THE RESURGENCE OF AN IDEAL WITH NOETHERIAN SYMBOLIC REES ALGEBRA IS RATIONAL

MICHAEL DIPASQUALE AND BEN DRABKIN

Abstract. We show that if an ideal in a polynomial ring has a Noetherian symbolic Rees algebra then two statistics which measure the relationship between its regular and symbolic powers are both rational; these are the resurgence and asymptotic resurgence.

1. Introduction

Suppose \( I \) is an ideal in a polynomial ring \( R = K[x_1, \ldots, x_n] \) over a field \( K \). There are different ways to take powers of \( I \). The ordinary power \( I^r \) is the ideal generated by all \( r \)-fold products of elements of \( I \); this retains many of the algebraic properties of \( I \). An alternative is the \( s \)-th symbolic power of \( I \), defined as \( I^{(s)} = \bigcap_{P \in \text{Ass}(I)} (I^s R_P \cap R) \), where \( \text{Ass}(I) \) is the set of associated primes of \( I \). The symbolic powers of \( I \) retain its geometry; if \( I \) is a radical ideal and \( K \) is algebraically closed of characteristic zero, the Zariski-Nagata theorem shows that \( I^{(s)} \) consists of all polynomials which vanish to order \( s \) along the variety defined by \( I \).

Symbolic and regular powers are at the center of the containment problem. The containment problem is to determine for which positive integers \( r, s \) we have \( I^{(s)} \subset I^r \). A seminal result is that \( I^{(hr)} \subset I^r \) if \( h \) is the maximum height of an associated prime of \( I \); this is proved in successively more generality in the papers of Ein, Lazarsfeld, and Smith [6], Hochster and Huneke [11], and Ma and Schwede [13].

To quantify the containment problem more precisely for individual ideals, Bocci and Harbourne introduced the notion of resurgence in [1], defined as

\[
\rho(I) = \sup \left\{ \frac{s}{r} \ \bigg| \ I^{(s)} \nsubseteq I^r \right\}.
\]

A coarsening of resurgence, called asymptotic resurgence, was introduced by Guardo, Harbourne, and Van Tuyl in [9]:

\[
\hat{\rho}(I) = \sup \left\{ \frac{s}{r} \ \bigg| \ I^{(st)} \nsubseteq I^{rt} \text{ for all } t \gg 0 \right\}.
\]

The definitions and remarks above give \( \hat{\rho}(I) \leq \rho(I) \leq h \).

A related construction which packages the behavior of symbolic powers in a single object is the symbolic Rees algebra

\[
R_s(I) := \bigoplus_{s \geq 0} I^{(s)} t^s \subset R[t],
\]

2010 Mathematics Subject Classification. 13F20, 13A15, 14C20.

Key words and phrases. symbolic powers, containment problem, resurgence, asymptotic resurgence, symbolic Rees algebra.

The second author is partially supported by NSF grant DMS-1601024 and Epscor grant OIA-1557417.
where by convention we take $I^{(0)} = R$. Our main result in this note is the following.

**Theorem 1.1.** Suppose $I \subset R = K[x_1, \ldots, x_n]$ is an ideal whose symbolic Rees algebra is Noetherian. Then both the resurgence and asymptotic resurgence of $I$ are rational.

If resolved positively, a conjecture of Nagata \[15\] would yield the existence of many ideals with irrational resurgence. We say more on this in Section 3.

**Remark 1.2.** Fix an ideal $I \subset R$ and suppose that

$$\rho(I) = \max_{0 \leq s \leq M, 1 \leq r \leq N} \left\{ \frac{s}{r} \mid I^{(s)} \not\subset I^r \right\}$$

for some $M, N \in \mathbb{N}$.

