CONSTRANED PATH-FINDING AND STRUCTURE FROM ACYCLICITY

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Abstract. This note presents several results in graph theory inspired by the author’s work in the proof theory of linear logic; these results are purely combinatorial and do not involve logic.

We show that trails avoiding forbidden transitions, properly arc-colored directed trails and rainbow paths for complete multipartite color classes can be found in linear time, whereas finding rainbow paths is NP-complete for any other restriction on color classes. For the tractable cases, we also state new structural properties equivalent to Kotzig’s theorem on the existence of bridges in unique perfect matchings.

Another result on graphs equipped with unique perfect matchings that we prove here is the combinatorial counterpart of a theorem due to Bellin in linear logic: a connection between blossoms and bridge deletion orders.

1. Introduction

Many problems which consist of finding a path or trail (see next page for the terminology) under some constraints between two given vertices are equivalent to the augmenting path problem for matchings, and thus tractable. The many-one reductions involved often preserve structural properties; Kotzig’s theorem on the existence of bridges in unique perfect matchings [Kot59] thus yields equivalent “structure from acyclicity” theorems associated to these problems (see [Sze04]). That is, the absence of constrained cycles or closed trails entails the positive existence of some structure in the graph. (Recall that a perfect matching is unique if and only if it admits no alternating cycle.)

A little-known fact is that this family of equivalent problems has a representative in proof theory, a fact which has led researchers in logic to independently rediscover equivalent versions of Kotzig’s theorem. This connection between the theory of proof net correctness in linear logic [Gir87] and perfect matchings was first noticed by Retoré [Ret03], but remained unexploited for a long time. We applied it in a previous paper [Ngu18] in order to make progress on open problems concerning proof nets. Conversely, here we demonstrate some benefits of this bridge between logic and graphs for pure graph theory. Some of the results we present here were directly motivated by logical applications and are used either in [Ngu18], in its upcoming journal version, or in our more recent work [Ngu20].

The paper can be divided into two distinct subthemes:

- The first concerns the structure of unique perfect matchings, in particular their “inductive” or “hereditary” characterization suggested by Kotzig’s theorem. We relate this with the blossoms [Edm65] which appear in combinatorial matching algorithms. This was discovered by directly transposing a theorem on proof nets, due to Bellin [Bel97], using the reductions defined in our previous work [Ngu18]; it turned out that a notion of “dependency” investigated by logicians corresponded exactly to blossoms. The result already appeared in the conference version of the latter paper, but we present it here in a stand-alone way for a broader audience of graph theorists, using the direct proof from [Ngu18, Appendix D].

Part of this work was carried out when the author was a student at École normale supérieure de Paris. After the author joined the Laboratoire d’informatique de Paris Nord and continued working on this paper, he was at first partially supported by the now finished ANR project Elica (ANR-14-CE25-0005).
• The second is a study of some constrained path-finding problems, which covers the results that we have previously announced at the CTW’18\(^1\) workshop. We show, for most of these studied problems, that they belong to the aforementioned family. This leads to algorithms to solve them and “structure from acyclicity” properties. In one case, we establish a “dichotomy theorem” between a tractable case – which is indeed equivalent to augmenting paths in matchings – and a NP-complete case. A key ingredient in all of those results is a simple \textit{edge-colored line graph} construction; here the input of proof nets is that this construction was discovered by dissecting Retoré’s aforementioned work \cite{Ret03}, in which it implicitly occurs. We also prove, at the end of the paper, a simple NP-completeness result that does not rely on this construction.

The first item will be treated in section \(2\) while the second one – whose results – will be the subject of the rest of the paper. For the remainder of the introduction, we will explain in more detail the contents of this second item, with the following table providing a summary of the discussion.

(Our contributions, marked in bold, fill some gaps and thus answer several natural questions.)

| Time complexity / additional results | Path avoiding forbidden transitions | Trail avoiding forbidden transitions | Properly colored path | Properly colored trail | Rainbow path/trail\(^2\) | Properly colored directed path | Properly colored directed trail | Alt. directed path for matching\(^3\) |
|-------------------------------------|-----------------------------------|------------------------------------|----------------------|------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|
| NP-complete with dichotomy result \cite{Sze03} | \textbf{Linear with structural theorem} | Linear with structural theorem (cf. \cite{BJG09}) | Linear with structural theorem \cite{ADF08} | \textbf{NP-complete} \cite{CFMY11}, with dichotomy result and structural theorem for the tractable case | Linear (already polynomial in \cite{GLMM13}) | \textbf{NP-complete} |

\textbf{An important terminological note:} following a common usage \cite{BJG09, Section 1.4}, a \textit{path} is a walk without repeating \textit{vertices} and a \textit{trail} is a walk without repeating \textit{edges}; a \textit{cycle} (resp. \textit{closed trail}) is a closed walk without repeating \textit{vertices} (resp. \textit{edges}). Paths (resp. cycles) are trails (resp. closed trails), but the converse does not always hold.

1.1. \textbf{Edge-colored graphs.} From an assignment of colors to the edges of a graph, one can define either \textit{local} or \textit{global} constraints:

• In a \textit{properly colored} (PC) path \cite{BJG09, Chapter 16}) or trail \cite{ADF08}, \textit{consecutive} edges must have different colors. Both can be found in linear time by reduction to augmenting paths, and conversely augmenting paths are a special case of both these problems. The structural result for PC cycles is Yeo’s theorem on cut vertices separating colors \cite{BJG09, §16.3}.

• In a \textit{rainbow} (also called \textit{heterochromatic} or \textit{multicolored}) path, all edges have different colors. Finding a rainbow path is NP-complete \cite{CFMY11} in the general case; there is also an algorithm running in \(2^k n^{O(1)}\) time and polynomial space for \(k\) colors and \(n\) vertices \cite{KL16}.

\(^1\)Cologne-Twente Workshop on Graphs and Combinatorial Optimization 2018.
\(^2\)The existence of a rainbow path is equivalent to the existence of a rainbow trail between two vertices.
\(^3\)Alternating directed trails for a perfect matching – with the right definition of the latter in a directed graph – are a special case of properly colored directed trail, so the linear algorithm mentioned in the penultimate row applies. However, NP-hardness works the other way around (while membership in NP is always trivial here): our result on alternating directed paths is slightly stronger than the one on properly colored paths in \cite{GLMM13} (though we derive the former quite easily from the latter).
Let us also mention the related subject of *rainbow connectivity*\(^4\), that has been an active area of research (see e.g. the PhD thesis [Lau16] since its introduction in [CJMZ08].

For rainbow paths, we investigate whether restrictions on the shape of the *color classes* – that is, the subgraphs induced by all edges of a given color – make the problem tractable, and we establish a dichotomy: there is a single case which is not NP-hard (theorem 5.2), and it can be solved in linear time. This tractable case also exhibits structure from acyclicity (theorem 5.4).

A special case of this structural theorem had been previously proved in Retoré’s PhD thesis [Ret93, Chapter 2], for a class of graphs he called “aggregates”, in an early attempt to extract the graph-theoretic content of the theory of proof net correctness. Aggregates are edge-colored graphs whose color classes are complete bipartite; we will call them *bipartite decompositions* instead, following [ABS91]. Our result generalizes this to *multipartite decompositions* (definition 5.3).

1.2. Forbidden transitions. A very general notion of *local* constraints is to simply forbid some pairs of edges from occurring consecutively in a path – what has been called a *path avoiding forbidden transitions* (we will also speak of *compatible paths*). Finding a compatible path has been proven to be NP-complete in general [Sze03], with a dichotomy theorem (the tractable case covers in particular properly colored cycles). However, the question for compatible trails does not seem to have been asked before in its full generality (despite previous work on *Eulerian trails avoiding forbidden transitions* [Fle90, Chapter VI]).

We show that it is, again, part of our family of equivalent problems; more precisely compatible trails can be found with time complexity *linear* in the number of *allowed* transitions (theorem 4.11). The associated structural result (theorem 4.12) entails, as a corollary, a new proof of the one for properly colored trails [ADF+08, Theorem 2.4].

1.3. Arc-colored directed graphs. The notion of properly colored path/trail also makes sense in *directed graphs* equipped with a coloring of their arcs (directed edges). We first look at PC directed trails and show that they can be found by a simple breadth-first search algorithm, in linear time (theorem 6.2). There was already a polynomial time algorithm for this problem in the literature [GLMM13, Theorem 1]; we examine more precisely its asymptotic complexity and argue that it is worse than ours (remark 6.3).

In contrast, deciding the existence of a PC directed path is NP-complete, even when we assume that the input has no PC circuit, as shown in [GLMM13, Theorem 5]. From this, we deduce that finding an alternating *circuit* (i.e. directed cycle) for a (certain notion of) perfect matching in a directed graph is NP-complete (theorem 6.7) (and as a corollary, this is also the case for directed paths). Though our reduction is pretty straightforward, and the statement itself might seem unsurprising, we have proved it as a step in a possibly much more significant result that we obtain in [Ngu20]: the refutation of an old conjecture in proof theory, namely the equivalence between pomset logic and system BV, see [Gug07] (though at the time of writing, we are not entirely sure that our refutation is correct). This is why we propose in [Ngu20] an alternative and more direct NP-hardness proof for alternating circuits by reduction from CNF-SAT, in order to make the proofreading of this latter paper more self-contained and avoid relying on [GLMM13].

