Multi-dimensional Virasoro algebra and quantum gravity

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February 1, 2008

Abstract

I review the multi-dimensional generalizations of the Virasoro algebra, i.e. the non-central Lie algebra extensions of the algebra vect(N) of general vector fields in N dimensions, and its Fock representations. Being the Noether symmetry of background independent theories such as N-dimensional general relativity, this algebra is expected to be relevant to the quantization of gravity. To this end, more complicated modules which depend on dynamics in the form of Euler-Lagrange equations are described. These modules can apparently only be interpreted as quantum fields if spacetime has four dimensions and both bosons and fermions are present.

In: Mathematical physics research at the leading edge
ed: Charles V. Benton, pp 91-111
2004 Nova Science Publishers, Inc.
ISBN 1-59033-905-3
1 Introduction

It is widely recognized that a candidate theory of quantum gravity must be general-covariant, i.e. it must carry a representation of the full spacetime diffeomorphism group [9]. Since tensor densities are the classical modules of this group [29, 30], this implies in particular that tensor calculus is the appropriate language of general relativity. However, experience with conformal field theory teaches us that the physically interesting representations are projective, i.e. that the corresponding Lie algebra acquires an extension. It is thus natural to look for a generalization of the Virasoro algebra to $N$ dimensions, i.e. an extension $Vir(N)$ of the algebra $Vect(N)$ of general vector fields (on the $N$-dimensional torus, say). Other names for $Vect(N)$ are diffeomorphism algebra or generalized Witt algebra; algebraists often denote it by $W_N$ in honor of Witt.

The first interesting representations of $Vir(N)$ were constructed by Rao and Moody [27], and the Fock/vertex operator modules were essentially understood in [2, 4, 18]. However, general covariance by itself is certainly not enough to describe gravity; information about the Einstein equations must somehow be introduced. In [21] a class of modules with this property was described. To each dynamical system, we can associate a family of representations of its Noether symmetry algebra; this is quantization in the sense that the brackets acquire non-trivial quantum corrections. In particular, choosing the dynamical system to be general relativity gives us a kind of quantum gravity, which is well-defined as a $Vir(N)$ module.

A crucial step in the construction of Fock modules is the replacement of all fields by $p$-jets, where $p$ is a finite integer. In order to have a field theory interpretation, it must be possible to take the limit $p \to \infty$. This limit is problematic, because the abelian charges (the higher-dimensional analogues of the central charge) diverge, but the leading divergences can be cancelled by a clever choice of field content. With some natural assumptions of the form on the Euler-Lagrange equations (second order for bosons and first order for fermions, and Noether identities of one order higher), it turns out that the finiteness conditions can only be satisfied if spacetime has four dimensions and there are two bosons for every three fermions with the naïve counting of degrees of freedom; this relation holds in the standard model coupled to gravity. Note that already the prediction of both bosons and fermions is quite remarkable without superalgebras. However, the same argument also requires new gauge symmetries, including fermionic ones, perhaps indicating the need for some new physics.

The reason why $Vir(N)$ went unnoticed for several decades is that it
is not a central extension. The fact that $\text{vect}(N)$ has no central extension when $N > 1$ has been rediscovered many times [6, 24, 25]; see [10] for the classification of central extensions of simple Lie superalgebras of vector fields. However, it does have two inequivalent abelian but non-central extensions which both reduce to the Virasoro algebra when $N = 1$ [16, 27].

Dzhumadildaev has classified all extensions of $\text{vect}(N)$ by modules of tensor densities [7, 19]. In contrast, the two Virasoro-like cocycles involve modules of closed $(N - 1)$-forms, which are not tensor modules but rather submodules thereof. However, the two Virasoro cocycles are closely related to his cocycles $\psi_3^W$ and $\psi_4^W$. It can be noted that most (possibly all) extensions by tensor modules are limiting cases of trivial extensions, in the sense that one can construct a one-parameter family of trivial cocycles reducing to the non-trivial cocycle for a critical value of the parameter. In contrast, the Virasoro-like cocycles are not limits of trivial cocycles, because modules of closed forms do not depend on any continuous parameters. They also arise naturally in Fock representations, and having a representation theory is of course essential for any application to physics.

2 Multi-dimensional Virasoro algebra

To make the connection to the Virasoro algebra very explicit, it is instructive to write down the brackets in a Fourier basis. Start with the Virasoro algebra $\text{Vir}$:

$$[L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n},$$

(2.1)

where $\delta_m$ is the Kronecker delta. When $c = 0$, $L_m = -i \exp(\imath mx) d/dx$, $m \in \mathbb{Z}$. The element $c$ is central, meaning that it commutes with all of $\text{Vir}$; by Schur’s lemma, it can therefore be considered as a $c$-number. Now rewrite $\text{Vir}$ as

$$[L_m, L_n] = (n - m)L_{m+n} + cm^2 n S_{m+n},$$

$$[L_m, S_n] = (n + m)S_{m+n},$$

$$[S_m, S_n] = 0,$$

$$mS_m = 0.$$  

(2.2)

It is easy to see that the two formulations of $\text{Vir}$ are equivalent (I have absorbed the linear cocycle into a redefinition of $L_0$). The second formulation immediately generalizes to $N$ dimensions. The generators are
\( L_\mu(m) = -i \exp(im_\mu x^\mu) \partial_\mu \) and \( S^\mu(m) \), where \( x = (x^\mu) \), \( \mu = 1,2,\ldots,N \) is a point in \( N \)-dimensional space and \( m = (m_\mu) \). The Einstein convention is used; repeated indices, one up and one down, are implicitly summed over.

The defining relations are

\[
\begin{align*}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n) \\
&\quad + (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m + n), \\
[L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m + n) + \delta_\nu^\rho m_\rho S^\rho(m + n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
\mu m S^\mu(m) &= 0.
\end{align*}
\]

(2.3)

This is an extension of \( \text{vect}(N) \) by the abelian ideal with basis \( S^\mu(m) \). This algebra is even valid globally on the \( N \)-dimensional torus \( T^N \). Geometrically, we can think of \( L_\mu(m) \) as a vector field and \( S^\mu(m) = \epsilon^{\mu\nu_2...\nu_N} S_{\nu_2...\nu_N}(m) \) as a dual one-form (and \( S_{\nu_2...\nu_N}(m) \) as an \((N-1)\)-form); the last condition expresses closedness.

The cocycle proportional to \( c_1 \) was discovered by Rao and Moody [27], and the one proportional to \( c_2 \) by myself [16]. There is also a similar multi-dimensional generalization of affine Kac-Moody algebras, presumably first written down by Kassel [13]. The multi-dimensional Virasoro and affine algebras are often referred to as “Toroidal Lie algebras” in the mathematics literature [1 2 3 4 5 22 26 28].

