FRAY JUAN DE ORTEGA’S APPROXIMATIONS, 500 YEARS AFTER

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ABSTRACT. In 1512, on December 30th, the first edition of Fray Juan de Ortega’s Arithmetic was published in Lyon. The last chapter, titled “Rules of Geometry”, deals with lower approximations of 14 square roots. In later editions of the Arithmetic on 1534, 1537 and 1542 in Seville, these values are replaced by upper approximations. Twelve of them verify the Pell’s equation, and so they are optimal. At this moment nobody knows the way they were obtained. In this paper we show how these approximations can be obtained through a method consistent with the mathematical knowledge at that time.

Résumé. Fray Juan de Ortega publia pour la première fois son Arithmétique le 30 Décembre 1512. Dans son dernier chapitre intitulé “Règles de la géométrie” il y a des approximations par défaut de 14 racines carrées. Dans les éditions suivantes de Séville de 1534, 1537 et 1542, ces valeurs ont été remplacées par des approximations par excès. Douze d’entre elles sont optimales, c’est à dire vérifient l’équation de Pell. Même aujourd’hui, personne ne sait encore comment elles ont été obtenues. Cet article décrit une méthode pour obtenir ces approximations, compatible avec les connaissances mathématiques de l’époque.

Resumen. El 30 de diciembre de 1512, Fray Juan de Ortega publicó en Lyon la primera edición de su Aritmética. En el último capítulo, “Reglas de geometría”, aparecen aproximaciones por defecto de 14 raíces cuadradas. En las ediciones de Sevilla de 1534, 1537 y 1542, se sustituyen estos valores por aproximaciones por exceso. Doce de ellas son óptimas (verifican la ecuación de Pell). Hasta la fecha se desconoce como fueron obtenidas. En este artículo, se expone un método por el que se obtienen todas estas aproximaciones y es coherente con los conocimientos matemáticos de la época.

1. Introduction

The first edition of Fray Juan de Ortega’s Arithmetic[n] was published on December 30th 1512 in Lyon. The book contains a collection of elemental arithmetic rules, as operations with integers and rational numbers, square roots..., some notions about commercial calculus such as proportions or equivalences between Spanish coins of that time. The rules are written in a practical and didactic way, so “there will not be fraud in the world about computing”.

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1 He was born in Palencia about 1480. He belonged to the religious order of preachers and was sent to the province of Aragón.
He devoted to the teaching of Mathematics in Spain and Italy.
Figure 1. Front cover. 1512 edition.
| Capítulo | Título                                                                 | Hojas |
|----------|------------------------------------------------------------------------|-------|
| 1.       | de numerar todas finas                                               | 2     |
| 2.       | de sumar todas finas por entero                                       | 4     |
| 3.       | de restar todas finas por entero                                      | 8     |
| 4.       | de multiplicar por entero                                             | 17    |
| 5.       | de partir por entero                                                  | 19    |
| 6.       | de tratar de progresiones                                            | 23    |
| 7.       | de las raíces cuadradas y cúbicas                                    | 27    |
| 8.       | de probar cualquiera cuenta                                          | 34    |
| 9.       | de reducir todo óbice roto                                            | 44    |
| 10.      | de sumar por óbice roto                                              | 47    |
| 11.      | de restar por óbice roto                                              | 51    |
| 12.      | de multiplicar por óbice roto                                         | 55    |
| 13.      | de partir por óbice roto                                              | 57    |
| 14.      | de sumar por óbice roto extra ordinario                              | 60    |
| 15.      | de restar por óbice roto extra ordinario                             | 65    |
| 16.      | de multiplicar extra ordinario                                       | 67    |
| 17.      | de partir por extra ordinario                                        | 69    |
| 18.      | de los nöbese que tienen regla uno                                  | 70    |
| 19.      | de eliminaciones                                                      | 75    |
| 20.      | de regla de tres en tiempo por entero                                 | 76    |
| 21.      | de regla de tres sin tiempo por entero                                | 79    |
| 22.      | de regla de libras y crías                                           | 86    |
| 23.      | de regla de tres con tiempo                                          | 88    |
| 24.      | de regla de tres con tiempo con tiempo                               | 91    |
| 25.      | de calcular por partir y multiplicar                                  | 94    |
| 26.      | de calcular por regla de tres                                        | 97    |
| 27.      | de calcular y de ganar                                               | 101   |
| 28.      | de regla de quadrada                                                  | 106   |
| 29.      | de regla de cópias                                                    | 109   |
| 30.      | de regla de cópias extra                                             | 154   |
| 31.      | de regla de cestas                                                    | 133   |
| 32.      | de regla de argenterias                                              | 149   |
| 33.      | de regla de viages                                                    | 169   |
| 34.      | de regla de polícios                                                 | 171   |
| 35.      | de regla de dos posiciones                                           | 134   |
| 36.      | de geometría                                                          | 193   |

**Figure 2.** Table of Contents. 1512 edition.
The text achieved great success in Europe. In 1515 the work was published and translated into Italian in Roma and it was also translated into French by Claude Plantin and published in Lyon. After this edition some new ones editions: in Messina (1522), in Seville (1534, 1537, 1542, 1552), in Grenade (1563) and in Cambray (1612). However, we remember this book, 500 years later, because of the approximations of the square roots written on the geometric applications at the end. In the last chapter, “Rules of Geometry”, Ortega solved exercises of elementary geometry and had to find some square roots. Some of them, in our language, is about to find the length of a field with a circular shape and whose area is equivalent to another one with a square shape, or to find the edge of an equilateral triangle so that its area is equivalent to a given square. Both problems are written in the Appendix II.

