Eugene Wigner and Translational Symmetries

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Abstract

As Einstein’s $E = mc^2$ unifies the energy-momentum relation for massive and massless particles, Wigner’s little group unifies their internal space-time symmetries. It is pointed out that translational symmetries play essential roles both in formulating the problem, and in deriving the conclusions. The input translational symmetry is the space-time translation, and the output translational symmetry is built in the $E(2)$-like little group for massless particles. The translational degrees of freedom in this little group are gauge degrees of freedom. It should be noted that a number of condensed matter physicists played important supporting and supplementary roles in Wigner’s unification of internal space-time symmetries of massive and massless particles.

1 Introduction

Eugene Wigner wrote many fundamental papers. However, if I am forced to name the most important paper he wrote, I have to mention his 1939 paper on representations of the inhomogeneous Lorentz group which is often called the Poincaré group [1]. In this paper, he introduces the “little group” which governs the internal space-time symmetry of relativistic particles. The physical implication of the little groups was not fully recognized when Wigner received the 1963 Nobel Prize in Physics. The best way to see the scientific value of Wigner’s little group is to compare it with Einstein’s work using the following table.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|-------------|----------------|
| $E = p^2/2m$  | Einstein’s $E = \sqrt{m^2 + p^2}$ | $E = cp$ |

$S_3$  
$S_1, S_2$  
Wigner’s Little Group  
$S_3$  
Gauge Trans.

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We shall call the above table \textit{Einstein-Wigner Table}. While Einstein’s $E = mc^2$ unifies the energy-momentum relations for massive and massless particles, Wigner’s little group unifies the internal space-time symmetries of massive and massless particles. A massive particle has three rotational degrees of freedom, and they are known as the spin degrees of freedom. A massless particle has one helicity degree of freedom. Unlike massive particles, massless particles have gauge degrees of freedom. This table was first published in 1986 \cite{2}, and I am the one who showed the table to Professor Wigner in 1985 before it was published. In his 1939 paper \cite{1}, Wigner noted that the little groups for massive and massless particles are locally isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean groups respectively. We shall call them the $O(3)$-like little group for massive particles and $E(2)$-like little group for massless particles.

Of course, the group of Lorentz transformations plays the central role in the little group formalism. But we need more. The purpose of the present report is to examine carefully the role of translational symmetries in completing the above table. The translational symmetry is very close to our daily life. When we walk, we perform translations on our body. Mathematically speaking, however, translations are very cumbersome operations. Indeed, there are many mathematical words associated with translations. To name a new, we use the words affine groups, inhomogeneous transformations, noncompact groups, semi-direct products, semi-simple Lie groups, invariant subgroup, induced representations, solvable groups, and more. Thus, it is not unreasonable to expect some non-trivial physical conclusions derivable from the translational symmetries.

In order to illustrate the mathematical complication from translational degrees of freedom, let us construct a rotation matrix applicable to a column vector $(x, y, 1)$:

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

Let us then consider the translation matrix:

$$T(a, b) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

If we take the product $T(a, b)R(\theta)$,

$$E(a, b, \theta) = T(a, b)R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

This is the Euclidean transformation matrix applicable to the two-dimensional $xy$ plane. The matrices $R(\theta)$ and $T(a, b)$ represent the rotation and translation subgroups respectively. The above expression is not a direct product because $R(\theta)$ does not commute with $T(a, b)$. The translations constitute an Abelian invariant subgroup because two different $T$ matrices commute with each other, and because

$$R(\theta)T(a, b)R^{-1}(\theta) = T(a', b'). \quad (4)$$

2
The rotation subgroup is not invariant because the conjugation

\[ T(a, b)R(\theta)T^{-1}(a, b) \]

does not lead to another rotation.

We can write the above transformation matrix in terms of generators. The rotation is generated by

\[ J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (5)

The translations are generated by

\[ P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}. \] (6)

These generators satisfy the commutation relations:

\[ [P_1, P_2] = 0, \quad [J_3, P_1] = iP_2, \quad [J_3, P_2] = -iP_1. \] (7)

This \( E(2) \) group is not only convenient for illustrating the groups containing an Abelian invariant subgroup, but also occupies an important place in constructing representations for the little group for massless particles, since the little group for massless particles is locally isomorphic to the above \( E(2) \) group.

