Transmuted Sushila Distribution and its Application to Lifetime Data

ADETUNJI ADEMOLA A

Department of Statistics, Federal Polytechnic, Ile-Oluji, Nigeria

Abstract

The Sushila distribution is generalized in this article using the quadratic rank transmutation map as developed by Shaw and Buckley (2007). The newly developed distribution is called the Transmuted Sushila distribution (TSD). Various mathematical properties of the distribution are obtained. Real lifetime data is used to compare the performance of the new distribution with other related distributions. The results shown by the new distribution perform creditably well.

Keywords: Sushila distribution, Lindley distribution, lifetime distribution, reliability function.

2010 MSC: 60E05, 63E10, 62G30, 62H10, 97K50.

1. Introduction

Last few years have witnessed generalizations of various lifetime distributions. This is achieved by compounding the distribution with any of new families of distributions. The process involves introduction of new shape parameter(s) to improve flexibility of the baseline distribution. Among well-known generalized families of distributions are: Marshall-Olkin family of distributions [1]; Beta G distributions [2]; Quadratic Transmuted family of distributions [3]; Kumaraswamy G distributions [4]; Gamma G distributions [5]; Exponentiated generalized G distributions [6]; Weibull G distributions [7]; and Alpha Power Transformation [8]. Researchers in sciences and engineering have applied these families of distribution to improve modelling of various lifetime data.

In this article, we generalize the Sushila distribution [9] using the Quadratic Transmuted family of distributions [3] and the new generalization is called the Transmuted Sushila Distribution (TSD). A random variable X is said to have the Sushila distribution if its distribution function (CDF) is given as:

\[ G(x) = 1 - \frac{\lambda (1 + \theta) + \theta x}{\lambda (1 + \theta)} e^{-\frac{\theta x}{\lambda}}. \]  (1.1)
It is observed that it is a special case of the Lindley distribution [10] when \( \lambda = 1 \).

Given a baseline distribution with the CDF \( G(x) \), the Quadratic Transmuted (QT) family of distributions has the cdf:

\[
F(x) = (1 + a)G(x) - a\left(G(x)^2\right).
\] (1.2)

QT was applied to some probability distributions by [11, 12] with the resulting distributions offering more flexibility. [13] also discussed some mathematical properties for the QT family of distributions. With generally acceptability, researchers have applied (1.2) and introduced different new members of the QT family for diverse lifetime distributions. List of some QT distributions was provided by [14].

2. Transmuted Sushila Distribution

The Transmuted Sushila Distribution (TSD) is obtained by putting (1.1) into (1.2). Hence, a random variable \( X \) is said to have the TSD, i.e. \( X \sim \text{TSD}(\theta, \lambda, a) \) if its CDF is given as:

\[
F(x) = (1 + a)\left(1 - \frac{\lambda(1 + \theta + \theta x - \theta) e^{-\theta x}}{\lambda(1 + \theta)}\right) - a\left(1 - \frac{\lambda(1 + \theta + \theta x) e^{-\theta x}}{\lambda(1 + \theta)}\right)^2.
\] (2.1)

Figures 1 illustrates the cdf of the TSD for some selected values of scale and shapes parameters. The probability distribution function (PDF) of TSD is obtained by differentiating (2.1) once. Hence, the PDF of \( X \sim \text{TSD}(\theta, \lambda, a) \) is:

\[
f(x) = \frac{\theta^2}{\lambda^3(1 + \theta)^2}(\lambda + x)\left(2a e^{-\theta x}(\lambda \theta + \theta x + \lambda) - \lambda(1 + \theta)(a - 1)\right) e^{-\theta x}.
\] (2.2)

Note:

(i) The TSD becomes the Sushila Distribution due to [9] if \( a = 0 \).
(ii) The TSD becomes the Lindley Distribution due to [10] if \( a = 0 \) and \( \lambda = 1 \).

The pdf of the TSD for some selected values of scale and shapes parameters is shown in figure 2.
3. Reliability Analysis

**Survival Function:** The probability of an item not failing prior to a particular time is defined by its reliability or survival function $S(x)$. This is defined by $S(x) = 1 - F(x)$. Therefore, if a random variable $X \sim \text{TSD} (\theta, \lambda, a)$, then its survival function is given by:

$$S(x) = 1 - (1 + a) \left( 1 - \frac{\lambda (1 + \theta) + \theta x}{\lambda (1 + \theta)} e^{-\frac{\theta x}{\lambda}} \right) + a \left( 1 - \frac{\lambda (1 + \theta) + \theta x}{\lambda (1 + \theta)} e^{-\frac{\theta x}{\lambda}} \right)^2. \quad (3.1)$$