In the first edition he approaches 14 lower square roots following a rule previously exposed. On the Seville editions in 1534, 1537 and 1542, those values were replaced, without any explanation, by upper approximations optimal in 12 cases out of 2.

In the edition of 1512, page 230, there are also two upper approximations \( \sqrt{1273} \approx 11 \frac{2}{7} \) and \( \sqrt{53} \approx 2 + \frac{1}{6} + \frac{1}{7} \), and they haven’t been changed in the following editions.

How did Ortega manage to obtain these values has been a mystery that has occupied the mind of many mathematicians and historians of science and has led to a lot of papers. Some of them are due to P. Tannery [14, 15, 16, 17], Cantor [4], Eneström [7], Perrot [8], J. Rey Pastor [9, 10, 11, 12, 13], and Barinaga [1, 2].

The method used by those authors to calculate the roots can not obtain some of the results presented by Ortega (bringing them to consider that or the author or the printer had a mistake).

Although the solutions obtained by Ortega in the three editions of Seville are optimal, they should be something very new to the point to change the values; when Gonzalo del Busto re-edit the work, he was forced to rectify the many mistakes he found in some previous impressions. Thus, the optimal approximations were replaced by those from the 1512 edition. In the following pages we will show all the roots and the approaches proposed by Ortega in different editions. And, according to the absence of explanations by the author, we suggest what could happen and we present a method to obtain all approaches consistent with the mathematical knowledge of the time.

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2The upper approximation \( \frac{x}{y} \) of \( \sqrt{n} \) is optimal if \( (\frac{x}{y})^2 - n = \frac{1}{y^2} \), these equations are known as Pell’s equations. A well known example of this equation is the problem of Archimedes’ cows. In the XII century, some similar equations were solved by Bascara.

3See Appendix II

4They also fail to discuss the upper approximations \( \sqrt{53} \approx 2 + \frac{1}{6} + \frac{1}{7} \) and \( \sqrt{1273} \approx 11 \frac{2}{7} \) which remain unmodified in all editions and that can be seen in Appendix II.
2. Relationship between the roots and the approaches proposed by Ortega

The roots that appear in the work are the following:

|   | Number | Editions ... | Remainder | Editions | Remainder |
|---|--------|--------------|-----------|----------|-----------|
| 1 | 128    | 11 1/2       | 11 1/2    | 11 1/2   | 11 1/2    |
| 2 | 80     | 8 3/4        | 8 3/4     | 8 3/4    | 8 3/4     |
| 3 | 297    | 297 1/2      | 17 23/24  | 17 23/24 | 17 23/24  |
| 4 | 300    | 300 1/2      | 17 23/24  | 17 23/24 | 17 23/24  |
| 5 | 375    | 375 1/2      | 19 1/2    | 19 1/2   | 19 1/2    |
| 6 | 135    | 135 1/2      | 11 23/24  | 11 23/24 | 11 23/24  |
| 7 | 75     | 75 1/2       | 8 1/2     | 8 1/2    | 8 1/2     |
| 8 | 756    | 756 1/2      | 27 1/2    | 27 1/2   | 27 1/2    |
| 9 | 611    | 611 1/2      | 24 1/2    | 24 1/2   | 24 1/2    |
| 10| 231    | 231 1/2      | 15 1/2    | 15 1/2   | 15 1/2    |
| 11| 800    | 800 1/2      | 28 1/2    | 28 1/2   | 28 1/2    |
| 12| 4100   | 4100 1/2     | 64 1/2    | 64 1/2   | 64 1/2    |
| 13| 2000   | 2000 1/2     | 44 1/2    | 44 1/2   | 44 1/2    |
| 14| 9600   | 9600 1/2     | 97 1/2    | 97 1/2   | 97 1/2    |
| 15| 127 1/11| 127 1/11    | 11 23/24  | 11 23/24 | 11 23/24  |
| 16| 5 1/2  | 2 + 1/2 + 1/2 | 2 + 1/2 + 1/2 | 2 + 1/2 + 1/2 | 2 + 1/2 + 1/2 |

In column I, the approximations of the first edition of 1512, those of 1515 and the approximations modified in editions of 1552 and 1563 are written.

All first 14 values are lower approximations. They have been obtained by applying the usual algorithm for square roots described in chapter 7 of the book titled “About square and cubic roots”. In our current language:

\[ a + \frac{r}{2a + 1} \leq \sqrt{n} \]

Where \( a = \lfloor \sqrt{n} \rfloor \) (integer part) and \( r = n - a^2 \), the remainder.

Thus \( \sqrt{128} = 11 + \frac{128 - 121}{22 + 1} = 11 + \frac{7}{23} \).

The last two values in column I are upper approximations and the way they were obtained is unknown. Both remain unchanged in later editions.

In column II all values that appeared in Seville’s editions in 1534, 1537 and 1542 are written. They are upper approximations and all of them are optimal except those that are written in rows 13, 14 and 15. In those cases we also indicate how the approximations can be done.

3. Our hypothesis about the method applied by Ortega

Rey Pastor in [10] page 80, point out the possibility that Ortega was inspired on Nicolas Chuquet’ *Triparty*, or on any book of Arabic origin, even on the Paciolo’ *Summa*.

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5We have not located any copy of the edition of 1612. But, at that time the author was died, so it is very reasonable to assume that this issue is a reprint of earlier editions.
Our hypothesis is that Ortega could obtained his approaches using a special method of “la regle des nombres mohines” (“mediation”\textsuperscript{6}, some kind of mean) written in the manuscript *Triparty en la sciencia des nombres*, from Chuquet \textsuperscript{5}. This book was finished in 1484\textsuperscript{7} in Lyon, the city where Ortega published the first edition of his work in 1512.

