Article
Dark Type Dynamical Systems: The Integrability Algorithm and Applications †

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† In memoriam of our friend and colleague Denis L. Blackmore (24 April 2022), who so loved to shed light on virtually dark mathematical problems.

Abstract: Based on a devised gradient-holonomic integrability testing algorithm, we analyze a class of dark type nonlinear dynamical systems on spatially one-dimensional functional manifolds possessing hidden symmetry properties and allowing their linearization on the associated cotangent spaces. We described main spectral properties of nonlinear Lax type integrable dynamical systems on periodic functional manifolds particular within the classical Floquet theory, as well as we presented the determining functional relationships between the conserved quantities and related geometric Poisson and recursion structures on functional manifolds. For evolution flows on functional manifolds, parametrically depending on additional functional variables, naturally related with the classical Bellman-Pontriagin optimal control problem theory, we studied a wide class of nonlinear dynamical systems of dark type on spatially one-dimensional functional manifolds, which are both of diffusion and dispersion classes and can have interesting applications in modern physics, optics, mechanics, hydrodynamics and biology sciences. We prove that all of these dynamical systems possess rich hidden symmetry properties, are Lax type linearizable and possess finite or infinite hierarchies of suitably ordered conserved quantities.

Keywords: dark type dynamical systems; evolution flows; conservation laws; Lax-Noether condition; asymptotic solutions; linearization; complete integrability

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1. Introduction

More than twenty years ago, a new class of nonlinear dynamical systems, called “dark equations” was introduced by Boris Kupershmidt [1,2] and shown to possess unusual hidden symmetry properties that were then not well understood at that time. Later, in related developments, some Burgers dark type [3] and also Korteweg-de Vries dark type [4,5] dynamical systems were studied in detail by Denis L. Blackmore [6–8] with collaborators, and it was stated that they have often a finite number of conservation laws, or an infinite hierarchy, yet strongly degenerate on functional submanifolds, are both linearizable and possessing the Lax type representations, among other properties. In what follows, we provide both a detail enough introductory setup of mathematical backgrounds of analytical description of the Lax type [9,10] integrable nonlinear dynamical systems and their application to testing hidden symmetry and integrability properties of a new class of the so called dark type nonlinear dynamical systems by means of a geometrically motivated [11–13] gradient-holonomic approach [14–16]. To do the presentation accessible for more wide audience, we described main spectral properties of nonlinear Lax type integrable dynamical systems on periodic spatially one-dimensional functional manifolds [14,16,17] within the classical Floquet theory, as well as we presented the determining
Algorithms 2022, 15, 266

functionality relationships between the conservation laws and related geometric Poisson and recursion structures [18,19] on functional manifolds. For evolution flows on functional manifolds, parametrically depending on additional functional variables and their natural relationship with the classical Bellman-Pontriagin optimal control problem theory, there is analyzed a wide class of dark type dynamical systems on spatially one-dimensional functional manifolds, which are both of diffusion and dispersion classes and can have interesting applications in modern physics, nonlinear optics, mechanics, hydrodynamics and biology sciences. These related and alleged applications concern in part practically important control problems of diffusion and convection processes in nuclear plant reactors, chemical reactions modeling, as well as plasma physics, where there is solved a problem of confining dispersive wave packets inside geometrically bounded constructions. Amongst the interesting evolution processes it is worth to mention the generalized Maccari [20,21] and Broer-Whitham nonlinear dynamical systems [22–24], whose dark type extensions can be of great importance for modeling phenomena in diverse nonlinear both optics and plasma media.

2. Nonlinear Dynamical Systems and the Lax Type Representation

2.1. Generalized Eigenvalue Problem

There is considered a uniform nonlinear dynamical system

\[ u_t = K[u] \]

(1)
on a 2\pi-periodic smooth functional manifold \( M \subset C'(\mathbb{R}/\{2\pi \mathbb{Z}\};\mathbb{R}^m) \), generated by a smooth vector field \( K: M \rightarrow T(M) \), where the nonlinear mapping \( K: \mathcal{F}(\mathbb{R}/\{2\pi \mathbb{Z}\};\mathbb{R}^m) \rightarrow \mathbb{R}^m \) for some finite \( r \in \mathbb{Z}_+ \) is a Frechet differentiable nonlinear mapping. In addition, we take into consideration a vector fibre bundle \( E(M; C(\mathbb{R}/\{2\pi \mathbb{Z}\};\mathbb{C})) \) over the manifold \( M \) with the space \( C(\mathbb{R}/\{2\pi \mathbb{Z}\};\mathbb{C}) \) as its bundle jointly with a linear smooth differential operator mapping \( L := L[u; \lambda]: L_\infty(\mathbb{R}; \mathbb{C}) \rightarrow L_\infty(\mathbb{R}; \mathbb{C}) \) of a finite order \( k \in \mathbb{N} \) with domain \( \text{Dom}(L) = C^k(\mathbb{R}; \mathbb{C}) \) and meromorphically depending on a complex spectral parameter number \( \lambda \in \mathbb{C} \). For this linear differential operator expression

\[ L[u; \lambda] := y^{(k)} + \sum_{s=1}^k l_k[u; \lambda]y^{(k-s)}, \]

(2)
one can define a generalized eigenvalue problem as follows: a number \( \lambda \in \mathbb{C} \) belongs to the spectrum \( \sigma(L) \) of this operator, if the solution \( y \in C^k(\mathbb{R}; \mathbb{C}) \) of the equation

\[ L[u; \lambda]y(x; \lambda) = 0 \]

(3)
is bounded, that is

\[ ||y||_\infty := \sup_{x \in \mathbb{R}} \sum_{s=0}^k |\lambda|^{(s)}(x; \lambda) < \infty, \]

where \( ||\cdot||_\infty \) is the norm in the Banach space \( L_\infty(\mathbb{R}; \mathbb{C}) \).

Let us suppose that the defined above spectrum \( \sigma(L) \subset \mathbb{C} \) is under the isospectral deformation by virtue of the dynamical system (1), i.e.,

\[ d\sigma(L)/dt = 0, \]

for all \( t \in \mathbb{R} \). As, in general, the spectrum \( \sigma(L) \) is a smooth 2\pi-periodic functional on \( M \), in the case under consideration, it will be invariant on \( M \). The sufficient condition for the considered above situation for the dynamical system (1) to occur consists in existence of depending on the admissible parameter \( \lambda \in \mathbb{C} \) another linear differential operator
Algorithm 2022, 15, 266

\[ A := A[u; \lambda] : L_\infty(\mathbb{R}; \mathbb{C}) \to L_\infty(\mathbb{R}; \mathbb{C}), \]

being a local functional on \( M \), such that for all \( u \in M \) and \( \lambda \in \mathbb{C} \) the following linear evolution equation

\[ \partial L / \partial t = [A, L] \]  

holds, where \([\cdot, \cdot]\) is the usual operator commutator. Then one easily ensures that the dynamical system (1) is the compatibility condition for the operator relationships (3) and (4) for all admissible \( \lambda \in \mathbb{C} \). The above representation of the dynamical system (1) in the form of (4) is called the Lax type representation.

2.2. Properties of the Periodic Spectral Problem: Floquet Theory Aspects

Let us study in detail the spectral properties of the problem (3) for the uniform and periodic dynamical system (1) within the classical [25,26] Floquet theory. To do this, we will consider the differential operator \( L[u; \lambda] : L_\infty(\mathbb{R}; \mathbb{C}) \to L_\infty(\mathbb{R}; \mathbb{C}) \), preliminarily equivalently represented in the following canonical differential-matrix form of the first order:

\[ \mathcal{L}[u; \lambda] = \partial / \partial x - l[u; \lambda], \]

where \( l := l[u; \lambda] \in \text{End}(L_\infty(\mathbb{R}; \mathbb{C}^k)) \) is the corresponding \( 2\pi \)-periodic matrix functional on the manifold \( M \). The Equation (3) can be then rewritten in the form of the linear matrix differential equation \( \mathcal{L}[u; \lambda]f = 0 \) in the space \( L_\infty(\mathbb{R}; \mathbb{C}^k) \), or equivalently as

\[ \partial f / \partial x = l[u; \lambda]f \]  

for \( f \in L_\infty(\mathbb{R}; \mathbb{C}^k) \). Let \( F(x, x_0; \lambda) \) be the fundamental solution of the Equation (6) normalized by the unit matrix at the point \( x = x_0 \in \mathbb{R} \), that is \( F(x_0, x_0; \lambda) = I \in \text{End} \mathbb{C}^k \) for all \( x_0 \in \mathbb{R}, \lambda \in \mathbb{C} \).

Evidently, any solution of Equation (6) can be represented in the form

\[ f = F(x, x_0; \lambda)f_0, \]

where \( f_0 \in \mathbb{C}^k \) is an initial value at the point \( x_0 \in \mathbb{R} \).

