Radion stabilization with(out) Gauss–Bonnet interactions and inflation

D. Konikowska $^a$, M. Olechowski $^{a,b}$, M.G. Schmidt $^b$

$^a$ Institute of Theoretical Physics, Warsaw University
ul. Hoża 69, PL–00–681 Warsaw, Poland

$^b$ Institut für Theoretische Physik, Universität Heidelberg
Philosophenweg 16, D–69120 Heidelberg, Germany

Abstract

Radion stabilization is analyzed in 5–dimensional models with branes in the presence of Gauss–Bonnet interactions. The Goldberger–Wise mechanism is considered for static and inflating backgrounds. The necessary and sufficient conditions for stability are given for the static case. The influence of the Gauss–Bonnet term on the radion mass and the inter–brane distance is analyzed and illustrated by numerical examples. The interplay between the radion stabilization and the cosmological constant problem is discussed.
1 Introduction

The idea that we live on a brane in a higher dimensional space–time has attracted a lot of attention since Hořava and Witten presented [1] their 11–dimensional model with 10–dimensional branes, motivated by M–theory. Many 5–dimensional models with 4–dimensional branes (some of them being effective models derived from the Hořava–Witten one) have been discussed since then. In any such model, the distance between the branes should be stabilized. A scalar field related to that distance is usually called the radion. Several, more or less precise, definitions of the radion are used in the literature. Of course, not every such a definition is reasonable. It was pointed out [2] that the radion should at least fulfill appropriate linearized Einstein equations of motion. It is easy to check that in models where only gravity propagates in the bulk the radion corresponds to the only gauge–invariant scalar perturbation of the metric. Moreover, such a radion is massless, which means that the distance between branes is not stabilized. Goldberger and Wise [3] proposed a class of simple models to stabilize this distance. They introduced a bulk scalar field with interactions described by some bulk and localized brane potentials. Perturbations of such a field mix with the scalar perturbations of the metric, giving rise to an infinite tower of KK states in an effective 4–dimensional theory. The radion can be defined as the lightest of those states. This lowest KK mode might sometimes have negative mass squared, which means that a given configuration is unstable. Situations where such a tachyonic radion may appear were discussed in several papers [4, 5, 6]. In [6] quite general criteria for such instabilities were presented.

One of the possible generalizations of this type of models takes into account interactions of higher order in the curvature tensor. Such interactions naturally appear in the $\alpha'$ expansion in string theories [7]. The simplest correction is the famous Gauss–Bonnet (GB) term. Many aspects of 5–dimensional models with GB interactions have been already discussed in the literature. However, the influence of GB interactions on the radion stabilization has not been addressed\(^1\). It seems that an appropriate way of modeling the radion is by introducing an additional bulk scalar field à la Goldberger and Wise. In this work we analyze the radion stabilization in models with GB interactions and with such a bulk scalar field. We consider 5–dimensional backgrounds with flat static 4–dimensional sections as well as with inflating ones. The case with inflation is especially interesting, because it was shown

\(^1\)The problem of radion stabilization in models with the GB term has been recently addressed in [8]. However, the definition of the radion used in that paper does not seem to be adequate. It is considered there as the (5,5) component of the metric. A potential for such a field is introduced and this explicitly breaks symmetries of general relativity.
that during inflation stabilization of the radion is more difficult. We find some generic conditions for radion stability in terms of background solutions. We discuss these conditions using analytical arguments as well as results of numerical calculations.

Models with scalar fields and Gauss–Bonnet interactions have been considered in the literature, but in contexts other than radion stabilization. Problems of singularities and fine tuning in such models were discussed in [10]. Phantom cosmology with GB corrections was investigated in [11, 12]. In [13], the interplay between the GB term and the quintessence scalar was analyzed. Experimental constraints on the quintessence-GB coupling were derived in [14]. Cosmological models with scalar dependent GB interactions were considered in [15]. In [16], a scalar field was used to support a smooth brane in an Einstein–GB gravity model. A cosmological model with GB interactions and two scalar fields was investigated in [17].

The organization of this paper is as follows: In the next section, we present the equations of motion and the boundary conditions for models with GB interactions and the Goldberger–Wise scalar field. The results are presented for an ansatz describing the 5–dimensional space–time with 4–dimensional inflating de Sitter sections and two branes. In section 3, equations of motion and boundary conditions for the scalar perturbations in such a model are derived and discussed. A variational formula for the radion mass is given in section 4. It is used later to derive the necessary and sufficient conditions for radion stability in the case of a vanishing Hubble constant. In section 5, we compare two classes of solutions present in models with the GB term. It is shown that the stability analysis strongly disfavors the so called ”new” solutions, which appear to be always unstable. Section 6 contains a discussion of the changes caused by the presence of GB interactions in the action. The relation between the radion stabilization and the cosmological constant problem is shortly discussed. Some results of numerical calculations are presented in section 7. Section 8 contains our conclusions.

## 2 Background solutions

We consider 5–dimensional models described by the action

\[
S = \int d^5 x \sqrt{-g} \left[ \frac{1}{2 \kappa^2} \left( R + \alpha R^2_{\text{GB}} \right) - \frac{1}{2} (\nabla \Phi)^2 - V(\Phi) - \sum_{i=1}^2 \delta(y - y_i)U_i(\Phi) \right],
\]

where \( R^2_{\text{GB}} \) is the Gauss–Bonnet term quadratic in the curvature tensor:

\[
R^2_{\text{GB}} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}.
\]
The scalar field $\Phi$ interacts via the bulk potential $V(\Phi)$ and two brane potentials $U_i(\Phi)$ located at $y_1$ and $y_2$. We do not include extrinsic curvature terms in the action, because we work in the up–stairs picture of the $S^1/Z_2$ orbifold, where all total derivatives integrate to zero. The boundary conditions are obtained by integrating the equations of motion around the branes. This method is equivalent to using the Israel junction conditions [18] after adding appropriate extrinsic curvature terms to the action [19].

We are interested in warped background solutions with the following ansatz for the metric and the scalar field:

$$ds^2 = a(y)^2 \left(-dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j + dy^2\right),$$

$$\Phi = \phi(y).$$

It describes the 5–dimensional space–time warped along the $y$–coordinate with the 4–dimensional de Sitter slices inflating with the Hubble constant equal to $H$. For $H = 0$, the same ansatz describes a warped geometry with flat Minkowski foliation. For simplicity, the 3–dimensional space has been chosen to be flat. Also for simplicity, we assume that inflation is driven by some brane fields which do not directly influence the bulk scalar perturbations, and that the inflaton and radion sectors can be treated separately. In this paper, the Hubble constant is treated just as a given parameter. Of course, analysis going beyond such a simplification would be very interesting.

