Fuzzy hyperspheres via confining potentials and energy cutoffs

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Abstract
We simplify and complete the construction of fully $O(D)$-equivariant fuzzy spheres $S^d_\Lambda$, for all dimensions $d \equiv D-1$, initiated in Fiore and Pisacane (2018 J. Geom. Phys. 132 423–51). This is based on imposing a suitable energy cutoff on a quantum particle in $\mathbb{R}^D$ subject to a confining potential well $V(r)$ with a very sharp minimum on the sphere of radius $r = 1$; the cutoff and the depth of the well diverge with $\Lambda \to \infty$. As a result, the noncommutative Cartesian coordinates $\pi^i$ generate the whole algebra of observables $A_\Lambda$ on the Hilbert space $\mathcal{H}_\Lambda$; applying polynomials in the $\pi^i$ to any $\psi \in \mathcal{H}_\Lambda$ we recover the whole $\mathcal{H}_\Lambda$. The commutators of the $\pi^i$ are proportional to the angular momentum components, as in Snyder noncommutative spaces. $\mathcal{H}_\Lambda$, as carrier space of a reducible representation of $O(D)$, is isomorphic to the space of harmonic homogeneous polynomials of degree $\Lambda$ in the Cartesian coordinates of (commutative) $\mathbb{R}^{D+1}$, which carries an irreducible representation $\pi_\Lambda$ of $O(D+1) \supset O(D)$. Moreover, $A_\Lambda$ is isomorphic to $\pi_\Lambda(Uso(D+1))$. We resp. interpret $\{\mathcal{H}_\Lambda\}_{\Lambda \in \mathbb{N}}$, $\{A_\Lambda\}_{\Lambda \in \mathbb{N}}$ as fuzzy deformations of the space $\mathcal{H}_\Lambda := L^2(S^d)$ of (square integrable) functions on $S^d$ and of the associated algebra $A_\Lambda$ of observables, because they resp. go to $\mathcal{H}_\Lambda, A_\Lambda$ as $\Lambda$ diverges (with $\hbar$ fixed). With suitable $\hbar = h(\Lambda) \to 0$, in the same limit $A_\Lambda$ goes to the (algebra of functions on the) Poisson manifold $T^*S^d$; more formally, $\{A_\Lambda\}_{\Lambda \in \mathbb{N}}$ yields a fuzzy quantization of a coadjoint orbit of $O(D+1)$ that goes to the classical phase space $T^*S^d$.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Noncommutative space(time) algebras are introduced and studied with various motivations, notably to provide an arena for regularizing ultraviolet divergences in quantum field theory (see e.g. [1]), reconciling quantum mechanics and general relativity in a satisfactory quantum gravity (QG) theory (see e.g. [2]), unifying fundamental interactions (see e.g. [3, 4]). Noncommutative geometry [5–8] has become a sophisticated framework that develops the whole machinery of differential geometry on noncommutative spaces. Fuzzy spaces are particularly appealing noncommutative spaces: a fuzzy space is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite-dimensional algebras such that $A_n \xrightarrow{n \to \infty} A \equiv$ algebra of regular functions on an ordinary manifold, with $\dim(A_n) \xrightarrow{n \to \infty} \infty$. They have raised a big interest in the high energy physics community as a non-perturbative technique in QFT based on a finite discretization of space(time) alternative to the lattice one: the main advantage is that the algebras $A_n$ can carry representations of Lie groups (not only of discrete ones). They can be used also for internal (e.g. gauge) degrees of freedom (see e.g. [9]), or as a new tool in string and D-brane theories (see e.g. [10, 11]). The first and seminal fuzzy space is the two-dimensional fuzzy sphere (FS) of Madore and Hoppe [12, 13], where $A_n \simeq M_n(\mathbb{C})$, which is generated by coordinates $x^i$ ($i = 1, 2, 3$) fulfilling

$$[x^i, x^j] = \frac{2i}{\sqrt{n^2 - 1}} \varepsilon^{ijk} x^k, \quad r^2 := x^i x^i = 1, \quad n \in \mathbb{N} \setminus \{1\} \quad (1)$$

(sum over repeated indices is understood); they are obtained by the rescaling $x^i = 2L_i/\sqrt{n^2 - 1}$ of the elements $L_i$ of the standard basis of $so(3)$ in the unitary irreducible representation (irrep) $(\pi^l, V^l)$ of dimension $n = 2l + 1$, i.e. where $V^l$ is the eigenspace of the Casimir $L^2 = L_i L^i$ with eigenvalue $l(l + 1)$. Reference [14, 15] first proposed a QFT based on it. Each matrix in $M_n$ can be expressed as a polynomial in the $x^i$ that can be rearranged as the expansion in spherical harmonics of an element of $C(S^2)$ truncated at level $n$. Unfortunately, such a nice feature is not shared by the FSs of dimension $d > 2$: the product of two spherical harmonics is not a combination of spherical harmonics, but an element in a larger algebra $A_n$. FSs of dimension $d = 4$ and any $d \geq 3$ were first introduced respectively in [16, 17]; other versions in $d = 3, 4$ or $d \geq 3$ have been proposed in [18–21].

The Hilbert space of a (zero-spin) quantum particle on configuration space $S^d$ and the space of continuous functions on $S^d$ carry the (same) irreducible representation of $O(D)$, with $D := d + 1$; this decomposes into irreducible representations (irreps) as follows

$$L^2(S^d) \simeq \bigoplus_{l=0}^{\infty} V^l_0 \simeq C(S^d), \quad (2)$$

where the carrier space $V^l_0$ is an eigenspace of the quadratic Casimir $L^2$ with eigenvalue

$$E_l := l(l + D - 2) \quad (3)$$

($V^l_0 \equiv V^l$). $C(S^d)$ can be seen as an algebra of bounded operators on $L^2(S^d)$. On the contrary, the mentioned fuzzy hyperspheres (including the Madore–Hoppe FS) are either based on sequences of irreps of $Spin(D)$ (so that $r^2$, which is proportional to $L^2$, is identically 1) parametrized by $n$ [12, 13, 16–19], or on sequences of reducible representations that are the
direct sums of small bunches of such irreps [20, 21]. In either case, even excluding the $n$ for
which the associated representation of $O(D)$ is only projective, the carrier space does not go
to (2) in the limit $n \to \infty$; we think this makes the interpretation of these FSS as fuzzy configuration
spaces $S^d$ (and of the $x'$ as spatial coordinates) questionable. For the Madore–Hoppe FSS
such an interpretation is even more difficult, because relations (1) are equivariant under $SO(3)$,
but not under the whole $O(3)$, e.g. not under parity $x' \mapsto -x'$, while the ordinary sphere $S^2$ is;
on the contrary, all the other mentioned FSS are $O(D)$-equivariant, because the commutators
$[x',x']$ are Snyder-like [1], i.e. proportional to angular momentum components $L_{ij}$.

The purpose of this work is to complete the construction [22, 23] of new, fully $O(D)$-
equivariant fuzzy quantizations of spheres $S^d$ of arbitrary dimension $d = D - 1 \in \mathbb{N}$ (thought
as configuration spaces) and of $T^*S^d$ (thought as phase spaces), in a sense that will be fully
clarified at the end of section 7; in the commutative limit the involved $O(D)$-representation
goes to (2). We also simplify and uniformize (with respect to $D$) the procedure of [22, 23].

We recall this procedure starting from the general underlying philosophy [22, 24, 25]. Consider
a quantum theory $\mathcal{T}$ with Hilbert space $\mathcal{H}$, algebra of observables on $\mathcal{H}$ (or with a domain
dense in $\mathcal{H}$) $\mathcal{A} \equiv \text{Lin}(\mathcal{H})$, Hamiltonian $H \in \mathcal{A}$. For any subspace $\mathcal{H}_1 \subset \mathcal{H}$
preserved by the action of $H$, let $\mathcal{P}_1 : \mathcal{H} \mapsto \mathcal{H}_1$ be the associated projector

$$\mathcal{A} \equiv \text{Lin}(\mathcal{H}_1) = \{ \mathcal{A} \equiv \mathcal{P}_1 \mathcal{A} \mathcal{P}_1 \mid A \in \mathcal{A} \};$$

the observable $\mathcal{A} \equiv \mathcal{P}_1 \mathcal{A} \mathcal{P}_1 \in \mathcal{A}_1$ will have the same physical interpretation as $A$. By construction
$\mathcal{H} = \mathcal{P}_1 \mathcal{H} = \mathcal{H}_1 \mathcal{P}_1$. The projected Hilbert space $\mathcal{H}_1$, algebra of observables $\mathcal{A}_1$ and Hamiltonian $H$
provide a new quantum theory $\mathcal{T}_1$. If $\mathcal{H}_1, H$ are invariant under some group $G$, then $\mathcal{P}_1, \mathcal{A}_1, H, \mathcal{T}_1$
will be as well. In general, the relations among the generators of $\mathcal{A}_1$ differ from those among
the generators of $\mathcal{A}$. In particular, if the theory $\mathcal{T}_1$ is based on commuting coordinates $x'$ (commative space) this will be in general no longer true for $\mathcal{T}$: $[x',x'] \neq 0$, and we have generated
a quantum theory on a NC space.

A physically relevant instance of the above projection mechanism occurs when $\mathcal{H}_1$ is the
subspace of $\mathcal{H}$ characterized by energies $E$ below a certain cutoff, $E < E$; then $\mathcal{T}_1$ is a low-energy
effective approximation of $\mathcal{T}$. What can it be useful for? If $\mathcal{A}_1$ contains all the observables corresponding to measurements that we can really perform with the experimental apparatus at our disposal, and the initial state of the system belongs to $\mathcal{H}_1$, then neither the dynamical evolution ruled by $H$, nor any measurement can map it out of $\mathcal{H}_1$, and we can replace $\mathcal{T}$ by the effective theory $\mathcal{T}_1$. Moreover, if at $E > E$ we even expect new physics not accountable by $\mathcal{T}$, then $\mathcal{T}_1$ may also help to figure out a new theory $\mathcal{T}'$ valid for all $E$.

For an ordinary (for simplicity, zero-spin) quantum particle in the Euclidean (configuration)
space $\mathbb{R}^D$ it is $\mathcal{H} = L^2(\mathbb{R}^d)$. Fixed a Hamiltonian $H(x,p)$, by standard wisdom the dimension of $\mathcal{H}_1$ is

$$\dim(\mathcal{H}_1) \approx \text{Vol}(\mathcal{B}_E)/\hbar^D,$$

where $\hbar$ is the Planck constant and $\mathcal{B}_E \equiv \{(x,p) \in \mathbb{R}^{2D} \mid H(x,p) \leq E\}$ is the classical phase space region with energy below $E$. If $H$ consists only of the kinetic energy $\Delta$, then this is infinite (figure 1 left). If $H = \Delta + V$, with a confining potential $V$, then this is finite (figure 1 right) at least for sufficiently small $\hbar$, and also the classical region $\mathcal{B}_E \subset \mathbb{R}^D$ in configuration space determined by the condition $V \leq E$ is bounded. In the sequel we rescale $x, p, H, V$ so that they are dimensionless and, denoting by $\Delta$ the Laplacian in $\mathbb{R}^D$, we can write

$$H = -\Delta + V.$$
Figure 1. Classical phase space regions with energy $E \leq \overline{E}$.

Figure 2. Three-dimensional plot of $V(r)$ in dimension $D = 2$.

The ‘dimensional reduction’ $\mathbb{R}^D \rightarrow S^d$ of the configuration space is obtained:

1. Assuming in addition that $V = V(r)$ depends only on the distance $r$ from the center of the sphere $S^d \subset \mathbb{R}^D$ and has a very sharp minimum, parametrized by a very large $k \equiv V''(1)/4$, on the sphere $S^d$ of equation $r = 1$, see figure 2.
2. Choosing $\overline{E}$ so low that all radial quantum excitations are ‘frozen’, i.e. excluded from $\overline{\mathcal{H}} \subset \mathcal{H}$; this makes $\mathcal{H}$ coincide with the Laplacian $L^2$ on $S^d$, up to terms $O(1/\sqrt{k})$.
3. Making both $k, \overline{E}$ depend on, grow and diverge with a natural number $\Lambda$. Thereby we rename $\overline{\mathcal{H}}, \overline{P}, \overline{A}$ as $\mathcal{H}_\Lambda, P_\Lambda, A_\Lambda$.

As $H$ is $O(D)$-invariant, so are $P_\Lambda, \mathcal{H} = P_\Lambda H$, and the projected theory is $O(D)$-equivariant. Technical details are given in sections 2 and 3. Section 2 fixes the notation and contains preliminaries partly developed in [23]. The representation-theoretical results of section 3, which deserve attention also on their own, allow to explicitly characterize the space $V^0_D$ as the space of harmonic homogeneous polynomials of degree $l$ in the Cartesian coordinates $x^i$ of $\mathbb{R}^D$ restricted to the sphere $S^d$; we determine such polynomials constructing the trace-free completely symmetric projector of $(\mathbb{R}^D)^{*}$ and applying it to the homogeneous polynomials of degree $l$ in $x^i$. The actions of the $L_{\Lambda i}$ and of the multiplication operators $x^i$ on such polynomials can be expressed by general formulae valid for all $D, l$; this allows to avoid the rather complicated
actions of the $L_{ab}$ on spherical harmonics (which also span $V^D_0$) used in [23]. It turns out that both $\mathcal{H}_A, V^A_{D+1}$ decompose into irreps of $O(D)$ as follows:

$$\mathcal{H}_A \simeq V^A_{D+1} = \bigoplus_{i=0}^{\Lambda} V^i_D.$$  

The second equality shows that, in contrast with the mentioned fuzzy hyperspheres, we recover (2) in the limit $\Lambda \to \infty$. The first equality suggests that the unitary irrep of the $\ast$-algebra $A_\Lambda$ on $\mathcal{H}_A$ is isomorphic to the irrep $\pi_\Lambda$ of $Uso(D+1)$ on $V^A_{D+1}$, what we in fact prove in section 5 (this had been proved for $D = 2, 3$ and conjectured for $D > 3$ in [22, 23]). The relations fulfilled by $\vec{r}, \vec{T}_{ab}$ are determined in section 4: the commutators $[\vec{x}, \vec{r}]$ are also Snyder-like [1], i.e. are proportional to $\vec{T}_{ab}/k$, with a proportionality factor that is the same constant on all of $\mathcal{H}_A$, except on the $l = \Lambda$ component of the latter, where it is a slightly different constant. $\vec{r}$ generate the whole $A_\Lambda$. The square distance $\vec{r}^2 \equiv \vec{x} \cdot \vec{x}$ is a function of $\vec{L}^2$ only, such that almost all its spectrum is very close to 1 and goes to 1 in the limit $\Lambda \to \infty$. In section 6 we show in which sense $\mathcal{H}_A, A_\Lambda$ go to $\mathcal{H}, A$ as $\Lambda \to \infty$, in particular how one can recover $f \in C(S^D) \subset A$, the multiplication operator of wavefunctions in $L^2(S^D)$ by a continuous function $f$, as the strong limit of a suitable sequence $f_\Lambda \in A_\Lambda$ (again, this had been only conjectured in [23]). In section 7 we discuss our results and possible developments in comparison with the literature; in particular, we point out that our pair $(\mathcal{H}_A, A_\Lambda)$ can be seen as a fuzzy quantization of a coadjoint orbit of $O(D)$ that can be identified with the cotangent space $T^*S^D$, the classical phase space over the $d$-dimensional sphere. Finally, we have concentrated most proofs in the appendix.

2. General setting

We choose a set of real Cartesian coordinates $x := (x^1, \ldots, x^D)$ of $\mathbb{R}^D$ and abbreviate $\partial_i \equiv \partial / \partial x^i$. We normalize $x^i, \partial_i$ and $H$ itself so as to be dimensionless. Then we can express $r^2 := x^2 \equiv x_i x^i, \Delta := \partial / \partial^2$ (sum over repeated indices understood), where actually $x_i = x^i$ and $\partial^i = \partial_i$ because the coordinates are real and Cartesian. The self-adjoint operators $x_i, -i \partial_i$ on $L^2(\mathbb{R}^D)$ fulfill the canonical commutation relations

$$[x^i, x^j] = 0, \quad [-i \partial_i, -i \partial_j] = 0, \quad [x^i, -i \partial_j] = i \delta^i_j,$$  

which are equivalent under all orthogonal transformations $Q$ (including parity $Q = -I$)

$$x^i \mapsto x'^i = Q x^i, \quad Q^{-1} = Q^T.$$  

All scalars $S$, in particular $S = \Delta, r^2, V, H$, are invariant. This implies $[S, L_{ij}] = 0$, where

$$L_{ij} := i(x^j \partial_i - x^i \partial_j)$$  

are the angular momentum components associated to $x$. These generate rotations of $\mathbb{R}^D$, i.e.

$$[iL_{ij}, \theta^k] = \theta^j \delta_i^k - \theta^i \delta_j^k$$  

hold for the components $\theta^k$ of all vector operators, in particular $\theta^j = \partial_j$, and close the commutation relations of $so(D)$,

$$[L_{ij}, iL_{hk}] = i(L_{ik} \delta_{jh} - L_{ik} \delta_{jh}) = i(L_{ik} \delta_{jh} - L_{ih} \delta_{kj} + L_{jk} \delta_{hi}).$$  

The $D$ derivatives $\partial_i$ make up a globally defined basis for the linear space of smooth vector fields on $\mathbb{R}^D$. As the $L_{ij}$ are vector fields tangent to all spheres $r = \text{const}$, the set $B = \{\partial_i, L_{ij} | i <$
This redundancy (unavoidable if $S^d$ is not parallelizable) will be no problem for our purposes. We shall assume that $V(r)$ has a very sharp minimum at $r = 1$ with very large $k = V''(1)/4 > 0$, and fix $V_0 := V(1)$ so that the ground state $\psi_0$ has zero energy, i.e. $E_0 = 0$ (see figure 3). We choose an energy cutoff $\tilde{E}$ fulfilling first of all the condition

$$V(r) \simeq V_0 + 2k(r - 1)^2 \quad \text{if } r \text{ fulfills } V(r) \leq \tilde{E},$$

(13)

so that we can neglect terms of order higher than two in the Taylor expansion of $V(r)$ around 1 and approximate the potential as a harmonic one in the classical region $v_\mathcal{T} \subset \mathbb{R}^D$ determined by the condition $V(r) \leq \tilde{E}$. By (13), $v_\mathcal{T}$ is approximately the spherical shell $|r - 1| \leq \sqrt{\frac{\tilde{E} - V_0}{2k}}$, when both $\tilde{E} - V_0$ and $k$ diverge, while their ratio goes to zero, then $v_\mathcal{T}$ reduces to the unit sphere $S^d$. We expect that in this limit the dimension of $\mathcal{H}_\mathcal{T}$ diverges, and we recover standard quantum mechanics on $S^d$. As we shall see, this is the case.