Consider the value \( f(x, x_0; \lambda) \) at \( x = x_0 \pm 2\pi N, x_0 \in \mathbb{R} \), where \( 2\pi \) is the period of the element \( u \in M \) and \( N \in \mathbb{Z}_+ \). By virtue of periodicity in the variable \( x \in \mathbb{R} \) of the matrix \( l[u; \lambda] \in \text{End} L_\infty(\mathbb{R}; \mathbb{C}^k) \), we have

\[ f(x_0 \pm 2\pi N, x_0; \lambda) = S^N(x_0; \lambda)f_0, \]

where \( S := S(x_0; \lambda) = F(x_0 + 2\pi, x_0; \lambda) \in \text{End} \mathbb{C}^k, x_0 \in \mathbb{R} \), is the so-called \textit{monodromy matrix} of the differential operator (3).

Let \( \xi(\lambda) \in \sigma(S) \subset \mathbb{C} \) be an eigenvalue of the matrix \( S(x_0; \lambda) \in \text{End} \mathbb{C}^k, x_0 \in \mathbb{R} \). Then from (8) it directly follows [14,15,27] that the solution \( f(x, x_0; \lambda) \in \mathbb{C}(\mathbb{R}; \mathbb{C}^{k}) \) is bounded on the whole axis \( \mathbb{R} \) if and only if any eigenvalue \( \xi(\lambda) \in \sigma(S) \) of the matrix \( S(x_0; \lambda) \) has the absolute value equal to one, that is \( |\xi(\lambda)| = 1 \).

**Lemma 1.** The spectrum \( \sigma(S) \subset \mathbb{C} \) of the periodic problem (6) does not depend on a point \( x_0 \in \mathbb{R} \).

**Proof.** To show that the eigenvalues \( \xi(\lambda) \in \sigma(S) \) don’t depend on a point \( x_0 \in \mathbb{R} \), we consider the corresponding differential equation for matrix \( S(x_0; \lambda) \in \text{End} \mathbb{C}^k : \)

\[ \partial S / \partial x_0 = [I, S], \]

where \([\cdot, \cdot]\) denotes here the usual matrix commutator in the space \( \text{End} \mathbb{C}^k \). From the Equation (9) we conclude that the traces \( \text{tr} S^m(x_0; \lambda), m \in \mathbb{Z} \), don’t depend on a point
We will show that the function \( \xi(\lambda) \in \sigma(S) \) of the matrix \( S(x_0;\lambda) \) does not depend on a point \( x_0 \in \mathbb{R} \), that is
\[
\frac{\partial \xi(\lambda)}{\partial x_0} = 0,
\]
proving the statement. \( \square \)

Let \( f \in C(\mathbb{R};\mathbb{C}^k) \) be an eigenfunction of the problem (6), satisfying the classical Bloch condition \([14,27]\)
\[
f(x + 2\pi, x_0; \lambda) = \xi(\lambda)f(x, x_0; \lambda),
\]
that is the vector-function \( f(x, x_0; \lambda) \in \mathbb{C}^k \) is an eigenfunction of the monodromy matrix \( S(x; \lambda) \):
\[
S(x; \lambda)f(x, x_0; \lambda) = \xi(\lambda)f(x, x_0; \lambda)
\]
for all \( x \in \mathbb{R} \). Under the additional assumption that the matrix \( l(x; \lambda) \in \text{End} \mathbb{C}^k \) depends meromorphically on the spectral parameter \( \lambda \in \mathbb{C} \) we get that the matrix function \( F(x, x_0; \lambda) \in \text{End} \mathbb{C}^k \) is also analytic in \( \lambda \in \mathbb{C} \). Based on Lemma 1 and the fact that the function \( \xi : \mathbb{C} \to \mathbb{C} \) is, in general, algebraic \([14,26–28]\) on a Riemannian surface \( \Gamma \) of finite or infinite genus, one states the following proposition.

**Proposition 1.** The function \( f(\lambda) \in \mathbb{C}^k \) is meromorphic in the parameters \( \lambda \in \Gamma \setminus \{\infty\} \), where we denoted by \( \{\infty\} \) its essentially singular points on the surface \( \Gamma \). The values \( \lambda \in \mathbb{C} \) for which \( |\xi(\lambda)| = 1 \) belong to the generalized spectrum \( \sigma(\mathcal{L}) \subset \mathbb{C} \) of the matrix operator (5), which, in general, is of zone structure on the surface \( \Gamma \).

For more detail examination of the generalized spectrum \( \sigma(\mathcal{L}) \subset \mathbb{C} \) properties it's necessary to concretize the form of the operator \( \mathcal{L}[u; \lambda] : L^\infty(\mathbb{R}; \mathbb{C}^k) \to L^\infty(\mathbb{R}; \mathbb{C}^k) \) for \( u \in M \) and its dependence on the parameter \( \lambda \in \mathbb{C} \), as it is presented in the classical manuals \([26,29–31]\).

2.3. Generative Function of Conserved Quantities

Assume the Equation (4) is a Lax type representation for the dynamical system (1). We will show that the function \( \xi(\lambda) \in \sigma(S) \) as a functional on the manifold \( M \) depending on the parameter \( \lambda \in \mathbb{C} \), doesn't depend on the evolution parameter \( t \in \mathbb{R} \) by virtue of the dynamical system (1). Equation (4) is evidently equivalent to the following system of differential equations
\[
\frac{\partial f}{\partial x} = l[u; \lambda]f, \quad \frac{\partial f}{\partial t} = p[u; \lambda]f
\]
(11)
for some matrix \( p[u; \lambda] \in \text{End} \mathbb{C}^k \), where \( f \in C(\mathbb{R}; \mathbb{C}^k) \) is the defined above Bloch eigenfunction of the operator \( \mathcal{L}[u; \lambda] : L^\infty(\mathbb{R}; \mathbb{C}^k) \to L^\infty(\mathbb{R}; \mathbb{C}^k) \) for a fixed \( u \in M \) and the parameter \( \lambda \in \mathbb{C} \). Now one can formulate the following important lemma.

**Lemma 2.** The spectrum \( \sigma(\mathcal{L}) \subset \mathbb{C} \) of the periodic problem (6) does not depend on the temporal parameter \( t \in \mathbb{R} \).

**Proof.** Having taken into consideration the property (10), one easily finds that
\[
\xi(\lambda) \frac{\partial f(x; \lambda)}{\partial t} + f(x; \lambda) \frac{\partial \xi(\lambda)}{\partial t} = \xi(\lambda) \frac{\partial f(x + 2\pi; \lambda)}{\partial t},
\]
whence, by virtue of (11), one follows that
\[
\frac{\partial \xi(\lambda)}{\partial t} = 0,
\]
that is the parametric functional \( \xi(\lambda) \in \mathcal{D}(M) \) is a meromorphic generative function of conserved quantities of dynamical system (1). \( \square \)
To construct analytically these conservation laws it’s convenient to make use of the Equation (6). To do this we consider the function
\[ \chi(x; \lambda) := \partial \ln f_1(x, x_0; \lambda) / \partial x, \tag{14} \]
where \( f_1 \in L_\infty(\mathbb{R}; \mathbb{C}) \) is the first component of the vector-function \( f \in L_\infty(\mathbb{R}; \mathbb{C}^k) \), normalized at \( x_0 \in \mathbb{R} \) by unity: \( f_1(x_0, x_0; \lambda) = 1 \) for all \( x_0 \in \mathbb{R} \).

Substituting recurrently components of the function \( f(x, x_0; \lambda) \in L_\infty(\mathbb{R}; \mathbb{C}^k) \) in such a form into the Equation (6), after simple calculations we make sure that the function \( \chi(x; \lambda) \in C^{k-1}(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{C}) \) satisfies an ordinary differential equation of the form
\[ \partial^{k-1} \chi / \partial x^{k-1} = R[u; \chi], \tag{15} \]
where \( R[u; \cdot] : f^{(k-2)}(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{R}^m) \rightarrow \mathbb{C} \) is a polynomial mapping, parametrically depending on a point \( u \in M \) and complex parameter \( \lambda \in \mathbb{C} \).

**Remark 1.** Remark here that at \( k = 2 \) the Equation (15) becomes the well-known Riccati equation.

Having supposed that the function \( \chi(x; \lambda) \in C^{k-1}(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{C}), x \in \mathbb{R}, \) as \( \lambda \rightarrow \{\infty\} \) can be represented by making use of the asymptotic expansion
\[ \chi(x; \lambda) \sim \lambda^{s(\chi)} \sum_{j \in \mathbb{Z}_+} \chi_j[u] \lambda^{-1} \tag{16} \]
in parameter \( \lambda \in \mathbb{C} \), where \( s(\chi) \in \mathbb{N} \) is some integer, after substitution (16) into the Equation (15) one ensues the recursion relations for densities \( \chi_j[u], j \in \mathbb{Z}_+ \), giving rise to their explicit expressions as local functionals on the functional manifold \( M \). In particular, all densities \( \chi_j[u], j \in \mathbb{Z}_+ \), prove to be smooth and periodic in variable \( x \in \mathbb{R} \) functionals on the manifold \( M \).

Moreover, as the obvious integral expression
\[ \exp \left[ \int_{x_0}^{x_0 + 2\pi} \chi(x; \lambda) dx \right] \rightarrow \xi(\lambda) \]
coincides with the eigenvalue \( \xi(\lambda) \in \mathbb{C} \) of the monodromy matrix \( S(x_0; \lambda) \in \text{End} \ \mathbb{C}^k \), from (13) and (16) we easily infer that all functionals
\[ \gamma_j = \int_{x_0}^{x_0 + j} \chi_j(x, \lambda) dx \tag{17} \]
for all \( j \in \mathbb{Z}_+ \) are conservation laws of the dynamical system (1). That is we have stated the following proposition.