The bulk equations of motion for the system described by action (1) and satisfying ansatz (3–4) are given by (we use units $\kappa = 1$)

$$\phi'' + 3 \frac{a'}{a} \phi' - a^2 V' = 0,$$

$$\left[\frac{a''}{a} - 2 \left(\frac{a'}{a}\right)^2 + H^2\right] \frac{\xi}{a^2} + \frac{1}{3} \phi''^2 = 0,$$

$$3 \left[\left(\frac{a'}{a}\right)^2 - H^2\right] \left(1 + \frac{\xi}{a^2}\right) - \frac{1}{2} \phi'' + a^2 V = 0.$$

The primes denote differentiations with respect to the appropriate (implicit) arguments, i.e., with respect to the scalar field $\Phi$ in the case of $V$, and with respect to the coordinate $y$ for $a$ and $\phi$. To simplify the formulae, a new function – depending on the warp factor, the Hubble constant and the coefficient of the GB term – has been introduced, namely

$$\xi = a^2 - 4\alpha \left(\left(\frac{a'}{a}\right)^2 - H^2\right).$$
The modifications of the background equations of motion caused by the GB term can be written in a simple way in terms of just this single function. It is interesting that, as we will show later, this also holds for almost all other modifications (equations of motion for the perturbations, stability conditions, most of the boundary conditions, etc.) considered in this paper. The only exception is one of the boundary conditions we discuss below.

The boundary conditions are obtained by integration of the equations of motion over infinitesimally small intervals around the branes. Of course, one must add to the eqs. (5–7) the contributions coming from the brane potentials $U_i$. Such integration of eq. (5) gives the boundary condition for the derivative of the scalar background:

$$
\lim_{y \to y_i^\pm} \frac{\phi'}{a} = \pm \frac{1}{2} U_i'.
$$

(9)

The corresponding condition for the warp factor $a(y)$ can be obtained from eq. (9) after using eq. (7) to eliminate $\phi'$. The result reads

$$
\lim_{y \to y_i^\pm} \left\{ \frac{a'}{a^4} \left[ a^2 - 4 \alpha \left( \frac{a'}{a} \right)^2 - H^2 \right] \right\} = \mp \frac{1}{6} U_i.
$$

(10)

The two above equations describe the jumps of the $\mathbb{Z}_2$ odd functions $a'(y)$ and $\phi'(y)$ at the $\mathbb{Z}_2$ fixed points $y_1$ and $y_2$. Observe that the expression in the square bracket of the last equation differs from $\xi$ defined in (8) just by a factor 1/3 in front of the term containing $a'^2$. The origin of this additional factor is the following: As the derivative $a'$ is a function discontinuous at the positions of the branes, the second derivative $a''$ contains Dirac delta contributions. Hence expressions of the type $a''(a')^2$ must be regularized before integration. Any regularization leads to the same result

$$
\int_{y_i^\pm}^{y_i^\pm+\epsilon} a''(a')^2 dy = \frac{1}{3} (a')^3 \bigg|_{y_i^\pm-\epsilon}^{y_i^\pm+\epsilon},
$$

(11)

which gives the factor 1/3 in eq. (10).

### 3 Scalar perturbations

The main interest in our analysis is the dynamics of scalar perturbations around the background metric given by the solutions to the equations of motion (5–7) with the boundary conditions (9–10). In general, one could perturb all the components of the metric, the scalar field and the positions...
of the branes. Some of such perturbations correspond to gauge degrees of freedom. This gauge dependence has been carefully discussed in the literature [20, 21, 22]. One can choose gauge independent combinations of perturbations or work in a specific gauge. In the following we will use the generalized longitudinal gauge. The scalar perturbations can be written in this gauge as

\[
ds^2 = a^2 \left[ (1 + 2F_1) \left( -dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j \right) + (1 + 2F_2) dy^2 \right],
\]

(12)

where \(a\) and \(\phi\) are the \(y\)-dependent background solutions to eqs. (5–7) and (9–10). The perturbations \(F_i\) are functions of all the 5 coordinates. This new ansatz has to be substituted into the equations of motion obtained from action (1) and expanded in powers of the (small) perturbations \(F_i\). The zeroth order equations are of course given by (5–7) and (9–10). The equations linear in the perturbations can be used to determine the mass eigenstates for such perturbations in a given background. Higher order terms of the expansion correspond to interactions which are more than bilinear in the perturbations. We are mainly interested in the masses of the 4-dimensional scalars, especially the radion, so we only need the linearized equations.

The off–diagonal components of the linearized Einstein equations give the following conditions, which must be satisfied in order to stay in the longitudinal gauge:

\[
\frac{\xi'}{\xi} F_1 + \frac{a'}{a} F_2 = 0,
\]

(14)

\[
(\xi F_1)' + \frac{1}{3} a^2 \phi' F_3 = 0.
\]

(15)

The diagonal Einstein equations, combined with the background equations of motion (5–7), give the dynamical equation for the scalar perturbations:

\[
\frac{\xi}{a^2} \left[ (\Box + 4H^2) F_1 + 4 \frac{a'}{a} F_1' - 4 \left( \frac{a'}{a} \right)^2 F_2 \right] + \frac{1}{3} \phi'^2 F_2
\]

\[
+ \left( \frac{1}{3} \phi'' + \frac{a'}{a} \phi' \right) F_3 - \frac{1}{3} \phi' F_3' = 0,
\]

(16)

where \(\Box\) is the 4-dimensional D’Alembert operator on the de Sitter slice of the 5-dimensional space–time (which will become the ordinary flat space D’Alembert operator when we later set the Hubble constant \(H\) to zero). The boundary conditions for the scalar perturbations read

\[
\lim_{y \to y_i^\pm} (F_3' - \frac{1}{2}a F_3 U''_i) = \pm \frac{1}{2} a F_3 U''_i.
\]

(17)
Equations (14) and (15) show that in the longitudinal gauge the three scalar perturbations are not independent. These equations can be used to eliminate $F_3$ and $F_2$, and to rewrite the dynamical equation of motion in terms of $F_1$ alone. In addition, taking into account the symmetry of the warped background (3–4), we can separate the variables in the equation of motion for $F_1$. Defining

$$F_1(t, \vec{x}, y) = \sum_{m^2} F_{m^2}(y) \left[ \int d^3k f_{(m^2,k)}(t) e^{i\vec{k}\vec{x}} \right],$$

we obtain the following equations

$$\ddot{f}_{(m^2,k)} + 3H \dot{f}_{(m^2,k)} + \left( a^{-2Ht} \vec{k}^2 + m^2 \right) f_{(m^2,k)} = 0,$$

$$F''_{m^2} + \left[ 2\frac{\xi'}{\xi} - \frac{a'}{a} - 2\frac{\phi''}{\phi'} \right] F'_{m^2} + \left[ \frac{\xi''}{\xi} - \frac{\xi' a'}{\xi a} - 2\frac{\xi' \phi''}{\xi \phi'} - \frac{a^3 \xi'}{3 a' \xi^2} (\phi')^2 + m^2 + 4H^2 \right] F_{m^2} = 0,$$

where the separation constant $m^2$ turns out to be the mass squared of scalars in the effective 4–dimensional description.

