A ZASSENHAUS-TYPE ALGORITHM SOLVES THE BOGOLIUBOV RECURSION

KURUSCH EBRAMI-FARD AND FRÉDÉRIC PATRAS

Abstract. This paper introduces a new Lie-theoretic approach to the computation of counterterms in perturbative renormalization. Contrary to the usual approach, the devised version of the Bogoliubov recursion does not follow a linear induction on the number of loops. It is well-behaved with respect to the Connes–Kreimer approach: that is, the recursion takes place inside the group of Hopf algebra characters with values in regularized Feynman amplitudes. (Paradigmatically, we use dimensional regularization in the minimal subtraction scheme, although our procedure is generalizable to other schemes.) The new method is related to Zassenhaus’ approach to the Baker–Campbell–Hausdorff formula for computing products of exponentials. The decomposition of counterterms is parametrized by a family of Lie idempotents known as the Zassenhaus idempotents. It is shown, inter alia, that the corresponding Feynman rules generate the same algebra as the graded components of the Connes–Kreimer β-function. This further extends previous work of ours (together with José M. Gracia-Bondía) on the connection between Lie idempotents and renormalization procedures, where we constructed the Connes–Kreimer β-function by means of the classical Dynkin idempotent.

PACS 2006: 03.70.+k; 11.10.Gh; 02.10.Hh; 02.10.Ox

Keywords: renormalization; Bogoliubov recursion; Lie idempotents; Zassenhaus idempotent; β-function; dimensional regularization; free Lie algebra; Hopf algebra; Rota–Baxter relation; Dynkin operator

Contents

Introduction
1. Rota–Baxter algebras and the Bogoliubov recursion
2. A Lie theoretic decomposition of characters
3. The Zassenhaus recursions and the descent algebra
4. On counterterms and renormalized characters
Acknowledgments
References

Introduction

Renormalization in perturbative quantum field theory (pQFT) is needed because amplitudes \( U(\Gamma) \) associated to Feynman graphs \( \Gamma \) are plagued by ultraviolet (UV) divergences. These divergences in general demand a regularization prescription, where by introducing extra parameters, the amplitudes \( U(\Gamma) \) become formally finite. For instance, in dimensional regularization (DR) they become Laurent series with a polar part encoding the UV divergences.

After regularization, a renormalization scheme has to be chosen. In general, the latter is characterized by an operator, here denoted by \( R \), which implements the extraction of the
UV divergence of a regularized Feynman graph amplitude. In the context of DR the minimal subtraction scheme operator $R$ picks out the polar part of the Laurent series expansion of the amplitude $U(\Gamma)$ associated to a graph $\Gamma$ in the neighbourhood of the physical space-time dimension of the theory. In the BPHZ renormalization prescription, the map $R$ picks out the terms in the Taylor expansion of the Feynman amplitude about zero momentum, usually up to an order determined by the overall (superficial) degree of divergence of the corresponding Feynman graph.

The naive procedure that would consist in defining the renormalized amplitude $U_{\text{ren}}(\Gamma)$ of a Feynman graph $\Gamma$ by subtracting $R(U(\Gamma))$, works only up to 1-loop order. Otherwise it leads to non-physical quantities: in particular, locality could be violated. The correct definition of the renormalized amplitude requires the preliminary treatment of the subdivergences associated to the subgraphs of a given Feynman graph (associated to a given theory). The combinatorics of the renormalization process is encoded in the Bogoliubov preparation map and the Bogoliubov recursion [1, 2, 14, 22, 24].

That recursion is the general subject of the present article. As in the previous article [6], we disclose further properties of renormalization schemes by methods inspired from the classical theory of free Lie algebras (FLAs). Our results build on Kreimer’s discovery of a Hopf algebra structure underlying the process of perturbative renormalization [12], and also on the Connes–Kreimer decomposition of Feynman rules [3] à la Birkhoff–Wiener–Hopf (BWH). Here we further exploit the interpretation of Feynman amplitudes as regularized Hopf algebra characters in terms of a Lie-theoretic version of Bogoliubov’s recursion. Indeed, the recursion now takes place inside the group of Hopf algebra characters (instead, as it is the case for the original Bogoliubov recursion, in the much larger space of linear functions on the set of forests of the theory). We show that, as Hopf algebra characters, the counterterm and the renormalized amplitude split into components naturally associated to a remarkable series of Lie idempotents, known as Zassenhaus idempotents, as well as to another closely related series, baptized accelerated Zassenhaus idempotents. The first series of idempotents, introduced by Krob, Leclerc and Thibon [13] in the setting of noncommutative symmetric functions, has been more recently studied by Duchamp, Krob and Vassilieva [5]. Their combinatorial and Lie-theoretic properties can be regarded as reflecting the Zassenhaus formula familiar from numerical analysis and physics applications. (In linear differential equations theory, similar to the Magnus expansion [15, 17], which is the continuous analogue of the Baker–Campbell–Hausdorff series, the Zassenhaus series has a continuous analogue called the Fer expansion [9, 23].) In this paper we concentrate on several properties of the Zassenhaus idempotents relevant for perturbative renormalization in QFT.

