De Broglie-Bohm Interpretation for Analytic Solutions of the Wheeler-DeWitt Equation in Spherically Symmetric Space-Time

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We discuss the implications of a wave function for quantum gravity that involves nothing but 3-dimensional geometries as arguments and is invariant under general coordinate transformations. We derive an analytic wave function from the Wheeler-DeWitt equation for spherically symmetric space-time with the coordinate system arbitrary. The de Broglie-Bohm interpretation of quantum mechanics is applied to the wave function. In this interpretation, deterministic dynamics can arise from a wave function in fully quantum regions as well as in semiclassical regions. By introducing a coordinate system, we obtain a cosmological black hole picture in compensation for the loss of general covariance.

§1. Introduction

The quantum theory of gravity is one of the most attractive subjects in particle physics and cosmology. The canonical formalism of gravity formulated by Arnowitt, Deser and Misner (ADM) and by Dirac describes the dynamics of gravity as a totally constraint system. For quantization, the constraints are used to construct possible quantum states. The equation that corresponds to the Hamiltonian constraint is called the Wheeler-DeWitt equation. A great deal of effort has been made to analyze this equation. In particular, the dynamics of spherically symmetric geometries have been studied extensively. In this case, although gravitational and electromagnetic waves cannot be realized, we can treat heuristic geometrical structures of space-time: black holes as small-scale structures and the expanding universe as the large scale structure.

In solving the Wheeler-DeWitt equation, there arise the problems of the operator ordering ambiguity and regularization. It has been argued that although the constraint algebra does close in certain operator ordering, it does not necessarily close for other orderings. In a previous paper, we considered a spherically symmetric geometry and a fixed operator ordering in the Hamiltonian, momentum and

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mass constraints, so that the algebra among these constraints is closed. For a consistent ordering, we also found an analytic wave function of the quantized spherically symmetric space-time as a simultaneous solution of the constraint equations.

To proceed further, in this paper, we attempt to extract quantum properties of the space-time from the analytic wave function.\(^{11}\),\(^{12}\) For this purpose, we use the de Broglie-Bohm (dBB) interpretation (quantum potential interpretation, or pilot wave approach) of quantum mechanics.\(^{13}\)-\(^{15}\) In this interpretation, a deterministic rigid trajectory in the configuration space, what we call a de Broglie-Bohm trajectory, is well-defined. The interpretation merely assigns the gradient of the phase of a wave function to the momentum of a quantum particle as

\[ m \frac{dx}{dt} = p \equiv \frac{\partial \Theta}{\partial x}, \tag{1.1} \]

where the wave function is expressed as \( \Psi = |\Psi| \exp(i\Theta) \). A dBB trajectory is obtained by integrating Eq. (1.1) with respect to \( t \). There can exist many trajectories specified by the configuration on which the quantum system is set at an initial time. The amplitude \( |\Psi|^2 \) is interpreted as the probability density in a statistical ensemble of the trajectories. For the potential problem of quantum particles, the equation to determine the phase of a wave function is derived from the Schrödinger equation. It is described as a particular form of the Hamilton-Jacobi equation, which includes a quantum potential given by

\[ V_Q = -\frac{\hbar^2}{2 |\Psi|} \frac{\partial^2}{\partial x^2} |\Psi|. \tag{1.2} \]

A similar equation can also be derived for gravity from the Wheeler-DeWitt equation.

The dBB interpretation of quantum mechanics is favorable, especially for a quantum theory of gravity, because this interpretation is able to resolve the conceptual problems of quantum gravity: the disappearance of the dynamics (the loss of time),\(^{16}\) and the problem of the observation.\(^{17}\) The dBB interpretation introduces the time parameter \( t \) through Eq. (1.1), which resolves the problem of time.\(^{18}\)-\(^{20}\) Further, a rigid trajectory can be traced without any observation, even in fully quantum regions, and connects the regions smoothly to semi-classical regions, where the quantum potential is negligible and the trajectory behaves like a classical path.

This paper is organized as follows. In §2, we prepare for the main part of the paper. First we carry out a canonical quantization of the Einstein-Maxwell theory with a cosmological constant in spherically symmetric space-time. Then we derive an analytic solution of the Wheeler-DeWitt equation. In the case of space-time with no electromagnetic field, we obtained such a solution in a previous paper.\(^{11}\) Section 3 is devoted to introducing the dBB interpretation to the analytic wave function. In §4, we present explicit expressions of the dBB trajectories. For this we must give coordinate conditions. A summary of the paper is given in §5.
§2. Canonical quantization of the Einstein-Maxwell theory in spherically symmetric space-time

In this section we consider a canonical quantization of the 4-dimensional Einstein-Maxwell theory with a cosmological term. Space-time is assumed to be spherically symmetric. We quantize space-time following our previous work,\(^{11}\) where we fixed the operator ordering in the Hamiltonian, momentum and mass constraints and showed that they form a closed algebra. We also derive an analytic wave function which satisfies these constraints. Here we extend our previous model\(^{11}\) to include an electromagnetic field. The inclusion of the electromagnetic field should be helpful for the examination of extreme black holes\(^^{21}\) and cosmological black holes.\(^^{22}\) We use the natural geometrical units \(c = \hbar = G = 1\) and adopt the conventions in Kuchař’s work\(^^{7}\) and our work.\(^^{11}\)

2.1. Canonical quantization

We start by considering a general spherically symmetric metric in the ADM decomposition,

\[
ds^2 = -N^2dt^2 + A^2(dr + N^r dt)^2 + R^2d\Omega^2,\tag{2.1}
\]

where \(d\Omega\) is a line element on the unit sphere, and the metric components \(N, N^r, A\) and \(R\) are functions of the time coordinate \(t\) and the radial coordinate \(r\). Hereafter we refer to \(N\) and \(N^r\) as the lapse and shift functions, respectively. These express the degrees of freedom in the choice of coordinate systems. The action of the Einstein-Maxwell theory has the form

\[
I = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R - 2\lambda - F_{\mu\nu}F^{\mu\nu} \right), \tag{2.2}
\]

where \(\lambda\) is a cosmological term, and \((4)R\) and \((4)g\) are the scalar curvature and the determinant of the metric tensor, respectively. Here the electromagnetic field strength is denoted by \(F_{\mu\nu}\). Spherical symmetry results in a reduction of the number of non-vanishing components of \(F_{\mu\nu}\). The nontrivial part is described as \(F_{01} = -F_{10} = \dot{A}_1 - A_0'\), where \(A_0\) and \(A_1\) are the components of the electromagnetic field. Hereafter the dot and prime denote derivatives with respect to \(t\) and \(r\), respectively.

