QUANTUM LIOUVILLE THEORY VERSUS QUANTIZED
TEICHMÜLLER SPACES

by

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Abstract. — This note announces the proof of a conjecture of H. Verlinde, according to which
the spaces of Liouville conformal blocks and the Hilbert spaces from the quantization of the Teichmüller
spaces of Riemann surfaces carry equivalent representations of the mapping class group.
This provides a basis for the geometrical interpretation of quantum Liouville theory in its relation to
quantized spaces of Riemann surfaces.

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1. Introduction

Quantum Liouville theory is a crucial ingredient for a variety of models for low dimen-
sional quantum gravity and noncritical string theories. In the case of two dimensional quantum
gravity or noncritical string theories this comes about due to the Weyl-anomaly [1], which
forces one to take into account the quantum dynamics of the conformal factor of the two-
dimensional metric. More recently it was proposed that Liouville theory also plays a crucial
role for three-dimensional quantum gravity in the presence of a cosmological constant in the
sense of representing a holographic dual for this theory, see e.g. [2, 3].

For all these applications it is crucial that the quantum Liouville theory has a geometric
interpretation as describing the quantization of spaces of two-dimensional metrics. Such an
interpretation is to be expected due to the close connections between classical Liouville the-
ory and the theory of Riemann surfaces. Having fixed a complex structure on the Riemann
surface one may represent the unique metric of negative constant curvature locally in the form
ds^2 = e^{2\varphi} dz d\bar{z} where \varphi must solve the Liouville equation \partial \bar{\partial} \varphi = \frac{1}{4} e^{2\varphi}. The relation between
the Liouville equation and the uniformization problem leads to beautiful connections between
classical Liouville theory and the theory of moduli spaces of Riemann surfaces [4].

There has been considerable recent progress on the Liouville quantum field theory as a
conformal field theory in its own right, see [5] and references therein. What was missing so
far is the interpretation of these results as a description of the quantum geometry of Riemann
2. Triangulation of the once-punctured torus.

The results that we want to present here may be seen as representing the “chiral half” of such an interpretation. More precisely, we would like to announce the proof of a conjecture of H. Verlinde, according to which the space of conformal blocks of the Liouville theory with its mapping class group representation is isomorphic to the space of states obtained by quantizing the Teichmüller spaces of Riemann surfaces.

2. Teichmüller spaces

The Teichmüller spaces are the spaces of deformations of the complex structures on Riemann surfaces. As there is a unique metric of constant curvature -1 associated to each complex structure one may identify the Teichmüller spaces with the spaces of deformations of the metrics with constant curvature -1. Coordinates for the Teichmüller spaces can therefore be obtained by considering the geodesics defined by the constant curvature metrics.

A particularly useful set of coordinates was introduced by R. Penner. They can be defined for Riemann surfaces that have at least one puncture. One may assume having triangulated the surface by geodesics that start and end at the punctures. As an example we have drawn in Figure 1 a triangulation of the once-punctured torus. The length of these geodesics will be infinite. In order to regularize this divergence one may introduce one horocycle around each puncture and measure only the length of the segment of a geodesic that lies between the horocycles. Assigning to an edge e its regularized length l_e gives coordinates for the so-called decorated Teichmüller spaces. These are fibre spaces over the Teichmüller spaces which have fibres that parametrize the choices of the “cut-offs” as introduced by the horocycles.

A closely related set of coordinates for the Teichmüller spaces themselves was introduced by Fock. The coordinate z_e associated to an edge e of a triangulation can be expressed in terms of the Penner-coordinates via z_e = l_a + l_c - l_b - l_d, where a, b, c and d label the other edges of the triangles that have e in its boundary as indicated in the left hand side of Figure 2. Instead of triangulations of the Riemann surfaces it is often convenient to consider the corresponding fat graphs, which are defined by putting a trivalent vertex into each triangle and by connecting these vertices such that the edges of the triangulation are in one-to-one correspondence to the edges of the fat-graph.

