Symmetries in Two Dimensional Conformal Field Theories and Related Integrable Models

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1 Introduction

The two-dimensional (2D) conformal field theories (CFT's) have a wide application in the description of the scaling and universality behaviour of the 2D statistical systems at the second order phase transition points. They play also a basic role in the description of the string theories providing the symmetry of the worldsheet. As was shown in [1] the specific properties of the reducible representations of the Virasoro algebra (describing the so called minimal models) allows one to solve the conformal bootstrap and to find the the exact critical exponents, the explicit multipoint correlation functions of the fields and finally to obtain the full structure of their associative operator product expansion (OPE) algebra. The different conformal models, characterized by the value of the central charge $c$ and their 2D operator content, describe different universal behaviours of the statistical systems near the critical point. In this sense the classification of the universality classes in two dimensions is equivalent to exhausting the minimal models of all possible extensions of the Virasoro algebra (e.g. supersymmetry, $W$-algebras etc.). The solution of this problem in the context of string models will provide us with all the possible classical string backgrounds. Another example of 2D CFT is the Liouville field theory (LFT). It has been studied actively for its relevance for non-critical string theories and 2D quantum gravity. This theory is also interesting on its own as an example of irrational CFT. Most of the CFT formalism developed for rational CFT’s do not apply to this class of theories mainly because they have continuously infinite number of primary fields. Various methods have been proposed to derive the structure constants and the reflection amplitudes, which are basic building blocks to complete the conformal bootstrap [2].

As mentioned above a natural generalization of the 2D CFT is its $N = 1$ supersymmetric extension. Two kinds of fields appear in this theory belonging to the so called Neveu-Swartz (NS) and Ramond (R) sectors respectively. An important technical problem is their full description in the supersymmetric minimal models. It turns out that the construction of the NS sector of these models requires a slight modification of the Virasoro algebra methods only [1]. The difficulties with the fusion rules and multipoint functions of the Ramond fields are connected with the double valuedness of the supercurrent $G(z)$: $G(e^{2\pi i}z) = e^{ik\pi}G(z)$, $k = 0, 1$ in the presence of the R fields. Because of that the direct generalization of of the NS sector null vector’s method to the R sector seems to be ineffective. A more constructive and certainly more powerful approach to the minimal models is the Coulomb gas representation proposed in [3] for the Virasoro algebra models. In the case of $N = 1$ superconformal minimal models the Coulomb gas method was developed in [4] and [5]. One could further consider $N = 1$ minimal models defined on hyperelliptic surfaces. It turns out that they also can be described by a generalized Coulomb gas representation. The partition function of the models on such $Z_2$ surfaces are then constructed in terms of the correlation functions of fields from the twisted sector of the corresponding branched sphere models. It is interesting to investigate also the renormalization group properties of the supersymmetric minimal models $SM_p$ perturbed by the least relevant field. The first order corrections were obtained in [6]. It was argued that there exists an infrared (IR) fixed point of the renormalization group
(RG) flow which coincides with the minimal superconformal model $SM_{p-2}$. It is interesting to check this result in the second order of the perturbation theory. Calculation up to the second order is always a challenge even in two dimensions. The problem is that one needs the corresponding four-point function which is not known exactly even in two dimensions. Fortunately, in the scheme proposed in [7] one needs the value of this function up to the zeroth order in the small parameter $\epsilon$ describing the dimension of the perturbing field. In addition to the minimal superconformal models one can consider the example of an irrational CFT - the $N = 1$ supersymmetric LFT (SLFT). This model has some motivations. It is applicable to the superstring theories and the 2D supergravity with fermionic matter fields. One can also understand the role of the extended conformal symmetry in the irrational CFT’s by studying this model. The methods used for the bulk LFT (in obtaining the 3-point functions for example) could be applied successfully to SLFT although the latter becomes algebraically more complicated. It is interesting to extend this formalism to the CFT defined in the two-dimensional space-time geometry with a boundary condition (BC) which preserves the conformal symmetry. It is known [8] that the conformally invariant BC’s can be associated with the primary fields for the case of rational CFT’s. It has been an issue whether this could be extended to the irrational theories. Another motivation is to understand open string theories in various nontrivial background space-time geometries.

The Vrasoro and the $N = 1$ superconformal minimal models are just the first two members of a more general infinite series of the so called coset models. More precisely, let us consider a model $M(k, l)$ based on the symmetric coset $\hat{s}u(2)_k \times \hat{s}u(2)_l / \hat{s}u(2)_{k+l}$, $k$ and $l$ are integers [9]. It is written in terms of the $\hat{s}u(2)_k$ WZNW models of level $k$. The WZNW model is a conformal theory with a stress-energy tensor given by the Sugawara construction:

$$T_k(z) = \frac{1}{k+2} J^2(z)$$

where $J^a(z)$ is the $\hat{s}u(2)$ current. The central charge of the corresponding Virasoro algebra is $c_k = \frac{3k}{k+2}$. The coset theory $M(k, l)$ is then also a conformal field theory with a stress tensor $T = T_k + T_l - T_{k+l}$. The resulting central charge can be read from this construction and is labeled by the two integers $k$ and $l$. The dimensions of the primary fields $\phi_{m,n}(l,k)$ of the minimal coset models are known [10] and are labelled by the integers $m, n$ and an additional integer number $s$. It is known [10, 11, 12] that the coset theory $M(k, l)$ possesses a larger parafermionic-like symmetry. The simplest (lowest dimensional) current $A(z)$ for example has a dimension $\Delta_A = \frac{l+4}{l+2}$. For $l = 2$ it coincides with the supercurrent $G(z)$ of the superconformal theory, for $l = 4$ the corresponding current generates the well known $4/3$-parafermionic series of models. In general, under this symmetry the primary fields are divided in sectors labelled by the integer $s$. The problem of the description of the coset theory in terms of the representations of the parafermionic algebra lies in the difficulty with its nonlocal nature. Because of that it is not well understood. We will present in Section 4 another recursive construction of $M(k, l)$ based on the product of lower level models. Another interesting problem is the investigation of the renormalization group properties of the coset models perturbed by the least relevant field, in the lines of what was done for the Vrasoro and $N = 1$ superconformal theories.

The conformal theories with $N = 2$ supersymmetry are natural generalizations of the $N = 0$ (Virasoro) and $N = 1$ superconformal theories. The extended $N = 2$ superconfor-
mal symmetry in two dimensions [13] plays an important role in the classification and the construction of the classical superstring vacua, giving rise to $N = 1$ SUSY models in four dimensions [14]. The different ways to realize explicitly such vacua and the corresponding low-energy effective action [15] are mainly based on the properties of appropriate $N = 2$ superconformal models. In principle all the properties of the effective four-dimensional theory we wish to have can be encoded as specific properties of the 2D $N = 2$ superconformal models realizing the corresponding superstring compactification. For example the fusion rules (FR’s) and the structure constants determine the Yukawa couplings of the massless low-energy particles and the N-point functions of the 2D fields (representing the vertices of these particles) are the main ingredients in the construction of the 4D scattering amplitudes. All these issues make it interesting the classification of the $N = 2$ superconformal models with $c \leq 9$ and the systematic study of their properties. An important point in the realization of this program is the full description of the minimal models or the $N = 2$ unitary discrete series [16, 17] and of the specific tensor products of these models. According to Gepner’s compactification scheme [18] they are related to certain Calabi-Yau non-linear sigma models. Different elements, needed for the explicit construction of the $N = 2$ minimal models can be found in [19]. The remaining open problems are mainly related to the appropriate description of the Ramond (R) and twisted (T) sectors of the minimal models, to the computation of the 4-point functions of all the fields in NS, R and T sectors, the corresponding structure constants and the FR’s.

The $N = 2$ SLFT is instead an example of an irrational CFT which has continuously infinite number of primary fields. In spite of the extended symmetry, it turns out that exact correlation functions of the $N = 2$ SLFT are much more difficult to derive than the $N = 0$ and $N = 1$ cases. The main reason is that the $N = 2$ SLFT has no strong-weak coupling duality. The invariance of the LFT and the $N = 1$ SLFT under $b \rightarrow 1/b$ is realized when the background charge changes to $b + 1/b$ from its classical value $1/b$ after quantum corrections [20]. All the physical quantities like the correlation functions depend on the coupling constant through this combination. This duality as well as the functional relations based on the conformal bootstrap methods are essential ingredients to obtain exact correlation functions uniquely for $N = 0$ [2] and $N = 1$ LFT [21] and their boundary extensions. Differently, the $N = 2$ SLFT is not renormalized and no duality appears. This non-renormalization is a general aspect of the $N = 2$ superconformal field theories in two dimensions. Without the self-duality the functional relations satisfied by the correlation functions cannot be solved uniquely. In [22], an $N = 2$ super-CFT has been proposed as a dual theory to the $N = 2$ SLFT under the transformation $b \rightarrow 1/b$.

It is of special interest to study the $N = 2$ SLFT in the presence of a (super) conformally invariant boundary. Computing the corresponding one-point functions is more complicated than in the case of $N = 0$ and $N = 1$ SLFT. The standard approach for the computation of the one-point functions is the conformal bootstrap method [23, 24, 25] which can generate functional relations using the conformally invariant boundary actions as boundary screening operators. The boundary action of the $N = 2$ SLFT has been derived in [26]. But the $N = 2$ SLFT with this boundary action is not self-dual either and one needs to know the boundary
action of the dual $N = 2$ theory. Without this one cannot solve the functional relations uniquely. Due to the non-locality of the bulk action of the dual $N = 2$ theory the method used in the $N = 2$ SLFT \cite{26} seems not be applicable. We need a different approach. One possibility is to use the so called modular bootstrap. The modular bootstrap method is a generalization of the Cardy formulation for the conformal BC’s to the irrational CFT’s. We can first derive the one point functions from the modular transformation properties. Then we relate them to the bulk and boundary actions of the $N = 2$ SLFT and its dual theory by the conformal bootstrap method.

It is known \cite{27} that in the LFT an infinite set of relations hold for the quantum operators. These equations relate different basic Liouville primary fields $V_\alpha(z)$ (represented by the vertex operators $\exp(\alpha \phi)$). They are parameterized by a pair of positive integers $(m,n)$ and are called ”higher equations of motion” (HEM), because the first one $(1,1)$ coincides with the usual Liouville equation of motion. Higher equations turn out to be useful in practical calculations. In particular, they were used to derive a general 4-point correlation function in the minimal Liouville gravity. Similar operator valued relations have been found also for the $N = 1$ SLFT \cite{28}. It is intriguing to understand if such relations appear also for the extended $N = 2$ SLFT.

The two dimensional integrable system is a classical or quantum field theory with the property of having an infinite number of local integrals of motion in involution (LIM). This kind of symmetry does not allow the determination of the most interesting features of the system because of its Abelian character. Instead, the presence of an infinite dimensional non-Abelian algebra could complete this Abelian algebra giving rise to the possibility of building its representations, i.e. the spectrum of local fields. One could call this non-commutative algebra a spectrum generating algebra. In different models the presence of this spectrum generating symmetry is often connected to the Abelian one. This is the case of the simplest integrable quantum theories—the 2D CFT’s—their common crucial property being their covariance under the infinite dimensional Virasoro symmetry. Indeed, the highest weight representations of this algebra classify all the local fields in the 2D CFT’s and turn out to be reducible because of occurrence of vectors of null Hermitian product with all the other vectors, the so called null-vector. The factorization by the modules generated over the null-vectors leads to a number of important algebraic-geometrical properties such as fusion algebras, differential equations for the correlation functions etc. Unfortunately, this beautiful picture collapses when one pushes the system away from criticality by perturbing the original CFT with some relevant local field. From the infinite dimensional Virasoro symmetry only the Poincare subalgebra survives the perturbation. Consequently, one of the most important open problems in 2D integrable quantum field theories (IQFT’s) is the construction of the spectrum of local fields and the computation of their correlation functions. Actually, the CFT possesses a bigger $W$-like symmetry and in particular it is invariant under an infinite dimensional Abelian subalgebra of the latter \cite{29}. With suitable deformations, this Abelian subalgebra survives the perturbation resulting in the so called LIM. As we noticed, being Abelian this symmetry does not carry sufficient information and in particular one cannot build the spectrum of an integrable theory by means of the LIM alone. It has
been conjectured in [30] that one could add to the LIM $I_{2m+1}$ non-commuting charges $J_{2m}$ in such a way that the resulting algebra would be sufficient to create all the states of a particular class of perturbed theory, the so-called restricted sine-Gordon theory. Therein it was also discovered that a sort of null-vector condition appears in the above procedure leading to certain equations for the form-factors.

One of the simplest integrable field theories is the 2D sine-Gordon (SG) model. It possesses an infinite number of conserved charges $I_{2n+1}, n \in \mathbb{Z}$ in involution. It is known [31] that the SG theory possesses also an infinite dimensional symmetry provided by the $sl(2)_q$ algebra. However, this symmetry connects the correlation functions of the fields in the same multiplet without giving a sufficient information about the functions themselves. It is known also to some extent that there should be another kind of symmetry present in the SG theory. Actually, it could be obtained as a particular scaling limit of the so-called XYZ-spin chain [32]. The latter is known to possess an infinite symmetry obeying the so-called Deformed Virasoro algebra (DVA) [33]. It is natural to suppose that in the scaling limit, represented by SG, there should be some infinite dimensional symmetry, a particular limit of DVA. What remains unclear is how this symmetry is realized in SG theory, for example what is the action of the corresponding generators on the exponential fields, what kind of restrictions it imposes on their correlation functions etc.

In a 2D integrable QFT which can be realized as a CFT perturbed by some relevant operator it is well known that any correlation function of local fields $O_a(x)$ in the short distance limit can be reduced down to the one-point functions $<O'_a(x)>$ by successive application of the operator product expansion [34]. These vacuum expectation values (VEV’s) contain important information about the IR environment. Important progress has been made concerning the evaluation of some VEV’s in different integrable QFT - for example the VEV’s of exponential fields in the sinh-Gordon and SG models [35] and in the Bullough-Dodd (BD) model with real and imaginary coupling [36]. However, the higher order corrections to the short distance expansion of the correlation functions involve the VEV’s of the descendent fields. The knowledge of these quantities improves the analytical prediction for the short distance expansion of the correlation functions which can be better compared with the results obtained from the numerical study of the model.

2 $N = 1$ superconformal theories

In this Section we present a detailed discussion of the Ramond sector of the superconformal models and the explicit structure of the OPE algebras of the fields of these models. It appears that all the elements of our construction: vertices, screening operators, 3- and 4-point functions etc. can be written in terms of of the Ising model fields $\sigma(z), \psi(z)$ and a free scalar field $\phi(z)$. These constructions allow us to find the fusion rules (FR’s) for the R and NS fields. The main advantage of the Ramond Coulomb gas method lies in the calculation of the multipoint correlation functions. An important step in the construction of the 2D models
(combining both the left and right chiral ones) is the requirement of the crossing symmetry of the 4-point functions. The solution of this problem allows us to calculate explicitly the operator content of the 2D $N = 1$ superconformal models and the exact structure constants of the OPE algebra of the fields of the model. Next, we describe also the superconformal minimal models on a restricted class of surfaces which can be represented as a double covering of the branched sphere. Such surfaces are known as hyperelliptic surfaces. The strategy we use is to reduce the genus $g$ problem to the corresponding $g = 0$ problem. The partition function is computed using a generalized Coulomb gas representation.

We are interested also in the renormalization group properties of the superconformal minimal models $SM_p$ perturbed by the last component of the superfield $\Phi_{1,3}$ in the second order of the perturbation theory. We present the computation of the conformal blocks in the NS sector and the mixed conformal blocks of NS and R fields. The computation of the beta-function and the IR fixed point confirms that it coincides up to second order with the model $SM_{p-2}$. The matrix of anomalous dimensions for certain NS and R fields are also computed. The results are in perfect agreement with the conjectured RG flow.

Finally, we consider the $N = 1$ supersymmetric LFT which is an example of irrational CFT. We propose exact expressions for the 3-point correlation functions in NS and R sectors of the theory. Using the reflection properties of the Liouville vertex operators we introduce the so called reflection amplitudes for the NS and R fields. We then extend our considerations to a supersymmetric LFT defined in the 2D space-time geometry with a boundary condition (BC) which preserves the (super-) conformal symmetry. We use the functional relation method for the boundary SLFT with super-conformal boundary action [23] to derive the one-point function of a bulk operator and correlation functions of two boundary operators for a given conformal BC. Here the conformal BC is denoted by a continuous parameter related to the coupling constant in the boundary action. Another development is to generalize this method to the boundary SLFT defined in the Lobachevskiy plane, or the pseudosphere. We show that in both cases the results are consistent with the Cardy formalism [8]. We also show that the boundary 2-point functions of the (NS) boundary operators satisfy the same relation as those of the LFT.

The results of this Section have been published in [5, 21, 25], [37]-[41], (1.-8.).

2.1 $N = 1$ minimal models

The infinite superconformal algebra in two dimensions (2D) splits into a direct sum of two algebras: a left one generated by the stress-energy tensor $T(z)$ (of dimension 2) and its
fermionic superpartner $G(z)$ (of dimension $\frac{3}{2}$) [4]:

\[ T(z_1)T(z_2) = \frac{c}{2z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{1}{z_{12}} \partial T(z_2) + \ldots \]  
\[ T(z_1)G(z_2) = \frac{3}{2z_{12}^2} G(z_2) + \frac{1}{z_{12}} \partial G(z_2) + \ldots \]  
\[ G(z_1)G(z_2) = \frac{2c}{3z_{12}^3} + \frac{2}{z_{12}} T(z_2) + \ldots \]

and a right one defined by the corresponding singular terms in the OPE of $\bar{T}(\bar{z})$ and $\bar{G}(\bar{z})$.

We shall restrict our discussion in the following to the "chiral", one-dimensional part, leaving the 2D construction for the end of this Section.

The two different boundary conditions for the supercurrent $G(e^{2\pi i z}) = \pm G(z)$ imply two different Laurent mode expansions:

\[ G^{(p)}(z) = \sum_{n \in \mathbb{Z}} \frac{G^{(p)}_{n+1/2}}{z^{n+2}} \]  
\[ G^{(a)}(z) = \sum_{n \in \mathbb{Z}} \frac{G^{(a)}_n}{z^{n+3/2}} \]

while the stress-energy tensor has the usual mode expansion:

\[ T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \]

Then the OPE’s (2.1) take the well known form of the Neveu-Schwarz (NS) and Ramond (R) algebras:

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0} \]  
\[ [L_n, G_\alpha] = \left( \frac{n}{2} - \alpha \right) G_{n+\alpha} \]  
\[ \{G_\alpha, G_\beta\} = 2L_{\alpha+\beta} + \frac{c}{3} (\alpha^2 - \frac{1}{4}) \delta_{\alpha+\beta,0} \]

where $\alpha, \beta \in \mathbb{Z} + 1/2$ for the NS sector and $\alpha, \beta \in \mathbb{Z}$ for the R sector. According to [4] we can define the primary states $|\Delta \rangle$ corresponding to the lowest weight representations of the superalgebra (3.124) requiring:

\[ L_n |\Delta \rangle = G_\alpha |\Delta \rangle = 0, \quad (n, \alpha > 0), \quad L_0 |\Delta \rangle = |\Delta \rangle \]

Since in the Ramond sector a $G(z)$ zero mode appears:

\[ [G_0, L_0] = 0, \quad G_0^2 = L_0 - \frac{c}{24} \]

the lowest energy Ramond state is doubly degenerate (for $\Delta \neq \frac{c}{24}$), i.e. both states $|\Delta, \pm \rangle$ defined as:

\[ G_0 |\Delta, + \rangle = |\Delta, - \rangle, \quad G_0 |\Delta, - \rangle = (\Delta - \frac{c}{24}) |\Delta, + \rangle \]
correspond to the same eigenvalue $\Delta$ of $L_0$. Introducing the invariant vacuum state $|0>$:

$$L_n|0> = G_\alpha|0> = 0, \quad n \geq -1, \alpha \geq -\frac{1}{2}$$

we can represent the NS state $|\Delta>$ in terms of the NS primary superfields $\Phi_\Delta(z, \theta) = \phi_\Delta(z) + \theta \psi_{\Delta+1/2}(z)$:

$$|\Delta> = \phi(0)|0>, \quad G_{-1/2}|\Delta> = \psi(0)|0>$$

In the OPE language these algebraic properties of the primary fields have the form:

$$T(z_1)\phi(z_2) = \frac{\Delta}{z_{12}^{\Delta}} \phi(z_2) + \frac{1}{z_{12}} \partial \phi(z_2) + \ldots, \quad (2.4)$$

$$G(z_1)\phi(z_2) = \frac{1}{z_{12}} \psi(z_2) + \ldots,$$

$$G(z_1)\psi(z_2) = \frac{2\Delta}{z_{12}^{\Delta}} \phi(z_2) + \frac{1}{z_{12}} \partial \phi(z_2) + \ldots$$

Using the mode expansion of $T(z)$ and $G(z)$ one obtains from (2.4) the Ward identities:

$$[L_n, \Phi(z, \theta)] = \left( z^{n+1} \partial + (n+1)z^n (\Delta + \frac{1}{2} \theta \frac{\partial}{\partial \theta}) \right) \Phi(z, \theta), \quad (2.5)$$

$$[G_{n+\frac{1}{2}}, \Phi(z, \theta)] = z^n \left( z \left( \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right) - 2\Delta(n+1)\theta \right) \Phi(z, \theta).$$

The $R$-primary states are created from the NS vacuum by the corresponding Ramond primary fields $R_\Delta^\pm(z)$:

$$|\Delta, \pm> = R_\Delta^\pm(0)|0> \quad (2.6)$$

The latter have the following OPE with the supercurrent:

$$G(z_1)R_\Delta^\pm(z_2) = \frac{a^\pm(\Delta)}{z_{12}^{3/2}} R_\Delta^\pm(z_2) + \ldots, \quad (2.7)$$

where:

$$a^+(\Delta) = 1, \quad a^-(\Delta) = \Delta - \frac{c}{24} \quad (2.8)$$

and the OPE with the stress-tensor is the same as for the NS fields. It is clear from the OPE’s (2.4) and (2.7) that the fields from the NS and R sectors realize different analytic behaviour of the supercurrent - periodic and antiperiodic respectively, which is a manifestation of the hidden $Z_2$ symmetry of of the $N = 1$ superconformal theories. Correspondingly, NS fields describe the even and R fields the odd sectors of such theories with respect to this discrete symmetry. We shall also use a diagonal basis for the Ramond fields:

$$G(z_1)\tilde{R}_\Delta(z_2) = \mp \frac{1}{z_{12}^{3/2}} \sqrt{\Delta - \frac{c}{24}} \tilde{R}_\Delta(z_2) + \ldots$$
\(\text{(for } \Delta \neq \frac{c}{24}\text{)}\) and we defined:

\[
\tilde{R}_\Delta = \sqrt{\Delta - \frac{c}{24}}R^+ \pm R^-.
\]

It is well known [4] that the (reducible) unitary representations of the \(N = 1\) superconformal algebras given by:

\[
c = \frac{3}{2} - \frac{12}{p(p + 2)} \quad \text{ (2.9)}
\]

\[
\Delta_{n,m} = \frac{(p + 2)n - pm)^2 - 4}{8p(p + 2)} + \frac{1}{32}(1 - (-1)^{n-m})
\]

\((n-m \in 2Z \text{ for the NS sector and } n-m \in 2Z+1 \text{ for the R sector})\) determine an infinite series of exactly solvable minimal models. The basic property of the superconformal representations (or superconformal families) \([\phi_{\Delta_{n,m}}]\) is that at level \(\frac{1}{2}nm\) there exist descendent fields \(\phi_{\Delta_{n,m}+\frac{1}{2}nm}\) which are again primary fields. Then the covariant condition \(\phi_{\Delta_{n,m}+\frac{1}{2}nm} = 0\) separates the irreducible part of the representations \([\phi_{\Delta_{n,m}}]\). In the NS sector these null vector conditions, together with the Ward identities (2.5) lead to differential equations for the \(n\)-point functions of the fields \(\phi_{\Delta_{n,m}}\). These equations allow one to find explicitly for example the corresponding fusion rules. This was explained in details in [42, 37].

The difficulties with the application of this method for the Ramond fields come from the branch cut singularity in the OPE (2.7). In [5] it was described a modification of the null vector’s method based on the specific analytic properties of the Ramond fields. It consists basically in defining an auxiliary correlation function in which the branch cut is cancelled by multiplying the original function with a suitable power of the coordinates. Then, the OPE (2.7) and the null vector condition lead to an equation for the original correlation function. We omit here the explicit calculations since the Coulomb gas construction that we present below turns out to be more powerful. We shall only consider in more details the computation of the Ising model 4-point functions, which is performed using a similar null vector technics, since they are important elements of the Ramond Coulomb gas construction [38].

Let us consider the semi-direct sum of the Virasoro algebra and the algebra of the Laurent coefficients \(\psi_n\) of the antiperiodic Majorana field:

\[
\psi(z) = \sum_{n \in Z} \frac{\psi_n}{z^{n+1/2}}, \quad \psi(e^{2\pi i}z) = -\psi(z),
\]

\[
[L_n, \psi_m] = -(\frac{n}{2} + m)\psi_{n+m}, \quad \{\psi_n, \psi_m\} = \delta_{n+m,0}
\]

(2.10)

which is an analog of the Ramond sector for the case of the Ising model. Because of the zero mode of the fermion:

\[
[L_0, \psi_0] = 0, \quad \psi_0^2 = \frac{1}{2}
\]

\(\Delta_{n,m} \in 2Z\) for the NS sector and \(\Delta_{n,m} \in 2Z+1\) for the R sector. The basic property of the superconformal representations (or superconformal families) \([\phi_{\Delta_{n,m}}]\) is that at level \(\frac{1}{2}nm\) there exist descendent fields \(\phi_{\Delta_{n,m}+\frac{1}{2}nm}\) which are again primary fields. Then the covariant condition \(\phi_{\Delta_{n,m}+\frac{1}{2}nm} = 0\) separates the irreducible part of the representations \([\phi_{\Delta_{n,m}}]\). In the NS sector these null vector conditions, together with the Ward identities (2.5) lead to differential equations for the \(n\)-point functions of the fields \(\phi_{\Delta_{n,m}}\). These equations allow one to find explicitly for example the corresponding fusion rules. This was explained in details in [42, 37].

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Let us consider the semi-direct sum of the Virasoro algebra and the algebra of the Laurent coefficients \(\psi_n\) of the antiperiodic Majorana field:

\[
\psi(z) = \sum_{n \in Z} \frac{\psi_n}{z^{n+1/2}}, \quad \psi(e^{2\pi i}z) = -\psi(z),
\]

\[
[L_n, \psi_m] = -(\frac{n}{2} + m)\psi_{n+m}, \quad \{\psi_n, \psi_m\} = \delta_{n+m,0}
\]

(2.10)

which is an analog of the Ramond sector for the case of the Ising model. Because of the zero mode of the fermion:

\[
[L_0, \psi_0] = 0, \quad \psi_0^2 = \frac{1}{2}
\]
the lowest energy state of this algebra, $|\sigma^\pm\rangle$, is doubly degenerate and has dimension $\Delta^\pm = \frac{1}{16}$. The "spin fields" corresponding to this "Ramond states" $|\sigma^\pm\rangle = \sigma^\pm(0)|0\rangle$ produce a branch cut singularity of the antiperiodic fermionic field:

$$\psi(z)\sigma^\pm(w) = \sqrt{\frac{1}{2(z-w)}}\sigma^\pm(w) + \ldots.$$ 

We shall also use a diagonal basis $\tilde{\sigma} = \frac{1}{\sqrt{2}}(\sigma^+ \mp \sigma^-)$ with the following OPE:

$$\psi(z)\tilde{\sigma}(w) = \mp \sqrt{\frac{1}{2(z-w)}}\tilde{\sigma}(w) + \ldots (2.11)$$

The $SL(2,R)$ invariance allows to write the 4-point function of the field $\sigma(z)$ in the form:

$$\mathcal{F}(z) = \lim_{w \to \infty} w^{1/8} <\sigma(w)\sigma(z)\sigma(1)\sigma(0)> = z^{1/8}(1-z)^{-1/8}f(z).$$

The "null vector method" used in the case of Ramond fields in [38] can be applied also for the above function in the Ising model. Namely, the singular part of the OPE (2.11) and the first level null vector of the Ising algebra:

$$\left(L_{-1} - \frac{1}{2}\psi_{-1}\psi_0\right)|\sigma> = 0$$

allow us to find a first order differential equation for the unknown function $f(z)$:

$$4zf' - \frac{1}{\sqrt{1 - \frac{1}{z}}}\left(1 - \sqrt{1 - \frac{1}{z}}\right)f(z) = 0.$$

The final solution reads:

$$\mathcal{F}(z) = \sqrt{\frac{1}{2}z^{1/8}(1-z)^{-1/8}}\sqrt{1 + \sqrt{1 - \frac{1}{z}}}.$$

(2.12) The normalization is fixed by the OPE (2.11) and by normalizing the two-point function of the spin field to one.

Repeating the same procedure for the 4-point function:

$$\tilde{\mathcal{F}}(z) = \langle \tilde{\sigma}(\infty)\sigma(z)\tilde{\sigma}(1)\sigma(0) >$$

we obtain a similar result:

$$\tilde{\mathcal{F}}(z) = \sqrt{\frac{1}{2}z^{1/8}(1-z)^{-1/8}}\sqrt{1 - \sqrt{1 - \frac{1}{z}}}.$$

(2.13)
This method, together with the OPE:

\[ \psi(z_1)\psi(z_2) = \frac{1}{z_{12}} + 2z_{12}T(z_2) + \ldots \]

leads to a recursive equation for the following correlation function:

\[ G^N(z, v_i) = \langle \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=1}^N \psi(v_i)\sigma(0) \rangle. \quad (2.14) \]

The solution of this equation can be written in the form of the Wick-like theorem:

\[ G^N(z, v_i) = (\prod_{i=1}^N f_i(v_i, N - i + 1) + \sum_{\text{all } i < j} \prod_{l \neq i, j} f_l g_{ij} + \ldots)F(z) \quad (2.15) \]

where:

\[ f_i(v_i, N) = \sqrt{\frac{1}{2}} \left( \frac{\sqrt{z - 1}}{1 - v_i} + (-1)^{i+1} \frac{\sqrt{z}}{v_i} \right) \sqrt{\frac{(1 - v_i)v_i}{z - v_i}}, \]

\[ g_{ij} = (-1)^{i+j-1} \frac{v_i(1 - v_i)(z - v_i)}{v_j(1 - v_j)(z - v_j)}. \]

As we will see below these functions are an important ingredient in the calculation of the 4-point functions of the Ramond fields.

The natural language for the description of the two-dimensional minimal models is the so called Coulomb gas construction. Its generalization to the case of of \( N = 1 \) superconformal models is based on the free scalar (chiral) superfields \( S(z, \theta) = \phi(z) + \theta \psi(z) \) (and antichiral \( \bar{S}(\bar{z}, \bar{\theta}) \)) with the action:

\[ A(S, \bar{S}) = \frac{2}{\pi} \int d\bar{z}d\bar{z} (\frac{1}{2} \partial \bar{\phi} \bar{\partial} \bar{\phi} - \psi \bar{\partial} \bar{\psi}). \quad (2.16) \]

It follows from this action that the propagator is given by:

\[ < S(z_1, \theta_1) \ S(z_2, \theta_2) >= -\ln \frac{\hat{z}_{12}}{R}, \quad (2.17) \]

\( (R \text{ is the infrared cut-off}) \). In this language the superfields of the conformal grid are constructed in terms of the so called NS vertices:

\[ V_\alpha(z, \theta) = e^{i\alpha S(z, \theta)}, \quad \alpha \in R. \quad (2.18) \]
where :: defines certain normal ordering.

Actually, the action (2.16) leads to a free theory with central charge $c = \frac{3}{2}$. The construction which leads to the anomalous central charge:

$$c = \frac{3}{2} - \frac{12}{p(p + 2)}$$

is generated by the modified action:

$$\mathcal{A}(S, \bar{S}) = \frac{2}{\pi} \int dz d\bar{z} d\theta d\bar{\theta} \left( \frac{1}{2} DS \bar{D} \bar{S} - 2i\alpha_0 \hat{R}(S + \bar{S}) \right)$$

(2.20)

where $\hat{R}$ is the curvature superfield. The latter can be chosen such that its inclusion in (2.20) is equivalent to an introduction of a vertex operator $\exp(-2i\alpha_0 S)$ at infinity. Then the correlation function:

$$<\prod_{i=1}^{N} V_{\alpha_i}(z_i, \theta_i) > = \int DS D\bar{S} \prod_{i=1}^{N} e^{i\alpha_i S(z_i, \theta_i)} e^{-\mathcal{A}(S, \bar{S})}$$

(2.21)

calculated with the modified action should satisfy the neutrality condition:

$$\sum_{i=1}^{N} \alpha_i = 2\alpha_0$$

(2.22)

(in order to cancel the cutoff dependence). For instance, the only non-zero two-point function is:

$$< V_{\alpha}(z_1, \theta_1) V_{2\alpha_0 - \alpha}(z_2, \theta_2) > = \hat{z}_{12}^{-\alpha(\alpha - 2\alpha_0)}$$

and therefore the vertices $V_{\alpha}$ and $V_{2\alpha_0 - \alpha}$ represent fields with the same dimension:

$$\Delta(\alpha) = \Delta(2\alpha_0 - \alpha) = \frac{1}{2} \alpha(\alpha - 2\alpha_0).$$

(2.23)

From the action (2.20) we can derive the expressions for the stress energy tensor and the supercurrent:

$$T(z) = -\frac{1}{2}((\partial \phi)^2 - \psi \partial \psi) + i\alpha_0 \partial^2 \phi,$$

$$G(z) = i\psi \partial \phi + 2\alpha_0 \partial \psi$$

(2.24)

and the central charge is a function of the charge at infinity $\alpha_0$:

$$c = \frac{3}{2} - 12\alpha_0^2.$$ 

(2.25)

Thus the different superconformal minimal models are parameterized by their charges at infinity:

$$\alpha_0^2 = \frac{1}{p(p + 2)}.$$ 

(2.26)
In the Coulomb gas construction the 4-point functions of the NS fields (of the same dimension) are defined as [4]:

\[
\langle \prod_{k=1}^{4} \phi(z_k, \theta_k) \rangle = \oint_{C} \prod_{i=1}^{n-1} d\zeta_i d\upsilon_i \oint_{C'} \prod_{j=1}^{m-1} d\eta_j d\omega_j
\]

\[
\langle V_{\alpha}(z_1, \theta_1)V_{\alpha}(z_2, \theta_2)V_{2\alpha_0-\alpha}(z_3, \theta_3)V_{\alpha}(z_4, \theta_4) \prod_{i=1}^{n-1} V_{\alpha_+}(v_i, \zeta_i) \prod_{j=1}^{m-1} V_{\alpha_-}(w_j, \eta_j) \rangle.
\]

The superinvariant dimensionless screening operators:

\[
J_{\pm} = \oint_{C_{\pm}} d\theta dz V_{\alpha}(z, \theta) \sim \oint_{C_{\pm}} dz \psi(z) e^{i\alpha_{\pm} \phi(z)}
\]

with charges and dimensions:

\[
\alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}, \quad \Delta(\alpha_{\pm}) = \frac{1}{2} \alpha_{\pm}(\alpha_{\pm} - 2\alpha_0) = \frac{1}{2}
\]

are introduced in (2.27) in order to screen the extra charge $2\alpha$. They generate non trivial solutions of the neutrality condition (2.22):

\[
\alpha_{n,m} = \frac{1}{2} ((1-n)\alpha_+ + (1-m)\alpha_-).
\]

This quantization of the charges of the superfields leads to the well known quantization of the dimensions of the minimal models:

\[
\Delta_{n,m} = \frac{1}{8} \left((n\alpha_+ + m\alpha_-)^2 - (\alpha_+ + \alpha_-)^2\right)
\]

which exactly coincides with the Kac formula (2.9) if $n - m \in 2\mathbb{Z}$.

The Ramond fields of the minimal models should have the same stress-energy tensor $T$ and the supercurrent $G$ as the NS fields. The only difference is that in this case $G(z)$ has to be an antiperiodic field and therefore we have to impose antiperiodic boundary conditions on the free Majorana field $\psi(z)$. Because of that the scalar field $\phi(z)$ and $\psi(z)$ cannot be combined in a superfield multiplet.

As it was explained in details in [5], the spin fields $\sigma$ and $\tilde{\sigma}$ (corresponding to the lowest energy states of $\psi(z)$) play an important role in the construction of the Ramond vertices. Namely, one can define the latter as follows:

\[
\tilde{R}_\alpha(z) = \tilde{\sigma}(z) : e^{i\alpha \phi(z)} :.
\]

The direct inspection based on the expression for the super stress-energy tensor (2.24), eq. (2.11) and the Wick theorem for the free fields shows that the Ramond vertices satisfy (2.4), (2.7) with dimensions given by:

\[
\Delta_R(\alpha) = \frac{1}{16} + \frac{1}{2} \alpha(\alpha - 2\alpha_0).
\]
In fact, we have:
\[ G(z_1)R_\alpha(z_2) = \frac{\alpha - \alpha_0}{\sqrt{2} (z_1 - z_2)^{3/2}} R_\alpha(z_2) + \ldots, \]
and a simple algebra gives:
\[ \sqrt{\frac{1}{2}}(\alpha - \alpha_0) = \pm \sqrt{\Delta R(\alpha) - \frac{c}{24}}. \]
Therefore the vertex \(2.31\) form a correct representation of the Ramond algebra.

Accepting that the screening operators \(J_\pm\) are the same as for the NS sector:
\[ J_\pm = \oint_{C_\pm} dz \psi(z) e^{i\alpha \pm \phi(z)} \]  
with antiperiodic \(\psi(z)\), we can construct the correlation function of four Ramond fields (of equal dimensions) modifying the NS average procedure \(2.27\):
\[ \langle \prod_{k=1}^{4} R_\Delta(z_k) \rangle = \oint_{C_i} \prod_{n=1}^{l-1} dv_i \oint_{C_j} \prod_{m=1}^{n-1} dw_j \]
\[ < R_\alpha(z_1)R_\alpha(z_2)R_{2\alpha_0 - \alpha}(z_3)R_\alpha(z_4) \prod_{i=1}^{n-1} < \psi e^{i\alpha \phi(v_i)} \prod_{j=1}^{m-1} \psi e^{i\alpha \phi(w_j)} > . \]

Since the neutrality condition implies that one has again the same charge quantization \(2.30\) the dimensions \(2.32\) are quantized in accordance with the Kac formula \(2.9\):
\[ \Delta^R_{n,m} = \frac{1}{16} + \frac{1}{8} ((n\alpha + m\alpha)^2 - (\alpha + \alpha)^2) \]  
where now \(n - m \in 2Z + 1\). This screening procedure works well also in the case of mixed R-NS correlation functions and in the general case of multi-point functions.

The screening procedure \(2.27\) and the neutrality condition \(2.22\) applied to the three-point functions generate the fusion rules for the fields of a given minimal model. In fact, the primary field \(\phi_{x,y}\) which enters the OPE of two given fields \(\phi_{n_1,m_1}\) and \(\phi_{n_2,m_2}\) should have a non-zero 3-point function:
\[ \langle \phi_{n_1,m_1}(z_1)\phi_{n_2,m_2}(z_2)\phi_{x,y}(z_3) \rangle . \]
The \(Z_2\) charge conservation implies the following qualitative description of the \(N = 1\) supersymmetric OPE algebra of fields:
\[ [R][R] \sim [NS], \quad [R][NS] \sim [R], \quad [NS][NS] \sim [NS]. \]

We begin with the fusion rules in the NS sector considering the correlation functions of three superfields. It is known that there exist two different structures in it, an even part and an odd one:
\[ \langle N(z_1,\theta_1)N(z_2,\theta_2)N(z_3,\theta_3) \rangle = \]
\[ = (\hat{z}_{12})^{\Delta_1-\Delta_2-\Delta_3}(\hat{z}_{13})^{\Delta_2-\Delta_1-\Delta_3}(\hat{z}_{23})^{\Delta_1-\Delta_2-\Delta_3}(\alpha_1 + a_2\eta), \]
\[ \eta = (\hat{z}_{12}\hat{z}_{13}\hat{z}_{23})^{-1/2}(\theta_1 \hat{z}_{23} + \theta_2 \hat{z}_{13} + \theta_3 \hat{z}_{12} + \theta_1 \theta_2 \theta_3) \]
(a_1 and a_2 are some constants). This structure gives rise to different fusion rules odd and even-generated by the corresponding odd and even parts of the function (2.36). This is explained in details in ref. [37].

In the Coulomb gas picture there exist three different ways to construct the 3-point function of the fields N_{n_1,m_1}, N_{n_2,m_2}, N_{n_3,m_3} depending on which vertex has conjugate charge. In each case there exists a number of screening operators which assure the neutrality condition (2.22). This screening procedure leads to a chain of equations for the unknown charge α_{x,y}. We have to take the common solution of these equations which is:

\[
x = |n_1 - n_2| + 1, |n_1 - n_2| + 3, \ldots, n_1 + n_2 - 1,
\]

\[
y = |m_1 - m_2| + 1, |m_1 - m_2| + 3, \ldots, m_1 + m_2 - 1.
\]

This gives the fusion rules for the fields N_{n_1,m_1}, N_{n_2,m_2} [37]:

\[
[N_{n_1,m_1}][N_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2|+1}^{m_1+m_2-1} [N_{x,y}]
\]

( ( ) stays for the conformal family of the corresponding field). The even fusion rules are recovered when there is an even number of screening operators, i.e. \( x + y \in 2\mathbb{Z}_+ \) while the odd ones correspond to an odd number of the latter \( x + y \in 2\mathbb{Z}_+ + 1 \) [37].

In order to find the fusion rules of two Ramond fields R_{n_1,m_1} and R_{n_2,m_2} we have to look at the non-zero 3-point functions with the NS superfield N_{x,y}: \( \langle R_{n_1,m_1} R_{n_2,m_2} N_{x,y} \rangle \). The same procedure as the one described above leads to the following FR’s of two R-fields:

\[
[R_{n_1,m_1}][R_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2|+1}^{m_1+m_2-1} [N_{x,y}].
\]

This result was first achieved in [5]. The corresponding mixed FR’s \([R][NS] \sim [R] \) are direct consequences of the FR’s we already found. The set of these FR’s completes the structure of the associative OPE algebra of fields of the corresponding supersymmetric minimal models.

Let us now turn to the computation of the four point functions of the Ramond fields. Using the vertex representation and the screening procedure we can express them in the form:

\[
\langle \prod_{k=1}^{4} R_{n_k,m_k}(z_k) \rangle = \oint C_{i} \prod_{i=1}^{n-1} dv_i \oint C_{j} \prod_{j=1}^{m-1} dw_j
\]

\[
\langle \sigma(z_1)\sigma(z_2)\sigma(z_3) \prod_{i=1}^{n-1} \psi(v_i) \prod_{j=1}^{m-1} \psi(w_j) \sigma(z_4) \rangle \times \langle e^{i\alpha_{n_1,m_1}\phi(z_1)e^{i\alpha_{n_2,m_2}\phi(z_2)}e^{i\alpha_{n_3,m_3}\phi(z_3)}e^{i(2\alpha_0-\alpha_{n_4,m_4})\phi(z_4)}} \prod_{i=1}^{n-1} e^{i\alpha_{-\psi(v_i)}} \prod_{j=1}^{m-1} e^{i\alpha_{+\phi(w_j)}} \rangle
\]

The second factor in the integrand is the well-known multi-point function of the modified Coulomb system:

\[
\langle \prod_{k=1}^{N} e^{i\alpha_k\phi(z_k)} \rangle = \prod_{l<n=1}^{N} (z_{ln})^{-\alpha_{ln}}, \quad \sum_{i=1}^{N} \alpha_i = 2\alpha_0.
\]
while the first factor is given by the solution to the recursive equation for the Ising model (2.15) (called below $G^{(n,m)}(z, v_i, w_j)$). Then the four point function of the Ramond fields (of same dimension and charge $\alpha = \alpha_{n,m}$) takes the form:

$$< R_{n,m}(\infty)R_{n,m}(z)R_{n,m}(1)R_{n,m}(0) > =$$

$$= z^{\alpha^2}(1 - z)^{\alpha(2\alpha_0 - \alpha)} \oint_{C_i} \prod_{i=1}^{n-1} dv_i \oint_{C_j} \prod_{j=1}^{m-1} dw_j G^{(n,m)}(z, v_i, w_j) \times$$

$$\times \prod_{l<k=2}^{n-1} v_{lk}^{\alpha^2} \prod_{s<t=2}^{m-1} w_{st}^{\alpha^2} \prod_{i=1}^{n-1} \prod_{j=1}^{m-1} ((v_i - z)v_i)^{\alpha - \alpha}(v_i - 1)^{\alpha - (2\alpha_0 - \alpha)} \times$$

$$\times ((w_j - z)w_j)^{\alpha + \alpha}(w_j - 1)^{\alpha + (2\alpha_0 - \alpha)}(v_i - w_j)^{\alpha - \alpha}.$$ 

The integration contours $C_i$ are fixed by the branch cut singularities of the integrand. Thus, for the general expression of the four point function of the Ramond fields we should take a linear combination of all four point functions corresponding to the possible independent choices of contours $C_i$. In the simplest example of one screening operator, say $J_+$, there are two independent contours, one from 0 to $z$ and the other from 1 to $\infty$, and the corresponding integrals are expressed in terms of hypergeometric function. As a result we get the following result for the 4-point function of the corresponding field $R_{1,2}$ (after taking the sum of $<RRRR>$ and $<RRRR>$):

$$G_{12}^p(z) = z^{-(p+12)/8p}(1 - z)^{(p+4)/8p} \sum_{i=1}^{4} A^i W_i(z)$$

where (using the notation $h = 1/p$ and $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$):

$$W_1(z) = \sqrt{1 + \sqrt{1 - \frac{1}{z}(B(1-h, h)\sqrt{z}F(1+h, -h, 1-2h; z) - B(-h, -h)\sqrt{1-z}F(1+h, -h, -2h; z))}},$$

$$W_2(z) = \sqrt{1 + \sqrt{1 - \frac{1}{z}(B(1+3h, -h)\sqrt{z}z^{2h}F(1+h, 1+h, 1+2h; z) + B(2+3h, -h)\sqrt{1-z}z^{-2-2h}F(1+h, 2+3h, 2+2h; z))}},$$

$$W_3(z) = \sqrt{1 - \sqrt{1 - \frac{1}{z}(B(1-h, h)\sqrt{z}F(1+h, -h, 1-2h; z) + B(-h, -h)\sqrt{1-z}F(1+h, -h, -2h; z))}},$$

$$W_4(z) = \sqrt{1 - \sqrt{1 - \frac{1}{z}(B(1+3h, -h)\sqrt{z}z^{2h}F(1+h, 1+h, 1+2h; z) - B(2+3h, -h)\sqrt{1-z}z^{-2-2h}F(1+h, 2+3h, 2+2h; z))}}.$$
the integrand one can see that there are two independent integration contours for each \( p \). Denoting \( a = 1/(p + 2) \) we have four independent solutions:

\[
G_{21}^p(z) = z^{-(p-10)/8(p+2)}(1 - z)^{(p-2)/8(p+2)} \sum_{i=1}^4 A^i Y_i(z) \quad (2.43)
\]

where:

\[
Y_1(z) = \sqrt{1 + \sqrt{1 - \frac{1}{z}}} B(a, 1 + a)\sqrt{z}F(1 - a, a, 1 + 2a; z) - B(a, a)\sqrt{1 - z}F(1 - a, a, 2a; z),
\]

\[
Y_2(z) = \sqrt{1 + \sqrt{1 - \frac{1}{z}}} B(1 - 3a, a)\sqrt{z}^{-2a}F(-a, 1 - 3a, 1 - 2a; z) + B(2 - 3a, a)\sqrt{1 - z}1^{-2a}F(1 - a, 2 - 3a, 2 - 2a; z),
\]

\[
Y_3(z) = \sqrt{1 - \sqrt{1 - \frac{1}{z}}} B(a, 1 + a)\sqrt{z}F(1 - a, a, 1 + 2a; z) + B(a, a)\sqrt{1 - z}F(1 - a, a, 2a; z),
\]

\[
Y_4(z) = \sqrt{1 - \sqrt{1 - \frac{1}{z}}} B(1 - 3a, a)\sqrt{z}^{-2a}F(-a, 1 - 3a, 1 - 2a; z) - B(2 - 3a, a)\sqrt{1 - z}1^{-2a}F(1 - a, 2 - 3a, 2 - 2a; z).
\]

Up to now we dealt with the one-dimensional, or chiral, fields. In the 2D minimal models the real two-dimensional fields are constructed as a product of left and right chiral fields \( \phi(z, \bar{z}) = \phi_\Delta(z)\bar{\phi}_\Delta(\bar{z}) \). Then the true 2D four-point correlation function should obey the crossing symmetry relations. For scalar fields, with \( \Delta = \bar{\Delta} \), they have the following form:

\[
G_{nm}^{nk}(z, \bar{z}) = G_{nl}^{mk}(1 - z, 1 - \bar{z}) = z^{-2\Delta}z^{-2\bar{\Delta}}G_{nk}^{lm}(\frac{1}{z}, \frac{1}{\bar{z}})
\]

where \( G_{nm}^{nk}(z, \bar{z}) = \langle \phi_k(\infty)\phi_l(1)\phi_n(z, \bar{z})\phi_m(0) \rangle \).

It is known [3] that the crossing symmetry is equivalent to the monodromy invariance of the 4-point functions. Denoting again by \( \{W_i(z)\} \) the set of functions corresponding to the possible independent contours (for the field \( R_{1,2} \)) one can write the 2D 4-point function in the form:

\[
G(z, \bar{z}) = \sum_{i,j} I_{i,j} W_i(z)\bar{W}_j(\bar{z}),
\]

where \( I_{i,j} \) are unknown coefficients. Since the functions \( \{W_i(z)\} \) have branch cut singularities in the points \( 0, 1, \infty \) they transform nontrivially along closed curves enclosing the singular points:

\[
W_i(z) \to (g_l)_{ik}W_k(z), \quad l = 0, 1, \infty.
\]
The matrices $g_l$ generate the monodromy group of the functions $W_i(z)$. The correlation functions of the scalar fields should be uniquely defined in the 2D space, i.e. they have to be monodromy invariant [3]:

$$G(z, \bar{z}) = \sum_{i,j} I_{ij} W_i \bar{W}_j = \sum_{i,j} \sum_{k,l} I_{ij} (g_l)_{ik} W_k (\bar{g}_l)_{jp} \bar{W}_p =$$

$$= \sum_{k,l} (\sum_{i,j} (g_l^t)_{ki} I_{ij} (\bar{g}_l)_{jp}) W_k \bar{W}_l.$$

Thus we obtain the following equation for the unknown coefficients $I_{ij}$:

$$I_{kp} = \sum_{ij} (g_l^t)_{ki} I_{ij} (\bar{g}_l)_{jp}.$$

These equations determine $I_{ij}$ up to an overall factor related to the normalization of the two-point functions. Solving the latter one obtains for example the 2D correlation function of four Ramond fields $R_1, R_2$ for any value of the central charge $c_p$:

$$G_{1,2}^p(z, \bar{z}) = \lambda_{1,2}(p) |z|^{-(p+12)/4p} |1 - z|^{(p+4)/4p} \times$$

$$\times \left[ W_1 \bar{W}_1 + W_3 \bar{W}_3 + (4 \cos^2 \left( \frac{\pi}{p} \right) - 1) (W_2 \bar{W}_2 + W_4 \bar{W}_4) \right].$$

(2.45)

where $W_i(z)$ are those defined in (2.42). In the case of the Ramond fields $R_{2,1}$ similar calculations lead to the following 4-point function:

$$G_{2,1}^p(z, \bar{z}) = \lambda_{2,1}(p) |z|^{-(10-p)/4(p+2)} |1 - z|^{(p-2)/4(p+2)} \times$$

$$\times \left[ Y_1 \bar{Y}_1 + Y_3 \bar{Y}_3 + (4 \cos^2 \left( \frac{\pi}{p+2} \right) - 1) (Y_2 \bar{Y}_2 + Y_4 \bar{Y}_4) \right].$$

(2.46)

and $Y_i$ were defined in (2.44).

Analogously, the monodromy invariant expression for the correlator involving the Ramond fields $R_{1,2}$ and $R_{2,1}$

$$G_{12,21}^p(z, \bar{z}) = \langle R_{1,2}(\infty) R_{1,2}(z, \bar{z}) R_{2,1}(1) R_{2,1}(0) \rangle$$

is

$$G_{12,21}^p(z, \bar{z}) = \frac{1}{8} |z|^{-3/4} |1 - z|^{-3/4} (N_1 \bar{N}_1 + N_2 \bar{N}_2),$$

(2.47)

$$N_1(z) = \sqrt{1 + \sqrt{1 - \frac{1}{z} \left( \frac{1 - 2z + 2 \sqrt{z(1 - z)}}{\sqrt{z}} \right)}},$$

$$N_1(z) = \sqrt{1 - \sqrt{1 - \frac{1}{z} \left( \frac{1 - 2z - 2 \sqrt{z(1 - z)}}{\sqrt{z}} \right)}}$$

with the usual normalization of the 2-point functions to one.
Looking at the singularities of the 4-point functions in the limit \( z \to 1 \) we can recover the OPE:

\[
R_{21} R_{21} \sim 1 + c_1(p) \Phi_{31} \\
R_{12} R_{12} \sim 1 + c_2(p) \Phi_{13} \\
R_{12} R_{21} \sim c_3(p) \Phi_{22}
\]

and the structure constants \( c_1, c_2, c_3 \). In fact, using the OPE:

\[
\phi_{\Delta_1}(z, \bar{z}) \phi_{\Delta_2}(1) = \sum_k \frac{c_k}{|1 - z|^{2(\Delta_1 + \Delta_2 - \Delta_k)}} \phi_k(1)
\]

with a normalization fixed by the identity, i.e. \( c_0 = 1 \), we have:

\[
\sum_k \frac{c_k}{|1 - z|^{2(\Delta_1 + \Delta_2 - \Delta_k)}} < R_{\Delta}(\infty) \Phi_k(1) R_{\Delta}(0) > \sim
\]

\[
\sum_k \frac{c_k^2}{|1 - z|^{2(\Delta_1 + \Delta_2 - \Delta_k)}}.
\]

Hence, the contribution of the different conformal families is identified by the power singularities.

In this way we obtain the normalization and the structure constants. Explicitly we get:

\[
\lambda_{2,1}(a) = \frac{1}{32 \cos^2(\pi a)} \left| \frac{\Gamma(-a)}{\Gamma(a) \Gamma(-2a)} \right|^2,
\]

\[
c_1(a) = \frac{1}{2} \frac{\Gamma(2a) \Gamma((p - 1)a)}{\Gamma(-2a) \Gamma((p + 3)a)} \sqrt{4 \cos^2(\pi a) - 1},
\]

\[
\lambda_{1,2}(h) = \frac{1}{32 \cos^2(\pi h)} \left| \frac{\Gamma(h)}{\Gamma(-h) \Gamma(2h)} \right|^2,
\]

\[
c_2(h) = \frac{3}{2} \frac{\Gamma(3h) \Gamma(-2h)}{\Gamma(2h) \Gamma(-h)} \sqrt{4 \cos^2(\pi h) - 1}.
\]

The mixed structure constant is simply \( c_3(p) = \frac{1}{2} \).

Using the mixed R-NS 4-point functions we can also extract the structure constants of the NS field \( \Phi_{31} \) with itself. For that purpose we compute the correlation function:

\[
G_{31}^p(z) = < \Phi_{3,1}(\infty) \Phi_{3,1}(1) R_{2,1}(z) R_{2,1}(0) > .
\]

In terms of the Coulomb gas construction it is given by an integral with a single screening
The usual OPE's and for 

$$T = \frac{\Gamma(3a)}{\Gamma(4a)\Gamma(1 - 4a)\Gamma(2a - 1)\Gamma(2 - 2a)}$$

$$\Gamma(1 - a)\Gamma(3a).$$

\[
G_{31}^p(z) = \sum_i \int_{C_i} dv < V_{2a_0 - \alpha_{31}}(\infty)V_{\alpha_{31}}(1)V_{\alpha_{21}}(z)V_{\alpha_{21}}(0)\Lambda_{a-}(v) > = \\
= z^{\alpha_{31}^2}(1 - z)^{\alpha_{31}\alpha_{21}}\sum_i \int_{C_i} dv\alpha\alpha_{21}(1 - v)^{\alpha_{31}\alpha_{21}}(z - v)^{\alpha_{31}\alpha_{21}} < \sigma(z)\psi(v)\sigma(0) > = \\
= z^{\alpha_{31}^2 + 3/8}(1 - z)^{\alpha_{31}\alpha_{21}}\sum_i \int_{C_i} dv\alpha\alpha_{21}^{-1/2}(1 - v)^{\alpha_{31}\alpha_{21}}(z - v)^{\alpha_{31}\alpha_{21} - 1/2}
\]

and is therefore expressed in terms of hypergeometric functions. The monodromy invariant solution with the usual normalization is:

\[
G_{31}^p(z, \bar{z}) = |z|^{(5p+6)/4(p+2)}|1 - z|^{p/(p+2)}\{z^{-2pa}F(pa, a, 2a, z)^2 + \\
+ \frac{s(2a)s(4a)}{s^2(a)}\left(\frac{\Gamma^2(2a)\Gamma(2pa)}{\Gamma^2(a)\Gamma((2p+2)a)}\right)^2|F((p+1)a, 2pa, (2p+2)a, z)|^2\}
\]

(\(s(x) = \sin(\pi x)\)). In the limit \(z \to 0\) the first term in (2.50) gives the contribution of the identity family while the second one - the contribution of the \(\Phi_{31}\) operator. In this way we get:

\[
G_{31}^p \sim c_1(p)|z|^{-2(2\Delta_{21} - \Delta_{31})} \Phi_{31}(\infty)\Phi_{31}(1)\Phi_{31}(0) \gg \\
\sim c_1(p)(a_1 + a_2|\eta|^2)|z|^{-2(2\Delta_{21} - \Delta_{31})}.
\]

Since the value of \(c_1(p)\) is known we can extract the structure constants \(a_1\) and \(a_2\) of the even and odd part of the field \(\Phi_{31}\):

\[
a_1 = 0 \\
a_2 = \frac{\Gamma^3(2a)\Gamma^2(2 - 4a)}{\Gamma(4a)\Gamma(1 - 4a)\Gamma(2a - 1)\Gamma(2 - 2a)}\sqrt{\frac{\Gamma(1 - a)\Gamma(3a)}{\Gamma^3(a)\Gamma(1 - 3a)}}.
\]

At the end of this Section we consider the case of the \(N = 1\) superconformal minimal models on \(Z_2\) hyperelliptic supersurfaces \(SX_g^{(2)}\) of genus \(g\). The problem is to find the appropriate minimal models on the branched supersphere which describe the \(N = 1\) minimal models on \(SX_g^{(2)}\). We consider the case of a split \(SX_g^{(2)}\) which can be defined as a double cover of the supersphere branched over \(2g + 2\) points \(a_i\). The split super hyperelliptic map is given by:

\[
w_i(z, \theta) = \sqrt{z - a_i}, \quad \chi_i(z, \theta) = \frac{\theta}{\sqrt{2(z - a_i)^{1/4}}}
\]

whose monodromy group is \(Z_4\).

Under (2.51) the stress tensor \(T(z)\) and the supercurrent \(G(z)\) on \(SX_g^{(2)}\) get mapped onto \(T^{(k)}(z)\) and \(G^{(k)}(z)\) \((k = 0, 1)\) defined on the corresponding sheets. For a given \(k\) they satisfy the usual OPE's and for \(k \neq m\) the OPE contains only regular terms. Consequently, the
central charge is twice that of the sphere one $\tilde{c} = 2c$. It is more useful to define diagonal currents through:

$$T = T^{(0)} + T^{(1)}, \quad T^\dagger = T^{(0)} - T^{(1)},$$

$$G = G^{(0)} - iG^{(1)}, \quad G^\dagger = G^{(0)} + iG^{(1)}.$$  (2.52)

The $Z_4$ charge of $T, G, T^\dagger$ and $G^\dagger$ is $0, 1, 2$ and $3$ respectively.

The primary states and fields are divided into four sectors $V_{[l]}(z) \ (l = 0, 1, 2, 3)$ according to the $Z_4$ boundary conditions for the generators. The primary fields $V_{[l]}(z)$ realize these boundary conditions through their OPE’s with the generators, defining in this way the modes of the latter in the various sectors. The direct comparison with the parafermionic algebras [43] shows that they represent the $Z_4$-disorder sectors of the generalized parafermionic $Z_p=4$ algebra. But for $SX_g^{(2)}$ the $Z_4$ monodromies do not exhaust all the discrete symmetries. In fact, the sheet-interchanging $Z_2$ symmetry acts as a charge conjugation for the generators, introducing in such way a C-disorder sector and leading to the $D_p=4$ parafermionic symmetry. As a consequence, we have to introduce, in addition to $V_{[l]}(z)$, new fields $W_{[l]}(z)$ representing this C-disorder sector. In conclusion, we are led to the statement that the $N = 1$ superconformal algebra on split $SX_g^{(2)}$ maps into the $D_p=4$ parafermionic algebra [44] on the branched supersphere. Therefore one can construct the superconformal models on $SX_g^{(2)}$ in terms of the minimal models of the parafermionic algebra. The unitary degenerate representations of this algebra can be obtained by the GKO method [9] which results in the quantization of the central charge: $\tilde{c} = 3 - 24/p(p + 2) = 2c, \quad p = 3, 4, \ldots$. The easiest way to construct these representations is by taking the usual (supersymmetric) Coulomb gas realization of $T^{(k)}$ and $G^{(k)}$ in terms of free Majorana fermions $\psi^{(k)}$ and free scalar fields $\phi^{(k)}$. It is actually more convenient to use again a diagonal basis $\phi^{(k)}$, $\psi^{(k)}$, analogous to that of (2.52).

We note that with respect to the Virasoro subalgebra generated by $T$ the Coulomb gas system splits into a sphere model with $c_{sp} = 1 - 24/p(p + 2)$ described by $\phi$, a $Z_2$-orbifold with $c_{orb} = 1$ corresponding to $\phi^{(1)}$ and the $X_g^{(2)}$ Ising model with $c_{Is} = 1$. This suggests that the vertex operators representing the different $D_p=2$ sectors can be constructed as products of $\exp(\alpha \phi)$, $\exp(\beta \phi^{(1)})$, the $Z_2$ twist fields $\sigma_\epsilon$ and certain Ising model fields. For the fields $V_{[l]}$ from the order sector we have for example:

NS and R sectors:

$$V_{[0]} = \exp(a_0 \phi + b_0 \phi^{(1)}), \quad V_{[2]} = V_{[2]}^\psi \exp(a_2 \phi + b_2 \phi^{(1)}),$$

$$V_{[1]} = V_{[1]}^\psi \sigma_0 \exp(\alpha \phi), \quad V_{[3]} = V_{[3]}^\psi \sigma_1 \exp(\alpha \phi)$$  (2.54)

and similarly for the C-disorder fields. Here we denoted by $V_{[1]}^\psi$ etc. the fields from the Ising model. They can be constructed in the same way as the one we discuss here specified to the simpler $N = 0$ non-supersymmetric theory [39].
We are going now to construct the screening operators. As usual, these operators are particular NS vertices whose contour integrals are invariant under the action of the generators. In our case these requirements are satisfied by:

$$
\hat{Q}_+^\pm = \oint dz (\psi + \psi^\dagger) \exp(\hat{a}_\pm (\phi + \phi^\dagger)), \quad \hat{Q}_-^\pm = \oint dz (\psi - \psi^\dagger) \exp(\hat{a}_\pm (\phi - \phi^\dagger)),
$$

$$\hat{a}_\pm^2 - \alpha_0 \hat{a}_\pm = \frac{1}{4}, \quad \hat{a}_\pm = \frac{1}{2}(\alpha_0 \pm \sqrt{\alpha_0^2 + 1}), \quad \hat{a}_+ + \hat{a}_- = \alpha_0, \quad \hat{a}_+ \hat{a}_- = -\frac{1}{4}.
$$

The vertices representing the primary fields in the minimal models have the form \(2.53\), \(2.54\) but with quantized charges. Of special interest for us here are the fields from the NS branched sectors \(l = 1\) and \(l = 3\), in which case we have:

$$\alpha_{n,m} = \frac{1}{2}(2 - n)\hat{a}_+ + \frac{1}{2}(2 - m)\hat{a}_-, \quad n - m \in 2\mathbb{Z},$$

$$\Delta^{[1]}_{n,m} = \Delta^{[3]}_{n,m} = \alpha_{n,m}^2 - 2\alpha_0 \alpha_{n,m} + \frac{3}{32} = \frac{(n(p + 2) - mp)^2 - 16 + 3}{16p(p + 2)}.
$$

Our final goal will be the construction of the partition function of \(N = 1\) minimal models on split \(SX^g_{2}\). It turns out that for this reason we need the lowest energy NS branching operators \(V^{(1)}_{1,1}\) and \(V^{(3)}_{1,1}\) only. In fact we have to calculate the vacuum expectation value \(<I>_g\) of the identity \(I\), \(\Delta_I = 0\) on \(SX^g_{2}\), with no other marked points on \(SX^g_{2}\), i.e. each point on \(SX^g_{2}\) represents the NS vacua \(V^g_{1,1}\), \(\Delta^g = 0\). By doing the map \(2.51\)

\(w^2(z) = z - \alpha, i = 1, \ldots, 2g + 2\) we produce \(2g + 2\) branching operators \(V^\epsilon_i(a_i) \leftarrow I\) and therefore:

$$Z_g(a_i) \equiv <I>_g = \prod_{i=1}^{2g+2} V^\epsilon_i(a_i)V^\epsilon_i(\hat{a}_i)|_{g=0}, \quad \sum \epsilon_i = 0 \mod 4. \quad (2.55)
$$

In this way the calculation of the partition function \(Z_g\) reduces to the problem of the construction of the \((2g + 2)\)-point function of the primary fields \(V^\epsilon_i(a_i)\).

We restrict ourselves to the case of \(g = 2\) split supersurface. The standard screening procedure leads to the following expression for the “chiral” part of the \(g = 2\) partition function:

$$Y^{g=2}_{G^\pm,p} = \prod_{i=1}^{6} V^\epsilon_i(a_i)(\hat{Q}_-^r(\hat{Q}_-^+)^{rp/2 - 1}) = \prod_{m=1}^{L} \prod_{m=1}^{M} \int_{c_i^\pm} du_i \int_{v_m} dv_m \left< \prod_{i=1}^{6} \exp\left(\frac{1}{2} \alpha_0 \phi(a_i)\right) \exp(\hat{a}_- \phi(u_i)) \exp(\hat{a}_+ \phi(v_m)) \right> \times \prod_{i=1}^{6} \sigma_i(a_i) \exp(\pm \hat{a}_- \phi^\dagger(u_i)) \exp(\pm \hat{a}_+ \phi^\dagger(v_m)) \prod_{i=1}^{6} V^\psi_i(a_i) \psi^{(k_i)}(u_i) \psi^{(k_m)}(v_m)
$$

(2.56)
where \( L = \frac{1}{2} r (p + 2) - 1 \), \( M = \frac{1}{2} r p - 1 \), \( r = 1 \) for even \( p \) and \( r = 2 \) for odd \( p \), \( \sum \epsilon_i = 0 \mod 4 \) and:

\[
\tilde{Q}_\pm = \int_C dx (\psi(x) \pm \psi^\dagger(x)) \exp(-\frac{1}{2}p\alpha_0(\phi(x) \pm \phi^\dagger(x))), \quad 2\tilde{a}_- = -p\alpha_0, \quad k_i = 0, 1,
\]

\( \psi^{(0)} \), \( \psi^{(1)} \) are Ising fermions on \( X^{(2)}_{g=2} \). It is clear that the integrand splits into a product of three correlation functions. The first one, involving the exponential fields is trivial. The correlation function \( \tilde{G}(p_a, a_i, x_k) \) of \( 2g + 2 = 6 \) twist fields \( \sigma_i \) and an arbitrary number of untwisted vertices \( \exp(q\phi^\dagger) \) can be computed using the methods explained in [45, 39]. It is expressed, up to power-like prefactors, in terms of abelian differentials of first and third kind. It remains to construct the multipoint Ising fermion correlation function on \( X^{(2)}_{g=2} \). As noticed above, this function is computable using the technics we used here for the case of \( N = 0 \) non-supersymmetric case with \( c = 1 \). Combining all these ingredients and satisfying the conditions for monodromy invariance of the six-point function (2.55) we obtain an integral representation for the \( g = 2 \) partition functions of the \( N = 1 \) superconformal minimal models \( Z_{g=2} \).

### 2.2 RG flow in \( N = 1 \) minimal models

In this section we consider a minimal superconformal theory \( SM_p \) perturbed by the least relevant field. Let us remind that the fields in the NS sector are organized in 2D superfields:

\[
\Phi(z, \bar{z}, \theta, \bar{\theta}) = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}\tilde{\phi}.
\]

The first (and the last) component of a spinless superfield of dimensions \( \Delta = \bar{\Delta} (\Delta + \frac{1}{2} = \bar{\Delta} + \frac{1}{2}) \) is expressed as a product of “chiral fields” depending on \( z \) and \( \bar{z} \), respectively. We use the same notations \( \phi \) and \( \tilde{\phi} \) below for these chiral components. If we fix the two-point function of the first component \( \phi \) to one, that of the second components is \( (2\Delta)^2 \) by supersymmetry. Since it is assumed that these functions are all equal to one in the renormalization procedure, we have to normalize the second component \( \tilde{\phi} \to \frac{1}{2\Delta}\tilde{\phi} \).

We will consider the superminimal model \( SM_p \) with \( p \to \infty \) perturbed by the least relevant field \( \tilde{\phi} = \tilde{\phi}_{1,3} \) of dimension \( \Delta = \Delta_{1,3} + \frac{1}{2} = 1 - \epsilon, \epsilon = \frac{2}{p+2} \to 0 \):

\[
\mathcal{L}(x) = \mathcal{L}_0(x) + \lambda\tilde{\phi}(x).
\]

It is obvious that this theory is also supersymmetric, since the perturbation can be written as a covariant super-integral over the superfield \( \Phi_{1,3} \).

The two-point function of arbitrary fields up to the second order is then given by:

\[
<\phi_1(x)\phi_2(0)> = <\phi_1(x)\phi_2(0)>_0 - \lambda \int <\phi_1(x)\phi_2(0)\tilde{\phi}(y)>_0 d^2y + \lambda^2 \int <\phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2)>_0 d^2x_1 d^2x_2 + \ldots
\]

25
where \( \phi_1, \phi_2 \) can be the first or the last components of a superfield or Ramond fields of dimensions \( \Delta_1, \Delta_2 \). Since the first order corrections were considered in \([6]\), we will focus on the second order.

One can use the conformal transformation properties of the fields to bring the double integral to the form:

\[
\int \langle \phi_1(x)\phi_2(0)\tilde{\phi}(x_1)\tilde{\phi}(x_2) \rangle_0 d^2x_1d^2x_2 = (x\bar{x})^{2-\Delta_1-\Delta_2-2\Delta} \int I(x_1) \langle \tilde{\phi}(x_1)\phi_1(1)\phi_2(0)\tilde{\phi}(\infty) \rangle_0 d^2x_1
\]

where:

\[I(x) = \int |y|^{2(a-1)}|1-y|^{2(b-1)}|x-y|^{2c}d^2y\]

and \( a = 2\epsilon + \Delta_2 - \Delta_1, b = 2\epsilon + \Delta_1 - \Delta_2, c = -2\epsilon \). It is well known that the integral for \( I(x) \) can be expressed in terms of hypergeometric functions:

\[
I(x) = \pi \frac{\gamma(b)\gamma(a+c)}{\gamma(a+b+c)}|F(1-a-b-c,-c,1-a-c,x)|^2 + \pi \frac{\gamma(1+c)\gamma(a)}{\gamma(1+a+c)}|x^{a+c}F(a,1-b,1+a+c,x)|^2
\]

This form is useful for evaluating \( I(x) \) near \( x = 0 \). Using the transformation properties of the hypergeometric functions, (2.60) can be rewritten as a function of \( 1-x \) and \( \frac{1}{x} \) which is suitable for the investigation of \( I(x) \) around the points 1 and \( \infty \), respectively.

It is clear that the integral (2.59) is singular. We follow the regularization procedure proposed in \([7]\). It consists basically in cutting discs in the two-dimensional surface of radius \( l \) (\( \frac{1}{l} \)) around singular points 0, 1 (\( \infty \)): \( D_{l,0} = \{ x \in C, |x| < l \} \), \( D_{l,1} = \{ x \in C, |x-1| < l \} \), \( D_{l,\infty} = \{ x \in C, |x| > 1/l \} \) with \( 0 < l_0 \ll l \ll 1 \) where \( l_0 \) is the ultraviolet cut-off. Clearly \( l \) should be canceled in the calculations and should not appear in the final result. We call the region outside these discs as \( \Omega_{l,l_0} \) where the integration is well-defined. It is useful to do this integration in radial coordinates. Since the correlation function exhibits poles only at the points 0 and 1, the phase integration can be performed by using residue theorem and the resulting rational integral in the radial direction is straightforward. Near the singular points one can use the OPE. In doing that it turns out that we count twice two lens-like regions around the point 1 so we have to subtract those integrals. Explicit formulas as well as a more detailed explanation can be found in \([7]\).

Let us start with the correlation function that enters in the integral (2.59) for the case of NS fields. As we explained in the previous section this could be done using the Coulomb gas construction. We will need however an explicit expression that could be integrated. So we will adopt here another strategy.

The basic ingredients for the computation of the four-point correlation functions are the conformal blocks. These are quite complicated objects in general and closed formula were
not known. It was argued that they coincide (up to factors) with the instanton partition
function of certain $N = 2$ YM theories on ALE spaces. This was proved by a recurrence
relation satisfied by the conformal blocks [46, 47] which we will use here. We need the
expressions for the first few levels conformal blocks in order to have a guess for the limit
$\epsilon \to 0$.