Substituting the metric (2.1) into Eq. (2.2), we transform the action into the form of the ADM decomposition,

\[
I = \int dt \int dr \left[ -N^{-1} \left( R(\dot{A} + (AN^r))(-\dot{R} + R'N^r) + \frac{1}{2}A(-\dot{R} + R'N^r)^2 \right) 
+ N \left( -A^{-1}RR'' - \frac{1}{2}A^{-1}R^2 + A^{-2}A'R' + \frac{1}{2}A(1 - \lambda R^2) \right) \right. 
+ \frac{1}{2}N^{-1}A^{-1}R^2(\dot{A}_1 - A_0)^2 \bigg] \tag{2.3}
\]

\[
= \int dt \int dr \left[ P_A \dot{A} + P_R \dot{R} + P_A \dot{A}_1 - N \left( \frac{1}{2}AR^{-2}P_A^2 - R^{-1}PRP_A + \frac{1}{2}R^2A^{-1} \right) 
+ RR''A^{-1} - RR'A^{-2}A' - \frac{A}{2}(1 - R^{-2}P_A^2 - \lambda R^2) \right)
\]
\[-N^r \left( R' P_R - A(P_A)' \right) - A_0 \hat{P}_A' \] ,

(2.4)

where $P_A$, $P_R$ and $P_A$ are the canonical momenta conjugated to $\Lambda$, $R$ and $A_1$, respectively.

We quantize the dynamical variables $\Lambda$, $R$ and $A_1$. In the Schrödinger picture, their canonical momenta are represented by functional differential operators:

\[
\hat{P}_\Lambda(r) = -i \frac{\delta}{\delta \Lambda(r)},
\]

\[
\hat{P}_R(r) = -i \frac{\delta}{\delta R(r)},
\]

\[
\hat{P}_A(r) = -i \frac{\delta}{\delta A_1(r)}.
\]

(2.5)

Here and in the rest of this section, we use hat to indicate differential operators and do not express the argument $t$ explicitly, because we always treat products of simultaneous operators only.

2.2. Constraint equations

The action (2.4) includes the non-dynamical variables $N$, $N^r$ and $A_1$. Variation with respect to them yields the classical constraint equations: the Hamiltonian, momentum constraints and the Gaussian law. The constraints are adapted to restrict a wave function of quantum gravity, $\Psi$, to satisfy the equations

\[
\hat{H}\Psi = 0,
\]

\[
\hat{H}_r\Psi = 0,
\]

\[
\hat{H}_A\Psi = 0,
\]

(2.6)

where the explicit expressions for the operators $\hat{H}$, $\hat{H}_r$ and $\hat{H}_A$ are given below. The first equation is called the Wheeler-DeWitt equation, and the third one corresponds to the conservation of electric charge. For spherically symmetric space-time with no matter, we can introduce the mass of a black hole, $M$, as a dynamical variable, \(^6,7,23\) which is a constant of motion. In quantum theory, we can construct mass eigenstates to satisfy

\[
\hat{M}\Psi = m\Psi,
\]

(2.7)

where $\hat{M}$ and $m$ are the quantized mass operator and a mass eigenvalue, respectively.

In a quantum theory of gravity, it is troublesome to fix the operator ordering. First, we introduce the Hamiltonian, momentum, electric charge and mass operators with ordering factors as

\[
\hat{H} = \frac{1}{2} R^{-2} A \hat{P}^{(A)}_A \hat{P}_A - R^{-1} \hat{P}_R A \hat{P}^{(A)}_A A^{-1}
\]

\[
- R^{-1} (R')^{-1} A (A') \hat{P}_A A^{-1} \hat{P}_A - A \hat{P}_A A^{-2} A' \hat{P}_A + \frac{1}{2} R^2 A^{-1}
\]

\[
+ R R' A^{-1} - R R' A^{-2} A' - \frac{A}{2} \left(1 - \hat{P}^2_A R^{-2} - \lambda R^2\right),
\]

(2.8)
\[
\hat{H}_r = R' \hat{P}_R - A(\hat{P}_A)',
\]
\[
\hat{H}_A = - (\hat{P}_A)',
\]
\[
\dot{M} - m = \frac{1}{2} R^{-1} \hat{P}_A^2 - \frac{1}{2} R(\chi - \hat{F}),
\]
where
\[
\chi \equiv R^2 A^{-2},
\]
\[
\hat{F} \equiv 1 - 2mR^{-1} + \hat{P}_A^2 R^{-2} - \frac{\lambda}{3} R^2.
\]
In Eqs. (2.8) and (2.11), the momentum \( \hat{P}_A \) is accompanied by an ordering function \( A \) as
\[
\hat{P}_A^2 (A) \Lambda \equiv A \hat{P}_A A^{-1}.
\]
In this ordering, \( \hat{M} \) is such that it satisfies the useful relation
\[
\dot{M}' = - \Lambda^{-1} R' \hat{H} - R^{-1} \hat{P}_A \Lambda^{-1} \hat{H}_r - R^{-1} \hat{P}_A \hat{H}_A,
\]
which guarantees that \( \hat{M} \) is spatially conserved, as in classical theory. This shows that the quantized constraints (2.8) – (2.11) have consistent ordering. It is worthwhile to note that the expressions of \( \hat{H} \) in Eq. (2.8) and \( \hat{M}' \) in Eq. (2.15) have operator orderings that differ from those in our previous paper. \(^{11}\) The important difference lies in the fact that the present expressions are derived without using products of Dirac’s delta function, \( \delta(0) \) and \( \delta'(0) \), which are not well defined mathematically. Detailed discussion of this point will appear in a separate paper.

