ON BISMUT FLAT MANIFOLDS

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Abstract. In this paper, we give a classification of all compact Hermitian manifolds with flat Bismut connection. We show that the torsion tensor of such a manifold must be parallel, thus the universal cover of such a manifold is a Lie group equipped with a bi-invariant metric and a compatible left invariant complex structure. In particular, isosceles Hopf surfaces are the only Bismut flat compact non-Kähler surfaces, while central Calabi-Eckmann threefolds are the only simply-connected compact Bismut flat threefolds.

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1. Introduction

In recent years, there has been much progress on the study of Hermitian differential geometry. Examples include the work of Jixiang Fu and others on non-Kähler Calabi-Yau manifolds and balanced manifolds, the work of Bo Guan on fully non-linear PDE with application to Hermitian manifolds, the work of Hermitian curvature flows by Streets and Tian, the study of general Hermitian geometry and various Ricci curvature tensors by Liu and Yang, and the recent solution to the Gauduchon Conjecture by Székelyhidi, Tosatti, and Weinkove. We refer the readers to [37], [9], [10], [11], [12], [13], [24], [41], [19], [16], [17], [18], [33], [34], [35], [36], [25], [26], [27], [39], [40], [38] and the references therein for some recent progress in this area.

Given a Hermitian manifold \((M^n, g)\), there are three well-known canonical connections associated with the metric, namely, the Riemannian (or Levi-Civita) connection \(\nabla\), the Chern (aka Hermitian) connection \(\nabla^c\), and the Bismut connection \(\nabla^b\). In [3], Bismut showed that on any Hermitian manifold, there exists a unique connection that is compatible with the metric \(g\) and the almost complex structure \(J\), and whose \((3,0)\) torsion tensor is skew-symmetric (see also [46]). This canonical connection is known as the Bismut connection.

Bismut connection has been playing an increasingly important role in the study of non-Kähler geometry in recent years. For instance, one version of the definitions of Calabi-Yau manifolds with torsion refers to compact non-Kähler Hermitian threefolds (with finite fundamental group) whose Bismut connection has \(SU(3)\) holonomy. As another example, in [35], Streets and Tian used Bismut connection to reinterpret their Hermitian curvature flow ([33], [34]) and exhibited a remarkable relationship to mathematical physics. They showed that, up to gauge equivalence, the flow is the renormalization group flow of a nonlinear sigma model with nonzero B-field.

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As a consequence they concluded that the flow is a gradient flow and discovered an entropy functional.

When $g$ is Kähler, all three connections coincide, and when $g$ is not Kähler, these three connections are mutually different, even though any one of them completely determines the other two. It is certainly very natural to study the curvature of these connections. In particular, one could ask: what kind of Hermitian manifolds are “space forms” with respect to each connection? The simplest case of course would be the everywhere zero curvature case, namely, to classify the flat spaces.

For the Chern connection, the classical result of Boothby [4] in 1958 states that, if $(M^n, g)$ is a compact Hermitian manifold whose Chern connection is flat, then the universal cover of $M$ is (holomorphically isometric to) a complex Lie group equipped with a left invariant Hermitian metric. In particular, there are no compact simply-connected Hermitian manifolds with zero Chern curvature. When $n \geq 3$, such manifolds need not be Kähler, as the famous Iwasawa manifold illustrates. An important subclass of these manifolds are the complex parallelizable manifolds, studied by H.-C. Wang ([42]) in 1954.

For the Riemannian (Levi-Civita) connection, the question is essentially to find all compatible complex structures on the flat torus $\mathbb{T}^2 \times \mathbb{R}^n$, since for any compact flat Riemannian manifold $M$, a finite unbranched cover of $M$ is a flat torus by the Bieberbach Theorem. For $n = 2$, the classification theory for compact complex surfaces implies that $M$ must be a complex torus, but when $n \geq 3$, there are non-Kählerian complex structures living on the flat torus. In a recent work [22], we were able to determine all orthogonal complex structures on flat 6-tori, thus solving the $n = 3$ case. The classification problem for Riemannian flat compact Hermitian manifold in complex dimension 4 or higher remains open at this point.

In light of the above results, it is natural to ask the following

**Question.** What kind of compact Hermitian manifolds will be Bismut flat, namely, the curvature of the Bismut connection is everywhere zero?

As Riemannian manifolds, the structure of such spaces are well known, as the classical theory of Cartan and Schouten ([7], [8]) states that the existence of a flat metric connection with skew-symmetric torsion would imply that the space is a Lie group or $S^7$ (or their products). See also the work of J. Wolf [43], [44] and the very nice new treatment by Agricola and Friedrich [1]. So the point here is to understand the complex structures compatible with these Riemannian metrics. In general, it is a challenging task to understand the set of all possible complex structures compatible with a given Riemannian metric, or in Simon Salamon’s term, all orthogonal complex structures (OCS), see [30]. For example, it is still unknown what is the set of all OCSs on a flat 8-torus, as we mentioned above. This set is known to be quite large, as it contains all the non-Kählerian warped complex structures given by Borisov, Salamon, and Viaclovsky [5].

Back to our Bismut flat manifolds, first let us see some examples of such spaces. It is not hard to see that any isosceles Hopf surface is Bismut flat (see §2). Also, recall that the “central” Calabi-Eckmann threefold is the manifold $M = S^3 \times S^3$, equipped with the product of (constant multiples of) the standard metric, with a compatible complex structure that is left invariant when $M$ is considered as the Lie group $SU(2) \times SU(2)$. This complex structure is the one constructed by Samelson [32] for even-dimensional compact Lie groups, and belongs to the family of complex structures constructed by Calabi and Eckmann [6] on the product of odd dimensional spheres. It is easy to check that the central Calabi-Eckmann threefolds are Bismut flat. More generally, Samelson showed [32] (see also §5 of the work of Alexandrov and Ivanov [2]) that any compact Lie group $G$ of even dimension admits left invariant complex structures that are compatible with a bi-invariant metric. It is well known that such a Hermitian manifold is Bismut flat (see for instance the work [2] or [20]). For the sake of convenience, let us introduce the following terminology:
Definition. A Samelson space is a Hermitian manifold \((G',g,J)\), where \(G'\) is a connected and simply-connected, even-dimensional Lie group, \(g\) a bi-invariant metric on \(G'\), and \(J\) a left invariant complex structure on \(G'\) that is compatible with \(g\).

By Milnor’s Lemma ([28], Lemma 7.5), a simply-connected Lie group \(G'\) with a bi-invariant metric must be the product of a compact semisimple Lie group with an additive vector group, namely, \(G' = G \times \mathbb{R}^k\), where \(0 \leq k \leq \dim G'\) and \(G\) is compact semisimple. Notice that \(G'\) and the compact Lie group \(G'' = G \times T^k\) (where \(T^k\) is the torus) share the same Lie algebra, so when \(G'\) is even dimensional, \(G''\) hence \(G'\) admits left invariant complex structures compatible with the bi-invariant metric.

Let \(\rho : \mathbb{Z}^k \rightarrow I(G)\) be a homomorphism from the free abelian group of rank \(k\) into the isometry group of \(G\). Then \(\Gamma_\rho \cong \mathbb{Z}^k\) acts on \(G \times \mathbb{R}^k\) by \(\gamma(x,y) = (\rho(\gamma)(x), y + \gamma)\) as isometries, and it acts freely and properly discontinuously, so we get a compact quotient \(M_\rho = (G \times \mathbb{R}^k)/\Gamma_\rho\).

Definition. Let \((G',g,J)\) be a Samelson space, where \(G' = G \times \mathbb{R}^k\). Let \(\rho : \mathbb{Z}^k \rightarrow I(G)\) be a homomorphism into the isometry group of \(G\), and \(M_\rho\) be the compact quotient defined as above. If the complex structure of \(G'\) is preserved by \(\Gamma_\rho\), then it descends down to \(M_\rho\) and makes it a compact manifold. In this case we will call the compact Hermitian manifold \(M_\rho\) a local Samelson space. Such a Hermitian manifold is Bismut flat, since its universal cover is so.

As we shall see in the proof of Theorem 1 below, \(M_\rho\) is always diffeomorphic to \(G \times T^k\), where \(T^k\) is the \(k\)-torus. However, \(M_\rho\) (or any finite unbranched cover of it) may not be a Lie group. Also, \(G\) (considered as left multiplications) is a proper subgroup of \(I(G)\) in general. When the image of \(\rho\) is contained in \(G\), then \(\Gamma_\rho\) acts as left multiplications in \(G''\) hence preserves the complex structure. In particular, \(M_0 = G \times T^k\) is always a compact Bismut flat manifold.

A somewhat surprising fact to us is that, compact Bismut flat manifolds actually form a rather small class, and they are essentially just these local Samelson spaces. To be precise, we have the following:

Theorem 1. Let \((M^n,g)\) be a compact Hermitian manifold whose Bismut connection is flat. Then there exists a finite unbranched cover \(M'\) of \(M\) such that \(M'\) is a local Samelson space \(M_\rho\) defined as above. Also, \(M_\rho\) is diffeomorphic to \(G \times T^k\), where \(T^k\) is the \(k\)-torus.

We remark that the finite unbranched cover \(M' \rightarrow M\) might not be Galois, even though there is always a finite sequence of Galois covers \(M_{i+1} \rightarrow M_i\), \(1 \leq i \leq r - 1\), such that \(M' = M_r\) and \(M_1 = M\). See §4 for more details.

Note that by a result of Pittie [29] (see also §5 of [2]), on an even dimensional Lie group \(G'\) with bi-invariant metric, any compatible left invariant complex structures must be those constructed by Samelson, from the root decompositions. (See §3 for a more detailed discussion of this). We remark that while the universal covering space \(M\) of a compact Bismut flat manifold is always a Samelson space, in general, however, the deck transformation group \(\Gamma\) might not be a subgroup of the Lie group \(M\) (see also the discussion in §4).

A property about Bismut flat manifolds worth mentioning is the following. This and other properties about such manifolds are actually used in the proof of Theorem 1. Recall that (see [14]) the Gauduchon 1-form \(\eta\) of a Hermitian manifold \(M^n\) is the global \((1,0)\) form on \(M\) determined by \(\partial \omega^{n-1} = -2\eta \wedge \omega^{n-1}\), where \(\omega\) is the Kähler (metric) form. The manifold is called balanced, if \(d(\omega^{n-1}) = 0\), or equivalently, \(\eta = 0\).

Theorem 2. Let \((M^n,g)\) be a Hermitian manifold which is Bismut flat. If it is balanced, then it is Kähler. If \(M\) is compact, then the equality

\[
\int_M |T^e|^2 \omega^n = 16 \int_M |\eta|^2 \omega^n
\]

holds, where \(T^e\) is the torsion of the Chern connection, and \(\eta\) is the Gauduchon 1-form which is the trace of the torsion.
Now let us consider the special case of \( n = 2 \). Recall that an *isosceles Hopf surface* is a compact complex surface \( M^2 \) with universal cover \( \mathbb{C}^2 \setminus \{0\} \) and deck transformation group a finite extension (by unitary rotations) of the infinite cyclic group \( \mathbb{Z} \), with \( f(z_1, z_2) = (az_1, bz_2) \), where \((z_1, z_2)\) is the Euclidean coordinate of \( \mathbb{C}^2 \) and \( 0 < |a| = |b| < 1 \).