The Teichmüller spaces carry a natural symplectic form, called Weil-Petersson symplectic form. We are therefore dealing with a family of phase-spaces, one for each topological type of the Riemann surfaces. One of the crucial virtues of the Penner/Fock-coordinates is the fact that
the Weil-Petersson symplectic form has a particularly simple expression in these coordinates. The corresponding Poisson-brackets are in fact constant for Fock’s variables $z_e$,

$$\{z_e, z_{e'}\} = n_{e,e'}, \quad \text{where} \quad n_{e,e'} \in \{0, 1, 2\}. \quad (1)$$

Changing the triangulation amounts to a change of coordinates for the (decorated) Teichmüller spaces. Any two triangulations can be related to each other by a sequence of elementary moves:

![Diagram of an elementary move between two triangulations](image)

**Figure 2.** The elementary move between two triangulations

The change of variables corresponding to the elementary move of Figure 2 is easy to describe:

- $z_a' = z_a - \phi(-z_e)$,
- $z_{e'} = z_{e'} - z_e$,
- $z_b' = z_b + \phi(z_e)$,
- $z_c' = -z_c - \phi(-z_e)$,
- $z_d' = z_d + \phi(z_e)$,
- $z_e' = -z_e$,

where $\phi(x) = \ln(e^x + 1)$. \quad (2)

and all other variables are left unchanged. These transformations generate a groupoid, the Ptolemy groupoid, that may be abstractly characterized by generators and relations: One has a generator $\omega_{ij}$ whenever the triangles labelled by $i$ and $j$ have an edge in common.

The mapping class group $MC_{\Sigma}$ consists of diffeomorphisms of the Riemann surface $\Sigma$ which are not isotopic to the identity. Elements $MC_{\Sigma}$ will map any graph drawn on the surface $\Sigma$, in particular any triangulation of $\Sigma$, into another one. Since any two triangulations can be connected by a sequence of elementary moves one may represent any element $MC_{\Sigma}$ by the corresponding sequence of flips. It is extremely useful to think of the algebraically complicated mapping class group $MC_{\Sigma}$ as being embedded into the Ptolemy groupoid.

Regarding the Teichmüller spaces as a collection of phase-spaces naturally leads one to look for suitable Hamiltonians. A natural choice is associated to pants decompositions of the Riemann surfaces. In the case of a Riemann surface $\Sigma_g^s$ of genus $g$ with $s$ boundary components this will be a collection of $3g - 3 + s$ closed geodesics $c_i$ on the surface such that cutting $\Sigma_g^s$ along $c_i$, $i = 1, \ldots, 3g - 3 + s$ decomposes it into a collection of three-holed spheres. The collection of lengths $l_i$ of the geodesics $c_i$ furnishes a set of functions on the Teichmüller space $T(\Sigma_g^s)$ that Poisson-commute with each other. $\{l_i, l_j\} = 0$. It is possible, but nontrivial to represent the lengths $l_i$ as polynomials in the variables $e^{\pm z_e/2}$. It seems natural to take $\{l_1, \ldots, l_{3g-3+s}\}$ as the set of Hamiltonians to turn $T(\Sigma_g^s)$ into an integrable system.

Having chosen the $\{l_1, \ldots, l_{3g-3+s}\}$ as the set of “action”-variables, it is amusing to note that the corresponding “angle”-variables are nothing but the twist-angles corresponding to the deformation of cutting $\Sigma_g^s$ along $c_i$ and twisting by some angle $\phi_i$ before gluing back along
The set \( \{ l_1, \ldots, l_{3g-3+s}, \varphi_1, \ldots, \varphi_{3g-3+s} \} \) gives another set of coordinates on the Teichmüller space \( T(\Sigma_g) \), the classical Fenchel-Nielsen coordinates. These coordinates are associated to trivalent graphs on Riemann surfaces which are such that the edges that do not end in some boundary component of \( \Sigma_g \) are cut by exactly one of the geodesics \( c_i \). The same type of graphs is used in the Moore-Seiberg formalism for conformal field theories \cite{12} to label spaces of conformal blocks. We will therefore call them Moore-Seiberg graphs. Again it is natural to consider the groupoid of coordinate changes generated by the transition from one Moore-Seiberg graph to another. A set of generators is pictorially represented in Figure 3.