The chiral components of the fields obey the OPEs:

$$\phi_1(x)\phi_2(0) = x^{\Delta - \Delta_1 - \Delta_2} \sum_{N=0}^{\infty} x^N C_N \phi_\Delta(0)$$  \hspace{1cm} (2.61)

$$\tilde{\phi}_1(x)\phi_2(0) = x^{\Delta - \Delta_1 - \Delta_2 - 1/2} \sum_{N=0}^{\infty} x^N \tilde{C}_N \phi_\Delta(0)$$

$$\phi_1(x)\tilde{\phi}_2(0) = x^{\Delta - \Delta_1 - \Delta_2 - 1/2} \sum_{N=0}^{\infty} x^N \tilde{C}'_N \phi_\Delta(0)$$

$$\tilde{\phi}_1(x)\tilde{\phi}_2(0) = x^{\Delta - \Delta_1 - \Delta_2 - 1} \sum_{N=0}^{\infty} x^N C'_N \phi_\Delta(0)$$

where $C_N$'s are polynomials of order $N$ in the generators of the superconformal algebra $L_{-k}$
and $G_{-\alpha}$ ($k, \alpha > 0$) with coefficients depending on the dimensions $\Delta$, $\Delta_1$, $\Delta_2$, which we
omitted, usually called chain vectors. Here $N$ runs over all nonnegative integers or half-
integers depending on the fusion rules of $SM_p$.

Acting by positive mode generators on the both sides of these OPEs and using the super-
conformal transformation properties gives the chain equations for $L$'s:

$$L_k C_N = (\Delta + k\Delta_1 - \Delta_2 + N - k) C_{N-k}$$  \hspace{1cm} (2.62)

(Here $C$ is any of of the chain vectors with the corresponding dimensions of the fields) and
for $G$'s:

$$G_k C_N = \tilde{C}_{N-k}$$  \hspace{1cm} (2.63)

$$G_{k} \tilde{C}_N = (\Delta + 2k\Delta_1 - \Delta_2 + N - k) C_{N-k}$$

$$G_{k} \tilde{C}'_N = C'_{N-k}$$

$$G_{k} C'_N = (\Delta + 2k\Delta_1 - \Delta_2 + N - k - \frac{1}{2}) \tilde{C}'_{N-k}$$

for $k > \frac{1}{2}$, and:

$$G_{\frac{1}{2}} \tilde{C}'_N = 2\Delta_2 C_{N-\frac{1}{2}} + C'_{N-\frac{1}{2}}$$

$$G_{\frac{1}{2}} C'_N = -2\Delta_2 \tilde{C}_{N-\frac{1}{2}} + (\Delta + \Delta_1 - \Delta_2 + N - 1) \tilde{C}'_{N-\frac{1}{2}}.$$  \hspace{1cm} (2.64)

There are two independent constants at the zeroth level in the OPEs (2.61), the other
two are expressible through them:

$$\tilde{C}'_0 = -\tilde{C}_0, \hspace{1cm} C'_0 = (\Delta - \Delta_1 - \Delta_2) C_0.$$
The above chain relations could be solved order by order. As mentioned before, in [46, 47] a recursion relation for the chain vectors can be also found.

The conformal blocks are readily obtained by the chain vectors. Presented as vectors in the basis of \( L \)'s and \( G \)'s, the conformal block can be expressed as:

\[
F(\Delta, \Delta_i) = \sum_{N=0}^{\infty} x^N F_N = \sum_{N=0}^{\infty} x^N C_N(\Delta, \Delta_3, \Delta_4) S_N^{-1} C_N(\Delta, \Delta_1, \Delta_2)
\]  

(2.65)

where \( S_N \) is the Shapovalov matrix at level \( N \). What of \( C_N \)'s appear depends on the external fields involved.

The conformal blocks are in general quite complicated objects. Fortunately, in view of the renormalization scheme and the regularization of the integrals, we need to compute them here only up to the zero-th order in \( \epsilon \). This simplifies significantly the problem.

Once the conformal blocks are known, the correlation function of spinless fields for our \( SM_p \) models is written as:

\[
\sum_n C_n |F(\Delta_n, \Delta_i)|^2
\]

where the range of \( n \) depends on the fusion rules and \( C_n \) is the corresponding structure constant. In what follows we compute the conformal blocks up to sufficiently high level and then check also the crossing symmetry and the behavior near the singular points 1 and \( \infty \).

We start with the computation of the \( \beta \)-function and the fixed point. For the computation of the \( \beta \)-function in the second order, we need the four-point function of the perturbing field. Here we consider a more general function:

\[
< \tilde{\phi}(x) \tilde{\phi}(0) \tilde{\phi}_{n+2}(1) \tilde{\phi}_n(\infty) > .
\]

(2.66)

There are three "channels" (or intermediate fields) in the corresponding conformal block: two even, corresponding to the identity and \( \phi_{1,5} \) and one odd - to \( \tilde{\phi} \) itself. From the procedure we explained above, we get the following expression for this correlation function:

\[
< \tilde{\phi}(x) \tilde{\phi}(0) \tilde{\phi}_{n+2}(1) \tilde{\phi}_n(\infty) > = \frac{1}{\left(1 - 2x + 7/3x^2 - 4/3x^3 + 1/3x^4\right)^2} + \frac{2(n+3)}{3(n+1)} \frac{(1 - 3/2x + 3/2x^2 - 1/2x^3)^2}{x(1-x)^2} + \frac{(n+3)(n+4)}{18n(n+1)} \frac{(1 - x + x^2)^2}{(1-x)^2}.
\]

(2.66)

We checked explicitly the crossing symmetry and the \( x \to 1 \) limit of this function. The function that enters the integral is obtained by the conformal transformation \( x \to 1/x \) (explicit formula is presented below).
The integration over the safe region far from the singularities yields \( I(x) \sim \frac{\pi}{\epsilon} \):

\[
\int_{\Omega_{1,0}} I(x) d^2x = \frac{-35\pi^2}{24\epsilon} + \frac{2\pi^2}{e l^2} + \frac{\pi^2}{2e l_0^2} - \frac{16\pi^2 \log l}{3\epsilon} - \frac{8\pi^2 \log 2l_0}{3\epsilon}
\]

and we omitted the terms of order \( l \) or \( l_0/l \).

We have to subtract the integrals over the lens-like regions since they would be accounted twice. Here is the result of that integration:

\[
\frac{\pi^2}{\epsilon} \left( -\frac{1}{l^2} + \frac{1}{2l_0^2} + \frac{61}{24} - \frac{8}{3} \log \frac{l}{2l_0} \right).
\]

Next we have to compute the integrals near the singular points 0, 1 and \( \infty \). For that purpose we can use the OPE of the fields and take the appropriate limit of \( I(x) \). Near the point 0 the relevant OPE is:

\[
\tilde{\phi}(x)\tilde{\phi}(0) = (x\bar{x})^{-2(\Delta_{1,3} + \frac{1}{2})}(1 + \ldots) + \tilde{\phi}^{(1,3)}_{(1,3)}(x\bar{x})^{-2(\Delta_{1,3} + \frac{1}{2})} (\tilde{\phi}(0) + \ldots). \tag{2.67}
\]

The channel \( \phi_{1,5} \) gives after integration a term proportional to \( l/l_0 \) which is negligible. The structure constants we need here and in the calculations below were computed in [48]. In the present case we have:

\[
\tilde{\phi}^{(1,3)}_{(1,3)} = \frac{2}{\sqrt{3}} - 2\sqrt{3}\epsilon \tag{2.68}
\]

to the first order in \( \epsilon \). The value of \( I(x) \) near 0 can be found by taking the limit in (2.60) written in terms of \( 1/x \). Finally one gets:

\[
\int_{D_{1,0}\setminus D_{0,0}} I(x) d^2x = -\frac{\pi^2}{l^2\epsilon} + \frac{2\pi^2}{3\epsilon^2} - \frac{16\pi^2}{\epsilon} - \frac{8\pi^2 \log l}{3\epsilon}.
\]

Since the integral near 1 gives obviously the same result, we just need to add the above result twice. To compute the integral near infinity, we use a relation

\[
<\phi_1(x)\phi_2(0)\phi_3(1)\phi_4(\infty) >= (x\bar{x})^{-2\Delta_1} <\phi_1(1/x)\phi_4(0)\phi_3(1)\phi_2(\infty)>
\tag{2.69}
\]

and \( I(x) \sim \frac{\pi}{\epsilon} (x\bar{x})^{-2\epsilon} \). This gives

\[
\int_{D_{1,\infty}\setminus D_{0,\infty}} I(x) d^2x = -\frac{\pi^2}{l^2\epsilon} + \frac{4\pi^2}{3\epsilon^2} - \frac{8\pi^2}{\epsilon} + \frac{8\pi^2 \log l}{3\epsilon}.
\]

Putting all together, we obtain the finite part of the integral:

\[
\frac{20\pi^2}{3\epsilon^2} - \frac{44\pi^2}{\epsilon}. \tag{2.70}
\]
Taking into account also the first order term (proportional to the above structure constant), we get the final result (up to the second order) for the two-point function of the perturbing field:

\[
G(x, \lambda) = \langle \tilde{\phi}(x) \tilde{\phi}(0) \rangle = (x\bar{x})^{-2+2}\left[ 1 - \lambda \frac{4\pi}{\sqrt{3}} \left( \frac{1}{\epsilon} - 3 \right) (x\bar{x})^\epsilon + \lambda^2 \frac{2}{3\epsilon^2} - \frac{44\pi^2}{\epsilon} \right] (x\bar{x})^{2\epsilon} + \ldots \tag{2.71}
\]

Now we introduce a renormalized coupling constant and a renormalized field \(\tilde{\phi}_g = \partial_g L\) which is normalized by \(\langle \tilde{\phi}_g(1) \tilde{\phi}_g(0) \rangle = 1\). Under the scale transformation \(x^\mu \rightarrow tx^\mu\), the Lagrangian transforms to the trace of the energy-momentum tensor \(\Theta\):

\[
\Theta(x) = \partial_t L = \beta(g) \partial_g L = \beta(g) \tilde{\phi}_g.
\]

Comparing these with the original bare Lagrangian where \(\tilde{\phi} = \partial_\lambda L\) and \(\Theta = \epsilon \lambda \tilde{\phi}\) leads to the \(\beta\)-function given by:

\[
\beta(g) = \epsilon \lambda \frac{\partial g}{\partial \lambda} = \epsilon \lambda \sqrt{G(1, \lambda)}, \tag{2.72}
\]

where \(G(1, \lambda)\) is given by \((2.71)\) with \(x = 1\). One can invert this and compute the bare coupling constant and the \(\beta\)-function in terms of \(g\):

\[
\lambda = g + g^2 \frac{\pi}{\sqrt{3}} \left( \frac{1}{\epsilon} - 3 \right) + g^3 \frac{\pi^2}{3} \left( \frac{1}{\epsilon^2} - \frac{5}{\epsilon} \right) + \mathcal{O}(g^4), \tag{2.73}
\]

\[
\beta(g) = \epsilon g - g^2 \frac{\pi}{\sqrt{3}} (1 - 3\epsilon) - \frac{2\pi^2}{3} g^3 + \mathcal{O}(g^4). \tag{2.74}
\]

In this calculations, we keep only the relevant terms by assuming the coupling constant \(\lambda\) (and \(g\)) to be order of \(\mathcal{O}(\epsilon)\).

A non-trivial IR fixed point occurs at the zero of the \(\beta\)-function:

\[
g^* = \frac{\sqrt{3}}{\pi} \epsilon(1 + \epsilon). \tag{2.75}
\]

It corresponds to the IR CFT \(SM_{p-2}\) as can be seen from the central charge difference:

\[
c^* - c = -8\pi^2 \int_0^{g^*} \beta(g) dg = -4\epsilon^3 - 12\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{2.76}
\]

The anomalous dimension of the perturbing field becomes:

\[
\Delta^* = 1 - \partial_g \beta(g)\big|_{g^*} = 1 + \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.77}
\]

which matches with that of the second component of the superfield \(\Phi^{\bar{p}p_{-2}}_{3,1}\) of \(SM_{p-2}\).

Now we turn to the computation of the mixing coefficients of the super-fields in the NS sector. Actually, using the second component of a super-field as a perturbing field guarantees
the preservation of super-symmetry along the RG flow. The dimension, which is close to 
\(1/2, 1/2\), and the fusion rules between the super-fields \(\Phi_{n,n±2}\) and \(D\bar{D}\Phi_{n,n}\) where \(D\) is the 
\(\)covariant super-derivative suggest that the operators mix along the RG-trajectory. We will 
compute the corresponding dilatation matrix for the anomalous dimensions of the second 
components while the mixing of the first ones is a consequence of the supersymmetry. For 
this purpose we compute the two-point functions and the corresponding integrals. Actually, 
the computation of the integrals goes along the same lines of that of the perturbing field 
itself. So we will present just the final result of the integration.

\[\text{Function } <\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0)>\]

The corresponding 4-point function in the second order of the perturbation was already 
written above (2.66). The integration goes in the same way as before. Combining all the 
integrals, we get:

\[- \frac{2\pi^2(-20 - 143n - 121n^2 - 33n^3 - 3n^4)}{3n(1 + n)(3 + n)^2\epsilon^2} - \frac{2\pi^2(5 + n)(8 + 151n + 143n^2 + 45n^3 + 5n^4)}{3n(1 + n)(3 + n)^2\epsilon}.
\]

We note that the final result is very similar to that of the Virasoro case [7]. This will be also 
the case with the next integrals.

\[\text{Function } <\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)>\]

The relevant four-point function in this case in the zeroth order of \(\epsilon\) is:

\[<\tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(\infty)> = \frac{1}{3} \sqrt{\frac{(-4 + n^2)}{n^2}} \left| \frac{1}{(1 - x)^2} (1 - x + x^2) \right|^2\]

\(\phi_{1,5}\) is only channel appearing here. Collecting again all the integrals gives:

\[-\frac{80(1 - 2\epsilon)\pi^2}{3\epsilon^2 n (-9 + n^2) \sqrt{-4 + n^2}}\]

\[\text{Function } <\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)>\]

The 4-point function computed in [11] is equal to:

\[<\tilde{\phi}(x)\tilde{\phi}(0)\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(\infty)> = \frac{2}{3} \sqrt{\frac{n + 2}{n}} |x|^{-2}\]

which leads after integration to:

\[-\frac{4(-1 + n)\pi^2}{3(3 + n)(5 + n)} \sqrt{\frac{n + 2}{n}} (11 + 3n + \epsilon(1 + n)(9 + 2n)) .\]
• Function $\langle \phi_{n,n}(1)\phi_{n,n}(0) \rangle$

Finally, we need the function $\langle \tilde{\phi}(x)\tilde{\phi}(0)\phi_{n,n}(1)\phi_{n,n}(\infty) \rangle$. This function happens to coincide exactly with the same function in the Virasoro case. Moreover, as we shall show in Section 3, it is the same for all levels $l$ of the general $\hat{s}\hat{u}(2)$ coset models. We shall present the exact expression in that Section. Because of this almost all integrals are the same. The only difference comes from the corresponding structure constants. Taking this into account, our final result here is:

$$\frac{(-1 + n^2)\pi^2}{6}(1 + \epsilon)$$

Since the dimension of the first component $\phi_{n,n}$ is close to zero, it doesn’t mix with other fields. Therefore, we need to compute only its anomalous dimension. Taking into account also the first order contribution, the final result for the two-point function is:

$$G_n(x, \lambda) = \langle \phi_{n,n}(x)\phi_{n,n}(0) \rangle = (x\bar{x})^{-2\Delta_{n,n}}\left[1 - \lambda\left(\frac{\sqrt{3}\pi}{6}(-1 + n^2)\epsilon(1 + 3\epsilon)\right)(x\bar{x})\epsilon + \frac{\lambda^2}{2}\left(\frac{\pi^2}{6}(1 + \epsilon)(-1 + n^2)\right)(x\bar{x})^{2\epsilon} + ...\right].$$

Computation of the anomalous dimension goes in exactly the same way as for the perturbing field:

$$\Delta_{n,n}^g = \Delta_{n,n} - \frac{\epsilon\lambda}{2}\partial_\lambda G_n(1, \lambda) = \Delta_{n,n} + \frac{\sqrt{3}\pi g}{12}\epsilon^2(1 + 3\epsilon)(-1 + n^2) - \frac{\pi^2 g^2}{12}\epsilon^2(-1 + n^2),$$

where we again kept the appropriate terms of order $\epsilon \sim g$. Then, at the fixed point (2.75), this becomes:

$$\Delta_{n,n}^g = \frac{(-1 + n^2)(\epsilon^2 + 3\epsilon^3 + 7\epsilon^4 + ...)}{8}$$

which coincides with the dimension of the first component of the superfield $\Phi^{(p-2)}_{n,n}$ of the model $SM_{p-2}$.

We are now in a position to compute the matrix of anomalous dimensions in the NS sector. The renormalization scheme we use is presented in [7] and is essentially a variation of that originally proposed by Zamolodchikov [10]. The renormalized fields are expressed through the bare ones by:

$$\phi^g_{\alpha} = B_{\alpha\beta}(\lambda)\phi_{\beta}\quad (2.78)$$

(here $\phi$ could be the first or last component). The two-point functions of the renormalized fields:

$$G^g_{\alpha\beta}(x) = \langle \phi^g_{\alpha}(x)\phi^g_{\beta}(0) \rangle, \quad G^g_{\alpha\beta}(1) = \delta_{\alpha\beta},\quad (2.79)$$
satisfy the Callan-Symanzic equation:

$$( x \partial_x - \beta(g) \partial_g ) G^g_{\alpha\beta} + \sum_{\rho=1}^{2} ( \Gamma_{\alpha\rho} G^g_{\rho\beta} + \Gamma_{\beta\rho} G^g_{\alpha\rho} ) = 0 \quad (2.80)$$

where the matrix of anomalous dimensions $\Gamma$ is given by:

$$\Gamma = B \hat{\Delta} B^{-1} - \epsilon \lambda B \partial_\lambda B^{-1} \quad (2.81)$$

where $\hat{\Delta} = \text{diag}(\Delta_1, \Delta_2)$ is a diagonal matrix of the bare dimensions. The matrix $B$ itself is computed from the matrix of the bare two-point functions we computed using the normalization condition (4.65) and requiring the matrix $\Gamma$ to be symmetric.

We computed above some of the entries of the $3 \times 3$ matrix of two-point functions in the second order. This matrix is obviously symmetric. It turns out also that the remaining functions $\langle \tilde{\phi}_{n,n-2}(1) \tilde{\phi}_{n,n-2}(0) \rangle$ and $\langle \phi_{n,n}(1) \tilde{\phi}_{n,n-2}(0) \rangle$ can be obtained from the computed ones $\langle \phi_{n,n+2}(1) \tilde{\phi}_{n,n+2}(0) \rangle$ and $\langle \phi_{n,n}(1) \tilde{\phi}_{n,n+2}(0) \rangle$ by just taking $n \to -n$. Let us denote for convenience the basis of fields:

$$\phi_1 = \tilde{\phi}_{n,n+2}, \quad \phi_2 = (2\Delta_{n,n}(2\Delta_{n,n} + 1))^{-1} \partial \bar{\partial} \phi_{n,n}, \quad \phi_3 = \tilde{\phi}_{n,n-2}$$

where we normalized the field $\phi_2$ so that its bare two-point function is 1. It is straightforward to modify the functions involving $\phi_2$ taking into account the derivatives and the normalization.

We can write the matrix of the two-point functions up to the second order in the perturbation expansion as:

$$G_{\alpha,\beta}(x, \lambda) = \langle \phi_{\alpha}(x) \phi_{\beta}(0) \rangle = (x \bar{x})^{-\Delta_{\alpha} - \Delta_{\beta}} \left[ \delta_{\alpha,\beta} - \lambda C^{(1)}_{\alpha,\beta}(x \bar{x})^{\epsilon} + \frac{\lambda^2}{2} C^{(2)}_{\alpha,\beta}(x \bar{x})^{2\epsilon} + \ldots \right]. \quad (2.82)$$

The two-point functions in the first order are proportional to the structure constants $[49]$:

$$C^{(1)}_{\alpha,\beta} = \tilde{C}_{(1,3)\alpha(\beta)} \frac{\pi \gamma(\epsilon + \Delta_{\alpha} - \Delta_{\beta}) \gamma(\epsilon - \Delta_{\alpha} + \Delta_{\beta})}{\gamma(2\epsilon)} \quad (2.83)$$

and are obviously symmetric. The second order entries $C^{(2)}_{\alpha,\beta}$ are the result of integration of the corresponding 4-point functions and were presented above.

Now, following the renormalization procedure, sketched above, we can obtain the matrix of anomalous dimensions (2.81). The bare coupling constant $\lambda$ is expressed through $g$ by
and the bare dimensions, up to order $\epsilon^2$. The results are:

\[
\begin{align*}
\Gamma_{1,1}^g &= \Delta_1 - \frac{(3 + n)(-1 + \epsilon(2 + n))\pi g}{\sqrt{3}(1 + n)} + \frac{4 g^2 \pi^2 (2 + n)}{3(1 + n)}, \\
\Gamma_{1,2}^g &= \Gamma_{2,1}^g = -\frac{(-1 + \epsilon)(-1 + n)\sqrt{\frac{2 + n}{3n}}\pi g}{(1 + n)} + \frac{2 g^2 (-1 + n)\sqrt{\frac{2 + n}{n}}}{3(1 + n)}, \\
\Gamma_{1,3}^g &= \Gamma_{3,1}^g = 0, \\
\Gamma_{2,2}^g &= \Delta_2 - \frac{2\sqrt{3\pi}(-2 + \epsilon + \epsilon n^2)g}{3(-1 + n^2)} + \frac{2 g^2 (3 + n^2)\pi^2}{3(-1 + n^2)}, \\
\Gamma_{2,3}^g &= \Gamma_{3,2}^g = -\frac{(-1 + \epsilon)\sqrt{\frac{-2 + n}{3n}}(1 + n)\pi g}{(-1 + n)}, \\
\Gamma_{3,3}^g &= \Delta_3 + \frac{(1 + \epsilon(-2 + n))(-3 + n)\pi g}{\sqrt{3(-1 + n)}} + \frac{4 g^2 \pi^2 (-2 + n)}{3(-1 + n)}.
\end{align*}
\]

where:

\[
\begin{align*}
\Delta_1 &= 1 - \frac{n + 1}{2} \epsilon + \frac{1}{8}(-1 + n^2)\epsilon^2, \\
\Delta_2 &= 1 + \frac{1}{8}(-1 + n^2)\epsilon^2, \\
\Delta_3 &= 1 + \frac{n - 1}{2} \epsilon + \frac{1}{8}(-1 + n^2)\epsilon^2.
\end{align*}
\]

Evaluating this matrix at the fixed point (2.75), we get:

\[
\begin{align*}
\Gamma_{1,1}^{g^*} &= 1 + \frac{(20 - 4n^2)\epsilon}{8(1 + n)} + \frac{(39 - n - 7n^2 + n^3)\epsilon^2}{8(1 + n)}, \\
\Gamma_{1,2}^{g^*} &= \Gamma_{2,1}^{g^*} = \frac{(-1 + n)\sqrt{\frac{2 + n}{n}}\epsilon(1 + 2\epsilon)}{n + 1}, \\
\Gamma_{1,3}^{g^*} &= \Gamma_{3,1}^{g^*} = 0, \\
\Gamma_{2,2}^{g^*} &= 1 + \frac{4\epsilon}{-1 + n^2} + \frac{(65 - 2n^2 + n^4)\epsilon^2}{8(-1 + n^2)}, \\
\Gamma_{2,3}^{g^*} &= \Gamma_{3,2}^{g^*} = \frac{\sqrt{-2 + n}(1 + n)\epsilon(1 + 2\epsilon)}{n - 1}, \\
\Gamma_{3,3}^{g^*} &= 1 + \frac{(-5 + n^2)\epsilon}{2(-1 + n)} + \frac{(-39 - n + 7n^2 + n^3)\epsilon^2}{8(-1 + n)}.
\end{align*}
\]

whose eigenvalues are (up to order $\epsilon^2$):

\[
\begin{align*}
\Delta_1^{g^*} &= 1 + \frac{1 + n}{2} \epsilon + \frac{7 + 8n + n^2}{8} \epsilon^2, \\
\Delta_2^{g^*} &= 1 + \frac{-1 + n^2}{8} \epsilon^2, \\
\Delta_3^{g^*} &= 1 + \frac{1 - n}{2} \epsilon + \frac{7 - 8n + n^2}{8} \epsilon^2.
\end{align*}
\]
This result coincides with dimensions $\Delta_{n+2,n}^{(p-2)} + 1/2, \Delta_{n,n}^{(p-2)} + 1$ and $\Delta_{n-2,n}^{(p-2)} + 1/2$ of the model $SM_{p-2}$ up to this order. The corresponding normalized eigenvectors should be identified with the fields of $SM_{p-2}$:

$$
\tilde{\phi}_{n+2,n}^{(p-2)} = \frac{2}{n(1+n)} \phi_1^g + \frac{2 \sqrt{2+n}}{1+n} \phi_2^g + \frac{\sqrt{-4+n^2}}{n} \phi_3^g
$$

$$
\phi_2^{(p-2)} = -\frac{2 \sqrt{2+n}}{1+n} \phi_1^g - \frac{5+n^2}{1+n^2} \phi_2^g + \frac{2 \sqrt{n-2}}{n-1} \phi_3^g
$$

$$
\tilde{\phi}_{n-2,n}^{(p-2)} = \frac{-\sqrt{-4+n^2}}{n} \phi_1^g - \frac{2 \sqrt{-2+n}}{n-1} \phi_2^g + \frac{2}{n(n-1)} \phi_3^g.
$$

We used as before the notation $\tilde{\phi}$ for the last component of the corresponding superfield and:

$$
\phi_2^{(p-2)} = \frac{1}{2\Delta_{n,n}^{p-2} (2\Delta_{n,n}^{p-2} + 1)} \partial \bar{\partial} \phi_{n,n}^{(p-2)}
$$

is the normalized derivative of the corresponding first component. We notice that these eigenvectors are finite as $\epsilon \to 0$ with exactly the same entries as in the Virasoro minimal models. We will show in Section 3 that this is the case also for any level $l$ of the general $\hat{su}(2)$ coset model.

Let us now consider the mixing of the fields in the Ramond sector.

The computation of the conformal blocks in the Ramond sector is more involved. A way of computing them was recently proposed in [50] where conformal blocks in the first few levels were shown to coincide with the instanton partition function of certain $N = 2$ YM theories in four dimensions by a generalized AGT correspondence up to prefactors.

Following [50] one can compute NS-R conformal blocks only for a special choice of the points. After that we can get the function necessary for the integration in the second order by using its conformal transformation properties.

The difficulties arise because of the branch cut in the OPE of Ramond fields with the supercurrent:

$$
G(z) R^\epsilon (0) = \frac{\beta R^{-\epsilon} (0)}{z^{4/7}} + \frac{G_{-1} R^{\epsilon} (0)}{z^{4/7}}
$$

(2.85)

where $\beta = \sqrt{\Delta - \frac{\hat{c}}{16}}$, $\hat{c} = \frac{2}{7} c$, $\epsilon = \pm 1$. Therefore one cannot obtain the usual commutation relations. Here the Ramond field $R^\epsilon$ is doubly degenerate because of the zero mode of $G$ in this sector.

The difficulty can be removed in the following way. Consider the OPE between NS and
Ramond fields:

\[ \phi_1(x) R_2^\epsilon(0) = x^{\Delta - \Delta_1 - \Delta_2} \sum_{N=0}^{\infty} x^N C_N(\Delta) R_2^\epsilon(0), \]

Here \( N \) runs over nonnegative integers as \( G \)'s have integer valued modes in the Ramond sector. Applying \( G_0 \) on both sides of (2.86) and taking into account (2.85), we obtain:

\[ G_0 C_N^\epsilon = \tilde{C}_N^\epsilon + \beta_2 C_N^{\epsilon - \epsilon}, \]

\[ G_0 \tilde{C}_N^\epsilon = (\Delta - \Delta_2 + N) C_N^\epsilon - \beta_2 C_N^{\epsilon - \epsilon}. \]

From the consistency conditions, \( C_0^\epsilon \) is given by:

\[ \tilde{C}_0^\epsilon = \beta C_0^\epsilon - \beta_2 C_0^{\epsilon - \epsilon}. \]

Acting with \( G_k \) with \( k > 0 \) gives chain relations:

\[ G_k C_N^\epsilon = \tilde{C}_{N-k}^\epsilon, \]

\[ G_k \tilde{C}_N^\epsilon = (\Delta + 2k\Delta_1 - \Delta_2 + N - k) C_N^{\epsilon - \epsilon}, \]

and \( L_k \) acts as usual with the appropriate dimensions (see [50] for the details).

One has to solve these chain relations order by order or to use the recursion formulae. Then the conformal block for the function \( < N(x) R(0) N(1) R(\infty) > (N \text{ here stays for the first or the last component of a NS field) is obtained in the same way as in the NS case:} \]

\[ F(x, \Delta, \Delta_i) = \sum_{N=0}^{\infty} x^N C_N(\Delta, \Delta_3, \Delta_4) S_N^{-1} C_N(\Delta, \Delta_1, \Delta_2) \]

where \( C_N \) could be actually \( C_N \) or \( \tilde{C}_N \) depending on the function in consideration. Finally the correlation function is constructed as:

\[ < N(x) R(0) N(1) R(\infty) >= \sum_n C_n |F_n(x)|^2, \]

where \( C_n \)'s are the structure constants and the range of \( n \) is dictated by the fusion rules. The function that enters the integral is then obtained by the conformal transformation.

As already mentioned, the conformal block in general is very complicated. Fortunately, it is sufficient to compute the finite term as \( \epsilon \to 0 \). We did the computation for the functions below up to high order and then check the behavior near the singular points. It turns out also that the two-point function do not depend on which of the fields \( R^\epsilon \) are involved. So we drop the subscript \( \epsilon \) from our notations in what follows.
• Function $< R_{n,n+1}(1)R_{n,n+1}(0) >$

Our calculation for the corresponding 4-point function gives:

$$< \tilde{\phi}(x)R_{n,n+1}(0)R_{n,n+1}(\infty)\tilde{\phi}(1) > = \frac{n^2 - 1}{12n^2} \left| \frac{1}{x(1-x)^2} \left(1 + \frac{n}{n+1}x - \frac{1}{n+1}x^2 \right) \right|^2 +$$

$$+ \frac{(2+n)^2}{48n^2} \left| \frac{1}{x(1-x)^2} \left(1 + \frac{2n}{n+2}x + \frac{n-2}{n+2}x^2 \right) \right|^2 + \frac{n+3}{12(n+1)} \left| \frac{1}{(1-x)^2} \left(1 + x \right) \right|^2.$$

To obtain the function that enters the integral, we use the conformal transformation properties. One can easily get:

$$< \tilde{\phi}(x)R_{n,n+1}(0)R_{n,n+1}(1)\tilde{\phi}(\infty) > = (x\tilde{x})^{-2\Delta_{1,3} - 1} < \tilde{\phi}(\frac{x-1}{x})R_{n,n+1}(0)R_{n,n+1}(\infty)\tilde{\phi}(1) >$$

$$= \frac{n^2 - 1}{12n^2} \left| \frac{(2x-1)(nx+1)}{(n+1)x(x-1)} \right|^2 + \frac{(2+n)^2}{48n^2} \left| \frac{(2x-1)(n(2x-1)+2)}{(n+2)x(x-1)} \right|^2 + \frac{n+3}{12(n+1)} \left| \frac{2x-1}{x} \right|^2.$$

As in the NS case we omit the detailed calculation of the integrals since it goes in the same way. The final result of the integration of this function is:

$$\pi^2 \left[ \frac{44 + 64n + 24n^2 + 3n^3 - 8\epsilon(1+n)(5+14n+7n^2+n^3)}{12\epsilon^2n(2+n)^2} \right].$$

• Function $< R_{n,n-1}(1)R_{n,n+1}(0) >$

The calculation of the four-point function with the perturbing fields can be done in the same way:

$$< \tilde{\phi}(x)R_{n,n+1}(0)\tilde{\phi}(1)R_{n,n-1}(\infty) > = \frac{\sqrt{n^2 - 1}}{12n} \left| \frac{1}{x(1-x)}(1 + x) \right|^2.$$

Performing the same transformation as in the previous case, the integrand becomes:

$$< \tilde{\phi}(x)R_{n,n+1}(0)R_{n,n-1}(1)\tilde{\phi}(\infty) > = \frac{\sqrt{n^2 - 1}}{12n} \left| \frac{2x-1}{x(1-x)} \right|^2.$$

Here is the final result in the second order:

$$\frac{4\pi^2\sqrt{n^2 - 1}(44 - 5n^2 - 2\epsilon(20 + n^2))}{3\epsilon^2n(n^2 - 16)(n^2 - 4)}.$$

The functions we computed above are enough for our computation since the other two functions $< R_{n,n-1}(1)R_{n,n-1}(0) >$ and $< R_{n,n+1}(1)R_{n,n-1}(0) >$ can be obtained from $< R_{n,n+1}(1)R_{n,n+1}(0) >$ and $< R_{n,n-1}(1)R_{n,n+1}(0) >$ by just changing $n \to -n$ as in the case.
of NS fields. Let us introduce again a basis: $R_1 = R_{n,n+1}$, $R_2 = R_{n,n-1}$. From the general formula (2.83) and the bare dimensions of the fields

\[
\Delta_1 = \frac{3}{16} - \left( \frac{n}{4} + \frac{1}{8} \right) \epsilon + \frac{1}{8} (n^2 - 1) \epsilon^2,
\]
\[
\Delta_2 = \frac{3}{16} + \left( \frac{n}{4} - \frac{1}{8} \right) \epsilon + \frac{1}{8} (n^2 - 1) \epsilon^2
\]

we can get the $2 \times 2$ matrix of two-point functions in the first order. Together with our results in the second order presented above and following the same procedure as in the NS case, we get for the matrix of anomalous dimensions up to order $\epsilon^2 \sim g^2$:

\[
\Gamma_{1,1} = \Delta_1 - \frac{g(n+2)(-1 - \epsilon + 2\epsilon n)\pi}{4\sqrt{3}n} + \frac{g^2\pi^2}{4},
\]
\[
\Gamma_{1,2} = \Delta_2 - \frac{(1 + \epsilon)g\sqrt{n^2 - 1}\pi}{2\sqrt{3}n},
\]
\[
\Gamma_{2,2} = \Delta_2 + \frac{g(n-2)(1 + \epsilon + 2\epsilon n)\pi}{4\sqrt{3}n} + \frac{g^2\pi^2}{4},
\]

which, at the fixed point (2.75), becomes:

\[
\Gamma_{g^*}^{1,1} = \frac{3}{16} + \frac{(4 + n - 2n^2)\epsilon}{8n} + \frac{(8 + n - 4n^2 + n^3)\epsilon^2}{8n},
\]
\[
\Gamma_{g^*}^{1,2} = \Gamma_{g^*}^{2,1} = \frac{\sqrt{n^2 - 1}\epsilon(1 + 2\epsilon)}{2n},
\]
\[
\Gamma_{g^*}^{2,2} = \frac{3}{16} + \frac{(-4 + n + 2n^2)\epsilon}{8n} + \frac{(-8 + n + 4n^2 + n^3)\epsilon^2}{8n}.
\]

The eigenvalues of this matrix are:

\[
\Delta_{g^*}^{1} = \frac{3}{16} + \left( \frac{1}{8} + \frac{n}{4} \right) \epsilon + \frac{1}{8} (1 + 4n + n^2) \epsilon^2,
\]
\[
\Delta_{g^*}^{2} = \frac{3}{16} + \left( \frac{1}{8} - \frac{n}{4} \right) \epsilon + \frac{1}{8} (1 - 4n + n^2) \epsilon^2.
\]

As expected, they coincide with the dimensions of the Ramond fields $\Delta^{(p-2)}_{n+1,n}$ and $\Delta^{(p-2)}_{n-1,n}$ of the $SM_{p-2}$. The corresponding fields are expressed as a (normalized) linear combination:

\[
R^{(p-2)}_{n+1,n} = \frac{1}{n} R_1^{g^*} + \frac{\sqrt{n^2 - 1}}{n} R_2^{g^*},
\]
\[
R^{(p-2)}_{n-1,n} = -\frac{\sqrt{n^2 - 1}}{n} R_1^{g^*} + \frac{1}{n} R_2^{g^*}.
\]

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2.3 Three-point correlation functions in $N = 1$ super-Liouville theory

In this section we consider the super Liouville field theory. We formulate the supersymmetric theory by the action:

$$S_{SL} = \frac{1}{4\pi} \int d^2z d^2\theta \left[ \frac{1}{2} D^a \Phi D^a \Phi - \hat{R} \Phi + \mu e^{\Phi} \right]$$

(2.89)

where the real superfield $\Phi$ possesses the expansion:

$$\Phi = \phi + \theta \psi + \bar{\theta} \bar{\psi} + \theta \bar{\theta} \tilde{\phi},$$

(the component fields here depend on both $z$ and $\bar{z}$), $\hat{R}$ is the scalar curvature corresponding to the background metric and $\mu$ in (2.89) is the cosmological constant. The classical equations of motion for (2.89) are:

$$D^a D^a \Phi = Q \hat{R} + \mu e^{\Phi}.$$  

The superspace notations that we shall use in the case of super Liouville theory are:

$$Z = (z, \theta) \quad Z_1 - Z_2 = z_1 - z_2 - \theta_1 \theta_2.$$ 

The super stress tensor of the super Liouville theory is expressed in terms of the real superfield $\Phi$:

$$T_{SL} = -\frac{1}{2} D\Phi \partial \Phi + \frac{Q}{2} D\partial \Phi,$$

the central charge of the super-Virasoro algebra with this normalization being given by:

$$\hat{c} = 1 + 2Q^2.$$  

(2.90)

Similarly to the minimal models, the superconformal primary fields are divided in two sectors depending on the boundary conditions of the supercurrent. In the Neveu-Schwarz sector they are represented by the vertex operators:

$$V_\alpha = e^{\alpha \Phi(z, \bar{z})}$$

(2.91)

of dimension $\Delta_\alpha = \frac{1}{2} \alpha (Q - \alpha)$ and in the Ramond sector by:

$$R_\alpha^\epsilon = \sigma^\epsilon e^{\alpha \Phi(z, \bar{z})}$$

(2.92)

where $\sigma^\epsilon$ is the so called spin field and $\Delta_\alpha = \frac{1}{2} \alpha (Q - \alpha) + \frac{1}{16} (\epsilon = \pm)$. The requirement for the cosmological term in (2.89) to be $(1/2,1/2)$ form in order to be able to integrate over the surface, gives a connection between $Q$ and $b$:

$$Q = b + \frac{1}{b}.$$  

(2.93)

\footnote{One can choose the background metric to be flat and therefore $\hat{R}$ will not be essential in the sequel.}
It is easy to see that the operators $V_\alpha = e^{\alpha \Phi}$ and $V_{Q-\alpha} = e^{(Q-\alpha)\Phi}$ have equal dimensions and therefore they are reflection image of each other (the same is true also for $R_\alpha$ and $R_{Q-\alpha}$).

In order to compute the 3-point function in the NS sector we shall follow the approach of [2]. Consider first the three-point correlation function of Liouville vertex operators from Neveu-Schwarz sector. The perturbative expansion in the cosmological constant $\mu$ is given by:

\[
\langle V_\alpha_1(Z_1)V_\alpha_2(Z_2)V_\alpha_3(Z_3) \rangle = \int D\Phi e^{-S_{SL}} e^{\alpha_1 \Phi(Z_1)} e^{\alpha_2 \Phi(Z_2)} e^{\alpha_3 \Phi(Z_3)} = \sum_{s=0}^{\infty} \left( \frac{\mu}{2\pi} \right)^s \frac{1}{s!} \int D\Phi e^{-S'_{SL}} \left( \int d^2 z d^2 \theta e^{b\Phi} \right)^s \prod_{i=1}^{3} e^{\alpha_i \Phi(Z_i)}. \tag{2.94}
\]

The free superfield action $S'_{SL}$:

\[
S'_{SL} = \frac{1}{4\pi} \int d^2 z d^2 \theta \left[ \frac{1}{2} D_\alpha \Phi D^\alpha \Phi - Q \hat{R} \Phi \right], \tag{2.95}
\]

is used above to compute the resulting integral.

Specializing to the case of correlation functions on the sphere we shall concentrate the curvature at infinity $(\infty,0)$. Therefore, we can use the super Coulomb gas formalism in order to evaluate the correlation function on the r.h.s in (2.94). As it is well known, (2.94) is nonzero only if:

\[
sb = Q - \sum_{i=1}^{3} \alpha_i \tag{2.96}
\]

for any order $s$ of the perturbation series. The result for the $s^{th}$ term in the expansion (2.94) is:

\[
\langle V_\alpha_1(Z_1)V_\alpha_2(Z_2)V_\alpha_3(Z_3) \rangle_s = \left( \frac{\mu}{2\pi} \right)^s \frac{1}{s!} \prod_{i<j}^{3} |Z_i - Z_j|^{-2\alpha_i \alpha_j} \int \prod_{j=1}^{s} D^2 Y_j \prod_{i=1}^{3} |Z_i - Y_j|^{-2\alpha_i} \prod_{i<j}^{s} |Y_i - Y_j|^{-2b^2} \tag{2.97}
\]

For $N = 1$ case there exists a supersymmetric extension of the Dotsenko-Fateev integrals and an analogous integral expression for the structure constants can be extracted. Applied to our problem this integral expression gives for the three-point function in the case of integer number of screening charges the following result:

\[
\langle V_\alpha_1(Z_1)V_\alpha_2(Z_2)V_\alpha_3(Z_3) \rangle_s = \left( \frac{\mu}{8\gamma} \left( \frac{b^2}{2} + \frac{1}{2} \right) \right)^s \prod_{i<j}^{3} |Z_i - Z_j|^{-2\delta_{ij}} \prod_{j=1}^{s} \gamma \left( \frac{j}{2} - \left[ \frac{j}{2} \right] - \frac{b^2}{2} \right) \times \prod_{j=0}^{s-1} \left( 1 - \frac{j}{2} + \left[ \frac{j}{2} \right] - b\alpha_i - \frac{b^2}{2} \right) \times \left\{ \begin{array}{ll} 1 & \text{if } s \in 2\mathbb{N} \\ \frac{1}{\eta^2} & \text{if } s \in 2\mathbb{N} + 1 \end{array} \right. \tag{2.97}
\]

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where:
\[
\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \quad \delta_{ij} = \Delta_i + \Delta_j - \Delta_k, \quad i \neq j \neq k, \quad \Delta_i = \alpha_i(Q - \alpha_i)
\]

In the above formula we reminded the definition of $\eta$ as the $SL(2|1)$ odd invariant for any given three points $(Z_1, Z_2, Z_3)$. In contrast to the bosonic case here the correlation function is different for $s \in 2\mathbb{N}$ and $s \in 2\mathbb{N} + 1$.

At this point we use the interpretation of (2.96) along the lines of [52]. It was suggested to consider (2.96) as a kind of ”on-mass-shell” condition for the exact correlation function. This condition means that the exact correlation function should satisfy:

\[
\text{res}_{s} \sum_{i=1}^{3} \alpha_i = Q - sb \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle = \left( -\mu \right)^{s} s! \left( \frac{D^2 Z V_0}{\sum_{i=1}^{3} \alpha_i = Q - sb} \right)^{3} \tag{2.98}
\]

when (2.96) holds for $s = 0, 1, 2, \ldots$. In general (2.98) alone seems to be unsufficient to determine $N$-point function, but for three Liouville vertex operators the situation is simple: the coordinate dependence on the left hand side and right hand side is as in the three-point function (2.94). Therefore we have the following ”on-mass-shell” condition for the structure constants:

\[
\text{res}_{s} = \sum_{i=1}^{3} \alpha_i = Q - sb \left( \frac{C^{\text{even}(odd)}(\alpha_1, \alpha_2, \alpha_3)}{I^{\text{even}(odd)}(\alpha_1, \alpha_2, \alpha_3)} \right) \tag{2.99}
\]

where we have denoted by $I^{\text{even}(odd)}(\alpha_1, \alpha_2, \alpha_3)$ the coordinate independent part of the $s^{th}$ term in the expansion (2.97).

Now we have to generalize the special function $\Upsilon(x)$ introduced in [2]. For both, bosonic and supersymmetric cases, we define the function $(0 < Re(x) < Q)$:

\[
\log \mathcal{R}(x, a) = \frac{1}{2} \int_{0}^{\infty} \frac{dt}{t} \left\{ \left[ \left( \frac{Q}{2} - x \right)^{2} + \left( \frac{Q}{2} - a \right)^{2} \right] e^{-t} \right. \right.
\]
\[
\left. \left. - 2sh^{2} \left[ \left( \frac{Q}{2} - x \right) + \left( \frac{Q}{2} - a \right) \right] \frac{t}{4} + sh^{2} \left[ \left( \frac{Q}{2} - x \right) - \left( \frac{Q}{2} - a \right) \right] \frac{t}{4} \right) \right\} . \tag{2.100}
\]

The simplest properties that are clear from (2.99) are:

\[
\mathcal{R}\left( \frac{Q}{2}, \frac{Q}{2} \right) = 1, \quad \mathcal{R}(x, a) = \mathcal{R}(Q - x, a), \quad \mathcal{R}(x, a) = \mathcal{R}(a, x). \tag{2.100}
\]

Let us define also:

\[
\mathcal{R}_0 = \frac{d\mathcal{R}(x, a)}{dx} \bigg|_{x=a=0}. \tag{2.100}
\]
We propose the following expression as an exact three-point function in the supersymmetric Liouville theory:

\[
C^{even}(\alpha_1, \alpha_2, \alpha_3) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) b^{-1-b^2} \right]^{\frac{2-\sum_i \alpha_i}{b}} \times \frac{R_0 R(2\alpha_1, 0) R(2\alpha_2, 0) R(2\alpha_3, 0)}{R(\alpha_1 + \alpha_2 + \alpha_3 - Q, 0) R(x_1, 0) R(x_2, 0) R(x_3, 0)}
\]

(2.101)

\[
C^{odd}(\alpha_1, \alpha_2, \alpha_3) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) b^{-1-b^2} \right]^{\frac{2-\sum_i \alpha_i}{b}} \times \frac{R_0 R(2\alpha_1, 0) R(2\alpha_2, 0) R(2\alpha_3, 0)}{R(\alpha_1 + \alpha_2 + \alpha_3 - Q, b) R(x_1, b) R(x_2, b) R(x_3, b)}
\]

(2.102)

where:

\[ x_i = \alpha_j + \alpha_k - \alpha_i; \quad i \neq j \neq k \]

Thus, we have in general:

\[
\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle = (C^{even}(\alpha_1, \alpha_2, \alpha_3) + \eta \bar{\eta} C^{odd}(\alpha_1, \alpha_2, \alpha_3)) \prod_{i<j} |Z_i - Z_j|^{\delta_{ij}}.
\]

(2.103)

This expression for the exact three-point function is based on the properties of the defined \( R(x, a) \) function described below.

We now pass to the Ramond sector. According to the explicit form of the Ramond vertex operator (2.92), its three-point function has the following perturbative expansion:

\[
\langle R^{\epsilon_1}_{\alpha_1} (z_1) R^{\epsilon_2}_{\alpha_2} (z_2) V_{\alpha_3} (Z_3) \rangle = \sum_{s=0}^{\infty} \left( \frac{\mu}{2\pi} \right)^s \int du_1 \ldots du_s \left( \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \prod_{j=1}^{s} e^{b\phi(u_j)} \right) - (2.104)
\]

\[
\times \langle \sigma^{\epsilon_1}(z_1) \sigma^{\epsilon_2}(z_2) \left( 1 + \theta_3 \bar{\theta}_3 \psi(z_3) \right) \prod_{j=1}^{s} \psi(u_j) \rangle
\]

As before, in order the free bosonic correlator to be nonzero we have to impose the condition (2.96). It can be interpreted again as a "on-mass-shell" condition for the exact correlation function. The explicit expression for the integrals in (2.104) can be extracted from the
corresponding formulae for the Ramond fields \cite{5,38,53}. The final result is:

\[
\langle R_{\alpha_1}^\epsilon(Z_1) R_{\alpha_2}^\epsilon(Z_2) V_{\alpha_3}(Z_3) \rangle_s = \left( \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) \right)^s \prod_{i<j}^3 |Z_i - Z_j|^{-2\Delta_i}
\]

\[
\times \prod_{j=1}^s \gamma \left( \frac{j}{2} - \left[ \frac{j}{2} \right] - j b_1^2 - \frac{b_1}{2} \right)^{s-1} \prod_{i=1}^2 \gamma \left( 1 + \frac{j}{2} - \left[ \frac{j}{2} \right] - b\alpha_i - j b_3^2 - \frac{1}{2} \right)
\]

\[
\times \gamma \left( 1 - \frac{j}{2} + \left[ \frac{j}{2} \right] - b\alpha_3 - j b_3^2 - \frac{1}{2} \right) \times A_{\epsilon_1, \epsilon_2}
\]

where:

\[
A_{\epsilon, \epsilon} = \begin{cases} 
1, & s = 2N \\
\theta_3 \bar{\theta}_3 \frac{|z_1 - z_2| (z_2 - z_3)}{|z_1 - z_2|}, & s = 2N + 1
\end{cases}
\]

\[
A_{\epsilon, -\epsilon} = \begin{cases} 
1, & s = 2N + 1 \\
\theta_3 \bar{\theta}_3 \frac{|z_1 - z_2| (z_2 - z_3)}{|z_1 - z_2|}, & s = 2N
\end{cases}
\]

and \(\gamma(x), \delta_{ij}, \Delta_i\) as in (2.98). Finally, we propose the following expression for the exact three-point function in the Ramond sector:

\[
\langle R_{\alpha_1}^\epsilon(Z_1) R_{\alpha_2}^\epsilon(Z_2) V_{\alpha_3}(Z_3) \rangle = \left( C_{\epsilon_1, \epsilon_2} + \theta_3 \bar{\theta}_3 \frac{|z_1 - z_3| |z_2 - z_3|}{|z_1 - z_2|} \tilde{C}_{\epsilon_1, \epsilon_2} \right) \prod_{i<j} |z_i - z_j|^{\delta_{ij}}
\]

where:

\[
C_{\epsilon, \epsilon}(\alpha_1, \alpha_2, \alpha_3) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) b_1^{1-b_1^2} \right] \left[ \prod_{i=1}^3 \mathcal{R}_{\alpha_i} \right] \frac{\mathcal{R}(2\alpha_1, b) \mathcal{R}(2\alpha_2, b) \mathcal{R}(2\alpha_3, 0)}{\mathcal{R}(\alpha_1 + \alpha_2 + \alpha_3 - Q, 0) \mathcal{R}(x_1, b) \mathcal{R}(x_2, b) \mathcal{R}(x_3, 0)}
\]

(2.105)

\[
C_{\epsilon, -\epsilon}(\alpha_1, \alpha_2, \alpha_3) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) b_1^{1-b_1^2} \right] \left[ \prod_{i=1}^3 \mathcal{R}_{\alpha_i} \right] \frac{\mathcal{R}(2\alpha_1, b) \mathcal{R}(2\alpha_2, b) \mathcal{R}(2\alpha_3, 0)}{\mathcal{R}(\alpha_1 + \alpha_2 + \alpha_3 - Q, 0) \mathcal{R}(x_1, 0) \mathcal{R}(x_2, 0) \mathcal{R}(x_3, b)}
\]

(2.106)

and \(\tilde{C}_{\epsilon_1, \epsilon_2}\) can be determined using the supersymmetry (\(x_i\) is as in (2.103)).

Now we are going to discuss the pole structure of the three-point correlation function and to define the so called reflection amplitudes. For this purpose we start with some
transformation properties and functional relations for $R(x,a)$ defined in (2.100). Using the integral representation of $R(x,a)$ (2.99) one can check that the following functional relation holds:

$$R(x+b,a) = b^{-bx+ab} \gamma \left( \frac{bx - ba + 1}{2} \right) R(x, a + b).$$

It is clear that due to the "self-duality" of $R(x,a)$ (i.e. the invariance under $b \to 1/b$) one can conclude that:

$$R(x+1/b,a) = b^{x-ba} \gamma \left( \frac{x-a+b}{2b} \right) R(x, a + 1/b).$$

It is easy to verify that, using the above properties, $\Upsilon_1(x) = R(x,0)$ and $\Upsilon_2(x) = R(x,b)$ are entire functions of $x$ with the following zero-structure:

$$\Upsilon_1(x) = 0 \quad \text{for} \quad x = -nb - \frac{m}{b}; \quad n - m = \text{even}$$

$$\Upsilon_2(x) = 0 \quad \text{for} \quad x = -nb - \frac{m}{b}; \quad n - m = \text{odd}$$

and due to (2.100):

$$\Upsilon_1(x) = 0 \quad \text{for} \quad x = (n+1)b + \frac{m+1}{b}; \quad n - m = \text{even}$$

$$\Upsilon_2(x) = 0 \quad \text{for} \quad x = (n+1)b + \frac{m+1}{b}; \quad n - m = \text{odd}$$

($n, m$ are non-negative integers).

Using all the above properties of $R(x,a)$ it is straightforward to check that the proposed exact three-point functions satisfy the "on-mass-shell" condition (2.96).

As in the bosonic case [2], the proposed correlators as a function of $\alpha = \sum_i^3 \alpha_i$ have more poles than expected: at $\alpha = Q - n/b - mb$ and at $\alpha = 2Q + n/b + mb$. They appear when more general multiple integrals are considered:

$$\sum_{a_i = Q - \frac{n}{b} - mb} res_{\alpha} \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle = \frac{\tilde{\mu}^n (\mu)^m}{n!m!} \left( \prod_{i=1}^3 V_{\alpha_i}(Z_i) \right) \left( \prod_{k=1}^n V_{1/b,X_k} \right) \left( \prod_{l=1}^m V_{5/4}(Y_l) \right)$$

(2.107)

where:

$$\frac{\tilde{\mu}}{8} \gamma \left( \frac{1}{2b^2} + \frac{1}{2} \right) = \left( \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) \right)^{\frac{1}{2}}.$$

We note that the correlation function (2.107) is self-dual with respect to $b \to \frac{1}{b}, \mu \to \tilde{\mu}$.

As it was mentioned above the Liouville vertex operators $V_\alpha$ and $V_{Q-\alpha}$ have the same dimensions and are interpreted as reflection images of each other. We shall use this property in order to define the so called reflection amplitudes in the supersymmetric case. We start with the reflection amplitude in the Neveu-Schwarz sector. We define the latter as:

$$C^{even(odd)}(Q - \alpha_1, \alpha_2, \alpha_3) = S^{NS}(\alpha_1)C^{even(odd)}(\alpha_1, \alpha_2, \alpha_3)$$
where \( S^{NS}(\alpha_1) \) is the reflection amplitude. Using the functional relations of \( R(x,a) \) we find that it equals to:

\[
S^{NS}(\alpha) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) \right]^{\frac{\alpha_1 - Q}{b}} b^{2\frac{Q - 2\alpha_1}{b}} \gamma \left( \frac{b\alpha - \frac{b^2}{2} + \frac{1}{2}}{2 - \frac{Q}{2} + \frac{1}{2(b^2 - b)}} \right). \tag{2.108}
\]

As in the bosonic case we associate the reflection amplitude with the two-point correlation function \[2\].

Now we pass to the Ramond sector and consider the correlation functions (2.105), (2.106) together. In these functions we have two Ramond fields \( (R_{\alpha_1}, R_{\alpha_2}) \) and one Neveu-Schwarz field \( (V_{\alpha_3}) \). Therefore the reflection of the first two fields will give us one reflection amplitude, but the reflection of the third field will differs from the first one. We find:

\[
C^{\epsilon, \pm \epsilon}(Q - \alpha_1, \alpha_2, \alpha_3) = S^R(\alpha_1) C^{\epsilon, \pm \epsilon}(\alpha_1, \alpha_2, \alpha_3)
\]

where:

\[
S^R(\alpha_1) = \left[ \frac{\mu}{8} \gamma \left( \frac{b^2}{2} + \frac{1}{2} \right) \right]^{\frac{\alpha_1 - Q}{b}} b^{2\frac{Q - 2\alpha_1}{b}} \gamma \left( \frac{b\alpha - \frac{b^2}{2} + \frac{1}{2}}{1 - \frac{Q}{2} + \frac{1}{2(b^2 - b)}} \right).
\]

For the reflection of the Neveu-Schwarz field we found:

\[
C^{\epsilon, \pm \epsilon}(\alpha_1, \alpha_2, Q - \alpha_3) = S^{NS}(\alpha_3) C^{\epsilon, \pm \epsilon}(\alpha_1, \alpha_2, \alpha_3)
\]

where \( S^{NS}(\alpha) \) is as in (2.108).

### 2.4 One-point function of \( N = 1 \) super-Liouville theory with boundary

Let us remind that the \( N = 1 \) SLFT describes a supermultiplet consisting of a bosonic field and its fermionic partner interacting with exponential potential. In this Section we prefer to use the component fields rather than the superfield language. In terms of the component fields, the Lagrangian can be expressed by:

\[
\mathcal{L}_{SL} = \frac{1}{8\pi}(\partial_a \phi)^2 - \frac{1}{2\pi}(\bar{\psi} \partial \bar{\psi} + \psi \partial \psi) + i\mu b^2 \psi \bar{\psi} \psi \partial \phi + \frac{\pi \mu^2 b^2}{2}(: \phi^{\phi} :)^2. \tag{2.109}
\]

The central charge in this normalization reads:

\[
c_{SL} = \frac{3}{2}(1 + 2Q^2). \tag{2.110}
\]

The NS and R primary fields are expressed again in terms of vertex operators (2.91) and (2.92) respectively. Their dimensions are the same as in the previous section. The physical states can be denoted by a real parameter \( P \) defined by:

\[
\alpha = \frac{Q}{2} + iP.
\]
In the first part of this Section we will be interested in the SLFT on a pseudosphere. This is a generalization of [24] where the LFT is studied in the geometry of the infinite constant negative curvature surface, the so-called Lobachevskiy plane, i.e. the pseudosphere. The equations of motion for the component fields of the SLFT are given by:

\[
\begin{align*}
\partial \bar{\partial} \phi &= 4\pi^2 \mu b^2 (\mu e^{b\phi} - i\bar{\psi}\psi) e^{b\phi} \\
\partial \bar{\psi} &= -i\mu e^{b\phi}\psi, \\
\bar{\partial} \psi &= i\mu e^{b\phi}\bar{\psi}.
\end{align*}
\]

We will assume that the fermion vanishes in the classical limit so that the background metric is determined by the bosonic field satisfying:

\[
e^\varphi(z) = \frac{4R^2}{(1 - |z|^2)^2},
\]

where \(\varphi = 2b\phi\) and \(R^{-2} = 4\pi^2 \mu^2 b^5\). The parameter \(R\) is interpreted as the radius of the pseudosphere in which the points at the circle \(|z| = 1\) are infinitely far away from any internal point. This circle can be interpreted as the “boundary” of the pseudosphere. In the same way as the LFT, we can now use the Poincaré model of the Lobachevskiy plane with complex coordinate \(\xi\) in the upper half plane.

We want to compute exact one-point functions of the (NS) and (R) bulk operators \(N_\alpha\) and \(R_\alpha\). Due to the superconformal invariance, these one-point functions are given by:

\[
\langle N_\alpha(\xi) \rangle = \frac{U^N(\alpha)}{|\xi - \bar{\xi}|^{2\Delta_N}}, \quad \langle R_\alpha(\xi) \rangle = \frac{U^R(\alpha)}{|\xi - \bar{\xi}|^{2\Delta_R}}.
\]

We will simply refer to the coefficients \(U^N(\alpha)\) and \(U^R(\alpha)\) as bulk one-point functions. To derive the functional relations satisfied by these one-point functions, we should consider the bulk degenerate fields which are defined by some differential equations with certain orders.

The degenerate fields in the (NS) sector are given by:

\[
N_{\alpha_{m,n}} = e^{\alpha_{m,n}\phi}, \quad \alpha_{m,n} = \frac{1}{2b}(1 - m) + \frac{b}{2}(1 - n), \quad \text{with} \quad m - n = \text{even}
\]

and those in the (R) sector by:

\[
R_{\alpha_{m,n}} = \sigma(\epsilon)e^{\alpha_{m,n}\phi}, \quad \text{with} \quad m - n = \text{odd}.
\]

One of the essential features of these fields is that the operator product expansion (OPE) of a degenerate field with any primary field is given by a linear combination of only finite number of primary fields and their descendents. The simplest degenerate fields are \(N_{-b}\) for the (NS) sector and \(R_{-b/2}\) for the (R) sector.

The OPE of \(R_{-b/2}\) with a (NS) primary field is given by:

\[
N_\alpha R_{-b/2} = C_+^{(N)}(\alpha) [R_\alpha_{-b/2}] + C_-^{(N)}(\alpha) [R_{\alpha+b/2}], \quad (2.111)
\]
where [...] stands for entire family of conformal descendants corresponding to a primary field. The structure constants can be computed using Coulomb gas integrals. One can set $C_+^{(N)} = 1$ since no screening insertion is needed. The other structure constant $C_-^{(N)}$ needs just one insertion of the SLFT interaction and can be computed to be:

$$C_-^{(N)}(\alpha) = \frac{\pi \mu b^2 \gamma \left(\alpha b - \frac{b^2}{2} - \frac{1}{2}\right)}{\gamma \left(\frac{1-b^2}{2}\right) \gamma(\alpha b)}.$$

Similarly, the OPE with the (R) primary field is

$$R_\alpha R_{-b/2} = C_+^{(R)}(\alpha) [N_{\alpha-b/2}] + C_-^{(R)}(\alpha) [N_{\alpha+b/2}]$$

where $C_+^{(R)} = 1$ as before and $C_-^{(R)}$ is given by:

$$C_-^{(R)}(\alpha) = \frac{\pi \mu b^2 \gamma \left(\alpha b - \frac{b^2}{2}\right)}{\gamma \left(\frac{1-b^2}{2}\right) \gamma(\alpha b + \frac{1}{2})}.$$

Now we consider the bulk two-point functions of the degenerate field $R_{-b/2}$ and a (NS) field $N_\alpha$,

$$G^{N}_\alpha, -b/2(\xi, \xi') = \langle N_\alpha(\xi') R_{-b/2}(\xi) \rangle.$$  

It is straightforward from (2.111) to show that the two-point function satisfy:

$$G^N_{\alpha,-b/2}(\xi, \xi') = C^{(N)}_{+}(\alpha) U^R_{\alpha-b/2} G^R_{\alpha-b/2}(\xi, \xi') + C^{(N)}_{-}(\alpha) U^R_{\alpha+b/2} G^R_{\alpha+b/2}(\xi, \xi')$$

where $G_{\pm}(\xi, \xi')$ are expressed in terms of some special conformal blocks:

$$G_{\pm}(\xi, \xi') = \frac{\eta^{2\Delta_\alpha - 2\Delta_{R-b/2}}}{|\xi - \xi'|^{4\Delta_\alpha}} F_{\pm}(\eta).$$

Here, the conformal blocks are given by hypergeometric functions (which are known) and:

$$\eta = \frac{(\xi - \xi')(\xi - \xi')}{(\xi - \xi')(\xi - \xi')}. $$

In the cross channel, an equivalent expression for the two-point function can be obtained as follows:

$$G^N_{\alpha,-b/2} = \frac{|\xi' - \bar{\xi}|^{2\Delta_\alpha - 2\Delta_{R-b/2}}}{|\xi - \bar{\xi}|^{4\Delta_\alpha}} \left[ B^{(N)}_{+}(\alpha) \tilde{F}_{+}(\eta) + B^{(N)}_{-}(\alpha) \tilde{F}_{-}(\eta) \right]$$

(2.114)
where $\tilde{F}(\eta)$ are again given by some known hypergeometric functions. The boundary structure constants $B^{(N)}_{-\pm}$ can be determined from the monodromy relations connecting $F(\eta)$ and $\tilde{F}(\eta)$.

The conformal block $\tilde{F}$ corresponds to the identity boundary operator with dimension 0 appearing in the boundary as the bulk operator $R_{-b/2}$ approaches the boundary with $\eta \to 1$. Another boundary operator $n_{-b}$ appearing as $R_{-b/2}$ approaches the boundary generates the $\tilde{F}_{+}$ block. As mentioned above, the geodesic distance to the boundary on the pseudosphere is infinite. Therefore, the two-point function in the LHS of (2.111) can be factorized into a product of two one-point functions and satisfies

$$B^{(N)}_{-}(\alpha) = U^{N}(\alpha)U^{R}(-b/2).$$

Combining all these and using (2.114), we obtain the following nonlinear functional equation in the $\eta \to 1$ limit:

$$\frac{\Gamma \left(1 - \frac{b^2}{2}\right) U^{N}(\alpha)U^{R} \left(-\frac{b}{2}\right)}{\Gamma(-b^2)\Gamma(\alpha b - \frac{b^2}{2} + \frac{1}{2})} = \frac{U^{R}(\alpha - \frac{b}{2})}{\Gamma(\alpha b - b^2)} + \frac{\pi \mu b^2 U^{R}(\alpha + \frac{b}{2})}{\Gamma(\alpha b + b^2 - \frac{1}{2})}.$$  (2.115)

Analysis of the other two-point function:

$$C^{R}_{\alpha,-b/2}(\xi,\xi') = \langle R_{\alpha}(\xi)R_{-b/2}(\xi') \rangle$$

goest along the same line and leads to the second functional equation:

$$\frac{\Gamma \left(1 - \frac{b^2}{2}\right) U^{R}(\alpha)U^{R} \left(-\frac{b}{2}\right)}{\Gamma(-b^2)\Gamma(\alpha b - \frac{b^2}{2} + \frac{1}{2})} = \frac{U^{N}(\alpha - \frac{b}{2})}{\Gamma(\alpha b - b^2 - \frac{1}{2})} + \frac{\pi \mu b^2 U^{N}(\alpha + \frac{b}{2})}{\Gamma(\alpha b + b^2 + \frac{1}{2})}.$$  (2.116)

The SLFT satisfies the duality $b \to 1/b$. This property requires considering another degenerate (R) operator $R_{-1/2b}$ which generates two more functional equations in addition to (2.115) and (2.116). These additional equations can be obtained by just replacing the coupling constant $b$ with $1/b$ and the parameter $\mu$ by the “dual” $\tilde{\mu}$ satisfying:

$$\pi \tilde{\mu} \gamma \left(\frac{Q}{2b}\right) = \left[\pi \mu b \left(\frac{bQ}{2}\right)\right]^{1/b^2}.$$  (2.117)

Therefore, the one-point functions $U^{N}(\alpha)$ and $U^{R}(\alpha)$ should satisfy four nonlinear functional equations.

We have found the solutions to these overdetemined nonlinear equations as follows:

$$U^{N}_{mn}(\alpha) = \frac{\sin \left(\frac{\pi Q}{2b}\right) \sin \left(\frac{n \pi b Q}{2}\right) \sin \left[m \pi \left(\frac{Q}{2b} - \frac{a}{2}\right)\right] \sin \left[n \pi \left(\frac{bQ}{2} - b\alpha\right)\right]}{\sin \left(m \pi \frac{b Q}{2b}\right) \sin \left(n \pi \frac{b Q}{2}\right) \sin \left[\pi \left(\frac{Q}{2b} - \frac{a}{2}\right)\right] \sin \left[\pi \left(\frac{bQ}{2} - b\alpha\right)\right]} U^{N}_{11}(\alpha)$$  (2.118)

$$U^{R}_{mn}(\alpha) = \frac{\sin \left(\frac{\pi Q}{2b}\right) \sin \left(\frac{n \pi b Q}{2}\right) \sin \left[m \pi \left(\frac{Q}{2b} - \frac{a}{2} + \frac{1}{2}\right)\right] \sin \left[n \pi \left(\frac{bQ}{2} - b\alpha + \frac{1}{2}\right)\right]}{\sin \left(m \pi \frac{b Q}{2b}\right) \sin \left(n \pi \frac{b Q}{2}\right) \cos \left[\pi \left(\frac{Q}{2b} - \frac{a}{2}\right)\right] \cos \left[\pi \left(\frac{bQ}{2} - b\alpha\right)\right]} U^{R}_{11}(\alpha)$$
where the ‘basic’ solutions are given by:

\[ U_{11}^N(\alpha) = \left[ \pi \mu \gamma \left( \frac{bQ}{2} \right) \right]^{-\alpha/b} \frac{\Gamma \left( \frac{bQ}{2} \right) \Gamma \left( \frac{Q}{2b} \right) Q/2}{\Gamma \left( -\alpha b + \frac{bQ}{2} \right) \Gamma \left( \frac{Q}{2b} \right) \frac{Q}{2} - \alpha} \]

\[ U_{11}^R(\alpha) = \left[ \pi \mu \gamma \left( \frac{bQ}{2} \right) \right]^{-\alpha/b} \frac{\Gamma \left( \frac{bQ}{2} \right) \Gamma \left( \frac{Q}{2b} \right) Q/2}{\Gamma \left( -\alpha b + \frac{bQ}{2} + \frac{1}{2} \right) \Gamma \left( -\frac{Q}{2b} + \frac{Q}{2} + \frac{1}{2} \right)}. \]

(2.119)

(2.120)

There are infinite number of possible solutions which are parametrized by two integers \((m,n)\). For these to be solutions, we find that the two integers should satisfy \(m - n = \text{even}\). The basic solutions, Eqs. (2.119) and (2.120), can be interpreted as the one-point functions of the bulk operators \(N_\alpha\) and \(R_\alpha\) corresponding to the vacuum boundary conditions (BC), the BC corresponding to the bulk vacuum operator \(N_0\). Then, the general solutions (2.118) can be identified with the one-point functions with the conformal BC \((m,n)\) classified by Cardy \[8\]. Since \(m - n = \text{even}\), the one-point functions we obtained correspond to the (NS)-type BCs only. This seems consistent with the fact that only the (NS) boundary operators arise when the (NS) or (R) bulk degenerate operators approach the boundary corresponding to the vacuum BC.

We also note that the solutions (2.118) satisfy the so-called bulk “reflection relations”:

\[ U_{m,n}^N(\alpha) = D^{(N)}(\alpha)U_{m,n}^N(Q - \alpha), \quad U_{m,n}^R(\alpha) = D^{(R)}(\alpha)U_{m,n}^R(Q - \alpha) \]

where \(D^{(N)}(\alpha)\) and \(D^{(R)}(\alpha)\) are the (NS) and the (R) reflection amplitudes computed in the previous section (called \(S^{(N,R)}\) there) and derived in \[21\].

Now we turn to the computation of the bulk one-point function in the presence of a boundary. We define the SLFT on half plane where superconformally invariant boundary action is imposed by choosing the following boundary action at \(y = 0\):

\[ \mathcal{L}_B = \frac{\mu_B}{2} e^{b\phi/2} a(\psi - i\gamma \bar{\psi})(x) \]

with \(\gamma = \pm 1\) and the fermionic zero-mode \(a\) satisfying \[54\]:

\[ \sigma^{(\pm)} = a\sigma^{(\mp)} \quad \text{and} \quad a^2 = 1. \]

This action includes additional boundary parameter \(\mu_B\) which generates continuous family of BCs. The boundary equations of motion are given by:

\[ \frac{1}{2\pi} \partial_y \phi = -\frac{1}{2} b\mu_B a(\psi - i\gamma \bar{\psi}) e^{b\phi/2} \]

\[ \frac{i}{2\pi} \bar{\psi} = \mu_B e^{b\phi/2} a, \quad \frac{i}{2\pi} \bar{\psi} = i\gamma \mu_B e^{b\phi/2} a. \]

Plugging these constraints back into the action, one can simplify the boundary action:

\[ \mathcal{L}_B = \mu_B e^{b\phi/2} a\bar{\psi}. \]
One can see that physical quantities should contain only even powers of $\mu_B$ because of the fermionic zero-mode. While the bulk properties of the boundary SLFT should be identical, we should define the boundary operators. As in the bulk, there are two sectors, the (NS) and (R) boundary operators:

$$n_\beta = e^{\beta\phi/2}(x), \quad r_\beta = \sigma^{(\epsilon)}e^{\beta\phi/2}(x).$$

The procedure to derive the functional equations satisfied by the bulk one-point functions are identical to that we used above. Major difference arises when the bulk degenerate operator $R_{-b/2}$ approaches the boundary as $z \to \infty$. The LHS of Eq.(2.113) can be evaluated by the boundary OPE which generates the boundary operator $n_0$ and $n_{-b}$. We choose the identity operator $n_0$, or the boundary vacuum state, since we are interested in the bulk one-point function. The fusion of the degenerate field $R_{-b/2}$ can be computed by a first order perturbation from the boundary action:

$$R^{(\epsilon)}(-b/2, Q) = -\mu_B \int dx \langle R^{(\epsilon)}_{-b/2}(x) \frac{i}{2} a\psi(x) e^{b\phi/2}(x) e^{Q\phi/2}(\infty) \rangle = \mu_B \int dx |x - i/2|^{b^2-1} = 2\pi\mu_B \frac{\Gamma(-b^2)}{\Gamma(\frac{1-b^2}{2})^2}. $$

Again, the dependence on the superindex $\epsilon$ disappears so that we can suppress it. With the vacuum state on the boundary, the two-point function becomes the bulk one-point function of the operator $N_\alpha$. Equating this with the RHS of Eq. (2.113) gives the functional equation:

$$\frac{2\pi\mu_B}{\Gamma(\frac{1-b^2}{2})} U^N(\alpha) = \frac{\Gamma \left( \frac{ab - \frac{b^2}{2} + \frac{1}{2}}{ab} \right)}{\Gamma \left( \frac{ab - \frac{b^2}{2}}{2} \right)} U^R \left( \alpha - \frac{b}{2} \right) + \frac{\pi\mu b^2\Gamma \left( \frac{ab - \frac{b^2}{2} - \frac{1}{2}}{2} \right)}{\gamma \left( \frac{1-b^2}{2} \right) \Gamma(\alpha)} U^R \left( \alpha + \frac{b}{2} \right). \quad (2.122)$$

Similar consideration for the $G_{\alpha,-b/2}$ leads to:

$$\frac{2\pi\mu_B}{\Gamma(\frac{1-b^2}{2})} U^R(\alpha) = \frac{\Gamma \left( \frac{ab - \frac{b^2}{2}}{2} \right)}{\Gamma \left( \frac{ab - b^2 - \frac{1}{2}}{2} \right)} U^N \left( \alpha - \frac{b}{2} \right) + \frac{\pi\mu b^2\Gamma \left( \frac{ab - \frac{b^2}{2} + \frac{1}{2}}{2} \right)}{\gamma \left( \frac{1-b^2}{2} \right) \Gamma(\alpha + \frac{1}{2})} U^N \left( \alpha + \frac{b}{2} \right). \quad (2.123)$$

As before, one should consider the dual equations coming from the dual degenerate operator $R_{-1/2b}$. 

