On the spectrum of S=1/2 XXX Heisenberg chain with elliptic exchange

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Abstract

It is found that the Hamiltonian of S=1/2 isotropic Heisenberg chain with N sites and elliptic non-nearest-neighbor exchange is diagonalized in each sector of the Hilbert space with magnetization $N/2 - M$, $1 < M \leq [N/2]$, by means of double quasiperiodic meromorphic solutions to the $M$-particle quantum Calogero-Moser problem on a line. The spectrum and highest-weight states are determined by the solutions of the systems of transcendental equations of the Bethe-ansatz type which arise as restrictions to particle pseudomomenta.
In recent years, much attention has been paid to studies of 1D lattice systems, due to their relevance to principal notions of field theory and experimental investigations of effectively low-dimensional crystals. Even the simplest lattice systems, namely isotropic S=1/2 Heisenberg chains, have unveiled rich structure and provided nontrivial examples of many-body interactions. The corresponding mathematical problem consists in finding the proper analytic tool for the diagonalization of the model Hamiltonian

$$H^{(s)} = \frac{J}{4} \sum_{1 \leq j \neq k \leq N} h(j - k)(\vec{\sigma}_j \vec{\sigma}_k - 1) \quad h(j) = h(j + N)$$  \hspace{1cm} (1)$$

where $\vec{\sigma}_j$ are Pauli matrices acting on spin at $j$th site.

At finite $N$, it has been successfully treated in the integrable cases of nearest-neighbor coupling solved by Bethe [1]

$$h(j) = \delta_{[j(\text{mod}N)]_1}$$  \hspace{1cm} (2)$$

and long-range trigonometric exchange proposed independently by Haldane and Shastry [2]

$$h(j) = \left( \frac{N}{\pi} \sin \frac{\pi j}{N} \right)^{-2}.$$  \hspace{1cm} (3)$$

At present, a number of impressive results are known for both these models. In particular, they include the additivity of the spectrum under proper choice of "rapidity" variables [1,3], the description of underlying symmetry [4,5], construction of thermodynamics in the limit $N \to \infty$ [6,3], the connection to the continuum integrable many-body problems [7,2], and closed-form expressions of correlations in the antiferromagnetic ground state. The rich collection of various generalizations and physical applications of Bethe and Haldane-Shastry models can be found in recent review papers [8,9].

Several years ago, I have introduced a more general one-parametric form of spin exchange which provides another example of integrable lattice Hamiltonian (1) [10]. It has been motivated by the similarity of the Lax representation of the Heisenberg equations of motion for continuum and lattice models. In the former case, the most general translationally-invariant integrable Hamiltonian with elliptic pairwise particle interaction has been found by Calogero [11] and Moser [12],

$$H_{CM} = \frac{1}{2} \left[ -\sum_{\beta=1}^{L} \frac{\partial^2}{\partial x_\beta^2} + \lambda(\lambda + 1) \sum_{\beta \neq \gamma} \phi(x_\beta - x_\gamma) \right].$$  \hspace{1cm} (4)$$

The existence of extra integrals of motion commuting with (4) has been demonstrated in [13]. Recently, the eigenvalue problem for the elliptic Calogero-Moser operator received
much attention due to its relation to the representations of double affine algebras and solutions of Knizhnik-Zamolodchikov-Bernard equations [14,15].

The lattice analog of (4) is given by (1) with

\[ h(j) = \left( \frac{\omega}{\pi} \sin \frac{\pi}{\omega} \right)^2 \left[ \varphi_N(j) + \frac{2}{\omega} \zeta_N \left( \frac{\omega}{2} \right) \right], \]

where \( \varphi_N(x), \zeta_N(x) \) are the Weierstrass functions defined on the torus \( T_N = \mathbb{C}/\mathbb{Z}N + \mathbb{Z}, \omega = i\kappa, \kappa \in \mathbb{R}_+ \). Remarkably, it turned out that the exchange (5) comprises both (2) and (3) [10]: in fact, the factor in (5) is chosen as to reproduce the nearest-neighbor coupling under periodic boundary conditions (2) in the limit \( \kappa \to 0 \) and the long-range exchange (3) in the limit \( \kappa \to \infty \).