1.4. The edge-colored line graph construction. The construction underlying these results (except for those on arc-colored digraphs) is the following. Given a set of forbidden transitions on a graph \(G\), one can consider the subgraph of its line graph \(L(G)\) containing only the edges corresponding to allowed transitions. To keep the information of the vertices in \(G\), one adds a coloring

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\(^4\)An edge-colored graph is rainbow connected if all pairs of vertices can be joined by a rainbow path; a common question is coloring the edges of a graph with a minimum number of colors to make it rainbow connected.

\(^5\)Though this paper [Gug07] was published in 2007, it has its origins in a technical report from the late 1990’s which already contained this conjecture.
of the edges of this subgraph of $L(G)$: this is what we call the edge-colored line graph $L_{EC}(G)$ (definition 4.3).

The results on trails avoiding forbidden transitions immediately follow from the properties of $L_{EC}(G)$ together with the known results on PC paths. For this specific purpose one can also use a variant $L_{PM}(G)$ defined using perfect matchings (definition 4.7). As will be explained in the journal version of [Ngu18], this $L_{PM}$ construction is at work implicitly Retoré’s paper [Ret03], and indeed, it is by attempting to understand and generalize Retoré’s reduction from proof nets to perfect matchings that we were led to define the edge-colored line graph – although it could undoubtedly have been discovered without this inspiration, since it seems to be a rather natural construction.

As for the dichotomy theorem for rainbow paths, it also relies mainly on the $L_{EC}$ construction, combined with proof techniques from [Sze03] and [CFMY11], in particular a characterization of complete multipartite graphs by excluded vertex-induced subgraphs [Sze03 Lemma 7].

1.5. **Other contributions.** For some of the aforementioned constraints, we investigate the problem of finding a constrained path/trail visiting some prescribed intermediate vertex/edge. The general pattern is NP-completeness in the general case and tractability under an acyclicity assumption. The initial motivation for this was as a subroutine to compute the “kingdom ordering” on unique perfect matchings studied in section 2 which has a rather meaningful logical counterpart (see [Ngu18]) as already mentioned.

We also make minor contributions to the theory of properly colored paths and cycles, using slight variations on pre-existing proofs, in section 3. Specifically, we show how to find properly colored cycles in linear time, and generalize a reduction from 2-edge-colored graphs to matchings.

1.6. **Acknowledgments.** Thanks a lot to Christoph Dürr for drawing my attention to the topic of constrained path-finding (accidentally, through his enthusiasm for programming contest problems), and for his feedback on early iterations (2016–2017) of this work. Thanks also to Nguyễn Kim Thắng for letting me work on this side project during my internship with Christoph and him.

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2. BRIDGES AND BLOSSOMS IN UNIQUE PERFECT MATCHINGS

As mentioned in the introduction, we present here a new graph-theoretic result directly coming from the theory of proof nets. We will also take the opportunity, in section 2.1 and section 2.3, to recall well-known properties of perfect matchings which the rest of the paper will rely on.

Let $G = (V, E)$ be a graph. Recall that a matching $M$ of $G$ is a subset of edges $M \subseteq E$ such that each vertex is incident to at least one edge. If each vertex is incident to exactly one edge, then $M$ is a perfect matching. We now recall a few properties of unique perfect matchings.

**Lemma 2.1** (Berge’s lemma for cycles\(^6\)). Let $G$ be a graph and $M$ be a perfect matching of $G$. If $C$ is an alternating cycle for $M$ – i.e. $C$ alternates between edges in $M$ and outside of $M$ – then the symmetric difference $M \triangle C$ is another perfect matching.

Conversely, if $M' \neq M$ is a different perfect matching, then $M \triangle M'$ is a vertex-disjoint union of cycles, which are alternating for both $M$ and $M'$.

Thus, a perfect matching is unique if and only if it admits no alternating cycle.

**Theorem 2.2** (Kotzig [Kot59]). Let $G$ be a graph with a unique perfect matching $M$. Then $M$ contains a bridge of $G$, i.e. an edge whose removal increases the number of connected components.

2.1. Decomposing unique perfect matchings. Kotzig’s theorem gives us a simple way to check that $M$ is the only perfect matching in $G$: find some $(u, v) \in M$ which is a bridge of $G$, delete its two endpoints from the graph, and repeat on the vertex-induced subgraph $G[V \setminus \{u, v\}]$, in which $M \setminus \{(u, v)\}$ is a perfect matching. The matching is unique if and only if we end up with the empty graph. Equivalently, we may state:

**Proposition 2.3.** Among the set of pairs $(G, M)$ of a graph $G$ and a perfect matching $M$ of $G$, the subset of those for which $M$ is the unique perfect matching of $G$ is the smallest subset:

- containing the empty graph;
- such that, if $(u, v) \in M$ is a bridge in $G = (V, E)$ and $(G[V \setminus \{u, v\}], M \setminus \{(u, v)\})$ is in the subset, then $(G, M)$ also is.

**Remark 2.4.** “The smallest subset such that...” suggests an inductive, “bottom-up” characterization: the graphs equipped with unique perfect matchings are obtained from the empty graph by successively adding matching edges and potentially joining their endpoints to different connected components. Precise statements of this kind are given in [Ret03 Theorem 1], [Ngu18 Theorem 2.4].

**Remark 2.5.** This iterative bridge deletion procedure admits a variant which does not look at $M$: for $G = (V, E)$, choose a bridge $(u, v) \in E$ such that the connected components of $u$ and $v$ in $(V, E \setminus \{(u, v)\})$ have an odd number of vertices (equivalently, the two new connected components in $G[V \setminus \{u, v\}]$ disconnected by the removal of $u$ and $v$ have an even number of vertices). If, by doing so iteratively, one reaches the empty graph, then the set of deleted bridges is the unique perfect matching of $M$. This gives a quasi-linear time algorithm for finding a unique perfect matching, see [GKT01 §2] and [HRT18 §1.2].

\(^6\)Though this statement does not appear in [Ber57], it is a simple variant of Berge’s lemma for paths (lemma 2.13).
We will be interested in the order of deletion of the matching edges for some execution of this procedure reaching the empty graph.

**Definition 2.6.** Let $G = (V, E)$ be a graph with a unique perfect matching $M$. A *bridge deletion ordering* is an ordering of the matching edges $M = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ ($|V| = 2n$) such that for all $i \in \{1, \ldots, n\}$, $(u_i, v_i)$ is a bridge in $G[V \setminus \{u_1, v_1, \ldots, u_{i-1}, v_{i-1}\}]$.

We define the *kingdom ordering* $\prec$, which is a partial order, as follows: $e \prec f$ if $e$ occurs before $f$ in all bridge deletion orderings of $M$ ($e, f \in M$).

Bridge deletion orderings in unique perfect matchings are somewhat analogous to perfect elimination orders in chordal graphs, with bridges instead of simplicial vertices. The terminology “kingdom ordering” is lifted straight from linear logic (see [Bel97]); it comes from the notion of *kingdom* whose graph-theoretic version we now define.

**Definition 2.7.** Let $e \in M$. The result of iteratively deleting the endpoints of bridges in $M$ except for $e$ is a non-empty vertex-induced subgraph of $G$ called the *kingdom* of $e$.

The idea is that the endpoints of $e$ must be deleted before any other matching edge in the kingdom can be deleted in the decomposition by bridge deletion. Thus, $e \prec f$ if and only if $f$ is in the kingdom of $e$, or equivalently, if the kingdom of $f$ is included in the kingdom of $e$.

### 2.2. Blossoms and Bellin’s theorem.

A *blossom* is a cycle such that all its vertices are matched within the cycle except for one, its *root*. The *stem* of a blossom is the matching edge incident to its root. We will characterize the kingdom ordering using the following notion:

**Definition 2.8.** We say that $e$ *blossom-binds* $f$, and write $e \rightarrow f$, when $e$ is the stem of some blossom $C$ such that $f \in M \cap C$.

We write $\rightarrow^+$ (resp. $\rightarrow^*$) for the transitive (resp. reflexive transitive) closure of $\rightarrow$.

The main theorem of this section can now be stated:

**Theorem 2.9.** Let $M$ be a unique perfect matching. $\forall e, f \in M$, $e \prec f \iff e \rightarrow^+ f$.

In a previous paper [Ngu18], we explained why this was equivalent to Bellin’s theorem on proof nets [Bel97]. This equivalence already establishes the truth of the above theorem, but we will give a direct proof without reference to logic. Indeed it was noted in [Ngu18] that translating to the setting of perfect matching simplifies the statement – which now involves the transitive closure of a single relation, instead of two – and so, accordingly, our proof should be simpler than Bellin’s.

**Proof.** If $e \rightarrow f$ then $e \prec f$: $f$ cannot become a bridge as long as the cycle $C$ survives, and the only way to cut $C$ from the outside is to delete $e$. By induction this establishes ($\Leftarrow$).