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The solutions of Eqs. (2.122) and (2.123) can be found as:

\[ U^N(\alpha) = \mathcal{N} b \left[ \pi \mu \gamma \left( \frac{bQ}{2} \right) \right]^{Q-2\alpha}{2\pi} \Gamma \left( \left( \alpha - \frac{Q}{2} \right) b \right) \Gamma \left( 1 + \left( \alpha - \frac{Q}{2} \right) \frac{1}{b} \right) \] (2.124)

\[ \times \cosh \left( \left( \alpha - \frac{Q}{2} \right) \pi s \right) \]

\[ U^R(\alpha) = \mathcal{N} \left[ \pi \mu \gamma \left( \frac{bQ}{2} \right) \right]^{Q-2\alpha}{2\pi} \Gamma \left( \left( \alpha - \frac{b}{2} \right) b \right) \Gamma \left( \left( \alpha - \frac{1}{2b} \right) \frac{1}{b} \right) \] (2.125)

\[ \times \cosh \left( \left( \alpha - \frac{Q}{2} \right) \pi s \right) , \]

where the normalization factor \( \mathcal{N} \) is given by:

\[ \mathcal{N} = \left[ \pi \mu \gamma \left( \frac{bQ}{2} \right) \right]^{-Q/2b} [b\Gamma(-Qb/2)\Gamma(1-Q/2b)\cosh(Q\pi s/2)]^{-1} \]

so that \( U^N(0) = 1 \). Here, the boundary parameter \( s \) is related to \( \mu_B \) by:

\[ \frac{\mu_B^2}{\mu b^2} \sin \left( \frac{\pi bQ}{2} \right) = \cosh^2 \left( \frac{\pi bs}{2} \right) . \]

Notice that the solutions (2.124) and (2.125) are self-dual if the parameter \( s \) is invariant and \( \mu \to \tilde{\mu} \) as Eq. (2.117). The continuous parameter \( s \) coming from \( \mu_B \) generates a continuous family of conformally invariant BCs. One can also check that these satisfy the bulk reflection relations Eq. (2.121).

3 \( N = 2 \) superconformal theories

This Section is devoted to the description of the \( N = 2 \) superconformal theories. As in \( N = 1 \), an effective approach to the \( N = 2 \) minimal models is the Coulomb gas representation. We present a detailed discussion of the Coulomb gas construction for NS, R and T fields. The basic ingredient of the \( N = 2 \) Coulomb gas representation is the system of two free scalar fields \( \phi, \phi^\dagger \) and two free fermionic fields \( \psi, \psi^\dagger \), with total central charge \( c = 3 \). In the NS sector they can be combined into two free dimensionless chiral \( N = 2 \) superfields \( S^\pm \). The Rmond and twisted primary fields can be represented by vertex operators using the spin \( \sigma^t_{1/8} \) and twisted \( \sigma^t_{1/16} \) fields of the \( c = 3 \) system. The non-trivial dynamics of the free field construction of the \( N = 2 \) minimal models (reflected in the non-integer central charge \( c_p \)) is carried by two background charges \( \beta, \bar{\beta} \) placed at infinity and by the corresponding screening operators.

Another approach to the \( N = 2 \) minimal models, based on the \( D_n \) parafermionic construction \[43, 44\] of the \( N = 2 \) superconformal algebra, has an advantage in comparison with the
Coulomb gas method in the calculation of the 4-point functions and the structure constants of the 2D OPE algebras. The reason for that is in the relation of the $D_n$ parafermionic models with the $su(2)$ Wess-Zumino-Witten (WZW) models [55]. Then the problem of the computation of the 4-point functions of the NS and R fields reduces to the simple problem to express these functions in terms of the $su(2)$ WZW functions and the 4-point functions of the free field vertices $e^{i\alpha\phi(z)}$. The same is true also for the structure constants and the corresponding OPE algebras. The twisted fields in this language are realized in terms of the so called C-disorder fields of the parafermionic models.

An important point in our discussion of the $N = 2$ minimal models is related to the origin of the $Z_{p+2}$ discrete symmetry [18]. It turns out that in each $c_p$ model there exist a set of NS superfields $N^p_m$ which together with the super-stress-energy tensor $\mathcal{W}(z, \theta^+, \theta^-)$ close an OPE algebra of the $Z_{p+2}$ parafermionic type. We describe in details this $N = 2$ super-parafermionic models and their relation to the corresponding $N = 2$ superconformal minimal models. It turns out that the super-parafermionic currents and the order-disorder fields are precisely the fields that play an important role in the construction of the vertices of the low-energy massless particles [18]. So these, as well as our general results for the FR’s, can be applied to the Gepner’s tensor product construction. More precisely, we consider in this Section the three-generation $1^116^3$ Gepner model of the heterotic string compactification [18]. The main point of the Gepner’s approach is the construction of the (modeled out) tensor products of $N = 2$ minimal models ($c = 1 + 3 \times 8/3 = 9$) having the same discrete symmetries as the maximally symmetric Calabi-Yau (C-Y) space. The generalized GSO projection relates specific combinations of the fields of these models with the free fields of the non-compactified degrees of freedom which reproduce the massless spectrum of the C-Y three-generation string model [18]. An important step in the consistency checks of this model is the explicit construction of the low-energy effective cubic superpotential. In this Section we derive the exact Yukawa couplings for the $1^116^3$ model.

The rest of this Section is devoted to the discussion of the $N = 2$ super Liouville field theory (SLFT). We show that it exhibits an interesting duality behaviour. Under the dual transformation $b \to 1/b$ the theory maps to a dual action which is another $N = 2$ super CFT. The $N = 2$ SLFT with a strong coupling can be described by the dual action perturbatively. We compute the reflection amplitudes (the 2-point functions) of the theory using functional relations derived from these actions. This procedure provides an exact relation between the two actions. An important generalization is the $N = 2$ SLFT in the presence of a boundary. We use the modular bootstrap method, which is a generalization of the Cardy formulation for the conformal BC’s to the irrational CFT’s. We compute the one-point function for general BC’s parameterized by a continuous parameter (the so called FZZT branes). Using the one-point functions, we rederive the bulk reflection amplitudes and compare them with those obtained before. Furthermore, we provide a conformal bootstrap analysis based on the $N = 2$ SLFT and its dual theory and confirm that the one-point functions obtained from the modular transformations are consistent with the bulk and boundary actions. As a byproduct, we obtain a relation between the continuous BC parameter and the boundary cosmological constants of the two dual theories. To find all the consistent conformal BC’s
of the $N = 2$ SLFT is important since they describe the D-branes moving in the black hole background. In addition to the FZZT and the vacuum BC’s there is an infinite number of discrete BC’s which are called ZZ branes in general. We find the ZZ brane solutions from the functional equations defined on the pseudosphere and discuss their implications for the $N = 2$ SLFT. We then perform modular bootstrap calculations for the degenerate fields of the theory which provide some consistency checks for the solutions.

At the end of this Section we find a set of higher equations of motion in $N = 2$ SLFT. In doing that we used the one point function obtained above. As already mentioned this is an interesting question because of the fact that this theory has actually few properties in common with the $N = 0, 1$ SLFT’s. One difference is the lack of the strong-week coupling duality mentioned above. Another important difference is in the spectrum of the degenerate representations $[16, 17, 56]$. We will show that the $N = 2$ SLFT still possesses higher equations of motion despite these differences.

The results of this Section have been published in $[22], [56]-[62], (9.-16.)$.

### 3.1 $N = 2$ minimal models

The set of generators of the $N = 2$ extended superconformal algebra contains in addition to the stress-energy tensor $T(z)$ and the supercurrent $G^1(z)$ (generating $N = 1$ SUSY) a $U(1)$ current $J(z)$ of conformal dimension 1 and its super-partner $G^2(z)$ of dimension 3/2 (and also the corresponding right-moving modes). It is convenient to use the $U(1)$ diagonal basis for the description of the NS and R sectors of the $N = 2$ algebra: $G^\pm(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)$ so that:

\[
T(z_1)G^\pm(z_2) = \frac{3}{2z_{12}^2}G^\pm(z_2) + \frac{1}{z_{12}}\partial G^\pm(z_2) + \ldots
\]

\[
J(z_1)G^\pm(z_2) = \pm \frac{1}{2z_{12}}G^\pm(z_2) + \ldots
\]

\[
G^+(z_1)G^-(z_2) = \frac{2c}{3z_{12}^3} + \frac{4}{z_{12}^2}J(z_2) + \frac{2}{z_{12}}(T(z_2) + \partial J(z_2)) + \ldots
\]

\[
G^+(z_1)G^+(z_2) = O(z_{12}) = G^-(z_1)G^-(z_2).
\]

Similarly to the case of $N = 1$ SUSY one may choose periodic or antiperiodic boundary conditions for the supercurrents with corresponding Laurent expansions:

\[
G^\pm(z) = \sum_{n \in \mathbb{Z}} z^{-n-2}G^\pm_{n+1/2} \quad \text{or} \quad G^\pm(z) = \sum_{n \in \mathbb{Z}} z^{-n-3/2}G^\pm_n.
\]
The coefficients $G^\pm_r$ of these expansions close NS and R parts of the $N = 2$ superalgebra respectively:

$$[L_m, G^\pm_r] = \left(\frac{m}{2} - r\right)G^\pm_{m+r}, \quad [J_n, G^\pm_r] = \pm G^\pm_{n+r},$$

$$\{G^+_r, G^-_s\} = 2L_{r+s} + (r - s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \quad \{G^\pm_r, G^\pm_s\} = 0,$$

$$[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = \frac{c}{3}m\delta_{m+n},$$

$r, s \in Z + 1/2$ for the NS sector, $r, s \in Z$ for the R sector and:

$$T(z) = \sum_{n \in Z} z^{-n-2}L_n, \quad J(z) = \sum_{n \in Z} z^{-n-1}J_n.$$

Due to the presence of the $U(1)$ current $J(z)$ we have one more possibility, that is to choose $Z_2$ twisted boundary conditions for this current $J(e^{2\pi i}z) = -J(z)$ and:

$$J(z) = \sum_{n \in Z} z^{-n-1/2}J_{n+1/2}.$$

Then, as a consequence, one of the currents $G^{1,2}(z)$ must have periodic and the other one antiperiodic boundary conditions. The two choices are equivalent so we can take for example:

$$G^1(z) = \sum_{n \in Z} z^{-n-2}G^1_{n+1/2}, \quad G^2(z) = \sum_{n \in Z} z^{-n-3/2}G^2_n$$

obeying the so called $N = 2$ twisted superconformal algebra.

The Cartan subalgebra of the NS and R algebras is two-dimensional and consists of the generators $L_0$ and $J_0$. Therefore, the corresponding lowest weight representations (LWR’s) for the fixed value of $c$ are labeled by two parameters: the conformal dimension $\Delta$ and the $U(1)$ charge $q$. The primary states $|\Delta, q>$ which generate these LWR’s satisfy the conditions:

$$L_0|\Delta, q> = \Delta|\Delta, q>, \quad J_0|\Delta, q> = q|\Delta, q>,$$

$$L_n|\Delta, q> = J_m|\Delta, q> = G^\pm_r |\Delta, q> = 0, \quad n, m, r > 0.$$

Let us consider the NS sector first. Introducing the $OSp(2|2)$ invariant vacuum of the the theory which belongs to the NS sector:

$$L_n|0> = J_{n+1}|0> = G^\pm_r |0> = 0, \quad n \geq -1, r \geq -1/2$$

we can realize the NS primary states $|\Delta, q>$ by acting on the vacuum with the NS primary superfields of dimension $\Delta$ and $U(1)$ charge $q$:

$$N(z, \theta^+, \theta^-) = \varphi(z) + \theta^+\psi^- (z) + \theta^-\psi^+(z) + \theta^+\theta^-\varphi(z).$$
The component fields have the following OPE’s with the generators:

\[ G^\pm(z_1) \varphi(z_2) = \frac{1}{z_{12}^2} \psi^\pm(z_2) + \ldots \]

\[ G^\pm(z_1) \psi^\pm(z_2) = \frac{2(\Delta \pm q)}{z_{12}^2} \varphi(z_2) + \frac{1}{z_{12}^2} (\partial \varphi(z_2) \pm \varphi(z_2)) + \ldots \]

\[ G^\pm(z_1) \tilde{\varphi}(z_2) = \mp \frac{2(\Delta \pm q)}{z_{12}^2} \psi^\pm(z_2) + \ldots \]

The LWR in the NS sector of the \( N = 2 \) superconformal algebra corresponding to the LW state \(|\Delta, q >\consists of all the linear combinations of the vectors of the form:

\[ L_{(-n_i)} J_{(-m_i)} G^+_{(-r_i)} G^-_{(-s_i)} |\Delta, q >, \quad n_i, m_i, r_i, s_i \geq 0 \]

where \( \{n_i\} \text{ etc.} \ is a multi-index and } k = \sum_i n_i + \sum_i m_i + \sum_i r_i + \sum_i s_i \text{ and } m = \frac{1}{2}(\#G^+ - \#G^-) \text{ are called level and relative charge of the state respectively. Because of the specific properties of the generators: } \{G^+_i, G^-_i\} = 0, \text{ it turns out that only the relative charges } 0 \text{ (at integer level } k \text{) and } \pm 1/2 \text{ (at half-integer level) are allowed.}

The two and three-point functions of the primary superfields \( N(z, \theta^+, \theta^-) \) are determined (up to an arbitrary constant) by the conditions of \( OSp(2|2) \) invariance i.e. the finite subalgebra of \( \mathfrak{osp}(2|2) \) spanned by \( L_0, L_{\pm 1}, J_0, G^\pm_{\frac{1}{2}} \). Using the corresponding superconformal Ward identities and the invariance of the vacuum, these conditions lead to a system of differential equations which can be solved (up to a constant). Then the corresponding two-point function for example has the form:

\[ < N_{1}(z_1, \theta^+_1, \theta^-_1) N(z_2, \theta^+_2, \theta^-_2) > = C_{2} z_{12}^{-2\Delta_1} \left( 1 + 2q_1 \frac{\theta^+_2 \theta^-_1}{z_{12}} \right) \delta_{\Delta_1 - \Delta_2} \delta_{q_1 + q_2}. \quad (3.2) \]

In the case of the 3-point functions one can see that there are three independent solutions corresponding to the following three possibilities for the total \( U(1) \) charge: \( q_1 + q_2 + q_3 = 0, \pm 1/2 \) which are dictated by the \( J_0 \) invariance. Thus we have one even 3-point function:

\[ < N_{\Delta_1}^{q_1} N_{\Delta_2}^{q_2} N_{\Delta_3}^{-q_1 - q_2} > \]

and two odd ones:

\[ < N_{\Delta_1}^{q_1} N_{\Delta_2}^{q_2} N_{\Delta_3}^{-q_1 - q_2 \pm 1/2} > \]

with three independent arbitrary structure constants \( C_3, C^+_3 \). The fact that we can have three different 3-point functions has an important consequence for the analysis of the fusion rules for the NS fields. As we shall show later it gives rise to three independent NS fusion rules - one even and two odd ones.

In this Section we will be interested in the so called degenerate unitary representations. Their main peculiarity is the existence of null-vectors at given levels and relative charges,
i.e. states which are again primary. This property gives rise to a certain relation between the parameters characterizing these representations: the central charge \(c\), the dimension \(\Delta\) and the \(U(1)\) charge \(q\), the so called Kac formula, which completely classify them. For the NS sector of the \(N=2\) superconformal algebra this formula reads \([16, 17]\):

\[
c = 3 - \frac{6}{p + 2}, \quad p = 1, 2, \ldots, \quad q_s = \frac{s}{2(p + 2)},
\]

\[
\Delta_{n1}^s = \frac{(p + 2 - n)^2 - s^2 - 1}{4(p + 2)}, \quad n = 0, 1, \ldots, p + 1, \quad |s| \leq p - n + 1,
\]

\[
\Delta_{n0}^s = \frac{(n + |s|)^2 - s^2 - 1}{4(p + 2)}, \quad n + |s| \leq p + 1, \quad n = 1, 3, 5, \ldots
\]

Correspondingly, one can introduce two kinds of degenerate NS superfields: \(N_{n1}^s\) which has a degeneracy at level \(n\) with zero relative charge and \(N_{n0}^s\) having level \(n/2\) degeneracy with relative charge \(\pm 1/2\).

The simplest null-vectors, at level 1/2 and relative charge \(\pm 1/2\), have the form:

\[
G_{-\frac{1}{2}}^\pm |\Delta, q > = 0
\]

provided \(\Delta = \pm q\). They generate the so called \(N = 2\) chiral (and anti-chiral) superfields:

\[
N^+(z, \theta^+, \theta^-) = \varphi(z) + \theta^+ \theta^- \partial \varphi(z), \quad D^+ N^+ = 0,
\]

\[
N^-(z, \theta^+, \theta^-) = \varphi(z) + \theta^- \theta^+ \partial \varphi(z), \quad D^- N^- = 0,
\]

\(D^+\) and \(D^-\) are the \(N = 2\) supercovariant derivatives.

In the R sector we have in addition to \(L_0\) and \(J_0\) the zero modes \(G_0^\pm\) of the supercurrents \(G^\pm(z)\). As a consequence, for each \(\Delta \neq c/24\) we have to consider two Ramond states \(|\Delta, q >\) and \(|\Delta, q + 1/2 >\). For the corresponding R fields \(R^q_\Delta\) creating the R states from the NS vacuum: \(|\Delta, q >_R = R^q_\Delta|0, 0 >\) we get the following WI:

\[
G^+(z_1) R^q_\Delta(z_2) = \sqrt{\frac{2\Delta - \frac{12}{z_{12}^3}}{z_{12}^3}} R^{q+1/2}_\Delta(z_2) + \ldots
\]

\[
G^-(z_1) R^{q+1/2}_\Delta(z_2) = \sqrt{\frac{2\Delta - \frac{12}{z_{12}^3}}{z_{12}^3}} R^q_\Delta(z_2) + \ldots
\]

\[
G^+(z_1) R^{q+1/2}_\Delta(z_2) = O(\sqrt{z_{12}}) = G^-(z_1) R^q_\Delta(z_2)
\]

The structure of the degenerate LWR representations in the R sector is very similar to that of the NS sector. The formula for the dimensions and charges of the degenerate fields is now:

\[
c = 3 - \frac{6}{p + 2}, \quad p = 1, 2, \ldots, \quad q_r^s = \frac{s - r}{2(p + 2)} + \frac{r}{4}, \quad r = \pm 1,
\]

\[
\Delta_{n1}^{sr} = \frac{(p + 2 - n)^2 - (s - r)^2 - 1}{4(p + 2)} + \frac{1}{8}, \quad n = 1, \ldots, p + 1, \quad |s| \leq p - n + 1,
\]

\[
\Delta_{n0}^{sr} = \frac{(n + |s - r|)^2 - (s - r)^2 - 1}{4(p + 2)} + \frac{1}{8}, \quad n + |s - r| \leq p + 1, \quad n = 0, 2, 4, \ldots
\]

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In the conformal family of the primary field $R^{sr}_{n1}(z)$ there is a degeneracy at level $n$ with relative charge 0, while the null-vector for the field $R^{sr}_{n0}(z)$ is at level $n/2$ (which is integer in this case) with relative charge $\pm 1/2$.

In the twisted sector, because of the antiperiodic boundary conditions for the $U(1)$ current $J(z)$, it has no zero mode and hence the primary states are labeled by the value of the conformal dimension only: $L_0|\Delta > = |\Delta >$ and $|\Delta >$ is annihilated by all the positive modes. Similarly to the R sector of the $N = 1$ SUSY $|\Delta >$ is doubly degenerated, i.e. to each primary state $|\Delta, + >$ there corresponds a state $|\Delta, - > \sim G^2_0|\Delta, + >$ with the same dimension $\Delta$ which is again primary. Due to the properties of $G^2_0$ we have for the corresponding primary fields:

$$G^2(z_1)T^\pm_\Delta(z_2) = \sqrt{\Delta - \frac{c}{24} \frac{1}{z_1^{3/2}}} T^\pm_\Delta(z_2) + \ldots$$

It is clear that the twisted $N = 2$ multiplet has more complicated structure. At level $1/2$ we have in general two independent descendants:

$$J_{-1/2}|\Delta, \pm > = J_{-1/2}T^\pm_\Delta(0)|0 > = t^\pm_{\Delta+1/2}(0)|0 >,$$

$$G^1_{-1/2}|\Delta, \pm > = T^\pm_{\Delta+1/2}(0)|0 >$$

which are not primary states of the full $N = 2$ algebra. Considering the $N = 1$ subalgebra (which is generated by the supercurrent $G^1(z)$) we can combine the fields $T^\pm_\Delta(z)$ and $T^\pm_{\Delta+1/2}(z)$ in a $N = 1$ superfield.

As in the NS and R sectors we are interested in the LWR’s only. The series of dimensions of the degenerate primary fields is given by:

$$\Delta_n = \left(\frac{p+2}{2} - n \right)^2 - 1 \cdot \frac{1}{8}, \quad n = 1, 2, \ldots, p + 2,$$

where the field $T_n(z)$ has degeneracy at level $n/2$ ($n \in \mathbb{Z}_+$).

Let us now pass to the Coulomb gas representation of the $N = 2$ supersymmetric models. It is slightly different from that we considered in the case of $N = 1$ ones. It is based on the theory of two $N = 2$ NS chiral superfields:

$$S^+(z, \theta^+, \theta^-) = \phi^+(z) + \theta^- \psi^+(z) + \theta^- \phi^+(z)$$

$$S^-(z, \theta^+, \theta^-) = \phi^-(z) + \theta^+ \psi^-(z) - \theta^+ \phi^-(z)$$

where $\phi^\pm$ and $\psi^\pm$ are free complex scalar and fermion fields respectively. The chirality is guaranteed by the covariant condition:

$$D^+ S^- = 0 = D^- S^+.$$

The free action of the theory:

$$A(S^+, S^-) = \frac{1}{2} \int d^2 z d^2 \theta d^2 \bar{\theta} S^+ S^-$$
(here we denoted by $S^{\pm}$ the two-dimensional superfield) gives the propagators of the chiral fields. In terms of components we have:

$$
<\psi^+(z_1)\psi^-(z_2)> = \frac{1}{z_{12}}, \quad <\phi^+(z_1)\phi^-(z_2)> = -\ln z_{12}.
$$

Analogously to the $N = 1$ case we define the NS vertices as:

$$
N_{\alpha,\bar{\alpha}}(z, \theta^+, \theta^-) = \exp i(\alpha S^-(z, \theta^+, \theta^-) + \bar{\alpha} S^+(z, \theta^+, \theta^-)).
$$

They are labeled by two real numbers $(\alpha, \bar{\alpha})$ called charges. The corresponding improved action:

$$
A(S^\pm, \tilde{R}) = A(S^+, S^-) + \int d^2zd^2\theta d^2\bar{\theta}(2\beta \tilde{R} S^+ + 2\bar{\beta} \tilde{R} S^-)
$$

where $\tilde{R}$ is the $N = 2$ supersymmetric curvature (which could be placed at infinity), leads effectively to the extra vertex at infinity with charges $(-2\beta, -2\bar{\beta})$. In what follows we made a special choice $\beta = \bar{\beta}$.

Similarly to the $N = 1$ case, in order to get a consistent $N$-point function of the NS vertices, we must impose the neutrality condition:

$$
\sum_{i=1}^{N} \alpha_i = 2\beta = \sum_{i=1}^{N} \bar{\alpha}_i.
$$

From the improved action one can extract the form of the generators of the $N = 2$ superconformal symmetry:

$$
J = \frac{1}{2}\psi^+\psi^- - i\beta \partial\phi^- + i\beta \partial\phi^+, \quad G^+ = -\sqrt{2}\psi^+\partial\psi^- + 2\sqrt{2}i\beta \partial\psi^+, \quad G^- = -\sqrt{2}\psi^-\partial\phi^+ + 2\sqrt{2}i\beta \partial\psi^-, \quad T = -\partial\phi^+\partial\phi^- + \frac{1}{2}(\psi^+\partial\psi^- + \psi^-\partial\psi^+) + i\beta \partial^2\phi^+ + i\beta \partial^2\phi^-.
$$

They close a $N = 2$ superconformal algebra with a central charge:

$$
c = 3 - 24\beta^2.
$$

Thus, in order to describe the minimal models, the charge at infinity should be quantized:

$$
\beta^2 = \frac{1}{4(p+2)}, \quad p = 1, 2, 3, \ldots
$$

We first discuss in more details the Coulomb gas representation of the NS sector. From the explicit form of the NS vertex (3.5) and the $N = 2$ generators (3.7) one can extract the conformal dimension $\Delta$ and the $U(1)$ charge $q$ of the NS primary field:

$$
\Delta(\alpha, \bar{\alpha}) = \alpha\bar{\alpha} - \beta(\alpha + \bar{\alpha}), \quad q(\alpha, \bar{\alpha}) = \beta(\alpha - \bar{\alpha}).
$$
We note some symmetries of these formulas which will be useful below. The change:
\[ \alpha \rightarrow 2\beta - \bar{\alpha}, \quad \bar{\alpha} \rightarrow 2\beta - \alpha, \]
leads to the same dimension and charge \( \Delta, q \) while the following two changes:
\[ \alpha \rightarrow 2\beta - \alpha, \quad \bar{\alpha} \rightarrow 2\beta - \bar{\alpha}, \]
and:
\[ \alpha \rightarrow \bar{\alpha}, \quad \bar{\alpha} \rightarrow \alpha, \]
give the same dimension \( \Delta \) but the opposite charge \(-q\).

Let us introduce the following NS vertices:
\[ V_{\alpha+, \bar{\alpha}+} = e^{i(\alpha_+ S^+ + \bar{\alpha}_+ S^+)}, \]
\[ V_{\alpha-, \alpha-} = e^{i\alpha_- S^-}, \quad V_{\bar{\alpha}_-, \bar{\alpha}_-} = e^{i\bar{\alpha}_- S^+}, \]
with charges \( \alpha_\pm, \bar{\alpha}_\pm \) chosen in such a way that the corresponding integrals (called screening operators):
\[ Q_+ = \oint_{C_+} dz d\theta^+ d\theta^- V_{\alpha+, \bar{\alpha}+}, \]
\[ Q_- = \oint_{C_-} dz d\theta^+ V_{\alpha-, \alpha-}, \quad \bar{Q}_- = \oint_{C_-} dz d\theta^- V_{\bar{\alpha}_-, \bar{\alpha}_-}, \]
are invariant under the \( N = 2 \) superconformal transformations, i.e. have zero dimension and zero charge. This condition determines the charges \( \alpha_\pm, \bar{\alpha}_\pm \) as a function of the charge at infinity:
\[ \alpha_+ = 2\beta = \bar{\alpha}_+, \quad \Delta(\alpha_+) = 0 = q(\alpha_+), \]
\[ \alpha_- = -\frac{1}{2\beta} = \bar{\alpha}_-, \quad \Delta(\alpha_-) = \frac{1}{2} = \Delta(\bar{\alpha}_-), \quad q(\alpha_-) = -\frac{1}{2} = -q(\bar{\alpha}_-). \]

The analysis of the null-vectors in the Coulomb gas picture (constructed with the help of the screening operators) leads to the realization of the fields in the minimal models through the vertex operators (3.5). It turns out that in this case the charges \( \alpha, \bar{\alpha} \) are quantized. For example, the fields \( N_{n_1}^s \) (degenerated at level \( n \) with relative charge zero) are represented by the vertex operator (3.5) with charges:
\[ \alpha_{n_1}^s = \frac{1}{2}(1 - n + s)\alpha_+ - \frac{1}{2}\alpha_-, \quad \bar{\alpha}_{n_1}^s = \frac{1}{2}(1 - n - s)\alpha_+ - \frac{1}{2}\alpha_- \]
Analogously, for the field \( N_{n_0}^s \) degenerated at level \( n/2 \) with relative charge \( 1/2 \) we obtain:
\[ \alpha_{n_0}^s = \frac{1}{2}(1 - n)\alpha_+, \quad \bar{\alpha}_{n_0}^s = \frac{1}{2}(1 - n - 2s)\alpha_+, \quad s > 0, \]
while for the one with relative charge $-1/2$ we have:

$$\alpha_{n0}^s = \frac{1}{2}(1 - n + 2s)\alpha_+, \quad \tilde{\alpha}_{n1}^s = \frac{1}{2}(1 - n)\alpha_+, \quad s < 0.$$ 

In the Coulomb gas picture the correlation functions of the fields are obtained by inserting in the corresponding correlators of the vertices (3.5) a proper number of screening operators (3.10). They ensure that the neutrality condition (3.6) is satisfied. Also, they do not destroy the $N = 2$ superconformal covariance of the functions since they are $N = 2$ superconformally invariant. In the case of the 3-point functions this screening procedure allows one to recognize which of them do not vanish, or equivalently to obtain the corresponding FR’s for the products of two arbitrary primary fields. As we discussed above, the general 3-point function of NS $N = 2$ superfields contains one even and two odd ingredients (with charges $\pm 1/2$ respectively). This specific structure gives rise to three different FR’s, one even and two odd, generated by the corresponding parts of this function. The even 3-point function is obtained by inserting an arbitrary number of even screening operators and an equal number of the two kinds of odd ones (in order to have an uncharged function). The odd 3-point function with charge $1/2$ ($-1/2$) can be obtain instead if we insert in the correlator an arbitrary number of even screening operators $Q_+$ but one more of the screenings $Q_-$ ($\bar{Q}_-$). Finally, combining the even and odd FR’s, we obtain the general ones, i.e. we obtain all the (families of) primary fields that appear in the OPE of two given superfields. We do not present here the explicit results for these FR’s. This is because, as we will explain below, in the case of minimal $N = 2$ models, there is a more convenient construction in terms of the so called parafermionic theories. The FR’s have more compact form in that description so we leave the explicit expressions for the discussion of that construction. The same concerns also the case of the 4-point functions.

We turn now to the discussion of the Ramond sector. The generators of the $N = 2$ superconformal algebra are the same as in the NS sector (3.7). The only difference is that in this case $G^\pm(z)$ are antiperiodic fields and therefore we have to impose antiperiodic boundary conditions for the fields $\psi^\pm(z)$ too. Then the fields are no more combined in supermultiplets. Similarly to the $N = 1$ case we define the R vertices as follows:

$$R_{\alpha, \bar{\alpha}}^r(z) = \sigma^r e^{i(\alpha \phi^+ + \bar{\alpha} \phi^-)}(z), \quad r = \pm$$

(3.12)

where the fields $\sigma^\pm(z)$ of dimension $\Delta = 1/8$ and charge $q = \pm 1/4$ correspond to the lowest energy states in the R sector of the algebra of the complex fermion field $\psi^\pm(z)$. For the dimension and charge of the field $R_{\alpha, \bar{\alpha}}^r$ we obtain:

$$\Delta(\alpha, \bar{\alpha}, r) = \alpha \bar{\alpha} - \beta(\alpha + \bar{\alpha}) + \frac{1}{8}, \quad q(\alpha, \bar{\alpha}, r) = \beta(\alpha - \bar{\alpha}) + \frac{r}{4}.$$ 

The symmetries of these formulas are similar to those of the NS sector:

$$\alpha \rightarrow 2\beta - \bar{\alpha}, \quad \bar{\alpha} \rightarrow 2\beta - \alpha, \quad r \rightarrow r$$
leads to $\Delta, q \rightarrow \Delta, q$, and:

$$\alpha \rightarrow 2\beta - \alpha, \quad \bar{\alpha} \rightarrow 2\beta - \bar{\alpha}, \quad r \rightarrow -r$$

or:

$$\alpha \rightarrow \bar{\alpha}, \quad \bar{\alpha} \rightarrow \alpha, \quad r \rightarrow -r$$

give the same dimension $\Delta$ but the opposite charge $-q$.

As in the $N = 1$ case, the screening operators have the same form as in the NS sector. The only difference is that $\psi^\pm(z)$ are now antiperiodic. So the null-vector construction goes in the same way. As a result we obtain that the vertices representing the degenerate fields in the Ramond sector have a similar form. More precisely, the charges of the vertices of the fields $R_{n1}^r(z)$ are obtained from those of $N_{n1}^s(z)$ by simply replacing $s$ with $s - r$. In the same way we obtain the charges of $R_{n0}^r(z)$ from those of $N_{n0}^s(z)$.

In order to obtain the FR’s for the R fields we have to recognize which of the 3-point functions $< R_1R_2N_x >$ are different from zero. In the Coulomb gas picture this means to obtain all the ways of screening such function. We have to consider separately the two cases: $< R^rR^{-r} >$ and $< R^rR^{+r} >$. In the first case the corresponding R fields are constructed with the help of the fields $\sigma^\pm_r$ respectively. The 3-point function is then proportional to:

$$< \sigma^r \sigma^{-r} \psi^+ \psi^- \ldots >.$$

It is different from zero only if the number of screenings $Q_-$ is equal to the number of $\bar{Q}_-$, the number of the even screenings $Q_+$ remains arbitrary. Therefore the screening procedure in this case is exactly the same as in the case of the even FR’s in the NS sector. In the second case the corresponding 3-point function is proportional to:

$$< \sigma^r \sigma^r \psi'^r \ldots >$$

where $\psi'^r$ is one of the fields $\psi^\pm$ which makes this function neutral. Here we have two possibilities. The first one is to put one more screening operator $Q_-$ or $\bar{Q}_-$ and hence to have the same procedure as in the case of odd FR’s in the NS sector. The second possibility is to implement the fact that the second component of the NS field (built with the help of some $\psi^\pm$) can contribute. The corresponding 3-point function is screened then with equal number of $Q_-$ and $\bar{Q}_-$. Therefore this possibility corresponds to the even screening procedure in the NS sector. As before, we do not present here the explicit expressions for the Ramond FR’s and the corresponding structure constants. We postpone this problem to the parafermionic description of the $N = 2$ minimal models.

Finally, we describe the construction of the twisted vertices. As we explained above we redefine the supercurrents such that $G^2 \sim G^+ - G^-$ is now antiperiodic. Also, the $U(1)$ current $J$:

$$J(z) = \frac{1}{4} \psi_1 \psi_2 + i\beta \partial \phi_2$$

is antiperiodic in this sector (here we introduced the field $\phi_2$ which is a linear combination of $\phi^\pm$). It follows that the scalar field $\phi_2$ should be antiperiodic too. The other scalar field
$\phi_1$ is periodic. Following the general idea, we define the twisted field $T(z)$ in terms of the lowest dimensional field in the twisted sector of the fermion $\psi_1$, $\psi_2$ theory called $\sigma_0^\psi$, the lowest twisted field of the $U(1)$ current $\partial \phi_2$ - $\sigma_0^\psi$ and the exponential of the free scalar field $\phi_1$:

$$T_\alpha(z) = \sigma_0^\psi(z) \sigma_0^\phi(z) e^{i\alpha \phi_1(z)}.$$ 

One can check that all the properties of the T-primary field are satisfied. In particular, its dimension is given by:

$$\Delta(\alpha) = \alpha^2 - 2\alpha\beta + \frac{1}{8}$$

As in the NS and R sectors, the analysis of the null-vectors gives the quantized charge of the degenerate fields:

$$\alpha_n = \frac{1}{2}(1 - n)\alpha_+ - \frac{1}{4}\alpha_-$$

and the corresponding field $T_n$ has a degeneracy at level $n/2$. The fusion rules are obtained analogously to NS and R sectors. The difference is that in the case of the lowest twisted field $\sigma_0^\psi$ we have the following FR’s:

$$\sigma_0^\psi \sigma_0^\psi \sim 1 + \sigma^\pm + \psi.$$ 

This confirms the fact that in the product of two twisted fields both Ramond and NS fields occur.

As we already mentioned above, the $N = 2$ superconformal minimal models admit a representation in terms of the $D_{2p}$ parafermionic (PF) theories. It is based on the observation [43, 44] of the fact that the generators of the $N = 2$ supersymmetric theory could be expressed in terms of the PF currents and a free scalar field:

$$T = T_p + T_\varphi, \quad J = \frac{p}{2} \sqrt{2p(p+2)} \partial \varphi, \quad (3.13)$$

$$G^+ = \sqrt{\frac{2p}{p+2}} \psi_1 \exp \left( i \frac{p + 2}{\sqrt{2p(p+2)}} \varphi \right),$$

$$G^- = \sqrt{\frac{2p}{p+2}} \psi_1^\dagger \exp \left( -i \frac{p + 2}{\sqrt{2p(p+2)}} \varphi \right)$$

where $\psi_1$ and $\psi_1^\dagger$ are parafermionic currents with $Z_p$ charges 1 and $p - 1$ respectively, and the reflection $C$ acts as: $\psi_1 \rightarrow \psi_1^\dagger$.

The central charge of these theories takes the values: $c_p = \frac{2p-1}{p+2}$, $p = 1, 2, \ldots$. In (3.13) $\varphi$ denotes a free scalar field with central charge 1. Therefore the central charge of the algebra of the currents (3.13) is:

$$c = 1 + c_p = \frac{3p}{p+2}, \quad p = 1, 2, \ldots$$

which coincides with that of the $N = 2$ minimal models.
The lowest dimensional fields $\phi^l_m$ of the parafermionic theory have dimensions:

$$d^l_m = d_l + \frac{l^2 - m^2}{4p}, \quad -l \leq m \leq l$$

where:

$$d_l = \frac{l(p - l)}{2p(p + 2)}, \quad l = 0, 1, \ldots, p$$

is the dimension of the corresponding order parameter fields $\sigma_l = \phi_l^l$. The dimension $d^l_m$ has the following symmetries:

$$l \rightarrow l, \quad m \rightarrow -m,$$

$$l \rightarrow p - l, \quad m \rightarrow p \pm m$$

(3.14)

where we identify $m = m + 2p$ since $m$ is the $Z_{2p}$ charge of the field $\phi^l_m$.

The primary fields in the $N = 2$ theories are constructed from the lowest fields of the PF theory and exponentials of the free scalar field $\phi$. For the NS sector we have:

$$N^l_m(z) = \phi^l_m(z) \exp\left(i \frac{m}{\sqrt{2p(p + 2)}} \phi(z)\right),$$

(3.15)

$$l = 0, 1, \ldots, p \quad m = -l, -l + 2, \ldots, l.$$  

The $U(1)$ charge of this field is:

$$q^l_m = \frac{m}{2(p + 2)}$$

and its dimension is simply the sum of the dimensions of the two ingredients:

$$\Delta^l_m = d^l_m + \frac{m^2}{2p(p + 2)} = \frac{l(l + 2)}{4(p + 2)} - \frac{m^2}{4(p + 2)}.$$  

To make a connection with the Coulomb gas representation of the NS sector we note that the first series $N^s_{n_1}$ is obtained from this construction as:

$$s = m, \quad n = p - l + 1$$

and for the second one $N^s_{n_2}$:

$$s = m, \quad n = l - |m| + 1.$$

The product with the supercurrents is:

$$G^{\pm}(z_1)N^l_m(z_2) = \sqrt{\frac{2p}{p + 2}} \frac{1}{z_{12}} \phi^{l}_{m \pm 2}(z_2) e^{i\frac{m \pm (p + 2)}{\sqrt{2p(p + 2)}} \phi(z_2)} + \ldots .$$

Note that, due to the symmetries (3.14), the second component of the field $N^l_m$ has the form:

$$(N^l_m)^{II \pm} \sim \phi^{p-l}_{m \pm (p+2)} e^{i\frac{m \pm (p + 2)}{\sqrt{2p(p + 2)}} \phi}$$
and therefore it looks just like \( N_{m\pm(p+2)}^{p-l} \) but without satisfying the conditions (3.15).

The primary fields of the Ramond sector are represented in a similar way:

\[
R_{l,m,\alpha}^l(z) = \phi_{l,m}^l(z) \exp \left( i \frac{m - \alpha p/2}{\sqrt{2p(p + 2)}} \varphi(z) \right), \quad l = 0, 1, \ldots, p \quad m = -l, -l + 2, \ldots, l, \quad \alpha = \pm 1. \tag{3.16}
\]

The \( U(1) \) charge is given by:

\[
q_{l,m,\alpha}^l = \frac{2m - \alpha p}{4(p + 2)}
\]

and the conformal dimensions are:

\[
\Delta_{l,m,\alpha}^l = \frac{l(l + 2)}{4(p + 2)} - \frac{(m + \alpha)^2}{4(p + 2)} + \frac{1}{8}.
\]

The fields (3.16) reproduce the right analytic behaviour of the supercurrents. For example:

\[
G^+(z_1)R_{l,m,\alpha}^l(z_2) = \sqrt{\frac{2p}{p + 2}} \frac{1}{z^{1+\alpha/2}_{12}} \phi_{m+2}^l(z_2) e^{i\left(\frac{m+2+(2-\alpha)p/2}{\sqrt{2p(p+2)}}\right)\varphi(z_2)} + \ldots
\]

shows that the fields \( R_{l,m,\alpha}^l \) produce the right branch cut singularities of the supercurrent \( G^+(z) \). Similar result holds also for the OPE with with \( G^-(z) \). The expressions for the \( U(1) \) charge and the dimension of the \( R \) fields coincide with those of the \( N = 2 \) minimal models with the following identification:

\[
s = m, \quad n = p - l + 1, \quad r = -\alpha
\]

for the series \( R_{n1}^{rs} \) and:

\[
s = m, \quad n = l - |m| + 1, \quad r = -\alpha
\]

for the second one \( R_{nb}^{rs} \).

In order to construct the twisted fields we have to consider the product of a primary field \( \varphi^{(s)}(z) \) from the so called C-disorder sector of the PF models and the twisted field \( \sigma_0^\xi(z) \) representing the lowest weight state of the \( Z_2 \) twisted current \( \partial \varphi(z) \):

\[
T_{\lambda}(z) = \varphi^{(s)}(z) \sigma_0^\xi(z). \tag{3.17}
\]

The fields \( \varphi^{(s)} \) are characterized by their OPE’s with the PF currents:

\[
\psi_1(z) \varphi^{(s)}(0) = z^{-\Delta_1} \sum_{n \in \mathbb{Z}} z^{-n/2} A_{n/2}^{(1)} \varphi^{(s)}(0)
\]

and have dimensions:

\[
\Delta^{(s)} = \frac{p - 2 + (p - 2s)^2}{16(p + 2)}, \quad s = 0, 1, \ldots |p/2|.
\]
The $\mathbb{Z}_2$ twisted field $\sigma_0^\varphi(z)$ has a dimension $1/16$ and is defined by the OPE:

$$\partial \varphi(z) \sigma_0^\varphi(0) = \frac{1}{\sqrt{2}} \sigma_1^\varphi(0) + \ldots$$

As it can be easily seen, the construction (3.17) leads to the well known dimension of the twisted fields of the discrete unitary series:

$$\Delta_s = \frac{(p-2s)^2 - 4}{16(p+2)} + \frac{1}{8}, \quad s = 0, 1, \ldots |p/2|$$

and indeed reproduces the correct branch cut of the $U(1)$ current:

$$J(z_1) T_s(z_2) = \frac{1}{\sqrt{z_{12}}} \sqrt{\frac{p}{2(p+2)}} i^{s+\frac{1}{2}} \varphi(z_2) + \ldots$$

where $t^{s+1/2}_{\Delta+1/2}(z) = \varphi^{(s)}(z) \sigma_1^\varphi(z)$.

In order to derive the FR’s of the NS and R fields we need the corresponding FR’s of the PF fields. The latter can be obtained using the relation between PF fields and the primary fields of the $su(2)$ WZW theory [43]:

$$\Phi_{jm}^l(z) = \phi_{lm}^j(z) e^{i\sqrt{j} \varphi(z)}.$$

Then the FR’s of the PF and consequently of the fields in the $N = 2$ superconformal theory follow directly from the known FR’s of the fields in the WZW theory.

Investigating the FR’s in the NS sector one must keep attention that they have more complicated structure due to the fact that there exist three different 3-point functions of the NS superfields - one even and two odd ones. The meaning of the odd FR’s in terms of component fields is that in the product of two first components of given superfields the second component of the RHS superfield appears. Taking all this into account we obtain the following FR’s in the NS sector:

$$N_{m_1}^{l_1} N_{m_2}^{l_2} = \sum_{l=|l_1-l_2|}^{L} [\Psi^l_m],
\quad L = \min (l_1 + l_2, 2p - l_1 - l_2)$$

where:

$$\Psi_m^l = (N_{m_1+m_2}^l)^{even}, \quad |m_1 + m_2| \leq l,$$

$$\Psi_m^l = (N_{m_1+2m_2+2(p+2)}^{l-p})^{odd}, \quad |m_1 + m_2| > l.$$
R fields:

\[ R_{m_1,\alpha}^l R_{m_2,-\alpha}^l = \sum_{l=|l_1-l_2|}^L [\Psi_{m_1+m_2}^l], \quad (3.20) \]

\[ R_{m_1,\alpha}^l R_{m_2,\alpha}^l = \sum_{l=|l_1-l_2|}^L [\Psi_{m_1+m_2-\alpha}^{p-l}], \]

where:

\[ \Psi_m^l = (N_m^l), \quad |m| \leq l, \]

\[ \Psi_m^l = (N_m^{p-l})^{\mp}, \quad |m| > l. \]

In the twisted sector the situation is more complicated. The product of two twisted \( U(1) \)
fields reproduces the exponents of the corresponding scalar field with the allowed charges.

The exact FR’s in the C-disorder PF sector however are not known exactly. We will not
need the explicit FR’s in the twisted sector in what follows so we omit the details.

We now turn to the computation of the 4-point correlation functions. In view of the
construction presented above the latter can be expressed in terms of of the 4-point functions
of the corresponding \( su(2) \) fields \( \Phi_m^j \). The most general correlation function of these
fields is calculated in [63]. It is proportional, up to a standard powers of \( z \), to a function
\( V_{j_1j_2j_3j_4}(z, \bar{z}; x, \bar{x}) \) with:

\[ z = \frac{2\alpha_j z_k}{\alpha_1 z_1^2} \] and similarly for \( x \) (here \( x \) is the isospin variable). This
function has the form:

\[ V_{j_1j_2j_3j_4} = N(j_1 \ldots j_4)|z|^{\frac{4j_1+j_2}{p+2}}|1-z|^{\frac{4j_3+j_4}{p+2}} \int \prod_{l=1}^{2j_1} dt_l \bar{t}_l |t_l - z|^{-\frac{2j_3}{p+2}} \times \]

\[ \times |t_l|^{-\frac{2j_3}{p+2}} |1-t_l|^{-\frac{2j_3}{p+2}} |x-t_l|^2 \prod_{i<j} |t_i - t_j|^{-\frac{4}{p+2}}, \]

\[ \beta_1 = j_1 + j_2 + j_3 + j_4 + 1, \quad \beta_2 = p + j_1 + j_2 - j_3 - j_4 + 1, \]

\[ \beta_3 = p + j_1 - j_2 + j_3 - j_4 + 1. \]

The constant \( N(j_1 \ldots j_4) \) is given by:

\[ N^2(j_1 \ldots j_4) = \left( \frac{\Gamma(\frac{1}{p+2})}{\Gamma(\frac{p+1}{p+2})} \right)^{4j_1+2} \frac{\Gamma(1-\frac{2j_1+1}{p+2})P^2(j_1 + j_2 + j_3 + j_4 + 1)}{\Gamma(\frac{2j_1+1}{p+2})P^2(2j_1)} \times \]

\[ \times \prod_{n=2}^4 \frac{\Gamma(1-\frac{2j_n+1}{p+2})}{\Gamma(\frac{2j_n+1}{p+2})} \frac{P^2(-j_1 + j_2 + j_3 + j_4 - 2j_n)}{P^2(2j_n)}, \]

where:

\[ P(j) = \prod_{n=1}^j \frac{\Gamma(\frac{n}{p+2})}{\Gamma(1-\frac{n}{p+2})}. \]
In two cases: \( j_1 = 1/2 \) and \( j_4 = \frac{p-1}{2} \) the integral in (3.21) can be expressed in terms of hypergeometric functions. Using the PF construction of the NS and R fields we can write down their most general 4-point function.

From the 4-point functions we can also extract the structure constants of the OPE algebra. They appear in the explicit form of the FR’s. In the NS case we have (we introduce here also the \( \bar{z} \) dependence):

\[
N^l_{m_1 \bar{m}_1}(z_1, \bar{z}_1)N^l_{m_2 \bar{m}_2}(z_2, \bar{z}_2) = \sum_l \sum_{m, \bar{m} = -l} C \left( \begin{array}{ccc} l & m & \bar{m} \\ l_1 & m_1 & \bar{m}_1 \\ l_2 & m_2 & \bar{m}_2 \\ l_3 & m_3 & \bar{m}_3 \end{array} \right) |z_{12}|^{2(\Delta_1 - \Delta_1 - \Delta_2)} N^l_{m \bar{m}}(z_2, \bar{z}_2) + \ldots.
\]

The meaning of the field \( N^l_{m \bar{m}} \) in the RHS is clear from the FR’s (3.19). The constant \( C \) in the above expression exactly coincides with the corresponding structure constant and is given by:

\[
C \left( \begin{array}{ccc} l & m & \bar{m} \\ l_1 & m_1 & \bar{m}_1 \\ l_2 & m_2 & \bar{m}_2 \\ l_3 & m_3 & \bar{m}_3 \end{array} \right) = \frac{\rho^2}{(l_1 + 1)(l_2 + 1)(l_3 + 1)} \frac{\Gamma(p+3)}{\Gamma(p+2)} \prod_{k=1}^{3} \frac{\Gamma(1 + \frac{l_k + 1}{p+2})}{\Gamma(1 + \frac{l_k + 1}{p+2})} \tilde{P}^2(l_1 + l_2 + l_3 + 1) \prod_{k=1}^{3} \frac{\tilde{P}^2(l_1 + l_2 + l_3 - l_k)}{\tilde{P}^2(l_k)}
\]

where the first two coefficients are the 3j-Wigner symbols and:

\[
\tilde{P}(l) = \prod_{k=1}^{l} \Gamma(1 + \frac{k}{p+2}) \Gamma^{-1}(1 - \frac{k}{p+2}).
\]

Exactly the same procedure goes also for the R sector. The structure constants are defined here by:

\[
R^{l_1}_{m_1 \bar{m}_1, \alpha}(1) R^{l_2}_{m_2 \bar{m}_2, -\alpha}(2) = \sum_l \sum_{m, \bar{m} = -l} C \left( \begin{array}{ccc} l & m & \bar{m} \\ l_1 & m_1 & \bar{m}_1 \\ l_2 & m_2 & \bar{m}_2 \end{array} \right) |z_{12}|^{2(\Delta_1 - \Delta_1 - \Delta_2)} N^{l_1}_{m \bar{m}}(2) + \ldots
\]

\[
R^{l_1}_{m_1 \bar{m}_1, \alpha}(1) R^{l_2}_{m_2 \bar{m}_2, \alpha}(2) = \sum_l \sum_{m, \bar{m} = -l} C \left( \begin{array}{ccc} l & m & \bar{m} \\ l_1 & m_1 & \bar{m}_1 \\ l_2 & m_2 & \bar{m}_2 \end{array} \right) |z_{12}|^{2(\Delta_1 - \Delta_1 - \Delta_2)} N^{l_1}_{m \bar{m}}(2) + \ldots
\]

and coincide with (3.22) (which NS field appear above is dictated again by the FR’s (3.20)).

We are going now to discuss in more details the discrete symmetries of the \( N = 2 \) minimal models which we will need in the next Section. The original \( Z_p \) symmetry, due to the PF construction, is lost because of the presence of the bigger \( U(1) \) symmetry. It is known however [18] that these models possess a bigger \( Z_{p+2} \) discrete symmetry. In our considerations this
fact can be explained as follows. Consider the special superfields \( N_{p}^{m} \) (i.e. \( l = p \)). They form a closed OPE algebra as it can be seen from the FR’s obtained above:

\[
N_{p}^{m_{1}} N_{p}^{m_{2}} = \left[ N_{p+2-(m_{1}+m_{2})}^{p} \right]^{\text{odd}}.
\]

Denoting all the fields of this type as \( N_{p}^{-(p+2)+2k} \), \( k = 1, 2, \ldots, p + 1 \) we can consider these FR’s as a multiplication law of the discrete group \( Z_{p+2} \) for representations with charges \( k_{1} \) and \( k_{2} \). Therefore these superfields (of fractional dimension \( \Delta_{k} = k(p+2-k)/(p+2) - 1/2 \)) generate a PF type symmetry of the \( N = 2 \) minimal models.

To make this statement more clear let us consider the \( N = 2 \) PF theory with a \( Z_{p+2} \) discrete symmetry. It contains, in addition to the \( N = 2 \) super stress-energy tensor \( W(z, \theta^{+}, \theta^{-}) \) (which includes \( T, J, G^{\pm}(z) \)), the PF supercurrents \( \Psi_{k}(z, \theta^{+}, \theta^{-}) \) carrying \( Z_{p+2} \) charge \( k = 1, 2, \ldots, p + 1 \) with dimension \( \Delta_{k} \) and \( U(1) \) charge \( q_{k} \). According to the \( Z_{p+2} \) symmetry they should close the following algebra:

\[
\Psi_{k_{1}}(z_{1}, \theta_{1}^{+}, \theta_{1}^{-}) \Psi_{k_{2}}(z_{2}, \theta_{2}^{+}, \theta_{2}^{-}) = C_{k_{1},k_{2}} \tilde{z}^{\Delta_{k_{1}}+k_{2}-\Delta_{k_{1}}-\Delta_{k_{2}}-1/2} D_{1} \Psi_{k_{1}+k_{2}}(z_{2}, \theta_{2}^{+}, \theta_{2}^{-}) + \ldots,
\]

\[
\Psi_{k}(z_{1}, \theta_{1}^{+}, \theta_{1}^{-}) \Psi_{k}^{\dagger}(z_{2}, \theta_{2}^{+}, \theta_{2}^{-}) = \tilde{z}^{-2\Delta_{k}} + \tilde{z}^{-2\Delta_{k}+1} D_{2} W(z_{2}, \theta_{2}^{+}, \theta_{2}^{-}) + \ldots
\]

where \( D_{1} \) and \( D_{2} \) are some \( N = 2 \) super-covariant derivatives (which can be explicitly constructed) and \( \Psi_{k}^{\dagger} \) is the conjugate of \( \Psi_{k} \) with \( Z_{p+2} \) charge \( p+2-k \), dimension \( \Delta_{k} \) and \( U(1) \) charge \( -q_{k} \).

These superconformal OPE’s are consistent with the \( Z_{p+2} \) symmetry if the dimensions \( \Delta_{k} \) obey certain monodromy condition. The elementary solution of the corresponding equation is:

\[
\Delta_{k} = k(p+2-k)/(p+2) - 1/2.
\]

In plus, we add the requirement that the \( Z_{p+2} \) symmetry is consistent with the bigger \( U(1) \) symmetry coming from the \( N = 2 \) superconformal algebra. This imposes a condition on the possible \( U(1) \) charges of the parafermions. Its simplest solution is:

\[
q_{k} = \frac{k}{p+2} - \frac{1}{2}.
\]

We note that the dimensions and charges of the super-parafermions given above coincide with those of the fields \( N_{p}^{-(p+2)+2k} \). This supports the suspect that these fields indeed generate a \( Z_{p+2} \) PF symmetry in the \( p \)-th minimal \( N = 2 \) superconformal model. To complete this discussion we need also to compute the central charge of the \( N = 2 \) PF theories. It turns out that it is exactly equal to:

\[
c = \frac{3p}{p+2},
\]

i.e. the central charge of these super-parafermionic models coincides with the central charge of the \( p \)-th \( N = 2 \) minimal model. This completes the proof of the equivalence between \( Z_{p+2} \) \( N = 2 \) super-parafermionic theories and \( p \)-th \( N = 2 \) minimal models.
At the end, we would like also to describe the spectrum of the super PF theory. It is clear that it should have NS and R order parameters corresponding to the different choices of boundary conditions for the supercurrents $G^{\pm}(z)$. The OPE of these order parameters with the PF currents is defined by their monodromy properties and is well known. We compare the latter with the OPE’s in the $\mathcal{N}=2$ minimal models:

$$
N_{-p}^p(z)N_k^k(0) = z^{-k/p+2}(N_{k+2}^k)^{II} + \ldots
$$

$$
N_{-p}^{p}(z)N_{-k}^{k}(0) = z^{k/p+2}(N_{k-2}^k)^{II} + \ldots
$$

where, according to our identification, the field $N_{-p}^p(z)$ corresponds to the first component of the super-parafermionic current $\Psi_1(z)$. The above OPE’s suggest that the chiral superfields $N_k^k$ and $N_{-k}^{k}$ are the NS order parameters with dimensions and charges:

$$
d_k = \frac{k}{p+2} = q_k, \quad \text{for} \quad N_{k}^{k} = \sigma_{NS}^{k},
$$

$$
d_k = \frac{k}{p+2} = -q_k, \quad \text{for} \quad N_{-k}^{k} = \sigma_{NS}^{k}.\n$$

Analogous calculation in the R sector leads to:

$$
N_{-p}^{p}(z)R_{k-1,1}^{k-1}(0) = z^{-k/p+2+1/2}(R_{k+1,1}^{k-1})^{II} + \ldots
$$

$$
N_{p}^{p}(z)R_{k-1,1}^{k-1}(0) = z^{k/p+2-1/2}(R_{k-3,1}^{k-1})^{II} + \ldots.
$$

Therefore, the fields that represent the order parameters of $Z_{p+2}$ charge $k$ are given in the minimal models by $R_{k-1,1}^{k-1}, k = 1, 2, \ldots, p + 1$. These are in fact all the Ramond fields with lowest dimension and $U(1)$ charge:

$$
d_k = \frac{c}{24}, \quad q_k = \frac{k}{2(p+2)} = \frac{1}{4}.
$$

We thus obtained that the spectrum of the $\mathcal{N}=2$ PF theories is given by the above expressions in the NS and R sectors. The other primary fields in the minimal models correspond to the descendants of the above primary fields of the PF theories with respect to the PF currents. Finally, we note that the fields in the twisted sector correspond to the C-disorder sector of the $\mathcal{N}=2$ super-PF theories.

At the end of this Section we would like to discuss the renormalization group properties of the $\mathcal{N}=2$ minimal models. In other words we would like to describe the RG flow of these models perturbed by the least relevant field. In the case of $\mathcal{N}=2$ minimal models the latter is constructed from the chiral and antichiral fields $N_{\pm}^{p}$ of dimension $\Delta = 1/2 - 1/(p+2)$ and $U(1)$ charge $q = \pm \Delta$. The suitable perturbation term, neutral and of dimension close to one, is therefore constructed out of the second components of such chiral fields. Explicitly we consider:

$$
\mathcal{L} = \mathcal{L}_0 + \int d^2z \Phi(z)
$$
where $\mathcal{L}_0$ represents the minimal model itself and the field $\Phi(z)$ is a combination of the second components:

$$\Phi = (N_p^p)^{II} + (N_{p-p}^p)^{II} \equiv \phi_+ + \phi_-.$$ 

It is neutral and has a dimension $\Delta = 1 - 1/(p+2) = 1 - \epsilon$. Similarly to what we did for the $N = 1$ superconformal theories, we consider the case $p \to \infty$ and assume $\epsilon = 1/(p+2)$ to be a small parameter. Also, according to our parafermionic construction, we can express the perturbing field in terms of the PF currents and exponents of the scalar field as follows:

$$(N_p^p)^{II} = \sqrt{\frac{2p}{p+2}} \psi_1^\dagger e^{-i \frac{2}{\sqrt{2p(p+2)}}} \equiv \phi_+,$$

$$(N_{p-p}^p)^{II} = \sqrt{\frac{2p}{p+2}} \psi_1 e^{i \frac{2}{\sqrt{2p(p+2)}}} \equiv \phi_-.$$ 

Our purpose now is to compute the beta-function of this theory and to check for an eventual fixed point. For that we need to compute the two-point function of the perturbing field up to a second order. The expansion was already written in (2.58). As in the case of $N = 1$ theory we need the 3- and 4-point functions of the perturbing field. We note that, due to the FR’s computed above, the 3-point function of the field $\Phi(z)$, and therefore the first term in (2.58), is identically zero. So we are left with the computation of the second order term only. This computation goes along the same lines as in the $N = 1$ case. We need to compute the 4-point function of $\Phi(z)$ up to zeroth order in $\epsilon$ and to integrate it in the safe region $\Omega_{l,l_0}$ far from the singularities. Near the singular points 0, 1 and $\infty$ we use the OPE’s that we computed above.

The 4-point function of the perturbing field $\Phi(z)$ is expressed through the corresponding functions of the parafermionic fields which are known [43] and the trivial power-like contribution of the exponents. The final result is (up to zeroth order in $\epsilon$):

$$< \Phi(x)\Phi(0)\Phi(1)\Phi(\infty) > = C|1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}|^2$$

where $C$ is some structure constant. We will not need its explicit expression here. The integration of this function over the safe region gives:

$$\frac{2\pi^2}{\epsilon} \left( \frac{31}{16} + \frac{1}{l^2} + \frac{1}{4l_0^2} \right).$$

From this we have to subtract the contribution of the lens-like region:

$$\frac{\pi^2}{\epsilon} \left( \frac{31}{16} - \frac{1}{l^2} + \frac{1}{2l_0^2} \right).$$

At the end, we add the result of the integration near the singular points:

$$2 \left( -\frac{\pi^2}{l^2\epsilon} + \frac{2\pi^2}{\epsilon} \left( -\frac{1}{2l^2} + \frac{1}{2l_0^2} \right) \right).$$
corresponding to the integrals around 0 (and 1) and $\infty$ respectively. Summing all the contributions we get finally as a result:

$$\frac{\pi^2}{\epsilon l_0^2}.$$  

Two comments are in order. First, this result contains only the cut-off parameter and could be cancelled by adding an appropriate counterterm in the action. Second, the finite contribution is identically zero. This means that there is no contribution to the beta-function neither in the first nor in the second order. One can speculate that this is the case also in higher orders. This result leads us to the conclusion that there do not exist a nontrivial fixed point of the beta-function close to the UV one. If such a fixed point exists it should be due to some non-perturbative effects.

### 3.2 Yukawa couplings for the three-generation string model

Let us describe briefly the derivation of the Yukawa couplings for the special C-Y manifold constructed as a hypersurface in $\mathbb{C}P^2 \times \mathbb{C}P^3$ [64]. It is defined by the zeroes of the polynomials:

\[
\begin{align*}
P_1 &= z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0, \\
P_2 &= z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0,
\end{align*}
\]

$z_i \in \mathbb{C}P^3, \quad x_i \in \mathbb{C}P^2$.

It represents a C-Y manifold of complex dimension 3 and Euler characteristic $\chi = -54$. We are particularly interested in the discrete symmetries of this manifold given by:

(a) $S_3$ group of permutations of the indexes $i = 1, 2, 3$: $z_i \rightarrow z_{p(i)}$, $x_i \rightarrow x_{p(i)}$;

(b) $Z_3 \times Z_9^3$ spanned by the transformations:

\[
\begin{align*}
z_0 &\rightarrow \exp{(2\pi i r_0/3)} z_0, \quad z_i \rightarrow \exp{(2\pi i r_i/3)} z_i, \\
x_i &\rightarrow \exp{(-2\pi r_i/9)} x_i.
\end{align*}
\]

The irrelevance of the overall phase in the discrete transformations of the polynomials determines the full group of the global discrete symmetries of this C-Y space to be $G = S_3 \times Z_3 \times Z_9^3/Z_9$. We denote the charges of the different objects under $Z_3 \times Z_9^3$ as a vector $m = (m_0, m_1, m_2, m_3)$.

It is known that the number of generations in the case of interest can be found enumerating the independent deformations of the complex structure of the corresponding C-Y manifold. They are given by all the possible (homogeneous) deformations of the defining polynomials. It turns out that they fit into 9 "families". Taking also into account their possible $Z_3 \times Z_9^3$ charges we have 35 independent polynomials and hence this C-Y sigma model has 35 generations. From the topological properties it follows that there are also 8 antigenerations.
The Yukawa couplings of the massless matter fields are given by some topological formula. The invariance condition with respect to the discrete $G$-transformations determines them (up to an appropriate normalization of the matter superfields) to be:

$$\lambda_{ijk} = \delta_{\text{mod}(3)}(2 + m_i^0 + m_j^0 + m_k^0) \prod_{r=1}^{3} \delta_{\text{mod}(9)}(2 + m_i^r + m_j^r + m_k^r)$$  \hspace{1cm} (3.24)

where \( \{m_i^r\} \) are the charges of the \( i \)-th polynomial. According to our normalization convention all the Yukawa couplings are either 1 or 0.

We now pass to the explicit evaluation of the Yukawa couplings in the framework of the \( N = 2 \) superconformal model \( 1^{163} \) describing the corresponding C-Y manifold \cite{18}. The compactified part \( (c = 9) \) of this model contains the tensor product (moded out by \( G \)-projection) of the \( N = 2 \) superconformal minimal models with \( p = 1 \) \( (c = 1) \) and three copies of \( p = 16 \) \( (c = 8/3) \) \cite{18}. Each of these models possesses \( Z_{p+2} \) discrete symmetry originated from its superparafermionic structure that we described in the previous Section. Then the total discrete symmetry group of the model \( 1^{163} \) is \( Z_3 \times Z_3^3 \). The co-compactified part of the corresponding massless string vertices is constructed in terms of the superparafermionic order parameters (we list them again here for convenience):

\[
\begin{align*}
NS & : \sigma_k^N = N_k, \quad \Delta_k = \frac{k}{2(p+2)}, \quad q_k = \frac{k}{2(p+2)} \\
R & : \sigma_{k+1}^R = R_k, \quad \Delta_k = \frac{c}{24}, \quad q_k = \frac{k + 1}{2(p+2)} - \frac{1}{4}
\end{align*}
\]

with \( Z_{p+2} \) charge \( k \), and:

\[
\begin{align*}
NS & : \sigma_k^N = N_k, \quad \Delta_k = \frac{k}{2(p+2)}, \quad q_k = \frac{k}{2(p+2)} \\
R & : \sigma_{k+1}^R = R_k, \quad \Delta_k = \frac{c}{24}, \quad q_k = \frac{k + 1}{2(p+2)} - \frac{1}{4}
\end{align*}
\]

with \( Z_{p+2} \) charge \( k + 1 \). Due to the \( G \)-projection only fields with even \( Z_{18} \) charge appear in the massless spectrum of the \( 1^{163} \) model and therefore the effective discrete group becomes \( Z_3 \times Z_3^3 \). The composite model obeys a new symmetry \( S_3 \) of permutation of the \( p = 16 \) models. Finally, the condition for an integer \( U(1) \) charge of the composite fields implies that the element \( g_0 = (1, 1, 1, 1) \in Z_3 \times Z_3^3 \) acts trivially. Therefore the full group of discrete symmetries of the \( 1^{163} \) model exactly coincides with the one of the C-Y model considered above, i.e. \( S_3 \times Z_3 \times Z_3^3 / Z_9 \).

According to the Gepner’s construction the massless matter fields appear as \( N = 1 \) space-time superfields in the \( 27 \) (for the generations) or in \( \overline{27} \) (for the antigenerations) of \( E_6 \). In fact the spinor \( 16(q) \) vector \( 10(q) \) and scalar \( 1(q) \) representations of \( SO(10) \times U(1) \) combine into the \( 27 \) and \( \overline{27} \) of \( E_6 \):

$$\begin{align*}
27 & = 1_{(-1)} \oplus 16_{(-1/4)} \oplus 10_{(1/2)}; \\
\overline{27} & = 1_{(1)} \oplus 16_{(1/4)} \oplus 10_{(-1/2)}.
\end{align*}$$

Let us discuss first the generations. The vertex operators for the space-time spinor component of the matter superfield \( S_a \) in \( 1_{(-1)} \) of \( SO(10) \times U(1) \) is given by:

$$\mathcal{V}_1(z, \bar{z}) = \left( R^k \prod_{i=1}^{3} R^l_i \right) (z) \left( N^m \prod_{i=1}^{3} N^l_i \right) (\bar{z})$$  \hspace{1cm} (3.25)
satisfying the following conditions:

1. \(6k + \sum l_i = 18\), i.e. \(U(1)\) charge to be \(q = -1/4\),

2. \(6m + \sum n_i = 36\), i.e. \(U(1)\) charge to be \(q = -1\),

3. \((m + k) \mod (3) = (n_i + l_i) \mod (9)\), \(i = 1, 2, 3\),

4. the left \((k, l_i)\)–right \((m, n_i)\) 2D constructions for each individual model are restricted by the exceptional modular invariant at level \(k = 16\) of the underlying \(su(2)\) Kac-Moody algebra \([18]\).

The corresponding spinor components of \(S_a\) in \(16_{(-1/4)}\) and \(10_{(1/2)}\) have a similar form:

\[
\begin{align*}
\mathcal{V}_{16}(z, \bar{z}) &= (R^1_1(R_{16}^{16})^3)(\bar{z})\mathcal{V}_1(z, \bar{z}), \\
\mathcal{V}_{10}(z, \bar{z}) &= (R^1_1(R_{16}^{16})^3)(\bar{z})\mathcal{V}_{16}(z, \bar{z}).
\end{align*}
\]

(3.26)

The vertex operators for the scalar components of \(S_a\) can be obtained by acting on the spinor ones with the space-time SUSY charge \(Q\) using the 2D OPE’s.

In order to compare the geometrical description of the model with the algebraic \((1^116^3)\) one we have to make a correspondence between the vertices \((3.25)\) and \((3.26)\) and the polynomials comparing their \(Z_3 \times Z_3^9\) charges. As argued in \([18]\) the charges \(\{Q_k\}\) to be compared are those of the scalars \(\mathcal{V}_{10}^{\text{scalar}}\) normalized as follows:

\[
m_0 = Q_0(\mod(3)), \quad m_i = 2Q_i(\mod(9)), \quad i = 1, 2, 3.
\]

(3.27)

In the table below the nine ”families” are represented by the spinor vertices \(\mathcal{V}_1(z, \bar{z})\) and the relevant charges of \(\mathcal{V}_{10}^{\text{scalar}}(z, \bar{z})\) and we used the notation \(R(0~ 6~ 6~ 6)N_{\pm}(1~ 10~ 10~ 10)\) for \((R^0_6(R_{10}^6)^3)(z)(N_{\pm 1}^1(N_{\pm 10}^{10})^3)(\bar{z})\), etc.

| "Family" | \(\mathcal{V}_1(z, \bar{z})\) | \(Z_3 \times Z_3^9\) charges of \(\mathcal{V}_{10}^{\text{scalar}}\) |
|-----------|----------------|-----------------------------|
| 1 | \(R(1~ 12~ 0~ 0)N_{-}(0~ 4~ 16~ 16)\) | \(1~ 6~ 0~ 0\) |
| 2 | \(R(1~ 8~ 4~ 0)N_{-}(0~ 8~ 12~ 16)\) | \(1~ -2~ -1~ 0\) |
| 3 | \(R(1~ 6~ 6~ 0)N_{-}(0~ 10~ 10~ 16)\) | \(1~ 3~ 3~ 0\) |
| 4 | \(R(1~ 4~ 4~ 4)N_{-}(0~ 12~ 12~ 12)\) | \(1~ -1~ -1~ -1\) |
| 5 | \(R(0~ 12~ 6~ 0)N_{-}(1~ 4~ 10~ 16)\) | \(0~ 3~ 6~ 0\) |
| 6 | \(R(0~ 10~ 8~ 0)N_{-}(1~ 6~ 8~ 16)\) | \(0~ 2~ -2~ 0\) |
| 7 | \(R(0~ 10~ 4~ 4)N_{-}(1~ 6~ 12~ 12)\) | \(0~ 2~ -1~ -1\) |
| 8 | \(R(0~ 8~ 6~ 4)N_{-}(1~ 8~ 10~ 12)\) | \(0~ 3~ -2~ -1\) |
| 9 | \(R(0~ 6~ 6~ 6)N_{-}(1~ 10~ 10~ 10)\) | \(0~ 3~ 3~ 3\) |

The condition of \(SO(10) \times U(1)\) invariance for the cubic superpotential restricts its form
as follows [65]:

\[ W = \sum_{ijk} (\lambda^{(1)}_{ijk} S^i_{10(-1/4)} S^j_{10(-1/4)} S^k_{10(1/2)} + \lambda^{(2)}_{ijk} S^i_{10(1/2)} S^j_{10(1/2)} S^k_{1(-1)}) \]

\((i, j, k)\) are the family indexes of the chiral superfields \( S^a \). The explicit construction of the vertices (3.25) and (3.26) and the specific properties of the fields \( R^{1}_{1} \) and \( R^{16}_{10} \), discussed in the previous section lead to the following important equality between the Yukawa couplings:

\[ \lambda^{(1)}_{ijk} = \lambda^{(2)}_{ijk}(\equiv \lambda_{ijk} = <ijk>). \]

The latter can be expressed as products of the 2D OPE structure constants of the \( N = 2 \) fields from the compactified part of the vertices \( V_{16}(z, \bar{z}) \) (see (3.26) and the above table):

\[ \lambda_{ijk} = N_{ijk} \delta(Q^0_i + Q^0_j + Q^0_k - 1) \prod_{r=1}^{3} \delta(Q^r_i + Q^r_j + Q^r_k - 16), \quad (3.28) \]

The \( \delta \)-function part in (3.28) is a direct consequence of the \( N = 2 \) fusion rules derived in the previous Section. The three-point functions:

\[ <\sigma^i_{n_1l}(\infty)\sigma^j_{n_2l}(1)\sigma^{k\dagger}_{n_1+n_2l}(0)> \]

represent the structure constants of the underlying parafermionic model. Putting together all these ingredients we get the following values for the non-vanishing Yukawa couplings:

\[ <994> = k_1^3, <973> = k_1^2 = <883>, \]
\[ <882> = k_1k_2, <884> = k_1k_2^2, \]
\[ <872> = k_2^2, <862> = k_2, \]
\[ <861> = <751> = <663> = <652> = 1, \]
\[ <852> = <554> = k_1 \]

where:

\[ k_1^2 = \frac{\Gamma(\frac{1}{18})\Gamma(\frac{13}{18})\Gamma(\frac{11}{18})^2}{\Gamma(\frac{17}{18})\Gamma(\frac{5}{18})\Gamma(\frac{7}{18})^2}, \]
\[ k_2^2 = \frac{\Gamma(\frac{1}{18})\Gamma(\frac{13}{18})^2}{\Gamma(\frac{17}{18})\Gamma(\frac{5}{18})}. \]

Let us compare now these algebraic results with the quasi-topological ones (3.24). Taking into account the normalization condition (3.27) it is easy to see that the Yukawa couplings of the C-Y model (3.24) and of the \( 1^116^3 \) Gepner model (3.28) are equal up to the constants
We can absorb these constants in the normalization of the corresponding polynomials. However, we have 9 families and 14 non-zero couplings and therefore the normalization should satisfy a non-trivial consistency condition. It turns out that in our case this condition is satisfied and the proper normalization can be chosen in the form:

\[
\begin{align*}
9^G &= k_1^{4/3}k_2^{-2/3}g_{C-Y}, & 8^G &= k_1^{1/3}k_2^{1/3}g_{C-Y}, \\
7^G &= k_1^{-2/3}k_2^{4/3}g_{C-Y}, & 6^G &= k_1^{2/3}k_2^{1/3}g_{C-Y}, \\
5^G &= k_1^{1/3}k_2^{-2/3}g_{C-Y}, & 4^G &= k_1^{1/3}k_2^{1/3}g_{C-Y}, \\
3^G &= k_1^{4/3}k_2^{-2/3}g_{C-Y}, & 2^G &= k_1^{1/3}k_2^{1/3}g_{C-Y}, \\
1^G &= k_1^{1/3}k_2^{-2/3}g_{C-Y}.
\end{align*}
\] (3.31)

Then all the couplings (3.24) and (3.28) exactly coincide.