The article is organized as follows. We first settle some notation and recall qualitative features of the Bogoliubov recursion. We then introduce directly the new Lie-theoretic version of the recursion. The by-product is the decomposition of the regularized Feynman character, and hence amplitudes, into an infinite product, whose components are indexed by the number of loops (that is, the degree in the Hopf algebra of graphs). In the third section, we recall the Zassenhaus recursion and its descent algebra interpretation. The last section relates this interpretation to the new recursion. We show that the exponential components of the counterterm and renormalized amplitudes (appearing in the new decomposition) satisfy universal Lie-theoretic properties. That is, they are linked to the global counterterm and the renormalized Feynman rules by means of universal formulas and coefficients pertinent to the Zassenhaus idempotents —in much the same way as the Connes–Kreimer $\beta$-function is underlaid by the Dynkin idempotent [6].
1. Rota–Baxter algebras and the Bogoliubov recursion

Let us recall the general principles of the Bogoliubov recursion in the Connes–Kreimer Hopf algebraic approach to renormalization in pQFT. For details, we refer to the original articles [12, 3], and their successive refinement and generalization in terms of Rota–Baxter algebras [7, 8, 10].

For a given renormalizable QFT (say, $\phi^4$-theory) the perturbative approach consists of expanding $n$-point Green functions in terms of Feynman diagrams with $n$ external legs. The latter are constructed out of propagators and 4-valent interaction vertices. We are not interested in questions concerning the overall convergence properties of those expansions. The perturbative expansions for the 2-point and 4-point Green functions are already not well defined due to the UV divergences appearing in the corresponding Feynman amplitudes beyond the tree-level. In the Hopf algebraic approach one uses the set $F$ of one-particle irreducible (1PI) 2-leg and 4-leg Feynman diagrams,\(^1\) to generate a polynomial algebra $H$ over the complex numbers $\mathbb{C}$ (the free commutative algebra over $F$). The algebra $H$ inherits a graded algebra structure from the decomposition of $F$ according to the number of loops of a given diagram. It receives a coproduct and a Hopf algebra structure from the process of extracting 1PI UV-divergent subdiagrams out of a given Feynman diagram. In summary, Feynman diagrams for any QFT can be organized into a graded connected commutative Hopf algebra $H$. We denote its product $m : H \otimes H \to H$; the coproduct $\Delta : H \to H \otimes H$; the unit map $u : \mathbb{C} \to H$; the co-unit $e : H \to \mathbb{C}$; and the antipode $S : H \to H$. They are such that:

$$m(H_p \otimes H_q) \subset H_{p+q}, \quad \Delta(H_n) \subset \sum_{p+q=n} H_p \otimes H_q, \quad \text{and} \quad S(H_n) \subset H_n.$$  

A Feynman diagram with $n$ loops belongs to $H_n$, the degree $n$ component of $H$.

The Feynman rules associating to each Feynman graph its corresponding amplitude, i.e. a multidimensional iterated integral, are seen in the framework of regularization as a map $\rho$ from $F$ to a commutative algebra $A$ over $\mathbb{C}$. Each set of Feynman rules extends uniquely to a character of $H$, that is, to a multiplicative map from $H$ to $A$. The set of characters $G(A)$ inherits a pro-unipotent group structure from the graded Hopf algebra structure of $H$ (a well-known phenomenon in algebraic geometry: as a functor from commutative algebras to groups, $G$ is the group scheme associated to the commutative Hopf algebra $A$). Similarly, the vector space of linear maps $\text{Lin}(H, A)$ inherits an algebra structure from the coalgebra structure of $H$: the product $\ast$ in $G(A)$ and $\text{Lin}(H, A)$ is called the convolution product and is given by:

$$\forall f, g \in \text{Lin}(H, A) : \quad f \ast g := \pi_A \circ (f \otimes g) \circ \Delta.$$  

The unit for $\ast$, written $e$ as well, is the composition of the unit of $A$ with the co-unit of $H$.

When the Feynman amplitudes are the “bare” ones, that is, when they depend on bare coupling constants of the theory, then $\varphi(\Gamma) = U(\Gamma)$ is ill-defined for a large class of diagrams $\Gamma \in H$. In DR together with minimal subtraction one introduces a regularizing parameter $\epsilon$, a deformation parameter for the space-time dimension of the theory. This allows to define a regularized Feynman rule as a map $\varphi$ from $F$ to the field of Laurent series $\mathcal{L} := \mathbb{C}[\epsilon^{-1}, \epsilon]]$. Now, $\mathcal{L}$ is a weight one Rota–Baxter algebra, that is, for any $(x, y) \in \mathcal{L}^2$, we have:

$$R_-(x)R_-(y) = R_-(xR_-(y)) + R_-(R_-(x)y) - R_-(xy)$$

\(^1\)The notion of $n$-particle irreducible diagrams corresponds to the maximal number, $n$, of propagators that can be cut without making the Feynman diagram disconnected. We refer the reader to aforementioned references.
where $R_-$ is the projection mapping each $x \in \mathbb{C}[\varepsilon^{-1}, \varepsilon]$ to its strict polar part, i.e. $R_-(x) \in \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$ of $\mathcal{L}$, orthogonally to the algebra of formal power series $\mathbb{C}[[\varepsilon]]$. We set $R_+ := 1 - R_-$, so that $R_+$ stands for the projection onto $\mathbb{C}[[\varepsilon]]$ orthogonally to $\varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$. Hence, the projectors $R_-$ and $R_+$ correspond to a splitting $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_+$ into two subalgebras with $1 \in \mathcal{L}_+$.

