Influence of the Gaia–Sausage–Enceladus on the Density Shape of the Galactic Stellar Halo Revealed by Halo K Giants from the LAMOST Survey

We present a study of the influence of the Gaia–Sausage–Enceladus (GSE) on the density shape of the Galactic stellar halo using 11,624 K giants from the LAMOST survey. Every star is assigned a probability of being a member of the GSE based on its spherical velocities and metallicity by a Gaussian mixture model. We divide the stellar halo into two parts by the obtained probabilities, of which one is composed of the GSE members and defined as the GSE-related halo, and the other one is referred to as the GSE-removed halo. Using a nonparametric method, the radial number density profiles of the two stellar halos can be well described by a single power law with a variable flattening $q (r = R^2 + [Z/q(r)]^2, \nu = \nu_0 r^{-\alpha})$. The index $\alpha$ is 4.92 ± 0.12 for the GSE-related halo and 4.25 ± 0.14 for the GSE-removed halo. Both of the stellar halos are vertically flattened at smaller radii but become more spherical at larger radii. We find that the GSE-related halo is less vertically flattened than the GSE-removed halo, and the difference of $q$ between the two stellar halos ranges from 0.07 to 0.15. However, after the consideration of the bias, it is thought to be within 0.08 at most of the radii. Finally, we compare our results with two Milky Way analogs that experience a significant major merger in the TNG50 simulation. The study of the two analogs also shows that the major merger–related stellar halo has a smaller ellipticity than the major merger–removed stellar halo.

Unified Astronomy Thesaurus concepts: Milky Way stellar halo (1060); Galaxy mergers (608); Milky Way formation (1053); Milky Way Galaxy physics (1056)

1. Introduction

In the Λ cold dark matter (ΛCDM) cosmological model, the galaxy structure forms hierarchically; galaxies grow by accretion from the mergers of smaller satellite galaxies (White & Rees 1978; Peebles 2020). According to the stellar mass ratio $\mu$ of the satellite to the host galaxy, mergers can be categorized into major ($\mu \geq 1/4$), minor ($1/4 \geq \mu \geq 1/16$), and very minor ($1/16 \geq \mu$) types. These mergers play an important role in driving the growth of the stellar components and determining their chemodynamical properties. With the advent of the large sky surveys, our understanding of the formation and evolution of the Galactic stellar halo has steadily advanced over the past few decades. Recent studies showed that the bulk of the accreted stellar halo is contributed by several massive dwarf galaxies. Observational evidence of the merger events can be found in the stellar kinematics (Nordström et al. 2004; Klement et al. 2008; Zhao et al. 2009, 2014; Liang et al. 2017; Deason et al. 2018; Gallart et al. 2019; Neeb et al. 2020; Feuillet et al. 2020; Zhao & Chen 2021; Wu et al. 2022), chemical abundance patterns (Das et al. 2020; Aguado et al. 2021; Buder et al. 2022; Feuillet et al. 2022), halo substructures (Janesh et al. 2016; Koppelman et al. 2019; Yang et al. 2019a; Naidu et al. 2020; Yuan et al. 2020; Carollo & Chiba 2021), and globular clusters brought by the satellites (Myeong et al. 2018; Kriujssen et al. 2019; Myeong et al. 2019; Woody & Schlaufman 2021). The merger process with a massive dwarf galaxy not only leaves observational relics like local tidal streams but also could affect the overall morphology of the stellar halo (Deason et al. 2013; Xue et al. 2015; Rodríguez-Gomez et al. 2016; Monachesi et al. 2019). Therefore, understanding the spatial distribution of halo stars, specifically the radial density profile and vertical flattening, is critical to disentangle the mass assembly history of the Milky Way.

Early studies attempting to measure the global structure of the Milky Way stellar halo had to rely on a small number of tracers and/or sky coverage, and they usually described the radial density profile by a single power law (SPL) with an index of 2.5–3.5 and a constant flattening $q$ of 0.5–1 (Harris 1976; Hawkins 1984; Sommer-Larsen 1987; Preston et al. 1991; Wetterer & McGraw 1996; Dehnen & Binney 1998). Recently, with the advent of large sky surveys, the density shape of the stellar halo can be measured out to a large distance using a variety of stellar tracers with an expanding number and sky coverage. Several studies seemed to favor a broken power law (BPL) with constant flattening using different tracers of main-sequence turnoff stars (Sesar et al. 2011; Pila-Diéz et al. 2015), blue stragglers and horizontal branch stars (Deason et al. 2011; Das et al. 2016), and RR Lyraes (Watkins et al. 2009; Sesar et al. 2010; Faccioli et al. 2014; Li & Binney 2022). They found a break radius $r_{\text{break}}$ between 19 and 30 kpc. Stars inside $r_{\text{break}}$ followed a power law with an index of about 2.5, while stars outside $r_{\text{break}}$ followed a much steeper one with an index of approximately 4. However, the existence of $r_{\text{break}}$ is controversial. Xue et al. (2015) found...
that the radial density profile of the SEGUE K giants can be fitted well by a BPL with constant flattening or an SPL with variable flattening, alternatively. Further studies also showed that a break radius is not necessarily needed, assuming that the halo density shape varies with radius, and the index $\alpha$ is about 4–5 using an oblate ellipsoid model (Das et al. 2016; Hernitschek et al. 2018; Xu et al. 2018) or 2.96 using a triaxial ellipsoid model (Iorio et al. 2018).

In this study, we aim to explore the influence of the Gaia–Sausage–Enceladus (GSE) on the density shape of the stellar halo using 11,624 halo K giants selected from the Large Sky Area Multi-Object Fiber Spectroscopic Telescope (LAMOST) survey. The data release of the Gaia satellite has contributed significantly to our understanding of the Galactic formation history. One of the most insightful findings is that our galaxy experienced a major merger event with a dwarf galaxy named the GSE (Belokurov et al. 2018; Helmi et al. 2018). The stellar debris of the GSE is characterized by the large eccentricity (ec) and low angular momentum along the z-axis ($L_z$), and the metallicity distribution function of the GSE members has a peak of $[\text{Fe}/\text{H}]$ ranging from $-1.4$ to $-1.2$ dex (Sahlholdt et al. 2019; Gallart et al. 2019; Naidu et al. 2020; Mackereuth & Bovy 2020; Das et al. 2020; Feuillet et al. 2020). By studying the chemodynamic properties of its possible members, the GSE is thought to be a large dwarf galaxy with a stellar mass of the order of $10^9–10^{10} M_\odot$ (Mackereth et al. 2019; Fatiah et al. 2019; Vincenzo et al. 2019; Deason et al. 2019). The head-on collision of the ancient GSE and the Milky Way is expected to have happened around 8–10 Gyr ago (Sahlholdt et al. 2019; Bignone et al. 2019; Bonaca et al. 2020).

