MOMENTS OF THE 2D SHE AT CRITICALITY

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Abstract. We study the stochastic heat equation in two spatial dimensions with a multiplicative white noise, as the limit of the equation driven by a noise that is mollified in space and white in time. As the mollification radius \( \varepsilon \to 0 \), we tune the coupling constant near the critical point, and show that the single time correlation functions converge to a limit written in terms of an explicit non-trivial semigroup. Our approach consists of two steps. First we show the convergence of the resolvent of the (tuned) two-dimensional delta Bose gas, by adapting the framework of [DR04] to our setup of spatial mollification. Then we match this to the Laplace transform of our semigroup.

1. Introduction and main result

In this paper, we study the Stochastic Heat Equation (SHE), which informally reads

\[ \partial_t Z = \frac{1}{2} \nabla^2 Z + \sqrt{\beta} \xi Z, \quad Z = Z(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \]

where \( \nabla^2 \) denotes the Laplacian, \( d \in \mathbb{Z}_+ \) denotes the spatial dimension, \( \xi \) denotes the spacetime white noise, and \( \beta > 0 \) is a tunable parameter. In broad terms, the SHE arises from a host of physical phenomena including the particle density of diffusion in a random environment, the partition function for a directed polymer in a random environment, and, through the inverse Hopf–Cole transformation, the height function of a random growth surface; the two-dimensional Kardar–Parisi–Zhang (KPZ) equation. We refer to [Cor12, Kho14, Com17] and the references therein.

When \( d = 1 \), the SHE enjoys a well-developed solution theory: For any \( Z(0, x) = Z_{ic}(x) \) that is bounded and continuous, and for each \( \beta > 0 \), the SHE (in \( d = 1 \)) admits a unique \( \mathcal{C}([0, \infty) \times \mathbb{R}) \)-valued mild solution, where \( \mathcal{C} \) denotes continuous functions, c.f., [Wal86, Kho14]. Such a solution theory breaks down in \( d \geq 2 \), due to the deteriorating regularity of the spacetime white noise \( \xi \), as the dimension \( d \) increases. In the language of stochastic PDE [Hai14, GIP15], \( d = 2 \) corresponds to the critical, and \( d = 3, 4, \ldots \) the supercritical regimes.

Here we focus on the critical dimension \( d = 2 \). To set up the problem, fix a mollifier \( \varphi \in C_c^\infty(\mathbb{R}^2) \), where \( C_c^\infty \) denotes smooth functions with compact support, with \( \varphi \geq 0 \) and \( \int \varphi \, dx = 1 \), and mollify the noise as

\[ \xi_\varepsilon(t, x) := \int \varphi_\varepsilon(x - y) \xi(t, y) \, dy, \quad \varphi_\varepsilon(x) := \frac{1}{\varepsilon^2} \varphi(\frac{x}{\varepsilon}). \]

Consider the corresponding SHE driven by \( \xi_\varepsilon \),

\[ \partial_t Z_\varepsilon = \frac{1}{2} \nabla^2 Z_\varepsilon + \sqrt{\beta_\varepsilon} \xi_\varepsilon Z_\varepsilon, \quad Z_\varepsilon = Z_\varepsilon(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{1.1} \]

with a parameter \( \beta_\varepsilon > 0 \) that has to be finely tuned as \( \varepsilon \to 0 \). The noise \( \xi_\varepsilon \) is white in time, and we interpret the product between \( \xi_\varepsilon \) and \( Z_\varepsilon \) in the Itô sense. Let \( p(t, x) := \frac{1}{2\pi t} \exp\left(-\frac{|x|^2}{2t}\right), \quad x \in \mathbb{R}^d \), denote the standard heat kernel in two dimensions. For fixed \( Z(0, x) = Z_{ic} \in \mathcal{L}^2(\mathbb{R}^2) \) and \( \varepsilon > 0 \), it is standard, though tedious, to show that the unique \( \mathcal{C}((0, \infty) \times \mathbb{R}^2) \)-valued mild solution of (1.1) is given by the chaos expansion

\[ Z_\varepsilon(t, x) = \int_{\mathbb{R}^2} p(t, x - x') Z_{ic}(x') \, dx' + \sum_{k=1}^{\infty} I_{\varepsilon,k}(t, x), \tag{1.2} \]

\[ I_{\varepsilon,k}(t, x) := \int \left( \prod_{s=1}^{k} p(\tau_{s+1} - \tau_s, x^{(s+1)} - x^{(s)}) \sqrt{\beta_\varepsilon} \xi_\varepsilon(\tau_s, x^{(s)}) \, d\tau_s \, dx^{(s)} \right) p(\tau_1, x^{(1)} - x') Z_{ic}(x') \, dx', \tag{1.3} \]

where the integral goes over all \( 0 < \tau_1 < \ldots < \tau_k < t \) and \( x', x^{(1)}, \ldots, x^{(k)} \in \mathbb{R}^2 \), with the convention \( x^{(k+1)} := x \) and \( \tau_{k+1} := t \).

From the expression (1.3) of \( I_{\varepsilon,k} \), it is straightforward to check that, for fixed \( \beta_\varepsilon = \beta > 0 \) as \( \varepsilon \to 0 \), the variance \( \text{Var}[I_{\varepsilon,k}] \) diverges, confirming the breakdown of the standard theory in \( d = 2 \). We hence seek to
tune \( \beta_\varepsilon \to 0 \) in a way so that a meaningful limit of \( Z_\varepsilon \) can be observed. A close examination shows that the divergence of \( \text{Var}[I_{\varepsilon,k}] \) originates from the singularity of \( p(t,0) = (2\pi t)^{-1} \) near \( t = 0 \), so it is natural to propose \( \beta_\varepsilon = \frac{\beta_0}{|\log \varepsilon|} \to 0 \), \( \beta_0 > 0 \). The \( \varepsilon \to 0 \) behavior of \( Z_\varepsilon \) for small values of \( \beta_0 \) has attracted much attention recently. For fixed \( \beta_0 \in (0,2\pi) \), [CSZ17] showed that the fluctuations of \( Z_\varepsilon(t,\cdot) \) converge (as a random measure) to a Gaussian field, more precisely, the solution of the two-dimensional Edwards–Wilkinson (EW) equation. For \( \beta_0 = \beta_{0,\varepsilon} \to 0 \), [Fen15] showed that the corresponding polymer measure exhibits diffusive behaviors. The logarithm \( h_\varepsilon(t,x) := \beta_\varepsilon^{-1/2} \log Z_\varepsilon(t,x) \) is also a quantity of interest; it describes the free energy of random polymers and the height function in surface growth phenomena which solves the two dimensional KPZ equation. The tightness of the centered height function was obtained in [CD18] for small enough \( \beta_0 \). It was then shown in [CSZ18b] that the centered height function converges to the EW equation for all \( \beta_0 \in (0,2\pi) \), and in [Gu18] for small enough \( \beta_0 \), i.e., the limit is Gaussian.

However, the \( \varepsilon \to 0 \) behavior of \( Z_\varepsilon \) goes through a transition at \( \beta_0 = 2\pi \). Recall that by Itô calculus, the \( n \)-th order correlation function of the solution of the mollified SHE (1.1) at a fixed time,

\[
u_\varepsilon(t,x_1,\ldots,x_n) := \mathbb{E} \left[ \prod_{i=1}^n Z_\varepsilon(t,x_i) \right],
\]

satisfies the \( n \) particle (approximate) delta Bose gas

\[
\partial_t \nu_\varepsilon(t,x_1,\ldots,x_n) = - (\mathcal{H}_\varepsilon \nu_\varepsilon)(t,x_1,\ldots,x_n), \quad x_i \in \mathbb{R}^2, \quad \nu_\varepsilon(0) = Z_{\varepsilon}^0 \otimes \mathbb{C},
\]

where \( \mathcal{H}_\varepsilon \) is the Hamiltonian

\[
\mathcal{H}_\varepsilon := -\frac{1}{2} \sum_{i=1}^n \nabla_i^2 - \beta_\varepsilon \sum_{1 \leq i < j \leq n} \delta_\varepsilon(x_i - x_j), \quad \delta_\varepsilon(x) := \varepsilon^{-2} \Phi(\varepsilon^{-2} x), \quad \Phi(x) := \int_{\mathbb{R}^2} \varphi(x+y) \varphi(y) \, dy.
\]

It can be shown (e.g., from [AGHKH88, Equation (I.5.56)]) that, for \( n = 2 \), the Hamiltonian \( \mathcal{H}_\varepsilon \) has a vanishing/diverging principal eigenvalue as \( \varepsilon \to 0 \), respectively for \( \beta_0 < 2\pi \) and \( \beta_0 > 2\pi \). This phenomenon in turn suggests a transition in behaviors of \( Z_\varepsilon \) at \( \beta_0 = 2\pi \). This transition is also demonstrated at the level of pointwise limit (in distribution) of \( Z_\varepsilon(t,x) \) as \( \varepsilon \to 0 \) by [CSZ17].

The preceding observations point to an intriguing question of understanding the behavior of \( Z_\varepsilon \) and \( u_\varepsilon \) at this critical value \( \beta_0 = 2\pi \). For the case of two particles (\( n = 2 \)), by separating the center-of-mass and the relative motions, the delta Bose gas can be reduced to a system of one particle with a delta potential at the origin. Based on this reduction and the analysis of the one-particle system in [AGHKH88, Chapter I.5], [BC98] gave an explicit \( \varepsilon \to 0 \) limit of the second order correlation functions (tested against \( L^2 \) functions). Further, given the radial symmetry of the delta potential, the one particle system (in \( d = 2 \)) can be reduced to an one-dimensional problem along the radial direction. Despite its seeming simplicity, this one-dimensional problem already requires sophisticated analysis. Although the final answer is non-trivial, it does not rule out a lognormal limit. For \( n > 2 \), these reductions no longer exist, and to obtain information on the correlation functions stands as a challenging problem; the only existing results are on third order (\( n = 3 \)): [Fen15] showed that for \( Z_\varepsilon \) the limiting ratio of the cube root of the third pointwise moment to the square root of the second moment is not what one would expect from a lognormal distribution, indicating (but not proving) non-trivial fluctuations. Using techniques developed in [CSZ18a] to control the chaos series, [CSZ18c] obtained the convergence of the third order correlations of \( Z_\varepsilon \) to a limit given in terms of a sum of integrals.

In this paper, we proceed through a different, functional analytic route, and obtain a unified description of the \( \varepsilon \to 0 \) limit of all correlation functions of \( Z_\varepsilon \). We now prepare some notation for stating our main result. Hereafter throughout the paper, we set

\[
\beta_\varepsilon := \frac{2\pi}{\log \varepsilon} + \frac{2\pi \beta_{\text{fine}}}{|\log \varepsilon|^2},
\]

where \( \beta_{\text{fine}} \in \mathbb{R} \) is a fixed, fine-tuning constant. This fine-tuning constant does not complicate our analysis, though the limiting expressions do depend on \( \beta_{\text{fine}} \). Let \( \gamma_{\text{EM}} = 0.577\ldots \) denote the Euler–Mascheroni constant, and, with \( \Phi \) as in (1.1) and (1.6), set

\[
\beta_* := 2 (2 \log 2 + \beta_{\text{fine}} - \beta_{\Phi} - \gamma_{\text{EM}}), \quad \beta_{\Phi} := \int_{\mathbb{R}^4} \Phi(x_1) \log |x_1 - x'_1| \Phi(x'_1) \, dx_1 \, dx'_1,
\]

where
and
\[
j(t, \beta_\alpha) := \int_0^\infty t^{\alpha-1} e^{\beta_\alpha \alpha} \frac{1}{\Gamma(\alpha)} \, d\alpha. \tag{1.9}
\]

We will often work with vectors \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), where \(x_i \in \mathbb{R}\), and similarly \(y = (y_2, \ldots, y_n) \in \mathbb{R}^{n-2}\), \(y_i \in \mathbb{R}\). We say \(x_i\) is the \(i\)-th component of \(x\). For \(1 \leq i < j \leq n\), consider a linear transformation \(S_{ij} : \mathbb{R}^{n-2} \to \mathbb{R}^n\),

\[
S_{ij}(y_2, \ldots, y_n) := (y_3, \ldots, y_2, \ldots, y_2, \ldots, y_n),
\]

and the induced operator \(S_{ij} : \mathcal{L}^2(\mathbb{R}^n) \to \mathcal{L}^2(\mathbb{R}^{n-2})\),

\[
(S_{ij} u)(y) := u(S_{ij} y). \tag{1.11}
\]

Let \(\mathcal{H}^\alpha(\mathbb{R}^n)\) denote the Sobolev space of degree \(\alpha \in \mathbb{R}\). As we will show in Lemma 4.1, (1.11) defines an unbounded operator \(\mathcal{L}^2(\mathbb{R}^n) \to \mathcal{L}^2(\mathbb{R}^{n-2})\), and there exists an adjoint \(S^*_{ij} : \mathcal{L}^2(\mathbb{R}^{n-2}) \to \mathcal{L}^2(\mathbb{R}^n)\). Adopting the shorthand notation \(\nabla_i^2 := \nabla_{x_i}^2\), we let

\[
\mathcal{P}_t := e^{\frac{t}{4} \nabla_i^2} \mathcal{P}_t, \tag{1.12}
\]

denote the heat semigroup on \(\mathcal{L}^2(\mathbb{R}^n)\); its integral kernel will be denoted \(P(t, x) := \prod_{i=1}^n \frac{1}{2\pi t} \exp\left(-\frac{|x_i|^2}{2t}\right)\). Define the operator \(\mathcal{P}^H_t : \mathcal{L}^2(\mathbb{R}^{n-2}) \to \mathcal{L}^2(\mathbb{R}^{n-2})\),

\[
\mathcal{P}^H_t := \int(t, \beta_\alpha) e^{\frac{t}{4} \nabla_i^2 + \frac{1}{2} \sum_{i=1}^n \nabla_i^2}. \tag{1.13}
\]

As we will show in Lemma 8.5, \(\mathcal{P}^H_t\) is a bounded operator on \(\mathcal{L}^2(\mathbb{R}^{n-2})\).

We need to prepare some index sets. Hereafter we write \(i < j\) for a pair of ordered indices in \(\{1, \ldots, n\}\), i.e., \(i < j \in \{1, \ldots, n\}\). For \(m \in \mathbb{Z}_+\), we consider \((i, j) = ((i_k, j_k))_{k=1}^m\) such that \((i_k < j_k) \neq (i_{k+1} < j_{k+1})\), i.e., \(m\) ordered pairs with consecutive pairs non-repeating. Let

\[
\text{Dgm}(n, m) := \{(i, j) : (1, \ldots, n)^m : (i_k < j_k) \neq (i_{k+1} < j_{k+1})\}, \tag{1.17}
\]

\[
\text{Dgm}(n) := \bigcup_{m=1}^\infty \text{Dgm}(n, m) \tag{1.18}
\]
denote the sets of all such indices, with the convention that \(\text{Dgm}(1, m) := \emptyset, m \in \mathbb{Z}_+.\) The notation \(\text{Dgm}(n)\) refers to ‘diagrams’, as will be explained in Section 2. Let

\[
\Sigma_m(t) := \sigma = (\tau_0, \tau_1, \ldots, \tau_m) \in \mathbb{R}_{++}^{m+1} : \tau_0 = \tau_1 + \ldots + \tau_m = t, \tag{1.19}
\]

so that for a fixed \(t \in \mathbb{R}_+\), the integral \(\int_{\Sigma_m(t)} \cdot \, d\varphi\) denotes a \((2m + 1)\)-fold convolution over the set \(\Sigma_m(t)\). For a bounded operator \(Q : \mathcal{H} \to \mathcal{H}’\) between Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}’\), let \(\|Q\|_{\text{op}} := \sup_{\|u\|_{\mathcal{H}} = 1} \|Qu\|_{\mathcal{H}’}\), denote the inherited operator norm. We use the subscript ‘op’ (standing for ‘operator’) to distinguish the operator norm from the vector norm, and omit the dependence on \(\mathcal{H}\) and \(\mathcal{H}’\), since the spaces will always be specified along with a given operator. The \(\mathcal{L}^2\) spaces in this paper are over \(\mathbb{C}\), and we write \(\langle f, g \rangle := \int_{\mathbb{R}_n} f(x) g(x) \, dx\) for the inner product. (Note our convention of taking complex conjugate in the first function.) Throughout this paper we use \(C(a, b, \ldots)\) to denote a generic positive finite constant that may change from line to line, but depends only on the designated variables \(a, b, \ldots\). We view the mollifier \(\varphi\) as fixed throughout this paper, so the dependence on \(\varphi\) will not be specified.

We can now state our main result.

**Theorem 1.1.**

(a) The operators

\[
\mathcal{P}_t + \mathcal{D}^{\text{Dgm}(n)}_t, \quad \mathcal{D}^{\text{Dgm}(n)}_t := \sum_{(i, j) \in \text{Dgm}(n)} \mathcal{D}^{(i, j)}_t, \quad t \geq 0, \tag{1.20}
\]

define a norm-continuous semigroup on \(\mathcal{L}^2(\mathbb{R}^n)\), where, for \(\mathcal{D}^{(i, j)}_t = (i_k, j_k))_{k=1}^m\),

\[
\mathcal{D}^{(i, j)}_t = \int_{\Sigma_m(t)} \mathcal{P}_{\tau_0} S_{i_1j_1}^* \left(4\pi \mathcal{P}^H_{\tau_{1/2}}\right) \left(\prod_{k=1}^{m-1} S_{i_kj_k} \mathcal{P}_{\tau_k} S_{i_{k+1}j_{k+1}}^* \left(4\pi \mathcal{P}^H_{\tau_{k+1/2}}\right)\right) S_{i_mj_m} \mathcal{P}_{\tau_m} \, d\varphi. \tag{1.21}
\]
The sum in (1.16) converges absolutely in operator norm, uniformly in \( t \) over compact subsets of \([0, \infty)\).

(b) Start the mollified SHE (1.1) from \( Z_\varepsilon(0, \cdot) = Z_{\varepsilon, \text{ic}}(\cdot) \in \mathcal{L}^2(\mathbb{R}^2)\). For any \( f(x) = f(x_1, \ldots, x_n) \in \mathcal{L}^2(\mathbb{R}^n) \), \( n \in \mathbb{Z}_+ \), we have

\[
\mathbb{E}[(f, Z_{\varepsilon, t}^{\otimes n})] := \mathbb{E} \left[ \int_{\mathbb{R}^{2n}} f(x) \prod_{i=1}^{n} Z_{\varepsilon, t}(x_i, x_i) \, dx \right] \rightarrow \left\langle f, (\mathcal{P}_t + \mathcal{D}_{\varepsilon, t}^{\text{Dgm}(n)}) Z_{\varepsilon, \text{ic}}^{\otimes n} \right\rangle \quad \text{as} \; \varepsilon \to 0, \tag{1.18}
\]

uniformly in \( t \) over compact subsets of \([0, \infty)\).

Remark 1.2. Our result does not apply to the flat initial condition \( Z_{\varepsilon, \text{ic}}(x) \equiv 1 \). We conjecture that Theorem 1.1 extends to such initial data, and leave this to future work.

Theorem 1.1 gives a complete characterization of the \( \varepsilon \to 0 \) limit of fixed time, correlation functions of the SHE with an \( \mathcal{L}^2 \) initial condition. We will show in Section 2 that \( \mathcal{D}_{(i,j)}^{(n)} \) permits an explicit integral kernel, for each \((i,j) \in \text{Dgm}(n)\). Hence the limiting correlation functions (i.e., r.h.s. of (1.18)) can be expressed as a sum of integrals. From this expression, we check (in Remark 2.1) that for \( n = 2 \) our result matches that of [BC98], and for \( n = 3 \), we derive (in Proposition 2.2) an analogous expression of [CSZ18b, Equations (1.24)–(1.26)].

A question of interest arises as to whether one can uniquely characterize the limiting process of \( Z_\varepsilon \). This does not follow directly from correlation functions, or moments, since we expect a very fast moment growth in \( n \) (see Remark 1.6). Still, as a simple corollary of Theorem 1.1, we are able to infer that every limit point of \( Z_\varepsilon \) must have correlation functions given by the r.h.s. of (1.18):

Corollary 1.3. Let \( Z_{\varepsilon, \text{ic}} \) and \( Z_{\varepsilon, t}(x) \) be as in Theorem 1.1, and, for each fixed \( t \), view \( Z_{\varepsilon, t}(x) \equiv \mu_{\varepsilon, t}(dx) \) as a random measure. Then, for any fixed \( t \in \mathbb{R}_+ \), the law of \( \{\mu_{\varepsilon, t}(dx)\}_{\varepsilon \in (0,1)} \) is tight in the vague topology, and, for any limit point \( \mu_{*, t}(dx) \) of \( \{\mu_{\varepsilon, t}(dx)\}_{\varepsilon \in (0,1)} \), and for any compactly supported, continuous \( f_1, \ldots, f_n \in \mathcal{C}^\infty(\mathbb{R}^2) \), \( n \in \mathbb{Z}_+ \),

\[
\mathbb{E} \left[ \prod_{i=1}^{n} \int_{\mathbb{R}^2} f_i(x_i) \mu_{*, t}(dx_i) \right] = \left\langle f_1 \otimes \cdots \otimes f_n, (\mathcal{P}_t + \mathcal{D}_{t}^{\text{Dgm}(n)}) Z_{\varepsilon, \text{ic}}^{\otimes n} \right\rangle. \tag{1.19}
\]

Furthermore, if \( Z_{\varepsilon, \text{ic}}(x), f(x) \geq 0 \) are nonnegative and not identically zero, then

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^2} f(x) \mu_{*, t}(dx) - \mathbb{E} \left[ \int_{\mathbb{R}^2} f(x) \mu_{*, t}(dx) \right] \right)^3 \right] > 0. \tag{1.20}
\]

Due to the critical nature of our problem (\( \beta_0 = 2\pi \) and \( d = 2 \)), the moments go through a non-trivial transition as \( \varepsilon \to 0 \). To see this, in (1.2), use the orthogonality \( \mathbb{E}[I_{\varepsilon,k}(t,x_1)I_{\varepsilon,k'}(t,x_2)] = 0 \), \( k \neq k' \), to express the second \((n = 2)\) moment as

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^2} Z_{\varepsilon, t}(x) f(x) \, dx \right)^2 \right] = \int_{\mathbb{R}^4} \sum_{i=1}^{2} p(t, x_1 - x'_i) f(x_i) Z_{\varepsilon, \text{ic}}(x'_i) \, dx_i + \sum_{k=1}^{\infty} \int_{\mathbb{R}^4} \mathbb{E} \left[ \prod_{i=1}^{2} I_{\varepsilon,k}(t,x_i) f(x_i) \right] \, dx_1 \, dx_2.
\]

As seen in [CSZ18b], the major contribution of the sum spans across a divergent number of terms — across all \( k \)'s of order \( |\log \varepsilon| \to \infty \). We are probing a regime where the limiting process ‘escapes’ to indefinitely high order chaos as \( \varepsilon \to 0 \), reminiscent of the large time behavior of the SHE/KPZ equation in \( d = 1 \).