The text contains the approximations of the square roots of the first natural numbers. Chuquet began with the integer part of the lower and upper square root and apply in a systematical way his rule. Here the approximations of square roots of 14 first natural numbers are computed\textsuperscript{8}.

For $\sqrt{6}$ he obtains the following approximations:

\begin{align*}
2 + \frac{2}{3} &< \sqrt{6} < 2 + \frac{1}{1}
\end{align*}

\begin{align*}
2 + \frac{1}{3} &< \sqrt{6} < 2 + \frac{1}{2}
\end{align*}

\begin{align*}
&\cdots
\end{align*}

| Number | Approximation of the square root | Upper error |
|--------|---------------------------------|-------------|
| 2      | $1 + \frac{1}{100}$            | 100464      |
| 3      | $1 + \frac{1}{700}$            | 609400      |
| 5      | $2 + \frac{1}{881}$            | 305121      |
| 6      | $2 + \frac{1}{1960}$           | 3841605     |
| 7      | $2 + \frac{1}{7873}$           | 118543364   |
| 8      | $2 + \frac{1}{985}$            | 1413721     |
| 10     | $3 + \frac{1}{787}$            | 74995864    |
| 10     | $3 + \frac{1}{881}$            | 1974025     |
| 11     | $3 + \frac{1}{1155}$           | 1432809     |
| 12     | $3 + \frac{1}{161}$            | 1432809     |
| 13     | $3 + \frac{1}{208}$            | 32400       |
| 14     | $3 + \frac{1}{2606}$           | 12931216    |

By applying the Chuquet’s method to 16 values in column II, we reach Ortega’s solution in 14 cases. However by this way we can’t obtain the solution for $\sqrt{9600}$ and, for $\sqrt{2000}$, the value obtained is $44\frac{189}{262}$, which is the result of simplifying by 11 the Ortega’s solution $44\frac{2079}{2882}$.

We suppose that Ortega could use the mediation rule taking as lower initial value the value given in the 1512 edition and the upper value which is obtained by increasing the denominator in one unit\textsuperscript{9} as we can see below:

In our language:

\begin{equation}
a + \frac{r}{2a + 1} \leq \sqrt{n} \leq a + \frac{r}{2a}.
\end{equation}

\textsuperscript{6}A mediation of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, $a, b, c, d > 0$, is the fraction $\frac{a+c}{b+d}$ between both fractions.
\textsuperscript{7}However the text was not printed until 1880 by Aristide Marre. See Chuquet \textsuperscript{5}.
\textsuperscript{8}This method provides solutions that verify the Pell’s equation.
\textsuperscript{9}Nowadays the inequality is well known, however the second part of the inequality (Heron’s formula) was not known until the XIX century. See \textsuperscript{14}. 
As above, \( a = \lfloor \sqrt{n} \rfloor \) and, \( r = n - a^2 \) the lower remainder. Thus for the first value of \( \sqrt{128} \) the following approximations are obtained:

\[
11 + \frac{7}{23} < \sqrt{128} < 11 + \frac{7}{22} \\
11 + \frac{14}{45} < \sqrt{128} < 11 + \frac{7}{22} \\
11 + \frac{21}{67} < \sqrt{128} < 11 + \frac{7}{22} \\
\cdots
\]

In the following tables we show the results obtained by this way for the 14 first values of column II. The red values are the solutions given by Ortega, even though we have continued the process to find the first optimal solution.

As we have seen, the method provides all of Ortega’s solutions, even the approximation of \( \sqrt{2000} \) which again appears simplified. Later we will see another way to approach \( \sqrt{2000} \) as Ortega did (without simplifying).

The approximations written in the two last rows \( \sqrt{127 \frac{3}{11}} \) and \( \sqrt{5 \frac{1}{3}} \) have three singular things in relation to those given above.

Those are upper approximation, remain unchanged in all editions and are not integers.

As the edition of 1512 does not gave a lower value to start the process described above, we have started from the lower and upper integer roots as Chuquet did.

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 11 + \( \frac{7}{23} \) | 11 + \( \frac{7}{22} \) | \frac{49}{484} |
| 11 + \( \frac{14}{45} \) | 11 + \( \frac{7}{22} \) | \frac{49}{484} |
| 11 + \( \frac{21}{67} \) | 11 + \( \frac{7}{22} \) | \frac{49}{484} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 8 + \( \frac{16}{11} \) | 9 + \( \frac{1}{1} \) | \frac{1}{1} |
| 8 + \( \frac{16}{11} \) | 8 + \( \frac{17}{18} \) | \frac{1}{1} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |

| Lower approximation | Upper approximation | Upper error |
|---------------------|--------------------|-------------|
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
| 17 + \( \frac{3}{19} \) | 17 + \( \frac{7}{19} \) | \frac{49}{784} |
|   | Lower approximation | Upper approximation | Upper error |
|---|---------------------|---------------------|-------------|
| 1 | $17 + \frac{11}{33}$ | $17 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 2 | $17 + \frac{11}{33}$ | $17 + \frac{21}{33}$ | $\frac{11}{12}$ |
| 3 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 4 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 5 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 6 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 7 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 8 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 9 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 10 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |
| 11 | $17 + \frac{11}{33}$ | $17 + \frac{15}{27}$ | $\frac{11}{12}$ |

