Prescribing Morse scalar curvatures: critical points at infinity

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Abstract

The problem of prescribing conformally the scalar curvature of a closed Riemannian manifold as a given Morse function reduces to solving an elliptic partial differential equation with critical Sobolev exponent. Two ways of attacking this problem consist in subcritical approximations or negative pseudo gradient flows. We show under a mild non degeneracy assumption the equivalence of both approaches with respect to zero weak limits, in particular a one to one correspondence of zero weak limit finite energy subcritical blow-up solutions, zero weak limit critical points at infinity of negative type and sets of critical points with negative Laplacian of the function to be prescribed.

Key Words: Conformal geometry, scalar curvature, subcritical approximation, critical points at infinity

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1 Introduction

Prescribing conformally the scalar curvature on a manifold as a given function falls into the class of variational problems, which lack compactness, as the underlying partial differential equation is critical with respect to Sobolev’s embedding. In particular the Palais-Smale condition is violated, which in classical variational theory allows the use of deformation lemmata, which in return are a fundamental pillar in the calculus of variations.

To overcome this lack of compactness one may try to restore compactness or study a hopefully only slightly different, yet compact situation and pass to the limit or return directly to the deformation lemmata themselves, hence studying non compact flows. The first approach is restrictive to e.g. symmetric situations with improved Sobolev embedding, the second one leads to the idea of compact approximation and the third one to the theory of critical points at infinity.
Let us comment on the corresponding ideas. First and famously in order to restore compactness the positive mass theorem has been used, cf. [23]. Here the argument is, that a certain sublevel set of the variational functional is shown to be compact, while, assuming sufficient flatness of \( K \) or even \( K \) to be constant, the positive mass term becomes dominant in the expansion of the energy of a specific test function pushing its energetic value below the threshold of the sublevel set, i.e. the test function already lies withing the latter, which is therefore not empty. Hence one can find a minimizer by direct methods.

For compact approximations, cf. [9], [13], [15], [16] in contrast the underlying equations can be solved classically, whereas the passage to the critical limit then has to be understood in detail. The advantage of this approach is, that one deals with a sequence of solutions to specific equations rather than with arbitrary Palais-Smale sequences. Of course there will be a lack of compactness, i.e. there will be, as we pass to the critical limit, solutions, which do not converge in the variational space. But one may hope to find at least some sequences, which remain compact, thus providing a solution to the critical equation itself.

Similarly in the context of studying non compact flows, cf. [4], [10], [18], i.e. returning to the study of energy deformation, we do not have to study arbitrary Palais-Smale sequences, but flow lines. And the liberty is, that we are not bound to study a specific, but an energy deformation of our choice. In particular given a flow exhibiting non compactness somewhere, we may hope to avoid the latter by adapting the former, as was done in [20]. While in [10] classical min-max schemes are established by excluding certain non compactness scenarios, in [4] the topological effect of non compact flow lines to sublevels sets is computed. The difference is, that while the first result is based on avoiding non compactness, the second one uses this non compactness by understanding its topological contribution directly, which is a central topic in the theory of critical points at infinity.

Evidently in case of compact, for instance subcritical approximations or the study of non compact flow lines one has to understand and describe the lack of compactness in absence of at least partial compactness as in [23] qualitatively. A natural question is, whether or not one can expect to find different results by means of subcritical approximation or the study of non compact flows, as the first describes subcritical non compact sequences of solutions and the latter non compact flow lines. Comparing Theorems 1 and 2 this does not seem to be the case.

1.1 Setting
Consider a closed Riemannian manifold
\[
M = (M^n, g_0) \quad \text{with} \quad n \geq 5,
\]
volume measure \( \mu_{g_0} \) and scalar curvature \( R_{g_0} \). We assume the Yamabe invariant
\[
Y(M, g_0) = \inf_{A} \int (c_n |\nabla u|_{g_0}^2 + R_{g_0} u^2) \, d\mu_{g_0}, \quad c_n = 4 \frac{n-1}{n-2},
\]
where
\[
A = \{u \in W^{1,2}(M, g_0) \mid u \geq 0, u \not\equiv 0\},
\]
to be positive positive. As a consequence the conformal Laplacian
\[
L_{g_0} = -c_n \Delta_{g_0} + R_{g_0}
\]
is a positive and self-adjoint operator. Without loss of generality we assume \( R_{g_0} > 0 \) and denote by
\[
G_{g_0} : M \times M \setminus \Delta(M) \longrightarrow \mathbb{R}_+ \quad \text{with} \quad \Delta(M) = \{(m_1, m_2) \in M \times M : m_1 = m_2\}
\]
the Green’s function of $L_{g_{0}}$. Considering a conformal metric $g = g_{a} = u^{-\omega} g_{0}$ there holds
\[ d\mu_{g_{a}} = u \frac{\omega}{\alpha} d\mu_{g_{0}} \quad \text{and} \quad R = R_{g_{a}} = u^{-\frac{\omega}{\alpha}}(-c_{n} \Delta_{g_{a}} u + R_{g_{a}} u) = u^{-\frac{\omega}{\alpha}} L_{g_{0}} u \]
due to conformal covariance of the conformal Laplacian, i.e.
\[ -c_{n} \Delta_{g_{a}} v + R_{g_{a}} v = L_{g_{a}} v = u^{-\frac{\omega}{\alpha}} L_{g_{0}} (uv). \]
So prescribing conformally the scalar curvature $R = K$ as a given function $K$ is equivalent to solving
\[ L_{g_{0}} u = Ku^{\frac{n+2}{n-2}} \quad (1.2) \]
and the Green’s function $G_{g_{a}}$ for $L_{g_{a}}$ transforms according to
\[ G_{g_{a}}(x, y) = u^{-1}(x)G_{g_{0}}(x, y)u^{-1}(y). \]
Moreover we may associate in a unique and smooth way to every $a \in M$ a suitable conformal metric
\[ g_{a} = g_{a} = u_{a}^{-\omega} g_{0} \quad \text{with} \quad u_{a} = 1 + O(d_{g_{0}}^{2}(a, \cdot)) \]
such, that in a geodesic normal coordinate system for $g_{a}$, which we call a conformal normal coordinate system for $g_{0}$, the volume element is locally euclidean, i.e.
\[ d\mu_{g_{a}} = dL^{n} \text{ close to } a \in M, \]
cf. [12]. In particular
\[ (\exp_{g_{a}}^{g_{0}})^{-} \circ \exp_{g_{a}}^{g_{0}}(x) = x + O(|x|^{3}) \]
for the exponential maps centered at $a$, which e.g. implies
\[ \nabla g_{0} K(a) = \nabla_{g_{a}} K(a), \quad \nabla^{2} g_{0} K(a) = \nabla^{2} g_{a} K(a), \quad (1.3) \]
and in case $\nabla g_{0} K(a) = 0$ also $\nabla^{3} g_{0} K(a) = \nabla^{3} g_{a} K(a)$. Then
\[ G_{a} = G_{g_{a}}(a, \cdot), \]
i.e. the Green’s function $G_{g_{a}}$ with pole at $a \in M$ for the conformal Laplacian
\[ L_{g_{a}} = -c_{n} \Delta_{g_{a}} + R_{g_{a}} \]
expands with $\omega_{n} = |S^{n-1}|$ denoting the unit volume as
\[ G_{a} = \frac{1}{4n(n-1)\omega_{n}}(r_{a}^{2} - n + H_{a}), \quad r_{a} = d_{g_{a}}(a, \cdot), \quad H_{a} = H_{r,a} + H_{s,a}, \]
where $r_{a}$ denotes the geodesic distance from $a$ with respect to the metric $g_{a},$
\[ H_{s,a} = O \left( \begin{array}{c} r_{a} \\ \ln r_{a} \\ r_{a}^{6-n} \end{array} \right) \quad \text{for} \quad \begin{array}{l} n = 5 \\ n = 6 \\ n \geq 7 \end{array} \]
and $H_{r,a} \in C^{2,\alpha}_{loc}$. As due to $R_{g_{0}} > 0$
\[ c\|u\|_{W^{1,2}(M, g_{0})}^{2} \leq \int u L_{g_{0}} u d\mu_{g_{0}} = \int \left( c_{n} |\nabla u|_{g_{0}}^{2} + R_{g_{0}} u^{2} \right) d\mu_{g_{0}} \leq C\|u\|_{W^{1,2}(M, g_{0})}^{2} \]
with positive constants $0 < c < C < \infty$, we may define and use
\[ \|u\|^2 = \|u\|_{L^2_{g_0}}^2 = \int u L_{g_0} u \, d\mu_{g_0} \]
as an equivalent norm on $W^{1,2}$. We then wish to study the scaling invariant functional
\[ J : \mathcal{A} \longrightarrow \mathbb{R} : u \longrightarrow \frac{\int L_{g_0} u \, d\mu_{g_0}}{\left( \int K u^{\frac{n+2}{n-2}} \, d\mu_{g_0} \right)^{\frac{n-2}{n}}} \text{ for } K > 0. \]
Since the conformal scalar curvature $R = R_u$ for $g = g_u = u^{\frac{4}{n-2}} g_0$ satisfies
\[ r = r_u = \int R \, d\mu_{g_u} = \int u L_{g_0} u \, d\mu_{g_0}, \]
we have
\[ J(u) = \frac{r}{k} \text{ with } k = k_u = \int K u^{\frac{n+2}{n-2}} \, d\mu_{g_0}. \quad (1.4) \]
The first and second order derivatives of the functional are given by
\[ \partial J(u)v = \frac{2}{k} \left[ \int L_{g_0} u v \, d\mu_{g_0} - \frac{r}{k} \int K u^{\frac{n+2}{n-2}} v \, d\mu_{g_0} \right], \]
hence and in particular (1.2) has variational structure, and
\[ \partial^2 J(u)w = \frac{2}{k} \left[ \int L_{g_0} u w \, d\mu_{g_0} - \frac{n+2}{n-2} \int K u^{\frac{n+2}{n-2}} w \, d\mu_{g_0} \right] \]
\[ - \frac{4}{k} \int L_{g_0} u w \, d\mu_{g_0} \int K u^{\frac{n+2}{n-2}} w \, d\mu_{g_0} + \int L_{g_0} u w \, d\mu_{g_0} \int K u^{\frac{n+2}{n-2}} w \, d\mu_{g_0} \]
\[ + \frac{2(n+2)}{k} \int K u^{\frac{n+2}{n-2}} w \, d\mu_{g_0} \int K u^{\frac{n+2}{n-2}} w \, d\mu_{g_0}. \]
Note, that $J$ is of class $C^{2,\alpha}(\mathcal{A} \cap \{c^{-1} < k < c\})$ for every $c > 1$ and that the scalar product
\[ \langle u, w \rangle = \langle u, w \rangle_{L_{g_0}} = \int L_{g_0} u w \, d\mu_{g_0} \]
induces the gradient $\nabla J = \nabla_{L_{g_0}} J$, i.e.
\[ \langle \nabla J(u), w \rangle = \langle \nabla_{L_{g_0}} J(u), w \rangle_{L_{g_0}} = \partial J(u)w \]
or in other words $\nabla_{L_{g_0}} J(u) = L_{g_0} \partial J(u)$ with $L_{g_0}$ denoting the inverse to
\[ L_{g_0} : W^{1,2}(M, g_0) \longrightarrow W^{-1,2}(M, g_0) \]
mapping $W^{1,2}(M, g_0)$ to its dual. Likewise and for the sake of brevity let us also write
\[ \nabla^k K = \nabla^k_{g_0} K \text{ for } k \geq 1 \text{ and } \Delta K = \Delta_{g_0} K. \]
1.2 Sub- and criticality

Let us review first the subcritical, non degenerate case.

**Definition 1.1.** We call a positive Morse function $K$ on $M$ non degenerate for $n \geq 5$, if
\[
\{ |\nabla K| = 0 \} \cap \{ \Delta K = 0 \} = \emptyset. \tag{1.5}
\]

We will always assume this non degeneracy, under which in [14] and [15] we proved for the subcritical approximation to (1.2), i.e.
\[
L_{g_0} u = K u \left( n + 2 - \frac{2}{n} - \tau \right) \text{ with } 0 < \tau \to 0 \tag{1.6}
\]
and subcriticality is understood with respect to Sobolev's embedding, the following uniqueness and existence result. Clearly $J = J_0$.

**Theorem 1 ([14], [15]).** Let $(M, g)$ be a compact manifold of dimension $n \geq 5$ with positive Yamabe invariant and $K : M \to \mathbb{R}$ be a positive Morse function satisfying (1.5). Let $x_1, \ldots, x_q$ be distinct critical points of $K$ with negative Laplacian.

Then there exists, as $\tau \to 0$ and up to scaling, a unique solution $u_{\tau, x_1, \ldots, x_q}$ to (1.6) developing a simple bubble at each $x_i$ and converging weakly to zero as $\tau \to 0$. Moreover and up to scaling
\[
m(J_\tau, u_{\tau, x_1, \ldots, x_q}) = (q - 1) + \sum_{i=1}^{q} (n - m(K, x_i)).
\]

Conversely all blow-up solutions of uniformly bounded energy and zero weak limit type are as above.

Here $m = m(\cdot, \cdot)$ denotes the Morse index, while blow-up refers to local concentration

\[
\forall 0 < \varepsilon \ll 1 \exists \lambda_\tau \to \infty : \sup_{x \in M} \int_{B_{\frac{\lambda_\tau}{\tau}}(x)} |\nabla u_\tau|^2 \geq \varepsilon
\]
of solutions $u_\tau \in \partial J_\tau = \emptyset$, cf. (3.2) of Proposition 3.1 in [14], and simple bubbling to
\[
\forall i = 1, \ldots, q \exists R_\tau \xrightarrow{\tau \to 0} \infty : \| u_{\tau, x_1, \ldots, x_q} - \alpha_i \varphi_{a, \lambda} \|_{W^{1, 2}(B_{R_\tau})} \xrightarrow{\tau \to \infty} 0.
\]

Precisely and with a multiplicative constant $\Theta$ reflecting the scaling invariance of $J_\tau$
\[
\alpha_j = \frac{\Theta}{K(x_j)^{\frac{n-2}{2}}} + o_\tau(1), \quad a_j \xrightarrow{\tau \to 0} x_j \text{ and } \lambda_j \simeq \tau^{-\frac{1}{2}}, \tag{1.7}
\]
cf. Remark 1.1 in [14]. Finally the functions
\[
\varphi_{a, \lambda} = u_a \left( \frac{\lambda}{1 + \lambda^2 \gamma_n G_a^{-\frac{1}{n}}} \right)^{\frac{n-2}{2}} \quad \text{with } G_a = G_{ga}(a, \cdot) \text{ and } \gamma_n = (4n(n-1)\omega_n)^{\frac{1}{n}}, \tag{1.8}
\]
to which we refer as bubbles, are zero weak limit almost solutions to (1.2), precisely
\[
\varphi_{a, \lambda} \xrightarrow{\lambda \to \infty} 0 \text{ weakly with } \limsup_{\lambda \to \infty} J(\varphi_{a, \lambda}) < \infty \text{ and } \| \partial J(\varphi_{a, \lambda}) \|_{\lambda \to \infty} \xrightarrow{\lambda \to \infty} 0
\]
uniformly, cf. [3.1] and Lemma 3.1 hence also for (1.6). We refer to Section 3 for precise statements.
The proof of Theorem 1 is based on considering the variational functional
\[
J_\tau(u) = \frac{\int L_{g_0} uu d\mu_{g_0}}{(\int K u^{p+1} d\mu_{g_0})^{\frac{p}{p+1}}} \quad \text{for } u \in A \quad \text{and with} \quad p = \frac{n+2}{n-2} - \tau
\]
corresponding to (1.6). As \(J_\tau\) is scaling invariant, we may restrict to
\[
X = \{0 \leq u \in W^{1,2}(M) \mid u \neq 0, \|u\| = 1\} \quad (1.9)
\]
then shows, that every zero weak limit Palais-Smale sequence has to be of the form of a finite sum
\[
u = \alpha^i \varphi_{a_i, \lambda_i} + v, \quad \|v\| \ll 1 \quad (1.10)
\]
with
(i) scaling parameters \(c < \alpha_i < C\)
(ii) high concentrations \(\lambda_i \to \infty\)
(iii) small error \(\|v\| \to 0\).

Vice versa such functions induce zero weak limit type Palais-Smale sequences. This representation is rendered unique by means of a minimisation problem, which provides certain orthogonality relations, when testing the derivative \(\partial J_\tau\) along
\[
\varphi_i, \lambda_i \partial_{\lambda_i} \varphi_i, \frac{\nabla_{a_i}}{\lambda_i} \varphi_i \quad \text{and} \quad v.
\]

Then a sophisticated combination of such testings provides a lower bound on \(|\partial J_\tau|\), which in return reduces to a high degree the possible configurations of the parameters \(\alpha_i, \lambda_i\) and \(a_i\) for zero weak limit blow-up solutions. In particular and necessarily
\[
a_i \to x_i \in \{\|\nabla K\| = 0\} \cap \{\Delta K < 0\} \quad \text{as} \quad \tau \to 0 \quad \text{and} \quad x_i \neq x_j \quad \text{for} \quad i \neq j, \quad (1.11)
\]
thus excluding tower bubbling, i.e. \(x_i = x_j\) for some \(i \neq j\). Finally in (1.15) and based on calculations of the second derivative \(\partial^2 J\) the sharpness of (1.11) is established, meaning that for every
\[
\{x_1, \ldots, x_q\} \subseteq \{\|\nabla K\| = 0\} \cap \{\Delta K < 0\}
\]
there exists a unique solution \(u \in \{\partial J_\tau = 0\}\) of type \(u = \sum_i \alpha_i \varphi_{a_i, \lambda_i} + v\) with
\[
\frac{1}{\lambda_i}, d(a_i, x_i), |1 - \sum_j a_j^2 K(a_i)\alpha_i \frac{1}{\lambda_i} - |, \|v\| \to 0 \quad \text{as} \quad \tau \to 0 \quad (1.12)
\]
and the latter convergence is understood to a high degree in \(\tau\). Hence Theorem 1.

Taking also the scaling invariance of \(J\) into account we consider again \(X\), cf. (1.9), as the variation space of \(J = J_0\) and will in the present work construct a semi-flow
\[
\Phi : \mathbb{R}_{\geq 0} \times X \to X
\]

rigorously defined in Section 4, which decreases the energy \( J \), and study its zero weak limit flow lines.

Roughly speaking and in analogy to the subcritical case the corresponding parabolic blow-up analysis and unique representation lead to the description \((1.10)\) of zero weak limit non compact flow lines for e.g. the strong gradient flow. In particular non compactness of a flow line corresponds to \( \forall \lambda_i \rightarrow \infty \) for at least a sequence in time. Then the flow \( \Phi \), while preserving \( X \), is finely tuned to a careful evaluation and combination of testings of the derivative \( \partial J \) with the scope to increase \( \max_i \lambda_i \) along \( \Phi \) as little as possible, whenever a flow line is of type \((1.10)\), while moving along the strong gradient flow otherwise. In particular this allows us to show, that in analogy to \((1.11)\) along zero weak limit flow lines of \( \Phi \) there necessarily holds

\[
a_i \rightarrow x_i \in \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \} \text{ as } t \rightarrow \infty \text{ and } x_i \neq x_j \text{ for } i \neq j.
\]

And again in analogy to Theorem 1 we show, that for every \( \{ x_1, \ldots, x_q \} \subseteq \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \} \) there exists a flow line

\[
u(t) = \Phi(t, u_0) \text{ of type } u = \sum_i \alpha_i \varphi_{a_i, \lambda_i} + v
\]

with

\[
\frac{1}{\lambda_i}, d(a_i, x_i), \sum \frac{\alpha_i^2}{\sum_j K(a_j) \alpha_j} |K(a_i)\alpha_i|, \|v\| \to 0 \text{ as } t \to \infty
\]

exponentially fast, cf. \((1.12)\).

Consequently zero weak limit flow lines for this energy decreasing flow \( \Phi \) and finite energy zero weak limit subcritical blow-up solutions display the same limiting behaviour. And, since from the computation of the second derivative \( \partial^2 J \) at the latter subcritical blow-up solutions the induced change of topology of sublevel sets is known according to their Morse index, the same change of topology is induced by corresponding critical points at infinity, for whose definition we refer to \((1.1)\) and Section 2.

**Theorem 2.** Let \((M, g)\) be a compact manifold of dimension \( n \geq 5 \) with positive Yamabe invariant and let \( K : M \rightarrow \mathbb{R} \) be a positive Morse function satisfying \((1.5)\). Let \( x_1, \ldots, x_q \) be distinct critical points of \( K \) with negative Laplacian.

Then there exists up to scaling a unique critical point at infinity \( c = u_{x_1, \ldots, x_q} \) for \( J \) of zero weak limit, energy decreasing type exhibiting a simple peak at each point \( x_i \). Moreover \( c = u_{x_1, \ldots, x_q} \) has index

\[
\text{ind}(J, u_{x_1, \ldots, x_q}) = (q - 1) + \sum_{i=1}^{q} (n - m(K, x_i)).
\]

Conversely all critical points at infinity of energy decreasing and zero weak limit type are as above.

Let us discuss the terminology and the practical impact on how to proceed. Consider in analogy to the negative gradient flow an energy decreasing deformation \( F \) generated by \( \partial_t u = G(u) \) with

\[
G \in C^{0,1}_{\text{loc}}(X, TX) \text{ such that } \partial J(u)G(u) \rightarrow 0 \implies |\partial J(u)| \rightarrow 0 \quad (1.13)
\]

Independently of a particular choice of \( G \) we then see, that

(1) for every flow line due to energy consumption necessarily

\[
|\partial J(u)| \rightarrow 0 \quad (1.14)
\]

up to a subsequence in time. Hence and generally one and only one of the three possibilities
Figure 1: Any point \((a, \lambda) \in J^{\epsilon_2 - \delta}\) with \(|a| \ll 1 \ll \lambda\) inevitably flows towards \((0, \infty)\)

(i) \(\|u\| \to \infty\)
(ii) \(u \rightharpoonup u_\infty\) weakly, but \(\|u - u_\infty\| \not\to 0\)
(iii) \(u \to u_\infty \in \{\partial J = 0\}\) strongly.

occurs up to a subsequence in time for every flow line. In cases (i) or (ii) we say, that \(u\) tends to leave the variational space or escapes to infinity, see Figure 1 for an illustration. While (i) may occur in case of the two dimensional analogon to the prescribed scalar curvature problem, i.e. the Gaussian one, in our setting (i) is ruled out, as \(X \subset \mathcal{A}\) is bounded. However, since energy is decreased, and by virtue of \((1.14)\) every flow line constitutes up to a subsequence in time a Palais-Smale sequence.

