VANISHING THEOREMS, RATIONAL CONNECTEDNESS AND INTERMEDIATE POSITIVITY

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Abstract. We establish a vanishing theorem for uniformly RC $k$-positive Hermitian holomorphic vector bundles, and show that the holomorphic tangent bundle of a compact complex manifold equipped with a positive $k$-Ricci curvature Kähler metric (or more generally a positive $k$-Ricci curvature Kähler-like Hermitian metric) is uniformly RC $k$-positive. Two main applications are presented. The first one is to deduce that spaces of some holomorphic tensor fields on such Kähler or Kähler-like Hermitian manifolds are trivial, generalizing some recent results. The second one is to show that a compact Kähler manifold whose holomorphic tangent bundle can be endowed with either a uniformly RC $k$-positive Hermitian metric or a positive $k$-Ricci curvature Kähler-like Hermitian metric is projective and rationally connected.

1. Introduction

A central topic in differential geometry is how the curvature conditions restrict the topology. In the case of Ricci curvature (abbreviated by “Ric”) in Kähler geometry this principle goes back to Bochner, who proved that ([Bo46], [Bo49], [YB53]) the condition of $\text{Ric} > 0$ or $\text{Ric} < 0$ on compact Kähler manifolds or the existence of Kähler-Einstein metrics imposes heavy restrictions on their holomorphic tensor fields. These conditions can now be reformulated in terms of the first Chern class, thanks to the celebrated Calabi-Yau theorem and the Aubin-Yau theorem ([Yau77]). In order to precisely state Bochner’s results, let (throughout this article) $T M$ and $T^* M$ be respectively the (holomorphic) tangent and cotangent bundle of a compact complex manifold $M$, and

$$\Gamma^p_q(M) := H^0(M, (T M)^{\otimes p} \otimes (T^* M)^{\otimes q}), \quad (p, q \in \mathbb{Z}_{\geq 0})$$

the space of $(p, q)$-type holomorphic tensor fields on $M$. Bochner’s results can be stated as follows ([Ko80-1], [Ko80-2], [KH83, p. 57]).

Theorem 1.1 (Bochner, Calabi-Yau, Aubin-Yau). Let $M$ be an $n$-dimensional compact Kähler manifold.

1. If $c_1(M)$ is quasi-positive, then there exists a positive constant $C = C(M)$ such that $\Gamma^p_q(M) = 0$ when $q > C \cdot p$, and consequently $\Gamma^0_q(M) = 0$ when $q \geq 1$. In particular the Hodge numbers $h^{q,0}(M) = 0$ when $1 \leq q \leq n$. 

2010 Mathematics Subject Classification. 53C55, 32L20, 32Q10, 14F17, 32Q05.

Key words and phrases. vanishing theorem, rational connectedness, projectivity, holomorphic sectional curvature, Ricci curvature, uniform RC $k$-positivity, $k$-Ricci curvature, $k$-scalar curvature, holomorphic tensor field, Chern-Kähler-like Hermitian metric.

The author was partially supported by the National Natural Science Foundation of China (Grant No. 12371066) and the Fundamental Research Funds for the Central Universities. The author states that there is no conflict of interest. This manuscript has no associated data.
(2) If $c_1(M)$ is quasi-negative, then there exists a positive constant $C = C(M)$ such that
\[ \Gamma^0_p(M) = 0 \] when $p > C \cdot q$, and consequently $\Gamma^p_0(M) = 0$ when $p \geq 1$.
(3) If $c_1(M) < 0$, then $\Gamma^0_p(M) = 0$ when $p > q$.

**Remark 1.2.**

1. The condition quasi-positivity (resp. quasi-negativity) of $c_1(M)$ means that there exists a closed $(1,1)$-form representing $c_1(M)$ which is nonnegative (resp. non-positive) everywhere and positive (resp. negative) somewhere.

2. It is well-known that a compact Kähler manifold with Hodge number $h^{2,0} = 0$ is projective ([MK71, p. 143]). So a direct consequence of Theorem 1.1 is that a compact Kähler manifold with quasi-positive $c_1$ must be projective.

It is well-known that the condition of Ric $> 0$, which is equivalent to be Fano due to the Calabi-Yau theorem, implies simple-connectedness ([Ko61]). This was further strengthened to be rationally connected by Campana ([Ca92]) and Kollár-Miyaoka-Mori ([KMM92]) independently, which was conjectured in Yau’s influential problem list ([Yau82, Problem 47]). Recall that a complex manifold is called rationally connected if any two points on it can be joined by a rational curve. Rational connectedness is an important tool/notion in algebraic geometry and for projective manifolds it implies simple-connectedness ([Deb01, Coro. 4.18]).

In view of the subtle relationship between Ric and the holomorphic sectional curvature $H$ ([Zh00, p. 181]), one may expect that the above-mentioned conclusions still hold when the condition on Ric is replaced by that on $H$. The simple-connectedness under the condition $H > 0$ is well-known ([Ts57]). A recent important result due to Wu-Yau, Tosatti-Yang and Diverio-Trapani ([WY16-1], [TY17], [DT19], [WY16-2]) implies that the quasi-negativity of $H$ implies $c_1(M) < 0$, and hence the conclusions of parts (2) and (3) in Theorem 1.1 remain true when $H$ is quasi-negative. Yau also conjectured in [Yau82, Problem 47] that a compact Kähler manifold with $H > 0$ is projective and rationally connected. Assuming the projectivity this was proved by Heier-Wong ([HW20]). Shortly afterwards the projectivity was proved by Yang in [Ya20] by showing that the Hodge number $h^{2,0} = 0$, which also provided an alternative proof of the rational connectedness therein. Recently Yang introduced in [Ya20] the notion of uniform RC-positivity on Hermitian holomorphic vector bundles, which is satisfied by the tangent bundle of a compact Kähler manifold with $H > 0$, and deduced that a compact Kähler manifold whose tangent bundle is uniformly RC-positive with respect to a (possibly different) Hermitian metric is projective and rationally connected ([Ya20, Thm 1.3]).

In contrast to the negative case, in general the condition $H > 0$ is not able to yield $c_1(M) > 0$, as exhibited by Hitchin ([Hi75]) that the Hirzebruch surfaces
\[ F := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}) \quad (k \geq 2) \]
admit Kähler metrics with $H > 0$ but $c_1(F)$ are not positive (cf. [Ya16, p. 949]). More related examples can be found in [AHZ18] and [NZ18]. Even so, we can still ask whether or not the conclusions of part (1) in Theorem 1.1 hold when $H > 0$. In fact, Yang showed that ([Ya18, Thm 1.7]) the condition $H > 0$ leads to the Hodge numbers $h^{q,0} = 0$ for $1 \leq q \leq n$ and thus provides positive evidence towards this validity. Recently the author showed that the conclusions of part (1) in Theorem 1.1 are true when $H > 0$ ([Li21]). Indeed what we proved in [Li21, Thm 1.5] is in a more general setting, i.e. for those Hermitian metrics whose Chern curvature tensors behave like the usual Kähler curvature tensors.

The $k$-Ricci curvatures $\text{Ric}_k$ for $1 \leq k \leq n$ were introduced by Ni ([Ni21-1]) in his study of the $k$-hyperbolicity of a compact Kähler manifold. $\text{Ric}_k$ coincides with $H$ when $k = 1$ and with
Ric when \( k = n \) and so they interpolate between \( H \) and Ric. Ni showed that the condition \( \text{Ric}_k > 0 \) for some \( k \in \{1, \ldots, n\} \) yields rational connectedness and \( h^{q,0} = 0 \) for \( 1 \leq q \leq n \), and hence the simple-connectedness and projectivity ([Ni21-2, Thms 1.1, 4.2]). Chu-Lee-Tam showed that ([CLT22, Thm 5.1]) the condition \( \text{Ric}_k < 0 \) for some \( k \in \{1, \ldots, n\} \) also implies \( c_1(M) < 0 \) (see [LNZ21] by Li-Ni-Zhu for an alternative proof). Hence the conclusions of parts (2) and (3) in Theorem 1.1 hold true under the condition \( \text{Ric}_k < 0 \) for some \( k \in \{1, \ldots, n\} \).

In view of the discussions above, we may wonder whether the conclusions of part (1) in Theorem 1.1 are true when \( \text{Ric}_k > 0 \) for some \( k \in \{1, \ldots, n\} \). One purpose in this article is, as an application of the main results established, to affirmatively prove it. Another major purpose is to introduce the notion of uniform RC \( k \)-positivity, which coincides with uniform RC-positivity when \( k = 1 \), and extend Yang’s differential-geometric criterion for rational connectedness from the case of uniform RC-positivity to that of uniform RC \( k \)-positivity for arbitrary \( k \).

2. Main results

Our main results and applications are stated in this section. The following concept is inspired by that of uniform RC-positivity introduced by Yang ([Ya20]) as well as BC \( k \)-positivity by Ni ([Ni21-2]).