|   | Lower approximation | Upper approximation | Upper error |
|---|---------------------|---------------------|-------------|
| 1 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 2 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 3 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 4 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 5 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 6 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 7 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 8 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 9 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 10 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 11 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 12 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 13 | $19 + \frac{11}{33}$ | $19 + \frac{11}{27}$ | $\frac{11}{12}$ |

|   | Lower approximation | Upper approximation | Upper error |
|---|---------------------|---------------------|-------------|
| 1 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 2 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 3 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 4 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 5 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
| 6 | $11 + \frac{11}{33}$ | $11 + \frac{11}{27}$ | $\frac{11}{12}$ |
### Lower approximation

| 75 | Lower approximation | Upper approximation | Upper error |
|----|---------------------|---------------------|-------------|
| 1  | $8 + \frac{1}{11}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 2  | $8 + \frac{1}{13}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 3  | $8 + \frac{1}{15}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 4  | $8 + \frac{1}{17}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 5  | $8 + \frac{1}{19}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 6  | $8 + \frac{1}{21}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 7  | $8 + \frac{1}{23}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 8  | $8 + \frac{1}{25}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 9  | $8 + \frac{1}{27}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 10 | $8 + \frac{1}{29}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 11 | $8 + \frac{1}{31}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 12 | $8 + \frac{1}{33}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |
| 13 | $8 + \frac{1}{35}$  | $8 + \frac{1}{23}$  | $\frac{2}{23}$ |
| 14 | $8 + \frac{1}{37}$  | $8 + \frac{1}{37}$  | $\frac{2}{37}$ |
| 15 | $8 + \frac{1}{39}$  | $8 + \frac{1}{23}$  | $\frac{2}{23}$ |
| 16 | $8 + \frac{1}{41}$  | $8 + \frac{1}{15}$  | $\frac{2}{15}$ |

### Upper approximation

| 756 | Lower approximation | Upper approximation | Upper error |
|-----|---------------------|---------------------|-------------|
| 1   | $27 + \frac{1}{11}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 2   | $27 + \frac{1}{13}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 3   | $27 + \frac{1}{15}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 4   | $27 + \frac{1}{17}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 5   | $27 + \frac{1}{19}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 6   | $27 + \frac{1}{21}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 7   | $27 + \frac{1}{23}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 8   | $27 + \frac{1}{25}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 9   | $27 + \frac{1}{27}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 10  | $27 + \frac{1}{29}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 11  | $27 + \frac{1}{31}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 12  | $27 + \frac{1}{33}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 13  | $27 + \frac{1}{35}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 14  | $27 + \frac{1}{37}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 15  | $27 + \frac{1}{39}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 16  | $27 + \frac{1}{41}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 17  | $27 + \frac{1}{41}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 18  | $27 + \frac{1}{43}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 19  | $27 + \frac{1}{45}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 20  | $27 + \frac{1}{47}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 21  | $27 + \frac{1}{49}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 22  | $27 + \frac{1}{51}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 23  | $27 + \frac{1}{53}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 24  | $27 + \frac{1}{55}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 25  | $27 + \frac{1}{57}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 26  | $27 + \frac{1}{59}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 27  | $27 + \frac{1}{61}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 28  | $27 + \frac{1}{63}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 29  | $27 + \frac{1}{65}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
| 30  | $27 + \frac{1}{67}$ | $27 + \frac{1}{15}$ | $\frac{2}{15}$ |
|   | Lower approximation | Upper approximation | Upper error |
|---|---------------------|---------------------|-------------|
| 1 | 24 + $\frac{1}{7}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 2 | 24 + $\frac{1}{4}$  | 24 + $\frac{1}{2}$  | 24 + $\frac{1}{2}$  |
| 3 | 24 + $\frac{1}{3}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 4 | 24 + $\frac{1}{2}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 5 | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 6 | 24 + $\frac{1}{2}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 7 | 24 + $\frac{1}{3}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 8 | 24 + $\frac{1}{4}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 9 | 24 + $\frac{1}{5}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 10| 24 + $\frac{1}{6}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 11| 24 + $\frac{1}{7}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 12| 24 + $\frac{1}{8}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 13| 24 + $\frac{1}{9}$  | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 14| 24 + $\frac{1}{10}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 15| 24 + $\frac{1}{11}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 16| 24 + $\frac{1}{12}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 17| 24 + $\frac{1}{13}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 18| 24 + $\frac{1}{14}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |
| 19| 24 + $\frac{1}{15}$ | 24 + $\frac{1}{1}$  | 24 + $\frac{1}{1}$  |

|   | Lower approximation | Upper approximation | Upper error |
|---|---------------------|---------------------|-------------|
| 1 | 15 + $\frac{1}{7}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 2 | 15 + $\frac{1}{6}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 3 | 15 + $\frac{1}{5}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 4 | 15 + $\frac{1}{4}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 5 | 15 + $\frac{1}{3}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 6 | 15 + $\frac{1}{2}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 7 | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 8 | 15 + $\frac{1}{8}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 9 | 15 + $\frac{1}{9}$  | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 10| 15 + $\frac{1}{10}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 11| 15 + $\frac{1}{11}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 12| 15 + $\frac{1}{12}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 13| 15 + $\frac{1}{13}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 14| 15 + $\frac{1}{14}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 15| 15 + $\frac{1}{15}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 16| 15 + $\frac{1}{16}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 17| 15 + $\frac{1}{17}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 18| 15 + $\frac{1}{18}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 19| 15 + $\frac{1}{19}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 20| 15 + $\frac{1}{20}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 21| 15 + $\frac{1}{21}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 22| 15 + $\frac{1}{22}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 23| 15 + $\frac{1}{23}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 24| 15 + $\frac{1}{24}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 25| 15 + $\frac{1}{25}$ | 15 + $\frac{1}{1}$  | 15 + $\frac{1}{1}$  |
| 231 | Lower approximation | Upper approximation | Upper error |
|-----|---------------------|---------------------|-------------|
| 26  | 15 + 31            | 15 + 31            | 20          |
| 27  | 15 + 31            | 15 + 31            | 20          |
| 28  | 15 + 31            | 15 + 31            | 20          |
| 29  | 15 + 31            | 15 + 31            | 20          |
| 30  | 15 + 31            | 15 + 31            | 20          |