Of course, the eigenfunctions of $H$ can be more easily determined in terms of polar coordinates $r, \theta_1, \ldots, \theta_d$, recalling that the Laplacian in $D$ dimensions decomposes as follows

$$\Delta = \partial_r^2 + \left(D - 1\right) \frac{1}{r} \partial_r - \frac{1}{r^2} L^2,$$

(14)

(see section 3.1) where $L^2 := L_\theta^2$ is the square angular momentum (in normalized units), i.e. the quadratic Casimir of $U_{so}(D)$ and the Laplacian on the sphere $S^d$. $L^2$ can be expressed in terms of angles $\theta_\phi$ and derivatives $\partial/\partial \theta_\phi$ only. The eigenvalues of $L^2$ are $l(l + D - 2)$, see section 3.1; we denote by $V^l_\mathcal{T}$ the $L^2 = E_l$ eigenspace within $L^2(S^d)$. Replacing the Ansatz
\[ \psi = f(r) Y_l(\theta), \] with \( f(r) = r^{-d/2} g(r), \) \( Y_l \in V_D^l, \theta \equiv (\theta_1, \ldots, \theta_d), \) transforms the Schrödinger PDE \( H \psi = E \psi \) into the Fuchsian ODE in the unknown \( g(r) \):

\[
-g''''(r) + \left[ \frac{D^2 - 4D + 3 + 4l(l + D - 2)}{4r^2} \right] g(r) = E g(r). \tag{15}
\]

The requirement \( \psi \in L^2(\mathbb{R}^D) \) implies that \( g \) belongs to \( L^2(\mathbb{R}^+) \), in particular goes to zero as \( r \to \infty \). The self-adjointness of \( H \) implies that it must be \( f(0) = 0 \); this is compatible [23] with Fuchs theorem provided \( r^2 V(r) \eta \to 0^+, T \in \mathbb{R}^+ \) [what is in turn compatible with (13)]. Since \( V(r) \) is very large outside the thin spherical shell \( v_T \) (a neighbourhood of \( S^d \)), \( g, f, \psi \) become negligibly small there, and, by condition (13), the lowest eigenvalues \( E \) are at leading order those of the one-dimensional harmonic oscillator approximation [23] of (15)

\[
-g''''(r) + g(r) k_i (r - \bar{r}_i)^2 = \bar{E}_i g(r),
\]

which is obtained neglecting terms \( O \left( (r - 1)^3 \right) \) in the Taylor expansions of \( 1/r^2, V(r) \) about \( r = 1 \). Here

\[
\begin{align*}
\bar{r}_i &:= 1 + \frac{b(l, D)}{3b(l, D) + 2k}, & \bar{E}_i &:= E - V_0 - \frac{2b(l, D)[k + b(l, D)]}{3b(l, D) + 2k}, \\
k_i &:= 2k + 3b(l, D), & b(l, D) &:= \frac{D^2 - 4D + 3 + 4l(l + D - 2)}{4}. \tag{17}
\end{align*}
\]

The (Hermite functions) square-integrable solutions of (16)

\[
g_{n,l}(r) = M_{n,l} \cdot e^{-\frac{V_0}{2}} (r - \bar{r}_i)^\frac{i}{k_i} \cdot H_n \left( \frac{r - \bar{r}_i}{k_i} \right) \quad \text{with } n \in \mathbb{N}_0
\]

(here \( M_{n,l} \) are normalization constants and \( H_n \) are the Hermite polynomials) lead to

\[
f_{n,l}(r) = \frac{M_{n,l}}{r^{\frac{d}{2}}} \cdot e^{-\frac{V_0}{2}} (r - \bar{r}_i)^\frac{i}{k_i} \cdot H_n \left( \frac{r - \bar{r}_i}{k_i} \right) \quad \text{with } n \in \mathbb{N}_0. \tag{18}
\]

The corresponding ‘eigenvalues’ in (16) \( \bar{E}_{n,l} = (2n + 1) \sqrt{k_i} \) lead to energies

\[
E_{n,l} = (2n + 1) \sqrt{k_i} + V_0 + \frac{2b(l, D)[k + b(l, D)]}{3b(l, D) + 2k}.
\]

As said, we fix \( V_0 \) requiring that the lowest one \( E_{0,0} \) be zero; this implies

\[
V_0 = -\sqrt{k_0} - \frac{2b(0, D)[k + b(0, D)]}{3b(0, D) + 2k} = -\sqrt{2k} - b(0, D) - \frac{3b(0, D)}{2\sqrt{2k}} + O \left( k^{-\frac{3}{2}} \right),
\]

and the expansions of \( E_{n,l} \) and \( \bar{r}_i \) at leading order in \( k \) become

\[
E_{n,l} = l(l + D - 2) + 2n \sqrt{2k} + O \left( k^{-\frac{1}{2}} \right), \quad \bar{r}_i = 1 + \frac{b(l, D)}{2k} + O \left( k^{-2} \right) \tag{19}
\]

where \( E_{0,l} \) coincide at lowest order with the desired eigenvalues \( E_l \) of the Laplacian \( L^2 \) on \( S^d \), while if \( n > 0 \) \( E_{n,l} \) diverge as \( k \to \infty \); to exclude all states with \( n > 0 \) (i.e. to ‘freeze’ radial oscillations; then all corresponding classical trajectories are circles) we impose the cutoff

\[
E_{n,l} \leq \bar{E}(\Lambda) := \Lambda(\Lambda + D - 2) < 2\sqrt{2k}, \quad \Lambda \in \mathbb{N}. \tag{20}
\]

\[1\] The present treatment of the equations in the case \( D = 2 \) is slightly different from the one adopted in reference [22], where the independent radial variable had been changed according to \( r \to \rho \equiv \ln r \); however this gives the same wavefunctions at lowest order in \( \rho \approx r - 1 \).
The right inequality is satisfied prescribing a suitable dependence \( k(\Lambda) \), e.g. \( k(\Lambda) = [\Lambda(\Lambda + D - 2)]^2 \); the left one is satisfied setting \( n = 0 \) and \( l \leq \Lambda \). Abbreviating \( f_l \equiv f_{0,l} \), we end up with eigenfunctions and associated energies (at leading order in \( 1/\Lambda \))

\[
\psi_l(r, \theta) = f_l(r) Y_l(\theta), \quad H\psi_l = E_l\psi_l, \quad l = 0, 1, \ldots, \Lambda. \tag{21}
\]

Thus \( \mathcal{H}_\Lambda \) decomposes into irreps of \( O(D) \) (and eigenspaces of \( L^2, H \)) as follows

\[
\mathcal{H}_\Lambda = \bigoplus_{l=0}^{\Lambda} \mathcal{H}^l_\Lambda, \quad \mathcal{H}^l_\Lambda := f_l(r) V^l_D.
\tag{22}
\]

We can express the projectors \( P^l_\Lambda : \mathcal{H}_\Lambda \to \mathcal{H}^l_\Lambda \) as the following polynomials in \( L^2 \):

\[
P^l_\Lambda = \prod_{n=0, n \neq l}^{\Lambda} \frac{\mathcal{L}^2 - E_n}{E_l - E_n}, \tag{23}
\]

In the commutative limit \( \Lambda \to \infty \) the spectrum \( \{E_{0,l}\}_{l=0}^{\Lambda} \) of \( \mathcal{H} \) goes to the whole spectrum \( \{E_l\}_{l \in \mathbb{N}_0} \) of \( L^2 \). If \( \phi, \phi' \in \mathcal{H} \equiv \mathcal{L}^2(\mathbb{R}^D) \) can be factorized into radial parts \( f(r), f'(r) \) and angular parts \( T, T' \in \mathcal{L}^2(S^D) \), i.e. \( \phi = fT, \phi' = f'T' \), then so can be their scalar product:

\[
\langle \phi, \phi' \rangle := \int_{\mathbb{R}^D} \, d^Dx \phi^*(x) \phi'(x) = \langle T, T' \rangle \int_0^{\infty} \, dr r^D f^*(r) f'(r).
\tag{24}
\]

Here we have denoted by \( \langle \cdot, \cdot \rangle \) the scalar product of \( \mathcal{L}^2(S^D) \),

\[
\langle T, T' \rangle := \int_{S^D} \, d\alpha \, T^* T',
\tag{25}
\]

where \( d\alpha \) is the \( O(D) \)-invariant measure on \( S^D \).

Assume that \( E := \{Y^m_l\}_{l(m) \in L} \) is an orthonormal basis of \( \mathcal{L}^2(S^D) \) consisting of eigenvectors of \( \mathcal{L}^2 \) (e.g. spherical harmonics),

\[
L^2 Y^m_l = E_l Y^m_l, \quad \langle Y^m_{l'}, Y^m_l \rangle = \delta_{mm'} \delta_{ll'};
\tag{26}
\]

here \( m \in I \) is a (multi-)index\(^3\) labelling the elements of an orthonormal basis \( \mathcal{B}_l \equiv \{Y^m_l\}_{m \in k} \) of \( V^l_D \), and \( I := \{(l, m) \mid l \in \mathbb{N}_0, m \in I_l \} \). Then, by appropriate choices of the normalization constants\(^4\) of (18), one obtains as orthonormal bases respectively of \( \mathcal{H} \) and \( \mathcal{H}^l_\Lambda \)

\[
\mathcal{B}' := \{ \psi_{n,l} := f_{n,l}(r) Y^m_l, \mid n \in \mathbb{N}_0, l \in I \}, \quad \mathcal{B}'^l := \{ \psi^m_{0,l} = f_{0,l}(r) Y^m_l \}_{m \in k}.
\]

The projector \( \bar{P}^l_\Lambda : \mathcal{H} \to \mathcal{H}^l_\Lambda \) acts by

\[
\bar{P}^l_\Lambda \phi (x) = \sum_{m \in k} \psi^m_{0,l}(x) \langle \psi^m_{0,l}, \phi \rangle \sum_{m \in k} \psi^m_{0,l}(x) \int_{\mathbb{R}^D} \, d^Dx' \psi^m_{0,l}(x') \phi(x').
\]

\(^2\) In terms of the angles \( \theta, d\alpha = [\sin^{D-1}(\theta_r) \sin^{D-2}(\theta_{r,l}) \cdots \sin(\theta_{l-1})] \, \, d\theta_0 \, \, d\theta_2 \cdots \, d\theta_{l-1} \).

\(^3\) If \( D = 2, 3 \), then \( k \subset \mathbb{Z} \); more precisely \( k_0 = \{0\} \), while for \( l > 0 \) it is \( k_l = \{+, -\} \) if \( D = 2, k_l = \{-l, l-1, \ldots, l\} \) if \( D > 3 \).

\(^4\) Choosing them positive, one easily finds \( \mathcal{M}_l = \sqrt{\mathcal{L}} \mathcal{P}_l \). In fact, by (26), (24) normalizing \( \psi^m_{0,l} \) amounts to \( 1 = \int_{\mathbb{R}^D} \, dr r^D f^*(r) f'(r) \approx L^2 \int_0^{\infty} \, dr \, r^D \mathcal{P}_l \int_{\mathbb{R}^D} \, dr \, e^{-\sqrt{\mathcal{L}}(r-\gamma)^2}; \) here \( \gamma \approx \) (due to the shift \( \gamma \to -\infty \) of the left integration extreme) means equality up to terms of the order \( 1/\sqrt{\mathcal{L}} e^{-\sqrt{\mathcal{L}}} \), which has zero asymptotic expansion in \( 1/\sqrt{\mathcal{L}} \), see [22].
If \( \phi \) has the form \( \phi(r, \theta) = \Theta_j(\theta)\phi(r) \), with \( \Theta_j \in \mathcal{V}_D^j \), then by (24) and (26) this simplifies to

\[
\left( \hat{P}_X^j \phi \right)(r, \theta) = \delta_{ij} \Theta_j(\theta) f_i(r) \int_0^\infty r^2 dr f_i^*(r) \phi(r),
\]

which is zero if \( j \neq i \), has the same angular dependence \( \Theta_j(\theta) \) as \( \phi \) if \( j = i \).

In next section we provide an explicit characterization of elements \( \Theta_j \in \mathcal{V}_D^j \) as polynomials in the coordinates \( t^i \) of points of the unit sphere \( S^d \), which fulfill the relation

\[
t^i t_i = 1 \quad \text{(sum over \( i \))},
\]
rather than as combinations of spherical harmonics \( Y^m_l(\theta), m \in \mathbb{N} \).

3. Representations of \( O(D) \) via polynomials in \( x^l, t^i \)

The differential operator \( \mathbf{L}^2 \) can be expressed as

\[
\mathbf{L}^2 = \eta(D - 2 + \eta) - r^2 \Delta, \quad \eta := x^i \partial_i.
\]

The ‘dilatation operator’ \( \eta \) and the Laplacian \( \Delta \) fulfill

\[
\eta x^i = x^i(\eta + 1), \quad \eta \partial_i = \partial_i(\eta - 1),
\]

\[
\Delta x^i = x^i \Delta + 2 \partial_i, \quad \Delta^2 = \eta^2 + 4 \eta + 2 D.
\]

In particular, the action of \( \eta \) on monomials in the \( x^i \) amounts to multiplication by their total degree. In terms of polar coordinates it is \( \eta = r \partial_r \), which replaced in (29) gives (14).

Let \( \mathbb{C}[x^1, \ldots, x^D] \) be the space of complex polynomial functions on \( \mathbb{R}^D \) and, for all \( l \in \mathbb{N}_0 \), let \( \mathcal{W}_D^l \) be the subspace of homogeneous ones of degree \( l \). The monomials of degree \( l \) \( x^{i_1}x^{i_2}\ldots x^{i_l} \in \mathcal{W}_D^l \) can be reordered in the form \( (x^1)^{i_1}(x^D)^{i_l} \) and make up a basis of \( \mathcal{W}_D^l \):

\[
\mathcal{B}_{\mathcal{W}_D^l} := \left\{ (x^1)^{i_1}\ldots(x^D)^{i_l} \mid (i_1, \ldots, i_l) \in \mathbb{N}_0^D, \sum_{i=1}^D i_l = l \right\}, \quad \dim(\mathcal{W}_D^l) = \binom{D + l - 1}{l}.
\]

the dimension of \( \mathcal{W}_D^l \) is the number of elements of \( \mathcal{B}_{\mathcal{W}_D^l} \). Clearly \( \mathcal{W}_D^l \) carries a representation of \( O(D) \) as well as \( Uso(D) \), but this is reducible if \( l \geq 2 \); in fact, the subspace \( r^2 \mathcal{W}_D^{l-2} \subset \mathcal{W}_D^l \) manifestly carries a smaller representation. We denote by \( \mathcal{V}_D^l \) the ‘trace-free’ component of \( \mathcal{W}_D^l \), namely the subspace such that \( \mathcal{W}_D^l = \mathcal{V}_D^l \oplus r^2 \mathcal{W}_D^{l-2} \). As a consequence,

\[
\dim(\mathcal{V}_D^l) = \dim(\mathcal{W}_D^l) - \dim(\mathcal{W}_D^{l-2}) = \frac{(l + D - 3)\ldots(l + 1)}{(D - 2)!}(D + 2l - 2)
\]

where \( \mathcal{V}_D^l \) carries the irreducible representation (irrep) \( \pi^l_D \) of \( Uso(D) \) and \( O(D) \) characterized by the highest eigenvalue of \( \mathbf{L}^2 \) within \( \mathcal{W}_D^l \), namely \( E_l \) (the eigenvalues of all other \( |D/2 - 1| \) Casimirs are determined by \( l \)). Abbreviating \( \chi_{l,k}^{hk} := (x^h \pm ix^k)^l \), this can be easily shown observing that for all \( h, k \in \{1, \ldots, D\} \) \( \chi_{l,k}^{hk} \in \mathcal{W}_D^l \) are annihilated by \( \Delta \) and are eigenvectors of \( \mathbf{L}^2 \) with that eigenvalue; moreover, they are eigenvectors of \( L_{hk} \) with eigenvalue \( \pm l \). Hence \( \chi_{l,k}^{hk} \).
$X^{hk}_{i-}$ can be used as the highest and lowest weight vectors of $V^i_D$. Since all the $L_i$ commute with $\Delta$, $V^i_D$ can be characterized also as the subspace of $W^i_D$, which is annihilated by $\Delta$. A complete set in $V^i_D$ consists of trace-free homogeneous polynomials $X^{hk}_{i-}$, which we will obtain below applying the completely symmetric trace-free projector $\mathcal{P}^i$ to the $x^i x^{j} \cdots x^{o}$'s.

We slightly enlarge $\mathbb{C}[x^1, \dots x^D]$ introducing as new generators $r, r^{-1}$ subject to the relations $r^2 = x^i x_i$ (sum over $i$), $rr^{-1} = 1$. Inside this enlarged algebra the elements

$$t^i := x^i / r$$

fulfill the relation (28) characterizing the coordinates of points of the unit sphere $S^D$. Choosing $g(r) = r^{-i}$ in (33)–(36) we obtain the same relations with $x^i$, $X^{hk}_{i-}$ replaced by $t^i$, $T^{hk}_{i-} := (t^i + it^i)^i$. We shall denote by $\text{Pol}_D$ the algebra of complex polynomials in such $t^i$, by $\text{Pol}_{D}^{\Lambda}$ : $\text{Pol}_{D} \to \text{Pol}_{D}^{\Lambda}$ the corresponding projector. $\text{Pol}_{D}^{\Lambda}$ endowed with the scalar product $\langle T, T' \rangle := \int_0^1 d\lambda T^i(t) T'^i(t)$ is a pre-Hilbert space; its completion is $L^2(S^D)$. We extend $\text{Pol}^{\Lambda}$ to all of $L^2(S^D)$ by continuity in the norm of the latter. Also $\text{Pol}^{\Lambda}_{D}$, $V^i_D$ are Hilbert subspaces of $L^2(S^D)$. $\text{Pol}^{\Lambda}_{D} := \mathcal{W}^{\Lambda - 1} \oplus \mathcal{W}^{\Lambda}_D$ carries a unitary reducible representation of $O(D)$ [and $\text{Uso}(D)$] which splits via $\text{Pol}^{\Lambda}_{D} = \bigoplus_{i=0}^{\Lambda} V^i_D$ into irreps carried by $V^0_D := V^i_D / r^i$. Its dimension is thus

$$\dim \left( \text{Pol}^{\Lambda}_{D} \right) = \sum_{i=0}^{\Lambda} \dim (V^i_D) = \sum_{i=0}^{\Lambda} \dim (W^i_D) - \sum_{i=2}^{\Lambda} \dim (W^i_D^{-2}) = \dim (W^0_D) + \dim (W^{\Lambda - 1}_D) = (D + \Lambda) (\Lambda + 1) / (D - 1)! + (D + \Lambda - 2) \cdots \Lambda / (D - 1)!$$

$$= (D + \Lambda - 2) \cdots (\Lambda + 1) / (D - 1)! (D + 2 \Lambda - 1) =: N$$

\begin{align}
\text{(32)} & \quad \equiv \dim \left( V^{\Lambda + 1}_D \right).
\end{align}

This suggests that $H^{\Lambda}_{\Lambda} \simeq \text{Pol}^{\Lambda}_{D} \simeq V^{\Lambda}_{D+1}$ as $\text{Uso}(D)$ (reducible) representations. We have proved the first isomorphism in section 2 and will prove the second in section 3.2.

5 In fact, in terms of Cartesian coordinates, using (29) and (31) we immediately find the following commutation relations among operators of multiplication $(x^i \pm ix^j)$ and differential operators

$$(\partial_i + i\partial_j)(x^i + ix^j)(\partial_j + i\partial_i) - (\partial_j - i\partial_i)(x^j - ix^i)(\partial_i - i\partial_j) = (x^i + ix^j)(\partial_i - i\partial_j)$$

\begin{align}
\Delta (x^i \pm ix^j) & \equiv (x^i \pm ix^j)\Delta + 2(\partial_i \pm i\partial_j), \\
\Delta x^{hk}_{i-} & \equiv x^{hk}_{i-}\Delta + 2\Lambda x^{hk}_{i-}\Delta (\partial_i \pm i\partial_j), \\
L_\alpha (r^i \pm ir^j) & \equiv (r^i \pm ir^j)(L_\alpha \pm 1), \\
L_\alpha (L_{i\alpha} \pm iL_{j\alpha}) & \equiv (L_{i\alpha} \pm iL_{j\alpha}) [L_\alpha \pm 1].
\end{align}

\begin{align}
\text{(33)} & \quad \equiv \Delta x^{hk}_{i-}
\end{align}

Consequently, we obtain the following functions at the rhs as results of the operator actions on functions at the lhs: $\Delta x^{hk}_{i-} = 0$ and, for all functions $g(r)$,

$$L^2 g(r) x^{hk}_{i-} = g(r) [\eta(D - 2 + \eta) - \Delta] x^{hk}_{i-} = E L g(r) x^{hk}_{i-}, \quad \text{(34)}$$

$$L_\alpha x^{hk}_{i-} g(r) = \pm i L x^{hk}_{i-} g(r), \quad L_\alpha x^{hk}_{i-} x^{hk}_{i-} g(r) = (I - m) x^{hk}_{i-} x^{hk}_{i-} g(r). \quad \text{(35)}$$

Denoting by $\tau = \left[ \frac{1}{2} \right]$ the rank of $\text{so}(D)$, as a basis of a Cartan subalgebra of $\text{so}(D)$ one can take any set $\{H_i \equiv L_{i\alpha}, H_2 \equiv L_{i\alpha}, \ldots, H_{\tau} \equiv L_{i\alpha} - \eta I, \ldots, \}$, with $i_1, i_2, \ldots, i_{\tau} \in \{1, \ldots, D\}$ all different from each other. If $(i_1, i_2) = (h, k)$, by (34) and (35) $X^{hk}_{i-}$ are the corresponding highest, lowest weight vectors, in the sense

$$H_i x^{hk}_{i-} g(r) = \pm i x^{hk}_{i-} g(r), \quad H_a x^{hk}_{i-} g(r) = 0 \text{ if } a > 1.$$
3.1. $O(D)$-irreps via trace-free completely symmetric projectors

Let $(\pi, \mathcal{E})$ be the $D$-dimensional irreducible unitary representation of $\text{Un}(D)$ and $O(D)$; the carrier space $\mathcal{E}$ is isomorphic to $V_D^*$. As a vector space $\mathcal{E} \simeq \mathbb{R}^D$; the set of coordinates $x := (x^1, \ldots, x^D) \in \mathbb{R}^D$ can be seen as the set of components of an element of $\mathcal{E}$ with respect to (w.r.t.) an orthonormal basis. The permutator on $\mathcal{E}^{\otimes 2} \equiv \mathcal{E} \otimes \mathcal{E}$ is defined via $\mathcal{P}(u \otimes v) = v \otimes u$ and linearly extended. In all bases it is represented by the $D^2 \times D^2$ matrix $\mathcal{P}_{jk} = \delta^k_l \delta_j^i$. The symmetric and antisymmetric projectors $\mathcal{P}^+, \mathcal{P}^-$ on $\mathcal{E}^{\otimes 2}$ are obtained as

$$\mathcal{P}^\pm = \frac{1}{2} \left( \mathbb{1}_{D^2} \pm \mathcal{P} \right);$$

