Faithful measure of Quantum non-Gaussianity via quantum relative entropy

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We introduce a measure of quantum non-Gaussianity (QNG) for those states not accessible by a mixture of Gaussian states in terms of quantum relative entropy. Specifically, we employ a convex-roof extension using all possible mixed-state decompositions beyond the usual pure-state decompositions. We prove that this approach brings a QNG measure fulfilling the properties desired as a proper monotone under Gaussian channels and conditional Gaussian operations. Furthermore, we explicitly calculate QNG for the noisy single-photon states and demonstrate that QNG coincides with non-Gaussianity of the state itself when the single-photon fraction is sufficiently large.

Introduction—Quantum mechanics provides a profound basis for many distinguished information-processing protocols which cannot be achieved in the classical world, such as quantum computation 1, quantum teleportation 2, and quantum cryptography 3. Such quantum protocols have been developed also using continuous variables (CVs) that can be usually described in terms of quasiprobability distributions like Glauber-Sudarshan P-function or the Wigner function in phase space 4 [5]. A wide range of states such as the coherent and the squeezed states is categorized as the so-called Gaussian states whose quasi-probability distributions take a Gaussian form and whose statistical properties are completely characterized by their first-order moments (amplitudes) and the second-order moments (covariances). Gaussian states and Gaussian operations are widely employed in many CV protocols due to their experimental feasibility in laboratory with their compact mathematical description [6]. Nevertheless, there exist numerous no-go theorems within Gaussian regime, which prevent Gaussian operations from performing several tasks such as universal quantum computation 7 [8], quantum error correction 9, and entanglement distillation 10 [12]. In such tasks, non-Gaussian states and non-Gaussian operations become essential resources.

In this respect, it is of crucial importance to identify quantum non-Gaussian states that cannot be produced by Gaussian resources and their statistical mixtures. Furthermore, it may provide a valuable framework for related studies to characterize quantum non-Gaussianity (QNG) under a proper quantitative measure. In a closely related context, several studies have investigated to quantify non-Gaussianity (NG) of quantum states 13 [15], which only represents the departure of a given state from Gaussian states. In particular, it was shown that relative entropy of NG exhibits important properties, for example, monotonicity under Gaussian channels 16. However, the measure is not convex because the set of Gaussian states is not convex. There indeed exist non-Gaussian states which can be simply generated using Gaussian operations and classical randomness, for example, a mixture of two different coherent states $|\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha|)/2$. These states, a simple mixture of Gaussian states, can be generated without quantum non-Gaussian operations and they are thus not suitable to perform quantum information tasks which require genuinely quantum non-Gaussian resources.

Recently some works have devoted to ruling out Gaussian mixtures and detecting genuinely quantum non-Gaussian states, i.e. $\rho \neq \sum p_i \rho_{G,i}$, where $\rho_{G,i}$ is a Gaussian state. Though a number of criteria have been developed to assess quantum non-Gaussian states 17–26, a faithful measure of quantum non-Gaussianity has not been reported yet. Recent studies in Refs. 27, 28 have remarkably adopted the Wigner negativity as a measure of QNG, which is a monotone under Gaussian protocols including classical mixing. However it is actually not a faithful measure because it cannot detect quantum non-Gaussian states with positive Wigner function, e.g. a highly noisy single-photon state $p|0\rangle|0\rangle + (1-p)|1\rangle|1\rangle$ with $p > 0.5$.

In this work, we propose a convex-roof measure of QNG based on quantum relative entropy. Our QNG measure is faithful because it always gives a positive value whenever a state cannot be described as a Gaussian mixture. We prove that our measure satisfies desirable properties as a proper measure of QNG including convexity, additivity, and monotonicity under Gaussian channels and conditional Gaussian operations. Furthermore, we show the explicit result of evaluating QNG for non-Gaussian mixed states.