**Proposition 2.** The functionals (17) generate a complete set of conserved quantities of the nonlinear dynamical system (1) on the functional manifold \( M \).

To study the properties of these conserved quantities, presenting conservation laws (17) for the evolution flow (1), we shall apply below to the parametric functional \( \xi(\lambda) \in \mathcal{D}(M) \) both the related differential geometric structures \([14, 32–34]\) on the functional manifold \( M \) and the recursive operator technique. The latter makes is possible to represent the dynamical system (1) as a Hamiltonian system on the functional manifold \( M \) and to state its Liouville-Arnold type complete integrability.
2.4. Gradient-Holonomic Integrability Analysis

Consider the Equation (6) on the fundamental solution $F := F(x, x_0; \lambda) \in \text{End } \mathbb{C}^k$:

$$\frac{\partial F}{\partial x} = l[u; \lambda] F,$$

(18)

where $F(x_0, x_0; \lambda) = I$ for all $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Since the fundamental solution $F(x, x_0; \lambda) \in \text{End } \mathbb{C}^k$ is also a functional on the linear manifold $M$, then from (18) one can easily obtain an equation for the variation $\delta F \in \text{End } \mathbb{C}^k$ of the fundamental solution, when the value $u \in M$ changes to $(u + \delta u) \in M$. Hence, we have

$$\partial(\delta Y)/\partial x = l[u; \lambda]\delta Y + \delta l[u; \lambda] F$$

(19)

under the evident condition $\delta F(x_0, x_0; \lambda) = 0$ for all $x_0 \in \mathbb{R}$. The solution of Equation (19) is

$$\delta F(x, x_0; \lambda) = \int_{x_0}^{x} F(x, y; \lambda)\delta l[u(y); \lambda] F(y, x_0; \lambda)dy$$

(20)

for all $x \in \mathbb{R}$. Taking now into consideration that $S(x_0; \lambda) = F(x_0 + 2\pi, x_0; \lambda)$, from relation (20) we find (see [14,27]) that

$$\delta S(x_0; \lambda) = \int_{x_0}^{x_0 + 2\pi} F(x_0 + 2\pi, x; \lambda)\delta l[u(x); \lambda] F(x, x_0; \lambda)dx.$$  

(21)

Consider now the functional

$$\gamma(\lambda) := \text{tr}S(x_0; \lambda),$$

which by virtue of (13) is a generating functional of conservation laws for the dynamical system (1). Having calculated the trace $\text{tr}S(x_0; \lambda)$, from (21) we obtain that

$$\delta \gamma(\lambda) = \int_{x_0}^{x_0 + 2\pi} \text{tr}[F(x_0 + 2\pi, x; \lambda)\delta l[u(x); \lambda] F(x, x_0; \lambda)]dx,$$

(22)

whence, making use of the Formula (22), we obtain for the gradient vector $\text{grad } \gamma(\lambda) \in T^*(M)$ the following expression:

$$\text{grad } \gamma(\lambda) = \text{tr}(l^* S),$$

(23)

or

$$\text{grad } \gamma(\lambda) = \text{tr}(\delta l/\partial u),$$

(24)

provided that the matrix $l[u; \lambda] \in \text{End } \mathbb{C}^k$ doesn’t depend in explicit way on the derivatives of the function $u \in M$. If the matrix $l[u; \lambda] \in \text{End } \mathbb{C}^k$ depends on derivatives $(u_{x_x}, u_{x_{xx}}, \ldots u_{x_{x^n}}) \in f^1(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{R}^m)$, $s \in \mathbb{Z}^+$, then Formula (22) will lead to a bit complicated than (24) expression for the gradient vector $\gamma(\lambda) \in T^*(M)$, where $T^*(M)$ denotes the cotangent space to the functional manifold $M$, being adjoint to the tangent space $T(M)$ to the manifold $M$ with respect to the standard bilinear form $(\cdot , \cdot) : T^*(M) \times T(M) \to \mathbb{R}$. To obtain this gradient vector $\gamma(\lambda) \in T^*(M)$, it is necessary to calculate the expression (23) separately, making use of the Equation (18) and the expression (23). For example, if $l[u; \lambda] = l(u, u_x; \lambda) \in \text{End } L_\infty(\mathbb{R}; \mathbb{C}^k)$, from the representation (23) we easily obtain that

$$\text{grad } \gamma(\lambda) = \text{tr}(l[l, l_{u_x}] - S\delta l_{u_x}/\partial x + S l_{u_x}),$$

(25)

where $l_{u_x}$ and $l_{u_{x^2}}$ are the usual partial derivatives with respect to the variables $(u, u_x) \in f^1(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{R}^m)$ of the matrix function $l(u, u_x; \lambda) \in \text{End } L_\infty(\mathbb{R}; \mathbb{C}^k)$. 

Obviously, in general case there exists such a matrix valued vector \(a = \{a_{ij}[u; \lambda] : i, j = 1, k\} \in (\text{End } \mathbb{C}^k)^m\), being a local functional on \(M\), that

\[
\text{grad } \gamma(\lambda) = \text{tr}(Sa[u; \lambda]).
\]

(26)

Suppose that the vector-matrix \(a \in (\text{End } \mathbb{C}^k)^m\) in (26) is non-trivial. Then making use of the system of differential Equation (9) for the independent matrix functionals \(S \in \text{End } \mathbb{C}^k\) and relation (26), one can derive the following recursion expression for the gradient vector \(\gamma(\lambda) \in T^*(M)\):

\[
\Lambda[u] \text{grad } \gamma(\lambda) = \lambda^{r(l)} \text{grad } \gamma(\lambda)
\]

(27)

for some integer \(r(l) \in \mathbb{N}\), where in general case the linear mapping \(\Lambda[u] : T^*(M) \rightarrow T^*(M)\) is an integro–differential operator, parametrically depending on \(u \in M\).

Observe now that, by definition,

\[
\gamma(\lambda) = \text{tr } S(x_0; \lambda) = \sum_{j=1}^k \exp[i\varphi_j(\lambda)],
\]

(28)

where \(\varphi_j(\lambda) \in D(M)\) are such that \(\xi_j(\lambda) = \exp[i\varphi_j(\lambda)], j = 1, k\), being the eigenvalues of the monodromy matrix \(S(x_0; \lambda) \in \text{End } \mathbb{C}^k\).

If turning the complex parameter \(\lambda \in \mathbb{C}\) to the singularity \(\{\infty\} \in \Gamma\) in some way and expanding the expression \(\text{grad } \gamma(\lambda) \in T^*(M)\) in an asymptotic series with respect to the parameter \(\lambda \in \mathbb{C}\), we will obtain from (27) the recursion relation

\[
\Lambda[u] \text{grad } \gamma_j = \text{grad } \gamma_{j+r(l)}
\]

(29)

for the gradient vectors \(\text{grad } \gamma_j \in T^*(M), j \in \mathbb{Z}_+,\) generated by the conservation laws for the dynamical system (1), obtained before via expressions (17).

2.5. Conservation Laws and the Related Involutive Properties

It is easy to observe that the operator \(\Lambda = \Lambda[u] : T^*(M) \rightarrow T^*(M)\), acting according to the rule (29), is hereditarily recursive, satisfying the following determining differential-functional equation:

\[
\partial \Lambda / \partial t = [\Lambda, K^*],
\]

(30)

following from the well known Lax differential-functional relationship

\[
\partial \varphi / \partial t + K^* \varphi = 0
\]

(31)

for the gradient vector \(\varphi := \text{grad } H \in T^*(M)\), corresponding to any smooth conservation law of the nonlinear dynamical system (1). Moreover, if the nonlinear dynamical system (1) possesses two compatible [14,32,33] Poisson operators \(\theta\) and \(\eta : T^*(M) \rightarrow T(M)\), that is their affine some \(\eta + \lambda \theta : T^*(M) \rightarrow T(M)\) persists to be Poisson for all \(\lambda \in \mathbb{R}\), the recursion operator \(\Lambda : T^*(M) \rightarrow T^*(M)\) can be factorized as

\[
\Lambda = \theta^{-1} \eta.
\]

(32)

In this connection the operators Poisson operators \(\theta\) and \(\eta : T^*(M) \rightarrow T(M)\) are then Noetherian and the nonlinear dynamical system (1) is a bi-Hamiltonian one, i.e.,

\[
u_t = -\theta \text{grad } H_\theta = -\eta \text{grad } H_\eta,
\]

(33)
where the conservation laws $H_\theta$ and $H_\eta \in \mathcal{D}(M)$ are, in general, linear combinations of a finite number of the conservation laws $\gamma_j \in \mathcal{D}(M), j \in \mathbb{Z}_+$, constructed above:

$$H_\theta = \sum_{j=-q}^m c_j \gamma_{j+q+r}, \quad H_\eta = \sum_{j=-q}^m c_j \gamma_{j+q+r}$$