Let us now rewrite the boundary conditions (17) in terms of $F_{m^2}$. The l.h.s. of (17) describes limits (from below or from above) of some expressions smooth in the bulk. As a result we can rewrite those expressions using the bulk equations (14) and (15) in order to replace $F_2$ and $F_3$ with $F_1$. However, this way the second derivative of $F_1$ appears in the boundary conditions. It can be eliminated with help of the dynamical bulk equation (20). The final result takes the form

$$\pm b_1^{(2)} \lim_{y \to y_1^+} \left( F'_{m^2} + \frac{\xi'}{\xi} F_{m^2} \right) + \left( m^2 + 4H^2 \right) \lim_{y \to y_1^-} (F_{m^2}) = 0,$$

$$\pm b_2^{(2)} \lim_{y \to y_2^-} \left( \frac{1}{2} aU''_{1} + \frac{a'}{a} - \frac{\phi''}{\phi'} \right),$$

where

$$b_1 = \lim_{y \to y_1^+} \left( \frac{1}{2} a U''_{1} + \frac{a'}{a} - \frac{\phi''}{\phi'} \right), \quad b_2 = \lim_{y \to y_2^-} \left( \frac{1}{2} a U''_{2} - \frac{a'}{a} + \frac{\phi''}{\phi'} \right).$$

The above boundary conditions become particularly simple for a new variable $Q_{m^2} = \xi F_{m^2}$ (we omit the subscript $m^2$ to simplify the notation):

$$b_1 Q'(y_1^+) + (m^2 + 4H^2) Q(y_1^+) = 0,$$

$$- b_2 Q'(y_2^-) + (m^2 + 4H^2) Q(y_2^-) = 0.$$
The dynamical equation of motion for $Q$ has also a very simple form

$$-(p Q')' + q Q = \lambda p Q,$$

(24)

where $p = 3/(2a\phi'^2)$, $q = (a^2\xi')/(2a'\xi^2)$ and $\lambda = (m^2 + 4H^2)$. This is the standard form of the Sturm–Liouville differential equation. The boundary conditions (23) can also be written in the usual (see e.g. [23]) form

$$\frac{\partial Q}{\partial n}(y_i) + \sigma(y_i)Q(y_i) = 0,$$

(25)

where in our case $\sigma(y_i) = -\lambda/b_i$. The $\partial/\partial n$ differentiation is in the direction of the outer normal at the boundary. For our 1–dimensional equation it is $(-d/dy)$ at $y_1$ and $(+d/dy)$ at $y_2$. The above boundary conditions are self–adjoint, but unfortunately quite unusual, because $\sigma$ depends on the eigenvalue $\lambda$. This dependence is caused by the procedure of obtaining the boundary conditions described before eq. (21).

With such unconventional boundary conditions (25) depending on eigenvalues, it is not possible to apply directly the analysis of the Sturm–Liouville systems from the standard textbooks [23]. However, it is possible to modify the standard arguments to obtain interesting results for the case at hand.

## 4 Stability conditions

For the problem of the inter–brane distance stabilization, the most interesting feature of the analyzed Sturm–Liouville equation is its lowest eigenvalue $\lambda_0 = m_0^2 + 4H^2$. There are three possibilities. First: the lowest eigenvalue corresponds to positive $m_0^2$, which means that the background solution is stable against scalar perturbations, and the positions of the branes are really stabilized. Second: the lowest eigenvalue gives vanishing $m_0^2$, the radion is massless and the inter–brane distance is not stabilized. Third: there is at least one tachyon signaling the instability of a given background configuration.

We start our analysis with identifying a variational problem which is equivalent to the differential equation (24) with the boundary conditions (25). Multiplying eq. (24) by $Q$ and integrating over the 5–th coordinate, one gets

$$\lambda \int pQ^2 = \int [-(pQ'_{\lambda})'Q_{\lambda} + qQ^2_{\lambda}]
= \int [pQ^2_{\lambda} + qQ^2_{\lambda}] + p\sigma Q^2_{\lambda} = \int [pQ^2_{\lambda} + qQ^2_{\lambda}] - \lambda p b^{-1}Q^2_{\lambda},$$

(26)
where the subscript $\lambda$ is added to stress the relation between an eigenstate and its eigenvalue. This equation can be used to express the eigenvalue $\lambda$ in terms of the corresponding eigenfunction $Q_\lambda$ as follows:

$$
\lambda = \frac{\int [pQ^2_\lambda + qQ^2_\lambda]}{\int [pQ^2_\lambda + b_1^{-1}(pQ^2_\lambda)|_{y_1} + b_2^{-1}(pQ^2_\lambda)|_{y_2}].}
$$

The smallest eigenvalue, $\lambda_0$, can be obtained by minimizing the r.h.s. of the above equation over all smooth functions $Q$ defined on the interval $[y_1, y_2]$

$$
\lambda_0 = \min_Q \left( \frac{\int [pQ^2 + qQ^2]}{\int [pQ^2 + b_1^{-1}(pQ^2)|_{y_1} + b_2^{-1}(pQ^2)|_{y_2}]} \right).
$$

Using the relation between $\lambda$ and $m^2$ one obtains the following formula for the radion mass

$$
m^2_0 = -4H^2 + \min_Q \left( \frac{\int [pQ^2 + qQ^2]}{\int [pQ^2 + b_1^{-1}(pQ^2)|_{y_1} + b_2^{-1}(pQ^2)|_{y_2}]} \right).\tag{29}
$$

For a general background (solution to eqs. (5–7)), the above expression cannot be minimized explicitly. But we can obtain some bound using just one trial function $Q$, namely the constant one. Recalling the definitions of $p$ and $q$, we get

$$
m^2_0 \leq -4H^2 + \frac{\int dy \left( \frac{a^2 \xi' / a' \xi^2}{3 \int dy (1/a' \xi^2) + \sum [b_i a(y_i) \phi^2(y_i)]^{-1}} \right).\tag{30}
$$

In the limit of $b_i \to \infty$ (very stiff brane potentials, i.e. potentials with very large second derivative) and $\xi \to a^2$ (no GB interactions), this expression simplifies to an analogous bound obtained in [2].

In models with inflating branes, i.e. for $H^2 > 0$, stability of the inter-brane distance occurs when $\lambda_0 > 4H^2$. The formula (28) can be used to check this condition, but in practically all models this can be done only by numerical calculations. However, some interesting analytic results can be obtained for the static branes for which $\lambda = m^2$. In such a case, the question of stability reduces to the issue of the sign of $\lambda_0$. In the rest of this section, we assume $H = 0$.

It will prove useful to rewrite the differential equation (24) in terms of other variables. Using the definitions

$$
u = \frac{1}{a^{1/2} \phi'} Q, \quad \theta = \frac{a'}{a^{3/2} \phi'},
$$

$^2$In the case of more conventional, $\lambda$–independent boundary conditions, the contributions from the boundaries appear with the opposite sign in the numerator [2].
and the background equations (5–7), the equation of motion (24) can be written in the following simple form

$$u'' + \left( \lambda - \frac{\theta''}{\theta} \right) u = 0. \quad (32)$$

The differential equations (20), (24) and (32) are not very convenient for background solutions with $\phi'$ vanishing at some $\tilde{y} \in [y_1, y_2]$. The reason is that some of the coefficients in those equations diverge for $\phi' = 0$. In such cases it is better to use a generalization of the Mukhanov variable [24], defined in our model as

$$v = a^{3/2} \left[ F_3 - \frac{a}{a'} F_1 \right], \quad (33)$$

for which the equation of motion reads

$$v'' + \left( \lambda - \frac{(1/\theta)''}{(1/\theta)} \right) v = 0. \quad (34)$$

