Some Gronwall–Bellman Inequalities on Time Scales and Their Continuous Forms: A Survey

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Abstract: Some generalizations of the Gronwall–Bellman (G–B) inequality are presented in this paper in continuous form and on time scales. After S. Hilger introduced the time scales theory in 1988, over the years many mathematicians have studied new versions of this inequality according to new results; the purpose of this paper is to present some of them. Therefore, in the Introduction, some generalizations of G–B inequality in continuous forms, linear and nonlinear are presented. In the second section, some important and interesting results on time scales theory are given. In the third and main part of our paper, G–B inequalities on time scales and their possible connection with G–B inequalities presented in the introduction are investigated. In particular, in the third section of this work, more attention is given to G–B type inequalities on time scales discussed in the last four years.

Keywords: Gronwall-Bellman inequality; integral inequalities; linear inequalities; nonlinear inequalities; time scales

1. Introduction

In this paper, we present the Gronwall–Bellman (G–B) inequality and some of its successive forms, which have been discussed and proved by many mathematicians since 1919. No new results are presented in this paper. We have selected some of the G–B inequalities presented in continuous forms, which are reported in this section, and we have found their corresponding forms on time scale theory, which are presented in the third and main part of this paper. The aim of this work is to highlight the link between these two forms of G–B inequalities, in order to show that inequalities presented on a time scale could be seen as a generalization of the inequalities in continuous form. We have chosen a small number of G–B inequalities, compared with the number of results available, in fact, we have decided to concentrate our attention only on inequalities in which bounded function, indicated in this paper by “u”, is a function with one variable. In addition, we have chosen to present some forms of G–B inequality that B.G. Pachpatte discussed in 1975, which we call Gronwall–Bellman–Pachpatte inequalities. Therefore, in this paper, Bihari’s inequalities and others generalizations of the G–B inequalities are not reported.

To emphasize the inequalities presented in this section, we have decided to present them in Theorems and named every result.

Before the presentation of the Gronwall inequality, we need to specify some notations we are going to use in this paper.

Notation 1. $\mathbb{R}^+ = \{ x \in \mathbb{R} | x \geq 0 \}$, $\mathbb{N}_0 = \{ 0, 1, 2, \ldots \}$, $\emptyset$ denotes the empty set, $I$ is an interval in $\mathbb{R}$ of the form $[t_0, t_1]$, $C^0(\mathbb{R}^+, \mathbb{R}^+) = \{ f : \mathbb{R}^+ \to \mathbb{R}^+ | f \text{ continuous} \}$.

In 1919, T.H. Gronwall [1] discussed and proved that if $u \in C^0(I, \mathbb{R}^+)$ and if

$$u(t) \leq \int_{t_0}^{t} [bu(x) + a] \, dx, \quad t \in I$$

then $u(t) = 0$ for all $t \in I$. In this inequality, $b \geq 0$ and $a \geq 0$. The inequality (1.1) is called the Gronwall inequality.
is satisfied, then
\[ u(t) \leq ah \cdot \exp(bh), \quad t \in I, \]
where \( a, b \) are real nonnegative constants and \( h = t_1 - t_0 \).

This result is called Gronwall inequality. All integrals that will appear below (and the one in Gronwall inequality) are Riemann integrals.

In past years, a great number of mathematicians have studied and generalized this type of inequality as well as T.H. Gronwall. In 1943, R.E. Bellman considered a new version of this inequality and presented it in an integral form, the following theorem is about this result.

**Theorem 1.** (Gronwall–Bellman inequality) ([2], Lemma 1) Let \( u, f \in C^0(\mathbb{R}^+, \mathbb{R}^+) \) and let \( c \) be a real positive constant, if
\[ u(t) \leq c + \int_0^t f(x)u(x)dx, \quad t \in \mathbb{R}^+ \]
is satisfied, then
\[ u(t) \leq c \cdot \exp\left(\int_0^t f(x)dx\right), \quad t \in \mathbb{R}^+. \]

In 1958, a generalization of this result was given by himself with L.C. Kenneth [3], even if R.E. Bellman and L.C. Kenneth proved their result in the case of function \( u \) is defined on \( I \) finite interval of \( \mathbb{R} \); therefore, we decided to present a much more general version, where \( u \) is defined on \( \mathbb{R}^+ \). B.G. Pachpatte presented this general version in 2001 as follows.

**Theorem 2.** (Gronwall–Bellman inequality’s general form) ([4], Lemma 2.1) Let \( u, f \in C^0(\mathbb{R}^+, \mathbb{R}^+) \) and
1. let \( \alpha \in C^0(\mathbb{R}^+, \mathbb{R}^+) \) be a nondecreasing function, if
\[ u(t) \leq \alpha(t) + \int_0^t f(x)u(x)dx, \quad t \in \mathbb{R}^+ \]
is satisfied, then
\[ u(t) \leq \alpha(t) \cdot \exp\left(\int_0^t f(x)dx\right), \quad t \in \mathbb{R}^+. \]
2. let \( \alpha \in C^0(\mathbb{R}^+, \mathbb{R}^+) \) be a nonincreasing function, if
\[ u(t) \leq \alpha(t) + \int_t^\infty f(x)u(x)dx, \quad t \in \mathbb{R}^+ \]
is satisfied, then
\[ u(t) \leq \alpha(t) \cdot \exp\left(\int_t^\infty f(x)dx\right), \quad t \in \mathbb{R}^+. \]

We can also have a discrete form of G–B inequality presented in Theorem 2 as follow.

**Theorem 3.** (Gronwall–Bellman inequality’s discrete general form) ([4], Lemma 2.5) Let \( u, f, \alpha \in \mathbb{R}^+ \) defined for \( n \in \mathbb{N}_0 \), \( \alpha \) is a nondecreasing function, if
\[ u(n) \leq \alpha(n) + \sum_{x=0}^n f(x)u(x), \quad n \in \mathbb{N}_0 \]
is satisfied, then
\[
U(n) \leq \alpha(n) \prod_{x=0}^{n} \left[ f(x) + 1 \right], \ n \in \mathbb{N}_0.
\]

From now on, we will consider only the G–B inequality type as the one in Theorem 2(1), because this is the form that R.E. Bellman and L.C. Kenneth proposed and proved and then B.G. Pachpatte generalized. In 1975 and in the subsequent years, B.G. Pachpatte employed the G–B inequality and proved various generalizations of it, we explicitly report only two of them in the following theorems that the author himself discussed in 1998.