**Hazard Rate Function (HRF):** This is the risk a system has in experiencing and event in an instantaneous time provided it has not experienced it at present time. It is a measure of proneness to an event and it is obtained as: $h(x) = \frac{f(x)}{S(x)}$. For a random variable that has the TSD, the hrf is given as:
\[ h(x) = \frac{\theta^2(x + \lambda)\left(2a \exp^{-\theta x} (\lambda \theta + \theta x + \lambda) - \lambda(1 + \theta)(a - 1)\right)}{\lambda(\lambda \theta + \theta x + \lambda)\left((\lambda \theta + \theta x)a \exp^{-\theta x}\right) - \lambda(1 + \theta)(a - 1)} \]  

Figure 4 shows the HRF of the TSD for some values of scale and shapes parameters.
Cumulative Hazard Rate Function, $H(x)$: This is the overall risk rate from the onset up to a given time. The measure is obtained as $H(x) = -\ln(S(x))$.

$$H(x) = -\ln \left( (\lambda \theta + \theta x + \lambda) \frac{\alpha \exp^{-\frac{\theta x}{\lambda}} (\lambda \theta + \theta x + \lambda - a \theta \lambda + \theta \lambda + \lambda)}{\lambda^2 (1 + \theta)^2} \exp^{-\frac{\theta x}{\lambda}} \right). \quad (3.3)$$

The trend depicting the cumulative HRF of the TSD for some values of scale and shapes parameters is shown in figure 5.

4. Mathematical Properties

4.1. Order Statistics

Given that $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ is a set of ordered random variable of size $n$, if $X \sim \text{TSD}(\theta, \lambda, a)$, then, the PDF of the $r^{th}$ order statistics is given as:

$$f_{r,n}(x) = \frac{n!}{(r-1)! (n-r)!} f(x) [F(x)]^{r-1} [1-F(x)]^{n-r}.$$

Therefore, the $r^{th}$ order statistics of $X$ is given as:

$$f_{r,n}(x) = \frac{n! \theta^2 (\lambda + x)}{(r-1)! (n-r)!} \left[ \frac{2ae^{-\frac{\theta x}{\lambda}} (\lambda \theta + \theta x + \lambda - a \theta \lambda - a \lambda + \theta \lambda - \lambda)}{\lambda^2 (1 + \theta)^2} \right] \left[ (1 + a) \left( 1 - \frac{\lambda (1 + \theta) + \theta x}{\lambda (1 + \theta)} e^{-\frac{\theta x}{\lambda}} \right) - a \left( 1 - \frac{\lambda (1 + \theta) + \theta x}{\lambda (1 + \theta)} e^{-\frac{\theta x}{\lambda}} \right)^2 \right]^{r-1} \quad (4.1)$$
If \( r = 1 \) and \( r = n \), the 1st and \( nth \) order statistic for \( X \) are respectively given in (4.2) and (4.3).

\[
f_{1,n}(x) = \frac{n!\theta^2(\lambda + x)}{(n-1)!} \left[ \frac{2ae^{-\frac{\theta}{\lambda}x}(\lambda\theta + \theta x + \lambda) - a\theta \lambda - a\lambda + \theta \lambda - \lambda}{\lambda^3(1+\theta)^2} e^{-\frac{\theta}{\lambda}x} \right] \\
\left[ 1 - (1 + a) \left( 1 - \frac{\lambda(1+\theta) + \theta x}{\lambda(1+\theta)} e^{-\frac{\theta}{\lambda}x} \right) + a \left( 1 - \frac{\lambda(1+\theta) + \theta x}{\lambda(1+\theta)} e^{-\frac{\theta}{\lambda}x} \right) \right]^{n-r}.
\]

\[
f_{n,n}(x) = \frac{n!\theta^2(\lambda + x)}{(n-1)!} \left[ \frac{2ae^{-\frac{\theta}{\lambda}x}(\lambda\theta + \theta x + \lambda) - a\theta \lambda - a\lambda + \theta \lambda - \lambda}{\lambda^3(1+\theta)^2} e^{-\frac{\theta}{\lambda}x} \right] \\
\left[ (1 + a) \left( 1 - \frac{\lambda(1+\theta) + \theta x}{\lambda(1+\theta)} e^{-\frac{\theta}{\lambda}x} \right) - a \left( 1 - \frac{\lambda(1+\theta) + \theta x}{\lambda(1+\theta)} e^{-\frac{\theta}{\lambda}x} \right) \right]^{n-1}.
\]

