The regularized 3D Boussinesq equations with fractional Laplacian and no diffusion

Hakima Bessaih * Benedetta Ferrario †

April 21, 2015

Abstract

In this paper, we study the 3D regularized Boussinesq equations. The velocity equation is regularized through a smoothing kernel of order $\alpha$ in the nonlinear term and with a $\beta$-fractional Laplacian; we are in the critical case $\alpha + \beta = \frac{5}{4}$. The temperature equation is a pure transport equation. We prove regularity results when the initial velocity is in $H^r$ and the initial temperature is in $H^{r-\beta}$ for $r > \max\left\{\frac{5}{2} - 2\alpha, \beta + 1\right\}$ with $\beta \geq \frac{1}{2}$ and $\alpha \geq 0$. This regularity is enough to prove uniqueness of solutions. We also prove a continuous dependence of solutions with respect to the initial conditions.

Keywords: Boussinesq equations, Leray-alpha models, Transport equation, Commutators.

Mathematics Subject Classification 2010: Primary 35Q35, 76D03; Secondary 35Q86.

1 Introduction

We consider the Boussinesq system in a $d$-dimensional space:

\begin{align*}
\partial_t v + (v \cdot \nabla)v - \nu \Delta v + \nabla p &= \theta e_d \\
\partial_t \theta + v \cdot \nabla \theta &= 0 \\
\nabla \cdot v &= 0
\end{align*}

(1.1)

*University of Wyoming, Department of Mathematics, Dept. 3036, 1000 East University Avenue, Laramie WY 82071, United States, bessaih@uwyo.edu
†Università di Pavia, Dipartimento di Matematica, via Ferrata 1, 27100 Pavia, Italy, benedetta.ferrario@unipv.it
where \( v = v(t, x) \) denotes the velocity vector field, \( p = p(t, x) \) the scalar pressure and \( \theta = \theta(t, x) \) a scalar quantity, which can represent either the temperature of the fluid or the concentration of a chemical component; \( e_d \) is the unit vector \((0, \ldots, 0, 1)\), the viscosity \( \nu \) is a positive constant. Suitable initial conditions \( v_0, \theta_0 \) and boundary conditions (if needed) are given.

For \( d = 2 \), the well posedness of system (1.1) in the plane has been studied by several authors under different assumptions on the initial data (see \([12, 7, 1, 11, 8, 9]\)). For \( d = 3 \), very little is known; it has been proven that there exists a local smooth solution. Some regularity criterions to get a global (in time) solution have been obtained in \([20, 10]\). Otherwise, in the particular case of axisymmetric initial data, \([2]\) shows the global well posedness for the Boussinesq system in the whole space.

To overcome the difficulties of the three-dimensional case, different models have been proposed. For instance, one can regularize the equation for the velocity by putting a fractional power of the Laplacian; this hyper-dissipative Boussinesq system takes the form

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v + \nu(-\Delta)\beta v + \nabla p &= \theta e_3 \\
\partial_t \theta + v \cdot \nabla \theta &= 0 \\
\nabla \cdot v &= 0
\end{align*}
\]

(1.2)

For \( \beta > \frac{5}{4} \), \([26]\) proved the global well posedness. This result has been improved by Ye \([25]\), allowing \( \beta = \frac{5}{4} \).

Notice that for zero initial temperature \( \theta_0 \), the Boussinesq system reduces to the Navier-Stokes equations. It is well known that the three-dimensional Navier-Stokes equations have either a unique local smooth solution or a global weak solution. The questions related to the local smooth solution being global or the global weak solution being unique are very challenging problems that are still open since the seminal work of Leray. For this reason, modifications of different types have been considered for the three-dimensional Navier-Stokes equations. On one side there is the hyper-viscous model, i.e. \([12, 23]\) with zero initial temperature; when \( \beta \geq \frac{5}{4} \), uniqueness of the weak solutions has been proved in \([16]\) (see Remark 6.11 of Chapter 1) and \([17]\). On the other hand, Olson and Titi in \([19]\) suggested to regularize the equations by modifying two terms. For a particular model of fluid dynamics, they replaced the dissipative term by a fractional power of the Laplacian and they regularized the bilinear term of vorticity stretching à la Leray. The well posedness of those equations is obtained by asking a balance between the modification of the nonlinearity and of the viscous dissipation; at least one of them has to be strong enough, while the other might be weak. Similarly,
Barbato, Morandin and Romito in [4] considered the Leray-$\alpha$ Navier-Stokes equations with fractional dissipation

\begin{equation}
\begin{aligned}
\partial_t v + (u \cdot \nabla) v + \nu(-\Delta)^\beta v + \nabla p &= 0 \\
v &= u + (-\Delta)^\alpha u \\
\nabla \cdot u &= \nabla \cdot v = 0
\end{aligned}
\end{equation}

and proved that this system is well posed when $\alpha + \beta \geq \frac{5}{4}$ (with $\alpha, \beta \geq 0$); even some logarithmic corrections can be included, but we do not specify this detail, since it is not related to our analysis. It is worth mentioning the result of the current authors with Barbato in [3], where a stochastic version of the associated inviscid system to (1.3) (when $\nu = 0$) has been studied. In fact, by choosing an appropriate stochastic perturbation to the system to be formally conservative, they were able to prove global existence and uniqueness of solutions in law for $\alpha > \frac{3}{4}$. This is a very strong result although the uniqueness is to be understood in law.

Similar regularization have been used for the MHD models, see for eg. [24] and the references therein. Since these models are quite different from the ones considered in the current paper, we don’t state their results and we refer interested readers to the literature related to these models.

Inspired by [4], in this paper we consider the modified Boussinesq system for $d = 3$, where the equation for the velocity has fractional dissipation whereas the temperature equation has no dissipation term; a Leray-regularization for the velocity appears in the quadratic terms. This system is

\begin{equation}
\begin{aligned}
\partial_t v + (u \cdot \nabla) v + \nu(-\Delta)^\beta v + \nabla p &= \theta e_3 \\
\partial_t \theta + u \cdot \nabla \theta &= 0 \\
v &= u + (-\Delta)^\alpha u \\
\nabla \cdot u &= \nabla \cdot v = 0
\end{aligned}
\end{equation}

Our goal is to generalize the results of Ye [25] and Barbato, Morandin, Romito [4], by proving well posedness of system (1.4) for $\alpha + \beta \geq \frac{5}{4}$ when $v_0, \theta_0$ are regular enough. So the interesting case is for $\beta < \frac{5}{4}$ with $\alpha = \frac{5}{4} - \beta > 0$; indeed, the result of Ye corresponds to $\alpha = 0$ and $\beta \geq \frac{5}{4}$ and that of Barbato, Morandin, Romito does not include the temperature equation, i.e. corresponds to our system (1.4) with $\theta_0 = 0$. We have to point out that the temperature satisfies a pure transport equation, without thermal diffusivity; hence, the uniqueness result for the unknown $\theta$ requires $v$ to be smooth enough. This imposes $\beta$ to be not too small.

We can summarize our result in the following
Theorem 1.1 Assume $\alpha \geq 0$, $\beta \geq \frac{1}{2}$ with
\[ \alpha + \beta \geq \frac{5}{4}. \]

Then, system (1.4) has a unique global smooth solution for every smooth initial conditions $v_0, \theta_0$.

Our proofs rely on the commutator estimates introduced in [14], also used in [25]. However in contrast to [25], we first prove global existence (for any $\alpha \geq 0$ and $\beta > 0$) and then uniqueness of these solutions; moreover we consider different order of space regularity for $v$ and $\theta$ ($H^r$-regularity for $v$ and $H^{r-\beta}$-regularity for $\theta$), whereas in [25] the same order of regularity for both $v$ and $\theta$ is considered. We point out that the requirement on the regularity on the initial data is needed only to guarantee uniqueness.

The paper is organized as follows. Section 2 is devoted to the mathematical framework. Our main functional spaces, the regularization operator $\Lambda^s$ with its properties given in Lemma 2.6 are defined. The bilinear operator of the Navier-Stokes equations, the transport operator and the commutator operator are defined and their properties are stated in Lemma 2.1, Remark 2.3 and Remark 2.4 and Lemma 2.5. The main system is then written in its abstract (operator) form and the definition of weak solutions is given. At the end of this section, we recall the Gagliardo-Nirenberg and Brezis-Gallouet-Wainger inequalities and some continuity results. In Section 3, we prove global existence of weak solutions with their uniform estimates. Slightly better estimates are performed. However, they are not enough to prove the uniqueness of solutions. The main result of the paper is stated in Section 4, Theorem 4.1, where we prove global existence of regular solutions; this regularity is enough to prove uniqueness of solutions and their continuous dependence with respect to the initial conditions, see Theorem 4.3 and Theorem 4.4. The main tool used is the commutator estimate. Let us point out that the results of Section 4 Theorem 1.1, i.e. every smooth initial data gives rise to a unique smooth solution. Section 5 is devoted to showing in more details the crucial estimates used in Section 4.

2 Mathematical framework

We consider the evolution for positive times and the spatial variable belongs to a bounded domain of $\mathbb{R}^3$; for simplicity and because of the lack of natural boundary conditions, we work on the torus, i.e. the spatial variable $x \in \mathbb{T} = [0, 2\pi]^3$ and periodic boundary conditions are assumed. We set $L_p = L^p(\mathbb{T})$. 
As usual in the periodic setting, we can restrict ourselves to deal with initial data with vanishing spatial averages; then the solutions will enjoy the same property at any fixed time \( t > 0 \).

Therefore we can represent any \( T \)-periodic function \( f : \mathbb{R}^3 \to \mathbb{R} \) as

\[
f(x) = \sum_{k \in \mathbb{Z}^3_0} f_k e^{ik \cdot x}, \quad \text{with } f_k \in \mathbb{C}, \ f_{-k} = \overline{f}_k \ \forall k.
\]

where \( \mathbb{Z}^3_0 = \mathbb{Z}^3 \setminus \{0\} \). For \( s \in \mathbb{R} \) we define the spaces

\[
H^s = \{ f = \sum_{k \in \mathbb{Z}^3_0} f_k e^{ik \cdot x} : f_{-k} = \overline{f}_k \text{ and } \sum_{k \in \mathbb{Z}^3_0} |f_k|^2 |k|^{2s} < \infty \}.
\]

They are a Hilbert spaces with scalar product

\[
\langle f, g \rangle_{V_s} = \sum_{k \in \mathbb{Z}^3_0} f_k g_{-k} |k|^{2s}.
\]

We simply denote by \( \langle f, g \rangle \) the scalar product in \( H^0 \) and also the dual pairing of \( H^s - H^{-s} \), i.e. \( \langle f, g \rangle = \sum_k f_k g_{-k} \).

