Existence of charge-density waves in two-dimensional ionic Hubbard model

Tadahiro Miyao
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail: miyao@math.sci.hokudai.ac.jp

Abstract

We rigorously investigated the charge-charge correlation function of the ionic Hubbard model in two dimensions by reflection positivity. We prove the existence of charge-density waves for large staggered potential \( \Delta \) (i.e., \( \frac{\Delta}{2} + 2V > U \)) at low temperatures, where \( U \) and \( V \) are the on-site and nearest-neighbor Coulomb repulsions, respectively. The results are consistent with previous numerical simulation results. We argue that the absence of charge-density waves for \( \Delta = 0 \) and \( U \) are large enough (i.e., \( U > \frac{\Delta}{2} + 2V \)).

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1 Introduction

The ionic Hubbard model was originally suggested to describe the charge-transfer organic salts 17, 34, although it was subsequently used to analyze ferroelectric perovskites 8, 10, 37. In these studies, the half-filled one-dimensional model was used to understand quantum phase transitions from band insulators to Mott insulators. The ionic Hubbard model comprises the usual Hubbard model with on-site Coulomb repulsion $U$ supplemented by an alternating one-particle potential of magnitude $\Delta$. Surprisingly, this model is reported to have two quantum critical points as $U$ is varied with fixed $\Delta$ 1, 19, 25, 36. For $U < U_1$, the system is a band insulator. In the intermediate regime $U_1 < U < U_2$, the system has a bond-order characterized by the ground-state expectation value of the staggered kinetic energy per bond. For $U > U_2$, the system is a Mott insulator.

Recently, the two-dimensional ionic Hubbard model has received significant attention both theoretically and experimentally. However, the phase diagram of this model remains a mystery. Conversely, recently developed experimental techniques make it possible to implement the ionic Hubbard model in an optical honeycomb lattice. As theoretical studies have suggested, for large $\Delta$, a charge-density wave appears. For large $U$, the charge-density wave is strongly suppressed 26. Whether bond-order exists in a two-dimensional system, however, remains unclear.

Many theoretical studies are based on numerical simulations that clarify the properties of the ionic Hubbard model in one and two dimensions; however, limited exact results are available. The aim of the present work was to study the extended ionic Hubbard model in a two-dimensional square lattice rigorously using reflection positivity. The results of this approach prove the existence of a staggered long-range charge order for $\frac{\Delta^2}{2} + 2V > U$, where $V$ is the nearest-neighbor Coulomb repulsion. This finding justifies a part of the phase diagram suggested by numerical simulations 5, 20, 33, 36. Although we do not address the bond-order in this paper, we discuss some rigorous results for $U > \frac{\Delta^2}{2} + 2V$. The work constitutes the first step in the rigorous study of the two-dimensional extended ionic Hubbard model.

A similar reflection-positive-based model for three and more dimensions was previously discussed 13. However, the present work is restricted to two-dimensional models.

The proposed method works well only for a sufficiently small hopping amplitude $t$. We are aware of no rigorous results when $t$ is large (i.e., $U \approx t$).

Reflection positivity originates from axiomatic quantum field theory 35. Glimm, Jaffe, and Spencer found that reflection positivity can be applied to the rigorous study of phase transitions 15, 16. In the 1970s, Dyson, Fröhlich, Israel, Lieb, Spencer, and Simon established the foundation of the methods of reflection positivity in statistical physics 7, 11, 12, 13, 14. Reflection positivity has been successfully applied to numerous models and is regarded as a crucial analysis method in condensed-matter physics 2, 8, 9, 38.
In the present work, we adapted the method by Fröhlich and Lieb [11] to the ionic Hubbard model. In [11], Fröhlich and Lieb proved the existence of long-range order in the two-dimensional models, including the anisotropic Heisenberg model. Their proof contains the following three parts:

(i) the Peierls argument;
(ii) the chessboard estimate;
(iii) the principle of exponential localization.

Reflection positivity is a basic input of the chessboard estimate. To use the chessboard estimate, the model was required to be defined on a square lattice with sides of length \(4M\) in [11]. In contrast, our reflection positivity arguments work well in a square lattice with sides of length \(4M+2\). This difference requires several extensions of the Fröhlich–Lieb method. Our methodological achievement is that we actually completed these extensions.

Note that our result could be proven by the quantum Pirogov–Sinai theory [4, 6]. However, as far as we know, there is no proof based on reflection positivity.

The organization of the paper is as follows: In Section 2, we present the definition of the ionic Hubbard model and state the main results.

In Section 3, we describe the strategy of the main theorem (Theorem 2.1).

Sections 4–7 are devoted to the proof of Theorem 2.1.

In Appendix A, we present the proof of an extension of the Dyson–Lieb–Simon (DLS) inequality.

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2 Results

Let \(\Lambda = [-L, L]^2 \cap \mathbb{Z}^2\) with \(L \in \mathbb{N}\). The extended ionic Hubbard model is

\[
H = -t \sum_{\langle i,j \rangle} \sum_{\sigma = \uparrow, \downarrow} (c^*_{i\sigma} c_{j\sigma} + c^*_{j\sigma} c_{i\sigma}) \\
+ U \sum_{j \in \Lambda} (n_j - 1)^2 + V \sum_{\langle i,j \rangle} (n_i - 1)(n_j - 1) + \frac{\Delta}{2} \sum_{j \in \Lambda} (-1)^{|\langle j |}(n_j - 1),
\]

(2.1)

where \(\sum_{\langle i,j \rangle}\) means a sum over nearest neighbors; \(t\) is the hopping amplitude between nearest-neighbor; \(U\) and \(V\) define the on-site and nearest-neighbor Coulomb interactions, respectively; \(\Delta\) denotes an alternating chemical potential; and \(c_{j\sigma}(c^*_{j\sigma})\) is the standard fermion annihilation (creation) operator on site \(j = (j_1, j_2) \in \Lambda\) with spin \(\sigma\).

The number operator \(n_j\) is defined by \(n_j = n_{j^\uparrow} + n_{j^\downarrow}\) with \(n_{j^\sigma} = c^*_{j\sigma} c_{j\sigma}\). \(H\) acts on the fermion Fock space \(\mathcal{F} = \mathcal{F}(\ell^2(\Lambda) \oplus \ell^2(\Lambda))\), where \(\mathcal{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \wedge^n \mathfrak{h}\). Here \(\wedge^n \mathfrak{h}\) is the \(n\)-fold antisymmetric tensor product of \(\mathfrak{h}\) with \(\wedge^0 \mathfrak{h} = \mathbb{C}\). We impose a periodic boundary condition, so \(L \equiv -L\).

In the present paper, we assume the following:

- \(\Delta \geq 0,\ V \geq 0,\ U \in \mathbb{R}\).
\[ \langle A \rangle_{\beta,\Lambda,H} = \text{Tr}_\Lambda [A e^{-\beta H}] / Z_{\beta,\Lambda,H}, \quad Z_{\beta,\Lambda,H} = \text{Tr}_\Lambda [e^{-\beta H}]. \] (2.2)

Note that if \( \Delta = 0 \), then the system is half-filled. Let \( q_j = n_j - 1 \). The charge-charge correlation function is given by

\[ \langle q_o q_j \rangle_{\beta,H} = \lim_{L \to \infty} \langle q_o q_j \rangle_{\beta,\Lambda,H}. \] (2.3)

The main result in this paper is the following:

**Theorem 2.1** If \( 2V - U + \frac{\Delta}{2} > 0 \), then, for sufficiently large \( \beta \) and small \( t \),

\[ \lim \inf_{|j| \to \infty} (-1)^{|j|} \langle q_o q_j \rangle_{\beta,H} > 0, \] (2.4)

where \( |j| = |j_1| + |j_2| \) for each \( j = (j_1, j_2) \in \Lambda \). Thus, there exists a long-range charge order.

**Remark 2.2** If \( V = 0 \), then our result are consistent with the phase diagrams obtained using numerical simulations \([5, 20, 36]\). On the other hand, if \( \Delta = 0 \), then Theorem 2.1 agrees with results predicted by numerical simulations, see, e.g., \([33]\).

**Remark 2.3** In three or more dimensions, we can prove the existence of long-range charge order by the method established in \([7, 12, 13, 32]\).

**Remark 2.4** Assume \( \Delta = 0 \). If \( 2V - U < 0 \), then we already know the following:

(i) For all \( \beta \) and \( t \),

\[ \lim_{|j| \to \infty} (-1)^{|j|} \langle q_o q_j \rangle_{\beta,H} = 0. \] (2.5)

Thus, there is no long-range charge order \([21, 29, 30]\).

(ii) If \( L \) finite, the ground state of \( H \) is unique and antiferromagnetic \([22, 28, 29]\).

### 3 Outline of proof of Theorem 2.1

#### 3.1 Preliminaries

Let

\[ v_{j\sigma} = \prod_{i \neq j} (-1)^{n_i} \left( c^*_{j\sigma} + c_{j\sigma} \right). \] (3.1)

It is not hard to check that

\[ v_{i\sigma} c_{j\sigma'} v^{-1}_{i\sigma} = \begin{cases} c^*_{i\sigma} & \text{if } (i, \sigma) = (j, \sigma') \\ c_{j\sigma'} & \text{if } (i, \sigma) \neq (j, \sigma'). \end{cases} \] (3.2)

Let \( \Lambda_e = \{ j \in \Lambda \mid |j| \text{ is even} \} \) and let \( \Lambda_o = \{ j \in \Lambda \mid |j| \text{ is odd} \} \).
Definition 3.1 A zigzag transformation is a unitary operator defined by

\[ \mathcal{V} = \prod_{j \in \Lambda_o} v_{j^\uparrow} v_{j^\downarrow}. \] (3.3)

We remark that

\[ \mathcal{V} c_j \sigma \mathcal{V}^{-1} = \begin{cases} c_j^\ast \sigma & \text{if } j \in \Lambda_o \\ c_j \sigma & \text{if } j \in \Lambda_e \end{cases}, \quad \mathcal{V} q_j \mathcal{V}^{-1} = (-1)^{|j|} q_j. \] (3.4)

Lemma 3.2 Let \( \tilde{H} = \mathcal{V} H \mathcal{V}^{-1} \). We have

\[ \tilde{H} = T + W, \quad T = \sum_{(i, j)} \sum_{\sigma = \uparrow, \downarrow} (-t) \left( c_{i\sigma}^c c_{j\sigma} + c_{j\sigma} c_{i\sigma} \right), \] (3.5)

\[ W = U \sum_{j \in \Lambda} q_j^2 - V \sum_{(i, j)} q_i q_j + \frac{\Delta}{2} \sum_{j \in \Lambda} q_j. \] (3.6)

Proof. Note that

\[ \sum_{(i, j)} \sum_{\sigma = \uparrow, \downarrow} (c_{i\sigma}^c c_{j\sigma} + c_{j\sigma} c_{i\sigma}) = \sum_{j \in \Lambda} \sum_{\sigma = \uparrow, \downarrow} \sum_{k = 1, 2} \epsilon = \pm 1 (c_{j\sigma} c_{j+\epsilon \delta_k \sigma} + \text{h.c.}), \] (3.7)

where \( \delta_1 = (1, 0) \) and \( \delta_2 = (0, 1) \). Thus, by using Eq. (3.4), we obtain Eqs. (3.5) and (3.6). \( \square \)

3.2 General strategy

By Lemma 3.2, we know that

\[ (-1)^{|j|} \langle q_o q_j \rangle_{\beta, H} = \langle q_o q_j \rangle_{\Lambda}, \] (3.8)

where \( \langle \cdot \rangle_{\beta, H} \) is abbreviated \( \langle \cdot \rangle_{\Lambda} \). Thus, the following theorem holds:

Theorem 3.3 Equation (2.4) is equivalent to

\[ \liminf_{|j| \to \infty} \langle q_j q_o \rangle > 0 \] (3.9)

for sufficiently large \( \beta \) and small \( t \), where \( \langle \cdot \rangle = \lim_{L \to \infty} \langle \cdot \rangle_{\Lambda} \).

