Central Limit Theorems for series of Dirichlet characters

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Abstract

For a given Dirichlet character \( \chi(n) = e^{i\theta_n} \), we prove central limit theorems for the series \( \sum_{p'} \cos \theta_{p'} \) for non-principal characters, and \( \sum_{p'} \cos(t \log p') \) for principal characters, where \( p' \) are integers based on a variant of Cramér’s random model for the primes. For non-principal characters, we use these results to show that the Generalized Riemann Hypothesis for the associated \( L \)-function is true with probability equal to one. For principal characters we propose how to extend these arguments to \( \Re(s) = t \to \infty \).

In memory of my daughter Alexandra LeClair, who passed away during the course of this work.

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I. INTRODUCTION

This article concerns the growth of certain infinite series defined over prime numbers. In this Introduction we explain the motivation for this study, define the series in question, and briefly summarize some of our results.

Let $\chi(n)$ denote a Dirichlet character, and as usual define the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (1)$$

where $s = \sigma + it$ is a complex variable. The Riemann zeta function $\zeta(s)$ corresponds to the $L$-function for the trivial character modulo 1 where all $\chi(n) = 1$. Due to the completely multiplicative property of the characters, $L$ enjoys an Euler product formula:

$$L(s, \chi) = \prod_{n=1}^{\infty} \left( 1 - \frac{\chi(p_n)}{p_n^s} \right)^{-1} \quad (2)$$

where $p_n$ is the $n$-th prime. The above formula is known to be valid for $\Re(s) > 1$ where both sides of the equation converge absolutely. Using the validity of the Euler product one can easily see that there are no zeros with $\Re(s) > 1$; in particular, $\log L$ is finite in this region since the series converges. If somehow the Euler product were valid for $\Re(s) > 1/2$, then the Generalized Riemann Hypothesis would follow by the same argument together with the functional equation that relates $L(s, \chi)$ to $L(1 - s, \overline{\chi})$. For Riemann $\zeta$ itself, and other $L$-functions based on principal Dirichlet characters, it is well understood that the Euler product in the above form is not valid for $1/2 < \Re(s) \leq 1$, essentially due to the pole at $s = 1$: domains of convergence of Dirichlet series are always half-planes, and due to this pole, the logarithm must be divergent for $\Re(s) \leq 1$. However for non-principal characters, the associated $L$-functions have no pole at $s = 1$, and thus the validity of the Euler product for $\Re(s) > 1/2$ is theoretically possible, although difficult to study.

For the moment, let us not distinguish between the cases of principal verses non-principal characters, although for the reasons discussed above there will subsequently be significant differences. Let the character have modulus $k$. The character $|\chi(n)| = 1$ if $(n, k) = 1$, i.e. $n, k$ are coprime, otherwise $\chi(n) = 0$. The non-zero characters are all roots of unity, so let us define the angles $\theta_n$:

$$\chi(n) = e^{i\theta_n}, \quad \forall \chi(n) \neq 0 \quad (3)$$
Now consider the series
\[ B_N(t, \chi) = \sum_{n=1}^{N} \cos (\theta_{p_n} + t \log p_n) \] (4)
where it is implicit that the terms corresponding to the finite number of primes for which \( \chi(p_n) = 0 \) are omitted in the sum. In [1, 2] it was proven that if \( B_N = O(\sqrt{N}) \) as \( N \to \infty \), then the Euler product formula is valid for \( \Re(s) > 1/2 \) because it converges in this region.

The proof involved an Abel transform for the logarithm of the Euler product, the Prime Number Theorem (PNT), and a bound on the sum over gaps between primes. Let us now specialize to non-principal characters. Because of the half-plane convergence property mentioned above, to establish validity of the Euler product formula for \( \Re(s) > \sigma_c \) for some \( \sigma_c \), it is sufficient to prove convergence at a single value of \( t \). Since there is no pole at \( s = 1 \), the simplest choice is \( t = 0 \). It is sufficient then to consider the series
\[ C_N = \sum_{n=1}^{N} \cos \theta_{p_n} \] (5)

As stated above, a proof that \( C_N = O(\sqrt{N}) \) would establish the validity of the Generalized Riemann Hypothesis for all non-principal characters. It is a completely deterministic series which depends on the actual primes which are largely unknown for large \( N \), thus it is difficult, if not impossible, to compute it for large enough \( N \). However if one is only interested in its growth as a function of \( N \), since \( C_N \) is a series, the fluctuations coming from the precise values of individual primes may not be important for determining this growth. In other words, the growth of \( C_N \) may only depend on some global properties of the set of primes, such as their average spacing, etc. In [2] it was conjectured that \( C_N = O(\sqrt{N}) \) based on the heuristic argument that it behaves like a random walk due to the multiplicative independence of the primes. In this article we apply methods of probability theory to further study this problem. The idea of using probability methods in number theory is certainly not new, and at least goes back to work of Cramér [5], which we will utilize.

