Black Holes in type IIA String on Calabi-Yau
Threefolds with Affine ADE Geometries and
q-Deformed 2d Quiver Gauge Theories

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Abstract

Motivated by studies on 4d black holes and q-deformed 2d Yang Mills theory, and borrowing ideas from compact geometry of the blowing up of affine ADE singularities, we build a class of local Calabi-Yau threefolds (CY\textsuperscript{3}) extending the local 2-torus model \( O(m) \oplus O(-m) \to T^2 \) considered in hep-th/0406058 to test OSV conjecture. We first study toric realizations of \( T^2 \) and then build a toric representation of \( X_3 \) using intersections of local Calabi-Yau threefolds \( O(m) \oplus O(-m - 2) \to \mathbb{P}^1 \). We develop the 2d \( N = 2 \) linear \( \sigma \)-model for this class of toric CY\textsuperscript{3}’s. Then we use these local backgrounds to study partition function of 4d black holes in type IIA string theory and the underlying q-deformed 2d quiver gauge theories. We also make comments on 4d black holes obtained from D-branes wrapping cycles in \( O(m) \oplus O(-m - 2) \to B_k \) with \( m = (m_1, \ldots, m_k) \) a \( k \)-dim integer vector and \( B_k \) a compact complex one dimension base consisting of the intersection of \( k \) 2-spheres \( S^2_i \) with generic intersection matrix \( I_{ij} \). We give as well the explicit expression of the q-deformed path integral measure of the partition function of the 2d quiver gauge theory in terms of \( I_{ij} \).

Key words: Black holes in string theory, OSV conjecture, q-deformed 2d quiver gauge theory, topological string theory.
# 1 Introduction

Few years ago, Ooguri, Strominger and Vafa (OSV) have made a conjecture [1] relating the microstates counting of 4d BPS black holes in type II string theory on Calabi-Yau threefolds $X_3$ to the topological string partition function $Z_{\text{top}}$ on the same manifold. The equivalence between the partition function $Z_{\text{brane}}$ of large $N$ D-branes, and that of the associated 4d BPS black hole $Z_{\text{BH}}$ leads to the correspondence $Z_{\text{brane}} = |Z_{\text{top}}|^2$ to all orders in $1/N$ expansion. OSV conjecture has brought important developments on this link: it provides the non-perturbative completion of the topological string theories [2]-[14] and gives a way to compute the corrections to 4d $\mathcal{N}=2$ Bekenstein-Hawking entropy [10, 15]. OSV relation has been extended in [7, 16] to open topological strings, which capture BPS states data information on D-branes wrapped on Lagrangian submanifolds of the Calabi-Yau 3-folds.

Evidence for OSV proposal has been obtained by using local Calabi-Yau threefolds and some known results on 2d $U(N)$ Yang-Mills theory [17]. It has been first tested in [2] by considering configurations of D-branes wrapped cycles $O(m) \oplus O(-m) \to T^2$ for $m$ a positive definite integer. Then it has been checked in [18] by using wrapped D-branes in a vector bundle of rank 2 non trivial fibration over a genus $g$-Riemann surface $O(2g + m - 2) \oplus O(-m) \to \Sigma_g$. It has been shown in these studies that the BPS black hole partition function localizes onto field configuration which are invariant under $U(1)$ actions of the fibers. In this way, the 4d gauge theory reduces effectively to q-deformed Yang-Mills theory on $\Sigma_g$. These works have been recently enlarged in [19] by considering local Calabi-Yau manifolds with torus symmetries such as local $\mathbb{P}^2 = O(-3) \to \mathbb{P}^2$, local $F_0 = O(-2, -2) \to \mathbb{P}_B^1 \times \mathbb{P}_P^1$ and $ALE \times C$. In the last example, the $A_k$ type ALE space is given by gluing together $(k+1)$ copies of $C^2$ viewed as a real two dimensional base fibered by torus $T^2$. Other related works have been developed in [20, 21, 22, 23, 24, 25, 26, 27, 28]. M-theory and $AdS_3/CFT_2$ interpretations of OSV formula have been also studied in [29].

In this paper we contribute to the program of testing OSV conjecture for massive 4d black holes on toric Calabi-Yau threefolds in connection with q-deformed quiver gauge theories in two dimensions. This study has been motivated by looking for a 2d $\mathcal{N}=2$ supersymmetric gauged linear sigma model of $O(m) \oplus O(-m) \to T^2$. More precisely we consider a special class of local Calabi-Yau threefolds which combine features of both $O(m) \oplus O(-m) \to T^2$ and $O(m) \oplus O(-m-2) \to S^2$, and involves toric graphs that look like affine Dynkin diagrams and beyond. Recall that affine Dynkin geometries are known to be described by elliptic fibration over $C^2$ and lead to $\mathcal{N}=2$ conformal quiver gauge theories in 4d space-time with gauge groups $G = \Pi_i U(s_i M)$, where the positive integers $s_i$ are the Dynkin weights of affine Kac-Moody algebras. Borrowing the method of the above quiver gauge theories [30, 31, 32] and using the results of [2], we engineer a new class of local Calabi-Yau threefolds that enlarges further the class of CY3s used before and that agrees with OSV conjecture.

Our study gives moreover an explicit toric representation of $O(m) \oplus O(-m) \to T^2$ especially if one recalls that $T^2$ viewed as $S^1 \times S^1$ does not have, to our knowledge, a simple nor unique toric realization. As we know real skeleton base of toric diagrams representing toric manifolds requires at least one 2-sphere $S^2$ which is not the case for the simplest $S^1 \times S^1$ geometry. In the analysis to be developed in this study, the 2-torus will be realized by using
special linear combinations \[\Delta_{n+1}\] of intersecting 2-spheres with same homology as a 2-torus. The positive integer \(n \geq 1\) refers to the arbitrariness in the number of \([S^2_i]\)’s one can use to get the elliptic curve class \([T^2]\). Among our main results, we mention that 2d quiver gauge theories, associated with BPS black holes in type IIA string on the local CY3’s we have considered, are classified by the ”sign” of the intersection matrix \(I_{ik}\) of the real 2-spheres \(S^2_i\) forming the compact base

\[
I_{ik} = [S^2_i] [S^2_k], \quad i, k = 0, ..., n.
\]

According to whether \(\sum_k I_{ik} u_k > 0, \sum_k I_{ik} u_k = 0\) or \(\sum_k I_{ik} u_k < 0\) for some positive integer vector \((u_k)\), we then distinguish three kinds of local models. For the second case

\[
\sum_k I_{ik} u_k = 0,
\]

the \(u_k\)’s are just the Dynkin weights of affine Kac-Moody algebras and the corresponding 2d quiver gauge theory is non-deformed in agreement with the result of \([2]\). The above relation corresponds then to the case where the genus g-Riemann surface is a 2-torus; i.e

\[
2g - 2 = 0, \quad \leftrightarrow \quad g = 1.
\]

For the two other cases, the gauge theory is q-deformed and recovers, as a particular case, the study of \([19]\) dealing with ALE spaces.

On the other hand, this study will be done in the type IIA string theory set up. So we shall also use our construction to complete partial results in literature on field theoretic realization using 2d \(\mathcal{N} = (2, 2)\) linear sigma model for affine ADE geometries with special focus on the \(\tilde{A}_n\) case. To our knowledge, this supersymmetric 2d field realization of local Calabi-Yau threefolds has not been considered before.

The organization of paper is as follows: In section 2, we begin by recalling general features on toric graphs. Then we study the toric realizations of local \(T^2\) using the techniques of blowing up of affine ADE geometries. We study also the field theoretic realization of the non trivial fibrations of local \(T^2\) which corresponds to implementing framing property \([38]\).

In section 3, we develop the \(\mathcal{N} = 2\) supersymmetric gauged linear sigma model describing the local torus geometry. This study gives an explicit field theoretic realization of geometric objects such as the surface divisors of the local 2-torus and their edge boundaries in terms of field equations of motion and vevs. In section 4, we construct the 4d BPS black holes in type IIA string by considering brane configurations using D0-D2-D4-branes in the non compact 4-cycles. We show, amongst others, that the gauge theory of the D-branes, which is dual to topological strings on the Calabi-Yau threefold, localizes to a ”q-deformed” 2d quiver gauge theory on the compact part of the affine ADE geometry and test OSV conjecture. More precisely, we show that the usual power \((2g - 2)\) of the weight of the deformed path integral measure for \(\mathcal{O}(2g + m - 2) \oplus \mathcal{O}(-m) \to \Sigma_g\) gets replaced, in case of 2d quiver gauge theories, by the intersection matrix \(I_{ij}\). There, we show that the property \((2g - 2) = 0\) for \(g = 1\) corresponds to the identity \(\sum_j I_{ij}s_j = 0\) of affine Kac-Moody algebras. Motivated by this link, we study 4d black holes based on D-branes wrapping cycles in \(\mathcal{O}(m) \oplus \mathcal{O}(-m - 2) \to B_k\) with \(m = (m_1, ..., m_k)\) an integer vector and where \(B_k\) is a complex one dimension base consisting of the intersection of \(k\) 2-spheres \(S^2_i\) with generic intersection matrix \(I_{ij}\). In section 5, we give our conclusion and outlook.
2 Toric realization of local $T^2$

In this section, we build toric representations of the class of local 2-torus $O(m) \oplus O(-m) \to T^2$ by developing the idea outlined in the introduction. There, it was observed that although, strictly speaking, $T^2 = S^1 \times S^1$ is not a toric manifold (base reduced to two points), it may nevertheless be realized by gluing several 2-spheres in very special ways. Before showing how this can be implemented in the above local CY3, recall that the study of local threefold geometry

$$O(m) \oplus O(-m) \to T^2 \quad (2.1)$$

is important from several views. It has been used in [2] to test OSV conjecture [1] and was behind the study of several generalizations, in particular

$$O(m + 2g - 2) \oplus O(-m) \to \Sigma_g. \quad (2.2)$$

The novelty brought by these class of local CY3s stems also from the use of the non trivial rank 2 fiber $O(p) \oplus O(-m)$, with $p = m + 2g - 2$ rather than $O(2g - 2) \oplus O(0)$. These non trivial fibers, which were motivated by implementing twisting by framing [38], turn out to play a crucial role in the study of 4d BPS black holes from type IIA string theory compactification. Generally, the class of local CY3s eq (2.2) which will be considered later (section 4), is mainly characterized by two integers $m$ and the genus $g$ and may be generically denoted as follows

$$X^{(k_3, k_2, k_1)}_3, \quad g = 0, 1, \ldots; \quad m \in \mathbb{Z}. \quad (2.3)$$

Here $k_1 = 2 - 2g$, $k_2 = -m$ and $k_3 = m + 2g - 2$ satisfy the Calabi-Yau condition $\sum_i k_i = 0$ leaving only two free integers $m$ and $g$. By making choices of these integers one picks up a particular local CY3. For $g = m = 0$ and $g = 1$, $m = 0$ for example, we have $O(0) \oplus O(-2) \to \mathbb{P}^1$ and $O(0) \oplus O(0) \to T^2$ respectively. The local CY3s (2.3) may be also viewed as a line bundle $\mathcal{L}^{(m+2g-2)}_{\mathcal{D}(m,g)}$ of the complex two dimensional divisor

$$[\mathcal{D}(m,g)] = O(-m) \to \Sigma_g. \quad (2.4)$$

This local complex surface $[\mathcal{D}(m,g)]$ has a compact curve $\Sigma_g$ with the following intersection number

$$[\mathcal{D}(m,g)] \cdot [\Sigma_g] = m + 2g - 2. \quad (2.5)$$

In the case where $\Sigma_g$ is a 2-torus ($g = 1$), the above two integers series of local CY3 reduces to the one integer threefold series $X^{(m,-m,0)}_3$. The previous non compact real 4-cycles are then given by [2]

$$[\mathcal{D}(m,g)] = O(-m) \to T^2, \quad (2.6)$$

and their intersection number with the 2-torus class $[T^2]$ is $[\mathcal{D}(m,g)] \cdot [T^2] = m$.