Let \( E_j(\cdot) \) be the spectral measure of \( q_j \). We set

\[ P_j^{(0)} = E_j(\{0\}), \quad P_j^{(+)} = E_j(\{0, +1\}), \quad P_j^{(-)} = E_j(\{-1\}). \] (3.10)

Theorem 3.4 For all \( j \in \Lambda, \beta > 0 \) and \( \Lambda \subset \mathbb{Z}^2 \),

\[ \langle q_o q_j \rangle_{\Lambda} \geq 1 - 3 \langle P_j^{(0)} \rangle_{\Lambda} - 2 \langle P_j^{(+)} P_j^{(-)} \rangle_{\Lambda} - 2 \langle P_j^{(-)} P_j^{(+)} \rangle_{\Lambda}. \] (3.11)
Proof. Let \( P_j^{\lambda=0} = E_j(\{0\}) \), \( P_j^{\lambda=+1} = E_j(\{+1\}) \) and \( P_j^{\lambda=-1} = E_j(\{-1\}) \). Note that \( P_j^{(0)} = P_j^{\lambda=0}, P_j^{(-)} = P_j^{\lambda=-1} \), but \( P_j^{(+)} \neq P_j^{\lambda=+1} \). By the spectral theorem,
\[
\langle \rho_0 \rho_j \rangle_\Lambda = \sum_{\lambda, \lambda' \in \{-1, 0, 1\}} \lambda \lambda' \langle E_\lambda(\{\lambda\}) E_j(\{\lambda'\}) \rangle_\Lambda
\]
\[
= \langle P_o^{\lambda=+1} P_j^{\lambda=+1} \rangle_\Lambda + \langle P_o^{\lambda=-1} P_j^{\lambda=-1} \rangle_\Lambda
- \langle P_o^{\lambda=+1} P_j^{\lambda=-1} \rangle_\Lambda - \langle P_o^{\lambda=-1} P_j^{\lambda=+1} \rangle_\Lambda.
\]
(3.12)
From
\[
P_j^{\lambda=+1} + P_j^{\lambda=-1} + P_j^{\lambda=0} = 1,
\]
(3.13)
it follows that
\[
\langle P_o^{\lambda=+1} P_j^{\lambda=+1} \rangle_\Lambda = \langle P_o^{\lambda=+1}(1 - P_j^{\lambda=-1} - P_j^{\lambda=0}) \rangle_\Lambda
\geq \langle P_o^{\lambda=+1} \rangle_\Lambda - \langle P_o^{\lambda=+1} P_j^{\lambda=-1} \rangle_\Lambda - \langle P_o^{\lambda=0} \rangle_\Lambda.
\]
(3.14)
where we have used the fact that \( \langle P_o^{\lambda=+1} P_j^{\lambda=0} \rangle_\Lambda \leq \langle P_o^{\lambda=0} \rangle_\Lambda \) (this inequality is an immediate consequence of the Schwartz inequality). Similarly, we obtain
\[
\langle P_o^{\lambda=-1} P_j^{\lambda=-1} \rangle_\Lambda \geq \langle P_o^{\lambda=-1} \rangle_\Lambda - \langle P_o^{\lambda=-1} P_j^{\lambda=+1} \rangle_\Lambda - \langle P_o^{\lambda=0} \rangle_\Lambda.
\]
(3.15)
From Eq. (3.14), we see that
\[
\langle P_o^{\lambda=+1} \rangle_\Lambda + \langle P_o^{\lambda=-1} \rangle_\Lambda = 1 - \langle P_o^{\lambda=0} \rangle_\Lambda.
\]
(3.16)
Thus, from Eqs. (3.14)-(3.16),
\[
\langle P_o^{\lambda=+1} P_j^{\lambda=+1} \rangle_\Lambda + \langle P_o^{\lambda=-1} P_j^{\lambda=-1} \rangle_\Lambda
\geq 1 - 3\langle P_o^{\lambda=0} \rangle_\Lambda - 2\langle P_o^{\lambda=+1} P_j^{\lambda=-1} \rangle_\Lambda - \langle P_o^{\lambda=-1} P_j^{\lambda=+1} \rangle_\Lambda.
\]
(3.17)
Inserting Eq. (3.17) into Eq. (3.12) gives
\[
\langle \rho_0 \rho_j \rangle_\Lambda \geq 1 - 3\langle P_o^{\lambda=0} \rangle_\Lambda - 2\langle P_o^{\lambda=+1} P_j^{\lambda=-1} \rangle_\Lambda - 2\langle P_o^{\lambda=-1} P_j^{\lambda=+1} \rangle_\Lambda.
\]
(3.18)
Since \( \langle P_o^{\lambda=+1} P_j^{\lambda=-1} \rangle_\Lambda \leq \langle P_o^{(+)} P_j^{(-)} \rangle_\Lambda \) and \( \langle P_o^{\lambda=-1} P_j^{\lambda=+1} \rangle_\Lambda \leq \langle P_o^{(-)} P_j^{(+)} \rangle_\Lambda \), we conclude (3.11). \( \Box \)

Thus, to prove Eq. (3.9), it suffices to show the following:

**Theorem 3.5** For arbitrary \( \varepsilon > 0 \), there exists \( \Lambda_0 \subset \mathbb{Z}^2, \beta_0 > 0 \) and \( t_0 \in (0, 1) \) such that if \( \Lambda \supseteq \Lambda_0, \beta > \beta_0 \) and \( 0 < t < t_0 \), then

(A) \( \langle P_o^{(+)} P_j^{(-)} \rangle_\Lambda \leq \varepsilon, \langle P_o^{(-)} P_j^{(+)} \rangle_\Lambda \leq \varepsilon, \)

(B) \( \langle P_o^{(0)} \rangle_\Lambda \leq \varepsilon. \)

The proof of Theorem 3.5 (A) is very complicated, so we present a concise strategy to prove it in Section 3.3. A proof of Theorem 3.5 (B) appears in Section 7.
3.3 Strategy of proof of Theorem 3.5 (A)

We only present a proof of the inequality \( \left\langle P_\alpha^{(+)} P_j^{(-)} \right\rangle_\Lambda \leq \varepsilon \), because the proof of the second inequality is quite similar.

The proof of Theorem 3.5 (A) consists of the following steps:

Step A-1: Find a key inequality.

Step A-2: Apply modified chessboard estimate.

Step A-3: Estimate \( R_{\text{Low}}^{\pm} \) and \( R_{\text{High}}^{\pm} \).

Step A-4: Apply principle of exponential localization of eigenvectors.

Step A-5: Complete the proof of Theorem 3.5 (A).

In what follows, we will explain each step.

3.3.1 Step A-1: Key inequality

**Definition 3.6** We regard \( \Lambda \) as a two-dimensional torus.

- The set of all connected sets \( \gamma \) in \( \Lambda \) is denoted \( \mathcal{S}_\Lambda = \{ \gamma \subseteq \Lambda \mid \gamma : \text{connected} \} \).
- By a *contour*, we mean the set \( \partial \gamma \) of the nearest neighbor pairs associated with the boundary of a set \( \gamma \in \mathcal{S}_\Lambda \) such that

\[
\partial \gamma = \left\{ \langle i_1, j_1 \rangle, \ldots, \langle i_\ell, j_\ell \rangle \mid i_k \in \gamma, j_k \notin \gamma \right\}.
\]

(3.19)

We present the proof of the following theorem in Section 4:

**Theorem 3.7**

\[
\left\langle P_m^{(+)} P_n^{(-)} \right\rangle_\Lambda \leq \sum_{\gamma \in \mathcal{S}_\Lambda, m, n \notin \gamma} \left\langle \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda.
\]

(3.20)

3.3.2 Step A-2: Modified chessboard estimate

Set \( L = 2M + 1 \). We define projections \( P_\Lambda^{(+)} \) and \( P_\Lambda^{(-)} \) by

\[
P_\Lambda^{(+)} = \left[ \prod_{m=1}^{M} \prod_{n=-L}^{L-1} P_{(-L+4m,n)}^{(+)} P_{(-L+4m+1,n)}^{(-)} P_{(-L+4m+2,n)}^{(-)} P_{(-L+4m+3,n)}^{(+)} \right] \partial P^{(+)}_\Lambda,
\]

(3.21)

\[
P_\Lambda^{(-)} = \left[ \prod_{m=1}^{M} \prod_{n=-L}^{L-1} P_{(-L+4m,n)}^{(-)} P_{(-L+4m+1,n)}^{(+)} P_{(-L+4m+2,n)}^{(+)} P_{(-L+4m+3,n)}^{(-)} \right] \partial P^{(-)}_\Lambda.
\]

(3.22)

1 We say that a subset \( \gamma \) of \( \Lambda \) is *connected* if any of its sites are linked by a path in \( \gamma \).
where
\[ \partial \mathcal{P}(\omega) = \prod_{n=-L}^{L-1} \mathcal{P}(\omega)_{(L-2,n)} \mathcal{P}(\omega)_{(L-1,n)}, \quad \omega = +, - . \] (3.23)

By using the modified chessboard estimate, we present the proof of the following theorem in Section 5.

**Theorem 3.8** Let \( \mathcal{P}_\Lambda = \max \{ \langle \mathcal{P}_\Lambda^{(+)} \rangle_\Lambda, \langle \mathcal{P}_\Lambda^{(-)} \rangle_\Lambda \} \). This gives
\[ \left\langle \prod_{(i,j) \in \partial \gamma} \mathcal{P}_\Lambda^{(+)}(i) \mathcal{P}_\Lambda^{(-)}(j) \right\rangle_\Lambda \leq \mathcal{P}_\Lambda^{(|\partial \gamma|/2|\Lambda|)} . \] (3.24)

**Corollary 3.9** There exists a \( C > 0 \) such that
\[ \langle \mathcal{P}^{(+)}_m \mathcal{P}^{(-)}_n \rangle_\Lambda \leq C \sum_{\ell=4}^{\infty} \ell^2 3^\ell \mathcal{P}_\Lambda^{(|\partial \gamma|/2|\Lambda|)} . \] (3.25)

**Proof.** By Theorems 3.7 and 3.8,
\[ \langle \mathcal{P}^{(+)}_m \mathcal{P}^{(-)}_n \rangle_\Lambda \leq \sum_{\gamma \in \mathcal{S}_\Lambda, m \in \gamma, n \notin \gamma} \mathcal{P}_\Lambda^{(|\partial \gamma|/2|\Lambda|)} , \] (3.26)

where \( |\partial \gamma| \) is the number of nearest neighbor pairs in \( \partial \gamma \). Because the smallest contours have surface 4, we have
\[ \text{RHS of (3.26)} = \sum_{\ell=4}^{\infty} \# \left\{ \gamma \in \mathcal{S}_\Lambda \mid m \in \gamma, n \notin \gamma, |\partial \gamma| = \ell \right\} \mathcal{P}_\Lambda^{(|\partial \gamma|/2|\Lambda|)} . \] (3.27)

By the standard Peierls argument, there exists a \( C > 0 \) such that
\[ \# \left\{ \gamma \in \mathcal{S}_\Lambda \mid m \in \gamma, n \notin \gamma, |\partial \gamma| = \ell \right\} \leq C \ell^2 3^\ell . \] (3.28)

Thus, the assertion in Corollary 3.9 holds. \( \square \)

### 3.3.3 Step A-3: Estimate \( R_{\text{Low}}^\pm \) and \( R_{\text{High}}^\pm \)
To prove Theorem 3.5 (A), showing that the right-hand side of Eq. (3.25) \( \leq \varepsilon \) suffices.

Let \( \mathcal{E}(\cdot) \) be the spectral measure of \( \mathcal{H} \). For each \( \delta > 0 \), we use
\[ \mathcal{E}_\delta = \mathcal{E}_\mathcal{H} (0, \varepsilon + \delta |\Lambda|), \quad \mathcal{E}_\delta^\perp = \mathbb{1} - \mathcal{E}_\delta , \] (3.29)

where \( \varepsilon = \min \text{spec}(\mathcal{H}) \) (i.e., the ground state energy). The choice of the quantity \( \delta \) will be addressed later.

We divide \( \langle \mathcal{P}_\Lambda^{(\pm)} \rangle_\Lambda \) into two pieces: \( \langle \mathcal{P}_\Lambda^{(\pm)} \rangle_\Lambda = \mathcal{R}_\text{Low}^{(\pm)} + \mathcal{R}_\text{High}^{(\pm)} \), where
\[ \mathcal{R}_\text{Low}^{(\pm)} = \left\langle \mathcal{E}_\delta \mathcal{P}_\Lambda^{(\pm)} \right\rangle_\Lambda, \quad \mathcal{R}_\text{High}^{(\pm)} = \left\langle \mathcal{E}_\delta^\perp \mathcal{P}_\Lambda^{(\pm)} \right\rangle_\Lambda . \] (3.30)

\(^2\) The factor \( 3^\ell \) comes from the fact that the number of connected surfaces \( N \) consisting of \( \ell \) blocks and containing a fixed block is bounded above by \( 3^\ell \cdot N \leq 3^\ell \). The factor \( C\ell^2 \) comes from the fact that each \( \gamma \) must contain \( m \).
**Theorem 3.10** We assert the following:

(i) \(|R_{\text{Low}}(\omega)| \leq 2|\Lambda|\left\{\text{Tr}_\delta [P^{(\omega)}_\Lambda E_\delta]\right\}^{1/2}\) for each \(\omega = +, -\).

(ii) \(|R_{\text{High}}(\omega)| \leq 4|\Lambda|e^{-\beta|\Lambda|}\) for each \(\omega = +, -\).

**Proof.** (i) By the Schwartz inequality, we have

\[\left|\text{Tr}_\delta \left[E_\delta P^{(\omega)}_\Lambda e^{-\beta \tilde{H}}\right]\right| \leq \|e^{-\beta \tilde{H}}\|_{\text{HS}}\|E_\delta P^{(\omega)}_\Lambda\|_{\text{HS}} \leq e^{-\beta \epsilon_\Lambda}2|\Lambda|\left\{\text{Tr}_\delta [P^{(\omega)}_\Lambda E_\delta]\right\}^{1/2},\]  

(3.31)

where \(\|A\|_{\text{HS}} := \left\{\text{Tr}_\delta [\|A\|^2]\right\}^{1/2}\), the Hilbert–Schmidt norm. Because \(Z_\Lambda(\beta) = \text{Tr}_\delta [e^{-\beta \tilde{H}}] \geq e^{-\beta \epsilon_\Lambda}\), we conclude part (i) of Theorem 3.10.

(ii) We observe that

\[\left|\text{Tr}_\delta \left[E_\delta P^{(\omega)}_\Lambda e^{-\beta \tilde{H}}\right]/Z_\Lambda(\beta)\right| \leq e^{-\beta(\epsilon_\Lambda + \beta|\Lambda|)}4|\Lambda|e^{-\beta \epsilon_\Lambda} = 4|\Lambda|e^{-\beta|\Lambda|}.\]  

(3.32)

This completes the proof. \(\square\)

### 3.3.4 Step A-4: Exponential localization of eigenvectors

By using the principle of exponential localization \([11]\), we show the following in Section 6:

**Theorem 3.11** Set \(S = 2V - U\). Let

\[\mathcal{J} = \begin{cases} \frac{1}{4}(\Delta + V) & \text{if } S \geq 0 \\ \frac{1}{2}(S + \frac{\Delta}{2}) + \frac{V}{\pi} & \text{if } S < 0. \end{cases}\]  

(3.33)

We choose \(\delta = \beta^{-\xi}\) with \(\xi \in (0, 1)\). If \(\Delta \mp S > 0\), then

\[\text{Tr}_\delta \left[P^{(\pm)}_\Lambda E_\delta\right] \leq 4|\Lambda|\gamma^d,\]  

(3.34)

where \(\gamma\) and \(d\) satisfy

\[\gamma = t \left(1 + 8 \mathcal{J}^{-1} \eta^{-1}\right) + O(\beta^{-\xi}),\]  

(3.35)

\[d > \frac{1 - \eta}{G} \mathcal{J}|\Lambda| + O(|\Lambda|^{1/2})\]  

(3.36)

with \(G = 6(|S| + V) + \Delta\) and \(\eta \in (0, 1)\).

