Ringing the Randall-Sundrum braneworld: metastable gravity wave bound states

Sanjeev S. Seahra

Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth, PO1 2EG, UK

(Dated: September 12, 2005)

In the Randall-Sundrum scenario, our universe is a 4-dimensional ‘brane’ living in a 5-dimensional bulk spacetime. By studying the scattering of bulk gravity waves, we show that this brane rings with a characteristic set of complex quasinormal frequencies, much like a black hole. To a bulk observer these modes are interpreted as metastable gravity wave bound states, while a brane observer views them as a discrete spectrum of decaying massive gravitons. Potential implications of these scattering resonances are discussed.

PACS numbers: 11.25.Wx, 04.30.Nk

Introduction. It is generally true that in many branches of physics, the best way to learn about an object is to hit it with something and analyze what comes back at you. Scattering experiments can either be real exercises done in the laboratory, such as in particle or condensed matter physics, or they can be of the virtual type, where the behaviour of radiation striking a target is simulated on a computer. The latter technique has proved to be an invaluable tool in the analysis of the fundamental structure of black holes, which are a class of targets not readily available in the lab. Numerical calculations of how gravity wave pulses bounce off black holes have revealed that they possess discrete quasinormal modes (QNMs) of vibration, which encode both important observational features for gravity wave detectors and valuable clues as to the canonical quantization of general relativity (GR).

The success achieved for black holes prompts us to look for other virtual gravity wave scattering targets. A prime candidate comes from the Randall-Sundrum (RS) braneworld scenario. This 5-dimensional phenomenological model is motivated by non-perturbative ideas from string theory, which postulate that we are living on a hypersurface (or ‘brane’) embedded within a higher-dimensional spacetime. One of the two variants of the model involves one flat brane and an infinite continuum of modes, with harmonics \( (i,j) \) that remain localized near the brane with definite lifetimes. After confirming the existence of these modes both analytically and numerically, we discuss how they might impact on various aspects of the braneworld scenario.

Gravity wave master equation. The RS scenario consists of an AdS5 spacetime with a geometric defect (brane) at \( y = 0 \). The 5-metric is

\[
ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \quad a(y) = e^{-|y|/\ell}.
\]

As is well known, in the RS gauge perturbations of the 5-metric are written as

\[
g_{\mu\nu} \rightarrow g_{\mu\nu} + \partial_\mu x^\kappa \partial_\nu x^\kappa h_{\mu\nu}, \quad \eta^{\mu\nu} h_{\mu\nu} = 0 = \partial^\mu h_{\mu\nu},
\]

subject to the boundary condition \( \partial_\mu h_{\mu\nu} = -2\ell^{-1}h_{\mu\nu} \) at \( y = 0 \). The simplest class of perturbations are the tensor modes, with harmonics \((i,j)\), \( i,j = 1, 2, 3 \);

\[
\bar{\nabla}^2 T^{(k)}_{ij} = -k^2 T^{(k)}_{ij}, \quad \partial^\mu T^{(k)}_{ij} = 0 = \text{Tr} T^{(k)}.
\]

Then, we express the metric perturbation as

\[
\delta h_{ij} = \int \frac{d^5k}{(2\pi)^3} e^{-|y|/2\ell} \Phi_k(t,y) T^{(k)}_{ij}, \quad \delta h_{0\mu} = 0.
\]

Putting this ansatz into the linearized Einstein equations \( \delta G_{ab} = (6/\ell^2)\delta g_{ab} \) yields that the \( \Phi_k \) expansion coefficients satisfy a master wave equation for \( y > 0 \):

\[
0 = \left[ \partial_\tau^2 - \partial_\tau^2 + V_k(x) \right] \Phi_k, \quad V_k(x) = k^2 + \frac{15}{4}x^{-2}, (5a)
\]

\[
0 = \left( \partial_x + \frac{3}{2} \right) \Phi_k \text{ at } x = 1. \quad (5b)
\]

1 We note that wave propagation (gravitational and otherwise) around 4-dimensional domain walls and supergravity ‘D-brane’ solutions has been considered by several authors.
The situation we are interested in is shown in Figure 1. We regard the spatial variable $\hat{x}$, defined via a delta-function and the imposition of mirror symmetry:

$$V_k(x) = \kappa^2 + \frac{15}{4} \hat{x}^2 - 3 \delta(\hat{x} - 1), \quad \hat{x} \equiv |x - 1| + 1.$$

So, the problem of perturbing the RS one-brane scenario is equivalent to the problem of wave propagation in a one-dimensional potential.