**Theorem 4.** (Gronwall–Bellman–Pachpatte inequality) ([5], Theorem 1.3.3) Let \( u, g, f \in C^0(I, \mathbb{R}^+) \), let \( \alpha \) be a positive, continuous and nondecreasing function defined on \( I \), if
\[
u(t) \leq \alpha(t) + g(t) \int_{0}^{t} f(x)u(x)dx, \ t \in I
\]
is satisfied, then
\[
u(t) \leq \alpha(t) \left[ 1 + g(t) \int_{0}^{t} f(x) \cdot \exp \left( \int_{s}^{t} f(y)g(y)dy \right) dx \right], \ t \in I.
\]

**Theorem 5.** (Gronwall–Bellman–Pachpatte inequality) ([5], Theorem 1.3.4) Let \( u, g, f, \alpha, \beta \in C^0(I, \mathbb{R}^+) \), if
\[
u(t) \leq \beta(t) + \alpha(t) \int_{0}^{t} f(x)u(x) + g(x)dx, \ t \in I
\]
is satisfied, then
\[
u(t) \leq \beta(t) + \alpha(t) \int_{0}^{t} \left[ f(x)\beta(x) + g(x) \right] \cdot \exp \left( \int_{s}^{t} f(y)\alpha(y)dy \right) dx, \ t \in I.
\]

After only two years, B.G. Pachpatte reported some inequalities in nonlinear form, we present one of them in the next result.

**Theorem 6.** (Bellman–Pachpatte nonlinear inequality) ([6], Theorem 1. (a1)) Let \( u, \alpha, \beta, g, f \in C^0(\mathbb{R}^+, \mathbb{R}^+) \) and \( p > 1 \) be a real constant, if
\[
u^p(t) \leq \beta(t) + \alpha(t) \int_{0}^{t} \left[ f(x)\beta(x) + g(x) \right]dx, \ t \in \mathbb{R}^+
\]
is satisfied, then
\[
u^p(t) \leq \beta(t) + \alpha(t) \int_{0}^{t} \left[ f(x)\beta(x) + g(x) \left( \frac{p-1}{p} + \frac{\beta(x)}{p} \right) \right] \cdot \exp \left( \int_{s}^{t} \alpha(\theta) \left( f(y) + \frac{g(y)}{p} \right) dy \right) dx, \ t \in \mathbb{R}^+.
\]

In 2007, S.K. Choi, B. Kang and N. Koo [7] discussed and proved a particular type of G–B inequality, we are going to present the form that S.K. Choi and N. Koo proposed in 2010 as follows.
We have that will definite it better in the next paragraph.

**Theorem 8.** (Kendre–Latpate–Ranmal nonlinear inequality) ([9], Theorem 2.1) Let \( u, f \in C^0(I, \mathbb{R}^+), \) let \( c \) be a real and nonnegative constant, let \( \kappa, \kappa_i \in C^0(I \times I, \mathbb{R}^+) \); if

\[
  u(t) \leq c + \int_{t_0}^{t} f(x) \left[ u(x) + \int_{t_0}^{x} \kappa(x, y) u(y) \, dy \right] \, dx, \ t \in I
\]

is satisfied, then

\[
  u(t) \leq c \left[ 1 + \int_{t_0}^{t} f(x) \cdot \exp \left( \int_{t_0}^{x} \left( f(y) + \kappa(y, y) + \int_{t_0}^{y} \kappa_i(y, z) \, dz \right) \, dy \right) \, dx \right], \ t \in I.
\]

In 2014, new important generalization of the Bellman–Pachpatte inequality were given and proved by S.D. Kendre, S.G. Latpate and S.S. Ranmal, who replaced the linear term by nonlinear term \( p \) as follows.

**Theorem 7.** (Choi–Koo inequality) ([8], Corollary 3.10) Let \( u, g, f, a' \in C^0(I, \mathbb{R}^+) \), let \( p \) be a real constant such that \( p \geq 1 \), if

\[
  u(t)^p \leq \alpha(t) + \int_{t_0}^{t} f(x) u(x) + \int_{t_0}^{t} g(x) u^p(x) \, dx, \ t \in I
\]

is satisfied, then

\[
  u(t)^p \leq \frac{\alpha(t_0)}{1 - R_1} \cdot \exp \left( \int_{t_0}^{t} m_1 f(z) \, dz \right) + \int_{t_0}^{t} \left( f_{y} \, m_1 f(z) \, dz \right) \, dx
\]

where \( k > 0, m_1 = \frac{1}{p} k^{-\frac{1}{p}} \), \( m_2 = \frac{1}{p} k^{\frac{1}{p}} \) and

\[
  R_1 = \int_{t_0}^{t} g(x) \cdot \exp \left( \int_{t_0}^{x} m_1 f(z) \, dz \right) \, dx < 1.
\]

All of these inequalities are even now an important instrument for the theory of differential equations on time scales; therefore, in our work, we concentrate our attention on the way the G–B inequality and the successive forms we introduce could be presented in this theory; to show it, we decided to present some of the most important versions of G–B inequality on time scales, with a bigger focus on results that have been discussed and proved in the last four years. We will focus on these inequalities in the third and last section of our treatment. Stefan Hilger was the first to discover the theory of time scales and presented it in 1988 in his PhD. thesis [10]; dynamic equations on time scales received a lot of attention after their introduction by S. Hilger, and the study of equations and inequalities on time scales became an interesting part of mathematics. The purpose of this theory is to unify continuous and discrete analysis; therefore, the domain of the function we called previously \( u \) is a particular set indicated by \( \mathbb{T} \) and called time scale. We will definite it better in the next paragraph.