This set of elementary moves generates a groupoid that will be called the Moore-Seiberg groupoid. The set of relations for the Moore-Seiberg groupoid is more complicated than the one for the Ptolemy groupoid \cite{12,13}.

3. Quantization of Teichmüller spaces

3.1. Algebra of observables and Hilbert space. — The simplicity of the Poisson brackets \eqref{1} makes part of the quantization quite simple. To each edge of a triangulation of a Riemann surface \( \Sigma_g \) associate a quantum operator \( z_e \). The algebra of observables \( A(\Sigma_g) \) will be the algebra with generators \( z_e \), relations

\[
[z_e, z_{e'}] = 2\pi i b^2 \{ z_e, z_{e'} \},
\]

and hermiticity assignment \( z_e^\dagger = z_e \). The algebra \( A(\Sigma_g) \) has a center with generators \( c_k \), \( k = 1, \ldots, s \) defined by \( c_k = \sum_{e \in E_k} z_e \) where \( E_k \) is the set of edges in the triangulation that emanate from the \( k \)th boundary component. Geometrically one may interpret \( l_k \) as the geodesic length of
the $k^{th}$ boundary component $[7][7]$. The representations of $\mathcal{A}(\Sigma_k^g)$ that we are going to consider will therefore be such that the generators $c_k$ are represented as the operators of multiplication by real positive numbers $l_k$. The tuple $\Lambda = (l_1, \ldots, l_5)$ of lengths of the boundary components will figure as a label of the representation $\pi(\Sigma_k^g, \Lambda)$ of the algebra $\mathcal{A}(\Sigma_k^g)$.

To complete the definition of the representation $\pi(\Sigma_k^g, \Lambda)$ by operators on a Hilbert space $\mathcal{H}(\Sigma_k^g)$ one just needs to find linear combinations $q_1, \ldots, q_{3g-3+s}$ and $p_1, \ldots, p_{3g-3+s}$ of the $z_e$ that satisfy $[q_i, p_j] = 2\pi ib^2 \delta_{ij}$. The representation of $\mathcal{A}(\Sigma_k^g, \Lambda)$ on $\mathcal{H}(\Sigma_k^g) := L^2(R^{3g-3+s})$ is then defined by choosing the usual Schrödinger representation for the $q_i, p_i$.

Let us discuss the example of a sphere with four holes. We shall consider the fat graph drawn in Figure 4 above. The algebra $\mathcal{A}(\Sigma_4^0)$ has six generators $z_i$ $i = 1, \ldots 6$ with nontrivial relations $[z_i, z_j] = 2\pi ib^2$ for

$$ (i, j) \in \{ (1, 2) , (1, 6) , (2, 3) , (2, 5) , (3, 4) , (3, 5) , (4, 1) , (4, 6) , (5, 2) , (5, 1) , (6, 3) , (6, 4) \}. $$

The four central elements corresponding to the holes in $\Sigma_{0,4}$ are

$$ c_1 = z_4 + z_6, \quad c_2 = z_1 + z_3 + z_5 + z_6, $$$$ c_3 = z_2 + z_5, \quad c_4 = z_1 + z_2 + z_3 + z_4. $$

After fixing the lengths of the four holes one is left with two variables, say $q \equiv z_3$ and $p \equiv z_5$. Choosing the Schrödinger representation for $q, p$ one simply finds $\mathcal{H}(\Sigma_4^0) \simeq L^2(\mathbb{R})$.

