A non-Standard Standard Model

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Abstract

This paper examines the Standard Model under the strong-electroweak symmetry group $SU_C(3) \times U_{EW}(2)$ subject to the Lie algebra condition $u_{EW}(2) \not\cong su_I(2) \oplus u_Y(1)$. Physically, the condition ensures that all electroweak gauge bosons interact with each other prior to symmetry breaking — as one might expect from $U(2)$ invariance. This represents a crucial shift in the identification of physical gauge bosons: Unlike the Standard Model which posits a change of Lie algebra basis induced by spontaneous symmetry breaking, here the basis is unaltered and $A, Z^0, W^\pm$ represent the physical bosons both before and after spontaneous symmetry breaking.

Our choice of $u_{EW}(2)$ requires some modification of the matter field representation of the Standard Model. For $U_{EW}(2)$, there are two pertinent representations $2$ and its $U(2)$-conjugate $2^c$ related by a global gauge transformation that squares to minus the identity. The product group structure calls for strong-electroweak degrees of freedom in the $(3,2)$ and the $(3,2^c)$ of $SU_C(3) \times U_{EW}(2)$ that possess integer electric charge just like leptons. These degrees of freedom play the role of quarks, and they lead to a modified Lagrangian that nevertheless reproduces transition rates and cross sections equivalent to the Standard Model.

The close resemblance between quark and lepton electroweak doublets in this picture suggests a mechanism for a speculative phase transition between quarks and leptons that stems from the product structure of the symmetry group. Our hypothesis is that the strong and electroweak bosons see each other as a source of decoherence. In effect, lepton representations get identified with the $SU(3)$-trace-reduced quark representations. This mechanism allows for possible extensions of the Standard Model that don’t require large inclusive multiplets of matter fields. As an example, we propose and investigate a model that turns out to have some promising cosmological implications.

1 Introduction

The present-day Standard Model (SM) started out as an electroweak (EW) theory of leptons.

Only later were hadrons tentatively incorporated by considering the known structure of charged hadronic currents. This led to a postulated hadronic composition by quarks along with their assumed weak isospin and hypercharge quantum numbers.

In the quark sector of the fledgling model, the canonical status enjoyed by isospin $I$ and hypercharge $Y$ was a consequence of the of the EW symmetry group $SU_I(2) \times U_Y(1)$ and
the success of the Gell-Mann/Nishijima relation \(Q \propto I + 1/2Y\) in classifying mesons and baryons in various approximate \(SU(2)\) and \(SU(3)\) flavor symmetry models. Historically, this led to the conclusion that the \((u, d, s)\) quarks possess fractional electric charge. But it was soon recognized that fermi statistics required additional degrees of freedom, and the idea of a local three-fold color symmetry group was born. Consequently, adding gauged \(SU_C(3)\), still assuming fractional electric charge, and using the Gell-Mann/Nishijima relation led to the assignment of quarks in the \((3, 2, 1/3)\) representation of gauged \(SU_C(3) \times SU_I(2) \times U_Y(1)\).

Our aim is to study how this historical picture changes if we start instead with the symmetry group \(SU_C(3) \times U_{EW}(2)\). There are good reasons to believe the \(EW\) group is \(U(2)\). First, if \(\rho\) is the representation of \(SU(2) \times U(1)\) furnished by the lepton fields of the SM, then \(\ker \rho = Z_2\) and therefore the lepton matter fields do not furnish a faithful representation.\([8]\) The group that does act effectively on all matter fields is \((SU(2) \times U(1))/Z_2 = U(2)\). Second, both \(SU(2) \times U(1)\) and \(U(2)\) have the same covering group \(SU(2) \times \mathbb{R}\). Representations of \(SU(2) \times \mathbb{R}\) will descend to representations of \(SU(2) \times U(1)\) or \(U(2)\) if the associated discrete factor groups are represented trivially by the unit matrix. For \((SU(2) \times U(1))/(SU(2) \times \mathbb{R})\) this requirement implies no relationship between isospin and hypercharge, but for \(U(2)\) it implies \(n = I + 1/2Y\) with \(n\) integer (\[27\], pg. 145). Identifying \(n\) with electric charge renders the Gell-Mann/Nishijima relation and electric charge quantization a consequence of \(U(2)\).

Third, symmetry reduction from \(U(2)\) to \(U(1)\) is less constrained and more natural than from \(SU(2) \times U(1)\).\([8, 9]\) Fourth, to the extent that a fiber bundle formulation over a paracompact base space is an appropriate physical model, the most general structure group for a complex matter field doublet in this case is \(U(2)\).\([10]\)

The point is that \(su(2) \oplus u(1) \cong u(2)\) as vector spaces, but as groups \(SU(2) \times U(1) \not\cong U(2)\). This subtle difference in symmetry groups has a big effect on the phenomenologically apposite basis of the associated Lie algebra that represents gauge fields.

The first task is to determine this basis. On physical grounds we demand that: 1) the gauge bosons are associated with a particular (up to group conjugation) ‘physical basis’ in a real subalgebra of \(u(2, \mathbb{C})\) that is endowed with a suitable \(Ad\)-invariant inner product (which in our case differs from the Killing inner product); and 2) all gauge bosons are allowed to take part in boson–boson interactions — as intuition suggests befits a \(U(2)\) symmetry group. These conditions restrict the starting gauge symmetry to a subgroup that we denote by \(U_{EW}(2) \subset U(2, \mathbb{C})\). A key feature of its Lie algebra \(u_{EW}(2)\) is that its generators do not form a basis for the algebra decomposition \(su_I(2) \oplus u_Y(1)\) of the SM. Instead, they comprise the ‘physical basis’ associated with the \(A, Z^0, W^\pm\) gauge bosons. This basis is supposed physically proper both before and after spontaneous symmetry breaking (SSB). Implementing this standpoint calls for some fairly moderate modifications of the SM.

To understand the motivation and implications of these modifications, in \(\S\ 2\) we embark on a somewhat pedestrian review\([11]\) of the adjoint and defining representations of Lie algebras of product groups. We propose that physical gauge bosons characterized by conserved charges should be identified with a distinguished Cartan basis with its concomitant root system. The Cartan basis of gauge bosons is ‘physical’ in the sense that we can identify.

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1 The subscript \(S\) stands for ‘strong’ and \(EW\) stands for ‘electroweak’. We will explain the significance of the distinction between the subscripts \(S, EW\) vs. \(C, I, Y\) later.
its roots (a.k.a. quantum numbers) with the conserved charges associated with coupling constants characterizing the Lie algebra inner product. We confirm that quantization and renormalization don’t spoil this identification, and so it is permitted to use the terms ‘root’, ‘quantum number’, and ‘charge’ interchangeably. (To be precise, the term ‘charge’ here refers to the eigenvalue of a charge operator acting on a gauge field.) Demanding that all generators of the Cartan subalgebra be gauge equivalent dictates the generators of $u_{EW}(2)$ and their associated charges.

In this Cartan-basis picture, in which the neutral gauge bosons span the Cartan subalgebra, elementary matter fields are identified with eigenfields of the neutral gauge bosons in the defining representation whose charges are determined by their associated weight system. This guarantees that matter fields and gauge bosons exchange the same types of charges — which are ultimately determined by the Cartan basis. Due to the product nature of the symmetry group, the eigenfields associated with neutral $SU_C(3) \times U_{EW}(2)$ gauge bosons evidently include both representations $(3,2)$ and $(3,2^c)$ where $2^c$ is $U_{EW}(2)$-conjugate via a global gauge transformation that squares to minus the identity. This means, that $(3,2)$ and $(3,2^c)$ each carry two types of conserved strong charge and two types of EW charge; the latter of which only one is conserved after SSB and corresponds to the integer unit of electric charge characterized by the (neutral) photon — in glaring contrast to quarks.

But, since $(3,2)$ and $(3,2^c)$ are supposed to be the building blocks of hadrons, an apparent contradiction arises: How can a triplet of these hadronic constituents (HC), each of which possess integer electric charge, combine to form hadrons with their observed electric charges? We show in the remainder of §2 that the charge carried by a matter eigenfield versus the coupling strength coming from its associated current are not necessarily equivalent.

How can this be? The defining representation $2$ of $U_{EW}(2)$ and its $U_{EW}(2)$-conjugate $2^c$ are related by a “small” gauge transformation; one that is homotopic to the identity. As such, they induce gauge equivalent field configurations, but they are not equivalent representations as they would be for $SU(2)$. The most general (quadratic with minimal coupling) effective Lagrangian that reflects this fact contains a linear combination of HC kinetic terms representing both $(3,2)$ and $(3,2^c)$. We will see that the ratio of scalar factors for the two terms is non-trivial: it can’t be absorbed in the normalization, it doesn’t get renormalized, and it doesn’t spoil the gauge symmetry. A non-trivial ratio renders scaled coupling constants that ultimately become coupling strengths in matter field currents. Consequently the charge is not always equivalent to the coupling strength between currents and gauge bosons.

This is an inceptive observation, and it inspires (what we consider) a relatively mild variation of the SM. With the physically relevant gauge bosons and their matter eigenfields in hand, we begin the explicit construction of this non-Standard Model (n-SM) in §3. After determining the defining representations of $U_{EW}(2)$ and $SU_C(3)$, a Lagrangian density is proposed. We stress that the only pieces that differ from the SM are the quark contributions — our’s include a linear combination of both $(3,2)$ and $(3,2^c)$ whose scalar factors are a priori free parameters, but anomaly cancellation fixes them. As a result, the strong and electroweak currents coming from the n-SM agree with those from the SM provided quark currents are identified with a pair of HC currents. In particular, the electromagnetic (EM) current contains the expected $2/3e$ and $-1/3e$ coupling strengths even though the HC have integer electric charge. For this reason, we call the HC “iquarks”.

Essentially, a SM quark doublet is interpreted as an average description of two n-SM iquark doublets. Insofar as experiments are not able to distinguish individual iquark currents, we prove spin averaged transition rates and cross sections of the n-SM coincide with those of the SM — they make equivalent experimental predictions. This may seem rather surprising given the different EW symmetry group and elementary particle interpretation\footnote{On the other hand, maybe it’s not so surprising: Referring to a well-known self-described folk theorem of Weinberg’s\cite{7,16}, the S-matrix is determined by unitarity, cluster decomposition, Lorentz invariance, and gauge invariance. Given the n-SM and SM have isomorphic Lie algebras, this argues the two theories’ S-matrices must be identical up to ‘elementary’ particle labels.} but it is welcome owing to the predictive success of the SM.

Perhaps the primary benefit of the n-SM is that it affords an alternative perspective for extensions. In the final section § 4 of the paper we explore some hypotheses — motivated by the iquark picture — that constitute a more substantial departure from the SM. The similarity between the iquark and lepton $U_{EW}(2)$ representations suggests they are related. So we hypothesize the possibility of a phase transition induced by the product group structure. Our idea is that gauge bosons of $SU(3)$ versus $U(2)$ see themselves as a quantum system and the other as an environment through their mutual interactions with matter fields. As a consequence, in restricted regions of phase space, their mutual decoherence is posited to precipitate a transmutation of matter field representations. In particular, the affected defining representation gets reduced to its trace due to decoherence. We interpret this representation transmutation as a phase transition, and it occurs without breaking symmetry. It is a non-perturbative dynamical effect not modeled by the Lagrangian; so our n-SM is an effective theory at the EW energy scale valid only in certain regions of phase space.

If this mechanism is viable, it allows otherwise excluded options for unifying symmetry groups, because iquarks and leptons are not required to belong to inclusive multiplets: We are not confined to imagine the elementary particles observed at low energy comprise the components of the defining representation of a large-dimension group at high energy. Based on this exemption, we consider the symmetry breaking scenario $U(4,\mathbb{C}) \rightarrow U_S(3) \times U_{EW}(2)$ followed by $U_S(3) \times U_{EW}(2) \rightarrow U_S(3) \times U_{EM}(1)$.\footnote{Unlike the Pati-Salam model\cite{17}, we do not combine a flavor of three (anti)quarks and one (anti)lepton into a quartet in the defining representation of $U(4,\mathbb{C})$. In our view, leptons are essentially decohered iquarks in the trivial representation of $U_S(3)$ so the nature of the matter eigenfields in the defining representation of $U(4,\mathbb{C})$ is a priori distinct from a simple collection of (anti)iquarks and (anti)leptons.} Representation transmutation associated with the first symmetry reduction renders the iquark and lepton content of the n-SM. Regarding gauge bosons; the first symmetry reduction, if spontaneous, gives rise to three (presumably semi-weakly interacting) massive gauge bosons. Dynamics dictates their fate, but at least there exists the potential for their associated on-shell particles to decouple from matter eigenfields of the final symmetry group $U_S(3) \times U_{EM}(1)$ or to form stable composites. If so, these cousins of the weak bosons could contribute to dark matter \cite{13,14}. Further, the massless gauge bosons gain another member and there are now three strong charges. With this addition, there exists a non-orthogonal superposition of gluons we call the ‘strong photon’ that generates a $u_S(1)$ subalgebra that mediates what we call ‘strong charge’ $e_S$ (by analogy to electric charge $e$); and it leads to a net strong charge carried by matter fields and certain strong-singlet states.
At first blush, postulating $U_S(3)$ as the strong symmetry group without symmetry breaking down to $SU_C(3)$ seems highly suspect and one is tempted to dismiss it outright. However, just like the EW group, the strong symmetry group is a subgroup $U_S(3) \subset U(4, \mathbb{C})$ determined by orthogonal gluon states with respect to a suitable $Ad$-invariant inner product on a suitable real subalgebra $u_S(3)$. Crucially, $u_S(3) \not\cong su_C(3) \oplus u_S(1)$ and the strong photon is not an orthogonal gluon state; consequently it couples to both baryon number and gluons.

Trouble is, the strong photon opposes gravitational attraction between ponderable matter. But, unlike the “fifth force” photon of [12], since the coupling includes the nuclear binding energy through gluon–gluon interaction it is not sensitive to the ratio of baryon number to nucleon/atomic mass-energy. In other words, the strong photon couples to mass-energy of ponderable matter (excluding electron binding energy) and combined with gravity it amounts to an effective attractive force concealed in effective quark/nucleon mass. Further, it opposes the formation of solid matter by acting against electromagnetism. However, a semi-quantitative estimate in §4.2.2 predicts a seemingly inconsequential $O(10^{-20})$ opposition relative to electromagnetic attraction/repulsion.

Certainly, $U_S(3)$ better agree with experimental QCD. So we need to compare relevant parameters. But since $u_S(3) \not\cong su_C(3) \oplus u_S(1)$, the standard QCD renormalization formulas do not necessarily apply and showing agreement with experiment is non-trivial. Nevertheless, we argue semi-quantitatively that $U_S(3)$ leads to physically reasonable phenomenology by comparing $U_S(3)$ vertex factors with QCD color factors and comparing running couplings.

Granted correct QCD phenomenology of $U_S(3)$, we go on to explore some cosmological implications of the chain $U(4, \mathbb{C}) \to U_S(3) \times U_{EW}(2) \to U_S(3) \times U_{EM}(1)$. Baryons/anti-baryons carry equal but opposite strong charge $e_S$; implying baryons repel. We perform an estimate of the baryon strong charge contribution to the Hubble parameter (idealizing a uniform baryon density throughout the universe) that indicates the strong photon could be a candidate for dark energy [15].

It is remarkable that $e_S$ is parametrized by the two coupling constants inherent in $U_S(3)$ in such a way (as we will see) that the strong-charge strength goes to infinity as the coupling constants approach each other — in clear distinction to the electric-charge strength. Since the strong charge is estimated by the RGE to diverge around $10^{17}$ GeV, any eventual matter/anti-matter imbalance near this scale would have resulted in violent repulsion accompanied by a rapid decrease in cosmic energy density and a consequent rapid decrease in $e_S$ strength. This scenario has the earmarks of inflation and later cosmic expansion, and we will explore this speculation briefly.

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4 This reaction is based on an extension of the SM (proposed some 30 years ago and dubbed the “fifth force”) by a $U_B(1)$ gauge symmetry that couples to baryon number [12]. The theory predicts an effective gravitational attraction that is sensitive to the ratio of baryon number to atomic mass. It is by now generally agreed that experiment has ruled this out.

5 This statement, which we will use several times, is succinct but imprecise: Technically, it means the orthogonal generators that span $u_S(3)$ cannot be decomposed into $su_C(3) \oplus u_S(1)$ with respect to a suitable inner product on the Lie algebra. Equivalently, the commutator subalgebra of the raising and lowering generators spans the entire Cartan subalgebra. The analogous condition was previously imposed on the electroweak algebra $u_{EW}(2) \not\cong su_I(2) \oplus u_Y(1)$.

6 According to our proposed extended Lagrangian, $e_S = 0$ for leptons so strong charge does not couple directly to electron atomic binding energy. This effect appears to be well beyond experimental observation.
The final bit of speculation in § 4 concerns the mass of the electric-neutral iquarks and leptons. If they are representation transmutations of each other and the strong group is $U_S(3)$, it is tempting to suppose their rest masses are dynamically generated by quantum radiative effects. It is known that this would have implications for the strong CP issue [16].

It should be mentioned that similarities between quarks and leptons have long been noticed. Besides inspiring ideas of unifying groups, they have prompted some attempts to endow leptons with an $SU(3)$ symmetry (see e.g. [24]), and they are getting attention in recent years under the program of quark-lepton complementarity which studies phenomenological similarities between the generation-mixing matrices [25]. Also, quarks with integer charge have been proposed before (see, e.g., [4, 17, 18], and the review of [19]). However, the symmetry groups of these models are not the $SU_C(3) \times SU_I(2) \times U_Y(1)$ of the SM, and the n-SM presented here is not related to these models. Also, the proposed iquarks are not “preons” or “pre-quarks” (see, e.g., [20, 21, 22] and the review of [23]). That is, conventional quarks are not composite states of the iquarks. Instead, within this framework, conventional quarks can be interpreted as a certain superposition of the iquarks.

Before getting into details, it is perhaps useful to recap the main features of the n-SM and the proposed extension $U(4, \mathbb{C}) \rightarrow U_S(3) \times U_{EW}(2) \rightarrow U_S(3) \times U_{EM}(1)$.

1. Replacing $SU_I(2) \times U_Y(1)$ with $U_{EW}(2)$ renders a faithful representation of matter fields and implies electric charge quantization.

2. The Lie algebra basis associated with physical gauge bosons is not altered by SSB. Instead, insisting that all $U(2)$ bosons interact and that the Lie algebra inner product, which is characterized by the two coupling constants $g_1$ and $g_2$, remains finite as $g_2 \rightarrow 0$ uniquely determines (after normalization) the $U_{EW}(2)$ matter-field representations. (Unique only up to the adjoint action, of course.)

3. The matter-field Lagrangian includes a linear combination of both $(3, 2)$ and $(3, 2^c)$ with relative contributions dictated by anomaly cancellation. These fields are the counterparts of quarks, but unlike quarks they possess integer electric charge. Yet their currents couple to photons with fractional electric coupling strengths.

4. In the strong sector, a pair of components of HC can be identified with up/down quarks possessing identical strong charges, fractional electric charge, and spin. Consequently, the n-SM and SM $SU(3) \times U(1)$ currents are identical. On the other hand, the weak sector is sensitive to the dual-field nature of HC, and we find two n-SM weak iquark currents correspond to one SM weak quark current. Nevertheless, given our hadron iquark assignments, the n-SM S-matrix for both QCD and EW iquark interactions is physically indistinguishable from its SM counterpart up to iquark/quark labels if individual iquark/quark currents are not directly observed. We conclude that transition rates and cross sections of the n-SM are equivalent to the SM for spin-averaged measurements.

5. The fact that HC and leptons share the same EW representation in the n-SM suggests the notion of representation transmutation precipitated by decoherence between strong
and EW bosons. In this picture, iquarks and leptons are merely different phases of the same underlying fields. This reduces the number of free parameters relative to the SM and implies relationships between quark and lepton parameters in the SM. In particular, it implies relationships between the CKM and PMNS matrices.

6. The small masses of neutrinos and the lepton/iquark correspondence suggest the neutral iquarks have relatively small masses, and it is tempting to suppose their masses are dynamically generated via strong dynamics. This would have the welcome effect of implying a vanishing $CP$ parameter $\theta = 0$ in the Lagrangian.

7. With guidance from our $U_{EW}(2)$ analysis, $SU_C(3)$ can be extended to the proposed $U_S(3)$ which is suggested by $U(4, \mathbb{C}) \to U_S(3) \times U_{EW}(2) \to U_S(3) \times U_{EM}(1)$. Although we do not propose a Yang-Mills QFT for the parent theory based on $U(4, \mathbb{C})$, for the sake of argument we do suppose that both symmetry reductions occur via SSB. The phenomenology of $U_S(3)$ agrees with $SU_C(3)$ in the sense that it yields values for $C_F$, $T_F$, and $C_A$ consistent with experiment. Moreover, it renders approximate coupled renormalization group equations (RGEs) for the $U_S(3)$ running couplings whose numerical solutions are consistent with QCD measurements and indicate convergence of the two coupling constants around $10^{17}$ GeV. This number, however, is very sensitive to the estimated initial values assigned to the coupling constants when solving the RGEs. Nevertheless, if we can expect convergence of the two coupling constants at the energy scale of $U(4, \mathbb{C}) \to U_S(3) \times U_{EW}(2)$, then the strength of the $e_S$-charge immediately after symmetry breaking would be very large.

8. The $U(4, \mathbb{C}) \to U_S(3) \times U_{EW}(2) \to U_S(3) \times U_{EM}(1)$ model implies: 1) three larger-mass cousins of $W^\pm, Z^0$ that perhaps contribute to dark matter, 2) the existence of a strong photon that couples to baryon mass-energy and is a candidate for dark energy if its associated Coulomb coupling strength is $k_S \sim 1 \cdot 10^{-9} \text{N}\cdot\text{m}^2$, 3) a rapidly increasing strong photon coupling (despite vanishing coupling constants) estimated to be near $10^{17}$ GeV that would encourage matter/anti-matter creation and annihilation followed by extreme repulsion in the presence of any matter/anti-matter asymmetry, 4) a concomitant rapid expansion of the volume occupied by matter accompanied by a simultaneous diminishing energy density and strong photon coupling — eventually rendering a greatly reduced expansion rate, 5) the inverse process which appears to offer a mechanism to halt gravitational collapse, and 6) given the estimate for $k_S$, a negligible counter force of $O(10^{-20})$ opposing EM in condensed matter.

We should add a few words of caution to temper the promising implications of $U_S(3)$. First, our comparison to the $T_F$, $C_F$, and $C_A$ values and to the QCD coupling constant over the currently accessible experimental range rests on an approximation: the usual renormalization results do not automatically carry over to $U(3)$ and our claims rely on using $u_S(3) \approx su_S(3) \oplus u_S(1)$ to calculate effective vertex factors $T_{eff}$ and $C_{eff}$ as well as coupled RGE for the running coupling constants. Second, we do not work through the details of gauge fixing in the $U_S(3)$ case. Obviously, pending detailed renormalization and gauge fixing analysis, the claimed implications regarding $U_S(3)$ can only be tentative.
2 Intrinsic v.s. Extrinsic Charge

2.1 Kinematical quantum numbers

Begin with a gauge field theory endowed with an internal symmetry group that is a direct product group $G = G_1 \times \cdots \times G_n =: \times_n G_i$ where $n \in \mathbb{N}$ and the $G_i$ with $i \in \{1, \ldots, n\}$ are Lie groups that mutually commute. Associated with each subgroup $G_i$ is a Lie algebra $\mathfrak{g}_i$ with basis $\{g_{a_i}\}_{a_i=1}^{\dim G_i}$. The full Lie algebra is $\mathfrak{g} := \oplus_n \mathfrak{g}_i$ (in obvious notation). Recall that the Lie algebra does not uniquely determine the Lie group.

Given that a physical system is invariant under the associated gauge group, it is possible to deduce some general properties or attributes of the associated gauge and matter fields based solely on the mathematics of the symmetry group and its representations \[26, 27\]. In particular, the mathematics identifies distinguished bases and associated eigenvalues (which we will identify with quantum numbers) in the vector spaces furnishing the adjoint representations of each $\mathfrak{g}_i$.

The distinguished bases are physically important in the sense that they describe the adjoint-representation counterpart of matter field irreducible representations. As such, they model the concept of particle/antiparticle and particle–particle interaction with charge exchange. For local symmetries, these special bases can be chosen independently at each spacetime point, essentially creating an unchanging structure by which to associate the unchanging quantum numbers of elementary particles — both bosons and fermions.

It is well-known, of course, that the distinguished bases are induced by the Cartan subalgebra. This is a well-worn story. But, as we are dealing with a product group $G = \times_n G_i$, we will spend a moment repeating it as a means of review and to set some notation.

Consider the adjoint representation $ad : \mathfrak{g}_i \rightarrow GL(\mathfrak{g}_i)$ on the complex extension of $\mathfrak{g}_i$. For a given element $c^{a_i} g_{a_i}$ (with $c^{a_i} \in \mathbb{C}$) in the Lie algebra, the adjoint representation yields a secular equation

$$\prod_{k_i=0}^{r_i} (\lambda - \alpha_{k_i})^{d_{k_i}} = 0 \quad (1)$$

where $\alpha_{k_i}$ are roots of the secular equation with multiplicity $d_{k_i}$ and $r_i$ is the rank of $\mathfrak{g}_i$. Since $\lambda = 0$ is always a solution, we put $\alpha_0 = 0$. Note that $\sum_{k_i=0}^{r_i} d_{k_i} = \dim \mathfrak{g}_i$. Associated with the roots $\alpha_{k_i}$ (which may not all be distinct in general) are $r_i$ independent eigenvectors.

The roots and their associated eigenvectors determine the Jordan block form of the element $ad(c^{a_i} g_{a_i})$. That is, there exists a non-singular transformation of $ad(c^{a_i} g_{a_i})$ into Jordan canonical form. With respect to the Jordan canonical form, the vector space that carries the representation $ad(\mathfrak{g}_i)$ decomposes into a direct sum of subspaces:

$$\mathfrak{g}_i = \oplus_{k_i} V_{\alpha_{k_i}} \quad (2)$$

with each $V_{\alpha_{k_i}}$ containing one eigenvector and $\dim V_{\alpha_{k_i}} = d_{k_i}$. This decomposition is with respect to any given element in the Lie algebra.

