Quantum-electrodynamic model of the finite-size electron and calculation of the fine-structure constant

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Abstract

We propose a model of a relativistic string formed by a scalar complex field, acting as electromagnetic field source. An axioymmetric solutions of the stationary equations for the scalar and electromagnetic fields are found numerically. The mass $m$ is calculated as a function of the charge $e$ and the magnetic moment $\mu$ of the system. The resulting toroidal structure is interpreted as an electron because the calculated ratio $e^3/(2mc^2\mu)$ coincides with the fine-structure constant $\alpha = e^2/(\hbar c) \approx e^3/(2mc^2\mu e)$.

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1 Introduction

Removing divergences in quantum electrodynamics by renormalization procedure allows calculating multiloop contributions to electrodynamics quantities [1]. As a result, the theory leads to an excellent agreement with experimental data. For example, taking into account radiation corrections into the electron’s magnetic moment leads to an agreement with experiment within 10 significant digits. However, the fine-structure constant, which enters all calculations, is not calculated within the theory. It is possible that one has to go beyond electrodynamics in order to calculate it. The possible value $1/(4 \pi)$ of the unifield coupling constant in the grand unification schemes at the energies from $10^{15}$ to $10^{19}$ GeV is still under discussion (see for example [2]). In addition, calculations connecting this constant with the fine-structure constant in the energy interval of 17 orders of magnitude seem to be ambiguous [3]. Alternative numerological approaches (see for example [4]) give good agreement with experiment but fail to make any predictions.
In quantum electrodynamics, the point-like electron is surrounded by a cloud of electron-positron pairs and is actually a finite-size structure. This fact suggests that it might be possible to use the finite size structure as a ground state approximation in the perturbation theory and thus avoid any divergences in the theory. The “combined” models, where two or more material fields are connected by the vector fields, are well known. For example, an electromagnetic field connects electrons with nuclei in atoms, gluon fields connect quarks in nucleons and mesons [5]. In contrast with this, in a model presented here we have only one material field as a source of electromagnetic field. The aim of the paper is to investigate the soliton-like states of the system.

In the previous paper [6], we proposed a classical model of the finite-size axisymmetric particle with charges and currents situated on a disk. These charges and currents were assumed to be proportional to the field potentials. Kaluza-Klein nonlinearity of electromagnetic field [7], [8] ensured the existence of stable solutions and finite values of all observables in non-trivial stationary states. However, the fine-structure constant calculated within the model turned out to be of order one.

In this paper we propose a new model of extended electron in the classical form of a flat ring current. Quantum fluctuations smear this string out to a tore, while the charge $e$ and the magnetic moment $\mu$ are invariant with respect to fluctuations. The basis of the relativistic Lagrangian formalism is a scalar complex material field, which acts as a source of the Maxwell electromagnetic field. The interacting fields form a stable axisymmetric soliton-like state. The aim of the paper is validation of the proposed relativistic Lagrangian formalism for the description of the electron as a finite-size toroidal structure. The system of equations derived from this formalism is solved numerically and the energy $E = mc^2$ of the stable state is evaluated. The energy is a function of the model parameters $e$ and $\mu$, but the dimensionless number $e^3/(2m\,c^2\,\mu)$ is independent of them and must coincide with the fine-structure constant $\alpha = e^2/(\hbar c) \approx e^3/(2m c^2 \mu_e)$ (if one neglects the electron’s anomalous magnetic moment). The demonstrated agreement with the experimental value serves as a validation of the proposed model.

2 Lagrangian and parameterization of solutions

We assume that the electromagnetic field is classical and the Lagrangian has the form

$$L_M = -\frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^3x,$$  \hspace{1cm} (1)