Using the background equations (5–7), one can show that

$$\frac{(1/\theta)''}{(1/\theta)} = \frac{15 a^2}{4 a^2} - 15 \frac{a^2 \phi'^2}{6 \xi} - \frac{1 a^3 \xi \phi'^2}{3 \xi^2 a'} + 2 \frac{a^6 \psi_4}{9 \xi^2 a'^2} + \frac{4 a^5 \phi' V'}{3 \xi a'} + a^2 V'', \quad (35)$$

which is explicitly regular for vanishing $\phi'$. The variables $v$ and $Q$ are related by

$$\lambda Q = -a^{1/2} \phi' \theta^{-1} (\theta v)' . \quad (36)$$

Now we check if there is a massless scalar in the model. Equation (32) is simple enough\(^3\) to be solved for vanishing $\lambda$ almost explicitly:

$$u_0(y) = \theta(y) \left[ c_1 + c_2 \int_{y_1}^{y} dy' \theta^{-2}(y') \right]. \quad (37)$$

Assuming that $\phi'(y) \neq 0$ for all $y$ in the bulk, one can rewrite (37) in terms of $Q$. Using also the boundary condition (23) at the first brane (for non–vanishing $b_1$), one finds the following result

$$Q_0(y) = \frac{\xi(y)}{\xi(y_1)} - \frac{a'(y)}{\xi(y_1) a^2(y)} \int_{y_1}^{y} dy' \frac{a^2(y') \xi'(y')}{a'(y')}, \quad (38)$$

$$Q'_0(y) = \frac{1}{3} [\phi'(y)]^2 \frac{a(y)}{\xi(y_1) \xi(y)} \int_{y_1}^{y} dy' \frac{a^2(y') \xi'(y')}{a'(y')}, \quad (39)$$

\(^3\)Unfortunately, for non–zero Hubble constant $H$ the equations of motion for the variables $u$ and $v$ can not be written in such a simple form as (32) and (34).
where the normalization has been chosen as $Q(y_1) = 1$. The boundary condition at the second brane, $b_2Q_0(y_2) = 0$, can be fulfilled in several ways. It seems that one of the possibilities could be the vanishing of the integral $\int_{y_1}^{y_2}(a^2\xi'/a')$. This, however, can not be realized because the integrand function must be positive for all $y$. The reason is that this function is closely related to the kinetic term for gravitons. This can be seen by considering (traceless and transverse) tensor perturbations $F_{\mu\nu}^{TT}$ of the background metric (3). By expanding action (1) to the second order in such perturbations we can obtain the appropriate kinetic term. We have found that, up to normalization and some total derivatives, this kinetic term is given by

$$-\frac{a^2\xi'}{a'}(\nabla F_{\mu\nu}^{TT})^2.$$  \hfill (40)

The coefficient in front of $(\nabla F_{\mu\nu}^{TT})^2$ must be negative, because only then one can define a tower of 4–dimensional KK graviton states with the standard sign of the kinetic terms (an effective action for the 4–dimensional KK states of scalar perturbations was obtained in [25]). Hence for all $y$

$$\frac{\xi'(y)}{a'(y)} > 0,$$  \hfill (41)

because otherwise some of the graviton KK states become ghost–like.

Let us now return to the discussion of the boundary condition for solutions given by eqs. (38) and (39). The positivity of the ratio $\xi'/a'$ means that the integral $\int_{y_1}^{y_2}(a^2\xi'/a') > 0$. Thus the boundary condition at the second brane can be fulfilled only when $b_2$ or $\phi'/(y_2)$ vanishes.

One can repeat the above reasoning starting from the boundary condition at the brane located at $y_2$. Choosing the integration constants in (37) in an appropriate way (for non–vanishing $b_2$), one finds that the massless mode exists in such a situation only when $b_1$ or $\phi'(y_1)$ is zero. Putting both cases together, we find that for $\phi'$ non–zero everywhere in the bulk the necessary and sufficient condition for existence of a massless mode is

$$b_1b_2\phi'(y_1)\phi'(y_2) = 0.$$  \hfill (42)

Now we identify conditions sufficient for the stability of the considered warped space–time with branes. From expression (27) it is obvious that all the eigenvalues are positive if $p(y)$, $q(y)$ and $b_i$ are positive. This conclusion is correct when the functions $p(y)$ and $q(y)$ are finite for all $y$, because otherwise our Sturm–Liouville system becomes singular, and a more careful treatment is necessary (quantities $b_i$ can be divergent, in the limit $b_i \to \infty$ the boundary
conditions (25) reduce to \( Q'(y_i) = 0 \). The function \( p(y) \) diverges at the points where \( \phi'(y) = 0 \). Using the definition of \( \xi \), we get

\[
\frac{\xi'}{a'} = 2\xi + \frac{4a}{3\xi}(\phi')^2;
\]

which shows that \( q(y) \) diverges at the points where \( \xi(y) = 0 \). The finiteness requirement for \( p(y) \) and \( q(y) \) is equivalent to the requirement that \( \phi' \) and \( \xi \) are non–zero for all \( y \). However, from the background equation of motion (6) we see that \( \phi' \) must be zero at the points where \( \xi \) is zero. Hence the condition \( p(y) \neq \pm\infty \) is stronger than the condition \( q(y) \neq \pm\infty \).

Thus the brane system is stable if, for all \( y \),

\[
 b_i > 0, \quad \frac{\xi'(y)}{a'(y)} > 0, \quad \phi'(y) \neq 0.
\]

These are the sufficient conditions for the stability, but we show below that they are in fact also the necessary ones.

One of the above conditions, namely the positivity of \( \xi'(y)/a'(y) \), we have already discussed. Any non–positive value of this ratio leads to some ghost–like states in the KK tower of the 4–dimensional gravitons. As a result, the second condition in eq. (44) is anyway necessary for the stability of the model.

Below we will use the following obvious principle of variational calculus. Let us consider two problems of minimalization of some functionals. If for every function the functional in the first problem is not bigger than the functional in the second problem then the first minimum can not be bigger than the second one.

Let us consider the following two problems. Both consist in minimizing the expression given in (28) with the same positive \( p(y) \) and \( q(y) \) but with different \( b_i \). Let the values of \( b_1 \) and/or \( b_2 \) be bigger in the second problem. Bigger \( b_i \) means smaller (at least not bigger) denominator in (28) and thus bigger (at least not smaller) the whole expression to be minimized. Applying the above mentioned rule of the variational calculus, we see that the lowest eigenvalue in the problem with bigger \( b_i \) is bigger (at least not smaller) than the lowest eigenvalue in the problem with smaller \( b_i \). The lowest eigenvalue, \( \lambda_0 \), is a monotonic, non–decreasing function of \( b_i \).

We apply the above reasoning to negative \( b_1 \) or \( b_2 \). The monotonic character of \( \lambda_0 \) as a function of \( b_i \) means that the lowest eigenvalue for any negative \( b_1 \) (\( b_2 \)) can not be bigger than the one for vanishing \( b_1 \) (\( b_2 \)), which was previously shown to be zero. We get \( \lambda_0 \leq 0 \) for \( b_1 < 0 \) or \( b_2 < 0 \). On the other hand, \( \lambda_0 \) can not be equal to zero, because we have shown that a
zero mode exists only if one of the $b_i$ vanishes (we consider a situation with $\phi'(y_i) \neq 0$, otherwise there is a zero mode for arbitrary $b_i$). A background with a negative value of any of the boundary parameters $b_i$ is unstable.