However, till now much less is known about the lattice model with the exchange (5) in comparison with its limiting forms due to the mathematical complexities caused by the presence of the Weierstrass functions. The family of the operators which commute with \( \mathcal{H}^{(s)} \) has been found only recently [16]. The simpler case of infinite chain \( N \to \infty \), \( h(j) \to \sinh(\pi/\kappa)/\sinh(\pi j/\kappa) \) has been considered in detail in [17]. As for finite \( N \), the description of the spectrum has been performed only for simplest two- and three-magnon excitations over ferromagnetic vacuum [10, 18, 19].

The aim of this Letter is to demonstrate the remarkable correspondence between the highest-weight eigenstates of the lattice Hamiltonian with the elliptic exchange (5) and double quasiperiodic meromorphic eigenfunctions of the Calogero-Moser operator (4) which allows to formulate the equations of the Bethe-ansatz type for calculating the whole spectrum.

The Hamiltonian (1) commutes with the operator of total spin \( \vec{S} = \frac{1}{2} \sum_{j=1}^{N} \vec{\sigma}_j \). Then the eigenproblem for it is decomposed into the problems in the subspaces formed by the common eigenvectors of \( S_3 \) and \( \vec{S}^2 \) such that \( S = S_3 = N/2 - M, 0 \leq M \leq \lfloor N/2 \rfloor \),

\[ \mathcal{H}^{(s)} |\psi^{(M)}\rangle = E_M |\psi^{(M)}\rangle. \]

The eigenvectors \( |\psi^{(M)}\rangle \) are written in the usual form

\[ |\psi^{(M)}\rangle = \sum_{n_1,..,n_M} \psi_M(n_1..n_M) \prod_{\beta=1}^{M} s_{n_{\beta}}^{-} |0\rangle, \]

where \( |0\rangle = |\uparrow\uparrow ... \uparrow\rangle \) is the ferromagnetic ground state with all spins up and the summation is taken over all combinations of integers \( \{n\} \leq N \) such that \( \prod_{\mu<\nu} (n_{\mu} - n_{\nu}) \neq 0 \). The substitution of (7) into (6) results in the lattice Schrödinger equation for completely
symmetric wave function $\psi_M$

$$
\sum_{s \neq n_1, \ldots, n_M}^{N} \sum_{\beta=1}^{M} \varphi_N(n_\beta - s)\psi_M(n_1, \ldots, n_{\beta-1}, s, n_{\beta+1}, \ldots, n_M)
$$

$$
+ \left[ \sum_{\beta \neq \gamma}^{M} \varphi_N(n_\beta - n_\gamma) - E_M \right] \psi_M(n_1, \ldots, n_M) = 0.
$$

(8)

The eigenvalues $\{E_M\}$ are given by

$$
E_M = J \left( \frac{\omega}{\pi} \sin \frac{\pi}{\omega} \right)^2 \left\{ E_M + \frac{2}{\omega} \left[ \frac{2M(2M-1) - N}{4} \zeta_N \left( \frac{\omega}{2} \right) - M\zeta_1 \left( \frac{\omega}{2} \right) \right] \right\},
$$

(9)

where $\zeta_1(x)$ is the Weierstrass zeta function defined on the torus $T_1 = C/Z + Z\omega$.

To find the solutions to (8), let us consider the following ansatz for $\psi_M$:

$$
\psi_M(n_1, \ldots, n_M) = \sum_{P \in \pi_M} \varphi^{(p)}_M(n_{P_1}, \ldots, n_{P_M}),
$$

(10)

$$
\varphi^{(p)}_M(n_1, \ldots, n_M) = \exp \left( -i \sum_{\nu=1}^{M} p_\nu n_\nu \right) \chi^{(p)}_M(n_1, \ldots, n_M),
$$

(11)

where $\pi_M$ is the group of all permutations $\{P\}$ of the numbers from 1 to $N$ and $\chi^{(p)}_M$ is the solution to the continuum quantum many-particle problem

$$
\left[ -\frac{1}{2} \sum_{\beta=1}^{M} \frac{\partial^2}{\partial x_\beta^2} + \sum_{\beta \neq \lambda}^{M} \varphi_N(x_\beta - x_\lambda) - E_M(p) \right] \chi^{(p)}_M(x_1, \ldots, x_M) = 0.
$$