(34)

where $r \in \mathbb{N}$ and $c_j \in \mathbb{R}, j = -q, m$, are some constants. Moreover, all evolution flows

$$u_t := -\theta \text{ grad } \gamma_j$$

(35)

with respect to temporal parameters $t_j \in \mathbb{R}, j \in \mathbb{Z}_+$, are functionally independent on the functional manifold $M$. As another important consequence of the factorization (32), we observe that the relationship (27) is equivalent to the following one:

$$\eta \text{ grad } \gamma(\lambda) = \lambda^{r(\lambda)} \text{ grad } \gamma(\lambda)$$

(36)

for all $\lambda \in \mathbb{C}$. From it one easily ensues that all invariants $\gamma_j \in \mathcal{D}(M), j \in \mathbb{Z}_+$, are commuting to each other with respect to two compatible Poisson brackets on $\mathcal{D}(M)$:

$$\{ \gamma_j, \gamma_k \}_\theta = 0 = \{ \gamma_j, \gamma_k \}_\eta,$$

(37)

where, by definition,

$$\{ \gamma, \mu \}_\theta := (\text{ grad } \gamma|_\theta \text{ grad } \mu), \quad \{ \gamma, \mu \}_\eta := (\text{ grad } \gamma|_\eta \text{ grad } \mu)$$

(38)

for any smooth functionals $\gamma, \mu \in \mathcal{D}(M)$. The latter, in particular, means that all evolution flows (35) are functionally independent bi-Hamiltonian and commuting to each other completely Liouville-Arnold type integrable flows on the functional manifold $M$. The obtained above result can be reformulated as the following proposition.

**Proposition 3.** The Lax type integrable dynamical system (1) is a bi-Hamiltonian flow with respect to two algebraically independent Poissonian structures $\theta$ and $\eta : T^*(M) \rightarrow T(M)$ on $M$ and generates an infinite hierarchy of functionally independent commuting to each other completely Liouville-Arnold type integrable bi-Hamiltonian flows.

### 2.6. Integrability Testing Algorithm

Consider now an nonlinear dynamical system (1) on the functional manifold $M$, which we will assume to be a priori Lax type integrable. The latter, in particular, means, as follows from the analysis in Section 2.1, that it possesses an infinite hierarchy of conserved quantities suitably ordered by powers a complex parameter $\lambda \in \mathbb{C}$ and whose complexified generating gradient vector function $\varphi := \varphi[u; \lambda] \in T^*(M) \otimes \mathbb{C}$, taken in the form (28), satisfies the Lax differential-functional equation

$$\partial \varphi / \partial t + K^* \varphi = 0.$$  

(39)

Moreover, as follows from the expression (28) and from the corresponding differential equation for matrix $S(x_0; \lambda) \in \text{End}^{\mathbb{C}}$, the gradient vector function $\varphi \in T^*(M) \otimes \mathbb{C}$ possesses the following asymptotic as $\lambda \rightarrow \infty$

$$\varphi = (1, a_1(x; \lambda), a_2(x; \lambda), ..., a_{m-1}(x; \lambda))^T \exp[\lambda^{s(\sigma)} t + \partial^{-1} \sigma(x; \lambda)],$$

(40)

where

$$\sigma(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \sigma_j[u] \lambda^{-j+s(n)}, \quad a_n(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \sigma_j[u] \lambda^{-j+s(n)}$$

(41)

for some integers $s(\sigma), s_n \in \mathbb{Z}_+, n = 1, m-1$, and which can be easily obtained upon substitution the asymptotic solution (40) into (39) from a resulting system of the recurrent
differential-functional relationships. The representation (40), in particular, generates an infinite hierarchy of functionals

$$\gamma_j = \int_0^{2\pi} \sigma_j |u| dx,$$

(42)

$j \in \mathbb{Z}_+$, on the manifold $M$, being conservation laws for the nonlinear dynamical system (1). This way obtained conservation laws can be effectively used for constructing additional structures inherent in the dynamic system (1). In particular, if a functional $H \in D(M) \in \text{span}_{\mathbb{R}} \{ \gamma_j \in D(M) : j \in \mathbb{Z}_+ \}$ is representable in the following scalar form: $H = \langle \psi | u \rangle$, and the element $\psi \in T^*(M)$ satisfies the modified Noether-Lax differential-functional equation

$$\frac{\partial \psi}{\partial t} + K^{l,\ast} \psi = \text{grad} \mathcal{L}$$

(43)

for some smooth functional $\mathcal{L} \in D(M)$ and $\psi' \neq \psi^{l,\ast}$, then our nonlinear dynamical system (1) is a priori Hamiltonian with respect to the Poisson structure $\theta = \psi' - \psi^{l,\ast} : T^*(M) \to T(M)$ and is representable as the following evolution flow

$$u_t = -\theta \text{grad} \langle \psi | K \rangle - \mathcal{L}$$

(44)

on the functional manifold $M$. The latter makes it possible in many practically important cases to obtain this way a next compatible Poisson structure $\eta : T^*(M) \to T(M)$ on the manifold $M$, which satisfies, owing to (26) and (27), the following determining recursion relationship

$$\eta \text{tr}(Sa[u; \lambda]) = \lambda^{r(l)} \theta \text{tr}(Sa[u; \lambda])$$

(45)

for all $\lambda \in \mathbb{C}$ and some integer $r(l) \in \mathbb{N}$, depending on the differential functional structure of the matrix mapping $A(u; \lambda) \in \left(\text{End} \mathbb{C}^k \right)^m$ depends explicitly on the Lax matrix $l[u; \lambda] \in \text{End} \mathbb{C}^k$, the obtained above differential-algebraic relationship (45) retrieve this matrix analytically, meaning thereby a solution of the Lax type representation reconstruction problem for the nonlinear dynamical system (1).

2.7. An Optimal Control Problem Aspect

Consider now a smooth nonlinear dynamical system

$$v_t = K[v, u]$$

(46)

on a $2\pi$-periodic functional manifold $M_v \subset C(\mathbb{R} / \{2\pi \mathbb{Z} \}; \mathbb{R}^m(v))$ and depending parametrically on a $2\pi$-periodic functional variable $u \in M_u \subset C(\mathbb{R} / \{2\pi \mathbb{Z} \}; \mathbb{R}^m[u])$ and pose the following Bellman-Pontriagin type optimal control problem [35,36] on a temporal interval $[0, T] \subset \mathbb{R}_+$:

$$\arg\inf_{(v,u) \in M_v \times M_u} \int_0^T dt \int_0^{2\pi} \mathcal{L}[v, u] dx = ?,$$

(47)

under condition that the evolution flow (46) possesses a smooth conserved quantity $\gamma = \int_0^{2\pi} \gamma[u, v] dx \in D(M_v \times M_u)$, that is $d\gamma/dt = 0$ on the combined manifold $M_v \times M_u$ for all $t \in [0, T]$. The latter, in particular, means that we need to determine such an additional evolution flow

$$u_t = F[v, u]$$

(48)

on the control manifold $M_v$, which ensures the existence of the mentioned above conserved quantity $\gamma \in D(M_v \times M_u)$. The problem above is solved by means of construction of the extended Lagrangian functional

$$\mathcal{L}_{\mu, \phi} := \int_0^T dt \left[ \int_0^{2\pi} \mathcal{L}[v, u] dx + \mu(t) \frac{d}{dt} \int_0^{2\pi} \gamma[u, v] dx + (\psi[v_t - K[v, u]] \right]$$

(49)
by means of the Lagrangian multipliers \( \mu \in C^1_0([0, T]; \mathbb{R}) \) and \( \psi \in C^1_0([0, T]; T^*(M_0)) \) almost everywhere with respect to the temporal parameter \( t \in [0, T] \), and finding its critical points:

\[
L_{\nu, \mu, \psi}^\ast(1) = 0 \sim \nabla_{\nu}[L - \mu(t)\gamma][v, u] - (\psi_t + K^\ast_\nu[v, u] \psi) = 0
\]

(50)

for all \((v, u) \in M_0 \times M_u\) jointly with the condition that \(d\gamma/dt = 0\). The obtained functional relationship (50) under the condition \( L = \mu(t)\gamma \) for \( t \in [0, T] \) reduces to the following Noether-Lax condition

\[
\psi_t + K^\ast_\nu[v, u] \psi = 0
\]

(51)

on the Lagrangian multiplier \( \psi \in C^1_0([0, T]; T^*(M_0)) \). The functional relationship (51) presents an evolution partial differential equation on the vector-function \( \psi \in C^1_0([0, T] \times \mathbb{R}/\mathbb{R} \otimes \mathbb{C}) \), which always possesses [37] an asymptotic as \( \lambda \to \infty \) solution in the following form:

\[
\psi = (1, a_1(x; \lambda), a_2(x; \lambda), \ldots, a_{m(v)-1}(x; \lambda)) \exp[\lambda^s(\psi)t + Z(x; \lambda)]
\]