Clearly if Equation (1) is satisfied then $\rho(I)$ is rational; however Equation (1) does not typically hold even if $I$ has Noetherian symbolic Rees algebra. For instance, if $I$ is a complete intersection then $I^{(s)} \not\subset I^r$ if and only if $s < r$, hence

$$\rho(I) = \sup \left\{ \frac{s}{r} \mid 0 \leq s < r \right\} = 1,$$

but clearly Equation (1) does not hold. More generally, it is shown in \[4\] that if $I$ is a normal ideal (that is, all of its powers are integrally closed), then $\rho(I) = \hat{\rho}(I)$ and $I^{(s)} \not\subset I^r$ if and only if $\frac{s}{r}$ is strictly less than $\rho(I)$.

The structure of this note is as follows. In Section 2 we prove Theorem 1.1. The proof progresses in two steps – we first show that the asymptotic resurgence of an ideal with Noetherian symbolic Rees algebra is rational. Then we prove in Proposition 2.6 that if the asymptotic resurgence is not equal to the resurgence, then Equation (1) in Remark 1.2 does hold. This latter result does not depend on the hypothesis of a Noetherian symbolic Rees algebra so may be of interest in a wider context than we address in this note. We conclude with several related remarks in Section 3.

### 2. Rationality of Resurgence and Asymptotic Resurgence

In this section we prove Theorem 1.1. The proof relies on a characterization of asymptotic resurgence using integral closures and valuations from \[4\]. We briefly recall the necessary background.

An element $f \in R$ is integral over an ideal $I$ if $x = f$ is a solution of an equation of the form $x^n + r_1 x^{n-1} + \cdots + r_0 = 0$, where $r_i \in I^i$ for $i = 0, \ldots, n$. The integral closure of $I$, written $\overline{I}$, is the set of all elements $f \in R$ which are integral over $I$. If $I = \overline{I}$ then we say $I$ is integrally closed.

If $K$ is a field, a discrete valuation on $K$ is a homomorphism $\nu : K^* \to \mathbb{Z}$ satisfying $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$. If $K$ is the fraction field of the polynomial ring $R = K[x_1, \ldots, x_n]$ then a valuation is determined by its values on $R$, so we abuse terminology by referring to valuations on $R$ instead of its field of fractions. We will be concerned only with valuations which are non-negative on $R$ – that is $\nu(f) \geq 0$ for all $f \in R$. In this case if $I$ is an ideal we write $\nu(I)$ for the minimum value which $\nu$ takes on $I$. We say $\nu$ is supported on $I$ if $\nu(I) \geq 1$.

In \[4\] it is shown that the limit

$$\hat{\nu}(I) := \lim_{s \to \infty} \frac{\nu(I^{(s)})}{s}$$

is an integer.
exists. These constants generalize the Waldschmidt constant $\alpha(I)$, where $\alpha$ is the valuation defined by $\alpha(f) = \text{deg}(f)$ and $\text{deg}(f)$ is the total degree of $f$. In [4] the following characterization of asymptotic resurgence is given. For the definition of Rees valuations, which appear below, see [16, Chapter 10].

**Theorem 2.1.** [4 Theorem 4.10] Let $I$ be an ideal and let $\nu_1, \ldots, \nu_r$ be the set of Rees valuations for $I$. Then

$$\hat{\rho}(I) = \max_i \left\{ \frac{\nu_i(I)}{\hat{v}_i(I)} \right\} = \sup_{\nu} \left\{ \frac{\nu(I)}{\hat{\nu}(I)} \right\},$$

where the maximum and supremum are taken over discrete valuations which are supported on $I$.

**2.1. Asymptotic resurgence.** To prove that the asymptotic resurgence is rational, we generalize a result on Waldschmidt constants [5, Theorem 3.6]. Let $\nu : R \to \mathbb{Z}$ be a valuation. For an ideal $I$, let $\nu(I)$ denote $\min\{\nu(x) \mid x \in I\}$. It follows from properties of valuations that, for any ideals $I$ and $J$, $\nu(I + J) = \min\{\nu(I), \nu(J)\}$ and $\nu(IJ) = \nu(I) + \nu(J)$.