**2. Preliminaries on Time Scales**

A **time scale**, denoted by \( \mathbb{T} \), is an arbitrary nonempty closed subset of the real numbers. We have that \( \mathbb{R}, \mathbb{Z}, \mathbb{N}, [0, 1] \cup [2, 3] \) are examples of time scales, while \( \mathbb{Q}, \mathbb{C}, \mathbb{R} \setminus \mathbb{Q}, (0, 1) \) are not time scales. When not specified, \( \mathbb{T} \) denotes an arbitrary time scale. We are now going to give some definitions and results on time scales (for a much more complete discussion, see [11,12]).
Definition 1. (Forward and backward jump operator) [11] Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by
\[
\sigma(t) = \inf \{s \in \mathbb{T}; s > t\},
\rho(t) = \sup \{s \in \mathbb{T}; s < t\}.
\]
We put
\[
\inf \emptyset = \sup \mathbb{T}, \text{ i.e., } \sigma(t) = t \text{ if } \mathbb{T} \text{ has a maximum } t,
\sup \emptyset = \inf \mathbb{T}, \text{ i.e., } \rho(t) = t \text{ if } \mathbb{T} \text{ has a minimum } t.
\]

Remark 1. We can observe that if $\mathbb{T} = \mathbb{R}$, we have that $\sigma(t) = \inf (t, \infty) = t$, for all $t \in \mathbb{R}$, while if $\mathbb{T} = \mathbb{Z}$, we have that $\sigma(t) = \inf \{t + 1, t + 2, \ldots \} = t + 1$, $\forall t \in \mathbb{R}$.

Definition 2. (Graininess) [11] The graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$.

Hence, the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$, while it is constant 1 if $\mathbb{T} = \mathbb{Z}$; however, a time scale $\mathbb{T}$ could have non-constant graininess.

Definition 3. [11] A point $t \in \mathbb{T}$ is said to be
1. left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$;
2. right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$;
3. left-scattered if $\rho(t) < t$;
4. right-scattered if $\sigma(t) > t$.

Points that are simultaneously right-dense and left-dense are said to be dense, while points that are simultaneously right-scattered and left-scattered are said to be isolated.

Definition 4. [11]
1. $\mathbb{T}^k$ is a set defined as follows: if $\mathbb{T}$ has a left-scattered maximum $M$, $\mathbb{T}^k = \mathbb{T} - M$; otherwise $\mathbb{T}^k = \mathbb{T}$.
2. $\mathbb{T}_k$ is a set defined as follows: if $\mathbb{T}$ has a right-scattered minimum $m$, $\mathbb{T}_k = \mathbb{T} - m$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 5. (Regulated function) [11] A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 6. (Delta derivative and delta differentiable) [11] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function and let $t \in \mathbb{T}^k$. Then, we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$, for some $\delta > 0$) such that
\[
|f\sigma(t) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \forall s \in U.
\]
We call $f^\Delta(t)$ the delta derivative of $f$ at $t$.

Moreover, we say that $f$ is delta differentiable (or differentiable) on $\mathbb{T}^k$, provided $f^\Delta(t)$ exists $\forall t \in \mathbb{T}^k$. Function $f^\Delta : \mathbb{T}^k \rightarrow \mathbb{R}$ is then called the delta derivative of $f$ on $\mathbb{T}^k$.

Remark 2. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$; if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t + 1) - f(t)$.

Definition 7. (Rd-continuous function) [11] A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called right-dense continuous (rd-continuous) provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.
The set of rd-continuous functions of the type \( f : T \to \mathbb{R} \) is denoted by \( C^0_{rd} = C^0_{rd}(T, \mathbb{R}) \) and the set of rd-continuous functions of the type \( f : T \to \mathbb{R}^+ \) is denoted by \( C^0_{rd^+} = C^0_{rd}(T, \mathbb{R}^+) \). The set of functions of the type \( f : T \to \mathbb{R} \) that are differentiable and whose derivative is rd-continuous is denoted by \( C^1_{rd} = C^1_{rd}(T, \mathbb{R}) \) and the set of functions of the type \( f : T \to \mathbb{R}^+ \) that are differentiable and whose derivative is rd-continuous is denoted by \( C^1_{rd^+} = C^1_{rd}(T, \mathbb{R}^+) \).

We want to cite an important result that S. Hilger reported in his work in 1990 and that connected rd-continuous functions to regulated functions as follows.

**Remark 3.** [12] continuous \( \implies \) rd-continuous \( \implies \) regulated.

**Definition 8.** (Antiderivative) [11] If \( F^\Delta(t) = f(t), \tau \in \mathbb{T}^k \), then \( F \) is called an antiderivative of \( f \), and in this case, we define the integral of \( f \) on \( T \) by

\[
\int_s^t f(\theta) \Delta \theta = F(t) - F(s),
\]

where \( s, t \in \mathbb{T} \).

**Lemma 1.** ([11], Theorem 2.2) Every rd-continuous function possesses an antiderivative.

**Definition 9.** (Regressive) [11] A function \( f : T \to \mathbb{R} \) is called regressive if

\[
1 + \mu(\tau) f(\tau) \neq 0, \forall \tau \in \mathbb{T},
\]

concerning initial value problem

\[
\begin{cases}
y^\Delta(\tau) = f(\tau) y(\tau), \\
y(\tau_0) = 1, \tau_0 \in \mathbb{T}.
\end{cases}
\]

\( \mathbb{R}_{rd} \) denotes the set of all regressive and right-dense continuous functions \( f : T \to \mathbb{R} \) and \( \mathbb{R}^{rd^+} = \{ f \in \mathbb{R}_{rd}, 1 + \mu(\tau) f(\tau) > 0, \forall t \in \mathbb{T} \} \); \( \mathbb{R}_{rd^+} \) denotes the set of all regressive and right-dense continuous functions \( f : T \to \mathbb{R}^+ \) and \( \mathbb{R}^{rd^+} = \{ f \in \mathbb{R}_{rd^+}, 1 + \mu(\tau) f(\tau) > 0, \forall t \in \mathbb{T} \} \).

**Lemma 2.** ([11], Theorem 3.1) If \( f \in \mathbb{R}_{rd} \) then (1) has an unique solution.

**Definition 10.** (Cylinder transformation) [11,12] For \( \ell \in \mathbb{R}^+ \) the function \( \xi_{\ell} : \{ z \in \mathbb{C}; \ell z + 1 \neq 0 \} \to \mathbb{C} \setminus \{ z \in \mathbb{C}; 1m(\ell z) = (zk + 1) \pi \} \) for a certain \( k \in \mathbb{Z} \) is called cylinder transformation and is defined by

\[
\xi_{\ell}(z) = \begin{cases} 
\text{Log}(1+\ell z) & \text{if } \ell \neq 0 \\
z & \text{if } \ell = 0;
\end{cases}
\]

where \( \text{Log}(\cdot) \) is the principal logarithm function.