The Bogoliubov recursion allows to disentangle the combinatorics of Feynman graphs. It is defined in terms of Bogoliubov’s preparation map $\varphi \mapsto \overline{\varphi}$, which sends the group $G(\mathcal{L})$ to the set of $\mathbb{C}$-linear maps from $H$ to $\mathcal{L}$. It can be defined, together with the counterterm character $\varphi_-$ and the renormalized character $\varphi_+$ by the set of equations:

$$
\overline{\varphi} := \varphi_+ \ast (\varphi - e) \quad \text{and} \quad \varphi_{\pm} = e \pm R_\pm(\overline{\varphi}).
$$

The Rota–Baxter algebra structure on $\mathcal{L}$ insures that $\varphi = \varphi_+^{-1} \ast \varphi_+$ as well as that $\varphi_-$ and $\varphi_+$ are characters: they are multiplicative maps respectively from $H$ to $\mathbb{C}[\varepsilon^{-1}]$ (in fact $\varphi_-$ maps $H^+$ to $\varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$) and from $H$ to $\mathbb{C}[[\varepsilon]]$. In our case scheme, when setting $\varepsilon = 0$ the renormalized character induces a well-defined, scalar-valued character $\varphi_{\text{ren}}$ from $H$ to $\mathbb{C}$, which values on Feynman graphs are the renormalized amplitudes of the theory.

The solution to the system $\overline{\varphi}$ is unique. It can be reached at by induction on the number of loops (hence the terminology “Bogoliubov recursion”). We pointed out already that the recursion takes place in $\text{Lin}(H, \mathcal{L})$ and not in $G(\mathcal{L})$ —since the preparation map is not a character. We refer to $[2]$ for the classical approaches to the Bogoliubov recursion (including its solution by means of Zimmermann’s forest formula) and to $[3]$ for a recent new approach.

2. A Lie theoretic decomposition of characters

Keep in mind the framework of pQFT. We consider the same renormalization scheme as before. In all likelihood, our results can be extended to more general settings —e.g. to other Rota–Baxter target algebras for the group of characters— but we refrain from seeking the utmost generality. Recall that, for an arbitrary commutative algebra $A$, an $A$-valued infinitesimal character is an element $\mu$ of the graded Lie algebra $\Xi(A) = \bigoplus_{n=1}^{\infty} \Xi_n(A)$ associated to the group $G(A)$, that is, an element of $\text{Lin}(H, A)$ that vanishes on $\mathbb{C} = H_0$ and on the square $(H^+)^2$ of the augmentation ideal of $H$.

**Definition 2.1.** An element $\varphi$ of $\bigoplus_{n \in \mathbb{N}} \text{Lin}(H_n, A)$ is $n$-connected if and only if it can be written $\varphi = e + \sum_{k \geq n} \varphi_k$, where $\varphi_k \in \text{Lin}(H_k, A)$. An element $\rho$ of $\Xi(A)$ is $n$-connected if and only if it can be written $\rho = \sum_{k \geq n} \rho_k$, where $\rho_k \in \text{Lin}(H_k, A)$.

In other terms, $\varphi_i = 0$ for $0 < i < n$, and equally $\rho_i = 0$ for $0 \leq i < n$. In the following we often identify implicitly a map such as $\rho_k$ in $\text{Lin}(H_k, A)$ with the corresponding map from $H$ to $A$ (i.e. the one equal to $\rho_k$ on $H_k$ and 0 on $H_i$, $i \neq k$). In particular, we will write $\rho_k \in \text{Lin}(H_k, A) \subset \text{Lin}(H, A)$. From now on in this section $A = \mathcal{L}$.

**Lemma 2.1.** Let $\varphi$ be an $n$-connected character. Then, there exist unique infinitesimal characters $\zeta_n^\varphi$ and $\mu_{n, 2n-1}^\varphi$, respectively in $\Xi_n(\mathcal{L})$ and $\bigoplus_{j=n}^{2n-1} \Xi_j(\mathcal{L})$ such that:

$$
\exp(-R_-(\zeta_n^\varphi)) \ast \varphi \ast \exp(-R_+(\zeta_n^\varphi))
$$

is an $n + 1$-connected character, respectively

$$
\exp(-R_-(\mu_{n, 2n-1}^\varphi)) \ast \varphi \ast \exp(-R_+(\mu_{n, 2n-1}^\varphi))
$$

is a $2n$-connected character.

---

$^2$Recall that for $\psi$ a character on $H$ its inverse is given by $\psi^{-1} := \psi \circ S$. 
Indeed, notice that, for $\rho$ any $n$-connected infinitesimal character, $R_-(\rho), R_+(\rho)$ and $\exp(\rho)$ are $n$-connected (respectively as infinitesimal characters and as a character). Notice also that, by the very definition of the logarithm, for any $n$-connected characters $\lambda, \beta$ and $\tau$, we have in the convolution algebra $\text{Lin}(H, A)$:

$$\log(\lambda) = \sum_{k \in \mathbb{N}^*} \frac{(-1)^{k-1}}{k} \left( \sum_{m \geq n} \lambda_m \right)^k,$$

$$\log(\lambda \ast \tau) = \sum_{k \in \mathbb{N}^*} \frac{(-1)^{k-1}}{k} \left( \sum_{m \geq n} \lambda_m + \sum_{m \geq n} \tau_m + \sum_{l, m \geq n} \lambda_l \ast \tau_m \right)^k,$$

and therefore, since $\lambda_l \ast \tau_m \in \text{Lin}(H_{l+m}, A) \subset \text{Lin}(H, A)$:

$$\log(\lambda)_{j} = \lambda_j, \ \log(\lambda \ast \tau)_{j} = \lambda_j + \tau_j, \ j = n, \ldots, 2n - 1.$$ 

It also follows that:

$$\log(\tau \ast \lambda \ast \beta)_{j} = \tau_j + \lambda_j + \beta_j, \ j = n, \ldots, 2n - 1.$$ 

The proof of the lemma follows by setting: $\zeta_n^\rho := \log(\varphi)_{n} = \varphi_{n}$ and $\mu_i^\rho := \log(\varphi)_{i} = \varphi_{i}, \ i = n, \ldots, 2n - 1$, where we write $\mu^\rho_i$ for the degree $i$ component of $\mu^\rho_{n,2n-1}$.