(12)

It is specified up to a normalization factor by the particle pseudomomenta $(p_1, \ldots, p_M)$. The standard argumentation of the Floquet-Bloch theory shows that due to periodicity of the potential term in (3) $\chi^{(p)}_M$ obeys the quasiperiodicity conditions [18]

$$
\chi^{(p)}_M(x_1, \ldots, x_\beta + N, \ldots, x_M) = \exp(ip_\beta N)\chi^{(p)}_M(x_1, \ldots, x_M),
$$

(13)

$$
\chi^{(p)}_M(x_1, \ldots, x_\beta + \omega, \ldots, x_M) = \exp(q_\beta(p) + ip_\beta \omega)\chi^{(p)}_M(x_1, \ldots, x_M), \quad 0 \leq \Im(q_\beta) < 2\pi
$$

(14)

$$
1 \leq \beta \leq M.
$$

The eigenvalue $E_M(p)$ is some symmetric function of $(p_1, \ldots, p_M)$. The set $\{q_\beta(p)\}$ is also completely determined by $\{p\}$. In this Letter I do not refer to the explicit form of these functions which is still unknown for $M > 3$.

The structure of the singularity of $\varphi_N(x)$ at $x = 0$ implies that $\chi^{(p)}_M$ can be presented in the form

$$
\chi^{(p)}_M = \frac{F^{(p)}(x_1, \ldots, x_M)}{G(x_1, \ldots, x_M)}, \quad G(x_1, \ldots, x_M) = \prod_{\alpha < \beta} \sigma_N(x_\alpha - x_\beta),
$$

(15)
where $\sigma_N(x)$ is the Weierstrass sigma function on the torus $T_N$. The only simple zero of $\sigma_N(x)$ on $T_N$ is located at $x = 0$. Thus $[G(x_1, ..x_M)]^{-1}$ absorbs all the singularities of $\chi_M^{(p)}$ on the hypersurfaces $x_\alpha = x_\beta$. The numerator $F^{(p)}$ in (15) is analytic on $(T_N)^M$ and obeys the equation

$$
\sum_{\alpha=1}^{M} \frac{\partial^2 F^{(p)}}{\partial x_\alpha^2} + \left[ 2E_M(p) - \frac{M}{2} \sum_{\alpha \neq \beta} (\varphi_N(x_\alpha - x_\beta) - \zeta_N^2(x_\alpha - x_\beta)) \right] F^{(p)}
$$

$$
= \sum_{\alpha \neq \beta} \zeta_N(x_\alpha - x_\beta) \left( \frac{\partial F^{(p)}}{\partial x_\alpha} - \frac{\partial F^{(p)}}{\partial x_\beta} \right).
$$

(16)

The regularity of the left-hand side of (16) as $x_\mu \to x_\nu$ implies that

$$
\left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x_\nu} \right) F^{(p)}(x_1, ..x_M) |_{x_\mu = x_\nu} = 0
$$

(17)

for any pair $(\mu, \nu)$.

The remarkable fact is that the properties (13-15,17) of $\chi_M^{(p)}$ allow one to validate the ansatz (10-11) for the eigenfunctions of the lattice Schrödinger equation (8). Substitution of (10) to (8) yields

$$
\sum_{P \in \pi_M} \left\{ \sum_{\beta=1}^{M} S_\beta(n_{P1}, ..n_{PM}) + \left[ \sum_{\beta \neq \gamma} \varphi_N(n_{P\beta} - n_{P\gamma}) - E_M \right] \varphi_M^{(p)}(n_{P1}, ..n_{PM}) \right\} = 0,
$$

(18)

where

$$
S_\beta(n_{P1}, ..n_{PM}) = \sum_{s \neq n_{P1}, ..n_{PM}} \varphi_N(n_{P\beta} - s) \hat{Q}_\beta^{(s)} \varphi_M^{(p)}(n_{P1}, ..n_{PM}).
$$

(19)

The operator $\hat{Q}_\beta^{(s)}$ in (19) replaces $\beta$th argument of the function of $M$ variables to $s$.