The space \( H^{s+\epsilon} \) is compactly embedded in \( H^s \) for any \( \epsilon > 0 \). Moreover, if \( 0 \leq s < \frac{3}{2} \) and \( \frac{1}{p} = \frac{1}{2} - \frac{s}{3} \), then \( H^s \subset L_p \) and there exists a constant \( C \) (depending on \( s \) and \( p \)) such that

\[
(2.1) \quad \|f\|_{L_p} \leq C\|f\|_{H^s}.
\]

Similarly, we define the spaces for the divergence free velocity vectors, which are periodic and have zero spatial average. For \( w : \mathbb{R}^3 \to \mathbb{R}^3 \) we write formally

\[
w(x) = \sum_{k \in \mathbb{Z}^3_0} w_k e^{ik \cdot x}, \quad \text{with } w_k \in \mathbb{C}^3, \ w_{-k} = \overline{w}_k, \ w_k \cdot k = 0 \ \forall k
\]

and for \( s \in \mathbb{R} \) define

\[
V^s = \{ w = \sum_{k \in \mathbb{Z}^3_0} w_k e^{ik \cdot x} : w_{-k} = \overline{w}_k, \ w_k \cdot k = 0 \text{ and } \sum_{k \in \mathbb{Z}^3_0} |w_k|^2 |k|^{2s} < \infty \}.
\]

This is a Hilbert space with scalar product

\[
\langle v, w \rangle_{V_s} = \sum_{k \in \mathbb{Z}^3_0} v_k \cdot w_{-k} |k|^{2s}.
\]
We define the linear operator \( \Lambda = (-\Delta)^{1/2} \), i.e.
\[
f = \sum_{k \in \mathbb{Z}^3_0} f_k e^{ik \cdot x} \implies \Lambda f = \sum_{k \in \mathbb{Z}^3_0} |k| f_k e^{ik \cdot x}
\]
and its powers \( \Lambda^s \): \( \Lambda^s f = \sum_{k \in \mathbb{Z}^3_0} |k|^s f_k e^{ik \cdot x} \); hence \( \Lambda^2 = -\Delta \). Note, in particular that \( \Lambda^s \) maps \( H^r \) onto \( H^{r-s} \).

For simplicity, we shall use the same notation for \( \Lambda \) in the scalar spaces \( s \) and in the vector spaces \( V^s \).

Let us denote by \( \Pi \) the Leray-Helmotz projection from \( L^2 \) onto \( V^0 \). The operators \( \Pi \) and \( \Lambda^s \) commute.

Finally we define the bilinear operator \( B : V^1 \times V^1 \rightarrow V^{-1} \) by
\[
\langle B(u,v), w \rangle = \int_T ((u \cdot \nabla)v) \cdot w \, dx
\]
i.e. \( B(u,v) = \Pi((u \cdot \nabla)v) \) for smooth vectors \( u, v \).

We summarize the properties of the nonlinear terms; these are classical results, see e.g. [23].

**Lemma 2.1** For any \( u, v, w \in V^1 \) and \( \theta, \eta \in H^1 \) we have
\[
\langle B(u,v), w \rangle = -\langle B(u,w), v \rangle, \quad \langle B(u,v), v \rangle = 0,
\]
\[
\langle v \cdot \nabla \theta, \eta \rangle = -\langle v \cdot \nabla \eta, \theta \rangle, \quad \langle v \cdot \nabla \theta, \theta \rangle = 0
\]
and more generally for any \( u, v, w \) giving a meaning to the trilinear forms above, as stated precisely in the following:
\[
\langle B(u,v), w \rangle \leq C \|u\|_{V^{m_1}} \|v\|_{V^{1+m_2}} \|w\|_{V^{m_3}}
\]
for any \( m_i \geq 0 \) with at least one of the three parameters positive and such that
\[
m_1 + m_2 + m_3 \geq \frac{3}{2}.
\]
Hereafter, we denote by the same symbol \( C \) different constants.

Now, we are ready to give the abstract formulation of problem (1.4); we apply the projection operator \( \Pi \) to the first equation in order to get rid of the pressure. In addition due to the periodic setting, we regularize \( u \) in a different, but equivalent way. Therefore, our system in abstract form is
\[
\begin{cases}
\partial_t v + B(u,v) + \nu \Lambda^{2\alpha} v = \Pi(\theta e_3) \\
\partial_t \theta + u \cdot \nabla \theta = 0 \\
v = \Lambda^{2\alpha} u
\end{cases}
\]
We focus our analysis on the unknowns $v$ and $\theta$. The pressure $p$ will be recovered by taking the curl of the equation for the velocity in (1.4), i.e. $p$ solves the equation $\Delta p = -\nabla \cdot [(u \cdot \nabla) v - \theta e_3] = -\nabla \cdot [(\Lambda^{-2\alpha}v) \cdot \nabla) v - \theta e_3]$. Therefore we give the following definition in terms of $v$ and $\theta$ only. The finite time interval $[0, T]$ is fixed throughout the paper.

**Definition 2.2** Let $\alpha \geq 0$ and $\beta > 0$. We are given $v_0 \in V^0, \theta_0 \in H^0$. We say that the couple $(v, \theta)$ is a weak solution to system (2.5) over the time interval $[0, T]$ if

$$v \in L^\infty(0, T; V^0) \cap L^2(0, T; V^\beta) \cap C_w([0, T]; V^0)$$

$$\theta \in L^\infty(0, T; H^0) \cap C_w([0, T]; H^0)$$

and, given any $\psi \in V^{\frac{5}{2}}, \phi \in H^{\frac{5}{2}},$ they satisfy

$$\langle v(t), \psi \rangle - \int_0^t \langle B(u(s), \psi), v(s) \rangle ds + \nu \int_0^t \langle \Lambda^{\beta} v(s), \Lambda^{\beta} \psi \rangle ds$$

$$= \langle v_0, \psi \rangle + \int_0^t \langle \theta(s)e_3, \psi \rangle ds$$

$$\langle \theta(t), \phi \rangle - \int_0^t \langle u(s) \cdot \nabla \phi, \theta(s) \rangle ds = \langle \theta_0, \phi \rangle$$

for every $t \in [0, T]$.

**Remark 2.3** In the weak formulations above, the trilinear terms are well defined; indeed, if $2\alpha + \beta \leq \frac{3}{2}$

$$|\langle B(u, \psi), v \rangle| \leq C\|u\|_{V^{2\alpha}} \|\psi\|_{V^{\frac{5}{2}-2\alpha-\beta}} \|v\|_{V^\beta} \quad \text{by } (2.4)$$

and if $2\alpha + \beta > \frac{3}{2}$

$$|\langle B(u, \psi), v \rangle| \leq C\|u\|_{L_\infty} \|\nabla \psi\|_{L_2} \|v\|_{L_2} \quad \text{by Hölder inequality}$$

$$\leq C\|u\|_{V^{2\alpha+\beta}} \|v\|_{V^\beta} \|\psi\|_{V^{1}}$$

$$\leq C\|v\|_{V^\beta} \|v\|_{V^0} \|\psi\|_{V^{1}}.$$  

Similarly for the temperature:

if $2\alpha + \beta < \frac{3}{2}$

$$|\langle u \cdot \nabla \phi, \theta \rangle| \leq \|u\|_{L_{p_1}} \|\nabla \phi\|_{L_{p_2}} \|\theta\|_{L_2}$$

$$\leq C\|u\|_{V^{2\alpha+\beta}} \|\nabla \phi\|_{H^{\frac{5}{2}-2\alpha-\beta}} \|\theta\|_{H^0}$$

$$\leq C\|v\|_{V^\beta} \|\phi\|_{H^{\frac{5}{2}}} \|\theta\|_{H^0}.$$
where we used first the Hölder inequality with \( \frac{1}{p_1} = \frac{1}{2} - \frac{2\alpha + \beta}{3}, \frac{1}{p_2} = \frac{1}{2} - \frac{1}{p_1} \) and then the embedding theorems;
if \( 2\alpha + \beta \geq \frac{3}{2} \)

\[
|\langle u \cdot \nabla \phi, \theta \rangle| \leq \|u\|_{L^4} \|\nabla \phi\|_{L^4} \|\theta\|_{L^2} \\
\leq C \|u\|_{V^{2\alpha + \beta}} \|\nabla \phi\|_{H^{\frac{3}{2}}} \|\theta\|_{H^0} \\
\leq C \|v\|_{V^{\beta}} \|\phi\|_{H^{\frac{3}{2}}} \|\theta\|_{H^0} \\
\leq C \|v\|_{V^{\beta}} \|\theta\|_{H^0} \|\phi\|_{H^{\frac{3}{2}}}
\]  
(2.11)

where we used first the Hölder inequality and then the embedding theorems

\( V^{2\alpha + \beta} \subset L_q \) for any finite \( q \), \( H^{\frac{3}{4}} \subset H^{\frac{3}{2}}, H^{\frac{3}{4}} \subset L_4 \).

For more regular solutions, the trilinear term \( \langle B(u, \psi), v \rangle \) is equal to

\(-\langle B(u, v), \psi \rangle \) and we recover the term appearing in the equation for the velocity. The same holds for the temperature.

Remark 2.4 We point out that the estimates by means of Sobolev embeddings need some restriction for the parameters; but, for bigger values of the parameters they are easier to prove and the details will be skipped. This means for instance that (2.8) with (2.2) gives

\[ \|B(u, v)\|_{V^{-\frac{3}{2}}} \leq C \|v\|_{V^0} \|v\|_{V^{\beta}} \]

assuming \( 2\alpha + \beta \leq \frac{3}{2} \), whereas for \( 2\alpha + \beta > \frac{3}{2} \) we get something stronger in (2.9):

\[ \|B(u, v)\|_{V^{-1}} \leq C \|v\|_{V^0} \|v\|_{V^{\beta}} \]

which is proven in another way. But for sure, from the proof of (2.8) one can say that \( \|B(u, v)\|_{V^{-\frac{3}{2}}} \leq C \|v\|_{V^0} \|v\|_{V^{\beta}} \) also for \( 2\alpha + \beta > \frac{3}{2} \) without proving it.

In this last part of the section, we summarize the technical tools to be used later on.