Now, regular Lie algebra elements are defined by the conditions that: (i) they lead to a decomposition that maximizes the distinct roots $\alpha_{k_i}$ (equivalently, minimize the dimension of $V_{\alpha_{k_i}}$), and (ii) they all determine the same $V_0$. For decomposition associated with regular
elements, the subspaces $V_{\alpha ki}$ have potentially useful properties for describing physical gauge bosons:

- $[V_{0i}, V_{0j}] \subseteq V_{0j}$ and hence $V_{0i}$ is a subalgebra. It is known as a Cartan subalgebra.
- The subspace $V_{0i}$ carries a representation of the Cartan subalgebra. Since its rank is 0, the Cartan subalgebra is solvable; in fact nilpotent.
- $[V_{0i}, V_{\alpha ki}] \subseteq V_{\alpha ki}$. Hence, each $V_{\alpha ki}$ is invariant with respect to the action of $V_{0i}$ and so carries a representation for $V_{0i}$. Moreover, since $V_{0i}$ is solvable, it has a simultaneous eigenvector contained in $V_{\alpha ki}$. More specifically, associated with the secular equation for an element of the subalgebra $V_{0i}$ with basis $\{ h_{s_i} \}_{s_i=1}^{d_0}$ is a set of $\dim V_{0i} = d_0$ roots, collectively denoted by $q_i := (q_{1i}, \ldots, q_{d_0})$, and a corresponding eigenvector $e_{\alpha ki} \in V_{\alpha ki}$ such that
  $$[h_{s_i}, e_{\alpha ki}] = q_{s_i} e_{\alpha ki},$$
  or more succinctly,
  $$[h_i, e_{\alpha ki}] = q_i e_{\alpha ki},$$
  In particular, this holds for $\alpha_0 = 0$. That is, there exists an $e_{0i} \in V_{0i}$ such that
  $$[h_i, e_{0i}] = 0.$$

- If $V_{0i}$ is contained in the derived algebra of $G_i$, then for $V_{\alpha ki}$, there is at least one $V_{\beta ki}$ such that $[V_{\alpha ki}, V_{\beta ki}] \subseteq V_{0ki}$. This implies that, for $q_i$ associated with each $e_{\alpha ki}$, there is at least one $e_{\beta ki}$ with roots $-q_i$. Additionally, any $q'_i \neq -q_i$ must be a rational multiple of $q_i \neq 0$.

These properties can be used to characterize ‘physical’ gauge bosons if we make one restriction: for $\alpha_{ki} \neq 0$, $\dim \bigoplus_{\alpha_{ki}} V_{\alpha ki} = \dim G_i - d_{0i} = r_i$. That is $\dim V_{\alpha ki} = 1$ for all $\alpha_{ki} \neq 0$. Without this restriction, there would be no means (mathematically) to distinguish between basis elements, and hence gauge bosons, in a given $V_{\alpha ki}$. As a consequence of this restriction, we must have $[V_{0i}, V_{0j}] = 0$ since otherwise $[[V_{0i}, V_{0j}], V_{\alpha ki}]$ in the Jacobi identity leads to a contradiction.

The commutativity of $V_{0i}$ is a necessary condition for $G_i$ to be the direct sum of one-dimensional abelian and/or simple algebras. Moreover, eventually the Lie algebra elements will be promoted to quantum fields so the adjoint carrier space is required to be Hilbert. Therefore, the inner product on the complex Lie algebra (or subspace thereof) is required to be Hermitian positive-definite. This implies that candidate symmetry group Lie algebras may be the direct sum of $u(1)$ and/or compact simple algebras since then the Killing inner product is positive-definite. However, it is consistent to allow a slightly larger class of algebras — reductive to be specific — provided they are endowed with $Ad$-invariant Hermitian inner products and commuting Cartan subalgebras.

It should be kept in mind that the class of Lie algebras under consideration up to now have been complex. However, the adjoint representation is real so the gauge bosons’ kinematical quantum numbers are real. Since we want the matter fields to be characterized by the same
real quantum numbers and we want to distinguish between field/anti-field, then on physical grounds we insist on symmetry groups generated by real, reductive Lie algebras endowed with a suitable $Ad$-invariant inner product — or subgroups and subalgebras thereof.

If the symmetry is not explicitly broken under quantization, then we can conclude that the quantized gauge fields associated with the Lie algebra $G_i$ describe gauge bosons characterized by the set of roots $q_i$. We will refer to these as kinetic quantum numbers for the gauge bosons. They correspond to conserved properties of the gauge bosons for unbroken symmetries, and they are physically relevant (though not necessarily observable) once a choice of matter field representations has been made (in the sense that the gauge bosons and matter fields are imagined to exchange an actual `charge').

Evidently, gauge bosons associated with the $h_{n_i}$ have vanishing kinetic quantum numbers while those associated with the $e_{n_i}$ carry the $q_i$ kinetic quantum numbers. Note that $e_{-n_i}$ carries $-q_i$. It is in this sense that the Lie algebra basis, defined by the (properly restricted) decomposition (2), characterizes the physical gauge bosons.

Turn now to the matter fields. We will confine our attention to Dirac spinors. (The spinor components of the matter fields will not be displayed since we work in Minkowski space-time and they play no role here in internal symmetries.) Let $V_{R_i}$ be a vector space with $\dim V_{R_i} = d_{R_i}$ that furnishes a representation of $G_i$ having basis $\{e_{i_{R_i}}\}_{1=1}^{d_{R_i}}$. And let $\rho^{(R_i)}: G_i \to GL(V_{R_i})$ denote a faithful irrep. The $R_i$ is a collection $(R_1, \ldots, R_{d_0})$ of $d_0$ numbers and serves to label the representation. (Recall $d_0 = \dim V_0$.)

Given some set $\{R_i\}$, suppose the corresponding set of fields $\{\Psi^{(R_i)}\}$ furnish inequivalent irreps $\rho^{(R_i)}(G_i)$ of the $G_i$. The associated tensor product representation

$$\rho^{\times R_n}(G_i) := \rho^{(R_1)}(G_1) \times \ldots \times \rho^{(R_n)}(G_n)$$

of the direct product group is also irreducible (where $\times R_n := (R_1, \ldots, R_n)$ denotes an element in the cartesian product $\{R_1\} \times \ldots \times \{R_n\}$). In fact for the class of groups under consideration, all irreps of $G$ are comprised of all possible combinations of relevant $\{R_i\}$ [26]. The corresponding Lie algebra representation

$$\rho_e^{\times R_n}(\oplus_R G_i) := \rho_e^{(R_1)}(\oplus_R G_1) \oplus \ldots \rho_e^{(R_n)}(\oplus_R G_n)$$

(where $\rho_e$ is the derivative map of the representation evaluated at the identity element) is likewise irreducible for all combinations of $\{R_i\}$ that are associated with irreps of the $G_i$.

Our supposition is that these representations have the potential to be realized in a physical system and so all combinations should be included in realistic models. The idea, of course, is these relevant combinations of irreps can be identified with elementary fields.

The representations $\rho^{(R_i)}(G_i)$ are largely a matter of choice depending on physical input. By assumption, the internal degrees of freedom associated with $G_i$ of elementary particles correspond to the basis elements $\{e_{i_{R_i}}\}_{1=1}^{d_{R_i}}$ spanning $V_{R_i}$. Hence, a given label $R_i$ characterizes the elementary particles (along with Lorentz labels). In particular, a basis is chosen such that the representation of the diagonal Lie algebra elements is (no summation implied)

$$\rho_e^{(R_i)}(h_{s_i}) e_{i_{R_i}}^{(R_i)} = i q_{s_i, d_i} e_{i_{R_i}}^{(R_i)}$$

(6)
where \(q^{(m)}_{s_i,l_i}\) are \((d_{s_i} \times d_{R_i})\) real numbers and the \((m)\) superscript indicates “matter”. In an obvious short-hand notation,

\[
\rho_e^{(R_i)}(h_{s_i})e^{(R_i)} = iq^{(m)}_s e^{(R_i)}.
\]

where \(q^{(m)}_{s_i} := (q^{(m)}_{s_i,1}, \ldots, q^{(m)}_{s_i,d_{R_i}})\) and \(e^{(R_i)} = (e^{(R_i)}_1, \ldots, e^{(R_i)}_{d_{R_i}})\) with no implied summation. Hence, \(q^{(m)}_{s_i}\) can serve to label the basis elements corresponding to elementary matter particle states for a given representation labelled by \(R_i\). In this sense, the elementary matter particles carry the kinematical quantum numbers \(q^{(m)}_{s_i}\).

Taking the complex conjugate of (6), gives

\[
[q^{(m)}_{s_i,l_i}]^* e^{(R_i)*} = -iq^{(m)}_s e^{(R_i)*}.
\]

Hence,

\[
e^{(R_i)*}[q^{(m)}_{s_i,l_i}]^* = -iq^{(m)}_s e^{(R_i)*}.
\]

So \(\{e^{(R_i)*}\}\) furnishes a dual complex-conjugate representation of \(G_i\) and is obverse to \(\{e^{(R_i)}_{l_i}\}\). That is, \(\{e^{(R_i)*}_{l_i}\}\) represents the internal degrees of freedom of the anti-\(G_i\)-particles associated with \(\{e^{(R_i)}_{l_i}\}\) since they are characterized by opposite quantum numbers.

The analysis in this subsection has yielded two insights that may be useful in model building. First, the Lie algebra possesses a distinguished basis, the Cartan basis, that is particularly suited to model gauge bosons and their physical attributes. Second, the matter field irreps for the direct product group \(G = \times_n G_i\) include all combinations of irreps of the subgroups \(G_i\). Therefore (and we want to emphasize this) all irrep combinations should be included in model Lagrangians, and this leads to the physical realization of elementary particles possessing all combinations of kinematical quantum numbers.

Of course kinematics is not the whole story, and we must somehow relate these kinematical quantum numbers to what is observed during quantum dynamics.

### 2.2 Dynamical quantum numbers

Again, the setup for gauge field theory is well-known, but we quickly review it here to set notation for a product symmetry group \(G = \times_n G_i\).

Consider a principal fiber bundle with structure group \(G_i\) and Minkowski space-time base space. Let \(A_i(x) := A^{a_i}(x) \otimes g_{a_i}\) be the local coordinate expression on the base space of the gauge potential (the pull-back under a local trivialization of the connection defined on the principal bundle). \(A^{a_i}(x)\) is a real one-form on the base space whose components \(A^{a_i}_\mu(x)\) represent gauge fields. The gauge field self-interactions are encoded in the covariant derivative of the gauge potential

\[
F_{\mu
u}(x) := D A_i(x) = dA_i(x) + \frac{1}{2}[A_i(x), A_i(x)] =: F^{a_i}(x)g_{a_i},
\]

where \(F^{a_i}\) is a two-form on the base space. In the special basis determined by the decomposition of the previous section, the commutator term describes interactions between gauge fields characterized by the kinematical quantum numbers \(q_i\) by virtue of (4).
Matter fields will be sections of a tensor product bundle $S \otimes V$. Here $S$ is a spinor bundle over space-time with typical fiber $\mathbb{C}^4$, and $V$ is a vector bundle associated to the gauge principal bundle with typical fiber $V_{x, R_n} := \otimes V_{R_i}$.

A basis element in $\mathbb{C}^4 \otimes V_{x, R_n}$ will be denoted $e_{x, l_n}^{(x R_n)} := \otimes e_{l_n}^{(R_i)}$. (For clarity, we will not make the spinor index explicit.) Vector space $V_{x, R_n}$ furnishes the representation $\rho^{(x R_n)}(x_n G_i)$. It is this representation that determines the gauge–matter field interactions via the covariant derivative $\not{D}$;

$$\not{D} \Psi^{(x R_n)}(x) = \left[ \not{\partial} + \rho^{(x R_n)}(\mathcal{A}) \right] \Psi^{(x R_n)}(x) \quad (11)$$

where $\mathcal{A} := i, \gamma A = \gamma_{\mu} A^\mu \in \oplus_n G_i$, and $\Psi^{(x R_n)}(x) := \psi^{x, l_n}(x) e^{(x R_n)}_{x, l_n}$.

There is a scale ambiguity that resides in the matter field covariant derivative. The inner product on $\rho'(G_i)$ for any faithful representation is proportional to the inner product on $G_i$. This implies the matrices in the covariant derivative (11) are determined only up to overall constants $\kappa_{G_i}$ — relative to the scale of the gauge fields. These constants are conventionally interpreted as coupling constants characterizing the gauge boson–matter field interaction. We choose the coupling constants so that, given gauge and matter field normalizations, the parameters in the matter field covariant derivative that characterize neutral gauge–matter field interactions coincide with the matter field kinematical quantum numbers $q^{(m)}_{G_i}$.

With this choice, the parameters characterizing couplings in both the gauge and matter field covariant derivatives are proportional to the kinematical quantum numbers associated with the weights of the associated representation.

Now, the (bare) Lagrangian density that determines the dynamics is comprised of the usual Yang-Mills terms, spinor matter field terms, ghost terms, and gauge fixing terms. The Yang-Mills terms are

$$-\frac{1}{2} \sum_i \mathcal{F}_i \cdot \mathcal{F}_i \quad (12)$$

where the dot product represents both the Minkowski metric and an $Ad(g_i)$ invariant inner product on each $G_i$. For reductive $G_i$, the $Ad$-invariant inner product on each subspace determined by span$_\mathbb{C}\{g_{a_i}\}$ is classified by two real constants.

The normalization chosen for the inner product effectively fixes the scale of $A^\mu_i(x)$ and hence also the gauge fields $A^\mu_i(x)$ given the standard Minkowski inner product.

The most general (quadratic, minimal coupling) matter field Lagrangian kinetic term consistent with the requisite symmetries is, according to the suggestion from the previous section, comprised of a sum over all the faithful irreps (including complex conjugates) of the elementary matter fields:

$$\mathcal{L}_m = i \sum_{x R_n} \kappa_{x R_n} \overline{\Psi}^{(x R_n)} \cdot \not{D} \Psi^{(x R_n)} + \text{mass terms} \quad (13)$$

---

Footnote 7: For any two elements $g_{a_i}$ and $g_{b_i}$ there are two independent trace combinations, viz. $\text{tr}(g_{a_i} g_{b_i}^\dagger)$ and $\text{tr}(g_{a_i}) \text{tr}(g_{b_i}^\dagger)$. So a general Hermitian bilinear form is a real linear combination of these two. Of course for $SU(N)$ the second combination vanishes identically, and the inner product is then classified by a single constant.
where $\Psi^{(xR_n)}$ is a section of the dual complex-conjugate bundle $\overline{S} \otimes V = S \otimes V$ and $\kappa_{xR_n}$ are positive real constants that are constrained by various consistency conditions; for example, anomaly considerations and CPT symmetry. It is clear that $\delta L_m = 0$ for $\Psi(x) \rightarrow \exp{\{\theta(x)^{a_i} \rho_e^m(g_{a_i})\}} \Psi(x)$ despite the presence of $\kappa_{xR_n}$ (assuming appropriate mass terms).

The dot product here represents a Lorentz and $\rho(g)$ invariant Hermitian matter field inner product; a bundle metric. Recall the matter fields furnish complex representations. For non-isomorphic representations, the scaling constants $\kappa_{xR_n}$ can be absorbed into the inner product on the representation space. However, for the special case of representations that are related via the adjoint action of a unitary symmetry group, their underlying vector spaces are isomorphic. Consequently their inner products are tied together and the associated ratios of $\kappa_{xR_n}$ can be non-trivial. This persists even after renormalization. The possibility of non-trivial factors $\kappa_{xR_n}$ in the matter field Lagrangian density will be a key element in our non-Standard Model.

For each individual subgroup $G_i$, the gauge and matter field terms in the Lagrangian density give rise to the conserved currents

$$J^\mu_{(a_i)} = -F^{\mu\nu}_{i} \cdot [g_{a_i}, A_{iv}] + j^\mu_{(a_i)}$$

where

$$j^\mu_{(a_i)} = \sum_{xR_n} \kappa_{xR_n} \Psi^{(xR_n)} \cdot \gamma^\mu \rho_e^{(xR_n)}(g_{a_i}) \Psi^{(xR_n)}$$

are the covariantly conserved matter field currents. In particular, the neutral conserved currents associated with $G_i$ are

$$J^\mu_{(s_i)} = -F^{\mu\nu}_{i} \cdot [h_{s_i}, A_{iv}] + j^\mu_{(s_i)}$$

$$= -q_{s_i} F^{\mu\nu}_{-\alpha_k} A_{\nu}^{\alpha_k} + \sum_{xR_n} \kappa_{xR_n} (q^{(m)}_{s_i}) \Psi^{(xR_n)} \cdot \gamma^\mu \Psi^{(xR_n)}.$$  \hfill (16)

The constants $\kappa_{xR_n} (q^{(m)}_{s_i})$ will be termed ‘coupling strengths’, and they represent the scale of gauge–matter field couplings given matter field normalizations. Evidently, not all matter field currents contribute to interactions on an equal basis if $\kappa_{xR_n}$ is non-trivial.

This is significant because some particles characterized by a set of kinematical quantum numbers may appear to have scaled coupling strengths when interacting with gauge bosons. However, in order to conclude this, we must first confirm that the normalization freedom in the Lagrangian density allows us to maintain equality between the renormalized parameters $q_{s_i}$ and $q^{(m)}_{s_i}$ appearing in the quantum field relations expressed below in equation (17) and the kinematical quantum numbers discussed in the previous section. Moreover, we must verify that non-vanishing scaling parameters $\kappa_{xR_n}$ do not destroy the assumed local symmetries.

To that end, consider the neutral quantum charge operators $Q_{(s_i)} := -i \int J^0_{(s_i)} dV$ associated with the currents given by (16). They encode dynamical quantum numbers in the sense that

$$[Q_{(s_i)}, A_{\perp}^{\alpha_k}] = q_{s_i} A_{\perp}^{\alpha_k} \delta_{ij}$$

$$[Q_{(s_i)}, \Psi^{(xR_n)}] = q^{(m)}_{s_i} \Psi^{(xR_n)}.$$  \hfill (17)
where the gauge and matter fields have been promoted to quantum operators and $A'^\alpha_i$ are the transverse gauge fields. The second relation follows because the conjugate momentum of $\Psi(x,R_n)$ is $\kappa \times R_n \overline{\Psi}^{(x,R_n)} \gamma^0$ as determined from (13).

Equations (17) are in terms of bare quantities, but they are required to be valid for renormalized quantities as well. Under the renormalizations

$$A_i^B \rightarrow Z^{1/2}_A A_i^R$$

and

$$\Psi^{(x,R_n)} \rightarrow Z^{1/2}_{\Psi^{(x,R_n)}} \Psi^{(x,R_n)^R},$$

the basis elements $g_{a_i}$ can be re-scaled so that $q_{B_i} = Z^{-1/2} q_{A_i}$. Likewise, the basis $e^R_{i_l}$ can be re-scaled so that $q_{,s_i} = Z^{-1/2} q_{,s_i} q_{s_i}$. Consequently the relations (17) will be maintained under renormalization. The renormalized form of equations (17) are to be compared to (3) and (6). That they are consistent is a consequence of: i) the covariant derivatives (10) and (11), ii) our choice of Lie algebra inner product, and iii) identifying the renormalized dynamical quantum numbers with the kinematical quantum numbers. This consistency ensures the renormalized gauge and matter fields appearing in the Lagrangian density can be identified with the elementary fields associated with the Lie algebra-induced quantum numbers $q_{,s_i}$. It should be emphasized that the gauge group coupling constants are implicit in $q_{B_i}$ and $q_{(m)}_{s_i}$, and non-trivial $\kappa \times R_n$ do not get renormalized; or, rather, non-trivial $\kappa \times R_n$ persist after renormalization of $\Psi^{(x,R_n)}$.

Now, to maintain the all-important local symmetries of the Lagrangian density, $Q_{(a_i)}$ and $\{g_{a_i}\}$, along with their associated commutation relations, must determine isometric algebras. Fortunately, we find

$$[\hat{J}^{0}_{(a_i)}, \hat{J}^{0}_{(b_j)}] = \delta_{ij} C^{c_j}_{a_i,b_j} \left\{ -\mathcal{F}^{\mu\nu}_{,i} \cdot [g_{c_j}, A^{\mu\nu}_{i}] + \sum_{x,R_n} \kappa \times R_n \overline{\Psi}^{(x,R_n)} \rho_{e^{(i)(n)}} (g_{c_j}) \Psi^{(x,R_n)} \right\}$$

$$= \delta_{ij} C^{c_j}_{a_i,b_j} \hat{J}^{0}_{(c_j)}$$

(20)

where $C^{c_j}_{a_i,b_j}$ are the structure constants of $G_i$.

It is crucial that the $\kappa \times R_n$ factors do not spoil the equality between the kinematical and dynamical quantum numbers or the local symmetries. Given these developments, it makes sense to refer to the two types of quantum numbers — renormalized dynamical quantum numbers and kinematical quantum numbers — by the common term intrinsic charges. On the other hand, the renormalized coupling strengths $\kappa \times R_n (q_{(m)}_{s_i})$ in the renormalized currents (16) will be called extrinsic charges.

The analysis in this subsection leads to the conclusion that, in some cases, the intrinsic charges of matter fields do not fully determine their coupling strengths to gauge bosons. Stated otherwise, the intrinsic and extrinsic charges of quantized matter fields are not necessary equivalent. (Evidently, the qualifier intrinsic/extrinsic is not necessary for gauge bosons.)
3 The non-Standard Model

Instead of the SM symmetry group $SU_C(3) \times SU_I(2) \times U_Y(1)$ based on color, isospin, and hypercharge; we choose the non-Standard Model (n-SM) symmetry group $SU_C(3) \times U_{EW}(2)$ with associated Lie algebra $su_C(3) \oplus u_{EW}(2) \cong su_C(3) \oplus u_{EW}(2)/u_{EM}(1) \oplus u_{EM}(1)$ based on strong charge (as opposed to color charge as will be explained below) and electric charge.

3.1 Gauge boson charges

In our picture, gauge boson charges are determined by the Cartan basis.

The root system for $SU_C(3)$ is two-dimensional implying two strong charges. Hence the Cartan decomposition yields two strong-charge-neutral gluons and three sets of oppositely strong-charged gluons. Accordingly, our interpretation of strong charge associated with $SU_S(3)$ differs from the standard assignment of color charge, because we associate charge with roots (quantum numbers) rather than degrees of freedom in the defining representation.

For book-keeping purposes, there is no essential difference between the two interpretations in the defining representation since there is a one-to-one correspondence between color degrees of freedom and matter field eigenstates with non-vanishing strong charge. (In other words, matter field eigenstates are kinematically gauge equivalent.) However, this is not the case for the adjoint representation where the interaction landscape differs because the Cartan subalgebra is invariant under inner automorphisms: Neutral and strong-charged gluons can interact but they don’t mix under gauge transformations, and this will be reflected in the dynamics.

Remark 3.1 Conventionally, gluons carry one color charge and one anti-color charge, and they exchange one of these color charges in a quark/gluon interaction. Presumptively, color charges stand on equal footing due to $SU(3)$ invariance. For us however, color simply labels the degrees of freedom in the defining representation and has nothing to do with strong charge — although it still plays a role in classifying irreducible representations in the usual way.

Our interpretation of strong charge differs substantially: Indeed, from our standpoint strong charges, being based on the Lie algebra root system, are hierarchical (in the sense of non-trivial charge ratios). This hierarchy will induce some self-energy differences among gluons since neutral gluons don’t directly interact.

The two neutral gluons, which apparently experience a quite different confining potential compared to charged gluons, have obvious implications regarding nuclear forces. Specifically, one can imagine the neutral gluons (at least partly) mediating the nuclear force.

Moving on to the $U_{EW}(2)$ subgroup, there will be two neutral gauge bosons and a pair of oppositely charged gauge bosons carrying two types of electroweak charge (electric and...
weak). However, in the broken symmetry sector, the only gauge boson that survives is a neutral boson that characterizes electric charge (and the mass and electric charge that now distinguish the broken symmetry generators stand in for their original electroweak charges). There is no compelling reason to introduce isospin and hypercharge since experiment dictates that the charge associated with the unbroken gauge boson is what we know as electric charge. Of course, in the unbroken symmetry phase, the Cartan basis characterizing electric charge is only unique modulo $U_{EW}(2)$ conjugation. Consequently, one can talk about any other gauge-equivalent charge combination consistent with the relations imposed by the Lie algebra decomposition described in § 2. Note, however, that isospin and hypercharge are not gauge equivalent to the two EW charges since the hypercharge generator commutes with every other generator.

### 3.2 Hadronic Constituents

We will consider only Dirac matter fields in the fundamental representation of $SU(3)$ and $U(2)$. Consequently, the matter fields are sections of an associated fiber bundle with typical fiber $\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. Since $su(3)$ and $u(2)$ are both rank two algebras, these matter fields can be labelled by four quantum numbers; two associated with $SU(3)$ and two with $U(2)$. According to the previous section, the relevant irreps of the direct product group are postulated to include the $(3, 2)$ and $(3, 2^c)$ and their anti-fields $(\bar{3}, 2)$ and $(\bar{3}, 2^c)$.

Spinor matter fields in the defining representation require the product bundle $S \otimes V_R$ where $S$ is a spinor bundle over Minkowski space-time. For example, given a trivialization of the bundle $S \otimes V_{(3,2)}$, let $\{e^{aA}\} := \{\psi^a \otimes e^A \otimes e^a\}$ be the chosen basis that spans the typical fiber $\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. (Indices are assumed to have the necessary ranges for any given representation.) Sections $\Psi = \psi^{aA}\Psi(a) e^{aA}$ of $S \otimes V_{(3,2)}$ constitute the elementary spinor fields in the $(3, 2)$ representation, and $e^A \otimes e^a$ encode the internal $SU(3) \times U(2)$ degrees of freedom.

For the $U_{EW}(2)$-conjugate representation, define $\rho^c := (i\tau_2)\rho^*(i\tau_2)^{-1}$ and $\Psi^c := (i\tau_2)\Psi^*$ where $i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Explicitly, $\Psi^c$ is a section of the bundle $S \otimes V_{SU(3)} \otimes V_{U(2)}^c$ which is the image under the bundle morphism

$$F : S \otimes V_{SU(3)} \otimes V_{U(2)} \rightarrow S \otimes V_{SU(3)} \otimes V_{U(2)}^c$$

$$(x, \psi_{aAa}(x)e^{aA}) \mapsto (x, [i\tau_2]^b\psi_{aAb}(x)e^{aA}) =: (x, \psi_{aAa}(x)(\psi^a \otimes e^A \otimes e^a))$$

in a given chart and trivialization. Note that $\{e^a\}^c = -\{e^a\}$. Meanwhile, the dual complex-conjugate representation is given by $\rho^T = (i\tau_2)^2\rho^*(i\tau_2)^{-2} = (i\tau_2)\rho^T(i\tau_2)^{-1}$ and

$$i\q^{(m)}_c \Psi^c = \rho_c^e(h_s)\Psi^c = (i\tau_2)(\rho_c^e(h_s)\Psi)^* = (i\tau_2)(-i\q^{(m)}_s \Psi^*) = -i\q^{(m)}_s \Psi^c. \quad (22)$$

So $\Psi$ and $\Psi^\dagger$ are minus the $U_{EW}(2)$-conjugate of a $U_{EW}(2)$-conjugate, and we interpret $\Psi^c$ as the “$U(2)$ anti-field” of $\Psi$ in the sense that $U_{EW}(2)$-conjugation yields a field with opposite $U_{EW}(2)$ charges with respect to a permuted basis. (We emphasize that there is no conjugation associated with the $SU(3)$ or spinor index here.)
The covariant derivatives acting on the matter fields in the \((3, 2)\) and \((3, 2^c)\) representations are
\[
(\nabla \Psi) = \{ \partial[1]_{Aa} + \mathcal{G}^\alpha [\lambda_\sigma]_A^{\alpha} \otimes [1]_a^b + [1]_A^{B} \otimes \mathcal{g}^\sigma [\lambda_\sigma]_A^b \} \Psi_{Bb} e^{Aa} \tag{23}
\]
and
\[
(\nabla^c \Psi^c) = \{ \partial[1]_{Aa} + \mathcal{G}^\alpha [\lambda_\sigma]_A^{\alpha} \otimes [1]_a^b + [1]_A^{B} \otimes \mathcal{g}^{\sigma*} [\lambda^C]_A^b \} \Psi_{Bb}^c e^{Aa} \tag{24}
\]
respectively with \(\lambda_\sigma\) and \(\Lambda_\alpha\) generating \(U_{EW}(2)\) and \(SU_C(3)\). These yield the kinetic matter field Lagrangian density;
\[
\mathcal{L}_m = i \kappa \overline{\Psi}_{A \alpha} \{ \partial[1]_{Aa} + \mathcal{G}^\alpha [\lambda_\sigma]_A^{\alpha} \otimes [1]_a^b + [1]_A^{B} \otimes \mathcal{g}^\sigma [\lambda_\sigma]_A^b \} \Psi_{Bb} \delta^{A'B} \delta^{a'a} + i \kappa c \overline{\Psi}^c_{A \alpha} \{ \partial[1]_{Aa} + \mathcal{G}^\alpha [\lambda_\sigma]_A^{\alpha} \otimes [1]_a^b + [1]_A^{B} \otimes \mathcal{g}^{\sigma*} [\lambda^C]_A^b \} \Psi^c_{Bb} \delta^{A'A} \delta^{a'a'} \tag{25}
\]
Note that it is not possible to absorb both \(\kappa\) and \(\kappa^c\) by separate field redefinitions because \(\Psi^c_{Aa} = [i \tau_2]^b A \Psi^*_{Aa} \) and \(e^a \cdot e^b = e^a \cdot e^c\); implying that the ratio \(\kappa^c/\kappa\) can be non-trivial in this example.

