Entanglement of Two-Superconducting-Qubit System Coupled with a Fixed Capacitor *

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We study thermal entanglement in a two-superconducting-qubit system in two cases, either identical or distinct. By calculating the concurrence of system, we find that the entangled degree of the system is greatly enhanced in the case of very low temperature and Josephson energies for the identical superconducting qubits and our result is in a good agreement with the experimental data.

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Entanglement has received much attention since it plays a central role in quantum information processing and quantum computing. There are several ways to generate entanglement through experiments. However, it is still an open question to generate very good entangled states. Current interest focuses on generating, maintaining and controlling precisely entangled states. As is well known, temperature and magnetic field can be prepared, control and maintain entanglement. One may ask a question: are there any other effective ways to control entanglement?

For micro-systems, entanglement between different kinds of qubits has been studied, for example, charge, flux and phase qubits. Of particular importance, the superconducting qubits take advantage of the two characteristic of superconducting and quantum and therefore become the most suitable candidates for quantum computing, which has been carried out in laboratory. For superconducting qubits, manipulation of quantum states has enabled scientists to generate partly entangled states. However, high quantum entangled states are required in such quantum technology. In the experimental aspect, entanglement has been generated for coupled charge qubits and coupled phase qubits, but the maximally entangled states are merely in theory. On the other hand, recent experiments have observed strong couplings between two superconducting qubits. As a consequence, they triggers the theoretical research on investigating superconducting qubits. We have carried out some research on the corresponding relations between the theory and experiment in quantum entanglement.

In this Letter, based on the experimental study of changing the entanglement degree by adjusting the capacitance and LC circuits, thermal entanglements are studied in two superconducting qubits, either identical or distinct. Different evolutions of the entanglement are observed. In the case that the superconducting qubits have the same Josephson energies, we investigate the effect of temperature and Josephson energies on entanglement. The result exhibits high quantum entangled states at low temperature. In addition, our theoretical results match with the experimental data very well, so the entangled qubits, which are made by making use of our data, should have better entangled nature. We hope that this would be confirmed experimentally in the future.

The present model is composed of two single cooper-pair box charge qubits, coupled with a fixed capacitor. This model has attracted much attention and research. The superconducting materials act as a superconductor with a suppressed transition temperature $T_c$ adjusted by using different materials, which is in mK range for practical operation in efficient and multiplex superconducting circuits. One good example is a superconductive material made from a superconducting Al/Ti/Au trilayer with respective thicknesses of 300, 200 and 200 Å, $T_C = 450$ mK. The Hamiltonian of two-superconducting-qubit system is given by

$$H = -\frac{1}{2}\left\{4E_{C1}\left(\frac{1}{2} - n_{q1}\right) + 2E_m\left(\frac{1}{2} - n_{q2}\right)\right\}\sigma_{z1} + 2E_m\left(\frac{1}{2} - n_{q1}\right)\sigma_{z2} + E_{J1}\sigma_{z1} + E_{J2}\sigma_{z2} - 2E_m\sigma_{zz},$$

where $E_{Cj}$ and $E_{Jj}$ are respectively the charging and Josephson energies and $E_m$ is the mutual coupling energy between the two qubits; $\sigma_{z1} = \sigma_x \otimes I$, $\sigma_{z2} =$ **

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important observation is that for the attractive case Hamiltonian is applied to study quantum gates too, which is independent of charging energy. This reduced the control gate capacitance and voltage, respectively.