2.1 Toric realization

Our main objectives in this subsection deals with the two following: (1) Build explicit toric realisations of the local 2-torus by using particular realisations of the real 2-cycle $[T^2]$. These realisations, which will be used later on, have been motivated from results on blowing
up of affine ADE singularities of ALE spaces and geometric engineering of 4d $\mathcal{N} = 2$ super QFTs \cite{30,31,32}. It gives a powerful tool for the explicit study of the special features of local 2-torus and allows more insight in the building of new classes of local Calabi-Yau threefolds for testing OSV conjecture.

(2) Use the result of the analysis of point (1) to complete partial results in literature on type IIA geometry with affine ADE singularities. More precisely, we construct the 2d $\mathcal{N} = 2$ supersymmetric gauged linear sigma model

$$
\mathcal{S}_{2d}^{\mathcal{N}=2} \sim \int d^2 \sigma d^4 \theta \left( \sum_i \Phi_i^+ e^{\sum_a q_i^a \Phi_i} + \sum_a \xi^a V_a \right),
$$

giving the field realization of local 2-tori. This construction to be developed further in section 3, will be used for the two following:

(a) Work out explicitly the results of (q-deformed) 2d YM theories on $T^2$ and give their extensions to 2d quiver gauge theories on the elliptic curve realized a linear combination of intersecting 2-spheres.

(b) Study partition function properties of 4d BPS black holes along the lines of \cite{29} and ulterior studies \cite{19,25,26,27,28} to test OSV conjecture.

2.1.1 General on toric graphs

In building toric realization of local CY$^3$s \cite{33,34,35,36,37}, one encounters few basic objects that do almost the complete job in striking analogy with the work done by Feynman graphs in perturbative QFTs. In particular, one has:

(i) "Propagators" given by the toric graph of the real 2-sphere $S^2$. It corresponds to the two points free field Green function in the language of quantum field theory (QFT). Recall that

![Figure 1](image.png)

Figure 1: (a) Toric graph of a compact 2-sphere with Kahler modulus $r \neq 0$. (b) The fattening of compact 2-sphere where the circle $S^1$ is represented. $S^1$ shrinks at the fixed points of the U(1) action.

the real 2-sphere $S^2$ is given by the compactification of the complex line $C$. The latter can be realized (polar coordinates) as the half line $R_+$ with fiber $S^1$ that shrinks at the origin. The compactification of $C$; i.e $\mathbb{P}^1$ the complex one dimensional projective space, is obtained by restricting $R_+$ to a finite straight line (a segment) which is interpreted as a propagator in the language of QFT Feynman graphs \cite{38}. Alike, we distinguish here also two situations:

(a) Finite (internal) lines which are associated with compact 2-spheres and are interpreted in terms of propagating closed string states.

(b) Infinite (external) lines with non compact 2-spheres (discs/ complex plane) and are used to implement open string state contributions to topological string amplitudes. These open string states end on D-branes.
With these propagators, one already build complex one and two dimensional toric manifolds as shown on figure below. To construct CY$^3$, we need 3-vertices which we discuss in the next.

Figure 2: (a) Toric quiver of two intersecting 2-spheres, (b) the corresponding fat toric graph where we have also represented the circles associated with the two U(1) toric actions.

(ii) The 3-vertex can be thought of locally as the intersection of the three complex lines of $C^3$ \[38\]. This vertex plays a crucial role in building local CY$^3$s. For instance $O(-3) \rightarrow \mathbb{P}^2$ has three the 3-vertices corresponding to the three fixed points of the $U^3(1)/U(1)$ toric actions. The edge propagators (2-spheres) are fixed under $U^2(1)$ subgroups of $U^4(1)/U(1)$

Figure 3: Toric graph of $O(-3) \rightarrow \mathbb{P}^2$. The compact part is $\mathbb{P}^2$. It consists of three intersecting $\mathbb{P}^1$'s and three vertices. (a) Figure (on left) represents real skeleton. (b) Figure (on right) gives its fattening.

With these objects one can build other local CY3s. Using fat propagators and vertices we can also have a picture on their internal topology. One may also describe its even dimensional homology cycles. Real 2-cycles $C_i$ of local CY3s are represented by linear combinations of segments and real 4-cycles $D_i$ (divisors of CY$^3$s) by 2d polygons: a triangle for $\mathbb{P}^2$, a rectangle for $\mathbb{P}^1_{fiber} \times \mathbb{P}^1_{base}$ and so on. Compact 4-cycles have then finite size.

The power of the toric quiver realization of threefolds comes also from its simplicity due to the fact that the full structure of toric CY$^3$ quiver diagrams is basically captured by the lines (2-cycles) since boundaries of divisors (4-cycles) are given by taking cross products of pairs of straight line generators.

Note that, though $\mathbb{P}^1$ is not a Calabi-Yau submanifold since its first Chern class is $c_1(\mathbb{P}^1) = 2$, this one dimensional complex projective space, together with the vertex, are the basic objects in drawing the 2d graphs of toric manifolds. Note also that torii $S^1$, $T^2$ and $T^3$ of CY$^3$ appear in this construction as fibers and play a fundamental role in the study of topological string theory amplitudes \[38, 39, 40\].
2.1.2 Toric realisations of $X_3^{(m,-m,0)}$

It has been shown in [2] that the one integer series of spaces $X_3^{(m,-m,0)} = \mathcal{O}(m) \oplus \mathcal{O}(-m) \to T^2$ is a toric local Calabi-Yau threefold used to check OSV conjecture. From the above study on toric graphs it follows that, a priori, one should be able to draw its corresponding toric quiver diagram. However, unlike the 2-sphere, the usual 2-torus $S^1 \times S^1$ has no simple toric graph realization. Then the question is what kind of 2-torii $T^2$ are involved in $X_3^{(m,-m,0)}$?

Here below, we would like to address this problem by using the graphic method of toric geometry. In particular, we develop a way to build toric graphs representing classes of $[T^2]$. This will be done by realizing $[T^2]$ in terms of intersecting 2-spheres $S^2_i$ by expressing $[T^2]$ as a "sum" of intersecting 2-spheres,

$$[T^2] \sim \sum_i u_i [S^2_i], \quad u_i \in \mathbb{Z}_+, \quad \text{and thinking about } X_3^{(m,-m,0)} \text{ as a special limit of a family of local CY3s}$$

$$X_3^{(m,-m,0)} \to X_3^{(m,-p,p-m)} = \mathcal{O}(m) \oplus \mathcal{O}(-p) \to \Delta_{n+1} \quad (2.8)$$

where $m$ and $p$ are some $(n+1)$-dimensional integer vectors given by

$$m = (m_0, m_0, ..., m_n), \quad p = (p_0, p_0, ..., p_n). \quad (2.9)$$

In this way the Calabi-Yau condition reads

$$p_i - m_i = 2, \quad i = 0, ..., n, \quad (2.10)$$

which we denote formally as $p - m = 2$. Then, we extend the obtained results to build the toric graphs describing $X_3^{(m,-p,2)}$ and their even real homology cycles. Here it is interesting to note the two following:

(a) This construction is important since once we have the toric quivers, one can use them for different purposes. For instance, we can use the toric graphs in the topological vertex method of [38, 40] to compute explicitly the partition functions of the topological string on $X_3^{(m,-p,2)}$.

(b) The local CY$^3$ we propose $X_3^{(m,-p,p-m)}$ is not exactly $X_3^{(m,-m,0)}$ considered in [2]; it is more general. These manifolds have different Kahler moduli spaces and their $U(1) \times U(1)$ isometry groups are realized differently. To fix the idea, think about the homogeneity group of the compact geometry $\Delta_{n+1}$ of eq.(2.8) as

$$U(1) \times U^{n+1}(1) U^n(1) \sim U(1) \times U(1), \quad n > 1. \quad (2.11)$$

To proceed we shall deal separately with base $\Delta_{n+1}$ and the two line fibers of $X_3^{(m,-p,p-m)}$. We first look for toric quivers to describe the $\Delta_{n+1}$ class and $\mathcal{O}(\pm k_0, ..., \pm k_n)$ independently. Then we use 3-vertex of $C^3$ and the Calabi-Yau condition to glue the various pieces. At the end, we get the right real 3-dimensional toric graph of our local CY$^3$s and, by fattening, the topology of $X_3^{(m,-p,2)}$. This approach is interesting since one can control completely the engineering of the toric quivers of local Calabi-Yau threefolds. Moreover, seen that $\mathcal{O}(\pm k_0, ..., \pm k_n)$ are line bundles, their toric graphs are mainly the toric graphs of $\mathcal{O}(\pm m)$ which are locally given by $C$
Figure 4: Figure (a) describes the toric graph for $\Delta_6$ with six intersecting 2-spheres. Figure (b) gives its fattening where the $S^1_i$ circles above the spheres $S^2_i$ and the intersection points $P_i$ are also represented.

patches. What remains is the determination of toric quiver of $\Delta_{n+1}$ class. In the figures given below, we develop two illustrating examples. The first one (figure 4) concerns the compact base $\Delta_6$ using six intersecting 2-spheres in the same spirit one uses for the blowing up of singularities of ALE complex surfaces. The second example (figure 5) has in addition external non compact lines.

Figure 5: This figure represents the fattening of the toric graph of $\Delta_6$ with 3-vertices and external legs ending on D branes discs.

In what follows, we show that these toric graphs and their fattening constitutes in fact particular topologies of infinitely many possible graphs.

(a) Solving the constraint equation of 2-torus homology:

As noted before, the question of drawing a toric graph for $T^2 = S^1 \times S^1$ seems to have no sense at first sight. This is because the basic (irreducible) real 2-cycle in toric geometry is $[\mathbb{P}^1] \sim S^2$ with self intersection

$$[\mathbb{P}^1] \cdot [\mathbb{P}^1] = -2.$$ (2.12)

Its toric graph is a straight line segment with length given by the size of the 2-sphere (Kahler modulus $r$). The 2-torus has a zero self-intersection

$$[T^2] \cdot [T^2] = 0,$$ (2.13)

and a priori has no simple nor unique toric diagram. Tori $T^n$ are generally speaking associated with $U^n (1)$ phases of complex variables. For instance, on complex line $C$ with local coordinate $z$, the unit circle $S^1$ is given by $|z| = 1$ and the $U (1)$ symmetry acts as $z \rightarrow e^{i\theta}z$. This circle and the associated $U (1)$ symmetry are exhibited when fattening toric graphs as shown on previous
To build a toric representation of $T^2$; but now viewed as $\Delta_{n+1}$, we use intersecting 2-spheres with particular combinations as

$$\Delta_{n+1} = \sum_{i=0}^{n} \epsilon_i \left[ C^i \right], \quad C^i \sim \mathbb{P}^1_i,$$

where the positive integers $\epsilon_i$ are obtained by solving eq(2.13). Denoting by $I_{ij}$ the intersection matrix of the 2-cycles $C^i$,

$$[C^i] \cdot [C^j] = I^{ij}, \quad I^{ii} = -2,$$

then the condition to fulfil eq(2.13) is

$$\sum_{i,j=0}^{n} \epsilon_i I^{ij} \epsilon_j = \sum_{i=0}^{n} \epsilon_i \left( \sum_{j=0}^{n} I^{ij} \epsilon_j \right) = 0, \quad \epsilon_k \in \mathbb{N}.$$

A solution of this constraint relation is given by taking

$$I^{ij} = -K_{ij}(\hat{g}),$$

as minus the generalized Cartan matrix of affine Kac-Moody algebras $\hat{g}$. In this case, the positive integers $\epsilon_j$ are interpreted as the Dynkin weights and the topology of the 2-torus is same as the affine Dynkin diagrams. In particular for the simplest case of simply laced affine ADE are reported below.

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**Figure 6:** Affine simply laced ADE Dynkin diagrams. In homology language, dots represent the 2-spheres and links the intersections. In toric language, the Poincaré dual of these diagrams are associated with toric graphs of 2-torii.

Therefore, there are infinitely many toric quivers realizing $T^2$ in terms of intersecting 2-spheres. But to fix the ideas we will mainly focus on the series based on affine $\hat{A}_n$ Kac-Moody algebras. In this case the elliptic curve is

$$\Delta_{n+1} = \sum_{i=0}^{n} \epsilon_i \left[ C^i \right], \quad \epsilon_i = 1.$$
It involves \((n + 1)\) real 2-cycles with intersection matrix
\[
C_iC_j = -\hat{K}_{ij}, \quad i, \ j = 0, 1, ..., n,
\]
\[
\hat{K}_{ij} = \delta_{i-1,j} - 2\delta_{ij} + \delta_{i+j}, \quad n + 1 \equiv 0.
\] (2.19)

The curve \(\Delta_{n+1}\) has the homology of a 2-torus realized by the intersection of \((n + 1)\) 2-spheres with the topology of Dynkin diagram of affine \(\hat{A}_n\).