Because of this, obtaining the \( \varepsilon \to 0 \) limit from chaos expansion requires elaborate and delicate analysis. In fact, just to obtain an \( \varepsilon \)-independent bound (for fixed \( Z_{\varepsilon, \text{ic}} \) and test functions \( f_i \)'s) from the chaos expansion is a challenging task. Such analysis is carried out for \( n = 2, 3 \) in [CSZ18b] (in a discrete setting and in the current continuum setting, both with \( Z_{\varepsilon, \text{ic}} \equiv 1 \)). Here, we progress through a different route. From (1.4), (1.5), and (1.6) obtaining the limit of the correlation functions is equivalent to obtaining the limit of the semigroup \( e^{-tH_{\varepsilon}} \), which reduces to the study of \( H_{\varepsilon} \) itself, or its resolvent.

The delta Bose gas enjoys a long history of study, motivated in part by phenomena such as unbounded ground-state energy and infinite discrete spectrum observed in \( d = 3 \). We do not survey the literature here, and refer to the references in [AGHK88]. Of most relevance to this paper is the work [DR04], which studied \( d = 2 \) with a momentum cutoff, and established the convergence of the resolvent of the Hamiltonian to an explicit limit [DR04, Equation (90)] (also (1.24)). Here, we follow the framework of [DR04], but instead of the momentum cutoff, we work with the space-mollification scheme as in (1.6), in order to connect the delta Bose gas to the SHE.
where \( p \) with domain \( \text{Dom}(\mathcal{J}_z) \).

**Proposition 1.4** (Limiting resolvent). There exists \( C < \infty \) such that, for \( z \in \mathbb{C} \) with \( \text{Re}(z) < -e^{Cn^2 + \beta} \),

(a) the following defines a bounded operator on \( \mathcal{L}^2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}^2(\mathbb{R}^{2n}) \):

\[
\mathcal{R}_z = G_z + \sum_{m=1}^{\infty} \sum_{(i,j) \in \text{Dgm}(n,m)} G_z S^*_{x_i} (4\pi (\mathcal{J}_z - \beta, I)^{-1}) \prod_{s=2}^{m} \left(S_{x_{i-s, j-s}} G_z S^*_{x_j} (4\pi (\mathcal{J}_z - \beta, I)^{-1}) \right) S_{x_{i,m}} G_z,
\]

where the sum converges absolutely in operator norm;

(b) when restricted to \( \mathcal{L}^2(\mathbb{R}^{2n}) \), the operator takes a simpler form,

\[
\mathcal{R}_z^{\text{sym}} := G_z + \frac{2}{n(n-1)} \left( \sum_{i<j} G_z S^*_{i,j} \right) \frac{1}{4\pi} (\mathcal{J}_z - \beta, I) - \frac{2}{n(n-1)} \sum_{i<j} S_{i,j} G_z S^*_{i,j} \right)^{-1} \left( \sum_{i<j} S_{i,j} G_z \right).
\]

The sum \( \sum_{i<j}^d \) is over distinct pairs \( (i,j) \neq (k,l) \).

**Theorem 1.5** (Convergence of the resolvent). There exist constants \( C_1 < \infty, C_2 (\beta_{\text{fine}}) > 0 \) such that for all \( \varepsilon \in (0, 1/C_2 (\beta_{\text{fine}})) \), and \( z \in \mathbb{C} \) with \( \text{Re}(z) < -e^{C_1 n^2 + \beta} \).

(a) \( (\mathcal{H}_z - z) \) has a bounded inverse \( \mathcal{L}^2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}^2(\mathbb{R}^{2n}) \);

(b) \( \mathcal{R}_{z, \varepsilon} := (\mathcal{H}_z - z)^{-1} \rightarrow \mathcal{R}_z \) strongly on \( \mathcal{L}^2(\mathbb{R}^{2n}) \), as \( \varepsilon \rightarrow 0 \).

**Remark 1.6.** Given Theorem 1.5, by the Trotter–Kato Theorem, c.f., [RS72, Theorem VIII.22], there exists an (unbounded) self-adjoint operator \( \mathcal{H} \) on \( \mathcal{L}^2(\mathbb{R}^{2n}) \), the limiting Hamiltonian, such that \( \mathcal{R}_z = (\mathcal{H} - z)^{-1}, \) \( \text{Im}(z) \neq 0. \) As implied by Theorem 1.5, the spectra of \( \mathcal{H}_z \) and \( \mathcal{H} \) are bounded below by \(-e^{Cn^2 + \beta}\). Such a bound is first obtained under the momentum cutoff in [DFT94]. Using a non-rigorous mean-field analysis, [Raj99] predicted that the lower end of the spectrum of \( \mathcal{H} \) should approximate \(-e^{c_n n^2}, \) for some \( c_0 \in (0, \infty) \).

**Remark 1.7** (SHE in \( d \geq 3 \)). In higher dimensions \( d \geq 3 \), the appropriate tuning parameter is \( \beta_\varepsilon = \beta_0 \varepsilon^{d-2} \). For small \( \beta_0 \), the studies on the EW-equation limit of the SHE/KPZ equation include [MU18, GRZ18, DGRZ18], and results on the pointwise fluctuations of \( Z_\varepsilon \) and the phase transition in \( \beta_0 \) can be found in [MSZ16, CL17, CCM18, CCM19, CN19]. For discussions on directed polymers in a random environment, we refer to [Com17] and the references therein.

**Outline.** In Section 2 we give an explicit expression for the limiting semigroup in terms of diagrams and use this to derive Corollary 1.1 from Theorem 1.1. In Section 3, we derive the key expression (3.6) for the resolvent \( \mathcal{R}_{z, \varepsilon} \), which allows the limit to be taken term by term: The limits are obtained in Sections 4 through 6, and these are used in Section 7 to prove Proposition 1.4(a),(b), Theorem 1.5(a),(b) and the convergence part of Theorem 1.1(b). In Section 8, we complete the proof of Theorem 1.1 by constructing the semigroup and matching its Laplace transform to the limiting resolvent \( \mathcal{R}_z \).
2. Diagram expansion

In this section, we give an explicit integral kernel $D^{(i,j)}(t, x, x')$ of the operator $D^i(t, x, x')$ in Theorem 1.1, and show how the kernel $D^{(i,j)}(t, x, x')$ can be encoded in terms of diagrams. This is then used to show how Corollary 1.3 follows from Theorem 1.1. The operators $S_{ij}P_t$, $P_tS_{ij}$ and $S_{ij}P_tS_{kl}^*$ have integral kernels

$$
(S_{ij}P_t u)(y) = \int_{\mathbb{R}^{2n}} P(t, S_{ij} y - x) u(x) \, dx, \quad y = (y_2, \ldots, y_n) \in \mathbb{R}^{2n-2},
$$

$$
(P_t S_{ij} v)(x) = \int_{\mathbb{R}^{2n-2}} P(t, x - S_{ij} y) v(y) \, dy, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n},
$$

$$
(S_{ij} P_t S_{kl}^* v)(y) = \int_{\mathbb{R}^{2n-2}} P(t, S_{ij} y - S_{kl} y') v(y') \, dy', \quad y = (y_2, \ldots, y_n) \in \mathbb{R}^{2n-2}.
$$

From this we see that $D^i(t, x, x')$ has integral kernel

$$
D^{(i,j)}(t, x, x') = \int_{\Sigma_m(t)} \, d^2 \int P(\tau_0, x - S_{ij} y^{(1/2)}) \, dy^{(1/2)} \cdot 4\pi P^J(\tau_1, y^{(1/2)} - y^{(1)}) \, dy^{(1)}
$$

$$
\cdot \prod_{k=1}^{m-1} \left( P(\tau_k, S_{ik} y^{(k)} - S_{ik+1} y^{(k+1/2)}) \, dy^{(k+1/2)} \cdot 4\pi P^J(\tau_{k+1/2}, y^{(k+1/2)} - y^{(k+1)}) \, dy^{(k+1)} \right)
$$

$$
\cdot P(\tau_m, S_{im} y^{(m)} - x'),
$$

where $\Sigma_m(t)$ is defined in (1.15), $x, x' \in \mathbb{R}^{2n}$, and $y^{(a)} \in \mathbb{R}^{2n-2}$ with $a \in (\mathbb{Z}/2) \cap (0, m]$.

We wish to further reduce (2.4) to an expression that involves only the two-dimensional heat kernel $p(\tau, x)$ and $j(\tau, \beta)$. Recall from (1.10) that $(S_{ij} y) := x$ is a vector in $\mathbb{R}^{2n}$ such that $x_i = x_j$. In (2.4), we write

$$
S_{ik} y^{(a)} = (y^{(a)}_1, \ldots, y^{(a)}_i, x_j, \ldots, y^{(a)}_n),
$$

and accordingly, $dy^{(a)} = d'x^{(a)}$, where $a = k - \frac{1}{2}, k$. The vector $x^{(a)}$ is in $\mathbb{R}^{2n}$, but the integrator $d'x^{(a)}$ is $(2n-2)$-dimensional due to the contraction $x^{(a)}_{ik} = x^{(a)}_{ik}$. More explicitly,

$$
d'x^{(a)} := \left( dx^{(a)}_{ik} \prod_{\ell \neq i, j} dx^{(a)}_{\ell} \right).
$$

We express $P$ as the product of two dimensional heat kernels, i.e., $P(\tau, x) = \prod_{\ell} p(\tau, x_{\ell})$ with $x = (x_1, \ldots, x_n)$, and similarly for $P^J(\tau, \star)$ (as in (8.6)). This gives

$$
D^{(i,j)}(t, x, x') := \int_{\Sigma_m(t)} \, d^2 \int \prod_{\ell} p(\tau_0, x_{\ell} - x^{(1/2)}_{\ell}) 1\{x^{(1/2)}_{\ell} = x^{(1/2)}_{\ell}\} d'x^{(1/2)}
$$

$$
\cdot \prod_{\ell=1}^m \left( p(\tau_1, x^{(1/2)}_{\ell} - x^{(1)}_{\ell}) 1\{x^{(1)}_{\ell} = x^{(1)}_{\ell}\} \right)
$$

$$
\cdot \prod_{k=1}^{m-1} \left( \prod_{\ell} 1\{x^{(k)}_{ik} = x^{(k)}_{jk}\} p(\tau_k, x^{(k)}_{ik} - x^{(k+1/2)}_{ik}) 1\{x^{(k+1/2)}_{ik} = x^{(k+1/2)}_{jk}\} d'x^{(k)}
$$

$$
\cdot \prod_{\ell=1}^m \left( p(\tau_{k+1/2}, x^{(k+1/2)}_{ik} - x^{(k+1)}_{ik}) 1\{x^{(k+1)}_{ik} = x^{(k+1)}_{jk}\} \right)
$$

$$
\cdot \prod_{\ell} p(\tau_m, x^{(m)}_{\ell} - x^{(k+1)}_{\ell}).
$$
This complicated looking formula can be conveniently recorded in terms of diagrams. Set $A := (1/2 \mathbb{Z}) \cap [0, m + 1/2]$, and adopt the convention $x^{(0)} := x$ and $x^{(m+1/2)} := x'$. We schematically represent spacetime $\mathbb{R}_+ \times \mathbb{R}^2$ by the plane, with the horizontal direction being the time axis $\mathbb{R}_+$, and the vertical direction representing space $\mathbb{R}^2$. We put dots on the plan representing $x^{(a)}_\ell$, $a \in A$. Dots with smaller $a$ sit to the left of those with bigger $a$, and those with the same $a$ lie on the same vertical line. The horizontal distance between $x^{(n-1/2)}_\ell$ and $x^{(a)}_\ell$, $a \in A$, represents a time lapse $\tau_a > 0$. We fix the time horizon between $x^{(0)}_\ell = x^{(a)}_\ell$ and $x^{(m+1/2)}_\ell = x'_\ell$ to be $t$, which forces $\tau_0 + \tau_1 + \ldots + \tau_m = t$. The points $x^{(a)}_\ell$, are generically represented by distinct dots, expect that $x^{(a)}_{i_k}$ and $x^{(a)}_{j_k}$ are joined for $k = a - 1/2, a$. In these cases we call the dot double, otherwise single. See Figure 1 for an example with $n = 4$ and $(i, j) = ((1 < 2), (2 < 3), (3 < 4))$.

Figure 1. Schematic representation of $x^{(a)}_\ell$, with $n = 4$ and $(i, j) = ((1 < 2), (2 < 3), (3 < 4))$. Each dot represents a point $x^{(a)}_\ell$, $a \in (1/2 \mathbb{Z}) \cap [0, 3 + 1/2]$, with the convention $x^{(0)}_\ell := x^{(0)}_\ell$ and $x^{(m+1/2)}_\ell := x^{(3+1/2)}_\ell$. In the figure, the $\ell$ indices are printed in back next to the dot, while the $a$ superscripts are put over the vertical, dashed line. The horizontal distances between dash lines represent time lapses $\tau_a$.

Next, connect dots that represent $x^{(a-1/2)}_\ell$ and $x^{(a)}_\ell$ together, by a ‘single’ line except for the case when both ends are double points, by a ‘double’ line otherwise. To each regular line we assign a two-dimensional heat kernel $p(\tau_\alpha, x^{(a-1/2)}_\ell - x^{(a)}_\ell)$, and to each double line assign the quantity $4\pi j(\tau_\alpha, \beta_\alpha)p(\tau_\alpha, x^{(a-1/2)}_\ell - x^{(a)}_\ell)$. The kernel $D^{(i,j)}(t, x, x')$ is then obtained by multiplying together the quantities assigned to the (regular and double) lines, and integrate the $x^{(a)}_\ell$’s and $\tau_\alpha$’s, with the points $x^{(0)}_\ell := x^{(0)}_\ell$ and $x^{(m+1/2)}_\ell := x^{(3+1/2)}_\ell$ being fixed. See Figure 2 for an example with $n = 4$ and $(i, j) = ((1 < 2), (2 < 3), (3 < 4))$.

In the follow two subsections, we examine the $n = 2, 3$ cases, and derive some useful formulas.

2.1. The $n = 2$ case. In this case, the only index is the singleton $(i, j) = ((1 < 2))$, whereby

\[ (P + D^{\text{reg}}(2))(t, x_1, x_2, x'_1, x'_2) = \sum_{\ell=1}^{2} p(t, x_\ell - x'_\ell) + \int_{\tau_0 + \tau_1 + \tau_2} d^2 \int_{\tau_0 + \tau_1 + \tau_2} p(\tau_0, x^{(1/2)}_\ell) dx^{(1/2)}_1 \]

\[ \cdot 4\pi j(\tau_1/2, \beta_1)p(\tau_1/2, x^{(1/2)}_1 - x^{(1)}_1) dx^{(1)}_1 \]

\[ \cdot \sum_{\ell=1}^{2} p(\tau_1, x^{(1)}_\ell - x'_\ell), \]

and the diagram of $D^{(1,2)}(t, x, x')$ is given in Figure 3.

In (2.6a), rewrite the products in the center-of-mass and relative coordinates,

\[ \prod_{\ell=1}^{2} p(\tau, x_\ell) = p(1/2 \tau, z_1 + z_2)p(2\tau, x_1 - x_2), \]
Figure 2. The diagram representation for $D^{(i,j)}(t,x,x')$, with $n = 4$ and $(i,j) = ((1 < 2), (2 < 3), (3 < 4))$. Each regular (single) line between dots is assigned $p(\tau, x_{\ell}^{(a-1/2)} - x_{\ell}^{(a)})$, while each double line is assigned $4\pi j(\tau, \beta_{*}) p(\frac{1}{2} \tau, x_{\ell}^{(a-1/2)} - x_{\ell}^{(a)})$, where $x_{\ell}^{(a-1/2)}$ and $x_{\ell}^{(a)}$ are represented by the dots at the two ends, and $\tau$ is the horizontal distance between these dots.

Figure 3. The diagram of $D^{((12))}(t,x,x')$.

and then integrate over $x_{1}^{(1/2)}, x_{1}^{(1)} \in \mathbb{R}^2$, using the semigroup property of $p(\bullet, \bullet)$. We then obtain

$$
(P + D^{(\text{gm}(2))})(t, x_{1}, x_{2}, x_{1}', x_{2}')
= p(\frac{1}{2} t, x_{c} - x_{c}') \left( p(2t, x_{d} - x_{d}') + \int_{\tau_{0} + \tau_{1/2} + \tau_{1} = t} d\tau p(2\tau, x_{d}) 4\pi j(\tau_{1/2}, \beta_{*}) p(2\tau, x_{d}') \right),
$$

(2.7)

where $x_{c} := \frac{x_{1} + x_{2}}{2}$, $x_{d} := x_{1} - x_{2}$, and similarly for $x_{c}'$.

Remark 2.1. The formula (2.7) matches [BC98, Equation (3.11)–(3.12)] after a reparametrization. Recall $\beta_{*}$ from (1.8). Comparing our parameterization (1.7) and with [BC98, Equation (2.6)], we see that $\beta_{*}$ here corresponds to $\log \beta$ in [BC98]. The expression in (2.7) matches [BC98, Equation (3.11)–(3.12)] upon replacing $(x_{d}, x_{d}') \mapsto (x, y)$, $\beta_{*} \mapsto \log \beta$, and using the identity:

$$
\int_{0}^{\tau} p(2(\tau - s), x_{d}) p(2s, x_{d}') ds = \frac{1}{8\pi^{2}\tau} \exp \left( -\frac{1}{4\tau}(|x_{d}|^{2} + |x_{d}'|^{2}) \right) \exp \left( -\frac{1}{4\tau}(u|x_{d}|^{2} + \frac{1}{4}|x_{d}'|^{2}) \right) K_{\nu}(\frac{|x_{d}|^{2} + |x_{d}'|^{2}}{2\tau}),
$$

(2.8)

where $K_{\nu}$ denotes the modified Bessel function of the second kind.

To prove (2.8), by scaling in $\tau$, without lost of generality we assume $\tau = 1$. On the l.h.s. of (2.8), factor out $\exp \left( -\frac{1}{4}(|x_{d}|^{2} + |x_{d}'|^{2}) \right)$, decompose the resulting integral into $s \in (0, 1/2)$ and $s \in (1/2, 1)$, for the former perform the change of variable $u = (1 - s)/s$, and for the latter $u = s/(1 - s)$. We have

$$
(\text{l.h.s. of (2.8)}) = \exp \left( -\frac{1}{4}(|x_{d}|^{2} + |x_{d}'|^{2}) \right) I_{*}, \quad I_{*} := 2 \int_{1}^{\infty} \frac{1}{(4\pi)^{2}u} e^{-\frac{1}{4}(u|x_{d}|^{2} + \frac{1}{4}|x_{d}'|^{2})} du.
$$
The integrand within the last integral stays unchanged upon the change of variable \( u \mapsto 1/u \), while the range maps to \((0,1)\). We hence replace \( 2 \int_0^\infty (\cdot) \, du \) with \( \int_0^\infty (\cdot) \, du \). Within the result, perform a change of variable \( v = 2u|x_d|^2 \), and from the result recognize \( \frac{1}{2\pi^2} e^{-rac{1}{\varepsilon} \langle |x_d|^2 |x_d'|^2 \rangle} = p(v, |x_d||x_d'|) \). We get
\[
I_* = \int_0^\infty \frac{1}{(4\pi)^2} e^{-\frac{|x_d|^2 |x_d'|^2}{4\varepsilon}} e^{-\frac{v}{8}} \, dv = \frac{1}{8\pi} G_{-\frac{1}{4}} \left( |x_d||x_d'| \right),
\]
where \( G_{-\frac{1}{4}}(|x|) = G_{-\frac{1}{4}}(x) \) denote two-dimensional Green’s function. As argued in the proof of Lemma 6.2, \( G_{-\frac{1}{4}}(x) = \frac{1}{8} K_0(\sqrt{-2\varepsilon}|x|) \). This gives (2.8).

2.2. The \( n = 3 \) case. Here we derive a formula for the limiting centered third moment. We say \( (i,j) = ((i_k < j_k))_{k=1}^m \in \text{Dgm}(n) \) is degenerate if \( \cup_{k=1}^m \{i_k, j_k\} \subseteq \{1, \ldots, n\} \), and otherwise nondegenerate. Let \( \text{Dgm}'(n) \) denote the set of all nondegenerate elements of \( \text{Dgm}(n) \), and, accordingly,
\[
\mathcal{D}_{t}^{\text{Dgm}'(n)} := \sum_{(i,j) \in \text{Dgm}'(n)} \mathcal{D}_{t}^{(i,j)}. 
\]

**Proposition 2.2.** Start the SHE from \( Z_{ic}(0, \cdot) = Z_{ic} \in L^2(\mathbb{R}^2) \). For any \( f \in L^2(\mathbb{R}^2) \),
\[
E \left[ \left( \langle f, Z_{ic}, t \rangle - E[\langle f, Z_{ic}, t \rangle] \right)^3 \right] \rightarrow \left\langle f^{\otimes 3}, \mathcal{D}_{t}^{\text{Dgm}'(3)} Z_{ic}^{\otimes 3} \right\rangle \quad \text{as } \varepsilon \to 0,
\]
uniformly in \( t \) over compact subsets of \([0, \infty)\).

