A Lower Bound for Nonadaptive, One-Sided Error Testing of Unateness of Boolean Functions over the Hypercube

Roksana Baleshzar\textsuperscript{*}  Deeparnab Chakrabarty\textsuperscript{†}  Ramesh Krishnan S. Pallavoor\textsuperscript{*}  Sofya Raskhodnikova\textsuperscript{*}  C. Seshadhri\textsuperscript{‡}

Abstract

A Boolean function $f : \{0, 1\}^d \mapsto \{0, 1\}$ is unate if, along each coordinate, the function is either nondecreasing or nonincreasing. In this note, we prove that any nonadaptive, one-sided error unateness tester must make $\Omega(d \log d)$ queries. This result improves upon the $\Omega(d \log 2^d)$ lower bound for the same class of testers due to Chen et al. (STOC, 2017).

1 Introduction

We study the problem of deciding whether a Boolean function $f : \{0, 1\}^d \mapsto \{0, 1\}$ is unate in the property testing model [7, 5]. A function is unate if, for each dimension $i \in [d]$, the function is either nondecreasing along the $i^{th}$ coordinate or nonincreasing along the $i^{th}$ coordinate. A property tester for unateness is a randomized algorithm that takes as input a proximity parameter $\varepsilon \in (0, 1)$ and has query access to a function $f$. If $f$ is unate, it must accept with probability at least $2/3$. If $f$ is $\varepsilon$-far from unate, it must reject with probability at least $2/3$. A tester has one-sided error if it always accepts unate functions. A tester is nonadaptive if it chooses all of its queries in advance; it is adaptive otherwise.

The problem of testing unateness was introduced by Goldreich et al. [4]. Following a result of Khot and Shinkar [6], Baleshzar et al. [1] settled the complexity of unateness testing for real-valued functions. Unateness can be tested with $O(\frac{d}{\varepsilon})$ queries adaptively and with $O(\frac{d \log d}{\varepsilon})$ queries nonadaptively. For constant $\varepsilon$, these complexities are optimal.

On the other hand, for the Boolean range, the complexity is far from settled. Baleshzar et al. [2] proved that $\Omega(\sqrt{d})$ queries are necessary for nonadaptive, one-sided error testers. Chen et al. [3] improved the lower bound for this class of testers to $\Omega(\frac{d}{\log d})$. They also proved a lower bound of $\Omega(\frac{\sqrt{d}}{\log d})$ for adaptive, two-sided error unateness testers.

In this note, we use a construction similar to the one used by Chen et al. [3] to get an $\Omega(\frac{d}{\log d})$ for nonadaptive, one-sided error unateness testers of Boolean functions over the hypercube. Our analysis of the lower bound construction is simpler and gives a better dependence on $d$. There is

\textsuperscript{*}Department of Computer Science and Engineering, Pennsylvania State University. rxb5410@cse.psu.edu, ramesh@psu.edu, sofya@cse.psu.edu. Partially supported by NSF award CCF-1422975.

\textsuperscript{†}Department of Computer Science, Dartmouth College. deeparnab@dartmouth.edu. Work done while at Microsoft Research, India.

\textsuperscript{‡}Department of Computer Science, University of California, Santa Cruz. sesh@ucsc.edu.
still a gap of $\log^2 d$ between the query complexity of the best known algorithm for this problem (from [1]) and our lower bound.

## 2 The Lower Bound

In this section, we prove the following theorem.

**Theorem 2.1.** Any nonadaptive, one-sided error unateness tester for functions $f : \{0, 1\}^d \mapsto \{0, 1\}$ with the distance parameter $\varepsilon \leq \frac{1}{3}$ must make $\Omega\left(\frac{d}{\log d}\right)$ queries.

**Proof.** We first define a hard distribution consisting of Boolean functions that are $\frac{1}{8}$-far from unate. By Yao’s minimax principle [8], it is sufficient to give a distribution on functions for which every deterministic tester fails with high probability. A deterministic nonadaptive tester is determined by a set of query points $Q \subseteq \{0, 1\}^d$. We prove that if $|Q| \leq \frac{d}{30 \log d}$, then the tester fails with probability more than $2/3$ over the hard distribution.

The hard distribution $D$ is defined as follows: pick 3 dimensions $a, b, c \in [d]$ uniformly at random and define $f_{a,b,c}(x) = x_a \cdot x_b + (1 - x_a) \cdot x_c$. We call $a, b, c$ the influential dimensions, since the value of the function depends only on them. The coordinate $x_a$ determines if $f_{a,b,c}(x)$ should be set to $x_b$ or $x_c$. If $x_a = 1$, then $f_{a,b,c}(x) = x_b$, otherwise, $f_{a,b,c}(x) = x_c$.

There are $\binom{d}{3}$ functions in the support of $D$. The next claim states that all of them are far from unate.

**Claim 2.2.** Every function $f_{a,b,c}$ in the support of $D$ is $\frac{1}{8}$-far from unate.

**Proof.** Consider an edge $(x, y)$ along the dimension $a$. We have $x_a = 0$ and $y_a = 1$, and $x_i = y_i$ for all $i \in [d] \setminus \{a\}$. By definition, $f_{a,b,c}(x) = x_c$ and $f_{a,b,c}(y) = y_b$. If $x_b = y_b = 1$ and $x_c = y_c = 0$, then $f_{a,b,c}$ is increasing along the edge $(x, y)$. On the other hand, if $x_b = y_b = 0$ and $x_c = y_c = 1$, then $f_{a,b,c}$ is decreasing along $(x, y)$. Thus, with respect to $f_{a,b,c}$, at least $2^{d-3}$ edges along the dimension $a$ are decreasing and at least $2^{d-3}$ edges along the dimension $a$ are increasing. Hence, at least $2^{d-3}$ function values of $f_{a,b,c}$ need to be changed to make it unate. Consequently, $f_{a,b,c}$ is $\frac{1}{8}$-far from unate.

