A scale-critical trapped surface formation criterion for the Einstein-Maxwell system

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Abstract

In this study we show that, from arbitrarily dispersed initial data, both the concentration of electromagnetic fields and the focusing of gravitational waves could lead to the formation of trapped surfaces. We establish a scale-critical semi-global existence result from the past null infinity for the Einstein-Maxwell system by assigning the signature for decay rates to geometric quantities and Maxwell components. This result generalizes an approach of the first author on studying Einstein vacuum equations by employing additional elliptic estimates and geometric renormalizations, and it also extends a result of Yu to the scale-critical regime.

1 Introduction

1.1 Background

In this paper, we study the evolution of the Einstein–Maxwell system for a $(3 + 1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ and an electromagnetic 2-tensor $F_{\alpha \beta}$:

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = T_{\mu \nu},$$

where

$$T_{\mu \nu} = F_{\mu \lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu \nu} F_{\lambda \tau} F^{\lambda \tau}.$$ (1.2)

Although a direct extension of Birkhoff’s theorem implies that the 2–parameter family of Reissner–Nördstrom electrovacuum spacetimes exhausts all possible spherically symmetric solutions to the Einstein-Maxwell system, in the absence of symmetry assumptions, the global dynamics of (1.1)-(1.2) are quite hard to study.

For the Einstein vacuum equations (where $T_{\mu \nu} = 0$), the study of (1.1)-(1.2) in the small data regime has been very successful. In a monumental work of Christodoulou and Klainerman [7], it is shown that the Minkowski spacetime is stable under small perturbations. Christodoulou and Klainerman showed that, for small perturbations of the trivial data, no singularity will form and all geodesics are complete. Later, Zipser [5] extended this result to the Einstein–Maxwell system.

One of the most fascinating aspects of the classical theory of general relativity is that it predicts the existence of a black hole. Historically, some notion of a black hole accompanies the theory of general relativity almost since its inception by Einstein in 1915. It was first encountered in an explicit solution to the Einstein vacuum equations and in particular the Schwarzschild solution $(\mathcal{M}, g)_{\text{Schw}}$, communicated by Schwarzschild to Einstein in a letter about one month after the latter presented his field equations of general relativity at the Prussian academy of sciences. However, it was neither Schwarzschild nor Einstein who understood that what would come to be known as a black hole region featured prominently in the Schwarzschild solution. It was Lemaître [13] who first observed, in 1932, that $(\mathcal{M}, g)_{\text{Schw}}$ contains a region $\mathcal{B}$ with the property that observers lying inside $\mathcal{B}$ cannot send signals to observers situated at an ideal conformal boundary at
infinity \( I^+ \) (this being defined in a rigorous and appropriate way). In the case of Schwarzschild solution, the existence of a non-empty \( \mathcal{B} \) is accompanied by another surprising, yet salient, feature:

Every observer in \( \mathcal{B} \) lives for finite proper time (future geodesic incompleteness).

When physicists and mathematicians first realized these two properties, they were hoping to be able to associate to them the characterization of an accident; a non-generic pathology, present only due to the high degree of symmetry imposed a priori on the solutions and that, in general solutions to the equations, such phenomena would not arise. Much to the surprise of the community, Penrose [19] in 1965 proved these hopes were ill–based through the following incompleteness theorem:

**Theorem 1.1** For a spacetime \((\mathcal{M}, g)\) containing a non–compact Cauchy hypersurface and \( g_{\mu\nu}, F_{\mu\nu} \) satisfying (1.1)–(1.2), if \( \mathcal{M} \) contains a compact trapped surface, then it is future causally geodesically incomplete.

A trapped surface is a 2–dimensional geometric object. Assume we are given a \((3 + 1)\)–dimensional, time–oriented Lorentzian manifold \((\mathcal{M}, g)\) and within it a closed, spacelike 2–surface \( S \). Since \( S \) has co–dimension 2, the tangent space at a point \( p \) on \( S \), \( T_pS \) has a 2–dimensional orthogonal complement in \( T_p\mathcal{M} \). Let \( l, l \) denote a null basis\(^1\) of this complement and extend \( l, l \) as vector fields. We define the following two fundamental forms \( \chi, \chi \) associated with the surface \( S \):

\[
\chi(X, Y) := g(\nabla_l X, Y), \quad \chi(X, Y) := g(\nabla_l Y, Y)
\]

where \( X \) and \( Y \) are vector fields tangent to \( S \). We look at the expansions \( \text{tr} \chi, \text{tr} \chi \). If both are pointwise negative on \( S \), then the surface is called trapped. A trapped surface is, therefore, a surface for which the area decreases for arbitrary infinitesimal displacements along the null generators of both null geodesic congruences normal to \( S \). Penrose’s theorem implies that the study of singularity formation for Einstein’s equations can, in some generality, be reduced to the problem of trapped surface formation. This problem had, again, remained open for a long time.

1.1.1 The Einstein vacuum case

In a breakthrough work in 2008, Christodoulou solved this long–standing open problem (trapped surface formation for Einstein vacuum equations) with a 587–page monumental work [6]. He designed an open set of large initial data, which encode a special structure, the short pulse ansatz. In particular, this ansatz allows one to consider a hierarchy of large and small quantities, parametrized by a small parameter \( \delta \). For such initial data, these quantities behave differently, being of various sizes in term of \( \delta \). Moreover, their sizes form a hierarchy. But for each quantity, surprisingly, its size is almost preserved by the nonlinear evolution. Therefore, once this hierarchy is satisfied at the level of initial data, it persists for later time. With this philosophy, despite it being a large data problem, a long–time global existence theorem can be established. Moreover, these initial conditions indeed lead to trapped–surface formation in finite time.

Effort was consequently put towards simplifying Christodoulou’s proof. In [12], an ingenious systematical approach was introduced by Klainerman–Rodnianski [12]. This approach was later extended by the first author in [1]. The Einstein vacuum equations are a nonlinear hyperbolic system, containing many unknowns. Christodoulou controlled all of them on a term–by–term basis. In [12], Klainerman and Rodnianski introduced a novel index \( s_1 \) which they termed signature for short pulse. With this index, Klainerman and Rodnianski systematically tracked the \( \delta \)–weights used in the estimates and gave a shorter, simplified proof of the almost–preservation of the \( \delta \)–hierarchy in a finite region. In [6], besides \( \delta \)–weights, Christodoulou also employed weights related to decay and proved his main theorem that a trapped surface could form dynamically with initial data prescribed arbitrary dispersed at past null infinity. In [1], the first author introduced a new index \( s_2 \) called signature for decay rates. With the help of this new index, the first author extended Klainerman and Rodnianski’s result [12] from a finite region to an infinite region and re–proved Christodoulou’s main theorem in [6] with around 120 pages.

\(^1\) That means \( g(l, l) = 0, \ g(l, \bar{Y}) = 0, \ g(l, \bar{D}) = -2 \).
Another important progress was made by the first author and Luk in [2]. By designing and employing a different hierarchy, in [2] they improved [6] and proved the first scale–critical result for the Einstein vacuum equations. With the same small parameter $\delta$, with relatively larger initial data, Christodoulou formed a trapped surface of radius 1; while with much smaller initial data, the first author and Luk formed a trapped surface of radius $\delta a$, where $a$ is a universal large constant like 1000. In [2] the first author and Luk want to form a tiny trapped surface with radius $\delta a$, hence they have to deal with the region very close to the center. In this region all the geometric quantities have growth rates. To bound these growth rates, they employ weighted estimates as well as several crucial geometric renormalizations.

Since [2] is scale critical, one can keep $a$ as a universal constant and let $\delta \to 0$. Hence a series of trapped surfaces (with radius shrinking to 0) are obtained. In [3], the first author further explored this idea. Together with an elliptic approach to identify the boundary, the first author showed that a whole black hole region could emerge dynamically from just a “point” $O$ in the spacetime. For an open set of initial data and appropriate control on all the derivatives of $\hat{\chi}_0$, this boundary (apparent horizon) is proved to be smooth except at $O$.

In early 2019, the first author [4] gave a different, 55–page proof of a trapped surface formation theorem that sharpens the previous results both of Christodoulou [6] and estimates in An–Luk [2]. The argument in [4] is based on a systematic extension of the scale–critical arguments in [2], connecting Christodoulou’s short–pulse method and Klainerman–Rodnianski’s signature counting argument to the peeling properties previously used in small–data results such as Klainerman–Nicolo. This in particular allows the author to avoid elliptic estimates and geometric renormalizations, and gives new technical simplifications.

1.1.2 The Einstein–Maxwell case

For the Einstein-Maxwell system (1.1)-(1.2), important progress was made by Yu [21, 22]. In a finite region, Yu extended the result of Klainerman-Rodnianski [12] and obtained trapped surface formation for the Einstein-Maxwell system by using signature for short-pulse. In the current paper, combining the new ingredients in [4], we will extend Yu’s results to obtain a scale-critical trapped surface formation criterion from past null infinity.

1.1.3 The Einstein–scalar field case

In a recent paper [14], Li and Liu studied the Einstein-scalar field system:

\[ \text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 T_{\mu\nu}, \]

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi, \]

and an almost scale-critical trapped surface formation criterion was obtained with singular initial data prescribed along the incoming null hypersurface. And renormalizations for scalar fields were used.

1.2 Main Results

We will introduce coordinates $u$ and $\underline{u}$ in $(\mathcal{M}, g)$ through a double null foliation, where $H_u$ and $\underline{H}_u$ are incoming and outgoing characteristic cones, respectively. With coordinates $u, \underline{u}$, characteristic initial data will be prescribed along incoming null hypersurface $H_0$, where $\underline{u} = 0$, and outgoing null hypersurface $H_{u_{\infty}}$, where $u = u_{\infty}$.

\[^2\text{Letting } a = \delta^{-1}, \text{ in a finite region they recover Christodoulou’s main result of [6].}\]

\[^3\text{The detailed construction of double null foliation will be explained in Section 2.1.}\]
Our main results can be summarized in three Theorems. The first one is a global existence result.

**Theorem 1.2** Given $I^{(0)}$, there exists a sufficiently large $a_0 = a_0(I^{(0)})$ such that the following holds. For any $0 < a_0 < a$ and with initial data $(\hat{\chi}, \alpha_F)$ satisfying

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{i}{2}} \| \nabla^k (|u|\nabla) \hat{\chi}, \alpha_F \|_{L^\infty(S_{u=\infty}), \infty} \leq I^{(0)}$ along $u = u_\infty$,
- Minkowskian initial data along $u = 0$,

then the Einstein-Maxwell system admits a unique smooth solution in the region

$$u_\infty \leq u \leq -a/4, \ 0 \leq u \leq 1.$$ 

The second one steps directly on the first one and is a formation of trapped surfaces statement.

**Theorem 1.3** Given $I^{(0)}$, there exists a sufficiently large $a_0 = a_0(I^{(0)})$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(M, g)$ of the Einstein–Maxwell system from Theorem 1.2 with initial data satisfying

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{i}{2}} \| \nabla^k (|u|\nabla) \hat{\chi}, \alpha_F \|_{L^\infty(S_{u=\infty}), \infty} \leq I^{(0)}$ along $u = u_\infty$,
- Minkowskian initial data along $u = 0$,
- $\int_0^1 |u_\infty|^2 \left( |\hat{\chi}_0|^2 + |\alpha_F_0|^2 \right) (u_\infty, u') du' \geq a$ uniformly for every direction along $u = u_\infty$

has a trapped surface at $S_{-\alpha/4, 1}$.

The third one is a rescaling of Theorem 1.3 and constitutes a criterion for trapped surface formation of the Einstein–Maxwell system in the region close to the center.

**Theorem 1.4** Given $I^{(0)}$ and a fixed $\delta > 0$, there exists a sufficiently large $a_0 = a_0(I^{(0)}, \delta)$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(M, g)$ of the Einstein-Maxwell equations from Theorem 1.2 with initial data satisfying

- $\sum_{i \leq 15, k \leq 3} a^{-\frac{i}{2}} \| \nabla^k (|u|\nabla) \hat{\chi}, \alpha_F \|_{L^\infty(S_{u=\infty}), \infty} \leq I^{(0)}$ along $u = u_\infty$,
- Minkowskian initial data along $u = 0$,
- $\int_0^\delta |u_\infty|^2 \left( |\hat{\chi}_0|^2 + |\alpha_F_0|^2 \right) (u_\infty, u') du' \geq a$ uniformly for every direction along $u = u_\infty$

has a trapped surface at $S_{-\delta \alpha/4, \delta}$. 


1.3 New Ingredients

Compared to the corresponding problem for the Einstein vacuum equations, the Einstein-Maxwell system gives rise to additional technical difficulties not encountered in the vacuum case. Here we list some important ones and outline the solutions.

1. We extend the systematical way of assigning signatures $s_2$ for the geometric quantities to the Maxwell field. To achieve this, we resort to the Bianchi equations, keeping the same $s_2$ values for the Ricci coefficients as those in the vacuum case. Crucially, the Maxwell equations expressed in a null frame have, in a sense, the same structure as the vacuum Bianchi equations. This means that such an assignment of signatures as in [1], [4] can be carried through in a cohesive and coherent way to the Einstein–Maxwell system.

2. The fact that the Ricci tensor is non–trivial for the Einstein–Maxwell system implies that the energy estimates will be best carried out using the Weyl tensor components instead of the Riemann tensor components. We thus work with the Weyl components $\alpha_W, \omega_W, \beta_W, \rho_W, \sigma_W$ and re-express all equations with respect to them. For simplicity, we still use $\alpha, \omega, \beta, \rho, \sigma$ to mean $\alpha_W, \omega_W, \beta_W, \rho_W, \sigma_W$.

3. We introduce and employ crucial renormalizations for $\beta_A$ and $\omega_A$. The reason for this stems from the Bianchi equations. For example, in the Bianchi equation involving $\nabla_4 \beta$, the right–hand side contains the term $D_1 R_{4A}$. This, in turn, contains the term $\nabla_4 \alpha F$. When one attempts to estimate $\nabla_4 \alpha F$, there is no available equation for it in the null Maxwell equations, thus making it difficult to estimate by itself. It is for this reason that we introduce the quantities

$$\tilde{\beta}_A := \beta_A - \frac{1}{2} R_{4A}, \quad \tilde{\omega}_A := \omega_A - \frac{1}{2} R_{3A}.$$.

We then rewrite the entire Bianchi equations in terms of those renormalized quantities and use them to do energy estimates. In this way, all the terms can be estimated directly through the null Maxwell equations and null Bianchi equations. In particular, terms like $\nabla_4 \alpha F$ and $\nabla_3 \omega F$ no longer appear this time.

4. The above renormalizations force us to introduce and use elliptic estimates in the scale–invariant framework. This is achieved in Section 6. Its purpose is to allow us to close the energy estimates for up to 11 derivatives of the Maxwell field components and up to 10 derivatives of the Weyl curvature components. In the process, a control on 11 derivatives of the Ricci coefficients is required. We find a non–trivial way (via energy estimates) to incorporate the elliptic estimates into the systematical approach via the signature for decay rates $s_2$.

2 Setting, equations and notations

2.1 Double Null Foliation

We construct a double null foliation in a neighbourhood of $S_{u=0}$ as follows:
We proceed to define the function \( \Omega \) to be 1 on \( S_{u,\infty} \) and extend \( \Omega \) as a continuous function along \( H_{u,\infty} \) and \( H_0 \). \(^4\) We consider vector fields

\[
L = \Omega^2 L' \quad \text{along} \quad H_{u,\infty}, \quad \text{and} \quad \underline{L} = \Omega^2 L'' \quad \text{along} \quad H_0,
\]

and define functions

\[
u \text{ on } H_{u,\infty} \text{ satisfying } L\nu = 1 \text{ and } \nu = 0 \text{ on } S_{u,\infty}, \\
u \text{ on } H_0 \text{ satisfying } L\nu = 1 \text{ and } \nu = u_{\infty} \text{ on } S_{u,\infty}.
\]

Let \( S_{u,\infty,w}' \) be the embedded 2-surface on \( H_{u,\infty} \), such that \( \underline{u} = u' \). Similarly, define \( S_{w',0} \) to be the embedded 2-surface on \( H_0 \), such that \( u = u' \). At each point \( q \) on 2-surface \( S_{u,\infty,w}' \), we already have the preferred outgoing null vector \( L_q' \) tangent to \( H_{u,\infty} \). Hence, at \( q \), we can also fix a unique incoming null vector \( L_q'' \) via requiring

\[
g(L_q', L_q') = 0 \quad \text{and} \quad g(L_q', L_q'') = -2\Omega^{-2}|q|.
\]

There exists a unique geodesic \( L_q \) emanating from \( q \) with direction \( L_q' \). We then extend \( L_q' \) along \( L_q' \) by imposing \( D_{L_q} L_q' = 0 \). Gathering all the \( \{L_q'\} \) for \( q \in S_{u,\infty,w}' \), we construct the incoming null hypersurface \( H_q'' \) emanating from \( S_{u,\infty,w}' \). Similarly, from \( S_{w',0} \) we also construct the outgoing null hypersurface \( H_{w'} \). We further define the 2-spheres \( S_{w',w} := H_{w'} \cap H_{w''} \).

At each point \( p \) of \( S_{w',w} \), we define the positive-valued function \( \Omega \) via

\[
g(L_p', L_p'') = -2\Omega^{-2}|p|.
\]

Note that \( L_p' \) is well-defined on \( H_{w'} \), along an outgoing null geodesic \( l \) passing through \( p \); \( L_p'' \) is also well-defined on \( H_{w''} \), along an incoming null geodesic \( l \) crossing \( p \).

These 3-dimensional incoming null hypersurfaces \( \{H_{w} \}_{0 \leq w \leq 1} \), along with the outgoing null hypersurfaces \( \{H_{w'} \}_{w \leq 0 \leq 1} \) and their pairwise intersections \( S_{w',w} = H_{w'} \cap H_{w''} \) give us the so-called double null foliation.

\(^4\)For a general double null foliation, we have the gauge freedom of choosing how to extend \( \Omega \) along \( H_{u,\infty} \) and \( H_0 \). In this paper, we extend \( \Omega \equiv 1 \) on both \( H_{u,\infty} \) and \( H_0 \).
On $S_{\bar{u}, \bar{u}}$, by (2.1) we have $g(L', L') = -2\Omega^{-2}$. Thus, $g(\Omega L', \Omega L') = -2$. Throughout this paper we will work with the normalized null pair $(e_3, e_4)$, namely
\[ e_3 := \Omega L', \quad e_4 := \Omega L', \quad g(e_3, e_4) = -2. \]

Moreover, for the imposition of our characteristic initial data we choose the following gauge:

\[ \Omega \equiv 1 \text{ on } H_{u_{\infty}} \text{ and } H_0. \]

**Remark 1** The functions $u$ and $\bar{u}$ defined above also satisfy the eikonal equations
\[ g^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad g^{\mu\nu} \partial_\mu \bar{u} \partial_\nu \bar{u} = 0. \]

And it is straightforward to check
\[ L'^\mu = -2g^{\mu\sigma} \partial_\sigma u, \quad L'^u = -2g^{\mu\sigma} \partial_\sigma \bar{u}, \quad Lu = 1, \quad L\bar{u} = 1. \]

Here $L := \Omega^2 L'$, $L' := \Omega^2 L$ are also called equivariant vector fields.

### 2.2 The Coordinate System

We shall use a coordinate system $(u, \bar{u}, \theta^1, \theta^2)$. Here $u$ and $\bar{u}$ are defined as above. To get $(\theta^1, \theta^2)$ for each point on $S_{u, \bar{u}}$, we follow the approach in Chapter 1 of [6]. We first define a coordinate system $(\theta^1, \theta^2)$ on $S_{u_{\infty}, 0}$. Since $S_{u_{\infty}, 0}$ is the standard 2-sphere in Minkowski spacetime, here we use the coordinates of stereographic projection. Then we extend this coordinate system to $H_0$ by requiring
\[ \mathcal{L}_L \theta^A = 0 \text{ on } H_0. \]

Here $\mathcal{L}_L$ is the restriction of the Lie derivative to $T S_{u, \bar{u}}$. In other words, given a point $p$ on $S_{u_{\infty}, 0}$, assuming $l_p$ is the incoming null geodesic on $H_0$ emanating from $p$, then all the points along $l_p$ are assigned the same angular coordinate $(\theta^1, \theta^2)$. We further extend this coordinate system from $H_0$ to the whole spacetime under the requirement
\[ \mathcal{L}_L \theta^A = 0, \]

i.e., that all the points along the same outgoing null geodesics (along $L$) on $H_u$ have the same angular coordinate. We have thus established a coordinate system in a neighborhood of $S_{u_{\infty}, 0}$. With this coordinate system, we can rewrite $e_3$ and $e_4$ as
\[ e_3 = \Omega^{-1} \left( \frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A} \right), \quad e_4 = \Omega^{-1} \frac{\partial}{\partial \bar{u}}. \]

The Lorentzian metric $g$ takes the form
\[ g = -2\Omega^2 (du \otimes du + d\bar{u} \otimes d\bar{u}) + \gamma_{AB} (d\theta^A - d^A du) \otimes (d\theta^B - d^B du). \quad (2.2) \]

We require $b^A$ to satisfy $b^A = 0$ on $H_0$.

### 2.3 The equations

In this paper, we study the Einstein-Maxwell equations for a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with signature $\{ -, +, +, + \}$
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}, \quad (2.3) \]

\hfill \footnote{On $H_0$, we have $\Omega = 1$ and $\mathcal{L}_L e^A = \frac{\partial}{\partial \bar{u}} e^A$.}
where
\[ T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau}. \]

Here \( F_{\alpha\beta} \) is an anti-symmetric 2–tensor representing the electromagnetic field. We introduce null tetrads \( \{e_a, e_b, e_3, e_4\} \) where \( a, b = 1, 2 \) and require
\[ g(e_a, e_b) = \delta_{ab}, \quad g(e_3, e_4) = -2, \quad g(e_a, e_3) = 0, \quad g(e_a, e_4) = 0. \]

For the Weyl curvature \( W_{\mu\nu\lambda\tau} \), we define the null Weyl curvature components:
\[
\begin{align*}
\alpha_{ab} &= W(e_a, e_4, e_b, e_4), \quad \alpha_{ab}^* = W(e_a, e_3, e_b, e_3), \\
\beta_a &= \frac{1}{2} W(e_a, e_4, e_3, e_4), \quad \beta_a^* = \frac{1}{2} W(e_a, e_3, e_4, e_4), \quad (2.4) \\
\rho &= \frac{1}{4} W(e_4, e_3, e_4, e_3), \quad \sigma = \frac{1}{4} * W(e_4, e_3, e_4, e_3).
\end{align*}
\]

Here \( *W \) is the Hodge dual of \( W \).

For the Riemann curvature \( R_{\mu\nu\lambda\tau} \), we define the null Riemann curvature components:
\[
\begin{align*}
(\alpha_R)_{ab} &= R(e_a, e_4, e_b, e_4), \quad (\alpha_R)_{ab}^* = R(e_a, e_3, e_b, e_3), \\
(\beta_R)_a &= \frac{1}{2} R(e_a, e_4, e_3, e_4), \quad (\beta_R)_a^* = \frac{1}{2} R(e_a, e_3, e_4, e_4), \\
\rho_R &= \frac{1}{4} R(e_4, e_3, e_4, e_3), \quad \sigma_R = \frac{1}{4} * R(e_4, e_3, e_4, e_3). \quad (2.5)
\end{align*}
\]

Here \( *R \) is the Hodge dual of \( R \).

Denote \( D_A := D e_A \). We define the Ricci coefficients:
\[
\begin{align*}
\chi_{AB} &= g(D Ae_4, e_B), \quad \chi_{AB}^* = g(D Ae_3, e_B), \\
\eta_A &= -\frac{1}{2} g(D_3 e_A, e_4), \quad \eta_A^* = -\frac{1}{2} g(D_4 e_A, e_3), \\
\omega &= -\frac{1}{4} g(D_4 e_3, e_4), \quad \omega^* = -\frac{1}{4} g(D_3 e_3, e_4), \\
\zeta_A &= \frac{1}{2} g(D_A e_4, e_3).
\end{align*}
\]

We decompose \( \chi \) and \( \chi^* \) into its trace and traceless parts. Denote by \( \hat{\chi}_{AB} \) and \( \hat{\chi}_{AB}^* \) the traceless part of \( \chi_{AB} \) and \( \chi_{AB}^* \) respectively.

We further define
\[
(\alpha_F)_a = F_{a4}, \quad (\alpha_F)_a = F_{a3}, \quad \rho_F = \frac{1}{2} F_{34}, \quad \sigma_F = F_{12}.
\]

Note that (2.3) implies
\[ R_{\mu\nu} = T_{\mu\nu} \quad \text{and} \quad R = 0. \]

Expressed in this double null frame, we have
\[
\begin{align*}
\nabla_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 &= -|\chi|^2 - 2 \omega \text{tr} \chi - \alpha^2, \quad (2.7) \\
\nabla_4 \hat{\chi} + \text{tr} \chi \hat{\chi} &= -2 \omega \hat{\chi} - \alpha, \quad (2.8) \\
\nabla_3 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 &= -|\chi|^2 - 2 \omega \text{tr} \chi - \alpha^2, \quad (2.9) \\
\nabla_3 \hat{\chi} + \text{tr} \chi \hat{\chi} &= -2 \omega \hat{\chi} - \alpha. \quad (2.10)
\end{align*}
\]
Note that

\[ \nabla_4 \chi + \frac{1}{2} \chi \nabla \chi = 2 \omega \nabla \chi + 2 \rho - \hat{\chi} \cdot \hat{\chi} + 2 \text{div} \eta + 2 |\eta|^2, \]

(2.11)

\[(\nabla_4 \hat{\chi})_{ab} + \frac{1}{2} \chi \nabla \hat{\chi}_{ab} = (\nabla \hat{\eta})_{ab} + 2 \omega \hat{\chi}_{ab} - \frac{1}{2} \chi \nabla \hat{\chi}_{ab} + (\hat{\eta} \nabla \eta)_{ab} - \frac{1}{2} (\alpha \hat{\omega} \hat{\omega})_{ab}, \]

(2.12)

\[\nabla_3 \chi + \frac{1}{2} \chi \nabla \chi = 2 \omega \nabla \chi + 2 \rho - \hat{\chi} \cdot \hat{\chi} + 2 \text{div} \eta + 2 |\eta|^2, \]

(2.13)

\[(\nabla_3 \hat{\chi})_{ab} + \frac{1}{2} \chi \nabla \hat{\chi}_{ab} = (\nabla \hat{\eta})_{ab} + 2 \omega \hat{\chi}_{ab} - \frac{1}{2} \chi \nabla \hat{\chi}_{ab} + (\hat{\eta} \nabla \eta)_{ab} - \frac{1}{2} (\alpha \hat{\omega} \hat{\omega})_{ab}, \]

(2.14)

Note that

\[\beta_a - \frac{1}{2} R_{a4} = (\beta_R)_a, \]

\[\beta_a + \frac{1}{2} R_{a3} = (\beta_R)_a, \]

\[\rho - \frac{1}{2} R_{a3} = \rho_R. \]

Moreover,

\[R_{11} = \frac{1}{2} \sigma^2 + \frac{1}{2} \rho^2 - (\alpha F)_1 (\frac{1}{4} \alpha F), \]

\[R_{22} = \frac{1}{2} \sigma^2 + \frac{1}{2} \rho^2 - (\alpha F)_2 (\frac{1}{4} \alpha F), \]

\[R_{4a} = R_{a4} = \rho F \alpha - \sigma F \epsilon_{ab} \alpha F b, \]

\[R_{3a} = R_{a3} = - \rho F (\alpha F)_a - \sigma F \epsilon_{ab} (\alpha F)_b, \]

\[R_{43} = \rho^2 + \sigma^2, \quad R_{44} = \alpha F \cdot \alpha F, \quad R_{33} = \alpha F \cdot \alpha F. \]

The other components satisfy the following transport equations:

\[\nabla_4 \eta_a = - \chi_{ab} \cdot (\eta - \eta)_b - \beta_a - R_{a4}, \]

(2.15)

\[\nabla_3 \eta_a = - \chi_{ab} \cdot (\eta - \eta)_b + \beta_a - R_{a3}, \]

(2.16)

\[\nabla_4 \omega = 2 \omega \omega - \eta \cdot \eta + \frac{1}{2} \eta^2 + \frac{1}{2} \rho + \frac{1}{4} R_{34}, \]

(2.17)

\[\nabla_3 \omega = 2 \omega \omega - \eta \cdot \eta + \frac{1}{2} \eta^2 + \frac{1}{2} \rho + \frac{1}{4} R_{34}, \]

(2.18)

as well as the constraint equations

\[\text{div} \hat{\chi} = \frac{1}{2} \nabla \chi - \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \chi) - \beta_R, \]

(2.19)

\[\text{div} \hat{\chi} = \frac{1}{2} \nabla \chi - \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \chi) - \beta_R, \]

(2.20)

\[\text{curl} \eta = - \text{curl} \eta = \sigma + \frac{1}{4} \chi \wedge \hat{\chi}, \]

(2.21)

\[K = - \rho_R - \frac{1}{4} \chi \nabla \chi + \frac{1}{2} \chi \cdot \hat{\chi}. \]

(2.22)

Here \(K\) is the Gauss curvature of the spheres \(S_{\omega, \eta}\). The null curvature components satisfy the null Bianchi equations

\[\nabla_4 \beta + 2 \chi \beta = \text{div} \alpha - 2 \omega \beta + \eta \cdot \alpha - \frac{1}{2} (D_A R_{44} - D_4 R_{A3}), \]

(2.23)

\[\nabla_3 \beta + \chi \beta = \nabla \rho + \nabla \sigma + 2 \hat{\chi} \cdot \beta + 2 \omega \beta + 3(\eta \rho + \eta \sigma) + \frac{1}{2} (D_A R_{43} - D_4 R_{A4}), \]

(2.24)
\[ \nabla_4 \beta + \text{tr} \chi \beta = -\nabla \rho + \ast \nabla \sigma + 2 \hat{\chi} \cdot \beta + 2 \omega \beta - 3(\eta \rho - \ast \eta \sigma) - \frac{1}{2}(D_4 R_{A3} - D_3 R_{A4}), \tag{2.25} \]

\[ \nabla_3 \beta + 2 \text{tr} \chi \beta = -\text{div} \alpha - 2 \omega \beta + p_a + \frac{1}{2}(D_4 R_{33} - D_3 R_{34}), \tag{2.26} \]

\[ \nabla_4 \alpha + \frac{1}{2} \text{tr} \chi \alpha = -\nabla \hat{\beta} + 4 \omega \alpha - 3(\hat{\chi} \rho - \ast \hat{\chi} \sigma) + (\zeta - 4 \eta) \hat{\beta} + \frac{1}{4}(D_4 R_{34} - D_4 R_{43}) g_{a b}, \tag{2.27} \]

\[ \nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha = \nabla \hat{\beta} + 4 \omega \alpha - 3(\hat{\chi} \rho + \ast \hat{\chi} \sigma) + (\zeta + 4 \eta) \hat{\beta} + \frac{1}{4}(D_4 R_{44} - D_4 R_{43}) g_{a b}, \tag{2.28} \]

\[ \nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho = \text{div} \beta - \frac{1}{2} \chi \cdot \alpha + \zeta \cdot \beta + \frac{1}{2}(D_4 R_{44} - D_4 R_{43}), \tag{2.29} \]

\[ \nabla_3 \rho + \frac{3}{2} \text{tr} \chi \rho = -\text{div} \beta - \frac{1}{2} \chi \cdot \alpha + \zeta \cdot \beta - \frac{1}{2}(D_4 R_{34} - D_4 R_{33}), \tag{2.30} \]

\[ \nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} \beta + \frac{1}{2} \chi \cdot \alpha - \zeta \cdot \beta - \frac{1}{2}(D_4 R_{44} - D_4 R_{43}) g_{a b}, \tag{2.31} \]

\[ \nabla_3 \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{div} \beta + \frac{1}{2} \chi \cdot \alpha - \zeta \cdot \beta + \frac{1}{2}(D_4 R_{34} - D_4 R_{33}) g_{a b}. \tag{2.32} \]

Here, the Schouten tensor $S_{\mu \nu}$ is equal to the Ricci tensor $R_{\mu \nu}$ because of the special form of the electromagnetic field tensor, namely the fact that the Ricci scalar $R$ vanishes. Finally, the Maxwell equations are equivalent to the null Maxwell equations

\[ \nabla_3 \Omega_F + \frac{1}{2} \text{tr} \chi \Omega_F = -\nabla \rho_F + \ast \nabla \sigma_F - 2 \eta \cdot \sigma_F - 2 \eta \cdot \rho_F + 2 \omega \Omega_F - \hat{\chi} \cdot \Omega_F, \tag{2.33} \]

\[ \nabla_3 \alpha_F + \frac{1}{2} \text{tr} \chi \alpha_F = \nabla \rho_F + \ast \nabla \sigma_F - 2 \eta \cdot \sigma_F + 2 \eta \cdot \rho_F + 2 \omega \Omega_F - \hat{\chi} \cdot \Omega_F, \tag{2.34} \]

\[ \nabla_4 \rho_F = \text{div} \alpha_F - \text{tr} \chi \rho_F - (\eta - \eta) \cdot \alpha_F, \tag{2.35} \]

\[ \nabla_4 \sigma_F = -\text{curl} \alpha_F - \text{tr} \chi \sigma_F + (\eta - \eta) \cdot \alpha_F, \tag{2.36} \]

\[ \nabla_3 \rho_F + \text{tr} \chi \rho_F = -\text{div} \alpha_F + (\eta - \eta) \cdot \alpha_F, \tag{2.37} \]

\[ \nabla_3 \sigma_F + \text{tr} \chi \sigma_F = -\text{curl} \alpha_F + (\eta - \eta) \cdot \alpha_F. \tag{2.38} \]

### 2.4 Integration

Let $U$ be a coordinate patch on $S_{u \omega}$ and let $p_U$ be the corresponding partition of unity. For a function $\phi$, we define its integral on $S_{u \omega}$ and along $H_u, H_\omega$ by

\[ \int_{S_{u \omega}} \phi := \sum_U \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \cdot p_U \cdot \sqrt{\det \gamma} \: d\theta^1 \: d\theta^2, \]

\[ \int_{H^{(1)}} \phi := \sum_U \int_{0}^{u} \int_{-\infty}^{\infty} \phi \cdot 2 p_U \cdot \Omega \cdot \sqrt{\det \gamma} \: d\theta^1 \: d\theta^2 \: du', \]

\[ \int_{H^{(1 \infty)}} \phi := \sum_U \int_{u_{\infty}}^{u} \int_{-\infty}^{\infty} \phi \cdot 2 p_U \cdot \Omega \cdot \sqrt{\det \gamma} \: d\theta^1 \: d\theta^2 \: du'. \]

Let $D_{u \omega}$ be the region $u_{\infty} \leq u' \leq u$, $0 \leq u' \leq u$. We define the integral of $\phi$ in the region $D_{u \omega}$ by

\[ \int_{D_{u \omega}} \phi := \sum_U \int_{u_{\infty}}^{u} \int_{0}^{u} \int_{-\infty}^{\infty} \phi \cdot p_U \cdot \Omega^2 \cdot \sqrt{\det g} \: d\theta^1 \: d\theta^2 \: du' \: du. \]

We proceed to define, for $1 \leq p < \infty$, the $L^p$-norms for an arbitrary tensorfield $\phi$:

\[ \| \phi \|^p_{L^p(S_{u \omega})} := \int_{S_{u \omega}} \langle \phi, \phi \rangle^p / 2, \]
we also define scale-invariant norms along null hypersurfaces

\[ \|\phi\|_{L^p(H_\omega)}^p := \int_{H_\omega} (\phi, \phi)_{\gamma}^{p/2}, \]

\[ \|\phi\|_{L^p(L_\omega)}^p := \int_{L_\omega} (\phi, \phi)_{\gamma}^{p/2}. \]

When \( p = \infty \), we define the \( L^\infty \) norm by

\[ \|\phi\|_{L^\infty(S_\omega)} := \sup_{\theta \in S_\omega} (\phi, \phi)_\gamma^{\frac{1}{2}}. \]

### 2.5 Definition of signatures

We give the following table for signatures throughout this work:

| \( s_2 \) | \( \alpha \) | \( \beta \) | \( \rho \) | \( \sigma \) | \( \beta \) | \( \alpha \) | \( \chi \) | \( \omega \) | \( \zeta \) | \( \eta \) | \( \eta \) | \( \text{tr} \chi \) | \( \omega \) | \( \alpha_F \) | \( \rho_F \) | \( \sigma_F \) | \( \alpha_F \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.5 | 1 | 1 | 1.5 | 2 | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 1 | 1 | 0 | 0.5 | 0.5 | 1 |

This comes from wanting to have \( s_2(\alpha_F) = s_2(\chi) = s_2(F_{\alpha}) = s_2(F_{\beta}) \) such that the null Maxwell equations conserve signature.

### 2.6 Scale-invariant norms

For any horizontal tensor-field \( \phi \) with signature \( s_2(\phi) \), we define the following scale-invariant norms on \( S_{u,\omega} \):

\[ \|\phi\|_{L_2^2(S_{u,\omega})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)+1} \|\phi\|_{L^2(S_{u,\omega})}, \]

\[ \|\phi\|_{L_2^4(S_{u,\omega})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)} \|\phi\|_{L^4(S_{u,\omega})}, \]

\[ \|\phi\|_{L_2^6(S_{u,\omega})} := a^{-s_2(\phi)}|u|^{2s_2(\phi)-1} \|\phi\|_{L^6(S_{u,\omega})}. \]

Along \( H_{\omega} \) and \( H_{u,\omega} \) we also define scale-invariant norms along null hypersurfaces

\[ \|\phi\|_{L_2^2(H_{\omega})} := \int_0^u \|\phi\|_{L_2^2(S_{\omega,u})} \, du', \]

\[ \|\phi\|_{L_2^4(H_{\omega})} := \int_0^{u,u} \frac{\alpha}{|u|^2} \|\phi\|_{L_2^4(S_{\omega,u})} \, du'. \]

### 2.7 Conservation of signatures

Notice that under the table of signatures in Section 2.5 and the fact that the induced metric on a 2–sphere \( \gamma_{\alpha\beta} \) has \( s_2(\gamma_{\alpha\beta}) = 0 \), the following remarkable property follows for tensorfields \( \phi_1 \) and \( \phi_2 \):

\[ s_2(\phi_1 \cdot \phi_2) = s_2(\phi_1) + s_2(\phi_2). \]

This ensures signature conservation for all null structure, Bianchi, constraint and null Maxwell equations. When working with scale-invariant norms, this key property enables us to treat all the terms on the right hand side as one term. For example, look at the null Maxwell equation for \( \nabla_{3} \alpha_F \):

\[ \nabla_3 \alpha_F + \frac{1}{2} \text{tr} \chi \alpha_F = -\nabla \rho_F + \nabla \alpha_F - 2 \nabla \eta \cdot \sigma_F + 2 \eta \cdot \rho_F + \frac{2}{2} \alpha_F - \hat{\chi} \cdot \alpha_F. \]

There holds

- \( s_2(\nabla_3 \alpha_F) = s_2 \alpha_F + 1 = 1 \),
- \( s_2(\text{tr} \alpha_F) = s_2(\text{tr} \alpha) + s_2(\alpha_F) = 1 \),
• \( s_2(\nabla \rho F, \nabla \sigma F) = \frac{1}{2} + s_2(\rho F, \sigma F) = \frac{1}{2} + \frac{1}{2} = 1 \),
• \( s_2(\eta \cdot \rho F, \cdot \eta \cdot \sigma F) = \frac{1}{2} + \frac{1}{2} = 1 \),
• \( s_2(\omega \alpha F) = 1 + 0 = 1 \),
• \( s_2(\chi \cdot \omega F) = 0 + 1 = 1 \).

Thus, throughout the equation, there is a balance of signature.

2.8 Hölder’s inequality in scale-invariant norms

Any two tensorfields satisfy the following scale-invariant Hölder inequalities:

\[
\| \phi_1 \cdot \phi_2 \|_{\mathcal{L}^p_S(S^n)} \leq \frac{1}{|u|} \| \phi_1 \|_{\mathcal{L}^p_S(S^n)} \| \phi_2 \|_{\mathcal{L}^p_S(S^n)},
\]

(2.39)

\[
\| \phi_1 \cdot \phi_2 \|_{\mathcal{L}^p_S(S^n)} \leq \frac{1}{|u|} \| \phi_1 \|_{\mathcal{L}^p_S(S^n)} \| \phi_2 \|_{\mathcal{L}^p_S(S^n)},
\]

(2.40)

\[
\| \phi_1 \cdot \phi_2 \|_{\mathcal{L}^p_S(S^n)} \leq \frac{1}{|u|} \| \phi_1 \|_{\mathcal{L}^p_S(S^n)} \| \phi_2 \|_{\mathcal{L}^p_S(S^n)}.
\]

(2.41)

Also, the following inequality holds

\[
\| \phi_1 \cdot \phi_2 \|_{\mathcal{L}^p_S(S^n)} \leq \frac{1}{|u|} \| \phi_1 \|_{\mathcal{L}^p_S(S^n)} \| \phi_2 \|_{\mathcal{L}^p_S(S^n)}.
\]

(2.42)

Crucially, in the region of study \( \frac{1}{|u|} \ll 1 \). This means, when measuring the size of a product of terms in scale-invariant norms, this size is very small compared to the scale-invariant norms of the individual terms. Essentially, it is this crucial fact that allows us to close all bootstrap arguments throughout this paper.

2.9 Norms

Let \( \psi \in \{\omega, \text{tr} \chi, \eta, \eta \cdot \omega\} \), \( \Psi \in \{\beta, \rho, \sigma, \beta, \alpha\} \) and \( \Psi' \in \{\rho, \sigma, \beta, \alpha\} \). Denote \( \tilde{\chi} = \text{tr} \chi + \frac{\omega}{|u|} \). Also, let \( \Upsilon \in \{\rho F, \sigma F, \alpha F\} \). For \( 0 \leq i \leq 6 \), we define

\[
\mathcal{O}_{i,6}(u, \omega) := \frac{1}{a^\frac{i}{2}} \| (a^\frac{i}{2} \nabla)^i \chi \|_{\mathcal{L}^{\infty}_S(S^n)} + \| (a^\frac{i}{2} \nabla)^i \psi \|_{\mathcal{L}^{\infty}_S(S^n)} + \frac{a^\frac{i}{2}}{|u|} \| (a^\frac{i}{2} \nabla)^i \chi \|_{\mathcal{L}^{\infty}_S(S^n)} + \frac{a^\frac{i}{2}}{|u|} \| (a^\frac{i}{2} \nabla)^i \psi \|_{\mathcal{L}^{\infty}_S(S^n)},
\]

(2.43)

\[
\mathcal{R}_{i,6}(u, \omega) = \frac{1}{a^\frac{i}{2}} \| (a^\frac{i}{2} \nabla)^i \alpha \|_{\mathcal{L}^{\infty}_S(S^n)} + \| (a^\frac{i}{2} \nabla)^i \psi \|_{\mathcal{L}^{\infty}_S(S^n)},
\]

\[
\mathcal{F}_{i,6}(u, \omega) = \frac{1}{a^\frac{i}{2}} \| (a^\frac{i}{2} \nabla)^i \alpha \|_{\mathcal{L}^{\infty}_S(S^n)} + \| (a^\frac{i}{2} \nabla)^i \Upsilon \|_{\mathcal{L}^{\infty}_S(S^n)}.
\]

For \( 0 \leq i \leq 9 \) and \( 0 \leq j \leq 10 \), we define

\[
\mathcal{O}_{j,2}(u, \omega) = \frac{1}{a^\frac{j}{2}} \| (a^\frac{j}{2} \nabla)^j \chi \|_{\mathcal{L}^{2}_S(S^n)} + \| (a^\frac{j}{2} \nabla)^j \psi \|_{\mathcal{L}^{2}_S(S^n)} + \frac{a^\frac{j}{2}}{|u|} \| (a^\frac{j}{2} \nabla)^j \chi \|_{\mathcal{L}^{2}_S(S^n)} + \frac{a^\frac{j}{2}}{|u|} \| (a^\frac{j}{2} \nabla)^j \psi \|_{\mathcal{L}^{2}_S(S^n)},
\]

(2.44)

\[
\mathcal{R}_{j,2}(u, \omega) = \frac{1}{a^\frac{j}{2}} \| (a^\frac{j}{2} \nabla)^j \alpha \|_{\mathcal{L}^{2}_S(S^n)} + \| (a^\frac{j}{2} \nabla)^j \Upsilon \|_{\mathcal{L}^{2}_S(S^n)};
\]

(2.44)
\[ F_{j,2}(u, w) = \left\| (a^j)^i \nabla^j \alpha F \right\|_{L^2(\sigma)} + \left\| (a^j)^i (\rho F, \sigma F, \Omega F) \right\|_{L^2(\sigma)} \].

