Turaev–Viro invariants and cabling operations

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Abstract

In this paper, we study the variation of the Turaev–Viro invariants for 3-manifolds with toroidal boundary under the operation of attaching a \((p, q)\)-cable space. We apply our results to a conjecture of Chen and Yang which relates the asymptotics of the Turaev–Viro invariants to the simplicial volume of a compact oriented 3-manifold. For \(p\) and \(q\) coprime, we show that the Chen–Yang volume conjecture is stable under \((p, q)\)-cabling. We achieve our results by studying the linear operator \(RT_r\) associated to the torus knot cable spaces by the Reshetikhin–Turaev \(SO_3\)-Topological Quantum Field Theory (TQFT), where the TQFT is well-known to be closely related to the desired Turaev–Viro invariants. In particular, our utilized method relies on the invertibility of the linear operator for which we provide necessary and sufficient conditions.

1 Introduction

For a compact 3-manifold \(M\), its Turaev–Viro invariants are a family of \(\mathbb{R}\)-valued homeomorphism invariants parameterized by an integer \(r \geq 3\) depending on a \(2r\)-th root of unity \(q\). We are primarily interested in the invariants when \(r\) is odd and \(q = e^{2\pi i/r}\).

In this paper, we study the variation of the Turaev–Viro invariants of a 3-manifold with toroidal boundary when we attach a \((p, q)\)-cable space.

Definition 1.1. Let \(V\) be the standardly embedded solid torus in \(S^3\), and let \(V'\) be a closed neighborhood of \(V\). For \(p\) and \(q\) coprime integers with \(q > 0\), let \(T_{p, q} \subset \partial V\) be the torus knot of slope \(p/q\). The \((p, q)\)-cable space, denoted \(C_{p, q}\), is the complement of the torus knot \(T_{p, q}\) in \(V'\). Let \(M\) be a 3-manifold with toroidal boundary. A manifold \(M'\) obtained from gluing a \((p, q)\)-cable space \(C_{p, q}\) to a boundary component of \(M\) along the exterior toroidal boundary component of \(C_{p, q}\) is called a \((p, q)\)-cable of \(M\).

Our main theorem is the following.

Theorem 1.2. Let \(M\) be a manifold with toroidal boundary, let \(p, q\) be coprime integers with \(q > 0\), and let \(r \geq 3\) be an odd integer coprime to \(q\). Suppose \(M'\) is a \((p, q)\)-cable of \(M\). Then there exists a constant \(C > 0\) and natural number \(N\) such that

\[
\frac{1}{Cr^N} TV_r(M) \leq TV_r(M') \leq Cr^N TV_r(M).
\]
Theorem 1.2 has notable applications to existing conjectures. In general, the Turaev–Viro invariants are difficult to compute; however, there is interest in the relationship between their $r$-asymptotic behavior and classical invariants of 3-manifolds. Chen and Yang [5] conjectured that the growth rate of the Turaev–Viro invariants for hyperbolic manifolds is related to the manifold’s hyperbolic volume. They also provided computational evidence for the conjecture in [5].

This conjecture should be compared to the well-known conjectures of Kashaev [12] and Murakami-Murakami [16] relating the Kashaev and colored Jones invariants of hyperbolic link complements to their hyperbolic volumes. Detcherry and Kalfagianni [7] restated the Turaev–Viro invariant volume conjecture more generally in terms of the simplicial volume for manifolds which are not necessarily hyperbolic. In order to state the conjecture, we will first introduce a slightly weaker condition for the growth rate of the Turaev–Viro invariants.

**Definition 1.3.** Define the following two asymptotics of the Turaev–Viro invariants for compact 3-manifolds as

$$lTV(M) := \liminf_{r \to \infty, \, r \text{ odd}} \frac{2\pi}{r} \log \left| TV_r \left( M; q = e^{2\pi i} \right) \right|,$$

and

$$LTV(M) := \limsup_{r \to \infty, \, r \text{ odd}} \frac{2\pi}{r} \log \left| TV_r \left( M; q = e^{2\pi i} \right) \right|.$$

Additionally, we will introduce the simplicial volume for compact orientable 3-manifolds with empty or toroidal boundary, originally defined by Gromov [9]. For $M$ a compact, orientable, irreducible 3-manifold, there is a unique collection of incompressible tori, up to isotopy, along which $M$ can be decomposed into atoroidal manifolds. This is known as the JSJ decomposition [10, 11]. By Thurston’s Geometrization Conjecture [23], famously completed in the work of Perelman [17, 18, 19], each of these atoroidal manifolds are either hyperbolic or Seifert-fibered, and by Thurston [22], the simplicial volume of $M$ coincides with the sum of the simplicial volumes of the resulting pieces. In the case where $M$ is hyperbolic, the simplicial volume is positive and is related to the hyperbolic volume of $M$ by

$$vol(M) = v_3\|M\|$$

where $vol(M)$ is the hyperbolic volume of $M$, $v_3 \approx 1.0149$ is the volume of the regular ideal tetrahedron, and $\|M\|$ is the simplicial volume of $M$.

This leads to a natural extension of Chen and Yang’s Turaev–Viro invariant volume conjecture [5].

**Conjecture 1.4 ([5], [7]).** Let $M$ be a compact oriented 3-manifold. Then

$$LTV(M) = v_3\|M\|,$$

where $v_3$ is the volume of the regular ideal tetrahedron and $\| \cdot \|$ is the simplicial volume.
If Conjecture 1.4 is true, since the simplicial volume is additive under the JSJ decomposition, it would imply that the asymptotics of the Turaev–Viro invariants are also additive under the decomposition. As an application of Theorem 1.2, we provide evidence of the additivity for the asymptotics of the Turaev–Viro invariants.

The asymptotic additivity property has been explored in several works. For a manifold $M$ which satisfies Conjecture 1.4, the property was proven for invertible cabling of $M$ by Detcherry and Kalfagianni [7], the figure-eight knot cabled with Whitehead chains by Wong [26], and an infinite family of manifolds with arbitrarily large simplicial volume by the authors of this paper [14]. Additionally, the property was proven to hold under the operation of attaching a $(p, 2)$-cable space by Detcherry [6], which we extend in this work.

A key property of the $(p, q)$-cable spaces is that they have simplicial volume zero. Theorem 1.2 provides a way to construct new manifolds without changing the simplicial volume while controlling the growth of the Turaev–Viro invariants. This leads to many examples of manifolds satisfying Conjecture 1.4.

To the authors’ knowledge, in all of the proven examples of Conjecture 1.4, the stronger condition that the limit approaches the simplicial volume is verified, as opposed to only the limit superior. Theorem 1.2 implies the following corollaries. See Section 3 for more details.

**Corollary 1.5.** Suppose $M$ satisfies Conjecture 1.4 and $\text{ltv}(M) = v_3 ||M||$. Then for any $p$ and $q$ coprime, any $(p, q)$-cable $M'$ also satisfies Conjecture 1.4.

Some examples which satisfy the hypothesis of Corollary 1.5 include the figure-eight knot and the Borromean rings by Detcherry-Kalfagianni-Yang [8], the Whitehead chains by Wong [25], the fundamental shadow links by Belletti-Detcherry-Kalfagianni-Yang [2], a family of hyperbolic links in $S^2 \times S^1$ by Belletti [1], a large family of octahedral links in $S^3$ by the first author of this paper [13], and a family of link complements in trivial $S^1$-bundles over oriented connected closed surfaces by the authors of this paper [14].

For general $p$ and $q$ coprime, Corollary 1.5 demonstrates the stability of Conjecture 1.4 under the $(p, q)$-cabling operation. However, in the case when $q = 2^n$ for $n \in \mathbb{N}$, we recover the full limit as shown in the following corollary.