To calculate the sum (19), let us introduce, following the consideration of the hyperbolic exchange in [17], the function of one complex variable $x$,

$$
W_p^{(s)}(x) = \sum_{s=1}^{M} \varphi_N(n_{P\beta} - s - x) \hat{Q}_\beta^{(s+x)} \varphi_M^{(p)}(n_{P1}, ..n_{PM}).
$$

(20)

As a consequence of (11), (13-14) it obeys the relations

$$
W_p^{(s)}(x + 1) = W_p^{(s)}(x), \quad W_p^{(s)}(x + \omega) = \exp(q_\beta(p)) W_p^{(s)}(x).
$$

(21)

The only singularity of $W_p^{(s)}$ on the torus $T_1 = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\omega$ is located at the point $x = 0$. It arises from the terms in (20) with $s = n_{P1}, ..n_{PM}$. The Laurent decomposition of (20) near $x = 0$ has the form

$$
W_p^{(s)}(x) = w_{-2}x^{-2} + w_{-1}x^{-1} + w_0 + O(x).
$$

(22)
The explicit expressions for $w_{-1}$ can be found from (20),

$$w_{-2} = \varphi_M^{(p)}(n_{P1}, .. n_{PM})$$  

$$w_{-1} = \frac{\partial}{\partial n_{P\beta}} \varphi_M^{(p)}(n_{P1}, .. n_{PM})$$  

$$+ (-1)^P G(n_1, .. n_M) \sum_{\lambda \neq \beta} T_{\beta\lambda}(n_{P1}, .. n_{PM}) \hat{Q}_{\beta}^{(n_{P\lambda})} \exp \left( -i \sum_{\nu=1}^{M} p_{\nu} n_{P\nu} \right) F^{(p)}(n_{P1}, .. n_{PM})$$  

$$w_0 = S_{\beta}(n_{P1}, .. n_{PM}) + \frac{1}{2} \frac{\partial^2}{\partial n_{P\beta} \partial n_{P\beta}} \varphi_M^{(p)}(n_{P1}, .. n_{PM}) + (-1)^P G(n_1, .. n_M)$$  

$$\times \sum_{\lambda \neq \beta} T_{\beta\lambda}(n_{P1}, .. n_{PM}) \left[ U_{\beta\lambda}(n_{P1}, .. n_{PM}) \hat{Q}_{\beta}^{(n_{P\lambda})} + \varphi_N(n_{P\beta} - n_{P\lambda}) \partial \hat{Q}_{\beta}^{(n_{P\lambda})} \right]$$  

$$\times \exp \left( -i \sum_{\nu=1}^{M} p_{\nu} n_{P\nu} \right) F^{(p)}(n_{P1}, .. n_{PM}),$$

where

$$T_{\beta\lambda}(n_{P1}, .. n_{PM}) = \sigma_N(n_{P\lambda} - n_{P\beta}) \prod_{\rho \neq \beta, \lambda} \sigma_N(n_{P\rho} - n_{P\lambda})$$

$$U_{\beta\lambda}(n_{P1}, .. n_{PM}) = \varphi_N'(n_{P\lambda} - n_{P\beta}) - \varphi_N(n_{P\beta} - n_{P\lambda}) \sum_{\rho \neq \beta, \lambda} \zeta_N(n_{P\rho} - n_{P\lambda}),$$

$(-1)^P$ means the parity of the permutation $P$ and the action of the operator $\partial \hat{Q}_{\beta}^{(n_{P\lambda})}$ on the function $Y$ of $M$ variables is defined as

$$\partial \hat{Q}_{\beta}^{(n_{P\lambda})} Y(z_1, .. z_M) = \frac{\partial}{\partial z_\beta} Y(z_1, .. z_M)|_{z_\beta = n_{P\lambda}}.$$  

The next step consists in writing the explicit expression for the function $W_P^{(\beta)}(x)$ obeying the relations (21) and (22) [17],

$$W_P^{(\beta)}(x) = \exp(a_\beta x) \frac{\sigma_1(r_\beta + x)}{\sigma_1(r_\beta - x)} \{w_{-2}(\varphi_1(x) - \varphi_1(r_\beta) + (w_{-2}(a_\beta + 2\zeta_1(r_\beta)) - w_{-1})$$

$$\times [\zeta_1(x - r_\beta) - \zeta_1(x) + \zeta_1(r_\beta) - \zeta_1(2r_\beta)]\}.$$  