We have thus shown that the first two conditions in eq. (44) are not only sufficient, but also necessary for the stability. We will now prove that the same is true also for the third condition, which means that all backgrounds with $\phi'$ vanishing somewhere in the bulk are unstable. Let us assume that $\phi'(\tilde{y}) = 0$ for some $y_1 < \tilde{y} < y_2$. One can prove that in such a case there is at least one negative eigenvalue in the problem defined by the differential equation (24) with the boundary conditions (25).

In order to show this, we consider the behavior of the solutions to eq. (24) for large negative $\lambda$. Such behavior can be found using the variable $v$ defined in (33). The differential equation for $v$ is regular also for vanishing $\phi'$, so the limit $\lambda \to -\infty$ can be obtained simply by dropping the $\theta$–dependent term in eq. (34). The corresponding solution reads

$$v_{-\infty}(y) = c_3 \exp[\sqrt{-\lambda}(y-y_1)] + c_4 \exp[-\sqrt{-\lambda}(y-y_1)].$$  

(45)

Using the relation (36), we obtain expressions for $Q(y)$ and $Q'(y)$ in the limit of large negative $\lambda$. The boundary condition (25) at $y_1$ can be fulfilled if $c_3 \approx c_4$ in the leading order in $1/\sqrt{-\lambda}$. Hence $c_3$ is not small compared to $c_4$, and away from the first brane the solution is dominated by the exponentially growing term

$$Q_{-\infty}(y) \approx -\frac{c_3 a^{1/2}(y)\phi'(y)}{\sqrt{-\lambda}} \exp[\sqrt{-\lambda}(y-y_1)].$$  

(46)

The important feature of this solution is that it is proportional to $\phi'(y)$ so it vanishes at $\tilde{y}$ (and changes sign if $\phi'(y)$ changes sign). This should be compared to the behavior of $Q_0$. As eqs. (38) and (39) imply that the derivative $Q_0'(y)$ is positive, $Q_0(y)$ is also positive for all $y$. The solution $Q_{\lambda}(y)$ changes continuously with $\lambda$ changing between 0 and $-\infty$. Hence there must exist a negative $\tilde{\lambda}$ for which $Q_{\tilde{\lambda}}$ has a zero point but is nowhere negative. It is easy to see that such a zero point must be at $y = y_2$, and that $Q_{\tilde{\lambda}}'(y_2) < 0$. (Both features result from the simple fact that, due to the equation of motion, $Q$ and $Q'$ may vanish at the same point only for $Q$ vanishing everywhere.) The above arguments may be summarized in the following form: if $\phi'$ vanishes at any point between the branes, there is a negative $\lambda$ for which

$$Q_0'(y_2) > 0, \quad Q_{\tilde{\lambda}}'(y_2) = 0, \quad Q_{\tilde{\lambda}}'(y_2) < 0.$$  

(47)

Thus (for non–vanishing $b_2$) neither $\lambda = 0$ nor $\lambda = \tilde{\lambda}$ is an eigenvalue of our problem, because in both cases the boundary condition at the second brane
reduces to $Q'(y_2) = 0$ and is not fulfilled. It follows also that the sign of the l.h.s. of the second condition in (23) is different for $\lambda = 0$ and for $\lambda = \tilde{\lambda}$. Thus there must be at least one value of $\lambda$ between $\tilde{\lambda}$ and 0 for which the l.h.s. of that equation vanishes. Such negative $\lambda$ is an eigenvalue of our problem. Hence the radion becomes tachyonic for any background solution for which $\phi'$ vanishes anywhere between the branes. The third condition in (44) is necessary for the stability. This completes the proof that all the conditions in (44) are not only sufficient but also necessary for the stability of our brane model.

This is a generalization of the results obtained in [6]4. The generalization is twofold. First: we have taken into account the possibility of GB interactions present in the action. Second: even in the case of the standard Einstein–Hilbert action we were able to analyze a bigger class of possible backgrounds. The authors of [6] assumed that $a'(y) \neq 0$ for all $y$, while we have been able to show that this assumption is in fact not necessary.

Let us discuss the last point in some detail. The equations of motion for the scalar perturbations $Q(y)$ are regular at the points where $a'(y) = 0$. The presence of $a'$ in the denominator in the definition of $q(y)$ may be misleading. Equation (43) shows that $q(y) = (a^2 \xi')/(2a'\xi^2)$ is in fact finite for $a' = 0$ (at such points $\xi = a^2 \neq 0$). Thus for all backgrounds for which $\phi'$ and $\xi$ are non–vanishing for all $y$, the Sturm–Liouville system defined by eqs. (24) and (25) is not singular and our discussion of the sign of its lowest eigenvalue is valid also if $a'$ vanishes at some point(s).

5 ”New” versus ”old” solutions

Action (1) contains the Gauss–Bonnet term whose strength is parameterized by a constant $\alpha$. The results for models with Einstein–Hilbert gravity can be obtained from the formulae presented in this paper by setting $\alpha = 0$ or, equivalently, by replacing $\xi$ with $a^2$. However, one should be careful because substituting $\alpha$ by zero is not necessarily equivalent to the limit $\alpha \to 0$. The reason is quite obvious: Taking, for example, equation (7), we see that it is linear in the combination $[(a'/a)^2 - H^2]$ for $\alpha = 0$, but is quadratic in this combination for any $\alpha \neq 0$. For any non–vanishing $\alpha$, there are two solutions to this equation (for fixed values of $\phi$ and $\phi'$). One of them converges to the $\alpha = 0$ solutions in the $\alpha \to 0$ limit. Such type or branch of solutions

4In [6], a different method for analyzing the equations of motion for the scalar perturbations was used. Some features of the phase space associated with the equations of motion were analyzed to find conditions for the lowest eigenvalue. The method used in the present paper seems to be simpler and easier to generalize to other models.
is sometimes called the “old” one. There are also “new” solutions which diverge for \( \alpha \to 0 \), but are well defined for any \( \alpha \neq 0 \). We will now show that all the solutions from this “new” branch are strongly disfavored by the stability requirements.