**Remark 3.12** Because \(\frac{\Delta}{2} + S > 0\), it holds that \(\mathcal{J} > 0\). \(\diamond\)
3.3.5 Step A-5: Completion of the proof of Theorem 3.5 (A).

By Theorems 3.10 and 3.11, we have

\[
\left\langle P_A^{(\pm)} \right\rangle_\Lambda \leq 4^{|\Lambda|} (e^{-A|\Lambda|} + e^{-\beta|\Lambda|}),
\]

where

\[
\mathcal{A} = \left\{ \frac{1 - \eta}{2G} J + \mathcal{O}(|\Lambda|^{-1/2}) \right\} \log t^{-1} \left\{ 1 + 8 J^{-1} \eta^{-1} + \mathcal{O}(\beta^{-\xi}) \right\}.
\]

Note that \( \mathcal{A} > 0 \), provided that \( t \) is sufficiently small. Let

\[
\mathcal{D} = \min\{\mathcal{A}, \beta \delta\}
\]

Recall that \( \delta = \beta - \xi \). We obtain

\[
P_\Lambda \leq 2 \cdot 4^{|\Lambda|} e^{-\mathcal{D}|\Lambda|}.
\]

Thus, by Corollary 3.9

\[
\left\langle P_0^{(+)} P_2^{(-)} \right\rangle_\Lambda \leq C \sum_{\ell=4}^\infty \ell^2 (24)^\ell \exp \left\{ - \frac{\mathcal{D}}{2} \ell \right\}.
\]

Because we can choose \( \mathcal{D} \) as large as we wish by using a sufficiently large \( \beta \) and sufficiently small \( t \), we obtain Theorem 3.5 (A).  \( \blacksquare \)

4 Proof of Theorem 3.7

Although Theorem 3.7 is proven in Ref. [11], we provide its proof here for the reader’s convenience.

**Definition 4.1**  

- A configuration \( c \) is a function on \( \Lambda \) with values in \( \{+, -\} \) such that \( c(m) = + \) and \( c(n) = - \). We denote by \( \mathcal{C} \) the set of all configurations on \( \Lambda \).

- For each \( c \in \mathcal{C} \), we set

\[
\Gamma(c) = \{ \partial \gamma = \{ (i_1, j_1), \ldots, (i_\ell, j_\ell) \} : \text{contour} |c(i_k) = +, c(j_k) = -, k = 1, \ldots, \ell \}\}.
\]

Note that the definition of \( \Gamma(c) \) is meaningful because of Remark 4.2 below.  \( \Diamond \)

**Remark 4.2**  

For all \( c \in \mathcal{C} \), there exists a unique smallest set \( \gamma(c) \in \mathcal{G}_\Lambda \) such that

- \( m \in \gamma(c) \),

- \( c(i) = + \) for all \( i \in \gamma(c) \).

**Lemma 4.3**

\[
\mathcal{C} = \bigcup_{\gamma \in \mathcal{G}_\Lambda} \{ c \in \mathcal{C} | \partial \gamma = \partial \gamma(c) \}.
\]

where \( \gamma(c) \) is given in Remark 4.2

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Proof. We denote by $\mathcal{K}$ the right-hand side of Eq. (4.2). Because $\mathcal{C} \supseteq \mathcal{K}$ is easy to see, we show the converse. For arbitrarily fixed $c_0 \in \mathcal{C}$, we set $\gamma_0 = \gamma(c_0)$, where $\gamma(c)$ is given in Remark 4.2. Showing that $c_0 \in \{ c \in \mathcal{C} | \partial \gamma_0 = \partial \gamma(c) \}$ is then trivial. Thus, $c_0 \in \mathcal{K}$, which implies $\mathcal{C} \subseteq \mathcal{K}$. \hfill $\Box$

Completion of proof of Theorem 3.7

Since $P_j^{(+)} + P_j^{(-)} = 1$, we have

$$\langle P_m^{(+)} P_n^{(-)} \rangle_\Lambda = \left( P_m^{(+)} P_n^{(-)} \prod_{j \not\in \Lambda} \left[ P_j^{(+)} + P_j^{(-)} \right] \right)_\Lambda = \sum_{c \in \mathcal{C}} \left( \prod_{j \in \Lambda} P_j^{(c(j))} \right)_\Lambda = \sum_{\gamma \in S_\Lambda} \sum_{c \in \{ c | \in \partial \gamma = \partial \gamma(c) \}} \left( \prod_{j \in \Lambda} P_j^{(c(j))} \right)_\Lambda \quad (\because \text{Lemma } 4.3)$$

$$< \sum_{\gamma \in S_\Lambda} \sum_{c \in \{ c | \in \partial \gamma = \partial \gamma(c) \}} \left( \prod_{j \in \Lambda} P_j^{(c(j))} \right)_\Lambda \quad (\because \{ c | \in \partial \gamma = \partial \gamma(c) \} \subseteq \{ c | \in \Gamma(c) \})$$

$$= \sum_{\gamma \in S_\Lambda} \sum_{c \in \{ c | \in \partial \gamma = \partial \gamma(c) \}} \left( \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right) \prod_{j \in \Lambda_{\partial \gamma}} P_j^{(c(j))} \left( \prod_{j \in \Lambda_{\partial \gamma}} P_j^{(c(j))} \right)_\Lambda$$

(4.3)

where $\Lambda_{\partial \gamma}$ is defined as follows: For each $\partial \gamma = \{ (i_1, j_1), \ldots, (i_k, j_k) \}$, let $[\partial \gamma] = \{ i_k \}_{k=1}^l \cup \{ j_k \}_{k=1}^l \subseteq \Lambda$. Then $\Lambda_{\partial \gamma}$ is given by $\Lambda_{\partial \gamma} = \Lambda \setminus \{ [\partial \gamma] \cup \{ m, n \} \}$. Note the following fact:

$$\sum_{c \in \{ c | \in \partial \gamma = \partial \gamma(c) \}} \prod_{j \in \Lambda_{\partial \gamma}} P_j^{(c(j))} = \prod_{j \in \Lambda_{\partial \gamma}} \left[ P_j^{(+)} + P_j^{(-)} \right] = 1. \quad (4.4)$$

Thus, because $P_m^{(+)} P_n^{(-)} \leq 1$, we have

$$\text{RHS of (4.3)} = \sum_{\gamma \in S_\Lambda} \left( \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right) P_m^{(+)} P_n^{(-)} \right)_{\Lambda} \leq \sum_{\gamma \in S_\Lambda} \left( \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right)_{\Lambda} \quad (4.5)$$

This completes the proof. \hfill $\Box$

5 Proof of Theorem 3.8

5.1 Reflection positivity

We use reflection positivity.

Let $\Lambda_L = \{ j = (j_1, j_2) \in \Lambda | j_1 \leq -1 \}$ and $\Lambda_R = \{ j = (j_1, j_2) \in \Lambda | j_1 \geq 0 \}$. Since $\ell^2(\Lambda) = \ell^2(\Lambda_L) \oplus \ell^2(\Lambda_R)$, we have

$$\ell^2(\Lambda) \oplus \ell^2(\Lambda) = \left( \ell^2(\Lambda_L) \oplus \ell^2(\Lambda_L) \right) \oplus \left( \ell^2(\Lambda_R) \oplus \ell^2(\Lambda_R) \right). \quad (5.1)$$
Thus, we obtain the following identification:
\[ \mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R, \]

where \( \mathcal{H}_L = \mathfrak{F}(\ell^2(\Lambda_L) \oplus \ell^2(\Lambda_L)) \) and \( \mathcal{H}_R = \mathfrak{F}(\ell^2(\Lambda_R) \oplus \ell^2(\Lambda_R)) \). Note that, by applying Eq. \( (5.2) \), the annihilation operator can be expressed as
\[ c_{j\sigma} = \begin{cases} c_{j\sigma} \otimes \mathbb{1} & j \in \Lambda_L \\ (-1)^{N_L} \otimes c_{j\sigma} & j \in \Lambda_R \end{cases}, \]

where \( N_L = \sum_{j \in \Lambda_L} n_j \).

**Lemma 5.1** According to Eq. \( (5.2) \), we have the following:

(i) \( T = T_L \otimes \mathbb{1} + \mathbb{1} \otimes T_R + T_{LR}, \) where
\[
T_L = \sum_{j \in \Lambda_L} \sum_{j_1 \leq -2} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( c_{j\sigma}^* c_{j_{1}\sigma} + \varepsilon \delta_k \sigma + \text{h.c.} \right),
\]
\[
T_R = \sum_{j \in \Lambda_L} \sum_{j_1 = 0} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( c_{j\sigma}^* c_{j_{1}\sigma} + \varepsilon \delta_k \sigma + \text{h.c.} \right),
\]
\[
T_{LR} = \sum_{j \in \Lambda_L} \sum_{j_1 = L-1} \sum_{\sigma = \uparrow, \downarrow} (-t) \left( ( \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( (-1)^{N_L} c_{j\sigma}^* c_{j_{1}\sigma} \right) \right) c_{j\sigma}^* + \text{h.c.}
\]

Here, \( \sum_{\varepsilon = \pm} \) refers to a sum over pairs \( \langle j, j_{1} + \varepsilon \delta_k \rangle \) such that \( j, j_{1} + \varepsilon \delta_k \in \Lambda_L \).

Similarly, \( \sum_{\varepsilon = \pm} \) refers to a sum over pairs \( \langle j, j_{1} + \varepsilon \delta_k \rangle \) such that \( j, j_{1} + \varepsilon \delta_k \in \Lambda_R \).

(ii) \( W = W_L \otimes \mathbb{1} + \mathbb{1} \otimes W_R + W_{LR}, \) where
\[
W_L = -S \sum_{j \in \Lambda_L} q_j^2 + \frac{V}{2} \sum_{\langle i,j \rangle \in \Lambda_L} (q_i - q_j)^2,
\]
\[
W_R = -S \sum_{j \in \Lambda_R} q_j^2 + \frac{V}{2} \sum_{\langle i,j \rangle \in \Lambda_R} (q_i - q_j)^2,
\]
\[
W_{LR} = -V \sum_{j \in \Lambda_L} q_{j_{1}} \otimes q_j - V \sum_{j \in \Lambda_L} q_{j+\delta_{1}} \otimes q_j.
\]

**Proof.** (i) Note
\[
T = \sum_{j \in \Lambda_L} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( c_{j\sigma}^* c_{j_{1}\sigma} + \varepsilon \delta_k \sigma + \text{h.c.} \right).
\]
Thus, Eqs. (5.1)–(5.6) are easily verified by Eq. (5.3). To show (ii), we note that $W$ can be expressed as

$$W = -S \sum_{j \in \Lambda} q_j^2 + \frac{V}{2} \sum_{(i,j)} (q_i - q_j)^2 + \frac{\Delta}{2} \sum_{j \in \Lambda} q_j. \quad (5.11)$$

For all $j \in \Lambda_L$, we define

$$a_{j\sigma} = c_{j\sigma}(-1)^{N_L}. \quad (5.12)$$

In terms of $a_{j\sigma}$, $T_L$ and $T_{LR}$ can be expressed as follows (note that we will use the fact $c_{j+\epsilon_\delta}c_{j\sigma} = -a_{j+\epsilon_\delta}a_{j\sigma}$).

**Proposition 5.2** We obtain

$$T_L = \sum_{j \in \Lambda_L, \ j_1 \leq -2} \sum_{\sigma = \uparrow, \downarrow} \sum_{k = 0}^\nu (t)(a_{j\sigma}^*a_{j+\delta k \sigma} + \text{h.c.}), \quad (5.13)$$

$$T_{LR} = \sum_{j \in \Lambda_L, \ j_1 = 0} \sum_{\sigma = \uparrow, \downarrow} (-t)(a_{j-\delta \sigma}^* \otimes c_{j\sigma}^* + \text{h.c.})$$

$$+ \sum_{j \in \Lambda_L, \ j_1 = L-1} \sum_{\sigma = \uparrow, \downarrow} (-t)(a_{j+\delta \sigma}^* \otimes c_{j\sigma}^* + \text{h.c.}). \quad (5.14)$$

**Remark 5.3** Because $q_j = \sum_{\sigma = \uparrow, \downarrow} a_{j\sigma}a_{j\sigma} - \mathbb{1}$, the expressions for $W_L$ and $W_{LR}$ are unchanged if we rewrite them in terms of $a_{j\sigma}$. \diamond

The reflection map $r_v : \Lambda_R \to \Lambda_L$ is defined by

$$r_v(j) = (-j_1 - 1, j_2), \quad j = (j_1, j_2) \in \Lambda_R. \quad (5.15)$$

Let $\vartheta_v$ be an antiunitary transformation\(^3\) from $\mathcal{H}_L$ to $\mathcal{H}_R$ such that

$$\vartheta_v \Omega_L = \Omega_R, \quad c_{j\sigma} = \vartheta_v a_{r_v(j)\sigma} \vartheta_v^{-1}, \quad j \in \Lambda_R, \quad (5.16)$$

where $\Omega_L$ ($\Omega_R$) is the Fock vacuum in $\mathcal{H}_L$ ($\mathcal{H}_R$).