The stellar debris of the GSE is first identified in the solar neighborhood and then shown to stretch from the inner to the outer stellar halo (Lancaster et al. 2019; Bird et al. 2021; Wu et al. 2022). Using different tracers of blue horizontal branch (BHB) stars, K giants, main-sequence stars, and RR Lyraes, recent studies have found that the GSE could deliver over half of the Galactic stellar halo (Necib et al. 2019; Lancaster et al. 2019; An & Beers 2021; Iorio & Belokurov 2021; Wu et al. 2022; Ablimit et al. 2022). Considering the large fraction of stars originating from the GSE, the global structure of the Galactic stellar halo is expected to be heavily influenced by this merger event. The influence of the major merger on the morphology of the host galaxy, including the dark matter (DM) and stellar halos, has been widely studied in many previous simulation works (Zhao et al. 2003; Vera-Ciro et al. 2011; Elias et al. 2018; Monachesi et al. 2019; Drakos et al. 2019; Pulsoni et al. 2021). The galaxy halo is elongated along the merger axis and becomes less concentrated due to the last major merger. Drakos et al. (2019) found that mergers on radial orbits lead to a prolate DM halo, while mergers on tangential orbits produce oblate remnants. Pulsoni et al. (2021) studied the stellar halo morphology of 1114 early-type galaxies in the TNG50 simulation. They also found that galaxies with a larger fraction of stars originating from the major merger events tend to be less vertically flattened, on average. Although it is hard to trace the evolution of the Galactic halo from the beginning of the major merger, the present density shapes of the accreted and in situ stellar halos may indicate the influence of the GSE. By studying the orbits of local metal-poor stars, Sato & Chiba (2022) showed that the global structure of the subsample of stars kinematically associated with the GSE is more spherical, with $q \sim 0.9$, than the general halo sample. Therefore, the Galactic stellar halo may evolve to be less vertically flattened due to the last major merger of the GSE.

This paper is organized as follows. In Section 2, we introduce the observational data of the halo K giants used in this study. Then we describe the Gaussian mixture model (GMM) and apply it to the observational data to select the possible GSE members in Section 3. The nonparametric method used for estimating the radial density profile is presented in Section 4. We analyze the density shapes of the GSE-related and GSE-removed halos in Section 5.1 and compare our results with two Milky Way analogs in the TNG50 simulation in Section 5.2. Finally, our conclusions are stated in Section 6.

2. Observational Data

The halo K giants used in this study are selected from the LAMOST DR5 catalog (Zhao et al. 2006; Cui et al. 2012; Luo et al. 2012; Zhao et al. 2012; Liu et al. 2015). These K giants are identified by a support vector machine classifier based on the spectral line features (Liu et al. 2014). As shown in Liu et al. (2014), the contamination by stars other than K giants is smaller than 2.5%. The distances of those K giants are estimated based on the fiducial isochrones using a Bayesian method presented in Xue et al. (2014). The photometric information used for correcting the selection effects was publicly released in the Two Micron All-Sky Survey (2MASS; Skrutskie et al. 2006). Line-of-sight velocity and stellar metallicity are derived by the well-calibrated LAMOST 1D pipeline (Wu et al. 2011, 2014). Proper motions are taken from Gaia EDR3 (Gaia Collaboration et al. 2021) by cross-matching with a radius of $\rho'$. The stellar sample provides 7D information for the halo stars, including equatorial coordinate information ($\alpha, \delta$), heliocentric distance $d$, line-of-sight velocity $v_{\text{los}}$, proper motion $(\mu_\alpha, \mu_\delta)$, and stellar metallicity $[\text{Fe}/\text{H}]$. The selected K giants have a mean distance precision $\langle \delta d / d \rangle$ of 13%, mean $v_{\text{los}}$ uncertainty of 7 km s$^{-1}$, and mean metallicity error of 0.14 dex. The proper-motion uncertainties of 95% of these halo stars are within 0.10 mas yr$^{-1}$.

The 7D information is transformed to the galactocentric Cartesian coordinates ($X, Y, Z$) and velocities ($U, V, W$) using the conventions in astropy (Astropy Collaboration et al. 2018). In galactocentric Cartesian coordinates, we use the default values of the solar galactocentric distance $r_{gc, 0} = 8.122$ kpc (Gravity Collaboration et al. 2018) and height $Z_0 = 20.8$ pc (Bennett & Bovy 2019). We adopt a solar motion of $(+12.9, +245.6, +7.78)$ km s$^{-1}$ (Reid & Brunthaler 2004; Gravity Collaboration et al. 2018; Drimmel & Poggio 2018). To transform to the galactocentric spherical coordinates $(r, \theta, \phi)$ and velocities $(v_r, v_\theta, v_\phi)$, we follow Equations (1)–(6) in Wu et al. (2022). The uncertainties in distance, line-of-sight radial velocity, and proper motion (including the measurement error and covariance of proper motions) are propagated using a Monte Carlo sampling to estimate the median, standard error, and covariance of the velocities $(v_r, v_\theta, v_\phi)$.

We first select stars with a relative distance precision better than 30% and retain stars with $|Z| > 2$ kpc and $[\text{Fe}/\text{H}] < -0.5$ to exclude possible disk stars. To eliminate egregious outliers, we require the absolute spherical velocities to be $<400$ km s$^{-1}$ and the uncertainties to be $<150$ km s$^{-1}$.

Wu et al. (2022) defined the stellar halo with the Sagittarius (Sgr) stream members removed as the Sgr-removed stellar halo. The unrelaxed Sgr stream (Ibata et al. 1994) is the most
obvious substructure in the Galactic halo, which may cause some bias in the GMM fitting (Lancaster et al. 2019; Wu et al. 2022). To obtain an unbiased measurement of the velocity ellipsoid and metallicity distribution, we remove the Sgr stream members published by Yang et al. (2019b), which contains about 1700 K giants selected from the LAMOST DR5. A detailed description of the integrals of motion method used for selecting the Sgr stream members can be found in X.-X. Xue et al. (2021, in preparation) and Wu et al. (2022). We use a less strict cut of $|Z|$ than Wu et al. (2022) in order to reduce the possible bias caused by the cut of $|Z|$ in estimating the stellar density. Figure 1 shows the spatial distributions (on the $\sqrt{X^2+Y^2}$–$Z$, $X$–$Y$, and $l$–$b$ planes) of our K giants, where most of them are located within $r_{gc}$ of 35 kpc. In the following, we will divide these halo K giants into two components by the GMM, of which one is composed of the possible GSE members and referred to as the GSE-related halo, and the other one is defined as the GSE-removed halo. The exploration of the difference in density shapes between the GSE-related and GSE-removed stellar halos will help us further understand the influence of the GSE on the global structure of the Galactic stellar halo.

3. Selection Methods and Results

In this section, we divide the Galactic stellar halo into two parts, the GSE-related and GSE-removed halos, by the GMM. The distributions of $L_s$ and $e_c$ are shown for the two selected halos. We apply the GMM to the mock data to check the accuracy of the selection in Section 3.2.