The corresponding \(U(2)\) and \(SU(3)\) currents are
\[
j^\mu_{(\sigma)} = \kappa \overline{\Psi^c_{A \alpha}} \gamma^\mu [\lambda_\sigma]_A^{\alpha} \Psi_{A} + \kappa c \overline{\Psi^c_{A \alpha}} \gamma^\mu [\lambda^C]_A^b \Psi_{B} \tag{26}
\]
and
\[
j^\mu_{(\alpha)} = \kappa \overline{\Psi}_{A \alpha} \gamma^\mu [\lambda_\sigma]_A^{\alpha} \Psi_{A} + \kappa c \overline{\Psi}_{A \alpha} \gamma^\mu [\lambda^C]_A^b \Psi_{B} \tag{27}
\]
respectively. Evidently, if \((\kappa + \kappa^c) = 1\) the original \(SU(3)\) coupling strength is preserved, i.e., the \(SU(3)\) intrinsic and extrinsic charges are equivalent. However, in this case, the \(U(2)\) external charges are fractional relative to the internal charges since \(\kappa, \kappa^c \neq 0\) by assumption. In a viable model, as we will see presently, the ratio is ultimately fixed by anomaly considerations.

The structure of these currents suggests to define hadronic constituents (HC) in the \((3, 2)\) representation by \(H^+ := H^+_A e^A\) and their \(U_{EW}(2)\)-conjugates \((3, 2^c)\) by \(H^- := H^-_A e^A\) where
\[
H^+_A := \Psi_{Aa} e^a = \Psi_{A1} e^1 + \Psi_{A2} e^2 =: (h^+ e^1)_A + (\xi^0 e^2)_A = \begin{pmatrix} h^+ \\ \xi^0 \end{pmatrix}_A \tag{28}
\]
and
\[
H^-_A := [i \tau_2]^b \Psi^*_{Aa} e^a = \Psi^*_{A2} e^1 - \Psi^*_{A1} e^2 = \Psi^*_{A1} e^1 + \Psi^*_{A2} e^2 =: \begin{pmatrix} \chi^0 \\ h^- \end{pmatrix}_A \tag{29}
\]
Here \(h^\pm, \xi^0, \chi^0\) are complex space-time Dirac spinor fields and \(e^{1,2}\) span \(\mathbb{C}^2\). The superscripts on the component fields denote electric charge only since weak charge is not relevant in the broken sector. Being components of hadronic constituents and possessing integer electric charge, we will give \(h^\pm, \xi^0, \chi^0\) the name ‘iquarks’. There are three copies of \(H^\pm\) accounting for the three quark generations. No generality is sacrificed by assuming \(H^\pm\) are normalized.

\footnote{The assignment \(h^- e^2 := -h^+ e^2\) follows from \((\text{6})\) and \((\text{30})\). Likewise, \(\xi^0 e^2 = \xi^0 e^1\). For example, the conjugate representation requires \(Q^c H^+_A = (i \tau_2) (Q^c)^\gamma (h^+ e^1 + \xi^0 e^2)_A = (h^+ e^2)_A\). On the other hand, from the explicit representation \((\text{36})\) of the electric charge generator on \(V^c_{\langle 3, 2 \rangle}\), we have \(Q^c H^-_A = -(h^- e^2)_A\).}
By assumption, both the left and right-handed quarks furnish the 3 of \( SU_C(3) \). Also by assumption, the left-handed quarks furnish the 2 and \( 2^c \) of \( U_{EW}(2) \) while the right-handed quarks furnish the 1\(^+\), 1\(^-\) and 1\(^0\). Thus, we have \( \bar{H}_L^+ := \frac{1}{2}(1 + \gamma_5)H^+, \bar{H}_L^-, h^-_R, \bar{h}^-_R, \) and \( \xi^0_R \) furnishing the \( (3, 2), (3, 2^c), (3, 1^+), (3, 1^-) \), and \( (3, 1^0) \) of \( SU_C(3) \times U_{EW}(2) \), respectively.

Let us work out the explicit form of these representations. Start with \( U_{EW}(2) \). Physically, in the broken symmetry regime characterized by matter fields with conserved electric charge, the Lie algebra \( u_{EW}(2) \) decomposes according to \( u_{EM}(1) \oplus (u_{EW}(2)/u_{EM}(1)) \). Consequently, the gauge bosons are also characterized by electric charge. This implies the broken symmetry map of the unbroken, electric charge generator. That is, the Lie algebra decomposition is \( u_{EW}(2) = u_{EM}(1) \oplus k \) such that \( u_{EM}(1) \cap k = 0 \) and \( ad(u_{EM}(1))k \subseteq k \).

Recall that we require, on physical grounds, a real subalgebra of \( u(2, \mathbb{C}) \) for the gauge bosons. Since \( U_{EW}(2) \) has rank 2, the relevant basis therefore obeys

\[
\begin{align*}
[\lambda_\pm, \lambda_\pm] &= \sum_i \pm c'_i \lambda_i, \\
[\lambda_\pm, \lambda_i] &= \pm c_i \lambda_\pm, \\
[\lambda_i, \lambda_j] &= 0,
\end{align*}
\tag{30}
\]

where \( c_i, c'_i \) are structure coefficients and \( i, j \in \{1, 2\} \). The real (compact) subalgebra that generates \( U_{EW}(2) \) is \( u_{EW}(2) := \text{span}_{\mathbb{C}} \{\lambda_+, \lambda_-, \lambda_1, \lambda_2\} \) where \( \mathbb{C}/\sim \) is the set of coefficients \((a^\pm, b_i, )\) such that \( a^- = (a^+)^* \) with \( a^\pm \in \mathbb{C} \) and \( b_i \in \mathbb{R} \). Explicitly, a general skew-hermitian element \( g \in u_{EW}(2) \) in this basis decomposes as \( g = a^+ \lambda_+ + a^- \lambda_- + b_1 \lambda_1 + b_2 \lambda_2 \).

The most general defining representation of \( u_{EW}(2) \) allowed by \( (30) \) is generated by

\[
\begin{align*}
T_+ &:= \rho'_e(\lambda_+) = i \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, & T_- &:= \rho'_e(\lambda_-) = i \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \\
T_0 &:= \rho'_e(\lambda_1) = i \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, & Q &:= \rho'_e(\lambda_2) = i \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix},
\end{align*}
\tag{31}
\]

where the representation is a unitary map \( \rho : U_{EW}(2) \to GL(\mathbb{C}^2) \), \( \rho'_e \) denotes the derivative of the representation evaluated at the identity element \( e \in U_{EW}(2) \), and \( r, s, t, u, v \) are real constants.

To proceed, an Ad-invariant form on \( u_{EW}(2) \) is required. In fact, there is a 2-dimensional real vector space of Ad-invariant Hermitian bilinear forms on \( u_{EW}(2) \) given by \( (3) \) pg. 114

\[
\langle \lambda_\sigma, \lambda_\rho \rangle := 2g_1^{-2}\text{tr}(\lambda_\sigma \lambda_\rho^\dagger) + (g_2^{-2} - g_1^{-2})\text{tr}\lambda_\sigma \cdot \text{tr}\lambda_\rho^\dagger \tag{32}
\]

where \( g_1 \) and \( g_2 \) are real parameters and the \( \dagger \) operation (complex-conjugate transpose) is defined since \( U(2) \) is a matrix Lie group.

Restricting to the half-space \( g_1^2 \geq g_2^2 \) yields a positive-definite inner product defined by

\[
g(\lambda_\sigma, \lambda_\rho) := 1/2\langle \lambda_\sigma, \lambda_\rho \rangle. \tag{33}
\]
Explicitly, in the basis defined by (30),

$$(g_{σρ}) = \begin{pmatrix} g_{W}^{-2} & 0 & 0 & 0 \\ 0 & g_{W}^{-2} & 0 & 0 \\ 0 & 0 & g_{Z}^{-2} & 0 \\ 0 & 0 & 0 & g_{Q}^{-2} \end{pmatrix} \quad (34)$$

where

$$g_{W}^{-2} := g(λ_±, λ_±), \quad g_{Z}^{-2} := g(λ_1, λ_1), \quad g_{Q}^{-2} := g(λ_2, λ_2). \quad (35)$$

The inner product can be put into canonical form by re-scaling the $$u_{EW}(2)$$ basis vectors by $$\lambda_± \rightarrow g_{W}λ_±, \lambda_1 \rightarrow g_{Z}λ_1,$$ and $$λ_2 \rightarrow g_{Q}λ_2.$$ The inner product on the Lie algebra is proportional to the inner product for any of its faithful representations. Hence, (28), (32), (33), together with the orthogonality condition $$u_{EM}(1) \cap k = 0$$ give the defining representation (with superscript $$+$$) and $$U_{EW}(2)$$-conjugate representation (with superscript $$-$$) for the doublet HC matter fields:

$$T_0^+ = \frac{ie}{2 \cos θ_W \sin θ_W} \begin{pmatrix} 2 \sin^2 θ_W - 1 & 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_0^- = \frac{-ie}{2 \cos θ_W \sin θ_W} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 \sin^2 θ_W - 1 & 0 \end{pmatrix}$$

$$T_+^+ = T_-^+ = \frac{ie}{\sqrt{2} \sin θ_W} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_-^- = T_+^- = \frac{-ie}{\sqrt{2} \sin θ_W} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_+^+ = ie \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_-^- = -ie \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (36)$$

where $$e$$ is the electric charge and $$θ_W$$ is the Weinberg angle defined by

$$g_1^2 g_2^2 / (g_1^2 + g_2^2) =: e^2 / \sin^2 θ_W$$

$$g_1^2 = \frac{e^2}{\sin^2 θ_W}$$

$$(g_1^2 + g_2^2) = \frac{e^2}{\sin^2 θ_W \cos^2 θ_W}. \quad (37)$$

For the $$^+$$-representation ($$^-$$-representation) $$r(s), t,$$ and $$u(v)$$ were absorbed into $$g_Q, g_W,$$ and $$g_Z.$$ This EW representation is identical to the SM lepton doublet representation — which is not surprising given the nature of the HC doublet construction.

The 1-dimensional representations for the charged right-handed iquarks are obtained by taking the trace of the 2-dimensional representations while maintaining proper normalization. For $$h^+_R$$ it amounts to $$a^+_R := \text{tr}(Q^+)$$ and $$t^+_R := \text{tr}(T^+_R).$$ Meanwhile, the electrically neutral $$ξ^0_R, λ^0_R$$ stem from the trivial representation and its conjugate.\(^{12}\)

This representation together with (30) implies that the structure coefficients $$c'_i$$ (in the defining representation) are functions of $$(g_1, g_2)$$ such that $$c'_i(g_1, g_2) \neq 0$$ for all $$i \in \{1, 2\}$$ for

\(^{12}\) Might $$ξ^0_R, λ^0_R$$ furnish the determinant representation instead?
generic values \((g_1, g_2)\). This has two important consequences for the generators of the Cartan subalgebra: 1) none of the Cartan generators can be proportional to the identity element, and 2) their norms are functions of both coupling constants. Notice that \(\text{span}_{\mathbb{C}/\sim}\{T_0, T_+ , T_- \}\) does not generate \(\mathfrak{su}(2)\) except when \(g_2 \to 0\). Evidently, \(u_{\text{EM}}(2)/u_{\text{EM}}(1) \not\cong \mathfrak{su}(2)\).

**Remark 3.2** Of course our assumed multiplet composition of the HC (with one charged and one neutral component) dictates the representation \([36]\). And in general one could construct a different representation consistent with the Cartan decomposition that contains the identity element as one of its generators. But then one of the \(c'_i(g_1, g_2)\) would have to alter the functions \(c'_i(g_1, g_2)\) in [36]. This is a plausible and reasonable picture.

However, an alternative picture is that the dynamics that induce SSB do nothing to change the functional form of the \(c'_i(g_1, g_2)\); and, owing to the \(U(2)\) invariance, neither of them vanishes identically at energy scales above SSB — since otherwise this would distinguish a particular direction in the Cartan subalgebra. Equivalently, we insist that \([\lambda_\pm, \lambda_\mp]\) spans the entire Cartan subalgebra. Physically, this means all \(U(2)\) bosons interact with each other.

Accordingly, the first equation in \([30]\) then leads to four conditions

\[
[T_\pm, T_\mp] = \pm c'_1 Q \pm c'_2 T_0 \Rightarrow \begin{cases} c'_1 = \frac{i^{2(u,v)}}{(ru - sv)} \neq 0 \\
 c'_2 = \frac{i^{2(r,s)}}{(ru - sv)} \neq 0
\end{cases} \Rightarrow \begin{cases} t \neq 0 \\
 r + s \neq 0 \\
 u + v \neq 0 \\
 ru - sv \neq 0
\end{cases} . \tag{38}
\]

Together with \(g(T_\pm, Q) = 0\) and \(g(T_0, Q) = 0\), this yields two possible solutions

\[
\left( u/v = \frac{g_2^2 - g_1^2}{g_1^2 + g_2^2}, \quad s = 0 \right) \quad \text{and} \quad \left( r/s = -\frac{g_2^2(u - v) + g_1^2(u + v)}{g_2^2(v - u) + g_1^2(u + v)} \right) . \tag{39}
\]

The first solution on the left is just the representation \([36]\) already found. The second solution on the right is under-determined. For the case \(u = v\), we have \(r/s = -1\) which then implies \(\text{span}_{\mathbb{C}/\sim}\{T_+, T_-, T_0, Q\} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)\) with respect to the inner product and this is unacceptable: besides this violates \(r + s \neq 0\). The case \(u \neq v\) is likewise unacceptable because we don’t recover \(\mathfrak{su}(2)\) when \(g_2 \to 0\) unless \(u = 0\) (or \(v = 0\)). So the second possibility is ruled out unless \(u = 0\): but then it is the same as the first solution with \((r, s) \leftrightarrow (v, u)\).

Hence, orthogonality together with \(\text{span}_{\mathbb{C}/\sim}\{T_+, T_-, T_0, Q\} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1)\) and the stipulation\(^{[3]}\) that \(\text{span}_{\mathbb{C}/\sim}\{T_+, T_-, T_0, Q\}\{g_2 \to 0\} \cong \mathfrak{su}(2)\) uniquely determine the representation.

\(^{[3]}\)SM convention has it that isospin and hypercharge are distinguished quantum numbers before spontaneous symmetry breaking (SSB) while electric charge \(e\) and the mixing angle \(\theta_W\) are preferred after SSB. Of course electric charge and \(\theta_W\) are running couplings. As \(g_2/g_1\) varies in the range \(0 < g_2/g_1 < 1\), the character of the neutral bosons’ interactions changes. In the limit \(g_2 \to 0\), electric charge vanishes and \(Q^\pm\) no longer participates in interactions. Meanwhile, \(T_0\) and \(T_\pm\) reduce to the standard fundamental \(SU(2)\) representation characterized by isospin. At the other extreme, when \(g_2/g_1 \to 1\) the relevant quantum number is electric charge since in this case \(\theta_W \to \pi/4\) and \(T_0 \to Q_\mp\). The point is, electric charge becomes more prominent as energy increases while isospin and hypercharge become more prominent as energy decreases.

\(^{[4]}\)This ensures the inner product remains finite as \(g_2 \to 0\) (which is required for a sensible theory) and that the \(g_1\) coupling constants appearing in the SM and our proposed n-SM are equivalent; which means consistent normalizations are maintained between the two theories and they can be compared directly.
Remark 3.3  What happens if we define different HC that possess the quark electric charges which produces a non-diagonal and degenerate quadratic Casimir (with respect to adjoint representation) represented by

\[
\begin{align*}
Ad_{\lambda_+} &= ie \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\cot \theta_W & 1 \\
\cot \theta_W & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} & Ad_{\lambda_-} &= ie \begin{pmatrix}
0 & 0 & \cot \theta_W & -1 \\
0 & 0 & 0 & 0 \\
0 & -\cot \theta_W & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \\
Ad_{\lambda_1} &= ie \begin{pmatrix}
-\cot \theta_W & 0 & 0 & 0 \\
0 & \cot \theta_W & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} & Ad_{\lambda_2} &= ie \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]

which produces a non-diagonal and degenerate quadratic Casimir (with respect to (33)) represented by

\[
C_{Ad} := \sum_{\sigma, \rho} g_{\sigma \rho} \rho_\sigma(\lambda_\sigma) \cdot \rho_\rho(\lambda_\rho)^\dagger = g_1^2 \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos^2 \theta_W & -\sin \theta_W \cos \theta_W \\
0 & 0 & -\sin \theta_W \cos \theta_W & \sin^2 \theta_W
\end{pmatrix}.
\]

Remark 3.3  What happens if we define different HC that possess the quark electric charges 2/3, −1/3 of the SM and re-run the analysis? In this case, orthogonality gives

\[
\begin{align*}
Q_{\text{quark}}^+ &= ie \begin{pmatrix}
2/3 & 0 \\
0 & -1/3
\end{pmatrix} & T_{0\text{quark}}^+ &= \frac{i(g_1^2 + 3g_2^2)}{2\sqrt{g_1^2 + 9g_2^2}} \begin{pmatrix}
-g_1^2 \frac{g_2^2}{g_1^2 + 3g_2^2} & 0 \\
0 & 1
\end{pmatrix}. 
\end{align*}
\]

The problem is, because of the numerical factors in front of \(g_2^2\), we get for \(\theta_W \simeq .23\)

\[
T_{0\text{quark}}^+ - T_0^+ \simeq ie \text{dia}(-2.48, 4)
\]

which clearly disagrees with experiment. In fact, the two representations agree only when \(\theta_W = 0\) which just takes us back to \(\mathfrak{su}(2) \oplus \mathfrak{u}_1(1)\) and the SM before SSB.\(^\text{15}\)

Our insistence that \(c_i^l(g_1, g_2) \neq 0\) for all \(i \in \{1, 2\}\) and generic \((g_1, g_2)\) is a crucial departure from the SM. Without it, the discarded solution found in the previous remark becomes valid; and the EW representation of HC can be constructed exactly as in the SM by specifying isospin and hypercharge quantum numbers — in which case iquarks no longer posses integer electric charge. So if we allow \(c_i^l(g_1, g_2) = 0\) for some \(i \in \{1, 2\}\), the only thing \(U_{EW}(2)\) brings to the table is the Gell-Mann/Nishijima relation as previously discussed.

Repeating the exercise for \(SU_C(3)\) using the inner product (again restricting to \(g_1^2 \geq g_2^2\))

\[
g(\Lambda_\alpha, \Lambda_\beta) := \frac{1}{6} \begin{bmatrix}
3g_1^{-2} \text{tr}(\Lambda_\alpha \Lambda_\beta^\dagger) + (g_2^{-2} - g_1^{-2}) \text{tr}\Lambda_\alpha \cdot \text{tr}\Lambda_\beta^\dagger
\end{bmatrix} = \frac{1}{2g_1^2} \text{tr}(\Lambda_\alpha \Lambda_\beta^\dagger),
\]

\(^\text{15}\)Note, however, that after SSB we no longer have \(T_{0\text{quark}}^+\) orthogonal to \(Q_{\text{quark}}^+\) — even if we use the Killing inner product instead of \(\text{diag}^\text{3} \).
the $Ad$-invariant inner product on the subalgebra $\mathfrak{su}_C(3) := \text{span}_C/\sim\{\mathbf{A}_i\}$ is

$$
\langle g_{\alpha\beta} \rangle = \begin{pmatrix}
g_{s_1}^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{s_1}^{-2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{s_2}^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_{s_2}^{-2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_{s_3}^{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_{h_1}^{-2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_{h_2}^{-2}
\end{pmatrix}
$$

(45)

with $g_{s_i}^{-2} := g((\mathbf{A}_i)_\pm, (\mathbf{A}_i)_\pm)$, $g_{h_1}^{-2} := g(\mathbf{A}_1, \mathbf{A}_1)$, and $g_{h_2}^{-2} := g(\mathbf{A}_2, \mathbf{A}_2)$.

An explicit defining representation is given by

$$
\begin{align*}
\mathbf{S}_1^+ &:= \rho'(((\mathbf{A}_1)_+) = igs_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{S}_1^- &:= \rho'((\mathbf{A}_1)_-) = igs_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{S}_2^+ &:= \rho'((\mathbf{A}_2)_+) = igs_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{S}_2^- &:= \rho'((\mathbf{A}_2)_-) = igs_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathbf{S}_3^+ &:= \rho'((\mathbf{A}_3)_+) = igs_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{S}_3^- &:= \rho'((\mathbf{A}_3)_-) = igs_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
\mathbf{H}_1 &:= \rho'(\mathbf{A}_7) = igh_1 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{H}_2 &:= \rho'(\mathbf{A}_8) = igh_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
$$

(46)

where $g_{s_i} = g_{h_1} = g_{h_2} = \tilde{g}_1$ is the strong coupling constant.

With this choice, the inner product in the defining representation becomes

$$
\langle g_{\alpha\beta} \rangle = 1/2 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}.
$$

(47)

This, of course, is not the standard representation or normalization. But it is convenient because the generators are all imaginary and it yields integer intrinsic charges.\footnote{As with the previous case of $\mathfrak{su}_E/\mathfrak{su}_F(2)$, to get Hermitian matrices we take $\mathfrak{su}_C(3) \equiv \text{span}_C/\sim\{\mathbf{S}_i^\pm, \mathbf{H}_i\}$. Standard normalization is readily achieved by putting $\sqrt{2}g_{h_1} = \sqrt{6}g_{h_2} = \tilde{g}_1$.} Explicitly, gluons posses the strong charges (as factors of $\tilde{g}_1$) $(\mathbf{A}_1)_\pm \rightarrow (\mp 2, 0)$, $(\mathbf{A}_2)_\pm \rightarrow (\mp 1, \mp 3)$, $(\mathbf{A}_3)_\pm \rightarrow (\pm 1, \mp 3)$, $\mathbf{A}_7 \rightarrow (0, 0)$, and $\mathbf{A}_8 \rightarrow (0, 0)$. Similarly, the three matter eigenfields in the defining representation posses strong charges $(\mp 1, \mp 1)$, $(\pm 1, \mp 1)$, and $(0, \pm 2)$.}
3.3 The Lagrangian density

The Yang-Mills, lepton, and Higgs contributions to the n-SM Lagrangian density are identical (up to our non-standard inner product and normalization) to the SM so we won’t display them. But, according to our previous discussion, the inequivalent quark irreps of the direct product group include the combinations \((3, 2), (3, 2^c), (3, 1^+), (3, 1^-),\) and \((3, 1^0)\) along with corresponding anti-particle combinations. Their contribution to the Lagrangian density is

\[
\mathcal{L}_{\text{quark}} = i \sum_s \kappa^+ (\overline{H}^+_{L,s} D^+ H^+_{L,s} + h^+_R h^+_{R,s} + \mathcal{O}_R \partial \mathcal{O}_{R,s}) \\
+ \kappa^- (\overline{H}^-_{L,s} D^- H^-_{L,s} + h^-_R h^-_{R,s} + \mathcal{O}_R \partial \mathcal{O}_{R,s})
\]

\( (48) \)

\[
\mathcal{L}_{\text{Yukawa}} = -\sum_s \kappa^+ (m^+_s h^+_{R,s} \Phi^+ H^+_{L,s} + n^+_s H^+_{L,s} \Phi^0 \mathcal{O}_{R,s}) \\
+ \kappa^- (m^-_s h^-_{R,s} \Phi^- H^-_{L,s} + n^-_s H^-_{L,s} \Phi^0 \mathcal{O}_{R,s}) + h.c.
\]

\( (49) \)

where \(s, t\) label quark generation and \(h.c.\) means Hermitian conjugate.

The covariant derivatives are

\[
D^+ H^+_L = (\partial + W^+ T^+_+, W^- T^-_+ + Z^0 T^0_+ + \mathcal{O}^+ + \mathcal{G}^+ \mathcal{L}^+ H^+_L
\]

\[
D^- H^-_L = (\partial + W^+ T^-_+ + W^- T^+ - + Z^0 T^0_+ + \mathcal{O}^+ + \mathcal{G}^+ \mathcal{L}^+ H^-_L
\]

\[
D^\pm h^\pm_R := \text{tr}[D^\pm] h^\pm_R, \quad D^\pm \xi^0_R = (\partial + \mathcal{G}^+ \mathcal{L}^+ \xi^0_R), \quad D^\pm \xi^0_R = (\partial + \mathcal{G}^+ \mathcal{L}^+ \xi^0_R)
\]

\( (50) \)

where the trace is only over \(U_{EW}(2)\) indices. The matrices \(m^\pm_s\) and \(n^\pm_s\) are generation-mixing complex matrices, and \(\Phi^+\) is the Higgs field

\[
\Phi^+ := \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} .
\]

Clearly \(\kappa^\pm\) in the Yukawa term can be absorbed into the mixing matrices so they play no role in the Yukawa term which can then be expressed in the unitary gauge as

\[
\mathcal{L}_{\text{Yukawa}} = -\sum_{s,t} (m^+_s h^+_R \Phi^+_t H^+_L + m^+_s h^-_R \Phi^-_t H^-_L) \\
+ (n^+_s h^+_R \xi^0_R + n^-_s h^-_R \Phi^-_t + \mathcal{G}^+ \mathcal{L}^+ \phi^0_R) + h.c.
\]

\( (52) \)

Hence, unitary transformations in generation space yield real, diagonal mass matrices for \(m^\pm_s\) and \(n^\pm_s\), and the HC can be redefined in terms of the mass eigenfields according to

\[
H^+ := \left( \frac{V^+_{(m)}}{V^+_{(n)}} \right)^A e_A, \quad H^- := \left( \frac{V^-_{(m)}}{V^-_{(n)}} \right)^A e_A
\]

\( (53) \)
where $V^\pm_{(m)}$ and $V^\pm_{(n)}$ are unitary CKM-like generation-mixing matrices and we have abused notation by re-using $h^+, h^-, \xi^0, \lambda^0$ to denote mass eigenfields.