To estimate an \( L_\infty \)-norm we use either the embedding theorem \( H^r \subset L_\infty \) with \( r > \frac{3}{2} \) or the Brézis-Gallouet-Wainger inequality (see [5, 6]):

for any \( r > \frac{3}{2} \) there exists a constant \( C \) such that

\[ \|g\|_{L_\infty} \leq C \|g\|_{H^\frac{3}{2}} \left( 1 + \sqrt{\ln(1 + \|g\|_{H^r})} \right). \]

Actually, we shall use the stronger form of this inequality, as given for instance in [25]: for any \( r > \frac{3}{2} \) there exists a constant \( C \) such that

\[ \|g\|_{L_\infty} \leq C \left( 1 + \|g\|_{H^\frac{3}{2}} + \|g\|_{H^\frac{3}{2}} \ln(e + \|g\|_{H^r}) \right). \]
Gagliardo-Niremberg inequality (see \[18\])
Let $1 \leq q, r \leq \infty$, $0 < s < m$, $\frac{s}{m} \leq a < 1$ and
\[
\frac{1}{p} = \frac{s}{3} + \left(\frac{1}{q} - \frac{m}{3}\right)a + \frac{1-a}{r}
\]
then there exists a constant $C$ such that
\[
\|\Lambda^a g\|_{L^p} \leq C \|g\|_{L^q}^{1-a} \|\Lambda^m g\|_{L^q}^a.
\]
(2.14)

Define the commutator
\[
[\Lambda^a, f]g = \Lambda^a (fg) - f \Lambda^a g.
\]

From \[14\], \[15\] we have

**Lemma 2.5 (Commutator lemma)** Let $s > 0$, $1 < p < \infty$ and $p_2, p_3 \in (1, \infty)$ be such that
\[
\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} \geq \frac{1}{p_3} + \frac{1}{p_4},
\]
Then
\[
\|[\Lambda^a, f]g\|_{L^p} \leq C \left( \|\nabla f\|_{L^{p_1}} \|\Lambda^a g\|_{L^{p_2}} + \|\Lambda^a f\|_{L^{p_3}} \|g\|_{L^{p_4}} \right).
\]

and

**Lemma 2.6** Let $s > 0$, $1 < p < \infty$ and $p_2, p_3 \in (1, \infty)$ be such that
\[
\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} \geq \frac{1}{p_3} + \frac{1}{p_4},
\]
Then
\[
\|\Lambda^a (fg)\|_{L^p} \leq C \left( \|f\|_{L^{p_1}} \|\Lambda^a g\|_{L^{p_2}} + \|\Lambda^a f\|_{L^{p_3}} \|g\|_{L^{p_4}} \right).
\]

We shall use the commutator acting also on vectors; in particular for $u, v \in \mathbb{R}^3, \theta \in \mathbb{R}$
\[
[\Lambda^a, u] \cdot \nabla \theta = \Lambda^a (u \cdot \nabla \theta) - u \cdot \nabla \Lambda^a \theta
\]
and
\[
[\Lambda^a, u] \cdot \nabla v = \Lambda^a ((u \cdot \nabla) v) - (u \cdot \nabla) \Lambda^a v.
\]
Therefore
\[ \langle \Lambda^s (u \cdot \nabla \theta), \Lambda^s \theta \rangle = \langle \left[ \Lambda^s, u \right] \cdot \nabla \theta, \Lambda^s \theta \rangle + \langle u \cdot \nabla \Lambda^s \theta, \Lambda^s \theta \rangle = 0 \text{ by (2.3)} \]
and
\[ \langle \Lambda^s ((u \cdot \nabla) v), \Lambda^s v \rangle = \langle \left[ \Lambda^s, u \right] \cdot \nabla v, \Lambda^s v \rangle + \langle (u \cdot \nabla) \Lambda^s v, \Lambda^s v \rangle = 0 \text{ by (2.2)} \]

About the continuity in time, we have the strong continuity result (see [21] or Lemma 1.4, Chap III in [22])

**Lemma 2.7** Let \( s \in \mathbb{R} \) and \( h > 0 \).
If \( v \in L^2(0, T; V^{s+h}) \) and \( \frac{dv}{dt} \in L^2(0, T; V^{s-h}) \), then \( v \in C([0, T]; V^s) \) and
\[
\frac{d}{dt} \| v(t) \|_{V^s}^2 = 2 \langle \Lambda^{-h} \frac{dv}{dt}(t), \Lambda^h v(t) \rangle_{V^s}
\]
and the weak continuity result (see [21]).

**Lemma 2.8** Let \( X \) and \( Y \) be Banach spaces, \( X \) reflexive, \( X \) a dense subset of \( Y \) and the inclusion map of \( X \) into \( Y \) continuous. Then
\[
L^\infty(0, T; X) \cap C_w(0, T; Y) = C_w(0, T; X).
\]

### 3 Existence of weak solutions

Existence of a global weak solution of system (2.5) can be obtained easily; the technique is very similar to that for the classical Boussinesq system. The equation for \( \theta \) is a pure transport equation; then the \( L_q \)-norm of \( \theta \) is conserved in time (for any \( q \leq +\infty \)). On the other hand, it is enough to have some regularization in the velocity equation (i.e. \( \beta > 0 \)) in order to get a weak solution as in Definition 2.2; moreover, this solution satisfies an energy inequality. Of course, the bigger are the parameters \( \alpha, \beta \), the more regular is the velocity \( v \).

**Theorem 3.1** Let \( \alpha \geq 0 \) and \( \beta > 0 \). For any \( v_0 \in V^0, \theta_0 \in H^0 \), there exists a weak solution \((v, \theta)\) of (2.5) on the time interval \([0, T]\). Moreover
\[
\theta \in C_w(0, T; L_q)
\]
for any \( q \geq 2 \) (including \( q = \infty \)).
Proof. We define the finite dimensional projector operator $\Pi_n$ in $V^0$ as $\Pi_n v = \sum_{0<|k|\leq n} v_k e^{ikx}$ for $v = \sum_{k \in \mathbb{Z}} v_k e^{ikx}$; similarly for the scalar case, i.e. $\Pi_n$ in $H^0$. We set $B_n(u, v) = \Pi_n B(u, v)$.

We consider the finite dimensional approximation of system (2.5) in the unknowns $v_n = \Pi_n v$, $u_n = \Pi_n u$ and $\theta_n = \Pi_n \theta$. This is the Galerkin approximation for $n = 1, 2, \ldots$

\begin{equation}
\begin{aligned}
    \partial_t v_n + B_n(u_n, v_n) + \nu \Lambda^{2\beta} v_n &= \Pi(\theta_n e_3) \\
    \partial_t \theta_n + \Pi_n(u_n \cdot \nabla \theta_n) &= 0 \\
    v_n &= \Lambda^{2\alpha} u_n
\end{aligned}
\end{equation}

We take the $L^2$-scalar product of the equation for the velocity $v_n$ with $v_n$ itself; bearing in mind (2.2) we get

$$
\frac{1}{2} \frac{d}{dt} \| v_n(t) \|^2_{V^0} + \nu \| v_n(t) \|^2_{V^0} = -\langle B_n(u_n(t), v_n(t)), v_n(t) \rangle + \langle \Pi(\theta_n(t)e_3), v_n(t) \rangle
$$

$$
= -\langle B(u_n(t), v_n(t)), v_n(t) \rangle + \langle \theta_n(t)e_3, v_n(t) \rangle
$$

$$
\leq \frac{1}{2} \| \theta_n(t) \|^2_{H^0} + \frac{1}{2} \| v_n(t) \|^2_{V^0}
$$

and similarly for the second equation

$$
\frac{d}{dt} \| \theta_n(t) \|^2_{H^0} = -\langle \Pi_n(u_n(t) \cdot \nabla \theta_n(t)), \theta_n(t) \rangle
$$

$$
= -\langle u_n(t) \cdot \nabla \theta_n(t), \theta_n(t) \rangle = 0.
$$

In both cases the trilinear forms vanish according to (2.2), (2.3).

Adding these estimates, by means of Gronwall’s lemma we get the basic $L^2$-energy estimate: there exists a constant $K_1$ independent of $n$ such that

$$
\sup_{0 \leq t \leq T} (\| v_n(t) \|^2_{V^0} + \| \theta_n(t) \|^2_{H^0}) + \nu \int_0^T \| v_n(t) \|^2_{V^0} dt \leq K_1
$$

for any $n$.

From the equation for the velocity $v_n$, one has that $\frac{dv_n}{dt}$ is expressed as the sum of three terms involving $v_n$, $u_n$ and $\theta_n$. In particular, the dissipative term $\Lambda^{2\beta} v_n \in L^2(0, T; V^{-\beta})$; by (2.8), (2.9) we have $B_n(u_n, v_n) \in L^2(0, T; V^{-s})$ for some finite $s \geq 1$. Therefore there exist constants $\gamma > 0$ and $K_2$ independent of $n$, such that

$$
\| \frac{dv_n}{dt} \|^2_{L^2(0, T; V^{-\gamma})} \leq K_2.
$$
This means that $v_n$ is bounded in $L^2(0, T; V^\beta) \cap W^{1,2}(0, T; V^{-\gamma})$ (with $\beta > 0$ and $\gamma > 0$), which is compactly embedded in $L^2(0, T; V^0)$ (see Lemma 2.2. in [22]). Hence we can extract a subsequence, still denoted by $\{v_n\}$ and $\{\theta_n\}$, such that

\[
v_n \rightharpoonup v \quad \text{weakly in} \quad L^2(0, T; V^\beta)
\]

\[
v_n \rightharpoonup^* v \quad \text{weakly}^* \text{ in} \quad L^\infty(0, T; V^0)
\]

\[
v_n \rightrightarrows v \quad \text{strongly in} \quad L^2(0, T; V^0)
\]

\[
\theta_n \rightharpoonup^* \theta \quad \text{weakly}^* \text{ in} \quad L^\infty(0, T; H^0).
\]

Using these convergences, it is a classical result to pass to the limit in the variational formulation (2.6) and (2.7) and prove that $(v, \theta)$ is solution of (2.5) and inherits all the regularity from $(v_n, \theta_n)$, i.e.

\[
v \in L^\infty(0, T; V^0) \cap L^2(0, T; V^\beta), \quad \theta \in L^\infty(0, T; H^0).
\]

Moreover, it is a classical result (see [25]) that

\[
\text{sup}_{0 \leq t \leq T} \|\theta_n(t)\|_{L_q} \leq \|\theta_0\|_{L_q}
\]

for any $q \leq \infty$.

Hence, the sequence $\{\theta_n\}_n$ is uniformly bounded in $L^\infty(0, T; L_q)$ which implies (up to a subsequence still denoted $\theta_n$) that

\[
\theta_n \rightharpoonup^* \theta \quad \text{weakly}^* \text{ in} \quad L^\infty(0, T; L_q)
\]

and

\[
\text{sup}_{0 \leq t \leq T} \|\theta(t)\|_{L_q} \leq \|\theta_0\|_{L_q}.
\]

Now, let us prove that $v \in C_w(0, T; V^0)$ and $\theta \in C_w(0, T; L_q)$. We integrate in time the equation for $v$:

\[
v(t) = v_0 + \int_0^t [-B(u(s), v(s)) - \nu \Lambda^2 v(s) + \Pi \theta(s) e_3] ds.
\]

Bearing in mind (2.2) and the estimates of Remark 2.3 we get that $B(u, v) \in L^2(0, T; V^{-\frac{m}{2}})$; therefore $v \in C(0, T; V^{-m})$ for some positive $m$. By Lemma 2.8 we get that $v \in C_w(0, T; V^0)$. 