NG—To begin with, for a mixed state $\rho$, one may define its non-Gaussianity (NG) in terms of quantum relative entropy with reference to its Gaussified state $\rho_G$ having the same first-order moments (average) and second-order moments (covariance) 14. That is, $N[\rho] \equiv S(\rho |\rho_G)$ where $S(\rho |\sigma) \equiv -Tr(\rho \log \sigma) + Tr(\rho \log \rho)$. $N[\rho]$ is quantum relative entropy. In particular, due to $-Tr(\rho \log \rho_G) = -Tr(\rho_G \log \rho_G)$, we have the relation $S(\rho |\rho_G) = S(\rho_G) - S(\rho)$, which highlights the fact that a Gaussian state among all states with the same covariance matrix takes a maximal entropy leading to the non-
negativity of the defined NG [29].

QNG—We are here interested in quantum non-Gaussianity (QNG) of states, which cannot be represented by a mixture of Gaussian states, namely, $\rho \neq \sum_i p_i \rho_i^G$. There can be several approaches to quantify the degree of QNG and we use the convex-roof extension of NG defined above. For a given state $\rho$, its QNG can be measured as

$$Q[\rho] \equiv \min_{\{\rho_i, p_i\}} \sum_i p_i S(\rho_i | \rho_{i,G})$$

where the minimization is taken over all possible decompositions of $\rho = \sum_i p_i \rho_i$. Note that this generalization includes the usual decomposition into pure-states only, $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, e.g. in [28]. By further allowing decompositions into mixed states, we may obtain a lower degree of QNG for a given state. We will illustrate it later by pointing out a range of noisy single-photon states whose QNG is given by a genuinely mixed-state decomposition.

Properties—We prove the following properties of the above-defined QNG.

N0- QNG is nonnegative.—This is obvious by its definition, as the relative entropies, and thus their average, are nonnegative.

N1-(faithfulness) QNG is strictly positive if and only if the state is not a mixture of Gaussian states. —This can also be readily understood. If $\rho = \sum_i p_i \rho_i^G$, then its QNG is null using the Gaussian component states. On the other hand, if the QNG is zero, it also means that the given state is a mixture of Gaussian states since any single non-Gaussian component state, if any, would give a strictly positive NG, thus a positive QNG.

N2-(convexity) QNG is convex with respect to state mixing, i.e. $Q[\lambda \rho_1 + (1-\lambda) \rho_2] \leq \lambda Q[\rho_1] + (1-\lambda) Q[\rho_2]$.

Proof: Let $\rho_1 = \sum_i p_i \rho_i^G$ and $\rho_2 = \sum_j q_j \sigma_j$ be the decompositions for their respective QNGs. Since $\sum_i \lambda_i \rho_i^G + \sum_j (1-\lambda_j) \sigma_j$ is one possible decomposition of the state $\sum_i \lambda_i \rho_i^G + (1-\lambda) \rho_2$, we have by definition

$$Q[\rho] \equiv \min_{\{\rho_i, p_i\}} \sum_i p_i S(\rho_i | \rho_{i,G})$$

and

$$\sum_j (1-\lambda_j) q_j S(\sigma_j | \sigma_{j,G})$$

$$= \lambda Q[\rho_1] + (1-\lambda) Q[\rho_2].$$

N3- QNG is invariant under Gaussian unitary operations.

Proof: For any fixed decomposition $\rho = \sum_i p_i \rho_i$, a Gaussian unitary operation leads to $\rho' = U_G \rho U_G^\dagger$. We also note that the relative entropy of each component NG is invariant under unitary operation, $S(\rho_i | \rho_{i,G}) = S(U(\rho_i U_i^\dagger) | U(\rho_{i,G} U_i^\dagger))$ and that the Gaussification of state commutes with Gaussian unitary operations. The latter property means that $U_G \rho_{i,G} U_G^\dagger$ is the Gaussified state of $\rho' = U_G \rho_i U_G^\dagger$. Therefore, $\sum_i p_i S(\rho_i | \rho_{i,G})$ is invariant under Gaussian unitary operations and so is QNG.

N4- QNG is not increasing under Gaussian channels.