**Proof.** Expand the l.h.s. of (2.9) into a sum of products of \( n' = 1, 2, 3 \) moments of \( \langle f, Z_{ic}, t \rangle \) as
\[
E \left[ \left( \langle f, Z_{ic}, t \rangle - E[\langle f, Z_{ic}, t \rangle] \right)^3 \right] = E[\langle f, Z_{ic}, t \rangle]^3 - 3E[\langle f, Z_{ic}, t \rangle^2 E[\langle f, Z_{ic}, t \rangle] + 2(E[\langle f, Z_{ic}, t \rangle])^3. \quad (2.10)
\]
For the \( n' = 1 \) moment, rewriting the SHE (1.1) in the mild (i.e., Duhamel) form and take expectation gives
\[
E[\langle f, Z_{ic}, t \rangle] = \langle f, p * Z_{ic} \rangle = \int_{x' \in \mathbb{R}^2} f(x') p(t, x' - x) Z_{ic}(x) \, dx \, dx',
\]
where \( * \) denotes convolution in \( x \in \mathbb{R}^2 \). Note that for \( n' = 2 \) the only index \( \text{Dgm}(2) = \{(1 < 2)\} \) is the singleton and that \( \langle f^{\otimes 2}, \mathcal{D}_{t}^{\text{Dgm}'(2)} Z_{ic}^{\otimes 2} \rangle = \langle f, p * Z_{ic} \rangle^{n'}. \) We then have
\[
\lim_{\varepsilon \to 0} E \left[ \langle f, Z_{ic}, t \rangle^3 \right] = \langle f, p * Z_{ic} \rangle^3 + \langle f^{\otimes 3}, \mathcal{D}_{t}^{\text{Dgm}'(3)} Z_{ic}^{\otimes 3} \rangle, \quad (2.11)
\]
\[
\lim_{\varepsilon \to 0} E \left[ \langle f, Z_{ic}, t \rangle^2 \right] = \langle f, p * Z_{ic} \rangle^2 + \langle f^{\otimes 2}, \mathcal{D}_{t}^{(12)} Z_{ic}^{\otimes 2} \rangle. \quad (2.12)
\]
Inserting (2.11)–(2.12) into (2.10) gives
\[
\lim_{\varepsilon \to 0} E \left[ \left( \langle f, Z_{ic}, t \rangle - E[\langle f, Z_{ic}, t \rangle] \right)^3 \right] = \left\langle f^{\otimes 3}, \mathcal{D}_{t}^{\text{Dgm}'(3)} Z_{ic}^{\otimes 3} \right\rangle - 3\langle f, p * Z_{ic} \rangle \langle f^{\otimes 2}, \mathcal{D}_{t}^{(12)} Z_{ic}^{\otimes 2} \rangle. \quad (2.13)
\]
For \( n' = 3 \), degenerate indices in \( \text{Dgm}(3) \) are the singletons \((1 < 2), (1 < 3), (2 < 3)\). This being the case, we see that the last term in (2.13) exactly cancels the contribution of degenerate indices in \( \langle f^{\otimes 3}, \mathcal{D}_{t}^{\text{Dgm}'(3)} Z_{ic}^{\otimes 3} \rangle \). The desired result follows. \( \square \)

2.3. **Proof of Corollary 1.3.** Here we prove Corollary 1.3 assuming Theorem 1.1 (which will proven in Section 8). Our first goal is to show \( Z_{ic}(t, x_1)dx_1 = \mu_{t, t, Z}(dx_1) \), as a random measure on \( \mathbb{R}^2 \), is tight in \( \varepsilon \), under the vague topology. By [Kal97, Lemma 14.15], this amounts to showing \( \int_{\mathbb{R}^2} g(x) \mu_{t, t, Z}(dx) = (g, Z_{ic}) \) is tight (as a \( \mathbb{C} \)-valued random variable), for each \( g \in C_c(\mathbb{R}^2) \). Apply Theorem 1.1 with \( n = 2 \), with \( Z_{ic}(x_1) \mapsto |Z_{ic}(x_1)| \in L^2(\mathbb{R}^2) \), and with \( f(x_1, x_2) = |g(x_1)g(x_2)| \). We obtain that \( E[|Z_{ic}, t, g|^2] \) is uniformly bounded in \( \varepsilon \), so \( \int_{\mathbb{R}^2} g(x) \mu_{t, t, Z}(dx) \) is tight.

Fixing a limit point \( \mu_{t, t, Z} \in C_c(\mathbb{R}^2) \), we proceed to show (1.19). Fix a sequence \( \varepsilon_k \to 0 \) such that \( \mu_{\varepsilon_k, t, Z} \to \mu_{t, t} \) vaguely, as \( k \to \infty \). The desired result (1.19) follows from Theorem 1.1 if we can upgrade the preceding vague convergence of \( \mu_{t, t, Z} \) to convergence in moments. For fixed \( f_1, \ldots, f_n \in C_c(\mathbb{R}^2) \), applying Theorem 1.1 with \( n \to 2n \), with \( Z_{ic}(x_1) \mapsto |Z_{ic}(x_1)| \in L^2(\mathbb{R}^2) \), and with \( f(x_1, \ldots, x_{2n}) = \prod_{i=1}^n |f_i(x_i) f_i(x_{n+i})| \), we obtain that
\[
E \left[ |Z_{ic}, t, g|^{2n+2} \right] = E \left[ |f_1 \otimes \cdots \otimes f_n, Z_{ic} |^{2n} \right] = E \left[ \prod_{i=1}^n \int_{\mathbb{R}^2} f_i(x_i) \mu_{t, t, Z}(dx_i) \right]^2.
\]
is uniformly bounded in $\varepsilon$. Hence $(\prod_{i=1}^n \int_{\mathbb{R}^2} f_i(x_i) \mu_{\varepsilon,t,Z}(dx_i))$ is uniformly integrable in $\varepsilon$ (as $C$-valued random variables), which guarantees the desired convergence in moments.

We now move on to showing (1.20). For $Z_{\varepsilon}(x_1), f_1(x_1) \geq 0$, both not identically zero, we apply Proposition 2.2 to obtain the $\varepsilon \to 0$ limit of the centered, third moment of $\int_{\mathbb{R}^2} f_1(x_1) \mu_{\varepsilon,t,Z}(dx_1)$. As just argued, such a limit is also inherited by $\mu_{\varepsilon,t,Z}$, whereby

$$
\mathbb{E}\left[\left( \int_{\mathbb{R}^2} f_1(x_1) \mu_{\varepsilon,t}(dx_1) \right) - \mathbb{E}\left[ \int_{\mathbb{R}^2} f_1(x_1) \mu_{\varepsilon,t}(dx_1) \right] \right]^3 = \langle f_1^{\otimes 3}, D_t^{\text{Dom}(3)} Z_{\varepsilon}^{\otimes 3} \rangle.
$$

As seem from (2.5), the operator $D_{(i,j)}$ has a strictly positive integral kernel. Under current assumption $Z_{\varepsilon}(x_1), f_1(x_1) \geq 0$ and not identically zero, we see that the r.h.s. of (2.14) is strictly positive.

3. Resolvent identity

In this section we derive the identity (3.6) for the resolvent $R_{\varepsilon,z} = (H_\varepsilon - z)^{-1}$ which is the key to our analysis. Let $H_{fr} := -\frac{1}{2} \sum_i \nabla_i^2$ denote the ‘free Hamiltonian’, and let $V_\varepsilon : \mathcal{L}^2(\mathbb{R}^{2n}) \to \mathcal{L}^2(\mathbb{R}^{2n})$

$$
V_\varepsilon u(x) := \sum_{i < j} \delta_\varepsilon(x_i - x_j) u(x)
$$
denote the operator of multiplication by the approximate delta potential, which is a bounded operator for each $\varepsilon > 0$. The Hamiltonian $H_\varepsilon$ is then an unbounded operator on $\mathcal{L}^2(\mathbb{R}^{2n})$ with domain $\mathcal{H}^2(\mathbb{R}^{2n})$ (the Sobolev space), i.e.,

$$
H_\varepsilon := H_{fr} - \beta_\varepsilon V_\varepsilon, \quad \text{Dom}(H_\varepsilon) := \mathcal{H}^2(\mathbb{R}^{2n}) \subset \mathcal{L}^2(\mathbb{R}^{2n}).
$$

The first step is to build a ‘square root’ of $V_\varepsilon$. More precisely, we seek to construct an operator $S_{i,j}$, indexed by a pair $i < j$, and its adjoint $S_{i,j}^*$ such that $V_\varepsilon = \sum_{i < j} S_{i,j}^* S_{i,j}$. To this end, for each $\varepsilon > 0$ and $1 \leq i < j \leq n$, consider the linear transformation $T_{i,j} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$
T_{i,j}(x_1, \ldots, x_n) := \left( \frac{x_i - x_j}{\varepsilon}, \frac{x_i + x_j}{\varepsilon}, x_{i+1}, \ldots, x_j \right),
$$

where $x \in \mathbb{R}^{2(n-2)}$ denotes the vector obtained by removing the $i,j$-th components from $x \in \mathbb{R}^{2n}$. In other words, the transformation $T_{i,j}$ places the relative distance (on the scale of $\varepsilon$) and the center of mass corresponding to $(x_i,x_j)$ in the first two components, while keeping all other components unchanged. The transformation $T_{i,j}$ has inverse $S_{i,j} := T_{i,j}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$
S_{i,j}(y_1, \ldots, y_n) := \left( y_3, \ldots, y_2 + \frac{\varepsilon y_2}{2}, \underbrace{y_3 + \frac{\varepsilon y_3}{2}, \ldots, y_n}_{i\text{ th}}, y_1, \ldots, y_{j-1}, \underbrace{y_j - \frac{\varepsilon y_j}{2}, \ldots, y_n}_{j\text{ th}} \right).
$$

Accordingly, we let $S_{i,j}$ and $S_{i,j}^*$ be the induced operators $\mathcal{L}^2(\mathbb{R}^{2n}) \to \mathcal{L}^2(\mathbb{R}^{2n})$,

$$
(S_{i,j}v)(y) := u(S_{i,j}y), \quad (S_{i,j}^*v)(x) := \varepsilon^{-2} v(T_{i,j}x).
$$

It is straightforward to check that $S_{i,j}^*$ is the adjoint of $S_{i,j}$, i.e., the unique operator for which $\langle S_{i,j}^*v, u \rangle = \langle v, S_{i,j}u \rangle$, $\forall u, v \in \mathcal{L}^2(\mathbb{R}^{2n})$. Since $S_{i,j}, T_{i,j}$ are both invertible, the operators $S_{i,j}, S_{i,j}^*$ are bounded for each $\varepsilon > 0$. $\Phi$ is even and non-negative, so we can set $\phi(x) := \sqrt{\Phi}(x)$ and view $\langle \phi v \rangle(y) := \phi(y_1)v(y_1, \ldots, y_n)$ as a bounded multiplication operator on $\mathcal{L}^2(\mathbb{R}^{2n})$. From (3.4), it is straightforward to check

$$
V_\varepsilon = \sum_{i < j} S_{i,j}^* \phi \phi S_{i,j}.
$$

Remark 3.1. We comment on how our setup compares to that of [DR04]. They work in $\mathcal{L}^2_{\text{sym}}(\mathbb{R}^{2n})$, corresponding to $n$ Bosons in $\mathbb{R}^2$, the key idea being to decompose the action of the delta potential $V_\varepsilon$ on $\mathcal{L}^2_{\text{sym}}(\mathbb{R}^{2n})$ into some intermediate actions from $\mathcal{L}^2_{\text{sym}}(\mathbb{R}^{2n})$ into an ‘auxiliary space’, consisting of $n-2$ Bosons and an ‘angle particle’. In our current setting, the auxiliary space is $\mathcal{L}^2(\mathbb{R}^{2n}) \ni v = v(y_1, y_2, y_3, \ldots, y_n)$. The components $y_1, \ldots, y_n$ correspond to the $n-2$ particles, the component $y_2$ corresponds to the angle particle, while $y_1$ is a ‘residual’ component that arises from our space-mollification scheme, and is not presented under the momentum-cutoff scheme of [DR04].
Given (3.5), the next step is to develop an expression for the resolvent \( R_{e,z} = (\mathcal{H}_e - zI)^{-1} \) that is amenable for the \( e \to 0 \) asymptotic. In the case of momentum cutoff, such a resolvent expression is obtained in (Eq (68) of) [DR04] by comparing two different ways of inverting a two-by-two (operator-valued) matrix. Here, we derive the analogous expression (i.e., (3.6)) using a more straightforward procedure — power-series expansion of (operator-valued) geometric series. Recall \( \text{Dgm}(n,m) \) from (1.13), recall that \( \|Q\|_{\text{op}} \) denotes the operator norm of \( Q \), and recall from (1.21) that \( G_e \) denotes the resolvent of the Laplacian.

**Lemma 3.2.** For all \( e \in (0,1) \) and \( z \in \mathbb{C} \) such that \( \text{Re}(z) < -\beta_e (1 + \sum_{i < j} \|S_{eij}\|_{\text{op}})^2 \), we have

\[
R_{e,z} := (\mathcal{H}_e - zI)^{-1} = G_e + \sum_{m=1}^{\infty} \sum_{(i,j) \in \text{Dgm}(n,m)} (G_e S_{eij})^{-1} \cdot \left( \beta_e^{-1} I - \phi_s S_{e12} G_e S_{e12}^* \right)^{-1} \cdot (\phi_s S_{e12} G_e S_{e12}^* \beta_e^{-1} I - \phi_s S_{e12} G_e S_{e12})^{-1} \cdot (\phi_s S_{e12} G_e S_{e12}).
\]

**Remark 3.3.** As stated, Lemma 3.2 holds for \( \text{Re}(z) < -C_1(e,n) \), with a threshold \( C_1(e,n) \) that depends on \( e \). This may not seem useful as \( e \to 0 \), however, as we will show later in Section 7, the r.h.s. of (3.6) is actually analytic (in norm) in \( \{z : \text{Re}(z) < -C_2(n)\} \), for some threshold \( C_2(n) < \infty \) that is independent of \( e \). It then follows immediately (as argued in Section 7) that (3.6) extends to all \( \text{Re}(z) < -C_2(n) \).

**Proof.** To simplify notation, set \( \mathbf{S}_{ij} := \beta_e^{1/2} \phi_s S_{eij}, \mathbf{S}_{ij}^* := (\mathbf{S}_{ij})^* = \beta_e^{1/2} S_{eij}^*, \) and \( \mathbf{S}_{ijkl} := S_{ijkl} G_e S_{ijkl}^* \). In (3.6b), factor \( \beta_e^{-1} \) from the inverse. Under the preceding shorthand notation, we rewrite (3.6) as

\[
R_{e,z} = G_e + \sum_{m=1}^{\infty} \sum_{(i,j) \in \text{Dgm}(n,m)} G_e \mathbf{S}_{ij}^* \cdot (I - G_{12}^2)^{-1} \prod_{k=2}^{m} (G_{k-1,jk-1} G_e S_{ijkl}^* G_e S_{ijkl})^{-1} \cdot \mathbf{S}_{im,jm} G_e.
\]

Our goal is to expand the inverse in (3.7), and then simplify the result to match \( (\mathcal{H}_e - zI)^{-1} \).

To expand the inverse in (3.7), we utilize the geometric series \( (I - Q)^{-1} = I + \sum_{k=1}^{\infty} Q^k \), valid for \( \|Q\|_{\text{op}} < 1 \). Indeed, \( \|G_e\|_{\text{op}} \leq 1/(-\text{Re}(z)) \), so under the assumption on the range of \( \text{Re}(z) \) we have \( \|\mathbf{S}_{ij}^2\|_{\text{op}} < 1 \). Using the geometric series for \( Q = G_{12}^2 \), and inserting the result into (3.7) gives

\[
R_{e,z} = G_e + \sum_{m=1}^{\infty} \sum_{(i,j) \in \text{Dgm}(n,m)} G_e \mathbf{S}_{ij}^* \cdot (I - G_{12}^2)^{-1} \prod_{k=2}^{m} (G_{k-1,jk-1} G_e S_{ijkl}^* G_e S_{ijkl})^{-1} \cdot \mathbf{S}_{im,jm} G_e.
\]

where the sum is over \( \ell_1, \ldots, \ell_m \geq 0, (i,j) \in \text{Dgm}(n,m), \) and \( m = 1, 2, \ldots \). The sum converges absolutely in operator norm by our assumption on \( z \). Since \( G_e \) acts symmetrically in the \( n \) components, we have \( G_{12}^2 = G_{ij}^2 \), for any pair \( i < j \). Use this property to rewrite (3.8) as

\[
R_{e,z} = G_e + \sum_{m=1}^{\infty} \sum_{(i,j) \in \text{Dgm}(n,m)} G_e \mathbf{S}_{ij}^* \cdot (I - G_{12}^2)^{-1} \prod_{k=2}^{m} (G_{k-1,jk-1} G_e S_{ijkl}^* G_e S_{ijkl})^{-1} \cdot \mathbf{S}_{im,jm} G_e.
\]

The summation can be reorganized as \( \sum_{m=1}^{\infty} \sum_{i < j} \mathbf{S}_{ij} \mathbf{S}_{ij}^* G_e \). To see this, recall from (1.13) that \( (i,j) \in \text{Dgm}(n,m) \) consists of pairs \( (i_k < j_k) \) under the constraint that consecutive pairs are non-repeating, i.e., \( (i_{k-1} < j_{k-1}) \neq (i_k < j_k) \). The r.h.s. of (3.9) replenishes all possible repeating of consecutive pairs, and hence lifts the constraints imposed by \( \text{Dgm}(n,m) \). In the resulting sum, express \( \mathbf{S}_{ijkl} G_e S_{ijkl} \) to get

\[
R_{e,z} = \sum_{m=0}^{\infty} G_e \left( \sum_{i < j} \mathbf{S}_{ij} \mathbf{S}_{ij}^* G_e \right)^m.
\]

From (3.5), we have \( \sum_{i < j} \mathbf{S}_{ij} \mathbf{S}_{ij}^* = \beta_e \mathcal{V}_e \), hence \( R_{e,z} = G_e (1 - \beta_e \mathcal{V}_e G_e)^{-1} \). Further \( G_e = (\mathcal{H}_{tr} - zI)^{-1} \) gives

\[
R_{e,z} = (\mathcal{H}_{tr} - zI)^{-1} \left(1 - \beta_e \mathcal{V}_e (\mathcal{H}_{tr} - zI)^{-1}\right)^{-1} = (\mathcal{H}_{tr} - zI - \beta_e \mathcal{V}_e)^{-1} = (\mathcal{H}_e - zI)^{-1}.
\]

This completes the proof. \( \square \)
The resolvent identity (3.6) is the gateway to the $\varepsilon \to 0$ limit. Roughly speaking, we will show that all terms in (3.6) converge to their limiting counterparts in the expression of $R_\varepsilon$ given in (1.23). The expression (1.23), however, does not expose such a convergence very well. This is so because some operators in (1.23) map one function space to a different one, (e.g., $S_{ij}$ maps functions of $n$ components to $n - 1$ components), while the operators in (3.6) always maps $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^{2n})$. We next rewrite (1.23) in a way that better compares with (3.6). To this end, consider the operators

$$\Omega_{ij} : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n-2}), \quad (\Omega_{ij}v)(y_{2-n}) := \int_{\mathbb{R}^2} \phi(y_1)v(y_1,y_{2-n})dy_1$$

(3.10)

$$\phi \otimes \cdot : L^2(\mathbb{R}^{2n-2}) \to L^2(\mathbb{R}^{2n}), \quad (\phi \otimes v)(y_1,y_{2-n}) := \phi(y_1)v(y_{2-n}).$$

(3.11)

Given that $\phi \in C_c^\infty(\mathbb{R}^2)$, it is readily checked that $\Omega_{ij}$ and $\phi \otimes \cdot$ are bounded operators. Note that from $\phi := \sqrt{\ell}$, $\phi$ has unit norm, i.e., $\int_{\mathbb{R}^2} \phi^2 dy = 1$. From this we obtain $\Omega_{ij}(\phi \otimes Q) = Q$, for a generic $Q : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n-2})$ or $Q : L^2(\mathbb{R}^{2n-2}) \to L^2(\mathbb{R}^{2n-2})$. Using this property, we rewrite (1.23) as

$$R_\varepsilon = G_\varepsilon + \sum_{m=1}^{\infty} \sum_{(i,j) \in Dgm(n,m)} \left( G_\varepsilon S_{ij}^* \Omega_{ij} \right)$$

(3.12a)

$$\cdot \left( \phi \otimes 4\pi(\mathcal{J}_z - \beta_s I)^{-1}\Omega_{ij} \right) \prod_{s=2}^m \left( \left( \phi \otimes S_{ij} G_\varepsilon S_{ij}^* \Omega_{ij} \right) \left( \phi \otimes 4\pi(\mathcal{J}_z - \beta_s I)^{-1}\Omega_{ij} \right) \right)$$

(3.12b)

$$\cdot \left( \phi \otimes S_{ij} G_\varepsilon \right).$$

(3.12c)

That is, we augment the missing $y_1$ dependence (in the operators $S_{ij}$, $S_{ij}^*$, etc.) along the subspace $\mathbb{C}\phi \subset L^2(\mathbb{R}^2)$. Equation (3.12) gives a better expression for comparison with (3.6).

For future references, let us setup some terminology for the operators in (3.6) and (3.12). We call the operators $S_{ij} G_\varepsilon$ or $\phi \otimes S_{ij} G_\varepsilon$ in (3.12c) the limiting incoming operators, and the operators $G_\varepsilon S_{ij}$ or $G_\varepsilon S_{ij}^* \Omega_{ij}$ in (3.12c) the limiting outgoing operators. Slightly abusing language, we will use these phrases interchangeably to infer operators with and without the action by $\phi \otimes \cdot$ or $\Omega_{ij}$. Similarly, we call the operators in (3.6c) the pre-limiting incoming operators, and the operators in (3.6a) the pre-limiting outgoing operators. Next, with $\mathcal{J}_z$ defined in (1.22) in the following, we refer to $(\mathcal{J}_z - \beta_s I)$ and $(\beta_s^{-1} I - S_{ij} G_\varepsilon S_{ij}^*)$ as the limiting and pre-limiting diagonal mediating operators, respectively, and refer to $S_{ij} G_\varepsilon S_{ij}^*$ and $S_{ijkl} G_\varepsilon S_{ijkl}^*$, with $(i < j) \neq (k < \ell)$, as the the limiting and pre-limiting off-diagonal mediating operators.

As we will show in Section 4, each pre-limiting incoming and outgoing operator converges to its limiting counterpart, and, as will show in Section 5, each off-diagonal mediating operator converges to its limiting counterpart. Diagonal mediating operators require a more delicate treatment because $\beta_s^{-1} I$ and $S_{ij} G_\varepsilon S_{ij}^*$ both diverge on their own, and we need to cancel the divergence (and also to take an inverse) to obtain a limit. This procedure, sometimes referred to as renormalization in the physics literature, will be carried out in Section 6.