**Proposition 2.1.** We have the asymptotic formulas in the group of characters:

$$\varphi = \lim_{n \to \infty} \exp(\lambda_1^+) \ast \cdots \ast \exp(\lambda_n^+) \ast \exp(\lambda_1^-) \ast \cdots \ast \exp(\lambda_1^-)$$

and

$$\varphi = \lim_{n \to \infty} \exp(\tau_1^-) \ast \cdots \ast \exp(\tau_n^-) \ast \exp(\tau_1^+) \ast \cdots \ast \exp(\tau_1^+),$$

with the recursive definitions:

$$\varphi_{(n-1)} := \exp(-\lambda_1^-) \ast \cdots \ast \exp(-\lambda_n^-) \ast \varphi \ast \exp(-\lambda_1^+) \ast \cdots \ast \exp(-\lambda_n^+);$$

$$\lambda_n^\pm := R_\pm(\zeta_n^\rho);$$

$$\varphi_{[n-1]} := \exp(-\tau_1^-) \ast \cdots \ast \exp(-\tau_n^-) \ast \varphi \ast \exp(-\tau_1^+) \ast \cdots \ast \exp(-\tau_n^+);$$

$$\tau_n^\pm := R_\pm(\mu_{n,2n-1}^{\rho}).$$

The proof follows from the previous lemma and the pro-unipotent nature of $G(\mathcal{L})$. We have constructed, for a set of Feynman rules corresponding to a perturbatively treated QFT with $\mathcal{L}$ as algebra of amplitudes, two decompositions

$$\varphi = \varphi_+^1 \ast \varphi_+ = \varphi_{(2)}^1 \ast \varphi_{(2)},$$

with

$$\varphi_+ := \lim_{n \to \infty} \exp(\lambda_n^+) \ast \cdots \ast \exp(\lambda_1^+) \ \text{resp.} \ \varphi_{(2)}^+ := \lim_{n \to \infty} \exp(\tau_1^+) \ast \cdots \ast \exp(\tau_n^+)$$

and

$$\varphi_- := \lim_{n \to \infty} \exp(\lambda_n^-) \ast \cdots \ast \exp(\lambda_1^-) \ \text{resp.} \ \varphi_{(2)}^- := \lim_{n \to \infty} \exp(\tau_1^-) \ast \cdots \ast \exp(\tau_n^-).$$

**Theorem 2.1.** We have $\varphi_- = \varphi_{(2)}^-$ and $\varphi_+ = \varphi_{(2)}^+$. Moreover, the $\varphi = \varphi_-^1 \ast \varphi_+$ decomposition agrees with the decomposition obtained from the Bogoliubov recursion.

The proof reduces to a simple unicity argument. Assume that, for $\phi \in G(\mathcal{L}), \phi = \phi^- \ast \phi_+$, where $\phi_\ast - \epsilon$ and $\phi_+ \ast \epsilon^{-1}$ are $C[\epsilon^{-1}]$ and $C[[\epsilon]]$-valued, respectively. Let us write $\alpha \ast \beta := \alpha \ast \beta - \alpha - \beta$. We have:

$$\phi_\ast \phi = \phi_+ - \phi_- - \phi.$$
and therefore \( \phi_- = e + R_-(\phi_-) = e + R_-(\phi_+ - \phi_+ \ast \phi) \) and \( \phi_+ = R_+(\phi_+) = R_+(\phi_- + \phi_- \ast \phi) \). That is:
\[
\phi_- = e - R_-(\phi_+ - \phi_- \ast \phi); \quad \phi_+ = e + R_+(\phi_+ - \phi_- \ast \phi).
\]
Hence:
\[
\phi_- = e - R_-(\phi_- \ast (\phi - e)) \quad \phi_+ = e + R_+(\phi_- \ast (\phi - e))
\]
which is (equivalent to) the Bogoliubov recursion formula. In particular, both decompositions \( \varphi = \varphi_-^{-1} \ast \varphi_+ \) and \( \varphi = \varphi_-^{-1}(2) \ast \varphi_+(2) \) solve the Bogoliubov recursion. Since the latter has a unique solution, the theorem follows.

3. The Zassenhaus Recursions and the Descent Algebra

In this section we review the definition and main properties of the descent algebra, together with its application to the usual Zassenhaus recursion. Our presentation of the latter stems from adapting the noncommutative symmetric functions definitions in [13, 5]; the reader is referred to [11, 18, 19, 20] for further details and proofs. The second series of Zassenhaus idempotents (termed “accelerated”) that we introduce below is new, to the best of our knowledge.

