Integral representation of the star product in Loop Quantum Cosmology

Jasel Berra–Montiel\textsuperscript{1,2} and Alberto Molgado\textsuperscript{1,2}

\textsuperscript{1} Facultad de Ciencias, Universidad Autónoma de San Luis Potosí
Campus Pedregal, Av. Parque Chapultepec 1610, Col. Privadas del Pedregal, San Luis Potosí, SLP, 78217, Mexico
\textsuperscript{2} Dual CP Institute of High Energy Physics, Mexico

E-mail: jasel.berra@uaslp.mx, alberto.molgado@uaslp.mx

Abstract. Guided by recent developments towards the implementation of the techniques of deformation quantization within the Loop Quantum Cosmology (LQC) formalism, in this paper we address the introduction of the integral representation of the star product for LQC. To this end, we consider the Weyl quantization map for cylindrical functions defined on the Bohr compactification of the reals. The integral representation contains all of the common properties that characterize a star product which, in the case under study here, stands for a deformation of the usual pointwise product of cylindrical functions. We also invoke a direct comparison with the integral representation of the Moyal product which may be reproduced from our formulation by judiciously substituting the appropriate characters that identify such representation. Further, we introduce a suitable star commutator that correctly reproduces both the quantum representation of the holonomy-flux algebra for LQC and, in the proper limit, the holonomy-flux classical Poisson algebra emerging in the cosmological setup. Finally, we propose a natural way to obtain the quantum dynamical evolution in LQC in terms of this star commutator for cylindrical functions. We expect that our findings may contribute to a better understanding of certain issues arising within the LQC program.

Keywords: Loop Quantum Cosmology, Deformation quantization, star product, Bohr compactification

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1. Introduction

Loop Quantum Cosmology (LQC) is a theory inspired by the quantization techniques developed within the Loop Quantum Gravity (LQG) program to bring together the principles of General Relativity and Quantum Mechanics. In particular, the LQC framework is associated with finite-dimensional minisuperspace models for which certain symmetries, such as homogeneity and isotropy, for the background spacetimes are assumed \cite{1, 2}. Within this framework, significant advances in the quantum gravity program have been successfully reported, being of particular relevance the analysis of
the microscopic ground for black hole entropies [3], [4], [5], the avoidance of classical singularities as a consequence of quantum bounces [2], [6], and the investigation of inhomogeneities imprinted on the cosmic expansion [7], [8]. Despite those remarkable developments, it is worth mentioning that certain technical issues are not completely well understood at the present time, as, for example, those concerning a detailed evolution of quantum states, the understanding of an appropriate semiclassical limit, the geometrical analysis of singularities, the emergence of effective theories, and the discussion of quantization ambiguities, among others. These topics remain as major open questions, and any contribution towards them may undoubtedly add to a realistic description of the LQC theory. (We refer the reader to [9] for a detailed review on these subjects from a modern perspective.)

In order to shed some light on the issues mentioned above, a different approach for the LQG program has been recently proposed [10], [11], [12]. This novel approach is based on the deformation quantization picture of quantum mechanics. Deformation quantization, also referred to as phase space quantum mechanics in the literature, consists of an alternative quantization procedure based on the idea that a quantum system is obtained by deforming the algebraic and geometrical structures defined in the classical phase space [13], [14]. Undeniable, one of the most prominent features of the deformation quantization approach is related to the algebra of quantum observables, in consideration of the fact that this algebra is not specified by a set of self-adjoint operators acting on a Hilbert space, as in ordinary quantum mechanics, but instead the observables are characterized by complex-valued functions defined on the classical phase space for which the conventional commutative pointwise product is replaced by an associative, non-commutative product, the so-called star product. As a consequence, the introduction of the star product induces a deformation of the Poisson algebra in such a way that the information hold in the quantum commutators of any pair of self-adjoint operators is mapped to the deformed algebraic classical structures. Until now, the formalism of deformation quantization has not only provided significant contributions in pure and applied mathematics [15], [16], but it also has proved to be an outstanding tool for the quantum analysis of a broad variety of physical systems [17], [18], [19], recently including the treatment of constrained systems [20], the coherent field quantization [21], [22], and the tomographic representation for fields [23].

