{(1)} \frac{\partial}{\partial r_i} \xi_k^{(1)} + \frac{\partial}{\partial r_k} (B_0 B_0) \frac{\partial}{\partial r_i} \xi_i^{(1)} \frac{\partial}{\partial r_k} \xi_j^{(1)} \right).$$

(A53)

The last term on the third line of equation (A52) and the term on the fourth line are summarized as $(\partial/\partial r)(B_0 B_0) (\partial/\partial r)(\xi^{(1)} \xi_k^{(1)} \text{div}\xi)$. This can be integrated by parts to give

$$- \frac{\partial^2}{\partial r_k \partial r_j} (B_0 B_0) \xi_i^{(1)} \xi_k^{(1)} \xi_j^{(1)} \xi_k^{(1)} \frac{\partial}{\partial r_j} \text{div}\xi.$$

(A54)

The last term is further integrated by parts by writing $(\partial/\partial r_j) \text{div}\xi$ as $(\partial/\partial r_j)(\partial \xi_k/\partial r_j)$. Then, equation (A54) becomes

$$- \frac{\partial^2}{\partial r_k \partial r_j} (B_0 B_0) \xi_i^{(1)} \xi_k^{(1)} \xi_j^{(1)} \xi_k^{(1)} \frac{\partial}{\partial r_j} \text{div}\xi.$$

(A55)

The first term of equation (A55) and the term on the seventh line of equation (A52) are cancelled out. The term on the sixth line of equation (A52) is also integrated by parts to give

$$- \frac{1}{2} \frac{\partial^2}{\partial r_i \partial r_k} (B_0 B_0) \left[ \xi_i^{(1)} \xi_k^{(1)} \frac{\partial}{\partial r_j} (\xi_k^{(1)} \frac{\partial}{\partial r_j} ) \right].$$

(A56)

The first term of this equation and the first term on the fifth line of equation (A52) are cancelled out.

Taking into account the above considerations we find that the terms below the second line of equation (A52) are summarized as

$$B_{C1} - \frac{\partial}{\partial r_j} (B_0 B_0) \frac{\partial}{\partial r_j} \xi_i^{(1)} \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_k} \xi_k^{(1)}$$

$$+ \frac{\partial}{\partial r_j} (B_0 B_0) \frac{\partial}{\partial r_j} \xi_i^{(1)} \frac{\partial}{\partial r_i} \xi_k^{(1)}$$

$$- \frac{\partial}{\partial r_j} (B_0 B_0) \frac{\partial}{\partial r_j} \xi_i^{(1)} \frac{\partial}{\partial r_i} \xi_k^{(1)}$$

$$= - \frac{1}{2} \frac{\partial^2}{\partial r_k \partial r_j} (B_0 B_0) \frac{\partial}{\partial r_k} \xi_i^{(1)} \xi_j^{(1)} \xi_k^{(1)} \xi_k^{(1)} \frac{\partial}{\partial r_j} \text{div}\xi.$$
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