(40)

here and below we denote by $\mathbb{1}_{D^2}$ the identity operator on $\mathcal{E}^{\otimes 2}$, which in all bases is represented by the $D^2 \times D^2$ matrix $1_{D^2} = \delta^k_l \delta_j^i$. The antisymmetric tensor product $\mathcal{P}^\otimes \mathcal{E}^{\otimes 2}$ is an irrep under $O(D)$, while the symmetrized one $\mathcal{P}^+ \mathcal{E}^{\otimes 2}$ contains two irreps: the 1-dim trace one and the trace-free symmetric one. The matrix representation of the 1-dim projector $\mathcal{P}^\alpha$ on the former is

$$\mathcal{P}^\alpha_{ij} = \frac{1}{D} \delta^\alpha_{ij} g_{ij}$$

(41)

where the $D \times D$ metric matrix $g_{ij}$ (in the chosen basis) is a $\text{so}(D)$-isotropic symmetric tensor, and $\delta^\alpha_{ij} g_{ij} = \delta^\alpha_{ij}$, whence $\delta^\alpha_{ij} g_{ij} = D$. Here we shall use an orthonormal basis of $\mathcal{E}$, whence $g_{ij} = \delta_{ij}$, and indices of vector components can be raised or lowered freely, e.g. $x_i = x^i$. The following relations hold:

$$\mathcal{P}^\alpha = \mathcal{P}^+ - \mathcal{P}^- = \frac{1}{2} \left( \mathbb{1}_{D^2} \pm \mathcal{P} \right) - \mathcal{P}' .$$

(42)

These projectors satisfy the equations

$$\mathcal{P}^\alpha \mathcal{P}^\beta = \delta^{\alpha \beta}, \quad \sum_\alpha \mathcal{P}^\alpha = \mathbb{1}_{D^2},$$

(43)

where $\alpha, \beta = -, s, t$. In the sequel we shall abbreviate $\mathcal{P} \equiv \mathcal{P}^s$. This implies in particular $\mathcal{P}' \mathcal{P}' = 0$, where we have introduced the new projector $\mathcal{P}' := \mathcal{P}^- + \mathcal{P}'$. $\mathcal{P}', \mathcal{P}^\alpha$ are symmetric matrices, i.e. invariant under transposition $^t$, and therefore also the other projectors are:

$$\mathcal{P}^T = \mathcal{P}, \quad \mathcal{P}^{\alpha T} = \mathcal{P}^{\alpha} .$$

(44)

Given a (linear) operator $M$ on $\mathcal{E}^{\otimes l}$, for all integers $l$, $h$ with $l > n$, and $1 \leq h \leq l + 1 - n$ we denote by $M_{h(h+1)\ldots(h+n-1)}$ the operator on $\mathcal{E}^{\otimes h}$ acting as the identity on the first $h-1$ and the last $l+1-n-h$ tensor factors, and as $M$ in the remaining central ones. For instance, if $M = \mathbb{P}$ and $l = 3$ we have $\mathbb{P}_{12} = \mathbb{P} \otimes 1_D, \mathbb{P}_{23} = 1_D \otimes \mathbb{P}$. It is straightforward to check

**Proposition 3.1.** *All the projectors $\mathcal{A} = \mathcal{P}^+, \mathcal{P}^-, \mathcal{P}', \mathcal{P}'$ fulfill the 'braid' relation*

$$A_{12} \mathcal{P}_{23} \mathcal{P}_{12} = \mathcal{P}_{23} \mathcal{P}_{12} A_{23} .$$

(45)

Moreover,

$$D \mathcal{P}_{23} \mathcal{P}_{12} = \mathcal{P}_{12} \mathcal{P}_{23} \mathcal{P}_{12}, \quad D \mathcal{P}_{12} \mathcal{P}_{23} \mathcal{P}_{12} = \mathcal{P}_{23} \mathcal{P}_{12} .$$

(46)

$$D \mathcal{P}_{12} \mathcal{P}_{23} = \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23}, \quad D \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23} = \mathcal{P}_{12} \mathcal{P}_{23} .$$

(47)

$$D \mathcal{P}_{23} \mathcal{P}_{12} = \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23}, \quad D \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23} = \mathcal{P}_{23} \mathcal{P}_{12} .$$

(48)
equations (45)–(48) hold also for \( l > 3 \), e.g. for all \( 2 \leq h \leq l - 1 \)

\[
A_{(h-1)k}P_{(h-1)h}P_{(h-1)k} = P_{(h-1)k}A_{(h-1)k}.
\]

**Proof.** Since \( A = 1_{P^l} \), \( P, P' \) fulfill (45), then also \( A = \mathcal{P}^+, \mathcal{P}^-, \mathcal{P}', \mathcal{P}'' \) do. One can immediately check the first equality in (46) via direct calculation; left multiplying the first by \( P_{12} \) one obtains the second. Equation (47) is obtained from (46) exchanging \( 1 \leftrightarrow 3 \) and using the symmetry of \( P, P'' \) under the flip. Equation (48) are obtained from (47) by transposition. \( \square \)

Next, we define and determine the completely symmetric trace-free projector \( P' \) on \( \mathcal{E}^{0|l} \) generalizing \( P^2 \equiv P \) to \( l > 2 \). It projects the tensor product of \( l \) copies of \( \mathcal{E} \) to the carrier space of the \( l \)-fold completely symmetric irrep \( \text{Us}(D) \), isomorphic to \( V_0 \), therein contained. It is uniquely characterized by the following properties:

\[
\begin{align*}
P_{n+1}P_{n+1} &= 0, & P_{n+1}P_{n+1} &= 0, & n = 1, \ldots, l - 1, \\
(P')^2 &= P'.
\end{align*}
\]

Consequently, it is also \( \text{tr}_1 \ldots (P') = \dim(V_0) \), which guarantees that \( P' \) acts as the identity (and not as a proper projector) on \( V_0 \). The right relations in (50) amount to

\[
\begin{align*}
P_{n+1}P_{n+1} &= 0, & P_{n+1}P_{n+1} &= 0, & n = 1, \ldots, l - 1, \\
(P')^2 &= P'.
\end{align*}
\]

Clearly the whole of (50) can be summarized as \( P_{n+1}P_{n+1} = 0 = P_{n+1}P_{n+1} \). It is straightforward to prove that the above properties imply also the ones

\[
P_{n+1}P_{n+1} = P', & n = 1, \ldots, l - 1; & P_{n+1}P_{n+1} = P', & h < l, 0 \leq i \leq l - h.
\]

**Proposition 3.2.** The projector \( P^{l+1} \) can be expressed as a polynomial in the permutators \( P_{12}, \ldots, P_{(l-1)l} \) and trace projectors \( P_{12}, \ldots, P_{(l-1)l} \) through either recursive relation

\[
P_{l+1} = P_{12} \ldots P_{l+1} \, \mathcal{M}_{l+1} \, P_{12} \ldots P_{l+1},
\]

\[
= P_{12} \ldots P_{l+1} \, \mathcal{M}_{12} \, P_{2(l+1)},
\]

where \( \mathcal{M} \equiv M(l+1) = \frac{1}{l+1} \left[ 1_{P^l} + P - \frac{2P}{D + 2l - 2} \right] \).

As a consequence, the \( P' \) are symmetric, \( (P')^T = P' \).

This the analog of proposition 1 in [26] for the quantum group \( \text{Us}(D) \) covariant symmetric projectors; the proof is in the appendix. By a straightforward computation one checks that

\[
P_{l+1}^{(h+1)\ldots (h+1)} = \frac{1}{l+1} \left[ D + l - \frac{2l}{D + 2l - 2} \right] P_{j_1 \ldots j_l}^{(h+1)\ldots (h+1)}
\]

\((h \text{ is summed over})\). Using (31) and (52) one easily shows that the homogeneous polynomials

\[
X_{i}^{(h+1)\ldots (h+1)} = P_{j_1 \ldots j_l}^{(h+1)\ldots (h+1)}
\]

are harmonic, i.e. satisfy \( \Delta X_{i}^{(h+1)\ldots (h+1)} = 0 \); using (29), we find that they are eigenvectors of \( L^2 \).

\[
L^2 X_{i}^{(h+1)\ldots (h+1)} = E_{i} X_{i}^{(h+1)\ldots (h+1)}.
\]

with eigenvalues (3). They make up a complete set in \( V_0 \), which can be thus also characterized as the subspace of \( W_0 \) that is annihilated by \( \Delta \), whereas \( \Delta \phi \neq 0 \) for all \( \phi \in \mathbb{P}^2 W_0^2 \). The \( X_{i}^{(h+1)\ldots (h+1)} \)
are not all independent, because they are invariant under permutations of \((i_1\ldots i_l)\) and by (52) fulfill the linear dependence relations
\[
\delta_{i_k,i_{k+1}}X_i^{i_1\ldots i_{l-n}} = 0, \quad n = 1,\ldots,l-1.
\] (60)

**Proposition 3.3.** In a compact notation,
\[
(P_{1\ldots l}^l - P_{1\ldots l}^{l+1}) x_{1\ldots l} = \zeta_{l+1} P_{l(l+1)}^l x_{1\ldots l},
\]

\[
(P_{2\ldots (l+1)}^l - P_{2\ldots (l+1)}^{l+1}) x_{1\ldots l} = \zeta_{l+1} P_{l2}^l x_{1\ldots l},
\]

\[
\zeta_{l+1} = \frac{Dl}{D + 2l - 2}.
\] (61)

The proof is in appendix ‘Proof of proposition 3.3’. More explicitly, (61) becomes
\[
x^h X_i^{i_1\ldots i_l} = \frac{\zeta_{l+1} r^2}{D} \cdot P_{i_1}^{i_2 \ldots i_l} x^i X_i^{i_1\ldots i_l}.
\] (62)

Contracting the previous relation with \(\delta_{i_1 i_l}\) and using (52) we obtain
\[
x^h X_i^{i_1\ldots i_l} = \frac{\zeta_{l+1} r^2}{D} \cdot P_{i_1}^{i_2 \ldots i_l} X_i^{i_1\ldots i_l}.
\] (63)

In the appendix we also prove

**Proposition 3.4.** The maps \(L_{hk} : V_D^l \to V_D^l\) explicitly act as follows:
\[
il_{hk} X_i^{i_1\ldots i_l} = \frac{\zeta_{l+1}}{\zeta_{l+1}} \left( P_{i_1}^{i_2 \ldots i_l} + P_{i_1}^{i_2 \ldots i_l} \right) X_i^{i_1\ldots i_l},
\]

\[
il_{hk} X_i^{i_1\ldots i_l} = \frac{r^2}{D} \left( D + l - \frac{2l - 2}{D + 2l - 4} \right) X_i^{i_1\ldots i_l}.
\] (64)

Dividing (59), (60) and (62)–(64) by the appropriate powers of \(r\) we find

**Proposition 3.5.** The \(T_i^{i_1\ldots i_l} := X_i^{i_1\ldots i_l}/r \in V_D^l\) belong to \(V_D^l\), because
\[
L^2 T_i^{i_1\ldots i_l} = E_l T_i^{i_1\ldots i_l},
\] (65)

they make up a complete set \(\{T_i\}\) in it, but not a basis, because they are invariant under permutations of \((i_1\ldots i_l)\) and by (60) fulfill the linear dependence relations
\[
\delta_{i_1 i_{k+1}} T_i^{i_1\ldots i_l} = 0, \quad n = 1,\ldots,l-1.
\] (66)

The actions of the operators \(t^h, \ nil_{hk}\) on \(T_i^{i_1\ldots i_l}\) explicitly read
\[
t^h T_i^{i_1\ldots i_l} = T_l^{i_1\ldots i_l} + \frac{l}{D + 2l - 2} P_l^{i_1\ldots i_l} T_i^{i_1\ldots i_l} \in V_D^l \oplus V_D^{l-1},
\] (67)

\[
t^h T_i^{i_1\ldots i_l} = \frac{1}{D + 2l - 2} \left( D + l - \frac{2l - 2}{D + 2l - 4} \right) T_i^{i_1\ldots i_l} \in V_D^{l-1},
\] (68)

\[
il_{hk} T_i^{i_1\ldots i_l} = \frac{\zeta_{l+1}}{\zeta_{l+1}} \left( P_{i_1}^{i_2 \ldots i_l} + P_{i_1}^{i_2 \ldots i_l} \right) T_i^{i_1\ldots i_l},
\]

\[
il_{hk} T_i^{i_1\ldots i_l} = \frac{r^2}{D} \left( D + l - \frac{2l - 2}{D + 2l - 4} \right) T_i^{i_1\ldots i_l}.
\] (69)

For all \(\phi \in \mathcal{H}_i \equiv \mathcal{L}^2(S^l)\) let
\[
\phi = \sum_{l=0}^{\infty} \sum_{m \in h} \phi_m^{i_l} Y_m^{i_l} = \sum_{l=0}^{\infty} \sum_{m \in h} \phi_m^{i_l} T_i^{i_1\ldots i_l}.
\] (70)

be its decompositions in the basis of spherical harmonics and in the complete set \(\{T_i\} \in \bigcup_{l=0}^{\infty} T_i\)
here the two sets of coefficients are related by \(\phi_m^{i_1\ldots i_l} = \sum_{m \in h} \phi_m^{i_1\ldots i_l} A_m^{i_1\ldots i_l}\), where \(A_m^{i_1\ldots i_l}\) are such
that $T^m_i = \sum A_i^m T^m_{i|\ldots|b}$. The $\phi^l_{i_{1}\ldots i_{l}}$ are uniquely determined if, as we shall assume, we choose them trace-free and completely symmetric, i.e.
fulfilling

$$\phi^l_{i_{1} \ldots i_{l}} = \phi^l_{j_{1} \ldots j_{l}} T^m_{i_{1} \ldots i_{l}; j_{1} \ldots j_{l}},$$

(71)

whence $\phi^l_{i_{1} \ldots i_{l}} \delta_{i_{l+1}} = 0$ for $n = 1, \ldots, l - 1$. Then (70) can be also written in the form

$$\phi = \sum_{l=0}^{\infty} \sum_{i_1, \ldots, i_l} \phi^l_{i_{1} \ldots i_{l}} t^1_i \ldots t^l_i.$$

(72)

The projector $P_\Lambda$ acts by truncation, $P_\Lambda \phi = \phi_\Lambda := \sum_{n_{l}=0}^{\Lambda} \sum_{m \leq b} \phi^l_{m} T_i^m = \sum_{n_{l}=0}^{\Lambda} \phi^l_{i_{1} \ldots i_{l}} T^m_{i_{1} \ldots i_{l}}$ (sum over $i_{1}, \ldots, i_{l}$). Clearly $\phi_\Lambda \sum_{l=0}^{\Lambda} \sum_{i_{1}, \ldots, i_{l}} \phi^l_{i_{1} \ldots i_{l}} t^1_i \ldots t^l_i$.

All completely symmetric, $O(D)$-isotropic tensors of even rank $N$ are proportional to

$$G^{\text{even}}_{N} := \delta_{i_{1}} \delta_{i_{2}} \ldots \delta_{i_{N}} + \text{permutations of } (i_1, \ldots, i_N).$$

(73)

**Proposition 3.6.** The ‘trace’ of $G^N_{N}$, $tr (G^N_{N}) := \phi_{i_{1} \ldots i_{l}} \delta_{i_{1}} \delta_{i_{2}} \ldots \delta_{i_{N}} G^{\text{even}}_{N}$, is equal to

$$tr (G^N_{N}) = N!! D(D+2) \ldots (D+N-2).$$

(74)

The $O(D)$-invariant integral over $S^l$ of the tensor $H^{i_{1} \ldots i_{l}} := t^1_i \ldots t^l_i$ is equal to

$$\int_{S^l} d\phi^l \ldots d\phi^l = C_N G^{\text{even}}_{N}, \quad C_N = \frac{\text{miss}(S^l)}{N!! D(D+2) \ldots (D+N-2)}.$$  

(75)

In terms of the decompositions (70b)–(71) the scalar product of generic $\phi, \psi \subset \mathcal{H}_s$ is equal to

$$\langle \phi, \psi \rangle = \sum_{l=0}^{\infty} Q_l (\phi^l_{i_{1} \ldots i_{l}})^* \psi^l_{i_{1} \ldots i_{l}}, \quad Q_l := C_l (l!)^2 = \frac{\text{miss}(S^l)}{(D+2l-2)! l!}.$$

(76)

In particular, (76) implies $\langle T^m_{i_{1} \ldots i_{l}}, \phi \rangle = Q_l \phi^l_{i_{1} \ldots i_{l}}$ and more generally

$$\langle T^m_{i_{1} \ldots i_{l}}, T^m_{i_{1} \ldots i_{l}} \rangle = \delta_{i_{1}} \delta_{i_{2}} \ldots \delta_{i_{N}} T^m_{i_{1} \ldots i_{l}},$$

(77)

the second equality holds only if (71) holds. Then, we have also

$$\| \phi \|^2 \equiv \langle \phi, \phi \rangle = \sum_{l=0}^{\infty} \sum_{i_{1}, \ldots, i_{l}} |\phi^l_{i_{1} \ldots i_{l}}|^2.$$  

(78)

We now determine the decomposition of $T^m_{i_{1} \ldots i_{l}} T^m_{i_{1} \ldots i_{l}}$.

**Theorem 3.7.** The product $T^m_{i_{1} \ldots i_{l}} T^m_{i_{1} \ldots i_{l}}$ decomposes as follows into $V^m_D$ components:

$$T^m_{i_{1} \ldots i_{l}} T^m_{i_{1} \ldots i_{l}} = \sum_{n \leq L_m} S^m_{k_{1} \ldots k_{n}} \ T^m_{k_{1} \ldots k_{n}},$$

(79)

where $L_m := |m| + |m| + 2, \ldots, |m|$ and, defining $r := \frac{l + m - n}{2} \in \{0, 1, \ldots, m\}$,

$$S^m_{k_{1} \ldots k_{n}} = \frac{N^m_{n}}{N_{n}^{k_{1} \ldots k_{n}}} \frac{V^m_{k_{1} \ldots k_{n}}}{V_{k_{1} \ldots k_{n}}} = \frac{(D + 2n - 2)! l!}{(D + 2n + 2r - 2)! (l - r)! (m - r)!}.$$  

(80)

The coefficients $S^m_{k_{1} \ldots k_{n}}$ are the analogs of the Clebsch–Gordan coefficients, which appear in the decomposition of a product of two spherical harmonics into a combination of spherical harmonics for $D = 3$. The first term of the sum (67) is $T^m_{i_{1} \ldots i_{l}} T^m_{i_{1} \ldots i_{l}}$. This is consistent
with the first term in the iterated application of (67). If \( r = m = 1 \), since \( P_{D}^{\pm l} \mid_{a_i} = \delta_{a_i}^l \), \( n = l - 1 \), the result is consistent with the second term in (67):
\[
S_{k_1,\ldots,k_{l-1}}^{i_1,\ldots,i_{l-1}} = \frac{l}{D + 2l - 2} P_{D}^{i_1,\ldots,i_{l}}.
\]

3.2. Embedding in \( \mathbb{R}^{D+1} \), isomorphism \( \text{End} (\text{Pol}_{D}^\bullet) \simeq \pi_{D+1}^\bullet [\text{UsO}(D + 1)] \)

We naturally embed \( \mathbb{C}[\mathbb{R}^D] \hookrightarrow \mathbb{C}[\mathbb{R}^D] \), where \( D \equiv D + 1 \). Henceforth we use real Cartesian coordinates \( (x') \) for \( \mathbb{R}^D \) and \( (x') \) for real \( \mathbb{R}^D \), \( h, i, j, k \in \{1, \ldots, D\} \), \( H, I, J, K \in \{1, \ldots, D\} \). We naturally embed \( O(D) \hookrightarrow SO(D) \) identifying \( O(D) \) as the subgroup of \( SO(D) \) which is the little group of the \( D \)th axis; its Lie algebra, isomorphic to \( \text{so}(D) \), is generated by the \( L_{ik} \). We shall add \( D \) as a subscript to distinguish objects in this enlarged dimension from their counterparts in dimension \( D \), e.g. the distance \( r_D \) from the origin in \( \mathbb{R}^D \), from its counterpart \( r \equiv r_D \) in \( \mathbb{R}^D \), \( P_{D}^l \) from \( P^l \equiv P_{D}^l \), and so on.