Let \( \mathbb{P} = \{p_1, p_2, \ldots\} \) denote the set of primes, where \( p_1 = 2, p_2 = 3 \), and so forth. We will consider replacing \( \mathbb{P} \) with the set \( \mathbb{P}' = \{p'_1, p'_2, \ldots\} \), which are a random, independent, ordered sequence of integers, and will study \( C_N' = \sum_{n=1}^{N} \cos \theta_{p'_n} \). The \( p'_n \) will be constrained to satisfy some global properties of the known primes, to be specified below. Since the \( p'_n \) are now random variables, we will consider \( \mathcal{P} = \{\mathbb{P}'\} \) which is the ensemble of all possible \( \mathbb{P}' \), i.e the set of sets \( \mathbb{P}' \). We will refer to \( \mathcal{P} \) as the pseudo-prime ensemble, and a specific
element \( P' \in P \) as a state of this ensemble. The actual primes \( P \) are then simply one state in the pseudo-prime ensemble. This terminology is borrowed from statistical mechanics in physics. For instance, for a gas of free particles, the states correspond to specific values for the positions and velocities of every individual particle, and various canonical ensembles are the set of all such states subject to certain constraints, such as the total number of particles or total energy held fixed. In light of this analogy, the idea we are pursuing here is that, for example, the macroscopic pressure of a gas of a large number of particles hardly depends on what specific state they are in, which is unknowable, and for the same reasons we expect the global (macroscopic) properties of \( C'_N \), in particular its growth as a function of \( N \), does not depend on the detailed properties of \( P' \). The aim of this article to make such statements precise using the theory of probability, as in statistical mechanics.

We need to be specific about the ensemble \( P \) and its probability measure. We will require two properties. The primes \( p_n \) are independent, more specifically they are multiplicatively independent. We thus require the \( p'_n \) to also be independent. Secondly, we require that the counting of \( p' \) is essentially equivalent to that implied by the Prime Number Theorem. Namely, as usual let \( \pi(x) \) be the number of primes less than \( x \). The PNT gives the leading behavior \( \pi(x) \approx \text{Li}(x) \approx \frac{x}{\log x} \). Let \( \pi'(x) \) be the analogous quantity for the primes \( p'_n \), i.e. the number of \( p' < x \). This counting function is now a random variable, and we require that its expectation value is approximately the leading term in \( \pi(x) \):

\[
E[\pi'(x)] \approx \pi(x) \approx \text{Li}(x) \approx \frac{x}{\log x} \tag{6}
\]

There are many possible choices of \( P \) compatible with these requirements. One particular interesting one is to take \( p'_n \) to be a random integer satisfying \( p_n \leq p'_n \leq p_{n+1} \). Remarkably, Grosswald and Schnitzer [4] proved that if one defines a \( \zeta \) function via an Euler product from the \( \{p'_n\} \), as in [2], then all of these possible \( \zeta \)'s can be analytically continued into the critical strip and have the same zeros as Riemann \( \zeta \) there. They proved a similar result for Dirichlet \( L \)-functions which will be discussed below. For our purposes however, this choice is more difficult to analyze than necessary. Instead we will use a variant of the Cramér model [5] which depends on the modulus of the character \( \chi \). For instance, in the simplest case of modulus \( k = 1 \), for each integer \( n \), the probability that \( n \in P' \) equals \( 1/\log n \). We will then prove that \( C'_N \) obeys a central limit theorem, i.e. when properly normalized, it has a normal distribution:
Theorem 1. For non-principal characters of modulus $k$,

$$\sqrt{1 + \frac{\log \log N}{\log N}} \frac{C'_N}{s^2N} \xrightarrow{d} \mathcal{N}(0, 1)$$

(7)

with

$$s^2 = a \frac{\varphi(k)}{k}$$

(8)

where $a = 1$ if the characters $\chi$ are all real, (i.e. all $\pm 1$), otherwise $a = 1/2$, $\varphi(k)$ is the Euler totient, and $\mathcal{N}(\mu, \sigma)$ is the normal distribution with mean $\mu$ and standard deviation $\sigma$.

