Representations of copointed Hopf algebras arising from the tetrahedron rack

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Abstract We study the copointed Hopf algebras attached to the Nichols algebra of the affine rack Aff (F_4, ω), also known as tetrahedron rack, and the 2-cocycle −1. We investigate the so-called Verma modules and classify all the simple modules. We conclude that these algebras are of wild representation type and not quasitriangular, also we analyze when these are spherical.

Keywords Copointed Hopf algebras · Representations of Hopf algebras · Liftings of Nichols algebras · Affine racks

Mathematics Subject Classification (2000) 16W30

1 Introduction

We work over an algebraically closed field k of characteristic zero. Let G be a finite non-abelian group and let k^G denote the algebra of functions on G. A Hopf algebra with coradical isomorphic to k^G for some G is called copointed. Nicolás Andruskiewitsch
and the second author began the study of the copointed Hopf algebras by classifying those finite-dimensional with $G = S_3$ in [8] and by analyzing the representation theory of them in [9].

Since $k^G$ is a commutative semisimple algebra, the representation theory of a copointed Hopf algebra over $k^G$ is studied in [9] by analogy with the representation theory of semisimple Lie algebras, with $k^G$ playing the role of the Cartan subalgebra and the induced modules from the simple one-dimensional $k^G$-modules as Verma modules.

There are few examples of Nichols algebras of finite dimension over non-abelian groups, see for instance [17, 19]. In particular, those arising from affine racks are only seven, including the tetrahedron rack. If $X$ is one of these affine racks, then all the liftings of the Nichols algebra $B(-1, X)$ over $k^G$ were classified in [18], where $G$ is any group admitting a principal YD-realization of $X$ with constant 2-cocycle $-1$. Also the liftings of $B(X, -1)$ over the group algebra $k^G$ were classified in [18].

The notation used in the following is explained in Sect. 3. Let $G$ be a finite group and $V \in k^G_{\text{YD}}$ arising from a faithful principal YD-realization of the tetrahedron rack with constant 2-cocycle $-1$. The Nichols algebra $B(V)$ has dimension 72. The ideal of relations of $B(V)$ is generated by four quadratic elements and a single extra one, of degree six, which we denote by $z$. By [18], the liftings of $B(V)$ over $k^G$ are the copointed Hopf algebras $\{A_{G, \lambda}\}_{\lambda \in k}$, in which the quadratic relations of $B(V)$ still hold and the 6-degree relation $z = 0$ deforms to $z = \lambda(1 - \chi^{-1}z) \in k^G$.

The goal of this paper is to investigate the representation theory of the family $\{A_{G, \lambda}\}_{\lambda \in k}$ following the strategy of [9]. We conclude that there are essentially two kinds of Verma modules. Here is an account of our main results which apply to any group $G$ admitting a faithful principal YD-realization of the tetrahedron rack with constant 2-cocycle $-1$:

- **Lemma 14.** If the element $z = \lambda(1 - \chi^{-1})$ annihilates the generator of the Verma modules $M_g$, then $M_g$ inherits a structure of $B(V)$-module such that it is a free $B(V)$-module of rank 1, see Lemma 14. Hence $M_g$ has a unique simple quotient of dimension 1 called $k^g$. 

- **Lemma 15.** Otherwise $M_g$ is the direct sum of six 12-dimensional non isomorphic simple projective modules $L_i^g$, see Lemma 15. Tables 1, 2, 3, 4, 5, 6 in the Appendix describe the simple modules $L_i^g$.

- **Proposition 17.** We prove that $A_{G, \lambda}$ is of wild representation type.

- **Lemma 8.** We give a necessary condition for a copointed Hopf algebra to be quasitriangular, see Lemma 8. As a consequence $A_{G, \lambda}$ is not quasitriangular, Proposition 12.

- **Proposition 18.** We characterize those $A_{G, \lambda}$ which are spherical Hopf algebras.

The other copointed Hopf algebras classified in [18] are defined by similar relations to $A_{G, \lambda}$, roughly speaking a set of quadratic ones and other single relation of bigger degree, but their dimensions are much bigger than $\dim A_{G, \lambda} = 72|G|$. To extend this work to the other copointed Hopf algebras in [18], a better understanding of the corresponding Nichols algebras is needed. We hope that our work will be useful for this purpose.

The paper is organized as follows. In Sect. 2 we analyze the representation theory of copointed Hopf algebras with emphasis in the weight spaces of the modules, we...
characterize the one-dimensional modules and describe the subalgebra corresponding to the elements of weight \(e \in G\). In Sect. 3, we present our main object of study: the algebras \(\mathcal{B}(V)\) and \(A_{G,\lambda}\). In Sect. 4 we concentrate our attention on representations of the algebras \(\{A_{G,\lambda}\}_{\lambda \in \mathbb{k}}\). A description of the simple \(A_{G,\lambda}\)-modules is in the Appendix.

1.1 Conventions and notation

We set \(\mathbb{k}^* = \mathbb{k}\{0\}\). If \(X\) is a set, \(\mathbb{k}X\) denotes the free vector space over \(X\).

Let \(A\) be a Hopf algebra. Then \(\Delta, \varepsilon, S\) denote respectively the comultiplication, the counit and the antipode. The group of group-like elements is \(G(A)\). Let \(\mathcal{A}^1\mathcal{YD}\) be the category of Yetter-Drinfeld modules over \(A\). The Nichols algebra \(\mathcal{B}(V)\) of \(V \in \mathcal{A}^1\mathcal{YD}\) is called the graded quotient \(T(V)/\mathcal{J}(V)\) where \(\mathcal{J}(V)\) is the largest Hopf ideal of \(T(V)\). Then \(\mathcal{B}(V)\) is generated as an ideal by homogeneous elements of degree \(\geq 2\) [7, 2.1].

Let \(\{A_{[n]}\}_{n \geq 0}\) denote the coradical filtration of \(A\). Assume \(H = A_{[0]}\) is a Hopf subalgebra. Let \(\text{gr} A\) be the graded Hopf algebra associated to the coradical filtration. Then \(\text{gr} A \simeq R\# H\) where \(R \in \mathcal{H}^1\mathcal{YD}\) is called the diagram of \(A\) and \(V = R_{[1]} \in \mathcal{H}^1\mathcal{YD}\) is the infinitesimal braiding [7, Definition 1.15].

Recall that two idempotents \(e, \tilde{e} \in A\) are orthogonal if \(\tilde{e}e = 0 = \tilde{e}e\). An idempotent is primitive if it is not possible to express it as the sum of two nonzero orthogonal idempotents. Let \(\{e_i\}_{i \in I}\) be a set of idempotents of \(A\). Assume \(\dim A < \infty\). Then \(A\) is a Frobenius algebra, see for instance [15, Lemma 1.5]. Let \(e\) be a primitive idempotent of \(A\). Then \(\text{top}(Ae) = Ae/\text{rad}(Ae)\) and the socle \(\text{soc}(Ae)\) of \(Ae\) are simple modules [12, Theorems 54.11 and 58.12]. Moreover, \(Ae\) is the injective hull of \(\text{soc}(Ae)\) and the projective cover of \(\text{top}(Ae)\) [12, page 400 and Theorem 58.14]. We denote by \(\text{Irr} A\) a set of representatives of simple \(A\)-modules.

2 Representations of copointed Hopf algebras

Let \(G\) be a finite group, \(\mathbb{k}G\) its group algebra and \(\mathbb{k}G\) the algebra of functions on \(G\). Let \(\{g : g \in G\}\) and \(\{\delta_g : g \in G\}\) be the dual basis of \(\mathbb{k}G\) and \(\mathbb{k}G\), respectively. The identity element of \(G\) will be denoted by \(e\).

If \(M\) is a \(\mathbb{k}G\)-module, then \(M[g] = \delta_g \cdot M\) is the isotypic component of weight \(g \in G\). We denote by \(\mathbb{k}_g\) the one-dimensional \(\mathbb{k}G\)-module of weight \(g\). We define

\[M^\times = \bigoplus_{g \neq e} M[g]\text{ and } \text{Supp} M = \{g \in G : M[g] \neq 0\}.\]

Through this section, \(A\) denotes a finite-dimensional copointed Hopf algebra over \(\mathbb{k}G\), i.e. its coradical is isomorphic to \(\mathbb{k}G\).

We consider \(A\) as a left \(\mathbb{k}G\)-module via the left adjoint action

\[ad\delta_t(a) = \sum_{s \in G} \delta_s a \delta_{t^{-1}s}, \forall t \in G, a \in A.\]
By [8, Lemma 3.1], $A = \oplus_{g \in G} A[g]$ is a $G$-graded algebra and

$$\delta_s a_s = a_s \delta_s^{-1} t \quad \forall a_s \in A[s], s, t \in G. \quad (1)$$

If $M$ is an $A$-module, then $M$ is a $k^G$-module by restriction. Hence

$$A[g] \cdot M[h] \subseteq M[gh] \quad \forall g, h \in G \text{ by (1).} \quad (2)$$

This means that $M$ is a $G$-graded $A$-module.

We denote by $A_k^G = A$ as right $k^G$-module via the right multiplication. Its isotypic components are $(A_k^G)[g] = A \delta_g$ for all $g \in G$. Note that $A$ is a $k^G$-bimodule with the above actions since $k^G \subseteq A[e]$.

Let $R \in k^G \forall D$ be the diagram of $A$. Then the multiplication in $A$ induces an isomorphism $R \otimes k^G \rightarrow A$ of $k^G$-bimodules [1, Lemma 4.1]. Hence we can think of $R$ as a left $k^G$-submodule of $A$ and therefore

$$A[g] = R[g] k^G \text{ and } (A_k^G)[g] = R \delta_g \quad \forall g \in G. \quad (3)$$

Let $g \in G$. As in [9], we define the Verma module of $A$ of weight $g$ as the induced module

$$M_g = \text{Ind}_{k^G}^{A_k^G} \delta_g = A \otimes_{k^G} k \delta_g.$$

Then $M_g$ is projective, being induced from a module over a semisimple algebra, and hence injective, because $A$ is Frobenius. By (1) and (3), the weight spaces satisfy

$$M_g[h] = R[hg^{-1}] \delta_g \quad \forall h \in G. \quad (4)$$

Also, $M_g = A \delta_g = R \delta_g$ and $A = \oplus_{g \in G} M_g$.