Hence, the main aim of this manuscript is to extend previous results obtained within the deformation quantization framework to the LQG context by explicitly addressing the integral representation of the star product for LQC. To this end, we start by considering the Weyl quantization map on a phase space given by the cylindrical functions defined on the Bohr compactification of the reals and, in analogy to the Moyal product, by conveniently introducing an associative and non-commutative star product for LQC that introduces an appropriate deformation of the pointwise product. As a consequence, we are able to identify an integral representation for this star product, which in turn allows us to define a suitable star commutator that not only reproduces the correct quantum representation of the holonomy-flux algebra for LQC, but it also duplicates, in
the relevant limit, the Poisson algebra emerging within the classical cosmology scenario. Our claim is thus that such integral representation for the star product in LQC may allow us to analyze, in a more succinct manner, several aspects related to both the semiclassical limit and the analysis of quantization ambiguities commonly encountered within the LQC framework. We also hope that the proposed star product for LQC, together with the introduction of a family of quasi-probability distributions, as developed in [12], would be advantageous in order to address, from an innovative perspective, the problem associated with the dynamical evolution of quantum states in LQC.

The paper is organized as follows: in section 2 we briefly review the basic concepts behind the Weyl quantization scheme for functions defined on the Bohr compactification of the reals. In section 3, we introduce the integral representation of the star product for LQC. In particular, in analogy with the Moyal’s prescription for quantum mechanics, we define the star commutator and, by means of the Wigner distribution for LQC, we establish the dynamical evolution for the quantum system. Finally, we present some concluding remarks in section 4.

2. The Wigner-Weyl correspondence in LQC

In order to obtain the integral representation of the star product for LQC, in this section, we briefly review the definition of the Weyl quantization map for the Bohr compactification of the reals, $b\mathbb{R}$. First, we start by considering some basic features of harmonic analysis regarding the analytical properties of functions defined on $b\mathbb{R}$ (for further details on this subject we refer the reader to [21], [23]). For the sake of simplicity, we restrict our attention to systems with one degree of freedom, although its generalization to a higher dimensional setup follows straightforwardly.

Let $\mathbb{R}$ be a locally compact Abelian group, given by the real numbers equipped with the usual addition operation and the standard topology. This means that the real numbers $\mathbb{R}$ forms a metrizable, $\sigma$-compact and locally compact Abelian group. In particular, the $\sigma$-compact property implies that there exists a sequence $K_n \subset K_{n+1}$ of compact subsets, such that $\mathbb{R} = \bigcup_n K_n$, i.e., the real numbers $\mathbb{R}$ comprises a compact exhaustion [20]. A character of the reals $\mathbb{R}$, is given by a continuous group homomorphism $h_\mu : \mathbb{R} \to \mathbb{T}$, to the unit torus, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, so $h_\mu$ is a map satisfying

$$h_\mu(a + b) = h_\mu(a)h_\mu(b),$$

for every $a, b \in \mathbb{R}$. Let us denote by $\hat{\mathbb{R}}$, the set of all characters of $\mathbb{R}$, labeled by $\mu \in \mathbb{R}$,

$$h_\mu(a) = e^{i\mu a}. \quad (2)$$

By using the $\sigma$-compact property of the reals, one can define a uniform metric on $\hat{\mathbb{R}}$ such that it endows the set of characters with the properties of a locally compact group, called the dual group of $\mathbb{R}$ [20]. The map given by (2) defines a group isomorphism $h_\mu : \mathbb{R} \to \hat{\mathbb{R}}$, hence, we can say that $\hat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$ as locally compact groups. Let us consider the dual group $\hat{\mathbb{R}}$, but now equipped with the discrete topology $\hat{\mathbb{R}}_{\text{discr}}$.
By definition, this group turns out to be discrete and Abelian. Then, by making use of the Pontryagin’s duality theorem, its dual group, denoted by \( \hat{\mathbb{R}}_{\text{discr}} \), forms a compact Abelian group, commonly named as the Bohr compactification of the reals, \( \mathbb{bR} \). On \( \mathbb{bR} \), the group of the real numbers \( \mathbb{R} \) can be embedded as a dense subgroup and the addition operator is extended uniquely to the continuous group operation given by the characters \[24\]. Finally, by applying the Pontryagin’s duality theorem once more, the dual group associated to the Bohr compactification, \( \hat{\mathbb{bR}} \), proves to be discrete, which means that no continuity requirements are demanded on its characters since this group is equipped with the discrete topology.