12
Now we look for the weak continuity of $\theta$. Assume that $\phi \in C^\infty_\#(\mathbb{T})$ which is the space of infinitely continuously differentiable functions on $\mathbb{T}$ that are periodic. Then for $t, s \in [0, T]$, we have that

$$\left| \langle \theta(t) - \theta(s), \phi \rangle \right| = \left| \int_s^t \langle u(r) \cdot \nabla \phi, \theta(r) \rangle dr \right|$$

$$\leq \int_s^t \| \nabla \phi \|_{L_\infty} \| u(r) \|_{L_2} \| \theta(r) \|_{L_2} dr$$

$$\leq \| \nabla \phi \|_{L_\infty} \| \theta \|_{L_\infty(0,T;H^0)} \int_s^t \| u(r) \|_{V_0} dr.$$ 

Using the density of $C^\infty_\#(\mathbb{T})$ in $L_q$ (with $\frac{1}{q} + \frac{1}{q'} \leq 1$), we deduce that

$$\lim_{t \to s} \langle \theta(t) - \theta(s), \phi \rangle = 0 \quad \forall \phi \in L_q$$

which means that $\theta \in C_{w}(0,T;L_q)$. A similar argument can be used for $q = \infty$ and this completes the proof. 

**Remark 3.2** Take $\alpha \geq 0$ and $\beta > 0$ such that

$$2\alpha + \beta \leq \frac{3}{2}, \quad \alpha + \beta \geq \frac{5}{4}.$$ 

For this to hold it is necessary that $\alpha$ is not too big ($\alpha \leq \frac{1}{4}$) and $\beta$ not too small ($1 \leq \beta \leq \frac{3}{2}$). Then, from the first estimate in Theorem 3.1 we get $B(u,v) \in L^2(0,T;V^{-\beta})$. Hence, going back to the proof of the previous theorem we get that $\frac{dv}{dt} \in L^2(0,T;V^{-\beta})$; by Lemma 2.7 this implies that $v \in C([0,T];V^0)$, which is stronger than the weak continuity result of Theorem 3.1.

In addition, for more regular initial data we have

**Theorem 3.3 (More regularity)** We are given $\alpha \geq 0$ and $\beta \geq \frac{1}{2}$ with

$$\alpha + \beta \geq \frac{5}{4}.$$ 

Let $s \geq 0$ with

$$1 - \beta \leq s \leq \beta.$$ 

Then, given $v_0 \in V^s, \theta_0 \in H^0$, any weak solution of (2.5) obtained in Theorem 3.1 is more regular; indeed, the velocity is more regular

$$v \in C([0,T];V^s) \cap L^2(0,T;V^{s+\beta}).$$
Proof. We look for a priori estimates for $v$. We proceed as before, but for more regular norms. We have

$$\frac{1}{2} \frac{d}{dt} \| v(t) \|^2_{V,s} + \nu \| v(t) \|^2_{V,s+\beta}$$

$$= - \langle B(u(t), v(t)), \Lambda^{2s} v(t) \rangle + \langle \Pi \theta(t), \Lambda^{s} v(t) \rangle$$

$$\leq \| \Lambda^s ((u(t) \cdot \nabla) v(t), \Lambda^s v(t)) \|_{L_2} + \| \Lambda^{s+\beta} \theta(t) \|_{L_2} \| \Lambda^{s+\beta} v(t) \|_{L_2}$$

$$\leq \| [\Lambda^s, u(t)] \cdot \nabla v(t) \|_{L_2} + \| \Lambda^s v(t) \|_{L_2} + C \| \theta(t) \|_{H^0} \| v(t) \|_{V,s+\beta}$$

where we used (2.16) and that $H^0 \subset H^{s-\beta}$.

We use the Commutator Lemma 2.5

$$(3.4) \quad \| [\Lambda^s, u] \cdot \nabla v \|_{L_2} \leq C \left( \| \Lambda u \|_{L_{p_1}} \| \Lambda^s v \|_{L_{p_2}} + \| \Lambda^s u \|_{L_{p_3}} \| \nabla v \|_{L_{p_4}} \right)$$

with

$$\frac{1}{p_1} = \frac{1}{2} - \frac{s+\beta-(1-2\alpha)}{3}, \quad \frac{1}{p_2} = \frac{1}{2} - \frac{\beta-s}{3}, \quad \frac{1}{p_3} = \frac{1}{2} - \frac{\beta-(s-2\alpha)}{3}, \quad \frac{1}{p_4} = \frac{1}{2} - \frac{\beta+s-1}{3}.$$ 

Notice that by our assumptions we get $s + \beta \geq 1 - 2\alpha, \beta \geq s, \beta \geq s - 2\alpha, s + \beta \geq 1$. Both conditions $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{2}$ and $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ are equivalent to $\alpha + \beta \geq \frac{5}{4}$. The choice of the $p_i$’s allows to use the Sobolev embedding theorem; we have

$$\begin{cases} 
\| \Lambda^{1-2\alpha} v \|_{L_{p_1}} \leq C \| v \|_{V,s+\beta} \\
\| \Lambda^s v \|_{L_{p_2}} \leq C \| v \|_{V,\beta} 
\end{cases}$$

and

$$\begin{cases} 
\| \Lambda^{s-2\alpha} v \|_{L_{p_3}} \leq C \| v \|_{V,\beta} \\
\| \nabla v \|_{L_{p_4}} \leq C \| v \|_{V,s+\beta} 
\end{cases}$$

Hence, we conclude that

$$(3.5) \quad \frac{1}{2} \frac{d}{dt} \| v(t) \|^2_{V,s} + \nu \| v(t) \|^2_{V,s+\beta}$$

$$\leq C \| v(t) \|_{V,\beta} \| v(t) \|_{V,s+\beta} \| v(t) \|_{V,s} + C \| \theta(t) \|_{H^0} \| v(t) \|_{V,s+\beta}$$

$$\leq \frac{\nu}{2} \| v(t) \|^2_{V,s+\beta} + C_{\nu} \| v(t) \|^2_{V,\beta} \| v(t) \|^2_{V,s} + C_{\nu} \| \theta(t) \|^2_{H^0}$$

by Young inequality. In particular,

$$\frac{d}{dt} \| v(t) \|^2_{V,s} \leq C_{\nu} \| v \|^2_{V,\beta} \| v \|^2_{V,s} + C_{\nu} \| \theta \|^2_{H^0}.$$
Since \( v \in L^2(0, T; V^\beta) \) and \( \theta \in L^\infty(0, T; H^0) \) from the previous theorem, we can proceed by means of Gronwall lemma to get the estimate for the \( L^\infty(0, T; V^\beta) \)-norm:

\[
\sup_{0 \leq t \leq T} \| v(t) \|_{V^\beta}^2 \leq \| v_0 \|_{V^\beta}^2 e^{C_v \int_0^T \| v(s) \|_{V^\beta}^2 ds} + C_\nu \int_0^T e^{C_v \int_0^s \| v(s) \|_{V^\beta}^2 ds} \| \theta(r) \|_{H^0}^2 dr.
\]

Integrating in time (3.5), we also get

\[
(3.6) \quad \frac{\nu}{2} \int_0^T \| v(t) \|_{V^{s+\beta}}^2 dt \leq \frac{1}{2} \| v_0 \|_{V^s}^2 + C_\nu \int_0^T \| v(t) \|_{V^\beta}^2 dt + C_\nu \int_0^T \| \theta(t) \|_{H^{\frac{1}{2}}}^2 dt.
\]

Summing up, we get that \( v \in L^\infty(0, T; V^s) \cap L^2(0, T; V^{s+\beta}) \). Now, we study the time regularity. We recall property (2.4) for the nonlinear term \( B(u, v) \) with \( m_1 = \frac{5}{2} - 2\beta, m_2 = s + \beta - 1 \geq 0, m_3 = \beta - s \geq 0 \); this can be used if \( \beta < \frac{5}{4} \). But for bigger values of \( \beta \) the estimate we are looking for is even easier to prove (see Remark [2.7]). We have

\[
\| \frac{dv}{dt}(t) \|_{V^{s-\beta}} = \sup_{\| \psi \|_{V^{s-\beta}} \leq 1} \left| \langle \frac{dv}{dt}(t), \psi \rangle \right|
\]

\[
\leq \sup_{\| \psi \|_{V^{s-\beta}} \leq 1} \left( C \| u(t) \|_{V^{\frac{s}{2} - 2\beta}} \| v(t) \|_{V^{s+\beta}} + \nu \| \Lambda^{s-\beta} v(t) \|_{H^0} \right) \| \psi \|_{V^{s-\beta}}
\]

\[
\leq C \| v(t) \|_{V^{\frac{s}{2} - 2\beta - 2\alpha}} \| v(t) \|_{V^{s+\beta}} + \nu \| v(t) \|_{V^{s+\beta}} + C \| \theta(t) \|_{H^0}
\]

since \( V^0 \subseteq V^{\frac{s}{2} - 2\beta - 2\alpha} \) for \( \alpha + \beta \geq \frac{5}{4} \).

Hence, using the regularity of \( v, \theta \) we get that

\[
\frac{dv}{dt} \in L^2(0, T; V^{s-\beta}).
\]

Now using Lemma [2.7] we deduce that \( v \in C([0, T]; V^s) \). \( \square \)

In particular we have

**Theorem 3.4** (s = \( \beta \)) **We are given** \( \alpha \geq 0 \) and \( \beta \geq \frac{1}{2} \) **with**

\[
\alpha + \beta \geq \frac{5}{4}.
\]

Then given \( v_0 \in V^\beta, \theta_0 \in H^0 \), any weak solution of (2.3) obtained in Theorem [3.1] is more regular; indeed, the velocity is more regular

\[
v \in C([0, T]; V^\beta) \cap L^2(0, T; V^{2\beta}).
\]
4 Regular solutions: global existence, uniqueness and continuous dependence on the initial data

The regularity of solutions from the previous section is not enough to prove uniqueness. To this end, we seek classical solutions. These are solutions for which the derivatives in the equations of (2.5) exist. Indeed, we shall get that \( v \in C([0,T];V^r) \cap L^2(0,T;V^{r+\beta}) \) and \( \theta \in C([0,T];H^{r-\beta}) \) with \( r \geq \frac{3}{2}, r \geq 2\beta, r - \beta \geq 1 \) (see Remark 4.2). The crucial point is to show that these regular solutions are defined on any given time interval \([0,T]\); their local existence is easy to prove.