As $N \to \infty$, the $\frac{\log \log N}{\log N}$ can of course be neglected, however we retain it in order to provide numerical evidence at large but finite $N$. We will use this theorem to say something precise about the growth of the original series $C_N$.

For principal characters all the angles $\theta_{p_n}$ are zero and one needs to consider now the series

$$B_N(t) = \sum_{n=1}^{N} \cos(t \log p_n)$$

(9)

Here, obviously we are interested in $t \neq 0$, which as explained above, in relation to the validity of the Euler product, this is due to the pole in $\zeta(s)$ at $s = 1$. As before we define $B'_N(t)$ as above with $p_n \to p'_n$. Below we will prove a central limit theorem for $B'_N(t)$ in the limit of large $t$ (Theorem 4 below).

II. NON-PRINCIPAL CASE

As explained in the Introduction, given a non-principal Dirichlet character $\chi$ of modulus $k$, we are interested in the series

$$C'_N = \sum_{n=1}^{N} \cos \theta_{p'_n}$$

(10)

where the angles $\theta_n = \text{Arg} \chi(n)$, and $\{p'_n\} = \mathbb{P}'$ is one state in the ensemble $\mathcal{P}$ appropriate to Cramér’s model.

We first describe how to implement the Cramér model and generate the states $\mathbb{P}'$ in a way that is simple to study both analytically and numerically. For simplicity we exclude
$p'_1 = 2$ from $\mathbb{P}'$. This does not affect the large $N$ result we will obtain. For each $n \geq 3$, let $r_n$ be a random variable uniformly distributed on the interval $[0, 1]$, and define $z_n$ as follows

$$z_n = \begin{cases} 1 & \text{if } r_n \leq \frac{1}{\log n} \\ 0 & \text{otherwise} \end{cases}$$

(11)

Then, by definition, for $n \geq 3$, $n \in \mathbb{P}'$ if $z_n = 1$. The $z_n$ are independent random variables, with probabilities $\Pr[z_n = 1] = 1/\log n$. We have excluded $n = 2$ since $1/\log 2 > 1$. The counting formula $\pi'(x)$ for the number of $p' \leq x$ is then simply

$$\pi'(x) = \sum_{n \leq x} z_n$$

(12)

Since $E[z_n] = 1/\log n$,

$$E[\pi'(x)] = \sum_{n=3}^{x} \frac{1}{\log n} \approx \int_{3}^{x} \frac{du}{\log u} \approx \frac{x}{\log x}$$

(13)

in accordance with (6).

In order to implement $(p'_n, k) = 1$ in (10), let us slightly modify the definition (11) to the following

$$z_{n,k} = \begin{cases} 1 & \text{if } r_n \leq \frac{1}{\log n} \text{ and } (n,k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(14)

for $n \geq 3$. The series $C'_N$ defined in the Introduction is now modeled as

$$C'_N = \sum_{n=1}^{p'_N} z_{n,k} \cos \theta_n$$

(15)

Summing over all integers up to $p'_N$ ensures there are $N$ terms in the sum when $k = 1$. The series $C'_N$ is now a random variable with a well-defined probability distribution. Let us now prove Theorem 1.

Proof. (of Theorem 1).

Let us write $C'_N = \sum_n c_n$ where $c_n = z_{n,k} \cos \theta_n$. The $c_n$ are independent random variables however they are not identically distributed, and thus the classical (Lindeberg-Lévy) central limit theorem (CLT) does not apply. However Lyapunov’s CLT does. More generally, let $x_n, n = 1, 2, \ldots, N$ be independent random variables with finite mean $\mu_n$ and variance $\sigma_n^2$, 

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which are allowed to vary with \( n \), and define the series \( X_N = \sum_{n=1}^{N} x_n \). Define \( m_N \) as the expectation value of \( X_N \),

\[
m_N = \mathbb{E}[X_N] = \sum_{n=1}^{N} \mu_n,
\]

and \( s_N^2 \) the sum of variances

\[
s_N^2 = \sum_{n=1}^{N} \sigma_n^2 \tag{17}
\]

