MAGNETIC-ELECTRIC FORMULATIONS FOR STATIONARY MAGNETOHYDRODYNAMICS MODELS

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Abstract. We discuss magnetic-electric fields based finite element schemes for stationary magnetohydrodynamics (MHD) systems with two types of boundary conditions. The schemes are unconditional well-posed and stable. Moreover, magnetic Gauss’s law $\nabla \cdot B = 0$ is preserved precisely on the discrete level. We establish a key $L^3$ estimate for divergence-free finite element functions for a new type of boundary condition. With this estimate and a similar one in [12], we rigorously prove the convergence of Picard iterations and the finite element schemes. These results show that the proposed finite element methods converge for singular solutions.

1. Introduction

Magnetohydrodynamics (MHD) models have various important applications in liquid metal industry, controlled fusion and astronomy etc. There have been extensive discussions on numerical methods for MHD models. However, due to the complicated nonlinear coupling and rich structures of MHD systems, the numerical simulation still remains a challenging and active research area. This paper is devoted to a new class of stable and structure-preserving finite element methods.

We consider the following stationary MHD system on a polyhedral domain $\Omega$:

$$
\begin{aligned}
(u \cdot \nabla)u - R_e^{-1}\Delta u - sj \times B + \nabla p &= f, \\
j - R_m^{-1}\nabla \times B &= 0, \\
\nabla \times E &= 0, \\
\nabla \cdot B &= 0, \\
\nabla \cdot u &= 0, \\
j &= E + u \times B.
\end{aligned} 
$$

Here $u$ is the fluid velocity, $p$ is the fluid pressure, $j$ is the current density, $E$ and $B$ are the electric and magnetic fields respectively.
We mainly consider the following type of boundary conditions:

\[ u = 0, \quad B \cdot n = 0, \quad E \times n = 0, \quad \text{on } \partial \Omega, \]

where \( n \) is the unit normal vector of \( \partial \Omega \). We also consider an alternative boundary condition:

\[ u = 0, \quad B \times n = 0, \quad E \cdot n = 0, \quad \text{on } \partial \Omega. \]

The divergence-free condition \( \nabla \cdot B = 0 \) plays an important role in both physics (nonexistence of magnetic monopole) and numerical simulations (c.f. [5, 8]). For time dependent problems, Faraday’s law reads:

\[ \frac{\partial B}{\partial t} + \nabla \times E = 0. \]

Taking divergence on both sides, we have \( \partial_t (\nabla \cdot B) = 0 \). Therefore \( \nabla \cdot B(t) = \nabla \cdot B(0) = 0 \), i.e. the divergence-free condition of the initial data implies that of any time \( t \). Consequently, the divergence-free condition \( \nabla \cdot B = 0 \) is not independent for the time-dependent system. Most numerical methods simply drop this equation. In this regard, magnetic Gauss’s law cannot be preserved or preserved only in a weak sense on the discrete level. As a remedy, there has been many existing studies, c.f. [11] and references therein.

A structure-preserving finite element scheme has been developed in [11]. Electric and magnetic fields are both retained on the same discrete de Rham sequence, discretized by the Nédélec and Raviart-Thomas (BDM) elements respectively. Due to the compatible properties, the magnetic Gauss’s law is preserved precisely, i.e. the identity

\[ \frac{B^n - B^{n-1}}{\Delta t} + \nabla \times E^n = 0 \]

holds in the strong sense. As a result, \( \nabla \cdot B^n = 0 \) holds for any \( n \geq 1 \), as long as this divergence-free condition holds for the initial data \( (n = 0) \). However, a straightforward analysis in [11] requires a time step size condition

\[ \Delta t \leq \frac{1}{8s} \| B^- \|_{0,\infty}^{-2} \]

for the well-posedness, where \( B^- \) is the magnetic field from the previous time step.

For stationary systems, Faraday’s law is reduced to

\[ \nabla \times E = 0. \]

In this case, the divergence-free condition \( \nabla \cdot B = 0 \) cannot be implied by other equations in the system. In [12], the authors developed a new technique based on proper Lagrange multipliers. In contrary to conventional finite element schemes where the magnetic fields and the Lagrange multipliers are discretized by the Nédélec edge elements and the Lagrange nodal elements (c.f. [16]) to guarantee the weak divergence-free condition

\[ \int_{\Omega} B \cdot \nabla s = 0, \quad \forall s \in H^1(\Omega)/\mathbb{R}, \]

the new scheme discretizes the current density \( j \), magnetic field \( B \) and the multiplier \( r \) by the Nédélec edge element, Raviart-Thomas (BDM) face element and piecewise polynomials on a discrete de Rham complex respectively. The use of the current density \( j \) is motivated by the energy law (it is \( j = R_m^{-1} \nabla \times B \), not \( E \) itself, appears in the energy estimate). Consequently, no extra conditions are required in the well-posedness analysis.
In this paper, we analyze the $B$-$E$ formulation for stationary MHD models. With a careful analysis based on reduced systems, we show that the schemes are unconditional stable and well-posed. To achieve this result for both types of boundary conditions, we also extend the key Hodge mapping and $L^3$ estimates established in [12] to a new type of boundary condition. We rigorously show the convergence of the Picard iterations and the finite element schemes.

With the $B$-$E$ formulation analyzed in this paper, we do not need extra variables to evaluate the nonlinear term $u \times B$ as in the $B$-$j$ formulation proposed in [12]. Moreover, in this paper we show another strategy to impose the strong divergence-free condition, instead of using Lagrange multipliers. We introduce an augmented term $(\nabla \cdot B, \nabla \cdot C)$ in the variational formulation. Thanks to the structure-preserving properties, these two approaches are actually equivalent and Faraday’s law $\nabla \cdot B = 0$ also holds precisely on the discrete level.

The remaining part of this paper will be organized as follows. In Section 2, we provide some preliminary settings. In Section 3, we give two types of $L^3$ estimates for the discrete magnetic field. In Sections 4, 5 and 6, we formulate the numerical method for the MHD models with boundary condition (1.2), show its Picard iterations are well-posed and convergent, and show the optimal convergence of approximations to the velocity field and magnetic field even for singular solutions. In Section 7, we generalize the numerical method for the MHD models with boundary condition (1.3), provide its basic properties and show the optimal convergence.

2. Preliminaries

We assume that $\Omega$ is a bounded Lipschitz polyhedron. For the ease of exposition, we further assume that $\Omega$ is contractable, i.e. there is no nontrivial harmonic form.

Using the standard notation for the inner product and the norm of the $L^2$ space

$$(u, v) := \int_{\Omega} u \cdot v \, dx, \quad \|u\| := \left( \int_{\Omega} |u|^2 \, dx \right)^{1/2}.$$  

The scalar function space $H^1$ is defined by

$$H^1(\Omega) := \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \}.$$  

For a function $u \in W^{k,p}(\Omega)$, we use $\|u\|_{k,p}$ for the standard norm in $W^{k,p}(\Omega)$. When $p = 2$ we drop the index $p$, i.e. $\|u\|_k := \|u\|_{k,2}$ and $\|u\| := \|u\|_{0,2}$. We define vector function spaces

$$H(\text{curl}, \Omega) := \{ v \in L^2(\Omega), \nabla \times v \in L^2(\Omega) \},$$

and

$$H(\text{div}, \Omega) := \{ w \in L^2(\Omega), \nabla \cdot w \in L^2(\Omega) \}.$$ 

With explicit boundary conditions, we define

$$H^1_0(\Omega) := \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \},$$

$$H_0(\text{curl}, \Omega) := \{ v \in H(\text{curl}, \Omega), \nabla \times v = 0 \text{ on } \partial \Omega \},$$

and

$$H_0(\text{div}, \Omega) := \{ w \in H(\text{div}, \Omega), \nabla \cdot w = 0 \text{ on } \partial \Omega \}.$$ 

We often use the following notation:

$$L^2_0(\Omega) := \left\{ v \in L^2(\Omega) : \int_{\Omega} v = 0 \right\}.$$
The corresponding norms in $H^1$, $H(\text{curl})$ and $H(\text{div})$ spaces are defined by
\[
\|u\|^2_1 = \|u\|^2 + \|\nabla u\|^2,
\|F\|^2_{\text{curl}} := \|F\|^2 + \|\nabla \times F\|^2,
\|C\|^2_{\text{div}} := \|C\|^2 + \|\nabla \cdot C\|^2.
\]

For a general Banach space $Y$ with a norm $\| \cdot \|_Y$, the dual space $Y^*$ is equipped with the dual norm defined by
\[
\|h\|_{Y^*} := \sup_{0 \neq y \in Y} \frac{\langle h, y \rangle}{\|y\|_Y}.
\]

For the special case that $Y = H^1_0(\Omega)$, the dual space $Y^* = H^{-1}(\Omega)$ and the corresponding norm is denoted by $\| \cdot \|_{-1}$, which is defined by
\[
\|f\|_{-1} := \sup_{0 \neq v \in [H^1_0(\Omega)]^3} \frac{\langle f, v \rangle}{\|\nabla v\|}.
\]

In this paper, we will use $C$ to denote a generic constant in inequalities which is independent of the exact solution and the mesh size. For instance, we will need the following Poincaré’s inequality:
\[
\|u\|_{0,6} \leq C\|\nabla u\|, \quad \forall u \in H^1_0(\Omega).
\]

Since the fluid convection frequently appears in subsequent discussions, we introduce a trilinear form
\[
L(w; u, v) := \frac{1}{2}[((w \cdot \nabla)u, v) - ((w \cdot \nabla)v, u)].
\]

Considering $w$ as a known function, $L(w; u, v)$ is a bilinear form of $u$ and $v$.