3.1. Gaussian Mixture Model

The Galactic stellar halo is thought to be heavily influenced by the ancient major merger of the GSE. To reveal the contribution of the GSE to the stellar halo, recent studies applied a GMM to describe the distributions of the spherical velocities and metallicity of halo stars (Necib et al. 2019; Lancaster et al. 2019; An & Beers 2021; Iorio & Belokurov 2021; Wu et al. 2022). The GMM divides the stellar halo into two components, of which one is a more metal-poor and isotropic stellar halo referred to as the GMM/isotropic component, and the other one represents a more metal-rich and radially biased stellar halo referred to as the GMM/anisotropic or GMM/Sausage component. In Wu et al. (2022), we explored the contribution of the GSE to the stellar halo by making use of the GMM and applying it to halo star samples of LAMOST K giants, SEGUE K giants (Xue et al. 2014), and Sloan Digital Sky Survey (SDSS) BHB stars (Xue et al. 2011). Here we briefly introduce the GMM.

For a star $D_i(v_{r,i}, v_{\phi,i}, v_{z,i}, [\text{Fe/H}]),$ the likelihoods of the isotropic and anisotropic stellar halos are shown in Equations (1) and (2),

\[
\mathcal{L}_{\text{iso}}(D_i) = \mathcal{N}(v_i|0, \sigma_{\text{iso}}^2)[\mathcal{N}([\text{Fe/H}]|\mu_{\text{iso}}, \sigma_{\text{iso}}^2)],
\]

\[
\mathcal{L}_{\text{an}}(D_i) = \frac{1}{2}[\mathcal{N}(v_i|\mathbf{v}^{\text{an}}, \sigma_{\text{an}}^2) + \mathcal{N}(v_i|\mathbf{v}^{\text{iso}}, \sigma_{\text{iso}}^2)]
\times \mathcal{N}([\text{Fe/H}]|\mu_{\text{an}}, \sigma_{\text{an}}^2),
\]

where $v_i$ and [Fe/H] are the galactocentric spherical velocity ($v_{r,i}, v_{\phi,i}, v_{z,i}$) and stellar metallicity of the star $i$, respectively.

In Equation (1), the spherical velocities of the isotropic halo are described by a 3D single Gaussian with mean spherical velocities set to zero. We enforce the velocity tensor in the tangential direction to be isotropic by setting $\sigma_{\text{iso}}^{v_{\phi}} = \sigma_{\text{iso}}^{v_{r}v_{\phi}}$, and $\sigma_{\text{iso}}^{v_{r}v_{r}}$ is used as a unified representation of the tangential velocity dispersion. The velocity dispersion $\sigma_{\text{iso}}^{v_{r}v_{r}}$ is a combination of the covariance matrix of measurement error $\Sigma_i = \text{diag}(\sigma_{v_{r,i}}^2, \sigma_{v_{\phi,i}}^2, \sigma_{v_{z,i}}^2)$ and the intrinsic dispersion $\sigma_{\text{iso}}^{v_{r}v_{r}} = \text{diag}(\sigma_{v_{r}v_{r},\text{iso}}^2, \sigma_{v_{\phi}v_{\phi},\text{iso}}^2, \sigma_{v_{z}v_{z},\text{iso}}^2)$. We define $\mu_{\text{an}}^\text{iso}$ as the mean metallicity. The metallicity dispersion $\sigma_{\text{an}}^{[\text{Fe/H}]}$ is a combination (in quadrature) of the individual measurement error $\sigma_{[\text{Fe/H}]}^{m,i}$ and the intrinsic dispersion $\sigma_{[\text{Fe/H}]}^{\text{iso}} = \sqrt{\sigma_{[\text{Fe/H}]}^{m,i} + \sigma_{[\text{Fe/H}]}^{\text{iso}}^2}$.

In Equation (2), the spherical velocities of the anisotropic halo are described by a symmetric bimodal Gaussian with two lobes of high positive and negative radial velocities. To track the dynamical state of the stellar halo, Belokurov et al. (2018) fitted the shape of the velocity ellipsoid of SDSS–Gaia main-sequence stars with a zero-mean multivariate Gaussian model. They found that the residuals show clear overdensities of stars with high positive and negative radial velocity, especially in the most metal-rich stellar halo. The two distinct high-$v_r$ lobes are proven to be the stellar debris of the GSE. We assume that the GMM/Sausage component is built from two Gaussians with an equal mixing fraction. To match the two $v_r$ lobes, the mean radial velocities of the two Gaussians are defined as $+(v_{\text{an}}^m, 0)$ and $-(v_{\text{an}}^m, 0)$. The mean rotation velocity $(v_{\phi}^m, v_{z}^m)$ of the Gaia–Sausage is not forced to be zero based on the analyses of Belokurov et al. (2018), Helmi et al. (2018), and Lancaster et al. (2019). In the two Gaussians, the mean velocity $\mathbf{v}^{\text{an}}$ is set to be $(v_{\text{an}}^m, 0, v_{\phi}^m)$, and $\mathbf{v}^{\text{iso}}$ is $-(v_{\text{an}}^m, 0, v_{\phi}^m)$. The velocity dispersion of the GMM/Sausage component $\Sigma_{\text{an}}^i$ is a

Figure 1. Spatial distributions of $R-Z$ (left), $X-Y$ (middle), and $l-b$ (right) for the LAMOST halo K giant sample.
combination of the covariance matrix of measurement error $\Sigma_n$ and the intrinsic dispersion $\Sigma_{an} = \text{diag}(\sigma_{v_{\text{cliktig}}}^2, \sigma_{\gamma_{\text{cliktig}}}^2, \sigma_{\phi_{\text{cliktig}}}^2)$. We define $\mu_{\text{Fe/H};i}$ as the mean metallicity of the anisotropic stellar halo. The full metallicity dispersion $\sigma_{\text{Fe/H};i}$ is a combination (in quadrature) of the measurement error $\sigma_{\text{Fe/H};i}$ and the intrinsic dispersion $\sigma_{\text{Fe/H};i}^2$. 

The likelihood of observing one star $D_i(v_{\text{cirs}}, v_{\text{rots}}, v_{\phi_{\text{rots}}}, [\text{Fe/H}])$ is defined as

$$L(D_i|\theta) = f_{\text{iso}} L_{\text{iso}}(D_i|\theta) + f_{\text{an}} L_{\text{an}}(D_i|\theta),$$

where $f_{\text{iso}}$ is the contribution of the isotropic component, and $f_{\text{an}}$ refers to the fraction of the anisotropic component, such that $f_{\text{iso}} + f_{\text{an}} = 1$.

There are 11 free parameters in the GMM. The likelihood of the GMM for a halo sample containing $N$ stars is defined as

$$L(D|\theta) = \prod_{i=1}^{N} L(D_i|\theta).$$

We allocate the K giants to different $r_{gc}$ bins and apply the GMM to each bin to obtain the best estimated parameters. To avoid the possible overfitting caused by the small number of stars in the outer halo ($r_{gc} > 30$ kpc), we concatenate the two $r_{gc}$ bins in Wu et al. (2022) together and use only one $r_{gc}$ bin in Table 1. The Markov Chain Monte Carlo (MCMC) fitting is accomplished by emcee, which is an implementation of the Goodman & Weare’s affine invariant MCMC ensemble sampler (Foreman-Mackey et al. 2013). We use 200 walkers and 1000 steps as burn-in, followed by 3000 steps. We define the 50th percentile of the marginalized posterior distributions as the best estimated value and the 16th and 84th as the uncertainties. The best estimated parameters shown in Table 1 are almost the same as the results of Wu et al. (2022).