### 3 Failure of the naïve approach to Fock representations

To construct Fock representations of the ordinary Virasoro algebra is straightforward:

- Start from classical modules, i.e. primary fields = scalar densities.
- Introduce canonical momenta.
- Normal order.

However, this approach does simply not work in several dimensions, because there are problems with normal ordering:

- It requires that at least a partial order has been introduced, which runs against the idea of diffeomorphism invariance.
• Normal ordering of bilinear expressions always results in a central extension, but the Virasoro cocycle is non-central when $N \geq 2$.

• It is ill defined. Formally, attempts to normal order result in an infinite central extension, which of course makes no sense.

This problem is the reason why the first interesting representations of the multi-dimensional Virasoro algebra only appeared a quarter century after their one-dimensional siblings. That it is a real problem can be seen by looking at the early (and failed) attempts in [8, 15, 16, 23]. It was only with the breakthrough in [27] that progress became possible. This was followed by a number of papers by different authors [1, 2, 3, 17, 18, 20], where the Rao-Moody construction was generalized in several ways and the underlying geometry explained.

The main idea, as described in the physics-flavored language of [18], is as follows:

• The arena is not just $N$-dimensional spacetime, but spacetime with a marked one-dimensional curve on it, the observer’s trajectory.

• All fields must be expanded in a Taylor series around the points of the observer’s trajectory, truncated at some arbitrary but fixed order $p$. In other words, we pass from the fields to the corresponding $p$-jets, or rather trajectories in jet space.

• We now have a classical realization on finitely many functions of a single variable (the parameter along the trajectory), which is precisely the situation where normal ordering applies.

• Introduce canonical momenta to the jets (not to the fields) and normal order with respect to frequency. This yields a realization of $Vir(N)$. Since the classical realization on jets is non-linear, the extension is non-central.

• The classical realization is highly reducible, since each point on the trajectory transforms independently of its neighbors. To lift this degeneracy, we introduce an additional $\text{vect}(1)$ factor, describing reparametrizations of the observer’s trajectory. The relevant algebra thus becomes the extension of $\text{vect}(N) \oplus \text{vect}(1)$ by its four Virasoro-like cocycles.

• The reparametrization symmetry can be eliminated with a constraint, but then one of the spacetime direction (“time”) is singled out [18]. Two of the four Virasoro-like cocycles of $DRO(N)$ transmute into
the complicated anisotropic cocycles found in [17]; these are collo-
quially known as the “messy cocycles”. By further specialization to
scalar-valued zero-jets on the torus, the results of Rao and Moody are
recovered.

The Kassel extension of current algebras can be treated along similar
lines.

4 DGRO algebra

By the arguments in the previous section one is lead to study the DGRO
(Diffeomorphism, Gauge, Reparametrization, Observer) algebra $\text{DGRO}(N, g)$,
whose ingredients are spacetime diffeomorphisms which generate $\text{vect}(N)$,
reparametrizations of the observer’s trajectory which form an additional
$\text{vect}(1)$ algebra, and gauge transformations which generate a cur-
tent algebra. Classically, the algebra is $\text{vect}(N) \times \text{map}(N, g) \oplus \text{vect}(1)$.

Let $\xi = \xi^\mu(x) \partial_\mu, x \in \mathbb{R}^N, \partial_\mu = \partial/\partial x^\mu$, be a vector field, with com-
mutator $[\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu$, and greek indices $\mu, \nu = 1, 2, ..., N$ label the spacetime coordinates. The Lie derivatives $L_\xi$ are the infinitesimal
diffeomorphisms, i.e. the generators of $\text{vect}(N)$.

Let $f = f(t) d/dt, t \in S^1$, be a vector field in one dimension. The com-
mutator reads $[f, g] = (f \dot{g} - g \dot{f}) d/dt$, where the dot denotes the $t$ derivative:
$\dot{f} \equiv df/dt$. We will also use $\partial_t = \partial/\partial t$ for the partial $t$ derivative. The choice
that $t$ lies on the circle is physically unnatural and is made for technical sim-
plicity only (quantities can be expanded in Fourier series). However, this
seems to be a minor problem at the present level of understanding. Denote
the reparametrization generators $L_f$.

Let $\text{map}(N, g)$ be the current algebra corresponding to the finite-dimen-
sional semisimple Lie algebra $g$ with basis $J^a$, structure constants $f_{ab}^c$, and
Killing metric $\delta^{ab}$. The brackets in $g$ are

$$[J^a, J^b] = if_{ab}^c J^c.$$  (4.1)

A basis for $\text{map}(N, g)$ is given by $g$-valued functions $X = X_a(x)J^a$ with
commutator $[X, Y] = if_{ab}^c X_a Y_b J^c$. The intertwining $\text{vect}(N)$ action is given
by $\xi X = \xi^\mu \partial_\mu X_a J^a$. Denote the $\text{map}(N, g)$ generators by $J_X$.

Finally, let $\text{Obs}(N)$ be the space of local functionals of the observer’s
tractory $q^\mu(t)$, i.e. polynomial functions of $q^\mu(t), \dot{q}^\mu(t), ..., d^k q^\mu(t)/dt^k, k$
finite, regarded as a commutative algebra. $\text{Obs}(N)$ is a $\text{vect}(N)$ module in
a natural manner.
DGRO$(N, g)$ is an abelian but non-central Lie algebra extension of $\text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1)$ by $\text{Obs}(N)$:

$$0 \rightarrow \text{Obs}(N) \rightarrow \text{DGRO}(N, g) \rightarrow \text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1) \rightarrow 0.$$ 

The brackets are given by

$$[L_\xi, L_\eta] = L_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \dot{q}^\mu(t) \left\{ c_1 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + c_2 \partial_\rho \partial_\mu \xi^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) \right\},$$

$$[L_\xi, J_X] = J_{\xi X},$$

$$[J_X, J_Y] = J_{[X, Y]} - \frac{c_5}{2\pi i} \delta^{ab} \int dt \dot{q}^b(t) \partial_\rho X_a(q(t)) Y_b(q(t)),$n

$$[L_f, L_\xi] = \frac{c_3}{4\pi i} \int dt (\dot{f}(t) - i\dot{f}(t)) \partial_\mu \xi^\mu(q(t)), \quad (4.2)$$

$$[L_f, L_g] = L_{[f, g]} + \frac{c_4}{24\pi i} \int dt (\dot{f}(t) \dot{g}(t) - \dot{f}(t) \dot{g}(t)),$$n

$$[L_\xi, q^\mu(t)] = \xi^\mu(q(t)),$n

$$[L_f, q^\mu(t)] = -f(t) \dot{q}^\mu(t),$$

$$[J_X, q^\mu(t)] = [q^\mu(s), q^\nu(t)] = 0,$$

extended to all of $\text{Obs}(N)$ by Leibniz’ rule and linearity. The numbers $c_1 - c_5$ are called abelian charges, in analogy with the central charge of the Virasoro algebra. In the references slightly more complicated extensions are considered, which depend on three additional abelian charges $c_6 - c_8$. However, these vanish automatically when $g$ is semisimple.