### 3.2. Representation of the Ptolemy groupoid. —

The first task is to find the quantum counterpart of the change of variables corresponding to the elements of the Ptolemy groupoid. A handy formulation of the solution [7][9] can be given in terms of the Fock-variables: The change of variables corresponding to the elementary move depicted in Figure 3 is given by

$$ z_a^\prime = z_a - \phi_a(-z_e), \quad z_b^\prime = z_b + \phi_b(+z_e), $$$$ z_c^\prime = -z_e, \quad z_c^\prime = -z_e. $$

This is not completely obvious, though: The Penner coordinates are defined only for punctures, corresponding to $l_e \equiv 0$. However, Fock’s construction [4] of the variables $z_e$ also works for surfaces with geodesic boundary, in which case the $l_e$ indeed measure the lengths of its components.
where the special function $\phi_b(x)$ is defined as

$$
\phi_b(z) = \frac{\pi b^2}{2} \int_{i0}^{i\infty} dw \frac{e^{-izw}}{\sinh(\pi w) \sinh(\pi b^2 w)}.
$$

(7)

$\phi_b(x)$ represents the quantum deformation of the classical expression given in (2). The formulae (6) define a representation of the Ptolemy groupoid by automorphisms of $A(\Sigma_g^e)$, see [9].

Let us recall that the mapping class group $MC_g^e$ can be embedded into the Ptolemy groupoid. Having realized the latter therefore gives a representation of $MC_g^e$ by automorphisms of $A(\Sigma_g^e)$. However, it turns out that one has to deal with subtleties related to projective phases if one wants to find a consistent representation of the mapping class by operators on $H(\Sigma_g^e)$. An elegant solution to this problem was given by Kashaev in [8]. It uses an enlarged set of variables, where pairs of variables are associated to the $2M = 4g - 4 + 2s$ triangles of a triangulation instead of its edges. The reduction to $(A(\Sigma_g^e), \mathcal{H}(\Sigma_g^e))$ can be described with the help of a simple set of constraints [8]. Kashaev’s formalism is particularly useful to control the projective phases in the relations of the mapping class group [14]. It turns out that the mapping class group is realized only projectively [14].

### 3.3. The length operators. —

Our aim is to make contact with the Liouville conformal field theory. In the Moore-Seiberg formalism for conformal field theories one considers bases for the space of conformal blocks that are associated to pants decompositions of Riemann surfaces. Our task may be seen as a quantum counterpart of the task to construct the change of variables from the Penner- to the Fenchel-Nielsen coordinates.

The first problem to address is of course the construction of quantum counterparts for the geodesic length functions on Teichmüller space. For a subset of the closed geodesics $\gamma$ on a Riemann surface $\Sigma$ one may find a fat graph $\Gamma$ that in an annular neighborhood $N$ of $\gamma$ looks as drawn in Figure 5. The subset of closed geodesics for which this is possible includes all closed geodesics in a Riemann surface of genus zero and all non-separating cycles [16]. In this case one finds according to [7] an expression for the hyperbolic cosine of the geodesic length function that is easy to quantize,

$$
L_\gamma = 2 \cosh 2\pi bp + e^{2\pi bx}, \quad 2\pi bx = \frac{1}{2}(z_a - z_b), \quad 2\pi bp = \frac{1}{2}(z_a + z_b).
$$

(8)

**Figure 5.** Neighborhood $N$ and the part of the fat graph $\Gamma$ contained in $N$
This finite difference operator can be shown to be self-adjoint and to have a nondegenerate spectrum given by the interval $(2, \infty)$ [15]. It follows that there exists a self-adjoint operator $l_\gamma$ with spectrum $\mathbb{R}_+$ such that $L_\gamma = 2 \cosh \frac{1}{2} l_\gamma$. $l_\gamma$ is the quantum operator corresponding to the hyperbolic length around the geodesic $\gamma$.