Write \(|z|^2\) for \(|z_1|^2 + |z_2|^2\). The standard Hermitian metric \( g \) on \( \mathbb{C}^2 \setminus \{0\} \) has the Kähler form
\[
\omega_g = \frac{1}{|z|^2} \partial \bar{\partial}|z|^2.
\]
When \(|a| = |b|\), this metric descends down to \( M^2 \). It is straightforward to check (see §2) that this metric is Bismut flat. So as a direct consequence of Theorem 1, we get the following

**Corollary 3.** Let \((M^2, g)\) be a compact Hermitian surface that is Bismut flat. Then either \( g \) is Kähler and \((M^2, g)\) is a flat complex torus or a flat hyperelliptic surface, or \( g \) is not Kähler and \((M^2, g)\) is an isosceles Hopf surface equipped with (a constant multiple of) the standard metric.

Next let us examine the 3 dimensional cases. Recall that a *central Calabi-Eckmann threefold* is the Lie group \( G = SU(2) \times SU(2) \) equipped with a left invariant complex structure \( J \) which is compatible with a bi-invariant metric. The bi-invariant metrics on \( G \) are unique up to scaling constants on the factors, namely, they are in the form \( c_1g_0 \times c_2g_0 \), where \( c_1, c_2 \) are positive constants and \( g_0 \) the standard metric on \( SU(2) = S^3 \) which has constant sectional curvature 1. As Hermitian manifolds, the holomorphic isometric class of such spaces are determined by the two scaling constants.

**Corollary 4.** Let \((M^3, g)\) be a compact Hermitian manifold of dimension 3 which is Bismut flat and non-Kähler. Then the universal cover of \( M \) is holomorphically isometric to either a central Calabi-Eckmann threefold, or \( (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \), equipped with the product of (a constant multiple of) the standard metric \( \omega_g \) and the flat metric. In particular, the only simply-connected compact Bismut flat threefolds are the central Calabi-Eckmann threefolds.

Notice that the central Calabi-Eckmann manifold \( S^3 \times S^3 \), being the only compact, simply-connected, three dimensional Bismut flat manifold, it just stands out as perhaps a perfect candidate for the hidden space of our universe in the non-Kähler Calabi-Yau theory.

Also, in the case when \( \tilde{M} = (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \), there might not be any finite unbranched cover \( M' \) of \( M \) such that \( M' \) is the product of an isosceles Hopf surface and an elliptic curve. In fact, \( M' \) might not even be an elliptic fibration over a Hopf surface or a Hopf surface fibration over an elliptic curve. We will see an example of such kind at the end of §4.

Using the fact that the only simply-connected compact simple Lie groups in dimension less than 14 are \( SU(2) \), \( SU(3) \), and \( Spin(5) \), one could have a similar discussion and classification on compact Bismut flat manifolds in complex dimension 6 or less.

Finally, it is natural to wonder about the compactness assumption in Theorem 1. By using a nice characterization of flat metric connection with skew-symmetric torsion on a Riemannian manifold, given by I. Agricola and T. Friedrich in [1], we obtain the following generalization of the main theorem:

**Theorem 5.** Let \((M^n, g)\) be a simply-connected Hermitian manifold whose Bismut connection is flat. Then there exists a Samelson space \((G, J, g_0)\), namely, \( G \) is a simply-connected even-dimensional Lie group, with \( g_0 \) a bi-invariant metric on \( G \) and \( J \) a compatible left invariant complex structure on \( G \), such that \( M \) is an open complex submanifold of \( G \) and \( g = g_0|_M \).

In other words, the universal cover of any Bismut flat manifold is always an open part of a Samelson space. In particular, any simply-connected, non-Kähler, Bismut flat surface is an open subset in \( (\mathbb{C}^2 \setminus \{0\}, cg) \), where \( c > 0 \) is a constant and \( \omega_g = \frac{1}{|z|^2} \partial \bar{\partial}|z|^2 \) is the standard metric.

In comparison, for manifolds with flat Chern connections, Boothby [4] observed that there are non-compact Chern flat surfaces whose torsion components are not constants. In §5, we will give more examples of such kind, including a complete one.
For Hermitian manifolds with flat Riemannian connection, there are also lots of non-compact examples with non-constant norm of torsion, even in complex dimension 2. It turns out that locally such structures are determined by three holomorphic functions. We will give some examples of such surfaces in the end. All such examples are necessarily incomplete, since a complete flat Riemannian 4-manifold is uniformized by the flat Euclidean space \( \mathbb{R}^4 \), and all orthogonal complex structures on \( \mathbb{R}^4 \) are the standard ones, by a result of Salamon and Viaclovsky ([31], Theorem 1.3).

The paper is organized as follows: In Section 2, we collect some preliminary results. In Section 3, we recall the construction of Samelson and Pittie on left invariant complex structures on even-dimensional compact Lie groups. In Section 4, we discuss the general properties of Bismut flat manifolds, and give the proofs of Theorem and the corollaries. In Section 5, we discuss the non-compact cases.

2. Preliminaries

In this section, we will collect some preliminary results and fix the notations and terminologies. More details can be found in our earlier work [45], but we will try to make things self-contained here for the convenience of the readers.

Let \((M^n, g)\) be a Hermitian manifold, with \( n \geq 2 \). We will denote by \( \nabla, \nabla^c, \) and \( \nabla^b \) the Riemannian, Chern, and Bismut connection of the metric \( g \), and by \( R, R^c, \) and \( R^b \) their curvatures, called the Riemannian, Chern, or Bismut curvature tensor, respectively. (In [45] we used the term Hermitian instead of Chern. The latter is a less ambiguous in this context).

Let \( T^{1,0}M \) be the bundle of complex tangent vector fields of type \((1, 0)\), namely, complex vector fields of the form \( v - \sqrt{-1}Jv \), where \( v \) is a real vector field on \( M \). Let \( \{e_1, \ldots, e_n\} \) be a local frame of \( T^{1,0}M \) in a neighborhood in \( M \). Write \( e = (e_1, \ldots, e_n) \) as a column vector. Denote by \( \varphi = (\varphi_1, \ldots, \varphi_n) \) the column vector of local \((1, 0)\)-forms which is the coframe dual to \( e \). For the Chern connection \( \nabla^c \) of \( g \), let us denote by \( \theta, \Theta \) the matrices of connection and curvature, respectively, and by \( \tau \) the column vector of the torsion 2-forms, all under the local frame \( e \). Then the structure equations and Bianchi identities are

\[
\begin{align*}
  d\varphi &= -\theta \wedge \varphi + \tau, \\
  d\theta &= \theta \wedge \theta + \Theta, \\
  d\tau &= -\theta \wedge \tau + \Theta \wedge \varphi, \\
  d\Theta &= \theta \wedge \Theta - \Theta \wedge \theta.
\end{align*}
\]

Note that under a frame change \( \tilde{e} = Pe \), the corresponding forms are changed by

\[
\tilde{\varphi} = \Psi^{-1}\varphi, \quad \tilde{\theta} = P\theta P^{-1} + dPP^{-1}, \quad \tilde{\Theta} = P\Theta P^{-1}, \quad \tilde{\tau} = \Psi^{-1}\tau
\]

In particular, the types of the 2-forms in \( \Theta \) and \( \tau \) are independent of the choice of the frame \( e \). Also, the compatibility of \( \nabla^c \) with the metric means that when \( e \) is unitary, both \( \theta \) and \( \Theta \) would be skew-Hermitian. Using these facts, and by taking \( e \) to be either holomorphic or unitary, we know that the entries of \( \Theta \) or \( \tau \) are always \((1, 1)\) or \((2, 0)\) forms, under any frame.

Let us write \( \langle \cdot, \cdot \rangle \) for the (real) inner product given by the Hermitian metric \( g \), and extend it bilinearly over \( \mathbb{C} \). Under the frame \( e \), let us denote the components of the Riemannian connection \( \nabla \) as

\[
\nabla e = \theta_1 e + \bar{\theta}_2 \sigma, \quad \nabla \sigma = \theta_2 e + \bar{\theta}_1 \sigma,
\]

then the matrices of connection and curvature for \( \nabla \) become:

\[
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}, \quad
\begin{bmatrix}
\Theta_1 \\
\Theta_2
\end{bmatrix}
\]
where

\begin{align}
\Theta_1 &= d\theta_1 - \theta_1 \wedge \theta_1 - \overline{\theta_2} \wedge \theta_2 \\
\Theta_2 &= d\theta_2 - \theta_2 \wedge \theta_1 - \overline{\theta_1} \wedge \theta_2 \\
d\varphi &= - \, \theta_1 \varphi - \theta_2 \wedge \overline{\varphi},
\end{align}

and under the frame change $\tilde{e} = Pe, \tilde{\varphi} = P\varphi$, the above matrices of forms are changed by

$$\tilde{\theta}_1 = P\theta_1 P^{-1} + dPP^{-1}, \quad \tilde{\theta}_2 = \overline{P\theta_2} P^{-1}, \quad \tilde{\Theta}_1 = P\Theta_1 P^{-1}, \quad \tilde{\Theta}_2 = \overline{P\Theta_2} P^{-1}$$

Following \cite{45}, we will write

$$\gamma = \theta_1 - \varphi.$$

We have $\dot{\gamma} = P\gamma P^{-1}$ under the frame change, so $\gamma$ represents a tensor. The compatibility of $\nabla_\gamma$ with the metric implies that when $e$ is unitary, both $\theta_2$ and $\Theta_2$ are skew-symmetric, while $\theta_1$, $\gamma$, or $\Theta_1$ are skew-Hermitian.

Let $\gamma = \gamma' + \gamma''$ be the decomposition of $\gamma$ into $(1, 0)$ and $(0, 1)$ parts. Denote by $T_{ij}^k = -T_{ji}^k$ the components of $\tau$:

$$\tau_k = \sum_{i,j=1}^n T_{ij}^k \varphi_i \wedge \varphi_j = \sum_{1 \leq i < j \leq n} 2 \, T_{ij}^k \varphi_i \wedge \varphi_j$$

Note that our $T_{ij}^k$ is only half of the components of the torsion $\tau$ used in some other literature where the second sigma term is used. As observed in \cite{45}, when $e$ is unitary, $\gamma$ and $\theta_2$ take the following simple forms:

$$\begin{align}
(\theta_2)_{ij} &= \sum_{k=1}^n \overline{T_{ij}^k} \varphi_k, \\
\gamma_{ij} &= \sum_{k=1}^n (T_{ik}^j \varphi_k - \overline{T_{jk}^i} \varphi_k)
\end{align}$$

Next, let us recall Gauduchon’s “torsion 1-form $\eta$ which is defined to be the trace of $\gamma'$ (\cite{14}). Under any frame $e$, it has the expression:

$$\eta = \text{tr}(\gamma') = \sum_{i,j=1}^n T_{ij}^i \varphi_j$$

A direct computation shows that

$$\partial \omega^{n-1} = -2 \, \eta \wedge \omega^{n-1},$$

where $\omega$ is the Kähler (or metric) form of $g$. The metric $g$ is said to be balanced if $\omega^{n-1}$ is closed. The above identity shows that $g$ is balanced if and only if $\eta = 0$. When $n = 2$, $\eta = 0$ means $\tau = 0$, so balanced complex surfaces are Kähler. But in dimension $n \geq 3$, $\eta$ contains less information than $\tau$.