Seen as locally compact groups, both the Bohr compactification, \( \mathbb{bR} \), and its dual group, \( \hat{\mathbb{bR}} \), carry a unique normalized Haar measure, respectively. Thus, the Haar measure \( dc \) on \( \mathbb{bR} \) is characterized by

\[
\int_{\mathbb{bR}} dc \, h_\mu(c) = \delta_{\mu,0},
\]

for any character \( h_\mu \), while the Haar measure \( d\mu \) defined on \( \hat{\mathbb{bR}} \) explicitly reads

\[
\int_{\hat{\mathbb{bR}}} d\mu \, \hat{f}_\mu = \sum_{\mu \in \mathbb{R}} \hat{f}_\mu,
\]

and, hence, it corresponds to the counting measure on \( \mathbb{R} \). Here \( \hat{f}_\mu \) stands for the Fourier transform on \( \mathbb{bR} \), that allows us to determine an isomorphism between \( L^2(\mathbb{bR}, dc) \) and \( L^2(\hat{\mathbb{bR}}, d\mu) \) such that

\[
\hat{f}_\mu = \int_{\mathbb{bR}} dc \, f(c) h_{-\mu}(c).
\]

By the Peter-Weyl theorem, the characters \( h_\mu \) form an orthonormal uncountable basis for \( L^2(\mathbb{bR}, dc) \), resulting in a non-separable Hilbert space structure. Furthermore, the Hilbert space \( L^2(\mathbb{bR}, dc) \) is isomorphic to the Hilbert, \( B^2(\mathbb{R}) \), given by the Besicovitch almost periodic functions on \( \mathbb{R} \) \[27\].

In order to define the Weyl quantization map on the Bohr compactification of the reals let us define the set of cylindrical functions, denoted by \( \text{Cyl}(\mathbb{bR}) \), as the finite span of characters on \( \mathbb{bR} \), and by \( \text{Cyl}(\hat{\mathbb{bR}}) \) the image of \( \text{Cyl}(\mathbb{bR}) \) under the Fourier transform. This means that any \( \psi \in \text{Cyl}(\mathbb{bR}) \) can be expressed in the form

\[
\psi(c) = \sum_{\mu} \tilde{\psi}_\mu h_\mu(c),
\]

where \( \tilde{\psi}_\mu \) represents the Fourier coefficient

\[
\tilde{\psi}_\mu = \int_{\mathbb{bR}} dc \, \psi(c) \overline{h_\mu(c)},
\]

which vanishes for all but countable \( \mu \in \mathbb{R} \). Therefore, the discrete sum depicted in (6) indicates that the space \( \text{Cyl}(\mathbb{bR}) \) is given by the set of all complex valued functions on \( \mathbb{R} \) that are vanishing almost everywhere except for a countable number of points.

The central difference between the Schrödinger representation of quantum mechanics and LQC lies on the choice of the Hilbert space. While, in the Schrödinger representation the Hilbert space is denoted by \( L^2(\mathbb{R}) \), in LQC the Hilbert space is
commonly designated as $L^2(b\mathbb{R}, dc)$. However, a distinctive difference between these Hilbert spaces relies on the fact that the position and momentum operators $\hat{q}$ and $\hat{p}$ are well-defined operators on $L^2(\mathbb{R})$, whereas in LQC this is no longer true. The reason stems from the full LQG theory, in which case a Hermitian operator corresponding to the analogue of the position operator, namely the gauge connection, does not exist [28], [29]. Nevertheless, since the holonomies associated with those connection variables are well-defined, instead one may consider as fundamental the operators $\hat{h}_\mu$ and $\hat{p}$ given by (in natural units where $\hbar = 1$)

$$\hat{h}_\mu \psi(c) := h_\mu(c)\psi(c), \quad \hat{p}\psi(c) := \sum_{\mu \in \mathbb{R}} \mu \tilde{\psi}_\mu h_\mu(c),$$