We look for the decompositions of each \( V_D^{ij} \) into irreps of such a \( \text{UsO}(D) \). Clearly, \( V_D^0 \simeq \mathbb{C} \simeq V_D^{0\prime} \), while \( V_D^j \equiv V_D^{ij} \oplus V_D^{ij} \), where \( V_D^0 \simeq V_D^0 \) is spanned by \( x^D \), and \( V_D^i \equiv V_D^j \) is spanned by the \( x^i \). The \( X_{D}^{ij} = x^i x^j - \frac{g^D}{D} \delta^{ij} / D \) span \( V_D^i \); the set of elements
\[
X_{D,2}^{ij} = x^i x^j - \frac{g^D}{D} \delta^{ij} / D, \quad X_{D,2}^j = x^j + \frac{g^D}{D} \delta^{ij} / D
\]
span carrier spaces \( V_D^{0,2}, V_D^{1,2}, V_D^{2,2} \) respectively isomorphic to \( V_D^0, V_D^1, V_D^2 \), and \( V_D^2 = V_D^{2,2} \oplus V_D^{0,2} \). More generally, \( V_D^{ij} \) is spanned by the \( x^i, x^j \), which are homogeneous polynomials of degree \( \Lambda \) in the \( x^l \) obtained by (58) in dimension \( D \), i.e.
\[
X_{D,\Lambda}^{ij} := P_{D,ij}^{l \ldots l} x^i x^j, \ldots, x^h \in V_D^{ij}
\]
where the projectors \( P_{D}^l \) are constructed as in proposition 3.2, but with \( D \) replaced by \( D \). Any pair of indices \( i, j \) appears either through the product \( x^i x^j \) or through \( r_D \delta^{ij} / D \). If we introduce \( r_D \) as a further generator constrained by the relation \( r_D^2 = x^i x^i \), then the \( X_{D,\Lambda}^{ij} \) can be seen also as homogeneous polynomials of degree \( \Lambda \) in \( x^l \). Since \( \delta^{ij} \) is spanned by the \( x^i, x^j \), any pair of indices \( i, j \) appears either through the product \( x^i x^j \) or through \( r_D \delta^{ij} / D \); any pair of indices \( j, k \) appears through the product \( x^j x^k \); any pair of indices \( j, k \) appears either through the product \( x^j x^k \) or through \( r_D \delta^{jk} / D \). By property (52), the latter terms completely disappear in any combination \( P_{D,\Lambda}^{i_1 \ldots i_l} x_{D,\Lambda}^{i_1 \ldots i_l} \), \( l \in \{2, 3, \ldots, \Lambda\} \). Therefore such a combination can be factorized as follows
\[
K_{D,\Lambda}^{i_1 \ldots i_l} := P_{D,ij}^{l \ldots l} x_{D,\Lambda}^{i_1 \ldots i_l} D = \rho_{\Lambda,l} x_{D,\Lambda}^{i_1 \ldots i_l},
\]
where \( \rho_{\Lambda,l} \) is a homogeneous polynomial of degree \( \Lambda - l \) in \( x^D, r_D \) of the form
\[
\rho_{\Lambda,l} = (x^D)^{\Lambda-l} + (x^D)^{\Lambda-l-2} r_D^2 b_{\Lambda,l+2} + (x^D)^{\Lambda-l-4} r_D^4 b_{\Lambda,l+4} + \ldots
\]
the coefficients \( b_{\Lambda,h} \) can be determined from \( \Delta_D K_{D,\Lambda}^{i_1 \ldots i_l} = 0 \), which follows from \( \Delta_D x_{D,\Lambda}^{i_1 \ldots i_l} = 0 \).

**Proposition 3.8.** \( V_D^{ij} \) decomposes into the following irreducible components of \( \text{UsO}(D) \):
\[
\check{V}_D^{ij} = \bigoplus_{\Lambda} \check{V}_D^{ij,\Lambda}.
\]
where $V_{D,A}^\lambda \simeq V_D^l$, is spanned by the $F_{D,A}^{li...j}$, since the latter are eigenvectors of $L^2$:
\begin{equation}
L^2 F_{D,A}^{li...j} = E_l F_{D,A}^{li...j}.
\end{equation}

Denoting by $|a\rangle$ the integral part of $a \in \mathbb{R}^+$, the coefficients of (84) are given by
\begin{equation}
b_{\lambda+2k} = (-)^k \frac{(\Lambda-I)!(2\Lambda-4-2k+D)!}{(\Lambda-I-2k)!(2k)!(2\Lambda-4+D)!}, \quad k = 1, 2, \ldots \left\lfloor \frac{\Lambda-I}{2} \right\rfloor.
\end{equation}

The $F_{D,A}^{li...j}$ transform under $L_{ab}$ as the $T_{l}^{li...j}$, and under $L_{ab}$ as follows:
\begin{equation}
iL_{ab} F_{D,A}^{li...j} = (\Lambda-I) F_{D,A}^{li...j} - \frac{l(\Lambda+l+D-2)}{D+2l-2} p_{lij...j} F_{D,A}^{li...j}.
\end{equation}

The proof is in appendix ‘Proof of proposition 3.8’. We now determine the decomposition of $V_D^\lambda := V_D^l/(r_\lambda)^l$ into irreps of $Uso(D)$, $V_D^0 \simeq V_D^0 \simeq \mathbb{C}$ is spanned by $1$. $V_D^l = V_{D,1}^0 \oplus V_{D,1}^l$, where $V_{D,1}^0 \simeq V_D^0$ is spanned by $r_\lambda$, and $V_{D,1}^l \simeq V_D^l$ is spanned by the $t'$ (here $t' := x^l/r_\lambda$). $V_D^l$ is spanned by the $T_{D,2}^{li}$; more explicitly,
\begin{equation}
T_{D,2}^{li} = T_{D,2}^{li} + \frac{\delta^{ij}}{D} T_{D,2}^{l} = T_{D,2}^{li} - \frac{\delta^{ij}}{D} T_{D,2}^{l} \text{ (sum over } h) = t' t - \frac{\delta^{ij}}{D} t_b.
\end{equation}

This shows the decomposition $V_D^\lambda = V_{D,2}^l \oplus V_{D,2}^{l'} \oplus V_{D,2}^0 \simeq Polo_D$ explicitly. More generally, let $F_{D,2}^{li...j} := T_{D,2}^{li...j}$ (which fulfill $F_{D,2}^{lj...j} = 0$) span an irreps $V_{D,2}^l$ isomorphic to $V_D^l$. The $F_{D,2}^{li...j}$ can be expressed as combinations of the $T_{D,2}^{li}$:
\begin{equation}
F_{D,2}^{li...j} = T_{D,2}^{li} + \frac{\delta^{ij}}{D} T_{D,2}^{l} = T_{D,2}^{li} - \frac{\delta^{ij}}{D} T_{D,2}^{l} \text{ (sum over } h) = t' t - \frac{\delta^{ij}}{D} t_b.
\end{equation}

As a direct consequence of proposition 3.8, dividing all relations by $(r_\lambda)^l$, we find
\begin{equation}
\text{Corollary 3.9. } V_D^\lambda \text{ decomposes into the following irreducible components of } Uso(D):
\end{equation}
\begin{equation}
\bigoplus_{l=0}^{\Lambda} V_{D,A}^l.
\end{equation}

where $V_{D,A}^l \simeq V_D^l$, is spanned by the $F_{D,A}^{li...j}$, The latter are eigenvectors of $L^2$,
\begin{equation}
L^2 F_{D,A}^{li...j} = E_l F_{D,A}^{li...j},
\end{equation}

transform under $L_{ab}$ as the $T_{l}^{li...j}$, and under $L_{ab}$ as follows:
\begin{equation}
iL_{ab} F_{D,A}^{li...j} = (\Lambda-I) F_{D,A}^{li...j} - \frac{l(\Lambda+l+D-2)}{D+2l-2} p_{lij...j} F_{D,A}^{li...j}.
\end{equation}

For convenience, we slightly enlarge $Uso(D)$ by introducing the new generator
\begin{equation}
\lambda = \frac{1}{2} \left[ \sqrt{(D-2)^2 + 4L^2} - D + 2 \right],
\end{equation}

\[16\]
which fulfills $\lambda(\lambda + D - 2) = L^2$, so that $V^D_\lambda$ is a $\lambda = l$ eigenspace, and $\lambda e^{D,\lambda} = 1 F^D,\Lambda$. 

**Proposition 3.10.** There exist a $\text{Uso}(D)$-module isomorphism $\chi \Lambda : \text{Pol}^D \rightarrow V^\lambda$ and a $\text{Uso}(D)$-equivariant algebra isomorphism $\hat{\chi} \Lambda : \text{End}(\text{Pol}^D) \rightarrow \pi^D_\Lambda [\text{Uso}(D)]$ such that

$$\chi \Lambda (aT) = \hat{\chi} \Lambda (a) \chi \Lambda (T), \quad \forall T \in \text{Pol}^D, \quad a \in \text{End}(\text{Pol}^D).$$

(95)

On the $T^i_{1\ldots l}$ (spanning $\text{Pol}^D$) and on generators $L_{hi}$, $P^\Lambda T^i$ of $\text{End}(\text{Pol}^D)$ they act by

$$\chi \Lambda (T^i_{1\ldots l}) := \hat{\chi} \Lambda (\lambda) \chi \Lambda (T^i_{1\ldots l}), \quad l = 0, 1, \ldots, \Lambda,$n

$$\hat{\chi} \Lambda (L_{hi}) := L_{hi}, \quad \hat{\chi} \Lambda (P^\Lambda T^i) := \hat{m} \Lambda (\lambda) L_{hi} \hat{m} \Lambda (\lambda),$$

(96)

(97)

where

$$\hat{m} \Lambda (l) = \frac{\Gamma(\frac{\Lambda + D - 1}{2}) \Gamma(\frac{\Lambda + l + 1}{2})}{\Gamma(\frac{l + 1}{2}) \Gamma(\frac{\Lambda + D + l - 1}{2})}$$

(98)

The proposition and its proof are obtained from theorem 5.1 and the associated proof by fixing $\Lambda$, taking $k$ independent of $\Lambda$ and letting $k \to \infty$.

**4. Relations among the $x^i, L_{hk}$**

Since for all fixed $l = 0, 1, \ldots, \Lambda$ the $T^i_{1\ldots l}$ make up a complete set $T^i_l$ in $V^D_\lambda$, then the functions

$$\psi^i_{1\ldots l} := T^i_{1\ldots l} f_i$$

(99)

$(i_h \in \{1, \ldots, D\}$ for $h \in \{1, \ldots, l\}$) make up a complete set $S^D_{\Lambda, \Lambda}$ in the eigenspace $\mathcal{H}^\Lambda_{\Lambda}$ of $H$, $L^2$, with eigenvalues $E_{0,0}$, $E_l$. They are not linearly independent, because they are completely symmetric under permutation of the indices and fulfill the relations

$$\delta_{n+i,i} \psi^i_{1\ldots l} = 0, \quad n = 1, \ldots, l - 1.$$n

(100)

$S^D_{\Lambda, \Lambda} := \bigcup_{l=0}^\Lambda S^D_{\Lambda, \Lambda}$ is a complete set in $\mathcal{H}_{\Lambda}$. By (99), (69) the $\hat{L}_{hi}$ act on the $\psi^i_{1\ldots l}$ via

$$\hat{L}_{hi} \hat{m} \Lambda (\lambda) L_{hi} \hat{m} \Lambda (\lambda) = \lambda \psi^i_{1\ldots l}$$

(101)

By (27) and (67), applying $P_\Lambda = \sum_{i=0}^\Lambda p^i_\Lambda$ to $x^i \psi^i_{1\ldots l} := t^i T^i_{1\ldots l} f_l (r)$ we find

$$P_\Lambda (x^i \psi^i_{1\ldots l}) = c_{l+1} \psi_{l+1}^i + c_l \frac{\lambda}{D} \psi^i_{l+i} + \lambda \psi^i_{1\ldots l},$$

where

$$c_l := \begin{cases} \sqrt{1 + \frac{(2D - 5)(D - 1)}{2l}} & \text{if } 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise}, \end{cases}$$

(102)

up to order $O \left( k^{-3/2} \right)$, see appendix ‘Evaluating a class of radial integrals, and proof of (103)’.

Hence, at the same order,

$$x^i \psi^i_{1\ldots l} = c_{l+1} \psi_{l+1}^i + c_l \frac{\lambda}{D + 2l - 2} \psi_{l+i}^i + \lambda \psi^i_{1\ldots l}.$$n

(103)
The $O(k^{-3/2})$ corrections depend on the terms proportional to $(r - 1)^k, k > 2$, in the Taylor expansion of $V$. These could be set rigorously equal to zero by a suitable choice of $V$. Henceforth we adopt (101)–(105) as exact definitions of $L_{ab}, x$. In the appendix we prove

**Proposition 4.1.** The $x, L_{ab}$ are self-adjoint operators generating the $N^2$-dimensional $*$-algebra $A_\Lambda := \text{End}(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C})$ of observables on $\mathcal{H}_\Lambda$; $N$ is given by (38). Abbreviating $x^2 := x^\dagger x, L^2 := L_{ij} L_{ij}/2, B := (2D - 5)(D - 1)/2, they fulfill, at leading order in $1/\sqrt{k}$,

$$[iL_{ij}, x^k] = x^\dagger \delta^k_{ij} - x^k \delta^i_j,$$

(104)

$$[iL_{ij}, L_{ab}] = i \left( L_{ik} \delta^j_h - L_{ih} \delta^j_k - L_{jk} \delta^i_h + L_{jh} \delta^i_k \right),$$

(105)

$$\varepsilon^{i_1 j_1 \ldots i_n j_n} x^{i_1} \cdots x^{i_n} L_{i_1 j_1} = 0, \quad D \geq 3;$$

(106)

$$\left((x^2 \pm i\hbar k)^\Lambda + 1\right)^{-1} = 0, \quad (L^{ij} + iL^{kj})^\Lambda = 0, \quad \text{if } h \neq j \neq k \neq h,$$

(107)

$$[x^\dagger, x] = \frac{-i}{k} + K P^A_\Lambda, \quad K := \frac{\Lambda + D - 2}{\Lambda + D - 2} \left[ 1 + \frac{B}{k} + \frac{\Lambda(\Lambda + D - 1)}{k} \right] P^A_\Lambda =: \chi(L^2).$$

(108)

$$x^2 = 1 + \frac{\overline{L}^2}{k} + B \frac{\Lambda + D - 2}{2\Lambda + D - 2} \left[ 1 + \frac{B}{k} + \frac{\Lambda(\Lambda + D - 1)}{k} \right] P^A_\Lambda =: \chi(L^2).$$

This is the analog of proposition 4.1 in [22]. We obtain a FS choosing $k$ as a function $k(\Lambda)$ fulfilling (20), e.g. $k = \Lambda^2(\Lambda + D - 2)^2/4$; the commutative limit is $\Lambda \rightarrow \infty$.

**Remarks.**

4.a Equation (106) is the analog of (12). By (108), it can be reformulated also in the form

$$\varepsilon^{i_1 j_1 \ldots i_n j_n} x^{i_1} \cdots x^{i_n} x^\dagger = 0.$$  

4.b $x^2$ is not a constant, but by (109), (23)$_{h=\Lambda}$ can be expressed as a polynomial $\chi$ in $L^2$ only, with the same eigenspaces $\mathcal{H}^\Lambda_\Lambda$. All its eigenvalues $r^\Lambda_i$, except $r^\Lambda_1$, are close to 1, slightly (but strictly) growing with $\Lambda$ and collapse to 1 as $\Lambda \rightarrow \infty$. Conversely, $L^2$ can be expressed as a polynomial $\chi$ in $x^2$, via $L^2 = \sum_{\ell=0}^\Lambda E_\ell P^\Lambda_\ell$ and $P^\Lambda_\ell = \prod_{\ell=0, n \neq \ell}^{\Lambda} I_{\ell-i}$.  

4.c By (108), (23)$_{h^\Lambda}$ the commutators $[x^\dagger, x]$ are Snyder-like, i.e. of the form $\alpha L_{ij}$; also $\alpha$ depends only on the $L_{ab}$, more precisely can be expressed as a polynomial in $L^2$.  

4.d Using (104), (105) and (108), all polynomials in $x, L_{ab}$ can be expressed as combinations of monomials in $x^\dagger, L_{ab}$ in any prescribed order, e.g. in the natural one

$$\left(x^\dagger\right)^{n_1} \cdots \left(x^\dagger\right)^{n_9} \left(L_{12}\right)^{n_{12}} \left(L_{13}\right)^{n_{13}} \cdots \left(L_{ab}\right)^{n_{ad}}, \quad n_i, n_{ij} \in \mathbb{N}_0;$$

(110)

the coefficients, which can be put at the right of these monomials, are complex combinations of 1 and $P^\Lambda_\Lambda$. Also $P^\Lambda_\Lambda$ can be expressed as a polynomial in $L^2$ via (23)$_{h=\Lambda}$. Hence a suitable subset (depending on $\Lambda$) of such ordered monomials makes up a basis of the $N^2$-dim vector space $A_\Lambda$.  

4.e Actually, $x$ generate the $*$-algebra $A_\Lambda$, because also the $L_{ij}$ can be expressed as non-ordered polynomials in the $x^\dagger$: by (108) $L_{ij} = [x^\dagger x]/\alpha$, and also $1/\alpha$, which depends only on $P^\Lambda_\Lambda$, can be expressed itself as a polynomial in $x^\dagger$, as shown above.

4.f Equations (104)–(109) are equivalent under the whole group $O(D)$, including the inversion $x^\dagger \rightarrow -x^\dagger$ of one axis, or (e.g. parity), contrary to Madore’s and Hoppe’s FS.
The operator norm of $X^2$ equals its highest eigenvalue, $\|X^2\|_{\text{op}} = r_{\Lambda-1}^2 = 1 + [B + (\Lambda - 1)(\Lambda + D - 3)]/k$. For all $\psi \in \mathcal{H}_\Lambda$ and $i = 1, \ldots, D$, we find $\|\psi\|^2 = (\psi, X^2 \psi) \leq \sum_{j=1}^{D} (\psi, X \psi) = (\psi, X^2 \psi) \leq \|X^2\|_{\text{op}} \|\psi\|^2$, whence

$$\|X\|_{\text{op}} \leq \sqrt{1 + \frac{B + (\Lambda - 1)(\Lambda + D - 3)}{k}} \leq 1 + \epsilon, \quad \epsilon := \frac{B + (\Lambda - 1)(\Lambda + D - 3)}{2k}.$$  

(111)

5. Isomorphisms of $\mathcal{H}_\Lambda$, $\mathcal{A}_\Lambda$, and $*$-automorphisms of $\mathcal{A}_\Lambda$

**Theorem 5.1.** There exist an $O(D)$-module isomorphism $\varkappa_\Lambda : \mathcal{H}_\Lambda \rightarrow V_\Lambda^D$ and an $O(D)$-equivariant algebra map $\kappa_\Lambda : \mathcal{A}_\Lambda \rightarrow \pi_\Lambda^D[Uso(D)]$, $D \equiv D + 1$, such that

$$\varkappa_\Lambda (\psi) = \kappa_\Lambda (a) \varkappa_\Lambda \psi, \quad \forall \psi \in \mathcal{H}_\Lambda, \quad a \in \mathcal{A}_\Lambda. \quad (112)$$

On the $\psi_i^1 \ldots^k$ (spanning $\mathcal{H}_\Lambda$) and on generators $L_{\theta_i}X^i$ of $\mathcal{A}_\Lambda$ they respectively act as follows:

$$\varkappa_\Lambda (\psi_i^1 \ldots^k) := a_{\Lambda,i} F_{\Lambda,\Lambda}^{i_1 \ldots i_k} = a_{\Lambda,i} p_{\Lambda,i} T_{\Lambda}^{i_1 \ldots i_k}, \quad i = 0, 1, \ldots, \Lambda, \quad (113)$$

$$\kappa_\Lambda (L_{\theta_i}) := \pi_\Lambda^D (L_{\theta_i}), \quad \kappa_\Lambda (x^i) := \pi_\Lambda^D \left[ m_{\Lambda,1} (\lambda) X^i m_{\Lambda,1} (\lambda) \right], \quad (114)$$

where $X^i := L_{\theta_i}, p_{\Lambda,i} = p_{\Lambda,i} (P)$ are the polynomials (90), and

$$a_{\Lambda,i} = a_{\Lambda,0} \frac{\Gamma (\Lambda + 2i) \Gamma (\Lambda - 1 + 2i) \Gamma (\Lambda + 1 + 2i)}{\Gamma (\Lambda + D + 1) \Gamma (\Lambda + D - 2) \Gamma (\Lambda - 2 + 2i) \Gamma (\Lambda + D - 2 + 2i)} \sqrt{k}; \quad (115)$$

$$m_{\Lambda,1} (x) = \frac{\Gamma (\Lambda + 2i)}{\Gamma (\Lambda + D + 1) \Gamma (\Lambda + D - 2) \Gamma (\Lambda + 2i + D)} \sqrt{k}; \quad (116)$$

here $A := \sqrt{k + (D - 1)(D - 3)/4}$, and $\Gamma$ is Euler gamma function.

The proof is in appendix 'Proof of proposition 5.1'. The theorem extends propositions 3.2 and 4.2 of [22] to $d > 2$. The claims for $d > 2$ were partially formulated, but not proved, in [23].