Under our notations, the components of the Chern and Riemannian curvature tensors are given by

$$R_{ijkl}^c = \sum_{p=1}^n \Theta_{ip}(e_k, \overline{e_l})g_{p\overline{c}}, \quad R_{abcd} = \sum_{e=1}^{2n} \hat{\Theta}_{ac}(e_c, e_d)g_{eb}$$

where $a, \ldots, e$ are between 1 and $2n$, with $e_{n+i} = \overline{e_i}$. Note that $g_{ij} = g_{i\overline{j}} = 0$, so we have

$$
\begin{align}
R_{ijkl} &= \sum_{p=1}^n (\Theta_{1}^{1,1})_{ip}(e_k, \overline{e_l})g_{p\overline{c}}, \\
R_{ijkl} &= \sum_{p=1}^n (\Theta_{2}^{2,0})_{ip}(e_k, e_l)g_{p\overline{c}} \\
R_{ijkl} &= R_k\overline{e_l} = \sum_{p=1}^n (\Theta_{2}^{1,1})_{ip}(e_k, \overline{e_l})g_{p\overline{c}} = \sum_{p=1}^n (\Theta_{1}^{0,2})_{ip}(e_k, e_l)g_{p\overline{c}} \\
R_{ijkl} &= R_{ijkl} = 0
\end{align}$$

where
The last line is because $\Theta^{0,2}_{z} = 0$ by Lemma 1 of [45], a property for general Hermitian metric discovered by Gray in [15] (Theorem 3.1 on page 603). The following lemma is taken from [45] (Lemma 7):

**Lemma 1.** Let $(M^n, g)$ be a Hermitian manifold. Let $e$ be a unitary frame in $M$, then

\begin{align}
2T^i_{jk} & = R^i_{jkl} - R^i_{jkl} \\
R_{ijkl} & = T^r_{ijkl} + T^r_{jikl} - T^r_{rijl} - T^r_{rjkl} - T^r_{rkjl} + T^r_{rkjl} \\
R_{ijkl} & = T^r_{ijkl} + 2T^r_{jikl} + T^r_{rijl} + T^r_{rjkl} - T^r_{rijl} - T^r_{rkjl} + T^r_{rkjl} - T^r_{rjkl}
\end{align}

where the index $r$ is summed over 1 through $n$, and the index after the comma stands for covariant derivative with respect to the Chern connection $\nabla^c$.

Note that these formula were the main computational tools used in [45]. However, in our present situations, we would prefer to use the Bismut connection $\nabla^b$ and Bismut covariant differentiation instead of $\nabla^c$. To give the precise formula, let us start with the description of $\nabla^b$ under our frame work. Again let us fix a Hermitian manifold $(M^n, g)$ with $n \geq 2$.

Recall that the Bismut connection $\nabla^b$ of $(M^n, g)$ is the unique connection that is compatible with the metric and the almost complex structure, and its $(3, 0)$ torsion is skew-symmetric. The existence and uniqueness of $\nabla^b$ is proved by Bismut in [3]. Again let us fix a local type $(1, 0)$ tangent frame $e$ and let $\varphi$ be its dual coframe. Since $\nabla^b J = 0$, we can write $\nabla^b e_i = \sum_{j=1}^n \theta^b_{ij} e_j$.

We have the following:

**Lemma 2.** Under any frame $e$ of type $(1, 0)$ tangent vectors, the components of the Bismut connection $\nabla^b$ are given by

\begin{equation}
\theta^b = \theta + 2\gamma
\end{equation}

Proof. Clearly, the connection $\nabla^b$ defined by $\theta + 2\gamma$ is compatible with the metric and the almost complex structure, so we just need to verify that its $(3, 0)$ torsion is skew-symmetric. We have $T^b(X, Y) = T^c(X, Y) + 2\gamma_X Y - 2\gamma_Y X$. Since $T^c(e_i, e_j) = 0$ and $T^c(e_i, e_j) = 2\sum_k T^b_{ijk} e_k$, so under any unitary frame $e$ by (10) we get

\begin{equation}
T^b(e_i, e_j) = -2 \sum_k T^b_{ijk} e_k, \quad T^b(e_i, e_j) = 2 \sum_k (T^b_{ijk} e_k - T^b_{jik} e_k),
\end{equation}

and from this it is easy to verify that $\langle T^b(X, Y), Z \rangle = -\langle T^b(X, Z), Y \rangle$ for any tangent vectors $X, Y, Z$. So by the uniqueness we know $\nabla^b$ must be the Bismut connection. \hfill \Box

Using Lemma 2, we can compute the curvature of the Bismut connection for a given Hermitian metric. Let us illustrate this by considering the following example:

**Lemma 3.** Consider the Hermitian metric on $\mathbb{C}^2 \setminus \{0\}$ with Kähler form $\omega = \frac{i}{2} \partial \bar{\partial} |z|^2$, where $z = (z_1, z_2)$ is the standard coordinate of $\mathbb{C}^2$, and $|z|^2 = |z_1|^2 + |z_2|^2$. We claim that the curvature of its Bismut connection is everywhere zero.

Proof. Let $e$ be the unitary frame $e_i = |z| \frac{\partial}{\partial z_i}$ for $i = 1, 2$, its dual coframe is $\varphi_i = \frac{1}{|z|} dz_i$. Under the frame $e$, we have

$\theta = (\bar{\partial} - \partial) \log |z| I, \quad \tau = -2\partial \log |z| \wedge \varphi.$

So we get $T^b_{12} = \frac{\partial}{\partial z_1}$ and $T^b_{21} = -\frac{\partial}{\partial z_2}$, thus

$\gamma = \frac{1}{2|z|^2} \begin{bmatrix}
\overline{z}_2 dz_2 - z_2 d\overline{z}_2 & z_2 d\overline{z}_1 - z_1 d\overline{z}_2 \\
\overline{z}_1 dz_1 - z_1 d\overline{z}_1 & z_1 d\overline{z}_2 - \overline{z}_2 dz_2
\end{bmatrix}, \quad \text{and} \quad \theta^b = \theta + 2\gamma = \frac{1}{2|z|^2} \begin{bmatrix}
A & 2B \\
-2B & -A
\end{bmatrix},$

where $A = z_1 d\overline{z}_1 + z_2 d\overline{z}_2 - z_2 d\overline{z}_2 - z_1 d\overline{z}_1, \quad B = z_2 d\overline{z}_1 - \overline{z}_1 d\overline{z}_2$. 

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Hopf surfaces are diffeomorphic to $S^3 \times S^1$, in the form $M_{a,b}$ or $M_{a;m}$ below, where $a$ and $b$ are complex numbers satisfying $0 < |a| \leq |b| < 1$ and $m \geq 2$ is an integer. Here

$$M_{a,b} = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z} \phi, \quad M_{a;m} = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z} \psi,$$

where

$$\phi(z_1, z_2) = (a z_1, b z_2), \quad \psi(z_1, z_2) = (a z_1, z_1^m + a^m z_2),$$

with $(z_1, z_2)$ the standard coordinate of $\mathbb{C}^2$.

We will call a Hopf surface $M$ covered by $M_{a,b}$ with $|a| = |b|$ an isosceles Hopf surface. Its fundamental group is an extension of $\mathbb{Z} \phi$ by a finite group $F$, where $\phi(z_1, z_2) = (a z_1, b z_2)$ with $0 < |a| = |b| < 1$, and $F$ is a finite subgroup of $U(2)$ (and in fact $F$ is a subgroup of $U(1) \times U(1)$ when $a \neq b$, see [21] for more details). Clearly, any element of $\pi_1(M)$ preserves $\omega$ in Lemma 3, so the metric descends down to $M$, and we get

**Lemma 4.** Any isosceles Hopf surface admits a Bismut flat Hermitian metric.

Conversely, as a consequence of our main theorem, we shall see that any compact Bismut flat Hermitian surface is either an isosceles Hopf surface when it is non-Kähler, or, when it is Kähler, a flat complex torus or hyperelliptic surface.

We conclude this section by stating the following well known result with a sketched proof.

**Lemma 5.** Let $(M^n, g)$ be a Hermitian manifold whose Bismut connection $\nabla^b$ is flat. Then given any $p \in M$, there exists a neighborhood $p \in U \subseteq M$ and type $(1, 0)$ unitary frame $e$ in $U$ which is $\nabla^b$-parallel, namely, $\nabla^b e_i = 0$ in $U$ for each $1 \leq i \leq n$.

**Proof.** Since the curvature of the connection $\nabla^b$ is everywhere zero, the entries of the connection matrix forms a completely integrable system, therefore there will be local frame of the real tangent bundle of $M$ which is $\nabla^b$-parallel. But $\nabla^b J = 0$, so we have type $(1, 0)$ complex tangent frame that is $\nabla^b$-parallel. □

### 3. The Samelson Spaces

In this section, let us recall Samelson’s construction [32] of left invariant complex structures on even-dimensional compact Lie groups, which comes from a choice of a maximal torus, a complex structure on the Lie algebra of the torus, and a choice of positive roots for the Cartan decomposition. We will also discuss Pittie’s theorem [29] which states that all left invariant complex structures on such groups are actually obtained this way. In §5 of [2], Alexandrov and Ivanov gave a nice description of both results. Here for the convenience of the readers, we include some of their arguments briefly, on those statements that we will need in our later discussions.

First let us begin with the following result, which was observed by Alexandrov and Ivanov ([2], p.263), and might be known to other experts as well. We include a proof here for the convenience of the readers.

**Lemma 6.** (Alexandrov-Ivanov) Let $G$ be an even dimensional connected Lie group equipped with a bi-invariant metric $g = (\cdot, \cdot)$ and a left invariant complex structure $J$ which is compatible with $g$. Then the Hermitian manifold $(G, J, g)$ is Bismut flat.
Proof. Let \( \{ e_1, \ldots, e_n \} \) be a unitary frame of left-invariant vector fields on \( G \) of type \((1, 0)\). It suffices to show that \( \nabla^b e_j = 0 \) for each \( j \). Since the metric is bi-invariant, it is well-known that the Riemannian connection \( \nabla \) is given by \( \nabla_X Y = \frac{1}{2}[X, Y] \) for left invariant vector fields. The integrability condition on the complex structure means that we have \( \langle e_i, e_j \rangle = \sum_k C^k_{ij} e_k \) for some constants \( C^k_{ij} \), and since

\[
\langle [e_i, e_j], e_k \rangle = \langle [e_j, e_k], e_i \rangle = C^i_{jk}
\]

we get \( [e_i, e_j] = \sum_k (C^i_{jk} e_k - C^j_{ik} e_k) \). By Lemma 2, we have

\[
\nabla^b e_j = \frac{1}{2} [e_i, e_j] + \sum T^k_{ij} e_k = \sum \left( \frac{1}{2} C^k_{ij} - T^k_{ij} \right) e_k
\]

\[
\nabla^b e_j = \nabla^c e_j + 2 \gamma e_j = \nabla e_j - \gamma e_j + 2 \gamma e_j
\]

\[
= \frac{1}{2} \left[ e_i, e_j \right] - \sum (T^k_{ji} e_k + T^k_{kj} e_j) - \sum (\frac{1}{2} C^k_{ij} - T^k_{ij}) e_k
\]

Since \( \nabla^b e_j = \sum_{k} \theta^b_{jk} [e_k] e_k \), we know from the last line above that \( \frac{1}{2} C^i_{jk} = T^i_{jk} \), hence \( \nabla^b e_j = 0 \) for each \( j \), and \( \nabla^b \) is flat.