(2) on the other hand an analysis of arbitrary Palais-Smale sequences shows, that

\[ u = u_\infty + \sum_{i=1}^{q} \alpha_i \varphi_{a_i, \lambda_i} + v \text{ for some } q \in \mathbb{N} \]

up to a subsequence in time eventually, where among other properties

(0) either \(0 < u_\infty \in \{\partial J = 0\}\) is a solution or \(u_\infty = 0\) and \(q \geq 1\)
(v) \(\|v\| \to 0\) is a vanishing perturbation
(\(\lambda\)) \(\forall i \lambda_i \to \infty\) and \(\varphi_{a_i, \lambda_i} \xrightarrow{\lambda_i \to \infty} 0\) weakly.
Note, that in case and only in case $u_\infty = 0$ we have

$$u \to 0 \text{ weakly, while } \|\alpha^i \varphi_{a_i, \lambda_i}\| \to 0 \text{ is impossible,}$$

since $\|\cdot\| = 1$ on $X$ by definition. In other words $u$ is of zero weak limit type and $u$ tends to leave the variational space $X$, i.e. escapes to infinity, via

$$\|u - \alpha^i \varphi_{a_i, \lambda_i}\| \to 0 \text{ with } \alpha_i \to \alpha_{i, \infty} \in \mathbb{R}^+, a_i \to a_{i, \infty} \in M \text{ and } \lambda_i \to \infty$$

(1.15)

up to a subsequence in time at least.

(3) based on an energy consumption argument relying on lower bound estimates on $|\partial J|$, if

$$u = \alpha^i \varphi_{a_i, \lambda_i} + v \text{ with } c < \alpha_i < C, \lambda_i \to \infty \text{ and } \|v\| \to 0$$

up to a subsequence in time, then also eventually, i.e. for $t \to \infty$, and

$$|\partial J(u)| \to 0 \text{ as } t \to \infty.$$

This is to say, that $u$ does not only escape to infinity, but also becomes critical at infinity. A priori however this does not imply a unique limiting profile of type (1.15) for $t \to \infty$.

Remark 1.1. In any case different flows may produce along their respective flow lines non coinciding sets of end configurations as in (1.15). For instance, as a pathological example in [20] shows, while the strong gradient flow exhibits zero weak limit non compact flow lines escaping to infinity, all of which have one and the same end configuration for $t \to \infty$, a slight variation of this flow is compact, i.e. is convergent or in other words does not have any flow lines escaping to infinity at all.

In order to avoid such issues we will

(i) define an energy decreasing flow $\Phi$ as in (1.13).

(ii) prove beyond (3) above, that every zero weak limit flow $u$ line of $\Phi$ escaping to infinity, i.e.

$$\partial_t u = A(u) \text{ with } u = \alpha^i \varphi_{a_i, \lambda_i} + v \text{ eventually,}$$

has a unique end configuration, informally

$$u \to c = \alpha^i_{\infty} \varphi_{a_i, \infty, \infty} \text{ as } t \to \infty$$

and $\varphi_{a_i, \infty, \infty} = \delta_i$ is a Dirac measure for $L^{\frac{n+2}{2}}$.

(iii) classify all these attained end configurations as $a_i, \infty = x_i$ for some

$$x_i \in \{\nabla K = 0\} \cap \{\Delta K < 0\} \text{ and } x_i \neq x_j \text{ for } i \neq j,$$

cf. (1.11), while $\alpha_i, \infty = \frac{\Theta}{K(x_i)}$ with a normalizing constant $\Theta > 0$, cf. (1.7).

(iv) determine homologically for a contractible neighbourhood

$$V_\varepsilon(c) = V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}$$

of $c$, cf. Definition 3.1 and 3.3, the change of topology as

$$H_k\{|J \leq J_\infty - \delta\} \cup V_\varepsilon(c), \{J \leq J_\infty - \delta\}) = \delta_{k,m} \text{ for } \delta \ll \varepsilon \ll 1 \text{ and } m = \text{ind}(J, u_\infty),$$

cf. Theorem 2 while this neighbourhood does not contain any solution, i.e.

$$V_\varepsilon(c) \cap \{\partial J = 0\} = \emptyset.$$
(v) show, that none of these end configurations can be avoided as an obstacle to energetic deformation, i.e. they are critical points at infinity, cf. Definition 2.6.

Whereas (i)-(iii) are seen by analysing one specific flow $\Phi$, the index in (iv) is justified from the subcritical approximation, while the minimality condition (v) follows from a Morse lemma at infinity, i.e. a faithful expansion of $J$ on $V_\varepsilon(c)$ of Morse type. And this is, how to prove Theorem 2.

Theorem 2 as a result, in particular and foremost the exclusion of tower bubbles along a suitable flow, is not new, we refer to Appendix 2 in [3] for the case of the sphere. While in the latter work the most important arguments are nicely displayed, there is an inaccuracy, which we shall discuss after the proof of Theorem 2 at the end of this work, whose motivation besides is threefold

1. the discourse fits well into the language and notation of [14], [15], [16], [18], [19], [20], [21] and the result demonstrates a natural equivalence of subcritical approximation versus critical points at infinity of energy decreasing type.

2. the flow, we study, is in contrast to previous explicit constructions, cf. [2], [3], [4], [5], norm and positivity preserving, hence provides a natural deformation of energy sublevels as subsets of the variational space $X$ for the variational functional $J$ on $X$. Conversely these properties hold true for Yamabe type, i.e. weak $L^2$-pseudo gradient flows, cf. [8], [18], [17], whose analysis relies on higher $L^p$ curvature norm controls, hence are not easy to adapt at infinity to exclude tower bubbles.

3. the construction of the flow as in Section 4 is explicit and keeps track of all the relevant quantities. In particular we move the blow-up points $a_i$ exactly along the stable manifolds of $K$, which will prove helpful for adaptations to describe the flow outside $V(q, \varepsilon)$, but still in a concentrated regime.

2 Critical points at infinity

While (i)-(v) above identify the critical points at infinity according to [1], their definition as in [1] is related to a pseudo gradient or more generally to a flow of type (1.13). And therefore this notion of a critical point at infinity is not intrinsic to the variational problem. On the other hand in some situations, cf. [20] and Remark 1.1 it is counter intuitive to identify a critical point at infinity with a non compact flow line of a specific flow, if any non compactness can be avoided by considering a different flow.

We wish to take a different view. Let us first define various objects related to Palais-Smale sequences. Strictly speaking Proposition 3.1 and Remark 3.1 describe the possible Palais-Smale end configurations as elements of (2.1) below, but in fact each such configuration can be easily obtained as a natural limit of a Palais-Smale sequences.

**Definition 2.1.** Let $(PS)$ denote the blow-up profiles arising from Palais-Smale sequences on $X$, i.e.

$$(PS) = \{ u_\infty + \sum_i \alpha_{i,\infty} \delta_{a_i,\infty} : a_i,\infty \in M \text{ and for some } \kappa_{\infty} > 0 \}
\quad \text{where} \quad L_{\log} u_\infty = \frac{K}{\kappa_{\infty}} u_\infty^{\frac{4n}{n-2}}, \quad \alpha_{i,\infty} = \frac{4n}{n-2} \sqrt{\frac{4n(n-1)\kappa_{\infty}}{K(a_i,\infty)}} \}.

(2.1)

We also

1. denote for $A \subseteq X$ by

$$(PS)^A = \{ c \in (PS) | \exists (u_n) \subseteq A \text{ Palais-Smale with } c \text{ as limiting profile } \}
\quad \text{the blow-up profiles arising from Palais-Smale sequences in } A \subseteq X.$$
(ii) denote for \( c \in \overline{(PS)} \) with corresponding \( \kappa_\infty > 0 \), cf. (2.1), by

\[
J(c) = \lim_{k \to \infty} J(u_k) = \kappa_\infty
\]

the unique limiting energy of any Palais-Smale sequence \( (u_k) \subset X \) with \( c \) as limiting profile.

(iii) call for \( c \in \overline{(PS)} \) an open set \( U = U_c \subseteq X \) an open neighbourhood of \( c \), if

\[
\forall (u_k) \subset X \text{ Palais-Smale with limiting profile } c : u_k \in U \text{ eventually } \land d(u_k, \partial U) \neq o(1),
\]

and call an open set \( V = V_A \subset X \) an open neighbourhood of \( A \subseteq \overline{(PS)} \), if

\[
\forall c \in A : V_A \text{ is a neighbourhood of } c.
\]

(iv) call \( O \subseteq \overline{(PS)} \) open, if

\[
\forall c \in O \exists U = U_c \text{ a neighbourhood of } c : \overline{(PS)} \setminus U \subseteq O,
\]

and \( C \subseteq \overline{(PS)} \) closed, if \( \overline{(PS)} \setminus C \) is open.

**Remark 2.1.** We remark, that

(i) as a fundamental property

\[
\forall L > 0 : \{ c \in \overline{(PS)} : J(c) \leq L \} \text{ is sequentially compact,}
\]

cf. (2.2), i.e. for any \( (c_k) \subset \overline{(PS)} \) with \( J(c_k) \leq L \) and up to a subsequence

\[
c_k = u_{\infty, k} + \sum_{i=1}^{q_k} \alpha_{i, \infty} \delta_{\alpha_{i, \infty, k}} \xrightarrow{k \to \infty} c_\infty = u_\infty + \sum_{i=0}^{q_\infty} \alpha_i \delta_{\alpha_i}
\]

in the sense of distributions. In fact by (3.7) the number of diracs \( q_k \) is bounded. And either \( u_{\infty, k} = 0 \) up to a subsequence or by (3.7) and (3.8) the sequence \( (u_{\infty, k}) \) of solutions constitutes a Palais-Smale sequence of bounded norm and energy, whence Proposition 3.1 is applicable.

(ii) finite intersections and arbitrary unions of open subsets of \( \overline{(PS)} \) are again open.

(iii) for \( c = u_\infty + \alpha \delta_{\alpha_{i, \infty}} \in \overline{(PS)} \) and \( \varepsilon > 0 \)

\[
U = U_c^\varepsilon = \left\{ \frac{u_\infty + \alpha \varphi_{\alpha_{i, \infty, \lambda_i}} + v}{\|u_\infty + \alpha \varphi_{\alpha_{i, \infty, \lambda_i}} + v\|} : \sum_{i,j} \varepsilon_{i,j} + |\alpha_i - \alpha_i| + \|v\| < \varepsilon \right\}
\]

is a natural neighbourhood of \( c \) in \( X \).

In this way the Palais-Smale closure of \( X \), i.e. \( E = X \cup \overline{(PS)} \) with the topology

\[
\text{Top}(E) = \{ V \cup (\overline{(PS)} \setminus (PS)^d V) : V \subseteq X \text{ open } \}
\]

becomes a separable Hausdorff space and we identify a neighbourhood \( O = O_C \in \text{Top}(E) \) of \( C \subseteq E \) with its part \( U = O \cap X \) in the variational space \( X \), cf. (iii) in Definition 2.1.
Definition 2.2. Let

\[ \Pi F = \{ \Psi = \Pi_{i=1}^k F_i \circ \tau_i : F_i, \tau_i \text{ as in (1) - (4) below } \} \]

denote the set of consecutive, energy decreasing deformations \( \Psi \), for which

(1) \( \forall 1 \leq i \leq k : F_i \) as in (1.3)
(2) \( \forall 1 \leq i \leq k - 1 : \tau_i : X \to \mathbb{R}_{\geq 0} \) Lipschitz
(3) during \([0, k - 1]\) we deform along

\[ (\Psi|_{[x \times [i - 1, i]})(v_0, t) = F_i(v_0, \tau_i(x) \cdot (t - i + 1)), \]

i.e. solving for \( 1 \leq i \leq k - 1 \) consecutively the initial value problems

\[ \begin{cases} v(t_0) = v_0 \\ \partial_t v = \tau_i(v_0) \cdot G_i(v) \end{cases} \quad \text{for } t_0 = i - 1 \]

\[ \begin{cases} v(t_0) = v_0 \\ \partial_t v = G_i(v) \end{cases} \quad \text{for } i - 1 \leq t \leq i \]

(4) and finally

\[ (\Psi|_{[x \times [k - 1, \infty]})(v_0, t) = F(v_0, t - k + 1), \]

i.e. solving the initial value problems

\[ \begin{cases} v(t_0) = v_0 \\ \partial_t v = G_i(v) \end{cases} \quad \text{for } t_0 = k - 1 \]

\[ \begin{cases} \partial_t v = G_i(v) \end{cases} \quad \text{for } t \geq k - 1 \].

Note, that every \( \Psi(\cdot, t) = \Pi_{i=1}^k F_i(\cdot, \tau_i(\cdot)t) \in \Pi F \) acts as a family of diffeomorphisms and along each flow line \( u = \Psi(u_0, \cdot) \) there holds almost always \( \partial_t J(u) \leq 0 \) and, cf. (1.13),

\[ \partial_t J(u) \to 0 \iff |\partial J(u)| \to 0. \]

Since ultimately critical points at infinity will be related to an obstacle to energetic deformation below a certain energy \( \sigma \), we introduce the subsequent notions.

Definition 2.3. For \( \sigma \in \mathbb{R} \) we call a closed subset \( W \subseteq X \) \( \sigma \)-reducible, if

\[ \forall \varepsilon > 0 \exists \Psi \in \Pi F \wedge T \geq 0 \forall w \in W : J(\Psi(w, T)) \leq \sigma + \varepsilon. \]

Clearly every closed subset of a \( \sigma \)-reducible set is \( \sigma \)-reducible, \( \{ J \leq \sigma \} \) is \( \sigma \)-reducible and, if \( W \) is \( \sigma_1 \)-reducible and \( \sigma_1 \leq \sigma_2 \), then \( W \) is \( \sigma_2 \)-reducible as well.

Definition 2.4. For \( \sigma \in \mathbb{R} \) we call a closed subset \( C \subseteq E_{\sigma} = E \cap \{ J = \sigma \} \) \( \sigma \)-capturing, if

\[ \forall U = U_C \exists \varepsilon > 0 \forall W \text{ \( \sigma \)-reducible } \exists \Psi \in \Pi F \wedge T \geq 0 \forall t \geq 0 : \Psi(W, T + t) \subset \{ J < \sigma - \varepsilon \} \cup U. \]

To clarify this definition

(i) consider the case, that \( C = \emptyset \) is \( \sigma \)-capturing. Then clearly \( E_{\sigma} = \{ J = \sigma \} \cap E \) as an energy level is not an obstacle to energetic deformation.

(ii) consider with \( D = [1, 1] \) the stretched maximum

\[ J : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} -(x + 1)^2 & \text{for } x \leq -1 \\ 0 & \text{for } -1 \leq x \leq 1 \\ -(x - 1)^2 & \text{for } x \geq 1 \end{cases}. \]

In this case \( C = D \) is \( \sigma \)-capturing, while no subset of \( D \) is \( \sigma \)-capturing. In fact suppose, that some \( C \subseteq D \) was \( \sigma \)-capturing. Then there exists \( U = U_C \) and
(1) $d \in D \setminus C, \varepsilon > 0$ such, that $B_{2\varepsilon}(d) \cap U_C = \emptyset$, since $C$ is closed.
(2) some $0 < \varepsilon < 0$ such, that for every $\sigma$-reducible $W$ we find $\Psi \in \Pi F, T \geq 0$ such, that
$$
\Psi(W, T) \subset \{ J < \sigma - \varepsilon \} \cup U.
$$
Combining then $\Psi$ with a flow $\Phi \in \Pi F$, along which for $t > T$
$$
\partial_t u = G(u) = \begin{cases} 
1 & \text{for } x \geq d + 2\varepsilon \\
0 & \text{for } d - \varepsilon \leq x \leq d + \varepsilon \\
-1 & \text{for } x \leq d - 2\varepsilon 
\end{cases}
$$
to a flow $\Theta$, we then find $\Theta(W, T + 3) \subset \{ J < \varepsilon \}$, while this is readily impossible, when choosing $W = \{ J \leq \sigma \} = \mathbb{R}$.

(iii) consider the function as in Figure 2 and observe, that

![Diagram of function](image)

Figure 2: Some function on $\mathbb{R}^2$

(1) every $\sigma$-reducible $W$, which by definition is closed, is a subset of some complement
$$
X \setminus (B_\varepsilon(m_1) \cup B_\varepsilon(m_2)), \ X = \mathbb{R}^2
$$
(2) evidently $D_\sigma = \bigcup_{i=1}^4 D_\sigma^i = (PS)_\sigma$ and for every $U = U_D$ and $\varepsilon > 0$ sufficiently small we may deform every $\sigma$-reducible $W$ along some $\Psi \in \Pi F$ onto $\{ J < \sigma - \varepsilon \} \cup U$, i.e. $D_\sigma$ is $\sigma$-capturing.
(3) we may deform along some $\Psi \in \Pi F$ suitably small neighbourhoods $U_{D_\sigma^i}$ of $D_\sigma^i$ in such a way, that $\Psi$ leaves $U_{D_\sigma^i} \cup \{ J < \sigma - \varepsilon \}$ invariant, while for some $T \geq 0$
$$
\Psi(U_{D_\sigma^1}, T), \Psi(U_{D_\sigma^2}, T) \subseteq \{ J < \sigma - \varepsilon \}
$$
(4) as a consequence also $C = D_\sigma^3 \cup D_\sigma^4$ is $\sigma$-capturing.
(5) in fact $C$ is minimal in the sense, that $C$ is $\sigma$-capturing and for every $\sigma$-capturing $D \subseteq E_\sigma$ also $C \cap D$ is $\sigma$-capturing, cf. Definition 2.4

We remark, that any flow, by which we push a neighbourhood of $D_\sigma^1$ below a certain energy $\sigma - \varepsilon$, requires diverging speed towards $D_{\sigma^1} = -\infty$, cf. (i) of Remark 2.2. A possible choice is $\partial_t x = x^2$ near $D_\sigma^3$. 

We refer to Figure 2
While the set $D$ in the aforegoing examples is naturally critical, the set $C$ is the one of variational interest, i.e. the obstacle to energetic deformation, and correctly identified as the unique and minimal strongly critical set as defined below.

**Definition 2.5.** We call $C \subseteq E_\sigma$ strongly critical, if $C$ is $\sigma$-capturing and

$$\forall \ D \subseteq E_\sigma \ \sigma\text{-capturing : } C \cap D \ \sigma\text{-capturing.}$$

Note, that this definition does not exclude the case, that $C = \emptyset$ is strongly critical. But if so, the situation is variationally trivial.

**Proposition 2.1.** $(PS)_\sigma = (PS) \cap \{J = \sigma\} \subseteq E_\sigma$ is strongly $\sigma$-critical.

**Proof.** Let $C = (PS)_\sigma$ and $U = U_C$ arbitrary. Then using

$$|\partial J| > \gamma \quad \text{on} \quad \{\sigma - \varepsilon < J < \sigma + \varepsilon\} \setminus U \quad \text{for some} \quad \varepsilon, \gamma > 0$$

and we consider for some $\delta > 0$ open subneighbourhoods of $C$ satisfying

$$U = V_3 \supset V_2 \supset V_1 \quad \text{with} \quad d(V_i, X \setminus V_{i+1}) > \delta.$$

Let $\varepsilon > 0$ such, that $\gamma \delta > 4\varepsilon$. Note, that to travel a distance $\delta > 0$ along a negative gradient flow line

$$u : [0, t] \rightarrow \{\sigma - \delta < J < \sigma + \varepsilon\} \setminus U$$

comes at an energetic cost $\Delta J \geq \gamma \delta$, since

$$\delta = d(u(t), u(0)) \leq \int_0^t |\partial J(u(\tau))| \, d\tau \leq \gamma^{-1} \int_0^t |\partial J(u(\tau))|^2 \, d\tau = \frac{J(u(0)) - J(u(t))}{\gamma}.$$

We therefore consider some arbitrary $\sigma$-reducible $W$ and choose $\Psi \in \Pi F$ such, that

$$\sup_{w \in W} J(\Psi(w, T_0)) < \sigma + \varepsilon$$

and $\Phi \in \Pi F$ given by $\Psi$ during $[0, T_0]$ and the negative gradient flow for $t > T_0$. We then show

$$\exists \ T \geq T_0 \ \forall \ t \geq T : \ \Phi(W, T + t) \subset \{J < \sigma - \varepsilon\} \cup U$$

in order to verify, that $C$ is $\sigma$-capturing. Hence consider some

$$u_0 \in \Psi(W, T_0) = \Phi(W, T_0) \subset \{J < \sigma + \varepsilon\}$$

as an initial data for the negative gradient flow line $u$. Then

(i) in case $u_0 \in X \setminus V_3$, the flow line $u$ can never reach $V_2 \cap \{J \geq \sigma - \varepsilon\}$, since otherwise $u$ would have to travel through $V_3$ bridging a distance $\delta > 0$, which comes at an energetic cost $\gamma \delta > 4\varepsilon$, while we have only an energetic gap of $2\varepsilon$ at disposition. As a consequence $u$ enters $\{J < \sigma - \varepsilon\}$ and does so in some finite time $T_1 \geq T_0$, which is uniformly upper bounded for all $u_0 \in X \setminus V_3$.

(ii) in case $u_0 \in V_3 \setminus V_2$ and by the same argument as above, the flow line can never reach $V_1 \cap \{J \geq \sigma - \varepsilon\}$ and thus enters $\{J < \sigma - \varepsilon\}$ in some uniformly upper bounded time $T_2$.

(iii) in case $u_0 \in V_2$, then the flow line $u$ can never leave $V_2 \cup \{J < \sigma - \varepsilon\}$ again by energy consumption.
As a consequence for $T = \max\{T_1, T_2\} \geq T_0$ we find

$$\forall t \geq T : \Phi(W, T + t) \subset \{J < \sigma + \varepsilon\} \cup V_3.$$ 

Recalling $U = V_3$, this shows, that $C = (PS)_{\sigma}$ is $\sigma$-capturing. To prove, that $C$ is even strongly $\sigma$-critical, we consider some arbitrary $D \cap \sigma$ that we may capture every $\varepsilon$ along the negative gradient flow. Hence this follows from energy consumption flowing by the negative gradient flow away from $C = (PS)_{\sigma}$.  