Unlike the previous section, here we will consider \( H^s\)-regularity for \( \theta(t) \) (with \( s > 0 \)). This will help prove the uniqueness of solutions.

**Theorem 4.1** We are given \( \alpha \geq 0, \beta \geq \frac{1}{2} \) with

\[
\alpha + \beta \geq \frac{5}{4}.
\]

Let

\[
r > \max \left( \frac{5}{2} - 2\alpha, \beta + 1 \right).
\]

Then, for any \( v_0 \in V^r, \theta_0 \in H^{r-\beta} \), there exists a solution \((v,\theta)\) to (2.5) such that

\[
v \in C([0,T];V^r) \cap L^2(0,T;V^{r+\beta}), \quad \theta \in C([0,T];H^{r-\beta}).
\]

**Proof.** We proceed as before. We take the \( L^2 \)-scalar product of the first equation of (2.5) with \( \Lambda^r v(t) \); then

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{V^r} + \nu \|v(t)\|^2_{V^{r+\beta}} = -\langle B(u(t),v(t)), \Lambda^r v(t) \rangle + \langle \theta(t)e_3, \Lambda^r v(t) \rangle
\]

\[
= -\langle B(\Lambda^{-2\alpha} v(t), v(t)), \Lambda^r v(t) \rangle + \langle \Lambda^{r-\beta} \theta(t)e_3, \Lambda^{r+\beta} v(t) \rangle
\]

\[
\leq C \|v(t)\|_{V^2} \|v(t)\|_{V^{r+\beta}} + C \|\theta(t)\|_{H^{r-\beta}} \|v(t)\|_{V^r} + C \|\theta(t)\|_{H^{r-\beta}}^2 \leq \frac{\nu}{4} \|v(t)\|^2_{V^{r+\beta}} + C_\nu \|v(t)\|^2_{V^2} + C \|\theta(t)\|_{H^{r-\beta}}^2
\]

where we used first Lemma 5.1 and then Young inequality.

Now for \( \theta \), we take the \( L^2 \)-scalar product of the second equation of (2.5) with \( \Lambda^{2r-2\beta} \theta(t) \); then

\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|^2_{H^{r-\beta}} = -\langle u(t) \cdot \nabla \theta(t), \Lambda^{2r-2\beta} \theta(t) \rangle
\]
We estimate the r.h.s. 
\[
\langle u \cdot \nabla \theta, \Lambda^{2r-2\beta} \rangle = \langle \Lambda^{-\beta}(u \cdot \nabla \theta), \Lambda^{r-\beta} \theta \rangle \\
= \langle [\Lambda^{r-\beta}, u] \cdot \nabla \theta, \Lambda^{r-\beta} \theta \rangle \quad \text{by (2.15)}
\]
\[
\leq \|[\Lambda^{r-\beta}, \Lambda^{-2\alpha} v] \cdot \nabla \theta\|_{L^2} \|\Lambda^{r-\beta} \theta\|_{L^2}
\]
and the Commutator Lemma 2.5 gives
\[
\leq C \left( \|\Lambda^{1-2\alpha} v\|_{L^\infty} \|\Lambda^{r-\beta} \theta\|_{L^2} + \|\Lambda^{r-\beta-2\alpha} v\|_{L^q} \|\Lambda \theta\|_{L^q} \right) \|\Lambda^{r-\beta} \theta\|_{L^2}
\]
with \( \frac{1}{q_3} + \frac{1}{q_4} \leq \frac{1}{2} \); we continue by means of the Brézis-Gallouet-Wainger estimate (2.13) (with \( g = \Lambda^{1-2\alpha} v \)) and Lemma 5.4
\[
\leq C \left( 1 + \|\Lambda^{\frac{5}{4}-2\alpha} v\|_{L^2} + \|\Lambda^{\frac{5}{4}} v\|_{L^2} \ln(e + \|v\|_{V^{m+1-2\alpha}}) \right) \|\theta\|_{H^{r-\beta}}^2 \\
+ C\|v\|_{V^{2\alpha}} \|v\|_{V^{r+\beta}}^{1-a} \|\theta\|_{L^q}^{1-a} \|\theta\|_{H^{r-\beta}}^{1+a}
\]
for any \( m > \frac{5}{4} \) and for suitable \( q > 2, \alpha \in (0, 1) \); \( m \) will be chosen later on. Finally we use that \( V^{2\beta} \subset V^{\frac{5}{2}-2\alpha} \) since \( \alpha + \beta \geq \frac{5}{4} \):
\[
\leq C \left( 1 + \|v\|_{V^{2\beta}} + \|v\|_{V^{2\beta}} \ln(e + \|v\|_{V^{m+1-2\alpha}}) \right) \|\theta\|_{H^{r-\beta}}^2 \\
+ C\|v\|_{V^{2\alpha}} \|v\|_{V^{r+\beta}}^{1-a} \|\theta\|_{L^q}^{1-a} \|\theta\|_{H^{r-\beta}}^{1+a}.
\]
Now, we use Young inequality:
\[
\|v\|_{V^{2\beta}}^\alpha \|v\|_{V^{r+\beta}}^{1-a} \|\theta\|_{L^q}^{1+a} \|\theta\|_{H^{r-\beta}}^{1+a} \leq \frac{\nu}{4} \|v\|_{V^{2\beta}}^2 + C\nu \|v\|_{V^{2\beta}}^2 \|\theta\|_{L^q}^{2(1-a)} \|\theta\|_{H^{r-\beta}}^2.
\]
Set \( \phi := \|v\|_{V^{2\beta}}^\alpha \|\theta\|_{L^q}^{2(1-a)} \); then \( \phi \in L^1(0, T) \) according to Theorem 3.4 and (3.3). Thus
\[
\frac{d}{dt}(\|v\|_{V^r}^2 - \|\theta(t)\|_{H^{r-\beta}}^2) \leq C \left( 1 + \|v(t)\|_{V^{2\beta}} + \|v(t)\|_{V^{2\beta}} \ln(e + \|v(t)\|_{V^{m+1-2\alpha}}) \right) \|\theta(t)\|_{H^{r-\beta}}^2 \\
+ \frac{\nu}{4} \|v(t)\|_{V^{r+\beta}}^2 + C\nu \phi(t) \|\theta(t)\|_{H^{r-\beta}}^2.
\]
Adding the estimates (4.2) for \( v \) and (4.3) for \( \theta \), we get
\[
\frac{d}{dt}(\|v(t)\|_{V^r}^2 + \|\theta(t)\|_{H^{r-\beta}}^2) + \nu \|v(t)\|_{V^{r+\beta}}^2 \leq C \|v(t)\|_{V^{2\beta}}^2 \|v(t)\|_{V^r}^2 \\
+ C \left( 1 + \|v(t)\|_{V^{2\beta}} + \|v(t)\|_{V^{2\beta}} \ln(e + \|v\|_{V^{m+1-2\alpha}}) + \phi(t) \right) \|\theta(t)\|_{H^{r-\beta}}^2.
\]
Recall that $r > \frac{5}{2} - 2\alpha$ by assumption; then there exists $m > \frac{3}{2}$ such that $V^r \subset V^{m+1-2\alpha}$. Thus, we get

\[(4.5) \quad \frac{d}{dt} \left( \|v(t)\|_{V^r}^2 + \|\theta(t)\|_{H^{r-\beta}}^2 + \nu \|v(t)\|_{V^{r+\beta}}^2 \right) \leq C \|v(t)\|_{V^{2\beta}}^2 \|v(t)\|_{V^r}^2 + C \left( 1 + \|v(t)\|_{V^{2\beta}} + \|v(t)\|_{V^{2\beta}} \ln(e + \|v(t)\|_{V^r}) + \phi(t) \right) \|\theta(t)\|_{H^{r-\beta}}^2 \]

Set $X(t) = \|v(t)\|_{V^r}^2 + \|\theta(t)\|_{H^{r-\beta}}^2$. Then, from (4.5) we easily get

\[
\frac{dX}{dt}(t) \leq C \left( 1 + \|v(t)\|_{V^{2\beta}} \ln(e + 1 + X(t)) + \|v(t)\|_{V^{2\beta}} + \phi(t) \right) X(t) 
\]

This implies that $Y(t) = \ln(e + 1 + X(t))$ satisfies

\[
Y'(t) \leq C \left( 1 + \|v(t)\|_{V^{2\beta}} Y(t) + \|v(t)\|_{V^{2\beta}} + \phi(t) \right). 
\]

By Gronwall lemma we get

\[
\sup_{0 \leq t \leq T} Y(t) \leq Y(0)e^{C\int_0^T \|v(s)\|_{V^{2\beta}} ds} + C \int_0^T e^{C\int_s^T \|v(r)\|_{V^{2\beta}} dr} (1 + \|v(s)\|_{V^{2\beta}}^2 + \phi(s)) ds.
\]

Since $v \in L^2(0, T; V^{2\beta})$ by Theorem 32 and $\phi \in L^1(0, T)$, we get that

\[
\sup_{0 \leq t \leq T} Y(t) \leq K_3
\]

and therefore going back to the unknown $X$

\[
\sup_{0 \leq t \leq T} X(t) \leq K_4;
\]

from (4.5), after integration on $[0, T]$ we get also

\[
\int_0^T \|v(t)\|_{V^{r+\beta}}^2 dt \leq K_5.
\]

Therefore we have proved that

\[
v \in L^\infty(0, T; V^r) \cap L^2(0, T; V^{r+\beta}), \quad \theta \in L^\infty(0, T; H^{r-\beta}).
\]

Now we consider the continuity in time. Lemma 2.6 (with $p = p_2 = 2$, $p_1 = \infty$) gives

\[
\|B(u, v)\|_{V^{r-\beta}} \leq C \left( \|u\|_{L_\infty} \|v\|_{V^{r-\beta+1}} + \|\Lambda^{r-\beta} u\|_{L_\infty} \|\Lambda v\|_{L_\infty^3} \right). 
\]
Take $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta + 2\alpha}{3}$, $\frac{1}{p_4} = \frac{1}{2} - \frac{\beta - 2\alpha}{3}$; by Sobolev embeddings we get

$$\|u\|_{L^\infty} \leq C\Lambda^{-2\alpha\beta} v \|_{L^\infty} \leq C\|v\|_{V^{r-2\alpha}} \leq C\|v\|_{V^r} \quad \text{since } r > \frac{5}{2} - 2\alpha$$

$$\|v\|_{V^{r-\beta+1}} \leq C\|v\|_{V^{r+\beta}} \quad \text{since } \beta \geq 1$$

$$\|\Lambda^{r-\beta} u \|^\infty_{L^p} \leq C\Lambda^{r-2\alpha\beta} v \|_{L^p} \leq C\|v\|_{V^r}$$

$$\|\Lambda v \|^\infty_{L^p} \leq C\|\Lambda v\|_{V^{\frac{r}{2} - 2\alpha}} = C\|v\|_{V^{\frac{r}{2} - 2\alpha}}.$$

Using that $V^{r+\beta} \subset V^{\frac{r}{2} - 2\alpha}$ when $\alpha + \beta \geq \frac{5}{4}$ and $r \geq 0$, we obtain that

$$\|B(u, v)\|_{V^{r-\beta}} \leq C\|v\|_{V^r} \|v\|_{V^{r+\beta}}.$$

This implies

$$\frac{dv}{dt} = -B(u, v) - \nu \Lambda^{2\beta} v + \Pi \theta e_3 \in L^2(0, T; V^{r-\beta})$$

By Lemma 2.7 we deduce that $v \in C([0, T]; V^r)$.