**Definition 11.** (Exponential function) [11] The unique solution of (1) is called exponential function and is denoted by \( e_f(\cdot, \tau_0) \). An explicit formula for \( e_f(t, s) \) is

\[
e_f(t, s) = \exp\left( \int_s^t e_{\mu(\theta)}(f(\theta)) \Delta \theta \right),
\]

where \( s, t \in \mathbb{T} \).

**Remark 4.** ([11], Example 4.1) We note that
1. if $T = \mathbb{R}$, then

\[
\begin{align*}
 e_f(t, t_0) &= \exp \left( \int_{t_0}^t f(\theta)d\theta \right), \\
 e_c(t, t_0) &= \exp (c(t - t_0)),
\end{align*}
\]

where $t_0, t \in \mathbb{R}, c \in \mathbb{R}, f \in C^0(\mathbb{R}, \mathbb{R});$

2. if $T = \mathbb{Z}$, then

\[
\begin{align*}
 e_f(t, t_0) &= \begin{cases} 
 \prod_{k=t_0}^{t-1} [1 + f(t)] & \text{if } t > t_0, \\
 1 & \text{if } t = t_0, \\
 \prod_{k=t_0}^{t-1} [1 + f(t)]^{-1} & \text{if } t < t_0,
\end{cases} \\
 e_c(t, t_0) &= \begin{cases} 
 \prod_{k=t_0}^{t-1} (1 + c) = (1 + c)^{t-t_0} & \text{if } t > t_0, \\
 1 & \text{if } t = t_0, \\
 \prod_{k=t_0}^{t-1} (1 + c)^{-t} & \text{if } t < t_0,
\end{cases}
\end{align*}
\]

where $t_0, t \in \mathbb{Z}, c \in \mathbb{R}, f \in C^0(\mathbb{Z}, \mathbb{R}).$

**Lemma 3.** ([13], Theorem 5.2; [14], Lemma 3.2) Let $f \in \mathbb{R}_{rd}$ then

1. $e_0(t, s) \equiv 1$ and $e_f(t, t) \equiv 1$;
2. if $f \in \mathbb{R}_{0}^{\geq 0}$ then $e_f(t, t_0) > 0 \forall t, t_0 \in \mathbb{T};$
3. $e_f(\sigma(t), s) = (1 + \mu(t)f(t))e_f(t, s);$  
4. $e_f(t, s) = \frac{1}{e_f(s, t)};$$
5. $e_f(t, s)e_f(s, r) = e_f(t, r);$  
6. $\int_{t_0}^{t_1} f(t)e_f(t_2, \sigma(t))\Delta t = -\int_{t_0}^{t_1} e_f(t_2, \cdot)\Delta(t) = e_f(t_2, t_0) - e_f(t_2, t_1).$  

We report a useful result for the introduction of the last inequalities in the next paragraph.

**Lemma 4.** Let $t_0 \in \mathbb{T}^k$ and $\kappa : \mathbb{T} \times \mathbb{T}^k \rightarrow \mathbb{R}$ be a function continuous at $(t, t)$, where $t \geq t_0, t \in \mathbb{T}^k$ with $t > t_0$. Assume that $\kappa^\Delta(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\epsilon > 0$, there exists a neighborhood $U$ of $t$, independent of $\tau \in [t_0, \sigma(t)]$, such that

\[
\left| \kappa(\sigma(t), \tau) - \kappa(s, \tau) - \kappa^\Delta(t, \tau)[\sigma(t) - s] \right| \leq \epsilon|\sigma(t) - s|, \forall s \in U,
\]

where $\kappa^\Delta$ denotes the delta derivative of $\kappa$ with respect to the first variable, then

\[
f(t) = \int_{t_0}^{t} \kappa(t, \tau)\Delta \tau
\]

implies

\[
f^\Delta(t) = \int_{t_0}^{t} \kappa^\Delta(t, \tau)\Delta \tau + \kappa(\sigma(t), t).
\]

For a better simplification, we put our notations in Table 1.
Table 1. Notations on T arbitrary time scale.

| Symbol         | Abbreviation | Definition                                                                 |
|----------------|--------------|-----------------------------------------------------------------------------|
| $C^0_d(T, \mathbb{R})$ | $C^0_d$       | The set of rd-continuous functions of the type $f : T \to \mathbb{R}$       |
| $C^0_d(T, \mathbb{R}^+)$ | $C^0_d^+$    | The set of rd-continuous functions of the type $f : T \to \mathbb{R}^+$    |
| $C^1_d(T, \mathbb{R})$ | $C^1_d$       | The set of functions of the type $f : T \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous |
| $C^1_d(T, \mathbb{R}^+)$ | $C^1_d^+$    | The set of functions of the type $f : T \to \mathbb{R}^+$ that are differentiable and whose derivative is rd-continuous |
| $\mathbb{R}^d_d(T, \mathbb{R})$ | $\mathbb{R}^d_d$ | The set of all regressive and right-dense continuous functions $f : T \to \mathbb{R}$ |
| $\mathbb{R}^d_d(T, \mathbb{R}^+)$ | $\mathbb{R}^d_d^+$ | The set of all regressive and right-dense continuous functions $f : T \to \mathbb{R}^+$ |
| $\mathbb{R}^{>0}_d(T, \mathbb{R})$ | $\mathbb{R}^{>0}_d$ | The set $\{ f | f \in \mathbb{R}^d_d, 1 + \mu(\tau)f(\tau) > 0, \forall t \in T \}$ |
| $\mathbb{R}^{>0}_d(T, \mathbb{R}^+)$ | $\mathbb{R}^{>0}_d^+$ | The set $\{ f | f \in \mathbb{R}^d_d^+, 1 + \mu(\tau)f(\tau) > 0, \forall t \in T \}$ |

We have to point out the following result to clarify Theorems and Remarks we are going to present in the third and main part of this paper.

**Remark 5.** We can note that if $f \in C^0_d^+, \forall t \in T$, then $f \in \mathbb{R}^{>0}_{rd^+}$.

### 3. Main Results on Time Scales

In this section, we are going to present some of the most important forms of the Gronwall–Bellman (G–B) inequality on time scales, in particular, we will put a bigger focus on new forms that are presented and proved in the past four years. We decided to select forms of G–B inequality type on time scale that are more simple than others, because we think this choice could better shown the link between this inequalities and inequalities we presented in our introduction.

We will always assume that $t \geq t_0$. First of all, we would like to present the G–B inequality on time scale, which B.G. Pachpatte reported in his paper in 2006.