Let $T_h$ be a triangulation of $\Omega$, and we assume that the mesh is regular and quasi-uniform, so that the inverse estimates hold [6]. We use $P_k(T_h)$ to denote the piecewise polynomial space of degree $k$ on $T_h$. The finite element de Rham sequence is an abstract framework to unify the above spaces and their discretizations, see e.g. Arnold, Falk, Winther [1, 2], Hiptmair [10], Bossavit [4] for more detailed discussions. Figure 1 and Figure 2 show the commuting diagrams we will use. The electric field $E$ and the magnetic field $B$ will be discretized in the middle two spaces respectively. Notice that though projections in Figure 1 can be different from corresponding ones in Figure 2, we don’t need to distinguish them in any analysis in this paper.

As we shall see, $H(\text{div})$ functions with vanishing divergence will play an important role in the study. So we define on the continuous level
\[
H_0(\text{div0}, \Omega) := \{C \in H_0(\text{div}, \Omega) : \nabla \cdot C = 0\},
\]
and the finite element subspace
\[
H_0^h(\text{div0}, \Omega) := \{C_h \in H_0^h(\text{div}, \Omega) : \nabla \cdot C_h = 0\}.
\]
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Figure 2. Continuous and discrete de Rham sequence - no boundary condition

We use $V_h$ to denote the finite element subspace of velocity $u_h$, and $Q_h$ for pressure $p_h$. There are many existing stable pairs for $V_h$ and $Q_h$, for example, the Taylor-Hood elements $[^9, ^3]$. Spaces $H^0_h(\text{grad}, \Omega)$ and $L^2_{0,h}(\Omega)$ are finite elements from the discrete de Rham sequence. For these spaces we use the explicit names for clarity, and use the notation $V_h$ and $Q_h$ for the fluid part to indicate that they may be different from $H^0_h(\text{grad}, \Omega)$ and $L^2_{0,h}(\Omega)$ in the de Rham sequence. We use $V_0^h$ to denote the discrete velocity space, i.e.

$$V_0^h := \{ v_h \in V_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q_h \}.$$

There is a unified theory for the discrete de Rham sequence of arbitrary order $[^3, ^1, ^2]$. In the case $n = 3$, the lowest order elements can be represented as:

$$
\begin{array}{cccccccc}
R & \subset & \mathcal{P}_3 \Lambda^0 & d & \mathcal{P}_2 \Lambda^1 & d & \mathcal{P}_1 \Lambda^2 & d & \mathcal{P}_0 \Lambda^3 & 0, \\
R & \subset & \mathcal{P}_2 \Lambda^0 & d & \mathcal{P}_1 \Lambda^1 & d & \mathcal{P}_1^- \Lambda^2 & d & \mathcal{P}_0 \Lambda^3 & 0, \\
R & \subset & \mathcal{P}_2 \Lambda^0 & d & \mathcal{P}_2^- \Lambda^1 & d & \mathcal{P}_1 \Lambda^2 & d & \mathcal{P}_0 \Lambda^3 & 0, \\
R & \subset & \mathcal{P}_1 \Lambda^0 & d & \mathcal{P}_1^- \Lambda^1 & d & \mathcal{P}_1^- \Lambda^2 & d & \mathcal{P}_0 \Lambda^3 & 0.
\end{array}
$$

The correspondence between the language of differential forms and classical finite element methods is summarized in Table 1.

To obtain compatible finite element schemes, below we require that the discrete spaces $H^0_h(\text{curl}, \Omega)$, $H^0_h(\text{div}, \Omega)$ and $L^2_{0,h}(\Omega)$ belong to the same finite element de Rham sequence.

| $k$ | $\Lambda^k_h(\Omega)$ | Classical finite element space |
|-----|------------------|-------------------------------|
| 0   | $\mathcal{P}_r \Lambda^0(\mathcal{T})$ | Lagrange elements of degree $\leq r$ |
| 1   | $\mathcal{P}_r \Lambda^1(\mathcal{T})$ | Nedelec 2nd-kind $H(\text{curl})$ elements of degree $\leq r$ |
| 2   | $\mathcal{P}_r \Lambda^2(\mathcal{T})$ | Nedelec 2nd-kind $H(\text{div})$ elements of degree $\leq r$ |
| 3   | $\mathcal{P}_r \Lambda^3(\mathcal{T})$ | discontinuous elements of degree $\leq r$ |
| 0   | $\mathcal{P}_r^- \Lambda^0(\mathcal{T})$ | Lagrange elements of degree $\leq r$ |
| 1   | $\mathcal{P}_r^- \Lambda^1(\mathcal{T})$ | Nedelec 1st-kind $H(\text{curl})$ elements of order $r - 1$ |
| 2   | $\mathcal{P}_r^- \Lambda^2(\mathcal{T})$ | Nedelec 1st-kind $H(\text{div})$ elements of order $r - 1$ |
| 3   | $\mathcal{P}_r^- \Lambda^3(\mathcal{T})$ | discontinuous elements of degree $\leq r - 1$ |

Table 1. Correspondences between finite element differential forms and the classical finite element spaces for $n = 3$ (from $[^1]$)
As we shall see, it is useful to group the spaces to define
\[ X_h := V_h \times H_0^b(\text{curl}, \Omega) \times H_0^b(\text{div}, \Omega), \]
and group \( Q_h \times L_0^{2,b}(\Omega) \) to define
\[ Y_h := Q_h \times L_0^{2,b}(\Omega). \]

For the analysis, we also need to define a reduced space, where \( E_h \) is eliminated:
\[ W_h := V_h \times H_0^b(\text{div}, \Omega). \]

Denote the kernel space
\[ X_{h0} := \left( V_0^b \times H_0^b(\text{div}, \Omega) \right) \times \left( V_0^b \times H_0^b(\text{curl}, \Omega) \right) \times H_0^b(\text{div}^0, \Omega), \]
and
\[ W_{h0} := V_0^b \times H_0^b(\text{div}^0, \Omega). \]

By definition, any \((u_h, B_h, q_h) \in W_{h0}\) satisfies
\[ (\nabla \cdot u_h, q_h) = 0, \quad \forall q_h \in Q_h \quad \text{and} \quad \nabla \cdot B_h = 0. \]

In order to define appropriate norms, we introduce the discrete curl operator on the discrete level.

For any \( C_h \in H_0^b(\text{div}, \Omega) \), define \( \nabla h \times C_h \in H_0^b(\text{curl}, \Omega) \) by:
\[ (\nabla h \times C_h, F_h) = (C_h, \nabla \times F_h), \quad \forall F_h \in H_0^b(\text{curl}, \Omega). \quad (2.2) \]

For any \( w_h \in H_0^b(\text{grad}, \Omega) \), we define \( \nabla h \cdot w_h \in H_0^b(\text{grad}, \Omega) \) by
\[ (\nabla h \cdot w_h, v_h) = -(w_h, \nabla v_h), \quad \forall v_h \in H_0^b(\text{grad}, \Omega). \quad (2.3) \]

We define \( \mathbb{P} : L^2(\Omega) \rightarrow H_0^b(\text{curl}, \Omega) \) to be the \( L^2 \) projection
\[ (\mathbb{P} \phi, F_h) = (\phi, F_h), \quad \forall F_h \in H_0^b(\text{curl}, \Omega), \phi \in L^2(\Omega). \]

We further define \( \| \cdot \|_d \) as a modified norm of \( H_0^b(\text{div}, \Omega) \) by
\[ \| C_h \|_d := \| C_h \|^2 + \| \nabla \cdot C_h \|^2 + \| \nabla h \times C_h \|^2. \]

Now we define the norms for various product spaces. For space \( Y_h \), we define
\[ \|(q, r)\|_{Y_h}^2 := \|q\|^2 + \|r\|^2. \quad (2.4) \]

For other product spaces, we define
\[ \|(v, F, C)\|_X^2 := \|v\|^2 + \|\nabla v\|^2 + \|\nabla \times F\|^2 + \|F + v \times B^-\|^2 + \|C\|^2 + \|\nabla \cdot C\|^2, \quad \forall (v, F, C) \in X_h, \quad (2.5) \]
and
\[ \|(u_h, B_h)\|_{W_h}^2 := \|u_h\|^2 + \|B_h\|_{d_t}^2, \quad \forall (u_h, B_h) \in W_h. \]

Here \( B^- \in H(\text{div}, \Omega) \) is a given function.

The constant \( sR_m^{-1} \) will appear in the discussions below frequently, therefore we denote
\[ \alpha := sR_m^{-1}. \]
3. Hodge mapping and $L^p$ estimates for divergence-free finite elements

In this section we present some key $L^3$ embedding results which are crucial for our analysis in the following sections.

**Theorem 1.** For any function $d_h \in H^0_d(\text{div}0, \Omega)$, we have

$$\|d_h\|_{0,3} \leq C\|\nabla_h \times d_h\|,$$

where the generic constant $C$ solely depends on $\Omega$.