In the case of antigenerations the vertex representing the spinor component in \(1(1)\) of \(SO(10) \times U(1)\) can be taken in the form:

\[
\tilde{V}_1(z, \bar{z}) = \left( R_k^3 \prod_{i=1}^{3} R_i^l \right) (z) \left( N_m^3 \prod_{i=1}^{3} N_n^m \right) (\bar{z}).
\] (3.32)

The conditions (1) and (3) below (3.26) remain unchanged but in the second we replace \(k\) to \(-k\) and \(l_i\) to \(-l_i\). The scalar component of the superfield is again obtained by acting with the space-time supercharge. The spinor vertices \(\tilde{V}_{16}\) and \(\tilde{V}_{10}\) are realized in terms of (3.32) as follows:

\[
\begin{align*}
\tilde{V}_{16}(z, \bar{z}) &= (R_0^3(R_0^0)^3)(\bar{z}) \tilde{V}_1(z, \bar{z}), \\
\tilde{V}_{10}(z, \bar{z}) &= (R_0^3(R_0^0)^3)(\bar{z}) \tilde{V}_{16}(z, \bar{z}).
\end{align*}
\] (3.33)

All this leads to the construction of 4 families representing 8 antigenerations listed in the table below:

| "Family" | \(\tilde{V}_1(z, \bar{z})\) | \(\tilde{V}_{10}\) charges | Number of vertices |
|----------|----------------|-----------------|------------------|
| 1        | \(R(1, 8, 2, 2)N_+(1, 14, 8)\) | \(2, 6, 3, 3\) | 3 |
| 2        | \(R(0, 4, 4, 4)N_+(0, 12, 12, 12)\) | \(0, 0, 0, 0\) | 1 |
| 3        | \(R(0, 8, 8, 2)N_+(0, 14, 14, 8)\) | \(1, 6, 6, 3\) | 3 |
| 4        | \(R(0, 6, 6, 6)N_+(1, 10, 10, 10)\) | \(0, 0, 0, 0\) | 1 |

Applying the \(N = 2\) fusion rules we get only two allowed Yukawa couplings for the above families of antigenerations: \(<\overline{4}\overline{4}\overline{2}>\) and \(<\overline{4}\overline{3}\overline{1}>\). It is straightforward to compute the first one using the same procedure as for the generations since it is connected to the correlation function of scalar \((\Delta = \overline{\Delta})\) \(su(2)\) fields. There is an obstruction in the evaluation of \(<\overline{4}\overline{3}\overline{1}>>\) since it involves left-right asymmetric fields \(\Phi_{7,4}(z, \bar{z}) (l = 7, l = 4)\) and \(\Phi_{4,1}(z, \bar{z})\) of spins \(s_{7,4} = \Delta - \overline{\Delta} = 2\) and \(s_{4,1} = 1\). However, using the correlation functions constructed in [66], one can find the values of the structure constants and the corresponding Yukawa couplings:

\[
<\overline{4}\overline{4}\overline{2}> = k_1^3, \quad <\overline{4}\overline{3}\overline{1}>> = k_3^3
\] (3.34)
where:
\[ k_3^2 = \frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{11}{18}\right)\Gamma\left(\frac{17}{18}\right)}{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{1}{6}\right)}. \]  
(3.35)

Up to now we have calculated and compared the Yukawa couplings for the 27-generation C-Y model [64] and the corresponding tensor product Gepner model [18]. Our final goal is to compute the Yukawa couplings for the three-generation 1\textsuperscript{1}16\textsuperscript{3} model. As it is observed in [64, 18], the global automorphism group \( G \) contains a subgroup \( H = \mathbb{Z}_3 \times \mathbb{Z}_3 \) generated by the elements \( h \) and \( g \):

\[ h : \quad z_i \rightarrow z_{i+1}, \quad x_i \rightarrow x_{i+1}, \]
\[ g \quad = \quad (0, 3, 6, 0) \in \mathbb{Z}_3 \times \mathbb{Z}_9. \]  
(3.36)

The three generation model is obtained by factoring out the 27-generation model by this \( H = \mathbb{Z}_3 \times \mathbb{Z}_3 \) subgroup.

Let us consider first the action of the element \( g \). By simply projecting the spectrum onto \( g \)-invariant states we find that 17 generations survive and all the antigenerations as well. More detailed analysis shows that the subgroup spanned by \( g \) does not act freely on the C-Y manifold. Therefore we have to complete the spectrum by the corresponding twisted states. It turns out that the only family of (six) twisted generations that appears is represented by the following spinor vertices:

\[ \mathcal{V}_t^1(z, \bar{z}) = R(0 \ 8 \ 8 \ 2)(z)N_-(1 \ 8 \ 14 \ 8)(\bar{z}). \]

The vertices of the twisted family of 6 antigenerations are given by:

\[ \mathcal{V}_t^6(z, \bar{z}) = R(0 \ 12 \ 0 \ 6)(z)N_+(1 \ 4 \ 16 \ 10)(\bar{z}). \]

Then the total number of generations becomes 23 = 17 + 6, the antigenerations are 14 = 8 + 6 and therefore at this stage the model contains 9 = 23 − 14 net generations.

The next step is to divide by the subgroup generated by \( h \). Since \( h \) acts freely we have only to project the space-time onto \( h \)-invariant states. The result is as follows:

Generations (\( \mathcal{V}_1 \) vertices):

\[ L_1 = R(1 \ 12 \ 0 \ 0)N_-(0 \ 4 \ 16 \ 16) + c.p., \]
\[ L_2 = R(1 \ 6 \ 6 \ 0)N_-(0 \ 10 \ 10 \ 16) + c.p., \]
\[ L_3 = R(1 \ 4 \ 4 \ 4)N_-(0 \ 12 \ 12 \ 12), \]
\[ L_4 = R(0 \ 12 \ 6 \ 0)N_-(1 \ 4 \ 10 \ 16) + c.p., \]
\[ L_5 = R(0 \ 6 \ 12 \ 0)N_-(1 \ 10 \ 4 \ 16) + c.p., \]
\[ L_6 = R(0 \ 10 \ 4 \ 4)N_-(1 \ 6 \ 12 \ 12) + c.p., \]
\[ L_7 = R(0 \ 6 \ 6 \ 6)N_-(1 \ 10 \ 10 \ 10), \]
\[ L_8^t = R(0 \ 8 \ 8 \ 2)N_-(1 \ 8 \ 14 \ 8) + c.p., \]
\[ L_9^t = R(0 \ 8 \ 8 \ 2)N_-(1 \ 14 \ 8 \ 8) + c.p. \]

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Antigenerations (\(\bar{V}_1\) vertices):

\[
L_1 = R(1\ 8\ 2\ 2)N_+(1\ 14\ 8\ 8) + c.p., \quad (3.38)
\]
\[
L_2 = R(1\ 4\ 4\ 4)N_+(0\ 12\ 12\ 12),
\]
\[
L_3 = R(0\ 8\ 8\ 2)N_+(0\ 14\ 14\ 8) + c.p.,
\]
\[
L_4 = R(0\ 6\ 6\ 6)N_+(1\ 10\ 10\ 10),
\]
\[
L_5^t = R(0\ 12\ 0\ 6)N_+(1\ 4\ 16\ 10) + c.p.,
\]
\[
L_6^t = R(0\ 12\ 6\ 0)N_+(1\ 4\ 16\ 10) + c.p..
\]

It is obvious that the vertices (3.37) and (3.38) represent the massless spectrum of the three-generation model (#\(L - \#\bar{L} = 3\)).

The vertices (3.37) and (3.38) correspond to the trivial embedding of the subgroup \(H = Z_3 \times Z_3\) in \(E_6\) which leaves it unbroken, i.e. the entire 27 (27) contribute to the massless spectrum. As it is explained in [65], in the case of a non-trivial embedding (Wilson lines) only part of 27 (27) survives the compactification. In this case \(E_6\) is broken to \(SU_c(3) \times SU_L(3) \times SU_R(3)\). The 27 of \(E_6\) decomposes as \((1,3,\bar{3}) \oplus (\bar{3},3,1) \oplus (\bar{3},1,3)\). The superfields belonging to the color singlet \((1,3,\bar{3})\) contain leptons and Higgses while the color triplets \((3,\bar{3},1)\) and \((\bar{3},1,3)\) - the quarks and antiquarks. Therefore the vertices \(L_i\) and \(\bar{L}_i\) given by (3.37) and (3.38) represent the leptons and Higgses from generations and antigenerations.

The corresponding quark and antiquark vertices are given by:

\[
Q_1 = R(1\ 4\ 8\ 0)N_-(0\ 12\ 8\ 16) + c.p., \quad (3.39)
\]
\[
\bar{Q}_1 = R(1\ 8\ 4\ 0)N_-(0\ 8\ 12\ 16) + c.p.,
\]
\[
Q_2 = R(0\ 10\ 8\ 0)N_-(1\ 6\ 8\ 16) + c.p.,
\]
\[
\bar{Q}_2 = R(0\ 8\ 10\ 0)N_-(1\ 8\ 6\ 16) + c.p.,
\]
\[
Q_3 = R(0\ 8\ 6\ 4)N_-(1\ 8\ 10\ 12) + c.p.,
\]
\[
\bar{Q}_3 = R(0\ 6\ 8\ 4)N_-(1\ 10\ 8\ 12) + c.p..
\]

The explicit construction of the vertices (3.37), (3.38) and (3.39) together with the exact values (3.29), (3.34) of the Yukawa couplings lead to the following general expression for the low-energy cubic superpotential:

\[
W = \sum_{ijk} [\lambda_{ijk} L_i L_j L_k + \bar{\lambda}_{ijk} \bar{L}_i \bar{L}_j \bar{L}_k +
\]
\[
+ (\mu_{ijk} Q_i Q_j Q_k + h.c.) + \rho_{ijk} \bar{Q}_i \bar{Q}_j L_k],
\]

where the new couplings \(\lambda, \bar{\lambda}, \mu, \rho\) are linear combinations of the old ones (3.29), (3.34):

1. **Leptons and Higgses**

   a. Generations:

   \[
   \lambda_{773} = k_1^3, \quad \lambda_{641} = 1 = \lambda_{651},
   \]
   \[
   \lambda_{762} = k_1^2, \quad \lambda_{652} = \lambda_{642} = \frac{1}{3} \lambda_{543} = k_1.
   \]
(b) Antigenerations:
\[ \bar{\lambda}_{652} = 3k_1, \quad \bar{\lambda}_{442} = k_1^3, \quad \bar{\lambda}_{431} = k_3^3. \]

(2) Quarks
\[ \mu_{133} = k_1k_2 = \mu_{133}, \quad \mu_{123} = k_2 = \mu_{123}. \]

(3) Quarks-antiquarks-Higgses
\[ \rho_{323} = k_1^2, \quad \rho_{333} = 3k_1k_2, \]
\[ \rho_{315} = k_1, \quad \rho_{316} = k_2^2 = \rho 36, \]
\[ \rho_{231} = \rho_{125} = \rho_{231} = \rho_{222} = \rho_{124} = 1. \]

In conclusion, we mention some specific features of the effective low-energy superpotential (3.40) for the three-generation Gepner model with Planck scale group $SU_c(3) \times SU_L(3) \times SU_R(3)$:

(a) the absence of quark antigenerations,
(b) small number of quark selfcouplings,
(c) the absence of Yukawa interactions for the twisted generations of leptons-Higgses.

### 3.3 Duality in $N = 2$ super-Liouville theory

The action of the $N = 2$ SLFT at the flat background is given by:
\[
\mathcal{A}_1(b) = \int d^2 z \left[ \int d^4 \theta SS^\dagger + \mu \int d^2 \theta e^{bS} + c.c. \right]
\]  
(3.41)

where $S$ is a chiral superfield satisfying:
\[ D_- S = \overline{D}_- S = 0, \quad D_+ S^\dagger = \overline{D}_+ S^\dagger = 0. \]

As in the LFT and the $N = 1$ SLFTs, one should introduce a background charge $1/b$ so that the second term in Eq. (3.41) becomes a screening operator of the conformal field theory (CFT). However, a fundamental difference arises since the background charge is unrenormalized due to the $N = 2$ supersymmetry. For the LFT and the $N = 1$ SLFTs, the background charge is renormalized to $Q = 1/b + b$ and the theories are invariant under the dual transformation $b \rightarrow 1/b$. This self-duality plays an essential role to determine various exact correlation functions of those Liouville theories. Unrenormalized, the $N = 2$ SLFT is not self-dual.

This theory is a CFT with a central charge:
\[ c = 3 + 6/b^2. \]  
(3.42)
The primary operators of the $N = 2$ SLFT are classified into Neveu-Schwarz (NS) and Ramond (R) sectors and can be written in terms of the (first) component fields as follows:

$$N_{\alpha \bar{\alpha}} = e^{\alpha \varphi^\dagger + \bar{\varphi}}, \quad R^\pm_{\alpha \bar{\alpha}} = \sigma^\pm e^{\alpha \varphi^\dagger + \bar{\varphi}},$$

where $\sigma^\pm$ are the spin operators. The conformal dimensions of these fields are given by:

$$\Delta^N_{\alpha \bar{\alpha}} = -\alpha \bar{\alpha} + \frac{1}{2b}(\alpha + \bar{\alpha}), \quad \Delta^R_{\alpha \bar{\alpha}} = \Delta^N_{\alpha \bar{\alpha}} + \frac{1}{8}$$

and the $U(1)$ charges are:

$$Q^N_{\alpha \bar{\alpha}} = -\frac{1}{2b}(\alpha - \bar{\alpha}), \quad Q^R_{\alpha \bar{\alpha}} = Q^N_{\alpha \bar{\alpha}} \pm \frac{1}{4}. \quad (3.45)$$

From these expressions one can notice that:

$$\alpha \to 1/b - \bar{\alpha}, \quad \bar{\alpha} \to 1/b - \alpha$$

do not change the conformal dimension and $U(1)$ charge. From the CFT point of view, this means that $N_{1/b-\bar{\alpha}, 1/b-\alpha}$ should be identified with $N_{\alpha \bar{\alpha}}$ and similarly for the (R) operators, up to normalization factors. The reflection amplitudes are determined by these normalization factors.

Without the self-duality, it is possible that there exists a ‘dual’ action to (3.41) whose perturbative (weak coupling) behaviour describes the $N = 2$ SLFT in the strong coupling region. This action should be another CFT. Our proposal for the dual action is as follows:

$$\mathcal{A}_\Pi(b) = \int d^2z \int d^4\theta \left[ SS^\dagger + \bar{\mu} e^{b(S + S^\dagger)} \right]$$

with the background charge $b$. The $N = 2$ supersymmetry is preserved because $S + S^\dagger$ is a $N = 2$ scalar superfield. One can see that this action is conformal invariant because the interaction term is a screening operator. Our conjecture is that the two actions, $\mathcal{A}_I(b)$ and $\mathcal{A}_\Pi(1/b)$ are equivalent. To justify this conjecture, we will compute the reflection amplitudes based on these actions and will compare them with some independent results.

As mentioned above, the reflection amplitudes of the Liouville-type CFT are defined by linear transformations between different exponential fields, corresponding to the same primary field of the chiral algebra. For simplicity, we will restrict ourselves to the case $\alpha = \bar{\alpha}$ in (3.43) where the $U(1)$ charge of the (NS) operators becomes 0. We will refer to this case as the ‘neutral’ sector. (From now on, we will suppress the second indices $\bar{\alpha}$.) The physical states in this sector are given by:

$$\alpha = \frac{1}{2b} + iP$$

where $P$ is a real parameter. This parameter is transformed by $P \to -P$ under the reflection relation (3.46) and can be thought of as a ‘momentum’ which is reflected off from a potential wall.
The two-point functions of the same operators can be expressed as:

\[
\langle N_\alpha(z, \bar{z}) N_\alpha(0, 0) \rangle = \frac{D^N(\alpha)}{|z|^{4\Delta^N_\alpha}}
\]

\[
\langle R^+_\alpha(z, \bar{z}) R^-_\alpha(0, 0) \rangle = \frac{D^R(\alpha)}{|z|^{4\Delta^R_\alpha}}
\]

where \(\Delta^N_\alpha, \Delta^R_\alpha\) are given by Eq. (3.44). The normalization factors \(D^N(\alpha), D^R(\alpha)\) define the reflection amplitudes and should satisfy:

\[
D^N(\alpha)D^N(1/b - \alpha) = 1, \quad D^R(\alpha)D^R(1/b - \alpha) = 1.
\]

To find these amplitudes explicitly, we consider the operator product expansions (OPE’s) with degenerate operators.

The NS and R degenerate operators in the neutral sector are \(N_{\alpha_n m}\) and \(R^{\pm}_{\alpha_n m}\) with integers \(n, m\) and:

\[
\alpha_{nm} = \frac{1 - n}{2b} - \frac{mb}{2}, \quad n, m \geq 0.
\]

The OPE of a NS field with a degenerate operator \(N_{-b/2}\) is simply given by:

\[
N_\alpha N_{-b/2} = N_{\alpha-b/2} + C^N_-(\alpha)N_{\alpha+b/2}.
\]

Here the structure constant can be obtained from the screening integral as follows:

\[
C^N_-(\alpha) = \kappa_1 \gamma(1 - \alpha b)\gamma(1/2 - \alpha b - b^2/2)\gamma(-1/2 + \alpha b)\gamma(\alpha b + b^2/2)
\]

where:

\[
\kappa_1 = \frac{b^2 b^4 \pi^2}{2} \gamma(-b^2 - 1)\gamma \left(1 + \frac{b^2}{2} \right) \gamma \left(\frac{b^2}{2} + \frac{3}{2}\right)
\]

with \(\gamma(x) = \Gamma(x)/\Gamma(1 - x)\) as usual.

To use this OPE, we consider a three-point function \(\langle N_{\alpha+b/2} N_\alpha N_{-b/2}\rangle\) and take the OPE of \(N_{-b/2}\) with either \(N_{\alpha+b/2}\) or \(N_\alpha\) using (3.48). This leads to a functional equation:

\[
C^N_-(\alpha)D^N(\alpha + b/2) = D^N(\alpha).
\]

This functional equation determines the NS reflection amplitude in the form:

\[
D^N(\alpha) = \left(\frac{\kappa_1}{b^4}\right)^{-2\alpha/b} \gamma(2\alpha/b - 1/b^2) \frac{\gamma(b\alpha + 1/2)}{\gamma(b\alpha)} f(\alpha),
\]

with an arbitrary function \(f(\alpha)\) satisfying \(f(\alpha) = f(\alpha + b)\). To fix this unknown function, we need an additional functional equation. It is natural that this relation is provided by the dual action \(A_{\Pi}(1/b)\).
For this purpose we consider OPE’s with another degenerate operator, namely:

\[ N_\alpha R_{\alpha - 1/2b} = R_{\alpha - 1/2b}^+ + \tilde{C}^N_\alpha R_{\alpha + 1/2b}^+ \]  
\[ R_\alpha R_{\alpha - 1/2b}^- = N_{\alpha - 1/2b} + \tilde{C}^R_\alpha N_{\alpha + 1/2b}. \]  

The structure constants can be computed by the screening integrals using the dual action \( \mathcal{A}_{11}(1/b) \). The result is:

\[ \tilde{C}^N_\alpha = \kappa_2(b) \frac{\gamma(2\alpha/b - 1/b^2)}{\gamma(2\alpha/b)}, \]
\[ \tilde{C}^R_\alpha = \kappa_2(b) \frac{\gamma(2\alpha/b - 1/b^2 + 1)}{\gamma(2\alpha/b + 1)} \]

where:

\[ \kappa_2(b) = \bar{\mu} \pi \gamma \left( \frac{1}{b^2} + 1 \right). \]

These results are consistent with the \( N = 2 \) minimal model results presented in Section 3.1.

Now we consider the three-point functions \( \langle R_{\alpha + 1/2b}^- N_{\alpha} R_{\alpha - 1/2b}^+ \rangle \) and \( \langle N_{\alpha + 1/2b} R_{\alpha} R_{\alpha - 1/2b}^- \rangle \). Taking the OPE of \( R_{\alpha + 1/2b}^- \) with one of the other two operators in the correlation functions and using the OPE relations (3.51) and (3.52), we obtain an independent set of functional relations as follows:

\[ \tilde{C}^N_\alpha D^R(\alpha + 1/2b) = D^N(\alpha), \]
\[ \tilde{C}^R_\alpha D^N(\alpha + 1/2b) = D^R(\alpha). \]

Solving for \( D^N(\alpha) \), we find that the most general solution of Eqs.(3.53) is:

\[ D^N(\alpha) = \kappa^{-2ab} \frac{\Gamma^2(ab + 1/2)}{\Gamma^2(ab)} \gamma(2\alpha/b - 1/b^2)g(\alpha) \]

where \( g(\alpha) \) is another arbitrary function satisfying \( g(\alpha) = g(\alpha + 1/b) \). Combining Eqs.(3.50) and (3.54), and requiring the normalization \( D^N(\alpha = \frac{1}{2b}) = 1 \), we can determine the NS reflection amplitude completely as follows:

\[ D^N(\alpha) = -\frac{2}{b^2} \kappa^{-2ab+1} \frac{\gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2}\right) \gamma\left(\alpha b + \frac{1}{2}\right)}{\gamma(1 - \alpha b)} \]

where the two parameters in the actions, \( \mu \) and \( \bar{\mu} \), are related by:

\[ \left( \frac{\kappa_1}{b^4} \right)^{1/b} = \kappa_2^b. \]

The R reflection amplitude can be obtained from (3.53):

\[ D^R(\alpha) = -\frac{b^2}{2} \kappa^{-2ab+1} \gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2} + 1\right) \gamma\left(-\alpha b + \frac{1}{2}\right) \gamma(\alpha b). \]
To justify the reflection amplitudes derived above and based on the conjectured action $A_{II}$, we can provide several consistency checks. It has been noticed that an integrable model with two parameters proposed in [67] can have $N = 2$ supersymmetry if one of the parameters take a special value. This means that one can compute the reflection amplitudes of the $N = 2$ SLFT independently as a special case of those in [67]. Indeed, we have confirmed that the two results agree exactly.

Furthermore, one can check the reflection amplitude for specific values of $\alpha$ directly from the action. For example when $\alpha = \frac{1}{2b} - \frac{1}{2}$ using the action $A_{I}(b)$. If instead $\alpha \to 0$, one can compute the two-point function directly from the action $A_{II}(1/b)$. Both results agree with (3.55).

### 3.4 One-point functions of $N = 2$ super-Liouville theory with boundary

The action of the $N = 2$ SLFT with boundary in terms of component fields is given by:

\[
S = \int d^2z \left[ \frac{1}{2\pi} \left( \partial \phi^- \bar{\partial} \phi^+ + \partial \phi^+ \bar{\partial} \phi^- + \psi^- \bar{\partial} \psi^+ + \psi^+ \bar{\partial} \psi^- + \bar{\psi}^- \partial \bar{\psi}^+ + \bar{\psi}^+ \partial \bar{\psi}^- \right) 
+ i\mu b^2 \psi^- \bar{\psi}^- e^{b\phi^+} + i\mu b^2 \psi^+ \bar{\psi}^+ e^{b\phi^-} + \pi \mu^2 b^2 e^{b(\phi^+ + \phi^-)} \right] + S_B, \tag{3.58}
\]

where the boundary action is derived in [26]:

\[
S_B = \int_{-\infty}^{\infty} dx \left[ -\frac{i}{4\pi} (\bar{\psi}^+ \psi^- + \bar{\psi}^- \psi^+) + \frac{1}{2} a^- \partial_x a^+ 
- \frac{1}{2} e^{b\phi^+ / 2} \left( \mu_B a^+ + \frac{\mu b^2}{4\mu_B} a^- \right) (\psi^- + \bar{\psi}^-) 
- \frac{1}{2} e^{b\phi^- / 2} \left( \mu_B a^- + \frac{\mu b^2}{4\mu_B} a^+ \right) (\psi^+ + \bar{\psi}^+) 
- \frac{2}{b^2} \left( \frac{\mu^2 b^4}{16\mu_B^2} \right) e^{b(\phi^+ + \phi^-) / 2} \right]. \tag{3.59}
\]

Note that we slightly changed the notations with respect to the previous Section.

The stress tensor $T$, the supercurrent $G^\pm$ and the $U(1)$ current $J$ are given by:

\[
T = -\partial \phi^- \bar{\partial} \phi^+ - \frac{1}{2} (\psi^- \bar{\partial} \psi^+ + \psi^+ \bar{\partial} \psi^-) + \frac{1}{2b} (\partial^2 \phi^+ + \partial^2 \phi^-), \tag{3.60}
\]

\[
G^\pm = \sqrt{2} i (\psi^\pm \partial \phi^\pm - \frac{1}{b} \partial \psi^\pm), \quad J = -\psi^- \psi^+ + \frac{1}{b} (\partial \phi^+ - \partial \phi^-). \tag{3.61}
\]

Using the mode expansions for the currents and their operator product expansion, one obtains the $N = 2$ super-Virasoro algebra with central charge $c = 3 + 6/b^2$. 

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The primary fields from the NS and R sectors are written in the new notations as follows:

\[ N_{\alpha\bar{\alpha}} = e^{\alpha\phi^+ + \bar{\alpha}\phi^-}, \quad R_{\alpha\bar{\alpha}}^\pm = \sigma^\pm e^{\alpha\phi^+ + \bar{\alpha}\phi^-}, \quad \] (3.62)

The conformal dimensions are given by (3.44). According to the normalization in this Section the \( U(1) \) charges of NS and R fields are rescaled:

\[ \omega = \frac{1}{b}(\alpha - \bar{\alpha}), \quad \omega^\pm = \omega \pm \frac{1}{2}. \] (3.63)

It is more convenient to use a 'momentum' defined by:

\[ \alpha + \bar{\alpha} = \frac{1}{b} + 2iP, \] (3.64)

and the \( U(1) \) charge \( \omega \) instead of \( \alpha, \bar{\alpha} \). In terms of these, the conformal dimensions are given by:

\[ \Delta^{NS} = \frac{1}{4b^2} + P^2 + \frac{b^2\omega^2}{4}. \] (3.65)

From now on, we will denote a NS primary state by \( |[P, \omega]\rangle \) and an R state by \( |[P, \omega, \epsilon]\rangle \) with \( \epsilon = \pm 1 \).

In this Section we compute exact one-point functions of the NS and R bulk operators \( N_{\alpha\bar{\alpha}} \) and \( R_{\alpha\bar{\alpha}}^\epsilon \) of the \( N = 2 \) SLFT with boundary. The one-point functions are defined by:

\[ \langle N_{\alpha\bar{\alpha}}(\xi, \bar{\xi}) \rangle = \frac{U^{NS}(\alpha, \bar{\alpha})}{|\xi - \bar{\xi}|^{2\Delta^{NS}_{\alpha\bar{\alpha}}}}, \quad \text{and} \quad \langle R_{\alpha\bar{\alpha}}^\epsilon(\xi, \bar{\xi}) \rangle = \frac{U^{R}(\alpha, \bar{\alpha})}{|\xi - \bar{\xi}|^{2\Delta^{R}_{\alpha\bar{\alpha}}}}, \] (3.66)

with the conformal dimensions given in (3.44). We will simply refer to the coefficients \( U^{NS}(\alpha, \bar{\alpha}) \) and \( U^{R}(\alpha, \bar{\alpha}) \) as the one-point functions.

According to Cardy’s formalism, one can associate a conformal BC with each primary state [8]. For the \( N = 2 \) SLFT, there will be an infinite number of conformal BCs. These BCs can be constructed by the fusion process and related to the one-point functions. Let us begin with the ‘vacuum’ BC which corresponds to the identity operator. First we introduce an amplitude as an inner product (or overlap) between the Isibashi state of a primary state and the conformal boundary state:

\[ \Psi_0^{NS}(P, \omega) = \langle \langle 0 \rangle | [P, \omega] \rangle \rangle. \]

Following Cardy and using the modular properties of the \( N = 2 \) characters we find that the amplitude satisfies the following relation:

\[ \Psi_0^{NS}(P, \omega)\Psi_0^{NS\dagger}(P, \omega) = S_{NS}(P, \omega) \] (3.67)

where:

\[ S_{NS}(P, \omega) = \frac{\sinh(2\pi bP)\sinh\left(\frac{2\pi P}{b}\right)}{2b^{-1}\cosh\left(\pi bP + \frac{i\pi b\omega}{2}\right)\cosh\left(\pi bP - \frac{i\pi b\omega}{2}\right)}. \]

---

2 We denote a conformal BC in ‘bold face’ like \( 0 \) and a conformal boundary state like \( |(0)\rangle \).
Since $\Psi^{NS\dagger}_0(P, \omega) = \Psi^{NS}_0(-P, \omega)$, one can solve this up to some unknown constant as follows:

$$\Psi^{NS}_0(P, \omega) = \sqrt{\frac{b}{2}} (X_{NS})^P \frac{\Gamma\left(\frac{1}{2} - ibP + \frac{b^2\omega}{2}\right) \Gamma\left(\frac{1}{2} - ibP - \frac{b^2\omega}{2}\right)}{\Gamma\left(-\frac{2P}{b}\right) \Gamma(1 - 2ibP)}.$$ (3.68)

The unknown constant $X_{NS}$ does not depend on $P, \omega$ and cannot be determined by the modular transformation alone. We will derive this constant later in this Section by comparing with the bulk reflection amplitudes.

Similarly, for the R sector we define the R amplitude by:

$$\Psi^R_0(P, \omega) = \langle (0) | [P, \omega, \epsilon] \rangle$$

which satisfies:

$$\Psi^R_0(P, \omega) \Psi^{R\dagger}_0(P, \omega) = S_R(P, \omega)$$ (3.69)

with:

$$S_R(P, \omega) = \frac{\sinh(2\pi bP) \sinh\left(\frac{2\pi P^2}{b}\right)}{2b^{-1} \sinh\left(\pi bP + \frac{i\pi b^2\omega}{2}\right) \sinh\left(\pi bP - \frac{i\pi b^2\omega}{2}\right)}.$$

The solution is up to an unknown constant:

$$\Psi^R_0(P, \omega) = -i \sqrt{\frac{b}{2}} (X_R)^P \frac{\Gamma\left(-ibP + \frac{b^2\omega}{2}\right) \Gamma\left(1 - ibP - \frac{b^2\omega}{2}\right)}{\Gamma\left(-\frac{2P}{b}\right) \Gamma(1 - 2ibP)}.$$ (3.70)

Again, the unknown constant $X_R$ will be fixed later.

Now we consider a continuous BC associated with a primary field. This field should be NS and its $U(1)$ charge should be zero because only the boundary neutral operators should appear. So, we consider the character of a (NS) primary state $|s\rangle \equiv |[s, 0]\rangle$ and its modular transformation. The parameter $s$ depends on the boundary parameter $\mu_B$ in (3.59). We define an inner product between the conformal boundary state and an Ishibashi state:

$$\Psi^{NS}_s(P, \omega) = \langle (s) | [P, \omega] \rangle.$$

Following the previous analysis of the modular transformation one can find that:

$$\Psi^{NS}_s(P, \omega) \Psi^{NS\dagger}_0(P, \omega) = b \cos(4\pi s P).$$

Now acting by $\Psi^{NS}_0(P, \omega)$ on this and using (3.67), we obtain

$$\Psi^{NS}_s(P, \omega) = b \Psi^{NS}_0(P, \omega) \frac{\cos(4\pi s P)}{S^{NS}_0(P, \omega)} = \sqrt{2b^2} (X_{NS})^P \frac{\Gamma\left(1 + \frac{2ibP}{P}\right) \Gamma\left(2ibP\right) \cos(4\pi s P)}{\Gamma\left(\frac{1}{2} + ibP + \frac{b^2\omega}{2}\right) \Gamma\left(\frac{1}{2} + ibP - \frac{b^2\omega}{2}\right)}.$$ (3.70)
One can follow the same steps for the R sector which leads to:

\[ \Psi_R^s(P, \omega) \Psi_0^R(P, \omega) = b \cos(4\pi sP), \]

where

\[ \Psi_R^s(P, \omega) = \langle (s)[P, \omega, \epsilon] \rangle. \]

Using (3.69) on this, we can obtain:

\[
\Psi_R^s(P, \omega) = b \Psi_0^R(P, \omega) \cos(4\pi sP) = \Psi_R^s(P, \omega) \cos(4\pi sP) = -i\sqrt{2}b^3 (X_R)_{iP} b \Gamma \left( \frac{1}{2} + ibP - \frac{b^2 \omega}{2} \right) \Gamma \left( \frac{1}{2} + ibP + \frac{b^2 \omega}{2} \right). \quad (3.71)
\]

The amplitudes (3.70) and (3.71) we have obtained are the one-point functions of the two sectors up to some normalization constants. To fix these constants, we recall the relation proved in [68]:

\[ U_k(\phi) = \frac{\langle (k)|\phi \rangle \rangle}{\langle (k)|0 \rangle} \]

where \( k \) is a conformal BC, \( \phi \) a primary field, and \( |\phi \rangle \rangle \), its Isibashi state. For the \( N = 2 \) SLFT, this relation means:

\[ U_{NS}^s(P, \omega) = \Psi_{NS}^s(P, \omega), \quad U_R^s(P, \omega) = \Psi_{NS}^s(-i/2b, 0). \]

From (3.70) and (3.71) we can obtain the one-point functions as follows:

\[
U_{NS}^s(P, \omega) = \frac{\Psi_{NS}^s(P, \omega)}{\Psi_{NS}^s(-i/2b, 0)}, \quad U_R^s(P, \omega) = \frac{\Psi_{NS}^s(P, \omega)}{\Psi_{NS}^s(-i/2b, 0)}. \quad (3.73)
\]

(3.74)

where the normalization coefficient \( \mathcal{N} \) can be fixed by:

\[ U_{NS}^s(-i/2b, 0) = 1 \rightarrow \mathcal{N} = \left[ (X_{NS})^{1/2b^2} \Gamma(1 + b^{-2}) \cosh \left( \frac{2\pi s}{b} \right) \right]^{-1}. \]

We will use now the reflection amplitudes found in the previous Section in order to fix the normalization of the one-point functions. Remind that the reflection amplitudes are defined by two-point functions of the same operators:

\[ \langle N_{\alpha\bar{\alpha}}(z, \bar{z}) N_{\alpha\bar{\alpha}}(0, 0) \rangle = \frac{\mathcal{D}_{NS}^{\alpha\bar{\alpha}}}{|z|^{4\Delta_{NS}^{\alpha\bar{\alpha}}}}, \quad \langle R_{\alpha\bar{\alpha}}^{+}(z, \bar{z}) R_{\alpha\bar{\alpha}}^{-}(0, 0) \rangle = \frac{\mathcal{D}_R^{\alpha\bar{\alpha}}}{|z|^{4\Delta_{NS}^{\alpha\bar{\alpha}}}} \]
Reflection properties imply that, in general:

\[ \langle N_{\alpha \pi}(z, \overline{z}) \ldots \rangle = D^{NS}(\alpha, \overline{\pi}) \langle N_{b - \pi, b - \alpha}(z, \overline{z}) \ldots \rangle \]  

(3.75)

and similarly for the R sector. Here the part \ldots can be any products of the primary fields. The reflection relations among the correlation functions can be used for the simplest case, namely, the one-point functions. In this case, the relation becomes:

\[ \langle N_{\alpha \pi}(z, \overline{z}) \rangle = D^{NS}(\alpha, \overline{\pi}) \langle N_{b - \pi, b - \alpha}(z, \overline{z}) \rangle, \]

\[ \langle R_{\alpha \pi}(z, \overline{z}) \rangle = D^{R}(\alpha, \overline{\pi}) \langle R_{b - \pi, b - \alpha}(z, \overline{z}) \rangle. \]

These lead to the following equations:

\[ \frac{U^{NS}_s(P, \omega)}{U^{NS}_s(-P, \omega)} = D^{NS}(P, \omega), \quad \frac{U^{R}_s(P, \omega)}{U^{R}_s(-P, \omega)} = D^{R}(P, \omega). \]

(3.76)

For the neutral sector \( \omega = 0 \), the reflection amplitudes has been derived in the previous Section:

\[ D^{NS}(P, 0) = -\kappa^{-2iP/b} \frac{\Gamma \left( 1 + \frac{2iP}{b} \right)}{\Gamma \left( 1 - \frac{2iP}{b} \right)} \frac{\Gamma \left( 1 + iPb \right)}{\Gamma \left( 1 - iPb \right)} \frac{\Gamma \left( \frac{1}{2} - iPb \right)}{\Gamma \left( \frac{1}{2} + iPb \right)}, \]

(3.77)

\[ D^{R}(P, 0) = \kappa^{-2iP/b} \frac{\Gamma \left( 1 + \frac{2iP}{b} \right)}{\Gamma \left( 1 - \frac{2iP}{b} \right)} \frac{\Gamma \left( 1 - iPb \right)}{\Gamma \left( 1 + iPb \right)} \frac{\Gamma \left( \frac{1}{2} + iPb \right)}{\Gamma \left( \frac{1}{2} - iPb \right)}. \]

(3.78)

where:

\[ \kappa = \frac{\mu^2 \pi^2}{2} \gamma(-b^2 - 1) \gamma \left( 1 + \frac{b^2}{2} \right) \gamma \left( \frac{b^2}{2} + 3 \right), \]

with \( \gamma(x) = \Gamma(x)/\Gamma(1 - x) \) and the bulk cosmological constant \( \mu \) defined in (3.38).

Inserting \( \omega = 0 \) and using (3.73) and (3.74), the reflection amplitudes in Eq. (3.76) are indeed in exact agreement with (3.77) and (3.78) if and only if we identify the constants:

\[ X_{NS} = X_R = \left[ 2^{2b^2} \kappa \right]^{-1}. \]

This provides a nontrivial check and completes our derivation for the one-point functions. Furthermore, we can use (3.76) to compute the reflection amplitudes for \( \omega \neq 0 \) case:

\[ D^{NS}(P, \omega) = (2^{2b^2} \kappa)^{-2iP/b} \frac{\Gamma \left( 1 + \frac{2iP}{b} \right)}{\Gamma \left( 1 - \frac{2iP}{b} \right)} \frac{\Gamma \left( 2ibP \right)}{\Gamma \left( -2ibP \right)} \frac{\Gamma \left( \frac{1}{2} - ibP + \frac{b^2 \omega}{2} \right)}{\Gamma \left( \frac{1}{2} + ibP + \frac{b^2 \omega}{2} \right)} \frac{\Gamma \left( \frac{1}{2} - ibP - \frac{b^2 \omega}{2} \right)}{\Gamma \left( \frac{1}{2} + ibP - \frac{b^2 \omega}{2} \right)}. \]

(3.79)

and:

\[ D^{R}(P, \omega) = (2^{2b^2} \kappa)^{-2iP/b} \frac{\Gamma \left( 1 + \frac{2iP}{b} \right)}{\Gamma \left( 1 - \frac{2iP}{b} \right)} \frac{\Gamma \left( 2ibP \right)}{\Gamma \left( -2ibP \right)} \frac{\Gamma \left( 1 - ibP - \frac{b^2 \omega}{2} \right)}{\Gamma \left( 1 + ibP - \frac{b^2 \omega}{2} \right)} \frac{\Gamma \left( -ibP + \frac{b^2 \omega}{2} \right)}{\Gamma \left( ibP + \frac{b^2 \omega}{2} \right)}. \]

(3.80)
These results can be compared with those from the two-parameter family models \cite{67} and we checked that the two independent results match exactly.

To complete our derivation of the one-point functions, we should relate the boundary parameter $s$ to the boundary cosmological constant $\mu_B$ in \cite{3.59}. For this, we consider one-point function of a neutral NS field $N_{\alpha\alpha}$:

\[
\text{residue } \left. \frac{U_{NS}(\alpha)}{\mathcal{N}} \right|_{\alpha=(b^{-1}-nb)/2} = \langle e^{\alpha(\phi^+ + \phi^-)} \rangle = \sum_{p,q} \frac{1}{p!q!} \langle e^{\alpha(\phi^+ + \phi^-)} V_{B}^{p} B^{q} \rangle_{0},
\]

where $V, B$ are the interaction terms in the bulk and boundary actions. If we choose $n = 1$ ($\alpha = 1/2b - b/2$), all terms vanish except $p = 0, q = 2$ which can be easily computed:

\[
\langle e^{\alpha(\phi^+ + \phi^-)} (i/2) B^{2} \rangle_{0} = 8\pi\mu_{B}^{2}\Gamma(-b^{2})\gamma \left( \frac{1+b^{2}}{2} \right) \sin \left( \frac{\pi+1+b^{2}}{2} \right)
\]

with:

\[
\mu_{B}^{2} = \mu_{B}^{2} + \mu_{b}^{2} \frac{b^{4}}{16\mu_{B}^{2}}.
\]

The residue of \cite{3.73} at $\alpha = \pi = 1/2b - b/2$ becomes:

\[
\frac{b}{2} (2^{2b^{2}} \kappa)^{1/2} \frac{\Gamma(-b^{2})}{\Gamma(\frac{1-b^{2}}{2})^{2}} \cosh(2\pi sb).
\]

Comparing these two, we find:

\[
\mu_{B}^{2} = \frac{\mu_{b}}{32\pi} \cosh(2\pi sb).
\]

We want now to compare these results with those obtained by the so called conformal bootstrap approach. It consists in deriving functional equations for the one-point functions in a way similar to what we discussed for $N = 1$ case in Section 2. Namely, consider two-point functions of the neutral operators:

\[
G_{\alpha}^{NS}(\xi, \xi') = \langle R_{-\frac{1}{2b}}^{+}(\xi) N_{\alpha}(\xi') \rangle, \quad G_{\alpha}^{R}(\xi, \xi') = \langle R_{-\frac{1}{2b}}^{+}(\xi) R_{\alpha}^{-}(\xi') \rangle
\]

where $R_{-1/2b}^{+}$ is a degenerate R operator, whose OPE’s are given by:

\[
R_{-\frac{1}{2b}}^{+} N_{\alpha} = \left[ R_{\alpha-\frac{1}{4b}}^{+} \right] + C^{NS}(\alpha) \left[ R_{\alpha+\frac{1}{4b}}^{+} \right],
\]

\[
R_{-\frac{1}{2b}}^{+} R_{\alpha}^{-} = \left[ N_{\alpha-\frac{1}{4b}}^{-} \right] + C^{R}(\alpha) \left[ N_{\alpha+\frac{1}{4b}}^{-} \right].
\]

Here the bracket $[\ldots]$ represents the conformal family of a given primary field and the structure constants have been computed in the previous Section based on the dual $N = 2$
where \( \tilde{\mu} \), the cosmological constant of the dual theory, has been related there to that of the \( N = 2 \) SLFT.

The two-point functions can be written as:

\[
G^\alpha_{NS}(\xi, \xi') = U^R \left( \alpha - \frac{b}{2} \right) G^+_{NS}(\xi, \xi') + C^NS(\alpha) U^R \left( \alpha + \frac{b}{2} \right) G^-_{NS}(\xi, \xi')
\]

\[
G^\alpha_{\alpha}(\xi, \xi') = U^NS \left( \alpha - \frac{b}{2} \right) G^+_{\alpha}(\xi, \xi') + C^\alpha U^NS \left( \alpha + \frac{b}{2} \right) G^-_{\alpha}(\xi, \xi')
\]

where \( G^\pm_{\alpha}(\xi, \xi') \)'s are expressed in terms of the special conformal blocks:

\[
G^\pm_{\alpha}(\xi, \xi') = \frac{|\xi' - \xi|^{2\Delta_a - 2\Delta - b/2}}{|\xi - \xi'|^{4\Delta_a}} \mathcal{F}_\pm(\eta), \quad G^\pm_{\alpha}(\xi, \xi') = \frac{|\xi' - \xi|^{2\Delta_r - 2\Delta - b/2}}{|\xi - \xi'|^{4\Delta_r}} \mathcal{F}_\pm(\eta)
\]

with

\[
\eta = \frac{(\xi - \xi')(\xi - \xi')}{(\xi - \xi')(\xi - \xi')}
\]

These conformal blocks are expressed in terms of some known hypergeometric functions.

On the other hand, one can compute the two-point functions as both \( R^+_{1/2b} \) and \( N_\alpha \) or \( R^-_{1/2b} \) approach the boundary. The fusion of the degenerate operator with the boundary is described by a special bulk-boundary structure constant which could be computed as a boundary screening integral with one insertion of the boundary interaction of the dual \( N = 2 \) theory if it were known. Since we can not fix it, we denote the unknown constant just as \( \mathcal{R}(-1/2b) \). Then, we can obtain the system of functional relations as follows:

\[
\mathcal{R} \left( -\frac{1}{2b} \right) U^{NS}(\alpha) = \frac{\Gamma(1 - \frac{1}{2b} + \frac{2\alpha}{b})\Gamma(-\frac{2\alpha}{b})}{\Gamma(1 - \frac{2\alpha}{b} + \frac{2\alpha}{b})\Gamma(1 - \frac{1}{2b})} U^R \left( \alpha - \frac{1}{2b} \right)
\]

\[
+ C^{NS}(\alpha) \frac{\Gamma(1 + \frac{1}{2b} - \frac{2\alpha}{b})\Gamma(-\frac{2\alpha}{b})}{\Gamma(1 - \frac{2\alpha}{b} + \frac{2\alpha}{b})\Gamma(1 - \frac{1}{2b})} U^R \left( \alpha + \frac{1}{2b} \right)
\]

\[
\mathcal{R} \left( -\frac{1}{2b} \right) U^{R}(\alpha) = \frac{\Gamma(\frac{2\alpha}{b} - \frac{1}{2b})\Gamma(-\frac{2\alpha}{b})}{\Gamma(\frac{2\alpha}{b} - \frac{2\alpha}{b})\Gamma(1 - \frac{1}{2b})} U^{NS} \left( \alpha - \frac{1}{2b} \right)
\]

\[
+ C^{R}(\alpha) \frac{\Gamma(\frac{1}{2b} - \frac{2\alpha}{b})\Gamma(-\frac{2\alpha}{b})}{\Gamma(\frac{1}{2b} - \frac{2\alpha}{b})\Gamma(1 - \frac{1}{2b})} U^{NS} \left( \alpha + \frac{1}{2b} \right).
\]

Although we do not know the bulk-boundary structure constant, we can eliminate it by taking the ratio of the above equations and find one relation which is completely fixed. It
can be shown that the one-point functions (3.73) and (3.74) indeed satisfy this relation. This means not only that the one-point functions obtained from the modular bootstrap procedures are consistent with the $N = 2$ SLFT actions, but also that the $N = 2$ theory proposed in the previous Section is indeed dual to the $N = 2$ SLFT. Furthermore, we can find the bulk-boundary structure constant as follows:

$$\frac{\mathcal{R} \left(-\frac{1}{2b}\right)}{\Gamma \left(-\frac{2}{b}\right)} \frac{\gamma \left(1 - \frac{1}{b^2}\right)}{\sqrt{\mu \gamma (1 + \frac{1}{b^2})}} = \cosh \left(\frac{2\pi s}{b}\right).$$

Along with (3.83), this equation relates the boundary cosmological constant of the $N = 2$ SLFT with that of the dual $N = 2$ theory.

### 3.5 ZZ-branes of $N = 2$ super-Liouville theory

Remind that the primary fields of NS and R sectors are expressed in terms of vertex operators:

$$N^{\alpha}_\alpha = e^{\alpha\phi^+ + \overline{\phi}^-}, \quad R^{(\pm)}_\alpha = \sigma^{\pm} e^{\alpha\phi^+ + \overline{\phi}^-}$$

(3.84)

where $\sigma^{\pm}$ are the spin operators. The conformal dimensions and the $U(1)$ charges of the primary fields $N^{\alpha}_\alpha$ and $R^{(\pm)}_\alpha$ were obtained above as (3.44) and (3.63) respectively.

Among the primary fields there is a series of degenerate fields of the $N = 2$ SLFT. In this Section we divide these fields into three classes. Class-I degenerate fields are given by:

$$N^\omega_{m,n} = N^{\alpha}_m \alpha^\omega_{n,m,n}, \quad R^{(\pm)}_{m,n} = R^{(\pm)}_{\alpha}_m \alpha^\omega_{n,m,n},$$

$$\alpha^\omega_{m,n} = \frac{1 - m + \omega b^2}{2b} - \frac{nb}{2}, \quad \overline{\alpha}^\omega_{m,n} = \frac{1 - m - \omega b^2}{2b} - \frac{nb}{2}, \quad m, n \in \mathbb{Z}_+.$$  

(3.85)

$N^\omega_{m,n}$ and $R^{(\pm)}_{m,n}$ are degenerate at level $mn$ where the corresponding null states turn out to be:

$$N^\omega_{m,-n}, \quad \text{and} \quad R^{(\pm)}_{m,-n}.$$  

(3.86)

As an example, consider the most simple case $N^\omega_{1,1}$ with the conformal dimension $b^2(\omega^2 - 1)/4 - 1/2$ and $U(1)$ charge $\omega$. After simple calculation, one can check that:

$$\left[\frac{b^2}{2} (1 - \omega^2) J_{-1} + G^+_{-1/2} G^-_{-1/2} - (1 - \omega) L_{-1}\right] |N^\omega_{1,1}\rangle$$

is annihilated by all the positive modes of the $N = 2$ super CFT. Since this state has the $U(1)$ charge $\omega$ and dimension +1 more than that of $N^\omega_{1,1}$, it corresponds to $|N^\omega_{1,-1}\rangle$ up to a normalization constant. One can continue this analysis to higher values of $m, n > 1$ to confirm the statement of Eq.(3.86). Notice that the null state structure changes dramatically for $\omega = \pm n$ case. The field $N^\omega_{m,n}$ has a null state $N^\pm_{m,-n}$ at level $mn$. This $N^\pm_{m,-n}$ field is in fact a class-II degenerate field which we will explain next and has infinite number of null states. Therefore, we exclude the case of $\omega = \pm n$ from class-I fields.
The second class of degenerate fields is denoted by $N^\omega_m$ and $R^{(\epsilon)}_m$ and comes in two subclasses, namely, class-IIA and class-IIB. These are given by:

Class − IIA :  
\[ N^\omega_m = N^\alpha_m, R^{(+)}_m = R^{(+)\alpha}_m, \quad \omega > 0 \]  
(3.87)

Class − IIB :  
\[ \tilde{N}^\omega_m = N^0\alpha_m, R^{(-)}_m = R^{(-)\alpha}_m, \quad \omega < 0. \]  
(3.88)

Here we have defined:

\[ \alpha^\omega_m \equiv \frac{1 - m + 2\omega b^2}{2b}, \quad \tilde{\alpha}^\omega_m \equiv \frac{1 - m - 2\omega b^2}{2b} \]

with $m$ a positive odd integer for the NS sector and even for the R sector.

These fields have null states at level $m/2$ which can be expressed again by Eq.(3.87) with $\omega$ shifted by +1 for class-IIA and by Eq.(3.88) with $\omega$ shifted by −1 for class-IIB. For $m = 1$, these fields become either chiral or anti-chiral field which are annihilated by $G^\pm_{-1/2}$, respectively. For $m = 3$, one can construct a linear combination of descendants:

\[ \left( \omega - \frac{2}{b^2} + 1 \right) \left( G^+_{-3/2} - G^+_{-1/2}L_{-1} + G^+_{-1/2}J_{-1} \right) |N^\omega_3\rangle \]  
(3.89)

which satisfies the null state condition. Since this state has $U(1)$ charge $\omega + 1$ and dimension 3/2 higher than that of $N^\omega_3$, it is straightforward to identify it as $N^{\omega+1}_3$ up to a normalization constant. However, it is not the end of the story in this case. The $N^{\omega+1}_3$ field is again degenerate at level 3/2 because a linear combination of its descendants, exactly Eq.(3.89) with $\omega$ shifted by +1, satisfies the null state condition. This generates $N^{\omega+2}_3$ and it continues infinitely. This infinite null state structure holds for any odd integer $m$.

This can be illustrated by semi-infinite sequence:

Class − IIA :  
\[ N^\omega_m \rightarrow N^{\omega+1}_m \rightarrow N^{\omega+2}_m \rightarrow \ldots \]

Class − IIB :  
\[ \tilde{N}^\omega_m \rightarrow \tilde{N}^{\omega-1}_m \rightarrow \tilde{N}^{\omega-2}_m \rightarrow \ldots \]

This works similarly for the R sector. For example, the null state of the $m = 2$ R field is given by:

\[ G^\pm_{-1/2} |R^{(\pm)\omega}_2\rangle. \]

We need to deal with class-II neutral ($\omega = 0$) NS fields separately. For example, consider the $N^0_3$ which has two null states:

\[ \left[ \left( 1 - \frac{2}{b^2} \right) G^\pm_{-3/2} - G^\pm_{-1/2}L_{-1} + G^\pm_{-1/2}J_{-1} \right] |N^0_3\rangle, \]  
(3.90)

which should be identified with $N^{1}_3$ and $\tilde{N}^{-1}_3$, respectively. We will call these neutral NS degenerate fields as class-III and denote them by:

Class − III :  
\[ N_m = N^{0\alpha}_m. \]  
(3.91)
The null state structure of the class-III fields has an infinite sequence in both directions:

\[
\text{Class} - \text{III} : \quad \ldots \leftarrow \tilde{N}_m^{-2} \leftarrow \tilde{N}_m^{-1} \leftarrow N_m \rightarrow N_m^1 \rightarrow N_m^2 \rightarrow \ldots \quad (3.92)
\]

The identity operator is the most simple class-III field with \( m = 1 \).

The degenerate fields are playing an essential role in both conformal and modular bootstraps. As we will see shortly, some simple degenerate fields satisfy relatively simple operator product expansions and make the conformal bootstrap viable.

In this Section we are interested in the \( N = 2 \) SLFT on Lobachevskiy plane or pseudosphere which is the geometry of the infinite constant negative curvature surface. As in the \( N = 1 \) case, using conformal bootstrap we derive and solve nonlinear functional equations which can provide discrete BCs.

The classical equations of motion for the \( N = 2 \) SLFT can be derived from the (bulk part of) Lagrangian \([3.58]\):

\[
\begin{align*}
\partial \bar{\partial} \phi^\pm &= \pi \mu b^3 \left[ \pi \mu e^{b(\phi^+ + \phi^-)} + i \psi^\pm \bar{\psi}^\mp e^{b\phi^-}\right] \\
\partial \bar{\psi}^\pm &= i \pi \mu b^2 e^{b\phi^\pm} \psi^\mp, \quad \bar{\partial} \psi^\pm = -i \pi \mu b^2 e^{b\phi^\pm} \bar{\psi}^\mp.
\end{align*}
\]

Assuming that the fermionic fields vanish in the classical limit, we can solve the bosonic fields classically:

\[
e^{\varphi(z)} = \frac{4R^2}{(1 - |z|^2)^2},
\]

where \( \varphi = b(\phi^+ + \phi^-) \) and \( R^{-2} = 4\pi^2 \mu^2 b^4 \). The parameter \( R \) is interpreted as the radius of the pseudosphere in which the points at the circle \( |z| = 1 \) are infinitely far away from any internal point. This circle can be interpreted as the "boundary" of the pseudosphere. This boundary has a different class of BC’s. For the \( N = 2 \) SLFT, we will call the discrete BCs as ZZ-branes following [24] and show that these correspond to the degenerate fields of the \( N = 2 \) SLFT.

Let us consider a two-point function of a neutral degenerate field \( N_{-b/2} \) and a general neutral field \( N_\alpha \):

\[
G_\alpha(\xi, \xi') = \langle N_{-b/2}(\xi) N_\alpha(\xi') \rangle. \quad (3.93)
\]

Using the OPE of the two fields, we can express this two-point function as:

\[
G_\alpha(\xi, \xi') = U^{NS} \left( \alpha - \frac{b}{2} \right) G_{1}^{NS}(\xi, \xi') + C_{--}(\alpha) U^{NS} \left( \alpha + \frac{b}{2} \right) G_{3}^{NS}(\xi, \xi')
\]

where the structure constant \( C_{--} \) is known and given by gamma-functions. The \( G_{i}^{NS}(\xi, \xi') \)'s are expressed in terms of the special conformal blocks:

\[
G_{i}^{NS}(\xi, \xi') = \frac{|\xi' - \bar{\xi}|^{2\Delta_{NS} - 2\Delta_{NS}^{b/2}}}{|\xi - \xi'|^{4\Delta_{NS}}}|\xi' - \bar{\xi}|^{2\Delta_{NS} - 2\Delta_{NS}^{b/2}} \mathcal{F}_{i}^{NS}(\eta), \quad i = 1, 2, 3
\]

\footnote{We will suppress one of the indices of the fields since \( \bar{\alpha} = \alpha \).}
These conformal blocks can be determined by two-fold Dotsenko-Fateev integrals [3], the index $i$ denotes the three independent integration contours between the branching points 0, $\eta$, 1, $\infty$. The conformal blocks $\mathcal{F}_{i}^{NS}(\eta)$ are regular at $\eta = 0$. Since we are interested in the limit $\eta \to 1$, we need to introduce another blocks $\tilde{\mathcal{F}}_{i}^{NS}(\eta)$ which are well defined in that limit. The monodromy relations between the conformal blocks are given by [3]:

$$\mathcal{F}_{i}^{NS}(\eta) = \sum_{j=1}^{3} \alpha_{ij} \tilde{\mathcal{F}}_{j}^{NS}(\eta),$$

where again $\alpha_{ij}$ are known and expressed in terms of gamma-functions.

On the pseudosphere geometry, as the two fields approach the boundary $\eta \to 1$, the distance between the two points become infinite due to the singular metric. This means that the two-point function is factorized into a product of two one-point functions. For example, the two-point function in (3.93) becomes:

$$G_{\alpha}(\xi, \xi') = \left|\frac{\xi' - \bar{\xi}'}{\xi - \bar{\xi}}\right|^{2\Delta_{NS}^2 - 2\Delta_{NS} - b/2} U_{NS}(-b/2) U_{NS}(\alpha) \tilde{\mathcal{F}}_{3}^{NS}(\eta).$$

Comparing these two results, we can obtain the following nonlinear functional equation for $U(\alpha)$:

$$\mathcal{C}_{1} U_{NS}(-b/2) U_{NS}(\alpha) = \frac{\Gamma(ab - \frac{b^2}{2}) \Gamma(ab + \frac{b}{2})}{\Gamma(ab) \Gamma(ab - \frac{b^2}{2} - \frac{b}{2})} U_{NS} \left(\alpha - \frac{b}{2}\right) + 2^{-2-2b^2} \pi^{2b^4} \mu \frac{\Gamma(ab - \frac{1}{2}) \Gamma(ab + \frac{b^2}{2})}{\Gamma(ab) \Gamma(ab - \frac{b^2}{2} + \frac{b}{2})} U_{NS} \left(\alpha + \frac{b}{2}\right)$$

(3.94)

with:

$$\mathcal{C}_{1} = \frac{\sqrt{\pi} \Gamma(-\frac{b^2}{2})}{\Gamma(-1) \Gamma(-\frac{b^2}{2} - \frac{b}{2})}.$$  

(3.95)

Similarly, using the OPE’s of the degenerate field $N^{-1}_{-\frac{b}{2}}$ with arbitrary primary fields $N_{\alpha, \bar{\alpha}}$ and $R_{\alpha, \bar{\alpha}}$ and the dual $N = 2$ action defined in Section (3.3), we can derive functional equations:

$$\mathcal{C}_{2} U_{NS}(-1/b, 0) U_{NS}(\alpha, \bar{\alpha}) = \frac{\Gamma(\frac{a+\bar{\alpha}}{b} - \frac{1}{b} + 1)}{(1 - ab) \Gamma(\frac{a+\bar{\alpha}}{b} - \frac{1}{b})} U_{NS} \left(\alpha - \frac{1}{b}, \bar{\alpha}\right) - \frac{\tilde{\mu}}{\Gamma(1 + \frac{a+\bar{\alpha}}{b})} U_{NS} \left(\alpha, \bar{\alpha} + \frac{1}{b}\right),$$

(3.96)

$$\mathcal{C}_{2} U_{NS}(-1/b, 0) U_{NS}(\alpha, \bar{\alpha}) = \frac{\Gamma(\frac{a+\bar{\alpha}}{b} - \frac{1}{b} + 1)}{(\frac{3}{2} - ab) \Gamma(\frac{a+\bar{\alpha}}{b} - \frac{1}{b})} U_{NS} \left(\alpha - \frac{1}{b}, \bar{\alpha}\right) - \frac{\tilde{\mu}}{\Gamma(1 + \frac{a+\bar{\alpha}}{b})} U_{NS} \left(\alpha, \bar{\alpha} + \frac{1}{b}\right).$$

(3.97)
with $C_2 = \Gamma(1 - \frac{1}{b^2})/\Gamma(-\frac{2}{b^2})$ and $\tilde{\mu}'$ is given by $\tilde{\mu}$, the cosmological constant of the dual $N = 2$ theory:

$$\tilde{\mu}' = 4\pi\tilde{\mu}\gamma(1 + b^{-2})b^{-4}.$$  

Here we have denoted one-point functions of the class-II degenerate field in terms of $\tilde{U}^{NS}$ since they are in principle different from the one-point functions of general fields. Notice that $C_1$ contains $\Gamma(-1)$ in Eq. (3.95) which arises from singular monodromy transformation. We can remove this singular factor by redefining $U^{NS(R)}/\Gamma(-1) \to U^{NS(R)}$. Notice that this redefinition does not change Eqs. (3.96) and (3.97) since they are linear in $U^{NS(R)}$ if assuming that $\tilde{U}^{NS}$ is regular.

The solutions to these equations can be expressed in terms of two integers $m, n \geq 1$ as follows:

$$U^{NS}_{mn}(\alpha, \bar{\alpha}) = N_{mn}(\pi\mu)^{-\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2} + 1}\Gamma(b(\alpha + \bar{\alpha}) - 1)$$

$$\times \sin \left[ \frac{\pi m}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right] \sin \left[ \frac{\pi nb}{2} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right],$$

(3.98)

$$U^{R}_{mn}(\alpha, \bar{\alpha}) = N_{mn}(\pi\mu)^{-\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2} + 1}\Gamma(b(\alpha + \bar{\alpha}) - 1)$$

$$\times \sin \left[ \frac{\pi m}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right] \sin \left[ \frac{\pi nb}{2} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right],$$

(3.99)

with the normalization factors given by:

$$N_{mn} = (-1)^n \frac{4b^2}{\Gamma(-1/b^2)} \frac{\cot(\pi nb^2)}{\sin(\pi m/b^2)}.$$  

This class of solutions will be associated with conformal BCs corresponding to the class-I neutral degenerate fields. It turns out that the conformal bootstrap equations do not allow discrete BCs corresponding to non-neutral degenerate fields. One possible explanation is that non-neutral BCs will introduce a boundary field which will not produce the identity operator when fused with bulk degenerate fields as they approach the boundary.

It is interesting to notice that the following one-point functions:

$$U^{NS}_{m}(\alpha, \bar{\alpha}) = N_{m}(\pi\mu)^{-\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2}} \Gamma(1 - \alpha b)\Gamma(1 - \bar{\alpha}b)$$

$$\times \sin \left[ \frac{\pi m}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right] \sin \left[ \frac{\pi}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right],$$

(3.100)

$$U^{R}_{m}(\alpha, \bar{\alpha}) = N_{m}(\pi\mu)^{-\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2}} \Gamma(\frac{3}{2} - \alpha b)\Gamma(\frac{1}{2} - \bar{\alpha}b)$$

$$\times \sin \left[ \frac{\pi m}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right] \sin \left[ \frac{\pi}{b} \left( \alpha + \bar{\alpha} - \frac{1}{b} \right) \right],$$

(3.101)

$$N_{m} = \frac{\pi}{\Gamma(-\frac{1}{b^2}) \sin(\pi m/b^2)}.$$  

(3.102)
satisfy Eqs. (3.96) and (3.97). Although they do not satisfy Eq. (3.94),
hence not complete solutions, this class of solutions turns out to be consistent
with modular bootstrap equations and we will show that they correspond to the
class-III BCs.

We want now to derive the modular bootstrap equations based on the modular
properties of degenerate characters. We will derive the boundary amplitudes
and show that they are consistent with the one-point functions derived before. Among
the conformal BC's of the $N = 2$ SLFT, we concentrate on those associated with the
degenerate fields. Following the modular bootstrap formulation, we can compute a boundary amplitude as the inner product
between the Ishibashi state of a primary state and the conformal boundary state.

Let us start with the class-I BC's. The boundary amplitudes are defined by:

$$
\Psi^{NS}_{mn}(P,\omega) = \langle m, n, \omega | N_{[P, \omega]} \rangle.
$$

From the modular transformation properties of the class-I character we can obtain:

$$
\Psi^{NS}_{mn}(P,\omega')\Psi^{NS\dagger}_{0}(P,\omega') = 2b \sinh(2\pi mP/b) \sinh(2\pi nbP) e^{-\pi ib^2\omega'}. 
$$

Since the vacuum boundary amplitude $\Psi^{NS}_{0}(P,\omega)$ was obtained in the previous Section, we get from (3.103) that:

$$
\Psi^{NS}_{mn}(P,\omega) = \sqrt{8b} \frac{\Gamma \left( \frac{2\pi b}{b} \right)}{\Gamma \left( \frac{1}{2} + ibP + \frac{b^2\omega}{2} \right) \Gamma \left( \frac{1}{2} + ibP - \frac{b^2\omega}{2} \right)} \sinh(2\pi mP/b) \sinh(2\pi nbP) e^{-\pi ib^2\omega'}. 
$$

This solution coincides with (3.98), the ZZ-brane solution with BC $(m, n, \omega = 0)$. This provides an important consistency check between the conformal and modular bootstraps.

We pass now to the class-II BC's. Denoting the class-II boundary state as $| m, \omega \rangle$, we can define the following boundary amplitude:

$$
\Psi^{NS}_{m}(P,\omega') = \langle m, \omega | N_{[P, \omega']} \rangle.
$$

Comparing this with the modular transformation of the class-II characters we obtain:

$$
\Psi^{NS}_{m}(P,\omega')\Psi^{NS\dagger}_{0}(P,\omega') = S_{m}(P,\omega'),
$$

where $S_{m}(P,\omega')$ is the modular $S$-matrix component. From this, one can solve for $\Psi^{NS}_{m}(P,\omega')$. Instead of presenting details for this case, we will analyze a more interesting case, namely the neutral $(\omega = 0)$ class-III BCs.

For a class-III (neutral) boundary state $| m \rangle$, we can define:

$$
\Psi^{NS}_{m}(P,\omega) = \langle m | N_{[P, \omega]} \rangle.
$$

In the same way as before it follows that:

$$
\Psi^{NS}_{m}(P,\omega) = \Psi^{NS}_{0}(P,\omega) \frac{\sinh(2\pi mP/b)}{\sinh(2\pi P/b)}. 
$$

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The solution (3.106) coincides with the one-point function (3.100).

One can perform similar analysis for the R sector. For the class-I amplitudes for example one obtains:

$$\Psi_{mn\omega}^R(P,\omega')\Psi_0^R(P,\omega') = 2b \sinh(2\pi mP/b) \sinh(2\pi nbP)e^{-\pi ib^2}\omega'$$

from which we can find:

$$\Psi_{mn\omega}^R(P,\omega') = -i\sqrt{8b}(\pi\mu)^{-\frac{2i\mu}{b}} \frac{\Gamma\left(\frac{2i\mu}{b}\right)\Gamma\left(1 + 2ibP\right)}{\Gamma\left(ibP + \frac{b\omega}{2}\right)\Gamma\left(1 + ibP - \frac{b\omega}{2}\right)} \times \sinh(2\pi mP/b) \sinh(2\pi nbP)e^{-\pi ib^2}\omega'$$

It is straightforward to continue this analysis for the class-II and class-III BCs and their mixed BCs for the R sector.

### 3.6 Higher equations of motion in $N = 2$ super-Liouville field theory

Let us remind that the $N = 2$ SLFT is based on the Lagrangian:

$$\mathcal{L} = \frac{1}{2\pi} \left( \partial \phi^- \bar{\partial} \bar{\phi}^+ + \partial \phi^+ \bar{\partial} \bar{\phi}^- + \psi^- \bar{\partial} \psi^+ + \psi^+ \bar{\partial} \psi^- + \bar{\psi}^- \partial \bar{\psi}^+ + \bar{\psi}^+ \partial \bar{\psi}^- \right) +$$

$$+ i\mu b^2 \psi^- \bar{\psi}^- e^{b\phi^+} + i\mu b^2 \psi^+ \bar{\psi}^+ e^{b\phi^-} + \pi \mu^2 b^2 e^{b\phi^+ + b\phi^-}$$

where $(\phi^\pm, \psi^\mp)$ are the components of a chiral $N = 2$ supermultiplet, $b$ is the coupling constant and $\mu$ is the cosmological constant. It is invariant under the $N = 2$ superconformal algebra (3.1) with central charge $c = 3 + \frac{6}{b^2}$. In (3.1) the left handed generators appear, there are in addition the right handed ones $\bar{L}_n$, $\bar{J}_n$, $\bar{G}_r^\pm$ closing the same algebra. The basic objects of interest here are the primary fields in the NS sector defined by the vertices $N_{\alpha,\bar{\alpha}} = e^{\alpha\phi^+ + \bar{\alpha}\phi^-}$, the corresponding states being annihilated by the positive modes. There are in addition also Ramond primary fields $R_{\alpha,\bar{\alpha}}$ but we will not be concerned with them in this Section. Let us remind also that the conformal dimension and the $U(1)$ charge of the primary fields are:

$$\Delta_{\alpha,\bar{\alpha}} = -\alpha\bar{\alpha} + \frac{1}{2b}(\alpha + \bar{\alpha}), \quad \omega = \frac{1}{b}(\alpha - \bar{\alpha}).$$

(3.107)

As we explained in the previous Section, among the primary fields there is a series of degenerate fields of the $N = 2$ SLFT. They are characterized by the fact that at certain level of the corresponding conformal family a new primary field (i.e. annihilated by all positive modes) appears. Such fields were divided there in three classes.