In an $r_{gc}$ bin, we define the probability of a star $D_i$ belonging to the GSE as $\text{prob}(an)$, in Equation (5),

$$\text{prob}(an) = \frac{f_{\text{an}} L_{\text{an}}(D_i|\theta)}{f_{\text{iso}} L_{\text{iso}}(D_i|\theta) + f_{\text{an}} L_{\text{an}}(D_i|\theta)},$$

where the model parameters ($\theta, f_{\text{an}}, f_{\text{iso}}$) are the best estimated parameters of the corresponding $r_{gc}$ bin. We select stars with relatively larger prob(an) from this bin according to the fraction of $f_{\text{an}}$. We collect the selected stars in all $r_{gc}$ bins together and define them as the GSE-related halo, and the other stars are defined as the GSE-removed halo. As shown in Figure 2, the main difference between the two stellar halos is reflected in the distributions of the tangential velocities and metallicity, which is consistent with the chemodynamic properties of the GSE as a highly radially biased and relatively more metal-rich subcomponent. Figure 3 shows a clear division in the distributions of prob(an) between the two stellar halos. Most of the GSE-related halo stars have a prob(an) larger than 0.8, while the prob(an) of 70% of the GSE-removed halo stars is smaller than 0.4. We apply the GMM to the GSE-related halo again to check the possible residual from the GMM/isotropic halo. The large
$f_{an} > 0.99$ suggests a negligible contamination of the stars from the GMM/isotropic halo.

Previous studies usually select the possible GSE members from halo stars by a robust kinematic cut of $|L_z| < 500$ kpc km s$^{-1}$ or $ec > 0.7$. Although the majority of the GSE stars have a large $ec$ and small $|L_z|$, there is still a certain amount of them located outside of the selection criteria, as revealed by the N-body simulation of the GSE stellar debris (Naidu et al. 2021). We show the normalized distributions of the $|L_z|$ of the two stellar halos in Figure 4. The stellar debris of the simulated GSE is added as a comparison. The fractions of stars satisfying $|L_z| > 500$ are 0.73 for the GSE-related halo, 0.77 for the simulated GSE, and 0.28 for the GSE-removed halo. Our selected GSE-related halo is consistent with the simulated GSE in the distribution of $L_z$. Figure 5 shows the number distributions of the $ec$ of the two stellar halos. We find a division in $ec = 0.67$, beyond which the GSE-related halo stars become the dominant part of the stellar halo.

### 3.2. Mock Data Test of the GMM Selection Method

In this subsection, we apply a mock data test to check the accuracy of the GMM selection under different GMM parameters and $f_{an}$. For each distance bin in Table 1, we use the best estimated model parameters to specify the velocity and metallicity distributions of the GMM/isotropic and GMM/Sausage components and then draw random samples from the specified distributions to make the mock data with 1 million stars by the random sampling function Numpy.random.multivariate_normal() in the Numpy package (Harris et al. 2020). Every sampled point is convolved with a matrix of measurement errors selected randomly from the observational data. After the construction of the mock data of the GMM/isotropic and GMM/Sausage components, we mix them with $f_{an}$ ranging from 0.1 to 0.9. We randomly draw the same number of fake stars as the observational data from each newly constructed mock data set. The GMM selection is applied to the mock data to obtain the true discovery rates of the GSE-related and GSE-removed halo stars. We generate 200 synthetic catalogs in this manner to obtain the mean discovery rate and standard deviation.

Figure 6 shows the discovery rates of the GSE-related and GSE-removed halo stars under different GMM models and $f_{an}$. 

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**Figure 3.** Histograms of the number distributions of prob(an) of the GSE-related (blue histogram) and GSE-removed (orange histogram) halos. Most K giants of the GSE-related halo have a prob(an) larger than 0.85, while the prob(an) of 70% of K giants of the GSE-removed halo is smaller than 0.4.

**Figure 4.** Histograms of the normalized distributions of the angular momentum $L_z$ for stars of the GSE-related halo (blue), the GSE-removed halo (orange), and the stellar debris of the simulated GSE of Naidu et al. (2021; green). The GSE-related halo has a much stronger distribution in $|L_z| < 500$ kpc km s$^{-1}$ than the GSE-removed halo, which conforms to the highly radial orbits of the stellar debris of the GSE. However, there is still a certain amount of stars with $|L_z| > 500$ kpc km s$^{-1}$ in both the observational and simulation data of the GSE. A robust cut of $L_z$ may lead to a deficiency of them.

**Figure 5.** Histograms of the number distributions of $ec$ for stars of the GSE-related (blue) and GSE-removed (orange) halos. We find a division in $ec = 0.67$, where the number of stars in either halo is equal.

**Figure 6.** Discovery rates of the GSE-related (blue dashed lines) and GSE-removed (orange solid lines) stellar halos under different GMM models and $f_{an}$ obtained by applying the GMM selection method to the mock data. Every dashed and solid line represents a model of the GMM/isotropic and GMM/Sausage components of a distance bin in Table 1, respectively.
It is clear that a higher $f_{an}$ results in a larger discovery rate of the GSE-related halo and a smaller discovery rate of the GSE-removed halo. The true value of $f_{an}$ obtained from our K giant sample is about 0.6–0.70, which corresponds to a typical discovery rate of 0.80–0.95 for the GSE-related halo and 0.70–0.85 for the GSE-removed halo. In general, it is a good result, especially for the GSE-removed halo, considering that the GSE is the dominant component. However, we note that the GMM is a simplified model where the metallicity distributions of the two components are simply described by a Gaussian function. Therefore, slight differences between the discovery rates of the observational and mock data may exist.

Figure 7 shows the distributions of the spherical velocities and the metallicity of the two stellar halos selected from the observational and mock data. These halos are consistent with the models of their corresponding GMM components, and the structure of the two $v_\phi$ lobes of the GSE-related halo is clearly seen in distance bins with small $r_{gc}$. We apply a Kolmogorov–Smirnov test to check the similarities between the stellar halos selected from the observational and mock data. From the $p$-values, we find that the GSE-related halos of the two data sets agree well with each other in both spherical velocities and metallicity distributions. For the GSE-removed halo, we find a good level of similarity in the spherical velocities, while some differences exist in the metallicity distribution. The GSE-
removed halo has a more complex formation mechanism than the GSE-related halo, which may originate from the in situ process, gas accretion, multiple minor mergers, the dissolution of globular clusters, etc. Therefore, a simple Gaussian function seems to be inadequate to model the distribution of globular clusters, etc. Therefore, a simple Gaussian function process, gas accretion, multiple minor mergers, the dissolution of the GSE-related halo has a more complex formation mechanism than the GSE-removed halo.

4. Fitting Method

In this section, we describe the method used for estimating the stellar density and apply it to derive the density shapes of the GSE-related and GSE-removed halos.