Theorem 1 and its proof can be found in [12, Theorem 1]. For the boundary condition given in (1.3), we have similar estimates.

**Theorem 2.** For any function $d_h \in H^0(\text{div}0, \Omega)$, we have

$$\|d_h\|_{0,3} \leq C\|\tilde{\nabla}_h \times d_h\|,$$

where $\tilde{\nabla}_h \times d_h \in H^0(\text{curl}, \Omega)$ satisfies

$$(\tilde{\nabla}_h \times d_h, F) = (d_h, \nabla \times F), \quad \forall F \in H^0(\text{curl}, \Omega).$$

The generic constant $C$ solely depends on $\Omega$.

**Proof.** We define $Z_0 = H^0_0(\text{curl}, \Omega) \cap H(\text{div}0, \Omega), \ Z_h^0 = H^0(\text{div}, \Omega) \cap H(\text{div}0, \Omega)$. Obviously, $d_h \in Z_h^0$.

We define an operator $H_d : Z_h^0 \to Z_0$ by

$$(\nabla \times (H_d d_h), \nabla \times v) = (\tilde{\nabla}_h \times d_h, \nabla \times v), \quad \forall v \in Z_0.$$

Obviously, $H_d$ is well defined. Since $H_d d_h \in Z_0$, we have

$$\|H_d d_h\|_{1+\delta} \leq C\|\nabla \times (H_d d_h)\| \leq C\|\tilde{\nabla}_h \times d_h\|,$$

where $\delta \in (0, \frac{1}{2}]$.

We use the projections $\Pi^\text{curl}$ and $\Pi^\text{div}$ in the commuting diagram in Figure 2.

Since $\nabla \cdot (d_h - \Pi^\text{div}(H_d d_h)) = 0$ in $\Omega$, there exists $\phi_h \in \{v \in H^0(\text{curl}, \Omega) : (v, \nabla s) = 0, \ \forall s \in H^0(\text{grad}, \Omega)\}$, such that

$$\nabla \times \phi_h = d_h - \Pi^\text{div}(H_d d_h).$$

We consider the auxiliary problem:

$$\nabla \times \nabla \times \psi = \nabla \times \phi_h \quad \text{in} \quad \Omega,$$

$$\nabla \cdot \psi = 0 \quad \text{in} \quad \Omega,$$

$$\psi \times n = 0 \quad \text{on} \quad \partial \Omega.$$

Since $\nabla \cdot (\nabla \times \phi_h) = 0$ in $\Omega$, the auxiliary problem (3.2) is well-posed. Obviously, $\nabla \times \psi$ satisfies

$$\nabla \times (\nabla \times \psi) = \nabla \times \phi_h \quad \text{in} \quad \Omega,$$

$$\nabla \cdot (\nabla \times \psi) = 0 \quad \text{in} \quad \Omega,$$

$$(\nabla \times \psi) \cdot n = 0 \quad \text{on} \quad \partial \Omega.$$

According to [10, Lemma 4.2], we have

$$\|\nabla \times \psi\|_{1+\delta} \leq C\|\nabla \times \phi_h\| = C\|d_h - \Pi^\text{div}(H_d d_h)\|.$$
We claim that
\[
\|\nabla \times \psi - \phi_h\| \leq Ch^{\frac{1}{2}+\delta}\|d_h - \Pi^{\text{div}}(H_d d_h)\|. \tag{3.4}
\]
Notice that by (3.2),
\[
\nabla \times \Pi^{\text{curl}}(\nabla \times \psi) = \Pi^{\text{div}}(\nabla \times \nabla \times \psi) = \Pi^{\text{div}}(\nabla \times \phi_h) = \nabla \times \phi_h.
\]
Since \(\Pi^{\text{curl}}(\nabla \times \psi), \phi_h \in H^h(\text{curl}, \Omega)\), there exists \(s_h \in H^h(\text{grad}, \Omega)\) such that
\[
\Pi^{\text{curl}}(\nabla \times \psi) - \phi_h = \nabla s_h \quad \text{in} \ \Omega.
\]
Since \((\nabla \times \psi) \cdot n = 0\) on \(\partial \Omega\), we have
\[
(\nabla \times \psi, \Pi^{\text{curl}}(\nabla \times \psi) - \phi_h) = (\nabla \times \psi, \nabla s_h) = 0.
\]
By the construction of \(\phi_h\), we have
\[
(\phi_h, \Pi^{\text{curl}}(\nabla \times \psi) - \phi_h) = (\phi_h, \nabla s_h) = 0.
\]
Thus
\[
(\nabla \times \psi - \phi_h, \Pi^{\text{curl}}(\nabla \times \psi) - \phi_h) = 0.
\]
So, by the above identity and (3.3), we have
\[
\|\nabla \times \psi - \phi_h\| \leq \|\nabla \times \psi - \phi_h\| - (\Pi^{\text{curl}}(\nabla \times \psi) - \phi_h)\|
\leq \|\nabla \times \psi - \Pi^{\text{curl}}(\nabla \times \psi)\|
\leq Ch^{\frac{1}{2}+\delta}\|d_h - \Pi^{\text{div}}(H_d d_h)\|.
\]
Therefore, the claim (3.4) is correct.

By the construction of \(H_d\) and the fact that \(\psi \in Z_0\),
\[
(\tilde{\nabla}_h \times d_h, \nabla \times \psi) = (\nabla \times (H_d d_h), \nabla \times \psi) = (H_d d_h, \nabla \times \nabla \times \psi) = (H_d d_h, \nabla \times \phi_h).
\]
By the fact that \(\phi_h \in H^h(\text{curl}, \Omega)\) and the above identity,
\[
(d_h, \nabla \times \phi_h) = (\tilde{\nabla}_h \times d_h, \phi_h)
= (\tilde{\nabla}_h \times d_h, \phi_h - \nabla \times \psi) + (\tilde{\nabla}_h \times d_h, \nabla \times \psi)
= (\tilde{\nabla}_h \times d_h, \phi_h - \nabla \times \psi) + (H_d d_h, \nabla \times \phi_h).
\]
Thus we have
\[
(d_h - H_d d_h, d_h - \Pi^{\text{div}}(H_d d_h)) = (d_h - H_d d_h, \nabla \times \phi_h) = (\tilde{\nabla}_h \times d_h, \phi_h - \nabla \times \psi).
\]
So we have
\[
\|d_h - H_d d_h\|^2
= (d_h - H_d d_h, d_h - \Pi^{\text{div}}(H_d d_h)) + (d_h - H_d d_h, \Pi^{\text{div}}(H_d d_h) - H_d d_h)
= (\tilde{\nabla}_h \times d_h, \phi_h - \nabla \times \psi) + (d_h - H_d d_h, \Pi^{\text{div}}(H_d d_h) - H_d d_h)
\leq \|\tilde{\nabla}_h \times d_h\| \cdot \|\phi_h - \nabla \times \psi\| + \|d_h - H_d d_h\| \cdot \|\Pi^{\text{div}}(H_d d_h) - H_d d_h\|.
\]
By applying (3.4) in the above inequality, we have
\[
\|d_h - H_d d_h\| \leq Ch^{\frac{1}{2}+\delta}\|\tilde{\nabla}_h \times d_h\|. \tag{3.5}
\]
Let $k_0$ be a positive integer such that $H^1_0(\text{div}0, \Omega) \subset [P_{k_0}(T_h)]^3$. We denote by $\Pi$ the standard $L^2$-orthogonal projection onto $[P_{k_0}(T_h)]^3$. Thus $\Pi d_h = d_h$. So, by the discrete inverse inequality and the fact that $\|\Pi v\|_{0,3} \leq C\|v\|_{0,3}$ for any $v \in [L^3(\Omega)]^3$, we have

$$\|d_h\|_{0,3} = \|\Pi d_h\|_{0,3} \leq \|\Pi(d_h - H_d d_h)\|_{0,3} + \|H_d d_h\|_{0,3} \leq C\left(h^{-\frac{1}{2}}\|\nabla \times d_h\| + \|H_d d_h\|_{0,3}\right).$$

Since $H_d d_h \in H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$,

$$\|H_d d_h\|_{0,3} \leq C\|\nabla \times (H_d d_h)\| \leq C\|\nabla \times d_h\|.$$

So, we can conclude that

$$\|d_h\|_{0,3} \leq C\|\nabla \times d_h\|.$$

This completes the proof. \(\square\)

### 4. Variational formulations

#### 4.1. Nonlinear scheme. We propose the following variational form for (1.1) with boundary condition (1.2):

**Problem 1.** Find $(u_h, E_h, B_h) \in X_h$ and $(p_h, r_h) \in Y_h$, such that for any $(v, F, C) \in X_h$ and $(q, s) \in Y_h$,

$$L(u_h; u_h, v) + R_e^{-1}(\nabla u_h, \nabla v) - s(j_h \times B_h, v) - (p_h, \nabla \cdot v) = \langle f, v \rangle, \quad (4.1a)$$

$$s(j_h, F) - \alpha(B_h, \nabla \times F) = 0, \quad (4.1b)$$

$$\alpha(\nabla \times E_h, C) + (r_h, \nabla \cdot C) = 0, \quad (4.1c)$$

$$-(\nabla \cdot u_h, q) = 0, \quad (4.1d)$$

$$\nabla \cdot B_h, s = 0, \quad (4.1e)$$

where $j_h$ is given by Ohm’s law: $j_h = E_h + u_h \times B_h$. Here $r_h$ is the Lagrange multiplier which approximates $r = 0$.