For simplicity, calculations are restricted at the degeneracy point, where \( n_g = n_q = 0.5 \), which is the condition of insensitivity to noise. Under this condition, the model Hamiltonian reduces to

\[
H = -\frac{1}{2}(E_{J1}\sigma_{x1} + E_{J2}\sigma_{x2} - 2E_m\sigma_{z2}),
\]

which is independent of charging energy. This reduced Hamiltonian is applied to study quantum gates too.\(^{[24]}\)

The eigenvalues and eigenvectors of Hamiltonian can be obtained,

\[
H|\Psi_1\rangle = -\frac{1}{2}\sqrt{A}|\Psi_1\rangle, \quad H|\Psi_2\rangle = \frac{1}{2}\sqrt{A}|\Psi_2\rangle,
\]

\[
H|\Psi_3\rangle = \frac{1}{2}\sqrt{B}|\Psi_3\rangle, \quad H|\Psi_4\rangle = -\frac{1}{2}\sqrt{B}|\Psi_4\rangle,
\]

where

\[
|\Psi_1\rangle = \frac{1}{\sqrt{N_1}}(|00\rangle - |11\rangle - a_1(|01\rangle - |10\rangle)),
\]

\[
|\Psi_2\rangle = \frac{1}{\sqrt{N_2}}(|00\rangle - |11\rangle + a_2(|01\rangle - |10\rangle)),
\]

\[
|\Psi_3\rangle = \frac{1}{\sqrt{N_3}}(|00\rangle + |11\rangle + a_3(|01\rangle + |10\rangle)),
\]

\[
|\Psi_4\rangle = \frac{1}{\sqrt{N_4}}(|00\rangle + |11\rangle - a_4(|01\rangle + |10\rangle)),
\]

\( A = (E_{J1} - E_{J2})^2 + 4E_m^2 \) and \( B = (E_{J1} + E_{J2})^2 + 4E_m^2 \). Here \( a_1 = (\sqrt{A} + 2E_m)/(E_{J1} - E_{J2}) \), \( a_2 = (\sqrt{A} - 2E_m)/(E_{J1} - E_{J2}) \), \( a_3 = (\sqrt{B} + 2E_m)/(E_{J1} + E_{J2}) \) and \( a_4 = (\sqrt{B} - 2E_m)/(E_{J1} + E_{J2}) \). \( N_i \) is the normalization coefficient of \( |\Psi_i\rangle \) (\( i = 1, 2, 3, 4 \)). For \( E_{J1}E_{J2} > 0 \), the ground state is \( |\Psi_4\rangle \) and \( |\Psi_1\rangle \) for \( E_{J1}E_{J2} < 0 \). An important observation is that for the attractive case of \( E_{J1}E_{J2} > 0 \), the degeneracy states in the ground state appear. In order to measure entanglement, concurrence has been proposed and is defined as\(^{[27,28]}\)

\[
C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\},
\]

where the parameters \( \lambda_i \) in decreasing order are the square roots of the eigenvalues of the operator

\[
\lambda = \rho(\sigma_x^y \otimes \sigma_x^y)\rho^*(\sigma_x^y \otimes \sigma_x^y),
\]

where \( \sigma_x^y \) are the Pauli spin matrix of the two qubits and \( \rho = (1/Z)\exp(-H/kT) \) is the density operator of the system at the thermal equilibrium, where \( Z = \text{Tr}[\exp(-H/kT)] \) is the partition function. The concurrence \( C \) ranges from 0 for a separable state to 1 for a maximally entangled state. Following the same method in the standard basis, the density matrix of the system is

\[
\rho(T) = \frac{1}{Z}\begin{pmatrix}
    m_1 - m_2 & m_3 - m_4 & m_3 + m_4 & m_5 + m_6 \\
    m_3 - m_4 & m_1 + m_2 & m_5 - m_4 & m_5 + m_4 \\
    m_3 + m_4 & m_5 - m_4 & m_1 + m_2 & m_5 - m_4 \\
    m_5 + m_6 & m_3 - m_4 & m_3 - m_4 & m_1 - m_2
\end{pmatrix},
\]

where \( m_1 = \frac{1}{2}(\cosh(x_A) + \cosh(x_B)) \), \( m_2 = E_m(\sinh(x_A) + \sinh(x_B)) \), \( m_3 = E_m \frac{E_{J1} - E_{J2}}{2\sqrt{A}} \sinh(x_A) \), \( m_4 = \frac{1}{2}(\cosh(x_A) - \cosh(x_B)) \), \( m_5 = E_m \frac{\sinh(x_A) - \sinh(x_B)}{\sqrt{A}} \) and \( Z = 4m_1 \), with \( x_A = \frac{\sqrt{A}}{2m} \) and \( x_B = \frac{\sqrt{B}}{2m} \). The concurrence can be easily calculated by Eqs. (5) and (6).