(8)

where $\psi \in \text{Cyl}(b\mathbb{R})$. In particular, one may observe from the first of these definitions that, since $h_\mu(c)\psi(c)$ proves to be non-differentiable on $L^2(b\mathbb{R}, dc)$, this prevents the existence of a position operator on the Hilbert space $L^2(b\mathbb{R}, dc)$, thus rendering a different representation of quantum mechanics by evoking the Stone-von Neumann uniqueness theorem [30]. In consequence, the operators defined in (8) provide the basic commutator relations

$$[\hat{h}_\mu, \hat{p}] \psi(c) = -\mu h_\mu \psi(c),$$

$$[\hat{h}_\mu, \hat{h}_\nu] \psi(c) = 0 = [\hat{p}, \hat{p}] \psi(c),$$

(9)

for all $\psi \in L^2(b\mathbb{R}, dc)$, which correspond to the LQC analogue of the quantum representation of the holonomy-flux algebra in LQG [2].

Given a distribution $g$ on Cyl$(b\mathbb{R} \times \hat{b}\mathbb{R})$ (also called symbol, according to the terminology adopted in harmonic analysis [24]), that is, $g \in \text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})^*$, where Cyl$(b\mathbb{R} \times \hat{b}\mathbb{R})^*$ denotes the dual space of Cyl$(b\mathbb{R} \times \hat{b}\mathbb{R})$ [31], and making use of the momentum representation with the purpose to avoid the issue of multiplying an element of $b\mathbb{R}$ by an arbitrary real number, the quantization map for LQC of the function $g(c, \mu)$, in the Weyl symmetric ordering, is defined as [12],

$$\hat{g}\psi(c) = Q^{LQC}(g)\psi(c) := \int_{b\mathbb{R} \times \hat{b}\mathbb{R}} d\mu d\nu \tilde{g} (\mu - \nu, \frac{\mu + \nu}{2}) h_\mu(c)\tilde{\psi}_\nu,$$

(10)

where $\psi \in L^2(b\mathbb{R}, dc)$, and $\tilde{g}$ denotes the partial Fourier transform of the function $g(c, \mu)$ with respect to the first variable. Following [12], after some straightforward calculations we may demonstrate that this quantization map $Q^{LQC}$, defined in (10), uniquely determines the holonomy and momentum operators stated in (8).

3. Star product representation of LQC

Let $f$ and $g$ denote distributions on Cyl$(b\mathbb{R} \times \hat{b}\mathbb{R})$, then, by means of the Weyl quantization map for LQC defined in the previous section, $Q^{LQC}(f)$ and $Q^{LQC}(g)$ correspond to operators acting on the Hilbert space $L^2(b\mathbb{R}, dc)$. Since the product $Q^{LQC}(f)Q^{LQC}(g)$ also determines an operator on $L^2(b\mathbb{R}, dc)$, in this section we introduce the so-called star product of symbols defined by

$$Q^{LQC}(f)Q^{LQC}(g) = Q^{LQC}(f \ast_{LQC} g),$$

(11)
where the star product $\star_{LQC}$, like the quantization map $Q^{LQC}$ itself, explicitly depends on $\hbar$ but, however, for the sake of simplicity we have suppressed this dependence on the notation. In order to compute $f \star_{LQC} g$, let us first express the Weyl quantization map \cite{10} in terms of an integral operator as

$$
\hat{g} \psi(c) = \int_{\mathfrak{b} \mathbb{R}} dB K_g(c, b) \psi(b),
$$

where the integral kernel $K_g(c, b)$ is given by

$$
K_g(c, b) = \int_{\mathfrak{b} \mathbb{R}^2} d\mu d\nu \tilde{g}(\mu, \nu) h^{\frac{\mu}{2}}(c + b) h^{\frac{\nu}{2}}(c - b).
$$