We verify some properties of the variational form Problem 1:

**Theorem 3.** Any solution for Problem 1 satisfies

1. **magnetic Gauss’s law:**

   $$\nabla \cdot B_h = 0.$$

2. **Lagrange multiplier** $r_h = 0$, and the strong form

   $$\nabla \times E_h = 0,$$
(3) energy estimates:
\[ R^{-1}_e \| \nabla u_h \|^2 + s \| j_h \|^2 = \langle f, u_h \rangle, \quad (4.2) \]
\[ \frac{1}{2} R^{-1}_e \| \nabla u_h \|^2 + s \| j_h \|^2 \leq \frac{R}{2} \| f \|^2, \quad (4.3) \]
\[ R^{-1}_m \| \nabla_h \times B_h \| \leq \| j_h \|, \quad (4.4) \]
\[ \| \nabla_h \times B_h \| \leq CR \bar{R} R^{-1}_m \| f \|^{-\frac{3}{2}}, \quad (4.5) \]
\[ \| E_h \| \leq CR \bar{R} R^{-1}_m \| f \|^{-\frac{1}{2}}. \quad (4.6) \]

Proof. The magnetic Gauss’s law is a direct consequence of (4.1c).

Taking \( C = \nabla \times E_h \) in (4.1c), we have \( \nabla \times E_h = 0 \). Therefore (4.1c) reduces to
\[ (r_h, \nabla \cdot C) = 0, \quad \forall C \in H^1_0(\text{div}, \Omega). \]

Since \( L^2_{0,h}(\Omega) = \nabla \cdot H^1_0(\text{div}, \Omega) \), we get \( r_h = 0 \).

To obtain the first energy estimate, we take \( v = u_h \), \( F = E_h \), \( C = B_h \) and \( q = p_h \) in (4.1a) - (4.1d) and add the equations together. The second energy estimate follows from the Young’s inequality
\[ \langle f, u_h \rangle \leq \| f \|^{-1}_1 \| \nabla u_h \| \leq \frac{1}{2} R^{-1}_e \| \nabla u_h \|^2 + \frac{1}{2} R^{-1}_e \| f \|^2. \]

Taking \( F = \nabla_h \times B_h \) in (4.1b) we have
\[ R^{-1}_m \| \nabla_h \times B_h \|^2 = R^{-1}_m (j_h, \nabla_h \times B_h) \leq \| j_h \| \| \nabla_h \times B_h \|, \]
which implies (4.4). Obviously, the estimate (4.5) is due to estimates (4.3) and (4.4).

Next we take \( F = E_h \) in (4.1b) and by the definition of \( j_h \) we have
\[ (E_h + u_h \times B_h, E_h) - R^{-1}_m (B_h, \nabla \times E_h) = 0. \]

By the fact that \( \nabla \times E_h = 0 \) and the generalized Hölder’s inequality we have
\[ \| E_h \|^2 = -(u_h \times B_h, E_h) \leq \| u_h \|_0 \| B_h \|_0 \| E_h \| \leq C \| \nabla u_h \| \| \nabla \times B_h \| \| E_h \|, \]
the last step is due to the Sobolev embedding results (2.1) and Theorem 1. The estimate (4.6) can be obtained by combining the above estimate with (4.3) and (4.5). This completes the proof. \( \square \)

Remark 1. From the above result we can see that the energy norm of the unknowns \( u_h, B_h, E_h \) solely depends on \( \| f \|^{-1}_1 \) and the physical constants \( R_m, R_e, s \). In addition, it is easy to verify that the exact solution satisfies the same stability estimate
\[ \| \nabla u \| \leq R_e \| f \|^{-1}_1, \quad (4.7) \]
\[ \| B \|_0 + \| \nabla \times B \| \leq CR^\frac{1}{2} R^{-1}_m s^{-\frac{1}{2}} \| f \|^{-1}_1, \]
\[ \| E \| \leq CR^\frac{3}{2} R^{-1}_m s^{-\frac{1}{2}} \| f \|^{-1}_1. \]

Theorem 4. Problem 1 is well-posed.

In the remaining part of this section we prove the well-posedness of Problem 1. We will first recast Problem 1 into an equivalent form ((4.9) and Problem 2) where \( E \) is formally eliminated. Then we demonstrate that this equivalent form is well-posed using the Brezzi theory and the key \( L^2 \) estimate (Theorem 5). Then we can conclude with the well-posedness of Problem 1.
Using (4.1b), we have
\[ E_h + \mathbb{P}(u_h \times B_h) = R_m^{-1} \nabla_h \times B. \]

Now the Lorentz force has an equivalent form
\[
-(j_h \times B_h, v) = (E_h + \mathbb{P}(u_h \times B_h), v \times B_h) + ((I - \mathbb{P})(u_h \times B_h), v \times B_h)
\]
\[ = R_m^{-1} (\nabla_h \times B_h, v \times B_h) + ((I - \mathbb{P})(u_h \times B_h), (I - \mathbb{P})(v \times B_h)). \quad (4.8) \]

Even though the velocity field \( u_h \) is smooth, the \( H(\text{div}) \) conformality of the magnetic field \( B_h \) cannot guarantee \( u_h \times B_h \in H(\text{curl}, \Omega) \). The term \((I - \mathbb{P})(u_h \times B_h)\) on the right-hand side of (4.8) measures the deviation of \( u_h \times B_h \) from \( H^b(\text{curl}) \) and \((I - \mathbb{P})(u_h \times B_h), (I - \mathbb{P})(u_h \times B_h)\) can be regarded as a penalty term.

Therefore (4.1) is equivalent to the following problem: Find \((u_h, B_h) \in W_h \) and \((p_h, r_h) \in Y_h \) such that for any \((v_h, C_h) \in W_h \) and \((q_h, s_h) \in Y_h \),

\[
\begin{align*}
L(w_h; u_h, v_h) + R_m^{-1}(\nabla h \times B_h, B_h \times v_h) &+ s (I - \mathbb{P})(u_h \times B_h), (I - \mathbb{P})(v \times B_h) - (p_h, \nabla \cdot v_h) = (f, v), \\
- \alpha(u_h \times B_h, \nabla h \times C_h) + sR_m^{-2}(\nabla h \times B_h, \nabla h \times C_h) + (r_h, \nabla \cdot C_h) &= 0, \\
(\nabla \cdot u_h, q_h) &= 0, \\
(\nabla \cdot B_h, s_h) &= 0. 
\end{align*} \quad (4.9)
\]

Denote \( \psi_h = (w_h, G_h), \xi_h = (u_h, B_h), \eta_h = (v_h, C_h) \) and \( x_h = (p_h, r_h), y_h = (q_h, s_h) \). Define
\[
a_s(\psi_h; \xi_h, \eta_h) := \frac{1}{2} [(w_h \cdot \nabla)u_h, v_h] + R_m^{-1}(\nabla u_h, \nabla v_h)
\]
\[
- \alpha(\nabla h \times B_h, G_h \times v_h) + s((I - \mathbb{P})(u_h \times G_h), (I - \mathbb{P})(v \times G_h))
\]
\[
- \alpha(u_h \times G_h, \nabla h \times C_h) + sR_m^{-2}(\nabla h \times B_h, \nabla h \times C_h),
\]

and
\[
b_s(\xi_h, y_h) := -(\nabla \cdot u_h, q_h) + (\nabla \cdot B_h, s_h).
\]

Equation (4.9) can be recast into a mixed system:

**Problem 2.** Given \( \theta \in W_h^* \) and \( \psi \in Y_h^* \), find \((\xi_h, x_h) \in W_h \times Y_h \), such that

\[
\begin{align*}
\mathbf{a}_s(\xi_h; \xi_h, \eta_h) + \mathbf{b}_s(\eta_h, x_h) &= \langle \theta, \eta_h \rangle, \quad \forall \eta_h \in W_h, \\
\mathbf{b}_s(\xi_h, y_h) &= \langle \psi, y_h \rangle, \quad \forall y_h \in Y_h. 
\end{align*} \quad (4.10)
\]

**Theorem 5.** Problem 2 is well-posed.

We prove the existence of solutions to the discrete variational form. To use the Brezzi theory and the fixed point theorem (see [9]), we need to show
- each term in (4.10) is bounded,
- the inf-sup condition for \( b_s \),
- coercivity of \( a_s \) on \( W_h^{00} \).

We establish these conditions in the subsequent lemmas.