For identical superconducting qubits, \( E_{J1} = E_{J2} = E_J \) and \( E_{C1} = E_{C2} \), the model Hamiltonian can be rewritten as

\[
H = -\frac{1}{2}(E_J\sigma_{x1} + E_J\sigma_{x2} - 2E_m\sigma_{z2}),
\]

where \( E_J \) and \( E_m \) can be adjusted by the experimental multiplexed capacitance in the circuits. Similar model was argued elsewhere for the choices of \( E_J \) as a magnetic field.\(^{[29]}\)

The eigenvalues and eigenvectors of Hamiltonian Eq. (7) read

\[
H|\psi_1\rangle = -E_m|\psi_1\rangle, \quad H|\psi_2\rangle = E_m|\psi_2\rangle,
\]

\[
H|\psi_3\rangle = \sqrt{D}|\psi_3\rangle, \quad H|\psi_4\rangle = -\sqrt{D}|\psi_4\rangle
\]

with \( D = E_m^2 + E_J^2 \); \( |\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \), \( |\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \), \( |\psi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \xi_+ |01\rangle + |10\rangle \) and \( |\psi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \xi_- |01\rangle + |10\rangle \), with \( \xi_\pm = \frac{E_m \pm \sqrt{D}}{E_J} \) and \( N_{\pm} \) are the normalization coefficients. Here \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are two of four Bell states, which are the maximally entangled states.

The density matrix can be obtained by the above same way in this case. Without lack of generality, the mutual coupling energy \( E_m \) is regarded as the energy unit and \( k = 1 \). Thus we can consider \( E_m/k = 1 \), whose unit is mK. For convenience, we only write its value as the same as \( E_J \). By making use of Eqs. (5) and (6), the concurrence can be calculated for the identical qubits. Especially, for \( E_J = 0 \) the Hamiltonian (7) only has the last item whose eigenvectors are the separable states, so that no thermal entanglement is present, namely, \( C = 0 \).

In the inset of Fig. 1, we show the concurrence as functions of Josephson energies and temperature, displaying nonmonotonic behavior for smaller \( E_J \) and lower temperature. However, in the limit \( E_J \to 0 \), \( E_m \approx \sqrt{D} \), the degeneracy states are present: \( \psi_1 \) and \( \psi_4 \); \( \psi_2 \) and \( \psi_3 \), namely, only two energy levels are populated. Thus there is an energy level crossing at the point \( E_J = 0 \), i.e., the ground state becomes the
degenerate states of \( \psi_4 \) and \( \psi_1 \). With an infinitesimally small increase of \( E_J \), the concurrence will increase sharply to a top in accordance with Ref. [29]. At the zero temperature, the entanglement primarily depends on \( |\psi_4\rangle \), i.e. on the ground state, which plays a major role. As \( T \) increases, the peaks fall, because the ground state will mix with excited states in thermodynamic equilibrium and mixing states combine the concurrence of the system. To illustrate this feature, Figs. 1 and 2 are plotted to show the behavior of \( C \) as a function of \( E_J \) and \( T \), respectively.

![Fig. 1](image1.png)

**Fig. 1.** (Color online) Two-dimensional plots of the concurrence vs Josephson energy \( E_J \) for different temperatures. Asterisks: the experimental data. Inset: three-dimensional plot of the concurrence as a function of \( E_J \) and \( T \).

![Fig. 2](image2.png)

**Fig. 2.** (Color online) Two-dimensional plots of the concurrence as a function of the temperature \( T \) for the four different Josephson energy.