4.2. Quantiles Function

The quantile function of a random variable \( X \sim \text{TSD}(\theta, \lambda, a) \), is given as:

\[
Q_{(u)} = -\left( \frac{\lambda}{\theta} \right) \left[ W \left( -\frac{-e^{-(1+\theta)(1+\theta)}(a-1+\sqrt{1+2a+a^2-4ua})}{2a} \right) + 1 + \theta \right]. \tag{4.4}
\]

Therefore, the first, the second, and the third quartiles for the random variable are obtained by respectively setting \( u \) to 0.25, 0.50, and 0.75. These are given by:

\[
Q_{(\frac{1}{4})} = -\left( \frac{\lambda}{\theta} \right) \left[ W \left( -\frac{e^{-(1+\theta)(1+\theta)}(a-1+\sqrt{1+2a+a^2})}{2a} \right) + 1 + \theta \right],
\]

\[
Q_{(\frac{1}{2})} = -\left( \frac{\lambda}{\theta} \right) \left[ W \left( -\frac{e^{-(1+\theta)(1+\theta)}(a-1+\sqrt{1+a^2})}{2a} \right) + 1 + \theta \right],
\]

\[
Q_{(\frac{3}{4})} = -\left( \frac{\lambda}{\theta} \right) \left[ W \left( -\frac{e^{-(1+\theta)(1+\theta)}(a-1+\sqrt{1+a^2-a})}{2a} \right) + 1 + \theta \right],
\]

where the Lambert function \( W \) is a complex function with multiple values which is defined as the solution for the equation \( W(u)e^{W(u)} = u \).

Equation (4.4) is obtained using Maple 2016 [15].
4.3. Skewness and Kurtosis

Variability in a data set can be investigated using skewness and kurtosis, and classical measures can be susceptible to outliers. Given the quantile function as in (4.4), the Moor’s Kurtosis [16] based on octiles is given as:

$$K_M = \frac{Q_{7/8} - Q_{5/8}}{Q_{6/8} - Q_{2/8}}.$$  (4.5)

Also, the Bowley’s measure of skewness [17] based on quartiles is given as:

$$SK_B = \frac{Q_{3/4} - 2Q_{1/4} + Q_{1/4}}{Q_{3/4} - Q_{1/4}}.$$  (4.6)

4.4. Moments

**Proposition 4.1.** If a random variable $X$ follows TSD with pdf as given in (2.2), then the $k$th raw moment is given by:

$$E(x^k) = \frac{\lambda^k k!}{\theta^k (1 + \theta)^k} \left( \frac{a\theta(1 + \theta)}{2^k} + \frac{a(1 + 2\theta)(k + 1)}{2^{k+1}} \right)$$

$$- (a\theta + a - \theta - 1)(\theta + k + 1) + \frac{a(k + 1)(k + 2)}{2^{k+2}}.  \tag{4.7}$$

**Proof.**

$$\mu_k = E(x^k) = \int_0^\infty x^k f(x) \, dx$$

$$= \int_0^\infty x^k \frac{\theta^2}{\lambda^3(1 + \theta)^2} (\lambda + x) \left( 2ae^{-\frac{\theta x}{\lambda}} (\lambda\theta + \theta x + \lambda) - \lambda(1 + \theta)(a - 1) \right) e^{-\frac{x}{\lambda}} \, dx$$

$$= \left( \frac{\theta^2}{\lambda^3(1 + \theta)^2} \right) \int_0^\infty \left[ \lambda x^k e^{-\frac{\theta x}{\lambda}} + x^{k+1} e^{-\frac{x}{\lambda}} \right]$$

$$\left( 2ae^{-\frac{\theta x}{\lambda}} (\lambda\theta + \theta x + \lambda) - \lambda(1 + \theta)(a - 1) \right) \, dx$$

$$= \left( \frac{\theta^2}{\lambda^3(1 + \theta)^2} \right) \int_0^\infty \left[ \lambda x^k e^{-\frac{\theta x}{\lambda}} + x^{k+1} e^{-\frac{x}{\lambda}} \right]$$

$$\left( 2ae^{-\frac{\theta x}{\lambda}} + \lambda 2ae^{-\frac{\theta x}{\lambda}} - a\theta\lambda - a\lambda + \theta\lambda + \lambda \right) \, dx$$

$$= \left( \frac{\theta^2}{\lambda^3(1 + \theta)^2} \right) \left[ 2a\theta\lambda^2 + 2a\lambda^2 \right] P_1 + \left[ 4a\theta\lambda + 2a\lambda \right] P_2$$