The Weierstrass functions $\varphi_1, \zeta_1$ and $\sigma_1$ in (25) are defined on the torus $T_1$ and the parameters $a_\beta, r_\beta$ are chosen as to satisfy the conditions (21),

$$a_\beta = (\pi i)^{-1} q_\beta(p) \zeta_1(1/2) \quad r_\beta = -(4\pi i)^{-1} q_\beta(p).$$

By expanding (25) in powers of $x$ one can find $w_0$ in terms of $w_{-2}, w_{-1}, q_\beta$ and obtain the explicit expression for $S_\beta(n_{P1}, .. n_{PM})$ with the use of (23a-c). It turns out that the equation (18) can be recast in the form

$$\sum_{P \in \pi_M} \left[ -\frac{1}{2} \sum_{\beta=1}^{M} \left( \frac{\partial}{\partial n_{P\beta}} - f_{\beta}(p) \right)^2 + \sum_{\beta \neq \gamma} \varphi_N(n_{P\beta} - n_{P\gamma}) - \mathcal{E}_M + \sum_{\beta=1}^{M} \varepsilon_{\beta}(p) \right] \varphi^{(p)}(n_{P1}, .. n_{PM})$$

$$= \mathcal{S}_\beta(n_{P1}, .. n_{PM}).$$

6
\[
\frac{1}{2} G(n_1, n_M) \sum_{P \in \pi_M} (-1)^P \sum_{\beta \neq \lambda} [Z_{\beta \lambda}(n_{P1}, \ldots n_{PM}) + Z_{\lambda \beta}(n_{P1}, \ldots n_{PM})],
\]  
(26)

where
\[
f_{\beta}(p) = (\pi i)^{-1} q_{\beta}(p) \zeta_1(1/2) - \zeta_1((2\pi i)^{-1} q_{\beta}(p)),
\]
(27)
\[
\varepsilon_{\beta}(p) = \frac{1}{2} \varphi_1((2\pi i)^{-1} q_{\beta}(p))
\]
(28)
and \(Z_{\beta \lambda}(n_{P1}, \ldots n_{PM})\) is defined by the relation
\[
Z_{\beta \lambda}(n_{P1}, \ldots n_{PM}) = T_{\beta \lambda}(n_{P1}, \ldots n_{PM}) \left[ U_{\beta \lambda}(n_{P1}, \ldots n_{PM}) \hat{Q}^{(n_{P\lambda})}_{\beta} + \varphi_N(n_{P\lambda} - n_{P \beta}) \right.
\]
\[
\times (\partial \hat{Q}^{(n_{P\lambda})}_{\beta} - f_{\beta}(p) \hat{Q}^{(n_{P\lambda})}_{\beta}) \right] \exp \left( -i \sum_{\nu=1}^{M} p_{\nu} n_{P \nu} \right) F^{(p)}(n_{P1}, \ldots n_{PM}).
\]
(29)

Turning to the definition (11) of \(\varphi^{(p)}\) one observes that each term of the left-hand side of (26) has the same structure as the left-hand side of the many-particle Schrödinger equation (12) and vanishes if \(E_M\) and \(f_{\beta}(p)\) are chosen as
\[
f_{\beta}(p) = -ip_{\beta}, \quad \beta = 1, \ldots M;
\]
(30)
\[
E_M = E_M(p) + \sum_{\nu=1}^{M} \varepsilon_{\beta}(p).
\]
(31)

Now let us prove that that the right-hand side of (26) also vanishes. The crucial observation is that the sum over permutations in it can be recast in the form
\[
\sum_{P \in \pi_M} (-1)^P \sum_{\beta \neq \lambda} [Z_{\beta \lambda}(n_{P1}, \ldots n_{PM}) - Z_{\lambda \beta}(n_{P1}, \ldots n_{PM})],
\]
where \(R\) is the transposition \((\beta \leftrightarrow \lambda)\) which leaves other numbers from 1 to \(M\) unchanged. The term in square brackets is simplified drastically with the use of the identities
\[
T_{\lambda \beta}(n_{PR1}, \ldots n_{PRM}) = T_{\beta \lambda}(n_{P1}, \ldots n_{PM}), \quad U_{\lambda \beta}(n_{PR1}, \ldots n_{PRM}) = U_{\beta \lambda}(n_{P1}, \ldots n_{PM})
\]
\[
\hat{Q}^{(n_{P\lambda})}_{\lambda} F(n_{PR1}, \ldots n_{PRM}) = \hat{Q}^{(n_{P\lambda})}_{\beta} F(n_{P1}, \ldots n_{PM}).
\]