**Proposition 2.2.** There exists a minimal, strongly critical $M \subseteq (PS)_{\sigma}$.  

**Proof.** We may assume, that $\emptyset \subseteq E_{\sigma}$ is not strongly critical. In particular and necessarily $(PS)_{\sigma} \neq \emptyset$, since otherwise for some $\gamma, \varepsilon > 0$ by (2.3)

$$|\partial J| > \gamma \text{ on } \{\sigma - \varepsilon < J < \sigma + \varepsilon\}$$

and this implies, that every $\sigma$-reducible $W$ can be brought down into $\{J < \sigma - \varepsilon\}$ for any $0 < 2\varepsilon < \varepsilon$ in finite time along the negative gradient flow. Hence $(PS)_{\sigma} \neq \emptyset$ and by virtue of Proposition 2.1, we may consider

$$P = \{D \subseteq (PS)_{\sigma} : D \text{ strongly critical } \} \neq \emptyset$$

as a by inclusion partially ordered set. Let us denote by $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ an arbitrary chain in $P$. Then the assertion follows from Zorn’s Lemma, provided

$$C = \cap_i C_i \subseteq (PS)_{\sigma}$$

is strongly critical, i.e. a lower bound for this chain in $P$. To see the latter we have to show, that

(i) $C$ is $\sigma$-capturing and

(ii) $C \cap D$ is $\sigma$-capturing, whenever some $D \subseteq E_{\sigma}$ is $\sigma$-capturing.

To prove (i) consider an arbitrary $U = U_C$. Then, as $C_k \subseteq (PS)_{\sigma}$ is closed and $(PS)_{\sigma}$ is sequentially compact, cf. (2.3), there exists $k \in \mathbb{N}$ such, that $U = U_{C_k}$ is a neighbourhood of $C_k$. Moreover, since $C_k$ is strongly critical, $C_k$ is in particular $\sigma$-capturing. Hence according to Definition 2.3 we find $\varepsilon > 0$, such that we may capture every $\sigma$-reducible $W$ in $\{J < \sigma - \varepsilon\} \cup U$ by some $\Psi \in \Pi F$ as desired. Therefore and, since $U = U_C$ is arbitrary, $C$ is $\sigma$-capturing itself.

To prove (ii) consider an arbitrary $\sigma$-capturing $D \subseteq E_{\sigma}$. Since $C_k$ is strongly critical, by definition $D \cap C_k \subseteq (PS)_{\sigma}$ is $\sigma$-capturing for every $k \in \mathbb{N}$. Arguing as for (i) we then find, that $D \cap C$ is $\sigma$-capturing as well.  

While Zorn’s Lemma, as we have seen, guarantees the existence of minimal strongly $\sigma$-critical sets, we have to show uniqueness of the latter separately.

**Lemma 2.1.** There exists a unique, minimal strongly critical $M_{\sigma} \subseteq (PS)_{\sigma}$.  

**Proof.** By Proposition 2.2 there exists some minimal, strongly critical $M_1 \subseteq (PS)_{\sigma}$. Suppose, there exists another minimal strongly critical $M_1 \neq M_2 \subseteq (PS)_{\sigma}$.

Then, since $M_2$ is strongly $\sigma$-critical, $M_2$ is by definition $\sigma$-capturing. And, since $M_1$ is strongly critical, we deduce, that $M_1 \cap M_2$ is $\sigma$-capturing as well.

Moreover consider some arbitrary $\sigma$-capturing $D \subseteq E_{\sigma}$. Then, since $M_2$ is strongly critical, also $D \cap M_2$ is $\sigma$-capturing. And, since $M_1$ is strongly critical, also $D \cap M_1 \cap M_2$ is $\sigma$-capturing.

We conclude, that $M_1 \cap M_2$ is strongly critical, which contradicts the minimality of $M_1$ and $M_2$.  

With Lemma 2.1 at hand we then define critical points at infinity as follows.
**Definition 2.6.** We call \( c \in E \) a critical point at infinity, if \( c \in \cup_{\sigma} M_{\sigma} \setminus X \).

Note, that the definition of \( M_{\sigma} \subseteq (PS)_{\sigma} \subseteq E_{\sigma} \) does only depend on \( J \) and the space \( \Pi F \) of admissible deformations, in particular does not depend on a specific flow \( \Phi \in \Pi F \) or for instance a presumed Morse structure around elements of \( (PS)_{\sigma} \).

Let us show, that the unique, minimal strongly critical sets \( M_{\sigma} \) are generically meaningful.

**Proposition 2.3.** Let \( c \in \{ J = \sigma \} \) be a non degenerate critical point of finite index. Then \( c \in M_{\sigma} \).

**Proof.** Arguing by contradiction, we suppose \( c \notin M_{\sigma} \). Then

\[
\exists U_{M_{\sigma}} \cap U_{c} : U_{M_{\sigma}} \cap U_{c} = \emptyset.
\]

Consider in a Morse chart around \( c = 0 \), e.g.

\[
J(u) = \sigma + |x^+|^2 - |x^-|^2 \quad \text{on} \quad V = V_c = B_c(0) \subset U_c,
\]

a sequence

\[
V \cap X^+ \supset (x_k^+) \rightarrow c \quad \text{as} \quad k \rightarrow \infty,
\]

to which for arbitrarily small \( \sqrt{\varepsilon} \ll \delta \ll \epsilon \) we attach \( m = \text{ind}(J,0) \)-dimensional disks

\[
B_{\delta M}(x_k^+) = \{(x^+,x^-) \in V : x^+ = x_k^+ \wedge |x^-| \leq 2\delta \} \subset V.
\]

with boundary \( \partial B_{\delta M}(x_k^+) \subset \{ J \leq \sigma - \varepsilon \} \). Then for degree reasons, see below,

\[
\forall \Psi \in \Pi F \wedge t \geq 0 \exists x_{k,t}^+ \in \Psi(B_{\delta M}(x_k^+),t) : x_{k,t}^+ \in V \cap X^+ \cap \{ \sigma \leq J \leq J(x_k^+) \}.
\]

(2.4)

However, since each \( B_{\delta M}(x_k^+) \) is \( \sigma \)-reducible and \( M_{\sigma} \) is \( \sigma \)-capturing, by definition we find

\[
\Psi(B_{\delta M}(x_k^+),T) \subset \{ J < \sigma - \gamma \} \cup U_{M_{\sigma}}.
\]

(2.5)

for suitable \( T, \gamma > 0 \) and \( \Psi \in \Pi F \). Then (2.4) and (2.5) lead to the obvious contradiction

\[
x_{k,t}^+ \in V \cap U_{M_{\sigma}} \subseteq U_c \cap U_{M_{\sigma}} = \emptyset.
\]

Hence we are left with proving (2.4). On the Morse chart \( V = B_c(0) \) consider the continuous map

\[
(x^+,x^-) \rightarrow x^- \rightarrow \left( \frac{\delta}{|x^-|^2} \mathbb{1}_{\{|x^-| \geq \delta\}} + \mathbb{1}_{\{|x^-| < \delta\}} x^- \right) \rightarrow \left( \frac{\delta}{|x^-|^2} \mathbb{1}_{\{|x^-| < \delta\}} + \mathbb{1}_{\{|x^-| > \delta\}} x^- \right) \sim,
\]

with \( \sim \) denoting the natural identification of the disk \( B^m_{\delta}(0) \) via

\[
x \sim y \iff \| x \| = \| y \| = \delta \quad \text{for} \quad x, y \in B^m_{\delta}(0)
\]

with the sphere \( S^m_{\delta} \) with south pole \( S = 0 \). After rescaling we hence obtain a continuous map

\[
\theta : B_c(0) \rightarrow S^m_{\delta}.
\]

Moreover for \( x = (x^+, x^-) \in \partial B_c(0) \cap \{ J \leq \sigma + \varepsilon \} \) and recalling \( 0 < \sqrt{\varepsilon} \ll \delta \ll \epsilon \) we have

\[
|x^+|^2 + |x^-|^2 = \epsilon^2 \quad \text{and} \quad |x^+|^2 \leq |x^-|^2 + \varepsilon,
\]

whence we may assume

\[
|x_-|^2 \geq \frac{\epsilon^2 - \varepsilon}{2} > 2\delta^2.
\]
Consequently and with $N$ denoting the north pole of $S^m_1$

$$\theta(\partial B_\epsilon(0) \cap \{ J \leq \sigma + \epsilon \}) = \{ N \},$$

whence we may extend continuously and restrict $\theta : \{ J \leq \sigma + \epsilon \} \rightarrow S^m_1$ by putting

$$\theta \equiv N \text{ on } \{ J \leq \sigma + \epsilon \} \setminus B_\epsilon(0). \quad (2.6)$$

We also find, that for all $k$ sufficiently large

$$\forall t \geq 0 : \theta(\partial \Psi(B_{2\delta}^-(x^+_k), t) = \{ N \} \quad (2.7)$$

for any flow $\Psi \in \Pi F$. In fact let $y_0 = (y^+_0, y^-_0) \in \partial B_{2\delta}^-(x^+_k) \subset B_\epsilon(0)$. Then by construction

$$\theta(y_0) = N \text{ and } J(y_0) = \sigma + |x^+_k|^2 - |y^-_0|^2 = \sigma - 4\delta^2 + \alpha_k^0(1).$$

Let $y = \Psi(y_0, \cdot)$ and suppose $\theta(y(t)) \neq N$ for some $t > 0$. Then by $(2.6)$ necessarily $y(t) \in B_\epsilon(0)$ and

$$\sigma - 4\delta^2 + \alpha_k^0(1) = J(y_0) \geq J(y) = \sigma + |y^+|^2 - |y^-|^2 \geq \sigma - |y^-|^2 \geq \sigma - \delta^2,$$

leading to a contradiction for $k$ sufficiently large. Then $(2.7)$ implies, that for the natural embedding

$$i : B^m_1(0) \xrightarrow{\sim} B_{2\delta}^-(x^+_k)$$

and for every $t \geq 0$ the composition $\theta \circ \Psi(\cdot, t) \circ i$ factorizes to a map

$$\theta \circ \Psi(\cdot, t) \circ \tilde{\theta} : S^m_1 \rightarrow S^m_1.$$

Since $\theta \circ \tilde{\theta} \simeq id_{S^m_1}$ as a homotopy equivalence, and by continuity and constancy of the degree on $S^m_1$

$$\forall t \geq 0 \exists s \in S^m_1 : \theta(\Psi(\tilde{\theta}(s), t)) = S.$$

Consequently $\forall t \geq 0 \exists x \in B_{2\delta}^-(x^+_k) : \Psi(x, t) = (x^+_k, 0)$. From this $(2.4)$ readily follows. \hfill $\Box$

Analogous arguments then show, that a finite index Morse structure at infinity leads to the same conclusion. The spaces following are real Banach.

**Lemma 2.2.** Let $c \in (PS)_c$ and suppose, that for a neighbourhood $U_c$ of $c$ we may parameterise

(i) with spaces $X^\pm$ and a neighbourhood $V = V_0$ of $0 \in X^+ \times X^-$

$$\overline{(PS)}_c^j \simeq \{(x^+, x^-) \in X^+ \times X^- \mid (x^+, x^-) \in V\}.$$

(ii) $U_C \simeq U \subset Y^+ \times Y^-$ open with spaces $Y^\pm = \Lambda^\pm \times X^\pm \times V^\pm$ and

$$\overline{(PS)}_c^j \simeq \{(\lambda^+, \lambda^-, x^+, x^-, v^+, v^-) : \lambda^\pm = \infty, (x^+, x^-) \in V, v^+_i = 0\}.$$

(iii) $J(u) = \sigma + |\lambda^+|^2 - |\lambda^-|^2 + |x^+|^2 - |x^-|^2 + |v^+|^2 - |v^-|^2$

Then

(1) $c \in M_\sigma \implies \Lambda^- = 0$

(2) $\Lambda^- = 0 \land \dim(Y^-) < \infty \implies c \in M_\sigma$.  

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Proof. As for (1) suppose $\Lambda^- \neq \emptyset$. We then decrease energy within $U \simeq U_c$ via

(a) decreasing $|x^+|$ and $|v^+|$ until $|x^+|^2 + |v^+|^2 < \varepsilon$ to find

$$J(u) < \sigma - |\lambda^-|^2 + |\lambda^+|^2 - |x^-|^2 - |v^-|^2 + \varepsilon \leq \sigma - |\lambda^-|^2 + |\lambda^+|^2 + \varepsilon$$

(\beta) increasing $|\lambda^+|$ until $|\lambda^+|^2 < \varepsilon$ to find $J(u) < \sigma - |\lambda^-|^2 + 2\varepsilon$

(\gamma) decreasing $|\lambda^-|$ until $|\lambda^-|^2 > 3\varepsilon$ to find $J(u) < \sigma - \varepsilon$.

Hence and by minimality of $M$ necessarily $c \not\in M_{\sigma}$, proving (1). (2) then follows exactly as Proposition 2.3 upon replacing $X^+$ by $Y^+$ and $X^-$ by $Y^- = X^- \times V^-$. \hfill \Box

Let us comment on these Morse structure results.

Remark 2.2. (i) Suppose, that as in Lemma 2.3 on a neighbourhood $U = U_c$ of $c \in (PS)_\sigma \setminus X$ we have a diffeomorphism

$$\Phi : U_c \longrightarrow \Phi(U_c) = U \subset Y^+ \times Y^-$$

such, that the functional takes form

$$J(u) = J(\Phi^- (y^+, y^-)) = \sigma + |\lambda^+|^2 - |\lambda^-|^2 + |x^+|^2 - |x^-|^2 + |v^+|^2 - |v^-|^2 = F(\lambda^-, \lambda^+, x^+, x^-).$$

Since $\lambda^\pm_1 = \infty$, $v^\pm_i = 0$ corresponds to the Palais-Smale limit, clearly

$$|\partial J|(u) = o_{\sum_i(|\lambda^-_i|^{-1} + |\lambda^+_i|^{-1} + |v^+_i| + |v^-_i|)}(1).$$

But evidently for $|x^+| + |x^-| \neq 0$

$$|\partial F|(\lambda^-, \lambda^+, x^+, x^-) \neq o_{\sum_i(|\lambda^-_i|^{-1} + |\lambda^+_i|^{-1} + |v^+_i| + |v^-_i|)}(1).$$

As a consequence $d\Phi$ must be degenerating and, while $J$ has a clear Morse structure at infinity, its derivative will not relate in a trivial way to that of the Morse representation. For instance consider

$$\Phi : \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \Phi(\mathbb{R}^n \times \mathbb{R}^+) \subset W^{1,2}(\mathbb{R}^n) : (a, \lambda) \longrightarrow \delta_{a, \lambda}$$

for $n \geq 5$ and the functional $J(\delta_{a, \lambda}) = \int K \delta^2_{a, \lambda}$, which expands as

$$J(\delta_{a, \lambda}) = c_1 K(a) + c_2 \frac{\Delta K(a)}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right).$$

The tangential space is given by $T_{\delta_{a, \lambda}} \Phi(\mathbb{R}^n \times \mathbb{R}^+) = \langle \frac{\nabla a}{\lambda} \delta_{a, \lambda}, \lambda \partial a \delta_{a, \lambda} \rangle$, whence for the derivative

$$|\partial J(\delta_{a, \lambda})| \simeq \frac{|\nabla K(a)|}{\lambda} + \frac{|\Delta K(a)|}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right).$$

In particular $(\infty, a) \in (PS)$ for every $a \in \mathbb{R}^n$. On the other hand $J(\delta_{a, \lambda})$ has under 1.5 readily a Morse structure for $a \in \{\nabla K = 0\}$ and $|\nabla a \Phi(a)| = \|\nabla a \delta_{a, \lambda}\| \simeq \lambda$.

(ii) The arguments leading to Proposition 2.3 and then to Lemma 2.3 do rely on a Morse structure of the functional, but not on a corresponding structure of the derivative, cf. (i) above.
(iii) Concerning the infinite index case, consider for instance the functional

\[ J(u) = \sigma - |y|^2 \quad \text{on a Hilbertspace } Y \quad \text{with } \dim(Y) = \infty. \]

Then \( c = 0 \in M_\sigma \). In fact either \( M_\sigma = \emptyset \) or \( M_\sigma = \{c\} \), since necessarily \( M_\sigma \subseteq (PS)_\sigma \). If \( M_\sigma = \emptyset \), then by definition and with \( W = \{J \leq \sigma\} \)

\[ \exists T, \varepsilon > 0 : \Psi \in \Pi F : \Psi(W, T) \subset \{J < \sigma - \varepsilon\}. \]

In particular \( \Psi(0, t) = 0 \) for all times is impossible, whence there exists \( t \geq 0 \) such that

\[ \Psi(0, t) = 0 \quad \text{and} \quad \partial_t \Psi(0, t) = G(\Psi(0, t)) = G(0) \neq 0. \]

Let \( y_0 = -\epsilon G(0) \) and compute \( y(t) = y_0 + tG(y_0) + o(t) \). We then find

\[ \partial_t J(y(t)) = |G(0)|^2(\epsilon - t) + o(t + \epsilon). \]

Hence for \( \epsilon > 0 \) sufficiently small the energy \( J \) is increasing along \( y \) for a short time. This of course contradicts (1.13), since \( y_0 \notin \partial J \).)

(iv) The case \( \dim(Y^-) = \infty \) in Lemma 2.3 and hence the question, whether or not a critical point at infinity can have an infinite index, is more delicate. For instance does \( \lambda = \infty \), \( y = 0 \) for

\[ J(u) = \sigma + \frac{1}{\lambda^2} - |y|^2 \quad \text{on a Hilbertspace } Y \quad \text{with } \dim(Y) = \infty \]

represent an obstacle to energetic deformation, i.e. for any energy decreasing flow of type \( \Psi \in \Pi F \)?

We conjecture, that the answer is no. In any case infinite indices do not occur in our framework.

We finally characterize \( M_\sigma \) as an obstacle to energetic deformation as follows.

**Proposition 2.4.** Let \( \sigma_1 \leq \sigma_2 \). Then every \( \sigma_2 \)-reducible \( W \) is also \( \sigma_1 \)-reducible, if and only if

\[ \forall \sigma_1 < \sigma \leq \sigma_2 : M_\sigma = \emptyset. \]

**Proof.** The case \( \sigma_1 = \sigma_2 \) trivially holds true. Hence let \( \sigma_1 < \sigma_2 \).

Suppose, that every \( \sigma_2 \)-reducible \( W \) is also \( \sigma_1 \)-reducible. Since for \( \sigma_1 < \sigma \leq \sigma_2 \) trivially every \( \sigma \)-reducible \( W \) is also \( \sigma_2 \)-reducible, we find, that every \( \sigma \)-reducible \( W \) is also \( \sigma_1 \)-reducible. Consider hence for \( \sigma_1 < \sigma \leq \sigma_2 \) an arbitrary \( \sigma \)-reducible \( W \) and choose \( \varepsilon > 0 \) such, that \( \sigma_1 + \varepsilon < \sigma - \varepsilon \). Since \( W \) is also \( \sigma_1 \)-reducible, we find \( \Psi \in \Pi F \) and \( T \geq 0 \) such, that \( J(\Psi(W, T)) \leq \sigma_1 + \varepsilon \), cf. Definition 2.3. As a consequence and, since \( \sigma_1 + \varepsilon < \sigma - \varepsilon \), the empty set \( \emptyset \subseteq E_\sigma \) is \( \sigma \)-capturing, cf. Definition 2.4 and then trivially strongly \( \sigma \)-critical as well, cf. Definition 2.4. By uniqueness of a minimal, strongly \( \sigma \)-critical \( M_\sigma \subseteq E_\sigma \) we conclude \( M_\sigma = \emptyset \).

Vice versa suppose, that \( \forall \sigma_1 < \sigma \leq \sigma_2 : M_\sigma = \emptyset \) and consider

\[ s = \inf\{\sigma_1 \leq \sigma \leq \sigma_2 \mid \forall W \sigma_2 \text{-reducible} : W \text{ } \sigma \text{-reducible.} \} \]

In view of Definition 2.3 we then have to show \( s = \sigma_1 \). Arguing by contradiction we assume \( \sigma_1 < s \leq \sigma_2 \) and find, that every \( \sigma_2 \)-reducible \( W \) is also \( s \)-reducible. Since \( M_s = \emptyset \) is strongly \( s \)-critical and by Definition 2.3 also \( s \)-capturing, we find \( \varepsilon > 0 \) and for every \( s \)-reducible \( W \) some \( \Psi \in \Pi F \) and \( T \geq 0 \) such, that \( \Psi(W, T) \subseteq \{J < s - 2\varepsilon\} \). But this implies, that every \( s \)-reducible \( W \) is also \( (s - \varepsilon) \)-reducible and therefore every \( \sigma_2 \)-reducible \( W \) is also \( (s - \varepsilon) \)-reducible. This contradicts the minimality of \( s \) leading to the desired contradiction.

From Proposition 2.3 we recover the classical deformation lemma.
Lemma 2.3. Let \( \sigma_1 < \sigma_2 \) and suppose, that
\[
\forall \sigma_1 \leq \sigma \leq \sigma_2 : M_\sigma = \emptyset.
\]
Then \( \{ J \leq \sigma_2 \} \rightarrow \{ J \leq \sigma_1 \} \) as a weak deformation retract.

Proof. Clearly \( \{ J \leq \sigma_2 \} \) is \( \sigma_2 \)-reducible and by virtue of Proposition 2.4 also \( \sigma_1 \)-reducible. Since \( M_{\sigma_1} = \emptyset \) is strongly \( \sigma_1 \)-critical and hence \( \sigma_1 \)-capturing, cf. Definitions 2.4, 2.5, we find \( \varepsilon > 0 \) and for \( W = \{ J \leq \sigma_2 \} \) a deformation \( \Psi \in \Pi F \) and \( T \geq 0 \) such that \( \Psi(W,T) \subseteq \{ J < \sigma_1 - \varepsilon \} \). And, since the flow \( \Psi \) does not increase energy, clearly \( \Psi : \{ J \leq \sigma_1 \} \times [0,T] \rightarrow \{ J \leq \sigma_1 \} \).

\( \square \)

3 Preliminaries

Let us start with a quantification of the deficit for some \( \varphi_{a,\lambda} \) from solving 1.2.

Lemma 3.1. There holds \( L_{g_0} \varphi_{a,\lambda} = O(\varphi_{a,\lambda}^{\frac{7}{2}}) \). More precisely on a geodesic ball \( B_a(a) \) for \( \alpha > 0 \) small
\[
L_{g_0} \varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{7}{2}} - 2nc n_r a^{-2}(n-1)H_a + r_a \partial_{r_a} H_a) \varphi_{a,\lambda}^{\frac{5}{2}} + \frac{u_a}{\lambda} R_{g_0} \varphi_{a,\lambda}^{\frac{3}{2}} + o(r_a^{-2}) \varphi_{a,\lambda}^{\frac{7}{2}},
\]
where \( r_a = d_{g_0}(a,\cdot) \). In particular

(i) \( L_{g_0} \varphi_{a,\lambda} = 4n(n-1)[1 - \frac{na}{2} r^{-2}(H_a + n \nabla H_a) \varphi_{a,\lambda}^{\frac{5}{2}} + O(\lambda^{-2} \varphi_{a,\lambda}) \) for \( n = 5 \);

(ii) \( L_{g_0} \varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{7}{2}} = 4n(n-1)[1 + \frac{na}{2} W(a) \ln r] \varphi_{a,\lambda}^{\frac{5}{2}} + O(\lambda^{-2} \varphi_{a,\lambda}) \) for \( n = 6 \);

(iii) \( L_{g_0} \varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{7}{2}} = O(\lambda^{-2} \varphi_{a,\lambda}) \) for \( n \geq 7 \).