**Corollary 1.6.** Suppose $M$ satisfies Conjecture 1.4. Then for any odd $p$ and $n \in \mathbb{N}$, any $(p, 2^n)$-cable $M'$ also satisfies Conjecture 1.4. Moreover, if $\text{ltv}(M) = v_3 ||M||$, then $\text{ltv}(M') = \text{ltv}(M') = v_3 ||M'||$.

As a direct result of Corollaries 1.5 and 1.6, we extend the work of Detcherry [6], where the author considers the operation of attaching a $(p, 2)$-cable space. This allows us to construct manifolds satisfying Conjecture 1.4 from manifolds with toroidal boundary where the conjecture is already known. This includes all previously mentioned examples.

The general method of proof for Theorem 1.2 follows from the work of Detcherry [6]. Considering the cable space $C_{p,q}$ as a cobordism between tori, the Reshetikhin–Turaev $SO_3$-TQFT at level $r$, denoted by $RT_r$, associates to it a linear operator

$$RT_r(C_{p,q}) : RT_r(T^2) \to RT_r(T^2).$$
For $p$ odd and $q = 2$, Detcherry presents $RT_r(C_{p,2})$ using the basis \{${e_1, e_3, \ldots, e_{2m-1}}$\}, which is equivalent to the orthonormal basis \{${e_1, e_2, \ldots, e_m}$\} for $RT_r(T^2)$ given in [4] under the symmetry $e_{m-i} = e_{m+i+1}$ for $0 \leq i \leq m - 1$. More details of the construction are given in Section 3. With this basis, $RT_r(C_{p,2})$ can be presented as a product of two diagonal matrices and one triangular matrix. This allows the author to directly write the inverse of $RT_r(C_{p,2})$. From the inverse of this linear operator, Detcherry establishes a lower bound of the Turaev–Viro invariants under attaching a $(p,2)$-cable space.

For general $q$, $RT_r(C_{p,q})$ does not have as simple a presentation under the same basis, making it more difficult to conclude that $RT_r(C_{p,q})$ is invertible. In order to resolve this, we present $RT_r(C_{p,q})$ using a different basis for $RT_r(T^2)$, defined in Section 4, that allows us to also show directly that $RT_r(C_{p,q})$ is invertible provided $r$ and $q$ are coprime. Following Detcherry’s argument, the invertibility of $RT_r(C_{p,q})$ is integral in finding the lower bound from Theorem 1.2; however, the invertibility of $RT_r(C_{p,q})$ is constrained by the condition that $r$ and $q$ are coprime, as outlined by Theorem 1.7.

**Theorem 1.7.** Let $p$ be coprime to some positive integer $q$. Then $RT_r(C_{p,q})$ is invertible if and only if $r$ and $q$ are coprime. Moreover, the operator norm $|||RT_r(C_{p,q})^{-1}|||$ grows at most polynomially.

As we will show in Section 3, the coprime condition between $r$ and $q$ leads to the discrepancy between recovering the limit superior in Corollary 1.5 versus the full limit in Corollary 1.6.

The paper is organized as follows: We recall properties of the Reshetikhin–Turaev $SO_3$-TQFTs, the $RT_r$ torus knot cabling formula, and relevant properties of the Turaev–Viro invariants in Section 2. In Section 3, we prove Theorem 1.2 assuming Theorem 1.7. In Section 4, the construction of the relevant basis for $RT_r(T^2)$ and the proof of Theorem 1.7 is given. Lastly, we consider future directions in Section 5.

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## 2 Preliminaries

### 2.1 Reshetikhin–Turaev TQFT

In this subsection, we outline relevant properties of the Reshetikhin–Turaev $SO_3$-TQFTs, which were defined by Reshetikhin and Turaev in [20]. Let $\mathcal{C}ob$ be the category of $(2+1)$-dimensional cobordisms and $\text{Vect}(\mathbb{C})$ be the category of $\mathbb{C}$-vector spaces. For an odd integer
For $r \geq 3$ and primitive $2r$-th root of unity $A$, one associates a $(2+1)$-dimensional TQFT $\mathcal{R}_r : \text{Cob} \to \text{Vect}(\mathbb{C})$. Blanchet, Habegger, Masbaum, and Vogel [4] gave a skein-theoretic framework for this $SO_3$-TQFT, and its main properties are the following:

1) For a closed oriented surface $\Sigma$, $\mathcal{R}_r(\Sigma)$ is a finite dimensional vector space over $\mathbb{C}$ with the natural Hermitian form. For a disjoint union $\Sigma \sqcup \Sigma'$, we have $\mathcal{R}_r(\Sigma \sqcup \Sigma') = \mathcal{R}_r(\Sigma) \otimes \mathcal{R}_r(\Sigma')$.

2) For an oriented closed 3-manifold $M$, $\mathcal{R}_r(M) \in \mathbb{C}$ is a topological invariant.

3) For an oriented compact 3-manifold $M$ with boundary $\partial M$, $\mathcal{R}_r(M) \in \mathcal{R}_r(\partial M)$ is a vector.

4) For a cobordism $(M, \Sigma_1, \Sigma_2)$, $\mathcal{R}_r(M) : \mathcal{R}_r(\Sigma_1) \to \mathcal{R}_r(\Sigma_2)$ is a linear map.

In [4], the authors also give explicit bases for any surface. However, we will focus on $\mathcal{R}_r(T^2)$, which can be considered as a quotient of the Kauffman skein module of the genus 1 handlebody $D^2 \times S^1$.

We begin by coloring the core $\{0\} \times S^1$ by the $(i-1)$-th Jones-Wenzl idempotent. This gives a family of elements $e_i$ of the Kauffman skein module of the solid torus. However, there are only finitely many Jones-Wenzl idempotents for a given odd $r = 2m + 1$ and $2r$-th root of unity $A$, namely $e_1, \ldots, e_{2m-1}$ [4]. We can consider these $e_i$’s as elements of the quotient $\mathcal{R}_r(T^2)$, giving us a basis.

**Theorem 2.1** ([4], Theorem 4.10). Let $r = 2m + 1 \geq 3$. Then the family $e_1, \ldots, e_m$ is an orthonormal basis for $\mathcal{R}_r(T^2)$. Moreover, the relation $e_{m-i} = e_{m+1+i}$ holds for $0 \leq i \leq m-1$.

The second part of the theorem implies that $\{e_1, e_3, \ldots, e_{2m-1}\}$ is just a reordering of the basis $\{e_1, \ldots, e_m\}$.

### 2.2 The Cabling Formula

Here, we will give an explicit description for the Reshetikhin–Turaev invariants of the torus knot cable spaces.

Let $p$ and $q$ be coprime integers where $q > 0$, and let $C_{p,q}$ be the $(p, q)$-cable space $C_{p,q}$. These spaces are Seifert-fibered and therefore have simplicial volume zero. For $r = 2m + 1 \geq 3$, we extend the vectors $e_i \in \mathcal{R}_r(T^2)$ to all $i \in \mathbb{Z}$ in the following way. Let $e_{-i} = -e_i$ for any $i \geq 0$, and let $e_{i+kr} = (-1)^k e_i$ for any $k \in \mathbb{Z}$. Note this means that $e_r = e_0 = 0$.