For \( 0 \leq i \leq 10 \) and \( 0 \leq j \leq 11 \) we define
\[ R_i(u, w) = \frac{1}{a^i} \left\| (a^j)^i \nabla^i \alpha F \right\|_{L^2(\sigma)} + \left\| (a^j)^i \nabla^i \psi \right\|_{L^2(\sigma)}, \]
\[ F_{j,2}(u, w) = \frac{1}{a^j} \left\| (a^j)^j \nabla^j \beta F \right\|_{L^2(\sigma)} + \left\| (a^j)^j \nabla^j \psi \right\|_{L^2(\sigma)}. \]

We now set \( O_{i, \infty}, O_{i, 2}, R_{i, \infty}, R_{i, 2}, F_{i, \infty}, F_{i, 2} \) to be the supremum over \( u, w \) in the spacetime region of the norms \( O_{i, \infty}(u, w), O_{i, 2}(u, w), R_{i, \infty}(u, w), R_{i, 2}(u, w), F_{i, \infty}(u, w) \) and \( F_{i, 2}(u, w) \) respectively. Finally, define \( O, R \) and \( F \):
\[ O = \sum_{i \leq 0} \left( O_{i, \infty} + R_{i, \infty} + F_{i, \infty} \right) + \sum_{0 \leq i \leq 9} R_{i, 2} + \sum_{0 \leq j \leq 10} O_{j, 2} + F_{j, 2}, \]
\[ R = \sum_{0 \leq i \leq 10} (R_i + R_i'), F = \sum_{0 \leq j \leq 11} (F_j + F_j'). \]

2.10 An explicit form of the Bianchi equations

Recall the Bianchi equations (2.23)-(2.32). We will work on a term-by-term basis to give the explicit forms of the right hand sides (RHS) of these equations. We will use the property
\[ (D_a R)_{\beta \gamma} = D_a (R_{\beta \gamma}) - R(D_a e_\beta, e_\gamma) - R(e_\beta, D_a e_\gamma), \]
as well as the following identities:
\[ D_A e_B = \nabla_A e_B + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \gamma_{AB} e_4, \]  
(2.44)
\[ D_A e_3 = \nabla_3 e_A + \eta_A e_3, \quad D_A e_4 = \nabla_4 e_A + \eta_A e_4, \]  
(2.45)
\[ D_A e_3 = \chi_A^{3B} e_B + \zeta_A e_3, \quad D_A e_4 = \chi_A^{4B} e_B - \zeta_A e_4, \]  
(2.46)
\[ D_A e_4 = 2\eta_A^{3A} e_A + 2\omega e_4, \quad D_A e_3 = 2\eta_A^{4A} e_A + 2\omega e_3, \]  
(2.47)
\[ D_A e_3 = -2\omega e_3, \quad D_A e_4 = -2\omega e_4. \]  
(2.48)

We therefore compute
\[ (D_A R)_{44} = 2\alpha F \cdot \nabla \alpha F - 2R(D_A e_4, e_4) = 2\alpha F \cdot \nabla \alpha F - (\psi, \tilde{\chi}) \cdot R(e_A, e_4) + \psi \cdot \alpha F \cdot \alpha F \]
\[ = 2\alpha F \cdot \nabla \alpha F + (\psi, \tilde{\chi}) \cdot \tilde{\chi} \cdot \alpha F + \psi \cdot \alpha^2 F, \]
\[ (D_A R)_{4A} = D_A (R_{4A}) - R(D_A e_4, e_A) - R(e_A, D_A e_4) = D_A (R_{4A}) + (\psi \cdot \tilde{\chi} \cdot \alpha F), \]
\[ (D_A R)_{43} = D_A (R_{43}) - R(D_A e_4, e_3) - R(D_A e_3, e_4) = \tilde{\chi} \cdot \nabla \tilde{\chi} + (\psi, \tilde{\chi} \cdot \tilde{\chi}) \cdot \tilde{\chi} \cdot (\tilde{\chi} \cdot \alpha F) \cdot \tilde{\chi}, \]
\[ (D_A R)_{33} = D_A (R_{33}) - R(D_A e_3, e_3) - R(D_A e_3, e_4) = \tilde{\chi} \cdot \nabla \tilde{\chi} + (\psi \cdot \tilde{\chi} \cdot (\tilde{\chi} \cdot \alpha F)). \]
\begin{align*}
\n&= \nabla (Y, \alpha_F) + (\psi, \tilde{\chi}) \cdot Y \cdot (Y, \alpha_F),
\end{align*}

since

\begin{align*}
\nabla_4 Y &= \nabla (Y, \alpha_F) + (\psi, \tilde{\chi}) \cdot Y, \\
\nabla_3 \alpha_F &= \nabla Y + (\psi, \text{tr}_X) \alpha_F + (\psi, \tilde{\chi}) \cdot Y.
\end{align*}

Continuing, we have

\begin{align*}
(D_3 R)_{34} &= D_3 (R_{34}) - R(D_3 e_3, e_3) = Y \cdot \nabla Y + (\psi, \tilde{\chi}) \cdot Y \cdot (Y, \alpha_F), \\
(D_3 R)_{3A} &= D_3 (R_{3A}) - R(D_3 e_3, e_4) = \nabla_3 (R_{3A}) + \psi \cdot Y \cdot Y, \\
(D_4 R)_{33} &= 2\alpha_F \cdot \nabla_3 \alpha_F - 2R(D_4 e_3, e_3) = 2\alpha_F \cdot \nabla_3 \alpha_F + \psi \cdot Y \cdot Y = \nabla Y + (\psi, \tilde{\chi}) \cdot (Y, \alpha_F) + \psi \cdot Y \cdot Y
\end{align*}

Continuing, we have

\begin{align*}
(D_A R)_{33} &= D_A (R_{33}) - 2R(D_A e_3, e_3) = Y \cdot \nabla Y + (\psi, \tilde{\chi}) \cdot Y \cdot (Y, \alpha_F), \\
(D_4 R)_{34} &= D_4 (R_{34}) - R(D_4 e_3, e_4) = Y \cdot \nabla_4 Y + \psi \cdot Y \cdot Y = \nabla Y + (\psi, \tilde{\chi}) \cdot Y \cdot (Y, \alpha_F), \\
(D_3 R)_{44} &= 2\alpha_F \cdot \nabla_3 \alpha_F - 2R(D_3 e_4, e_4) = 2\alpha_F \cdot \nabla_3 \alpha_F + \psi \cdot \alpha_F \cdot (\alpha_F, Y)
\end{align*}

The expressions for \((D_A R)_{4B}\) and \((D_A R)_{3B}\) are as follows:

\begin{align*}
(D_A R)_{4B} &= D_A (R_{4B}) - R(D_A e_4, e_B) - R(D_A e_B, e_4) = (\alpha_F \cdot \nabla Y + Y \cdot \nabla Y) \\
&+ ((\psi, \tilde{\chi}) \cdot Y \cdot Y + \psi \cdot \alpha_F \cdot Y) + ((\psi, \tilde{\chi}) \cdot Y + (\psi, \tilde{\chi}) \cdot \text{tr}_X) \cdot (\alpha_F \cdot Y)
\end{align*}

\begin{align*}
(D_A R)_{3B} &= D_A (R_{3B}) - R(D_A e_3, e_B) - R(D_A e_B, e_3) = Y \cdot \nabla Y + (\psi, \tilde{\chi}) \cdot \text{tr}_X \cdot Y.
\end{align*}
2.10.1 An important renormalization

A novel ingredient in our analysis is the introduction of renormalized quantities for $\beta$ and $\beta$. The motivation behind the introduction of these two quantities stems from the Bianchi equations. Take for example the identity for $\beta$:

$$\nabla_{\alpha}^{4} \beta + 2 \text{tr}_{\chi} \beta = \text{div}_{\alpha} - 2 \omega_{\beta} + \eta \cdot \alpha - \frac{1}{2} (D_{A} R_{44} - D_{4} R_{4A}).$$

Strictly speaking, this is an equality of 1-forms. In particular, if we evaluate on a vector $e_{A}$, we get

$$(\nabla_{\alpha} \beta)(e_{A}) + 2 \text{tr}_{\chi} \beta A = (\text{div}_{\alpha})(e_{A}) - 2 \beta_{\alpha} + (\eta \cdot \alpha)_{A} - \frac{1}{2} (D_{A} R_{44} + \frac{1}{2} (D_{4} R_{4A}) = 0 \Rightarrow$$

$$(\nabla_{\alpha} \beta)(e_{A}) - \frac{1}{2} \nabla_{\alpha} (R_{44}(\chi))(e_{A}) = \nabla \alpha + \psi \cdot \Psi + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot \Upsilon \cdot \alpha F + \psi \cdot \alpha_{F} + \psi \cdot \alpha_{F} \Rightarrow$$

$$\nabla_{\alpha} (\beta - \frac{1}{2} R_{44}(\chi)) = \nabla \alpha + \psi \cdot (\Psi, \alpha) + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot \Upsilon \cdot \alpha F + \psi \cdot \alpha_{F}.$$ 

This motivates us to define the normalized curvature component

$$\tilde{\beta} := \beta - \frac{1}{2} R_{44}(\chi).$$

(2.53)

Similarly, we need to define

$$\tilde{\beta} := \beta + \frac{1}{2} R_{34}(\chi).$$

(2.54)

The gain from (2.53) and (2.54) is that the Bianchi equations are now expressed in a way that the right-hand sides of the equations are controllable in terms of the Ricci coefficients and the curvature and Maxwell components.

2.10.2 The Bianchi equations in schematic form for the renormalized components

In this section we give, in schematic form, the Bianchi equations expressed in terms of $\{\alpha, \omega, \tilde{\beta}, \tilde{\alpha}, \rho, \sigma\}$. We explain our way of obtaining these. Take for example the transport equation for $\beta$:

$$\nabla_{\alpha}^{4} \beta + 2 \text{tr}_{\chi} \beta = \text{div}_{\alpha} - 2 \omega_{\beta} + \eta \cdot \alpha - \frac{1}{2} (D_{A} R_{44} - D_{4} R_{4A}).$$

Then

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = (\nabla_{\alpha} \beta + 2 \text{tr}_{\chi} \beta) - \frac{1}{2} \nabla_{\alpha} (R_{44}(\chi))(e_{A}) - \text{tr}_{\chi} R_{44}.$$

Working similarly, we obtain

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \alpha + \psi \cdot (\alpha, \Psi) + \alpha F \cdot \nabla \alpha F + \psi \cdot (\alpha F, \alpha F) + (\psi, \chi) \cdot \Upsilon \cdot \alpha F,$$

(2.55)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + (\psi, \chi) \cdot \Psi + \Upsilon \nabla (\alpha F, \alpha F) + (\alpha F, \Upsilon) \Upsilon + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.56)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + (\psi, \chi) \cdot \Psi + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.57)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + \psi \cdot \Psi + \Upsilon \nabla (\alpha F, \alpha F) + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.58)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + \psi \cdot \Psi + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.59)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + \psi \cdot \Psi + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.60)

$$\nabla_{\alpha} \tilde{\beta} + 2 \text{tr}_{\chi} \tilde{\beta} = \nabla \Psi + \psi \cdot \Psi + \alpha F \cdot \nabla \alpha F + (\psi, \chi) \cdot (\alpha F, \Upsilon) \cdot \Upsilon,$$

(2.61)
\[ \nabla_3 \rho + \frac{3}{2} \nabla \Psi = \nabla Y + (\psi, \hat{\chi}) \cdot \nabla Y + (\psi, \hat{\chi}) \cdot Y + (\psi, \hat{\chi}) \cdot (\nabla \alpha_F) \cdot Y, \quad (2.62) \]
\[ \nabla_4 \sigma + \frac{3}{2} \nabla \chi \sigma = \nabla Y + (\psi, \hat{\chi}) \cdot (\chi, Y) + \alpha_F \cdot \nabla Y + \nabla \alpha F + (\psi, \hat{\chi}) \cdot (\nabla \alpha_F) \cdot (\nabla \alpha_F) \cdot Y, \quad (2.63) \]
\[ \nabla_4 \sigma + \frac{3}{2} \nabla \chi \sigma = \nabla Y + \nabla Y + (\psi, \hat{\chi}) \cdot \Psi + (\psi, \hat{\chi}) \cdot Y + (\psi, \hat{\chi}) \cdot (\nabla \alpha_F) \cdot Y. \quad (2.64) \]

3 The preliminary estimates

3.1 Setting up the bootstrap argument

We shall employ a bootstrap argument to derive uniform upper bounds on \( O, R, F, \hat{F} \) for the nonlinear Einstein-Maxwell equations. Along \( H_{\infty} \) and \( H_0 \), by analysing the characteristic initial data, we can obtain the bounds

\[ O^{(0)} + R^{(0)} + R^{(0)} + F^{(0)} + \hat{F}^{(0)} \lesssim I^{(0)}. \quad (3.1) \]

Our goal is to show that in \( D = \{(u, \bar{u}) \mid u_{\infty} \leq u \leq -a/4, 0 \leq \bar{u} \leq 1\} \) there holds

\[ O(u, \bar{u}) + R(u, \bar{u}) + R(u, \bar{u}) + F(u, \bar{u}) + \hat{F}(u, \bar{u}) \lesssim \left( I^{(0)} \right)^4 + \left( I^{(0)} \right)^2 + I^{(0)} + 1. \quad (3.2) \]

Once these uniform bounds are obtained, by a standard local existence result, the solutions can always be extended a bit towards the future direction of \( u \). Hence, the uniform estimate (3.2) for \( u_{\infty} \leq u \leq -a/4 \) would imply that a solution to the Einstein-Maxwell equations exists in the slab \( D \). To derive the uniform bound (3.2), we make the bootstrap assumptions

\[ O(u, \bar{u}) \leq O, \quad R(u, \bar{u}) + R(u, \bar{u}) \leq R, \quad F(u, \bar{u}) + \hat{F}(u, \bar{u}) \leq F, \quad (3.3) \]

for large numbers \( O, R \) and \( F \) such that

\[ \left( I^{(0)} \right)^4 + \left( I^{(0)} \right)^2 + I^{(0)} + 1 \ll \min \left\{ O, R, F \right\}, \]

but also such that

\[ (O + R + F)^{2\alpha} \ll \frac{a}{4 \pi}. \]

Define the set \( B = \{u \mid u_{\infty} \leq u \leq -a/4 \text{ and } (3.3) \text{ holds for every } 0 \leq \bar{u} \leq 1\} \). We are hoping to prove that \( B \) is in fact equal as a set to the entire interval \([u_{\infty}, -a/4]\). To do this, we take advantage of the topology of the unit interval. In particular, since it is connected, it suffices to show that the set \( B \) is both closed and open.

By assumption, at \( u = u_{\infty} \), we have (3.1). By continuity of solutions (via local existence), there exists a small \( \epsilon > 0 \) such that it holds for \( u_{\infty} \leq u \leq u_{\infty} + \epsilon \) we have

\[ O^{(0)} \lesssim I^{(0)} \ll O, \quad R^{(0)} + R^{(0)} \lesssim I^{(0)} \ll R, \quad F^{(0)} + \hat{F}^{(0)} \lesssim I^{(0)} \ll F, \]
\[ O(u, \bar{u}) \lesssim 2I^{(0)} \ll O, \quad R(u, \bar{u}) + R(u, \bar{u}) \lesssim 2I^{(0)} \ll R, \quad F(u, \bar{u}) + \hat{F}(u, \bar{u}) \lesssim 2I^{(0)} \ll F. \]

This implies in particular that \( B \) is not empty and in fact \([u_{\infty}, u_{\infty} + \epsilon] \subseteq B \). At the same time, naturally there holds \( \bar{B} \subseteq [u_{\infty}, -\frac{a}{4}] \). If we are able to prove that \( B \) as a set is both open and closed, we can conclude that in fact \( B = [u_{\infty}, -\frac{a}{4}] \). Indeed, in the remainder of the paper we show the following estimates

\[ O(u, \bar{u}) \lesssim I^{(0)} + R(u, \bar{u}) + R(u, \bar{u}) + F(u, \bar{u}) + \hat{F}(u, \bar{u}), \]
\[ F(u, \bar{u}) + \hat{F}(u, \bar{u}) \lesssim R^2(u, \bar{u}) + R^2(u, \bar{u}) + \left( I^{(0)} \right)^2 + I^{(0)} + 1, \]
\[ R(u, \bar{u}) + R(u, \bar{u}) \lesssim \left( I^{(0)} \right)^2 + I^{(0)} + 1. \]

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These are improvements of the upper bounds in bootstrap assumptions. By the continuity of solutions and local existence arguments, \( B \) can be extended a bit towards larger \( u \). This implies that \( B \) is open. Together with closedness or \( B \), we conclude that \( B ≡ [u_∞, −a/4] \) and in \( B \) the desired bounds hold.

### 3.2 Estimates for the metric components

**Proposition 3.1** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have

\[
\| \Omega - 1 \|_{L^∞(S_{u, \varphi})} \lesssim \frac{O}{|u|}.
\]

**Proof.** Consider the equation

\[
\omega = -\frac{1}{2} \nabla_4 (\log \Omega) = \frac{1}{2} \frac{\partial}{\partial u} (\Omega^{-1}).
\]

We integrate with respect to \( du \). Since on \( H_0 \) we have \( \Omega^{-1} = 1 \), we can obtain

\[
\| \Omega^{-1} - 1 \|_{L^∞(S_{u, \varphi})} \lesssim \int_0^u \| \omega \|_{L^∞(S_{u, \varphi})} du' \lesssim \frac{O}{|u|}.
\]

Here we have used the bootstrap assumption. Finally, notice that

\[
\| \Omega - 1 \|_{L^∞(S_{u, \varphi})} \lesssim \| \Omega \|_{L^∞(S_{u, \varphi})} \| \Omega^{-1} - 1 \|_{L^∞(S_{u, \varphi})} \lesssim \frac{O}{1 + \frac{O}{|u|}} \lesssim \frac{O}{|u|}.
\]

\[\blacksquare\]

We now control the induced metric \( γ \) on \( S_{u, \varphi} \):

**Proposition 3.2** Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have for the metric \( γ \) on \( S_{u, \varphi} \):

\[
c' \leq \det γ \leq C',
\]

where the two constants depend only on the initial data. Moreover, in \( D \), there holds

\[
|γ_{AB}| + |(γ^{-1})^{AB}| \leq C'.
\]

**Proof.** We employ the first variation formula \( \mathcal{L}_L γ = 2 \Omega χ \). In coordinates, this rewrites as

\[
\frac{\partial}{\partial u} γ_{AB} = 2 \Omega χ_{AB}.
\]

This implies

\[
\frac{\partial}{\partial u} (\log(\det γ)) = 2 \Omega \tr χ.
\]

Let \( γ_0(u, \varphi, θ^1, θ^2) = γ(u, 0, θ^1, θ^2) \). Then with \( 2 \Omega \tr χ \lesssim \frac{O}{|u|} \), we have

\[
\frac{\det γ}{\det γ_0} = e^{\frac{2Ω}{|u|} \tr χ} du' \lesssim e^{\frac{O}{|u|}}.
\]

Via Taylor expansion, this implies

\[
|\det γ - \det γ_0| \lesssim \frac{O}{a}.
\]

This gives uniform upper and lower bounds for \( \det γ \). Let \( \Lambda \) be the larger eigenvalue of \( γ \). We have

\[
\Lambda \leq \sup γ_{AB},
\]

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\[ \sum_{A,B=1,2} |\chi_{AB}| \leq \Lambda \|\chi\|_{L^\infty(S_{u,0})}, \]

\[ |\gamma_{AB} - (\gamma_{0})_{AB}| \leq \int_{0}^{u} |\chi_{AB}| du' \leq \Lambda \frac{a^2}{|u|} O \lesssim \frac{O}{a^2}. \]

We will also need the following:

**Proposition 3.3** We continue to work under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3). Fix a point \((u, \theta)\) on the initial hypersurface \(H_0\). Along the outgoing null geodesics emanating from \((u, \theta)\), define \(\Lambda(u)\) and \(\lambda(u)\) to be the larger and smaller eigenvalue of \(\gamma^{-1}(u,0,\theta) \gamma(u,u,\theta)\). Then there holds

\[ |\Lambda(u) - 1| + |\lambda(u) - 1| \lesssim \frac{1}{a^2}. \]

**Proof.** Define \(\nu(u) := \sqrt{\frac{\Lambda(u)}{\lambda(u)}}\). Following the derivation of (5.93) in [6], by (3.5), we have

\[ \nu(u) \leq 1 + \frac{1}{2} \int_{0}^{u} \Omega \hat{\chi}(u') \nu(u') du'. \]

Via Grönwall's inequality, this implies

\[ |\nu(u)| \lesssim 1 \quad \text{and} \quad |\nu(u) - 1| \leq \frac{a^2}{|u|^2} O \lesssim \frac{1}{a}. \quad (3.7) \]

The desired estimate follows from (3.6) and (3.7). □

The above two propositions also imply

**Proposition 3.4** Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), in the slab of existence \(D\) we have

\[ \sup_{u} |\text{Area}(S_{u,2}) - \text{Area}(S_{u,0})| \lesssim \frac{O}{a^2} |u|^2. \]

**Proof.** This follows from the definitions in Subsection 2.4 and the estimate (3.6). □

### 3.3 Estimates for transport equations

In the sections to follow, we will employ the following propositions for the transport equations:

**Proposition 3.5** Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), for an arbitrary \(S\)-tangent tensor \(\phi\) of arbitrary rank, we have

\[ \|\phi\|_{L^2(S_{u,0})} \lesssim \|\phi\|_{L^2(S_{u,0'})} + \int_{u'}^{u} \|\nabla_4 \phi\|_{L^2(S_{u,0''})} du'', \quad (3.8) \]

\[ \|\phi\|_{L^2(S_{u,2})} \lesssim \|\phi\|_{L^2(S_{u,2'}')} + \int_{u'}^{u} \|\nabla_3 \phi\|_{L^2(S_{u,2''})} du''. \quad (3.9) \]

**Proof.** Here we first prove (3.8). For a scalar function \(f\), by variation of area formula, we have

\[ \frac{d}{du} \int_{S_{u,2}} f = \int_{S_{u,2}} \left( \frac{df}{du} + \Omega \text{tr} \chi f \right) = \int_{S_{u,2}} \Omega (\epsilon_4 f) + \text{tr} \chi f. \]

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Taking \( f = |\phi|^2 \), using Cauchy-Schwarz inequality on the sphere and \( L^\infty \) bounds for \( \Omega \) and \( \text{tr} \chi \), we obtain
\[
2 \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \cdot \frac{d}{du} \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \lesssim \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \cdot \| \nabla_\omega \phi \|_{L^2(S_{\omega_\infty}, \omega)} + O(\omega) \| \phi \|_{L^2(S_{\omega_\infty}, \omega)}^2.
\]
This implies
\[
\frac{d}{du} \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \lesssim \| \nabla_\omega \phi \|_{L^2(S_{\omega_\infty}, \omega)} + O(\omega) \| \phi \|_{L^2(S_{\omega_\infty}, \omega)}.
\]
And (3.8) can be concluded by applying Grönwall’s inequality for \( u \) variable.

Inequality (3.9) can be proved in a similar fashion. For a scalar function \( f \), we arrive at
\[
\| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \cdot \frac{d}{du} \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} = \int_{S_{\omega_\infty}} \Omega \left( e_3(f) + \text{tr} \chi f \right)
\]
Taking \( f = |\phi|^2 \), using Cauchy-Schwarz inequality on the sphere and the fact \( \Omega > 0, \text{tr} \chi < 0 \), we obtain
\[
2 \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \cdot \frac{d}{du} \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \lesssim \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \cdot \| \nabla_\omega \phi \|_{L^2(S_{\omega_\infty}, \omega)}.
\]
This implies \( \frac{d}{du} \| \phi \|_{L^2(S_{\omega_\infty}, \omega)} \lesssim \| \nabla_\omega \phi \|_{L^2(S_{\omega_\infty}, \omega)} \) and (3.9) follows.

We then rewrite the above inequalities in scale invariant norms:

**Proposition 3.6** There holds
\[
\| \phi \|_{L^2_{(\omega)}(S_{\omega_\infty}, \omega)} \lesssim \| \phi \|_{L^2_{(\omega)}(S_{\omega, \omega})} + \int_0^u \| \nabla_\omega \phi \|_{L^2_{(\omega)}(S_{\omega, \omega})} \, du',
\]
\[
\| \phi \|_{L^2_{(\omega)}(S_{\omega_\infty}, \omega)} \lesssim \| \phi \|_{L^2_{(\omega)}(S_{\omega_\infty}, \omega)} + \int_0^{u_\infty} \frac{a}{u^2} \| \nabla_\omega \phi \|_{L^2_{(\omega)}(S_{\omega', \omega})} \, du'.
\]

For equations along the incoming direction, sometimes the borderline terms necessitate more precise estimates. Typically, a borderline term contains \( \text{tr} \chi \). It turns out that the coefficients in front of \( \text{tr} \chi \) play an important role.

**Proposition 3.7** We continue to work under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3). Let \( v \) and \( T \) be \( S_{u, \omega} \)–tangent tensor fields of rank \( k \) satisfying the transport equation
\[
\nabla_3 v A_1... A_k + \lambda_0 \text{tr} \chi v A_1... A_k = T A_1... A_k.
\]

If we define \( \lambda_1 = 2\lambda_0 - 1 \), we have
\[
|u|^{\lambda_1} \| v \|_{L^2(S_{\omega_\infty}, \omega)} \lesssim \| u \|^{\lambda_1} \| v \|_{L^2(S_{\omega_\infty}, \omega)} + \int_0^u |u'|^{\lambda_1} \| T \|_{L^2(S_{\omega', \omega}, \omega)} \, du'
\]
where the implicit constant is allowed to depend on \( \lambda_0 \).

**Proof.** We use variation of area formula for equivariant vector \( L \)\(^6\) and a scalar function \( f \):
\[
\frac{d}{du} \int_{S_{\omega_\infty}} f = \int_{S_{\omega_\infty}} (Lf + \Omega \text{tr} \chi f) = \int_{S_{\omega_\infty}} \Omega \left( e_3(f) + \text{tr} \chi f \right).
\]

\(^6\)Recall \( \Omega = \Omega e_3 \).
With this identity, we obtain
\[
\mathcal{L} \int_{S_{u,\underline{u}}} |u|^{2\lambda_1} |\phi|^2 \\
= \int_{S_{u,\underline{u}}} \Omega \left( -2\lambda_1 |u|^{2\lambda_1 - 1} (e_3 u) |\phi|^2 + 2|u|^{2\lambda_1} < \phi, \nabla_3 \phi > + \text{tr}_\lambda |u|^{2\lambda_1} |\phi|^2 \right) \\
= \int_{S_{u,\underline{u}}} \Omega |u|^{2\lambda_1} \left( -2\lambda_1 (e_3 u) |u| + (1 - 2\lambda_0) \text{tr}_\lambda \right) |\phi|^2.
\]
Observe that we have
\[
-2\lambda_1 (e_3 u) = -2\lambda_1 \Omega^{-1} + (1 - 2\lambda_0) \text{tr}_\lambda \\
\leq -2\lambda_1 (\Omega^{-1} - 1) + (1 - 2\lambda_0) (\text{tr}_\lambda + \Omega^{-1}) - \frac{2\lambda_1 + 2 - 4\lambda_0}{|u|} \\
\lesssim \frac{O}{|u|^2}.
\]
For the last inequality, we employ (3.4), the bootstrap assumption and the fact that \(|\text{tr}_\lambda + \frac{3}{|u|}||\infty(S_{u,\underline{u}}) \leq \frac{O}{|u|^2}\), and \(\lambda_1 = 2(\lambda_0 - 1/2)\).

Using Cauchy-Schwarz for the first term and applying Grönwall’s inequality for the second term, we obtain
\[
|u|^{\lambda_1} \|\phi\|_{L^2(S_{u,\underline{u}})} \\
\lesssim C |u|^{-2} \|\phi\|_{L^2(S_{u,\underline{u}})} + \int_{u}^{u} |u'|^{\lambda_1} \|F\|_{L^2(S_{u',\underline{u}})} \, du' \\
\lesssim |u|^{\lambda_1} \|\phi\|_{L^2(S_{u,\underline{u}})} + \int_{u}^{u} |u'|^{\lambda_1} \|F\|_{L^2(S_{u',\underline{u}})} \, du'.
\]
In the last step, we use \(O\|u|^{-2}\|_{L^1} \lesssim O/a \lesssim 1\). \(\Box\)

### 3.4 Sobolev embedding

With the derived estimates for the metric \(\gamma\), we can obtain a bound on the isoperimetric constant for a 2–sphere \(S\):

\[
I(S) = \sup_{U \subset S \subset \mathcal{C}^1} \frac{\min \left\{ \text{Area}(U), \text{Area}(U^c) \right\}}{(\text{Perimeter}(\partial U))^2}.
\]

**Proposition 3.8** Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), the isoperimetric constant obeys the bound

\[
I(S_{u,\underline{u}}) \leq \frac{1}{\pi}
\]

for \(u_\infty \leq u \leq -a/4\) and \(0 \leq \underline{u} \leq 1\).
Proof. Fix \( u \). Given \( U_u \subset S_{u,0} \), denote by \( U_0 \subset S_{u,0} \) the pullback image of \( U_u \) under the diffeomorphism generated by the equivariant vector field \( L \). Using Propositions 3.2 and 3.3 we can obtain that

\[
\frac{\text{Perimeter} (\partial U_u)}{\text{Perimeter} (\partial U_0)} \geq \sqrt{\inf_{S_{u,0}} \lambda(u)}.
\]

\[
\frac{\text{Area} U_u}{\text{Area} U_0} \leq \sup_{S_{u,0}} \det(\gamma_u), \quad \frac{\text{Area} U_u^*}{\text{Area} U_0^*} \leq \sup_{S_{u,0}} \det(\gamma_u).
\]

Using the fact that \( I(S_{u,0}) = \frac{1}{\pi^2} \), as it is the standard sphere in Minkowski spacetime, the bounds in Propositions 3.2 and 3.3 yield the conclusion. \( \blacksquare \)

We shall be employing an \( L^2 - L^\infty \) embedding statement in this paper quite often. To derive it, in addition to 3.8, we require the two propositions below, whose proof is found in [6].

**Proposition 3.9** Suppose \((S, \gamma)\) is a Riemannian 2- manifold. There holds

\[
\left( \text{Area}(S) \right)^{-\frac{1}{p}} \| \phi \|_{L^p(S)} \leq C_p \sqrt{\max \left\{ I(S), 1 \right\}} \left( \| \nabla \phi \|_{L^2(S)} + \left( \text{Area}(S) \right)^{-\frac{1}{2}} \| \phi \|_{L^2(S)} \right),
\]

for any \( 2 < p < \infty \) and any tensor \( \phi \).

**Proposition 3.10** Suppose \((S, \gamma)\) is a Riemannian 2- manifold. There holds

\[
\| \phi \|_{L^\infty(S)} \leq C_p \sqrt{\max \left\{ I(S), 1 \right\}} \left( \text{Area}(S) \right)^{\frac{1}{2} - \frac{1}{p}} \left( \| \nabla \phi \|_{L^{p'}(S)} + \left( \text{Area}(S) \right)^{-\frac{1}{2}} \| \phi \|_{L^{p'}(S)} \right),
\]

for any \( 2 < p < \infty \) and any tensor \( \phi \).

Given Proposition 3.4, we know that \( \text{Area}(S_{u,0}) = |u|^2 \). Substituting this into Propositions 3.9 and 3.10 and taking into account Proposition 3.8, we have the following \( L^2 - L^\infty \) Sobolev embedding inequality:

**Proposition 3.11** Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), it holds

\[
\| \phi \|_{L^\infty(S_{u,0})} \lesssim \sum_{0 \leq i \leq 2} \left\| |u|^{i-1} \nabla^i \phi \right\|_{L^2(S_{u,0})}.
\]

In scale invariant norms:

\[
\| \phi \|_{C^0_{\gamma}(S_{u,0})} \lesssim \sum_{0 \leq i \leq 2} \left\| (a^\frac{1}{2} \nabla)^i \phi \right\|_{C^0_{\gamma}(S_{u,0})}.
\]

### 3.5 Commutation formulae

We list some useful commutation formulae that shall be used to give a schematic representation of repeated commutations.

**Proposition 3.12** For a scalar function \( f \), there holds

\[
|\nabla_4, \nabla| f = \frac{1}{2}(\eta + \nu) \nabla_4 f - \chi \cdot \nabla f,
\]

\[
|\nabla_3, \nabla| f = \frac{1}{2}(\eta + \nu) \nabla_3 f - \chi \cdot \nabla f.
\]

**Proposition 3.13** For an \( S_{u,0} \)-tangent 1-form \( U_b \), there holds

\[
|\nabla_4, \nabla_a| U_b = -\chi_{ac} \nabla_c U_b + \epsilon_{ac}^* \beta_{Rb} U_c + \frac{1}{2}(\eta_a + \eta_n) \nabla_4 U_b - \chi_{ac} \eta_b U_c + \chi_{ab} \eta \cdot U,
\]

\[
|\nabla_3, \nabla_a| U_b = -\chi_{ac} \nabla_c U_b + \epsilon_{ac}^* \beta_{Rb} U_c + \frac{1}{2}(\eta_a + \eta_n) \nabla_3 U_b - \chi_{ac} \eta_b U_c + \chi_{ab} \eta \cdot U.
\]
Proposition 3.14 For an $S_{u,a}$-tangent 2-form $V_{bc}$, there holds

$$\left[ \nabla_4, \nabla_a \right] V_{bc} = \frac{1}{2}(\eta_a + \eta_b) \nabla_4 V_{bc} - \eta_a V_{dc} \chi_{ad} - \eta_b V_{bd} \chi_{ad} - \epsilon_{bd}^* \beta_{R_a} V_{dc} - \epsilon_{cd}^* \beta_{R_b} V_{bd} + \chi_{ac} V_{bd} \eta_d + \chi_{ab} V_{dc} \eta_d - \chi_{ad} \nabla_a V_{bc}.$$  

$$\left[ \nabla_3, \nabla_a \right] V_{bc} = \frac{1}{2}(\eta_a + \eta_b) \nabla_3 V_{bc} - \eta_b V_{dc} \chi_{ad} - \eta_c V_{bd} \chi_{ad} - \epsilon_{bd}^* \beta_{R_a} V_{dc} - \epsilon_{cd}^* \beta_{R_b} V_{bd} + \chi_{ac} V_{bd} \eta_d + \chi_{ab} V_{dc} \eta_d - \chi_{ad} \nabla_a V_{bc}.$$  

Proposition 3.15 Assume $\nabla_4 \phi = F_0$. Let $\nabla_4 \nabla^i \phi = F_i$. Then

$$F_i = \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} F_0 + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \beta_{R} \nabla^{i_4} \phi + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \chi \nabla^{i_4} \phi.$$  

Assume now that $\nabla_3 \phi = G_0$. Let $\nabla_3 \nabla^i \phi = G_i$. Then

$$G_i + \frac{i}{2} \text{tr}\chi \nabla^i \phi = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} G_0 + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \beta_{R} \nabla^{i_4} \phi + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \nabla^{i_4} \phi.$$  

Finally, we can replace $\beta_{R}, \bar{\beta}_{R}$ by expressions involving Ricci coefficients, under the Codazzi equations:

$$\beta_{R} = -\text{div} \chi + \frac{1}{2} \text{tr} \chi \nabla - \frac{1}{2}(\eta - \eta) \cdot (\bar{\chi} - \frac{1}{2} \text{tr} \chi),$$  

$$\bar{\beta}_{R} = \text{div} \chi - \frac{1}{2} \text{tr} \chi - \frac{1}{2}(\eta - \eta) \cdot (\bar{\chi} - \frac{1}{2} \text{tr} \chi).$$

That way, we arrive at the following:

Proposition 3.16 Suppose $\nabla_4 \phi = F_0$. Let $\nabla_4 \nabla^i \phi = F_i$. Then

$$F_i = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} F_0 + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \beta_{R} \nabla^{i_4} \phi.$$  

Similarly, suppose $\nabla_3 \phi = G_0$. Let $\nabla_3 \nabla^i \phi = G_i$. Then

$$G_i + \frac{i}{2} \text{tr}\chi \nabla^i \phi = \sum_{i_1+i_2+i_3=i} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} G_0 + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \beta_{R} \nabla^{i_4} \phi + \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1}(\eta + \eta)^{i_2} \nabla^{i_3} \nabla^{i_4} \phi.$$  

4 $L^2(S_{u,a})$-estimates for Ricci coefficients and Maxwell components

We start with some useful estimates. Let

$$\psi \in \left\{ \frac{\partial}{\partial x}, \text{tr} \chi, \omega, \eta, \eta, \bar{\omega}, \bar{\omega}, \frac{1}{2} \text{tr} \chi, \frac{1}{2} \text{tr} \chi, \frac{1}{2} \text{tr} \chi \right\}, \Psi \in \left\{ \frac{\partial}{\partial x}, \beta, \rho, \sigma, \beta, \bar{\omega} \right\} \text{ and } \Upsilon \in \left\{ \frac{\partial}{\partial x}, \beta, \rho, \sigma, \Delta \right\}.$$  

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Proposition 4.1 Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have

\[
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1} \psi \|^2_{L^2(S_{\infty})} \leq |u|, \\
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1+2} \psi \|^2_{L^2(S_{\infty})} \leq O, \\
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1+2} \psi \|^2_{L^2(S_{\infty})} \leq \frac{O^2}{|u|^2}, \\
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1+3} \psi \|^2_{L^2(S_{\infty})} \leq O, \\
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1+3} \psi \|^2_{L^2(S_{\infty})} \leq \frac{O^2}{|u|^3}, \\
\sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1+4} \psi \|^2_{L^2(S_{\infty})} \leq \frac{O^2}{|u|^3}.
\]

Proof. We focus on the last five statements. The ones before are similar and their proof can be found in Section 4 of [4].

- For the first one we distinguish two cases: If \(i_2 = 0\), then the result holds trivially as \(\|1\|^2_{L^2(S_{\infty})} = |u|\).
  
  For \(i_2 \geq 1\), we can rewrite \(\nabla^{i_1} \Upsilon^{i_2}\) as a product of \(i_2\) terms

\[\nabla^{i_1} \Upsilon^{i_2} = \nabla^{j_1} \Upsilon \ldots \nabla^{j_{i_2}} \Upsilon, \text{ with } j_1 + \cdots + j_{i_2} = i_1.\]

Assume that \(j_{i_2}\) is the largest number. Then we rewrite

\[(a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1} \Upsilon^{i_2} = (a^\frac{1}{2})^{i_1} \cdot (a^\frac{1}{2})^{j_{i_2}} \Upsilon \cdot \prod_{k=1}^{i_2-1} (a^\frac{1}{2})^{j_k} \Upsilon.\]

Now we bound \((a^\frac{1}{2})^{j_{i_2}} \Upsilon \) in \(L^2(S_{\infty})\) and the rest of the terms in \(L^\infty(S_{\infty})\). We then have

\[
\frac{1}{|u|} \sum_{i_1+i_2 \leq 10} \| (a^\frac{1}{2})^{i_1+i_2} \nabla^{i_1} \Upsilon \|^2_{L^2(S_{\infty})}.
\]

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\[
\leq \frac{1}{|u|} \sum_{i_1+i_2 \leq 10} \frac{(a^2)^{i_2}}{|u|^{i_2-1}} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \prod_{k=1}^{i_2-1} \|(a^{\frac{2}{i_2}})j_{k} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq \frac{(a^2)^{i_2} O^{i_2}}{|u|^{i_2}} \leq 1.
\]

- For the second one, if \(i_2 = 0\), then the statement is true because of the definition of \(O\). If \(i_2 \geq 1\), then assume \(i_1 = j_1 + \cdots + j_{i_2+1}\). Assume \(j_{i_2+1}\) is the largest. Then, as above,

\[
\sum_{i_1+i_2 \leq 10} \|(a^{\frac{2}{i_2}})j_{i_2+1} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq \sum_{i_1+i_2 \leq 10} \frac{(a^2)^{j_{i_2+1}}}{|u|^{i_2}} \|(a^{\frac{2}{i_2}})j_{i_2+1} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq \frac{(a^2)^{j_{i_2+1}} O^{j_{i_2+1}}}{|u|^{i_2}} \leq O.
\]

- We have

\[
|u| \sum_{i_1+i_2 \leq 10} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq |u| \cdot \frac{1}{|u|} \sum_{i_1+i_2 \leq 10} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq O \cdot O = O^2.
\]

The proof of this is the same as the item above. If \(i_3 \leq 6\) we bound \((a^{\frac{2}{i_3}})j_{i_3} \mathbf{T} \) in \(L_{(\sigma)}^{\infty}(S_u, \omega)\), otherwise we bound it in \(L_{(\sigma)}^{2}(S_u, \omega)\) and the rest of the terms in \(L_{(\sigma)}^{\infty}(S_u, \omega)\).