Note that any one-sided error tester for unateness must accept if the query answers are consistent with a unate function. Let $f_{|Q}$ denote the restriction of the function $f$ to the points in $Q$. We say that $f_{|Q}$ is extendable to a unate function if there exists a unate function $g$ such that $g_{|Q} = f_{|Q}$. For $f \sim D$, we show that if $|Q| \leq \frac{d}{30 \log d}$, then, with high probability, $f_{|Q}$ is extendable to a unate function. Consequently, the tester accepts with high probability.

Next, we define a conjunctive normal form (CNF) formula $\phi(f_{|Q})$. Intuitively, each pair $(x, y)$ of domain points on which $f$ differs imposes a constraint on $f$ (assuming that $f$ is unate). Specifically, at least one of the dimensions on which $x$ and $y$ differ must be consistent (i.e., nondecreasing or nonincreasing) with the change of the function value between $x$ and $y$. This constraint is formalized in the definition of $\phi(f_{|Q})$ as follows. For each dimension $i$, we have a variable $z_i$ which is true if $f$ is nondecreasing along the dimension $i$, and false if it is nonincreasing along that dimension. For each $x, y \in Q$ such that $f(x) = 1$ and $f(y) = 0$, create a clause (think of $x, y$ as sets where $i \in x$ iff $x_i = 1$)

$$c_{x,y} = \bigvee_{i \in x \setminus y} z_i \lor \bigvee_{i \in y \setminus x} \overline{z_i}.$$
Set $\phi(f|_Q) = \bigwedge_{x, y \in Q : f(x) = 1, f(y) = 0} c_{x, y}$.

**Observation 2.3.** The restriction $f|_Q$ is a certificate for non-unateness iff $\phi(f|_Q)$ is unsatisfiable.

Now we need to show that, with probability greater than 2/3 over $f \sim D$, the CNF formula $\phi(f|_Q)$ is satisfiable. This follows from Claims 2.4 and 2.5.

The width of a clause is the number of literals in it; the width of a CNF formula is the minimum width of a clause in it.

**Claim 2.4.** With probability at least $2/3$ over $f \sim D$, the width of $\phi(f|_Q)$ is at least $3 \log d$.

**Proof.** Consider a graph $G$ with vertex set $Q$, and an edge between $x, y \in Q$ if $|x \Delta y| \leq 3 \log d$ (Here, $x \Delta y$ is the symmetric difference between the sets $x$ and $y$). Take an arbitrary spanning forest $F$ of $G$. Observe that for any edge $(u, v)$ of $G$, we have $u \Delta v \subseteq \bigcup_{(x, y) \in F} x \Delta y$. Note that $F$ has at most $\frac{d}{30 \log d}$ edges. Let $C = \bigcup_{(x, y) \in F} x \Delta y$, the set of dimensions captured by $Q$. We have $|C| \leq \sum_{(x, y) \in F} |x \Delta y| \leq \frac{d}{30 \log d} \cdot 3 \log d \leq \frac{d}{10}$. Over the distribution $D$, the probability that at least one of the influential dimensions, $\{a, b, c\}$, is in $C$ is at most $3/10$ which is less than $1/3$. Hence, with probability at least 2/3, no $(u, v) \in G$ contributes a clause to $\phi(f|_Q)$. Therefore, the width of $\phi(f|_Q)$ is at least $3 \log d$. \qed

**Claim 2.5.** Any CNF that has width at least $3 \log d$ and at most $d^2$ clauses is satisfiable.

**Proof.** Apply the probabilistic method. A clause is not satisfied by a random assignment with probability at most $1/d^3$. Hence, the expected number of unsatisfied clauses is at most $\frac{d^2}{d^3} < 1$. \qed

Thus, $f|_Q$ is a certificate for non-unateness with probability at most 1/3 when $|Q| \leq \frac{d}{30 \log d}$, which completes the proof of Theorem 2.1.

**References**

[1] Roksana Baleshzar, Deeparnab Chakrabarty, Ramesh Krishnan S. Pallavoor, Sofya Raskhodnikova, and C. Seshadhri. Optimal unateness testers for real-valued functions: Adaptivity helps. In *Proceedings, International Colloquium on Automata, Languages and Processing (ICALP)*, 2017.

[2] Roksana Baleshzar, Meiram Murzabulatov, Ramesh Krishnan S. Pallavoor, and Sofya Raskhodnikova. Testing unateness of real-valued functions. *CoRR*, abs/1608.07652, 2016.

[3] Xi Chen, Erik Waingarten, and Jinyu Xie. Beyond talagrand functions: New lower bounds for testing monotonicity and unateness. *CoRR*, abs/1702.06997, 2017. To appear in STOC 2017.

[4] Oded Goldreich, Shafi Goldwasser, Eric Lehman, Dana Ron, and Alex Samorodnitsky. Testing monotonicity. *Combinatorica*, 20:301–337, 2000.

[5] Oded Goldreich, Shafi Goldwasser, and Dana Ron. Property testing and its connection to learning and approximation. *J. ACM*, 45(4):653–750, 1998.

[6] Subhash Khot and Igor Shinkar. An $\tilde{O}(n)$ queries adaptive tester for unateness. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2016*, pages 37:1–37:7, 2016.
[7] Ronitt Rubinfeld and Madhu Sudan. Robust characterizations of polynomials with applications to program testing. *SIAM J. Comput.*, 25(2):252–271, 1996.

[8] Andrew Chi-Chih Yao. Probabilistic computations: Toward a unified measure of complexity (extended abstract). In *Proceedings, IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 222–227, 1977.