As far as the continuity in time for $\theta$ is concerned, we have that $\theta$ satisfies a transport equation

$$\partial \theta + u \cdot \nabla \theta = 0$$

where the velocity is given and in particular $u \in C([0, T]; V^{r+2\alpha})$. [13] considers this equation in $\mathbb{R}^2$; but a straightforward modification of Lemma 4.4 of [13] allows to prove in the three dimensional case that given $u \in C([0, T]; V^\rho)$ with $\rho > \frac{5}{2}$ and $\theta_0 \in H^k$ with $0 \leq k < [\rho]$, then there exists a unique solution $\theta \in C([0, T]; H^k)$. Taking $\rho = r + 2\alpha$ and $k = r - \beta$, we get the continuity result for $\theta$. \hfill \Box

**Remark 4.2** Let $\beta \geq \frac{1}{2}$. As far as the range of values of $r$ is concerned, we have that when $\alpha + \beta \geq \frac{5}{4}$

$$\max \left\{ \frac{5}{2} - 2\alpha, \beta + 1 \right\} \equiv \max \left( 2\beta, \beta + 1 \right) = \begin{cases} 
\beta + 1 & \text{for } \frac{1}{2} \leq \beta \leq 1 \\
2\beta & \text{for } \beta \geq 1
\end{cases}$$

Therefore we have $r > \frac{3}{2}$ and $r - \beta > 1$ at least. In addition, $r > 2\beta$.

This regularity is enough to get uniqueness.

**Theorem 4.3 (Uniqueness)** We are given $\alpha \geq 0$ and $\beta \geq \frac{1}{2}$ with

$$\alpha + \beta \geq \frac{5}{4}.$$
Let
\[ r > \max \left( \frac{5}{2} - 2\alpha, \beta + 1 \right). \]

Then, the solutions given in Theorem 4.1 are unique.

**Proof.** Let \((v_1, \theta_1)\) and \((v_2, \theta_2)\) be two solutions given by Theorem 4.1. We define \(V = v_1 - v_2, U = u_1 - u_2\) and \(\Phi = \theta_1 - \theta_2\). Using the bilinearity we have that they satisfy
\[
\begin{cases}
\partial_t V + \nu \Lambda^{2\beta} V + B(u_1, V) + B(U, v_2) = \Pi \Phi e_3 \\
\partial_t \Phi + U \cdot \nabla \theta_1 + u_2 \cdot \nabla \Phi = 0
\end{cases}
\]

As before, using (2.2) we get
\[
\frac{1}{2} \frac{d}{dt} \|V(t)\|_{V^0}^2 + \nu \|V(t)\|_{V^\beta}^2 \\
= -\langle B(u_1(t), V(t)), V(t) \rangle - \langle B(U(t), v_2(t)), V(t) \rangle + \langle \Pi \Phi(t) e_3, V(t) \rangle \\
\leq -\langle B(U(t), v_2(t)), V(t) \rangle + \|\Pi \Phi(t)\|_{H^0} \|V(t)\|_{V^0}.
\]

And similarly, using (2.3)
\[
\frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_{\vec{u}^0}^2 = -\langle (U(t) \cdot \nabla \theta_1(t)), \Phi(t) \rangle - \langle u_2(t) \cdot \nabla \Phi(t), \Phi(t) \rangle \\
= -\langle (U(t) \cdot \nabla \theta_1(t)), \Phi(t) \rangle.
\]

Let us estimate the terms on the right hand side of each of the relationships above. For the velocity equation, we proceed as usual by means of the Sobolev embeddings:
\[
|\langle B(U, v_2), V \rangle| \leq \|(U \cdot \nabla)v_2\|_{L_2} \|V\|_{L_2} \\
\leq \|U\|_{L_{p_1}} \|\nabla v_2\|_{L_{p_2}} \|V\|_{V^0} \quad \text{if } \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{2} \\
\leq C \|U\|_{V^{\beta+2\alpha}} \|v_2\|_{V^{2\beta}} \|V\|_{V^0} \quad \text{if } \frac{1}{p_1} = \frac{1}{2} - \frac{\beta + 2\alpha}{3}, \frac{1}{p_2} = \frac{1}{2} - \frac{2\beta - 1}{3} \\
\leq C \|V\|_{V^\beta} \|v_2\|_{V^{2\beta}} \|V\|_{V^0} \\
\leq \frac{\nu}{4} \|V\|_{V^\beta}^2 + C_\nu \|v_2\|_{V^{2\beta}}^2 \|V\|_{V^0}^2
\]

and the condition \(\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{2}\) is equivalent to \(\frac{3}{2} \beta + \alpha \geq \frac{5}{4}\), which is trivially satisfied.
Let \( v_r > \beta \) (given in Theorem 4.1 (\( r > \frac{5}{2} - 2\alpha \) and \( r > \beta + 1 \)) hold true).

Theorem 4.4 (Continuous dependence on the initial data) We are given \( \alpha \geq 0 \) and \( \beta \geq \frac{1}{2} \) with

\[
\alpha + \beta \geq \frac{5}{4}.
\]

Let

\[
r \geq \beta + 2.
\]

Then, given any initial conditions \( v_{1,0}, v_{2,0} \in V^r \) and \( \theta_{1,0}, \theta_{2,0} \in H^{r-\beta} \) we have

\[
\| v_1 - v_2 \|_{L^\infty(0,T;V^r)} + \| v_1 - v_2 \|_{L^2(0,T;V^{r+1-\beta})} + \| \theta_1 - \theta_2 \|_{L^\infty(0,T;H^{r-\beta-1})} 
\leq C \left( \| v_{1,0} - v_{2,0} \|_{V^{r-1}} + \| \theta_{1,0} - \theta_{2,0} \|_{H^{r-\beta-1}} \right)
\]

where the constant \( C \) depends on \( T \), \( \| \theta_1 \|_{L^\infty(0,T;H^{r-\beta})} \), \( \| v_1 \|_{L^2(0,T;V^{r+\beta-1})} \) and \( \| v_i \|_{L^\infty(0,T;V^r)} \).

**Proof.** We begin by pointing out that, under the assumptions on \( \alpha \) and \( \beta \), if \( r \geq \beta + 2 \) then the conditions on \( r \) given in Theorem 4.4 (\( r > \frac{5}{2} - 2\alpha \) and \( r > \beta + 1 \)) hold true.

Similarly, for the temperature equation:

\[
\langle (U \cdot \nabla \theta_1, \Phi) \rangle \leq \| U \cdot \nabla \theta_1 \|_{L^2} \| \Phi \|_{L^2} 
\leq \| U \|_{L^p} \| \nabla \theta_1 \|_{L^{p_4}} \| \Phi \|_{H^0} 
\leq C \| U \|_{V^\beta} \| \theta_1 \|_{H^{r-\beta}} \| \Phi \|_{H^0} 
\leq C \| V \|_{V^\beta} \| \theta_1 \|_{H^{r-\beta}} \| \Phi \|_{H^0} 
\leq \frac{\nu}{4} \| V \|_{V^\beta}^2 + C_\nu \| \theta_1 \|_{H^{r-\beta}}^2 \| \Phi \|_{H^0}^2.
\]

Summing up, we have obtained

\[
\frac{d}{dt} \| V(t) \|_{V^\beta}^2 + \nu \| V(t) \|_{V^\beta}^2 + \frac{d}{dt} \| \Phi(t) \|_{H^0}^2 
\leq C \| v_2(t) \|_{V^\beta}^2 \| V(t) \|_{V^\beta}^2 + \| \theta_1(t) \|_{H^{r-\beta}}^2 \| \Phi(t) \|_{H^0}^2 + \| \Phi(t) \|_{H^0}^2 + \| V(t) \|_{V^\beta}^2.
\]

If we define \( Z(t) = \| V(t) \|_{V^\beta}^2 + \| \Phi(t) \|_{H^0}^2 \), we have \( Z(0) = 0 \) and

\[
Z'(t) \leq C \left( \| v_2(t) \|_{V^\beta}^2 + \| \theta_1(t) \|_{H^{r-\beta}}^2 + 1 \right) Z(t).
\]

By Gronwall lemma we get \( Z(t) = 0 \) for all \( t \), and this completes the proof. \( \square \)
Using the same setting as in the proof of Theorem 4.3, we get

\[
\frac{1}{2} \frac{d}{dt} \|V(t)\|_{V^{r-1}_v}^2 + \nu \|V(t)\|_{V^{r-1+\beta}_\nu}^2 = -\langle B(\Lambda^{-2\alpha}v_1(t), V(t)), \Lambda^{2r-2}V(t) \rangle \\
- \langle B(\Lambda^{-2\alpha}V(t), v_2(t)), \Lambda^{2r-2}V(t) \rangle + \langle \Lambda^{r-\beta-1}\Phi(t)e_3, \Lambda^{r-1+\beta}V(t) \rangle.
\]

We estimate the first two terms of r.h.s. by means of Lemma 5.2

\[
|\langle B(\Lambda^{-2\alpha}v_1(t), V(t)), \Lambda^{2r-2}V(t) \rangle| + |\langle B(\Lambda^{-2\alpha}V(t), v_2(t)), \Lambda^{2r-2}V(t) \rangle| \\
\leq C(\|v_1\|_{V^r} \|V\|_{V^{r-1}} + \|v_1\|_{V^{r+\beta}} \|V\|_{V^{r+\beta-1}}) \|V\|_{V^{r-1}} \\
+ C\|V\|_{V^{r-1}} \|v_2\|_{V^{r+\beta-1}} \|V\|_{V^{r+\beta-1}}.
\]