Conversely, let $G$ be a graph with a unique perfect matching $M$. We may assume w.l.o.g. that $e$ is the only bridge of $G$ in $M$, by restricting to the kingdom of $e$. Then $e$ is minimum for $\prec$ and the goal becomes to show that for all $f \neq e$, $e \rightarrow^+ f$. Removing the edge $e$, but not its endpoints, results in two connected components which both have a unique near-perfect matching (leaving one vertex unmatched) containing no bridge. If both these components have a single vertex, then the theorem is vacuously true; else, we have reduced it to the following proposition [2.11].

**Definition 2.10.** We say that a vertex $u$ *blossoms-binds* a matching edge $f$, which we write $u \rightarrow f$, when $f$ is contained in a blossom with root $u$.

**Proposition 2.11.** Let $G$ be a graph with a near-perfect matching $M$ and let $u$ be the unmatched vertex. Suppose $G$ has no bridge in $M$ and no alternating cycle for $M$. Then for all $f \in M$, there exists $g \in M$ such that $u \rightarrow g \rightarrow^* f$.
The proof of this proposition relies on the blossom shrinking operation: starting from the graph $G$ with a matching $M$, this consists in taking the quotient graph $G'$ where all the vertices of the blossom have been identified; $M$ induces a matching $M'$ in $G'$. This operation is also central in combinatorial matching algorithms, starting with Edmonds’s blossom algorithm [Edm65].

**Lemma 2.12.** Under the hypotheses of the proposition, if $M \neq \emptyset$, then:

1. There exists a blossom in $G$ for $M$ with root $u$.
2. Let $G'$ be the graph obtained by shrinking this blossom, with induced matching $M' \subset M$. There is no bridge in $M'$ and no alternating cycle for $M'$.
3. Let $u'$ be the exposed vertex in $G'$, corresponding to the shrunk blossom. For all $f \in M'$ with $u' \rightarrow f$ in $G'$, there exists $g \in M$ such that $u \rightarrow g \rightarrow^* f$ in $G$.

**Proof of proposition 2.11 using lemma 2.12.** By induction on the size of $G$.

Let us take a blossom using lemma (1). If it contains $f$, then $u \rightarrow f$ and we are done. Else, we shrink the blossom and get $G'$, $M'$ and $u'$; by lemma (2), they satisfy the assumptions of the proposition. By the induction hypothesis, there exists $g$ such that $u' \rightarrow g \rightarrow^* f$ in $G'$. Thanks to lemma (3), $u' \rightarrow g$ entails $u \rightarrow h \rightarrow^* g$ in $G$ for some $h \in M$. Also, $g \rightarrow^* f$ in $G'$ entails $g \rightarrow^* f$ in $G$ because the (possibly empty) sequence of blossoms which binds $f$ to $g$ in $G'$ cannot contain the vertex $u'$, and therefore lifts to exactly the same edges in $G$. Thus, $u \rightarrow h \rightarrow^* g \rightarrow^* f$ and therefore $u \rightarrow h \rightarrow^* f$ in $G$.

**Proof of lemma 2.12 (1).** The absence of alternating cycle amounts to saying that $M$ is the unique perfect matching of $G[V \setminus \{u\}]$ where $V$ is the vertex set of $G$. (Note that $M \neq \emptyset \Leftrightarrow V \setminus \{u\} \neq \emptyset$.) By Kotzig’s theorem, $M$ contains a bridge $e$ of $G[V \setminus \{u\}]$; let $V_1$ and $V_2$ be the connected components created by the removal of $e$ (but keeping its endpoints) from $G[V \setminus \{u\}]$. We create a new graph $H$ by starting from $G[V \setminus \{u\}]$, adding two new vertices $u_1$ and $u_2$, and adding the edges $(u_i, v)$ for all $v \in V_i$ with $v$ adjacent to $u$ in $G (i = 1, 2)$, and the edge $(u_1, u_2)$.

The perfect matching $M \cup \{(u_1, u_2)\}$ of $H$ contains no bridge of $H$: since $e$ is not a bridge of $G$, there is at least one edge between $u_1$ and $V_1$ and one edge between $u_2$ and $V_2$, so $(u_1, u_2)$ is not a bridge in $H$; and any edge of $M$ would be a bridge of $G$ if it were a bridge of $H$. Let us apply Kotzig’s theorem again: this perfect matching admits an alternating cycle, which cannot be contained in $H[V \setminus \{u\}] = G[V \setminus \{u\}]$. Therefore, it contains an alternating path from $u_1$ to $u_2$, from which we retrieve a blossom with root $u$ in $G$.

**Proof of lemma 2.12 (2).** If there existed a bridge $e \in M'$ of $G'$, then $G' \setminus \{e\}$ would be disconnected while $G \setminus \{e\}$ would be connected; this is impossible. An alternating cycle for $M'$ would not visit $u'$ because it is unmatched, and therefore would be an alternating cycle for $M$ in $G$.

**Proof of lemma 2.12 (3).** Let $B$ be the blossom with root $u$ in $G$ that has been shrunk, and $B'$ be the blossom with root $u'$ in $G'$ containing $f$. There are two non-matching edges $e_1'$ and $e_2'$ in $B'$ incident to $u'$; let $e_1 = (u_1, v_1)$ be a preimage of $e_1'$ and $e_2 = (u_2, v_2)$ be a preimage of $e_2'$ in $G$, with $u_1, u_2 \in B$.

The blossom $B$ can be decomposed into $P_1 \cup Q \cup P_2$, where $P_1$ is an alternating path from $u$ to $u_1$ (possibly empty, if $u = u_1$), $Q$ is an alternating path from $u_1$ to $u_2$ (possibly empty, if $u_1 = u_2$), and $P_2$ is an alternating path from $u_2$ to $u$. As for $B'$, it lifts to an alternating path $R$ between $u_1$ and $u_2$ starting and ending with a non-matching edge, so that $|R|$ is odd and $f \in R$. We proceed by case analysis on the parity of $|P_1|$ and $|P_2|$.

- If they are both even, then $P_1 \cup R \cup P_2$ is a blossom: $u \rightarrow f \rightarrow^* f$.
- If $|P_1|$ is even and $|P_2|$ is odd, then $Q \cup R$ is a blossom with root $u_1$. Either $u_1 = u$ and then $u \rightarrow f_1$, or there is an edge $g \in B \cap M$ incident to $u_1$ and then $u \rightarrow g \rightarrow f$.
- The case $|P_1|$ even and $|P_2|$ odd is symmetric to the previous one.
- If they were both odd, $Q \cup R$ would be an alternating cycle.
2.3. Algorithmic aspects. One interest of characterizing the kingdom ordering through blossom-binding is that the later can be decided in polynomial time; thus, it shows that computing the kingdom ordering is tractable.

To expound on this, we need to recall some bits of the classical theory around matching algorithms. We start with a version of Berge’s lemma which is relevant to the maximum matching problem, complementing lemma 2.1.

**Lemma 2.13** (Berge’s lemma for paths [Ber57]). Let \( G \) be a graph and \( M \) be a matching of \( G \). If \( P \) is an augmenting path for \( M \) – i.e. an alternating path whose endpoints are unmatched – then \( \triangle P \) is a matching and \( |\triangle P| = |M| + 1 \). Conversely, if \( M \) is a matching with \( |M'| > |M| \), then \( \triangle M' \) is a vertex-disjoint union of:

- \(|M'| - |M|\) augmenting paths for \( M \);
- some (possibly zero) cycles which are alternating for both \( M \) and \( M' \).

**Theorem 2.14.** There are algorithms with running time linear in the edges of the input for:

- finding an augmenting path for a matching, or detecting that the matching is maximum;
- deciding whether a perfect matching is unique, and finding an alternating cycle if it is not.

**Proof.** See [GT85] and [Tar83, Section 9.4] for the first result, [GKT01] for the second. Note that both of these algorithms use blossom shrinking.

Since a significant portion of the algorithms we will see in this paper ultimately proceed by a reduction to an augmenting path or alternating cycle problem, the above results play a central role.

To decide the blossom-binding relation, we need to find blossoms with both the stem and one intermediate edge prescribed. A first step towards this will be to find augmenting paths crossing a given edge. We will see later that this is NP-complete in general (corollary 3.12). However, the problem is tractable in the absence of alternating cycles (plus a condition which will always be satisfied in the various reductions to augmenting paths used in this paper).

**Lemma 2.15.** Let \( G = (V, E) \) be a graph and \( M \) be a matching of \( G \). Suppose that:

- there are no alternating cycles for \( M \) – equivalently, \( M \) is the unique perfect matching of the subgraph induced by the vertices matched by \( M \);
- there are exactly two unmatched vertices.

Then the existence of an augmenting path for \( M \) crossing a prescribed matching edge \( e \in M \) can be reduced to the existence of a perfect matching; and such a path can be found in linear time.

**Proof.** Let \( u, v \in V \) be the unmatched vertices. If there is an augmenting path for \( M \) in \( G \), its endpoints must be \( u \) and \( v \), and this is equivalent to the existence of a perfect matching in \( G \). Let \( e = (a, b) \), \( G' = (V, E \setminus \{e\}) \) and \( M' = M \setminus \{e\} \).

Suppose \( G' \) admits a perfect matching \( M'' \). Then the symmetric difference \( M' \triangle M'' \) consists of two vertex-disjoint alternating paths for \( M' \) whose endpoints are \( \{u, v, a, b\} \), by Berge’s lemma for paths (lemma 2.13); indeed, our assumptions prevent the existence of alternating cycles for \( M' \), and therefore for \( M' \subset M \) as well.