4. INCOMING AND OUTGOING OPERATORS

In this section we obtain the $\varepsilon \to 0$ limit of $\phi S_{ij} G_\varepsilon$ and $G_\varepsilon S_{ij}^* \phi$ to $\phi \otimes (S_{ij} G_\varepsilon)$ and $G_\varepsilon S_{ij}^* \Omega_{ij}$. The main result is stated in Lemma 4.4.

Recall the linear transformation $S_{ij}$ and its induced operator $\mathcal{S}_{ij}$ from (1.10)–(1.11). Comparing (3.3) and (1.10), we see that $S_{ij}(y_1,\ldots,y_{2n}) \to S_{ij}(y_2,\ldots,y_{2n})$ as $\varepsilon \to 0$. Namely, $S_{ij}$ is the pointwise limit of $S_{ij}$. This observation hints that $\varepsilon S_{ij}$ should be the limit of $S_{ij}$, and the $\varepsilon \to 0$ limit of the incoming operator $\phi S_{ij} G_\varepsilon$ should be obtained by replacing $S_{ij}$ with $\varepsilon S_{ij}$. Note that, however, the operator $\varepsilon S_{ij}$ is unbounded, because, unlike $S_{ij}$, $\varepsilon S_{ij}$ maps between spaces of different dimensions; the $y_1$ dependence in $S_{ij}(y_1,\ldots,y_{2n})$ ‘vanishes’ as $\varepsilon \to 0$ (c.f., (3.3)).

As the first step of building the limiting operators, we construct the domain of $S_{ij}$, along with its adjoint $S_{ij}^*$. In the following we will often work in the Fourier domain. Let $\mathcal{F}(\phi) := \int_{\mathbb{R}^d} e^{-iy \cdot \phi} \frac{dy}{(2\pi)^{d/2}}$ denote Fourier transform of functions on $\mathbb{R}^d$; the inverse Fourier transform then reads $\phi(y) = \int_{\mathbb{R}^d} e^{iy \cdot \phi} \frac{dy}{(2\pi)^{d/2}}$.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions. In our subsequential analysis, $d$ is typically $2n$ or $2(n-1)$. 


Consider the (invertible) linear transformation \( \mathbb{R}^{2n} \to \mathbb{R}^{2n} \):

\[
M_{ij}q := (q_3, \ldots, \frac{1}{2}q_2 + q_1, \ldots, \frac{1}{2}q_2 - q_1, \ldots, q_n).
\]

(4.1)

For \( q \in \mathbb{R}^{2n} \), we write \( q_{i-j} := (q_i, \ldots, q_j) \in \mathbb{R}^{2(j-i+1)} \), and recall that \( q_{ij} \in \mathbb{R}^{2n-4} \) is obtained from removing the \( i \)-th and \( j \)-th components of \( q \).

**Lemma 4.1.**

(a) The operator \( S_{ij} \), given by equation (1.11), is unbounded from \( \mathcal{L}^2(\mathbb{R}^{2n}) \) to \( \mathcal{L}^2(\mathbb{R}^{2n-2}) \), with

\[
\text{Dom}(S_{ij}) := \left\{ f \in \mathcal{L}^2(\mathbb{R}^{2n}) : \int_{\mathbb{R}^2} |\hat{f}(M_{ij}(q_1, \cdot))|dq_1 \in \mathcal{L}^2(\mathbb{R}^{2n-2}) \right\} \subset \mathcal{L}^2(\mathbb{R}^{2n}),
\]

and for \( f \in \text{Dom}(S_{ij}) \), we have

\[
\hat{S}_{ij}f(q_{2-n}) = \int_{\mathbb{R}^2} \hat{f}(M_{ij}q_1) \frac{dq_1}{2\pi}.
\]

(4.3)

In addition, for all \( a > 1 \), we have \( \mathcal{H}^a(\mathbb{R}^{2n}) \subset \text{Dom}(S_{ij}) \).

(b) The operator

\[
\hat{S}_{ij}^*g(p) := \frac{1}{2\pi} \hat{g}(p_i + p_j, p_{ij})
\]

maps \( \mathcal{L}^2(\mathbb{R}^{2n-2}) \to \cap_{a>1} \mathcal{H}^{-a}(\mathbb{R}^{2n}) \), and is adjoint to \( S_{ij} \) in the sense that

\[
\langle S_{ij}^*g, f \rangle = \langle g, S_{ij}f \rangle, \quad g \in \mathcal{L}^2(\mathbb{R}^{2n-2}), \quad f \in \mathcal{H}^a(\mathbb{R}^{2n}), \quad a > 1.
\]

(4.5)

**Proof.** (a) Let us first show (4.3) for \( f \in \mathcal{S}(\mathbb{R}^{2n}) \). On the Fourier transform of \( f \), perform the change of variables \( x = S_{ij}y \), where \( S_{ij} = S_{ij} \varepsilon_1 \), and the substitute \( p = M_{ij}q \). From (3.3), it is readily checked that \( \text{det}(S_{ij}) = 1 \), and from (4.1), we have \( \langle S_{ij}y \cdot (M_{ij}q) = y \cdot q, \) so

\[
\hat{f}(M_{ij}q) = \int_{\mathbb{R}^{2n}} e^{-iy \cdot q} f(S_{ij}y) \frac{dy_{2-n}}{(2\pi)^n}.
\]

(4.6)

Our goal is to calculate the Fourier transform of \( f(S_{ij} \varepsilon_1) \). Comparing (1.10) and (3.3) for \( \varepsilon = 1 \), we see that \( \langle S_{ij}y \rangle_{y_1=0} = S_{ij}(y_{2-n}). \) It is hence desirable to ‘remove’ the \( y_1 \) variable on the r.h.s. of (4.6). To this end, apply the identity

\[
\int_{\mathbb{R}^{2n-2}} g(0, y_{2-n}) e^{-iy_{2-n} \cdot y_{2-n}} \frac{dy_{2-n}}{(2\pi)^{n-1}} = \int_{\mathbb{R}^2} \hat{g}(q) \frac{dq_1}{2\pi}, \quad g \in \mathcal{S}(\mathbb{R}^{2n})
\]

with \( g(\cdot) = f(S_{ij} \varepsilon_1) \) to obtain

\[
\int_{\mathbb{R}^{2n-2}} \hat{f}(M_{ij}q) \frac{dq_1}{2\pi} = \int_{\mathbb{R}^{2n-2}} e^{-iy_{2-n} \cdot q_{2-n}} f(S_{ij}y) \frac{dy_{2-n}}{(2\pi)^{n-1}} = \int_{\mathbb{R}^{2n-2}} e^{-iy_{2-n} \cdot q_{2-n}} f(S_{ij}(y_{2-n})) \frac{dy_{2-n}}{(2\pi)^{n-1}}.
\]

The last expression is \( \hat{S}_{ij}f(q_{2-n}) \) by definition. We hence conclude (4.3) for \( f \in \mathcal{S}(\mathbb{R}^{2n}) \). By approximation, it follows that \( S_{ij} \) extends to an unbounded operator with domain (4.2), and the identity (4.3) extends to \( f \in \text{Dom}(S_{ij}) \).

Fix \( a > 1 \), we proceed to show \( \mathcal{H}^a(\mathbb{R}^{2n}) \subset \text{Dom}(S_{ij}) \). For \( f \in \mathcal{H}^a(\mathbb{R}^{2n}) \), it suffices to bound

\[
\int_{\mathbb{R}^{2n-2}} \left| \int_{\mathbb{R}^2} \hat{f}(M_{ij}q) \frac{dq_1}{2\pi} \right|^2 dq_{2-n}.
\]

(4.7)

Within the integrals, multiply and divide by \( \left(\frac{1}{2}|M_{ij}q|^2 + 1\right)^{\frac{a}{2}} \). Use \( \frac{1}{2}|M_{ij}q|^2 \geq |q_1|^2 \) (as readily checked from (4.1)) and apply the Cauchy–Schwarz inequality over the integral in \( q_1 \). We then obtain

\[
(4.7) = \int_{\mathbb{R}^{2n-2}} \left( \frac{1}{2}|M_{ij}q|^2 + 1 \right)^{\frac{a}{2}} \left( \frac{1}{2}|M_{ij}q|^2 + 1 \right)^{\frac{a}{2}} \left| \hat{f}(M_{ij}q) \right| dq_1 \left| dq_{2-n} \right.
\]

\[
\leq \int_{\mathbb{R}^2} \left( \frac{1}{|q_1|^2 + 1} \right)^a dq_1 \left\| f \right\|_{\mathcal{H}^a(\mathbb{R}^{2n})} \leq \frac{C}{a-1} \left\| f \right\|_{\mathcal{H}^a(\mathbb{R}^{2n})}.
\]

(4.8)

This verifies \( \mathcal{H}^a(\mathbb{R}^{2n}) \subset \text{Dom}(S_{ij}) \).

(b) That \( S_{ij}^* \) maps \( \mathcal{L}^2(\mathbb{R}^{2n-2}) \to \cap_{a>1} \mathcal{H}^{-a}(\mathbb{R}^{2n}) \) is checked by similar calculations as in (4.8). To check (4.5), calculate the inner product \( \langle S_{ij}^*g, f \rangle \) in Fourier variables from (4.4). Within the resulting integral, perform
a change of variable $p = M_{ij}q$, and use $|\det(M_{ij})| = 1$ and $(p_i + p_j, p_{ij}) = [M_{ij}^{-1}p]_{2-n}$ (as readily checked from (4.1)). We then obtain

$$\langle S^*_{ij}g, f \rangle = \int_{\mathbb{R}^n} \frac{g(p_i + p_j, p_{ij})}{2\pi} \hat{f}(p) \frac{dp}{2\pi} = \int_{\mathbb{R}^n} \frac{g(q_{2-n})}{2\pi} \hat{f}(M_{ij}q) \frac{dq}{2\pi}.$$  

From (4.3), we see that the last expression matches $\langle g, S_{ij}f \rangle$.

Recall that, for each $\text{Re}(z) < 0$, $\mathcal{G}_z(\mathcal{L}^2(\mathbb{R}^{2n})) = \mathcal{H}^2(\mathbb{R}^{2n})$. This together with Lemma 4.1 implies that $S_{ij}\mathcal{G}_z$ is defined on the entire $\mathcal{L}^2(\mathbb{R}^{2n})$, with image in $\mathcal{L}^2(\mathbb{R}^{2n-2})$, and that $\mathcal{G}_zS^*_{ij}$ is defined on $\mathcal{L}^2(\mathbb{R}^{2n-2})$, with image in $\mathcal{L}^2(\mathbb{R}^{2n})$. Informally, $\mathcal{G}_z$ increases regularity by 2, while $S_{ij}$ and $S^*_{ij}$ both decrease regularity by $-(1^+) = 1^+ > 0$.

We now establish a quantitative bound on the operator norm of the limiting operators $S_{ij}\mathcal{G}_z$ and $\mathcal{G}_zS^*_{ij}$.

**Lemma 4.2.** For $1 \leq i < j \leq n$ and $\text{Re}(z) < 0$, $\|S_{ij}\mathcal{G}_z\|_{op} = \|\mathcal{G}_zS^*_{ij}\|_{op} \leq C (\text{Re}(z))^{-1/2}$.

**Proof.** That $\|S_{ij}\mathcal{G}_z\|_{op} = \|\mathcal{G}_zS^*_{ij}\|_{op}$ follows by (4.5), so it is enough to bound $\|S_{ij}\mathcal{G}_z\|_{op}$. Fix $u \in \mathcal{L}^2(\mathbb{R}^{2n})$ and apply (4.3) for $u' = \mathcal{G}_z u$ to get

$$\langle S^*_{ij}\mathcal{G}_z u(q_{2-n}) \rangle = \int_{\mathbb{R}^n} \frac{\hat{u}(M_{ij}q)}{2\pi} |M_{ij}|^2 \frac{dq_1}{2\pi}.$$  

Calculate the norm of $S_{ij}\mathcal{G}_z u$ from (4.9) and by the same argument as in (4.8) we get

$$\|S_{ij}\mathcal{G}_z u\|^2 = \int_{\mathbb{R}^{2n-2}} \int_{\mathbb{R}^n} \frac{\hat{u}(M_{ij}q)}{2\pi} |M_{ij}|^2 \frac{dq_1}{2\pi} \int_{\mathbb{R}^{2n-2}} \|u\|^2 \|q_1\|^2 \frac{dq_2}{2\pi} \leq \int_{\mathbb{R}^{2n-2}} \|u\|^2 \|q_1\|^2 \frac{dq_1}{2\pi} \int_{\mathbb{R}^{2n-2}} \|u\|^2 \|q_2\|^2 \frac{dq_2}{2\pi},$$

Applying the Cauchy–Schwarz inequality over the $q_1$ integration, we conclude

$$\|S_{ij}\mathcal{G}_z u\|^2 \leq \left( \int_{\mathbb{R}^n} \frac{1}{|q_1|^2} \frac{dq_1}{2\pi} \right)^2 \|u\|^2 \leq \left( \int_{\mathbb{R}^n} \frac{1}{(|q_1|^2 + \text{Re}(z))^2} \frac{dq_1}{2\pi} \right)^2 \|u\|^2.$$  

The last integral over $q_1$ can be evaluated in polar coordinate to be $\frac{1}{2\pi} \text{Re}(z)$. This completes the proof. \hfill \Box

Having built the limiting operator, our next step is to show the convergence. In the course of doing so, we will often use a partial Fourier transform in the last $n-1$ components:

$$\hat{f}(q_1, q_{2-n}) := \int_{\mathbb{R}^{2n-2}} e^{-iy(q_2, \ldots, q_n)} f(q_1, \ldots, q_n) \prod_{i=2}^n \frac{dq_i}{2\pi}.$$  

(4.10)

Recall $S_{eij}$ from (3.4). To prepare for the proof of the convergence, we establish an expression of $S_{eij}u$ in partial Fourier variables.

**Lemma 4.3.** For every $1 \leq i < j \leq n$ and $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\widehat{S_{eij}u}(y_1, q_{2-n}) = \int_{\mathbb{R}^2} e^{i\eta \cdot y_1} \hat{u}(M_{ij}q) \frac{dq_1}{2\pi}.$$  

(4.11)

**Proof.** A partial Fourier transform can be obtained by inverting a full transform in the first component:

$$\widehat{S_{eij}f}(y_1, q_{2-n}) = \int_{\mathbb{R}^2} \widehat{S_{eij}f}(\eta) e^{iy_1 \cdot \eta} \frac{d\eta}{2\pi}.$$  

(4.12)

We write the full Fourier transform as $\widehat{S_{eij}f}(\eta) = \int_{\mathbb{R}^{2n}} e^{-i\eta \cdot q} \hat{f}(S_{eij}y) \frac{dq}{(2\pi)^n}$. We wish to perform a change of variable $x = S_{eij}y$. Doing so requires understanding how $(y \cdot q)$ transform accordingly. Defining

$$M_{eij}q := (q_3, \ldots, \frac{1}{2}q_2 + \varepsilon^{-1}q_1, \ldots, \frac{1}{2}q_2 - \varepsilon^{-1}q_1, \ldots, q_n),$$

it is readily checked that $y \cdot q = (M_{eij}q) \cdot (S_{eij}y)$. Given this, we perform the change of variable $x = S_{eij}y$. With $|\det(S_{eij})| = \varepsilon^2$, we now have

$$\widehat{S_{eij}f}(\eta) = \varepsilon^{2} - \int_{\mathbb{R}^{2n}} e^{-i(M_{eij}q) \cdot x} f(x) \frac{dx}{(2\pi)^n} = \varepsilon^{-2} \hat{f}(M_{ij}q).$$  

(4.13)

Inserting (4.13) into the r.h.s. of (4.12), and performing a change of variable $q_1 \rightarrow \varepsilon q_1$, under which $M_{eij}q \rightarrow M_{ij}q$, we conclude the desired result (4.11). \hfill \Box
We now show the convergence. Recall $\Omega_\phi$ from (3.10).

**Lemma 4.4.** For each $i < j$ and $\text{Re}(z) < 0$, we have\[ \left\| \phi S_{ij} G_z - \phi \otimes (S_{ij} G_z) \right\|_{op} + \left\| G_z S^*_{ij} \phi - G_z S^*_{ij} \Omega_\phi \right\|_{op} \leq C \varepsilon^{\frac{1}{2}} (-\text{Re}(z))^{-1/4} \to 0, \text{ as } \varepsilon \to 0. \]

**Proof.** It suffices to consider $\phi S_{ij} G_z$ since $G_z S^*_{ij} \phi = (\phi S_{ij} G_z)^*$ and $G_z S^*_{ij} \Omega_\phi = (\phi \otimes (S_{ij} G_z))^*$. Fix $u \in \mathscr{S}^\prime(\mathbb{R}^2)$, and, to simplify notation, let $u' := (\phi S_{ij} G_z - \phi \otimes (S_{ij} G_z))u$. We use (4.9) and (4.11) to calculate the partial Fourier transform of $u'$ as\[ \hat{u}'(y_1, q_2 - n) = \phi(y_1) \int_{\mathbb{R}^2} \frac{e^{iy_1 \cdot q_1} - 1}{\sqrt{2}} \hat{g}(M_{ij} q) \frac{dq_1}{2\pi}. \]

From this we calculate the norm of $u'$ as\[ \|u'\|^2 = \int_{\mathbb{R}^{2n}} \left| \hat{u}'(y_1, q_2 - n) \right|^2 dy_1 dq_2 - n = \int_{\mathbb{R}^{2n}} \left| \phi(y_1) \int_{\mathbb{R}^2} \frac{e^{iy_1 \cdot q_1} - 1}{\sqrt{2}} \hat{g}(M_{ij} q) \frac{dq_1}{2\pi} \right|^2 dy_1 dq_2 - n. \]

Recall that, by assumption, $\phi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ is fixed, so $|\phi(y_1)| \leq C1_{\{|y_1| \leq C\}}$. For $|y_1| \leq C$ we have $|e^{iy_1 \cdot q_1} - 1| \leq C((|y_1|)^\land 1)$. Using this and $|M_{ij} q|^2 \geq 2|q_1|^2$ (as verified from (4.1)), we have\[ \|u'\|^2 \leq C \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^2} \frac{\left|\frac{\varepsilon|q_1|}{2\pi} \hat{g}(M_{ij} q)\right|^2 dq_1}{|q_1|^2 - \text{Re}(z)} \right)^2 dq_2 - n \leq C \|u\|^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{\left|\frac{\varepsilon|q_1|}{2\pi} \hat{g}(M_{ij} q)\right|^2 dq_1}{|q_1|^2 - \text{Re}(z)} \right)^2 dq_1. \]

Set $-\text{Re}(z) = a > 0$ to simplify notation. We perform a change of variable $q_1 \mapsto \sqrt{a}q_1$ in the last integral to get $\frac{1}{a^2} \int_{\mathbb{R}_a^{2n}} \left( \frac{\varepsilon|q_1|}{|q_1|^2 + a} \right)^{\land 1} dq_1$. Decompose it according to $|q_1| < \varepsilon a^{1/4}$ and $|q_1| > \varepsilon a^{1/4}$. For the former use $\left( \frac{\varepsilon|q_1|}{|q_1|^2 + a} \right)^{\land 1} \leq 1$, and for the latter use $\left( \frac{\varepsilon|q_1|}{|q_1|^2 + a} \right)^{\land 1} \leq \left( \frac{\varepsilon|q_1|}{|q_1|^2 + a} \right)^2$. It is readily checked that the integrals are both bounded by $C\varepsilon a^{-1/2}$. \qed

5. Off-diagonal mediating operators

To get a rough idea of how the mediating operators (those in (3.6b)) should behave as $\varepsilon \to 0$, we perform a regularity exponent count similar to the discussion just before Lemma 4.2. Recall that $G_z$ increases regularity by 2, while $S_{ij}$ and $S^*_{ij}$ decrease regularity by $-1^+$. **Formally** the regularity of $S_{ij} G_z S^*_{kl}$ adds up to $2 - (1^+) - (1^+) = 0 < 0$. This being the case, one might expect $S_{ij} G_z S^*_{kl}$ to diverge, in a somewhat marginal way, as $\varepsilon \to 0$.

As we will show in the next section, the diagonal operator $S_{i12} G_z S^*_{12}$ diverges logarithmically in $\varepsilon$. This divergence, after a suitable manipulation, cancels the relevant, leading order divergence in $\beta_\varepsilon^{-1} I$ (recall from (1.7) that $\beta_\varepsilon^{-1} \to \infty$). On the other hand, for each $(i < j) \neq (k < l)$, the off-diagonal operator $S_{ij} G_z S^*_{kl}$ converges. This is not an obvious fact, cannot be teased out from the preceding regularity counting, and is ultimately due to an inequality derived in [DFT94, Equation (3.2)]. We treat the off-diagonal terms in this section.

We begin by building the limiting operator.

**Lemma 5.1.** Fix $(i < j) \neq (k < l)$ and $\text{Re}(z) < 0$. We have that $G_z S^*_{kl} : \mathcal{L}^2(\mathbb{R}^{2n-2}) \to \text{Dom}(S_{ij})$, so $S_{ij} G_z S^*_{kl}$ maps $\mathcal{L}^2(\mathbb{R}^{2n-2})$ to $\mathcal{L}^2(\mathbb{R}^{2n-2})$. Furthermore, $\|S_{ij} G_z S^*_{kl}\|_{op} \leq C$ and\[ \langle g, S_{ij} G_z S^*_{kl} f \rangle = \int_{\mathbb{R}^{2n}} \frac{g(p_i + p_j, P_{ij})}{\sqrt{2}|p|^2 - z} \tilde{f}(p_k + p_l, P_{kl}) \frac{dp}{(2\pi)^2}, \quad (5.1) \]

for $f, g \in \mathcal{L}^2(\mathbb{R}^{2n-2})$.