- We have

\[
|u| \sum_{i_1+i_2+i_3+i_4 \leq 10} \|(a^{\frac{2}{i_4}})j_{i_4} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq |u| \cdot \frac{1}{|u|} \cdot O \cdot \sum_{i_1+i_2+i_3 \leq 10} \|(a^{\frac{2}{i_3}})j_{i_3} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)}
\]

since one of \(i_3, i_4\), without loss of generality say \(i_4\), has to be at most 6 so that we can bound the term \(\nabla^{i_4} \mathbf{T} \) in \(L_{(\sigma)}^{\infty}(S_u, \omega)\). We have

\[
\sum_{i_1+i_2+i_3+i_4 \leq 10} \|(a^{\frac{2}{i_4}})j_{i_4} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq O \cdot \sum_{i_1+i_2 \leq 10} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} + O \cdot \sum_{i_1+i_2 \leq 3} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq O \cdot \sum_{i_1+i_2 \leq 10} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} + O \cdot \sum_{i_1+i_2 \leq 5} \|(a^{\frac{2}{i_2}})j_{i_2} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq O^2 \cdot |u| = O.
\]

Here we have distinguished between the cases where \(i_3\) is at most 6, in which case we bound it in \(L_{(\sigma)}^{\infty}(S_u, \omega)\) and the case where \(7 \leq i_3 \leq 10\), in which case we bound the term \(\nabla^{i_3} \mathbf{T} \) in \(L_{(\sigma)}^{2}(S_u, \omega)\) and use the Sobolev embedding theorem to bound \((a^{\frac{2}{i_3}})j_{i_3} \mathbf{T} \) in \(L_{(\sigma)}^{\infty}(S_u, \omega)\). Putting everything together, we arrive at

\[
\sum_{i_1+i_2+i_3+i_4 \leq 10} \|(a^{\frac{2}{i_4}})j_{i_4} \mathbf{T} \|_{L_{(\sigma)}^s(S_u, \omega)} \leq O^2 \cdot |u|.
\]
4.1 $L^2(S_u, \omega)$-estimates for the Ricci coefficients

**Proposition 4.2** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have

$$\sum_{i \leq 10} \| (a^i \nabla)^i \omega \|_{L^2(S_u, \omega)} \lesssim \frac{a^i}{|u|^2} (R[\rho] + 1).$$

**Proof.** We use the following schematic null structure equation for $\omega$:

$$\nabla_3 \omega = \frac{1}{2} \rho + \psi \psi + YY.$$

Commuting $i$ times with $\nabla$ using Proposition 3.16, we arrive at

$$\nabla_3 \nabla^i \omega + \frac{i}{2} \text{tr} \nabla^i \omega$$

$$= \sum_{i_1 + i_2 + i_3 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\rho + \psi \psi + YY) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi, \chi, \text{tr} \chi) \nabla^4 \omega$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi, \chi, \text{tr} \chi) \nabla^4 \omega$$

$$= \nabla^i \rho + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi)$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi, \chi, \text{tr} \chi) \nabla^4 \omega$$

Now notice that for any $j$ and $S$-tangent tensorfields $\phi_1, \phi_2$ we have the schematic equality $\nabla^j (\phi_1 \cdot \phi_2) = \sum_{j_1 + j_2 = j} \nabla^{j_1} \phi_1 \nabla^{j_2} \phi_2$. We can thus write

$$\nabla_3 \nabla^i \omega + \frac{i}{2} \text{tr} \nabla^i \omega$$

$$= \nabla^i \rho + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi)$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^i \nabla^1 \nabla^2 \nabla^3 (\psi, \chi, \text{tr} \chi) \nabla^4 \omega$$

Rewrite the above as

$$\nabla_3 \nabla^i \omega + \frac{i}{2} \text{tr} \nabla^i \omega = G.$$

Applying Proposition 3.7, there holds

$$|u|^i \| \nabla^i \omega \|_{L^2(S_u, \omega)} \leq |u_{\infty}|^{i-1} \| \nabla^i \omega \|_{L^2(S_u, \omega)} + \int_{u_{\infty}}^{u} |u'|^{i-1} \| G \|_{L^2(S_u, \omega)} \, du'.$$

Multiplying both sides by $|u|$ and using $|u| \leq |u'|, |u| \leq |u_{\infty}|$ we get

$$|u|^i \| \nabla^i \omega \|_{L^2(S_u, \omega)} \leq |u_{\infty}|^{i-1} \| \nabla^i \omega \|_{L^2(S_u, \omega)} + \int_{u_{\infty}}^{u} |u'|^{i-1} \| G \|_{L^2(S_u, \omega)} \, du'.$$  \hspace{1cm} (4.1)
From the signature table we get

\[ s_2(G) = s_2(\nabla_3 \nabla^i \omega) = 1 + s_2(\nabla^i \omega) = \frac{i}{2} + 1. \]

Using the definition of the scale-invariant norms \( \mathcal{L}^2_{(s)}(S_{u \omega}) \) we have

\[ \| \phi \|_{\mathcal{L}^2_{(s)}(S_{u \omega})} = a^{-s_2(\phi)}|u|^{2s_2(\phi)} \| \phi \|_{L^2(S_{u \omega})} \]

and thus

\[ \| \nabla^i \omega \|_{\mathcal{L}^2_{(s)}(S_{u \omega})} = a^{-\frac{i}{2}}|u|^i \| \nabla^i \omega \|_{L^2(S_{u \omega})}, \quad \| G \|_{\mathcal{L}^2_{(s)}(S_{u \omega})} = a^{-\frac{i}{2}-1}|u|^{i+2} \| G \|_{L^2(S_{u \omega})}. \]

Equivalently,

\[ |u|^i \| \nabla^i \omega \|_{L^2(S_{u \omega})} = \|(a^{\frac{i}{2}} \nabla)^i \omega\|_{\mathcal{L}^2_{(s)}(S_{u \omega})}, \quad |u|^i \| G \|_{L^2(S_{u \omega})} = \frac{a}{|u|^2} \|(a^{\frac{i}{2}})^i G\|_{\mathcal{L}^2_{(s)}(S_{u \omega})}. \]

We can now write (4.1) in scale-invariant norms as

\[
\|(a^{\frac{i}{2}} \nabla)^i \omega\|_{\mathcal{L}^2_{(s)}(S_{u \omega})} \leq \|(a^{\frac{i}{2}} \nabla)^i \omega\|_{\mathcal{L}^2_{(s)}(S_{u \omega})} + \int_{u=1}^{u} \frac{a}{|u'|^2} \|(a^{\frac{i}{2}} \nabla)^i \rho\|_{\mathcal{L}^2_{(s)}(S_{u' \omega})} du' \\
+ \int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=3} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du' \\
+ \int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=3} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du' \\
+ \int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+1=4} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du'.
\]

We look at each term separately. For the first one, since \( \Omega = 1 \), we note that \( \omega = -\frac{1}{2} \nabla_4 (\log \Omega) \), we have \( \|(a^{\frac{i}{2}} \nabla)^i \omega\|_{\mathcal{L}^2_{(s)}(S_{u \omega})} = 0 \). For the second and third terms, we have

\[
\int_{u=1}^{u} \frac{a}{|u'|^2} \|(a^{\frac{i}{2}} \nabla)^i \rho\|_{\mathcal{L}^2_{(s)}(S_{u' \omega})} du' + \int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=3} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du' \\
\leq \left( \int_{u=1}^{u} \frac{a}{|u'|^2} \|(a^{\frac{i}{2}} \nabla)^i \rho\|_{\mathcal{L}^2_{(s)}(S_{u' \omega})} du' \right)^{\frac{1}{2}} \left( \int_{u=1}^{u} \frac{a}{|u'|^2} du' \right)^{\frac{1}{2}} + \int_{u=1}^{u} \frac{a}{|u'|^2} \frac{a^{\frac{i}{2}}}{|u|^2} \cdot O^2 du' \\
= \|(a^{\frac{i}{2}} \nabla)^i \rho\|_{\mathcal{L}^2_{(s)}(S_{u' \omega})} \cdot \frac{a^{\frac{i}{2}}}{|u|^2} + \frac{a^{\frac{i}{2}}}{|u|^2} O^2 \leq \frac{a^{\frac{i}{2}}}{|u|^2} (\mathcal{R}(\rho) + 1).
\]

For the next term, we have

\[
\int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=3} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du' \\
\leq \int_{u=1}^{u} \frac{a}{|u'|^2} O^2 du' = \frac{a^{\frac{i}{2}}}{|u|^2} O^2 \leq \frac{a^{\frac{i}{2}}}{|u|^2} O^2 \leq \frac{a^{\frac{i}{2}}}{|u|^2}.
\]

For the last two terms, we have

\[
\int_{u=1}^{u} \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+1=4} (a^{\frac{i}{2}})^{i_1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \|\| \mathcal{L}^2_{(s)}(S_{u' \omega}) du' 
\]

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Under the assumptions of Theorem 1.2 and the bootstrap assumption 

Moreover, 

Rewriting the above equation as 

Gathering all the estimates above and letting \( a \) be sufficiently large, we obtain 

\[
\sum_{i \leq 10} \|(a^{\frac{2}{d}} \nabla)^i \hat{\omega}\|_{L^2(S_{u,\omega})} \lesssim \frac{a^{\frac{2}{d}}}{|u|^d} (R|\rho| + 1).
\]

\( \square \)

Proposition 4.3 Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have 

\[
\sum_{i \leq 10} \frac{a^{\frac{2}{d}}}{|u|} \|(a^{\frac{2}{d}} \nabla)^i \hat{\omega}\|_{L^2(S_{u,\omega})} \lesssim 1.
\]

Proof. We look at the \( \nabla_3 \)-equation for \( \hat{\omega} \): 

\[
\nabla_3 \hat{\omega} + \text{tr} \nabla \hat{\omega} = -2\omega \hat{\omega} - \hat{\alpha}.
\]

Commuting with \( i \) angular derivatives and using Proposition 3.6 we arrive at 

\[
\nabla_3 \nabla_i \hat{\omega} + \frac{i + 2}{2} \text{tr} \nabla \nabla_i \hat{\omega}
\]

\[
= \nabla_i \hat{\alpha} + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2 + 1} \nabla^{i_3} \hat{\omega} + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla \hat{\omega}) \nabla^{i_4} \hat{\omega}
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2 + 1} \nabla^{i_3} \text{tr} \nabla \nabla^{i_4} \hat{\omega}.
\]

Rewriting the above equation as 

\[
\nabla_3 \nabla_i \hat{\omega} + \frac{i + 2}{2} \text{tr} \nabla \nabla_i \hat{\omega} = F,
\]

an application of Proposition 3.7 gives us 

\[
|u|^{i+1} \|\nabla_i \hat{\omega}\|_{L^2(S_{u,\omega})} \leq |u_x|^{i+1} \|\nabla_i \hat{\omega}\|_{L^2(S_{u,\omega})} + \int_{u_x}^u |u'|^{i+1} \|F\|_{L^2(S_{u',\omega})} \, du'.
\]

(4.2)

Rewriting (4.2) in scale-invariant norms, we arrive at 

\[
\frac{a}{|u|} \|(a^{\frac{2}{d}} \nabla)^i \hat{\omega}\|_{L^2(S_{u,\omega})} \leq \frac{a}{|u_x|} \|(a^{\frac{2}{d}} \nabla)^i \hat{\omega}\|_{L^2(S_{u,\omega})} + \int_{u_x}^u \frac{a^2}{|u'|^d} \|(a^{\frac{2}{d}})^i F\|_{L^2(S_{u',\omega})} \, du'.
\]
Multiplying this equation by $a^{-\frac{1}{2}}$ we get

$$\frac{a^\frac{3}{2}}{|u|} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \leq \frac{a^\frac{1}{2}}{|u|} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) + \int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'$$

$$+ \int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \sum_{i_1+i_2+i_3+1=i} (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \| \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'$$

$$+ \int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \sum_{i_1+i_2+i_3+1=i} (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \| \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'.$$

The initial data term is directly bounded by $T^{(0)}(u) \lesssim 1$. For the terms containing $\alpha$, we have

$$\int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du' + \int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \sum_{i_1+i_2+i_3+1=i} (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \| \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'$$

$$\leq \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \cdot \frac{a^\frac{1}{2}}{|u|} + \frac{a^2 \cdot O^2}{a^\frac{3}{2}} \leq 1.$$

The last two terms can be bounded as follows:

$$\int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \sum_{i_1+i_2+i_3+1=i} (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \| \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'$$

$$+ \int_{u_{\infty}}^{u} \frac{a^\frac{3}{2}}{|u'|^3} \sum_{i_1+i_2+i_3+1=i} (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \| \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du' \leq \frac{O^2 + O^3}{a^\frac{3}{2}} \leq 1.$$

\[\square\]

**Proposition 4.4** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have

$$\sum_{i \leq 10} \frac{1}{a^\frac{1}{2}} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \lesssim R[\alpha] + 1.$$

**Proof.** We look at the schematic equation

$$\nabla_4 \hat{\chi} = \psi \cdot \hat{\chi} + \alpha. \quad (4.3)$$

Commuting (4.3) with $i$ angular derivatives we arrive at

$$\nabla_i \nabla^i \hat{\chi} = \nabla_4 \hat{\chi} + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1} \psi^{i_2} \psi^{i_3} \nabla^{i_4} \hat{\chi} + \sum_{i_1+i_2+i_3+1=i} \nabla^{i_1} \psi^{i_2} \psi^{i_3} \nabla^{i_4} \hat{\chi}. \quad (4.4)$$

We thus have, passing to scale-invariant norms,

$$\frac{1}{a^\frac{3}{2}} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega})$$

$$\leq \frac{1}{a^\frac{1}{2}} \int_{0}^{u} \| (a^\frac{4}{2} \nabla) \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du' + \sum_{i_1+i_2+i_3+1=i} \frac{1}{a^\frac{1}{2}} \int_{0}^{u} \| (a^\frac{4}{2})^{i_1} \psi^{i_2} \psi^{i_3} \nabla^{i_4} \hat{\chi} \| L^2_{(\alpha)}(S_{u',\omega}) \, du'.$$
Commuting this equation with \(i\) angular derivatives, using Proposition 3.16, we obtain
\[
\nabla_4 \omega = \rho + \psi \cdot \psi + \Upsilon \cdot \Upsilon
\]
Commuting this equation with \(i\) angular derivatives, using Proposition 3.16, we obtain
\[
\nabla_i \nabla_i \omega = \nabla_i \rho + \sum_{i_1+i_2+i_3+i_4=1} \nabla_i \psi^{i_1} \nabla^{i_2} \nabla^{i_3} \rho + \sum_{i_1+i_2+i_3+i_4=1} \nabla_i \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon
\]
Multiplying by \((a^\perp)^i\) and using Proposition 3.6 we get
\[
\frac{1}{a^\perp} \int \| (a^\perp)^i \nabla \nabla i \omega \|^2_{L^2(S_u \omega, \omega)} + \sum_{i_1+i_2+i_3+i_4=1} \int \| (a^\perp)^i \nabla_i \psi^{i_1} \nabla^{i_2} \nabla^{i_3} \rho \|^2_{L^2(S_u \omega, \omega)} \, d\omega' + \sum_{i_1+i_2+i_3+i_4=1} \int \| (a^\perp)^i \nabla_i \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \rho \|^2_{L^2(S_u \omega, \omega)} \, d\omega' + \sum_{i_1+i_2+i_3+i_4=1} \int \| (a^\perp)^i \nabla_i \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} \Upsilon \|^2_{L^2(S_u \omega, \omega)} \, d\omega' \]
The result follows.

We proceed with estimates for \(\omega\).

**Proposition 4.5** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), there holds
\[
\sum_{i \leq 10} \| (a^\perp)^i \omega \|^2_{L^2(S_u \omega, \omega)} \lesssim \mathcal{R}[\rho] + 1.
\]

**Proof.** We have the schematic null structure equation
\[
\nabla_4 \omega = \rho + \psi \cdot \psi + \Upsilon \cdot \Upsilon
\]
We begin by recalling the structure equation (2.15) for \( \psi \) working in scale-invariant norms, we get

\[
\| (a_x^2 \nabla) \rho \|_{L^2_t(S_u, \omega)}^2 + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \int_0^\infty a_x^2 \left\| \left( a_x^2 \nabla \right) \psi^{i_2} \nabla^{i_3} \left( \frac{\psi}{a_x^2}, \frac{\tilde{\chi}}{a_x^2} \right) \nabla^{i_4} \psi \right\|_{L^2_t(S_u, \omega)}^2 \, du' \\
\leq \| (a_x^2 \nabla) \rho \|_{L^2_t(S_u, \omega)}^2 + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \frac{a_x^2}{|u|} \leq \mathcal{R}[\bar{\beta}] + 1.
\]

Here and throughout we have made use of Proposition 4.1.

**Proposition 4.6** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have

\[
\sum_{i \leq 10} \| (a_x^2 \nabla) \psi \|_{L^2_t(S_u, \omega)}^2 \lesssim \mathcal{R}[ar{\beta}] + 1.
\]

**Proof.** We begin by recalling the structure equation (2.15) for \( \eta \):}

\[
\nabla_4 \eta_a = -\chi_{ab} \cdot (\eta - \frac{\eta}{u})_b - \frac{1}{2} R_{a4}.
\]

Also recall that \( \bar{\beta} = \beta - \frac{1}{4} R_{44} \). We can therefore rewrite (2.15) in terms of \( \bar{\beta} \) as follows:

\[
\nabla_4 \eta_a = -\chi_{ab} \cdot (\eta - \frac{\eta}{u})_b - \bar{\beta}_a - R_{a4}.
\]

This leads us to the following schematic null structure equation:

\[
\nabla_4 \eta = \bar{\beta} + \psi \cdot (\psi, \tilde{\chi}) + (\rho_F, \sigma_F) \cdot \alpha_F.
\]

Commuting with \( i \) angular derivatives, using Proposition 3.16, we have

\[
\nabla_4 \nabla^i \eta = \nabla^i \bar{\beta} + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \bar{\beta}
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} \psi
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F) \nabla^{i_4} \alpha_F.
\]

Working in scale-invariant norms, we get

\[
\| (a_x^2 \nabla) \psi \|_{L^2_t(S_u, \omega)}^2 \\
\leq \int_0^\infty \left\| (a_x^2 \nabla) \bar{\beta} \right\|_{L^2_t(S_u, \omega)}^2 \, du' + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \| (a_x^2) \psi^{i_2} \nabla^{i_3} \bar{\beta} \|_{L^2_t(S_u, \omega)}^2 \, du' \\
+ \sum_{i_1 + i_2 + i_3 + i_4 = 1} a_x^2 \left\| (a_x^2) \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \left( \frac{\psi}{a_x^2}, \frac{\tilde{\chi}}{a_x^2} \right) \right\|_{L^2_t(S_u, \omega)}^2 \, du' \\
+ \sum_{i_1 + i_2 + i_3 + i_4 = 1} a_x^2 \left\| (a_x^2) \psi^{i_2} \nabla^{i_3} \left( \frac{\alpha_F}{a_x^2} \right) \right\|_{L^2_t(S_u, \omega)}^2 \, du' \\
\leq \mathcal{R}[\bar{\beta}] + \frac{a_x^2}{|u|} \leq \mathcal{R}[\bar{\beta}] + 1.
\]
Proposition 4.7 Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have
\[
\sum_{i \leq 10} \left\| (a^{\frac{2}{3}} \nabla)^i \text{tr} \chi \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \lesssim (R[a] + E[\rho_F, \sigma_F] + 1)^2.
\]

Proof. We again start by considering the schematic equation
\[
\nabla_4 \text{tr} \chi = \dot{\chi} \cdot \dot{\chi} + \alpha_F \cdot \alpha_F + \psi \hat{\psi}.
\]

By commuting with $i$ angular derivatives, we arrive at
\[
\nabla_4 \nabla^i \text{tr} \chi = \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \chi + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \alpha_F \nabla^{i_4} \alpha_F
\]
\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \psi
\]
\[
= \sum_{i_1 + i_2 = i} \nabla^{i_1} \dot{\chi} \nabla^{i_2} \chi + \sum_{i_1 + i_2 + i_3 + i_4 + 1 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \chi
\]
\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \psi
\]
\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \psi.
\]

Taking this into account\textsuperscript{7}, we have
\[
\left\| (a^{\frac{2}{3}} \nabla)^i \text{tr} \chi \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \leq \sum_{i_1 + i_2 = i} \int_0^2 a \left\| (a^{\frac{2}{3}})^i \nabla^{i_1} \left( \frac{\dot{\chi}}{a^{\frac{2}{3}}} \cdot \alpha_F \right) \nabla^{i_2} \left( \frac{\dot{\chi}}{a^{\frac{2}{3}}} \cdot \alpha_F \right) \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \, du'
\]
\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^2 a \left\| (a^{\frac{2}{3}})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \dot{\chi} \nabla^{i_4} \chi \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \, du'
\]
\[
\leq \frac{a}{|u|} O_2[\dot{\chi}, \alpha_F] \cdot O_{\infty}[\dot{\chi}, \alpha_F] + \frac{a}{|u|^2} O^3 + \frac{a^{\frac{2}{3}}}{|u|^2} O^2
\]
\[
\leq O_2[\dot{\chi}, \alpha_F] \cdot O_{\infty}[\dot{\chi}, \alpha_F] + 1 \lesssim (R[a] + E[\rho_F, \sigma_F] + 1)^2,
\]
by using the estimates on $\dot{\chi}$ proved in Proposition 4.4 and the estimates on $\alpha_F$ that will be shown in Proposition 4.10 in the following subsection.

We move on to estimates for tr$\chi$.

Proposition 4.8 Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have
\[
\sum_{i \leq 10} \frac{a}{|u|} \left\| (a^{\frac{2}{3}} \nabla)^i (\text{tr} \chi + \frac{2}{|u|}) \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \lesssim R[\rho] + R[\rho] + 1, \quad \sum_{i \leq 10} \frac{a}{|u|^2} \left\| (a^{\frac{2}{3}} \nabla)^i \text{tr} \chi \right\|_{L^2_{(\psi^{\alpha})}(S_{u^{\alpha}})} \lesssim \frac{R[\rho]}{|u|} + 1.
\]

\textsuperscript{7}In the following, even though we do not encounter cross terms of the form $\nabla^{i_1} \dot{\chi} \nabla^{i_2} \alpha_F$, we do not lose any control on the inequality by grouping the terms together and controlling schematically terms of the form $\nabla^{i_1} (\dot{\chi} \cdot \alpha_F) \nabla^{i_2} (\dot{\chi} \cdot \alpha_F)$.  

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Proof. For $\tilde{\chi} = \chi + \frac{\alpha_F}{|u|}$, we have the schematic null structure equation

$$\nabla_\beta \tilde{\chi} + \chi \tilde{\chi} = \frac{2}{|u|^2} (\Omega^{-1} - 1) + \tilde{\chi} \tilde{\chi} + \psi \tilde{\chi} - |\tilde{\chi}|^2 - |\alpha_F|^2.$$  

Commuting this equation with $i$ angular derivatives, we have

$$\nabla_\beta \nabla_i \tilde{\chi} + \frac{i+2}{2} \chi \tilde{\chi}$$

$$= \sum_{i_1+i_2+i_3=1} \nabla_i \nabla_i \nabla_i \left( \frac{2}{|u|^2} (\Omega^{-1} - 1) + \tilde{\chi} \tilde{\chi} + \psi \tilde{\chi} - |\tilde{\chi}|^2 - |\alpha_F|^2 \right)$$

$$+ \sum_{i_1+i_2+i_3+i_4=i} \nabla_i \nabla_i \nabla_i (\psi, \tilde{\chi}) \nabla_i \tilde{\chi} + \sum_{i_1+i_2+i_3+i_4+1=i} \nabla_i \nabla_i \nabla_i \tilde{\chi} \nabla_i \tilde{\chi}$$

$$= \tilde{F}.$$  

Rewriting in terms of scale-invariant norms,

$$\frac{a}{|u|^2} \| (a \nabla^i \tilde{\chi}) \|_{L^2(S_{u, w})} \leq \frac{a}{|u|^2} \| (a \nabla^i \tilde{\chi}) \|_{L^2(S_{u, w})} + \int_{u}^\infty \frac{a^2}{|u'|^2} \| (a \nabla^i \tilde{\chi}) \|_{L^2(S_{u', w})} du'$$

$$= \frac{a}{|u|^2} \| (a \nabla^i \tilde{\chi}) \|_{L^2(S_{u, w})} + I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{u}^\infty \frac{a^2}{|u'|^2} \| (a \nabla^i \psi) \|_{L^2(S_{u', w})} du'$$

$$= \int_{u}^\infty \frac{a^2}{|u'|^2} \| (a \nabla^i \psi) \|_{L^2(S_{u', w})} du'$$

$$\lesssim O^2(\tilde{\chi}) + \frac{a^2}{|u|^2} O^2(\alpha_F) + 1 \lesssim 1.$$  

There holds

$$I_2 = \int_{u}^\infty \frac{a^2}{|u'|^3} \| (a \nabla^i \psi) \|_{L^2(S_{u', w})} du'$$

$$= \int_{u}^\infty \frac{a^2}{|u'|^3} \| (a \nabla^i \psi) \|_{L^2(S_{u', w})} du'$$

$$\leq \int_{u}^\infty \frac{a^2}{|u'|^3} \| (a \nabla^i \tilde{\chi}) \|_{L^2(S_{u', w})} du'$$

$$\lesssim O^2(|\tilde{\chi}|) + O^2(\alpha_F) + 1 \lesssim R[\tilde{\chi}] + 1$$ (by Proposition 4.5 and the fact that $\frac{a^2}{|u|^2} \| \tilde{\chi} \|_{L^2(S_{u, w})} \lesssim 1$).
There also holds
\[
\mathcal{I}_3 = \int_{u_0}^u \frac{a^2}{|u'|^3} \left( (a^\frac{\hat{x}}{x})^i \right) \sum_{i_1+i_2+i_3 = 1} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \left( \frac{\Omega^{-1} - 1}{|u'|^2} \right) \| \xi_{(i)}^2 (S_{\omega'}) \| du'
\]
\[
= \int_{u_0}^u |u'|^{i+1} \sum_{i_1+i_2+i_3 = 2} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \left( \frac{\Omega^{-1} - 1}{|u'|^2} \right) \| L^2 (S_{\omega'}) \| du' \quad \text{(in standard norms)}
\]
\[
= \int_{u_0}^u |u'|^{i+1} \sum_{i_1+i_2+i_3 = 2} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \left( \frac{\Omega^{-1} - 1}{|u'|^2} \right) \| L^2 (S_{\omega'}) \| du' \quad \text{Using } \frac{\partial}{\partial \Omega} \Omega^{-1} = 2 \omega \Rightarrow \nabla_4 \Omega^{-1} = 2 \Omega^{-1} \omega
\]
\[
= \int_{u_0}^u |u'|^{i+1} \sum_{i_1+i_2+i_3 = 2} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \left( \frac{1}{|u'|^2} \right) \int_0^u 2 \omega (u', u', \theta^1, \theta^2) du' \| L^2 (S_{\omega'}) \| du'
\]
\[
\leq |u'|^{i+1} \sum_{i_1+i_2+i_3 = 2} \left( \frac{1}{|u'|^{i+1}} \cdot \frac{1}{|u'|^2} \cdot \frac{1}{|u'|^3} \cdot \frac{a^\hat{x}}{|u'|^2} \right) \left( \mathcal{R} [\rho] + 1 \right) du'
\]
\[
\leq \int_{u_0}^u \frac{a^\hat{x}}{|u'|^2} \left( \mathcal{R} [\rho] + 1 \right) du' \lesssim \mathcal{R} [\rho] + 1.
\]

Finally, there holds
\[
\mathcal{I}_4 = \int_{u_0}^u \frac{a^2}{|u'|^3} \left( (a^\hat{x})^i \right) \sum_{i_1+i_2+i_3 = i+1} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \| \xi_{(i)}^2 (S_{\omega'}) \| du'
\]
\[
= \int_{u_0}^u a^\hat{x} \left( (a^\hat{x})^{i-1} \sum_{i_1+i_2+i_3 = i+1} \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} \left( \frac{a}{|u'|^2} \right) \| \xi_{(i)}^2 (S_{\omega'}) \| du'
\]
\[
\leq \int_{u_0}^u a^\hat{x} \cdot \frac{O^3}{|u'|^4} du' \leq 1 \quad \text{(by Proposition 4.1)}.
\]

In summary, we have obtained
\[
\sum_{i \leq 10} \frac{a}{|u'|} \left( (a^\hat{x})^i \right) \| \nabla (a^\hat{x}) \| L^2_{(i)} (S_{\omega'}) \lesssim \mathcal{R} [\rho] + \mathcal{R} [\rho] + 1,
\]
which implies
\[
\sum_{i \leq 10} \frac{a}{|u'|} \left( (a^\hat{x})^i \right) \| \xi_{(i)}^2 (S_{\omega'}) \| L^2_{(i)} (S_{\omega'}) \lesssim \frac{\mathcal{R} [\rho] + \mathcal{R} [\rho]}{|u'|} + 1.
\]

\[
\square
\]

**Proposition 4.9** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have
\[
\sum_{i \leq 10} \| (a^\hat{x})^i \| L^2_{(i)} (S_{\omega'}) \lesssim \mathcal{R} [\tilde{\beta}] + \mathcal{R} [\tilde{\beta}] + 1.
\]

**Proof.** We use the following schematic null structure equation for $\eta$:
\[
\nabla_3 \eta + \frac{1}{2} \text{tr} \chi \eta = \tilde{\beta} + \text{tr} \chi \eta + \tilde{\psi} \cdot \psi + \Upsilon \cdot \Upsilon.
\]

Commuting with $i$ angular derivatives, we have
\[
\nabla_i \nabla^i \eta + \frac{i + 1}{2} \text{tr} \chi \nabla^i \eta.
\]

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Proposition 4.10  
Under the assumptions of Theorem 1.2 and the bootstrap assumption

\begin{align*}
&= \nabla^1 \tilde{\beta} + \sum_{i_1+i_2+i_3+i_4+1 = i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\beta} + \text{tr}_X \nabla^1 \eta + \sum_{i_1+i_2+1 = i} \nabla^{i_1+1} \text{tr} \nabla^{i_2} (\eta, \eta) \\
&+ \sum_{i_1+i_2+i_3+i_4+1 = i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \psi \nabla^{i_4} \text{tr}_X + \sum_{i_1+i_2+1 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\tilde{\chi}, \text{tr}_X) \\
&+ \sum_{i_1+i_2+i_3+i_4+1 = i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\eta} \nabla^{i_4} \tilde{\eta}.
\end{align*}

By passing to scale-invariant norms we have

\begin{align*}
&\frac{1}{|u|} \|(a^1 \nabla)^1 \eta\|_{L^2(S_{u, \omega})} \\
&\leq \frac{1}{|u|_\infty} \|(a^1 \nabla)^1 \eta\|_{L^2(S_{u, \omega})} + \int_{u_\infty}^u \frac{a}{|u'|^3} \|(a^1 \nabla)^1 \tilde{\beta}\|_{L^2(S_{u', \omega})} \, du' \\
&+ \int_{u_\infty}^u \frac{a}{|u'|^3} \sum_{i_1+i_2+1 = i} (a^1 \nabla)^1 \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \tilde{\beta}\|_{L^2(S_{u', \omega})} \, du' \\
&+ \int_{u_\infty}^u \frac{a}{|u'|^3} \sum_{i_1+i_2+1 = i} (a^1 \nabla)^1 \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \psi \nabla^{i_4} \text{tr}_X \|_{L^2(S_{u', \omega})} \, du' \\
&+ \int_{u_\infty}^u \frac{a}{|u'|^3} \sum_{i_1+i_2+1 = i} (a^1 \nabla)^1 \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\tilde{\chi}, \text{tr}_X) \|_{L^2(S_{u', \omega})} \, du' \\
&+ \int_{u_\infty}^u \frac{a}{|u'|^3} \sum_{i_1+i_2+i_3+i_4+1 = i} (a^1 \nabla)^1 \psi^{i_2+1} \nabla^{i_3} \tilde{\eta} \nabla^{i_4} \tilde{\eta} \|_{L^2(S_{u', \omega})} \, du' \\
&\leq \frac{1}{|u|_\infty} + \frac{|R[\tilde{\beta}]| + |R[\tilde{\beta}]| + 1}{|u|} + \int_{u_\infty}^u \frac{a}{|u'|^3} \cdot \frac{|O^2|}{|u'|} \, du' \leq \frac{|R[\tilde{\beta}]| + |R[\tilde{\beta}]| + 1}{|u|}.
\end{align*}

This concludes the $L^2$-estimates on Ricci coefficients.

4.2 $L^2(S_{u, \omega})$-estimates for the Maxwell components

Proposition 4.10  Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have

\[ \sum_{i \leq 10} \frac{1}{|u|} \|(a^1 \nabla)^1 \alpha_F\|_{L^2(S_{u, \omega})} \leq |F[\rho_F, \sigma_F]| + 1. \]

Proof. We have the schematic equation

\[ \nabla_3 \alpha_F + \frac{1}{2} \text{tr}_X \alpha_F = \nabla(\rho_F, \sigma_F) + \psi \cdot (\rho_F, \sigma_F) + \tilde{\chi} \cdot \alpha_F + \psi \cdot \alpha_F. \]

Commuting with $i$ angular derivatives, we get

\[ \nabla_i \nabla^i \alpha_F + \frac{i+1}{2} \text{tr}_X \alpha_F \]

\[ = \nabla^{i+1}(\rho_F, \sigma_F) + \sum_{i_1+i_2+i_3+i_4+1 = i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \psi \nabla^{i_4}(\rho_F, \sigma_F). \]
\[
+ \sum_{i_1 + i_2 = i} \nabla^{i_1+1} \operatorname{tr} \nabla^{i_2} \alpha_F + \sum_{i_1 + i_2 = i} \nabla^{i_1} \nabla^{i_2} \alpha_F + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\rho_F, \sigma_F)
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \alpha_F + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \alpha_F
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2+1} \operatorname{tr} \nabla^{i_3} \nabla^{i_4} \alpha_F
\]

Denote the right-hand side of the above as \( G \). We then have
\[
a^{-\frac{1}{2}}|u|^i \| \nabla^i \alpha_F \|_{L^2(S_{u=\infty})} \leq a^{-\frac{1}{2}}|u_{\infty}|^i \| \nabla^i \alpha_F \|_{L^2(S_{u=\infty})} + \int_{u_{\infty}}^u a^{-\frac{1}{2}} |u'|^i \| G \|_{L^2(S_{u=\infty})} du'.
\]

From the signature table one can read off
\[
s_2(\alpha_F) = 0 \Rightarrow s_2(\nabla^i \alpha_F) = 0 + i \cdot \frac{1}{2} = \frac{i}{2}.
\]

By conservation of signatures,
\[
s_2(G) = s_2(\nabla^i \alpha_F) = \frac{i}{2}.
\]

Taking into account now that
\[
\| \phi \|_{C^2((S_{u=\infty}))} := a^{-s_2(\phi)}|u|^{2s_2(\phi)} \| \phi \|_{L^2(S_{u=\infty})},
\]
we can conclude that
\[
a^{-\frac{1}{2}}|u|^i \| \nabla^i \alpha_F \|_{L^2(S_{u=\infty})} = a^{-\frac{1}{2}} \| (a^{\frac{i}{2}} \nabla)^i \alpha_F \|_{C^2((S_{u=\infty}))},
\]
\[
a^{-\frac{1}{2}}|u|^i \| G \|_{L^2(S_{u=\infty})} = a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i G \|_{C^2((S_{u=\infty}))}.
\]

Therefore,
\[
a^{-\frac{1}{2}}|u|^i \| \nabla^i \alpha_F \|_{L^2(S_{u=\infty})} \leq a^{-\frac{1}{2}}|u|^i \| \nabla^i \alpha_F \|_{L^2(S_{u=\infty})} + \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} (\rho_F, \sigma_F) \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} (\rho_F, \sigma_F) \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'+
+ \int_{u_{\infty}}^u a^{-\frac{1}{2}} \| (a^{\frac{i}{2}})^i \nabla^{i_1+1} \nabla^{i_2+1} \nabla^{i_3+1} \nabla^{i_4+1} \alpha_F \|_{C^2((S_{u'=\infty}))} du'.
\]
For the first term, we have

\[ a^{-\frac{1}{2}} |u|^\alpha \| \nabla \alpha_F \|_{L^2(S_{u'\infty})} \leq F(0)(u) \lesssim 1. \]

For the two terms involving the highest number of derivatives, we have

\[
\int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1}(\rho^\frac{1}{2}, \sigma_F) \|_{L^2(S_{u'\infty})}^2 \text{d}u' + \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \sum_{i_1+i_2+i_3=1} \frac{(a^\frac{1}{2})^i \nabla^{i+1}(\rho^\frac{1}{2}, \sigma_F) \|_{L^2(S_{u'\infty})}^2 \text{d}u'} \\
\leq \left( \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1}(\rho^\frac{1}{2}, \sigma_F) \|_{L^2(S_{u'\infty})}^2 \text{d}u' \right)^{\frac{1}{2}} \left( \int_{u_{\infty}}^u \frac{1}{|u'|^2} \text{d}u' \right)^{\frac{1}{2}} + \int_{u_{\infty}}^u \frac{1}{|u'|^2} \| a^\frac{1}{2} \cdot O^2 \text{d}u' \\
= \| (a^\frac{1}{2})^i \nabla^{i+1}(\rho^\frac{1}{2}, \sigma_F) \|_{L^2(S_{u'\infty})}^2 \cdot \frac{1}{|u'|^2} + \frac{a^\frac{1}{2}}{|u'|^2} \cdot O^2 \\
\leq F(\rho^\frac{1}{2}, \sigma_F) + 1 \lesssim F(\rho^\frac{1}{2}, \sigma_F) + 1.
\]

For the next two terms, we have

\[
\int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \sum_{i_1+i_2+i_3=1} \frac{(a^\frac{1}{2})^i \nabla^{i+1} \nabla \chi \nabla^{i+2} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u'} + \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \sum_{i_1+i_2+i_3=1} \frac{(a^\frac{1}{2})^i \nabla^{i+1} \nabla \chi \nabla^{i+2} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u'} \\
\leq \left( \int_{u_{\infty}}^u \frac{1}{|u'|^2} \sum_{i_1+i_2+i_3=1} \frac{(a^\frac{1}{2})^i \nabla^{i+1} \nabla \chi \nabla^{i+2} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' \right)^{\frac{1}{2}} \left( \int_{u_{\infty}}^u \frac{1}{|u'|^2} \text{d}u' \right)^{\frac{1}{2}} + \int_{u_{\infty}}^u \frac{1}{|u'|^2} \cdot O^2 \text{d}u' \\
\leq \int_{u_{\infty}}^u \frac{1}{|u'|^2} \cdot O^2 \text{d}u' + \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \cdot O^2 \text{d}u' \leq 1.
\]

For the sixth term, notice

\[
\int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1} \nabla^{i+2} \nabla \chi \nabla^{i+4} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' \leq \int_{u_{\infty}}^u \frac{a}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1} \nabla^{i+2} \nabla \chi \nabla^{i+4} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' \leq \frac{a^2 O^2}{|u'|^2} \leq 1.
\]

The seventh term can be absorbed schematically under the last term. For the last two terms, we can write

\[
\int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1} \nabla^{i+2} \nabla \chi \nabla^{i+4} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' + \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1} \nabla^{i+2} \nabla \chi \nabla^{i+4} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' \\
= \int_{u_{\infty}}^u \frac{a^\frac{1}{2}}{|u'|^2} \| (a^\frac{1}{2})^i \nabla^{i+1} \nabla^{i+2} \nabla \chi \nabla^{i+4} \alpha_F \|_{L^2(S_{u'\infty})}^2 \text{d}u' \\
\leq \frac{a^2 O^2}{|u'|^2} \leq 1.
\]

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\[
\begin{align*}
= & \int_{u_{\infty}}^{u} \left\| (a_{*}^{i}) \right\| \sum_{i_{1}+i_{2}+i_{3}+i_{4} = 1} \nabla^{i_{1}} \psi^{i_{2}+1} \nabla^{i_{3}} \left( \frac{a}{u_{*}^{i}} \gamma \nabla \right)^{i_{4}} \left( \frac{a_{*}^{i}}{a_{*}^{i}} \right) \left\| L^{i_{1}}(S_{u_{*}^{i}}) \right\| \, du' \\
+ & \int_{u_{\infty}}^{u} \left( a_{*}^{i} \right) \sum_{i_{1}+i_{2}+i_{3}+i_{4} = 1} \nabla^{i_{1}} \psi^{i_{2}+1} \nabla^{i_{3}} \left( \frac{a_{*}^{i}}{u_{*}^{i}} \gamma \nabla \right)^{i_{4}} \left( \frac{a_{*}^{i}}{a_{*}^{i}} \right) \left\| L^{i_{1}}(S_{u_{*}^{i}}) \right\| \, du' \\
\leq & \int_{u_{\infty}}^{u} \frac{O}{|u|^{2}} \, du' + \int_{u_{\infty}}^{u} \frac{O^{2}}{|u|^{2}} \, du' \leq O^{3} \left\| \frac{a_{*}^{i}}{u_{*}^{i}} \right\| \leq 1.
\end{align*}
\]

\[\square\]

**Proposition 4.11** Under the assumptions of Theorem 1.2 and the bootstrap assumptions 3.3, there holds

\[
\sum_{i \leq 10} \left\| (a_{*}^{i}) \gamma \right\| L^{i_{1}}(S_{u_{*}^{i}}) \lesssim F[\alpha] + 1.
\]

**Proof.** We have the following equations along the incoming direction:

\[
\begin{align*}
\nabla \rho \gamma + \gamma \rho \gamma &= \text{div} \, \rho \gamma + (\eta - \eta) \cdot \rho \gamma, \quad (4.6) \\
\nabla \sigma \gamma + \gamma \sigma \gamma &= -\gamma_{\text{curl}} \rho \gamma + (\eta - \eta) \cdot \gamma \rho \gamma. \quad (4.7)
\end{align*}
\]

Schematically, we can rewrite the above as

\[
\nabla \rho \gamma + \gamma \rho \gamma = \nabla \rho \gamma + \gamma \cdot \gamma.
\]

Commuting with \( i \) angular derivatives, we arrive at

\[
\begin{align*}
\nabla \rho \nabla \rho \gamma + \frac{i + 2}{2} \gamma \rho \nabla \rho \\
= & \nabla \rho \nabla \rho \gamma + \sum_{i_{1}+i_{2}+i_{3}+i_{4} = 1} \nabla^{i_{1}} \psi^{i_{2}+1} \nabla^{i_{3}} \rho \gamma + \sum_{i_{1}+i_{2}+i_{3}+i_{4} = 1} \nabla^{i_{1}} \psi^{i_{2}+1} \nabla^{i_{3}} \rho \gamma + \sum_{i_{1}+i_{2}+i_{3}+i_{4} = 1} \nabla^{i_{1}} \psi^{i_{2}+1} \nabla^{i_{3}} \rho \gamma \\
= & G.
\end{align*}
\]

Applying Proposition 3.7 with \( \lambda_{i} = i + 1 \), we have

\[
|u|^{i+1} \left\| \nabla \rho \gamma \right\| L^{i}(S_{u_{*}^{i}}) \lesssim |u_{\infty}|^{i+1} \left\| \nabla \rho \gamma \right\| L^{i}(S_{u_{*}^{i}}) + \int_{u_{\infty}}^{u} |u|^{i+1} \left\| G \right\| L^{2}(S_{u_{*}^{i}}) \, du'.
\]

We have

\[
s_{2} \left( \nabla \rho \gamma \right) = \frac{i + 1}{2}, \quad s_{2}(G) = \frac{i + 3}{2}.
\]

Therefore,

\[
\left\| (a_{*}^{i}) \gamma \right\| L^{i}(S_{u_{*}^{i}}) = (a_{*}^{i})^{i} \cdot a^{- \frac{i+1}{2}} \left\| \nabla \rho \gamma \right\| L^{i}(S_{u_{*}^{i}}),
\]

so that

\[
|u|^{i+1} \left\| \nabla \rho \gamma \right\| L^{i}(S_{u_{*}^{i}}) = (a_{*}^{i})^{i} \cdot a^{- \frac{i+1}{2}} |u|^{i+1} \left\| \nabla \rho \gamma \right\| L^{i}(S_{u_{*}^{i}}),
\]

as well as

\[
\left\| (a_{*}^{i})^{i} G \right\| L^{i}(S_{u_{*}^{i}}) = (a_{*}^{i})^{i} a^{- \frac{i+1}{2}} |u|^{i+3} \left\| G \right\| L^{2}(S_{u_{*}^{i}}),
\]

whence we get

\[
|u|^{i+1} \left\| (a_{*}^{i})^{i} G \right\| L^{i}(S_{u_{*}^{i}}) = \frac{a_{*}^{i}}{|u|^{2}} \left\| (a_{*}^{i})^{i} G \right\| L^{i}(S_{u_{*}^{i}}).
\]
Passing therefore to scale-invariant norms we have

\[
\| (a \frac{2}{3} \nabla)^i (\mathcal{F}_F, \sigma_F) \|_{L^2(\Sigma_{\infty})}^2 \\
\lesssim \| (a \frac{2}{3} \nabla)^i (\mathcal{F}_F, \sigma_F) \|_{L^2(\Sigma_{\infty})}^2 + \int_{u_{\infty}}^u \frac{a}{|u'|^2} \| (a \frac{2}{3} \nabla)^i \mathcal{F}_F \|_{L^2(\Sigma_{\infty})}^2 \, du' \\
\lesssim \| (a \frac{2}{3} \nabla)^i (\mathcal{F}_F, \sigma_F) \|_{L^2(\Sigma_{\infty})}^2 + \int_{u_{\infty}}^u \frac{a}{|u'|^2} \| (a \frac{2}{3} \nabla)^{i+1} \mathcal{F}_F \|_{L^2(\Sigma_{\infty})}^2 \, du'
\]

\[
+ \int_{u_{\infty}}^u \frac{a}{|u'|^2} \left\| (a \frac{2}{3})^i \sum_{i_1 + i_2 + i_3 + 1 = i} \nabla^{i_1} \psi^{i_2 + 1 + i_3 + 1} \psi \nabla^{i_4} F \right\|_{L^2(\Sigma_{\infty})}^2 \, du'
\]

\[
+ \int_{u_{\infty}}^u \frac{a}{|u'|^2} \left\| (a \frac{2}{3})^i \sum_{i_1 + i_2 + i_3 + 1 = i} \nabla^{i_1} \psi^{i_2 + 1} \nabla^{i_3} \nabla^{i_4} F \right\|_{L^2(\Sigma_{\infty})}^2 \, du'
\]

\[
+ \int_{u_{\infty}}^u \frac{a}{|u'|^2} \left\| (a \frac{2}{3})^i \sum_{i_1 + i_2 + i_3 + 1 = i} \nabla^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \nabla^{i_4} (\mathcal{F}_F, \sigma_F) \right\|_{L^2(\Sigma_{\infty})}^2 \, du'
\]

\[
+ \int_{u_{\infty}}^u \frac{a}{|u'|^2} \left\| (a \frac{2}{3})^i \sum_{i_1 + i_2 + i_3 + 1 = i} \nabla^{i_1} \psi^{i_2 + 1} \nabla^{i_3} \nabla^{i_4} (\psi, \chi) \nabla^{i_4} (\mathcal{F}_F, \sigma_F) \right\|_{L^2(\Sigma_{\infty})}^2 \, du'
\]

\[= J_1 + \ldots + J_7.\]

We treat \( J_1, \ldots, J_7 \) on a term-by-term basis.