This integral is not necessarily absolutely convergent, and should be understood as a partial Fourier transform with respect to $L^2(\mathfrak{b} \mathbb{R}, d\mu)$. We can use \cite{13} to recover the partial Fourier transform of the symbol $g$ from the kernel $K_g(c, b)$ as

$$
\tilde{g}(\mu, \nu) = \int_{\mathfrak{b} \mathbb{R}^2} \frac{da}{2\pi} db d\alpha d\beta K_g(\alpha, \beta) h^{\frac{\alpha + \beta - \mu}{2}}(a + b) h^{\frac{\beta}{2}}(c - b) h^{\frac{\nu}{2}}(b - a) \psi(a),
$$

where in this case, the integral kernel reads

$$
K_{fg}(c, a) = \int_{\mathfrak{b} \mathbb{R}^2} \frac{da}{2\pi} db d\alpha d\beta d\mu d\nu \tilde{f}(\alpha, \beta) \tilde{g}(\mu, \nu) h^{\frac{\alpha + \beta - \mu}{2}}(a + b) h^{\frac{\beta}{2}}(c - b) h^{\frac{\nu}{2}}(b - a).
$$

Substituting this expression on \cite{14}, and after computing several integrations using properties \cite{3} and \cite{4}, we obtain that the symbol associated to the product of the operators $\hat{f}$ and $\hat{g}$, denoted as $f \star_{LQC} g$, is obtained as follows

$$
(f \star_{LQC} g)(c, \mu) = \int_{\mathfrak{b} \mathbb{R}^2} da db d\rho d\sigma f(a, \rho) g(b, \sigma) h^{2(\rho - \sigma)}(c) h^{2(\rho - \mu)}(b) h^{2(\sigma - \mu)}(a).
$$

This non-commutative bilinear product between two symbols $f(c, \mu)$, $g(c, \mu) \in \text{Cyl}(\mathfrak{b} \mathbb{R} \times \mathfrak{b} \mathbb{R})^*$, thus corresponds to the integral representation of the star product in the context of LQC. Moreover, as a consequence of the associativity of the product of operators on a Hilbert space, i.e., $(\hat{f} \hat{g}) \hat{h} = \hat{f}(\hat{g} \hat{h})$, one may easily demonstrate that the star product \cite{17} also defines an associative product for the symbols, that is,

$$
((f \star_{LQC} g) \star_{LQC} h)(c, \mu) = (f \star_{LQC} (g \star_{LQC} h))(c, \mu).
$$

Besides, it is worth noticing that if in the integral formula \cite{17} we replace the characters $h_{\mu}(c)$ of the Bohr compactification of the reals $\mathfrak{b} \mathbb{R}$ with the characters associated with the group of the real numbers $\mathbb{R}$, which in the latter case are merely given by exponentials, we may recognize that the star product stated in \cite{17} corresponds to the integral representation of the Moyal product obtained in ordinary quantum mechanics \cite{32}. This
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means that the star product $\star_{\text{LQC}}$ may be thought of as a generalization of the Moyal product between functions defined on the Schwartz space $\mathcal{S}(\mathbb{R})$ [30] by allowing the incorporation of symbols defined on $\text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})^\ast$.

In complete analogy to the Moyal product, the star product $\star_{\text{LQC}}$ thus determines a deformation of the ordinary pointwise product of functions on $\text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})^\ast$. In order to appreciate such deformation, let us start by recovering the $\hbar$ dependence in the notation for the integral formula (17), and explicitly compute its Fourier transform which reads
\[
(\tilde{f} \star_{\text{LQC}} \tilde{g})(c,\mu) = \int_{b\mathbb{R} \times \hat{b}\mathbb{R}} da d\mu \tilde{f}(c-a,\mu-\nu)\tilde{g}(a,\nu)h_{\frac{\mu}{\hbar}}(c)h_{\frac{\nu}{\hbar}}(a).
\] (19)
Note that if we set $\hbar \to 0$, the formula for $(\tilde{f} \star_{\text{LQC}} \tilde{g})(c,\mu)$ reduces to the convolution of $\tilde{f}$ and $\tilde{g}$, which is equal to the Fourier transform of the pointwise product $fg$. This means that
\[
\lim_{\hbar \to 0}(f \star_{\text{LQC}} g)(c,\mu) = (fg)(c,\mu),
\] (20)
and thus our claim is valid.