**Quasinormal resonances** The situation we are interested in is shown in Figure 1. We regard the spatial wavenumber $\kappa$ of the perturbation as given, and we consider a compact pulse of right-moving gravitational radiation centered about $x_0 > 1$ at $t = 0$. This pulse will of course strike the brane and scatter, and we are interested in the late-time features of the reflected waveform. Now, since the potential governing gravity waves is centered about an attractive delta-function, we can reasonably expect some of the incident wave to become trapped and remain localized near the brane forever, while the rest of energy will bounce off and propagate to infinity.

However, all the reflected energy need not be promptly reflected away from the brane. A well-known phenomenon in spectral theory is that of a scattering resonance. This occurs when certain frequency components of the initial pulse become temporarily trapped within the potential. One can think of these modes as metastable bound states: they are initially localized, but they eventually leak out to infinity. In black hole perturbation theory, these metastable states are precisely the QNMs. In that problem, the master equation has the form of a Schrödinger-like equation:

$$\omega^2 \phi_k = -\phi_k'' + V_k(x) \phi_k.$$

If $\omega^2 \neq \kappa^2$, this admits an exact solution that satisfies the boundary condition at the brane:

$$\phi_k(x) = \sqrt{2} \left[ H_1^{(1)}(\mu) H_2^{(2)}(\mu \hat{x}) - H_1^{(2)}(\mu) H_2^{(1)}(\mu \hat{x}) \right].$$

Here, $H_n^{(1)}$ and $H_n^{(2)}$ are Hankel functions of the first and second kind respectively, and $\nu$ is their order. The dimensionless $\mu$ parameter is defined via the dispersion relation

$$\mu^2 = \omega^2 - \kappa^2, \quad |\text{Arg}(\mu)| < \pi.$$

Conventionally, $\mu$ is assumed to be real and positive, and is related to the conventional Kaluza-Klein mass of the fluctuation via $\mu = m \ell$. However, the key to finding a resonant mode is to let $\mu$ range over the complex plane. Consider the asymptotic expansion of the Hankel functions:

$$\phi_k(x) \propto H_1^{(1)}(\mu) e^{-i\mu x} - H_1^{(2)}(\mu) e^{+i\mu x} \quad \text{as } x \to \infty.$$

Depending on the sign of $\text{Re } \mu$ relative to the sign of $\text{Re } \omega$, each term corresponds to either a left or right moving wave at infinity. The next step in the QNM programme is to demand that the asymptotic form of $\phi_k$ is ‘purely outgoing’, which in this context means that we need to set the coefficient of the left-moving wave to zero.

Formally, we are searching for a complex pole of the scattering matrix $S_k(\omega)$, which is defined as the ratio of the amplitudes of outgoing and ingoing radiation at infinity. Clearly, for a pole we need $\mu$ to be a zero of either $H_1^{(1)}$ or $H_1^{(2)}$. For reference, we quote the first four zeros of $H_1^{(1)}$:

$$\mu_1 = -0.419 - 0.577 i, \quad \mu_2 = -3.832 - 0.355 i, \quad \mu_3 = -7.016 - 0.350 i, \quad \mu_4 = -10.174 - 0.348 i.$$

The corresponding zeros of $H_1^{(2)}$ are simply the complex conjugates $\bar{\mu}_n$, and it is easy to confirm that all zeros of both Hankel functions have negative real parts. Therefore, if $\text{Re } \omega > 0$ we must set $H_1^{(1)}(\mu) = 0$ in order to have a purely outgoing wave; that is, we choose $\mu \in \{\mu_n\}$. Conversely, if $\text{Re } \omega < 0$ we need $H_1^{(2)}(\mu) = 0$, or in other words $\mu \in \{\bar{\mu}_n\}$. From this, it follows that there are two families of complementary quasinormal frequencies:

$$\omega_n^{(1)}(\kappa) = + \sqrt{\mu_n^2 + \kappa^2}, \quad H_1^{(1)}(\mu_n) = 0,$$

$$\omega_n^{(2)}(\kappa) = - \sqrt{\mu_n^2 + \kappa^2}, \quad H_1^{(2)}(\bar{\mu}_n) = 0.$$

2 The fact that QNMs are defined by a boundary condition ‘at infinity’ tells us that they are not relevant to two-brane scenarios where the extra dimension is effectively finite. Generally speaking, in order for a system to exhibit metastable resonances there must be some type of mechanism to remove energy from the system, which only really happens in the one-brane case.
FIG. 2: The first twenty quasinormal frequencies of the brane as the spatial wavenumber $\kappa$ is varied from 0 (red) to 25 (purple). Those in the right half of the complex plane correspond to $\omega^{(1)}_n$, while those in the left half are $\omega^{(2)}_n$.