If the Lyapunov condition is satisfied, namely if for some \( \delta > 0 \)

\[
\lim_{N \to \infty} \frac{1}{s_N^{2+\delta}} \sum_{i=1}^{p'_N} \mathbb{E}[|x_{n_i} - \mu_{n_i}|^{2+\delta}] = 0, \tag{18}
\]

then Lyapunov’s theorem states that

\[
\frac{1}{s_N} (X_N - m_N) \xrightarrow{d} \mathcal{N}(0,1) \tag{19}
\]

Let us now apply this to \( X_N = C'_N \). First consider \( m_N \). For non-principal characters, one has

\[
\sum_{n=1}^{k-1} \chi(n) = 0 \tag{20}
\]

Thus the angles \( \theta_n \) are equally spaced on the unit circle. If the \( p'_n \) are random, then \( \sum \cos \theta_{p'_n} \) is always close to zero, and on average is zero. We will only need the weaker statement that \( m_N = O(1) \).

Let us now turn to \( s_N^2 \):

\[
s_N^2 = \sum_{n=3}^{p'_N} \mathbb{E}\left[z_{n,k}^2 \cos^2 \theta_n\right] \tag{21}
\]

Let us invoke an Abel transform (integration by parts) to re-express the above series in terms of \( \sum \cos^2 \theta_n \). If the characters are all real, then \( \cos^2 \theta_n = 1 \) for all \( n \). On the other hand if they are complex and equally spaced on the unit circle the average of \( \cos^2 \theta_n \) is 1/2. Let us distinguish these two cases by defining \( a = 1 \) and \( a = 1/2 \) respectively. For \( 1 \leq n \leq k \), there are exactly \( \varphi(k) \) non-zero characters \( \chi(n) \). Since the characters are periodic, \( \chi(n+k) = \chi(n) \), the fraction of non-zero terms in the above sum is \( \varphi(k)/k \). One clearly has \( \Pr[z_{n,k}^2 = 1] = 1/\log n \), which implies

\[
s_N^2 = s^2 \sum_{n=3}^{p'_N} \frac{1}{\log n} \tag{22}
\]
FIG. 1: Numerical evidence for Theorem 1 based on the character (24). We fixed $N=5,000$. Displayed is a normalized histogram for 10,000 states $P'$. The red curve is the fit to the data, which is the normal distribution $\mathcal{N}(0.000500253, 1.0051)$. The blue curve is the prediction $\mathcal{N}(0, 1)$ which is nearly invisible since it is indistinguishable from the fit.

where $s^2$ is defined in (8). Next we use $p'_N \approx N \log N$ to obtain in the limit of large $N$:

$$s^2_N \approx s^2 N \log N \int_{\log 3}^{N \log N} \frac{du}{\log u} \approx s^2 N \left(1 + \frac{\log \log N}{\log N}\right)^{-1}$$

(23)

The Lyapunov condition is easily verified for integer $\delta$, since $s_N = O(\sqrt{N})$ and the expectation in (18) is $O(N)$ for any $\delta$. The theorem then follows from Lyapunov’s result (19), using $\lim_{N \to \infty} m_N/s_N = 0$.

In Figure 1 we present compelling numerical evidence for Theorem 1. We chose the following character with $k = 7$:

$$\chi(1), \ldots, \chi(7) = 1, e^{2\pi i/3}, e^{\pi i/3}, e^{-2\pi i/3}, e^{-\pi i/3}, -1, 0$$

(24)

Here, $a = 1/2$ and $\varphi(7) = 6$. We fixed $N = 5,000$ and generated 10,000 states $P'$ numerically according to (14); displayed is a normalized histogram. Performing a fit to a normal distribution gave $\mathcal{N}(0.000500253, 1.0051)$.

Although the true primes are obviously special, they fall well within the bell curve, which is to say they are rather “normal”. Namely, for the special state $P'$ equal to the actual primes $\mathbb{P}$, the LHS of (7) for the series $C_N$ equals $-0.145$ for $N = 5000$. 

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Theorem 2. For any $\epsilon > 0$, in the limit $N \to \infty$,

$$C'_N = O(N^{1/2+\epsilon})$$

with probability equal to 1.