Next, let \( G \) be a connected Lie group equipped with a bi-invariant metric \( \langle , \rangle \). Denote by \( \mathfrak{g} \) the Lie algebra of \( G \), and also denote by \( \langle , \rangle \) the inner product on \( \mathfrak{g} \) induced by the metric of \( G \). Since the metric is bi-invariant, we have

\[
\langle [X, Y], Z \rangle = -\langle [X, Z], Y \rangle
\]

for any vectors \( X, Y, Z \) in \( \mathfrak{g} \). So if \( a \subset \mathfrak{g} \) is an ideal, denote by \( a^\perp \) its perpendicular compliment in \( \mathfrak{g} \). By letting \( Y \in a \) and \( Z \in a^\perp \) in the above identity, we see that \( a^\perp \) is also an ideal in \( \mathfrak{g} \). So we can write the Lie algebra as the orthogonal direct sum of simple ideals. This leads to the following Milnor’s Lemma (see Lemma 7.5 of [28]):

**Lemma 7. (Milnor)** Let \( G \) be a simply-connected Lie group with a bi-invariant metric \( \langle , \rangle \). Then \( G \) is isomorphic and isometric to the product \( G_1 \times \cdots \times G_r \times \mathbb{R}^k \) where each \( G_i \) is a simply-connected compact simple Lie group and \( \mathbb{R}^k \) is the additive vector group with the flat metric. Here \( 0 \leq k \leq \dim(G) \).

As is well known, the simply-connected compact simple Lie groups are fully classified, they are:

\[
A_n = SU(n+1), n \geq 1, \dim(A_n) = n(n+2);
\]

\[
B_n = Spin(2n+1), n \geq 2, \dim(B_n) = n(2n+1);
\]

\[
C_n = Sp(2n), n \geq 3, \dim(C_n) = n(n+1);
\]

\[
D_n = Spin(2n), n \geq 4, \dim(D_n) = n(2n-1);
\]

\[
E_6, \dim(E_6) = 78;
\]

\[
E_7, \dim(E_7) = 133;
\]

\[
E_8, \dim(E_8) = 248;
\]

\[
F_4, \dim(F_4) = 52;
\]

\[
G_2, \dim(G_2) = 14.
\]

Note that the only ones in dimension less than 14 are \( M^3 = SU(2) \), \( M^8 = SU(3) \), and \( M^{10} = Spin(5) \). So only those three could appear in a compact Bismut flat manifold of complex dimension less than or equal to 6.

Since any bi-invariant metric on a compact simple Lie group is a constant multiple of the Killing form, the bi-invariant metric on \( G \) is unique up to constant multiples on the compact factors, and each \( G_i \) is an Einstein manifold with positive Ricci curvature.
Let us fix a simply-connected Lie group $G$ with a bi-invariant metric $(\cdot, \cdot)$. We have $G = G_1 \times \cdots \times G_r \times \mathbb{R}^k$ as above. Let $G' = G_1 \times \cdots \times G_r \times T_k$ where $T_k$ is the torus. Then $G$ is the covering group of $G'$, and they share the same Lie algebra $\mathfrak{g}$.

Now we assume that $\dim(G)$ is even. Note that the left invariant complex structures on $G$ or $G'$ are both in one-one correspondence with left invariant complex structures on $\mathfrak{g}$, which are linear maps $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -I$ and
\begin{equation}
J([X,Y] - [JX, JY]) = [JX, Y] + [X, JY]
\end{equation}
for any $X, Y$ in $\mathfrak{g}$.

Samelson constructed left invariant complex structures $J$ on $G$ that is compatible with the metric, by choosing a maximal torus $K$ in $G'$, a complex structure on the Lie algebra $\mathfrak{k}$ of $K$, and a choice of positive roots for the Cartan decomposition of $\mathfrak{g}$. In [2], §5, Alexandrov and Ivanov give a nice description of Samelson’s construction. Here we include a brief account of it for the convenience of the readers.

Denote by $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$. The existence of a compatible left invariant complex structure $J$ is equivalent to the existence of a complex subspace $\mathfrak{s} \subset \mathfrak{g}^c$, such that $\langle \mathfrak{s}, \mathfrak{s} \rangle = 0$, $\mathfrak{s} \cap \mathfrak{g} = 0$, and $\mathfrak{s} \oplus \mathfrak{r} = \mathfrak{g}^c$. Such a subspace is called a Samelson subalgebra of $\mathfrak{g}^c$.

Now let $K$ be a maximal torus of $G'$, and $\mathfrak{k}$ its Lie algebra. Denote by $\mathfrak{k}^c$ the complexification of $\mathfrak{k}$. When a set of positive roots $\alpha_1, \ldots, \alpha_m$ is chosen, then it is well-known that one has the $\text{ad}(K)$-invariant decomposition
\begin{equation}
\mathfrak{g}^c = \mathfrak{k}^c \oplus \sum_{j=1}^m \mathfrak{g}_{\alpha_j} \oplus \sum_{j=1}^m \mathfrak{g}_{-\alpha_j},
\end{equation}
where
\begin{equation}
\mathfrak{g}_{\pm \alpha_j} = \{ Y \in \mathfrak{g}^c \mid [X, Y] = \pm 2\pi \sqrt{-1} \alpha_j(X)Y, \ \forall X \in \mathfrak{k} \}
\end{equation}
are the root spaces.

Since $\dim(G)$ is even, we know that the abelian Lie algebra $\mathfrak{k}$ is even dimensional. So we can choose an almost complex structure on $\mathfrak{k}$ that is compatible with the metric. This means, we have a complex subspace $\mathfrak{a} \subset \mathfrak{k}^c$ such that $\langle \mathfrak{a}, \mathfrak{a} \rangle = 0$, $\mathfrak{a} \cap \mathfrak{k} = 0$, and $\mathfrak{a} \oplus \mathfrak{r} = \mathfrak{k}^c$.

Now one could simply take
\begin{equation}
\mathfrak{s} = \mathfrak{a} \oplus \sum_{j=1}^m \mathfrak{g}_{\alpha_j}
\end{equation}
to be the Samelson subalgebra. So on any even-dimensional Lie group $G$ equipped with a bi-invariant metric, there always exists compatible left invariant complex structures on $G$, constructed by an arbitrary choice of an almost complex structure (compatible with the metric) on the Cartan subalgebra plus the choice of a set of positive roots in the root decomposition.

Conversely, Pittie [29] proved that, any left invariant complex structure on $G$ is obtained this way. Again a nice description of this is given by Alexandrov and Ivanov in §5 of [2], and we also include a brief account of their argument here for readers’ convenience.

Let $\mathfrak{s}$ be a Samelson subalgebra of $\mathfrak{g}^c$ corresponding to a left invariant complex structure $J$ on the compact Lie group $G'$. Let

\[ \mathfrak{k} = \{ X \in \mathfrak{g} \mid \text{ad}(X)(\mathfrak{s}) \subseteq \mathfrak{s} \} \]

be the set of all elements in $\mathfrak{g}$ that preserves the decomposition $\mathfrak{g}^c = \mathfrak{s} \oplus \mathfrak{r}$. Then it is easy to see that $\mathfrak{k}$ is a $J$-invariant subalgebra of $\mathfrak{g}$, hence $\mathfrak{k}$ is a complex Lie algebra. Let $K$ be a closed connected Lie subgroup of $G'$ corresponding to $\mathfrak{k}$. Then $K$ is a compact complex Lie group, thus a torus and is abelian. We have an $\text{ad}(K)$-invariant orthogonal decomposition (26) with $\mathfrak{g}_{\alpha_j}$ given by (25) and
\[ \mathfrak{a} = \{ Y \in \mathfrak{s} \mid [X, Y] = 0, \ \forall X \in \mathfrak{k} \}. \]
It follows from the definition of $\mathfrak{t}$ that $\mathfrak{t}^c = \mathfrak{a} \oplus \mathfrak{a}$, so $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and $K$ is a maximal torus. So (24) is satisfied, and $\pm \alpha_1, \ldots, \pm \alpha_m$ are all the roots. Since $[s, s] \subseteq s$, we know that if $\alpha_1 + \alpha_j$ is a root, then $\alpha_1 + \alpha_j = \alpha_l$ for some $l$. So we can take $\{\alpha_1, \ldots, \alpha_m\}$ to be our set of positive roots. This shows that any left invariant complex structure on a compact Lie group is determined by a choice of a maximal torus, a choice of a complex structure on the Lie algebra of the maximal torus, and a choice of positive roots.

As an application of the above characterization, let us consider left invariant complex structures $J$ on a simply-connected Lie group $G$ equipped with a bi-invariant metric, in the special cases when $\dim(G) = 2n$ is small. Note that when $G = \mathbb{R}^{2n}$, the left invariant complex structures are just those identifications of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. In this case the metric is Kähler, and vice versa. So for our discussion below let us assume that $G$ is not the vector group.

First let us start with $n = 2$. In this case, $G$ has only one choice: $SU(2) \times \mathbb{R}$. Denote by $W$ a unit vector in the factor $\mathbb{R}$. Note that in the Lie algebra $\mathfrak{su}(2)$, the bracket is given by (twice of) the usual cross product, namely, if $X, Y, Z$ forms a (positively oriented) orthonormal basis of it, then

$$[X,Y] = 2Z, \quad [Y,Z] = 2X, \quad [Z,X] = 2Y.$$

Therefore, a compatible left invariant complex structure $J$ on $G$ is determined by the choice of a unit vector $X$ in $\mathfrak{su}(2)$, as the image of $W$ under $J$, and we must have $JY = Z$ if $\{X,Y,Z\}$ forms a positive orthonormal basis. Since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, we know that such $J$ are all isomorphic to each other. In other words, when $n = 2$, the universal cover of compact, non-Kähler Bismut flat surfaces is unique (up to the change on the metric by a constant multiple): they are all holomorphically isometric to $\mathbb{C}^2 \setminus \{0\}$ equipped with the metric $c \frac{1}{|z|^2} \partial \overline{\partial}|z|^2$, where $c$ is a positive constant.

Now let us look at the $n = 3$ case. $G$ is either $SU(2) \times \mathbb{R}^3$ or $SU(2) \times SU(2)$. Let $J$ be a compatible left invariant complex structure on $G$. By the results of Samelson and Pittie, we know that $JV \cap V \neq 0$ for the $V = \mathfrak{su}(2)$ factor in $\mathfrak{g}$. So in the case when $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}^3$, the complex structure must be in the form: $JY = Z, JX = W_1$, and $JW_2 = W_3$, where $\{X,Y,Z\}$ is an orthonormal basis of $\mathfrak{su}(2)$ and $\{W_1, W_2, W_3\}$ is an orthonormal basis of $\mathbb{R}^3$. This means that $G$ is holomorphically isometric to $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$.