Proof:

$$Q[\rho] = \min_{\{\rho_i, p_i\}} \sum_i p_i S(\rho_i | \rho_{i,G})$$

$$\geq \sum_i p_i S(\rho^{G} | \rho_{i,G}) = Q[\rho^{G}],$$

where the first inequality is due to the contraction property of relative entropy under an arbitrary quantum channel. Note again that $\rho^{G} | \rho_{i,G}$ is equivalent to the Gaussified state of $\rho_{i,G}$, and that $\sum_i p_i \rho^{G} | \rho_{i,G}$ is one of possible decompositions of $\rho_{i,G}$, which leads to the second inequality.

N5- QNG is not increasing on average under conditional Gaussian maps.

For its proof, we first introduce two preliminary tools.

Preliminary—(1) Takagi and Zhuang in [27] have identified a general conditional Gaussian map as the one attaching an ancillary (multi-mode) vacuum to the system followed by a global unitary Gaussian operation and homodyne detection. The conditional map results from implementing a Gaussian map conditioned on the measurement outcome. That is, with $\rho_{SE} = U_G |0\rangle \langle 0| \otimes \rho_{S'} U_{G'}^\dagger$, we obtain $\rho' = \sum_k |k\rangle \langle k| \otimes \rho_{k'} = \sum_k k |k\rangle \langle k| \otimes \rho_{k'}$, where $\rho_{k'} = (|k\rangle \langle k| \otimes E)_{\rho_{SE}}$ is an orthonormal state conditioned on the homodyne outcome $k$ with $\rho_{k'} = \text{Tr}_E(\rho_{k'})$. The final conditional map reads $\rho'' = \sum_k k |k\rangle \langle k| \otimes E_{\rho_{SE}}(\rho_{k'})$.

Preliminary—(2) For two mixed states $\rho = \sum_j p_j^{(1)} |j\rangle \langle j| \otimes \rho_j^{(1)}$ and $\sigma = \sum_j p_j^{(2)} |j\rangle \langle j| \otimes \sigma_j$ where $|j\rangle$'s are orthonormal states for subsystem $A$, the relative entropy $S(\rho | \sigma)$ turns out to be

$$S(\rho | \sigma) = H(p^{(1)} | p^{(2)}) + \sum_j p_j^{(1)} S(\rho_j | \sigma_j).$$

where $H$ is the Shannon relative entropy. Using these properties, we have the following proof.

Proof: We first note that the QNG of $\rho_{SE} = U_G |0\rangle \langle 0| \otimes \rho_{S'} U_{G'}^\dagger$ is the same as that of $\rho_{SE}$, since neither adding an ancillary Gaussian state nor a unitary Gaussian operation changes QNG. Let $\rho_{SE} = \sum_i p_i \rho_i$ be the decomposition yielding its QNG, i.e. $Q[\rho_{SE}] = \sum_i p_i S(\rho_i | \rho_{i,G})$, where $\rho_i$ belongs to a larger Hilbert space of $\{SE\}$.

We may introduce a further extended state of $\rho_{SE'}$ as $\rho_{SE'} = \sum_i p_i |i\rangle \langle i| \otimes \rho_i$, where $\rho_{SE'} = \text{Tr}_E(\rho_{SE})$ and $|i\rangle$'s are orthonormal basis states for $E'$. With its “Gaussified” version $\sigma_{SE'} = \sum_i p_i |i\rangle \langle i| \otimes \rho_{i,G}$, we have $Q[\rho_{SE}] = S(\rho_{SE} | \sigma_{SE'})$ due to Preliminary (2), that is, expressed in terms of the relative entropy of the total states without decompositions.

Let us now take a homodyne measurement with basis $|k\rangle$ on subsystem $E$ for the two states $\rho_{SE'}$ and $\sigma_{SE'}$. 