In Sec. 2, we give a historical review of Wigner’s little groups. In Sec. 3, we shall see how the little group for a massive particle can becomes a little group for massless particles in the infinite-momentum/zero-mass limit. In Sec. 4, we review how the gauge degree of freedom of a massless particle is associated with the translation-like degrees of freedom in the \( E(2) \)-like little group for a massless particle.

## 2 Historical Review of Wigner’s Little Groups

Wigner was the first one to introduce the rotation group to the quantum mechanics of atomic spectra [3]. Since he had a strong background in chemistry [4], he became also interested in condensed matter physics. He had a graduate student named Frederick Seitz at Princeton University. Wigner and Seitz together published a paper on the constitution of metallic sodium in 1933 [5].

Seitz then started developing his own research line on space groups applicable to solid crystals. In his 1936 paper [6], Seitz discussed the augmentation of translation degrees of freedom to the three-dimensional rotation group, just as we did for the two-dimensional plane in Sec. 1 of the present paper. The results Seitz obtained were new at that time. This seminal paper opened up an entirely new research line on symmetries in crystals. I am not competent enough to write a review article on this research line, but I can say that the physicists belonging to this genealogy later made a decisive contribution to the physics of Wigner’s little groups. The purpose of this report is to discuss what role the condensed matter physicists played in completing the table given in Sec. 1.
In his 1939 paper [1], Wigner replaced the three-dimensional rotation group Seitz’s paper by the four-dimensional Lorentz group. By bringing in the translational degrees of freedom to the group of Lorentz transformations, Wigner observed the importance of the four-momentum in constructing representations. He then focused his attention to the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. The maximal subgroup which leaves the four-momentum invariant is called the little group. The point is that a relativistic particle has its Lorentz-covariant internal space-time structure, in addition to its four-momentum.

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation? The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation!

The rest-particle can now be boosted, and it will pick up non-zero space-like momentum components. The generators of the \( O(3) \) little group will also be boosted. The boost will take the form of conjugation by the boost operator. This boost will not change the Lie algebra of the rotation group and the boosted little group will still leave the boosted four-momentum invariant. We call this the \( O(3) \)-like little group.

It is not possible to bring a massless particle to its rest frame. In his 1939 paper [1], Wigner observed that the little group for a massless particle along the \( z \) axis is generated by the rotation generator around the \( z \) axis, and two other generators. If we use the four-vector coordinate \((x, y, z, t)\), the rotation around the \( z \) axis is generated by

\[
J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

(8)

and the other two generators are

\[
N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(9)

If we use \( K_i \) for the boost generator along the \( i \)-th axis, these matrices can be written as

\[
N_1 = K_1 - J_2, \quad N_2 = K_2 + J_3.
\]

(10)

Very clearly, the generators \( J_3, N_1 \) and \( N_2 \) generate Lorentz transformations (boosts and rotations), and they satisfy the commutation relations:

\[
[N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1.
\]

(11)

If we replace \( N_1 \) and \( N_2 \) by \( P_1 \) and \( P_2 \), then the above set of commutation relations becomes the set given for the \( E(2) \) group given in Eq.(7). This is the reason why the little group for massless particles is \( E(2) \)-like.
It is not difficult to associate the rotation generator $J_3$ with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the $N_1$ and $N_2$ generators? Wigner left this problem as a homework problem for younger generations. Before attempting to solve this problem, let us note that there are at least two more homework problems contained in Wigner’s paper [1].

- First, it is possible to interpret the Dirac equation in terms of Wigner’s representation theory [7]. Then, why is it not possible to find a place for Maxwell’s equations in the same theory?

- Second, as is shown by Inonu and Wigner [8], the rotation group $O(3)$ can be contracted to $E(2)$. Does this mean that the $O(3)$-like little group can become the $E(2)$-like little group in a certain limit?