The representation of $L_\gamma$ in terms of the Fock variables associated to another fat graph $\Gamma'$ can become complicated. It has to be found by applying the automorphism of $A(\Sigma'_g)$ that corresponds to the element of the Ptolemy groupoid which relates $\Gamma$ and $\Gamma'$. As an example let us quote the expressions for the operators $L_s$ and $L_t$ representing the lengths of the geodesics isotopic to the s- and t-cycles drawn in Figure 3 respectively.

$$L_s = 2 \cosh \left( p + \frac{1}{2}(c_4 - c_3) \right) + e^{-\frac{1}{2}q} \left[ 2 \cosh \frac{1}{2} \left( p + \frac{1}{2}(c_1 + c_4 - c_3) \right) \right] 2 \cosh \frac{1}{2} q \right] e^{-\frac{1}{2}q}$$

$$L_t = 2 \cosh \left( p - \frac{1}{2}(c_3 - c_2) \right) + e^{\frac{1}{2}q} \left[ 2 \cosh \frac{1}{2} \left( p - \frac{1}{2}(c_1 + c_3 - c_4) \right) \right] 2 \cosh \frac{1}{2} (p - c_3) \right] e^{\frac{1}{2}q}.$$  

(9)

Complete sets of eigenfunctions for the operators $L_s$ and $L_t$ were found in [17]. They will be denoted by

$$\Psi_l^{(l_i, l_j)}(q) \quad \text{and} \quad \Psi_l^{(l_i, l_j)}(q)$$

respectively.

(10)

It will be shown [16] that the definition of the length operators $l_\gamma$ can be extended to arbitrary closed geodesics $\gamma$ such that

- $l_\gamma$ is self-adjoint with spectrum $\mathbb{R}_+$,
- $[l_\gamma, l_\gamma] = 0$ if $\gamma \cap \gamma' = \emptyset$.

Moreover, diagonalization of the length operator $l_\gamma$ for a closed geodesic $\gamma \subset \Sigma$ leads to a factorization of $\pi(\Sigma, \Lambda)$ in the following sense. Let $\Sigma'_\gamma = \Sigma \setminus \gamma$ be the possibly disconnected Riemann surface obtained by cutting along $\gamma$. The coloring $\Lambda$ of the boundary components of $\Sigma$ can be naturally extended to a coloring $\Lambda'_\gamma$ for $\Sigma'_\gamma$ by assigning the number $l \in \mathbb{R}_+$ to the two new boundary components that were created by cutting along $\gamma$. The spectral representation for $l_\gamma$ then yields the following representation for $\pi(\Sigma, \Lambda)$.

$$\pi(\Sigma, \Lambda) \simeq \int_{\mathbb{R}_+} dl \pi(\Sigma'_\gamma, \Lambda'_\gamma).$$

(11)

The corresponding representations of the mapping class group factorize/ restrict accordingly [16]. This allows one to construct bases (in the sense of generalized functions) for $\mathcal{H}(\Sigma)$ labelled by the assignments of lengths to the closed geodesics $c_1, \ldots, c_{3g-3+g}$ that define a pants decomposition.

3.4. Realization of the Moore-Seiberg groupoid. — Thanks to the factorization properties of the quantized Teichmüller spaces one may indeed construct a realization of the Moore-Seiberg groupoid by associating unitary operators to the elementary moves depicted in Figure 3 [16].
In the case of the A-move, the unitary operator will simply be the one that describes the change of basis between the eigenfunctions of the length operators for s- and t-cycles of the four-holed sphere (see Figure 3) respectively. The unitary operator representing the A-move has matrix elements $F_{ll'}$ given by the overlap of $\Psi_l$ and $\Psi_{l'}$:

$$F_{ll'}^{[l_1 l_2 l_3]} = \int dq \frac{\Psi_l^{[l_1 l_2 l_3]}(q)}{\Psi_{l'}^{[l_1 l_2 l_3]}(q)}.$$

The matrix elements $F_{ll'}$ essentially coincide with the b-Racah-Wigner coefficients for the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ studied and calculated in [17].