- The initial data term \( J_1 \) is bounded by \( \mathcal{F}^{(0)}(u) \lesssim 1. \)

- We have

\[
J_2 \lesssim \frac{1}{|u|} \| (a \frac{2}{3})^i \nabla^{i+1} \mathcal{F}_F \|_{L^2(\Sigma_{\infty})} \lesssim \frac{1}{|u|} \mathcal{F}(\mathcal{F}_F) \lesssim \mathcal{F}(\mathcal{F}_F).
\]

- For \( J_3 \) we have

\[
J_3 \lesssim \frac{a}{|u|} \cdot O^2 \lesssim 1.
\]

- For \( J_4 \) we can similarly bound the term by 1.

- For \( J_5 \), there holds

\[
J_5 = \int_{u_{\infty}}^u \frac{1}{|u'|^2} \left\| (a \frac{2}{3})^i \sum_{i_1 + i_2 + 1 = i} \nabla^{i_1 + 1} \left( \frac{a}{|u'|} \nabla^{i_2} \nabla^{i_3} \nabla^{i_4} (\mathcal{F}_F, \sigma_F) \right) \right\|_{L^2(\Sigma_{\infty})}^2 \, du' \lesssim \frac{O^2}{|u|} \lesssim 1.
\]

Note that we have crucially used the fact that in \( J_5 \) there exists at least one derivative on \( \nabla \chi \), allowing us to rewrite the expression in terms of \( \nabla \chi \) since \( \nabla \nabla \chi = \nabla \chi \).

- There holds

\[
J_6 \lesssim \frac{O^3}{a} \lesssim 1.
\]

- There holds

\[
J_7 \lesssim \frac{a^2}{|u|} \cdot O^2 \lesssim 1.
\]

Combining these estimates together, we arrive at the desired result.
Proposition 4.12 Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have

\[ \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla^i) F \|_{L^2(x)(S_{u,w})} \lesssim \mathcal{F}[\rho_F, \sigma_F] + \mathcal{F}[\rho_F, \sigma_F] + 1. \]

Proof. For \( \Psi \in \{ \rho_F, \sigma_F, \alpha_F \} \) we have the following schematic equation

\[ \nabla \Psi = \nabla (\rho_F, \sigma_F) + (\hat{\Psi}, \hat{\psi}) \cdot (\Psi, \alpha_F). \]

Commuting this with \( i \) angular derivatives, we get

\[ \nabla_{i} \nabla (\Psi) = \nabla (\rho_F, \sigma_F) \]

By multiplying with \( (a^{\frac{1}{2}})^i \) on both sides and passing to scale-invariant norms, we have

\[ \| (a^{\frac{1}{2}} \nabla^i F) \|_{L^2(x)(S_{u,w})} \]

In the last inequality we have used the refined estimates on \( \hat{\Psi} \) and \( \alpha_F \) from Propositions 4.3 and 4.10 respectively.

\[ \square \]

5 \( L^2(S_{u,w}) \)-estimates for curvature

Proposition 5.1 Under the assumptions of Theorem 1.2 and the bootstrap assumption (3.3), we have

\[ \sum_{i \leq 9} \| (a^{\frac{1}{2}} \nabla^i \alpha) \|_{L^2(x)(S_{u,w})} \lesssim 1. \]
Proof. Reading off equation (2.60), we have

$$\nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha = \nabla \beta + \alpha F \cdot \nabla \Upsilon + \Upsilon \cdot (\nabla \alpha F, \nabla \Upsilon) + (\psi, \chi) \cdot \Psi + \psi \cdot \alpha + (\tilde{\psi}, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \cdot (\alpha F, \Upsilon) \cdot (\alpha F, \Upsilon).$$

Commuting with $i$ angular derivatives, we obtain

$$\nabla_3 \nabla^i \alpha + \frac{i + 1}{2} \text{tr} \chi \nabla^i \alpha = \nabla^{i+1} \beta + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \beta + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \alpha F \nabla^{i_4+1} \Upsilon$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} (\alpha F, \Upsilon) + \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} \Upsilon$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} \alpha F + \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4+1} (\alpha F, \Upsilon).$$

Denote the above as

$$\nabla_3 \nabla^i \alpha + \frac{i + 1}{2} \text{tr} \chi \nabla^i \alpha = G.$$

Using the definition of the $L^2_{(x)}(u, u)$-norms we have

$$\| \nabla^i \alpha \|_{L^2_{(x)}(S_{\omega, \omega})} = a^{-\frac{i}{2}} |u|^i \| \nabla^i \alpha \|_{L^2(S_{\omega, \omega})}, \quad \| G \|_{L^2_{(x)}(S_{\omega, \omega})} = a^{-\frac{i+2}{2}} |u|^2 \| G \|_{L^2(S_{\omega, \omega})},$$

which translates to

$$a^{-\frac{i}{2}} |u|^i \| \nabla^i \alpha \|_{L^2(S_{\omega, \omega})} = a^{-\frac{i}{2}} \| (a^{\frac{1}{2}} \nabla)^i \alpha \|_{L^2_{(x)}(S_{\omega, \omega})}, \quad a^{-\frac{i}{2}} |u|^i \| G \|_{L^2(S_{\omega, \omega})} = a^{-\frac{i}{2}} |u|^2 \| (a^{\frac{1}{2}} G) \|_{L^2_{(x)}(S_{\omega, \omega})}.$$

Hence we have

$$a^{-\frac{i}{2}} \| (a^{\frac{1}{2}} \nabla)^i \alpha \|_{L^2_{(x)}(S_{\omega, \omega})} \leq a^{-\frac{i}{2}} \| (a^{\frac{1}{2}} \nabla)^i \alpha \|_{L^2_{(x)}(S_{\omega, \omega})} + \int_{u_0}^u \frac{1}{|u'|^2} \| (a^{\frac{1}{2}} \nabla)^{i+1} \beta \|_{L^2_{(x)}(S_{\omega, \omega})} du'$$

$$+ \int_{u_0}^u \frac{a^{\frac{i}{2}}}{|u'|^2} \| \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} (a^{\frac{1}{2}}) \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4+1} \Psi \|_{L^2_{(x)}(S_{\omega, \omega})} du'$$

$$+ \int_{u_0}^u \frac{a^{\frac{i}{2}}}{|u'|^2} \| \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4+1} (\alpha F, \Upsilon) \|_{L^2_{(x)}(S_{\omega, \omega})} du'$$

$$+ \int_{u_0}^u \frac{a^{\frac{i}{2}}}{|u'|^2} \| \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4+1} (\alpha F, \Upsilon) \|_{L^2_{(x)}(S_{\omega, \omega})} du'$$

$$+ \int_{u_0}^u \frac{a^{\frac{i}{2}}}{|u'|^2} \| \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4+1} \Psi \|_{L^2_{(x)}(S_{\omega, \omega})} du'$$

$$+ \int_{u_0}^u \frac{a^{\frac{i}{2}}}{|u'|^2} \| \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4+1} (\alpha F, \Upsilon) \|_{L^2_{(x)}(S_{\omega, \omega})} du'.$$
\[ + \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \|(a^2)^i \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} \alpha \nabla^{i_4} \alpha \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ := T_1 + T_2 + \cdots + T_9. \]

The first term can be bounded by the initial data, since

\[ T_1 = a^{-\frac{1}{2}} \|(a^2)^i \alpha \|_2^2(S_{u, \infty}) \leq T(0)(u) \lesssim 1. \]

For the terms involving \( \beta \), we have

\[ T_2 + T_3 = \int_{u \to \infty}^u \frac{1}{|u'|^2} \|(a^2)^i \beta \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ + \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=1} (a^2)^i \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} \beta \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ \leq \left( \int_{u \to \infty}^u \frac{1}{|u'|^2} \|(a^2)^i \beta \|_2^2(S_{u', u}) \mathrm{d}u' \right)^{\frac{1}{2}} \left( \int_{u \to \infty}^u \frac{1}{|u'|^2} \mathrm{d}u' \right)^{\frac{1}{2}} + \int_{u \to \infty}^u \frac{a^2}{|u'|^2} O^2 \mathrm{d}u' \]

\[ \leq a^{-\frac{1}{2}} \|(a^2)^i \beta \|_2^2(S_{u, \infty}(u_{\to \infty})) \cdot 1 \|u|^{-\frac{1}{2}} + \frac{a^2}{|u|^2} O^2 \lesssim R(\beta) \frac{1}{|u|^2} + \frac{a^2}{|u|^2} O^2 \lesssim 1. \]

Notice that the curvature term actually vanishes. For the next two terms, we need to treat the cases where all the weight falls on \( i_4 \) separately. Look, first, at

\[ T_4 = \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \|(a^2)^i \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} (\alpha_F, \chi) \nabla^{i_4} + \chi \|_2^2(S_{u', u}) \mathrm{d}u'. \]

If \( i = 1 \), then we can bound the term \( (\alpha_F, \chi) \) below in \( L^\infty(S_{x, \infty}) \) to get

\[ T_4 = \int_{u \to \infty}^u \frac{1}{|u'|} \|(a^2)^i \chi \cdot \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} (\alpha_F, \chi) \nabla^{i_4} + \chi \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ \leq \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \|O \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ \leq \int_{u \to \infty}^u \frac{a^2}{|u'|^2} O^2 \mathrm{d}u' \lesssim a^2 \cdot O^2 \lesssim 1. \]

If \( i < 1 \leq 9 \) we distinguish two cases:

- There holds \( i + 1 \leq 6 \). We then write

\[ (a^2)^i \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} (\alpha_F, \chi) \nabla^{i_4} \chi = (a^2)^i \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} (\alpha_F, \chi) \nabla^{i_4} \chi \]

and bound \( (a^2)^i \chi \nabla^{i_4} \chi \) in \( L^\infty(S_{x, \infty}) \). We have

\[ \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \|(a^2)^i \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} (\alpha_F, \chi) \nabla^{i_4} \chi \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ \leq \int_{u \to \infty}^u \frac{a^2}{|u'|^2} \|O \|_2^2(S_{u', u}) \| \sum_{i_1+i_2+i_3+i_4=1} (a^2)^i i_1 + i_2 + i_3 + i_4 \nabla^{i_1} \psi i^{i_2} \nabla^{i_3} \chi \|_2^2(S_{u', u}) \mathrm{d}u' \]

\[ \leq a^2 \cdot O^2 \leq \frac{O^2}{a^2} \leq 1. \]
There holds $7 \leq i_4 + 1 \leq 9$. We then bound $(a\hat{\tau} \nabla)^{i_4 + 1} \Psi$ in $L^2_{(sc)}$ and the rest of the terms in $L^\infty_{(sc)}$. This gives

$$
\int_{u_0}^u \frac{a\hat{\tau}}{|u'|} \| (a\hat{\tau})^{i_4} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \Psi) \nabla^{i_4} \Psi \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau}}{|u'|} \cdot \frac{1}{|u'|} \| (a\hat{\tau} \nabla)^{i_4 + 1} \Psi \|_{L^2_{(sc)}(S_u, \omega)} \| \sum_{i_1+i_2+i_3 \leq 9} (a\hat{\tau})^{i_4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \|_{L^\infty_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \cdot \frac{1}{|u'|} \cdot O \cdot \frac{1}{a\hat{\tau}} \| (a\hat{\tau} \nabla)^{i_4 + 1} \alpha F \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \frac{a\hat{\tau} O^2}{|u|^2} \leq 1.
$$

We move on to $T_5$. If $i_4 = i$, we have

$$
\int_{u_0}^u \frac{a\hat{\tau}}{|u'|} \| (a\hat{\tau})^{i_4} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \cdot \frac{1}{|u'|} \| (a\hat{\tau} \nabla)^{i_4} \alpha F \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \frac{a\hat{\tau} O^2}{|u|^2} \leq 1.
$$

If $i_4 < i \leq 9$ we again distinguish two cases:

- There holds $i_4 + 1 \leq 6$. We then write
  
  $$(a\hat{\tau})^{i_4} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \alpha F = (a\hat{\tau})^{i_4 + 1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \left( \frac{\alpha F}{a\hat{\tau}} \right)
  $$

  and bound $\nabla^{i_4 + 1} \left( \frac{\alpha F}{a\hat{\tau}} \right)$ in $L^\infty_{(sc)}$. We have

$$
\int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \| (a\hat{\tau})^{i_4} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \alpha F \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \cdot \frac{1}{|u'|} \| (a\hat{\tau} \nabla)^{i_4 + 1} \left( \frac{\alpha F}{a\hat{\tau}} \right) \|_{L^\infty_{(sc)}(S_u, \omega)} \| \sum_{i_1+i_2+i_3 \leq 9} (a\hat{\tau})^{i_4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \|_{L^\infty_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau} O^2}{|u|^2} \leq 1.
$$

- There holds $7 \leq i_4 + 1 \leq 9$. We then bound $\nabla^{i_4 + 1} \left( \frac{\alpha F}{a\hat{\tau}} \right)$ in $L^2_{(sc)}$ and the rest of the terms in $L^\infty_{(sc)}$. We then have

$$
\int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \| (a\hat{\tau})^{i_4 + 1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \left( \frac{\alpha F}{a\hat{\tau}} \right) \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \cdot \frac{1}{|u'|} \| (a\hat{\tau} \nabla)^{i_4 + 1} \left( \frac{\alpha F}{a\hat{\tau}} \right) \|_{L^2_{(sc)}(S_u, \omega)} \| \sum_{i_1+i_2+i_3 \leq 2} (a\hat{\tau})^{i_4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Psi \nabla^{i_4} \Psi \|_{L^\infty_{(sc)}(S_u, \omega)} \, du' 
\leq \int_{u_0}^u \frac{a\hat{\tau} O^2}{|u|^2} \leq 1.
$$

For $T_6$, we have

$$
\int_{u_0}^u \frac{a\hat{\tau}}{|u'|^2} \| (a\hat{\tau})^{i_4} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\hat{\chi}) \nabla^{i_4} \Psi \|_{L^2_{(sc)}(S_u, \omega)} \, du' 
$$

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\[
= \int_{u_0}^{u} \frac{a}{|u'|^2} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^2} \frac{\tilde{\chi}}{a^2}\right) \nabla^{i_4} \Psi \|C_{\text{loc}}^2(S_{u',\omega}) \, du'
\]
\[
\leq \int_{u_0}^{u} \frac{a}{|u'|^2} \frac{a}{O^2} \, du' = \frac{a}{|u|} \leq 1.
\]

For the term $T_7$, which contains a triple anomaly, we have
\[
\int_{u_0}^{u} \frac{a}{|u'|^2} \frac{a}{|u|^2} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^2} \frac{\tilde{\chi}}{a^2}\right) \nabla^{i_4} \nabla^{i_5} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
= \int_{u_0}^{u} \frac{\frac{O^3}{|u'|}}{a \frac{O^2}{|u|^2}} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
\leq \int_{u_0}^{u} \frac{O^3}{|u'|} \frac{a}{|u|} \frac{O^2}{|u|^2} \, du' = \frac{a}{|u|} \leq 1.
\]

For $T_8$, there holds
\[
\int_{u_0}^{u} \frac{a}{|u'|^2} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\psi}{a^2} \frac{\tilde{\chi}}{a^2}\right) \nabla^{i_4} \nabla^{i_5} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
= \int_{u_0}^{u} \frac{1}{|u'|} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
\leq \int_{u_0}^{u} \frac{1}{|u'|} \frac{a}{|u|} \frac{O^2}{|u|^2} \, du' = \frac{a}{|u|} \leq 1.
\]

Finally, we can bound $T_9$ as follows:
\[
\int_{u_0}^{u} \frac{a}{|u'|^2} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \nabla^{i_5} \left(\frac{\psi}{a^2} \frac{\tilde{\chi}}{a^2}\right) \nabla^{i_6} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
= \int_{u_0}^{u} \frac{1}{|u'|} \left(\frac{a^2}{u'}\right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 + i_6 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \nabla^{i_5} \left(\frac{\psi}{a^2} \frac{\tilde{\chi}}{a^2}\right) \nabla^{i_6} \left(\frac{\alpha F, Y}{a^2}ight) \|C_{\text{loc}}^1(S_{u',\omega}) \, du'
\]
\[
\leq \int_{u_0}^{u} \frac{1}{|u'|} \frac{a}{|u|} \frac{O^3}{|u|^2} \, du' \leq 1.
\]

In the last inequality above we have used Proposition 4.1. Putting all the estimates together, the result follows.

We move on to estimates for the curvature components $\beta, \rho, \sigma, \tilde{\beta}, \tilde{\alpha}$.

**Proposition 5.2** Let $\Psi \in \{\beta, \rho, \sigma, \tilde{\beta}, \tilde{\alpha}\}$. Under the assumptions of Theorem 1.2 and (3.3), we have
\[
\sum_{i \leq 9} \|(a^2 \nabla) i \Psi\|C_{\text{loc}}^2(S_{u,\omega}) \leq R[\alpha] + E[\rho F, \sigma F] + 1.
\]

**Proof.** The terms $\Psi$ satisfy the following schematic equations:
\[ \nabla_4 \Psi = \nabla (\Psi, \alpha) + (\psi, \tilde{\chi}) \cdot (\Psi, \alpha) + (\alpha F, Y) \cdot \nabla (\alpha F, Y) + (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \cdot (\alpha F, Y) \cdot (\alpha F, Y). \] (5.1)

Commuting (5.1) with \( i \) angular derivatives and using Proposition 3.16, we have

\[
\nabla_4 \nabla^i \Psi = \nabla^{i+1} (\Psi, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3+1} (\Psi, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} (\Psi, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} (\alpha F, Y) \nabla^{i_5} (\alpha F, Y) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} (\Psi, \alpha).
\]

Applying Proposition 3.6 and multiplying both sides by \((a^{i_2})^j\) we get

\[
\begin{aligned}
&\| (a^{i_2})^j \nabla \Psi \|_{L^2_{(\alpha)}(S_u \omega)}^2 \\
\leq &\int_0^\infty \left\| (a^{i_2})^j \nabla^{i+1} (\Psi, \alpha) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' + \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^j \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3+1} (\Psi, \alpha) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^j \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} (\Psi, \alpha) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^j \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} (\alpha F, Y) \nabla^{i_5} (\alpha F, Y) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^j \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} (\Psi, \alpha) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
\leq & \mathcal{R} [\alpha] + \frac{1}{a^{i_2}} \mathcal{R} [\Psi] + \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^{i+1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3+1} \left( \frac{\Psi}{a^{i_2}} - \frac{\alpha}{a^{i_2}} \right) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^{i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left( \frac{\psi}{a^{i_2}} \psi_i \frac{\alpha}{|u|^2 \alpha} \right) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^{i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left( \frac{\alpha F}{a^{i_2}} \frac{\alpha}{a^{i_2}} \frac{\alpha}{a^{i_2}} \right) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
&+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^\infty \left\| (a^{i_2})^{i+1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left( \frac{\alpha F}{a^{i_2}} \frac{\alpha}{a^{i_2}} \frac{\alpha}{a^{i_2}} \right) \right\|_{L^2_{(\alpha)}(S_u \omega)}^2 \, du' \\
:= & \sum_{k=1}^7 J_k.
\end{aligned}
\]

We focus on each \( J_k \)-term separately.

- We have \( J_1 + J_2 \leq \mathcal{R} [\alpha] + 1. \)
We have
\[ J_3 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \int_0^\infty \left\| (a_{\frac{2}{a}})^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \left( a_{\frac{2}{a}} \frac{\psi}{a} \frac{\alpha}{a^2} \right) \right\|_{L^p(S_u \omega')} \, du' \leq \frac{a_{\frac{2}{a}} \cdot O^2}{|u|}. \quad (5.2) \]

We have
\[ J_4 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \int_0^\infty \left\| (a_{\frac{2}{a}})^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \left( a_{\frac{2}{a}} \frac{\psi}{a} \frac{\alpha}{a^2} \right) \right\|_{L^p(S_u \omega')} \, du' \]
\[ \leq \frac{|u|}{a_{\frac{2}{a}}} \left( O \left[ O \left[ \frac{1}{|u|} \cdot O[\alpha] + 1 \right] \right] \right) \leq O[\alpha] + 1 \leq 1. \]

Note that here we have made use of Proposition 4.3 and Proposition 5.1 and used the improved (compared to the bootstrap assumptions) bounds on \( \hat{\chi} \) and \( \alpha \).

We have
\[ J_5 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \int_0^\infty \left\| (a_{\frac{2}{a}})^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \left( a_{\frac{2}{a}} \frac{\psi}{a} \frac{\alpha}{a^2} \right) \right\|_{L^p(S_u \omega')} \, du' \]
\[ \leq \int_0^\infty \left\| \left( a_{\frac{2}{a}} \right)^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \left( a_{\frac{2}{a}} \frac{\psi}{a} \frac{\alpha}{a^2} \right) \right\|_{L^p(S_u \omega')} \, du' \]
\[ \leq \frac{a_{\frac{2}{a}} \cdot O}{|u|} \cdot \left( F[\alpha F], \frac{F[\nabla^2 \chi]}{a_{\frac{2}{a}}} \right) \]
\[ \leq \frac{a_{\frac{2}{a}} \cdot O}{|u|} \cdot \left( \frac{F[\alpha F]}{a_{\frac{2}{a}}} \right) + \frac{a_{\frac{2}{a}} \cdot O}{|u|} + \frac{a_{\frac{2}{a}}}{|u|} \cdot O \cdot \left( O + \sum_{1 \leq i_2 \leq 8} \frac{a_{\frac{2}{a}}}{|u|^2} \cdot O[i_2 + 1] \right) \leq 1. \]

We have
\[ J_6 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \frac{|u|^2}{a} \]
\[ \times \int_0^\infty \left\| (a_{\frac{2}{a}})^{i_1 + 1} \nabla^{i_2} \nabla^{i_3} \left( a_{\frac{2}{a}} \frac{\psi}{a} \frac{\alpha}{a^2} \right) \right\|_{L^p(S_u \omega')} \, du' \]
\[ \leq \frac{a_{\frac{2}{a}} \cdot O^2}{|u|^2}. \]
We recall here the definition of divergence and curl for a symmetric, covariant tensor of arbitrary rank:

\[ \nabla \cdot \phi = \nabla^B \phi_{BA_1...A_r}, \]

\[ \nabla \times \phi = f^{BC} \nabla_B \phi_{CA_1...A_r}. \]

The trace of such a tensor is defined by

\[ (\text{tr} \phi)_{A_1...A_{r-1}} = (\gamma^{-1})^{BC} \phi_{BCA_1...A_{r-1}}. \]

The main elliptic estimate that will be used here is the following:

**Proposition 6.1** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), let \( \phi \) be a totally symmetric \((r + 1)-\)covariant tensorfield on a metric 2–sphere \((S^2, \gamma)\), satisfying

\[ \text{div} \phi = f, \quad \text{curl} \phi = g, \quad \text{tr} \phi = h. \]

\[ \leq \frac{|u|^2}{a} \cdot a \cdot \left( \frac{O[\text{tr} \phi] \cdot O_{i_4} \left[ \frac{\alpha F}{a^2} \right] \cdot O_{i_5} \left[ \frac{\alpha F}{a^2} \right]}{|u|^2} \right) + 1. \]

The logic behind the bound above is as follows. If the term we wish to bound, schematically, is not in the form of a triple anomaly, then the estimates are not borderline and the term is bounded above by 1. The worst term is when we wish to bound

\[ \sum_{i_1+i_2+i_3+i_4+i_5=1} \left| \frac{|u|^2}{a} \int_0^2 \left( \frac{a^2}{a^2} \right)^{i+1} \nabla^i \psi \nabla^i \nabla^i \left( \frac{\psi}{a^2} \right) \nabla^i \psi \right|_{L^2_{(i)}(S_{u, u'})} \, du'. \]

This term can only be bounded by \( O[\text{tr} \phi] \cdot O_{i_4} \left[ \frac{\alpha F}{a^2} \right] \cdot O_{i_5} \left[ \frac{\alpha F}{a^2} \right] \). We now use the improved bounds from Propositions 4.8 and 4.10 to bound \( O[\text{tr} \phi] \leq 1 \) and \( O_{i_4} \left[ \frac{\alpha F}{a^2} \right] \cdot O_{i_5} \left[ \frac{\alpha F}{a^2} \right] \leq (F[\rho F, \sigma F] + 1) \cdot 1 \). This is because at least one of the indices \( i_4, i_5 \) will not be of top order, hence the estimate from Proposition 4.10 for that term will be better. Combining these estimates, we arrive at

\[ J_6 \leq F[\rho F, \sigma F] + 1. \]

- The final term \( J_7 \) is handled as follows:

\[ \sum_{i_1+i_2+i_3+i_4=1} \int_0^2 \left| \left( \frac{a^2}{a^2} \right)^{i+1} \nabla^i \psi \nabla^i \left( \frac{\psi}{a^2} \right) \nabla^i \psi \right|_{L^2_{(i)}(S_{u, u'})} \, du' \]

\[ \leq \int_0^2 a^2 \cdot O^2 \, du' \leq \frac{a^2}{|u|} \leq 1. \]

Combining all the estimates above, we arrive at the desired conclusion.

\[ \square \]

### 6 Elliptic estimates for top-order derivatives of Ricci coefficients

#### 6.1 General elliptic estimates for Hodge systems

We recall here the definition of divergence and curl for a symmetric, covariant tensor of arbitrary rank:

\[ (\text{div} \phi)_{A_1...A_r} = \nabla^B \phi_{BA_1...A_r}, \]

\[ (\text{curl} \phi)_{A_1...A_r} = f^{BC} \nabla_B \phi_{CA_1...A_r}. \]

The trace of such a tensor is defined by

\[ (\text{tr} \phi)_{A_1...A_{r-1}} = (\gamma^{-1})^{BC} \phi_{BCA_1...A_{r-1}}. \]

We need to improve Proposition 4.10 like Proposition 5.1.
Then, for $1 \leq i \leq 11$, we have
\[
\| (a^\frac{i}{i})^i \phi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} \lesssim a^\frac{i}{i} \sum_{j=0}^{i-1} \| (a^\frac{i}{i})^j (f, g) \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} + \sum_{j=0}^{i-1} \| (a^\frac{i}{i})^j (\phi, h) \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})}.
\]

Proof. Recall the following identity from Chapter 7 in [Christodoulou] that, for $\phi, f, g$ and $h$ as above, there holds
\[
\int_{s_{n,\omega}} \left( |\nabla \phi|^2 + (r + 1)K |\phi|^2 \right) d\mu_r = \int_{s_{n,\omega}} \left( |f|^2 + |g|^2 + rK |h|^2 \right) d\mu_r. \quad (6.1)
\]
Here $K$ denotes the Gauss curvature of the sphere. To prove the lemma for the case $i = 1$ first, we need to control $K$ in $L^\infty$. To that end, we will prove the following stronger lemma:

Lemma 6.1 For $0 \leq k \leq 7$, there holds $\| (a^\frac{i}{i})^k K \|_{\mathcal{L}_{(\kappa)}^\infty(s_{n,\omega})} \lesssim 1$.

Proof. We begin by recalling that
\[
K = -\rho_R - \frac{1}{4} \tr \chi \tr \chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} = -\rho - \frac{1}{4} \tr \chi \tr \chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} + \left( |\rho|^2 + |\sigma|^2 \right)
\]
and $s_2(K) = 1$. By virtue of the scale-invariant version of the $L^2 - L^\infty$ Sobolev embedding inequality from Proposition 3.11, there holds
\[
\sum_{0 \leq k \leq 7} \| (a^\frac{i}{i})^k K \|_{\mathcal{L}_{(\kappa)}^\infty(s_{n,\omega})} \lesssim \sum_{0 \leq j \leq 9} \| (a^\frac{i}{i})^j \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})}. \quad (6.2)
\]
We proceed to estimate, for a fixed $0 \leq i \leq 9$, the term $\| (a^\frac{i}{i})^k K \|_{\mathcal{L}_{(\kappa)}^\infty(s_{n,\omega})}$. We have
\[
\| (a^\frac{i}{i})^k K \|_{\mathcal{L}_{(\kappa)}^\infty(s_{n,\omega})} \lesssim \sum_{0 \leq j \leq 9} \| (a^\frac{i}{i})^j \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} + \sum_{i_1 + i_2 = i} \| (a^\frac{i}{i})^i \tr \chi \tr \chi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} + \sum_{i_1 + i_2 = i} \| (a^\frac{i}{i})^i \tr \chi \tr \chi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})}.
\]
The first term above can be bounded by 1, by Proposition 5.2. For the second term, we have
\[
\sum_{i_1 + i_2 = i} \| (a^\frac{i}{i})^i \tr \chi \tr \chi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} \lesssim \frac{|a|^2}{|a|} \sum_{i_1 + i_2 = i} \| (a^\frac{i}{i})^i \tr \chi \tr \chi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} \lesssim \frac{|a|^2}{|a|} \cdot \left( O_{\infty} \tr \chi + O_2 \tr \chi + O_2 \tr \chi \right).
\]
In the above inequality we have conditioned on the number of derivatives that fall on $\tr \chi$ and those that fall on $\tr \chi$. Notice that, from Proposition 4.8, there holds $O_{\infty} \tr \chi + O_2 \tr \chi \lesssim 1$. For $O_2 \tr \chi$, from Proposition 4.7, we read off (4.5) that
\[
O_2 \tr \chi \leq \frac{a}{|a|} O[\hat{\chi}, \alpha F] \cdot O[\hat{\chi}, \alpha F] + \frac{a}{|a|^2} O^3 + \frac{a^2}{|a|} O^2.
\]
Plugging this inequality in (6.3) and using $O[\hat{\chi}, \alpha F] \lesssim 1$ from Propositions 4.4 and 4.10 (remember crucially that we work with up to 9 derivatives at most, so the top order terms $\mathcal{R}[a]$ and $\mathcal{E}[\rho, \sigma F]$ used to estimate $\hat{\chi}$ and $\alpha F$ are redundant on the right-hand side) we arrive at
\[
\sum_{i_1 + i_2 = i} \| (a^\frac{i}{i})^i \tr \chi \tr \chi \|_{\mathcal{L}_{(\kappa)}^2(s_{n,\omega})} \lesssim 1.
\]
For the third term, there holds
\[ \sum_{i_1 + i_2 = i} \left\| (a^i) \nabla^{i_1} \chi \nabla^{i_2} \hat{\chi} \right\|_{L^2(\mathcal{C})} = |u| \sum_{i_1 + i_2 = i} \left\| (a^i) \nabla^{i_1} \left( \frac{\hat{\chi}}{|u|^{\frac{3}{2}}} \right) \nabla^{i_2} \left( \frac{a^i}{|u|^{\frac{3}{2}}} \right) \right\|_{L^2(\mathcal{C})} \lesssim O_\infty[\hat{\chi}] \cdot O_2[\chi] + O_2[\hat{\chi}] \cdot O_\infty[\chi] \lesssim 1 \cdot 1 = 1. \]

Here we have used Proposition 4.3 and Proposition 4.4, where we achieve the better bound \( O_2[\hat{\chi}] \lesssim 1 \), given that we work up to 9 derivatives, not 10, which means that we can bound the curvature term in the \( L^2(\mathcal{C}) \) norm instead of the \( H^{\infty}(\mathcal{C}) \) norm. The bound on \( \alpha \) is then invoked by Proposition 5.1. Finally, the fourth term is bounded by \( O^2/|u| \lesssim 1 \), using proposition 4.1. This concludes the proof of the lemma.

By applying the scale-invariant version of Hölder's inequality for \( K|h|^2 \) and using Lemma 6.1, we obtain the result for \( i = 1 \). For \( i > 1 \), the symmetrized angular derivative of \( \phi \) defined by

\[ (\nabla \phi)^s_{BA_1...A_{r+1}} = \frac{1}{r+2} \left( \nabla_B \phi_{A_1...A_r} + \sum_{i=1}^{r+1} \nabla_{A_i} \phi_{A_1...A_r B...A_{r+1}} \right) \]

satisfies the div-curl system

\[
\begin{align*}
\text{div} (\nabla \phi)^s &= (\nabla f)^s - \frac{1}{r+2} (\text{curl } \nabla g)^s + (r+1) K \phi - \frac{2K}{r+1} (\gamma \otimes^s h), \\
\text{curl} (\nabla \phi)^s &= \frac{r+1}{r+2} (\nabla f)^s + (r+1) K (\gamma \phi)^s, \\
\text{tr}(\nabla \phi)^s &= \frac{r}{r+2} f + \frac{r}{r+2} (\nabla h)^s,
\end{align*}
\]  

(6.4)

where

\[ \gamma \otimes^s h := \gamma_{A_i A_j} \sum_{i \leq j = 1, \ldots, r+1} h_{A_1...A_i} \ldots A_j \ldots A_{r+1} \]

and

\[ (\gamma \phi)^s_{A_1...A_{r+1}} := \frac{1}{r+1} \left\{ \sum_{i=1}^{r+1} f_{A_i} \phi_{A_1...A_r B...A_{r+1}} \right\}. \]

Using (6.1) and iterating, we obtain that for \( i \leq 11 \) there holds

\[
\begin{align*}
\| \nabla^i \phi \|^2_{L^2(S_{u,\omega})} &\leq \| \nabla^{i-1} (f, g) \|^2_{L^2(S_{u,\omega})} + \| K[(\nabla^{i-2} (f, g))^2 + |\nabla^{i-1} (\phi, h)|^2 ] \|^2_{L^1(S_{u,\omega})} \\
&+ \left\| K \left( \sum_{i_1 + 2i_2 + i_3 = i-1} \nabla^{i_1} K^{i_2+1} \nabla^{i_3} (\phi, h) \right) \right\|^2_{L^1(S_{u,\omega})} \\
&+ \sum_{i_1 + 2i_2 + i_3 = i-2} \| \nabla^{i_1} K^{i_2+1} \nabla^{i_3} (\phi, h) \|^2_{L^2(S_{u,\omega})} + \sum_{i_1 + 2i_2 + i_3 = i-3} \| \nabla^{i_1} K \nabla^{i_2} (f, g) \|^2_{L^2(S_{u,\omega})},
\end{align*}
\]  

(6.5)

where we have adopted the convention that \( \sum_{i \leq -1} = 0 \). Whenever a \( K \)-term appears with at most 7 derivatives, we estimate it in \( L^\infty \) or equivalently in \( L^\infty(\mathcal{C}) \). Whenever a \( K \)-term contains between 8 and 9 derivatives we shall estimate it in \( L^2 \) and the rest of the terms in \( L^\infty \), noting that we can estimate terms of the form \( \| \nabla^i (f, g, \phi, h) \|_{L^\infty} \) with \( i \leq 7 \) by the corresponding norms in \( L^2 \) through the standard Sobolev embedding. By Lemma 6.1, after translating back to standard \( L^p \) norms, there holds

\[
\sum_{i \leq 7} \| |u|^i \nabla^i K \|_{L^\infty(S_{u,\omega})} + \sum_{j \leq 9} \| |u|^j \nabla^j K \|_{L^2(S_{u,\omega})} \lesssim 1.
\]
Therefore, for $i \leq 11$, we have

$$\|u|^{i} \nabla^{i} \phi\|_{L^{2}(S_{n, \omega})}^{2} \lesssim \sum_{j \leq i-1} \left(\|u|^{i+1} \nabla^{j} (f, g)\|_{L^{2}(S_{n, \omega})}^{2} + \|u|^{i} \nabla^{j} (\phi, h)\|_{L^{2}(S_{n, \omega})}^{2}\right).$$

Translating the above equation into scale-invariant norms and then multiplying it by $\|u|^{2(i+\varepsilon_{r})}$, we arrive at

$$\|(a^{\frac{i}{2}} \nabla)^{i} \phi\|_{L^{2}_{(r)}(S_{n, \omega})}^{2} \lesssim \sum_{j \leq i-1} \left(\|(a^{\frac{i}{2}})^{i+1} \nabla^{j} (f, g)\|_{L^{2}_{(r)}(S_{n, \omega})}^{2} + \|(a^{\frac{i}{2}} \nabla)^{j} (\phi, h)\|_{L^{2}_{(r)}(S_{n, \omega})}^{2}\right). \quad (6.6)$$

Taking square roots above yields Proposition 6.1.

Finally, for the special case where $\phi$ is a symmetric, traceless 2–tensor, we need only know its divergence:

**Proposition 6.2** Suppose $\phi$ is a symmetric, traceless 2–tensor satisfying

$$\text{div} \phi = f.$$

Then, under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), for $1 \leq i \leq 11$, there holds

$$\|(a^{\frac{i}{2}} \nabla)^{i} \phi\|_{L^{2}_{(r)}(S_{n, \omega})}^{2} \lesssim \sum_{j \leq i-1} \left(\|(a^{\frac{i}{2}})^{i+1} \nabla^{j} f\|_{L^{2}_{(r)}(S_{n, \omega})}^{2} + \|(a^{\frac{i}{2}} \nabla)^{j} \phi\|_{L^{2}_{(r)}(S_{n, \omega})}^{2}\right).$$

**Proof.** This is a direct application of Proposition 6.1, by noticing that

$$\text{curl} \phi = * f.$$

This is a straightforward calculation, using that the 2–tensor $\phi$ is symmetric and traceless. \hfill \Box

### 6.2 Elliptic estimates for 11 derivatives of Ricci coefficients

We start this section with the following auxiliary bootstrap assumption. Introduce the top-order quantity

$$O_{11, 2}(u, v) = \frac{1}{a^{2}} \|(a^{\frac{i}{2}})^{10} \nabla^{11} \chi\|_{L^{2}_{(r)}(H^{0, \omega})}^{2} + \|(a^{\frac{i}{2}})^{10} \nabla^{11} (\text{tr} \chi, \omega)\|_{L^{2}_{(r)}(H^{0, \omega})}^{2} + \|a^{\frac{i}{2}} \nabla^{11} \eta\|_{C^{0}_{(r)}(H^{0, \omega})}^{2} + \frac{a}{|u|} \|a^{\frac{i}{2}} \nabla^{11} (\eta, \eta)\|_{C^{0}_{(r)}(H^{0, \omega})}^{2} + \int_{u_{\infty}}^{u} a^{\frac{i}{2}} \|a^{\frac{i}{2}} \nabla^{11} \chi\|_{C^{0}_{(r)}(S_{n, \omega})}^{2} \, du + \|a^{\frac{i}{2}} \nabla^{11} \omega\|_{C^{0}_{(r)}(H^{0, \omega})}^{2} + \int_{u_{\infty}}^{u} a^{\frac{i}{2}} \|a^{\frac{i}{2}} \nabla^{11} \chi\|_{C^{0}_{(r)}(S_{n, \omega})}^{2} \, du.$$ \quad (6.7)

Throughout this section we assume

$$O_{11, 2} \leq O_{11} \lesssim a^{\frac{11}{38}}. \quad (6.8)$$

We begin with estimates for $\text{tr} \chi$ and $\chi$.

**Proposition 6.3** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), there holds

$$\|(a^{\frac{i}{2}})^{10} \nabla^{11} \text{tr} \chi\|_{L^{2}_{(r)}(S_{n, \omega})} \lesssim 1 + \mathcal{R}[\bar{\beta}] + \mathcal{R} [\alpha] + \mathcal{F} [\alpha F],$$

$$\frac{1}{a^{\frac{i}{2}}} \|(a^{\frac{i}{2}})^{10} \nabla^{11} \chi\|_{L^{2}_{(r)}(S_{n, \omega})} \lesssim 1 + \mathcal{R}[\bar{\beta}] + \mathcal{R} [\alpha].$$
Proof. Consider the following equation:
\[
\nabla_4 \text{tr} \chi + \frac{1}{2}(\text{tr} \chi)^2 = -|\hat{\chi}|^2 - |\alpha_F|^2 - 2\omega \text{tr} \chi.
\]
(6.9)
Commuting with angular derivatives \(i\) times, we arrive at
\[
\nabla_4 \nabla_4^i \text{tr} \chi = \sum_{i_1+i_2+i_3+i_4=i} \nabla_1^{i_1} \psi^{i_2} \nabla_3^{i_3} \hat{\chi}^{i_4} \hat{\chi} + \sum_{i_1+i_2+i_3+i_4=i} \nabla_1^{i_1} \psi^{i_2} \nabla_3^{i_3} \alpha_F \nabla_4 \alpha_F
\]
\[
+ \sum_{i_1+i_2+i_3+i_4=i} \nabla_1^{i_1} \psi^{i_2} \nabla_3^{i_3} (\psi, \hat{\chi}) \nabla_4 \psi.
\]
Passing the above to scale-invariant norms and applying the triangle inequality, we have
\[
\|(a^{\frac{1}{2}})^{10} \nabla_1^{i_1} \text{tr} \chi\|_{L^2(S_{s,u})} \leq \int_0^{u} \|a^{\frac{1}{2}}\| \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}} \cdot \alpha_F + \frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla_1^{i_1} \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}} \cdot \alpha_F + \frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla_4 \alpha_F \, du' + \sum_{i_1+i_2+i_3+i_4=11} \int_0^{\frac{a}{u}} \|a^{\frac{1}{2}}\| \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}} \cdot \alpha_F + \frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla_1^{i_1} \nabla_3^{i_2} \nabla_4 \chi \, du' + \sum_{i_1+i_2+i_3+i_4=11} \int_0^{\frac{a}{u}} \|a^{\frac{1}{2}}\| \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}} \cdot \alpha_F + \frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla_1^{i_1} \nabla_3^{i_2} \nabla_4 \chi \, du' \]
(6.10)
\[
\leq \frac{a}{|u|} O[\chi] \cdot \int_0^{\frac{a}{u}} \|a^{\frac{1}{2}}\| \left(\frac{\hat{\chi}}{a^{\frac{1}{2}}} \cdot \alpha_F + \frac{\alpha_F}{a^{\frac{1}{2}}}\right) \nabla_4 \alpha_F \, du' + \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + \frac{a}{|u|} O[\chi, \alpha_F] \cdot O[\alpha_F] + \frac{a}{|u|} O[\alpha_F] \cdot O[\alpha_F] + \frac{a}{|u|} O \cdot O_1 + \frac{a}{|u|} O \cdot O_1
\]
\[
\leq \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + \frac{a}{|u|} O[\chi, \alpha_F] \cdot O[\alpha_F] + \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + 1.
\]
For \(\hat{\chi}\), we have
\[
\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} (\eta - \eta) \cdot (\hat{\chi} - \frac{1}{2} \text{tr} \chi \gamma) - \hat{\beta} + \frac{1}{2} R_{\alpha_4}.
\]
Schematically,
\[
\text{div} \hat{\chi} - \frac{1}{2} \nabla \text{tr} \chi + \hat{\beta} = \psi \cdot \psi + \alpha_F \cdot Y.
\]
Applying Proposition 6.2, we arrive at
\[
\begin{align*}
&\|(a^{\frac{1}{2}})^{10} \nabla^{11} \hat{\chi} \|_{L^2_{(\omega)}(S_{\omega})} \\
&\lesssim \sum_{i \leq 10} \frac{1}{a^{\frac{i}{2}}} \| (a^{\frac{1}{2}} \nabla)^i \text{tr} \chi \|_{L^2_{(\omega)}(S_{\omega})} + \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \hat{\beta} \|_{L^2_{(\omega)}(S_{\omega})} \\
&+ \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \psi \nabla^{i_1} \psi \|_{L^2_{(\omega)}(S_{\omega})} + \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \nabla^{i_1} \nabla^{i_2} \chi \|_{L^2_{(\omega)}(S_{\omega})} \\
&+ \frac{1}{a^{\frac{1}{2}}} \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \hat{\chi} \|_{L^2_{(\omega)}(S_{\omega})}. 
\end{align*}
\]
(6.11)

By using the estimate on \(\| (a^{\frac{1}{2}} \nabla)^i \text{tr} \chi \|_{L^2_{(\omega)}(S_{\omega})}\) from Proposition 4.7 and applying Grönwall’s inequality, we get
\[
\begin{align*}
&\begin{align*}
\| (a^{\frac{1}{2}})^{10} \nabla^{11} \hat{\chi} \|_{L^2_{(\omega)}(S_{\omega})} \\
&\lesssim \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + R[\alpha] + 1 + \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \hat{\beta} \|_{L^2_{(\omega)}(S_{\omega})} \\
&+ \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \psi \nabla^{i_1} \psi \|_{L^2_{(\omega)}(S_{\omega})} + \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \nabla^{i_1} \nabla^{i_2} \chi \|_{L^2_{(\omega)}(S_{\omega})} \\
&+ \frac{1}{a^{\frac{1}{2}}} \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \hat{\chi} \|_{L^2_{(\omega)}(S_{\omega})}. 
\end{align*}
\end{align*}
(6.12)

Raising (6.12) to the square and integrating along \(u\), we arrive at
\[
\begin{align*}
\int_0^u \| (a^{\frac{1}{2}})^{10} \nabla^{11} \hat{\chi} \|_{L^2_{(\omega)}(S_{\omega})}^2 \, du' \\
\lesssim \left( \frac{a}{|u|} O[\alpha_F] \cdot F[\alpha_F] + R[\alpha] + 1 \right)^2 + \int_0^u \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \hat{\beta} \|_{L^2_{(\omega)}(S_{\omega})}^2 \, du' \\
+ \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \psi \nabla^{i_1} \psi \|_{L^2_{(\omega)}(S_{\omega})}^2 \, du' \\
+ \sum_{i \leq 10} \sum_{i_1 + i_2 = i} \| (a^{\frac{1}{2}} \nabla)^i \nabla^{i_1} \nabla^{i_2} \chi \|_{L^2_{(\omega)}(S_{\omega})}^2 \, du' + \int_0^u \sum_{i \leq 10} \| (a^{\frac{1}{2}} \nabla)^i \left( \frac{\hat{\chi}}{a^{\frac{1}{2}}} \right) \|_{L^2_{(\omega)}(S_{\omega})}^2 \, du'.
\end{align*}
(6.13)

Taking the square roots of the above inequality, we arrive at
\[
\begin{align*}
\frac{1}{a^{\frac{1}{2}}} \| (a^{\frac{1}{2}} \nabla)^{11} \hat{\chi} \|_{L^2_{(\omega)}(H^6_{\omega})} \lesssim R[\hat{\beta}] + R[\alpha] + F[\alpha_F] + 1.
\end{align*}
(6.14)

By plugging this back to (6.10) and applying Hölder’s inequality, we get
\[
\| a^{\frac{1}{2}} \nabla^{11} \text{tr} \chi \|_{L^2_{(\omega)}(S_{\omega})} \lesssim \frac{a^{\frac{1}{2}}}{|u|} (R[\alpha] + R[\hat{\beta}] + 1) + \frac{a}{|u|} O[\alpha_F] F[\alpha_F] + R[\alpha] + 1
\lesssim 1 + R[\hat{\beta}] + R[\alpha] + F[\alpha_F].
(6.15)

Squaring and taking \(L^2_{(\omega)}\) in the \(u\)-direction, we arrive at
\[
\| a^{\frac{1}{2}} \nabla^{11} \text{tr} \chi \|_{L^2_{(\omega)}(H^6_{\omega})} \lesssim R[\alpha] + F[\alpha_F] + 1.
(6.16)
\]
We now proceed with estimates for the highest number of derivatives in $\omega$. Define the following Hodge operators acting on the leaves $S_u$ of our double null foliation

- The operator $D_1$ maps a 1-form $F$ to the pair of functions $(\text{div} F, \text{curl} F)$,
- The operator $D_2$ maps an $S$-tangent, symmetric traceless tensor $F$ into the $S$-tangent one-form $\text{div} F$,
- The operator $*D_1$ maps a pair of scalar functions $(F_1, F_2)$ to the $S$-tangent 1-form $-\nabla F_1 + *\nabla F_2$,
- The operator $*D_2$ maps a 1-form $F$ to the 2-covariant, symmetric, traceless tensor $-\frac12 \mathcal{L}_F \gamma$, where $\mathcal{L}_F \gamma_{ab} = \nabla_a F_b + \nabla_b F_a - (\text{div} F) \gamma_{ab}$.