| 800 | Lower approximation | Upper approximation | Upper error |
|-----|---------------------|---------------------|-------------|
| 1   | 28 + 28            | 28 + 28            | 1           |
| 2   | 28 + 28            | 28 + 28            | 1           |
| 3   | 28 + 28            | 28 + 28            | 1           |
| 4   | 28 + 28            | 28 + 28            | 1           |
| 5   | 28 + 28            | 28 + 28            | 1           |
| 6   | 28 + 28            | 28 + 28            | 1           |
| 7   | 28 + 28            | 28 + 28            | 1           |
| 8   | 28 + 28            | 28 + 28            | 1           |
| 9   | 28 + 28            | 28 + 28            | 1           |
| 10  | 28 + 28            | 28 + 28            | 1           |
| 11  | 28 + 28            | 28 + 28            | 1           |
| 12  | 28 + 28            | 28 + 28            | 1           |
| 13  | 28 + 28            | 28 + 28            | 1           |
| 14  | 28 + 28            | 28 + 28            | 1           |
| 15  | 28 + 28            | 28 + 28            | 1           |
| 16  | 28 + 28            | 28 + 28            | 1           |

| 4100 | Lower approximation | Upper approximation | Upper error |
|------|---------------------|---------------------|-------------|
| 1    | 64 + 4             | 64 + 4             | 1/4         |

| 2000 | Lower approximation | Upper approximation | Upper error |
|------|---------------------|---------------------|-------------|
| 1    | 44 + 41            | 44 + 41            | 1/1000      |
| 2    | 44 + 41            | 44 + 41            | 1/1000      |
| 3    | 44 + 41            | 44 + 41            | 1/1000      |
| 4    | 44 + 41            | 44 + 41            | 1/1000      |
| 5    | 44 + 41            | 44 + 41            | 1/1000      |
| 6    | 44 + 41            | 44 + 41            | 1/1000      |
| 7    | 44 + 41            | 44 + 41            | 1/1000      |
| 8    | 44 + 41            | 44 + 41            | 1/1000      |
| 9    | 44 + 41            | 44 + 41            | 1/1000      |
| 10   | 44 + 41            | 44 + 41            | 1/1000      |
| 11   | 44 + 41            | 44 + 41            | 1/1000      |
| 12   | 44 + 41            | 44 + 41            | 1/1000      |
| 13   | 44 + 41            | 44 + 41            | 1/1000      |
| 14   | 44 + 41            | 44 + 41            | 1/1000      |
| 15   | 44 + 41            | 44 + 41            | 1/1000      |
| 16   | 44 + 41            | 44 + 41            | 1/1000      |
| 17   | 44 + 41            | 44 + 41            | 1/1000      |
| 18   | 44 + 41            | 44 + 41            | 1/1000      |
| 19   | 44 + 41            | 44 + 41            | 1/1000      |
| 20   | 44 + 41            | 44 + 41            | 1/1000      |
In the ninth iteration appears \( \frac{44}{11} + \frac{189}{262} \), which is the result after simplify by 11 the approximation \( \frac{2079}{2882} \). The first upper optimal approximation does not appear until the 20th iteration.

| 9600 | Lower approximation | Upper approximation | Upper error |
|------|---------------------|---------------------|-------------|
| 1    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 36381       |
| 2    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 70623       |
| 3    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 103331      |
| 4    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 137529      |
| 5    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 167886      |
| 6    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198829      |
| 7    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 8    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 9    | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 10   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 11   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 12   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 13   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 14   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 15   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 16   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 17   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 18   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 19   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 20   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 21   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 22   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 23   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 24   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 25   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 26   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 27   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 28   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 29   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 30   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 31   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 32   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 33   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 34   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 35   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 36   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 37   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 38   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 39   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
| 40   | 97 + \( \frac{97}{19} \) | 97 + \( \frac{97}{19} \) | 198509      |
The approximation $97 \frac{191}{194}$ appears in the first iteration and the first upper optimal approximation appears in the iteration number 48.
Approximation $11\frac{3}{11}$ appears in the fifth iteration and the first upper approximation is in the iteration number 46.

| $11\frac{3}{11}$ | Lower approximation | Upper approximation | Upper error |
|------------------|---------------------|---------------------|-------------|
| 21               | $\pm 1199$          | $\pm 252$           | $\pm 51$    |
| 22               | $\pm 1199$          | $\pm 1888$          | $\pm 219$   |
| 23               | $\pm 778$           | $\pm 547$           | $\pm 205$   |
| 24               | $\pm 778$           | $\pm 435$           | $\pm 341$   |
| 25               | $\pm 778$           | $\pm 396$           | $\pm 341$   |
| 26               | $\pm 778$           | $\pm 305$           | $\pm 261$   |
| 27               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 28               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 29               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 30               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 31               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 32               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 33               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 34               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 35               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 36               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 37               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 38               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 39               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 40               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 41               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 42               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 43               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 44               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 45               | $\pm 778$           | $\pm 259$           | $\pm 259$   |
| 46               | $\pm 778$           | $\pm 259$           | $\pm 259$   |

Approximation $5\frac{4}{3}$ appears in the first iteration and the first upper approximation is in the iteration number 10.
To reach $44\cfrac{2079}{2882}$ as approximation of $\sqrt{2000}$ we can proceed as follows:

The Chuquet’s approximation of $\sqrt{5}$ is $15\cfrac{682}{1000}$, We can write so that $2000 = 10000 \cdot 5$, and using the above approximation of $\sqrt{5}$, we get $44 + \cfrac{44}{61} = 44 + \cfrac{44}{61}$, which is a lower approximation.