Recall that the tensor algebra \( T(X) = \bigoplus_{n \in \mathbb{N}} T_n(X) \) over a countable set \( x_1, \ldots, x_n, \ldots \) is, as a vector space, the linear span of words \( y_1 \cdots y_n, \ y_i \in X \) over \( X \). It is provided with a graded connected cocommutative Hopf algebra structure defined in terms of concatenation of words as the product \( y_1 \cdots y_n \cdot z_1 \cdots z_p := y_1 \cdots y_n z_1 \cdots z_p \), and the unshuffling map as the coproduct \( \Delta : T(X) \to T(X) \otimes T(X) \):
\[
\Delta(y_1 \cdots y_n) := \sum_{l=(i_1, \ldots, i_k) \subseteq [n]} y_{i_1} \cdots y_{i_k} \otimes y_{j_1} \cdots y_{j_{n-k}},
\]
where \( J = \{j_1, \ldots, j_{n-k}\} = [n] - I \) and \( [n] = \{1, \ldots, n\} \). A word \( y_1 \cdots y_n \) is an element of \( T_n(X) \), the degree \( n \) component of \( T(X) \). There is a natural isomorphism between \( T(X) \) and its graded dual \( T^*(X) := \bigoplus_{n \in \mathbb{N}} T^*_n(X) \) induced by the scalar product on \( T(X) \) for which the words form an orthonormal basis. This defines another, graded connected commutative, Hopf algebra structure on \( T^*(X) \) that can be described, once again, by means of operations on words: the product is now the shuffle product, whereas the coproduct is the deconcatenation of words. Since \( T^*(X) \) is a Hopf algebra, the set \( \text{End}(T^*(X)) \) of linear endomorphisms of \( T^*(X) \) is naturally provided with an associative convolution product \( * \).

**Definition 3.1.** The descent algebra \( \mathcal{D} \) is the convolution subalgebra of \( \bigoplus_{n \in \mathbb{N}} \text{End}(T^*_n(X)) \subset \text{End}(T^*(X)) \) generated by the graded projections \( p_n : T^*(X) \to T^*_n(X) \).

The descent algebra inherits a graduation \( \mathcal{D} = \bigoplus_{n \in \mathbb{N}} \mathcal{D}_n \) from the graded algebra structure on \( \bigoplus_{n \in \mathbb{N}} \text{End}(T^*_n(X)) \) induced by the convolution product —since, for \( f \in \text{End}(T^*_n(X)) \subset \text{End}(T^*(X)) \) and \( g \in \text{End}(T^*_{m}(X)) \subset \text{End}(T^*(X)) \), \( f \ast g \in \text{End}(T^*_{n+m}(X)) \). It embeds naturally into the direct sum of the symmetric group algebras \( \mathcal{S} := \bigoplus_{n \in \mathbb{N}} \mathbb{C}[S_n] \), where \( \mathbb{C}[S_n] \) is viewed as a subset of \( \text{End}(T^*_n(X)) \) by letting \( \sigma \in S_n \) act on \( T^*_n(X) \) by letter permutation:
\[
\sigma(y_1 \cdots y_n) := y_{\sigma^{-1}(1)} \cdots y_{\sigma^{-1}(n)}.
\]
This embedding property follows, e.g. from the fact that \( \mathcal{S} \) is closed under the convolution product in \( \text{End}(T^*(X)) \); see [16].

**Proposition 3.1.** The degree \( n \)-component \( \mathcal{D}_n \) of the descent algebra is a subalgebra of \( \text{End}(T^*_n(X)) \) for the composition of maps, denoted as usual by \( \circ \). It is a free associative algebra over the polynomial ring \( p_n \) for the convolution product \( * \) and carries naturally a Hopf algebra structure.
for which the \( p_n \) behave as a series of divided powers; that is, if we write \( \delta \) for the coproduct on \( D \):

\[
\delta(p_n) = \sum_{i \leq n} p_i \otimes p_{n-i}.
\]

The first part of the Proposition is known as the Solomon fundamental theorem in the case of Coxeter groups of type \( A_n \). It generalizes to arbitrary (finite) Coxeter groups \([21]\). The second part follows directly from the computation of convolution products of the \( p_n \) in terms of descent sets of permutations, see e.g. \([20]\) Chap.9.

**Definition 3.2.** The Zassenhaus series (or left Zassenhaus series) \((Z_n)_{n \in \mathbb{N}}\) (respectively the accelerated or left accelerated Zassenhaus series \((Z_n^{(2)})_{n \in \mathbb{N}}\)) is the series of elements \( Z_n \in D_n \) (respectively \( Z_n^{(2)} \in D_n \)) defined as the (necessarily unique) solution of:

\[
Id_{T^\ast(X)} = \exp(Z_1) \ast \exp(Z_2) \ast \cdots \ast \exp(Z_n) \ast \cdots,
\]

respectively by:

\[
Id_{T^\ast(X)} = \exp(Z_1^{(2)}) \ast \exp(Z_2^{(2)} + Z_3^{(2)}) \ast \cdots \ast \exp(Z_{2n}^{(2)} + \cdots + Z_{2n+1}^{(2)}) \ast \cdots.
\]

The right Zassenhaus and right accelerated Zassenhaus series are defined similarly by:

\[
Id_{T^\ast(X)} = \ast \exp(\tilde{Z}_n) \ast \cdots \ast \exp(\tilde{Z}_2) \ast \exp(\tilde{Z}_1),
\]

respectively by:

\[
Id_{T^\ast(X)} = \ast \exp(\tilde{Z}_{2n}^{(2)} + \cdots + \tilde{Z}_{2n+1}^{(2)}) \ast \cdots \ast \exp(\tilde{Z}_2^{(2)} + Z_3^{(2)}) \ast \exp(\tilde{Z}_1^{(2)}).
\]