## 5 Trajectories in jet space

The classical representations of the DGRO algebra are tensor fields over $\mathbb{R}^N \times S^1$ valued in $g$ modules. The basis of a classical DGRO module $Q$ is thus a field $\phi_\alpha(x, t), x \in \mathbb{R}^N, t \in S^1$, where $\alpha$ is a collection of all kinds of indices. The $\text{DGRO}(N, g)$ action on $Q$ can be succinctly summarized as

$$[L_\xi, \phi_\alpha(x, t)] = -\xi^\mu(x) \partial_\mu \phi_\alpha(x, t) - \partial_\nu \xi^\mu(x) T_{\alpha\mu}^{\beta} \phi_\beta(x, t),$$

$$[J_X, \phi_\alpha(x, t)] = -X_a(x) J_{\alpha}^{\beta a} \phi_\beta(x, t), \quad (5.1)$$

$$[L_f, \phi_\alpha(x, t)] = -f(t) \partial_\mu \phi_\alpha(x, t) - \lambda(\dot{f}(t) - i\dot{f}(t)) \phi_\alpha(x, t).$$

7
Here $J^a = (J^a_{\beta})$ and $T^\mu_\nu = (T^\mu_\nu_{\beta\rho})$ are matrices satisfying $g$ (1.1) and $gl(N)$, respectively:

\[
[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\tau T^\mu_\nu - \delta^\mu_\nu T^\sigma_\tau.
\] (5.2)

The crucial idea in [18] is to expand all fields in a Taylor series around the observer’s trajectory and truncate at order $p$, before introducing canonical momenta. Hence e.g.,

\[
\phi_\alpha(x, t) = \sum_{|m| \leq p} \frac{1}{m!} \phi_{\alpha, m}(t)(x - q(t))^m,
\] (5.3)

where $m = (m_1, m_2, ..., m_N)$, all $m_\mu \geq 0$, is a multi-index of length $|m| = \sum_{\mu=1}^N m_\mu$, $m! = m_1!m_2!...m_N!$, and

\[
(x - q(t))^m = (x^1 - q^1(t))^{m_1}(x^2 - q^2(t))^{m_2}...(x^N - q^N(t))^{m_N}.
\] (5.4)

Denote by $\mu$ a unit vector in the $\mu$:th direction, so that $m + \mu = (m_1, ..., m_\mu + 1, ..., m_N)$, and let

\[
\phi_{\alpha, m}(t) = \partial_m \phi_{\alpha}(q(t), t) = \partial_{m_1}...\partial_{m_\mu}...\partial_{m_N} \phi_{\alpha}(q(t), t)
\] (5.5)

be the $|m|$:th order derivative of $\phi_{\alpha}(x, t)$ evaluated on the observer’s trajectory $q^\mu(t)$. Such objects transform as

\[
[L_\xi, \phi_{\alpha, m}(t)] = \partial_m ([L_\xi, \phi_{\alpha}(q(t), t)]) + [L_\xi, q^\mu(t)] \partial_\mu \partial_m \phi_{\alpha}(q(t), t)
\]

\[
\equiv - \sum_{|n| \leq |m|} T^{\beta n}_{\alpha m}(\xi(q(t))) \phi_{\beta, n}(t),
\]

\[
[J_X, \phi_{\alpha, m}(t)] = \partial_m ([J_X, \phi_{\alpha}(q(t), t)])
\]

\[
\equiv - \sum_{|n| \leq |m|} J^{\beta n}_{\alpha m}(X(q(t))) \phi_{\beta, n}(t),
\]

\[
[L_f, \phi_{\alpha, m}(t)] = -f(t) \dot{\phi}_{\alpha, m}(t) - \lambda(\dot{f}(t) - if(t)) \phi_{\alpha, m}(t),
\]

where

\[
T^m_n(\xi) \equiv (T^\alpha_{\beta n}(\xi))
\]

\[
= \binom{n}{m} \partial_{n-m+\rho} \xi^\mu T^\mu_\rho + \binom{n}{m-\mu} \partial_{n-m+\mu} \xi^\mu - \delta^m_{\rho n} \xi^\mu,
\] (5.7)

\[
J^m_n(X) \equiv (J^\alpha_{\beta n}(X)) = \binom{n}{m} \partial_{n-m} X_a J^a,
\]
and

\[
\begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!} = \binom{m_1}{n_1} \binom{m_2}{n_2} \cdots \binom{m_N}{n_N}.
\]  

(5.8)

We thus obtain a (non-linear) realization of \( \text{vect}(N) \) on the space of trajectories in the space of tensor-valued \( p \)-jets \( \mathcal{J}^p \). Note that \( \mathcal{J}^p \) is spanned by \( q^\mu(t) \) and \( \{ \phi_{\alpha,m}(t) \}_{|m| \leq p} \) and thus not a DGRO\( (N, g) \) module by itself, because diffeomorphisms act non-linearly on \( q^\mu(t) \), as can be seen in (4.2). However, the space \( \mathcal{C}(\mathcal{J}^p) \) of functionals on \( \mathcal{J}^p \) (local in \( t \)) is a module, because the action on a \( p \)-jet can never produce a jet of order higher than \( p \). The space \( \mathcal{C}(q) \otimes q \mathcal{J}^p \), where only the trajectory itself appears non-linearly, is a submodule.