Any background solution must fulfill three equations at each brane. These are the two boundary conditions (11) and (12), as well as the bulk equation (7). Fixing the normalization of our metric to be \( a(y_1) = 1 \) and eliminating \( \phi' (y_1) \) with help of (9), we get the following two equations

\[
6 \left( [a'(y_1)]^2 - H^2 \right) \left( 1 - 2\alpha \left( [a'(y_1)]^2 - H^2 \right) \right) = \frac{1}{4} U'_1(\phi(y_1)) - V(\phi(y_1)),
\]

(48)

\[
a'(y_1) \left[ 1 - 4\alpha \left( \frac{1}{3} [a'(y_1)]^2 - H^2 \right) \right] = - \frac{1}{6} U_1(\phi(y_1))
\]

(49)

for the two quantities \( a'(y_1) \) and \( \phi(y_1) \). For general potentials \( V(\phi) \) and \( U_1(\phi) \), this set may have several solutions which can be found only numerically. However, quite interesting results can be obtained analytically in the approximation of a stiff brane potential of the form

\[
U_1(\phi) = \frac{1}{2} \mu_1 (\phi - v_1)^2 + \ldots
\]

(50)

with very large \( \mu_1 \). In such an approximation, the difference \( \phi(y_1) - v_1 \) is small for each solution. This small quantity appears linearly in \( U'_1 \) and quadratically in \( U_1 \). Hence in the leading order in \( 1/\mu_1 \) the possible values of \( a' \) at the brane are given approximately by the solutions of

\[
a'(y_1) \left[ 1 - 4\alpha \left( \frac{1}{3} [a'(y_1)]^2 - H^2 \right) \right] + \frac{1}{6} U_1(v_1) \approx 0.
\]

(51)

The l.h.s. of this equation is cubic in \( a'(y_1) \) and has two extrema. It is easy to calculate that those extrema are at the values of \( a'(y_1) \) for which \( \xi(y_1) \), being a quadratic function of \( a'(y_1) \), vanishes\(^5\). Thus there is at most one solution with positive \( \xi(y_1) \). This is an ”old” solution which reduces to the \( \alpha = 0 \) solution in the \( \alpha \to 0 \) limit (if it exists for the given parameters). The ”new” solutions behave like \( a'(y_1) \sim \alpha^{-1/2} \) in the small \( \alpha \) limit, and always correspond to negative \( \xi(y_1) \).

The above result is very important for the stability of the model. By eq. (13), the ratio \( \xi'/a' \) is typically negative for negative \( \xi \), which leads to an instability of the model. Comparing eqs. (8) and (13), we see that positive

\(^5\)The factor of 1/3 present in (51) and discussed after eq. (10) is crucial for this relation between the extrema of (51) and the sign of \( \xi \).
$\xi'/a'$ and negative $\xi$ are possible only for large negative $\alpha$ and large $H$. Large values of $\alpha$ are not realistic from the phenomenological point of view. Moreover, equation (23) shows that large values of the Hubble constant tend to destabilize the model even more.

We have shown above that in the stiff brane potential approximation the "new" solutions are unstable. Relaxing this approximation is not likely to improve the stability of such solutions. It was argued in the previous section that the radion mass squared is a monotonic function of the parameters $b_i$. Equations (22) show that the $b_i$ decrease when we go away from the stiff brane potentials approximation. We conclude that all the "new" solutions are strongly disfavored by the stability conditions.

6 Role of Gauss–Bonnet interactions

In the previous section it was shown that the additional "new" solutions, which appear after including the GB term in the action, are probably all unstable. Only the "old" solutions can have a radion with a positive mass squared. These "old" solutions converge to the $\alpha = 0$ (no GB term) solutions in the $\alpha \to 0$ limit. This does not, however, mean that the addition of GB interactions is unimportant. The numerical analysis shows that solutions with small but non–zero $\alpha$ can substantially differ in some aspects from the solutions with $\alpha = 0$. The main features of the solutions can be understood analytically in the stiff brane potentials approximation. For simplicity, we again consider the case with a vanishing Hubble constant $H$.

We start with the boundary conditions at the first brane. As discussed in the previous section, the value of $a'(y_1)$ is given by the solution to eq. (51). For small $\alpha$, it is

$$a'(y_1) \approx -\frac{1}{6} U_1(v_1) \left[ 1 + \frac{\alpha}{27} U_1^2(v_1) \right].$$

(52)

Substituting the above result into eq. (52), we find

$$\phi'(y_1) \approx -\frac{1}{3} U_1^2(v_1) \left[ 1 + \frac{\alpha}{54} U_1^2(v_1) \right] + 2V(v_1).$$

(53)

The last two equations show that the absolute values of $a'(y_1)$ and $\phi'(y_1)$ grow with $\alpha$. The change of $a$ and $\phi$ with $y$ is faster (slower) for positive (negative) $\alpha$ compared to the $\alpha = 0$ case. One can see that the same is true also away from the branes. For sufficiently big $y$ the equations of motion (50-7) are dominated by large derivatives and the contributions from the bulk.
potential $V$ become subdominant. In this regime, eqs. (5) and (7) give

$$\phi'(y) \approx \frac{c}{a(y)^3},$$  \hspace{1cm} (54)

$$[a'(y)]^2 \approx \frac{a^4(y)}{4\alpha} \left(1 \pm \sqrt{1 - \frac{2\alpha c^2}{3a^8(y)}}\right).$$  \hspace{1cm} (55)

Expanding the r.h.s. of the last equation in $\alpha$, one can see that the background fields change faster for positive $\alpha$ and slower for negative $\alpha$. As the expansion parameter is proportional to $\alpha/a^8$ the expansion breaks down for sufficiently small $a$. The behavior of the r.h.s. of (55) for small $a(y)$ depends crucially on the sign of $\alpha$. For negative $\alpha$, it tends to a constant, and $a(y)$ can be arbitrarily small. For positive $\alpha$, there is a minimal value of $a(y)$

$$a^8_{\text{min}} \approx \frac{2}{3} \alpha c^2,$$  \hspace{1cm} (56)

for which $a'$ is still real. It is easy to find the behavior of different quantities when $a \to a_{\text{min}}$, namely: $a'' \to -\infty$, $\xi \to 0$, $\xi'/a' \to +\infty$. A solution with positive $\alpha$ ends (if there is no brane before, i.e. at a smaller value of $y$) at a singularity with a finite value of the warp factor. The solution can not be extended beyond such point.

The solution for $\alpha = 0$ has a different behavior. It ends at a singularity for finite $y$ at which $a \to 0$, $a' \to \text{const}$, $\phi' \to -\infty$. Solutions with negative $\alpha$ also end at the singularity $a = 0$, and the position of this singularity increases with increasing $|\alpha|$. However, the position of the second brane can not be arbitrarily close to such a singularity. The reason is that the ratio $\xi'/a'$ becomes negative before the position of a singularity, and the model is unstable if $\xi'/a' < 0$ somewhere between the branes. The behavior of $\xi'/a'$ can be obtained from the following expansion

$$\left(\frac{\xi'}{a'}\right)' = 2a' \left(1 - 4\alpha \frac{a'^2}{a^4} - \frac{20}{3} \alpha \frac{\phi'^2}{a^2} + \frac{8}{3} \alpha \frac{a}{a'} \phi' V'\right) + \mathcal{O}(\alpha^2).$$  \hspace{1cm} (57)

For sufficiently big $y$, this expression is dominated by the $\phi'^2$ term which for small $a$ grows approximately as $a^{-8}$. It has the same sign as $\alpha$ ($a' < 0$ because by definition the first brane is the positive tension one). Our numerical results confirm that indeed $\xi'/a'$ becomes negative before a singularity for $\alpha < 0$ solutions. For each background solution, there is a maximal possible distance between the branes. For $\alpha \geq 0$, it is given by the position of a singularity, while for $\alpha < 0$ it is given by the requirement that $\xi'/a'$ must be positive. The precise value of such a maximal distance depends, of course,
on the details of the potentials present in action (11). This maximal distance
between the branes is a decreasing function of \( \alpha \geq 0 \). For negative \( \alpha \), it is
more model dependent.