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu.$$  

The wave function $\psi$ is normalized as

$$\frac{i\,e\,\lambda}{2} \int (\psi^* \frac{\partial}{\partial x_0} \psi - \psi \frac{\partial}{\partial x_0} \psi^*) d^3x = e,$$ \hspace{1cm} (2)
where $\lambda$ is the fundamental length of the model determined later. The interaction Lagrangian containing only linear terms in potentials is

$$ L_{\text{int}} = -\frac{ie\lambda}{2} \int (\psi^* \frac{\partial}{\partial x_\nu} \psi - \psi \frac{\partial}{\partial x_\nu} \psi^*) A_\nu \, d^3x, \quad (3) $$

the Lagrangian of the "free" field $\psi$ has the form

$$ L_0 = e^2 \lambda \int (\lambda^2 \frac{\partial}{\partial x_\nu} \psi^* \frac{\partial}{\partial x_\nu} \psi - \psi^* \psi) \, d^3x. \quad (4) $$

We note the parameters of the model are the charge $e$ and the fundamental length $\lambda$. Note also that there are no bilinear in electromagnetic potentials terms in Lagrangian components (1, 3, 4), which are required by the gauge invariance. Thus, conservation of electromagnetic current does not follow from Lagrange field equations.

We seek a solution for $\psi$, which is axiymmetric (around $z$-axis) and has a plane of symmetry $z = 0$ in the cylindrical coordinate system centered in the center of the charge. Conservation of electromagnetic current can be ensured by choosing the wave function to be an eigenfunction of the operator $ie\lambda \frac{d}{dr}$ and operator $ie\lambda/r \frac{d}{d\phi}$, such that the phase depends only on $t$ and $\phi$, and the amplitude depends only on $r$ and $z$: $\psi = 1/(2 \pi)^{1/2} \psi(r, z) e^{i\phi - i t/\lambda}$.

The magnetic moment $\mu = \mu_e$ can be obtained by setting $\lambda = \mu_e/e = \hbar/(2 m_e c) = 1.93 \cdot 10^{-13} \text{ m}$, i.e. 4 times smaller than the Compton wave length of electron. The parameter $\lambda$ can be interpreted as a ring radius of the string current at which the current four-vector is light-like. Because the fine-structure constant $e^2/(2 m c^2 \mu)$ we are calculating is dimensionless, we set $e = 1$, $\lambda = 1$, $c = 1$ and calculate the mass $m = E$.

Because $\psi^*(r, z) = \psi(r, z)$, the currents components $J_r$ and $J_z$ are equal to zero, and the electromagnetic field has only two non-zero components $A_0(r, z)$ and $A_\phi(r, z)$. Taking all these assumptions into account, calculating the integrals with respect to $\phi$ and varying the Lagrangian over $A_0(r, z)$, $A_\phi(r, z)$, and $\psi(r, z)$, we get the Maxwell equations and the wave function equation:

$$ \nabla^2 A_0(r, z) = -2 \psi^2(r, z), \quad (5) $$

$$ \nabla^2 A_\phi(r, z) = -2 \psi^2(r, z)/r, \quad (6) $$

$$ (-1/2 \nabla^2 + U(r, z)) \psi(r, z) = 0, \quad (7) $$

$$ U(r, z) = \frac{1}{2r^2} + A_0(r, z) - \frac{A_\phi(r, z)}{r}. $$

The amplitude of the wave function is normalized as

$$ \int \psi^2(r, z) \, dS = 1, \quad (8) $$

where $dS = r \, dr \, dz$. 

3
Because of the symmetry, the boundary conditions at $z = 0$ require that all functions are even functions of $z$. The electric field potential is an even function of $r$, and the wave function and the magnetic potential are odd functions of $r$. At large $r$ and $z$, the wave function should vanish, and the asymptotic behavior of $A_0$ and $A_\phi$ is respectively determined by $e$ and $\mu$.

The wave function equation (7) has the form of the stationary Schrödinger equation with zero self-energy. The magnetic field creates the potential well surrounded by the barrier created by the electric field at large distances and by the "centrifugal" forces at small distances. The depth of the well, which is necessary for existence of a stable solution, can be achieved at a characteristic radius of the current that is much smaller than the fundamental length. This leads to a small value of the fine-structure constant.