(52)

for some fixed \( s(\psi) \in Z_+ \), where \( z(x; \lambda) \sim \sum_{j \in Z_+} z_j(x)\lambda^{-j+s(Z)} \), \( s(Z) \in Z_+ \), and whose coefficients \( z_j \in C^\infty(\mathbb{R}^m; \mathbb{R}) \), \( j \in Z_+ \), can be easily calculated by means of the respectively derived recurrent differential-functional relationships, depending exclusively on the derivatives \( \partial z_j/\partial t, \partial z_j/\partial x \), \( j \in Z_+ \). If, in addition, the coefficients \( z_j \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^m; \mathbb{R}^m(u)); \mathbb{R}) \), \( j \in Z_+ \), then there exist a priori local quantities \( \gamma_j := \gamma_j[v, u] \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^m; \mathbb{R}^m(u)); \mathbb{R}) \), \( j \in Z_+ \), such that \( \partial \gamma_j[v, u]/\partial t = \partial (\partial z_j/\partial t)/\partial x \), simply meaning the quantities \( \gamma_j := \int_0^{2\pi} \gamma_j[u, v]dx \), \( j \in Z_+ \), are conserved, that is \( \frac{d}{dt} \int_0^{2\pi} \gamma_j[u, v]dx = 0 \) for all \( j \in Z_+ \). Moreover, as the evolution flow on the functional manifold \( M_0 \) are assumed to possess the conserved quantity \( \gamma \in D(M_0 \times M_u) \), its gradient \( \psi := \text{grad}_{\nu} \gamma[v, u] \in T^*(M_0) \) a priori satisfies the Noether-Lax differential-functional Equation (51), that is

\[
\psi_t + K^\ast_\nu[v, u] \psi,
\]

(53)

ensuring the existence of the asymptotic solution (52). The latter condition, as the first step, makes it possible to regularly check the existence of mentioned above hidden symmetry properties, in particular local conserved quantities and, by the second step, to construct the searched additional evolution flow (48), solving our optimal control problem (47). Moreover, if the conserved quantities obtained this way prove to be suitably ordered, this case will strictly correspond to the completely integrable evolution flow

\[
\begin{align*}
\nu_t &= K[v, u] \\
u_t &= F[v, u, p] \\
p_t &= F[v, u, p]
\end{align*}
\]

(54)

on the extended functional manifold \( M_0 \times M_u \times M_p \), where \( p \in M_p \) is a suitably introduced supplementing functional variable.

Below we will apply the devised above optimal control problem solving algorithm to a new interesting class of the so called dark type nonlinear dynamical systems, whose analytical studies of the hidden symmetry properties were initiated by B. Kupershmidt [1,2], and who demonstrated their interesting mathematical properties.

3. Nonlinear Integrable Dark Type Dynamical Systems: Hidden Symmetry Properties and Integrability Testing Algorithm

3.1. Dark Type Dynamical Systems: Introductory Setting

We begin with studying integrability properties of a certain class of nonlinear dynamical systems of the form

\[
\nu_t = K[v, u],
\]

(55)
on a suitably chosen [17] smooth $2\pi$-periodic functional manifold $M_\varphi \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^{m(\varphi)})$, $m(\varphi) \in \mathbb{N}$, where $t \in \mathbb{R}$ is the evolution parameter and $K : M_\varphi \to T(M_\varphi)$ is a smooth vector field on $M_\varphi$, with values in its tangent space $T(M_\varphi)$, represented by means of polynomial functions on the related jet-space $J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^{m(\varphi)} \times m(\varphi))$ of a finite order, depending parametrically on a functional variable $u \in M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^{m(u)})$, $m(u) \in \mathbb{N}$. Moreover, we will assume that the vector field on $M_\varphi$ satisfies the following additional functional constraint: the flow on $M_\varphi$ generated by the vector field (55), possesses internal hidden symmetry properties, for instance an infinite hierarchy of suitably ordered conservation laws (they may be almost all nontrivial, or except finite, trivial), satisfying the related recursion relationships. The latter gives rise to some functional-analytic constraints on the parametric variable $u \in M_u$, which in general can be represented as the following supplementing (55) dynamical system:

\[
\begin{align*}
  u_t &= F[v, u, p], \\
  p_t &= P[v, u, p]
\end{align*}
\tag{56}
\]

on an extended functional manifold $M_u \times M_p \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^{m(u)} \times m(p))$, where the additional vector fields $F : M_u \to T(M_u)$ and $P : M_p \to T(M_p)$ are smooth and, by definition, finitely-component, that is the functional manifold $M_p \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^{m(p)})$, where the dimension $p \in \mathbb{Z}_+$ has to be finite. The resulting combined dynamical system

\[
\begin{align*}
  v_t &= K[v, u] \\
  u_t &= F[v, u, p] \\
  p_t &= P[v, u, p]
\end{align*}
\tag{57}
\]

is closed and determines a completely integrable Lax type linearized flow on the joint functional manifold $M_\varphi \times M_u \times M_p$. The latter makes it possible to formulate for the dynamical system (55) the following definition.

**Definition 1.** A dynamical system (55), allowing the finitely-component and completely integrable Lax-type linearized extension (57), is called the dark type system.

This definition proves to be constructive enough and allows by means of the gradient-holonomic approach [14–16,27] to classify many linear and nonlinear dynamical systems of dark type, presenting, in addition, a great interest for applications in modern physics, mechanics, hydrodynamics and biology sciences. To be more specific, we will turn to the analysis of some interesting examples of dark type dynamical systems on functional manifolds.

### 3.2. Integrability Analysis: Diffusion Class Dynamical Dark Type Systems

#### 3.2.1. A First Diffusion Class Dynamical System

As a simple example, let us analyze a nonlinear dynamical system of the diffusion class

\[
\begin{align*}
  v_t &= u(v_{xx} - v^2) := K[v, u] 
\end{align*}
\tag{58}
\]

on the manifold $M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$, parametrically dependent on a functional variable $u \in M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$. To determine whether it is of dark type, we need to study the related existence of an infinite hierarchy of suitably ordered conservation laws, what can be done by means of the gradient-holonomic scheme [14–16,27], based on constructing special asymptotic solutions $\varphi \in T^*(M_\varphi) \otimes \mathbb{C}$ to the corresponding Lax–Noether equation

\[
\varphi_t + K_{\varphi}^{\leftarrow} [v, u] \varphi = 0
\tag{59}
\]
on the complexified cotangent space $T^* (M_u) \otimes \mathbb{C}$, asymptotically as $\lambda \to \infty$ depending on the complex parameter $\lambda \in \mathbb{C}$. So, this determining Lax-Noether equation looks as

$$\varphi_t + \left[ uv (dv/dv) + v^{-1} (dv/dx) \right] \varphi = 0,$$

whose asymptotic as $\lambda \to \infty$ solution

$$\varphi = \exp \left[ -\lambda^2 x + \partial^{-1} \sigma (x, \lambda) \right], \quad \sigma (x, \lambda) \sim \sum_{j \in \mathbb{Z} \cup \{-1\}} \lambda^{-j} \sigma_j [u, v],$$

easily reduces to solving the following infinite hierarchy of recurrent differential-algebraic relationships:

$$\delta_{j=2} + \partial^{-1} \sigma_{j+2} + 2v^{-1} (dv/dx) \sigma_j + u (\sigma_j, x + \sum_{k \in \mathbb{Z} \cup \{-2, -1\}} \sigma_{j-k} \sigma_k) = 0$$

for $j \in \mathbb{Z} \cup \{-2, -1\}$. Whence one ensues that $\sigma_{-1} = u^{-1/2}$, which should satisfy a first important constraint:

$$\partial \sigma_{-1} / \partial t = -\partial \eta_{-1} [u, v] / \partial x$$

for some mapping smooth $\eta_{-1} : f^4 (\mathbb{R} / \{2 \pi \mathbb{Z}\}; \mathbb{R}^2) \to \mathbb{R}$ on the jet-space $f^4 (\mathbb{R} / \{2 \pi \mathbb{Z}\}; \mathbb{R}^2)$. The latter can be easily satisfied if to put, by definition,

$$\eta_{-1} [u, v] := 2v^{-1} (dv/dx) u^{-1/2} - 1/2u_x u^{-1/2},$$

entailing, in addition, the quantities

$$\sigma_j = 0$$

for all $j \in \mathbb{Z}$. As a result of expressions (63) and (64) one easily obtains that

$$u_t = -3uu_x - 4uv^{-1}v_x := F[v, u].$$

Moreover, owing to the conditions (65) one easily derives the exact linearization of the diffusion class dynamical system (58). The obtained result one can formulate as the following theorem.

**Theorem 1.** The joint nonlinear dynamical system

$$\begin{align*}
    v_t & = u (v_{xx} - v_x^2) \\
    u_t & = -3uu_x - 4uv^{-1}v_x
\end{align*}$$

$$Q[u, v]$$

presents a completely integrable dark type flow on the functional manifold $M_u \times \mathbb{M}_v$, being Lax type linearized and possessing only a finite number of conservation laws.