Class I degenerate fields $N_{m,n}^\omega = N_{\alpha m,\bar{\alpha} m}^\omega$ are given by (3.85) where $m, n$ are positive integers. $N_{m,n}^\omega$ is degenerate at level $mn$ and relative $U(1)$ charge zero. The irreducibility of the corresponding representations is assured by imposing the null-vector condition.
examples for class IIA fields:

The null-operators for class IIB fields are obtained from (3.109) by changing $D_{m,n} = 0$, $\bar{D}_{m,n} N^\omega_{m,n} = 0$, where $\bar{D}_{m,n}$ is a polynomial of the generators in (3.11) of degree $mn$ and has $U(1)$ charge zero. It is normalized by choosing the coefficient in front of $(L_{-1})^m$ to be 1. Let us give some examples of the corresponding null-operators:

$$D_{1,1}^\omega = L_{-1} - \frac{1}{2} b^2 (1 + \omega) J_{-1} + \frac{1}{\omega - 1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}},$$

$$D_{1,2}^\omega = L_{-1}^2 + b^2 L_{-2} - b^2 (1 + \omega) L_{-1} J_{-1} + \frac{b^2}{2} (1 + \omega - b^2 (2 + \omega)) J_{-2} + \frac{b^4}{4} \omega (\omega + 2) J_{-1}^2 + \frac{2}{\omega - 2} L_{-1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} - \frac{b^2 \omega}{\omega - 2} J_{-1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} - \frac{b^2}{2} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} + \frac{b^2}{2} \omega + 2 \omega G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}},$$

$$D_{2,1}^\omega = L_{-1} + \frac{1}{b^2} L_{-2} - b^2 (1 + \omega) L_{-1} J_{-1} + \frac{1}{2} (b^2 (1 + \omega) - \omega - 2) J_{-2} + \frac{1}{b^4} (b^4 (\omega + 1)^2 - 1) J_{-1}^2 + \frac{2 b^4 \omega}{b^4 (\omega - 1)^2 - 1} L_{-1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} - \frac{b^2 + b^4 (\omega^2 - 1)}{b^4 (\omega - 1)^2 - 1} J_{-1} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} + \frac{b^4 (\omega + 1) + b^2 - 2}{2 + 2 b^2 (\omega - 1)} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} + \frac{2 - b^2 + b^4 (\omega - 1) (1 + b^2 (\omega + 1)) G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}}}{2 b^4 (\omega - 1)^2 - 1}. \quad (3.108)$$

The second class of degenerate fields is denoted by $N^\omega_m$ and comes in two subclasses IIA and IIB introduced above in (3.87) and (3.88) respectively. Here $m$ is an odd positive integer number and the level of degeneracy of $N^\omega_m$ is $\frac{m}{2}$, relative charge $\pm 1$. In this case the operator $D^\omega_m$ is a polynomial of “degree” $m/2$, the coefficient in front of $L_{-1}^m G^+_{-\frac{1}{2}}$ is chosen to be 1. Analogously to the class I we have to impose $D^\omega_m N^\omega_m = \bar{D}^\omega_m N^\omega_m = 0$. Here are the first examples for class IIA fields:

$$D_{1}^\omega = G^+_{-\frac{1}{2}},$$

$$D_{3}^\omega = L_{-1} G^+_{-\frac{1}{2}} - J_{-1} G^+_{-\frac{1}{2}} + \left( \frac{2}{b^2} - \omega \right) G^+_{-\frac{1}{2}},$$

$$D_{5}^\omega = L_{-1}^2 G^+_{-\frac{1}{2}} + \left( \frac{4}{b^2} - \omega - 1 \right) L_{-2} G^+_{-\frac{1}{2}} - 3 L_{-1} J_{-1} G^+_{-\frac{1}{2}} + 2 J_{-1}^2 G^+_{-\frac{1}{2}} + \left( \frac{5}{2} - \frac{6}{b^2} + \frac{3}{2} \omega \right) J_{-2} G^+_{-\frac{1}{2}} + \left( 1 + \frac{6}{b^2} - 2 \omega \right) L_{-1} G^+_{-\frac{1}{2}} + 4 \left( \omega - \frac{3}{b^2} \right) J_{-1} G^+_{-\frac{1}{2}} + \frac{24}{b^4} - \frac{14 \omega}{b^2} + 2 \omega^2 - 1 \right) G^+_{-\frac{1}{2}}. \quad (3.109)$$

The null-operators for class IIB fields are obtained from (3.109) by changing $G^+ \rightarrow G^\mp$ and $\omega \rightarrow -\omega$.

We remind that a special case of Class IIA (B) fields are the chiral (antichiral) fields with
\[ m = 1. \] The Class II fields having \( U(1) \) charge zero are classified in a separate Class III fields. The simplest \( m = 1 \) field here represents the identity operator.

Let us now consider, for a further use, the norms of the states created by applying the null-operators on primary states \(|\alpha\rangle\). As explained above, such states should vanish at \( \alpha = \alpha_M^\omega \)

Taking the first terms in the corresponding Taylor expansion, we define:

\[
\begin{align*}
\bar{r}_M^\omega &= \partial_\alpha \langle \alpha, \bar{\alpha} | D_M^{\omega \dagger} D_M^{\omega} | \alpha, \bar{\alpha} \rangle |_{\alpha = \alpha_M^\omega, \bar{\alpha} = \bar{\alpha}_M^\omega}, \\
\bar{r}_M^\omega &= \partial_\alpha \langle \alpha, \bar{\alpha} | D_M^{\omega \dagger} D_M^{\omega} | \alpha, \bar{\alpha} \rangle |_{\alpha = \alpha_M^\omega, \bar{\alpha} = \bar{\alpha}_M^\omega} 
\end{align*}
\]

for both classes of representations, \( M = m \) or \( (m, n) \), where \( D_M^{\omega \dagger} \) is the corresponding null-operator and \( D_M^{\omega} \) is defined as usual through \( L_n^1 = L_{-n}^1, J_n^1 = J_{-n}, (G^\pm_r)^\dagger = G^\mp_{-r}. \)

One can compute "by hand" the first few \( r \)'s. With the use of the explicit form of the null-operators (3.108) we find for the class I fields:

\[
\begin{align*}
\bar{r}_{1,1}^\omega &= \frac{1}{b} \frac{(1 + b^2)(1 + \omega)}{(-1 + \omega)}, \\
\bar{r}_{1,2}^\omega &= \frac{-2}{b} \frac{(1 - b^2)(1 + b^2)(1 + 2b^2)(2 + \omega)}{(-2 + \omega)}, \\
\bar{r}_{1,3}^\omega &= \frac{12}{b} \frac{(1 - 2b^2)(1 - b^2)(1 + b^2)(1 + 2b^2)(1 + 3b^2)(3 + \omega)}{(-3 + \omega)}, \\
\bar{r}_{2,1}^\omega &= \frac{2}{b^5} \frac{(1 - b^2)(1 + b^2)(2 + b^2)(-1 + b^2 + b^2\omega)(1 + b^2 + b^2\omega)}{(-1 - b^2 + b^2\omega)(1 - b^2 + b^2\omega)} 
\end{align*}
\]

and \( \bar{r}_{m,n}^\omega = \bar{r}_{m,n}^\omega \) for all the examples above. Based on these expression we propose for the general form of \( \bar{r}_{m,n}^\omega \):

\[
\bar{r}_{m,n}^\omega = \bar{r}_{m,n}^\omega = \prod_{l=1-m}^m \prod_{k=1-n}^n \left( \frac{l - kb}{b} \right) \prod_{l=1-m, \text{mod } 2}^{m-1} \left( \frac{l - (n + \omega)b^2}{l + (n - \omega)b^2} \right). \tag{3.111}
\]

Similarly, from (3.109) we have for the class IIA:

\[
\begin{align*}
\bar{r}_1^\omega &= 2 \left( \frac{1}{b} - \omega b \right), \\
\bar{r}_3^\omega &= \frac{2}{b^5} (2 - b^2\omega)(3 - b^2\omega)(2 - b^2 - b^2\omega), \\
\bar{r}_5^\omega &= \frac{8}{b^9} (3 - b^2\omega)(4 - b^2\omega)(5 - b^2\omega)(3 - b^2 - b^2\omega)(4 - b^2 - b^2\omega), \\
\bar{r}_m^\omega &= 0, \quad m = 1, 3, 5, 7.
\end{align*}
\]

These expressions can be fitted in a general form of \( r_m^\omega \) and \( \bar{r}_m^\omega \):

\[
\begin{align*}
\bar{r}_m^\omega &= 0, \\
\bar{r}_m^\omega &= 2 (1 + \frac{m+1}{2}) b^{1-m} \prod_{l=m+1}^m \left( \frac{l - b\omega}{b} \right) \prod_{l=m+1}^{m-1} \left( \frac{l - b(\omega + 1)}{l} \right). \tag{3.112}
\end{align*}
\]
For the class IIB fields one obtains \( \bar{r}_m^\omega = 0 \) and \( r_m^\omega \) is as \( \bar{r}_m^\omega \) in (3.112) with the change \( \omega \to -\omega \).

Let us now introduce the so called logarithmic fields. They are defined as:

\[
N'_{\alpha, \bar{\alpha}} = \partial_{\alpha} N_{\alpha, \bar{\alpha}}, \quad \tilde{N}'_{\alpha, \bar{\alpha}} = \partial_{\alpha} \tilde{N}_{\alpha, \bar{\alpha}}.
\]

One can introduce also the logarithmic primary fields corresponding to degenerate fields by:

\[
N_M^{\omega} = N'_{\alpha, \bar{\alpha}}|_{a = \alpha_{\bar{\alpha}}', \bar{\alpha} = \bar{\alpha}_{\alpha}'}, \quad \tilde{N}_M^{\omega} = \tilde{N}'_{\alpha, \bar{\alpha}}|_{a = \alpha_{\bar{\alpha}}', \bar{\alpha} = \bar{\alpha}_{\alpha}'} \tag{3.113}
\]

where \( M \) is \((m, n)\) for class I and \( M \) is \( m \) for class II fields respectively. The basic statement about the fields \((3.113)\) is that:

\[
\tilde{N}_M^{\omega} = \tilde{D}_M^{\omega} D_M^{\omega} N_M^{\omega}, \quad \tilde{N}_M^{\omega} = \tilde{D}_M^{\omega} D_M^{\omega} N_M^{\omega} \tag{3.114}
\]

with \( D_M^{\omega}, \tilde{D}_M^{\omega} \) as in (3.108), (3.109) are again primary. The proof of this statement goes along the same lines as for \( N = 0, 1 \) SLFT \[27, 28\].

Comparing the dimension and \( U(1) \) charge for class I fields: \( \tilde{\Delta}_{m,n} = \Delta_{m,n} + mn, \tilde{\omega} = \omega \) we conclude that the fields \((3.114)\) are proportional to \( N_{m,-n}^{\omega} \). Thus, we arrive at the higher equations of motion (HEM) for the class I fields:

\[
\tilde{D}_m^{\omega} D_m^{\omega} N_m^{\omega} = B_m^{\omega} N_m^{\omega}, \quad \tilde{D}_m^{\omega} D_m^{\omega} N_m^{\omega} = \tilde{B}_m^{\omega} N_m^{\omega} \tag{3.115}
\]

For class IIA (B) the dimension of the resulting primaries in \((3.114)\) is \( \tilde{\Delta}_m^\omega = \Delta_m^\omega + m/2 \), the \( U(1) \) charges are \( \tilde{\omega} = \omega + 1 \) \((\tilde{\omega} = \omega - 1)\) respectively, and the HEMs in this case are:

\[
\tilde{D}_m^{\omega} D_m^{\omega} N_m^{\omega} = B_m^{\omega} N_m^{\omega}, \quad \tilde{D}_m^{\omega} D_m^{\omega} N_m^{\omega} = \tilde{B}_m^{\omega} N_m^{\omega} \tag{3.116}
\]

Computation of \( B_{m,n}^{\omega}(\tilde{B}_{m,n}^{\omega}) \) and \( B_m^{\omega}(\tilde{B}_m^{\omega}) \) is the final goal of this Section. HEMs \((3.115)\) and \((3.116)\) are to be understood in an operator sense, i.e. they should hold for any correlation function. Here we will insert them into the simplest one-point function on the so called Poincaré disk (or Lobachevski plain) \[24\]. In this case we have:

\[
\langle B_1 | \tilde{D}_m^{\omega} D_m^{\omega} N_m^{\omega} | \tilde{N}_M^{\omega} \rangle = \langle B_1 | \tilde{N}_M^{\omega} \rangle, \quad \langle B_1 | D_m^{\omega} D_m^{\omega} \tilde{N}_M^{\omega} | \tilde{N}_M^{\omega} \rangle = \langle B_1 | \tilde{N}_M^{\omega} \rangle.
\]

The boundary state \( \langle B_1 \rangle \) corresponds to the identity boundary conditions on the Poincaré disc. It enjoys \( N = 2 \) superconformal invariance:

\[
\lambda B_1 | G_r^\pm \rangle = -i \lambda B_1 | G_r^\mp \rangle, \quad \lambda B_1 | \bar{L}_n, \lambda B_1 | (L_n)^\dagger \rangle, \quad \lambda B_1 | \bar{J}_n \rangle = \lambda B_1 | (J_n)^\dagger \rangle.
\]

(so called A-type boundary conditions, see e.g. \[26\]).

With the definition of \( r \)'s in \((3.110)\) the HEMs \((3.115)\) and \((3.116)\) take the form:

\[
r_{m,n}^{\omega} U_1(m, n; \omega) = B_{m,n}^{\omega} U_1(m, -n; \omega), \quad \bar{r}_{m,n}^{\omega} U_1(m, n; \omega) = \bar{B}_{m,n}^{\omega} U_1(m, -n; \omega) \tag{3.117}
\]
for class I, and:

\[ r^\omega_m U_1(m, \omega) = iB^\omega_m U_1(m, \omega + 1), \quad r^\omega_m U_1(m, \omega) = i\bar{B}^\omega_m U_1(m, \omega + 1) \quad (3.118) \]

for class II. Here \( U_1 \) is the one-point function for “identity boundary conditions” of the corresponding field. In (3.118) the factor \( i \)’s appear because the class II null-operators are fermionic, and + (−) refers to class IIA (IIB).

The one-point function on the Poincaré disk for identity (or vacuum) boundary conditions in \( N = 2 \) SLFT was obtained in the previous Section. Let us remind its general form:

\[ U_1(\alpha, \bar{\alpha}) = \Gamma(b^-2)(\pi\mu)^{-\frac{1}{2}(\alpha + \bar{\alpha})} \frac{\Gamma(1 - \alpha b)\Gamma(1 - \bar{\alpha}b)}{\Gamma(-\frac{\alpha + \bar{\alpha}}{b} + \frac{1}{b^2})\Gamma(2 - b(\alpha + \bar{\alpha}))}. \]

With the specific values (3.85) the ratio of one-point functions of class I fields then is:

\[ \frac{U_1(m, n; \omega)}{U_1(m, -n; \omega)} = (\pi\mu)^2n^2\gamma(1 + m - nb^2) \prod_{k=-n}^{n-1} (\frac{m}{b} + k) \prod_{l=-m}^{m} (l + nb^2) \gamma(1 - m + (n - \omega)b^2) \times \prod_{l=1-m, \mod 2}^{m-1} \frac{l + (n - \omega)b^2}{l - (n + \omega)b^2} \]

and for the HEM coefficient we obtain:

\[ B^\omega_{m,n} = B^\omega_{m,n} = \gamma^\omega_m \frac{U_1(m, n; \omega)}{U_1(m, -n; \omega)} = (\pi\mu)^2n^{2b^2} \gamma(m - nb^2) \frac{\gamma(1 + m + (n - \omega)b^2)}{\gamma(1 - m + (n + \omega)b^2)} \prod_{l=1-m}^{m-1} \prod_{k=-n}^{n-1} (\frac{l}{b} + kb) \quad (3.119) \]

where we impose that \((k, l) = (0, 0)\) is excluded in the product.

Analogously for class IIA fields:

\[ \frac{U_1(m, \omega)}{U_1(m, \omega + 1)} = \pi\mu b \frac{\prod_{l=m+1}^{m-1} (\frac{l}{b} - b(\omega + 1))}{\prod_{l=m+1}^{m+1} (\frac{l}{b} - b\omega)} \]

and:

\[ B^\omega_m = 0, \quad \bar{B}^\omega_m = -i r^\omega_m \frac{U_1(m, \omega)}{U_1(m, \omega + 1)} = -2\pi i\mu b^{2-m}\Gamma^2 \left(\frac{m+1}{2}\right) \prod_{l=m+1}^{m-1} \left(\frac{l}{b} - b(\omega + 1)\right)^2 \quad (3.120) \]

For class IIB \( B \) and \( \bar{B} \) are exchanged and \( \omega \) is replaced by \(-\omega\). Equalities (3.119) and (3.120) are the main results of this Section.
We want now to check these results in the classical limit. The latter is defined as \( b \to 0 \): \( b\phi \to \varphi, \; b\beta \psi \to \psi, \; \pi \mu b^2 \to M \) the Lagrangian \( \mathcal{L} \to \frac{1}{2m^2} \mathcal{L} \). The corresponding equations of motion are given by:

\[
\bar{\partial} \psi^{\pm} = -i M \bar{\psi}^{\pm} e^{\varphi^{\pm}}, \quad \partial \psi^{\pm} = i M \psi^{\mp} e^{\varphi^{\mp}}, \quad \partial \bar{\varphi}^{\pm} = i M \bar{\psi}^{\mp} e^{\varphi^{\mp}} + M^2 e^{\varphi^{+}+\varphi^{-}}. \tag{3.121}
\]

The holomorphic currents:

\[
T^{\pm} = -\partial \varphi^{-} \partial \varphi^{+} - \frac{1}{2}(\psi^{+} \partial \psi^{+} + \psi^{-} \partial \psi^{-}) + \frac{1}{2}(\partial^2 \varphi^{+} + \partial^2 \varphi^{-}), \quad S^{\pm} = -i \sqrt{2}(\psi^{\pm} \partial \varphi^{\pm} - \partial \psi^{\pm}), \quad J = \partial \varphi^{-} - \partial \varphi^{+} - \psi^{-} \psi^{+}, \tag{3.122}
\]

are conserved by \( \bar{\partial} T = \bar{\partial} S^{\pm} = \bar{\partial} J = 0 \) on the equations of motion and similarly for the antiholomorphic ones. One has to introduce also the generators of \( N = 2 \) supersymmetry \( G^{\pm} \) and \( G^{\pm} \):

\[
G^{\pm} \varphi^{\mp} = i \sqrt{2} \psi^{\pm}, \quad G^{\pm} \varphi^{\pm} = 0, \quad G^{\pm} \varphi^{\mp} = 0
\]

obeying the algebra:

\[
\{G^{+}, G^{-}\} = 2\partial, \quad \{G^{\pm}, G^{\pm}\} = \{G^{\pm}, \bar{G}^{\pm}\} = 0, \quad \{G, \bar{G}\} = 0. \tag{3.124}
\]

For the class IIA fields only the chiral fields, \( N^{\omega}_{1^0} = e^{\omega \phi^{+}} \), has a classical limit. Their HEMs take the form:

\[
\bar{G}^{+}_{-\frac{1}{2}} G^{+}_{-\frac{1}{2}} \phi^{+} N^{\omega}_{1^0} = 0, \quad \bar{G}^{+}_{-\frac{1}{2}} G^{+}_{-\frac{1}{2}} \phi^{-} N^{\omega}_{1^0} = B^{\omega}_{1} N^{\omega+1}_{1},
\]

where \( B^{\omega}_{1} = -2\pi \mu b \) can be read from (3.120). In the classical limit along with the analogous HEMs for class IIB anti-chiral fields with \( \omega = 0 \), these become:

\[
\bar{G}^{\pm} G^{\pm} \varphi^{\mp} = -2i M e^{\varphi^{\pm}}.
\]

Together with (3.123) and the algebra (3.124) these relations encode the equations of motion (3.121).

From the class I fields only the series \( N^{\omega}_{1,\alpha} \) has a classical limit, the simplest “classical null-operators” being:

\[
D^{\omega(c)}_{1,1} = \partial - \frac{1}{2}(\omega + 1)J + \frac{1}{\omega - 1} G^{+} G^{-},
\]

\[
D^{\omega(c)}_{1,2} = \partial^2 - (\omega + 1) J \partial - \frac{1}{2}(\omega + 2) \partial J + \frac{1}{4} \omega (\omega + 2) J^2 + \frac{2}{\omega - 2} G^{+} G^{-} \partial - \frac{\omega}{\omega - 2} J G^{+} G^{-} - \frac{1}{2} S^{-} G^{+} + \frac{1}{2} \frac{\omega + 2}{\omega - 2} S^{+} G^{-}.
\]

It is easy to check, using the algebra (3.124) and the explicit form of the currents (3.122), that the classical expressions of the corresponding null-vector conditions are:

\[
D^{\omega(c)}_{1,1} e^{\frac{1}{2}(\omega - 1) \varphi^{+} - \frac{1}{2}(\omega + 1) \varphi^{-}} = 0, \quad D^{\omega(c)}_{1,2} e^{\frac{1}{2}(\omega - 2) \varphi^{+} - \frac{1}{2}(\omega + 2) \varphi^{-}} = 0.
\]
precisely, we propose a construction of minimal models. In what follows is that a general level of lower level models can be expressed explicitly in terms of the corresponding data from the Virasoro minimal models only. In this sense, all the minimal models - can be expressed explicitly in terms of the corresponding data from the Virasoro minimal models. By iterating this recursive construction we arrive at the projected product of Virasoro minimal models. The crucial role is played by the projection. This is in a perfect agreement with (3.115) if we take into account that the classical limit, $b \rightarrow 0$, of $B_{1,n}^\omega = B_{1,n}^\omega$ from (3.119) is:

$$B_{1,n}^\omega \rightarrow (-1)^{n+1} \frac{\omega + n}{\omega - n} n! (n - 1)! b^{-1} (\pi \mu b^2)^{2n}.$$  

4 General $\hat{su}(2)$ coset models

We start this Section with the description of the "fine structure" of the $\hat{su}(2)$ coset family of minimal models $M(k,l) = \hat{su}(2)_k \times \hat{su}(2)_l / \hat{su}(2)_{k+l}$, $k,l = 1,2,\ldots$. The main statement in what follows is that a general $l$'th family of minimal models $M(k,l), l > 1, k = 1,2,\ldots$ can be realized as a projected tensor product of consequent Virasoro minimal models $M(k,l) \equiv M(k)$. We show that all the data for the general $M(k,l)$ model - the primary fields, the conformal blocks and the 4-point functions, the structure constants, the fusion algebra etc. - can be expressed explicitly in terms of the corresponding data from the Virasoro minimal models only. In this sense, all the minimal models $M(k,l)$ with $l > 1$ are reducible. More precisely, we propose a construction of $M(k,l)$ in terms of a recursive projected product of lower level $l$ models. By iterating this recursive construction we arrive at the projected product of Virasoro minimal models. The crucial role is played by the projection $P$, i.e. the restriction of the products of Virasoro primary fields of the form $\phi_{p_1}^{k+1} \phi_{p_2}^{k+l-1}$ only. In particular, in computing the four-point functions only the products of conformal blocks corresponding to such products of fields are allowed. Still, we show that this is enough to construct monodromy invariant correlation functions. In this way we obtain the corresponding structure constants as products of the structure constants of the Virasoro models. One could wonder how general this procedure of reducing and solving a general coset model in terms of lower level coset models only is. Based on our experience with a variety of other coset constructions, our conjecture for an arbitrary (symmetric) coset series of models $G(k,l) = \hat{g}_k \times \hat{g}_l / \hat{g}_{k+l}$ ($\hat{g}_k$ denotes level $k$ of the affine algebra $\hat{g}$) is that $G(k,l)$ is reducible to the products of the first level models only.

We are next interested in the calculation of the matrix of anomalous dimensions and the corresponding mixing of certain fields in the case of general $\hat{su}(2)$ coset models perturbed by the least relevant field in the second order of the perturbation theory. In this Section we extend the results of [7] and [11] (presented in Section 2) to these models denoted as $M(k,l)$. The first order corrections are obtained in [18]. It is shown that there exists an infrared (IR)fixed point of the renormalization group flow which coincides with the model...
$M(k-l,l)$. As it was demonstrated in Section 2 the calculation up to second order is difficult. The problem is that one needs the corresponding 4-point functions which are not known exactly. Basic ingredients for the computation of the 4-point functions are the conformal blocks. They are quite complicated objects and a close form is not known. In this Section we use the strategy explained above. Namely, we use the fact that the structure constants and the conformal blocks for the general $\hat{su}(2)$ coset models $M(k,l)$ at some level $l$ can be obtained recursively from those of those of the lower levels or finally from the Virasoro minimal models by certain projected tensor product. We use this construction here to define the perturbing field and the other fields in consideration. It turns out that we are able to compute the necessary structure constants and conformal blocks up to the desired order. There is an alternative approach to the calculation of the mixing matrix in the perturbed CFT models, the so called RG domain wall [69, 70]. It was shown in [7, 71] for the Virasoro case and in [72] for the supersymmetric extension that there is an agreement between the results obtained by such construction and the perturbative calculations up to the second order. Moreover, as it was shown in Section 2, this mixing matrix do not depend on $\epsilon$ and is exactly the same in both theories. We show here that this is the case also for the general $\hat{su}(2)$ coset models perturbed by the least relevant field.

The results of this Section have been published in [48], [73]-[76], (17.-21.).

### 4.1 Fusion of conformal models

We start with the description of the $\hat{su}(2)$ coset family of minimal models:

$$M(k,l) = \frac{\hat{su}(2)_k \times \hat{su}(2)_l}{\hat{su}(2)_{k+l}}$$

(4.1)

where $k, l = 1, 2, \ldots$ [9]. They are conformal field theories with a central charge given by:

$$c = \frac{3kl(k+l+4)}{(k+2)(l+2)(k+l+2)} = \frac{3l}{l+2} \left(1 - \frac{2(l+2)}{(k+2)(k+l+2)}\right).$$

(4.2)

The main statement in this Section is that a general $l$-th family of minimal models $M(k,l)$ with $l > 1, k = 1, 2, \ldots$ can be realized as a projected tensor product of consequent Virasoro minimal models $M(k,1) \equiv M(k)$. We will show that all the data for a general $M(k,l)$ model - the primary fields, the conformal blocks and the four-point functions, the structure constants, the fusion algebra etc. - can be expressed explicitly in terms of the corresponding data from the Virasoro minimal models only. In this sense all the minimal models $M(k,l)$ are reducible.

More precisely, for general $l$ we state that:

$$M(l-1,1) \times M(k,l) = P(M(k,1) \times M(k+1,l-1)).$$

(4.3)
We introduced here explicitly the projection $P$. In terms of primary fields it projects from the space of all product of fields to the subspace where only a product of fields with the same internal indexes are allowed. By iterating eq. (4.3) we arrive at:

$$M(k,l) \times P(M(1,1) \times M(2,1) \times \ldots \times M(l-1,1)) =$$

$$= P(M(k,1) \times M(k+1,1) \times \ldots \times M(k+l-1,1)). \quad (4.4)$$

Eq. (4.4) means that any model $M(k,l), l > 1$ can be constructed and explicitly solved in terms of Virasoro models only. Note that we have imposed the projection $P$ too. Our statement is that Eqs. (4.3) and (4.4) imply that for any field from $M$ one can find fields from $M$ such that the (projected) products of the fields have the same correlation functions. Furthermore, where there is no projection $P$ (like between $M(k,l)$ and $M(k,l)$), the monodromy invariant 2D correlation function of the product of the fields factorizes into the product of the correlation functions. A crucial role is played by the projection $P$. In particular, in computing the 4-point functions only the products of of conformal blocks corresponding to the projected product of fields are allowed. We will show that this is enough to construct monodromy invariant correlation functions. In this way we will obtain the the corresponding structure constants as products of the structure constants of the Virasoro models.

One could wonder what is the origin of the reducibility of the models $M(k,l), l > 1$. A formal answer is that it follows from the obvious coset identities:

$$\frac{\hat{s}_u(2)_1 \times \hat{s}_u(2)_{l-1}}{\hat{s}_u(2)_l} \times \frac{\hat{s}_u(2)_l \times \hat{s}_u(2)_k}{\hat{s}_u(2)_{k+l}}$$

$$= \frac{\hat{s}_u(2)_1 \times \hat{s}_u(2)_k}{\hat{s}_u(2)_{k+1}} \times \frac{\hat{s}_u(2)_{l-1} \times \hat{s}_u(2)_{k+1}}{\hat{s}_u(2)_{k+l}}.$$

We will give a precise meaning to this statement below.

Consider first the problem of the realization of the $M(k,l)$ chiral algebra and its field representations in the space $P(M(k) \times M(l-1,k+1))$. We begin with the simplest case $l = 2$ (i.e. the superconformal models). The natural candidates for for the generators of the $N = 1$ superconformal algebra in $P(M(k) \times M(k+1))$ are the fields $\psi_p = \phi_p^k \phi^{k+1}_p, p = 2, 3 \ldots$ with dimensions $\Delta_p = \frac{1}{2}(p-1)^2$, their derivatives, and the stress-energy tensors $T^{(k)}, T^{(k+1)}$ of $M(k)$ and $M(k+1)$. Define the following field combinations:

$$\psi = \phi_{12}^k \phi_{21}^{k+1}, \quad (4.5)$$

$$T^I = \frac{k + 2}{4(k+5)} T^{(k)} + \frac{k + 4}{4(k+1)} T^{(k+1)} + \frac{1}{2} \sqrt{\frac{3k(k+6)}{4(k+1)(k+5)}} \phi_{13}^k \phi_{31}^{k+1},$$

$$G = i \sqrt{-\frac{1}{(k+2)(k+4)}} \left( k \phi_{12}^k \phi_{21}^{k+1} - (k+6) \partial \phi_{12}^k \phi_{21}^{k+1} \right), \quad (4.6)$$

$$T^{SUSY} = \frac{3(k+6)}{4(k+5)} T^{(k)} + \frac{3k}{4(k+1)} T^{(k+1)} - \frac{1}{2} \sqrt{\frac{3k(k+6)}{4(k+1)(k+5)}} \phi_{13}^k \phi_{31}^{k+1}.$$
We want to show that:

(i) $T^I$ and $\psi$ generate the usual Ising model algebra with central charge $c = \frac{1}{2}$;

(ii) $T^{SUSY}$ and $G$ are the generators of the $N = 1$ superconformal algebra with $c = 3/2 - 12/(k + 2)(k + 4)$;

(iii) the $(T^I, \psi)$ and $(T^{SUSY}, G)$ algebras are in direct product.

Let us start with (i). Using the OPE’s of $M(k)$ and $M(k + 1)$ models we have to prove that $\psi$ and $T^I$ given by (4.5) satisfy the well known OPE’s:

\[
T^I(z_1)T^I(z_2) = \frac{1}{4z_{12}^2} + \frac{2}{z_{12}^2}T^I(z_2) + \frac{1}{z_{12}} \partial T^I(z_2) + \ldots,
\]

\[
T^I(z_1)\psi(z_2) = \frac{1}{2z_{12}^2}\psi(z_2) + \frac{1}{z_{12}} \partial \psi(z_2) + \ldots,
\]

\[
\psi(z_1)\psi(z_2) = \frac{1}{z_{12}} + 2z_{12}T^I(z_2) + \ldots.
\]

To do this we have to implement the projection $P$ in the OPE’s and in the construction of the conformal blocks of the primary fields in terms of $M(k) \times M(k + 1)$ blocks. We address here the specific problem of constructing the 4-point functions and the OPE’s of the currents using the conformal blocks of the ingredients $\phi_{lp}^k$ and $\phi_{pl}^{k+1}$, $p = 1, 2, 3 \ldots$. According to the construction (4.5), the 4-point function of $\psi(z)$ can be written as a sum of products of the conformal blocks $I_i^k$ of $\phi_{12}^k$ and $I_j^{k+1}$ of $\phi_{21}^{k+1}$:

\[
F_\psi(z) \equiv \langle \psi(0)\psi(z)\psi(1)\psi(\infty) \rangle = (z(1 - z))^{-1}\sum_{i,j=1}^{2} Y_{ij}I_i^k(z)I_j^{k+1}(z).
\]

The condition for the monodromy invariance of $F_\psi(z)$ at $z = 0$ implies that $Y_{12} = 0 = Y_{21}$ and we obtain:

\[
F_\psi(z) = (z(1 - z))^{-1}\left(F\left(-\frac{k}{k + 3}, \frac{1}{k + 3}, \frac{2}{k + 3}; z\right)F\left(-\frac{k + 6}{k + 3}, \frac{1}{k + 3}, \frac{2}{k + 3}; z\right) +
\right.
\]

\[
+ Y_{22}z^2F\left(\frac{k + 2}{k + 3}, \frac{1}{k + 3}, \frac{2k + 4}{k + 3}; z\right)F\left(\frac{k + 4}{k + 3}, \frac{1}{k + 3}, \frac{2k + 8}{k + 3}; z\right)\right),
\]

where $Y_{11} = 1$ is a normalization condition. Considering the small distance behaviour $z \to 0$ of Eq. (4.7) we conclude that the first term gives rise to $\phi_{11}^k\phi_{11}^{k+1}(0)$ in the OPE $\psi(z)\psi(0)$ and the second one to $\phi_{13}^k\phi_{31}^{k+1}(0)$, i.e. the terms $\phi_{11}^k\phi_{31}^{k+1}(0)$ and $\phi_{13}^k\phi_{11}^{k+1}(0)$ are projected out. We therefore see that in this case applying the projection $P$ is the same as requiring monodromy invariance around $z = 0$ for the 4-point functions.

The monodromy invariance around $z = 1$ fixes:

\[
Y_{22} = C_{(12)(12)(13)}^k C_{(21)(21)(31)}^{k+1} = \frac{3k(k + 6)}{4(k + 1)(k + 5)}
\]
where \( C_{(12)(13)}^k \) and \( C_{(21)(31)}^{k+1} \) are the well known Virasoro structure constants. One can then show, using some nontrivial identities between the hypergeometric functions, that the correlation function \((4.7)\) coincides with the 4-point function of the free Majorana field \( \psi(z) \) given by:

\[
< \psi(0) \psi(z) \psi(1) \psi(\infty) >= \left( z(1-z) \right)^{-1}(1 - z + z^2).
\]

From here we can obtain the following OPE:

\[
\psi(z) \psi(0) = \frac{1}{z} + 2z \left( \frac{\Delta_{12}^k}{c(k)} T^k(0) + \frac{\Delta_{21}^{k+1}}{c(k+1)} T^{k+1}(0) + \frac{1}{2} \sqrt{Y_{22}^k} \phi_{13}^k \phi_{31}^{k+1}(0) \right) + \ldots.
\]

We see that the structure constant in front of the \( \phi_{13}^k \phi_{31}^{k+1} \) term is \( \sqrt{C_{(12)(13)}^k C_{(21)(31)}^{k+1}} \), a square root of what one would naively expect. To understand this, remember that the OPE’s should always be thought of as operations performed within well-defined correlation functions. Since the currents are distinguished from the usual scalar fields by having well defined 1D (dependent only on \( z \), i.e. with only left-moving fields) correlation functions, their 1D OPE’s are well-defined. In the present context, the currents are realized as sums of products of ordinary conformal fields whose only well-defined correlation functions (and therefore OPE’s and structure constants) are two-dimensional. Still, it can be proved that the particular combinations used to construct the currents can have well-defined 1D correlation functions. The monodromy invariance around \( z = 1 \) of the 1D 4-point functions of the currents results in the structure constants of 1D OPE being constructed from the square roots of the standard 2D structure constants. Heuristically, one could think of the square root appearing since only the left moving fields contribute to the OPE.

Let us return to the proof of the statement (i) and consider the OPE \( T^I \psi \). Keeping in mind the above discussion and using the Virasoro Ward identities we find that:

\[
T^I(z) \psi(0) = \left( \frac{\Delta_{12}^k}{c(k)} + \frac{\Delta_{21}^{k+1}}{c(k+1)} + \frac{3k(k+6)}{8(k+1)(k+5)} \right) \frac{1}{z^2} \psi(0) + \ldots
\]

In proving (ii) and (iii) we follow the same procedure, i.e. we start with the constructions \((4.5)\) and \((4.6)\) and perform the OPE’s. In these OPE’s we keep only the terms consistent with the projection \( P \) and use the square roots of the 2D structure constants in 1D OPE’s. Applying this to the product of two supercurrents we obtain the well-known OPE:

\[
G(z) G(0) = \frac{k(k+6)}{(k+2)(k+4)} \frac{1}{z^3} + \frac{2}{z} \left( \frac{3(k+6)}{4(k+5)} T^k(0) + \frac{3k}{4(k+1)} T^{k+1}(0) - \frac{1}{2} \sqrt{Y_{22}^k} \phi_{13}^k \phi_{31}^{k+1}(0) \right) + \ldots
\]

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implying \( c(2, k) = 3/2 - 12/(k + 2)(k + 4) \). Analogous calculations for \( \psi(z)G(0) \) and \( T^I(z)T^{SUSY}(0) \) show that no singular terms appear in these OPE’s, i.e. the Ising model algebra \((T^I, \psi)\) and the \( N = 1 \) superconformal algebra are in fact in direct product.

It remains to consider the supercurrent Ward identities and the properties of the primary fields. In terms of the latter \( \mathbf{P} \) projects from the space of all products of fields \( \{\phi^k_p \phi^{k+1}_p\} = M(k) \times M(k + 1) \) to the subspace:

\[
\mathbf{P}(M(k) \times M(k + 1)) = \{\phi^k_p \phi^{k+1}_p\}, \quad p = 1, \ldots, k + 2,
\]

which is isomorphic to the representation space \( M(1) \times M(2, k) \). This isomorphism is based on the following simple relations between the dimensions of the primary fields from two consequent Virasoro minimal models, \( N = 1 \) superconformal models and the Ising model \( M(1) \):

\[
\Delta_{rp}(1, k) + \Delta_{ps}(1, k + 1) - \Delta_{rs}^{NS}(2, k) = \frac{1}{2}(p - \frac{1}{2}(r + s))^2, \quad r - s \in 2\mathbb{Z},
\]

\[
\Delta_{rp}(1, k) + \Delta_{ps}(1, k + 1) - \Delta_{rs}^{R}(2, k) = \frac{1}{2}(p - \frac{1}{2}(r + s))^2 - \frac{1}{16}, \quad r - s \in 2\mathbb{Z} + 1.
\]

This leads us to suggest the following construction:

\[
N_{rs} = \phi^k_{r/2(r+s)} \phi^{k+1}_{1/2(r+s)},
\]

\[
\sigma R^i_{rs} = \phi^k_{r/2(r+s)} \phi^{k+1}_{1/2(r+s+1)}, \quad i = 1, 2.
\]

Here \( \sigma \) is the Ising field with dimension \( \Delta = 1/16 \), \( N_{rs} \) and \( R^i_{rs} \) are the NS and R fields of the \( N = 1 \) superconformal minimal model. The rest of the products \( \phi^k_p \phi^{k+1}_p \) for \( p \neq (r + s)/2 \) or \( r + s \equiv 1)/2 \) correspond to the descendants of the primary fields \((4.10)\).

Let us consider now the transformation properties of the fields \((4.10)\). Starting with the NS sector, we have to find a realization of the second component \( N^{II}_{rs} \) with dimension \( \Delta_{rs}(2, k) + 1/2 \), consistent with the constructions for \( G \) and \( N_{rs} \), i.e. satisfying the OPE’s:

\[
G(z)N_{rs}(0) = \frac{1}{z}N^{II}_{rs}(0) + \ldots, \quad r - s \in 2\mathbb{Z},
\]

\[
G(z)N^{II}_{rs}(0) = \frac{2\Delta_{rs}(2, k)}{z^2}N_{rs}(0) + \frac{1}{z}\partial N_{rs}(0) + \ldots.
\]

The result is:

\[
N^{II}_{rs}(k) = a_-(k)\phi^k_{r/2(r+s)-1} \phi^{k+1}_{1/2(r+s)-1} + a_+(k)\phi^k_{r/2(r+s)+1} \phi^{k+1}_{1/2(r+s)+1},
\]

\[
a_\pm(k) = \frac{1}{\sqrt{(k+2)(k+4)}}(k(\Delta^{k+1}_{21}(r+s+1), s - \Delta^{k+1}_{2}(r+s), s) -
\]

\[
- (k + 6)(\Delta^{k}_{21}(r+s+1) - \Delta^{k}_{12} - \Delta^{k}_{12}(r+s))) \times
\]

\[
\times \left(C^k_{(12)}(r, \frac{1}{2}(r+s))C^{k+1}_{(21)}(r, \frac{1}{2}(r+s), s)(\frac{1}{2}(r+s) + 1, s)\right)^{1/2}.
\]
For example, for $r = 1, s = 3$ we obtain the field driving the RG flow $M(2, k) \to M(2, k-2)$ described in Section 2:

$$N_{13}^{II} = \sqrt{\frac{k}{k+4}} \left( \sqrt{\frac{k}{2(k+1)}} \phi_{11}^{k+1} \phi_{13}^k + \sqrt{\frac{k+2}{2(k+1)}} \phi_{13}^k \phi_{33}^{k+1} \right).$$

(4.11)

Note that there is one more field with the same dimension:

$$\psi N_{13} = \frac{1}{2(k+1)} ((k+2)\phi_{11}^k \phi_{13}^{k+1} + k\phi_{13}^k \phi_{33}^{k+1}),$$

but only $N_{13}^{II}$ defined as (4.11) has all the properties of the second component of $N_{13}$.

To conclude the discussion of the supercurrent Ward identities we turn to the R sector. Using (4.10) one can show that:

$$G(z) R_{rs}^{1(2)}(0) = \sqrt{\Delta_{rs} - \frac{c}{24} \frac{1}{z^{3/2}}} R_{rs}^{2(1)}(0) + \ldots.$$

What we have done so far is still not enough to prove that $N_{rs}$ and $R_{rs}$ constructed above obey all the required null-vector properties. We have to show that their fusion rules, structure constants and 4-point functions coincide with the ones for the $N = 1$ minimal models. We will address these questions below.

In extending the discussion of the current algebra and the Ward identities to the higher level coset models one encounters some difficulties. This motivates the change of the strategy we will use for $l > 2$ which will entail abandoning the study of the current algebra and focusing on the direct construction of monodromy invariants. One difficulty comes from the fact that even the dimensions of all the currents are not known - for $l \geq 5$ there seem to exist additional currents over the stress-tensor $T$ and the well known current $A(z)$ of dimension $\Delta_A = (l+4)/(l+2)$. Another difficulty is that the dimension of $A(z)$ stops being a multiple of $1/2$ for $l > 2$. As a consequence its own algebra is not well understood.

Therefore, for the study of the higher level models we adopt a different strategy. We will start by constructing all the primary fields for any $l$ in terms of the projected products of the Virasoro fields since their conformal blocks, structure constants, etc., are fully understood and explicitly calculated. Then we proceed with the calculation of the corresponding conformal blocks and their monodromy-invariant combinations for higher levels. That will allow us to obtain their fusion rules and the structure constants. In the cases where there are previous results to compare with, e.g. $l = 2, 4$, our results will be confirmed.

We will limit ourselves to the fields which are primary with respect to the stress-tensor and the eventual additional currents present in the higher level models. The descendants will be considered later on. The primary fields of the model $k$ at level $l$ are $\phi_{mn}(l,k)$ with
conformal dimensions given by [10] [11]:

\[ \Delta_{m,n}(l,k) = \frac{((k + 2 + l)m - (k + 2)n)^2 - l^2}{4l(k + 2)(k + 2 + l)} + \frac{s(l - s)}{2l(l + 2)}, \]  
\[ s = |m - n (\text{mod}(l))|, \quad 0 \leq s \leq l, \]
\[ 1 \leq m \leq k + 1, \quad 1 \leq n \leq k + l + 1. \]

If \( n - m \in l\mathbb{Z} \) the expression for \( \Delta_{mn} \) simplifies since \( s(l - s) = 0 \). For \( l = 2 \) such fields belong to the NS sector, for general \( l \) we will call such sector the ”vacuum sector”. Since it is significantly simpler we present the construction first for these fields.

It is easy to check that:

\[ \Delta_{mn}(l,k) = \Delta_{mx}(1,k) + \Delta_{xn}(l-1,k+1), \]

if \( x = (1/l)(n + (l - 1)m) \). This identity leads us to write:

\[ \phi_{mn}(l,k) = \phi_{mx}(1,k)\phi_{xn}(l-1,k+1). \]  \hspace{1cm} (4.13)

Two remarks are in order: first, note that \( M(l - 1, 1) \) from the LHS of eq. (4.3) contributes the identity field to the LHS of (4.13); second, as will become clear later on, the products \( \phi_{mt}(1,k)\phi_{tn}(l-1,k+1) \) with \( t \neq x \) represent (part of) descendants of \( \phi_{mn}(l,k) \). Since:

\[ n - \frac{n + (l - 1)m}{l} \in (l - 1)\mathbb{Z} \]

we can immediately iterate (4.13) and finally obtain \( \phi_{mn}(l,k) \) written in terms of the Virasoro fields:

\[ \phi_{mn}(l,k) = \prod_{i=0}^{l-1} \phi_{ki,k_{i+1}}(1,k+i), \]  
\[ k_i = \frac{ln + (l - i)m}{l}, \quad n - m \in l\mathbb{Z}. \] \hspace{1cm} (4.14)

Furthermore, starting from eq. (4.14) one can reach any other projected product:

\[ \phi_{mk_1,k_2}(1,k+1)\phi_{k_{i+1}}(1,k+1) \ldots \phi_{k_{i-1}}(1,k+1) \]  
\[ \phi_{mk_i}(1,k)\phi_{k_{1}k_{2}}(1,k+1) \ldots \phi_{k_{i-1}n}(1,k+l-1) \] \hspace{1cm} (4.15)

by changing \( k_1 \) into \( \hat{k}_1 \), \( k_2 \) into \( \hat{k}_2 \), etc. Similarly to eq. (4.9), the dimension of the field (4.15) is higher than the one of (4.13) by a multiple of 1/2. We interpret products as (part of) descendants of \( \phi_{mn}(l,k) \) with respect to \( T, G(l = 2) \), any other additional currents or a product of these currents.

To summarize, among all the products (4.15) we search for the one with the lowest dimension to identify it with the primary field \( \phi_{mn}(l,k) \). Minimizing the dimension is equivalent to minimizing \( S = \sum_{i=0}^{l-1}(k_i - k_{i+1})^2 \). If \( m - n \in l\mathbb{Z} \) there is a unique solution (with
$k_0 = m, k_l = n$) that gives $S = lK^2 (K = (1/l)(n-m))$, namely equidistant $k_i$'s, $k_i = m+iK$, as in (1.14).

Turning to the nonvacuum sector (Ramond and analogous), namely fields $\phi_{mn}(l, k)$ with $m-n \notin l\mathbb{Z}$, we will see that things stop being so simple. We begin by deriving the expression for a product of Virasoro fields having the required (minimal) dimension. Unfortunately, it will become obvious that for nonvacuum sectors that expression is not unique. We will study for a product of Virasoro fields having the required (minimal) dimension. Unfortunately, it

\[ \Delta_{s,s+1}(1, l-1) + \Delta_{mn}(l, k) = \Delta_{my}(1, k) + \Delta_{yn}(l-1, k+1), \]

where $y = (1/l)((l-1)m + n \mp (l-s))$. Therefore we write:

\[ \phi_{s,s+1}(1, l-1)\phi_{mn}(l, k) = \phi_{my}(1, k)\phi_{yn}(l-1, k+1). \] (4.16)

This time $M(1, l-1)$ contributes a nontrivial field $\phi_{s,s+1}(1, l-1)$. Analogously to the discussion for the vacuum sector, $n-m \in l\mathbb{Z} \mp s$ implies:

\[ n - \frac{n+(l-1)m \mp (l-s)}{l} \in (l-1)\mathbb{Z} \mp (s-1). \]

The field from $M(k+1, l-1)$ is again from nonvacuum sector, even though one step closer to the vacuum sector. We again iterate the process and after $s$ steps arrive at the general formula for the nonvacuum sector fields:

\[ \phi_{12}(1, l-s)\phi_{23}(1, l-s+1) \ldots \phi_{s,s+1}(1, l-1)\phi_{mn}(l, k) = \]

\[ = \phi_{1,s+1}(s, l-s)\phi_{mn}(l, k) = \prod_{i=0}^{l-1} \phi_{k_ik_{i+1}}(1, k+i), \]

\[ n-m \in l\mathbb{Z} \mp s, \quad 1 \leq s \leq l-1 \] (4.17)

where $k_i = \frac{in+(l-i)m+d_i^s}{l}$ and $d_i^s = \mp i(l-s)$ if $i \leq s$, $d_i^s = \mp s(l-i)$ if $i > s$.

After some simple combinatorics one can notice that there are $\binom{l}{s}$ different products of the Virasoro fields that have the same dimension as those of $\phi_{1,s+1}(s, l-s)\phi_{mn}(l, k)$. To understand the origin of this degeneracy, note that $\phi_{1,s+1}(s, l-s)$ in (4.17) represents actually (according to (1.14)) the product:

\[ \phi_{11}(1, 1) \ldots \phi_{11}(1, l-s-1)\phi_{12}(1, l-s)\phi_{23}(1, l-s+1) \ldots \phi_{s,s+1}(1, l-1). \]

There are exactly $\binom{l}{s}$ such projected products of the fields from:

\[ \mathbf{P}(M(1, 1) \times (M(1, 1) \times \ldots \times M(l-1, 1)) \]

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that have the same dimension. Moreover, we could view the original field \( \phi_{mn}(l,k) \) as belonging to the \( l-s \) sector. In that case the field \( \phi_{1,l-s+1}(l-s,s) \) would appear in (4.17) which in its turn would represent \( \binom{l-1}{l-s} \) products like:

\[
\phi_{11}(1,1) \ldots \phi_{11}(1,s-1)\phi_{12}(1,s)\phi_{23}(1,s+1) \ldots \phi_{l-s,l-s+1}(1,l-1). 
\]

Finally, since:

\[
\binom{l}{s} = \binom{l-1}{s} + \binom{l-1}{l-s}
\]

we conclude that all the degeneracy of the RHS of (4.17) is accounted for by the degeneracy of the LHS.

We now turn to the explicit construction of the four-point correlation functions for arbitrary fields from a higher level model. We start with the simplest example of the function of the same fields:

\[
G(z, \bar{z}) = \langle \phi_{mn}(l,k)(0)\phi_{mn}(l,k)(z)\phi_{mn}(l,k)(1)\phi_{mn}(l,k)(\infty) \rangle \tag{4.18}
\]

where \( n-m \in l\mathbb{Z} \). As already mentioned there are two basic steps in the calculation of \( G \). First, one obtains the conformal blocks, i.e. the linearly independent solutions of the differential equations obeyed by the correlation function, and second, one combines them in a monodromy invariant expression which is the correlation function. According to (4.14) \( \phi_{mn}(l,k) \) is a product of Virasoro fields and therefore the conformal blocks for (4.18) will be products of the Virasoro conformal blocks. Of course, only certain products of conformal blocks will survive the projection \( P \).

It is known that the conformal blocks of the correlation function of the Virasoro fields:

\[
G_V(z, \bar{z}) = \langle \phi_{rs}(1,k)(0)\phi_{rs}(1,k)(z)\phi_{rs}(1,k)(1)\phi_{rs}(1,k)(\infty) \rangle
\]

can be obtained with Coulomb gas technic as certain multi-contour integrals \[3\], denoted as \( I^k_{ij}(a, a'; z) \), \( i = 1, \ldots, r, j = 1, \ldots, s \) where:

\[
a = 2\alpha_- \alpha_r, \quad a' = 2\alpha_+ \alpha_r, \\
\alpha_r = \frac{1}{2}((1-r)\alpha + (1-s)\alpha_-), \\
\alpha_+^2 = \frac{k+3}{k+2}, \quad \alpha_-^2 = \frac{k+2}{k+3}, \quad \alpha_+ \alpha_- = -1.
\]

In order to preserve the projection \( P \) in the intermediate channel we allow only products of conformal blocks of the form:

\[
I^k_{i_0 i_1} I^{k+1}_{i_1 i_2} \ldots I^{k+l-1}_{i_l i_0} . \tag{4.19}
\]

Having obtained the conformal blocks, we want to construct their monodromy invariant combinations. We start with the simple example \( l = 2 \), i.e. 4-point functions of of NS
fields $\phi_{mn}(2, k) = \phi_{mz}(1, k) \phi_{zn}(1, k + 1)$, $x = \frac{1}{2}(m + n)$. The task is to find the coefficients $X_{i_0i_1j_2j_0j_1j_2}$ such that:

$$G(z, \bar{z}) = \sum_{i_0, j_0 = 1, \ldots, m} X_{i_0i_1j_2j_0j_1j_2} I^k_{i_0i_1} I^{k+1}_{i_1i_2}(z) I^k_{j_0j_1} I^{k+1}_{j_1j_2}(\bar{z}).$$

is monodromy invariant. In other words, we want $G(z, \bar{z})$ to be well-defined, that is, single valued in the complex plane. Since the conformal blocks only have poles at $z = 0, 1$ and $\infty$, $G(z, \bar{z})$ will be single valued everywhere if it is invariant under analytic continuation in $z$ along a contours surrounding $z = 0$ and $z = 1$.

As usual, the calculation around $z = 0$ is straightforward and leads to the following form of the 4-point function:

$$G(z, \bar{z}) = \sum_{i_0 = 1, \ldots, m} X_{i_0i_1j_2j_0j_1j_2} I^k_{i_0i_1} I^{k+1}_{i_1i_2}(z) I^k_{j_0j_1} I^{k+1}_{j_1j_2}(\bar{z}).$$

(4.20)

We turn to the analytic continuation around $z = 1$. First, we remind that the Virasoro conformal blocks $I^{(rs)}_{ij}(a, a'; z)$ could be rewritten as:

$$I^{(rs)}_{ij}(a, a'; z) = \sum_{p=1}^{r} \sum_{q=1}^{s} \alpha^{rs}_{ij, pq}(a, a') \tilde{I}^{(rs)}_{pq}(a, a'; 1 - z)$$

where the $\alpha$-matrices are well known [3]. To study (4.20) under $(1 - z) \rightarrow (1 - z) e^{2\pi i}$ we use these $\alpha$-matrices. The result is:

$$G(z, \bar{z}) = \sum_{i_0, j_0 = 1, \ldots, m} X_{i_0i_1j_2j_0j_1j_2} \alpha^{k+1}_{i_0i_1, ef} \alpha^{k+1}_{i_1i_2, gh} \alpha^{k+1}_{i_0j_1, rs} \alpha^{k+1}_{j_1j_2, tu} I^k_{i_0i_1} I^{k+1}_{i_1i_2}(z) I^k_{j_0j_1} I^{k+1}_{j_1j_2}(\bar{z})$$

(summation over the repeated indexes is assumed). There are two requirements that have to be satisfied. First, the $\alpha$-transformation should not take us outside of the subspace defined by $P$. Second, $G((z, \bar{z})$ should be of the same form with respect to $I(1 - z)$ as eq. (4.20) is to $I(z)$ is order to insure invariance under the monodromy transformation around $z = 1$. Using some special properties of the $\alpha$-matrices [3], we thus arrive at the final form of our monodromy invariant correlation function:

$$G(z, \bar{z}) = \sum_{i_0, j_0 = 1, \ldots, m} X^k_{i_0i_1} X^{k+1}_{j_1j_2} I^k_{i_0i_1} I^{k+1}_{i_1i_2}(z) I^k_{j_0j_1} I^{k+1}_{j_1j_2}(\bar{z})$$

(4.21)

where the coefficients $X^k_{ij}$ are those defining the monodromy invariant Virasoro 4-point function [3]. Since the $X$’s (up to certain normalization constants) define the structure constants,
we can already see that the NS structure constants for $M(k, 2)$ will be given by certain products of the Virasoro structure constants for $M(k, 1)$ and $M(k + 1, 1)$.

Before turning to the structure constants we would like to generalize the simple example that led to (4.21). Let us consider the correlation function (4.18). The relevant conformal blocks are of the form (4.19). An analysis similar to the one for $l = 2$ shows that only the terms in the correlation function of the form:

$$I_{i_1j_1}^k I_{i_2j_2}^{k+1} \cdots I_{i_lj_l}^{k+l-1}(z) I_{j_1j_2}^{k+1} \cdots I_{j_lj_l}^{k+l-1}(z)$$

are invariant under $z \to z e^{2\pi i}$. Furthermore, the properties of the $\alpha$-matrices that led us to (4.21) do not depend on the level, indexes or the value of $k$. Thus, we conclude that the monodromy invariant correlation function will be again of the same form:

$$G(z, \bar{z}) = \sum X_{a_1}^k X_{j_1j_2}^{k+1} \cdots X_{j_lj_l}^{k+l-1} I_{i_1j_1}^k \cdots I_{i_lj_l}^{k+l-1}(z) I_{j_1j_2}^{k+1} \cdots I_{j_lj_l}^{k+l-1}(z)$$

(4.22)

We can generalize further and discuss asymmetrical correlation functions:

$$G_a(z, \bar{z}) = \left< \prod_{a=1}^{4} \phi_{m_a n_a} (l, k)(z_a, \bar{z}_a) \right>, \quad m_a - n_a \in lZ.$$ 

Now $I, \alpha$ and $X$ depend on three sets of parameters:

$$a_i = 2\alpha_- \alpha_{m_i n_i}, \quad a'_i = 2\alpha_+ \alpha_{m_i n_i}, \quad i = 1, 2, 3.$$ 

It is straightforward to go over the arguments and convince ourselves that there are no significant changes.

Turning to the nonvacuum sectors, we want to calculate the 4-point function of $\phi_{mn}(l, k)$ where $n - m \in lZ \neq s, 1 \leq s \leq l - 1$. From (4.17) we know that the product $\phi_{1,s+1}(s, l-s)\phi_{mn}(l, k)$ can be expressed as various products of the Virasoro fields. The construction of the 4-point functions of these products of Virasoro fields proceeds as above and we conclude that the 4-point function of $\phi_{1,s+1}(s, l-s)\phi_{mn}(l, k)$ has the form (4.22). Furthermore, since there is no projection between $\phi_{1,s+1}(s, l-s)$ and $\phi_{mn}(l, k)$, the 4-point function of the product factorizes into the product of the 4-point functions of the corresponding fields.

Now we want to use the construction of the monodromy invariant 4-point functions performed above for the study of the fusion algebras and the structure constants for the higher level models. We will limit our considerations here to the vacuum sector fields only.

Let us start with the $N = 1$ supersymmetric theory, i.e. take $l = 2$. The NS fields (vacuum sector) are constructed as:

$$\phi_{mn}(2, k) = \phi_{m,\frac{1}{2}(m+n)}^k \phi_{\frac{1}{2}(m+n), n}^{k+1}, \quad n - m \in 2Z.$$ 

(4.23)

All the other combinations $\phi_{mp}^k \phi_{pm}^{k+1}$ belong to the descendants of $\phi_{mn}(2, k)$ with the dimension $\Delta_{mn}(2, k) + \frac{1}{2}(p - \frac{1}{2}(m + n))^2$. Since we have seen that the conformal blocks used in
our constructions are projected products of \((k,1)\) and \((k+1,1)\) conformal blocks, the fusion algebra will follow the same recipe.

For the study of the fusion rules it suffices to consider only the diagonal terms \((i_1 = j_1)\) in (1.21). The other terms correspond to descendants with respect to the stress tensor or \(G(z)G(\bar{z})\) which will be already accounted for in the fusion rules. Considering the diagonal terms only is equivalent to using the Virasoro fusion rules [1]:

\[
\phi^k_{m_1 n_1} \phi^k_{m_2 n_2} = \sum_{r=|m_1-m_2|+1}^{\text{min}(m_1+m_2-1,2(k+2)-m_1-m_2-1)} \sum_{s=|n_1-n_2|+1}^{\text{min}(n_1+n_2-1,2(k+3)-n_1-n_2-1)} \phi^k_{r,s},
\]

\((r \text{ and } s \text{ advance in steps of } 2), \text{ for each of the fields in (1.23) and then imposing the projection by identifying the middle indices. The result is:}\)

\[
\phi_{m_1 n_1}(2, k)\phi_{m_2 n_2}(2, k) = \sum_{p=|m_1-m_2|+1}^{\text{min}(m_1+m_2-1,2(k+2)-m_1-m_2-1)} \sum_{q=|\frac{1}{2}(m_1+n_1)-\frac{1}{2}(m_2+n_2)|+1}^{\text{min}(n_1+n_2-1,2(k+4)-n_1-n_2-1)} \times \phi^k_{pq} \phi^{k+1}_{qr}.
\]

The remaining problem is to identify the products \(\phi^k_{pq} \phi^{k+1}_{qr}\) with the super-Virasoro fields. First we note that:

\[
r - p = |n_1 - n_2| - |m_1 - m_2|(\text{mod}(2)) = n_1 - m_1 - (n_2 - m_2)(\text{mod}(2)) \in 2Z.
\]

Therefore \(\phi^k_{pq} \phi^{k+1}_{qr}\) stands for a NS field \(\phi_{pr}(2, k)\) or its descendant. It is a simple exercise to fix the range of \(q\) and identify its minimal values. This will distinguish between the primary field or the second component of a latter. The final conclusion is that the following NS fusion rules hold:

\[
\phi_{m_1 n_1}(2, k)\phi_{m_2 n_2}(2, k) = \sum_{r=|m_1-m_2|+1}^{\text{min}(m_1+m_2-1,2(k+2)-m_1-m_2-1)} \sum_{s=|n_1-n_2|+1}^{\text{min}(n_1+n_2-1,2(k+4)-n_1-n_2-1)} \phi^{(II)}_{r,s}(2, k)
\]

where \(n_i - m_i \in 2Z\) and \(\phi^{(II)}_{rs} = N_{rs}\) if \(r+s-|m_1+n_1-m_2-n_2| \in 4Z+2\), \(\phi^{(II)}_{rs} = N_{rs}^{II}\) if \(r+s-|m_1+n_1-m_2-n_2| \in 4Z\) in agreement with our results in Section 2.

Let us sketch briefly the calculations for the next level \(l = 3\). We study the fusions of two vacuum sector fields like:

\[
\phi^{k}_{mn}(3, k) = \phi^{k}_{m,\frac{3}{2}(n+2m)} \phi^{k+1}_{\frac{3}{4}(n+2m),\frac{3}{4}(2n+4m)} \phi^{k+2}_{\frac{3}{8}(2n+4m),n}, \quad n-m \in 3Z.
\]
Detailed analysis, similar to that of the previous \( l = 2 \) case, leads to a new situation for \( l = 3 \), namely, it happens that in the product of two vacuum sector fields a nonvacuum sector field, or a descendant of such field, appears. Omitting the details we present the final result:

\[
\phi_{m_1n_1}(3, k)\phi_{m_2n_2}(3, k) = \\
\sum_{r = |m_1 - m_2| + 1}^{\min(m_1 + m_2 - 1, 2(k+2) - m_1 - m_2 - 1)} \sum_{s = |n_1 - n_2| + 1}^{\min(n_1 + n_2 - 1, 2(k+5) - n_1 - n_2 - 1)} \phi^{(d)}_{rs}(3, k)
\]

where \( r \) and \( s \) advance in steps of 2 and \( \phi^{(d)}_{rs}(3, k) \) is the primary field \( \phi_{rs}(3, k) \) if \( k - l \in 3\mathbb{Z} \) and its descendant with respect to the current \( A(z) \) (of dimension 7/5 here) otherwise.

One proceeds in the same way for the next levels \( l = 4, 5, \ldots \) We will not present the explicit calculations for them here. Finally, we arrive at the following general vacuum sector fusion rules:

\[
\phi_{m_1n_1}(l, k)\phi_{m_2n_2}(l, k) = \\
\sum_{r = |m_1 - m_2| + 1}^{\min(m_1 + m_2 - 1, 2(k+2) - m_1 - m_2 - 1)} \sum_{s = |n_1 - n_2| + 1}^{\min(n_1 + n_2 - 1, 2(k+l+2) - n_1 - n_2 - 1)} \phi^{(d)}_{rs}(l, k)
\]

where \( m_i - n_i \in l\mathbb{Z}, r, s \) advance in steps of 2, and \( \phi^{(d)}_{rs}(l, k) \) is the primary field \( \phi_{rs}(l, k) \) if \( r - s = lK, K \in \mathbb{Z} \) and \( K - 1/l (|n_1 + (l-1)m_1 - (n_2 + (l-1)m_2)| - |m_1 - m_2|) \in 2\mathbb{Z} \), and its descendant with respect to \( A(\ell+4)/\ell+2 \), one of the additional currents appearing for \( l \geq 5 \), or some product of those currents otherwise.

All the results obtained above demonstrate that, as long as we stay within the part of the fusion rules that maps the primary vacuum sector fields into the primary vacuum sector fields, we have full control, for any \( l \). We will use that now to obtain explicit expressions for the structure constants connecting three vacuum sector primary fields for any \( l \).

The structure constants appear as a limit of the modromy invariant 4-point functions. The conformal blocks whose limit one is taking are nothing but the products of the Virasoro conformal blocks. Since for the primary fields from the vacuum sector the mapping from the \( l\)-th level fields into the products of the Virasoro fields remains strictly one-to-one and does not involve any nontrivial fields from \( M(1, l-1) \) and/or additional currents, it is obvious that the \( l\)-th level vacuum sector structure constants are given by the products of the Virasoro structure constants. Explicitly, for the fields \( \phi_{m_an_a}(l, k), a = 1, 2, 3 \) from (4.14) where:

\[
(n_3 - m_3) - (|n_1 + (l - 1)m_1 - (n_2 + (l - 1)m_2)| - |m_1 - m_2|) \in 2l\mathbb{Z}
\]

the structure constants are given by:

\[
C_{(m_1n_1)(m_2n_2)(m_3n_3)} = \prod_{i=0}^{l-1} C_{(k_1^{i}k_{i+1}^{i})(k_2^{i}k_{i+1}^{i})(k_3^{i}k_{i+1}^{i})}(1, k + i).
\]
4.2 Second order RG flow

In this Section we want to discuss the renormalization group properties of the $\hat{su}(2)$ coset models $M(k,l)$ defined by (4.1) (we assume here that $k$ and $l$ are integers and $k > l$). We remind that it is written in terms of $\hat{su}(2)_k$ WZNW models with current $J^a$, $k$ is the level. The latter are CFT’s with a stress tensor expressed through the currents by the Sugawara construction:

$$T_k(z) = \frac{1}{k+2} \left( (J^0)^2 + \frac{1}{2} J^+ J^- + \frac{1}{2} J^- J^+ \right).$$

(4.27)

The central charge of the corresponding Virasoro algebra is $c_k = \frac{3k}{k+2}$. The energy momentum tensor of the coset (4.1) is then given by:

$$\hat{T} = T_k + T_l - T_{k+l}$$

in obvious notations. It defines a Virasoro algebra with central charge that can be read from this construction and is given by (4.2). The dimensions of the primary fields $\phi_{m,n}(l,p)$ of the "minimal models" (rational CFT) were written in (4.12) ($m, n$ are integers). We want to slightly change the notations in what follows, introducing $p = k + 2$. The dimensions then become:

$$\Delta_{m,n}(l,p) = \left( (p+1) m - pm \right)^2 - l^2 \quad \text{for} \quad \frac{4l(p+l)}{2l(l+2)}$$

(4.28)

As we mentioned in the previous Section, it is known [10, 11, 12] that the theory $M(k,l)$ possesses a symmetry generated by a "parafermionic current" $A(z)$ of dimension $\Delta_A = \frac{l+4}{l+2}$. We shall present an explicit construction of this current below. Here we just mention that under this symmetry the primary fields (4.28) are divided in sectors labeled by the integer $s$. The branching of the current $A(z)$ on the field (or state) of sector $s$ can be written symbolically as [77, 78]:

$$A_{-m-\frac{(l+2)}{l+2}}|s> = |s+2>, \quad A_{-m}|s> = |s>, \quad A_{-m-\frac{(l+2)}{l+2}}|s> = |s-2>.$$

(4.29)

In this Section we prefer to use the description of the theory $M(k,l)$ presented above, namely we will define the fields, correlation functions, structure constants etc. using the construction (4.3) and the specific projection $P$.

Let us now define the model. We consider the CFT $M(k,l)$ perturbed by the least relevant field. Our goal here is to find the $\beta$-function and investigate its eventual fixed point up to second order in the perturbation theory. In addition, we want to describe also the mixing of certain fields under the RG flow.

Let us briefly sketch the constructions. The perturbed theory is described by the Lagrangian:

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \lambda \tilde{\phi}_{1,3}(x)$$

where $\mathcal{L}_0(x)$ describes the theory $M(k,l)$ itself. We identify the field $\tilde{\phi}_{1,3}$ with the first descendent of the corresponding primary field (4.28) with respect to the current $A(z)$. In
fact, in view of (4.28) \( \phi_{1,3} \) belongs to the sector \( |2> \) and has a descendent belonging to sector \( |0> \) due to the last of (4.29). The dimension of this first descendent is therefore (for \( s = 2 \)):

\[
\Delta = \Delta_{1,3} + \frac{l}{l+2} = 1 - \frac{2}{p+l} = 1 - \epsilon. \tag{4.30}
\]

In this and in the next Section we consider the case \( p \rightarrow \infty \) and assume that \( \epsilon = \frac{2}{p+l} \ll 1 \) is a small parameter.

Following our constructions of the previous Section we find it more convenient here to define the field \( \tilde{\phi}_{1,3} \) alternatively in terms of lower level fields:

\[
\tilde{\phi}_{1,3}(l,p) = a(l,p)\phi_{1,1}(1,p)\tilde{\phi}_{1,3}(l-1, p+1) + b(l,p)\phi_{1,3}(1,p)\phi_{3,3}(l-1, p+1). \tag{4.31}
\]

Here the field \( \phi_{3,3}(l, p) \) is just a primary field constructed as:

\[
\phi_{3,3}(l, p) = \phi_{3,3}(1, p)\phi_{3,3}(l-1, p+1) \tag{4.32}
\]

with dimension from (4.28). It is straightforward to check that the field (4.31) has a correct dimension (4.33). The coefficients \( a(l,p) \) and \( b(l,p) \) as well as the structure constants of the fields involved in the constructions (4.31) and (4.32) can be found by demanding the closure of the fusion rules:

\[
\begin{align*}
\tilde{\phi}_{1,3}(l,p)\tilde{\phi}_{1,3}(l,p) &= 1 + C_{(13)(13)}^{(13)}\tilde{\phi}_{1,3}(l,p) + C_{(13)(13)}^{(15)}\tilde{\phi}_{1,5}(l,p), \\
\phi_{3,3}(l,p)\phi_{3,3}(l,p) &= 1 + C_{(33)(33)}^{(13)}\tilde{\phi}_{1,3}(l,p) + C_{(33)(33)}^{(15)}\tilde{\phi}_{3,3}(l,p) + C_{(33)(33)}^{(15)}\tilde{\phi}_{1,5}(l,p).
\end{align*} \tag{4.33}
\]

We found that:

\[
a = \sqrt{\frac{(l-1)(p-2)}{l(p-1)}}, \quad b = \sqrt{\frac{p-l-2}{l(p-1)}},
\]

the structure constants are just a special case of those listed in Appendix A.

We introduced explicitly here the descendent field:

\[
\tilde{\phi}_{1,5}(l,p) = x'(l,p)\phi_{1,1}(1,p)\tilde{\phi}_{1,5}(l-1, p+1) + y'(l,p)\phi_{1,3}(1,p)\tilde{\phi}_{3,5}(l-1, p+1). \tag{4.34}
\]

of dimension \( \tilde{\Delta}_{1,5} = 2 - \frac{6}{p+l} \). The coefficients and the structure constants involving this field are found from the closure of (4.33):

\[
\begin{align*}
x' &= \sqrt{\frac{(l-2)(p-3)}{l(p-1)}}, \quad y' = \sqrt{\frac{2(p+l-3)}{l(p-1)}}, \tag{4.35} \\
C_{(33)(33)}^{(15)}(l,p) &= -\sqrt{\frac{2l(l-1)}{(p-2)(p-3)(p+l-3)(p+l-4)}}\tilde{G}_3(p+l-1), \\
C_{(13)(13)}^{(15)}(l,p) &= (p+l-2)\sqrt{\frac{2(l-1)(p-3)}{l(p+l-3)(p+l-4)(p-2)}}\tilde{G}_3(p+l-1)
\end{align*}
\]
where the function $\tilde{G}_n(p + l - 1)$ is defined in Appendix A.

The mixing of the fields along the RG flow is connected to the two-point function. Up to the second order of the perturbation theory it is given by (2.58):

\[
< \phi_1(x) \phi_2(0) > = \phi_1(x) \phi_2(0) >_0 - \lambda \int \phi_1(x) \phi_2(0) \tilde{\phi}(y) >_0 d^2 y + \frac{\lambda^2}{2} \int \phi_1(x) \phi_2(0) \tilde{\phi}(x_1) \tilde{\phi}(x_2) >_0 d^2 x_1 d^2 x_2 + \ldots
\]

where $\phi_1, \phi_2$ can be arbitrary fields of dimensions $\Delta_1, \Delta_2$.

As it was was explained in Section 2.2 one can use the transformation properties of the fields to bring the double integral to the semi-factorized form (2.59) where $I(x)$ (2.60) is expressed through the hypergeometric functions which are fully under control. Also, in regularizing the integral we follow the procedure described in Section 2.2 [7]. For convenience we will just change the notations for the additional parameter and the ultraviolet cut-off introduced there and call them $r$ and $r_0$ respectively in what follows (this is since we want to preserve the notation $l$ for the level of the coset model). Thus, for example, the safe region, far from singularities will be denoted as $\Omega_{r,r_0}$.

Let us consider the correlation function that enters the integral (2.59). The basic ingredients for the computation of the four-point correlation functions are the conformal blocks. These are quite complicated objects in general and closed formulae were not known. Recently, it was argued that they coincide (up to factors) with the instanton partition function of certain $N = 2$ YM theories. Here we adopt another strategy, namely, we find the expressions for the conformal blocks up to a sufficiently high level in order to have a guess for the limit $\epsilon \to 0$.

Let us remind that, according to the construction (4.3) presented in the previous Section, any field $\phi_{m,n}(l, p)$ (or its descendant) can be expressed recursively as a product of lower level fields. Therefore the corresponding conformal blocks will be a product of lower level conformal blocks. Due to the RHS of (4.3) only certain products of conformal blocks will survive the projection $P$. We would like to be more explicit here, so let us define the conformal block at level $l$ by:

\[
F_l(r, s) = \phi_{i_1,j_1}(x) \phi_{i_2,j_2}(0)|_{r,s} \phi_{i_3,j_3}(1) \phi_{i_4,j_4}(\infty) >_l
\]

where in the notation we omitted the ”external” fields and $r, s$ stands for the internal channel field $\phi_{r,s}$. The latter could be a primary field from (4.28) or a descendant like those defined in (4.31) and (4.34) (which could be identified with some descendant with respect to the current $A$). Which internal field can appear in the conformal block is defined by the fusion rules. The latter can be obtain recursively as it was explained in the previous Section.

The conformal block is a chiral object, i.e. it depends only on the chiral coordinate $x$. It
can be expanded as:

\[ F(x) = x^{\Delta_{rs} - \Delta_{i_1,j_1} - \Delta_{i_2,j_2}} \sum_{N=0}^{\infty} x^N F_N \]  

(4.36)

where \( N \) is called level (not to be confused with the level \( l \) of \( M(k,l) \)) and we omitted the indexes.

