Periodic orbits for a class of galactic potentials

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Abstract In this work, applying general results from averaging theory, we find periodic orbits for a class of Hamiltonian systems $H$ whose potential models the motion of elliptical galaxies.

Keywords Periodic orbit · Averaging theory · Hamiltonian system

1 Introduction and statement of the main results

Galactic dynamics is a branch of Astrophysics whose development started only around sixty years ago, when it was possible to have a view of the physical world beyond the integrable and near integrable systems (Contopoulos 2002). Even the importance of the analysis of galactic potentials, the global dynamics of galaxies is not a simple question and represents a big challenge for the researches in the field (Contopoulos 1988). Most of the work in the analysis of galaxies is numerical, in this paper we present an analytical technique, the averaging theory, which allows to find periodic orbits of a differential system.

In the last years, great quantity of the research on galactic dynamics has been focused on models of elliptical galaxies. In most of these models the terms in the potential are of even order, so we have adopted this fact in the Hamiltonian system that we are analyzing. Another important point that appears in these kind of potentials is that the existence of periodic orbits is a useful tool for constructing new and more complicated self consistent models. One way to identify periodic orbits is to localize the central fixed points on the surfaces of constant energy. In Patsis and Zachilas (1990), the authors study the localization of periodic orbits and their linear stability for a particular two-component galactic potential. In fact, in our days the study of individual orbits in some galactic potentials is a new branch of galactic dynamics (see for instance the articles Caranicolas (2000), El-Sabaa and Sherief (1990), Greiner (1987)).

The calculation of particular orbits in some analytical potentials modeling elliptical galaxies, indicates that relatively small symmetry breaking corrections can increase dramatically the number of stochastic orbits, showing the importance of the study of perturbations of simple models (Habib et al. 1997). The class of potentials studied in this paper have not chosen with the aim of modeling some particular galaxies, our objective is to study systems which are generic in their basic properties.

In Pucacco et al. (2008), the authors study the galactic potential

$$H = \frac{1}{2} (p_x^2 + p_y^2) + V(x^2, y^2).$$

These kind of potentials are important in the modeling of elliptic galaxies, as for instance we can mention the potentials $V_L = \log (1 + X^2 + Y^2/q)$ and $V_C = \sqrt{1 + X^2 + Y^2/q} - 1$, where the parameter $q$ gives the eccentricity of the elliptic.
galaxy. In this paper we deal with the Hamiltonian

\[
H = \frac{1}{2}(P_x^2 + X^2) + \frac{1}{2q}(P_y^2 + Y^2) + (aX^4 + bX^2Y^2 + cY^4),
\]

and its respective Hamilton’s equation

\[
\begin{align*}
\dot{X} &= P_x, \\
\dot{Y} &= \frac{P_y}{q}, \\
\dot{P}_x &= -X - (4aX^3 + 2bXY^2), \\
\dot{P}_y &= -\frac{Y}{q} - (2bX^2Y + 4cY^3),
\end{align*}
\]

the matrix of the linear part of this system at the origin of coordinates is

\[
M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{1}{q} \\
-1 & 0 & 0 & 0 \\
0 & -\frac{1}{q} & 0 & 0
\end{pmatrix},
\]

with eigenvalues \(\pm i, \pm i/q\) where \(i = \sqrt{-1}\).

In order to obtain periodic orbits for these kind of potentials we will apply averaging theory, in this way we re-parametrize the coordinates by the factor \(\sqrt{\varepsilon}\) for \(\varepsilon\) positive small enough (a similar change of coordinates has been used in Llibre and Roberto (2013)), that is we do the change

\[
(X, Y, P_x, P_y) \rightarrow \sqrt{\varepsilon}(x, y, p_x, p_y).
\]

After straightforward computations we get the Hamilton’s equations

\[
\begin{align*}
\dot{x} &= p_x, \\
\dot{y} &= \frac{p_y}{q}, \\
\dot{p}_x &= -x - \varepsilon(4ax^3 + 2bxy^2), \\
\dot{p}_y &= -\frac{y}{q} - \varepsilon(2bX^2y + 4cy^3),
\end{align*}
\]

which have the same linear part at the origin than the previous one, the structure of the new Hamiltonian is identity with (1) in the new variables. Our goal is to study which periodic orbits for \(\varepsilon = 0\) (the unperturbed system) persists for \(\varepsilon\) positive and small enough (the perturbed system).