The crucial observation is that the jet space \( \mathcal{J}^p \) consists of finitely many functions of a single variable \( t \), which is precisely the situation where the normal ordering prescription works. After normal ordering, denoted by double dots : :, we obtain a Fock representation of the DGRO algebra:

\[
\begin{align*}
\mathcal{L}_\xi &= \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): - \sum_{|n| \leq |m| \leq p} T^{\beta n}_{\alpha m}(\xi(q(t))):\phi_{\beta,n}(t)\pi^{\alpha,m}(t): \right\}, \\
\mathcal{J}_X &= -\int dt \left\{ \sum_{|n| \leq |m| \leq p} J^{\beta n}_{\alpha m}(\xi(q(t))):\phi_{\beta,n}(t)\pi^{\alpha,m}(t): \right\}, \\
L_f &= \int dt \left\{ -f(t):\dot{\phi}_{\alpha,m}(t)\pi^{\alpha,m}(t): -\lambda(f(t) - if(t)):\phi_{\alpha,m}(t)\pi^{\alpha,m}(t): \right\},
\end{align*}
\]

(5.9)

where we have introduced canonical momenta \( p_\mu(t) = \delta/\delta q^\mu(t) \) and \( \pi^{\alpha,m}(t) = \delta/\delta \phi_{\alpha,m}(t) \). The field \( \phi_\alpha(x, t) \) can be either bosonic or fermionic but the trajectory \( q^\mu(t) \) is of course always bosonic.

Normal ordering is defined with respect to frequency; any function of \( t \in S^1 \) can be expanded in a Fourier series, e.g.

\[
p_\mu(t) = \sum_{n=-\infty}^{\infty} p_\mu(n)e^{-int} \equiv p_\mu^< (t) + p_\mu^>(t),
\]

(5.10)

\(^1\) \( p \)-jets are usually defined as an equivalence class of functions: two functions are equivalent if all derivatives up to order \( p \), evaluated at \( q^\mu \), agree. However, each class has a unique representative which is a polynomial of order at most \( p \), namely the Taylor expansion around \( q^\mu \), so we may canonically identify jets with truncated Taylor series. Since \( q^\mu(t) \) depends on a parameter \( t \), we deal in fact with trajectories in jet space, but these will also be called jets for brevity.
where \( p^<(\mu)(t) \) (\( p^>(\mu)(t) \)) is the sum over negative (positive) frequency modes only. Then

\[
: \xi^\mu(q(t)) p_\mu(t) : \equiv \xi^\mu(q(t)) p^<(\mu)(t) + p^>(\mu)(t) \xi^\mu(q(t)),
\]

where the zero mode has been included in \( p^<(\mu)(t) \).

It is clear that (5.9) defines a Fock representation for every \( gl(N) \) irrep \( \rho \) and every \( g \) irrep \( M \); denote this Fock space by \( J^pF_p \), which indicates that it also depends on the truncation order \( p \). Namely, introduce a Fock vacuum \( |0\rangle \) which is annihilated by half of the oscillators, e.g. \( q^<(\mu)(t), p^\geq(\mu)(t), \phi^<_{\alpha,m}(t) \) and \( \pi^>_{\alpha,m}(t) \). Then \( DGRO(N, g) \) acts on the space of functionals \( C(q^<(\mu), p^\geq(\mu), \phi^\geq_{\alpha,m}, \pi^<_{\alpha,m}) \) of the remaining oscillators; this is the Fock module.

Define numbers \( k_0(\rho), k_1(\rho), k_2(\rho) \) and \( y_M \) by

\[
\begin{align*}
\text{tr}_q T^\mu_{\nu} &= k_0(\rho) \delta^\mu_{\nu}, \\
\text{tr}_q T^\mu_{\nu} T^\sigma_{\tau} &= k_1(\rho) \delta^\mu_{\nu} \delta^\sigma_{\tau} + k_2(\rho) \delta^\mu_{\nu} \delta^\sigma_{\tau}, \\
\text{tr}_M J^a J^b &= y_M \delta^{ab}.
\end{align*}
\]

The values of the abelian charges \( c_1 - c_5 \) (4.2) were calculated in [18], Theorems 1 and 3, and in [20], Theorem 1:

\[
\begin{align*}
c_1 &= 1 - u \left( \frac{N + p}{N} \right) - x \left( \frac{N + p + 1}{N + 2} \right), \\
c_2 &= -v \left( \frac{N + p}{N} \right) - 2w \left( \frac{N + p}{N + 1} \right) - x \left( \frac{N + p}{N + 2} \right), \\
c_3 &= 1 + (1 - 2\lambda)(w \left( \frac{N + p}{N} \right) + x \left( \frac{N + p}{N + 1} \right)), \\
c_4 &= 2N - x(1 - 6\lambda + 6\lambda^2) \left( \frac{N + p}{N} \right), \\
c_5 &= y \left( \frac{N + p}{N} \right).
\end{align*}
\]

where

\[
\begin{align*}
u &= \mp k_1(\rho) \dim M, \\
x &= \mp \dim \rho \dim M, \\
v &= \mp k_2(\rho) \dim M, \\
y &= \mp \dim \rho y_M, \\
w &= \mp k_0(\rho) \dim M,
\end{align*}
\]

and the sign factor depends on the Grassmann parity of \( \phi_\alpha \); the upper sign holds for bosons and the lower for fermions, respectively. The \( p \)-independent contributions to \( c_1, c_3 \) and \( c_4 \) come from the trajectory \( q^\mu(t) \) itself. We will henceforth set \( \lambda = 0 \).
6 Dynamics and the KT complex

The modules constructed in the previous section show that a quantum generalization of tensor calculus exists. However, this is not by itself a theory of quantum gravity any more than tensor calculus determines general relativity. Somehow information about dynamics must be included into the picture. A natural candidate is found in the physics of gauge theories, as formulated cohomologically in the anti-field formalism \[11\].

The goal of classical physics is to find the stationary surface \(\Sigma\), i.e. the set of solutions to the Euler-Lagrange (EL) equations, viewed as a submanifold embedded in the space of all field configurations \(Q\). Dually, one wants to construct the function algebra \(C(\Sigma) = C(Q)/I\), where \(I\) is the ideal generated by the EL equations. \(C(\Sigma)\) will evidently carry a representation of the Noether symmetries of the action \(S = \int d^Nx \mathcal{L}(\phi)\).

We can thus regard classical physics as the representation theory of its Noether symmetries. The conventional next step would be to identify gauge-equivalent configurations by passing to the orbit space, and thus obtain a covariant description of phase space as the space of solutions to the EL equation modulo gauges \[9\]. However, this would be rather uninteresting from an algebraic point of view, since the gauge symmetries by construction act trivially on the orbit space. More importantly, as we have seen above, normal ordering gives rise to non-trivial abelian extensions, which could be interpreted as an anomaly; a putative BRST charge would not remain nilpotent or even well defined. One can equivalently consider the full stationary surface together with the action of the gauge symmetries. Classically, and in the absence of anomalies, the two formulations are equivalent; the passage to the orbit space can always be performed if so desired. The abelian extension becomes harmless\[3\] from this point of view; the algebra merely acquires its full quantum form but the representation theory remains well defined. Indeed, all interesting representations of the the diffeomorphism algebra, at least in one dimension, have non-zero extensions. This issue is further discussed in the conclusion.