**Theorem 9.** ([15], Theorem 3.1) Let $u, f, \alpha \in C^0_d^+$, and let $a$ be a nondecreasing function, if

$$u(t) \leq \alpha(t) + \int_{t_0}^t f(\theta)u(\theta)\Delta \theta, \ t \in T^k$$

is satisfied, then

$$u(t) \leq \alpha(t)\epsilon_f(t, t_0), \ t \in T^k,$$

where $\epsilon_f(t, t_0)$ is a solution of the problem (1) in Definition 9 (in fact, $f \in \mathbb{R}^{>0}_{rd^+}$).

**Remark 6.** If $T = \mathbb{R}$, in Theorem 9, we can observe (using Remark 4(1) and others results we presented in previous section) that the inequality on time scale obtained in Theorem 9 is reduced to the continuous inequality we have in Theorem 2(1) (even if we have a result much more general than in Theorem 2, in fact the functions $u, f, \alpha : \mathbb{R} \to \mathbb{R}^+$, instead in Theorem 2 the functions $u, f, \alpha : \mathbb{R}^+ \to \mathbb{R}^+$).

In 2001, an important form of G–B inequality was presented by R.P. Agarwal, M. Bohner and A. Peterson, whom proved the following result on time scale.

**Theorem 10.** ([13], Theorem 5.6) Let $u, \alpha \in \mathbb{R}^d_d$ and $f \in \mathbb{R}^{>0}_{rd^+}$, then

$$u(t) \leq \alpha(t) + \int_{t_0}^t u(\theta)f(\theta)\Delta \theta, \ t \in T$$
implies
\[ u(t) \leq \alpha(t) + \int_{t_0}^{t} e_f(t, \sigma(\theta)) \alpha(\theta) f(\theta) \Delta \theta, \quad t \in \mathbb{T}. \]

In 2005, E. Akin-Bohner, M. Bohner and F. Akin discussed and proved the Gronwall–Bellman–Pachpatte inequality, presented in Theorem 4, on a time scale as follows.

**Theorem 11.** ([16], Theorem 3.1) Let \( u, \alpha, \beta \in C_{rd}^0, \) let \( g, f \in C_{rd}^0, \) if
\[ u(t) \leq \alpha(t) + g(t) \int_{t_0}^{t} [f(\theta) u(\theta) + \beta(\theta) u(\theta)] \Delta \theta, \quad t \in \mathbb{T}, \]
is satisfied, then
\[ u(t) \leq \alpha(t) + g(t) \int_{t_0}^{t} [\alpha(\theta) f(\theta) + \beta(\theta) e_{fg}(t, \sigma(\theta))] \Delta \theta, \quad t \in \mathbb{T}, \]
where \( e_{fg}(t, \cdot) \) is a solution of initial value problem (1) in Definition 9 with \( f \) replaced by \( fg \) (in fact, \( fg \in \mathbb{R}_{rd}^0 \)).

**Corollary 1.** ([16], Remark 3.3(ii)) Let \( u, \alpha \in C_{rd}^0, \) let \( g, f \in C_{rd}^0, \) \( \alpha \) be a nondecreasing function, if
\[ u(t) \leq \alpha(t) + g(t) \int_{t_0}^{t} f(\theta) u(\theta) \Delta \theta, \quad t \in \mathbb{T}, \]
is satisfied, then
\[ u(t) \leq \alpha(t) \left[ 1 + g(t) \int_{t_0}^{t} f(\theta) e_{fg}(t, \sigma(\theta)) \Delta \theta \right], \quad t \in \mathbb{T}, \]
where \( e_{fg}(t, \cdot) \) is a solution of the initial value problem (1) in Definition 9 with \( f \) replaced by \( fg \).

**Remark 7.** If \( \mathbb{T} = \mathbb{R} \) in Corollary 1, we can observe (using Remarks 1 and 4(1) and others results we presented in the previous section) that the inequality on the time scale obtained in Corollary 1 is reduced to the continuous inequality we have in Theorem 4 (even if we have a result much more general than in Theorem 4). In fact, the functions \( u, \alpha : \mathbb{R} \rightarrow \mathbb{R} \) and \( f, g : \mathbb{R} \rightarrow \mathbb{R}^+ \), instead in Theorem 4 the functions \( u, f, g : I \rightarrow \mathbb{R}^+ \) and \( \alpha \) is a positive function defined on \( I \).

In 2007, W.N. Li and W. Sheng proposed, as the first of their main results, the following theorem.

**Theorem 12.** ([17], Theorem 3.2) Let \( u, \alpha, \beta, g, f \in C_{rd}^0, \) let \( p, q \) be real constants, if
\[ u^p(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} [f(\theta) u^p(\theta) + g(\theta) u^q(\theta)] \Delta \theta, \quad t \in \mathbb{T}, \]
is satisfied, then
\[ u^p(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} \left[ f(\theta) \beta(\theta) + g(\theta) \left( \frac{K(p-q) + q\beta(\theta)}{p^{\frac{p-1}{p}}} \right) \right] e_{A}(t, \sigma(\theta)) \Delta \theta, \quad t \in \mathbb{T}, \]
where \( K > 0 \) and
\[ A(t) = \alpha(t) \left( f(t) + \frac{qg(t)}{p^{\frac{p-1}{p}}} \right). \]
Theorem 14. ([8], Theorem 3.4) Let \( \alpha, \beta, g, f \in C_{rd}^{0} \), let \( p, q \) be real constants, if
\[
u_{\alpha}(t) = \beta(t) + \left( f(t) \frac{p - q}{p} K^{\frac{p-1}{p}} + \frac{q g(\tau)}{p K^{\frac{p-1}{p}}} \right) e_{B}(t, \sigma(\tau)) \Delta \tau, \quad t \in \mathbb{T}
\]
is satisfied, then
\[
u_{\alpha}(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} \left[ f(\sigma(\tau)) \frac{p - q}{p} K^{\frac{p-1}{p}} + \frac{q g(\tau)}{p K^{\frac{p-1}{p}}} \right] e_{B}(t, \sigma(\tau)) \Delta \tau, \quad t \in \mathbb{T}
\]
where \( K > 0 \) and
\[B(t) = \frac{q \alpha(t) f(t)}{p K^{\frac{p-1}{p}}}, \quad t \in \mathbb{T}.
\]

Remark 8. If \( \mathbb{T} = \mathbb{R}, p > 1, K = q = 1 \) and \( t_0 = 0 \) in Theorem 12, we can observe (using Remarks 1 and 4(1) and others results we presented in previous section) that the inequality on the time scale obtained in Theorem 12 is reduced to the continuous inequality we have in Theorem 6 (even if we have a result much more general than in Theorem 6, in fact the functions \( u, f, \alpha : \mathbb{R} \to \mathbb{R}^{+}, \) instead in Theorem 6 the functions \( u, f, \alpha : \mathbb{R}^{+} \to \mathbb{R}^{+} \).