Proof. Using the normal distribution of Theorem 1 in the limit $N \to \infty$ one has

$$\Pr [C'_N \leq s\kappa N^{1/2+\epsilon}] = 1 - \frac{e^{-\kappa^2 N^{2\epsilon}}}{\sqrt{2\pi\kappa N^\epsilon}} \left(1 - O\left(\frac{1}{\kappa^2 N^{2\epsilon}}\right)\right)$$

For any $\epsilon > 0$,

$$\lim_{N \to \infty} \Pr [C'_N = O(N^{1/2+\epsilon})] = 1$$

Given any particular state $\mathbb{P}'$, we can define the function

$$L'(s, \chi) = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p'_n)}{(p'_n)^s}\right)^{-1}$$

(27)

Theorem 3. With probability equal to 1, all the functions $L'$ have no zeros with $\Re(s) > 1/2 + \epsilon$ for any $\epsilon > 0$.

Proof. Consider the limit $\epsilon \to 0^+$ in Theorem 2. It was shown in [2] that if $C'_N = O(N^{1/2+\epsilon})$, then the logarithm of the product on the RHS of (27) converges for $\Re(s) > 1/2 + \epsilon$. Thus the very definition of $L'$ as an Euler product provides an analytic continuation for $\Re(s) > 1/2 + \epsilon$. The product is convergent and never zero because its logarithm is finite, thus there are no zeros to the right of the critical line since $\epsilon$ can be taken arbitrarily small.

Corollary 1. The Dirichlet L-function built on the actual primes $\mathbb{P}$ is known to satisfy a functional equation that relates $L(s, \chi)$ to $L(1-s, \overline{\chi})$. Thus Theorem 3 implies that the Generalized Riemann Hypothesis for non-principal characters is true with probability equal to 1.

Remark 1. Define $\mathcal{P}_{gs}$ as the ensemble of states $\mathbb{P}'$ where $p'_n$ is a random integer satisfying

$$p_n \leq p'_n \leq p_n + K, \quad p'_n = p_n \text{ mod } k$$

(28)

where $K$ is an integer. Grosswald and Schnitzer proved that the functions $L'(s, \chi)$ can be analytically continued to $\Re(s) > 0$ and remarkably have the same zeros as the $L$-function [2] inside the critical strip [3]. Corollary 1 implies that all these random $L'$-functions based on $\mathcal{P}_{gs}$ satisfy the Riemann Hypothesis with probability equal to 1 if $\mathcal{P}_{gs} \subset \mathcal{P}$.
III. PRINCIPAL CASE

We now consider the case of the principal character of modulus \( k \), where by definition \( \chi(n) = 1 \) if \((n, k) = 1\), otherwise \( \chi(n) = 0 \). As explained in the Introduction, we are interested in the series

\[
B'_N(t) = \sum_{n=1}^{N} \cos(t \log p'_n) \tag{29}
\]

where \( t \neq 0 \). As in the previous section, this can be modeled as

\[
B'_N(t) = \sum_{n=3}^{p'_N} z_{n,k} \cos(t \log n) \tag{30}
\]

**Theorem 4.** In the limit of large \( t \to \infty \),

\[
\sqrt{\frac{1 + \frac{\log \log N}{\log N}}{s^2 N}} \left( B'_N(t) - m_N(t) \right) \xrightarrow{d} \mathcal{N}(0, 1) \tag{31}
\]

where \( s^2 = \varphi(k)/2k \) and

\[
m_N(t) \approx \Re\left( \left[ (1 + it) \log(N \log N) \right] Ei \right) \tag{32}
\]

**Proof.** As in the non-principal case of the last section, the proof is based on the Lyapunov CLT. Let \( \mu_n, \sigma_n \) be the mean and standard deviation of each term in the sum (30). Then

\[
m_N(t) = \sum_n \mu_n = \sum_{n=3}^{p'_N} \frac{1}{\log n} \cos(t \log n) \approx \frac{\varphi(k)}{k} \int_3^{p'_N} \frac{du}{\log u} \cos(t \log u) \tag{33}
\]

The above integral can be expressed in terms of the exponential integral function \( Ei \):

\[
\int_{x}^{\infty} \frac{du}{\log u} \cos(t \log u) = \Re \left( Ei \left[ (1 + it) \log x \right] \right) \tag{34}
\]

Using \( p'_N \approx N \log N \), we obtain (32).