A few remarks are in order:

- In the lepton sector, putting $H^- \to i\tau_2 H^+$ transforms $\mathcal{L}_{\text{lepton}}$ in the form of [18] into the usual SM quark Lagrangian density. This happens because

$$i\bar{H}^- D_{\text{lepton}}^- H^- = (i\bar{H}^- D_{\text{lepton}}^- H^-)^* \to i\bar{H}^+ D_{\text{lepton}}^+ H^+ + \text{total derivative} \quad (54)$$

and we get $\mathcal{L}_{\text{lepton}} = (\kappa^+ + \kappa^-) i\bar{H}^+ D_{\text{lepton}}^+ H^+$. In contrast, in the iquark sector $D^\pm$ contains $SU_C(3)$ terms $\mathcal{G}_A$ that do not transform under $U_{EW}(2)$-conjugation so $(i\tau_2)D^-(i\tau_2)^{-1} \neq D^{\pm*}$ and then $(i\bar{H}^- D^- H^-)^* \neq i\bar{H}^+ D^+ H^+ + \text{total derivative}$. The two theories, with just $H^+$ v.s. $H^+$ and $H^-$, are not equivalent in the iquark sector.

- Re-scaling the iquark fields cannot cancel the relative scale difference between $H^+$ and $H^-$ since they are $U_{EW}(2)$-conjugate to each other (unless $\kappa^+ = \kappa^-)$. Consequently, these factors are not trivial and it was already shown that their ratio is not altered by renormalization. The effect of the constants $\kappa^+$ and $\kappa^-$ is to re-scale the charge $e$ in $\mathcal{L}_{\text{iquark}}$. Note that the $SU_C(3)$ coupling strengths are not altered as long as $\kappa^+ + \kappa^- = 1$.

That is, $\kappa^+ + \kappa^- = 1$ guarantees $SU_C(3)$ intrinsic and extrinsic charge equality under $U(2)$ gauge transformations.

- $\mathcal{L}_{\text{iquark}}$ is not invariant under distinct $U_{EW}(2)$ gauge transformations of $H^+$ and $H^-$ for two reasons. First, the Yukawa term forbids it. Second, the $U_{EW}(2)$-conjugate representation is reached through the adjoint action of $C = i\tau_2$ with det $C = 1$ and $C^2 = -1$, and this induces a “small” gauge transformation. Hence, a gauge transformation $R$ on span$_R\{e_a\}$ induces a conjugate transformation on span$_R\{e_a^\pm\}$ given by $R^c = (i\tau_2)R^c(i\tau_2)^{-1}$. But the gauge algebra is closed with respect to the adjoint action, so $R^c$ is also a small gauge transformation — different but not independent.

- There is an approximate discrete symmetry under $H^+ \leftrightarrow H^-$ as long as $\kappa^+ / \kappa^-$ is non-trivial and not too small. In consequence, there is an approximate global symmetry $U(2) \times Z_2 \simeq SU(2) \times U(1)$, and in the limit of vanishing masses this extends to $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$ (just like in the SM).

- The $\xi^0, \lambda^0$ fields completely decouple from the $U_{EW}(2)$ gauge bosons. However, they do couple to the $SU_C(3)$ gauge bosons. They also have an induced mass due to the Higgs interaction included in $\mathcal{L}_{\text{Yukawa}}$ (if $n_{st} \neq 0$).

---

17 Consider an infinitesimal gauge transformation $U(x) = 1 + i\alpha^a(x)\lambda_a + i\alpha^i(x)\lambda_i + O(\alpha^2)$. For the defining representation, make use of the identity det $A = 1/2((tr A)^2 - tr A^2)$ valid in two dimensions. Then, to first order in $\alpha$ calculate $(tr U(x))^2 = (tr 1 + i\alpha^a(x)tr \lambda_a)^2 \simeq (tr 1)^2 + 2i\alpha^a(x)tr \lambda_a$ as well as $(tr U(x))^2 = tr (1 + 2i\alpha^a(x)\lambda_a + 2i\alpha^i(x)\lambda_i) = tr (1 + 2i\alpha^a(x)\lambda_a).$ Hence, $det U(x) \simeq (tr 1)^2 - tr 1 > 0$ to first order in $\alpha$. Conclude that $det U(x) > 0$ implies a gauge transformation homotopic to the identity.
3.4 Comparing to the SM

Recall that $e^a \cdot e^a = e^a_x \cdot e^a_x$ together with (27) imply each eigenfield contribution to the $SU_C(3)$ current gets multiplied by the harmless constant $\kappa^+ + \kappa^- = 1$. In other words, the strong interaction cannot distinguish the difference between $\pm$ iquarks. In consequence, although there are $4 \times 3$ elementary iquarks, the strong interaction in hadrons “effectively sees” only $2 \times 3$ iquarks carrying the total requisite (same as the SM) strong charges. Hence, the QCD sector of the n-SM agrees precisely with the SM.

It suffices to compare the quark versus iquark content of the Yukawa and EW sectors since everything else is unaltered. The Yukawa term describes the same Higgs interaction as the SM; albeit with different elementary fields and double the mass parameters. To compare EW sectors, start with the n-SM EW currents.

3.4.1 Currents and Anomalies

Using (28), (36), (48), and (53) the $U_{EW}(2)$ currents for each iquark generation are

$$
\begin{align*}
J^0_Z(\mu) &= \frac{e}{2 \sin \theta_W \cos \theta_W} \left[ \kappa^+ \left(2 \sin^2 \theta_W - 1\right) \bar{h}^+_L \gamma_\mu h^+_L - \kappa^- \left(2 \sin^2 \theta_W - 1\right) \bar{h}^-_L \gamma_\mu h^-_L 
+ \kappa^+ 2 \sin^2 \theta_W \bar{h}^+_R \gamma_\mu h^+_R + \kappa^- \bar{\xi}^0_L \gamma_\mu \xi^0_L 
- \kappa^- 2 \sin^2 \theta_W \bar{h}^-_R \gamma_\mu h^-_R \right], \\
J^0_A(\mu) &= \kappa^+ \bar{e} h^+ \gamma_\mu h^+ - \kappa^- \bar{e} h^- \gamma_\mu h^-,
\end{align*}
$$

$$
\begin{align*}
J^-_\mu &= \left[ \kappa^+ \bar{h}^+_L \gamma_\mu V^+ \xi^0_L + \kappa^- \bar{\chi}^0_L \gamma_\mu h^-_L \right], \\
J^+_\mu &= \left[ \kappa^+ \bar{\xi}^0_L V^+ \gamma_\mu h^+_L + \kappa^- \bar{h}^-_L \gamma_\mu \chi^0_L \right],
\end{align*}
$$

where we used $W^\pm = W^\pm$ for the $\kappa^-$ terms in the two charged currents, $V^+ := V^\dagger_{(m)} V^+_{(n)}$, $V^- := V^-_{(m)} V^-_{(n)}$, and summation over $SU_C(3)$ indices is implicit.

As is well known, for a consistent quantum version of this model to exist, the anomalies associated with these currents must cancel the lepton anomalies. Because the iquarks furnish the same EW representation as the leptons and because there are three degrees of freedom of each, one would not expect the anomalies in this model to cancel trivially.

In contrast to the SM, the Yukawa and EW terms in the n-SM include twice as many mass parameters ($m^+_L$ and $n^+_L$) and CKM parameters ($V^+$ and $V^-$) due to the elementary content being $(\mathbf{3}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}^c)$. Since individual quark/iquark currents are not observed, the extra parameters don’t seem to be inferable without additional assumptions. One possibility rests on the model proposed in §4 which implies a relation between iquarks and leptons that would (in principle) relate the iquark/lepton masses and generation-mixing parameters; thus reducing the free mass and mixing parameters to the same as the SM.
To check this, it must be kept in mind that the $U_{EM}(1)$ quantities which enter into the anomaly calculation are not the intrinsic electric charges of the matter fields, per se, but the coupling strengths in the photon-matter field current $j_{\mu}^{(A)}$. The $U_{EM}(1)$, $U_{EW}(2)$, and $SU_C(3)$ contributions of the left-handed matter fields are given in Table 1.

| fermions | $(h^+, \xi^0)_L$ | $(\chi^0, h^-)_L$ | $h^+_R$ | $h^-_R$ | $\xi^0_R$ | $\chi^0_R$ | $(\nu^0, l^-)_L$ | $l^-_R$ | $\nu^0_R$ (?) |
|----------|-----------------|-----------------|--------|--------|--------|--------|-----------------|--------|----------------|
| $U(1)$   | $(\kappa^+, 0)$ | $(0, -\kappa^-)$ | $-\kappa^+$ | $\kappa^-$ | 0 | 0 | $(0, -1)$ | 1 | 0 |
| $U(2)$   | 2 | 2c | $\bar{1}$ | $\bar{1}c$ | $\bar{1}$ | $\bar{1}c$ | 2 | 1 | 1 |
| $SU(3)$  | 3 | 3 | $\frac{3}{3}$ | $\frac{3}{3}$ | $\frac{3}{3}$ | 1 | 1 | 1 |

Table 1: Anomaly contributions for left-handed fermionic matter fields.

There are only four cases to check including the gravitational anomaly [16]: $[U(2)]^2 U(1)$, $[SU(3)]^2 U(1)$, $[U(1)]^3$, and $[G]^2 U(1)$. (The $[SU(3)]^3$ case vanishes trivially since the representation is real.) In that order, the relevant terms are

\[
\sum_{\text{doublets}} p = 3(\kappa^+) + 3(-\kappa^-) + (-1) = 0 , \quad (56a)
\]

\[
\sum_{\text{triplets}} p = (\kappa^+) + (-\kappa^-) + (-\kappa^+) + (\kappa^-) + 0 + 0 = 0 , \quad (56b)
\]

\[
\sum_{\text{all}} p^3 = 3(\kappa^+)^3 + 3(-\kappa^-)^3 + 3(-\kappa^+)^3 + 3(\kappa^-)^3 + (-1)^3 + (1)^3 = 0 , \quad (56c)
\]

\[
\sum_{\text{all}} p = 3(\kappa^+) + 3(-\kappa^-) + 3(-\kappa^+) + 3(\kappa^-) + (-1) + (1) = 0 , \quad (56d)
\]

where $ep$ denotes the $U_{EM}(1)$ coupling parameter for the quark currents. With the exception of (56a), the anomaly conditions are null rather trivially. From (56a) and the condition $\kappa^+ + \kappa^- = 1$, there will be no anomaly associated with the gauge symmetries for the choice

\[
\kappa^+ = \frac{2}{3} , \quad \kappa^- = \frac{1}{3} . \quad (57)
\]

Now turn to the chiral anomaly and the decay rate of $\pi^0 \to 2\gamma$ associated with the global chiral transformation

\[
\delta_\lambda H^A_+ = \lambda \Xi H^A_+ \quad \delta_\lambda H^A_- = \lambda \Xi' H^A_-
\]

where $\Xi = \begin{pmatrix} i\gamma_5 & 0 \\ 0 & -i\gamma_5 \end{pmatrix}$ and $\Xi' = (i\tau_2)\Xi^*(i\tau_2)^\dagger = \Xi$. The anomaly is proportional to

\[
\text{tr}_{U(2)} \left[ (\kappa^+ Q^+)^2 \Xi + (\kappa^- Q^-)^2 \Xi' \right] \propto \text{tr}_{U(2)} \left[ (\kappa^+ Q^+ + \kappa^- Q^-)^2 \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right]
\]

which, for three colors and using (57), yields the SM result

\[
3 \left( \frac{2}{3} \right)^2 - 3 \left( \frac{1}{3} \right)^2 = 1 .
\]
3.4.2 Quarks v.s. iquark

To make contact with SM phenomenology, we need to associate the conventional generation-mixed quark mass eigenstates with \( h^\pm, \xi^0, \chi^0 \). Inspection of (55) suggests that the familiar fractionally charged quark mass eigenstates should be associated with a pair of HC. Thus, make the following correspondence:

\[
\begin{align*}
\left( \begin{array}{c}
\frac{1}{3}u^+ \\
-\frac{1}{3}d^-
\end{array} \right) & \leftrightarrow \left( \begin{array}{c}
h^+ \\
\xi^0
\end{array} \right) + \left( \begin{array}{c}
\chi^0 \\
h^-
\end{array} \right)
\end{align*}
\]

(61)

where \( u \) and \( d \) represent up and down quark fields respectively. More accurately, the neutral quark bilinears are identified with a pair of neutral HC bilinears

\[
\begin{align*}
\overline{u^+} \gamma_\mu \gamma_\mu u^+ & \leftrightarrow \left( \overline{h^+} \gamma_\mu h^+ ; \overline{\xi^0} \gamma_\mu \chi^0 \right) \\
\overline{d^-} \gamma_\mu d^- & \leftrightarrow \left( \overline{\xi^0} \gamma_\mu \xi^0 ; \overline{h^-} \gamma_\mu h^- \right)
\end{align*}
\]

(62)

and the charged quark bilinears are identified with charged HC pairs

\[
\begin{align*}
\overline{u^+} \gamma_\mu V d^- & \leftrightarrow \left( \overline{h^+} \gamma_\mu V^+ \xi^0 ; \overline{\chi^0} V^{-1} \gamma_\mu h^- \right) \\
\overline{d^-} V^\dagger \gamma_\mu u^+ & \leftrightarrow \left( \overline{\xi^0} V^\dagger \gamma_\mu h^+ ; \overline{h^-} \gamma_\mu V^- \chi^0 \right)
\end{align*}
\]

(63)

where \( V \) is the CKM matrix.

So, the neutral and charged EW currents can be compared in the two different pictures with the help of (55):

\[
\begin{align*}
\frac{2}{3}\overline{u^+} \gamma_\mu \gamma_\mu u^+ & \sim \frac{2}{3}\overline{h^+} \gamma_\mu h^+ , \\
-\frac{1}{3}\overline{d^-} \gamma_\mu d^- & \sim -\frac{1}{3}\overline{h^-} \gamma_\mu h^- ;
\end{align*}
\]

(64)

\[
\begin{align*}
\left( \frac{4}{3} \sin^2 \theta - 1 \right) \overline{u^+} \gamma_\mu \gamma_\mu u^+ & \sim \left( \frac{4}{3} \sin^2 \theta - \frac{2}{3} \right) \overline{h^+} \gamma_\mu h^+ - \frac{1}{3} \overline{\chi^0} \gamma_\mu \chi^0 , \\
\left( \frac{4}{3} \sin^2 \theta \right) \overline{u^+} \gamma_\mu \gamma_\mu u^+ & \sim \left( \frac{4}{3} \sin^2 \theta \right) \overline{h^+} \gamma_\mu h^+ ;
\end{align*}
\]

(65)

and

\[
\overline{u^+} \gamma_\mu V d^- \sim \frac{2}{3} \overline{h^+} \gamma_\mu V^+ \xi^0 + \frac{1}{3} \overline{\chi^0} V^{-1} \gamma_\mu h^- .
\]

(66)

There are analogous relations for the currents \( \overline{d^-} \gamma_\mu d^- \) and \( \overline{d^-} V^\dagger \gamma_\mu u^+ \).

To the extent that (62) and (63) are justified, the weak currents in (55) agree precisely with the SM currents. Graphically, the correspondence associates one-particle quark currents and their vertex factors with an equivalent two-particle iquark current whose vertex factor is the sum of the individual vertex factors of the constituent one-particle currents. Physically, the correspondence constitutes an average description in the sense that individual quarks/iquarks cannot be resolved.
Observe, however, that the iquark parton distribution and structure functions of charged lepton–hadron DIS interactions will differ from the standard parton model since a (presumably small) portion of the spin-1/2 mass-energy of the hadron comprised of $\xi^0$, $\chi^0$ does not directly interact with the lepton via $U_{EM}(1)$: To the extent that the strong coupling between HC decreases, the spin of the neutral iquarks will not participate in charged lepton DIS. This may have implications regarding the proton spin puzzle (see e.g. [38, §5]) and EMC effect.

With the n-SM EW currents and quark/iquark bilinear correspondences in hand, we can compare the SM and n-SM phenomenology, but first we need to exhibit the iquark content of hadrons.

### 3.4.3 Hadrons

According to [55], iquark EW currents include iquarks belonging to both $H^+$ and $H^-$. This gives a hint about how to assign iquark content to hadrons. Based on the relationship between quark v.s. iquark EW currents and the circumstance that charged weak interactions interchange the up and down components of $H^\pm$, it is convenient to define valence HC composed of up-type and down-type iquark pairs in order to characterize hadrons

$$
H^\uparrow_s := h^\uparrow_s + \chi^0_s = P^\uparrow(H^\uparrow_s + H^\downarrow_s) \\
H^\downarrow_s := \xi^0_s + h^\downarrow_s = P^\downarrow(H^\uparrow_s + H^\downarrow_s).
$$

(67)

where $P^\uparrow$ projects onto the up/down $U_{EW}(2)$ component.

The n-SM currents seem to be telling us that we should think of $H^\uparrow_s$ like up/down quarks. But will this lead to acceptable spin content when we try to form hadrons? Let’s enumerate the spin possibilities: denote the spin content by $|J\rangle$. For meson composites $H^\uparrow_s H^\uparrow_t := P^\uparrow H^\uparrow_s P^\uparrow H^\uparrow_t + P^\downarrow H^\downarrow_s P^\downarrow H^\downarrow_t$, the possible spin combinations are

$$
(|\frac{1}{2}, \pm \frac{1}{2}); |\frac{1}{2}, \pm \frac{1}{2}) \otimes (|\frac{1}{2}, \pm \frac{1}{2}); |\frac{1}{2}, \pm \frac{1}{2}) = \text{(singlet; singlet)} + \text{(triplet; singlet)} + \text{(singlet; triplet)} + \text{(triplet; triplet)}
$$

(68)

where singlet $\equiv |0, 0\rangle$ and triplet $\equiv |1, 1\rangle, |1, 0\rangle, |1, -1\rangle$. The total average spin associated with each of these four sets of spin states is $(0 \pm 0)/2, (1 \times 3 \pm 0)/(2 \times 3), (0 \pm 1 \times 3)/(2 \times 3)$, and $(1 \times 3 \pm 1 \times 3)/(2 \times 3)$, but the middle two are excluded by spin statistics. So we can expect (for zero orbital angular momentum) pseudoscalar and vector $H^\uparrow_s H^\uparrow_t$. Similarly, for baryons $H^\uparrow_s H^\uparrow_t H^\uparrow_u := P^\uparrow H^\uparrow_s P^\uparrow H^\uparrow_t P^\uparrow H^\uparrow_u + P^\downarrow H^\downarrow_s P^\downarrow H^\downarrow_t P^\downarrow H^\downarrow_u$ (antisymmetry in color indices implied), the average spin combinations turn out to be $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. In this case, spin statistics rule out the integers and we can expect (for zero orbital angular momentum) $J = \frac{1}{2}$ and $J = \frac{3}{2}$.

**Remark 3.4** In both composites the only combinations that lead to acceptable spin statistics derive from iquark pairs in the same spin state. That is, in hadrons, on average the two components of valence $H^\uparrow_s$ have aligned spins. As far as the low-energy effective theory based on $SU_C(3) \times U_{EM}(1)$ is concerned then, there is really no material difference between valence quarks and valence $H^\uparrow_s$ — they posses equivalent strong charges, extrinsic electric charges,
and spin. On the other hand, the EW interaction is based on $H^{\pm}$. Heuristically speaking, we might say that HC (in hadrons at least) have a split personality because $H^{\pm}$ want to couple to massless bosons while $H^{\mp}$ want to couple to massive bosons.

The mesons and baryons are composites of $H_1^1H_1^1$ and $H_2^2H_2^2$ respectively where $s,t,u$ label iquark generation and the up/down arrow denotes either/or. Table 2 below contains proposed assignments for pseudoscalar mesons comprised of the first two generations of composites $H_1^1H_1^1$, $H_1^1H_2^1$, and $H_2^2H_2^2$ (spin content is ignored).

| HC composite | Iquark composite | Meson | $J^{PC}$ |
|--------------|------------------|-------|---------|
| $H_1^1H_1^1$ | $h_1^1 \xi_1^0 + \chi_1^0 h_1^1$ | $\pi^+$ | $\pi^-$ |
| $H_1^1H_1^1$ | $h_1^1 \chi_1^0 + \xi_1^0 h_1^1$ | | 0^- |
| $H_1^1H_1^1$ + $H_1^1H_1^1$ | $h_1^1 h_1^1 + \chi_1^0 \chi_1^0 + h_1^1 h_1^1 + \xi_1^0 \xi_1^0$ | $\pi^0$ | 0^0 |
| $H_1^1H_2^1$ | $h_1^1 \xi_1^0 + \chi_1^0 h_2^1$ | $K^+$ | $K^-$ |
| $H_1^1H_2^1$ | $h_1^1 \chi_1^0 + \xi_1^0 h_2^1$ | | 0^- |
| $H_1^1H_2^1$ + $H_1^1H_2^1$ | $(h_1^1 h_2^1 + \xi_1^0 \xi_2^0) \pm (h_2^1 h_1^1 + \xi_2^0 \xi_1^0)$ | $K_L^0, K_S^0$ | 0^- |
| $H_1^1H_2^1$ + $H_2^2H_2^2$ | $h_1^1 h_2^1 + \chi_1^0 \chi_2^0 + h_2^1 h_2^1 + \xi_2^0 \xi_2^0$ | $\eta$ | 0^- |
| $H_1^1H_2^1$ + $H_2^2H_2^2$ | $h_1^1 h_2^1 + \xi_1^0 \xi_2^0 + h_2^1 h_2^1 + \xi_1^0 \xi_2^0$ | $\eta'$ | 0^- |
| $H_1^1H_2^1$ | $h_1^1 \xi_1^0 + \chi_1^0 h_2^1$ | $D^+$ | $D^-$ |
| $H_1^1H_2^1$ | $h_1^1 \chi_1^0 + \xi_1^0 h_2^1$ | | 0^- |
| $H_1^1H_2^1$ + $H_2^2H_2^2$ | $(h_1^1 h_2^1 + \chi_1^0 \chi_2^0) \pm (h_2^1 h_1^1 + \chi_2^0 \chi_1^0)$ | $D_{+}^0, D_{-}^0$ | 0^- |
| $H_1^1H_2^1$ | $h_1^1 \xi_2^0 + \chi_1^0 h_2^1$ | $D_{+}^0$ | $D_{-}^0$ |
| $H_1^1H_2^1$ | $h_1^1 \chi_2^0 + \xi_1^0 h_2^1$ | | 0^- |
| $H_2^2H_2^1$ | $h_2^2 \xi_1^0 + \chi_2^0 h_2^1$ | | 0^- |
| $H_2^2H_2^1$ | $h_2^2 \chi_1^0 + \xi_2^0 h_2^1$ | | 0^- |
| $H_2^2H_2^1$ + $H_2^2H_2^1$ | $h_2^2 h_2^2 + \chi_2^0 \chi_2^0 + h_2^2 h_2^2 + \xi_2^0 \xi_2^0$ | $\eta_c$ | 0^- |

Table 2: Proposed HC assignments for selected mesons. Field superscripts denote intrinsic electric charge, subscripts denote iquark generation, and the overbar denotes anti-particle. $J$ is total momentum, $P = (-1)^{L-1}$ is parity, and $C$ indicates $SU_C(3) \times U_{EM}(1)$-conjugation.

As evidenced from the table, it is straightforward to establish the iquark content of the charged mesons, and it is natural to guess that neutral mesons are linear combinations of
both $H^+$ and $H^\dagger$. With the exception of $\eta$, $\eta'$, and $\eta^{19}$, this recipe yields iquark flavor content that matches the quark flavor assignments of the standard quark model if we identify the up-type quark/antiquark composites with $H_s^+H_s^\dagger$, down-type composites with $H_s^+H_s^\dagger$, and mixed-type with $H_s^+H_s^\dagger$. Note that the two mixed-generation neutral combinations give rise to both $CP$-odd and $CP$-even states. It is interesting that $D^0_+ = D^0 + \bar{D}^0$ and $D^0_+ = D^0 - \bar{D}^0$ seem to exhibit very little mass and lifetime asymmetry implying no $CP$ violation$^{20}$ despite their $CP$ asymmetry (unlike $K^0_L$ and $K^0_S$)$^{37}$.

Moving on, Table 3 contains proposed assignments of selected spin 1/2 and 3/2 baryons to HC composites $H_1^+H_1^+H_1^+$, $H_1^+H_1^+H_2^+$, $H_1^+H_2^+H_2^+$, and $H_2^+H_2^+H_2^+$.

| HC composite | Iquark composite | Baryon | $J^P$ |
|--------------|-----------------|--------|------|
| $H_1^+H_1^+H_1^+$ | $h_1^+ h_1^+ \xi_1^0 + \chi_1^0 \chi_1^0 h_1^-$ | $p$ | 1/2$^+$ |
| $H_1^+H_1^+H_1^+$ | $h_1^+ \xi_1^0 \xi_1^0 + \chi_1^0 h_1^- h_1^-$ | $n$ | 1/2$^+$ |
| $H_1^+H_1^+H_1^+$ | $\xi_1^0 \xi_1^0 \xi_1^0 + h_1^+ h_1^- h_1^-$ | $\Delta^-$ | 3/2$^+$ |
| $H_1^+H_1^+H_1^+$ | $h_1^+ h_1^+ h_1^- + \chi_1^0 \chi_1^0 \chi_1^0$ | $\Delta^{++}$ | 3/2$^+$ |
| $H_1^+H_1^+H_2^+$ | $h_1^+ h_1^+ \xi_2^0 + \chi_1^0 \chi_1^0 h_2^-$ | $\Sigma^+$ | 1/2$^+$ |
| $H_1^+H_2^+H_2^+$ | $h_1^+ \xi_1^0 \xi_2^0 + \chi_1^0 h_1^- h_2^-$ | $\Sigma^0$, $\Lambda^0$ | 1/2$^+$ |
| $H_1^+H_2^+H_2^+$ | $h_1^+ \xi_1^0 \xi_2^0 + \chi_1^0 h_1^- h_2^-$ | $\Sigma^+_c$, $\Lambda^+_c$ | 1/2$^+$ |
| $H_1^+H_2^+H_2^+$ | $\xi_1^0 \xi_1^0 \xi_2^0 + h_1^+ h_1^- h_2^-$ | $\Sigma^0$ | 1/2$^+$ |
| $H_1^+H_1^+H_2^+$ | $h_1^+ h_1^+ \xi_2^0 + \chi_1^0 h_1^- h_2^-$ | $\Xi^0$ | 1/2$^+$ |
| $H_1^+H_1^+H_2^+$ | $h_1^+ h_1^+ \xi_2^0 + \chi_1^0 h_1^- h_2^-$ | $\Xi^0_c$, $\Xi^0_c$ | 1/2$^+$ |
| $H_1^+H_2^+H_2^+$ | $h_1^+ \xi_1^0 \xi_2^0 + \chi_1^0 h_2^- h_1^{-0}$ | $\Xi^0_c$, $\Xi^0_c$ | 1/2$^+$ |
| $H_1^+H_2^+H_2^+$ | $h_1^+ \xi_1^0 \xi_2^0 + \chi_1^0 h_2^- h_1^{-0}$ | $\Xi^0_c$, $\Xi^0_c$ | 1/2$^+$ |

Table 3: HC and iquark assignments for selected spin 1/2 and 3/2 baryons.

$^{19}$Simple combinatorics suggest the meson assignments in the table, but the HC content can clearly be adjusted to match the standard quark model.

$^{20}$Or perhaps $CP$ is violated but the effect is suppressed by the expected relative large mass of $h_2^\pm$. Note Added: Direct $CP$ violation for the $D^0_+, D^0_-$ pair has now been observed.$^{59}$. 

30
Whether one adheres to color or strong charge as $SU_C(3)$ quantum numbers, it is easy to check that the iquark composites contribute the requisite (vanishing) strong charges and extrinsic electric charge of the corresponding baryon.