**Proposition 6.4** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3)–(6.8), there holds

$$\|a^5 \nabla^{11} \omega\|_{L^2_{(T)}}(H^{10}_u \omega) \lesssim \mathcal{R} \beta + 1.$$  

**Proof.** Introduce $\omega^\dagger$, defined as the solution to

$$\nabla_3 \omega^\dagger = \frac{1}{2} \sigma$$

with zero initial data on $H_{u,\omega}$. Introduce the pair of scalars $\langle \omega \rangle = (-\omega, \omega^\dagger)$ and define $\kappa$ by

$$\kappa := *D_1 \langle \omega \rangle - \frac12 \beta = \nabla \omega + *\nabla \omega^\dagger - \frac12 \beta.$$  

We need to derive a transport equation for $*D_1 \langle \omega \rangle$. To this end, recall the commutation formula

$$[\nabla_3, \nabla] f = -\frac12 \text{tr} \chi * \nabla f - \nabla f + \frac12 (\eta + \gamma) \nabla_3 f,$$

$$[\nabla_3, *\nabla] g = -\frac12 \text{tr} \chi * \nabla g + \nabla g + \frac12 (\eta + \gamma) \nabla_3 g.$$  

Therefore,

$$[\nabla_3, *D_1](f, g) = -\frac12 \text{tr} \chi *D_1(f, g) + \nabla f + *\nabla g - \frac12 (\eta + \gamma) \nabla_3 f + \frac12 (\eta + \gamma) \nabla_3 g.$$  

Now recall that

$$\nabla_3 \omega = \frac{1}{2} \rho + \psi \psi + \Upsilon \Upsilon := \frac{1}{2} \rho + \mathcal{E}.$$  

This means that

$$\nabla_3 *D_1 \langle \omega \rangle = *D_1 \left( -\frac12 \rho - \mathcal{E} + \frac12 \sigma \right) + [\nabla_3, *D_1] \langle \omega \rangle$$

$$= \frac{1}{2} \nabla \rho + \frac{1}{2} *\nabla \sigma + \nabla \mathcal{E} - \frac{1}{2} \text{tr} \chi *D_1 \langle \omega \rangle + \nabla (-\nabla \omega + *\nabla \omega^\dagger)$$

$$- \frac{1}{2} (\eta + \gamma) \left( \frac{1}{2} \rho + \mathcal{E} \right) + \frac{1}{4} (\eta + \gamma) \sigma.$$  

Schematically, we reduce this to the equation

$$\nabla_3 *D_1 \langle \omega \rangle + \frac{1}{2} \text{tr} \chi *D_1 \langle \omega \rangle - \frac{1}{2} (\nabla \rho + *\nabla \sigma)$$

$$= \psi \nabla \langle \omega, \omega^\dagger, \eta, \eta \rangle + \nabla \langle \omega, \omega^\dagger \rangle + (\rho_F, \sigma_F) \nabla (\rho_F, \sigma_F) + \psi \cdot \Psi + \psi \cdot \psi \cdot \psi + \psi \cdot \Upsilon \cdot \Upsilon. \quad (6.17)$$  

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Recall also the $\nabla_3$-direction schematic equation for $\tilde{\beta}$:

$$\nabla_3 \tilde{\beta} + \text{tr} \chi \tilde{\beta} - \nabla \rho - \text{tr} \nabla \sigma = (\psi, \chi) \Psi + \alpha_F \nabla \alpha_F + \alpha_F \nabla \alpha_F + \left( \rho_F, \sigma_F \right) \nabla \left( \rho_F, \sigma_F \right) + (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \cdot (\alpha_F, \Upsilon) \cdot \Upsilon.$$  (6.18)

From (6.17) and (6.18) we see that $\kappa$ obeys the following schematic equation:

$$\nabla_3 \kappa + \frac{1}{2} \text{tr} \nabla \kappa = (\psi, \tilde{\chi}) \Psi + (\psi, \tilde{\chi}) \nabla \psi + \text{tr} \nabla \left( \Upsilon, \alpha_F \right) + (\alpha_F, \Upsilon) \nabla \left( \psi, \psi, \psi \right).$$  (6.19)

By commuting (6.19) with $i \leq 10$ angular derivatives, we arrive at

$$\nabla_3 \nabla^i \kappa + \frac{i + 1}{2} \text{tr} \nabla \nabla^i \kappa = \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4} \Psi + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}) \nabla^{i_4+1} \psi + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \text{tr} \nabla (\Upsilon, \alpha_F) + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \Upsilon) \nabla^{i_4+1} \Upsilon + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_5} \Upsilon + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} \psi \nabla^{i_5} \psi + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \text{tr} \chi, \tilde{\chi}) \nabla^{i_4} \kappa + \sum_{i_1 + i_2 + i_3 + i_4 = i \leq 10} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \text{tr} \chi \nabla^{i_4} \kappa := G.$$  (6.20)

Applying Proposition 3.7 with $\lambda_0 = \frac{i + 1}{2}$, we get

$$|u|^i \| \nabla^i \kappa \|_{L^2(S_u, \omega)} \lesssim |u|_{\infty}^i \| \nabla^i \kappa \|_{L^2(S_u, \omega)} + \int_{u_{\infty}}^u |u|^i \| G \|_{L^2(S_{u'}, \omega)} \, du'.$$

Now by definition, we have $s_2(\kappa) = s_2(\nabla \omega, \tilde{\beta}) = 0.5$. So $s_2(\nabla^i \kappa) = \frac{i + 1}{2}$. This means that

$$\| (a^{\frac{1}{2}} \nabla)^i \kappa \|_{L^2(S_u, \omega)} = a^{\frac{i - 1}{2}} |u|^{i + 1} \| \nabla^i \kappa \|_{L^2(S_u, \omega)} = (a^{\frac{i - 1}{2}} |u|) \cdot |u|^i \| \nabla^i \kappa \|_{L^2(S_u, \omega)} \lesssim (a^{\frac{i - 1}{2}} |u|) \cdot \left( |u|_{\infty}^i \| \nabla^i \kappa \|_{L^2(S_{u'}), \omega} + \int_{u_{\infty}}^u |u'|^i \| G \|_{L^2(S_{u'}, \omega)} \, du' \right) \lesssim a^{\frac{i - 1}{2}} |u|_{\infty}^i + 1 \| \nabla^i \kappa \|_{L^2(S_{u'}), \omega} + \int_{u_{\infty}}^u a^{\frac{i - 1}{2}} |u'|^{i + 1} \| G \|_{L^2(S_{u'}, \omega)} \, du'.$$

In the last inequality we have used the facts that $|u| \leq |u|_{\infty}$, $|u| \leq |u'|$ for $|u'|$ in the range given above. From this we conclude that

$$\| (a^{\frac{1}{2}} \nabla)^i \kappa \|_{L^2(S_u, \omega)} \lesssim \| (a^{\frac{1}{2}} \nabla)^i \kappa \|_{L^2(S_{u'}), \omega} + \int_{u_{\infty}}^u \frac{\alpha}{|u|^2} \| (a^{\frac{1}{2}} G \|_{L^2(S_{u'}, \omega)} \, du' \right).$$
\[
\left\langle (a^2 \nabla)^j \kappa \right\rangle_{L^2_{(y)}(S_{\infty} \omega_1)} \lesssim \int_{u_{\infty}}^{u} \frac{a}{|u|^2} \sum_{i_1 + i_2 + i_3 + i_4 = 1} \left\| \left( a^2 \right)^j \nabla^{i_1} \psi \nabla^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4} \psi \right\|_{L^2_{(y)}(S_{u'} \omega_1)} du' + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \left\| \left( a^2 \right)^j \nabla^{i_1} \psi \nabla^{i_2} \psi \nabla^{i_3} (\psi, \chi, \nabla^{i_4} \psi) \right\|_{L^2_{(y)}(S_{u'} \omega_1)} du'
\]

By raising the above to the second power and integrating in \( u \) we get

\[
\int_{0}^{u} \left\langle (a^2 \nabla)^j \kappa \right\rangle_{L^2_{(y)}(S_{u'} \omega_1)}^2 du' \leq \left\langle (a^2 \nabla)^j \kappa \right\rangle_{L^2_{(y)}(H_{u_{\infty}})}^2 + \int_{0}^{u} \frac{a}{|u|^2} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left\| \left( a^2 \right)^j \nabla^2 \right\|_{L^2_{(y)}(S_{u'} \omega_1)}^2 du' du
\]

We bound each term separately.
There holds
\[ T_1 = \| (a^{\frac{1}{2}} \nabla)^i \|_{L^2_{(x_0)}(H_{\infty})}^2 \leq \| (a^{\frac{1}{2}} \nabla)^i \|_{L^2_{(x_0)}(H_{\infty})}^2 + \frac{1}{a} \sum_{i \leq 11} \| (a^{\frac{1}{2}} \nabla)^i \|_{L^2_{(x_0)}(H_{\infty})}^2 \leq R[\beta]^2 + \left( \mathcal{I}^{(0)} \right)^2 + 1 \leq R[\beta]^2 + 1. \]

There holds
\[ T_2 = \int_0^u \sum_{i_1+i_2+i_3+i_4=i} \| (a^{\frac{1}{2}})^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \chi) \nabla^{i_4} \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' \leq \frac{a \cdot O^4}{|u|^2} + \int_0^u \left( \frac{\psi}{a^2}, \frac{\chi}{a^2} \right) \cdot (a^{\frac{1}{2}} \nabla)^{10} \Psi \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' \leq \frac{a \cdot O^4}{|u|^2} + \frac{a \cdot O^2}{|u|^2} R[\Psi]^2. \]

Therefore
\[ T_2 \leq \frac{a}{|u|} \int_{u, \infty} a \| u' \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' \leq \frac{a^3 \cdot O^4}{|u|^4} + \frac{a^3 \cdot O^2 \cdot R^2}{|u|^4} \leq 1. \] 

For the third term
\[ \int_0^u \frac{a}{|u|} \int_{u, \infty} a \| u' \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' d u' \]
we bound separately the cases where \( i_4 = 10 \) and where not. In the former case, we need to distinguish the subcases where the \( \psi \)-term in \( \nabla^{i_4+1} \psi \) belongs to those components that are bounded in the \( \| \cdot \|_{L^2_{(x_0)}(H_0^{(0,0)})} \)-norm and those bounded in the \( \| \cdot \|_{L^2_{(x_0)}(H_0^{(0,0)})} \)-norm in the bootstrap assumption (6.8).

- When \( i_4 < 10 \) we can bound \( T_3 \) by 1, using Proposition 4.1.
- When \( i_4 = 10 \) we have
\[ \int_0^u \frac{a}{|u|} \int_{u, \infty} a \| u' \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' d u' \leq \frac{a^2 \cdot O^4}{|u|^4} + \frac{a^2 \cdot O^2 \cdot F}{|u|^4} \cdot |u|^4. \]

In the last inequality we have distinguished the cases according to the exact form of \( \psi \) in (6.8), so that we are able to bound it by \( O_{11} \).

There holds
\[ T_4 = \int_0^u \frac{a}{|u|} \int_{u, \infty} a \| u' \|_{L^2_{(x_0)}(S_{u', u'})}^2 \sum_{i_1+i_2+i_3+i_4=i} \| (a^{\frac{1}{2}})^{i_1} \psi^{i_2} \nabla^{i_3}(\chi \alpha_F) \|_{L^2_{(x_0)}(S_{u', u'})}^2 d u' d u' \leq \frac{a^2 \cdot O^4}{|u|^4} + \frac{a^2 \cdot O^2 \cdot F}{|u|^4} \cdot |u|^4. \]

Here we have calculated explicitly all the possible pairs that appear in the schematic \( \chi \nabla(\chi \alpha_F) \) and those are \( \rho_F \nabla \rho_F, \sigma_F \nabla \sigma_F \) and \( \alpha_F \nabla \alpha_F \).
Similarly, there holds

\[
T_5 = \int_0^\infty \frac{a}{|u|} \int_{u=0}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=i} \left\| (a^2)^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, Y) \nabla^{i_4} Y \right\|^2_{L^2(\omega)} a' \, du' \, du \\
\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \int_0^\infty \frac{a}{|u|} \int_{u=0}^u \frac{a^2}{|u'|^2} \left\| (a^2)^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, Y) \nabla^{i_4} Y \right\|^2_{L^2(\omega)} a' \, du' \, du \\
\lesssim \frac{a^2 \cdot O^4}{|u|^4} + \frac{a^3 \cdot O^2 \cdot F^2[\alpha_F]}{|u|^4} + \frac{a^2 \cdot O^2 \cdot F^2[\alpha_F, \sigma_F]}{|u|^4}.
\]

Here we have calculated explicitly all the possible pairs that appear in the schematic \((\alpha_F, Y) \nabla Y\) and those are \(\alpha_F \nabla \alpha_F, \rho_F \nabla \rho_F\) and \(\sigma_F \nabla \sigma_F\).

There holds

\[
T_6 = \int_0^\infty \frac{a}{|u|} \int_{u=0}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+i_5=i} \left\| (a^2)^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \tilde{\chi}, \nabla \chi, \chi) \nabla^{i_4} (\alpha_F, Y) \nabla^{i_5} Y \right\|^2_{L^2(\omega)} a' \, du' \, du \\
\lesssim \int_0^\infty \frac{a}{|u|} \int_{u=0}^u \frac{a}{|u'|^2} \frac{O^6}{a} \, du' \, du = \frac{a \cdot O^6}{|u|^2}.
\]

Here we have estimated

\[
\int_{u=0}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+i_5=i} \left\| (a^2)^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{\chi}, \tilde{\chi}, \nabla \chi, \chi) \nabla^{i_4} (\alpha_F, Y) \nabla^{i_5} Y \right\|^2_{L^2(\omega)} a' \, du' \\
= \int_{u=0}^u \frac{a}{|u'|^2} \cdot \frac{|a'|^4}{a^2} \cdot \frac{O^6}{a^5} \, du' \leq \frac{O^6}{u}.
\]

There holds

\[
T_7 = \int_0^\infty \frac{a}{|u|} \int_{u=0}^u \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4+i_5=i} \left\| (a^2)^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi^{i_4} \nabla^{i_5} \psi \right\|^2_{L^2(\omega)} a' \, du' \\
\lesssim \frac{a^2 \cdot O^6}{|u|^4}.
\]

The final two terms can be absorbed to the left by Grönwall’s inequality, by virtue of schematically containing the term \(\nabla^{i_4} \kappa\).

From the following div – curl system

\[
\begin{align*}
\text{div } \nabla \omega &= \text{div } \kappa + \frac{1}{2} \nabla \tilde{\beta}, \\
\text{curl } \nabla \omega &= 0, \\
\text{div } \nabla \omega^\dagger &= \kappa + \frac{1}{2} \text{curl } \tilde{\beta}, \\
\text{curl } \nabla \omega^\dagger &= 0
\end{align*}
\]

and Proposition 6.1 we have that

\[
\left\| a^2 \nabla^{i_1} (\omega, \omega^\dagger) \right\|_{L^2(\omega)} \leq \frac{1}{56}
\]
Moreover,

\[ \sum_{j=0}^{10} \| (a^j \nabla)^j \kappa \|_{L^2_{\om}(S_{u, \om})} + \| (a^j \nabla)^j \beta \|_{L^2_{\om}(S_{u, \om})} + \frac{1}{a^j} \sum_{j=0}^{10} \| (a^j \nabla)^j (\omega, \omega') \|_{L^2_{\om}(S_{u, \om})}. \]

Passing to \( \| \|_{L^2_{\om}(H^9_{u, \om})} \)-norms, we arrive at

\[ \| (a^j \nabla)^{10} (\omega, \omega') \|_{L^2_{\om}(H^9_{u, \om})} \lesssim R[\tilde{\beta}] + 1. \]  \hfill (6.22)

We move on to top order estimates for \( \eta \).

**Proposition 6.5** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3)-(6.8), there holds

\[ \frac{a}{|u|} \| a^{5} \nabla^{11} \eta \|_{L^2_{\om}(H^9_{u, \om})} + \| a^{5} \nabla^{11} \eta \|_{L^2_{\om}(H^9_{u, \om})} \lesssim R + 1. \]

**Proof.** Introduce the quantity

\[ \mu = -\text{div} \eta - \rho. \]

Our goal is to derive a \( \nabla_4 \)-transport equation for \( \mu \). Recall the commutation formula, for a \( 1 \)-form \( U \)

\[ [\nabla_4, \text{div}] U = -\frac{1}{2} \text{tr} \chi \text{div} U - \chi \cdot \nabla U - \tilde{\beta} \cdot U + \frac{1}{2} (\eta + \rho) \cdot \nabla_4 U - \eta \cdot \chi \cdot U - \frac{1}{2} \text{tr} \chi \eta \cdot U + \text{tr} \chi \eta \cdot U. \]

In particular

\[ \nabla_4 \text{div} \eta = \text{div} (\nabla_4 \eta) + [\nabla_4, \text{div}] \eta = \text{div} (\chi, \text{tr} \chi \gamma) \cdot (\eta - \rho) + (\chi, \text{tr} \chi \gamma) \cdot (\eta - \rho) - \text{div} \tilde{\beta} - \frac{1}{2} \text{tr} \chi \eta \cdot \eta - \frac{1}{2} \text{tr} \chi \eta \cdot \eta - \frac{1}{2} \text{tr} \chi \eta \cdot \eta - \frac{1}{2} \text{tr} \chi \eta \cdot \eta. \]

Schematically, this rewrites as

\[ \nabla_4 (\text{div} \eta) + \text{div} \tilde{\beta} = (\psi, \chi) \cdot (\nabla (\eta, \eta) - \psi \nabla (\psi, \chi) + \psi \cdot (\Psi, \alpha) - (\psi, \chi, \chi) + (\psi, \alpha) \cdot (\alpha, \chi) \cdot \eta + \psi \cdot (\psi, \chi, \chi) \cdot (\alpha, \chi) \cdot \eta. \]

(6.23)

Moreover,

\[ \nabla_4 \rho - \text{div} \tilde{\beta} = (\psi, \chi) \cdot (\alpha, \Psi) + \alpha F \cdot (\rho F, \sigma F) + (\rho F, \sigma F) \cdot (\nabla (\alpha F, \rho F, \sigma F) + (\psi, \chi, \text{tr} \chi \gamma) \cdot (\alpha F, \chi) \cdot (\alpha, \chi) \cdot \eta. \]

Consequently, \( \mu \) satisfies the following transport equation:

\[ \nabla_4 \mu = (\psi, \chi) \cdot (\nabla (\eta, \eta) - \psi \nabla (\psi, \chi) + \alpha F \cdot (\rho F, \sigma F) + (\rho F, \sigma F) \cdot \nabla (\alpha F, \rho F, \sigma F) + (\psi, \chi, \text{tr} \chi \gamma) \cdot (\alpha F, \chi) \cdot (\alpha, \chi) \cdot \eta + \psi \cdot (\psi, \chi, \chi) \cdot (\alpha, \chi) \cdot (\alpha, \chi) \cdot \eta. \]

(6.24)

Commuting (6.24) with \( i \leq 10 \) angular derivatives we arrive at

\[ \nabla_4 \nabla^i \mu = \sum_{i_1 + i_2 + i_3 = i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} \psi \nabla^{i_4+1} \eta + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4+1} \psi \]
We now pass to scale-invariant norms. Noticing that
\[\sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4} \mu.\]

We obtain
\[
\left\| (a^{\frac{\xi}{2}} \nabla)^j \mu \right\|_{L^2_{(\xi)}(S_{u,\omega})} \\
\lesssim \int_0^\infty \sum_{i_1+i_2+i_3+i_4=1} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4+1} (\eta, \omega) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \\
+ \int_0^\infty \sum_{i_1+i_2+i_3+i_4=2} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4+1} (\rho_F, \sigma_F) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \\
+ \int_0^\infty \sum_{i_1+i_2+i_3+i_4=2} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\rho_F, \sigma_F) \nabla^{i_4+1} (\alpha_F, \sigma_F) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \\
+ \int_0^\infty \sum_{i_1+i_2+i_3+i_4=2} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \sigma_F) \nabla^{i_4+1} (\alpha_F, \sigma_F) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \\
+ \int_0^\infty \sum_{i_1+i_2+i_3+i_4=2} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \sigma_F) \nabla^{i_4+1} (\alpha_F, \sigma_F) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \\
+ \int_0^\infty \sum_{i_1+i_2+i_3+i_4=2} \left\| (a^{\frac{\xi}{2}})^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \sigma_F) \nabla^{i_4+1} (\alpha_F, \sigma_F) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega'.
\]

\[= I_1 + \cdots + I_8.\]

- We have
\[
I_1 \lesssim \int_0^\infty \left\| (a^{\frac{\xi}{2}})^{i_1+1} \left( \frac{\psi}{a^{\frac{\xi}{2}}}, \frac{\chi}{a^{\frac{\xi}{2}}} \right) \nabla^{i_4+1} (\eta, \omega) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \lesssim \frac{\tilde{a} \cdot O}{|u|}. \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim 1.
\]

- We have
\[
I_2 \lesssim \int_0^\infty \left\| (a^{\frac{\xi}{2}})^{i_1+1} \psi \nabla^{11} \left( \frac{\psi}{a^{\frac{\xi}{2}}}, \frac{\chi}{a^{\frac{\xi}{2}}} \right) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim 1.
\]

- We have
\[
I_3 \lesssim \int_0^\infty \left\| (a^{\frac{\xi}{2}})^{i_1+1} \chi \nabla^{11} \nabla \left( \frac{\alpha F}{a^{\frac{\xi}{2}}} \right) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim 1.
\]

- Similarly, we have
\[
I_4 \lesssim \int_0^\infty \left\| \chi (a^{\frac{\xi}{2}})^{i_1+1} \nabla \left( \frac{\alpha F}{a^{\frac{\xi}{2}}} \right) \right\|_{L^2_{(\xi)}(S_{u,\omega})} \, d\omega' \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim \frac{\tilde{a} \cdot O}{|u|} \cdot \frac{\alpha \cdot O^2}{|u|} \lesssim 1.
\]
• There holds
\[
I_5 \lesssim \int_0^u \frac{|u'|}{a^2} \left\| \left( \frac{a^2}{u^3} \psi, \frac{a^2}{u} \chi \right) \cdot (a^2 \nabla)^{10}(\alpha, \Psi) \right\|_{L^2_{(\alpha, \omega')} (S_{u \omega'})} du' + 1 \\
\lesssim \int_0^u \frac{u}{a^2} \cdot O_{\infty}[\hat{\chi}] \| (a^2 \nabla)^{10}(\alpha, \Psi) \|_{L^2_{(\alpha, \omega')} (S_{u \omega'})} du' + 1 \lesssim R[\alpha] + 1,
\]

since \( O_{\infty}[\hat{\chi}] \lesssim 1 \) by Proposition 4.3.

• There holds
\[
I_6 \lesssim \frac{O^6}{|u| \cdot a^2}.
\]  

(6.28)

• There holds
\[
I_7 \lesssim 1,
\]  

(6.29)

just as in the term \( J_6 \) in the proof of Proposition 5.2.

• Finally, the term \( I_8 \), after expanding \( \mu = -\text{div} \eta - \rho \), can be controlled by \( I_1 + I_5 \).

Consequently,
\[
\|(a^2 \nabla)^{11}\mu\|_{L^2_{(\alpha, \omega')}} \lesssim R[\alpha] + 1.
\]  

(6.30)

Now observe the div–curl system (the second equation is given schematically):
\[
\text{div} \eta = -\mu - \rho, \\
\text{curl} \eta = \sigma + \hat{\chi} \wedge \hat{\chi} + \Upsilon \cdot \Upsilon.
\]

So that, applying Proposition 6.1, we have
\[
\|(a^2 \nabla)^{11} \eta\|_{L^2_{(\alpha, \omega')}} \lesssim \left\{ \begin{array}{ll}
\frac{1}{u} \sum_{i \leq 10} \left( \left\| (a^2 \nabla)^{11} \mu \right\|_{L^2_{(\alpha, \omega')}} + \left\| (a^2 \nabla)^{11} (\rho, \sigma) \right\|_{L^2_{(\alpha, \omega')}} \\
+ u \cdot \left\| (a^2 \nabla)^{11} \left( \frac{a^2}{u} \hat{\chi} + \frac{\hat{\chi}}{a^2} \right) \right\|_{L^2_{(\alpha, \omega')}} + \left\| (a^2 \nabla)^{11} (\Upsilon \cdot \Upsilon) \right\|_{L^2_{(\alpha, \omega')}} \right. \\
+ \sum_{i \leq 10} \left\| (a^2 \nabla)^{11} \eta \right\|_{L^2_{(\alpha, \omega')}}. 
\end{array} \right.
\]

(6.31)

Integrating along the \( u \)-direction and raising to the second power, we arrive at
\[
\frac{a}{|u|} \left\| a^5 \nabla^{11} \eta \right\|_{L^2_{(\alpha, H^0 u \omega')}} \lesssim R + 1.
\]  

(6.32)

In a similar way, using (6.31), we get
\[
\left\| a^5 \nabla^{11} \eta \right\|_{L^2_{(\alpha, H^0 u \omega')}} \lesssim R + 1.
\]
We move on to estimates for $\tilde{\eta}$.

**Proposition 6.6** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3)-(6.8), there holds

$$\frac{a}{|u|} \|a^5 \nabla^{11} \eta \|_{L^2((H^3_\omega, \omega))} \lesssim \mathcal{R} + 1.$$ 

**Proof.** Introduce $\mu$ defined by

$$\mu = -\text{div} \tilde{\eta} - \rho.$$ 

We then have the Hodge system for $\tilde{\eta}$

$$\text{div} \eta = -\mu - \rho,$$
$$\text{curl} \eta = -\sigma - \frac{3}{2} \tilde{\chi} \wedge \tilde{\chi}.$$ 

For a 1–form $U_b$ we have

$$[\nabla_3, \text{div}] U = -\frac{1}{2} \text{tr}_\chi \text{div} U - \tilde{\chi} \cdot \nabla U - \tilde{\beta} \cdot U + \frac{1}{2}(\eta + \bar{\eta}) \nabla_3 U - \eta \cdot \tilde{\chi} \cdot U - \frac{1}{2} \text{tr}_\chi \eta \cdot U + \text{tr}_\chi \eta \cdot U.$$ 

Consequently,

$$\nabla_3 \text{div} \tilde{\eta} = \text{div} (\nabla_3 \tilde{\eta}) + [\nabla_3, \text{div}] \tilde{\eta}$$

$$= \text{div} \left(-\tilde{\chi} \cdot (\eta - \eta) - \frac{1}{2} \text{tr}_\chi \cdot (\eta - \eta) + \tilde{\beta} - \text{grad} \cdot (\rho F, \sigma_F)\right) + [\nabla_3, \text{div}] \tilde{\eta}$$

$$= -\left(\text{div} \tilde{\chi} \cdot (\eta - \eta) - \tilde{\chi} \cdot \text{div} (\eta - \eta) - \frac{1}{2} \text{tr}_\chi \text{div} \eta + \frac{1}{2} \text{tr}_\chi \text{div} \tilde{\eta}\right)$$

$$- \frac{1}{2}(\eta - \eta) \text{div} (\text{tr}_\chi) + \text{div} \tilde{\beta} - \text{grad} \cdot (\rho F, \sigma_F) - (\rho F, \sigma_F) \text{div} \rho F$$

$$- \frac{1}{2} \text{tr}_\chi \text{div} \eta - \tilde{\chi} \cdot \text{curl} \eta - \tilde{\beta} \cdot \eta + \frac{1}{2}(\eta + \bar{\eta}) \cdot \left(\psi \cdot (\tilde{\chi}, \text{tr}_\chi + \tilde{\beta} + \Upsilon \cdot \Upsilon)\right) + \frac{1}{2} \text{tr}_\chi \cdot \eta \cdot \eta.$$ 

We thus have the (semi–)schematic identity

$$\nabla_3 \text{div} \eta + \text{tr}_\chi \text{div} \eta - \text{div} \tilde{\beta} = \psi \cdot \text{div} \tilde{\chi} + \tilde{\chi} \text{curl} (\eta, \eta) + \text{tr}_\chi \text{curl} \eta + \text{tr}_\chi \text{div} (\text{tr}_\chi) + \Upsilon (\rho F, \sigma_F)$$

$$+ \Upsilon \text{div} (\rho F, \sigma_F) + \psi \cdot \tilde{\beta} + \psi \cdot (\tilde{\chi}, \text{tr}_\chi) \cdot \psi + \psi \cdot \Upsilon \cdot \Upsilon.$$ 

Also,

$$\nabla_3 \rho + \text{tr}_\chi \rho + \text{div} \tilde{\beta} = -\frac{1}{2} \text{tr}_\chi \cdot \rho + \tilde{\chi} \cdot \rho + \psi \cdot \tilde{\beta} + (\rho F, \sigma_F) \text{div} \rho F$$

$$+ \text{grad} (\rho F, \sigma_F) + (\psi, \text{tr}_\chi) \cdot \Upsilon + (\psi, \tilde{\chi}) \cdot (\Upsilon, \tilde{\chi}) \cdot \Upsilon.$$ 

Combining the above two equations, $\tilde{\eta}$ satisfies the following transport equation:

$$\nabla_3 \tilde{\eta} + \text{tr}_\chi \tilde{\eta}$$

$$= \psi \cdot \nabla \tilde{\chi} + \tilde{\chi} \text{curl} (\eta, \eta) + \text{tr}_\chi \text{curl} \eta + \psi \text{tr}_\chi + \text{grad} (\rho F, \sigma_F)$$

$$+ (\rho F, \sigma_F) \text{div} \rho F + \psi \cdot \tilde{\beta} + \text{tr}_\chi \cdot \rho + \psi \cdot (\tilde{\chi}, \text{tr}_\chi) \cdot \psi + (\psi, \text{tr}_\chi) \cdot \Upsilon + (\psi, \tilde{\chi}) \cdot (\Upsilon, \tilde{\chi}) \cdot \Upsilon.$$ 

By commuting (6.33) with $i \leq 10$ angular derivatives, we arrive at
\[ \nabla_3 \nabla_i \mu + \frac{i + 2}{2} \text{tr} \chi \mu = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_3} \nabla^{i_3} ; \nabla^{i_3} \chi + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_3} \nabla^{i_3} \nabla^{i_4} \psi^{i_4}(\eta, \psi) \]

+ \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_3} \nabla^{i_3} \chi \nabla^{i_4} \psi^{i_4} + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_3} \nabla^{i_3} \psi^{i_4} \nabla^{i_4} \chi + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_3} \nabla^{i_3} \chi \nabla^{i_4} \psi^{i_4} \nabla^{i_4} \chi \nabla^{i_4} \psi^{i_4}(\chi, \mu) \]

By Proposition 3.7 we can bound

\[ |u|^{i+1} \| \nabla_i \mu \|_{L^2(S_{u, \omega})} \lesssim |u|^{i+1} \| \nabla_i \mu \|_{L^2(S_{u, \omega})} + \int_{S_{u, \omega}} |u'|^{i+1} \| G \|_{L^2(S_{u', \omega})} \, du'. \]

We have \( s_2(\nabla_i \mu) = \frac{d_2^2}{u} \) and \( s_2(G) = \frac{d_2}{u} \). By passing to scale-invariant norms, we have

\[ \frac{a}{|u|} \| (a^2 \nabla)^j \mu \|_{C^2_{(s)}(S_{u, \omega})} \leq \frac{a}{|u|} \| (a^2 \nabla)^j \mu \|_{C^2_{(s)}(S_{u, \omega})} + \int_{u_0}^u \frac{a^2}{|u'|} \| (a^2 \nabla)^j \mu \|_{C^2_{(s)}(S_{u', \omega})} \, du'. \]

\[ \leq \frac{a}{|u|} \| (a^2 \nabla)^j \mu \|_{C^2_{(s)}(S_{u, \omega})} + \int_{u_0}^u \frac{a^2}{|u'|} \| (a^2 \nabla)^j \mu \|_{C^2_{(s)}(S_{u', \omega})} \, du'. \]
\[
+ \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\hat{\nabla} \nabla^{i_5} \nabla^{i_6}) \right) \left( \mathcal{L}_{(\varphi)} (S_{\varphi'}) \right) \, du' \\
+ \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\nabla^{i_5} \nabla^{i_6} \nabla^{i_7}) \right) \left( \mathcal{L}_{(\varphi)} (S_{\varphi'}) \right) \, du' \\
+ \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\nabla^{i_5} \nabla^{i_6} \nabla^{i_7}) \right) \left( \mathcal{L}_{(\varphi)} (S_{\varphi'}) \right) \, du' \\
+ \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\nabla^{i_5} \nabla^{i_6} \nabla^{i_7}) \right) \left( \mathcal{L}_{(\varphi)} (S_{\varphi'}) \right) \, du' \\
:= T_1 + \cdots + T_{15}.
\]

We bound $T_1$ to $T_{15}$ individually.

- Given that $\mu = -\text{div} \, \mathbf{n} - \rho$ and the fact that $\mathcal{T}^{(0)}$ bounds up to 14 derivatives for $\mathbf{n}$ and $\rho$, there holds

  \[
  T_1 = \frac{a}{|u_{\infty}|} \|(a^\frac{3}{2} \nabla) \| \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \lesssim \mathcal{T}^{(0)} \lesssim 1. \quad (6.34)
  \]

  \[
  \text{There holds}
  \]

  \[
  T_2 = \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} + \nabla^{i_5} \nabla^{i_6} \nabla^{i_7} \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  \lesssim \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \|(a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} + \nabla^{i_5} \nabla^{i_6} \nabla^{i_7} \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  + \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} + \nabla^{i_5} \nabla^{i_6} \nabla^{i_7} \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  \lesssim \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \|(a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} + \nabla^{i_5} \nabla^{i_6} \nabla^{i_7} \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  + \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( (a^\frac{3}{2})^i \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \psi \nabla^{i_4} + \nabla^{i_5} \nabla^{i_6} \nabla^{i_7} \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  \lesssim 1 + \frac{a}{|u_{\infty}|} \cdot O \cdot O(1) \lesssim 1.
  \]

Here we have made use of Proposition 6.8.

- There holds

  \[
  T_3 \lesssim a^2 O^2 + \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( a^5 \nabla^{i_1} \psi \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' \\
  \lesssim \int_{u_{\infty}}^{u} \frac{a^2}{|u'|^{|i|}} \left( a^5 \nabla^{i_1} \psi \right) \left( \mathcal{L}_{(\varphi)} (S_{u_{\infty}, \varphi'}) \right) \, du' + 1. \quad (6.35)
  \]

This term is controlled by Propositions 6.5 and 6.6.
• There holds

\[
T_4 \lesssim \frac{a^\frac{7}{2} \cdot O}{|u|^2} + \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \frac{|u'|^2}{a} \cdot \frac{O_{\infty}[\text{tr} \chi]}{|u'|} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu' + 1
\]

\[
\lesssim \int_{|u'|^2}^u \frac{a \cdot O_{\infty}[\text{tr} \chi]}{|u'|} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(\mathcal{S}_a')}}(S_{a'}') d\nu' + 1
\]

\[
\lesssim \frac{a}{|u|} \|a^5 \nabla^{11} \eta\|_{\mathcal{L}^2_{(\mathcal{S}_a')}}(S_{a'}') + 1.
\]

(6.36)

Here we have used the fact that $O_{\infty}[\text{tr} \chi] \lesssim 1$, shown in Section 4.

• There holds

\[
T_5 \lesssim \frac{a^\frac{7}{2} \cdot O}{|u|^2} + \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \frac{|u'|}{a} \cdot \frac{O}{|u'|} \|a^5 \nabla^{11} \tilde{\chi} \|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu'
\]

\[
\lesssim \frac{a^\frac{7}{2} \cdot O}{|u|^2} + \frac{a^\frac{7}{2} \cdot O}{|u|^2} \|a^5 \nabla^{11} \tilde{\chi} \|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') (\mu_{(u')^a})
\]

\[
\lesssim \frac{O}{|u|^2} \cdot a^\frac{7}{2} \cdot O_{11}[\text{tr} \chi] + 1.
\]

(6.37)

• There holds

\[
T_6 \lesssim \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \cdot \frac{1}{a^\frac{2}{5}} \cdot O^2 d\nu' + \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \frac{O}{|u'|} \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu'
\]

\[
\lesssim \frac{a^\frac{7}{2} \cdot O^2}{|u|^3} + \left( \int_{|u'|^2}^u \frac{a^\frac{7}{2}}{|u'|^2} \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu' \right) \lesssim 1 + \frac{a^\frac{7}{2} \cdot O^2}{|u|^3} \cdot \mathcal{F}[\rho_F, \sigma_F] \lesssim 1.
\]

• Similarly, there holds

\[
T_7 \lesssim \frac{a^\frac{7}{2} \cdot O^2}{|u|^3} + \frac{a^\frac{7}{2} \cdot O}{|u|^2} \cdot \|a^5 \nabla^{11} \alpha_F\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu'
\]

\[
\lesssim \left( \sum_{k \leq 10} \int_{|u'|^2}^u \frac{a^\frac{7}{2}}{|u'|^2} \|a^5 \nabla^{11} \alpha_F\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu' \right) \lesssim \frac{a^\frac{7}{2} \cdot O^2}{|u|^3} + \frac{a^\frac{7}{2} \cdot O}{|u|^2} \cdot \mathcal{F}[\alpha_F] \lesssim 1.
\]

(6.38)

• There holds

\[
T_8 \lesssim \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \cdot \frac{O}{|u'|} \sum_{k \leq 10} \|a^\frac{7}{2} \nabla^k \|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu'
\]

\[
\lesssim \left( \sum_{k \leq 10} \int_{|u'|^2}^u \frac{a^\frac{7}{2}}{|u'|^2} \|a^\frac{7}{2} \nabla^k \|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu' \right) \lesssim \frac{a^\frac{7}{2} \cdot O^2}{|u|^3} \cdot \mathcal{F}[\beta] \lesssim 1.
\]

• There holds (this is the most marginal term)

\[
T_9 \lesssim \int_{|u'|^2}^u \frac{a^2}{|u'|^{\frac{7}{2}}} \frac{O}{|u'|} \sum_{k \leq 10} \|a^\frac{7}{2} \nabla^k \rho\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu'
\]

\[
\lesssim \left( \sum_{k \leq 10} \int_{|u'|^2}^u \frac{a^\frac{7}{2}}{|u'|^2} \|a^\frac{7}{2} \nabla^k \rho\|_{\mathcal{L}^{2}_{(\mathcal{S}_a')}}(S_{a'}') d\nu' \right) \lesssim \frac{a^\frac{7}{2} \cdot O[\text{tr} \chi]}{|u|^\frac{7}{2}} \cdot \mathcal{R}[\rho] \lesssim \mathcal{R}[\rho].
\]
• There holds

\[ T_{11} + T_{12} = \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ + \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ \lesssim \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ + \int_{u \to \infty} a^3 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ \lesssim \int_{u \to \infty} \left( \frac{a^3}{|u|^2} + \frac{a^3}{|u|^2} \right) \cdot O^3 \left( \frac{a^3}{|u|^2} \right) \lesssim \frac{a^3 O^3}{|u|^2} \lesssim 1. \]

• The last three terms can be controlled by Grönwall’s inequality. Indeed,

\[ \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ + \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ + \int_{u \to \infty} a^2 \left( \left( \frac{a}{u} \right)^i \right)^i \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \nabla) \nabla^{i_4} \nabla^{i_5} \nabla \|_{L^2_v(S_v^\omega)} \] 

\[ \lesssim \int_{u \to \infty} \left( \frac{a^3}{|u|^2} + \frac{a^3}{|u|^2} \right) \cdot O^3 \left( \frac{a^3}{|u|^2} \right) \lesssim \frac{a^3 O^3}{|u|^2} \lesssim 1. \]

When \( i_1 = i \) the three terms in the parenthesis of the line above are integrable with respect to \( u \) and so Grönwall’s inequality allows us to control the term. When \( i_1 < i \), we can use the definition \( \mu = -\nabla \eta - \rho \) and bound the terms by the already established estimates of the previous sections.