Regarding the cable space $C_{p,q}$ as a cobordism between tori, the Reshetikhin–Turaev $SO_3$-TQFT gives a linear map

$$\mathcal{R}_r(C_{p,q}) : \mathcal{R}_r(T^2) \to \mathcal{R}_r(T^2).$$

The map $\mathcal{R}_r(C_{p,q})$ sends the element $e_i$ to the element of $\mathcal{R}_r(T^2)$ corresponding to a $(p, q)$-torus knot embedded in the solid torus and colored by the $(i-1)$-th Jones-Wenzl idempotent. Morton [15] gives the following formula for the image of the basis elements under $\mathcal{R}_r(C_{p,q})$. 

$$
\mathcal{R}_r(C_{p,q}) : \mathcal{R}_r(T^2) \to \mathcal{R}_r(T^2)
$$

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$$
\mathcal{R}_r(C_{p,q}) : \mathcal{R}_r(T^2) \to \mathcal{R}_r(T^2)
$$
Theorem 2.2 ([15], Section 3, Cabling Formula).

\[ RT_r(C_{p,q})(e_i) = A^{pq(i^2-1)/2} \sum_{k \in S_i} A^{-2pk(qk+1)} e_{2qk+1}, \]

where \( S_i \) is the set

\[ S_i = \left\{ -\frac{i-1}{2}, -\frac{i-3}{2}, \ldots, \frac{i-3}{2}, \frac{i-1}{2} \right\}. \]

As we will see in Subsection 2.3, the Reshetikhin–Turaev TQFT is closely related to the Turaev–Viro invariants for 3-manifolds. By using their relationship, the explicit formula given in Theorem 2.2 will allow us to obtain a lower bound on the Turaev–Viro invariants under the cabling operation.

2.3 Properties of the Turaev–Viro invariants

In this subsection, we discuss properties of the Turaev–Viro invariants [24] as well as an important characterization in terms of the Reshetikhin–Turaev invariants.

The Turaev–Viro invariants were defined by Turaev and Viro [24] in terms of state sums over triangulations of a 3-manifold \( M \), but they are also closely related to the Reshetikhin–Turaev invariants. The following identity was originally proven for closed 3-manifolds by Roberts [21] and then extended to compact manifolds with boundary by Benedetti and Petronio [3].

Theorem 2.3 ([3, 21]). Let \( r \geq 3 \) be an odd integer, and let \( q \) be a primitive \( 2r \)-th root of unity. Then for a compact oriented manifold \( M \) with toroidal boundary,

\[ TV_r(M; q) = \left\| RT_r(M; q^2) \right\|^2 \]

where \( \| \cdot \| \) is the natural Hermitian norm on \( RT_r(\partial M) \).

We note that this identity holds more generally, but we have restricted to manifolds with toroidal boundary for simplicity.

In [7], Detcherry and Kalfagianni proved that the growth rate of the Turaev–Viro invariants has properties reminiscent of simplicial volume. We summarize their results in the following theorem.

Theorem 2.4 ([7]). Let \( M \) be a compact oriented 3-manifold, with empty or toroidal boundary.

1) If \( M \) is a Seifert manifold, then there exist constants \( B > 0 \) and \( N \) such that for any odd \( r \geq 3 \), we have \( TV_r(M) \leq Br^N \) and \( LTV(M) \leq 0 \).

2) If \( M \) is a Dehn-filling of \( M' \), then \( TV_r(M) \leq TV_r(M') \) and \( LTV(M) \leq LTV(M') \).

3) If \( M = M_1 \bigcup_T M_2 \) is obtained by gluing two 3-manifolds along a torus boundary component, then \( TV_r(M) \leq TV_r(M_1)TV_r(M_2) \) and \( LTV(M) \leq LTV(M_1) + LTV(M_2) \).
3 Bounding the Invariant Under Cabling

In this section, we will prove Theorem 1.2 with the assumption of a key theorem, and we reserve the technical details for Section 4. We remark that the major components of our argument follow from the work of Detcherry [6] where the case when \( p \) is odd and \( q = 2 \) was proven. For convenience, we will restate the main theorem.

**Theorem 1.2.** Let \( M \) be a manifold with toroidal boundary, let \( p, q \) be coprime integers with \( q > 0 \), and let \( r \geq 3 \) be an odd integer coprime to \( q \). Suppose \( M' \) is a \((p, q)\)-cable of \( M \). Then there exists a constant \( C > 0 \) and natural number \( N \) such that

\[
\frac{1}{C r^N} TV_r(M) \leq TV_r(M') \leq Cr^N TV_r(M).
\]

We will now assume Theorem 1.7, which we also restate for convenience.

**Theorem 1.7.** Let \( p \) be coprime to some positive integer \( q \). Then \( RT_r(C_{p,q}) \) is invertible if and only if \( r \) and \( q \) are coprime. Moreover, the operator norm \( \|RT_r(C_{p, q})^{-1}\| \) grows at most polynomially.

**Proof of Theorem 1.2.** As mentioned previously, the case when \( p \) is odd and \( q = 2 \) was shown by Detcherry [6], and our approach follows closely in structure. We let \( M \) be a manifold with toroidal boundary, \( p \) an integer, \( q > 0 \) an integer coprime to \( p \), \( r \geq 3 \) odd and coprime to \( q \), and \( M' \) a \((p, q)\)-cable of \( M \). We will proceed to prove Theorem 1.2 by showing the upper inequality of

\[
\frac{1}{C r^N} TV_r(M) \leq TV_r(M') \leq Cr^N TV_r(M)
\]

followed by the lower inequality, where \( C > 0 \) and \( N \in \mathbb{N} \). To obtain the upper inequality, we first remark that \( M' = C_{p,q} \bigcup_T M \). By Theorem 2.3 this implies that

\[
TV_r(M') \leq TV_r(C_{p,q}) TV_r(M).
\]

Since \( C_{p,q} \) is a Seifert manifold, we have that

\[
TV_r(C_{p,q}) \leq C_1 r^{N_1}
\]

for some \( C_1 > 0 \) and \( N_1 \in \mathbb{N} \) also by Theorem 2.4. This leads to the upper inequality

\[
TV_r(M') \leq C_1 r^{N_1} TV_r(M).
\]

For the lower inequality, we will use Theorem 1.7. From the properties of the Reshetikhin–Turaev \( SO_3 \)-TQFT, we consider the linear map

\[
RT_r(C_{p,q}) : RT_r(T^2) \to RT_r(T^2).
\]

If \( M \) only has one boundary component, then

\[
RT_r(M') = RT_r(C_{p,q}) RT_r(M)
\]
by the properties of a TQFT. If $M$ has other boundary components, then the invariant associated to any coloring $i$ of the other boundary components may be computed as

$$RT_r(M', i) = RT_r(C_{p,q})RT_r(M, i).$$

By the invertibility of $RT_r(C_{p,q})$ from Theorem 1.7 we have the inequality

$$||RT_r(M)|| ≤ ||RT_r(C_{p,q})^{-1}|| \cdot ||RT_r(M')||$$

where $||·||$ is the norm induced by the Hermitian form of the TQFT and $|||·|||$ is the operator norm. Since the operator norm grows at most polynomially by Theorem 1.7, we obtain the inequality

$$\frac{1}{C_2 r^{N_2}}||RT_r(M)|| ≤ ||RT_r(M')||$$

for some $C_2 > 0$ and $N_2 ∈ \mathbb{N}$. Lastly, by Theorem 2.3 the norm of the Reshetikhin–Turaev invariant is related to the Turaev–Viro invariant such that we arrive to the desired inequality

$$\frac{1}{C_3 r^{N_3}}TV_r(M) ≤ TV_r(M')$$

for some $C_3 > 0$ and $N_3 ∈ \mathbb{N}$. This leads to

$$\frac{1}{C r^{N}}TV_r(M) ≤ TV_r(M') ≤ Cr^N TV_r(M)$$

where $C > 0$ and $N ∈ \mathbb{N}$. □

As discussed in Section 1, the following corollaries follow from Theorem 1.2.

**Corollary 1.5.** Suppose $M$ satisfies Conjecture 1.4 and $lTV(M) = v_3 ||M||$. Then for any $p$ and $q$ coprime, any $(p,q)$-cable $M'$ also satisfies Conjecture 1.4.