**Lemma 5.4**

(i) $T_R = \vartheta_v T_L \vartheta_v^{-1}$.

(ii) $T_{LR} = \sum_{j \in \Lambda_L, \ j_1 = 0} \sum_{\sigma = \uparrow, \downarrow} (-t)(a_{j-\delta_1 \sigma}^* \otimes \vartheta_v a_{j-\delta_1 \sigma}^* \vartheta_v^{-1} + \text{h.c.})$

$$+ \sum_{j \in \Lambda_L, \ j_1 = L-1} \sum_{\sigma = \uparrow, \downarrow} (-t)(a_{j+\delta_1 \sigma}^* \otimes \vartheta_v a_{j+\delta_1 \sigma}^* \vartheta_v^{-1} + \text{h.c.}). \quad (5.17)$$

(iii) $W_R = \vartheta_v W_L \vartheta_v^{-1}$.

\(^3\) Namely, $\vartheta_v$ is a bijective antilinear map that satisfies $\langle \vartheta_v \varphi | \vartheta_v \psi \rangle = \langle \varphi | \psi \rangle$ for all $\varphi, \psi \in \mathcal{H}_L$. 13
where  

\[ W_{LR} = -V \sum_{j \in \Lambda_v, j_k=0} q_{j-\delta_1} \otimes \vartheta_v q_{j-\delta_1} \vartheta_v^{-1} - V \sum_{j \in \Lambda_v, j_k=L-1} q_{j+\delta_1} \otimes \vartheta_v q_{j+\delta_1} \vartheta_v^{-1}. \]  

(5.18)

**Proof.** Items (ii), (iii), and (iv) are easy to check. To verify item (i), note that \( T_L \) can be expressed as

\[ T_L = \sum_{j \in \Lambda_v, j_1 \leq -1} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (+t) \left( \alpha_{\vartheta_v}^{\ast}(j, \sigma) \alpha_{\vartheta_v}^{\ast}(j+\varepsilon \delta_k, \sigma) + \text{h.c.} \right) \vartheta_v^{-1} \]

Thus, by Eqs. (5.5) and (5.16), we obtain

\[ T_R = \vartheta_v \sum_{j \in \Lambda_v, j_1 \geq 0} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( \alpha_{\vartheta_v}^{\ast}(j, \sigma) \alpha_{\vartheta_v}^{\ast}(j-\varepsilon \delta_k, \sigma) + \text{h.c.} \right) \vartheta_v^{-1} \]

\[ = \vartheta_v \sum_{X=(X_1, X_2) \in \Lambda_v, X_1 \leq -1} \sum_{\sigma = \uparrow, \downarrow} \sum_{k=1,2} \sum_{\varepsilon = \pm} (-t) \left( \alpha_{\vartheta_v}^{\ast}(j, \sigma) \alpha_{\vartheta_v}^{\ast}(j+\varepsilon \delta_k, \sigma) + \text{h.c.} \right) \vartheta_v^{-1} \]

\[ = \vartheta_v T_L \vartheta_v^{-1}. \] 

(5.20)

Here, we have used the fact that if \( j \in \Lambda_v \), then \( r_v(j) \in \Lambda_v \). □

By Theorem [A.1] and Lemma [5.4], we immediately obtain the following:

**Theorem 5.5** Let \( A, B \in \mathcal{B}(\mathcal{H}_L) \), where \( \mathcal{B}(\mathcal{X}) \) is the set of all linear operators on \( \mathcal{X} \). The following then holds:

(i) \( \langle A \otimes \vartheta_v A \vartheta_v^{-1} \rangle_{\Lambda} \geq 0 \),

(ii) \( \left| \langle A \otimes \vartheta_v B \vartheta_v^{-1} \rangle_{\Lambda} \right| \leq \langle A \otimes \vartheta_v A \vartheta_v^{-1} \rangle_{\Lambda} \langle B \otimes \vartheta_v B \vartheta_v^{-1} \rangle_{\Lambda} \).

Theorem 5.5 is reflection positivity associated with a vertical line \( j_1 = -\frac{1}{2} \). We can also construct reflection positivity associated with a horizontal line \( j_2 = -\frac{1}{2} \).

Let \( \Lambda^U = \{ j = (j_1, j_2) \in \Lambda \mid j_2 \geq 0 \} \) and \( \Lambda^L = \{ j = (j_1, j_2) \in \Lambda \mid j_2 \leq -1 \} \). As before, we obtain the following identification:

\[ \mathcal{H} = \mathcal{H}^L \otimes \mathcal{H}^U, \]

(5.21)

where \( \mathcal{H}^L = \mathcal{F}(\ell^2(\Lambda^L) \oplus \ell^2(\Lambda^L)) \) and \( \mathcal{H}^U = \mathcal{F}(\ell^2(\Lambda^U) \oplus \ell^2(\Lambda^U)) \). The reflection map \( r_h : \Lambda^L \to \Lambda^U \) is given by

\[ r_h(j) = (j_1, -j_2 - 1), \quad j = (j_1, j_2) \in \Lambda^L. \] 

(5.22)

Let \( \vartheta_h \) be an antiunitary transformation from \( \mathcal{H}^L \) to \( \mathcal{H}^U \) such that

\[ \vartheta_h \Omega^L = \Omega^U, \quad c_{j\sigma} = \vartheta_h a_{r_h(j)\sigma} \vartheta_h^{-1}, \quad j \in \Lambda^U, \] 

(5.23)

where \( \Omega^L \) (\( \Omega^U \)) is the Fock vacuum in \( \mathcal{H}^L \) (\( \mathcal{H}^U \)). By a parallel argument, we can prove the following theorem:
Theorem 5.6 Let $A, B \in \mathcal{B}(\mathfrak{H}^L)$. Then the following holds:

(i) $\langle A \otimes \vartheta h A \vartheta h^{-1} \rangle_A \geq 0$,

(ii) $\left| \left( A \otimes \vartheta h B \vartheta h^{-1} \right)^2 \leq \langle A \otimes \vartheta h A \vartheta h^{-1} \rangle_A \langle B \otimes \vartheta h B \vartheta h^{-1} \rangle_A \right|$. 

5.2 Modified chessboard estimate

To prove Theorem 3.8, the chessboard estimate established from [11] can be employed. Unfortunately, the idea of Ref. [11] cannot be directly applied because it works well only if $L$ is even. However, our argument of reflection positivity requires that $L$ be odd. Therefore, it is necessary to extend the chessboard estimate to overcome this difficulty.

First, we recall the original chessboard estimate.

Theorem 5.7 Let $\mathfrak{A}$ be a vector space with antilinear involution $J$. Let $\omega$ be a multilinear functional on $\mathfrak{A}^{2L}$. Assume the following:

(i) $\omega(A_1, \ldots, A_{2L}) = \omega(A_2, \ldots, A_{2L}, A_1)$.

(ii) $\omega(A_1, \ldots, A_L, JA_L, \ldots, JA_1) \geq 0$.

(iii)

$$\omega(A_1, \ldots, A_{2L}) \leq \omega(A_1, \ldots, A_L, JA_L, \ldots, JA_1)^{1/2} \times \omega(JA_{2L}, \ldots, JA_{L+1}, A_{L+1}, \ldots, A_{2L})^{1/2}. \quad (5.24)$$

Then the following holds:

$$\omega(A_1, \ldots, A_{2L}) \leq \prod_{j=1}^{2L} \omega(JA_j, A_j, JA_j, A_j)^{1/2L}. \quad (5.25)$$

The following is a modified version of Theorem 5.7. In the present work, we use both Theorems 5.7 and 5.8.

Theorem 5.8 Let $\mathfrak{A}$ be a vector space with antilinear involution $J$. Let $\omega$ be a multilinear functional on $\mathfrak{A}^{2M+1}$ with $M$ even. Assume the following:

(i) $\omega(A_1, \ldots, A_{2M+1}) = \omega(A_2, \ldots, A_{2M+1}, A_1)$.

(ii) There exist real linear maps $T_+$ and $T_-$ from $\mathfrak{A}$ to $\mathfrak{A}$ such that

(a) For each $A$ in $\mathfrak{A}$,

$$T_\alpha(T_\beta(A)) = T_\beta(A), \quad \alpha, \beta = +, -. \quad (5.26)$$

(b) $\omega(A_1, \ldots, A_M, T_+(A_{M+1}), JA_M, \ldots, JA_1) \geq 0$.

(c)

$$\omega(A_1, \ldots, A_{2M+1}) \leq \omega(A_1, \ldots, A_M, T_+(A_{M+1}), JA_M, \ldots, JA_1)^{1/2} \times \omega(JA_{2M+1}, \ldots, JA_{M+2}, T_-(A_{M+1}), A_{M+2}, \ldots, A_{2M+1})^{1/2}. \quad (5.27)$$
For each $A_1, \ldots, A_{2M+1} \in \mathfrak{A}$, we have\footnote{By (a) and (b), we have $\omega(JA, A, \ldots, JA, A, T_+ (JA)) \geq 0$ for all $A \in \mathfrak{A}$.}

\[
\left| \omega(A_1, \ldots, A_{2M+1}) \right| \leq \prod_{j=1}^{2M+1} \omega(JA_j, A_j, \ldots, JA_j, A_j, T_+ (JA_j))^{1/2M+1} \tag{5.28}
\]

and

\[
\omega(JA_j, A_j, \ldots, JA_j, A_j, T_+ (JA_j)) = \omega(JA_j, A_j, \ldots, JA_j, A_j, T_- (A_j)) \tag{5.29}
\]

for all $j = 1, \ldots, 2M+1$.

**Proof.** First, we present the proof of Eq. (5.28). Without loss of generality, we assume that

\[
\omega(JA_j, A_j, \ldots, JA_j, A_j, T_+ (JA_j)) = 1 \tag{5.30}
\]

for all $j = 1, \ldots, 2M+1$. We set

\[
JA_j = A_{j+2M+1}, \quad T_+(A_j) = A_{j+4M+2}, \quad T_-(A_j) = A_{j+6M+3},
\]

\[
T_+(JA_j) = A_{j+8M+4}, \quad T_-(JA_j) = A_{j+10M+5}, \quad j = 1, \ldots, 2M+1.
\]

A configuration $c$ is a function on $\{1, \ldots, 2M+1\}$ with value in $\{1, \ldots, 12M+6\}$.

Let $z = \max_c |\omega(A_{c(1)}, \ldots, A_{c(2M+1)})|$, and let $\tilde{c}$ be a configuration such that $z = |\omega(A_{\tilde{c}(1)}, \ldots, A_{\tilde{c}(2M+1)})|$. It suffices to show that $z = 1$. It is easy to see that $1 \leq z$. We show $z \leq 1$. Toward this end, set $\tilde{c}(1) = j$. Then we have

\[
z = \left| \omega(A_{\tilde{c}(1)}, \ldots, A_{\tilde{c}(2M+1)}) \right|
\]

\[
\leq \omega(A_j, A_{\tilde{c}(M)}, \ldots, A_{\tilde{c}(M+1)}), JA_{\tilde{c}(M+1)}, \ldots, JA_j \right)^{1/2}
\]

\[
\times \omega(JA_{\tilde{c}(2M+1)}, \ldots, JA_{\tilde{c}(2M+2)}), T_-(A_{\tilde{c}(M+1)}), A_{\tilde{c}(M+2)}, \ldots, A_{\tilde{c}(2M+1)} \right)^{1/2}
\]

\[
\leq z^{1/2} \omega(JA_j, A_{\tilde{c}(2)}, \ldots, A_{\tilde{c}(M-1)}, A_{\tilde{c}(M)}, T_+(A_{\tilde{c}(M+1)}), \ldots, JA_{\tilde{c}(2)})^{1/2}
\]

\[
\leq z^{3/4} \omega(JA_j, JA_j, A_{\tilde{c}(2)}, \ldots, A_{\tilde{c}(M-1)}, A_{\tilde{c}(M)}, T_+(A_{\tilde{c}(M+1)}), \ldots, JA_{\tilde{c}(2)})^{1/4}
\]

\[
\vdots
\]

\[
\leq z^{1-2^{-(m-2)}} \omega(JA_j, A_j, \ldots, JA_j, A_j, A_{\tilde{c}(2)}, \ast, \ldots, \ast) \right)^{2^{-(m-2)}}
\]

\[
\leq z^{1-2^{-(m-1)}} \omega(JA_j, A_j, \ldots, JA_j, A_j, T_+(A_{\tilde{c}(2)}), JA_j, A_j, \ldots, JA_j, A_j) \right)^{2^{-(m-1)}}
\]

\[
\leq z^{1-2^{-m}} \omega(JA_j, A_j, \ldots, JA_j, A_j, T_+(JA_j), JA_j, A_j, \ldots, JA_j, A_j) \right)^{2^{-m}}
\]

\[
= z^{1-2^{-m}}. \tag{5.31}
\]

Thus, we conclude that $z \leq 1$, which implies Eq. (5.28).
To show Eq. (5.29), we observe that
\[
\begin{align*}
z &= \omega(JA_j, A_j, \ldots, JA_j, T_+(JA_j)) \\
&= \omega(T_+(JA_j), A_j, \ldots, JA_j, A_j, JA_j, JA_j, \ldots, JA_j, A_j) \\
&\leq z^{1/2} \omega(JA_j, A_j, \ldots, JA_j, T_-(A_j), JA_j, A_j, \ldots, JA_j, A_j)^{1/2} \\
&= z^{1/2} \omega(JA_j, A_j, \ldots, JA_j, A_j, T_-(A_j))^{1/2}.
\end{align*}
\]
Thus, we conclude Eq. (5.29).