Notice that if $L$ is a simple $A$-module and $0 \neq v \in L[g]$, then $L$ is a quotient of $M_g$ via $\delta_g \mapsto \delta_g \cdot v = v$.

Let $e \in A$ be an idempotent. We say that $e$ is a $g$-idempotent if $e \in R[e] \delta_g$. A set $\{e_i\}_{i \in I}$ of $g$-idempotents is called complete if $\delta_g = \sum_{i \in I} e_i$. Next lemma ensures that there always exists a complete set of orthogonal primitive $g$-idempotents.

**Lemma 1** Let $g \in G, e$ be a $g$-idempotent and $\mathcal{E}_g = \{e_i\}_{i \in I}$ be a set of orthogonal idempotents of $A$ such that $\delta_g = \sum_{i \in I} e_i$.

(a) $\mathcal{E}_g$ is a complete set of $g$-idempotents.

(b) $e$ is primitive if and only if it is not possible to express $e$ as a sum of orthogonal $g$-idempotents.

(c) There is a complete set of orthogonal primitive $g$-idempotents in $A$.

(d) $e \cdot M = e \cdot M[g] \subseteq M[g]$ for any $A$-module $M$.

(e) If $\# \mathcal{E}_g = \dim R[e]$, then $e_i$ is primitive for all $i \in I$. Moreover, if $e$ is primitive, then $e = e_i$ for some $i \in I$.

(f) If $\# \mathcal{E}_g = \dim R[e]$, then $A e_i \not\cong A e_j$ if $i \neq j$. 
Proof (a) We have to prove that \( e_i \) is a \( g \)-idempotent for all \( i \in I \). Fix \( i \in I \) and set 
\[
\alpha = e_i \quad \text{and} \quad \beta = \sum_{i \neq j \in I} e_j .
\]
If \( t \in G \) and \( t \neq g \), then \( 0 = \delta_g \delta_t = \alpha \delta_t + \beta \delta_t \).
Since \( \alpha \) and \( \beta \) are orthogonal, \( \alpha \delta_t = 0 \). Hence \( \alpha = \alpha \delta_g \) because \( 1 = \sum_{g \in G} \delta_g \).
Similarly \( \alpha = \delta_g \alpha \). Let \( a_s \in R[\delta] \) such that \( \alpha = \sum_{s \in G} a_s \delta_g \). Then \( \alpha = \delta_g \alpha = \sum_{s \in G} \delta_g a_s \delta_g = \sum_{s \in G} a_s \delta_{g^{-1}} \delta_g = a_e \delta_g \).
That is, \( \alpha = e_i \) is a \( g \)-idempotent.

(b) The first implication is obvious. For the second implication, we proceed as in (a).
(c) follows from (a) and (d). (d) holds because \( e \in R[e] \delta_g \).

(e) is a consequence of the fact that \( E_g \) is a basis of \( R[e] \delta_g \). Indeed, pick \( \alpha = e_i \in E_g \) and suppose \( \alpha = a + b \) with \( a \) and \( b \) orthogonal \( g \)-idempotents of \( A \). Then \( (Aa)[e] \oplus (Ab)[e] = (A\alpha)[e] = (\delta \alpha \delta_g) \alpha = \delta \alpha \) and therefore \( a = 0 \) or \( b = 0 \).
For the second statement, we write \( e = \sum_{i \in I} a_i e_i \) with \( a_i \in \delta, i \in I \). Since \( e^2 = e, a_i = 0 \) or \( 1 \) for all \( i \in I \) and hence \( e = e_i \) for some \( i \in I \).

(f) \( (Ae_i)[e] = \delta \epsilon_i \neq (Ae_j)[e] = \delta \epsilon_j \) if \( i \neq j \). Hence \( Ae_i \neq Ae_j \).

Given a set of idempotents \( E \) and an \( A \)-module \( M \), we write 
\[
\text{Supp}_E M = \{ e \in E : e \cdot M \neq 0 \} .
\]
By [12, Theorem 54.16] if \( L \) is a simple \( A \)-module and \( e \in \text{Supp}_E L \), then 
\[
top (Ae) \simeq L .
\]
This allows us to analyze the dimension of the weight spaces of the simple \( A \)-modules using \( g \)-idempotents.

Lemma 2 Let \( g \in G \) and \( E_g = \{ e_i \}_{i \in I} \) be a complete set of orthogonal primitive \( g \)-idempotents. Let \( L \) be a simple \( A \)-module.

(a) \( \dim L[g] = \# \text{Supp}_{E_g} L \).
(b) If \( \# E_g = \dim R[e] \) or 1, then \( \dim L[g] = 1 \) or 0.
(c) \( E_g = \bigcup_{L \in \text{Irr} A} \text{Supp}_{E_g} L \) is a partition.
(d) \( \dim R[e] \geq \sum_{L \in \text{Irr} A} (\dim L[g])^2 = \sum_{L \in \text{Irr} A} (\# \text{Supp}_{E_g} L)^2 \geq \# E_g \).

Proof (a) By [12, Theorem 54.16], \( \dim e_i \cdot L = 1 \) for all \( e_i \in \text{Supp}_{E_g} L \). Pick \( w_i \in e_i \cdot L - \{ 0 \} \) for each \( i \in I \). Then \( \{ w_i : i \in I \} \) is a basis of \( L[g] \) since
\[
v = \delta_g \cdot v = \sum_{e_i \in \text{Supp}_{E_g} L} e_i \cdot v \quad \text{for all} \ v \in L[g] .
\]
(b) If \( \# E_g = 1 \), then \( \dim L[g] = 1 \) or 0 by (a) If \( \# E_g = \dim R[e] \), the statement follows from (a) and Lemma 1 (f).
(c) is clear. (d) follows from (a) and (c) since
\[
R[e] \delta_g = \bigoplus_{i \in I} R[e] e_i = \bigoplus_{L \in \text{Irr} A} \bigoplus_{e_i \in \text{Supp}_{E_g} L} R[e] e_i .
\]
In some cases, the simple \( A \)-modules can be distinguished by their weight spaces.

Lemma 3 Let \( g \in G \) and \( E_g = \{ e_i \}_{i \in I} \) be a complete set of orthogonal primitive \( g \)-idempotents. Assume that \( \top (Ae_i) \) and \( \top (Ae_j) \) are not isomorphic as \( \delta \delta_g \)-modules
for all \( i \neq j \). Let \( L \) be a simple \( A \)-module. Then \( L \simeq \text{top}(Ae_i) \) as \( A \)-modules if and only if \( L \simeq \text{top}(Ae_j) \) as \( k^G \)-modules.

**Proof** If \( L \simeq \text{top}(Ae_i) \) as \( k^G \)-modules, then \( g \in \text{Supp} L \). Hence \( L \simeq \text{top}(Ae_j) \) for some \( j \). Then \( i = j \) because \( \text{top}(Ae_i) \) and \( \text{top}(Ae_j) \) are not isomorphic as \( k^G \)-modules for \( i \neq j \). The other implication is obvious.

For each \( g \in G \), let \( E_g \) be a complete set of orthogonal primitive \( g \)-idempotents. If \( e, \tilde{e} \in E_g \) and \( eA\tilde{e} \neq 0 \), it is said that \( e \) and \( \tilde{e} \) are linked. This is an equivalence relation [12, Definition 55.1]. Let \( E_g = \bigcup_{i \in I_g} B_i \) be the corresponding partition. The subalgebra \( A[e] = \mathbb{R}[e]k^G \) can be used to compute the simple \( A \)-modules, see for instance [22, Theorem 2.7.2].

**Lemma 4** Let \( g \in G \) and \( E_g = \bigcup_{i \in I_g} B_i \) be as above. Then \( \bigoplus_{e \in B_i} A[e]e \) is a subalgebra and a set of representatives of its simple modules is

\[
\text{Irr} \left( \bigoplus_{e \in B_i} A[e]e \right) = \{ L[g] : L \in \text{Irr} A \text{ and } B_i \cap \text{Supp}_e L \neq \emptyset \}.
\]

Moreover as algebras

\[
A[e] = \prod_{g \in G, i \in I_g} \bigoplus_{e \in B_i} A[e]e.
\]

**Proof** By (1), \( e\tilde{e} = 0 = \tilde{e}e \) if either \( e \in E_g \) and \( \tilde{e} \in E_h \) with \( g \neq h \) or \( e, \tilde{e} \in E_g \) but are not linked. Clearly, \( B_i \) is a complete set of orthogonal primitive idempotents of \( \bigoplus_{e \in B_i} A[e]e \). Also \( \text{top}(A[e]e) = L[g] \) since \( L[g] = \text{top}(Ae)[g] = A[e]e \) for all \( e \in E_g \).

For \( g \in G \), we define the linear map \( \chi_g : A \longrightarrow \mathbb{R} \) by

\[
\chi_g rf = \varepsilon(r) f(g) \quad \forall rf \in A = \mathbb{R}k^G.
\]

(5)

If \( \chi_g \) is an algebra map, then \( \mathbb{R}_g \) is also an \( A \)-module.

**Lemma 5** Let \( G \) be a finite group, \( A \) a finite-dimensional copointed Hopf algebra over \( \mathbb{R}^G \) with diagram \( R \in \mathbb{R}^G \mathcal{YD} \) and \( \chi \in G(A^*) \). If \( R \) is generated by \( R^\times \) as an algebra, then \( \chi = \chi_g \) for some \( g \in G \). Moreover, the map

\[
G(A^*) \longrightarrow G, \quad \chi_g \longmapsto g
\]

is an injective group homomorphism.

In particular, if \( R \) is a Nichols algebra, then \( R \) is generated by \( R^\times \).
Proof Let $g \in G$ such that $\chi(f) = f(g)$ for all $f \in \mathbb{k}^G$. By (1), $\chi(R^\times) = 0$. Hence $\chi = \chi_g$.