We then obtain
\[
\rho_{\text{SEE}} \rightarrow \rho'_{\text{SEE}} = \sum_{i,k} p_i |i⟩⟨i| \otimes |k⟩⟨k| \otimes |ρ_i|k⟩k
\]
\[
= \sum_{i,k} p_i p_{k|i} |i⟩⟨i| \otimes |k⟩⟨k| \otimes \hat{ρ}_{k|i}, \quad (5)
\]
where \(\hat{ρ}_{k|i}\) is the normalized state conditioned on the measurement outcome \(k\) starting with the state \(ρ_i\) and \(p_{k|i} = \text{Tr}[k|ρ_i|k]\) is the corresponding conditional probability. The product \(p_i \hat{ρ}_{k|i}\) defines a joint probability as such. Similarly, we obtain the state after measurement for the system conditioned on the measurement outcome \(k\), i.e. \(ρ_{k|i} \equiv \text{Tr}[k|ρ_i|k]\) is not necessarily the same as \(p_k|k⟩⟨k|\otimes \hat{ρ}_{k|i}\). Nevertheless, with
\[
σ_{\text{SEE}} \rightarrow σ'_{\text{SEE}} = \sum_{i,k} p_i |i⟩⟨i| \otimes |k⟩⟨k| \otimes |ρ_i,G⟩k, \quad (6)
\]
and since a measurement on a partial system is a CP map (its action is actually to eliminate all off-diagonal elements in the subsystem), we have \(S(σ_{\text{SEE}}||σ'_{\text{SEE}}) \geq S(ρ_{\text{SEE}}||ρ'_{\text{SEE}})\). Using the Preliminary (2) again, the latter quantity is given by
\[
H(p_{k|i}|ρ_{k|i}^G) + \sum_{i,k} p_k S(ρ_{k|i,G}) \geq \sum_k p_k S_k. \quad (H \text{ is nonnegative.})
\]
We have here defined \(p_k = \sum_i p_{k|i}\) and \(S_k = \sum_{i,k} p_{k|i} S(ρ_{k|i,G})\).

Noting that \(ρ_k = \frac{1}{p_k} \sum_i p_{k|i}\hat{ρ}_{k|i}\) is the state of system conditioned on the measurement outcome \(k\) on \(E\), we have therefore proved \(Q[ρ] \equiv Q[ρ_{\text{SEE}}] = S(ρ_{\text{SEE}}||σ_{\text{SEE}}) \geq S(ρ'_{\text{SEE}}||ρ'_{\text{SEE}}) \geq \sum_k p_k S_k \geq \sum_{i,k} p_k Q[ρ_i] \geq \sum_{i,k} p_k Q[ρ_i,G]\).

**CASE OF PURE STATES**

If the state is pure, \(ρ = |Ψ⟩⟨Ψ|\), the state itself is the only possible decomposition of it. Therefore, its QNG coincides with its NG, \(Q[ρ] = N[ρ]\). Next we consider the case of mixed states.

**NOISY SINGLE-PHOTON STATE**

We here obtain the QNG of a noisy single-photon state, i.e. \(ρ = |1⟩⟨1| + (1 - p)|0⟩⟨0|\) as follows. To begin with, the NG of a noisy single-photon state in a general form of \(ρ = |1⟩⟨1| + (1 - p)|0⟩⟨0| + re^{iθ}|1⟩⟨1| + re^{-iθ}|1⟩⟨0|\) is readily obtained by
\[
N[ρ] = (\bar{n}_{\text{th}} + 1) \log(\bar{n}_{\text{th}} + 1) - \bar{n}_{\text{th}} \log \bar{n}_{\text{th}} + \lambda_+ \log \lambda_+ + \lambda_- \log \lambda_-,
\]
where \(\bar{n}_{\text{th}} = \sqrt{(\frac{p}{2} + p)(\frac{2}{3} + p^2 - 2r^2)} - \frac{1}{2}\) and \(λ_+ = \frac{1}{2} \pm \sqrt{(\frac{1}{2} - p)^2 + r^2}\) are the thermal photon number of the Gaussified state \(ρ_G\) and the eigenvalues of the state \(ρ\), respectively (See Appendix A for details). Note that the NG of the state \(ρ\) is independent of the phase \(θ\), which is indeed due to the invariance property under Gaussian unitary operations (phase rotation in this case), i.e. \(N[ρ] = N[e^{iθ}ρe^{-iθ}]\).