The positivity of \( \xi'/a' \) is one of the conditions for the stability. Another
condition is the positivity of the \( b_i \) parameters defined in eq. (22). We can
easily see how \( b_2 \) changes with \( \alpha \). From definition (22) and equation (31) with
the \( V' \)–term neglected, one obtains

\[
b_2 \approx \frac{1}{2} a U'_2 + 4 \frac{|a'|}{a}.
\]  (58)

The exact behavior of the r.h.s. of this equation depends on the details of a
model, but one feature is rather model independent. We have shown that the
warp factor evolves faster for bigger values of \( \alpha \). This means that typically
\( b_2 \) increases with increasing \( \alpha \).

The main consequences of adding the Gauss–Bonnet term to the action
are discussed in the two previous paragraphs. Such a term with a positive
coefficient \( \alpha \) causes the inter–brane distance to decrease, and the the radion
mass squared to increase (improving the stability of the brane positions). The
results of adding the GB term with a negative coefficient are more model–
dependent, but in general the stability seems to be worse (the radion becomes
lighter, and can eventually become tachyonic).

Let us now discuss the inter–brane distance in more detail. Of course the
second brane can be placed at \( y_2 \) smaller than the maximal value discussed
after eq. (56). A very important point is that the boundary conditions (9)
and (10) must be fulfilled at that second brane. The values of the l.h.s.
of both the equations are, for a given background solution, known at each
\( y \). For a fixed form of the potential \( U_2 \), also the r.h.s. of these equations
are known for each \( y \). For a general value of \( y \) none of those two boundary
conditions is fulfilled. The idea of the Goldberger–Wise mechanism [3] is
that there is a special value of \( y \) for which both equations (9) and (10)
are fulfilled, and the inter-brane distance is dynamically determined to be
\( y_2 - y_1 \). Of course, in general this can not be achieved. The reason is very
simple: there are two conditions but only one parameter to be adjusted,
namely \( y_2 \). So we need a mechanism which adjusts the parameters of the
Lagrangian in such a way that one combination of the boundary conditions
is fulfilled ”automatically”. This is just another form of the cosmological
constant problem. Unfortunately, the result given by the Goldberger–Wise
mechanism depends very strongly on the details of the unknown solution to
that problem. In the next section some results of numerical calculations are
presented to illustrate our analytical analysis.
We consider a simple model similar to that discussed in [6]. The bulk and brane potentials have the following forms

\[ V(\phi) = \Lambda + \frac{1}{2}M^2 \phi^2, \quad U_i(\phi) = \lambda_i + \frac{1}{2}\mu_i(\phi - v_i)^2. \]  

(59)

We chose the parameters in \( V \) and \( U_1 \) as follows: \( \Lambda = -15, M = 1, \lambda_1 = 10, \mu_1 = 100, v_1 = 1 \) (in \( \kappa = 1 \) units). For 5 values of \( \alpha \), we solve the boundary condition at \( y_1 = 0 \) numerically, and integrate the background equations of motion. Two stability conditions, \( b_1 > 0 \) and \( \phi'(y) \neq 0 \), are fulfilled in all these cases. The parameter \( \xi'/a' \) as a function of \( y \) is shown in fig. 1.

The results are consistent with our qualitative discussion: away from the first brane, \( \xi'/a' \) is bigger for bigger values of \( \alpha \). This effect increases with \( y \). For positive \( \alpha \), the ratio \( \xi'/a' \) becomes very large when we approach the singularity of the background solution. For the Einstein–Hilbert theory (\( \alpha = 0 \)), this ratio decreases but stays positive up to the singularity. For negative \( \alpha \), it decreases faster and becomes negative before the corresponding singularity.

![Figure 1: Ratio \( \xi'/a' \) as a function of \( y \) for different values of \( \alpha \): 0.01 (a); 0.005 (b); -0.005 (c); -0.01 (d). The curve without a label corresponds to a model without the Gauss–Bonnet term (\( \alpha = 0 \)).](image)

Then, for each value of \( y \), we solve the boundary conditions numerically as
if the second brane were positioned at that $y$. There are two such boundary conditions and three parameters in $U_2$ so one of the parameters can be fixed. First, we fix $\lambda_2 = -\lambda_1$ in order to compare our results with those of [6]. The obtained values of $b_2$ are plotted in fig. 2. The model can be stable only for positive $b_2$, so the second brane can be placed only at the values of $y$ for which $b_2 > 0$.

![Figure 2](image)

Figure 2: Parameter $b_2$ defined in eq. (22) as a function of the inter–brane distance for $\lambda_2 = -\lambda_1$. Labels are as in fig. 1. Curves for $\alpha > 0$ end at the background solutions singularities, curves for $\alpha < 0$ end at values of $y$ for which $\xi'/a'$ becomes negative.

We have discussed the values of $\xi'/a'$ and $b_2$. In section 4, it has been shown how they are related to the radion mass. To illustrate this relation, we have calculated the radion mass numerically for some cases. For the parameters given above and for $y_2 - y_1 = 0.15$, we have found that $m_0^2$ equals -0.12, 0.07, 0.40, 1.01, 2.43 for 5 values of $\alpha$: -0.01, -0.005, 0, 0.005, 0.01, respectively. Comparing this with figs. 1 and 2, one can see that indeed $m_0^2$ grows with the values of $\xi'/a'$ and $b_2$. A negative value of $m_0^2$ for $\alpha = -0.01$ reflects the fact that in this case $b_2 < 0$.

We confirm the result of [6] that in this particular set–up there is a minimal inter–brane distance below which the model is unstable. With GB interactions, this minimal distance decreases (increases) for positive (negative) $\alpha$. But there is also an upper bound on the inter–brane distance. For
\( \alpha \geq 0 \), it is determined by the singularity of the background solution\(^6\) while for \( \alpha < 0 \) it is given by value of \( y \) for which \( \xi'/a' \) becomes negative.

One should ask the question: is the above result typical for the considered model? The answer is: no, it is not. To illustrate this, we show in figs. 3 and 4 the \( y \)-dependence of the parameter \( b_2 \) for models where the value of \( \lambda_2 \) is changed by 10\%. Such small changes in one of the parameters have an important influence on the possible inter–brane distance. There is no lower bound on \( y_2 - y_1 \) for \( \lambda_2 = -0.9\lambda_1 \) or for \( \lambda_2 = -1.1\lambda_1 \). But in the second case not all values of \( y_2 - y_1 \) between 0 and the maximal value are allowed. There is a window of forbidden values of \( y \) for which the model becomes unstable while being stable outside.

In fact, the functional behavior of \( b_2 \) shown in figs. 3 and 4 is much more typical than that in fig. 2. For general values of \( \lambda_2 \) there is no lower bound on \( y_2 - y_1 \). For \( \lambda_2 \leq -\lambda_1 \), there are two ranges of \( y_2 - y_1 \) for which the model can be stable. The upper range shrinks when we decrease \( \lambda_2 \), and eventually disappears for large enough \( |\lambda_2| \). The lower bound on \( y_2 - y_1 \) exists only when \( \lambda_2 \approx -\lambda_1 \) up to a few percent. The common feature of all the models is the existence of an upper bound on the inter–brane distance above which the radion can not be stabilized.