The expansions stated above persist upon taking \( \lambda \partial_{\lambda} \) and \( \nabla^\alpha \) derivatives.

Proof. Cf. Lemma 2.1 in [14].

Thereby we may describe the blow-up behaviour of Palais-Smale sequences for 1.2.

Proposition 3.1. Let \((u_m)_m \subset W^{1,2}(M,g_0)\) be a sequence with \( u_m \geq 0 \) and \( k_m = k_{u_m} = 1 \) satisfying
\[
J(u_m) = r_m \rightarrow r_\infty \text{ and } \partial J(u_m) \rightarrow 0 \text{ in } W^{-1,2}(M,g_0)
\]
for some \( r_\infty > 0 \). Then up to a subsequence there exist
\[
u_\infty : M \rightarrow [0,\infty) \text{ smooth with } \partial J(u_\infty) = 0,
\]
\( q \in \mathbb{N}_0 \) and for \( i = 1, \ldots, q \) sequences
\[
\alpha_i,m \subset \mathbb{R}_+ \text{ and } M \ni (a_{i,m}) \rightarrow a_{i,\infty}, \quad R_+ \ni \lambda_{i,m} \rightarrow \infty \text{ as } m \rightarrow \infty
\]
for some \( a_{i,\infty} \in M \) such that \( u_m = u_\infty + \sum_i \alpha_i,m \varphi_{a_{i,m},\lambda_{i,m}} + v_m \) and
\[
\|v_m\| \rightarrow 0 \text{ and } \frac{r_m K(a_{i,m}) \alpha_{i,m}^{-\frac{1}{2}}}{4n(n-1)} \rightarrow 1
\]
for each pair \( 1 \leq i < j \leq q \) there holds
\[
\varepsilon_{i,j} = \frac{\lambda_{i}^{\frac{1}{2}}}{\lambda_{i}^{\frac{1}{2}}} + \frac{\lambda_{j}^{\frac{1}{2}}}{\lambda_{j}^{\frac{1}{2}}} + \lambda_i \lambda_j \gamma_n G_{g_0} \varphi_{a_{i,m}}^{\frac{7}{2}}(a_{i,j}) \varphi_{a_{i,j}}^{\frac{7}{2}}, \quad \gamma_n = (4n(n-1))^{\frac{7}{2}}.
\]
Proof. Cf. Proposition 3.1 in [14].

Remark 3.1. We remark, that

(i) \( u_m \to u_\infty \) weakly implies, that necessarily and in addition to (3.2) there holds
\[
L_{g_0}u_\infty = r_{u_m} K_{u_\infty}^{\frac{n+2}{2}} + o(1),
\]

i.e. \( L_{g_0}u_\infty = r_\infty K_{u_\infty}^{\frac{n+2}{2}} \) and therefore \( J(u_\infty) = r_\infty \left( \int K_{u_\infty}^{\frac{n+2}{2}} d\mu_{g_0} \right)^{\frac{n}{2}}. \)

(ii) for the limiting energy \( J_\infty = \lim_{m \to \infty} J(u_m) \) we then obtain
\[
J_\infty = \int L_{g_0}u_\infty u_\infty d\mu_{g_0} + c_n \sum_i \alpha_{i_\infty}^2
\]
where
\[
c_n = \lim_{\lambda \to \infty} \int L_{g_0}\varphi_{a,\lambda}\varphi_{a,\lambda} d\mu_{g_0} = 4n(n-1) \int_{\mathbb{R}^n} \frac{1}{(1+r^2)^n},
\]

\[\text{cf. (3.8) and Lemma 3.1, and (3.2).}\]

 Inserting (3.6) into (3.5) we conclude
\[
J_\infty = (J(u_\infty) + c_n \sum_i \left( \frac{4n(n-1)}{K(a_{i_\infty})^{\frac{n-2}{2}}} \right) \frac{1}{2})^{\frac{n}{2}},
\]

(iii) restricting to \( X = \{ \| \cdot \| = 1 \} \) instead of normalising to \( k = 1 \) is the same up to

\[
\text{(1) } J(u_m) = k_{u_m}^{\frac{2n}{n+2}} \to k_{u_\infty}^{\frac{2n}{n+2}}, \text{ cf. (3.1)}
\]

\[\text{(2) } K(a_{i,m}) \frac{a_{i,m}^{\frac{n+2}{2}}}{4n(n-1) k_{u_m}} = 1 + o(1), \text{ cf. (3.2)}\]

\[\text{(3) } L_{g_0}u_\infty = \frac{K}{k_{u_m}} u_{\infty}^{\frac{n+2}{2}} + o(1), \text{ cf. (3.4)}\]

and \( \| \cdot \| = 1 \) necessitates
\[
1 = c_n \sum_i \alpha_i^2 + \int L_{g_0}u_\infty u_\infty d\mu_{g_0},
\]

(iv) as a consequence of (3.1) the number of bubbles and as a consequence of (3.7) and (3.8) norm and energy of the weak limit \( u_\infty \) of a Palais-Smale sequence are bounded.

Proposition 3.1 justifies to consider the following subset of peaked function and look for zero weak limit Palais-Smale sequences thereon only.

Definition 3.1. For \( \varepsilon > 0, q \in \mathbb{N} \) and \( u \in W^{1,2}(M, g_0) \) let

(i) \( A_u(q, \varepsilon) = \{ (\alpha^i, \lambda_i, a_i) \in \mathbb{R}^q \times \mathbb{R}_+^q \times M^q \mid \forall i \neq j, \lambda_i = 1, \lambda_j = 1, a_i, a_j, \varepsilon_i, \varepsilon_j \}
\]

\[\text{with } 1 - \frac{r a_i}{4n(n-1) k} \leq |1 - \frac{r a_i}{4n(n-1) k}|, \| u - \alpha^i \varphi_{a_i, \lambda_i} \| \leq \varepsilon.\]
Proposition 3.2. For every $\varepsilon_0 > 0$ there exists $\varepsilon_1 > 0$ such, that for $u \in V(q, \varepsilon)$ with $\varepsilon < \varepsilon_1$

$$\inf_{(\tilde{a}, a, \lambda) \in A_u(q, 2\varepsilon_0)} \| u - \tilde{a}^i \varphi_{a_i, \lambda_i} \|^2$$

admits a unique minimizer $(\alpha_i, a_i, \lambda_i) \in A_u(q, \varepsilon_0)$ depending smoothly on $u$ and we set

$$\varphi_i = \varphi_{a_i, \lambda_i}, \quad v = u - \alpha^i \varphi_i, \quad K_i = K(a_i). \quad (3.9)$$

Proof. Cf. Appendix A in [4].

For the sake of brevity and recalling (1.3) we denote e.g.

$$K_i = K(a_i), \quad \nabla K_i = \nabla K(a_i), \quad \Delta K_i = \Delta K(a_i)$$

for a set of points $\{a_i\} \subset M$ and for $k, l = 1, 2, 3$ and $\lambda_i > 0, a_i \in M, i = 1, \ldots, q$ we let

(i) $\varphi_i = \varphi_{a_i, \lambda_i}$ and $(d_{1, i}, d_{2, i}, d_{3, i}) = (1, -\lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i} \nabla a_i)$

(ii) $\phi_{1, i} = \varphi_i, \quad \phi_{2, i} = -\lambda_i \partial_{\lambda_i} \varphi_i, \quad \phi_{3, i} = \frac{1}{\lambda_i} \nabla a_i \varphi_i$, in particular $\phi_{k, i} = d_{k, i} \varphi_i$

(iii) $\langle \phi_{k, i} \rangle = \text{span}\{\phi_{k, i} : k = 1, 2, 3 \text{ and } 1 \leq i \leq q\}$,

in particular $\nabla_{a_i} \varphi_i = (\nabla_{g_0})_{a_i, \varphi_{a_i, \lambda_i}}$ pointwise, i.e.

$$\forall \ x \in M \text{ and } w \in TM : \langle \nabla_{a_i} \varphi_i(x), w(a_i) \rangle = d_{a_i, \varphi_{a_i, \lambda_i}}(x)w(a_i).$$

With this notation the term

$$v = u - \alpha^i \varphi_i$$

from Proposition [3.2] is orthogonal to $\langle \phi_{k, i} \rangle$ with respect to the scalar product

$$\langle \cdot, \cdot \rangle_{L_{g_0}} = \langle L_{g_0} \cdot, \cdot \rangle_{L_{g_0}^2}$$

and we define for $u \in V(q, \varepsilon)$ its complement

$$H_u(q, \varepsilon) = \langle \phi_{k, i} \rangle_{L_{g_0}^2}^\perp.$$ 

A precise analysis of $J$ on $V(q, \varepsilon)$ was performed in [14] by testing the variation $\partial J$ separately with the bubbles $\varphi_i$ and their derivatives $-\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i$ on the one hand and orthogonally to them, i.e. with elements of $H_u(q, \varepsilon)$ on the other.

Recalling [3.3] we collect below some principal interactions over various integrals involving $\phi_{k, i}$, which clearly appear in the gradient testing or expansion of the energy $J$ itself. Note, that $\| \phi_{k, i} \| \simeq 1$.

Lemma 3.2. For $k, l = 1, 2, 3$ and $i, j = 1, \ldots, q$ and we have with constants $b_k, c_k > 0$

(i) $|\phi_{k, i}|, |\lambda_i \partial_{\lambda_i} \phi_{k, i}|, \frac{1}{\lambda_i} \nabla_{a_i} \phi_{k, i} | \leq C \varphi_i$

(ii) $\int \varphi_i^4 \phi_{k, i} \phi_{k, i} d\mu_{g_0} = c_k \cdot id + O(\frac{1}{\lambda_i}), \quad c_k > 0$
(iii) for \( i \neq j \)
\[
\int \phi_i^{\frac{n+2}{n+2}} d\mu_{g_0} = b k d_i \varepsilon_{i,j} = \int \phi_i d_{k,j} \phi_j^{\frac{n+2}{n+2}} d\mu_{g_0}
\]

(iv) \( \int \phi_i^{\frac{n-2}{n+2}} \phi_{k,i} d\mu_{g_0} = O(\frac{1}{\lambda_i^2}) \) for \( k \neq l \) and \( \int \phi_i^{\frac{n+2}{n-2}} \phi_{k,i} d\mu_{g_0} = O \left( \frac{\lambda_i^{2-n}}{\lambda_i^2} \right) \) for \( n \leq 5 \), \( \frac{\ln \lambda_i}{\lambda_i^2} \) for \( n = 6 \), \( \frac{\lambda_i^{2-n}}{\lambda_i^2} \) for \( n \geq 7 \) for \( k = 2, 3 \).

(v) \( \int \phi_i^{\alpha, \beta} d\mu_{g_0} = O(\varepsilon_{i,j}^2) \) for \( i \neq j \), \( \alpha + \beta = \frac{2n}{n-2} \), \( \alpha > \frac{n-2}{\lambda_i} \) \( > \beta \geq 1 \)

(vi) \( \int \phi_i^{\frac{n+2}{n-2}} \phi_{j,i} d\mu_{g_0} = O(\varepsilon_{i,j}^2 \ln \varepsilon_{i,j}) \) for \( i \neq j \)

(vii) \( (1, \lambda_i \partial_{\lambda_i}, \frac{\nabla a_{i}}{\lambda_i}) \varepsilon_{i,j} = O(\varepsilon_{i,j}) \) for \( i \neq j \).

Proof. Cf. Lemma 3.4 in [18] or Lemma 2.2 in [14].

Let us comment on the following lemmata, which describe the testing of \( \partial J \). First a testing in an orthogonal direction is due to orthogonalities small.

Lemma 3.3. For \( u \in V(q, \varepsilon) \) with \( k = 1 \) and \( \nu \in H_u(q, \varepsilon) \) there holds
\[
\partial J(\alpha^i \varphi_i) \nu = O \left( \left[ \sum_r \frac{\nabla K_r}{\lambda_r} + \sum_r \frac{1}{\lambda_r^2} + \sum_{r \neq s} \varepsilon_{r,s}^{\frac{n+2}{n}} \right] \| \nu \| \right).
\]

Proof. Cf. Proposition 4.4 in [18] or Lemma 4.1 in [14].

In combination with the well known uniform positivity of the second variation on the orthogonal space \( \langle \phi_{k,i} \rangle^\perp \), cf. [22], this allows us to estimate \( v \) itself in terms of the aforegoing quantities.

Lemma 3.4. For \( u \in V(q, \varepsilon) \) with \( k = 1 \) and \( v \) is as in (3.9) there holds
\[
\| v \| = O \left( \sum_r \frac{\nabla K_r}{\lambda_r} + \sum_r \frac{1}{\lambda_r^2} + \sum_{r \neq s} \varepsilon_{r,s}^{\frac{n+2}{n}} + | \partial J(u) | \right).
\]

Proof. Cf. Corollary 4.6 in [18] or Lemma 4.2 in [14].

The latter smallness estimate will turn out to be sufficient to consider \( v \) as a negligible quantity in the sense, that \( v \) is not responsible for a blow-up.

Let us turn to the testing in the directions of the bubbles and their derivative as had been performed carefully in low dimensions \( n = 3, 4, 5 \) in Section 4 of [13]. We note, that for each bubble we have three quantities associated, namely \( \alpha, a \) and \( \lambda \). The \( \alpha \)-direction then corresponds to a testing with a bubble itself, since \( \alpha \partial_{\alpha}(\alpha \varphi_{a, \lambda}) = \alpha \varphi_{a, \lambda} \). Again for the sake of brevity let us define the quantities
\[
\alpha^2 = \sum_i \alpha_i^2 \quad \text{and} \quad \alpha_{K_i}^{\frac{2n}{n-2}} = \sum_i K_i \alpha_i^{\frac{2n}{n-2}}, \quad (3.10)
\]
which are the principal terms in the nominator and denominator of \( J \), cf. [14].
Lemma 3.5. For \( u \in V(q, \varepsilon) \) and \( \varepsilon > 0 \) sufficiently small the three quantities \( \partial J(u) \phi_{1,j}, \partial J(\alpha^i \phi_i) \phi_{1,j} \),
\( \partial_\alpha J(\alpha^i \phi_i) \) can be written as

\[
\frac{\alpha_j}{(\alpha_K^i \alpha_j)} \left( \tilde{c}_j \left( 1 - \frac{\alpha_j^2}{\alpha_K^i} K_j \alpha_j^2 \right) - \tilde{c}_k \left( \frac{\Delta K_k^j}{K_j \lambda_j^2} \right) - \sum_k \frac{\Delta K_k^j \alpha_j^2}{K_k \lambda_k^2} \right)
+ b_1 \left( \sum_{k \neq j} \frac{\alpha_k \alpha_l}{\alpha_j} \varepsilon_{k,l} - \sum_{j \neq i} \frac{\alpha_l}{\alpha_j} \varepsilon_{i,j} \right) - \tilde{d}_1 \left( \begin{array}{ll}
\frac{H_j}{\lambda_j^3} & \text{for } n = 5 \\
\frac{W_j \ln \lambda_j}{\lambda_j} & \text{for } n = 6 \\
0 & \text{for } n \geq 7
\end{array} \right)
\]

with positive constants \( \tilde{c}_0, \tilde{c}_2, \tilde{b}_1, \tilde{d}_1 \) and up to some \( O\left( \sum_{r \neq s} \frac{||\nabla K_r^j||^2}{\lambda_j^2} + \frac{1}{\lambda_j^2} + \varepsilon_{r,s} + |\partial J(u)|^2 \right) \).

Proof. Cf. Lemma 5.1 in [14], for instance see also Lemma A.4.3 in [3] or Proposition 5.1 in [7]. \( \square \)

Evidently the principal term due to largeness of the concentration parameters \( \lambda_i \) and smallness of the interaction terms \( \varepsilon_{i,j} \) in the above expansion is the one related to \( \tilde{c}_0 \) forcing \( \alpha_j \) into a certain regime.

Lemma 3.6. For \( u \in V(q, \varepsilon) \) and \( \varepsilon > 0 \) sufficiently small the three quantities \( \partial J(u) \phi_{2,j}, \partial J(\alpha^i \phi_i) \phi_{2,j} \)
and \( \frac{\Delta}{\alpha_j \lambda_j} \partial_\lambda J(\alpha^i \phi_i) \) can be written as

\[
\frac{\alpha_j}{(\alpha_K^i \alpha_j)} \left( \tilde{c}_2 \frac{\Delta K_j^i}{K_j \lambda_j^2} - \tilde{b}_2 \sum_{j \neq i} \frac{\alpha_l \lambda_j}{\alpha_j} \partial_\lambda \varepsilon_{i,j} + \tilde{d}_1 \left( \begin{array}{ll}
\frac{H_j}{\lambda_j^3} & \text{for } n = 5 \\
\frac{W_j \ln \lambda_j}{\lambda_j} & \text{for } n = 6 \\
0 & \text{for } n \geq 7
\end{array} \right) \right)
\]

with positive constants \( \tilde{c}_1, \tilde{c}_2, \tilde{b}_2, \tilde{d}_2 \) and up to some \( O\left( \sum_{r \neq s} \frac{||\nabla K_r^j||^2}{\lambda_j^2} + \frac{1}{\lambda_j^2} + \varepsilon_{r,s} + |\partial J(u)|^2 \right) \).

Proof. Cf. Lemma 5.2 in [14], for instance see also Lemma A.4.3 in [3] or Proposition 5.1 in [7]. \( \square \)

Here at least in high dimensions the principal terms are the ones related to \( \tilde{c}_1 \) and \( \tilde{b}_2 \). The first one turns out to be responsible for a potential diverging flow line within \( V(q, \varepsilon) \) depending on the sign of \( \Delta K \), the latter one, measuring interactions, may be relatively strong or weak depending on, whether the corresponding \( a_j \) are close to \( a_i \) or not. In any case these interaction terms will turn out to be responsible for excluding tower bubbling, i.e. multiple bubbles concentrating at the same point along a flow line, just as they prevent tower bubbling in the subcritical case, cf. [15]. The location of a bubble \( \phi_{a,\lambda} \) on \( M \) in the sense of the centre \( a \) is principally determined from the \( a \)-testing below.

Lemma 3.7. For \( u \in V(q, \varepsilon) \) and \( \varepsilon > 0 \) sufficiently small the three quantities \( \partial J(u) \phi_{3,j}, \partial J(\alpha^i \phi_i) \phi_{3,j} \)
and \( \frac{\nabla}{\alpha_j \lambda_j} \partial_\lambda J(\alpha^i \phi_i) \) can be written as

\[
\frac{\alpha_j}{(\alpha_K^i \alpha_j)} \left( \tilde{c}_3 \frac{\nabla K_j^i}{K_j \lambda_j^3} + \tilde{c}_4 \frac{\nabla \Delta K_j^i}{K_j \lambda_j^3} + \tilde{b}_3 \sum_{j \neq i} \frac{\alpha_l}{\alpha_j} \frac{\nabla a_l}{\lambda_j} \varepsilon_{i,j} \right)
\]

with positive constants \( \tilde{c}_3, \tilde{c}_4, \tilde{b}_3 \) and up to some \( O\left( \sum_{r \neq s} \frac{||\nabla K_r^j||^2}{\lambda_j^2} + \frac{1}{\lambda_j^2} + \varepsilon_{r,s} + |\partial J(u)|^2 \right) \).

Proof. Cf. Lemma 5.3 in [14], for instance see also Lemma A.4.3 in [3] or Proposition 5.1 in [7]. \( \square \)
Evidently the principal terms are the one related to $\hat{c}_3$, trying to force the centres of concentration to be close to critical points of $K$, and the one related to $b_j$.

Of course the source of delicacy is, that the principle terms above are related by their error terms.

**Proposition 3.3.** For $\varepsilon > 0$ sufficiently small there holds uniformly on $V(q, \varepsilon) \cap \{k = 1\}$

$$\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \left| 1 - \frac{\alpha^2 K_r \alpha_r}{\alpha_K^2} \right| + \varepsilon_{r,s} \lesssim |\partial J| \lesssim \sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \left| 1 - \frac{\alpha^2 K_r \alpha c^{-2}}{\alpha_K^2} \right| + \varepsilon_{r,s} + \|v\|.$$  

**Proof.** The lower bound is due to Theorem 2 in [14], the upper bound due to Lemma 6.1 in [14].

Here and later on we use for two functions $a, b \geq 0$ the shorthand notation

$$a \lesssim b :\iff \exists C > 0 : a \leq C b.$$  

We note, that the latter gradient estimates evidently prevent the existence of a solution in $V(q, \varepsilon)$ and allow us to compare the quantities appearing to $|\partial J|$ and vice versa. Finally we may perform an expansion of the energy itself on $V(q, \varepsilon)$, which reads as

**Lemma 3.8.** For $u = \alpha^i \varphi_i + v \in V(q, \varepsilon)$ and $\varepsilon > 0$, both $J(u)$ and $J(\alpha^i \varphi_i)$ can be written as

$$\frac{\hat{c}_0 \alpha^2}{(\alpha_K^2)^{n-2}} \left( 1 - \hat{c}_2 \sum_i \frac{\Delta K_i \alpha^2}{K_i \lambda_i^2} \alpha^2 + \hat{b}_1 \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\alpha^2} \varepsilon_{i,j} - \hat{d}_1 \sum_i \frac{\alpha^2}{\alpha^2} \left( \begin{array}{c} H_i \lambda_i^2 \alpha_i \lambda_i^2 \varepsilon_{i,j} \\ \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\alpha^2} \varepsilon_{i,j} \end{array} \right) \right)$$

with positive constants $\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{b}_1, \hat{d}_1$ and up to some $O(\sum_r \frac{\nabla K_r|}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \sum_{r \neq s} \varepsilon_{r,s} + |\partial J|)^2$

**Proof.** Cf. Proposition 5.1 in [14], see also Proposition 5.6 in [2].

In the following we will work with the normalisation to the unit sphere, i.e. on $\{\|\cdot\| = 1\}$, whereas in [14] and [13] we have been restricting to $\{k = 1\}$, i.e. to the unit sphere with respect to the conformal $K$-volume, cf. (1.4). However, along an energy decreasing flow line $u$ we have

$$0 < c < J(u) = \frac{r}{k^{\frac{n-2}{n}}} = \frac{\|u\|^2}{k^{\frac{n-2}{n}}} < J(u_0)$$

thanks to the positivity of the Yamabe invariant, cf. (1.1), and hence a control of $k$ via $\|u\|$ and vice versa. Moreover on $V(q, \varepsilon)$ there holds

$$J(u) = \frac{\alpha^2}{(\alpha_K^2)^{n-2}} + o(1) \quad \text{and} \quad \frac{\alpha^2}{\alpha_K^2} K_i \alpha_i^{-2} = 1 + o(1),$$

cf. [3,10], whence, as an easy computation shows, we have uniform energy control on each $V(q, \varepsilon)$ via

$$J(u) = \sum_i (K_i^{\frac{2-n}{n}})^{\frac{2}{n}} + o(1) \quad \text{for} \quad u \in V(q, \varepsilon).$$

In particular the foregoing Lemmata are still applicable, when working on $\{\|\cdot\| = 1\}$ instead of $\{k = 1\}$. 