We shall come back to the main question in Sec. 4, and we shall deal with the second question Sec. 3. Since the physics of $N_1$, and $N_2$ was not known, there had been a tendency in the past to construct representations for massless particles without these degrees of freedom. Indeed, in 1964 [9], Weinberg found a place for the electromagnetic tensor in Wigner’s representation theory. He accomplished this by constructing from the $SL(2,c)$ spinors all the representations of massless fields which are invariant under the translation-like transformations of the $E(2)$-like little group. Weinberg stated in 1964 that the $N$-invariant state vectors are gauge-invariant states, indicating that the $N$ operators generate gauge transformations. Weinberg did not elaborate on this point in his 1964 papers [9]. We shall return to the question of the $N_1$ and $N_2$ in Sec. 4.

3 Contraction of $O(3)$-like Little Group to $E(2)$-like Little Group

The contraction of $O(3)$ to $E(2)$ is well known and is often called the Inonu-Wigner contraction [8]. The question is whether the $E(2)$-like little group can be obtained from the $O(3)$-like little group. In order to answer this question, let us closely look at the original form of the Inonu-Wigner contraction. We start with the generators of $O(3)$. The $J_3$ matrix is given in Eq.(5), and

\[
J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (12)

The Euclidean group $E(2)$ is generated by $J_3, P_1$ and $P_2$, and their Lie algebra has been discussed in Sec.(I).

Let us transpose the Lie algebra of the $E(2)$ group. Then $P_1$ and $P_2$ become $Q_1$ and $Q_2$ respectively, where

\[
Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}.
\] (13)
Together with \( J_3 \), these generators satisfy the same set of commutation relations as that for \( J_3, P_1 \), and \( P_2 \) given in Eq.(7)

\[
\begin{align*}
\{Q_1, Q_2\} &= 0, \\
\{J_3, Q_1\} &= iQ_2, \\
\{J_3, Q_2\} &= -iQ_1.
\end{align*}
\] (14)

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by \( J_3 \). The matrices \( Q_1 \) and \( Q_2 \) generate translations along the direction of \( z \) axis. The group generated by these three matrices is called the cylindrical group \([10, 11, 12]\).

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

\[
\begin{align*}
P_1 &= \frac{1}{R} B^{-1} J_2 B, \\
P_2 &= -\frac{1}{R} B^{-1} J_1 B,
\end{align*}
\] (15)

and

\[
\begin{align*}
Q_1 &= -\frac{1}{R} B J_2 B^{-1}, \\
Q_2 &= \frac{1}{R} B J_1 B^{-1},
\end{align*}
\] (16)

where

\[
B(R) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & R
\end{pmatrix}.
\] (17)

The vector spaces to which the above generators are applicable are \((x, y, z/R)\) and \((x, y, Rz)\) for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively \([10]\).

Since \( P_1(P_2) \) commutes with \( Q_2(Q_1) \), we can consider the following combination of generators.

\[
F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2.
\] (18)

Then these operators also satisfy the commutation relations:

\[
\begin{align*}
\{F_1, F_2\} &= 0, \\
\{J_3, F_1\} &= iF_2, \\
\{J_3, F_2\} &= -iF_1.
\end{align*}
\] (19)

However, we cannot make this addition using the three-by-three matrices for \( P_i \) and \( Q_i \) to construct three-by-three matrices for \( F_1 \) and \( F_2 \), because the vector spaces are different for the \( P_i \) and \( Q_i \) representations. We can accommodate this difference by creating two different \( z \) coordinates, one with a contracted \( z \) and the other with an expanded \( z \), namely \((x, y, Rz, z/R)\). Then the generators become

\[
\begin{align*}
P_1 &= \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
P_2 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\end{align*}
\] (20)

\[
\begin{align*}
Q_1 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
Q_2 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\] (21)
Then \( F_1 \) and \( F_2 \) will take the form:

\[
F_1 = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad F_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{22}
\]

The rotation generator \( J_3 \) takes the form

\[
J_3 = \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{23}
\]

These four-by-four matrices satisfy the \( \text{E}(2) \)-like commutation relations of Eq.\((19)\).