**Proof.** By Theorem 2.4 Part (1), $LTV(C_{p,q}) ≤ 0$, and thus by Theorem 2.4 Part (3), $LTV(M') ≤ LTV(M)$. Since $lTV(M) = LTV(M) = v_3 ||M||$, the limit exists, and any subsequence also converges to $v_3 ||M||$. By Theorem 1.2 along odd $r$,

$$\limsup_{r→∞, (r,q)=1} \frac{2π}{r} \log |TV_r(−)| = \limsup_{r→∞, (r,q)=1} \frac{2π}{r} \log |TV_r(M)| = LTV(M) = v_3 ||M||,$$

where

$$\limsup_{r→∞, (r,q)=1} \frac{2π}{r} \log |TV_r(−)|$$

is the limit superior of the subsequence along which $r$ and $q$ are coprime.

Since

$$v_3 ||M|| = \limsup_{r→∞, (r,q)=1} \frac{2π}{r} \log |TV_r(M')| ≤ LTV(M') ≤ LTV(M) = v_3 ||M||,$$

we have

$$LTV(M') = v_3 ||M|| = v_3 ||M'||,$$

where the final equality follows from the fact that the simplicial volume does not change under attaching a $(p,q)$-cable space. □
Corollary 1.6. Suppose $M$ satisfies Conjecture 1.4. Then for any odd $p$ and $n \in \mathbb{N}$, any $(p, 2^n)$-cable $M'$ also satisfies Conjecture 1.4. Moreover, if $lT V(M) = v_3||M||$, then $lT V(M') = LTV(M') = v_3||M'||$.

Proof. Since $r$ is odd, $(r, 2^n) = 1$ for any $n \geq 1$, which means Theorem 1.2 holds for any $(p, 2^n)$-cable of $M$ provided $p$ is odd. Since $||M|| = ||M'||$, this implies that $lT V(M') = LTV(M') = v_3||M'|| = LTV(M')$.}

\section{Proof of Supporting Theorem}

In this section, we will provide a proof of Theorem 1.7, which we restate here for convenience.

**Theorem 1.7.** Let $p$ be coprime to some positive integer $q$. Then $RT_r(C_{p,q})$ is invertible if and only if $r$ and $q$ are coprime. Moreover, the operator norm $|||RT_r(C_{p,q})^{-1}|||$ grows at most polynomially.

We will use the following supporting proposition for the proof of Theorem 1.7 which is given in Subsection 4.1. The proof of this proposition is given in Subsection 4.2. We also use a couple of technical lemmas which are subsequently proven in Subsection 4.3. We begin by constructing a basis over which $RT_r(C_{p,q})$ admits a simpler expression.

By the cabling formula given by Theorem 2.2,

$$RT_r(C_{p,q})(e_i) \in \operatorname{Span}\{e_1, e_{q+1}, e_{q-1}\}_{l=1}^{m-1}$$

where $m = \frac{r-1}{2}$. Let $F_m := \{f_l\}_{l=0}^{m-1}$, where

$$f_0 := e_1$$  
$$f_l := e_{q+1} - A^{2pl} e_{q-1} \quad l = 1, \ldots, m.$$  

Define $\tilde{f}_l \in \operatorname{Span}\{e_1, \ldots, e_m\}$ to be the reduction of $f_l$ under the quotient induced by the symmetries $e_{-i} = -e_i$ for any $i \geq 0$ and $e_{i+nr} = (-1)^n e_i$ for any $n \in \mathbb{Z}$. Note that for each $l, ql \pm 1 = kr + j$ for some non-negative integers $k, j$ where $0 \leq j < r$. This means that up to sign, these symmetries imply

$$e_{ql \pm 1} = e_{ql \pm kr \pm 1} = e_j \quad \text{for } 0 \leq j \leq m \quad (1)$$

$$e_{ql \pm 1} = e_{(k+1)r-ql \pm 1} = e_{r-j} \quad \text{for } m + 1 \leq j < r. \quad (2)$$
Finally, define $\tilde{F}_m := \{\tilde{f}_l\}_{l=0}^{m-1}$, and let $R_m$ be the $(m \times m)$-matrix with columns corresponding to the reduced vectors $\tilde{f}_l$, for $l = 0, \ldots, m - 1$. In particular, $\tilde{f}_l$ corresponds to $\text{col}(l+1)$ of $R_m$, and the rows of $R_m$ correspond to the original orthonormal basis $\{e_1, \ldots, e_m\}$ spanning $RT_r(T^2)$.

**Remark 4.1.** We note that $F_m$, $\tilde{F}_m$, and $R_m$ are also dependent on $p$ and $q$, but these dependencies are suppressed to avoid unwieldy notation.

The following proposition will be used to prove Theorem 1.7.

**Proposition 4.2.** Let $r = 2m + 1 \geq 3$ be coprime to $q$. Then $R_m$ is a change of basis from $\tilde{F}_m \to \{e_1, \ldots, e_m\}$ and the operator norm $||R_m^{-1}||$ grows at most polynomially in $m$. Moreover, for $i \in \{1, \ldots, m\}$,

$$RT_r(C_{p,q})(e_i) = A^{\frac{q^2}{2}(i^2-1)} \sum_{l \in T_i} A^{-p\left(\frac{q^2}{2}l^2+1\right)} \tilde{f}_l,$$

(3)

where $T_i = \{0, 2, \ldots, i - 1\}$ for odd $i$ and $T_i = \{1, 3, \ldots, i - 1\}$ for even $i$.

The idea of the proof is to leverage symmetric properties of the $\tilde{f}_l$ to give a presentation of $R_m^{-1}$ and bound its operator norm. The assumption that $(r, q) = 1$ is necessary for invertibility, as indicated by the following proposition.

**Proposition 4.3.** Suppose $r = 2m + 1 \geq 3$ is odd and not coprime to $q$. Then $R_m$ is singular.

The proofs of Propositions 4.2 and 4.3 will be given in Subsection 4.2.

### 4.1 Proof of Theorem 1.7

We can now proceed with the proof of Theorem 1.7, assuming Proposition 4.2.

**Proof of Theorem 1.7.** We begin with the necessary condition. Suppose $(r, q) = d > 1$. Then there are coprime $q', r'$ such that $q = dq'$ and $r = dr'$. We claim that row$(nd)$ of $RT_r(C_{p,q})$ consists of only zeros for each $n$. Suppose some $e_{q'd+1} = e_{kr'+j}$, where $0 \leq j \leq m$, reduces to $e_j = e_{nd}$. Then by Equation (1), $q'l - kr \pm 1 = nd$, which means $d(q'l - r'k - n) = \mp 1$, which is a contradiction. Similarly, if $e_{q'd+1} = e_{kr'+j}$, where $m + 1 \leq j < r$, reduces to $e_{r-j} = e_{nd}$. Then by Equation (2), $d((1+k)r' - q'l - n) = \pm 1$, which is also a contradiction. This means that row$(nd) = [0, \ldots, 0]$, thus $RT_r(C_{p,q})$ is singular.

For sufficiency, suppose $(r, q) = 1$. By Proposition 4.2, we can write $RT_r(C_{p,q})$ as a product of two diagonal matrices with an upper-triangular matrix and the change of basis $R_m$:  

\[ R_m := \{\tilde{f}_l\}_{l=0}^{m-1}, \] 

and let $R_m$ be the $(m \times m)$-matrix with columns corresponding to the reduced vectors $\tilde{f}_l$, for $l = 0, \ldots, m - 1$. In particular, $\tilde{f}_l$ corresponds to $\text{col}(l+1)$ of $R_m$, and the rows of $R_m$ correspond to the original orthonormal basis $\{e_1, \ldots, e_m\}$ spanning $RT_r(T^2)$. 

**Remark 4.1.** We note that $F_m$, $\tilde{F}_m$, and $R_m$ are also dependent on $p$ and $q$, but these dependencies are suppressed to avoid unwieldy notation.