The existence of such an upper bound can cause problems for model building. The most appealing rationale for considering the 5–dimensional brane models comes for the Ho\v{r}ava–Witten construction motivated by M–theory\(^1\). One of the main features of those models is a relatively big length of the 5–th dimension necessary to obtain a large enough 4–dimensional Planck scale. Hence one has to check whether an upper bound on the inter–brane distance in a given model is compatible with phenomenological constraints. This is especially important after including GB interactions, which typically cause the maximal inter–brane distance to decrease.

We would like also to stress (again) that radion stabilization is quite sensitive to the unknown mechanism of solving the cosmological constant problem. In the examples presented in this work we fixed \( \lambda_2 \) and used two other parameters in \( U_2 \) to fix the brane position and to set the cosmological constant to zero. Of course, we do not know whether these two particular parameters have anything to do with the cosmological constant problem. Hence one can consider situations where parameters other that \( \mu_2 \) and \( v_2 \) are adjustable. We have checked that, for example, fixing \( v_2 \) gives three classes of solutions for \( b_2 \) similar to those shown in figs. 2 to 4. However, details of a given solution depend on the actual value chosen for \( v_2 \).

\(^6\)Such an upper bound exists also in the absence of the Gauss–Bonnet term, which was not observed in ref. \(^2\).
Figure 3: As figure 2 but for $\lambda_2 = -0.9 \lambda_1$.

Figure 4: As figure 2 but for $\lambda_2 = -1.1 \lambda_1$. 
8 Conclusions

We have found the equations of motion for scalar perturbations in the presence of the Gauss–Bonnet interactions for an inflating background. The expression for the mass of the lightest of such perturbations, which can be identified as the radion, has a form very similar to that for the standard Einstein–Hilbert gravity. However, the presence of the Gauss–Bonnet term with a positive coefficient typically increases the actual radion mass. This can help to stabilize the radion in an inflating background.

The necessary and sufficient conditions for the radion stabilization have been found for the case of a vanishing Hubble constant. One additional condition appears when GB interactions are taken into account. The form of the other conditions does not change after adding those interactions. However, the parameters of the model for which those conditions are fulfilled can change substantially, because the character of the background solutions changes when the GB term is present in the action.

There are two classes of solutions in the presence of GB interactions. In the limit where the GB coefficient goes to zero, solutions from one class just converge to the solutions in the Einstein–Hilbert theory, while solutions from the second class do not. It has been shown that the solutions from the second class are strongly disfavored by a stability analysis.

Even without inflation, numerical calculations are in general necessary to check whether the branes can be stabilized for a given model. The results for the radion stabilization are quite sensitive to all (mainly unknown) details of models. They depend, for example, on assumptions made about the mechanism leading to an acceptably small cosmological constant. This is true for models both with and without the Gauss–Bonnet interactions.

However, some typical features of the solutions can be found using analytical arguments. The most important of them is the existence of an upper bound on the inter–brane distance for each given model. General changes caused by the GB term can also be identified. Typically, the values of some parameters appearing in the formula for the radion mass change with $\alpha$ in such a way that this mass grows with growing $\alpha$. Hence (with all other parameters fixed) stability increases if the GB term with a positive coefficient is present in the action. At the same time, the maximal value of the inter–brane distance becomes smaller.
Acknowledgments

This work was partially supported by the EU 6th Framework Program MRTN-CT-2004-503369 “Quest for Unification”. M.O. was partially supported by the DFG Schwerpunktprogramm: “Stringtheorie in Kontext von Teilchenphysik, Quantenfeldtheorie, Quantengravitation, Kosmologie und Mathematik” Schm 561/2-3 and the Polish MEiN grant 1 P03B 099 29 for years 2005-2007.

References

[1] P. Hořava and E. Witten, Nucl. Phys. B 460 (1996) 506 arXiv:hep-th/9510209; Nucl. Phys. B 475 (1996) 94 arXiv:hep-th/9603142.

[2] C. Charmousis, R. Gregory and V. A. Rubakov, Phys. Rev. D 62 (2000) 067505 arXiv:hep-th/9912160.

[3] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. 83 (1999) 4922 arXiv:hep-ph/9907447.

[4] T. Tanaka and X. Montes, Nucl. Phys. B 582 (2000) 259 arXiv:hep-th/0008151.

[5] C. Csáki, M. L. Graesser and G. D. Kribs, Phys. Rev. D 63 (2001) 065002 arXiv:hep-th/0008151.

[6] J. Lesgourgues and L. Sorbo, Phys. Rev. D 69 (2004) 084010 arXiv:hep-th/0310007.

[7] R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B 191 (1987) 354; D. J. Gross and J. H. Sloan, Nucl. Phys. B 291 (1987) 41.

[8] G. L. Alberghi and A. Tronconi, Phys. Rev. D 73 (2006) 027702 arXiv:hep-ph/0510267.

[9] A. V. Frolov and L. Kofman, Phys. Rev. D 69 (2004) 044021 arXiv:hep-th/0309002.

[10] N. E. Mavromatos and J. Rizos, Phys. Rev. D 62 (2000) 124004 arXiv:hep-th/0008074; Int. J. Mod. Phys. A 18 (2003) 57 arXiv:hep-th/0205299; P. Binetruy, C. Charmousis, S. C. Davis and J. F. Dufaux, Phys. Lett.
B 544 (2002) 183 [arXiv:hep-th/0206089];
A. Jakóbek, K. A. Meissner and M. Olechowski, Nucl. Phys. B 645 (2002) 217 [arXiv:hep-th/0206254].

[11] S. Nojiri, S. D. Odintsov and M. Sasaki, Phys. Rev. D 71 (2005) 123509 [arXiv:hep-th/0504052].

[12] G. Calcagni, S. Tsujikawa and M. Sami, Class. Quant. Grav. 22 (2005) 3977 [arXiv:hep-th/0505193].

[13] L. Amendola, C. Charmousis and S. C. Davis, arXiv:hep-th/0506137.

[14] G. Esposito-Farese, arXiv:gr-qc/0306018.

[15] B. M. N. Carter and I. P. Neupane, arXiv:hep-th/0512262.

[16] N. Deruelle and C. Germani, Nuovo Cim. 118B (2003) 977 [arXiv:gr-qc/0306116].

[17] I. P. Neupane, arXiv:hep-th/0602097.

[18] W. Israel, Nuovo Cim. B 44S10 (1966) 1 [Erratum-ibid. B 48 (1967 NUCIA.B44,1.1966) 463].

[19] S. C. Davis, Phys. Rev. D 67 (2003) 024030 [arXiv:hep-th/0208205].

[20] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D 62 (2000) 123515 [arXiv:hep-th/0005032].

[21] S. Mukohyama, Phys. Rev. D 65 (2002) 084036 [arXiv:hep-th/0112205].

[22] S. Mukohyama and L. Kofman, Phys. Rev. D 65 (2002) 124025 [arXiv:hep-th/0112115].

[23] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience Publishers, 1953.

[24] V. F. Mukhanov, Sov. Phys. JETP 67 (1988) 1297 [Zh. Eksp. Teor. Fiz. 94N7 (1988) 1].

[25] L. Kofman, J. Martin and M. Peloso, Phys. Rev. D 70 (2004) 085015 [arXiv:hep-ph/0401189].