The Lagrangian \(\mathcal{L}(\phi)\) is a local functional of \(\phi\), i.e. a function of \(\phi_\alpha(x)\) and its derivatives \(\partial_\mu \phi_\alpha(x)\), \(\partial_\mu \partial_\nu \phi_\alpha(x)\), etc., up to some finite order, all evaluated at the same point \(x\). In practice, the Lagrangian only depends on

\[2\] Note that a basis for this “configuration space” \(Q\) is given by all fields in spacetime, not just in space.

\[3\] The extension would be very harmful if representations of the extended algebra were missing.
first-order derivatives. The EL equations,

\[ \mathcal{E}^\alpha(x) \equiv \frac{\delta S}{\delta \phi_\alpha(x)} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha(x)}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_\alpha(x)}(x) = 0, \quad (6.1) \]

generate an ideal \( \mathcal{I} \subset C(\mathcal{Q}) \), and the factor space \( C(\Sigma) = C(\mathcal{Q})/\mathcal{I} \) is still a \( \text{vect}(N) \times \text{map}(N, g) \) module due to the invariance assumption. This factor space is most conveniently described as a resolution of a certain Koszul-Tate (KT) complex. For each field \( \phi_\alpha(x) \), introduce an antifield \( \phi^{*\alpha}(x) \) transforming as the corresponding EL equation \( \mathcal{E}^\alpha(x) \). We then consider the extended configuration space \( \mathcal{Q}^* \) as the span of \( \phi_\alpha(x) \) and \( \phi^{*\alpha}(x) \). Now consider the space of local functionals on \( \mathcal{Q}^* \):

\[ C(\mathcal{Q}^*) = C(\mathcal{Q}) \oplus \bigoplus_{g=0}^{\infty} C^g(\mathcal{Q}^*). \quad (6.2) \]

The KT complex takes the form

\[ 0 \leftarrow \delta \rightarrow C^0(\mathcal{Q}^*) \leftarrow \delta \rightarrow C^1(\mathcal{Q}^*) \leftarrow \delta \rightarrow C^2(\mathcal{Q}^*) \leftarrow \delta \rightarrow \ldots \quad (6.3) \]

where the KT differential \( \delta \) is defined by

\[ \delta \phi_\alpha(x) = 0, \quad \delta \phi^{*\alpha}(x) = \mathcal{E}^\alpha(x). \quad (6.4) \]

By a standard argument \cite{11}, the cohomology groups \( H^g(\delta) = 0 \) unless \( g = 0 \), and \( H^0(\delta) = C(\mathcal{Q})/\mathcal{I} = C(\Sigma) \). We have thus obtained a resolution of the space of functionals on the stationary surface, as desired.

Introduce canonical momenta \( \pi^\alpha(x) = \delta/\delta \phi_\alpha(x) \) and \( \pi^{*\alpha}(x) = \delta/\delta \phi^{*\alpha}(x) \), with antifield numbers \( \text{afn} \pi^\alpha = 0 \), \( \text{afn} \pi^{*\alpha} = -1 \). The KT differential can then be written as a bracket: \( \delta F = [Q, F] \), where

\[ Q = \int d^N x \mathcal{E}^\alpha(x)\pi^{*\alpha}(x). \quad (6.5) \]

Let \( \mathcal{P} \) be the space spanned by \( \phi_\alpha(x) \) and \( \pi^\alpha(x) \), and let \( \mathcal{P}^* \) be the span of \( \phi_\alpha(x) \), \( \phi^{*\alpha}(x) \), \( \pi^\alpha(x) \) and \( \pi^{*\alpha}(x) \). The expression \( (6.5) \) defines a differential, also denoted by \( Q \), which acts on the space \( C(\mathcal{P}^*) \) of local functionals on \( \mathcal{P}^* \). Note that \( C(\mathcal{P}^*) \) is a non-commutative ring, which can be thought of as the
algebra of differential operators on $Q^*$. The decomposition into subspaces of fixed antifield number now extends indefinitely in both directions:

$$C(P^*) = \oplus_{g = -\infty}^{\infty} C^g(P^*). \quad (6.6)$$

Accordingly, we obtain the two-sided complex

$$\ldots \xrightarrow{Q} C^{-1}(P^*) \xrightarrow{Q} C^0(P^*) \xrightarrow{Q} C^1(P^*) \xrightarrow{Q} \ldots \quad (6.7)$$

The cohomology group $H^0(Q)$ can be thought of as the space of differential operators on the stationary surface $\Sigma$. However, I do not know if (6.7) is a resolution, i.e. if the other cohomology groups vanish.

There is a problem: the EL equations may be dependent, i.e. there may be relations of the form

$$r^a(x) = r^a_\alpha(x)\phi^{*\alpha}(x) \equiv 0, \quad (6.8)$$

where $r^a_\alpha(x)$ is some functional of $\phi_\alpha(x)$. Then $H^1(Q) \neq 0$, because $r^a_\alpha(x)\phi^{*\alpha}(x)$ is KT closed: $[Q, r^a_\alpha(x)\phi^{*\alpha}(x)] = 0$. The standard way to kill this unwanted cohomology is to introduce a second-order antifield $b^a(x)$. Let $[Q, b^a(x)] = r^a_\alpha(x)\phi^{*\alpha}(x)$, which makes the latter expression exact and thus makes it vanish in cohomology. To obtain the explicit expression for $Q$, introduce the second-order antifield momentum $c_a(x) = \delta/\delta b^a(x)$. The full KT differential is now

$$Q = \int d^N x \left( \mathcal{E}^\alpha(x)\pi_\alpha^a(x) + r^a_\alpha(x)\phi^{*\alpha}(x)c_a(x) \right). \quad (6.9)$$

There can in principle be relations also among the $r^a_\alpha(x)$ of the form $Z^A(x) = Z^A_\alpha(x)r^a_\alpha(x) \equiv 0$. If so, it is necessary to introduce higher-order antifields to eliminate the unwanted cohomology. However, we will assume that the gauge symmetries are irreducible, i.e. that no non-trivial higher-order relations exist, since this is the case in all experimentally established theories of physics.