The Kahler parameter \(r\) of the curve \(\Delta_{n+1}\) defined as
\[
r = \int_{\Delta_{n+1}} \omega,
\] (2.20)
with \(\omega\) being the usual real Kahler 2-form, is given by the sum over the Kahler parameters \(r^i\) of the 2-cycles \(C^i\) making up \(\Delta_{n+1}\). We have
\[
\omega = \sum \ r^i \omega_i, \quad \int_{C^i} \omega_j = \delta^i_j
\] (2.21)
and so the Kahler modulus of \(\Delta_{n+1}\) is given by
\[
r = \sum_{i=0}^{n} \int_{C_i} \omega_i = \sum_{i=0}^{n} r_i.
\] (2.22)

In this computation we have ignored the volume of the intersection points of the 2-spheres \(C^i\) and \(C^j\) since they are isolated points and moreover their volumes vanish in any case. To fix the ideas, we shall set
\[
r \geq r_0 \geq r_1 \geq ... \geq r_n \geq 0,
\] (2.23)
and for special computation, in particular when we study the path integral of the partition function on quiver gauge theory on \(\Delta_{n+1}\) dual to 4d black hole (section 4), we will in general sit at the moduli space point where
\[
r_i = \frac{r}{n + 1}.
\] (2.24)

In all these cases, the Kahler moduli \(r_i\) are positive. We shall also suppose that we are away from \(r = 0\) describing the singularity of the curve \(\Delta_{n+1}\) and where full non abelian gauge symmetry of quiver theory is restored.

(b) 4-cycles of \(O(m) \oplus O(-p) \to \Delta_{n+1}\)

Using the above realization, one can go ahead and build the 4-cycle and the local Calabi-Yau threefold. Viewed as a whole, the non compact 4-cycle is
\[
D: O(-p) \to \left( \sum_{i=0}^{n} \left[ S_i^2 \right] \right),
\] (2.25)
with toric graph as given below.

Now using eq(2.13), the relation \([D] \cdot [\Delta_{n+1}] = m\) giving the intersection number which reads as
\[
[D] \cdot \left( \sum_{i=0}^{n} \left[ S_i^2 \right] \right) = m
\] (2.26)
Figure 7: Toric graph of a non compact 4-cycle of the local Calabi-Yau Threefold. The compact part consists of eight intersecting 2-spheres.

and splitting $m$ as

$$m = \sum_{i=0}^{n} (m_i - 2), \quad p_i = m_i - 2. \tag{2.27}$$

It is not difficult to see that $[\mathcal{D}]$ can be decomposed as follows:

$$[\mathcal{D}] = \sum_{i=0}^{n} [\mathcal{D}_i], \quad \mathcal{D}_i = \mathcal{O}(-p_i) \rightarrow S_i^2 \tag{2.28}$$

with the property

$$[\mathcal{D}] \cdot [S_i^2] = m_i - 2. \tag{2.29}$$

If we take $m$ a positive integer, then we should have

$$\sum_{i=0}^{n} m_i > 2(n + 1) \tag{2.30}$$

as required by positivity of the intersection number.

(c) Toric graph of $X^3_{m,-p,2}$

Similarly, we can build the toric graph of the local Calabi-Yau threefold $X^3_{m,-p,2}$. Using the realization eq(3.20) and the splitting eq(2.27), the above local CY$_3$ reads as

$$\mathcal{O}(m_0, \ldots, m_n) \oplus \mathcal{O}(-p_0, \ldots, -p_n) \rightarrow \left( \sum_{i=0}^{n} S_i^2 \right), \tag{2.31}$$

with intersection matrix

$$[S_i^2] \cdot [S_j^2] = -\hat{A}_{ij}. \tag{2.32}$$

The fibers $\mathcal{O}(\pm k_0, \ldots, \pm k_n)$ carry charges under the various $U(1)$ gauge symmetries of the individual 2-spheres $S_i^2$. The total charge is given by eq(2.27). With these results at hand, we are now in position to proceed forward and study the field theoretical representation of the above class of local CY$_3$s by using the method of 2d $\mathcal{N} = 2$ supersymmetric gauged linear sigma model.
2.2 Supersymmetric field model

Here we develop the study of type IIA geometry of $X_3^{(m,-p,2)}$. To do so, we use known results on $2d \mathcal{N} = 2$ supersymmetric gauged linear sigma model formulation and take advantage of our construction to also complete partial ones on type IIA geometries based on standard affine models as well as non trivial fibrations.

To begin, recall that in the $2d \mathcal{N} = 2$ supersymmetric sigma model framework, the field equations of motion of the auxiliary fields $D_a$ of the gauge supermultiplets $V_a$

$$\frac{\delta S^{\mathcal{N}=2}_{2d}}{\delta V_a} = \sum_i q_i^a \Phi_i \Phi_i^* - r^a = 0 \quad (2.33)$$

define the type IIA geometry. This method has been used in literature to deal with K3 surfaces with ADE geometries. But here we would like to extend this method to the case of the local geometry $X_3^{(m,-p,2)}$. To that purpose, we shall proceed as follows:

1. Study first the field realization on two special examples. This analysis, which will be given in present subsection, allows to set up the procedure. Then we give useful tools and illustrate the method.

2. Develop, as a next step, the general field theoretical $2d \mathcal{N} = 2$ supersymmetric gauged linear sigma model of eq(2.31). This is a more extensive and will be given in next section. Actually this is one of the results of the present study.

2.2.1 Type IIA model for $\mathcal{O}(m) \oplus \mathcal{O}(-m - 2) \rightarrow S^2$

To start note that for $m = 0$, this local Calabi-Yau threefold describes just the usual A1 geometry on the complex line. The variety has been studied extensively in literature; see [41] for instance. For non zero $m$ the situation is, as far as we know, new and its supersymmetric linear sigma model can be obtained by considering a $U(1)$ gauge field $V$ and four chiral superfields $\Phi_i$ with gauge symmetry

$$\Phi_i \rightarrow \Phi_i' = e^{iq_i A} \Phi_i, \quad i = 1, \ldots, 4, \quad (2.34)$$

with charges

$$q_i = (1, 1, -2 - m, m) \quad (2.35)$$

satisfying the CY condition $\sum_{i=1}^4 q_i = 0$. The gauge invariant superfield action $S^{\mathcal{N}=2}_{2d}$ of this model reads as

$$S^{\mathcal{N}=2}_{2d} \sim \int d^2 \sigma d^4 \theta \sum_{i=1}^4 \Phi_i^+ e^{q_i V} \Phi_i - r \int d^2 \sigma d^4 \theta V \quad (2.36)$$

and the field equation of motion $\frac{\delta S^{\mathcal{N}=2}_{2d}}{\delta V_a} = 0$ of the gauge field leads to

$$|\phi_1|^2 + |\phi_2|^2 - (2 + m) |\phi_3|^2 + m |\phi_4|^2 = r. \quad (2.37)$$

This equation describes indeed $\mathcal{O}(m) \oplus \mathcal{O}(-m - 2) \rightarrow S^2$. It has four special divisors $\phi_i = 0$ while the base 2-sphere corresponds to

$$|\phi_1|^2 + |\phi_2|^2 = r. \quad (2.38)$$
In case where \( m \) is positive definite, this geometry can be also viewed as describing the line bundle \( \mathcal{O}(-m - 2) \) over the weighted projective space \( \mathbb{WP}^2_{(1,1,m)} \). Note that for \( m = 1 \), one gets the normal bundle of \( \mathbb{P}^2 \) and the local Calabi-Yau threefold coincides with

\[
\mathcal{O}(-3) \to \mathbb{P}^2.
\]

(2.39)

Note also that for \( m = -1 \), one has the resolved conifold

\[
\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1.
\]

(2.40)

It is interesting to note here that resolved conifold can be realized from normal bundle on \( \mathbb{P}^2 \) just by sending to infinity one of the edges of \( \mathbb{P}^2 \). Note finally that for \( m = 0, -2 \), one has the \( A_1 \) geometry fibered on the complex line \( C \).

### 2.2.2 Example: \( \mathcal{O}(m_1,m_2) \oplus \mathcal{O}(-p_1,-p_2) \to C_2 \)

Following the same method, one can also build the type IIA sigma model for this local CY\(^3\) \((p_i = m_i + 2)\) based on two intersecting 2-spheres \( S^2_1 \) and \( S^2_2 \)

\[
C_2 = S^2_1 + S^2_2
\]

(2.41)

with intersection \( S^2_1 \cap S^2_2 = \{ \text{a point } P \} \) modulo gauge transformations. The sigma model involves two \( U(1) \) gauge fields \( V_1, V_2 \) and five chiral superfields \( \Phi_i \). Denoting by \( X_1, X_2, X_3 \) the bosonic field components parameterising the compact 2-cycle \( C_2 \) and by \( Y_1, Y_2 \) the complex variables parameterising \( \mathcal{O}(-m_1 - 2, -m_2 - 2) \) and \( \mathcal{O}(m_1, m_2) \) respectively, the two sigma model equations are given by

\[
|X_1|^2 + |X_2|^2 - (2 + m_1) |Y_1|^2 + m_1 |Y_2|^2 = r_1
\]

\[
|X_2|^2 + |X_3|^2 - (2 + m_2) |Y_1|^2 + m_2 |Y_2|^2 = r_2.
\]

(2.42)

In these equations, one recognizes two \( SU(2) / U(1) \) relations describing the 2-spheres \( S^2_1 \) and \( S^2_2 \) associated with taking \( Y_1 = Y_2 = 0 \); i.e

\[
|X_1|^2 + |X_2|^2 = r_1
\]

\[
|X_2|^2 + |X_3|^2 = r_2.
\]

(2.43)

Notice that eqs(2.42) involve five complex (10 real) variables which are not all of them free since they are constrained by two real constraint relations (the \( D_1 \) and \( D_2 \) auxiliary field equations of motion) and \( U(1) \times U(1) \) gauge symmetry

\[
X_1 \equiv X_1 \exp(i \vartheta_1), \quad X_2 \equiv X_2 \exp(i \vartheta_1 + i \vartheta_2), \quad X_3 \equiv X_3 \exp(i \vartheta_2),
\]

(2.44)

where \( \vartheta_i \) are the two gauge group parameters. At the end one is left with \((10 - 2 - 2)\) degrees of freedom which can be described by three independent complex variables. Notice also that the spheres \( S^2_1 \) and \( S^2_2 \) intersect at

\[
(X_1, X_2, X_3) = (r_1, 0, r_2).
\]

(2.45)
$X_2 = 0$ is the fixed point under $U(1) \times U(1)$ gauge symmetry (toric action) and up to a gauge transformation, the same is valid for $X_1 = r_1$ and $X_3 = r_2$. Indeed if parameterising $X_1 = |X_1|e^{-i\varphi}$ and $X_3 = |X_3|e^{-i\psi}$, one can usually set $\varphi = \vartheta_1$ and $\psi = \vartheta_2$ by using $U(1) \times U(1)$ gauge invariance. Then setting $X_2 = 0$ in eqs (2.43), one discovers eq (2.45). Notice finally that this construction generalizes easily to the case of an open chain

$$C_n = \sum_{i=1}^{n} S_i^2, \quad C_nC_n = -2, \quad (2.46)$$

involving several intersecting 2-spheres $S_i^2$ with $[S_i^2] \cdot [S_j^2] = -2$ and $[S_i^2] \cdot [S_{i\pm 1}^2] = 1$ otherwise zero. The sigma model field equations describing this complex one dimension curve read as

$$|X_i|^2 + |X_{i+1}|^2 = r_i, \quad 1 \leq i \leq n, \quad (2.47)$$

involving $(n+1)$ complex field variables $X_i$ constrained by $n$ complex constraint equations.