3.2.2. A Second Diffusion Class Dynamical System

As an interesting enough another example, let us analyze the next of diffusion class dynamical system

$$v_t = u^{-2} (v_{xx} - v^{-1} v_x^2) := K[v, u]$$

on the manifold $M_v$, parameterized by a functional variable $u \in M_u$. The determining Lax-Noether Equation (59) on the extended cotangent space $T^* (M_u) \otimes \mathbb{C}$ looks as

$$\varphi_t + \left[ u^{-2} (dv/dv) + v^{-1} \partial^2 u^{-2} \varphi \right] \varphi = 0$$

and allows the following asymptotic as $\lambda \to \infty$ solution:

$$\varphi = \exp \left[ -\lambda^2 x + \partial^{-1} \sigma (x, \lambda) \right], \quad \sigma (x, \lambda) \sim \sum_{j \in \mathbb{Z} \cup \{-1\}} \lambda^{-j} \sigma_j [u, v].$$
From this expressions above one ensues the infinite hierarchy of recurrent differential-algebraic relationships:

\[-\delta_{j+2} + \partial^{-1}\sigma_{j,t} + u^{-2}\sigma_{j,x} + u^{-2}\sum_{k \in \mathbb{Z}_+ \cup \{-1\}} \sigma_k \sigma_{j-k}^+ + 2u^{-2}v^{-1}\sigma_{j} + 2u^{-2}v^{-1}\sigma_{j,0} - u^{-2}v^{-2}\delta_{j,0} = 0\]  

(71)

for \( j \in \mathbb{Z}_+ \cup \{-2, -1\} \), whose solutions

\[
\sigma_{-1} = u, \quad \sigma_0 = -\frac{1}{2}u\partial^{-1}u_t - v^{-1}v_x - \frac{1}{4}u^{-1}u_x = -1/2p - v^{-1}v_x - 1/2u^{-1}u_x = -1/2(p + \ln(uv^2))_x,
\]

(72)
give rise to the expressions \( \sigma_j = 0 \) for all \( j \in \mathbb{Z}_+ \), iff the following nonlinear and smooth evolution flow

\[
\begin{align*}
    u_t &= -u^{-2}u_x p_x + u^{-1} p_{xx} := F[u, p], \\
    p_t &= -u^{-2}p_x + 1/2u^{-2}p_x^2 + 2u^{-3}u_x p_x - u^{-3}u_{xx} + 3/2u^{-4}u_x^2 := P[u, p]
\end{align*}
\]

(73)

(75)
on the extended functional manifold \( M_p \subset \mathbb{C}[^{2\pi\mathbb{Z}}; \mathbb{R}] \).

It is easy now to observe that the nonlinear evolution flows (73) and (75) are closed on the extended functional manifold \( M_\sigma \times M_p \subset \mathbb{C}[^{2\pi\mathbb{Z}}; \mathbb{R}^2] \), posing an additional question about their joint integrability. To demonstrate that this is the case, we applied once more yet now to the joint nonlinear dynamical system

\[
\begin{align*}
    u_t &= -u^{-2}u_x p_x + u^{-1} p_{xx} \\
    p_t &= -2u^{-2}p_{xx} + 1/2u^{-2}p_x^2 + 2u^{-3}u_x p_x - u^{-3}u_{xx} + 3/2u^{-4}u_x^2 \\
\end{align*}
\]

(76)
on the extended functional manifold \( M_\sigma \times M_p \) the mentioned above gradient-holonomic approach and proved the following lemma.

**Lemma 3.** The nonlinear dynamical system (76) determines a completely integrable Lax linearized diffusion type flow on the extended functional manifold \( M_\sigma \times M_p \), possessing only a finite number of suitably ordered conservation laws.

The latter guarantees the integrability of the combined nonlinear diffusion flow (58) with that of (76):

\[
\begin{align*}
    v_t &= -u^{-2}v_{xx} - u^{-2}v^{-1}v_x^2 \\
    u_t &= -u^{-2}u_x p_x + u^{-1} p_{xx} \\
    p_t &= -2u^{-2}p_{xx} + 1/2u^{-2}p_x^2 + 2u^{-3}u_x p_x - u^{-3}u_{xx} + 3/2u^{-4}u_x^2
\end{align*}
\]

(77)
on the extended functional manifold \( M_\sigma \times M_\sigma \times M_p \). The obtained result can be now formulated as the following theorem.

**Theorem 2.** The joint nonlinear dynamical system (77) presents a completely integrable dark type flow on the extended functional manifold \( M_\sigma \times M_\sigma \times M_p \), being Lax type linearized and possessing only a finite number of suitably ordered conservation laws.
As an interesting remark, one can observe that the first two evolution flows of the vector field \((77)\) are of classical diffusion class with positive definite diffusion coefficients \(D_\varepsilon(x, t) = u^{-2}\) and \(D_u(x, t) = u^{-1}\), yet the third evolution flow of this vector field is of diffusion class with the negative diffusion coefficient \(D_p(x, t) = -2u^{-2}\), eventually meaning that the corresponding \(p\)-component describes an inverse anti-diffusion process, which results in the aggregation of a certain substance in space.

3.2.3. A Third Diffusion Class Dynamical System: Not Dark Type Example

Herewith we will consider then next nonlinear dynamical system [6] of diffusion class

\[
v_t = -vv_x - (u^{-2}v)_x := K[v, u], \quad (78)
\]

which is defined on a functional manifold \(M_0 \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})\), depends parametrically on a functional parameter \(u \in M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})\) proves to be of not-dark type. The corresponding Lax-Noether equation

\[
\phi_t + v\phi_x + (u^{-2}\phi)_x = 0, \quad (79)
\]

on the extended cotangent space \(T^*(M_0) \otimes \mathbb{C}\) possesses the asymptotic as \(\lambda \to \infty\) solution

\[
\varphi = \exp(-\lambda^2t + \partial^{-1}\sigma), \quad (80)
\]

where \(\sigma \simeq \sum_{j \in \mathbb{Z} \cup \{-1\}} \sigma_j\lambda^{-j}\), whose coefficients satisfy an infinite hierarchy of the recurrent relationships

\[
-\delta_{j,-2} + \partial^{-1}\sigma_{j} + v\sigma_j + (u^{-2}\sigma_j)_x + u^{-2}\sigma_{j-k}\sigma_k = 0 \quad (81)
\]

for \(j \in \mathbb{Z} \cup \{-2, -1\}\). In particular at \(j = -2\) we obtain \(\sigma_{-1} = u \in M_u\) and at \(j = -1\)

\[
\partial^{-1}u_t + uv - u^{-2}u_x + 2u^{-1}\sigma_0 = 0. \quad (82)
\]

To determine the conserved quantity \(\sigma_0 : J(\mathbb{R}; \mathbb{R}^2) \to \mathbb{R}\), we put, by definition,

\[
u_t = -(uv)_x + \left(u^{-1}p\right)_x - \left(u^{-1}\right)_{xx}, \quad (83)
\]

where have introduced an additional functional variable \(p \in M_p \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})\), allowing to determine from the recurrent hierarchy \((81)\) the next density \(\sigma_0 : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^2) \to \mathbb{R}\):

\[
\sigma_0 = -p/2. \quad (84)
\]

Herewith, to close the system of two evolution flows \((78)\) and \((83)\) we assume that the additional functional variable \(p \in M_p \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})\) satisfies the following closing evolution relationship:

\[
p_t := P[v, u, p] = 4\left(u^{-1}h_x\right)_x - (vp)_x - (pu^{-2})_x + (u^{-1}p^2/2)_x, \quad (85)
\]

where a functional variable \(h \in M_h \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})\) makes it possible to determine the trivial conserved density \(\sigma_1 = h_x\) on the extended functional manifold \(M_0 \times M_u \times M_p \times M_h \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^4)\). The latter gives rise to the next alleged conserved density

\[
\sigma_2 = \left(u^{-1}h_xp - u(u^{-2}h_x)_x - uvh_x + uh_t\right)/2, \quad (86)
\]

which can be compatibly retrieved if and only if \(\sigma_2 = \partial \eta_2/\partial x\) for some density mapping \(\eta_2 : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^4) \to \mathbb{R}\) for the deemed evolution flow

\[
h_t = H[v, u, p, h], \quad (87)
\]
generated by some smooth mapping \( H : \mathcal{I}(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^4) \to \mathbb{R} \). As a result of simple calculations one easily ensured that the density \( \sigma_2 = -f[u]_x/2 \) and

\[
H[v, u, p, h] = u^{-2}h_x p - (u^{-2}h_x)_x - vh_x + u^{-1}f[u]_x
\]

for some smooth mapping \( f : \mathcal{I}(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \to \mathbb{R} \). Moreover, based on the recurrent hierarchy (81), one obtains successively that the next density

\[
\sigma_3 = (uf[u] + uvf[u]_x + u(u^{-2}f[u]_x)_x - 2u^{-1}h^2_x)/4
\]

is not conserved for any smooth mapping \( f : \mathcal{I}(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \to \mathbb{R} \), that is \( \sigma_{3,t} \neq \partial \eta_3/\partial x \) on the functional manifold \( M_0 \times M_u \times M_p \times M_h \). Thus we have stated that the closed dynamical system

\[
\begin{align*}
\nu_t &= -uv_x + (u^{-2}v_x)_x \\
\phi_t &= -(uv)_x + (u^{-1}p)_x - (u^{-1}h^2_x)_x \\
\eta_t &= 4(u^{-1}h_x)_x - (vp)_x - (pu^{-2})_x + (u^{-1}p^2/2)_x \\
\lambda_t &= u^{-2}h^2_x - (u^{-2}h_x)_x - vh_x + u^{-1}f[u]_x
\end{align*}
\]

(90)

is not integrable and not Lax type linearizable on the extended functional manifold \( M_0 \times M_u \times M_p \times M_h \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^4) \). The latter simultaneously also means that the infinite recurrent hierarchy (81) is not compatible, that is equivalent to the fact that the combined dynamical system (90) is not of dark type, as it is not linearized and not completely integrable on the extended functional manifold \( M_0 \times M_u \times M_p \times M_h \). The obtained result we can formulate as the following theorem.