Notice there are four exceptional entries in this table of the form $(H^u_s H^d_s)H^t_t$ with $s \neq t$. Each is associated with two particles of unequal mass. In the standard quark model, this is attributed to alignment or anti-alignment of isospin. It is noteworthy that the larger mass state primarily decays into its smaller mass partner plus either $\gamma$ (for $\Sigma^0$, $\Xi^+_c$, and $\Xi^0_c$) or $\pi^0$ (for $\Sigma^+$). Can the iquark picture explain this?

Recall that $SU_C(3) \times U_{EM}(1)$ sees $H^t_t$ like an up/down quark with definite strong charges, extrinsic electric charge, and spin. Consider the case $J = \frac{1}{2}$, then $(H^u_s H^d_s)H^t_t$ has either $H^u_s H^d_s$ or $H^u_s H^t_t$ with aligned spins. We propose to attribute the mass splitting (between the two baryons in each of the four exceptional entries) to two effects: First, when $J = \frac{1}{2}$ there must be spin/anti-spin coupling between either the same or different generations. Second, the pair $H^u_s H^t_t$ contains the iquark content of $H^+$ and $H^-$ in the same generation. But (with spins aligned) this is precisely what is required to couple to $A$ and $Z^0$. On the other hand, while $H^u_s H^t_t$ also contains the iquark content of $H^+$ and $H^-$, the mixed-generation iquarks can only couple to $W^\pm$ which is a much slower process. The point is, dynamics associated with spin coupling together with two different (same-generation v.s mixed-generation) EW current couplings render two possible states. Since the neutral current interactions have shorter mean lifetimes relative to charged current interactions, we expect a mass splitting between the two states and different decay rates.

**Remark 3.5** The combinatorics used to tabulate the hadrons can instead be performed in the context of approximate $SU(4)$ flavor symmetry in the same way that quarks are combined in the standard quark model of hadrons. This would require the assignment of associated quantum numbers to $H^t_t$ and revised HC content that would mirror the quark model classification scheme given the identifications $H^1_u \sim u$, $H^1_d \sim d$, $H^1_c \sim c$, and $H^2_s \sim s$. There is certainly precedence in favor of such a scheme. However, (as we discuss in the next subsection) the two approaches make different predictions about the nature of $\Omega^0_{cb} \equiv csb \equiv H^1_u H^1_d H^2_s$: our classification predicts a two-mass state while the standard quark model predicts a single state due to zero isospin. There is not yet definitive particle data so the jury is still out on which classification scheme is correct on this account.

### 3.4.4 Weak phenomenology

It is instructive to look at some specific iquark interactions in the particle picture.

**Pseudoscalar meson decays**

- $\pi^0$ decay: Iquark content is $(h^+_1, \chi^0_1, \overline{h}^+_1, \chi^0_1, h^-_1, h^-_1, \xi^0_1, \overline{h}^-_1, \overline{\xi}^0_1)$. Kinematically, $\pi^0$ must decay via $U_{EW}(2)$. As far as the EW interaction is concerned, $\pi^0$ contains neutral currents that couple primarily to $A$ through $h^\pm$. Coupling to $Z^0$ can also occur, but it requires sufficiently localized charged and neutral iquarks and is far weaker. Hence,

21 Notice that $Z^0$ is very happy to decay into $\pi^0$. 

31
the primary decay mode in the iquark picture includes annihilation of $\chi^0$, $\overline{\chi}^0$ and $\xi^0$, $\overline{\xi}^0$ into gluons via the strong interaction accompanied by $h^\pm$, $\overline{h}^\pm$ annihilation into $\gamma\gamma$ via the EW interaction. The $\chi^0$, $\overline{\chi}^0$ and $\xi^0$, $\overline{\xi}^0$ annihilation obviously proceeds on a much faster time scale.

- $\eta$, $\eta'$, $\eta_c$ decay: Iquark content of $\eta$ for example is ($h_1^+, \chi_0^0$, $h_1^0$, $\overline{h}_1^+, \overline{\chi}_1^0$, $h_2^0$, $\xi^0$, $h_2^-, \overline{\xi}^0$). There are two modes of decay available: The first is the same as that for $\pi^0$ with the valence iquarks contributing to neutral currents coupling primarily to $H$ by $\chi$. The two decay modes are not mutually exclusive, but decays into $n\pi\pi$ primary EM decay mode and process are the same as for $H$. The two decay modes are not mutually exclusive, but decays into $n\pi\gamma\gamma$ are prohibited by $C$ invariance of EM. However, there are kinematically allowed secondary decay modes of $n\pi^0\gamma\gamma$ and $\pi^-\pi^-\gamma$ as well as intermediate decays into heavier mesons for $\eta'$ and $\eta_c$.

- $m^0_{st} := K^0_L$, $K^0_S$, $D^0_+$, $D^0_-$ decay: Iquark content of $K^0$ is ($h_1^+, \chi_0^0$, $h_2^0$, $\xi_1^0$, $h_1^-$, $\overline{h}_2^0$, $\xi_1^0$, $\overline{h}_1^-$). Likewise, for $D^0$ it is ($h_1^+, \chi_0^0$, $h_2^0$, $\xi_1^0$, $h_1^-$, $\overline{h}_2^0$, $\xi_1^0$, $\overline{h}_1^-$). Evidently $m^0_{st}$ has just the right valence iquark content to exhibit generation-changing decay primarily through the process $H_2^{\downarrow\uparrow} \rightarrow H_1^{\downarrow\uparrow} + \overline{H}_1^{\uparrow\downarrow} H_1^{\uparrow\downarrow}$ via $W^\pm$, and the resulting set of valence HC can combine into various collections of mesons and/or leptons. Note that $m^0_{st}$, being a mixed-generation meson by construction, represents both $CP$-even and $CP$-odd states. $CP$-even states can decay into an even number of mesons while the $CP$-odd states decay into an odd number of mesons and/or leptons (to the extent that $CP$ is conserved).

- $m_{st}^\pm := \pi^\pm$, $K^\pm$, $D^\pm$, $D_S^\pm$ decay: Iquark content of $\pi^+$ for example is ($h_1^+, \chi_1^0$, $h_1^-$, $\xi_1^0$). Note that $m_{st}^\pm$ couples to $A$ with total extrinsic electric charge $\pm 2/3 \equiv (-1/3)$. Since $m_{st}^\pm$ iquark weak currents are comprised of both $H^\uparrow$ and $H^\downarrow$ constituents, there are potentially two modes available: Scattering of $H^\uparrow$, $H^\downarrow$ into $W^\pm$ and decay of $H_2^{\downarrow\uparrow}$ if present. So, $m_{st}^\pm$ will decay via $W^\pm$ into mesons comprised of both up-type and down-type iquarks and/or leptons. So, for example, the possible leptonic decay modes are $m_{st}^+ \rightarrow$ mesons + $l^+_u\nu_e$ and $m_{st}^- \rightarrow$ mesons + $l^-_u\overline{\nu}_e$ or $m_{st}^\pm \rightarrow$ $n\pi^0$ + $m\pi^\pm$.

**Baryon EW decays**

- $n$ decay: The only primary decay mode available is $H_1^\downarrow \rightarrow H_1^\uparrow +$ leptons via $W^-$. The same decay mode is not kinematically available to $p \equiv H_1^\uparrow H_1^\uparrow H_1^\uparrow$ because the resulting particle $H_1^\downarrow H_1^\uparrow H_1^\uparrow$ requires aligned spins forming a higher energy $J = \frac{3}{2}$ state.

- $b(s,t) := (H_s^\uparrow H_t^\downarrow) H_1^\uparrow$ with $s \neq t$ decay. We have already considered these exceptional cases and concluded their peculiar behavior stems from a dynamical interplay between spin coupling and EW current coupling of mixed-generations. If we include the third generation of iquarks, we expect $udb \equiv H_1^\uparrow H_1^\uparrow H_1^\uparrow$ to be manifested as two particles $\Sigma^0_6$ and $\Lambda^0_5$ (as does the standard quark model). However, $csb \equiv H_2^\uparrow H_2^\uparrow H_3^\uparrow$ is also predicted.
to manifest as two particles $\Omega^{0}_{cb}$ and $\Omega^{0}_{db}$ in the iquark picture even though both states have zero isospin (in which case the standard quark model predicts a single state). The particle data is still inconclusive: the first case seems to hold but neither particle in the second case has been observed.

- $b_{(st)q} := (H^+_s H^+_t) H^+_s$ with $s \neq t$ decay. Here we expect the iquark content from $H^+_s H^+_t$ to favor weak decays unless $H^+_s$ makes strong decay modes available kinematically. For example $\Sigma^+$ and $\Xi^0$ have $H^+_1$ and $H^+_2$ respectively, so our heuristic suggests the primary decay mode is weak. On the other hand, $\Sigma^+_c$ and $\Xi^+_c$ have $H^+_1$ and $H^+_2$ so we expect strong decay. These expectations are borne out for the first three, but the $\Xi^+_c$ decay mode has not been established.

### 3.4.5 Weak S-matrix

We have already argued that the QCD sector of the SM and the n-SM are identical. It remains to establish equivalence\footnote{To be clear: we mean equivalence with the SM as a Yang-Mills QFT not the standard quark model: We have already seen that the iquark picture in the n-SM does not completely agree with the approximate-flavor-symmetry classification scheme of the standard quark model.} between the n-SM and the SM S-matrix amplitudes for weak interactions. So it’s time to display explicit details regarding the relevant QFT aspects\footnote{We will follow the notation and conventions of Weinberg\cite{Weinberg} in this subsection.}

To remind, our focus is on the iquark Lagrangian

\[
\mathcal{L}_{\text{iquark}} = i \sum_s \left( H^+_L \hat{\mathcal{D}}^+ H^+_L + H^+_R \hat{\mathcal{D}}^+ H^+_R + \xi^0_{R,s} \hat{\mathcal{D}} \xi^0_{R,s} \right) + \left( H^+_L \hat{\mathcal{D}}^- H^+_L + H^+_R \hat{\mathcal{D}}^- H^+_R + \chi^0_{R,s} \hat{\mathcal{D}} \chi^0_{R,s} \right)
\]

(69)

where we have absorbed the scaling constants $\kappa^\pm$ into the covariant derivatives $\hat{\mathcal{D}}^\pm := \kappa^\pm \mathcal{D}^\pm$ which is conceptually appropriate. The $H^+$ field iquark components are

\[
h^+_i(x) := (2\pi)^{-3/2} \sum_{\sigma} \int d^3 p \left[ u_i(p, \sigma, n_{h^+}) a(p, \sigma, n_{h^+}) e^{ip \cdot x} + v_i(p, \sigma, n_{h^+}) a^\dagger(p, \sigma, n_{h^+}^c) e^{-ip \cdot x} \right]
\]

(70)

and

\[
\xi^0_i(x) := (2\pi)^{-3/2} \sum_{\sigma} \int d^3 p \left[ s_i(p, \sigma, n_{\xi^0}) a(p, \sigma, n_{\xi^0}) e^{ip \cdot x} + t_i(p, \sigma, n_{\xi^0}) a^\dagger(p, \sigma, n_{\xi^0}^c) e^{-ip \cdot x} \right]
\]

(71)

where $n_{(\cdot)}$ denotes the $SU_C(3) \times U_{EM}(1)$ quantum numbers and mass of the indicated iquark.

There are analogous fields for $H^-$, and they all combine to give the up/down iquark fields

\[
H^+_i(x) := (2\pi)^{-3/2} \sum_{\sigma} \int d^3 p \left\{ \left[ u_i(p, \sigma, n_{h^+}) a(p, \sigma, n_{h^+}) + s_i(p, \sigma, n_{\xi^0}) a(p, \sigma, n_{\xi^0}) \right] e^{ip \cdot x} + \left[ v_i(p, \sigma, n_{h^+}) a^\dagger(p, \sigma, n_{h^+}^c) + t_i(p, \sigma, n_{\xi^0}) a^\dagger(p, \sigma, n_{\xi^0}^c) \right] e^{-ip \cdot x} \right\}
\]

(72)
and

\[ H_{\uparrow}^{\pm}(x) := (2\pi)^{-3/2} \sum_{\sigma} \int d^3p \ \{ [u_l(p, \sigma, n_{h^-})a(p, \sigma, n_{h^-}) + s_l(p, \sigma, n_{\xi^0})a(p, \sigma, n_{\xi^0})]e^{ip.x} \\
+ [v_l(p, \sigma, n_{h^-})a^\dagger(p, \sigma, n_{h^-}) + t_l(p, \sigma, n_{\xi^0})a^\dagger(p, \sigma, n_{\xi^0})]e^{-ip.x} \} . \]

(73)

With the weak interaction turned off, there is no distinction between \( H_{L,s}^{\pm} \) and \( H_{L,s}^{-} \) so \( L_{\text{iquark}} \) reduces to the fermion kinetic term \( L_{\text{quark}} \) of QCD in this limit. Consequently iquarks and quarks share identical free-field propagators in their respective perturbative QFTs. Now turn on the weak interaction and compare the EW interaction Feynman rules for \( H_{\uparrow}^{\pm} \) v.s. up/down quarks \( U, D \):

| n-SM iquark/neutral-boson | SM quark/neutral-boson |
|--------------------------|------------------------|
| \( (h^+; \chi^0) \)     | \( U \)               |
| \( A \sim i(\frac{2}{3}e; -\frac{1}{3}0)\gamma^\mu \) | \( A \sim \frac{2}{3}e\gamma^\mu \) |
| \( (h^-; \xi^0) \)     | \( D \)               |
| \( A \sim i(-\frac{1}{3}e; +\frac{2}{3}0)\gamma^\mu \) | \( A \sim -\frac{1}{3}e\gamma^\mu \) |
| \( (h^+; \chi^0) \)     | \( U \)               |
| \( Z^0 \sim i(\frac{2}{3}F(\theta)g_{Z^0}; -\frac{1}{3}g_{Z^0})\gamma^\mu \) | \( Z^0 \sim i(\frac{2}{3}F(\theta) - \frac{1}{3})g_{Z^0}\gamma^\mu \) |
| \( (h^-; \xi^0) \)     | \( D \)               |
| \( Z^0 \sim i(-\frac{1}{3}F(\theta)g_{Z^0}; +\frac{2}{3}g_{Z^0})\gamma^\mu \) | \( Z^0 \sim i(\frac{2}{3} - \frac{1}{3}F(\theta))g_{Z^0}\gamma^\mu \) |

where \( F(\theta_W) := 2\sin^2(\theta_W) - 1 \), and the coupling constant is \( g_{Z^0} := e/(2\cos\theta_W \sin\theta_W) \). The left column actually represents two diagrams; one for each iquark in the ordered pair \( (\cdot; \cdot) \).
Similarly, the charged couplings are

\[
\begin{align*}
\text{n-SM iquark/charged-boson} & \quad \text{SM quark/charged-boson} \\
(h^+; \chi^0) & \quad \overline{U} \\
(\xi^0; h^-) & \quad W^- \sim i(\frac{2}{3}V^+ g_W + \frac{1}{3}V^- g_W)\gamma^\mu \\
(\xi^0; h^-) & \quad D \\
(h^+; \chi^0) & \quad W^+ \sim i(\frac{2}{3}V^+ g_W + \frac{1}{3}V^- g_W)\gamma^\mu \\
& \quad \overline{D} \\
& \quad U
\end{align*}
\]

where \( g_W = e/\sqrt{2} \sin \theta_W \).

As previously remarked, each up/down quark coupling is identified with the sum of two corresponding up/down iquark couplings. Given this observation, our aim is to show that, up to a re-definition of elementary-particle content, the n-SM and SM predict identical S-matrix amplitudes for EW interactions if we impose \( \frac{2}{3}V^+ + \frac{1}{3}V^- \equiv V \) or \( V^+ = V^- \equiv V \).

Expanded in terms of individual iquarks, the n-SM interaction Hamiltonian density is of the form

\[
\mathcal{H}_{n-\text{SM}}(x) = \sum_{klm} \phi_{k0}(x) g_{klm} \psi_\up^\dagger(x) \psi_m(x)
\]

where \( g_{klm} \) are coupling strengths encoded in (36) and (69), \( \phi_{k0}(x) \) represents EW gauge bosons and \( \psi_l(x) \) represents iquarks. The subscripts indicate Lorentz indices, particle type, and particle generation when relevant.

Notice that \( g_{klm} \) vanishes for any \( lm \) that mix \( \pm \)iquark fields. In terms of up/down iquarks,

\[
\mathcal{H}_{n-\text{SM}}(x) = \sum_{k0lm} \phi_{k0}(x) \left( g_{k0lm}^\up \psi_\up^\dagger(x) \psi_m(x) \right) + \sum_{lm} \left( \phi_{k-}(x) g_{k-lm}^- \psi_\up^\dagger(x) \psi_m(x) + \phi_{k+}(x) g_{k+l+}^\dagger \psi_m^\dagger(x) \psi_\up^\dagger(x) \right) . \tag{74}
\]

Likewise, assuming equivalent gauge fixing, the SM counterpart has the same form

\[
\mathcal{H}_{\text{SM}}(x) = \sum_{klm} \phi_{k0}(x) \tilde{g}_{klm} \psi_\up^\dagger(x) \tilde{\psi}_m(x)
\]

where \( \tilde{g}_{klm} \) are the usual SM coupling constants and \( \psi_l(x) \) represents quarks. In terms of up/down quarks,

\[
\mathcal{H}_{\text{SM}}(x) = \sum_{k0lm} \phi_{k0}(x) \left( \tilde{g}_{k0lm}^\up U_\up^\dagger(x) U_m(x) + \tilde{g}_{k0lm}^\dagger D_\up^\dagger(x) D_m(x) \right) + \sum_{lm} \left( \phi_{k-}(x) \tilde{g}_{k-lm}^- U_\up^\dagger(x) D_m(x) + \phi_{k+}(x) \tilde{g}_{k+l+}^\dagger D_\up^\dagger(x) U_m(x) \right) . \tag{75}
\]
The two theories are congruent in the sense that matrix elements between their respective hadronic composites are identical. For example, consider \( \pi^+ \equiv H_1^+ H_1^- = h_1^+ \xi_1^0 + \chi_1^0 h_1^- \equiv u d \). Then
\[
\left( H_1^+ H_1^-, \mathcal{H}_{n-SM}(x) H_1^+ H_1^- \right) = i g_W W^+(x) \left[ \frac{2}{3} V^{+\dagger} \left( h_1^+ \xi_1^0, h_1^- \xi_1^0 \right) + \frac{1}{3} V^- \left( \chi_1^0 h_1^-, \chi_1^0 h_1^- \right) \right].
\]

where we have imposed \( \frac{2}{3} V^+ + \frac{1}{3} V^- \equiv V \) (or \( V^+ = V^- \equiv V \)). On the other hand,
\[
(ud, \mathcal{H}_{SM}(x) ud) = i g_W W^+(x) V^+. \tag{76}
\]

Referring to the Feynman diagrams, it is not difficult to see that the matrix elements between any two hadron states are identical:
\[
(\text{hadron}', \mathcal{H}_{n-SM}(x)\text{hadron}) \cong (\text{hadron}, \mathcal{H}_{SM}(x)\text{hadron}). \tag{77}
\]

We are using the term ‘identical’ and the symbol \( \cong \) because each side contains spinor functions that carry iquark/quark labels. So mathematically, the two sides can be distinguished; but since no iquark/quark has been observed they are physically indistinguishable.

More generally, for matrix elements of a state \( \Phi_\alpha \) comprised of leptons, mesons, and baryons labeled in either theory
\[
\Phi_\alpha(\psi^\dagger_T) := \text{leptons} + \sum_{lm} m_{lm}^\dagger \psi^\dagger_m \psi^\dagger_l + \sum_{lm} b_{lm}^\dagger \psi^\dagger_m \psi^\dagger_l,
\]

\[
\Phi_\alpha(\bar{\psi}^\dagger_T) := \text{leptons} + \sum_{lm} \bar{m}_{lm}^\dagger \bar{\psi}^\dagger_m \bar{\psi}^\dagger_l + \sum_{lm} \bar{b}_{lm}^\dagger \bar{\psi}^\dagger_m \bar{\psi}^\dagger_l, \tag{79}
\]

we have two physically indistinguishable descriptions of the \( S \)-matrix \( (\Phi_\beta, S \Phi_\alpha) =: S_{\alpha \beta} \) because
\[
\left( \Phi_\beta(\psi^\dagger_T), T\{\mathcal{H}_{n-SM}(x_1) \cdots \mathcal{H}_{n-SM}(x_N)\} \Phi_\alpha(\psi^\dagger_T) \right) = \left( \Phi_\beta(\bar{\psi}^\dagger_T), T\{\mathcal{H}_{SM}(x_1) \cdots \mathcal{H}_{SM}(x_N)\} \Phi_\alpha(\bar{\psi}^\dagger_T) \right). \tag{80}
\]

Relation (80) follows from straightforward induction on \( N \), inserting complete sets of states appropriately, and using \( (\Phi_\alpha(\psi^\dagger_T), \mathcal{H}_{n-SM}(x) \Phi_\alpha(\psi^\dagger_T)) \cong (\Phi_\alpha(\bar{\psi}^\dagger_T), \mathcal{H}_{SM}(x) \Phi_\alpha(\bar{\psi}^\dagger_T)). \)

Since individual iquarks/quarks are not observed, \( |S_{\alpha \beta}|^2 \) includes an average over the associated spins. This eliminates the spinors that carry iquark/quark labels, and everything (sans leptons) boils down to momenta, masses, and electric charges of \( \Phi_\alpha \) and \( \Phi_\beta \) — no direct link to either \( \psi^\dagger_T \) or \( \bar{\psi}^\dagger_T \) remains. For states comprised of leptons and hadrons then, the \( S \)-matrix for EW transitions in the \( n \)-SM and SM are physically indistinguishable and they yield equivalent transition rates and cross sections. This is not to say that parametrized, model-dependent predictions will necessarily agree: As we have already mentioned, PDFs will certainly differ for charged lepton–hadron DIS phenomenology.
4 Some Speculation

It is noteworthy that the SM and the n-SM have different gauge group and elementary particle content, and yet they make equivalent experimental predictions. The virtue of the n-SM is that it offers new avenues and suggestions for speculative modifications.

4.1 Representation transmutation

It is believed that pure QCD has the potential to exhibit Coulomb, confining, and Higgs phases — the confining phase being manifest at typical terrestrial energies. With matter included, the phases (as a function of matter field chemical potential and temperature) appear to include the hadronic phase, a quark/gluon plasma phase, a quark liquid phase, and a CFL superfluid phase.

With this backdrop, a notable feature of the n-SM is the equivalent EW representation of quarks and leptons. Recall this is a direct consequence of a product symmetry group. One wonders if the product structure exerts influence elsewhere.

We propose that the product group structure can lead to a phase that does not belong to one of the phase classes described above. Our hypothesis is based on the observation that a trace over the SU_C(3) components of the covariant derivative leads to transmuted representations $(3, 2)_L, (3, 2^c)_L \rightarrow (1, 2)_L, (1, 2^c)_L$ and $(3, 1^\pm)_R, (3, 1^0)_R \rightarrow (1, 1^\pm)_R, (1, 1^0)_R$ (and their $SU_C(3)$ conjugates). In as much as $\kappa^+ + \kappa^- = 1$, such a representation transmutation would look like a phase change from quarks to leptons — without $SU_C(3)$ symmetry breaking. In effect, QCD is posited to have a leptonic phase. We will call this hypothetical transition from hadronic phase to leptonic phase “trans-representation” for short.

But what mechanism could possibly trigger trans-representation? One idea is to view the gauge bosons of the product group $SU_C(3) \times U_{EW}(2)$ as a combined system/environment bridged by their mutual coupling to matter fields. Then one could imagine that the EW gauge–matter interaction might see the strong gauge–matter interaction as a source of decoherence. And, under suitable position/momentum/particle-content conditions, the decoherence might accumulate to the point of a phase transition as described. One can even imagine the influence goes the other way. Perhaps at the same or different energy scale the strong gauge–matter interaction sees the EW gauge–matter interaction as an environment. The same representation reduction might occur for $SU_C(3)$, and the eventual EW-energy-scale quark and lepton content would ultimately spring from $(3, 2) \oplus (3, 2^c)$.

To be more explicit, assume the domain of the Lagrangian (48) has been restricted to the physical state space which is endowed with a Hilbert structure. The n, since the Lagrangian is a closed and symmetric form, there exist unique self-adjoint operators that represent the covariant derivatives (50). So we can go over to an operator picture.

At the beginning of the electroweak epoch, we assume matter fields in representation $(3, 2) \oplus (3, 2^c)$ and boson states $|\phi_{SU(3)} \otimes |\phi_{U(2)} \rangle \in H_{SU(3)} \otimes H_{U(2)} \subset H_{\text{gauge}} \oplus H_{\text{matter}} =: H$ where $H$ is the total physical Hilbert space.

Of course the $SU(3)$ and $U(2)$ gauge bosons do not interact directly. However, they do share common matter fields (including Higgs), and over a sufficiently coarse-grained phase space some of the matter–gauge interactions can be viewed as boson–boson scattering (viz.
diagrams with internal matter loops). Adopting this picture, we will assume it can be described by a total scattering operator that can be decomposed as

\[ S_{\text{Tot}} = S_0 + S_1 + S_2 + S_3 \] (81)

where \( S_0 \) is diagonal in the Cartan basis of both \( SU(3) \) and \( U(2) \) (so it looks like segregated gauge boson scattering), \( S_1 \) (resp. \( S_2 \)) is diagonal in the Cartan basis of \( SU(3) \) (resp. \( U(2) \)), and \( S_3 \) is not diagonal in either. These individual components would presumably only be manifest in various subspaces of the total phase space.

For example, \( S_1 \) represents an operator corresponding to a scattering event that does not mix the \( SU(3) \) Cartan basis states. Then \( S_1 \) has the (assumed) form

\[ S_1 := \sum_\alpha |\alpha\rangle \langle \alpha| \otimes S_\alpha \] (82)

where \( |\alpha\rangle \) represents the Cartan basis in \( su_C(3) \) and \( S_\alpha \) are scattering operators representing the influence of gauge boson \( |\alpha\rangle \) coupling to \( |\phi_{U(2)}\rangle \) via virtual matter fields. Likewise,

\[ S_0 := \sum_{\alpha,\sigma} |\alpha\rangle \langle \alpha| \otimes |\sigma\rangle \langle \sigma| \]

\[ S_2 := \sum_\sigma S_\sigma \otimes |\sigma\rangle \langle \sigma| \]

\[ S_3 := \sum_{\alpha,\sigma} S_\sigma \otimes S_\alpha \] (83)

Consider a single scattering event governed by \( S_1 \). On a time scale much larger than the interaction time, the initial gauge field joint density is \( \rho_{SU(3)} \otimes \rho_{U(2)} \). The final reduced density will then be

\[ \rho^R = S_1 \left( \rho_{SU(3)} \otimes \rho_{U(2)} \right) S_1^\dagger \]

\[ = \sum_{\alpha,\beta} p_\alpha p_\beta |\alpha\rangle \langle \alpha| \rho_{SU(3)} |\beta\rangle \langle \beta| \otimes S_{\alpha,\beta} \] (84)

where \( p_\alpha, p_\beta \) are probabilities and \( S_{\alpha,\beta} := S_\alpha \rho_{U(2)} S_\beta^\dagger \). The \( SU(3) \) contribution to the reduced final density matrix is therefore

\[ \rho^R_{SU(3)} = \text{tr}_{U(2)}(\rho^R) \]

\[ = \sum_{\alpha,\beta} p_\alpha p_\beta |\alpha\rangle \langle \alpha| \rho_{SU(3)} |\beta\rangle \left[ \text{tr}_{U(2)}(S_{\alpha,\beta}) \right] |\beta\rangle \langle \beta|. \] (85)

To the extent that \( \left[ \text{tr}_{U(2)}(S_{\alpha,\beta}) \right] \) diminishes \( |\rho^R_{SU(3)}| \), the off-diagonal matrix elements become ‘decoherent’ in the sense that superpositions of \( SU(3) \) gauge bosons that might be induced by the collision can become highly suppressed. On the other hand, for \( \alpha = \beta \) the right-hand side of \( \rho^R \) collapses to \( \rho_{SU(3)} \left[ \text{tr}_{U(2)}(\rho_{U(2)}) \right] \) since \( S_1 \) is unitary. Together, these indicate the canonical status of gauge boson states associated with the \( SU(3) \) Cartan basis are not disrupted by the collisions encoded in \( \rho^R_{SU(3)} \) and the diagonal of \( \rho^R \). This is crucial
for on-shell iquarks trying to couple to gluons. Recall that iquarks are eigenfields relative to the Cartan basis: they can’t exchange charge with a statistical ensemble of gluon states (which would carry an associated mixture of charges). So, after the scattering, on-shell iquarks in the defining representation can only effectively couple to \( \rho_{SU(3)}^R \) and the diagonal part of \( \rho^R \).