Next, for an arbitrary couple of functions on $f, g \in \text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})^\ast$ we may define the star commutator as $[f, g]_{\star_{\text{LQC}}} := f \star_{\text{LQC}} g - g \star_{\text{LQC}} f$. In particular, by choosing the functions $f(c,\mu) = h_\mu(c)$ and $g(c,\mu) = \mu$, we can observe that
\[
[h_\mu(c), \mu]_{\star_{\text{LQC}}} = -\mu \hbar h_\mu(c),
\]
\[
[h_\mu(c), h_\nu(c)]_{\star_{\text{LQC}}} = 0 = [\mu, \nu]_{\star_{\text{LQC}}},
\] (21)
which is nothing but the quantum representation of the holonomy-flux algebra depicted in (9). Furthermore, by means of the property (20) of the star product $\star_{\text{LQC}}$, the star commutator satisfies
\[
\lim_{\hbar \to 0} \frac{1}{i\hbar} [h_\mu(c), \mu]_{\star_{\text{LQC}}} = i\mu h_\mu(c) = \{h_\mu(c), \mu\},
\]
\[
\lim_{\hbar \to 0} \frac{1}{i\hbar} [h_\mu(c), h_\nu(c)]_{\star_{\text{LQC}}} = 0 = \{h_\mu(c), h_\nu(c)\},
\]
\[
\lim_{\hbar \to 0} \frac{1}{i\hbar} [\mu, \mu]_{\star_{\text{LQC}}} = 0 = \{\mu, \nu\}.
\] (22)
It should be noted that the right hand side of the identities (22) agrees with the holonomy-flux Poisson algebra occurring in classical cosmology [2]. Thus, we conclude that the Weyl quantization map $Q_{\text{LQC}}$ defined in (10) and the star product $\star_{\text{LQC}}$, provides a one-parameter, associative deformation of the holonomy-flux classical algebra. Within the deformation quantization approach [32], it is expected that the induced deformations become irrelevant at large scales, hence providing a natural mechanism to understand how the classical description of mechanics emerges from the quantum world.

Finally, in order to establish a complete parallelism with Moyal’s prescription for quantum mechanics, we propose that the quantum dynamical evolution in LQC is governed by the natural extension of Moyal’s equation [32] that incorporates the $\star_{\text{LQC}}$ bracket defined above and thus, within this context, we introduce the LQC equation
\[
\frac{\partial \rho^{\text{LQC}}(c,\mu)}{\partial t} = \frac{1}{i\hbar} [H(c,\mu), \rho(c,\mu)]_{\star_{\text{LQC}}},
\] (23)
where $\rho^{LQC}(c, \mu)$ corresponds to the Wigner quasi-probability distribution for LQC [12], [31], while $H(c, \mu) \in \text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})^*$ stands for the classical Hamiltonian. For a given state $\psi \in L^2(b\mathbb{R}, dc)$, the Wigner distribution $\rho^{LQC}(c, \mu)$ consists of the complex valued function on $b\mathbb{R} \times \hat{b}\mathbb{R}$, given by

$$
\rho^{LQC}(c, \mu) = \int_{\hat{b}\mathbb{R}} d\nu \bar{\psi}_{(\mu-\nu)/2} \psi_{(\mu+\nu)/2} h(\nu) \quad (24).
$$

Note that the dynamical equation for LQC (23), as it happens for the Moyal equation, results eminently comparable to the Heisenberg’s equation of motion in quantum mechanics, with the significant exception that for the case of our interest here $H(c, \mu)$ and $\rho^{LQC}(c, \mu)$ are functions taking values on $\text{Cyl}(b\mathbb{R} \times \hat{b}\mathbb{R})$, instead of operators in a Hilbert space. Further, the star product $\star^{LQC}$ in the LQC equation embodies all of the non-commutative behavior that characterizes the quantum system. Thus, we claim that within the LQC framework, equation (23) should be fundamental in order to elucidate the concealed link between quantum operators and the Poisson structure. Further research is needed to address this last issue.