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4 Flow construction

Lemma 4.1. For bounded $v \in \langle \varphi_i, \lambda_i \partial_{\lambda_i} \varphi_i, \frac{\nabla \varphi_i}{\lambda_i} \rangle_{L^2}$ and

$$\beta_{\alpha, a, \lambda} = (\beta_{\alpha_i}, \beta_{a_i}, \beta_{\lambda_i}) \in C^0(V(q, \varepsilon))$$

there exist $b_\alpha, b_v \in C^0(V(q, \varepsilon))$ with

(i) $b_\alpha = \frac{\sum \beta_{\alpha_i} \alpha_i^2 \Vert \varphi_i \Vert^2}{\sum \alpha_i \Vert \varphi_i \Vert^2} + O(\sum_{r \neq s} \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + \|v\|^2 + \|\nu\|^2);$

(ii) $b_v = O(||v|| (\sum |\beta_{\lambda_i}| + |\beta_{a_i}|) \sum_{r} \varphi_r) \in \langle \phi_{k,i} \rangle$

such that moving $u = \alpha^i \phi_{a_i} + \nu \in V(q, \varepsilon)$ and $\nu \in \{ \alpha \in \mathbb{R} \}$ along

$$\frac{\lambda_i}{\dot{\lambda}_i} = \beta_{\alpha_i} + b_\alpha, \quad \frac{\lambda_i}{\dot{\lambda}_i} = \beta_{\lambda_i}, \quad \lambda_i \dot{\alpha}_i = \beta_{a_i}, \quad \dot{\partial} = b_v + \nu$$

there holds $\partial_i \|u\|^2 = 0$ and $v \in \langle \varphi_i, \lambda_i \partial_{\lambda_i} \varphi_i, \frac{\nabla \varphi_i}{\lambda_i} \rangle_{L^2}.$

Remark 4.1. Lemma 4.1 simply tells us, that for every principal movement

$$\langle \alpha_i, \lambda_i \rangle$$

and $\nu$ we may

(i) preserve the movement of $\langle \alpha_i, \lambda_i \rangle$ along $(\beta_{\alpha_i}, \beta_{\lambda_i})$

(ii) modify the movement of $\langle \alpha_i, \nu \rangle$ along $(\beta_{\alpha_i}, \nu)$ only slightly by $(b_\alpha, b_v)$ such that we

(iii) ensure, that writing

$$u = \alpha^i \phi_{a_i} + \nu = \alpha^i \varphi_i + v$$

remains compatible with the representation on $V(q, \varepsilon)$ in the sense of proposition 3.2 and

(iv) preserve the norm $\|u\| = \|u\|_{L^2}$ of $u = \alpha^i \varphi_i + v$.

Proof. Let us suppose a $\langle v, \phi_{k,i} \rangle_{L^2}$ 0 orthogonality preserving evolution

$$\partial_t u = \partial_t (\alpha^i \varphi_i) + \partial_t v = \beta_{\alpha_i} \alpha^i \varphi_i + \alpha^i \varphi_i + \beta_{\lambda_i} \alpha^i \lambda_i \partial_{\lambda_i} \varphi_i + \beta_{a_i} \alpha^i \frac{\nabla \varphi_i}{\lambda_i} + \alpha^i \varphi_i + b_v + \nu$$

exists. Then the preservation of orthogonality, i.e. $\partial_t \langle v, \phi_{k,i} \rangle = 0$ for $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2},$ necessitates

(i) in the $a$-variable

$$0 \equiv \partial_t \langle v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle = \langle b_v + v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \langle v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \beta_{a_i} \lambda_i \partial_{\lambda_i} \varphi_i$$

(ii) in the $\lambda$-variable

$$0 \equiv \partial_t \langle v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle = \langle b_v + v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \langle v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \beta_{a_i} \lambda_i \partial_{\lambda_i} \varphi_i$$

(iii) in the $a$-variable

$$0 \equiv \partial_t \langle v, \frac{\nabla \varphi_i}{\lambda_i} \rangle = \langle b_v + v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \langle v, \lambda_i \partial_{\lambda_i} \varphi_i \rangle + \beta_{a_i} \left( \frac{\nabla \varphi_i}{\lambda_i} \right)^2$$
Since $|φ_{k,i}| ≤ Cφ_i$ pointwise and, as follows from Lemmata 3.1 and 3.2,

$$\langle φ_{k,i}, φ_{l,j} \rangle = c_{k,i,l,j} δ_{k,i,l,j} + O(\sum_{r \neq s} \frac{1}{λ_r^2} + ε_{r,s})$$

for some $c_{k,i,l,j} < C$, we may choose some $O(φ_j) = φ^*_j ∈ \langle φ_{k,i} \rangle$ dual to $φ_{k,i} = (φ_i, λ_i δ_{k,i → φ_i}, \frac{α_i}{λ_i} φ_k,i)$ such that

$$\langle φ^*_j, φ_{k,i} \rangle = δ_{k,i,l,j}$$

and $\|φ^*_j - φ_{k,i}\| = O(\sum_{r \neq s} \frac{1}{λ_r^2} + ε_{r,s}).$ \hspace{1cm} (4.1)

We then may solve (i)-(iii) via $b_v = b^k,i φ^*_k,i$, $b_{l,i} = 0$ and for $k = 2, 3$

$$b_{k,i} = - β_{λ_i}(v, λ_i δ_{k,i → φ_i}) + β_{a_i}(v, \frac{α_i}{λ_i} φ_k,i).$$ \hspace{1cm} (4.2)

Conversely solving $∂_t v = b_v + ν$ with the above choice of $b_v$ we find

$$\langle v, φ_{k,i} \rangle_{t=r} = \langle v, φ_{k,i} \rangle_{t=0} + \int^r_0 ∂_t \langle v, φ_{k,i} \rangle = \int^r_0 \langle b_v, φ_{k,i} \rangle + \langle v, ∂_t φ_{k,i} \rangle = \int^r_0 b_{k,i} = β_{λ_i}(v, λ_i δ_{k,i → φ_i}) + β_{a_i}(v, \frac{α_i}{λ_i} φ_k,i) = 0.$$

Hence the statement on $b_v$ follows. Therefore and in particular due to $b_{l,i} = 0$ we find as well

$$∂_t \|u\|^2 = 2(α^i φ_i + \frac{α_i}{λ_i} φ_i + \frac{λ_i}{α_i} λ_i δ_{k,i → φ_i} + λ_i α_i δ_{k,i → φ_i} + b_v + v)$$

$$= 2 \sum_i a_i^2 (β_{a_i} + b_α) \|φ_i\|^2 + O(\sum_{r \neq s} \frac{1}{λ_r^2} + ε_{r,s} + \|v\|^2 + \|ν\|^2),$$

 whence $∂_t \|u\|^2 = 0$ is equivalent to putting

$$b_α = - \sum_i a_i^2 \frac{β_{a_i} \|φ_i\|^2}{\|φ_i\|^2} + O(\sum_{r \neq s} \frac{1}{λ_r^2} + ε_{r,s} + \|v\|^2 + \|ν\|^2),$$

noticing that $c < α_i, \|φ_i\| < C$ on $V(q, ε) \cap \{\|\cdot\| = 1\}$. \hspace{1cm} \Box

**Definition 4.1.** Let $q ∈ N$. Then for arbitrary constants

$$0 < ε = ε_q ≪ 1$$

and $κ_α = κ_{α,q}, κ_λ = κ_{λ,q}, κ_α = κ_{α,q}, κ_v = κ_{v,q} ≫ 1$ \hspace{1cm} (4.3)

consider on $V(q, 2ε) \cap \{\|\cdot\| = 1\}$ the subsets

(i) \hspace{1cm} $A_{v,κ_α} = \{\|v\| ≥ κ_α(\sum_{r \neq s} \frac{|∇_{K_r}|}{λ_r} + \frac{1}{λ_r^2} + |1 - \frac{α_i}{α_r} K_r α_r \frac{4}{r} | + ε_{r,s})\}$;

(ii) \hspace{1cm} $A_{κ_α} = \{κ_α(\sum_{r \neq s} \frac{|∇_{K_r}|}{λ_r} + \frac{1}{λ_r^2} + \frac{4}{r} | + ε_{r,s})\}$;

(iii) \hspace{1cm} $A_{a_i,κ_α} = \{\|∇_{K_r}\| ≥ \frac{κ_α}{λ_r^2}\}$;

(iv) \hspace{1cm} $A_{κ_λ} = \{\sum_{r \neq s} ε_{r,s} ≥ \frac{κ_λ}{λ_r^2}\}$;

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(v) \( A_{\lambda_j, \kappa, \lambda_i}^\leq = \{ \sum_{r \neq s} \varepsilon_{r,s} \leq \frac{\kappa^2}{\lambda_j^2} \} \)

and to each of these subsets a corresponding cut-off functions

\( \eta \in C^\infty (V(q, \varepsilon), [0, 1]) \)

satisfying

(i) \( \eta_v \Bigr|_{A_{\nu, \kappa, \nu}} = 1 \) and \( \eta_v \Bigr|_{(A_{\nu, \kappa, \nu})^c} = 0; \)

(ii) \( \eta_{\alpha} \Bigr|_{A_{\alpha, \kappa, \alpha}} = 1 \) and \( \eta_{\alpha} \Bigr|_{(A_{\alpha, \kappa, \alpha})^c} = 0; \)

(iii) \( \eta_{\alpha_j} \Bigr|_{A_{\alpha_j, \kappa, \alpha_j}} = 1 \) and \( \eta_{\alpha_j} \Bigr|_{(A_{\alpha_j, \kappa, \alpha_j})^c} = 0; \)

(iv) \( \eta_{\lambda_j, \kappa, \lambda_i}^\geq \Bigr|_{A_{\lambda_j, \kappa, \lambda_i}^\geq} = 1 \) and \( \eta_{\lambda_i, \kappa, \lambda_i}^\geq \Bigr|_{(A_{\lambda_i, \kappa, \lambda_i})^c} = 0; \)

(v) \( \eta_{\lambda_i, \kappa, \lambda_i}^\leq \Bigr|_{A_{\lambda_i, \kappa, \lambda_i}^\leq} = 1 \) and \( \eta_{\lambda_i, \kappa, \lambda_i}^\leq \Bigr|_{(A_{\lambda_i, \kappa, \lambda_i})^c} = 0. \)

Moreover for some monotone cut-off function \( \eta \in C^\infty (R, [0, 1]) \) with

\( \eta|_{(-\infty, \frac{1}{4})} = 0, \eta|_{[1, \infty)} = 1 \)

define

\( m_{\lambda_j, \lambda_i} = \eta \left( \frac{\lambda_j}{\lambda_i} \right) \) and \( m_{\lambda_j} = \kappa \sum_{r \neq i} m_{\lambda_j, \lambda_i}, \kappa \gg 1. \)

As set out in Lemma 4.1 we then evolve on \( V(q, 2\varepsilon) \) according to

(a) \( \frac{\partial v}{\partial t} = \beta_{\alpha_j} + b_{\alpha} = -\eta_{\alpha}(1 - \eta_{\alpha}) ||\varphi_j||^{-2}(1 - \frac{\alpha^2}{\alpha_K}K_j\alpha_j^\frac{1}{2}) + b_{\alpha}; \)

(a) \( \lambda_j \partial_{\lambda_j} = \beta_{\alpha_j} = (1 - \eta_{\alpha})(1 - \eta_{\alpha})\eta_{\alpha_j} \frac{\nabla K_j}{|\nabla K_j|}; \)

(\lambda) \( \frac{\lambda_j}{\lambda_j} = \beta_{\lambda_j} = -(1 - \eta_{\alpha})(1 - \eta_{\alpha})(\eta_{\lambda_i}^\leq \Pi_i(1 - \eta_{\alpha_i}) m_{\lambda_j, \lambda_i} \frac{\Delta K_j}{|\Delta K_j|} + \eta_{\lambda_i}^\geq m_{\lambda_j}) ; \)

(v) \( \partial_t v = b_v + \nu = b_v - C_v v \)

with a constant \( C_v = C_{v,q} \gg 1, i.e. \)

\( \partial_t u = \partial_t (\alpha^\lambda \varphi_{\alpha, \lambda} + \partial_t v = A(q, a, \alpha, v), \)

where

\[
A(q, a, \alpha, \lambda, v) = -\eta_{\alpha}(1 - \eta_{\alpha}) ||\varphi_j||^{-2}\alpha_j^2(1 - \frac{\alpha^2}{\alpha_K}K_j\alpha_j^\frac{1}{2}) \varphi_j + b_{\alpha}\varphi_i
\]

\[
+ (1 - \eta_{\alpha})(1 - \eta_{\alpha})\eta_{\alpha_j} \alpha_j^\lambda \frac{\Delta K_j}{|\Delta K_j|} \left( \frac{\nabla K_j}{|\nabla K_j|} - \frac{\nabla \lambda_j}{\lambda_j} \varphi_j \right)
\]

\[
- (1 - \eta_{\alpha})(1 - \eta_{\alpha})\alpha_j^\lambda \Pi_i(1 - \eta_{\alpha_i}) m_{\lambda_j, \lambda_i} \frac{\Delta K_j}{|\Delta K_j|} + \eta_{\lambda_i}^\geq m_{\lambda_j} \lambda_j \partial_{\lambda_j} \varphi_j
\]

\[
+ b_v - C_v v.
\]

Remark 4.2.  (i) There holds

\[
b_{\alpha} = o(||v||) + O(\sum_{r \neq s} \frac{1}{\lambda_j^2} + \varepsilon_{r,s}),
\]

(4.5)
cf. Lemma 4.1, since from

\[ \frac{\dot{\alpha}_i}{\alpha_j} = -\eta_a (1 - \eta_v) \| \varphi_j \|^2 (1 - \frac{\alpha^2}{\alpha_K} K_j \alpha_j^{-\frac{1}{2}}) + b_j = \beta_{\alpha_j} + b_j \]

we find

\[ \sum_i \beta_{\alpha_i} \alpha_i^2 \| \varphi_i \|^2 = 0, \]

cf. (3.10). Secondly, since \( \nu = -C_v v \), we find \( \| \nu \|^2 = o(\| v \|) \), provided

\[ \| \nu \| = C_v \| v \| \leq C \varepsilon = o(1), \]

which we may assume as \( \varepsilon \to \infty \).

(ii) The purpose of introducing \( C_v \) is to obtain \( \partial_t \| v \|^2 \leq 0 \) and in fact \( b_v \) does not depend on \( C_v \). Moreover from (4.4) and (1.5) it is clear, that

\[ \partial_t u \geq -C \alpha \varphi_i - C_v \]

for a universal \( C_{\kappa, q} > 0 \) independent of \( C_v \). Hence \( \partial_t u \geq -C_v u \) by fixing \( C_v \gg C \).

(iii) Whereas the movement in \( \alpha \) and \( a \) are obviously well defined, we note, that

\[ \{ \eta_{\lambda_j} \| \Pi_i (1 - \eta_{\alpha_i}) \| m_{\lambda_j, \lambda_i} \neq 0 \} \subseteq \{ \eta_{\alpha_i} \neq 1 \} = \| \nabla K_j \| \leq \frac{K_{\lambda_j}}{\lambda_j}, \]

whence by non degeneracy, cf. (4.10), also the movement in \( \lambda \) is well defined.

(iv) The union \( \cup A_{\kappa, \lambda} \) covers \( V(q, 2\varepsilon) \) for \( \varepsilon > 0 \) sufficiently small. Indeed we have

\[ \forall 1 \leq i, j \leq q : \frac{1}{\lambda_i^2} \lesssim \sum_{r \neq s} \varepsilon_{r, s} \approx \frac{1}{\lambda_j^2} \text{ and } |\nabla K_j| \lesssim \frac{1}{\lambda_j} \]

on \( A^c = V(q, 2\varepsilon) \setminus \cup A_{\kappa, \lambda} \), whence \( \frac{1}{\lambda_i^2} \approx \frac{1}{(\lambda_i \lambda_j)^{\frac{1}{2}}} \approx \frac{1}{\lambda_j} \), a contradiction for \( n \geq 5 \).

(v) In view of Lemmata (3.8, 3.9 and 3.7) we call

\[ \begin{align*}
(\alpha) & \quad P(\partial J(u) \varphi_j) = (1 - \frac{\alpha^2}{\alpha_K} K_j \alpha_j^{-\frac{1}{2}}); \\
(\lambda) & \quad P(\partial J(u) \lambda_j \partial \lambda_j) = \frac{\Delta K_j}{\lambda_j}; \quad \sum_{j \neq i} \lambda_j \partial \lambda_j \varepsilon_{i, j}; \\
(a) & \quad P(\partial J(u) \frac{\nabla \lambda_j}{\lambda_j}) = \frac{\nabla K_j}{\lambda_j}.
\end{align*} \]

the principal terms in \( \alpha, \lambda \) and \( a \). The flow is designed in such a way, that whenever the principal terms are dominant in the expansion of the lemmata above, their corresponding movement in \( \alpha, \lambda \) and \( a \) will decrease energy. Notably we move \( \lambda = \max \lambda_i \) as little as possible by the Laplacian of \( K \).

From Lemma 4.1 and Definition 4.1 it is clear that in particular, we may generate a flow.

**Lemma 4.2.** On \( X = \{ 0 \leq u \in W^{1,2}(M) \mid u \neq 0, \| u \| = 1 \} \) the evolution

\[ \partial_t u = \eta \sum_q A_q(\alpha, a, \lambda, v) - (1 - \eta) \nabla J(u), \quad \nabla = \nabla_{L^2} \]

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for a cut-off function $\eta \in C^\infty_{loc}(X,[0,1])$ satisfying

$$\eta|_{\cup_{q}V(q,c_{q})} = 1 \quad \text{and} \quad \eta|_{X\backslash \cup_{q}V(q,2c_{q})} = 0$$

induces a semi-flow

$$\Phi: \mathbb{R}_{\geq 0} \times X \to X,$$

i.e. the flow exists for all times, remains non negative and preserves $\| \cdot \| = 1$, provided

$$\forall q \in \mathbb{N} : \kappa_{v} \gg 1, \kappa_{a} \gg 1, \kappa_{\alpha} \gg \kappa_{\lambda}^{2}, \sqrt{\kappa_{\lambda}} \gg \kappa_{v}^{2}, \kappa \gg 1 \quad \text{and} \quad 0 < \varepsilon \ll 1,$$

cf. (4.3).

**Proof.** Since $\nabla J = \nabla_{L_{g_{0}}} J$ is the $L_{g_{0}}$-gradient, we may write $\partial_{t}u = f(u)$ and $f$ is a locally smooth vectorfield on $W^{1,2}(M, \mathbb{R}_{\geq 0}) \backslash \{0\}$. So we have short time existence and due to

$$\partial_{t}\|u\|^{2} = -2(1 - \eta)\langle \nabla J(u), u \rangle_{L_{g_{0}}} + 2\eta\langle \sum_{q} A_{q}(\alpha, a, \lambda, v), u \rangle_{L_{g_{0}}} = 0,$$

as $\langle A_{q}(\alpha, a, \lambda, v), u \rangle_{L_{g_{0}}} = 0$ on $V(q,2\varepsilon)$ by construction and

$$\langle \nabla J(u), u \rangle_{L_{g_{0}}} = \partial J(u)u = 0$$

by scaling invariance of $J$, the $L_{g_{0}}$-norm is preserved. Moreover, as Proposition 4.1 will show, there holds $\partial_{t}J(u) \leq 0$, whence in combination with the positivity of the Yamabe invariant we have

$$c_{K} < J(u) = \frac{\int L_{g_{0}} u \mu_{g_{0}}}{(\int K u^{\frac{n-2}{2}} d\mu_{g_{0}})^{\frac{n-2}{2}}} = \frac{\|u\|^{2}}{k^{\frac{n-2}{2}}} = \frac{1}{k^{\frac{n-2}{2}}} < J(u_{0})$$

(4.6)

along each flow line for its time of existence. Hence $\nabla J$ and thus $f$ are uniformly bounded along each flow line and, as is easy to see, locally smooth. Therefore every flow line exists for all times. Moreover, as $|\phi_{k,i}| \lesssim \varphi_{i}$, we find from (4.4), (1.5) and (ii) of Lemma 4.1 that

$$A_{q}(\alpha, a, \lambda, v) \geq -C(\alpha^{4} \varphi_{i}) - C_{v}v \geq C_{v}u \quad \text{on} \quad V(q,2\varepsilon) \quad \text{for some} \quad C > 0$$

provided $C_{v} > C$, i.e. $C_{v} > 0$ is sufficiently large. Hence

$$\nabla J(u) = L_{g_{0}}^{-}(\partial J(u)) = \frac{2L_{g_{0}}^{-}r}{k^{\frac{n+2}{2}}(L_{g_{0}} u - r K u^{\frac{n+2}{2}}}) < \frac{2u}{k^{\frac{n+2}{2}}}$$

by definition of the $L_{g_{0}}$-gradient and positivity of $L_{g_{0}}^{-} \simeq G_{g_{0}} > 0$. We conclude $\partial_{t}u > -cu$ using (4.6), so every initially non negative or positive flow line becomes or remains positive for all times to come.

We point out, that the long time existence part is not critical, since the flow is based on a strong gradient, i.e. the gradient corresponding to the metric on the variational space, and thus falls into the class of ordinary differential equations, cf. [8] or [18], in contrast to Yamabe type flows as in [8] or [18].