Let $\chi_g, \chi_h \in G(A^*)$ for some $g, h \in G$. Then $\chi_g \ast \chi_h$ is an algebra map and $\chi_g \ast \chi_h(f) = f(gh)$ for all $f \in \mathbb{k}^G$. Hence $\chi_g \ast \chi_h = \chi_{gh}$ and $G(A^*) \to G, \chi_g \mapsto g$ is an injective group homomorphism.

Finally, if $R$ is a Nichols algebra, then $R$ is generated by $R[1]$. Moreover, $R[1] \subset R^\times$ by [8, Lemma 3.1 (f)]. In particular, $R$ is generated by $R^\times$.

Example 1 Let $V \in \mathbb{k}^G \otimes \mathcal{D}$ with finite-dimensional Nichols algebra $\mathcal{B}(V)$. Then $\{\delta_g : g \in G\}$ is a complete set of orthogonal primitive idempotents of $\mathcal{B}(V) \# \mathbb{k}^G$ and therefore $\{\delta_g : g \in G\}$ are its simple modules.

Let $\int_A^\ell$ (resp. $\int_A^\ell$) denote the space of right (resp. left) integrals, see for example [21]. Since $A$ is finite-dimensional, the space of right (left) integrals is one-dimensional. Let $t \in \int_A^\ell$. Then there exists a unique $\alpha \in G(A^*)$, called the distinguished group-like element, such that $at = \alpha(a)t$ for all $a \in A$.

Lemma 6 Let $G$ be a finite group, $A$ a finite-dimensional copointed Hopf algebra over $\mathbb{k}^G$ and $\alpha \in G(A^*)$ the distinguished group-like element. Assume that there is $g \in G$ such that $\alpha(f) = f(g)$ for all $f \in \mathbb{k}^G$. Hence

$$\text{Supp}(\text{soc}(Ae)) = g \text{ Supp}(\text{top}(Ae))$$

for any primitive idempotent $e \in A$.

In particular, $\int_A^\ell = \text{soc}(Ae_{g^{-1}}) \subset R[g]e_{g^{-1}}$ where $e_{g^{-1}}$ is the primitive $g^{-1}$-idempotent such that $\text{top}(Ae_{g^{-1}}) \simeq \mathbb{k}^G$ as $\mathbb{k}^G$-modules.

Proof Let $\eta : A \to A$ be the Nakayama automorphism. If $M$ is an $A$-module, then $\overline{M}$ denotes the vector space $M$ with action $a \cdot m = \eta^{-1}(a)m$ for all $a \in A, m \in M$. By [15, Lemma 1.5],

$$\eta^{-1}(\delta_t) = (\alpha^{-1}, S^2(\delta_t), S^2(\delta_t)) \delta_t$$

for all $t \in G$. Therefore $\overline{M}[h] = M[gh]$ for all $h \in G$. By [23, Lemma 2], $\text{top}(Ae) = \text{soc}(Ae)$ and hence $\text{Supp}(\text{soc}(Ae)) = g \text{ Supp}(\text{top}(Ae))$.

In particular, we obtain that $\int_A^\ell = \text{soc}(Ae_{g^{-1}}) \subset R[g]e_{g^{-1}}$, the inclusion follows from (4).

We include the next lemma for completeness.

Lemma 7 Let $A$ be an algebra and $a_1, \ldots, a_n$ be idempotents of $A$ such that $a_i a_j = a_j a_i$ for all $i, j = 1, \ldots, n$. Set

$$e_i = a_i + a_i \sum_{\ell=1}^{i-1} (-1^\ell \sum_{1 \leq j_1 < \cdots < j_\ell \leq i-1} a_{j_1} \cdots a_{j_\ell}.$$ 

Then $e_i e_j = \delta_{i,j} e_i$ for all $i, j = 1, \ldots, n$. 

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Proof For \( j < i \), we write
\[
e_i = a_i + a_i \sum_{\ell=1}^{i-1} (-1)^\ell \sum_{1 \leq j_1 < \cdots < j_\ell \leq i-1, \ j_\ell \neq j} a_{j_1} \cdots a_{j_\ell} + a_i \sum_{\ell=1}^{i-1} (-1)^\ell \sum_{1 \leq j_1 < \cdots < j_\ell \leq i-1} a_{j_1} \cdots a_{j_\ell}.
\]
Then \( a_j e_i = 0 \) and hence \( e_j e_i = \delta_{i,j} e_i \) for all \( i,j = 1, \ldots, n \).

The order of the set \( \{a_i\} \) alters the result of the above lemma. Moreover, it can produce \( e_i = 0 \) for some \( i \). For example: \( \{1, a\} \) and \( \{a, 1\} \) with \( a \) an idempotent.

2.1 Quasitriangular copointed Hopf algebras

Let \( G \) be a non-abelian group and \( A \) be a quasitriangular finite-dimensional copointed Hopf algebra over \( \mathbb{k}^G \) with \( R \)-matrix \( Q \in A \otimes A \), that is, \( Q \) is an invertible element which satisfies [24, (QT.1)–(QT.4)] and
\[
Q \Delta(x) = \Delta^{cop}(x) Q, \quad \text{for all } x \in A.
\] (6)

Let \( (A_Q, Q) \) be its unique minimal subquasitriangular Hopf algebra [24, p. 292]. Then \( A_Q = HB \) with Hopf subalgebras \( H, B \subseteq A \) such that \( B \simeq H^{*cop} \) by [24, Proposition 2 and Theorem 1].

Lemma 8 \( H, B \) and \( A_Q \) are pointed Hopf algebras over abelian groups. Moreover, \( A_Q \) is neither a group algebra nor the bosonization of its diagram with \( G(A_Q) \).

Proof Since \( H_0 = H \cap A_0 \) and \( B_0 = B \cap A_0 \), there are group epimorphisms \( G \to G_H \) and \( G \to G_B \) such that \( H_0 = \mathbb{k}^{G_H} \) and \( B_0 = \mathbb{k}^{G_B} \). Then there is an epimorphism of Hopf algebras \( B \overset{\sim}{\longrightarrow} H^{*cop} \longrightarrow \mathbb{k}G_H \). By [21, Corollary 5.3.5], the restriction \( B_0 = \mathbb{k}G_B \to \mathbb{k}G_H \) is surjective. Thus \( G_H \) is an abelian group. Mutatis mutandis, we see that \( G_B \) is also an abelian group. Hence \( H \) and \( B \) are generated by skew-primitives and group-likes elements by [6, Theorem 2] and therefore is also \( A_Q = HB \). Then \( A_Q = HB \), \( H \) and \( B \) are pointed Hopf algebras over abelian groups. Set \( \Gamma = G(A_Q) \).

Now we assume \( A_Q = \mathbb{k}^\Gamma \) and let \( \delta_g \in \mathbb{k}^{G \setminus \Gamma} \). It must hold \( Q \Delta(\delta_g) = \Delta^{cop}(\delta_g) Q \) by (6). However, this is not possible since \( Q \) is invertible and \( \mathbb{k}^G \) is commutative but not cocommutative. Then \( A_Q \neq \mathbb{k}^\Gamma \).

Finally, we assume that \( A_Q = B(V)\#\mathbb{k}^\Gamma \) where \( B(V) \) is the diagram of \( A_Q \) which is a Nichols algebra by [6, Theorem 2]. Let \( Q_0 \in \mathbb{k}^\Gamma \otimes \mathbb{k}^\Gamma \) and \( Q^+ \in B(V)^+ \# \mathbb{k}^G \otimes \mathbb{k}^G \) such that \( Q = Q_0 + Q^+ \). Then \( Q_0 \) is invertible since \( Q \) is so and \( B(V)^+ \) is nilpotent. If \( \delta_g \in \mathbb{k}^{G \setminus \Gamma} \), then it must hold \( Q_0 \Delta(\delta_g) = \Delta^{cop}(\delta_g) Q_0 \) by (6). As above, this is not possible. Therefore \( A_Q \neq B(V)\#\mathbb{k}^\Gamma \).
3 The tetrahedron rack and their associated algebras

Let $\mathbb{F}_4$ be the finite field of four elements and $\omega \in \mathbb{F}_4$ such that $\omega^2 + \omega + 1 = 0$. The tetrahedron rack is the affine rack $\text{Aff}(\mathbb{F}_4, \omega)$. That is, the set $\mathbb{F}_4$ with operation $a \triangleright b = \omega b + \omega^2 a$.

Let $(\cdot, g, \chi_G)$ be a faithful principal YD-realization of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$ over a finite group $G$ [5, Definition 3.2]. Recall that

- $\triangleright$ is an action of $G$ over $\mathbb{F}_4$,
- $\cdot$ is a multiplication function such that $g_{h \cdot i} = hg_i h^{-1}$ and $g_i \cdot j = i \triangleright j$ for all $i, j \in \mathbb{F}_4, h \in G$,
- $\chi_G : G \rightarrow \mathbb{K}^*$ is a multiplicative character such that $\chi_G (g_i) = -1$ for all $i \in \mathbb{F}_4$; we can consider such a $\chi_G$ by [5, Lemma 3.3(d)].

These data define a structure on $\text{Aff}(\mathbb{F}_4, \omega)$.

We can calculate in [18] using previous results of [16] for the pointed case. Namely, we fix a faithful principal YD-realization $\text{Aff}(\mathbb{F}_4, \omega)$.

We obtain (7) using the fact that the categories $\mathbf{k}^G\mathcal{YD}$ and $\mathbf{k}^G\mathcal{YD}$ are braided equivalent [3, Proposition 2.2.1], see [18, Subsection 3.2] for details.

We denote by $G'$ the subgroup of $G$ generated by $\{g_i\}_{i \in \mathbb{F}_4}$. Then $G'$ is a quotient of the enveloping group of $\text{Aff}(\mathbb{F}_4, \omega)$ [13, 20]:

$$G_{\text{Aff}(\mathbb{F}_4, \omega)} = \langle g_0, g_1, g_2, g_3 | g_1 g_j = g_i \triangleright g_i, i, j \in \mathbb{F}_4 \rangle.$$  

Let $m \in \mathbb{N}$. We denote by $C_m$ the cyclic group of order $m$ generated by $t$. The semidirect product group $\mathbb{F}_4 \rtimes_{\omega} C_{6m}$ is given by $t \cdot i = \omega i$ for all $i \in \mathbb{F}_4$.