$$- \lambda^2 \left[ a\theta + a - \theta - 1 \right] P_3 + 2a\theta P_4 - \lambda \left[ a\theta + a - \theta - 1 \right] P_5,$$

where

$$P_1 = \int_0^\infty x^k e^{-\frac{2\theta x}{\lambda}} \, dx = \frac{\lambda^{k+1} k!}{2^{k+1} \theta^{k+1}}; \quad P_2 = \int_0^\infty x^{k+1} e^{-\frac{\theta x}{\lambda}} \, dx = \frac{\lambda^{k+2} (k + 1)!}{2^{k+2} \theta^{k+2}};$$
\[ P_3 = \int_0^\infty x^k e^{-\theta x} \, dx = \frac{\lambda^{k+1}k!}{\theta^{k+1}}; \quad P_4 = \int_0^\infty x^{k+2} e^{-\theta x} \, dx = \frac{\lambda^{k+3}k!}{2^{k+2}\theta^{k+2}}; \]

\[ P_5 = \int_0^\infty x^{k+1} e^{-\theta x} \, dx = \frac{\lambda^{k+2}(k+1)!}{\theta^{k+2}}. \]

Therefore,

\[
E(x^k) = \left( \frac{\theta^2}{\lambda^2(1+\theta)^2} \right) \left( \frac{2a\theta \lambda^2 + 2a\lambda^2}{2^{k+1}\theta^{k+1}} \right) + \left( \frac{4a\theta \lambda + 2a\lambda}{2^{k+2}\theta^{k+2}} \right) - \lambda^2 \left( a\theta + a - \theta - 1 \right) \left( \frac{\lambda^{k+1}k!}{\theta^{k+1}} \right) + 2a\theta \left( \frac{\lambda^{k+3}k!}{2^{k+2}\theta^{k+2}} \right) - \lambda \left( a\theta + a - \theta - 1 \right) \left( \frac{\lambda^{k+2}(k+1)!}{\theta^{k+2}} \right).
\]

Hence,

\[
E(x^k) = \frac{\lambda^k k!}{\theta^{k}(1+\theta)^2} \left[ \frac{a\theta(1+\theta)}{2^k} + \frac{a(1+2\theta)(k+1)}{2^{k+1}} - (a\theta + a - \theta - 1)(\theta + k + 1) + \frac{a(k+1)(k+2)}{2^{k+2}} \right].
\]

**Mean and Variance of TSD:** The mean of a random variable \( X \) follows TSD is obtained by putting \( k = 1 \) in (4.7) above. Hence the mean is given as:

\[
E(x) = \frac{\lambda}{4\theta(1+\theta)^2} \left( 2a\theta^2 + 6a\theta - 4\theta^2 + 3a - 12\theta - 8 \right)
\]

Also, the variance of \( X \) is obtained as \( \text{var}(x) = E(x^2) - (E(x))^2 \), where \( E(x^2) \) is obtained by equating \( k = 2 \) in (4.7).

\[
E(x^2) = \frac{\lambda^2}{4\theta^2(1+\theta)^2} \left( 6a\theta^2 + 24a\theta - 8\theta^2 + 15a - 32\theta - 24 \right)
\]

Hence, the variance of \( X \) is given by:

\[
\text{var}(x) = \frac{\lambda^2}{4\theta^2(1+\theta)^2} \left[ \frac{[2a\theta^2 + 6a\theta - 4\theta^2 + 3a - 12\theta - 8]^2}{4(1+\theta)^2} \right] + (6a\theta^2 + 24a\theta - 8\theta^2 + 15a - 32\theta - 24).
\]
4.5. Moment Generating Function

**Proposition 4.2.** The moment generating function of a random variable $X$ that follows the TSD is given as:

$$
E(e^{tx}) = \left( \frac{\theta^2}{(1+\theta)^2} \right) \left( \frac{2a(1+\theta) + 2a(1+2\theta)}{2\theta - \lambda t} + \frac{4a\theta}{(2\theta - \lambda t)^2} \right) 
- \frac{\lambda^2}{(\theta - \lambda t)^2} 
\left( a\theta + a - \theta - 1 \right) \left( \theta - \lambda t + 1 \right) \right). 
\quad (4.8)
$$