The existence and uniqueness of a solution to these equations follow by the same arguments as the ones we developed to construct an exponential solution to the Bogoliubov recursion. The \( Z_n \) and the \( Z_n^{(2)} \) are primitive elements in the Hopf algebra \( D \) and, equivalently, the \( \exp(Z_n) \) are group-like elements; since the logarithm and the exponential map are bijections between group-like and primitive elements, this follows by induction on \( n \) from the equations:

\[
Z_n = p_n \circ \log\left(\exp(-Z_{n-1}) \ast \cdots \ast \exp(-Z_1) \ast Id_{T^\ast(X)}\right),
\]

and

\[
Z_n^{(2)} = p_n \circ \log\left(\exp(-Z_{2n}^{(2)}) - \cdots - Z_{2n+1}^{(2)} \ast \cdots \ast \exp(-Z_1) \ast Id_{T^\ast(X)}\right),
\]

where \( m \) is the highest integer such that \( 2^{m+1} \leq n \). Analogous statements hold for the right series.

The link between left and right series can be made explicit as follows. Let us recollect first some classical facts. Let \( S \) be the convolution inverse of \( Id_{T^\ast(X)} \) (\( S \) is the antipode of \( T^\ast(X) \) in the language of Hopf algebras). We have:

\[
Id_{T^\ast(X)} = \ast \exp(\tilde{Z}_n) \ast \cdots \ast \exp(\tilde{Z}_2) \ast \exp(\tilde{Z}_1),
\]

so that

\[
S = Id_{T^\ast(X)}^{-1} = \exp(-\tilde{Z}_1) \ast \exp(-\tilde{Z}_2) \ast \cdots \ast \exp(-\tilde{Z}_n) \ast \cdots.
\]

However, since \( S = Id_{T^\ast(X)}^{-1} \), and since \( T^\ast(X) \) is commutative, \( S \) is a multiplicative map:

\[
S(xy) = S(x)S(y).
\]

This follows, e.g. from the observation that \( S \), being the convolution inverse of \( Id_{T^\ast(X)} \), belongs to the group \( G(H) \) of \( H \)-valued characters of \( H \). Also, we have \( S^2 = Id_{T^\ast(X)} \). Therefore:

\[
S \circ S = Id_{T^\ast(X)} = S \circ \exp(-\tilde{Z}_1) \ast S \circ \exp(-\tilde{Z}_2) \ast \cdots \ast S \circ \exp(-\tilde{Z}_n) \ast \cdots
\]

\[
= \exp\left(S \circ (-\tilde{Z}_1)\right) \ast \exp\left(S \circ (-\tilde{Z}_2)\right) \ast \cdots \ast \exp\left(S \circ (-\tilde{Z}_n)\right) \ast \cdots
\]

Finally, we note that the \( \delta(p_n) \) behave as a series of divided powers for the \( \delta \) on \( D \):

\[
\delta(p_n) = \sum_{i \leq n} p_i \otimes p_{n-i}.
\]
that is, by uniqueness of the decomposition, \( S \circ (-\tilde{Z}_i) = Z_i \).

(For convenience, our conventions for the Zassenhaus series \( Z_n \) are slightly different from the ones in [13]⁵; they write \( \tilde{Z}_n \) for our \( Z_n \).)

**Corollary 3.1.** The \( Z_k \) and the \( Z_k^{(2)} \) are dual to Lie quasi-idempotents: that is, the dual elements in \( \text{End}(T(X)) \) are, up to multiplication by a scalar, projections from \( T(X) \) onto \( \text{Lie}(X) = \text{Prim}(T(X)) \).

For the Zassenhaus series, the quasi-idempotence property was proved in [13]⁵, using a characterization of Lie quasi-idempotents in the descent algebras [11]: the property follows from the \( Z_k \) being primitive elements in \( D \). The same argument applies to the right and left accelerated Zassenhaus series.

**Proposition 3.2.** The descent algebra is freely generated as an associative algebra by the series of Zassenhaus elements \( Z_n \) (respectively by the series \( Z_n^{(2)}, \tilde{Z}_n, Z_n^{(2)} \)).

This follows by induction on \( n \) from the observation that \( (Z_n - p_n) \) is a noncommutative polynomial in the \( Z_i, i \leq n \), and from the fact that the \( p_n \) generate freely \( D \) as an associative algebra. The same proof holds for the other series.

Recall that the Dynkin operator (of degree \( n \)) is the element of \( \text{End}(T(X)) \) mapping the word \( y_1 \cdots y_n \) to the iterated Lie bracket \([\cdots [y_1, y_2] \cdots, y_n] \). The dual operators \( D_n \) on \( T(X)^* \), that we still call slightly abusively the Dynkin operators, belong to the descent algebra and generate freely the descent algebra as an associative algebra, see [11]⁵.

**Corollary 3.2.** The Zassenhaus elements (respectively the Dynkin operators \( D_n \)) can be expressed as noncommutative polynomials in terms of the Dynkin operators (respectively the Zassenhaus elements).