---

**Figure 8:** Open chain describing a typical compact base. It involves several intersecting 2-spheres with one intersection point. Figure in top involves straight lines and figure in bottom its fattening by representing the circles above the 2-spheres.

---

### 3 More on 2d $\mathcal{N} = 2$ sigma model description

So far we have considered special examples of type IIA realization of local CY$^3$ as an introduction to the important case where the previous (compact) open chain $C_n$ gets closed by the adjunction of an extra 2-sphere $S_0^2$,

$$C_n \rightarrow \hat{C}_{n+1} \quad (3.1)$$

which we denote also as $\Delta_{n+1}$. In this section, we give the type IIA description of these kinds of local CY$^3$s.

To proceed, we start from the previous open chain $C_n$ and add an extra 2-sphere $S_0^2$,

$$\Delta_{n+1} = S_0^2 + C_n, \quad n > 2, \quad (3.2)$$

with the following features,

$$[S_0^2] \cdot [S_0^2] = -2, \quad [S_0^2] \cdot [C_n] = 2. \quad (3.3)$$
Figure 9: Toric graph of $\Delta_{n+1}$ realized by using affine $\tilde{A}_n$ Dynkin diagrams. It is obtained by adding an extra 2-sphere in same manner as we do in building affine Dynkin diagrams from ordinary ones.

In this case, we have

$$[\Delta_{n+1}] : [\Delta_{n+1}] = 0,$$

(3.4)

as required by the homology property of the 2-torus. One of the solutions of eq(2.42) is given by

$$[S_0^2] : [S_1^2] = 1, \quad [S_0^2] : [S_n^2] = 1,$$
$$[S_0^2] : [S_i^2] = 0, \quad i \neq 1, n.$$  (3.5)

Other solutions are possible and are associated with Dynkin diagrams of affine Kac-Moody algebras. To write down the explicit sigma model field equations of the geometry $O(m) \oplus O(-p) \to \Delta_{n+1}$, we shall first write down the equations for $\Delta_{n+1}$ and then give the general result.

### 3.1 Sigma model eqs for $\Delta_{n+1}$

A way to get the sigma model field equations for the 2-torus $[\Delta_{n+1}]$, preserving the constraint equations; in particular the dimensionality of $\Delta_{n+1}$ and $[\Delta_{n+1}] : [\Delta_{n+1}] = 0$, is to embed it in the $C^{n+3}$ with $(n+2)$ constraint complex relations resulting from $(n+2)$ real equations of motion and $(n+2)$ abelian gauge symmetries. The construction is done as follows: Start from eqs(2.47) describing the open chain $C_n$ together with the field equation of motion

$$|Z_0|^2 + |Z_1|^2 = r_0,$$  (3.6)

describing the extra 2-sphere $S_0^2$ with Kahler modulus $r_0$. The meaning of the complex variables $Z_0$ and $Z_1$ will be specified later. Then glue $S_0^2$ and $C_n$ by implementing the constraint relations (3.5). A priori this could be achieved by setting $Z_1 = X_1$ and $Z_0 = X_{n+1}$ so that eq(3.6) becomes

$$|X_{n+1}|^2 + |X_1|^2 = r_0,$$  (3.7)

In this way $S_0^2$ intersects once the 2-sphere $S_1^2$ defined by $|X_1|^2 + |X_2|^2 = r_1$ as well as $S_n^2$ with eq $|X_n|^2 + |X_{n+1}|^2 = r_n$. But strictly speaking there is still a problem although the resulting
geometry looks having the topology of a 2-torus. This construction does not exactly work. The
point is that by combining eqs(2.47,3.7), we cannot have the right dimension since the $(n + 1)$
complex variables $\{X_i, 1 \leq i \leq n + 1\}$ are constrained by $(n + 1)$ complex constraint relations.
As mentioned before, we have $(n + 1)$ real relations coming from the field equations of motion
(2.47-3.7) and an equal number following from $U(1)^{n+1}$ gauge symmetry acting on the field

$$X_i \rightarrow X_i e^{i \sum_{a=1}^{n+1} q^a \vartheta_a}$$

(3.8)

with $i = 1, ..., n + 1$ and the $\vartheta_a$’s being the $U(1)$ group with charges

$$q_i^1 = (1, 1, 0, 0, 0, ..., 0, 0, 0)$$
$$q_i^2 = (0, 1, 1, 0, 0, ..., 0, 0, 0)$$
$$q_i^3 = (0, 0, 1, 1, 0, ..., 0, 0, 0)$$

...  

$$q_i^n = (0, 0, 0, 0, 0, ..., 1, 1, 0)$$
$$q_i^{n+1} = (1, 0, 0, 0, 0, ..., 0, 0, 1).$$

This dimensionality problem can be solved in different, but a priori equivalent, ways. Let us
describe below the key idea behind these solutions.

A natural way to do is to start from the complex two dimension ALE geometry with blown up
$A_n$ singularity and make an appropriate dimension reduction down to one complex dimension.
More precisely, start from equations

$$|X_{j-1}|^2 - 2|X_j|^2 + |X_{j+1}|^2 = r_j, \quad j = 1, ..., n$$

(3.10)

and add the following extra constraint relation reducing the dimensionality by one

$$|X_{n}|^2 - 2|X_{n+1}|^2 + |X_{0}|^2 = r_0.$$  

(3.11)

There is also an extra $U(1)$ gauge invariance giving charges to $(X_n, X_{n+1}, X_0)$. This picture
involves $(n + 2)$ complex variables constrained by $(n + 1)$ relations. The compact geometry is
determined from eqs[3.10,3.11] by restricting to the compact divisors $X_i = 0$. This gives

$$|X_{i-1}|^2 + |X_{i+1}|^2 = r_i \quad i = 1, ..., n$$
$$|X_{0}|^2 + |X_{n}|^2 = r_0.$$  

(3.12)

The second way is to use the correspondence

$$\alpha_i \leftrightarrow S_i^2$$

(3.13)

between roots of Lie algebras $\alpha_i$, which in present analysis are given in terms of unit basis
vectors of $R^{n+1}$ as,

$$\alpha_i = e_i - e_{i+1}$$

(3.14)

and the $S_i^2$ 2-sphere homology of ALE space with blowing up singularities. To have the 2-
torus, one should consider affine Kac-Moody symmetries and use its correspondence between
the imaginary root $\delta = \alpha_0 - \sum_i \alpha_i$

$$\delta \leftrightarrow T^2$$

(3.15)
Below, we shall develop an other way to do relying on the following correspondence

\[
e_i^2 \quad \leftrightarrow \quad \frac{1}{\rho} |X_i|^2,
\]

\[
e_i \cdot e_j \quad \leftrightarrow \quad \frac{1}{\rho} |Y_{ij}|^2, \quad (3.16)
\]

where \( e_i \) is as in eqs \((3.14)\). In the case where \( \{e_i\} \) are orthogonal; i.e \( e_i \cdot e_j = 0 \), there is no \( Y_{ij} \) variable and this is interpreted as just a divisor equation. In this correspondence, roots \( \alpha_i \) are associated with cotangent bundle of \( P^1 \) since by computing \( \alpha_{2i} = e_{2i} - 2e_i \cdot e_{i+1} + e_{i+1}^2 + 1 + e_{2i+1}^2 \) and using above correspondence,

\[
|X_i|^2 - 2|Y_{i,i+1}|^2 + |X_{i+1}|^2 = \rho \quad (3.17)
\]

For \( Y_{i,i+1} = 0 \), one recovers the usual 2-sphere. The link between these ways initiated here will be developed in \([42]\).

We start from eq \((2.47)\) and modify it as follows: (i) an extra complex variable \( X_0 \) so that the new system involves the following complex variables \( \{X_0, X_1, ..., X_{n+1}\} \). The extra variable charged under the \( U_1(1) \) abelian gauge symmetry,

\[
X'_0 = X_0 e^{-2i\theta_0}, \quad X'_1 = X_1 e^{i\theta_0}, \quad X'_2 = X_2 e^{i\theta_0}. \quad (3.18)
\]

(ii) an extra \( U_0(1) \) gauge symmetry acting as

\[
X'_0 = X_0 e^{i\theta_0}, \quad X'_1 = X_1 e^{i\theta_0} \quad (3.19)
\]

and trivially on the remaining others. (iii) modify eq \((2.47)\) as

\[
-2|X_0|^2 + |X_1|^2 + |X_2|^2 = r_1 \quad (3.20)
\]

whose compact part \( X_0 = 0 \) is just the 2-sphere \( |X_1|^2 + |X_2|^2 = r_1 \) of eq \((2.47)\), and the remaining \((n - 1)\) others unchanged

\[
|X_2|^2 + |X_3|^2 = r_2 \\
... = ... \\
|X_{n-1}|^2 + |X_n|^2 = r_{n-1}, \quad (3.21)
\]

\[
|X_n|^2 + |X_{n+1}|^2 = r_n.
\]

(iv) Finally add moreover

\[
|X_{n+1}|^2 + |X_0|^2 = r_0. \quad (3.22)
\]

These relations involves \((n + 2)\) complex variables \( X_i \) subject to \((n + 1)\) real constraint equations and \((n + 1)\) \( U(1) \) symmetries. They describe exactly the elliptic curve \( \Delta_{n+1} \). Note that at the level eq \((3.20)\) the variable \( X_0 \) parameterises a complex space \( C \), which in the language of toric graphs, is represented by a half line. The relation \((3.22)\) describes then the compactification of non compact complex space \( C \) with variable \( X_0 \) to the complex one projective space (real 2-sphere).

In sigma model language, this corresponds to having \((n + 2)\) chiral superfields \( \Phi_i \) with leading bosonic component fields,

\[
X_0, \ X_1, \ X_2, \ ..., \ X_{n-1}, \ X_n, \ X_{n+1}, \quad (3.23)
\]
charged under \((n + 1)\) Maxwell gauge superfields,

\[
V_0, \ V_1, \ V_2, \ ..., \ V_{n-1}, \ V_n, \tag{3.24}
\]

with the \(U^{n+1}(1)\) charges \(q^a_i = (q^a_0, q^a_1, 0, ... , q^a_{n+1})\) as follows,

\[
U_0(1) : q^0_i = (1, 0, 0, 0, 0, ..., 0, 0, 1)
\]

and

\[
U_1(1) : q^1_i = (-2, 1, 1, 0, 0, ..., 0, 0, 0) \\
U_2(1) : q^2_i = (0, 0, 1, 1, 0, ..., 0, 0, 0) \\
... \\
U_{n-1}(1) : q^{n-1}_i = (0, 0, 0, 0, 0, ..., 1, 1, 0) \\
U_n(1) : q^n_i = (1, 0, 0, 0, 0, ..., 0, 1, 1) . \tag{3.25}
\]

Below, we use this construction to build our class of local Calabi-Yau threefold using the elliptic curve \(\Delta_{n+1}\) as the compact part.

### 3.2 Local 2-torus

Implementing the fibers \(O(m) \oplus O(-m)\) which, in our present realization, take the form \(O(m_0, ..., m_n)\) and \(O(-p_0, ..., -p_n)\) and extending the construction of subsection 2.2, one can write down the sigma model relations. We have

\[
|X_1|^2 + |X_2|^2 - (2 + m_1) |X_0|^2 + m_1 |Y_2|^2 = r_1 \\
|X_2|^2 + |X_2|^2 - (2 + m_2) |Y_1|^2 + m_2 |Y_2|^2 = r_2 \\
... \\
|X_n|^2 + |X_{n+1}|^2 - (2 + m_n) |Y_n|^2 + m_n |Y_2|^2 = r_n \\
|X_{n+1}|^2 + |X_0|^2 - (2 + m_0) |Y_1|^2 + m_0 |Y_2|^2 = r_0 , \tag{3.26}
\]

where \(Y_1\) and \(Y_2\) are the fiber variables and carry non trivial charges under \(U^{n+1}(1)\) gauge symmetries.