The boundedness of the variational form is a direct consequence of the key \( L^2 \) estimate.
Lemma 1. The trilinear form $a_s(\cdot, \cdot, \cdot)$ and the bilinear form $b_s(\cdot, \cdot)$ are bounded, i.e. there exists a positive constant $C$ such that
\[
a_s(\psi_h, \xi_h, \eta_h) \leq C \|\psi_h\|_W \|\xi_h\|_W \|\eta_h\|_W, \quad \forall \psi_h, \xi_h, \eta_h \in W_h,
\]
and
\[
b_s(\eta_h, y_h) \leq C \|\eta_h\|_W \|y_h\|_Y, \quad \forall \eta_h \in W_h, y_h \in Y_h.
\]

Since we have used a stronger norm for $B_h, C_h \in H^1_0(\text{div}, \Omega)$, the inf-sup condition for the bilinear form $b_s(\cdot, \cdot)$ becomes more subtle. Following a similar proof as shown in [12] for the $B\cdot j$ formulation, we get:

Lemma 2. (inf-sup conditions for $b_s(\cdot, \cdot)$) There exists a positive constant $\gamma$ such that
\[
\inf_{\eta_h \in Y_h} \sup_{y_h \in W_h} \frac{b_s(\eta_h, y_h)}{\|\eta_h\|_W \|y_h\|_Y} \geq \gamma > 0.
\]

The coercivity of $a_s(\cdot, \cdot, \cdot)$ holds on the kernel space $W_h^{00}$.

Lemma 3. On $W_h^{00}$ we have
\[
a_s(\xi_h, \xi_h, \xi_h) \geq \gamma \|\xi_h\|^2_W,
\]
where $\gamma$ is a positive constant.

Proof. We note that
\[
a_s(\xi_h, \xi_h, \xi_h) = R^{-1} \|\nabla u_h\|^2 + s((I - P)(u_h \times B_h))^2 + sR_h^{-2} \|\nabla \times B_h\|^2.
\]

Discrete Poincaré’s inequality holds on $W_h^{00}$:
\[
\|B_h\| \lesssim \|\nabla \times B_h\|.
\]

This completes the proof. \hfill \Box

By Lemma 1, Lemma 2 and Lemma 3, the nonlinear variational form (4.10) is well-posed. Therefore (4.9) has at least one solution. For suitable source and boundary data, the solution is also unique.

4.2. Picard iterations. We propose the following Picard type iterations for Problem 1:

Algorithm 1 (Picard iterations for nonlinear schemes). Given $(u^{n-1}, B^{n-1})$, find $(u^n, E^n, B^n) \in X_h$ and $(p^n, r^n) \in Y_h$, such that for any $(v, F, C) \in X_h$ and $(q, s) \in Y_h$,
\[
L(u^{n-1}; u^n, v) + R^{-1}_e(\nabla u^n, \nabla v) - s(j_{n-1}^n \times B^{n-1}, v) - (p^n, \nabla \cdot v) = \langle f, v \rangle, \quad (4.11)
\]
\[
s(j_{n-1}^n, F) - \alpha(B^n, \nabla \times F) = 0, \quad (4.12)
\]
\[
\alpha(\nabla \times E^n, C) + (r^n, \nabla \cdot C) = 0, \quad (4.13)
\]
\[
-(\nabla \cdot u^n, q) = 0, \quad (4.14)
\]
\[
(\nabla \cdot B^n, s) = 0, \quad (4.15)
\]

where $j_{n-1}^n$ is defined by $j_{n-1}^n = E^n + u^n \times B^{n-1}$.

The divergence-free property, compatibility and energy estimates can be obtained in an analogous way:

Theorem 6. For any possible solution to Algorithm 1:
(1) magnetic Gauss’s law holds precisely:
\[ \nabla \cdot B^n = 0. \]
(2) the Lagrange multiplier \( r^n = 0 \), therefore (4.13) has the form
\[ \nabla \times E^n = 0. \]
(3) the energy estimates hold:
\[ R_e^{-1}\|\nabla u^n\|^2 + s\|j_{n-1}^n\|^2 = (f, u^n), \]
and
\[ \frac{1}{2} R_e^{-1}\|\nabla u^n\|^2 + s\|j_{n-1}^n\|^2 \leq \frac{1}{2} R_e\|f\|_{-1}^2. \quad (4.16) \]

We will use the Brezzi theory to prove the well-posedness of the Picard iterations. We first
recast Picard iterations (Algorithm 1) as follows. Given \( (u^-; B^-) \in W_h \). For \( \Psi = (u, E, B) \),
\( \Omega = (v, F, C) \in X_h \) and \( (p, r), (q, s) \in Y_h \), define bilinear forms \( a_{s, s}(:, :) \) and \( b(:, :) \):
\[ a_{s, s}(\Psi, \Psi) := \frac{1}{2} L (u^-; u, v) + R_e^{-1}(\nabla u, \nabla v) + s(E + u \times B^-; F + v \times B^-) \]
\[ - \alpha(B, \nabla \times F) + \alpha(\nabla \times E, C). \]

Given a nonlinear iterative step, the mixed form of the iterative scheme in Algorithm 1 can be
written as: for any \( h = (f, r, l) \in X_h \) and \( g \in Y_h^* \), find \( (\Psi, y) \in X_h \times Y_h^* \), such that for any
\( (\Psi, y) \in X_h \times Y_h^* \),
\[ \begin{aligned}
& a_{s, s}(\Psi, \eta) + b_s(\Psi, x) = (h, \eta), \\
& b_s(\Psi, y) = (g, y). \end{aligned} \quad (4.17) \]

To prove the well-posedness of (4.17) based on the Brezzi theory, we need to verify the boundedness
of each term, the inf-sup condition of \( b_s(:, :) \) and the coercivity of \( a_{s, s}(:, :) \) on \( X_h^{00} \).

For the inf-sup condition of \( b_s(:, :) \), we have:

**Lemma 4.** (inf-sup conditions of \( b_s(:, :) \)) There exists a positive constant \( \gamma \) such that
\[ \inf_{\Psi \in \Omega} \sup_{\Psi \in \Omega} \frac{b_s(\Psi, y)}{\|\Psi\|_X \|y\|_Y} \geq \gamma > 0. \]

**Proof.** There exists a positive constant \( \gamma_0 > 0 \) such that
\[ \inf_{q \in Q_h} \sup_{v \in \Omega} -\langle \nabla \cdot v, q \rangle \geq \gamma_0 > 0. \]
Consequently, for any \( q \in Q_h \) there exists \( v_q \in \Omega_h \), such that
\[ -\langle \nabla \cdot v_q, q \rangle \geq \gamma_0 \|q\|^2, \]
and
\[ \|v_q\|_1 = \|q\|. \]

For the magnetic multiplier, we have \( \nabla \cdot H_0^2(\text{div}, \Omega) = L_{0, h}^2(\Omega) \). For any \( s \in L_{0, h}^2(\Omega) \), there exists
\( C_s \in H_0^2(\text{div}, \Omega) \) such that \( \nabla \cdot C_s = s, \|C_s\|_{\text{div}} \leq C\|s\| \), where \( C \) is a positive constant.

For any \( \Psi = (q, s) \), take \( y = (v_q, C_s) \). Then
\[ b_s(\Psi, y) = -\langle \nabla \cdot v_q, q \rangle + \langle \nabla \cdot C_s, s \rangle \geq \gamma_0 \|q\|^2 + \|s\|^2 \geq \min(\gamma_0, 1) \|y\|_Y^2, \]
and
\[ \|v_q\|^2_1 + \|C_s\|_{\text{div}}^2 \leq \|q\|^2 + C^2 \|s\|^2 \leq \max(1, C^2)\|y\|^2_{\mathcal{Y}}. \]

This completes the proof. \hfill \Box

**Theorem 7.** Problem (4.17), therefore Algorithm 1, is well-posed with the norms defined by (2.5) and (2.4).

**Proof.** The boundedness of the variational form is obvious from the definition of \( \| \cdot \|_{\mathcal{X}} \). Moreover, we note that \( a_{s, L}(u, u) = R^{-1}\|\nabla u\| + s\|E + u \times B^{-}\|^2 \). Therefore the bilinear form \( a_{s, L}(\cdot, \cdot) \) is coercive on \( X_h^{00} \).

Combining the boundedness of the variational form, the inf-sup condition of \( b_s(\cdot, \cdot) \) (Lemma 4) and the coercivity of \( a_{s, L}(\cdot, \cdot) \) on \( X_h^{00} \), we complete the proof. \hfill \Box

From the triangular inequality and Hölder’s inequality, we have
\[ \|E\| \leq \|E + u \times B^{-}\| + \|u \times B^{-}\| \lesssim \|E + u \times B^{-}\| + \|u\|_1\|B^{-}\|_{0,3}. \]

In Picard iterations (Algorithm 1), function \( B^{-} \) is given by the magnetic field from the previous iterative step, i.e. \( B^{-} = B^{n-1} \). We have the following estimate:
\[ \|B^{-}\|_{0,3} = \|B^{n-1}\|_{0,3} \lesssim \|\nabla h \times B^{n-1}\| \lesssim \|f\|_{-1}, \tag{4.18} \]
where the last equality is due to the energy law.

Therefore the \( L^2 \) norm of the electric field \( E \) can be bounded by \( \|(u, E, B)\|_{X} \) and given data, i.e., norm \( \|(u, E, B)\|_{X} \) is equivalent to the decoupled norm
\[ (\|u\|^2 + \|E\|^2_{\text{curl}} + \|B\|^2_{\text{div}})^{\frac{1}{2}}. \]

The constants involved in the equivalence depend on \( \|B^{-}\|_{0,3} \) which further depends on \( \|f\|_{-1} \).

### 4.3. Schemes without magnetic Lagrange multipliers

Thanks to the structure-preserving properties of the discrete de Rham complex, we can design a finite element scheme for stationary MHD problems without using magnetic multipliers. The resulting scheme is equivalent to (4.1), therefore magnetic Gauss’s law is precisely preserved.