In order to preserve the projection \( P \) in the intermediate channel, we allow only products of conformal blocks of the form:

\[ \begin{align*}
&< \phi_{i_1,j_1}(x)\phi_{i_2,j_2}(0)\phi_{i_3,j_3}(1)\phi_{i_4,j_4}(\infty) >_1 \times \\
&\times < \phi_{k_1,l_1}(x)\phi_{k_2,l_2}(0)\phi_{k_3,l_3}(1)\phi_{k_4,l_4}(\infty) >_{l-1} \times \\
&\times \sqrt{C_{rt(i_1,j_1)(i_2,j_2)}C_{rt(i_3,j_3)(i_4,j_4)}C_{ts(k_1,l_1)(k_2,l_2)}C_{ts(k_3,l_3)(k_4,l_4)}}.
\end{align*} \]

(4.37)

Namely, only products of conformal blocks that involve the same internal indexes are allowed. Note that we included explicitly the corresponding structure constants. This is needed because they give different relative contribution on the subsequent levels in the expansion (4.36). The overall constant will define the actual structure constant. Also, as it was discussed above, we take square roots of the structure constants because our considerations are chiral, i.e. depend only on the chiral coordinate \( x \). Then, the true structure constant will be a square of the resulting one in (4.37).

Actually, we consider below descendent fields which are some linear combinations like (4.31). Therefore we will have a linear combinations of products (4.37). We give more details of the explicit construction of the conformal blocks in consideration in Appendixes B and C.

The conformal blocks are in general quite complicated objects. Fortunately, in view of the renormalization scheme and the regularization of the integrals, we need to compute them here only up to the zero-th order in \( \epsilon \). This simplifies significantly the problem.

Once the conformal blocks are known, the correlation function of spinless fields for our \( M(k,l) \) models is written as:

\[ \sum_{r,s} C_{rs} |F(r,s)|^2 \]

where the range of \((r,s)\) depends on the fusion rules and \( C_{rs} \) is the corresponding structure constant (we omitted the external indexes). The structure constants for the fields of interest are listed in Appendix A.

Our strategy here is to compute the conformal blocks recursively up to sufficiently high level. In addition we impose the condition of the crossing symmetry of the corresponding correlation function and the correct behaviour near the singular points 1 and \( \infty \).

We turn now to the computation of the \( \beta \)-function and the fixed point. For the computation of the \( \beta \)-function up to the second order, we need the four-point function of the
perturbing field. As explained in Appendix B there are three “channels” (or intermediate fields) in this conformal block corresponding to the identity $\phi_{1,1}$, $\tilde{\phi}_{1,5}$ and to $\tilde{\phi}$ itself. The explicit expression for the correlation function is (B.8):

$$
< \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty) > = 
\left| \frac{(1-2x + \frac{5}{3} + \frac{4}{3l})x^2 - (\frac{2}{3} + \frac{4}{3l})x^3 + \frac{1}{3}x^4}{x^2(1-x)^2} \right|^2 + \frac{16}{3l^2} \left| \frac{(1 - \frac{3}{2}x + \frac{(l+1)x^2 - l}{2x(1-x)^2})}{x(1-x)^2} \right|^2 + 
\frac{5}{9} \left( \frac{2(l-1)}{l} \right)^2 \left| \frac{(1-x + \frac{l}{2(l-1)x^2})}{(1-x)^2} \right|^2.
$$

In Appendix B we checked explicitly the crossing symmetry and the $x \rightarrow 1$ limit of this function. In order to compute the $\beta$-function and the fixed point to the second order we just have to integrate the above function.

The integration over the safe region $\Omega_{r,r_0}$ goes in exactly the same way as for the $N = 1$ ($l = 2$) case so we omit here the details. To compute the integrals near the singular points 0, 1 and $\infty$ we use again the OPE:

$$
\tilde{\phi}(x)\tilde{\phi}(0) = (x\bar{x})^{-2\Delta}(1 + \ldots) + C_{(1,3)(1,3)}^{(1,3)}(x\bar{x})^{-\Delta}(\tilde{\phi}(0) + \ldots)
$$

which follows from the definition (4.33). The channel $\tilde{\phi}_{1,5}$ gives after integration a term proportional to $r/r_0$ which is negligible. The structure constant is a particular case of those presented in Appendix A. Its value is:

$$
C_{(1,3)(1,3)}^{(1,3)} = \frac{4}{l\sqrt{3}} - 2\sqrt{3} \epsilon
$$

to the first order in $\epsilon$.

Putting altogether, we obtain the finite part of the integral:

$$
\frac{80\pi^2}{3l^2\epsilon^2} - \frac{88\pi^2}{l\epsilon}.
$$

We notice that, although the single integrals give different results, the final answer matches perfectly the known $l = 1$ [7] and $l = 2$ (Section 2.2) cases.

Taking into account also the first order term (whose calculation is straightforward and proportional to the above structure constant), we get the final result (up to the second order) for the two-point function of the perturbing field:

$$
G(x, \lambda) = < \tilde{\phi}(x)\tilde{\phi}(0) > = 
(x\bar{x})^{-2+2\epsilon} \left[ 1 - \lambda \cdot \frac{4\pi}{\sqrt{3}} \left( \frac{2}{l\epsilon} - 3 \right) (x\bar{x})^\epsilon + \frac{\lambda^2}{2} \left( \frac{80\pi^2}{3l^2\epsilon^2} - \frac{88\pi^2}{l\epsilon} \right) (x\bar{x})^{2\epsilon} + \ldots \right].
$$
The expression for the $\beta$-function in this renormalization scheme was already given in Section 2.2 and reads:

$$\beta(g) = \epsilon \lambda \frac{\partial g}{\partial \lambda} = \epsilon \lambda \sqrt{G(1, \lambda)}$$

where $G(1, \lambda)$ is given by (4.38) with $x = 1$. One can invert this and compute the bare coupling constant and the $\beta$-function in terms of the renormalized coupling constant $g$:

$$\lambda = g + g^2 \frac{\pi}{\sqrt{3}} \left( \frac{2}{l \epsilon} - 3 \right) + g^3 \frac{\pi^2}{3} \left( \frac{4}{l^2 \epsilon^2} - \frac{10}{l \epsilon} \right) + O(g^4), \quad (4.39)$$

$$\beta(g) = \epsilon g - g^2 \frac{\pi}{\sqrt{3}} \left( \frac{2}{l} - 3 \epsilon \right) - \frac{4\pi^2}{3l} g^3 + O(g^4).$$

In this calculations, we keep only the relevant terms by assuming the coupling constant $\lambda$ (and $g$) to be order of $O(\epsilon)$. A non-trivial IR fixed point occurs at the zero of the $\beta$-function:

$$g^* = \frac{l\sqrt{3}}{2\pi} \epsilon (1 + \frac{l}{2} \epsilon). \quad (4.40)$$

It corresponds to the IR CFT $M(k-l, l)$ as can be seen from the central charge difference:

$$c^* - c = -\frac{4(l + 2)}{l} \pi^2 \int_0^{g^*} \beta(g) dg = -l(1 + \frac{l}{2} \epsilon)^3 - \frac{3l^2}{4} (l + 2) \epsilon^4 + O(\epsilon^5).$$

The anomalous dimension of the perturbing field becomes:

$$\Delta^* = 1 - \partial_g \beta(g)|_{g^*} = 1 + \epsilon + le^2 + O(\epsilon^3)$$

which matches with that of the field $\phi_{3,1}(l, p-l)$ of $M(k-l, l)$ (a particular case of the fields defined immediately below).

Let us now define recursively, in analogy with the fields $\tilde{\phi}_{1,3}(l, p)$ and $\phi_{3,3}(l, p)$, the following descendent fields:

$$\tilde{\phi}_{n,n+2}(l, p) = x(l, p) \phi_{n,n}(1, p) \tilde{\phi}_{n,n+2}(l-1, p+1) + y(l, p) \phi_{n,n+2}(1, p) \phi_{n+2,n+2}(l-1, p+1), \quad (4.41)$$

$$\tilde{\phi}_{n,n-2}(l, p) = \tilde{x}(l, p) \phi_{n,n}(1, p) \tilde{\phi}_{n,n-2}(l-1, p+1) + \tilde{y}(l, p) \phi_{n,n-2}(1, p) \phi_{n-2,n-2}(l-1, p+1)$$

and the primary field

$$\phi_{n,n}(l, p) = \phi_{n,n}(1, p) \phi_{n,n}(l-1, p+1). \quad (4.42)$$

The dimensions of these fields are:

$$\tilde{\Delta}_{n,n\pm 2} = 1 + \frac{n^2 - 1}{4p} - \frac{(2 \pm n)^2 - 1}{4(p + l)} = 1 - \frac{1 \pm n}{2} \epsilon + O(\epsilon^2), \quad (4.43)$$

$$\Delta_{n,n} = \frac{n^2 - 1}{4p} - \frac{n^2 - 1}{4(p + l)} = \frac{(n^2 - 1)l}{16} \epsilon^2 + O(\epsilon^3).$$
They are analogs of the (descendants of the) NS fields of the $N = 1$ super conformal theory ($l = 2$) and the fields from $S$ or $D$-sectors of 4/3-parafermionic theory ($l = 4$).

Two remarks are in order. First, similarly to $\tilde{\phi}_{1,3}(l, p)$ and $\phi_{3,3}(l, p)$ the fields defined above belong to the zero charge, or ”vacuum sector” of the current $A(z)$. The arguments for that go along the same lines. Second, the fields (4.41) and the derivative of (4.42) have dimensions close to one and therefore can mix. To ensure this we ask that their fusion rules with the perturbing field are closed. This requirement defines the coefficients in (4.41) and the corresponding structure constants. So we impose the conditions:

$$\tilde{\phi}_{1,3}(l, p)\tilde{\phi}_{n,n+2}(l, p) = C^{(nn)}_{(13)(nn+2)}(l, p)\phi_{n,n}(l, p) + C^{(nn+2)}_{(13)(nn+2)}(l, p)\tilde{\phi}_{n,n+2}(l, p),$$

$$\phi_{3,3}(l, p)\phi_{n,n}(l, p) = C^{(nn+2)}_{(33)(nn)}(l, p)\tilde{\phi}_{n,n+2}(l, p) + C^{(nn)}_{(33)(nn)}(l, p)\phi_{n,n}(l, p).$$

Using the constructions (4.31), (4.32) and (4.41), (4.42), we obtain functional equations for the coefficients and the structure constants:

$$a \times C^{(nn+2)}_{(13)(nn+2)}(l - 1, p + 1) + bx C^{(nn)}_{(13)(nn)}(1, p)C^{(nn+2)}_{(13)(nn+2)}(l - 1, p + 1) +$$

$$+ by C^{(nn+2)}_{(13)(nn+2)}(l - 1, p + 1) = yC^{(nn+2)}_{(13)(nn+2)}(l, p),$$

$$a \times y C^{(n+2nn+2)}_{(13)(nn+2+2)}(l - 1, p + 1) + bx C^{(nn+2+2)}_{(13)(nn+2)}(1, p)C^{(n+2nn+2)}_{(13)(nn+2)}(l - 1, p + 1) +$$

$$+ by C^{(nn+2+2)}_{(13)(nn+2)}(l - 1, p + 1) = yC^{(nn+2+2)}_{(13)(nn+2)}(l, p),$$

$$a \times (C^{(nn)}_{(13)(nn+2)}(l - 1, p + 1) + bx C^{(nn)}_{(13)(nn+2)}(1, p)C^{(nn)}_{(33)(nn+2)}(l - 1, p + 1) +$$

$$+ by C^{(nn)}_{(13)(nn+2)}(l - 1, p + 1) = C^{(nn)}_{(13)(nn+2)}(l, p),$$

from the first of (4.44) and

$$C^{(nn)}_{(33)(nn)}(1, p)C^{(nn+2)}_{(33)(nn)}(l - 1, p + 1) = xC^{(nn+2)}_{(33)(nn)}(l, p),$$

$$C^{(nn+2)}_{(33)(nn)}(1, p)C^{(n+2nn+2)}_{(33)(nn)}(l - 1, p + 1) = yC^{(n+2nn+2)}_{(33)(nn)}(l, p),$$

$$C^{(nn)}_{(33)(nn)}(1, p)C^{(nn)}_{(33)(nn)}(l - 1, p + 1) = C^{(nn)}_{(33)(nn)}(l, p)$$

from the second one. In all these equations $x$, $y$, $a$ and $b$ are at values $(l, p)$. Note that $x^2 + y^2 = 1$ (as well as $a^2 + b^2 = 1$) by normalization.

In order to solve these functional equations we use the fact that we know the value of the structure constants $C(1, p)$, i.e. the Virasoro ones. Also, by construction, the fields $\phi_{3,3}(l, p)$ and $\phi_{n,n}(l, p)$ are primary. Therefore their structure constants are just a product of lower level ones, as can be seen from the last of the equations (4.48). Finally, one can use the knowledge of the solutions for $l = 1, 2, 4$ and (53, 79). With all this, we can make a guess and check it directly. We will present the result for the structure constants in Appendix A.
Our goal in this Section is the computation of the matrix of anomalous dimensions and the corresponding mixing matrix of the fields (4.41) and (4.42) up to the second order of the perturbation theory. For that purpose we compute their two-point functions up to second order and the corresponding integrals.

- **Function** \(< \tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0) >\)

  The corresponding function in the second order of the perturbation theory can be found in Appendix C (C.1). After transformation \(x \rightarrow 1/x\) it becomes:

  \[
  \begin{align*}
  < \tilde{\phi}(x)\tilde{\phi}_{n,n+2}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}(\infty) > &= \left\{ \frac{l - (2l + 4)x + (5l + 4)x^2 - 6lx^3 + 3lx^4}{3lx^2(1-x)^2} \right\}^2 + \\
  &+ \frac{8(n + 3)}{3l^2(n + 1)} \left\{ \frac{l - 2(l + 1)x + 6x^2 - 4x^3}{4x^2(1-x)^2} \right\}^2 + \\
  &+ \left( \frac{2(l - 1)}{l} \right)^2 \frac{(n + 3)(n + 4)}{18n(n + 1)} \left\{ \frac{(l + 2(1-l)x + 2(l - 1)x^2)}{2(l-1)x^2(1-x)^2} \right\}^2.
  \end{align*}
  \]

  The integration of this function is very similar to that we did in the case of the computation of the \(\beta\)-function. It goes along the same lines of the \(l = 1\) and \(l = 2\) cases so we do not present here the detailed calculation. The only difference is in the structure constants needed in the OPE’s around 0, 1 and \(\infty\). They are given in Appendix A:

  \[
  (C_{(nn)}^{(nn-2)})^2 = \frac{4(n + 3)^2}{3l^2(n + 1)^2} - \frac{4(n + 2)(n + 3)^2\epsilon}{3l(n + 1)^2} + O(\epsilon^2), \quad (4.49)
  \]

  \[
  (C_{(13)(nn-2)})^2 = \frac{n + 2}{3n} + O(\epsilon^2).
  \]

  The final result of the integration is:

  \[
  \frac{8\pi^2(20 + 143n + 121n^2 + 33n^3 + 3n^4)}{3l^2(n + 1)(n + 3)^2\epsilon^2} - \frac{4\pi^2(n + 5)(8 + 151n + 143n^2 + 45n^3 + 5n^4)}{3\ln(n + 1)(n + 3)^2\epsilon}.
  \]

  This is in perfect agreement with \(l = 1\) and \(l = 2\) cases.

- **Function** \(< \tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0) >\)

  The relevant four-point function in this case in the zeroth order of \(\epsilon\) is given by (C.3). Transforming \(x \rightarrow 1/x\), one obtains:

  \[
  < \tilde{\phi}(x)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(0)\tilde{\phi}(\infty) > = \frac{1}{3} \sqrt{\frac{(n^2 - 4)}{n^2}} \left. \frac{1}{lx^2(1-x)^2} \right|_{l-2(l-1)x+2(l-1)x^2}^2.
  \]

  Again, the integration over the safe region and lens-like region is very similar to \(l = 1\) and \(l = 2\) cases. The same is true also for the singular points where we have to take the structure constant:

  \[
  C_{(13)(nn-2)}^{(nn)} = \frac{n - 2}{3n} + O(\epsilon^2).
  \]
Collecting all the integrals leads to the final result:
\[
\frac{320(1 - l\epsilon)\pi^2}{3l^2\epsilon^2n(n^2 - 9)\sqrt{n^2 - 4}}
\]
which again matches with Virasoro and superconformal cases.

- **Function** \(<\phi_{n,n}(1)\tilde{\phi}_{n,n+2}(0)>

  The four point function differs only in the structure constant (C.4):
  \[
  <\tilde{\phi}(x)\phi_{n,n}(1)\phi_{n,n+2}(0)\phi(\infty)> = \frac{4}{3l}\sqrt{\frac{n + 2}{n}}|x|^{-2}.
  \]

  Therefore the calculations are exactly the same. Also, the necessary structure constants for the calculation around singular points were already presented above. This leads to a final result:
  \[
  \frac{4(n - 1)\pi^2}{3l(n + 3)(n + 5)}\sqrt{\frac{n + 2}{n}}\left[-22 - 6n + \epsilon(-2(n + 5)(3n + 11) + l(46 + n(n + 15)))\right].
  \]

- **Function** \(<\phi_{n,n}(1)\phi_{n,n}(0)>

  Finally, we need the function \(<\tilde{\phi}(x)\phi_{n,n}(1)\phi_{n,n}(0)\phi(\infty)>\). As it is shown in Appendix C this function happens to coincide exactly with the one found in [7] and in Section 2.2 and is given explicitly by (C.5). Therefore almost all integrals are the same. The only exception is the integral around \(\infty\) due to the different structure constants:
  \[
  C_{(mn)}^{(13)} C_{(nn)}^{(13)} = \frac{(n^2 - 1)\epsilon^2}{6}(1 - (l - 2)\epsilon).
  \]

  With this, the result is:
  \[
  \frac{(n^2 - 1)\pi^2}{12}(2 + (8 - 3l)\epsilon).
  \]

  Since the dimension of the field \(\phi_{n,n}\) is close to zero, it doesn’t mix with other fields. Therefore, we need to compute only its anomalous dimension. Taking into account also the first order contribution, the final result for the two-point function is:
  \[
  G_n(x, \lambda) = <\phi_{n,n}(x)\phi_{n,n}(0)> = (x\bar{x})^{-2\Delta_{n,n}} \left[ 1 - \lambda \left( \frac{\sqrt{3l}\pi}{24}(n^2 - 1)\epsilon(2 + (l + 4)\epsilon) \right) (x\bar{x})^\epsilon \right.\]
  \[
  + \left. \frac{\lambda^2}{2} \left( \frac{\pi^2}{12}(2 + (8 - 3l)\epsilon)(n^2 - 1) \right) (x\bar{x})^{2\epsilon} + \ldots \right].
  \]
Computation of the anomalous dimension goes in exactly the same way as for the perturbing field:

\[
\Delta_{g}^{n,n} = \Delta_{n,n} - \frac{\epsilon \lambda}{2} \partial_{\lambda} G_{n}(1, \lambda) = \\
= \Delta_{n,n} + \frac{3\pi g l}{48} \epsilon^{2} (2 + (l + 4)\epsilon)(n^{2} - 1) + \frac{\pi^{2} g^{2}}{24} \epsilon^{2}(l - 4)(n^{2} - 1)
\]

where we again kept the appropriate terms of order $\epsilon \sim g$. Then, at the fixed point (4.40), this becomes:

\[
\Delta_{g}^{\ast n,n} = \frac{(n^{2} - 1) l(4\epsilon^{2} + 6l\epsilon^{3} + 7l^{2}\epsilon^{4} + \ldots)}{64}
\]

which coincides up to the desired order with the dimension of the field $\phi_{n,n}(l, p - l)$ of the model $M(k - l, l)$.

We turn now to the computation of the matrix of anomalous dimensions. We use the same renormalization scheme that was presented in Section 2.2 for the case of $N = 1$ supersymmetric models ($l = 2$). Let us remind that the matrix of anomalous dimensions is defined as:

\[
\Gamma = B\hat{\Delta}B^{-1} - \epsilon\lambda B\partial_{\lambda}B^{-1}
\]

where $\hat{\Delta} = diag(\Delta_{1}, \Delta_{2})$ is a diagonal matrix of the bare dimensions. The matrix $B$, defined as the multiplicative renormalization $\phi_{\alpha} = B_{\alpha\beta}(\lambda)\phi_{\beta}$, is computed from the matrix of the bare two-point functions (see Section 2.2).

We computed above some of the entries of the $3 \times 3$ matrix of two-point functions in the second order. This matrix is obviously symmetric. It turns out also that the remaining functions $<\tilde{\phi}_{n,n-2}(1)\tilde{\phi}_{n,n-2}(0)>$ and $<\phi_{n,n}(1)\phi_{n,n-2}(0)>$ can be obtained from the computed ones $<\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(0)>$ and $<\phi_{n,n}(1)\phi_{n,n+2}(0)>$ by just taking $n \rightarrow -n$.

As we did for $l = 2$ case, let us combine the fields in consideration in a vector with components:

$$
\phi_{1} = \tilde{\phi}_{n,n+2}, \quad \phi_{2} = (2\Delta_{n,n}(2\Delta_{n,n} + 1))^{-1}\partial\bar{\partial}\phi_{n,n}, \quad \phi_{3} = \tilde{\phi}_{n,n-2}.
$$

The field $\phi_{2}$ is normalized so that its bare two-point function is 1. It is straightforward to modify the functions involving $\phi_{2}$ taking into account the derivatives and the normalization.

The matrix of the two-point functions up to the second order in the perturbation expansion was written in (2.82) where the first order term is proportional to the structure constant (2.83).
Collecting all the dimensions and structure constants, we get:

\[
C^{(1)}_{1,1} = \frac{2(n + 3)(-2 + l\epsilon(n + 2))\pi}{\sqrt{3l\epsilon(n + 1)}}, \quad C^{(1)}_{1,2} = \frac{8(-2 + l\epsilon)\sqrt{\frac{n+2}{n}}}{\sqrt{3l\epsilon(n + 1)(n + 3)}}, \quad C^{(1)}_{1,3} = 0,
\]

\[
C^{(1)}_{2,2} = \frac{16\pi}{\sqrt{3l(n^2 - 1)\epsilon}} - \frac{4(n^2 + 1)\pi}{\sqrt{3(n^2 - 1)}}, \quad C^{(1)}_{2,3} = \frac{8(-2 + l\epsilon)\sqrt{\frac{n+2}{n}}}{\sqrt{3l\epsilon(n - 3)(n - 1)}},
\]

\[
C^{(1)}_{3,3} = \frac{-2(n - 3)(-2 + l\epsilon(2 - n))\pi}{\sqrt{3l\epsilon(n - 1)}},
\]

for the first order, and:

\[
C^{(2)}_{1,1} = \frac{8(20 + 143n + 121n^2 + 33n^3 + 3n^4)\pi^2}{3l^2n(n + 1)(n + 3)^2\epsilon^2} - \frac{4(n + 5)(8 + 151n + 143n^2 + 45n^3 + 5n^4)\pi^2}{3ln(n + 1)(n + 3)^2\epsilon},
\]

\[
C^{(2)}_{1,2} = -\frac{64\sqrt{\frac{n+2}{n}(3n + 11)\pi^2}}{3l^2(n + 1)(n + 3)(n + 5)\epsilon^2} + \frac{32\sqrt{\frac{n+2}{n}(57 + 18n + n^2)\pi^2}}{3ln(n + 1)(n + 3)(n + 5)\epsilon},
\]

\[
C^{(2)}_{1,3} = \frac{320(1 - l\epsilon)\pi^2}{3l^2\epsilon^2 n(n^2 - 9)\sqrt{n^2 - 4}},
\]

\[
C^{(2)}_{2,2} = \frac{128\pi^2}{3l^2(n^2 - 1)\epsilon^2} - \frac{16(n^2 + 19)\pi^2}{3l(n^2 - 1)\epsilon},
\]

\[
C^{(2)}_{2,3} = -\frac{64\sqrt{\frac{n-2}{n}(3n - 11)\pi^2}}{3l^2n(n - 1)(n - 3)(n - 5)\epsilon^2} - \frac{32\sqrt{\frac{n-2}{n}(57 - 18n + n^2)\pi^2}}{3ln(n - 1)(n - 3)(n - 5)\epsilon},
\]

\[
C^{(2)}_{3,3} = -\frac{8(-20 + 143n - 121n^2 + 33n^3 - 3n^4)\pi^2}{3l^2n(n - 1)(n - 3)^2\epsilon^2} + \frac{4(n - 5)(8 - 151n + 143n^2 - 45n^3 + 5n^4)\pi^2}{3ln(n - 1)(n - 3)^2\epsilon},
\]

for the second one.

Now we can apply the renormalization procedure of Section 2.2 and obtain the matrix of anomalous dimensions (4.50). The bare coupling constant \( \lambda \) is expressed through \( g \) by (4.39) and the bare dimensions, up to order \( \epsilon^2 \). The computation goes analogously to \( l = 2 \) case so we omit the details here.
Evaluating this matrix at the fixed point \((4.40)\), we get:

\[
\begin{align*}
\Gamma^\ast_{1,1} &= 1 + \frac{(20 - 4n^2)\epsilon}{8(n + 1)} + \frac{l(39 - n - 7n^2 + n^3)\epsilon^2}{16(n + 1)}, \\
\Gamma^\ast_{1,2} &= \Gamma^\ast_{2,1} = \frac{(n - 1)\sqrt{\frac{n^2}{n}}\epsilon(1 + \epsilon)}{n + 1}, \\
\Gamma^\ast_{1,3} &= \Gamma^\ast_{3,1} = 0, \\
\Gamma^\ast_{2,2} &= 1 + \frac{4\epsilon}{n^2 - 1} + \frac{l(65 - 2n^2 + n^4)\epsilon^2}{16(n^2 - 1)}, \\
\Gamma^\ast_{2,3} &= \Gamma^\ast_{3,2} = \frac{\sqrt{\frac{n-2}{n}(n + 1)\epsilon(1 + \epsilon)}}{n - 1}, \\
\Gamma^\ast_{3,3} &= 1 + \frac{(n^2 - 5)\epsilon}{2(n - 1)} + \frac{l(-39 - n + 7n^2 + n^3)\epsilon^2}{16(n - 1)},
\end{align*}
\]

whose eigenvalues are (up to order \(\epsilon^2\)):

\[
\begin{align*}
\Delta^\ast_1 &= 1 + \frac{1 + n\epsilon}{2} + \frac{l(7 + 8n + n^2)\epsilon^2}{16}, \\
\Delta^\ast_2 &= 1 + \frac{l(n^2 - 1)}{16}\epsilon^2, \\
\Delta^\ast_3 &= 1 + \frac{1 - n\epsilon}{2} + \frac{l(7 - 8n + n^2)\epsilon^2}{16}.
\end{align*}
\]

This result coincides with the dimensions \(\tilde{\Delta}_{n+2,n}(l, p - l)\), \(\Delta_{n,n}(l, p - l) + 1\) and \(\tilde{\Delta}_{n-2,n}(l, p - l)\) of the model \(M(k - l, l)\) up to this order. The corresponding normalized eigenvectors should be identified with the fields of \(M(k - l, l)\):

\[
\begin{align*}
\tilde{\phi}_{n+2,n}(l, p - l) &= \frac{2}{n(n + 1)}\phi_1^\ast + \frac{2\sqrt{\frac{n+2}{n}}}{n + 1}\phi_2^\ast + \frac{\sqrt{n^2 - 4}}{n}\phi_3^\ast, \\
\phi_2(l, p - l) &= -\frac{2\sqrt{\frac{n+2}{n}}}{n + 1}\phi_1^\ast - \frac{n^2 - 5}{n^2 + 1}\phi_2^\ast + \frac{2\sqrt{\frac{n-2}{n}}}{n - 1}\phi_3^\ast, \\
\tilde{\phi}_{n-2,n}(l, p - l) &= \frac{\sqrt{n^2 - 4}}{n}\phi_1^\ast - \frac{2\sqrt{\frac{n-2}{n}}}{n - 1}\phi_2^\ast + \frac{2}{n(n - 1)}\phi_3^\ast.
\end{align*}
\]

We used as before the notation \(\tilde{\phi}\) for the descendent field defined as in \((4.41)\) and:

\[
\phi_2(l, p - l) = \frac{1}{2\Delta_{n,n}(2\Delta_{n,n} + 1)}\partial\bar{\partial}\phi_{n,n}(l, p - l)
\]

is the normalized derivative of the corresponding primary field. We notice that these eigenvectors are finite as \(\epsilon \to 0\) with exactly the same entries as in \(l = 1\) and \(l = 2\) minimal models. We will show in the next Section that they are also in agreement with those computed using the domain wall construction. This is our main result in this Section.
4.3 RG domain wall

In the previous Section we proved that the coset CFT $M(k, l)$ perturbed by the field $\tilde{\phi}_{1,3}$ has a nontrivial fixed point corresponding to $M(k - l, l)$ up to the second order of the perturbation theory. We also found the mixing coefficients for certain fields between the UV $T_{UV} = M(k, l)$ and the IR $T_{IR} = M(k - l, l)$ theories.

Few years ago Gaiotto constructed a nontrivial conformal interface (RG domain wall) encoding the UV-IR map resulting through the RG flow described above [69]. Let us briefly recall the construction. Gaiotto considered a theory consisting of a IR $M(k - l, l)$ theory in the upper half plane and a UV $M(k, l)$ in the lower one. The conformal interface between the two CFT models is equivalent to some conformal boundary for the direct product of the theories $T_{UV} \times T_{IR}$:

\[ \hat{su}(2)_k \times \hat{su}(2)_l \times \hat{su}(2)_{k+l} \sim \hat{su}(2)_{k-l} \times \hat{su}(2)_l \times \hat{su}(2)_{k+l} \]

Note that two factors of $\hat{su}(2)_l$ appear at the RHS and therefore the theory possesses a natural $Z_2$ symmetry. In [69] it was shown that the desired boundary of the theory:

\[ T_B = \frac{\hat{su}(2)_{k-l} \times \hat{su}(2)_l \times \hat{su}(2)_l}{\hat{su}(2)_{k+l}} \]

acts as a $Z_2$ twisting mirror. Explicitly, this RG boundary is given by:

\[ |\tilde{B} > = \sum_{s, t} \sqrt{S_{1,t}^{(k-l)} S_{1,s}^{(k+l)}} \sum_d |t, d, d; B, Z_2 > \]

where the indices $t, d, s$ of the Ishibashi states refer to the representations of $\hat{su}(2)_{k-l}$, $\hat{su}(2)_l\hat{su}(2)_{k+l}$ respectively and $S_{n,m}^{(k)}$ are the modular matrices of the $\hat{su}(2)_k$ WZNW model:

\[ S_{n,m}^{(k)} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi nm}{k+2} \]

In this construction, the coefficients (4.51) of the UV-IR map are expressed in terms of the one point functions of the theory $T_{UV} \times T_{IR}$ in the presence of the RG boundary. So we need the explicit expression of the states corresponding to the fields $\phi_{IR}^\phi \phi_{UV}$ in terms of the states of the coset theory $T_B$.

Basic ingredient of the latter is the $\hat{su}(2)_k$ WZNW with a current $J$. As we mentioned above it is a CFT with central charge $c_k = \frac{3k}{k+2}$. The primary fields $\phi_{j,m}$ and the corresponding states $|j, m >$ are labeled by the (half)integer spin $j$ and its projection $m = -j, -j + 1, ..., j$. Their conformal dimensions are given by:

\[ \Delta_j = \frac{j(j+1)}{k+2} \]

(4.53)
The representations are defined by the action of the currents on these states:

\[ J^\pm_j |j, m> = \sqrt{j(j + 1) - m(m + 1)} |j, m \pm 1>, \]  
\[ J^0_0 |j, m> = m |j, m>. \]  

(4.54)

Following [72] let us denote by \( K(z) \) and \( \tilde{K}(z) \) the WZNW currents of \( \hat{su}(2)_l \) entering the cosets of the IR and UV theories respectively. We reserve the notion \( J(z) \) for the current of \( \hat{su}(2)_{k-l} \) entering the IR coset. The corresponding energy momentum tensors can be expressed in terms of these currents using (4.27). For example we can write symbolically the IR stress tensor as:

\[ T_{ir} = \frac{1}{k-l+2} J^2 + \frac{1}{l+2} K^2 - \frac{1}{k+2} (J + K)^2 \]  

(4.55)

and similarly for the UV one. Finally, we impose the condition that the state of the coset \( T_B \) be a highest weight state of the diagonal current \( J + K + \tilde{K} \).

Now we are in a position to compare the mixing coefficients in (4.51) with the corresponding one-point functions of the domain wall construction. Actually, we found it easier to compute the one-point functions of the other components of the corresponding multiplets. Namely, we shall consider the mixing of the "first components" given by the primary fields \( \phi_{n,n} \) and the first descendent of \( \phi_{n,n} \) with respect to the current \( A(z) \). Indeed, since \( \phi_{n,n} \) belongs to the "vacuum sector" the current \( A(z) \) is not branched around it and the dimension of the descendent \( \tilde{\phi}_{n,n} = A_{\frac{2}{l+2}} \phi_{n,n} \) is:

\[ \tilde{\Delta}_{n,n} = \frac{2}{l+2} + \frac{n^2 - 1}{4p} - \frac{n^2 - 1}{4(p+l)}. \]

So all these fields have dimension close to \( \frac{2}{l+2} \) in the limit \( p \rightarrow \infty \). Suppose they mix in the same way like it was in the case \( l = 2 \) for example [72]. We want to compare the corresponding one point functions with the coefficients in (4.51).

We shall need therefore the explicit construction for the current \( A(z) \). It goes in a way very similar to that of [72] (see also [12]). Consider for example the IR model. As in [72] we take:

\[ A(z) = C_a J^a(z) \phi_{1,-a}(z) + D_a K^a_{-1} \phi_{1,-a}(z) \]  

(4.56)

where \( \phi_{1,m}(z) \) is a spin 1 field of the level \( l \) WZNW theory with a current \( K(z) \) and there is a summation over the index \( a = \pm 1, 0 \). Indeed, the dimension of this current is:

\[ \Delta_A = 1 + \frac{2}{l+2} = \frac{l+4}{l+2}. \]

The coefficients \( C_a, D_a \) are fixed by the requirement that the respective state be the highest weight state of the diagonal current algebra \( J + K \). We get:

\[ D_+ = \frac{\kappa}{\sqrt{2}}, \quad D_0 = \kappa, \quad D_- = -\frac{\kappa}{\sqrt{2}} \]  
\[ C_+ = -\kappa \frac{l+4}{(k-l)\sqrt{2}}, \quad C_0 = -\kappa \frac{l+4}{(k-l)}, \quad C_- = \kappa \frac{l+4}{(k-l)\sqrt{2}} \]  

(4.57)
where \( \kappa \) is a normalization constant. Since below we shall normalize the corresponding states we don’t need it explicitly here. It is straightforward to make a similar construction for the UV coset with obvious change of currents and levels.

Now we can pass to the computation of the one-point functions of the fields \( \phi_{ir} \phi_{uv} \) and compare them with the corresponding coefficients in (4.51).

Let us first start though with the field \( \phi_{uv} \) itself. As we showed above it flows to the field \( \phi_{ir} \) in the infrared. So we need to find the state in \( \mathcal{T}_B \) corresponding to \( \phi_{ir} \phi_{uv} \). For this we need to match their conformal dimensions and to ensure that the state is a highest weight state of the diagonal current \( J + K + \tilde{K} \). The dimension of the primary field \( \phi_{n,n} \) can be read from (4.43). For the product of the IR and UV fields we have:

\[
\Delta_{ir}^{n,n} + \Delta_{uv}^{n,n} = \frac{n^2 - 1}{4(k - l + 2)} - \frac{n^2 - 1}{4(k + l + 2)}. \tag{4.58}
\]

It is easy to identify the corresponding state with:

\[
| \frac{n - 1}{2}, \frac{n - 1}{2} > | 0, 0 > | 0, 0 >
\]

where the three states correspond to \( su(2) \) of levels \( k - l \) (with current \( J \)), IR level \( l \) (with current \( K \)) and UV level \( l \) (with current \( \tilde{K} \)) respectively. Indeed, this state is obviously a spin \( \frac{n - 1}{2} \) highest weight state of \( J + K + \tilde{K} \) and its dimension:

\[
\Delta_{J\frac{1}{2}}^{n-1} + \Delta_{0}^{K} + \Delta_{0}^{\tilde{K}} - \Delta_{\frac{1}{2}}^{J+K+\tilde{K}}
\]

coincides with (4.58). It is obvious that this state is invariant under the \( Z_2 \) action, i.e. the exchange of the second and third factors. So the overlap of this state with its \( Z_2 \) image is just equal to 1 and therefore:

\[
< \phi_{ir}^{n,n} \phi_{uv}^{n,n} | RG > = \sqrt{\frac{S_{1,n}^{(k-l)} S_{1,n}^{(k+l)}}{S_{1,n}^{(k)}}} = 1 + \frac{3l^2}{4k^2} + O(\frac{1}{k^3}). \tag{4.59}
\]

This confirms that up to the leading order in \( k \to \infty \) the field \( \phi_{uv} \) flows to \( \phi_{ir} \).

Let us now find, for example, the state corresponding to \( \phi_{n+2,n}^{ir} \phi_{n,n+2}^{uv} \). The dimensions can be found from (4.43) and we have:

\[
\Delta_{ir}^{n+2,n} + \Delta_{uv}^{n,n+2} = \frac{4}{l + 2} + \frac{(n + 1)(n + 3)}{4(k - l + 2)} - \frac{(n + 1)(n + 3)}{4(k + l + 2)}.
\]

The corresponding state should have the form:

\[
\sum_{\alpha,\beta = \pm 1,0} C_{\alpha\beta} | \frac{n + 1}{2}, \frac{n + 1}{2} - \alpha - \beta > | 1, \alpha > | 1, \beta > . \tag{4.60}
\]
The coefficients $C_{\alpha\beta}$ are obtained by imposing the condition that (4.60) has a correct IR dimension and is a highest weight state of $J + K + \tilde{K}$. We obtain:

$$C_{++} = -\frac{1}{\sqrt{n}} C_{0+}, \quad C_{-+} = -\sqrt{\frac{n+1}{2}} C_{0+}$$

and all the other coefficients vanish. The overall normalization fixes:

$$C^2_{0+} = \frac{2n}{(n+1)(n+2)}.$$

Taking the overlap of the state (4.60) with its $Z_2$ image we find:

$$\langle \phi_{n+2,n}^{ir} | \phi_{n,n+2}^{uv} | RG \rangle = \frac{2}{(n+1)(n+2)} \frac{\sqrt{S_{1,n+2}^{(k-l)} S_{1,n+2}^{(k+l)}}}{S_{1,n}^{(k)}} = \frac{2}{n(n+1)} + O(\frac{1}{k^2}). \quad (4.61)$$

The other calculations go in the same way, we present here just the result:

$$\langle \phi_{n,n-2,n}^{ir} | \phi_{n,n-2,n}^{uv} | RG \rangle = \frac{2}{(n-1)(n-2)} \frac{\sqrt{S_{1,n-2}^{(k-l)} S_{1,n-2}^{(k+l)}}}{S_{1,n}^{(k)}} = \frac{2}{n(n-1)} + O(\frac{1}{k^2}), \quad (4.62)$$

$$\langle \phi_{n+2,n}^{ir} | \phi_{n,n+2}^{uv} | RG \rangle = \frac{\sqrt{S_{1,n+2}^{(k-l)} S_{1,n+2}^{(k+l)}}}{S_{1,n}^{(k)}} = \frac{n^2-4}{n} + O(\frac{1}{k^2}),$$

$$\langle \phi_{n,n-2,n}^{ir} | \phi_{n,n+2}^{uv} | RG \rangle = \sqrt{S_{1,n-2}^{(k-l)} S_{1,n+2}^{(k+l)}} = \frac{n^2-4}{n} + O(\frac{1}{k^2}).$$

Consider in more details the functions involving the descendent field $\tilde{\phi}_{n,n}$. Let us first consider $\phi_{n,n}^{ir}\phi_{n,n+2}^{uv}$:

$$\Delta_{n,n+2}^{ir} + \Delta_{n,n+2}^{uv} = \frac{2}{l+2} + \frac{n^2-1}{4(k-l+2)} - \frac{(n+1)(n+3)}{4(k+l+2)}. \quad (4.63)$$

The corresponding state is:

$$\left| \frac{n-1}{2}, \frac{n-1}{2} > |0,0 > |1,1 > \quad (4.64)$$

(because the spin 1 term in (4.63) refers to UV level $l$ current $\tilde{K}$). Using the explicit expression of the current it is easy to find that for the descendent we have:

$$A_{n,n+2}^{a}\left| \frac{n-1}{2}, \frac{n-1}{2} > |0,0 > |1,1 > = C_{a} J_{0}^{a}\left| \frac{n-1}{2}, \frac{n-1}{2} > |1,-a > |1,1 > +$$

$$+ D_{a} K_{0}^{a}\left| \frac{n-1}{2}, \frac{n-1}{2} > |1,-a > |1,1 >$$
where the coefficients are given by (4.57). This gives:

\[
A_{n, \frac{n-1}{2}, \frac{n-1}{2}} |0,0 \rangle \langle 1,1| = \kappa \frac{(l + 4)}{(k-l)} \sqrt{\frac{n-1}{2}} \gamma^{\frac{n-1}{2}, n-1, \frac{n-3}{2}, \frac{n-1}{2}} |0,0 \rangle \langle 1,1| - \kappa \frac{(l + 4)}{(k-l)} \gamma^{\frac{n-1}{2}, \frac{n-1}{2}} |1,0 \rangle \langle 1,1|. 
\]

The normalization condition is:

\[
\frac{\kappa^2 (l + 4)^2 n^2 - 1}{(k-l)^2} = 1. 
\]

Thus, for the one-point function we get:

\[
< \tilde{\phi}_{ir, n, n}^w \tilde{\phi}_{ir, n, n}^w | RG > = 2 \sqrt{\frac{S_{1,n}^{(k-l)} S_{1,n+2}^{(k+l)}}{S_{1,n}^{(k)}}} = 2 \sqrt{n+2} + O\left(\frac{1}{k^2}\right). 
\]

The other calculations are similar and finally we get:

\[
< \tilde{\phi}_{ir, n, n}^w \tilde{\phi}_{ir, n, n-2}^w | RG > = -2 \sqrt{\frac{S_{1,n}^{(k-l)} S_{1,n-2}^{(k+l)}}{S_{1,n}^{(k)}}} = -2 \sqrt{n-2} + O\left(\frac{1}{k^2}\right), 
\]

\[
< \tilde{\phi}_{ir, n-2, n}^w \tilde{\phi}_{ir, n, n}^w | RG > = -2 \sqrt{\frac{S_{1,n-2}^{(k-l)} S_{1,n}^{(k+l)}}{S_{1,n}^{(k)}}} = -2 \sqrt{n-2} + O\left(\frac{1}{k^2}\right), 
\]

\[
< \tilde{\phi}_{ir, n+2, n}^w \tilde{\phi}_{ir, n, n}^w | RG > = 2 \sqrt{\frac{S_{1,n+2}^{(k-l)} S_{1,n}^{(k+l)}}{S_{1,n}^{(k)}}} = 2 \sqrt{n+2} + O\left(\frac{1}{k^2}\right), 
\]

\[
< \tilde{\phi}_{ir, n, n+2}^w \tilde{\phi}_{ir, n, n}^w | RG > = 2 \sqrt{\frac{S_{1,n+2}^{(k-l)} S_{1,n}^{(k+l)}}{S_{1,n}^{(k)}}} = 2 \sqrt{n+2} + O\left(\frac{1}{k^2}\right). 
\]

We see that all these results (4.60), (4.62), (4.66) and (4.67) are in a perfect agreement with the leading order calculations (4.51) presented in the previous section.

5 Perturbation of 2D CFT and hidden symmetries of the related 2D integrable field theories

In this Section we present a general framework to investigate the symmetries and related charges in 2D integrable field theories. We start with the construction of some non-trivial conserved quantities in simple models like the massive free Majorana fermions and their
$O(N)$ generalization. A more powerful strategy put forward in [80] consists in trying to understand the link between the particle description and the field theory one by implementing a quantum inverse scattering method (QISM) for CFT. Indeed, off-criticality the Virasoro symmetry is lost, and the QISM results in the most hopeful method applicable to compute physical quantities of the theory. The proposal in [80] is then first to map the CFT data into a QISM structure at criticality and later study how to leave the critical point by suitable perturbation of this structure.

As mentioned in the Introduction, of special interest is the calculation of the VEV’s of the descendants fields in the integrable theories providing the next to leading order in the UV limit of the two-point function. In this Section we calculate the second and third order descendant fields in the Bullough-Dodd (BD) model. It should be stressed that differently from the SG model the third level VEV is non-zero due to the existence of a local conserved current of spin 3 in the BD model. This model has attracted big interest, in particular in connection with perturbed minimal models: $c < 1$ minimal CFT perturbed by the operators $\Phi_{12}, \Phi_{21}$ or $\Phi_{15}$ can be obtained by a quantum group restriction of the imaginary BD model [81, 82] with special value of the coupling. We use this property to deduce the VEV’s $\langle L_{-2}L_{-2}\Phi_{lk} \rangle$ and $\langle L_{-3}L_{-3}\Phi_{lk} \rangle$ in the mentioned perturbed minimal models.

We further present another idea for the investigation of symmetries and corresponding charges in the 2D integrable theories. It is based on a generalization of the so called dressing symmetry transformations [83, 84, 85]. In fact, our basic objects will be the transfer matrix $T(x, \lambda)$ which generates the dressing and the resolvent $Z(x, \lambda)$, the dressed generator of the underlying symmetry. Although it is clear from the construction that our method is applicable to any generalized KdV hierarchy [86], we will be concerned with the semiclassical limit [87, 88, 80] of minimal CFT’s [1], namely the $A^{(1)}_1$-and $A^{(2)}_2$-KdV systems [80]. We present an alternative approach to the description of the spectrum of local fields in the classical limit of certain 2d integrable theories. It is possible to generalize the aforementioned dressing transformations. The idea is that we may dress not only the generators of the underlying Kac-Moody algebra but also differential operators in the spectral parameter $\lambda^n \partial^n$ forming a $w_\infty$ algebra. The corresponding vector fields close a $w_\infty$ algebra as well with a Virasoro subalgebra (for $n = 1$) made up of quasi-local and non-local transformations. Finally, we present a construction of a Virasoro symmetry directly in the sine-Gordon theory. Although we are of course interested in the quantum theory, we restrict ourselves to the classical picture. Also, we are mainly concerned here with the construction of this symmetry in the case of the $N$-soliton solutions. One of the reasons for this is that the symmetry in this case is much simpler realized - in particular it becomes local contrary to the field theory realization.

The results of this Section have been published in [89]-[97], (22.-30.).
5.1 Off critical current algebras

The conformal models belong to the big family of the relativistic integrable models. The key point in the construction of the spectrum and correlation functions of the fields in this class of integrable models (IM’s) is the appearance of the infinite dimensional symmetry provided by the Virasoro algebra. One could wonder whether analogous infinite symmetries algebra approach works in the case of the nonconformal IM’s, say - sin-Gordon, massive fermions, affine Toda models etc. As it is known, the integrability of all these models is based on the existence of an infinite set of conserved charges (CC):

\[ P_s = \oint T_{2s} dz - \oint \Theta_{2s-2} d\bar{z}, \quad \bar{P}_s = \oint \bar{T}_{2s} d\bar{z} - \oint \Theta_{2s-2} dz \]  \hspace{1cm} (5.1)

\[ T_{\mu_1...\mu_{2s}} = (T_{2s}, \bar{T}_{2s}, \Theta_{2s-2}), \quad \bar{\partial}T_{2s} = \partial\Theta_{2s-2} \]

they have. However, the algebra of the \( P_s (\bar{P}_s) \) is abelian: \([P_s, P_s'] = 0 = [\bar{P}_s, \bar{P}_s']\) and this is an obstacle in using these symmetries for the calculation of the exact correlation functions of the model. Therefore, the question one has to answer first is whether \( P_s \) exhaust all the conserved charges of these models, i.e.

1) are there more (nontrivial) conservation laws?

2) if so, is it the algebra of the new conserved charges nonabelian?

In this Section we give an explicit construction of noncommuting CC’s describing such infinite symmetries of the IM’s. Our starting point is the fact that almost all relativistic (nonconformal) IM’s can be represented as an appropriate perturbation of certain conformal models [98], i.e.:

\[ S_{IM} = S_{conf} + g \int \Phi_{\Delta}(z, \bar{z}) d^2z. \]  \hspace{1cm} (5.2)

This suggests that the desired new charges (if they exist) should be realized as specific combinations of the higher momenta of the conserved tensors (\( T_{2s}, \bar{T}_{2s}, \Theta_{2s-2} \)):

\[ \mathcal{F}^{(n)}_{2s} = \sum_{k=1}^{s} \left\{ \alpha_k(g)z^{2k-1+n-\Delta}z^{\gamma(k,n)}T_{2k}(z, \bar{z}) + \bar{\alpha}_k(g)z^{2k-1+n-\Delta}z^{\bar{\gamma}(k,n)}\bar{T}_{2k}(z, \bar{z}) \right\} \]  \hspace{1cm} (5.3)

such that:

\[ \bar{\partial}\mathcal{F}^{(n)}_{2s} = \partial\mathcal{G}^{(n)}_{2s-2} \]

where \( \alpha_k(g) = \alpha_k g^{\frac{s-k}{1-\Delta}}, \quad \beta_k(g) = \beta_k g^{\frac{s-k}{1-\Delta}}. \)

The crucial observation that simplifies the construction of the new conservation laws \( \mathcal{F}^{(n)}_{2s} \) is the following criterion of existing of such quantities: If the conservation laws of the
spin - 2s ( s > 1 ) tensors \( T_{2s} \) are in the form:

\[
\bar{\partial} T_{2s} = \partial^{2s-1} \Theta + g^p \sum_{l=1}^{s-1} A_l \partial^{2(s-l)-1} T_{2l}
\]

\[
\partial \bar{T}_{2s} = \bar{\partial}^{2s-1} \Theta + g^p \sum_{l=1}^{s-1} A_l \bar{\partial}^{2(s-l)-1} \bar{T}_{2l}
\]

\[
\bar{\partial} T = \partial \Theta, \quad \partial T = \bar{\partial} \Theta, \quad p = \frac{1}{1 - \Delta}.
\]

then there exist 4s - 3 new conserved currents \( \mathcal{F}_{2s}^{(n)} \) \((n = 1, 2, \ldots, 4s - 3)\) for each fixed \( s = 1, 2, \ldots \). In words, the existence of new conserved charges:

\[
\begin{align*}
L_{-n}^{(2s)} &= \int \mathcal{F}_{2s}^{(n)} d\bar{z} - \int \mathcal{G}_{2s-2}^{(n)} d\bar{z} \\
\bar{L}_{-n}^{(2s)} &= \int \bar{\mathcal{F}}_{2s}^{(n)} - \int \bar{\mathcal{G}}_{2s-2}^{(n)} d\bar{z}
\end{align*}
\]

is hidden in the specific form of the traces \( \Theta_{2s-2} \) of the traditional conserved currents \( T_{2s} \):

\[
\Theta_{2s-2} = \partial^{2s-2} \Theta + g^p \sum_{l=1}^{s-1} A_l \partial^{2(s-l)-2} T_{2l}.
\]

Turning back to our problem of constructing noncommuting conserved charges for the IM’s given by (5.2) we have to check whether exist models which satisfy our criterion, i.e. their standard \( T_{2s} \) - conservation laws to be in the form (5.4).

The simplest case is the set of models obtained by \( \Phi_{\Delta, 3} \) perturbations of the conformal minimal models \( (c_p = 1 - \frac{6}{(p+1)(p+2)}, \Delta_{1,3}(p) = \frac{p}{p+2}) \) (see [98]). They have all the \( T_{2s}, \bar{T}_{2s}, s = 1, 2, \ldots \) conserved. The first model \( (p = 2) \) of this set is the thermal perturbation of the Ising model which in the continuum limit coincides with the theory of free massive Majorana fermion \( (\psi, \bar{\psi}) \):

\[
\begin{align*}
\bar{\partial} \psi &= m \bar{\psi} \\
\partial \bar{\psi} &= -m \psi
\end{align*}
\]

\[
T = \frac{1}{2} \psi \partial \psi, \quad \bar{T} = \frac{1}{2} \bar{\psi} \bar{\partial} \bar{\psi}, \quad \Theta = m \bar{\psi} \psi.
\]

To find the explicit form of \( \Theta_{2s-2} \) in this case it is better to use the equation of motion (5.6) instead of the conformal perturbative technics. The corresponding conservation tensors of spin 2s can be taken in the form \( T_{2s} = \psi \partial^{2s-1} \psi, \quad s = 2, 3, \ldots \).

Simple computations based on the eq. (5.6) leads to the following desired form of the \( T_{2s} \) - conservation laws:

\[
\begin{align*}
\bar{\partial} T_4 &= \partial^3 \Theta + 2m^2 \partial T \\
\bar{\partial} T_6 &= \partial^5 \Theta + m^2 (\partial T_4 + 4 \partial^3 T) \\
\bar{\partial} T_8 &= 2 \bar{\partial}^2 T + m^2 (\partial T_6 + 3 \partial T_4 + 2 \partial^3 T)
\end{align*}
\]
etc. The conclusion is that this model satisfies our criterion and therefore it has \(4s - 3\) new conservation laws for each \(s = 1, 2, \ldots\). The corresponding conserved charges \(L_{-n}^{(2s)}\), \(\bar{L}_{-n}^{(2s)}\), \(0 \leq n \leq 2s - 1\), can be derived as integrals of certain momenta of the conserved currents. One could try to construct them order by order but this turns to be inconvenient for deriving (or guessing) the general form of \(L_{-n}^{(2s)}\) and for computing their algebra as well. For these purposes it is better to have \(L_{-n}^{(2s)}\)'s as differential operators acting on \(\psi\) and \(\bar{\psi}\). One can do this in few steps. We first exclude the time derivatives \(\partial_t \psi\) and then take \(t = 0\) \((z = t + x, \bar{z} = t - x, \partial = \partial_x)\) in the first few examples. The next step is to derive the momentum space form of \(L_{-n}^{(2s)}\) by substituting the standard creation and annihilation operators \(a^\dagger(p)\) decomposition of \(\psi\) and \(\bar{\psi}\). This allows us to make a conjecture about the general form of all the \(L_{-k}^{(2s)}\) \((0 \leq k \leq 2s - 1)\):

\[
[L_{-k}^{(2s)}, \psi] = \frac{-i}{2} \left[ (\bar{z}\partial - z\partial + 1)_{2s-1-k} + (z\bar{\partial} - z\partial - 2s + 1)_{2s-1-k} \right] \partial^k \psi, \tag{5.9}
\]

where \((A)_p = A(A + 1) \ldots (A + p - 1)\). In order to prove our conjecture we have to be able to derive from (5.9) the integral form of \(L_{-k}^{(2s)}\) and to show that the integrands are conserved quantities. It exists, however, an indirect way to prove that (5.9) are conserved charges, namely we can prove that they are generators of symmetries of the action (5.10). Let us first check whether the simplest nontrivial charge \(L_{-2}^{(4)}\) leaves invariant the action:

\[
S = \int \left( -\frac{1}{2} \psi \bar{\partial} \psi + \frac{1}{2} \bar{\psi} \partial \bar{\psi} + m \bar{\psi} \psi \right) d^2 z \equiv \int \mathcal{L} d^2 z. \tag{5.10}
\]

By using (5.9) and:

\[
[L_{-2}^{(4)}, \bar{\psi}] = (\bar{z}\partial - z\partial - \frac{1}{2}) \partial^2 \bar{\psi}
\]

one can verify easily that:

\[
[L_{-2}^{(4)}, \mathcal{L}] = \partial A + \bar{\partial} B
\]

for some \(A\) and \(B\). Therefore \(L_{-2}^{(4)}\) is a generator of a specific new symmetry of (5.10). The same is true for \(\bar{L}_{-2}^{(4)}\). Together with the Lorentz rotation generator:

\[
L_0 = \int (zT + \bar{z}\Theta) dz - \int (\bar{z}T + z\Theta) d\bar{z}
\]

they close an \(SL(2, R)\) - algebra. One can repeat this calculation with \(L_{-1}^{(4)}, \bar{L}_{-1}^{(4)}, L_0^{(4)}, \bar{L}_{-2}^{(4)}\) etc. and the result is always that these charges commute with the action (5.10). As it becomes clear from this discussion, the proof that \(L_{-k}^{(2s)}\) given by (5.9) are conserved charges is equivalent to the statement that \([L_{-k}^{(2s)}, S] = 0\). To prove it we have to make one more conjecture, namely:

\[
[L_{-k}^{(2s)}, \bar{\psi}] = \frac{-i}{2} \left[ (\bar{z}\partial - z\partial + 1)_{2s-1-k} + (z\bar{\partial} - z\partial - 2s + 2)_{2s-1-k} \right] \partial^k \bar{\psi} \tag{5.11}
\]

\(0 \leq k \leq 2s - 1\).
The remaining part of the proof is a straightforward but tedious higher derivative calculus.

To make complete our study of the conserved charges of the off-critical Ising model we have to find the general form of the “conjugated charges” $\bar{L}_{-k}^{(2s)}$. By arguments similar to the ones presented above we arrive to the following result:

$$\bar{L}_{-k}^{(2s)} = \frac{1}{2} \left[ (\bar{z} \bar{\partial} - z \partial + \bar{\alpha} - 2s + k + 2)_{2s-1-k} + (\bar{z} \bar{\partial} - z \partial + \alpha + k + 1)_{2s-1-k} \right] \bar{\partial}^k, \quad (5.12)$$

where $\bar{\alpha} = -1$ for $\psi$ and $\alpha = 0$ for $\bar{\psi}$. Our claim is that $(5.9)$, $(5.11)$ and $(5.12)$ do exhaust all the local symmetries (i.e. local conserved charges) of the action $(5.10)$.

Studying the conformal limit of $L_{-k}^{(2s)}$ we have realized that the “conformal” $W_\infty(V)$ algebra has a specific subalgebra $\mathcal{P}W_\infty(V)$ spanned by:

$$\mathcal{L}_{-k}^{(2s)} = \frac{1}{2} \left[ (\bar{L}_0 - \frac{1}{2})_{2s-1-k} + (\bar{L}_0 + 2s - \frac{3}{2})_{2s-1-k} \right] L_{-1}^k, \quad 0 \leq k \leq 2s - 1, \quad \bar{L}_0 = z \partial + \frac{1}{2}. \quad (5.13)$$

Having at hand the explicit form $(5.9)$, $(5.11)$ and $(5.12)$ of $L_{-k}^{(2s)}$ and $\bar{L}_{-k}^{(2s)}$ we are prepared to compute their algebra. As we have already mentioned $L_{-2}^{(4)}$, $L_{-1}^{(4)}$ and $L_0$ close an $SL(2, R)$ algebra. Two more $SL(2, R)$ algebras are spanned by $L_{-1}$, $L_{-2}$, $L_0$ and $L_{-1}$, $L_{-2}$, $L_0$. Passing to the general case let us first try to find the structure of the “left” algebra, i.e.:

$$\left[ L_{-k_1}^{(2s_1)}, L_{-k_2}^{(2s_2)} \right] = \sum_{r=1}^{s_1+s_2-1} g_{2r}^{s_1s_2}(k_1, k_2) L_{-k_1-k_2}^{2(s_1+s_2-r)}. \quad (5.14)$$

The simplest way to prove $(5.13)$ and to compute the structure constants $g_{2r}^{s_1s_2}(k_1, k_2)$ is based on the following “conformal” decomposition of the generators $L_{-k}^{(2s)}$ in terms of the conformal generators $\mathcal{L}_{-k}^{(2s)}$:

$$L_{-k}^{(2s)} = \sum_{l=0}^{2s-1-k} \binom{2s - 1 - k}{l} (\bar{z} \bar{\partial}^l + \alpha \bar{\partial}^l) \mathcal{L}_{-k-l}^{(2s)} (\bar{z} \partial)^{-l} \quad (5.14)$$

The fact that the operators $S_l = (\bar{z} \bar{\partial}^l + \alpha \bar{\partial}^l)$ and $S_l = (\bar{z} \bar{\partial}^l + \alpha \bar{\partial}^l)$ are commuting, i.e. $[S_l, S_l] = 0$ reduces the computation of the structure constants $g_{2r}^{s_1s_2}(k_1, k_2)$ to the conformal ones $C_{2r}^{s_1s_2}(k_1, k_2) (k_1 \leq 2s_1 - 1)$:

$$\left[ \mathcal{L}_{-k_1}^{(2s_1)}, \mathcal{L}_{-k_2}^{(2s_2)} \right] = \sum_{r=1}^{s_1+s_2-1} C_{2r}^{s_1s_2}(-k_1, -k_2) \mathcal{L}_{-k_1-k_2}^{2(s_1+s_2-r)}. \quad (5.15)$$

Actually, it turns out that:

$$g_{2r}^{s_1s_2}(k_1, k_2) = C_{2r}^{s_1s_2}(-k_1, -k_2)$$

$$0 \leq k_i \leq 2s_i - 1$$
The identical statement holds for the algebra of $L_{-k}^{(2s)}$'s as well. The conclusion is that the algebra we are looking for has as subalgebras two incomplete $(0 \leq k \leq 2s - 1)$ $W_\infty$ algebras which do not commute between themselves.

The general structure of the remaining “left - right” commutators:

$$\left[ L_{-k}^{(2s)} , \bar{L}_{-l}^{(2p)} \right] = \sum_{r=0}^{s+p-k-2} \bar{g}_{2r}^{sp}(k,l)(m^2)^k \bar{L}_{k-l}^{2(s+p-k-1-r)}$$  \hspace{1cm} (5.16)

(if $k < l$) is a consequence of the explicit form (5.9) and (5.12) of the generators. In order to calculate $\bar{g}_{2r}^{sp}(k,l)$ we first commute $L_{k-1}$ and $\bar{L}_{l-1}$ to the right and then expand the both sides of (5.16) in powers of $X = L_0 + \frac{1}{2}$. In doing this we have to know the coefficients $B_{m}^{N,a}$ in the power expansion of $(X + a)^{N+1}$:

$$(X + a)^{N+1} = \sum_{m=0}^{N+1} B_{m}^{N,a} X^m.$$  

A simple combinatorial analysis leads to the following form of $B_{m}^{N,a}$:

$$B_{m}^{N,a} = \frac{1}{3.2^{m+1}} (N + 2 - m) m A_{m}^{N,a} (N + 2a).$$

The general solution for $A_{m}^{N,a}$ is a specific linear combination of the Bernuli polynomials of degree $m$.

The LHS of (5.16) contains 8 terms of the form:

$$(X + a)^{N+1}(X + a)^{M+1} = \sum_{k=0}^{N+M+2} Y_{R}^{(N+M+2-k)}(aN|bM)X^k$$

where:

$$Y_{L}^{(m)}(aN|bM) = \sum_{k=0}^{N+M+2} B_{k}^{a,N} B_{m-k}^{b,M}.$$  

Denote by $Y_{L}^{(m)}$ the sum of the contributions of all the 8 terms in the LHS. One can perform the same calculations for the RHS. Equating the left and right hand sides one can derive the following recursive relations for the structure constants:

$$Y_{L}^{(2n-1)} = 2 \sum_{r=0}^{n-1} \bar{g}_{2r}^{sp}(k,l) Y_{R}^{(2n-2-2r)}.$$  

Inspired by the observation that the off-critical $XY$- model has as dynamical symmetries two different Virasoro algebras we now look for Virasoro algebras generated by specific combinations of $L_{-k}^{(2s)}$ and $\bar{L}_{-k}^{(2s)}$. One can easily verify that $\mathcal{L}_n$ given by:

$$\mathcal{L}_n = [z\bar{\partial} - z \partial - \frac{1}{2}] n + \partial^n, \hspace{1cm} n \geq -1$$

$$[A]_k = A(A - 1) \ldots (A - k + 1), \hspace{1cm} [A]_0 = 1$$

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(note that $\partial^{-1}\psi = -1/m^2\partial\psi$) close a (incomplete) Virasoro algebra. One more incomplete Virasoro algebra $\bar{V}$ is generated by:

$$\mathcal{L}_n = \bar{z}\partial - z\partial + \frac{1}{2}i_{n+1}\bar{\partial}^n, \quad n \geq -1.$$ 

The third Virasoro algebra $V_c$ spanned by:

$$\frac{(-m^2)^{1-s}}{2}L^{(2s)}_{-2s+2} = \frac{(-m^2)^{1-s}}{2}(\bar{z}\partial - z\partial - \frac{2s-1}{2})\partial^{2s-2} \equiv \frac{(-m^2)^{1-s}}{2}(L_0 - s + 1)L^{2s-2}_{-1}$$

$$\frac{(-m^2)^{1-s}}{2}\bar{L}^{(2s)}_{-2s+2} = \frac{(-m^2)^{1-s}}{2}(\bar{z}\partial - z\partial + \frac{2s-3}{2})\partial^{2s-2} \equiv \frac{(-m^2)^{1-s}}{2}(L_0 + s - 1)\bar{L}^{2s-2}_{-1}$$

$s = 1, 2, \ldots,$ plays in our opinion the major role for the exact integrability. Using once more the formal identity $\bar{L}_{-1} = -m^2(L_{-1})^{-1}$ we can rewrite the $V_c$ - generators in an unique formula:

$$V_n = \frac{1}{2}(-m^2)^n(L_0 - n)L^{2n}_{-1}, \quad -\infty \leq n \leq \infty.$$ 

As in the case of the massive Dirac fermion [99] one is expecting that $V_c$ has nonzero central charge. One could check this by calculating the commutator $[L^{(6)}_{-4}, \bar{L}^{(6)}_{-4}]$. The result is:

$$[L^{(6)}_{-4}, \bar{L}^{(6)}_{-4}] = -8m^8L_0 + m^8.$$ 

In terms of the Virasoro algebra it means that the central charge is:

$$c = \frac{1}{2},$$

i.e. the massive Majorana fermion has the same central charge as the massless one. This is in agreement with the result $c = 1$ for the Dirac fermions.

An important consequence of the fact that $L^{(2s)}_{-k}$ and $\bar{L}^{(2s)}_{-k}$ are generators of the symmetries of the action (5.10) is the following infinite set of Ward identities for the $n$-point functions of $\psi(z, \bar{z})$ and $\bar{\psi}(z, \bar{z})$:

$$\left\langle 0 \left| L^{(2s)}_{-k}\Pi^M_{i=1}\psi(z_i, \bar{z}_i)\Pi^N_{j=1}\bar{\psi}(z_j, \bar{z}_j) \right| 0 \right\rangle = 0,$$

$$\left\langle 0 \left| \Pi^M_{i=1}\psi(z_i, \bar{z}_i)\Pi^N_{j=1}\bar{\psi}(z_j, \bar{z}_j)\bar{L}^{(2s)}_{-k} \right| 0 \right\rangle = 0.$$ 

The condition for the invariance of the vacuum: $L^{(2s)}_{-k}|0\rangle = 0 = \langle 0|\bar{L}^{(2s)}_{-k}$ together with eqs. (5.9), (5.11) and (5.12) lead to the following system of differential equations for $G_{MN}(z_i, \bar{z}_l) = \langle 0 | \Pi^M_{i=1}\psi(z_i, \bar{z}_i)\Pi^N_{j=1}\bar{\psi}(z_j, \bar{z}_j) | 0 \rangle$:

$$\left\{ \sum_{i=1}^{M} \left[ (\bar{z}_i\partial_i - z_i\partial_i)_{2s-1-k} + (\bar{z}_i\partial_i - z_i\partial_i - 2s + 1)_{2s-1-k} \right] \partial_i + \sum_{j=1}^{N} \left[ (\bar{z}_j\partial_j - z_j\partial_j + 1)_{2s-1-k} + (\bar{z}_j\partial_j - z_j\partial_j - 2s + 2)_{2s-1-k} \right] \partial_j \right\} G_{MN}(z_i, \bar{z}_l) = 0.$$
A similar set of equations can be obtained from the condition of \( \bar{L}_{-k}^{(2s)} \) - symmetry of \( G_{MN} \).

Restricting ourselves to the case of 2 - point functions \( (M + N = 2) \) we are going to demonstrate that the Poincare invariance \( (L_{-1}, L_0, \bar{L}_{-1}) \) and the new \( SL(2,R) \) symmetries \( (L_{-2}^{(4)}, \bar{L}_{-2}^{(4)}, L_0) \) are sufficient to fix uniquely \( G_{20}, G_{02} \) and \( G_{11} \) - functions. The relativistic invariance requires:

\[
G_{20} = m\sqrt{\bar{z}}g_{20}(y), \quad G_{02} = m\sqrt{\bar{z}}g_{02}(y), \quad G_{11} = \text{im}g_{11}(y), \quad y = m\sqrt{-4z\bar{z}}.
\]

The condition of \( L_{-2}^{(4)} \) - invariance of \( G_{20} \) leads to the following third order differential equation:

\[
y^2g'''_{20} + 2yg''_{20} - y(y^2 + 1)g'_{20} - (y^2 + 1)g_{20} = 0.
\]  

(5.19)

It happens that one can solve (5.19) in terms of \( K_1(y) \) - Bessel function. This reflects the fact that (5.19) can be obtained as a consequence of the \( K_1 \) - Bessel equation:

\[
y^2g'''_{20} + yg'_{20} - (y^2 + 1)g_{20} = 0
\]

and a specific third order equation:

\[
y^3g'''_{20} - y(y^2 + 3)g'_{20} + (y^2 + 3)g_{20} = 0.
\]  

(5.20)

The eq.(5.20) follows from the standard recursive relations for \( K_{\pm 1}, K_0 \) and \( K_2 \) - Bessel functions. The \( L_{-2}^{(4)} \) - Ward identity imposes the eq. (5.20) only. Repeating the same analysis for \( G_{02} \) and \( G_{11} \) we find that \( g_{02}(y) = K_1(y) \) and that \( g_{11} \) satisfy the \( K_0 \) - Bessel equation:

\[
yg''_{11} + g'_{11} - yg_{11} = 0,
\]

i.e. \( g_{11} = K_0(y) \).

To make complete our discussion of the off-critical Ising model we have to mention that as in the conformal case the WI’s (5.17), (5.18) are fixing uniquely the 2- and 3-point functions only. The calculation of, say, the 4- point function (using only the symmetries of the model) requires more information about the representations of the algebra (5.13), (5.16) we are using. One could expect that the null-vector conditions for the off-critical Virasoro algebra spanned by \( L_{-2s+2}^{(2s)}, L_0 \) and \( \bar{L}_{-2s+2}^{(2s)} \) will be sufficient to fix uniquely the corresponding 4-point functions \( (M + N = 4) \).

As a generalization, let us consider a \( k = 1 \) \( O(n) \)-WZW models which represent free fermions [100]. Their massive perturbation is described by the action:

\[
S = \int \frac{1}{2} \left( i\bar{\psi}i \phi \psi^i + m\bar{\psi}i \psi^i \right) d^2z.
\]  

(5.21)

Our problem is to construct explicitly all the conserved charges of the models given by (5.21), i.e. - \( O(n) \)-Majorana massive fermions \( \bar{\psi}^i(z, \bar{z}) \) \( (i = 1, \cdots , n) \). One could expect that the case of \( n \) massive fermions in the \( O(n) \)-vector representation is a straightforward
generalization of the results for one massive fermion. There exist however few important
differences. The first is that together with $T_{2s} = \delta_{ij} T_{2s}^{ij}$ and $J_{2s-1}^{ij}$ we have to consider all
components of the symmetric conserved tensor $T_{2s}^{ij}$. The second very important point is that
the algebra of the standard conserved charges:

$$P_s^{ij} = \int T_{2s}^{ij} dz - \int \delta_{ij} T_{2s-2}^{ij} d\zeta , \quad \overline{P}_s^{ij} = \int \overline{T}_{2s}^{ij} d\zeta - \int \delta_{ij} \overline{T}_{2s-2}^{ij} dz$$  \hspace{1cm} (5.22)

$$Q_s^{ij} = \int J_{2s-1}^{ij} dz - \int \overline{J}_{2s-3}^{ij} d\zeta , \quad \overline{Q}_s^{ij} = \int \overline{J}_{2s-1}^{ij} d\zeta - \int \overline{J}_{2s-3}^{ij} dz$$

is nonabelian. Its abelian subalgebra is spanned by $P_s = \delta_{ij} P_s^{ij}$ and $\overline{P}_s = \delta_{ij} \overline{P}_s^{ij}$. In order
to find this algebra, it is better to realize $P_s^{ij}$, $Q_s^{ij}$ etc in terms of differential operators.
Following the standard massive fermion technology we start with the conserved tensors:

$$T_{2s}^{ij} = \psi^i \partial^{2s-1} \psi^j + \psi^j \partial^{2s-1} \psi^i , \quad T_{2}^{ij} = \frac{1}{2} (\psi^i \partial \psi^j + \psi^j \partial \psi^i)$$  \hspace{1cm} (5.23)

and similar expressions for $\overline{T}_{2s}^{ij}$ and $\overline{J}_{2s-1}^{ij}$. Using the equations of motion:

$$\overline{\partial} \psi = m \overline{\psi} , \quad \overline{\partial} \overline{\psi} = - m \psi$$

one can show that (5.23) are indeed conserved tensors. Using as before the formal identity
$\overline{\partial} = - m^2 \partial^{-1}$, one can write $(m^2)^s Q_s^{ij} \equiv \overline{Q}_s^{ij}$ and $(m^2)^{s-1} T_{2s}^{ij} \equiv \overline{T}_s^{ij}$ as a unique object $\overline{Q}_s^{ij} (-\infty \leq s \leq \infty)$. One can show that the latter generates the $O(n)$-Kac-Moody algebra.
The total algebra is a subalgebra $GL(n, R)_{mod 2}$ of the $GL(n, R)$-Kac-Moody algebra spanned
by $\overline{P}_{2s-1}^{ij} \equiv P_s^{ij}$ and $\overline{Q}_{2s} = Q_s^{ij}$, i.e. the closed subalgebra of symmetric generators $P_s^{ij}$ with odd indices and antisymmetric generators $Q_s^{ij}$ with even indices.