4.1. Estimation of the Stellar Density

Liu et al. (2017) and Xu et al. (2018) introduced a statistic method to derive the stellar density profile along each line of sight of a spectroscopic survey. The core of this method is to treat the contribution of a star as a probability density function extending along a line of sight. Selection effects due to the target selection and distance estimation are considered in the estimation. Using this method, we can construct a density map of the Galactic stellar halo with varying vertical flattening. We will briefly introduce the nonparametric method hereafter, and more details can be found in Liu et al. (2017).

We assume that the photometric survey is complete for stars brighter than the limiting magnitude. Compared to photometric surveys, spectroscopic surveys are less complete and have smaller coverage due to the target selection. To correct the bias generated by selecting subsamples, we need the selection function $S$ in which the probability of a star being included in the observation is given by their Galactic coordinates $(l, b)$, colors $(c)$, and apparent magnitudes $(m)$. For a given line of sight, the relation between the densities derived from the photometric ($\nu_{\text{ph}}$) and spectroscopic ($\nu_{\text{sp}}$) data is assumed to be

$$\nu_{\text{ph}}(D; c, m, l, b) = \nu_{\text{sp}}(D; c, m, l, b) S^{-1}(c, m, l, b),$$

where $D$ is the distance along the given line of sight.

The selection function $S(c, m, l, b)$ is defined as

$$S(c, m, l, b) = \frac{n_{\text{sp}}(c, m, l, b)}{n_{\text{ph}}(c, m, l, b)},$$

where $n_{\text{sp}}(c, m, l, b)$ and $n_{\text{ph}}(c, m, l, b)$ are the star counts of the spectroscopic and photometric data in the color–magnitude diagram. The selection function of the LAMOST K giants is provided by Liu et al. (2017), which is based on the photometric survey of 2MASS.

After the integration over $c$ and $m$, we can obtain the stellar density profile for a given line of sight as

$$\nu_{\text{ph}}(D; l, b) = \int \nu_{\text{sp}}(D; c, m, l, b) S^{-1}(c, m, l, b) dcdm.$$

To obtain $\nu_{\text{ph}}$, we need to first obtain $\nu_{\text{sp}}$ from the spectroscopic data. We apply a kernel density estimation method to derive $\nu_{\text{sp}}$ along a given line of sight. We treat the contribution of a star $i$ as a probability density function $p_i$ extending along the corresponding line of sight, and $p_i$ is defined as

$$p_i(D) = \frac{N(D; D_i, \sigma_D^2)}{\int_{D_{\text{min}}}^{D_{\text{max}}} N(D; D_i, \sigma_D^2) dD},$$

where $N$ is a normal function, $D_i$ is the estimated distance of star $i$, and $\sigma_D$ is the uncertainty of the estimated distance. We set $D_{\text{min}} = 0$ kpc and $D_{\text{max}} = 200$ kpc.

We take into account the contribution of all stars around $c$ and $m$. The density profile $\nu_{\text{sp}}$ is defined as

$$\nu_{\text{sp}}(D; c, m, l, b) = \frac{1}{\Omega D^2} \sum_i n_{\text{sp}}(c, m, l, b) p_i(D),$$

where $\Omega$ is the solid angle of the given line of sight. Since it is a constant value and has no influence on the obtained density shape, we normalize it to 1. Combining Equations (8) and (10), we obtain the derived $\nu_{\text{ph}}$ as a continuous function of $D$ for a given line of sight. The LAMOST survey has thousands of plates (lines of sight) to be observed, and each plate has a derived $\nu_{\text{ph}}$. However, as Xu et al. (2018) pointed out, the derived $\nu_{\text{ph}}$ may suffer from large uncertainties at the position with no stars. Therefore, we will only adopt $\nu_{\text{ph}}$ where we can find a star. In other words, every halo K giant is assigned a derived $\nu_{\text{ph}}$ that represents an approximation of the stellar density at this position.

The nonparametric method requires no analytic density profiles, such as a Einasto function, before the fitting process. The independence of any preset stellar density model enables Xu et al. (2018) to study the variation of the vertical flattening of the Galactic stellar halo. We will use the same nonparametric method as Xu et al. (2018) to obtain the density maps of the GSE-related and GSE-removed halo samples in the following.

4.2. Fitting of the Density Shape

We remove stars dimmer than the 2MASS limiting magnitude $K$ of 14.3 mag to ensure the completeness of the photometric survey. This final selection retains 7872 stars in the GSE-related halo and 3752 stars in the GSE-removed halo.

There is not much difference in the sky area ($l–b$ panel) between the two stellar halos. We apply the nonparametric method to the two stellar halos and derive the density $\nu$ of their K giants, respectively. Figure 8 shows the median value of $\nu$ of the K giants at each $R–|Z|$ pixel. We can see that both the GSE-related and GSE-removed stellar halos tend to be more vertically flattened at smaller radii, and the GSE-removed halo seems to have a smaller ellipticity.

In order to characterize the vertical flattening of the density shapes quantitatively, we fit the isodensity contour of the derived density $\nu$ of these K giants with an ellipsoidal profile, defined as

$$\nu(r) = \nu_0 r^{-\alpha},$$

$$r = \sqrt{R^2 + (Z/q(r))^2},$$

where $r$ is the semimajor axis of the ellipse, and $q$ is defined as the ratio of the semiminor to the semimajor axis. In this study, we assume a ratio of the semi-intermediate to semimajor axis $p$ to be 1 ($R = X^2 + Y^2/p^2$, $p = 1$). A recent study of the density shape of the Galactic stellar halo by RR Lyrae of Gaia DR2 shows a $p$-value of 0.77, and the major axis is tilted by a $\phi$ of...
$21^\circ$ with respect to the $y$-axis (Iorio & Belokurov 2019). We note that the values of $p$ and $\phi$ are obtained from the study of the whole Galactic stellar halo by RR Lyraes, which could be different for the GSE-related and GSE-removed halos in this study. We also try to add $p$ and $\phi$ to our density model as two free parameters. However, a larger halo star sample with more complete sky coverage is needed to obtain a reliable result of $p$ and $\phi$ as a function of $r$. Therefore, we will use the simplified density model and only focus on the variation of $q$ in this study.

We divide the K giants into 23 $\ln(\nu)$ bins with a bin size of 0.25 in the range of $-15.525 < \ln(\nu) < -9.875$. A total of 7389 stars in the GSE-related halo and 3500 stars in the GSE-removed halo remain in this range. The specific star numbers and best estimated parameters of the two stellar halos in each $\ln(\nu)$ bin are recorded in Table 2. We fit the spatial distribution of the K giants with an ellipse in each $\ln(\nu)$ bin and then obtain the best estimated spherical radius $r$ and flattening $q$. The detailed fitting method is introduced as follows.
In the ellipsoidal coordinates, $R$ and $Z$ are defined as
\begin{align}
R &= r \cos(\eta), \\
Z &= r q \sin(\eta), \\
r_{gc} &= \sqrt{R^2 + Z^2}, \\
\sin(\theta) &= Z/r_{gc}.
\end{align}

Combining Equations (13)–(15), we rewrite $r_{gc}$ as
\begin{equation}
r_{gc} = r \sqrt{1 + (q^2 - 1)\sin^2(\eta)}.
\end{equation}

Using Equation (16), we write $\sin(\eta)$ as
\begin{equation}
\sin(\eta) = \frac{r_{gc} \sin(\theta)}{rq}.
\end{equation}

Substituting Equation (18) into Equation (17), we can fit the isodensity contour in $r_{gc} - \sin(\theta)$ space with Equation (19),
\begin{equation}
r_{gc} = rq \sqrt{\frac{1}{q^2 - (q^2 - 1)\sin^2(\theta)}}.
\end{equation}

5. Results

In this section, we show the radial density profiles of the GSE-related and GSE-removed stellar halos. We find two Milky Way–like galaxies in the TNG50 simulation and analyze the density shapes of the major related and removed stellar halos.