The situation is summarized in the following table:

| $g$ | Field | Momentum | Ideal |
|-----|-------|----------|-------|
| 0   | $\phi_\alpha(x)$ | $\pi^\alpha(x)$ | $-$   |
| 1   | $\phi^{*\alpha}(x)$ | $\pi^a_\alpha(x)$ | $\mathcal{E}^\alpha(x) \approx 0$ |
| 2   | $b^a(x)$ | $c_a(x)$ | $r^a_\alpha(x)\phi^{*\alpha}(x) \approx 0$ |

(6.10)

As a preparation for normal ordering, we must now add the variable $t$, i.e. replace $\phi_\alpha(x) \to \phi_\alpha(x,t)$. The EL equations (6.1) now read $\mathcal{E}^\alpha(x,t) = 0$, 13
and the KT charge \((6.9)\) is replaced by

\[
Q = \int d^N x \, dt \left( \mathcal{E}^\alpha(x,t)\pi^\alpha(x,t) + r_\alpha^a(x,t)\phi^{*\alpha}(x,t)c_a(x,t) \right). \tag{6.11}
\]

Since the space of functionals over \(\phi_\alpha(x,t)\) is larger than \(C(Q)\), we must factor out a larger ideal to obtain a resolution of the same space \(C(\Sigma)\). It is easy to see that the necessary additional requirement is \(\partial_\mu \phi_\alpha(x,t) \approx 0\); to implement this constraint in cohomology, we introduce the antifield \(\phi_\alpha(x,t)\) with canonical momentum \(\pi^\alpha(x,t)\). Since \(\mathcal{E}^\alpha(x,t)\) depends on \(\phi_\alpha(x,t)\) only, we now have \(\partial_\mu \mathcal{E}^\alpha(x,t) \equiv 0\), which generates unwanted cohomology. This is eliminated by introducing a second-order antifield \(\phi^{*\alpha}(x,t)\). Finally, the other second-order antifield \(b^a(x,t)\), associated with the gauge symmetry, is now reducible. Correct this by introducing a third-order antifield \(\phi^{**\alpha}(x,t)\).

The situation is summarized in the following table:

| \(g\) | Field          | Momentum       | Ideal                    |
|------|----------------|----------------|--------------------------|
| 0    | \(\phi_\alpha(x,t)\) | \(\pi^\alpha(x,t)\) | \(-\)                     |
| 1    | \(\phi^{*\alpha}(x,t)\) | \(\pi^\alpha(x,t)\) | \(\mathcal{E}^\alpha(x,t) \approx 0\) |
| 1    | \(\phi_\alpha(x,t)\) | \(\pi^\alpha(x,t)\) | \(\partial_\mu \phi_\alpha(x,t) \approx 0\) |
| 2    | \(b^a(x,t)\)      | \(c_a(x,t)\)    | \(r_\alpha^a(x,t)\phi^{*\alpha}(x,t) \approx 0\) |
| 3    | \(\phi^{**\alpha}(x,t)\) | \(\pi^\alpha(x,t)\) | \(\partial_\mu \phi^{*\alpha}(x,t) \approx 0\) |
|      | \(\phi^{**\alpha}(x,t)\) | \(\pi^\alpha(x,t)\) | \(\partial_\mu b^a(x,t) \approx 0\) |

### 7 KT complex in jet space and quantization

In order to construct the jet space version of the KT complex, we expand not only the fields but also the EL equations and the anti-fields in multi-dimensional Taylor series. Set \(\mathcal{E}_m^\alpha(t) = \partial_m \mathcal{E}^\alpha(q(t),t)\) and \(\phi_m^{*\alpha}(t) = \partial_m \phi^{*\alpha}(q(t),t)\). What must be noted is that we can only define \(\mathcal{E}_m^\alpha(t)\) for \(|m| \leq p - o_\alpha\), where \(o_\alpha\) is the order of the EL equation \(\mathcal{E}^\alpha(x)\). This is because \(\mathcal{E}_m^\alpha(t)\) is a function of \(\phi_{\alpha,n}(t)\) for all \(|n| \leq |m| + o_\alpha\), and \(\phi_{\alpha,n}(t)\) is undefined for \(|n| > p\). Similarly, the relations \((6.8)\) and the corresponding second-order anti-fields \(b^a(x)\) give rise to the jets \(r_m^\alpha(x) = \partial_m (r_\alpha^a(q(t),t)\phi^{*\alpha}(q(t),t))\) and \(b_m^a(t) = \partial_m b^a(q(t),t)\), respectively. If the relations \(r_\alpha^a(x)\) are of order \(c_\alpha\) in the derivatives, \(r_m^\alpha(t)\) and \(b_m^a(t)\) is only defined for \(|m| \leq p - c_\alpha\).

The ideals of type \(\partial_\mu \phi_\alpha(x,t) \approx 0\) translate into:

\[
\begin{align*}
D_t \phi_{\alpha,m}(t) & \equiv \dot{\phi}_{\alpha,m}(t) - \dot{q}^\mu(t)\phi_{\alpha,m+\mu}(t) \approx 0, \\
D_t \phi^{*\alpha}_m(t) & \equiv \dot{\phi}^{*\alpha}_m(t) - \dot{q}^\mu(t)\phi^{*\alpha}_{m+\mu}(t) \approx 0, \\
D_t b^a_m(t) & \equiv \dot{b}^a_m(t) - \dot{q}^\mu(t)b^a_{m+\mu}(t) \approx 0.
\end{align*}
\tag{7.1}
\]
A crucial observation is that the KT charge $Q$ and the cohomology groups of this complex are well defined $DGRO(N, g)$ modules since $Q$ commutes with the module action. It would not be possible to construct a similar complex with a BRST charge, because normal ordering would then ruin nilpotency.
8 Finiteness condition

The modules obtained in this fashion are well defined for all finite values of the jet order $p$, but in order to have a field theory interpretation, it must be possible to reconstruct the original field by means of the Taylor series \([5.3]\), i.e. to take the limit $p \to \infty$. A necessary condition for taking this limit is that the abelian charges have a finite limit. Taken at face value, the prospects for succeeding appear bleak. When $p$ is large, $(m+p)_n \approx p^n/n!$, so the abelian charges \([5.12]\) diverge; the worst case is $c_1 \approx c_2 \approx p^{N+2}/(N+2)!$, which diverges in all dimensions $N > -2$. In \([20]\) a way out of this problem was devised: consider a more general realization by taking the direct sum of operators corresponding to different values of the jet order $p$. Take the sum of $r+1$ terms like those in \([5.9]\), with $p$ replaced by $p, p-1, \ldots, p-r$, respectively, and with $g$ and $M$ replaced by $g^{(i)}$ and $M^{(i)}$ in the $p-i$ term.