$44 + \cfrac{44}{61} = 44 + \cfrac{11}{15}$ is an upper approximation. From $44 + \cfrac{44}{61}$ and $44 + \cfrac{11}{15}$, using the rule above exposed and simplifying by 2, we can obtain $44\cfrac{2079}{2882}$ in the 48th iteration. The first upper optimal approximation appears in the iteration number 81.
| Lower approximation | Upper approximation | Upper error |
|---------------------|---------------------|-------------|
| 36                  | 44 + 24/1           | 44 + 12621  |
| 37                  | 44 + 44/1           | 44 + 1499   |
| 38                  | 44 + 64/1           | 44 + 1733   |
| 39                  | 44 + 84/1           | 44 + 1968   |
| 40                  | 44 + 104/1          | 44 + 2202   |
| 41                  | 44 + 124/1          | 44 + 2437   |
| 42                  | 44 + 144/1          | 44 + 2672   |
| 43                  | 44 + 164/1          | 44 + 2907   |
| 44                  | 44 + 184/1          | 44 + 3142   |
| 45                  | 44 + 204/1          | 44 + 3377   |
| 46                  | 44 + 224/1          | 44 + 3612   |
| 47                  | 44 + 244/1          | 44 + 3847   |
| 48                  | 44 + 264/1          | 44 + 4082   |
| 49                  | 44 + 284/1          | 44 + 4317   |
| 50                  | 44 + 304/1          | 44 + 4552   |
| 51                  | 44 + 324/1          | 44 + 4787   |
| 52                  | 44 + 344/1          | 44 + 5022   |
| 53                  | 44 + 364/1          | 44 + 5257   |
| 54                  | 44 + 384/1          | 44 + 5492   |
| 55                  | 44 + 404/1          | 44 + 5727   |
| 56                  | 44 + 424/1          | 44 + 5962   |
| 57                  | 44 + 444/1          | 44 + 6197   |
| 58                  | 44 + 464/1          | 44 + 6432   |
| 59                  | 44 + 484/1          | 44 + 6667   |
| 60                  | 44 + 504/1          | 44 + 6902   |
| 61                  | 44 + 524/1          | 44 + 7137   |
| 62                  | 44 + 544/1          | 44 + 7372   |
| 63                  | 44 + 564/1          | 44 + 7607   |
| 64                  | 44 + 584/1          | 44 + 7842   |
| 65                  | 44 + 604/1          | 44 + 8077   |
| 66                  | 44 + 624/1          | 44 + 8312   |
| 67                  | 44 + 644/1          | 44 + 8547   |
| 68                  | 44 + 664/1          | 44 + 8782   |
| 69                  | 44 + 684/1          | 44 + 9017   |
| 70                  | 44 + 704/1          | 44 + 9252   |
| 71                  | 44 + 724/1          | 44 + 9487   |
| 72                  | 44 + 744/1          | 44 + 9722   |
| 73                  | 44 + 764/1          | 44 + 9957   |
| 74                  | 44 + 784/1          | 44 + 10192  |
| 75                  | 44 + 804/1          | 44 + 10427  |
| 76                  | 44 + 824/1          | 44 + 10662  |
| 77                  | 44 + 844/1          | 44 + 10897  |
| 78                  | 44 + 864/1          | 44 + 11132  |
| 79                  | 44 + 884/1          | 44 + 11367  |
| 80                  | 44 + 904/1          | 44 + 11602  |
| 81                  | 44 + 924/1          | 44 + 11837  |
4. Conclusions

It is clear that with this method we can obtain all approximations. They are consistent with the mathematical knowledge at that time and may be with the Ortega’s knowledge, because the book was printed in Lyon, the same place where Chuquet was living and the same city in which he published his text “Triparty” where we can find the “regle des nombres mohines” for computing approximations of square roots.

Nevertheless, computing $\sqrt{2000}$ (see appendix III) is too long and written $\sqrt{5\frac{1}{3}}$ as $2 + \frac{1}{6} + \frac{1}{7}$ instead of $2\frac{13}{17}$, may suggest some doubt about this hypothesis. In any case, consistency and accuracy of the results leads us to refuse that Ortega or the publisher had made a mistake in the approximations as some authors maintain.