After two years, W.N. Li and M. Han presented and discussed a similar result to the one just reported as follows.

Theorem 13. ([18], Theorem 2.4) Let \( u, \alpha, \beta, g, f \in C_{rd}^{0} \), let \( p, q \) be real constants, if
\[
u_{\alpha}(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} \left[ f(\theta) \frac{p - q}{p} K^{\frac{p-1}{p}} + \frac{q g(\theta)}{p K^{\frac{p-1}{p}}} \right] e_{B}(t, \sigma(\theta)) \Delta \theta, \quad t \in \mathbb{T}
\]
is satisfied, then
\[
u_{\alpha}(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} \left[ f(\sigma(\tau)) \frac{p - q}{p} K^{\frac{p-1}{p}} + \frac{q g(\tau)}{p K^{\frac{p-1}{p}}} \right] e_{B}(t, \sigma(\tau)) \Delta \tau, \quad t \in \mathbb{T}
\]
where \( K > 0 \) and
\[B(t) = \frac{q \alpha(t) f(t)}{p K^{\frac{p-1}{p}}}, \quad t \in \mathbb{T}.
\]

Remark 9. If \( \mathbb{T} = \mathbb{R}, p > 1, K = q = 1 \) and \( t_0 = 0 \) in Theorem 12, we observe (using Remarks 1 and 4(1) and others results we presented in previous section) that the inequality on the time scale obtained in Theorem 12 is reduced to the continuous inequality we have in Theorem 6 (even if we have a result much more general than in Theorem 6, in fact the functions \( u, f, \alpha : \mathbb{R} \to \mathbb{R}^{+}, \) instead in Theorem 6 the functions \( u, f, \alpha : \mathbb{R}^{+} \to \mathbb{R}^{+} \).

In 2010, S.K. Choi and N. Koo unified previous results and discussed a particular form of the G-B inequality on time scales, we present it in the following Theorem. Before giving it, we consider that \( x^{\Delta} \) denotes the delta derivative of \( x \) with respect to the first variable.

Theorem 14. ([8], Theorem 3.4) Let \( \mathbb{T} = \mathbb{R} \), let \( u, f \in C_{rd}^{0} \), let \( c \) be a real and nonnegative constant, let \( v(t, \theta) \) be defined as in Lemma 4 such that \( v(\sigma(t), t), v^{\Delta}(t, \theta) \in C_{rd}^{0} \) for \( t, \theta \in \mathbb{T} \) with \( t \geq \theta \); if
\[
u(t) \leq c + \int_{t_0}^{t} f(\theta) \left[ u(\theta) + \int_{t_0}^{\theta} v(\theta, \tau) u(\tau) \Delta \tau \right] \Delta \theta, \quad t \in \mathbb{T}
\]
is satisfied, then
\[
u(t) \leq c \left[ 1 + \int_{t_0}^{t} f(\theta) e_{p}(\theta, t_0) \Delta \theta \right], \quad t \in \mathbb{T}
\]
where
\[p(t, t_0) = f(t) + v(\sigma(t), t) + \int_{t_0}^{t} v^{\Delta}(t, \theta) \Delta \theta.
\]

Remark 10. If \( v(t, \theta) = 0 \) in the Theorem 14 and we assume in Theorem 9 that \( \alpha(t) = c \forall t \in \mathbb{T} \) with \( c \in \mathbb{R}^{+} \), then we have that Theorem 14 is reduced to Theorem 9.

Remark 11. If \( \mathbb{T} = \mathbb{R} \) in Theorem 14, we can observe (using Remarks 1, 2 and 4(1) and others results we presented in previous section) that the inequality on the time scale obtained in Theorem 14 is reduced to the continuous inequality we have in Theorem 7 (even if we have a result much more general...
Therefore, Theorem 16 is a generalization of Theorem 14. 

In 2013, the authors themselves proposed a different form of the G–B inequality on time scale as follows.

**Theorem 15.** ([19], Lemma 2.2) Let \( u, f, g \in C_{0,1}^{0}, -f \in R_{n,1}^{0} \), let \( c \) be a real nonnegative constant, let \( p \) be a real positive constant, \( p \neq 1 \), and \( q = 1 - p \); if

\[
    u(t) \leq c + \int_{t_0}^{t} f(\theta)u(\theta)\Delta \theta + \int_{t_0}^{t} g(\theta)u^p(\theta)\Delta \theta, \quad t \in T_0,
\]

is satisfied, then

\[
    u(t) \leq \frac{1}{e^{-f(t_0)}} \left[ c^d + q \int_{t_0}^{t} g(\theta)e^{\beta f(\theta,t_0)}\Delta \theta \right]^\frac{1}{q^t}, \quad t \in [t_0, \beta] \cap T,
\]

where

\[
    \beta = \sup \left\{ t \in T_0 ; \ c^d + q \int_{t_0}^{t} g(\theta)e^{\beta f(\theta,t_0)}\Delta \theta > 0 \right\}.
\]

In 2017, B. Ben Nasser, K. Boukerrioua, M. Defoort, M. Djemai and M. A. Hammami investigated Bellman–Pachpatte-type inequalities on time scales and proposed an interesting result as follows.