**Definition 2.1.** A Hermitian holomorphic vector bundle \((E, h)\) over an \( n \)-dimensional Hermitian manifold \((M, \omega)\) is called uniformly RC \( k \)-positive (resp. uniformly RC \( k \)-negative) \((1 \leq k \leq n)\) at \( x \in M \) if there exists a \( k \)-dimensional subspace \( \Sigma \subset T_x M \) such that for every nonzero vector \( u \in E_x \) (the fiber of \( E \) at \( x \)),

\[
R^{(E, h)}(\Sigma; u, \pi) := \sum_{i=1}^{k} R^{(E, h)}(E_i, \overline{E_i}, u, \pi) > 0 \quad \text{(resp.} < 0),
\]

where \( R^{(E, h)} \) is the Chern curvature tensor of \((E, h)\) and \( \{E_1, \ldots, E_k\} \) a unitary basis of \( \Sigma \) with respect to \( \omega \). If this holds for any \( x \in M \), then it is called uniformly RC \( k \)-positive (resp. uniformly RC \( k \)-negative).

**Remark 2.2.**

(1) The uniform RC-positivity in [Ya20] is exactly the uniform RC 1-positivity in our notion, whose definition is indeed irrelevant to the metric \( \omega \). Nevertheless, the definition of uniform RC \( k \)-positivity for \( k \geq 2 \) relies on the metric \( \omega \). A different but closely related notion called BC \( k \)-positivity was introduced by Ni in [Ni21-2, p. 282]. We refer to Definition 6.5 and Remark 6.6 for more details on various positivity concepts.

(2) It is obvious that \((E, h)\) is uniformly RC \( k \)-positive if and only if the dual bundle \((E^*, h^*)\) is uniformly RC \( k \)-negative, where \( h^* \) is the metric induced from \( h \).
for all \( p, q, m \in \mathbb{Z}_{\geq 0} \) with \( q > C_1 \cdot p + C_2 \cdot m \).

By taking \( E = TM \) or \( T^*M \) and \( F \) trivial in Theorem 2.3, we have

**Corollary 2.4.** Let \((M, \omega)\) be an \( n \)-dimensional compact Hermitian manifold such that \((TM, \omega)\) is uniformly RC \( k \)-positive (resp. \( k \)-negative) over \((M, \omega)\) for some \( k \in \{1, \ldots, n\} \). Then there exists a positive constant \( C = C(\omega) \) such that \( \Gamma^p_q(M) = 0 \) when \( q > C \cdot p \) (resp. \( p > C \cdot q \)), and consequently \( \Gamma^0_q(M) = 0 \) (resp. \( \Gamma^p_0(M) = 0 \)) when \( q \geq 1 \) (resp. \( p \geq 1 \)). In particular in the former case the Hodge numbers \( h^{p,0}(M) = 0 \) when \( 1 \leq q \leq n \).

The main motivation to introduce the concept of uniform RC \( k \)-positivity is that the tangent bundle of a compact Kähler manifold \((M, \omega)\), or more generally a compact Kähler-like Hermitian manifold \((M, \omega)\), with the condition \( \text{Ric}_k(\omega) > 0 \) (resp. \( \text{Ric}_k(\omega) < 0 \)) turns out to be uniformly RC \( k \)-positive (resp. negative). To this end, let us recall the following notion, which was first proposed and investigated in detail by B. Yang and Zheng in [YZ18].

**Definition 2.5.** Let \((M, \omega)\) be a Hermitian manifold and \( R \) the Chern curvature tensor of \( \omega \). The Hermitian metric \( \omega \) is called Chern-Kähler-like (abbreviated by CKL) if

\[
(2.3) \quad R(X, \overline{Y}, Z, \overline{W}) = R(Z, \overline{Y}, X, \overline{W})
\]

for any \((1,0)\)-type tangent vectors \( X, Y, Z \) and \( W \).

**Remark 2.6.**

1. When \( \omega \) is Kähler, \( R \) is the usual Kähler curvature tensor and (2.3) is satisfied. By taking the complex conjugation, (2.3) implies that

\[
R(X, \overline{Y}, Z, \overline{W}) = R(Y, \overline{X}, W, \overline{Z}) = R(W, \overline{X}, Y, \overline{Z}) = R(X, W, Z, \overline{Y}).
\]

Therefore the condition (2.3) ensures that \( R \) obeys all the symmetries satisfied by a Kähler metric and thus the term CKL is justified.

2. There are many non-Kähler Hermitian metrics which are CKL ([YZ18, p. 1197]). On the other hand, Yang-Zheng showed that a CKL Hermitian metric \( \omega \) must be balanced, i.e., \( dw^{n-1} = 0 \) ([YZ18, Thm 3]). Hence CKL Hermitian metrics interpolate between Kähler and balanced metrics.

With this notion understood, our second main result is the following

**Theorem 2.7.** Let \((M, \omega)\) be an \( n \)-dimensional compact Chern-Kähler-like Hermitian manifold and \( \text{Ric}_k(\omega) > 0 \) (resp. \( \text{Ric}_k(\omega) < 0 \)) for some \( k \in \{1, \ldots, n\} \). Then \((TM, \omega)\) is uniformly RC \( k \)-positive (resp. uniformly RC \( k \)-negative) over \((M, \omega)\).

Combining Corollary 2.4 with Theorem 2.7, it yields the following desired vanishing theorem, which extends [Li21, Thm 1.5] from the case \( \text{Ric}_1 \) to arbitrary \( \text{Ric}_k \).

**Theorem 2.8.** Let \((M, \omega)\) be an \( n \)-dimensional compact Chern-Kähler-like Hermitian manifold.

1. If \( \text{Ric}_k(\omega) > 0 \) for some \( k \in \{1, \ldots, n\} \), then there exists a positive constant \( C = C(\omega) \) such that \( \Gamma^p_q(M) = 0 \) when \( q > C \cdot p \), and consequently \( \Gamma^0_q(M) = 0 \) when \( q \geq 1 \). In particular the Hodge numbers \( h^{p,0}(M) = 0 \) when \( 1 \leq q \leq n \).

2. If \( \text{Ric}_k(\omega) < 0 \) for some \( k \in \{1, \ldots, n\} \), then there exists a positive constant \( C = C(\omega) \) such that \( \Gamma^p_q(M) = 0 \) when \( p > C \cdot q \), and consequently \( \Gamma^p_0(M) = 0 \) when \( p \geq 1 \).
Remark 2.9. When $\omega$ is Kähler, part (2) in Theorem 2.8 follows from Theorem 1.1 and the results due to Wu-Yau et al. and Chu-Lee-Tam, as mentioned above. Nevertheless, if the CKL metric $\omega$ is non-Kähler, the results in part (2) are still new.

Our second major application is a differential-geometric criterion for the rational connectedness of compact Kähler manifolds. The following nice criterion for rational connectedness was established in [CDP15, Thm 1.1], on which both [Ya20, Thm 1.3] and [Ni21-2, Thm 1.1] are based.

**Theorem 2.10 (Campana-Demailly-Peternell).** Let $M$ be an projective manifold. Then $M$ is rationally connected if and only if for any ample line bundle $L$ on $M$, there exists a positive constant $C = C(L)$ such that

$$H^0(M, (T^*M)^{\otimes q} \otimes L^{\otimes m}) = 0$$

for $q, m \in \mathbb{Z}_{\geq 0}$ with $q > C \cdot m$.

We now have the following criterion for projectivity and rational connectedness.

**Theorem 2.11.** Let $M$ be an $n$-dimensional compact Kähler manifold. Then it is projective and rationally connected (and hence simply-connected) provided one of the following two conditions holds true.

1. There exists a (possibly different) Hermitian metric $\omega$ such that $(TM, \omega)$ is uniformly RC $k$-positive over $(M, \omega)$ for some $k \in \{1, \ldots, n\}$.
2. $M$ has a Chern-Kähler-like Hermitian metric $\omega$ with $\text{Ric}_k(\omega) > 0$ for some $k \in \{1, \ldots, n\}$.

**Proof.** By Corollary 2.4 and Theorem 2.8, either condition implies that the Hodge number $h^{2,0} = 0$ and hence the manifold is projective. Since either condition also implies that $(TM, \omega)$ is uniformly RC $k$-positive over $(M, \omega)$ and hence the condition in Theorem 2.10 is satisfied by taking $E = TM$, $p = 0$ and $F = L$ in (2.2).

**Remark 2.12.** (1) Part (1) in Theorem 2.11 extends [Ya20, Thm 1.3] from the case of $k = 1$ to arbitrary $k$. Part (2) technically improves on [Ni21-2, Thm 1.1] from Kähler metrics to Chern-Kähler-like Hermitian metrics.

(2) The Hodge number $h^{2,0} = 0$ and hence projectivity can be derived via a somewhat weaker BC 2-positivity by Ni ([Ni21-2, p. 280-281] (see Definition 6.5 and Remark 6.6 for more details), and the proof in [Ni21-2, Thm 4.6] can be adopted to give an alternative one of [Ya20, Thm 1.3] (cf. [Ni21-2, p. 285]).

The rest of this article is organized as follows. Some necessary background materials are collected in Section 3. Sections 4 and 5 are devoted to the proofs of Theorems 2.3 and 2.7 respectively. In Section 6 some related questions and remarks shall be discussed, in which various positivity notions are proposed and their relations are briefly discussed for possible further study in the future.