### 4 Brane theory and 4d black holes in type II string

In this section we consider type string IIA compactification on the class of local Calabi-Yau threefolds constructed in previous sections

\[
X_3^{(m, -m-2, 2)} = O(m) \oplus O(-m - 2) \to \Delta_{n+1} . \tag{4.1}
\]

Then, we develop a field theoretical method to study 4d large black holes by using the 2d q-deformed quiver gauge theory living on \(\Delta_{n+1}\). Large black holes in four dimensional space-time are generally obtained by using configurations of type II or M-theory branes on cycles of the internal manifolds. In type IIA framework, an interesting issue is given by BPS configurations involving, amongst others, \(N\) D4-branes wrapping non-compact divisors of the local CY\(^3\) giving
rise to a dual of topological string. Our construction follows more a less the same method used in [2]; the difference comes mainly from the structure of the internal manifold $X_3^{(m,-m-2,2)}$ and the engineering of the quiver gauge theory living on $\Delta_{n+1}$.

We first study the D-brane formulation of the BPS 4d black hole in the framework of type IIA string compactification on local $\Delta_{n+1}$. Then we study the reduction of $\mathcal{N} = 4$ twisted topological theory on 4-cycles to 2d quiver gauge theory, represented by ADE Dynkin diagrams.

### 4.1 Brane theory in $X_3^{(m,-m-2,2)}$ background

In type IIA string compactification on local Calabi-Yau threefolds $X_3^{(m,-m-2,2)}$, the effective 4d, $\mathcal{N} = 2$ supersymmetric theory has massive BPS particles coming from D-branes wrapping cycles in $X_3^{(m,-m-2,2)}$. Under some assumptions, BPS states based on a special D-branes configuration may be interpreted in terms of 4d space-time black holes. This configuration involves D0-D4 and D2-D4 brane bound states but no D6 due to the reality of the string coupling constant $g_s$. The D0-particles couple the RR type IIA 1-form $A_1$ while the D2- and D4-branes couple to the RR 3-form $C_3$. Their respective charges $Q_0$, $Q_2a$ and $Q_4^a$ give the following expression of the macroscopic entropy of the black hole [6, 19, 15]

$$ S_{BH} = \frac{1}{4} \sqrt{\frac{1}{6} C_{abc} Q_a^b Q_b^c (Q_0 - \frac{1}{2} C^{ab} Q_{2a} Q_{2b})} \quad (4.2) $$

with

$$ C_{abc} = \int_{CY^3} \omega_a \wedge \omega_b \wedge \omega_c, \quad C_{abc} = C_{abc} Q_4^c, \quad C^{ab} C_{bc} = \delta_{bc}^a. \quad (4.3) $$

The above 4d black hole construction can be made more precise for our present study. Here the local threefold $X_3^{(m,-m-2,2)}$ is given by

$$ \mathcal{O}(m_0,..,m_n) \oplus \mathcal{O}(-p_0,..,-p_n) \to \left( \sum_{i=0}^{n} S_i^2 \right), \quad (4.4) $$

with $p_i = m_i + 2$. The 2-cycles basis $\{ [C_i], i = 1,..,h^{1,1}(X) \}$ of $H_2(X,Z)$ is given by the compact 2-spheres $S_i^2$ with kahler modulus $r_i$ and the following supersymmetric linear sigma field theoretical realization

$$ C^i : |X_i|^2 + |X_{i+1}|^2 = r_i, \quad i = 0,..,n, \quad (4.5) $$

with the identification $S_0^2 \equiv S_{n+1}^2$ and $n + 1 = h^{1,1}(X)$. The components $[\mathcal{D}_i]$ of the dual basis of 4-cycles of $H_4(X,Z)$ is given by the non compact complex surfaces

$$ [\mathcal{D}_i] = \mathcal{O}(-p_0,..,-p_n) \to C^i, \quad i = 0,..,n, \quad (4.6) $$

with generic equations

$$ |X_i|^2 + |X_{i+1}|^2 - (2 + m_i) |Z|^2 + m_i |Y_2|^2 = r_i, \quad i = 0,..,n, \quad (4.7) $$
where $Z$ stands either for $X_0$ or $Y_1$ as given in eqs(3.26). These dual 2- and 4-cycles determine a basis for the $(n + 1)$ abelian vector fields $B^i = B^i(t, r)$ obtained by integrating the RR 3-form $C_3$ on the 2-cycles $C_i$ as shown below

$$B^i = \int_{C_i} C_3, \quad \int_{C_i} \omega_j = \delta^i_j.$$  \hspace{1cm} (4.8)

Under these $B^i$ abelian gauge fields, the D2-branes in the class $[C] \in H_2(X, Z)$ and D4-branes in the class $[D] \in H_4(X, Z)$ are given by

$$[C] = \sum_{i=0}^n Q_{2i}[C_i], \quad [D] = \sum_{i=0}^n Q_4^i[D_i],$$ \hspace{1cm} (4.9)

and carry respectively $Q_{2i} (Q_{2i} = M_i)$ electric and $Q_4^i (Q_4^i = N_i)$ magnetic charges. We also have D0-brane charge $Q_0$ that couple the extra $U(1)$ vector field originating from RR 1-form. D6-brane charges are turned off.

Following [2], the indexed degeneracy $\Omega (Q_0, Q_{2i}, Q_4^i)$ of BPS particles in space-time with charges $Q_0$, $Q_{2i}$, $Q_4^i$ can be computed by counting BPS states in the Yang-Mills theory on the D4-brane. This is computed by the supersymmetric path integral of the four dimensional theory on $D$ in the topological sector of the Vafa-Witten maximally supersymmetric $\mathcal{N} = 4$ theory on $D$ [2, 13, 19],

$$Z_{\text{Brane}} = \int [DA] \exp \left( -\frac{1}{2g_\nu} \int_D Tr F \wedge F - \frac{\theta}{g_\nu} \int_D \omega \wedge Tr F \right).$$ \hspace{1cm} (4.10)

Up to an appropriate gauge fixing, this relation can be written, by using the chemical potentials $\varphi_0 = \frac{4\pi}{g_\nu}$ and $\varphi_1 = \frac{2\pi \theta}{g_\nu}$ for D0 and D2-branes respectively, as follows

$$Z_{\text{Brane}} [N_i, \varphi^0, \varphi^i] = \sum_{Q_0, M_i} \Omega (Q_0, M_i, N_i) \exp (-Q_0 \varphi^0 - M_i \varphi^i)$$ \hspace{1cm} (4.11)

where we have used

$$Q_0 = \frac{1}{8\pi^2} \int_D Tr (F \wedge F), \quad M_i = \frac{1}{2\pi} \int_D Tr F \wedge \omega_i.$$ \hspace{1cm} (4.12)

The above relation may be expanded in series of $e^{-g_\nu}$ due to S duality of underlying $\mathcal{N} = 4$ theory that relates strong and weak coupling expansions [19]. Recall that the world-volume gauge theory on the $N$ D4-branes is the $\mathcal{N} = 4$ topological $U(N)$ YM on $D$. Turning on chemical potentials for D0-brane and D2-brane correspond to introducing the observables

$$S = \frac{1}{2g_\nu} \int_D Tr F \wedge F + \frac{\theta}{g_\nu} \int_D \omega \wedge Tr F$$ \hspace{1cm} (4.13)

where $\omega$ is the unit volume form of $\Delta_{n+1}$. The topological theory (4.13) is invariant under turning on the massive deformation

$$\delta S = \sum_i \frac{p_i}{2} \int \Phi_i^2$$ \hspace{1cm} (4.14)

which simplifies the theory. By using further deformation which correspond to a $U(1)$ rotation on the fiber, the theory localizes to modes which are invariant under the $U(1)$ and effectively reduces the 4d theory to a gauge theory on $\Delta_{n+1}$. 
\section{2d quiver gauge theories on $\Delta_{n+1}$}

Note first that from four dimensional space-time view, the wrapped $N$ D-branes on $\mathcal{D} = \mathcal{O}(-p) \to \mathcal{C}$ describe a point-particle with dynamics governed by 4d $\mathcal{N} = 2$ supergravity coupled to $U(N)$ super-Yang Mills. On the D4-branes live a (4+1) space-time $\mathcal{N} = 4$ $U(N)$ supersymmetric gauge theory and on its reduced topological sector one has a 4d $\mathcal{N} = 4$ topological theory twisted by massive deformations.

In our present study, the 2-cycle $\mathcal{C}$ is represented by the closed chain $\Delta_{n+1}$ with multitoric actions and the line $\mathcal{O}(-p)$ is a non trivial fiber capturing charges under these abelian symmetries. It will be denoted as $\mathcal{O}(-p)$,

$$ p = (p_1, \ldots, p_n). \quad (4.15) $$

As a consequence of the topology of $\Delta_{n+1}$ which is given by $(n+1)$ intersecting 2-spheres, the previous $U(N)$ gauge invariance gets broken down to

$$ U(N) \to U(N_0) \times U(N_1) \times \cdots \times U(N_n), \quad (4.16) $$

with the following condition on the group ranks

$$ N = \sum_{i=0}^{n} N_i. \quad (4.17) $$

This symmetry breaking phenomenon requires non zero Kahler moduli (2.23) of the various 2-spheres of the base $\Delta_{n+1}$,

$$ r_i \neq 0, \quad i = 0, 1, \ldots, n. \quad (4.18) $$

Viewed from 4d space-time, the effective theory of type IIA string low energy limit on local $\text{CY}^3$ with D-branes wrapping cycles is given by a 4d $\mathcal{N} = 2$ supergravity coupled to 4d $\mathcal{N} = 2$ quiver gauge theory with gauge group $G = U(N_0) \times \cdots \times U(N_n)$. This is the general picture of the string low energy effective field approximation.

By requiring 4d space-time $\mathcal{N} = 2$ superconformal invariance, the vanishing conditions $\beta_i = 0$ of one loop beta functions on the gauge group factors $U(N_i)$ put a strong constraint on the ranks $N_i$ of these $U(N_i)$’s. These conditions have been approached for different purposes; in particular in the context of geometric engineering of 4d space-time $\mathcal{N} = 2$ superconformal theories embedded in type IIA string theory on local $\text{CY}^3$s with blown up affine ADE singularities. There, these conditions take the remarkable form

$$ \sum_{i=0}^{n} \hat{K}_{ij} N_j = 0, \quad (4.19) $$

and are solved by taking $N_i$ ranks as $N_i = s_i M$, where the $s_i$’s are the Dynkin weights introduced earlier. In the case of affine $\hat{A}_n$ model, the $s_i$’s are equal to unity and so the $U(N)$ gauge group gets reduced, in the superconformal case, to

$$ U(N) \to G_{\text{scft}} = U(M)^{n+1} \quad (4.20) $$

with

$$ N = (n+1) M. \quad (4.21) $$
On the 4-cycle $\mathcal{D}$ of the local Calabi-Yau threefold, the theory is a $\mathcal{N} = 4$ topologically twisted gauge theory; but using the result of [2], this theory can be simplified by integrating gauge field configuration on fiber $\mathcal{O}(-p)$ and fermionic degrees freedom to end with a 2d bosonic quiver gauge theory on $\Delta_{n+1}$. This theory has $\prod_i U(N_i)$ as a gauge symmetry group and involves:

(i) Gauge fields $A_i$, $i = 0, 1, \ldots, n$, for each gauge group factor $U(N_i)$ with field strength $F_i = dA_i + A_i \wedge A_i$,

$$F_i = \sum_{a_i=1}^{N_i} H_{a_i} F_{i,a_i} + \sum_{\beta_i} E_{\beta_i}^+ F_{i}^{-\beta_i} + E_{\beta_i}^- F_{i}^{+\beta_i}, \quad (4.22)$$

where $\{H_{a_i}, E_{\beta_i}^\pm\}$ is the Cartan basis of $U(N_i)$ and where $\beta_i$ are the positive roots of the $i$-th gauge group factor. The above relation contains as a closed subset the usual $U^{N_i}(1)$ abelian part $dA_i = \sum_{a_i=1}^{N_i} H_{a_i} dA_{i,a_i}$ where the $H_{a_i}$’s are the commuting Cartan generator.