5.3 Proof of Theorem 3.8: Step 1

We now are ready to present the proof of Theorem 3.8. The proof is divided into three steps.

Definition 5.9 Let \( \partial \gamma \) be the contour associated with \( \gamma \in \mathcal{S}_\Lambda \).

- \( \partial_{\gamma h} = \{ (i, j) \in \partial \gamma \mid \exists a \in \mathbb{Z} \text{ s.t. } i - j = (a, 0) \} \).
- \( \partial_{\gamma v} = \partial \gamma \setminus \partial_{\gamma h} = \{ (i, j) \in \partial \gamma \mid \exists b \in \mathbb{Z} \text{ s.t. } i - j = (0, b) \} \).
- For each \( (i, j) \in \partial_{\gamma h} \), \( i \wedge j \) denotes the site with smaller 1-coordinate.
- For each \( (i, j) \in \partial_{\gamma v} \), \( i \wedge j \) denotes the site with smaller 2-coordinate.
- For each \( \alpha = h, v \), we set
  \[
  \partial_{\alpha, \beta} = \{ (i, j) \in \partial \gamma \mid i \wedge j : \text{even} \}, \quad \partial_{\alpha, o} = \{ (i, j) \in \partial \gamma \mid i \wedge j : \text{odd} \}. \tag{5.33}
  \]

Note that \( \partial \gamma = \partial_{\gamma h} \cup \partial_{\gamma v} = (\partial_{\gamma h, e} \cup \partial_{\gamma h, o}) \cup (\partial_{\gamma v, e} \cup \partial_{\gamma v, o}) \).

Lemma 5.10 To prove Theorem 3.8 it suffices to show
\[
\left\langle \prod_{(i,j) \in \partial_{\gamma, \alpha, \beta}} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda \leq P^{2|\partial_{\gamma, \alpha, \beta}|/|\Lambda|}_\Lambda \tag{5.34}
\]
for all \( \alpha = h, v \) and \( \beta = e, o \).

Proof. By the Schwartz inequality, we have
\[
\begin{align*}
\left\langle \prod_{(i,j) \in \partial \gamma} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda &\leq \left\langle \prod_{(i,j) \in \partial_{\gamma h}} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda \left\langle \prod_{(i,j) \in \partial_{\gamma v}} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda^{1/2} \\
&\leq \prod_{\alpha = h, v} \prod_{\beta = e, o} \left\langle \prod_{(i,j) \in \partial_{\alpha, \beta}} P_i^{(+)} P_j^{(-)} \right\rangle_\Lambda^{1/4}.
\end{align*}
\]
This finishes the proof.
5.4 Proof of Theorem 3.8: Step 2

The Hilbert space for a single electron is $\ell^2(\Lambda) \oplus \ell^2(\Lambda)$. Note the identification

$$\ell^2(\Lambda) \oplus \ell^2(\Lambda) = \bigoplus_{j \in \Lambda} (C \oplus C).$$  \hspace{1cm} (5.36)

Recall the well-known property of fermion Fock space:

$$\mathfrak{F}(h_1 \oplus h_2) = \mathfrak{F}(h_1) \otimes \mathfrak{F}(h_2).$$  \hspace{1cm} (5.37)

By Eqs. (5.36) and (5.37), the fermion Fock space can be identified as

$$H = \bigotimes_{j \in \Lambda} H_j,$$  \hspace{1cm} (5.38)

where $H_j = \mathfrak{F}(C \oplus C)$. Therefore, $H$ can be expressed as

$$H = \bigotimes_{k=-L}^{L-1} \tilde{\mathfrak{H}}_h(k),$$  \hspace{1cm} (5.39)

where $\tilde{\mathfrak{H}}_h(k) = \bigotimes_{j=(j_1, j_2) \in \Lambda} \tilde{\mathfrak{H}}_j$. Let $\tilde{\mathfrak{H}}_h = \tilde{\mathfrak{H}}_h(k = 0)$. For each $A \in \mathcal{B}(\tilde{\mathfrak{H}}_h)$, we define a linear operator on $\tilde{\mathfrak{H}}$ by

$$\tau_j(A) = \bigotimes_{L+j+1}^{2L} A \otimes \bigotimes_{j=-L}^{L-1} \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \quad j = -L, \ldots, L - 1.$$  \hspace{1cm} (5.40)

Here, the tensor products correspond to Eq. (5.39). We want to apply Theorem 5.7 with

$$A = \mathcal{B}(\tilde{\mathfrak{H}}_h), \quad \omega(A_{-L}, \ldots, A_{L-1}) = \left( \prod_{j=-L}^{L-1} \tau_j(A_j) \right) \Lambda.$$  \hspace{1cm} (5.41)

To this end, we have to choose a suitable $J$. Let $\xi_h$ be an antiunitary operator on $\tilde{\mathfrak{H}}_h$ defined by

$$\xi_h \Omega_h = \Omega_h, \quad \xi_h \tilde{c}_{j\sigma} \xi_h^{-1} = \tilde{c}_{j\sigma}(-1)^{N_h},$$  \hspace{1cm} (5.42)

where $\tilde{c}_{j\sigma}$ is the annihilation operator on $\tilde{\mathfrak{H}}_h$, $\Omega_h$ is the Fock vacuum in $\tilde{\mathfrak{H}}_h$, and $N_h = \sum_{j, \text{s.t. } j_2=0} \tilde{n}_j$ with $\tilde{n}_j = \tilde{c}_{j\sigma} \tilde{c}_{j\sigma}^\dagger$. Now, $J$ is defined by $JA = \xi_h A \xi_h^{-1}$ for all $A \in \mathcal{B}(\tilde{\mathfrak{H}}_h)$. It is easy to check that $J \tilde{n}_{j\sigma} = \tilde{n}_{j\sigma}$.

Note the following relationship: Let $\tilde{j} = (j_1, j_2) \in \Lambda^U$ such that $j_2 = k > 0$. Then

$$\vartheta_{h}^{-1} \left[ \bigotimes_{\mathcal{B}(\tilde{\mathfrak{H}}^U)} n_{j\sigma} \right] \vartheta_{h} = \bigotimes_{\mathcal{B}(\tilde{\mathfrak{H}}^L)} \cdots \otimes \bigotimes_{\mathcal{B}(\tilde{\mathfrak{H}}^L)} \cdots \otimes \bigotimes_{\mathcal{B}(\tilde{\mathfrak{H}}^L)} \left[ \bigotimes_{\mathcal{B}(\tilde{\mathfrak{H}}^U)} J \tilde{n}_{j\sigma} \right]$$  \hspace{1cm} (5.43)

where $\vartheta_h$ is defined by Eq. (5.23).
Proposition 5.11 Let $A_{-L}, A_{-L+1}, \ldots, A_{L-1} \in \mathcal{B}(\hat{H}_h)$. We have
\[
\left| \prod_{j=-L}^{L-1} \tau_j(A_j) \right|_{\Lambda} \leq \prod_{j=-L}^{L-1} \left( \tau_{-L}(JA_j)\tau_{-L+1}(A_j) \cdots \tau_{L-2}(JA_j)\tau_{L-1}(A_j) \right)^{1/2L}_{\Lambda}. \tag{5.44}
\]

Proof. Assumption (i) in Theorem 5.7 is fulfilled by the translational invariance of the model. Assumptions (ii) and (iii) in Theorem 5.7 are satisfied by Theorem 5.6. \(\square\)

For each $\ell = -L, \ldots, L - 1$, we define
\[
\partial \gamma_{h,e}(\ell) = \{ (i,j) \in \partial \gamma_{h,e} \mid i_2 = j_2 = \ell \}. \tag{5.45}
\]
Trivially, we have $\partial \gamma_{h,e} = \bigcup_{\ell=-L}^{L-1} \partial \gamma_{h,e}(\ell)$.

Proposition 5.12 We have
\[
\left| \prod_{(i,j) \in \partial \gamma_{h,e}} P_i^{(+)} P_{j}^{(-)} \right|_{\Lambda} \leq \prod_{\ell : \partial \gamma_{h,e}(\ell) \neq \emptyset} \left| \prod_{(i,j) \in \partial \gamma_{h,e}(\ell)} P_{(i,k)}^{(+)} P_{(j,k)}^{(-)} \right|^{1/2L}_{\Lambda}. \tag{5.46}
\]
Here, $i_1$ and $j_1$ on the right-hand side of Eq. (5.46) are related to $i$ and $j$ by $i = (i_1, \ell)$ and $j = (j_1, \ell)$.

Proof. Let $c_{\sigma}$ ($c^*_{\sigma}$) be the annihilation(creation) operator in $\mathfrak{S}(\mathbb{C} \oplus \mathbb{C})$. The number operator is $n = \sum_{\sigma=\uparrow, \downarrow} n_{\sigma}$ with $n_{\sigma} = c^*_{\sigma} c_{\sigma}$. Let $q = n - 1$. Corresponding to Eq. (5.38), we have $q_j = \bigotimes_{k \in \Lambda} q^{kj}$, where $q^{kj} = q$ if $i = j$, $q^{kj} = \mathbb{1}$ if $i \neq j$. Let $E_q(\cdot)$ be the spectral measure of $q$. We set $P^{(+)} = E_q(\{0,+1\})$, $P^{(-)} = E_q(\{-1\})$. Trivially, we have $P_j^{(\omega)} = \bigotimes_{k \in \Lambda} (P^{(\omega)})^{\delta_{kj}}$.

Let $A_h = \{-L, -L+1, \ldots, L-1\}$. Let $A$ be a linear operator in $\mathfrak{S}(\mathbb{C} \oplus \mathbb{C})$. For each $\mathcal{J} \subseteq A_h$, we define $[A]_{\mathcal{J}}^{\mathcal{F}} \in \mathcal{B}(\mathfrak{S}(\mathbb{C} \oplus \mathbb{C}))$ by
\[
[A]_k^{\mathcal{F}} = \begin{cases} A & \text{if } k \in \mathcal{J} \\ \mathbb{1} & \text{if } k \notin \mathcal{J}. \end{cases} \tag{5.47}
\]
In this proof, an operator of the form $\bigotimes_{k \in A_h} [A]_k^{\mathcal{J}} [B]_k^{\mathcal{J}'}$ will play an important role.

Set
\[
B_{\ell} = \left\{ \prod_{(i,j) \in \partial \gamma_{h,e}(\ell)} P_i^{(+)} P_{j}^{(-)} \right\} \text{ if } \partial \gamma_{h,e}(\ell) \neq \emptyset \tag{5.48}
\]
\[
\prod_{(i,j) \in \partial \gamma_{h,e}(\ell)} P_i^{(+)} P_{j}^{(-)} = \prod_{\ell=-L}^{L-1} B_{\ell}. \tag{5.49}
\]
We rewrite $B_{\ell}$ by using the notation of Eq. (5.37), because the new expression is convenient for our proof. To this end, write $\partial \gamma_{h,e}(\ell) = \{ (i^{(1)}, j^{(1)}), \ldots, (i^{(m)}, j^{(m)}) \}$. 

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We also have \( t^{(\alpha)} = (t_1^{(\alpha)}, \ell) \) and \( j^{(\alpha)} = (j_1^{(\alpha)}, \ell) \) for each \( \alpha = 1, \ldots, m \). Let \( \Theta_+^{(\ell)} = \{ t_1^{(1)}, \ldots, t_1^{(m)} \} \), \( \Theta_-^{(\ell)} = \{ j_1^{(1)}, \ldots, j_1^{(m)} \} \). Now, let

\[
A_{\ell} = \bigotimes_{k \in \Lambda_b} \left[ P^+(\ell)_k \right] \Theta_+^{(\ell)} \left[ P^-(\ell)_k \right] \Theta_-^{(\ell)} \in \mathcal{B}(\hat{\mathcal{H}}_h).
\]  

(5.50)

If \( \partial_{\gamma_{h,e}}(\ell) = \emptyset \), we simply set \( A_{\ell} = 1 \). \( B_{\ell} \) can be expressed as \( B_{\ell} = \tau(\ell)(A_{\ell}) \). Note that \( JA_{\ell} = A_{\ell} \). Now, we apply Proposition 5.11. Because

\[
\left\langle \prod_{k=-L}^{L-1} \tau_k(A_{\ell}) \right\rangle \Lambda = \left\langle \prod_{k=-L}^{L-1} \prod_{(i,j) \in \partial_{\gamma_{\hat{h},e}}(\ell)} P^+(\ell)(i,k) P^-(\ell)(j,k) \right\rangle \Lambda \quad \text{if } \partial_{\gamma_{\hat{h},e}}(\ell) \neq \emptyset,
\]

(5.51)

we obtain the assertion in the proposition. \( \square \)

5.5 Proof of Theorem 3.8: Step 3

In this step, we identify \( \hat{\mathcal{H}} \) as

\[
\hat{\mathcal{H}} = \bigotimes_{k=1}^{L} \hat{\mathcal{H}}_v(k),
\]

(5.52)

where \( \hat{\mathcal{H}}_v(k) = \bigotimes_{j=1}^{L} \hat{\mathcal{H}}_j \). Recall that \( L = 2M + 1 \). We suppose that \( M \) is even. Set \( \hat{\mathcal{H}}_v = \hat{\mathcal{H}}_v(k = \frac{L+1}{2}) \). Note that \( \hat{\mathcal{H}}_v = \mathfrak{h}_L \otimes \mathfrak{h}_R \), where \( \mathfrak{h}_L = \bigotimes_{j \in \Lambda, j_1 = -1} \hat{\mathcal{H}}_j \) and \( \mathfrak{h}_R = \bigotimes_{j \in \Lambda, j_1 = 0} \hat{\mathcal{H}}_j \).