**Proof.**

$$
M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x) \, dx 
= \int_0^\infty e^{tx} \frac{\theta^2}{\lambda^3(1+\theta)^2} (\lambda + x) \left( 2ae^{-\frac{\theta}{\lambda}} (\lambda t - \theta x + \lambda) - \lambda(1+\theta)(a-1) \right) e^{-\frac{\theta}{\lambda} x} \, dx 
= \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) \int_0^\infty (\lambda + x) e^{-\frac{\theta}{\lambda} x} \left( 2a\theta \lambda e^{-\frac{\theta}{\lambda} x} + 2a\theta x e^{-\frac{\theta}{\lambda} x} + 2a\lambda e^{-\frac{\theta}{\lambda} x} \right) 
- \lambda(a\theta + a - \theta - 1) \right) \, dx 
= \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) \left( 2a\theta \lambda^2 + 2a\lambda^2 \right) + \left( 4a\theta \lambda \right) + \left( 2a\lambda \right) P_2 - \lambda^2 \left( a\theta + a - \theta - 1 \right) P_3 + 2a\theta P_4 - \lambda \left( a\theta + a - \theta - 1 \right) P_5,
$$

where

$$
P_1 = \int_0^\infty e^{-\frac{\theta}{\lambda} x} \, dx = \frac{\lambda}{2\theta - \lambda t}; \quad P_2 = \int_0^\infty xe^{-\frac{\theta}{\lambda} x} \, dx = \frac{\lambda^2}{(2\theta - \lambda t)^2};
$$

$$
P_3 = \int_0^\infty e^{-\frac{\theta}{\lambda} x} \, dx = \frac{\lambda}{(\theta - \lambda t)^2}; \quad P_4 = \int_0^\infty x^2 e^{-\frac{\theta}{\lambda} x} \, dx = \frac{2\lambda^3}{(2\theta - \lambda t)^3};
$$

$$
P_5 = \int_0^\infty xe^{-\frac{\theta}{\lambda} x} \, dx = \frac{\lambda^2}{(\theta - \lambda t)^2}.
$$
Therefore,
\[
\mathbb{E}(e^{tx}) = \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) (2a\theta\lambda^2 + 2a\lambda^2) \left( \frac{\lambda}{2\theta - \lambda t} \right) + (4a\theta\lambda + 2a\lambda) \left( \frac{\lambda}{(2\theta - \lambda t)} \right)^2 \\
-\lambda^2(a\theta + a - \theta - 1) \left( \frac{\lambda}{\theta - \lambda t} \right) + 4a\theta \left( \frac{2\lambda}{(2\theta - \lambda t)} \right)^3 - \lambda(a\theta + a - \theta - 1) \left( \frac{\lambda}{(\theta - \lambda t)} \right)
\]
\[
= \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) \left[ 2a\lambda^2(1+\theta) \left( \frac{\lambda}{2\theta - \lambda t} \right) \\
+ 2a\lambda(1+2\theta) \left( \frac{\lambda}{2\theta - \lambda t} \right)^2 + 4a\theta \left( \frac{\lambda}{2\theta - \lambda t} \right)^3 \\
- (a\theta + a - \theta - 1)\lambda^3 \left( 1 + \frac{1}{\theta - \lambda t} \right) \right].
\]

Hence
\[
\mathbb{E}(e^{tx}) = \left( \frac{\theta^2}{(1+\theta)^2} \right) \left( \frac{2a(1+\theta)}{2\theta - \lambda t} + \frac{2a(1+2\theta)}{(2\theta - \lambda t)^2} + \frac{4a\theta}{(2\theta - \lambda t)^3} - \frac{(a\theta + a - \theta - 1)(\theta - \lambda t + 1)}{(\theta - \lambda t)^2} \right).
\]

\[\square\]

5. Parameter Estimation

**Proposition 5.1.** Given that \(X_i, i = 1, 2, ..., n\) are iid random variables from TSD, then the log-likelihood function of \(X\) is defined as:

\[
\log L = 2n \left( \log(\theta) - \log(1+\theta) \right) - 3n \log(\lambda) + \sum_{i=1}^{n} \log(\lambda + x_i) \\
+ \sum_{i=1}^{n} \log \left( 2ae^{-\theta x_i}(\lambda\theta + \theta x_i + \lambda) - a\theta\lambda + \theta\lambda + \lambda \right) - \frac{\lambda}{\theta} \sum_{i=1}^{n} x_i.
\]
Proof. The likelihood function of a random variable $X$ that follows TSD is:

$$L = \prod_{i=1}^{n} \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) (\lambda + x_i) \left( 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\theta \lambda - a\lambda + \theta \lambda + \lambda \right) e^{-\frac{\theta}{\lambda} x_i}$$

$$= \left( \frac{\theta^2}{\lambda^3(1+\theta)^2} \right) \prod_{i=1}^{n} (\lambda + x_i) \prod_{i=1}^{n} \left( 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\theta \lambda - a\lambda + \theta \lambda + \lambda \right) e^{-\frac{\theta}{\lambda} \sum_{i=1}^{n} x_i}$$

$$= \left( \frac{\theta}{1+\theta} \right)^{2n} \lambda^{-3n} \prod_{i=1}^{n} (\lambda + x_i) \prod_{i=1}^{n} \left( 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\theta \lambda - a\lambda + \theta \lambda + \lambda \right) e^{-\frac{\theta}{\lambda} \sum_{i=1}^{n} x_i}.$$