(ii) 2d scalars $\Phi_i = \Phi_i(z, \bar{z})$ in the adjoint for each factor $U(N_i)$ having a similar expansion as in (4.22); but which we reduce to its $U^{N_i}(1)$ diagonal form

$$\Phi_i = \sum_{a_i=1}^{N_i} H_{a_i} \Phi_{i,a_i}^{a_i}, \quad \Phi_i^{\pm \beta_i} = 0, \quad i = 0, \ldots, n. \quad (4.23)$$

These fields are obtained by integration of the 4d gauge field strengths $F_i(z, \bar{z}, y, \bar{y})$ on the fiber $\mathcal{O}(-p)$

$$\Phi_i(z, \bar{z}) = \int_{\mathcal{O}(-p)} d^2y \ F_i(z, \bar{z}, y, \bar{y}) \quad (4.24)$$

which, as usual, can be put as

$$\Phi_i(z, \bar{z}) = \int_{S^1, |y| \to \infty} A_i(z, \bar{z}, y, \bar{y}), \quad z \in S^2_i$$

where the loop $S^1_i$ can be thought of as a circle at the infinity ($|y| \to \infty$) of the non compact fiber $\mathcal{O}(p_i) \sim C$ parameterized by the complex variable $y$.

(iii) 2d matter fields $\Phi_{ij}$ in the bi-fundamentals of the quiver gauge group living on the intersection of $\Delta_{n+1}$ patches with, in general, a leg on $S^2_i$ and the other on $S^2_j$.

In the language of the representations of the gauge symmetry $U(N_i) \times U(N_j)$, these fields belongs to $(N_i, \overline{N}_j)$ and describe the link between the gauge theory factors living on the irreducible 2-cycles making $\Delta_{n+1}$.

In the language of topological string theory using caps, annuli and topological vertex [38], these bi-fundamentals can be implemented in the topological partition function thought insertion operators type

$$S_{\mathcal{R}_i, \overline{\mathcal{R}}_{i+1}} \quad (4.25)$$

using sums over representations $\mathcal{R}_i, \overline{\mathcal{R}}_{i+1}$ of the gauge invariance $U(N_i) \times U(N_{i+1})$. This construction has been studied recently in [19] for particular classes of local CY3 such as $\mathcal{O}(-3) \to \mathbb{P}^2$ and $\mathcal{O}(-2, -2) \to \mathbb{P}^1 \times \mathbb{P}^1$. We will not develop this issue here. Below we

---

1. We have used the Greek letter $\beta$ to refer to the roots of the of the gauge group $U(N)$. Positive roots of the $U(N_i)$ are denoted by $\beta_i$ and should not confused with simple roots $\alpha_i$ used in the intersection matrix ($K_{ij} = \alpha_i \alpha_j$) of the 2-cycles of the base $\Delta_n$ of local CY3.

2. As argued in [19], the matter fields localized at the intersection point $P_i$ of the 2-spheres $S^2_i$ and $S^2_{i+1}$ corresponds to inserting the operator $V = \sum_{\alpha} T_{\alpha} V_{i^{-1}} T_{\alpha} V_{i+1}$ with $V_i = \exp \left( i \Phi_i - i \int A_i \right)$ and $V_{i+1} = e^{i \Phi_{i+1}}$ where the integral contour is a small loop around $P_i$.
shall combine however field theoretical analysis and representation group theoretical method to deal with bi-fundamentals.

4.2.1 Derivation of 2d quiver gauge field action

Here we construct the 2d field action $S_{\Delta_{n+1}}$ describing the localization of the topological gauge theory of the BPS D4-, D2-, D0-brane configurations on the non compact divisor $[D] = \mathcal{O}(-p) \to \Delta_{n+1}$ of the local CY$^3$. This action can be obtained by following the same method as done for the case $\mathcal{O}(-p) \to \Sigma_g$.

One starts from eq(4.13), describing the gauge theory on the $N$ D4-branes wrapping $D$ with D0-D4 and D2-D4 bound states

$$ S_{4d} = \frac{1}{2g_s} \int [\mathcal{D}] \text{Tr} (F \wedge F) + \frac{\theta}{g_s} \int [\mathcal{D}] \text{Tr} F \wedge \omega. \quad (4.26) $$

In this equation, the parameters $g_s$ and $\theta$ are related to the chemical potentials $\varphi_0$ and $\varphi_1$ for D0 and D2-branes respectively as $\varphi_0 = \frac{4\pi^2}{g_s}$ and $\varphi_1 = \frac{2\pi \theta}{g_s}$. The field $F$ is the 4d $U(N)$ gauge field strength $F = dA + A \wedge A$. It is a hermitian 2-form with gauge connexion $A$.

Moreover, like $A$, the 2-form field $F$ is valued in the Lie algebra of $U(N)$ gauge symmetry and so can be expanded as

$$ F = \sum_{\alpha=1}^{N} H_{\alpha} F^\alpha + \sum_{\text{positive roots } \beta}^{\text{of } U(N)} E_{\beta}^+ F^{-\beta} + E_{\beta}^- F^{+\beta}, \quad (4.28) $$

where $\{H_{\alpha}, E_{\beta}^\pm\}$ is the Cartan basis of $U(N_i)$. The above relation contains as a closed subset the usual $U^N(1)$ abelian part

$$ dA = \sum_{\alpha=1}^{N} H_{\alpha} dA^\alpha, \quad [H_{\alpha}, H_{\beta}] = 0, \quad (4.29) $$

which plays a crucial role in the computation of Wilson loops. In eq(4.26), $\omega$ is the 2-form on the compact cycle $\Delta_{n+1}$ on which the D2-branes lives and is normalized as

$$ \int_{\Delta_{n+1}} \omega = 1. \quad (4.30) $$

On the other hand, using eq(2.18-2.21), we can put the right hand side of the above relation as follows

$$ \int_{\Delta_n} \omega = \sum_{i} \left( \int_{S_i^1} \omega \right) - \frac{1}{2} \sum_{i \neq j} \left( \int_{S_i^1 \cap S_j^2} \omega \right) \quad (4.31) $$

We have used the formula $\int_{A\cup B} K = \int_A K + \int_B K - \int_{A \cap B} K$ for Kahler modulus of two intersecting surfaces.
Note that since \( \int_{S^2 \cap S^2_j} \omega \) vanishes for the intersections \( S^2_i \cap S^2_j \) which are given by a set of separated points, we can simplify this expression further into

\[
\int_{\Delta_n} \omega = \frac{1}{r} \sum_{i=0}^{n} r_i = 1, \quad r \neq 0. \tag{4.32}
\]

This shows that on \( \Delta_{n+1} = \sum_{i=0}^{n} C_i \), the Kahler form splits as \( \omega = \frac{1}{r} \sum_{i=0}^{n} r_i \omega_i \) with \( \int_{C_i} \omega_j = \delta^i_j \).

The next step is to perform integration on the fiber variables \( y \) and \( \overline{y} \). The topological theory (4.26) localizes to modes which are invariant under \( U^{n+1}(1) \) symmetries and effectively reduces to a gauge theory on the base \( \Delta_{n+1} \). Let us give details by working out explicitly these steps:

(i) First, we have

\[
\int_{[\mathcal{D}_4]} \text{Tr} (F \wedge F) = \int_{\Delta_{n+1}} \text{Tr} \left( \int_{\mathcal{O}(\mathbf{p}) \to z} F \wedge F \right) = 2 \int_{\Delta_n} \text{Tr} (\Phi F) \tag{4.33}
\]

where we have set

\[
\Phi (z, \overline{z}) = \int_{\mathcal{O}(\mathbf{p}) \to z} d^2 y F (y, \overline{y}, z, \overline{z}) \tag{4.34}
\]

and where we have restricted \( F \) to its values in the \( U^{N_i}(1) \) abelian subsalgebra eq(4.29) in order to put \( \Phi (z, \overline{z}) \) in the Wilsonian form

\[
\Phi (z, \overline{z}) = \oint_{S^1_{|y|} \to z, |y| \to \infty} A (y, \overline{y}, z, \overline{z}) , \quad z \in \Delta_{n+1}. \tag{4.35}
\]

The same thing can be done for the second term of eq(4.26). We find

\[
\int_{[\mathcal{D}]} \omega \wedge (\text{Tr} F) = \int_{\Delta_{n+1}} \omega \text{Tr} (\Phi). \tag{4.36}
\]

The 4d action (4.26) reduces then to the following 2d one

\[
S_{2d} = \frac{1}{g_s} \int_{\Delta_{n+1}} \text{Tr} (\Phi F) + \frac{\theta}{g_s} \int_{\Delta_{n+1}} \omega \text{Tr} \Phi. \tag{4.37}
\]

Now using the fact that \( \Delta_{n+1} = \sum_{i=0}^{n} \mathbb{P}_i^1 \) combined with eq(4.31), we see that, depending on the patches of \( \Delta_{n+1} \) where the Wilson field \( \Phi \) is sitting, we get either adjoint 2d scalars \( \Phi_i \) or bi-fundamentals \( \Phi_{ij} \) as shown below

\[
\Phi (z) \equiv \Phi_i (z), \quad z \in \mathbb{P}_i^1, \\
\Phi (z) \equiv \Phi_{ij} (z), \quad z \in \mathbb{P}_i^1 \cap \mathbb{P}_j^1. \tag{4.38}
\]

Note that the \( \Phi_i \) fields are valued in \( U^{N_i}(1) \) maximal abelian group. They parameterise the maximal \( T^{N_i} \) torii of the Lie group \( U (N_i) \). So they should be compact and undergo periodicity conditions. This means that the linear expansion

\[
\Phi_i = \sum_{a=1}^{N_i} \Phi_{ai} H_{ai}, \quad \Phi_{ai} \sim Tr (H_{ai} \Phi_i), \tag{4.39}
\]
should be understood as
\[ U_i = \exp i\Phi_i, \quad i = 0, \ldots, n, \]  
(4.40)
and so the 2d field components \( \Phi_{ai} \) are constrained as,
\[ \Phi_{ai} = \Phi_{ai} + 2\pi m_{ai}, \quad m_{ai} \in \mathbb{Z}, \]  
(4.41)
leaving \( U_i \) invariant.

Now substituting eq (4.38) in the relation (4.37), we obtain after implementing the hermiticity condition \( \Phi = \frac{1}{2} (\Phi + \Phi^* ) \) and \( F = \frac{1}{2} (F + F^* ) \), the following
\[
\int_{\Delta_n} \text{Tr} (\Phi F) = \sum_i \int_{P_i} \text{Tr} (\Phi_i F_i) - \frac{1}{8} \sum_{i \neq j} \int_{P_i \cap P_j} \text{Tr} (\Phi_{ij} F_{ji} + \text{cc})
\]
(4.42)
where we have set \( F_{ji} = F_{ij}^* \) and where we have disregarded the terms \( \Phi_{ij} F_{ij} \) transforming in the \((N_i^{\alpha \beta}, N_j^{\gamma \delta})\) representation of \( U(N_i) \times U(N_j) \). These terms do not preserve the abelian subsymmetry \( U^{n+1} (1) \) of the quiver gauge group \( U(N_0) \times \ldots \times U(N_n) \). These 2d field configurations have a group theoretical interpretation. They correspond to splitting the adjoint representation of \( U(N) \) with \( N = N_0 + \ldots + N_n \) in terms of representation of \( U(N_0) \times \ldots \times U(N_n) \)
\[
\text{Adj} U(N) = \sum_{i=0}^n \text{Adj} U(N_i) \oplus \sum_{i \neq j} (N_i, N_j).
\]
(4.43)
The terms of the first sum are associated with \( \Phi_i (z) \) while the other are associated with \( \Phi_{ij} \). Obviously since in present case only \( P_i \cap P_{i \pm 1} \) which are non trivial, there are no bi-fundamentals \( \Phi_{ij} \) for \( j \neq i \pm 1 \).

For the term \( \int_{\Delta_n} \omega \text{Tr} (\Phi) \) (4.37), we get
\[
\int_{\Delta_{n+1}} \text{Tr} (\Phi) \omega = \sum_{i=0}^n \frac{r_i}{r} \int_{P_i} \text{Tr} (\Phi_i) \omega_i.
\]
(4.44)
Let us first discuss these configurations separately and then give the general result.