As already recalled, the group of $*$-automorphisms of $\mathcal{A}_\Lambda \simeq M_N (\mathbb{C})$ is inner and isomorphic to $SU(N)$, i.e. of the type

$$a \mapsto g a g^{-1}, \quad a \in M_N (\mathbb{C}), \quad (117)$$

with $g$ an unitary $N \times N$ matrix with unit determinant. We can identify in $SU(N)$ a subgroup $\simeq SO(D)$, acting via the $N$-dimensional representation $(V_0^D, \pi_0^D)$; namely, it consists of matrices of the form $g = \pi_0^D (e^{i\alpha})$, where $\alpha \in \text{so}(D)$. Choosing $\alpha \in \text{so}(D) \subset \text{so}(D)$ the automorphism amounts to a $SO(D) \subset SO(D)$ transformation, i.e. a rotation in the $x \equiv (x^1, \ldots, x^D) \in \mathbb{R}^D$ space. $O(D) \subset SO(D)$ transformations with determinant $-1$ in this space keep the same form in the $X \equiv (x^1, \ldots, x^D)$ space (where $X \equiv L_{\theta_i}$) and by (114) also in the $X \equiv (x^1, \ldots, x^D)$ space. In particular, those inverting one or more axes of $\mathbb{R}^D$ (i.e. changing the sign of one or more $x^i$, and thus also of $X, \bar{X}$), e.g. parity, can be also realized as $SO(D)$ transformations, i.e. rotations in $\mathbb{R}^D$. This shows that (114) is equivariant under parity and the whole $O(D)$, which plays the role of isometry group of this FS.
6. Fuzzy spherical harmonics, and limit $\Lambda \rightarrow \infty$

In this section we suppress Einstein’s summation convention over repeated indices. The previous results allow to define $Uso(D)$-module Hilbert space isomorphisms

\[
\sigma_\Lambda : \text{Pol}_D^\Lambda = \bigoplus_{i=0} \mathcal{V}_D^i \rightarrow \mathcal{H}_\Lambda = \bigoplus_{i=0} \mathcal{H}_\Lambda^i, \quad \psi_{l_1\ldots l_{i-1}}^m \mapsto \phi_{l_1\ldots l_{i-1}}^m, \tag{118}
\]

where $\psi_{l_1\ldots l_{i-1}}^m := \sum \Lambda_{l_1\ldots l_{i-1}} \psi_{l_1\ldots l_{i-1}}^m$. The objects at the right of the arrows are our fuzzy analogs of the objects at the left. In the limit $\Lambda \rightarrow \infty$ the above decomposition of $\mathcal{H}_\Lambda$ into irreducible components under $O(D)$ becomes isomorphic to the decomposition of $\text{Pol}_D \simeq \mathcal{H}_\Sigma$.

We define the $O(D)$-equivariant embedding $\mathcal{I} : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_\Sigma$ by setting $\mathcal{I}(\psi_{l_1\ldots l_{i-1}}^m) := T_{l_1\ldots l_{i-1}}^\Lambda$ and applying linear extension. Below we drop $\mathcal{I}$ and identify $\psi_{l_1\ldots l_{i-1}}^m = T_{l_1\ldots l_{i-1}}^\Lambda$, or equivalently $\psi_{l_1\ldots l_{i-1}}^m \equiv Y_{l_1\ldots l_{i-1}}^m$, as elements of the Hilbert space $\mathcal{H}_\Lambda$. For all $\phi \equiv \sum_{i=0}^{\infty} \phi_{l_1\ldots l_{i-1}} T_{l_1\ldots l_{i-1}}^\Lambda \in \mathcal{L}^2(S^2)$ and $\Lambda \in \mathbb{N}$ let $\phi_{l_1\ldots l_{i-1}} := P_{\Lambda} \phi = \sum_{i=0}^{\Lambda} \phi_{l_1\ldots l_{i-1}} T_{l_1\ldots l_{i-1}}$ be its projection to $\mathcal{H}_\Lambda$ (or $\Lambda$th truncation). Clearly $\phi_{l_1\ldots l_{i-1}} \rightarrow \phi$ in the $\mathcal{H}_\Lambda$-norm $\| \cdot \|$. In this simplified notation, $\mathcal{H}_\Lambda$ ‘invades’ $\mathcal{H}_\Sigma$ as $\Lambda \rightarrow \infty$;

$\mathcal{I}$ induces the $O(D)$-equivariant embedding of operator algebras $\mathcal{J} : \mathcal{A}_\Lambda \rightarrow \mathcal{B}(\mathcal{H}_\Sigma)$ by setting $\mathcal{J}(a)\mathcal{I}(\psi) := \mathcal{I}(a\psi)$; here $B(\mathcal{H}_\Sigma)$ stands for the $\ast$-algebra of bounded operators on $\mathcal{H}_\Sigma$. By construction, $\mathcal{A}_\Lambda$ annihilates $\mathcal{H}_\Lambda$. In particular, $\mathcal{J}(L_{ab}) = L_{ab} \mathcal{P}_\Lambda$, and $L_{ab} \phi_{l_1\ldots l_{i-1}} \rightarrow L_{ab} \phi$ for all $\phi \in \mathcal{I}(\mathcal{H}_\Lambda)$ in the domain of $L_{ab}$. More generally, $f(L_{ab}) \rightarrow f(L_{ab})$ strongly on $D[f(L_{ab})] \subset \mathcal{H}_\Sigma$, for all measurable functions $f(s)$.

Continuous functions $f$ on $S^2$, acting as multiplication operators $f : \phi \in \mathcal{H}_\Sigma \mapsto f\phi \in \mathcal{H}_\Sigma$, make up a subalgebra $C(S^2)$ of $B(\mathcal{H}_\Sigma)$. Clearly, $f$ belongs also to $\mathcal{H}_\Sigma$. Since $\text{Pol}_D$ is dense in both $\mathcal{H}_\Sigma$, $C(S^2)$, $f_N$ converges to $f$ as $N \rightarrow \infty$ in both the $\mathcal{H}_\Sigma$ and the $C(S^2)$ norm.

We define the $\Lambda$th fuzzy analogs of the $T_{l_1\ldots l_{i-1}}^\Lambda$ (seen as an element of $C(S^2)$), i.e. acting by multiplication on $\psi \in \mathcal{H}_\Sigma$, $l \leq \Lambda$, by replacing $\mathcal{I}$ by $\mathcal{I}$ in the definition of $T_{l_1\ldots l_{i-1}}^\Lambda$, i.e. by

\[
\tilde{T}_{l_1\ldots l_{i-1}} := \mathcal{P}_{l_1\ldots l_{i-1}}^1 \mathcal{P}_{l_1\ldots l_{i-1}}^2 \ldots \mathcal{P}_{l_1\ldots l_{i-1}}^m,
\]

sum over repeated indices (cf proposition 3.5); the $\tilde{T}_{l_1\ldots l_{i-1}}$ fulfill again (66). Having identified $\psi_{l_1\ldots l_{i-1}}^m \equiv T_{l_1\ldots l_{i-1}}^\Lambda$, we rewrite (67) and (103) in the form

\[
T_{l_1\ldots l_{i-1}} = T_{l_1\ldots l_{i-1}}^0 + d_l \mathcal{P}_{l_1\ldots l_{i-1}}^l, \quad d_l := \frac{1}{D + 2l - 2}, \tag{120}
\]

\[
\tilde{T}_{l_1\ldots l_{i-1}} = c_{l_1+1} T_{l_1+1}^0 + c_l d_l \mathcal{P}_{l_1\ldots l_{i-1}}^lT_{l_1-1}^l. \tag{121}
\]

Using these formulae in appendix ‘Proof of theorem 6.1’ we prove the fuzzy version of proposition 3.7:

**Theorem 6.1.** The action of the $\tilde{T}_{l_1\ldots l_{i-1}}^m \equiv \mathcal{P}_{l_1\ldots l_{i-1}}^m \mathcal{F}_{l_1\ldots l_{i-1}}^m \mathcal{F}_{l_1\ldots l_{i-1}}^m$ on $\mathcal{H}_\Lambda$ is given by

\[
\tilde{T}_{l_1\ldots l_{i-1}}^m T_{l_1\ldots l_{i-1}}^n = \sum_{n,m} \mathcal{N}_{l_1\ldots l_{i-1}}^m \mathcal{P}_{a_1\ldots a_{l_1}}^{m_1\ldots m_{l_1}-a_1\ldots a_{l_1}} \mathcal{P}_{a_2\ldots a_{l_2}}^{m_2\ldots m_{l_2}} \ldots \mathcal{P}_{a_{l_1\ldots l_{i-1}+1}}^{m_{l_1\ldots l_{i-1}+1}} \mathcal{P}_{c_1\ldots c_n}^{m_{l_1\ldots l_{i-1}+1}} \mathcal{P}_{a_1\ldots a_{l_1}}^{m_{l_1\ldots l_{i-1}+1}}, \tag{122}
\]

with suitable coefficients $\mathcal{N}_{l_1\ldots l_{i-1}}^m$ related to their classical limits $N_{l_1\ldots l_{i-1}}^m > 0$ of formula (67) by

\[
\mathcal{N}_{l_1\ldots l_{i-1}}^m = 0 \quad \text{if } l - m > \Lambda, \quad N_{l_1\ldots l_{i-1}}^m \leq \mathcal{N}_{l_1\ldots l_{i-1}}^m \leq (c_\Lambda)^l \quad \text{otherwise}. \tag{123}
\]
As a fuzzy analog of the vector space $C(S^d)$ we adopt
\[
C_\Lambda := \left\{ \hat{f}_{2\Lambda} := \sum_{l=0}^{2\Lambda} \sum_{i_1, \ldots, i_l} f_{i_1, \ldots, i_l} \tilde{T}^{i_1, \ldots, i_l}_l \in C \right\} \subset \mathcal{A}_\Lambda \subset B(\mathcal{H}_x) \tag{124}
\]
here the highest $l$ is 2$\Lambda$ because by (123) the $\tilde{T}^{i_1, \ldots, i_l}_l$ annihilate $\mathcal{H}_\Lambda$ if $l > 2\Lambda$. By construction,
\[
C_\Lambda = \bigoplus_{l=0}^{2\Lambda} \mathcal{V}_D, \quad \mathcal{V}_D := \left\{ \sum_{i_1, \ldots, i_l} f_{i_1, \ldots, i_l} \tilde{T}^{i_1, \ldots, i_l}_l \in C \right\} \tag{125}
\]
is the decomposition of $C_\Lambda$ into irreducible components under $O(D)$. $\mathcal{V}_D$ is trace-free for all $l > 0$. In the limit $\Lambda \to \infty$ (125) becomes the decomposition of $C(S^d)$. As a fuzzy analog of $f \in C(S^d)$ we adopt the sum $\hat{f}_{2\Lambda}$ appearing in (124) with the coefficients of the expansion (70) of $f$ up to $l = 2\Lambda$. In appendix 6.2 we prove

**Theorem 6.2.** For all $f, g \in C(S^d)$ the following strong limits as $\Lambda \to \infty$ hold: $\hat{f}_{2\Lambda} \to f_\Lambda, (\hat{f}\hat{g})_{2\Lambda} \to fg$ and $\hat{f}_{2\Lambda} g_{2\Lambda} \to f g$.

The last statement says that the product in $\mathcal{A}_\Lambda$ of the approximations $\hat{f}_{2\Lambda}, \hat{g}_{2\Lambda}$ goes to the product in $B(\mathcal{H}_x)$ (the algebra of bounded operators on $\mathcal{H}_x \equiv L^2(S^d)$) of $f_\Lambda, g_\Lambda$. We point out that $\hat{f}_{2\Lambda}$ does not converge to $f$ in operator norm, because the operator $\hat{f}_{2\Lambda}$ (a polynomial in the $x^i$) annihilates $\mathcal{H}_\Lambda$ (the orthogonal complement of $\mathcal{H}_\Lambda$), so since do the $x^i = P^i A^i \Lambda$. Essentially the same claims of this theorem were proved for $d = 1, 2$ in [22] requiring that $k(\Lambda)$ diverges much faster than required by (20), and were formulated (without proof) for $d > 2$ in theorem 7.1 of [23] with the same strong assumptions on the divergence of $k(\Lambda)$.

### 7. Outlook, discussion and conclusions

In this paper we have completed our construction [22, 23] of FSs $S^d_\Lambda$ that be equivariant under the full orthogonal group $O(D)$, $D \equiv d + 1$, for all $d \in \mathbb{N}$. The construction procedure consists (sections 1, 2) in starting with a quantum particle in $\mathbb{R}^D$ configuration space subject to a $O(D)$-invariant potential $V(r)$ with a very sharp minimum on the sphere of radius $r = 1$ and projecting the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^D)$ to the subspace $\mathcal{H}_\mathcal{E}$ with energy below a suitable cutoff $\mathcal{E}$; $\mathcal{E}$ is sufficiently low to exclude all excited radial modes of $\mathcal{H}$ (this can be considered as a quantum version of the constraint $r = 1$), so that on $\mathcal{H}_\mathcal{E}$ the Hamiltonian essentially reduces to the square angular momentum $L^2$ (the Laplacian, i.e. the free Hamiltonian, over the sphere $S^d$). By making both the confining parameter $k \equiv V''(1)/4$ and $\mathcal{E}$ depend on $\Lambda \in \mathbb{N}$, and diverge with it, we have obtained a sequence $\{(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda)\}_{\Lambda \in \mathbb{N}}$ of $O(D)$-equivariant approximations of a quantum particle on $S^d$. $\mathcal{H}_\Lambda$ is the $\Lambda$th projected Hilbert space of states and $\mathcal{A}_\Lambda \equiv \text{End}(\mathcal{H}_\Lambda)$ is the associated $*$-algebra of observables. The projected Cartesian coordinates $\vec{x}$ no longer commute (section 4); their commutators $[\vec{x}, \vec{\tau}]$ are of Snyder type, i.e. proportional to the angular momentum components $\vec{L}_\xi$, $\mathcal{A}_\Lambda$ is spanned by ordered monomials (110) in $\vec{x}, \vec{L}_\xi$ (of appropriately bounded degrees), in the same way as the algebra $A_\Lambda$ of observables on $\mathcal{H}_\Lambda$ is spanned by ordered monomials in $\vec{x}, \vec{L}_\xi$. However, while $\vec{x}$ generate the whole $\mathcal{A}_\Lambda$, because $[\vec{x}, \vec{x}] \propto \vec{L}_\xi$, this has no analog $A_\xi$. The square distance $\vec{r}^2$ from the origin is not identically 1, but a function of $L^2$ such that its spectrum is very close to 1 and collapses to 1 as $\Lambda \to \infty$. We have also constructed (section 6) the subspace $C_\Lambda \subset A_\Lambda$ of completely symmetrized trace-free polynomials in the $\vec{x}$; this is also spanned by the fuzzy analogs of spherical harmonics (thought as multiplication operators on $\mathcal{H}_\Lambda$). $H_\Lambda, A_\Lambda, C_\Lambda$ carry reducible representations of $O(D)$; as $\Lambda \to \infty$ their decompositions into irreps respectively go to the decompositions of
\[ \mathcal{H}_x \equiv L^2(S^d), \text{ of } \mathcal{A}_x \text{ and of } C(S^d) \subset \mathcal{A}_x \text{ (the abelian subalgebra of continuous functions on } S^d \text{ acting as operators on } \mathcal{H}_x) \text{ [see (6), (125)]. There are natural embeddings } \mathcal{H}_\Lambda \rightarrow \mathcal{H}_x, \mathcal{C}_\Lambda \rightarrow C(S^d) \text{ and } \mathcal{A}_\Lambda \rightarrow \mathcal{A}_x \text{ such that } \mathcal{H}_\Lambda \rightarrow \mathcal{H}_x \text{ in the norm of } \mathcal{H}_x, \text{ while } \mathcal{C}_\Lambda \rightarrow C(S^d), \mathcal{A}_\Lambda \rightarrow \mathcal{A}_x \text{ strongly as } \Lambda \rightarrow \infty \text{ (section 6).}

A basis of \( \mathcal{A}_\Lambda \) consists of a suitable (\( \Lambda \)-dependent) subset \( S_\Lambda \) of ordered monomials (110). Since \( L_0 \psi_0 = 0 \) for all \( i, j \leq D \), the subset \( S'_\Lambda \) of \( S_\Lambda \) with all \( n_{ij} = 0 \) is a basis of \( \mathcal{C}_\Lambda \), and \( \mathcal{H}_\Lambda = \mathcal{C}_\Lambda \psi_0; \mathcal{H}_\Lambda, \mathcal{C}_\Lambda, \text{Pol}_D^2 \) carry the same reducible representation of \( O(D) \). As \( \Lambda \rightarrow \infty \): (i) \( S_\Lambda \) becomes a basis \( S \) of \( \mathcal{A}_\Lambda \) consisting of ordered monomials in \( \ell^0, L_0 \); (ii) \( S'_\Lambda \) becomes a basis \( S' \) of \( C(S^d) \) consisting of ordered monomials in \( \ell^0 \); (iii) \( S'_\Lambda \psi_0 \) becomes a basis of \( L^2(S^d) = C(S^d)\psi_0 \).

The structure of the pairs \( (\mathcal{H}_\Lambda, \mathcal{A}_\Lambda) \) is made transparent by the discovery (section 5) that these are isomorphic to \( \left( V^\Lambda_D, \pi_\Lambda(\text{Us}(D)) \right) \), \( D \equiv D + 1 \), also as \( O(D) \)-modules; \( \pi_\Lambda \) is the irrep of \( \text{Us}(D) \) on the space \( V^\Lambda_D \) of harmonic polynomials of degree \( \Lambda \) on \( \mathbb{R}^D \), restricted to \( S^0 \).

If we reintroduce \( h \) and the physical angular momentum components \( l_i := hL_i \), and we define as usual the quantum Poisson bracket as \( \{ f, g \} = \{ f, g \}/ih \), then in the \( h \rightarrow 0 \) limit \( \mathcal{A}_x \) goes to the (commutative) algebra \( \mathcal{F} \) of (polynomial) functions on the classical phase space \( T^*S^d \), which is generated by \( \ell, l_0 \). We can directly obtain \( \mathcal{F} \) from \( \mathcal{A}_\Lambda \) adopting a suitable \( \Lambda \)-dependent \( h \) going to zero as \( \Lambda \rightarrow \infty \).\(^6\) Using the isomorphism \( \mathcal{A}_\Lambda \simeq \pi_\Lambda^h(\text{Us}(D)) \), we now show that, more formally, we can see \( \{ \mathcal{A}_\Lambda \}_{\Lambda \in \mathbb{N}} \) as a fuzzy quantization of a coadjoint orbit of \( O(D) \) that goes to the classical phase space \( T^*S^d \).

We recall that given a Lie group \( G \), a coadjoint orbit \( \mathcal{O}_\lambda \), for \( \lambda \) in the dual space \( g^* \) of the Lie algebra \( g \) of \( G \), may be defined either extrinsically, as the actual orbit \( Ad^*_c \lambda \) of the coadjoint action \( Ad^*_c \) inside \( g^* \) passing through \( \lambda \), or intrinsically as the homogeneous space \( G/G_\lambda \), where \( G_\lambda \) is the stabilizer of \( \lambda \) with respect to the coadjoint action (this distinction is worth making since the embedding of the orbit may be complicated). Coadjoint orbits are naturally endowed with a symplectic structure arising from the Killing form; we can resp. rewrite these definitions in the form

\[
\mathcal{O}_\Lambda := \left\{ g \lambda g^{-1} \mid g \in G \right\} \subset g^*, \quad \mathcal{O}_\Lambda := G/G_\lambda \quad \text{where } G_\lambda := \left\{ g \in G \mid g \lambda g^{-1} = \lambda \right\}.
\]

(126)

Clearly, \( G_{\Lambda\Lambda} = G_\lambda \) for all \( \Lambda \in \mathbb{C} \setminus \{0\} \). Denoting as \( \mathcal{H}_\Lambda \) the (necessarily finite-dimensional) carrier space of the irrep with highest weight \( \lambda \), one can regard (see e.g. (27)) the sequence of \( \{ \Lambda_\Lambda \}_{\Lambda \in \mathbb{N}} \), with \( \Lambda_\Lambda := \text{End}(\mathcal{H}_\Lambda) \), as a fuzzy quantization of the symplectic space \( \mathcal{O}_\Lambda \simeq G/G_\lambda \). We recall that the Killing form \( B \) of \( so(D) \) gives \( B(L_{HK}, L_{JK}) = 2(D - 2) (\delta^H_J \delta^K_I - \delta^K_J \delta^H_I) \) for all \( H, J, K \in \{1, 2, \ldots, D \} \equiv D + 1 \). Let \( \sigma := [\frac{D}{2}] \equiv \text{rank of } so(D) \). As the basis of the Cartan subalgebra \( h \) of \( so(D) \) we choose \( \{ h_\sigma \}_{\sigma = 1} \), where

\[
H_\sigma := L_{DB}, \quad H_{\sigma - 1} := L_{(d - 1)D}, \quad \ldots, \quad H_1 = \begin{cases} L_{12} & \text{if } D = 2\sigma \\ L_{23} & \text{if } D = 2\sigma + 1 \end{cases}
\]

(127)

We choose the irrep of \( \text{Us}(D) \) on \( V^\Lambda_D \) as \( \mathcal{H}_\Lambda \); as the highest weight vector \( \Omega^\Lambda_D \in V^\Lambda_D \) we choose \( \Omega^\Lambda_D := (i^D + i^D)^{\Lambda} \). (for brevity we do not work down the associated partition of roots of \( so(D) \) into positive and negative). The associated weight in the chosen basis, i.e. the joint spectrum of \( H := (H_1, \ldots, H_\sigma) \), is \( \Lambda = (0, \ldots, 0, \Lambda) \). Identifying weights \( \lambda \in h^* \) with elements \( H_\Lambda \in h \) via the Killing form, we find that \( H_\Lambda \propto H_\sigma = L_{DB} \). The stabilizer of the latter in \( SO(D) \) is \( SO(2) \times SO(d) \), where \( SO(2), SO(d) \) have Lie algebra respectively spanned by \( L_{DB} \) and by the

\(^6\) More precisely, to obtain the classical Poisson brackets from (104)-(108) it suffices that \( h(\Lambda)k(\Lambda) \) keeps diverging; if e.g. \( k = N^2(\Lambda + D - 2)^2/4 \), then \( h(\Lambda) = O(\Lambda^{-\alpha}) \) with \( 0 < \alpha < 4 \) is enough. Setting \( f_i := h_\sigma \), in this limit \( \pi f_i \rightarrow f_i, l_0 \rightarrow f_i, l_0 \) respectively.
Let $L_j$ with $i, j < D$. Therefore the corresponding coadjoint orbit $SO(D)/\{SO(2) \times SO(d)\}$ has dimension

$$\frac{D(D+1)}{2} - 1 - \frac{(D-2)(D-1)}{2} = 2(D-1) = 2d,$$

which is also the dimension of $T^* S^d$, the cotangent space of the $d$-dimensional sphere $S^d$ (or phase space over $S^d$). This is consistent with the interpretation of $A_\Lambda$ as the algebra of observables (quantized phase space) on the FS. It would have not been the case if we had chosen some other generic irrep of $Uso(D)$: the coadjoint orbit would have been some other equivariant bundle over $S^d$ [27].