Hence, the log-likelihood function of a random variable $X$ that follows TSD is:

$$\log L = 2n \left( \log(\theta) - \log(1+\theta) \right) - 3n \log(\lambda) + \sum_{i=1}^{n} \log(\lambda + x_i)$$

$$+ \sum_{i=1}^{n} \log \left( 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\lambda + \theta \lambda + \lambda \right) - \frac{\lambda}{\theta} \sum_{i=1}^{n} x_i.$$

The MLE of $(\theta, \lambda, a)$ can be obtained by maximizing (5.1). This gives the set of normal equations below:

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{2n}{1+\theta} - \frac{1}{\lambda} \sum_{i=1}^{n} x_i$$

$$+ \sum_{i=1}^{n} \left( -\frac{1}{\lambda^2} 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) + 2ae^{-\theta x_i}(\lambda + x_i + \lambda) - a\lambda + \lambda \right) 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\lambda + \theta \lambda + \lambda$$

$$= 0.$$

$$\frac{\partial \log L}{\partial \lambda} = -\frac{3n}{\theta} + \sum_{i=1}^{n} \frac{1}{\lambda + x_i} + \frac{\theta}{\lambda^2} \sum_{i=1}^{n} x_i$$

$$+ \sum_{i=1}^{n} \left( -\frac{1}{\lambda^2} 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) + 2ae^{-\theta x_i}(1+\theta) - a\theta - a + \theta + 1 \right) 2ae^{-\theta x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - a\lambda + \theta \lambda + \lambda$$

$$= 0.$$
\[
\frac{\partial \log L}{\partial a} = \sum_{i=1}^{n} \frac{2ae^{-\frac{\theta}{2}x_i}(\lambda \theta + \theta x_i + \lambda) - \theta \lambda - \lambda}{2ae^{-\frac{\theta}{2}x_i}(\lambda \theta + \theta x_i + \lambda) - a\theta \lambda - \theta \lambda + \lambda} = 0.
\]

Obtaining solutions for the set of normal equations analytically is tedious. Using the \textsc{MaxLik} function in R language [18], the solutions are obtained numerically using algorithms like Newton-Raphson.

5.1. Asymptotic Confidence Bounds of TSD

Using the variance-covariance matrix \( I_{\theta}^{-1} \), a \((100 - \alpha)\%\) confidence intervals of the parameters \( \theta, \lambda, a \) can be obtained. \( I_{\theta}^{-1}[\Psi] \) is the inverse of the observed information matrix [19] given by:

\[
\begin{bmatrix}
I_{\theta\theta} & I_{\theta\lambda} & I_{\theta a} \\
I_{\lambda\theta} & I_{\lambda\lambda} & I_{\lambda a} \\
I_{a\theta} & I_{a\lambda} & I_{aa}
\end{bmatrix} = \begin{bmatrix}
\text{var}(\hat{\theta}) & \text{covar}(\hat{\theta}, \hat{\lambda}) & \text{covar}(\hat{\theta}, \hat{a}) \\
\text{covar}(\hat{\theta}, \hat{\lambda}) & \text{var}(\hat{\lambda}) & \text{covar}(\hat{\lambda}, \hat{a}) \\
\text{covar}(\hat{\theta}, \hat{a}) & \text{covar}(\hat{\lambda}, \hat{a}) & \text{var}(\hat{a})
\end{bmatrix}
\]

\[
I_{\theta\theta} = \frac{\partial^2 \log L}{\partial \theta^2}; I_{\lambda\lambda} = \frac{\partial^2 \log L}{\partial \lambda^2}; I_{aa} = \frac{\partial^2 \log L}{\partial a^2}; I_{\theta\lambda} = \frac{\partial^2 \log L}{\partial \theta \partial \lambda}; I_{\theta a} = \frac{\partial^2 \log L}{\partial \theta \partial a}; I_{a\lambda} = \frac{\partial^2 \log L}{\partial a \partial \lambda}
\]