Next let us turn to \( s^2_N \):

\[
s^2_N(t) = \sum_{n=3}^{p'_N} \sigma_n^2 = \sum_{n=3}^{p'_N} \mathbb{E} \left[ z_{n,k}^2 \cos^2(t \log n) \right] - \mu_n^2
\]

\[
= \sum_{n=3}^{p'_N} \left( \frac{1}{\log n} - \frac{1}{\log^2 n} \right) \cos^2(t \log n) \tag{35}
\]
We can neglect the $1/\log^2 n$ term since it is of lower order. Approximating the sum by an integral as in (33), one has

$$s_N^2(t) \approx \frac{\varphi(k)}{2k} \left( \text{Li}(p'_N) + \Re(\text{Ei}[(1 + 2it) \log p'_N]) \right)$$

The factor of $1/2$ in the leading Li term is a reflection that the average of $\cos^2$ is $1/2$. In the limit of large $t$ the Ei term can be neglected since it is smaller by a factor of $O(1/t)$ (see the approximation in (39)). Again using $p'_N \approx N \log N$,

$$\lim_{t \to \infty} s_N^2(t) \approx \frac{\varphi(k)}{2k} \left( \frac{N}{\log \log N} \right) \left( t + t^2 \right) \sin \left( t \log \left( N \log N \right) \right)$$

For the same reasons as in Theorem 1, the Lyapunov condition (18) is satisfied. The theorem then follows from the CLT (19).

Note that for fixed $N$, $\lim_{t \to \infty} m_N(t) = 0$ (see the approximation (39) below). In Figure 2 we provide numerical evidence for Theorem 4. As for the non-principal case, for the state $\mathbb{P}'$ corresponding to the actual primes $\mathbb{P}$, the series is well within the bell curve, namely the LHS of (31) for the original series $B_N(t)$ equals $-0.280$.

**Remark 2.** Theorem 4 is similar, but not identical, to a theorem of Kac. For the latter, the $p'_n$ are the true primes and thus not random. Rather, randomness is introduced by making $t$ a random variable, in contrast to Theorem 4 where $t$ is not random and fixed. Kac’ CLT is valid for $t \in [T, 2T]$ in the limit $T \to \infty$.

**Theorem 5.** If $t > \sqrt{N}$, then with probability equal to one,

$$B'_N(t) = O(N^{1/2+\epsilon})$$

for any $\epsilon > 0$ in the limit of large $N$.

**Proof.** For large $t$ and $N$, to a very good approximation

$$m_N(t) \approx \frac{\varphi(k)}{k} \left( \frac{N}{\log \log N} \right) \left( t + t^2 \right) \sin \left( t \log \left( N \log N \right) \right)$$

If $t > \sqrt{N}$, then $m_N(t) = O(\sqrt{N})$. Using this, and repeating the arguments of Theorem 2 proves the theorem.

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FIG. 2: Numerical evidence for Theorem 4. We fixed \( N = 5,000 \) and \( t = 1000 \). Displayed is a normalized histogram for 10,000 states \( \mathbb{P}' \). The red curve is the fit to the data, which is the normal distribution \( \mathcal{N}(-0.02681, 1.00325) \). The blue curve is the prediction \( \mathcal{N}(0,1) \).

Remark 3. The above theorem implies that the Riemann Hypothesis is true with probability equal to one in the limit \( t \to \infty \). The argument is the same as in Theorem 3. The condition \( t \to \infty \) makes this a weaker statement than in the non-principal case. In order to deal with finite \( t \), it was proposed in [1, 3, 7] that a truncated Euler product is a good approximation to the \( \zeta \) function and can be used to study the Riemann Hypothesis. Namely the following formula is valid:

\[
\zeta(s) = \prod_{n=1}^{N(t)} \left( 1 - \frac{1}{p_n^s} \right)^{-1} \exp(R_N(s)) \tag{40}
\]

where \( N(t) \sim t^2 \) and \( R_N(s) \sim 1/t^{2\sigma-1} \). Thus in the limit \( t \to \infty \), \( R_N \) can be neglected if \( \sigma > 1/2 \). The above formula would rule out zeros to the right of the critical line since the RHS is then never zero.

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