Taking into account the relations (29-30), one finds
\[
Z_{\beta \lambda}(n_{P1}, \ldots n_{PM}) - Z_{\lambda \beta}(n_{P1}, \ldots n_{PM}) = T_{\beta \lambda}(n_{P1}, \ldots n_{PM}) \varphi_N(n_{P\lambda} - n_{P \beta})
\]
\[
\times \exp \left[ -i \left( (p_{\beta} + p_{\lambda}) n_{P\lambda} + \sum_{\rho \neq \beta, \lambda}^{M} p_{\rho} n_{P \rho} \right) \right] \left( \frac{\partial}{\partial n_{P \beta}} - \frac{\partial}{\partial n_{P \lambda}} \right) F^{(p)}(n_{P1}, \ldots n_{PM})|_{n_{P \beta} = n_{P \lambda}}.
\]
(32)

The last factor in (32) vanishes due to the condition (17) imposed by the regularity of the left-hand side of the Schrödinger equation (16).
It remains to show that the states of the spin lattice given by (7) with the functions $\psi_M$ of the form (10-11) are highest-weight states with $S = S_3$. This statement is equivalent to the relation $S_+ |\psi^{(M)} > = 0$, which can be rewritten as

$$\sum_{\beta=1}^M \sum_{P \in \pi_M^{(\beta)}} \sum_{s \neq n_1 \ldots n_{M-1}} \hat{G}^{(s)}(p) \phi_M^{(p)}(n_{P1}, \ldots n_{PM}) = 0, \quad (33)$$

where $\{\pi_M^{(\beta)}\}$ are the subsets of $\pi_M$: $P \in \pi_M^{(\beta)} \leftrightarrow P \beta = M$. The sums over $s$ in (33) can be reduced and presented in the closed form by using the technique described above. It turns out that the left-hand side of (33) contains the factors similar to the last factor in (32) and vanishes due to the condition (17).

The descendant states with $S_3 < S$ are obtained by acting with $S_-$ on the basic states $|\psi^{(M)} >$ (7). Thus the present consideration allows, in principle, to reproduce all the eigenvectors of $\mathcal{H}^{(s)}$ for the exchange (5) as it has been done by Bethe [1] for nearest-neighbor spin coupling. The equations (30) for the pseudomomenta $\{p\}$ constitute the analog of the usual Bethe ansatz. The spectrum is given by the relations (9) and (31).

In conclusion, it is demonstrated that the procedure of the exact diagonalization of the lattice Hamiltonian with the non-nearest-neighbor elliptic exchange can be reduced in each sector of the Hilbert space with given magnetization to the construction of the special double quasiperiodic eigenfunctions of the many-particle Calogero-Moser problem on a continuous line. The equations of the Bethe-ansatz form appear very naturally as a set of restrictions to the particle pseudomomenta. The proof of this correspondence between lattice and continuum integrable models is based only on analytic properties of the eigenfunctions. One can expect that the set of spin lattice states constructed by this way is complete. This is supported by exact analytic proof in the two-magnon case.

The analysis of explicit form of the equations (30) available for $M = 2, 3$ shows that the spectrum of the lattice Hamiltonian with the exchange (5) is not additive being given in terms of pseudomomenta $\{p\}$ or phases which parametrize the sets $\{p, q\}$ [10,19]. The problem of finding appropriate set of parameters which gives the "separation" of the spectrum remains open. It would be also of interest to consider various limits $(N \to \infty, \kappa \to 0, \infty)$ so as to recover the results of the papers [1,3,17] and prove the validity of the approximate methods of asymptotic Bethe ansatz after finding explicit form of the functions $q_\beta(p)$ and $E_M(p)$.

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