Example 2 [18, Proposition 4.1] Let $k, m \in \mathbb{N}$, $0 \leq k < m$. The $(m, k)$-affine realization of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$ over $\mathbb{F}_4 \rtimes_{\omega} C_{6m}$ is defined by

- $g : \mathbb{F}_4 \rightarrow \mathbb{F}_4 \rtimes_{\omega} C_{6m}, i \mapsto g_i = (i, i^{6k+1})$;
- $\cdot : \mathbb{F}_4 \rtimes_{\omega} C_{6m} \rightarrow \mathbb{F}_4$ is $h \cdot i = j$, if $h g_i h^{-1} = g_j$;
- $\chi_{\mathbb{F}_4 \rtimes_{\omega} C_{6m}} : \mathbb{F}_4 \rtimes_{\omega} C_{6m} \rightarrow \mathbb{K}^*, (j, t^r) \mapsto (-1)^{r}, \forall i, j \in A, s \in \mathbb{N}.$

3.1 A Nichols algebra over $\text{Aff}(\mathbb{F}_4, \omega)$

From now on, we fix a faithful principal YD-realization $(\cdot, g, \chi_G)$ over a finite group $G$ of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$. Let $V \in \mathbf{k}^G\mathcal{YD}$ be as in (7).

In [18, Subsection 3.1] it was discussed how braided functors modify the Nichols algebras. As a consequence the defining relations of the Nichols algebra $B(V)$ were calculated in [18] using previous results of [16] for the pointed case. Namely, $B(V)$ is the quotient of $T(V)$ by the ideal $\mathcal{J}(V)$ generated by

$$x_j^2, \quad x_j x_l + x_l x_j (x_{(0+1)i+oj} + x_{(0+1)i+oj} x_j \quad \forall i, j \in \mathbb{F}_4 \quad \text{and} \quad (8)$$

$$z := (x_{(0)})^2 + (x_{1,0}x_0)^2 + (x_0x_1x_0)^2. \quad \text{and} \quad (9)$$
In fact, [18, Proposition 4.4 (b)] states that $J(V)$ is generated by the elements in (8) and $z'_{(-1,4,\omega)} = (x_\omega x_\omega^2 x_0)^2 + (x_1 x_\omega^2 x_\omega)^2 + (x_0 x_\omega^2 x_1)^2$. An straightforward computation shows that $z - z'_{(-1,4,\omega)}$ belongs to the ideal generated by the elements in (8). Hence, we can take $z$ as a generator of $J(V)$ instead of $z'_{(-1,4,\omega)}$.

Let $B$ be the subset of $B(V)$ consisting of all possible words $m_1 m_2 m_3 m_4 m_5$ such that $m_i$ is an element in the $i$th row of the next list

1, $x_0$, 1, $x_1$, $x_1 x_0$, 1, $x_\omega x_0 x_1$, 1, $x_\omega$, $x_\omega x_0$, 1, $x_\omega^2$.

By (7) the weight of a monomial $x_{i_1} \cdots x_{i_\ell} \in T(V)$ is $g^{-1}_{i_1} \cdots g^{-1}_{i_\ell}$. Set $g_{top} = g^{-1}_0 g^{-1}_1 g^{-1}_2 g^{-1}_3 g^{-1}_4 g^{-1}_5 g^{-1}_6$ and

$$m_{top} = x_0 x_1 x_0 x_\omega x_0 x_1 x_\omega x_0 x_\omega^2 \in \mathbb{B}[g_{top}]$$

Lemma 9 The set $\mathbb{B}$ is a basis of $B(V)$ and $m_{top}$ is an integral.

Proof The faithful principal YD-realization $(\cdot, g, \chi_G)$ over $G$ of $\text{Aff}(\mathbb{F}_4, \omega, -1)$ also defines a Yetter-Drinfeld module $W \in \mathbb{F}_4^{\ell G} YD$ with basis $\{y_{i}\}_{i \in \mathbb{F}_4}$, see for instance [18, (7)]. By [4, Theorem 6.15], the ideal defining the Nichols algebra $B(W)$ is generated by

$$y_i^2, \quad y_i y_j + y_{(\omega+1)i+\omega j} y_i + y_{j} y_{(\omega+1)i+\omega j} \quad \forall i, j \in \mathbb{F}_4$$

$$+ (y_\omega y_1 y_0)^2 + (y_0 y_\omega y_1)^2 + (y_1 y_0 y_\omega)^2.$$ 

Let $\phi : W \rightarrow V$ be the linear map defined by $\phi(y_0) = x_1$, $\phi(y_1) = x_0$, $\phi(y_\omega) = x_\omega$ and $\phi(y_{\omega^2}) = x_{\omega^2}$. By (8) and (9), $\phi$ induces an algebra isomorphism $\phi' : B(W) \rightarrow B(V)$. Also, [4, Theorem 6.15] gives a basis $B$ of $B(W)$ which consists of all possible words $m_1 m_2 m_3 m_4 m_5$ such that $m_i$ is an element in the $i$th row of the next list

1, $x_1$, 1, $x_0$, $x_0 x_1$, 1, $x_\omega x_0 x_1$, 1, $x_\omega$, $x_\omega x_0$, 1, $x_\omega^2$.

Then $\phi'(B)$ is a basis of $B(V)$. Since $x_1 x_0 x_1 = x_0 x_1 x_0$ in $B(V)$ by (8), $\mathbb{B}$ also is a basis of $B(V)$.

Finally, the space of integrals of a finite-dimensional Nichols algebra is the homogeneous component of bigger degree, see for instance [4, p. 227]. Therefore $m_{top}$ is an integral.
Lemma 10 Let $G$ be a finite group with a faithful principal YD-realization $(\cdot, g, \chi_G)$ of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$. Then

(a) $\text{Supp} \mathcal{B}(V) = \text{Supp} \mathcal{B} \subset G'$.

(b) $G' \mapsto \mathbb{F}_4 \rtimes_{\omega} C_6, g_i \mapsto (i, t)$ is an epimorphism of groups.

(c) If $z \in T(V)[e]$, then $\mathcal{B}[e] = \{1, b_1, b_2, b_3, b_4, b_5\}$ where

\[
\begin{align*}
b_1 &= x_0x_1x_0x_0x_0x_0, & b_2 &= x_0x_0x_0x_1x_0x_0, & b_3 &= x_1x_0x_0x_0x_1x_0, \\
b_4 &= x_1x_0x_0x_1x_0x_0, & b_5 &= x_0x_1x_0x_0x_1x_0.
\end{align*}
\]

(d) Let $y = \sum_{i \in \mathbb{F}_4} x_i$ and $U = \mathbb{k}[x_0 - x_1, x_0 - x_0, x_0 - x_0^2]$. Then $\mathbb{k}y$ and $U$ are simple $\mathbb{k}^G$-comodules such that $V = \mathbb{k}y \oplus U$.

Proof (a) holds since the elements of $\mathcal{B}$ are $\mathbb{k}^G$-homogeneous and $\mathcal{B}(V)$ is a $\mathbb{k}^G$-module algebra.

(b) By [4, Lemma 1.9 (1)], the quotient of $G'$ by its center $Z(G')$ is isomorphic to $\text{Inn}_G \text{Aff}(\mathbb{F}_4, \omega) = \mathbb{F}_4 \rtimes_{\omega} C_3$ via $\overline{g_i} \mapsto (i, t), i \in \mathbb{F}_4$. Then $G'/(Z(G') \cap \ker \chi_G) \cong \mathbb{F}_4 \rtimes_{\omega} C_3 \times C_2 \cong \mathbb{F}_4 \rtimes_{\omega} C_6$.

(c) If $z \in \mathcal{B}[e]$, then $\{1, b_1, b_2, b_3, b_4, b_5\} \subset \mathcal{B}[e]$ since $g_i g_j = g_i \cdot g_j g_i$. Let $w = x_{i_1} \cdots x_{i_s} \in \mathcal{B}[e]$. Applying the epimorphism of (b) to the weight of $w$, we see that $w = 1$ or $s = 6$. If $w \neq 1$, we can check that $w = b_i$ for some $i$.

(d) is equivalent to prove that $\mathbb{k}y$ and $U$ are simple $\mathbb{k}G$-modules via the action $g \cdot x_i = \chi_G(g) x_{g \cdot i}, i \in \mathbb{F}_4$. Clearly, $\mathbb{k}y$ and $U$ are $\mathbb{k}G$-submodules and $\mathbb{k}y$ is $\mathbb{k}G$-simple. Moreover, it is an straightforward computation to show that $U$ is $\mathbb{k}G'$-simple and therefore $\mathbb{k}G$-simple.

3.2 Copointed Hopf algebras over $\text{Aff}(\mathbb{F}_4, \omega)$

The copointed Hopf algebras over $\mathbb{k}^G$ whose infinitesimal braiding arises from a principal YD-realization of the affine rack $\text{Aff}(\mathbb{F}_4, \omega)$ with the constant 2-cocycle $-1$ are classified in [18] as follows.

By (7) the smash product Hopf algebra $T(V) \# \mathbb{k}^G$ is defined by

\[
\delta_t x_i = x_i \delta_{g_i t} \quad \text{and} \quad \Delta(x_i) = x_i \otimes 1 + \sum_{t \in G} \chi_G(t) \delta_{t^{-1}} \otimes x_{t i} \quad \forall t \in G, i \in X. \tag{10}
\]

Definition 1 Let $\lambda \in \mathbb{k}$ and assume $z \in T(V)[e]$. The Hopf algebra $A_{G, \lambda}$ is the quotient of $T(V) \# \mathbb{k}^G$ by the ideal generated by (8) and $z - f$ where

\[
f = \lambda(1 - \chi_z^{-1}) \quad \text{and} \quad \chi_z = \chi_G^6.
\]

Notice that if either $\lambda = 0$ or $\chi_z = 1$, then $A_{G, \lambda} = \mathcal{B}(V) \# \mathbb{k}^G$.
Theorem 11 Let $A$ be a copointed Hopf algebra over $k^G$ whose infinitesimal braiding arises from a principal YD-realization of the affine rack $\text{Aff}(\mathbb{F}_4, \omega)$ with the constant 2-cocycle $-1$.