The following proposition will be used to prove Theorem 1.7.

**Proposition 4.2.** Let $r = 2m + 1 \geq 3$ be coprime to $q$. Then $R_m$ is a change of basis from $\tilde{F}_m \to \{e_1, \ldots, e_m\}$ and the operator norm $||R_m^{-1}||$ grows at most polynomially in $m$. Moreover, for $i \in \{1, \ldots, m\}$,

$$RT_r(C_{p,q})(e_i) = A^{\frac{q^2}{2}(i^2-1)} \sum_{l \in T_i} A^{-p\left(\frac{q^2}{2}l^2+1\right)} \tilde{f}_l,$$

(3)

where $T_i = \{0, 2, \ldots, i - 1\}$ for odd $i$ and $T_i = \{1, 3, \ldots, i - 1\}$ for even $i$.

The idea of the proof is to leverage symmetric properties of the $\tilde{f}_l$ to give a presentation of $R_m^{-1}$ and bound its operator norm. The assumption that $(r, q) = 1$ is necessary for invertibility, as indicated by the following proposition.

**Proposition 4.3.** Suppose $r = 2m + 1 \geq 3$ is odd and not coprime to $q$. Then $R_m$ is singular.

The proofs of Propositions 4.2 and 4.3 will be given in Subsection 4.2.

### 4.1 Proof of Theorem 1.7

We can now proceed with the proof of Theorem 1.7 assuming Proposition 4.2.

**Proof of Theorem 1.7.** We begin with the necessary condition. Suppose $(r, q) = d > 1$. Then there are coprime $q', r'$ such that $q = dq'$ and $r = dr'$. We claim that row$(nd)$ of $RT_r(C_{p,q})$ consists of only zeros for each $n$. Suppose some $e_{q'd+1} = e_{kr'+j}$, where $0 \leq j \leq m$, reduces to $e_j = e_{nd}$. Then by Equation (1), $q'l - kr \pm 1 = nd$, which means $d(q'l - r'k - n) = \mp 1$, which is a contradiction. Similarly, if $e_{q'd+1} = e_{kr'+j}$, where $m + 1 \leq j < r$, reduces to $e_{r-j} = e_{nd}$. Then by Equation (2), $d((1+k)r' - q'l - n) = \pm 1$, which is also a contradiction. This means that row$(nd) = [0, \ldots, 0]$, thus $RT_r(C_{p,q})$ is singular.

For sufficiency, suppose $(r, q) = 1$. By Proposition 4.2, we can write $RT_r(C_{p,q})$ as a product of two diagonal matrices with an upper-triangular matrix and the change of basis $R_m$:  

\[ R_m := \{\tilde{f}_l\}_{l=0}^{m-1}, \] 

and let $R_m$ be the $(m \times m)$-matrix with columns corresponding to the reduced vectors $\tilde{f}_l$, for $l = 0, \ldots, m - 1$. In particular, $\tilde{f}_l$ corresponds to $\text{col}(l+1)$ of $R_m$, and the rows of $R_m$ correspond to the original orthonormal basis $\{e_1, \ldots, e_m\}$ spanning $RT_r(T^2)$.
Note that the columns of the middle upper triangular matrix correspond to the index sets $T_i$ of the sum in Equation (3). Inverting this product, we have

$$RT_r(C_{p,q})^{-1} = R_m \left( \begin{array}{ccccccc}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & A_p((2-1+\frac{q}{2}(2-1)^2) & \cdots & \vdots & \vdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\
0 & \cdots & 0 & A_p((m-1+\frac{q}{2}(m-1)^2) & \cdots & \vdots & \vdots \\
\end{array} \right) \times \left( \begin{array}{ccccccc}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & A_p\left(2^{2-1}\right) & \cdots & \vdots & \vdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\
0 & \cdots & 0 & A_p\left(m^2-1\right) & \cdots & \vdots & \vdots \\
\end{array} \right).$$

By Proposition 4.2, $\|R_m^{-1}\|$ grows at most polynomially in $m$, so it is bounded polynomially in $r$. For the total bound, observe that both of the diagonal matrices are isometries, and the upper triangular matrix has operator norm bounded above by a polynomial in $r$ by the Cauchy-Schwartz inequality.

4.2 Proof of Propositions 4.2 and 4.3

We give proofs of Propositions 4.2 and 4.3 in this subsection. The following definitions and lemmas will be useful in the proofs.

By the symmetries $e_{-i} = -e_i$ for any $i \geq 0$ and $e_{i+k}\cdot r = (-1)^k e_i$ for any $k \in \mathbb{Z}$, we may extend the definition of $f_l$ to all $l \in \mathbb{Z}$ using the following symmetries:
• \( f_l = e_{1+ql} + A^{2pl}e_{1-ql} \) for any \( l \in \mathbb{Z} \),

• \( f_{l+r} = (-1)^q f_l \) for any \( l \in \mathbb{Z} \), and

• \( f_1 = A^{2pl} f_{-l} \).

The following Lemma will be used to present \( R_{m^-1} \).

**Lemma 4.4.** Let \( r = 2m + 1 \geq 3 \) be coprime to \( q \), and let \( q^* \) be the inverse of \( q \) modulo \( r \). Then for \( l \in \{0, \ldots, m - 1\} \),

\[
e_{l+1} = \begin{cases} f_0 & \text{if } l = 0 \\ f_{q^*} & \text{if } l = 1 \\ f_{q^*l} + \sum_{k=1}^{[l/2]} A^{2pq^* (kl - \sum_{i=0}^{k-1} 2i)} f_{q^*(l-2k)} & \text{if } l > 1. \end{cases} \tag{4}
\]

Moreover, for \( i, j \in \{0, \ldots, r - 1\} \), \( q^*i \equiv q^*j \mod r \) if and only if \( i = j \).

**Proof.** Since \((r, q) = 1\), there is a unique \( q^* \in \mathbb{Z}_r \) such that \( qq^* \equiv 1 \mod r \). Using the symmetries of \( f_1 \) and substituting in \( q^*l \), we have

\[
e_{1+l} = f_{q^*l} - A^{2pq^*} e_{1-l} = f_{q^*l} + A^{2pq^*} e_{l-1} = f_{q^*l} + A^{2pq^*} e_{1+(l-2)}. \tag{5}
\]

We can then apply Equation (5) iteratively to express the \( e_i \)'s in terms of the \( f_j \)'s.

\[
e_{l+1} = f_{q^*l} + A^{2pq^*} f_{q^*(l-2)} + A^{2pq^* (2l-2)} f_{q^*(l-4)} + \cdots + A^{2pq^* \left( \left[ l/2 \right]l - \sum_{i=0}^{\left[ l/2 \right]-1} 2i \right)} f_{q^*(l-2\left[ l/2 \right])}
\]

for \( l \in \{2, \ldots, m - 1\} \). When \( l = 0 \), by definition, \( e_1 = f_0 \). When \( l = 1 \), Equation (5) yields that \( e_2 = f_{q^*} \). For any \( l \in \{0, \ldots, m - 1\} \), the iterative use of Equation (5) to express \( e_{l+1} \) terminates when the final term is a scalar multiple of either \( e_1 \) or \( e_2 \), depending on the parity of \( l \).

For the final statement, note that \((q^*, r) = 1\). This means there is a group isomorphism between the cyclic groups \( \{q^*k \mod r | k \in \mathbb{Z}_r \} \) and \( \mathbb{Z}_r \) sending the indices \( q^*k \mod r \) in Equation (4) to distinct \( j \) for \( j \in \{0, 1, \ldots, r - 1\} \). Since \( q^*k \mod r \) are distinct for \( k \in \mathbb{Z}_r \), this shows that \( q^*i \equiv q^*j \mod r \) if and only if \( i = j \).