Now the \( B \) matrix of Eq.\((17)\), can be expanded to

\[
B(R) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R & 0 \\
0 & 0 & 0 & 1/R
\end{pmatrix}. \tag{24}
\]

If we make a similarity transformation on the above form using the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\
0 & 0 & 1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}, \tag{25}
\]

which performs a 45-degree rotation of the third and fourth coordinates, then this matrix becomes

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \eta & \sinh \eta \\
0 & 0 & \sinh \eta & \cosh \eta
\end{pmatrix}, \tag{26}
\]

with \( R = e^\eta \). This form is the Lorentz boost matrix along the \( z \) direction. If we start with the set of expanded rotation generators \( J_3 \) of Eq.\((23)\), and

\[
J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{27}
\]

perform the same operation as the original Inonu-Wigner contraction given in Eq.\((13)\), the result is

\[
N_1 = \frac{1}{R} B^{-1} J_2 B, \quad N_2 = -\frac{1}{R} B^{-1} J_1 B, \tag{28}
\]

where \( N_1 \) and \( N_2 \) are given in Eq.\((14)\).

We are ultimately interested in giving the physical interpretation to the Wigner row in the Einstein-Wigner table given in Sec. 4. In the meantime, it is clear now from this Section that \( N_1 \) and \( N_2 \) are Lorentz-boosted rotation generators \( J_2 \) and \( J_1 \) respectively. All we have to do next is to give a physical interpretation to these operators.
4 Translations and Gauge Transformations

As was noted in Sec. 2, it is possible to get the hint that the $N$ operators generate gauge transformations from Weinberg’s 1964 papers \[9, 13\]. But it was not until 1971 when Janner and Janssen explicitly demonstrated that they generate gauge transformations \[14, 15\]. In order to fully appreciate their work, let us compute the transformation matrix generated by $N_1$, which takes the form

$$\exp(-iuN_1) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - u^2/2 & u^2/2 \\ u & 0 & -u^2/2 & 1 + u^2/2 \end{pmatrix}. \quad (29)$$

If we apply this matrix to the four-vector to the four-momentum vector

$$p = (0, 0, \omega, \omega) \quad (30)$$

of a massless particle, the momentum remains invariant. It therefore satisfies the condition for the little group. If we apply this matrix to the electromagnetic four-potential

$$A = (A_1, A_2, A_3, A_0) \exp(i(kz - \omega t)), \quad (31)$$

the result is a gauge transformation. This is what Janner and Janssen discovered in their 1971 and 1972 papers \[14\].

In order to see this without going through the generators, let us rotate the four-momentum of Eq.(30) around the $y$ axis using the matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (32)$$

We can come back to the original four-momentum by applying the inverse $R(\theta)$, but the result will be trivial. However, we can also come back to the original momentum by a boost. The boost matrix needed for this process is

$$\begin{pmatrix} 1 + 2[\sin(\theta/2)]^2 & 0 & -[\sin(\theta/2)]^2 \tan(\theta/2) & -2 \tan(\theta/2) \\ 0 & 1 & 0 & 0 \\ -[\sin(\theta/2)]^2 \tan(\theta/2) & 0 & 1 + 2[\tan(\theta/2)]^2 \sin(\theta/2)^2 & 2[\tan(\theta/2)]^2 \\ -2 \tan(\theta/2) & 0 & -2[\tan(\theta/2)]^2 & 1 + 2[\tan(\theta/2)]^2 \end{pmatrix}. \quad (33)$$

The rotation $R(\theta)$ followed by the above boost matrix leaves the four-momentum $p$ of Eq.(30) invariant. The resulting transformation matrix is

$$D(u) = \begin{pmatrix} 1 & 0 & \tan(\theta/2) & -\tan(\theta/2) \\ 0 & 1 & 0 & 0 \\ -\tan(\theta/2) & 0 & 1 - [\tan(\theta/2)]^2/2 & [\tan(\theta/2)]^2/2 \\ -\tan(\theta/2) & 0 & -[\tan(\theta/2)]^2/2 & 1 + [\tan(\theta/2)]^2/2 \end{pmatrix}. \quad (34)$$
If we replace \( \tan(\theta/2) \) by \(-u\), then the result will be the transformation matrix given in Eq. (29) generated by \( N_1 \). We can carry out a similar calculation with \( N_2 \), and the conclusion will be the same.