Such a sum of contributions arises naturally from the KT complex, because the antifields are only defined up to an order smaller than $p$ (e.g. $p-o_a$ or $p-\varsigma_a$). Denote the numbers $u, v, w, x, y$ in the modules $g^{(i)}$ and $M^{(i)}$, defined as in \([5.13]\), by $u_i, v_i, w_i, x_i, y_i$, respectively. Of course, there is only one contribution from the observer’s trajectory. Then it was shown in \([20]\), Theorem 3, that

\[c_1 = -U \binom{N+p-r}{N-r}, \quad c_2 = -V \binom{N+p-r}{N-r},\]
\[c_3 = W \binom{N+p-r}{N-r}, \quad c_4 = -X \binom{N+p-r}{N-r},\]
\[c_5 = Y \binom{N+p-r}{N-r},\] (8.1)

where $u_0 = U, v_0 = V, w_0 = W, x_0 = X$ and $y_0 = Y$, provided that the following conditions hold:

\[i \quad u_i + (-)^i \binom{r-2}{i-2} X = (-)^i \binom{r}{i} U,\]
\[ii \quad v_i - 2(-)^i \binom{r-1}{i-1} W = (-)^i \binom{r-2}{i-2} X = (-)^i \binom{r}{i} V,\]
\[iii \quad w_i - (-)^i \binom{r-1}{i-1} X = (-)^i \binom{r}{i} W,\]
\[iv \quad x_i = (-)^i \binom{r}{i} X,\]
\[v \quad y_i = (-)^i \binom{r}{i} Y.\]

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The contributions from the observer’s trajectory have also been eliminated by antifields coming from the geodesic equation [21]; this is not important in the sequel because these contributions were finite anyway.

9 Solutions to the finiteness conditions

Let us now consider the solutions to (8.2) for the numbers $x_i$, which can be interpreted as the number of fields and anti-fields. First assume that the field $\phi_{\alpha,m}(t)$ is fermionic with $x_F$ components, which gives $x_0 = x_F$. We may assume, by the spin-statistics theorem, that the EL equations are first order, so the bosonic antifields $\phi^*_{\alpha,m}(t)$ contribute $-x_F$ to $x_1$. The barred antifields $\bar{\phi}_{\alpha,m}(t)$ are also defined up to order $p-1$, and so give $x_1 = -x_F$, and the barred second-order antifields $\bar{\phi}^{*,\alpha}_m(t)$ give $x_2 = x_F$. Further assume that the fermionic EL equations have $x_S$ gauge symmetries, i.e. the second-order antifields $b^a_m(t)$ give $x_2 = x_S$. In established theories, $x_S = 0$, but we will need a non-zero value for $x_S$. Finally, the corresponding barred antifields give $x_3 = -x_S$.

For bosons the situation is analogous, with two exceptions: all signs are reversed, and the EL equations are assumed to be second order. Hence $\phi^*_{\alpha,m}(t)$ yields $x_2 = x_B$ and the gauge antifields $b^a_m(t)$ give $x_3 = -x_G$. Accordingly, the barred antifields are one order higher.

The situation is summarized in the following tables, where the upper half is valid if the original field is fermionic and the lower half if it is bosonic:

\[
\begin{array}{ccc}
g & \text{Jet} & \text{Order} & x \\
0 & \phi_{\alpha,m}(t) & p & x_F \\
1 & \bar{\phi}_{\alpha,m}(t) & p-1 & -x_F \\
1 & \phi^*_{\alpha,m}(t) & p-1 & -x_F \\
2 & \bar{\phi}^{*,\alpha}_m(t) & p-2 & x_F \\
2 & b^a_m(t) & p-2 & x_S \\
3 & \bar{b}^a_m(t) & p-3 & -x_S \\
0 & \phi_{\alpha,m}(t) & p & -x_B \\
1 & \bar{\phi}_{\alpha,m}(t) & p-1 & x_B \\
1 & \phi^*_{\alpha,m}(t) & p-2 & x_B \\
2 & \bar{\phi}^{*,\alpha}_m(t) & p-3 & -x_B \\
2 & b^a_m(t) & p-3 & -x_G \\
3 & \bar{b}^a_m(t) & p-4 & x_G \\
\end{array}
\]

If we add all contributions of the same order, we see that relation iv in (8.2)
can only be satisfied provided that
\[
p : \quad x_F - x_B = X \\
p - 1 : \quad -2x_F + x_B = -rX, \\
p - 2 : \quad x_B + x_F + x_S = \binom{r}{2} X, \\
p - 3 : \quad -x_B - x_S - x_G = -\binom{r}{3} X, \\
p - 4 : \quad x_G = \binom{r}{4} X, \\
p - 5 : \quad 0 = -\binom{r}{5} X, ...
\] (9.2)

The last equation holds only if \( r \leq 4 \) (or trivially if \( X = 0 \)). On the other hand, if we demand that there is at least one bosonic gauge condition, the \( p - 4 \) equation yields \( r \geq 4 \). Such a demand is natural, because both the Maxwell/Yang-Mills and the Einstein equations have this property. Therefore, we are unambiguously guided to consider \( r = 4 \) (and thus \( N = 4 \)). The specialization of (9.2) to four dimensions reads
\[
p : \quad x_F - x_B = X \\
p - 1 : \quad -2x_F + x_B = -4X, \\
p - 2 : \quad x_B + x_F + x_S = 6X, \\
p - 3 : \quad -x_B - x_S - x_G = -4X, \\
p - 4 : \quad x_G = X.
\] (9.3)

Clearly, the unique solution to these equations is
\[
x_F = 3X, \quad x_B = 2X, \quad x_S = X, \quad x_G = X.
\] (9.4)

The solutions to the remaining equations in (8.2) are found by analogous
reasoning. The result is

\begin{align*}
  u_B &= 2U \\
  u_F &= 3U \\
  u_S &= U - X \\
  u_G &= U - X \\
  v_B &= 2V + 2W \\
  v_F &= 3V + 2W \\
  v_S &= V + 2W + X \\
  v_G &= V + 2W + X \\
  w_B &= 2W + X \\
  w_F &= 3W + X \\
  w_S &= W + X \\
  w_G &= W + X \\
  y_B &= 2Y \\
  y_F &= 3Y \\
  y_S &= Y \\
  y_G &= Y
\end{align*}

(9.5)

This result expresses the twenty parameters $x_B - w_G$ in terms of the five parameters $X, Y, U, V, W$. For this particular choice of parameters, the abelian charges in (8.1) are given by

\begin{align*}
  c_1 &= -U, \\
  c_2 &= -V, \\
  c_3 &= W, \\
  c_4 &= -X, \\
  c_5 &= Y,
\end{align*}

(9.6)

independent of $p$. Hence there is no manifest obstruction to the limit $p \to \infty$.