**Proof.** The inequalities derived in [DFT94, Equations (3.1), (3.3), (3.4), (3.6)] translate, under our notation, into\[ \sup_{\alpha > 0} \int_{\mathbb{R}^{2n}} \frac{|\tilde{g}(p_i + p_j, P_{ij})||\tilde{f}(p_k + p_l, P_{kl})|}{|p|^2 + \alpha} dp \leq C \|g\| \|f\|, \quad (5.2) \]

for all $(i < j) \neq (k < l)$ and $f, g \in \mathcal{L}^2(\mathbb{R}^{2n-2})$. Also, from (4.4) we have\[ \tilde{S^*_{ij}} g(p) = \frac{1}{2\pi} \tilde{g}(p_i + p_j, P_{ij}), \quad \tilde{S^*_{kl}} f(p) = \frac{1}{2\pi} \tilde{f}(p_k + p_l, P_{kl}). \quad (5.3) \]
A priori, we only have \( G_z S_{\ell \ell} f \in \mathcal{L}^2(\mathbb{R}^{2n}) \) from Lemma 4.1. Given (5.2)–(5.3) together with \( \text{Re}(z) < 0 \), we further obtain
\[
\int_{\mathbb{R}^{2n}} \left| \tilde{g}(p_i + p_j, p_{\overline{ij}}) \right| \frac{1}{|p|^2 - z} S_{\ell \ell} f(p) \, dp = \int_{\mathbb{R}^{2n}} \left| \tilde{g}(q_2 - n) \right| \frac{1}{|M_{ij} q|^2 - z} S_{\ell \ell} f(M_{ij} q) \, dq \leq C \| g \| \| f \|, \tag{5.4}
\]
where, in deriving the equality, we apply a change of variable \( g = M_{ij}^{-1} p \), together with \( (p_i + p_j, p_{\overline{ij}}) = \left( M_{ij}^{-1} p_2 - n, \det(M_{ij}) \right) = 1 \) (as readily verified from (4.1)). Referring to the definition (4.2) of \( \text{Dom}(S_{ij}) \), since (5.4) holds for all \( g \in \mathcal{L}^2(\mathbb{R}^{2n-2}) \), we conclude \( G_z S_{\ell \ell} f \in \text{Dom}(S_{ij}) \) and further that \( |\langle g, S_{ij} G_z S_{\ell \ell} f \rangle| = |\langle S_{ij}^* g, G_z S_{\ell \ell} f \rangle| \leq C \| g \| \| f \| \). The desired identity (5.1) now follows from (5.3). \( \square \)

We next derive the \( \varepsilon > 0 \) analog of (5.1). Recall that \( \tilde{\mathcal{V}}(y_1, q_{2-n}) \) denotes partial Fourier transform in the last \( n - 1 \) components.

Lemma 5.2. For (not necessarily distinct) \((i < j), (k < \ell), \text{Re}(z) < 0, \) and \( v, w \in \mathcal{S}(\mathbb{R}^{2n}), \)
\[
\langle w, S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* v \rangle = \int_{\mathbb{R}^{2n}} \hat{w}^*(\varepsilonvik(p_i - p_j), p_i + p_j, p_{\overline{ij}}) \frac{1}{|p|^2 - z} \hat{v}(\varepsilonvik(p_k - p_\ell), p_k + p_\ell, p_{\overline{\ell \ell}}) \, dp \tag{5.5a}
\]
where \( \varepsilon = \frac{1}{2pi} \). Our goal is to expressing the last quantity in Fourier variables, which amounts to expressing \( S_{\varepsilon \ell \ell}^* v \) and \( S_{\varepsilon ij}^* w \) in Fourier variables. Recall (from (3.4)) that \( S_{\varepsilon ij} \) acts on \( \mathcal{L}^2(\mathbb{R}^{2n}) \) by \( v(\cdot) \mapsto \varepsilon^{-2}v(T_{\varepsilon ij} \cdot) \), where \( T_{\varepsilon ij} \) is the invertible linear transformation defined in (3.2). Write
\[
S_{\varepsilon ij}^* w(p) = \int_{\mathbb{R}^{2n}} e^{-ip \cdot x} \varepsilon^{-2} w(T_{\varepsilon ij} x) \frac{dx}{(2\pi)^n}.
\]
We wish to perform a change of variable \( T_{\varepsilon ij} x = y \). Doing so requires understanding how \( (p \cdot x) \) transform accordingly. Defining \( \tilde{M}_{\varepsilon ij} p := (\varepsilonvik(p_i - p_j), p_i + p_j, p_{\overline{ij}}), \) it is readily checked that \( p \cdot x = \tilde{M}_{\varepsilon ij} p \cdot (T_{\varepsilon ij} x) \). Given this, we perform the change of variable \( T_{\varepsilon ij} x = y \). With \( |\det(T_{\varepsilon ij})| = \varepsilon^{-2} \), we now have
\[
S_{\varepsilon ij}^* w(p) = \int_{\mathbb{R}^{2n}} e^{-i\tilde{M}_{\varepsilon ij} p \cdot y} w(y) \frac{dy}{(2\pi)^n} = \hat{w}(\tilde{M}_{\varepsilon ij} p) = \hat{w}(\varepsilonvik(p_i - p_j), p_i + p_j, p_{\overline{ij}}),
\]
and similarly \( \hat{S}_{\varepsilon \ell \ell}^* v(p) = \hat{v}(\varepsilonvik(p_k - p_\ell), p_k + p_\ell, p_{\overline{\ell \ell}}) \). From these expressions of \( S_{\varepsilon \ell \ell}^* v \) and \( S_{\varepsilon ij}^* w \) we conclude (5.5a). The identity (5.5b) follows from (5.5a) by writing \( \tilde{\mathcal{V}}(y_1, q_{2-n}) = \int_{\mathbb{R}^2} e^{i y_1 \cdot p} \tilde{v}(p) \frac{dp}{2\pi} \) and (similarly for \( w \)). \( \square \)

A useful consequence of Lemma 5.2 is the following norm bound.

Lemma 5.3. For distinct \((i < j) \neq (k < \ell), \text{Re}(z) < 0, \) and \( \varepsilon \in (0, 1), \)
\[
\| \phi S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* \phi \|_{\text{op}} \leq C.
\]
Proof. In (5.5b), apply (5.2) with \( f(\cdot) = \phi(y_1) \tilde{\mathcal{V}}(y_1, \cdot) \) and \( g(\cdot) = \phi(y_1') \tilde{\mathcal{V}}(y_1', \cdot) \). and integrate the result over \( y_1, y_1' \). We have
\[
|\langle \phi w, S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* \phi \rangle| \leq C \int_{\mathbb{R}^2} \| v(y_1, \cdot) \| \phi(y_1) dy_1 \int_{\mathbb{R}^2} \| w(y_1', \cdot) \| \phi(y_1') dy_1'.
\]
The last expression, upon an application of the Cauchy–Schwarz inequality in \( y_1 \) and in \( y_1' \), is bounded by \( C \| v \| \| w \| \). From this we conclude \( \| \phi S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* \phi \|_{\text{op}} \leq C. \) \( \square \)

We are now ready to establish the convergence of the operator \( \phi S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* \phi \) for distinct pairs. Recall \( \Omega_\phi \) from (3.10).

Lemma 5.4. For each \((i < j) \neq (k < \ell), \text{Re}(z) < 0, \) we have \( \phi S_{\varepsilon ij} G_z S_{\varepsilon \ell \ell}^* \phi \to \phi \otimes (S_{ij} G_z S_{\ell \ell}^* \Omega_\phi) \) strongly as \( \varepsilon \to 0. \)
Proof. Our goal is to show \( \phi S_{ij} G_z S_{kl}^{\phi v} \rightarrow \phi \otimes S_{ij} G_z S_{kl}^{\phi v} \), for each \( v \in L^2(\mathbb{R}^{2n}) \). As shown in Lemmas 5.1 and 5.3, the operators \( (S_{ij} G_z S_{kl}^{\phi v}) \) and \( (S_{ij} G_z S_{kl}^{\phi v}) \) are norm-bounded, uniformly in \( \varepsilon \). Hence it suffices to consider Schwartz \( v \). Fix \( v \in L^2(\mathbb{R}^{2n}) \), and, to simplify notation, set \( u_\varepsilon := (\phi S_{ij} G_z S_{kl}^{\phi v})v \) and \( w := (\phi \otimes S_{ij} G_z S_{kl}^{\phi v})v \). The strategy of the proof is to express \( \| u_\varepsilon - u \|^2 \) as an integral, and use the dominated convergence theorem.

The first step is to obtain expressions for the partial Fourier transforms of \( u_\varepsilon := (\phi S_{ij} G_z S_{kl}^{\phi v})v \) and \( u := (\phi \otimes S_{ij} G_z S_{kl}^{\phi v})v \). To this end, fix \( v, w \in L^2(\mathbb{R}^{2n}) \), in (5.1), set \( (f(\cdot, g (\cdot )) = (\phi)(y_1) v(y_1, \cdot, \phi(y_1) w(y_1, \cdot)) \), and integrate over \( y_1, y_1' \). Note that \( \hat{f}(p_2- n) = \phi(y_1) \nabla y_1 p_2- n \) (and similar for \( g \)). We have

\[
\langle w, u \rangle = \int_{\mathbb{R}^{2n}} \frac{\phi(y_1, p_1 + p_j, p_{j-1})}{2\pi|p|^2 - z} \phi(y_1) \frac{1}{2\pi} \left( \frac{1}{2\pi} \right)^2.
\]

(5.1')

Similarly, in (5.5b), substitute \( (v, w) = (\phi v, \phi w) \) to get

\[
\langle w, u_\varepsilon \rangle = \int_{\mathbb{R}^{2n}} \frac{\phi(y_1, p_1 + p_j, p_{j-1})}{2\pi|p|^2 - z} \phi(y_1) \frac{1}{2\pi} \left( \frac{1}{2\pi} \right)^2.
\]

(5.5b')

Equations (5.1') and (5.5b') express the inner product (against a generic \( w \)) of \( u_\varepsilon \) and \( u \) in partial Fourier variables. From these expressions we can read off \( \nabla y_1 p_2- n \) and \( \nabla y_1 p_2- n \). Specifically, we perform a change of variable \( q = M_{ij} p = (\frac{1}{2}(p_1 - p_j), p_i + p_j, p_{j-1}) \) in (5.1) and (5.5b'), so that \( \hat{w} \) takes variables \( (y_1, q_2- n) \) instead of \( (y_1, p_1 + p_j, p_{j-1}) \). From the result we read off

\[
\nabla y_1 p_2- n = \int_{\mathbb{R}^4} f_{z,v} y_1 d y_1 d q_1, \quad \nabla u_\varepsilon y_1 p_2- n = \int_{\mathbb{R}^4} E_z f_{z,v} y_1 d y_1 d q_1.
\]

(5.6)

Here \( E_z \) and \( f_{z,v} \) are (rather complicated-looking) functions of \( q, y_1, y_1' \), given in the following. The precisely functional forms of \( f_{z,v} \) and \( E_z \) will be irrelevant. Instead, we will explicitly signify what properties of these functions we are using whenever doing so. We have \( E_z := e^{i\pi z_1 y_1 - i\pi[M_{ij}^{-1} M_{ij} q_{2- n}]}, y_1 \) and

\[
f_{z,v} := \phi(y_1) \frac{1}{2\pi|p|^2 - z} \phi(y_1) \frac{1}{2\pi} \left( \frac{1}{2\pi} \right)^2.
\]

Additionally, we will need an auxiliary function \( v' \in L^2(\mathbb{R}^{2n}) \) such that \( \hat{v'}(y_1, p) = |\nabla y_1 p| \). Such a function \( v' = v'(y) \) is obtained by taking inverse Fourier of \( |\nabla y_1 p_2- n| \) in \( q_2- n \). Note that \( \| v' \| = \| v' \| < \infty \). Set \( a := -\text{Re}(z) > 0 \) and \( v' := (\phi \otimes S_{ij} G_z S_{kl}^{\phi v})v' \). We have

\[
\nabla v'(y_1, q_2- n) = \int_{\mathbb{R}^4} f_{-a, v'} y_1 d q_1, \quad f_{-a, v'} \geq |f_{z,v}| > 0.
\]

(5.7)

Now, use (5.6) and (5.7) to write

\[
\| u_\varepsilon - u \|^2 \leq \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^4} |f_{z,v}| \| E_z - 1 | y_1 d q_1 \right)^2 d y_1 d q_2- n.
\]

(5.8)

\[
\| u' \|^2 = \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^4} |f_{-a, v'}| y_1 d q_1 \right)^2 d y_1 d q_2- n.
\]

(5.9)

View (5.8)–(5.9) as integrals over \( \mathbb{R}^{k+2n} \), i.e.,

\[
\text{r.h.s. of (5.8)} := \int_{\mathbb{R}^{k+2n}} g_e d(\ldots), \quad \text{r.h.s. of (5.9)} := \int_{\mathbb{R}^{k+2n}} g d(\ldots).
\]

We now wish to apply the dominated convergence theorem on \( g_e \) and \( g \). To check the relevant conditions, note that: since \( |E_z| \leq 1 \) and \( |f_{z,v}| \leq f_{-a, v'} \), we have \( 0 \leq g_e \leq g \); since \( |E_z| \to 0 \) pointwisely on \( \mathbb{R}^{k+2n} \), we have \( g_e \to 0 \) pointwisely on \( \mathbb{R}^{k+2n} \); the integral of \( g \) over \( \mathbb{R}^{k+2n} \) evaluates to \( \| u' \|^2 = \| (\phi \otimes S_{ij} G_z S_{kl}^{\phi v})v' \|^2 \), which is finite since the operators \( S_{ij} G_z S_{kl}^{\phi v} \), \( (\phi \otimes \cdot) \), and \( \Omega_\phi \) are bounded. The desired result \( \int_{\mathbb{R}^{k+2n}} g_e d(\ldots) = \| u_\varepsilon - w \|^2 \to 0 \) follows.
6. Diagonal mediating operators

The main task here is to analyze the asymptotic behavior of the diagonal part \( \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \), which diverges logarithmically. We begin by deriving an expression for \( \langle w, \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \rangle \) that exposes such an \( \varepsilon \to 0 \) behavior. Let \( \mathbb{G}_c(x), x \in \mathbb{R}^2 \) denote the Green’s function in two dimensions. Recall that \( |p|_2^2 := \frac{1}{2} |p_2|^2 + |p_3|^2 + \ldots + |p_n|^2 \).

Lemma 6.1. For \( v, w \in \mathcal{L}^2(\mathbb{R}^{2n}) \), we have
\[
\langle w, \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \rangle = \int_{\mathbb{R}^{2n}} \frac{|v(y_1', p_2-n)|}{2 \mathbb{G}_c(\frac{1}{2} z - \frac{1}{2} |p|_2^2)} (y_1' - y_1) \phi(y_1) \mathbb{V}(y_1, p_2-n) \ dy_1 dy'_1 dp_{2-n}.
\]

(6.1)

Proof. Apply Lemma 5.2 for \( (i < j) = (k < \ell) = (1 < 2) \) and for \( (v, w) \mapsto (\phi v, \phi w) \), and perform a change of variable \( (\frac{v-p_2}{2}, p_1 + p_2) \mapsto (p_1, p_2) \) in the result. We obtain
\[
\langle w, \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \rangle = \int_{\mathbb{R}^{2+n}} \frac{|v(y_1', p_2-n)|}{|p|^2 + \frac{1}{2} |p|_2^2 - z} \phi(y_1) \mathbb{V}(y_1, p_2-n) \ dy_1 dy'_1 dp_1 dp_{2-n}.
\]

(6.2)

and we recognize \( \int_{\mathbb{R}^2} \frac{e^{i p_1 \cdot z}}{|p|^2 - \frac{1}{2} |p|_2^2} \ dp_1 \) as the Fourier transform of the two-dimensional Green’s function \( \mathbb{G}_c \).

Lemma 6.1 suggests analyzing the behavior of \( \mathbb{G}_c(x) \) for small \( |z| \):

Lemma 6.2. Take the branch cut of the complex-variable functions \( \sqrt{z} \) and \( \log z \) to be \( (-\infty, 0) \), let \( \gamma_{\text{EM}} \) denote the Euler–Mascheroni constant. For all \( x \neq 0 \) and \( z \in \mathbb{C} \setminus [0, \infty) \), we have
\[
\mathbb{G}_c(x) = \frac{-1}{\pi} \log \frac{\sqrt{-2\pi|x|}}{\sqrt{2}} - \frac{1}{\pi} \gamma_{\text{EM}} + A(\sqrt{-zx}),
\]

(6.3)

for some \( A(\cdot) \) that grows linearly near the origin, i.e., \( \sup_{|z| \rightarrow 0} |z|^{-1} |A(z)| \leq C(a), \) for all \( a < \infty \).

The proof follows from classical special function theory. We present it here for the convenience of the readers.

Proof. Write the equation \( \left(-\frac{1}{2} \nabla^2 - z\right) \mathbb{G}_c(x) = 0, x \neq 0 \), in radial coordinate, compare the result to the modified Bessel equation [AS65, 9.6.1], and note that \( \mathbb{G}_c(x) \) vanishes at \( |x| \to \infty \). We see that \( \mathbb{G}_c(x) = c K_0(\sqrt{-2\pi|x|}), \) for some constant \( c \), where \( K_0 \) denotes the modified Bessel function of second kind. To fix \( c \), compare the know expansion of \( K_0(r) \) around \( r = 0 \) [AS65, 9.6.4] (noting that \( I_0(0) = 1 \) therein), and use \( -\pi \frac{d}{dr} G_z(|r|) = 1 \) (because \( -\frac{1}{2} \nabla^2 G_z(x - z) = \delta(x) \)) for \( r \to 0 \). We find \( c = \frac{1}{\pi} \). The claim now follows from [AS65, 9.6.4].

For subsequent analysis, it is convenient to decompose \( \mathcal{L}^2(\mathbb{R}^{2n}) \) into a ‘projection onto \( \phi \)’ and its orthogonal complement. More precisely, recall \( \Omega_\phi \) from (3.10), and that \( \int \phi^2 = 1 \), we define the projection
\[
\Pi_\phi := \phi \otimes \Omega_\phi : \mathcal{L}^2(\mathbb{R}^{2n}) \to \mathcal{L}^2(\mathbb{R}^{2n}), \quad \phi \otimes \Omega_\phi(v)(y) := \phi(y_1) \int_{\mathbb{R}^2} \phi(y'_1) v(y_1', y_2-n) \ dy'_1.
\]

(6.4)

Returning to the discussion about the \( \varepsilon \to 0 \) behavior of \( \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \), inserting (6.3) into (6.1), we see that \( \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \) has a divergent part \( (\frac{1}{\pi} \log \|z\|) \Pi_\phi \). The coefficient \( \frac{1}{\pi} \log \varepsilon \) matches the leading order of \( \beta_\varepsilon^{-1} \) (see (1.7)), so \( \Pi_\phi \) cancels the divergence \( \beta_\varepsilon^{-1} \Pi_\phi \) on the subspace \( \text{Im}(\Pi_\phi) \), but still leaves the remaining part \( \beta_\varepsilon^{-1} I_{\text{Im}(\Pi_\phi)} = \beta_\varepsilon^{-1} (I - \Pi_\phi) \) divergent. However, recall that \( \beta_\varepsilon^{-1} I - \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \) appears as an inverse in the resolvent identity (3.6). Upon taking inverse, the divergent part on \( \text{Im}(\Pi_\phi) \) becomes a vanishing term.

We now begin to show the convergence of \( \beta_\varepsilon^{-1} I - \phi S_{12} \mathbb{G} \mathbb{S}^*_c \phi \). Doing so requires a technical lemma. To setup the lemma, consider a collection of bounded operators \( \{ T_{\varepsilon; p} : \mathcal{L}^2(\mathbb{R}^2) \to \mathcal{L}^2(\mathbb{R}^2) \} \), indexed by \( \varepsilon \in (0, 1) \) and \( p \in \mathbb{R}^{2n-2} \), such that for each \( \varepsilon > 0 \), \( \sup_{p \in \mathbb{R}^{2n-2}} \| T_{\varepsilon; p} \|_{\text{op}} < \infty \). Note that here, unlike in the preceding, here \( p = (p_2, \ldots, p_n) \in \mathbb{R}^{2n-2} \) denotes a vector of \( n-1 \) components. For each \( \varepsilon \in (0, 1) \), construct a bounded operator \( T_{\varepsilon} \) as
\[
T_{\varepsilon} : \mathcal{L}^2(\mathbb{R}^2) \to \mathcal{L}^2(\mathbb{R}^2), \quad T_{\varepsilon} u(\cdot) := T_{\varepsilon; p} u(\cdot, p).
\]

Roughly speaking, we are interested in an operator \( T_{\varepsilon} \) that acts on \( y_1 \in \mathbb{R}^2 \) in a way that depends on the partial Fourier components \( p = (p_2, \ldots, p_n) \in \mathbb{R}^{2n-2} \). The operator \( T_{\varepsilon; p} \) records the action of \( T_{\varepsilon} \) on \( y_1 \) per
fixed \( p \in \mathbb{R}^{2n-2} \). We are interested in obtaining the inverse \( T_{\varepsilon}^{-1} \) and its strong convergence (as \( \varepsilon \downarrow 0 \)). The following lemma gives the suitable criteria in terms of each \( T_{\varepsilon,p} \).

**Lemma 6.3.** Let \( \{T_{\varepsilon,p}\} \) and \( T_{\varepsilon} \) be as in the preceding. If each \( T_{\varepsilon,p} \) is invertible with

\[
\sup \{ \|T_{\varepsilon,p}^{-1}\|_{\text{op}} : \varepsilon \in (0,1), p \in \mathbb{R}^{2n-2} \} := b < \infty,
\]

and if each \( T_{\varepsilon,p}^{-1} \) permits a norm limit, i.e., there exists \( T'_{p} : L^{2}(\mathbb{R}^{2}) \rightarrow L^{2}(\mathbb{R}^{2}) \) such that

\[
T_{\varepsilon,p}^{-1} \rightarrow T'_{p} \quad \text{in norm as} \quad \varepsilon \rightarrow 0,
\]

then \( T_{\varepsilon} \) is invertible with \( \sup_{\varepsilon \in (0,1)} \|T_{\varepsilon}^{-1}\|_{\text{op}} \leq b < \infty \),

\[
T_{\varepsilon}^{-1} \rightarrow T', \quad \text{strongly, as} \quad \varepsilon \rightarrow 0,
\]

and \( \|T'\|_{\text{op}} \leq b < \infty \), where the operator \( T' : L^{2}(\mathbb{R}^{2n}) \rightarrow L^{2}(\mathbb{R}^{2n}) \) is built from the limit of each \( T_{\varepsilon,p}^{-1} \) as \( T'_{p}(u\star,\star,p) := T'_{p} T_{p}(u\star,\star,p) \).