The reduced final density can alternatively be expressed as

\[
\rho_{SU(3)}^R = \sum_{\sigma, \varepsilon} p_\sigma \langle \varepsilon \big| S_1 | \sigma \rangle \rho_{SU(3)} \langle \sigma \big| S_1^\dagger \rangle \varepsilon
\]

\[= \sum_{\sigma, \varepsilon} p_\sigma(s_1(\sigma, \varepsilon) \rho_{SU(3)} S_1(\sigma, \varepsilon)^\dagger \tag{86}\]

where \( S_1(\sigma, \varepsilon) = \sum_\alpha |\alpha\rangle \langle \alpha | \sigma S_\alpha | \varepsilon \rangle \). Unitary \( S_1 \) implies

\[
\sum_{\sigma, \varepsilon} p_\sigma S_1(\sigma, \varepsilon) I_{SU(3)} S_1(\sigma, \varepsilon)^\dagger = 1, \tag{87}\]

so \( S_1(\sigma, \varepsilon) \) represents an effective scattering operator on \( H_{SU(3)} \). We want to associate this operator with the covariant derivatives acting on the \((3, 1^\pm)\) and \((3, 1^0)\) matter fields. Reversing the roles of \( SU(3) \) and \( U(2) \), similar conclusions apply for \( \rho_{U(2)}^R \). The effective scattering operators \( S_2(\alpha, \beta) \) on \( H_{U(2)} \) would lead to the lepton covariant derivative for \((1, 2)\). Finally, \( S_3 \) would yield the lepton covariant derivatives for \((1, 1^\pm)\) and \((1, 1^0)\).

**Remark 4.1** A reasonable guess of how this would work is that meson production along with \( S_\alpha \) would lead to \((3, 2) \rightarrow (3, 2)_L, (3, 1^\pm)_R, (3, 1^0)_R \) and \((3, 2^c) \rightarrow (3, 2^c)_L, (3, 1^\pm^c)_R, (3, 1^0)_R \). Likewise for \((\bar{3}, \bar{2})\) and \((\bar{3}, \bar{2}^c)\). Either before, during, or after this would be accompanied by \( S_\sigma \) induced transitions \((3, 2) \rightarrow (1, 2)\) and \((3, 1^\pm), (3, 1^0) \rightarrow (1, 1^\pm), (1, 1^0)\). One can verify that inserting these lepton fields into the iquark Lagrangian and using \( \kappa^+ + \kappa^- = 1 \) yields the usual lepton Lagrangian of the SM. A more detailed accounting of this is included in appendix A.4.

We know the matter field representations supposedly induced by \( S_1 \) and \( S_3 \). At energy scales above \( Q_{EW} \), we might expect \( S_0 \) and \( S_2 \) induced states to contribute to the Lagrangian according to \( (133) \) and \( (135) \). Ostensibly, such states would not survive the decoherence and/or electroweak symmetry breaking.

One might guess that \( S_\sigma \) and \( S_\alpha \) are the origin of generation mixing and chiral asymmetry. If so and assuming a massless pair \((3, 2), (3, 2^c)\), there would be an approximate global \((U(2)_L \times U(2)_R) \times \mathbb{Z}_2 \) outer automorphism symmetry in \( S_0 \) that would be spontaneously broken via \( S_\alpha \) to an approximate global \( SU_V(2) \times U_V(1) \) in \( S_1 \) and \( S_4 \) — resulting in pseudo-Goldstone bosons. A possible connection with Higgs \( 35 \) merits further investigation of this point.

Evidently \( B + L \) is conserved across all energy scales in this picture. But, between the hypothetical energy scale of \( SU(3) \times U(2) \) and the electroweak SSB scale, baryon and lepton numbers may not be separately conserved because iquarks and leptons can be transmuted. This is potentially problematic for proton decay, but the next section offers a remedy.
4.2 \( U(3) \) strong symmetry group?

The notion of trans-representation allows one to imagine a high-energy gauge group whose adjoint and defining representations contain the low-energy gauge bosons of the SM but not necessarily matter multiplets comprised of low-energy quarks and leptons; since these can be transmuted according to our hypothesis. For example, consider the symmetry breaking chain \( U(4, \mathbb{C}) \rightarrow U_S(3) \times U_{EW}(2) \rightarrow U_S(3) \times U_{EM}(1) \) where supposedly the middle regime \( U_S(3) \times U_{EW}(2) \) would harbor strong and EW interactions leading to trans-representation. For this model to have a chance, there better be no anomalies and it better behave as expected at currently accessible energy scales. So we need to establish the \( U_S(3) \) strong-charge structure, check it for anomalies, and make sure it is physically reasonable.

At the level of \( U(4, \mathbb{C}) \) there are sixteen massless gauge bosons. The first symmetry reduction (which seems to indicate a loss of \( C \) invariance) presumably would occur around \( 10^{26} \) eV (see appendix A.5) and, if spontaneously broken, would lead to three heavier cousins\(^2\) of the massive electroweak \( W^\pm, Z^0 \) which in turn are induced by the second symmetry reduction near 100 GeV. There would remain the \( 8+1 \) gauge bosons associated with the decomposition \( u_S(3)/u_S(1) \oplus u_{EM}(1) \) plus a ‘ninth gluon’. After verifying there are no anomalies, we will examine some phenomenological implications of the ‘ninth gluon’.

The first business is to construct the matter field representation consistent with the Cartan decomposition. Recall from the discussion of the \( U_{EW}(2) \) representation that we assume the identity element is not one of the generators of the Cartan subalgebra. In consequence, for the present case of \( U_S(3) \) the two coupling constants will get mixed and the dynamics will not resemble simple \( SU_C(3) \times U_S(1) \).

Return to the explicit \( SU_C(3) \) defining representation. According to appendix A.1, to construct a suitable matter field representation of \( U_S(3) \) we can keep the \( S^\pm \) and replace the diagonal generators \( H_1 \) and \( H_2 \) by

\[
\begin{align*}
\tilde{H}_1 & := ig_{h_1} \begin{pmatrix} -a-1 & 0 & 0 \\ 0 & -a+1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\tilde{H}_2 & := ig_{h_2} \begin{pmatrix} -b+1 & 0 & 0 \\ 0 & b+1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\
\tilde{H}_3 & := ig_{h_3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}
\end{align*}
\]  

(88)

where \( 2g_{h_1} = g_{h_2} = 2g_{h_3} = \tilde{g}_1 \). The Cartan basis determines \( a = b = c = \pm \sqrt{3} \tilde{g}_2/\sqrt{\tilde{g}_1^2 - \tilde{g}_2^2} \) provided \( \tilde{g}_1^2 \geq \tilde{g}_2^2 \). Note the normalization here. It has been chosen so that when \( \tilde{g}_2 \rightarrow 0 \), which enforces \( \text{Tr}(A_\alpha) = 0 \), the covariant derivative reduces to our original \( SU_C(3) \) covariant

\(^2\) An important question to answer about these heavier cousins is the fate of any on-shell particles after the second symmetry breaking — might they decouple from \( U_S(3) \times U_{EM}(1) \) matter eigenfields or form stable composites?
We come to an important point here: this parametrization yields

\[
\begin{align*}
\text{tr}[\tilde{H}_1] &= \mp i \frac{\sqrt{3} \tilde{g}_1 \tilde{g}_2}{\sqrt{\tilde{g}_1^2 - \tilde{g}_2^2}} =: \mp i 2 e_S \\
\text{tr}[\tilde{H}_2] &= 0 \\
\text{tr}[\tilde{H}_3] &= \pm i \frac{\tilde{g}_1 \tilde{g}_2}{\sqrt{2} \sqrt{\tilde{g}_1^2 - \tilde{g}_2^2}} = \pm i e_S
\end{align*}
\]

and

\[
\begin{align*}
g(\tilde{H}_1, \tilde{H}_1) &= \frac{3 \tilde{g}_1^2}{4 \tilde{g}_1^2 - 4 \tilde{g}_2^2} = \frac{1}{4} \left( \frac{4 e_S^2}{\tilde{g}_2^2} \right) =: \frac{1}{4} \Delta^2 \\
g(\tilde{H}_2, \tilde{H}_2) &= \frac{3 \tilde{g}_1^2}{\tilde{g}_1^2 - \tilde{g}_2^2} = \frac{4 e_S^2}{\tilde{g}_2^2} = \Delta^2 \\
g(\tilde{H}_3, \tilde{H}_3) &= \frac{3 \tilde{g}_1^2}{8 \tilde{g}_1^2 - 8 \tilde{g}_2^2} = \frac{1}{8} \left( \frac{4 e_S^2}{\tilde{g}_2^2} \right) = \frac{1}{8} \Delta^2.
\end{align*}
\]

We come to an important point here: \( \tilde{H}_1 \) and \( \tilde{H}_3 \) have unequal norms. But we are not free to adjust their normalizations independently, because we insist that the covariant derivative \( \mathcal{D}_{U_S(3)} \rightarrow \mathcal{D}_{SU_C(3)} \) when \( \tilde{g}_2 \rightarrow 0 \). Even worse, if one does normalize \( \tilde{H}_i \), the symmetry group reduces to a direct product \( U_S(3) \rightarrow SU_S(3) \times U_S(1) \) (see appx. A.3). With unequal norms, a rotation in the 1–3 plane of the Cartan subalgebra will generally lead to non-orthogonal generators. We will see this explicitly later (in appx A.2), and it has significant physical consequences regarding the strong photon mediator of the strong charge \( e_S \) which makes its first appearance in \( (89) \).

**Remark 4.2** What happens if we normalize the representation of the Cartan subalgebra basis \( \tilde{H}_i \rightarrow \tilde{H}_i^\prime \) according to convention so that \( g(\tilde{H}_i^\prime, \tilde{H}_i^\prime) = 1/2 \) for \( i \in \{1, 2, 3\} \)? First, taking the limit \( \tilde{g}_2 \rightarrow 0 \) yields \( \tilde{H}_2^\prime|_{\tilde{g}_2=0} = \tilde{H}_2 \) but \( (\tilde{H}_1^\prime + \tilde{H}_3^\prime)|_{\tilde{g}_2=0} = \frac{1}{\sqrt{6}} (1 + \frac{2}{\sqrt{3}})^{-1} \tilde{H}_1 \). This resulting renormalization of the \( SU_C(3) \) Cartan basis is legitimate, but it doesn’t allow a direct comparison of the interaction couplings between \( U_S(3) \) and QCD since the covariant derivatives differ. Comparison is still possible, but it takes a little more effort.

Second, and more problematic; the normalized basis \( \tilde{H}_i^\prime \) is also isomorphic to a basis that isolates the strong photon, and one can see from the bi-linearity of the inner product that the strong photon generator will be orthogonal to the other basis elements. Moreover, the brackets \( [S_i^\pm, S_j^\mp] \) will not include the strong photon generator. Hence the \( u(1) \) portion of the algebra decouples, and the symmetry group reduces to \( SU_S(3) \times U_S(1) \) which is unacceptable.

As in the \( U_{EW}(2) \) case, this ensures consistent normalizations between \( SU_C(3) \) and \( U_S(3) \) so they can be compared directly. Although our chosen normalization is convenient for avoiding a proliferation of square root factors, in appendix A.2 the normalization of the generators will be standardized so that vertex factors for Feynman graphs can be compared to QCD color factors. The standard normalization for the inner product obtains for the choice \( 2\sqrt{2} \tilde{g}_1 = \sqrt{6} \tilde{g}_2 = 2\sqrt{2} \tilde{g}_3 = \tilde{g}_1 \).
Evidently, the semi-direct product group with a positive-definite inner product implies a non-orthogonal center $Z(u_S(3))$; implying the gauge field associated with the identity matrix is not an orthogonal state.

The gauge boson and matter eigenfield quantum numbers are now (still as factors of $\tilde{g}_1$)

$$
(A_1)_\pm \rightarrow \left( \pm 1, \pm 4 \frac{e_S}{\tilde{g}_1}, \pm 1 \right)
$$

$$
(A_2)_\pm \rightarrow \left( \frac{1}{2} \left( 1 + \frac{2e_S}{\tilde{g}_1} \right), \mp \frac{1}{2} \left( 3 - \frac{2e_S}{\tilde{g}_1} \right), \pm \frac{1}{2} \left( 1 + \frac{2e_S}{\tilde{g}_1} \right) \right)
$$

$$
(A_3)_\pm \rightarrow \left( \mp \frac{1}{2} \left( 1 - \frac{2e_S}{\tilde{g}_1} \right), \mp \frac{1}{2} \left( 3 + \frac{2e_S}{\tilde{g}_1} \right), \mp \frac{1}{2} \left( 1 - \frac{2e_S}{\tilde{g}_1} \right) \right)
$$

$$
\tilde{\Lambda}_7 \rightarrow (0, 0, 0)
\tilde{\Lambda}_8 \rightarrow (0, 0, 0)
\tilde{\Lambda}_9 \rightarrow (0, 0, 0)
$$

and

$$
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \rightarrow \left( -\frac{1}{2} \left( 1 + \frac{2e_S}{\tilde{g}_1} \right), 1 - \frac{2e_S}{\tilde{g}_1}, -\frac{1}{2} \right)
$$

$$
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} \rightarrow \left( \frac{1}{2} \left( 1 - \frac{2e_S}{\tilde{g}_1} \right), 1 + \frac{2e_S}{\tilde{g}_1}, \frac{1}{2} \right)
$$

$$
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \rightarrow \left( 0, -2, \frac{12e_S}{2\tilde{g}_1} \right)
$$

As expected, the iquarks and charged gluons carry a third charge partly characterized by a new coupling constant $\tilde{g}_2$. Similar to the $U_{EW}(2)$ case, $\tilde{g}_2$ always appears together with $\tilde{g}_1$ in the combination we have defined as $e_S$. What is unexpected is the fact that the strong photon is not an orthogonal state and the strong charge $e_S$ blows up as $\tilde{g}_2/\tilde{g}_1 \rightarrow 1$.

Does this lead to sensible strong-force physics?

- Starting with $(3, 2) \oplus (\bar{3}, \bar{2}) \oplus (\bar{3}, \bar{2}) \oplus (3, 2c)$ matter fields, trans-representation eventually leads to a low-energy sector in phase space where leptons carry no $e_S$ charge if we assume leptons manifest the trivial $U_S(3)$ matter-field representation. For the iquark sector, anomaly analysis for iquark strong charge shows there will be no anomaly if the iquark Lagrangian (48) is augmented to $L_{\text{iquark}} + s.c.$ where $s.c.$ means strong complex conjugate. Appendix A.4 gives the details, but it is evident that left-handed and right-handed strong complex conjugate iquark anomaly contributions will cancel separately.
• Since strong charge $e_S$ is conserved and baryons and leptons are strong-singlet states, $B$ and $L$ are separately conserved below the scale where representation transmutation terminates. In this picture, protons can’t experience semi-leptonic decay because (presumably) leptons carry zero strong charge. 
(appx. A.4)

• Contrary to the $SU_S(3)$ case, inspection of \[ (90) \] reveals the values of $T_F$, $C_F$, and $C_A$ depend on the energy scale through the ratio $\tilde{g}_2/\tilde{g}_1$ (due to our chosen normalization). From appendix A.2 we get

$$T_F(\tilde{g}_2/\tilde{g}_1) = \frac{6 + \left(\frac{\Delta^2}{4} + \frac{\Delta^2}{3} + \frac{\Delta^2}{8}\right)}{2 \times 9}; \quad C_F(\tilde{g}_2/\tilde{g}_1) = \frac{6 + \left(\frac{\Delta^2}{4} + \frac{\Delta^2}{3} + \frac{\Delta^2}{8}\right)}{2 \times 3}$$  \[ (93) \]

and

$$C_A(\epsilon) = \frac{1}{9} \left(2 \left(6 + 1 + \frac{2\epsilon^2}{3} + \frac{(1 + \epsilon)^2}{4} + \frac{(1 - \epsilon)^2}{4} + \frac{(3 + \epsilon)^2}{6} + \frac{(3 - \epsilon)^2}{6}\right)ight.$$  
$$+ \frac{\Delta^2}{4} \left(8 + 2(\epsilon + 1)^2 + 2(\epsilon - 1)^2\right) + \frac{\Delta^2}{3} \left(\frac{12\epsilon^2 + 3(\epsilon + 3)^2 + 3(\epsilon - 3)^2}{2(\epsilon^2 + 3)^2}\right)$$  
$$+ \frac{\Delta^2}{8} \left(32 + 8(\epsilon + 1)^2 + 8(\epsilon - 1)^2\right)\right).$$  \[ (94) \]

As evidenced by the non-trivial form of \[ (94) \], gauge-field dynamics for $U(3)$ is far more complicated than for $SU(3)$.

Isolating the $u_S(1)$ subalgebra associated with the strong photon, we approximate the Lie algebra as $u_S(3) \approx su_S(3) \oplus u_S(1)$ and define “effective” factors $T_F^{\text{eff}}$ and $C_F^{\text{eff}}$ for the $su_S(3)$ contributions to quark vertices that can be compared to their QCD counterparts for $su_C(3)$. We find (see appendix A.2)

$$T_F^{\text{eff}}(\tilde{g}_2/\tilde{g}_1) = \frac{6 + \frac{3\Delta^2}{16} + \frac{3\Delta^2}{8} + 2\frac{\Delta^2}{16}}{2 \times 8}; \quad C_F^{\text{eff}}(\tilde{g}_2/\tilde{g}_1) = \frac{6 + \frac{3\Delta^2}{16} + \frac{3\Delta^2}{8} + 2\frac{\Delta^2}{16}}{2 \times 3}. \quad (95)$$

Taking the limit $\tilde{g}_2 \to 0$ gives $T_F(0) = .45$ and $C_F(0) = 1.35$ while $T_F^{\text{eff}}(0) = .496$ and $C_F^{\text{eff}}(0) = 1.32$. In the same limit, $C_A(0) = 3.0$. All of these are consistent with QCD.

• We don’t have precise reference values for $\tilde{g}_1$ and $\tilde{g}_2$, but if we can trust \[ (96) \] we can estimate a range as follows: The measured value of $1.2 \leq C_F^{\text{meas}} \leq 1.4$ along with \[ (93) \] gives $.03 \leq \tilde{g}_2^2/\tilde{g}_1^2 \leq .48$. Then, using $26\alpha_1(m_Z) + \alpha_S(m_Z) \approx \alpha_{QCD}(m_Z) \approx .1184$ with $m_Z = 92$ GeV and noting that the valence HC triplet couples to the strong photon with extrinsic charge $e_S/4$ gives $1.22 \geq \tilde{g}_1(m_Z) \geq 1.15$ and $.21 \leq \tilde{g}_2(m_Z) \leq .8$ (see appx. A.2 and A.5).

For these ranges, $3.02 \leq C_A(m_Z) \leq 3.63$ and $1.36 \leq C_F(m_Z) \leq 1.58$ which yield the ratio range $2.219 \leq C_A(m_Z)/C_F(m_Z) \leq 2.300$. This is consistent with the experimental values of $C_A^{\text{meas}} = 2.89 \pm .24$ and $C_F^{\text{meas}} = 1.3 \pm .1$ (29) p. 140 which $\alpha_{QCD}$.

\[26\]Our $\alpha_S = (e_S/4)^2/4\pi$ is not the strong coupling constant of QCD which instead will be denoted by $\alpha_{QCD}$.
yield $2.208 \leq C_{A}^{\text{meas}}/C_{F}^{\text{meas}} \leq 2.236$. Using this measured ratio, we can further restrict to $1.22 \geq \tilde{g}_{1}(m_{Z}) \geq 1.20$ and $.21 \leq \tilde{g}_{2}(m_{Z}) \leq .49$ and the best fit value $C_{A}^{\text{meas}}/C_{F}^{\text{meas}} = 2.223$ gives $\tilde{g}_{1}(m_{Z}) = 1.21$ and $\tilde{g}_{2}(m_{Z}) = .32$. However, $T_{F}, C_{F}, C_{A}$ are not particularly sensitive to $\tilde{g}_{2}/\tilde{g}_{1}$ in this energy range so this best fit estimate is pretty rough. Nevertheless, the ranges are consistent with QCD well above $m_{Z}$.

• What is the fate of asymptotic freedom if $T_{F}, C_{F}, C_{A}$ depend on the energy scale? To answer this, the standard QCD renormalization formulas can’t be used directly since $u_{S}(3) \not\cong su_{S}(3) \oplus u_{S}(1)$. However, in the spirit of our semi-quantitative analysis, we can hope to use the approximation

$$
\beta_{US(1)}(\tilde{a}_{1}, \tilde{a}_{2}) \sim \frac{2N_{f}}{3\pi} \tilde{a}_{S}^{2} \\
\beta_{SU(3)}(\tilde{a}_{1}, \tilde{a}_{2}) \sim -\frac{11C_{A}(\tilde{a}_{1}, \tilde{a}_{2}) - 4N_{f}T_{F}^{\text{eff}}(\tilde{a}_{1}, \tilde{a}_{2}) - N_{H}}{6\pi} \tilde{a}_{1}^{2}
$$

(96)

where there are $N_{f}$ of $H_{s}^{+}$ and $N_{H}$ Higgs degrees of freedom at a given energy scale, and $\tilde{a}_{S} = (e_{S}/4)^{2}/4\pi$ since each $H_{s}^{+}$ degree of freedom couples to the strong photon with strength $e_{S}/4$ (see appx. A.2). These lead to coupled differential equations for $\tilde{a}_{1}, \tilde{a}_{2}$. Qualitatively, as $\tilde{g}_{2}/\tilde{g}_{1} \rightarrow 1$, the Casimir $C_{A}(\tilde{a}_{1}, \tilde{a}_{2})$ grows much faster than $T_{F}^{\text{eff}}(\tilde{a}_{1}, \tilde{a}_{2})$. To the extent that using these beta functions is justified, it appears $\tilde{a}_{1}, \tilde{a}_{2} \rightarrow 0$ as energy increases. On the other hand, $\tilde{a}_{S}$ increases with the energy scale.

Without the precise beta function for $U_{S}(3)$, it is difficult to say if the running couplings $\tilde{a}_{1}(Q), \tilde{a}_{S}(Q)$ are consistent with current measurements of the QCD coupling constant $\alpha_{QCD}(Q)$. But, since the best fit estimate $\tilde{g}_{2}(m_{Z})/\tilde{g}_{1}(m_{Z}) = .07$, it is plausible that the effects of $\tilde{a}_{S}(Q)$ relative to $\tilde{a}_{1}(Q)$ are not discernable in measurements of $\alpha_{QCD}(Q)$; at least up to $Q_{EW}$. However, given the expectation that the ratio $\tilde{g}_{2}/\tilde{g}_{1} \rightarrow 1$ near the energy scale of $U(4, C) \rightarrow U_{S}(3) \times U_{EW}(2)$, then clearly $e_{S}^{2} \rightarrow \infty$ at very high energy scales. So somewhere above $Q_{EW}$ the running strength of the strong force (as embodied in $\alpha_{QCD}$) is expected to reverse course and increase. Presumably, as $\tilde{a}_{1}, \tilde{a}_{2}$ derive from $U(4, C)$, they should remain finite even as $e_{S}^{2}$ diverges.

To make this more quantitative, in appendix A.5 we assume the validity of (96) and numerically solve the coupled RGEs. Fitting to $\alpha_{QCD}(500 \text{ MeV})$ gives the estimates $\tilde{g}_{1}(m_{Z}) \approx 1.208$ and $\tilde{g}_{2}(m_{Z}) \approx .37$, and we get $\tilde{a}_{1}(500 \text{ MeV}) + \tilde{a}_{S}(500 \text{ MeV}) \approx 1$ and $\tilde{a}_{1}(m_{Z}) + \tilde{a}_{S}(m_{Z}) \approx .1184$. Far above $Q_{EW}$ we find $\tilde{a}_{1}(10^{26} \text{ eV}) + \tilde{a}_{S}(10^{26} \text{ eV}) \approx 1$ confirming the qualitative features discussed above.

The key point: since gluons carry the strong charge $e_{S}$, this implies a rapidly increasing strong force at very high energy scales in spite of $\tilde{a}_{1} \rightarrow 0$ and $\tilde{a}_{2} \rightarrow 0$. Whether this leads to confinement of quarks in this energy region is an open question.

• It is plausible that quark masses are at least partially dynamically generated by $U_{S}(3)$. But according to the effective theory modeled by our Lagrangian, lepton currents carry zero $e_{S}$-charge. Might such masses survive trans-representation?
• One can imagine scenarios in which a $U(1)$ subgroup of $U_S(3)$ is distinguished due to dynamical SSB in (for example) ponderable atomic matter. Such a massive vector boson has been invoked repeatedly to explain various as yet statistically inconclusive experimental anomalies.

### 4.2.1 Cosmological implications

Assuming the QCD phenomenology is sensible, $U_S(3)$ baryon singlets would carry a total strong charge of $e_S$. Of course, baryons with a non-vanishing strong charge flies in the face of convention so we need to check its viability.

In the very early universe, the ramification would be a collective repulsive force due to any matter/anti-matter asymmetry. Since $\tilde{g}_1$ and $\tilde{g}_2$ run obversely to each other with energy scale, it is not hard to imagine they were nearly equivalent around the beginning of the postulated trans-representation epoch somewhere around the $U(4, \mathbb{C}) \rightarrow U_S(3) \times U_{EW}(2)$ scale (which according to appx. A.5 is around $10^{26}$ eV). The ensuing $e_S$-charge repulsion between quark states would have been extreme as soon as quark/anti-quark asymmetry took hold — leading to a rapid expansion accompanied by a decrease of cosmic energy density and a corresponding decrease in the relative strength of $e_S$.