We claim that these paths either go from \( u \) to \( a \) and \( b \) to \( v \), or from \( u \) to \( b \) and \( a \) to \( v \). Otherwise, there would be an alternating path from \( a \) to \( b \) for \( M' \) in \( G' \), and together with \( (a, b) = e \in M \), this would give us an alternating cycle for \( M \) in \( G \).

In both cases, let us join the two paths together by adding \( e \). We get a path starting with \( u \), ending with \( v \), crossing \( e \) and alternating for \( M \) in \( G \). Conversely, from such a path, one can get a perfect matching in \( G' \).

For the linear time complexity, we exploit the fact that we already have at our disposal a matching \( M' \) of \( G' \) which leaves only 4 vertices unmatched. A perfect matching can then be found as follows: find a first augmenting path \( P \) for \( M' \) in linear time, and then a second one \( P' \) for \( M' \triangle P \), both
steps being done in linear time by theorem 2.14. If both augmenting paths exist, then \( M \triangle P \triangle P' \) is a perfect matching, and conversely, if \( G' \) admits a perfect matching, then the procedure succeeds in finding some \( P \) and \( P' \). (This does not mean that \( P \) and \( P' \) are the same as the paths in the previous part of the proof, since they may not be vertex-disjoint.) □

**Theorem 2.16.** Let \( M \) be the unique perfect matching of a graph \( G \), \( u \in V \) and \( e, f \in M \). The blossom-binding relations \( u \rightarrow e \) and \( e \rightarrow f \) can both be decided in linear time.

**Proof.** For \( e = (v, w) \), \( e \rightarrow f \) if and only if \( v \rightarrow f \) or \( w \rightarrow f \), so it suffices to treat the case \( u \rightarrow e \). This is done by a reduction to the previous lemma.

We build a graph \( G' \) by adding two new vertices \( s, t \) to \( G \); the neighbors of \( s \) in \( G' \) are exactly those of \( u \) in \( G \), and the same goes for \( t \). If \( P \) is an augmenting path in \( G' \), then by replacing both endpoints \( s \) and \( t \) by \( u \), we get a blossom in \( G \) whose root is \( u \); the converse also holds. □

**Corollary 2.17.** The kingdom ordering of a unique perfect matching can be decided in time \( O(n^2 m) \) for a graph with \( n \) vertices and \( m \) edges.

**Proof.** The previous result allows us to compute the blossom-binding relation between all pairs of matching edges in time \( O(n^2 m) \) (there are \( n/2 \) matching edges). The conclusion follows from theorem 2.9 and the \( O(n^3) \) Floyd–Warshall algorithm for transitive closure. □

3. A FEW REMARKS ON PROPERLY COLORED PATHS AND CYCLES

In this section, we mostly focus on recalling known result on edge-colored graphs and provide some minor improvements which will be useful in the remainder of the paper. Novelties such as the edge-colored line graph and its applications are to be found in the next sections, not here.

As usual, there are both an efficient algorithm for finding properly colored paths, and an associated “structure from acyclicity” theorem.

**Theorem 3.1.** Let \( G \) be an edge-colored graph and \( u, v \) be two vertices in \( G \). A properly colored path between \( u \) and \( v \) can be found in linear time.

**Definition 3.2.** In an edge-colored graph, the chromatic degree of a vertex is the number of different colors among its incident edges.

**Theorem 3.3** (Yeo [Yeo97]). Let \( G \) be an edge-colored graph with no properly colored cycles. Note \( V \) for the vertex set of \( G \). Then, \( G \) has a color-separating vertex \( u \in V \): for each connected component \( C \) in \( G[V \setminus \{u\}] \), all edges between \( C \) and \( u \) have the same color. In particular, if \( u \) has chromatic degree \( \geq 2 \), then it is a cut vertex.

The algorithmic part (theorem 3.1) was actually first proven in Szeider’s paper on forbidden transitions [Sze03], by reduction to augmenting paths. A more general presentation of this reduction, parameterized by a choice of “P-gadget”, was later given in [GK09] (see also the exposition in [BJG09, Section 16.4]).

There is a much simpler reduction to matchings in the case of 2-edge-colored graphs, i.e. edge-colored graphs using only two colors. It seems to have been first published in [Man95], where it is attributed to Edmonds. The existence of this reduction also makes the 2-edge-colored case of Yeo’s theorem above immediately deducible from Kotzig’s theorem (theorem 2.2); that said, this case was first proved in [GH83] without using Kotzig’s theorem.

Our two small contributions here are the following:

- To our knowledge, no linear time algorithm for finding properly colored cycles appears in the literature. (A slightly worse than linear algorithm is proposed in [BJG09, Corollary 16.4.3].) We repair this omission in this section, by a mostly straightforward adaptation of the reduction for paths in [GK09] – the main subtlety being that we need to tweak the definition of “P-gadgets” for our purpose.
We generalize the reduction for the 2-edge-colored case to cover edge-colored graphs with chromatic degree $\leq 2$, and unify it with the well-known correspondence between bipartite matchings and directed graphs. This is achieved by extending it to a combinatorial bijection between graphs equipped with perfect matchings and a new object we call “locally 2-colored graphs”.

3.1. Finding properly colored cycles in linear time.

**Definition 3.4.** A $P$-gadget on the vertices $V$ is a graph $G = (V', E)$ equipped with a unique perfect matching $M$ such that $V \subseteq V'$ and, for each nonempty $U \subseteq V$, $G[V' \setminus U]$ has at least one perfect matching if and only if $|U| = 2$.

This differs slightly from the definition in $[GK09]$:

- $M$ does not merely exist, but is included as part of the data (this is a minor detail);
- more importantly, we require this perfect matching $M$ to be unique.

This last condition will help us for cycle existence problems.

**Remark 3.5.** Szeider proved the equivalence between Kotzig’s and Yeo’s theorems (theorems 2.2 and 3.3) in $[Sze04]$ using a particular $P$-gadget; the proof adapts using any $P$-gadget according to our definition, making use of this uniqueness condition.

**Lemma 3.6.** For every finite vertex set $V$, there exists a $P$-gadget on $V$ with size $O(|V|)$ and it can be constructed in time $O(|V|)$.

**Proof.** Three possible constructions are given in $[BJG09]$ Subsection 16.4.1] and one can check that they fit our altered definition of a $P$-gadget. □

**Theorem 3.7.** A properly colored cycle in an edge-colored graph can be found in linear time.

**Proof.** We proceed by a linear-time reduction to the problem of finding an alternating cycle and apply theorem 2.14.

Let $G = (V, E)$ be an edge-colored graph. Without loss of generality, we can assume that $G$ has no isolated vertices.

For each $v \in V$, let $c_1, \ldots, c_k$ be the colors used by the edges incident to $v$, $k$ being the chromatic degree of $v$. Introduce fresh vertices $v_{c_1}, \ldots, v_{c_k}$ and construct a $P$-gadget $(G_v, M_v)$ on $\{v_{c_1}, \ldots, v_{c_k}\}$.

Build a graph $G'$ by first, taking the disjoint union of the $G_v$ for $v \in V$ and then, for each original edge $e = (u, v) \in E$ with color $c$, adding an edge between $u_c$ and $v_c$ in $G'$. The target of the reduction is this graph $G'$ equipped with the perfect matching $M' = \bigcup_v M_v$. Thanks to lemma 3.6 this reduction is indeed linear-time.

We claim that this perfect matching is unique if and only if $G$ has no properly colored cycle. First, notice that the $P$-gadgets are disconnected from each other by the removal of $E'$ from $G'$.

Thus, from the uniqueness condition in our definition of $P$-gadgets, it follows that $M'$ is the only perfect matching in $G'$ which does not intersect $E'$.

Suppose $M'' \neq M'$ is a perfect matching. For each $v \in V$, let $V_v$ be the vertices of $G_v$, and $U_v$ be those covered by an edge in $M'' \cap E'$. $U_v \subseteq \{v_{c_1}, \ldots, v_{c_k}\}$ and, since $M''$ induces a perfect matching on $G_v[V_v \setminus U_v], |U_v| \in \{0, 2\}$. $M'' \cap E'$ therefore lifts to a non-empty set $C \subseteq E$ of edges in $G$ such that every vertex of $G$ is incident either to no edge, or to two edges with different colors, of $C$. In other words, $C$ is the edge set of a disjoint union of properly colored cycles.

Thus, by finding an alternating cycle $C'$ for $M'$ in $G'$, we can obtain such an $M''$ as the symmetric difference $C' \triangle M'$, which allows us to retrieve a properly colored cycle in $G$.

Conversely, properly colored cycles in $G$ map to alternating cycles for $M'$, up to a choice, for each visited vertex $v$, of a perfect matching of $G_v[V_v \setminus \{v_c, v_{c'}\}]$ where $c$ and $c'$ are the colors of the two edges of the cycle incident to $v$. □
3.2. **Locally 2-colored graphs.** As mentioned before, we now introduce a new object and put it in bijection with graphs equipped with perfect matchings. This can be thought of as a new way to see the latter, whose benefit is to make some reductions to matchings more obvious.

We write \( \partial(u) \) for the set of incident edges of a vertex \( u \).