Consequently, there holds

\[ \frac{a}{|u|} \| (a \nabla \eta) \|_{L^2_v(S_v^\omega)} \lesssim \int_{u \to \infty} a^5 \nabla^{11} (\eta, \eta) \|_{L^2_v(S_v^\omega)} \] 

\[ + \frac{a}{|u|} \| a^5 \nabla^{11} \eta \|_{L^2_v(H^0_v)} + R |\rho| + 1. \]

(6.39)

Now observe the \( \text{div} - \text{curl} \) system (the second equation is given schematically):

\[ \text{div} \ \eta = -\mu - \rho, \]

\[ \text{curl} \ \eta = -\sigma - \tilde{\chi} + \nabla \cdot \nabla. \]

(6.40, 6.41)

Consequently,
\[
\frac{a}{|u|} \| a^5 \nabla^{11} u \|_{L^2_{t,x}(H^{11})} \lesssim \sum_{i < 10} \left( \frac{a}{|u|} \| (a^2 \nabla) \|_{L^2_{t,x}(H^{11})} + \| (a^2 \nabla)^i \|_{H^{11}} \right) + \| (a^2 \nabla)^i (\cdot \cdot \cdot) \|_{L^2_{t,x}(H^{11})} + \left( \frac{1}{a^2} \sum_{i < 10} \| (a^2 \nabla)^i \|_{L^2_{t,x}(H^{11})} \right).
\]

By raising the above to the second power, integrating along \( u \) and using (6.39) along with Grönwall’s inequality we can get that

\[
\frac{a}{|u|} \| a^5 \nabla^{11} u \|_{L^2_{t,x}(H^{11})} \lesssim R + 1.
\]

We now prove the highest order bounds for \( \omega \).

**Proposition 6.7** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3)-(6.8), we have

\[
\| a^5 \nabla^{11} \omega \|_{L^2_{t,x}(H^{11})} \lesssim 1 + R + F.
\]

**Proof.** Define the auxiliary function \( \omega^\dagger \) by

\[
\nabla_4 \omega^\dagger = \frac{1}{2} \sigma
\]

and trivial initial data along \( H^0 \). We then define \( \kappa \) by

\[
\kappa = -\nabla_4 \omega^\dagger - \frac{1}{2} \frac{\beta}{\omega^\dagger}.
\]

We need to obtain a transport equation for \( \kappa \). Notice that if we set \( \langle \omega \rangle = (\omega, \omega^\dagger) \), then

\[
\kappa = \nabla_4 \langle \omega \rangle - \frac{1}{2} \frac{\beta}{\omega^\dagger}.
\]

Recall the commutation formulae

\[
[\nabla_4, \nabla] f = -\frac{1}{2} \text{tr} \chi \nabla f - \chi \cdot \nabla f + \frac{1}{2} (\eta + \eta) \nabla_4 f, \\
[\nabla_4, \nabla^*] g = -\frac{1}{2} \text{tr} \chi^* \nabla^* g + \chi \cdot \nabla^* g + \frac{1}{2} (\eta + \eta) \nabla_4 g.
\]

Thus, for a pair of scalars \((f, g)\), there holds

\[
[\nabla_4, \nabla] f \langle \omega \rangle = -\frac{1}{2} \text{tr} \chi \nabla \langle \omega \rangle + \chi \cdot \nabla \langle \omega \rangle + \frac{1}{2} (\eta + \eta) \nabla_4 \langle \omega \rangle + \frac{1}{2} (\eta + \eta) \nabla_4 g. \quad (6.43)
\]

Now recall that

\[
\nabla_4 \omega^\dagger = \frac{1}{2} \rho + \psi \cdot \psi + \Psi \cdot \Psi := \frac{1}{2} \rho + F.
\]

Therefore,

\[
\nabla_4 \langle \omega \rangle + \frac{1}{2} \nabla \rho - \frac{1}{2} \nabla \omega^\dagger = (\psi, \chi) \cdot \nabla \psi + \Psi \cdot \Psi + \psi \cdot \psi + \psi \cdot \psi + \Psi \cdot \Psi. \quad (6.44)
\]

Moreover,

\[
\nabla_4 \beta + \text{tr} \chi^\dagger \beta + \nabla \rho - \nabla \sigma = (\psi, \chi) \cdot \Psi + (\alpha_F, \Psi) \cdot \nabla \Psi + (\psi, \chi, \chi, \text{tr} \chi) \cdot (\alpha_F, \Psi) \cdot \Psi. \quad (6.45)
\]

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Consequently,

\[
\nabla_4 \hat{\mathbf{k}} = (\psi, \hat{x}) \cdot \Psi + (\alpha_F, \mathbf{T}) \cdot \nabla \mathbf{T} + (\psi, \hat{\chi}) \cdot \nabla \mathbf{\psi} + (\psi, \hat{x}, \hat{\chi}, \nabla_x \mathbf{\chi}) \cdot (\alpha_F, \mathbf{T}) \cdot \mathbf{T} + \psi \cdot \psi \cdot \psi.
\]

(6.46)

Commuting with \( i \leq 10 \) angular derivatives, we get

\[
\nabla_4 \hat{\mathbf{k}}
\]

\[
= \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} + \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\alpha_F, \mathbf{T}) \nabla^{i_4+1} \mathbf{T}
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}, \hat{\chi}, \nabla_x \mathbf{\chi}) \nabla^{i_4} \mathbf{T} \nabla^{i_5} \mathbf{T}
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} + \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi}.
\]

Passing to scale-invariant norms, there holds

\[
\| (a^n) \nabla \|_{\mathcal{L}_2(S_n \cdot \omega)}^2
\]

\[
\lessgtr \int_0^\infty \sum_{i_1 + i_2 + i_3 + i_4 = i} \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 \, du'
\]

\[
+ \int_0^\infty \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\alpha_F, \mathbf{T}) \nabla^{i_4+1} \mathbf{T} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 \, du'
\]

\[
+ \int_0^\infty \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4+1} \mathbf{T} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 \, du'
\]

\[
+ \int_0^\infty \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}, \hat{\chi}, \nabla_x \mathbf{\chi}) \nabla^{i_4} \mathbf{T} \nabla^{i_5} \mathbf{T} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 \, du'
\]

\[
+ \int_0^\infty \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 \, du'
\]

\[
:= J_1 + \cdots + J_6.
\]

We again estimate term by term.

- We have

\[
\| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2
\]

\[
\lesssim \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\psi, \hat{x}) \nabla^{i_4} \mathbf{\psi} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2 + \| (a^n)^{i_1} \psi^{i_2} \nabla^{i_3}(\alpha_F, \mathbf{T}) \nabla^{i_4+1} \mathbf{T} \|_{\mathcal{L}_2(S_n \cdot \omega)}^2
\]

\[
\lesssim \frac{O^2}{a^2} \| (a^n) \psi \|_{\mathcal{L}_2(S_n \cdot \omega)}^2.
\]

Consequently,

\[
J_1 \lesssim \frac{O^2}{a^2} + \frac{O_{\infty} [\hat{x}] [R \psi]}{a^2} \lesssim 1.
\]
• We have

\[
\| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} (\alpha F, Y) \nabla^{14} \psi \nabla^{15} \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \\
= a^{\frac{2}{3}} \cdot \| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} \left( \frac{\alpha F}{a^{\frac{2}{3}}} \cdot Y \right) \nabla^{14} \psi \nabla^{15} \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \]
\]

(6.48)

Consequently,

\[
J_2 \lesssim \frac{a^{\frac{2}{3}} \cdot O^2}{|u|} + \frac{a^{\frac{2}{3}} \cdot O \cdot |Y| + O [\alpha F] \cdot |Y|}{u}.
\]

• We have

\[
\| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} (\psi, \tilde{\chi}) \nabla^{14} \psi \nabla^{15} \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \\
= a^{\frac{2}{3}} \cdot \| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} \left( \psi, \tilde{\chi} a^{\frac{2}{3}} \right) \nabla^{14} \psi \nabla^{15} \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \]
\]

(6.49)

Then \(J_3\) satisfies the same bound.

• We have

\[
\| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} (\psi, \tilde{\chi}, \tilde{\chi}, \tilde{\chi}, \tilde{\chi}) \nabla^{14} (\alpha F, Y) \nabla^{15} Y \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \\
= \frac{u^2}{a} \cdot a^{\frac{2}{3}} \cdot \| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} \left( \frac{\alpha F}{u^2} \psi, \frac{\alpha F}{u^2} \chi, \frac{\alpha F}{u^2} \chi, \frac{\alpha F}{u^2} \chi \right) \nabla^{14} \left( \frac{\alpha F}{a^{\frac{2}{3}}} \cdot Y \right) \nabla^{15} Y \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \]
\]

(6.50)

Thus \(J_4\) satisfies, upon integration, the same bound.

• There holds

\[
\sum_{i_1+i_2+i_3+i_4+i_5=1} \| (a^{\frac{2}{3}})^4 \psi^{12} \psi \nabla^{13} \psi \nabla^{14} \psi \nabla^{15} \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \lesssim \frac{O^3}{|u|^2}
\]

• The final term can be absorbed to the left by a Grönwall-type argument.

Overall,

\[
\| (a^{\frac{2}{3}})^4 \psi \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \lesssim 1 + \mathcal{R} + \mathcal{F}.
\]

By the following div-curl system

\[
\begin{align*}
\text{div} \ \nabla \omega &= -\text{div} \ \kappa - \frac{1}{2} \text{div} \ \tilde{\beta}, \\
\text{curl} \ \nabla \omega &= 0, \\
\text{curl} \ \omega^\dagger &= \text{curl} \ \kappa + \frac{1}{2} \text{curl} \ \tilde{\beta}, \\
\text{div} \ \nabla \omega^\dagger &= 0,
\end{align*}
\]

applying Proposition 6.1, we have

\[
\| (a^{\frac{2}{3}})^{10} \nabla^{11} \omega \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \lesssim \sum_{j=0}^{10} \left( \| (a^{\frac{2}{3}})^{j} \nabla \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} + \| (a^{\frac{2}{3}})^{j} \tilde{\beta} \|_{\mathcal{L}^2_{\psi}(S_u \omega^\cdot)} \right)
\]

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We proceed to estimate $I$.

Commuting this equation with Proposition 6.8

Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3)-(6.8), there holds

\[ \int_{u_*}^{u} \frac{a^2}{|u|^3} \|a^5 \nabla_{\Omega}^i \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \, du' \lesssim \mathcal{R} + \mathcal{R} + 1, \]

\[ \int_{u_*}^{u} \frac{a^2}{|u|^3} \|a^5 \nabla_{\Omega} \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \, du' \lesssim 1. \]

**Proof.** We begin with the equation

\[ \nabla_3 \nabla^i \nabla \chi + \frac{i + 2}{2} \nabla \nabla \chi = \sum_{i_1 + i_2 + i_3 = i} \nabla_{i_1} \psi_{i_2} \nabla_{i_3} \left( \frac{2}{|u|^2} (\Omega^{-1} - 1) + \nabla \nabla \chi + 2 \nabla \nabla \chi - |\nabla \chi|^2 - |\alpha_F|^2 \right) \]

Commuting this equation with $i$ angular derivatives, we get

\[ \nabla_3 \nabla^i \nabla \chi + \frac{i + 2}{2} \nabla \nabla \chi = \sum_{i_1 + i_2 + i_3 = i} \nabla_{i_1} \psi_{i_2} \nabla_{i_3} \nabla \nabla \nabla \chi + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla_{i_4} \psi_{i_3} \nabla_{i_2} \nabla \nabla \nabla \chi. \]

Rewriting in terms of scale-invariant norms,

\[ \frac{a}{|u|} \|(a^\frac{1}{2})^{i-1} \nabla \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \leq \frac{a}{|u|} \|(a^\frac{1}{2})^{i-1} \nabla \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) + \int_{u_*}^{u} \frac{a^2}{|u|^3} \|(a^\frac{1}{2})^{i-1} \nabla \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \, du' \]

where

\[ \frac{a}{|u|} \|(a^\frac{1}{2})^{i-1} \nabla \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \leq \frac{a}{|u_*|} \|(a^\frac{1}{2})^{i-1} \nabla \nabla \chi \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \lesssim 1. \]

We proceed to estimate $I_1$ to $I_3$.

- We can rewrite $I_1 = I_{11} + I_{12} + I_{13} + I_{14} + I_{15}$ in the obvious way. We further decompose $I_{11} = I_{111} + I_{112}$.

There holds

\[ I_{111} = \int_{u_*}^{u} \frac{a^2}{|u|^3} \|(a^\frac{1}{2})^{i-1} \sum_{i_1 + i_2 + i_3 = i} \nabla_{i_1} \psi_{i_2} \nabla_{i_3} + \frac{2}{|u|^2} (\Omega^{-1} - 1) \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \, du' \]

\[ = \int_{u_*}^{u} \frac{a^2}{|u|^3} \|(a^\frac{1}{2})^{i-1} \sum_{i_1 + i_2 = i-1} \nabla_{i_1} \psi_{i_2} + 1 \|_{C^2_{(\omega, \omega')}}(S_{u, \omega}) \, du' \lesssim \frac{a^2}{|u|^4}. \]
Also,
\[
I_{112} = \int_{u_0}^u \frac{a^2}{|u'|^3} \left\| (a^\frac{2}{3})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} (\frac{\Omega^{-1} - 1}{|u'|^2}) \right\|_{L^2_{\tilde{e}(S_{u'}, S_{u', \omega})}} du'
\]
\[
= \int_{u_0}^u \left| u' \right|^{i_1+1} \left| \frac{1}{|u'|^2} \right| \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{\Omega^{-1} - 1}{|u'|^2} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du' \quad (\text{in standard norms})
\]
\[
= \int_{u_0}^u \left| u' \right|^{i_1+1} \left| \frac{1}{|u'|^2} \right| \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{\Omega^{-1} - 1}{|u'|^2} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du' \quad (\text{Using } \frac{\partial}{\partial u} \frac{1}{|u'|^2} = 2 \omega \Leftrightarrow \nabla_4 \Omega^{-1} = 2 \Omega^{-1} \omega)
\]
\[
= \int_{u_0}^u \left| u' \right|^{i_1+1} \left| \frac{1}{|u'|^2} \right| \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{1}{|u'|^2} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du' \quad (\text{Using } \frac{\partial}{\partial u} \frac{1}{|u'|^2} = 2 \omega \Leftrightarrow \nabla_4 \Omega^{-1} = 2 \Omega^{-1} \omega)
\]
\[
\leq \int_{u_0}^u \left| u' \right|^{i_1+1} \left| \frac{1}{|u'|^2} \right| \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{1}{|u'|^2} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
\leq \int_{u_0}^u \frac{a^2}{|u'|^3} \left( R[\rho] + 1 \right) du' \lesssim R[\rho] + 1.
\]
Similarly, there holds
\[
I_{12} = \int_{u_0}^u \frac{a^2}{|u'|^3} \left\| (a^\frac{2}{3})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \tilde{\chi}^{i_3} \nabla^{i_4} \tilde{\chi} \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
= \int_{u_0}^u \frac{a^2}{|u'|^3} \left\| (a^\frac{2}{3})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|^2} \tilde{\chi} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
\lesssim \int_{u_0}^u \frac{a^2}{|u'|^3} \frac{|u'|^2}{|u'|} \cdot \frac{O^2}{|u'|^3} du' + \int_{u_0}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|}{|u'|} \cdot \frac{O}{|u'|} \cdot \left\| a^5 \nabla^{i_1} \tilde{\chi} \right\|_{L^2_{\tilde{e}(s_{u', S_{u', \omega})}} du'.
\]
Also,
\[
I_{13} = \int_{u_0}^u \frac{a^2}{|u'|^3} \left\| (a^\frac{2}{3})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \tilde{\chi} \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
= \int_{u_0}^u \frac{a^2}{|u'|^3} \frac{|u'|^2}{a} \cdot \left\| (a^\frac{2}{3})^i \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|^2} \tilde{\chi} \right) \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
\lesssim \int_{u_0}^u \frac{a^2}{|u'|^3} \frac{|u'|^2}{a} \cdot \frac{O^2}{|u'|^3} du' + \int_{u_0}^u \frac{a^2}{|u'|^3} \cdot \frac{|u'|^2}{a} \cdot \frac{O}{|u'|} \cdot \left\| a^5 \nabla^{i_1} \tilde{\chi} \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
+ \int_{u_0}^u \frac{a^2}{|u'|^3} \frac{|O^2}{|u'|^3} \cdot \frac{O^2}{a} \cdot \left\| \frac{O}{|u'|} \cdot \left\| a^5 \nabla^{i_1} \tilde{\chi} \right\|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du'
\]
\[
\lesssim \frac{a^2 \cdot O^2}{|u'|^3} + \int_{u_0}^u \frac{a}{|u'|^2} \| a^5 \nabla^{i_1} \tilde{\chi} \|_{L^2_{\tilde{e}(S_{u', S_{u', \omega})}} du' + (a \text{ term handled by Grönwall’s inequality}).
\]
There also holds

\[ I_{14} = \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \tilde{\chi} \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \frac{|u'|^2}{a} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \frac{a^2}{|u|^2} \cdot O^2 \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

(6.52)

Finally, for the term \( I_{15} \) we have

\[ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \tilde{\chi} \nabla^{i_4} \tilde{\chi} \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \frac{|u'|^2}{a} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \frac{a^2}{|u|^2} \cdot O \cdot a^5 \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

(6.53)

- There holds

\[ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \tilde{\chi} \nabla^{i_4} \tilde{\chi} \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \frac{|u'|^2}{a} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

+ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

\[ \leq \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

+ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left( \left( \frac{a}{|u'|} \right)^{i-1} \sum_{i_1+i_2+i_3+i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} \left( \frac{a}{|u'|} \tilde{\chi} \right) \right) \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' \]

Noting that \( \psi \in \{ \eta, \eta \} \) and recalling Propositions 6.5 and 6.6, using Grönwall’s inequality, we can bound this term by

\[ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \| \mathcal{L}^2_{(\xi)} (S_{u', \omega}) \| du' + 1. \]
For the last term there holds
\[ \int_{u_\infty}^{u} \frac{a^2}{|u|^3} \left\| (a^{\frac{2}{\gamma}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \nabla^{i_4} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' \]
\[ = \int_{u_\infty}^{u} \frac{a^2}{|u|^3} \left\| (a^{\frac{2}{\gamma}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left( \frac{a}{|u'|^2} \nabla \chi \right) \nabla^{i_4} \left( \frac{a}{|u'|^2} \nabla \chi \right) \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' \]
\[ \lesssim \int_{u_\infty}^{u} \frac{a^2}{|u|^3} \cdot \frac{|u'|^2}{a} \cdot \left\| (a^{\frac{2}{\gamma}})^{i-1} \sum_{i_1+i_2+i_3+i_4+1=i} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left( \frac{a}{|u'|^2} \nabla \chi \right) \nabla^{i_4} \left( \frac{a}{|u'|^2} \nabla \chi \right) \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' \]
\[ \lesssim \frac{O^5}{a^2} \cdot |u| \lesssim 1. \] (6.54)

We thus have, using Grönwall’s inequality,
\[ \frac{a}{|u|} \left\| a^5 \nabla^{11} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) \lesssim 1 + |R|^2 + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| a^5 \nabla^{11} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' + \int_{u_\infty}^{u} \frac{a^2}{|u'|^2} \left\| a^5 \nabla^{11} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du'. \] (6.55)

For $\hat{\chi}$, we have the constraint equation
\[ \text{div} \; \hat{\chi} = \frac{1}{2} \nabla \nabla \chi - \frac{1}{2} (\eta - \eta)(\hat{\chi} - \frac{1}{2} \nabla \chi) + \frac{\beta}{a}. \]

Consequently,
\[ \left\| a^5 \nabla^{11} \hat{\chi} \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) \lesssim \sum_{j \leq 10} \left( \left\| (a^{\frac{2}{\gamma}})^{j} \nabla^{j+1} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) + \left\| (a^{\frac{2}{\gamma}}) \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) \right) \]
\[ + \left\| (a^{\frac{2}{\gamma}})^{j} \sum_{j_1 + j_2 = j} \nabla^{j_1} (\eta, \eta) \nabla^{j_2} (\hat{\chi}, \nabla \chi) \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) \]
\[ + \sum_{j \leq 10} \left( \left\| (a^{\frac{2}{\gamma}})^{j} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u, \omega}) \right). \]

Integrating this along the incoming direction, we have
\[ \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| a^5 \nabla^{11} \hat{\chi} \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' \]
\[ \lesssim \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \sum_{j \leq 10} \left( \left\| (a^{\frac{2}{\gamma}})^{j} \nabla^{j+1} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) + \left\| (a^{\frac{2}{\gamma}}) \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \right) \]
\[ + \frac{|u'|^2}{a} \left( \left\| (a^{\frac{2}{\gamma}})^{j} \sum_{j_1 + j_2 = j} \nabla^{j_1} (\eta, \eta) \nabla^{j_2} (\hat{\chi}, \nabla \chi) \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \right) \, du' \]
\[ + \int_{u_\infty}^{u} \frac{a}{|u'|^2} \sum_{j \leq 10} \left( \left\| (a^{\frac{2}{\gamma}}) \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \right) \, du' \lesssim \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| a^5 \nabla^{11} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' + \frac{a^2}{|u'|^2} \cdot |R| \]
\[ + \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \cdot \frac{|u'|^2}{a} \cdot O^2 \, du' + \int_{u_\infty}^{u} \frac{a}{|u'|^3} \cdot \frac{|u|}{a^2} \, du' \]
\[ \lesssim \int_{u_\infty}^{u} \frac{a^2}{|u'|^3} \left\| a^5 \nabla^{11} \nabla \chi \right\| \mathcal{E}^{2}_{(\omega_{\pi}, \omega)}(S_{u', u}) \, du' + 1. \]

Plugging this back to (6.55) and using Grönwall’s inequality, we get
\[
\frac{a}{|u|} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u=\omega})} \lesssim 1 + \mathcal{R}[\rho] + \int_{u^\infty}^u \frac{a}{|u'|^2} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u'=\omega})} \, du' \\
\lesssim 1 + \mathcal{R}[\rho] + \frac{a}{|u|} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u'=\omega})} \lesssim \mathcal{R} + \mathcal{R} + 1.
\]

Integrating in the \( u \)-direction we obtain
\[
\int_{u^\infty}^u \frac{a^2}{|u'|^3} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u'=\omega})} \lesssim \mathcal{R} + \mathcal{R} + 1.
\]

**Remark 2** In a similar fashion, we can obtain the following estimates:
\[
\left( \int_{u^\infty}^u \frac{a^3}{|u'|^4} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u'=\omega})} \, du \right)^{\frac{1}{2}} + \left( \int_{u^\infty}^u \frac{a^2}{|u'|^3} ||a^5 \nabla^{11} \tilde{\chi}||_{L^2(S_{u'=\omega})} \, du \right)^{\frac{1}{2}} \lesssim \mathcal{R} + \mathcal{R} + 1. \tag{6.57}
\]

**7 Energy estimates**

In this section with scale invariant norms we will derive energy estimates for curvature components and their angular derivatives. Our goal is to show that
\[
\mathcal{R} + \mathcal{R} + \mathcal{F} + \mathcal{F} \lesssim \left( I^{(0)} \right)^4 + \left( I^{(0)} \right)^2 + I^{(0)} + 1.
\]

We begin with the integration by parts formula.

**7.1 Integration by parts**

The following holds. Define \( D_{u,\omega} := (u^\infty, u) \times (0, u) \). A direct computation yields the following:

**Proposition 7.1** Suppose \( \phi_1, \phi_2 \) are \( r \)-tensorfields. Then there holds
\[
\int_{D_{u,\omega}} \phi_1 \nabla_\omega \phi_2 + \int_{D_{u,\omega}} \phi_2 \nabla_\omega \phi_1 = \int_{H_{u,\omega}^{(0)}} \phi_1 \phi_2 - \int_{H_{u^\infty,\omega}^{(0)}} \phi_1 \phi_2 + \int_{D_{u,\omega}} (2\omega - \text{tr} \chi) \phi_1 \phi_2.
\]

**Proposition 7.2** Given an \( r \)-tensorfield \( (^{(1)} \phi) \) and an \( (r-1) \)-tensorfield \( (^{(2)} \phi) \), there holds
\[
\int_{D_{u,\omega}} (^{(1)} \phi) \nabla_\omega A_r \cdot (^{(2)} \phi) A_{r-1} + \int_{D_{u,\omega}} \nabla_\omega (^{(1)} \phi) A_{r-1} \cdot (^{(2)} \phi) A_{r} = -(\eta + \varphi) (^{(1)} \phi)(^{(2)} \phi).
\]

Moreover, we shall require the following bound:

**Proposition 7.3** Suppose \( \phi \) is an \( r \)-tensorfield and let \( \lambda_1 = 2\lambda_0 - 1 \). Then
\[
2 \int_{D_{u,\omega}} |u|^2 \lambda_1 \phi (\nabla_\omega + \lambda_0 \text{tr} \chi) \phi = \int_{H_{u,\omega}^{(0)}} |u|^2 \lambda_1 \phi^2 - \int_{H_{u^\infty,\omega}^{(0)}} |u| \lambda_1 \phi^2 + \int_{D_{u,\omega}} |u|^2 \lambda_1 f \phi^2,
\]

where \( f \) satisfies the bound
\[
f \lesssim \frac{O}{|u|^2}.
\]
Proof. There holds
\[
\frac{d}{du} \left( \int_{S_{u,\omega}} |u|^{2\lambda_1} \Omega |\phi|^2 \right) = \mathcal{L} \left( \int_{S_{u,\omega}} |u|^{2\lambda_1} \Omega |\phi|^2 \right)
\]
\[
= \int_{S_{u,\omega}} \Omega^2 \left( 2|u|^{2\lambda_1} \langle \phi, \nabla_3 \phi + \lambda_0 \lambda_\chi \phi \rangle \right) + \int_{S_{u,\omega}} \Omega^2 \left( |u|^{2\lambda_1} \left(-\frac{2\lambda_1 (e_3(u))}{|u|} + (1 - 2\lambda_0)\lambda_\chi - 2\omega \right) |\phi|^2 \right).
\]
Here we have used that \( \mathcal{L} = \Omega e_3 = \frac{\partial}{\partial u} + b^A \frac{\partial}{\partial y^A} \). Immediate calculations imply that
\[
\left| -\frac{2\lambda_1 (e_3(u))}{|u|} + (1 - 2\lambda_0)\lambda_\chi - 2\omega \right| \lesssim \frac{O}{|u|^2}.
\]
The proposition then follows by integrating in the slab \( D_{u,\omega} \) and using the fundamental theorem of calculus. □

7.2 The Hodge structure as an aid for energy estimates

Observe that for \( (\mathcal{Q}_1, \mathcal{Q}_2) \in \left\{ (\alpha, \beta, (\tilde{\alpha}, (\rho, \sigma), ((\rho, \sigma), (\tilde{\alpha}, (\rho, \sigma))) \cup \left\{ (\alpha_F, (\rho_F, \sigma_F)), ((\rho_F, \sigma_F), \alpha_F) \right\} \right\} \) we can write the equations for \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) in the following form:
\[
\nabla_3 \mathcal{Q}_1 + \left( \frac{1}{2} + s_2(\mathcal{Q}_1) \right) \lambda_\chi \mathcal{Q}_1 - D\mathcal{Q}_1 = P_0, \quad (7.1)
\]
\[
\nabla_3 \mathcal{Q}_2 - *D\mathcal{Q}_1 = Q_0. \quad (7.2)
\]
Here by \( D \) we denote a differential operator on \( S_{u,\omega} \) and by \(*D\) its \( L^2 \)-adjoint. By commuting the above equations \( i \) times, we arrive at
\[
\nabla_3 \nabla^i \mathcal{Q}_1 + \left( \frac{i + 1}{2} + s_2(T_1) \right) \lambda_\chi \nabla^i \mathcal{Q}_1 - D\nabla^i \mathcal{Q}_1 = P_i, \quad (7.3)
\]
\[
\nabla_3 \nabla^i \mathcal{Q}_2 - *D\nabla^i \mathcal{Q}_1 = Q_i. \quad (7.4)
\]
The purpose of this section is to prove the following:

Proposition 7.4 Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3) and given a pair \( (\mathcal{Q}_1, \mathcal{Q}_2) \) satisfying
\[
\nabla_3 \nabla^i \mathcal{Q}_1 + \left( \frac{i + 1}{2} + s_2(T_1) \right) \lambda_\chi \nabla^i \mathcal{Q}_1 - D\nabla^i \mathcal{Q}_1 = P_i,
\]
\[
\nabla_3 \nabla^i \mathcal{Q}_2 - *D\nabla^i \mathcal{Q}_1 = Q_i,
\]
the following inequality holds:
\[
\int_{H^{0,\omega}_{\mathcal{Q}_1}} \|\nabla^i \mathcal{Q}_1\|_{L^2(S_u,\omega)}^2 \, du' + \int_{H^{0,\omega}_{\mathcal{Q}_2}} \frac{a}{|u'|^2} \|\nabla^i \mathcal{Q}_2\|_{L^2(S_u,\omega)}^2 \, du' \n\]
\[
\lesssim \int_{H^{0,\omega}_{\mathcal{Q}_1}} \|\nabla^i \mathcal{Q}_1\|_{L^2(S_u,\omega)}^2 \, du' + \int_{H^{0,\omega}_{\mathcal{Q}_2}} \frac{a}{|u'|^2} \|\nabla^i \mathcal{Q}_2\|_{L^2(S_u,\omega)}^2 \, du' + \int_{D_{u,\omega}} \frac{a}{|u'|^2} \|\nabla^i \mathcal{Q}_1\|_{L^2(S_u,\omega)} \, du' \, du'' + \int_{D_{u,\omega}} \frac{a}{|u'|^2} \|\nabla^i \mathcal{Q}_2\|_{L^2(S_u,\omega)} \, du' \, du''.
\]

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Proof. The Hodge structure will play a crucial role: For a pair \((\mathcal{Q}_1, \mathcal{Q}_2)\) or a pair \((\nabla^i \mathcal{Q}_1, \nabla^i \mathcal{Q}_2)\), the angular derivative operator \(D\) and its \(L^2\) adjoint operator \(D^*\) form a Hodge system. Through Proposition 7.2, we have

\[
\int_{\mathbb{R}^n} \mathcal{Q}_1 D^* \mathcal{Q}_2 + \mathcal{Q}_2 D^* \mathcal{Q}_1 = - \int_{\mathbb{R}^n} (\eta + \varphi) \mathcal{Q}_1 \mathcal{Q}_2, \tag{7.5}
\]

We now move forward and apply Proposition 7.3 for \(\nabla^i \mathcal{Q}_1\). With

\[
\lambda_0 = \frac{1 + i}{2} + s_2(\mathcal{Q}_1), \quad \lambda_1 := 2\lambda_0 - 1 = i + 2s_2(\mathcal{Q}_1),
\]

we get

\[
2 \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_1|^2 - \int_{D_{n,\omega}} |u|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_1|^2 + \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} f |\nabla^i \mathcal{Q}_1|^2,
\]

where \(|f| \leq O/|u'|^2\).

We also use Proposition 7.1, plugging in \(\phi_1 = \phi_2 = |u|^{i+2s_2(\mathcal{Q}_1)} \nabla^i \mathcal{Q}_2\)

\[
2 \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_2|^2 - \int_{D_{n,\omega}} |u|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_2|^2
\]

Adding (7.6) and (7.7), we obtain

\[
2 \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_1|^2 - \int_{D_{n,\omega}} |u|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_1|^2 + \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} f |\nabla^i \mathcal{Q}_1|^2
\]

\[
+ \int_{H_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_2|^2 - \int_{H_{n,\omega}} |u|^{2i+4s_2(\mathcal{Q}_1)} |\nabla^i \mathcal{Q}_2|^2
\]

\[
+ \int_{D_{n,\omega}} |u'|^{2i+4s_2(\mathcal{Q}_1)} (2\omega - \text{tr}\chi) |\nabla^i \mathcal{Q}_2|^2.
\]
We now take (7.3) and (7.4) into account. With the help of (7.5), we then arrive at

\[
\int_{H^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
= \int_{H^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
+ 2 \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1 \cdot P + 2 \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2 \cdot Q \\
- 2 \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} (= \eta + \eta) |\nabla^i \mathcal{Q}|_1 |\nabla^i \mathcal{Q}|_2 \\
+ \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} f |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} (2 \omega - \text{tr} \chi) |\nabla^i \mathcal{Q}|_2^2.
\]

Using \( |(\eta + \eta) |\nabla^i \mathcal{Q}|_1 |\nabla^i \mathcal{Q}|_2 | \leq |\eta + \eta| \cdot (|\nabla^i \mathcal{Q}|_1^2 + |\nabla^i \mathcal{Q}|_2^2) \), and the fact

\[
|\eta + \eta| \leq a^{\frac{1}{2}} O / |u|^2, \quad |f| \leq O / |u|^2, \quad |2 \omega - \text{tr} \chi| \leq O / |u|
\]

by applying Grönwall’s inequality twice (one for du, one for du), we obtain

\[
\left. \int_{H^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
\leq \int_{H^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
+ 2 \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1 \cdot P + 2 \int_{D_{\eta,\omega}} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2 \cdot Q.
\]

Multiplying by \( a^{-i-2s_2(\mathcal{Q})} \) on both sides, we get

\[
\left. \int_{H^{0,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
\leq \int_{H^{0,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1^2 \left. + \right. \int_{L^{0,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2^2 \\
+ 2 \int_{D_{\eta,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_1 \cdot P + 2 \int_{D_{\eta,\omega}} a^{-i-2s_2(\mathcal{Q})} |u|^{2i+4s_2(\mathcal{Q})} |\nabla^i \mathcal{Q}|_2 \cdot Q.
\]

Taking into account the signature identities

\[
s_2(\nabla^i \mathcal{Q})_1 = \frac{i}{2} + s_2(\mathcal{Q})_1, \quad s_2(\nabla^i \mathcal{Q})_2 = \frac{i + 1}{2} + s_2(\mathcal{Q})_2, \\
s_2(P) = s_2(\nabla_3 \nabla^i \mathcal{Q})_1 = \frac{i + 2}{2} + s_2(\mathcal{Q})_1, \quad s_2(Q) = s_2(D^* \nabla^i \mathcal{Q})_1 = \frac{i + 1}{2} + s_2(\mathcal{Q})_1,
\]

and definitions

\[
\|\phi\|_{L^2_x(S_u \omega)} = a^{-s_2(\phi)} |u|^{2s_2(\phi)} \|\phi\|_{L^2_x(S_u \omega)}, \\
\|\phi\|_{L^1_x(S_u \omega)} = a^{-s_2(\phi)} |u|^{2s_2(\phi) - 1} \|\phi\|_{L^1_x(S_u \omega)},
\]

we rewrite (7.8) as

\[
\left. \int_{H^{0,\omega}} \|\nabla^i \mathcal{Q}\|_{L^2_x(S_u \omega)}^2 \left. + \right. \int_{L^{0,\omega}} a \|\nabla^i \mathcal{Q}\|_{L^2_x(S_u \omega)}^2 \\
\leq \int_{H^{0,\omega}} \|\nabla^i \mathcal{Q}\|_{L^2_x(S_u \omega)}^2 \left. + \right. \int_{L^{0,\omega}} a \|\nabla^i \mathcal{Q}\|_{L^2_x(S_u \omega)}^2 \\
+ 2 \int_{D_{\eta,\omega}} a \|\nabla^i \mathcal{Q}\|_1 \cdot P \|L^2_x(S_u \omega) \left. + \right. 2 \int_{D_{\eta,\omega}} a \|\nabla^i \mathcal{Q}\|_2 \cdot Q \|L^2_x(S_u \omega).
\]

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Recalling the definitions
\[
\|\phi\|^2_{\mathcal{L}^2_{\omega}(H^{(0,\omega)})} := \int_0^\infty \|\phi\|^2_{\mathcal{L}^2_{\omega}(S_{\omega,\omega})} du' ,
\]
\[
\|\phi\|^2_{\mathcal{L}^2_{\omega}(H^{(\infty,\omega)})} := \int_{u_\infty}^u \frac{a}{|u'|^2} \|\phi\|^2_{\mathcal{L}^2_{\omega}(S'_{\omega,\omega})} du' ,
\]
and substituting them in the above, we arrive at the desired result.

\[\square\]

7.3 Energy estimates on the Maxwell components

Recall the null Maxwell equations
\[
\nabla_4 \mathbf{A} + \frac{1}{2} \text{tr} \chi \mathbf{F} = -\nabla \rho_F - \nabla \sigma_F - 2 \nabla \cdot \mathbf{A}_F, \tag{7.9}
\]
\[
\nabla_3 \mathbf{A} + \frac{1}{2} \text{tr} \chi \mathbf{F} = -\nabla \rho_F - \nabla \sigma_F - 2 \nabla \cdot \mathbf{A}_F, \tag{7.10}
\]
\[
\nabla_4 \rho_F = -\text{div} \mathbf{A}_F - \text{tr} \chi \mathbf{F} - (\eta - \bar{\eta}) \cdot \mathbf{A}_F, \tag{7.11}
\]
\[
\nabla_4 \sigma_F = -\text{curl} \mathbf{A}_F - \text{tr} \chi \mathbf{F} + (\eta - \bar{\eta}) \cdot \mathbf{A}_F, \tag{7.12}
\]
\[
\nabla_3 \rho_F + \text{tr} \chi \mathbf{F} = \text{div} \mathbf{A}_F + (\eta - \bar{\eta}) \cdot \mathbf{A}_F, \tag{7.13}
\]
\[
\nabla_3 \sigma_F + \text{tr} \chi \mathbf{F} = -\text{curl} \mathbf{A}_F + (\eta - \bar{\eta}) \cdot \mathbf{A}_F. \tag{7.14}
\]

Notice that for the pair \((Y_1, Y_2) = \{\alpha_F, (\rho_F, \sigma_F)\}\) we have
\[
\nabla_3 Y_1 + \left(\frac{1}{2} + s_2(Y_1)\right) \text{tr} \chi Y_1 - \text{tr} \mathbf{A}_F = \psi \cdot \mathbf{T} + \psi \cdot (Y, \alpha_F), \tag{7.15}
\]
\[
\nabla_4 Y_2 + \text{tr} \mathbf{A}_F = \psi \cdot (Y, \alpha_F), \tag{7.16}
\]
while for the pair \((Y_1, Y_2) = \{\rho_F, -\sigma_F, \mathbf{A}_F\}\) we have
\[
\nabla_3 Y_1 + \left(\frac{1}{2} + s_2(Y_1)\right) \text{tr} \chi Y_1 - \text{tr} \mathbf{A}_F = \psi \cdot \mathbf{T}, \tag{7.17}
\]
\[
\nabla_4 Y_2 - \text{tr} \mathbf{A}_F = \psi \cdot (Y, \alpha_F). \tag{7.18}
\]

We introduce the following proposition

**Proposition 7.5** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3) and given a pair \((Y_1, Y_2)\) satisfying
\[
\nabla_3 \nabla^i Y_1 + \left(\frac{i+1}{2} + s_2(Y_1)\right) \text{tr} \chi \nabla^i Y_1 - \nabla \nabla^i Y_2 = P,
\]
\[
\nabla_4 \nabla^i Y_2 - *\nabla \nabla^i Y_1 = Q,
\]
the following inequality holds:
\[
\int_{H^{(0,\omega)}} \|\nabla^i Y_1\|^2_{\mathcal{L}^2_{\omega}(S_{\omega,\omega})} du' + \int_{H^{(\infty,\omega)}} \frac{a}{|u'|^2} \|\nabla^i Y_2\|^2_{\mathcal{L}^2_{\omega}(S'_{\omega,\omega})} du'
\]
\[
\leq \int_{H^{(0,\omega)}} \|\nabla^i Y_1\|^2_{\mathcal{L}^2_{\omega}(S_{\omega,\omega})} du' + \int_{H^{(\infty,\omega)}} \frac{a}{|u'|^2} \|\nabla^i Y_2\|^2_{\mathcal{L}^2_{\omega}(S'_{\omega,\omega})} du'
\]
\[
+ \int_{D_u} \frac{a}{|u'|^2} \|\nabla^i Y_1 \cdot P\|_{\mathcal{L}^2_{\omega}(S'_{\omega,\omega})} du' du' + \int_{D_u} \frac{a}{|u'|^2} \|\nabla^i Y_2 \cdot Q\|_{\mathcal{L}^2_{\omega}(S'_{\omega,\omega})} du' du'.
\]

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We begin with the pair \((\alpha_F, (-\rho_F, \sigma_F))\).

**Proposition 7.6** Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), for \(i \leq 11\), we have

\[
\frac{1}{a^2} \|(a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(H^0_u, \mathcal{W})} + \frac{1}{a^2} \|(a^\frac{2}{3})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_c(H^{\infty, \infty})} \\
\leq \frac{1}{a^2} \|(a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(H^{0, 0}_u, \mathcal{W})} + \frac{1}{a^2} \|(a^\frac{2}{3})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_c(H^{\infty, \infty})} + \frac{1}{a^2}.
\]

**Proof.** We schematically have

\[
\nabla_3 \alpha_F + \frac{1}{2} \text{tr} \nabla \alpha_F - \mathcal{D}(-\rho_F, \sigma_F) = (\psi, \hat{\psi}) \cdot \nabla \psi + \psi \cdot (\nabla \alpha_F), \\
\nabla_4 (-\rho_F, \sigma_F) - \mathcal{D} \alpha_F = \psi \cdot (\nabla \alpha_F).
\]

Commuting with \(i\) angular derivatives we arrive at

\[
\nabla_4 \nabla^i \alpha_F + \frac{i + 1}{2} \text{tr} \nabla \alpha_F - \mathcal{D} \nabla^i (\rho_F, \sigma_F) \\
= \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} (\eta, \eta) \nabla^{i_4} (\rho_F, \sigma_F) + \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} \hat{\nabla^i \alpha_F} \\
+ \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} \chi \nabla^{i_4} \alpha_F + \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} \nabla \chi \nabla^{i_4} \alpha_F \\
+ \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} (\psi, \hat{\psi}) \nabla^{i_4} \alpha_F \\
:= P_1,
\]

while for \((\rho_F, \sigma_F)\) we similarly obtain

\[
\nabla_4 \nabla^i (\rho_F, \sigma_F) - \mathcal{D} \nabla^i \alpha_F \\
= \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} (\eta, \eta) \nabla^{i_4} (\rho_F, \sigma_F) + \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} \nabla \chi \nabla^{i_4} (\rho_F, \sigma_F) \\
+ \sum_{i_1, i_2, i_3, i_4 = i} \nabla^{i_1, i_2, i_3} (\eta, \eta) \nabla^{i_4} (\rho_F, \sigma_F) \\
:= Q_1.
\]

We arrive at

\[
\|(a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(H^0_u, \mathcal{W})} + \|(a^\frac{2}{3})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_c(H^{\infty, \infty})} \\
\leq \|(a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(H^{0, 0}_u, \mathcal{W})} + \|(a^\frac{2}{3})^{i-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_c(H^{\infty, \infty})} + N_1 + M_1,
\]

where

\[
N_1 = \int_0^u \int_{a_{\infty}}^u \frac{a}{|w|} \|(a^\frac{2}{3})^{i-1} P_1 \cdot (a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(S_{u, u'}, \mathcal{W})} du' du', \\
M_1 = \int_0^u \int_{a_{\infty}}^u \frac{a}{|w|} \|(a^\frac{2}{3})^{i-1} Q_1 \cdot (a^\frac{2}{3})^{i-1} \nabla^i \alpha_F\|_{L^2_c(S_{u, u'}, \mathcal{W})} du' du'.
\]

Let us focus on the term \(N_1\) first. Using the scale-invariant version of Hölder’s inequality we get
\[ N_1 \leq \int_0^\mu \int_0^{u_0} \frac{a}{|u|^2} \| (a^\perp)^{-1} P_1 \|_{L^2(S_{u', w})} \| (a^\perp)^{-1} \nabla^i \alpha_F \|_{L^2(S_{u', w})} \, du' \, du \\
\leq \int_0^\mu \frac{a}{|u|^2} \left( \int_0^\mu \| (a^\perp)^{-1} P_1 \|_{L^2(S_{u', w})}^2 \, du' \right)^{\frac{1}{2}} \, du' \cdot \sup_{u'} \| (a^\perp)^{-1} \nabla^i \alpha_F \|_{L^2(H_{u', w})}, \]

where

\[ P_1 = \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} (\eta, \frac{\eta}{\alpha}) \nabla^{i_4} (\rho_F, \sigma_F) + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} \nabla^{i_4} \alpha_F \\
+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} \nabla^{i_4} \alpha_F \\
+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} (\psi, \frac{\alpha}{\alpha'}) \nabla^{i_4} \alpha_F. \]

Denote

\[ H_1 = \int_0^\mu \| (a^\perp)^{-1} P_1 \|_{L^2(S_{u', w})}^2 \, du' \]

We further have

\[ H_1 = \int_0^\mu \| (a^\perp)^{-1} P_1 \|_{L^2(S_{u', w})}^2 \, du' \leq 5 \sum_{j=1}^5 \int_0^\mu \| (a^\perp)^{-1} P_{1j} \|_{L^2(S_{u', w})}^2 \, du' \]

We treat each of those five terms separately.