**Proof.** We begin by constructing the inverse of \( T_{\varepsilon} \). By assumption each \( T_{\varepsilon,p} \) has inverse \( T_{\varepsilon,p}^{-1} \), from which we define \( T_{\varepsilon}^{-1}(u\star,\star,p) := T_{\varepsilon,p}^{-1} T_{p}(u\star,\star,p) \). It is readily checked that \( \|T'\|_{\text{op}} \leq \sup_{\varepsilon,p} \|T_{\varepsilon,p}^{-1}\| \leq b \), and the operator \( T'_{\varepsilon} \) actually gives the inverse of \( T_{\varepsilon} \), i.e., \( T'_{\varepsilon} T_{\varepsilon} = T_{\varepsilon} T'_{\varepsilon} = I \). Note that, for each \( p \in \mathbb{R}^{2n-2} \), the operator \( T'_{p} \) inherits a bound from \( T_{\varepsilon,p}^{-1} \), i.e., \( \sup_{p} \|T'_{p}\|_{\text{op}} \leq \|T_{\varepsilon,p}^{-1}\|_{\text{op}} \leq b \). Together with the definition of \( T' \) we also have \( \|T'\|_{\text{op}} \leq b \).

It remains to check the strong convergence. For each \( u \in L^{2}(\mathbb{R}^{2}) \) we have

\[
\|T_{\varepsilon}^{-1}u - T'u\|^{2} = \int_{\mathbb{R}^{2n-2}} \left( \int_{\mathbb{R}^{2}} |T_{\varepsilon,p}^{-1} \mathcal{U}(y,p) - T'_{p} \mathcal{U}(y,p)|^{2} \, dy \right) \, dp
\]

\[
\leq \int_{\mathbb{R}^{2n-2}} \|T_{\varepsilon,p}^{-1} - T'_{p}\|_{\text{op}}^{2} \int_{\mathbb{R}^{2}} |\mathcal{U}(y,p)|^{2} \, dy \, dp.
\]

The integrand within the last integral converges to zero pointwisely, and is dominated by \( 4b^{2} \|\mathcal{U}(y,p)\|^{2} \), which is integrable over \( \mathbb{R}^{2n} \). Hence by the dominated convergence theorem \( \|T_{\varepsilon}^{-1}u - T'u\|^{2} \rightarrow 0 \).

With Lemma 6.3, we next establish the norm boundedness and strong convergence of \( (\beta_{\varepsilon}^{-1}I - \phi S_{12}G_{z}S_{12}^*\phi)^{-1} \) in two steps, first for fixed \( p \in \mathbb{R}^{2n-2} \). Slightly abusing notation, in the following lemma, we also treat \( \Pi_{\phi} \) (defined in (6.4)) as its analog on \( L^{2}(\mathbb{R}^{2}) \), namely the projection operator \( \Pi_{\phi} f(y) := \phi(y) \int_{\mathbb{R}^{2}} \phi(y') f(y') \, dy' \).

**Lemma 6.4.** For each \( p \in \mathbb{R}^{2n-2} \), define an operator \( T_{\varepsilon,p} : L^{2}(\mathbb{R}^{2}) \rightarrow L^{2}(\mathbb{R}^{2}) \),

\[
T_{\varepsilon,p} f(y) := \beta_{\varepsilon} f(y) - \phi(y) \int_{\mathbb{R}^{2}} \frac{1}{2} G_{z}(\frac{1}{2} p + \frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} - \frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} |(y - y')\phi(y') f(y') \, dy'.
\]

Then, there exist constants \( C_{1} < \infty \), \( C_{2}(\beta_{\text{fine}}) > 0 \) such that, for all \( \text{Re}(z) < -\beta_{\ast} + C_{1} \) and \( \varepsilon \in (0,1/C_{2}(\beta_{\text{fine}})) \),

\[
\left\| (T_{\varepsilon,p})^{-1} \right\|_{\text{op}} \leq C \left( \log(-\text{Re}(z)) - \beta_{\ast} \right)^{-1},
\]

\[
(T_{\varepsilon,p})^{-1} \rightarrow \frac{4\pi}{\log(\frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} - z)} \Pi_{\phi}, \quad \text{in norm as} \quad \varepsilon \rightarrow 0, \quad \text{for each fixed} \quad p \in \mathbb{R}^{2n-2}.
\]

**Proof.** Throughout the proof, we say a statement holds for \( \text{Re}(z) \) large enough, if the statement holds for all \( \text{Re}(z) > e^{2\varepsilon + C} \), for some fixed constant \( C < \infty \), and we say a statement holds for all \( \varepsilon \) small enough, if the statement holds for all \( \varepsilon < 1/C(\beta_{\text{fine}}) \), for some constant \( C(\beta_{\text{fine}}) < \infty \) that depends only on \( \beta_{\text{fine}} \).

Our first goal is to show \( T_{\varepsilon,p} \) is invertible and establish bounds on \( \|T_{\varepsilon,p}^{-1}\|_{\text{op}} \). We do this in two separate cases: \( i \) \( \frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} - z \leq \varepsilon^{-2} \) and \( ii \) \( \frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} - z > \varepsilon^{-2} \).

\( i \) The first step here is to derive a suitable expansion of \( T_{\varepsilon,p} \). Recall that, we have abused notation to write \( \Pi_{\phi} \) (defined in (6.4)) for the projection operator \( \Pi_{\phi} f(y) := \phi(y) \int_{\mathbb{R}^{2}} \phi(y') f(y') \, dy' \). Applying Lemma 6.2 yields

\[
T_{\varepsilon,p} = \beta_{\varepsilon}^{-1} I + \left( -\frac{1}{4\pi} \log|\varepsilon| + \frac{1}{4\pi} \log(\frac{1}{2} |p|_{\mathbb{Z}_{-2n}}^{-1} - z) - \frac{1}{4\pi} \log 2 + \frac{1}{4\pi} \gamma_{\text{EM}} \right) \Pi_{\phi} + \mathcal{L}_{\phi} - \mathcal{A}_{\varepsilon,z,p}, \quad (6.6)
\]
where $\mathcal{L}_\phi$ and $\mathcal{A}_{\varepsilon,z,p}$ are integral operators $\mathcal{L}^2(\mathbb{R}^2) \rightarrow \mathcal{L}^2(\mathbb{R}^2)$ defined as
\begin{equation}
(\mathcal{L}_\phi f)(y_1) := \frac{1}{2\pi} \phi(y_1) \int_{\mathbb{R}^2} \log |y_1 - y'_1| \phi(y'_1) f(y'_1) \, dy'_1,
\end{equation}
\begin{equation}
(\mathcal{A}_{\varepsilon,z,p} f)(y_1) := \frac{1}{2} \phi(y_1) \int_{\mathbb{R}^2} \left( \frac{1}{2} \log \left| y_1 - y'_1 \right| + \sqrt{\frac{1}{2} |p|_{2-n}^2 - z} \right) \phi(y'_1) f(y'_1) \, dy'_1,
\end{equation}
and the function $A(\cdot)$ is the remainder term in Lemma 6.2. Let $\Pi_{\perp} := I - \Pi_\phi$ denote the orthogonal projection onto $(\mathbb{C}\phi)^1$ in $\mathcal{L}^2(\mathbb{R}^2)$ and recall $\beta_\varepsilon$ from (1.7). In (6.6), decomposing $\beta_\varepsilon^{-1} I = \beta_\varepsilon^{-1} \Pi_{\perp} + \frac{1}{4\pi} (|\log \varepsilon| - \beta_{\varepsilon,\text{fine}}) \Pi_\phi$, where $\beta_{\varepsilon,\text{fine}} := |\log \varepsilon| - |\log \varepsilon| (1 + \frac{\beta_{\varepsilon,\text{fine}}}{|\log \varepsilon|})^{-1}$, we rearrange terms to get
\begin{equation}
\mathcal{T}_{\varepsilon,p} = \beta_\varepsilon^{-1} \Pi_{\perp} + \frac{1}{4\pi} (\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}) \Pi_\phi + \mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p},
\end{equation}
where $\beta_{\varepsilon,\varepsilon} := 2 (\log 2 + \varepsilon_{\varepsilon,\text{fine}} - \gamma_{\text{EM}})$. We next take the inverse of $\mathcal{T}_{\varepsilon,p}$ from (6.9), utilizing
\begin{equation}
(\mathcal{Q} - \tilde{\mathcal{Q}})^{-1} = \sum_{m=0}^{\infty} \mathcal{Q}^{-1} (\tilde{\mathcal{Q}} \mathcal{Q}^{-1})^m,
\end{equation}
valid for operators $\mathcal{Q}$, $\tilde{\mathcal{Q}}$ such that $\mathcal{Q}$ is invertible with $\|\mathcal{Q}^{-1}\|_{\text{op}} \|\tilde{\mathcal{Q}}\|_{\text{op}} < 1$. Our choice will be $\mathcal{Q} := \beta_\varepsilon^{-1} \Pi_{\perp} + \frac{1}{4\pi} (\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}) \Pi_\phi$ and $\tilde{\mathcal{Q}} := -\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p}$.

From (6.7), we have $\|\mathcal{Q}\|_{\text{op}} < \infty$. Under our current assumption $|\frac{1}{2} |p|_{2-n}^2 - z| \leq \varepsilon^{-2}$, from (6.8) and the property of $A(\cdot)$ stated in Lemma 6.2, we have $\|\mathcal{A}_{\varepsilon,z,p}\|_{\text{op}} \leq C < \infty$. Hence
\begin{equation}
\| -\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p}\|_{\text{op}} \leq C.
\end{equation}
With $\Pi_{\perp}$ and $\Pi_\phi$ being projection operators orthogonal to each other, we calculate
\begin{equation}
(\beta_\varepsilon^{-1} \Pi_{\perp} + \frac{1}{4\pi} (\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}) \Pi_\phi)^{-1} = \beta_\varepsilon \Pi_{\perp} + 4\pi \left( \log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon} \right)^{-1} \Pi_\phi.
\end{equation}
The operator norm of this inverse is thus bounded by $\max\{\beta_\varepsilon, \frac{4\pi}{\log(-\text{Re}(z)) - \beta_{\varepsilon,\varepsilon}}\}$. Since $\beta_{\varepsilon,\varepsilon} \to \beta_\varepsilon + 2\beta_\phi$ and $\beta_\varepsilon \to 0$, this allows us to get a convergent series (6.10) for $-\text{Re}(z)$ large enough and $\varepsilon$ small enough, with $\|\mathcal{T}_{\varepsilon,p}\|_{\text{op}} \leq C(\log(-\text{Re}(z)) - \beta_\varepsilon^{-1})$.

ii) We apply (6.10) again to (6.5) with $\mathcal{Q} := \beta_\varepsilon^{-1} I$. To check the relevant condition, we write the operator $\mathcal{T}_{\varepsilon,p}$ (in (6.5)) in a coordinate-free form as $\mathcal{T}_{\varepsilon,p} = \beta_\varepsilon^{-1} I - \frac{1}{2} G_{\varepsilon^{(n=1)}}(\frac{1}{2} z - \frac{1}{4} |p|_{2-n}^2)^n \phi$, where $G_{\varepsilon^{(n=1)}}$ denotes the two-dimensional Laplace resolvent. Recall that $\text{Re}(z) < -\varepsilon^2+\text{C}_1 < 0$, so $\text{Re}(\frac{1}{2} z - \frac{1}{4} |p|_{2-n}^2) < 0$, which gives $\|G_{\varepsilon^{(n=1)}}(\frac{1}{2} z - \frac{1}{4} |p|_{2-n}^2)\|_{\text{op}} \leq \varepsilon^2(\frac{1}{2} z - \frac{1}{4} |p|_{2-n}^2)^{-1}$. Under the current assumption $|\frac{1}{2} |p|_{2-n}^2 - z| \geq \varepsilon^{-2}$, this is bounded by $\frac{1}{2}$, so
\begin{equation}
\|G_{\varepsilon^{(n=1)}}(\frac{1}{2} z - \frac{1}{4} |p|_{2-n}^2)\phi\| \leq C.
\end{equation}
Since $\beta_\varepsilon^{-1} \to \infty$, (6.10) applied to (6.5) with $\mathcal{Q} = \beta_\varepsilon^{-1} I$, show that $\mathcal{T}_{\varepsilon,p}^{-1}$ exists with $\|\mathcal{T}_{\varepsilon,p}^{-1}\|_{\text{op}} \leq C(\log \varepsilon)^{-1}$, for all $\varepsilon$ small enough.

Having obtained $\mathcal{T}_{\varepsilon,p}^{-1}$ and its bound, we next show the norm convergence. The condition $|\frac{1}{2} |p|_{2-n}^2 - z| \leq \varepsilon^{-2}$ holds for all $\varepsilon \leq C(\rho)$, whence we have from (6.10) that
\begin{equation}
\mathcal{T}_{\varepsilon,p}^{-1} = \left( \beta_\varepsilon \Pi_{\perp} + \frac{4\pi}{\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}} \Pi_\phi \right) \sum_{m=0}^{\infty} \left( (-\mathcal{L}_\phi + \mathcal{A}_{\varepsilon,z,p}) \left( \beta_\varepsilon \Pi_{\perp} + \frac{4\pi}{\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}} \Pi_\phi \right) \right)^m.
\end{equation}

We now take termwise limit in (6.13). Referring to (6.8), with $p \in \mathbb{R}^{2n-2}$ being fixed, the linear growth property of $A(\cdot)$ in Lemma 6.2 gives that $\mathcal{A}_{\varepsilon,z,p}$ converges to 0 in norm. Since $\beta_\varepsilon \to 0$,
\begin{equation}
\beta_\varepsilon \Pi_{\perp} + \frac{4\pi}{\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_{\varepsilon,\varepsilon}} \Pi_\phi \to \frac{4\pi}{\log(\frac{1}{2} |p|_{2-n}^2 - z) - \beta_\varepsilon - 2\beta_\phi} \Pi_\phi, \text{ in norm.}
\end{equation}
Further, the bound (6.11) guarantees that, for all \(-\text{Re}(z)\) large enough, the series (6.13) converges absolutely in norm, uniformly for all \(\varepsilon\) small enough. From this we conclude \(T_{ε,p}^{-1} \rightarrow T_p^{-1}\) in norm, where
\[
T_p := \frac{4\pi}{\log(\frac{1}{2}|p|_2^2 - z) - \beta_* - 2\beta_F} \sum_{m=0}^{∞} \Pi_{φ} \left( \frac{4\pi}{\log(\frac{1}{2}|p|_2^2 - z) - \beta_* - 2\beta_F} (-L_φ) \Pi_{φ} \right)^m.
\]
(6.14)
This expression can be further simplified using \(Π_{φ}^m = Π_{φ}\) and \(Π_{φ} L_φ Π_{φ} = \frac{b_ε}{ππ} Π_{φ}\),
\[
T_p = \frac{4\pi}{\log(\frac{1}{2}|p|_2^2 - z) - \beta_* - 2\beta_F} \sum_{m=0}^{∞} Π_{φ} \left( \frac{-2β_F}{\log(\frac{1}{2}|p|_2^2 - z) - \beta_* - 2β_F} Π_{φ} \right)^m = \frac{4\pi}{\log(\frac{1}{2}|p|_2^2 - z) - \beta_*} \Pi_{φ}.
\]
This completes the proof.

Recall \(J_z\) from (1.22). Combining Lemmas 6.3–6.4 immediately gives the main result of this section:

**Lemma 6.5.** There exist constants \(C_1 < 0, C_2(β_{\text{line}}) > 0\) such that, for all \(\text{Re}(z) < -e^{β_c + C_1}\), and for all \(ε \in (0, 1/C_2(β))\), the inverse \((β_ε^{-1}I - φ S_{12}G_z S_{12}φ)^{-1} : \mathcal{L}^2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}^2(\mathbb{R}^{2n})\) exists, with
\[
\left\| \left( β_ε^{-1}I - φ S_{12}G_z S_{12}φ \right)^{-1} \right\|_\text{op} ≤ C (\log(\text{Re}(z)) - β_c)^{-1},
\]
(6.15)
\[
(β_ε^{-1}I - φ S_{12}G_z S_{12}φ)^{-1} \rightarrow 4πφ \otimes \left( (J_z - β, I)^{-1} Ω_φ \right), \text{ strongly as } ε \to 0.
\]
(6.16)

### 7. Convergence of the Resolvent

In this section we collect the results of Sections 3–6 to prove Proposition 1.4(a)–(b) and Theorem 1.5(a)–(b) and the convergence part of Theorem 1.1(b).

Proposition 1.4(a) and Theorem 1.5(a) follow from the bounds obtained in Lemmas 4.2–4.4, 5.3, and 6.5. We now turn to Theorem 1.5(b). Recall that Lemma 3.2, as stated, applies only for \(\text{Re}(z) < -C(n, ε)\), for some threshold \(C(n, ε)\) that depends on \(ε\). Here we argue that the threshold can be improved to be independent of \(ε\). Given the bounds from Lemmas 4.2–4.4, 5.3, and 6.5, we see that the r.h.s. of (3.6) defines an analytic function (in operator norm) in \(B := \{\text{Re}(z) < -e^{Cn^2 + β_c}\}\). On the other hand, we also know that the l.h.s. \((R_{ε,z} - G_z)\) is analytic in \(z\) off \(σ(H_z) \cup [0, ∞)\), where \(σ(H_z) \subset \mathbb{R}\) denotes the spectrum of \(H_z\). Consequently, both sides must match on \(B \setminus σ(H_z)\). We now argue \(B \cap σ(H_z) = \emptyset\), so the matching actually holds on the entire \(B\). Assuming the contrary, we fix \(z_0 \in B \cap σ(H_z)\), take a sequence \(z_k \in B\) and approaches \(z_k \to z_0\) along the vertical axis. Along this sequence \((R_{ε,z_k} - G_{z_k})\) is bounded, contradicting \(z_0 \in σ(H_z)\).

We now show the convergence of the resolvent, i.e. (3.6) to (3.12). As argued previously, both series (3.6) and (3.12) converge absolutely in operator norm, uniformly over \(ε\). It hence suffices to show termwise convergence. By Lemmas 4.4, 5.4, and 6.5, each factor in (3.6a)–(3.6c) converges to its limiting counterparts in (3.12a)–(3.12c), strongly or in norm. Using this in conjunction with the elementary, readily checked fact
\[
Q_zQ′_z \to Q′ Q \text{ strongly if } Q_z, Q′_z \text{ are uniformly bounded and } Q_z \to Q, Q′_z \to Q′ \text{ strongly},
\]
we conclude the desired convergence of the resolvent, Theorem 1.5(b).

Next we prove Proposition 1.4(b). First, given the bounds from Lemmas 4.2–4.4, 5.3, and 6.5, we see that \(R_{z}^{\text{sym}}\) in (2.4) defines a bounded operator on \(\mathcal{L}^2(\mathbb{R}^{2n})\) for all \(\text{Re}(z) < -e^{β_c + n^2 C}\). Our goal is to match \(R_{z}^{\text{sym}}\) to \(R_z\) on \(\mathcal{L}^2(\mathbb{R}^{2n})\), for these values of \(z\). Apply (6.10) with \(Q = \frac{1}{4π}(J_z - β, I)\) and with \(\Omega = \frac{2}{α(π - 1)} \sum_{j=1}^{d} S_{ij} G_z S_{ik}\) for the prescribed values of \(z\) (so that the condition for (6.10) to apply checks). We obtain
\[
R_{z}^{\text{sym}} = G_z + ∑ S^{i,j}_{k,k} G_z \left( 4π(J_z - β, I)^{-1} \Pi_{s=2}^{m} \left( \frac{2}{n(n-1)} S_{k,k} G_z S^{i,j}_{k,k} 4π(J_z - β, I)^{-1} \right) \right) \frac{2}{n(n-1)} G_z S_{k,m+1} k_{m+1},
\]
(7.1)
where the sum is over all pairs \((i_1 < j_1), (k_2 < ℓ_2) \neq (i_2 < j_2), \ldots, (k_m < ℓ_m) \neq (i_m < j_m), (k_{m+1} < k_{m+1}), and all \(m\).