Next, we fix the ordering function \( A \) so that \( \hat{H}, \hat{H}_r \) and \( \hat{M} \) form a closed algebra. According to the results of Ref. 11), the commutators of \( \hat{H} \) and \( \hat{H}_r \), or \( \hat{H} \) and \( \hat{H}_r \), are evaluated from that of \( \hat{M} \) and \( \hat{M} \) or \( \hat{M} \) and \( \hat{H}_r \), and \( A \) is shown to take the form
\[
A = A_Z(Z) \bar{A}(R, \chi),
\]
where \( A_Z \) and \( \bar{A} \) are arbitrary functions, and the argument \( Z \) is defined as
\[
Z \equiv \int dr A f(R, \chi) = \int dr \int^A dA \bar{f}(R, \chi),
\]
with arbitrary functions \( f \) and \( \bar{f} \). Here \( f \) and \( \bar{f} \) are related as
\[
f(R, \chi) = - \frac{\chi^{1/2}}{2} \int^\chi dx x^{-3/2} \bar{f}(R, x).
\]
In the proof of the closure of the constraint algebra, we use the property that
\[
[Z, H_r(r)] = i \left( \frac{R'(r)}{\delta R(r)} \Lambda(r) \left( \frac{\delta Z}{\delta A(r)} \right) \right) = 0.
\]
Tsamis and Woodard \(^{10}\) pointed out that the closure of the constraints in quantum gravity is ill-defined in the sense that formal evaluation of the commutator...
yields a product of multiple delta functions, and that the theory must be regulated. In particular, the formal adoption of calculational rules for the delta function to the product of multiple delta functions with the same argument even leads to contradictory results. In our evaluation of the commutator,\(^{11}\) on the other hand, we only have to utilize the commutators of \(\mathcal{M}\), and thus we need not treat the product of multiple delta functions with the same argument. The only products of multiple delta functions in our calculation consist of \(\delta(0)\), which come from the commutation of coincident operators. Further, all such products that are not canceled are multiplied by the constraints in the canonical gravity. Thus, we can hopefully proceed further through the formal manipulation of the unregulated theory.

2.3. Solutions of constraint equations

The wave function \(\Psi\) is a functional of the electromagnetic field \(A_1\) and the geometrical fields \(\Lambda\) and \(R\). The wave function is assumed to be in the separable form

\[
\Psi = \Psi_{EM}[A_1] \Psi_G[\Lambda, R] .
\]

The functions \(\Psi_{EM}[A_1]\) and \(\Psi_G[\Lambda, R]\) are constructed as follows. The quantized Gaussian law \(\hat{H}_A \Psi_{EM}[A_1] = 0\) is trivially satisfied by the equation

\[
\hat{P}_A \Psi_{EM} = Q \Psi_{EM},
\]

whose solution is

\[
\Psi_{EM}[A_1] = \exp \left( i \int dr \, Q A_1(r) \right).
\]

Here \(Q\) is an eigenvalue of the conserved charge. Considering Eq. (2.19), a solution of the momentum constraint \(\hat{H}_r \Psi_G[\Lambda, R] = 0\) is

\[
\Psi_G = \Psi_G(Z),
\]

where \(\Psi_G(Z)\) on the right-hand side is an arbitrary function of \(Z\).

From Eq. (2.15), we see that if \(\Psi_G(Z)\) satisfies the mass constraint (2.7), it simultaneously satisfies the Wheeler-DeWitt equation. Thus we only have to consider the mass constraint, instead of the Wheeler-DeWitt equation. With regard to the ordering function \(\bar{A}\) in Eq. (2.16), we assume that

\[
\frac{\delta Z}{\delta \bar{A}} (\equiv \bar{f}) = \bar{A} \quad \text{and} \quad \bar{A}^2 = R^2 (\chi - F(R)),
\]

where \(F \equiv \hat{F} (\hat{P}_A \to Q)\). This assumption reduces the differential functional equation (2.7) with Eq. (2.11) to an ordinary differential equation on \(Z\):

\[
\frac{d^2 \Psi_G}{dZ^2} - \frac{1}{A_Z} \frac{dA_Z}{dZ} \frac{d \Psi_G}{dZ} + \Psi_G = 0 .
\]

The variable \(Z\) in Eq. (2.17) is given by

\[
Z = \int dr \int^A d\Lambda \, R \sqrt{\chi - F(R)}
\]

\[
= \int dr \, R A \left( \sqrt{\chi - F(R)} + \frac{\sqrt{\chi}}{2} \ln \left| \frac{\sqrt{\chi - F(R)} - \sqrt{\chi}}{\sqrt{\chi - F(R)} + \sqrt{\chi}} \right| \right) .
\]
Finally, we choose the rest of the ordering factors, $A_Z$, such that the solution of Eq. (2.25) becomes a special function.

(i) Bessel-type solutions

As the simplest case, we consider

$$A_Z = Z^{2\nu - 1}. \tag{2.27}$$

Then the solution of Eq. (2.25) is given by the Hankel (or Bessel) functions as

$$\Psi_G^{(\nu)}(Z) = Z^{\nu} (a_1 H_1^{(1)}(Z) + a_2 H_2^{(2)}(Z)), \tag{2.28}$$

where $a_1$ and $a_2$ are integration constants.

(ii) Hypergeometric type solutions

If we choose

$$A_Z = Z^{\sigma}(Z - 1)^{\delta}, \tag{2.29}$$

the solution of Eq. (2.25) is given by the hypergeometric functions as

$$\Psi_G^{(\sigma,\delta)}(Z) = Z^{\sigma + 1} (Z - 1)^{\delta + 1} \left\{ b_1 F(\alpha, \beta, \gamma; Z) + b_2 Z^{1 - \gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; Z) \right\}, \tag{2.30}$$

where $b_1$ and $b_2$ are integration constants, and $\alpha$, $\beta$ and $\gamma$ are constants under the constraints

$$\alpha \beta = \sigma + \delta + 2, \quad \alpha + \beta = \sigma + \delta + 3, \quad \gamma = \sigma + 2. \tag{2.31}$$

The functions in Eqs. (2.28) and (2.30) each is a mass eigenstate with a mass eigenvalue $m$. We also mention the generality of the wave functions (2.28) and (2.30). For the Hamilton-Jacobi equation, a complete solution contains the same number of arbitrary constants as of dynamical variables. In our model, there are two dynamical variables, $R$ and $\Lambda$, at each point of $r$. However, the Wheeler-DeWitt equation acts as if it were a kind of Hamilton-Jacobi equation with zero energy. Therefore, the zero energy condition is forced at each point of $r$. Further, we set the mass constraint equation (2.7) at each point. As a result, our solutions contain only the universal mass eigenvalue as an arbitrary constant, unlike a complete solution of the Hamilton-Jacobi equation.

§3. De Broglie-Bohm interpretation

In the previous section, we obtained the wave function for spherically symmetric space-time. It should be noted that, although some assumptions were imposed in order to obtain the analytic formula, no coordinate condition on $N$ or $N^r$ has been imposed. For classical relativity, we need to fix the coordinate system in order to get an explicit expression of a space-time geometry. By using the dBB interpretation of quantum mechanics, we attempt to extract physical meaning from the wave function of quantum gravity. For ordinary quantum systems, as briefly discussed in §1, the
dBB interpretation gives us deterministic rigid trajectories with no ambiguity, instead of wave functions. In this section, we show how the dBB interpretation allows a rigid space-time picture in the quantum theory of gravity.