By the form of the matrix \(M\) we observe the necessity to split the analysis for the periodic orbits in two cases

- \(q\) is an irrational number. Here the linear part of system (4) has two planes foliated by periodic orbits. In the first one the orbits have period \(2\pi\), each periodic orbit on this plane is of the form

\[
PO_1 = (x_0 \cos t + px_0 \sin t, 0, px_0 \cos t - x_0 \sin t, 0).
\]

In the second one, the orbits have period \(2\pi q\), each periodic orbit on this plane is of the form

\[
PO_2 = (0, y_0 \cos (t/q) + py_0 \sin (t/q), 0, py_0 \cos (t/q) - y_0 \sin (t/q)).
\]

- \(q\) is a rational number. Here the linear part of system (4) has a 4-dimensional space filled of periodic orbits of period \(2\pi r\) if \(q = r/s\) with \((r, s) = 1\), where each periodic orbit is of the form

\[
PO_3 = (x_0 \cos t + px_0 \sin t, y_0 \cos (st/r) + py_0 \sin (st/r), px_0 \cos t - x_0 \sin t, py_0 \cos (st/r) - y_0 \sin (st/r)).
\]

When \(q\) is an irrational number our main result is:

**Theorem 1.1** For \(q\) an irrational number, we have that in every energy level \(H = h > 0\) the Hamiltonian system (2) has

(a) at least one periodic solution \((X(t), Y(t), P_x(t), P_y(t))\) such that when \(\varepsilon \rightarrow 0\), we have that \((X(0), Y(0), P_x(0), P_y(0))\) tends to \((0, 0, 0, 0)\);

(b) at least one periodic solution \((X(t), Y(t), P_x(t), P_y(t))\) such that when \(\varepsilon \rightarrow 0\), we have that \((X(0), Y(0), P_x(0), P_y(0))\) tends to \((0, 0, 0, 0)\).

So, for \(q\) irrational, we obtain that in every energy level \(H = h > 0\) the perturbed Hamiltonian system has at least 2 periodic orbits.

**Remark 1** We note that the periodic orbits found in the statements of Theorem 1.1 are in fact degenerate Hopf bifurcations periodic orbits, since they born from the equilibrium point localized at the origin of coordinates. Unfortunately we cannot obtain periodic solutions when \(q\) is a rational number, see Remark 2.

The paper is organized as follows. In Sect. 2 we present the theorem from averaging theory necessary to prove our main result. In Sect. 3 we give the proof of Theorem 1.1.

## 2 Some results from averaging theory

In order to have a self contained paper, in this section we present the basic results from the averaging theory that are necessary for proving the main results of this paper.

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2.1 Results from averaging theory

We consider the problem of the bifurcation of $T$-periodic solutions from the differential system

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x), \quad (5)$$

where the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ are of class $C^2$ functions, $T$-periodic in the first variable, and $\Omega$ is an open subset of $\mathbb{R}^n$. When $\varepsilon = 0$ we get the unperturbed system

$$x'(t) = F_0(t, x). \quad (6)$$

One of the main assumptions on the above system is that it has a submanifold of periodic solutions. A solution of system (5), for $\varepsilon$ sufficiently small is given using the averaging theory. For a general introduction to the averaging theory see the books of Sanders and Verhulst (1985), and of Verhulst (1991).

Let $x(t, z)$ be the solution of the unperturbed system (6) such that $x(0, z) = z$. We write the linearization of the unperturbed system along the periodic solution $x(t, z)$ as

$$y' = D_x F_0(t, x(t, z)) y. \quad (7)$$

In what follows we denote by $M_{\varepsilon}(t)$ some fundamental matrix of the linear differential system (7), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first $k$ coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

**Theorem 2.1** Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0 : \text{Cl}(V) \to \mathbb{R}^{n-k}$ be a $C^2$ function. We assume that

(i) $Z = \{ z_\alpha = (\alpha, \beta_0(\alpha)), \ \alpha \in \text{Cl}(V) \} \subset \Omega$ and that for each $z_\alpha \in Z$ the solution $x(t, z_\alpha)$ of (6) is $T$-periodic;

(ii) for each $z_\alpha \in Z$ there is a fundamental matrix $M_{\varepsilon}(t)$ of (7) such that the matrix $M_{\varepsilon}(t)^{-1}(0) - M_{\varepsilon}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix $\Delta_\alpha$ with $\det(\Delta_\alpha) \neq 0$.

We consider the function $F : \text{Cl}(V) \to \mathbb{R}^k$

$$F(\alpha) = \xi\left( \int_0^T M_{\varepsilon}^{-1}(t) F_1(t, x(t, z_\alpha)) \, dt \right). \quad (8)$$

If there exists $\alpha \in V$ with $F(\alpha) = 0$ and

$$\det(\delta F / \delta \alpha)(\alpha) \neq 0,$$

then there is a $T$-periodic solution $\varphi(t, \varepsilon)$ of system (5) such that $\varphi(0, \varepsilon) \to z_\alpha$ as $\varepsilon \to 0$.