**Proposition 4.1.** Under $u = \Phi(\cdot, u_{0})$ there holds

$$\partial_{t}J(u) \leq -c_{q}(\sum_{r \neq s} |\nabla K_{r}|^{2} \lambda^{2}) + 1 - \frac{\alpha^{2}}{\alpha^{2}_{K} - 2} K_{r} \alpha^{2}_{r} \lambda^{2} |\varphi|^{2} + \varepsilon_{r,\lambda}^{2} + \|v\|^{2}$$

on $V(q,2\varepsilon)$

for some $c_{q} > 0$, provided

$$\kappa_{v} \gg 1, \kappa_{a} \gg 1, \kappa_{\alpha} \gg \kappa_{\lambda}^{2}, \sqrt{\kappa_{\lambda}} \gg \kappa_{v}^{2}, \kappa \gg 1 \quad \text{and} \quad 0 < \varepsilon \ll 1,$$

cf. (4.3), while $\partial_{t}J(u) = -|\nabla J(u)|^{2}$ on $(\cup_{q}V(q,2\varepsilon))^{c}$. 30
Proof. First consider the flow $\partial_t u = A_q(\alpha, a, \lambda, v)$ on $V(q, 2\varepsilon)$ for some $q \in \mathbb{N}$. We then have

$$I_v = \partial J(u) \partial_t v = \partial J(u)(-C_v v + b_v).$$

From $4.11$ and $4.12$ we have

$$\partial J(u)b_v = O(\|v\|\left(\sum_i |\beta_{\lambda_i}| + |\beta_{\alpha_i}|\right)) \left(\sum_i (|\partial J(u)\phi_{2,i}| + |\partial J(u)\phi_{3,i}|) + O(\|\partial J(u)\|)(\sum_{r \neq s} \left|\frac{1}{\lambda_r^2} + \varepsilon_{r,s}\right|\right),$$

whence by virtue of Lemmata 3.6, 3.7 and Proposition 3.3

$$\partial J(u)b_v = O(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + 1 - \frac{\alpha^2}{\alpha_K^2} K_r \alpha_r^{\frac{1}{2}} |v|^4 + \|\|v\|\|v|)$$

and thus

$$\partial J(u)b_v = O\left(\sum_{r} \frac{|\nabla K_r|^2}{\lambda_r} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s} + |v|^3\right) + O\left(\sum_{r \neq s} \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s}\right).$$

Moreover by the well known positivity of $\partial^2 J(\alpha^i \varphi_i) > 0$ on $H_u(q, 2\varepsilon) = \langle \phi_{k,s} \rangle$ we find by expansion

$$-\partial J(u)v \simeq -\|v\|^2 + O\left(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}\right),$$

where we made use of Lemma 3.3. Hence

$$I_v \simeq -\|v\|^2 + O\left(\sum_{r} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s} + |v|^3\right) + O\left(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}\right).$$

Secondly for

$$I_\alpha = \alpha_i^i \partial J(u)\alpha_i \varphi_i = \partial J(u)(1 - \eta_\alpha(1 - \eta_\alpha))||\varphi_j||^{-2} \alpha^2 (1 - \frac{\alpha^2}{\alpha_K^2} K_j \alpha_j^{\frac{1}{2}}) \varphi_j + b_\alpha \alpha^i \varphi_i$$

we have due to $\partial J(u)u = 0$ by scaling invariance of $J$ and due to $4.6$, $4.7$

$$b_\alpha \partial J(u)\alpha^i \varphi_i = O\left(\sum_{r \neq s} \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + \|v\|\|\partial J(u)v = O(\|v\|^3) + O\left(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}\right).$$

Moreover Lemma 3.5 and Proposition 3.3 show

$$-||\varphi_j||^{-2} \alpha^2 (1 - \frac{\alpha^2}{\alpha_K^2} K_j \alpha_j^{\frac{1}{2}}) \partial J(u)\varphi_j \lesssim -\sum_j |1 - \frac{\alpha^2}{\alpha_K^2} \frac{1}{K_j \alpha_j^{\frac{1}{2}}} |^2$$

and we obtain

$$I_\alpha \lesssim -\eta_\alpha(1 - \eta_\alpha) \sum_j |1 - \frac{\alpha^2}{\alpha_K^2} K_j \alpha_j^{\frac{1}{2}} |^2$$

$$+ O\left(\sum_{r} \frac{1}{\lambda_r^2} + \varepsilon_{r,s} + |v|^3\right) + O\left(\sum_{r \neq s} \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}\right).$$

(4.9)
Therefore combining (4.8) and (4.9)

\[ I_\alpha + I_v \lesssim -\|v\|^2 - \eta_\alpha (1 - \eta_v) \sum_j \left| 1 - \frac{\alpha^2}{\alpha \lambda_j} K\alpha_i \alpha_j \right|^2 \]

\[ + O \left( \sum_r \left| 1 - \frac{\alpha^2}{\alpha \lambda_j} K\alpha_i \alpha_j \right|^3 + \|v\|^3 \right) + O(\sum_{r \neq s} \frac{\|K\|_r^2}{\lambda_r^2} + \frac{1}{\lambda_s^2} + \varepsilon_{r,s}) \]

and recalling the definitions of \( \eta_v \) and \( \eta_\alpha \) from Definition 3.1, we conclude

\[ I_\alpha + I_v \lesssim -\|v\|^2 - \eta_\alpha (1 - \eta_v) \sum_j \left| 1 - \frac{\alpha^2}{\alpha \lambda_j} K\alpha_i \alpha_j \right|^2 \]

\[ + (1 - \eta_\alpha)(1 - \eta_v)O(\sum_{r \neq s} \frac{\|K\|_r^2}{\lambda_r^2} + \frac{1}{\lambda_s^2} + \varepsilon_{r,s}) \]  

(4.10)

We turn to the \( \lambda \) and \( \alpha \) evolution. By Proposition 3.3 and the definitions of \( \eta_v \) and \( \eta_\alpha \) we find up to some

\[(1 - \eta_\alpha)(1 - \eta_v)O(\sum_{r \neq s} \frac{\|K\|_r^2}{\lambda_r^2} + \frac{1}{\lambda_s^2} + \varepsilon_{r,s}) \]

from Lemma 3.6 the relation

\[ I_\lambda = \tilde{\lambda}_j \partial J(u) \alpha_j \lambda_j \partial \lambda_j \varphi_j = -(1 - \eta_\alpha)(1 - \eta_v) \eta_\lambda \Pi_i (1 - \eta_{\lambda_i})^{m_{\lambda_j}} \lambda_j \frac{\Delta K_j}{\|K\|_j} + \tilde{\lambda}_j \alpha_j \frac{\eta_\lambda}{\lambda_j} \lambda_j \varphi_j \]

\[ + \sum_{j \neq i} \left( \frac{\alpha_i \lambda_j}{\alpha K^2} \right) K_j \lambda_j \alpha_j \lambda_j \partial \lambda_j \varphi_{i,j} + O \left( \frac{1}{\lambda_j} \right) \]

\[ + \sum_{j \neq i} \left( \frac{\alpha_i \lambda_j}{\alpha K^2} \right) K_j \lambda_j \lambda_j \partial \lambda_j \varphi_{i,j} + O \left( \frac{1}{\lambda_j} \right) \]

(4.12)

and likewise from Lemma 3.7

\[ I_\alpha = \lambda_j \partial J(u) \alpha_j \lambda_j \partial \alpha_j \varphi_j = -(1 - \eta_\alpha)(1 - \eta_v) \eta_\alpha \frac{\alpha^2 \alpha_j \lambda_j}{\alpha K^2} \frac{\partial K_j}{\|K\|_j} \varphi_j \]

\[ + \sum_{j \neq i} \left( \frac{\alpha_i \alpha_j}{\alpha K^2} \right) \lambda_j \partial \lambda_j \varphi_{i,j} + O \left( \frac{1}{\lambda_j} \right) + \sum_{j \neq i} \left( \frac{\alpha_i \alpha_j}{\alpha K^2} \right) \lambda_j \partial \lambda_j \varphi_{i,j} + O \left( \frac{1}{\lambda_j} \right) \]

Thus recalling (iii) of Remark 4.2 and the definition of \( \eta_\alpha \), we have for some positive constants \( c_i > 0 \)

\[ I_\lambda + I_\alpha \leq -c_1 (1 - \eta_\alpha)(1 - \eta_v) \sum_j \eta_\lambda \frac{\alpha_j^2}{\alpha K^2} \frac{\partial K_j}{\|K\|_j} \left( \varphi_j + O \left( \sum_{j \neq i} \varepsilon_{i,j} \right) \right) \]

\[ - c_2 (1 - \eta_\alpha)(1 - \eta_v) \sum_j \eta_\lambda \Pi_i (1 - \eta_{\lambda_i}) \frac{\alpha_i^2}{\alpha K^2} \frac{\partial K_j}{\|K\|_j} \left( \varepsilon_{i,j} + O \left( \sum_{j \neq i} \varepsilon_{i,j} \right) \right) \]

\[ + c_3 (1 - \eta_\alpha)(1 - \eta_v) \sum_j \eta_\lambda \frac{\alpha_i \alpha_j}{\alpha K^2} \lambda_j \partial \lambda_j \varepsilon_{i,j} + O \left( \frac{1}{\lambda_j} \right) \]

(4.11)

\[ + c_4 (1 - \eta_\alpha)(1 - \eta_v) \sum_j \eta_\lambda \frac{\alpha_i \alpha_j}{\alpha K^2} \lambda_j \partial \lambda_j \varepsilon_{i,j} + O \left( \frac{1}{\lambda_j} \right) \]

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up to some \((1 - \eta_\alpha)(1 - \eta_v)O(\sum_{r \neq s} \frac{||\nabla K_r||^2}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \frac{\kappa_{\alpha}^2}{\kappa_{\alpha}})\). Let us now suppose

\[ \exists 1 \leq i \leq q : \frac{||\nabla K_i||}{\lambda_i} = \max_j \frac{||\nabla K_j||}{\lambda_j} \geq \frac{\kappa_{\alpha}}{2\lambda_i}, \] where \(\lambda = \min \lambda_i\).

In particular we may assume \(\eta_{\alpha i} \geq \frac{1}{2}\), cf. Definition 4.1, and this implies

\[ \exists 1 \leq i \leq q : \sum_{j} \eta_{\alpha j} \frac{||\nabla K_j||}{\lambda_j} \simeq \max_j \{ \eta_{\alpha j} \frac{||\nabla K_j||}{\lambda_j} \} \simeq \frac{||\nabla K_i||}{\lambda_i} \simeq \sum_j ||\nabla K_j||. \]

We thus infer from (4.11), that up to some

\[ (1 - \eta_\alpha)(1 - \eta_v)O(\sum_{r \neq s} \frac{||\nabla K_r||^2}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \frac{\kappa_{\alpha}^2}{\kappa_{\alpha}}) \]

and with possibly different constants \(c_i > 0\)

\[ I_\lambda + I_a \leq -c_1(1 - \eta_\alpha)(1 - \eta_v)(\sum_j \frac{||\nabla K_j||}{K_j\lambda_j}) + \kappa_\alpha + O(\sum_{j} m_{\lambda_j}) + O(\sum_{r \neq s} \varepsilon_{r,s}) \]

\[ + c_3(1 - \eta_\alpha)(1 - \eta_v) \sum_{j \neq i} \eta_{\alpha j} \frac{m_{\lambda_j}}{(\lambda_i K_i)^{\frac{n}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \]

\[ \leq -c_1(1 - \eta_\alpha)(1 - \eta_v)(\sum_j \frac{||\nabla K_j||}{K_j\lambda_j}) + \kappa_\alpha + O(\sum_{j} \kappa_{\lambda_j} m_{\lambda_j}) + O(\sum_{r \neq s} \varepsilon_{r,s}) \]

\[ + c_3(1 - \eta_\alpha)(1 - \eta_v) \sum_{j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\lambda_i K_i)^{\frac{n}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \] (4.12)

recalling the definition of \(\eta_{\lambda_i}^\geq\), cf. Definition 4.1, for the last inequality. Note, that

\[ - \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = \frac{n - 2}{2} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} + \lambda_i \lambda_j \gamma_n \frac{G_{90}^{m_{\lambda_i}}}{2} (a_i, a_j) \] (4.13)

cf. (4.33) and recalling Definition 4.1, there holds

(i) \(m_{\lambda_j} \geq m_{\lambda_i}\) for \(\lambda_j \geq \lambda_i\);

(ii) \(m_{\lambda_j} \geq \kappa m_{\lambda_i}\) for \(\lambda_j \geq 2\lambda_i\).

Therefore

\[ \sum_{j \neq i} m_{\lambda_j} \alpha_i \alpha_j \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \lesssim - \sum_{i \neq j} m_{\lambda_i} \varepsilon_{i,j} \]

and, since \(m_{\lambda_i} \geq \kappa \gg 1\), cf. Definition 4.1, plugging this into (4.12), we conclude

\[ I_\lambda + I_a \lesssim (1 - \eta_\alpha)(1 - \eta_v)(\sum_j \frac{||\nabla K_j||}{K_j\lambda_j} + \frac{1}{\lambda_j} + \sum_{j \neq i} \varepsilon_{i,j} + O(\sum_{r \neq s} \frac{||\nabla K_r||^2}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \frac{\kappa_{\alpha}^2}{\kappa_{\alpha}})), \] (4.14)

provided \(\kappa_{\alpha} \gg \sum_{j} \kappa_{\lambda_j} m_{\lambda_j} \gg 1\). We now assume contrarily and in addition

\[ \forall 1 \leq j \leq q : \frac{||\nabla K_j||}{\lambda_j} \leq \frac{\kappa_{\alpha}}{2\lambda_i} \text{ and } \sum_{r \neq s} \varepsilon_{r,s} \leq \frac{1}{\kappa_{\lambda} \max_{r} m_{\lambda_r} \lambda_r^2}. \]
Then from (4.11) and recalling the definition of $\eta^{\tilde{\lambda}}_{\Lambda}$ we find with possibly different constants $c_i > 0$

$$I_{\lambda} + I_a \leq c_1 (1 - \eta_a)(1 - \eta_v)\left( \sum_{j} \eta_{a_j} \frac{\nabla K_j}{K_j \lambda_j} + O(\max_{r \neq s} m_{\lambda_r} \sum r \neq s \varepsilon_{r,s}) \right)$$

$$- c_2 (1 - \eta_a)(1 - \eta_v) \sum_{j} \eta^{\leq \lambda}_i \Pi_i (1 - \eta_{a_i})^{\Theta_{i,j}} \left( \frac{|\Delta K_j|}{K_j \lambda_j^2} \right)$$

up to some $(1 - \eta_a)(1 - \eta_v)O(\sum_{r \neq s} \frac{\nabla K_r^2}{\lambda_r^2} + \frac{1}{\lambda^2} + \varepsilon_{r,s})$. Note, that

$$\sum_{r \neq s} \varepsilon_{r,s} \leq \frac{1}{\kappa_{\lambda} \max_r m_{\lambda_r} \lambda^2} \quad \Rightarrow \quad \sum_{r \neq s} \varepsilon_{r,s} \leq \max_r m_{\lambda_r} \sum_{r \neq s} \varepsilon_{r,s} \leq \frac{1}{\kappa_{\lambda} \lambda^2}$$

and in particular $\eta^{\leq \lambda}_{\Lambda} = 1$. Hence up to the same error as above

$$I_{\lambda} + I_a \lesssim (1 - \eta_a)(1 - \eta_v)\left( \sum_{j} \eta_{a_j} \frac{\nabla K_j}{K_j \lambda_j} + \frac{\Pi_i (1 - \eta_{a_i})^{\Theta_{i,j}}}{\lambda^2} + O(\frac{1}{\kappa_{\lambda} \lambda^2}) \right).$$

Recalling Definition 4.1 there holds

$$\Pi_i (1 - \eta_{a_i})^{\Theta_{i,j}} = o(1) \quad \Rightarrow \quad \exists \Lambda \leq \lambda_i \leq 2 \Lambda : (1 - \eta_{a_i}) = o(1)$$

and in the latter case $\eta_{a_i} \approx 1$, i.e. $\frac{|\nabla K_i|}{\lambda_i} \geq \frac{\kappa_{a_i}}{2 \Lambda^2} \geq \frac{\kappa_{a_i}}{\lambda_i^2}$. Hence we deduce, that in any case

$$I_{\lambda} + I_a \lesssim (1 - \eta_a)(1 - \eta_v)\left( \frac{1}{\lambda_j^2} + \frac{1}{\lambda_r^2} + \frac{1 + \varepsilon_{r,s}}{\lambda} \right)$$

and (4.11) follows again, provided $\kappa_{\lambda} \gg 1$ is sufficiently large. We finally consider the remaining case

$$\forall 1 \leq j \leq q : \frac{|\nabla K_j|}{\lambda_j} < \frac{\kappa_{a_i}}{2 \lambda^2} \quad \text{and} \quad \sum_{r \neq s} \varepsilon_{r,s} \geq \frac{1}{\kappa_{\lambda} \max_r m_{\lambda_r} \lambda^2}.$$

Then recalling again the definition of $\eta^{\tilde{\lambda}}_{\Lambda}$ we find from (4.11)

$$I_{\lambda} + I_a \leq c_3 (1 - \eta_a)(1 - \eta_v)\left( \sum_{j \neq i} \eta^{\leq \lambda}_{\Lambda} \frac{\alpha_i \lambda_i}{\kappa_a} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} + O((1 + \max_r m_{\lambda_r} \sqrt{\kappa_{\lambda}} \sum_{r \neq s} \varepsilon_{r,s})) \right)$$

up to some

$$(1 - \eta_a)(1 - \eta_v)O(\sum_{r \neq s} \frac{\nabla K_r^2}{\lambda_r^2} + \frac{1}{\lambda_r^4} + \varepsilon_{r,s}).$$

Let us decompose for some $\Lambda \gg 1$ and with a slight abuse of notation

$$q = q_1 + q_2, \quad \text{where} \quad q_1 = \{1 \leq i \leq q : \lambda_i > \Lambda \lambda \} \quad \text{and} \quad q_2 = q \setminus q_1.$$ 

In particular for $i, j \in q_2$ we have

$$\Lambda^{-1} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda \quad \text{and} \quad |\nabla K_i| \leq \frac{\kappa_{a_i}}{2 \Lambda} \varepsilon \to 0.$$

Hence, if $a_i, a_j$ for $i, j \in q_2$ were close to the same critical point, we find, cf. 5.33 the contradiction

$$0 \leq \varepsilon_{i,j} \sim (\frac{1}{\Lambda + \Lambda^2 \frac{\lambda_j^2}{(\lambda_i \Lambda^2)^2}}) \sim \frac{1}{(\kappa_{a_i} \Lambda^2)^{n-2}} \not\to 0 \quad \text{as} \quad \varepsilon \to 0.$$
Hence for $i, j \in q_2$ we may assume, that $a_i, a_j$ are close to different critical points of $K$, whence
\[ \varepsilon_{i,j} \approx \left( \frac{1}{\lambda_i \lambda_j} \right)^{\alpha \frac{2^s}{2}} \leq \frac{1}{\lambda^{n-2}}. \]

Moreover for $j \in q_1$
\[ \{ \eta_{\lambda_j} \lesssim 1 \} = \{ \sum_{r \neq s} \varepsilon_{r, s} < \frac{\kappa_a}{\lambda_j} \} \subseteq \{ \sum_{r \neq s} \varepsilon_{r, s} < \frac{\kappa_a}{\Lambda^2 \lambda_j^2} \}. \]

Consequently
\[
I_\lambda + I_a \leq c_3 (1 - \eta_\alpha) (1 - \eta_v) \sum_{q_1 \ni j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}
+ c_3 (1 - \eta_\alpha) (1 - \eta_v) \sum_{q_2 \ni j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}
+ (1 - \eta_\alpha) (1 - \eta_v) O(\frac{\kappa_a \max \lambda_r m_{\lambda_r}}{\Lambda^2 \lambda_j^2} + \frac{\max \lambda_r m_{\lambda_r}}{\Lambda^{n-2}} + (1 + \frac{\max \lambda_r m_{\lambda_r}}{\sqrt{\kappa_a}}) \sum_{r \neq s} \varepsilon_{r,s})
\]
up to some
\[ (1 - \eta_\alpha) (1 - \eta_v) O(\sum_{r \neq s} \frac{\|\nabla K_r\|^2}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s}^2) \]
and rearranging this we obtain up to the same error
\[
I_\lambda + I_a \leq c_3 (1 - \eta_\alpha) (1 - \eta_v) \sum_{q_1 \ni j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}
+ c_3 (1 - \eta_\alpha) (1 - \eta_v) \sum_{q_2 \ni j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}
+ (1 - \eta_\alpha) (1 - \eta_v) O(\frac{\kappa_a \max \lambda_r m_{\lambda_r}}{\Lambda^2 \lambda_j^2} + \frac{\max \lambda_r m_{\lambda_r}}{\Lambda^{n-2}} + (1 + \frac{\max \lambda_r m_{\lambda_r}}{\sqrt{\kappa_a}}) \sum_{r \neq s} \varepsilon_{r,s}).
\]

Recalling (4.13) and from Definition 4.1
(i) $m_{\lambda_j} \geq m_{\lambda_i}$ for $\lambda_j \geq \lambda_i$;
(ii) $m_{\lambda_j} \geq \kappa m_{\lambda_i}$ for $\lambda_j \geq 2 \lambda_i$,
we find
\[
\sum_{q_1 \ni j \neq i} m_{\lambda_j} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \lesssim - \sum_{q_1 \ni j \neq i} m_{\lambda_i} \varepsilon_{i,j}
\]
and using $\lambda_j \geq \lambda_i$ for $j \in q_1$ and $i \in q_2$
\[
\sum_{q_2 \ni j \neq i} \frac{\alpha_i \alpha_j}{(\alpha_K^{-1})^{\frac{2s}{2}}} (m_{\lambda_j} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} + \eta_{\lambda_j}^2 m_{\lambda_j} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}) \lesssim \sum_{q_1 \ni j \neq i} m_{\lambda_i} \varepsilon_{i,j}.
\]

Therefore and recalling $\varepsilon_{i,j} \lesssim \frac{1}{\lambda_i}$ for $i, j \in q_2$
\[
I_\lambda + I_a \lesssim - (1 - \eta_\alpha) (1 - \eta_v) \sum_{i \neq j} m_{\lambda_i} \varepsilon_{i,j} + O(\frac{\kappa_a \max \lambda_r m_{\lambda_r}}{\Lambda^2 \lambda_j^2} + \frac{\max \lambda_r m_{\lambda_r}}{\Lambda^{n-2}} + (1 + \frac{\max \lambda_r m_{\lambda_r}}{\sqrt{\kappa_a}}) \sum_{r \neq s} \varepsilon_{r,s}))
\]
up to some
\[ (1 - \eta_\alpha) (1 - \eta_v) O(\sum_{r \neq s} \frac{\|\nabla K_r\|^2}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s}^2). \]
Consequently (4.14) follows again and thus in any case from $\kappa^q \geq \max_r m_\lambda$, and upon choosing

$$\sqrt{\lambda} \gg \sqrt{\kappa} \gg \kappa^q.$$  

We therefore conclude combining (4.10) and (4.14), that on $V(q, \varepsilon)$ for $\varepsilon > 0$ sufficiently small

$$I_\alpha + I_\lambda + I_a + I_v \leq \|v\|^2 - \eta_\alpha (1 - \eta_\alpha) \sum_j \left(1 - \frac{\alpha^2}{\alpha_n^2} \right) K_i \alpha_i^{n-2} |\eta_j \partial J_{\alpha_i}| + \frac{1}{\lambda_j} + \sum_{j \neq i} \eta_{i,j} + O\left(\sum_{r \neq s} \|\nabla K_r\|_2 \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \varepsilon_{r,s}\right),$$

which recalling the definitions of $\eta_\alpha$, $\eta_\alpha$, cf. Definition 4.1 simplifies to

$$I_\alpha + I_\lambda + I_a + I_v
\leq \|v\|^2 - \sum_j \left(1 - \frac{\alpha^2}{\alpha_n^2} \right) K_i \alpha_i^{n-2} |\eta_j \partial J_{\alpha_i}| + \frac{1}{\lambda_j} + \sum_{j \neq i} \eta_{i,j}$$
\quad (4.15)

As for the gluing with the gradient flow on some $V(q, 2\varepsilon) \setminus V(q, \varepsilon)$ via $\eta$, i.e.

$$\partial_t u = \eta A_q(\alpha, a, \lambda, v) - (1 - \eta) \nabla J(u),$$

we remark, that with a suitably small constant $c_q > 0$ from (4.14) we now have

$$\eta \partial J(u) A_q(\alpha, a, \lambda, v) \leq -c_q \eta \sum_{r \neq s} \|\nabla K_r\|_2 \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^2} + 1 - \frac{\alpha^2}{\alpha_n^2} K_i \alpha_i^{n-2} |\eta_j \partial J_{\alpha_i}|^2 + \varepsilon_{r,s} + \sum_{j \neq i} \eta_{i,j}$$

on $V(q, 2\varepsilon)$, but also

$$(1 - \eta) \nabla J(u)^2 \geq c_q (1 - \eta) \sum_{r \neq s} \|\nabla K_r\|_2 \frac{1}{\lambda_r^2} + \frac{1}{\lambda_r^2} + 1 - \frac{\alpha^2}{\alpha_n^2} K_i \alpha_i^{n-2} |\eta_j \partial J_{\alpha_i}|^2 + \varepsilon_{r,s} + \sum_{j \neq i} \eta_{i,j}$$

due to Proposition 4.3 and Lemma 4.4. Thence the proposition follows.