We express a wave function of quantum gravity in polar coordinates as

\[ \Psi(Z) = |\Psi(Z)| \exp(i\Theta(Z)), \]  

(3.1)

where the phase \( \Theta(Z) \) and the amplitude \( |\Psi(Z)| \) depend only on \( Z \). In analogy to the case of Eq. (1.1), the derivatives of the phase \( \Theta \) are identified with the canonical momenta \( P_\Lambda, P_R, \) and \( P_A \), which are conjugate to \( \Lambda, R, \) and \( A_1 \). Evaluating the derivatives as

\[ \frac{\delta \Theta}{\delta \Lambda} = \frac{\delta Z}{\delta \Lambda} \frac{d\Theta}{dZ} = \bar{f} \frac{d\Theta}{dZ}, \]

(3.2)

\[ \frac{\delta \Theta}{\delta R} = \frac{\delta Z}{\delta R} \frac{d\Theta}{dZ} = \frac{\Lambda}{R} \frac{d\Theta}{dZ}, \]

\[ \frac{\delta \Theta}{\delta A_1} = Q, \]

we obtain the equations to determine the dBB trajectory of the space-time geometry and the electromagnetic field as

\[ \dot{R} - R'N^r = -\frac{N}{R} \bar{f} \frac{d\Theta}{dZ}, \]

(3.3)

\[ R(\dot{\Lambda} - (AN^r)'), + \Lambda(\dot{R} - R'N^r) = -\frac{NA}{R} \bar{f} \frac{d\Theta}{dZ}, \]

(3.4)

\[ \dot{A}_1 - A_1' = QN\Lambda R^{-2}, \]

(3.5)

where \( \bar{f} = R\sqrt{\chi - F(R)} \), from the assumption (2.24), and \( \chi \equiv R^2\Lambda^{-2} \), by the definition (2.12). In the evaluation (3.2), we have used Eq. (2.19). Equations (3.3) – (3.5) form simultaneous differential equations with respect to the time and radial coordinates. By comparing a solution of these equations with the classical solution which is derived from the classical equation of motion, we can determine quantum gravity effects on the space-time geometry qualitatively. When the difference between them is negligible, we only have to assert that the quantum space-time is reduced to the classical one spontaneously, or the classical state is realized without any observer.

We should note that the quantum effect is usually represented by the quantum potential [Eq. (1.2)] which is the difference between the real part of the Wheeler-DeWitt equation and the Hamilton-Jacobi equation.\(^{18,20,27}\) In our approach, the factor \( d\Theta/dZ + 1 \) is proportional to the quantum potential and therefore represents the quantum effect. If the condition \( d\Theta/dZ + 1 = 0 \) holds, the de Broglie-Bohm equations (3.3) – (3.5) reduce to the Einstein equations and the system becomes classical.

Before integrating Eqs. (3.3) – (3.5), let us first consider them as they stand. It is seen that the equation for the electromagnetic part, (3.5), is the same as the classical equation. By taking the ratio of Eq. (3.4) to Eq. (3.3), we obtain

\[ \frac{\dot{A}}{A} + \frac{\dot{R}}{R} - \frac{\dot{R}f'}{Rf} = N'r' + N'r \left( \ln \frac{AR}{f} \right)'. \]

(3.6)
We note that Eq. (3.6) does not include the phase $\Theta$, and the solution of Eq. (3.6) does not depend on the explicit form of the wave function. For the gravitational part, thus, the correlation between the dynamical variables $\Lambda$ and $R$ corresponds to the classical path in configuration space. By contrast to Eq. (3.6), Eq. (3.3) includes a functional form of $\Theta$ and the time slicing function $N$. Combined with Eq. (3.6), Eq. (3.3) is used to determine the dBB trajectory of the space-time geometry which is parametrized by the time $t$.

In order to proceed to obtain dBB trajectories explicitly, we specify a wave function for our analysis. We adopt the Bessel-type solution (2.28) with $a_1 = 0$,

$$\Psi_G^{(\nu)}(Z) = a_2 Z^{\nu} H_{\nu}^{(2)}(Z),$$

since its asymptotic behavior is simple, and thus may allow us to derive physical meanings from the wave function. For the wave function (3.7), we can evaluate $d\Theta/dZ$ analytically by using the relations $H_{\nu}^{(1)}(Z)^* = H_{\nu}^{(2)}(Z)$ for the real $Z$ and $H_{\nu}^{(2)}(dH_{\nu}^{(1)}/dZ) - H_{\nu}^{(1)}(dH_{\nu}^{(2)}/dZ) = 4i/(\pi Z)$. The result is

$$n(Z)^{-1} \equiv -\frac{d\Theta}{dZ} = \frac{2}{\pi Z |H_{\nu}^{(2)}(Z)|^2}. \quad (3.8)$$

Thus $n(Z)$ is given asymptotically as

$$n(Z) \to 1 \quad \text{for} \quad Z \to \infty. \quad (3.9)$$

Although $\Psi_G^{(\nu)}(Z)$ is an analytic function in the complex $Z$, the phase $\Theta(Z)$ is not. Therefore, we confine $n(Z)$ to real value of $Z$ and refrain from extending $n(Z)$ analytically to the complex plane. We note that the wave function (3.7) satisfies Vilenkin’s boundary condition, since the time derivative of $R$ has a positive sign as seen from Eq. (3.3). Using the wave function with Vilenkin’s boundary condition [Eq. (3.7)] and the asymptotic behavior in Eq. (3.9), we see that the Einstein equations are recovered from the de Broglie-Bohm equations (3.3) – (3.5) asymptotically.

It may be worthwhile to mention other choices of the wave function. If $b_1 = b_2$ or $b_1 = -b_2$, the wave function remains real or purely imaginary, and the dBB interpretation yields no dynamics. For the case $b_2 = 0$, the sign of the phase in the wave function is opposite to that in the wave function (3.7), and thus the direction of the time parameter $t$ is reversed.