## 3. Background materials

We collect in this section some basic facts on Hermitian holomorphic vector bundles and Hermitian and Kähler manifolds in the form we shall use to prove our main results. A thorough treatment can be found in [Ko87] and [Zh00].
3.1. **Hermitian holomorphic vector bundles.** Let \((E, h) \to M\) be a Hermitian holomorphic vector bundle of rank \(r\) over an \(n\)-dimensional complex manifold \(M\) with canonical Chern connection \(\nabla\). The Chern curvature tensor

\[
R = R^{(E, h)} := \nabla^2 \in \Gamma(\Lambda^{1,1} M \otimes E^* \otimes E)
\]

Here and throughout this article \(\Gamma(\cdot)\) is used to denote the space of smooth sections for holomorphic vector bundles and the notation \(\Gamma^p_q(M)\) in (1.1) is reserved throughout this article to denote the space of \((p, q)\)-type holomorphic tensor fields on \(M\).

Take a local frame field \(\{s_1, \ldots, s_r\}\) of \(E\), whose dual coframe field is denoted by \(\{s_1^*, \ldots, s_r^*\}\), and a local holomorphic coordinates \(\{z^1, \ldots, z^n\}\) on \(M\). With the Einstein summation convention adopted here and in what follows, the Chern curvature tensor \(R\) and the Hermitian metric \(h\) can be written as

\[
R =: \Omega^\beta_{\alpha \beta} s_\beta^* \otimes s_\alpha,
\]

\[
h = (h_{\alpha \bar{\beta}}) := (h(s_\alpha, s_{\bar{\beta}})),
\]

\[
R_{ij\alpha\bar{\beta}} := R_{ij\alpha} h_{\gamma\bar{\beta}}.
\]

The Hermitian metric \(h(\cdot, \cdot)\) and the induced metrics on various vector bundles arising naturally from \(E\) are sometimes denoted by \(\langle \cdot, \cdot \rangle\).

For \(u = u^\alpha s_\alpha \in \Gamma(E)\), \(v = v^\alpha s_\alpha \in \Gamma(E)\), \(X = X^i \frac{\partial}{\partial z^i}\), and \(Y = Y^i \frac{\partial}{\partial z^i}\), we have

\[
R(u) = (\Omega^\beta_{\alpha \beta} s_\beta^* \otimes s_\alpha) (u^\gamma s_\gamma) = \Omega^\beta_{\alpha} u^\alpha s_\beta \in \Gamma(\Lambda^{1,1} M \otimes E),
\]

\[
R_{XY}(u) = \Omega^\beta_{\alpha} (X, Y) u^\alpha s_\beta = R^\beta_{ij\alpha} X^i Y^j u^\alpha s_\beta \in \Gamma(E),
\]

and therefore,

\[
R(X, Y, u, v) := \langle R_{XY}(u), v \rangle = \langle R^\beta_{ij\alpha} X^i Y^j u^\alpha v^\gamma s_\beta, v^\gamma s_\gamma \rangle = R^\gamma_{ij\alpha} X^i Y^j u^\alpha v^\gamma h_{\beta\bar{\gamma}} = R_{ij\alpha\bar{\beta}} X^i Y^j u^\alpha v^\beta.
\]

Here and in what follows we always use capital letters to denote vectors in \(TM\) and lowercase letters to denote vectors in \(E\), so as to distinguish them.

The formula (3.1) yields an easy but useful fact, which we record in the next lemma for our later reference.

**Lemma 3.1.** The linear map \(R_{X\bar{X}}(\cdot)\) is a Hermitian transformation:

\[
< R_{X\bar{X}}(u), v > = < u, R_{X\bar{X}}(v) >, \quad \text{for any } u, v \in \Gamma(E).
\]

Hence \(R_{X\bar{X}}(\cdot)\) is diagonalizable and its eigenvalues are all real at any point in \(M\).
3.2. The Ricci and scalar \( k \)-curvatures. Let \( (M, \omega) \) be an \( n \)-dimensional Hermitian manifold, \( R = R^{(TM, \omega)} \) the Chern curvature tensor. Denote by \( T_xM \) the \((1,0)\)-type tangent space at \( x \in M \) and \( \{E_1, \ldots, E_n\} \) a unitary basis of \( T_xM \). For \( X \in T_xM \), let \( H(X), \text{Ric}(X, \overline{X}) \) and \( S(x) \) be respectively the holomorphic sectional curvature, Chern-Ricci curvature and Chern scalar curvature:

\[
\begin{aligned}
H(X) &:= \frac{R(X, \overline{X}, X, \overline{X})}{|X|^4}, \\
\text{Ric}(X, \overline{X}) &:= \sum_{i=1}^n R(X, \overline{X}, E_i, \overline{E_i}), \\
S(x) &:= \sum_{i,j=1}^n R(E_i, \overline{E_i}, E_j, \overline{E_j}).
\end{aligned}
\]

When \( \omega \) is Kähler, \( H(X), \text{Ric}(X, \overline{X}) \) and \( S(x) \) are the usual notions of holomorphic sectional curvature, Ricci curvature and scalar curvature in Kähler geometry.

For \( k \in \{1, \ldots, n\} \), pick a \( k \)-dimensional subspace \( \Sigma \subset T_xM \) and choose a unitary basis \( \{E_1, \ldots, E_k\} \) of \( \Sigma \). Define

\[
(3.3) \quad \text{Ric}_k(x, \Sigma)(X, \overline{X}) := \sum_{i=1}^k R(X, \overline{X}, E_i, \overline{E_i}), \quad \text{for any } X \in \Sigma,
\]

and call it the \( k \)-Ricci curvature of \( \omega \), which was introduced by Ni in [Ni21-1]. The \( k \)-Ricci curvatures \( \text{Ric}_k \) interpolate \( H \) and \( \text{Ric} \) in the sense that

\[
\text{Ric}_1(x, CX)(X, \overline{X}) = |X|^2 H(X) \quad \text{and} \quad \text{Ric}_n = \text{Ric}.
\]

We say that \( \text{Ric}_k(x) > 0 \) if \( \text{Ric}_k(x, \Sigma)(X, \overline{X}) > 0 \) for any \( k \)-dimensional subspace \( \Sigma \) in \( T_xM \) and any nonzero \( X \in \Sigma \), and \( \text{Ric}_k = \text{Ric}_k(\omega) > 0 \) if \( \text{Ric}_k(x) > 0 \) for any \( x \in M \).

Closely related to \( \text{Ric}_k \) are the \( k \)-scalar curvatures \( S_k = S_k(\omega) \) \((1 \leq k \leq n)\) introduced in [NZ22] by Ni-Zheng for Kähler metrics, which is the average of \( H \) on \( k \)-dimensional subspaces of \((1,0)\)-type tangent spaces. These \( \{S_k\} \) interpolate between \( H \) \((k = 1)\) and the usual scalar curvature \((k = n)\). They extended Yang’s results by showing that the condition \( S_k > 0 \) for a Kähler metric implies that the Hodge numbers \( h^{q,0} = 0 \) for \( k \leq q \leq n \) ([NZ22, Thm 1.3]). In particular, the condition \( S_2 > 0 \) is enough to guarantee the projectivity.

Here we shall define \( S_k = S_k(\omega) \) for \( k \in \{1, \ldots, n\} \) in the Hermitian situation, which turns out to be the same as the original one in [NZ22] when \( \omega \) is Kähler (see Lemma 3.3). As above, for \( x \in M \), a \( k \)-dimensional subspace \( \Sigma \subset T_xM \) and a unitary basis \( \{E_1, \ldots, E_k\} \) of \( \Sigma \), define

\[
(3.4) \quad S_k(x, \Sigma) := \sum_{i,j=1}^k R(E_i, \overline{E_i}, E_j, \overline{E_j}).
\]

It is obvious that in the Hermitian case these \( \{S_1, \ldots, S_n\} \) interpolate between \( H \) \((k = 1)\) and the Chern scalar curvature \( S(x) \) \((k = n)\). Similarly we say that \( S_k(x) > 0 \) if \( S_k(x, \Sigma) > 0 \) for any \( k \)-dimensional subspace \( \Sigma \) in \( T_xM \), and \( S_k = S_k(\omega) > 0 \) if \( S_k(x) > 0 \) for any \( x \in M \).

3.3. Integral formulas. The trick of the proof in the next lemma is usually attributed to Berger, who first applied it to show that for Kähler metrics the sign of \( H_x(\cdot) \) determines that of \( S(x) \).
Lemma 3.2. Let the notation be as above and
\[ f(\cdot, \cdot) : T_xM \times T_xM \to \mathbb{C}, \quad g(\cdot, \cdot, \cdot, \cdot) : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{C} \]
be two smooth maps such that the first variable of \( f \) (resp. the first and third variables of \( g \)) is (resp. are) linear and the second variable of \( f \) (resp. the second and fourth variables of \( g \)) is (resp. are) conjugate-linear. Then
\[
\int_{Y \in \Sigma, |Y| = 1} f(Y, Y) d\theta(Y) = \frac{V(S^{2k-1})}{k} \sum_{i=1}^{k} f(E_i, E_i), \tag{3.5}
\]
and
\[
\int_{Y \in \Sigma, |Y| = 1} g(Y, Y, Y, Y) d\theta(Y) = \frac{V(S^{2k-1})}{k(k+1)} \sum_{i,j=1}^{k} \left[ g(E_i, E_i, E_j, E_j) + g(E_i, E_j, E_j, E_i) \right], \tag{3.6}
\]
where \( d\theta(Y) \) is the spherical measure on \( \{ Y \in \Sigma, |Y| = 1 \} \cong S^{2k-1} \) and \( V(S^{2k-1}) \) the volume with respect to it.