(a) If $G = G'$, then $A \simeq B(V)\# k^G$.

(b) If $z \in T(V)^\times$, then $A \simeq B(V)\# k^G$.

(c) If $z \in T(V)[e]$, then $A \simeq A_{G,\lambda}$ for some $\lambda \in k$.

(d) $A_{G,\lambda}$ is a cocycle deformation of $A_{G,\lambda'}$, for all $\lambda, \lambda' \in k$.

(e) $A_{G,\lambda}$ is a lifting of $B(V)$ over $k^G$ for all $\lambda, \lambda' \in k$.

(f) $A_{G,\lambda} \simeq A_{G,1} \neq A_{G,0}$ for all $\lambda \in k^*$.

We are specially interested in the case that $A_{G,\lambda}$ is not isomorphic to $B(V)\# k^G$. The next faithful principal YD-realization gives such a $A_{G,\lambda}$.

Example 3 Suppose that $m | 6k + 1$. Let $G_1$ be a finite group with a multiplicative character $\chi_{G_1}: G_1 \to k^*$ such that $\chi_{G_1}^6 \neq 1$. Then the $(m, k)$-affine realization, recall Example 2, is extended to a principal YD-realization over $G = \mathbb{F}_4 \rtimes_\omega C_{6m} \times G_1$ setting $G_1 \cdot i = i$ and $\chi_G = \chi_{\mathbb{F}_4 \rtimes_\omega C_{6m}} \times \chi_{G_1}$. Note that $z \in T(V)[e]$ and $\chi_z = \chi_G^6 \neq 1$.

The next example will be necessary in Lemma 14.

Example 4 Let $G' \leq G_1 \leq G$ be finite groups. If $(\cdot, g, \chi_G)$ is a faithful principal YD-realization of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$ over $G$, then $(\cdot, g, (\chi_G)_{G_1})$ is a faithful principal YD-realization of $(\text{Aff}(\mathbb{F}_4, \omega), -1)$ over $G_1$. For instance, $G_1 = \ker \chi_z$.

We think of $A_{G,\lambda}$ as an algebra presented by generators $\{x_i, \delta_g : i \in \mathbb{F}_4, g \in G\}$ and relations:

$$
\delta_g x_i = x_i \delta_g g, \quad x_i^2 = 0, \quad \delta_g \delta_h = \delta_g (h) \delta_g, \quad 1 = \sum_{g \in G} \delta_g,
$$

$$
x_0 x_\omega + x_\omega x_1 + x_1 x_0 = 0 = x_0 x_\omega^2 + x_\omega^2 x_\omega + x_\omega x_0,
$$

$$
x_1 x_\omega^2 + x_0 x_1 + x_\omega x_0 = 0 = x_\omega x_\omega^2 + x_1 x_\omega + x_\omega^2 x_1 \quad \text{and}
$$

$$
x_\omega x_0 x_1 x_\omega x_0 x_1 + x_1 x_\omega x_0 x_1 x_\omega x_0 + x_0 x_1 x_\omega x_0 x_1 x_\omega = f,
$$

for all $i \in \mathbb{F}_4$ and $g \in G$. Since $\chi_z(g_1) = 1$, it holds that

$$
f x_i = x_i f \quad \forall i \in \mathbb{F}_4.
$$

A basis for $A_{G,\lambda}$ is $A = \{x \delta_g | x \in B, g \in G\}$ and a basis for the Verma module $M_g$ is $\mathcal{M} = \{x_{i_1} \cdots x_{i_n} \delta_g \in \mathcal{B}_g | \delta_g\}$.

Proposition 12 $A_{G,\lambda}$ is not quasitriangular.

Proof Let $A$ be a pointed Hopf subalgebra of $A_{G,\lambda}$ with abelian group of group-like elements. By Lemma 8, the proposition follows if we show that $A$ is either a group algebra or the bosonization of its diagram with $G(A)$.

Note that $A$ is generated by skew-primitives and group-like elements by [6, Theorem 2].

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Let \( y = \sum_{i \in \mathbb{F}_4} x_i \). The space of skew-primitives of \( \mathcal{A}_{G,\lambda} \) is \( \mathbb{k}G(\mathcal{A}_{G,\lambda}) \oplus \mathbb{k}\{yg \mid g \in G(\mathcal{A}_{G,\lambda})\} \) by Lemma 10 (d). Also, \( y^2 = 0 \) by (11). Hence \( A = G(A) \) or \( A = (\mathbb{k}[y]/(y^2)) \# \mathbb{k}G(A) \).

4 Representation theory of \( \mathcal{A}_{G,\lambda} \)

Let \((\cdot, g, \chi_G)\) be a faithful principal YD-realization of \((\text{Aff}(\mathbb{F}_4, \omega), -1)\) over a fixed finite group \( G \). Let \( V \in \mathbb{k}_G \mathcal{YD} \) be as in (7).

We are interested in the representation theory of the liftings of the Nichols algebra \( \mathcal{B}(V) \) over \( \mathbb{k}^G \). By Theorem 11, these liftings are the Hopf algebras \( \mathcal{A}_{G,\lambda}, \lambda \in \mathbb{k} \), recall Definition 1. We begin by classifying the simple modules.

If \( \mathcal{A}_{G,\lambda} \) is isomorphic to the bosonization \( \mathcal{B}(V) \# \mathbb{k}^G \), then the simple modules are the one-dimensional modules \( \mathbb{k}_g, g \in G \), where the Nichols algebra acts by zero, see Example 1.

From now on, we fix \( \lambda \in \mathbb{k}^* \) and assume that \( \mathcal{A}_{G,\lambda} \) is not isomorphic to the bosonization \( \mathcal{B}(V) \# \mathbb{k}^G \).

In this case, \( z \in T(V)[e] \) and \( \chi_z \neq 1 \) by Theorem 11 and Definition 1. Let \( f = \lambda(1 - \chi_z^{-1}) \) as in Definition 1. For \( g \in G \setminus \ker \chi_z \), we define

\[
\begin{align*}
\epsilon_1^g &= -\frac{1}{f(g)} b_1 \delta_g, & \epsilon_2^g &= -\frac{1}{f(g)} b_2 \delta_g, & \epsilon_3^g &= \frac{1}{f(g)} b_3 \delta_g, \\
\epsilon_4^g &= \frac{1}{f(g)} (b_4 - b_3) \delta_g, & \epsilon_5^g &= \frac{1}{f(g)} (b_5 + b_1) \delta_g & \text{and} \\
\epsilon_6^g &= \delta_g + \frac{1}{f(g)} (b_2 - b_4 - b_5) \delta_g,
\end{align*}
\]

where \( b_1, b_2, b_3, b_4, b_5 \in \mathcal{A}_{G,\lambda} \) are as in Lemma 10 (c).

Lemma 13 A complete set of orthogonal primitive idempotents of \( \mathcal{A}_{G,\lambda} \) is

\[
\mathcal{E} := \{ \delta_h, \epsilon_1^g, \epsilon_2^g, \epsilon_3^g, \epsilon_4^g, \epsilon_5^g, \epsilon_6^g \mid h \in \ker \chi_z, g \in G \setminus \ker \chi_z \}.
\]

Proof By Lemma 10 (c), \( \{b_i \delta_i \mid 1 \leq i \leq 6\} \) is a basis of \( \mathcal{B}(V)[e] \delta_g \) for all \( g \in G \). By (11) and (12), it holds that:

\[
\begin{align*}
& b_1^2 = -b_1 f, & b_2 b_1 = 0, & b_1 b_2 = 0, & b_1 b_3 = 0, & b_1 b_4 = 0, & b_1 b_5 = b_1 f, \\
& b_2 b_1 = 0, & b_2 b_2 = 0, & b_2 b_3 = 0, & b_2 b_4 = 0, & b_2 b_5 = 0, \\
& b_3 b_1 = 0, & b_3 b_2 = 0, & b_3 b_3 = b_3 f, & b_3 b_4 = b_3 f, & b_3 b_5 = 0, \\
& b_4 b_1 = 0, & b_4 b_2 = 0, & b_4 b_3 = b_3 f, & b_4 b_4 = b_4 f, & b_4 b_5 = 0, \\
& b_5 b_1 = b_1 f, & b_5 b_2 = 0, & b_5 b_3 = 0, & b_5 b_4 = 0, & b_5 b_5 = b_5 f.
\end{align*}
\]
Therefore \( \mathcal{E}_h = \{ \delta_h \} \) is a complete set of orthogonal primitive \( h \)-idempotents for all \( h \in \ker \chi_z \). If \( g \in G \setminus \ker \chi_z \), we apply Lemma 7 to the ordered set

\[
\left\{ -\frac{1}{f(g)} b_1 \delta_g, -\frac{1}{f(g)} b_2 \delta_g, -\frac{1}{f(g)} b_3 \delta_g, -\frac{1}{f(g)} b_4 \delta_g, -\frac{1}{f(g)} b_5 \delta_g, \delta_g \right\}
\]

and hence \( \mathcal{E}_g = \{ e_i^g \} [1 \leq i \leq 6] \) is a complete set of orthogonal primitive \( g \)-idempotents. Then \( \mathcal{E} = \bigcup_{g \in G} \mathcal{E}_g \) is a complete set of orthogonal primitive idempotents.

Let \( M \) be an \( \mathcal{A}_{G, \lambda} \)-module. Since \( \mathcal{A}_{G, \lambda} \) is a quotient of \( T(V)\# \mathbb{K}^G \), \( M \) is also a \( T(V)\# \mathbb{K}^G \)-module. Moreover, \( M \) is a \( T(V)\# \ker \chi_z \)-module if \( \text{Supp} M \subseteq \ker \chi_z \) since \( T(V)\# \ker \chi_z \) is a subalgebra of \( T(V)\# \mathbb{K}^G \), cf. Example 4.