### 4. On Counterterms and Renormalized Characters

In the present section we show how the four Zassenhaus series of Lie quasi-idempotents connect to the renormalization process and the \( \beta \)-function in pQFT, in the sense of Connes and Kreimer [4]. The series do actually provide universal formulas to split the counterterm and the renormalized characters into products of exponentials of infinitesimal characters of increasing degrees. Since the splittings of \( \varphi_{-1} \) and \( \varphi_+ \) follow the same pattern up to a duality (the two splittings are obtained as left-to-right respectively right-to-left infinite products), we will develop only the case of the counterterm; the renormalized character can be treated similarly, replacing the left Zassenhaus series by the right ones.

Let us recall first a few facts on Hopf algebras and descent algebras from [18]. For \( H \) an arbitrary graded connected commutative Hopf algebra, there is an algebra map \( \alpha_H \) from \( D \) to \( \text{End}(H) \). The map is defined by

\[
\alpha_H(p_{n_1} \cdots p_{n_k}) := p_{n_1}^H \cdots p_{n_k}^H,
\]

where \( p_n^H \) is the projection from \( H \) to the \( n \)-th graded component \( H_n \). It is an algebra morphism for both the convolution and the composition product in \( D \) and \( \text{End}(H) \) [18 Thm.II.7]. The map also behaves nicely with respect to the coalgebra structures on \( H \) (see [19]⁶): namely, for \( h, h' \) in \( H \) and \( f \in D \), we have:

\[
\alpha_H(f)(hh') = \alpha_H(f(1))(h)\alpha_H(f(2))(h')
\]

with the (enhanced) Sweedler notation for the coproduct: \( \delta(f) = f(1) \otimes f(2) \). From now on, for notational convenience, we will often omit \( \alpha_H \) and identify implicitly an element in \( D \).
with its image in $\text{End}(H)$. In particular, we will consider the Zassenhaus series as a series of elements in $\text{End}(H)$.

So, let $h$ be an arbitrary element of $H$. From the identity

$$\text{Id}_{T^\circ(X)} = \exp(Z_1) \cdots \exp(Z_n) \cdots$$

in $\mathcal{D}$, and since, by its very definition, $\alpha_H(\text{Id}_{T^\circ(X)}) = \text{Id}_H$, we obtain:

$$\text{Id}_H = \exp(Z_1) \cdots \exp(Z_n) \cdots$$

Now, for $\mu \in G(A)$ with $A$ an arbitrary commutative algebra, and for $f, g \in \text{End}(H)$ and $h \in H$, we have:

$$\mu(f \ast g)(h) = \mu(f(h^{(1)}) \cdot g(h^{(2)})) = (\mu \circ f \ast \mu \circ g)(h),$$

so that, in particular:

$$\mu \circ \exp(f) = \exp(\mu \circ f).$$

It yields:

$$\mu = \mu \circ \text{Id}_H = \exp(\mu \circ Z_1) \cdots \exp(\mu \circ Z_n) \cdots$$

However, by construction $Z_i \in \mathcal{D}_i$. Therefore, $\mu \circ Z_i \in \text{Hom}(H_i, A) \subset \text{Hom}(H, A)$ or, in other terms, $\mu \circ Z_i$ is zero on the components $H_k$ of $H$ for $k \neq i$. Also, for any $i > 0$, since $\delta(Z_i) = Z_i \otimes 1 + 1 \otimes Z_i$,

$$Z_i(hh') = Z_i(h) \cdot e(h') + e(h) \cdot Z_i(h')$$

which is equal to 0 for any $h, h' \in H^+$. In particular, $\mu \circ Z_i$ is an infinitesimal character of $H$.

Applying this construction to $\varphi^{-1}_-$, the counterterm associated to a set of Feynman rules, it ensues:

$$\varphi^{-1}_- = \exp(\varphi^{-1}_- \circ Z_1) \cdots \exp(\varphi^{-1}_- \circ Z_n) \cdots$$

with $\varphi^{-1}_- \circ Z_i$ an infinitesimal character in $\text{Hom}(H_i, \mathbb{C}[\mathbb{C}[\epsilon^{-1}])$. The uniqueness of such a decomposition insures that $\varphi^{-1}_- \circ Z_i = \lambda_i$.  

**Proposition 4.1.** Let $\varphi$ be a Feynman rule character and $\varphi = \varphi^{-1}_- \ast \varphi^+$ its corresponding decomposition into a counterterm and a renormalized character. Then:

$$\varphi^{-1}_- = \exp(\varphi^{-1}_- \circ Z_1) \cdots \exp(\varphi^{-1}_- \circ Z_n) \cdots$$

with $Z_i$ the $i$-th Zassenhaus idempotent. This decomposition agrees with the exponential decomposition introduced in the first part of the article. Analogous statements hold for the accelerated series; respectively for the renormalized character and the two right Zassenhaus series.

The $\beta$-function associated to a given Feynman rule character controls the flow of the renormalized Feynman rule character with respect to the t’Hooft mass scale, see [6, Sect.7] for details. Algebraically, the $\beta$-function reads:

$$\beta = \varphi_- \circ D$$

or $\beta_n = \varphi_- \circ D_n$ with $D = \sum_n D_n$ the Dynkin operator. Since $D_n$ can be written uniquely as a noncommutative polynomial in the $\tilde{Z}_i$ (and conversely), there are coefficients $c_{i_1, \ldots, i_k}$ with:

$$D_n = \sum_{i_1 + \cdots + i_k = n, \ i_j > 0} c_{i_1, \ldots, i_k} \tilde{Z}_{i_1} \cdots \tilde{Z}_{i_k}.$$
and it follows that
\[ \beta_n = \sum_{i_1 + \cdots + i_k = n, \ i_j > 0} c_{i_1, \ldots, i_k} \varphi_- \circ \tilde{Z}_{i_1} \ast \cdots \ast \varphi_- \circ \tilde{Z}_{i_k}. \]