5. Appendix I EDITIONS OF THE WORK.

• 1512. Lyon.
  “Siguese una composicion de la arte de la aritmetica y Juntamente de geometria: fecha y ordenada por fray Juan de ortega de la orden de santo domingo: de los predicadores.”
  Imprimido a Leon : en casa de maistro Nicolau de Benedictis : por Joannes trinxer librero de barcelona
  Reference: Fondo Histórico de la Universidad de Salamanca
  http://brumario.usal.es/
  Free access digitized copy:
  http://gredos.usal.es/jspui/handle/10366/83271

• 1515. Lyon.
  Oeuvre tres subtile et profitable de l’art de science de aristm´ eticque et ´ eom´ etrie, translat´ e nouvellement d’espaignol en fran¸ coys [de fr` ere Jehan de Lortie, de l’ordre Saint Dominique]... Ayez ce livre, n’y faillez nullement ; Symon Vincent si vous en fournira, en rue Merciere o` u il est demourant....
  “A la fin” : Imprim´ e a Lyon, par maistre Estienne Baland, l’an mil cinq cens et quinze, le XXIII. jour de octobre
  Traduit par fr` ere Claude Platin, humble religieux de l’ordre de Saint Anthoine en Viennoys
  Reference: Bibliothèque nationale de France
  http://catalogue.bnf.fr/ark:/12148/cb31041178s/PUBLIC

• 1515. Roma.
  Suma de arithmetica, geometria pratica, utilissima, ordinata per Johane de Ortega, Spagnolo Palentino.
  Impresso in Roma: per Maestro Stephano Guillieri de Lorena, anno del nostro Signor 1515 adì 10 de Noue[m]bre regnante Leone Papa decimo in suo anno tertio.
  Although this work has a Latin title, it is actually the author’s Italian translation and adaptation of his Spanish original.
  References:
  – http://ccuc.cbuc.cat/
  – http://catalogue.bnf.fr/ark:/12148/cb310411794/PUBLIC
  – http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=1231388
– http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=6437294
– http://galenet.galegroup.com/servlet/MOME?af=RN&ae=U106932055&srchtp=akste=14&locID=konink

• 1522. Mesina.
  [Sequitur la quarta opera de arithmetica & geometria / facta et ordinata per Johanne de Ortega ...].
  Stampata in la nobili citati di Misina [Messina] : Per Giorgi & Petrucio Spera patri & figlio
  Misinisi, lanno dela Incarnatione del Signore. M.D. XX. II. adi. xxiii.
  Di Décem[bro]. (1522)
  Reference: Columbia University.
  http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=6189299

• 1534. Sevilla.
  Tratado subtilissimo de Arismetica y de Geometria cópuesto y ordenado por el reuerendo padre fray Juan de Ortega de la orden de los predicadores
  En ... Sevilla en casa de Juã Cröberger
  Reference: Biblioteca Nacional de España. (BNE).
  Free access digitized copy:
  http://bibliotecadigitalhispanica.bne.es/view/action/singleViewer.do?dvs=1352480892777~646&locale=es_ES&VIEWER_URL=/view/action/singleViewer.do?&DELIVERY_RULE_ID=10&frameId=1&usePid1=true&usePid2=true

• 1537. Sevilla.
  Tratado subtilissimo de arismetica y de geometria cópuesto y ordenado por el reuer endo padre fray Juã de Ortega de la orden de los predicadores.
  Agora nueuam ete corregido y emendado En ... Sevilla en casa de Juã Cröberger
  Reference: Biblioteca Nacional de España. (BNE).
  http://bibliotecadigitalhispanica.bne.es:80/webclient/DeliveryManager?pid=2688375&custom_att_2=simple_viewer

• 1542. Sevilla.
  Tratado subtilissimo de arismetica y de geometria cópuesto y ordenado por el reuerendo padre fray Juã de Ortega de la orden de los predicadores.
  Fue impresso el presente libro ... agora nueumète corregido y emendado en casa d Jacom Cröberger en la muy noble y muy leal ciudad de Seuilla, 1542.
  – http://cisne.sim.ucm.es/record=b2338782*spl
  – http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=1232813
  – http://books.google.com/books/ucm?vid=UCM5322482689&printsec=frontcover
  – http://ccuc.cbuc.cat/

• 1552. Sevilla.
  Tractado subtilissimo d’arismetica y de geometria, compuesto por el reuer edo padre fray Juan de Hortega de la orden de los predicadores.
  Ahora de nuevo enmendado ... por Gonçalo Busto.
Fue impresso è la muy noble muy leal ciudad de Seuilla, por Juà canalla... Acabose... año de nuestro criador y redéptor Jesu Christo de mill quinientos cinquenta y dos años... 1552.

- http://ccuc.cbuc.cat/
- Biblioteca Nacional de España (BNE).
- http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=1231391
- Biblioteca Nacional de Portugal (BNP).

- 1563. Granada

Tractado subtilissimo ã arismetica y geometria cópuesto por el reuerédo padre fray Juà de Hortega; agora de nueuo emendado ã mucha diligécia por Juan Lagarto y antes por Gonçalo Busto de muchos errores ã auia en algunas impressiones passadas; van annadidas en esta impression las prueuas desde reduzir hasta partir quebrados, y en las mas de las figuras de geometria sus prueuas, con ciertos auisos subjectos al algebra. Va añadido en esta postrera impresión vn Tractado del bachiller Juà Perez de Moya: trata reglas para cótar sin pluma y de reduzir vnas monedas castellanas en otras

Fue impresso en la muy noble, nóbrrada grá ciudad de Granada: en casa de Rene Rabut, impressor de libros, junto alos hospitales del Corpus Christi: a costa de Juà dias mercader de libros, 1563, en ocho dias del mes de abril

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  http://brumario.usal.es/
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  http://clio.cul.columbia.edu:7018/vwebv/holdingsInfo?bibId=1231391

- 1612. Cambray.
  Cited by [18].
6. **Appendix II Problems in which appear** $\sqrt{\frac{127}{11}}$ y $\sqrt{\frac{51}{3}}$.