**Theorem 16.** ([20], Theorem 8) Let \( u, f \in C_{d}^{0} \), let \( c \) be a real nonnegative constant, let \( \omega \) be a positive and continuous function defined on \( T \times \mathbb{R}^+ \) and \( \kappa \in C_{d}^{0}(\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+) \) be functions satisfying

\[
    0 \leq \kappa(t,x) - \kappa(t,y) \leq \omega(t,y)(x - y),
\]

for \( t \in T^k, x \geq y \geq 0 \), and

\[
    \kappa^A(t,0) \geq 0, \quad \omega^A(t,0) \geq 0,
\]

for \( t \in T^k \); let \( v(t, \theta) \) be defined as in Lemma 4 such that \( v(t, \theta) \geq 0 \) and \( v^A(t, \theta) \geq 0 \) for \( t, \theta \in T \) with \( t \geq \theta \); if

\[
    u(t) \leq c + \int_{t_0}^{t} f(\theta) \left[ \kappa(\theta,u(\theta)) + \int_{t_0}^{\theta} v(\theta,\tau)\kappa(\tau,u(\tau))\Delta \tau \right] \Delta \theta, \quad t \in T^k
\]

is satisfied, then

\[
    u(t) \leq c + \int_{t_0}^{t} f(\theta) \left[ (\omega(t_0,0)c + \kappa(t_0,0))e_{\lambda}(\theta,t_0) + \int_{t_0}^{\theta} e_{\lambda}(\theta,\sigma(\tau))\mu(\tau)\Delta \tau \right] \Delta \theta, \quad t \in T^k
\]

where

\[
    \lambda(t) = \omega(\sigma(t),0)f(t) + \frac{\omega^A(t,0)}{\omega(t,0)} + v(\sigma(t),t) + \int_{t_0}^{t} v^A(t,\tau)\Delta \tau,
\]

\[
    \mu(t) = \kappa^A(t,0) + v(\sigma(t),t)\kappa(t,0) + \int_{t_0}^{t} v^A(t,\tau)\kappa(\tau,0)\Delta \tau.
\]

**Remark 12.** If we assume that \( \kappa(t,u(t)) = u(t), \ v(\theta,t) = \alpha(t), \ \omega(\theta,t) \equiv 1 \ \forall t \in T \) in Theorem 16, we have that \( \lambda = f + \alpha \) and Theorem 16 is reduced to Theorem 14 with \( p = \lambda \). Therefore, Theorem 16 is a generalization of Theorem 14.
Remark 13. If $\mathbb{T} = \mathbb{R}$ in Theorem 16 and we assume all conditions in Remark 12, we can observe (using Remarks 1, 2 and 4(1) and others results we presented in the previous section) that the inequality on the time scale obtained in Theorem 16 is reduced to the continuous inequality we have in Theorem 7, and if we assume all conditions in Remark 10 too, we have that Theorem 16 is reduced to the continuous inequality in Theorem 9.

In 2018, A.A. El-Deeb discussed G–B inequality with nonlinearity on a time scale and proved the following results.

**Theorem 17.** ([21], Theorem 3.1) Let $t_0, t_1 \in \mathbb{T}_k$ with $t_0 < t_1$, let $u, g, f, \alpha \in C^0_{rd}([t_0, t_1]_{\mathbb{T}}, \mathbb{R}^+)$, $a$ be delta-differentiable on $\mathbb{T}$ with $a^\lambda(t) \in \mathbb{R}^+$ and let $p$ be a real constant such that $p \geq 1$; if

$$u^p(t) \leq u(t) + \int_{t_0}^{t} f(\theta) u(\theta) d\theta + \int_{t_0}^{t} g(\theta) u^p(\theta) d\theta, \quad t \in [t_0, t_1]_{\mathbb{T}},$$

is satisfied, then

$$u^p(t) \leq \frac{u(t_0) + \int_{t_0}^{t} g(\theta) \left( \int_{t_0}^{\theta} [a^\lambda(\tau) + m_2 f(\tau)] \cdot e_{m_1 f}(\theta, \sigma(\tau)) d\tau \right) d\theta}{1 - R_2} \cdot e_{m_2 f}(t, t_0) + \int_{t_0}^{t} \left[ a^\lambda(t) + m_2 f(t) \right] e_{m_1 f}(t, \sigma(\theta)) d\theta$$

where $K > 0, m_1 = \frac{1}{p} K^{\frac{1}{p}}, m_2 = \frac{p - 1}{p} K^{\frac{1}{p}}, R_2 = \int_{t_0}^{t} g(\theta) e_{m_2 f}(\theta, t_0) d\theta < 1$.

**Remark 14.** If $\mathbb{T} = \mathbb{R}$ in Theorem 17, we can observe (using Remarks 1, 2 and 4(1) and other results we presented in previous section) that the inequality on time scale obtained in Theorem 17 is reduced to the continuous inequality we have in Theorem 8 (even if we have a result much more general than in Theorem 8, in fact the functions $u, g, f, \alpha : \mathbb{R} \rightarrow \mathbb{R}$, instead in Theorem 8 the functions $u, f, g, \alpha : I \rightarrow \mathbb{R}^+$).

**Theorem 18.** ([21], Theorem 3.2) Let $t_0, t_1 \in \mathbb{T}_k$ with $t_0 < t_1$ and $u, g, f, \alpha \in C^0_{rd}([t_0, t_1]_{\mathbb{T}}, \mathbb{R}^+)$, $a$ be delta-differentiable on $\mathbb{T}$ with $a^\lambda(t) \in \mathbb{R}^+$ and $k(t, \theta), a^\lambda(t, \theta) \in C^0_{rd}([t_0, t_1]_{\mathbb{T}} \times [t_0, t_1]_{\mathbb{T}}, \mathbb{R}^+)$ for $t_0 \leq \theta \leq t \leq t_1$ and let $p$ be a real constant such that $p \geq 1$; if

$$u^p(t) \leq \alpha(t) + \int_{t_0}^{t} k(t, \theta) u(\theta) d\theta + \int_{t_0}^{t} g(\theta) u^p(\theta) d\theta, \quad t \in [t_0, t_1]_{\mathbb{T}},$$

is satisfied, then

$$u^p(t) \leq \frac{\alpha(t_0) + \int_{t_0}^{t} g(\theta) \left( \int_{t_0}^{\theta} [a^\lambda(\tau) + m_2 \eta(\tau)] \cdot e_{m_1 \eta}(\theta, \sigma(\tau)) d\tau \right) d\theta}{1 - R_2} \cdot e_{m_2 \eta}(t, t_0) + \int_{t_0}^{t} \left[ a^\lambda(t) + m_2 \eta(t) \right] e_{m_1 \eta}(t, \sigma(\theta)) d\theta$$

where

$$K > 0, m_1 = \frac{1}{p} K^{\frac{1}{p}}, m_2 = \frac{p - 1}{p} K^{\frac{1}{p}},$$

$$\eta(t) = k(\sigma(t), t) + \int_{t_0}^{t} a^\lambda(t, \tau) d\tau,$$
Theorem 20 is reduced to the continuous inequality we have in Theorem 6. [14], Theorem 3.4