**Definition 3.8.** A **locally 2-colored graph** is a graph in which each vertex \( u \) comes with a partition \( \partial(u) = R_u \sqcup B_u \) ("red" and "blue" edges) of its incident edges. We allow not only simple graphs, but also multigraphs, under the condition that any pair of parallel edges has at least one endpoint where they differ in color.

A path is said to be **compatible** if, for every intermediate vertex \( u \), one of its incident edges in the path is in \( R_u \) and the other is in \( B_u \). Compatible cycles are defined analogously.

This generalizes properly colored paths in 2-edge-colored graphs, as well as in any edge-colored graph with chromatic degree bounded by 2 – the latter case will arise in the next section. Local 2-colorings are in turn a particular case of forbidden transitions (definition 4.1), see remark 4.2.

**Theorem 3.9.** There is a bijection (modulo isomorphism) between graphs equipped with a perfect matching and locally 2-colored graphs, through which alternating paths (resp. cycles) correspond to compatible paths (resp. cycles).

**Proof.** Let \( G = (V, E, R, B) \) be a locally 2-colored graph; we will build a graph \( G' = (V', E') \) with a perfect matching \( M \) from \( G \).

For each vertex \( u \in V \), add two new vertices \( u_R, u_B \) to \( V' \) and a edge \( e_u = (u_R, u_B) \) to \( E' \). Let \( M = \{ e_u \mid u \in V \} \). Then, for each edge \( e = (u, v) \), add an edge \( (u_X, v_Y) \) to \( E' \), where \( X, Y \in \{R, B\} \) are determined uniquely by the condition \( e \in X_u \cap Y_v \). The restriction on parallel edges in the definition of locally 2-colored (multi)graphs ensures that the resulting graph \( G' \) is simple.

The inverse bijection is given by contracting the edges in a perfect matching into vertices. As for the correspondence between compatible and alternating paths, it holds for the same reason as in the 2-edge-colored case [Man95, Lemma 1.1]. \( \square \)

An example of this bijection is shown in fig. 1.

![Graph with chromatic degree ≤ 2](image)

![The corresponding graph with perfect matching](image)
Corollary 3.10. In a locally 2-colored graph, finding a compatible path between two given vertices, or a compatible cycle, can be done in linear time.

Furthermore, if the graph has no compatible cycle, then a compatible path joining two given endpoints and visiting a third given intermediate vertex can be found in linear time.

Proof. Let \( G \) be a locally 2-colored graph. The corresponding graph with perfect matching \((G', M)\) can be computed in linear time.

If we are looking for a compatible cycle in \( G \), it suffices to find an alternating cycle for \( M \) in \( G' \). To find a path, suppose that \( e, f \in M \) are the edges of \( G' \) corresponding to the required endpoints in \( G \); we add two auxiliary unmatched vertices \( u, v \) to \( G' \), join \( u \) to both endpoints of \( e \), and similarly for \( v \) and \( f \), so that augmenting paths will correspond to compatible paths. In both cases, the result follows from theorem 2.14. For the path visiting a prescribed vertex, we apply lemma 2.15. □

Using an existing result on 2-edge-colored graphs, we can now show that the acyclicity assumption in the above corollary and in lemma 2.15 is necessary.

Theorem 3.11 \((\text{CMM}^{+94})\). For a 2-edge-colored graph \( G \) and three vertices \( s, t \) and \( u \) in \( G \), the existence of a properly colored path between \( s \) and \( t \) containing \( u \) is an NP-complete problem.

Corollary 3.12. Finding an augmenting path crossing a prescribed matching edge is an NP-complete problem, even when there are only two unmatched vertices.

Proof. NP-hardness is established by the same reduction as the previous corollary, and the problem is in NP (guess the path non-deterministically). □

Beyond its algorithmic applications, this combinatorial bijection also allows us to recover a well-known equivalence. A directed graph induces a local 2-coloring on the underlying undirected (multi)graph, which distinguishes incoming and outgoing arcs at each vertex; directed paths then correspond to compatible paths. On the other hand, finding an augmenting path for a matching in a bipartite graph amounts to traversing a directed graph; indeed, this is implicitly what happens when applying the Ford–Fulkerson flow algorithm to find a maximum matching, and it is also why historically bipartite maximum matchings were solved before the general case.

Proposition 3.13. The previous bijection sends bipartite graphs equipped with a perfect matching on the locally 2-colored encodings of directed graphs and vice versa.

Proof. In a locally 2-colored graph coming from a directed graph, each edge has two different colors since it is incoming for one of its endpoints and outgoing for the other one. Conversely, any such locally 2-colored graph can be realized as a directed graph: simply orient each edge from its blue side to its red side.

Now, this means that in the corresponding graph with perfect matching, every edge is between \( V_R = \{u_R \mid u \in V\} \) and \( V_B = \{v_B \mid v \in V\} \) (reusing the notations from a previous proof). Therefore, the graph is bipartite, with the partition of the vertices being \( V' = V_R \sqcup V_B \).

Remark 3.14. One could actually develop in a straightforward way, using the machinery of P-gadgets (previous subsection), a theory of “locally \( k \)-colored graphs” for \( k \geq 2 \), with linear time algorithms and a general formulation of the “local Yeo’s theorem”. We will not detail this generalization since it will not be needed in the rest of the paper.

4. Graphs with forbidden transitions

As mentioned in the introduction, we will turn our attention to a very general notion of local constraints on paths and trails.
**Definition 4.1** ([Sze03]). Let $G = (V, E)$ be a graph. A transition graph for a vertex $v \in V$ is a graph whose vertices are the edges incident to $v$: $T(v) = (\partial(v), E_v)$. A transition system on $G$ is a family $T = (T(v))_{v \in V}$ of transition graphs. A graph equipped with a transition system is called a graph with forbidden transitions.

A path (resp. trail) $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$ is said to be compatible if for $i = 1, \ldots, k-1$, $e_i$ and $e_{i+1}$ are adjacent in $T(v_{i+1})$. For a cycle (resp. closed trail), we also require $e_{k-1}$ and $e_1$ to be adjacent in $T(v_1) = T(v_k)$.

(That is, the edges of the transition graphs specify the allowed transitions, i.e. the pairs of edges which may occur consecutively in a compatible path or trail.)

**Remark 4.2.** An edge-coloring of a graph induces a transition system made of complete multipartite graphs: for each vertex $v$, two edges of $\partial(v)$ are adjacent in $T(v)$ if and only if they have different colors. Properly colored paths (resp. trails) are exactly the paths (resp. trails) compatible with this transition system.

For a similar reason, locally 2-colored graphs (definition 3.8) correspond exactly\(^7\) to graphs with forbidden transitions whose transition graphs are all complete bipartite, the notion of compatible path / cycle / (closed) trail being the same.

4.1. **The edge-colored line graph.** We now introduce a key construction of this paper.

**Definition 4.3.** Let $G = (V, E)$ be a graph and $T$ be a transition system on $G$. The EC-line graph $L_{EC}(G, T)$ is formed by taking the line graph of $G$, coloring its edges so that the clique corresponding to $v$ is given the color $v$ (using the vertices of $G$ as the set of colors), and deleting the edges corresponding to forbidden transitions.

Formally, $L_{EC}(G, T)$ is defined as the graph with vertex set $E$ and edge set $F = \bigcup_{v \in V} T(v)$, equipped with an edge coloring $c : E' \to V$ with values in $V$: for $f \in F$, $c(f)$ is the unique vertex such that $f \in T(c(f))$.

Consider a vertex in the line graph, corresponding to an edge $e = (u, v)$ in the original graph. Its neighbors are the edges which share a vertex with $e$. Equipping the line graph with an edge coloring allows us to distinguish between the neighbors of $e$ incident to $u$ and those incident to $v$. The following proposition encapsulates the usefulness of this additional information.

**Proposition 4.4.** The compatible cycles (resp. closed trails) of a graph with forbidden transitions correspond bijectively to rainbow (resp. properly colored) cycles in its EC-line graph.

**Proof.** What happens here is fairly intuitive. Let us however pinpoint a crucial role played by the edge coloring: it excludes monochromatic sub-paths of length $> 1$ in transition graphs, which could allow forbidden transitions to be taken. For instance, suppose we have a vertex with three incident edges $e, f, g$ such that the allowed transitions are $e \leftrightarrow f$ and $f \leftrightarrow g$; then a path in the EC-line graph containing the sub-path $e \to f \to g$ would translate, in the original graph, to a path where $e$ and $g$ occur consecutively.

This explains why we want properly colored cycles – rainbow cycles being, in particular, properly colored. As for the distinction between compatible cycles and closed trails, notice that the global condition of non-repetition of vertices translates in the EC-line graph to requiring a rainbow cycle, whereas a repeated edge in the original graph would correspond to a repeated vertex in the EC-line graph.

The situation for paths and trails is analogous to the above, a few precisions being necessary to ensure bijectivity.

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\(^7\)Modulo the fact that in a locally 2-colored graph, a pair of vertices may have two parallel edges joining them, if they are colored differently on one endpoint.
Proposition 4.5. Let $G$ be a graph with transition system $T$, and $s,t$ be two distinct vertices of $G$. The compatible paths between $s$ and $t$ correspond bijectively to rainbow paths in $L_{EC}(G,T)$ between some vertex of $\partial(s)$ (identified with a subset of vertices in $L_{EC}(G,T)$) and some vertex of $\partial(t)$ which do not cross any edge with color $s$ or $t$.