Consider the following weak form:

**Problem 3.** Find \( (u_h, E_h, B_h) \in X_h \) and \( \rho_h \in Q_h \), such that for any \( (v, F, C) \in X_h \) and \( q \in Q_h \),
\[
\begin{align*}
L(u_h; u_h, v) + R^{-1}(\nabla u_h, \nabla v) - s(j_h \times B_h, v) - (\rho_h, \nabla \cdot v) &= \langle f, v \rangle, \\
\alpha(\nabla \times E_h, C) + \alpha(\nabla \cdot B_h, \nabla \cdot C) &= 0, \\
\alpha(j_h, F) - \alpha(B_h, \nabla \times F) &= 0, \\
\alpha(\nabla \cdot u_h, q) &= 0,
\end{align*}
\]  

(4.19)

where \( j_h \) is given from Ohm’s law: \( j_h = E_h + u_h \times B_h \).

Compared with Problem 1, the magnetic Lagrange multiplier has been removed and we augment the variational formulation by introducing \( (\nabla \cdot B_h, \nabla \cdot C) \) term. Next we verify some properties of the proposed schemes.

**Theorem 8.** Any solution to Problem 3 satisfies
(1) **magnetic Gauss’s law in the strong sense:**
\[ \nabla \cdot B_h = 0, \]

(2) **the discrete energy law:**
\[ R_e^{-1} \| \nabla u_h \|^2 + s \| j_h \|^2 = (f, u_h), \]
and
\[ \frac{1}{2} R_e^{-1} \| \nabla u_h \|^2 + s \| j_h \|^2 \leq \frac{R_e}{2} \| f \|^{-1}. \]

**Proof.** The proof of the discrete energy law is almost the same as Problem 1. Therefore we only prove the magnetic Gauss’s law.

Taking \( C = \nabla \times E_h \) in (4.19), we have \( \nabla \times E_h = 0 \). Therefore
\[ (\nabla \cdot B_h, \nabla \cdot C_h) = 0, \quad \forall C_h \in H^0_0(\text{div}, \Omega). \]
This implies that \( \nabla \cdot B_h = 0 \).

To verify the well-posedness, we can formally eliminate \( E_h \) to get a system with \( u_h, p_h \) and \( B \). For the Lagrange multiplier \( p_h \), one can verify the inf-sup condition of the \( (\nabla \cdot u, q) \) pair. We can also verify the boundedness and coercivity in \( V_h^0 \times H^0_0(\text{curl}, \Omega) \times H^0_0(\text{div}, \Omega) \) for other terms. Consequently, we have the well-posedness result:

**Theorem 9.** Problem 3 has at least one solution \( (u_h, E_h, B_h, p_h) \in X_h \times Q_h \). With suitable data, the solution is unique.

We can similarly define Picard iterations: For \( n = 1, 2, \cdots \), given \( (u^{n-1}, B^{n-1}) \in W_h \), find \( (u^n, E^n, B^n) \in X_h \) and \( p^n \in Q_h \), such that for any \( (v, F, C) \in X_h \) and \( q \in Q_h \),
\[
L(u^{n-1}, u^n, v) + R_e^{-1}(\nabla u^n, \nabla v) - s(j^n_{n-1} \times B^{n-1}, v) - (p^n, \nabla \cdot v) = (f, v),
\]
\[
s(j^n_{n-1}, F) - \alpha(B^n, \nabla \times F) = 0,
\]
\[
\alpha(\nabla \times E^n, C) + \alpha(\nabla \cdot B^n, \nabla \cdot C) = 0,
\]
\[
-(\nabla \cdot u^n, q) = 0, \quad \tag{4.20}
\]
where \( j^n_{n-1} \) is given by Ohm’s law: \( j^n_{n-1} = E^n + u^n \times B^{n-1} \). One can similarly verify the following properties:

**Theorem 10.** Any solution to Problem 4.20 satisfies

1. **magnetic Gauss’s law in the strong sense:**
\[ \nabla \cdot B^n = 0, \quad n = 1, 2, \cdots, \]

2. **the discrete energy law:**
\[ R_e^{-1} \| \nabla u^n \|^2 + s \| j^n_{n-1} \|^2 = (f, u^n), \]
and
\[ \frac{1}{2} R_e^{-1} \| \nabla u^n \|^2 + s \| j^n_{n-1} \|^2 \leq R_e \| f \|^{-1}. \]

Analogous to Theorem 5, we can verify the well-posedness:

**Theorem 11.** Variational form (4.20) has a unique solution \( (u^n, E^n, B^n, p^n) \in X_h \times Q_h \).
5. Convergence of Picard iterations

**Theorem 12.** If both $R_2 \|f\|_{-1}$ and $R_c R_m \|f\|_{-1}$ are small enough, then the method (4.1) (Problem 1) with the boundary condition (1.2) has a unique solution, and the solution of the Picard iteration (Algorithm 1) converges to it with respect to the norms defined by (2.5) and (2.4).

We skip the proof of Theorem 12, since it is a simpler version of the proofs of the following Theorem 13.

6. Convergence of finite element methods

In this section, we present the error estimates of the method (4.1), which is for the boundary condition (1.2). Our analysis is based on the minimum regularity assumption on the exact solutions (c.f. [16]). Namely, we assume

\[
\mathbf{u} \in [H^{1+\sigma}(\Omega)]^3, \quad \mathbf{B}, \nabla \times \mathbf{E} \in [H^\sigma(\Omega)]^3, \quad p \in H^\sigma(\Omega) \cap L^2_0(\Omega),
\]

here $\sigma > \frac{1}{2}$. Next we introduce notations used in the analysis. For a generic unknown $\mathcal{U}$ and its numerical counterpart $\mathcal{U}_h$ we split the error as:

\[
\mathcal{U} - \mathcal{U}_h = (\mathcal{U} - \Pi \mathcal{U}) + (\Pi \mathcal{U} - \mathcal{U}_h) := \delta_U + e_U.
\]

Here $\Pi \mathcal{U}$ is a projection of $\mathcal{U}$ into the corresponding discrete space that $\mathcal{U}_h$ belongs to. Namely, for $(\mathbf{E}, r)$ we use the projections $(\Pi^{\text{curl}} \mathbf{E}, \Pi^0 r)$ in the commuting diagram in Figure 1. For $\mathbf{B}$ and $p$ we define the $L^2$ projection $\Pi^D \mathbf{B}, \Pi^0 p$ into $H^1_0(\text{div}0, \Omega), Q_h$ respectively. Notice here $r = 0$ implies that $\Pi^0 r = 0$ and hence $\delta_r = 0$. Finally, for the velocity $\mathbf{u}$ we define $(\Pi^V \mathbf{u}, \tilde{\mathbf{p}}_h) \in V_h \times Q_h$ be the unique numerical solution of the Stokes equation:

\[
(\nabla \Pi^V \mathbf{u}, \nabla \mathbf{v}) + (\tilde{\mathbf{p}}_h, \nabla \cdot \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}),
\]

\[
(\nabla \cdot \Pi^V \mathbf{u}, q) = 0,
\]

for all $(\mathbf{v}, q) \in V_h \times Q_h$. Notice that $(\mathbf{u}, 0)$ is the exact solution of the Stokes equations:

\[
-\Delta \mathbf{\tilde{u}} + \nabla \tilde{p} = -\Delta \mathbf{u},
\]

\[
\nabla \cdot \mathbf{\tilde{u}} = 0,
\]

with $\mathbf{\tilde{u}} = 0$ on $\partial \Omega$. Hence, if $V_h \times Q_h$ is a stable Stokes pair, we should have optimal approximation for the above equation:

\[
\|\mathbf{u} - \Pi^V \mathbf{u}\|_1 \leq C \inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_1.
\]

Immediately we can see that

\[
(\delta_\mathbf{u}, q) = 0 \quad \text{for all} \quad q \in Q_h.
\]

Since $\mathbf{B}, \Pi^D \mathbf{B}, \mathbf{B}_h \in H_0(\text{div}0, \Omega)$ and $\mathbf{E}, \mathbf{E}_h, \Pi^{\text{curl}} \mathbf{E} \in H_0(\text{curl}0, \Omega)$ we have

\[
\nabla \cdot \epsilon \mathbf{B} = \nabla \cdot \delta \mathbf{B} = 0, \quad \nabla \times \epsilon \mathbf{E} = \nabla \times \delta \mathbf{E} = 0.
\]

In addition, since $\nabla \times H^1_0(\text{curl}, \Omega) \subset H^1_0(\text{div}0, \Omega)$ we have

\[
(\delta \mathbf{B}, \nabla \times \mathbf{F}) = 0 \quad \text{for all} \quad \mathbf{F} \in H^1_0(\text{curl}, \Omega).
\]
Let $\Pi^{\text{div}}$ be the $H(\text{div})$-conforming projection in the commuting diagram in Figure 1. Obviously, $\Pi^{\text{div}} B \in H^h_0(\text{div}0, \Omega)$. Then, due to the construction of $\Pi^D$, we have

$$\|\Pi^D B - B\| = \inf_{C \in H^h_0(\text{div}0, \Omega)} \|B - C\| \leq \|\Pi^{\text{div}} B - B\| \leq C \inf_{C \in H^h_0(\text{div}, \Omega)} \|B - C\|. \quad (6.8)$$

Now we are ready to present the error equations for the error estimates. Notice that the exact solution $(u, E, B, r, p)$ also satisfies the discrete formulation (4.1). Subtracting two systems, with the splitting of the errors and above properties of the projections (6.5), (6.6) and (6.7), we arrive at:

$$(L(u; u, v) - L(u_h; u_h, v)) + R_\epsilon^{-1}(\nabla e_u, \nabla v) - s(j \times B - j_h \times B_h, v) - (e_p, \nabla v) = -R_\epsilon^{-1}(\nabla \delta u, \nabla v) + (\delta_p, \nabla \cdot v), \quad (6.9)$$

$$s(j - j_h, F) - \alpha(e_B, \nabla \times F) = 0, \quad (6.10)$$

$$\alpha(\nabla \times e_E, C) + (e_p, \nabla \cdot C) = - (\delta_r, \nabla \cdot C), \quad (6.11)$$

$$-(\nabla \cdot e_u, q) = 0, \quad (6.12)$$

$$(\nabla \cdot e_B, s) = 0, \quad (6.13)$$

for all $(v, F, C) \in X_h$ and $(q, s) \in Y_h$.