4. Conclusions

As it is well-known, the LQC program relies on a quantization on the Bohr compactification of the reals which naturally brings about discreteness, together with the failure to simultaneously introduce well-defined position and momentum operators acting on the associated Hilbert space. In consequence, we are confronted with a different representation that is clearly unequivalent to standard Schrödinger quantum mechanics. Recently, a proposal to introduce the techniques of deformation quantization within the LQC formalism has been under analysis, being of particular relevance the manner in which the structures associated with the Bohr compactification adapt systematically to the non-commutative formulation emerging in the phase space quantization. Following previous results in this direction, as reported in [10], [11], [12], we addressed here the introduction of the integral representation of the star product of LQC. To that end, we started by considering, in close analogy to the formulation of the Moyal product, the Weyl quantization map explicitly defined for cylindrical functions defined on the Bohr compactification of the reals. As discussed in [12], such quantization map allows us to uniquely determine the holonomy and momentum operators. Therefore, we were able to identify the integral representation for this star product which, in particular, allowed us to demonstrate in a straightforward manner the non-commutative, bilinear and associative properties, which also, as described above, stands for a fiducial deformation of the usual pointwise product of cylindrical functions. Further, the introduced integral representation for the star product granted us a direct comparison with the integral representation of the Moyal product, as the latter may be obtained by substituting the characters of the Bohr compactification of the reals with the characters associated with the group of the real numbers. In this sense, our resulting integral representation settle a legitimate extension of the Moyal product.
between functions defined on the Schwartz space by incorporating symbols defined on the space of cylindrical functions.

The integral representation for the star product also favored the definition of a suitable star commutator that correctly reproduces the quantum representation of the holonomy-flux algebra for LQC. Besides, by considering the appropriate classical limit, it was shown that this star commutator replicates the Poisson algebra emerging in the cosmological setup, thus introducing a one-parameter, associative deformation of the holonomy-flux classical algebra. In sight of this, we proposed a natural way to obtain the quantum dynamical evolution in LQC in terms of this star commutator and a suitable Wigner quasi-probability distribution for LQC. This dynamical equation extends the analogous in Moyal’s approach by incorporating cylindrical functions within our formulation for LQC.

Our claim is thus that the introduced integral representation for the star product within the deformation quantization program for LQC may shed some light to a better comprehension of technical issues ranging from the understanding of an appropriate semiclassical limit to the evolution of quantum states, among others. In particular, the former issue may be natural within the context of deformation quantization, while the latter may be appropriately tackled by considering the proposed LQC dynamical equation.

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References

[1] Bojowald M., Loop quantum cosmology, Living Rev. Relativ. 8 11 (2005), arXiv:gr-qc/0601085.
[2] Ashtekar A. and Singh P., Loop Quantum Cosmology: A Status Report, Class. Quantum Grav. 28 213001 (2011), arXiv:1108.0893 [gr-qc].
[3] Rovelli C., Black hole entropy from loop quantum gravity, Phys. Rev. Lett. 77 3288–91 (1996), arXiv:gr-qc/9603063.
[4] Ashtekar A., Baez J., Corichi A. and Krasnov K., Quantum geometry and black hole entropy, Phys. Rev. Lett. 80 904–7 (1998), arXiv:gr-qc/9710007.
[5] Domagala M. and Lewandowski J., Black-hole entropy from quantum geometry, Class. Quantum Grav. 21 5233–43 (2004), arXiv:gr-qc/0407051.
[6] Bojowald M., Quantum nature of cosmological bounces, Gen. Relativ. Gravit. 40 2659–83 (2008), arXiv:0801.4001 [gr-qc].
[7] Bojowald M., Loop quantum cosmology and inhomogeneities, Gen. Rel. Grav. 38 1771–1795 (2006), arXiv:gr-qc/0609034.
[8] Agullo I. and Singh P., Loop Quantum Cosmology: A Brief Review, 100 Years of General Relativity (Loop Quantum Gravity: The First 30 Years vol 4) eds. Ashtekar A. and Pullin J. (Singapore: World Scientific, 2017), arXiv:1612.01236 [gr-qc].
[9] Bojowald M., Critical evaluation of common claims in loop quantum cosmology, Universe 6 36 (2020), arXiv:2002.05703 [gr-qc].
Integral representation of the star product in Loop Quantum Cosmology

[10] Berra-Montiel J. and Molgado A., *Polymer quantum mechanics as a deformation quantization*, Class. Quantum Grav. **36** 025001 (2019), arXiv:1805.05943 [gr-qc].

[11] Berra-Montiel J., *The Polymer representation for the scalar field: a Wigner functional approach*, Class. Quantum Grav. **37** 055009 (2020), arXiv:1911.00945 [gr-qc].

[12] Berra-Montiel J. and Molgado A., *Quasi-probability distributions in Loop Quantum Cosmology*, Class. Quantum Grav. **37** 215003 (2020), arXiv:2007.01324 [gr-qc].