The wave function decreases at large distances. Therefore, the asymptotic behavior of the electromagnetic field potentials are the same as those of the point-like particle. This in turn allows calculating the wave function asymptotic behavior:

$$\psi = \exp\left(-\left(8^{1/2} (r^2+z^2)^{1/4}\right) \frac{r^{1/4}}{(r^2+z^2)^{1/4}}\left(1+1.5468/(r^2+z^2)^{1/4}+\ldots\right)\right). \quad (9)$$

The relatively slow decay of the wave function (in comparison with Gaussian decay $\exp(-x^2/2)$ for the harmonic oscillator wave function) is due to the fast decay at the boundaries of the potential well and its smallness in the vicinity of the barrier. This fact validates the quasi-classical approximation for the ground state. We give a quasi-classical estimate of the toroidal structure radius $r_0$. The potential well $A_\phi/r = 1/(2 r_0^3)$ and kinetic energy $(\pi/(2 r_0))^2$ must be approximately equal. This implies $r_0 = 2/\pi^2$. The energy of the bounded state is

$$E = \int \partial L/\partial (\partial_0 \psi^*) \partial_0 \psi^* \, d^3x + \int \partial L/\partial (\partial_0 \psi) \partial_0 \psi \, d^3x - L_M - L_0 - L_{int} \quad (10)$$

and taking the normalization condition into account, we obtain that

$$E = 1 - L_M - L_0 - L_{int}. \quad (11)$$

For $\psi$ and $A$ satisfying the equations of motion, we have $L_0 + L_{int} = 0$ and $L_{int} = -2 \star L_M$. This results in

$$E = 1 + \frac{1}{2} \int \psi^2 A_\phi/r \, d^3x. \quad (12)$$

Integrating over $\phi$ leads to the final result for the energy of the bounded state,

$$E = 1 + \frac{1}{2} \int \psi^2 (r, z) A_\phi (r, z) /r \, dS. \quad (13)$$

So, in the quasi-classical approximation we find $E = \pi^6/16$ and $1/\alpha = \pi^6/8 \approx 120$. The fact that such a crude estimate gives a very reasonable result is encouraging.
3 Numerical solution algorithm

We propose the following change of variables in order to take the non-polynomial (exponential) form of the wave function into account effectively:

\[ r = L_r f(\rho), \]
\[ z = L_z g(\zeta), \]
where \( L_r \) and \( L_z \) are the scale factors, \( f(\rho) \) and \( g(\zeta) \) are some logarithmic functions which transform infinite quadrant \( r = 0..\infty, z = 0..\infty \) into the unit square \( \rho = 0..1, \zeta = 0..1: \)

\[ f(\rho) = s \rho + (-\ln(1 - \rho^t))^{(t-1)/t}/\rho^{t-2}, \]
\[ g(\zeta) = (-\ln(1 - \zeta^v))^{(v-1)/v}/\zeta^{v-2}. \]

After this transformation (at fixed \( s, t, \) and \( v \)) the wave function is interpolated by polynomials, which are easy to integrate and differentiate. The problem then split into two parts:

1) calculating \( L_z \) at fixed \( L_r/L_z, f(\rho) \) and \( g(\zeta) \), which leads to a convergent solution and

2) choosing an "optimal" solution by optimizing \( L_r/L_z, f(\rho) \) and \( g(\zeta) \).

3.1 Calculation of the scale factor

In order to calculate \( L_z \), we note that the nonhomogeneous Maxwell equations have nontrivial solution for any wave function in the right hand side, whereas the homogeneous Schrödinger equation has only trivial solution with the approximate potentials obtained from the approximate solution of Maxwell equations. We construct an iterative process by adding nonhomogeneous term proportional to \( \psi_i \) to eq. (7) for \( \psi_{i+1} \):

\[ (-1/2 \nabla^2 + U_i(r, z)) \psi_{i+1}(r, z) + U_i(r, z) \psi_i(r, z) = 0. \]

We start with some initial value of the normalized wave function \( \psi = \psi_0 \), solve nonhomogeneous Maxwell equations, and then find the next iteration \( \psi_{i+1} \) from eq. (18). The resonance increase of the norm of \( \psi_{i+1} \) (by 7-8 orders of magnitude) indicates that the nonhomogeneous term is small in comparison with the analogous term containing \( \psi_{i+1} \), and therefore, the iterations can be stopped (after each iteration, the solution is normalized according to eq. (8)). In order to calculate the eigenfunction of the wave function equation, we vary the scale factor \( L_z \) (at fixed \( L_r/L_z \)) until we find the resonance in the norm of \( \psi \). The energy value is then calculated.