**Lemma 5.** We have the energy identity:

$$R_\epsilon^{-1} \|\nabla e_u\|^2 + \alpha \|\nabla_h \times e_B\|^2 = -(�(u; u, e_u) - L(u_h; u_h, e_u)) + (\delta_p, \nabla \cdot e_u) - R_\epsilon^{-1}(\nabla \delta u, \nabla e_u) + s(j \times B - j_h \times B_h, e_u) + s(j - j_h, \nabla \times e_B).$$

**Proof.** Taking $v = e_u, F = -\nabla_h \times e_B, q = e_p$ in (6.9), (6.10) and (6.12) and adding these equations, we can obtain the above identity by rearranging terms in the equation. \qed

From the above result we can see that it suffices to bound the terms on the right hand side of the energy identity to get the error estimates in the energy norm. The first four terms can be handled with standard tools for Navier-Stokes equations, see [9, 19] for instance. In particular, we need the following continuity result for the advection term, see [19]:

**Lemma 6.** For any $u, v, w \in [H^1_0(\Omega)]^3$, we have

$$L(w; u, v) \leq C \|\nabla w\| \|\nabla u\| \|\nabla v\|,$$

where $C$ solely depends on the domain $\Omega$.

In order to bound the last two terms, we need the following auxiliary results:

**Lemma 7.** If the regularity assumption (6.1) is satisfied, we have

$$\|u \times B - u_h \times B_h\| \leq C \left(\|u\|_{0, \infty} \|\delta_B\| + R_\epsilon f_{-1} \|\nabla_h \times e_B\| + R_\epsilon^2 R_m s^{-1} f_{-1} \|e_u\|_{1} + \|\delta_u\|_{1}\right),$$

$$\|e_E\| \leq \|\delta_E\| + \|u \times B - u_h \times B_h\|.$$

**Proof.** For $\|u \times B - u_h \times B_h\|$, we have

$$\|u \times B - u_h \times B_h\| = \|u \times \delta_B + u \times e_B + (\delta_u + e_u) \times B_h\|$$

$$\leq \|u \times \delta_B\| + \|u \times e_B\| + \|\delta_u + e_u\| \times B_h\|$$

$$\leq \|u\|_{0, \infty} \|\delta_B\| + \|u\|_{0, \infty} \|e_B\|_{0, 1} + (\|\delta_u\|_{0, 1} + \|e_u\|_{0, 1}) \|B_h\|_{0, 1}.$$
the last step is due to Hölder’s inequality. By (2.1) and Theorem 1, we have
\[ \| \mathbf{u} \times \mathbf{B} - \mathbf{u}_h \times \mathbf{B}_h \| \leq C(\| \mathbf{u} \|_{0, \infty} \| \delta \mathbf{B} \| + \| \mathbf{u}_1 \| \| \nabla \times \mathbf{e} \mathbf{B} \| + (\| \nabla \delta \mathbf{u} \| + \| \nabla \mathbf{e} \mathbf{u} \|) \| \nabla \times \mathbf{B}_h \|). \]

Finally we can obtain the estimate for this term by the stability result in Theorem 3 and Remark 1.

Next, taking \( \mathbf{F} = \mathbf{e} \mathbf{E} \) in (6.10), by (6.6), we have
\[ (j - j_h, \mathbf{e} \mathbf{E}) = 0. \]

By the definition of \( j, j_h \), we obtain:
\[ \| \mathbf{e} \mathbf{E} \|_2^2 = - (\delta \mathbf{E}, \mathbf{e} \mathbf{E}) - (\mathbf{u} \times \mathbf{B} - \mathbf{u}_h \times \mathbf{B}_h, \mathbf{e} \mathbf{E}). \]

The proof is completed by Cauchy-Schwarz inequality.

Now we are ready to give our first error estimate:

**Theorem 13.** If the regularity assumption (6.1) holds, in addition, both \( R^3_2 \| \mathbf{f} \|_{-1} \) and \( R_c R^3_m \| \mathbf{f} \|_{-1} \) are small enough, then we have
\[ R_c^{-\frac{3}{2}} \| \nabla \mathbf{e} \mathbf{u} \| + \alpha^{\frac{1}{2}} \| \mathbf{e} \mathbf{B} \|_{0.3} + \alpha^{\frac{1}{2}} \| \nabla \times \mathbf{e} \mathbf{B} \| \leq C(\| \delta_p \| + \| \nabla \delta \mathbf{u} \| + (\| \mathbf{u} \|_{1+\sigma} + \| \nabla \times \mathbf{B} \|_{\sigma}) \| \mathbf{e} \mathbf{B} \| + \| \delta \mathbf{E} \|), \]
where \( C \) depends on all the parameters \( R_m, R_c, s \) and \( \| \mathbf{f} \|_{-1} \).

**Proof.** Since \( \nabla \cdot \mathbf{e} \mathbf{B} = 0 \) by (6.6), we can apply Theorem 1 to obtain
\[ \| \mathbf{e} \mathbf{B} \|_{0.3} \leq C \| \nabla \times \mathbf{e} \mathbf{B} \|. \]

By Lemma 5, it suffices to bound terms on the right hand side in the energy identity. The two bilinear terms can be bounded by using Cauchy-Schwarz inequality as,
\[ (\delta_p, \nabla \cdot \mathbf{e} \mathbf{u}) \leq \| \delta_p \| \| \nabla \mathbf{e} \mathbf{u} \|, \]
\[ R_c^{-1}(\nabla \delta \mathbf{u}, \nabla \mathbf{e} \mathbf{u}) \leq R_c^{-1} \| \nabla \delta \mathbf{u} \| \| \nabla \mathbf{e} \mathbf{u} \|. \]