Similarly, when $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have orthonormal basis $\{X,Y,Z\}$ for the first factor and $\{X_1,Y_1,Z_1\}$ for the second factor, such that $JY = Z, JY_1 = Z_1$, and $JX = X_1$. This is one particular complex structure in the family of complex structures on $S^3 \times S^3$ given by Calabi-Eckmann [6], and for lack of better terminologies, we will call it a central Calabi-Eckmann threefold (see §1). Note that as Hermitian manifolds, such spaces are unique up to the choice of two positive constants $c, c'$, so the metric on the manifold is $g = (cg_0) \times (c'g_0)$, where $g_0$ is the standard metric on $SU(2) = S^3$ with constant sectional curvature 1.

For $n = 4$, we have $G = SU(2) \times \mathbb{R}^5$, or $SU(2) \times SU(2) \times \mathbb{R}^2$, or $SU(3)$. In the first case, since $J$ has to have a non-trivial invariant part in the $\mathfrak{su}(2)$ factor, there is only one direction in the Euclidean factor that is $J$-involuted with $\mathfrak{su}(2)$, so $G$ is holomorphically isometric to the product of $\mathbb{C}^2 \setminus \{0\}$ with $\mathbb{C}^2$. In the case $SU(2) \times SU(2) \times \mathbb{R}^2$, a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ would consist of one direction from each $\mathfrak{su}(2)$ plus the two dimensional Euclidean factor. The choice of $J$ on $\mathfrak{t}$ may or may not respect the original splitting, e.g., $J$ could be chosen to be

$$JX = aY + bZ, \quad JY = -aX - bW, \quad JZ = -bX + aW, \quad JW = bY - aZ,$$

where $\{Z,W\}$ is an orthonormal basis of $\mathbb{R}^2$ and $X, Y$ are unit vectors from the two $\mathfrak{su}(2)$ factors, perpendicular to the $J$-invariant part, and $a, b$ are real constants satisfying $a^2 + b^2 = 1$. Note that for such a $J$ (when $ab \neq 0$), $G$ is not holomorphically isometric to either the product of two copies of $\mathbb{C}^2 \setminus \{0\}$, or the product of a central Calabi-Eckmann threefold and $\mathbb{C}$.

One could apply similar analysis on $G$ in other small dimensions. In our opinion, Samelson spaces provide an interesting class of complex manifolds, whose differential geometric aspects could be further studies and exploited.
4. The Bismut flat metrics

In this section, let us assume that \((M^n, g)\) is a Hermitian manifold whose Bismut connection \(\nabla^b\) is flat. We are only interested in the case when \(g\) is not Kähler.

By Lemma 5, locally there will always be \(\nabla^b\)-parallel frames. Such frames are obviously unique up to changes by constant matrices. Let us fix a \(\nabla^b\)-parallel, unitary local frame \(e\), and denote by \(\varphi\) its dual coframe. By Lemma 2, we have \(\theta = -2\gamma\), and the structure equations and the first Bianchi identity for \(\nabla^c\) and \(\nabla\) specialize into the following

**Lemma 8.** On a Bismut flat Hermitian manifold \((M^n, g)\), under a local unitary \(\nabla^b\)-parallel frame \(e\), it holds that

\[
\begin{align*}
\partial \varphi &= -\tau = \gamma' \wedge \varphi \\
\mathcal{F} \varphi &= -2 \gamma' \wedge \varphi \\
\partial \gamma' &= -2 \gamma' \wedge \gamma' \\
0 &= \gamma' \wedge \gamma' \wedge \varphi \\
0 &= \bar{\partial} \gamma' \wedge \varphi - 2 \partial \gamma' \wedge \varphi + 2 \gamma' \wedge \varphi + 2 \gamma' \wedge \gamma \wedge \varphi
\end{align*}
\]

**Proof.** The first two identities are immediate from the structure equations and the fact \(\theta = -2\gamma\) since \(e\) is \(\nabla^b\)-parallel. The third one is due to the fact that the \((2, 0)\) part of \(\Theta\) is zero, and the last two are direct consequence of the first Bianchi identity under the circumstance. \(\square\)

Using the expression \(\gamma'_{ij} = \sum_k T^j_{ik} \varphi_{k}\), we can rewrite the last three identities of Lemma 8 in terms of the torsion components \(T^j_{ik}\) and their covariant derivatives with respect to \(\nabla^b\):

\[
\begin{align*}
T^j_{ik,l} - T^j_{il,k} &= 2 \sum_r (T^r_{ik} T^j_{rl} + T^r_{il} T^j_{rk} + T^r_{kl} T^j_{ri}) \\
0 &= \sum_r (T^r_{ij} T^l_{rk} + T^r_{jk} T^l_{ri} + T^r_{kl} T^l_{ij}) \\
T^j_{kl,j} + T^k_{ij,l} - T^l_{ij,k} &= 2 \sum_r (T^r_{lr} T^k_{jr} - T^r_{jr} T^k_{lr} - T^r_{lr} T^k_{ir} - T^r_{jr} T^k_{ir} + T^r_{kl} T^l_{ir} - T^r_{kl} T^l_{ir})
\end{align*}
\]

for any \(1 \leq i, j, k, l \leq n\). Note that when \(i, j, k\) are not all distinct, the right hand side of the middle equality is automatically zero, so this line holds true even when \(n = 2\). From the first two, we know that \(T^j_{ik,l} = T^j_{il,k}\), which implies \(T^j_{ik,l} = 0\) for all indices, since any trilinear form which is skew-symmetric with respect to its first two positions while symmetric with respect to its last two positions must be zero, as illustrated by

\[
C_{ij,k} = -C_{ji,k} = -C_{jk,i} = C_{kj,i} = C_{ki,j} = -C_{ik,j} = -C_{ij,k}.
\]

For the last identity, let us denote the right hand side of the equality by \(A^j_{kl}\). It is skew-symmetric in \(ij\), namely, \(A^j_{il} + A^j_{kl} = 0\). So the identity implies that \(T^j_{kl,j} = -A^j_{kl}\), thus its left hand side is equal to \(T^j_{kl,j} + 2T^k_{ij,l}\). Also, since \(A^j_{ij} = A^j_{ik}\), we know that \(T^j_{kl,j} = T^j_{ij,l} - \frac{1}{3} A^j_{kl}\). In summary, we have the following

**Lemma 9.** On a Bismut flat Hermitian manifold \((M^n, g)\), under a local unitary \(\nabla^b\)-parallel frame \(e\), it holds

\[
\begin{align*}
0 &= T^j_{ik,l} \\
0 &= \sum_r (T^r_{ij} T^l_{rk} + T^r_{jk} T^l_{ri} + T^r_{kl} T^l_{ij})
\end{align*}
\]
\begin{align}
(34) \quad T^i_{kl,\mathcal{J}} &= - T^j_{kl,\mathcal{T}} = T^b_{ij,\mathcal{T}} \\
(35) \quad T^i_{kl,\mathcal{J}} &= \frac{2}{3} \sum_r \left( T^I_{lr} T^k_{jr} - T^I_{kr} T^l_{jr} - T^I_{lr} T^k_{ir} + T^I_{kr} T^l_{ir} - T^r T^k_{ir} - T^r T^l_{ir} \right) \\
(36) \quad \sum r \eta_r,\mathcal{r} &= \frac{2}{3} (|T|^2 - 2|\eta|^2)
\end{align}

for any \(1 \leq i, j, k, l \leq n\), where \(r\) is summed from 1 to \(n\), and the index after the comma means covariant derivative with respect to \(\nabla^b\).

Note that the last identity is obtained by letting \(i = k, j = l\), and sum up in (35).

Write \(\eta = \sum_i \eta_i \varphi_i\). By (28), we have \(\bar{\partial} \eta = - \sum_{i,j=1}^n (\eta_{i,\mathcal{T}} + 2 \sum_p \eta_p T^p_{ij}) \varphi_i \wedge \varphi_j\), so

\[
\sqrt{-1} \bar{\partial} \eta \wedge \omega^{n-1} = - \sum_i (\eta_{i,\mathcal{T}} + 2|\eta_i|^2) \frac{\omega^n}{n},
\]

where \(\omega\) is the Kähler form of the metric of \(M^n\). On the other hand, by (12), we have

\[
\partial \bar{\partial} \omega^{n-1} = 2(\partial \eta + 2\eta \wedge \bar{\eta}) \wedge \omega^{n-1},
\]

thus by (36) we get the following:

**Lemma 10.** On a Bismut flat manifold \((M^n, g)\), it holds

\[
(37) \quad - \sqrt{-1} \bar{\partial} \omega^{n-1} = \frac{2}{n} \left( \sum_i \eta_i \varphi_i \right) \omega^n = \frac{4}{3n} (|T|^2 - 2|\eta|^2) \omega^n
\]

From this identity, we immediately get that, if the Bismut flat manifold \(M\) is balanced, then \(T = 0\), i.e., it is Kähler. Also, when \(M\) is compact, the integral of the right hand side of the above equation is zero. Note that under the frame \(\{e, \mathcal{T}\}\), the torsion tensor \(T^c\) of the Chern connection takes the form

\[
T^c(e_i, e_j) = 2 \sum_k T^k_{ij} e_k, \quad T^c(e_i, \mathcal{T}_j) = 0, \quad T^c(\mathcal{T}_i, \mathcal{T}_j) = 2 \sum_k \overline{T^k_{ij}} e_k,
\]

so \(|T^c|^2 = 8 \sum_{i,j,k} |T^k_{ij}|^2 = 8|T|^2\), thus Theorem 2 is proved.

Note that when \(n = 2\), the torsion tensor has only two components: \(T^1_{12}\) and \(T^2_{12}\). The Gauduchon 1-form has coefficients \(\eta_1 = - T^1_{12}\) and \(\eta_2 = T^2_{12}\), and we always have \(|T|^2 = 2|\eta|^2\) when \(n = 2\). So \(\eta_{1,\mathcal{T}} + \eta_{2,\mathcal{I}} = 0\) by (36). On the other hand, by (34), \(\eta_{1,\mathcal{I}} = - T^2_{12,\mathcal{I}} = T^1_{12,\mathcal{I}} = \eta_{2,\mathcal{I}}\), so both are zero, and we get \(T^1_{jk,\mathcal{I}} = 0\) for all indices. Hence both \(T^1_{12}\) and \(T^2_{12}\) are constants. This leads to a proof of Theorem 5 in the \(n = 2\) case if we follow the proof of Theorem 1 in the next page.

When a Bismut flat manifold \((M^n, g)\) is compact, however, we will show that all the \(T^i_{jk}\) (under a local Bismut parallel unitary frame) are indeed constants. The reason is due to the following simple observation that, the globally defined function \(|T|^2 = \sum_{i,j,k} |T^k_{ij}|^2\) on \(M\) is plurisubharmonic. Note that the sum is independent of the choice of the local unitary frames, so the function is globally defined.