**Theorem 3.** The joint nonlinear dynamical system (90) presents an evolution equation of not dark type, it contains only two conservation laws and is not linearized and not integrable flow on the extended functional manifold \( M_0 \times M_u \times M_p \times M_h \).

3.3. Integrability Analysis: Dispersion Class Dynamical Dark Type Systems

3.3.1. A First Dispersion Class Dynamical System

We consider the dispersion class dynamical system

\[
\nu_t = -uv_x + v_{xxx} = K[v, u]
\]

(91)

of the Korteweg-de Vries type on a functional manifold \( M_0 \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \), parametrically depending on a functional variable \( u \in M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \). The determining Lax-Noether equation

\[
\phi_t - \phi_{xxx} + (u\phi)_x = 0
\]

(92)

on the complexified cotangent space \( T^*(M_0) \otimes \mathbb{C} \) allows the following asymptotic as \( \lambda \to \infty \) solution

\[
\phi = \exp(\lambda^3 t + \partial^{-1} \sigma),
\]

(93)

where \( \sigma \sim \sum_{j \in \mathbb{Z}_+ \cup \{-1\}} \sigma_j \lambda^{-j} \), whose coefficients satisfy the following recurrent relationships:

\[
\delta_{j,-3} + \partial^{-1} \sigma_{j,t} - \sigma_{j,xx} - 3\sigma_{j-k} \sigma_{k,x} - \sigma_{j-k} \sigma_{k-s} \sigma_{s} + u \sigma \delta_{j,0} + uw \sigma_j = 0
\]

(94)

for \( j \in \mathbb{Z}_+ \cup \{-3, -2, -1\} \). From (94) one easily obtains that

\[
\sigma_{-1} = 1, \sigma_0 = 0, \sigma_1 = \frac{u}{3}, \sigma_2 = 0
\]

(95)

and

\[
\partial^{-1} \sigma_1 - \sigma_{xx} - 9\sigma_3 = 0.
\]

(96)
Since the quantity $\gamma_3 := \int \sigma_3 dx$ should be a conservation law, we put the functional coefficient $\sigma_3 := p/9 \in M_p \subset C(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{R})$. Then we obtained a first supplementing dynamical system

$$u_t = u_{xxx} + p_x$$

(97)
on the functional manifold $M_u$. Following the above recurrent scheme at $j = 2$, one finds that

$$\sigma_4 = -\left( \frac{1}{18} u^2 + \frac{p}{9} \right).$$

(98)

The simplest way to close the hierarchy consists in making the density (98) trivial, putting by definition, that

$$p = -u^2/2.$$  

(99)

The latter gives rise to the next conservation law density:

$$\sigma_5 = \frac{1}{27} \partial_t p_t + \frac{1}{9} (uu_x)_x - \frac{1}{81} u^3 - \frac{1}{27} p u + \frac{2}{27} p_{xx} = \frac{1}{18} (u_x^2 + u^3/3),$$

(100)

generating a true conservation law $\gamma_5 := \int \sigma_5 dx$ if the following functional constraint

$$u_t = u_{xxx} - uu_x := P[u]$$

(101)

holds on the functional manifold $M_u$. Respectively on easily calculates the next densities:

$$\sigma_6 = -\frac{1}{9} uu_{ux}, \quad \sigma_8 = \frac{1}{9} (uu_x^2 - u_{xxx}),$$

$$\sigma_7 = \frac{1}{18} [(u_x^2)_x + u_{xx}^2 - 5/3 uu_x^2 - 5/36 u^4]...$$

and so on, that is the functional constraint (99) proves to be compatible for the whole infinite hierarchy of recurrent relationships (94). Thus, the resulting dynamical system

$$v_t = -uu_x + v_{xxx}$$

$$u_t = u_{xxx} - uu_x := Q[u, u]$$

(102)

is compatible and presents a closed nonlinear dynamical system on the extended functional manifold $M_v \times M_u \subset C(\mathbb{R}/\{2\pi \mathbb{Z}\}; \mathbb{R}^2)$. In addition, as it can be verified by means of the gradient-holonomic integrability scheme, the supplementing closed dynamical system (102) proves to be completely integrable on the functional manifold $M_v \times M_u$. The latter makes it possible to formulate subject to the combined dynamical system (102) the following theorem.

**Theorem 4.** The combined nonlinear dynamical system (102) presents a completely integrable dark type flow on the extended functional manifold $M_v \times M_u$, it is Lax type linearized and possesses an infinite number of suitably ordered conservation laws.

The second interesting case arises when the density (98) is not trivial. Then one obtains

$$\sigma_5 = \frac{1}{27} \partial_t p_t - \frac{1}{81} u^3 - \frac{1}{27} p u + \frac{1}{9} (uu_x)_x + \frac{2}{27} p_{xx},$$

(103)

which can be rewritten as

$$\frac{1}{27} \partial_t p_t - \frac{1}{81} u^3 - \frac{1}{27} p u = \frac{1}{27} g[u, p],$$

(104)

or, equivalently, as

$$p_t = g_x + u^2 u_x + (pu)_x$$

(105)
where an introduced mapping \( g : f(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}^2) \to \mathbb{R} \) should be determined from the constraint \( \sigma_6 = 0 \), that is
\[
\sigma_6 = \left( -\frac{1}{9} uu_{xx} - \frac{1}{18} u_x^2 - \frac{2}{27} up - \frac{1}{27} p_{xx} - \frac{2}{27} g \right)_x = 0,
\]
reducing to the expression
\[
g = -\frac{3}{2} uu_{xx} - \frac{3}{4} u_x^2 - \frac{1}{2} p_{xx} - pu.
\]
Based now on expressions (103), (106) and (107) we can calculate the next nontrivial conservation density
\[
\sigma_7 = -\frac{1}{36} uu_{xxxx} - \frac{5}{36} u_x uu_{xxx} - \frac{5}{36} u_x^2 u_{xx} + \frac{1}{18} u_x^2 u_{xx} + \frac{1}{12} uu_x^2 + \frac{5}{36} p_x u_x - \frac{1}{18} u_x p_{xx} - \frac{1}{324} a^4 - \frac{1}{34} p^2
\]
and similarly all other ones, which prove to be compatible with the recurrent hierarchy (94).
Summarizing calculations above, we can construct the next closed dynamical system
\[
\begin{align*}
u_t &= (p + u_{xx})_x \\
p_t &= (-\frac{3}{2} uu_{xx} - \frac{3}{4} u_x^2 + \frac{1}{2} u^3 - \frac{1}{2} p_{xx})_x = Q[u, p]
\end{align*}
\]
on the functional manifold \( M_u \times M_p \), which also proves to be completely integrable, that we can easily check by means of the gradient-holonomic scheme, applied to the vector field (109). Moreover, we can combine the dynamical system (91) with that of (109) and obtain the joint completely integrable dynamical system
\[
\begin{align*}
v_t &= -uv_x + v_{xxx} \\
u_t &= (p + u_{xx})_x \\
p_t &= (\frac{3}{2} uu_{xx} - \frac{3}{4} u_x^2 + \frac{1}{2} u^3 - \frac{1}{2} p_{xx})_x = K[v, u, p]
\end{align*}
\]
on the extended functional manifold \( M_v \times M_u \times M_p \).
The obtained above new integrability results subject to the combined dynamical system (110) can be formulated as the following theorem.

**Theorem 5.** The combined nonlinear dynamical system (110) presents a completely integrable dark type flow on the extended functional manifold \( M_v \times M_u \times M_p \), it is Lax type linearized and possesses an infinite number of suitably ordered conservation laws.

3.3.2. A Second Diffusion-Dispersion Class Dynamical System

This diffusion-dispersion class dynamical system
\[
\begin{align*}
v_t &= v_{xx} - (v^2 u)_x + 2a(u + v)_x := K[v, u] \\
u_t &= \frac{1}{2} F[v, u, p],
\end{align*}
\]
is defined on a functional manifold \( M_v \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}) \), where \( u \in M_u \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}) \) is a functional parameter and \( a \in \mathbb{R} \) is arbitrary. To determine a kind of evolution
\[
u_t + \varphi_{xx} + 2(\nu v - a)\varphi_x = 0,
\]
on the functional manifold \( M_v \), ensuring the existence for (111) conservation laws within the gradient-holonomic integrability scheme, one needs to find an asymptotic as \( \lambda \to \infty \) solution \( \varphi \in T^* (M_v) \otimes \mathbb{C} \) to the corresponding Lax-Noether equation.
where
\[ \varphi = \exp(-\lambda^2 t + \partial^{-1} \sigma), \quad \sigma \sim \sum_{j \in \mathbb{Z}_+ \cup \{-1\}} \sigma_j \lambda^{-j}. \] (114)