Proof. Let \( Y = \sum_{i=1}^{k} Y^i E_i \in \Sigma \) and recall the following two classical identities on spherical measure:
\[
\int_{Y \in \Sigma, |Y| = 1} Y^i Y^j d\theta(Y) = \frac{\delta_{ij}}{k} \cdot V(S^{2k-1}), \tag{3.7}
\]
and
\[
\int_{Y \in \Sigma, |Y| = 1} Y^i Y^j Y^r Y^s d\theta(Y) = \frac{\delta_{ij} \delta_{rs} + \delta_{is} \delta_{jr}}{k(k+1)} \cdot V(S^{2k-1}), \tag{3.8}
\]
where \( \delta_{ij} \) is the Kronecker delta.
Then
\[
\int_{Y \in \Sigma, |Y| = 1} f(Y, Y) d\theta(Y) = \int_{Y \in \Sigma, |Y| = 1} f(Y^i E_i, Y^j E_j) d\theta(Y)
= f(E_i, E_j) \int_{Y \in \Sigma, |Y| = 1} Y^i Y^j d\theta(Y)
= \frac{V(S^{2k-1})}{k} \sum_{i=1}^{k} f(E_i, E_i), \quad \text{(by (3.7))}
\]
and
\[
\int_{Y \in \Sigma, |Y| = 1} g(Y, Y, Y, Y) d\theta(Y)
= \int_{Y \in \Sigma, |Y| = 1} g(Y^i E_i, Y^j E_j, Y^r E_r, Y^s E_s) d\theta(Y)
= g(E_i, E_j, E_r, E_s) \int_{Y \in \Sigma, |Y| = 1} Y^i Y^j Y^r Y^s d\theta(Y)
= \frac{V(S^{2k-1})}{k(k+1)} \sum_{i,j=1}^{k} \left[ g(E_i, E_i, E_j, E_j) + g(E_i, E_j, E_j, E_i) \right]. \quad \text{(by (3.8))}
\]
□
Applying Lemma 3.2 to (2.1), (3.3) and (3.4) produces the following alternative definitions as integrals over the unit sphere in $\Sigma$.

**Corollary 3.3.** Let the notation be as above. We have

$$R^{(E,h)}(\Sigma; u, \overline{u}) = \frac{k}{V(S^{2k-1})} \int_{Y \in \Sigma, |Y|=1} R^{(E,h)}(Y, \overline{Y}, u, \overline{u}) d\theta(Y),$$

(3.9)

$$\text{Ric}_k(x, \Sigma)(X, \overline{X}) = \frac{k}{V(S^{2k-1})} \int_{Y \in \Sigma, |Y|=1} R(X, \overline{X}, Y, \overline{Y}) d\theta(Y),$$

(3.10)

and

$$S_k(x, \Sigma) = \frac{k(k+1)}{2V(S^{2k-1})} \int_{Y \in \Sigma, |Y|=1} H(Y) d\theta(Y),$$

(3.11) when $\omega$ is CKL.

**Remark 3.4.** In [NZ22] the identity (3.11) was taken as the definition of $S_k$ for Kähler metrics. Either formula of $\text{Ric}_k$ or $S_k$ has its own advantage. For instance, from (3.3) and (3.4) the condition $\text{Ric}_k(x) > 0$ (resp. $\text{Ric}_k > 0$) implies $S_k(x) > 0$ (resp. $S_k > 0$). On the other hand, (3.11) tells us that $S_k > 0$ (resp. $S_k < 0$) implies $S_{k+1} > 0$ (resp. $S_{k+1} < 0$). In contrast to it, in general the sign of $\text{Ric}_k$ is independent from that of $\text{Ric}_l$ when $k \neq l$, as illustrated by Hitchin’s examples (1.2).

### 4. Proof of Theorem 2.3

#### 4.1. Two lemmas.**

We prepare two crucial lemmas in order to establish Theorem 2.3. The following result was obtained in [Li21, Lemma 2.1] by adopting some arguments in [NZ22] and [Ya18], whose proof is to apply a $\partial\bar{\partial}$-Bochner formula and the maximum principle to part of directions.

**Lemma 4.1.** Let $u \in \Gamma(E)$ be a holomorphic section of the Hermitian holomorphic vector bundle $(E, h)$ over a compact complex manifold $M$, and the maximum of $|u| := < u, u >^\frac{1}{2}$ is attained at $x \in M$. Then

$$< R_X \overline{X}(u), u > \big|_x \geq 0,$$

(4.1)

for all $X \in T_x M$.

**Remark 4.2.** The use of a $\partial\bar{\partial}$-Bochner formula, together with the maximum principle to part of directions, was recently revived by some works ([An12], [AC11], [Liu16], [Ni13], [NZ22], [Ya18]).

The next lemma is parallel to [Ya20, Prop. 2.9].

**Lemma 4.3.** Let $(E, h)$ be a uniformly RC $k$-positive Hermitian holomorphic vector bundle over a compact Hermitian manifold $(M, \omega)$. Then there exists a constant $C = C(h, \omega) > 0$ such that for any $x \in M$, there exists a $k$-dimensional subspace $\Sigma_x \subset T_x M$ such that

$$R^{(E,h)}(\Sigma_x; u, \overline{u}) \geq C,$$

(4.2)

for any unit vector $u \in E_x$.

**Proof.** For $x \in M$, let

$$G_k(T_x M) := \left\{ \Sigma \mid \Sigma \text{ are } k\text{-dimensional subspaces in } T_x M \right\}$$

be the associated complex Grassmannian of $T_x M$ and

$$SE_x := \{ u \in E_x, |u| = 1 \}$$

be the set of unit vectors in $E_x$. Then...
the unit sphere of $E_x$. Let
\begin{equation}
(4.3)
C = C(h, \omega) := \min_{x \in M} \max_{\Sigma \in \mathbb{G}_k(T_x M)} \min_{u \in SE_x} R^{(E, h)}(\Sigma; u, \overline{u}),
\end{equation}
which is well-defined due to the compactness of $M$, $\mathbb{G}_k(T_x M)$ and $SE_x$.

First note that such defined $C$ satisfies (4.2). In fact, by definition (4.3) implies that for any $x \in M$,
\begin{equation}
\max_{\Sigma \in \mathbb{G}_k(T_x M)} \min_{u \in SE_x} R^{(E, h)}(\Sigma; u, \overline{u}) \geq C.
\end{equation}
This in turn yields that there exists a $\Sigma_x \in \mathbb{G}_k(T_x M)$ such that
\begin{equation}
\min_{u \in SE_x} R^{(E, h)}(\Sigma_x; u, \overline{u}) \geq C,
\end{equation}
which leads to (4.2).

It suffices to show $C > 0$. Suppose on the contrary that $C \leq 0$. Then (4.3) implies that there exists an $x_0 \in M$ such that
\begin{equation}
\max_{\Sigma \in \mathbb{G}_k(T_{x_0} M)} \min_{u \in SE_{x_0}} R^{(E, h)}(\Sigma; u, \overline{u}) = C \leq 0.
\end{equation}
This leads to
\begin{equation}
\min_{u \in SE_{x_0}} R^{(E, h)}(\Sigma; u, \overline{u}) \leq 0
\end{equation}
for any $\Sigma \in \mathbb{G}_k(T_{x_0} M)$, which in turn yields that for each $\Sigma \in \mathbb{G}_k(T_{x_0} M)$ there exists a $u(\Sigma) \in SE_{x_0}$ such that $R^{(E, h)}(\Sigma; u(\Sigma), u(\Sigma)) \leq 0$.

In summary, under the assumption of $C \leq 0$, we derive a conclusion that there exists an $x_0 \in M$ such that, for every $\Sigma \in \mathbb{G}_k(T_{x_0} M)$, there exists $u(\Sigma) \in SE_{x_0}$ such that $R^{(E, h)}(\Sigma; u(\Sigma), u(\Sigma)) \leq 0$, which exactly contradicts to the condition of uniform RC $k$-positivity of $(E, h) \rightarrow (M, \omega)$! \hfill \qed

4.2. Proof of Theorem 2.3. Let $T \in H^0(M, E^{op} \otimes (E^*)^{ot} \otimes F^{ot})$, $x \in M$ and a unit vector $X \in T_x M$. Arbitrarily choose a Hermitian metric $h_F$ on $F$ and simply write $R$ to denote the Chern curvature tensors of various Hermitian holomorphic vector bundles involved in the proof.