**Proposition 2.2.** Let $I$ be a homogeneous ideal such that $R_s(I)$ is generated in degree at most $n$ and $\nu : R \to \mathbb{Z}$ a discrete valuation. For each $i \in \{1, \ldots, n\}$, let $\nu_i := \nu(I^{(i)})$. Then

$$\hat{\nu}(I) = \lim_{m \to \infty} \frac{\nu_1 y_1 + \nu_2 y_2 + \cdots + \nu_n y_n}{m} = \min_{m \leq n} \frac{\nu(I^{(m)})}{m}$$

where $y_1, \ldots, y_n$ in the first equality are positive integers minimizing $\nu_1 y_1 + \nu_2 y_2 + \cdots + \nu_n y_n$ with respect to the constraint $y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m$.

**Proof.** Since $R_s(I) = R[I, I^{(2)}]$, we have that

$$I^{(m)} = \sum_{y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m} y_1 (I^{(2)})^{y_2} (I^{(3)})^{y_3} \cdots (I^{(n)})^{y_n}.$$

Thus, by the properties mentioned above,

$$\nu(I^{(m)}) = \min \left\{ \nu\left( \sum_{y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m} y_1 (I^{(2)})^{y_2} (I^{(3)})^{y_3} \cdots (I^{(n)})^{y_n} \right) \mid y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m \right\}.$$

Setting $\nu_i = \nu(I^{(i)})$ gives

$$\nu(I^{(m)}) = \min \{ \nu_1 y_1 + \nu_2 y_2 + \cdots + \nu_n y_n \mid y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m \}.$$

Dividing by $m$ and taking $\lim_{m \to \infty}$ proves the first equality. For the second equality, consider the rational linear program

- minimize $\nu_1 y_1 + \nu_2 y_2 + \cdots + \nu_n y_n$
- subject to $y_i \geq 0$ for $i = 1, \ldots, n$ and $y_1 + 2y_2 + 3y_3 + \cdots + ny_n = m$.

These constraints define an $(n-1)$-simplex with vertices $(m, 0, \ldots, 0)$, $(0, m/2, 0, \ldots, 0)$, \ldots, $(0, \ldots, m/n)$; hence the minimum of the linear functional $\nu_1 y_1 + \nu_2 y_2 + \cdots + \nu_n y_n$ is attained at one of these vertices, say $(0, \ldots, m/k, \ldots, 0)$. It follows that the solution to the above linear program is $\frac{\nu_{min}}{k}$ for some $1 \leq k \leq n$. Whenever $k \mid \nu_{min}$ this solution will also be integral. As this happens for infinitely many $m$, dividing by $m$ and taking $\lim_{m \to \infty}$ proves the second equality. \hfill $\square$
Remark 2.3. If we only aim to prove the rationality of \( \hat{\rho}(I) \) we can use the well-known fact that \( R_s(I) \) is Noetherian if and only if there is an integer \( c \) so that \( I^{(cn)} = (I^{(c)})^n \) for all \( n \geq 1 \) (see for instance [10, Theorem 2.1]). From this it is straightforward to see that \( \hat{\rho}(I) = \lim_{k \to \infty} \frac{\nu(I^{(k+c)})}{k} = \frac{\nu(I^{(c)})}{c} \).

Proposition 2.2 offers a slight computational advantage, however, as the value of \( c \) guaranteed by the Noetherian property of \( R_s(I) \) is not always bounded above by the maximum generating degree of \( R_s(I) \). The following example illustrates this fact.

Example 2.4. The following example appears as [10, Example 5.5]. Suppose \( R = \mathbb{K}[x_1, \ldots, x_7] \) and \( I \) is the monomial ideal generated by the monomials below:

\[
\begin{align*}
&x_1x_2 \quad x_1x_3 \quad x_1x_4 \quad x_2x_3 \quad x_2x_4 \quad x_1x_5 \\
&x_2x_5 \quad x_3x_5 \quad x_1x_6 \quad x_2x_6 \quad x_3x_6 \quad x_4x_6 \\
&x_1x_7 \quad x_2x_7 \quad x_4x_7 \quad x_5x_7
\end{align*}
\]

Then \( R_s(I) \) is generated as an \( R \)-algebra by 52 monomials of degree at most 7 (see [10, Example 5.5]). However we can check in Macaulay2 [7] that \((I^{(i)})^2 \neq I^{(2i)}\) for any \( i = 1, \ldots, 7 \). Thus the \( c \) in Remark 2.3 is strictly larger than the generating degree of \( R_s(I) \).