**Adjoint 2d scalars**

Putting eq (3.22) back into (4.26) and focusing on the patches \( P^1_i \) by substituting \( \Phi_{\text{diag}} = \sum_i \Phi_i (z) \), we get the diagonal part of the topological 2d quiver gauge field action
\[
S_{\text{diag}} = \sum_{i=0}^n S_i
\]
(4.45)
with
\[
S_i = \frac{1}{g_s} \int_{P^1_i} \text{Tr} (\Phi_i F_i) + \frac{\theta}{g_s} \frac{r_i}{r} \int_{P^1_i} \text{Tr} \Phi_i + \frac{p_i}{2g_s} \int_{S^2_i} \text{Tr} \Phi_i^2
\]
(4.46)
and where we have added the topologically invariant point-like observables \( \text{Tr} \Phi_i^2 \) at the points \( z \in P^1_i \). Upon integrating out fermions and adjoint scalars using \( \Phi_i = \frac{-E_i - \theta}{p_i} \) and following [2], this topologically twisted theory is equivalent to the bosonic 2d Yang-Mills theory
\[
S_i = - \int_{P^1_i} \frac{1}{2g_s^2} \text{Tr} (F_i^2) - \frac{\theta Y_i^M}{g_s^2} \int_{P^1_i} \text{Tr} F_i
\]
(4.47)
with Yang-Mills gauge coupling constants \( g_i Y M_i \equiv g_i \) and \( \theta_i Y M \equiv \theta_i \) terms given by,

\[
g_i^2 = p_i g_s \frac{r_i}{r}, \quad \theta_i Y M = \theta \frac{r_i}{r}. \tag{4.48}
\]

Here the \( r_i \)'s are the Kahler parameters of the 2-sphere constituting \( \Delta_{n+1} \). Note that these gauge coupling constants and \( \theta_i \)'s are not completely independent and are related amongst others as

\[
\frac{(n + 1) r}{g_s} = \sum_{i=0}^{n} p_i r_i g_s^2, \quad \sum_{i=0}^{n} \theta_i Y M = \theta. \tag{4.49}
\]

The first equation should be compared with the standard relation \( \frac{1}{g_s} = \sum_{i=0}^{n} g_i^{-2} \) appearing in the geometric engineering of quiver gauge theories.

### 2d bi-fundamentals

To get the field action describing the contribution of bi-fundamentals, it is interesting to proceed in steps as follows:

Start from the topological field action on the 4-cycle \([D_4]\),

\[
S_{4d} = \frac{1}{2g_s} \sum_{i=1}^{n} \int_{[D_4]} \text{Tr} (\Phi_i \wedge F) + \frac{\theta}{g_s} \sum_{i=1}^{n} \int_{[D_4]} \text{Tr} F \wedge \omega \tag{4.50}
\]

and think about \( F \) as a field strength valued in the maximal non abelian gauge group \( U(N_0 + \ldots + N_n) \).

Then expand the real field \( F \) as

\[
F = \sum_{i=0}^{n} F_i + \sum_{i<j} (F_{ij} + F_{ji}), \quad F_{ji} = (F_{ij}) \tag{4.51}
\]

with \( F_{ji} = (F_{ij}) \). The \( F_i \)'s are the real field strengths of the gauge fields \( A_i \) valued in the adjoints of \( U(N_i) \) factors with generators \( \{T_{a_i}\} \)

\[
F_i = \sum_{a_i=1}^{N_i^2} F_i^{a_i} T_{a_i}, \quad i = 1, \ldots, n, \quad F_i = (F_i). \tag{4.52}
\]

The \( F_{ij} \)'s are the field strengths of the gauge fields \( A_{ij} \) valued in the Lie algebra associated to the coset

\[
\frac{U(N_0 + \ldots + N_n)}{U(N_0) \times \ldots \times U(N_n)}. \tag{4.53}
\]

Obviously the group \( U(N_0 + \ldots + N_n) \) is not a full gauge invariance of the \( \mathcal{N} = 4 \) topological gauge theory since the gauge fields \( A_{ij} \) part get non zero masses \( m_{ij} \sim (r_i - r_j) \) after breaking \( U(N_0 + \ldots + N_n) \) down to \( U(N_0) \times \ldots \times U(N_n) \).

The next step is to use the same trick as before by integrating partially over the variables of the fiber \( O(-p_0, \ldots, -p_n) \). We get

\[
S_{4d} = \frac{1}{g_s} \sum_{i} \int_{P_i} \text{Tr} (\Phi_i \wedge F) + \frac{\theta}{g_s} \sum_{i} \int_{P_i} \text{Tr} \Phi_i \wedge \omega
- \frac{1}{g_s} \sum_{i \neq j} \int_{P_i \cap P_j} \text{Tr} (\Phi_{ij} \wedge F) - \frac{\theta}{g_s} \sum_{i \neq j} \int_{P_i \cap P_j} \text{Tr} \Phi_{ij} \wedge \omega. \tag{4.54}
\]

Now using the expansion (4.51) and the property

\[
\text{Tr} (T_{a_i} T_{b_j}) \sim \delta_{a_i b_j} \tag{4.55}
\]
we have
\begin{align*}
\text{Tr} (\Phi_i \wedge F) &= \text{Tr} (\Phi_i \wedge F_i), \\
\text{Tr} (\Phi_{ij} \wedge F) &= \text{Tr} (\Phi_{ij} \wedge F_{ji}), \\
\text{Tr} (\Phi_{ij}) &= 0.
\end{align*}
(4.56)

Then we can bring eq.(4.54) into the following reduced form
\begin{align*}
S_{2d} &= \frac{1}{g_s} \sum_i \int_{P^1_i} \text{Tr} (\Phi_i \wedge F_i) + \frac{\theta_i}{g_s} \sum_i \int_{P^1_i} \text{Tr} \Phi_i \wedge \omega + \frac{p_i}{2g_s} \int_{S^2_i} \text{Tr} \Phi_i^2 \\
&\quad \quad \quad - \frac{1}{g_s} \sum_{i \neq j} \int_{P^1_i \cap P^1_j} \text{Tr} (\Phi_{ij} \wedge F_{ji}) + \sum_{i \neq j} \frac{p_{ij}}{2g_s} \int_{P^1_i \cap P^1_j} \text{Tr} (\Phi_{ij} \Phi_{ji}),
\end{align*}
(4.57)

where we have added the typical mass deformations \(p_i\) and by analogy \(p_{ij}\) integers which a priori should be related to the \(p_i\) degrees of the line bundle. Integrating out the scalar fields, one ends with
\begin{align*}
S_{2d} &= - \sum_i \int_{P^1_i} \frac{1}{2g^2_i} \text{Tr} (F_i^2) - \sum_i \frac{\theta_i^Y M}{g_i^2} \int_{P^1_i} \text{Tr} F_i - \sum_{i \neq j} \int_{P^1_i \cap P^1_j} \frac{1}{2G^2_{ij}} \text{Tr} (F_{ij} F_{ji})
\end{align*}
(4.58)

where \(g_i^2\) and \(\theta_i^Y M\) are as in (4.48) and where
\begin{align*}
G^2_{ij} &= \frac{p_{ij} g_s}{r} \text{vol} (P^1_i \cap P^1_j).
\end{align*}
(4.59)

Note that for the base \(\Delta_{n+1}\) realizing the elliptic curve in terms of intersecting 2-spheres, the intersection \(P^1_i \cap P^1_j\) is given by a finite and discrete set of points \(P_{ij}\) of \(\Delta_{n+1}\). These points have zero volumes \(\text{vol} (P^1_i \cap P^1_j)\) and so
\begin{align*}
G^2_{ij} \simeq 0.
\end{align*}
(4.60)

In the present case where \(\Delta_{n+1}\) is taken as \(\sum_i P^1_i\), we have \((n + 1)\) intersection points \(P_{i,i+1}\). The non zero intersection numbers is between neighboring spheres \(P^1_i\) and \(P^1_{i\pm 1}\),
\begin{align*}
\left( [P^1_i] \cap [P^1_j] \right) = \delta_{j,i\pm 1}.
\end{align*}
(4.61)

Implementing this specific data, the last term of eq.(4.58) reduces then to a sum of integrals over the following field densities
\begin{align*}
\frac{1}{2G_i^2} \text{Tr} \left( |F_{i,i+1}|^2 \right), \quad F_{n,n+1} = F_{n,0},
\end{align*}
(4.62)

which diverge as long as \(|F_{i,i+1}|^2 \neq 0\). This property is not strange and was in fact expected. It has the behavior of a Dirac function one generally use for implementing insertions. To exhibit this feature, denote by \(P^1_i \cap P^1_{i+1} = \{ P_i \}\), the points where the 2-spheres intersect. Then we have
\begin{align*}
\int_{P^1_i \cap P^1_{i+1}} \frac{1}{2G_i^2} \text{Tr} \left( |F_{i,i+1}|^2 \right) = \int_{P^1_i} \frac{1}{2G_i^2} \delta (P - P_i) \text{Tr} \left( |F_{i,i+1}|^2 \right)
\end{align*}
(4.63)
where $\delta (P - P_i^+)$ is a Dirac delta function. Combining the above results, one ends with the following field action of the 2d bosonic quiver gauge theory describing the brane configuration on the non compact 4-cycle $[D_4] = \mathcal{O}(-p_0, ..., -p_n) \to \Delta_{n+1}$ of the local CY$^3$,

$$S_{\Delta_n} = \sum_{i=0}^{n} \int_{\mathbb{P}_i^1} \left( \frac{1}{2g_i^2} \text{Tr} (F_i^2) + \frac{\theta_{YM}}{g_i^2} \text{Tr} F_i + \frac{1}{2G_i^2} \delta (P - P_i) \text{Tr} \left( |F_{i,i+1}|^2 \right) \right). \quad (4.64)$$

In this relation, the coupling constants $g_i^2$ and $G_i^2$ are expressed in terms of the string coupling $g_s$, the Kahler moduli of the 2-spheres of the base $\Delta_{n+1}$ and the degrees of the fiber $\mathcal{O}(-p_0, ..., -p_n)$. The $F_i$'s are the $U (N_i)$ gauge field strengths and

$$\delta (P - P_i) \times F_{i,i+1} \quad (4.65)$$

are insertion operators in bi-fundamental representations and are needed to glue the spheres. The last term may be rewritten in different forms. For example like $\sum_{i=0}^{n} \frac{1}{2G_i^2} \times \text{Tr} \left( |F_{i,i+1} (P_i)|^2 \right)$ and it depends on the $\mathbb{P}_i$'s.

### 4.2.2 Path integral measure in 2d q-deformed quiver gauge theory

Here we want to study the structure of the measure in the path integral description of the partition function of the quiver gauge field action $S_{2d}$ eq(4.57,4.64) which we rewrite as

$$S_{2d} = \sum_i \frac{p_i}{2g_s} \int_{\mathbb{P}_i^1} \text{Tr} \Phi_i^2 + \frac{1}{g_s} \sum_i \int_{\mathbb{P}_i^1} \text{Tr} (\Phi_i \wedge F_i) + \sum_i \frac{\theta_{YM}}{g_s} \int_{\mathbb{P}_i^1} \text{Tr} \Phi_i \wedge \omega \left( \Phi_i \wedge F_i \right) + \frac{1}{g_s} \sum_i \int_{\mathbb{P}_i^1} \text{Tr} \Phi_i \wedge \omega \left( \Phi_i \wedge F_i \right) \quad (4.66)$$

We will give arguments indicating that bi-fundamentals contribute as well to the deformation of path integral measure and in a very special manner. More precisely, we give an evidence that adjoints and bi-fundamentals altogether deform the measure by the quantity

$$J_{(a_1, ..., a_n)} (\Phi) = \prod_{i,j=0}^{n} \prod_{a_i < a_j=1}^{N_i} \left( \sqrt{[\Phi_{a_i} - \Phi_{a_j}]} g_{ij} \frac{[\Phi_{a_j} - \Phi_{a_j}]}{g_{ij}} \right)^{-I_{ij}}. \quad (4.67)$$

In this relation, the $\Phi_{a_i}$'s are as in eq(4.39) and where $I_{ij}$ is the intersection matrix of the 2-spheres of the $\Delta_{n+1}$ base. It is equal to minus the generalized Cartan matrix of affine $\hat{A}_n$.