Indeed, the \( N \) matrices generate gauge transformations. If we combine this with the result of Sec. 3, the conclusion is that the Lorentz-boosted \( J_1 \) and \( J_2 \) become the generators of gauge transformations in the limit of infinite momentum and/or zero-mass \([16]\). The operator \( J_3 \) remains invariant and keeps serving as the helicity generator. These results lead to the Einstein-Wigner table given in Sec. 1.

**Acknowledgments**

I would like to thank the organizers of this conference, particularly Professor Tadeusz Lulek and Professor Wojciech Florek, for inviting me to present this paper. The Einstein-Wigner table given in Sec. 1 is important to all physicists, but more so to particle theorists, and I am one of them. For this reason, it is a gratifying experience for me to recognize the contributions made by condensed matter physicists toward the construction of the table. In 1987, I had a telephone conversation with Professor Frederick Seitz. He encouraged me to develop a conference series which these days is called the International Wigner Symposium. The fifth meeting of this Symposium will be held in Vienna in 1997.

I met Professor Aloysio Janner for the first time in 1978 during the Seventh International Colloquium on Group Theoretical Methods in Physics held in Austin, Texas. Since then he has given me many valuable advices as a senior group theoretician. However, he was too modest to tell me about his own papers on Wigner’s little group. He was still modest when I told him last month (July 1996) in Germany about my plan to present this paper at this conference. His modesty made my job of finding his contribution very difficult, and I did not know about his papers on this subject until I wrote with M. E. Noz on the Poincaré group which was published in 1986 [17], even though we managed to quote one of his papers in the book. It is indeed my personal pleasure to emphasize at this conference his role in completing the Einstein-Wigner table given in Sec. 1. Professor Janner of course has a brilliant publication record in the field of application of group theory to crystal physics. I hope to learn more from his papers in the future.

I am not known as a condensed matter theorist, but I am here because I published seven papers with Eugene Wigner, and many people ask me how I did it. I met Professor Wigner while I was a graduate student at Princeton University from 1958 to 1961. I stayed there for one more year as a post-doctoral fellow before joining the faculty of the University of Maryland in 1962. My advisor at Princeton was Sam Treiman, and I wrote my PhD thesis on dispersion relations. However, I did my extra-curricular activity on Wigner’s papers, particularly on his 1939 paper on representations of the Poincaré group [1]. It is not uncommon for one’s extra-curricular activity to become his/her life-time job. Indeed, by 1985, I had completed the manuscript for the above-mentioned book entitled *Theory and Applications of the Poincaré Group* [17] with Marilyn Noz who has been my closest colleague since 1970.

After writing this book in 1985, I approached Wigner again and asked him whether I could start working on edited volumes of all the papers he had written, but he had a better
idea. Wigner told me that he was interested in writing new papers and that he had been looking for a younger person who could collaborate with him. He was interested in many interesting problems. However, I was only able to assist him on two subjects. One was on group contractions. Wigner was particularly eager to establish a connection between the Inonu-Wigner contraction and his work on the Poincaré group. He was also interested in constructing Wigner functions which can be Lorentz-transformed. I was well prepared for the first problem, but I did not have a strong background in the Wigner function. On the other hand, I had a formalism of harmonic oscillators which can be Lorentz-transformed, and this formalism forms the back-bone of my book with Noz [17]. It is not difficult to translate the oscillator formalism into the language of Wigner functions. This was how I was able to write papers with him.

During the period 1986 - 1991, I went to Princeton regularly to work with him. It takes three hours by train to go to Princeton from the University of Maryland. After this period, I became much smarter. I am eternally grateful to Professor Wigner.

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