For instance, the four-dimensional FSs introduced in [16], as well as the ones of dimension $d \geq 3$ considered in [17–19], are based on $\text{End}(V^\Lambda)$, where the $V^\Lambda$ carry irreducible representations of both $\text{Spin}(D)$ and $\text{Spin}(D+1)$, and therefore of both $Uso(D)$ and $Uso(D)$. Then: (i) $i$ for some $\Lambda$ these may be only projective representations of $O(D)$; (ii) in general (106) will not be satisfied; (iii) as $\Lambda \to \infty$ $V^\Lambda$ does not go to $L^2(S^d)$ as a representation of $Uso(D)$, in contrast with our $\mathcal{H}_\Lambda \simeq V^\Lambda$.

The $X^i := L_{0i}$ play the role of fuzzy coordinates. As $x^2 \equiv XX'$ is central, it can be set $x^2 = 1$ identically. The commutation relations are also $O(D)$-covariant and Snyder-like, except for the case of the Madore–Hoppe FS [12, 13]. The corresponding coadjoint orbit for $d = 4$ is the 6-dimensional $CP^3$ [20, 21], which can be seen as a $so(5)$-equivariant $S^2$ bundle over $S^4$ (while [17] does not identify coadjoint orbits for generic $d$).

In [20, 21] the authors consider also constructing a fuzzy four-sphere $S^4$ through a reducible representation of $Uso(5)$ on a Hilbert space $V$ obtained decomposing an irrep $\pi$ of $Uso(6)$ characterized by a highest weight triple $(N, 0, n)$ with respect to $(H'_{\lambda_1}, H'_{\lambda_2}, H'_{\lambda_3})$, where

$$H'_{\lambda_i} := i \left( L_{34} + L_{12} + L_{56} \right), \quad H'_{\lambda_2} := i L_{56}, \quad H'_{\lambda_3} := i \left( L_{34} - L_{12} + L_{56} \right).$$

The $X^i \equiv L_{0i}$ ($i = 1, \ldots, 5$), which make up a basis of the vector space $so(6) \setminus so(5)$, still play the role of noncommuting Cartesian coordinates. If $n = 0$ then $x^2 \equiv 1$ ($V$ carries an irrep of $O(5)$), and one recovers the $so(5)$-equivariant $S^2$ bundle over the) fuzzy four-sphere of [16]. If $n > 0$, then the $O(5)$-scalar $x^2 \equiv XX'$ is no longer central, but its spectrum is still very close to 1 provided $N \gg n$, because then $V$ decomposes only in few irreducible $SO(5)$-components, all with eigenvalues of $x^2$ very close to 1. The associated coadjoint orbit is ten-dimensional and can be seen as a $so(5)$-equivariant $CP^3$ bundle over $CP^3$, or a $so(5)$-equivariant twisted bundle over either $S^6$ or $S^4$. On the contrary, with respect to $(H'_{\lambda_1}, H'_{\lambda_2}, H'_{\lambda_3})$ the highest weight triple of the irrep $V^\Lambda$ considered here is $(\Lambda, \Lambda, \Lambda)$; as said, $x^2 \equiv XX' \simeq 1$ is guaranteed by adopting as noncommutative Cartesian coordinates the $x^i = m_\Lambda(L^i) X^i m_\Lambda(L^i)$, with a suitable function $m_\Lambda$, rather than the $X^i$, and the associated coadjoint orbit has dimension 8, which is also the dimension of $T^* S^4$, as wished.

We now clarify in which sense we have provided a $O(D)$-equivariant fuzzy quantization of $T^* S^d$ and $S^d$—the phase space and the configuration space of our particle.

Although $A_\Lambda$ is generated by all the $\hat{t}^i, L_{ij}$ with $h \leq i, j \leq D$ (subject to the relations (10), (11) and (28), $\varepsilon_{i_1 j_1 \cdots i_9 j_9} L_{i_1 j_1} = 0$ due to (12), $t^R = \hat{t}^R$, and $C(S^d)$ is generated by the $t^R$ alone, the $\pi'$ (or the simpler generators $X'$) alone generate the whole $A_\Lambda \simeq \pi_0^R[Uso(D)]$, which contains $C_\Lambda$ as a proper subspace, but not as a subalgebra. Thus the Hilbert–Poincaré series of the algebra generated by the $\pi'$ (or $X'$), $A_\Lambda$, is larger than that of $Pol_2^R$ and $C_\Lambda$. If by a ‘quantized space’ we understand a noncommutative deformation of the algebra of functions

7 That the $\pi'$ do has been explained in the Remarks after proposition 4.1; that the $X' = L_{0i}$ do follows from (11), which implies $L_{ij} = i[X_i, X_j]$, and proposition 5.1.
on that space preserving the Hilbert–Poincaré series, then \( \{ A_\lambda \}_{\lambda \in \mathbb{N}} \) is a \((O(D))\)-equivariant, fuzzy) quantization of \( T^* S^d \), the phase space on \( S^d \), while \( \{ C_\lambda \}_{\lambda \in \mathbb{N}} \) is not a quantization of \( S^d \), nor are the other FSs, except the Madore–Hoppe fuzzy two-dimensional sphere: all the others, as ours, have the same Hilbert–Poincaré series of a suitable equivariant bundle on \( S^d \), i.e. a manifold with a dimension \( n > d \) (in our case, \( n = 2d \)). (Incidentally, in our opinion also for the Madore–Hoppe FS the most natural interpretation is of a quantized phase space, because the \( h \to 0 \) limit of the quantum Poisson bracket endows its algebra with a nontrivial Poisson structure.)

Therefore we understand \( \mathcal{H}_\lambda, C_\lambda \) as fuzzy ‘quantized’ \( S^d \) in the following weaker sense. \( \mathcal{H}_\lambda \) is the quantization of the space \( L^2(S^d) \) of square integrable functions, and the space \( C_\lambda \) of fuzzy spherical harmonics is the quantization of the space \( C(S^d) \) of continuous functions, seen as operators acting on the former, because the whole \( \mathcal{H}_\lambda \) is obtained applying to the ground state \( \psi_0 \) (or any other \( \psi \in \mathcal{H}_\lambda \)) the polynomials in the \( \mathfrak{t}^* \) alone, or equivalently (by proposition 5.1) the polynomials in the \( \mathfrak{t}^* = L^*_B \) alone, or the space \( C_\lambda \), in the same way as the Hilbert \( L^2(S^d) \) is obtained (modulo completion) by applying \( C(S^d) \) or \( \text{Pol}_D \), i.e. the polynomials in the \( \mathfrak{t}^* = x^*/r \), to the ground state, i.e. the constant function on \( S^d \). These quantizations are \( O(D) \)-equivariant because \( \mathcal{H}_\lambda, C_\lambda \) not only have the same dimension, but carry also the same reducible representation of \( O(D) \), that of the space (and commutative algebra) \( \text{Pol}_D \) of polynomials of degree \( \Lambda \) in the \( \mathfrak{t}^* = x^*/r \). Identifying \( \mathcal{H}_\lambda, C_\lambda \) with \( \text{Pol}_D \) as \( O(D) \)-modules, in the \( \Lambda \to \infty \) the latter becomes dense in both \( C(S^d) \), \( L^2(S^d) \), and its decomposition into irreps of \( O(D) \) becomes that (2) of \( C(S^d) \), \( L^2(S^d) \). This is not the case for the other FSs.

Many aspects of these new FSs deserve investigations: e.g. space uncertainties, optimally localized states and coherent states also for \( d > 2 \), as done in [24, 28, 29] for \( d = 1, 2 \); a distance between optimally localized states (as done e.g. in [30] for the FS); extending the construction to particles with spin\(^8\); QFT based on our FSs; application of our FSs to problems in QG, or condensed matter physics; etc. It would be also interesting to investigate whether our procedure can be applied (or generalized) to other symmetric compact submanifolds\(^9\) \( S \subset \mathbb{R}^D \) that are level sets of smooth or polynomial function(s) \( \rho(x) \).

Finally, we point out that a different approach to the construction of noncommutative submanifolds of noncommutative \( \mathbb{R}^D \), equivariant with respect to a ‘quantum group’ (twisted Hopf algebra) has been proposed in [31, 32]; it is based on a systematic use of Drinfel’d twists.

**Data availability statement**

No new data were created or analysed in this study.

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\(^8\) This should be possible adopting as the starting Hilbert space \( \mathcal{H} \simeq L^2(\mathbb{R}^D) \otimes \mathbb{C}^n, n := 2[\frac{D}{2}] \).

\(^9\) By (4), if \( S \) is not compact then the corresponding Hilbert spaces \( \mathcal{H}_\lambda \) will have infinite dimension, and therefore will not lead to fuzzy spaces in the sense given in the Introduction.
Appendix

Proof of proposition 3.2

Our Ansatz is (54) with $M \equiv M(l+1)$ a $O(D)$-invariant matrix to be determined. The most general one is

$$M(l+1) = \alpha_{l+1} (1 + \beta_{l+1} \sigma + \gamma_{l+1} \sigma^5).$$

We first determine the coefficients $\beta_{l+1}, \gamma_{l+1}$ by imposing the conditions (50). By the recursive assumption, only the condition with $m = l$ is not fulfilled automatically and must be imposed by hand. Actually, it suffices to impose just (50a), due to the symmetry of the Ansatz (54) and of the matrices $P'$ under transposition. Abbreviating $M' \equiv M(l)$ this amounts to

$$0 \equiv P^{l+1} P'_{l(l+1)} (54) = P^l_{l+1} M(l+1) P^l_{l+1} P'_{l(l+1)}$$

$$= P^l_{l+1} M(l+1) P^l_{l+1} M(l+1) P^l_{l+1} P'_{l(l+1)} = P^l_{l+1} M(l+1) M(l+1) P^l_{l+1} P'_{l(l+1)}$$

$$= P^l_{l+1} \left[ \left( D^{l+1} + \beta_{l+1} P_{l+1} + \gamma_{l+1} P'_{l+1} \right) \left( 1 + \frac{1}{\beta_{l+1} + \beta_{l+1} D_{l+1} + \gamma_{l+1} D'_{l+1}} \right) \right] P'_{l(l+1)}$$

$$= P^l_{l+1} \left[ 2 P'_{l(l+1)} - \left( 1 + \frac{1}{\beta_{l+1} + \beta_{l+1} D_{l+1} + \gamma_{l+1} D'_{l+1}} \right) \right] P'_{l(l+1)}$$

$$+ \beta_{l+1} D_{l+1} P'_{l(l+1)} + \gamma_{l+1} D'_{l+1} P'_{l(l+1)}$$

where we have used also the relations

$$P P' = \left( P' + P' \right) P' = \left( P' - P' \right) P' = \left[ 2 P' - 1 \right] P'$$

$$P'_{l(l+1)} P'_{l(l+1)} \equiv \left[ 2 P'_{l(l+1)} - 1 \right] P'_{l(l+1)}$$

The conditions that the three square brackets vanish

$$1 + \beta_{l} + \beta_{l+1} = 0,$$

$$2 \beta_{l+1} + \gamma_{l+1} + \beta_{l+1} \frac{2}{D} = 0,$$

$$\gamma_{l+1} - \beta_{l+1} \gamma_{l+1} D + \gamma_{l+1} D' = 0,$$
are recursively solved, starting from \( l = 1 \) with initial input \( \beta_1 = 0 = \gamma_1 \) (since \( P^1 = 1_D \)), by

\[
\beta_{l+1} = l, \quad \gamma_{l+1} = -\frac{2Dl}{D + 2I - 2}.
\]

We determine the coefficient \( \alpha_{l+1} \) by imposing the condition (51). This gives

\[
0 \overset{!}{=} P^{l+1} \left( P^{l+1} - 1_D^{l+1} \right) \overset{(54)}{=} P^{l+1} \left( P^{l+1}_{1, ...} - 1_D^{l+1} \right)
\]

\[
\overset{(53),(56)}{=} P^{l+1} \left[ \alpha_{l+1} (1 + \beta_{l+1}) P^{l+1}_{1, ...} - 1 \right] \overset{(53)}{=} P^{l+1} \left[ \alpha_{l+1} (1 + \beta_{l+1}) - 1 \right].
\]

The condition that the square bracket vanishes is recursively solved, starting from \( l = 0 \) with initial input \( \alpha_0 = 1 \), by \( \alpha_{l+1} = 1/(l+1) \). This makes (128) into (56) [yielding back (42) if \( l = 2 \). We have thus proved that the Ansatz (54) fulfills (50) and (51). Similarly one proves that also the Ansatz (55) does the same job.

**Proof of proposition 3.3**

For \( l = 1 \) it is \( \zeta_2 = 1, P^1 = 1_D, P^2 = P \), and the claim (61) is true by (7) and (42). We now show that (61) \( \alpha_{l+1} \) implies (61):

\[
(P^{l+1}_{1, ...} - P^{l+1}) x_1 ... x_{l+1}
\]

\[
\overset{(54)}{=} P^{l+1}_{1, ...} \left[ \frac{1}{l+1} (\frac{1}{l+1} + P_{1}(l+1) + \gamma_{l+1} P_{1}(l+1)) \right] x_1 ... x_{l+1}
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \frac{1}{l+1} - \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
\overset{(51)}{=} \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

\[
= \frac{1}{l+1} P^{l+1}_{1, ...} \left[ \left( \gamma_{l+1} P_{1}(l+1) \right) \left( P_{l+1}^{l+1} - P_{l+1}^{l+1} + P_{l+1}^{l+1} \right) x_1 ... x_{l+1} \right]
\]

namely the left equality in (61) is fulfilled if the \( \zeta_l \) satisfy the recursion relation

\[
\zeta_{l+1} = \frac{1}{l+1} \left[ -\gamma_{l+1} + \frac{\zeta_l}{D} \frac{P_{1}(l+1)}{\gamma_{l+1}} \right] = \frac{l}{(l+1)(D+2l-2)} \left[ 2D + \zeta_l(D+2l-4) \right],
\]

which setting \( \zeta_2 = 1 \) is actually solved by \( \zeta_l = \frac{D(l-1)}{(D+2l-2)} = -\frac{l}{2} \gamma_l \), as claimed.
Proof of proposition 3.4

Using proposition 3.3 we easily find

\[
\text{id}_{\mathfrak{h}_{k}} (\chi^{i_1,...,i_l}) = x^i \partial^i - x^i \partial^i \mathfrak{p}^j_{i_1} \mathfrak{p}^j_{i_2} \cdots \mathfrak{p}^j_{i_l} = (x^i g^{jkl} - x^j g^{ihi}) \mathfrak{p}^{jhi} \mathfrak{p}^{jhi} \cdots \mathfrak{p}^{jhi} = i \mathfrak{D} \left( \chi^{i_1,...,i_l} \right) = \frac{\mathfrak{D} \left( \chi^{i_1,...,i_l} \right)}{\mathfrak{D} \left( \chi^{i_1,...,i_l} \right)} \mathfrak{g}^{jhi} \mathfrak{g}^{jhi} \cdots \mathfrak{g}^{jhi}.
\]

Alternatively, from (133) we obtain also (64), because

\[
i \mathfrak{h}_{k} (\chi^{i_1,...,i_l}) = (x^i g^{jkl} - x^j g^{ihi}) \mathfrak{p}^{jhi} \mathfrak{p}^{jhi} \cdots \mathfrak{p}^{jhi} = \frac{\mathfrak{D} \mathfrak{g}^{jhi} \mathfrak{g}^{jhi} \cdots \mathfrak{g}^{jhi}}{\mathfrak{D} \mathfrak{g}^{jhi} \mathfrak{g}^{jhi} \cdots \mathfrak{g}^{jhi}} = (h \leftrightarrow k).
\]

Multiplying the previous relations by \(1/r^2\), which commutes with \(L_{\mathfrak{h}_k}\), (64) give (69).

Proof of proposition 3.6 and theorem 3.7

The right-hand side (rhs) of (73) is the sum of \(N!\) terms; in particular, \(G^i_2 = \delta^i + \delta^i = 2\delta^i.\)

\(G_{\pi} G_{\pi-2}\) are related by the recursive relation

\[G_{\pi} \cdots \pi = (\delta \Delta) G_{\pi-2} \cdots \pi + (\delta \Delta) G_{\pi-2} \cdots \pi + \cdots + (\delta \Delta) G_{\pi-2} \cdots \pi.\]

The rhs is the sum of \((\overline{N} - 1)\) products \(\delta \pi \pi G_{\pi-2} \). The ‘trace’ of \(G_{\pi}\) equals \(\text{tr}(G_2) = 2D\) for \(\overline{N} = 2\) and by (134) fulfills the recursive relation \(\text{tr}(G_{\pi}) = \overline{N}(D + \overline{N} - 2)\text{tr}(G_{\pi-2})\). In fact, each of the \(\overline{N}\) products \(\delta \pi G_{\pi-2} \pi, \delta \pi G_{\pi-2} \pi, \delta \pi G_{\pi-2} \pi, \cdots, \delta \pi G_{\pi-2} \pi\) contributes by \(2D\) to \(G_{\pi-2}\), while each of the \((\overline{N} - 2)\) remaining ones contributes by \(\text{tr}(G_{\pi-2})\).

The recursion relation is solved by (74). The integral over \(S^l\) of \(H^l_{\pi} = : t_1, \ldots, t_l \pi \pi \pi \pi\) is \(O(D)^{-l}\)-invariant and therefore must be proportional to \(G^l_{\pi} \pi \pi \pi \pi\); the proportionality coefficient \(C_{\pi}\) is found by consistency contracting both sides with \(\delta_{\pi_1 \pi} \delta_{\pi_2 \pi} \cdots \delta_{\pi_{l-1} \pi} \pi\), and using (28) and (74).

The scalar product of \(\phi^j, \psi^i \in V_{\pi} \subset \mathfrak{h}_i\) is given by

\[
\langle \phi^j, \psi^i \rangle = \int_{\mathfrak{h}_i} d\alpha \phi^j \psi^i = \left( \phi^j_{l_1 \cdots l_l} \right)^* \psi_{j_1 \cdots j_l} \int_{\mathfrak{h}_i} d\alpha t_1 \cdots t_l \psi^j \phi^j \cdots \phi^j = C_{2l} \left( \phi^j_{l_1 \cdots l_l} \right)^* \psi_{j_1 \cdots j_l} G_{2l}^{j_1 j_2 \cdots j_l}.
\]

The sum has \((2l)!\) terms. In fact, all terms where both indices of at least one Kronecker \(\delta\) contained in \(G_{2l}\) are contracted with the two indices of the coefficients \(\phi^j_{l_1 \cdots l_l}\), or of the \(\psi^j_{l_1 \cdots l_l}\), vanish, by (71). The remaining \((2l)!\) terms arise from the \((2l)!\) products contained in \(G_{2l}\) of the type \(\delta^\pi \pi \pi \pi \ldots, \delta^\pi \pi \pi \pi \ldots, \delta^\pi \pi \pi \pi \ldots, \delta^\pi \pi \pi \pi \ldots\), where \(\pi, \pi' \) are permutations of \((1,...,l)\), and the other ones which are obtained exchanging the order of the indices \(i_{\pi(k)} j_{\pi(k)} \rightarrow j_{\pi'(k)} i_{\pi(k)}\) in one or more of these Kronecker \(\delta\)’s; they are all equal, again by (71). Hence,
\[ \langle \phi^j, \psi^i \rangle = Q_i(\phi^j_{i_1 \ldots i_j})^* \psi^i_{i_1 \ldots i_j}. \]  

By the orthogonality \( V_D^f \perp V_D^{l'} \) for \( f \neq l' \) we find that the scalar product of generic \( \phi, \psi \subset H \) is given by (76). This concludes the proof of proposition 3.6.

Applying \( m \) times (67) and absorbing into a suitable combination of \( T_n \)’s \( (n \in \mathbb{N}) \) the \( m \)-degree monomials in \( t \) whose combination gives \( T_n^{k_1 \ldots k_n} \), we find that (67) must hold with suitable coefficients \( S_{k_1 \ldots k_n} \in \mathbb{R} \). We determine these coefficients faster using (67) as an Ansatz, making its scalar product with \( T_n^{l_1 \ldots l_n} \) and using (77). One finds

\[ Q_n S_{k_1 \ldots k_n} = \left\langle T_n^{k_1 \ldots k_n}, T_n^{l_1 \ldots l_n} \right\rangle = P_{n_1 \ldots n_m} \int d\phi^i_{a_1 \ldots a_n} P^n_{i_1 \ldots i_m} \phi^a_{c_1 \ldots c_n} \delta^s(a_1 \ldots a_n), \]

where \( N = l + m = n + r = 2s \) is even. Due to the form of \( G_{\pi} \), the sum has \( \mathcal{N} \) terms, each containing a product of \( s \) Kronecker \( \delta \)’s. All terms where both indices of some Kronecker \( \delta \) contained in \( G_{\pi} \) are contracted with two indices of \( P^i \), or \( P^m \), or \( P^n \), vanish by (52).