The next corollary follows immediately from Proposition 2.2 and Theorem 2.1.

Corollary 2.5. Suppose \( I \subset R \) is an ideal with Noetherian symbolic Rees algebra generated in degree at most \( n \) and Rees valuations \( \nu_1, \ldots, \nu_r \). Then

\[
\hat{\rho}(I) = \max_{1 \leq i \leq r, 1 \leq j \leq n} \left\{ \frac{j\nu_i(I)}{\nu_j(I^{(j)})} \right\}.
\]

In particular, \( \hat{\rho}(I) \) is rational.

2.2. Resurgence. We now strengthen [4, Proposition 4.19], showing that \( \rho(I) \) satisfies Equation (11) if \( \hat{\rho}(I) < \rho(I) \). Consequently we will see that \( \rho(I) \) is rational if \( \hat{\rho}(I) < \rho(I) \). In this section we do not use the hypothesis that \( I \) has a Noetherian symbolic Rees algebra.

For an ideal \( I \subset R \), write \( b = b(I) \) for the minimum of the integers \( k \) satisfying \( I^{r+k} \subset I^r \) for all \( r \geq 1 \). The Briançon-Skoda theorem [10, Theorem 13.3.3] guarantees that \( b \leq n - 1 \), where \( n \) is the number of variables of \( S \).

Proposition 2.6. Suppose that \( I \subset R \) is an ideal and suppose \( \hat{\rho}(I) < \rho(I) \). Then there exist positive integers \( s_0, r_0 \) so that \( I^{(s_0)} \not\subset I^r \) and \( \frac{s_0}{r_0} > \hat{\rho}(I) \). Put

\[ N = \frac{b\hat{\rho}(I)}{s_0/r_0 - \hat{\rho}(I)}. \]

Then

\[
\rho(I) = \max_{1 \leq r < N, r \leq s < (r+b)\hat{\rho}(I)} \left\{ \frac{s}{r} \mid I^{(s)} \not\subset I^r \right\}.
\]

At worst, we may take \( b = (n-1) \) in the definition of \( N \) by the Briançon-Skoda theorem. Consequently if \( \hat{\rho}(I) < \rho(I) \) then \( \rho(I) \) is rational.

Proof. If \( \hat{\rho}(I) < \rho(I) \), it is clear that there must exist positive integers \( s_0, r_0 \) so that \( I^{(s_0)} \not\subset I^r \) and \( \frac{s_0}{r_0} > \hat{\rho}(I) \). Now set \( b = b(I) \) and suppose \( I^{(s)} \not\subset I^r \). Then, since
$I^{r+b}$ is the homogeneous ideal in $K[x, y, z]$ defining a set of $n = \binom{s+2}{2}$ distinct generic points. It follows from work of Nagata [15] that the symbolic Rees algebra of $I$ is not finitely generated, but Bocci and Harbourne compute that its resurgence is $\rho(I) = \frac{n+1}{k}$ [1] Corollary 1.3.1] (this also coincides with the asymptotic resurgence). The smallest example of this is the ideal of $36 = \binom{5}{2}$ general points in $\mathbb{P}^2$, which has resurgence of $\rho(I) = \frac{5}{6} = \frac{4}{3}$.

**Remark 3.2.** Bocci and Harbourne explain in [1] that a (still open) conjecture of Nagata would imply that the ideal $I$ of $n = \binom{s+2}{2}$ generic points in $\mathbb{P}^2$ has resurgence $\rho(I) = \frac{s+1}{\sqrt{n}}$. (As Remark 3.1 indicates, this is known if $n$ is square.)