5.1. Number Density Shapes and Profiles of the Observational Data

We take the fitting results of the bin with $\ln(\nu) = -12.25$ as an example. We fit the distributions of all K giants of $-12.50 < \ln(\nu) < -12.00$ with Equation (19) in $r_{gc} - |\sin(\theta)|$ space. We use the scipy.optimize.curve_fit function in the fitting. Figure 9 shows the best fitting results of the GSE-related and GSE-removed halos at $\ln(\nu) = -12.25$. The median values of $r_{gc}$ in each $\sin(\theta)$ bin are very well in accordance with the best fitting results. When turning to the $R-|Z|$ space, we find an ellipse of $r = \sqrt{R^2 + Z^2/q^2}$ clearly shown by the median values and the fitting results. Both of the stellar halos are vertically flattened at $\ln(\nu) = -12.25$, but the GSE-related halo is proven to be more spherical than the GSE-removed halo from the best estimated $q$.

The fitting results of all $\ln(\nu)$ bins are shown in Figure 10 for the GSE-related halo and Figure 11 for the GSE-removed halo. In general, the distributions of the K giants in $r_{gc} - |\sin(\theta)|$ space can be well described by Equation (19). However, due to the lack of enough data points and the large uncertainties of the distance estimation in the outer halo, a certain degree of dispersion exists at severe bins with the lowest $\ln(\nu)$, for example, $\ln(\nu) = -15.25$ and $-15.62$. A larger data sample including more outer halo stars is needed to improve the accuracy of the estimation in the future. Table 2 shows the best estimated values and the errors. We remove the outliers at $\ln(\nu) = -10.25$ in Figure 12 to ensure the reliability of the fitting.

The upper panel of Figure 12 shows the variation of $q$ with the spherical radius $r$. Both of the stellar halos are vertically flattened where $q$ ranges from 0.4 to 0.5 at the inner radii ($r < 15$ kpc). The two stellar halos become more spherical with increasing $r$. At larger radii ($r > 25$ kpc), the values of $q$ are about 0.8–0.9 for the GSE-related halo and 0.6–0.8 for the GSE-removed halo. We define $\Delta(q)$ as the difference of the average $q$ between the two stellar halos in a spherical radius bin with a bin size of 5 kpc. From Figure 13, we find that the GSE-related halo is less vertically flattened than the GSE-removed halo, and $\Delta(q)$ is within 0.15 in most bins. Our results are consistent with those of Sato & Chiba (2022) that the global structure of the subsample of stars possibly associated with the GSE is more spherical. Similar cases can also be found in the stellar halos of Au8 and Au25 from the Auriga simulation (Monachesi et al. 2019). Au8 and Au25 experienced interactions with massive satellites 4 and 0.9 Gyr ago, respectively. Monachesi et al. (2019) found that the accreted stellar halos of Au8 and Au25 are less vertically flattened than the whole stellar halo, and the difference of $q$ is up to 0.2. The difference of $q$ between the two stellar halos indicates that the whole Galactic stellar halo may become less vertically flattened due to the GSE merger. This result corresponds to the hydrodynamical simulation that stellar halos of galaxies with a larger fraction of accreted stars from the major mergers tend to have a larger $q$, on average (Pulsoni et al. 2021).

The lower panel of Figure 12 shows the relation between $\ln(r)$ and $\ln(\nu)$. We fit it with an SPL as Equation (11). The radial density profiles of the two stellar halos can be described well by an SPL with $\alpha = 4.97 \pm 0.12$ for the GSE-related halo and $\alpha = 4.25 \pm 0.14$ for the GSE-removed halo. The steeper density profile of the GSE-related halo is mainly due to the decline of the contribution of the GSE in the outer halo. We also try to fit the two density profiles with a BPL. We obtain an $r_{\text{break}}$ of 12.32 kpc for the GSE-related halo and 12.11 kpc for the GSE-removed halo. Considering that only one or two points are located within $r < 13$ kpc, the obtained $r_{\text{break}}$ is not credible and more like an overfitting. Therefore, our results are more inclined to an SPL, where no $r_{\text{break}}$ is necessarily needed.

In this study, we divide the Galactic stellar halo into two components and find that the GSE-related halo is more spherical than the GSE-removed halo. However, a check of the nonparametric method shows that the difference of $q$ may be exaggerated due to the different star numbers in the two stellar halos. We select 7872 stars randomly from our K giant sample (corresponding to the star number of the GSE-related halo), and 3752 stars are left after the random selection (corresponding to the star number of the GSE-removed halo). We define the halo sample with 7872 stars as halo-7872, and the other one is referred to as halo-3752. We then apply the nonparametric method to the two halo samples and obtain $\Delta(q)$ as the difference of the average $q$ between the GSE-related halo and GSE-removed halo in each $r$ bin. We repeat this 200 times and find that halo-7872 tends to be less vertically flattened than halo-3752, on average. This bias is mainly caused by the special treatment used in Equations (8)–(10). In Equation (8), every K giant is assigned a derived $\nu_{\text{ph}}$ that includes the contribution of this K giant and other stars in the same line of sight. The decrease of the star number reduces the contribution of other stars and causes the decline of $\nu_{\text{ph}}$, which will further lead to the decline of $\nu_{\text{ph}}$. The calculated density shape will remain unchanged if the decline of $\nu_{\text{ph}}$ is the same everywhere. However, due to the vertical flattening, stars in our halo sample tend to be closer to each other in a line of sight pointing to a higher galactic latitude, which makes the contribution of other stars more important. Therefore, the decrease of the star number tends to cause a larger decline of...
\(\nu_p\) and \(\nu_{ph}\) in a line of sight pointing to a higher latitude. We define the median \(\Delta(q)\) of halo-7872 and halo-3752 as the signal introduced by the different star numbers and the standard deviation as the 1σ error. The median \(\Delta(q)\) caused by the different star numbers is about 0.03–0.06, as shown in Figure 13, which may partly contribute to the \(\Delta(q)\) of the GSE-related and GSE-removed halos. To reduce this bias much as possible, we randomly select a subsample of 3752 stars from the GSE-related halo and compare it with the GSE-removed halo. We repeat this 200 times and obtain the median \(\Delta(q)\) \(\langle q/\text{subsamp}\rangle - \langle q/\text{GSE-removed}\rangle\) in each \(r\) bin. In Figure 13, we find that this median \(\Delta(q)\) slowly increases from 0.04 to 0.07 at \(r \sim 10–30\) kpc and suddenly increases up to 0.14 at \(r \sim 30–35\) kpc. Although it still supports the view that the GSE-related halo is less vertically flattened than the GSE-removed halo, the intrinsic difference of \(q\) between the two stellar halos is probably less obvious than the result shown in Figure 12. The long-time dynamical evolution of the GSE members in the Galactic halo may erase their features in the spatial distribution, which makes the GSE-related halo less distinguishable from \(q\) than the accreted stellar halos of Au8 and Au25.