§4. De Broglie-Bohm trajectory

In this section, we solve the differential equations (3.3) and (3.4) and obtain the dBB trajectories hidden in the wave function (3.7). For the equations to be simplified and analytically integrable, we assume that $\chi$ defined in Eq. (2.12) depends on $r$ only through $R$:

$$\chi \equiv R^2/A^2 = \bar{\chi}(R), \quad (4.1)$$

where $\bar{\chi}(R)$ is a function only of $R$. This assumption requires that $\bar{f}$ defined in Eq. (2.24) is also a function only of $R$. If we choose the coordinate condition $N^r = 0$,
Eq. (3.6) is reduced to the simple form
\[ \dot{\Lambda} + \frac{\dot{R}}{\Lambda} - \frac{\dot{f}}{f} = 0. \] (4.2)

By integrating Eq. (4.2), we obtain a relation
\[ \bar{f} = c_0(r) R \Lambda, \] (4.3)
where \( c_0 \) is an arbitrary function of \( r \), and is the constant of integration for the integration over \( t \). From the definition (2.24), we can determine \( \Lambda \) through \( R \):
\[ \Lambda = \sqrt{\bar{\chi}(R) - F(R)/c_0(r)}. \] (4.4)

From Eq. (4.1), on the other hand, \( R' \) can be expressed as
\[ R' = \sqrt{\bar{\chi}(R) \Lambda} = \sqrt{\bar{\chi}(R)(\bar{\chi}(R) - F(R))/c_0(r)}. \] (4.5)

If the right-hand side of Eq. (4.5) is nonzero, we obtain the solution
\[ G(R) = \int \frac{dR}{\sqrt{\bar{\chi}(R)(\bar{\chi}(R) - F(R))}} = \int \frac{dr}{c_0(r)} + \phi(t), \] (4.6)
which determines implicitly the dependence of \( R \) on \( r \). Here \( \phi \) is an arbitrary function of \( t \), and is the constant of integration for the integration over \( r \). The remaining equation of motion, (3.3), is used to determine the time dependence of \( \phi(t) \):
\[ \dot{\phi} = \frac{dG(R)}{dR} \dot{R} = \frac{N}{n(Z)} \frac{1}{\sqrt{\bar{\chi}(R)}}. \] (4.7)

It is noted that if we use \( N^r = 0 \) as the coordinate condition, the ansatz (4.1) is compatible with canonical evolution and, resultantly, corresponds to the condition for picking up a special solution.

In the following, we evaluate dBB trajectories by giving explicit functional forms of \( \bar{\chi}(R) \). We use \( ds_{\text{dBB}} \) and \( Z_{\text{dBB}} \) to represent the line element of the rigid geometry described by the dBB picture and the value of \( Z \) that is evaluated on the rigid geometry from Eq. (2.26).

Case A: \( \bar{\chi}(R) \equiv F(R) \geq 0 \)

This is a special case in which \( \dot{f} = 0 \), and therefore, Eqs. (4.4) and (4.6) do not apply. Directly from Eqs. (3.3) and (3.4) with \( N^r = 0 \), we find
\[ \Lambda = \dot{R} = 0, \] (4.8)
which means that there is no dynamical evolution. This conclusion is confirmed by observing that the integrand of \( Z \) in Eq. (2.17) or Eq. (2.26) vanishes, and thus the wave function becomes a constant. To maintain the relation \( \bar{\chi}(R) \equiv F(R) \), or \( R'^2 = \Lambda^2 F(R) \), during the time evolution we require the condition
\[ \frac{\partial}{\partial t} R' = 0. \] (4.9)
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If we set \( R' = R \) (that is, \( R = \exp r \)), which satisfies the consistency condition (4.9), then we obtain \( A^2 = R^2 F(R)^{-1} \) and

\[
\frac{ds_{\text{dBB}}^2}{dt^2} = -N^2 dt^2 + F(R)^{-1} dR^2 + R^2 d\Omega^2.
\]  

(4.10)

Here, the lapse function \( N \) remains an arbitrary function. The Reissner-Norström-de Sitter metric corresponds to the case of the choice \( N = \sqrt{F(R)} \).

Case B: \( \bar{\chi}(R) \equiv 0 \)

First, we take the integration constant in Eq. (4.3) as \( c_0 = 1 \). From Eq. (4.4), when \( F(R) \leq 0 \), \( \Lambda \) is given by

\[
\Lambda^2 = -F(R).
\]  

(4.11)

From Eq. (4.5), we have \( R' = 0 \). Therefore,

\[
R = \phi(t),
\]  

(4.12)

where \( \phi(t) \) is an arbitrary function of \( t \), and is the constant of integration for the integration over \( r \). Equations (4.11) and (4.12) define an initial configuration to pick up a dBB trajectory.

If the lapse function \( N \) is chosen as

\[
N = \frac{1}{\sqrt{-F(R)}} n(Z),
\]  

(4.13)

then Eq. (3.3) is reduced to \( \dot{\phi} = 1 \), and the dBB trajectory becomes

\[
R(r, t) = t.
\]  

(4.14)

As a result, the line element \( ds_{\text{dBB}} \) is

\[
\frac{ds_{\text{dBB}}^2}{dt^2} = \frac{1}{F(t)} n(Z_{\text{dBB}})^2 dt^2 - F(t) dr^2 + t^2 d\Omega^2,
\]  

(4.15)

where

\[
Z_{\text{dBB}} = -\int dr RF(R) = -t F(t) R_0,
\]  

(4.16)

and \( R_0 \) is the world size. For large \( t \), the metric (4.15) approaches the inside geometry of the Reissner-Nordström-de Sitter black hole, which is discussed in Ref. 12, and, when \( Q = \lambda = 0 \), approaches that of the Schwarzschild black hole, which is discussed in Refs. 20 and 25). These classical solutions are generalizations of the Kantowski-Sachs metric.26

Case C: \( \bar{\chi}(R) \equiv 1 - 2m/R + Q^2/R^2 \geq 0 \)

The ansatz of Eq. (4.1) causes Eq. (2.26) to reduce to

\[
Z = \int dr R^2 \left( \frac{\sqrt{\lambda}}{3} R + \frac{\sqrt{\bar{\chi}(R)}}{2} \ln \left| \frac{\sqrt{\lambda / 3} R - \sqrt{\bar{\chi}(R)}}{\sqrt{\lambda / 3} R + \sqrt{\bar{\chi}(R)}} \right| \right)
\]  

(4.17)
If we set $c_0(r) = \sqrt{\lambda/3}$ in Eq. (4.3), then $\Lambda$ is given as

$$\Lambda = R,$$

(4.18)

from Eq. (4.4), and Eq. (4.5) becomes

$$R' = R \sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}.$$  \hspace{1cm} (4.19)

Integrating Eq. (4.19), we obtain

$$R = x \left( 1 + \frac{m}{x} + \frac{m^2 - Q^2}{4x^2} \right),$$ \hspace{1cm} (4.20)

where $x \equiv \exp \left( r + \sqrt{\frac{\lambda}{3}} \phi(t) \right)$, and $\phi(t)$ is an arbitrary function of $t$, and is the constant of integration for the integration over $r$. Equations (4.18) and (4.20) describe an initial configuration.