Theorem 20.  Let \( u, a \) be defined as in Theorem 18, \( \kappa_1(t, \theta), \kappa_2(t, \theta), \kappa_3(t, \theta) \) and \( \kappa_4(t, \theta) \in C^0_{rd}(\{t_0, t_1\} \times [t_0, t_1], \mathbb{R}^+) \) for \( t_0 \leq \theta \leq t \leq t_1 \), let \( t_0, t_1 \in T_k \) with \( t_0 < t_1 \) and let \( p \) be a real constant such that \( p > 1 \); if

\[
R_2 = \int_{t_0}^{t_1} g(\theta) e_{m_1}(\theta, t_0) \Delta \theta < 1.
\]

is satisfied, then

\[
u^p(t) \leq \alpha(t) + \int_{t_0}^{t} \kappa_1(t, \theta) u(\theta) \Delta \theta + \int_{t_0}^{t_1} \kappa_2(t, \theta) u^p(\theta) \Delta \theta, \quad t \in [t_0, t_1]
\]

is satisfied, then

\[
u^p(t) \leq \frac{\alpha(t) + \int_{t_0}^{t_1} \kappa_2(t_0, \theta) \left( \int_{t_0}^{t} \left[ \kappa(\theta) + m_2(\theta) \right] e_{m_1}(\theta, \tau(\theta)) \Delta \tau \right) \Delta \theta}{1 - R_3}
\]

\[
\cdot e_{m_1}(\theta, t_0) + \int_{t_0}^{t} \left[ \kappa(\theta) + m_2(\theta) \right] e_{m_1}(\theta, t, \sigma(\theta)) \Delta \theta
\]

where

\[
K > 0, \quad m_1 = \frac{1}{p} K^{1 - p}, \quad m_2 = \frac{1}{p} K^{1 - p}.
\]

\[
E_1(t) = \kappa_1(\sigma(t), t) + \int_{t_0}^{t} \kappa_3(t, \theta) \Delta \theta,
\]

\[
E_2(t) = \int_{t_0}^{t} \kappa_4(t, \theta) \Delta \theta,
\]

\[
R_3 = \int_{t_0}^{t_1} \kappa_2(t, t_0) e_{m_1}(\theta, t_0) \Delta \theta < 1.
\]

In 2019, A.A. El-Deeb discussed and proved other inequalities on time scale; we decided to propose the one that can have a link with the Bellman–Pachpatte inequality we enunciated in the introduction of this paper.

Theorem 20.  [14], Theorem 3.4

Let \( T_0 = [t_0, \infty) \cap T \), let \( u, g, f, a, b, \kappa_1(t, \theta), \kappa_2(t, \theta), \kappa_3(t, \theta) \), and \( \kappa_4(t, \theta) \in C^0_{rd}(T_{0} \times T_{0}, \mathbb{R}^+) \) and let \( p \) be a real constant such that \( p > 1 \); if

\[
u^p(t) \leq \beta(t) + \alpha(t) \left( \int_{t_0}^{t} f(\theta) u^p(\theta) + g(\theta) u(\theta) + \kappa(\theta, \sigma(\theta)) \Delta \theta \right), \quad t \in T_0
\]

is satisfied, then

\[
u^p(t) \leq \beta(t) + \alpha(t) \int_{t_0}^{t} \zeta(\theta) e_{m_1}(\theta, t_0) \Delta \theta,
\]

where

\[
\zeta(t) = f(t) \beta(t) + g(t) \left( \frac{p - 1}{p} + \frac{\beta(t)}{p} \right) + \kappa(\sigma(t), t) + \int_{t_0}^{t} \kappa_4(t, \theta) \Delta \theta,
\]

\[
\omega(t) = \alpha(t) \left( f(t) + \frac{g(t)}{p} \right).
\]

Remark 15.  If \( T = \mathbb{R} \) and \( \kappa(t, \theta) = 0 \) in Theorem 20, we can observe (using Remarks 1, 2 and 4(1) and other results we presented in the previous section) that the inequality on time scale obtained in Theorem 20 is reduced to the continuous inequality we have in Theorem 6.
4. Conclusions

The purpose of this paper is to present some of the most studied Gronwall–Bellman (G-B) type inequalities and show the connection between them and G–B inequalities on time scales, as we have said. Therefore, in our work, we do not prove new results but present some inequalities of the G–B type that have been discussed and proved since 1919. We have decided to choose inequalities on time scales that could better show this link in an immediate way, in particular A.A. El-Deeb discussed in [14,16] inequalities that are useful in this regard. Additionally, we have decided to focus our attention only on function “\( u \)” of one variable (where \( u \) are the bounded function appears in inequalities), even if there are a lot of interesting works regarding functions of two variables, two of them are [22,23]. During our work, we have tried to present a clear paper that not only could collect G–B inequalities on \( I, \mathbb{R}^+ \) and \( T \), but even present some of the most important results on time scales.

We can not hide that there were some difficulties to select the right results, first of all, the fact that we have analyzed only open access journals; therefore, a lot of results and interesting works are not taken into account and this fact has limited the possibility to report inequalities that could better fit our aim. Another problem was finding results on a time scale that could have a connection with G–B inequalities presented in continuous forms and at the same time that could have different forms with each other. In addition to this, this paper is supposed to be a small survey; therefore, we have tried to select results on time scales, which were published after just a few years. We have tried to present results on a time scale in forms which better fit with the G–B inequalities we presented in our introduction. Obviously, with simple operations, our results can be returned to their original form.

The purpose of the theory on time scale was to unify the continuous forms of equations and inequalities with results defined on the \( T \) time scale; therefore, in this paper, we have tried to emphasize this aim. Additionally, we have tried to mention that finding the generalized form of G–B inequality on a time scale is a very interesting part of mathematics and constitutes another reason for investigating further (we decided to present Theorem 14 and the following remark to report this aspect).

We think that could be useful to conduct other surveys on G–B inequalities with a bounded function with two or more variables, or analyze Gronwall–Belman–Bihari results on a time scale and presented them in a similar way as we have done. Furthermore, we think that complete this survey with sophisticated results on time scales could be an interesting and useful work to present.

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