As a warm-up, consider first the case \( n = l + m \). Renaming for convenience \( b_1, \ldots, b_m \) as \( a_{l+1}, \ldots, a_n \), the remaining \( 2^n(n!)^2 \) terms arise from the \( (n!)^2 \) products of the type

\[ \delta^s(a_1 \ldots a_n) \delta^s(a_{l+1} \ldots a_n), \]

contained in \( G_{\pi} \), where \( \pi, \pi' \) are permutations of \( (1, \ldots, h) \), and the other ones which are obtained exchanging the order of the indices \( e_{\pi(b)} a_{\pi(c)} \rightarrow a_{\pi(b)} e_{\pi(c)} \), in one or more of these \( n \) Kronecker \( \delta \)’s; by the complete symmetry of \( P^m \), they are all equal to the term where \( \pi, \pi' \) are the trivial permutations; correspondingly, the product is \( \delta^s(a_1 \ldots a_n) \). Hence,

\[ Q_n S_{k_1 \ldots k_n} = C_{2s} 2^n(n!)^2 \int d\phi_{a_1 \ldots a_n} P^n_{i_1 \ldots i_m} \phi^a_{c_1 \ldots c_n} \delta^s(a_1 \ldots a_n) = C_{2s} 2^n(n!)^2 \int d\phi_{a_1 \ldots a_n} P^n_{i_1 \ldots i_m} \phi^a_{c_1 \ldots c_n}, \]

implying \( S_{k_1 \ldots k_n} = P^n_{i_1 \ldots i_m} \), i.e. the term of highest rank at the rhs (67) is \( T_n^{k_1 \ldots k_n} \).

This is consistent with the first term in the (even iterated) application of (67).

For generic \( n \in \mathbb{N} \), (136) becomes

\[ Q_n S_{k_1 \ldots k_n} = C_{2s} F_n P^n_{i_1 \ldots i_m} \phi^{a_{l+1} \ldots a_n} \phi_{c_1 \ldots c_n} \delta^s(a_1 \ldots a_n), \]

where \( r := \frac{l+m-n}{2}, s = \frac{l+m+n}{2}, F_n = \frac{2^n(n!)^2}{(l-r)!(m-r)!(n-r)!} = \frac{2^n(n!)^2}{(l-r)!(m-r)!(n-r)!} \). In fact, one of the products of Kronecker \( \delta \)’s contained in \( G_{\pi} \) and yielding a nonzero contribution is displayed in the second line. The number \( F_n^{l,m} \) of such products can be determined as follows, starting from the same product with all indices removed: there are \( l \) ways to pick out the first index from the set \( \{a_1, \ldots, a_l\} \), \( m \) ways to pick out the second index from the set \( \{b_1, \ldots, b_m\} \), hence \( lm \) ways to pick out the first index from \( A_l \) and the second from \( B_m \); similarly, there are \( lm \) ways to pick out the second index from \( A_l \) and the first from \( B_m \); altogether, there are \( 2lm \) ways to pick out one of the first two indices from \( A_l \) and the other one from \( B_m \). After anyone of these corresponding sets of indices \( A_{l-1}, B_{m-1} \) will have \( l-1, m-1 \) indices respectively; therefore there are \( 2(l-1)(m-1) \) ways to pick out one among the third, fourth indices from \( A_{l-1} \) and the other from \( B_{m-1} \). And so on. Therefore there are \( \frac{2^n(n!)^2}{2^{l+m-n}} \) ways to pick out \( r \) indices appearing in the first \( r \) \( \delta \)’s (one for each \( \delta \)) from \( A_l \) and the \( r \) remaining ones from \( B_m \). After anyone of these choices the corresponding sets of indices \( A_{l-r}, B_{m-r} \) will have \( l-r, m-r \) indices respectively, and \( D_n := A_{l-r} \cup B_{m-r} \) will have \( l-r + m-r = n \) indices. Reasoning as in the case \( n = l+m \), we find that there are \( 2^n(n!)^2 \) ways to pick out \( n \) indices appearing in the remaining \( n \) \( \delta \)’s (one for each \( \delta \)) from \( D_n \) and the remaining \( h \) ones

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from \( C_n := \{c_1, \ldots, c_n\} \). Consequently, so far there are \( \frac{2^{n+r}n!m^r}{(r-1)!m^r} (n!)^2 \) ways to do these operations. The remaining ways are obtained allowing that the \( r \) pair of indices picked one out of \( A_r \) and the other out of \( B_m \) appear not necessarily in the first \( r \) \( \delta \)'s, but in any subgroup of \( r \) \( \delta \)'s out of the totality of \( s \); hence we have to multiply the previous number by the number \( \frac{1}{(r-1)!} \) of such subgroups, and we finally obtain \( F_{n,m} = \frac{2^{n+r}n!m^r}{(r-1)!m^r} (n!)^2 \) (because \( n + r = s \), as claimed. Thus (137) implies the following relation, which gives (80):

\[
S_{k_1 \ldots k_n} = \frac{(D + 2n - 2)!!}{(D + 2s - 2)!!} n! (2s)!! \frac{2^s s! l! m!}{(I - r)! (m - r)!} \prod_{b_i \neq b_j} \prod_{c_i \neq c_j} \prod_{c_i \neq b_j} \prod_{b_i \neq c_j} \prod_{b_i \neq b_j} \prod_{c_i \neq c_j} \frac{\bar{a} \bar{b} \bar{c} \ldots \bar{a} \bar{b} \bar{c}}{\bar{a} \bar{b} \bar{c} \ldots \bar{a} \bar{b} \bar{c}},
\]

**Proof of proposition 3.8**

Since \( L_{ab} \) commute with scalars, and \([L_{ab}, X^D_i] = 0\), \( p_{\Lambda,l} \) commutes with all the \( L_{ab} \) and therefore also with \( L^2 \). Hence, using (59), we find \( L^2 \bar{p}_{\Lambda,l} X_{i_1^{\Lambda l}} = \bar{p}_{\Lambda,l} L^2 X_{i_1^{\Lambda l}} = E_i \bar{p}_{\Lambda,l} X_{i_1^{\Lambda l}} \), i.e (86). To compute the coefficients \( \bar{b}_{\Lambda,l+2} \) we preliminarily note that

\[
\Delta(x)_{b} = (x)^b \Delta + 2h(x)^b - 1 \partial - 1 h^2 - 2, \\
\Delta(\bar{x}) = \bar{r}^b \Delta + h^b - 2(2h + D + h - 2), \\
\Delta D(x)^h = (x)^h \Delta + 2h(x)^h - 1 \partial - 1 h^2 - 2, \\
\Delta D(\bar{x}) = \bar{r}^b \Delta D + h^b - 2(2h + D + h - 2),
\]

\[
\partial b_{\Lambda,l} = x^l \left( \frac{D}{2} \right)^{l-2} b_{\Lambda,l+2} + 4 \left( x^D \right)^{l-4} \left( r^b \bar{b}_{\Lambda,l+4} \right) + \ldots
\]

\[
\Delta D p_{\Lambda,l} = (\Lambda - l)(\Lambda - l - 1) \left( x^D \right)^{l-2} + \left[ \frac{r^b D^2 + 2(2h + D)}{r^b D + 2} \right] \left( x^D \right)^{l-4} b_{\Lambda,l+2} + \ldots
\]

\[
= (\Lambda - l)(\Lambda - l - 1) \left( x^D \right)^{l-2} + \left[ \frac{r^b D^2 + 2(2h + D)}{r^b D + 2} \right] \left( x^D \right)^{l-4} b_{\Lambda,l+2} + \ldots
\]

We now impose that the \( F_{l,\Lambda,l}^{b_{\Lambda,l}} \) are harmonic in dimension \( D \). By a direct calculation

\[
0 = \Delta D F_{l,\Lambda,l}^{b_{\Lambda,l}} = \Delta D p_{\Lambda,l} + 2 \left( \left( x^D \right)^{l-2} b_{\Lambda,l+2} + 4 \left( x^D \right)^{l-4} r^b \bar{b}_{\Lambda,l+4} \right)
\]

\[
= \left( x^D \right)^{l-2} \left( \Lambda - l \right)(\Lambda - l - 1) + 2b_{\Lambda,l+2} \left( \frac{2 \left( \Lambda - l - 2 \right) + D}{2} \right) + \ldots
\]

\[
= \left( x^D \right)^{l-4} \left( \Lambda - l \right)(\Lambda - l - 1) + 2 \left( \Lambda - l - 2 \right) + D \right) + \ldots
\]

the vanishing of the coefficient of each monomial \( (x^D)^{l-2-2h} r^b \) implies
\[ b_{\Lambda,j+2} = -\frac{(\Lambda - l)(\Lambda - l - 1)}{2(2\Lambda - 4 + D)}, \]
\[ b_{\Lambda,j+4} = -\frac{(\Lambda - l - 2)(\Lambda - l - 3)}{4(2\Lambda - 6 + D)}, \]
\[ b_{\Lambda,j+2} = \frac{(\Lambda - l)(\Lambda - l - 1)(\Lambda - l - 2)(\Lambda - l - 3)}{2 \cdot 4(2\Lambda - 4 + D)(2\Lambda - 6 + D)}, \]
\[ \ldots. \]

namely, more compactly, we obtain (87).

The \( F_{\Lambda,l}^{j-i} \) transform as the \( X_{li}^{j-i} \) under the action of the \( L_{\partial x} \), because the latter commute with \( \partial_{\Lambda,j} \). Using (7) and (83), and the fact that \( \partial_{\Sigma} \) annihilates all polynomials in the \( x^i \), we find

\[ i\lambda_{\Sigma\Lambda} F_{\Lambda,l}^{j-i} \]
\[ = (x^b \partial_x - x^b \partial_x) \partial_{\Lambda,j} X_{li}^{j-i} \]
\[ = \left\{ x^b (x^b) \Lambda - l - 1 (\Lambda - l) + x^b (x^b) \Lambda - l - 2 \partial_{\Sigma} - (x^b) \Lambda - l - 1 \partial_{\Sigma} \right\} + \]
\[ + b_{\Lambda,j+2} \partial_{\Sigma} \left[ x^b (x^b) \Lambda - l - 1 (\Lambda - l) + x^b (x^b) \Lambda - l - 2 \partial_{\Sigma} - (x^b) \Lambda - l - 1 \partial_{\Sigma} \right] \]
\[ + b_{\Lambda,j+4} \partial_{\Sigma} \left[ x^b (x^b) \Lambda - l - 1 (\Lambda - l) + x^b (x^b) \Lambda - l - 2 \partial_{\Sigma} - (x^b) \Lambda - l - 1 \partial_{\Sigma} \right] \]
\[ = x^b X_{li}^{j-i} \tilde{N}_{\Lambda,j+1} - l x^b \partial_{\Lambda,j} X_{li}^{j-i} \]
\[ = \tilde{N}_{\Lambda,j+1} = 8(2\Lambda - 4 + D)(2\Lambda - 6 + D) \partial_{\Sigma} (x^b) \Lambda - l - 3 \]
\[ = (\Lambda - l)b_{\Lambda,j+1} \]

(138)

(here \( l' \equiv \Lambda - l \)) and \( \tilde{Q}_{\Lambda,j-1} \) is the homogeneous polynomial of degree \( l' + 1 \) in \( x^b, \partial_x \)

\[ \tilde{Q}_{\Lambda,j-1} := l_x \partial_{\Lambda,j} - \frac{q_{j+1}}{D} \partial_{\Sigma} \tilde{N}_{\Lambda,j+1} = l_x \partial_{\Lambda,j} - \frac{l}{D + 2 + 2} \left[ \frac{q_{j}}{D} - (x^b)^2 \right] \tilde{N}_{\Lambda,j+1} \]
\[ = (x^b)^{l' + 1} \left[ \frac{l' + 1}{D + 2 + 2} \right] - \frac{q_{j}}{D} (x^b)^{l' - 1} l' \left[ \frac{1}{D + 2 + 2} \left[ \frac{l' + 1}{D + 2} \right] + \frac{l' - 1}{2(2\Lambda - 4 + D)} \right] \]
\[ + \frac{q_{j}}{D} (x^b)^{l' - 1} l' \left( \frac{l' + 1}{D + 2 + 2} \right) \left[ \frac{1}{D + 2 + 2} \left[ \frac{l' + 1}{D + 2} \right] + \frac{l' - 3}{2(2\Lambda - 4 + D)} \right] \]
\[ = l(\Lambda + l + D - 2) \partial_{\Sigma} b_{\Lambda,j-1}. \]

(139)

(equalities (138) and (139) are proved by direct calculations), whence (88).

**Evaluating a class of radial integrals, and proof of (103)**

Given a smooth \( h(r) \) not depending on \( k \), formula (98) of [22] gives the (asymptotic expansion of) the radial integral of its product with \( g_\ell(r) g_L(r) \) \((\ell, L \in \mathbb{N}_0)\) at lowest order in \( 1/k \):

\[ \int_0^{\infty} g_L(r) g_\ell(r) h(r) dr = e^{\sqrt{\Lambda^2 + k^2} S_{\Lambda,k}} \sum_{\ell=0}^{\infty} \frac{h(2\ell) (\tilde{p}_{\ell,L})}{(2\ell)!! (\sqrt{k^2 + \sqrt{\Lambda^2}})} \]

(140)
where \( \tilde{\gamma}_j, k_l \) were defined in (17), while \( \tilde{\gamma}_{j,l} := \frac{\sqrt{\xi_j} + \sqrt{\xi_{k_l}}}{\sqrt{\xi_j} + \sqrt{\xi_{k_l}}}, h^{(n)}(r) \equiv d^n h/dr^n. \) Up to \( O(k^{-3/2}) \) the exponential in (140) is 1, because by explicit computation

\[
\sqrt{\xi_j} = \sqrt{2} \left( 1 + \frac{3b_L(x)}{4k} \right) + O(k^{-1}), \quad \tilde{\gamma}_j = \frac{b_L(x)}{3b_L(x) + 2k} - \frac{b_L(x)}{3b_L(x) + 2k} = O(k^{-1}),
\]

\[
\frac{\sqrt{\xi_j} - \tilde{\gamma}_j}{2} = 1 + \frac{b_L(x) + b_L(x)}{4k} + O(k^{-1}), \quad \frac{\sqrt{\xi_j} - \tilde{\gamma}_j}{2} = O(k^{-1}).
\]

By (27), (67), applying \( P_A = \sum_{l=0}^A P_A \) to \( x^j \psi^{l \cdots k} = i^l T^{l \cdots k}_j(r) \) we find

\[
P_A (x^j \psi^{l \cdots k}_j) = \tilde{P}_{l+1} \big[ T^{l \cdots k}_{j+1}(r) \big] + \tilde{P}_{l-1} \bigg[ \frac{\gamma_l + 1}{D} g^l \frac{\partial T^{l \cdots k}_j}{\partial y_{j \cdots k}} T^{l \cdots k}_j \bigg] f_l(r)
\]

The last integral is of the type (140) with \( L = l - 1 \) and \( h(r) = r. \) By explicit computation,

\[
\rho_{l-1,l} := \int_0^r dr f_{l-1}(r) f_l(r) = \int_0^r dr g_{l-1}(r) g_l(r) = \int_0^r dr g_l(r) g_l(r) = O(k^{-1})
\]

By setting the \( O(k^{-3/2}) \) term equal to zero we finally arrive at (102) and (103).

**Proof of proposition 4.1**

The equations in (104)–(106) follow from (10)–(12) and \([L_{ij}, P_A] = 0 \).

By (34–35), \( \psi^{l \cdots k}_m = L \psi^{l \cdots k}_m \) are eigenvectors with the highest and lowest eigenvalues, \( \pm A. \) Given a basis \( \{ \psi_m \} \) of eigenvectors of \( L_{mk} \), \( \psi^{l \cdots k}_m = m \psi^{l \cdots k}_m \) with \( m \in \{ -A, 1 - A, \ldots, A \} \) (\( a \) are some extra labels), by (33) we find that for all \( m, A(x) \psi^{l \cdots k}_m \) is either zero or an eigenvector of \( L_{mk} \) with eigenvalue \( j + m \), which must be \( \leq A \). Similarly, \( (L_{ij} + i L_{ij}) \psi^{l \cdots k}_m \) is either zero or an eigenvector of \( L_{mk} \) with eigenvalue \( j + m \), which must be \( \leq A \). Therefore for \( j > 2A \) such vectors must be zero, and we obtain (107).
Applying (103) twice we find
\[
\tilde{x}^2 \psi_{ij}^{l_1 \ldots l_i} = \tilde{x} \left( c_{i+1} \psi_{i+1}^{l_1 \ldots l_i} + c_i \frac{\zeta_{i+1}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} \right)
\]
\[
= c_i \frac{\zeta_{i+1}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} + c_{i+1} \frac{\zeta_{i+2}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} + c_i \frac{\zeta_{i+1}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} + c_{i+1} \frac{\zeta_{i+2}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j}
\]
\[
= c_{i+1} \psi_{i+1}^{l_1 \ldots l_i} + c_i \left( \frac{\zeta_{i+1}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} \right)
\]
\[
+ (c_{i+1})^2 \frac{\zeta_{i+2}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} + c_{i+1} \frac{\zeta_{i+2}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j}.
\]

(102)

Taking the difference of (142) and (142) with \(i, j\) exchanged we find
\[
\tilde{x}^2 \psi_{ij}^{l_1 \ldots l_i} = \tilde{x} \tilde{x} \psi_{ij}^{l_1 \ldots l_i} - (i \leftrightarrow j)
\]
\[
= \frac{(c_{i+1})^2}{D} \left( \psi_{i-1}^{l_1 \ldots l_i} + c_i \frac{\zeta_{i+1}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j} \right)
\]
\[
+ (c_{i+1})^2 \frac{\zeta_{i+2}}{D} g^{ih} p_{l_1 \ldots l_i h_j} \psi_{i-1}^{l_1 j}
\]
\[
= \left( \frac{(c_{i+1})^2}{D} \right) \tilde{x} \tilde{x} \psi_{ij}^{l_1 \ldots l_i}
\]

(143)

up to order \(O(k^{-3/2})\), whence (108), using (141). Contracting the indices \(ij\) of (142) we find how \(\tilde{x}^2 = \tilde{x} \tilde{x}\) acts on the \(H_A\) basis elements:
\[
\tilde{x}^2 \psi_{ij}^{l_1 \ldots l_i} = g_{ij} \tilde{x} \tilde{x} \psi_{ij}^{l_1 \ldots l_i}
\]
\[
= \frac{(c_{i+1})^2}{D} \tilde{x} \tilde{x} \psi_{ij}^{l_1 \ldots l_i}
\]
\[
= \left( \frac{(c_{i+1})^2}{D} \right) \tilde{x} \tilde{x} \psi_{ij}^{l_1 \ldots l_i}
\]

(143)
\[
\begin{align*}
\psi_{j}^{0,\ldots,k} \quad & = \begin{cases} 
(c_{j+1})^{2} + \frac{l[(c_{j})^{2} - (c_{j+1})^{2}]}{D + 2l - 2} \end{cases} \psi_{j}^{0,\ldots,k} \\
& \quad = \begin{cases} 
1 + \frac{(2D - 5)(D - 1)}{2k} \end{cases} \psi_{j}^{0,\ldots,k}, \quad \text{if } l < \Lambda.
\end{align*}
\]

**Proof of proposition 5.1**

We first show that indeed \( A_{\Lambda} \) is generated by the \( T_{0j}^{} \). Since the action of \( so(D) \) (which is spanned by the \( T_{0j}^{} \)) is transitive on each irreducible component \( H_{\Lambda}^{0} \simeq V_{j} \) \((l \in \{0,1,\ldots,\Lambda\})\) contained in \( H_{\Lambda}^{0} \), it remains to show that some \( \psi_{j}^{0} \in H_{\Lambda}^{0} \) can be mapped into some element of \( H_{\Lambda}^{m} \) for all \( m \neq l \) by applying polynomials in \( \mathfrak{t} \). For all \( \psi_{j} \in H_{\Lambda}^{l} \), we have \( \pi_{j}^{0} \psi_{j} \in H_{\Lambda}^{l+1} \oplus H_{\Lambda}^{l-1} \); \( \pi_{j}^{0} \psi_{j} \in H_{\Lambda}^{l+1} \) automatically; if \( l < \Lambda \) one can map \( H_{\Lambda}^{l} \) into \( H_{\Lambda}^{l+1} \) by the contracted multiplication of (68), while applying \( (\pi_{j}^{0} + i\bar{\pi}_{j}^{0}) \). To \( (l + i\bar{\pi}_{j}^{0}) \psi_{j} \in H_{\Lambda}^{l+1} \), one obtains a vector proportional to \( (l + i\bar{\pi}_{j}^{0})^{l+1} \psi_{j} \in H_{\Lambda}^{l+1} \).