Therefore, at specified level of significance, \((100 - \alpha)\%\) confidence intervals of \((\theta, \lambda, a)\) are respectively given as: \( \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{\theta})}; \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{\lambda})}; \quad \hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{\text{var}(\hat{a})}. \)

6. Application

The QT technique has been applied to obtain new set of distributions by compounding the Lindley distribution. The Transmuted Lindley distribution (TLD) was introduced by [20]. [21] introduced Transmuted Quasi-Lindley distribution (TQLD) while [22] introduced the Transmuted Two-Parameter Lindley distribution (TTPLD). The Transmuted Generalized Quasi Lindley distribution (TGQLD) was introduced by [23]. All these newly transmuted distributions have the Lindley distribution has special case. This attribute is also shared by the TSD newly introduced in this research. The probability distribution functions of all the models compared for data application are presented in table 1 below.

The data set used to observe the performance of TSD is the remission times (months) of a sample of 128 bladder cancer patients. This data has been applied in various survival analysis [19, 20].

\begin{tabular}{cccccccccccccccc}
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 17.16, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69
\end{tabular}
Table 1: Probability distribution functions of compared models

| Distribution | PDF |
|--------------|-----|
| TSD          | \(\frac{\theta^2}{\lambda(1+\theta)(1+\theta+\lambda)}(\lambda x + \lambda - \lambda(1+\theta)(\alpha - 1) e^{-\frac{\theta x}{\lambda}})\) |
| SD           | \(\frac{\theta^2}{\lambda(1+\theta)} \left(1 + \frac{\lambda x}{\theta}\right) e^{-\frac{\theta x}{\lambda}}\) |
| LD           | \(\frac{\theta^2}{\lambda} (1 + \theta)e^{-\frac{\theta x}{\lambda}}\) |
| TLD          | \(\frac{\theta^2}{\lambda} (1 + \theta)e^{-\frac{\theta x}{\lambda}}\left(1 - \lambda + 2\lambda(1+\theta) \frac{1}{1+\lambda}\right)\) |
| TQLD         | \(\frac{\theta^2}{\lambda(1+\theta)} (\alpha + \theta x) e^{-\frac{\theta x}{\lambda}} \left(1 - \lambda + 2\lambda e^{-\frac{\theta x}{\lambda}} \frac{\theta x}{\lambda}\right)\) |
| TTPLD        | \(\frac{\theta^2}{\lambda(1+\theta)} (\alpha + \theta x) e^{-\frac{\theta x}{\lambda}} \left(1 + \lambda - 2\alpha \left(1 - \alpha \frac{\theta x}{\lambda} e^{-\frac{\theta x}{\lambda}}\right)\right)\) |
| TGQLD        | \(\frac{\theta^2}{\lambda(1+\theta)} (\alpha + \theta x) e^{-\frac{\theta x}{\lambda}} \left(1 - e^{-\frac{\theta x}{\lambda}} \left(1 + \alpha \frac{\theta x}{\lambda}\right)^{\alpha-1} \left(1 + \lambda - 2\frac{\theta x}{\lambda} e^{-\frac{\theta x}{\lambda}}\right)^{\alpha}\right)\) |

Table 2: Descriptive Statistics for remission times in months

| Min. | Q1  | Median | Q3  | Max  | Mean | Variance | Skewness | Kurtosis |
|------|-----|--------|-----|------|------|----------|----------|----------|
| 0.080| 3.348| 6.395  | 11.838| 79.050| 9.366| 110.425  | 3.287    | 15.483   |

The descriptive statistics for the data set is presented in table 2 while table 3 shows the parameter estimates and model comparison criteria. To compare the models, -2LL (negative 2 Log-Likelihood), AIC (Akaike Information Criterion), and CAIC (Corrected Akaike Information Criterion), are used. Distribution with lowest criterion is the "best". Results from table 3 shows that the newly proposed distribution (TSD) performs creditably well for the lifetime dataset presented.

7. Conclusion

We introduced a three-parameter generalization of the Sushila distribution using the Quadratic Transmuted technique championed by [3]. Shape of the distribution function and hazard rate function are investigated and various mathematical properties of the new distribution are presented. Since the Sushila distribution [9] is a special case of the Lindley distribution, performance of the new distribution is compared with the Sushila distribution and other transmuted Lindley distributions using data on cancer remission (in moths) of patients. Results show that the Transmuted Sushila Distribution (TSD) givers a better fit to the data set among the competing distributions.

Competing Interest

Author has declared that no competing interest exist.