5.2. Milky Way Analogs in TNG50

The lack of enough observational data prevents us from obtaining a full morphology of the accreted stellar halo. In this subsection, we select two Milky Way analogs that experience a significant major merger event from the TNG50 simulation and study the density shapes of the major merger–related and major merger–removed stellar halos. Using cosmological hydrodynamical simulations can help us disentangle the impact of the major merger event on the stellar halo without the bias caused by the selection effect and distance estimation.

The TNG50 is the highest-resolution volume in the IllustrisTNG cosmological hydrodynamical simulations (Pillepich et al. 2018; Nelson et al. 2019). The simulation follows the evolution of \(2 \times 2160^3\) initial resolution elements. The mass of the DM particle is \(4.5 \times 10^3\ M_\odot\), and the mean mass of the gas and the star particles is \(8.5 \times 10^4\ M_\odot\).

Emami et al. (2021a, 2021b) selected a sample of 25 Milky Way–like galaxies by the total subhalo mass and the mass fraction of stars satisfying \(e > 0.7\) from the TNG50 simulation. All of the selected galaxies are required to have a disklike, rotationally supported morphology that is similar to the Milky Way. We first check the merger histories of those 25 selected Milky Way–like galaxies by their merger trees (Rodriguez-Gomez et al. 2015). We require the host galaxy to have experienced at least one major merger with a satellite having a stellar mass larger than \(10^9\ M_\odot\), at redshift \(z < 4\). We remove the galaxy with ID 529365 because its major merger–related stars are mainly located within \(r_{gc} < 5\) kpc. We finally obtain two Milky Way analogs with IDs 506720 and 505586 at redshift \(z = 0\). Galaxy 505586 experienced a merger with galaxy 231738 (ID at the snapshot of 42 in the TNG50 simulation) of a stellar mass of \(2.4 \times 10^{10}\ M_\odot\) 9.0 Gyr ago, and galaxy 506720 experienced a merger with galaxy 270761 (ID at the snapshot of 46) of a stellar mass of \(1.0 \times 10^{10}\ M_\odot\) 8.4 Gyr ago. We define stars originating from the massive

Figure 9. Fitting results of the GSE-related (upper panels) and GSE-removed (lower panels) halos at \(\ln(\nu) = -12.25\). The left panels show the histograms of the deviation of the fitting \((r_{gc} - r_0)\). The middle panels show the distribution of the K giants (blue points) and the fitting results (black lines) in \(r_{gc} - |\sin(\theta)|\) space. The orange triangles represent the median values of \(r_{gc}\) in each \(\sin(\theta)\) bin, which conform to the best-fitting lines very well. The right panels show the K giants (blue points) and the fitting results (black lines) in \(R - |Z|\) space. Although the distributions of the halo K giants have a large dispersion, the median values (orange triangles) and the best fitting results show a clear ellipse, which suggests that both of the stellar halos are vertically flattened. Compared to the GSE-removed halo, the GSE-related stellar halo is less vertically flattened at \(\ln(\nu) = -12.25\).
satellite of the last major merger as the major related stellar component and the other stars as the major removed stellar component.

We use a kinematic definition of the stellar halo as in previous numerical works (Aumer et al. 2013; Gómez et al. 2017; Monachesi et al. 2019). We first define the net rotation direction \( \mathbf{j}_{\text{net}} \) (\( z \)-axis) as

\[
\mathbf{j}_{\text{net}} = \frac{\sum m_i \mathbf{r}_i \times \mathbf{v}_i}{\sum m_i},
\]

where \( i \) refers to a star particle in the host galaxy. The orbital circularity parameter \( \epsilon_i \) for a star \( i \) is defined as

\[
\epsilon_i = \frac{J_{z,i}}{J_{c}(E_i)} = \sqrt{\frac{GM(<r_c)\, r_c}{E_i}},
\]

where \( J_{z,i} \) is the magnitude of angular momentum along the \( z \)-axis, and \( J_{c}(E_i) \) is the maximum specific angular momentum at the same specific binding energy \( E_i \). A more detailed introduction of the circular radius \( r_c \) and binding energy \( E_i \) can be found in Emami et al. (2021a). We keep the subset of star particles satisfying \( \epsilon_i < 0.7 \) as the spheroidal component and view it as the stellar halo.

The completeness of the simulation data allows us to study the full morphology of the stellar halo, including the semi-intermediate to semimajor axis ratio \( p \). Motivated by Emami et al. (2021a), we calculate the stellar density shape using a local iterative shell method (LISM). We split the stellar halo into 100 logarithmic radial thin shells with a spherical radius \( r_{\text{sph}} \) ranging from \( r_{\text{sph}}^{\text{min}} = 5 \) to \( r_{\text{sph}}^{\text{max}} = 40 \) or 60 kpc. The \( r_{\text{sph}}^{\text{min}} = 5 \) kpc is used to eliminate the bulge stars. We assume that the spatial distribution of star particles in each shell follows a triaxial ellipsoid. For a given \( r_{\text{sph}} \), the axis lengths and orientations are the eigenvalues and eigenvectors of the reduced inertia tensor, defined as

\[
I_{ij} = \frac{1}{M_{\text{tot}}} \sum_{n=1}^{N} m_n x_{n,i} x_{n,j}, \quad i, j = 1, 2, 3,
\]

where \( M_{\text{tot}} = \sum_{n=1}^{N} m_n \), and \( N \) is the total number of star particles in this thin shell. Here \( m_n \) represents the mass of the \( n \)-th particle, and \( x_{n,i} \) is the \( i \)-th coordinate of the \( n \)-th particle. The reduced inertia tensor has been widely used in calculating the density shapes of DM and stellar components in previous numerical works (Jing & Suto 2002; Schneider et al. 2012; Shao et al. 2021). The normalization of the particle positions, \( R_n(r_{\text{sph}}) \), is the elliptical of the \( n \)-th particle defined in terms of the halo axis length,

\[
R_n^2(r_{\text{sph}}) = \frac{x_n^2}{a^2(r_{\text{sph}})} + \frac{y_n^2}{b^2(r_{\text{sph}})} + \frac{z_n^2}{c^2(r_{\text{sph}})},
\]

where \( (a, b, c) \) represent the axis length of the triaxial halo. We require that star particles in this thin shell satisfy

Figure 10. Fitting results of the GSE-related stellar halo in each \( \ln\nu \) bin. The blue points represent the K giants in \( r_{gc} - |\sin(\theta)| \) space, and the black lines describe the best fitting results. The median values of \( r_{gc} \) in each \( \sin(\theta) \) bin are indicated by the orange inverted triangles, and the bin size is 0.1.