In order to prove Proposition 4.2, we will use the following lemma. The proof of this lemma is given in Subsection 4.3.

**Lemma 4.5.** Suppose \( r = 2m + 1 \geq 3 \) is coprime to \( q \). Then

\[
\tilde{f}_m = \sum_{j=0}^{m-1} C_j f_j,
\]

where \( C_j \in \mathbb{C} \) such that \(|C_j| = 1\) for \( j \in \{0, \ldots, m - 1\} \).
Proof of Proposition 4.2. It suffices to show that $R_m$ is nonsingular, in which case $R_m$ corresponds to the basis transformation $\tilde{F}_m \to \{e_1, \ldots, e_m\}$. To establish nonsingularity, we will give a presentation of $R_m^{-1}$ by expressing $e_i$, for $i \in \{1, \ldots, m\}$, in terms of $f_j$ where $j \in \{0, \ldots m - 1\}$.

By Lemma 4.4, each $e_i$, for $i \in \{1, \ldots, m\}$, can be written in terms of $f_j$ where $j \in \mathbb{Z}$. These $f_j$'s reduce to $f_l$'s, where $l \in \{0, \ldots, m\}$, using the above symmetries. This means that $\text{Span}\{e_1, \ldots, e_m\}$ of dimension $m$ is contained in $\text{Span}\{f_0, \ldots, f_m\}$, a vector space of dimension at most $m + 1$.

Lemma 4.5 implies that $\tilde{f}_m \in \text{Span}\{f_0, \ldots, f_{m-1}\}$, which means that $\text{Span}\{f_0, \ldots, f_{m-1}\} = \text{Span}\{e_1, \ldots, e_m\}$. This means we may write

$$e_i = \sum_{j=0}^{r-1} B^i_j f_j,$$

where $B^i_j$ is either zero or a root of unity and the summands correspond to the reduction of each index modulo $r$. We remark that since $B^i_j$ is either zero or a root of unity, $|B^i_j| \leq 1$.

Now after applying the symmetry $f_l = A^{2pl}f_{-l} = A^{2pl}f_{r-l}$ for any $l > m$, we may express $e_i$ as

$$e_i = \sum_{j=0}^{m} (B^i_j + A^{2p(r-j)}B^i_{r-j}) f_j = \sum_{j=0}^{m} D^i_j f_j,$$

where $D^i_j = B^i_j + A^{2p(r-j)}B^i_{r-j}$ and $|D^i_j| \leq 2$. Additionally, by Lemma 4.5, we know that the coefficient of any summand of $f_m$ in terms of the basis $\{f_0, \ldots, f_{m-1}\}$ is $C^i_j$ with $|C^i_j| = 1$. This means we may write
\[ e_i = \left( \sum_{j=0}^{m-1} D_j^i f_j \right) + D_m^i f_m \]
\[ = \left( \sum_{j=0}^{m-1} D_j^i f_j \right) + D_m^i \left( \sum_{k=0}^{m-1} C_k^i f_k \right) \]
\[ = \sum_{j=0}^{m-1} \left[ D_m^i C_j^i + D_j^i \right] f_j. \]
\[ = \sum_{j=0}^{m-1} E_j^i f_j \]

where \( E_j^i = D_m^i C_j^i + D_j^i \). Note that since \(|D_j^i| \leq 2\) and \(|C_j^i| = 1\), we have
\[ |E_j^i| = |D_m^i C_j^i + D_j^i| \leq |D_m^i C_j^i| + |D_j^i| = |D_m^i||C_j^i| + |D_j^i| \leq 4. \]

Hence every entry of \( R_m^{-1} \) has modulus bounded above by 4. For any complex unit vector \( v = [v_0, \ldots, v_{m-1}]^T \) such that \(|v_i| \leq 1\) for \( i \in \{0, \ldots, m-1\} \), the Cauchy-Schwarz inequality implies that
\[
\|R_m^{-1}v\| = \left\| \left[ \sum_{i=0}^{m-1} E_0^i v_i, \ldots, \sum_{i=0}^{m-1} E_{m-1}^i v_i \right]^T \right\|
\leq \left( \sum_{i=0}^{m-1} |E_0^i v_i|^2 + \cdots + \sum_{i=0}^{m-1} |E_{m-1}^i v_i|^2 \right)^{\frac{1}{2}}
\leq \left( \sum_{i,j=0}^{m-1} |E_j^i|^2 |v_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i,j=0}^{m-1} |E_j^i|^2 \right)^{\frac{1}{2}}
\leq \left( \sum_{i,j=0}^{m-1} 4 \right) \leq \left( \sum_{i,j=0}^{m-1} 16 \right)^{\frac{1}{2}} = \left( 16m^2 \right)^{\frac{1}{2}} = 4m.
\]

This shows that
\[ ||R_m^{-1}v|| \leq O(m), \]
so the operator norm \( ||R_m^{-1}|| \) is bounded polynomially.

Lastly, by the Cabling Formula in Theorem 2.2 and the definition of \( f_l \), the coefficient of \( f_l \) in \( RT_r(C_{p,q})(e_i) \) is given by
\[
RT_r(C_{p,q})(e_i) = A^{\frac{i^2-1}{2}} \sum_{l \in T_i} A^{-p\left(\frac{i^2-1}{2}+l\right)} \tilde{f}_l,
\]
where \( T_i = \{0, 2, \ldots, i-1\} \) for odd \( i \) and \( T_i = \{1, 3, \ldots, i-1\} \) for even \( i \).
In order to prove Proposition 4.3, we establish the following definitions.

For \( 1 \leq l \leq m \), define \( f_l^{\pm} := e_{ql \pm 1} \). Observe that \( f_l = f_l^+ - A^{2pl} f_l^- \) for \( 1 \leq l \leq m \). In addition, define \( \hat{f}_l^{\pm} \) to be the quotient of \( f_l^{\pm} \) under the symmetries \( e_i = -e_i \) for any \( i \geq 0 \) and \( e_{i+kr} = (-1)^k e_i \) for any \( k \in \mathbb{Z} \). We will use the convention that \( f_0^+ = f_0 = e_1 \) and \( f_0^- = 0 \).

Recall that for each \( l \), \( ql \pm 1 = kr + j \) for some non-negative integers \( k, j \) where \( 0 \leq j < r \).

This means that up to sign, \( \hat{f}_l^{\pm} = \begin{cases} e_j = e_{ql - kr \pm 1} & 0 \leq j \leq m \\ e_{r-j} = e_{(k+1)r - ql \mp 1} & m + 1 \leq j < r. \end{cases} \) (6)

We can now prove Proposition 4.3.

**Proof of Proposition 4.3** We will repeat the argument given in the proof of Theorem 1.7. Suppose \( (r, q) = d > 1 \), then there are coprime \( q' \) and \( r' \) such that \( q = dq' \) and \( r = dr' \). We claim that \( row(nd) \) of \( R_m \) consists of only zeros for each \( n \). Suppose some \( \hat{f}_l^{\pm} = e_j = e_{nd} \), then \( ql - kr \pm 1 = nd \). This implies that \( d(q'l - r'k - n) = \mp 1 \) which is a contradiction. Similarly, if \( \hat{f}_l^{\pm} = e_{r-j} = e_{nd} \), then \( d((1+k)r' - ql - n) = \pm 1 \) which is also a contradiction. This means that \( row(nd) = [0, \ldots, 0] \), thus \( R_m \) is singular.

4.3 Proof of Lemma 4.5

In this subsection, we provide a proof for Lemma 4.5. We use the notation introduced in Subsection 4.2.

**Remark 4.6.** For the following arguments, we use the convention that equalities between vectors \( e_i \) are necessarily taken up to sign. This ultimately has no effect on the arguments for Proposition 4.2 and Theorem 1.7.