Figure 1 clearly shows that no entanglement is present for \( E_J = 0 \). As \( E_J \) increases, the entanglement first reaches sharply to the maximum, then decays rapidly, finally reduces slowly and asymptotically to a stable value. Moreover, lower temperature and lower Josephson energy will cause the entanglement richer. Considering the data of the sample 2 for the identical qubits in Ref. [22] and \( E_{m} \) as the energy unit, we obtain \( E_{J1}/k = E_{J2}/k = 3.625 \). For \( T = 20 \text{ mK} \) and \( E_J = 3.625 \), the theoretical prediction is \( C = 0.26593 \), which shows the excellent agreement with \( C = 0.27 \) observed experimentally. [22]

In Fig. 2, \( C \) vs \( T \) for different Josephson energy is presented. For lower Josephson energy, for example \( E_J = 0.5 \), the concurrence will vary dramatically, but is not so apparent for bigger Josephson energy. When the temperature \( T = 0 \), only the ground state \( |\psi_4\rangle \) exists and \( C \approx 0.9 \). With the increase of \( E_J \), \( C_{|\psi_4\rangle} \) will decrease, so the intersections of the curve and \( C \)-axis decline. For a fixed smaller \( E_J \), as the temperature rising, the ground state and three excited states mix, \( C \) will decrease sharply. On the contrary, for the larger Josephson energy, the change behavior of \( C \) becomes very slow and finally \( C \) tends stably at \( T \leq T_C \). Thus, the concurrence is very susceptible to small Josephson energy at lower temperature (see Fig. 2).

![Fig. 3](image3.png)

**Fig. 3.** (Color online) Three-dimensional plot of the concurrence as functions of temperature \( T \) and Josephson energy \( E_{J1} \) with \( E_{J2} = 17.2 \).

For the distinct superconducting qubits, the concurrence can be calculated through Eqs. (5) and (6). To distinguish different influences of identical and distinct superconducting qubits on the entanglement at the same temperature, Figure 3 shows the evolution of the concurrence as functions of \( E_{J1} \) and \( T \) with \( E_{J2} = 17.2 \). Obviously \( C \) is smaller than that of the identical case at low temperature.

The contour figure of \( C \) is plotted in Fig. 4. It is
worth noting that two Josephson energies are smaller and closer, the concurrence decreases more slowly and the peak is higher. This proves that $C$ is maximal at $E_{J1} = E_{J2}$ for the stable temperature. When the values of $E_{J1}$ stay away from $E_{J2}$, $C$ will decay. According to the data in Ref. 22, by taking $E_{J1} = 13.6$ and $E_{J2} = 17.2$, our theoretical result is $C = 0.064$, which matches with the experimental $C = 0.06$.[22] 

To compare in more detail, evolutions of concurrences are shown in Fig. 5 for the two cases. One can clearly find the difference between them. The apparent difference is the maximal concurrence. The maximum value of $C$ for $E_{J1} = E_{J2}$ is much larger than $E_{J1} \neq E_{J2}$. That is to say, choosing the proper superconducting qubits can enhance the entanglement at low temperature. Two Josephson energies are smaller and closer, then the maximum value of concurrence will be larger.

In conclusion, we have investigated the effect of Josephson energies on the thermal entanglement in the two-superconducting-qubit model. In the two cases, i.e., identical and distinct superconducting qubits, we have presented the evolution of concurrence with respect to the Josephson energy and temperature. Comparing the results of these two cases, we conclude that the entanglement may be enhanced under the identical superconducting qubits for the same temperature. When the temperature and Josephson energies are lower, the entangled degree of the system is greatly enhanced. Our theoretical prediction is in good agreement with the experiments and provides a new way to enhance and to control the entanglement degree of the system by adjusting the Josephson energies, which can be realized experimentally by changing the capacitance and LC circuits. Utilizing the results of calculation and investigation, we may generate better entangled and stable states, which could have wide applications in the quantum communication and physical experiments.

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Fig. 5. (Color online) Two-dimensional plots of the concurrence as a function of Josephson energy $E_{J1}$ for $T = 20$ mK.