"
At every $r_{\text{sph}}$, we start with $a = b = c = r_{\text{sph}}$ in the first iteration. Then we use the eigenvalues and eigenvectors of the reduced inertia tensor to deform the above shell while keeping the volume within the ellipsoid fixed. The new

\begin{align*}
\mathbf{R} & = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} \\
\mathbf{R}^{-1} & = \frac{1}{abc^2} \begin{bmatrix} c^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} \\
\mathbf{A} & = \mathbf{R}^{-1} \mathbf{M} \mathbf{R} 
\end{align*}

where $\mathbf{M}$ is the inertia tensor of the initial ellipsoid. The eigenvalues and eigenvectors of $\mathbf{A}$ are used to deform the initial shell $\mathbf{X}$ to the new shell $\mathbf{X}'$ as follows:

\begin{align*}
\mathbf{X}' & = \mathbf{R} \mathbf{X} \\
\mathbf{X}' & = \mathbf{R} \mathbf{A} \mathbf{R}^{-1} \mathbf{X} \\
\mathbf{X}' & = \mathbf{R} \begin{bmatrix} a^2 \mathbf{X} \\ b^2 \mathbf{X} \\ c^2 \mathbf{X} \end{bmatrix} \\
\mathbf{X}' & = \begin{bmatrix} a \mathbf{X} \\ b \mathbf{X} \\ c \mathbf{X} \end{bmatrix} 
\end{align*}

where $\mathbf{X}$ is the initial shell. The new shell $\mathbf{X}'$ is then used as the initial shell for the next iteration.

\begin{align*}
\mathbf{X}' & = \mathbf{X} + \mathbf{X}' \\
\mathbf{X}' & = \mathbf{X} + \mathbf{R} \mathbf{A} \mathbf{R}^{-1} \mathbf{X} \\
\mathbf{X}' & = \mathbf{X} + \begin{bmatrix} a^2 \mathbf{X} \\ b^2 \mathbf{X} \\ c^2 \mathbf{X} \end{bmatrix} \\
\mathbf{X}' & = \begin{bmatrix} a \mathbf{X} \\ b \mathbf{X} \\ c \mathbf{X} \end{bmatrix} 
\end{align*}

where $\mathbf{X}$ is the new shell. The new shell $\mathbf{X}'$ is then used as the initial shell for the next iteration.
axis lengths \((a, b, c)\) are rescaled and given by

\[
\begin{align*}
& a = \frac{r_{\text{sph}}}{(abc)^{\frac{1}{3}}} \sqrt{\lambda_1}, \\
& b = \frac{r_{\text{sph}}}{(abc)^{\frac{1}{3}}} \sqrt{\lambda_2}, \\
& c = \frac{r_{\text{sph}}}{(abc)^{\frac{1}{3}}} \sqrt{\lambda_3},
\end{align*}
\]

where \(\lambda_i\) \((i = 1, 2, 3; \lambda_1 \geq \lambda_2 \geq \lambda_3)\) are the absolute eigenvalues of the reduced inertia tensor. In the \(m\)th iteration, we define the intermediate to major axis ratio \(p_m\) as \(\frac{b}{a}\) and minor to major axis ratio \(q_m\) as \(\frac{c}{b}\). The iteration algorithm is terminated when either the residuals of the shape parameters converge to a given tolerance \((\text{Max}\ [(|p_m - p_{m-1}|)/p_m]^2, (|q_m - q_{m-1}|)/q_m]^2) \leq 10^{-3}\) or the total number of star particles inside the shell is smaller than 1000.

An advantage of the simulation data is that we can directly select stars from the massive satellite by the provided particle IDs. We apply the LISM to the major related and removed stellar halos of the two Milky Way analogs at redshifts \(z = 0\). The density shape parameters \((p, q)\) at redshifts \(z = 0\) are shown as functions of \(r_{\text{sph}}\) in Figure 14. The main difference is the value of \(q\). It is evident that the major related stellar halo is less vertically flattened than the major removed stellar halo. For the galaxy 505586, the difference of \(q\) peaks at \(r_{\text{sph}} = 5\) kpc with a value of 0.35. Then it declines progressively with the increasing \(r_{\text{sph}}\) and finally keeps in the range of 0.11–0.18 at \(r_{gc} > 17\) kpc. This tendency is mainly caused by the decrease of \(q\) in the major related stellar halo, while the \(q\) of the major removed halo hardly changes. Things are different for the galaxy 506720, where the major removed stellar halo has a larger variety of \(q\) than the major related stellar halo. The difference of \(q\) is within 0.1 at \(r_{gc} = 10–29\) kpc, which is much smaller than the galaxy 505586 and more similar to the Milky Way. At the outer halo, this difference increases with a peak around 0.18 at \(r_{gc} = 33\) kpc and finally stays in the range of 0.10–0.15 at \(r_{gc} = 40–60\) kpc. Our study of the two Milky Way analogs also suggests that the global structure of stars associated with the major merger event is less vertically flattened than the other stars.

6. Summary

In this study, we dived the halo K giants into two components, the GSE-related and GSE-removed stellar halos, by the GMM. The radial density profiles of the two stellar halos are determined through a nonparametric method that allows the variation of the vertical flattening with the spherical radius. Our results are compared with two Milky Way analogs that are significantly influenced by the last major merger. We summarize our main results as follows.

1. Both of the stellar halos are vertically flattened at the inner radii and become more spherical with the increasing spherical radius. The GSE-related stellar halo is less vertically flattened than the GSE-removed halo, which suggests that the Galactic stellar halo may become more spherical due to the last major merger. The difference of \(q\) between the two stellar halos is about 0.07–0.15. After considering the possible bias caused by the different star numbers, this difference is thought to be within 0.08 in most of the spherical radii expect for \(r = 30–35\) kpc.
2. The radial density profiles of the two stellar halos can be fitted well with an SPL assuming that the halo density shape varies with the spherical radius. The index $\alpha$ of the SPL is $4.92 \pm 0.12$ for the GSE-related halo and $4.25 \pm 0.14$ for the GSE-removed halo. The steeper density profile of the GSE-related stellar halo is mainly caused by the decline of the fraction $f_{\text{in}}$ in the outer halo.

3. Using the LISM method, we study the stellar density shape parameters ($\rho, q$) of the two Milky Way analogs in the TNG50 simulation. Our result shows that the major related stellar halo is less vertically flattened than the major removed stellar halo for the two analogs.

In this study, we explore the influence of the GSE on the stellar halo by comparing the density shapes of the GSE-related halo with the GSE-removed halo. The difference in vertical flattening between the two stellar halos is exaggerated due to the signal introduced by the special treatment in the nonparametric method. We suspect that this bias will be reduced by making use of a halo star sample with a larger star number and more complete sky coverage. The lack of enough data points also prevents us from determining the semi-intermediate to major removed stellar halo for the two analogs.

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