From the non-Gaussianity in Eq. (7), we may find the minimum of \(N[ρ]\) for all states with a fixed \(p\) as
\[
\mathcal{M}(p) \equiv \min_{r} N[ρ], \quad (8)
\]
which can be obtained by solving
\[
\frac{d}{dr}N[ρ] = 4r \left( \frac{\tanh^{-1}(2λ_+ - 1)}{2λ_+} - \frac{1 + 2p}{2\bar{n}_{\text{th}}} \tanh^{-1} \frac{1}{2\bar{n}_{\text{th}}} \right) = 0,
\]
and comparing the extrema. We plot the minimum \(\mathcal{M}(p)\) and the corresponding optimal parameter \(r_{\text{opt}}\) as a function of \(p\) in Fig. 1. The minimum NG is given by the maximally mixed states (\(r_{\text{opt}} = 0\)) and partially mixed states (\(0 < r_{\text{opt}} < \sqrt{p(1 - p)}\) for \(p > 0.062\) and \(p \leq 0.062\), respectively.

**FIG. 1.** (a) Minimum \(\mathcal{M}(p)\) as a function of the single photon fraction \(p\). (b) Optimal parameter \(r_{\text{opt}}\) for minimum \(\mathcal{M}(p)\).

Using the above result, we obtain the QNG of \(ρ_{p} = p|1⟩⟨1| + (1 - p)|0⟩⟨0|\) as follows. Given a state \(ρ_{p}\), our task is to find a decomposition yielding \(Q[ρ_{p}] = \min_{q_k} \sum_k q_k N[ρ_k]\) among all decompositions \(ρ_{p} = \sum_k q_k ρ_k\). In particular, we let \(ρ_k\) be the state with single-photon fraction \(p_k\) thus satisfying the constraint \(p = \sum_k f_k p_k\). The idea of optimization here is to find values \(p_k\) with the constraint \(p = \sum f_k p_k\) to have a minimum \(\sum f_k M[p_k]\), where \(M[p]\) is the function given in Fig. 1. This optimization actually corresponds to the lower convex envelope of \(\mathcal{M}(p)\) defined by
\[
\tilde{M}(p) = \sup \{ f(p)|f \text{ is convex and } f \leq M \text{ in } [0,1] \},
\]
which is obtained below.

Investigating \(\tilde{M}''(p)\), we find that \(\tilde{M}(p)\) itself is convex on the two intervals \([0,c]\) and \([c,1]\) individually with \(c \approx 0.062\), but not in the whole interval (inset of Fig. 1 (a)). Then, we may construct the lower convex envelope by
finding a tangent line to $\mathcal{M}(p)$ with care. If there exists a solution to the equation

$$y = \mathcal{M}'(p)(p - p') + \mathcal{M}(p') = \mathcal{M}(p),$$  \quad (11)$$

the line is tangent to both intervals together with the condition $\mathcal{M}'(p) = \mathcal{M}'(p')$. Indeed we find the solution $p' \simeq 0.0559$ and $p \simeq 0.0701$, respectively. Therefore, we obtain the QNG of $\rho_a$ as

$$Q[\rho_p] = \begin{cases} 
\mathcal{M}(p) & \text{for } 0 \leq p \leq a, \\
\frac{p - a}{b - a} \mathcal{M}(b) + \frac{b - p}{b - a} \mathcal{M}(a) & \text{for } a \leq p \leq b, \\
\mathcal{M}(p) & \text{for } b \leq p \leq 1,
\end{cases}$$  \quad (12)$$

where $a \simeq 0.0559$ and $b \simeq 0.0701$.