Due to Lemma 3.1 denote by the real eigenvalues of the Hermitian transformations $R^{X\overline{X}}(\cdot)$ on $(E_x, h(x))$ and $(F_x, h_F(x))$ by $\lambda_i = \lambda_i(x, X)$ $(1 \leq i \leq r_1)$ and $\mu_j = \mu_j(x, X)$ $(1 \leq j \leq r_2)$ respectively, where $r_1$ and $r_2$ are the ranks of $E$ and $F$. Let $\{e_1, \ldots, e_{r_1}\}$ and $\{s_1, \ldots, s_{r_2}\}$ be unitary bases of $E_x$ and $F_x$ such that
\begin{equation}
(4.4)
\begin{cases}
R^{X\overline{X}}(e_i) = \lambda_i e_i, & (1 \leq i \leq r_1) \\
R^{X\overline{X}}(s_j) = \mu_j s_j, & (1 \leq j \leq r_2)
\end{cases}
\end{equation}

Let $\{\theta^1, \ldots, \theta^{r_1}\}$ be the unitary basis of $E_x^*$ dual to $\{e_i\}$. Then the induced action of $R^{X\overline{X}}(\cdot)$ on $\{\theta^i\}$ is given by
\begin{equation}
(4.5)
R^{X\overline{X}}(\theta^i) = -\lambda_i \theta^i, \quad 1 \leq i \leq r_1.
\end{equation}

Write
\[ T^a_{\beta_1 \cdots \beta_q} \equiv T^a_{\beta_1 \cdots \beta_q} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \theta^{\beta_1} \otimes \cdots \otimes \theta^{\beta_q} \otimes s_{\gamma_1} \otimes \cdots \otimes s_{\gamma_m}. \]
Then (4.4) and (4.5) imply
\[
R_{X,T} \left| \frac{X}{x}, T \right| = \sum_{\alpha_1, \ldots, \alpha_p \atop \beta_1, \ldots, \beta_q \atop \gamma_1, \ldots, \gamma_m} \left( \sum_{i=1}^{p} \lambda_{\alpha_i} - \sum_{j=1}^{q} \lambda_{\beta_j} + \sum_{l=1}^{m} \mu_{\gamma_l} \right) T_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q \gamma_1 \ldots \gamma_m}^p \varepsilon_{\alpha_1} \otimes \cdots \otimes \varepsilon_{\alpha_p} \otimes \theta^\beta_1 \otimes \cdots \otimes \theta^\beta_q \otimes s_{\gamma_1} \otimes \cdots \otimes s_{\gamma_m},
\]
which yields that
\[
< R_{X,T} \left| \frac{X}{x}, T \right| >
= \sum_{\alpha_1, \ldots, \alpha_p \atop \beta_1, \ldots, \beta_q \atop \gamma_1, \ldots, \gamma_m} \left( \sum_{i=1}^{p} \lambda_{\alpha_i} - \sum_{j=1}^{q} \lambda_{\beta_j} + \sum_{l=1}^{m} \mu_{\gamma_l} \right)^2.
\]

Let
\[
\begin{cases}
\lambda_{\text{max}}(x,X) := \max_{1 \leq i \leq r_1} \{ \lambda_i(x,X) \} \\
\lambda_{\text{min}}(x,X) := \min_{1 \leq i \leq r_1} \{ \lambda_i(x,X) \} \\
\lambda_{\text{max}} := \max_{x \in M, X \in ST_x M} \lambda_{\text{max}}(x,X) \\
\lambda_{\text{min}} := \min_{x \in M, X \in ST_x M} \lambda_{\text{min}}(x,X)
\end{cases}
\]
and similarly define \( \mu_{\text{max}}(x,X), \mu_{\text{min}}(x,X), \mu_{\text{min}} \) and \( \mu_{\text{max}} \). We remark that in general \( \lambda_{\text{max}}(x,X) \) and \( \lambda_{\text{min}}(x,X) \) are only continuous functions and may not be smooth. Nevertheless, continuity is enough to guarantee that \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are both well-defined real numbers as the maximum and minimum in (4.7) are over the unit sphere bundle of \( TM \), which is compact. Clearly \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) depend only on \( h \) and \( \omega \). Moreover, continuity is also enough to do integration as we shall see in the following lemma. Accordingly, for various \( \mu \)'s similar properties hold.

An efficient upper bound estimate for (4.6) is exhibited by the next result.

**Lemma 4.4.** Let \( (E,h) \to (M,\omega) \) be a uniformly RC \( k \)-positive Hermitian holomorphic vector bundle. Let \( C(h,\omega) \) be the positive constant and \( \Sigma_x \) the desired \( k \)-dimensional subspace in \( T_xM \) in Lemma 4.3. Then for any \( x \in M \), we have
\[
\int_{X \in \Sigma_x, |X| = 1} < R_{X,T} \left| \frac{X}{x}, T \right| > d\theta(X) \\
\leq \left[ \lambda_{\text{max}} \cdot p - \frac{C(h,\omega)}{k} \cdot q + \mu_{\text{max}} \cdot m \right] \cdot V(S^{2k-1}) \cdot |T|^2.
\]

**Proof.** To simplify the notation, denote by \( \oint f(X) \) the integral of the function \( f(X) \) over \( \{ X \in \Sigma_x \mid |X| = 1 \} \cong S^{2k-1} \) with respect to the spherical measure \( d\theta(X) \).

First we claim that
\[
\oint \mu_i(x,X) \leq \mu_{\text{max}} \cdot V(S^{2k-1})
\]
and

\[(4.10) \quad 0 < \frac{C(h, \omega)}{k} \cdot V(S^{2k-1}) \leq \oint \lambda_i(x, X) \leq \lambda_{\text{max}} \cdot V(S^{2k-1}).\]

In fact, (4.9) and the last inequality in (4.10) are obvious as \(\mu_i(x, X) \leq \mu_{\text{max}}\) and \(\lambda_i(x, X) \leq \lambda_{\text{max}}\). On the other hand, by definition

\[\lambda_i(x, X) \overset{(4.4)}{=} < R_X(x, e_i), e_i > = R(X, \overline{X}, e_i, e_i).\]

Thus

\[
\oint \lambda_i(x, X) = \oint R(X, \overline{X}, e_i, e_i)
\]

\[= \frac{V(S^{2k-1})}{k} R(\Sigma_x; e_i, e_i) \quad \text{(by (3.9))}
\]

\[\geq \frac{V(S^{2k-1})}{k} \cdot C(h, \omega), \quad \text{(by (4.2))}
\]

which gives the desired one in (4.10).

Taking integrals \(\oint (\cdot)\) on both sides of (4.6) yields

\[
\oint < R_X(X) \big|_x T \big|_x >
\]

\[= \sum_{\alpha_1, \ldots, \alpha_p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma_1, \ldots, \gamma_m} \left| T^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q \gamma_1 \cdots \gamma_m} \right|^2 \oint \left( \sum_{i=1}^p \lambda_{\alpha_i} - \sum_{j=1}^q \lambda_{\beta_j} + \sum_{l=1}^m \mu_{\gamma_l} \right)
\]

\[\leq \sum_{\alpha_1, \ldots, \alpha_p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma_1, \ldots, \gamma_m} \left| T^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q \gamma_1 \cdots \gamma_m} \right|^2 \left[ \lambda_{\text{max}} \cdot p - \frac{C(h, \omega)}{k} \cdot q + \mu_{\text{max}} \cdot m \right] \cdot V(S^{2k-1}) \quad \text{(by (4.9) and (4.10))}
\]

\[= \left[ \lambda_{\text{max}} \cdot p - \frac{C(h, \omega)}{k} \cdot q + \mu_{\text{max}} \cdot m \right] \cdot V(S^{2k-1}) \cdot |T|^2 \big|_x.
\]

This gives the desired estimate (4.8). \(\Box\)

We are ready now to complete the proof of Theorem 2.3 in the following lemma.

**Lemma 4.5.** Let \((E, h) \to (M, \omega)\) be a uniformly RC \(k\)-positive Hermitian holomorphic vector bundle and \(F\) any holomorphic vector bundle over \(M\). With the notation above understood and set two positive constants by

\[(4.11) \quad C_1(h, \omega) := \frac{k \cdot \lambda_{\text{max}}}{C(h, \omega)} \quad C_2 = C_2(F, h, \omega) := \begin{cases} \frac{k \cdot \mu_{\text{max}}}{C(h, \omega)}, & \text{if } \mu_{\text{max}} > 0 \\ 1, & \text{if } \mu_{\text{max}} \leq 0 \end{cases}
\]

Then for all \(p, q, m \in \mathbb{Z}_{\geq 0}\) with \(q > C_1 \cdot p + C_2 \cdot m\), we have

\[H^0(M, E^\otimes p \otimes (E^*)^\otimes q \otimes F^\otimes m) = 0
\]

and hence Theorem 2.3 holds.
Proof. First note that the positivity of $\lambda_{\text{max}}$ and hence $C_1$ follows from the inequality (4.10). Let $T \in H^0(M, E^{\otimes p} \otimes (E^*)^{\otimes q} \otimes F^{\otimes m})$ and $|T|$ attains its maximum at $x_0 \in M$. On the one hand, Lemma 4.1 yields

\begin{equation}
\int_{X \in \Sigma_{x_0}, \ |X| = 1} < R_{X X}(T) \bigg|_{x_0}, T \bigg|_{x_0} > d\theta(X) \geq 0.
\end{equation}