Next we choose the time coordinate condition

$$N = \sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}} n(Z) = \left( 1 - \frac{m^2 - Q^2}{x^2} \right) \left( 1 + \frac{2m}{x} + \frac{m^2 - Q^2}{x^2} \right)^{-1} n(Z).$$ \hspace{1cm} (4.21)

Then Eq. (4.7) is reduced to $\dot{\phi} = 1$. Therefore, the parameter $x$ is determined to be

$$x = a_0 \exp \left( \sqrt{\frac{\lambda}{3}} t + r \right) \equiv a(t) \rho(r),$$ \hspace{1cm} (4.22)

where $a_0$ is a constant, and $a(t)$ corresponds to the scale factor of an expanding de Sitter universe with cosmological term $\lambda$. As a result, the line element $ds_{\text{dBB}}$ is

$$ds_{\text{dBB}}^2 = -\left( 1 - \frac{2m}{R} + \frac{Q^2}{R^2} \right) n(Z_{\text{dBB}})^2 dt^2 + R^2 (dr^2 + d\Omega^2)$$

$$= - \left( 1 - \frac{m^2 - Q^2}{4x^2} \right)^2 n(Z_{\text{dBB}})^2 dt^2$$

$$+ a(t)^2 \left( 1 + \frac{m}{x} + \frac{m^2 - Q^2}{4x^2} \right)^2 (d\rho^2 + \rho^2 d\Omega^2).$$ \hspace{1cm} (4.23)

Here we study some limiting cases. First we consider the special case $m = Q = 0$. Then, Eq. (4.23) is reduced to

$$ds_{\text{dBB}}^2 = -n(Z_{\text{dBB}})^2 dt^2 + a(t)^2 (d\rho^2 + \rho^2 d\Omega^2),$$ \hspace{1cm} (4.24)

whose asymptotic form behaves like a classical de Sitter universe. Horiguchi\textsuperscript{27} showed that quantization of the homogeneous universe with a cosmological term
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gives a Bessel type wave function of order $\nu = 1/3$ (the Airy function) with the argument

$$Z = \int d\rho \rho^2 a(t)^3 = a(t)^3 V_0,$$

(4.25)

where $V_0$ is the world volume. The argument (4.25) is different from our $Z$, given in Eq. (4.17), in that it lacks the second term. This is due to the difference in the order of the two procedures, quantization and reduction of the number of degrees of freedom. In our case, the cosmological isotropic symmetry is imposed after quantization, while it is imposed before quantization in Ref. 27). The property of quantum fluctuations depends on the manner in which the mini-superspaces are constructed. Next, we consider the special case $m = Q \neq 0$. Then, the asymptotic line element is

$$ds_{\text{dBB}}^2 \to -\left(1 + \frac{m}{x}\right)^{-2} dt^2 + \rho^2(1 + \frac{m}{x})^2 (d\rho^2 + \rho^2 d\Omega^2),$$

(4.26)

where $x = a(t)\rho$. This is called an “extreme black hole”. In classical theory, extension of the Majumdar-Papetrou geometry to a cosmological black hole is discussed in Ref. 22).

It is interesting to study the case of a test charged particle in the background of a quantum extreme black hole, because the gravitational force and electromagnetic force may exert different influences on the test particle. This is a problem for future study.

Case D: $\bar{\chi}(R) \equiv \frac{1}{2}(F(R) + \sqrt{F(R)^2 + 4R^2})$

The ansatz of Eq. (4.1) yields

$$\Lambda^2 = \frac{1}{2} \left(-F(R) + \sqrt{F(R)^2 + 4R^2}\right)$$

(4.27)

and $R' = R$ from Eqs. (4.4) and (4.5), respectively. Here $c_0 = 1$ has been chosen. Thus the initial $R$ takes the simple form

$$R = \exp(r + \phi(t)), \quad (4.28)$$

where $\phi(t)$ is an arbitrary function of $t$, and is the constant of integration for the integration over $r$.

If we take the lapse function $N$ to satisfy

$$N^2 = \frac{\lambda}{6} \left(F(R) + \sqrt{F(R)^2 + 4R^2}\right) n(Z)^2,$$

(4.29)

Eq. (3.3) is reduced to $\ddot{\phi} = \sqrt{\lambda} 3$. Therefore the form of $R$ is determined as

$$R = a_0 \exp \left(\sqrt{\frac{\lambda}{3}} t + r\right) \equiv a(t) \rho(r), \quad (4.30)$$

where $a_0$ is a constant. As a result, the line element $ds_{\text{dBB}}$ is given by

$$ds_{\text{dBB}}^2 = -\frac{\lambda}{6} \left(\sqrt{F(R)^2 + 4R^2} + F(R)\right) n(Z_{\text{dBB}})^2 dt^2$$
\[ + a^2(t) \left( \frac{2 d\rho^2}{\sqrt{F(R)^2 + 4R^2} + F(R)} + \rho^2 d\Omega^2 \right). \] (4.31)

The classical limit of Eq. (4.31) corresponds to a cosmological black hole geometry in standard form, which contrasts with the isotropic form in the case C.

In the asymptotic region, where \( n_{\text{dBB}} \to 1 \), or the classical limit \( \bar{\h} \to 0 \), Eqs. (4.15), (4.23) and (4.31) corresponds to the same classical Reissner-Nordström-de Sitter space-time. They differ merely in regard of the coordinate system with respect to which they are described and the region which they cover in the fully extended space-time. In Eq. (4.15) of the case B, the coordinate \( t \) plays the role of the Schwarzschild radial coordinate. The associated condition \( F(R) \leq 0 \) shows that the corresponding classical geometry covers only the dynamical region, or one patch bounded by the horizons. On the other hand, the asymptotic form of Eq. (4.23) is described with respect to the cosmological isotropic coordinates and covers the region from the inside of the event horizon of the black hole to the outside of the event horizon of the de Sitter universe. The transformation between the cosmological isotropic coordinates and the static coordinates is given by

\[ a\Omega \rho = \bar{\rho}, \ t = \bar{t} - \frac{\lambda}{3} \int \frac{1}{F(\bar{\rho})} \frac{\bar{\rho}^2}{\sqrt{\bar{\rho}^2 - 2m\bar{\rho} + Q^2}} d\bar{\rho}, \] (4.32)

where \( \bar{t} \) and \( \bar{\rho} \) are the time and radial coordinates in the static Reissner-Nordström-de Sitter space-time. We find that the coordinate \( t \) plays the role of an advanced null coordinate. Alternatively, the coordinate transformation

\[ a\rho = \bar{\rho}, \ t = \bar{t} - \int \frac{1}{F(\bar{\rho})} \sqrt{\bar{\rho}^2 - F(\bar{\rho})^2} d\bar{\rho}, \] (4.33)

relates the asymptotic form of Eq. (4.31) with that of Eq. (4.15) to extend the dynamical region to the static region.