The variation of the first twenty QNMs with $\kappa$ is depicted in Figure 2. Note that in order to evaluate the square root, we need to take the same branch cut as that of the Hankel function; i.e., the negative real axis. In particular, this means $\sqrt{\mu_n^2} = -\mu_n$.

Before moving on, we should mention that the Randall-Sundrum model is not the only 5-dimensional scenario that exhibits resonant behaviour. The Dvali-Gabadadze-Porrati (DGP) model features a Minkowski bulk surrounding a brane with ‘induced gravity’ [7]. There is no zero-mode in this model, but the massive modes sum together in such a way to approximately recover Newton’s law on small scales. This superposition of massive modes is interpreted as a metastable 4-dimensional graviton that decays into the bulk after a finite streaming length along the brane. To the best of our knowledge, the QNM spectrum of the DGP model has not been specifically investigated, although for certain branch cut choice one can find DGP resonances with $\text{Im} \omega < 0$; i.e., unstable fluctuations [8]. An interesting and perhaps fruitful exercise would be to repeat the current calculation for that model.

**Numerical results** So far, we have concentrated on modes that are outgoing at spatial infinity. But the RS potential also supports a normalizable bound state:

$$\Phi^{(0)}_k(\tau, x) = e^{i\omega \tau} e^{-3/2}, \quad \omega = \pm \kappa. \tag{11}$$

The existence of this so-called ‘zero-mode’ is what gives the RS model its main appeal, since it gives rise to GR at low energies. How does the zero mode manifest itself in our scattering problem? To answer, we first look at a numeric simulation of a Gaussian pulse incident on the brane in Figure 3. This is obtained by numerically integrating (5a) for $x > 1$ and with the boundary condition (5b). Our assumed initial data is at $x = 10$ and has a full-width at half maximum (FWHM) of 5, and we take $\kappa = \pi/10$. We see that after the pulse hits the brane, part of the energy is reflected and propagates off to infinity while the other part remains trapped on the brane at $x = 1$. Indeed, a close examination of the brane perturbation shows that after the initial pulse, the waveform is an undamped sinusoid with frequency $\pi/10$. On the other hand, the waveform at $x = 100$ shows that the reflected pulse damps to zero as $\tau \to \infty$. In other words, the incident pulse excites a perpetual and spatially localized perturbation on the brane; i.e., the zero-mode.

The one thing missing from the situation depicted in Figure 3 is any hint of resonant phenomena. The Gaussian pulse immediately excites the zero mode when it strikes the brane, which suggests any quasinormal resonances are sub-dominant to the undamped bound state. Some experimentation reveals that this is true for many different types of Gaussians and values of $\kappa$. The reason is simple: in Fourier space a Gaussian pulse always has a significant low frequency component that will efficiently excite the zero mode. In order to see the QNM influence directly, we should really consider coherent plane wave initial data with a frequency high enough to preferentially excite the first QNM. But we cannot model an infinite plane wave numerically, so we instead use a finite wavetrain, which is essentially a segment of the left-moving wave $\sin(\omega_0 \tau + \mu_0 x)$, with $\omega_0^2 = \mu_0^2 + \kappa^2$. If such a wavetrain is $N$ cycles long, its spectral width is on the order of $\Delta \omega = \omega_0/N$. It is also useful to look at the reflected wave from an off-brane vantage point, because the influence of the zero mode is strongest at $x = 1$. 

FIG. 3: Numeric waveform for $\kappa = \pi/10$ and an incident Gaussian pulse centered at $x = 10$ with a FWHM of 5.
In Figure 4, we show the perturbation evaluated at \( x = 10 \) for this type of initial data with \( \kappa = \pi/10 \) and \( \omega_0 = 0.6 \). The left-hand edge of the wavetrain is initially at \( x = 10 \), and it is 10 wavelengths long. For this choice of \( \kappa \), the first quasinormal mode is at \( \omega_1 = 0.463 + 0.523i \). In this case, the gravitational waveform is analogous to the response of a damped mechanical oscillator subject to a driving force. The natural frequency of the system is that of the zero mode to a driving force. The natural frequency of the system is that of the zero mode \( \omega = \pi/10 \sim 0.314 \), so the wavetrain represents a high frequency forcing term. As the wavetrain reflects off the brane, the perturbation is obliged to oscillate at \( \omega = 0.6 \). By \( t \sim 150 \), the entire reflected wavetrain has travelled past our vantage point, and the driving force is effectively ‘switched off.’ But the waveform does not immediately revert to zero mode vibrations, there is instead a ‘ringdown phase’ where the brane radiates away the high frequency components of the driving force. The upper inset of Figure 4 shows the amplitude squared of the waveform’s Fast Fourier Transform (FFT) during this ringdown. The power spectrum is peaked at the real part of the first QNM at 0.463, which is fairly clear evidence of quasinormal ringing of the brane. (Note that the damping of a QNM means that its power spectrum has a significant width.) After the ringdown phase, the FFT (not shown) is sharply peaked about \( \pi/10 \), suggesting strong domination of the zero mode. In the standard lexicon of black hole perturbation theory, the part of the waveform that follows quasinormal ringing is known as the ‘wave-tail.’ So in the RS braneworld, the wave tail essentially consists of zero mode oscillations. The lower inset in Figure 4 shows that the amplitude of the zero mode signal becomes constant only for late times. Note that a slow variation in the amplitude of oscillatory wave-tails is a familiar feature of potentials that are non-vanishing at infinity. In the future, we will report on the Randall-Sundrum wave-tail in detail.