- There holds

\[ \int_0^\mu \| (a^\perp)^{-1} P_{i1} \|_{L^2(S_{u', w})}^2 \, du' \]

\[ = \int_0^\mu \| (a^\perp)^{-1} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} (\eta, \frac{\eta}{\alpha}) \nabla^{i_4} (\rho_F, \sigma_F) \|_{L^2(S_{u', w})}^2 \, du' \]

\[ \leq \frac{O^4}{a^2} \int_0^\mu \| a^{5} \nabla^{11} (\eta, \frac{\eta}{\alpha}) \|_{L^2(S_{u', w})}^2 \, du' + \frac{O^2}{|u|^2} \int_0^\mu \| a^{5} \nabla^{11} (\rho_F, \sigma_F) \|_{L^2(S_{u', w})}^2 \, du' \]

Here we have used Propositions 6.5 and 6.6 as well as the bootstrap bounds \((3.3)\).

- There holds

\[ \int_0^\mu \| (a^\perp)^{-1} P_{i2} \|_{L^2(S_{u', w})}^2 \, du' \]

\[ = \int_0^\mu \| (a^\perp)^{-i} \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi_j^2 \nabla^{i_3} \nabla^{i_4} (\alpha_F, \frac{\alpha}{\alpha'}) \|_{L^2(S_{u', w})}^2 \, du' \]

\[ \leq \frac{O^4}{|u|^2} + \frac{a \cdot O^2 \cdot \int_0^\mu \| a^{5} \nabla^{11} \|_{L^2(S_{u', w})}^2 \, du' + \frac{a \cdot O^2}{|u|^2} \int_0^\mu \| a^{5} \nabla^{11} a_F \|_{L^2(S_{u', w})}^2 \, du' \]
There holds
\[
\int_0^\infty \|(a^{\hat{\beta}})^{i-1} P_{i3}\|^2_{L^2_v(S_{\nu',\psi})} \, du' = \int_0^\infty \|(a^{\hat{\beta}})^i \sum_{i_1+i_2+i_3+i_4=3} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left( \frac{\dot{\psi}}{a^{\hat{\beta}}} \right) \nabla^{i_4} \varphi F \|^2_{L^2_v(S_{\nu',\psi})} \, du' \\
\lesssim \frac{O^4}{|u'|^2} + \frac{a \cdot O^2}{|u'|^2} \int_0^\infty \|a^5 \nabla^{i_1} \| \|_{L^2_v(S_{\nu',\psi})} \, du' + \frac{a \cdot O^2}{|u'|^2} \int_0^\infty \|a^5 \nabla^{i_1} \varphi F \|^2_{L^2_v(S_{\nu',\psi})} \, du'.
\]
Here we have made use of Proposition 6.3.

The last two terms can be bounded above by
\[
F^2[\alpha_F] \cdot O[\dot{\varphi}]^2 + 1,
\]
using Grönwall’s inequality and the elliptic estimates.

We arrive at the bound
\[
N_1 \lesssim \left(F[\alpha_F] \cdot O[\dot{\varphi}] + 1 \right) \sup_{u'} \| (a^{\hat{\beta}})^{i-1} \nabla^i \varphi F \|_{L^2_v(S_{\nu',\psi})}. \tag{7.20}
\]

For the term $M_1$ we follow the same procedure. We have
\[
M_1 = \int_0^\infty \int_{u_{\infty}}^u \frac{a}{|u'|^2} \|(a^{\hat{\beta}})^{i-1} Q_1 \cdot (a^{\hat{\beta}})^{i-1} \nabla^i (\rho_F, \sigma_F) \|_{L^2_v(S_{\nu',\psi})} \, du' \, du'' \\
\leq \int_0^\infty \int_{u_{\infty}}^u \frac{a}{|u'|^2} \|(a^{\hat{\beta}})^{i-1} Q_1 \|_{L^2_v(S_{\nu',\psi})} \|(a^{\hat{\beta}})^{i-1} \nabla^i (\rho_F, \sigma_F) \|_{L^2_v(S_{\nu',\psi})} \, du' \, du'' \\
\leq \left( \int_{u_{\infty}}^u \frac{a}{|u'|^2} \|(a^{\hat{\beta}})^{i-1} Q_1 \|_{L^2_v(S_{\nu',\psi})}^2 \, du' \right)^{\frac{1}{2}} \cdot \sup_{u'} \| (a^{\hat{\beta}})^{i-1} \nabla^i (\rho_F, \sigma_F) \|_{L^2_v(S_{\nu',\psi})} \\
\leq \left( \int_{u_{\infty}}^u \frac{a}{|u'|^2} \|(a^{\hat{\beta}})^{i-1} Q_1 \|_{L^2_v(S_{\nu',\psi})}^2 \, du' \right)^{\frac{1}{2}} \sup_{u'} \| (a^{\hat{\beta}})^{i-1} \nabla^i (\rho_F, \sigma_F) \|_{L^2_v(S_{\nu',\psi})},
\]
Here we recall that
\[
Q_1 = \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \tilde{\eta}) \nabla^{i_4} \varphi F + \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \varphi \nabla^{i_4} (\rho_F, \sigma_F) \\
+ \sum_{i_1+i_2+i_3+i_4=4} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \tilde{\eta}) \nabla^{i_4} (\rho_F, \sigma_F)
\]
\[
:= Q_{11} + Q_{12} + Q_{13}.
\]
Let
\[
J_1 = \int_0^\infty \int_{u_{\infty}}^u \frac{a}{|u'|^2} \|(a^{\hat{\beta}})^{i-1} Q_1 \|_{L^2_v(S_{\nu',\psi})}^2 \, du' \, du'' := J_{11} + J_{12} + J_{13}.
\]

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We have

$$J_1 \lesssim \sum_{j=1}^{3} \int_{0}^{u} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_{ij} \|^2_{L^2_{x}(S_{\omega' \omega})} \, du \, du'. \quad (7.21)$$

Then, separating the cases where \( i_4 \neq i \) and \( i_4 = i \) and treating each of the three terms separately, we get

- There holds
  
  $$J_{11} = \int_{0}^{u} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \eta') \nabla^{i_4} \alpha_{F}\|_{L^2_{x}(S_{\omega' \omega})}^2 \, du \, du'$$
  
  $$= \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( \int_{0}^{u} \|(a^{\frac{1}{2}})^{i} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \eta') \nabla^{i_4} \left( \frac{\alpha_{F}}{a^{\frac{1}{2}}} \right)\|_{L^2_{x}(S_{\omega' \omega})}^2 \right) \, du'$$
  
  $$\lesssim \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( [O^4] \frac{O^2 \cdot F[\alpha_F]^2}{|u'|^2} + \frac{O^2 \cdot (1 + R)^2}{a} \right) \, du' \lesssim 1.$$

We have made use of Propositions 6.5 and 6.6 here.

- There holds
  
  $$J_{12} = \int_{0}^{u} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} (\rho_F, \sigma_F)\|_{L^2_{x}(S_{\omega' \omega})}^2 \, du \, du'$$
  
  $$= \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( \int_{0}^{u} \|(a^{\frac{1}{2}})^{i} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \nabla^{i_4} (\rho_F, \sigma_F)\|_{L^2_{x}(S_{\omega' \omega})}^2 \right) \, du'$$
  
  $$\lesssim \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( [O^4] \frac{O^2 \cdot F[\rho_F, \sigma_F]^2}{|u'|^2} + \frac{O^2 \cdot (1 + R)^2}{|u'|^2} \right) \, du' \lesssim 1.$$

We have made use of Propositions 6.5 and 6.6 as well as the bootstrap assumptions (3.3).

- There holds
  
  $$J_{13} = \int_{0}^{u} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\eta, \eta') \nabla^{i_4} (\rho_F, \sigma_F)\|_{L^2_{x}(S_{\omega' \omega})}^2 \, du \, du'$$
  
  $$= \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( \int_{0}^{u} \|(a^{\frac{1}{2}})^{i} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \left( \frac{\eta \cdot \eta'}{a^{\frac{1}{2}}} \right) \nabla^{i_4} (\rho_F, \sigma_F)\|_{L^2_{x}(S_{\omega' \omega})}^2 \right) \, du'$$
  
  $$\lesssim \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \left( [O^4] \frac{O^2 \cdot F[\rho_F, \sigma_F]^2}{|u'|^2} + \frac{O^2 \cdot (1 + R)^2}{|u'|^2} \right) \, du' \lesssim 1.$$

Here we have made use of Propositions 6.3, 6.5 and 6.6 as well as the bootstrap bounds 3.3.

Hence

$$M_1 \leq \sup_{u} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i} (\rho_F, \sigma_F)\|_{L^2_{x}(S_{\omega' \omega})}^2. \quad (7.22)$$

Taking the bounds (7.20) and (7.22) into account, we get

$$a^{-1} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i} \alpha_{F}\|_{L^2_{x}(H^{(\omega')}_{\omega, \omega})}^2 + a^{-1} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i} (\rho_F, \sigma_F)\|_{L^2_{x}(H^{(\omega')}_{\omega, \omega})}^2 \leq a^{-1} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i} \alpha_{F}\|_{L^2_{x}(H^{(\omega')}_{\omega, \omega})}^2 + a^{-1} \|(a^{\frac{1}{2}})^{i-1} \nabla^{i} (\rho_F, \sigma_F)\|_{L^2_{x}(H^{(\omega')}_{\omega, \omega})}^2 + a^{-1} (N_1 + M_1)$$

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We recall the following schematic equations for the pair $(\mathcal{Y}_{1,2})$:

\begin{align*}
\nabla_3 \mathcal{Y}_1 + \left( \frac{1}{2} + s_2(\mathcal{Y}_1) \right) \text{tr} \mathcal{Y}_1 - \mathcal{D}_1 \mathcal{Y}_2 &= (\eta, \eta) \cdot \alpha_F, \\
\nabla_4 \mathcal{Y}_2 - *\mathcal{D}_1 \mathcal{Y}_1 &= (\eta, \eta) \cdot (\rho_F, \sigma_F) + \omega \cdot \mathcal{A}_F + \hat{\chi} \cdot \alpha_F.
\end{align*}

(7.24)

(7.25)

Commuting these equations $i$ times with angular derivatives $\nabla$ we arrive at the equation

\begin{align*}
\nabla_3 \nabla^i \mathcal{Y}_1 + \left( \frac{i+1}{2} + s_2(\mathcal{Y}_1) \right) \text{tr} \nabla^i \mathcal{Y}_1 - \mathcal{D} \nabla^i \mathcal{Y}_2 \\
= \sum_{i_1+i_2+i_3=i} \nabla^{i_1+i_2} \nabla^{i_3} (\eta, \eta) \nabla^{i_4} \mathcal{A}_F + \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1+i_2+i_3+i_4} \nabla^{i_4} (\rho_F, \sigma_F) \\
+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1+i_2+i_3+i_4} (\eta, \eta, \hat{\chi}, \text{tr} \mathcal{Y}_2) \nabla^{i_4} (\rho_F, \sigma_F) \\
:= P_2,
\end{align*}

as well as the equation

\begin{align*}
\nabla_4 \nabla^i \mathcal{Y}_2 - *\mathcal{D}_1 \nabla^i \mathcal{Y}_1 \\
= \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1+i_2} \nabla^{i_3} (\eta, \eta) \nabla^{i_4} (\rho_F, \sigma_F) + \sum_{i_1+i_2+i_3=i} \nabla^{i_1+i_2+i_3} \omega \nabla^{i_4} \mathcal{A}_F \\
+ \sum_{i_1+i_2+i_3=i} \nabla^{i_1+i_2+i_3} \hat{\chi} \nabla^{i_4} \alpha_F \\
+ \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1+i_2+i_3+i_4} (\eta, \eta, \hat{\chi}) \nabla^{i_4} \alpha_F \\
:= Q_2.
\end{align*}

Applying the proposition, we arrive at
\begin{align}
\|(a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)}^2 & + \|(a \hat{\psi})^{-1} \nabla^i \bs{\alpha}_{F}\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)}^2
\leq & \|(a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)}^2 + \|(a \hat{\psi})^{-1} \nabla^i \bs{\alpha}_{F}\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)}^2 + N_2 + M_2,
\end{align}

where

\begin{align}
N_2 &= \int_0^u \int_{u_\infty}^u \frac{a}{|u'|} \|(a \hat{\psi})^{-1} P_2 \cdot (a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' du', \\
M_2 &= \int_0^u \int_{u_\infty}^u \frac{a}{|u'|} \|(a \hat{\psi})^{-1} Q_2 \cdot (a \hat{\psi})^{-1} \nabla^i \bs{\alpha}_{F}\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' du'.
\end{align}

We focus on $N_2$ first. Using the same reasoning as for $N_1$, we have

\begin{align}
N_2 &= \int_0^u \int_{u_\infty}^u \frac{a}{|u'|} \|(a \hat{\psi})^{-1} P_2 \cdot (a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' du'
\leq & \int_0^u \int_{u_\infty}^u \frac{a}{|u'|} \|(a \hat{\psi})^{-1} P_2 \|_{L^2_{\mathcal{E}}(S_{u', \omega})} \cdot \|(a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' du'.
\end{align}

Recall at this point the form that $P_2$ assumes:

\begin{align*}
P_2 &= \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi_{i_2} \nabla^{i_3} \bs{\eta} \nabla^{i_4} \bs{\alpha}_{F} + \sum_{i_1 + i_2 + i_3 + i_4 + 1 = i} \nabla^{i_1} \psi_{i_2} \nabla^{i_3} (\hat{x}, \bs{\tau} \bs{\chi}) \nabla^{i_4} (\rho_F, \sigma_F)
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi_{i_2} \nabla^{i_3} (\bs{\eta}, \bs{\chi} \bs{\tau}) \nabla^{i_4} (\rho_F, \sigma_F) := P_{21} + P_{22} + P_{23}.
\end{align*}

Consequently, we have the bound

\begin{align}
N_2 \leq & \sum_{j=1}^3 \int_0^u \int_{u_\infty}^u \frac{a}{|u'|} \|(a \hat{\psi})^{-1} P_{2j} \|_{L^2_{\mathcal{E}}(S_{u', \omega})} \cdot \|(a \hat{\psi})^{-1} \nabla^i (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' du' \\
&= N_{21} + N_{22} + N_{23}.
\end{align}

We estimate each term separately.

- There holds

\begin{align}
N_{21} &= \int_0^u \int_{u_\infty}^u \frac{a}{|u'|^2} \int_0^u \frac{O^4}{a \cdot |u'|^2} + \int_0^u \frac{O^2}{|u'|^2} \|a^5 \nabla^{11} (\bs{\eta}, \bs{\eta})\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)}^2 \cdot \|(a \hat{\psi})^{-1} (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)} du' du'
\leq & \int_0^u \frac{O^4}{|u'|^2} \cdot \|(a \hat{\psi})^{-1} (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)} du' + \int_0^u \frac{O^2}{|u'|^2} \|a^5 \nabla^{11} \bs{\alpha}_{F}\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' \cdot \|(a \hat{\psi})^{-1} (\rho_F, \sigma_F)\|_{L^2_{\mathcal{E}}(H^0_\infty, \omega)} \cdot \bs{F} du'
\leq & \int_0^u \frac{O^4}{|u'|^2} \cdot \bs{F} du' + \int_0^u \frac{O^2}{|u'|^2} \|a^5 \nabla^{11} \bs{\alpha}_{F}\|_{L^2_{\mathcal{E}}(S_{u', \omega})} du' \cdot \bs{F} du'.
\end{align}
Here $F$ is the bootstrap constant appearing in (3.3). Notice that what we have done in the above is to separate between three cases. The first is when neither $(\eta, \tilde{\eta})$ nor $\alpha_F$ have 11 derivatives, the second is when 11 derivatives fall on $(\eta, \tilde{\eta})$ and finally the third is for when 11 derivatives fall on $\alpha_F$. The reason for this distinction is that we use Hölder’s inequality in different directions, depending on what elliptic estimates and Maxwell norms we have. Making use of Propositions 6.5 and 6.6, we conclude that the last two terms can be bounded above by 1, using the section on elliptic estimates. In particular,

$$N_{21} \lesssim 1.$$  

- There holds

$$N_{22} = \int_{u_\infty}^{u} \frac{a \cdot F}{|u'|^2} \left( \int_0^{|a|} (a^{\frac{1}{2}})^{-1} \sum_{i_1 + i_2 + i_3 + i_4 + 1 = 1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} (\tilde{\chi}, \text{tr}\chi) \nabla^{i_4} (\rho_F, \sigma_F) \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})}^2 du' \right)^{\frac{1}{2}} du'$$

$$= \int_{u_\infty}^{u} \left( \int_0^{|a|} (a^{\frac{1}{2}})^{-1} \sum_{i_1 + i_2 + i_3 + i_4 + 1 = 1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} \left( \frac{a}{|u|^2} \tilde{\chi}, \frac{a}{|u|^2} \text{tr}\chi \right) \nabla^{i_4} (\rho_F, \sigma_F) \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})}^2 du' \right)^{\frac{1}{2}} du' \cdot F$$

$$\lesssim \int_{u_\infty}^{u} \frac{1}{a^{\frac{3}{2}}} \frac{|u|^2}{|u'|^2} du' \cdot F \lesssim 1.$$  

- There holds

$$N_{23} = \int_{u_\infty}^{u} \int_0^{|a|} \frac{a}{|u|^2} (a^{\frac{1}{2}})^{-1} P_{23} \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})} (a^{\frac{1}{2}})^{-1} \nabla^{i} (\rho_F, \sigma_F) \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})} du'.$$

We first distinguish between the cases where $i < 11$ and $i = 11$. For the case $i < 11$, there holds

$$\| (a^{\frac{1}{2}})^{-1} \sum_{i_1 + i_2 + i_3 + i_4 + 1 = 1} \nabla^{i_1} \psi^{i_2+1} \nabla^{i_3} (\eta, \tilde{\eta}, \tilde{\chi}, \text{tr}\chi) \nabla^{i_4} (\rho_F, \sigma_F) \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})} \lesssim \frac{1}{a^{\frac{3}{2}}} \cdot \frac{|u|}{|u'|^2} \cdot O^2 \lesssim \frac{O^2}{a}.$$  

Moreover, since $i < 11$, we can bound

$$\| (a^{\frac{1}{2}})^{-1} \nabla^{i} (\rho_F, \sigma_F) \|_{\mathcal{L}^{2}_{(\tilde{\eta}, \chi)}(S_{u', \omega'})} \lesssim \frac{O}{a^{\frac{3}{2}}}.$$  

Hence, when $i < 11$, it is easy to establish that $N_{23} \lesssim 1$. When $i = 11$, we distinguish between four cases. The first one is when neither $(\eta, \tilde{\eta}, \tilde{\chi}, \text{tr}\chi)$ nor $(\rho_F, \sigma_F)$ have 11 derivatives. The second is when 11 derivatives fall on $(\rho_F, \sigma_F)$. The third is when 11 derivatives fall on $(\eta, \tilde{\eta})$ and the fourth is when 11 derivatives fall on $(\tilde{\chi}, \text{tr}\chi)$. We treat these cases below:
\[ N_{23} \lesssim \int_{u_{\infty}}^{\mu} \int_{0}^{u} \frac{a}{|u'|^2} \cdot \frac{\partial^2}{a^2} \cdot \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ + \int_{u_{\infty}}^{\mu} \int_{0}^{u} \frac{a}{|u'|^2} \cdot \frac{\partial}{|u'|} \cdot \frac{O}{a} \cdot \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|^2_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ + \int_{u_{\infty}}^{\mu} \int_{0}^{u} \frac{a}{|u'|^2} \cdot \frac{\partial}{|u'|} \cdot \|a^5 \nabla^{11} (\eta, \tilde{x})\|_{L^2(\mathcal{S}^{(a)})} \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ \lesssim 1 + \int_{u_{\infty}}^{\mu} \frac{a \cdot O}{|u'|} \cdot \|a^5 \nabla^{11} (\eta, \tilde{x})\|_{L^2(H^{(a)}_\omega)} \cdot \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ + \sup_{\bar{u}} \left( \int_{u_{\infty}}^{\mu} \frac{a^2 O^2}{a^2} \cdot \|a^5 \nabla^{11} (\eta, \tilde{x})\|^2_{L^2(\mathcal{S}^{(a)})} \, du' \right)^\frac{1}{2} \cdot \|a^5 \nabla^{11} (\rho_F, \sigma_F)\|_{L^2(\mathcal{S}^{(a)})} \]
\[ \lesssim 1 + \mathcal{T}^{(0)} \cdot (\mathcal{R} + \mathcal{R} + 1) \lesssim (\mathcal{T}^{(0)})^2 + \mathcal{R}^2 + \mathcal{R}^2 + 1. \]

(7.29)

Here we have used the fact that \( O_{\infty} [\rho_F, \sigma_F] \lesssim 1 \) from Proposition 4.11, the energy estimates on \( \mathcal{F}[\rho_F, \sigma_F] \) to bound the term by the initial data, as well as Proposition 6.8 and in particular (6.57).

Combining these estimates, we arrive at

\[ N_{23} \lesssim (\mathcal{T}^{(0)})^2 + \mathcal{R}^2 + \mathcal{R}^2 + 1. \]

Putting everything together, there holds

\[ N_2 \lesssim (\mathcal{T}^{(0)})^2 + \mathcal{R}^2 + \mathcal{R}^2 + 1. \]

We move on to \( M_2 \). We have

\[ \int_{u_{\infty}}^{\mu} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} Q_2 \cdot (\partial^4)^{-1} \nabla^4 \mathbf{F}_2 \|_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ \lesssim \int_{u_{\infty}}^{\mu} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} Q_2 \|_{L^2(\mathcal{S}^{(a)})} \cdot (\partial^4)^{-1} \nabla^4 \mathbf{F}_2 \|_{L^2(\mathcal{S}^{(a)})} \, du' \, du' \]
\[ \lesssim \left( \int_{u_{\infty}}^{\mu} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} Q_2 \|_{L^2(\mathcal{S}^{(a)})} \right)^\frac{1}{2} \left( \int_{u_{\infty}}^{\mu} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} \nabla^4 \mathbf{F}_2 \|_{L^2(\mathcal{S}^{(a)})} \right)^\frac{1}{2} \, du' \]
\[ \lesssim \left( \int_{u_{\infty}}^{\mu} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} Q_2 \|_{L^2(\mathcal{S}^{(a)})} \right)^\frac{1}{2} \cdot \sup_{\bar{u}} (\partial^4)^{-1} \nabla^4 \mathbf{F}_2 \|_{L^2(\mathcal{S}^{(a)})}. \]

Denote

\[ H_2 = \int_{u_{\infty}}^{\mu} \int_{u_{\infty}}^{u} \frac{a}{|u'|^2} \cdot (\partial^4)^{-1} Q_2 \|_{L^2(\mathcal{S}^{(a)})} \, du' \, du'. \]

Recall at this point that
Thus,

\[ Q_2 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \xi) \nabla^{i_4} \rho_F, \sigma_F + \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \omega \nabla^{i_4} \alpha_F \]

\[ + \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \xi \nabla^{i_4} \alpha_F + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3}(\eta, \xi, \chi) \nabla^{i_4} \alpha_F := Q_{21} + Q_{22} + Q_{23} + Q_{24}. \]

Thus,

\[ H_2 \leq \int_0^u \int_{u_{\infty}} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} Q_2\|_{L^2(S_u, \omega)}^2 \, du' \, du'. \]  

(7.30)

We estimate term by term.

- The first two terms

\[ \int_0^u \int_{u_{\infty}} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} (Q_{21}, Q_{22})\|_{L^2(S_u, \omega)}^2 \, du' \, du' \]

can be bounded by 1 as before.

- For the third term, there holds

\[ \int_0^u \int_{u_{\infty}} \frac{a}{|u'|^2} \|(a^{\frac{1}{2}})^{i-1} \sum_{i_1 + i_2 + i_3 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \omega \|_{L^2(S_u, \omega)}^2 \, du' \, du' \]

\[ \leq \int_0^u \int_{u_{\infty}} \frac{1}{a} \frac{|u'|^2}{|u|^2} \cdot a \cdot \frac{O(|\omega|)^2 |\omega_F|^2}{|u'|^2} \, du' + \int_{u_{\infty}}^u \frac{|u'|^2}{a} \cdot \frac{O^2}{|u|} \left( \int_0^u \|(a^{\frac{1}{2}})^{i-1} \|_{L^2(S_u, \omega)}^2 \, du' \right) \, du' \]

\[ + \int_0^u \left( \int_{u_{\infty}}^u \frac{a}{|u'|^2} \cdot \frac{O}{|u'|} \cdot \frac{|u'|^2}{a} \cdot \|a^{\frac{1}{2}} \nabla^{i} \left( \frac{a^{\frac{1}{2}}}{|u'|^{\frac{1}{2}}} \right) \|_{L^2(S_u, \omega)}^2 \, du' \right) \, du' \]

\[ \leq 1 + (R^2 + R^2 + 1) \cdot (F^2 |\omega_F| + F^2 |\rho_F, \sigma_F| + 1) + \int_0^u \int_{u_{\infty}} \frac{a^2 \cdot O_{\infty} |\omega_F|^2}{|u|^4} \cdot \|a^{\frac{1}{2}} \nabla^{i} \|_{L^2(S_u, \omega)}^2 \, du' \, du' \]

We focus on the term

\[ \int_{u_{\infty}}^u \frac{a^2 \cdot O_{\infty} |\omega_F|^2}{|u'|^4} \|a^{\frac{1}{2}} \nabla^{i} \|_{L^2(S_u, \omega)}^2 \, du' \leq \int_{u_{\infty}}^u \frac{a^2}{|u'|^4} \|a^{\frac{1}{2}} \nabla^{i} \|_{L^2(S_u, \omega)}^2 \, du'. \]

Recall that, from the proof of Proposition 6.8, there holds

\[ \|a^{\frac{1}{2}} \|_{L^2(S_u, \omega)}^2 \leq \sum_{j \leq 10} \left( \left\| (a^{\frac{1}{2}})^{j} \nabla^{j+1} t \right\|_{L^2(S_u, \omega)}^2 + \left\| (a^{\frac{1}{2}}) \nabla^{j} \right\|_{L^2(S_u, \omega)}^2 \right) \]

\[ + \left\| (a^{\frac{1}{2}})^{j} \sum_{j_1 + j_2 = j} \nabla^{j_1}(\eta, \xi) \nabla^{j_2}(\chi, \delta) \right\|_{L^2(S_u, \omega)}^2 + \sum_{j \leq 10} \frac{1}{a^2} \cdot \left\| (a^{\frac{1}{2}})^{j} \right\|_{L^2(S_u, \omega)}^2. \]

This implies that

\[ \|a^{\frac{1}{2}} \|_{L^2(S_u, \omega)}^2 \]

\[ \leq \sum_{j \leq 10} \left( \left\| (a^{\frac{1}{2}})^{j} \nabla^{j+1} t \right\|_{L^2(S_u, \omega)}^2 + \left\| (a^{\frac{1}{2}}) \nabla^{j} \right\|_{L^2(S_u, \omega)}^2 \right) \]

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We also know that the last term can be absorbed by the left-hand side. Thus, consequently, we have

\[
\begin{align*}
\frac{u^2}{a^2} \sum_{j_1 + j_2 = j} \nabla^{j_1}(\eta, \eta) \nabla^{j_2} \left( \frac{a}{|u|^2} \nabla^2 \chi \right) \| \nabla^{2j} \|_{L^2(\mathbb{R}^n)}^2 + \sum_{j \leq 10} \| (a^{\frac{1}{2}} \nabla)^j \left( \frac{\chi}{a^{\frac{1}{2}}} \right) \|_{L^2(S_{\mathbb{R}^n})}^2 \\
\lesssim \frac{|u|^2}{a^2} (R + R^2 + 1)^2 + \| (a^{\frac{1}{2}} \nabla)^j \|_{L^2(S_{\mathbb{R}^n})}^2 + \frac{|u|^4}{|u|^2} + \frac{1}{a^2} \cdot \frac{|u|^2}{a^2} \\
\lesssim \frac{|u|^2}{a^2} R^2 + \| (a^{\frac{1}{2}} \nabla)^j \|_{L^2(S_{\mathbb{R}^n})}^2 + \frac{|u|^4}{|u|^2} + \frac{1}{a^2} \cdot \frac{|u|^2}{a^2}.
\end{align*}
\]

Multiplying the above by \( \frac{a^2}{|u|^2} \) and taking the integral in the incoming direction, we have

\[
\int_{i\infty}^{u} \frac{a^2}{|u|^2} \| a^{\frac{1}{2}} \nabla t \chi \|_{L^2(S_{\mathbb{R}^n})}^2 \, du' \lesssim \int_{i\infty}^{u} \frac{R^2}{|u|^2} \, du' + \int_{i\infty}^{u} \frac{a^2}{|u|^2} \| (a^{\frac{1}{2}} \nabla)^j \|_{L^2(S_{\mathbb{R}^n})}^2 \, du' \\
+ \int_{i\infty}^{u} \frac{O^4 + 1}{|u|^2} \, du' \lesssim R^2 + O^4 + 1 + \frac{16}{a^2} R |\tilde{\beta}|^2 \lesssim 1.
\]

Here we have used Propositions 4.10 and 6.8 as well as the bootstrap bounds (3.3).

- The fourth term can be bounded by 1 using the same procedures as above.

Consequently,

\[
H_2 \leq 1 + (R^2 + R^2 + 1) \cdot (\mathcal{F}^2[\sigma_F] + \mathcal{F}^2[\rho_F, \sigma_F] + 1)
\]

and also

\[
M_2 \leq H_2 \| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})} \| \leq H_2 + \frac{1}{4} \| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})}.
\]

We also know that \( N_2 \leq 1 \). Putting everything together, we have

\[
\| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})}^2 \leq H_2 + \frac{1}{4} \| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})}.
\]

The last term can be absorbed by the left-hand side. Thus,

\[
\| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})}^2 \leq H_2 + \frac{1}{4} \| (a^{\frac{1}{2}})^{i-1} \nabla^i \chi \|_{L^2(S_{\mathbb{R}^n})}.
\]

Thus, we arrive at the energy inequality

\[
\mathcal{F}^2[\rho_F, \sigma_F] + \mathcal{F}^2[\rho_F] \leq \mathcal{F}^2[\rho_F, \sigma_F] + \mathcal{F}^2[\rho_F] + (1 + R^2 + R^2) (\mathcal{F}^2[\sigma_F] + \mathcal{F}^2[\rho_F, \sigma_F] + a^{-\frac{1}{2}})
\]

\[
\leq (\tau^0)^2 + (1 + R^2 + R^2) ((\tau^0)^2 + \frac{1}{a^2}) \leq (\tau^0)^2 + (R^4 + R^4 + 1) + ((\tau^0)^4 + \frac{1}{a^2})
\]

\[
\lesssim R^4 + R^4 + (\tau^0)^4 + (\tau^0)^2.
\]

Combining (7.31) and (7.34) we get the following
Theorem 7.1 Under the assumptions of Theorem 1.2 and the bootstrap bounds (3.3), there holds
\[ \mathcal{F} + \mathcal{E} \lesssim \mathcal{R}^2 + \mathcal{R}^2 + (\mathcal{I}^{(0)})^2 + (\mathcal{I}^{(0)}) + 1. \]

7.4 Energy estimates for curvature

Again, for \((\Psi_1, \Psi_2) \in \{(\alpha, \beta), (\hat{\beta}, (\rho, \sigma)), ((\rho, \sigma), \tilde{\beta}), (\tilde{\beta}, \alpha)\}\) we have the following:

Proposition 7.8 Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), assuming we have a pair \((\Psi_1, \Psi_2)\) satisfying
\[
\nabla_3 \nabla^i \Psi_1 + \left( \frac{1 + i}{2} + s_2(\Psi_1) \right) \text{tr} \nabla \nabla^i \Psi_1 - \mathcal{D} \nabla^i \Psi_2 = P, \\
\nabla_4 \nabla^i \Psi_2 - * \mathcal{D} \nabla^i \Psi_1 = Q, 
\]
with \((\mathcal{D}, * \mathcal{D})\) forming a Hodge dual, it follows
\[
\int_{\Omega} \| \nabla^i \Psi_1 \|^2_{L^2(\mathcal{D}_u^\infty \omega)} \, du' + \int_{\Omega} \| \nabla^i \Psi_2 \|^2_{L^2(\mathcal{D}_u^\infty \omega)} \, du' 
\leq \int_{\Omega} \| \nabla^i \Psi_1 \|^2_{L^2(\mathcal{D}_u^\infty \omega)} \, du' + \int_{\Omega} \| \nabla^i \Psi_2 \|^2_{L^2(\mathcal{D}_u^\infty \omega)} \, du' 
+ \int_{\Omega} \frac{a}{|u'|} \| \nabla^i \Psi_1 \cdot P \| L^1(\mathcal{D}_u^\infty \omega) \, du' \, du' + \int_{\Omega} \frac{a}{|u'|} \| \nabla^i \Psi_2 \cdot Q \| L^1(\mathcal{D}_u^\infty \omega) \, du' \, du'.
\]

With this in mind, we begin by considering the pair \((\alpha, \tilde{\beta})\):

Proposition 7.9 Under the assumptions of Theorem 1.2 and the bootstrap assumptions (3.3), we have for \(i \leq 10\) the following:
\[
\frac{1}{a^2} \| (a^2 \nabla)^i \alpha \|_{L^2(\mathcal{D}_u^\infty \omega)} + \frac{1}{a^2} \| (a^2 \nabla)^i \tilde{\beta} \|_{L^2(\mathcal{D}_u^\infty \omega)} 
\leq \frac{1}{a^2} \| (a^2 \nabla)^i \alpha \|_{L^2(\mathcal{D}_u^\infty \omega)} + \frac{1}{a^2} \| (a^2 \nabla)^i \tilde{\beta} \|_{L^2(\mathcal{D}_u^\infty \omega)} + \frac{1}{a^2}.
\]

Proof. We have the schematic equations
\[
\nabla_4 \tilde{\beta} - * \mathcal{D} \alpha = \psi(\tilde{\beta}, \alpha) + \alpha_F \cdot \nabla \alpha_F + (\psi, \hat{\chi}) \cdot (\chi, \alpha_F) + \psi \cdot (\chi, \alpha_F, \alpha_F) \quad (7.35)
\]
\[
\nabla_3 \alpha + \frac{1}{2} \text{tr} \chi \alpha - \mathcal{D} \beta = (\psi, \hat{\chi}) \cdot (\Psi, \beta, \alpha) + (\rho, \sigma, \sigma_F) \cdot \nabla (\rho, \sigma, \sigma_F, \alpha_F) + \alpha_F \cdot \nabla (\rho, \sigma, \sigma_F) + (\psi, \hat{\chi}, \chi) \cdot (\chi, \alpha_F, \alpha_F). 
\]

Commuting the above equations \(i\) times with \(\nabla\) we arrive at
\[
\nabla_3 \nabla^i \alpha + \frac{1 + i}{2} \text{tr} \chi \nabla^i \alpha - \mathcal{D} \nabla^i \tilde{\beta} 
= \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2 + 1} \nabla^{i_3} (\hat{\chi}, \text{tr} \chi) \nabla^{i_4} \alpha 
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \hat{\chi}, \hat{\chi}, \text{tr} \chi) \nabla^{i_4} (\Psi, \beta, \alpha) 
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\chi, \nabla (\chi, \alpha_F) + \alpha_F \cdot \nabla \chi + (\psi, \hat{\chi}, \chi) \cdot (\chi, \alpha_F, \alpha_F)) 
\]

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\[
\frac{\partial}{\partial \xi} u + \nabla \cdot (A(x) \nabla u) = f(x)
\]
We work on the term $H_1$.

$$H_1 = \int_0^u \|(a^\frac{7}{6})^i F_i \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$\leq \int_0^u a^{-1} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i (\Psi, \beta, \alpha) \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \int_0^u \frac{|u'|^2}{a} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i (\Psi, \beta, \alpha) \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \int_0^u \frac{|u'|^2}{a} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i (\Psi, \beta, \alpha) \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \int_0^u \frac{|u'|^2}{a} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i (\Psi, \beta, \alpha) \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \int_0^u \frac{|u'|^2}{a} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i (\Psi, \beta, \alpha) \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

- The sum of the terms $J_1 + J_2 + J_3$ can be controlled just as in the vacuum case by

$$\mathcal{R}^2[\alpha] \cdot (O^2[\chi]) + O^6 + O^4.$$

- For the terms $J_4$ and $J_5$ we have

$$J_4 + J_5 = \sum_{i_1 + i_2 + i_3 + i_4 = i} \int_0^u \frac{a^i}{|u'|} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$+ \sum_{i_1 + i_2 + i_3 + i_4 + i_5 = i} \int_0^u \frac{a^i}{|u'|} \left\| \frac{a^\frac{7}{6}}{|u'|} \psi \frac{a^\frac{7}{6}}{|u'|} \sqrt{1 + \frac{a^\frac{7}{6}}{|u'|}} \right\|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}} \| (a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}}^2 \, du'$$

$$\lesssim O^2 \left( |u'|^2 + \frac{O^2}{|u'|} \cdot (\mathcal{F}[\alpha^2] + \mathcal{F}[\rho F, \sigma F]^2) \right) \lesssim O^2 \left( |u'|^2 + \frac{O^2}{|u'|} \cdot (R^4 + 1) \right).$$

where we have used Theorem 6.1 in the last inequality.

- The final term $J_6$ can be bounded by $O^6$.

Therefore, putting everything together, we have

$$H_1 \leq \mathcal{R}^2[\alpha] \cdot (O^2[\chi]) + O^2[\chi] + \frac{a \cdot O^2}{|u'|^2} \cdot (R^4 + (\mathcal{I}^0)^4 + (\mathcal{I}^0)^2 + 1) + O^6 + O^4,$$

which translates to

$$N_1 \leq \left( \mathcal{R}[\alpha] \cdot (O[\chi]) + \frac{a \cdot O}{|u'|} \cdot (R^2 + (\mathcal{I}^0)^2 + (\mathcal{I}^0)^2 + 1) + O^3 + O^2 \right) \cdot \sup_{u'} \|(a^\frac{7}{6})^i \chi \|_{L^2_{(\omega,\omega')}}^2 \cdot (H_{i\omega}^4) .$$

We continue with the term $M_1$ in the same way. We have

$$M_1 = \int_0^u \int_0^u \frac{a}{|u'|} \| ((a^\frac{7}{6})^i G_i \cdot (a^\frac{7}{6}) \chi \|_{L^2_{(\omega,\omega')}} \, du' \, du'$$

(7.38)
At this point we recall that
\[
G_1 = \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\tilde{\psi}, \tilde{\chi}) \nabla^{i_4} (\tilde{\beta}, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4} (\tilde{\gamma}, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4} (\gamma, \alpha) + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \chi) \nabla^{i_4} (\gamma, \alpha).
\]

Define
\[
K_1 := \int_0^u \int_{u'} \frac{a}{|u'|^2} \|(a^{i_1}) G_1 \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du.
\]

We then have
\[
K_1 \leq \int_0^u \int_{u'} \frac{a^2}{|u'|^2} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

- The sum $I_1 + I_2$ can be bounded as in the vacuum case by $R^2 \cdot O^2 + O^4$.
- We have
\[
I_3 + I_4 = \int_0^u \int_{u'} \frac{a}{|u'|^2} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
+ \int_0^u \int_{u'} \frac{a}{|u'|^2} \sum_{i_1 + i_2 + i_3 + i_4 = i} \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \, du'
\leq \int u \frac{a}{|u'|^2} \cdot O^2 \left( \int_0^u \|(a^{i_1}) \|_{L^2_{\text{loc}}(S_{\text{loc} \omega'}, \omega')}^2 \, du' \right) \, du' + \int_{u'} \frac{a}{|u'|^2} \cdot \frac{a \cdot O^4}{|u'|^2} \, du' \, du'
\leq \frac{a^2 \cdot O^2}{|u'|^2} \cdot F(a)^2 + 1 \lesssim O^2 \cdot (T(0))^2 + 1.
\]

(7.39)

- We have
Putting everything together, we have

$$K_1 \leq R^2 \cdot O^2 + O^6 + \frac{O^2}{|u|^2} (R^4 + (T^0)^4 + (T^0)^2 + 1),$$

so that

$$M_1 \leq \left( R \cdot O + O^2 + O^3 + \frac{O}{|u|^2} (R^2 + (T^0)^2 + T^0 + 1) \right) \cdot \sup_{\bar{\omega}} \| (a\hat{z} \nabla)^3 \hat{\beta} \|_{L^2_{(co)}(H^{(\infty, \infty)}_w)},$$

Putting (7.38) and (7.43) together, we finally get

$$\frac{1}{a^2} \| (a\hat{z} \nabla)^3 \alpha \|_{L^2_{(co)}(H^{(0,0)}_w)} + \frac{1}{a^2} \| (a\hat{z} \nabla)^3 \hat{\beta} \|_{L^2_{(co)}(H^{(\infty, \infty)}_w)} \leq \frac{1}{a^2} \| (a\hat{z} \nabla)^3 \alpha \|_{L^2_{(co)}(H^{(0,0)}_w)} + \frac{1}{a^2} \| (a\hat{z} \nabla)^3 \hat{\beta} \|_{L^2_{(co)}(H^{(\infty, \infty)}_w)} + \frac{1}{a^2} N_1 + \frac{1}{a^2} M_1$$

$$\leq \frac{1}{a^2} \| (a\hat{z} \nabla)^3 \alpha \|_{L^2_{(co)}(H^{(0,0)}_w)} + \frac{1}{a^2} \| (a\hat{z} \nabla)^3 \hat{\beta} \|_{L^2_{(co)}(H^{(\infty, \infty)}_w)} + \frac{1}{a^2} \left( |R[a] \cdot \langle O[\hat{\chi}] + O[\hat{\chi}] \rangle + \frac{a^2 \cdot O}{|u|^2} \cdot (R^2 + (T^0)^2 + T^0 + 1) + O^3 + O^2 \right) \cdot \sup_{\bar{\omega}} \| (a\hat{z} \nabla)^3 \alpha \|_{L^2_{(co)}(H^{(0,0)}_w)}$$

$$\left( R \cdot O + O^2 + O^3 + \frac{O}{|u|^2} (R^2 + (T^0)^2 + T^0 + 1) \right) \cdot \sup_{\bar{\omega}} \| (a\hat{z} \nabla)^3 \hat{\beta} \|_{L^2_{(co)}(H^{(\infty, \infty)}_w)},$$

from which our result follows.
7.4.1 Estimates on the remaining components

We proceed to show the estimates for the pair \((\Psi, \bar{\Psi}) = (\tilde{\beta}, (\rho, \sigma))\). The other two pairs are similar.