Returning now to quantum theory, we find that the rigid space-time picture is not covariant under general coordinate transformations. For example, under the coordinate transformation (4.33), Eq. (4.15) is not equivalent to Eq. (4.31) due to the presence of the quantum gravity factor \( n(Z_{\text{dBB}}) \). In other words, the quantum effect appears differently depending on the choice of coordinate systems. This may seem strange. Indeed, a quantum state in the superspace is constructed to be invariant. However, the deterministic picture labels quantum fluctuations in the superspace by introducing the new time coordinate. In this way, the dynamics special to quantum mechanics realizes.

By giving the initial and coordinate conditions explicitly, in this section we have obtained various space-times that are related with each other by coordinate transformations. For other conditions, we can also perform analytic evaluations. In the Appendix, some calculations are given. As a nontrivial case, we here consider a space-time with positive constant curvature. For \( m = Q = 0 \), the configuration variables \( (R, \Lambda) = (a(t)r, a(t)/\sqrt{1 - Kr^2}) \) satisfy Eq. (3.6) when we set the shift function as \( N^\nu = 0 \). Here \( K \) denotes the curvature. The variable \( a(t) \) is determined.
by Eq. (3.3) as
\[ \dot{a} = \frac{N}{n(Z)} \sqrt{\frac{\lambda}{3}} a^2 - K. \tag{4.34} \]
When we fix the lapse function as \( N = n(Z) \), the solution is
\[ a(t) = \sqrt{\frac{3K}{\lambda}} \cosh \sqrt{\frac{\lambda}{3}} t. \tag{4.35} \]
In the classical limit \( \hbar \to 0 \), this solution corresponds to the closed de Sitter universe, whose scale factor is Eq. (4.35). In this case, the corresponding classical geometry cannot be related to the cases A–D by any coordinate transformation.

In this section, several examples of dBB trajectories (cases A–D and the de Sitter universe) are obtained after fixing the coordinate conditions. One of the merits of our dBB approach is that it allows the quantum solutions and the classical solutions to be treated in a parallel manner. The difference between them represents the quantum effect and is represented by the factor \( n^{-1} = -d\Theta/dZ \). The evaluation of this quantum effect will be carried out in a separate paper. \( ^{29} \)

§5. Summary

We have studied canonical quantization of the Einstein-Maxwell theory with a cosmological term in spherically symmetric space-time from the viewpoint of the de Broglie-Bohm interpretation. First, we constructed the canonical formalism of the spherically symmetric geometry and quantized it. To resolve the operator ordering ambiguity, we followed the procedure proposed in our previous work. \( ^{11} \) We obtained an analytic wave function, which is a simultaneous eigenstate of the mass and electric charge operators. It is noted that, at this stage, no coordinate fixing had been made, and the lapse and shift functions \( N \) and \( N^r \) were still arbitrary. Next, the dBB interpretation of quantum mechanics was applied to the wave function, as if the Wheeler-DeWitt equation were the Schrödinger equation with zero energy. The equations of motion for the quantized space-time geometry were given as deterministic partial differential equations. Using the assumption specifying the initial geometry and choosing a coordinate system, we integrated the equations analytically. In the asymptotic limit, the obtained rigid metrics correspond to the classical solutions of the Einstein-Maxwell theory, or the various representations of the Reissner-Norström-de Sitter space-time and the closed de Sitter universe.

As discussed in §4 in detail, comparison between the rigid geometries (in the form of the dBB trajectories) shows that the dBB trajectory picture is not covariant under coordinate transformations. When we translate a quantum state from the wave function picture to the dBB trajectory representation, we need additionally a coordinate system that describes the canonical evolution. If, once we choose a coordinate condition and obtain the rigid space-time picture, the metric of the space-time cannot be transformed into others induced by different choices of the coordinate system under any coordinate transformation. In other words, realization of the quantum world breaks the covariance under general coordinate transformations,
while a quantum state as a bundle of possible rigid geometries possesses invariance. This reminds us of the effective action at a spontaneous symmetry breaking.

In our analysis, a coordinate system is chosen a priori. However, if this choice has important meaning for quantum gravity, it must be treated as physical process fundamentally. In order to make our analysis more plausible, we have to throw probes into quantum-mechanically fluctuating space-times rather than choose a particular coordinate condition as a special one. Such a probe will play the role of the coordinate system in the sense that it provides the method of measuring a space-time structure. The dBB trajectory picture constructed using the information obtained from the probe will allow us to determine what quantum gravitational state is realized. We may regard the probe and the observable information as a quantum field and the Hawking radiation, for example. Analysis along this line is needed to solve the conceptual problem.

There is another purpose of our work regarding model analysis. Using the model discussed here, we can construct the quantum mechanics of extreme black holes, in particular of cosmological extreme black holes, and analyze the gravitational fluctuations near the event horizon for a \( N' \neq 0 \) gauge analogous to the Vaidya metric. This provides a different viewpoint for considering Hawking radiation.

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Appendix A

\[ \text{Ansatz: } \bar{\chi}(R) \equiv -\frac{\lambda}{3} R^2 \]

Following the analysis in §4, we present other calculations concerning the dBB trajectory according to the choice of the function in Eq. (4.1).

We assume \( \bar{\chi}(R) \equiv -\frac{\lambda}{3} R^2 \) and treat only the region \( \bar{\chi}(R) - F(R) < 0 \). Choosing \( c_0(r) = 1 \) in Eq. (4.3), we find

\[ \Lambda^2 = -\left( 1 - \frac{2m}{R} + \frac{Q^2}{R^2} \right). \tag{A.1} \]