10 Comparison with known physics

All experimentally known physics is well described by quantum theory, gravity, and the standard model in four dimensions. We have already seen that quantum general covariance more or less dictates that spacetime has $N = 4$ dimensions (9.2). It is therefore interesting to investigate to what extent the particle content matches (9.4); recall that $x = \text{tr} \ 1$ equals the number of field components.

The bosonic content of the theory is given by the following table. Standard notation for the fields is used, and one must remember that it is the na"ive number of components that enters the equation, not the gauge-invariant physical content. E.g., the photon is described by the four components $A_\mu$ rather than the two physical transverse components. Also, the

\footnote{After this work was completed, I realized that the negative signs of $c_1, c_2$ and $c_4$ imply problems with unitarity. Finiteness seems to be a more pressing problem, however.}
gauge algebra $su(3) \oplus su(2) \oplus u(1)$ has $8 + 3 + 1 = 12$ generators.

| Field  | Name            | EL equation       | $x_B$ |
|--------|-----------------|-------------------|-------|
| $A^a_\mu$ | Gauge bosons  | $D_\nu F^{a\mu} = j^{a\mu}$ | $12 \times 4 = 48$ |
| $g_{\mu\nu}$ | Metric  | $G^{\mu\nu} = \frac{1}{8\pi} T^{\mu\nu}$ | 10 |
| $H$ | Higgs field  | $g^{\mu\nu} \partial_\mu \partial_\nu H = V(H)$ | 2 |

The total number of bosons in the theory is thus $x_B = 48 + 10 + 2 = 60$, which implies $X = 30$ by (9.4). The number of gauge conditions is $x_G = 16$, which implies $X = 16$. There is certainly a discrepancy here.

The fermionic content in the first generation is given by

| Field  | Name            | EL equation       | $x_F$ |
|--------|-----------------|-------------------|-------|
| $u$ | Up quark  | $\bar{\psi} u \gamma^\mu D_\mu \phi = \ldots$ | $2 \times 3 = 6$ |
| $d$ | Down quark | $\bar{\psi} d \gamma^\mu D_\mu \phi = \ldots$ | $2 \times 3 = 6$ |
| $e$ | Electron  | $\bar{\psi} e \gamma^\mu D_\mu \phi = \ldots$ | 2 |
| $\nu_L$ | Left-handed neutrino | $\bar{\psi} \nu_L \gamma^\mu D_\mu \phi = \ldots$ | 1 |

The number of fermions in the first generation is thus $x_F = 6 + 6 + 2 + 1 = 15$. Counting all three generations and anti-particles, we find that the total number of fermions is $x_F = 2 \times 3 \times 15 = 90$, which implies $X = 30$. There are no fermionic gauge conditions, so $x_S = 0$, which implies $X = 0$.

It is clear that the predictions for $X$ (30, 16, 30, 0) are not mutually consistent. However, to cancel the leading terms, of order $p$ and $p - 1$, it is only necessary that $2x_F = 3x_B$, which is indeed the case in known physics. It is therefore tempting to speculate that known physics is a first approximation of a more elegant theory, which has the same field content but more gauge conditions, including fermionic ones. An attractive possibility, suggested by Kac [12], would be to replace the standard model symmetries by one of the recently discovered exceptional Lie superalgebras, whose irreps are in 1-1 correspondence with $su(3) \oplus su(2) \oplus u(1)$ irreps.

It is important to check that the results remain the same if the same physical situation is described with a different, but equivalent, set of fields. Typically, such spurious degrees of freedom have algebraic EL equations. Denote the original (bosonic, say) $x_B$ fields by $\phi_{\alpha,m}(t)$ and let $\psi_{i,m}(t)$ be $x_A$ spurious fields, defined for $|m| \leq p$. The contribution to $x_0$ from the bosonic fields is thus $-x_B - x_A$. There are also $x_A$ new EL equations $E^i_{m}(t)$,
defined for $|\mathbf{m}| \leq p$ because they are algebraic; $E^{i}_{\mathbf{m}}(t)$ contains $\psi_{j,n}(t)$ for all $|\mathbf{n}| \leq |\mathbf{m}|$, but not of higher order. The corresponding anti-fields $\psi^{*i}_{\mathbf{m}}(t)$ add $x_A$ to $x_0$. The total result is $x_0 = -x_B - x_A + x_A = -x_B$, as before.

An example is given by the gravitational field in vielbein formalism. Instead of the ten components of the metric $g_{\mu\nu} = g_{\nu\mu}$ we have the sixteen vielbein components $e^{i}_{\mu}$. However, the requirement that the metric $g_{\mu\nu} = e^{i}_{\mu}e^{i}_{\nu}$ be symmetric gives rise to six algebraic conditions, so the contribution to $x_0$ is still ten.

11 Conclusion

There are two key lessons to be learnt from twentieth century physics:

- General relativity teaches us the importance of diffeomorphism invariance. Physics is fully relational; there is no background stage over which physics takes place, but geometry itself participates actively in the dynamics. Note that this is very different from mere coordinate invariance, because there is no compensating background metric.

- Quantum theory teaches us the importance of projective lowest-energy representations; the passage from Poisson brackets to commutators makes normal ordering necessary, and the brackets typically acquire quantum corrections.

The successful construction of a quantum theory of gravity will probably combine these two insights. It seems obvious that the correct way to combine diffeomorphism invariance and projective representations is to consider projective representations of the diffeomorphism group, which on the Lie algebra level gives rise to the DGRO algebra.

A common objection is that the presence of an extension makes diffeomorphism symmetry anomalous. Although anomaly cancellation is certainly a valuable mechanism which is experimentally confirmed in the standard model, it is not so natural from an algebraist’s point of view; in particular, all mathematically interesting (= non-trivial, irreducible, unitary) representations of the Virasoro algebra have a positive value of the central charge, something that is also necessary for locality, i.e. decaying correlation functions. An intriguing recent observation is that post-Newtonian corrections seem to violate general covariance [14]; this is possibly related because an extension is the simplest way to relax diffeomorphism symmetry in a mathematically consistent way. Moreover, the Schwinger terms arising in the
standard model are quite different from the multi-dimensional Virasoro algebra. They give rise to Mickelsson-Faddeev algebras, which are known not to possess fully quantum representations [23]. As shown in this paper, not only does the DGRO algebra possess quantum representations, but one can associate a family of such representations (labelled by the jet order $p$) to every general-covariant dynamical system, which can probably be viewed as a kind of quantization.

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