Similarly, the compatible trails between $s$ and $t$ where neither $s$ nor $t$ appear as intermediate vertices correspond bijectively to properly colored paths in $L_{EC}(G,T)$ between some vertex of $\partial(s)$ and some vertex of $\partial(t)$ which do not cross any edge with color $s$ or $t$.

From now on until the end of this section, we will focus mostly on the correspondence between compatible trails and properly colored paths. To apply the results of section 4.2, we remark that:

Proposition 4.6. EC-line graphs have chromatic degree $\leq 2$, and can therefore be seen as locally 2-colored paths.

Composing the EC-line graph construction with the bijection from theorem 3.9, we end up with a direct reduction to matchings. It turns out to be simple and clean, because we exploit the low chromatic degree; otherwise, we would have had to use the reduction based on P-gadgets (section 3.1) instead of the bijection, which would have given a messier result.

Definition 4.7. The PM-line graph $L_{PM}(G,T)$ is defined as the graph:

- with vertex set $\{u_e \mid e \in E, u$ is an endpoint of $e\}$;
- with edge set $M \sqcup E'$, where
  - $M = \{(u_e, u_v) \mid e = (u, v) \in E\}$;
  - $E' = \{(u_e, u_f) \mid u \in V, e, f \in \partial(u)$ are adjacent in $T(u)\}$;
- equipped with the perfect matching $M$.

Proposition 4.8. Compatible closed trails in a graph with forbidden transitions correspond bijectively to alternating cycles of its PM-line graph.

Proof. By combining theorem 3.9 with the relevant half of proposition 4.4.

Naturally, a similar correspondence holds for compatible trails between two given vertices. However, a disadvantage of $L_{PM}$ with respect to $L_{EC}$ is that the former does not provide a counterpart to paths avoiding forbidden transitions.

Remark 4.9. This PM-line graph construction appears implicitly in Retoré’s work [Ret03] on proof nets. This remark was indeed the starting point of the present work; we refer the reader back to the introduction for further discussion of this point.

4.2. Algorithms for trails avoiding forbidden transitions. With the tools now at our disposal, we are in a position to give an algorithm for finding compatible trails.

Theorem 4.10. Deciding whether, in a graph $G$ with transition system $T$, two given vertices can be joined by a compatible trail, and computing such a trail, can be done in linear time (in the size of the input, including $T$).

Similarly, a compatible closed trail in $G$ can be found in linear time.

To be more precise about what we mean by “size of the input”, let us say we represent each transition graph $T(v)$ using an adjacency list. Then the size $|T(v)|$ of this representation is the number of edges in $T(v)$ and can go up to $\deg(v)^2$. Thus, the total size of the input is $\Theta(|T|)$, where $|T| = \sum_v |T(v)| \leq \sum_v \deg(v)^2$ (we have $|E| = O(|T|)$ and, assuming there is no isolated vertex, $|V| = O(|T|)$).

For an edge-colored graph, writing down the explicit transition graphs may result in a non-linear increase in size. A linear-time algorithm for finding a properly colored trail between two given vertices of an edge-colored graph is given in [ADF+08] Corollary 2.3]
Proof of theorem 4.10. Thanks to proposition 4.6 it suffices to apply the algorithm of corollary 3.10 to $L_{EC}(G, T)$, the running time being therefore linear in the size of $L_{EC}(G, T)$. Since the edge set of $L_{EC}(G, T)$ is $\bigcup_{v \in V} T(v)$, the size is also $\Theta(|T|)$. □

Theorem 4.11. For a graph $G$, a transition system $T$ on $G$, two vertices $s, t$ and an edge $e$ in $G$, the existence of a compatible trail from $s$ to $t$ crossing $e$:

- is a NP-complete problem in general;
- can be decided in linear time in the size of the input, including $T$, when $G$ contains no compatible closed trail.

In the latter case, the linear time algorithm also solves the corresponding search problem.

Proof. Finding an augmenting path for a matching crossing a given edge is an instance of this problem: indeed, an “augmenting trail” cannot visit the same vertex $u$ twice, since it would have to go through the unique matching edge incident to $u$ twice. Thus, the NP-completeness result from corollary 3.12 applies.

Suppose now that $G$ contains no compatible closed trail. Using the EC-line graph construction, the problem reduces to finding a path in $L_{EC}(G, T)$ between $\partial(u)$ and $\partial(v)$ visiting $e$, knowing that $L_{EC}(G, T)$ has no properly colored cycle. This can be done in linear time by the second half of corollary 3.10 thanks again to proposition 4.6. □

4.3. A structural theorem. As for alternating cycles in perfect matchings, the absence of closed trails avoiding forbidden transitions leads to the existence of a bridge – provided we add another assumption.

Theorem 4.12. Let $G$ be a graph with transition system $T$, with at least one edge. If, for all vertices $v$ in $G$, the transition graph $T(v)$ is connected, and $G$ has no compatible closed trail, then $G$ has a bridge.

The connectedness assumption is not too restrictive: if $T(v)$ has multiple connected components, then $v$ can be split into multiple vertices corresponding to those connected components without changing the reachability by compatible trails. However, the bridges of the graph after this transformation are not necessarily bridges of the original graph.

Proof. Again, our proof uses the EC-line graph structure (definition 4.3). Without loss of generality, we assume $G$ to be connected (as a graph, without taking the transition system into consideration). Since $G$ has at least one edge, $L_{EC}(G, T)$ is not the empty graph; since the transition graphs are connected, it can be seen $L_{EC}(G, T)$ is connected as well; and the absence of compatible closed trail in $G$ means that it has no properly colored cycle. Therefore, by Yeo’s theorem (theorem 3.3), there is a color-separating vertex $e$ in $L_{EC}(G, T)$ – the notation reminding us that it corresponds to an edge of $G$.

If $e$ has chromatic degree 0 (resp. 1) in $L_{EC}(G, T)$, it means that both its endpoints are (resp. one of its endpoints is) a degree 1 vertex in $G$. In both cases, $e$ is a bridge of $G$ and we are done.

Else, $e$ has chromatic degree 2, and since it is color-separating, it is a cut-vertex of $L_{EC}(G, T)$. But we have $L_{EC}(G, T) \setminus \{e\} \simeq L_{EC}(G \setminus \{e\}, T)$, and so, again using the connectedness of the transition graphs, since $L_{EC}(G, T) \setminus \{e\}$ is disconnected, $G \setminus \{e\}$ is disconnected. In other words, $e$ is a bridge of $G$.

The same argument can be replayed by applying Kotzig’s theorem (theorem 2.2) to $L_{PM}(G, T)$ (definition 1.3). □

8Taking a vertex-induced subgraph on the left of the $\simeq$ sign, and removing an edge while leaving the vertices intact on the right.
As a corollary, we obtain a new proof of the "structure from acyclicity" property for properly colored trails. (The original proof [ADF+08] applies Yeo’s theorem to a construction which is rather different from our EC-line graph and which does not generalize to forbidden transitions.)

Corollary 4.13 ([ADF+08, Theorem 2.4]). Let $G$ be an edge-colored graph such that every vertex of $G$ is incident with at least two differently colored edges. Then, if $G$ does not have a properly colored closed trail, then $G$ has a bridge.

Proof. Let $T$ be the transition system induced by the edge coloring of $G$, and $v$ be any vertex of $G$. $T(v)$ is a complete $k$-partite graph, $k$ being the chromatic degree of $v$. Since $v$ is incident to at least two differently colored edges, $k \geq 2$ making $T(v)$ connected. Thus, the previous theorem can be applied in this case. □

To conclude this section, we show that our theorem on closed trails avoiding forbidden transitions actually implies to Kotzig’s theorem (theorem 2.2) – and therefore, is equivalent to both Kotzig’s and Yeo’s theorems.

Proof of Kotzig’s theorem from theorem 4.12. We use a proof by contradiction. Let $G = (V, E)$ be a minimal counterexample, and $M$ be its unique perfect matching. It has no degree 1 vertex since then the incident matching edge would be a bridge. Thus, every vertex is incident to exactly one matching edge and at least one edge outside the matching.

Since alternating paths a matching $M$ are the same as properly colored closed trails for the induced 2-edge-coloring, corollary 4.13 applies to show $G$ has at least one bridge. The bridges of $G$ being outside the matching, after removing them, which disconnects the 2-edge-connected components of $G$, $M$ is still the unique perfect matching of $G$.

Now, let $G' = (V', E')$ a 2-edge-connected component of $G$. Then $G'$ is strictly smaller than $G$, and since $G'$ is a connected component of a spanning subgraph of $G$ which contains $M$ as a perfect matching, $M \cap E'$ is a perfect matching of $G'$. It is also unique, or else $M$ would not be unique in $G$. Thus, $G'$ is a counterexample to Kotzig’s theorem, contradicting the minimality of $G$. □

5. Finding rainbow paths

The EC-line graph construction connects forbidden transitions and edge-colored graphs. The previous section applied this connection to the former; here we are interested in drawing the consequences for the latter.

A color class in an edge-colored graph is the set of all edges with some common color; it may also refer to the subgraph edge-induced by such a set. The purpose of this section is to study the complexity of the following problem.