A realization of the B-move in terms of the generators of the Ptolemy groupoid was already discussed in [15], where it was found to be related to the R-operator for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that was proposed by Faddeev in [18]. It turns out [19] that the corresponding operator becomes diagonal in a basis that diagonalizes the three length operators $l_i$, $i = 1, 2, 3$ corresponding to the boundaries of the three-holed sphere with eigenvalues $l_i$. A quick way to find the expression for the eigenvalue $B(l_1, l_2, l_3)$ of the operator that represents the B-move is to combine the observation from [15] with [19, Theorem 6]. The result is

$$B^T(l_1, l_2, l_3) = \exp(i\pi(\Delta_1 - \Delta_2 - \Delta_3)), \quad \Delta_i = \frac{1}{4b^2} \left(1 + b^2\right)^2 + \left(\frac{l}{2\pi}\right)^2. \quad (13)$$

The data $(F^T, B^T)$ turn out [19] to be sufficient to construct a projective representation of the Moore-Seiberg groupoid on the quantized Teichmüller spaces $\mathcal{T}_g^T$. The expression for the generator of the S-move in terms of $(F^T, B^T)$ is similar to the one that was found for rational conformal field theories in [20].

4. Relation to Liouville theory

The spectrum of quantum Liouville theory can be represented as follows:

$$\mathcal{H}^L \simeq \int_{\mathbb{S}^1} d\alpha \ \mathcal{V}_\alpha \otimes \check{\mathcal{V}}_\alpha, \quad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+. \quad (14)$$

In (14) we used the notation $\mathcal{V}_\alpha$, $\check{\mathcal{V}}_\alpha$ for the unitary highest weight representations of the Virasoro algebras which are generated from the modes of the holomorphic and anti-holomorphic parts of the energy-momentum tensor respectively. The central charge of the representations is given by the parameter $b$ via $c = 1 + 6Q^2$, $Q = b + b^{-1}$, and the highest weight of the representations $\mathcal{V}_\alpha$, $\check{\mathcal{V}}_\alpha$ is parametrized as $\Delta_\alpha = \alpha(Q - \alpha)$.

Quantum Liouville theory in genus zero may be characterized by the set of n-point functions $V_\alpha(z, \bar{z})$ which are the quantized exponential functions $e^{2\pi q(z, \bar{z})}$ of the Liouville field. The construction [5] of n-point functions $\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_n}(z_n, \bar{z}_n) \rangle$ can be conveniently formulated within the Moore-Seiberg formalism. A basis for the space of Liouville conformal blocks can be associated to each pants decomposition of the n-punctured sphere. The elements of this basis are constructed by taking matrix elements of compositions of chiral vertex operators [5]. A labelling of the elements of such a basis will be obtained by assigning numbers $\alpha_i \in \frac{Q}{2} + i\mathbb{R}^+$ to the curves $c_i$ that define the pants decomposition. Changing
the pants decomposition simply amounts to a change of basis in the space of conformal blocks. The corresponding transformations can again be constructed out of the representatives for the elementary A-moves and B-moves depicted in Figure 3. These moves are represented in terms of the fusion coefficients $F_{\alpha \alpha}^{L_{\alpha \alpha} \alpha}$ and the elementary braiding phase $B_{\alpha \alpha}^{L_{\alpha \alpha} \alpha}$. The explicit expressions for the data $(F^L, B^L)$ were determined in [5]. We would like to emphasize that the definition of the data $(F^L, B^L)$ involves nothing but the representation theory of the Virasoro algebra.

The main observation to be made is the following. Upon choosing a natural normalization for the Liouville conformal blocks one finds

$$(F^T, B^T) \equiv (F^L, B^L) \quad \text{provided that} \quad \alpha_l = Q + i \frac{l}{4\pi b}. \quad (15)$$

It follows that the spaces of conformal blocks of Liouville theory and the Hilbert spaces from the quantization of the Teichmüller spaces are indeed isomorphic as representations of the mapping class group. So far the conformal blocks of Liouville theory were only constructed in genus zero, but our results also imply that the corresponding mapping class group representation has a consistent and essentially unique extension to higher genus. Moreover, the geodesic length $l$ of a closed geodesic on the Riemann surface $\Sigma$ is indeed directly related to the representation $V_{\alpha}$ that “flows through that cycle” as was anticipated in [6].

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