From the above analysis, we can also identify an optimal decomposition of $\rho_p$ straightforwardly. For $p \geq b$ we have $\mathcal{M}(p) = \mathcal{N}[p]$, which means that the state $\rho_p$ itself is the optimal decomposition attaining minimum convex roof QNG. This is a clear example for which the mixed-state decomposition becomes optimal rather than the pure-state decomposition. For $p \leq a$, the equal mixture of two optimal states $\rho^\pm_p = p|1\rangle\langle 1| + (1 - p)|0\rangle\langle 0|$ achieves the bound. For the remaining case, i.e., $b \geq p \geq a$, the optimal decomposition becomes

$$\{\rho^p_a, \rho^p_b, b|1\rangle\langle 1| + (1 - b)|0\rangle\langle 0|\}$$

with the probability distribution $\{\frac{b - p}{b - a}, \frac{1}{2}, \frac{p - a}{b - a}\}$.

**DISCUSSION**

We have proposed a measure of quantum non-Gaussianity based on quantum relative entropy. Specifically, we have introduced a convex-roof extension of non-Gaussianity using all possible mixed-state decompositions beyond the typical pure-state decompositions. This enables us to come up with properties desired as a proper measure of QNG including convexity and monotonicity under Gaussian channels and conditional Gaussian operations. We furthermore studied the case of noisy-single photon state, which is a practically important QNG resource for many applications like linear-optical quantum computation, and identified its QNG rigorously. By doing so, we illustrated that there exist a wide range of state whose QNG is given by a mixed-state decomposition, not a pure-state one, and furthermore that their QNG actually coincides with NG if the single-photon fraction is sufficiently large.

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APPENDIX A: NON-GAUSSIANITY FOR A NOISY SINGLE PHOTON STATE

For a noisy single photon state \( \rho = p|1\rangle\langle 1| + (1 - p)|0\rangle\langle 0| + re^{i\theta}|0\rangle\langle 1| + re^{-i\theta}|1\rangle\langle 0| \), we have \( \langle \hat{a} \rangle = \langle \hat{a}^\dagger \rangle^* = re^{-i\theta}, \langle \hat{a}^2 \rangle = \langle (\hat{a}^\dagger)^2 \rangle = 0 \) and \( \langle \hat{a}\hat{a}^\dagger \rangle = \langle \hat{a}^\dagger\hat{a} \rangle + 1 = p + 1 \) which yield \( \langle \hat{q} \rangle = \sqrt{2}r \cos \theta, \langle \hat{p} \rangle = -\sqrt{2}r \sin \theta, \langle \hat{q}^2 \rangle = \langle \hat{p}^2 \rangle = \frac{1}{2} + p \) and \( \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle = 0 \) where \( \hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \) and \( \hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}} \). The covariance matrix of \( \rho \) is given by

\[
\Gamma = \begin{pmatrix}
\frac{1}{2} + p - 2r^2 \cos^2 \theta & 2r^2 \sin \theta \cos \theta \\
2r^2 \sin \theta \cos \theta & \frac{1}{2} + p - 2r^2 \sin^2 \theta
\end{pmatrix},
\]  

(13)

where the covariance matrix elements are defined as \( \Gamma_{ij} = \frac{1}{2} \langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle \) with \( \hat{x}_1 = \hat{q} \) and \( \hat{x}_2 = \hat{p} \). It determines the quantum entropy of the reference Gaussian state \( \rho_G \) as

\[
S(\rho_G) = (\bar{n}_{th} + 1) \log(\bar{n}_{th} + 1) - \bar{n}_{th} \log \bar{n}_{th},
\]  

(14)

where

\[
\bar{n}_{th} = \sqrt{\det \Gamma} - \frac{1}{2} = \sqrt{\left(\frac{1}{2} + p\right)\left(\frac{1}{2} + p - 2r^2\right)} - \frac{1}{2}.
\]  

(15)

The non-Gaussianity \( NG_\rho \) is given by

\[
NG_\rho = (\bar{n}_{th} + 1) \log(\bar{n}_{th} + 1) - \bar{n}_{th} \log \bar{n}_{th}
\]  

\[
+ \lambda_+ \log \lambda_+ + \lambda_- \log \lambda_-,
\]  

(16)

where \( \lambda_\pm = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2} - p\right)^2 + r^2} \) are the eigenvalues of \( \rho \).