At this point we need to use the symmetry of \(\mathcal{L}^2(\mathbb{R}^{2n})\). Given that \(G_z\) acts symmetrically in the \(n\) components, for the incoming operator we have
\[
S_{i,j} G_z \big|_{\mathcal{L}^2(\mathbb{R}^{2n})} = S_{i,j} G_z \big|_{\mathcal{L}^2(\mathbb{R}^{2n})}, \quad \forall (i < j), (i' < j').
\]
(7.2)
A similar symmetry holds for the off-diagonal mediating operator. To state it, note that \( G_z \) maps \( L^2_{\text{sym}}(\mathbb{R}^{2n}) \) into a subspace of \( L^2(\mathbb{R}^{2n-2}) \) that consists of functions \( v(y_2, \ldots, y_n) \) symmetric in the last \((n-2)\) components. More explicitly,

\[
G_z(L^2_{\text{sym}}(\mathbb{R}^{2n})) \subset L^2_{\text{sym}}(\mathbb{R}^{2n-2}) := \{ v \in L^2(\mathbb{R}^{2n-2}) : v(y_2, y_\sigma(3), \ldots, y_\sigma(n)) = y(y_2, y_3, \ldots, y_n) \sigma \in S_{n-2} \}.
\]

Also, this space \( L^2_{\text{sym}}(\mathbb{R}^{2n-2}) \) is invariant under the action of mediating operators (both diagonal and off-diagonal). From (5.1) we have

\[
S_{k\ell} G_z S_{ij}^* \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n-2})} = S_{\sigma(k)\sigma(\ell)} G_z S_{\sigma(i)\sigma(j)}^* \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n-2})}, \quad \forall (i < j) \neq (k < \ell), \quad \sigma \in S_n.
\]

In (7.1), use (7.3) to rearrange the sum over \((k < \ell) \neq (i_2 < j_2)\) as

\[
\frac{2}{n(n-1)} \sum_{(k < \ell) \neq (i_2 < j_2)} S_{k\ell} G_z S_{ij}^* \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n-2})} = \sum_{i_2 < j_2} S_{i_2 j_2}^* G_z S_{i_2 j_2} \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n-2})} 1_{\{(i_2 < j_2) \neq (i_1 < i_1)\}}.
\]

That is, we use (7.3) for some \( \sigma \in S_n \) such that \((\sigma(k_2) < \sigma(k_2)) = (i_1 < j_1)\). Doing so reduces the sum over double pairs \((k_2 < \ell_2) \neq (i_2 < j_2)\) into a sum over a single pair \((i_2 < j_2)\) with \((i_2 < j_2) \neq (i_1 < j_1)\), and the counting in this reduction cancels the prefactor \(2/(n(n-1))\). Continue this procedure inductively from \(s = 2\) through \(s = m\), and then, at the \(m + 1\) step, similarly use (7.2) to write

\[
\frac{2}{n(n-1)} \sum_{k_m+1 < \ell_m+1} S_{k_m+1 \ell_m+1} G_z \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n})} = S_{t_m j_m} G_z \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n})}.
\]

We then conclude Proposition 1.4(b),

\[
R_z \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n})} = R_z \big|_{L^2_{\text{sym}}(\mathbb{R}^{2n})}.
\]

We now turn to the convergence of the fixed time correlation functions in Theorem 1.1(b). Given Theorem 1.5, applying the Trotter–Kato Theorem, c.f., [RS72, Theorem VIII.22], we know that there exists an (unbounded) self-adjoint operator \( \mathcal{H} \) on \( L^2(\mathbb{R}^{2n}) \), such that \( R_z \) (in (1.23)) is the resolvent for \( \mathcal{H} \), i.e., \( R_z = (\mathcal{H} - zI)^{-1} \), for all \( \text{Im}(z) \neq 0 \). Theorem 1.5 also guarantees that the spectra of \( \mathcal{H} \) and \( \mathcal{H} \) are bounded below, uniformly in \( z \). More precisely, \( \sigma(\mathcal{H}), \sigma(\mathcal{H}) \subset (-C_1(n, \beta), \infty) \), for all \( \varepsilon \in (0, 1/C_2^*(\beta_{\text{fine}})) \), for some \( C_1(n, \beta) < \infty \) and \( C_2(\beta_{\text{fine}}) > 0 \). Fix \( t \in \mathbb{R}_+ \). We now apply [RS72, Theorem VIII.20], which says that if self-adjoint operators \( \mathcal{H}_e \to \mathcal{H} \) in the strong resolvent sense, and \( f \) is bounded and continuous on \( \mathbb{R} \) then \( f(\mathcal{H}_e) \to f(\mathcal{H}) \) strongly. We use \( f(\lambda) = e^{-(\lambda/\varepsilon)C_1(n, \beta_*)}, \) which is bounded and continuous, and from what we have proved, \( f(\mathcal{H}_e) = e^{-t\mathcal{H}_e} \) and \( f(\mathcal{H}) = e^{-t\mathcal{H}} \). Hence

\[
e^{-t\mathcal{H}_e} \to e^{-t\mathcal{H}} \text{ strongly on } L^2(\mathbb{R}^{2n}), \text{ for each fixed } t \in \mathbb{R}_+.
\]

For Theorem 1.1(b), we wish to upgrade this convergence to be uniform over finite intervals in \( t \). Given the lower bound on the spectra, we have the uniform (in \( z \)) norm continuity:

\[
\| e^{-t\mathcal{H}_e} - e^{-s\mathcal{H}_e} \|_o^p + \| e^{-t\mathcal{H}} - e^{-s\mathcal{H}} \|_o^p \leq C_2(n, \beta_*) |t - s| e^{C_2(n, \beta_*)(tv/s)},
\]

for all \( \varepsilon \in (0, 1/C_2(\beta_{\text{fine}})) \) and \( s, t \in [0, \infty) \). This together with (7.5) gives

\[
\lim_{\varepsilon \to 0} \sup_{t \in [0, \tau]} \| e^{-t\mathcal{H}_e} u - e^{-t\mathcal{H}} u \| = 0, \quad u \in L^2(\mathbb{R}^{2n}), \quad \tau < \infty.
\]

Comparing this with (1.4), we now have, for each fixed \( g \in L^2(\mathbb{R}^{2n}) \),

\[
\mathbb{E} \left[ \int_{0}^{\tau} e^{-t\mathcal{H}_e} \right] \to \mathbb{E} \left[ \int_{0}^{\tau} e^{-t\mathcal{H}} \right], \quad \text{uniformly over finite intervals in } t.
\]

What is missing for the proof of Theorem 1.1 is the identification of the semigroup \( e^{-t\mathcal{H}} \) with the explicit operators defined in (1.16), (1.17). This is the subject of the next section.
8. Identification of the limiting semigroup

The remaining task is to match $e^{-tH}$ to the operator $\mathcal{P}_t + D_t^{\text{Dgm}(n)}$ on r.h.s. of (1.18). This matching can be heuristically seen by taking the inverse Laplace transform of $\mathcal{R}_z$ in (1.23) in $z$. At a formal level, doing so turns the operators $\mathcal{G}_*$ and $(\mathcal{J} - \beta_* \mathcal{I})^{-1}$ into $\mathcal{P}_*$ and $\mathcal{P}_*^c$, and the products of operators in $z$ become the convolutions in $t$.

To rigorously perform this matching, it is more convenient to operate in the forward Laplace transform, i.e., going from $t$ to $z$. Doing so requires establishing bounds on the relevant operators in (1.17), and verifying the semigroup property of $\mathcal{P}_t + D_t^{\text{Dgm}(n)}$, defined in (1.16). The bounds will be established in Section 8.1, and, as the major step toward verifying the semigroup property, we establish an identity in Section 8.2.

8.1. Bounds and Laplace transforms. We begin with the incoming and outgoing operators. We now establish a quantitative bound on the norms of $S_{ij}\mathcal{P}_t$ and $\mathcal{P}_t S_{ij}^*$, and match them to the corresponding Laplace transform.

Lemma 8.1.  

(a) For each pair $i < j$ and $t \in \mathbb{R}_+$, $S_{ij}\mathcal{P}_t : \mathcal{L}^2(\mathbb{R}^n) \to \mathcal{L}^2(\mathbb{R}^{2n-2})$ and $\mathcal{P}_t S_{ij}^* : \mathcal{L}^2(\mathbb{R}^{2n-2}) \to \mathcal{L}^2(\mathbb{R}^n)$ are bounded with 

$$\|S_{ij}\mathcal{P}_t\|_{\text{op}} + \|\mathcal{P}_t S_{ij}^*\|_{\text{op}} \leq Ct^{-1/2},$$

(b) For each pair $i < j$, $\text{Re}(z) < 0$, $u \in \mathcal{L}^2(\mathbb{R}^n)$, and $v \in \mathcal{L}^2(\mathbb{R}^{2n-2})$,

$$\int_{\mathbb{R}_+} e^{tz} \langle v, S_{ij}\mathcal{P}_tu \rangle \, dt = \int_{\mathbb{R}_+ \times \mathbb{R}^{4n-2}} e^{tz} \mathcal{P}_t(\mathcal{P}_t y - x)u(x) \, dt \, dy \, dx = \langle u, S_{ij}\mathcal{G}_z v \rangle,$$

$$\int_{\mathbb{R}_+} e^{tz} \langle u, \mathcal{P}_t S_{ij}^* v \rangle \, dt = \int_{\mathbb{R}_+ \times \mathbb{R}^{4n-2}} e^{tz} \mathcal{P}_t(x - S_{ij}y)v(y) \, dt \, dy \, dx = \langle u, \mathcal{G}_z S_{ij}^* v \rangle,$$

where the integrals converge absolutely (over $\mathbb{R}_+$ and over $\mathbb{R}_+ \times \mathbb{R}^{2n-4}$).

Proof. It suffices to consider $S_{ij}\mathcal{P}_t$ since $\mathcal{P}_t S_{ij}^* = (S_{ij}\mathcal{P}_t)^*$.

(a) Fix $u \in \mathcal{L}^2(\mathbb{R}^n)$, we use (4.3) to bound

$$\|S_{ij}\mathcal{P}_t u\|^2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \mathcal{P}_t u(M_{ij}q) \, dq \right)^2 \, dq_{2-n} = \int_{\mathbb{R}^{2n-2}} \left( \int_{\mathbb{R}^n} e^{-\frac{t}{2}|M_{ij}q|^2} \mathcal{G}(M_{ij}q) \, dq \right)^2 \, dq_{2-n}.$$ 

On the r.h.s., bound $|M_{ij}q|^2 \geq \frac{1}{2} |q_1|^2$ (as checked from (4.1)), and applying the Cauchy–Schwarz inequality in the $q_1$ integral. We conclude the desired result

$$\|S_{ij}\mathcal{P}_t u\|^2 \leq C \int_{\mathbb{R}^n} \left( e^{-\frac{t}{2}|q_1|^2} \right)^2 \, dq_1 \|u\|^2 \leq \frac{C}{t} \|u\|^2.$$

(b) Fix $\text{Re}(z) < 0$, integrate $\langle v, S_{ij}\mathcal{P}_tu \rangle$ against $e^{zt}$ over $t \in (0, \infty)$, and use (4.3) to get

$$\int_0^\infty e^{zt} \langle v, S_{ij}\mathcal{P}_tu \rangle \, dt = \int_0^\infty \int_{\mathbb{R}^{2n}} \mathcal{P}_t(\mathcal{P}_t y - x)u(x) \, dt \, dy \, dx = \lim_{t \to \infty} \int_{\mathbb{R}^{2n}} \mathcal{P}_t(\mathcal{P}_t y - x)u(x) \, dt \, dy \, dx.$$ 

This integral converges absolutely since $\|S_{ij}\mathcal{P}_t\|_{\text{op}} \leq Ct^{-1/2}$ and $\text{Re}(z) < 0$. This being the case, we swap the integrals and evaluate the integral over $t$ to get

$$\int_0^\infty e^{zt} \langle v, S_{ij}\mathcal{P}_tu \rangle \, dt = \int_{\mathbb{R}^{2n}} \frac{1}{\mathcal{G}(q_{2-n})} \int_{\mathbb{R}^n} \frac{1}{\mathcal{G}(M_{ij}q)} \, dq \, dq_{2-n}.$$ 

The last expression matches $\langle v, S_{ij}\mathcal{G}_z u \rangle$, as seen from (4.9). \hfill \Box

Lemma 8.2.

(a) For distinct pairs $(i < j) \neq (k < \ell)$, $t \in \mathbb{R}_+$, $\mathcal{P}_t S_{k\ell}^*(\mathcal{L}^2(\mathbb{R}^{2n-2})) \subset \text{Dom}(S_{ij})$, so the operator $S_{ij}\mathcal{P}_t S_{k\ell}^*$ maps $\mathcal{L}^2(\mathbb{R}^{2n-2}) \to \mathcal{L}^2(\mathbb{R}^{2n-2})$. Further

$$\|S_{ij}\mathcal{P}_t S_{k\ell}^*\|_{\text{op}} \leq C t^{-1}.$$
(b) For distinct pairs \((i < j) \neq (k < l)\), \(v, w \in \mathcal{L}^2(\mathbb{R}^{2n-2})\), and \(\text{Re}(z) < 0\),
\[
\int_{\mathbb{R}^n \times \mathbb{R}^{2n-4}} e^{zt} \mathcal{W}(y) P(t, S_{ij} y - S_{kl} y') v(y') \, dt \, dy' = \langle w, S_{ij} G_z S_{kl} v \rangle,
\]
where the integral converges absolutely.

**Remark 8.3.** Unlike in the case for incoming and outgoing operators, here our bound on \(Ct^{-1}\) on the mediating operator does not ensure the integrability of \(\|S_{ij} P_t S_{kl}^*\|\) near \(t = 0\). Nevertheless, the integral in (8.1) still converges absolutely.

**Proof.** Fix distinct pairs \((i < j) \neq (k < l)\) and \(v, w \in \mathcal{L}^2(\mathbb{R}^{2n-2})\).

(a) As argued just before Lemma 8.1, we have \(P_t S_{kl}^* v \in \mathcal{L}^2(\mathbb{R}^{2n})\). To check the condition \(P_t S_{kl}^* v \in \text{Dom}(S_{ij})\), consider
\[
\int_{\mathbb{R}^n} \left| \tilde{w}(q_{2-n}) e^{-\frac{i}{2} |M_{ij}|^2 S_{ij}^\top v(M_{ij} q)} \right| \frac{dq}{2\pi} = \int_{\mathbb{R}^n} \left| \tilde{w}(p_i + p_j + p_{ij}) e^{-\frac{i}{2} |p|^2 S_{ij}^\top v(p)} \right| \frac{dp}{2\pi},
\]
where the equality follows by a change of variable \(q = M_{ij}^{-1} p\), together with \(|p_i + p_j + p_{ij}| = |M_{ij}^{-1} p|_{2-n}\) and \(|\det(M_{ij})| = 1\) (as readily verified from (4.1)). In (8.2), bound \(e^{-\frac{1}{2} |p|^2} \leq C \|p\|^{-1}\) and use (5.2) to get
\[
\|v\| \leq C t^{-1} \|w\|.
\]

(b) To prove (8.1), assume for a moment \(z = -\lambda \in (-\infty, 0)\) is real, and \(v(y), w(y) \geq 0\) are positive. In (8.1), express the integral over \(y, y'\) as \(\langle w, S_{ij} P_t S_{kl}^* v \rangle = \langle S_{ij}^\top w, P_t S_{kl}^* v \rangle\), and use (5.3) to get
\[
\int_{\mathbb{R}^n \times \mathbb{R}^{2n-4}} e^{zt} \mathcal{W}(y) P(t, S_{ij} y - S_{kl} y') v(y') \, dt \, dy' = \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}^{2n}} \tilde{w}(p_i + p_j + p_{ij}) e^{-\frac{i}{2} |p|^2 \tilde{v}(p_k + p_l + p_{kl})} \, dp \right) \, dt.
\]
The integral on the r.h.s. converges absolutely over \(\mathbb{R}_+ \times \mathbb{R}^n\), i.e., jointly in \(t, p\). This follows by using (5.2) together with \(\int_0^\infty e^{-\lambda t} e^{-2\frac{1}{2} |p|^2} \, dt = \frac{\lambda}{\lambda + 2}\). Given the absolute convergence, we swap the integrals over \(t\) and \(p\), and evaluate the former to get the expression for \(\langle w, S_{ij} G_z S_{kl}^* v \rangle\) on the right hand side of (5.1). For general \(v(y), w(y)\), the preceding calculation done for \(\langle v(y), w(y) \rangle \mapsto (|v(y)|, |w(y)|)\) and for \(z \mapsto \text{Re}(z)\) guarantees the relevant integrability.

Recall \(j(t, \beta_\ast)\) from (1.9). For the diagonal mediating operator, let us first settle some properties of \(j\).

**Lemma 8.4.** For each \(\text{Re}(z) < -e^{-\beta_\ast}\), the Laplace transform of \(j(t, \beta_\ast)\) evaluates to
\[
\int_0^\infty e^{zt} j(t, \beta_\ast) \, dt = \frac{1}{\log(-z) - \beta_\ast},
\]
where the integral converges absolutely, and \(j(t, \beta_\ast)\) has the following pointwise bound
\[
j(t, \beta_\ast) = |j(t, \beta_\ast)| \leq C t^{-1} \| \log(t \wedge \frac{1}{2}) \|^{2} e^{(\beta_\ast + 1) Ct}, \quad t \in \mathbb{R}_+.
\]

**Proof.** To evaluate the Laplace transform, assume for a moment that \(z \in (\infty, -e^{-\beta_\ast})\) is real. Integrate (1.9) against \(e^{zt}\) over \(t\). Under the current assumption that \(z\) is real, the integrand therein is positive, so we apply Fubini’s theorem to swap the \(t\) and \(\alpha\) integrals to get
\[
\int_0^\infty e^{zt} j(t, \beta_\ast) \, dt = \int_0^\infty \frac{e^{zt} \alpha^\beta_\ast}{\Gamma(\alpha)} \left( \int_0^\infty t^{\alpha - 1} e^{-(zt)} \, dt \right) d\alpha.
\]
The integral over \(t\), upon a change of variable \(z t \rightarrow t\), evaluates to \(\Gamma(\alpha)/(\alpha/\alpha)^\alpha\). Canceling the \(\Gamma(\alpha)\) factors and evaluating the remaining integral over \(z\) yields (8.4) for \(z \in (\infty, -e^{-\beta_\ast})\). For general \(z \in \mathbb{C}\) with \(\text{Re}(z) < -e^{-\beta_\ast}\), since \(|e^{zt}| = e^{\text{Re}(z)t}\), the preceding result guarantees integrability of \(|e^{-zt + \alpha \beta_\ast} t^{\alpha - 1} \Gamma(\alpha)|\) over \((t, \alpha) \in \mathbb{R}_+^2\). Hence Fubini’s theorem still applies, and (8.4) follows.

To show (8.5), in (1.9), we separate the integral (over \(\alpha \in \mathbb{R}_+\)) into two integrals over \(\alpha > 1\) and over \(\alpha < 1\), denoted by \(I_+\) and \(I_-\), respectively. For \(I_+\), we use the bound \(\exp(-\log \Gamma(\alpha)) \leq \frac{\alpha}{2} \log \alpha - C \alpha\) (c.f., [AS65, 6.1.40]) to write \(I_+ \leq \int_1^\infty \exp(-\alpha \left(\frac{1}{2} \alpha \log \alpha - C + \beta_\ast \right) - \alpha) \, d\alpha\). It is now straightforward to check that \(I_+ \leq e^{(\beta_\ast + 1) Ct}\). Using \(\frac{1}{\Gamma(\alpha)} \leq C \alpha\) \(\alpha \in (0, 1)\) (c.f., [AS65, 6.1.34]), we bound \(I_-\) as \(I_- \leq Ct^{-\beta_\ast} \int_0^1 \alpha^\alpha d\alpha\).
For all \( t \geq \frac{1}{2} \), the last integral is indeed bounded by \( e^{(\beta + 1)Ct} \). For \( t < \frac{1}{2} \), we write \( t^n = e^{-\alpha |\log t|} \) we perform a change of variable \( \alpha |\log t| \rightarrow t \) to get \( I_\alpha \leq C t^{-1} e^{\beta |\log t|^{-2}} \int_0^{|\log t|} e^{-\alpha} d\alpha \leq C t^{-1} e^{\beta |\log t|^{-2}} \). Collecting the preceding bounds and adjusting the constant \( C \) gives (8.5). 

Referring to the definition (1.12) of \( \mathcal{P}_t^{\beta} \), we see that this operator has an integral kernel

\[
(\mathcal{P}_t^{\beta} v)(y) = \int_{\mathbb{R}^{n-2}} P^{\beta}(t, y, y') v(y') dy', \quad P^{\beta}(t, y, y') := j(t, \beta_*) p(t, y_2 - y'_2) \prod_{i=3}^{n} p(t, y_i - y'_i). \tag{8.6}
\]

**Lemma 8.5.**

(a) For each \( t \in \mathbb{R}_+ \), \( \mathcal{P}_t^{\beta} : \mathcal{L}^2(\mathbb{R}^{n-2}) \to \mathcal{L}^2(\mathbb{R}^{n-2}) \) is a bounded operator with

\[
\|\mathcal{P}_t^{\beta}\|_{\text{op}} \leq C (t \wedge \frac{1}{2})^{-1} |\log(t \wedge \frac{1}{2})|^{-2} e^{(\beta + 1)Ct}. \tag{8.7}
\]

(b) Further, for each \( v, w \in \mathcal{L}^2(\mathbb{R}^{n-2}) \) and \( \text{Re}(z) < -\beta \),

\[
\int_{\mathbb{R}_+} e^{zt} \langle w, \mathcal{P}_t^{\beta} v \rangle dt = \int_{\mathbb{R}_+ \times \mathbb{R}^{n-4}} e^{zt} \overline{w(y)} P^{\beta}(t, y, y') v(y') dt dy' = \langle w, (\mathcal{J}_z - \beta I)^{-1} v \rangle, \tag{8.8}
\]

where the integrals converge absolutely (over \( \mathbb{R}_+ \) and over \( \mathbb{R}_+ \times \mathbb{R}^{n-4} \)).

**Proof.** Part (a) follows from (8.5) and the fact that heat semigroups have unit norm, i.e., \( \|e^{-at\nabla^2_i}\|_{\text{op}} = 1 \), \( a \geq 0 \). For part (b), we work in Fourier domain and write

\[
\int_{\mathbb{R}^{n-4}} \overline{w(y)} P^{\beta}(t, y, y') v(y') dt dy' = \hat{j}(t, \beta_*) \int_{\mathbb{R}^{n-2}} \overline{w(p)} e^{-\frac{1}{4} t |p|_\beta^2 - \phi(p)} dp,
\]

where, recall that \( |p|_\beta^2 = |p_2|^2 + |p_3|^2 + \ldots + |p_n|^2 \). Integrate both sides against \( e^{zt} \) over \( t \in \mathbb{R}_+ \), and exchange the integrals over \( p \) and over \( t \). The swap of integrals are justified the same way as in the proof of Lemma 8.2, so we do not repeat it here. We now have

\[
\int_{\mathbb{R}_+ \times \mathbb{R}^{n-4}} e^{zt} \overline{w(y)} P^{\beta}(t, y, y') v(y') dt dy' = \int_{\mathbb{R}^{n-2}} \left( \int_0^\infty e^{zt - \frac{1}{4} t |p|_\beta^2 - \phi(p)} dt \right) \overline{w(p)} \hat{\phi}(p) dp.
\]

Applying (8.4) to evaluate the integral over \( t \) yields the expression in (1.22) for \( \langle w, (\mathcal{J}_z - \beta I)^{-1} v \rangle \). \( \square \)

### 8.2. An identity for the semigroup property.

Our goal is to prove Lemma 8.8 in the following. Key to the proof is the identity (8.12). It depends on a cute fact about the \( \Gamma \) function. Set

\[
p_k(\alpha) := \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + 1)} = (\alpha + k) \cdots (\alpha + 1) \alpha, \quad \alpha \geq 0. \tag{8.9}
\]

with the convention \( p_{-1} := 1 \).