On the other hand, applying Lemma 4.4 to $x_0$ leads to

\begin{equation}
\int_{X \in \Sigma_{x_0}, \ |X| = 1} < R_{X X}(T) \bigg|_{x_0}, T \bigg|_{x_0} > d\theta(X)
\leq - \frac{C(h, \omega)}{k} \cdot \left[ q - \left( \frac{k \lambda_{\text{max}}}{C(h, \omega)} \cdot p + \frac{k \mu_{\text{max}}}{C(h, \omega)} \cdot m \right) \cdot V(S^{2k-1}) \cdot |T|^2 \right]_{x_0}
\leq - \frac{C(h, \omega)}{k} \cdot \left[ q - (C_1 \cdot p + C_2 \cdot m) \right] \cdot V(S^{2k-1}) \cdot |T|^2 \bigg|_{x_0}. \quad \text{(by (4.11))}
\end{equation}

If $q > C_1 \cdot p + C_2 \cdot m$, (4.12) and (4.13) together imply

\begin{equation}
\int_{X \in \Sigma_{x_0}, \ |X| = 1} < R_{X X}(T) \bigg|_{x_0}, T \bigg|_{x_0} > d\theta(X) = 0,
\end{equation}

which, in turn via (4.13) tells us that the only possibility is $|T|^2 \bigg|_{x_0} = 0$. The maximum of $|T|$ at $x_0$ then implies $T \equiv 0$. This completes the proof of this lemma and hence that of Theorem 2.3. \qed

5. Proof of Theorem 2.7

5.1. A lemma and the proof of Theorem 2.7. The proof of Theorem 2.7 is based on the next result, which was essentially obtained by Ni-Zheng in [NZ22, Prop. 3.1], although the conclusion there was stated for Kähler manifolds.

Lemma 5.1 (Ni-Zheng). Let $(M, \omega)$ be a compact Chern-Kähler-like Hermitian manifold, $x \in M$, and the $k$-dimensional subspace $\Sigma \subset T_x M$ minimize (resp. maximize) the $k$-scalar curvature $S_k(x, \cdot)$ at $x$. Then for any $Y \in \Sigma$ and $Z \in \Sigma^\perp := \{W \in T_x M \mid W \perp \Sigma\}$, we have

\begin{equation}
\int_{X \in \Sigma, \ |X| = 1} R(x, X, Y, Z) d\theta(X) = \int_{X \in \Sigma, \ |X| = 1} R(x, X, Z, Y) d\theta(X) = 0,
\end{equation}

and

\begin{equation}
\int_{X \in \Sigma, \ |X| = 1} R(x, X, Z, Z) d\theta(X) \geq \frac{V(S^{2k-1})}{k(k + 1)} \cdot S_k(x, \Sigma) \cdot |Z|^2.
\end{equation}

Remark 5.2. Lemma 5.1 can be viewed as a $k$-dimensional generalization of [Ya18, Lemma 6.1] for holomorphic sectional curvature in the Kähler situation, which still holds true for CKL Hermitian metrics ([Li21, Lemma 2.4]). The proof in [NZ22, Prop. 3.1], in spite of the similar principle to the one in [Ya18, Lemma 6.1], is skillful. For the reader’s convenience as well as for completeness, we include a detailed proof below. We shall see in the process of the proof that what we really need is various Kähler-type symmetries of the Chern curvature tensor, which, as explained in Remark 2.6, is satisfied by the CKL Hermitian metrics.
We first explain how Lemma 5.1, together with the materials in Section 3, leads to Theorem 2.7 and postpone its proof to the next subsection.

Proof of Theorem 2.7.

Proof. Assume that \((M, \omega)\) is an \(n\)-dimensional compact Chern-Kähler-like Hermitian manifold and \(Ric_k(\omega) > 0\) (resp. \(Ric_k(\omega) < 0\)) for some \(k \in \{1, \ldots, n\}\). Let

\[
D := \min_{x \in M} \min_{\Sigma \in G_k(T_x M)} \Ric_k(x, \Sigma)(X, \overline{X}) > 0 \quad \text{(resp. } D := \max \cdots < 0).\tag{5.3}
\]

The definitions of \(Ric_k(\omega)\) and \(S_k(\omega)\) in (3.3) and (3.4) then imply that

\[
\min_{\Sigma \in G_k(T_x M)} S_k(x, \Sigma) \geq kD > 0 \quad \text{(resp. } \max \cdots \leq kD < 0).\tag{5.4}
\]

For any \(x \in M\), let the \(k\)-dimensional subspace \(\Sigma_x \subset T_x M\) minimize (resp. maximize) the \(k\)-scalar curvature \(S_k(x, \cdot)\) at \(x\), as required in Lemma 5.1. We shall show that such \(\Sigma\) satisfies the inequality (2.1) and hence \((TM, \omega)\) is uniformly RC \(k\)-positive (resp. uniformly RC \(k\)-negative) over \((M, \omega)\). In fact, for any \(X \in T_x M\), decompose \(X\) as \(X = X_1 + X_2\), where \(X_1 \in \Sigma_x\) and \(X_2 \in \Sigma_x^\perp\). Like before, denote by \(\int f(Y)\) the integral of \(f(Y)\) over \(\{Y \in \Sigma \mid |Y| = 1\}\) with respect to the spherical measure. Then

\[
R(TM, \omega)(\Sigma; X, \overline{X})
\]

\[
= \frac{k}{V(S^{2k-1})} \int R(Y, \overline{Y}, X, \overline{X}) \quad \text{(by (3.9))}
\]

\[
= \frac{k}{V(S^{2k-1})} \int R(Y, \overline{Y}, X_1 + X_2, \overline{X_1} + \overline{X_2})
\]

\[
= \frac{k}{V(S^{2k-1})} \int \left[ R(Y, \overline{Y}, X_1, \overline{X_1}) + R(Y, \overline{Y}, X_2, \overline{X_2}) \right] \quad \text{(by (5.1))}
\]

\[
= \Ric_k(x, \Sigma)(X_1, \overline{X_1}) + \frac{k}{V(S^{2k-1})} \int R(Y, \overline{Y}, X_2, \overline{X_2}) \quad \text{(by (3.10))}
\]

\[
\geq \Ric_k(x, \Sigma)(X_1, \overline{X_1}) + \frac{S_k(x, \Sigma)}{k+1} |X_2|^2 \quad \text{(by (5.2))}
\]

\[
\geq D |X_1|^2 + \frac{kD}{k+1} |X_2|^2 \quad \text{(by (5.3) and (5.4))}
\]

\[
\geq \frac{kD}{k+1} |X|^2.
\]

This completes the proof of Theorem 2.7. \(\square\)

5.2. Proof of Lemma 5.1. The proof here basically follows the strategy in [NZ22, Prop. 3.1], but with more clear presentation in places.

Let \(U(n)\) be the isometry group of \((T_x M, \omega_x = <, >)\) and

\[
u(n) := \left\{ a \in \text{Hom}(T_x M) \mid < a(W_1), W_2 > + < W_1, a(W_2) > = 0, \forall W_1, W_2 \in T_x M \right\}
\]

the (real) Lie algebra of $U(n)$. It is well-known that
\[ e^{ta} := \exp(ta) := \text{id} + \sum_{i=1}^{\infty} \frac{(ta)^i}{i!} \in U(n) \quad \text{and} \quad \frac{d^i}{dt^i}(e^{ta}) = a^i e^{ta}, \quad \text{for any } t \in \mathbb{R}. \]

As before denote by $\oint f(X) d\theta(X)$ the integral of $f$ over $\{X \in \Sigma, |X| = 1\}$, where $\Sigma$ is the $k$-dimensional subspace in $T_xM$ minimizing (resp. maximizing) $S_k(x, \cdot)$ at $x$.

For any $a \in u(n)$ consider the following real-valued function
\[ f(t) := \oint H(e^{ta}X) d\theta(X), \quad t \in \mathbb{R}. \]

The minimum (resp. maximum) of $f(t)$ at $t = 0$ implies that $f'(t) = 0$ and $f''(t) \geq 0$ (resp. $\leq 0$). Direct calculations, together with the fact that the Chern curvature tensor $R$ satisfy Kähler-like symmetries ensured by the condition of CKL, lead to
\[
\oint \left[ R(a(X), X, X, X) + R(X, a(X), X, X) \right] d\theta(X) = 0
\]
and
\[
\oint \left[ R(a^2(X), X, X, X) + R(X, a^2(X), X, X) + 4R(a(X), a(X), X, X) \right.
+ R(a(X), a(X), X, a(X)) + R(X, a(X), a(X)) \left. \right] d\theta(X) \geq 0.
\]

We may assume that the two vectors $Y \in \Sigma$ and $Z \in \Sigma^\perp$ are both nonzero as otherwise (5.1) and (5.2) trivially hold. Let
\[ a(\cdot) := \sqrt{-1}( < \cdot, Y > Z + < \cdot, Z > Y ) \]
and it can be easily checked that $a(\cdot) \in u(n)$. Then
\[
a(X) = \sqrt{-1} < X, Y > Z, \quad a^2(X) = - < X, Y > |Z|^2 Y.
\]

Apply (5.6) to (5.8) and also the one with $Z$ being replaced by $\sqrt{-1}Z$, and sum the two we have
\[
\oint < X, Y > R(Z, X, X, X) d\theta(X) = 0.
\]

We take a unitary basis of $\Sigma$ as follows
\[
\{ E_1 = Y/|Y|, E_2, \ldots, E_k \},
\]
and simply write $V := V(S^{2k-1})$. Then
\[
0 = \oint < X, Y > R(Z, X, X, X) d\theta(X) \quad \text{(by (5.9))}
\]
\[
= \frac{V}{k(k+1)} \sum_{i,j=1}^{k} \left[ < E_i, Y > R(Z, E_i, E_j, E_j) + < E_i, Y > R(Z, E_j, E_j, E_i) \right] \quad \text{(by (3.6))}
\]
\[
= \frac{2V}{k(k+1)} \sum_{j=1}^{k} R(Z, Y, E_j, E_j) \quad \text{(by (5.10) and $\omega$ CKL)}
\]
\[
= \frac{2}{k+1} \oint R(Z, Y, X, X) d\theta(X). \quad \text{(by (3.5))}
\]
This and its conjugate lead to (5.1).