**Lemma 11.** On a Bismut flat manifold \((M^n, g)\), the square norm of the torsion tensor (for the Chern connection) is plurisubharmonic, and under a local unitary Bismut parallel frame \(e\), it holds that

\[
\partial \bar{\partial} |T|^2 = \sum_{i,j,k,l,m} T^i_{jk,\mathcal{I}} \overline{T^j_{kl,m}} \varphi_m \wedge \overline{\varphi_l}
\]

In particular, if \(M\) is compact, then all \(T^i_{jk}\) are constants.
Proof. Let $e$ be a local tangent frame of type $(1,0)$ vector fields, that is unitary and $\nabla^b$-parallel. Let $\varphi$ be the coframe of $(1,0)$ forms dual to $e$. Denote by $T^i_{jk}$ the components under the frame $e$ of the torsion tensor of the Chern connection. From the proof of Lemma 6, we have

$$[e_m, e_l] = 2 \sum_p (T^p_{ip} e_p - T^p_{mp} e_p).$$

So by (32), we get

$$T^i_{jk,lm} = [e_m, e_l]T^i_{jk} = -2 \sum_p T^p_{mp} T^i_{jk,p}.$$ 

Also, by (28) in Lemma 8, we know that

$$\partial \varphi = -2 \sum_l \gamma_{pl} \varphi_l = -2 \sum m,i T^i_{pm} \varphi_m \wedge \varphi_l.$$ 

So by (32), we have

$$\partial \bar{\partial}|T|^2 = \partial \sum T^i_{jk,l} T^j_{ik} \partial \varphi_l$$

$$= \sum T^i_{jk,l} T^j_{ik,m} \varphi_m \wedge \bar{\varphi} + \sum T^i_{jk,l} T^j_{ik-p} \varphi_m \wedge \bar{\varphi} + \sum T^i_{jk,l} T^j_{ik} \partial \varphi_l$$

$$= \sum T^i_{jk,l} T^j_{ik,m} \varphi_m \wedge \bar{\varphi} - 2 \sum (T^i_{pm} + T^i_{pm}) T^j_{jk,p} T^j_{ik} \varphi_m \wedge \bar{\varphi}$$

$$= \sum T^i_{jk,l} T^j_{ik,m} \varphi_m \wedge \bar{\varphi} \geq 0$$

When $M$ is compact, using any Gauduchon metric $\tilde{\omega}$ on $M$, we know that the function $|T|^2$ has to be a constant, so $T^i_{jk,l} = 0$ for all indices, thus all $T^i_{jk,l}$ are constants. \hfill $\Box$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $(M^n, g)$ be a compact Bismut flat manifold. Given any $p \in M$, let $e$ be a unitary $\nabla^b$-parallel frame of $(1,0)$ tangent vectors in a neighborhood of $p$, with $\varphi$ the dual coframe. By Lemma 11, all the components $T^i_{jk}$ of the torsion tensor under $e$ are constants. Since $\nabla^b e_i = 0$, we get from (22) in the proof of Lemma 2 the following

$$[e_i, e_j] = -T^k_{ij} e_k = 2 \sum T^k_{ij} e_k$$ 

$$[e_i, e_j] = -T^k_{ij} e_k = 2 \sum (T^k_{jk} e_k - T^k_{ik} e_k)$$

It is easy to verify that

$$(X, Y, Z) = -([X, Z], Y)$$

(39)

hold for any $X, Y, Z$ in $\{e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n\}$. If we write $\varphi_i = \frac{1}{\sqrt{2}} (\phi_i + \sqrt{-1} \phi_{n+i})$, then it is straight forward to check that $\{\phi_i\}_{i=1}^n$ form the left invariant forms for a local Lie group, with left invariant metric and complex structure, and by (39) we see that the metric is actually bi-invariant.

So lifting the metric and complex structure to the universal covering space $\tilde{M}$ of $M$, we know that $\tilde{M}$ is a connected, simply-connected Lie group of even (real) dimension, equipped with a bi-invariant metric, and a compatible left invariant complex structure. In other words, $\tilde{M}$ is a Samelson space.

Let us denote by $\Gamma$ the deck transformation group. By Milnor’s Lemma, we know that $\tilde{M}$ is isomorphic and isometric to the product $G \times \mathbb{R}^k$, where $G$ is a simply-connected compact semisimple Lie group, equipped with a bi-invariant metric, and $\mathbb{R}^k$ is the vector group, with the flat Euclidean metric. Note that for each simple factor of $G$, the bi-invariant forms are all proportional to the Killing form, so as a Riemannian manifold it is Einstein with positive Ricci curvature. So the $\mathbb{R}^k$ corresponds to the kernel foliation of the Riemannian curvature tensor, the so-called nullity foliation.
Since the elements of $\Gamma$ are isometries, they preserve the nullity foliation and its perpendicular compliment, therefore we know that each $\gamma$ in $\Gamma$ must be in the form $\gamma(x, y) = (\gamma_1(x), \gamma_2(y))$ for any $(x, y) \in G \times \mathbb{R}^k$, with $\gamma_1 \in I(G)$ and $\gamma_2 \in I(\mathbb{R}^k)$ in the isometry group of the factors.

For $i = 1, 2$, let us denote by $\pi_i : \Gamma \rightarrow \Gamma_i$ the projection maps, with $\Gamma_i$ the image group.

Denote by $A$ the kernel of $\pi_2 : \Gamma \rightarrow \Gamma_2$. Since $\Gamma$ has discrete orbit, and $G$ is compact, we know that $A$ must be a finite subgroup of $G$. For any $\gamma \in \Gamma$ and any $a \in A$, we have $\pi_2(\gamma a \gamma^{-1}) = 1$, so the map $\iota_\gamma(a) = \gamma a \gamma^{-1}$ is an automorphism of $A$, and we get a group homomorphism $\iota : \Gamma \rightarrow \text{Aut}(A)$. Since $\text{Aut}(A)$ is finite, we may replace $\Gamma$ by the kernel of $\iota$, a normal subgroup of finite index, which amounts to replacing $M$ by a finite unbranched cover of it, in this way we may assume that $\iota$ is trivial, that is, $A$ in contained in the center of $\Gamma$.

Now since both $M$ and $G$ are compact, it is easy to see that $\Gamma_2$ acts discretely and compactly on $\mathbb{R}^k$. So by Bieberbach Theorem, there exists a normal subgroup $\Gamma'_2 \subseteq \Gamma_2$ of finite index, such that $\Gamma'_2 \cong \mathbb{Z}^k$ is a lattice. If we replace $\Gamma$ by $\pi_2^{-1}(\Gamma'_2)$, which amounts to replacing $M$ by another finite unbranched cover of it, we may assume that $\Gamma_2 \cong \mathbb{Z}^k$ is a lattice in $\mathbb{R}^k$. In particular, $\Gamma_2$ is abelian. So now we have the exact sequence

$$1 \rightarrow A \rightarrow \Gamma \rightarrow \Gamma_2 \rightarrow 1,$$

where $\Gamma_2 \cong \mathbb{Z}^k$ and $A$ is a finite group contained in the center of $\Gamma$.

Note that the commutator group $[\Gamma, \Gamma]$ is contained in $A$ since $\Gamma_2$ is abelian. For any $b, c \in \Gamma$, we have $bcb^{-1} = (ac)^n$ for some elements $a \in A$. From this, we know that for any positive integer $n$, $b^n b^{-1} = (ac)^n = a^n c^n$, so

$$b^n c^n b^{-n} = b^{n-1}(bc b^{-1})b^{-(n-1)} = a^n b^{n-1}e^{n-1} = \cdots = a^n c^n.$$ 

Therefore, $[b^n, c^n] = [b, c]^n$.

Now let $\{\gamma_1, \ldots, \gamma_k\}$ be a subset in $\Gamma$, such that $\{t_1, \ldots, t_k\}$ is a set of generators in $\Gamma_2 \cong \mathbb{Z}^k$, where $t_i = \pi_2(\gamma_i)$. Let $n$ be a positive integer that is a multiple of the order of $A$. Let $\Gamma'_2 \subseteq \Gamma_2$ be generated by $\{nt_1, \ldots, nt_k\}$, and let $\Gamma'' = \pi_2^{-1}(\Gamma'_2)$. Then $\Gamma''$ is generated by the set $A \cup \{\gamma_1^n, \ldots, \gamma_k^n\}$. The commutators of any two elements of this union set is trivial by the above identity. So $\Gamma''$ is abelian, and there is a homomorphism from it onto its torsion part.

In summary, we can replace the original deck transformation group $\Gamma$ by a finite sequence of successive normal subgroup of finite index, so in the end we may assume that the map $\pi_2 : \Gamma \rightarrow \Gamma_2$ is injective, and $\Gamma_2 \cong \mathbb{Z}^k$ is a lattice in $\mathbb{R}^k$. By letting $\rho = \pi_1 \circ \pi_2^{-1}$, we get a homomorphism from $\mathbb{Z}^k$ into $\Gamma_1 \subseteq I(G)$ such that the elements of $\Gamma \cong \mathbb{Z}^k$ take the form

$$\gamma_t(x, y) = (\rho(t)(x), y + t), \quad \forall (x, y) \in G \times \mathbb{R}^k,$$

where $t \in \mathbb{Z}^k$. We will denote this group by $\Gamma_\rho$, and write $M_\rho = (G \times \mathbb{R}^k)/\Gamma_\rho$. $M_\rho$ is a finite unbranched cover of the original $M$ that we started with.

To see that $M_\rho$ is diffeomorphic to $G \times T^k$, where $T^k = \mathbb{R}^k/\mathbb{Z}^k$ is the torus, let us start from the isometry group $I(G)$ of $G$. Since $G$ is compact, $I(G)$ is a compact Lie group. Let $\{v_1, \ldots, v_k\}$ be a set of generators of $\mathbb{Z}^k$. Then any $y \in \mathbb{R}^k$ can be uniquely written as $y = t_1 v_1 + \cdots + t_k v_k$, where $t_1, \ldots, t_k$ are real numbers. For each $\rho(v_i)$ in $I(G)$, let $\psi_i^t, t \in \mathbb{R}$, be a 1-parameter subgroup of $I(G)$, such that $\psi_i^1 = \rho(v_i)$.

Define a diffeomorphism $\Psi$ from $G \times \mathbb{R}^k$ onto itself by letting $\Psi(x, y) = (\psi_1^t \circ \cdots \circ \psi_k^t(x), y)$, where $y = t_1 v_1 + \cdots + t_k v_k$. Then $\Psi(x, y + v_i) = \gamma_i \circ \Psi(x, y)$, where $\gamma_i = (\rho(v_i), v_i) \in \Gamma_\rho$. So $\Psi$ descends down to a diffeomorphism from $G \times T^k$ onto the manifold $M_\rho$. This completes the proof of Theorem 1.

Note that when the image of $\rho$ is finite, then we can use its kernel to be the new deck transformation group, thus reducing to the $\rho = 0$ case. In this case a finite cover of $M$ becomes the compact Lie group $G \times T^k$. When the image of $\rho$ is infinite, since it is abelian, we can ignore the torsion part (again by lifting to a finite cover) and assume that $\mathbb{Z}^k$ is the direct sum of two
free abelian groups, with one summand being the kernel of $\rho$, and with $\rho$ being injective on the other summand.