Theorem 2.1 goes back to Malkin (1956) and Roseau (1966), for a shorter proof see Buică et al. (2007).

3 Proof of the main theorem

In this section we give the proof of our main result Theorem 1.1.

3.1 Proof of Theorem 1.1

We know that the periodic orbits of a Hamiltonian system always appear in cylinders foliated by periodic orbits, each periodic orbit corresponds to a different value of the energy $h$, see for more details (Abraham et al. 1988). In order to have isolated periodic orbits and be able to apply the averaging theory we fix the total energy $H = h$. Computing $p_x$ in the energy level $H = h$ we get

$$p_x = \pm \sqrt{2h - \frac{p_y^2}{q} - x^2 - \frac{y^2}{q} - \varepsilon(2ax^4 + 2bx^2y^2 + 2cy^4)}. \quad (9)$$

The fix value $h$ of the total energy is determined by the initial periodic orbit, which in our case for the periodic orbit $PO_1$ it corresponds to $h = \frac{1}{2}(y_{x_0}^2 + x_0^2)$, choosing the sign $+$ for $p_x$, and expanding around $\varepsilon = 0$ we obtain

$$p_x = \sqrt{p_{x_0}^2 - x^2 + x_0^2 - \frac{y^2 + p_y^2}{q} - \varepsilon \frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - \frac{y^2 + p_y^2}{q}}}. \quad (10)$$

The equations of motion on the energy level $H = (p_{x_0}^2 + x_0^2)/2$ are

$$\dot{x} = \sqrt{p_{x_0}^2 - x^2 + x_0^2 - \frac{(y^2 + p_y^2)}{q}} - \varepsilon \frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - \frac{(y^2 + p_y^2)}{q}}},$$

$$\dot{y} = \frac{p_y}{q},$$

$$\dot{p}_y = -\frac{y}{q} - \varepsilon(2bx^2y + 4cy^3). \quad (11)$$

In order to apply Theorem 2.1 to system (11), let

$$x = (x, y, p_y),$$

$$F_0(t, x) = \left( \sqrt{p_{x_0}^2 - x^2 + x_0^2 - \frac{(y^2 + p_y^2)}{q}}, \quad p_y/q, \quad -y/q \right). \quad (12)$$
\[ F_1(t, x) = \left( -\frac{ax^4 + bx^2y^2 + cy^4}{\sqrt{p_{x_0}^2 - x^2 + x_0^2 - (y^2 + p_y^2)/q}} \right), \]

\[ 0, -(2bx^2y + 4cy^3) \].

The set \( \mathcal{O} = \{(x, y, p_x)|q(p_{x_0}^2 - x^2 + x_0^2 - y^2 - p_y^2 \neq 0)\} \) is an open subset of \( \mathbb{R}^3 \). Clearly the above functions are of class \( C^2(\mathcal{O}) \). The set \( V \) of Theorem 2.1 is given by

\[ V = \{ z = (x_0, 0, 0) : |x_0| < \rho \} \quad \text{for some } \rho \text{ large enough.} \]

Let \( x(t, z) \) be the solution of the unperturbed system (6) such that \( x(0, z) = z \). The variational equations of the unperturbed system along the periodic solution \( PO_1 \) are

\[ y' = D_xF_0(t, x(t, z))y, \tag{13} \]

where \( y \) is a \( 3 \times 3 \) matrix.

The fundamental matrix \( M(t) \) of the differential system (13) such that \( M(0) \) is the identity matrix of \( \mathbb{R}^3 \) takes the simple form

\[ M(t) = \begin{pmatrix} \cos t - x_0 \sin t/p_{x_0} & 0 & 0 \\ 0 & \cos (t/q) & \sin (t/q) \\ 0 & -\sin (t/q) & \cos (t/q) \end{pmatrix}, \]

whose inverse is given by

\[ M^{-1}(t) = \begin{pmatrix} p_{x_0}/p_{x_0}\cos t - x_0 \sin t & 0 & 0 \\ 0 & \cos (t/q) & -\sin (t/q) \\ 0 & \sin (t/q) & \cos (t/q) \end{pmatrix}. \tag{14} \]

An easy computation shows that

\[ M^{-1}(0) = M^{-1}(2\pi) \]

we choose the sign + for \( p_y \) and expand around \( \varepsilon = 0 \) getting

\[ p_y = \pm \sqrt{2qh - q(p_x^2 + x_0^2 - y^2 - \varepsilon(2aqx^4 + 2bqx^2y + 2cxy^2 + 4cy^3))}. \tag{15} \]