**Lemma 14** Let \( h \in \ker \chi_z \).

(a) If \( M \) is an \( \mathcal{A}_{G, \lambda} \)-module with \( \text{Supp} M \subseteq \ker \chi_z \), then \( M \) is a module over \( B(V)\# \ker \chi_z \).

(b) \( M_h \) is a free \( B(V) \)-module of rank 1 generated by \( \delta_h \).

(c) \( \chi_h : \mathcal{A}_{G, \lambda} \to \mathbb{K} \) is an algebra map.

(d) \( \text{Top}(M_h) \simeq \mathbb{K} \) and \( \text{soc}(M_h) \simeq \mathbb{K} \text{top} h \).

(e) \( \int_{\mathcal{A}_{G, \lambda}} = \text{soc}(M \text{top}^{-1}) \) and \( \chi \text{top} \) is the distinguished group-like element.

**Proof** (a) Since \( M \) is a \( T(V)\# \ker \chi_z \)-module, we have to see that the elements in (8) and \( z \) act by zero over \( M \). This is true for the first elements because they are zero in \( \mathcal{A}_{G, \lambda} \). If \( h \in \ker \chi_z \), then \( f \delta_h = 0 \) and hence \( z \cdot M[h] = f \cdot (\delta_h \cdot M) = 0 \). (b) follows from (a). (c) is clear. (d) and (e) follows from (b) and Lemma 6.

For each \( e_i^g \in \mathcal{E} \), we set \( L_i^g = \mathcal{A}_{G, \lambda} e_i^g \).

**Lemma 15** (a) \( L_i^g \) is an injective and projective simple module of dimension 12 for all \( e_i^g \in \mathcal{E} \).

(b) There exist \( \mathbb{K}^G \)-submodules \( L_1, \ldots, L_6 \subseteq B(V) \) such that \( B(V) = L_1 \oplus \cdots \oplus L_6 \) and \( L_i^g = L_i^g \delta_g \) for all \( i = 1, \ldots, 6 \) and \( g \in G \).

(c) \( \text{Supp} L_i \neq \text{Supp} L_j \) and \( \text{Supp} L_i^g = (\text{Supp} L_i) g \) for all \( 1 \leq i, j \leq 6 \) and \( g \in G \).

(d) \( L_i^g \simeq L_j^h \) if and only if \( \text{Supp} L_i \) \( g = (\text{Supp} L_j) h \).

**Proof** (a) Let \( v = e_i^g \in \text{Top}(L_i^g) \). Since \( f(g) v = z \cdot v = (x_{\omega} x_0 x_1)^2 \cdot v + b_4 \cdot v + b_5 \cdot v \neq 0 \), there are \( x_{i_6}, \ldots, x_{i_1} \in \mathcal{A}_{G, \lambda} \) such that \( x_{i_6} \cdots x_{i_1} \cdot v \neq 0 \) for all \( \ell = 1, \ldots, 6 \). We claim that \( \dim \text{Top}(L_i^g) \geq 11 \). In fact, if \( 1 \leq \ell < 6 \), then by (8)

\[
-x_{i_{\ell+1}} x_{(\omega+1)\ell+\omega+\ell+1} \cdots x_{i_1} \cdot v \\
= -x_{i_{\ell+1}} x_{(\omega+1)\ell+\omega+\ell+1} \cdots x_{i_1} \cdot v \\
= x_{(\omega+1)\ell+\omega+\ell+1} x_{i_{\ell+1}} \cdots x_{i_1} \cdot v \neq 0
\]

and hence \( x_{(\omega+1)\ell+\omega+\ell+1} \cdots x_{i_1} \cdot v \neq 0 \) or \( x_{i_{\ell+1}} \cdots x_{i_1} \cdot v \neq 0 \). Applying the epimorphism given by Lemma 10 (b), we find 11 elements with different weights belong to \( \text{Top}(L_i^g) \). Then \( \# \text{Supp} \text{Top}(L_i^g) \geq 11 \).
Now, we show that \( L_i^g = soc(L_i^g) = top(L_i^g) \) and (a) follows. Otherwise, \( \dim L_i^g \geq 22 \) since \( \dim top(L_i^g) = \dim soc(L_i^g) \) by [12, Lemma 58.4]. But the above claim holds for all \( i \) and hence \( 72 = \dim M_g \geq 22 + 5 \cdot 11 \), a contradiction.

(b) follows from Tables 1, 2, 3, 4, 5, 6 in Appendix. (c) If \( G' = \mathbb{F}_4 \times C_6 \), then \( \text{Supp} L_i \neq \text{Supp} L_j \) by Table 7 in Appendix and therefore for any \( G' \) by Lemma 10 (b). By (b), \( \text{Supp} L_i^g = (\text{Supp} L_i)_g \). (d) follows from (c) and Lemma 3.

\( \square \)

We consider the product set \( \{1, 2, 3, 4, 5, 6\} \times G \) with the equivalence relation \( i \times g \sim j \times h \) if and only if \( (\text{Supp} L_i)_g = (\text{Supp} L_j)_h \). Let \( \mathcal{X} \) be the set of equivalence classes of \( \sim \). We denote by \([i, g]\) the equivalence class of \( i \times g \). By Lemma 15 (d), we can define \( L_{[i, g]} = L_i^g \).

**Theorem 16** Every simple \( A_{G, \lambda} \)-module is isomorphic to either

\[ \mathbb{k}_g \text{ for a unique } g \in \ker \chi_z \text{ or } \]

\[ L_{[i, g]} \text{ for a unique } [i, g] \in \mathcal{X}. \]

In particular, there are (up to isomorphism) \( |\ker \chi_z| \) one-dimensional simple \( A_{G, \lambda} \)-modules and \( \frac{(|G| - |\ker \chi_z|)}{2} \) 12-dimensional simple \( A_{G, \lambda} \)-modules.

**Proof** It follows from Lemmata 13, 14 and 15. \( \square \)

**Example 5** Assume \( G' = \mathbb{F}_4 \times C_6 \) and let \( g \in G \setminus \ker \chi_z \). The set \( \mathcal{X} \) is completely defined by the equivalence class \([1, g]\) which is

\[
\left\{ \begin{array}{l}
1 \times g, \, 2 \times (1, t^2)g, \, 3 \times (0, t)g, \, 4 \times (\omega, t^2)g, \\
5 \times (1, t)g, \, 6 \times (\omega, 1)g, \, 1 \times (0, t^3)g \\
2 \times (1, t^5)g, \, 3 \times (0, t^4)g, \, 4 \times (\omega, t^5)g, \, 5 \times (1, t^4)g, \, 6 \times (\omega, t^3)g
\end{array} \right\}.
\]

Therefore

\[
\begin{align*}
L_{[1, g]} &= L_1^g \simeq L_2^{(1, t^2)g} \simeq L_3^{(0, t)g} \simeq L_4^{(\omega, t^2)g} \simeq L_5^{(1, t)g} \simeq L_6^{(\omega, 1)g} \\
L_1^{(0, t^3)g} \simeq L_2^{(1, t^5)g} \simeq L_3^{(0, t^4)g} \simeq L_4^{(\omega, t^5)g} \simeq L_5^{(1, t^4)g} \simeq L_6^{(\omega, t^3)g}.
\end{align*}
\]

Note that \( i \times g \sim i \times (0, t^3)g \) for all \( i \), hence \( L_i^g \simeq L_i^{(0, t^3)g} \).

In fact, \( (\text{Supp} L_2)(1, t^2) = \text{Supp} L_1 \), see Tables 1 and 2 in Appendix. Then \( L_1^g \simeq L_2^{(1, t^2)g} \) by Lemma 15 (d). The other isomorphisms are obtained in the same way.

4.1 Decomposition of the category of \( A_{G, \lambda} \)-modules

We fix \( \lambda \in \mathbb{k}^* \) and assume that \( A_{G, \lambda} \) is not isomorphic to the bosonization \( B(V)\#\mathbb{k}^G \). Let \( I \subset \{1, 2, 3, 4, 5, 6\} \times G \) be a set of representatives of the equivalence classes of \( \sim \). Let \( M \) be an \( A_{G, \lambda} \)-module.
If \( i \times g \in I \), then \( d_{i,g}^M = \dim(\epsilon_i^g \cdot M) \) is the number of composition factors of \( M \) which are isomorphic to \( L_{i,g} \) [12, Theorem 54.16]. The number \( d_{i,g}^M \) can be calculated by Lemma 1 (d). Since \( L_{i,g} \) is projective and injective by Lemma 15, there is a submodule \( N \subseteq M \) such that \( \text{Supp} \ N \subseteq \ker \chi_z \) and

\[
M = N \oplus \bigoplus_{j \in I} (L_j)^{d_{i,j}^M}.
\]

Moreover, \( N \) is a \( B(V) \# k^\ker \chi_z \)-module by Lemma 14 (a).

4.2 Representation type of \( A_{G,\lambda} \)

From now on, \( A_{G,\lambda} \) is any lifting of \( B(V) \) over \( k \). It can be isomorphic to \( B(V) \# k \) or not. Let \( k_g \) and \( k_h \) be one-dimensional \( A_{G,\lambda} \)-modules such that \( g = g_i^{-1} h \in \ker \chi_z \) for some \( i \in \mathbb{F}_4 \). We define the \( A_{G,\lambda} \)-module \( M_{g,h} = k\{w_h, w_g\} \) by \( k\{w_g\} \simeq k \) as \( A_{G,\lambda} \)-modules, \( w_h \in M[h] \) and \( x_j w_h = \delta_{j,i} w_g \) for all \( j \in \mathbb{F}_4 \).

**Proposition 17** The extensions of one-dimensional \( A_{G,\lambda} \)-modules are either trivial or isomorphic to \( M_{g,h} \) for some \( g, h \in \ker \chi_z \). Hence \( A_{G,\lambda} \) is of wild representation type.