**Implications** As mentioned above, the principle implication of the existence of QNMs in the RS one-brane scenario is the prediction of a discrete set of metastable massive modes within the continuous Kaluza-Klein tower.\(^3\) If \( \ell = 0.1 \) mm, which is an upper value set by experimental limits on deviations from Newton’s law, the lowest order resonance has mass \( 8.386 \times 10^{-4} \) eV and a half life of \( 4.002 \times 10^{-11} \) seconds; i.e., it is light and short-lived. Ideally, one would look for such resonances in particle accelerators, but in the RS one-brane case the coupling between matter and the Kaluza-Klein states is quite small since the 5-dimensional Planck mass is \( \gtrsim 10^5 \) TeV. This scale suggests that the best place to look for QNMs is in the high-energy regime associated with the early universe; in particular, one can expect the stochastic gravity wave background\(^1\) to carry potentially observable echoes of metastable masses. Of course, the expansion of the brane universe at that epoch will certainly change the spectrum of quasinormal resonances. However, it is reasonable to assume that very high-frequency QNMs will ‘see’ a moving brane to be stationary, and hence our results from the static case will be applicable.

It is interesting, and perhaps instructive, to see what contribution the quasinormal modes make to the effective braneworld Einstein equations. Using the Shiromizu-Maeda-Sasaki formalism\(^12\), we see that QNMs induce ‘dark radiation’ on the brane—essentially the pullback of the 5-dimensional Weyl tensor onto the 4-dimensional spacetime. The QNM contribution to the stress energy of this effective matter field is

\[
\varepsilon_{\mu\nu}^{QNM}(k) = \sum_n A_n \mu_n e^{i\omega_n \tau} T_{\mu\nu}(\vec{k}),
\]  

where \( A_n \) are constants describing the degree to which mode is excited. Note the explicit dependence on the complex mass \( \mu_n \); the massless zero-mode does not enter into the dark radiation. Hence, the quasinormal resonances can be interpreted as a discrete set of brane matter fields prone to decay by tunnelling into the bulk. Returning to the cosmological case, this implies that the principal high frequency bulk effects on early universe cosmological perturbations are via ‘matter’ contributions of the form (12). The quantitative effects of these fields on the cosmic microwave background and large scale structure are yet to be calculated.

Finally, Eq. (12) hints at a possible holographic interpretation of these resonances. According to the AdS/CFT conjecture, the Randall-Sundrum model is

---

\(^3\) Alternatively, one could view the resonant modes as spin-2 fields obeying a complex-valued dispersion relation; i.e., travelling in an absorptive medium. Details of such an interpretation can be found in Ref. 13.
dual to a large-$N$ conformal field theory (CFT) with cutoff living on the brane \[14\]. Specifically, the retarded bulk graviton Green’s function is identified with the generating functional of the CFT. Now, the metastable resonances found here are complex poles of the Green’s function, and hence will play an important role in the boundary theory. In particular, note that $\mathcal{E}_{\mu \nu}$ in the gravitational theory is precisely the stress-energy tensor of the CFT, which suggests how the gravitational QNMs describe how the boundary theory relaxes to equilibrium after some initial disturbance, and also give the timescale for such a relaxation.

**Summary** Just like a black hole, the RS braneworld ‘rings’ with a discrete set of complex frequencies. These metastable bound states, which look like decaying massive gravitons to 4-dimensional observers, should play an important role in any brane phenomena involving bulk degrees of freedom. An interesting question is: how do QNMs manifest themselves in generalizations of the RS scenario? We have recently reported on the resonant behaviour of a spherical Einstein static brane encircling a Schwarzschild-AdS$_5$ black hole \[17\], but the key challenge is characterizing the quasinormal excitations of a moving cosmological brane. We hope to investigate this in the future.

**Acknowledgements** I would like to thank C. Clarkson, K. Koyama, R. Maartens, A. Mennim, F. Piazza, and D. Wands for discussions and NSERC for financial support.