We put the described procedure to a test by calculating of the odd wave function of the first exited state of the 1-d harmonic oscillator. A few first odd moments of the probability density coincided with the analytical values within a few tenths of percent (due to the interpolation over 8 points).
3.2 "Optimal" lattice

The search of the resonance value of $L_z$ was repeated after changing the ratio $L_r/L_z$ or the functions $f(\rho)$ and $g(\zeta)$. The energy was found to depend (within a few percents) on the particular choice of $L_r/L_z$, $f(\rho)$ and $g(\zeta)$, suggesting a need for an increase of the number of parameters of the wave function. Such an approach leads to a prohibitive calculation time. An alternative approach used in this paper is to choose the ratio $L_r/L_z$ and functions $f(\rho)$ and $g(\zeta)$ in such a way that the result achieves its asymptotic value with a relatively small number of parameters (of order of thousand) determining the wave function and the potentials. For this purpose, we introduce additional requirements on functions (14, 15), which reduce arbitrariness and increase the accuracy of the result.

1) We require a balance of contributions to the norm of the wave function from large and small distances along $r$ axis and a balance of contributions from large distances between $r$ and $z$ axis. This requires that the probability density as a function of $2\rho - 1, \zeta$ should be close to axi-symmetric function (i.e., look like the up-side down cup). This is analogous to the requirement that a few first moments should be zero (we limited ourselves with the dipole, quadrupole and octopole moments):

$$\int (2\rho - 1)^2 \psi^2(r, z) \, dS = 0,$$

$$\int ((2\rho - 1)^2 - z^2) \psi^2(r, z) \, dS = 0,$$

$$\int (2\rho - 1)^3 \psi^2(r, z) \, dS = 0.$$

2) We characterize the fields by their values on the lattice with coordinates defined by the zeros of the Legendre polynomials in order to use the Gaussian quadratures numerical integration method. We can also estimate the accuracy of obtained solution by calculating the probability of the total square deviation of the full solution from the solution $\psi_4$ interpolated over each fourth point of the lattice:

$$\delta P_4 = \int (\psi(r, z) - \psi_4(r, z))^2 \psi^2(r, z) \, dS.$$

The minimum of this deviation, i.e. the good approximation of the solution by the low order polynomial, suggests a high precision of numerical integration and differentiation.

3) We can impose one more condition on the ratio $L_r/L_z$: we assume that the characteristic volume $L_r^2 L_z$ is less sensitive to numerical errors and require that its derivative with respect to the ratio $L_r/L_z$ (for fixed $f(\rho)$, $g(\zeta)$ and resonance $L_z$) is zero:

$$\frac{\partial}{\partial(L_r/L_z)} L_r^2 (L_r/L_z) L_z (L_r/L_z) = 0.$$

We should stress that conditions (19, 23) are not physical equations but serve as a criterion for the most accurate solution at fixed lattice size. We impose
these conditions one by one and calculate the ratio $L_r/L_z$ and the energy $H$ for fixed $f(\rho)$ and $g(\zeta)$. The stability of results (minimum of relative dispersion of $L_r/L_z$) with respect to the choice of conditions 19..23 is considered as an evidence of a good choice for functions $f(\rho)$ and $g(\zeta)$.

The most time-consuming part of the calculation is solving the system of linear equations for the field values on the lattice sites. The calculation time grows as the cube of the number of lattice sites. It also grows with the increasing precision of numerical differentiation. Therefore we differentiate the wave function using its interpolation over 9 or 11 points and electromagnetic potentials over 5 points because they are smoother than the wave function. Numerical integration using Gaussian quadratures is exact for polynomials of order of twice number of points on the lattice. In this work we use lattices $20 \times 10$, $24 \times 12$ and $28 \times 14$ along the respective $\rho$ and $\zeta$ axis.