Using Young inequality, we get

\[
(4.7) \quad \frac{1}{2} \frac{d}{dt} \|V(t)\|_{V^{r-1}_v}^2 + \nu \|V(t)\|_{V^{r-1+\beta}_\nu}^2 \leq \frac{\nu}{2} \|V(t)\|_{V^{r+\beta-1}_\nu}^2 + C\nu \|\Phi(t)\|_{H^{r-1}_\nu} \\
+ C\nu (\|v_1(t)\|_{V^r} + \|v_1(t)\|_{V^{r+\beta-1}} + \|v_2(t)\|_{V^{r+\beta-1-1}}) \|V(t)\|_{V^{r-1}}^2.
\]

Similarly, for the temperature difference; we use Lemma 5.3 and Young inequality

\[
\frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_{H^{r-1}_\nu}^2 = -\langle (U(t) \cdot \nabla \theta_1(t), \Lambda^{2r-2\beta-2}\Phi(t)) - \langle u_2(t) \cdot \nabla \Phi(t), \Lambda^{2r-2\beta-2}\Phi(t) \rangle \\
\leq C\|V(t)\|_{V^{r-1}} \|\theta_1(t)\|_{H^{r-\beta}} \|\Phi(t)\|_{H^{r-\beta-1}} + C\|v_2(t)\|_{V^r} \|\Phi(t)\|_{H^{r-\beta}} \\
\leq C\|V(t)\|_{V^{r-1}}^2 + C(\|\theta_1(t)\|_{H^{r-\beta}}^2 + \|v_2(t)\|_{V^r}) \|\Phi(t)\|_{H^{r-\beta}}^2.
\]

Finally, we consider the sum \( \|V(t)\|_{V^{r-1}}^2 + \|\Phi(t)\|_{H^{r-\beta}}^2 := W(t) \) and define \( a(t) = 1 + \|\theta_1(t)\|_{H^{r-\beta}}^2 + \|v_1(t)\|_{V^{r+\beta-1}}^2 + \|v_2(t)\|_{V^{r+\beta-1}}^2 + \|v_1(t)\|_{V^r} + \|v_2(t)\|_{V^r} \); we have \( a \in L^1(0,T) \) and

\[
W'(t) + \nu \|V(t)\|_{V^{r+\beta-1}}^2 \leq Ca(t)W(t).
\]

Gronwall lemma applied to

\[
W'(t) \leq Ca(t)W(t)
\]

gives

\[
\sup_{0 \leq t \leq T} W(t) \leq W(0)e^{C\int_0^T a(t) \, dt}.
\]

Integrating in time \( (4.8) \) and using the latter result we get the estimate for \( \int_0^T \|V(t)\|_{H^{r+\beta-1}}^2 \, dt \). This concludes the proof. \( \square \)
5 Auxiliary results

In this section we prove the lemma used in the proofs of the previous section.

**Lemma 5.1** Let $\alpha \geq 0$, $\beta \geq \frac{1}{2}$ with $\alpha + \beta \geq \frac{5}{4}$. Then for any $r > 0$ there exists a constant $C > 0$ such that

$$\|\langle B(\Lambda^{-2\alpha} v, v), \Lambda^{2r} v \rangle \| \leq C\|v\|_{V^{2\beta}}\|v\|_{V^{r+\beta}}\|v\|_{V^r}.$$  

**Proof.** Set $u = \Lambda^{-2\alpha} v$. First

$$\langle B(u, v), \Lambda^{2r} v \rangle = \langle \Lambda^r ((u \cdot \nabla)v), \Lambda^{r} v \rangle = \langle [\Lambda^{r}, u] \cdot \nabla v, \Lambda^{r} v \rangle \quad \text{by} \quad (2.16)$$

$$\leq \| [\Lambda^{r}, u] \cdot \nabla v \|_{L^2} \| \Lambda^{r} v \|_{L^2}.$$  

Then, we use the Commutator Lemma 2.5 with $p = 2$:

$$\| [\Lambda^{r}, u] \cdot \nabla v \|_{L^2} \leq C \left( \| \Lambda^{1-2\alpha} u \|_{L^{p_1}} \| \Lambda^{r} v \|_{L^{p_2}} + \| \Lambda^{r} u \|_{L^{p_3}} \| \Lambda^{1-2\alpha} v \|_{L^{p_4}} \right)$$

$$\leq C \| \Lambda^{2\beta} v \|_{L^2} \| \Lambda^{r+\beta} v \|_{L^2}.$$  

For the latter estimate we have used the embeddings

(5.1) \[
\begin{cases}
\| \Lambda^{1-2\alpha} u \|_{L^{p_1}} \leq C \| \Lambda^{2\beta} v \|_{L^2} & \text{for } \frac{1}{p_1} = \frac{1}{2} - \frac{2\beta - (1-2\alpha)}{3} \geq 1 - 2\alpha \\
\| \Lambda^{r} v \|_{L^{p_2}} \leq C \| \Lambda^{r+\beta} v \|_{L^2} & \text{for } \frac{1}{p_2} = \frac{1}{2} - \frac{\beta}{3} \geq 1
\end{cases}
\]

The condition $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{2}$ is equivalent to $\alpha + \frac{3}{2} \beta \geq \frac{5}{3}$, which holds when $\alpha + \beta \geq \frac{5}{4}$. Moreover $\alpha + \beta \geq \frac{5}{4}$ implies that the condition $2\beta \geq 1 - 2\alpha$, needed for the embedding, holds true.

Similarly for the other two terms:

(5.2) \[
\begin{cases}
\| \Lambda^{r-2\alpha} v \|_{L^{p_3}} \leq C \| \Lambda^{r+\beta} v \|_{L^2} & \text{for } \frac{1}{p_3} = \frac{1}{2} - \frac{r + \beta - (r-2\alpha)}{3} \geq \frac{1}{2} - \frac{2\beta - 1}{3} \geq 1 \\
\| \Lambda^r v \|_{L^{p_4}} \leq C \| \Lambda^{2\beta} v \|_{L^2} & \text{for } \frac{1}{p_4} = \frac{1}{2} - \frac{2\beta - 1}{3} \geq 1
\end{cases}
\]

The condition $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ is again equivalent to $\alpha + \frac{3}{2} \beta \geq \frac{5}{3}$. \qed

**Lemma 5.2** Let $\alpha \geq 0$, $\beta \geq \frac{1}{2}$ with $\alpha + \beta \geq \frac{5}{4}$. If

$$r > \frac{5}{2} - 2\alpha \quad \text{and} \quad r \geq 1 + \beta,$$

then
then there exists a constant $C > 0$ such that

$$|\langle B(\Lambda^{-2\alpha}w, v), \Lambda^{2r-2}v \rangle| \leq C(\|w\|_{V^r} \|v\|_{V^{r-1}} + \|w\|_{V^{r+\beta-1}} \|v\|_{V^{r+\beta-1}}) \|v\|_{V^{r-1}}$$

and

$$|\langle B(\Lambda^{-2\alpha}v, w), \Lambda^{2r-2}v \rangle| \leq C\|v\|_{V^{r-1}} \|w\|_{V^{r+\beta-1}} \|v\|_{V^{r+\beta-1}}.$$

**Proof.** First, notice that we also have $r \geq 1$ and $r \geq 2 - \beta$.

To prove the first inequality, we use the Commutator Lemma 2.5 with $p = p_2 = 2$, $p_1 = \infty$, $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta + 2\alpha}{3}$, $\frac{1}{p_4} = \frac{1}{2} - \frac{r+\beta - 2}{3}$; the condition $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ is equivalent to $\alpha + \beta + \frac{r}{2} \geq \frac{7}{4}$, which is fulfilled if $\alpha + \beta \geq \frac{5}{2}$ and $r \geq 1$. Then we use the Sobolev embeddings:

$$|\langle B(\Lambda^{-2\alpha}w, v), \Lambda^{2r-2}v \rangle| = |\langle \Lambda^{r-1} (\Lambda^{-2\alpha}w \cdot \nabla)v, \Lambda^{r-1}v \rangle|$$

by (2.16)

$$\leq C(\|\Lambda^{-2\alpha}w\|_{L^p_{\infty}} \|\Lambda^{r-1}v\|_{L^2} + \|\Lambda^{r-1-2\alpha}w\|_{L^{p_3}} \|\Lambda v\|_{L^{p_4}}) \|\Lambda^{r-1}v\|_{L^2}$$

$$\leq C(\|w\|_{V^r} \|v\|_{V^{r-1}} + \|w\|_{V^{r+\beta-1}} \|v\|_{V^{r+\beta-1}}) \|v\|_{V^{r-1}}$$

when $r > \frac{5}{2} - 2\alpha$.

For the second inequality, we use Lemma 2.6 with $p = 2$, $\frac{1}{p_1} = \frac{1}{2} - \frac{\beta + 2\alpha}{3}$, $\frac{1}{p_2} = \frac{1}{2} - \frac{\beta + 2\alpha}{3}$, $\frac{1}{p_3} = \frac{1}{2} - \frac{r+\beta - 2}{3}$, the conditions $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ and $\frac{1}{p_3} \leq \frac{1}{2}$ are again equivalent to $\alpha + \beta + \frac{r}{2} \geq \frac{7}{4}$. Then we use the Sobolev embeddings:

$$|\langle B(\Lambda^{-2\alpha}v, w), \Lambda^{2r-2}v \rangle| = |\langle \Lambda^{r-1-\beta} (\Lambda^{-2\alpha}v \cdot \nabla)w, \Lambda^{r+\beta-1}v \rangle|$$

$$\leq C(\|\Lambda^{-2\alpha}v\|_{L^{p_1}} \|\Lambda^{r-\beta}v\|_{L^{p_2}} + \|\Lambda^{r-1-2\alpha}v\|_{L^{p_3}} \|\Lambda w\|_{L^{p_4}}) \|v\|_{V^{r+\beta-1}}$$

$$\leq C\|v\|_{V^{r-1}} \|w\|_{V^{r+\beta-1}} \|v\|_{V^{r+\beta-1}}.$$

\[ \square \]