For each \( A \in \mathcal{B}(\hat{\mathcal{H}}_v) \), we define a linear operator on \( \hat{\mathcal{H}} \) by

\[
\eta_j(A) := \prod_{j_1 = -L+2, -L+2k-1}^{2M + 1} \prod_{\Lambda} A \otimes \mathbb{I} \otimes \mathbb{I}, \quad j = 1, \ldots, 2M + 1.
\]

(5.53)

Here, the tensor products in Eq. (5.53) correspond to Eq. (5.52). Let \( \xi_v \) be an antiunitary transformation from \( \mathfrak{h}_L \) onto \( \mathfrak{h}_R \) that is defined by

\[
\xi_v \hat{\Omega}_L = \hat{\Omega}_R, \quad \xi_v^{-1} \hat{c}_{j\sigma} \xi_v = \hat{c}_{r_v(j)\sigma} (-1)^{N_L}, \quad j \in \Lambda \text{ s.t. } j_1 = 0,
\]

(5.54)

where \( \hat{c}_{j\sigma} \) is the annihilation operator on \( \hat{\mathcal{H}}_v \), \( \hat{\Omega}_L \) (\( \hat{\Omega}_R \)) is the Fock vacuum in \( \mathfrak{h}_L \) (\( \mathfrak{h}_R \)), and \( N_L = \sum_{\sigma} j \sum_{j_1 = 1} N_j \) with \( \hat{n}_j \) of \( \hat{c}_{j\sigma} \xi_v \). Here, \( r_v \) is defined by Eq. (5.13)\(^6\).

For each \( A \in \mathcal{B}(\mathfrak{h}_L) \) and \( B \in \mathcal{B}(\mathfrak{h}_R) \), we set

\[
T_+(A \otimes B) = A \otimes (\xi_v A \xi_v^{-1}), \quad T_-(A \otimes B) = (\xi_v^{-1} B \xi_v) \otimes B.
\]

(5.55)

\( T_\pm \) are real linear maps on \( \mathcal{B}(\hat{\mathcal{H}}_v) \) that satisfy \( T_\alpha \circ T_\beta = T_\beta \). We define an antilinear involution \( J \) on \( \mathcal{B}(\hat{\mathcal{H}}_v) \) by

\[
J(A \otimes B) = (\xi_v^{-1} B \xi_v) \otimes (\xi_v A \xi_v^{-1}).
\]

(5.56)\(^6\)

To be precise, \( r_v \) in Eq. (5.13)\(^6\) is a restriction of \( r_v \) to \( \{ j = (j_1, j_2) \in \Lambda \mid j_1 = 0 \} \).
Let \( j \in \Lambda \) such that \( j_1 = -1 \). We have
\[
n_{j_1} \vartheta_v n_{j_1} \vartheta_v^{-1} = \left( \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \hat{n}_j \right) \otimes \left( \xi_v \hat{n}_j \xi_v^{-1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \right);
\]
let \( \langle i, j \rangle \in \gamma_{h,e} \) such that \( i \wedge j = i_1 = -2 \). We have
\[
\vartheta_v n_i n_j \vartheta_v^{-1} = \left( \xi_v^{-1} \hat{n}_i \xi_v \otimes \xi_v \hat{n}_j \xi_v^{-1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \right),
\]
where \( \vartheta_v \) is defined by Eq. (5.15).

**Proposition 5.13** Let \( A_1, \ldots, A_{2M+1} \in \mathfrak{B}(\hat{H}_v) \). We have
\[
\left| \prod_{\alpha=1}^{2M+1} \eta_\alpha(A_\alpha) \right|_\Lambda \leq \prod_{\alpha=1}^{2M+1} \left( \eta_1(JA_\alpha) \eta_2(A_\alpha) \cdots \eta_{2M-1}(JA_\alpha) \eta_{2M}(A_\alpha) \eta_{2M+1}(T_+(JA_\alpha)) \right)^{1/2M+1}.
\]

**Proof.** We apply Theorem 5.8 with \( A = \mathfrak{B}(\hat{H}_v) \), \( \omega(A_1, \ldots, A_{2M+1}) = \prod_{\alpha=1}^{2M+1} \eta_\alpha(A_\alpha) \). Note that the assumptions (b) and (c) in Theorem 5.8 are satisfied by Theorem 5.5. \( \square \)

For each \( \alpha = 1, \ldots, 2M+1 \), we set
\[
\partial_{\gamma_{h,e}}(\ell; \alpha) = \{ (i, j) \in \partial_{\gamma_{h,e}}(\ell) \mid i \wedge j = -L - 2 + 2\alpha \}.
\]

Note that \( \# \partial_{\gamma_{h,e}}(\ell; \alpha) = 0 \) or 1. Let us define a linear operator \( C_\alpha \) by
\[
C_\alpha = \begin{cases} 
\prod_{k=-L}^{L-1} P_{(i_1, k)}^{(+)} P_{(j_1, k)}^{(-)} & \text{if } \partial_{\gamma_{h,e}}(\ell; \alpha) \neq \emptyset, \\
\mathbb{1} & \text{if } \partial_{\gamma_{h,e}}(\ell; \alpha) = \emptyset,
\end{cases}
\]
where \( i_1, j_1 \) satisfy \( \min\{i_1, j_1\} = -L - 2 + 2\alpha \) and \( \partial_{\gamma_{h,e}}(\ell; \alpha) = \{ (i_1, \ell), (j_1, \ell) \} \). Note that
\[
\prod_{(i, j) \in \partial_{\gamma_{h,e}}(\ell)} \prod_{k=-L}^{L-1} P_{(i_1, k)}^{(+)} P_{(j_1, k)}^{(-)} = \prod_{\alpha=1}^{2M+1} C_\alpha.
\]

\( ^6 \# S \) is the cardinality of set \( S \).
We consider the case where \( \partial \gamma_{h,e}(\ell; \alpha) \neq \emptyset \). Write \( \partial \gamma_{h,e}(\ell; \alpha) = \{ [i^{[a]}, j^{[a]}] \} \). Suppose first that \( j_1^{[a]} > i_1^{[a]} = -L - 2 + 2\alpha \). Let

\[
\mathcal{P}^{(+)} = \bigotimes_{k=-L}^{L-1} P^{(+)}_{(i,k)}, \quad \mathcal{P}^{(-)} = \bigotimes_{k=-L}^{L-1} P^{(-)}_{(j,k)},
\]

where \( P^{(\pm)} \) is defined in the proof of Proposition \[\text{5.12}\]. Let us define a linear operator \( A \in \mathfrak{B}(\mathfrak{H}) \) by \( A = \mathcal{P}^{(+)} \otimes \mathcal{P}^{(-)} \). Here, we regard \( \mathcal{P}^{(+)} \) (\( \mathcal{P}^{(-)} \)) as a linear operator on \( \mathfrak{B}(h_L) \) (\( \mathfrak{B}(h_R) \)). Evidently, \( C_\alpha \) can be expressed as \( C_\alpha = \eta_\alpha(A) \). Because \( JA = \mathcal{P}^{(-)} \otimes \mathcal{P}^{(+)} \) and \( T_+(JA) = \mathcal{P}^{(-)} \otimes \mathcal{P}^{(-)} \), we have

\[
\left\langle \eta_1(JA)\eta_2(A) \cdots \eta_{2M-1}(JA)\eta_{2M}(A)\eta_{2M+1}(T_+(JA)) \right\rangle_\Lambda = \left\langle \mathcal{P}^{(-)} \right\rangle_\Lambda. \tag{5.65}
\]

Conversely, if \( j_1^{[a]} > i_1^{[a]} = -L - 2 + 2\alpha \), we see that \( C_\alpha \) can be expressed as \( C_\alpha = \eta_\alpha(B) \) with \( B = \mathcal{P}^{(-)} \otimes \mathcal{P}^{(+)} \). Moreover,

\[
\left\langle \eta_1(JB)\eta_2(B) \cdots \eta_{2M-1}(JB)\eta_{2M}(B)\eta_{2M+1}(T_+(JB)) \right\rangle_\Lambda = \left\langle \mathcal{P}^{(+)} \right\rangle_\Lambda. \tag{5.66}
\]

We apply Proposition \[\text{5.13}\] with \( A_\alpha = A \) if \( j_1^{[a]} > i_1^{[a]} \), \( A_\alpha = B \) if \( j_1^{[a]} < i_1^{[a]} \). If \( \partial \gamma_{h,e}(\ell; \alpha) \neq \emptyset \), then at least one \( \alpha \) exists such that \( \partial \gamma_{h,e}(\ell; \alpha) \neq \emptyset \). Thus, by Eqs. \[\text{5.65}\], (\[\text{5.66}\]) and Proposition \[\text{5.13}\] we have

\[
\left\langle \prod_{(i,j) \in \partial \gamma_{h,e}(\ell)} P^{(+)}_{(i, k)} P^{(-)}_{(j, k)} \right\rangle_\Lambda \leq \mathcal{P}_{1/2M+1}. \tag{5.67}
\]

Thus, by Proposition \[\text{5.12}\] we obtain

\[
\left\langle \prod_{(i,j) \in \partial \gamma_{h,e}} P^{(+)}_i P^{(-)}_j \right\rangle_\Lambda \leq \mathcal{P}_{2^{\partial \gamma_{h,e} / |\Lambda|}}. \tag{5.68}
\]

This finishes the proof of Eq. \[\text{5.34}\] for the case where \( \alpha = h \) and \( \beta = e \). In a similar way, Eq. \[\text{5.34}\] can be proven in the remaining three cases. \( \square \)

6  Proof of Theorem \[\text{3.11}\]

We apply the principle of exponential localization that was established by Fröhlich and Lieb \[\text{[II]}\], which is stated as follows.

**Theorem 6.1** Let \( A \) and \( B \) be self-adjoint operators on a Hilbert space \( \mathcal{H} \) such that

(i) \( A \geq 0 \),

(ii) \( \pm B \leq \varepsilon A \) with \( \varepsilon \in [0,1) \).
Suppose that
\[(A + B)\psi = \lambda \psi, \quad \|\psi\| = 1. \tag{6.1}\]
Choose some \(\rho > \lambda\) such that \(\gamma := \varepsilon (\rho - \lambda)^{-1} < 1\). Let \(P_\rho = E_A [\rho, \infty)\), the spectral measure of \(A\) corresponding to \([\rho, \infty)\), and let \(M_\rho = \text{ran} P_\rho\). Finally, let \(\mathcal{N}\) be the closed subspace such that
\[(iii) \{B(A - \lambda)^{-1}\} \mathcal{N} \subseteq M_\rho, \quad j = 1, \ldots, d - 1 \text{ with } d \geq 1.\]
Then \(\langle \psi | P_\mathcal{N} \psi \rangle \leq \gamma^d\), where \(P_\mathcal{N}\) is the orthogonal projection onto \(\mathcal{N}\).

Theorem 3.11 can be proven by applying Theorem 6.1. We begin with the following lemma:

**Lemma 6.2** Denote by \(\varepsilon^W\) the lowest eigenvalue for \(W\). We obtain the following:

(i) \(\varepsilon^W = -(S + \frac{\Delta}{2})|A|\).

(ii) The Fock vacuum \(\Omega\) is the ground state of \(W\): \(W\Omega = \varepsilon^W \Omega\).

*Proof.* Use Eq. (5.11). \(\square\)

We choose \(A\) and \(B\) as
\[A = W - \varepsilon(t = 1), \quad B = T = (t - 1) \sum_{(i,j)} \sum_{\sigma = \uparrow, \downarrow} (c_i^\sigma c_j^\sigma + c_j^\sigma c_i^\sigma), \tag{6.2}\]
where \(\varepsilon(t = 1)\) is the lowest eigenvalue for \(\tilde{H}\) with \(t = 1\). In the remainder of this section, we present checks of every assumption in Theorem 6.1.

**Lemma 6.3** We have the following:

(i) \(A \geq 0\)

and

(ii) \(\pm B \leq tA\).

*Proof.* (i) By Lemma 6.2 we have \(\langle \Omega | W \Omega \rangle = \varepsilon^W\). Because \(\langle \Omega | B \Omega \rangle = 0\), we have
\[\varepsilon^W = \langle \Omega | W \Omega \rangle = \langle \Omega | \tilde{H}^{t = 1} \Omega \rangle \geq \varepsilon(t = 1). \tag{6.3}\]
Thus, we obtain item (i).

(ii) We see that
\[A + t^{-1} B = \tilde{H}^{t = 1} - \varepsilon(t = 1) \geq 0, \tag{6.4}\]
which implies \(-B \leq tA\).

Let \(u = \exp \left\{ i \pi \sum_\sigma \sum_{j \in \Lambda_0} n_{j\sigma} \right\} \). We have \(uBu^{-1} = -B\) and \(uAu^{-1} = A\). Thus,
\[0 \leq u(A + t^{-1} B)u^{-1} = A - t^{-1} B, \tag{6.5}\]
which implies \(B \leq tA\). \(\square\)

7 More precisely, for all \(\phi \in \mathcal{N}, \{B(A - \lambda)^{-1}\} \phi \in M_\rho\) for \(j = 1, \ldots, d\).
Lemma 6.4 We have $e(t = 1) = -8|\Lambda| - S|\Lambda| - \frac{\Delta}{2}|\Lambda|$.