**Proof** Let \( M \) be an extension of \( k_h \) by \( k_g \). Then \( M = M[g] \oplus M[h] \) as \( k \)-modules and \( M[g] \simeq k \) as \( A_{G,\lambda} \)-modules. Since \( x_i \cdot M[h] \subseteq M[g_i^{-1} h] \), the first part follows.

For the second part we can easily see that \( \text{Ext}^1_{A_{G,\lambda}}(k_g, k_h) \) is either 1 or 0 for all \( g, h \in \ker \chi_z \). Then the separated quiver of \( A_{G,\lambda} \) is wild. The details for this proof are similar to [9, Proposition 26].

4.3 Is \( A_{G,\lambda} \) spherical?

A Hopf algebra \( H \) is spherical [10] if there is \( \omega \in G(H) \) such that

\[
S^2(\omega) = \omega \omega^{-1} \quad \forall \omega \in H \quad \text{and} \quad tr_V(\omega) = tr_V(\omega^{-1}) \quad \forall V \in \text{Irr} \ H \quad \text{by \ [AAGTV Proposition 2.1].}
\]

**Proposition 18** \( B(V) \# k^G \) is spherical iff \( \chi_G^2 = 1 \). Moreover, \( (A_{G,\lambda}, \chi_G) \) with \( \lambda \neq 0 \) is spherical iff \( (\chi_G|_{\ker \chi_z})^2 = 1 \).

**Proof** It is a straightforward computation to see that \( \chi_G \) satisfies (14) using (10). Let \( V \in \text{Irr} \ A_{G,\lambda} \). If \( \dim V = 12 \), then \( V \) is projective and therefore \( tr_V(\chi_G^{\pm 1}) = 0 \) [11, Proposition 6.10]. If \( V = k_h \) with \( h \in \ker \chi_z \), then (15) holds iff \( \chi_G(h) = \pm 1 \).

**Example 6** Let \( (\cdot, g, \chi_G) \) be the faithful principal YD-realization in Example 3. Then \( (A_{G,\lambda}, \chi_G) \) is a spherical Hopf algebra with non involutory pivot.

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Any spherical Hopf algebra $H$ has an associated tensor category $\text{Rep}(H)$ which is a quotient of $\text{Rep}(H)$, see [1, 10, 11] for the background of this subject. Moreover, $\text{Rep}(H)$ is semisimple but rarely is a fusion category in the sense of [14], i.e. $\text{Rep}(H)$ rarely has a finite number of irreducibles. One hopes to find new examples of fusion categories as tensor subcategories of $\text{Rep}(H)$ for a suitable $H$. However, this is not possible for $H = A_{G,\lambda}$, see below.

**Remark 19** Assume that $(A_{G,\lambda}, \chi_G)$ is spherical. Then only the one-dimensional simple modules survive in $\text{Rep}(A_{G,\lambda})$ since the other simple modules are projective. Then $\text{Rep}(A_{G,\lambda})$ is equivalent to $\text{Rep}(B(V)\#\text{ker}\chi_G)$ by Subsection 4.1, where the pivot $\chi_G|_{\text{ker}\chi_G}$ is involutory. Hence any fusion subcategory of $\text{Rep}(A_{G,\lambda})$ is equivalent to $\text{Rep}(K)$, with $K$ a semisimple quasi-Hopf algebra, by [2, Proposition 2.12].

**Acknowledgments** The authors thank professor Nicolás Andruskiewitsch for proposing this problem and useful suggestions for this article. The first author also thanks Carolina Renz for her hospitality during her stay in Córdoba. Bárbara Pogorelsky was partially supported by Capes-Brazil. Cristian Vay was partially supported by ANPCyT-Foncyt, CONICET, MinCyT (Córdoba) and Secyt (UNC).

**Appendix**

The next tables describe the structure of the 12-dimensional simple modules of $A_{G,\lambda}$. These were used in Lemma 15.

See Tables 1, 2, 3, 4, 5, 6 and 7

**Table 1** Action of the generators $x_j$ on $L^g_1 = A_{G,\lambda}e^g_1$

| Linear basis of $L^g_1$ | $x_0^1$ | $x_1^1$ | $x_\omega^1$ | $x_{\omega^2}^1$ |
|--------------------------|---------|---------|--------------|----------------|
| $c_1 = x_0x_1x_0x_0x_0x_1x_0x_0x_0x_0x_0x_0\delta_g$ | 0       | 0       | $-f(g)c_6$  | $-f(g)c_{10}$ |
| $c_2 = x_0x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g = -f(g)e^g_1$ | 0       | 0       | $-c_5$      | $-c_9$         |
| $c_3 = x_0x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | 0       | $c_1$  | $f(g)c_{12}$ | 0              |
| $c_4 = x_0x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | 0       | $c_2$  | $c_{11}$    | 0              |
| $c_5 = x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | 0       | $c_7$  | 0           | $-c_3$         |
| $c_6 = x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | 0       | $c_8$  | 0           | $-c_4$         |
| $c_7 = x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_1$  | 0       | 0           | $-f(g)c_{12}$ |
| $c_8 = x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_2$  | 0       | 0           | $c_{11}$       |
| $c_9 = x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_3$  | 0       | $-c_7$      | 0              |
| $c_{10} = x_1x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_4$  | 0       | $-c_8$      | 0              |
| $c_{11} = x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_5$  | $c_9$  | 0           | 0              |
| $c_{12} = x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_6$  | $c_{10}$ | 0           | 0              |
Table 2  Action of the generators $x_j$ on $L_2^g = A_{G,\lambda}e_2^g$

| Linear basis of $L_2^g$ | $x_0^\cdot$ | $x_1^\cdot$ | $x_\omega^\cdot$ | $x_{\omega^2}^\cdot$ |
|--------------------------|-------------|-------------|-----------------|----------------|
| $c_1 = x_0x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | 0           | $c_6$           | $-f(g)c_{10}$ |
| $c_2 = x_0x_1x_0x_0x_0x_1x_0x_0^2\delta g$ | 0           | 0           | $-c_5$          | $-c_9$        |
| $c_3 = x_0x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | $c_1$       | $-c_{12}$       | 0             |
| $c_4 = x_0x_1x_0x_0x_0^2\delta g$ | 0           | $c_2$       | $c_{11}$        | 0             |
| $c_5 = x_0x_0x_0x_0x_1x_0x_0x_0^2\delta g = f(g)e_2^g$ | 0           | $c_7$       | 0               | $-c_3$        |
| $c_6 = x_0x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | 0           | $-f(g)c_8$  | 0               | $f(g)c_4$    |
| $-x_0x_0x_0x_0^2\delta g$ |               |             |                 |               |
| $c_7 = x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | $c_1$       | 0           | 0               | $-c_{12}$     |
| $c_8 = x_1x_0x_0x_0^2\delta g$ | 0           | 0           | 0               | $c_{11}$      |
| $c_9 = x_1x_0x_0x_0^2\delta g$ | 0           | $c_3$       | 0               | $-c_7$        |
| $c_{10} = x_1x_0^2\omega^2\delta g$ | 0           | $c_4$       | 0               | 0             |
| $c_{11} = x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | $c_5$       | 0               | 0             |
| $c_{12} = x_1x_0x_0x_0x_0x_0x_0x_0^2\delta g = x_0x_0x_0x_0^2\delta g$ | 0           | $-f(g)c_{10}$ | 0               | 0             |

Table 3  Action of the generators $x_j$ on $L_3^g = A_{G,\lambda}e_3^g$

| Linear basis of $L_3^g$ | $x_0^\cdot$ | $x_1^\cdot$ | $x_\omega^\cdot$ | $x_{\omega^2}^\cdot$ |
|--------------------------|-------------|-------------|-----------------|----------------|
| $c_1 = x_0x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | 0           | $c_6$           | $-c_{10}$     |
| $c_2 = x_0x_1x_0x_0x_0^2\delta g$ | 0           | 0           | $-c_5$          | $-c_9$        |
| $c_3 = x_0x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | $c_1$       | $c_{12}$        | 0             |
| $c_4 = x_0x_1x_0x_0^2\delta g$ | 0           | $c_2$       | $c_{11}$        | 0             |
| $c_5 = x_0x_0x_0x_0x_1x_0x_0x_0^2\delta g$ | 0           | $c_7$       | 0               | $-c_3$        |
| $c_6 = x_0x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | 0           | $-f(g)c_8$  | 0               | $f(g)c_4$    |
| $-f(g)x_0x_0x_0^2\delta g$ |               |             |                 |               |
| $c_7 = x_1x_0x_0x_0x_1x_0x_0x_0^2\delta g = f(g)e_3^g$ | $c_1$       | 0           | 0               | $c_{12}$     |
| $c_8 = x_0x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | $-f(g)c_2$  | 0           | 0               | $-f(g)c_{11}$ |
| $-f(g)x_1x_0x_0x_0^2\delta g$ |               |             |                 |               |
| $c_9 = x_1x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | $c_3$       | 0           | 0               | $-c_7$        |
| $c_{10} = x_0x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | $-f(g)c_4$  | 0           | 0               | $c_8$        |
| $-f(g)x_1x_0x_0x_0^2\delta g$ |               |             |                 |               |
| $c_{11} = x_0x_0x_0x_0x_1x_0x_0x_0x_0^2\delta g$ | $c_5$       | $c_9$       | 0               | 0             |
| $c_{12} = x_1x_0x_0x_0x_0x_0x_0x_0x_0^2\delta g$ | $-c_6$       | $-c_{10}$   | 0               | 0             |