**Lemma 5.3** Let $\alpha \geq 0$ and $\beta > 0$.

If

$$r > \frac{5}{2} - 2\alpha \text{ and } r \geq \beta + 2,$$

then there exists a constant $C > 0$ such that

$$|\langle \Lambda^{-2\alpha}v \cdot \nabla \theta, \Lambda^{2r-2\beta-2} \phi \rangle| \leq C\|v\|_{V^{r-1}} \|\theta\|_{H^{r-\beta}} \|\phi\|_{H^{r-\beta-1}}$$

and

$$|\langle \Lambda^{-2\alpha}v \cdot \nabla \theta, \Lambda^{2r-2\beta-2} \theta \rangle| \leq C\|v\|_{V^r} \|\theta\|_{H^{r-\beta-1}}^2.$$
Proof. To prove the first inequality, we use Lemma 2.6 with $p = p_2 = 2$, $p_1 = \infty$, $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta + 2\alpha + 1}{3}$, $\frac{1}{p_4} = \frac{1}{2} - \frac{r - \beta - 2}{3}$; the condition $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ is equivalent to $r \geq \frac{5}{2} - 2\alpha$. Then the Sobolev embeddings:

$$||\langle \Lambda^{-2\alpha} v \cdot \nabla \theta, \Lambda^{2r - 2\beta - 2} \phi \rangle|| = ||\langle \Lambda^{-\beta - 1}(\Lambda^{-2\alpha} v \cdot \nabla \theta), \Lambda^{\beta - 1} \phi \rangle||$$

$$\leq ||\Lambda^{\beta - 1}(\Lambda^{-2\alpha} v \cdot \nabla \theta)||_{L^2} \phi||_{H^{\beta - 1}}$$

$$\leq C(||\Lambda^{-2\alpha} v||_{L^\infty} ||\Lambda^{\beta} \theta||_{L^2} + ||\Lambda^{\beta - 1 - 2\alpha} v||_{L^p_3} ||\Lambda \theta||_{L^p_4}) \phi||_{H^{\beta - 1}}$$

$$\leq C(||\Lambda^{-2\alpha} v||_{V^{r + 2\alpha - 1}} \theta||_{H^{\beta - 1}} + ||v||_{V^r} ||\theta||_{H^{\beta - 1}} \phi||_{H^{\beta - 1}}$$

since $V^{r+2\alpha-1} \subset L^\infty$ for $r > \frac{5}{2} - 2\alpha$.

For the second inequality, we use the Commutator Lemma 2.5 with $p = p_2 = 2$, $p_1 = \infty$, $\frac{1}{p_3} = \frac{1}{2} - \frac{\beta + 2\alpha + 1}{3}$, $\frac{1}{p_4} = \frac{1}{2} - \frac{r - \beta - 2}{3}$; the condition $\frac{1}{p_3} + \frac{1}{p_4} \leq \frac{1}{2}$ is equivalent again to $r \geq \frac{5}{2} - 2\alpha$. Then we use the Sobolev embeddings as before:

$$||\langle \Lambda^{-2\alpha} v \cdot \nabla \theta, \Lambda^{2r - 2\beta - 2} \theta \rangle|| = ||\langle \Lambda^{-\beta - 1}(\Lambda^{-2\alpha} v \cdot \nabla \theta), \Lambda^{\beta - 1} \theta \rangle||$$

$$= ||\langle [\Lambda^{\beta - 1}, \Lambda^{-2\alpha} v] \cdot \nabla \theta, \Lambda^{\beta - 1} \theta \rangle||$$

by (2.15)

$$\leq C(||\Lambda^{1-2\alpha} v||_{L^\infty} ||\Lambda^{\beta - 1} \theta||_{L^2} + ||\Lambda^{\beta - 1 - 2\alpha} v||_{L^p_3} ||\Lambda \theta||_{L^p_4}) \theta||_{H^{\beta - 1}}$$

$$\leq C(||\Lambda^{1-2\alpha} v||_{V^{r + 2\alpha - 1}} \theta||_{H^{\beta - 1}} + ||v||_{V^r} ||\theta||_{H^{\beta - 1}} \theta||_{H^{\beta - 1}}$$

$$= C||v||_{V^r} \theta||^2_{H^{\beta - 1}}.$$

Lemma 5.4 Let $\alpha \geq 0, \beta > 0$ with $\alpha + \beta \geq \frac{5}{4}$ and $r > \beta + 1$. Then, there exist $q_3, q_4 > 2$ with $\frac{1}{q_3} + \frac{1}{q_4} \leq \frac{1}{2}$ and $q > 2$, $a \in (0, 1)$, $C > 0$ such that

$$||\Lambda^{\beta - 2\alpha} v||_{L^{q_3}} \Lambda \theta||_{L^{q_4}} \theta||_{H^{\beta - 1}} \leq C ||v||_v^{\alpha} \theta||_v^{\beta} ||v||_v^{1 - a} \theta||_v^{1 + a} \theta||_{H^{\beta - 1}}.$$

Proof. We use Sobolev embedding theorem, interpolation theorem and the Gagliardo-Nirenberg inequality; then for some $a \in (0, 1)$ and $q \geq 2$ to be defined later on we look for

$$\begin{cases} ||\Lambda^{\beta - 2\alpha} v||_{L^{q_3}} \leq C ||\Lambda^{\beta - r\alpha + \beta} v||_{L^{q_3}} \leq C ||v||_v^{\beta} \theta||_v^{1 - a} \theta||_v^{1 + a} \theta||_{L^2} \\ ||\Lambda \theta||_L \leq C \theta||^{1 - a}_{L^q} \theta||^{\beta}_{L^{q_3}} \theta||_L \end{cases}$$

under the conditions

$$\begin{cases} r + \beta - r\alpha + \beta a \geq r - \beta - 2\alpha \\ \frac{1}{q_3} + \frac{1}{q_4} \leq \frac{1}{2} \\ \frac{1}{r - \beta} \leq a < 1 \end{cases}$$

25
equivalent to (since \( r > \beta \) by assumption)

\[
\begin{align*}
\left\{ \begin{array}{l}
a \leq \frac{2\alpha + \beta}{r - \beta} \\
\frac{1}{3} + \frac{a}{2} + \frac{1-a}{q} \leq \frac{2}{3}(\alpha + \beta) \\
\frac{1}{r - \beta} \leq a < 1
\end{array} \right.
\end{align*}
\]

The second equation is satisfied for some \( q \) (big enough) when \( \frac{1}{3} + \frac{a}{2} < \frac{2}{3}(\alpha + \beta) \); therefore we need to find \( a \in (0, 1) \) such that

\[
\frac{1}{r - \beta} \leq a < \min \left( \frac{2\alpha + \beta}{r - \beta}, \frac{2(2\alpha + \beta) - 1}{3} \right).
\]

Assuming \( \alpha + \beta \geq \frac{5}{4} \), we have \( 2(\alpha + \beta) \geq \frac{5}{2} > 1 \) and \( 2^2(\alpha + \beta) - 1 \geq 1 \); this implies that, under the conditions \( r - \beta > 1 \) there exists \( a \in (0, 1) \) satisfying (5.4).

**Acknowledgements** The research of Hakima Bessaih was supported by the NSF grants DMS-1416689 and DMS-1418838. Part of this research started while Hakima Bessaih was visiting the Department of Mathematics of the University of Pavia and was partially supported by the GNAMPA-INDAM project ”Regolarità e dissipazione in fluidodinamica”; she would like to thank the hospitality of the Department.

**References**

[1] H. Abidi, T. Hmidi: On the global well-posedness for Boussinesq system, *J. Differential Equations* 233 (2007), no. 1, 199-220

[2] H. Abidi, T. Hmidi, S. Keraani: On the global regularity of axisymmetric Navier-Stokes-Boussinesq system, *Discrete Contin. Dyn. Syst.* 29 (2011), no. 3, 737-756

[3] D. Barbato, H. Bessaih, B. Ferrario: On a stochastic Leray-\( \alpha \) model of Euler equations, *Stochastic Process. Appl.* 124 (2014), no. 1, 199-219

[4] D. Barbato, F. Morandin, M. Romito: Global regularity for a slightly supercritical hyperdissipative Navier-Stokes system, *arxiv*:1407.6734v1 (2014)

[5] H. Brézis, T. Gallouet: Nonlinear Schrödinger evolution equations, *Nonlinear Anal.* 4 (1980), no. 4, 677-681
[6] H. Brézis, S. Wainger: A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* 5 (1980), no. 7, 773-789

[7] D. Chae: Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Adv. Math.* 203 (2006), no. 2, 497-513

[8] R. Danchin, M. Paicu: Existence and uniqueness results for the Boussinesq system with data in Lorentz spaces, *Phys. D* 237 (2008), no. 10-12, 1444-1460

[9] R. Danchin, M. Paicu: Global well-posedness issues for the inviscid Boussinesq system with Yudovich’s type data, *Comm. Math. Phys.* 290 (2009), no. 1, 1-14

[10] J. Geng, J. Fan: A note on regularity criterion for the 3D Boussinesq system with zero thermal conductivity, *Appl. Math. Lett.* 25 (2012), no. 1, 63-66

[11] T. Hmidi, Keraani: On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity, *Adv. Differential Equations* 12 (2007), no. 4, 461-480

[12] T. Y. Hou, C. Li: Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dyn. Syst.* 12 (2005), no. 1, 1-12

[13] T. Kato, G. Ponce: Well posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p(R^2)$, *Rev. Mat. Iberoam.* 2 (1986), no. 1-2, 73-88

[14] T. Kato, G. Ponce: Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.* 41 (1988), no. 7, 891-907

[15] C. Kenig, G. Ponce, L. Vega: Well-posedness of the initial value problem for the Korteweg-de-Vries equation, *J. Amer. Math. Soc.* 4 (1991), 323-347

[16] J.-L. Lions: *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Gauthier-Villars, Paris 1969

[17] J. C. Mattingly, Ya. G. Sinai: An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations, *Commun. Contemp. Math.* 1 (1999), no. 4, 497-516
[18] L. Nirenberg: On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa 13 (1959), no. 2, 115-162

[19] E. Olson, E. S. Titi: Viscosity versus vorticity stretching: global well-posedness for a family of Navier-Stokes-alpha-like models, Nonlinear Anal. 66 (2007), no. 11, 2427-2458

[20] H. Qiu, Y. Du, Z. Yao: A blow-up criterion for 3D Boussinesq equations in Besov spaces, Nonlinear Anal. 73 (2010), no. 3, 806-815

[21] W. A. Strauss: On continuity of functions with values in various Banach spaces, Pacific J. Math. 19 (1966), 543-551

[22] R. Temam: Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications 2, North-Holland Publishing Co., Amsterdam-New York, 1979.

[23] R. Temam: Navier-Stokes equations and nonlinear functional analysis, Second edition. CBMS-NSF Regional Conference Series in Applied Mathematics, 66. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[24] K. Yamazaki: On the global regularity of generalized Leray-alpha type models, Nonlin. Anal. 75 (2012), 503–515

[25] Z. Ye: A note on global well-posedness of solutions to Boussinesq equations with fractional dissipation, Acta Math. Sci. Ser. B Engl. Ed. 35 (2015), no. 1, 112-120

[26] Xiang Zhaoyin, Yan Wei: Global regularity of solutions to the Boussinesq equations with fractional diffusion, Adv. Differential Equations 18 (2013), no. 11-12, 1105-1128