[13] Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D., *Deformation theory and quantization I. Deformations of symplectic structures*, Ann. Phys., NY **111** 61–110 (1978).

[14] Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D., *Deformation theory and quantization. II. Physical applications*, Ann. Phys., NY **111** 111–51 (1978).

[15] Kontsevich M., *Deformation quantization of poisson manifolds*, Lett. Math. Phys. **66** 157–216 (2003), arXiv:q-alg/9709040.

[16] Waldmann S., *Recent developments in deformation quantization*, in Proc. Regensburg Conf. 2014 on Quantum Mathematical Physics, eds. Finster F., Kleiner J., Röken C. and Tolksdorf J. (Birkhäuser, 2016) pp. 421–439, arXiv:1502.00097 [math.QA].

[17] Fredenhagen K. and Rejzner K., *Perturbative construction of models of algebraic quantum field theory*, in Advances in Algebraic Quantum Field Theory, eds. Brunetti R., Dappiaggi C., Fredenhagen K. and Yngvason J. (Springer Cham, Switzerland, 2015), pp. 31–74, arXiv:1503.07814 [math-ph].

[18] Garcia-Compean H., Plebansky J. F., Przanowski M. and Turrubiates F. J., *Deformation quantization of classical fields*, Int. J. Mod. Phys. **A16** 2533–2558 (2001), arXiv:hep-th/9909206.

[19] Cordero R., Garcia-Compean H., and Turrubiates F. J., *Deformation quantization of cosmological models*, Phys. Rev. **D83** 125030 (2011), arXiv:1102.4379 [hep-th].

[20] Berra-Montiel J. and Molgado A., *Deformation quantization of constrained systems: a group averaging approach*, Class. Quantum Grav. **37** 055009 (2020), arXiv:1911.00945 [gr-qc].

[21] Berra-Montiel J. and Molgado A., *Coherent representation of fields and deformation quantization*, Int. J. Geom. Meth. Mod. Phys. **17** 11 2050166 (2020), arXiv:2005.14333 [quant-ph].

[22] Berra-Montiel J., *Star product representation of coherent state path integrals*, (2020), arXiv:2007.02483 [quant-ph].

[23] Berra-Montiel J. and Cartas R., *Deformation quantization and the tomographic representation of quantum fields*, to appear in Int. J. Geom. Meth. Mod. Phys. (2020), arXiv:2005.14333 [quant-ph].

[24] Folland G. B., *An Abstract Course in Harmonic Analysis*, 2nd edn. (London: Taylor and Francis, 2016).

[25] Reiter H. and Stegeman J. D., *Classical Harmonic Analysis and Locally Compact Groups* (London Mathematical Society Monographs) 2nd edn (Oxford: Clarendon, 2000).

[26] Deitmar A. and Echterhoff S., *Principles of Harmonic Analysis*, 2nd edn. (Switzerland: Springer, 2014).

[27] Chojnaki W., *Almost periodic Schrödinger operators in $L^2(\mathbb{R})$ whose point spectrum is not all of the spectrum*, J. Fun. Anal. **65** 236 (1986).

[28] Ashtekar A., Bojowald M. and Lewandowski J, *Mathematical structure of loop quantum cosmology*, Adv. Theor. Math. Phys. **7** 233 (2003), arXiv:gr-qc/0304074.

[29] Ashtekar A., Corichi A. and Singh P., *Robustness of key features of loop quantum cosmology*, Phys. Rev. **D77** 024046 (2008), arXiv:0710.3565 [gr-qc].

[30] Takhtajan L. A., *Quantum Mechanics for Mathematicians (Graduate Studies in Mathematics, Vol 95)* (Rhode Island: American Mathematical Society, 2008).

[31] Fewster C. J. and Sahlmann H., *Phase space quantization and loop quantum cosmology: a Wigner function for the Bohr-compactified real line*, Class. Quantum Grav. **25** 225015 (2008), arXiv:0804.2541 [math-ph].

[32] Zachos C. K., Fairlie D. B. and Curtright T. L., *Quantum Mechanics in Phase Space: An Overview with Selected Papers*, (Singapore: World Scientific, 2005).