However, as the strong-coulomb force weakened and atomic matter formed much later, this picture gets more complicated. Crucially, since the strong photon couples to other gluons, the strong-coulomb force depends on baryon number and nuclear binding energy. For atomic matter then, $e_S$ couples to baryonic mass-energy. Without access to the gluon dynamics, nothing quantitative can be concluded. But if we are permitted to draw a parallel with EM photons interacting with quasi-bosons in condensed matter, we might expect ‘strong photons’ of energy $E_S$ to develop an effective interaction range characterized by $e^{-r/r_S(Z_A)}$ where the the decay distance $r_S(Z_A) = h v(Z_A)/E_S$ depends on the atomic number $Z_A$ through the phase velocity $v(Z_A)$. Then the baryon–baryon coulomb interaction would presumably go like $k_S e_S^2 e^{-r/r_S}/r$ for some unknown constant $k_S$. According to this picture, the observed attraction between atomic matter would constitute an effective description of gravity/strong-coulomb with the relative strength of the opposing forces concealed in the atomic mass. To the extent that the experimental precision of the gravitation constant $G$ cannot distinguish electron binding energy, the strong charge $e_S$ looks to be (qualitatively at least) consistent with classical gravity.

A model of a $U(1)$ gauge field $A_\mu$ coupled to a fermionic current $J_\mu$ in de-Sitter and slightly deformed de-Sitter space was considered in [40]. They found that the time-like interaction $A_0 J^0$ propels space-time inflation if one assumes a spatially uniform background gauge field plus its quantum perturbations. Inflationary dynamics result from the fact that

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27 The reverse is also true, and quark repulsion would offer a counter-force to gravitational collapse (if the gravitational coupling grows slower than $e_S$) that could plausibly prevent a black hole singularity. If so, one can imagine black hole evaporation might eventually allow internal hadronic matter to escape (or perhaps even destroy) the event horizon — releasing information in the process.

28 Of course it is quite possible that there is no exponential decay: the point of including it is to emphasize that even in this case there is a strong-coulomb interaction coming from IR ‘strong photons’.

29 It is certainly an idealization to assume a spatially uniform background gauge field, so the resulting space-time inflation distribution is likely to have some multipole components.
field-field and field-current interactions yield an isotropic (in space-time) energy density in their model; which in turn comes from the fact that \( A_0 \propto a(t) \) while \( J^0 \propto 1/a(t) \) and \( \vec{J} = 0 \) where \( a(t) \) is a conformal factor. In light of this, strong charge, which approximately behaves like a \( U(1) \) gauge field coupled to fermionic currents, has dark energy implications.

Let’s go to the Coulomb gauge and get an estimate of the strong-charge energy density from dimensional analysis in SI units. The strong charge \( e_S \) is a dimensionless fundamental constant, but we will give it the unit symbol \( C \) for very deep IR strong photons (i.e. \( r_S(\Lambda) \rightarrow \infty \)), \( u_S = \frac{1}{2} k_S^2 \) where \( k_S \) is the strong force constant and \( \sigma_B \) is a constant surface charge density. By analogy with electromagnetism, put \( k_S \equiv \mu_S c^2 / 4\pi \) where \( \mu_S \) has units \( \text{kg} \cdot \text{m} / C^2 \).

We take \( \sigma_S = 4\pi n_B \sigma_B \) to be the surface charge density on a unit sphere where \( n_B \) is the number of baryons per unit sphere. Then assuming \( u_S \) remains nearly constant during expansion, we find \( \mu_S \approx 1.6 \cdot 10^{-25} \text{kg} \cdot \text{m}/C^2 \) and \( k_S \approx 1 \cdot 10^{-9} \text{N} \cdot \text{m}^2 \) by imposing \( \Omega_S = \Omega_{\Lambda} \);

\[
\Omega_S := \frac{u_S}{\rho_c \cdot c^2} \approx \frac{\mu_S \sigma_B^2}{8\pi \rho_c} = \frac{\Omega_{\Lambda}}{2} \approx .69
\]

where \( \rho_c \approx 9 \cdot 10^{-27} \text{kg/m}^3 \) is the critical density and we used \( n_B \approx .25/\text{m}^3 \) and \( e_S = .31 \). Hence, given our assumed form of \( \sigma_S \) together with the model of \([40]\) implies a contribution of \( \Omega_S a(t)^0 \) to the Hubble parameter if our reasoning and approximations are sensible.

4.2.2 Microscopic implications

Below atomic distance scales, baryon repulsion would oppose the strong force responsible for nuclei formation and the electromagnetic attraction responsible for atomic bonds. Since nuclei formation is beyond perturbation analysis, we will just assume the strong-charge exchange of neutral gluons between nucleons wins out over \( e_S \)-charge repulsion until about .7 fm. From the previous subsection, the \( e_S \)-coulomb force in ponderable matter is presumed to go like \( F_S(r) = k_S^2 \sigma_B^2 e^{-r} / r^2 \). If we put \( F_S(\cdot.7 \text{ fm}) \sim O(1) \text{ N} \), then \( r_S \sim .015 \text{ fm} \) and \( F_S(1 \text{ fm}) \sim O(10^{-9}) \text{ N} \).

It remains to estimate opposition to atomic bonding through atom/atom \( e_S \)-charge repulsion: According to the numerical study in appendix A.5, the baryon strong charge is estimated to be comparable to the electric charge below the QCD scale; \( \tilde{a}_S \approx .25 \alpha_{EM} \) for \( Q \leq Q_{QCD} \). The relevant quantity for strong photon v.s. EM force is the ratio \( f_S / f_{EM} = k_S a_S / k_{EM} \alpha \approx O(10^{-20}) \) which appears to be negligible.

\[^{30}\text{The calculation in}[40]\text{ did not take into account a running coupling constant. But since } .1 < e_S^2(Q) < .2 \text{ for } 10^{-12} \text{eV} < Q < 10^{16} \text{eV}, \text{ this does not have much of an effect below } Q_{EW}.\]
4.3 Strong CP

We might expect that \( m_h/m_\xi \approx m_h/m_\chi \approx m_l/m_\nu \) at tree level, and if this continues to hold (more or less) for renormalized mass ratios, then electric-neutral iquarks would appear to be nearly massless compared to their electric-charged counterparts. Perhaps, then, \( m_\xi, m_\chi \) are dynamically generated due to their strong charge, in which case it is natural to speculate that \( n_{st} = 0 \) in \( \mathcal{L}_{\text{Yukawa}} \). It has been well-argued that this vanishing mass can be employed to set the CP-violating parameter \( \theta = 0 \) (see e.g. [16]).

5 Wrap-up

We want to repeat for emphasis that the primary purpose of this paper is to explore some foundational aspects of the SM under relatively mild modifications. Specifically these include electroweak \( U_{\text{EW}}(2) \) and a modified elementary particle content which lead to what we call the n-SM. Developing the n-SM leads to some interesting departures from the SM — particularly regarding iquark quantum numbers and hadronic constituents. Despite the differences, the SM and the n-SM contain identical phenomenology as long as quarks/iquarks remain unobserved. In retrospect, since the two theories share isomorphic Lie algebras and they employ consistent elementary matter-field representations, their \( S \)-matrices must agree up to iquark/quark labels.

But the modifications leading to the n-SM motivate a more substantial departure from the SM: They suggest the ideas of representation transmutation and the model symmetry group \( U(4, \mathbb{C}) \rightarrow U_S(3) \times U_{\text{EW}}(2) \) which offer physics beyond the SM that appears promising. In particular, the strong photon spawns some interesting cosmological phenomenology. Admittedly, the evidence presented in §4 contains a fair amount of speculation; but the “quotient” of potentiality/modification is encouraging, and we hope it spurs further examination and development.

A Supporting Calculations

A.1 \( U_S(3) \) Representation

The universal covering group of \( U(3) \) is \( SU(3) \times \mathbb{R} \). So \( U(3) \cong (SU(3) \times U(1))/\mathbb{Z}_3 \). In the same way that \( U(2) \) implies the Gell-Mann/Nishijima relation[27], reduction of \( SU(3) \times \mathbb{R} \) down to \( U(3) \) imposes one (integer) condition relating the three quantum numbers that label representations of \( SU(3) \times U(1) \). Insofar as \( U(3) \) is unbroken, the allowed integer values are not particularly relevant here. What is important is that it justifies the condition that the commutators of the raising and lowering generators of \( u(3) \) span the entire Cartan subalgebra: Otherwise, a \( U(1) \) subgroup completely decouples and the symmetry group degenerates into \( SU(3) \times U(1) \). Explicitly, the condition reads

\[
[A^+_i, A^+_j] = \sum_j \tilde{c}^{ij}_i h^j ; \quad \tilde{c}^{ij}_i \neq 0 \forall \ i, j \in \{1, 2, 3\} .
\]
This is a very natural condition in the sense that all the neutral gluons couple to charged
gluons and they are interchangeable via gauge invariance.

We learned from the $U_{EW}(2)$ case that the HC electroweak doublet representation could
be derived (not assumed) by imposing $[30]$ and span$_{c/\sim} \{Q^\pm, T_0^\pm, T_+^\pm, T_-^\pm \}|_{g_2 \rightarrow 0} \cong \mathfrak{su}(2)$. We
will use this lesson to guide our search for the strong triplet representation.

To see the implications for HC, consider a particular parametrized defining representation

$$
S_i^+ = i\tilde{g}_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_i^- = i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{H}_1 = i\tilde{g}_1 \begin{pmatrix} a_1 - 1 & 0 & 0 \\ 0 & b_1 + 1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}
$$

$$
S_i^+ = i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_i^- = i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{H}_2 = i\tilde{g}_1 \begin{pmatrix} a_2 + 1 & 0 & 0 \\ 0 & b_2 + 1 & 0 \\ 0 & 0 & c_2 - 2 \end{pmatrix}
$$

$$
S_i^+ = i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_i^- = i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{H}_3 = in_3 \begin{pmatrix} a_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix}
$$

(99)

where $(a_i, b_i, c_i)$ are real constants (to be determined) that may depend on $(\tilde{g}_1, \tilde{g}_2)$ and $n_3$ is
a normalization constant. This parametrization is convenient but not unique. The choice of
additive instead of multiplicative diagonal parameters in $\tilde{H}_1$ and $\tilde{H}_2$ is suggested from the
final form of $T_0^\pm$ obtained in the $U_{EW}(2)$ case.

By re-scaling $\tilde{g}_1 \rightarrow \frac{2}{2-c_2} \tilde{g}_1$, we can arrange for

$$
\tilde{H}_2 = i\tilde{g}_1 \begin{pmatrix} a'_2 + 1 & 0 & 0 \\ 0 & b'_2 + 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
$$

(100)

where $a'_2 := \frac{2a_2 + c_2}{2-c_2}$ and $b'_2 := \frac{2b_2 + c_2}{2-c_2}$. The $\tilde{H}_1$ parameters get likewise adjusted maintaining the
form of $\tilde{H}_1$ so we will drop the prime on all of them. To maintain the coupling constant for $S_i^+$,
we absorb the re-scaling factor into $g(A_i^\pm, A_j^\pm)$. Essentially, we are using the freedom to re-
scale the $\mathfrak{su}_3(3)$ coupling constant to set $c_2 = 0$, and so the $g_1, g_2$-dependent normalization\footnote{We still have to multiply $\mathbf{H}_i$ by suitable scalars to compare with standard $SU(3)$ normalization.} for
these generators is now fixed. The remaining normalization $n_3$ will be fixed later.

The bracket constraints give nine non-vanishing relations among the parameters and one
more for a common denominator that obviously cannot vanish. Their explicit forms are not
particularly illuminating, and it turns out to be simpler to start by imposing orthogonality.
The $S_i^\pm$ are already orthogonal. The orthogonality constraints on $\langle \tilde{H}_i, \tilde{H}_j \rangle = 0$ for $i \neq j$ yield
an under-determined linear system for $(a_i, b_i, c_i)$ even after normalizing. To make headway,
we test the hypothesis that the zero eigenvalue of $\tilde{H}_1$ is not shifted, i.e., $c_1 = 0$. Then from the
bracket constraints we learn that $a_1 + b_1 \neq 0$ and $c_3 \neq 0$. Further, if $c_3|_{\tilde{g}_2 \rightarrow 0} = 0$ then $a_3|_{\tilde{g}_2 \rightarrow 0} \rightarrow 0$ and $b_3|_{\tilde{g}_2 \rightarrow 0} \rightarrow 0$, on the other hand if $(a_3 + b_3)|_{\tilde{g}_2 \rightarrow 0} \rightarrow 0$ then $c_3|_{\tilde{g}_2 \rightarrow 0} \rightarrow 0$. However, reduction to $\mathfrak{su}(3)$ disqualifies the latter possibility.

\footnote{We still have to multiply $\mathbf{H}_i$ by suitable scalars to compare with standard $SU(3)$ normalization.}
Using Mathematica, we learn that there are seventeen families of potential solutions to the orthogonality constraints for $c_1 = 0$ but only two of them turn out to be consistent with $\text{span}_{C^2} \{ S^z \pm, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3 \} \mid \tilde{g}_2 \to 0 \cong \text{su}(3)$ and one is disqualified by $a_1 + b_1 \neq 0$. We are left with one family

$$
\begin{align*}
\alpha_{b_3} &= a_3 (a_2 + b_2)^2 \\
\beta_{b_3} &= a_3 (a_2 (a_2 + 6) - 4b_2 (a_2 + 3) + b_2^2) \\
\delta_{b_3} &= 2a_3 (a_2^2 - a_2 (b_2 - 3) + b_2 (b_2 + 3) + 9) \\
\rho_{b_3} &= (a_2 + b_2)^2 \\
\sigma_{b_3} &= (a_2^2 - 4a_2 (b_2 + 3) + b_2 (b_2 + 6)) \\
\gamma_{b_3} &= -2(a_2^2 - a_2 (b_2 - 3) + b_2 (b_2 + 3) + 9)
\end{align*}
$$

where

$$
\alpha_{c_3} = -2a_3 (a_2 + b_2)^2 \\
\beta_{c_3} = -a_3 (5a_2^2 - 2a_2 (b_2 - 6) + b_2 (5b_2 + 12)) \\
\delta_{c_3} = 2a_3 (a_2^2 - a_2 (b_2 - 3) + b_2 (b_2 + 3) + 9) \\
\rho_{c_3} = \rho_{b_3} \\
\sigma_{c_3} = \sigma_{b_3} \\
\gamma_{c_3} = \gamma_{b_3}
$$

and the condition $\text{span}_{C^2} \{ S^z \pm, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3 \} \mid \tilde{g}_2 \to 0 \cong \text{su}(3)$ implies $a_1 \mid \tilde{g}_2 \to 0 \to 0$, $b_1 \mid \tilde{g}_2 \to 0 \to 0$, and $c_3 \mid \tilde{g}_2 \to 0 \to 0$ while $a_3 \mid \tilde{g}_2 \to 0 = -b_3 \mid \tilde{g}_2 \to 0 \to 0$.

Of this family, we only need to exhibit one valid solution; which we can’t expect to be unique owing to the gauge invariance. The trick is to find a particular solution that will allow for some physical interpretation and satisfy the bracket conditions. A particularly simple (and it will turn out apposite) solution obtains by restricting to a subspace of the free-parameter space determined by $a_1 = b_1$, $a_2 = -b_2$, and $a_3 = -b_3$. These relations are satisfied for $\tilde{g}_2 \to 0$, so we arepositing they hold for all $\tilde{g}_2$. From $a_1 = b_1$ and $a_2 = -b_2$ we learn $a_1 = a_2$. Together with $a_3 = -b_3$ these finally imply

$$
a_1 = b_1 = a_2 = -b_2 = c_3 / a_3 = \mp \frac{\sqrt{3} \tilde{g}_2}{\sqrt{\tilde{g}_1^2 - \tilde{g}_2^2}} =: \mp \varepsilon .
$$

In retrospect, the restriction $c_1 = 0$ turns out to be a particularly fortunate choice because it leads to $a_1 = b_1 = a_2 = -b_2 = -c_3 = \mp \varepsilon$ where we absorb $a_3$ into the normalization of $\tilde{H}_3$ according to $n_3 \equiv -\tilde{g}_1 / a_3$ to satisfy the covariant derivative condition. This equivalence greatly simplifies the physical interpretation of the defining representation.

At this point, a justifiable criticism of the preceding is the seemingly arbitrary starting parametrization in $\{ 92 \}$, and there is no guarantee there does not exist a different parametrization that looks more like the $U(2)$ case or something completely different. In reply, our claim is that we have found one interesting defining representation that satisfies

$^{32}$Mathematica 9.0 was used for most matrix manipulations in this section.
our starting conditions. There is no claim of uniqueness, and in that sense it should be regarded as an ansatz.\footnote{Outside of what is reported here, we have attempted to follow the $U(2)$ derivation more closely by starting with general parametrized diagonals and relying on normalization to help fix parameters, but we have not been able to find a consistent representation that doesn’t reduce to $SU_S(3) \times U_S(1)$. Of course, that is not to say that one does not exist.}

## A.2 $U_S(3)$ Vertex factors

To aid in comparison with QCD color factors, we use the standard $SU(3)$ normalization in this appendix. Note that this yields properly normalized $su_S(3)$ when $\hat{g}_2 \to 0$ — in the sense that it agrees with the QCD $su_C(3)$ covariant derivative with its standard normalization. Recall the $u_S(3)$ $Ad$-invariant inner product and quark representation\footnote{While the $U_S(3)$ quark representation was motivated in the previous subsection, we could instead start with (105) as an ansatz and simply define $U_S(3)$ to be its exponentiation as a real algebra.}

\[ g(\Lambda_\alpha, \Lambda_\beta) := \frac{1}{6} \left[ 3\hat{g}_1^{-2} \text{tr}(\Lambda_\alpha \Lambda_\beta^\dagger) + (\hat{g}_2^{-2} - \hat{g}_1^{-2}) \text{tr} \Lambda_\alpha \cdot \text{tr} \Lambda_\beta^\dagger \right], \]  

and

\[
\begin{align*}
S^+_1 &= ig_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & S^-_1 &= ig_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{H}_1 &= i \frac{\hat{g}_1}{2\sqrt{2}} \begin{pmatrix} -\epsilon -1 & 0 & 0 \\ 0 & -\epsilon +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
S^+_2 &= ig_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & S^-_2 &= ig_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{H}_2 &= i \frac{\hat{g}_1}{\sqrt{6}} \begin{pmatrix} -\epsilon +1 & 0 & 0 \\ 0 & \epsilon +1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \\
S^+_3 &= ig_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & S^-_3 &= ig_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{H}_3 &= i \frac{\hat{g}_1}{2\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix},
\end{align*}
\]  

where $\epsilon := e_S/\hat{g}_1 := \sqrt{3\hat{g}_2/\hat{g}_1^2 - \hat{g}_2^2}$. Observe that $u_S(3) \not\simeq su_S(3) \oplus u_S(1)$ since we have

\[ \hat{H}_1 = \frac{1}{2\sqrt{2}} \left[ H_1 - \frac{\epsilon}{3} H_2 - \frac{\epsilon}{3} \right], \quad \hat{H}_2 = \frac{1}{\sqrt{6}} \left[ H_2 - \epsilon H_1 \right], \quad \text{and} \quad \hat{H}_3 = \frac{1}{2\sqrt{2}} \left[ H_1 - \frac{\epsilon}{3} H_2 + \frac{\epsilon}{3} \right]. \]

In the quark representation the inner product is

\[
(g_{\alpha\beta}) = 1/2 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]  

where $\Delta^2 := \epsilon^2_S/\hat{g}_2^2$. Notably, insisting that $u_S(3) \to su_C(3)$ when $\hat{g}_2 \to 0$ leads to inequivalent norms in the Cartan subalgebra — quite unlike the $U_{EW}(2)$ case. Of course, the
generators \{\tilde{H}_1, \tilde{H}_2, \tilde{H}_3\} can be normalized. But we will see in the next subsection that a normalized Cartan subalgebra will imply an orthogonal decomposition \(u_S(3) = su_S(3) \oplus u_S(1)\) leading to \(SU_S(3) \times U_S(1)\).

The commutators for the Cartan subalgebra clearly vanish: the remaining commutators are

\[
\begin{align*}
[\tilde{H}_1, S_1^+] &= \mp i \tilde{g}_1 \frac{1}{\sqrt{2}} S_1^+, \\
[\tilde{H}_2, S_1^+] &= \mp i \tilde{g}_1 \frac{1}{2\sqrt{2}} S_2^+, \\
[\tilde{H}_3, S_1^+] &= \mp i \tilde{g}_1 \frac{1}{2\sqrt{2}} S_3^+
\end{align*}
\]

(107)

\[
\begin{align*}
[S_1^+, S_1^+] &= 0, \\
[S_2^+, S_2^+] &= 0, \\
[S_3^+, S_3^+] &= 0
\end{align*}
\]

(108)

where

\[
D_1 := i \tilde{g}_1 \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} = \frac{2\sqrt{2}}{\epsilon^2 + 3} \tilde{H}_1 + \frac{\sqrt{6} \epsilon}{\epsilon^2 + 3} \tilde{H}_2 + \frac{4\sqrt{2}}{\epsilon^2 + 3} \tilde{H}_3,
\]

(110)

\[
D_2 := i \tilde{g}_1 \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \frac{\sqrt{2} (\epsilon + 1)}{\epsilon^2 + 3} \tilde{H}_1 + \frac{\sqrt{2} (\epsilon - 3)}{\epsilon^2 + 3} \tilde{H}_2 + \frac{2\sqrt{2} (\epsilon + 1)}{\epsilon^2 + 3} \tilde{H}_3,
\]

(111)

\[
D_3 := i \tilde{g}_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \frac{\sqrt{2} (\epsilon - 1)}{\epsilon^2 + 3} \tilde{H}_1 - \frac{\sqrt{2} (\epsilon + 3)}{\epsilon^2 + 3} \tilde{H}_2 + \frac{2\sqrt{2} (\epsilon - 1)}{\epsilon^2 + 3} \tilde{H}_3.
\]

(112)
For completeness, we list the anti-commutators;

\[
\{\tilde{H}_1, S^\pm_1\} = -i\tilde{g}_1 \frac{\epsilon}{\sqrt{2}} S^\pm_1 \quad \{\tilde{H}_2, S^\pm_1\} = i\tilde{g}_1 \frac{\sqrt{2}}{\sqrt{3}} S^\pm_1 \quad \{\tilde{H}_3, S^\pm_1\} = 0
\]

\[
\{\tilde{H}_1, S^\pm_2\} = -i\tilde{g}_1 \frac{1+\epsilon}{2\sqrt{2}} S^\pm_2 \quad \{\tilde{H}_2, S^\pm_2\} = -i\tilde{g}_1 \frac{1+\epsilon}{\sqrt{6}} S^\pm_2 \quad \{\tilde{H}_3, S^\pm_2\} = -i\tilde{g}_1 \frac{(1-\epsilon)}{2\sqrt{2}} S^\pm_2
\]

\[
\{\tilde{H}_1, S^\pm_3\} = i\tilde{g}_1 \frac{(1-\epsilon)}{2\sqrt{2}} S^\pm_3 \quad \{\tilde{H}_2, S^\pm_3\} = -i\tilde{g}_1 \frac{(1-\epsilon)}{\sqrt{6}} S^\pm_3 \quad \{\tilde{H}_3, S^\pm_3\} = i\tilde{g}_1 \frac{(1+\epsilon)}{2\sqrt{2}} S^\pm_3
\]

\[
(113)
\]

\[
\{S^\pm_1, S^\pm_1\} = 0 \quad \{S^\pm_1, S^\pm_2\} = 0 \quad \{S^\pm_1, S^\pm_3\} = i\tilde{g}_1 S^\pm_2
\]

\[
\{S^\pm_2, S^\pm_1\} = 0 \quad \{S^\pm_2, S^\pm_2\} = 0 \quad \{S^\pm_2, S^\pm_3\} = 0
\]

\[
\{S^\pm_3, S^\pm_1\} = i\tilde{g}_1 S^\pm_2 \quad \{S^\pm_3, S^\pm_2\} = 0 \quad \{S^\pm_3, S^\pm_3\} = 0
\]

\[
(114)
\]

\[
\{S^\pm_1, S^\pm_1\} = i\tilde{g}_1 \tilde{D}_1 \quad \{S^\pm_1, S^\pm_2\} = i\tilde{g}_1 S^\pm_3 \quad \{S^\pm_1, S^\pm_3\} = 0
\]

\[
\{S^\pm_2, S^\pm_1\} = i\tilde{g}_1 S^\pm_3 \quad \{S^\pm_2, S^\pm_2\} = i\tilde{g}_1 \tilde{D}_2 \quad \{S^\pm_2, S^\pm_3\} = i\tilde{g}_1 S^\pm_1
\]

\[
\{S^\pm_3, S^\pm_1\} = 0 \quad \{S^\pm_3, S^\pm_2\} = i\tilde{g}_1 S^\pm_1 \quad \{S^\pm_3, S^\pm_3\} = i\tilde{g}_1 \tilde{D}_3
\]

\[
(115)
\]

where

\[
\tilde{D}_1 := i\tilde{g}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{2\sqrt{2}(\epsilon^2 + 2)}{\epsilon(\epsilon^2 + 3)} \tilde{H}_1 + \frac{\sqrt{6}}{\epsilon(\epsilon^2 + 3)} \tilde{H}_2 + \frac{4\sqrt{2}}{\epsilon(\epsilon^2 + 3)} \tilde{H}_3 ,
\]

\[
(116)
\]

\[
\tilde{D}_2 := i\tilde{g}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{\sqrt{2}(\epsilon^2 + \epsilon + 4)}{\epsilon(\epsilon^2 + 3)} \tilde{H}_1 - \frac{\sqrt{3}(\epsilon + 1)}{\sqrt{2}(\epsilon^2 + 3)} \tilde{H}_2 + \frac{2\sqrt{2}(\epsilon^2 - \epsilon + 2)}{\epsilon(\epsilon^2 + 3)} \tilde{H}_3 ,
\]

\[
(117)
\]

\[
\tilde{D}_3 := i\tilde{g}_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{\sqrt{2}(\epsilon^2 - \epsilon + 4)}{\epsilon(\epsilon^2 + 3)} \tilde{H}_1 + \frac{\sqrt{3}(\epsilon - 1)}{\sqrt{2}(\epsilon^2 + 3)} \tilde{H}_2 + \frac{2\sqrt{2}(\epsilon^2 + \epsilon + 2)}{\epsilon(\epsilon^2 + 3)} \tilde{H}_3 .
\]

\[
(118)
\]

The $US(1)$ subgroup associated with the $e_S$-charge mediator can be extracted by an energy-independent orthonormal change of basis in the Cartan subalgebra;

\[
\tilde{H}_1' = \frac{1}{\sqrt{2}} \left( -\tilde{H}_3 - \tilde{H}_1 \right) = \frac{-i\tilde{g}_1}{4} \begin{pmatrix} -\epsilon - 2 & 0 & 0 \\ 0 & -\epsilon + 2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}
\]

\[
(119)
\]

and

\[
\tilde{H}_3' = \frac{1}{\sqrt{2}} \left( \tilde{H}_3 - \tilde{H}_1 \right) = \frac{i\tilde{g}_1}{4} \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} = \frac{i\epsilon_S}{4} 1 .
\]

\[
(120)
\]
In this new basis, \( \text{Tr}(\tilde{H}_1' + \tilde{H}_3') = i\epsilon_S \), and the inner product becomes
\[
(\tilde{g}'_{\alpha\beta}) = \frac{1}{2} \left( \begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta^2 - \frac{3\Delta^2}{16} & \frac{\Delta^2}{16} \\
0 & 0 & 0 & 0 & 0 & \Delta^2 & \frac{3\Delta^2}{16} \\
0 & 0 & 0 & 0 & 0 & \Delta^2 & \frac{3\Delta^2}{16} \\
0 & 0 & 0 & 0 & 0 & \Delta^2 & \frac{3\Delta^2}{16} \\
\end{array} \right) .
\]  
(121)

Evidently \( u_5(3)/u_5(1) \not\cong u_{C}(3) \) except when \( \epsilon \to 0 \). Note that \( \tilde{H}_3' \) commutes with all the other generators, and it couples equally to each iquark degree of freedom. The strength of the strong photon coupling to baryon singlet states is \( \epsilon_S \) since \( \text{Tr}(\tilde{H}_1' + \tilde{H}_3') = i\epsilon_S \). So it is correct to identify \( \epsilon_S \) as the strong-charge counterpart to electric charge. Crucially, the commutators reveal that \( \tilde{H}_3' \) still couples to gluon dynamics.