Proof. Because $\|B\| \leq 8t|\Lambda|$ and $e^W = -(S + \Delta/2)|\Lambda|$, we obtain the result. □

Let
\[
x = \frac{1}{|\Lambda|} \{ e^W - e(t = 1) \}, \quad \rho = e^W - e(t = 1) + n\delta|\Lambda|.
\]
(6.6)

By Lemma 6.4, we have $x \leq 8$. Thus, we obtain the following:

Lemma 6.5 For each $\lambda$ with $\lambda \leq x|\Lambda| + \delta|\Lambda|$, we define $\gamma = t\frac{\rho}{\rho - \lambda}$. Then we have
\[
\gamma \leq t\left\{ 1 + \frac{8 + \delta}{(n - 1)\delta} \right\}.
\]
(6.7)

Let
\[
E^A\left(P^{(\omega)}_\Lambda \right) = \min \text{spec} \left(P^{(\omega)}_\Lambda W P^{(\omega)}_\Lambda \right) - e(t = 1), \quad \omega = +, -.
\]
(6.8)

Lemma 6.6 We have $E^A\left(P^{(\omega)}_\Lambda \right) - \rho \geq (\mathcal{J} - n\delta)|\Lambda| + \mathcal{O}(\mathcal{|\Lambda|^{1/2}})$ for each $\omega = +, -$, where $\mathcal{J}$ is defined by Eq. (3.33).

Proof. Recall Eq. (3.10). We can check that
\[
\min \text{spec} \left(P^{(\pm)}_\Lambda W P^{(\pm)}_\Lambda \right) \geq \begin{cases} 
-\left(S + \Delta \right)|\Lambda| + \frac{V}{4}|\Lambda| + \mathcal{O}(\mathcal{|\Lambda|^{1/2}}) & \text{if } S \geq 0 \\
-\frac{1}{2}\left(S + \Delta \right)|\Lambda| + \frac{V}{4}|\Lambda| + \mathcal{O}(\mathcal{|\Lambda|^{1/2}}) & \text{if } S < 0
\end{cases},
\]
(6.9)

which gives the desired result. □

Because $\mathcal{J} > 0$ by the assumption that $S + \frac{\Delta}{2} > 0$, we obtain the following:

Corollary 6.7 Let $\mathcal{N}^{(\pm)} = \text{ran} \left(P^{(\pm)}_\Lambda \right)$. If $|\Lambda|$ is sufficiently large such that
\[
(\mathcal{J} - n\delta)|\Lambda| + \mathcal{O}(\mathcal{|\Lambda|^{1/2}}) \geq 0,
\]
(6.10)

then we have $\mathcal{N}^{(\pm)} \subseteq \mathcal{M}_\rho$.

Proposition 6.8 Let $G = 6(|S| + V) + \Delta$. We have $B\{\text{ran}E_A[e, \infty]\} \subseteq \text{ran}E_A[e - G, \infty]$. That is, if $\psi \in \text{ran}E_A[e, \infty]$, then $B\psi \in \text{ran}E_A[e - G, \infty]$.

Proof. For each $m = \{m_j\}_{j \in \Lambda} \in \{-1, 0, 1\}^\Lambda$, we set $\mathcal{H}(m) = \text{ran} \left[ \prod_{j \in \Lambda} E_{\mathcal{H}_j}(\{m_j\}) \right]$. We have $\mathcal{H} = \bigoplus_{m \in \{-1, 0, 1\}^\Lambda} \mathcal{H}(m)$ and
\[
W \upharpoonright \mathcal{H}(m) = e(m), \quad e(m) = -S \sum_{j \in \Lambda} m_j^2 + \frac{V}{2} \sum_{(i,j)} (m_i - m_j)^2 + \frac{\Delta}{2} \sum_{j \in \Lambda} m_j.
\]
(6.11)
where $W \upharpoonright \mathcal{H}(m)$ is the restriction of $W$ onto $\mathcal{H}(m)$. Since \text{spec}(W) = \{e(m) | m \in (-1,0,1)^A\}, we have
\[\text{ran}E_A[e, \infty) = \bigoplus_{m \in (-1,0,1)^A} \mathcal{H}(m).\] (6.12)

To discuss how the linear operator $B$ maps $\mathcal{H}(m)$ for each $m \in (-1,0,1)^A$, we introduce the notation $\mathcal{M} = \{\emptyset, m \mid m \in (-1,0,1)^A\}$. Note that the operator $B$ consists of $c_{ij}^s c_{js}^s$ and $c_{ij} c_{is}$. For each $m \in \mathcal{M}$, there exists an $m' \in \mathcal{M}$ such that $c_{ij}^s c_{js}^s \mathcal{H}(m) \subseteq \mathcal{H}(m')$, where $\mathcal{H}(m') = \{0\}$ if $m' = \emptyset$. More precisely, $m'$ is of the form $m' = \{m_k \pm \delta_k \pm \delta_{kj}\}_{k \in A}$, where $\delta_{ki}$ is the Kronecker delta. If $m_k \pm \delta_{ki} \pm \delta_{kj} = \pm 2$ for $k = i$ or $j$, then we understand that $m' = \emptyset$. A naive estimate tells us that $|e(m) - e(m')| \leq 6(|J| + V) + \Delta$. Thus, if $\psi \in \text{ran}E_A[e, \infty)$, then $B\psi \in \text{ran}E_A[e - G, \infty)$.
\[\square\]

**Corollary 6.9** Suppose that $\mathcal{J} - n\delta > 0$. For sufficiently large $\Lambda$, we have $\{B(A - \lambda)^{-1}\}^j \mathcal{N}(\pm) \subseteq \mathcal{M}_\rho, \; j = 1, \ldots, d - 1$ with $d \in \mathbb{N}$ which satisfies
\[d > \frac{1}{G} \left(\mathcal{J} - n\delta\right) |\Lambda| + \mathcal{O}(\Lambda^{1/2}).\] (6.13)

**Proof.** By Proposition 6.8 we remark that if $\psi \in \text{ran}E_A \left[\mathcal{E}^A \left(P^\omega_A\right), \infty\right)$, then $B^\ell \psi \in \text{ran}E_A \left[\mathcal{E}^A \left(P^\omega_A\right) - G\ell, \infty\right)$ for each $\ell \in \mathbb{N}$. If $d$ satisfies
\[\mathcal{E}^A \left(P^\omega_A\right) - Gd < \rho \leq \mathcal{E}^A \left(P^\omega_A\right) - G(d - 1),\] (6.14)
then it holds that $\{B(A - \lambda)^{-1}\}^j \mathcal{N}(\pm) \subseteq \mathcal{M}_\rho$ for $j = 1, \ldots, d - 1$, where $\mathcal{N}(\pm) = \text{ran}(P^\pm_A)$. Thus, we have, by Lemma 6.6
\[d > \frac{1}{G} \left(\mathcal{J} - n\delta\right) |\Lambda| + \mathcal{O}(\Lambda^{1/2})\]. (6.15)

This completes the proof. \[\square\]

**Completion of proof of Theorem 5.17**

Note that $\text{Tr}_\mathcal{H}[P^\omega_A E_\delta] = \sum_n \langle \psi_n | P^\omega_A | \psi_n \rangle$, where $\{\psi_n\}_n$ is a complete orthonormal system of $\text{ran}E_\delta$. By Theorem 6.1 and Corollary 6.9, we have $\langle \psi_n | P^\omega_A | \psi_n \rangle \leq \gamma^d$ for all $n$. Therefore, it holds that $\text{Tr}_\mathcal{H}[P^\omega_A E_\delta] \leq 4^{\left|\mathcal{A}\right|}\gamma^d$. Finally, we choose $\delta = \beta^{-\xi}$ and $n = \mathcal{J} \eta \beta^k$ for arbitrary $\xi, \eta \in (0,1)$. \[\square\]

**7 Proof of Theorem 3.5 (B)**

We divide the proof into two cases. In this section, we present proofs of the following Theorems:
Theorem 7.1 Assume that $S \geq 0$ and $\frac{\Delta}{2} + S > 0$. For any $\varepsilon > 0$, we have $\langle P_0^{(0)} \rangle_\Lambda < \varepsilon$, provided that $\beta$ and $t^{-1}$ are sufficiently large.

and

Theorem 7.2 Assume that $S < 0$. Moreover, assume that $\frac{\Delta}{2} - |S| > 0$. For any $\varepsilon > 0$, we have $\langle P_0^{(0)} \rangle_\Lambda < \varepsilon$, provided that $\beta$ and $t^{-1}$ are sufficiently large.

7.1 Proof of Theorem 7.1

Lemma 7.3 Assume that $|1 - \langle q_\Lambda^2 \rangle_\Lambda| < \varepsilon$. We have $\langle P_j^{(0)} \rangle_\Lambda < \varepsilon$.

Proof. Note that, by the spectral theorem, we have

$$\langle q_\Lambda^2 \rangle_\Lambda = \langle P_\Lambda^{\lambda=+1} \rangle_\Lambda + \langle P_\Lambda^{\lambda=-1} \rangle_\Lambda.$$  (7.1)

Since $P_0^{(0)} + P_\Lambda^{\lambda=+1} + P_\Lambda^{\lambda=-1} = 1$, we obtain the desired result. \(\square\)

Lemma 7.4 We have $\langle q_\Lambda^2 \rangle_\Lambda^{1/2} \geq 1 - \frac{8t}{S + \frac{\Delta}{2}} - \frac{\ln 4}{\beta(S + \frac{\Delta}{2})}$.

Proof. Because $\frac{1}{2} \sum_{i,j} (q_i - q_j)^2 \geq 0$, we have

$$\frac{\langle -\tilde{H} \rangle_\Lambda}{|\Lambda|} \leq 8t + S\langle q_\Lambda^2 \rangle_\Lambda - \frac{\Delta}{2}\langle q_\Lambda \rangle_\Lambda.$$  (7.2)

By using the fact $|\langle q_\Lambda \rangle_\Lambda| \leq \langle q_\Lambda^2 \rangle_\Lambda^{1/2}$, we obtain

$$\frac{\langle -\tilde{H} \rangle_\Lambda}{|\Lambda|} \leq 8t + \left(S + \frac{\Delta}{2}\right)\langle q_\Lambda^2 \rangle_\Lambda^{1/2},$$  (7.3)

where we have used the fact that $x \leq \sqrt{x}$ for all $x \in [0, 1]$.

Because $\Omega$ is the ground state of $W$ and $\langle \Omega | T \Omega \rangle = 0$, we have $\langle \Omega | \tilde{H} \Omega \rangle = -\left(S + \frac{\Delta}{2}\right)|\Lambda|$. Thus, by the Peierls–Bogoliubov inequality [39], we have

$$\text{Tr}_S \left[ e^{-\beta \tilde{H}} \right] \geq e^{-\beta \langle \Omega | \tilde{H} \Omega \rangle} = e^{\beta(S + \frac{\Delta}{2})},$$  (7.4)

Conversely, because of the convexity of $\ln \text{Tr}[e^{-A}]$, we have

$$\langle -\beta \tilde{H} \rangle_\Lambda \geq \ln \text{Tr}_S \left[ e^{-\beta \tilde{H}} \right] - |\Lambda| \ln 4.$$  (7.5)

Combining Eqs. (7.3), (7.4), (7.5), we arrive at the result in the lemma. \(\square\)

Completion of proof of Theorem 7.1

Theorem 7.1 immediately follows from Lemmas 7.3 and 7.4. \(\square\)
7.2 Proof of Theorem 7.2

Because \( U \sum_{j \in \Lambda} q_j^2 \geq 0 \), we have, from Eq. (3.6),

\[
\langle -\tilde{H} \rangle_\Lambda |\Lambda| \leq 8t + 4V \langle q_0 q_0 \rangle_\Lambda - \frac{\Delta}{2} \langle q_0 \rangle_\Lambda. \tag{7.6}
\]

Because \( |\langle q_0 q_δ \rangle| \leq \langle q_0^2 \rangle, \langle |q_0| \rangle \leq \langle q_0^2 \rangle^{1/2} \), we obtain

\[
\langle -\tilde{H} \rangle_\Lambda |\Lambda| \leq 8t + \left( 4V + \frac{\Delta}{2} \right) \langle q_0^2 \rangle^{1/2}. \tag{7.7}
\]

We can easily verify that \( \langle \Omega | -\tilde{H} | \Omega \rangle = \left( |S| - \frac{\Delta}{2} \right) |\Lambda|. \) Thus, by the Peierls–Bogoliubov inequality, we have

\[
\ln \text{Tr}_{\mathcal{X}_L \otimes \mathcal{X}_R} [ e^{-\beta \tilde{H}} ] \geq \beta \left( \frac{\Delta}{2} - |S| \right) |\Lambda|. \tag{7.5}
\]

By combining this result with Eqs. (7.5) and (7.7), we arrive at

\[
\langle q_0^2 \rangle^{1/2} \geq \frac{\Delta - |S|}{4} + \frac{8t}{4V} + \frac{2 \ln 4}{\beta (\frac{\Delta}{2} + 4V)}. \tag{7.8}
\]

Thus, for any \( \varepsilon > 0 \), we have \( 1 - \langle q_0^2 \rangle^2 < \varepsilon \), provided that \( \beta \) and \( t^{-1} \) are sufficiently large. The application of Lemma 7.3 concludes the assertion. \( \Box \)

A Dyson–Lieb–Simon inequality

Let \( \mathcal{X}_L \) and \( \mathcal{X}_R \) be complex Hilbert spaces. For simplicity, we suppose that \( \dim \mathcal{X}_L = \dim \mathcal{X}_R < \infty \). Let \( \vartheta \) be an antiunitary transformation from \( \mathcal{X}_L \) onto \( \mathcal{X}_R \). Let \( A, B_1, \ldots, B_n \in \mathfrak{B} (\mathcal{X}_L) \). Assume that \( A \) is self-adjoint. Here we address the self-adjoint operator defined by

\[
H = A \otimes \mathbb{I} + \mathbb{I} \otimes \vartheta A \vartheta^{-1} - \sum_{j=1}^n \left( B_j \otimes \vartheta B_j \vartheta^{-1} + B_j^* \otimes \vartheta B_j^* \vartheta^{-1} \right). \tag{A.1}
\]

As usual, thermal expectation associated with \( H \) is given by

\[
\langle \bullet \rangle = \frac{\text{Tr}_{\mathcal{X}_L \otimes \mathcal{X}_R} [ \bullet e^{-\beta H} ]}{\text{Tr}_{\mathcal{X}_L \otimes \mathcal{X}_R} [ e^{-\beta H} ]}. \tag{A.2}
\]

The following theorem is a generalized version of the DLS inequality:

**Theorem A.1** Let \( C, D \in \mathfrak{B} (\mathcal{X}_L) \). We have

(i) \( \langle C \otimes \vartheta C \vartheta^{-1} \rangle \geq 0 \),

(ii) \( \left| \langle C \otimes \vartheta D \vartheta^{-1} \rangle \right| \leq \langle C \otimes \vartheta C \vartheta^{-1} \rangle \langle D \otimes \vartheta D \vartheta^{-1} \rangle \).