Following the previous discussion we are interested mainly in the possible Virasoro subalgebras of full algebra of symmetries. How to construct the Virasoro charges for one massive
fermion we already know from the off-critical Ising model case. In order to generalize it for
the $O(n)$-massive fermions we have to find specific combinations of the “higher momenta” of
the $T_s^{ij}, J_s^{ij}, \theta_s^{ij}$ and $\overline{\theta}_s^{ij}$ to be conserved. It turns out that we can construct $(4s-3)^{\frac{n(n+1)}{2}}$ new symmetric charges $L_{-s}^{ij(2s)}, T_{-s}^{ij(2s)} (0 \leq n \leq 2s-1)$ and $(4s-5)^{\frac{n(n-1)}{2}}$ antisymmetric
ones $Q_{-s}^{ij(2s-1)}, \overline{Q}_{-s}^{ij(2s-1)} (0 \leq k \leq 2s-2)$ for each $s = 2, 3, \ldots$. The simplest one is the
generalization of the Lorentz rotation $L_0 = \frac{1}{2} \delta^{ij} L_0^{ij}$:

$$L_0^{ij} = \int (z T_2^{ij} + \overline{z} \theta^{ij}) dz - \int (\overline{z} T_2^{ij} + z \theta^{ij}) d\zeta$$

The next ones are straightforward $O(n)$-matrix generalizations of the corresponding one
fermion charges $L_{-2s+2}^{(2s)} = \frac{1}{2} \delta^{ij} L_{-2s+2}^{ij(2s)}$ and we can take them in the following differential
form:

\[
\begin{align*}
\left[ L_{-2s+2}^{ij(2s)}, \psi^{k}(z, \bar{z}) \right] & = -i(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})(z \partial - z \partial - \frac{2s - 1}{2}) \partial^{2s-2}\psi^{l} \\
\left[ T_{-2s+2}^{ij(2s)}, \psi^{k}(z, \bar{z}) \right] & = -i(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})(z \partial - z \partial + \frac{2s - 3}{2}) \partial^{2s-2}\psi^{l}
\end{align*}
\]

(5.24)

The proof that they are indeed the conserved charges we are looking for is again based on the fact that they do generate new symmetries of the action (5.21): i.e.

\[
[L_{-2s+2}^{ij(2s)}; S] = 0 = [T_{-2s+2}^{ij(2s)}; S].
\]

The question about the algebra of these new symmetries is now in order. By direct calculations, using (5.24) one can see that \(L_{-2s+2}^{ij(2s)}\) and \(T_{-2s+2}^{ij(2s)}\) does not close an algebra. It is necessary to consider together with them the first momenta \(Q_{-2s+3}^{ij(2s-1)}\) of the current \(J_{2s-1}^{ij}\). Before doing this we should mention that the traces \(L(\bar{L})_{-2s+2}^{ij(2s)} = \frac{1}{2}\delta_{ij}L(\bar{L})_{-2s+2}^{ij(2s)}\) do close an algebra which coincides with the off-critical Virasoro algebra \(V_{c}\) of the off-critical Ising model. One could wonder what is then the algebra of \(\tilde{Q}_{s}^{ij}\) and these Virasoro generators:

\[
V_{k} = \frac{1}{4}(-m^{2})^{k}\delta_{ij}L_{-2k}^{ij(2k+2)}, \quad V_{-k} = \frac{1}{4}(-m^{2})^{k}\delta_{ij}L_{-2k}^{ij(2k+2)}.
\]

As one could expect, the result of simple computations is the larger current algebra \(V_{c} \otimes \hat{O}_{n}\):

\[
\begin{align*}
\left[ V_{m_{1}}, V_{m_{2}} \right] & = (m_{1} - m_{2})V_{m_{1} + m_{2}} + \frac{n}{24}m_{1}(m_{1}^{2} - 1)\delta_{m_{1} + m_{2}} \\
\left[ V_{m_{1}}, \tilde{Q}_{m_{2}}^{ij} \right] & = -m_{2}\tilde{Q}_{m_{1} + m_{2}}^{ij} \\
\left[ \tilde{Q}_{m_{1}}^{ij}, \tilde{Q}_{m_{2}}^{kl} \right] & = \delta^{ik}\tilde{Q}_{m_{1} + m_{2}}^{jl} + \delta^{jl}\tilde{Q}_{m_{1} + m_{2}}^{ik} - \delta^{il}\tilde{Q}_{m_{1} + m_{2}}^{jk} - \delta^{jk}\tilde{Q}_{m_{1} + m_{2}}^{il} + \frac{n}{2}m_{1}\delta_{m_{1} + m_{2}}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}).
\end{align*}
\]

(5.25)

We have enlarged in this way the known symmetries of the action (5.21) to the \(V_{c} \otimes \hat{O}(n)\)-algebra.

Turning back to our problem of constructing the first momenta of the current \(J_{2s-1}^{ij}\) we arrive at the following general differential form for \(Q(\bar{Q})_{-2s+3}^{ij(2s-1)}\):

\[
\begin{align*}
\left[ Q_{-2s+3}^{ij(2s-1)}, \psi^{k}(z, \bar{z}) \right] & = -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(z \partial - z \partial - s + 1)\partial^{2s-3}\psi^{l}(z, \bar{z}) \\
\left[ \bar{T}_{-2s+3}^{ij(2s-1)}, \psi^{k}(z, \bar{z}) \right] & = -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(z \partial + z \partial - 2)\partial^{2s-3}\psi^{l}(z, \bar{z}).
\end{align*}
\]

(5.26)

Considering \(Q(\bar{Q})_{-2s+3}^{ij(2s-1)}\) together with \(L(\bar{L})_{-2s+2}^{ij(2s)}, Q(\bar{Q})_{s}^{ij} \equiv Q(\bar{Q})_{-2s+2}^{ij(2s-1)}, P(\bar{P})_{s}^{ij} \equiv L(\bar{L})_{-2s+1}^{ij(2s)}\) we are expecting them to close an algebra. However this is not the case. One
can easily check using (5.26) that the commutator $[Q_{ij}^{(2s_1-1)}, Q_{kl}^{(2s_2-1)}]$ contains higher momenta of $J_{2s-1}^{ij}$ and $T_{2s}^{ij}$ as well. Therefore the algebra of the first momenta of $T_{2s}^{ij}$ and $J_{2s-1}^{ij}$ is not closed. Involving the higher momenta of $T_{2s}^{ij}$, $J_{2s-1}^{ij}$ we are constructing in this way an algebra of the $W_{\infty}(\hat{G}_n)$-type. We address here the question about its subalgebras. Up to now we have constructed two such subalgebras: $\hat{GL}(n, R)_{\text{mod}2}$ and Vir $\otimes \hat{O}(n)$ of eq. (5.25).

Deriving the missing commutator:

$$[V_{m_1}, \tilde{P}_{m_2}^{ij}] = -(m_2 - 1/2)\tilde{P}_{m_1+m_2}^{ij}$$

we can unify them in an unique current algebra, namely: Vir $\otimes \hat{GL}(n, R)_{\text{mod}2}$. Are there more subalgebras of this type? As in the cases of one and two fermions one could expect to find two incomplete ($n \geq -1$) Virasoro subalgebras. In our case they are generated by a specific combination of $\delta^{ij}L(\bar{\bar{L}})^{(2k)}_{ij}$ and $\delta^{ij}L(\bar{\bar{L}})^{ij}_{-1} \equiv P(\bar{P})_1$:

$$(-)^n \frac{(-)}{n+1} \frac{z\partial - z\partial \pm 1/2}{\partial^n}$$

Do they have an $\hat{O}(n)$-Kac-Moody counterpart? Actually, one can easily guess the general form of the $\hat{O}(n)$ generators:

$$(Q_s^{ij})_{kl} = (Q_0^{ij})_{kl} [z\partial - z\partial - 1]_n \partial^n , \quad n \geq 0$$

which indeed close $\hat{O}(n)$-Kac-Moody algebra. Similarly, the conserved charges:

$$(\bar{Q}_s^{ij})_{kl} = (Q_0^{ij})_{kl} [\bar{z}\partial - \bar{z}\partial]_n \bar{\partial}^n , \quad n \geq 0$$

generate one more $\hat{O}(n)$-current algebra. These two algebras however do not mutually commute.

The algebras of symmetries of (5.21) we have described up to now are sufficient for the calculation of the correlation functions. We shall mention here the following simple and remarkable fact: the $Q^{ij}_{-1}$ (or $\bar{Q}^{ij}_{-1}$) W.I.’s for the 2-point function:

$$g^{lm}(z_1, z_2 | \bar{z}_1, \bar{z}_2) = \langle \psi^l(z_1, \bar{z}_1) \psi^m(z_2, \bar{z}_2) \rangle$$

coincide with the $K_1$-Bessel equation.

Taking into account the Poincaré invariance (i.e., $L_0, L(\bar{\bar{L}})_{-1} = \frac{1}{2} \delta^{ij}L(\bar{\bar{L}})^{ij}_{-1}$) we get:

$$g^{lm}(z_1, z_2 | \bar{z}_1, \bar{z}_2) = \delta^{lm} \sqrt{\frac{z_{12}}{\bar{z}_{12}}} K(x), \quad x = \sqrt{-4m^2 z_{12} \bar{z}_{12}}.$$ 

We next require the $Q^{ij}_{-1}$-Ward identity:

$$\langle Q^{ij}_{-1} \psi^l(z_1, \bar{z}_1) \psi^m(z_2, \bar{z}_2) \rangle = 0.$$ 

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As a consequence of (5.26) and $Q^{ij}_{-1}$-invariance of the vacua we obtain the following equation:

$$\left(\frac{z_{12}}{m} + \frac{2}{m^2}\partial_{12} + \frac{1}{m^2}z_{12}\partial_{12}^2\right)\sqrt{\frac{z_{12}}{z_{12}^2}}K(x) = 0$$

which is equivalent to the $K_1$-Bessel equation:

$$x^2K''(x) + xK'(x) - (x^2 + 1)K(x) = 0.$$

### 5.2 Quantum inverse scattering description of conformal minimal models

In this Section we present an alternative description of CFT which is a variation of the quantum inverse scattering method. The basic objects are the monodromy matrix and its trace generating the integrals of motion and encoding the basic CFT data. It is known [87, 88] that the $A^{(1)}_1$ KdV system describes the classical limit ($c \to -\infty$) of the 2D CFT. There exists one more possible description of this theory. It is based on the generalized KdV system attached to $A^{(2)}_2$ which also yields the classical limit of Virasoro as Poisson bracket structure.

Consider the generalized KdV equations corresponding to the two vertices $c_0$ and $c_1$ of the Dynkin diagram of $A^{(2)}_2$ in the Drinfeld-Sokolov [86] classification:

$$
\begin{align*}
    c_0 : & \quad \partial_t U = \partial_u^3 U + 5\partial_u^2 U + 5\partial_u U \partial_u^2 U + 5U^2 \partial_u U \\
    c_1 : & \quad \partial_t U = \partial_u^5 U + 10U \partial_u^3 U + 25\partial_u U \partial_u^2 U + 20U^2 \partial_u U 
\end{align*}
\tag{5.28}
$$

As the usual KdV, both equations (5.28) are Hamiltonian. Their second Hamiltonian structures are associated with the Hamiltonians:

$$
H^{(0)} = 3(\partial_u U)^2 - 16U^3, \quad H^{(1)} = 3(\partial_u U)^2 - U^3.
$$

Here and in the following, the superscript in parenthesis $^{(0)}$ and $^{(1)}$ refer to the $c_0$ and $c_1$ cases respectively. The crucial observation is that the Poisson bracket algebra of the fields $U(u)$ corresponding to these two second hamiltonian structures coincides with the classical ($c \to -\infty$) limit of the Virasoro algebra:

$$
\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u - v) + \delta''(u - v).
\tag{5.29}
$$

The systems (5.28) describe isospectral deformations of third order differential operators:

$$
\begin{align*}
    L^{(0)} = \partial_u^3 + U\partial_u + \partial_u U - \lambda^3, \quad & L^{(1)} = \partial_u^3 + U\partial_u - \lambda^3. 
\end{align*}
\tag{5.30}
$$

Eqs. (5.28) can be obtained directly by reduction of the Boussinesq equation, which describes the classical limit of CFT having extended $W_3$-algebra symmetry [88]. There are two consistent reductions of Boussinesq equation: $W = \partial_u U$ and $\hat{W} = 0$, leading to the first and
second equation of (5.28) respectively. However this observation is valid only at the classical level, since \((A_2^{(2)})_q\), which is relevant for the quantum case, is an essentially nonlinear deformation of \(A_2^{(2)}\), and not just a twist of \((A_2^{(1)})_q\). Being integrable, the equations (5.28) possess an infinite number of conserved IM \(I^{(i)}_s\), \(i = 0, 1\) having spin \(s = 1, 5 \mod 6\) \cite{29}. One can compute them using the Lax pair representations of (5.28) and show that the Poisson bracket algebra they close is abelian \(\{I^{(i)}_k, I^{(i)}_l\} = 0, i = 0, 1\). These IM should obviously be the classical limit of the corresponding quantum conserved charges of CFT, and indeed they happen to coincide with the classical limit of the quantum IM written in \cite{88} for the Boussinesq system, once the reductions \(W = \partial_u U\) (for \(c_0\)) and \(W = 0\) (for \(c_1\)) are enforced.

Let us consider the first order matrix realization of (5.30):

\[
\mathcal{L} = \partial_u - \phi'(u)h - (e_0 + e_1)
\]

(5.31)

where \(\phi(u)\) is related to \(U(u)\) by the Miura transformation \cite{111} \(U(u) = -\phi'(u)^2 - \phi''(u)\). Written in the canonical gradation of \(A_2^{(2)}\) \cite{86} eq. (5.31) defines the Lax representation for the generalized modified KdV (mKdV) corresponding to the algebra \(A_2^{(2)}\) and \(h, e_0, e_1\) are the Cartan-Chevalley generators of \(A_2^{(2)}\) level 0 algebra:

\[
e_0 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

(5.32)

By choosing instead to represent the \(h, e_0, e_1\) matrices in the two possible standard gradations \((c_0\) or \(c_1)\), one obtains that the first component of eq. (5.31) satisfies the first and second of (5.30) respectively.

The expressions (5.30) are obtained if one takes \(h, e_0, e_1\) in the fundamental representation. One can however give meaning to (5.31) for general representations of \(A_2^{(2)}\). The irreducible representations \(\pi_s\) relevant here are labelled by an integer \(s \geq 0\). From the solution to the equation \(\mathcal{L}\Psi(u) = 0\), the monodromy matrix can be easily written:

\[
M_s(\lambda) = \pi_s \left\{ e^{2\pi ikh} \mathcal{P} \exp \lambda \int_0^{2\pi} du (e^{-2\phi(u)}e_0 + e^{\phi(u)}e_1) \right\}.
\]

(5.33)

Its “improved” form \(L_s(\lambda) = \pi_s(e^{-i\pi kh})M_s(\lambda)\) satisfies the Poisson bracket algebra:

\[
\{L_s(\lambda) \otimes L_{s'}(\mu)\} = [r_{s,s'}(\lambda \mu^{-1}), L_s(\lambda) \otimes L_{s'}(\mu)]
\]

(5.34)

where \(r_{s,s'}\) is the classical r-matrix associated with \(A_2^{(2)}\) \cite{101}. It follows from (5.34) that the trace of the monodromy matrix \(T_s(\lambda) = \text{Tr} M_s(\lambda)\) closes an abelian Poisson bracket algebra

\[
\{T_s(\lambda), T_{s'}(\mu)\} = 0
\]

(5.35)

One can check that this \(T\)-operator in the fundamental representation \((T_1)\) is indeed the generating function of the infinite number of classical IM of the \(A_2^{(2)}\) mKdV.
Let us turn now to the quantum case. The quantization procedure [102] consists essentially in using the quantum deformations \((A_2^{(2)})_{q=\pm}\) instead of \(A_2^{(2)}\), where:

\[
q_{\pm} = e^{i\pi\beta_{\pm}} \quad , \quad \beta_{\pm} = \sqrt{\frac{1-c}{24}} \pm \sqrt{\frac{25-c}{24}} \quad , \quad \beta_+ = \frac{1}{\beta_-}
\]

and a free scalar field:

\[
\phi(u) = Q + Pu + i \sum_{n \neq 0} \frac{a_n}{n} e^{imu} \tag{5.36}
\]

\[
[Q, P] = i\frac{\beta_+^2}{2} \quad , \quad [a_n, a_m] = \frac{\beta_+^2}{2} n \delta_{n,-m}.
\]

The Miura transformation translates, at the quantum level, into the celebrated Feigin-Fuchs construction of the CFT through the screened free boson \((5.36)\):

\[
-\beta_+^2 T(u) =: \phi'(u)^2 : + (1 - \beta_+^2) \phi''(u) + \frac{\beta_+^2}{24}.
\]

Following [102, 80] we define the quantum monodromy matrix and the \(L\)-operator as follows

\[
L_s(\lambda) = \pi_s \left\{ e^{i\pi Ph} \exp \lambda \int du \left( e^{-2\phi} : q^h e_0 + : e^\phi : q^{-h/2} e_1 \right) \right\}, \tag{5.37}
\]

\[
M_s(\lambda) = \pi_s (e^{i\pi Ph}) L_s(\lambda)
\]

where \(\phi(u)\) is a free massless scalar field like \((5.36)\), and \(e_0, e_1, h\) are now Cartan-Chevalley generators of the affine quantum algebra \((A_2^{(2)})_q\) for \(q = e^{i\pi\beta^2}\):

\[
[e_i, f_j] = \delta_{ij} [h_j] \quad , \quad [h_i, e_j] = a_{ij} e_j \quad , \quad [h_i, f_j] = -a_{ij} f_j \quad , \quad i, j = 0, 1
\]

\[
h = h_0 = -2h_1 \quad , \quad a_{00} = a_{11} = 2 \quad , \quad a_{01} = -4 \quad , \quad a_{10} = -1
\]

where \([a] = \frac{a^2 - q^{-2}}{q - q^{-1}}\). We shall comment later on the relation between \(\beta\) and \(c\). Similarly to the classical case we can give meaning to \((5.37)\) in any irreducible representation of \((A_2^{(2)})_q\).

We briefly describe here these representations. Denote the basic vector of the representation \(\pi_s\) as \(|j, m\rangle\), \(j = 0, \frac{1}{2}, 1, \ldots, \frac{s}{2}\), \(m = -j, -j + 1, \ldots, j\). We define the action of the generators of \((A_2^{(2)})_q\) on this basis by:

\[
\begin{align*}
   h|j, m\rangle &= 2m|j, m\rangle \\
e_0|j, m\rangle &= \sqrt{|j - m|}[j + m + 1]|j, m + 1\rangle \\
f_0|j, m\rangle &= \sqrt{|j + m|}[j - m + 1]|j, m - 1\rangle \\
e_1|j, m\rangle &= \sqrt{e(j)[j - m + 1]}|j + 1, m - \frac{1}{2}\rangle + \sqrt{e(j - \frac{1}{2})[j + m]}|j - 1, m + \frac{1}{2}\rangle \\
f_1|j, m\rangle &= \sqrt{e(j)[j + m + 1]}|j + 1, m + \frac{1}{2}\rangle + \sqrt{e(j - \frac{1}{2})[j - m]}|j - 1, m - \frac{1}{2}\rangle
\end{align*}
\]
and:
\[ e(j) = \frac{[j + 1][j + \frac{1}{2}]}{[\frac{1}{2}][2j + 2][2j + 1]} \left\{ \left[ \frac{s}{2} + 1 \right] + \left[ \frac{s}{2} + \frac{1}{2} \right] - [j + 1] - [j + \frac{1}{2}] \right\} \]
is the solution of the recursive equation:
\[ \frac{[2j]}{[j]}e(j - \frac{1}{2}) - \frac{[2j + 2]}{[j + 1]}e(j) = 1 \quad , \quad e\left(\frac{s}{2}\right) = 0. \] (5.39)

One can verify by direct calculation that the definition (5.38) indeed ensures the closing of the \((A_2^{(2)})_q\) algebra provided (5.39) is satisfied.

Let us now return to the operator (5.37). It can be shown that \(L_s(\lambda)\) so constructed satisfies the quantum Yang-Baxter equation:
\[ R_{ss'}(\lambda\mu^{-1})(L_s(\lambda) \otimes 1)(1 \otimes L_{s'}(\mu)) = (1 \otimes L_{s'}(\mu))(L_s(\lambda) \otimes 1)R_{ss'}(\lambda\mu^{-1}) \] (5.40)

where now \(R_{ss'}\) is the quantum \(R\)-matrix associated with \((A_2^{(2)})_q\).

The definition (5.37) is understood in terms of power series expansion in \(\lambda\):
\[ L_s(\lambda) = \pi_s \left\{ e^{i\pi P h} \sum_{k=0}^{\infty} \lambda^k \int_{2\pi u_1 \geq \ldots \geq u_k \geq 0} du_1 \ldots du_k K(u_1) \ldots K(u_k) \right\} \] (5.41)
where:
\[ K(u) = : e^{-2\phi(u)} : q^{\frac{h}{2}} e_0^{\phi(u)} + : e^{\phi(u)} : q^{-\frac{h}{2}} e_1 \]

Similarly to the case considered in [80], an estimate of the singularity properties of the integrands shows that the integrals in (5.41) should be convergent for \(\beta^2 < \frac{1}{2}\) and need regularization for \(\beta^2 \geq \frac{1}{2}\). The analytic properties of the eigenvalues of \(T_s\) are strongly influenced by this regularization.

A direct consequence of (5.40) is that the trace of the quantum monodromy matrix:
\[ T_s(\lambda) \equiv \text{Tr}M_s(\lambda) \] (5.42)
defines a commuting operator \([T_s(\lambda), T_{s'}(\mu)] = 0\) which is the generator of quantum local and non-local IM. In the case of the fundamental representation \(\pi_1\), one easily computes \(T_1(\lambda)\) in terms of power series expansion around \(\lambda = 0\):
\[ T_1(\lambda) = 2 \cos 2\pi P + \sum_{n=1}^{\infty} \lambda^{3n} Q_n \] (5.43)
where:
\[ Q_n = q^{3n/2} \int_{2\pi u_1 \geq \ldots \geq u_{3n} \geq 0} du_1 \ldots du_{3n} \times \]
\[ \times \left\{ e^{2i\pi P} : e^{-2\phi(u_1)} : e^{\phi(u_2)} : e^{\phi(u_3)} : \ldots : e^{-2\phi(u_{3n-2})} : e^{\phi(u_{3n-1})} : e^{\phi(u_{3n})} : \right. \]
\[ + \left. e^{-2i\pi P} : e^{\phi(u_1)} : e^{\phi(u_2)} : e^{-2\phi(u_3)} : \ldots : e^{\phi(u_{3n-2})} : e^{\phi(u_{3n-1})} : e^{-2\phi(u_{3n})} : \right\} \] (5.44)
are the non-local IM. As a consequence of (5.43), \( T_1(\lambda) \) is an entire function of \( \lambda^3 \). One can show that it also exhibits an essential singularity at infinity. The analysis of the corresponding asymptotic expansion should involve a hard Bethe Ansatz calculation. Our expectation, of course, is that the coefficients in this expansion should be given by the quantum version of the local IM [88].

In the general case, eq.(5.42) can be computed using the so-called R-fusion procedure [103]. Here we give only the first non-trivial terms in the \( \lambda \)-expansion:

\[
T_s(\lambda) = \sin \frac{s+2}{2}x \sin \frac{s+1}{2}x + \lambda^3 A_s(x, a)Q_1 + O(\lambda^6) \tag{5.45}
\]

where \( x = 2\pi P, a = \pi \beta^2 \) and:

\[
A_s(x, a) = \sum_{l=0}^{s} \frac{1}{8 \sin x \sin a \sin \frac{a}{4}} \left[ \frac{\sin(x-a)(l+1)}{\sin(x+a)} - \frac{\sin(x+a)(l+1)}{\sin(x-a)} \right] \times \frac{\cos \frac{a}{2} \sin \frac{a}{2}(s+\frac{3}{2}) - \cos \frac{a}{2} \sin \frac{a}{2}(l+1)}{\cos \frac{a}{2}(l+1) \cos \frac{a}{2}}. \tag{5.46}
\]

One can show, using the explicit form (5.45) that \( T_s(\lambda) \) satisfies (at least to order \( \lambda^3 \)) the fundamental relation:

\[
T_s(q^{1/6}\lambda)T_s(q^{-1/6}\lambda) = T_{s+1}(\lambda)T_{s-1}(\lambda) + T_s(q^{1/3}\beta^2\lambda). \tag{5.47}
\]

The very nice result is that this equation coincides with the one conjectured in a completely different fashion in [104]. Conversely, it is interesting to note that, assuming (5.47) as correct and expanding in \( \lambda \), one gets, for each order in the expansion, new curious identities.

The possible choices of \( \beta \) in the \( A_2^{(2)}_q \) case are dictated by adapting the classical limit of the \( A_1 \) Feigin-Fuchs construction to the two possible choices of Miura transformations, labeled by \( c_0 \) and \( c_1 \). Moreover, the classical limit can be realized in two ways, sending \( \beta_+ \to \infty \) or \( \beta_- \to \infty \). This gives 4 possibilities in total. One can see that both operators \( \int_{0}^{2\pi} du : e^{-2\phi} : \) and \( \int_{0}^{2\pi} du : e^{\phi} : \) commute with the IM [88]. Following the same reasoning as in the \( A_1^{(1)} \) case, if one chooses : \( e^{-2\phi} : \) as screening operator, then : \( e^{\phi} : \) is the perturbing field. The two parametrizations with \( \beta_\pm \) correspond to the two possible choices for the screening operator and give : \( e^{\phi} := \phi_{1,2} \) and : \( e^{\phi} := \phi_{2,1} \) respectively. However, one is also free to choose : \( e^{\phi} : \) as screening operator. This leads to the identification of : \( e^{-2\phi} : \) with \( \phi_{1,5} \) for \( \beta_- \) and \( \phi_{5,1} \) for \( \beta_+ \).

In general (5.47) could be considered as a recursive relation for \( T_s(\lambda) \). For \( q \) root of 1 however the quantum group truncation operates and (5.47) becomes a closed system of functional equations. This important fact allows one to do a crucial conjecture: the solutions of (5.47) having the suitable asymptotic behavior and analytic properties (see [80] for details) are the whole set of eigenvalues \( t_s(\lambda) \) of \( T_s(\lambda) \) in the Hilbert space of the model. Therefore the system (5.47), together with its analog presented in [80], provide a complete description of the (chiral) Hilbert space of \( c < 1 \) RCFT’s, i.e. minimal models. These ideas were further developed in [105]-[108].
5.3 Expectation values of descendent fields in the Bullough-Dodd model and related perturbed conformal field theories

The purpose of this Section is to calculate the VEV’s of the simplest non-trivial descendent fields in the Bullough-Dodd (BD) model which is generally described by the following action in the Euclidean space:

\[ A_{BD} = \int d^2x \left[ \frac{1}{16\pi} (\partial_\nu \phi)^2 + \mu e^{b\phi} + \mu' e^{-\frac{b}{2}\phi} \right]. \]  

Here, the parameters \( \mu \) and \( \mu' \) are introduced, as the two operators do not renormalize in the same way, on the contrary to any simply-laced affine Toda field theory. This model has attracted over the years a certain interest, in particular in connection with perturbed minimal models: \( c < 1 \) minimal CFT perturbed by the operators \( \Phi_{12}, \Phi_{21} \) or \( \Phi_{15} \) can be obtained by a quantum group (QG) restriction of imaginary Bullough-Dodd model [81, 82, 36] with special values of the coupling. We will use this property to deduce the VEV’s of the descendents in the following perturbed minimal models:

\[ A = M_{p/p'} + \lambda \int d^2x \Phi_{12}, \]  

\[ \hat{A} = \mathcal{M}_{p/p'} + \hat{\lambda} \int d^2x \Phi_{21}, \]  

or \( \tilde{A} = \mathcal{M}_{p/p'} + \tilde{\lambda} \int d^2x \Phi_{15}, \)

where we denote respectively \( \Phi_{12}, \Phi_{21} \) and \( \Phi_{15} \) as specific primary operators of the unperturbed minimal model \( \mathcal{M}_{p/p'} \) and the parameters \( \lambda, \hat{\lambda} \) and \( \tilde{\lambda} \) characterize the strength of the perturbation.

Similarly to the ShG model [109], the BD model can be regarded as a relevant perturbation of a Gaussian CFT. We remind that in this free field theory, the field is normalized such that:

\[ < \varphi(z, \bar{z})\varphi(0, 0)>_{Gauss} = -2 \log(z\bar{z}). \]

and we have the classical equation of motion:

\[ \partial \bar{\partial} \varphi = 0. \]  

Instead of considering the action (5.48) we turn directly to the case of an imaginary coupling constant which is the most interesting for our purpose. The perturbation is then relevant if \( 0 < \beta^2 < 1 \) (\( b = i\beta \)). Although the model (5.48) for real coupling is very different from the one with imaginary coupling in its physical content, there are good reasons to believe that the expectation values obtained in the real coupling case provide also the expectation values for the imaginary coupling. The calculation of the VEVs in both cases (\( b \) real or imaginary) within the standard perturbation theory agree through the identification \( b = i\beta \) [36]. With
this substitution in (5.43), the general short distance OPE for two arbitrary primary fields 
$e^{iα_1φ}(x)$ and $e^{iα_2φ}(y)$ takes the form :

$$e^{iα_1φ}(x)e^{iα_2φ}(y) = \sum_{n=0}^{∞} \left\{ C_{α_1α_2}^{n,0}(r)e^{i(α+β)φ}(y) + \ldots \right\}$$

$$+ \sum_{n=1}^{∞} \left\{ C'_{α_1α_2}^{n,0}(r)e^{i(α-β)φ}(y) + \ldots \right\}$$

$$+ \sum_{n=1}^{∞} \left\{ D_{α_1α_2}^{n,0}(r)e^{i(α+(n-β)β)φ}(y) + \ldots \right\}$$

(5.53)

where $α = α_1 + α_2$, $r = |x - y|$ and the dots in each term stand for the contributions of the descendants of each field. The different coefficients in eq. (5.53) are computable within the conformal perturbation theory (CPT) [34 3]. We obtain :

$$C_{α_1α_2}^{n,0}(r) = μ^r r^{4α_1α_2+4nβ(α_1+α_2)+2n(1-β^2)+2n^2β^2} f_{α_1α_2}^{n,0}(μ(μ')^2 r^{6-3β^2})$$

(5.54)

$$C'_{α_1α_2}^{n,0}(r) = μ' r^{4α_1α_2-2nβ(α_1+α_2)+2n(1-β^2)+3n^2β^2} f_{α_1α_2}^{n,0}(μ(μ')^2 r^{6-3β^2})$$

$$D_{α_1α_2}^{n,0}(r) = μ' r^{4α_1α_2+4n(1-β)β(α_1+α_2)+2n(1-2β^2)+2n^2β^2} g_{α_1α_2}^{n,0}(μ(μ')^2 r^{6-3β^2})$$

where any function $h \in \{ f, f', g \}$ admits a power series expansion :

$$h_{α_1α_2}^{n,0}(t) = \sum_{k=0}^{∞} h_{k}(α_1, α_2)t^k.$$  (5.55)

Each coefficient in (5.54) is expressed in terms of Coulomb type integrals. The corresponding leading terms are respectively given by :

$$f_{0}^{n,0}(α_1, α_2) = j_n(α_1β, α_2β, β^2) \quad \text{for} \quad n \neq 0;$$

$$f'_{0}^{n,0}(α_1, α_2) = j_n(-\frac{α_1β}{2}, -\frac{α_2β}{2}, \frac{β^2}{4});$$

$$g_{0}^{n,0}(α_1, α_2) = F_{n,1}(α_1β, α_2β, β^2)$$

where we introduced the Dotsenko-Fateev integrals $j_n(a, b, ρ)$ and $F_{n,m}(a, b, ρ)$ [3]. The integrals $j_n(a, b, ρ)$ have been evaluated explicitly in [3] with the result :

$$j_n(a, b, ρ) = \pi^n(γ(ρ))^{-n} \prod_{k=1}^{n} γ(kρ) \times$$

$$\prod_{k=0}^{n-1} γ(1 + 2a + kρ)γ(1 + 2b + kρ)γ(-1 - 2a - 2b - (n - 1 + k)ρ).$$  (5.57)

As we already mentioned, the next sub-leading terms in (5.53) involve the descendant fields. There are four independent second-level descendant fields in BD :

$$(\partialφ)^2(\overline{∂φ})e^{iαφ}; \quad (\partialφ)^2(\overline{∂φ})e^{iαφ};$$

$$(\overline{∂φ})(\overline{∂φ})^2e^{iαφ}; \quad (\overline{∂φ})(\overline{∂φ})e^{iαφ}.\quad (5.58)$$
Similarly to the SG (or ShG) case, using (5.52) it is easy to show that linear combinations of these descendent fields can be written in terms of total derivatives of local fields. As a result, the VEVs of the composite fields (5.58) can all be expressed in terms of a single VEV, say:

\[
< (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i\alpha \phi} >_{BD}.
\]

Let us make an important observation. The second sub-leading terms in the OPE (5.53) appear to be the third order descendants of the primary fields. Analogously to the previous discussion linear combinations of them can be expressed in terms of total derivatives of some local fields. As before, all the corresponding VEVs can be expressed through \(< (\partial \phi)^3 (\bar{\partial} \phi)^3 e^{i\alpha \phi} >\). Unlike the SG case, it is non-vanishing due to the absence of a conserved charge of spin 3 in the BD model. We will consider in more details this VEV later in this Section.

One can now write the short-distance expansion for the two-point function:

\[
G_{\alpha_1 \alpha_2}(r) = < e^{i\alpha_1 \phi(x)} e^{i\alpha_2 \phi(y)} >_{BD} \quad \text{with} \quad r = |x - y|
\]

by taking the expectation value of the r.h.s. of the OPE (5.53) in the BD model with imaginary coupling. Due to the previous discussion, the first non-vanishing contribution of the VEVs of lowest descendent fields in the r.h.s. of the VEV of (5.53) correspond to the following terms:

\[
C_{n,2}^{\alpha_1 \alpha_2}(r) < (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i(\alpha + n\beta) \phi} >_{BD};
\]

\[
C'_{n,2}^{\alpha_1 \alpha_2}(r) < (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i(\alpha - n\beta) \phi} >_{BD};
\]

\[
D_{n,2}^{\alpha_1 \alpha_2}(r) < (\partial \phi)^2 (\bar{\partial} \phi)^2 e^{i(\alpha + (n-\frac{1}{2})\beta) \phi} >_{BD},
\]

respectively. These coefficients also admit expansion similar to eqs. (5.54), (5.55) and (5.56). Finally, the short-distance \((r \to 0)\) expansion of the two-point correlation function in the
BD model with imaginary coupling writes:

\[ \mathcal{G}_{\alpha_1\alpha_2}(r) = \mathcal{G}_{\alpha_1} \sum_{n=1}^{\infty} \mu^r n^{4\alpha_1\alpha_2+4n\frac{\beta}{2}(\alpha_1+\alpha_2)+2n(1-\beta^2)+2n^2\beta^2} \times \mathcal{G}_{\alpha_2+n\frac{\beta}{2}} \left\{ 1 + O\left( \mu^{r6-3\beta^2} \right) \right\} \]

where we defined \( \mathcal{H}(\alpha) \) and \( \mathcal{K}(\alpha) \) by the ratios:

\[ \mathcal{H}(\alpha) = \frac{\langle (\partial \varphi)^2 (\overline{\partial} \varphi)^2 e^{i\alpha \varphi} \rangle_{BD}}{\langle e^{i\alpha \varphi} \rangle_{BD}} \]

\[ \mathcal{K}(\alpha) = \frac{\langle (\partial \varphi)^3 (\overline{\partial} \varphi)^3 e^{i\alpha \varphi} \rangle_{BD}}{\langle e^{i\alpha \varphi} \rangle_{BD}} \]

and \( \mathcal{G}_\alpha = \langle e^{i\alpha \varphi} \rangle_{BD} \) is the VEV of the exponential field in the BD model. A closed analytic expression for this latter VEV has been proposed in [36]:

\[ \langle e^{i\alpha \varphi} \rangle_{BD} = \left[ \frac{\mu^r 2^{-\beta^2} \Gamma(1+\beta^2) \Gamma(1-\frac{\beta^2}{4})}{\mu\Gamma(1-\beta^2) \Gamma(1+\beta^2)} \right] \times \exp \left[ \int_0^{+\infty} \frac{dt}{t} \frac{\sinh((2-\beta^2)t) \Psi(t, \alpha)}{\sinh(2(2-\beta^2)t) \sinh(2t) \sinh(\beta^2t)} - 2\alpha^2 e^{-2t} \right] \]

where:

\[ \Psi(t, \alpha) = -\sinh(2\alpha \beta t) \left( \sinh((4-2\beta^2-2\alpha \beta) t) - \sinh((2-2\beta^2+2\alpha \beta) t) + \sinh((2-\beta^2-2\alpha \beta) t) - \sinh((2-\beta^2+2\alpha \beta) t) - \sinh((2+\beta^2-2\alpha \beta) t) \right) \]

Its integral representation is well defined if:

\[ -\frac{1}{2\beta} < \Re(\alpha) < \frac{1}{\beta} \]

and obtained by analytic continuation outside this domain.
It is then straightforward to obtain the result associated with the action (5.48) i.e. for real values of the coupling constant $b$ which follows from the obvious substitutions:

$$\beta \to -ib; \quad \alpha_1 \to -ia_1; \quad \alpha_2 \to -ia_2;$$
$$\mu \to -\mu; \quad \mu' \to -\mu'. \quad (5.65)$$

In the (Gaussian) free field theory, the composite fields (5.58) are spinless with scale dimension:

$$D \equiv \Delta + \bar{\Delta} = 2\alpha^2 + 4. \quad (5.66)$$

For generic value of the coupling $\beta$ some divergences arise in the VEVs of the fields (5.58) due to the perturbation in (5.48) with imaginary coupling. They are generally cancelled if we add specific counterterms which contain spinless local fields with cut-off dependent coefficients. For $0 < \beta^2 < 1$ the perturbation becomes relevant and a finite number of lower scale dimension counterterms are then sufficient. However, this procedure is regularization scheme dependent, i.e. one can always add finite counterterms. For generic values of $\alpha$ this ambiguity in the definition of the renormalized expression for the fields (5.58) can be eliminated by fixing their scale dimensions to be (5.66). In the BD model with imaginary coupling, this situation arises if two fields, say $O_\alpha$ and $O_{\alpha'}$, satisfy the resonance condition:

$$D_\alpha = D_{\alpha'} + 2n(1 - \beta^2) + 2m(1 - \beta^2/4) \quad \text{with} \quad (n, m) \in \mathbb{N}$$

associated with the ambiguity:

$$O_\alpha \rightarrow O_\alpha + \mu^n \mu'^m O_{\alpha'}. \quad (5.67)$$

In this specific case one says that the renormalized field $O_\alpha$ has an $(n|m)$-th resonance with the field $O_{\alpha'}$. Due to the condition (5.64) and using (5.66) we find immediately that a resonance can appear between the descendent field $(\partial \varphi)^2(\bar{\partial} \varphi)^2 e^{i\alpha \varphi}$ and the following primary fields:

(i) $e^{i(\alpha + \beta)\varphi}$ i.e. $(n|m) = (1|0)$ for $\alpha = \frac{1}{2\beta}$;

(ii) $e^{i(\alpha + 2\beta)\varphi}$ i.e. $(n|m) = (2|0)$ for $\alpha = \frac{-\beta}{2}$;

(iii) $e^{i(\alpha - \beta)\varphi}$ i.e. $(n|m) = (0|2)$ for $\alpha = \frac{\beta}{4}$;

(iv) $e^{i(\alpha + \frac{\beta}{2})\varphi}$ i.e. $(n|m) = (1|1)$ for $\alpha = \beta$.

If we now look at the expression (5.60), we notice that the contribution (5.61), brought by the second level descendent field, and that of any of the exponential fields in (i), (ii), (iii) and (iv), have the same power behaviour in $r$ ($r^{4\alpha_1\alpha_2 + 4}$) at short-distance for the corresponding values of $\alpha$ in (5.68). The integrals which appear in these contributions and their
corresponding poles are, respectively:

\[ j_1(\alpha_1\beta, \alpha_2\beta, \beta^2) \quad \text{with the pole} \quad \alpha = \frac{1}{2\beta} \]  

\[ j_2(\alpha_1\beta, \alpha_2\beta, \beta^2) \quad \text{with the pole} \quad \alpha = -\frac{\beta}{2} \]  

\[ j_2\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}, \frac{\beta^2}{4}\right) \quad \text{with the pole} \quad \alpha = \frac{\beta}{4} \]  

\[ \mathcal{F}_{1,1}(\alpha_1\beta, \alpha_2\beta, \beta^2) \quad \text{with the pole} \quad \alpha = \beta. \]

By analogy with the SG (or ShG) model, one expects that the VEV (5.61) (and similarly for the real coupling case) exhibits, at least, the same poles in order that the divergent contributions compensate each other. This last requirement leads for instance to the relations:

\[ (i') \quad \text{Res}_{\alpha = \frac{1}{2\beta}} \mathcal{H}(\alpha) = 8\pi\beta^3 \mu \frac{G_{a+\beta} G_{a}}{G_{a}}|_{\alpha = \frac{1}{2\beta}}; \]  

\[ (ii') \quad \text{Res}_{\alpha = -\frac{\beta}{4}} \mathcal{H}(\alpha) = -32\pi^2 \beta^3 \mu^2 \gamma(\beta^2) \gamma(-1 - \beta^2) \frac{G_{a+2\beta} G_{a}}{G_{a}}|_{\alpha = -\frac{\beta}{4}}; \]  

\[ (iii') \quad \text{Res}_{\alpha = \frac{\beta}{4}} \mathcal{H}(\alpha) = 4\pi^2 \beta^3 \mu^2 \gamma(\beta^2/2) \gamma(-1 - \beta^2/4) \frac{G_{a-\beta} G_{a}}{G_{a}}|_{\alpha = \frac{\beta}{4}}; \]  

\[ (iv') \quad \text{Res}_{\alpha = \beta} \mathcal{H}(\alpha) = -\frac{4}{(\alpha_1\alpha_2)^2} \mu \mu' \frac{G_{a+\frac{\beta}{2}} G_{a}}{G_{a}}|_{\alpha = \beta} \text{Res}_{\alpha = \beta} \mathcal{F}_{1,1}(\alpha_1\beta, \alpha_2\beta, \beta^2). \]

These last conditions will be used to fix the normalization of the VEV (5.61). Let us now turn to the evaluation of (5.61) which plays an important role in the two-point function (5.60).

The BD model (5.48) can be regarded as two different perturbations of the Liouville field theory [36]. First, one can consider the Liouville action:

\[ \mathcal{A}_L^{(1)} = \int d^2x \left[ \frac{1}{16\pi} (\partial \varphi)^2 + \mu e^{b\varphi} \right]. \]  

The perturbation is then identified with \( e^{-\frac{\varphi}{2}} \). Alternatively, we can take:

\[ \mathcal{A}_L^{(2)} = \int d^2x \left[ \frac{1}{16\pi} (\partial \varphi)^2 + \mu' e^{-\frac{\varphi}{4}} \right] \]  

as the initial action and consider \( e^{b\varphi} \) as a perturbation. Using the first picture, the holomorphic stress-energy tensor:

\[ T(z) = -\frac{1}{4} (\partial \varphi)^2 + \frac{Q}{2} \partial^2 \varphi \]  

ensures the local conformal invariance of the Liouville field theory (5.71) and similarly for the anti-holomorphic part. The exponential fields \( e^{\varphi} \) are spinless primary fields with conformal dimension:

\[ \Delta = a(Q - a). \]
The property of reflection relations which relates operators with the same quantum numbers is a characteristic of the CFT. Using the conformal perturbation theory (CPT) framework, one expects that similar relations are also satisfied in the perturbed case (5.48). With the change \( b \to -b/2 \) in (5.73) and using the second picture (5.72), one assumes that the VEV of the exponential field \( \langle e^{a\varphi} \rangle_{BD} \) satisfies simultaneously the following two functional equations:

\[
\begin{align*}
\langle e^{a\varphi} \rangle_{BD} &= R(a) \langle e^{(Q-a)\varphi} \rangle_{BD} ; \\
\langle e^{-a\varphi} \rangle_{BD} &= R'(a) \langle e^{(-Q'+a)\varphi} \rangle_{BD}
\end{align*}
\]  

(5.74)

with:

\[
Q = \frac{1}{b} + b \quad \text{and} \quad Q' = \frac{2}{b} + \frac{b}{2}.
\]  

(5.75)

The functions \( R(a) \), \( R'(a) \) are the “reflection amplitudes”. An exact expression for \( R(a) \) was presented in [23]. \( R'(a) \) is obtained from \( R(a) \) by the substitutions \( b \to \frac{b}{2} \) and \( \mu \to \mu' \).

Under certain assumptions about the analytic properties of the VEV, the system (5.74) was solved and the VEV for these exponential fields was derived in [36].

Let us denote the descendent fields:

\[
L_{[n]} \overline{L}_{[m]} e^{a\varphi} \equiv L_{-n_1} \ldots L_{-n_K} \overline{L}_{-m_1} \ldots \overline{L}_{-m_K} e^{a\varphi}
\]  

(5.76)

where \([n] = [-n_1, \ldots, -n_K] \) and \([m] = [-m_1, \ldots, -m_K] \) are arbitrary strings and \( L_n, \overline{L}_n \) are the standard Virasoro generators. The descendent fields (5.76) and the ones obtained after the reflection \( a \to Q - a \) possess the same quantum numbers. Consequently, using the arguments of [36], [109] based on the CPT framework, one also expects that their VEVs in the perturbed theory (5.48) satisfy the following “reflection relation”:

\[
\langle L_{[n]} \overline{L}_{[m]} e^{a\varphi} \rangle_{BD} = R(a) \langle L_{[n]} \overline{L}_{[m]} e^{(Q-a)\varphi} \rangle_{BD}.
\]  

(5.77)

However, it is more convenient to use the basis:

\[
(\partial^{n_1} \varphi) \ldots (\partial^{n_K} \varphi)(\overline{\partial}^{m_1} \varphi) \ldots (\overline{\partial}^{m_K} \varphi)e^{a\varphi}.
\]  

(5.78)

Using (5.52) we get:

\[
\langle L_{-2} \overline{L}_{-2} e^{a\varphi} \rangle_{BD} = \frac{1}{16} (1 + 2a(Q + 2a))^2 \langle (\partial \varphi)^2 (\overline{\partial} \varphi)^2 e^{a\varphi} \rangle_{BD}
\]  

(5.79)

which leads to the following reflection relation:

\[
(1 + 2a(Q + 2a))^2 \langle (\partial \varphi)^2 (\overline{\partial} \varphi)^2 e^{a\varphi} \rangle_{BD} = (1 + 2(Q - a)(3Q - 2a))^2 \langle (\partial \varphi)^2 (\overline{\partial} \varphi)^2 e^{(Q-a)\varphi} \rangle_{BD}
\]  

(5.80)

One can also consider the second picture (5.72) where the Liouville theory has coupling \(-\frac{b}{2}\) instead of \( b \) and is perturbed by \( e^{b\varphi} \). If we define the analytic continuation of (5.61):

\[
H(a) = \frac{\langle (\partial \varphi)^2 (\overline{\partial} \varphi)^2 e^{a\varphi} \rangle_{BD}}{\langle e^{a\varphi} \rangle_{BD}},
\]  

(5.80)
then the two different pictures provide us the following two functional relations:

\[
H(a) = \left[ \frac{(2b + 3/b - 2a)(3b + 2/b - 2a)}{(b + 2a)(1/b + 2a)} \right]^2 H(Q - a), \tag{5.81}
\]

\[
H(-a) = \left[ \frac{(b + 6/b - 2a)(3b/2 + 4/b - 2a)}{(b/2 + 2a)(2/b + 2a)} \right]^2 H(-Q' + a).
\]

Notice that these equations are invariant with respect to the symmetry \( b \rightarrow -2b \) with \( a \rightarrow -a \) in agreement with the well-known self-duality of the BD-model.

As was shown above, the solution of these functional equations should exhibit, at least, the poles (5.69) through the identification \( b = i\beta \) and \( a = i\alpha \). Since the solution of (5.81) is defined up to a multiplication constant, we naturally choose to fix it by imposing eqs. (5.70). We find that the “minimal” solution which follows from these constraints is:

\[
H(a) = -\left[ \frac{m \Gamma(b^2/h)^2}{\Gamma(1/3)^2 \Gamma(2/h)^2} \right]^4 \times \frac{\gamma^2(1/3)}{\gamma(2b^2/h) \gamma(1/3)} \times \frac{\gamma(2b^2 + 4/h)}{\gamma(2b^2 - 2/h)} \frac{\gamma(2ba + 3 + b^2)}{\gamma(-2ba - 1/h)} \times \frac{\gamma(2ba + 2b^2)}{\gamma(2ba - 2/h)} \frac{\gamma(-2ba + 2 + 3b^2/2)}{\gamma(-2ba - 2b^2/2/h)} \gamma(2ba - 2b^2/2/h), \tag{5.82}
\]

where \( h = 6 + 3b^2 \) is the “deformed” Coxeter number \([110]\). Here we have used the exact relation between the parameters \( \mu \) and \( \mu' \) in the action \([5.48]\) and the mass of the particle \( m \) \([36]\):

\[
m = \frac{2\sqrt{3}\Gamma(1/3)}{\Gamma(1 + b^2/h) \Gamma(2/h)} \left( -\mu \pi \gamma(1 + b^2) \right)^{1/h} \left( -2\mu' \pi \gamma(1 + b^2/4) \right)^{2/h}. \tag{5.83}
\]

It is easy to see (taking into account also \([5.61]\)) that for \( b = i\beta \) and \( a = i\alpha \), \( H(\alpha) \) possess poles located at:

\[
\alpha_0 \in \{-\beta, \frac{1}{2\beta}, \frac{\beta}{4}, \beta\}. \tag{5.84}
\]

Accepting the conjecture \([5.82]\) and using eq. \([5.79]\) for \( a = 0 \) one can easily deduce for instance:

\[
<T^T >_{BD} = <L_{-2}T_{-2}^\perp >_{BD} = -\pi^2 f_{BD}^2 \tag{5.85}
\]

where:

\[
f_{BD} = \frac{m^2}{16\sqrt{3}\sin(\frac{\pi b^2}{h}) \sin(\frac{2\pi}{h})}
\]

is the bulk free energy of the BD model \([36]\).
Let us now turn to the computation of the expectation values of the descendent fields in $\Phi_{12}$, $\Phi_{21}$ and $\Phi_{15}$ perturbed minimal models.

For imaginary value of the coupling $b = i\beta$, $\mu \rightarrow -\mu$ and $\mu' \rightarrow -\mu'$ the action of the BD model (5.48) becomes complex. Whereas it is not clear if it can be defined as a QFT, this model is known to be integrable and its $S$-matrix was constructed in [81]. It is known that this model possess a quantum group symmetry $U_q(A_2^{(2)})$ with deformation parameter $q = e^{i\pi/2}$ [81]. An important role is played by one of its subalgebras $U_q(sl_2) \subset U_q(A_2^{(2)})$. Following [81], we can restrict the Hilbert space of states of the complex BD model at special values of the coupling constant, more precisely when $q$ is a root of unity, i.e. for:

$$\beta^2 = \frac{p}{p'} \quad \text{or} \quad \beta^2 = \frac{p'}{p} \quad \text{with} \quad 1 < p < p'$$

relative prime integers, in which case the complex BD is identified with the perturbed minimal models (5.49) or (5.50), respectively. In the following, $\Phi_{lk}$ will denote a primary field of the minimal model $M_{p/p'}$.

It is then straightforward to get the VEV in the model associated with the action (5.49):

$$\langle 0_s | L_{-2} \mathcal{T}^{-2} \Phi_{lk} | 0_s \rangle = -\frac{\sqrt{3}\pi(\xi + 2)M\Gamma(1 + \frac{2 + 2\xi}{3\xi + 6})}{\Gamma(\frac{1}{3})2^{2/3+1/2}\Gamma(\frac{\xi}{3\xi + 6})} \gamma(\frac{1}{3})^{2(1/3)} \gamma(-\frac{2\xi}{3\xi + 6})\gamma(\frac{4 + 4\xi}{3\xi + 6})$$

$$\times \mathcal{W}_{12}((\xi + 1)l - \xi k) \quad (5.86)$$

where we denote:

$$\xi = \frac{p}{p' - p}. \quad (5.87)$$

Here:

$$\mathcal{W}_{12}(\eta) = \frac{1}{\xi^2(\xi + 1)^2} \times w(\eta; 5 + 4\xi, 4 + 2\xi, -1 - 2\xi, 1 + \xi/2; 3\xi + 6)$$

and we introduce the useful notation:

$$w(\eta; a_1, a_2, a_3, a_4; g) = \prod_{i=1}^{4} \gamma \left( \frac{a_i + \eta}{g} \right) \gamma \left( \frac{a_i - \eta}{g} \right).$$

We also use the particle-breather identification:

$$m = 2M \sin \left( \frac{\pi \xi}{3\xi + 6} \right). \quad (5.88)$$

Here $|0_s \rangle$ is one of the degenerate ground states of the QFT (5.49). Taking $\Phi_{lk}$ in (5.86) to be the identity operator, it is easy to get:

$$\langle TT \rangle = -\frac{\pi^2 M^4}{48} \frac{\sin^2(\frac{\pi \xi}{3\xi + 6})}{\sin^2(\frac{\pi (2\xi + 2)}{3\xi + 6})}. \quad (5.87)$$
In the second restriction $\beta^2 = p'/p$, which leads to the action (5.50). Along the same line as for the $\Phi_{12}$ perturbation we obtain the following expression for the VEV in the model associated with this action:

$$< 0_s | L_{-2} \Phi_{lk} | 0_s > = \frac{\sqrt{3\pi}(1 - \xi)M \Gamma(1 - \frac{2\xi}{3 - 3\xi})^4 \gamma^2(1/3)}{\Gamma(1/3)2^{2/3 + 1/2} \Gamma(-\frac{4\xi}{3 - 3\xi})} \times W_{21}((\xi + 1)l - \xi k)$$

with:

$$W_{21}(\eta) = \frac{1}{\xi^2(\xi + 1)^2} \times w(\eta; 1 - 4\xi, 2 - 2\xi, 1 + 2\xi, 1/2 - \xi/2; 3 - 3\xi)$$

where $|0_s>$ is one of the degenerate ground states of the QFT (5.50).

Another subalgebra of $U_q(A^{(2)}_2)$ is the subalgebra $U_q(sl_2)$. One can again restrict the phase space of the complex BD with respect to this subalgebra for a special value of the coupling:

$$\beta^2 = \frac{4p}{p'} \quad \text{with} \quad 2p < p'$$

relative prime integers. Then, for this value of the coupling, the BD model is identified with the perturbed minimal model with the action (5.51). Taking the ratio of the VEV of the descendent field of $\Phi_{lk}$ associated with the action (5.51) and the VEV of the primary field itself, one obtains:

$$< 0_s | L_{-2} \Phi_{lk} | 0_s > = \frac{m\xi \Gamma(1 + \frac{1 + \xi}{3 - 3\xi})\Gamma(-\frac{2\xi}{3 - 3\xi})^4 \gamma^2(1/3)}{\Gamma(1/3)\sqrt{3}2^{2/3 + 1/2} \Gamma(-\frac{4\xi}{3 - 3\xi})} \times W_{15}((\xi + 1)l - \xi k)$$

with:

$$W_{15}(\eta) = \frac{1}{\xi^2(\xi + 1)^2} \times w(\eta; \xi + 5, 4 - 4\xi, -1 - 5\xi, 1 - \xi; 6 - 6\xi)$$

We want now to calculate the contribution of the VEV of the third level descendent fields $K(\alpha) = \frac{< (\partial \varphi)^3(x)^3 e^{i\alpha \varphi} >_{BD}}{< e^{i\alpha \varphi} >_{BD}}$ to (5.60).

In the (Gaussian) free field theory, the composite fields $(\partial \varphi)^3(\partial \varphi)^3 e^{i\alpha \varphi}$ are spinless with scale dimension:

$$D \equiv \Delta + \overline{\Delta} = 2\alpha^2 + 6$$

(5.89)

For $0 < \beta^2 < 1$ the perturbation is relevant and, similarly to the second level, a finite number of lower scale dimension counterterms are sufficient to cancel the divergences arising in the
VEVs of third level descendent fields. As before, we are looking for \((n|n')\) resonances with some primary fields (5.67). One can easily find that a resonance can appear between the third level descendent field \((\partial \varphi)^3(\partial \varphi)^3 e^{i\alpha \varphi}\) and the following primary fields:

\[
\begin{align*}
(i) & \quad e^{i(\alpha-\beta)\varphi} \quad \text{i.e.} \quad (n|n') = (1|4) \quad \text{for} \quad \alpha = \frac{1}{\beta} - \frac{\beta}{2}; \\
(ii) & \quad e^{i(\alpha+3\beta)\varphi} \quad \text{i.e.} \quad (n|n') = (3|0) \quad \text{for} \quad \alpha = -\beta; \\
(iii) & \quad e^{i(\alpha-\frac{\beta}{2})\varphi} \quad \text{i.e.} \quad (n|n') = (0|1) \quad \text{for} \quad \alpha = -\frac{2}{\beta}; \\
(iv) & \quad e^{i(\alpha-\frac{3\beta}{2})\varphi} \quad \text{i.e.} \quad (n|n') = (0|3) \quad \text{for} \quad \alpha = \frac{\beta}{2}.
\end{align*}
\]

If we now look at the expression (5.60), we notice that the contribution brought by the third level descendent field in (5.62), and that of any of the exponential fields in (i), (ii), (iii) and (iv), have the same power behavior in \(r \leftrightarrow a^{\alpha_1}\alpha_2 + b\) at short-distance for the corresponding values of \(\alpha\). The integrals which appear in these contributions are, respectively:

\[
\begin{align*}
(i) & \quad F_{1,4}(\alpha_1\beta, \alpha_2\beta, \beta^2), \\
(ii) & \quad j_3(\alpha_1\beta, \alpha_2\beta, \beta^2), \\
(iii) & \quad j_1(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}), \\
(iv) & \quad j_3(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}).
\end{align*}
\]

One can see that \(\mathcal{K}(\alpha)\) (and similarly for the real coupling case) exhibits the same poles in order that the divergent contributions compensate each other. This last requirement leads for instance to a set of relations for \(\mathcal{K}(\alpha)\). The third one reads:

\[
\frac{\alpha_1^2\alpha_2^2(\alpha_1 - \alpha_2)^2}{144} \text{Res}_{\alpha=-\frac{\beta}{2}} \mathcal{K}(\alpha) = \mu \frac{G_{\alpha-\beta/2}}{G_\alpha}|_{\alpha=-\frac{\beta}{2}} \text{Res}_{\alpha=-\frac{\beta}{2}} j_1(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}),
\]

which is used to fix the \(\alpha\)-independent part (normalization) of \(\mathcal{K}(\alpha)\).

On the other hand, to determine the explicit form of the \(\alpha\)-dependent part of \(\mathcal{K}(\alpha)\), we use again the reflection relations method. The calculations go along the same line as for the second level descendent. Consequently, if we denote:

\[
K(a) = \frac{\langle(\partial \varphi)^3(\partial \varphi)^3 e^{i\alpha \varphi}\rangle_{BD}}{\langle e^{i\alpha \varphi}\rangle_{BD}},
\]

then we obtain the following two functional relations:

\[
\begin{align*}
K(a) &= \left[\frac{(b+1/b-a)(b+2/b-a)(2b+1/b-a)}{a(a+1/b)(a+b)}\right]^2 K(Q-a), \\
K(-a) &= \left[\frac{(b/2+2/b-a)(b/2+4/b-a)(b+2/b-a)}{a(a+2/b)(a+b/2)}\right]^2 K(-Q'+a).
\end{align*}
\]

Notice that these equations are invariant with respect to the symmetry \(b \rightarrow -\frac{b}{2}\) with \(a \rightarrow -a\) in agreement with the well-known self-duality of the BD-model. Assuming that \(K(a)\) is a
meromorphic function in $a$, we find that the “minimal” solution which follows from (5.91), (5.93) is:

$$K(a) = \frac{-1}{a^2} \left[ \frac{m \Gamma \left( \frac{b^2}{2} \right) \Gamma \left( \frac{2}{a} \right)}{\Gamma \left( \frac{1}{a} \right) \sqrt{3}} \gamma \left( \frac{2ba + b^2 + 2}{h} \right) \gamma \left( \frac{-2ba - 2}{h} \right) \gamma \left( \frac{2ba - b^2 + 4}{h} \right) \right] \times$$

$$\times \gamma \left( \frac{-2ba - 2b^2}{h} \right) \gamma \left( \frac{-2ba + 2b^2 - 2}{h} \right) \gamma \left( \frac{2ba - 4}{h} \right) \gamma \left( \frac{-2ba + 2b^2 + 2}{h} \right) \gamma \left( \frac{2ba - b^2}{h} \right)$$

where $h = 6 + 3b^2$ is the “deformed” Coxeter number [110]. Here we have used the exact relation between the parameters $\mu$ and $\mu'$ and the mass of the fundamental particle $m$ [36].

Notice that $K(a)$ is invariant under the duality transformation $b \rightarrow -2/b$ as expected, and contains all the expected poles. Accepting this conjecture and taking $a = 0$, we obtain for instance:

$$\langle L_{-3}\bar{L}_{-3}\Phi_{l+k} | 0_s \rangle_{BD} = -\frac{m^2}{2^{10/3} \gamma(1/2 + 2/h) \gamma(1/2 + b^2/h) \gamma(1/3 + 6/h) \gamma(1/3 + 3b^2/h)} f_{BD}$$

where $f_{BD}$ is the bulk free energy of the Bullough-Dodd model, obtained in [36].

In the same way as above we can apply these results for the corresponding perturbed conformal field theories. Let us consider, for example, the first case i.e. the $\Phi_{12}$ perturbation, obtained for $\beta^2 = p/p'$ with $1 < p < p'$ relative prime integers. Using the particle-breather identification $m = 2M \sin \left( \frac{\pi \xi}{3\xi + 6} \right)$ and parameter $a = i \left( \frac{1 - \xi}{2\beta} - \frac{k-1}{2} \right)$ in $K(a)$ it is straightforward to get the VEV:

$$\frac{\langle 0_s | L_{-3}\bar{L}_{-3}\Phi_{l+k} | 0_s \rangle}{\langle 0_s | \Phi_{l+k} | 0_s \rangle} = -\left[ \frac{2^{2/3} \pi MT(\frac{2+2\xi}{3\xi+6})} {\sqrt{3} \Gamma(\frac{1}{3}) \Gamma(\frac{\xi}{3\xi+6}) (1 + \xi)} \right]^6 \frac{1}{\xi^2(1 + \xi)^2(3\xi + 6)^2} \times$$

$$\times \frac{\gamma(\frac{2-4\xi-3}{3\xi+6}) \gamma(\frac{-\eta-4\xi-3}{3\xi+6}) \gamma(\frac{\eta+1+\xi}{3\xi+6}) \gamma(\frac{-\eta+1+\xi}{3\xi+6}) \gamma(\frac{\eta+2\xi+3}{3\xi+6}) \gamma(\frac{-\eta+2\xi+3}{3\xi+6}) \gamma(\frac{-\eta+2\xi+1}{3\xi+6}) \gamma(\frac{-\eta+2\xi+1}{3\xi+6})} {\gamma(\frac{\eta+2\xi+2}{3\xi+6}) \gamma(\frac{-\eta+2\xi+2}{3\xi+6}) \gamma(\frac{-\eta+2\xi+1}{3\xi+6}) \gamma(\frac{-\eta+2\xi+1}{3\xi+6})}$$

The calculations for the other perturbations are straightforward so we will not report them here.

### 5.4 Hidden local, quasi-local and non-local symmetries in integrable systems

As observed in [87], [88], the classical limit ($c \rightarrow -\infty$) of CFT’s is described by the second Hamiltonian structure of the (usual) KdV which corresponds to $A_1^{(1)}$ in the Drinfeld-Sokolov scheme [86]. The KdV variable $u(x,t)$ is related to the mKdV variable $v(x,t)$ by the Miura transformation $u = -v^2 + v'$, which is the classical counterpart of the Feigin-Fuchs transformation [111]. In fact the mKdV equation is:

$$\partial_t v = \frac{3}{2} v^2 v' - \frac{1}{4} v'''$$

(5.94)
and the mKdV field \( v = -\phi' \) is the derivative of the Darboux field \( \phi \). The equation (5.94) can be re-written as a null curvature condition \([\partial_t - A_t, \partial_x - A_x] = 0\) for connections belonging to the \( A_1^{(1)} \) loop algebra:

\[
A_x = -vh + (e_0 + e_1), \quad A_t = \lambda^2 (e_0 + e_1 - vh) - \frac{1}{2} ((v^2 - v') e_0 + (v^2 + v') e_1) - \frac{1}{2} (v'' - v^3) h
\]

(5.95)

where the generators \( e_0, e_1, h \) are chosen in the fundamental representation and canonical gradation of the \( A_1^{(1)} \) loop algebra:

\[
e_0 = \lambda E = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad e_1 = \lambda F = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad h = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(5.96)

Of special interest for us will be the so called transfer matrix which performs the parallel transport along the \( x \)-axis, and is thus the solution of the associated linear problem:

\[
\partial_x T(x; \lambda) = A_x(x; \lambda)T(x; \lambda).
\]

(5.97)

The formal solution of the previous equation is given by:

\[
T(x, \lambda) = e^{H\phi(x)} \mathcal{P} \exp \left( \lambda \int_0^x dy (e^{-2\phi(y)} E + e^{2\phi(y)} F) \right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

(5.98)

where the expansions of the entries are:

\[
A(x; \lambda) = e^{\phi(x)} + O(\lambda^2), \quad B(x; \lambda) = \lambda e^{\phi(x)} \int_0^x dy e^{-2\phi(y)} + O(\lambda^3), \quad C(x; \lambda) = A(-\phi(x)), \quad D(x; \lambda) = B(-\phi(x)).
\]

(5.99)

Note that the first terms of the expansion (5.99) are exactly the classical limits of the two elementary vertex operators. Besides, the expression (5.98) defines \( T(x, \lambda) \) as an entire function of \( \lambda \) with an essential singularity at \( \lambda = \infty \) where it is governed by the corresponding asymptotic expansion. The two expansions give rise to different algebraic and geometric structures, as we shall see below.

Let us first consider the regular expansion. In our case the formal solution (5.98) can be expressed as an expansion in positive powers of \( \lambda \) with an infinite radius of convergence and non-local coefficients (similarly to what we did in Section 5.2 for the \( A_2^2 \) case):

\[
T(x; \lambda) = e^{H\phi(x)} \sum_{k=0}^{\infty} \lambda^k \int_{x\geq x_1 \geq x_2 \geq \ldots \geq x_k \geq 0} K(x_1)K(x_2)\ldots K(x_k) dx_1 dx_2 \ldots dx_k
\]

(5.100)

where \( K(x) = e^{-2\phi(x)} E + e^{2\phi(x)} F \). After calculating the expression (5.100) for \( x = L \) and taking the trace, we obtain the regular expansion for \( \tau(\lambda) = trT(L; \lambda) \) in terms of the non-local conserved charges in involution (we slightly changed the notations here with respect to Section 5.2 where we had \( L = 2\pi \) and the trace was denoted as \( T(\lambda) \)). However, one
may obtain a larger number of non-local conserved charges not in involution, i.e. so that the charges commute with local hamiltonian of the mKdV \((5.94)\) but not between themselves. This can be carried out by means of the dressing techniques in the following way. By assuming the regular expansion \((5.100)\), let us construct the generic resolvent by dressing one of the generators \(X = H, E, F\) \((5.96)\):

\[
Z^X(x, \lambda) = (TXT^{-1})(x, \lambda) = \sum_{k=0}^{\infty} \lambda^k Z_k^X. \tag{5.101}
\]

\(Z^X(x, \lambda)\) is clearly a resolvent for the operator \(L = \partial_x - A_x\) \((5.95)\) since by construction it satisfies:

\[
[\mathcal{L}, Z^X(x; \lambda)] = 0. \tag{5.102}
\]

The foregoing property of the resolvent assures that, once we define the gauge connection of the dressing symmetries:

\[
\Theta^X_n(x; \lambda) = (\lambda^{-n} Z^X(x; \lambda)) = -\sum_{k=0}^{n-1} \lambda^{k-n} Z_k^X, \tag{5.103}
\]

the commutator \([\mathcal{L}, \Theta^X_n(x; \lambda)]\) is of degree zero in \(\lambda\). Therefore it is possible to construct the gauge transformation:

\[
\begin{align*}
\delta^X_n A_x &= -\delta^X_n \mathcal{L} = -[\Theta^X_n(x; \lambda), \mathcal{L}], \\
\delta^X_n A_t &= -[\Theta^X_n(x; \lambda), \partial_t - A_t],
\end{align*} \tag{5.104}
\]

which preserves the zero curvature condition by construction. It will also be a true symmetry of the model in case the last term in \((5.104)\) is proportional to \(H\):

\[
\delta^X_n A_x = H \delta^X_n \phi'. \tag{5.105}
\]

This depends, for \(X\) fixed, on whether \(n\) is even or odd. Indeed, by directly substituting the regular expansion \((5.100)\) in \((5.101)\), one can obtain:

\[
\begin{align*}
Z^H_{2n}(x) &= a^H_{2m}(x) H, & Z^H_{2m+1}(x) &= b^H_{2m+1}(x) E + c^H_{2m+1}(x) F \\
Z^E_{2n}(x) &= b^E_{2m}(x) E + c^E_{2m}(x) F, & Z^E_{2n+1}(x) &= a^E_{2m+1}(x) H \\
Z^F_{2p}(x) &= b^F_{2p}(x) E + c^F_{2p}(x) F, & Z^F_{2p+1}(x) &= a^F_{2p+1}(x) H.
\end{align*} \tag{5.106}
\]

The variation \((5.104)\) can be explicitly calculated as:

\[
\delta^X_n A_x = [Z^X_{n-1}, E + F] \tag{5.107}
\]

and hence it is clear that \(Z^X_{n-1}\) cannot contain any term proportional to \(H\). The conclusions are that:

- in the \(\Theta^H_n\) case \(n\), in \((5.104)\), must be even,
• in the $\Theta^E_n$ and $\Theta^F_n$ case $n$ must conversely be odd.

Besides, it is possible to show by direct calculation that these infinitesimal transformation generators form a representation of a (twisted) Borel subalgebra $A_1 \otimes \mathbb{C}$ (of the loop algebra $A_1^{(1)}$):

$$[\delta^X_n, \delta^Y_n] = \delta^{[X,Y]}_{m+n}, \quad X, Y = H, E, F.$$  \hfill (5.108)

The first generators of this algebra are explicitly given by:

$$\begin{align*}
\delta^E_1 \phi'(x) &= e^{2\phi(x)} \\
\delta^F_1 \phi'(x) &= -e^{-2\phi(x)} \\
\delta^H_2 \phi'(x) &= e^{2\phi(x)} \int_0^x dy e^{-2\phi(y)} + e^{-2\phi(x)} \int_0^x dy e^{2\phi(y)}
\end{align*}$$  \hfill (5.109)

and the rest are derived from these by commutation. Note that they are essentially non-local (this is true also for the higher ones).

At this point we want to make an important observation. Consider the KdV variable $x$ as a space direction $x_-$ of some more general system (and $\partial_\equiv \partial_x$ as a space derivative). Introduce the time variable $x_+$ and the corresponding evolution defining:

$$\partial_+ \equiv (\delta^E_1 + \delta^F_1).$$  \hfill (5.110)

It is then obvious from (5.109) that the equation of motion for $\phi$ becomes:

$$\partial_+ \partial_- \phi = 2 \sinh(2\phi),$$  \hfill (5.111)

i.e. the sine-Gordon equation! We consider this observation very important since it provides a global introduction of sine-Gordon dynamics in the KdV system.

Around the point $\lambda = \infty$ the system is governed by the asymptotic expansion. It can be obtained through the procedure described in [86]. Namely, the asymptotic expansion for a solution of (5.97) can be written as:

$$T(x; \lambda) = KG(x; \lambda) e^{-\int_0^x dy D(y)},$$  \hfill (5.112)

in terms of a constant matrix $K = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, a diagonal matrix:

$$D(x; \lambda) = d(x; \lambda) H, \quad d(x; \lambda) = \sum_{k=-1}^{\infty} \lambda^{-k} d_k(x)$$  \hfill (5.113)

and, finally, of the off-diagonal matrices $G_j(x), j > 0$:

$$G(x; \lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j(x)$$
with entries \( (G_j(x))_{12} = g_j(x) \) and \( (G_j(x))_{21} = (-1)^{j+1} g_j(x) \). It can be shown that the latter satisfy certain recursion relations. Note that the \( d_{2n}(x) \) are exactly the charge densities (of the mKdV equation) resulting from the asymptotic expansion of \( \tau(\lambda) = trT(\lambda) \).