**Lemma 8.6.** For \( m \in \mathbb{Z}_{\geq 0} \),

\[
p_m(\alpha) = \int_0^\alpha \sum_{k=0}^m \binom{m+1}{m-k+1} (m-k)! p_{k-1}(\alpha_k) d\alpha_k.
\]

**Proof.** Taking derivative gives \( \frac{d}{d\alpha} p_m(\alpha) = \sum_{i=0}^m \prod_{j=0}^i (\alpha+i) \), where \( \prod_{j=0}^i \) denotes a product over \( i \in \{0, \ldots, m\} \setminus \{j\} \). Our goal is to express this derivative in terms of \( p_{m-1}(\alpha), p_{m-2}(\alpha), \ldots \). The \( j = m \) term skips the \( (\alpha + m) \) factor, and is hence exactly \( p_{m-1}(\alpha) \). For other values of \( j \), we use \( (\alpha - m) \) to compensate the missing \( (\alpha + j) \) factor. Namely, writing \( (\alpha + m) = (\alpha + j + (m-j)) \), we have

\[
\prod_{j=0}^m (\alpha+i) = p_{m-1}(\alpha) + (m-j) \prod_{j=m}^{m-1} (\alpha+i). \tag{8.10}
\]

This gives

\[
\frac{d}{d\alpha} p_m(\alpha) = \sum_{j=0}^m \prod_{j=0}^m (\alpha+i) = \sum_{j=0}^m p_{m-1}(\alpha) + \sum_{j=0}^m (m-j) \prod_{j=0}^{m-1} (\alpha+i).
\]
In (8.10), we have reduced $\prod_{\ell=1}^{m}(\alpha+i)$ to $\prod_{\ell=1}^{m-1}(\alpha+i)$, i.e., the same expression but with $m$ decreased by 1. Repeating this procedure yields

$$\frac{d}{d\alpha}p_m(\alpha) = \sum_{\ell=1}^{m} p_{m-\ell}(\alpha) \left( \sum_{j=0}^{m} (m-j)_+ (m-j-1)_+ \cdots (m-j-\ell)_+ \right) = \sum_{\ell=1}^{m} p_{m-\ell-1}(\alpha) \sum_{j=0}^{m} \prod_{i=0}^{\ell-1} (j-i)_+ = \sum_{\ell=1}^{m} p_{m-\ell-1}(\alpha) \left( \frac{m+1}{\ell+1} \right)!,$$

(8.11)

where $\prod_{i\in \emptyset}(\cdot) := 1$. Within the last equality, we have used the identity $\sum_{j=0}^{m} (m-j)_+ \cdot (m-j-\ell)_+ \cdot \cdots \cdot (m-j-\ell)_+ = \frac{(m+1)!}{\ell!}$. In (8.11), perform a change of variable $m-\ell := k$, and integrate in $\alpha$, using $p_m(0) = 0$ to get the result. \qed

**Lemma 8.7.** For $s < t \in \mathbb{R}_+$, $i < j$, we have

$$j(t, \beta_*) = \int_{0<s<t} \int_{s<t} j(t_1, \beta_*) (t_2 - t_1)^{-1} j(t - t_2, \beta_*) \, dt_1 \, dt_2.$$  

(8.12)

**Proof.** Write $j(t, \beta_*) = j(t)$ to simplify notation. Let the r.h.s. of (8.12) be denoted by $F(s, t)$. It is standard to check that $F(s,t)$ is continuous on $0 < s < t < \infty$. Hence it suffices to show

$$\int_{0}^{t} F(s, t) s^m \, ds = j(t) \int_{0}^{t} s^m \, ds = j(t) (m + 1)^{-1} t^{m+1}, \quad m \in \mathbb{Z}_{\geq 0}.$$  

(8.13)

From (8.5), it is readily checked both sides of (8.13) grow at most exponentially in $t$. Taking Laplace transform on both sides of (8.5), the problem is further reduced to showing, for some $C(m, \beta_*) < \infty$,

$$\int_{0}^{\infty} \int_{0}^{t} e^{-\lambda t} F(s, t) s^k \, dt \, ds = \int_{0}^{\infty} e^{-\lambda t} j(t) (m + 1)^{-1} t^{m+1} \, dt, \quad \lambda > C(m, \beta_*).$$  

(8.14)

The left hand side can be computed

$$\text{l.h.s. of (8.14)} = \int_{0}^{\infty} \frac{e^{\beta_* \lambda} \alpha^{-\alpha-1}}{m+1} \left( \int_{0}^{\alpha} \sum_{k=0}^{m} \left( \frac{m+1}{m-k+1} \right) (m-k)! p_{k-1}(\alpha_1) d\alpha_1 \right) d\alpha.$$  

(8.15)

The integral (8.15) is indeed finite for large enough $\lambda \geq C(\beta_*, m)$. The right hand side is given by

$$\text{r.h.s. of (8.14)} = \int_{0}^{\infty} \frac{e^{\beta_* \lambda} \alpha^{-\alpha-1}}{m+1} p_m(\alpha) \, d\alpha.$$  

(8.16)

By Lemma 8.6 the two coincide. \qed

**Lemma 8.8.** For $t' < s < t \in \mathbb{R}_+$, $i < j$, we have

$$\int_{t' < t_1 < s} \int_{s < t_2 < t} (4\pi P_{t_1 - t_2}) S_{ij} P_{t_2 - t_1} S_{ij}^* (4\pi P_{t_1 - t_2}) \, dt_1 \, dt_2 = \mathcal{P}_{t-t'}^J.$$  

(8.17)

**Remark 8.9.** The integral (8.17) converges absolutely in operator norm. This is seen by writing $S_{ij} P_{t_2 - t_1} S_{ij}^* = (S_{ij} P_{s-t_1}) (P_{t_2 - s} S_{ij}^*)$, and by using the bounds from Lemmas 8.1(a) and 8.5(a).

**Proof.** For $\tau > 0$, the operator $S_{ij} P_{\tau} S_{ij}^*$ has an integral kernel $P(\tau, S_{ij}(y-y')) = (p(\tau, y_2 - y_2)) \prod_{i=3}^{n} p(\tau, y_i - y_i)$, where $p$ denotes the two-dimensional heat kernel. From this and $(p(\tau, y))^2 = \frac{1}{4\pi\tau} p(\tau, y, y)$, we have $S_{ij} P_{\tau} S_{ij}^* = \frac{1}{4\pi\tau} \exp(-\frac{\tau}{4} \sum_{i=3}^{n} \nabla_i^2)$. Recall that $P_{\tau}^J := j(\tau, \beta_*) \exp(-\frac{\tau}{4} \sum_{i=3}^{n} \nabla_i^2)$. We obtain

$$\text{l.h.s. of (8.17)} = 4\pi e^{-\frac{\tau}{4} \nabla_2^2 - \frac{\tau}{2} \sum_{i=3}^{n} \nabla_i^2} \int_{t' < t_1 < s} \int_{s < t_2 < t} j(t_1 - t') (t_2 - t_1)^{-1} j(t - t_2) \, dt_1 \, dt_2.$$  

The desired result now follows from Lemma 8.12. \qed
8.3. Proof of Theorem 1.1. We begin with a quantitative bound on \( D_{t}^{(i,j)} \).

Lemma 8.10. For \((i,j) = (i_{k}, j_{k})\)_{k=1}^{m} \in \text{Dgm}(n, m), t \in \mathbb{R}_{+}\) and \( \lambda \geq 2 \), we have
\[
\| D_{t}^{(i,j)} \|_{\text{op}} \leq C(\log (\frac{t}{2m+1} + \frac{1}{2}))^{-1} m^{2} e^{\lambda C(\beta, t) t} (C/\lambda)^{m-1}.
\] (8.18)

Proof. To simplify notation, we index the incoming and outgoing operators by \( 0 \) and by \( m \): \( Q^{(0)} := P_{\tau_{0}} S_{a_{1}j_{1}} \), \( Q^{(m)} := S_{\pi_{m}j_{m}} P_{\tau_{m}} \), index the diagonal mediating operators by half integers: \( Q^{(a)} := 4 \pi P_{\tau_{a}} S_{a_{m+1}j_{m+1}} \), \( a \in (\frac{1}{2} + \mathbb{Z}) \cap (0, m) \), and index the off-diagonal mediating operators by integers: \( Q^{(a)} := S_{\pi_{a}j_{a}} P_{\tau_{a}} S_{a_{m+1}j_{m+1}} \), \( a \in \mathbb{Z} \cap (0, m) \). Under these notation
\[
D_{t}^{(i,j)} = \int_{\sum(t)} Q^{(0)} Q^{(1)} \cdots Q^{(m)} \, d\bar{\tau}.
\] (1.17')

Recall that the incoming, outgoing, and mediating operators all have positive integral kernels, c.f., (2.1)–(2.2), (2.3), (8.6). Accordingly, we interpret the r.h.s. of (1.17') as an integral operator, with integrand consisting of a convolution of the aforementioned kernels. We seek to bound
\[
\int_{\sum(t)} \left| \langle u', \prod_{a \in A} Q^{(a)} u \rangle \right| \, d\bar{\tau} = \int_{\sum(t)} \left| \langle u', \prod_{a \in A} Q^{(a)} u \rangle \right| \, d\bar{\tau},
\] (8.19)
for \( u, u' \in L^{2} (\mathbb{R}^{2n}) \).

An undesirable feature of (8.19) is the constraint \( \tau_{0} + \tau_{1} + \cdots + \tau_{m} = t \) from \( \sum(t) \). To break such a constraint, fix \( \lambda \geq 2 \). In (8.19), multiply and divide by \( e^{\lambda \beta_{*} t} \), and use \( \sum(t) \subset (\cup_{\tau \in \mathbb{Z}_{n} \cap (0, t)} \{ \tau : \tau \geq \frac{t}{2m+1} \}) \cap (0, t)^{2m+1} \) to obtain
\[
\| D_{t}^{(i,j)} \|_{\text{op}} \leq e^{\lambda \beta_{*} t} \sum_{a \in A} \left( \sup_{\tau \in \sum(t)} e^{\lambda \beta_{*} \tau} \| Q^{(a)} \|_{\text{op}} \right) \prod_{a' \in A} \left( \int_{0}^{t} e^{-\lambda \beta_{*} \tau} Q^{(a')} d\tau \right)_{\text{op}}.
\] (8.20)

To bound the ‘sup’ term in (8.20), forgo the exponential factor (i.e., \( e^{-\lambda \beta_{*} \tau} \leq 1 \)), and use the bound on \( \| Q^{(a)} \|_{\text{op}} \) from Lemmas 8.1(a), 8.2(a), and 8.5(a). We have
\[
\sup_{\tau \in \sum(t)} e^{\lambda \beta_{*} \tau} \| Q^{(a)} \|_{\text{op}} \leq C \begin{cases} (t/m)^{-1/2}, & \text{for } a = 0, m, \\ (t/m)^{-1} \left( \log (\frac{t}{2m+1} + \frac{1}{2}) \right)^{-2} e^{C^{(1+\beta_{*})} \tau}, & \text{for } a \in (\frac{1}{2} + \mathbb{Z}) \cap (0, m), \end{cases}
\] (8.21)
\[
\| \int_{0}^{t} e^{-\lambda \beta_{*} \tau} Q^{(a')} d\tau \|_{\text{op}} \leq C \| Q^{(a') \text{op}} \| \leq C \left( \frac{t}{2m+1} + \frac{1}{2} \right)^{-1} e^{C^{(1+\beta_{*})} \tau}, \text{ for } a' \in (\frac{1}{2} + \mathbb{Z}) \cap (0, m).
\] (8.23)

The bound (8.23) gives a useful logarithmic decay in \( t \to 0 \), but has an undesirable exponential growth in \( t \to \infty \). We will also need a bound that does not exhibit the exponential growth. For \( a' \in (\frac{1}{2} + \mathbb{Z}) \cap (0, m) \), we use the fact that \( Q^{(a')} \) is an integral operator with a positive kernel to write
\[
\| \int_{0}^{t} e^{-\lambda \beta_{*} \tau} Q^{(a')} d\tau \|_{\text{op}} \leq \int_{0}^{\infty} e^{-\lambda \beta_{*} \tau} Q^{(a')} d\tau.
\] The last expression is a Laplace transform, and has been evaluated in Lemmas 8.2(b) and 8.5(b), whereby
\[
\| \int_{0}^{t} e^{-\lambda \beta_{*} \tau} Q^{(a')} d\tau \|_{\text{op}} \leq \| S_{\beta} \mathcal{F}_{\lambda_{a}, \beta_{a}} \|_{\text{op}}, \text{ for } a' \in (0, m) \cap \mathbb{Z},
\] (8.24)
Here \((i < j) \neq (k < \ell)\) corresponds to the index \(a'\). Using the bounds on \(\|S_{ij}G_{i}S_{kj}\|_{\text{op}}\) from Lemma 5.1 and the bound \(\|(\sigma_{-a}, - \beta_{e})^{-1}\| \leq 1/\log \lambda\) (c.f., (1.22)) we have

\[
\left\| \int_{0}^{t} e^{-\lambda \beta_{e} \tau} Q_{\tau} d\tau \right\|_{\text{op}} < C \left\{ \begin{array}{ll}
1 & \text{for } a' \in (0, m) \cap \mathbb{Z}, \\
(\log \lambda)^{-1} & \text{for } a' \in (0, m) \cap (\frac{1}{2}, \frac{1}{2} + \mathbb{Z}).
\end{array} \right.
\]  

(8.24)

For \(a \in \frac{1}{2} \mathbb{Z}\), inserting the bounds (8.21)–(8.22), (8.24) into (8.20) gives

\[
F_{a} \leq C \exp(\lambda C(\beta_{e}+1) t)(\log(t^{2}m_{2+1} + \frac{1}{2}))^{-1} \lesssim t^{-1/2+1/2} + \frac{1}{2} \exp(\lambda m_{1}m_{2+1}).
\]

For \(a \notin \frac{1}{2} \mathbb{Z}\), in (8.20), use the bound (8.21) for the sup term, use (8.23) for \(a' = \frac{1}{2}\), and use (8.22) and (8.24) for other \(a'\). This gives

\[
F_{a} \lesssim C \exp(\lambda C(\beta_{e}+1) t)(\log(t^{2}m_{2+1} + \frac{1}{2}))^{-1} \lesssim t^{-1/2+1/2} \left( \begin{array}{c}
\sum_{a \in \mathbb{Z}} \exp(\lambda m_{1}m_{2+1}) \\lesssim \exp(\lambda m_{1}m_{2+1}).
\end{array} \right)
\]

Inserting these bounds on \(F_{a}\) into (8.20), we conclude the desired result (8.18).

\(\square\)

Proof of Theorem 1.1(a). Sum the bound (8.18) over \((i,j) \in \text{Dgm}(n)\), and note that \(|\text{Dgm}(n, m)| \leq (n(n-1)/2)^{m}\) (c.f., (1.13)). In the result, choose \(\lambda = Cn^{2}\) for some large but fixed \(C < \infty\), we have

\[
\left\| D_{t}^{\text{Dgm}(n)} \right\|_{\text{op}} \lesssim \sum_{m=1}^{\infty} m^{2} n^{2} \log(\frac{t^{2}m_{2+1} + \frac{1}{2}}{1})^{-1} \exp(\lambda n^{2}(\beta_{e} + 1)t)
\]

\[
\lesssim n^{2} \exp(\lambda n^{2}(\beta_{e} + 1)t).
\]

This verifies that \(D_{t}^{\text{Dgm}(n)}\) defines a bounded operator on \(L^{2}(\mathbb{R}^{2n})\).

To show the semigroup property, we fix \(s \leq t \in \mathbb{R}_{+}\) and calculate \((P_{s} + D_{t}^{\text{Dgm}(n)})(P_{t-s} + D_{t-s}^{\text{Dgm}(n)})\), which boils down to calculating \(P_{s}P_{t-s} = P_{t}D_{t-s}^{\text{Dgm}(n)}P_{t-s}\), \(P_{s}D_{t-s}^{\text{Dgm}(n)} = D_{t-s}^{\text{Dgm}(n)}P_{t-s}\), \(P_{s}P_{t-s} = D_{t-s}^{\text{Dgm}(n)}D_{t-s}^{\text{Dgm}(n)}\), for \((i,j) \in \text{Dgm}(n, m)\) and \((i', j') \in \text{Dgm}(n, m')\). To streamline notation, we relabel time variables as \(t_{k} := \tau_{0} + \ldots + \tau_{k-2}/2\), and set

\[
B^{(i,j)}(\bar{t}) := \prod_{i,k<j} \left( 4 \pi \int_{t_{k+1}}^{t_{k+2}} S_{i,j} S_{i+1,j} (4 \pi \int_{t_{k+2}}^{t_{k+3}} S_{i,j}) \right) S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j} S_{i,j})
\]

Using (1.17) and the semigroup property of \(P_{s}\), we have \(P_{s}P_{t-s} = P_{t}\).

\[
P_{s}D_{t-s}^{(i',j')} = \prod_{(i',j') \in \text{Dgm}(n)} \left( \int_{(i',j') \in \text{Dgm}(n)} B^{(i',j')}(\bar{t}) d\bar{t} \right),
\]

(8.27)

\[
P_{s}D_{t-s}^{(i,j)} = \prod_{(i,j) \in \text{Dgm}(n)} \left( \int_{(i,j) \in \text{Dgm}(n)} B^{(i,j)}(\bar{t}) d\bar{t} \right),
\]

(8.28)

\[
P_{s}P_{t-s} = \prod_{(i,j) \in \text{Dgm}(n)} \left( \int_{(i,j) \in \text{Dgm}(n)} B^{(i,j)}(\bar{t}) d\bar{t} \right),
\]

(8.29)

where \((a,b)^{k} := \{\bar{t} \in (a,b)^{k} : a < t_{1} < \ldots < t_{k} < b\}\), \(\Omega_{k}(s,t) := \{\bar{t} \in (0, s) : t_{k} < s < t_{k+1} < \ldots < t_{k+\ell} < t\}\), and \((i''', j''')\) is obtained by concatenating \((i,j)\) and \((i', j')\), i.e.,

\[
(i''', j''') = (i_{k}+j_{k})_{k=1}^{m'+m'} := ((i_{1} < j_{1}), \ldots, (i_{m} < j_{m}), (i_{1} < j_{1}), \ldots, (i_{m'} < j_{m})).
\]

Such an index is not necessarily in \(\text{Dgm}(n)\), because we could have \((i_{m} < j_{m}) = (i_{1} < j_{1})\). When this happens, applying Lemma 8.8 with \((i,j) = (i_{m}, j_{m})\) and with \((i', j') = (t_{2m-1}, t_{2m+2})\) gives

\[
P_{s}D_{t-s}^{(i,j)} = \prod_{(i,j) \in \text{Dgm}(n)} \left( \int_{\Omega_{2m-1,2m+2}(s,t)} B^{(i'',j'')} d\bar{t} \right),
\]

(8.29')

where \((i'', j'')\) is obtained by removing \((i_{1} < j_{1})\) from \((i''', j''')\), i.e.,

\[
(i'', j'') := ((i_{1} < j_{1}), \ldots, (i_{m} < j_{m}), (i_{2} < j_{2}), \ldots, (i_{m'} < j_{m})).
\]

Summing (8.27)–(8.29), (8.29') over \((i,j), (i', j') \in \text{Dgm}(n)\) verifies the desired semigroup property:

\[
P_{s}P_{t-s} + (P_{s}D_{t-s}^{\text{Dgm}(n)}) + D_{s}^{\text{Dgm}(n)}P_{t-s} + D_{s}^{\text{Dgm}(n)}D_{s}^{\text{Dgm}(n)} = P_{t} + D_{t-s}^{\text{Dgm}(n)}.
\]
We now turn to norm continuity. Given the semigroup property, it suffices to show continuity at \( t = 0 \). The heat semigroup \( \mathcal{P}_t \) is indeed continuous at \( t = 0 \). As for \( \mathcal{D}_t^{\text{Dgm}(n)} \), we have \( \mathcal{D}_0^{\text{Dgm}(n)} := 0 \), and from (8.25) \( \lim_{t \to 0} \| \mathcal{D}_t^{\text{Dgm}(n)} \|_{\text{op}} = 0 \).

Proof of Theorem 1.1(b). Given (7.6), proving Part (b) amounts to showing \( \lim_{t \to 0} \| \mathcal{P}_t + \mathcal{D}_t^{\text{Dgm}(n)} \|_{\text{op}} \), which is justified because each \( \mathcal{P}_t \) is the Laplace transform of \( u \), \( \mathcal{D}_t^{\text{Dgm}(n)} \) for sufficiently large values of \( \lambda \) of the Laplace variable, i.e., to show that the Laplace transform of \( u \), \( \mathcal{P}_t + \mathcal{D}_t^{\text{Dgm}(n)} \) is the resolvent \( \langle u', \mathcal{R}_{-\lambda} u \rangle \).

To evaluate the Laplace transform of \( \langle u', (\mathcal{P}_t + \mathcal{D}_t^{\text{Dgm}(n)}) u \rangle \), assume for a moment \( u(x), u'(x) \geq 0 \), we integrate (1.17) (viewed as in integral operator) against \( e^{-\lambda t} \mathcal{Q}_t^{(n)}(x) u(x') \) over \( t \in \mathbb{R}^+ \) and \( x, x' \in \mathbb{R}^{2n} \), and sum the result over all \( (i,j) \in \text{Dgm}(n) \). This gives

\[
\int_0^\infty e^{-\lambda t} \langle u', \mathcal{P}_t + \mathcal{D}_t^{\text{Dgm}(n)} u \rangle dt = \sum_{(i,j) \in \text{Dgm}(n)} \left\langle u', \left( \prod_{a \in A} \int_0^\infty e^{-\lambda t} \mathcal{Q}_t^{(a)} dt \right) u \right\rangle,
\]

where, the operator \( \mathcal{Q}_t^{(a)} \) are indexed as described in the preceding. In deriving (8.30), we have exchanged sums and integrals, which is justified because each \( \mathcal{Q}_t^{(a)} \) has a positive kernel, and \( u(x'), u'(x) \geq 0 \) under the current assumption. On the r.h.s. of (8.30), the Laplace transforms \( \int_0^\infty e^{-\lambda t} \mathcal{Q}_t^{(n)} dt \) are evaluated as in Lemmas 8.1(b), 8.2(b), and 8.5(b). Putting together the expressions from these lemmas, and comparing the result to (3.12), we now have

\[
\int_0^\infty e^{-\lambda t} \langle u', (\mathcal{P}_t + \mathcal{D}_t^{\text{Dgm}(n)}) u \rangle dt = \left\langle u', \left( \text{r.h.s. of (1.23)} \right) \right|_{z=-\lambda} u \rangle = \langle u', \mathcal{R}_{-\lambda} u \rangle.
\]

For general \( u, u' \in \mathcal{L}^2(\mathbb{R}^{2n}) \), the preceding calculation done for \( (u(x), u'(x')) \rightarrow (|u(x)|, |u'(x')|) \) guarantees the relevant integrability, and justifies the exchange of sums and integrals.

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