Definition 5.1. The problem $\mathcal{A}$-Rainbow Path (for $\mathcal{A}$ a class of graphs) is defined as:

- **Input**: an edge-colored graph $G$, whose color classes all induce graphs belonging to $\mathcal{A}$ (up to isomorphism), and two vertices $s$ and $t$ in $G$.
- **Output**: a rainbow path between $s$ and $t$ in $G$.

We will also use the same name to refer to the decision problem which asks for the existence of such a path.

Using the notations from [Sze03], we write $\mathcal{A}^{\text{ind}}$ for the closure of $\mathcal{A}$ under taking vertex-induced subgraphs, and $K_2 + K_2$, $P_4$ and $L_4$ refer to the graphs shown in fig. 2. We also write $K_2$ for the complete graph on 2 vertices.

The algorithmic results of this section can be summarized as follows.

Theorem 5.2. If $\mathcal{A}^{\text{ind}}$ contains $K_2 + K_2$, $P_4$ or $L_4$, then the problem $\mathcal{A}$-Rainbow Path is NP-complete.
Else, every graph in $A$ is the union of a complete multipartite graph and of isolated vertices, and $A$-Rainbow Path can be solved in linear time.

The case singled out as tractable by this theorem deserves a name, which we take from [ABS91].

**Definition 5.3.** A multipartite decomposition of a graph is an edge coloring whose color classes are all complete multipartite.

Again, we have a “structure from acyclicity” result:

**Theorem 5.4.** Let $G$ be a rainbow acyclic graph whose edge coloring is a multipartite decomposition. Suppose $G$ has at least one edge. Then there exists a color class $H$ with vertex partition $U_1, \ldots, U_k$, such that removing the edges of $H$ disconnects $U_1, \ldots, U_k$ – or in other words, such that for all $v \in U_i$ and $w \in U_j$ with $i \neq j$, $v$ and $w$ are not connected in $G \setminus H$.

The special case of bipartite decompositions has been considered in a number of works on combinatorics focusing on the minimum number of colors needed for a bipartite decomposition of a given graph. For instance, a well-known result is that all bipartite decompositions of the complete graph $K_n$ use at least $n - 1$ colors [GP71].

As for rainbow paths and cycles in bipartite decompositions, as mentioned in the introduction, they were considered under the name “aggregates” in [Ret93, Chapter 2], their study being motivated by linear logic. A proof of theorem 5.4 for this special case, which does not rely on another result such as Kotzig’s or Yeo’s theorems (respectively theorems 2.2 and 3.3), was the main result of this chapter [Ret93, Theorem 2.4]. Retoré would later adapt this into a direct proof of Kotzig’s theorem in [Ret03, Appendix].

### 5.1. Hardness results on rainbow paths

Let us establish the first half of theorem 5.2. In the following proofs of NP-completeness, we only treat the NP-hard part, since it is clear that the problems belong to NP.

**Proposition 5.5.** If $P_4 \in A_{\text{ind}}$ or $L_4 \in A_{\text{ind}}$, then $A$-Rainbow Path is NP-complete.

**Proof.** Let us call $A$-Compatible Path the problem of finding a path avoiding forbidden transitions between two given vertices when the transition graphs are all in $A$. There is a polynomial-time reduction from $A$-Compatible Path to $A$-Rainbow Path by proposition 4.4. The former problem was shown to be NP-complete when $P_4 \in A_{\text{ind}}$ or $L_4 \in A_{\text{ind}}$ in [Sze03, Theorem 1]. □

**Lemma 5.6.** $\{K_2, K_2 + K_2\}$-Rainbow Path is NP-complete.

**Proof.** The NP-completeness of the general rainbow path problem is proved in [CFMY11, Theorem 2.3] by a reduction from from 3-Sat. Examining the proof reveals that the instances generated by the reduction are edge-colored graphs whose color classes are all isomorphic either to $K_2$ (a single edge) or $K_2 + K_2$. Thus, this proof actually provides a polynomial reduction from 3-Sat to $\{K_2, K_2 + K_2\}$-Rainbow Path. □

**Proposition 5.7.** If $K_2 + K_2 \in A_{\text{ind}}$, then $A$-Rainbow Path is NP-complete.
Proof. We proceed by reduction from \(\{K_2, K_2 + K_2\}\)-Rainbow Path.

Suppose \(K_2 + K_2 \in \mathcal{A}^{\text{ind}}\). We can then choose a graph \(\Gamma \in \mathcal{A}\) with a set of four vertices \(W\) such that the induced graph \(\Gamma[W]\) is isomorphic to \(K_2 + K_2\).

Now, let \(G\) be an instance of \(\{K_2, K_2 + K_2\}\)-Rainbow Path. Consider a color class \(H\) in \(G\) such that \(H \simeq K_2 + K_2\). We build a new graph \(G'\) by removing the edges of \(H\), adding (a copy of) \(\Gamma\) with the same color as \(H\), and identifying the vertices of \(H\) with (the copy of) \(W\). This is equivalent to adding the vertices in \(\Gamma[\partial W]\) and adding some edges with the same color as \(H\), with every new edge having at least one endpoint among the new vertices, and without removing anything. This modification does not affect the set of rainbow paths between \(s\) and \(t\), in particular, it does not change its emptiness or nonemptiness, since any rainbow path starting at \(s\) and crossing a new edge would end up stuck in a new vertex.

Thus, by induction, we can replace all color classes isomorphic to \(K_2 + K_2\) with copies of our gadget \(\Gamma\), without affecting the existence of a rainbow path between \(s\) and \(t\). The same operation can be carried out to replace all the color classes consisting of single edges by copies of \(\Gamma\): indeed, if \(K_2 + K_2\) is a vertex-induced subgraph of \(\Gamma\), then \(K_2\) is as well. In the end, all color classes are isomorphic to \(\Gamma\); we have constructed an instance of \(\mathcal{A}\)-Rainbow Path.

All we have left to prove is that our reduction can be computed in polynomial time. This is clear if we remember that the size of \(\Gamma\) is \(O(1)\), since it depends only on \(\mathcal{A}\) and not on the input. \(\square\)

5.2. Solving the tractable case. The next 2 propositions cover the remaining half of theorem 5.2.

Proposition 5.8. If neither \(K_2 + K_2, P_4\) nor \(L_4\) are in \(\mathcal{A}^{\text{ind}}\), then every graph in \(\mathcal{A}\) is the union of a complete multipartite graph with isolated vertices.

Proof. This is shown as part of the proof of [Sze03, Lemma 7]. \(\square\)

In fact, since edge-induced subgraphs cannot have isolated vertices, an instance for \(\mathcal{A}\)-Rainbow Path for such a class \(\mathcal{A}\) can only have complete multipartite graphs as color classes, that is, it is necessarily a multipartite decomposition.

Proposition 5.9. There are linear-time algorithms for finding a rainbow path or cycle in a multipartite decomposition.

Proof. We treat here the case of paths; the case of cycles uses the same reduction.

Let \(G\) be a graph with a multipartite decomposition of its edges, \(V\) be its set of vertices, and \(H_1, \ldots, H_k\) be its color classes. Define \(G'\) as the graph with vertices \(V \sqcup W\), where \(W = \{w_1, \ldots, w_k\}\) is a set of with one fresh vertex per color class, and with edges \((v, w_i)\) for all \(v \in V\) and \(i \in \{1, \ldots, k\}\) such that \(v \in H_i\). Note that \(G'\) is bipartite with partition \((V, W)\). Color the edges of \(G\) in such a way that \((u, w_i)\) and \((v, w_j)\) have the same color if and only if \(i = j\) and \(u\) is not adjacent to \(v\) in \(H_i\), or equivalently, if \(u\) and \(v\) are in the same part of the vertex partition of \(H_i\).

We claim that properly colored paths in \(G'\) between vertices of \(V\) correspond bijectively to rainbow paths for \(G\). Let \(s = v_1, w_1, \ldots, w_{n-1}, v_n = t\) be a properly colored path between \(s \in V\) and \(t \in V\) in \(G'\), with \(v_i \in V\) and \(w_i \in W\) for all \(i \in \{1, \ldots, n - 1\}\). Our choice of coloring, together with the fact that the path is properly colored, ensures that for all \(i\), \((v_i, v_{i+1})\) is an edge in \(H_i\). Thus, the path corresponds to a path \(P\) in \(G\). By definition, a path has no repeated vertices, so the \(w_i -\) and therefore the \(H_i -\) are distinct, which means that \(P\) is actually a rainbow path. It is clear that this defines a bijection.

Thus, our algorithm is to build \(G'\) and then find a properly colored path in it. The time complexity is linear thanks to the following facts:

- the uncolored graph \(G'\) can be constructed in linear time, since the number of edges to add between \(V\) and \(W\) is at most the sum of the numbers of vertices of each \(H_i\), which is itself bounded by twice the number of edges of \(G\);
• the edges of \( G' \) can be colored in linear time – in fact, this requires computing the vertex partition of a complete multipartite graph in linear time, which is non-trivial, see lemma 5.10;
• properly colored paths can be found in linear time (theorem 3.1).

\[
\square
\]

**Lemma 5.10.** The vertex partition of a complete multipartite graph can be computed in linear time.

**Proof.** Recall that the class of cographs \([\text{CLB81}]\) is the smallest class of graphs containing the one-vertex graph and closed under disjoint unions and complementation. Equivalently, a cograph is a graph which can be described by a cotree: a rooted tree whose leaves are labeled with the vertices of the graph, and whose internal nodes are labeled with either \( \land \) or \( \lor \), such that two vertices are adjacent iff the lowest common ancestor of the corresponding leaves is a \( \land \) node. If we require all paths from the root to the leaves to alternate between \( \land \) and \( \lor \), this makes the cotree for a cograph unique.