To begin recall that the partition function $Z_{Y,M} (\Sigma_g)$ of topological 2d q-deformed $U (M)$ YM on a genus $g$-Riemann surface $\Sigma_g$ is given by

$$Z_{Y,M} (\Sigma_g) = \frac{1}{M!} \int' \left( \prod_{a=1}^{M} (D\phi_a) \right) \left( \Delta_H (\phi_a) \right)^{2-2g} \prod_{b=1}^{M} e^{g_{YM} \int_{\Sigma_g} (\phi_b^2 + \theta \phi_a + \frac{\bar{\theta} \phi_b^2})}. \quad (4.68)$$

where the $\phi_a$'s are the diagonal values of $U (M)$ unitary gauge symmetry. In this relation, $\Delta_H (\phi_a)$ is given by,

$$\Delta_H (\phi) = \prod_{a<b=1}^{M} \left( 2 \sin \left( \frac{\phi_a - \phi_b}{2} \right) \right), \quad (4.69)$$

and is invariant under the periodic changes (4.41). It can take the following form

$$[\Delta_H (\phi)]^{2g-2} = \prod_{a<b=1}^{M} (|x_{ab}|_g)^{2-2g} \quad (4.70)$$
with
\[
x_{ab} = \frac{2i (\phi_a - \phi_b)}{g_s}, \quad [x]_q = \left( q^\frac{\phi_a - \phi_b}{2} - q^{-\frac{\phi_a - \phi_b}{2}} \right), \quad q = e^{-gs}.
\] (4.71)

Using this relation, we see that, on each 2-sphere \(S^2_i\) of \(\Delta_{n+1}\), the correction to the path integral measure is
\[
J_{S^2_i} = \prod_{a_i < b_i=1}^{N_i} \left[ 2 \sin \left( \frac{\phi_{a_i} - \phi_{b_i}}{2} \right) \right]^2
\]
which by setting \(q_i = \exp\left(-g_s \frac{\phi_{a_i} - \phi_{b_i}}{2}\right)\) can be put in the equivalent form
\[
J_{S^2_i} = \prod_{a_i < b_i=1}^{N_i} \left( [\Phi_{a_i} - \Phi_{b_i}]_{q_i} \right)^2, \quad i = 0, \ldots, n.
\] (4.72)

The power 2 in the right hand of above relation can be interpreted in terms of the entries of the intersection matrix \(I_{ii} = -K_{ii}\) of the \(i\)-th 2-spheres of \(\Delta_{n+1}\). This property is visible on eq(4.68) where the power \(2 - 2g\) (Euler characteristics) is just the self-intersection of the Riemann surface \(\Sigma_g\). As such, the above relation can be put in the form
\[
J_{S^2_i} = \prod_{a_i < b_i=1}^{N_i} \left( \sqrt{[\Phi_{a_i} - \Phi_{b_i}]_{q_i} [\Phi_{a_j} - \Phi_{b_j}]_{q_j}} \right)^{K_{ii}}, \quad K_{ii} = 2, \quad i = 0, \ldots, n.
\] (4.73)

This relation is very suggestive, it lets understand that this feature is a special property of a more general situation where appears number intersection. More precisely, the structure of the deformation of the path integral measure for local Calabi-Yau threefolds with some base \(B\) made of 2-cycles \(C_i\) with intersection matrix \(I_{ij} = [C_i] \cdot [C_j]\) should be as follows
\[
J_B = \prod_{i,j} \prod_{a_i < b_i=1}^{N_i} \left( \sqrt{[\Phi_{a_i} - \Phi_{b_i}]_{q_i} [\Phi_{a_j} - \Phi_{b_j}]_{q_j}} \right)^{-I_{ij}}.
\] (4.74)

In our concern, the 2d manifold is given by the base manifold \(\Delta_{n+1}\) of the local Calabi-Yau threefold. In this case the intersection matrix of the 2-cycles is given by \(I_{ij} = -K_{ij}\). So the partition function of the q-deformed 2d quiver gauge theory reads in general as
\[
Z_{\text{Quiver}} = \int' \left( \prod_{i=0}^{n} \prod_{a_i=1}^{N_i} \frac{[D\phi_{a_i}]}{N_i!} \right) \left( \prod_{i,j=0}^{n} \prod_{a_i < b_i}^{N_i} \left( \sqrt{[\Phi_{a_i} - \Phi_{b_i}]_{q_i} [\Phi_{a_j} - \Phi_{b_j}]_{q_j}} \right) K_{ij} \right) e^{-S_{\text{Quiver}}}
\] (4.75)

where \(S_{\text{Quiver}}\) is the action given by eq(4.64,4.66). Of course, here \(\Delta_{n+1}\) is an elliptic curve and so one should have \(J_B = 1\). This condition can be turned around and used rather as a consistency condition to check the formula (4.74). Indeed, for the case of the 2-torus \(T^2 = S^1 \times S^1\), we know that
\[
[T^2] \cdot [T^2] = 0,
\] (4.76)
and so no q-deformation in agreement with eq(4.68). The same property is valid for \([\Delta_{n+1}]\). But at this level, one may ask what is then the link between the two realisations \(T^2 = S^1 \times S^1\) and \([\Delta_{n+1}] = \sum_{j=0}^{n} S^2_j\). The answer is that in the second case the role of the condition
\[
(2 - 2g) = 0
\] (4.77)
that is obeyed by $S^1 \times S^1$ is now played by the vanishing property,
\[
\sum_{j=0}^{n} K_{ij} s_j = 0
\]  
(4.78)
for affine Kac Moody algebras (with $s_j = 1$ for affine $\hat{A}_n$). Let us check that $J_{\Delta_{n+1}}$ eq(4.74) is indeed equal to unity. We will do it in two ways:

First consider the simplest case given by the superconformal model with gauge symmetry as in eq(4.20) and specify the Kahler parameters at the moduli space point where all the 2-spheres have the same area ($r_i = \frac{1}{n+1}$). In this case the quantity $[\Phi_{a_i} - \Phi_{b_i}] q_i$ is independent of the details of $\Delta_{n+1}$ and so the above formula reduces to,
\[
J_{SCFT}^{\Delta_{n+1}}(\Phi) = \left( \prod_{a<b=1}^{M} [\Phi_a - \Phi_b] q \right) \sum_{i=0}^{n} \sum_{j=0}^{n} K_{ij}
\]  
(4.79)
which is equal unity ($J(\Phi) = 1$) due to the relation $\sum_{j=0}^{n} K_{ij} = 0$. In the general case where the gauge group factors are arbitrary and for generic points in the moduli space, the identity (4.74) holds as well due to the same reason. For the instructive case $n = 2$ for instance, we have
\[
J_{QFT}^{\Delta_{3}}(\Phi) = \left( \prod_{a<b=1}^{N_0} [\Phi_{a_0} - \Phi_{b_0}] q_0 \right)^2 \left( \prod_{a_1<b_1=1}^{N_1} [\Phi_{a_1} - \Phi_{b_1}] q_1 \right)^2 \left( \prod_{a_2<b_2=1}^{N_2} [\Phi_{a_2} - \Phi_{b_2}] q_2 \right)^2
\]
\[
\times \left( \prod_{a_0<b_0=1}^{N_0} [\Phi_{a_0} - \Phi_{b_0}] q_0 \right) \left( \prod_{a_1<b_1=1}^{N_1} [\Phi_{a_1} - \Phi_{b_1}] q_1 \right)^{-2} \theta^2
\]
\[
\times \left( \prod_{a_1<b_1=1}^{N_1} [\Phi_{a_1} - \Phi_{b_1}] q_1 \right) \left( \prod_{a_2<b_2=1}^{N_2} [\Phi_{a_2} - \Phi_{b_2}] q_2 \right)^{-2} \theta^2
\]
\[
\times \left( \prod_{a_2<b_2=1}^{N_2} [\Phi_{a_2} - \Phi_{b_2}] q_2 \right) \left( \prod_{a_0<b_0=1}^{N_0} [\Phi_{a_0} - \Phi_{b_0}] q_0 \right)^{-2} \theta^2
\]  
(4.80)
As we see, the diagonal terms of the first line of the right hand side of the above relation are compensated by the off diagonal terms. Thus $J_{QFT}^{\Delta_{3}}(\Phi)$ reduces exactly to unity and so the 2d quiver gauge theory is not deformed. Nevertheless, one should keep in mind that this would be a special property of a general result for 4d black holes obtained from BPS D-branes in type IIA superstring moving on the following general local Calabi-Yau threefolds
\[
\mathcal{O}(\textbf{m}) \oplus \mathcal{O}(-\textbf{m} - 2) \rightarrow \mathcal{B}_k.
\]  
(4.81)
Here $\textbf{m} = (m_1, ..., m_k)$ is an integer vector and $\mathcal{B}_k$ is a complex one dimension base consisting of the intersection of $k$ 2-spheres $S^2_i$ with some intersection matrix $I_{ij}$. Using Vinberg theorem [44, 51, 45, 46], the possible matrices $I_{ij}$ may be classified basically into three categories. In the language of Kac-Moody algebras, these correspond to: (i) Cartan matrices of finite dimensional Lie algebras satisfying
\[
\sum_{j} I_{ij} u_j > 0
\]  
(4.82)
for some positive integer vector \((u_j)\). In this case the resulting 2d quiver gauge theory is q-deformed. This theory has been also studied in \([19]\). (ii) Cartan matrices for affine Kac-Moody algebras including simply laced ADE ones

\[\sum_j I_{ij} u_j = 0,\]  

where now the \(u'_j\)s are just the Dynkin weights. In this case, the 2d quiver gauge theory is un-deformed due to the identity \(\sum_j I_{ij} s_j = 0\) where the \(s'_j\)s are the Dynkin weights. (iii) Cartan matrices for indefinite Kac-Moody algebras where the intersection matrix satisfies the condition

\[\sum_j I_{ij} u_j < 0,\]  

for some positive integer vector \((u_j)\). Here the 2d quiver gauge theory is q-deformed.

5 Conclusion

In this paper, we have studied 4d black holes in type IIA superstring theory on a particular class of local Calabi-Yau threefolds with compact base made up of intersections of several 2-spheres. This study aims to test OSV conjecture for the case of stacks of D-brane configurations on CY\(^3\) cycles involving q-deformed 2d quiver gauge theories with gauge symmetry \(G\) having more than one \(U(N_i)\) gauge group factor. The class of local threefolds we have considered with details is given by \(X_3^{(m,-m-2,2)} = O(m) \oplus O(-m-2) \rightarrow \Delta_{n+1}\) where \(m\) is a \((n+1)\) integer vector \((m_0, ..., m_n)\). The \(m_i\) components capture the non trivial fibration of rank 2 line bundles of the local CY\(^3\). They also define the \(U^{n+1}(1)\) charges of the corresponding chiral superfields in the supersymmetric gauged linear sigma model field realization. The compact elliptic curve \(\Delta_{n+1}\) is generally given by \((n+1)\) intersecting spheres according to affine Dynkin diagrams.

This study has been illustrated in the case of the local affine \(\hat{A}_n\) model; but may a priori be extended to the other affine models especially for DE simply laced series and beyond. Black holes in four dimensions are realized by using D-brane configurations in type IIA superstring compactified on \(X_3^{(m,-m-2,2)}\). The topological twisted gauge theory on D4-brane wrapping 4-cycles in the local CY\(^3\)s is shown to be reduced down to a 2d quiver gauge theory on the base \(\Delta_{n+1}\) and agrees with OSV conjecture. This agreement is ensured by the results on \(O(2g + m - 2) \oplus O(-m) \rightarrow \Sigma_g\) obtained in \([2, 18]\). It is interestingly remarkable that bi-fundamentals and adjoint scalars contribute to the deformed path integral measure with opposite powers and compensate in the case of affine geometries as shown in the example \((4.80)\).

In developing this analysis, we have taken the opportunity to complete partial results on the 2d \(\mathcal{N} = 2\) supersymmetric gauged linear sigma model realization of the resolution of affine singularities and the local Calabi-Yau threefold with non trivial fibrations. We have also given comments on other black hole models testing OSV conjecture. They concern the class of 4d black holes with D-branes wrapping cycles in local threefolds with complex one dimension base manifolds \(B_k\) classified by Vinberg theorem \([44]\). The latter is known to classify Kac-Moody algebras in three main sets: (i) ordinary finite dimensional, (ii) affine Kac-Moody and (iii) indefinite set.
In the end, we would like to note that the computation given here can be also done by using the topological vertex method. Aspects of this approach have been discussed succinctly in present study. More details on this powerful method as well as other features related to 2d quiver gauge theories and topological string will be considered elsewhere.

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