+ $x_0x_0x_0x_0x_0x_0x_0x_0^2\delta g - f(g)x_0x_0^2\delta g$
Table 4  Action of the generators $x_i$ on $L_4^g = A_{G,\lambda}e_4^g$

| Linear basis of $L_4^g$ | $x_0^\cdot$ | $x_1^\cdot$ | $x_\omega^\cdot$ | $x_{\omega^2}^\cdot$ |
|-------------------------|--------------|--------------|------------------|-------------------|
| $c_1 = x_0x_1x_0x_0x_0\delta_g$ | 0            | 0            | $-c_6$           | $-c_{10}$         |
| $c_2 = x_0x_1x_0x_0x_0x_1x_0x_0\delta_g$ | 0            | 0            | $-f(g)c_5$      | $-c_9$            |
| $c_3 = x_0x_1x_0x_0x_0\delta_g - x_0x_1x_0x_0x_0x_0^2\delta_g$ | 0            | $c_1$        | $c_{12}$         | 0                 |
| $c_4 = x_0x_1x_0x_0x_0x_0\delta_g - x_0x_1x_0x_0x_0x_1x_0x_0\delta_g$ | 0            | $c_2$        | 0                | $c_{11}$          |
| $c_5 = x_0x_0x_0x_0\delta_g$ | 0            | $c_7$        | 0                | $-c_3$            |
| $c_6 = x_0x_0x_0x_0x_0x_0\delta_g$ | 0            | $c_8$        | 0                | $-c_4$            |
| $c_7 = x_0x_0x_0x_0\delta_g$ | $c_1$        | 0            | 0                | $-c_{12}$         |
| $c_8 = x_0x_0x_0x_0x_1x_0x_0\delta_g$ | $c_2$        | 0            | 0                | $-c_{11}$         |
| $c_9 = x_1x_0x_0x_0\delta_g - x_1x_0x_0x_0\delta_g$ | 0            | $c_3$        | 0                | 0                 |
| $c_{10} = x_1x_0x_0x_0x_0x_0x_0\delta_g - x_1x_0x_0x_0x_1x_0\delta_g$ | 0            | $c_4$        | 0                | $-c_8$            |
| $= f(g)\mathbf{e}_4^g$ |             |              |                  |                   |
| $c_{11} = x_0x_1x_0x_1x_0x_0x_0x_0\delta_g - f(g)x_0x_0x_0\delta_g$ | $c_5$        | $c_9$        | 0                | 0                 |
| $+ f(g)x_0x_0x_0\delta_g$ |             |              |                  |                   |
| $c_{12} = -x_0x_0x_0x_0x_1x_0\delta_g + x_0x_0x_0x_0x_0x_0\delta_g$ | $c_6$        | $c_{10}$    | 0                | 0                 |

Table 5  Action of the generators $x_i$ on $L_5^g = A_{G,\lambda}e_5^g$

| Linear basis of $L_5^g$ | $x_0^\cdot$ | $x_1^\cdot$ | $x_\omega^\cdot$ | $x_{\omega^2}^\cdot$ |
|-------------------------|--------------|--------------|------------------|-------------------|
| $c_1 = x_0x_1x_0x_0x_0\delta_g$ | 0            | 0            | $-c_6$           | $c_{10}$         |
| $c_2 = x_0x_1x_0x_0x_0x_0x_0x_0\delta_g$ | 0            | 0            | $-c_5$           | $c_9$            |
| $c_3 = x_0x_1x_0x_0x_0x_0x_1x_0x_0\delta_g$ | 0            | $f(g)c_1$   | $-f(g)c_{12}$   | 0                 |
| $+ f(g)x_0x_1x_0x_0\delta_g$ |             |              |                  |                   |
| $c_4 = x_0x_1x_0x_0x_0x_0x_0x_0\delta_g - x_0x_1x_0x_0x_0x_0x_0^2\delta_g$ | 0            | $c_2$        | $c_{11}$         | 0                 |
| $= f(g)\mathbf{e}_5^g$ |             |              |                  |                   |
| $c_5 = x_0x_1x_0x_0x_0x_0x_0x_0\delta_g + f(g)x_0x_0x_0\delta_g$ | 0            | $f(g)c_7$   | 0                | $c_3$            |
| $c_6 = x_0x_0x_0x_0x_0x_1x_0x_0\delta_g - f(g)x_0x_0x_0\delta_g$ | 0            | $c_8$        | 0                | $c_4$            |
| $c_7 = x_1x_0x_0x_0\delta_g$ | $c_1$        | 0            | 0                | $c_{12}$         |
| $c_8 = x_0x_0x_0x_0x_1x_0x_0\delta_g$ | $c_2$        | 0            | 0                | $c_{11}$         |
| $c_9 = x_1x_0x_0x_0x_0x_0x_0x_0\delta_g + f(g)x_1x_0x_0\delta_g$ | 0            | $c_3$        | 0                | $-f(g)c_7$      |
| $c_{10} = x_1x_0x_0x_0x_0x_0\delta_g - x_1x_0x_0x_0x_0x_0\delta_g$ | 0            | $c_4$        | 0                | $-c_8$           |
| $c_{11} = x_0x_0x_0x_0x_0x_0x_0x_0\delta_g$ | $c_5$        | $c_9$        | 0                | 0                 |
| $+ x_1x_0x_0x_0x_0x_0x_0x_0\delta_g + f(g)x_0x_0\delta_g$ |             |              |                  |                   |
| $c_{12} = x_0x_0x_0x_0x_0x_0x_0x_0\delta_g - x_0x_0x_0x_0x_0x_0\delta_g$ | $c_6$        | $c_{10}$    | 0                | 0                 |
Table 6  Action of the generators $x_i$ on $L_6^g = A_{G, x} e_6^g$

| Linear basis of $L_6^g$ | $x_0^g$ | $x_1^g$ | $x_{ω}^g$ | $x_{ω^2}^g$ |
|-------------------------|---------|---------|-----------|-----------|
| $c_1 = x_0 x_1 x_0 g$   | 0       | 0       | $-c_6$    | $-c_{10}$ |
| $c_2 = x_0 x_1 x_0 g_2$ | 0       | 0       | $-c_5$    | $c_9$     |
| $c_3 = x_0 x_1 x_0 x_0 x_1 x_0 g_2$ | 0       | $f(g)c_1$ | $c_{12}$ | 0         |
| $c_4 = x_0 x_1 x_0 x_0 x_1 g_2$ | 0       | $c_2$   | $c_{11}$ | 0         |
| $c_5 = -x_0 x_1 x_0 x_0 x_1 x_0 g_2$ | 0       | $c_7$   | 0         | $c_3$     |
| $c_6 = x_0 x_0 x_0 x_0 g_2$ | 0       | $c_8$   | 0         | $-c_4$    |
| $c_7 = -x_0 x_1 x_0 x_0 x_1 x_0 g_2$ | 0       | $f(g)c_1$ | 0         | $c_{12}$ |
| $c_8 = x_1 x_0 x_0 x_1 g_2$ | $c_2$   | 0       | 0         | $c_{11}$ |
| $c_9 = x_1 x_0 x_0 x_0 x_1 x_0 g_2$ | 0       | $-c_7$  | 0         | 0         |
| $-x_0 x_1 x_0 x_0 x_1 x_0 g_2 + f(g)x_1 g_2$ | $c_{10}$ | 0       | $-c_8$    | 0         |
| $c_{11} = x_0 x_0 x_0 x_1 x_0 x_0 g_2$ | 0       | $c_9$   | 0         | 0         |
| $-x_0 x_1 x_0 x_0 x_1 x_0 g_2 + f(g)g_2 = f(g)e_6^g$ | $f(g)c_6$ | $c_{10}$ | 0         | 0         |
| $+f(g)x_0 x_0 x_0 x_1 x_0 x_0 g_2$ | $f(g)c_6$ | $c_{10}$ | 0         | 0         |

Table 7  Weight of the vectors $c_i$ in the case $G' = F_4 \times C_6$

| $L_1^g$ | $L_2^g$ | $L_3^g$ | $L_4^g$ | $L_5^g$ | $L_6^g$ |
|---------|---------|---------|---------|---------|---------|
| $c_1$   | $(0, t^3)g$ | $(ω, t^4)g$ | $(0, t^5)g$ | $(ω^2, t)g$ | $(ω^2, t^2)g$ | $(ω, t^3)g$ |
| $c_2$   | $g$      | $(ω, t)g$  | $(0, t^5)g$ | $(ω^2, t^4)g$ | $(ω^2, t^5)g$ | $(ω, 1)g$   |
| $c_3$   | $(1, t^4)g$ | $(ω, t^3)g$ | $(1, 1)g$  | $(0, t^2)g$  | $(0, t^3)g$  | $(ω, t^4)g$ |
| $c_4$   | $(1, t)g$ | $(ω, t^3)g$ | $(1, t^3)g$ | $(0, t^5)g$  | $(ω, t)g$   | $(ω, t^2)g$ |
| $c_5$   | $(1, t^5)g$ | $g$       | $(1, t)g$  | $(ω^2, t^3)g$ | $(ω^2, t^4)g$ | $(0, t^5)g$ |
| $c_6$   | $(1, t^2)g$ | $(0, t^3)g$ | $(1, t^4)g$ | $(ω^2, 1)g$  | $(ω^2, t)g$  | $(0, t^2)g$ |
| $c_7$   | $(0, t^4)g$ | $(ω^2, t^5)g$ | $g$       | $(1, t^2)g$  | $(1, t^3)g$  | $(ω^2, t^4)g$ |
| $c_8$   | $(0, t)g$  | $(ω^2, t^2)g$ | $(0, t^3)g$ | $(1, t^5)g$  | $(1, 1)g$   | $(ω^2, t)g$ |
| $c_9$   | $(ω, t^5)g$ | $(ω^2, 1)g$ | $(ω, t)g$  | $(0, t^3)g$  | $(0, t^4)g$  | $(ω^2, t^5)g$ |
| $c_{10}$ | $(ω, t^2)g$ | $(ω^2, t^3)g$ | $(ω, t)g$  | $g$       | $(0, t)g$   | $(ω^2, t^2)g$ |
| $c_{11}$ | $(ω, 1)g$  | $(0, t)g$  | $(ω, t^5)g$ | $(1, t^4)g$  | $(1, t^5)g$  | $g$         |
| $c_{12}$ | $(ω, t^3)g$ | $(0, t^4)g$ | $(ω, t^5)g$ | $(1, t)g$   | $(1, t^2)g$  | $(0, t^3)g$ |
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