The Ansatz (113) with generic coefficients \( a_{s,j}^{\Lambda} \) is manifestly \( Uso(D) \)-equivariant, i.e. fulfilles (112) for all \( a = L_{0j} \); it is also invariant under permutations of \((i_{1}, i_{2}, \ldots, i_{l})\) and fulfills relations (100) (both sides give zero when contracted with any \( \delta_{i_{a}l} \)). More explicitly, (113)

\[
\begin{align*}
f_{0} & \quad \mapsto \quad a_{0,0}^{\Lambda} F_{0,0}^{\Lambda} = a_{0,0} \psi_{0}^{\Lambda}, \\
T_{ij}^{\Lambda} f_{i} & \quad \mapsto \quad a_{i,1}^{\Lambda} F_{1,0}^{\Lambda} = a_{i,1} \psi_{1}^{\Lambda}, \\
T_{ij}^{\Lambda} f_{i} & \quad \mapsto \quad a_{i,2}^{\Lambda} F_{2,0}^{\Lambda} = a_{i,2} \psi_{2}^{\Lambda}, \\
\ldots & \quad \mapsto \quad \ldots
\end{align*}
\]

Similarly, the Ansatz (114) with a generic function \( m_{s}(\lambda) \) of a real nonnegative variable \( s \) is manifestly \( Uso(D) \)-equivariant, and by (92), (93) we find

\[
m_{s}^{\Lambda}(\lambda) L_{ab} m_{s}^{\Lambda}(\mu) F_{b}^{\Lambda} = i \mu_{s}(l) \psi_{b}^{\Lambda} - i \mu_{s}^{\Lambda}(l - 1) \frac{l(l + l + 2)}{D + 2l - 2} \psi_{b}^{\Lambda},
\]

(144)

where we have abbreviated \( \mu_{s}(l) \equiv m_{s}(l) m_{s}^{\Lambda}(l + 1) \). We determine the unknown \( m_{s}^{\Lambda}, a_{s,j}^{\Lambda} \) requiring (112) for \( a = \pi_{j}^{0} \). Applying \( \sigma_{\Lambda} \) to equation (103) with \( l < \Lambda \), and imposing (112) we obtain

\[
c_{l+1} a_{\Lambda,l+1} F_{b+1}^{\Lambda} + c_{l} a_{\Lambda,l} F_{b}^{\Lambda} = \frac{l}{D + 2l - 2} \psi_{b}^{\Lambda},
\]

(144)

which implies the recursion relations for the coefficients \( a_{s,j}^{\Lambda} \)

\[
c_{l+1} a_{\Lambda,l+1} = i (l + 1) a_{\Lambda,l} \mu_{s}(l), \quad c_{l} a_{\Lambda,l} = -i (l + l + 2) a_{\Lambda,l} \mu_{s}^{\Lambda}(l - 1).
\]

(145)

Multiplying (146a) by (146b), i.e. by \( c_{l+1} a_{\Lambda,l} \), we find

\[
c_{l+1} a_{\Lambda,l+1} a_{\Lambda,l+1} = a_{\Lambda,l+1} \mu_{s}(l) a_{\Lambda,l} \mu_{s}^{\Lambda}(l + l + 2) - a_{\Lambda,l+1} \mu_{s}(l)(l + l + 2) = c_{l+1}^{2}.
\]

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implying
\[
m^*_n(l+1)m_n(l) = \mu_n(l) = \frac{\epsilon_n\xi_{l+1}^{(l)}}{\sqrt{(\Lambda - l)(\Lambda + l + D - 1)}}, \quad a_n\xi_{l+1}^{(l)} = i\alpha_n\epsilon_n^{(l)}\sqrt{\frac{\Lambda - l}{(\Lambda + l + D - 1)}}
\]
(146)

where \(\epsilon_n\), \(a_n\) are arbitrary phase factors. Choosing \(\epsilon_n = 1\), \(m_n(s)\) positive-definite on \(s \in \mathbb{R}^+\), and using the property \(\Gamma(z+1) = z\Gamma(z)\) of the Euler gamma function we find that (146) is solved by (116). Applying \(p_{\Lambda}^{LM}\) to equation (103) with \(l = \Lambda\), and imposing (112) we obtain
\[
\epsilon_n^{\Lambda} / D + 2\Lambda - 2 \left( \sum_{\Lambda=1}^{\Lambda} \sum_{L=1}^{\Lambda} \sum_{M=1}^{\Lambda} \sum_{n=0}^{L-1} \right) m_n^{(l+1)} m_n^{(l)} \left( \frac{D}{\Lambda + l + D - 1} \right) = \frac{1}{D + 2\Lambda - 2} \left( \sum_{\Lambda=1}^{\Lambda} \sum_{L=1}^{\Lambda} \sum_{M=1}^{\Lambda} \sum_{n=0}^{L-1} \right) m_n^{(l+1)} m_n^{(l)} \left( \frac{D}{\Lambda + l + D - 1} \right)
\]
which also is satisfied by (116).

**Proof of theorem 6.1**

The proof is recursive. By (119) \(T_{l+1}^{n} h_{l+1} = \sum_{n=0}^{D} \sum_{L=1}^{\Lambda} \sum_{M=1}^{\Lambda} \sum_{n=0}^{L-1} \left( \frac{D}{\Lambda + l + D - 1} \right) m_n^{(l+1)} m_n^{(l)} \left( \frac{D}{\Lambda + l + D - 1} \right) \), which applied to \(T_{l}^{n} h_{l+1}\) gives
(147)

Now we note that
\[
\sum_{n=0}^{D} \sum_{L=1}^{\Lambda} \sum_{M=1}^{\Lambda} \sum_{n=0}^{L-1} \left( \frac{D}{\Lambda + l + D - 1} \right) m_n^{(l+1)} m_n^{(l)} \left( \frac{D}{\Lambda + l + D - 1} \right)
\]

for the last equality we have used the symmetry of \(p_{l+1}^{n+1} h_{l+1} h_{l+1}^{n+1}\) under the exchange \(b \leftrightarrow b_1\). Iterating the procedure we find
\[
\sum_{n=0}^{D} \sum_{L=1}^{\Lambda} \sum_{M=1}^{\Lambda} \sum_{n=0}^{L-1} \left( \frac{D}{\Lambda + l + D - 1} \right) m_n^{(l+1)} m_n^{(l)} \left( \frac{D}{\Lambda + l + D - 1} \right)
\]

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which replaced in (147) gives

\[
\tilde{T}\tilde{T}_{m+1} - T_m^b = \sum_{n \in L^{b_{n+1}}} \tilde{T}_{n+1} \left[ \prod_{l=1}^{n+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} \prod_{l=2}^{n+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} + \frac{(n-l+r)[D + 2(m-l+r) - 4]}{n(D + 2n - 4) + D + 2n - 2} \right] 
\]

\[
\times \prod_{n \in L^{b_{n+1}}} \left[ c_n \prod_{l=1}^{n+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} + \frac{(n-l+m)[D + n-l+m-4]}{2(D + 2n - 2)(D + 2n - 4)} \right] 
\]

\[
\equiv \sum_{n \in L^{b_{n+1}}} \tilde{T}_{n+1} \left[ c_n \prod_{l=1}^{n+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} + \frac{(n-l+m)[D + n-l+m-4]}{2(D + 2n - 2)(D + 2n - 4)} \right] 
\]

renaming \( n' \equiv n - 1 \) in the first sum, \( n' \equiv n + 1 \) in the second, this becomes

\[
\tilde{T}\tilde{T}_{m+1} - T_m^b = \sum_{n' \in L^{b_{n'+1}}} \tilde{T}_{n'+1} \left[ c_{n'} \prod_{l=1}^{n'+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} + \frac{(n'-l+m)[D + n'-l+m-3]}{2(D + 2n' - 2)(D + 2n' - 2)} \right] 
\]

\[
\times \prod_{n' \in L^{b_{n'+1}}} \left[ c_{n'} \prod_{l=1}^{n'+1} \frac{p_{l+1}^{b_1 \ldots b_{l+1}}}{a_1 \ldots a_{l+1} b_{l+1}} + \frac{(n'-l+m)[D + n'-l+m-3]}{2(D + 2n' - 2)(D + 2n' - 2)} \right] 
\]

where \( r' := \frac{l+1+m-n'}{2} \), we have used that \( 2(n-l+r) = n-l+m \) and we have set

\[
\tilde{T}_{n+1}^{(b_{n'+1})} = \tilde{T}_{n+1}^{(b_{n'+1})} + \tilde{T}_{n+1}^{(b_{n'+1})} \left[ \frac{(n-l+m+1)[D + n-l+m-3]}{2(D + 2n - 2)(D + 2n - 2)} \right] 
\]

for all \( n \in L^{(b_{n'+1})} \). The relation among \( \tilde{T}_{n+1}^{(b_{n'+1})}, \tilde{T}_{n+1}^{(b_{n'+1})}, \tilde{T}_{n+1}^{(b_{n'+1})} \) is obtained replacing \( c_b \rightarrow 1 \) and removing. Since \( c_b < c_\Lambda \), by the induction hypothesis (123), we find (123)_b+1, as claimed:

\[
\tilde{T}_{n+1}^{(b_{n'+1})} \leq c_\Lambda \left[ \tilde{T}_{n+1}^{(b_{n'+1})} + \tilde{T}_{n+1}^{(b_{n'+1})} \left[ \frac{(n-l+m+1)[D + n-l+m-3]}{2(D + 2n - 2)(D + 2n - 2)} \right] \right] \leq (c_\Lambda)^{b+1} \tilde{T}_{n+1}^{(b_{n'+1})}.
\]
Proof of theorem 6.2

Setting $\phi^I_A := \phi - \phi_A$ we find

$$
(f \!-\! \tilde{f}_2)\phi = (f \!-\! f_2)\phi + (f_2 - \tilde{f}_2)\phi \\
= (f \!-\! f_2)\phi + (f_2 - \tilde{f}_2)\phi^I_A + (f_2 - \tilde{f}_2)\phi_A \cdot \| (f \!-\! \tilde{f}_2)\phi \|
$$

$$
\leq \| (f \!-\! f_2)\phi \| + \| (f_2 - \tilde{f}_2)\phi^I_A \| + \| (f_2 - \tilde{f}_2)\phi_A \|
$$

$$
\leq \| f \!-\! f_2 \|_{op} \cdot \| \phi \| + \| f_2 - \tilde{f}_2 \|_{op} \cdot \| \phi^I_A \| + \| (f_2 - \tilde{f}_2)\phi_A \|.
$$

(149)

Since $f \in C(S^d)$, then $\| f_2 - f \|_{op} \equiv \sup_{p \in S^d} | f_2(p) - f(p) |$, which goes to zero because $f_2 \to f$ uniformly over $S^d$ (in fact, $f_2 - f$ is a continuous function over the compact manifold $S^d$, and therefore by Heine–Cantor theorem is also uniformly continuous); therefore the first term at the rhs (149) goes to zero as $\Lambda \to \infty$. As $\Lambda \to \infty$ also the second term at the rhs (149) goes to zero because so does $\| \phi^I_A \| = \| \phi - \phi_A \|$, while $\| f_2 - f_2 \|_{op}$ certainly is bounded (actually one can easily show that this goes to zero as well).

To show that the lhs (149) goes to zero as $\Lambda \to \infty$ we now show that the last term at the rhs does, as well. Using the decompositions (70) and (71) for $\phi_A, f_2, \tilde{f}_2, \phi$ we can write

$$
\tilde{f}_2 = f - f_2 + \| \phi - \phi_A \| \geq \| (f \!-\! f_2)\phi \| \geq \| (f \!-\! f_2)\phi \| \geq \| f_2 - \tilde{f}_2 \|_{op} \cdot \| \phi - \phi_A \| 
$$

and taking the square norm in $H_\alpha$,

$$
\| (f \!-\! f_2)\phi \| = \sum_{l=0}^{2\Lambda} \sum_{m=0}^{\Lambda} \sum_{n \in \mathbb{L}_m} \left( \tilde{f}_2 \right)_{l,m,n}^2
$$

(150)
By (102), (111) it is \((c_A)^l \leq (1 + \epsilon)^{l/2} < e^{l/2} \leq e^{\Lambda c}\), because we are using \(l \leq 2\Lambda\); hence by theorem 6.1 we find

\[
0 \leq (\hat{N}_n^m - N_n^m) \leq N_n^m [(c_A)^l - 1] < N_n^m (e^{\Lambda c} - 1).
\]

Replacing these inequalities in the result above we obtain

\[
\left\| \hat{f}_{2\Lambda} - f_{2\Lambda} \right\|^2 < (e^{\Lambda c} - 1)^2 \sum_{l=0}^{2\Lambda} \sum_{m=0}^{\Lambda} N_n^m N_{n'}^{m'} Q_n
\]

\[
= (e^{\Lambda c} - 1)^2 \left\| f_{2\Lambda} \phi_A \right\|^2. \tag{151}
\]

The last equality holds because the sum is nothing but \(\|f_{2\Lambda} \phi_A\|^2\), i.e. what we would obtain from (150) replacing \(\hat{N}_n^m\) \(\rightarrow \) 0, and therefore \(f_{2\Lambda} \rightarrow 0\). Moreover, \(\|f_{2\Lambda} \phi_A\| \leq \|f_{2\Lambda}\|_\text{op} \|\phi_A\|\); but both factors have \(\Lambda\)-independent bounds: \(\|\phi_A\| \leq \|\phi\|\), while by the triangular inequality, \(\|f_{2\Lambda}\|_\text{op} \leq \|f\|_\text{op} + \|f_{2\Lambda} - f\|_\text{op}\), and the second term goes to zero as \(\Lambda \rightarrow \infty\), hence is bounded by some \(\eta \geq 0\). So we end up with

\[
\left\| \hat{f}_{2\Lambda} - f_{2\Lambda} \right\| < (e^{\Lambda c})(1) \left( \|f\|_\text{op} + \eta \right) \|\phi\| \xrightarrow{\Lambda \rightarrow \infty} 0; \tag{152}
\]

the limit is zero because by (20) \(k(\Lambda)\) diverges at least as \(\Lambda^4\), then by (111) \(\Lambda c(\Lambda)\) goes to zero as \(\Lambda^{-1}\). Replaced in (149) this yields for all \(f \in C(S^d)\)

\[
\left\| (f - \hat{f}_{2\Lambda}) \phi \right\| \xrightarrow{\Lambda \rightarrow \infty} 0 \tag{153}
\]

i.e. \(\hat{f}_{2\Lambda} \rightarrow f\) strongly, as claimed. Replacing \(f \rightarrow fg\), we find also that \((\hat{fg})_{2\Lambda} \rightarrow (fg)\cdot\) strongly for all \(f, g \in C(S^d)\), as claimed. On the other hand, since \(\|\phi_A\| \leq \|\phi\|\) (because \(\phi = \phi_A + \phi_A^\perp\) with \(\langle \phi_A, \phi_A^\perp \rangle = 0\)), relations (149) and (147) imply also

\[
\left\| (f - \hat{f}_{2\Lambda}) \phi \right\| \leq \left( \left\| f - f_{2\Lambda} \right\|_\text{op} + \left\| f_{2\Lambda} - \hat{f}_{2\Lambda} \right\|_\text{op} + (e^{\Lambda c}(\Lambda) - 1) (\|f\|_\text{op} + \eta) \right) \|\phi\| \leq F \|\phi\|,
\]

where \(F > 0\) is an upper bound for the expression in the square bracket; hence

\[
\|\hat{f}_{2\Lambda} \phi\| \leq \|f_{2\Lambda} - f\| + \|f\| + \|f\|_\infty \|\phi\| \leq F + \|f\|_\infty \|\phi\|
\]

i.e. the operator norms \(\|f_{2\Lambda}\|_\text{op}\) of the \(\hat{f}_{2\Lambda}\) are uniformly bounded: \(\|f_{2\Lambda}\|_\text{op} \leq (F + \|f\|_\infty) \|\phi\|\). Therefore (153) implies the last claim of the theorem

\[
\left\| (fg - \hat{f}_{2\Lambda} g_{2\Lambda})\phi \right\| \leq \left\| (f - \hat{f}_{2\Lambda}) g\phi \right\| + \left\| \hat{f}_{2\Lambda} (g - g_{2\Lambda}) \phi \right\|
\]

\[
\leq \left\| (f - \hat{f}_{2\Lambda}) (g\phi) \right\| + \left\| \hat{f}_{2\Lambda} (g - g_{2\Lambda}) \phi \right\| \xrightarrow{\Lambda \rightarrow \infty} 0. \tag{154}
\]

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References

[1] Snyder H S 1947 Quantized space-time Phys. Rev. 71 38

[2] Doplicher S, Fredenhagen K and Roberts J E 1994 Spacetime quantization induced by classical gravity Phys. Lett. B 331 39–44

[3] Connes A and Lott J 1990 Particle models and noncommutative geometry Nucl. Phys. B 331 29–47

[4] Chamseddine A H and Connes A 2010 Noncommutative geometry as a framework for unification of all fundamental interactions including gravity, Part I Fortschr. Phys. 58 553

[5] Connes A 1995 Noncommutative Geometry (New York: Academic)

[6] Gracia-Bondia J M, Figueroa H and Varilly J 2000 Elements of Non-Commutative Geometry (Besel: Birkh"auser)

[7] Landi G 1997 An Introduction to Noncommutative Spaces and Their Geometries (Lecture Notes in Physics vol 51) (Berlin: Springer)

[8] Madore J 1999 An Introduction to Noncommutative Differential Geometry and its Physical Applications (Cambridge: Cambridge University Press)

[9] Aschieri P, Steinacker H, Madore J, Manousselis P and Zoupanos G 2007 Fuzzy extra dimensions: dimensional reduction, dynamical generation and renormalizability 6th Workshop on Supersymmetries and Quantum Symmetries (SQS’05) (Dubna, Russia, 27–31 July 2005) pp 25–42 and references therein

[10] Alekseev A Y, Recknagel A and Schomerus V 1999 Non-commutative world-volume geometries: D-branes on SU(2) and fuzzy spheres J. High Energy Phys. JHEP09(1999)023

[11] Hikida Y, Nozaki M and Sugawara Y 2001 Formation of spherical d2-brane from multiple D0-branes Nucl. Phys. B 617 117

[12] Madore J 1991 Quantum mechanics on a fuzzy sphere J. Math. Phys. 32 332

[13] Madore J 1992 The fuzzy sphere Class. Quantum Grav. 9 6947

[14] Hoppe J 1982 Quantum theory of a massless relativistic surface and a 2-dimensional bound state problem PhD Thesis Massachusetts Institute of Technology

[15] Hoppe J and Nicolai H 1988 Nucl. Phys. B 305 545

[16] Grosse H and Madore J 1992 A noncommutative version of the Schwinger model Phys. Lett. B 283 218

[17] Grosse H, Klimcik C and Presnajder P 1996 Towards finite quantum field theory in non-commutative geometry Int. J. Theor. Phys. 35 231–44

[18] Grosse H, Klimcik C and Presnajder P 1996 On finite 4D quantum field theory in non-commutative geometry Commun. Math. Phys. 180 429–38

[19] Ramgoolam S 2001 On spherical harmonics for fuzzy spheres in diverse dimensions Nucl. Phys. B 610 461–88

[20] Ramgoolam S 2002 Higher dimensional geometries related to fuzzy odd-dimensional spheres J. High Energy Phys. JHEP10(2002)064 and references therein

[21] Dolan B P and O’Connor D 2003 A Fuzzy three sphere and fuzzy tori J. High Energy Phys. JHEP10(2003)060

[22] Dolan B P, O’Connor D and Presnajder P 2004 Fuzzy complex quadrics and spheres J. High Energy Phys. JHEP2(2004)055

[23] Steinacker H 2016 Emergent gravity on covariant quantum spaces in the IKKT model J. High Energy Phys. JHE12(2016)156

[24] Sperling M and Steinacker H 2017 Covariant 4-dimensional fuzzy spheres, matrix models and higher spin J. Phys. A: Math. Theor. 50 375202

[25] Fiore G and Pisacane F 2018 New fuzzy spheres through confining potentials and energy cutoffs J. Geom. Phys. 132 423–51

[26] Fiore G and Pisacane F 2020 O(D)-equivariant fuzzy hyperspheres (arXiv:2002.01901)

[27] Hawkins E 1999 Quantization of equivariant vector bundles Commun. Math. Phys. 202 517–46
[28] Fiore G and Pisacane F 2020 The $x_i$-eigenvalue problem on some new fuzzy spheres  *J. Phys. A: Math. Theor.* **53** 095201

[29] Fiore G and Pisacane F 2020 On localized and coherent states on some new fuzzy spheres  *Lett. Math. Phys.* **110** 1315–61

[30] D’Andrea F, Lizzi F and Martinetti P 2014 Spectral geometry with a cut-off: Topological and metric aspects  *J. Geom. Phys.* **82** 18–45

[31] Fiore G, Franco D and Weber T 2020 Twisted quadrics and algebraic submanifolds in $\mathbb{R}^n$  *Math. Phys. Anal. Geom.* **23** 38

[32] Fiore G and Weber T 2021 Twisted submanifolds of $\mathbb{R}^n$  *Lett. Math. Phys.* **111** 76