Here we allow \( \Lambda \) to become purely imaginary only if \( Z \) is purely real. From Eq. (4.6), we obtain

\[ R = y \left( 1 + \frac{m}{y} + \frac{m^2 - Q^2}{4y^2} \right), \tag{A.2} \]

where \( y \equiv \exp \left( \sqrt{\frac{2}{3}} r + \phi(t) \right) \), and \( \phi(t) \) is an arbitrary function of \( t \), a constant of integration. If we adopt the coordinate condition

\[ N^2 = -y^2 \left( 1 + \frac{m}{y} + \frac{m^2 - Q^2}{4y^2} \right)^2 n(Z)^2, \tag{A.3} \]

where the evolution of the variables takes place in the purely imaginary time \( t \),
Eq. (3.3) is reduced to $\dot{\phi} = 1$, and thus $y = a_0 \exp\left(\sqrt{\frac{\lambda}{3}}r + t\right)$, where $a_0$ is a constant. As a result, the line element $ds_{dBB}$ becomes

$$ds_{dBB}^2 = -\left(\frac{1 - \frac{m^2 - Q^2}{4y^2}}{1 + \frac{m}{y} + \frac{m^2 - Q^2}{4y^2}}\right)^2 dr^2 + y^2 \left(\frac{1 + \frac{m}{y} + \frac{m^2 - Q^2}{4y^2}}{4y^2}\right) (n(Z_{dBB})^2 dt^2 + d\Omega^2).$$  \hspace{1cm} (A.4)

For $N^r = 0$, we define a space-time dual transformation as

$$r \rightarrow t, \quad t \rightarrow r, \quad \Lambda \rightarrow iN \quad \text{and} \quad N \rightarrow -i\Lambda. \hspace{1cm} (A.5)$$

Under this transformation, the action $(2.4)$ is invariant and the classical solutions (asymptotic forms) derived in §4 are related to each other as follows:

\begin{equation*}
\text{case A} \leftrightarrow \text{case B} \quad \text{and} \quad \text{case C} \leftrightarrow \text{case in Appendix} \hspace{1cm} (A.6)
\end{equation*}

However, the dBB trajectory representation is not covariant under the space-time dual transformation.

**Appendix B**

--- Ansatz: $\bar{\chi}(R) \equiv 0 \quad (N^r \neq 0)$ ---

In §4, we used the shift function $N^r = 0$ as a coordinate condition. Here we present an example of a nonzero shift function.

From the ansatz $\bar{\chi}(R) \equiv 0$, we find $R' = 0$, and Eq. (2.26) is reduced to

$$Z = \int dr \Lambda R \sqrt{-F(R)}. \hspace{1cm} (B.1)$$

By evaluating the derivative of $Z$ with respect to $R$, we have

$$\frac{\delta \Theta}{\delta R} = \Lambda \left(\sqrt{-F(R)} + R \frac{d}{dR} \sqrt{-F(R)}\right) \frac{d\Theta}{dZ}. \hspace{1cm} (B.2)$$

The equations of motion for the dBB trajectory are reduced to

$$\dot{R} = \frac{N}{n(Z)} \sqrt{-F(R)} \quad \text{and} \quad \dot{\Lambda} - (N^r \Lambda)' = \frac{N}{n(Z)} \Lambda \frac{d}{dR} \sqrt{-F(R)}. \hspace{1cm} (B.3)$$

If we set $N = n(Z)/\sqrt{-F(R)}$ and $(N^r)' = 0$, then we find $R = t$ and

$$\Lambda = G\left(r + \int N^r dt\right) \sqrt{-F(t)}, \hspace{1cm} (B.4)$$

where $G$ is an arbitrary function. Thus the line element $ds_{dBB}$ is

$$ds_{dBB}^2 = \frac{n(Z_{dBB})}{F(t)} dt^2 - F(t)G(r + \int N^r dt)^2 (dr + N^r dt)^2 + t^2 d\Omega^2, \hspace{1cm} (B.5)$$
where
\[ Z_{\text{dBB}} = t \sqrt{-F(t)} \int dr G(r + \int N r \, dt) . \] (B.6)

Equation (B.5) can be related to Eq. (4.15) using the coordinate transformation \( \bar{t} = t \) and \( \bar{r} = \int G(\alpha) \, d\alpha \), where \( d\alpha = dr + N r \, dt \), and \( \bar{t} \) and \( \bar{r} \) denote the coordinates in the static Reissner-Norström-de Sitter metric.

References

1) R. Arnowitt, S. Deser and C. W. Misner, in Gravitation: An Introduction to Current Research, ed. L. Witten (Wiley, New York, 1962).
2) P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York, 1964).
3) J. A. Wheeler, in Batelle Rencontres: 1967 Lectures in Mathematics and Physics, ed. C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).
4) B. S. DeWitt, Phys. Rev. 160 (1967), 1113.
5) K. V. Kuchař, gr-qc/9304012.
6) W. Fischler, D. Morgan and J. Polchinski, Phys. Rev. D42 (1990), 4042.
7) K. V. Kuchař, Phys. Rev. D50 (1994), 3961.
8) B. K. Berger, D. M. Chitre, V. E. Moncrief and Y. Nutku, Phys. Rev. D5 (1972), 2467.
9) J. Schwinger, Phys. Rev. 132 (1963), 1317.
10) A. Komar, Phys. Rev. D20 (1979), 830.
11) M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, Phys. Rev. D36 (1987), 3641.
12) J. L. Friedman and I. Jack, Phys. Rev. D37 (1988), 3495.
13) M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, Phys. Rev. D59 (1999), 124004.
14) M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, Int. J. Mod. Phys. A15 (2000), 2059.
15) D. Bohm, Phys. Rev. 85 (1952), 166, 180.
16) J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, England, 1987).
17) P. R. Holland, The Quantum Theory Of Motion (Cambridge University Press, Cambridge, England, 1993).
18) C. J. Isham, Report Imperial/TP/91-92/25, gr-qc/9210011, 1992.
19) J. J. Halliwell, Phys. Rev. D36 (1987), 3626.
20) S. P. de Alwis and D. A. MacIntire, Phys. Rev. D50 (1994), 5164.
21) M. Kenmoku, K. Otsuki, K. Shigemoto and K. Uehara, Class. Quantum Grav. 13 (1996), 1751.
22) M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, Phys. Rev. D57 (1998), 4925.
23) J. B. Hartle and S. W. Hawking, Commun. Math. Phys. 26 (1972), 87.
24) D. Kastor and J. Traschen, Phys. Rev. D47 (1993), 5370.
25) Y. Nambu and M. Sasaki, Prog. Theor. Phys. 79 (1988), 96.
26) A. Vilenkin, Phys. Rev. D37 (1988), 888.
27) K. Nakamura, S. Konno, Y. Oshiro and A. Tomimatsu, Prog. Theor. Phys. 90 (1993), 861.
28) R. Kantowski and R. Sachs, J. Math. Phys. 7 (1966), 443.
29) T. Horiguchi, Mod. Phys. Lett. A9 (1994), 1429.
30) S. W. Hawking, Commun. Math. Phys. 43 (1975), 199.
31) M. Kenmoku, T. Matsuyama, R. Sato and S. Uchida, in preparation.