For obvious reasons, a general and direct comparison of \( U_S(3) \) with QCD is not possible. Specific processes can be compared of course, but such analysis lies beyond our present scope. Instead, we opt for a semi-quantitative comparison of vertex/color factors.

The first thing to note is the role of the strong photon. Regarding interactions with iquarks, it behaves much like the EM photon. So it’s a reasonable approximation for the iquark–gluon vertices to view the \( \tilde{g}'_{99} \) contribution to the dynamics as an abelian degree of freedom and the remaining terms — including the off-diagonal ones — as an effective \( SU_S^{\text{eff}}(3) \) with eight non-abelian degrees of freedom. On the other hand, the strong photon clearly couples to the other gluons both through the off-diagonal terms\(^{35}\) in \( g'_{\alpha\beta} \) and the commutators of the charged gluons \( \left[ S^+_i, S^-_i \right] \). Consequently, for gluon–gluon vertices we will have to compare the full \( U_S(3) \) gluon Casimir directly with \( SU_C(3) \).

With this understanding, we can compute an “effective” normalization \( T_F^{\text{eff}} \) and a second-order Casimir \( C_F^{\text{eff}} \) in the defining representation of \( SU_S^{\text{eff}}(3) \) to compare with the QCD color factor counterparts. For the iquark–gluon interaction, (121) gives
\[
T_F^{\text{eff}}(\tilde{g}_2/\tilde{g}_1) = \left( \frac{3\Delta^2}{16} + \frac{\Delta^2}{3} + \frac{2\Delta^2}{16} \right) \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} \rightarrow \lim_{\tilde{g}_2 \to 0} \frac{496}{7} 
\]
\[
C_F^{\text{eff}}(\tilde{g}_2/\tilde{g}_1) = \left( \frac{3\Delta^2}{16} + \frac{\Delta^2}{3} + \frac{2\Delta^2}{16} \right) \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} \rightarrow \lim_{\tilde{g}_2 \to 0} \frac{1.32}{7} .
\]  
(122)

These are minimum values when \( \tilde{g}_2 = 0 \): They compare well with the \( SU_C(3) \) derived \( T_F = 1/2 \) and \( C_F = 4/3 \).

\(^{35}\)The off-diagonal terms can be eliminated by defining \( \tilde{H}'_1 := \tilde{H}'_1 - \frac{1}{2} \tilde{H}_3' \). In which case one gets \( \text{Tr}(\tilde{H}'_1) = 0 \) and \( g(\tilde{H}'_1, \tilde{H}'_2) = \frac{1}{2} + \frac{1}{3} \left( \frac{2\epsilon_S}{\tilde{g}_2} \right)^2 \). Evidently \( \{ \tilde{H}'_1, \tilde{H}_2, \tilde{H}_3 \} \) reduces to \( \{ H_1, H_2 \} \) when \( \tilde{g}_2 \to 0 \). However, this will lead to \( [S^+_i, S^-_i] = c'_{i,1} \tilde{H}'_1 + c'_{i,2} \tilde{H}_2 \) and \( c_{i,3} = 0 \) identically — which does not satisfy (98).
A less trivial check is the second-order Casimir for the adjoint representation. Here we use (106) and (107)–(112) to find

\[ N^2 C_A(\epsilon) := \sum_{\alpha, \beta} \tilde{g}_\gamma \rho f_{\gamma \beta}^\alpha f^\beta_{\mu} \]

\[ = 2 \left( 6 + 1 + \frac{2\epsilon^2}{3} + \frac{1 + \epsilon}{4} + \frac{(1 - \epsilon)^2}{4} + \frac{1}{6} + \frac{(3 + \epsilon)^2}{6} + \frac{(3 - \epsilon)^2}{6} \right) \]

\[ + \Delta^2 \frac{4}{3} \left( \frac{8 + 2(\epsilon + 1)^2 + 2(\epsilon - 1)^2}{(\epsilon^2 + 3)^2} \right) + \Delta^2 \frac{3}{3} \left( \frac{12\epsilon^2 + 3(\epsilon + 3)^2 + 3(\epsilon - 3)^2}{2(\epsilon^2 + 3)^2} \right) \]

\[ + \Delta^2 \frac{8}{8} \left( \frac{32 + 8(\epsilon + 1)^2 + 8(\epsilon - 1)^2}{(\epsilon^2 + 3)^2} \right). \]

(123)

Setting \( N = 3 \) and \( \tilde{g}_2 = 0 \) gives the lower bound \( C_A(0) = 3.0 \) to be compared to \( C_A^{QCD} = 3.0 \).

### A.3 Normalized \( \Lambda_\alpha \)

In this subsection, we follow the consequences of normalizing the generators of the Cartan subalgebra. Strictly speaking, we fixed the normalizations already at the beginning. But let’s see what happens if we ignore that. Normalizing with respect to (104) yields

\[ \hat{H}_1 := \frac{i \tilde{g}}{\sqrt{6}} \begin{pmatrix} -\epsilon - 1 & 0 & 0 \\ 0 & -\epsilon + 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{H}_2 := \frac{i \tilde{g}}{\sqrt{6}} \begin{pmatrix} -\epsilon + 1 & 0 & 0 \\ 0 & \epsilon + 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \hat{H}_3 := \frac{i \tilde{g}}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \]

(124)

where \( \tilde{g} := \sqrt{g_1^2 - g_2^2} \). It is evident that the covariant derivative \( \mathcal{D}_{\text{Us}(3)} \to \mathcal{D}_{\text{SU}_c(3)} \) when \( g_2 \to 0 \).

The non-vanishing commutators are

\[ [\hat{H}_1, S^+_1] = \pm \tilde{g}_1 \tilde{g} \frac{\sqrt{2}}{\sqrt{3}} S^+_1 \quad [\hat{H}_2, S^+_1] = \pm \tilde{g}_1 \tilde{g} \frac{2}{\sqrt{3}} \epsilon S^+_1 \quad [\hat{H}_3, S^+_1] = \pm \tilde{g}_1 \tilde{g} \frac{2}{\sqrt{3}} \epsilon S^+_1 \]

\[ [\hat{H}_1, S^+_2] = \pm \tilde{g}_1 \tilde{g} \frac{1 + \epsilon}{\sqrt{6}} S^+_2 \quad [\hat{H}_2, S^+_2] = \mp \tilde{g}_1 \tilde{g} \frac{1}{\sqrt{6}} S^+_2 \quad [\hat{H}_3, S^+_2] = \pm \tilde{g}_1 \tilde{g} \frac{2}{\sqrt{3}} (1 + \epsilon) S^+_2 \]

\[ [\hat{H}_1, S^+_3] = \mp \tilde{g}_1 \tilde{g} \frac{1 - \epsilon}{\sqrt{6}} S^+_3 \quad [\hat{H}_2, S^+_3] = \mp \tilde{g}_1 \tilde{g} \frac{1}{\sqrt{6}} S^+_3 \quad [\hat{H}_3, S^+_3] = \mp \tilde{g}_1 \tilde{g} \frac{2}{\sqrt{3}} (1 - \epsilon) S^+_3 \]

(125)

\[ [S^+_1, S^+_1] = \mp \tilde{g}_1 D_1 \quad [S^+_1, S^+_2] = \mp \tilde{g}_1 S^+_3 \quad [S^+_1, S^+_3] = \mp \tilde{g}_1 S^+_2 \]

\[ [S^+_2, S^+_1] = \mp \tilde{g}_1 S^+_3 \quad [S^+_2, S^+_2] = \mp \tilde{g}_1 D_2 \quad [S^+_2, S^+_3] = \pm \tilde{g}_1 S^+_1 \]

\[ [S^+_3, S^+_1] = \mp \tilde{g}_1 S^+_2 \quad [S^+_3, S^+_2] = \pm \tilde{g}_1 S^+_1 \quad [S^+_3, S^+_3] = \mp \tilde{g}_1 D_3 \]

(126)
where

\[
D_1 := i\tilde{g}_1 \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} = \frac{\sqrt{2g}}{\sqrt{3}g_1} \hat{H}_1 + \frac{\sqrt{2g} \epsilon}{\sqrt{3}g_1} \hat{H}_2 + \frac{2g}{\sqrt{3}g_1} \hat{H}_3 ,
\]

\[
D_2 := i\tilde{g}_1 \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} = \frac{\hat{g}(\epsilon + 1)}{\sqrt{6}g_1} \hat{H}_1 + \frac{\hat{g}(\epsilon - 3)}{\sqrt{6}g_1} \hat{H}_2 + \frac{\hat{g}(\epsilon + 1)}{\sqrt{3}g_1} \hat{H}_3 ,
\]

\[
D_3 := i\tilde{g}_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \frac{\hat{g}(\epsilon - 1)}{\sqrt{6}g_1} \hat{H}_1 - \frac{\hat{g}(\epsilon + 3)}{\sqrt{6}g_1} \hat{H}_2 + \frac{\hat{g}(\epsilon - 1)}{\sqrt{3}g_1} \hat{H}_3 .
\]

An orthogonal transformation

\[
\hat{H}_1' := -\sqrt{\frac{1}{3}} \hat{H}_3 - \sqrt{\frac{2}{3}} \hat{H}_1 ; \quad \hat{H}_3' := \sqrt{\frac{2}{3}} \hat{H}_3 - \sqrt{\frac{1}{3}} \hat{H}_1
\]

leads to

\[
\hat{H}_1' = \frac{-\hat{g}}{3\sqrt{2}} \begin{pmatrix}
-\epsilon - 3 & 0 & 0 \\
0 & -\epsilon + 3 & 0 \\
0 & 0 & 2\epsilon
\end{pmatrix} ; \quad \hat{H}_3' = \frac{\hat{g}}{3} \begin{pmatrix}
\epsilon & 0 & 0 \\
0 & \epsilon & 0 \\
0 & 0 & \epsilon
\end{pmatrix} .
\]

with \(g(\hat{H}_1', \hat{H}_1') = 1/2\) and \(g(\hat{H}_3', \hat{H}_3') = 1/2\) as expected.

In this basis, \(D_{SU(3)}|g_2\rangle \rightarrow D_{SU_C(3)}\) but the brackets \([S^+_i, S^-_j]\) have \(D_1 \sim \alpha_i \hat{H}_1' + \beta_i \hat{H}_2\) while the inner products \(g(\hat{H}_1', \hat{H}_1') = g(\hat{H}_2', \hat{H}_3') = 0\). Hence, the strong photon completely decouples from the rest of the gauge fields and the symmetry group reduces to a direct product \(SU_C(3) \times U_S(1)\). Conclude that normalizing (105) destroys the semi-direct product group structure of \(U_S(3)\).

### A.4 Trans-representation and anomaly cancellation

This subsection offers a possible scenario of how SSB and trans-representation might unfold.

Assume the starting symmetry group is \(U(4, \mathbb{C})\). Immediately following the conjectured symmetry breaking \(U(4, \mathbb{C}) \rightarrow U_S(3) \times U_{EW}(2)\), we propose the matter field content consists of \((3, 2), (3, 2^*), (\bar{3}, \bar{2})\), and \((\bar{3}, \bar{2}^*)\). Suppose, then, the \(U_{EW}(2)\) environment induces trans-representation \((3, 2) \rightarrow (3, 2)_L \oplus (3, 1^+_R) \oplus (3, 1^0_R)\) where the subscript denotes chirality and the superscript denotes electric charge. Subsequently, the \(U_S(3)\) environment induces \((3, 2)_L \oplus (3, 1^+_R) \oplus (3, 1^0_R) \rightarrow (1, 2)_L^0 \oplus (1, 1^+_R) \oplus (1, 1^0_R)\) where the superscript outside the parentheses indicates strong charge \(e_S\). There are analogous representations for the other representations. As discussed in subsection 4.1, they are associated with the scattering operators \(S_0, S_1, S_2, S_3\).

Recall that \(S_\alpha\) encodes \(U_S(3)\) coupling to its \(U_{EW}(2)\) environment and vice versa for \(S_\sigma\), and they are responsible for the transitions

\[
S_0 \xrightarrow{S_\sigma} S_2 \xrightarrow{S_\alpha} S_3
\]

\[
S_0 \xrightarrow{S_\alpha} S_1 \xrightarrow{S_\sigma} S_3 .
\]

55
Clearly $S_1$ acts on the quark representation while $S_3$ acts on the lepton representation. It is tempting to think that $S_2$ governs new particle types appearing somewhere above the EW scale and likewise for $S_0$ near the scale of $U(4, \mathbb{C}) \to U_S(3) \times U_{EW}(2)$. On the other hand, perhaps the transition represented by the top line does not occur.

The idea is that each scattering operator governs a subspace of phase space where only its associated matter field representation is manifest. The only gauge bosons that mediate between these subspaces are the electric photon and the strong photon, because evidently they survive the decoherence. Gauge invariance then rests on anomaly cancellation, and this condition fixes the relative contribution of the matter fields in each subspace to the total Lagrangian. The possible Lagrangian content is tabulated below.

| $S_0$ | $S_1$ | $S_2$ | $S_3$ |
|-------|-------|-------|-------|
| $\kappa^+$ | $(3, 2)_{LR}$, $(3, 1^+)_R$, $(3, 1^0)_R$ | $(1, 2)^{c0}, (1, 2, 0^0)$ | $(1, 2)^{00}, (1, 1^+, 0^0)_R$, $(1, 1^0)_R$ |
| $\kappa^-$ | $(3, 2^c)_{LR}, (3, 1^-)_R$, $(3, 0^0)_R$ | $(1, 2^{c0}), (1, 2^c, 0^0)$ | $(1, 1^0, 0^0)_R$, $(1, 1^0)^0_R$ |
| $\bar{\kappa}^+$ | $(3, 2)_{LR}$, $(3, 1^+)_R$, $(3, 1^0)_R$ | $(1, 2)^{c0}$, $(1, 2, 0^0)$ | $(1, 1^+)^0_R$, $(1, 1^0)^0_R$ |
| $\bar{\kappa}^-$ | $(3, 2^c)_{LR}, (3, 1^-)_R$, $(3, 1^0)_R$ | $(1, 2^{c0})$, $(1, 2^c, 0^0)$ | $(1, 1^0, 0^0)_R$, $(1, 1^0)^0_R$ |

Table 4: Fermion field content for effective Lagrangian densities.

To emphasize; the scale factors $\kappa^+, \kappa^-, \bar{\kappa}^+, \bar{\kappa}^-$ (possibly different in respective phase space regions) are partly determined by anomaly cancellation, and not all terms in the table necessarily contribute to the Lagrangian.

Let’s write down matter field contributions for each phase space region. For $S_0$, let $H^{++}$ represent an element in the $(3, 2)$ representation and $D^{++}_0$ its associated covariant derivative. The superscripts refer to the sign of the strong and electric charges (in that order). Assuming there are $s$ fermion generations,

$$S_0 : \sum_s \kappa_0^+ \overline{H}_s^{++} D^{++}_0 H^{++}_s + \bar{\kappa}_0^- \overline{H}_s^{-} D_0^- H^{-}_s$$

$$+ \kappa_0^+ \overline{H}_s^{-} D_0^+ H^{++}_s + \kappa_0^+ \overline{H}_s^{++} D_0^- H^{-}_s$$

$$= \sum_s \overline{H}_s^{++} D^{++}_0 H^{++}_s + \overline{H}_s^{-} D_0^- H^{-}_s$$

$$+ \overline{H}_s^{-} D_0^+ H^{++}_s + \overline{H}_s^{++} D_0^- H^{-}_s.$$  \hspace{1cm} (133)

A real-valued Lagrangian density requires $\kappa_0^+ = \bar{\kappa}_0^-$ and $\kappa_0^- = \bar{\kappa}_0^+$. The second equality in (133) follows because the $U_S(3)$ complex conjugate representations are inequivalent so we can arrange for $\kappa_0^+ = \bar{\kappa}_0^- = 1$ by suitable normalization of the carrying vector spaces.
The iquark contribution is

$$S_1: \sum_s \kappa_1^+ \left( H_{L,s}^+ \varphi_0^{++} H_{L,s}^{++} + h_{R,s}^+ \varphi_1^{++} h_{R,s}^{++} + \xi_{R,s}^+ \varphi_1^0 \xi_{R,s}^{0+} \right)$$

$$+ \kappa_1^- \left( H_{L,s}^- \varphi_0^{--} H_{L,s}^{--} + h_{R,s}^- \varphi_1^{--} h_{R,s}^{--} + \xi_{R,s}^- \varphi_1^0 \xi_{R,s}^{0-} \right)$$

$$+ \kappa_1^0 \left( H_{L,s}^{0+} \varphi_0^{0+} H_{L,s}^{0+} + h_{R,s}^{0+} \varphi_1^{0+} h_{R,s}^{0+} + \xi_{R,s}^{0+} \varphi_1^0 \xi_{R,s}^{0+} \right)$$

(134)

where $\varphi_0^{+\pm} = tr_{U_{EW}(2)} \varphi_0^{+\pm}$ and $\varphi_1^{0\pm}$ is the restriction of $\varphi_0^{+\pm}$ to the trivial representation of $U_{EW}(2)$. Except for the contribution of the ninth gluon in the covariant derivatives and the representation strong charge, this is the same as the iquark Lagrangian for $SU_C(3) \times U_{EW}(2)$.

Re-scaling the representation bases leaves the undetermined ratios $\kappa_1^+ / \kappa_1^-$ and $\kappa_1^+ / \kappa_1^-$ which are fixed by $\kappa_1^+ + \kappa_1^- = 1$, $\kappa_1^+ + \kappa_1^- = 1$, and anomaly cancellation.

Next, again a real-valued Lagrangian density implies $\kappa_2^{+\pm} = \tilde{\kappa}_2^{+\pm}$ and inequivalent complex conjugate representations imply $\kappa_2^{\pm\pm} = \tilde{\kappa}_2^{\pm\pm}$ so

$$S_2: \sum_s \left( h_s^+ \varphi_2^{++} h_s^{++} + \overline{h_s^+} \varphi_2^{0+} h_s^{0+} + \overline{h_s^+} \varphi_2^{++} h_s^{++} + \overline{h_s^+} \varphi_2^{0+} h_s^{0+} \right)$$

$$+ \left( h_s^- \varphi_2^{--} h_s^{--} + \overline{h_s^-} \varphi_2^{0-} h_s^{0-} + \overline{h_s^-} \varphi_2^{--} h_s^{--} + \overline{h_s^-} \varphi_2^{0-} h_s^{0-} \right)$$

(135)

where $\varphi_2^{+\pm} = tr_{U_{S}(3)} \varphi_2^{+\pm}$, and $\varphi_2^{0\pm}$ (which contains no $U_S(3)$ generator) is the covariant derivative in the trivial $U_S(3)$ representation. Again we absorbed $\kappa_2^{+\pm}$ into the $U_S(3)$ complex conjugate normalizations.

Finally, the lepton contribution is (there is no need to distinguish between $\kappa$ and $\tilde{\kappa}$)

$$S_3: \sum_s \kappa_3^+ \left( h_{L,s}^+ \varphi_3^{0+} h_{L,s}^{0+} + h_{R,s}^+ \varphi_3^{0+} h_{R,s}^{0+} + \xi_{R,s}^0 \varphi_3^0 \xi_{R,s}^{0+} \right)$$

$$+ \kappa_3^- \left( h_{L,s}^- \varphi_3^{0-} h_{L,s}^{0-} + h_{R,s}^- \varphi_3^{0-} h_{R,s}^{0-} + \xi_{R,s}^0 \varphi_3^0 \xi_{R,s}^{0-} \right)$$

(136)

where $\varphi_3^- = tr_{U_{EW}(2)} \varphi_3^{0-}$ and $\varphi_3^0 = \varphi$. Experiment dictates $\kappa_3^- = 1$ and $\kappa_3^+ = 0$ so

$$S_3: \sum_s \overline{h_{L,s}^-} \varphi_3^{0-} \overline{h_{L,s}^-} + \overline{h_{R,s}^-} \varphi_3^{0-} \overline{h_{R,s}^-} + \overline{\xi_{R,s}^0} \varphi_3^0 \overline{\xi_{R,s}^0}$$

(137)

Given this matter field content, we need to verify there are no anomalies. The $S_0$ and $S_2$ sectors exhibit chiral symmetry so they are safe. For the remaining $S_1$ and $S_3$ sectors, we already have the $U_{EM}(1)$ case settled; no anomaly if $\kappa_1^+ = \kappa_1^- = 2/3$ and $\kappa_1^+ = \kappa_1^- = 1/3$.

Moving on to the $U_S(1)$ anomaly, the $S_3$ sector furnishes the trivial representation of $U_S(3)$ so its matter fields carry no strong charge. For the $S_1$ sector, $\kappa_1^+ = \kappa_1^-$ and $\kappa_1^- = \kappa_1^-$ ensures that the anomaly contributions of oppositely $e_S$-charged $U_{EW}(2)$ left-handed doublet and right-handed singlet iquarks separately cancel — so no anomalies.
A.5 Running couplings

Keeping in mind that the beta functions in (96) are just approximations, we want to use the associated coupled RGEs to learn what kind of behavior we might expect from $\tilde{\alpha}_1(Q)$ and $\tilde{\alpha}_2(Q)$. The coupled equations are

$$
\mu \frac{d\tilde{\alpha}_S(\mu)}{d\mu} = \frac{2N_f}{3\pi} (\tilde{\alpha}_S^2(\mu))^2 = \frac{2N_f}{3\pi} \left( \frac{1}{4\pi} \left( \frac{\sqrt{\tilde{g}_1(\mu) \tilde{g}_2(\mu)}}{\sqrt{\tilde{g}_1^2(\mu) - \tilde{g}_2^2(\mu)}} \right) \right)^2
$$

$$
= \frac{2N_f}{3\pi} \left( \frac{3}{4\pi \cdot 16} \left( \frac{\tilde{g}_1^2(\mu) \tilde{g}_2^2(\mu)}{\tilde{g}_1^2(\mu) - \tilde{g}_2^2(\mu)} \right) \right)^2
$$

$$
\mu \frac{d\tilde{\alpha}_1(\mu)}{d\mu} = -\frac{11C_A(\tilde{g}_1(\mu), \tilde{g}_2(\mu)) - 4N_f T_F^{eff}(\tilde{g}_1(\mu), \tilde{g}_2(\mu)) - N_H}{6\pi} \left( \frac{\tilde{g}_1^2(\mu)}{4\pi} \right)^2
$$

(138)

where $N_f(N_H)$ is the number of $H^{\uparrow \downarrow}$(Higgs) degrees of freedom at energy $\mu$ and the $\tilde{g}_1, \tilde{g}_2$ dependence of $\tilde{\alpha}_S$ comes from (120). For comparing to QCD, we use

$$
\alpha_{QCD}(\mu) = \frac{.1184}{1 + .1184 \left( \frac{33-2N_f-N_H}{12\pi} \right) \log \left( \frac{\mu^2}{\Lambda^2} \right)}
$$

(139)

with $\Lambda = m_Z$ for $N_f \leq 5$ and then matching at the top quark mass 173 GeV gives $\Lambda = 84$ GeV for $N_f \leq 6$.

We numerically solve (138) using the Mathematica 9.0 routine NDSolve with initial values $\tilde{g}_1(m_Z) = 1.208$ and $\tilde{g}_2(m_Z) = .37$ (to enforce $\tilde{\alpha}_1(m_Z) + \tilde{\alpha}_S(m_Z) = .1184 = \alpha_{QCD}(m_Z)$). The two tables below compare $\tilde{\alpha}_1(\mu) + \tilde{\alpha}_S(\mu)$ against $\alpha_{QCD}(\mu)$ in experimentally accessible energy regions.

| GeV | .5   | .7   | .9   | 1    | 2    |
|-----|------|------|------|------|------|
| $\alpha_{QCD}$ | 1.024 | .6858 | .5500 | .5079 | .3376 |
| $\tilde{\alpha}_1 + \tilde{\alpha}_S$ | 1.022 | .6842 | .5489 | .5068 | .3370 |

Table 5: Iquark v.s. QCD strong coupling for 500 MeV – 2 GeV with $N_f = 3$ and $N_H = 0$.

| GeV | 5    | 10   | 50   | 92   | 200  |
|-----|------|------|------|------|------|
| $\alpha_{QCD}$ | .2044 | .1743 | .1298 | .1184 | .1065 |
| $\tilde{\alpha}_1 + \tilde{\alpha}_S$ | .2045 | .1743 | .1298 | .1184 | .1065 |

Table 6: Iquark v.s. QCD strong coupling for 5 GeV – 200 GeV with $N_f = 5$ and $N_H = 0$. 
Table 7: Iquark v.s. QCD strong coupling for 1 TeV – 10^4 TeV with N_f = 6 and N_H = 1.

Tables A.5–A.7 show \( \tilde{\alpha}_1 + \tilde{\alpha}_S \cong \alpha_{QCD} \) well above the EW scale. But table A.8 below reveals that near 10^5 TeV they begin to separate and \( \tilde{\alpha}_1 + \tilde{\alpha}_S \) gets increasingly larger than \( \alpha_{QCD} \).

Evidently something interesting is happening for \( \mu > 10^{16} \) eV. In this region, we would expect the first wave of transmutations represented in (132) to unfold if that mechanism is valid. It is unclear how many iquark flavors exist throughout this range so in the next table we have simply taken \( N_f = 6 \). There is a “desert” between 10^16 eV and 10^{24} eV where (presumably) the dynamics of transmutation play out, and beyond that \( \tilde{\alpha}_1 + \tilde{\alpha}_S \) diverges rapidly after 10^{26} eV.

Table 8: Iquark v.s. QCD strong coupling for 10^5 TeV – 10^{13} TeV with \( N_f = 6 \) and \( N_H = 1 \).

The final table A.9 shows what happens to \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) above the 1 TeV scale. Since the RGE are coupled, they do not follow linear trajectories; and as expected they asymptotically approach each other with \( \tilde{\alpha}_2 \) gradually rising to meet the decreasing \( \tilde{\alpha}_1 \). But near 10^{26} eV both \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) turn down sharply and eventually decrease at a near vertical slope close to the divergence point of \( \tilde{\alpha}_S \) where they seem to converge to zero.

Table 9: Running of \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) between 1 TeV – 10^{13} TeV with \( N_f = 6 \) and \( N_H = 1 \).

We want to stress that the intention of presenting the numerics in this subsection is not to point to specific numbers — after all, the RGE we used are just approximate. Moreover, the divergence point is very sensitive to the initial values of \( \tilde{g}_1(m_Z) \) and \( \tilde{g}_2(m_Z) \), and it shifts a bit when higher order terms are included. Nevertheless, we expect the trends reflected in the numbers to hold for the exact treatment.

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\(^{36}\)The numerics likely can’t be trusted all the way to zero.
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