For the convection term, by Lemma 6 we have
\[ L(\mathbf{u}; \mathbf{u}, \mathbf{e} \mathbf{u}) - L(\mathbf{u}_h; \mathbf{u}_h, \mathbf{e} \mathbf{u}_h) = L(\mathbf{u} - \mathbf{u}_h; \mathbf{u}, \mathbf{e} \mathbf{u}) + L(\mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{e} \mathbf{u}_h) \]
\[ \leq C(\| \nabla \delta \mathbf{u} \| + \| \nabla \mathbf{e} \mathbf{u} \| + \| \nabla \mathbf{u} \| \| \nabla \mathbf{e} \mathbf{u} \| + C(\| \nabla \delta \mathbf{u} \| + \| \nabla \mathbf{e} \mathbf{u} \|) \| \nabla \mathbf{u}_h \| \| \nabla \mathbf{e} \mathbf{u} \| \]
\[ \leq CR^2_c \| \mathbf{f} \|_{-1}(R_c^{-1} \| \nabla \mathbf{e} \mathbf{u} \|^2 + R_c^{-1} \| \nabla \delta \mathbf{u} \| \cdot \| \nabla \mathbf{e} \mathbf{u} \|), \]
the last step is by the stability result (4.7) in Remark 1. In order to obtain the convergent result, we need \( R^2_c \| \mathbf{f} \|_{-1} \) to be small enough.
Next we need to bound the last two terms in Lemma 5. By Cauchy-Schwarz inequality we have
\[
s(j - j_h, \nabla \times eB) \leq s\|j - j_h\|\|\nabla h \times eB\|
\]
\[
= s\|E + u \times B - (E_h + u_h \times B_h)\|\|\nabla h \times eB\|
\]
\[
\leq s(\|\delta_E\| + \|e_E\| + \|u \times B - u_h \times B_h\|)\|\nabla h \times eB\|
\]
\[
\leq Cs(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + \|\nabla h \times eB\|
\]
\[
+ R_e^2 R_m s^{\frac{1}{2}} \|f\|_{-1}\|e_u\|_{1} + \|\delta_u\|_{1})\|\nabla h \times eB\|
\]
\[
= Cs(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + R_e^2 R_m s^{\frac{1}{2}} \|f\|_{-1}\|e_u\|_{1})\|\nabla h \times eB\|
\]
\[
+ C(R_e s\|f\|_{-1}\|\nabla h \times eB\|^2 + R_e^2 R_m s^{\frac{1}{2}} \|f\|_{-1}\|e_u\|_{1})\|\nabla h \times eB\|
\]
\[
\leq Cs(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + R_e^2 R_m s^{\frac{1}{2}} \|f\|_{-1}\|e_u\|_{1})\|\nabla h \times eB\|
\]
\[
+ C(R_e R_m\|f\|_{-1}(\|\nabla h \times eB\|^2) + R_e R_m^2 \|f\|_{-1}(R_e^{-1}\|\nabla e_u\|^2 + \alpha\|\nabla h \times eB\|^2)).
\]
In order to obtain the convergent result, we need \(R_e R_m^2\|f\|_{-1}\) to be small enough. Finally, for the last term we begin by splitting the term into three terms and applying the generalized Hölder’s inequality to have
\[
s(j \times B - j_h \times B_h, e_u) = s(j \times \delta B, e_u) + s(j \times e_B, e_u) + s((j - j_h) \times B_h, e_u)
\]
\[
= T_1 + T_2 + T_3.
\]
By the fact that \(j = R_m^{-1}\nabla \times B\), we can further apply the generalized Hölder’s inequalities, Sobolev embedding inequalities, \(H^s(\Omega) \hookrightarrow L^3(\Omega)\), (2.1) and Theorem 1 for \(T_1\), \(T_2\) and \(T_3\) as:
\[
T_1 \leq s\|j\|_{0,3}\|\delta_B\|\|e_u\|_{0,6} \leq Cs R_m^{-1}\|\nabla \times B\|\|\delta_B\|\|\nabla e_u\|,
\]
\[
T_2 \leq s\|j\|\|e_B\|_{0,3}\|e_u\|_{0,6} \leq Cs R_m^{-1}\|\nabla \times B\|\|\nabla h \times e_B\|\|\nabla e_u\| \leq CR_m^{\frac{1}{2}} s^{\frac{1}{2}} \|f\|_{-1}\|\nabla h \times e_B\|\|\nabla e_u\|
\]
\[
\leq CR_e R_m\|f\|_{-1}(R_e^{-1}\|\nabla e_u\|^2 + \alpha\|\nabla h \times e_B\|^2),
\]
\[
T_3 \leq s\|j - j_h\|\|B_h\|_{0,3}\|e_u\|_{0,6} \leq Cs\|j - j_h\|\|\nabla h \times B_h\|\|\nabla e_u\| \leq CR_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}\|j - j_h\|\|\nabla e_u\|
\]
\[
\leq CR_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + R_e\|f\|_{-1}\|\nabla h \times e_B\| + R_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}(\|e_u\|_{1} + \|\delta_u\|_{1}))\|\nabla e_u\|
\]
\[
\leq CR_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + R_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}\|\delta_u\|_{1})
\]
\[
+ CR_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}\|e_u\|_{1}\|\nabla h \times e_B\| + CR_e R_m\|f\|_{2,1}\|e_u\|_{1}^2
\]
\[
\leq CR_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}(\|\delta_E\| + \|u\|_{0,\infty}\|\delta_B\| + R_m^{\frac{1}{2}} R_m s^{\frac{1}{2}} \|f\|_{-1}\|\delta_u\|_{1})
\]
\[
+ R_e R_m\|f\|_{2,1}(R_e^{-1}\|\nabla e_u\|^2 + \alpha\|\nabla h \times e_B\|^2) + CR_m^2 R_m\|f\|_{2,1}(R_e^{-1}\|\nabla e_u\|^2).
\]
Referring to \(T_2\) and \(T_3\), we need \(R_e R_m^2\|f\|_{-1}\), \(R_e R_m\|f\|_{2,1}\) and \(R_m^2 R_m\|f\|_{2,1}\) to be small enough such that convergent results can be obtained.

So, if \(R_e^\frac{1}{2}\|f\|_{-1}\) and \(R_e R_m\|f\|_{2,1}\) are both small enough, we have
\[
R_e^\frac{1}{2}\|\nabla e_u\| + \alpha^\frac{1}{2}\|\nabla \times e_B\| \leq C(\|\delta_p\| + \|\nabla \delta_u\| + (\|u\|_{1+\sigma} + \|\nabla \times B\|_{\sigma})\|\delta_B\| + \|\delta_E\|).
\]
Here \(C\) depends on all the parameters \(R_m, R_e, s\) and \(f\|_{-1}\). This completes the proof. \(\square\)
7. Nonlinear scheme for the alternative boundary condition

We propose the following variational form for (1.1) with boundary condition (1.3):

**Problem 4.** Find \((u_h, E_h, B_h) \in \tilde{X}_h\) and \((p_h, r_h) \in Y_h\), such that for any \((v, F, C) \in \tilde{X}_h\) and \((q, s) \in Y_h\),

\[
L(u_h; u_h, v) + R_c^{-1}(\nabla u_h, \nabla v) - s(j_h \times B_h, v) - (p_h, \nabla \cdot v) = (f, v),
\]

\[
s(j_h, F) - \alpha(B_h, \nabla \times F) = 0,
\]

\[
\alpha(\nabla \times E_h, C) + (r_h, \nabla \cdot C) = 0,
\]

\[
- (\nabla \cdot u_h, q) = 0,
\]

\[
(\nabla \cdot B_h, s) = 0,
\]

where \(j_h\) is given by Ohm’s law: \(j_h = E_h + u_h \times B_h\) and \(r_h\) is the Lagrange multiplier which approximates \(r = 0\), and \(\tilde{X}_h = V_h \times H^1(\text{curl}, \Omega) \times H^1(\text{div}, \Omega)\).

Similar to Theorem 3, we have Theorem 14, whose proof is the same as that of Theorem 3.

**Theorem 14.** Any solution for Problem 4 satisfies

1. magnetic Gauss’s law:

\[
\nabla \cdot B_h = 0.
\]

2. Lagrange multiplier \(r = 0\), and the strong form

\[
\nabla \times E_h = 0,
\]

3. energy estimates:

\[
R_c^{-1}\|\nabla u_h\|^2 + s\|j_h\|^2 = (f, u_h),
\]

\[
\frac{1}{2}R_c^{-1}\|\nabla u_h\|^2 + s\|j_h\|^2 \leq \frac{R_c}{2}\|f\|_{-1}^2,
\]

\[
R_m^{-1}\|\nabla j_h \times B_h\| \leq \|j_h\|,
\]

\[
\|\nabla j_h \times B_h\| \leq CR_m^2 R_m s^{-\frac{1}{2}}\|f\|_{-1},
\]

\[
\|E_h\| \leq CR_m^2 R_m s^{-\frac{1}{2}}\|f\|_{-1}.
\]

Similar to the argument in Section 4.1, we can conclude that Problem 4 is well-posed.

We define \(e_u\), \(\delta_u\), \(e_p\), \(\delta_p\), \(e_r\), \(\delta_r\) the same as those in Section 6. We use \(\Pi_{\text{curl}}\) in Figure 2 for the electric field \(E\). We define \(e_E = \Pi_{\text{curl}} E - E_h\) and \(\delta_E = E - \Pi_{\text{curl}} E\). For the magnetic field \(B\), we define the \(L^2\)-projection \(\Pi^\delta\) into \(H^1(\text{div}, \Omega)\). We denote \(e_B = \Pi^\delta B - B_h\) and \(\delta_B = B - \Pi^\delta B\). It is easy to see that

\[
\nabla \cdot e_B = 0,
\]

\[
(B - \Pi^\delta B, \nabla \times F) = 0 \quad \text{for all} \ F \in H^1(\text{curl}, \Omega),
\]

\[
\|B - \Pi^\delta B\| \leq C \inf_{C \in H^1(\text{div}, \Omega)} \|B - C\|.
\]

Thus by using Theorem 2 to replace Theorem 1, we can use the same argument in Section 6 to obtain Theorem 15.
Theorem 15. If the regularity assumption (6.1) holds, in addition, both $R_m^2\|f\|_{-1}$ and $ReR_m^2\|f\|_{-1}$ are small enough, then we have
\[
Re^{\frac{1}{2}}\|\nabla e_u\| + \alpha^2\|e_B\|_{0,3} + \alpha^2\|\nabla_h \times e_B\| \leq C(\|\delta_p\| + \|\nabla \delta_u\| + (\|u\|_{1+\sigma} + \|\nabla \times B\|_\sigma)\|\delta_B\| + \|\delta_E\|),
\]
where $C$ depends on all the parameters $R_m, Re, s$ and $\|f\|_{-1}$.

8. Conclusion

We proposed a mixed finite element scheme for the stationary MHD system where both the electric and the magnetic fields were discretized on a discrete de Rham complex. Two types of boundary conditions were considered. Thanks to the structure-preserving properties, we rigorously established the energy law, the well-posedness and the magnetic Gauss’s law $\nabla \cdot B_h = 0$ on the discrete level. The convergence of the finite element schemes was proved based on minimal regularity assumptions.

We expect that the electric-magnetic mixed formulation (also see [11, 12]) and the technical tools developed in this paper could also shed some light on a broader class of plasma models and numerical methods, for example, compressible MHD models and discontinuous Galerkin methods (c.f. [15, 13, 18, 17]).

The theoretical analysis in this paper also lays a foundation for further investigation of block preconditioners for stationary MHD systems (c.f. [14, 7]).

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