Substitution of (114) into (113) gives rise to the following infinite hierarchy of recurrent equations:
\[ \partial^{-1} \sigma_{j,t} + \sum_{k \in \mathbb{Z}_+ \cup \{-1\}} \sigma_{j-k} \sigma_k + \sigma_{j,x} + 2(\nu v - \alpha) \delta_{j,-1} + 2uv \sigma_j = 0 \] (115)
for \( j \in \mathbb{Z}_+ \cup \{-1\} \), whose first two coefficients are
\[ \sigma_{-1} = 1, \quad \sigma_0 = \alpha - uv, \quad \sigma_1 = [\partial^{-1}(uv)]_t + (uv)^2 - \alpha^2 + (uv)_x]/2. \] (116)

Now we will take into account that the quantity \( \gamma_1 := \int \sigma_1 dx \) should be a conservation law for the combined dynamical system
\[ \begin{align*}
\nu_t &= \nu_{xx} - (\nu^2 u)_x \\
u_t &= F[v, u]
\end{align*} \] (117)
The latter condition is equivalent to the following functional constraint:
\[ (uv)_t = 2\alpha uv_x + 2\alpha uu_x + \nu uv_x - 2\nu^2 v_x - uuv_x + vF[v, u] = \delta \rho_0[v, u] / \partial x \] (118)
for some smooth density \( \rho_0 : \{\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2 \} \rightarrow \mathbb{R} \). Having applied to the left-hand side of (118) the gradient operators with respect to the functional variables \( (u, v) \in M_u \times M_v \), one easily obtain the following system of differential-functional relationships:
\[ F_u^* \cdot v = -2\alpha v_x - v_{xx} + 2\nu vv_x, \] \[ F_v^* \cdot v = 2\alpha u_x - u_{xx} - 2\nu uu_x - A[v, u], \] (119)
where \( A : \{\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2 \} \rightarrow \mathbb{R} \) is some smooth jet-mapping, \( F_u^* : T(M_u) \rightarrow T(M_u) \) and \( F_v^* : T(M_v) \rightarrow T(M_u) \) are the corresponding Frechet derivatives of the smooth mapping \( F : M_u \times M_v \rightarrow T(M_u) \) and the star “\(^*\)” denotes their corresponding adjoint mappings.

From the first relationship of (119) one easily derives that the linear operator
\[ F_u^* v = -\partial^2 + c_1 uv \partial + c_2 uv_x + A_u^* \partial v^{-1}, \] (120)
where \( c_1, c_2 \in \mathbb{R} \) are constant parameters, satisfying the constraint \( c_1 + c_2 = 2 \), or, equivalently,
\[ F_u^* = 2\alpha \partial - \partial^2 - c_1 \partial \circ uv + c_2 \nu \partial v_x - v^{-1} \partial \circ A_u^*, \] (121)
From the expressions (121) one easily ensues that
\[ F[v, u] = -u_{xx} - \frac{1}{2} c_1 (u^2 v)_x + \frac{1}{2} c_2 u^2 v_x + 2\alpha u_x - v^{-1} A_x + H \] (122)
satisfies the second relationship of (119) identically, if the following functional-operator constraint
\[ H_u^* v + H = 0, \] (123)
holds on the functional manifold \( M_v \). Having assumed, for simplicity, that \( H : \{\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R} \} \rightarrow \mathbb{R} \), we obtain right away that
\[ H(v, v_x) = k(v)v_x \] (124)
for arbitrary smooth mapping \( k : J^0(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \in \mathbb{R} \). Thus, the resulting dynamical system looks as follows

\[
\begin{align*}
    u_t &= -u_{xx} - \frac{1}{2}c_1 (u^2)_x + \frac{1}{2}c_2 u^2 v_x + 2\alpha u_x - v^{-1} A_x + v^{-1} k'(v) v_x, \\
    v_t &= v_{xx} - (v^2 u)_x + 2\alpha (u + v)_x,
\end{align*}
\]

(125)

depending on the up to now not determined smooth mapping \( A : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^2) \rightarrow \mathbb{R} \) and which can be formally retrieved from the necessary condition that the next functional \( \gamma_1 := \int \sigma_1 [u, v] dx \) generates a conserved quantity, that is \( \partial \gamma_1 / \partial t = 0 \) along the evolution flow (125) on the functional manifold \( M_0 \times M_\alpha \). As this conserved quantity depends a priori on the unknown smooth mapping \( A : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^2) \rightarrow \mathbb{R} \), this problem can be strongly simplified, if to extend the present phase space \( M_0 \times M_\alpha \) of the evolution flow (125) to the phase space \( M_0 \times M_\alpha \times M_p \), having introduced an additional functional variable \( p := A + \int_0^x k(v) dv \in M_p \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \). Then, it is enough to find such an evolution flow

\[
p_t := p[v, u, p]
\]

(126)
on the functional manifold \( M_p \), for which the functional \( \gamma_1 := \int \sigma_1 [u, v] dx \) persists to be conserved on the extended phase space \( M_0 \times M_\alpha \times M_p \) already with respect to the combined evolution flow

\[
\begin{align*}
    u_t &= -u_{xx} - \frac{1}{2}c_1 (u^2)_x + \frac{1}{2}c_2 u^2 v_x + 2\alpha u_x - v^{-1} p_x, \\
    v_t &= v_{xx} - (v^2 u)_x + 2\alpha (u + v)_x, \quad p_t = p[v, u, p],
\end{align*}
\]

(127)
on the functional manifold \( M_0 \times M_\alpha \times M_p \) under the condition that \( c_j \in \mathbb{R}, j = \overline{1,2} \) and \( c_1 + c_2 = 2 \). Taking into account that the density

\[
\sigma_1 = \frac{\alpha^2}{2} + \frac{c_1\alpha}{2} u^2 + uv_x - \frac{c_1}{2} u^2 v^2 - \frac{1}{2} u^2 v^2 - \frac{1}{2} p_x
\]

(128)
is conserved if and only if its temporal derivative satisfies the relationship

\[
\partial \sigma_1 / \partial t = \partial \sigma_1 / \partial x
\]

(129)
for some jet-density mapping \( \rho_1 : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}^2) \rightarrow \mathbb{R} \), it is enough to apply to the left-hand side of (129) the gradient operator and to check that it vanishes on the extended manifold \( M_0 \times M_\alpha \times M_p \):

\[
\sigma_1''(1) = \left( \begin{array}{ccc}
-\frac{p''(1)}{2} + 2\alpha \frac{p''(1)}{v^2} + \frac{p''(1)}{v} + u p_x \\
-\frac{p''(1)}{2} + \sigma(p) - 2\alpha \frac{p''(1)}{v} \\
-\frac{p''(1)}{2} + 2\alpha (\frac{u}{v})_x + (\frac{u}{v})_x - (u v)_x
\end{array} \right) = 0.
\]

(130)
System (130) proves to be compatible if

\[
P = -4\alpha uv^{-1} p_x + 2p_{xx} \ln v + 2uv p_x,
\]

(131)
giving rise to the following closed nonlinear dark type integrable dynamical system

\[
\begin{align*}
    u_t &= -u_{xx} - \frac{1}{2}c_1 (u^2)_x + \frac{1}{2}c_2 u^2 v_x + 2\alpha u_x - v^{-1} p_x, \\
    v_t &= v_{xx} - (v^2 u)_x + 2\alpha (u + v)_x, \\
    p_t &= -4\alpha uv^{-1} p_x + 2p_{xx} \ln v + 2uv p_x
\end{align*}
\]

(132)
on the extended manifold \( M_0 \times M_\alpha \times M_p \), if \( c_j \in \mathbb{R}, j = \overline{1,2} \) and \( c_1 + c_2 = 2 \).
As a simplest inference from the flow (132) one easily observes that the functional reduction on the submanifold \{ (v, u, p) ∈ M_v × M_u × M_p : p = 0 ∈ M_p \} is compatible, thus determining a nonlinear dynamical system on the reduced phase \( M_v × M_u \). The obtained above new integrability results can be now formulated as the following theorem.

**Theorem 6.** The combined nonlinear dynamical system (132) presents a completely integrable dark type flow on the extended functional manifold \( M_v × M_u × M_p \), it is Lax type linearized and possesses for any \( c_j ∈ \mathbb{R}, j = 1, 2, c_1 + c_2 = 2 \), an infinite number of suitably ordered conservation laws. Moreover, its reduction on the submanifold \{ (v, u, p) ∈ M_v × M_u × M_p : p = 0 ∈ M_p \} is invariant, determining a Lax type integrable nonlinear dynamical system on the reduced phase \( M_v × M_u \).

We can observe here that the constructed above nonlinear dynamical system (132) is closely related [38] with the so called completely integrable modified nonlinear Schrödinger equation, having applications in plasma physics.

4. Conclusions

In the work we described theoretical backgrounds of the gradient-holonomic integrability testing algorithm and applied it to study hidden symmetries of a series of nonlinear dark type dynamical systems of diffusion and diffusion-dispersive kinds on spatially one-dimensional functional manifolds. Based on the devised algorithm, we were able to prove that these dark type dynamical systems are completely integrable and possess hierarchies of conserved quantities, which generate either finite or infinite number of suitably ordered conservation laws. This class of nonlinear dark type dynamical systems possesses many interesting dynamical properties and can be useful for modeling different diffusion and dispersion type processes, having interesting applications in modern physics, nonlinear optics, mechanics, hydrodynamics and biology sciences.

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