In other terms, the knowledge of the beta function is equivalent to the knowledge of the (right) exponential decomposition of \( \varphi_- \):
\[ \varphi_- = \cdots \ast \exp(\varphi_- \circ \tilde{Z}_n) \ast \cdots \ast \exp(\varphi_- \circ \tilde{Z}_1). \]

However, by the argument preceding the corollary in the previous section, we also have
\[ -\varphi_- \circ \tilde{Z}_i = \varphi_-^{-1} \circ Z_i, \]
for any \( i > 0 \). Thus the knowledge of the \( \beta \)-function is also algebraically equivalent to that of the left exponential decomposition of the counterterm Feynman character.

The analysis of the physical meaning of the new processes and quantities introduced in the present article is postponed to further work.

Acknowledgments. We thank the organizers of the 7th International Workshop “Lie Theory and Its Applications in Physics”, 18-24 June 2007, Varna, Bulgaria, for giving one of us (FP) the opportunity to present parts of our recent research results, at that memorable occasion. We also like to thank J. M. Gracia-Bondía for useful discussions and a very careful reading of the manuscript leading to several improvements. We also thank D. Manchon for useful comments and discussions.

References

[1] W. E. Caswell and A. D. Kennedy, Simple approach to renormalization theory, Phys. Rev. D 25 (1982) 392–408.

[2] J. C. Collins, Renormalization, Cambridge University Press, Cambridge, 1984.

[3] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Commun. Math. Phys. 210 (2000) 249–273. arXiv:hep-th/9912092

[4] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The \( \beta \)-function, diffeomorphisms and the renormalization group, Commun. Math. Phys. 216 (2001) 215–241. arXiv:hep-th/0003188

[5] G. Duchamp, D. Krob and E. A. Vassilieva, Zassenhaus Lie idempotents, \( q \)-bracketing and a new exponential/logarithm correspondence, J. Alg. Comb. 12 (2000) 251–277.

[6] K. Ebrahimi-Fard, J. Gracia-Bondía and F. Patras, A Lie theoretic approach to renormalization, Commun. Math. Phys. 276 (2007) 519–549. arXiv:hep-th/0609035

[7] K. Ebrahimi-Fard, J. Gracia-Bondía and F. Patras, Rota–Baxter algebras and new combinatorial identities, Letters in Mathematical Physics 81 (2007) 61–75. arXiv:math/0701031

[8] K. Ebrahimi-Fard, D. Manchon and F. Patras, A noncommutative Bohnenblust–Spitzer identity for Rota–Baxter algebras solves Bogoliubov’s recursion, preprint (2007), arXiv:0705.1265.

[9] K. Ebrahimi-Fard and D. Manchon, A Magnus- and Fer-type formula in dendriform algebras, preprint (2007), arXiv:0707.0607.

[10] K. Ebrahimi-Fard and D. Manchon, The combinatorics of Bogoliubov’s recursion in renormalization, preprint (2007), arXiv:0710.3675.

[11] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348. arXiv:hep-th/9407124

[12] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv. Theor. Math. Phys. 2 (1998) 303–334. arXiv:q-alg/9707029

[13] D. Krob, B. Leclerc and J.-Y. Thibon, Noncommutative symmetric functions II: Transformations of alphabets, Int. J. of Alg. and Comput. 7 (1997) 181–264.

[14] J. H. Lowenstein, BPHZ renormalization, in Renormalization theory (Proceedings NATO Advanced Study Institute, Erice, 1975), NATO Advanced Study Institute Series C: Math. and Phys. Sci. Vol. 23, Reidel, Dordrecht, 1976.
[15] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. 7 (1954) 649–673.

[16] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra 177 (1995) 967–982.

[17] B. Mielnik and J. Plebański, *Combinatorial approach to Baker–Campbell–Hausdorff exponents* Ann. Inst. Henri Poincaré A XII (1970) 215–254.

[18] F. Patras, *L’algèbre des descentes d’une bigèbre graduée*, J. Algebra 170 (1994) 547–566.

[19] F. Patras and C. Reutenauer, *On Dynkin and Klyachko idempotents in graded bialgebras*, Adv. Appl. Math. 28 (2002) 560–579.

[20] C. Reutenauer, *Free Lie algebras*, Oxford University Press, Oxford, 1993.

[21] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra 41 (1976) 255–268.

[22] A. N. Vasilev, *The field theoretic renormalization group in critical behavior theory and stochastic dynamics*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[23] R. M. Wilcox, *Exponential operators and parameter differentiation in quantum physics*, J. Math. Phys. 8 (1967) 962–982.

[24] W. Zimmermann, *Convergence of Bogoliubov’s method of renormalization in momentum space*, Commun. Math. Phys. 15 (1969) 208–234.

Laboratoire MIA, Université de Haute Alsace, 4 rue des Frères Lumière, 68093 Mulhouse, France

E-mail address: kurusch.ebrahimi-fard@uha.fr
URL: http://www.th.physik.uni-bonn.de/th/People/fard/

Laboratoire J.-A. Dieudonné UMR 6621, CNRS, Parc Valrose, 06108 Nice Cedex 02, France

E-mail address: patras@math.unice.fr
URL: www-math.unice.fr/~patras