Recall \( R_m \) is the \((m \times m)\)-matrix with columns corresponding to \( \hat{F}_m = \{f_0, \ldots, f_{m-1}\} \). We also define \( S_m \) to be the \((m \times (m + 1))\)-matrix obtained by appending the column corresponding to \( f_m \) to \( R_m \). The following technical lemmas will be used in the proof of Lemma 4.5.

**Lemma 4.7.** Suppose \( r = 2m + 1 \geq 3 \) is coprime to \( q \), and let \( q^* \) be the multiplicative inverse of \( q \) in the ring \( \mathbb{Z}_r \). Then

(i) Each column of \( S_m \) has at most two nonzero entries. Moreover, for each column with two nonzero entries, their corresponding row indices differ by at most 2.

(ii) Let \( l^* := \begin{cases} q^* & \text{if } q^* \leq m \\ r - q^* & \text{if } q^* > m. \end{cases} \)

Then in \( S_m \), \( col(1) = [1, 0, \ldots, 0]^T \) and \( col(l^* + 1) = [0, D_{l^*}, 0, \ldots, 0]^T \) where \( D_{l^*} \) is a root of unity. Moreover, every other column of \( S_m \) has exactly two nonzero entries which are roots of unity.
Proof. Part (2): Each column of $S_m$ corresponds to the reduced vector $\tilde{f}_i$, $0 \leq l \leq m$. Since $f_i$ is a linear combination of at most two vectors in $Span\{e_1, e_{ql+1}, e_{ql-1}\}_{l=1}^{m-1}$, there are at most two nonzero entries in $col(l+1)$.

Now suppose the index of $f_i^+$ is $ql + 1 = kr + j$, where $0 \leq j < r$. Then the index of $f_i^-$ is $ql - 1 = kr + j - 2 = k'r + j'$, where either $(k', j') = (k-1, r+j-2)$ (for $j \in \{0, 1\}$) or $(k', j') = (k, j-2)$ (for $j \geq 2$). We split into cases:

- If $j = 0$, then $j' = r - 2$, $f_i^+ = e_0 = 0$, and $f_i^- = e_2$.
- If $j = 1$, then $j' = r - 1$, $f_i^+ = e_{ql-kr+1} = e_1$, and $f_i^- = e_{(k+1)r-ql+1} = e_1$. This implies that $l = \frac{k}{r}$. Since $l \in \mathbb{Z}$ and $r, q$ are coprime, $k = qn$, for some $n \geq 0$. However, if $n \geq 1$, we have $l \geq 1 + r > m$, which is a contradiction. Thus $n = 0$, so $l = 0$, corresponding to $col(1)$.
- If $2 \leq j \leq m$, then $\tilde{f}_i^+ = e_j \neq e_{j-2} = \tilde{f}_i^-$. If $j = m + 1$, then $j' = m - 1$ and $\tilde{f}_i^+ = e_m \neq e_{m-1} = \tilde{f}_i^-$. If $j = m + 2$, then $j' = m$ and $\tilde{f}_i^+ = e_{m-2} \neq e_m = \tilde{f}_i^-$. If $m + 3 \leq j < r$, then $\tilde{f}_i^+ = e_{r-j} \neq e_{r-j+2} = \tilde{f}_i^-$. This implies that the row indices of the nonzero entries in each column differ by at most two for every column except $col(1)$. In particular, the only case where the row indices differ by exactly 1 occurs when $j = m + 1$.

Part (1): Since $e_{-i} = -e_i$, we have $f_i = e_{1+ql} + A^{ql}e_{1-ql}$ for $1 \leq l \leq m$. Note that $col(l+1)$ has exactly one nonzero entry if and only if one of the following occurs:

1. Either $1 + ql$ and $1 - ql$ are equal or opposite modulo $r$.
2. Either $1 + ql$ or $1 - ql$ vanishes modulo $r$.

Case (1) occurs if and only if $l = 0$, corresponding to $\tilde{f}_0 = f_0 = e_0$. In this case, $col(1) = [1, 0, \ldots, 0]^T$.

Case (2) occurs if and only if either $l = q^*$ or $l = -q^*$ modulo $r$. Define

$$l^* := \begin{cases} q^* & \text{if } q^* \leq m \\ r - q^* & \text{if } q^* > m. \end{cases}$$

Note that if $ql \pm 1$ vanishes, then $|ql \pm 1| = 2$. Define $D_{l^*}$ to be the coefficient of the vector $e_{|ql\pm1|} = e_2$ obtained from Equation (5). This means that $col(l^* + 1)$ is the unique column with exactly one nonzero entry except for $col(1)$.

Finally, the conclusion follows from the uniqueness of $l^*$ and Part (2). \(\square\)

The second technical lemma makes use of Lemma 4.7 in its proof.

**Lemma 4.8.** Suppose $r = 2m + 1 \geq 3$ is coprime to $q$. Then
(i) Each row of $S_m$ has exactly two nonzero entries.

(ii) There is a unique $l'$, $1 \leq l' \leq m$, such that $\text{col}(l' + 1) = [0, \ldots, 0, D_{l'}, E_{l'}]^T$, where $D_{l'}, E_{l'}$ are roots of unity.

The following lemma will be useful in the proof of Lemma 4.8.

**Lemma 4.9.** Suppose $r \geq 3$ is coprime to $q$, and let $g_i^\pm := ql - kr \pm 1$ and $h_i^\pm = (1 + k)r - ql \mp 1$. Then for $0 \leq l_1, l_2 \leq m$ with $l_1 \neq l_2$,

(i) $g_i^\pm = g_i^\mp$, $g_i^\pm = h_i^\mp$, and $h_i^\pm = h_i^\mp$ do not have integer solutions,

(ii) $g_i^\pm = g_i^\mp$, $g_i^\pm = h_i^\mp$, and $h_i^\pm = h_i^\mp$ may each have integer solutions.

**Proof.** Note $g_i^\pm$ and $h_i^\pm$ encode the two families of indices of the reduced vectors $\tilde{f}_i^\pm$ given in Equation (6). There are six equations relating pairs of expressions in $\{g_i^+, g_i^-, h_i^+, h_i^-\}$.

**Part (i):** This follows from the fact that $(r, q) = 1$ and the bounds on $l_1$ and $l_2$. We show the case $g_i^+ = g_i^-$ and note that the other two cases follow analogously. Assume for distinct $l_1, l_2 \in \{0, \ldots, m\}$ and $k_1, k_2 \in \mathbb{Z}$ that $ql_1 - k_1r = 1 = ql_2 - k_2r \pm 1$. This implies

$$k_1 - k_2 = \frac{q(l_1 - l_2)}{r} \in \mathbb{Z}.$$

Since $l_1 \neq l_2$ and $(r, q) = 1$, $l_1 - l_2$ must have a nontrivial factor of $r$, which contradicts the bounds on $l_1$ and $l_2$.

**Part (ii):** We have the following:

- $g_i^\pm = g_i^\mp$ if and only if $q(l_1 - l_2) = (k_1 - k_2)r \mp 2$,
- $g_i^\pm = h_i^\mp$ if and only if $q(l_1 + l_2) = (1 + k_1 + k_2)r \mp 2$, and
- $h_i^\pm = h_i^\mp$ if and only if $q(l_1 - l_2) = (k_1 - k_2)r \mp 2$.

All three of these equations may have integer solutions for $l_1, l_2 \in \{0, \ldots, m\}$. \qed

**Proof of Lemma 4.8.** We will use the same notation as in the proof of Lemma 4.7 and in Lemma 4.9.

**Part (i):** It is a corollary of Lemma 4.9 that every row of $S_m$ has at most two nonzero entries. In particular, let $(l_1, l_2)$ be an integral solution to one of the equations of Lemma 4.9 Part (ii). Suppose $l_3 \in \{0, \ldots, m - 1\}$ is such that $(l_1, l_3)$ and $(l_2, l_3)$ are both solutions to equations in Lemma 4.9 Part (ii). Then by Lemma 4.9 Part (ii), either $l_3 = l_1$ or $l_3 = l_2$.