For a compact Bismut flat manifold $M^n$, since the local unitary $\nabla^b$-parallel frames are unique up to changes by constant unitary matrices, we get the Bismut holonomy map which is a homomorphism $\gamma : \pi_1(M) \to U(n)$. When the image group of $h$ is finite, then a finite unbranched cover $M'$ of $M^n$ has a global unitary $\nabla^b$-parallel frame, thus is a compact Lie group. For this $M'$, the deck transformation group $\pi_1(M')$ is a normal subgroup of the Lie group $\tilde{M} = G \times \mathbb{R}^k$. As in the proof of Theorem 1, by passing to a finite cover of $M'$ if necessary, we may assume that $\Gamma_2 \cong \mathbb{Z}^k$ and the deck transformation group is given by $\Gamma_\rho$ where $\rho : \mathbb{Z}^k \to \Gamma_1$. The normality of $\Gamma_\rho$ in $G \times \mathbb{R}^k$ implies that $\Gamma_1$ is in the center of $G$, thus is finite. So when the Bismut holonomy group $h(\pi_1(M))$ is finite, the map $\rho$ has finite image, which means that $M$ is covered by $G \times T^k$. Conversely, when $M$ is covered by $G \times T^k$, then both $\rho$ and $h$ has finite image of course. To summaries, we have the following

**Lemma 12.** Let $(M^n, g)$ be a compact Bismut flat manifold. Let $\tilde{M} = G \times \mathbb{R}^k$ be its universal cover, where $G$ is compact semisimple. Let $h : \pi_1(M) \to U(n)$ be the Bismut holonomy map, and let $\rho : \mathbb{Z}^k \to I(G)$ be the homomorphism constructed in the proof of Theorem 1, namely, a subgroup of finite index in $\pi_1(M)$ which takes the form $\Gamma_\rho \cong \mathbb{Z}^k$ with elements
\[
\gamma_k(x, y) = (\rho(t)(x), y + t), \quad \forall (x, y) \in G \times \mathbb{R}^k, \quad \forall t \in \mathbb{Z}^k.
\]
Then the following are equivalent:

1. The image of $h$ is finite.
2. The image of $\rho$ is finite.
3. A finite unbranched cover of $M$ is a compact Lie group.
4. A finite unbranched cover of $M$ is $G \times T^k$, where $T^k = \mathbb{R}^k/\mathbb{Z}^k$ is the torus.

To illustrate the role of the deck transformation groups, let us examine the isosceles Hopf surface case. In this case, the universal cover is the space $\mathbb{C}^2 \setminus \{0\} = SU(2) \times \mathbb{R}$, where the identification map is $\phi(z) = (A_z, \log |z|)$. Here $z = (z_1, z_2)$, $|z|^2 = |z_1|^2 + |z_2|^2$, and
\[
A_z = \frac{1}{|z|} \begin{bmatrix} z_1, & -\bar{z}_2 \\ \bar{z}_2, & \bar{z}_1 \end{bmatrix} \in SU(2).
\]
The deck transformation group $\Gamma$ is a finite extension (by unitary rotations) of the infinite cyclic group $\mathbb{Z}f$ where $f(z_1, z_2) = (az_1, bz_2)$, with $0 < |a| = |b| < 1$. On $SU(2) \times \mathbb{R}$, the action of the generator is $\gamma(A_z, y) = (\rho(f)(A_z), \log |a| + y)$ where
\[
\rho(f)(A_z) = \frac{1}{|a| \cdot |z|} \begin{bmatrix} az_1, & -\bar{b}z_2 \\ b\bar{z}_2, & \bar{a}z_1 \end{bmatrix}.
\]
Note that $\rho(f)$ is always in the isometry group $I(G)$ of $G = SU(2)$, but it will be in $G$ (as left multiplications) if and only if $b = \bar{a}$. So in general, the image of $\rho$ is not contained in $G$ itself. Also, the image of $\rho$ (or equivalently the image of the holonomy map $h$) is finite if and only if both $\frac{a}{|a|}$ and $\frac{b}{|b|}$ are roots of unity. So for a generic choice of $|a| = |b|$, the primary isosceles Hopf surface $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}f$ does not have finite Bismut holonomy, and the image of $\rho$ are not all left multiplications of $G$.

Next, let us give an example of a compact Bismut-flat threefold in Corollary 4, whose universal cover is $SU(2) \times \mathbb{R}^3 = (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$, but none of the finite unbranched covers of $M$ can be the product of a Hopf surface and an elliptic curve.

Let us consider the homomorphism $\rho : \mathbb{Z}^3 \to SU(2)$ defined by
\[
\rho(1, 0, 0) = A, \quad \rho(0, 1, 0) = \cos \alpha I + \sin \alpha A, \quad \rho(0, 0, 1) = \cos \beta I + \sin \beta A,
\]
where $\alpha, \beta$ are real numbers and
\[
A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}.
\]
Note that $A^2 = -I$, and $\rho(0,n,m) = \cos(n\alpha + m\beta)I + \sin(n\alpha + m\beta)A$ for any integer $n$ and $m$. Let us take the values of $\alpha$ and $\beta$ so that $n\alpha + m\beta$ is not a rational multiple of $\pi$ for any $n$, $m \in \mathbb{Z}$. This would be the case if we take $\alpha = \sqrt{2}\pi$ and $\beta = \sqrt{3}\pi$ for instance.

Let us now consider the group $\Gamma_\rho \cong \mathbb{Z}^3$ which acts on $SU(2) \times \mathbb{R}^3$ by $\gamma_t(x,y) = (\rho(t)(x),y+t)$ for any $(x,y) \in SU(2) \times \mathbb{R}^3$, where $t \in \mathbb{Z}^3$. Let $M^3_\rho = (SU(2) \times \mathbb{R}^3)/\Gamma_\rho$. Since both the metric and the complex structure on $SU(2) \times \mathbb{R}^3$ are left invariant, the elements of $\Gamma_\rho$ are holomorphic isometries, so $M_\rho$ is a compact Bismut flat threefold, and it is diffeomorphic to $SU(2) \times \mathbb{R}^3$. However, for any subgroup $\Gamma' \subseteq \Gamma_\rho$ with finite index, the (free part of the) abelian group $\rho(\Gamma')$ still has rank 2. Thus any finite unbranched cover of $M_\rho$ cannot be the product of a Hopf surface and an elliptic curve.

5. The non-compact case

In this section, let us discuss non-compact Hermitian manifolds that are Bismut flat. It turns out that the compactness assumption in Theorem 1 can be dropped, thanks to a nice property about flat metric connections with skew-symmetric torsion on a Riemannian manifold, given by Agricola and Friedrich ([1], Prop. 2.1). The result states that on a Riemannian manifold, if a metric connection $\nabla''$ with skew-symmetric torsion is flat, then the torsion of $\nabla''$ is parallel with respect to another metric connection $\nabla = \frac{2}{3}\nabla + \frac{1}{3}\nabla''$. In our notation, apply this result to the flat connection $\nabla^b$, we get the following:

**Lemma 13. (Agricola-Friedrich)** Let $(M^n,g)$ be a Hermitian manifold with flat Bismut connection $\nabla^b$. Then the torsion tensor $T^b$ of $\nabla^b$ is parallel with respect to the metric connection $\nabla = \frac{2}{3}\nabla + \frac{1}{3}\nabla''$, where $\nabla$ is the Riemannian (Levi-Civita) connection.

By using formulae (32)-(35) in Lemma 9, one can also check directly that $\nabla' T^b = 0$. Now we are ready to prove Theorem 5.

**Proof of Theorem 5.** Let $(M^n,g)$ be a simply-connected Bismut flat manifold. Let $e$ be a local unitary $\nabla^b$-parallel frame. Under such an $e$, we have

$$T^b(e_i,e_j) = -2 \sum_k T^b_{ij} e_k$$

$$T^b(\overline{e_i}, e_j) = 2 \sum_k (\overline{T^b_{ik} e_k} - \overline{T^b_{jk} e_k})$$

So the square norm $|T^b|^2 = 24 \sum_{i,j,k} |T^b_{ij}|^2 = 24 |T|^2$. Since $T^b$ is $\nabla'$-parallel by Lemma 13, we know that the square norm $|T^b|^2$ is a constant on $M$. Thus the left hand side of formula (38) in Lemma 11 is identically zero, which implies that $T^b_{ij} = 0$ for any indices. So under any local unitary $\nabla^b$-parallel frame $e$, the components $T^b_{ij}$ of the torsion of the Chern connection are all constants. By the proof of Theorem 1, we know that $M$ is an open subset of a Samelson space, and Theorem 5 is proved.

Theorem 5 suggests that the flatness of the Bismut connection is perhaps more restrictive than the flatness of some other metric connections on a Hermitian manifold. For instance, in [4], Boothby pointed out that in the non-compact case, a Chern flat metric doesn’t have to have parallel Chern torsion, even in complex dimension 2. Below let us give another example in complex dimension 2.

Let $\Omega \subseteq \mathbb{C}^2$ be a domain, and $f$, $h$ be holomorphic functions on $\Omega$, and $(z_1, z_2)$ be the standard coordinates of $\mathbb{C}^2$. Consider the Hermitian metric $g$ on $\Omega$ given by

$$\omega_g = \sqrt{-1}(e^{f+\overline{f}} dz_1 \wedge d\overline{z}_1 + e^{h+\overline{h}} dz_2 \wedge d\overline{z}_2).$$
The Chern connection of \((\Omega, g)\) is flat, since it has a holomorphic unitary frame \(e_1 = e^{-f} \frac{\partial}{\partial z_1}\) and \(e_2 = e^{-h} \frac{\partial}{\partial z_2}\). It is Kähler if and only if both \(\frac{\partial^2 f}{\partial z_1^2}\) and \(\frac{\partial^2 h}{\partial z_1^2}\) are identically zero, and the components of the torsion of the Chern connection under the frame \(e\) are

\[
T_{12}^1 = -\frac{1}{2} \frac{\partial f}{\partial z_2} e^{-h}, \quad T_{12}^2 = \frac{1}{2} \frac{\partial h}{\partial z_1} e^{-f}.
\]

For generic choices of \(f\) and \(h\), clearly the Chern torsion does not have constant norm, thus can not be parallel under any metric connection.

There are also complete examples of this kind. For instance, consider the Chern flat Hermitian metric \(g\) on \(\mathbb{C}^2\) given by

\[
\omega_g = \sqrt{-1}(\varphi_1 \wedge \bar{\varphi_1} + \varphi_2 \wedge \bar{\varphi_2}),
\]

where \(\varphi_1 = dx, \varphi_2 = dy - 2xydx\), and \((x, y)\) is the standard coordinate of \(\mathbb{C}^2\). The torsion components under the unitary coframe \(\varphi\) are \(T_{12}^1 = 0, T_{12}^2 = x\). So the norm of the Chern torsion \(|T|^2 = 16|x|^2\) is not a constant.