Complete \( k \)-partite graph for are a particular case of cographs: for \( k \geq 2 \) (resp. \( k = 1 \)) they are the cographs whose canonical cotree has depth 2 (resp. 1) and whose root has label \( \land \) (resp. \( \lor \)). In this case, the immediate subtrees of the root describe the vertex partition. Fortunately, there are several algorithms for computing a cotree in linear time, e.g. \([\text{CPS85, BCHP08}]\).

\[
\square
\]

Using the same reduction as proposition 5.9, we can also prove “structure from acyclicity” for rainbow acyclic multipartite decompositions. The main ingredient in our proof will be Yeo’s theorem (theorem 3.3).

**Proof of theorem 5.4.** We proceed by strong induction on the size of \( G \). Let \( G' \) be the corresponding edge-colored graph with only stars as color classes, as constructed in the algorithm of proposition 5.9; we reuse the notations from its proof, writing \( V \) and \( W \) respectively for the vertices of \( V \) and the additional vertices in \( G' \) corresponding to color classes.

Since \( G \) is rainbow acyclic, \( G' \) has no properly colored cycle. By Yeo’s theorem, \( G' \) has a color-separating vertex \( u \in V \sqcup W \). It remains to do a case analysis on this vertex:

• If \( u \in W \), then it corresponds to a color class \( H \). The color separation property for \( u \) means exactly that the removal of \( H \) in \( G \) disconnects its vertex partition, which is what we wanted.
• Else, \( u \in V \), and it can be seen that \( u \) is also a color-separating vertex of \( G \). From this point on, we can forget the construction \( G' \). Choose a color \( c \) used by some edges incident to \( u \), and let \( H \) be the color class of \( c \) in \( G \) and \( U \) be the vertices of \( H \) except \( u \).
  – If \( H \) is a star with center \( c \) (which includes the case of a single edge), then by color separation, removing the edges of \( H \) indeed disconnects \( u \) from \( U \).
  – Else, let \( H' \) be the color class of \( c \) in \( G[V \setminus \{u\}] \); \( H' \) has at least two vertices and is connected. Let \( F \) be the connected component of \( H' \) in \( G[V \setminus \{u\}] \). \( F \) is smaller than \( G \), is also rainbow acyclic, and has at least one edge, so by the induction hypothesis, it has a color class with color \( c' \) whose removal disconnects its vertex partition in \( F \).
    * If \( c' \neq c \), note that the color component of \( c' \) in \( G \) is connected – it is a complete multipartite graph – and has no edge incident to \( u \) – again by color separation – so it is actually included in \( F \). The removal of this color component thus disconnects its vertex partition.
    * If \( c' = c \), then removing \( H \) from \( G \) disconnects both \( u \) from \( U \) and the vertices of \( U \) which are not in the same part of the partition from each other; thus, we have what we wanted.

\[
\square
\]
6. Arc-colored directed graphs

6.1. Properly colored directed trails and closed trails. We now consider directed graphs (a.k.a. digraphs) and trails therein. It is important to note that given two vertices \( u \) and \( v \) in a digraph, if the arcs \((u, v)\) and \((v, u)\) both exist, they are considered to be different arcs, and therefore they can both appear in the same trail. (This follows the definitions used in [GLMM13].)

In particular, this means that a directed trail in a symmetric digraph – symmetry means that the arc \((u, v)\) is present if and only if \((v, u)\) is – does not necessarily induce a trail in the corresponding undirected graph.

This should help understand how the following property can hold while its counterpart in edge-colored undirected graphs is completely wrong (the quintessential counterexample being alternating walks for perfect matchings that go through blossoms).

**Proposition 6.1.** Let \( G \) be an arc-colored digraph and \( s, t \) be two vertices of \( G \). From any properly colored directed walk from \( s \) to \( t \) one can extract a subset of arcs which forms a properly colored directed trail from \( s \) to \( t \).

**Proof.** Consider a walk \( s, e_1, \ldots, v_i, e_{i+1}, \ldots, v_j, e_{j+1}, v_{j+1}, \ldots, e_n, t \) where the \( e_k \) are arcs, such that \( e_{i+1} = e_{j+1} \). Since these arcs are directed, their sources are equal and their targets are equal: \( v_i = v_j \), \( v_{i+1} = v_{j+1} \). The following is therefore a legal walk: \( s, e_1, \ldots, v_i, e_{i+1}, v_{j+1}, \ldots, e_n, t \). If the initial walk was properly colored, then so is the new one (\( e_{i+1} = e_{j+1} \) so in particular they have the same color). By iterating this process until we reach a minimal subwalk, we obtain a properly colored trail.

This means that finding a PC directed trail is a particularly simple algorithmic problem:

**Theorem 6.2.** There is a linear time algorithm for finding directed properly colored trails.

**Proof.** This can be done using a breadth-first search. Indeed, a BFS can be used to find PC walks of minimum length since the notion of PC walk, unlike that of PC path or trail, is “history-free” (paths and trails involve a global non-repetition constraint, while the constraints on PC walks are purely local). By the previous proposition, since this PC walk is minimal, it is a trail.

**Remark 6.3.** Let us compare this with the polynomial time algorithm given in [GLMM13, Theorem 1] (whose statement does not give a precise exponent). That algorithm ultimately proceeds by the following sequence of reductions:

PC directed trail \( \rightarrow \) minimum reload+weight directed trail \( \rightarrow \) shortest weighted path/trail

The second reduction is treated in [AGM11, Proposition 1] and involves a potentially quadratic blowup of the instance size. So the bounds that we can infer for the algorithm of [GLMM13, Theorem 1] are quadratic – worse than our linear time result.

Note by the way that in the end, this algorithm relies on solving the shortest path problem via e.g. the classical Dijkstra algorithm, which is much simpler than the refinement of Edmonds’s blossom algorithm used for most of our other results. This supports our idea that the problem is simple enough to be tackled by a mere breadth-first search.

6.2. Alternating circuits and directed paths. Concerning properly colored directed paths, a NP-completeness result is known even with only 2 colors and an acyclicity condition:

**Theorem 6.4 (GLMM13, Theorem 5).** Deciding whether a 2-arc-colored digraph contains a properly colored path between two given vertices is NP-complete, even when the input is restricted to digraphs with no properly colored circuit.

We can deduce NP-hardness for PC circuits (directed cycles without vertex repetitions).
Figure 3. A 2-arc-colored digraph and its translation into a digraph with a perfect matching.

Corollary 6.5. Finding a properly colored circuit in a 2-arc-colored digraph is NP-complete.

Proof sketch. We prove NP-hardness by reduction from the previous problem (while membership in NP is trivial). To any 2-arc-colored digraph with designated source $s$ and designated target $t$, glue an acyclic gadget to $s$ and $t$ to add a properly arc-colored path from $t$ to $s$ with any starting color and any ending color. If the original digraph had no properly colored circuit, then the new one admits a properly colored circuit if and only if there was a properly colored path from $s$ to $t$ in the original, whose concatenation with a new path from $t$ to $s$ results in a circuit.

Our goal here will be to show that a special case of this problem – the alternating circuit problem – is already NP-complete. To define this restricted problem, we must first explain what our notion of perfect matching is in the setting of digraphs.

Definition 6.6. A perfect matching $M$ of a directed graph is a subset of arcs such that:
- any vertex $u \in V$ has exactly one outgoing arc in $M$ and exactly one incoming arc in $M$ (i.e. there is exactly one pair $(v, w) \in V^2$ such that $(u, v) \in M$ and $(w, u) \in M$);
- for all $u, v \in V$, $(u, v) \in M \iff (v, u) \in M$ – morally, $M$ consists of undirected edges.

Alternating paths and alternating circuits are defined as expected.

Theorem 6.7. Detecting alternating circuits for perfect matchings in digraphs is NP-complete.

Proof. It suffices to adapt in the obvious way Edmonds’s reduction from 2-edge-colored graphs to undirected graphs equipped with perfect matchings – recall that in section 3.2, we presented and generalized this reduction. See fig. 3 for an example.

As previously mentioned, we propose in [Ngu20] an alternative proof which, instead of going through corollary 6.5, proceeds by direct reduction from CNF-SAT.

As we said in the introduction, while this theorem is not difficult and may seem anecdotal, it has a potentially significant application in logic [Ngu20]. For this application, it is the hardness result for circuits that is important. That said, for the sake of completeness, one can also establish NP-completeness for alternating directed paths.

Corollary 6.8. Finding an alternating directed path between two given vertices is NP-complete.

Proof. Intuitively speaking, it is obvious that an oracle for alternating directed paths can be used to find alternating circuits. But to satisfy the definition of NP-hardness, we must exhibit a proper many-one reduction on instances. TODO.

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