**Remark 3.3.** In general it is quite difficult to determine if the symbolic Rees algebra of an ideal is Noetherian, but this is known for some interesting classes of ideals. In [12] Lyubeznik shows that $R_s(I)$ is Noetherian if $I$ is a squarefree monomial ideal. This is extended to arbitrary monomial ideals by Herzog, Hibi, and Trung in [10]. The Noetherian property of the symbolic Rees algebra of a monomial curve has also been extensively studied. An example of an ideal $I$ defining a monomial curve so that $R_s(I)$ is not Noetherian is provided in [14]. Cutkosky gives several criteria for monomial curves in [3]; perhaps the simplest of these is that if $P$ is the kernel of the homomorphism $k[x, y, z] \rightarrow k[t]$ with $x \rightarrow t^a, y \rightarrow t^b, z \rightarrow t^c$, then $R_s(P)$ is Noetherian if $(a+b+c)^2 > abc$. 

**Proof of Theorem 1.1** Let $I$ be an ideal of $R$ so that the symbolic Rees algebra $R_s(I)$ is Noetherian. Corollary 2.5 shows that if $I^{(s_0)} \not\subset I^r$ for some $s_0 > \tilde{\rho}(I)$, then Proposition 2.6 guarantees $\rho(I) = \tilde{\rho}(I)$. Hence we consider $\rho(I)$ if $\rho(I) = \tilde{\rho}(I)$, then we are done. Otherwise Proposition 2.6 guarantees $\rho(I) = \tilde{\rho}(I)$. This proves the proposition. □
Remark 3.4. Theorem 1.1 holds for more general Noetherian rings than polynomial rings. The criterion of [4] is stated for polynomial rings, but the proof holds if \( R \) is a normal Noetherian domain in which the symbolic and adic topologies of the ideal \( I \) are equivalent (and perhaps more generally). Certainly if \( R \) is regular and either local or standard graded then the criterion of [4] holds along with Theorem 1.1.

Remark 3.5. Following up from Remark 1.2 and Proposition 2.6, it seems natural to ask when Equation (1) holds. The following example, which comes from combinatorial optimization, shows that Equation (1) can hold even when \( \hat{\rho}(I) = \rho(I) = 1 \).

Consider the squarefree monomial ideal \( I = \langle abd, ace, bcf, def \rangle \) in the polynomial ring \( K[a, b, c, d, e, f] \). One can show, for instance using vertex cover algebras from [10], that \( I^{(r)} = I^r + (abcdef)I^{r-2} \) for \( r \geq 2 \). From this equality it is straightforward to show that \( I^{(r)} \neq I^r \) for \( r \geq 2 \) but \( I^{(r+1)} \subset I^r \) for \( r \geq 1 \). Thus \( \rho(I) = 1 \), and is attained as a maximum (indeed we can take \( M = N = 2 \) in Equation (1)). This example also provides a negative answer to [2, Question 5.2].

Remark 3.6. The methods of this paper may have implications for the stable Harbourne Conjecture (see [8]), which is an intriguing avenue for future research. We thank Alexandra Seceleanu and Eloisa Grifo for bringing this connection to our attention.

4. Acknowledgements

We thank Daniel Hernández for suggesting that \( \rho(I) \) might be a maximum of finitely many values if \( \hat{\rho}(I) < \rho(I) \). We also thank Alexandra Seceleanu, Eloisa Grifo, Vivek Mukundan, Adam Van Tuyl, Chris Francisco, Jay Schweig, Jeff Mermin, and Elena Guardo for insightful comments and helpful discussions.

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**Michael DiPasquale, Department of Mathematics, Colorado State University**

E-mail address: Michael.DiPasquale@colostate.edu

URL: https://midipasq.github.io/

**Ben Drabkin, Department of Mathematics, University of Nebraska-Lincoln**

E-mail address: benjamin.drabkin@huskers.unl.edu

URL: http://www.math.unl.edu/~bdrabkin2/