Now, we write the equations of motion on the energy level \( H = (p_{x_0}^2 + x_0^2)/2q = h \) in the order \( (y, x, p_y) \), they are given by the system

\[ \dot{y} = \sqrt{2qh - q(p_x^2 + x^2 - y^2)/q}. \tag{16} \]
An easy computation shows that
\[
F_1(t, x(t, z)) = \left( -\varepsilon \frac{ax^4 + bx^2 y^2 + 2y^4}{\sqrt{2qh - q(p_{x_0}^2 + x^2) - y^2}}, 0, 0 \right).
\]

In order to apply Theorem 2.1 to system (17) we are using the same notations and definitions (with the obvious changes) than in the previous case.

Let \( x(t, z) \) be the solution of the unperturbed system (6) such that \( x(0, z) = z \). The variational equations of the unperturbed system along the periodic solution \( PO_2 \) are
\[
y' = D_x F_0(t, x(t, z)) y,
\]
where \( y \) is a \( 3 \times 3 \) matrix.

The fundamental matrix \( M(t) \) of the differential system (18) such that \( M(0) \) is the identity matrix of \( \mathbb{R}^3 \) takes the simple form
\[
M(t) = \begin{pmatrix}
\cos (t/q) - y_0 \sin (t/q)/(p_{x_0}^2 + 2y^2) & 0 & 0 \\
0 & \cos t & \sin t \\
0 & -\sin t & \cos t
\end{pmatrix},
\]
whose inverse is given by
\[
M^{-1}(t) = \begin{pmatrix}
p_{x_0} \cos (t/q) - y_0 \sin (t/q) & 0 & 0 \\
0 & \cos t & \sin t \\
0 & -\sin t & \cos t
\end{pmatrix}.
\]

An easy computation shows that
\[
M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 \sin^2 (\pi q) & 2 \sin^2 (2\pi q) \\
0 & -\sin (2\pi q) & 2 \sin^2 (2\pi q)
\end{pmatrix}.
\]

We observe that this matrix has two zeros in the upper right corner of size \( 1 \times 2 \); the determinant of the 2 \( \times \) 2 matrix which appears in the lower right corner is \( 4 \sin^2 (\pi q) \neq 0 \) because \( q \) is an irrational number. Consequently all the assumptions of Theorem 2.1 are satisfied. Therefore we must compute the simple zeroes of the function \( F \) defined in Theorem 2.1.

A straightforward computation shows that
\[
M^{-1}(t) F_1(t, x(t, z)) = \left( -\frac{cp_{x_0} [y_0 \cos (t/q) + p_{x_0} \sin (t/q)]^4}{(p_{x_0} \cos (t/q) - y_0 \sin (t/q))^2}, 0, 0 \right).
\]

Therefore we have
\[
M^{-1}(t) F_1(t, x(t, z)) = \left( -\frac{cp_{x_0} [y_0 \cos (t/q) + p_{x_0} \sin (t/q)]^4}{(p_{x_0} \cos (t/q) - y_0 \sin (t/q))^2}, 0, 0 \right).
\]

Let
\[
f_1(y_0) = \frac{1}{2\pi} \int_0^{2\pi} \left( -\frac{cp_{x_0} [y_0 \cos (t/q) + p_{x_0} \sin (t/q)]^4}{(p_{x_0} \cos (t/q) - y_0 \sin (t/q))^2} \right) dt
\]
\[
= 3cp_{x_0} (p_{x_0}^2 + y_0^2)/2 = 3c h^2 q^2 \sqrt{2h^2 - y_0^2}.
\]

In the last equality we have gotten \( p_{x_0} \) from the energy relation \( h = \frac{(p_{x_0}^2 + y_0^2)^2}{2} \). So the solutions of \( f_1 = 0 \) are \( y_0 = \pm \sqrt{2h^2} \), which are simple zeroes. On the other hand we can verify that both zeroes generate the same periodic orbit.

Doing the rescaling (3) we get the statement (b) of Theorem 1.1.

Therefore we have proved that for \( q \) an irrational number, in every energy level \( h > 0 \), the Hamiltonian system (2) has at least 2 periodic orbits, so Theorem 1.1 holds.

**Remark 2** Using the methods of averaging theory studied in this paper, we could not obtain any periodic orbit for the Hamiltonian system (2) when \( q \) is a rational number. We have tried to get some information in two different ways, using Cartesian coordinates as in statement (a) and using a modified kind of polar coordinates in two different planes. In the first way we have obtained the variational equations, but unfortunately we could not solve them. In the second way we have obtained that one of the equations that we must solve for obtain the periodic solutions is identically zero.

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