**Proof.** We have the schematic equation

\[
\nabla_3 \tilde{\beta} + \text{tr} \tilde{\beta} - D(-\rho, \sigma) = (\psi, \bar{\chi}) \Psi + \alpha_F \nabla(\alpha_F, \rho_F, \sigma_F) + (\rho_F, \sigma_F) \nabla(\rho_F, \sigma_F, \alpha_F) + (\psi, \bar{\chi}, \text{tr} \bar{\chi}) \cdot (\alpha_F, \Upsilon) \cdot \Upsilon,
\]

\[
\nabla_4((-\rho, \sigma)) - \text{tr} \tilde{\beta} = (\psi, \bar{\chi})(\Psi, \alpha) + \alpha_F \nabla \Upsilon + \Upsilon \nabla(\Upsilon, \alpha_F) + (\psi, \bar{\chi}, \text{tr} \bar{\chi})(\alpha_F, \Upsilon)(\alpha_F, \Upsilon).
\]

(7.44)

(7.45)

Commuting with \(i\) angular derivatives we get

\[
\nabla_3 \nabla^i \tilde{\beta} + \frac{i + 2}{2} \text{tr} \nabla^i \tilde{\beta} - D \nabla^i((-\rho, \sigma))
\]

\[
= \sum_{i_1 + i_2 + i_3 + i_4 + i = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\tilde{\chi}, \text{tr} \bar{\chi}) \nabla^{i_4} \Psi + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \bar{\chi}, \tilde{\chi}, \tilde{\chi}) \nabla^{i_4} \Psi
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 + i = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \Upsilon) \nabla^{i_4} \Upsilon + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} (\Upsilon, \alpha_F)
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \bar{\chi}, \text{tr} \bar{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_4} (\alpha_F, \Upsilon)
\]

\[:= F_2,\]

while

\[
\nabla_4 \nabla^i((-\rho, \sigma)) - \text{tr} \nabla^i \tilde{\beta}
\]

\[
= \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \bar{\chi}, \tilde{\chi}) \nabla^{i_4} (\Psi, \alpha)
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha_F, \Upsilon) \nabla^{i_4} \Upsilon + \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \Upsilon \nabla^{i_4} (\Upsilon, \alpha_F)
\]

\[
+ \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \bar{\chi}, \text{tr} \bar{\chi}) \nabla^{i_4} (\alpha_F, \Upsilon) \nabla^{i_4} (\alpha_F, \Upsilon)
\]

\[:= G_2.\]

We have

\[
\| (a\tilde{\beta} \nabla)^i \tilde{\beta} \|^2_{L^2_{(\text{loc})}(H^0_{\text{loc}} \omega)} + \| (a\tilde{\beta} \nabla)^i (\rho, \sigma) \|^2_{L^2_{(\text{loc})}(H^0_{\text{loc}} \omega)} \leq\| (a\tilde{\beta} \nabla)^i \tilde{\beta} \|^2_{L^2_{(\text{loc})}(H^0_{\text{loc}} \omega)} + \| (a\tilde{\beta} \nabla)^i (\rho, \sigma) \|^2_{L^2_{(\text{loc})}(H^0_{\text{loc}} \omega)} + N_2 + M_2,
\]

where

\[
N_2 = \int_0^u \int_{u_{\text{loc}}}^u a \| (a\tilde{\beta} \nabla)^i F_2 \cdot (a\tilde{\beta} \nabla)^i \tilde{\beta} \|_{L^2_{(\text{loc})}(S_{u'} \omega)} \, du' \, du'',
\]

(7.46)

\[
M_2 = \int_0^u \int_{u_{\text{loc}}}^u a \| (a\tilde{\beta} \nabla)^i G_2 \cdot (a\tilde{\beta} \nabla)^i (\rho, \sigma) \|_{L^2_{(\text{loc})}(S_{u'} \omega)} \, du' \, du''.
\]

(7.47)

We have

\[
N_2 = \int_0^u \int_{u_{\text{loc}}}^u a \| (a\tilde{\beta} \nabla)^i F_2 \cdot (a\tilde{\beta} \nabla)^i \tilde{\beta} \|_{L^2_{(\text{loc})}(S_{u'} \omega)} \, du' \, du''
\]
\[ \leq \int_{a_\infty}^u \frac{a}{|u'|^2} \left( \int_0^{|u|} (a^\frac{1}{2})^i F_2^2 \| \nabla \psi_i \|_{L^2(S_{\omega', \omega})} \, du' \right)^\frac{1}{2} \, du' \cdot R, \]

where we recall

\[ F_2 = \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\tilde{x}, \tilde{\chi}) \nabla^{i_4} \tilde{x} + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\psi, \tilde{x}, \tilde{\chi}) \nabla^{i_4} \tilde{x} \]

\[ + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \tilde{Y}) \nabla^{i_4} \tilde{Y} + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \tilde{Y} \nabla^{i_4} (\tilde{Y}, \alpha F) \]

\[ + \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\tilde{x}, \tilde{\chi}) \nabla^{i_4} \tilde{Y} \] 

Define \( H_2 = \int_0^{|u|} (a^\frac{1}{2})^i (a^\frac{1}{2})^j \| \nabla \psi_i \|_{L^2(S_{\omega', \omega})} \, du' \). The first two terms are handled like in the vacuum case. For the last three terms, we have

- For the first of the last three terms, we control

\[ \int_0^{|u|} (a^\frac{1}{2})^i \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \tilde{Y}) \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

\[ \leq \int_0^{|u|} (a^\frac{1}{2})^i \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \tilde{Y}) \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

(7.48)

- For the middle term we have

\[ \int_0^{|u|} (a^\frac{1}{2})^i \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \tilde{Y} \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

\[ \leq \int_0^{|u|} (a^\frac{1}{2})^i \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} \tilde{Y} \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

(7.49)

- For the last term we have

\[ \int_0^{|u|} (a^\frac{1}{2})^i \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \tilde{Y}) \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

\[ \leq \int_0^{|u|} \left( \frac{|u|^2}{a} \right)^i (a^\frac{1}{2})^j \sum_{i_1 + i_2 + i_3 + i_4 = 1} \nabla^{i_1} \psi^{i_2} \nabla^{i_3} (\alpha F, \tilde{Y}) \nabla^{i_4} \tilde{Y} \|_{L^2(S_{\omega', \omega})} \, du' \]

\[ \leq \int_0^{|u|} \left( \frac{|u|^2}{a} \right)^i \frac{O^6}{a} = \frac{O^6}{a}. \]
This completes the bounds on $N_2$. For $M_2$, we have

$$M_2 = \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i G_2 \cdot (a\hat{z})^j \nabla^j \Psi \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$\leq \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i G_2 \right\|_{L^2(S') \cap \Delta} \left\| (a\hat{z})^j \nabla^j \Psi \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$\leq \left( \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i G_2 \right\|_{L^2(S') \cap \Delta} \, du' \right)^{\frac{1}{2}} \left( \int_{0}^{u} \int_{u_{\infty}}^{\infty} \left\| (a\hat{z})^j \nabla^j \Psi \right\|_{L^2(S') \cap \Delta} \, du' \right)^{\frac{1}{2}}$$

It is already clear that the last term above will eventually be handled by Grönwall’s inequality. The following lemma will allow this:

**Lemma 7.1 (Lemma 14.8 in [12])** Let $f(x, y), g(x, y)$ be positive functions defined on the rectangle $U := \{(x, y) \mid 0 \leq x \leq x_0, 0 \leq y \leq y_0\}$. Suppose there exist nonnegative constants $J, c_1, c_2$ such that $f$ and $g$ verify the inequality

$$f(x, y) + g(x, y) \leq J + c_1 \int_{0}^{x} f(x', y) \, dx' + c_2 \int_{0}^{y} g(x, y') \, dy'$$

for all $(x, y) \in U$. Then there holds

$$f(x, y) + g(x, y) \leq J e^{c_1 x + c_2 y}, \quad \forall (x, y) \in U.$$

Before applying Grönwall, we first define

$$K_2 = \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i G_2 \right\|_{L^2(S') \cap \Delta} \, du' \, du'.$$

We then have

$$K_2 \leq \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i \right\|_{L^2(S') \cap \Delta} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \hat{z}^{i_2} \nabla^{i_3} \psi \nabla^{i_4} (\Psi, \alpha)^{2} \left\| \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$+ \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i \right\|_{L^2(S') \cap \Delta} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \hat{z}^{i_2} \nabla^{i_3} \nabla^{i_4} (\alpha F, \chi \nabla^{i_4} (\Psi, \alpha)) \left\| \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$+ \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i \right\|_{L^2(S') \cap \Delta} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \hat{z}^{i_2} \nabla^{i_3} \nabla^{i_4} (\alpha F, \chi \nabla^{i_4} (\Psi, \alpha)) \left\| \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$+ \int_{0}^{u} \int_{u_{\infty}}^{\infty} \frac{a}{|u'|^2} \left\| (a\hat{z})^i \right\|_{L^2(S') \cap \Delta} \sum_{i_1 + i_2 + i_3 + i_4 = i} \nabla^{i_1} \hat{z}^{i_2} \nabla^{i_3} \nabla^{i_4} (\alpha F, \chi \nabla^{i_4} (\Psi, \alpha)) \left\| \right\|_{L^2(S') \cap \Delta} \, du' \, du'$$

$$:= K_{21} + K_{22} + K_{23} + K_{24}.$$
\[
K_{21} \leq \int_0^u \int_{u_\infty}^u \frac{a}{|u'|^2} \left\| (\frac{1}{2})^i (\psi, \tilde{\chi}, \tilde{\chi}) \nabla^i (\Psi, \alpha) \right\|^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
+ \int_0^u \int_{u_\infty}^u \left( \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi z^2 \nabla^{i_2} (\psi, \tilde{\chi}, \tilde{\chi}) \nabla^{i_3} (\Psi, \alpha) \right)^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
\leq \int_0^u \int_{u_\infty}^u \frac{a}{|u'|^2} \left\| \psi, \tilde{\chi}, \tilde{\chi} \right\|^2_{L^\infty_x(S_{u', \infty})} \left( \frac{a}{|u'|^2} \right)^i \nabla^i (\Psi, \alpha)^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
+ \int_0^u \int_{u_\infty}^u \left( \frac{a}{|u'|^2} \sum_{i_1+i_2+i_3+i_4=1} \nabla^{i_1} \psi z^2 \nabla^{i_2} (\psi, \tilde{\chi}, \tilde{\chi}) \nabla^{i_3} (\Psi, \alpha) \right)^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
\leq \int_0^u \int_{u_\infty}^u a^{-1} \left( \frac{a}{|u'|^2} \sum_{i} \left( \frac{a}{|u'|^2} \right)^i \nabla^i (\psi, \tilde{\chi}, \tilde{\chi}) \right)^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
+ \int_0^u \int_{u_\infty}^u \left( \frac{a}{|u'|^2} \sum_{i} \left( \frac{a}{|u'|^2} \right)^i \nabla^i (\psi, \tilde{\chi}, \tilde{\chi}) \right)^2_{L^2_x(S_{u', \infty})} du' du''
\]
\[
\leq a^{-1} \cdot O^2 \cdot R^2 \left( O^2[\hat{\chi}] + O^2[\hat{\chi}] + 1 \right) \cdot \left( \mathcal{K}^2[\beta] + \frac{1}{a} \right)
\]
\[
+ \frac{a}{|u|} \left( O^2[\hat{\chi}] + O^2[\hat{\chi}] + 1 \right) \cdot \left( R^2[\alpha] + O^2[\alpha] + \frac{a}{|u|} \cdot O^4 \right)
\]
\[
\leq \frac{1}{a^4} \cdot \left( R^2[\alpha] + \mathcal{K}^2[\beta] + \mathcal{K}^2[\beta] + 1 \right) \leq \left( \mathcal{Z}^{(0)} \right)^4 + 1.
\]

The sum $K_{22} + K_{23}$ can be controlled by
\( K_{22} + K_{23} \)
\[
= \int_0^u \int_{u' | u'|^2} (\tilde{a}^\frac{i}{2})^j \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi \nabla^{i_2} (\alpha F, Y) \nabla^{i_3+1} Y \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
+ \int_0^u \int_{u' | u'|^2} (\tilde{a}^\frac{i}{2})^j \sum_{i_1+i_2+i_3+i_4=i} \nabla^{i_1} \psi \nabla^{i_2} Y \nabla^{i_3+1} \alpha F \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
\lesssim \int_0^u \int_{u' | u'|^2} a^2 \left( \alpha F, \frac{Y}{a^\frac{i}{2}} \right) \nabla^{i_1} Y \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
+ \int_0^u \int_{u' | u'|^2} (\tilde{a}^\frac{i}{2})^{j+1} \nabla^{i_1} \psi \nabla^{i_2} \left( \frac{\alpha F}{a^\frac{i}{2}} \right) \nabla^{i_3+1} Y \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
\leq \int_0^u \int_{u' | u'|^2} a (O^4 + O^2 \cdot F^2) \text{d}u' \text{d}u'' \lesssim 1.
\]

The final term \( K_{24} \) can be controlled in the same way.
\[
= \int_0^u \int_{u' | u'|^2} (\tilde{a}^\frac{i}{2})^j \sum_{i_1+i_2+i_3+i_4+i_5=i} \nabla^{i_1} \psi \nabla^{i_2} \left( \psi_i \chi \text{tr} \chi \right) \nabla^{i_3} (\alpha F, Y) \nabla^{i_5+1} (\alpha F, Y) \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
= \int_0^u \int_{u' | u'|^2} a \cdot |u'|^2 \cdot (\tilde{a}^\frac{i}{2})^j \sum_{i} \nabla^{i_1} \psi \nabla^{i_2} \left( \frac{a \text{tr} \chi}{|u'|^\frac{i}{2}} \right) \nabla^{i_3} \left( \frac{(\alpha F, Y)}{a^\frac{i}{2}} \right) \nabla^{i_5+1} \left( \frac{(\alpha F, Y)}{a^\frac{i}{2}} \right) \|_{L^2(S^\omega, \omega)}^2 \text{d}u' \text{d}u''
\]
\[
\lesssim \sup \int_0^u a \cdot |u'|^2 \cdot O^2 [\text{tr} \chi] O^4 [\alpha F] \text{d}u' \lesssim O^2 [\text{tr} \chi] O^4 [\alpha F].
\]

Now \( O[\text{tr} \chi] \lesssim 1 \) by Proposition 4.8 and \( O[\alpha F] \lesssim \mathcal{E}[\rho F, \sigma F] + 1 \lesssim 2 \mathcal{E}[\rho F, \sigma F] + \frac{1}{\alpha^\frac{i}{2}} \), where the first inequality we use the scale–invariant \( L^2 \)-estimates on the Maxwell components and in the second inequality the energy estimates. In particular, the inequality can be traced back to the initial data and this is a key point.

Finally, we use Grönwall for \( M_2 \) and conclude. In a similar way, we obtain
\[
\| (\tilde{a}^\frac{i}{2} \nabla)^j (\rho, \sigma, \beta) \|_{L^2(S^\omega, \omega)} + \| (\tilde{a}^\frac{i}{2} \nabla)^j (\beta, \omega) \|_{L^2(S^\omega, \omega)} \lesssim (\mathcal{I}^{(0)})^2 + \mathcal{I}^{(0)} + 1.
\]

The result follows. \( \square \)
8 The formation of trapped surfaces

In this section, we prove

**Theorem 1.3** Given $\mathcal{I}(0)$, there exists a sufficiently large $a_0 = a_0(\mathcal{I}(0))$ such that the following holds. For any $0 < a_0 < a$, the unique smooth solution $(\mathcal{M}, g)$ of the Einstein-Maxwell equations from Theorem 1.2 with initial data satisfying

- $\sum_{i<j, k<l \leq 3} u^{-\frac{i}{2}} \| \nabla_1^I (|u_\infty| \nabla) (\tilde{\chi}, \alpha_F) \|_{L^\infty(S_\infty, \omega)} \leq \mathcal{I}(0)$ along $u = u_\infty$,
- Minkowskian initial data along $u = 0$,
- $\int_0^1 |u_\infty|^2 (|\tilde{\chi}_0|^2 + |\alpha_F|^2) (u_\infty, u') du' \geq a$ uniformly for every direction along $u = u_\infty$.

has a trapped surface at $S_{-\alpha / 4, 1}$.

Proof. We first derive pointwise estimates for $|\tilde{\chi}|^2$. Fix $(\theta^1, \theta^2) \in S^2$. We consider the following null structure equation

$$\nabla_3 \tilde{\chi} + \frac{1}{2} \nabla_1 \tilde{\chi} = \nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}.$$ 

We contract this 2-tensor with another 2-tensor $\tilde{\chi}$ and get

$$\frac{1}{2} \nabla_3 |\tilde{\chi}|^2 + \frac{1}{2} \nabla_1 |\tilde{\chi}|^2 - 2 |\nabla_1 \tilde{\chi}|^2 = \tilde{\chi} (\nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}).$$ 

Employing the fact $\omega = -\frac{1}{2} \nabla (\log \Omega) = -\frac{1}{2} \Omega^{-1} \nabla_3 \Omega$, we rewrite (8.1) as

$$\nabla_3 (\Omega^2 |\tilde{\chi}|^2) + \nabla_1 \Omega^2 |\tilde{\chi}|^2 = 2 \Omega^2 \tilde{\chi} (\nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}).$$

Using $\nabla_3 = \frac{1}{\Omega} (\frac{\partial}{\partial u} + b^A \frac{\partial}{\partial \theta^A})$, we rewrite the above equation as

$$\frac{\partial}{\partial u} (\Omega^2 |\tilde{\chi}|^2) + \Omega \nabla_1 \cdot \Omega^2 |\tilde{\chi}|^2 = 2 \Omega^2 \tilde{\chi} (\nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}) - b^A \frac{\partial}{\partial \theta^A} (\Omega^2 |\tilde{\chi}|^2).$$

Substitute $\Omega \nabla_1 \tilde{\chi}$ with

$$\Omega \nabla_1 \tilde{\chi} = \Omega (\nabla_1 + \frac{2}{|u|} - \Omega \frac{2}{|u|} = \Omega (\nabla_1 + \frac{2}{|u|} - (\Omega - 1) \frac{2}{|u|} \frac{2}{|u|},$$

we have

$$\frac{\partial}{\partial u} (\Omega^2 |\tilde{\chi}|^2) - \frac{2}{|u|} \Omega^2 |\tilde{\chi}|^2 = 2 \Omega^2 \tilde{\chi} (\nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}) - b^A \frac{\partial}{\partial \theta^A} (\Omega^2 |\tilde{\chi}|^2) - \Omega (\nabla_1 + \frac{2}{|u|}) (\Omega^2 |\tilde{\chi}|^2) + (\Omega - 1) \frac{2}{|u|} \cdot (\Omega^2 |\tilde{\chi}|^2).$$

This gives

$$\frac{\partial}{\partial u} \left( u^2 \Omega^2 |\tilde{\chi}|^2 \right) = 2 |u|^2 \cdot \Omega^2 \tilde{\chi} (\nabla \tilde{\eta} - \frac{1}{2} \nabla_1 \tilde{\chi} + \tilde{\eta} \tilde{\eta}) - |u|^2 \cdot b^A \frac{\partial}{\partial \theta^A} (\Omega^2 |\tilde{\chi}|^2) - |u|^2 \cdot \Omega (\nabla_1 + \frac{2}{|u|}) (\Omega^2 |\tilde{\chi}|^2) + |u|^2 \cdot (\Omega - 1) \frac{2}{|u|} \cdot (\Omega^2 |\tilde{\chi}|^2).$$ (8.2)

For $b^A$, we have equation

$$\frac{\partial b^A}{\partial u} = -4 \Omega^2 \zeta^A,$$
which is from
\[ [L, L] = \frac{\partial b^A}{\partial u} \frac{\partial}{\partial \theta^A}. \]
Applying the identity \( \zeta_A = \frac{1}{2} \eta_A - \frac{1}{2} \omega_A \), Proposition 3.1.1 and the derived estimates on \( \eta, \omega \), we conclude that there holds in \( D_u \)
\[ \|b^A\|_{L^\infty(S_u \omega)} \leq \frac{a}{|u|^2}. \]
For the right hand side of (8.5), we have
\[ \|2 \cdot |u|^2 \cdot \Omega^2 \hat{\chi} (\nabla \hat{\eta} - \frac{1}{2} \text{tr} \chi \hat{\eta} + \hat{\chi} \hat{\eta})\|_{L^\infty(S_u \omega)} \leq \|u\|^2 \cdot \frac{a}{|u|^2} \cdot \frac{a}{|u|^2} + \frac{a}{|u|^2} \leq \frac{a}{|u|^2}, \]
\[ \| - |u|^2 \cdot \Omega (\text{tr} \chi + \frac{2}{|u|}) (\Omega^2 \hat{\chi}^2)\|_{L^\infty(S_u \omega)} \leq \|u\|^2 \cdot \frac{1}{|u|^2} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2}, \]
\[ \| |u|^2 \cdot (\Omega - 1) \cdot \frac{2}{|u|} \cdot (\Omega^2 \hat{\chi}^2)\|_{L^\infty(S_u \omega)} \leq \|u\|^2 \cdot \frac{1}{|u|^2} \cdot \frac{2}{|u|} \cdot \frac{a}{|u|^2} \leq \frac{a}{|u|^2}. \]
In summary, we have
\[ \frac{\partial}{\partial u} \left( u^2 \Omega^2 \hat{\chi}^2 \right) = M, \quad \text{and} \quad |M| \leq \frac{a}{|u|^2} \leq \frac{a}{|u|^2}, \]
which implies
\[ \frac{a}{|u|^2} + \frac{a}{|u|} \leq \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u, \theta^1, \theta^2) - \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u, \theta^1, \theta^2). \]
Recall \( \Omega(u, u, \theta^1, \theta^2) = 1 \). We hence have
\[ \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u, \theta^1, \theta^2) \geq \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u, \theta^1, \theta^2) - \frac{a}{|u|^2}. \]
Integrating with respect to \( u \), we further have for \( u \leq -a/4 \)
\[ \int_0^1 \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u', \theta^1, \theta^2) du' \geq \int_0^1 \|u\|^2 \Omega^2 \hat{\chi}^2 (u, u, \theta^1, \theta^2) du' - \frac{a}{|u|^2}. \] (8.3)
In the same fashion, we derive pointwise estimates for \( |\alpha \eta|^2 \). Consider the null Maxwell equation
\[ \nabla_3 \alpha \eta + \frac{1}{2} \text{tr} \chi \alpha \eta - 2 \omega \alpha \eta = - \nabla \rho \alpha - \nabla \sigma \alpha - 2 \eta \cdot \sigma \alpha + 2 \eta \cdot \rho \alpha - \hat{\chi} \cdot \alpha \eta. \]
We contract this 1-form with another 1-form \( \alpha \eta \) and get
\[ \frac{1}{2} \nabla_3 |\alpha \eta|^2 + \frac{1}{2} \text{tr} \chi |\alpha \eta|^2 - 2 \omega |\alpha \eta|^2 = \alpha \eta \cdot \nabla \rho \alpha + \nabla \sigma \alpha - 2 \eta \cdot \sigma \alpha + 2 \eta \cdot \rho \alpha - \hat{\chi} \cdot \alpha \eta. \] (8.4)
Employing the fact \( \omega \equiv - \frac{1}{2} \nabla_3 (\log \Omega) = - \frac{1}{2} \Omega^{-1} \nabla_3 \Omega \), we rewrite (8.4) as
\[ \nabla_3 (\Omega^2 |\alpha \eta|^2) + \Omega \text{tr} \chi \cdot |\alpha \eta|^2 = 2 \Omega^2 \alpha \eta \cdot (\nabla \rho \alpha + \nabla \sigma \alpha - 2 \eta \cdot \sigma \alpha + 2 \eta \cdot \rho \alpha - \hat{\chi} \cdot \alpha \eta). \]
Using \( \nabla_3 = \frac{1}{\Omega} (\partial / \partial u + b^A \partial / \partial \theta^A) \), we rewrite the above equation as
\[ \frac{\partial}{\partial u} (\Omega^2 |\alpha \eta|^2) + \Omega \text{tr} \chi \cdot \Omega^2 |\alpha \eta|^2 = 2 \Omega^3 \alpha \eta \cdot (\nabla \rho \alpha + \nabla \sigma \alpha - 2 \eta \cdot \sigma \alpha + 2 \eta \cdot \rho \alpha - \hat{\chi} \cdot \alpha \eta) - b^A \frac{\partial}{\partial \theta^A} (\Omega^2 |\alpha \eta|^2). \]
Substitute $\Omega \text{tr}_A$ with

$$\Omega \text{tr}_A = \Omega(\text{tr}_A + \frac{2}{|u|}) - \Omega(\text{tr}_A + \frac{2}{|u|}) = (\Omega - 1) \frac{2}{|u|} - \frac{2}{|u|}$$

we have

$$\frac{\partial}{\partial u}(\Omega^2 |\alpha_F|^2) - \frac{2}{|u|} \Omega^2 |\alpha_F|^2 = 2\Omega^2 |\alpha_F|^2 \left(- \nabla \rho_F + \frac{1}{2} \nabla \sigma_F - 2 \frac{1}{2} \sigma_F + 2 \eta^2 \frac{1}{2} \sigma_F - \tilde{\chi} \cdot \alpha_F \right) - b^A \frac{\partial}{\partial \theta_A}(\Omega^2 |\alpha_F|^2)$$

$$- \Omega(\text{tr}_A + \frac{2}{|u|}) (\Omega^2 |\alpha_F|^2) + (\Omega - 1) \frac{2}{|u|} (\Omega^2 |\alpha_F|^2).$$

This gives

$$\frac{\partial}{\partial u} \left( \Omega^2 |\alpha_F|^2 \right) = 2 |u|^2 \Omega^2 |\alpha_F|^2 \left(- \nabla \rho_F + \frac{1}{2} \nabla \sigma_F - 2 \frac{1}{2} \sigma_F + 2 \eta^2 \frac{1}{2} \sigma_F - \tilde{\chi} \cdot \alpha_F \right) - |u|^2 \cdot b^A \frac{\partial}{\partial \theta_A}(\Omega^2 |\alpha_F|^2)$$

$$- |u|^2 \cdot \Omega(\text{tr}_A + \frac{2}{|u|}) (\Omega^2 |\alpha_F|^2) + |u|^2 \cdot (\Omega - 1) \frac{2}{|u|} (\Omega^2 |\alpha_F|^2).$$

(8.5)

For $b^A$, we have the equation

$$\frac{\partial b^A}{\partial u} = -4\Omega^2 \zeta A,$$

which is from

$$[L, L] = \frac{\partial b^A}{\partial u} \frac{\partial b^A}{\partial \theta A}.$$

Applying the identity $\zeta_A = \frac{1}{2} \gamma_A - \frac{1}{2} \gamma_A$, Propositions 3.1, derived estimates of $\eta, \eta$, it holds in $D_{u,\Omega}$

$$\|b^A\|_{L^\infty(S_{\gamma,\omega})} \leq \frac{a^2}{|u|^2}.$$

For the right hand side of (8.5), we have

$$\|2 |u|^2 \cdot a^2 \Omega^2 \left(- \frac{1}{2} \nabla \rho_F + \frac{1}{2} \nabla \sigma_F - \frac{1}{2} \sigma_F + 2 \frac{1}{2} \eta \cdot \sigma_F - \tilde{\chi} \cdot \alpha_F \right)\|_{L^\infty(S_{\gamma,\omega})} \leq |u|^2 \cdot \frac{a^2}{|u|^2} \left( \frac{a^2}{|u|^2} + \frac{a^2}{|u|^2} \right) \leq \frac{a^4}{|u|^2} + \frac{a^2}{|u|^2},$$

$$\|\frac{\partial}{\partial \theta A}(\Omega^2 |\alpha_F|^2)\|_{L^\infty(S_{\gamma,\omega})} \leq |u|^2 \cdot \frac{a^2}{|u|^2} \left( \frac{a^2}{|u|^2} + \frac{a^2}{|u|^2} \right) \leq \frac{a^4}{|u|^2} + \frac{a^2}{|u|^2};$$

$$\| |u|^2 \cdot \Omega(\text{tr}_A + \frac{2}{|u|}) (\Omega^2 |\alpha_F|^2)\|_{L^\infty(S_{\gamma,\omega})} \leq |u|^2 \cdot \frac{1}{|u|^2} \cdot \frac{a^2}{|u|^2} \leq \frac{a^2}{|u|^2};$$

$$\| |u|^2 \cdot (\Omega - 1) \frac{2}{|u|} (\Omega^2 |\alpha_F|^2)\|_{L^\infty(S_{\gamma,\omega})} \leq |u|^2 \cdot \frac{1}{|u|^2} \cdot \frac{2}{|u|^2} \leq \frac{a^2}{|u|^2}.$$

In summary, we have

$$\frac{\partial}{\partial u} \left( u^2 \Omega^2 |\alpha_F|^2 \right) = M,$$

and $|M| \lesssim \frac{a^2}{|u|^2} \ll \frac{a^2}{|u|^2},$

which implies

$$- \frac{a^2}{|u|} + \frac{a^2}{|u|^2} \leq |u|^2 \Omega^2 |\alpha_F|^2 |u, u^1 \theta^1, \theta^2| - |u|^2 \Omega^2 |\alpha_F|^2 |u, u^1 \theta^1, \theta^2|.$$
Integrating with respect to $u$, we further have for $u_\infty \leq u \leq -a/4$

$$\int_0^1 |u|^2 \Omega^2 |\alpha_F|^2(u, u', \theta^1, \theta^2) du' \geq \int_0^1 |u_\infty|^2 |\alpha_F|^2(u_\infty, u', \theta^1, \theta^2) du' - \frac{a^2}{|u|}. $$

Together with (8.3)

$$\int_0^1 |u|^2 \Omega^2 |\chi|^2(u, u', \theta^1, \theta^2) du' \geq \int_0^1 |u_\infty|^2 |\chi|^2(u_\infty, u', \theta^1, \theta^2) du' - \frac{a^2}{|u|}. $$

We conclude that

$$\int_0^1 |u|^2 \Omega^2 (|\chi|^2 + |\alpha_F|^2)(u, u', \theta^1, \theta^2) du' \geq \int_0^1 |u_\infty|^2 (|\chi|^2 + |\alpha_F|^2)(u_\infty, u', \theta^1, \theta^2) du' - \frac{2a^2}{|u|} \geq a - \frac{2a^2}{|u|} \geq a - \frac{8a^2}{a} \geq \frac{7a}{8}. $$

Pick $u = -a/4$. With the fact $\|\Omega - 1\|_{L^\infty(S, u)} \lesssim 1/a$, for sufficiently large $a$, we hence have

$$\left(-\frac{a}{4}\right)^2 \int_0^1 (|\chi|^2 + |\alpha_F|^2) u', \theta^1, \theta^2 du' \geq \frac{6}{7} \int_0^1 \left(-\frac{a}{4}\right)^2 \Omega^2 (|\chi|^2 + |\alpha_F|^2) u', \theta^1, \theta^2 du' \geq \frac{6}{7} \cdot \frac{7a}{8} = \frac{3a}{4} \cdot \frac{16}{a} = \frac{12}{a}. $$

This implies

$$\int_0^1 (|\chi|^2 + |\alpha_F|^2) u', \theta^1, \theta^2 du' \geq \frac{3a}{4} \cdot \frac{16}{a} = \frac{12}{a}. \tag{8.6}$$

We now consider the outgoing null structure equation for $\text{tr} \chi$,

$$\nabla_4 \text{tr} \chi + \frac{1}{2}(\text{tr} \chi)^2 = -|\chi|^2 - 2\omega \text{tr} \chi - |\alpha_F|^2. $$

Using $\omega = -\frac{1}{2} \nabla_4 (\log \Omega)$, we have

$$\nabla_4 \text{tr} \chi + \frac{1}{2}(\text{tr} \chi)^2 = -|\chi|^2 - 2\omega \text{tr} \chi - |\alpha_F|^2$$

$$= -|\chi|^2 + \nabla_4 (\log \Omega) \text{tr} \chi - |\alpha_F|^2 = -|\chi|^2 + \frac{1}{\Omega} \nabla_4 \Omega \cdot \text{tr} \chi - |\alpha_F|^2. $$

Hence,

$$\nabla_4 (\Omega^{-1} \text{tr} \chi) = -\Omega^{-2} \nabla_4 \Omega \cdot \text{tr} \chi + \Omega^{-1} \nabla_4 \text{tr} \chi$$

$$= \Omega^{-1} (\nabla_4 \text{tr} \chi - \Omega^{-1} \cdot \nabla_4 \Omega \cdot \text{tr} \chi) = \Omega^{-1} \left(-\frac{1}{2}(\text{tr} \chi)^2 - |\chi|^2 - |\alpha_F|^2\right). $$

With the fact $e_4 = \Omega^{-1} \frac{\partial}{\partial u}$, we have

$$\frac{\partial}{\partial u} (\Omega^{-1} \text{tr} \chi) = -\frac{1}{2}(\text{tr} \chi)^2 - |\chi|^2 - |\alpha_F|^2. \tag{8.7}$$

For every $(\theta^1, \theta^2) \in S^2$, along $\mathcal{H}_u$, we have

$$\Omega^{-1} \text{tr} \chi (\frac{a}{4}, 0, \theta^1, \theta^2) = 1^{-1} \cdot \frac{2}{a/4} = \frac{8}{a}. $$

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We then integrate \((8.7)\). Using \((8.6)\) we obtain
\[
(\Omega^{-1}\text{tr}\chi)\left(-\frac{a}{4}, 1, \theta^1, \theta^2\right)
\leq (\Omega^{-1}\text{tr}\chi)\left(-\frac{a}{4}, 0, \theta^1, \theta^2\right) - \int_0^1 (|\dot{x}|^2 + |\alpha F|^2)(-\frac{a}{4}, \theta^1, \theta^2)\, du'
\leq \frac{8}{a} - \frac{12}{a} < 0.
\]
Recall, finally, that in \(D_{u,u}\) the following estimate holds
\[
\|\text{tr}\chi + \frac{2}{u}\|_{L^\infty(S_{u,u})} \leq \frac{1}{|u|^2}
\]
In particular, this implies
\[
\text{tr}\chi(-\frac{a}{4}, 1, \theta^1, \theta^2) < 0 \text{ for every } (\theta^1, \theta^2) \in S^2.
\]
Therefore, we conclude that \(S_{-\frac{a}{4},1}\) is a trapped surface.

9 A Scaling Argument

In this article, we use coordinate system \((u, u', \theta^1, \theta^2)\) based on double null foliations, where \((\theta^1, \theta^2)\) are stereographic coordinates on \(S^2\). In these coordinates, we study spacetime region
\[
u_\infty \leq u \leq \frac{a}{4}, \quad 0 \leq u' \leq 1.
\]
The Lorentzian metric \(g\) satisfies ansatz
\[
g = -2\Omega^2(du \otimes du + du \otimes du) + \gamma_{AB}(d\theta^A - dA du) \otimes (d\theta^B - dB du).
\]

9.1 A Spacetime Rescaling

Following [4], we use a new coordinate system \((u', u', \theta'^1, \theta'^2)\), where
\[
u' = \delta u, \quad \nu' = \delta u, \quad \theta'^1 = \delta \theta^1, \quad \theta'^2 = \delta \theta^2.
\] (9.1)
Note that coordinates \((\theta^1, \theta^2)\) on \(S_{a,u}\) are set up through stereographic projection. Assume \((x_1, x_2, x_3)\) satisfying \(x_1^2 + x_2^2 + x_3^2 = \rho^2\) and lying on the upper hemisphere of \(S_{-\rho,0}\) (with radius \(\rho\)). It then has stereographic projection \((\zeta_1, \zeta_2) = \left(\frac{ax_1}{a + x_3}, \frac{ax_2}{a + x_3}\right)\). Scale down the length by a factor \(\delta\), we then have \(x'_1 = \delta x_1, x'_2 = \delta x_2, x'_3 = \delta x_3, (x'_1)^2 + (x'_2)^2 + (x'_3)^2 = \delta^2 x^2\) and \((x'_1, x'_2, x'_3)\) has stereographic projection
\[
(\zeta'_1, \zeta'_2) = \left(\frac{\delta x_1}{\delta a + x_3}, \frac{\delta x_2}{\delta a + x_3}\right) = \left(\frac{\delta a \cdot \delta x_1}{\delta a + \delta x_3}, \frac{\delta a \cdot \delta x_2}{\delta a + \delta x_3}\right) = \left(\frac{\delta a}{a + x_3}, \frac{\delta ax_2}{a + x_3}\right).
\]
Therefore, the rescaled coordinates \((\theta^1, \theta^2) = (\delta \theta^1, \delta \theta^2)\) on \(S_{a', u'}\) make perfect sense since 2-sphere \(S_{a', u'} = S_{\delta u, \delta u}\) is scaled down from \(S_{a, u}\) by a factor \(\delta\).

Under the rescaling (9.1), it follows
\[
g'(u', u', \theta'^1, \theta'^2) = \delta^2 \cdot g(u, u, \theta^1, \theta^2).
\]
In \((u', u', \theta'^1, \theta'^2)\) coordinates, we let
\[
g'(u', u', \theta'^1, \theta'^2) = -2\Omega'^2(du' \otimes du' + du' \otimes du') + \gamma'_{AB}(d\theta'^A - dA' du') \otimes (d\theta'^B - dB' du').
\]
Compare with
\[
g(u, u, \theta^1, \theta^2) = -2\Omega^2(du \otimes du + du \otimes du) + \gamma_{AB}(d\theta^A - dA du) \otimes (d\theta^B - dB du).
\]
Here we have
\[ du' = \delta \cdot du, \quad du'' = \delta \cdot du, \quad d\theta^A' = \delta \cdot d\theta^A \text{ for } A = 1, 2, \]
\[ \Omega^2(u', u', \theta', \theta'') = \Omega^2(u, u, \theta, \theta'), \quad \gamma'_{AB'}(u', u', \theta', \theta'') = \gamma_{AB}(u, u, \theta, \theta'), \]
\[ dA'(u', u', \theta', \theta'') = dA(u, u, \theta, \theta'), \]
\[ e_3'(u', u', \theta', \theta'') = \Omega^{-1}(\partial \frac{\partial}{\partial u'} + dA' \frac{\partial}{\partial u'}) = \delta^{-1} \Omega^{-1}(\partial \frac{\partial}{\partial u} + dA \frac{\partial}{\partial u}) = \delta^{-1} \cdot e_3(u, u, \theta, \theta'), \]
\[ e_4'(u', u', \theta', \theta'') = \Omega^{-1} \frac{\partial}{\partial u} = \delta^{-1} \cdot e_4(u, u, \theta, \theta'), \quad (9.2) \]
\[ e_4'(u', u', \theta', \theta'') = \delta^{-1} \cdot e_A(u, u, \theta, \theta'), \quad (9.3) \]

9.2 Rescaled Geometric Quantities

As usual, with frame \{e_3', e_4', e_A', e_B'\}, we define
\[ \chi_{A'B'}' = g'(D_A' e_3', e_B'), \quad \chi'_{A'B'} = g'(D_A' e_3', e_B'), \]
\[ \eta_A' = -\frac{1}{2} g'(D_A' e_3', e_3'), \quad \eta_A' = -\frac{1}{2} g'(D_A' e_3', e_3'), \]
\[ \omega' = -\frac{1}{4} g'(D_A' e_3', e_3'), \quad \omega' = -\frac{1}{4} g'(D_A' e_3', e_3'). \]

With \( \gamma'_{A'B'} \) being the induced metric on \( S'_{u, u} \), we further decompose \( \chi', \chi' \) into
\[ \chi_{A'B'}' = \frac{1}{2} \text{tr} \chi' \cdot \gamma'_{A'B'} + \chi_{A'B'}', \quad \chi'_{A'B'} = \frac{1}{2} \text{tr} \chi' \cdot \gamma'_{A'B'} + \chi'_{A'B'}. \]

Here \( D_{e_3'} e_3' := \Gamma_{\mu'\nu'}^{\lambda} e_3' \) and \( \Gamma_{\mu'\nu'}^{\lambda} := \frac{1}{2} g'_{\alpha'\beta'} \left( \frac{\partial g'_{\alpha'\beta'}}{\partial x_{\mu'}} + \frac{\partial g'_{\beta'\mu'}}{\partial x_{\alpha'}} - \frac{\partial g'_{\alpha'\mu'}}{\partial x_{\beta'}} \right). \)

In [4], we have

**Proposition 9.1** For \( \Gamma \in \{ \chi, \text{tr} \chi, \chi, \eta, \eta, \zeta, \omega, \omega \} \) written in two different coordinates \( (u', u', \theta', \theta'') \) and \( (u, u, \theta, \theta') \), it holds that
\[ \Gamma'_\Lambda(u', u', \theta', \theta'') = \delta^{-1} \cdot \Gamma(u, u, \theta, \theta'). \]

And

**Proposition 9.2** For \( \Psi \in \{ \alpha, \beta, \rho, \sigma, \beta, \alpha \} \) written in coordinates \( (u', u', \theta', \theta'') \) and \( (u, u, \theta, \theta') \), the following identity is true
\[ \Psi'_\Lambda(u', u', \theta', \theta'') = \delta^{-2} \cdot \Psi(u, u, \theta, \theta'). \]

Besides above geometric quantities, under the rescaling (9.1), for the Maxwell field we have
\[ F'_{\mu\nu}(u', u', \theta', \theta'') = \delta \cdot F_{\mu\nu}(u, u, \theta, \theta'). \quad (9.4) \]

And we define
\[ (\alpha'F')_\Lambda = F'_{\alpha' e_3'}, \quad (\alpha'F')_\Lambda = F'_{\alpha' e_3'}, \quad \rho'F' = \frac{1}{2} F'_{\rho' e_3'}, \quad \sigma'F' = F'_{\sigma' e_3'}, \]

It holds that

**Proposition 9.3** For \( \Gamma_F \in \{ \alpha_F, \beta_F, \rho_F, \sigma_F \} \) written in two different coordinates \( (u', u', \theta', \theta'') \) and \( (u, u, \theta, \theta') \), it holds that
\[ \Gamma'_F(u', u', \theta', \theta'') = \delta^{-1} \cdot \Gamma_F(u, u, \theta, \theta'). \]

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Proof. We first calculate \((a' F)_A\). Using definition of \((a' F)_A\) and (9.4), we have
\[
(a' F)_A(u', u', \theta^1, \theta^2) = F'_{\epsilon'}e_4 = \delta \cdot F'\eta = F\eta\cdot (a F)_A(u, u, \theta^1, \theta^2).
\]
The rest Maxwell components are treated in the same way. \(\square\)

9.3 Rescaled Uniform Bounds

Applying Proposition 9.1 and Proposition 9.2, next we establish the connection to [2]. Take \(\chi\) as an example. With Proposition 9.1, estimates derived for \(O_{\gamma, \infty}[\chi]\) and \(u' = \delta u\), we have
\[
|\chi_A(u', u', \theta^1, \theta^2)| = \delta^{-1} \cdot |\chi_A(u, u, \theta^1, \theta^2)| \leq \delta^{-1} \cdot \frac{a^{\frac{2}{3}}}{|u|} = \frac{a^{\frac{2}{3}}}{|u'|}.
\]
In the same fashion, we have
\[
|\chi_A(u', u', \theta^1, \theta^2)| = \delta^{-1} \cdot |\chi_A(u, u, \theta^1, \theta^2)| \leq \delta^{-1} \cdot \frac{a^{\frac{2}{3}}}{|u|} = \frac{a^{\frac{2}{3}}}{|u'|},
\]
\[
|\chi(u', u', \theta^1, \theta^2)| = \delta^{-1} \cdot |\chi(u, u, \theta^1, \theta^2)| \leq \delta^{-1} \cdot \frac{a^{2}}{|u|} = \frac{a^{2}}{|u'|},
\]
\[
|\chi(u', u', \theta^1, \theta^2)| = \delta^{-1} \cdot |\chi(u, u, \theta^1, \theta^2)| \leq \delta^{-1} \cdot \frac{a^{2}}{|u|} = \frac{a^{2}}{|u'|}.
\]
For the estimates of \(u'\) and \(\text{tr} u'\), we use \(|u'| \geq \delta a^2\). In the same manner, by Proposition 9.2 and with the help that \(|u| \geq \delta a/4\) we have
\[
|\beta_A(u', u', \theta^1, \theta^2)| = \delta^{-2} \cdot |\beta_A(u, u, \theta^1, \theta^2)| \leq \delta^{-2} \cdot \frac{a^{\frac{2}{3}}}{|u|} = \frac{a^{\frac{2}{3}}}{|u'|},
\]
\[
|\beta(u', u', \theta^1, \theta^2)| = \delta^{-2} \cdot |\beta(u, u, \theta^1, \theta^2)| \leq \delta^{-2} \cdot \frac{a^{2}}{|u|} = \frac{a^{2}}{|u'|},
\]
\[
|\gamma(u', u', \theta^1, \theta^2)| = \delta^{-2} \cdot |\gamma(u, u, \theta^1, \theta^2)| \leq \delta^{-2} \cdot \frac{a^{2}}{|u|} = \frac{a^{2}}{|u'|},
\]
\[
|\beta_A(u', u', \theta^1, \theta^2)| = \delta^{-2} \cdot |\beta_A(u, u, \theta^1, \theta^2)| \leq \delta^{-2} \cdot \frac{a^{2}}{|u|} = \frac{a^{2}}{|u'|}.
\]
(9.5)
Similarly by Proposition 9.2 and with the help that $|u'|\geq\delta a/4$ we have

$$
|\alpha'_{A'B'}(u',u',\theta^1,\theta^2)| = \delta^{-2} \cdot |\alpha_{AB}(u,u,\theta^1,\theta^2)| \leq \delta^{-2} \frac{a^2}{|u|} = \delta^{-1} \frac{a^2}{|u|} = \delta^{-1} \frac{a^2}{|u'|}.
$$

By repeating the arguments as in Section 8, we therefore obtain Theorem 1.4.

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