Let us show, that a flow line, which at least up to a sequence in time concentrates for some $q$ eventually for every $\varepsilon > 0$ in $V(q, \varepsilon)$, then the whole flow line will eventually stay for every $\varepsilon > 0$ in $V(q, \varepsilon)$. In particular such a flow line will be eventually governed by the prescribed movements $(\alpha, a, \lambda)$ and $(v)$ as in Definition 4.1 i.e. the patching with the gradient flow will be irrelevant.

Lemma 4.3. If for a flow line $u = \Phi(\cdot, u_0)$

$$u_{t_k} = \Phi(t_k, u_0) \in V(q, \varepsilon_k) \text{ with } \varepsilon_k \rightarrow 0,$$

then for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$u = \Phi(t, u_0) \in V(q, \varepsilon) \text{ for all } t > T_\varepsilon.$$  

Proof. If the statement was false, there would exist $0 < t_1 < t'_1 < t_2 < t'_2 < \ldots < \infty$ such, that

$$u \in V(q, \varepsilon) \setminus V(q, \frac{\varepsilon}{2}) \text{ during } [t_i, t'_i], \text{ } u_{t_i} \in \partial V(q, \varepsilon) \text{ and } u_{t'_i} \in \partial V(q, \frac{\varepsilon}{2})$$
for some arbitrarily small $\varepsilon > 0$. Thus the flow has during $[t_i, t_i']$ to travel a distance
\[ c_{0, \varepsilon} = d(\partial V(q, \varepsilon), \partial V(q, \frac{\varepsilon}{2})) > 0 \]
with bounded speed $|\partial_t u| \leq c_{1, \varepsilon}$ and energy decay $\partial_t J(u) \leq -c_{2, \varepsilon}$ due to proposition 4.1. So the time for this travelling is lower bounded, i.e. $|t_i' - t_i| \geq \frac{c_{0, \varepsilon}}{c_{1, \varepsilon}}$, and thus we consume at least a quantity of energy
\[ J(u_{t_i}) - J(u_{t_i'}) = -\int_{t_i}^{t_i'} \partial_t J(u) \geq \frac{c_{0, \varepsilon}c_{2, \varepsilon}}{c_{1, \varepsilon}}. \]
Clearly this leads to a contradiction, as the lower bounded energy is never increased.

\[ \square \]

5 Non compact flow lines

Since every flow line can be considered as a Palais-Smale sequence, when restricted to a sequence in time, every non compact zero weak limit flow line has by Proposition 3.1 to enter every $V(q, \varepsilon)$ and by lemma 4.3 to remain therein eventually. Let us study such a flow line $u = \alpha^i \varphi_{a_i, \lambda_i}$, which then satisfies

\begin{itemize}
  \item[(a)] $\frac{\partial}{\partial t} = -\eta_0 (1 - \eta_0) \|\varphi\|^2 \left(1 - \frac{\alpha^2}{\alpha^2 + \frac{1}{\lambda_i}} K_i \alpha^4 \right) + b_{\alpha}$;
  \item[(a)] $\lambda_j \lambda_j = (1 - \eta_0) (1 - \eta_0) \eta_0 \left(\frac{\partial K_i}{\partial K_i}\right)$;
  \item[(a)] $\frac{\partial}{\partial t} = - (1 - \eta_0) (1 - \eta_0) (\eta \lambda_i \Pi_j (1 - \eta_0)^{m_{x_j, \lambda_j}} + \eta \lambda_j m_{A_i})$;
  \item[(a)] $\partial_t v = b_v - C_v v$
\end{itemize}

for all times to come. First note, that from (v) we have $\partial_t \|v\|^2 \leq -\|v\|^2$, while from (a) and (λ)

\[ \partial_t \frac{1}{\lambda_i^2} = O\left(\frac{1}{\lambda_i^2}\right), \quad \partial_t \frac{\nabla K_i}{\lambda_i^2} = O\left(\frac{\nabla K_i}{\lambda_i^2} + \frac{1}{\lambda_i^2}\right) \quad \text{and} \quad \partial_t \sum_{i \neq j} \frac{\alpha^2}{\lambda_i^2} = O\left(\sum_{i \neq j} \frac{\nabla K_i}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \alpha \lambda_i^2 \right) \]

and finally from (a), (a) and 4.1, cf. also 4.10,

\[ \partial_t \sum_{i \neq j} \left(1 - \frac{\alpha^2}{\lambda_i^2} K_i \alpha^4 \right)^2 = O\left(\sum_{i \neq j} \left(1 - \frac{\alpha^2}{\lambda_i^2} K_i \alpha^4 \right)^2 \right) \]

\[ + O\left(\sum_{i \neq j} \frac{\nabla K_i}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \frac{\alpha^2}{\lambda_i^2} K_i \alpha^4 \right) \]

Recalling the definition of $\eta_v$, cf. Definition 4.1 we then have for $\kappa_v$ sufficiently large

\[ \partial_t \|v\|^2 \ll \partial_t \sum_{i \neq j} \frac{\nabla K_i}{\lambda_i^2} + \frac{1}{\lambda_i^2} + \left(1 - \frac{\alpha^2}{\lambda_i^2} K_i \alpha^4 \right)^2 + \frac{\eta_v^2}{\lambda_i^2} \quad \text{on} \quad \{\eta_v \geq 0\} \]

and may thus assume, that eventually, hence from now on and for all times to come

\[ \eta_v \equiv 0. \] (5.1)

We turn to describing the movement in $a_i$ and $\lambda_i$. Clearly we may assume

\[ \forall i : \lambda_i \xrightarrow{t \to \infty} \infty \]
due to Lemma 4.3 and so at least for a time sequence $t_k \rightarrow \infty$

$$\partial_t \lambda_i|_{t=t_k} > 0, \text{ where } \lambda = \max_i \lambda_i.$$ 

Hence (\lambda) and \forall i : m_{\lambda_i, \lambda} = 1, cf. Definition 4.1, show, that necessarily

$$\forall i : \eta_{\lambda_i} < 1 \text{ at } t = t_k \text{ and hence } \forall i : \frac{|\nabla K_i|}{\lambda_i} \leq \frac{\kappa_a}{\lambda_i^\alpha}.$$ 

Since we assume $K$ to be Morse, we conclude, that at least for a sequence in time

$$\forall i : a_i \rightarrow x_i \in \{ |\nabla K| = 0 \}.$$ 

On the other hand due to (a) all $a_i$ move exclusively along the gradient of $K$, whence necessarily

$$a_i \xrightarrow{t \rightarrow \infty} x_i \in \{ |\nabla K| = 0 \}$$ 
and $a_i$ has to move along the stable manifold of $x_i$ with respect to the positive gradient flow for $K$, hence

$$\forall i : a_i \in W_s(x_i) = W_s^{\nabla K}(x_i).$$

But the only possibility for $\lambda_i$ to increase is $\Delta K_i < 0$, cf. (\lambda), whence necessarily as a first consequence

$$\forall i : W_s(x_i) \ni a_i \xrightarrow{t \rightarrow \infty} x_i \in \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \}.$$ 

In particular we may assume $\Delta K_i < 0$ from now on. Secondly, since for $\lambda_j \geq \lambda_k$

$$\eta_{\lambda_j}^< \leq \eta_{\lambda_k}^<, \eta_{\lambda_j}^> \geq \eta_{\lambda_k}^> \text{ and } \forall i : m_{\lambda_j, \lambda_i} \geq m_{\lambda_k, \lambda_i} \text{ and } m_{\lambda_j} \geq m_{\lambda_k},$$

cf. Definition 4.1 we derive from (\lambda) using $\frac{\Delta K_i}{|\nabla K_i|} = -1$ and $\eta_v = 0$

$\frac{\dot{\lambda}_j}{\lambda_j} = (1 - \eta_\lambda)(\eta_{\lambda_j}^< \Pi_i (1 - \eta_{\lambda_i})^{m_{\lambda_j, \lambda_i} - \eta_{\lambda_j}^> m_{\lambda_j}) \leq (1 - \eta_\lambda)(\eta_{\lambda_k}^< \Pi_i (1 - \eta_{\lambda_i})^{m_{\lambda_k, \lambda_i} - \eta_{\lambda_k}^> m_{\lambda_k}) = \frac{\dot{\lambda}_k}{\lambda_k},$

whence $\partial_t \frac{\lambda_j}{\lambda_k} \leq 0$ for $\lambda_j \geq \lambda_k$ and we may therefore assume from now on

$$1 \leq \frac{\lambda_i}{\Delta} \leq C = C(u_0) \text{ for } \lambda = \max_i \lambda_i \text{ and } \Delta = \min_i \lambda_i.$$ 

(5.2)

Thirdly, as we had said, $\lambda$ has to grow at times $t = t_k$, and then we have

$$\forall i : \eta_{\lambda_i} < 1 \text{ and } \frac{|\nabla K_i|}{\lambda_i} \leq \frac{\kappa_a}{\lambda_i^\alpha},$$ 

(5.3)

whereas generally there holds due to (a) and (\lambda)

$$\frac{\partial_t}{2}(\lambda_i^2 |\nabla K_i|^2) = \frac{\dot{\lambda}_i}{\lambda_i} \lambda_i^2 |\nabla K_i|^2 + \lambda_i |\nabla K_i| \langle 2K_i, \frac{\nabla K_i}{|\nabla K_i|}, \lambda_i \dot{a}_i \rangle \leq (1 - \eta_\lambda)(1 - \eta_{\lambda_i}) \lambda_i^2 |\nabla K_i|^2 + \eta_\lambda \lambda_i |\nabla K_i| \langle 2K_i, \frac{\nabla K_i}{|\nabla K_i|}, \lambda_i \dot{a}_i \rangle, \leq (1 - \eta_\lambda)[(1 - \eta_{\lambda_i}) \lambda_i^2 |\nabla K_i|^2 + \eta_\lambda \lambda_i |\nabla K_i| \langle 2K_i, \frac{\nabla K_i}{|\nabla K_i|}, \lambda_i \dot{a}_i \rangle],$$

where we used $m_{\lambda_i, \lambda_i} = 1$. Since $K$ is Morse and $a_i$ is close to $x_i$ and moves along $W_s(x_i)$, we have

$$\langle 2K_i, \frac{\nabla K_i}{|\nabla K_i|}, \lambda_i \dot{a}_i \rangle < -\delta \text{ for some } \delta > 0.$$ 

(5.4)
and consequently
\[ \frac{\partial}{\partial t}(\lambda_i^2|\nabla K_i|^2) \leq (1 - \eta_\alpha)(1 - \eta_\alpha, \lambda_i^2|\nabla K_i|^2 - \delta \eta_\alpha, \lambda_i|\nabla K_j|). \] (5.5)

In particular (5.3) and (5.5) imply, that we may assume from now on and for all times to come
\[ \forall i : \eta_\alpha < 1 - \epsilon \quad \text{and} \quad \frac{d_{g_0}(a_i, x_i)}{\lambda_i} \simeq \frac{|\nabla K_i|}{\lambda_i} < \frac{\kappa_a}{\lambda_i^2} \] (5.6)

for some fixed \( \epsilon > 0 \). From (5.2) and (5.6) we may exclude tower bubbling, i.e.
\[ a_i \rightarrow x_i = x_j \leftarrow a_j. \]

Indeed in the latter case
\[ d_{g_0}^2(a_i, a_j) \leq (d_{g_0}(a_i, x_i) + d_{g_0}(a_j, x_j))^2 \lesssim (\lambda_i^{-1} + \lambda_j^{-1})^2 \lesssim \lambda_i^{-2} + \lambda_j^{-2} \]
and \( \lambda_i \simeq \lambda_j \), whence
\[ 0 \xrightarrow{t \rightarrow \infty} \epsilon_{i,j} \simeq \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_{g_0}^2(a_i, a_j) \xrightarrow{2r} (\lambda_i + \lambda_j) \xrightarrow{2r} 0 \quad \text{as} \quad \epsilon \rightarrow 0, \]
a contradiction. Hence we may assume \( x_i \neq x_j \), thus \( d(a_i, a_j) \not\rightarrow 0 \) and therefore
\[ \forall i : \sum_{r \neq s} \epsilon_{r,s} \simeq \sum_{r \neq s} (\lambda_r \lambda_s) \xrightarrow{2r} \frac{1}{\lambda_i^0} \quad \text{and} \quad \sum_{r \neq s} \frac{\mu_i^r}{\lambda_r^2} = o\left(\frac{1}{\lambda_i^2}\right) \] (5.7)
for all times to come due to \( \lambda_r \simeq \lambda_s \simeq \lambda_i \), cf. (5.2). In particular (a) and (\( \lambda \)) therefore simplifies to

(a) \( \lambda_j \partial_j = (1 - \eta_\alpha)\eta_{a_j} |\nabla K_j|/|\nabla K_i| \),

(\( \lambda \)) \( \frac{\lambda_i}{\lambda_j} = (1 - \eta_\alpha)\Pi_i (1 - \eta_\alpha) \simeq \eta_i (1 - \eta_\alpha) \)
due to (5.6), (5.7) and, cf. Definition 1.1 \( m_{\lambda_j, \lambda_i} \in [0, 1] \) and

(i) \( \eta_{\lambda_j}^< = 1 \text{ on } \{ \sum_{r \neq s} \epsilon_{r,s} \leq \frac{\kappa_a}{\lambda_i^2} \} ; \)

(ii) \( \eta_{\lambda_j}^\geq = 0 \text{ on } \{ \sum_{r \neq s} \epsilon_{r,s} \leq \frac{\kappa_a}{\lambda_i^2} \} . \)

We turn our attention to the movement in \( \alpha_i \). Since we may assume by now \( \eta_\alpha = 0 \), hence
\[ \|v\| \lesssim \sum_{r \neq s} \frac{|\nabla K_s|}{\lambda_r} + \frac{1}{\lambda_i^2} + |1 - \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}}| + \frac{n+2}{\lambda_i^r} \lesssim \sum_{r} \frac{1}{\lambda_r^2} + |1 - \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}}|, \]

cf. Definition 4.1, 5.6 and 5.7, and due to 4.3, we have

(a) \( \frac{\alpha_j}{\alpha_j} = -\frac{\alpha_j}{\alpha_j} (1 - \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}}) + o(\sum_r |1 - \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}}|) + O(\sum_r \frac{1}{\lambda_i^r}), \)

where we made use of \( \|\varphi_j\| = \tilde{c}_0 + O\left(\frac{1}{\lambda_i^2}\right) \). Note, that due to (a) we have
\[ \partial_t \left( \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}} - \alpha_j^2 \right) = \sum_i \alpha_i \partial_i \left( \frac{\alpha_j^2}{\alpha_K^2} \alpha_j \frac{4}{K^2} \right) + \alpha_j \partial_j \left( \frac{\alpha_j^2}{\alpha_K^2} K_j \alpha_j^{\frac{4}{K^2}} - \alpha_j \partial_j \alpha_j \alpha_j^{\frac{4}{K^2}} \right) \]
up to some $O(\sum_r \frac{\|K_r\|}{\lambda_r^2})$. Recalling (5.4), we then find, that up to some $\sum_i \alpha_i \partial_{\alpha_i} \alpha_i^2 \frac{\partial_{\alpha_i}}{\alpha_i} - 0$ and secondly

$$\sum_i \alpha_i \partial_{\alpha_i} \alpha_i^2 \frac{\partial_{\alpha_i}}{\alpha_i} = \frac{2n}{n-2} \eta \sum_i \left( \frac{\alpha_i^2}{\alpha_i^4} K_i \alpha_i^2 - \partial_{\alpha_i} \left( \frac{\alpha_i^2}{\alpha_i^4} K_i \alpha_i^2 - 1 \right) \right) = 0.$$  

Hence we obtain up to the same error

$$\partial_t \left( \alpha_i^2 \frac{\partial_{\alpha_i}}{\alpha_i} K_j \alpha_j^2 - \alpha_j^2 \right) = \eta \sum_i \left( \frac{\alpha_i^2}{\alpha_i^4} K_i \alpha_i^2 - \partial_{\alpha_i} \left( \frac{\alpha_i^2}{\alpha_i^4} K_i \alpha_i^2 - 1 \right) \right) \geq \eta \left( \frac{\alpha_i^2}{\alpha_i^4} K_i \alpha_i^2 - \alpha_j^2 \right).$$

Consequently we have

$$\partial_t \sum_i \left( \alpha_i^2 \frac{\partial_{\alpha_i}}{\alpha_i} K_j \alpha_j^2 - \alpha_j^2 \right)^2 \geq 0 \quad \text{and} \quad \partial_t \frac{\|
abla K_r\|}{\lambda_r^2} \leq 0 \quad (5.8)$$

as long as $u \in \{\eta > \frac{1}{2}\}$ due to (\lambda) and (a), cf. (5.4). Thence necessarily

$$\eta < \frac{1}{2} \quad (5.9)$$

for all times to come. Indeed we may assume

(i) $\{\eta > \frac{1}{2}\} = \{ \sum_j \left| 1 - \frac{\alpha_i^2}{\alpha_i^4} K_j \alpha_j^2 \right| \geq \frac{\kappa_{\alpha, \frac{1}{4}}}{\kappa_{\alpha, \frac{1}{4}}} \left( \sum_{r \neq j} \frac{\|
abla K_r\|}{\lambda_r^2} + \frac{\eta}{\lambda_r^2} \right) \}$

(ii) $\{\eta > \frac{1}{2}\} = \{ \sum_j \left| 1 - \frac{\alpha_i^2}{\alpha_i^4} K_j \alpha_j^2 \right| \geq \frac{\kappa_{\alpha, \frac{1}{4}}}{\kappa_{\alpha, \frac{1}{4}}} \left( \sum_{r \neq j} \frac{\|
abla K_r\|}{\lambda_r^2} + \frac{\eta}{\lambda_r^2} \right) \}$

with

$$\kappa_{\alpha, \frac{1}{4}}, \kappa_{\alpha, \frac{1}{4}} \in (\sqrt{\kappa_{\alpha, \frac{1}{4}}}, \kappa_{\alpha, \frac{1}{4}}) \text{ and } \frac{\kappa_{\alpha, \frac{1}{4}}}{\kappa_{\alpha, \frac{1}{4}}} \rightarrow \infty \text{ as } \kappa_{\alpha} \rightarrow \infty.$$  

We then find from (5.4), since $\delta < \alpha_i < \delta^{-1}$ on $V(q, r) \cap \{\| \cdot \| = 1\}$, that

$$\exists c_\delta > 0 \\forall u_0 \in \{\eta > \frac{1}{2}\} : \sum_j \left| 1 - \frac{\alpha_i^2}{\alpha_i^4} K_j \alpha_j^2 \right| \geq c_\delta \kappa_{\alpha, \frac{1}{4}} \left( \sum_{r \neq j} \frac{\|
abla K_r\|}{\lambda_r^2} + \frac{1}{\lambda_r^2} \right)$$

as long as $u \in \{\eta > \frac{1}{2}\}$, and thus by virtue of (5.7)

$$\sum_j \left| 1 - \frac{\alpha_i^2}{\alpha_i^4} K_j \alpha_j^2 \right| \geq \frac{c_\delta}{2} \kappa_{\alpha, \frac{1}{4}} \left( \sum_{r \neq j} \frac{\|
abla K_r\|}{\lambda_r^2} + \frac{1}{\lambda_r^2} + \frac{n+2}{r^{-2}} \right).$$
Consequently and, since we may assume
\[ \frac{c_k}{2} \kappa_\alpha \frac{\kappa_\alpha}{2} > \kappa_\alpha \frac{\kappa_\alpha}{2}, \]
provided \( \kappa_\alpha \) is sufficiently large, we will stay in \( \{ \eta_\alpha > \frac{1}{4} \} \) for all times, whence by virtue of (5.8)
\[ \sum_i \left| \frac{\alpha^2}{\alpha_i K_n} \right| K_n \alpha_i \frac{\kappa_\alpha}{2} - 1 \right| \right| \leq \sqrt{\sum_i \left| \frac{\alpha^2}{\alpha_i K_n} \right| K_n \alpha_i \frac{\kappa_\alpha}{2} - \alpha_i^2 |^2} \not\rightarrow 0 \quad \text{as} \ t \rightarrow \infty, \]
a contradiction. We thus conclude from (5.2), (5.6), (5.9), and (5.11), that eventually
\[ \forall i : \lambda_i = \lambda_j, \ e^{-ct} \leq 1, d_{g_k}(a_i, x_i), |1 - \frac{\alpha^2}{\alpha_i K_n} K_n \alpha_i \frac{\kappa_\alpha}{2} |, ||v|| = e^{-ct} \]
for some \( 0 < c < C < \infty \). We have thus derived the non trivial part of

**Proposition 5.1.** A zero weak limit flow line is non compact, if and only if eventually

(i) \( ||v|| \) decays exponentially and

(ii) \( |\alpha^2 K_i \alpha_i \frac{\kappa_\alpha}{2} - \alpha_i^2 | \) decays exponentially and

(iii) \( \lambda_i \) increases exponentially and

(iv) \( a_i \rightarrow x_i \in \{ \nabla K = 0 \} \cap \{ \nabla K < 0 \} \) along \( W_a(x_i) \) exponentially fast, where \( x_i \neq x_j \)
for all \( i \neq j = 1, \ldots, q \). Conversely flow lines satisfying (i)-(iv) exist.