In the 1512 edition there are two upper approximations that remain unchanged in the following editions. The approximations are: $\sqrt{\frac{127}{11}} \approx 11\frac{2}{7}$ and $\sqrt{\frac{51}{3}} \approx 2 + \frac{1}{6} + \frac{1}{7}$, we can find them in page 230:

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Un hóbre tiene vna torre quadrada la qual tiene por cada vn cadrágulo.10.canas este hóbre quiere trocar esta tierra quadrada a otra tierra redóda: demando que quàtas canas terna por circuito la tal tierra redóda: faras ansi multiplica por si

---
las.10 canas que tiene la tierra 4drada por cada cadr~gulo y m~tar~a.100.y tantas
canas diras q tiene la tierra quadrada:pues busca vn nombre que qando le sus tres
catorzenes qued~a.100:el qual hallaras enesta manera como por vna falsa posicion
que buscaras vn nombre que quitandole su septima parte la tal pte sea.10.el qual
n~bre hallaras que s~o.70.pues toma la septima parte que son.10.y despues toma la
mitad destos.10.que s~o.5.y ponlos c~olos mesmos.10.y ser~a.15.y estos.15.son los \( \frac{3}{14} \)
porq \( \frac{3}{14} \) son vn setabo y medio:pues quita estos.15.delos.70.y qued~ra.55.despues
di por regla de.3si.55.son restados de.70.de qui~ estaran.100.multiplica y parte
como te he ense~nado por regla de.3.y hallaras q restaran de.127 \( \frac{3}{14} \) y este es el
n~bre que quit~doles vn setabo y medio:o.3.catorzenes q todo es vno restar~a.100.
Pues quita la raiz quadrada q son.11 \( \frac{2}{7} \) a causa del roto y t~tas canas terna el di-
ametro. Pues multiplica estos.11 \( \frac{2}{7} \) por.3 \( \frac{1}{7} \) y verna ala multiplicaci~on.35.canas y \( \frac{22}{39} \)
de cana:y t~tas canas terna la tal tierra red~da por circuito:y ansi diras que tanbien
terna la tal tierra red~da.100.canas. Si lo quieres ver toma la mitad delas.11.canas
y \( \frac{2}{7} \) de cana que tiene la tierra por diametro que son.5.canas y \( \frac{9}{14} \) de caray multi-
plica c~ellos.17 \( \frac{20}{49} \) que es la mitad delas.35.canas y \( \frac{22}{39} \) de cana que tiene por circuito:y
hallaras q montan.100.canas.

Un h~bre tiene vna tierra quadrada que tiene por cada quadr~gulo.10.la q
tierra tiene.100.canas: este h~bre quiere trocar esta tierra a otra que esta fecha
en triangulo:dem~do q quantas canas terna la tal
tierra. Faras ansi:multiplica las.10.canas que tiene cada quadrangulo por si y montar~a.100.los cuales dobla los y montaran.200.despues
to ima el \( \frac{1}{3} \) delos.100.canas y montaran.50. Si escases de poquita
cosas ponlos los.200.canas y montaran.250. y si restaran la
raiz quadrada que qes.15. y tantas cantas diras que tiene cada las 3
del triangulo como veis figurado.
To compute the square of the edge of an equilateral triangle as a function of
his area he have multiply the area by \( \frac{4}{\sqrt{3}} = \sqrt{\frac{51}{3}} \), and to do that he used the
approximation \( 2 + \frac{1}{6} + \frac{1}{7} \) in all editions.

7. APPENDIX III. THE APPROXIMATION OF \( \sqrt{2000} \)

We can use the result:

\[
44 + \frac{44}{61} < \sqrt{2000} < 44 + \frac{11}{15},
\]

hence, the “mediation” between

\[
44 + \frac{44}{61} \text{ and } 44 + \frac{11}{15}
\]

is \( 44 + \frac{55}{76} \).

To check if \( 44 + \frac{55}{76} \) is a lower or upper approximation, we can see if \( (44 + \frac{55}{76})^2 - 2000 \)
is less or greater than 0, and we do it in the following way

\[
\left(44 + \frac{55}{76}\right)^2 - 2000 = (44 + \frac{44}{61} + \frac{11}{61 \cdot 76})^2 - 2000 = \frac{-16}{61^2} + \frac{5456 \cdot 11}{61^2 \cdot 76} + \frac{11^2}{61^2} \cdot \frac{1}{76^2} > 0
\]

since \( \frac{5456 \cdot 11}{61^2 \cdot 76} > 16 \).

Since \( \frac{5456 \cdot 11}{61^2 \cdot 76} > 16 \) and the “mediation” between \( 44 + \frac{44}{61} \) and \( 44 + \frac{55}{76} \) is \( 44 + \frac{99}{137} \).

\[
\frac{99}{137} - \frac{44}{61} = \frac{11}{61 \cdot 137}
\]

\[
(44 + \frac{99}{137})^2 - 2000 = (44 + \frac{44}{61} + \frac{11}{61 \cdot 137})^2 - 2000 = \frac{-16}{61^2} + \frac{5456 \cdot 11}{61^2 \cdot 137} + \frac{11^2}{61^2} \cdot \frac{1}{137^2} > 0
\]

And if \( x < \frac{5456 \cdot 11}{61^2 \cdot 137} = 3751 \) approximation \( 44 + \frac{44}{61} + \frac{11}{61 \cdot x} \) is an upper approximation
of \( \sqrt{2000} \).

So that, all “mediations” to reach \( 44 + \frac{2079}{2882} \) are upper aproximationns.
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