### 4 Numerical results

We use transformations (14, 15) and seek a resonance value of the scale parameter $L_z$, which leads to a convergent solution of Maxwell equations (5, 6) and the wave equation (7) for fixed functions (16,17) and different values of the ratio $L_r/L_z$. Using five additional criteria (19..23) for the optimal lattice, we calculate five different values of the ratio $L_r/L_z$. A small scatter of these values $\delta L/L = \delta(L_r/L_z)/(L_r/L_z)$ is considered as a criterion for the optimal choice of functions (16, 17). The table below lists the results for three lattices of different size where the minimum value of $\delta L/L$ is below 3%.

| Lattice sites | Num. diff. | $\delta P_k$ | $\delta L/L$ | $2 m_e c^2 \mu_e/e^3$ | $\bar{r}$ | $\sqrt{z^2}$ |
|---------------|------------|-------------|--------------|----------------------|--------|----------|
| $20 \times 10$ | 9 pt       | 0.1576      | 2.1 %        | 134.13± 0.14         | 0.1362 | 0.0744   |
| $24 \times 12$ | 11 pt      | 0.0144      | 1.4 %        | 136.55± 0.20         | 0.1363 | 0.0735   |
| $28 \times 14$ | 11 pt      | 0.0073      | 0.3 %        | 136.82± 0.05         | 0.1363 | 0.0718   |

The experimental value of the quantity $2 m_e c^2 \mu_e/e^3$ is 137.19 (it is slightly higher than $1/\alpha = 137.03$ because of the anomalous magnetic moment of the electron). So, the calculations on the $28 \times 14$ lattice agree with the experimental value within few tenths of percent, and the discrepancy decreases with increasing the lattice size. As expected, the characteristic size of the toroidal structure is almost an order of magnitude smaller than the fundamental length (see two last columns of the table, which list the ratios of the average radius and height of the toroidal structure to the fundamental length) and two orders of magnitude...
smaller than the Compton length of electron. Its typical values are of order $2 \cdot 10^{-14}$ m. The correction to the second bounded state of the toroidal structure in non-uniform proton field is 7 orders of magnitude smaller than the second level energy of a hydrogen atom and 10 times smaller than the Lamb shift. Thus, the contribution from the finite size of the totoidal structure is comparable with the radiation corrections, at least at small energies.

5 Conclusions

We propose a relativistic Lagrangian formalism, which lead to a system of equations for the complex scalar and electromagnetic fields. Assuming axial and mirror symmetries and specific angular and time dependence of the wave function, we obtain a system of three equations for the stationary amplitude of the wave function and two electromagnetic potentials as functions of two spatial variables. We construct an iteration procedure for solving these equations on a lattice and the algorithm for selecting an optimal lattice. The numerical solution of this system allow calculating the energy (or the mass) of the soliton-like state. We interpret the obtained toroidal structure as an electron because the value of dimensionless constant $e^3/(2mc^2\mu)$ coincides with the experimental value $e^3/(2mc^2\mu_e) \approx e^2/\bar{h}c = \alpha$ within few tenths of percent. Further increase of numerical precision is desirable.

One of possible confirmations of the proposed Lagrangian formalism and the analogous axial symmetry might be the construction of another soliton-like solution with larger mass and smaller magnetic moment that can be interpreted as $\mu$-meson.

The correspondence principle requires constructing the effective Lagrangian of toroidal structure, investigating it’s properties, including gauge invariance, and solving the well known electrodynamic problems on its basis. One could try to find electron stationary states in external fields, in particular, to calculate bound states of toroidal electron in the field of the proton (hydrogen atom).

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References

[1] Richard Feynman, *QED: The strange theory of light and matter*, Princeton Univ. Press., (1985)

[2] Vladimir Ritus, *arXiv: hep-th/0509209*

[3] David J. Gross, *Physics Today*, Dec. 1989.

[4] Anastas Anastasov, *arXiv: hep-physics/9712044*

[5] Griffiths, David J. *Introduction to Elementary Particles*. Wiley, John and Sons, Inc. ISBN 0-471-60386-4, (1987)
[6] E.P. Likhtman arXiv: hep-th/0210130
[7] Th. Kaluza. Sitzungsber. Preuss. Acad. Wiss., K1, 966, (1921)
[8] O. Klein. Z. Phys., 37, 895, (1926)