References

[1] Marshall AW, Olkin IA (1997). New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families. Biometrika, 84: 641-652. https://doi:10.1093/biomet/84.3.641.
Table 3: Parameter estimates and model selection criteria

| Distribution | $\theta$ | $\lambda$ | $a$ | $\alpha$ | $-2\text{ LL}$ | AIC | CAIC |
|--------------|---------|----------|-----|---------|----------------|-----|-----|
| TSD          | 0.742   | 6.323    | 0.712 | 825.884  | 831.884        | 832.037 |
| SD           | 52.650  | 484.078  |       | 828.694  | 834.694        | 834.842 |
| LD           | 0.196   |          |       | 839.040  | 845.040        | 845.194 |
| TLID         | 0.156   | 0.617    |       | 830.310  | 836.310        | 836.464 |
| TQLD         | 0.061   | 1138.000 | 0.862 | 826.994  | 832.995        | 833.147 |
| TPLD         | 0.132   | 0.531    |       | 828.087  | 834.087        | 834.240 |
| TGQLD        | 0.121   | 1138.000 | 0.862 | 826.994  | 832.995        | 833.147 |

[2] Eugene N, Lee C, Famoye F (2002). Beta-Normal Distribution and Its Applications. Communications in Statistics-Theory and Methods, 31:497-512. https://doi:10.1081/sta-120003130.
[3] Shaw W, Buckley I (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.
[4] Cordeiro GM, Castro M (2011). A New Family of Generalized Distributions. Journal of Statistical Computation and Simulation, 81:883-898. https://doi:10.1080/00949650903530745
[5] Ristic MM, Balakrishnan N (2012). The Gamma Exponentiated Exponential Distribution. Journal of Statistical Computation and Simulation. 82:1191-1206. https://doi:10.1080/00949655.2011.574633
[6] Cordeiro GM, Ortega EMM, Cunha DCC (2013). The Exponentiated Generalized Class of Distributions. Journal of Data Science. 11:1-27.
[7] Alzaatreh A, Lee C, Famoye F (2013). A New Method for Generating Families of Continuous Distributions. Metron. 71:63-79. https://doi:10.1007/s40300-013-0007-y.
[8] Shanker R, Shambhu S, Uma S, Ravi S (2013). Sushila distribution and its application to waiting times data. International Journal of Business Management. 3(2):1-11.
[9] Lindley DV (1958). Fiducial distributions and Bayes-Theorem. Journal of the Royal Statistical Society, B(20):102-107.
[10] Aryal GR, Tsokos CP (2009). On the transmuted extreme value distribution with application. Nonlinear Analysis: Theory, Methods and Applications, 71:1401-1407. https://doi:10.1016/j.na.2009.01.168.
[11] Aryal GR, Tsokos CP (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. European Journal of Pure and Applied Mathematics. 4:89-102.
[12] Bourguignon M, Ghosh I, Cordeiro GM (2016). General Results for the Transmuted Family of Distributions and New Models. Journal of Probability and Statistics. 7208425. https://doi:10.1155/2016/7208425.
[13] Tahir MH, Cordeiro GM (2016). Compounding of distributions: a survey and new generalized classes. Journal of Statistical Distributions and Applications, 3:1-35. https://doi.org/10.1186/s40488-016-0052-1.
[14] Maple Incorporated, a division of Waterloo Maple Incorporated, 2016.
[15] Kenney J, Keeping E (1962). Mathematics of Statistics. Volume 1, Third edition, Princeton, Van Nostrand, New Jersey.
[16] R Development Core Team (2009). R: A Language and Environment for Statistical Computing. Foundation for Statistical Computing, Vienna, Austria.
[17] Lawless JF (2003). Statistical Models and Methods for lifetime data. John Wiley and Sons, New York, 20:1108-1113
[18] Merovci F (2013). Transmuted Lindley distribution. Int. J. Open Problems Comput. Math., 6:63-72.
[19] Elbatal I, Elgarhy M (2013). Transmuted quasi-Lindley distribution: A Generalization of the Quasi Lindley distribution. Int. J. Pure Appl. Sci. Technol. 18:59-70.
[20] Al-khazaleh M, Al-Omari AI, Al-khazaleh AMH (2016). Transmuted Two-Parameter Lindley Distribution. Journal of Statistics Applications & Probability. 5(3):421-432 http://dx.doi.org/10.18576/jsap/050306
[21] Elgarhy M, Elbatal I, Diab LS, Hwas HK, Shawki AW (2017). Transmuted Generalized Quasi Lindley Distribution. International Journal of Scientific Engineering and Science. 1(7):1-8