**Proof.** By what we have seen above, every non compact zero weak limit flow line has to satisfy (i)-(iv) above eventually and clearly every flow line satisfying (i)-(iv) is non compact with zero weak limit. Hence we are left with showing their existence. Let us choose for simplicity as initial data
\[ v = 0, a_i = x_i \quad \text{and} \quad \frac{\alpha^2}{\alpha_i K_n} K_n \alpha_i \frac{\kappa_\alpha}{2} = 1 \]
for \( x_i \in \{ \nabla K = 0 \} \cap \{ \nabla K < 0 \} \) and \( x_i \neq x_j \) for \( i \neq j \). Recalling \( b_v = O(||v||) \), cf. Lemma 4.1, we have
\[ \partial ||v||^2 = -C_v ||v||^2 + O(||v||^2), \ C_v \gg 1 \]
cf. (v) and hence \( v = 0 \) is preserved. Secondly due to (a) also \( a_i = x_i \) is preserved. Thirdly
\[ \partial_t \left( \frac{\alpha^2}{\alpha_i K_n} K_n \alpha_i \frac{\kappa_\alpha}{2} \right) = 0 \]
follows from \( \dot{\alpha_i} = b_\alpha \) and \( \dot{a_i} = 0 \), cf. (a), hence also \( \frac{\alpha^2}{\alpha_i K_n} K_n \alpha_i \frac{\kappa_\alpha}{2} = 1 \) is preserved. In particular
\[ \eta_\alpha, \eta_\alpha, \eta_\alpha = 0 \]
are preserved, as long as we do not leave some \( V(q, \varepsilon) \), upon which (a), (a), (\lambda) and (v) are valid. Thus
\[ \frac{\dot{\lambda}_j}{\lambda_j} = (\eta_\lambda, \Pi_i - \eta_\lambda, \eta_\lambda) \]

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due to (λ). Since \( a_i = x_i \) and \( x_i \neq x_j \) for \( i \neq j \) by assumption, there holds for \( \Lambda = \min_i \lambda_i \)
\[
\sum_{n \neq 0} \varepsilon_{r,s} \lesssim \left( \frac{1}{\lambda_r \lambda_s} \right)^{n/2} \lesssim \frac{1}{\lambda^{n-2}} = o(\frac{1}{\lambda^n})
\]
whence \( \eta^\prec \Lambda = 1 \) and \( \eta^\succ \Lambda = 0 \), so \( \partial \Lambda = 1 \). Hence the above \( V(q, \varepsilon) \) will never be left and \( \Lambda \nearrow \infty \).

**Proof of Theorem** Consider the flow introduced in Lemma 4.2, i.e.
\[
\Phi : \mathbb{R}_{\geq 0} \times X \to X \quad \text{with} \quad X = \{ 0 \leq u \in W^{1,2}(M) \mid u \neq 0, \|u\| = 1 \},
\]
which decreases the energy \( J \) according to Proposition 4.1. Then by energy reasoning every flow line \( u(t) = \Phi(t, \cdot) \) induces upon choice of a subsequence in time a Palais-Smale sequence \( u_{t_k} = \Phi(t_k, \cdot) \).

Indeed \( J \) is by positivity of the Yamabe invariant strictly positive, while by virtue of Propositions 3.3 and 4.1 every flow line consumes energy as long as \( |\partial J| > 0 \). Then Proposition 5.1 and the comment following it show, that upon a subsequence \( u_{t_k} \) is of zero weak limit, if and only if \( u_{t_k} \) concentrates in the sense
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall k \geq N : u_{t_k} \in V(q, \varepsilon),
\]
in which case \( u \) is of zero weak limit itself, as Lemma 4.3 shows. Hence according to Proposition 5.1 the full flow line concentrates simply with limiting profile and energy
\[
\sum_{i} \frac{c}{K_i^{2/2 - n}} \delta_{x_i} \quad \text{and} \quad \lim_{t \to \infty} J(u(t)) = J_{x_1, \ldots, x_q} = c(\sum_i K_i^{2-n})^\frac{n}{2}
\]
respectively, where \( c > 0 \) is a dimensional constant and
\[
x_i, \ldots, x_q \in \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \}
\]
are distinct. Hence zero weak limit sequences along a flow line are classified with respect to their end configuration, which corresponds one to one to subsets of \( \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \} \) on the one hand and to finite energy and zero weak limit subcritical blow-up solutions on the other, cf. [15]. And of course flow lines of the latter type do exist by Proposition 5.1.

So let us consider
\[
x_i, \ldots, x_q \in \{ |\nabla K| = 0 \} \cap \{ \Delta K < 0 \} \quad \text{with} \quad x_i \neq x_j
\]
and denote correspondingly by
\[
u_{\tau,x_1,\ldots,x_q} \in \{ \partial J_{\tau} = 0 \}
\]
the unique, zero weak limit subcritical blow-up solution from [15] of the same limiting profile and energy
\[
\sum_{i} \frac{c}{K_i^{2-n}} \delta_{x_i} \quad \text{and} \quad \lim_{\tau \to 0} J(u_{\tau,x_1,\ldots,x_q}) = J_{x_1,\ldots,x_q}.
\]
Then by virtue of Proposition 3.1 in [15] there exists \( \varepsilon > 0 \) such, that for all for \( 0 < \tau \ll \varepsilon \)
\[
\{ u_{\tau,x_1,\ldots,x_q} \} = \{ \partial J_{\tau} = 0 \} \cap V(q, \varepsilon) \cap \{ d(a_i, x_i) \ll 1 \},
\]
i.e. uniqueness as a solution on some \( V(q, \varepsilon) \cap \{ d(a_i, x_i) \ll 1 \} \). Hence
\[
\{ \partial J_{\tau} = 0 \} \cap V(q, \varepsilon) \cap \{ d(a_i, x_i) \ll 1 \}
\]
for some \( \varepsilon > 0 \) and any \( 0 < \tau \ll \varepsilon \) contains exactly one element as a subcritical solution with
\[
m_{x_1,\ldots,x_m} = m(J_{\tau, u_{\tau,x_1,\ldots,x_q}}) = q - 1 + \sum_{i=1}^{q} (n - m(K, x_i))
\]
as Morse index. And we have a homotopy equivalence by attaching a cell, cf. Figure 3

\[ J_{x_1, \ldots, x_q - \delta}^J \cup (V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}) \simeq J_{x_1, \ldots, x_q - \delta}^J \# C, \quad \dim C = m_{x_1, \ldots, x_q} \]

along the unstable manifold of and suspended at \( u_{x_1, \ldots, x_q} \) with energy

\[ J_t(u_{x_1, \ldots, x_q}) = J_{x_1, \ldots, x_q} + o_1(1). \]

Since \( J^s_t = \{J_t \leq s\} \subset \{J \leq s\} = J^s \) due to Hölder’s inequality, we then find

\[
\begin{align*}
H_k \left( J_{x_1, \ldots, x_q - \delta}^J \cup (V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}) \right) &
= H_k \left( J_{x_1, \ldots, x_q - \delta}^J \cup (J_{x_1, \ldots, x_q - \delta}^J \cup (V(q, \varepsilon) \cap \{|d(a_i, x_i| \leq 1\})) \right), J_{x_1, \ldots, x_q - \delta}^J \right) \\
&= H_k \left( J_{x_1, \ldots, x_q - \delta}^J \cup (J_{x_1, \ldots, x_q - \delta}^J \# C), J_{x_1, \ldots, x_q - \delta}^J \right) \\
&= H_k \left( J_{x_1, \ldots, x_q - \delta}^J \# C, J_{x_1, \ldots, x_q - \delta}^J \right) = \delta_{k, m_{x_1, \ldots, x_q}}
\end{align*}
\]

for the \( k \)-th relative singular homology \( H_k(A, B) \) with \( k = m_{x_1, \ldots, x_q} \) of the pair

\[ J_{x_1, \ldots, x_q - \delta}^J = B \subset A = J_{x_1, \ldots, x_q - \delta}^J \cup (V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}), \]

see [11]. Thus we observe a change of topology of the sublevel sets of \( J \) on

\[ V(q, \varepsilon) \cap \{|d(a_i, x_i| \leq 1\}, \quad (5.11) \]

while thanks to Proposition [3.3] we know, that for \( \varepsilon > 0 \) sufficiently small \( \partial J = 0 \cap V(q, \varepsilon) = \emptyset \).

In other words this change of topology happens on \( (5.11) \) as a *neighbourhood* of the limiting profile \( (5.11) \) of a non compact, energy decreasing, zero weak limit flow line of \( \Phi \). And in fact this limiting profile does correspond to a *critical point at infinity*, as \( J \) exhibits a correct Morse structure on this neighbourhood, cf. Proposition [5.2] and Lemma [2.2]. Hence we may justly

(i) say, that this change of topology is induced by this critical point at infinity

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(ii) associate to this critical point at infinity the index

\[
\text{ind}(J, u_\infty, x_1, \ldots, x_q) = m_{x_1, \ldots, x_q} = m(J, u_{\tau, x_1, \ldots, x_q}) = (q - 1) + \sum_{i=1}^{q} (n - m(K, x_i)).
\]  

(5.12)

This completes the proof. \[\Box\]

To establish the Morse structure at infinity, we first require a further orthogonalization.

**Lemma 5.1.** For every \(\alpha^i \varphi_i \in V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}\), there exists a unique minimizer for \(\bar{v} \in H\)

\[
J(\alpha^i \varphi_i + \bar{v}) = \min_{v \in H_{\alpha^i \varphi_i}} J(\alpha^i \varphi_i + v),
\]

provided \(\varepsilon > 0\) is sufficiently small, and, if \(x_i \neq x_j\) for \(i \neq j\), there holds \(\|\bar{v}\| = O(\sum_r \frac{|\nabla K_i|}{\lambda_r} + \sum_r \frac{1}{\lambda_r^2})\).

**Proof.** Existence follows from uniform positivity of \(\partial^2 J\) on \(H\), cf. [22]. Moreover by Lemma 4.1 in [14]

\[
\partial J(\alpha^i \varphi_i) \bar{v} = O(\sum_r \frac{|\nabla K_i|}{\lambda_r} + \sum_r \frac{1}{\lambda_r^2} \|\bar{v}\|),
\]

(5.13)

since \(\tau, \theta = 0, n \geq 5\) and the blow-up points \(a_i\) are far from each other, cf. [33]. Expanding

\[0 = \partial J(\alpha^i \varphi_i + \bar{v}) \bar{v} = \partial J(\alpha^i \varphi_i) \bar{v} + \partial^2 J(\alpha^i \varphi_i) \bar{v}^2 + o(\|\bar{v}\|^2),\]

the claimed estimate follows by positivity of the second variation on \(H\) and absorption. \[\Box\]

Clearly \(\bar{v} = \bar{v}\), and we may represent every \(u \in V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}\) uniquely as

\[u = \alpha^i \varphi_i + \bar{v} + \bar{v}.\]

Then by construction

\[J(u) = J(\alpha^i \varphi_i + \bar{v} + \bar{v}) = J(\alpha^i \varphi_i + \bar{v}) + \partial^2 J(\alpha^i \varphi_i + \bar{v}) \bar{v}^2 + o(\|\bar{v}\|^2)\]

and by positivity of \(\partial^2 J\) on \(H\) and smallness of \(\bar{v}\) we find

\[J(u) = J(\alpha^i \varphi_i + \bar{v}) + O^+ (\|\bar{v}\|^2) \text{ with } 0 < c \leq O^+ (1) \leq C < \infty,\]

(5.14)

which is to say, that the \(\bar{v}\)-direction is a positive, i.e. energy increasing one. Moreover

\[J(\alpha^i \varphi_i + \bar{v}) = J(\alpha^i \varphi_i) + o_{\varepsilon}(\sum_r \frac{1}{\lambda_r^2}),\]

(5.15)

as follows by expansion using Lemma 5.1 and 5.13.

**Proposition 5.2.** For \(x_i \neq x_j\) for \(i \neq j\) and \(\varepsilon > 0\) sufficiently small there holds

\[
J(u) = c_0 (\sum_i \frac{1}{K^{n-2}(x_i)})^\# \left( 1 - \sum_{i=1}^{q-1} (1 + o_{\varepsilon}(1)) \hat{a}_i^2 \right. \]

\[
- \sum_{i=1}^{q} (1 + o_{\varepsilon}(1)) \left( \sum_j |\hat{a}_{ij}^+|^2 - \sum_j |\hat{a}_{ij}^-|^2 \right) + \sum_{i=1}^{q} \frac{1}{\lambda_i^2} ) \right) + O^+(\|\bar{v}\|^2),
\]

for \(u \in V(q, \varepsilon) \cap \{|a_i - x_i| \leq \varepsilon\}\), where upon rescaling.
(i) \( \hat{\lambda}_i = \lambda_i \)

(ii) \( \hat{a}_{i,j}^+, \hat{a}_{i,j}^- \) are local coordinates of \( a_i \) in a Morse chart around \( x_i \), upon which

\[
K(a_i) = K(x_i) + \sum_j \hat{a}_{i,j}^+ - \sum_j \hat{a}_{i,j}^-
\]

(iii) \( \hat{a}_i \) are the eigenvectors negative eigenvalues of

\[
A_{i,j} = -\frac{4}{n-2}(\delta_{i,j} - \frac{K_i^{-\frac{n-2}{2}}(x_i)K_j^{-\frac{n-2}{2}}(x_i)}{\sum_i K_i^{-\frac{n-2}{2}}(x_i)})
\]

**Remark 5.1.** At this point, cf. (5.12), it is hardly surprising, that for the number of negative directions

\[
(q - 1) - z(\hat{a}_{i,j}^+) = (q - 1) - \sum_{i=1}^1 \text{coindex}K(x_i) = \text{ind}(J,u_{\infty,x_1,...,x_q}).
\]

**Proof of Proposition 5.2.** We clearly have to study \( J \) at \( \alpha^i \varphi_i \) only. From Proposition 5.1 in [14] we have

\[
J(\alpha^i \varphi_i) = \frac{\hat{c}_0 \sum_i \alpha_i^2}{(\sum_i K(a_i)\alpha_i^{\frac{n-2}{2}})^{\frac{n-2}{2}}} \left( 1 - \hat{c}_2 \frac{\sum_i \Delta K_i}{K_i \lambda_i^4} \right) + o(1)
\]

with positive constants \( \hat{c}_0, \hat{c}_2 \), noting that

(i) there is no \( O(|\partial J(u)|^2) \) in the remainder in case \( u = \alpha^i \varphi_i \)

(ii) \( n \geq 5 \) and \( \tau = 0 \)

(iii) the blow-up points are far from each other, since

(iv) \( |a_i - x_i| \leq \varepsilon \) and \( x_i \neq x_j \) for \( i \neq j \), in particular \( |\nabla K_i| = o(1) \)

Moreover and, since \( |1 - \frac{r K_i \alpha_i^{\frac{1}{2}}}{\lambda_i^2}| \leq \varepsilon \), we obtain

\[
J(\alpha^i \varphi_i) = \frac{\hat{c}_0 \sum_i \alpha_i^2}{(\sum_i K(x_i)\alpha_i^{\frac{n-2}{2}})^{\frac{n-2}{2}}} \left( 1 - \hat{c}_2 \frac{\sum_i (1+o(1))\Delta K_i}{K_i \lambda_i^4} \right)
\]

recalling the non degeneracy assumption (1.5). Passing to the Morse charts and expanding we get

\[
J(\alpha^i \varphi_i) = \frac{\hat{c}_0 \sum_i \alpha_i^2}{(\sum_i K(x_i)\alpha_i^{\frac{n-2}{2}})^{\frac{n-2}{2}}} \left( 1 - \hat{c}_1 \frac{\sum_i (1+o(1))\sum_j \hat{a}_{i,j}^+ - \sum_j \hat{a}_{i,j}^- |^2}{\sum_j K_{\pm}^{\frac{n-2}{2}}(x_j)} - \hat{c}_2 \frac{\sum_i (1+o(1))\Delta K_i}{K_i \lambda_i^4} \right),
\]

where \( \hat{c}_1 = \frac{n-2}{2} \). Finally consider the scaling invariant function

\[
f(\alpha_i) = \frac{\sum_i \alpha_i^2}{(\sum_i K(x_i)\alpha_i^{\frac{n-2}{2}})^{\frac{n-2}{2}}},
\]

whose restriction to

\[
X_\alpha = \{ \sum_i K(x_i)\alpha_i^{\frac{n-2}{2}} = 1 \}
\]
reflects the restriction of \( J \) to \( X_a \). Then \( f|_{X_a} \) has a unique, strict and non-degenerate maximum in
\[
\alpha_i = \frac{\Theta}{K^{\frac{2-n}{2}}(x_i)} \quad \text{and there} \quad \frac{1}{2} \partial^{2}_{\alpha_i} f(\alpha) = -\frac{4}{n-2} \frac{\delta_{i,j} - \frac{\alpha_i \alpha_j}{\sum_r K(x_r) \alpha_r^{\frac{4}{n-2}}}}{(\sum_r K(x_r) \alpha_r^{\frac{4}{n-2}})^{\frac{n}{n-2}}}. 
\]
In particular \( \Theta^{\frac{2-n}{2}} = \sum_i K^{\frac{2-n}{2}}(x_i) \) and due smallness of \( |\partial J| \) on \( V(q, \varepsilon) \) necessarily
\[
\alpha_i = \frac{\Theta}{K^{\frac{2-n}{2}}(x_i)} + o_2(1)
\]
Denoting hence by \((\tilde{\alpha}_i) \in \mathbb{R}^q \) for \( i = 1, \ldots, q-1 \) the eigenvector with negative eigenvalue \( \lambda_{\tilde{\alpha}_i} \) of
\[
A_{i,j} = \frac{4}{n-2} (\delta_{i,j} - K^{\frac{2-n}{2}}(x_i) K^{\frac{2-n}{2}}(x_j))
\]
and \((\alpha_q) = (K^{\frac{2-n}{2}}(x_1), \ldots, K^{\frac{2-n}{2}}(x_q))\) with \( \alpha_q = 0 \), we conclude with
\[
J(\alpha^i \varphi_i) = \hat{c}_0 \left( \sum_i \frac{1}{K^{\frac{n-2}{2}}(x_i)} \right)^{\frac{1}{2}} \left( 1 - \sum_i \left(1 + o_2(1)\right) \lambda_{\tilde{\alpha}_i} \tilde{\alpha}_i^2 - \hat{c}_1 \sum_i \frac{\left(1 + o_2(1)\right) \Delta K(x_i)}{K^{\frac{n-2}{2}}(x_i) \lambda_i^2} \right)
\]
Recalling \( \Delta K(x_i) < 0, \) (5.14) and (5.15), the proposition follows.

Let us conclude with a discussion of the inaccuracy in [3], namely, that the deformation constructed in its Appendix 2, cf. also [2], leaves the variational space
\[
X = \{0 \leq u \in W^{1,2}(M) \mid \|u\| = 1\}
\]
by not preserving non-negativity \( u \geq 0 \) and not preserving the normalisation \( \|u\| = 1 \). While the first violation is not an issue as exposed in [2], the latter has to be addressed. Let us discuss some possibilities.

(i) **Naive renormalisation** The most simple approach would be to let flow and renormalise afterwards, which however might lead to a lack of well definedness as a flow on \( X = \{\|\cdot\| = 1\} \).

(ii) **Brute force normalisation** The construction in [3] is an adaptation at infinity, i.e. flow lines of type
\[
u = \alpha^i \varphi_a, \lambda_i + v.
\]
Leaving the \( v \)-part aside, the constructed vectorfield prescribes a movement in \( a_i \) and \( \lambda_i \) keeping the scaling parameters \( \alpha_i \) invariant. Hence one might adjust the \( \alpha_i \) dynamically, e.g. along
\[
\frac{\dot{\alpha}_i}{\alpha_i} = \beta_\alpha, \quad \beta_\alpha = \beta_\alpha(a_i, \lambda_i)
\]
in order to preserve \( \|u\| = 1 \). This however adds error terms of type
\[
\alpha \sum_r \left( 1 - \frac{\sum_s \alpha_s^2}{\sum_r K(a_r) \alpha_r^{\frac{4}{n-2}}(a_r \alpha_r^{\frac{4}{n-2}})} \right),
\]
when verifying energy decreasing, i.e. \( \partial_t J(u) \leq 0 \), and hence necessitates a dynamical control of these quantities, which is not available from [3] or [2]. We perform this argument here.
(iii) \textit{Geometric normalisation} The most intuitive way to adjust the vectorfield $w_0$ constructed in [3] in order to preserve $\|u\| = 1$ is passing from $\partial_t u = w_0(u)$ to

$$\partial_t u = w(u) = w_0(u) - \frac{\langle w_0(u), u \rangle}{\|u\|^2} u.$$  

(5.16)

Of course the flow lines for $w_0$ and $w$ are then not evidently related and passing from $w_0$ to $w$ perturbs the movements in $a_i$ and $\lambda_i$. But then the statement of Proposition A2 in [3], that away from the \textit{critical points at infinity} $\lambda = \max_i \lambda_i$ is non increasing, requires justification.

Since there has been a variety of scientific research relying on [3] and in particular its Appendix 2, we would like to point out, that in our opinion and based on the availability of better estimates on $v$ the errors induced by the necessity to normalise the flow are not critical at least in low dimensions $n = 3, 4, 5$. Also note, that for instance [2] describing the positivity and norm preserving gradient flow, is not affected.

\begin{center}
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\end{center}

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