**Remark A.2** (i) In the original DLS inequality [7], all of the matrix elements of \( A \) and \( B_j \) are assumed to be real. However, noted in Refs. [24, 27, 32], we can weaken this assumption.
As noted in Ref. [31], item (i) can be regarded as a non-commutative version of the Griffiths inequality.

Theorem A.1 can be proven by applying a series of lemmas. The basic idea of our proof comes from Ref. [27].

We begin with the following observation:

**Lemma A.3** For each $A, B \in \mathcal{B}(X_L)$, we have

$$\text{Tr}_{X_L \otimes X_R} [A \otimes \vartheta B \vartheta^{-1}] = \text{Tr}_{X_L} [\text{Tr}_{X_L} [B]]^*.$$  

(A.3)

In particular, we have

$$\text{Tr}_{X_L \otimes X_R} [A \otimes \vartheta A \vartheta^{-1} - 1] = \left| \text{Tr}_{X_L} [A] \right|^2 \geq 0.$$  

(A.4)

**Proof.** It suffices to prove that $\text{Tr}_{X_R} [\vartheta B \vartheta^{-1}] = (\text{Tr}_{X_L} [B])^*$. Let $\{e_i\}_i$ be a complete orthonormal system in $X_R$. Then $\{\vartheta^{-1}e_i\}_i$ is a complete orthonormal system in $X_L$ as well. We have

$$\text{Tr}_{X_R} [\vartheta B \vartheta^{-1}] = \sum_i \langle e_i | \vartheta B \vartheta^{-1} e_i \rangle = \sum_i \langle \vartheta^{-1}e_i | \vartheta B \vartheta^{-1} e_i \rangle = \sum_i (\langle \vartheta^{-1}e_i | B \vartheta^{-1} e_i \rangle)^* = (\text{Tr}_{X_L} [B])^*.$$  

(A.5)

This completes the proof. □

Let $\mathcal{C}_0$ be a convex cone defined by

$$\mathcal{C}_0 = \text{Coni} \left\{ A \otimes \vartheta A \vartheta^{-1} \mid A \in \mathcal{B}(X_L) \right\},$$  

(A.6)

where $\text{Coni}(S)$ is the conical hull of $S$. Let $\mathcal{C}$ be the closure of $\mathcal{C}_0$ under the operator norm topology. A linear operator $X$ on $X_L \otimes X_R$ is called reflection positive if $X$ belongs to $\mathcal{C}$. If $X$ is reflection positive, then we write $X \succeq 0$.

By Lemma A.3 we have the following:

**Lemma A.4** If $X \succeq 0$, then $\text{Tr}_{X_L \otimes X_R} [X] \geq 0$.

The following lemma is often useful:

**Lemma A.5** If $X \succeq 0, Y \succeq 0$, then we have the following

(i) $XY \succeq 0$;

(ii) $aX + bY \succeq 0$ for all $a, b \geq 0$.

**Proposition A.6** We have $e^{-\beta H} \succeq 0$ for all $\beta \geq 0$.

**Proof.** Let

$$H_0 = A \otimes 1 + 1 \otimes \vartheta A \vartheta^{-1}, \quad V = \sum_{j=1}^n (B_j \otimes \vartheta B_j \vartheta^{-1} + B_j^* \otimes \vartheta B_j^* \vartheta^{-1}).$$  

(A.7)
First, observe that \( e^{-\beta H_0} = e^{-\beta A} \otimes \vartheta e^{-\beta A} \vartheta^{-1} \succeq 0 \) for all \( \beta \in \mathbb{R} \). Conversely, because \( V \succeq 0 \), we have
\[
e^{-\beta V} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} V^n \succeq 0
\] (A.8)
for all \( \beta \geq 0 \). By the Trotter–Kato product formula, we obtain
\[
e^{-\beta H} = \lim_{n \to \infty} \left( e^{-\beta H_0/n} e^{\beta V/n} \right)^n \succeq 0
\] (A.9)
for all \( \beta \geq 0 \). \( \square \)

**Proposition A.7** Assume that \( X \succeq 0 \). For each \( C, D \in \mathcal{B}(X_L) \), we have
\[
\left| \text{Tr}_{X_L \otimes x_R} \left[ C \otimes \vartheta D \vartheta^{-1} X \right] \right|^2 \leq \text{Tr}_{X_L \otimes x_R} \left[ C \otimes \vartheta C \vartheta^{-1} X \right] \text{Tr}_{X_L \otimes x_R} \left[ D \otimes \vartheta D \vartheta^{-1} X \right].
\] (A.10)

**Proof.** For simplicity, we assume that \( X \in \mathcal{C}_0 \). Thus, we can write \( X \) as \( X = \sum_{j=1}^{N} E_j \otimes \vartheta E_j \vartheta^{-1} \). By Lemma A.3, we have
\[
\text{Tr}_{X_L \otimes x_R} \left[ C \otimes \vartheta D \vartheta^{-1} X \right] = \sum_{j=1}^{N} \text{Tr}_{X_L \otimes x_R} \left[ CE_j \otimes \vartheta DE_j \vartheta^{-1} \right] = \sum_{j=1}^{N} \text{Tr}_{X_L} \left[ CE_j \right] \left( \text{Tr}_{X_L} \left[ DE_j \right] \right)^*.
\] (A.11)

Hence, by the Schwartz inequality, we obtain
\[
\left| \text{Tr}_{X_L \otimes x_R} \left[ C \otimes \vartheta D \vartheta^{-1} X \right] \right|^2 \leq \sum_{j=1}^{N} \left| \text{Tr}_{X_L} \left[ CE_j \right] \right|^2 \sum_{j=1}^{N} \left| \text{Tr}_{X_L} \left[ DE_j \right] \right|^2 \leq \text{Tr}_{X_L \otimes x_R} \left[ C \otimes \vartheta C \vartheta^{-1} X \right] \text{Tr}_{X_L \otimes x_R} \left[ D \otimes \vartheta D \vartheta^{-1} X \right].
\] (A.12)

This finishes the proof. \( \square \)

**Proof of Theorem A.1**

By Proposition A.6, we have \( C \otimes \vartheta C \vartheta^{-1} e^{-\beta H} \succeq 0 \) for all \( \beta \geq 0 \). Therefore, by Lemma A.4, we conclude item (i).

Item (ii) immediately follows from Propositions A.6 and A.7. \( \square \)

**References**

[1] C. D. Batista, A. A. Aligia, Exact Bond Ordered Ground State for the Transition between the Band and the Mott Insulator. Phys. Rev. Lett. 92, 246405(2004).
[2] M. Biskup, R. Kotecky, Forbidden gap argument for phase transitions proved by means of chessboard estimates. Comm. Math. Phys. 264 (2006), 631-656.

[3] M. Biskup, L. Chayes, S. Starr, Quantum spin systems at positive temperature. Comm. Math. Phys. 269 (2007), 611-657.

[4] C. Borgs, R. Kotecky, D. Ueltschi, Low temperature phase diagrams for quantum perturbations of classical spin systems. Comm. Math. Phys. 181 (1996), 409-446.

[5] K. Bouadim, N. Paris, F. Hebert, G. G. Batrouni, R. T. Scalettar, Metallic phase in the two-dimensional ionic Hubbard model. Phys. Rev. B 76, 085112(2007).

[6] N. Datta, R. Fernandez, J. Fröhlich, Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely-many ground states. J. Stat. Phys. 84 (1996), 455-534.

[7] F. J. Dyson, E. H. Lieb, B. Simon, Phase transitions in quantum spin systems with isotropic and nonisotropic interactions. J. Stat. Phys. 18 (1978), 335-383.

[8] T. Egami, S. Ishihara, M. Tachiki, Lattice effect of strong electron correlation: Implication for ferroelectricity and Superconductivity, Science. 261(1993), 1307-1310.

[9] A. C. D. van Enter, S. B. Shlosman, Provable first-order transitions for nonlinear vector and gauge models with continuous symmetries. Comm. Math. Phys. 255 (2005), 21-32.

[10] M. Fabrizio, A. O. Gogolin, A. A. Nersesyan, From Band Insulator to Mott Insulator in One Dimension, Phys. Rev. Lett. 83 (1999), 2014-2017.

[11] J. Fröhlich, E. H. Lieb, Phase transitions in anisotropic lattice spin systems. Comm. Math. Phys. 60 (1978), 233-267.

[12] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon, Phase transitions and reflection positivity. I. General theory and long range lattice models. Comm. Math. Phys. 62 (1978), 1-34.

[13] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon, Phase transitions and reflection positivity. II. Lattice systems with short-range and Coulomb interactions. J. Stat. Phys. 22 (1980), 297-347.

[14] J. Fröhlich, B. Simon, T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking. Comm. Math. Phys. 50 (1976), 79-95.

[15] J. Glimm, A. Jaffe, T. Spencer, Phase transitions for $\varphi_2^4$ quantum fields. Comm. Math. Phys. 45 (1975), 203-216.

[16] J. Glimm, A. Jaffe, Quantum physics. A functional integral point of view. Second edition. Springer-Verlag, New York, 1987.

[17] J. Hubbard, J. B. Torrance, Model of the Neutral-Ionic Phase Transformation, Phys. Rev. Lett. 47 (1981), 1750-1754.
[18] J. Jedrzejewski, Electron charge ordering in the extended Hubbard model. Z. Phys. B Condensed Matter 48 (1982), 219-225.

[19] A. P. Kampf, M. Sekania, G. I. Japaridze1, Ph. Brune, Nature of the insulating phases in the half-filled ionic Hubbard model. J. Phys.: Condens. Matter 15 (2003), 5895-5907.

[20] S. S. Kancharla, E. Dagotto, Correlated Insulated Phase Suggests Bond Order between Band and Mott Insulators in Two Dimensions. Phys. Rev. Lett. 98, 016402(2007).

[21] K. Kubo, T. Kishi, Rigorous bounds on the susceptibilities of the Hubbard model. Phys. Rev. B 41 (1990), 4866-4868.

[22] E. H. Lieb, Two theorems on the Hubbard model. Phys. Rev. Lett. 62 (1989), 1201-1204.

[23] E. H. Lieb, Flux phase of the half-filled band. Phys. Rev. Lett. 73 (1994), 2158-2161.

[24] E. H. Lieb, B. Nachtergaele, Stability of the Peierls instability for ring-shaped molecules. Phys. Rev. B 51 (1995), 4777-4791.

[25] S. R. Manmana, V. Meden, R. M. Noack, K. Schönhammer, Quantum critical behavior of the one-dimensional ionic Hubbard model. Phys. Rev. B 70, 155115(2004).

[26] M. Messer, R. Desbuquois, T. Uehlinger, G. Jotzu, S. Huber, D. Greif, T. Esslinger, Exploring Competing Density Order in the Ionic Hubbard Model with Ultracold Fermions. Phys. Rev. Lett. 115, 115303(2015).

[27] T. Miyao, Self-dual cone analysis in condensed matter physics. Rev. Math. Phys. 23 (2011), 749-822.

[28] T. Miyao, Ground state properties of the SSH model. J. Stat. Phys. 149 (2012), 519-550.

[29] T. Miyao, Rigorous results concerning the Holstein–Hubbard model model. arXiv:1402.5202 Ann. Henri Poincare (in press).

[30] T. Miyao, Upper bounds on the charge susceptibility of many-electron systems coupled to the quantized radiation field. Lett. Math. Phys. 105 (2015), 1119-1133.

[31] T. Miyao, Quantum Griffiths Inequalities, arXiv:1507.05355 J. Stat. Phys. (in press).

[32] T. Miyao, Long-range charge order in the extended Holstein–Hubbard model, arXiv:1601.00765

[33] M. Murakami, Possible Ordered States in the 2D Extended Hubbard Model, J. Phys. Soc. Jpn. 69 (2000), 1113-1124.
[34] N. Nagaosa, J. Takimoto, Theory of Neutral-Ionic Transition in Organic Crystals. I. Monte Carlo Simulation of Modified Hubbard Model. J. Phys. Soc. Jpn. 55 (1986), 2735-2744.

[35] K. Osterwalder, R. Schrader, Axioms for Euclidean Green’s functions. Comm. Math. Phys. 31 (1973), 83-112. Axioms for Euclidean Green’s functions. II. With an appendix by Stephen Summers. Comm. Math. Phys. 42 (1975), 281-305.

[36] N. Paris, K. Bouadim, F. Hebert, G. G. Batrouni, R. T. Scalettar, Quantum Monte Carlo Study of an Interaction-Driven Band-Insulator-to-Metal Transition. Phys. Rev. Lett. 98, 046403(2007).

[37] R. Resta, S. Sorella, Many-Body Effects on Polarization and Dynamical Charges in a Partly Covalent Polar Insulator. Phys. Rev. Lett. 74 (1995), 4738-4741.

[38] S. Shlosman, Y. Vignaud, Dobrushin interfaces via reflection positivity. Comm. Math. Phys. 276 (2007), 827-861.

[39] B. Simon, The Statistical Mechanics of Lattice Gases, Volume I. Princeton Univ Press, 1993