To see that \(g\) is complete, let \(\sigma : [0, \infty) \to \mathbb{C}^2\) be a smooth curve that goes to infinity. Write \(\sigma(t) = (x(t), y(t))\). Its length under \(g\) is

\[
L_g(\sigma) = \int_0^\infty \sqrt{|x'|^2 + |y' - 2xy'|^2} \, dt.
\]

Assume that \(L_g(\sigma) < \infty\). Then \(\int_0^\infty |x'| \, dt < \infty\), so \(|x(t)| < C\) for some constant \(C\). Let \(z(t) = e^{-x^2(t)}y(t)\), then

\[
|z'| = |e^{-x^2}(y' - 2xy')| \leq e^{C^2} |y' - 2xy'|
\]

whose integral over \([0, \infty)\) is finite. So \(z(t)\) stays bounded, which implies that \(y(t)\) also stays bounded, as \(|y| \leq e^{C^2}|z|\). But this is impossible as \((x(t), y(t))\) needs to go to infinity when \(t \to \infty\). So \(L_g(\sigma)\) must be \(\infty\), and this shows the completeness of the metric \(g\).

For Hermitian surfaces \((M^2, g)\) with flat Riemannian connection, there are also lots of non-compact examples, but there are no complete ones. In fact, if we assume that \((M^2, g)\) is a complete Hermitian manifold with flat Riemannian connection, then its universal cover \(\tilde{M}\) as a Riemannian manifold is just the flat Euclidean space \(\mathbb{R}^4\). In \([31]\) (Theorem 1.3), Salamon and Viaclovsky showed that any orthogonal complex structure on \(\mathbb{R}^4\) (or \(\mathbb{R}^4\) deleting a subset with zero 1-dimensional Hausdorff measure) must be the standard one, namely, \(\tilde{M}\) is holomorphically isometric to the flat \(\mathbb{C}^2\). In contrast, Borisov, Salamon, and Viaclovsky in \([5]\) were able to construct infinitely many nonstandard orthogonal complex structures on the Euclidean space \(\mathbb{R}^6\).

It is well-known that Hermitian surfaces \((M^2, g)\) with flat Riemannian connection correspond to holomorphic maps from \(M^2\) into the space \(Z\) of all almost complex structures on \(\mathbb{R}^4\) compatible with the Euclidean metric and (a fixed) orientation. However, it is not necessarily easy to write down such metrics explicitly in terms of the complex Euclidean coordinate \((z_1, z_2)\). Here we observe that such surfaces are locally determined by three holomorphic functions, and using this characterization, we can write down lots of explicit examples of such metrics.

Let \(U\) be a complex manifold of complex dimension 2, and \(u, v, f\) are holomorphic functions in \(U\). Let us denote by

\[
\begin{align*}
\varphi_1 &= \frac{1}{\sqrt{2\sqrt{\lambda}}} du + \frac{1}{\sqrt{2\sqrt{\lambda}}} (\overline{v} - \overline{u} \overline{f}) \, df \\
\varphi_2 &= \frac{1}{\sqrt{2\sqrt{\lambda}}} dv - \frac{1}{\sqrt{2\sqrt{\lambda}}} (\overline{u} + v \overline{f}) \, df
\end{align*}
\]

where \(\lambda = 1 + |f|^2\). We have the following:

**Lemma 14.** Let \((M^2, g)\) be a Hermitian surface of flat Riemannian connection. Then for any \(p \in M\), there exists a neighborhood \(p \in U \subseteq M\) and three holomorphic functions \(u, v, f\) in
$U$, such that $\{\varphi_1, \varphi_2\}$ given in $(40), (41)$ forms a unitary coframe in $U$. Conversely, given any three holomorphic functions $u, v, f$ in a complex surface $U$ such that $\varphi_1 \wedge \varphi_2$ is nowhere zero, the Hermitian metric $g$ using $\varphi$ as unitary coframe has flat Riemannian connection.

Proof. Let $(x_1, \ldots, x_4)$ be the standard coordinate of $\mathbb{R}^4$ and write $\epsilon = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}\}$. Under the natural frame $\epsilon$, the elements $J$ in $Z$ are represented by matrices

$$J_z = \begin{bmatrix} aE & bE - cI \\ bE + cI & -aE \end{bmatrix}, \quad \text{where } E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$z = x + iy \in \mathbb{C} \cup \{\infty\}$, and $(a, b, c) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{2z^2 - 1}{|z|^2 + 1}\right)$, which identifies $Z \cong S^2$ with $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Namely, we have $J(\epsilon) = \epsilon J \epsilon$.

In order to get an explicit expression of a local unitary frame, we look for a local orthonormal frame $\tilde{\epsilon} = P\epsilon$ such that $J(\tilde{\epsilon}) = J_0 \tilde{\epsilon}$, or equivalently, $P^{-1} J_0 P = J_z$. While $P$ is highly non-unique, the following symmetric matrix

$$P = \frac{1}{\sqrt{|z|^2 + 1}} \begin{bmatrix} xI & yI - E \\ yI + E & -xI \end{bmatrix}$$

is clearly orthogonal and satisfies the condition $P^{-1} J_0 P = J_z$.

Write

$\epsilon' = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$, $\epsilon'' = \begin{bmatrix} \epsilon_3 \\ \epsilon_4 \end{bmatrix}$, $\epsilon' = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$, $\epsilon'' = \begin{bmatrix} \epsilon_3 \\ \epsilon_4 \end{bmatrix}$,

then we have

$$\epsilon' = \frac{1}{\sqrt{|z|^2 + 1}}(x \epsilon' + (yI - E) \epsilon''), \quad \epsilon'' = \frac{1}{\sqrt{|z|^2 + 1}}((yI + E) \epsilon' - \epsilon'').$$

\[ \text{From this, we can form a local unitary frame } e = \{\epsilon_1, \epsilon_2\} \text{ by } \]

$$e = \frac{1}{\sqrt{2}}(\epsilon' - i \epsilon'') = \frac{1}{\sqrt{2 |z|^2 + 1}} \left((i |z|^2 - iE) \epsilon' + (i |z|^2 - E) \epsilon''\right)$$

Let $\{\varphi_1, \varphi_2\}$ be the local unitary coframe on $M$ dual to $e$, then we have

$$(43) \quad \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \frac{1}{\sqrt{2} \sqrt{|z|^2 + 1}} \left((zI + iE) \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - (izI + E) \begin{bmatrix} dx_3 \\ dx_4 \end{bmatrix}\right).$$

Now if $(M^2, g)$ is a Hermitian surface with flat Riemannian connection. Fix any $p \in M$, we can choose a small neighborhood $U$ of $p$ and a local coordinate $(x_1, \ldots, x_4)$ centered at $p$, such that the natural frame $\epsilon = \frac{\partial}{\partial x}$ is orthonormal and parallel under the Riemannian connection.

The complex structure on $M$ gives a smooth map $f$ from $U$ into $Z \cong \mathbb{P}^1$, such that the almost complex structure of $M$ at $q \in U$ corresponds to $J_{f(q)} \in Z$. As is well-known, the integrability of $J$ is equivalent to the holomorphicity of $f$.

From the formula $(43)$ above, we get a local unitary coframe $\varphi$ in $U$:

$$(44) \quad \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \frac{1}{\sqrt{2} \sqrt{|f|^2 + 1}} \left((fI + iE) \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - (ifI + E) \begin{bmatrix} dx_3 \\ dx_4 \end{bmatrix}\right).$$

Write $\lambda = 1 + |f|^2$, $t_1 = x_1 + ix_3$, $t_2 = x_2 + ix_4$, then the above formula can be rewritten as

$$(45) \quad \varphi_1 = \frac{1}{\sqrt{2} \sqrt{\lambda}}(fd\bar{t}_1 + idt_2)$$

$$(46) \quad \varphi_2 = \frac{1}{\sqrt{2} \sqrt{\lambda}}(fd\bar{t}_2 - idt_1)$$

Since the $(0,1)$-components of $\varphi_1, \varphi_2$ are zero, we know that $u = \bar{f}\bar{t}_1 + it_2$ and $v = f\bar{t}_2 - it_1$ are both holomorphic functions. Expressing $\bar{t}_1, t_2$ in terms of $u$ and $\bar{u}$, we get

$$(47) \quad t_1 = \frac{1}{\lambda}(f\bar{u} + iv), \quad t_2 = \frac{1}{\lambda}(f\bar{v} - iu).$$
Plugging them into (45) and (46), we get the expressions (40) and (41). This proved the first part of the lemma.

Conversely, if we start with three holomorphic functions \( u, v, f \) in \( U \) with \( \varphi_1 \wedge \varphi_2 \) nowhere zero, then the Hermitian metric \( g \) with metric form

\[
\omega = i(\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2)
\]

will have flat Riemannian connection. This is because if we define \( t_1 \) and \( t_2 \) by (47), and let \( x_1 \) and \( x_3 \) (respectively \( x_2 \) and \( x_4 \)) be the real and imaginary parts of \( t_1 \) or \( t_2 \), then the formula (40) and (41) becomes (45) and (46), and then (44). From this, it is easy to compute that the matrices of the Riemannian connection are \( \theta_1 = \alpha I \) and \( \theta_2 = \beta E \), where

\[
\alpha = \frac{1}{2\lambda}(f\bar{f} - \bar{f}f), \quad \beta = -\frac{if}{\lambda}.
\]

Clearly, \( \alpha = -\alpha \), and \( d\beta = \beta \bar{\beta}, \ d\beta = 2\beta \alpha \). This means \( \Theta_1 = \Theta_2 = 0 \), so the Riemannian connection of \( \omega \) is everywhere flat. This completes the proof of the lemma. \( \square \)

Note that \( \beta \bar{\beta} = \partial \bar{\partial} \log(1 + |f|^2) \) is globally defined, so \( \log(1 + |f|^2) \) is defined up to an additive pluriharmonic function, but \( f \) itself is not globally defined.

Using Lemma 14, we can easily produce lots of explicit examples of Hermitian metrics with flat Riemannian connections. For instance, if we take \( u = z_1, v = 0, f = z_2 \), we get a Hermitian metric \( g_1 \) on \( \mathbb{C}^* \times \mathbb{C} \):

\[
\omega_{g_1} = \frac{\sqrt{-1}}{(1 + |z_2|^2)^2} \{(1 + |z_2|^2)dz_1 \wedge d\bar{z}_1 + |z_1|^2dz_2 \wedge d\bar{z}_2 - z_1\bar{z}_2dz_1 \wedge d\bar{z}_2 - z_2\bar{z}_1dz_2 \wedge d\bar{z}_1\}
\]

If we let \( u = z_1, v = z_2 \) and \( f = \sqrt{-1}z_1z_2 \), then we get a Hermitian metric \( g_2 \) on \( \mathbb{C} \times \Omega \), where \( \Omega \subseteq \mathbb{C} \) is any domain not intersecting the unit circle \( |z_2| = 1 \), by

\[
\omega_{g_2} = \frac{\sqrt{-1}}{(1 + |z_1z_2|^2)^2} \{(1 - |z_2|^2)^2dz_1 \wedge d\bar{z}_1 + (1 + |z_1|^2)^2dz_2 \wedge d\bar{z}_2\}.
\]

We will leave it to the readers to verify that the square norm \( |T^{\ast}|^2 \) of the Chern torsion tensor for \( g_1 \) or \( g_2 \) is not a constant. Note that for a given explicit metric such as \( g_2 \), it is a rather tedious task to compute its Riemannian curvature, without knowing a convenient unitary coframe a priori.

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