We now derive (5.2). Apply (5.7) to the above (5.8) and also the one being replaced by \( \sqrt{-1}Z \), and sum the two we have

\[
4 \oint | <X,Y>|^2 R(Z, \overline{Z}, X, \overline{X}) d\theta(X) \geq \oint \left[ <X,Y> R(Y, \overline{X}, X, \overline{X}) + <Y,X> R(X, \overline{Y}, X, \overline{X}) \right] |Z|^2 d\theta(X).
\]

(5.12)

Arbitrarily choose a unitary basis \( \{E_1, \ldots, E_k\} \) of \( \Sigma \), which is irrelevant to (5.10). Then

\[
\oint \left[ \text{LHS of (5.12)} \right] d\theta(Y) = \frac{4V}{k} \sum_{i=1}^{k} \oint | <X, E_i>|^2 R(Z, \overline{Z}, X, \overline{X}) d\theta(X) \quad \text{(by (3.5))}
\]

\[
= \frac{4V^2}{k^2(k+1)} \sum_{i,j=1}^{k} \left[ |<E_j, E_i>|^2 R(Z, \overline{Z}, E_i, \overline{E}_i) + <E_j, E_i>|<\overline{E}_i, E_i>R(Z, \overline{Z}, E_i, \overline{E}_i) \right] \quad \text{(by (3.6))}
\]

(5.13)

\[
= \frac{4V^2}{k^2} \sum_{i=1}^{k} R(Z, \overline{Z}, E_i, \overline{E}_i)
\]

= \frac{4V}{k} \oint R(Z, \overline{Z}, X, \overline{X}) d\theta(X). \quad \text{(by (3.5))}
\]

Analogous to (5.13) it can be derived that

\[
\oint \left[ \text{RHS of (5.12)} \right] d\theta(Y) = \frac{4V^2}{k^2(k+1)} S_k(x, \Sigma) |Z|^2.
\]

(5.14)

Putting (5.12), (5.13) and (5.14) together yields the desired (5.2).

6. Further questions and remarks

In the process of proofs we have seen that the strict positivity or negativity in Theorems 2.3 and 2.7 is needed. In view of Theorem 1.1, it is natural to wonder whether the conclusions remain true if only assuming quasi-positivity or quasi-negativity of various curvature conditions. Therefore the following question can be proposed (see [Ya20, Conjecture 1.9]).

**Question 6.1.** Whether the conclusions in Theorems 2.3, 2.7, 2.8 and 2.11 hold true if various positivity or negativity conditions are weakened respectively to quasi-positivity or quasi-negativity?

**Remark 6.2.** What Heier-Wong showed in [HW20] is indeed that a projective Kähler manifold with quasi-positive holomorphic sectional curvature is rationally connected, which provides some positive evidence towards Question 6.1. The condition of quasi-positivity in this case was further strengthened by Matsumura in [Ma22, Thm 1.2]. By combining this with some arguments in [Ya20] and [HLWZ18], S. Zhang and X. Zhang recently showed that the quasi-positivity of \( H = \text{Ric}_1 \) for a compact Kähler manifold leads to projectivity and rational-connectedness ([ZZ23]).
Among various geometric positivity concepts for Hermitian holomorphic vector bundles, the Griffiths positivity ([Gr69]) may be the best-known. In view of the results proved in this article, it may be interesting to propose the following notion.

**Definition 6.3.** A rank $r$ Hermitian holomorphic vector bundle $(E, h)$ over an $n$-dimensional Hermitian manifold $(M, \omega)$ is called Griffiths $(k, l)$-positive $(1 \leq k \leq n, \ 1 \leq l \leq r)$ if at each point $x \in M$, and for any $\Sigma \in \mathcal{G}_k(T_x M)$ and any $\sigma \in \mathcal{G}_l(E_x)$, we have

$$R^{(E, h)}(\Sigma; \sigma) := \sum_{1 \leq i \leq k \atop 1 \leq j \leq l} R^{(E, h)}(E_{ij}, e_i, e_j) > 0,$$

where $\{E_1, \ldots E_k\}$ (resp. $\{e_1, \ldots , e_l\}$) is a unitary basis of $\Sigma$ (resp. $\sigma$). Griffiths $(k, l)$-negativity and various quasi-versions can be similarly defined.

**Remark 6.4.**

1. By (3.5), an alternative definition for $R^{(E, h)}(\Sigma; \sigma)$ is

$$R^{(E, h)}(\Sigma; \sigma) = \frac{k^2}{V(S^{2k-1})V(S^{2l-1})} \int_{\{u \in \Gamma, \ |u| = 1\} \atop \{X \in \Sigma, \ |X| = 1\}} R^{(E, h)}(X, \overline{X}, u, \overline{u}) d\theta(X) d\theta(u).$$

2. Griffiths $(1, 1)$-positivity is the original Griffiths positivity, in which case the metric $\omega$ is irrelevant. By definition Griffiths $(k, l)$-positivity (resp. negativity) implies Griffiths $(k + 1, l)$-positivity and $(k, l + 1)$-positivity (resp. negativity). Thus the condition Griffiths $(k, l)$-positivity becomes weaker as $k$ or $l$ increases (compare to the $k$-scalar curvatures in Remark 3.4).

3. The subject of cohomology vanishing theorems for Griffiths positive holomorphic vector bundles occupies a central role in several complex variables and algebraic geometry ([Dem12, §7], [SS85, §6]). It seems to be possible to generalize these to Griffiths $(k, l)$-positive vector bundles.

Another two versions of $(k, l)$-positivity (resp. negativity) related to Definition 2.1 are as follows.

**Definition 6.5.** Let $(E, h)$ be a rank $r$ Hermitian holomorphic vector bundle over an $n$-dimensional Hermitian manifold $(M, \omega)$, and $k \in \{1, \ldots , n\}$ and $\ell \in \{1, \ldots , r\}$.

1. It is called **RC $(k, l)$-positive** (resp. **BC $(k, l)$-positive**) at $x \in M$ if for any $\Sigma \in \mathcal{G}_k(T_x M)$ (resp. $\Sigma \in \mathcal{G}_k(T_x M)$), there exists $\Sigma \in \mathcal{G}_k(T_x M)$ (resp. $\Sigma \in \mathcal{G}_l(E_x)$) such that $R^{(E, h)}(\Sigma; \sigma) > 0$. If this holds at each $x \in M$, then it is called **RC $(k, l)$-positive** (resp. **BC $(k, l)$-positive**).

2. It is called uniformly **RC $(k, l)$-positive** (resp. uniformly **BC $(k, l)$-positive**) at $x \in M$ if there exists $\Sigma \in \mathcal{G}_k(T_x M)$ (resp. $\Sigma \in \mathcal{G}_k(T_x M)$) such that for every $\Sigma \in \mathcal{G}_k(T_x M)$ (resp. $\Sigma \in \mathcal{G}_k(T_x M)$), $R^{(E, h)}(\Sigma; \sigma) > 0$. If this holds at each $x \in M$, then it is called uniformly **RC $(k, l)$-positive** (resp. uniformly **BC $(k, l)$-positive**).

(Uniform) RC $(k, l)$-negativity or BC $(k, l)$-negativity and various quasi-versions can be similarly defined.

**Remark 6.6.**

1. Uniform RC $(k, 1)$-positivity is the one in Definition 2.1, RC $(1, 1)$-positivity is the RC-positivity in [Ya18], and BC $(1, l)$-positivity is the BC $l$-positivity in [Ni21-2, p. 280]. For CKL Hermitian manifold $(M, \omega)$, $(TM, \omega)$ is (uniformly) RC
(k, l)-positive (resp. negative) if and only if it is (uniformly) BC (l, k)-positive (resp. negative).

(2) It turns out that BC k-positivity of the tangent bundle of a Hermitian metric for some k implies \( h^{k,0} = 0 \) ([Ni21-2, Coro. 4.4]). It may be interesting to explore possible consequences of geometric importance for general uniform RC (resp. BC) (k, l)-positivity.