It is likewise known \([86]\) that the construction of the mKdV flows goes through the definition of a resolvent \( Z(x; \lambda) \) defined through the following property of its asymptotic expansion

\[
[\mathcal{L}, Z(x; \lambda)] = 0, \quad Z(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_k, \quad Z_0 = E + F. \tag{5.114}
\]

From the definition \(\text{(5.114)}\) it is possible to derive the resolvent \( Z \), obtained by dressing the generator \( H \) with the asymptotic expansion of \( T (5.112) \):

\[
Z(x, \lambda) = (THT^{-1})(x, \lambda). \tag{5.115}
\]

The modes of the \( \lambda \)-expansion are given by:

\[
Z_{2k}(x) = b_{2k}(x)E + c_{2k}(x)F, \quad Z_{2k+1}(x) = a_{2k+1}(x)H, \tag{5.116}
\]

where for example

\[
a_1 = -v, \quad a_3 = \frac{1}{4} v^3 - \frac{1}{8} v'' \\
b_2 = \frac{1}{4} v^2 + \frac{1}{4} v', \quad b_4 = -\frac{3}{16} v^4 + \frac{1}{8} v'' v - \frac{1}{16} v'^2 - \frac{3}{8} v' v^2 + \frac{1}{16} v''', \text{etc..} \tag{5.117}
\]

As in the regular case, the system enjoys a gauge symmetry of the form \(\text{(5.104)}\) with the constraint \(\text{(5.105)}\):

\[
\delta_{2k+1} A_x = -[\theta_{2k+1}(x; \lambda), \mathcal{L}],
\]

where the \( \theta_{2k+1} \) are the Lax connections associated to \( A_x \)

\[
\theta_{2k+1}(x; \lambda) = (\lambda^{2k+1} Z(x; \lambda))_+ = \sum_{j=0}^{2k+1} \lambda^{2k+1-j} Z_j(x).
\]

It happens that these transformations coincide exactly with the commuting higher mKdV flows (or mKdV hierarchy):

\[
\delta_{2k+1} \phi'(x) = \partial a_{2k+1}(x) \tag{5.118}
\]

and are therefore local in contrast with the regular ones. It turns out that the other entries of the resolvent \( b_{2n}(x) \) are exactly the conserved densities, namely:

\[
\delta_{2k+1} \phi'(x) = \{ I_{2k+1}, \phi'(x) \} \quad , \quad I_{2k-1} = \int_0^L dx b_{2k}(x). \tag{5.119}
\]

They differ from \( d_{2k+1} \) \(\text{(5.113)}\) by a total derivative. For example:

\[
b_2 = -d_1 + \frac{1}{2} \phi'', \quad b_4 = \frac{3}{4} d_3 + \partial(\frac{7}{32} \phi'' \phi' + \frac{1}{16} (\phi')^3 + \frac{1}{16} \phi''') \text{ etc.} \tag{5.120}
\]
Let us note that it can be shown that these two kind of symmetries (regular and asymptotic) commute with each other. In this sense the non-local regular transformations provide a true symmetry of the KdV hierarchy. One can construct also the flows deriving from $Z^E = TET^{-1}$ and $Z^F = TFT^{-1}$ and no more commuting with the $\delta_{2k+1}$ of the hierarchy, but rather closing with them a spectrum generating algebra.

Finally, one can use the above constructions to propose an alternative description of the spectrum of local fields. Namely, we use as basic objects the entries of the resolvent $Z(x; \lambda)$ modulo the gauge transformations described above. A number of constraints, or classical null vectors, appear in this picture coming from the equation of motion $\delta_{2k+1}Z = [\theta_{2k+1}, Z]$ of the resolvent and the obvious constraint $Z^2 = 1$ [94].

One can show that our approach is easily applicable to other integrable systems. We can consider for example the case of the $A_2^{(2)}$-KdV equation. The reason is that it can be considered as a different classical limit of the CFT’s as was discussed in the previous Section. It turns out that all the constructions described above go perfectly well also in this case.

We want to show now that one can construct in a natural way more general kinds of dressing-like symmetries. It is well known that the vector fields $l_m = \lambda^{m+1}\partial_\lambda$ on the circle realize the centerless Virasoro algebra:

$$[l_m, l_n] = (m - n)l_{m+n}.$$ 

A very natural dressing is represented by the resolvent:

$$Z^V_{-m} = T_{\text{reg}}l_{-m}T_{\text{reg}}^{-1}, \quad m > 0$$

where $T_{\text{reg}}$ indicates the regular expansion of the transfer matrix [5.98] and $m$ is a positive integer. Of course, this dressed generator satisfies the usual property (5.102) of being a resolvent:

$$[\mathcal{L}, Z^V_{-m}(x; \lambda)] = 0.$$  (5.122)

As in the previous cases, the property (5.122) allows us to calculate the expansion modes $Z_n$ of:

$$Z^V_{-1} = T_{\text{reg}}l_{-1}T_{\text{reg}}^{-1} = \sum_{n=0}^{\infty} \lambda^n Z_n - \partial_\lambda$$

and thus the expansion modes of the more general Virasoro resolvent (5.121). In the same way, (5.122) authorizes us to define a gauge connection:

$$\theta^V_{-m} = (Z^V_{-m}) - = \sum_{n=0}^{m} \lambda^{n-1-m} Z_n - \partial_\lambda$$

and the relative gauge transformation:

$$\delta^V_{-m}A_x = -[\theta^V_{-m}(x; \lambda), \mathcal{L}].$$  (5.123)
Finally, we have to verify the consistency of this gauge transformation requiring
\( \delta_{-m} A_x = H \delta_{-m} \phi \). It is easy to see that this requirement imposes \( m \) to be even. Explicit examples of the first flows are:

\[
\begin{align*}
\delta_{-2} \phi &= e^{2\phi(x)} \int_0^x dy e^{-2\phi(y)} - e^{-2\phi(x)} \int_0^x dy e^{2\phi(y)} = e^{2\phi(x)} B_1 - e^{-2\phi(x)} C_1 \\
\delta_{-4} \phi &= e^{2\phi(x)} (3B_3(x) - A_2(x)B_1(x)) - e^{-2\phi(x)} (3C_3(x) - D_2(x)C_1(x)) \\
\delta_{-6} \phi &= e^{2\phi(x)} (5B_5(x) - 3A_4(x)B_1(x) + A_2(x)B_5(x)) - e^{-2\phi(x)} (5C_5(x) - 3D_4(x)C_1(x) + D_2(x)C_5(x)) \quad (5.124)
\end{align*}
\]

where \( A_i, B_i, C_i, D_i \) stand for the coefficients in the \( \lambda \)-expansion of the matrix \( \delta_{-2} \). We stress that these infinitesimal variations have a form very similar to that of the regular dressing flows \((5.104)\) with \( X = H \). Nevertheless, in spite of the commutativity \([\delta_{-2r}, \delta_{-2s}] = 0\) one can check by direct calculation that instead the flows \((5.124)\) obey Virasoro commutation relations: \([\delta_{-2}, \delta_{-4}] = \delta_{-6}\). Actually, this is true also in the general case:

\[
[\delta_{-2m}, \delta_{-2n}] = (2n - 2m) \delta_{-2m-2n}. \quad (5.125)
\]

From \((5.124)\) the transformations of the classical primary fields \( e^\phi \) follow. For example:

\[
\begin{align*}
\delta_{-2} e^\phi &= (D_2 - A_2) e^\phi \\
\delta_{-4} e^\phi &= [(3D_4 - C_3 B_1) - (3A_4 - B_3 C_1)] e^\phi.
\end{align*}
\]

It is understood of course that these fields are primary with respect to the usual space-time Virasoro symmetry.

In the same way it is quite natural to generate a resolvent by dressing the remaining vector fields \( l_m = \lambda^{m+1} \partial_\lambda \), \( m \geq 0 \):

\[
Z_m^V = T_{asy} l_m T_{asy}^{-1}, \quad m \geq 0 \quad (5.126)
\]

through the asymptotic expansion of the transfer matrix \((5.112)\) \( T_{asy} \). Now we have:

\[
Z_{-1}^V = T_{asy} l_{-1} T_{asy}^{-1} = \sum_{n=0}^{\infty} \lambda^{-n} Z_n - \partial_\lambda.
\]

In general:

\[
Z_{2n} = \beta_{2n} E + \gamma_{2n} F, \quad Z_{2n+1} = \alpha_{2n+1} H
\]

where for example \( \beta_0 = x = \gamma_0, \alpha_1 = 2x g_1, \beta_2 = -xb_2 - g_1 + f^x d_1, \gamma_2 = -xc_2 + g_1 + f^x d_1 \) etc. In the same manner we define a gauge connection:

\[
\theta_m^V = (Z_m^V)_+ = \sum_{n=0}^{m+1} \lambda^{m+1-n} Z_n - \partial_\lambda
\]

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and the relative gauge transformation $\delta^V_m A_x = -[\theta^V_m(x; \lambda), \mathcal{L}]$. The consistency condition of this gauge transformation, $\delta^V_m A_x = H\delta^V_m \phi'$, impose $m$ to be even in this case too. Actually, the first transformation:

$$\delta^V \phi'(x) = (x \partial + 1)\phi'(x) \quad (5.127)$$

is exactly the scale transformation - it counts the dimension (or level). The first non-trivial examples are:

$$\delta^V_2 \phi' = 2xa'_3 - (\phi')^3 + \frac{3}{4} \phi'' + 2a'_1 \int_0^x d_1,$$

$$\delta^V_4 \phi' = 2xa'_5 + (\phi')^5 - 5 \frac{1}{2} \phi'''(\phi')^2 - 27 \frac{1}{8} (\phi'')^2 \phi' + \frac{5}{16} \phi + 2a'_3 \int_0^x d_1 + 6a'_1 \int_0^x d_3. \quad (5.128)$$

We note that these depend explicitly on $x$ and are quasi-local (they contain some indefinite integrals). For further reference we presented the integrands in (5.128) explicitly in terms of the entries of the basic objects $T(x, \lambda)$ and $Z(x, \lambda)$, defined in (5.113), (5.116). Furthermore, one can find the transformation of the resolvent and therefore the transformation of the conserved densities $\delta_{2k}b_{2n}(x)$. In particular the first nontrivial transformations of the KdV variable $u = b_2$ read:

$$\delta^V_2 b_2 = \delta^V_2 u = 2xb'_4 + u'' - 2u^2 - \frac{1}{2}u' \int_0^x u,$$

$$\delta^V_4 b_2 = \delta^V_4 u = 2xb'_6 + 2u^3 + 3uu'' + \frac{17}{8}(u')^2 + \frac{3}{8}u'^4 + u' \int_0^x b_4 + b'_4 \int_0^x u. \quad (5.129)$$

One can check by direct calculation that the first flows (5.128) obey Virasoro commutation relations. Actually, in general one can show that:

$$\delta_{2n}^V Z_{2m}^V = [\theta_{2n}^V, Z_{2m}^V] - (2n - 2m)Z_{2n+2m}^V$$

$$\delta_{2n}^V \theta_{2m}^V - \delta_{2m}^V \theta_{2n}^V = [\theta_{2n}^V, \theta_{2m}^V] - (2n - 2m)\theta_{2n+2m}^V. \quad (5.130)$$

From these it is not difficult to see that the asymptotic flows also close (half) the Virasoro algebra:

$$[\delta_{2m}^V, \delta_{2n}^V] = (2m - 2n)\delta_{2m+2n}^V, \quad m, n \geq 0. \quad (5.130)$$

An important question arises at this point: what are the commutation relations between the asymptotic and regular transformations? This is a very nontrivial question in view of the different character of the corresponding vector fields - the asymptotic ones are quasilocal (they can be made local by differentiating a certain number of times), the regular instead are essentially non-local being expressed in terms of vertex operators. We recall here that the (proper) regular dressing symmetries (5.104) commute with all the mKdV flows (5.118). We
shall see that this is not the case here. In fact it is easy to compute the most simple relations: 
\[ [\delta_0, \delta_{2n}] = -2n\delta_{2n}, \quad n \in \mathbb{Z} \] (i.e. \( \delta_0 \) counts the dimension or level). Using the explicit formulae presented above one can also compute the first nontrivial commutator: 
\[ [\delta_{2}, \delta_{-2}] = 4\delta_0. \] In fact, one can show that in general \[96\]:
\[ \delta_{V}\theta_{-n} - \delta_{-n}\theta_{V} = [\theta_{V}\theta_{-n}] - (m + n)\theta_{m-n}. \]

From these it is easy to deduce: 
\[ [\delta_{V}, \delta_{-n}] = (m + n)\delta_{V-m-n}, \text{ and therefore:} \]
\[ [\delta_{2m}, \delta_{2n}] = (2m - 2n)\delta_{2m+2n}, \quad m, n \in \mathbb{Z}. \quad (5.131) \]

We want to stress once more that this Virasoro symmetry is different from the space-time one and is essentially non-local. The additional symmetries coming from the regular dressing are very important for applications. They complete the asymptotic ones forming an entire Virasoro algebra and provide a possibility of a central extension.

With the aim of understanding the classical and quantum structure of the mKdV system we present here the complete algebra of symmetries. The Virasoro flows commute neither with the mKdV hierarchy \[5.118\] nor with the (proper) regular dressing flows \[5.104\]. In fact one can show that \[96\]:
\[ [\delta_{2k+1}, \delta_{2m}] = (2k + 1)\delta_{2k+1+2m}, \quad [\delta_{n}, \delta_{2m}] = n\delta_{n+2m}. \quad (5.132) \]

Note that the indices of r.h.s. can become negative, for the first, or positive, for the second of equations \[5.132\]. Explicit calculation show that in these cases the commutator is exactly equal to zero. This fact confirms the self-consistency of the construction.

\section*{5.5 Hidden Virasoro symmetry of (soliton solutions of) the sine-Gordon theory}

We would like now to restrict the Virasoro symmetry to the soliton solutions of the (m)KdV theory. One can expect that in this case it simplifies considerably.

We start with a brief description of the well known soliton solutions of (m)KdV. They are best expressed in terms of the so-called tau-function. In the case of N-soliton solution of (m)KdV it has the form:
\[ \tau(X_1, ..., X_N|B_1, ..., B_N) = \det(1 + V) \quad (5.133) \]
where \( V \) is a matrix:
\[ V_{ij} = 2 \frac{B_iX_i(x)}{B_i + B_j}, \quad i, j = 1, ..., N. \]
The m-KdV field is then expressed as:

\[ e^{\phi} = \frac{\tau_-}{\tau_+}, \quad \text{(5.134)} \]

where:

\[ \tau_\pm(x) = \tau(\pm X(x) | B) \quad \text{(5.135)} \]

and \( X_i(x) \) is simply given by:

\[ X_i(x) = X_i \exp(2B_i x). \]

The variables \( B_i \) and \( X_i \) are the parameters describing the solitons: \( \beta_i = \log B_i \) are the so-called rapidities and \( X_i \) are related to the positions. The integrals of motion, restricted to the N-soliton solutions have the form:

\[ I_{2n+1} = \sum_{i=1}^{N} B_i^{2n+1}, \quad n \geq 0. \quad \text{(5.136)} \]

It is well known that (m)KdV admits a non-degenerate symplectic structure. One can find the corresponding Poisson brackets between the basic variables \( B_i \) and \( X_i \) [85]. The (m)KdV flows are then generated by (5.136) via:

\[ \delta_{2n+1}^* = \left\{ \sum_{i=1}^{N} B_i^{2n+1}, \right\}, \quad n \geq 0. \quad \text{(5.137)} \]

Our final goal is the quantization of solitons and of the Virasoro symmetry. It was argued in [112] that this is best performed in another set of variables \( \{A_i, B_i\} \). The latter are the soliton limit of certain variables describing the more general quasi-periodic finite-zone solutions of (m)KdV and are known as analytical variables.

Explicitly, the change of variables is given by:

\[ X_j \prod_{k \neq j} \frac{B_j - B_k}{B_j + B_k} = \prod_{k=1}^{N} \frac{B_j - A_k}{B_j + A_k}, \quad j = 1, \ldots, N. \]

The non-vanishing Poisson brackets expressed in terms of these new variables take the form:

\[ \{A_i, B_j\} = \frac{\prod_{k \neq i} (B_j^2 - A_k^2) \prod_{k \neq j} (A_i^2 - B_k^2)}{\prod_{k \neq i} (A_i^2 - A_k^2) \prod_{k \neq j} (B_j^2 - B_k^2)} (A_i^2 - B_j^2). \quad \text{(5.138)} \]

The corresponding tau-functions have also a very compact form in terms of the analytical variables:

\[ \tau_+ = 2^N \prod_{j=1}^{N} B_j \left\{ \frac{\prod_{i \neq j} (A_i + A_j) \prod_{i \leq j} (B_i + B_j)}{\prod_{i \neq j} (B_i + A_j)} \right\}, \]

\[ \tau_- = 2^N \prod_{j=1}^{N} A_j \left\{ \frac{\prod_{i \neq j} (A_i + A_j) \prod_{i \leq j} (B_i + B_j)}{\prod_{i \neq j} (B_i + A_j)} \right\}. \quad \text{(5.139)} \]
Therefore, from the explicit form of the m-KdV field in terms of the tau-functions (5.134) we obtain the following very simple expression:

$$e^{\phi} \equiv \frac{\tau_-}{\tau_+} = \prod_{j=1}^{N} \frac{A_j}{B_j}. \quad (5.140)$$

The equation of motion of the $A_i$ variable is given by:

$$\partial_x A_i \equiv \delta_1 A_i = \{I_1, A_i\} = \prod_{j=1}^{N} (A_i^2 - B_j^2) \prod_{j \neq i} \frac{1}{(A_i^2 - A_j^2)}. \quad (5.141)$$

One can verify that, as a consequence, the usual KdV variable $u$ is expressed as:

$$b_2 \equiv u = \frac{1}{2} (\phi')^2 + \frac{1}{2} \phi'' = \sum_{j=1}^{N} A_j^2 - \sum_{j=1}^{N} B_j^2. \quad (5.142)$$

One can restrict also the higher KdV flows to the soliton solutions. For example it is clear from (5.137) that:

$$\delta_{2n+1} B_i = 0, \quad n \geq 0. \quad (5.144)$$

The variation of the $A_i$ variables can be easily computed as:

$$\delta_{2n+1} A_i = \{I_{2n+1}, A_i\}, \quad n \geq 0 \quad (5.143)$$

using the Poisson brackets (5.138).

Now, we want to restrict the Virasoro symmetry of (m)KdV constructed in the previous Section to the case of soliton solutions. We shall be only interested in the positive part of the latter. The transformation of the rapidities can be easily deduced as a soliton limit of the Virasoro action on the finite-zone solutions described in [113]:

$$\delta_{2n} B_i = B_i^{2n+1}, \quad n \geq 0. \quad (5.144)$$

What remains is to obtain the transformations of the $A_i$ variables. We found it quite difficult to deduce them as a soliton limit of the corresponding transformations of [113]. Instead, we propose here another approach. Namely, we use the transformation of the fields $\delta_{2n} \phi$, $\delta_{2n} \phi'$, $\delta_{2n} u$ etc. which we found before, restricted to the soliton solutions using (5.140), (5.142). The problem is simplified by the fact that the Virasoro algebra is freely generated, i.e. we need to compute only the $\delta_0$, $\delta_2$ and $\delta_4$ transformations, the remaining ones are then obtained by commutation. In practice, we perform the computation for the first few cases of $N = 1, 2, 3$ solitons and then proceed by induction.

Let us make an important observation. As we have stressed, the transformation of the basic objects in the field theory of (m)KdV are quasi-local – they contain certain indefinite integrals. It happens that the corresponding integrands become total derivatives when
restricted to the soliton solutions. For example:

\[
b_2 \equiv u = \partial_x \sum_{i=1}^{N} A_i(x),
\]

\[
b_4 = \partial_x \sum_{i=1}^{N} A_i^3 - \frac{1}{2} u' \equiv \partial_x \left[ \sum_{i=1}^{N} \left( A_i^3 - \frac{1}{2} \partial_x A_i \right) \right].
\]

(5.145)

Therefore the Virasoro transformations become \textit{local} when restricted to the soliton solutions!

The calculation is straightforward but quite tedious so we present here only the final result:

\[
\delta_0 A_i = (x \partial_x + 1) A_i,
\]

\[
\delta_2 A_i = \frac{1}{3} x \delta_3 A_i + A_i^3 - \left( \sum_{j=1}^{N} A_j \right) \partial_x A_i,
\]

\[
\delta_4 A_i = \frac{1}{5} x \delta_5 A_i + A_i^5 - \left\{ \sum_{j \neq i} A_j (A_i^2 - A_j^2) + \sum_{j=1}^{N} A_j \sum_{k=1}^{N} B_k^2 \right\} \partial_x A_i,
\]

(5.146)

where the KdV flows read explicitly:

\[
\frac{1}{3} \delta_3 A_i = \left( \sum_{j=1}^{N} B_j^2 - \sum_{k \neq i} A_k^2 \right) \partial_x A_i,
\]

\[
\frac{1}{5} \delta_5 A_i = \left( \sum_{j=1}^{N} B_j^4 - \sum_{k \neq i} A_k^4 \right) \partial_x A_i - \sum_{j \neq i} (A_i^4 - A_j^4) \partial_x A_i \partial_x A_j.
\]

(5.147)

As we already mentioned, the remaining transformations can be obtained by commutation, for example:

\[2 \delta_6 A_i = [\delta_4, \delta_2] A_i, \text{ etc.}\]

Now we pass to the most important part of this Section. We would like to extend the construction presented above in (m)KdV theory to the case of sine-Gordon. For this purpose one has to find a way of extending the mKdV dynamics up to the sine-Gordon one. It is to some extent known how this can be done in the case of the soliton solutions [112]. The idea is close to what we proposed in the previous Section directly in the field theory of (m)KdV.

Namely, let us consider the KdV variable \(x\) as a \textit{space} variable of some more general system and call it \(x_\text{-}\) (and \(\partial\equiv\partial_x\) correspondingly). We would like to introduce a new \textit{time} variable \(x_+\) and the corresponding time dynamics. In the case of the \(N\)-soliton solutions the latter is generated by the Hamiltonian:

\[I_{-1} = \sum_{i=1}^{N} B_i^{-1}\]

(essentially the inverse power of the momentum) so that the \textit{time} flow is given by:

\[\partial_+ * = \delta_{-1} * = \{ I_{-1}, * \}\]

(5.148)
using again the Poisson brackets (5.138). In particular:
\[
\partial_+ A_i = \prod_{j=1}^{N} \frac{A_j^2 - B_j^2}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{(A_i^2 - A_j^2)}.
\]
(5.149)

One can check, using (5.140, 5.141), that with this definition the resulting equation for the field $\phi$ is:
\[
\partial_+ \partial_- \phi = 2 \sinh(2\phi)
\]
or under the change $\phi \to i\phi$:
\[
\partial_+ \partial_- \phi = 2 \sin(2\phi)
\]
i.e. the sine-Gordon equation. In a similar manner one can introduce the rest of the sine-Gordon Hamiltonians:
\[
I_{-2n-1} = \sum_{i=1}^{N} B_i^{-2n-1}, \quad n \geq 0.
\]
(5.152)

They generate the “negative KdV flows” via the Poisson brackets (5.138):
\[
\delta_{-2n-1} B_i = 0, \\
\delta_{-2n-1} A_i = \{I_{-2n-1}, A_i\}, \quad n \geq 0.
\]
(5.153)

Now we arrive at the main conjecture of this Section. Having in mind the symmetric role the derivatives $\partial_-$ and $\partial_+$ are playing in the sine-Gordon equation we would like to suppose that one can obtain another half Virasoro algebra by using the same construction as above but with $\partial_-$ interchanged with $\partial_+$!

So let us define as before:
\[
\delta_{-2n} B_i = -B_i^{-2n+1}, \quad n \geq 0
\]
(5.154)

(note the additional – sign in the r.h.s. which is needed for the self-consistency of the construction). Following our conjecture we construct the negative flows of the $A_i$ variable in the same way as before but with the change $\partial_- \to \partial_+$. We have for example:
\[
\delta_{-2} \phi = x_+(2a_3^+) + b_2^+ - 2a_1^+ \int_0^{x_+} b_2^+, \\
\delta_{-2} b_2^+ \equiv \delta_{-2} u = \frac{1}{3} x_+ \delta_{-3} u + (\partial_+ \phi - \int_0^{x_+} b_2^+) \partial_- e^{2\phi},
\]
(5.155)

where $\delta_{-3} u \equiv \{I_{-3}, u\}$ etc. In (5.155) the + subscript means that we take the same objects as defined in (5.113), (5.116) but with $\partial_-$ changed by $\partial_+$. For example:
\[
b_2^+ = \frac{1}{2}(\partial_+ \phi)^2 + \frac{1}{2} \partial_+^2 \phi, \\
a_3^+ = -\frac{1}{4}(\partial_+ \phi)^3 + \frac{1}{8} \partial_+^3 \phi \quad \text{etc.}
\]
(5.156)
At this point we want to make an important remark. Very non-trivially, it happens again that the integrands in the expressions (5.155) and similar become total derivatives when restricted to the N-soliton solutions. So that again the (negative) Virasoro symmetry is \textit{local} in the case of solitons! We present below the first examples of this phenomenon:

\begin{equation}
\begin{aligned}
b_2^+ &= \partial_+ \left\{ \sum_{i,j=1}^{N} \frac{A_i A_j}{B_i^2 B_j^2} \sum_{i=1}^{N} A_i - \partial_- \sum_{i,j=1}^{N} \frac{A_i A_j}{B_i^2 B_j^2} \right\}, \\
b_4^+ &= \partial_+ \left\{ \sum_{i,j=1}^{N} \frac{A_i A_j}{B_i^4 B_j^4} \sum_{i=1}^{N} A_i^3 + b_2^- \partial_- \sum_{i,j=1}^{N} \frac{A_i A_j}{B_i^4 B_j^4} - \partial_- b_2^- \sum_{i,j=1}^{N} \frac{A_i A_j}{B_i^4 B_j^4} \right\} \quad \text{etc.}
\end{aligned}
\tag{5.157}
\end{equation}

We then proceed as in the case of the positive Virasoro flows, i.e. we restrict the transformations of the fields thus obtained to the soliton solutions. As we explained, it is enough to find only the first transformations \( \delta_{-2} A_i \) and \( \delta_{-4} A_i \) and the remaining ones are found by commutation. Following our approach we do the computation explicitly in the case of \( N = 1, 2, 3 \) solitons and then proceed by induction. Here we give the final results only:

\begin{equation}
\begin{aligned}
\delta_{-2} A_i &= \frac{1}{3} x_+ \delta_{-3} A_i - A_i^{-1} - \left( \sum_{j=1}^{N} A_j^{-1} \right) \partial_+ A_i, \\
\delta_{-4} A_i &= \frac{1}{5} x_+ \delta_{-5} A_i - A_i^{-3} - \\
&\quad - \left( \sum_{j\neq i} \frac{1}{A_i} \left( \frac{1}{A_j} - \frac{1}{A_j} \right) + \sum_{j=1}^{N} \frac{1}{A_j} \sum_{k=1}^{N} \frac{1}{B_k^2} \right) \partial_+ A_i, \\
\end{aligned}
\tag{5.158}
\end{equation}

where as before \( \delta_{-3} A_i = \{ \sum_{j=1}^{N} B_j^{-3} , A_i \} \) etc. As stated above, we then can compute \( 2\delta_{-6} A_i = [\delta_{-2} , \delta_{-4}] A_i \) etc.

Now, we come to the important problem of the commutation relations between the two half Virasoro algebras so constructed. This is a non-trivial question in view of the different way we obtained them. In fact, it is clear that, by construction, the positive (negative) Virasoro flows commute with the corresponding \( \partial_- \) ( \( \partial_+ \) ) derivatives:

\begin{equation}
\begin{aligned}
[\delta_{2n} , \partial_-] A_i &= 0, \\
[\delta_{-2n} , \partial_+] A_i &= 0 , \quad n \geq 0.
\end{aligned}
\tag{5.159}
\end{equation}

It is easy to see that this is not true for the “cross commutators”. Actually, one finds in this case:

\begin{equation}
\begin{aligned}
[\delta_{2n} , \partial_+] A_i &= -\delta_{2n-1} A_i, \\
[\delta_{-2n} , \partial_-] A_i &= -\delta_{-2n+1} A_i , \quad n \geq 0.
\end{aligned}
\tag{5.160}
\end{equation}
It is clear that we are interested in a true symmetry of the sine-Gordon theory. We must therefore obtain transformations that commute with the $\partial_+$ and $\partial_-$ flows and as a consequence with the corresponding Hamiltonians. It is obvious from (5.159), (5.160) that this is achieved by a simple modification of the flows, i.e. let us define:

$$
\delta'_{2n} = \delta_{2n} - x_+ \delta_{2n-1},
$$

$$
\delta'_{-2n} = \delta_{-2n} - x_- \delta_{-2n+1}, \quad n \geq 0.
$$

Then, for the modified transformation we obtain:

$$
[\delta'_{2n}, \partial_{\pm}] A_i = 0, \quad n \in \mathbb{Z}.
$$

Finally, one can show that, with this modification, the commutation relations between the positive and negative parts of the transformations close exactly the whole Virasoro algebra:

$$
[\delta'_{2n}, \delta'_{2m}] A_i = (2n - 2m) \delta'_{2n+2m} A_i, \quad n, m \in \mathbb{Z}.
$$

6 Contributions

1. We present an explicit construction of the Ramond sector of the superconformal minimal models in terms of the Coulomb gas representation. The basic ingredients are written in terms of the Ising model variables (the order-disorder parameter fields and a free Majorana fermion) and a free scalar field. This allows the explicit construction of the fusion rules in all sectors. We compute also the four-point functions and the structure constants of the simplest Ramond fields.

2. We compute the genus $g = 2$ partition function for the $N = 1$ superconformal minimal models on $\mathbb{Z}_2$ hyperelliptic supersurfaces. The latter are mapped onto the minimal models of the $D_4^{p=2}$ parafermion algebra on the branched sphere. The partition functions are written in terms of the multi-point Ising correlation function on the ordinary hyperelliptic surface and the $\mathbb{Z}_2$ orbifold correlation functions.

3. We describe the renormalization group flow of the $N = 1$ superconformal minimal models perturbed by the least relevant field. For that purpose we compute the conformal blocks and the corresponding four-point functions of certain fields in NS and R sectors in the leading order. The anomalous dimensions and the mixing matrix of these fields are obtained. It turns out that the latter is finite and coincides exactly with that found in the non-supersymmetric case.

4. We compute the exact three-point correlation functions of the NS and R fields in the $N = 1$ supersymmetric Liouville field theory. They are expressed in terms of some generalized special function. Using the properties of the NS and R fields we obtain also the reflection amplitudes (or two-point functions) for the supersymmetric case.
5. We compute the exact one-point function of the $N = 1$ super-Liouville field theory with appropriate boundary conditions. Exact results are derived both for the theory defined on a pseudosphere with discrete (NS) boundary conditions and for the theory with explicit boundary action which preserves the superconformal symmetry. We also show that these one-point functions can be related to a generalized Cardy conditions along with corresponding modular S-matrices.

6. We present the Coulomb gas representation of the $N = 2$ superconformal minimal models. The basic ingredient of the construction is the system of two scalar fields and two free fermionic ones. We show that the dynamics of the models is generated by two different kinds of screening operators - one based on chiral (antichiral) superfields and the other on a scalar superfield. The Ramond and twisted primary fields are represented by vertex operators involving the lowest dimensional spin (for fermions) and twisted (for scalars) fields.

7. We use the parafermionic construction of the $N = 2$ superconformal algebra to derive the fusion rules, four-point functions and structure constants in all sectors of the $N = 2$ minimal models. This is used to reveal the origin of the $Z_{p+2}$ symmetry of the $p$-th minimal model. We show that it is generated by specific $N = 2$ superfields which, together with the super-stress tensor, close an $N = 2$ super-parafemionic algebra.

8. We find the general form and the exact Yukawa coupling constants of the low-energy effective superpotential for the three-generation Gepner’s tensor product model.

9. We propose a dual action for the $N = 2$ super-Liouville field theory based on a scalar superfield. We claim that it realizes the strong-weak coupling duality ($b \rightarrow 1/b$) of the theory. We compute the reflection amplitudes (or two-point functions) of the NS and R fields based on the conjectured dual action and show that the results are consistent with the known results.

10. We find the conformal boundary conditions and the corresponding one-point functions of the $N = 2$ super-Liouville theory. This is done using the conformal and the modular bootstrap methods. We find both continuous (FZZT branes) and discrete (ZZ branes) boundary conditions.

11. We present an infinite set of higher equations of motion in $N = 2$ super-Liouville field theory. They are in one to one correspondence with the degenerate representations and are enumerated by the $U(1)$ charge and by a pair of positive integers. We check that in the classical limit these equations hold as relations among the classical fields.

12. We show that the higher level $\hat{su}(2)$ coset models can be represented as projected tensor products of lower level models or, finally, as products of Virasoro models. We construct the monodromy invariant correlation functions for arbitrary level fields and calculate some of the structure constants.
13. We describe a RG flow in a general $\hat{su}(2)$ coset model perturbed by the least relevant field. Using our (projected) tensor product construction we obtain the structure constants and the four-point functions in the leading order. This allows us to compute the mixing coefficients among the fields in the UV and the IR theory up to the second order in the perturbation theory. It turns out that they are finite and exactly the same for all levels and, in that sense, universal.

14. An RG flow in a general $\hat{su}(2)$ coset model perturbed by the least relevant field is considered using the (non-perturbative) RG domain wall construction. The mixing matrix between the UV and the IR fields in this construction is expressed in terms of one-point functions of these fields in the presence of a special boundary condition. We compute these one-point functions and show that the result agrees with the perturbative calculation up to second order.

15. We propose an alternative description of the two-dimensional conformal field theory in terms of quantum inverse scattering. It is based on the generalized KdV system attached to $A_2^{(2)}$, yielding the classical limit of Virasoro as Poisson bracket structure. We classify the primary operators of the minimal models that commute with all the integrals of motion, and are therefore candidates to perturb the model by keeping the conservation laws. For our $A_2^{(2)}$ structure these happen to be the fields $\phi_{1,2}, \phi_{2,1}, \phi_{1,5}$.

16. We calculate the exact vacuum expectation values (VEV’s) of the second and third order level descendent fields in the Boulough-Dodd model. By performing quantum group restrictions we obtain the VEV’s of the corresponding descendants of primary fields in the $\phi_{1,2}, \phi_{2,1}, \phi_{1,5}$ perturbed minimal models.

17. We propose an alternative description of the spectrum of local fields in the classical limit of the integrable quantum field theories. It is essentially a variation of the inverse scattering method and is based on the so called dressing symmetry transformations. Our approach provides a systematic way of deriving the null-vectors that appear in this construction.

18. We generalize the dressing symmetry construction in the mKdV hierarchy. This leads to non-local vector fields (expressed in terms of vertex operators) closing a Virasoro algebra. We argue that this algebra should play an important role in the study of the two-dimensional integrable field theories and in particular should be related to the deformed Virasoro algebra when the construction is perturbed out of the critical theory.

19. We present a construction of a Virasoro symmetry in the sine-Gordon theory. It is a dynamical one and is not related to the space-time Virasoro symmetry of 2D CFT. We are mainly concerned with the corresponding N-soliton solutions. We present explicit expressions for the infinitesimal transformations and show that they are local in this case.
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Appendices

A Structure constants in general $\hat{su}(2)$ coset models

In this Appendix we present the solution of the equations for the structure constants written in Section 4.2. Here is the list of the structure constants we need:

\begin{align}
C_{(33)(mn)}^{(nn)}(l,p) &= \frac{G_n(p + l - 1)}{G_n(p - 1)}, \\
C_{(33)(mn)}^{(n+2n+2)}(l,p) &= \frac{\tilde{G}_n(p + l - 1)}{G_n(p - 1)}, \\
C_{(33)(mn)}^{(n+2)}(l,p) &= \sqrt{\frac{l}{(p - n - 1)(p + l - n - 1)}} \frac{\tilde{G}_n(p + l - 1)}{G_n(p - 1)}, \\
C_{(33)(nn+2)}^{(n+2n+2)}(l,p) &= -2 \sqrt{\frac{l}{(p - n - 1)(p + l - n - 1)}} \frac{G_{n+2}(p + l - 1)}{G_n(p - 1)},
\end{align}

\begin{align}
C_{(13)(mn)}^{(nn)}(l,p) &= -(n - 1) \sqrt{\frac{l}{(p + l - 2)(p - 2)}} \frac{G_n(p + l - 1)}{G_n(p - 1)}, \\
C_{(13)(nn+2)}^{(n+2)}(l,p) &= \sqrt{\frac{(p + l - 2)(p - n - 1)}{(p + l - n - 1)(p - 2)}} \frac{G_n(p + l - 1)}{G_{n+2}(p - 2)}, \\
C_{(13)(nm+2)}^{(n+2)}(l,p) &= \left(- l(n + 1) + \frac{2(p + l - 2)(p - n - 1)}{p + l - n - 1}\right) \frac{G_n(p + l - 1)}{\sqrt{l(p + l - 2)(p - 2)}} \\
C_{(33)(nm+2)}^{(nn+2)}(l,p) &= (1 - \frac{2l}{(p - n - 1)(p + l - n - 1)}) \frac{G_{n+2}(p + l - 1)}{G_{n+2}(p - 2)}
\end{align}

where we introduced the functions:

\begin{align}
G_n(p) &= \left[\frac{\gamma^3(p + 1)^2}{p + 1}\right]^\frac{1}{2}, \\
\tilde{G}_n(p) &= \left[\frac{\gamma(p + 1)^2}{p + 1}\right]^\frac{1}{2}.
\end{align}
We want to stress that the "structure constants" thus obtained are actually square roots of the true structure constants \( C \). The reason is that our construction makes use of "chiral" one-dimensional fields instead of the real two-dimensional ones (see Section 4.1). Therefore the true structure constants are squares of those in (A.1) and (A.2).

The coefficients in the construction (4.41) are given by:

\[
x = \sqrt{\frac{(l-1)(p-n-1)}{l(p-n)}} \quad y = \sqrt{\frac{p+l-n-1}{l(p-n)}}.
\]

In exactly the same way one obtains the structure constants (and the coefficients \( \tilde{x}, \tilde{y} \)) involving the field \( \tilde{\phi}_{n,n-2}(l,p) \). It turns out that they are obtained from the corresponding ones for \( \tilde{\phi}_{n,n+2}(l,p) \) by simply changing \( n \to -n \). This anticipates our observation in the main text that the two-point functions involving the field \( \tilde{\phi}_{n,n-2}(l,p) \) are obtained from those of \( \tilde{\phi}_{n,n+2}(l,p) \) by the same substitution.

Finally \( C_{(n-2)(n+2)}^{(n-2)}(l,p) = 0 \) as can be seen by examining recursively the OPEs and fusion rules of the fields.

**B  Correlation function** \( <\tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty)> \)

In this Appendix we present the calculation of the correlation function of the field \( \tilde{\phi}(x) \) defined in Section 4.2:

\[
<\tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}(1)\tilde{\phi}(\infty)> = \prod_{i=1}^{4} \left( a(l,p)\phi_{1,1}(1,p)\tilde{\phi}_{1,3}(l-1,p+1) - b(l,p)\phi_{1,3}(1,p)\phi_{3,3}(l-1,p+1) \right) (x_i) >.
\]

It defines the \( \beta \)-function and the fixed point up to a second order of the perturbation theory. As explained in Section 4 the conformal blocks corresponding to (B.1) are linear combinations of products of conformal blocks at levels 1 and \( l-1 \) (4.37). There are in general 16 terms in (B.1). Some of them are absent because of the fusion rules in each intermediate channel. Here there are three channels: identity \( \phi_{1,1} \), the field \( \tilde{\phi}_{1,3} \) itself and \( \tilde{\phi}_{1,5} \) which was defined in (4.34). We present the calculation of the corresponding conformal blocks separately. Our strategy here is to compute the conformal blocks up to a sufficiently high order and to make a guess. For \( l = 1 \) this was done in [7]. For \( l-1 \) we proceed recursively and use the fact that we know the result for \( l = 2,3,5 \). The calculations are simplified significantly by the fact that we need the result in the leading order in \( \epsilon \to 0 \).

- **Channel** \( \phi_{1,1} \)
The possible internal channels in the product (4.37) are $r, t = 1, n$ and $t, s = n, 1$ with $n = 1, 3, 5, \ldots$ (odd integer) corresponding to descendants at higher level as in (4.36). We examine the various terms that enter the sum (B.1) and call for simplicity the corresponding conformal block at level $l$ $F_l$ omitting the indexes. We do not present here the detailed calculations which are straightforward but quite tedious [75]. As a result, we get a recursive equation for the conformal block at level $l$:

$$F_l = a^4 F_{l-1} + b^4 F_1 + 2a^2 b^2 + 2a^2 b^2 x^2 C_{(13)(33)}^{(31)} (l - 1) \left(1 + \frac{1}{(1-x)^2}\right)$$  \hspace{1cm} (B.2)

(note that we dropped the overall factor $x^{-2}$ for the time being). The values of the coefficients in (B.2) in the leading order are:

$$a \sim \sqrt{\frac{l-1}{l}}, \quad b \sim \sqrt{\frac{1}{l}}, \quad C_{(13)(33)}^{(31)} \sim \frac{1}{3}.$$

Also:

$$F_1 = \frac{1}{(1-x)^2} \left(1 - 2x + 3x^2 - 2x^3 + 1/3x^4\right)$$

as computed in [7]. Introducing the useful notation:

$$\tilde{F}_l = (1-x)^2 F_l$$

the recursion equation (B.2) becomes:

$$l^2 \tilde{F}_l = (l - 1)^2 \tilde{F}_{l-1} + \tilde{F}_1 + 2(l - 1) f(x)$$  \hspace{1cm} (B.3)

where we defined:

$$f(x) = (1-x)^2 + \frac{x^2}{3} (1 + (1-x)^2).$$  \hspace{1cm} (B.4)

The solution of this equation is given by:

$$\tilde{F}_l = \frac{1}{l} \tilde{F}_1 + \frac{l-1}{l} f(x).$$

Inserting $f(x)$ and returning to the initial notations (and restoring the overall $x^{-2}$) we get the final result for the conformal block:

$$< \tilde{\phi}_{1,3}(x) \tilde{\phi}_{1,3}(0) \tilde{\phi}_{1,3}(1) \tilde{\phi}_{1,3}(\infty) > = $$  \hspace{1cm} (B.5)

$$= \frac{1}{x^2(1-x)^2} \left[ 1 - 2x + \left(\frac{5}{3} + \frac{4}{3l}\right)x^2 - \left(\frac{2}{3} + \frac{4}{3l}\right)x^3 + \frac{1}{3}x^4 \right].$$

This result is in perfect agreement with $l = 1$ [7] and $l = 2$ (Section 2.2).

- **Channel $\tilde{\phi}_{1,5}$**
The field $\tilde{\phi}_{1,5}$ was defined in (4.34) and has a dimension close to 2. Therefore the possible internal channels in the product (4.37) are $r, t = 1, n$ and $t, s = n, 5$ with $n = 1, 3, 5, \ldots$. Following the same logic as before we arrive at a solution similar to the one we obtained above:

$$\tilde{F}_i = \frac{1}{l} x^2 \tilde{F}_1 + \frac{l - 1}{l} f(x)$$

where now $\tilde{F}_i = (1 - x)^2 c^{(15)}_{(13)(13)}(l) F_i$ and:

$$f(x) = \frac{\sqrt{5}}{3} (1 + (1 - x)^2) = \frac{\sqrt{5}}{3} (2 - 2x + x^2).$$

Now we use the fact that we know the conformal block $\tilde{F}_1 = \frac{\sqrt{5}}{3}$ from [7]. Restoring the initial notations we obtain for the conformal block with internal channel $\tilde{\phi}_{1,5}$ (note that here the overall power of $x$ is simply $x^0 = 1$):

$$\langle \tilde{\phi}_{1,5}(x) \tilde{\phi}_{1,3}(0) |_{15} \tilde{\phi}_{1,3}(1) \tilde{\phi}_{1,3}(\infty) \rangle = \frac{1}{(1 - x)^2} \left[ 1 - x + \frac{l}{2(l - 1)} x^2 \right]. \quad (B.6)$$

**Channel $\tilde{\phi}_{1,3}$**

One can proceed in the same way as for the previous channels. It turns out however that in this case some of the conformal blocks that enter the sum (B.1) are divergent as $p \to \infty$. These divergences are exactly compensated by the zeros of the corresponding structure constants in (4.37). Since the analysis similar to the above channels is more complicated here we adopt another strategy. Namely, we use the crossing symmetry of the correlation function (B.1). We ask that it is invariant under the transformation $x \to 1/x$ and use the explicit form of the remaining conformal blocks that we obtained above. This leads to linear equations for the coefficients in the $x$-expansion of the desired conformal block. The result is:

$$\langle \tilde{\phi}_{1,3}(x) \tilde{\phi}_{1,3}(0) |_{13} \tilde{\phi}_{1,3}(1) \tilde{\phi}_{1,3}(\infty) \rangle = \frac{1}{x(1-x)^2} \left[ 1 - \frac{3}{2} x + \frac{l + 1}{2} x^2 - \frac{l}{4} x^3 \right]. \quad (B.7)$$

Combining altogether we finally obtain the 2D correlation function:

$$\langle \tilde{\phi}(x) \tilde{\phi}(0) \tilde{\phi}(1) \tilde{\phi}(\infty) \rangle = \left| \frac{1}{x^2(1-x)^2} \left[ 1 - 2x + \left( \frac{5}{3} + \frac{4}{3l} \right) x^2 - \left( \frac{2}{3} + \frac{4}{3l} \right) x^3 + \frac{1}{3} x^4 \right] \right|^2 +$$

$$+ \frac{16}{3l^2} \left| \frac{1}{x(1-x)^2} \left[ 1 - \frac{3}{2} x + \frac{l + 1}{2} x^2 - \frac{l}{4} x^3 \right] \right|^2 +$$

$$+ \frac{5}{9} \left( \frac{2(l - 1)}{l} \right)^2 \left| \frac{1}{(1-x)^2} \left[ 1 - x + \frac{l}{2(l - 1)} x^2 \right] \right|^2. \quad (B.8)$$

We used this function in Section 4.2 for the computation of the $\beta$-function and the fixed point.
In this Appendix we present the calculation of the other correlation functions we used in Section 4.2 to describe the mixing of the fields.

First, we notice that the computation of the function \( \langle \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(\infty) \rangle \) goes in the same way as that of the function of the perturbing field itself, the latter being just a special case \( n = 1 \). There are again the same three internal channels. It turns out that the corresponding conformal blocks are exactly the same, in agreement with \( l = 1 \) and \( l = 2 \) cases. The difference is only in the structure constants. Omitting the details we present the final result:

\[
\begin{align*}
\langle \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n+2}(\infty) \rangle &= \left( C.1 \right) \\
&= \frac{1}{x^2(1-x)^2} \left[ 1 - 2x + \left( \frac{5}{3} + \frac{4}{3l} \right)x^2 - \left( \frac{2}{3} + \frac{4}{3l} \right)x^3 + \frac{1}{3}x^4 \right]^2 + \\
&+ \frac{8}{3l^2} \frac{n+3}{n+1} \frac{1}{x^2(1-x)^2} \left[ 1 - \frac{3}{2}x + \frac{l+1}{2}x^2 - \frac{l}{4}x^3 \right]^2 + \\
&+ \left( \frac{2(l-1)}{l} \right)^2 \frac{(n+3)(n+4)}{18n(n+1)} \frac{1}{(1-x)^2} \left[ 1 - x + \frac{l}{2(l-1)}x^2 \right]^2.
\end{align*}
\]

- **Function** \( \langle \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(\infty) \rangle \)

The only internal channel in this function corresponds to the field \( \tilde{\phi}_{1,5} \). Denoting again \( \tilde{F}_t = C_tF_t \) we obtain explicitly the recursion equation:

\[
\tilde{F}_t = \frac{(l-1)^2}{l^2} \tilde{F}_{t-1} + \frac{1}{l^2} x^2 \tilde{F}_1 + 2 \frac{(l-1)}{l^2} C f(x)
\]

where:

\[
f(x) = 1 + \frac{1}{(1-x)^2}, \\
C = \sqrt{\frac{1}{3n} \sqrt{n^2 - 4}}
\]

(\( C \) turns to coincide exactly with \( C_1 \)). The solution of the equation then in terms of \( \tilde{F}_t = (1-x)^2 \tilde{F}_t \) is:

\[
\tilde{F} = C \left( \frac{2(l-1)}{l} - \frac{2(l-1)}{l} x + x^2 \right).
\]

Returning back to the original notations we can write finally the result for the correlation function:

\[
\begin{align*}
\langle \tilde{\phi}(x)\tilde{\phi}(0)\tilde{\phi}_{n,n+2}(1)\tilde{\phi}_{n,n-2}(\infty) \rangle &= \left( C.2 \right) \\
&= \frac{1}{3n} \sqrt{n^2 - 4} \left( \frac{2(l-1)}{l} \right)^2 \left( \frac{1}{(1-x)^2} \right)^2 \left( 1 - x + \frac{l}{2(l-1)}x^2 \right)^2.
\end{align*}
\]
Function $\langle \tilde{\phi}(x)\tilde{\phi}(0)\phi_{n,n}(1)\phi_{n,n+2}(\infty) \rangle$

There is only one relevant intermediate internal channel in the leading order in this function corresponding to $\tilde{\phi}_{1,3}$. Inserting the values of the structure constants from Appendix A in the leading order we get for this correlation function:

$$\langle \tilde{\phi}(x)\tilde{\phi}(0)\phi_{n,n}(1)\phi_{n,n+2}(\infty) \rangle = \frac{4}{3l} \sqrt{\frac{n+2}{n}} |x|^{-2}.$$  \hspace{1cm} (C.3)

Function $\langle \tilde{\phi}(x)\tilde{\phi}(0)\phi_{n,n}(1)\phi_{n,n}(\infty) \rangle$

It was mentioned in Section 4.2 that this correlation function is exactly equal to those of $l = 1$ and $l = 2$. Here we want to explain in more details what is the reason for that. Note that, as mentioned in [7], we have to keep terms up to order $\epsilon^2$ in this correlation function. Since the correlation function is quadratic in the conformal blocks we keep in the latter only terms up to order $\epsilon$.

Since $\phi_{n,n}$ is just a primary field only four terms appear in this correlation function. There are two relevant contributions in the intermediate channels corresponding to $\phi_{1,1}$ and $\tilde{\phi}_{1,3}$.

Let us consider first the contribution from $\phi_{1,1}$. There are two terms proportional to square roots of products of the constants $C_{(nn)(nn)}^{(13)}C_{(nn)(nn)}^{(31)}$ which are of order $\epsilon^4$. As explained above we drop them. Inserting the values of the structure constants in the leading order in the remaining two terms gives:

$$F_l = a^2 F_{l-1} + b^2 F_1 = \frac{l-1}{l} F_{l-1} + \frac{1}{l} F_1.$$  

This equation is easily solved recursively:

$$F_l = F_1$$

(where $F_1$ is that of [7]).

Similarly, in the channel corresponding to $\tilde{\phi}_{1,3}$ there remain two terms, the other being of order $\epsilon^2$ so we drop them. Then the equation reads:

$$\tilde{F}_l = a^2 F_{l-1} \sqrt{C_{(13)(13)}^{(13)}(l-1)C_{(nn)(nn)}^{(13)}(l-1)} + b^2 F_1 \sqrt{C_{(33)(33)}^{(13)}(l-1)C_{(nn)(nn)}^{(33)}(l-1)} = \sqrt{\frac{2(n^2-1)}{3p^2}} (a^2 F_{l-1} + b^2 F_1)$$

and we inserted the values of the structure constants. We see that the overall constant do not depend on $l$ so that this equation is very similar to the previous one and again the solution
is:
\[ \hat{F}_l = \hat{F}_1 = \sqrt{\frac{2(n^2 - 1)}{3p^2}} F_1. \]

As a result the correlation function is the same for all \( l \) and reads (up to order \( \epsilon^2 \)):
\[
< \tilde{\phi}(x) \tilde{\phi}(0) \tilde{\phi}_{n,n}(1) \tilde{\phi}_{n,n}(\infty) > = |F_1(1,1)|^2 + \frac{2(n^2 - 1)}{3p^2} |F_1(1,3)|^2 = (C.4)
\]
\[
= |x|^{-4} + \frac{(n^2 - 1)\epsilon^2}{12} |x|^{-4} \left( \frac{x^2}{2(1 - x)} + \frac{\bar{x}^2}{2(1 - \bar{x})} + (\log(1 - x) + \log(1 - \bar{x}))^2 \right).
\]

References

[1] A. Belavin, A. Polyakov, A. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B241 (1984) 333.

[2] A. Zamolodchikov, Al. Zamolodchikov, Structure constants and conformal bootstrap in Liouville field theory, Nucl. Phys. B477 (1996) 577; arXiv:hep-th/9506136; J. Teschner, On the Liouville three-point function, Phys. Lett. B363 (1995) 65; arXiv:hep-th/9507109.

[3] V. Dotsenko, V. Fateev, Conformal algebra and multipoint correlation functions in 2D statistical models, Nucl. Phys. B240 (1984) 312; V. Dotsenko, V. Fateev, Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge \( c \leq 1 \), Nucl. Phys. B251 (1984) 691.

[4] M. Bershadsky, V. Knizhnik, M. Teitelman, Superconformal symmetry in two dimensions, Phys. Lett. B151 (1985) 31; D. Friedan, Z. Qiu, S. Shenker, Superconformal invariance in two dimensions and the tricritical Ising model, Phys. Lett. B151 (1985) 37.

[5] G. Mussardo, G. Sotkov, M. Stanishkov, Ramond sector of the supersymmetric minimal models, Phys. Lett. B195 (1987) 397.

[6] R. Poghossian, Study of the Vicinities of Superconformal Fixed Points in Two-dimensional Field Theory, Sov.J.Nucl.Phys. 48 (1988) 763.

[7] R. Poghossian, Two Dimensional Renormalization Group Flows in Next to Leading Order, JHEP 1401 (2014) 167; arXiv:hep-th/1303.3015.

[8] J. Cardy, Conformal invariance and surface critical behaviour, Nucl. Phys. B240 (1984) 514.

[9] P. Goddard, A. Kent, D. Olive, Virasoro algebras and coset space models, Phys. Lett. B152 (1985) 88.
[10] D. Kastor, E. Martinec, Z. Qiu, Current Algebra and Conformal Discrete Series, Phys. Lett. B200 (1988) 434.

[11] F. Ravanini, An Infinite Class of New Conformal Field Theories With Extended Algebras, Mod. Phys. Lett. 3A (1988) 397.

[12] P. Argyres, J. Grochocinski, S. Tye, Structure Constants of the Fractional Supersymmetry Chiral Algebras, Nucl. Phys. B367 (1991) 217; arXiv:hep-th/9110052.

[13] M. Ademollo et. al., Dual string with U(1) colour symmetry, Nucl. Phys. B111 (1976) 77.

[14] T. Banks, L. Dixon, D. Friedan, E. Martinec, Phenomenology and conformal field theory or can string theory predict the weak mixing angle, Nucl. Phys. B299 (1988) 613.

[15] L. Dixon, J. Harvey, C. Vafa, E. Witten, Strings on orbifolds (II), Nucl. Phys. B274 (1986) 285;
D. Friedan, E. Martinec, S. Shenker, Conformal invariance, supersymmetry and string theory, Nucl. Phys. B271 (1986) 93.

[16] W. Boucher, D. Friedan, A. Kent, Determinant formulae and unitarity for the N = 2 superconformal algebras in two dimensions or exact results on string compactification, Phys. Lett. B172 (1986) 316;
V. Dobrev, Characters of the unitarizable highest weight modules over the N = 2 superconformal algebras, Phys. Lett. B186 (1987) 43.

[17] P. Di Vecchia, J. Petersen, M. Yu, H. Zheng, Explicit construction of unitary representations of the N = 2 superconformal algebra, Phys. Lett. B174 (1986) 280.

[18] D. Gepner, Space-time supersymmetry in compactified string theory and superconformal models, Nucl. Phys. B296 (1988) 757;
D. Gepner, Exactly solvable string compactifications on manifolds of SU(N) holonomy, Phys. Lett. B199 (1987) 380;
D. Gepner, String Theory on Calabi-yau Manifolds: The Three Generations Case, Preprint-88-0085 (Princeton); arXiv:hep-th/9301089.

[19] D. Gepner, Z. Qiu, Modular invariant partition functions for parafermionic field theories, Nucl. Phys. B285 (1987) 423;
P. Christe, R.Flume, The four-point correlations of all primary operators of the d = 2 conformally invariant SU(2) -model with Wess-Zumino term, Nucl. Phys. B282 (1987) 466;
M. Yu, H. Zheng, N = 2 superconformal invariance in two-dimensional quantum field theories, Nucl. Phys. B288 (1987) 275.

[20] T. Curtright, C. Thorn, Conformally Invariant Quantization of the Liouville Theory, Phys.Rev.Lett 48 (1982) 1309.
[21] R. Rashkov, M. Stanishkov, Three point correlation functions in $N = 1$ superLiouville theory, Phys. Lett. B380 (1996) 49; arXiv:hep-th/9602148.

[22] C. Ahn, C. Kim, C. Rim, M. Stanishkov, Duality in $N = 2$ super-Liouville theory, Phys.Rev. D69 (2004) 106011; arXiv:hep-th/0210208.

[23] V. Fateev, A. Zamolodchikov, Al. Zamolodchikov, Boundary Liouville field theory I. Boundary state and boundary two-point function, arXiv:hep-th/0001012.

[24] A. Zamolodchikov, Al. Zamolodchikov, Liouville field theory on a pseudosphere, arXiv:hep-th/0101152.

[25] C. Ahn, C. Rim, M. Stanishkov, Exact one point function of $N = 1$ super-Liouville theory with boundary, Nucl. Phys. B636 (2002) 497; arXiv:hep-th/0202043.

[26] C. Ahn, M. Yamamoto, Boundary action of $N = 2$ super-Liouville theory, Phys.Rev. D69 (2004) 026007; arXiv:hep-th/0310046.

[27] Al. Zamolodchikov, Higher Equations of Motion in Liouville Field Theory , Int.J.Mod.Phys. A19 (2004) 510; arXiv:hep-th/0312279.

[28] A. Belavin, Al. Zamolodchikov, Higher equations of motion in $N = 1$ SUSY Liouville field theory, JETP Lett. 84 (2006) 418; arXiv:hep-th/0610316.

[29] R. Sasaki, I. Yamanaka, Field theoretical construction of an infinite set of quantum commuting operators related with soliton equations, Adv.Stud.Pure Math.16 (1988) 271; B. Feigin, E. Frenkel, Free field resolutions in affine Toda field theories, Phys. Lett. B276 (1992) 79.

[30] O. Babelon, D. Bernard, F. Smirnov, Null-Vectors in Integrable Field Theory, Comm. Math. Phys. 186 (1997) 601; arXiv:hep-th/9606068.

[31] D. Bernard, A. LeClair, The fractional supersymmetric sine-Gordon models, Phys. Lett. B247 (1990) 309; D. Bernard, A. LeClair, Quantum group symmetries and non-local currents in 2D QFT, Comm. Math. Phys. 142 (1991) 99.

[32] A. Luther, Eigenvalue spectrum of interacting massive fermions in one dimension, Phys.Rev.B14 (1976) 2153.

[33] S. Lukyanov, A note on the deformed Virasoro algebra, Phys. Lett. B367 (1996) 121; arXiv:hep-th/9509037.

[34] Al. Zamolodchikov, Two-point correlation function in scaling Lee-Yang model Nucl. Phys. B348 (1991) 619.

[35] S. Lukyanov, A. Zamolodchikov, Exact expectation values of local fields in quantum sine-Gordon model, Nucl. Phys. B493 (1997) 571; arXiv:hep-th/9611238.
[36] V. Fateev, S. Lukyanov, A. Zamolodchikov, Al. Zamolodchikov, Expectation values of local fields in Bullough-Dodd model and integrable perturbed conformal field theories Nucl. Phys. B516 (1998) 652; arXiv:hep-th/9709034.

[37] G. Sotkov, M. Stanishkov, N = 1 Superconformal Operator Product Expansions and Superfield Fusion Rules, Phys. Lett. B177 (1986) 361.

[38] G. Mussardo, G. Sotkov, M. Stanishkov, Fine Structure of the Supersymmetric Operator Product Expansion Algebras, Nucl. Phys. B305 (1988) 69.

[39] C. Crnkovic, G. Sotkov, M. Stanishkov, Minimal Models On Hyperelliptic Surfaces, Phys. Lett. B220 (1989) 397.

[40] C. Crnkovic, G. Sotkov, M. Stanishkov, Genus Two Partition Functions For Superconformal Minimal Models, Phys. Lett. B222 (1989) 217.

[41] C. Ahn, M. Stanishkov, On the Renormalization Group Flow in Two Dimensional Superconformal Models, Nucl. Phys. B885 (2014) 713; arXiv:hep-th/1404.7628.

[42] Z. Qiu, Supersymmetry, two-dimensional critical phenomena and the tricritical Ising model, Nucl. Phys. B270 (1986) 205.

[43] A. Zamolodchikov, V. Fateev, Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in ZN-symmetric statistical systems, Sov.Phys.JETP 62 (1985) 215.

[44] A. Zamolodchikov, V. Fateev, Disorder fields in two-dimensional conformal quantum-field theory and N=2 extended supersymmetry, Sov.Phys.JETP 63 (1986) 913.

[45] Al. Zamolodchikov, Conformal scalar field on the hyperelliptic curve and critical Ashkin-Teller multipoint correlation functions, Nucl. Phys. B285 (1987) 481.

[46] A. Belavin, V. Belavin, A. Neveu, Al. Zamolodchikov, Bootstrap in Supersymmetric Liouville Field Theory. I. NS Sector, Nucl. Phys. B784 (2007) 202; arXiv:hep-th/0703084.

[47] V. Belavin, N=1 supersymmetric conformal block recursion relations, Theor.Math.Phys. 152 (2007) 1275; arXiv:hep-th/0611295.

[48] C. Crnkovic, G. Sotkov, M. Stanishkov, Renormalization Group Flow for General SU(2) Coset Models, Phys. Lett. B226 (1989) 297.

[49] A. Zamolodchikov, Renormalization group and perturbation theory about fixed points in two-dimensional field theory, Sov.J.Nucl.Phys. 46 (1987) 1090.

[50] A. Belavin, B. Mukhametzhanov, N=1 superconformal blocks with Ramond fields from AGT correspondence, JHEP 1301 (2013) 178; arXiv:hep-th/1210.7454.

[51] M. Goulian, M. Li, Correlation functions in Liouville theory, Phys. Rev. Lett.66 (1991) 2051.
[52] A. Polyakov, Self-tuning fields and resonant correlations in 2d-gravity, Mod. Phys. Lett. \textbf{A6} (1991) 635;  
    P. Di Francesco, D. Kutasov, Correlation functions in 2D string theory, Phys. Lett. \textbf{B261} (1991) 385.

[53] R. Poghossian, A. Zamolodchikov, Operator algebra in two-dimensional superconformal field theory, Sov.J.Nucl.Phys. \textbf{47} (1988) 929.

[54] S. Ghoshal, A. Zamolodchikov, Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory, Int. J. Mod. Phys. \textbf{A9} (1993) 3841.

[55] J. Wess, B. Zumino, Consequences of anomalous Ward identities, Phys. Lett. \textbf{B37} (1971) 95;  
    E. Witten, Non-abelian bosonisation in two dimensions, Comm. Math. Phys. \textbf{92} (1984) 455.

[56] G. Mussardo, G. Sotkov, M. Stanishkov, $N = 2$ Superconformal Minimal Models, Int. J. Mod. Phys. \textbf{A4} (1989) 1135.

[57] G. Mussardo, G. Sotkov, M. Stanishkov, Fusion Rules, Four Point Functions and Discrete Symmetries of $N = 2$ Superconformal Models, Phys. Lett. \textbf{B218} (1989) 191.

[58] G. Sotkov, M. Stanishkov, Yukawa Couplings for the Three Generation String Model, Phys. Lett. \textbf{B215} (1988) 674.

[59] E. Gava, M. Stanishkov, On the renormalization group flow in $N = 2$ superconformal models, Mod. Phys. Lett. \textbf{A5} (1990) 2261.

[60] C. Ahn, M. Stanishkov, M. Yamamoto, One point functions of $N = 2$ super-Liouville theory with boundary, Nucl. Phys. \textbf{B683} (2004) 177; arXiv:hep-th/0311169.

[61] C. Ahn, M. Stanishkov, M. Yamamoto, ZZ-branes of $N = 2$ super-Liouville theory, JHEP \textbf{0407} (2004) 057; arXiv:hep-th/0405274.

[62] C. Ahn, M. Stanishkov, M. Stoilov, Higher Equations of Motion in N=2 Superconformal Liouville Field Theory, Phys. Lett. \textbf{B695} (2011) 501; arXiv:hep-th/1010.5843.

[63] A. Zamolodchikov, V. Fateev, Operator algebra and correlation functions in the two-dimensional $SU(2) \times SU(2)$ chiral Wess-Zumino model, Sov.J.Nucl.Phys. \textbf{43} (1986) 657.

[64] R. Schimmrigk, A New Construction of a Three Generation Calabi-yau Manifold, Phys. Lett. \textbf{B193} (1987) 175;  
    P. Candelas, A. Dole, C. Lutken, R. Schimmrigk, Complete Intersection Calabi-Yau Manifolds, Nucl. Phys. \textbf{B298} (1988) 493.

[65] M. Green, J. Schwarz, E. Witten, Superstring theory, vol. 2 (Cambridge U.P., Cambridge, 1987).
[66] V. Petkova, Two-dimensional (half-) integer spin conformal theories with central charge $c < 1$, Int. J. Mod. Phys. A3 (1988) 2945.

[67] V. Fateev, The sigma model (dual) representation for a two-parameter family of integrable quantum field theories, Nucl. Phys. B473 (1996) 509; P. Baseilhac, V. Fateev, Expectation values of local fields for a two-parameter family of integrable models and related perturbed conformal field theories, Nucl. Phys. B532 (1998) 567; arXiv:hep-th/9906010.

[68] J. Cardy, D. Lewellen, Bulk and boundary operators in conformal field theory, Phys. Lett. B259 (1991) 274.

[69] D. Gaiotto, Domain walls for two-dimensional Renormalization group flows, JHEP 1212 (2012) 103; arXiv:hep-th/1201.0767.

[70] I. Brunner, D. Roggenkamp, Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds, JHEP 0804 (2008) 001; arXiv:hep-th/0712.0188.

[71] A. Poghosyan, H. Poghosyan, Mixing with descendant fields in perturbed minimal CFT models, JHEP 1310 (2013) 131; arXiv:hep-th/1305.6066.

[72] A. Poghosyan, H. Poghosyan, RG domain wall for the $N = 1$ minimal superconformal models, JHEP 1505 (2015) 043; arXiv:hep-th/1412.6710.

[73] C. Crnkovic, R. Paunov, G. Sotkov, M. Stanishkov, Fusions of Conformal Models, Nucl. Phys. B336 (1990) 637.

[74] M. Stanishkov, RG Domain Wall for the General $\hat{su}(2)$ Coset Models, JHEP 1608 (2016) 096; arXiv:hep-th/1606.03605.

[75] M. Stanishkov, Second order RG flow in general $\hat{su}(2)$ coset models, JHEP 1609 (2016) 040; arXiv:hep-th/1606.04328.

[76] M. Stanishkov, On the RG Flow in the Two-Dimensional Coset Models, Bulg.J.Phys 44 (2017) 057; arXiv:hep-th/1612.01867.

[77] Z. Kakushadze, S. Tye, Kac and new determinants for fractional superconformal algebras, Phys. Rev. D49 (1994) 4122; arXiv:hep-th/9310160.

[78] N. Wyllard, Coset conformal blocks and N=2 gauge theories, arXiv:hep-th/1109.4264.

[79] R. Poghossian, Operator Algebra in Two Dimensional Conformal Quantum Field Theory Containing Spin 4/3 Parafermionic Coserved Currents, Int. J. Mod. Phys. A6 (1991) 2005.

[80] V. Bazhanov, S. Lukyanov, A. Zamolodchikov, Integrable structure of conformal field theory, quantum KdV theory and Thermodynamic Bethe Ansatz, Comm. Math. Phys. 177 (1996) 381; V. Bazhanov, S. Lukyanov, A. Zamolodchikov, Integrable quantum field theories in finite volume: Excited state energies, Nucl. Phys. B489 (1997) 487; arXiv:hep-th/9607099.
[81] F. Smirnov, Exact S matrices for phi(1,2) perturbated minimal models of conformal field theory, Int. J. Mod. Phys. A6 (1991) 1407.

[82] M. Martins, Constructing a S matrix from the truncated conformal approach data, Phys. Lett. B262 (1991) 39;
A. Koubek, M. Martins, G. Mussardo, Scattering matrices for Phi(1,2) perturbed conformal minimal models in absence of kink states, Nucl. Phys. B368 (1992) 591.

[83] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Landau-Lifshitz equation: solitons, quasiperiodic solutions and infinite-dimensional Lie algebras, J. Phys. A16 (1983) 221.

[84] M. Semenov-Tian-Shansky, Dressing transformations and Poisson group actions, RIMS Kyoto Univ. 21 (1985) 1237.

[85] O. Babelon, D. Bernard, Dressing symmetries, Comm. Math. Phys. 149 (1992) 279; arXiv:hep-th/9111036.

[86] V. Drinfeld, V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, J.Sov.Math. 30 (1984) 1975.

[87] J.-L. Gervais, Infinite Family of Polynomial Functions of the Virasoro Generators with Vanishing Poisson Brackets, Phys. Lett. B160 (1985) 277.

[88] B. Kupershmidt, P. Mathieu, Quantum Korteweg-de Vries Like Equations and Perturbed Conformal Field Theories, Phys. Lett. B227 (1989) 245.

[89] G. Sotkov, M. Stanishkov, Off critical \( W_\infty \) and Virasoro algebras as dynamical symmetries of the integrable models, In:”Integrable Quantum Field Theories”. NATO ASI Series B, vol.310, Como, Italy, 1992, Plenum, 1993, pp. 217-234; arXiv:hep-th/9301066.

[90] E. Abdalla, M.C. Abdalla, G. Sotkov, M. Stanishkov, Off critical current algebras, Int. J. Mod. Phys. A10 (1995) 1717; arXiv:hep-th/9302002.

[91] D. Fioravanti, F. Ravanini, M. Stanishkov, Generalized KdV and quantum inverse scattering description of conformal minimal models, Phys. Lett. B367 (1996) 113, arXiv:hep-th/9510047.

[92] P. Baseilhac, M. Stanishkov, Expectation values of descendent fields in the Bullough-Dodd model and related perturbed conformal field theories, Nucl. Phys. B612 (2001) 373; arXiv:hep-th/0104220.

[93] P. Baseilhac, M. Stanishkov, On the third level descendent fields in the Bullough-Dodd model and its reductions, Phys. Lett. B554 (2003) 217; arXiv:hep-th/0212342.

[94] D. Fioravanti, M. Stanishkov, On the null vectors in the spectra of the 2-D integrable hierarchies, Phys. Lett. B430 (1998) 109; arXiv:hep-th/9806090.

[95] D. Fioravanti, M. Stanishkov, Nonlocal Virasoro symmetries in the mKdV hierarchy, Phys. Lett. B447 (1999) 277; arXiv:hep-th/9810046.
[96] D. Fioravanti, M. Stanishkov, Hidden local, quasilocal and nonlocal symmetries in integrable systems, Nucl. Phys. B577 (2000) 500; arXiv:hep-th/0001151.

[97] D. Fioravanti, M. Stanishkov, Hidden Virasoro symmetry of (soliton solutions of) the Sine-Gordon theory, Nucl. Phys. B591 (2000) 685; arXiv:hep-th/0005158.

[98] A. Zamolodchikov, Integrals of Motion in Scaling Three State Potts Model Field Theory, Int. J. Mod. Phys. A3 (1988) 743.

[99] H. Itoyama, H. Thacker, Integrability and Virasoro Symmetry of the Noncritical Baxter-Ising Model, Nucl. Phys. B320 (1989) 541.

[100] E. Witten, Nonabelian Bosonization in Two-Dimensions, Comm. Math. Phys. 92 (1984) 455.

[101] M. Jimbo, Quantum $r$ Matrix for the Generalized Toda System, Comm. Math. Phys. 102 (1986) 537.

[102] V. Fateev, S. Lukyanov, Poisson Lie groups and classical W algebras, Int. J. Mod. Phys. A7 (1992) 853.

[103] P. Kulish, N. Reshetikhin, E. Sklyanin, Yang-Baxter Equation and Representation Theory. 1, Lett. Math. Phys. 5 (1981) 393.

[104] A. Kuniba, J. Suzuki, Functional relations and analytic Bethe Ansatz for twisted quantum Affine algebras, J.Phys. A28 (1995) 711; arXiv:hep-th/9408135.

[105] D. Fioravanti, M. Rossi, From the braided to the usual Yang-Baxter relation, J.Phys. A34 (2001) L567; arXiv:hep-th/0107050.

[106] D. Fioravanti, M. Rossi, A braided Yang-Baxter algebra in a theory of two coupled lattice quantum KdV: Algebraic properties and ABA representations, J.Phys. A35 (2002) 3647; arXiv:hep-th/0104002.

[107] D. Fioravanti, M. Rossi, Exact conserved quantities on a cylinder 1: Conformal case, JHEP 0307 (2003) 031; arXiv:hep-th/0211094.

[108] D. Fioravanti, M. Rossi, Exact conserved quantities on a cylinder 2: Off critical case, JHEP 0308 (2003) 042; arXiv:hep-th/0302220.

[109] V. Fateev, D. Fradkin, S. Lukyanov, A. Zamolodchikov, Al. Zamolodchikov, Expectation values of descendent fields in the sine-Gordon model, Nucl. Phys. B540 (1999) 587; arXiv:hep-th/9807236.

[110] G. Delius, M. Grisaru, D. Zanon, Exact $S$ matrices for nonsimply laced affine Toda theories, Nucl. Phys. B382 (1992) 365; arXiv:hep-th/9201067; E. Corrigan, P. Dorey, R.Sasaki, On a generalized bootstrap principle, Nucl. Phys. B408 (1993) 579; arXiv:hep-th/9304065.
[111] R. Miura, Korteweg de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation, J.Math.Phys. 9 (1968) 1202.

[112] O. Babelon, D. Bernard, F. Smirnov, Quantization of solitons and the restricted Sine-Gordon model, Comm. Math. Phys. 182 (1996) 319; arXiv:hep-th/9603010.

[113] P. Grinevich, A. Orlov, Virasoro action on Riemann surfaces, Grassmannians, det (delta-bar(J)) and Segal-Wilson tau function, 9th International Conference (Spring School) on the Problems of Quantum Field Theory, Alushta, Springer-Verlag, 1989, p. 86-106;
P. Grinevich, Nonisospectral symmetries of the KdV equation and the corresponding symmetries of the Whitham equations, In: Singular Limits of Dispersive Waves, NATO ASI Series, Ser. B: Physics, Vol. 320, Plenum, 1994, p.67-88; arXiv: solv-int/9509004.