(3) By the proof of Lemma 4.3, it is easy to see that RC (k, l)-positivity or BC (k, l)-positivity on a compact Hermitian manifold amounts to

\[
\min_{x \in M} \min_{\sigma \in G_l(E_x)} \max_{\Sigma \in G_k(T_x M)} R^{(E,h)}(\Sigma; \sigma) > 0
\]

or

\[
\min_{x \in M} \min_{\Sigma \in G_k(T_x M)} \max_{\sigma \in G_l(E_x)} R^{(E,h)}(\Sigma; \sigma) > 0,
\]

and uniform RC (k, l)-positivity or uniform BC (k, l)-positivity amounts to

\[
\min_{x \in M} \max_{\sigma \in G_l(E_x)} \min_{\Sigma \in G_k(T_x M)} R^{(E,h)}(\Sigma; \sigma) > 0
\]

or

\[
\min_{x \in M} \max_{\Sigma \in G_k(T_x M)} \min_{\sigma \in G_l(E_x)} R^{(E,h)}(\Sigma; \sigma) > 0
\]

respectively.

(4) Besides the curvatures mentioned in this article, some other interesting curvature notions were also introduced by Ni and Ni-Zheng. We refer the reader to their survey article [NZ19] for more details.

References

[AHZ18] A. Alvarez, G. Heier, F.-Y. Zheng: On projectivized vector bundles and positive holomorphic sectional curvature, Proc. Amer. Math. Soc. 146 (2018), 2877-2882.

[An12] B. Andrews: Noncollapsing in mean-convex mean curvature flow, Geom. Topol. 16 (2012), 1413-1418.

[AC11] B. Andrews, J. Clutterbuck: Proof of the fundamental gap conjecture, J. Amer. Math. Soc. 24 (2011), 899-916.

[Bo46] S. Bochner: Vector fields and Ricci curvature, Bull. Amer. Math. Soc. 52 (1946), 776-797.

[Bo49] S. Bochner: Curvature and Betti numbers. II, Ann. of Math. (2) 50 (1949), 77-93.

[Ca92] F. Campana: Connexion rationnelle des variétés de Fano, Ann. Sci. Éc. Norm. Supér. (4) 25 (1992), 539-545.

[CDP15] F. Campana, J.-P. Demailly, P. Peternell: Rationally connected manifolds and semipositivity of the Ricci curvature, in: Recent advances in algebraic geometry, London Math. Soc. Lecture Note Ser. 417, Cambridge University Press, Cambridge (2015), 71-91.

[CLT22] J. Chu, M.-C. Lee, L.-F. Tam: Kähler manifolds and mixed curvature, Trans. Amer. Math. Soc. 375 (2022), 7925-7944.

[Deb01] O. Debarre: Higher-dimensional algebraic geometry, Universitext, Springer-Verlag, New York, 2001.

[Dem12] J.-P. Demailly: Complex analytic and differential geometry, available at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[DT19] S. Diverio, S. Trapani: Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle, J. Differential Geom. 111 (2019), 303-314.

[Gr69] P. A. Griffiths: Hermitian differential geometry, Chern classes and positive vector bundles, Global Analysis (papers in honor of K. Kodaira), Princeton Univ. Press, Princeton, (1969), 181-251.

[HLWZ18] G. Heier, S.S.Y Lu, B. Wong, F.Y. Zheng: Reduction of manifolds with semi-negative holomorphic sectional curvature, Math. Ann. 372 (2018), 951-962.

[HW20] G. Heier, B. Wong: On projective Kähler manifolds of partially positive curvature and rational connectedness, Doc. Math. 25 (2020), 219-238.
[Hi75] N. Hitchin: *On the curvature of rational surfaces*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, R. I., 1975, pp. 65-80. 2

[Ko61] S. Kobayashi: *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. of Math. (2) 74 (1961), 570-574. 2

[Ko80-1] S. Kobayashi: *The first Chern class and holomorphic symmetric tensor fields*, J. Math. Soc. Japan 32 (1980), 325-329. 1

[Ko80-2] S. Kobayashi: *First Chern class and holomorphic tensor fields*, Nagoya Math. J. 77 (1980), 5-11. 1

[Ko87] S. Kobayashi: *Differential Geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Kanô Memorial Lectures, 5, Princeton University Press, Princeton, N.J., 1987. 5

[KH83] S. Kobayashi, C. Horst: *Topics in Complex Differential Geometry*, Complex differential geometry, 4-66, DMV Sem., 3, Birkhäuser, Basel, 1983. 1

[KMM92] J. Kollár, Y. Miyaoka, S. Mori: *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. 36 (1992), 765-779. 2

[LNZ21] C. Li, L. Ni, X.H. Zhu: *An application of a $C^2$-estimate for a complex Monge-Ampère equation*, Internet. J. Math. 32 (2021), Paper No. 2140007, 13 pp. 3

[Li21] P. Li: *Vanishing theorems on compact Chern-Kähler-like Hermitian manifolds*, arXiv:2112.02367, to appear in Math. Res. Lett. 2, 4, 9, 13

[Liu16] G. Liu: *Three-circle theorem and dimension estimate for holomorphic functions on Kähler manifolds*, Duke Math. J. 165 (2016), 2899-2919. 9

[Ma22] S. Matsumura: *On projective manifolds with semi-positive holomorphic sectional curvature*, Amer. J. Math. 144 (2022), 747-777. 16

[MK71] J. Morrow, K. Kodaira: *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971. 2

[Ni13] L. Ni: *Estimates on the modulus of expansion for vector fields solving nonlinear equations*, J. Math. Pures Appl. (9) 99 (2013), 1-16. 9

[Ni21-1] L. Ni: *Liouville theorems and a Schwarz lemma for holomorphic mappings between Kähler manifolds*, Comm. Pure Appl. Math. 74 (2021), 1100-1126. 2, 7

[Ni21-2] L. Ni: *The fundamental group, rational connectedness and the positivity of Kähler manifolds*, J. Reine Angew. Math. 774 (2021), 267-299. 3, 5, 17, 18

[NZ18] L. Ni, F.Y. Zheng: *Comparison and vanishing theorems for Kähler manifolds*, Calc. Var. Partial Differential Equations 57 (2018), Paper No. 151, 31 pp. 2

[NZ22] L. Ni, F.Y. Zheng: *Positivity and the Kodaira embedding theorem*, Geom. Topol. 26 (2022), 2491-2505. 7, 9, 13, 14

[NZ19] L. Ni, F.-Y. Zheng: *On Orthogonal Ricci Curvature*, In Advances in Complex Geometry, Contemp. Math. 735, pp. 203-215, Providence, RI: Amer. Math. Soc., 2019. 18

[SS85] B. Shiffman, A. J. Sommese: *Vanishing theorems on complex manifolds*, vol. 56 of Progress in Mathematics. Boston, MA: Birkhäuser Boston Inc., 1985. 17

[TY17] V. Tosatti, X.-K. Yang: *An extension of a theorem of Wu-Yau*, J. Differential Geom. 107 (2017), 573-579. 2

[Ts57] Y. Tsukamoto: *On Kählerian manifolds with positive holomorphic sectional curvature*, Proc. Japan Acad. 33 (1957), 333-335. 2

[WY16-1] D.-M. Wu, S.-T. Yau: *Negative Holomorphic curvature and positive canonical bundle*, Invent. Math. 204 (2016), 595-604. 2

[WY16-2] D.-M. Wu, S.-T. Yau: *A remark on our paper “Negative Holomorphic curvature and positive canonical bundle”*, Comm. Anal. Geom. 24 (2016), 901-912. 2

[YZ18] B. Yang, F.Y. Zheng: *On curvature tensors of Hermitian manifolds*, Comm. Anal. Geom. 26 (2018), 1195-1222. 4

[Ya16] X.K. Yang: *Hermitian manifolds with semi-positive holomorphic sectional curvature*, Math. Res. Lett. 23 (2016), 939-952. 2

[Ya18] X.K. Yang: *RC-positivity, rational connectedness and Yau’s conjecture*, Camb. J. Math. 6 (2018), 183-212. 2, 9, 13, 17

[Ya20] X.K. Yang: *RC-positive metrics on rationally connected manifolds*, Forum Math. Sigma 8 (2020), Paper No. e53, 19 pp. 2, 3, 5, 9, 16
[YB53] K. Yano, S. Bochner: *Curvature and Betti Numbers*, Annals of Mathematics Studies, No. 32. Princeton University Press, Princeton (1953)

[Yau77] S.T. Yau: *Calabi’s conjecture and some new results in algebraic geometry*, Proc. Natl. Acad. Sci. USA, 74 (1977), 1798-1799.

[Yau82] S.T. Yau: *Problem Section*, Seminar on Differential Geometry, pp 669-706, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[ZZ23] S. Zhang, X. Zhang: *Compact Kähler manifolds with quasi-positive holomorphic sectional curvature*, arXiv:2311.18779.

[Zh00] F.Y. Zheng: *Complex differential geometry*, AMS/IP Studies in Advanced Mathematics 18, American Mathematical Society, Providence, RI 2000.

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