Note that by Lemma 4.7 Part (ii), $S_m$ has exactly $2m$ nonzero entries since there are 2 in each column other than $\text{col}(1)$ and $\text{col}(l^* + 1)$, which each have exactly 1. This means that every row of $S_m$ must have exactly 2 nonzero entries.

**Part (ii):** In the proof of Lemma 4.7 Part (ii), we saw that the only value of $j$ corresponding to a column with the nonzero row entry indices differing by 1 is $j = m + 1$. By Part (ii), $\text{row}(m)$ of $S_m$ has exactly 2 nonzero entries. This implies that there are some $l_1, l_2$
such that \( \text{col}(l_1 + 1) \) has nonzero entries in \( \text{row}(m) \) and \( \text{row}(m - 1) \) and \( \text{col}(l_2 + 1) \) has nonzero entries in \( \text{row}(m) \) and \( \text{row}(m - 2) \). Take \( l' = l_1 \). Finally, define \( D_V \) and \( E_V \) to be the coefficients of the vectors \( e_{m-1} \) and \( e_m \) defined by Equation (3), respectively. Note that if \( m = 2, l_2 = l^* \) and \( \text{col}(l_2 + 1) \) has only 1 nonzero entry.

Lastly, we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. The last column \( \text{col}(m + 1) \) of the matrix \( S_m \) represents the reduced vector \( f_m \) written in terms of the basis \( \{e_1, \ldots, e_m\} \). We will prove Lemma 4.5 by showing that \( \text{col}(m + 1) \) can be written as a linear combination of the first \( m \) columns. From this linear combination, we will see that the coefficients will have the required bounds from the statement.

We claim that \( \text{col}(m + 1) \) of \( S_m \) can be written as a linear combination of elements in \( \{f_0, \ldots, f_{m-1}\} \). From Lemma 4.7 Part (i), if \( l^* = m \), then \( \text{col}(m+1) \) has exactly one nonzero entry, and if \( l^* < m \), then \( \text{col}(m+1) \) has exactly two nonzero entries.

Case 1:

We first consider the case \( l^* = m \). Here, the nonzero entry of \( \text{col}(m+1) \) lies in \( \text{row}(2) \). This implies that \( \tilde{f}_m \in \text{Span}\{e_1, \ldots, e_m\} \) has a scalar of \( e_2 \) as a summand. By Lemma 4.8 Part (i), we know that there is exactly one other nonzero entry in \( \text{row}(2) \) in some column \( j_1 \). From the argument of Lemma 4.7 Part (ii), there exists a nonzero entry in \( \text{row}(4) \) of \( \text{col}(j_1) \). Lemma 4.8 Part (ii) implies there exists a nonzero entry in some column \( j_2 \) and \( \text{row}(4) \). From the argument of Lemma 4.7 Part (ii), there exists a nonzero entry in \( \text{row}(6) \) of \( \text{col}(j_2) \). Again, we pick the other nonzero entry of \( \text{row}(6) \) which lies in some column \( j_3 \). Note that \( \text{col}(j_3) \) cannot be equal to any of the previous columns. If it were a previous column, it would contradict our bound on the number of nonzero entries in a column. We continue this iteration until we reach either \( \text{row}(m - 1) \) or \( \text{row}(m) \), depending on the parity of \( m \).

If \( m - 1 \) is even, by Lemma 4.8 Part (i), the next corresponding row with a nonzero entry will be \( \text{row}(m) \) where \( m \) is odd. Similarly, if \( m \) is even, by Lemma 4.8 Part (ii), the next corresponding row with a nonzero entry will be \( \text{row}(m - 1) \) where \( m - 1 \) is odd. Now when we continue the algorithm, our subsequent row indices will be odd and decrease by 2 until we reach \( \text{row}(1) \). By Lemma 4.8 Part (ii) and Lemma 4.7 Part (ii), there exists a nonzero entry in \( \text{row}(1) \) of \( \text{col}(1) \), and it is the only nonzero entry in \( \text{col}(1) \). Since every entry of our matrix is a root of unity by Lemma 4.7 Part (ii) and terminates at \( \text{row}(1) \), scalars by roots of unity of the columns appearing in our sequence gives \( f_m \) as a linear combination of elements of \( \{f_0, \ldots, f_{m-1}\} \) where all coefficients are roots of unity.

Case 2:

Now suppose \( \text{col}(m+1) \) has exactly two nonzero entries. We denote the row indices of these entries by \( i_1^- \) and \( i_1^+ \), where \( i_1^- < i_1^+ \). By Lemma 4.8 Part (i), \( \text{row}(i_1^-) \) has another nonzero entry in some other column \( j_1^- \). Similarly, \( \text{row}(i_1^+) \) has another nonzero entry in some column \( j_1^+ \). We make the following claim, which we prove at the end.

Claim: \( j_1^- \neq j_1^+ \).

We will proceed similarly to the first case. Consider the column \( \text{col}(j_1^+) \), which has exactly two nonzero entries and cannot correspond to either \( \text{col}(1) \) or \( \text{col}(l^* + 1) \) since \( i_1^+ > 2 \). By
Lemma 4.8 Part (ii) and the claim, there exists another nonzero entry in some row $i^*_2$ of \( col(j^*_j) \) such that \((i^*_2 - i^*_1) \in \{-1, 2\}\). The case when \((i^*_2 - i^*_1) = -1\) corresponds to \(i^*_1 = m\), and the case when \((i^*_2 - i^*_1) = 1\) corresponds to \(i^*_1 = m - 1\). We now implement the same argument as the case with one entry in \(col(m + 1)\). Note that, in this procedure, we do not utilize any rows with index less than \(i^*_1\) with the same parity as \(i^*_1\). If \(i^*_1 = m - 1\) and \(i^*_1 = m\), we will have different parities. In the other case, \(i^*_1\) will have the parity of \(i^*_k\) until we have a \(k\) such that \((i^*_k - i^*_k) \in \{-1, 1\}\). This implies that for all \(k' \geq k\), \(i^*_k\) will have opposite parity to \(i^*_1\).

We now follow the same algorithm beginning with row \(i^*_1\). By the claim, the indices of our subsequent rows \(i^*_k\) must be decreasing. Otherwise, this would contradict Lemma 4.8 Part (ii).

Since both cases in total utilize every row exactly once, \(f_m\) is given by a linear combination of elements of \(\{f_0, \ldots, f_{m-1}\}\) where, by Lemma 4.7 Part (ii), all coefficients are roots of unity.

**Proof of Claim:**

It now suffices to prove that \(j^*_1 \neq j^*_1\). By contradiction, let us assume that \(j^*_1 = j^*_1\), and we will denote \(i^*_1 = i^*_1\).

If \(i^*_1 = m\), then either \(i^*_1 = m - 1\) or \(i^*_1 = m - 2\). If \(i^*_1 = m - 1\), then since \(j^*_1 = j^*_1\), we will have two columns with nonzero entries in the last two rows. This contradicts Lemma 4.8 Part (ii), which states that there is a unique such column. If \(i^*_1 = m - 2\), then there are two distinct columns with nonzero entries in row \((m - 2)\) and row \((m)\). By Lemma 4.8 Part (ii), there must exist a different column with nonzero entries in row \((m - 1)\) and row \((m)\), which contradicts that being at most 2 entries in row \((m)\).

If \(i^*_1 = m - 1\), then \(i^*_1 < m - 1\), and there are no other columns with nonzero entries in row \((m - 1)\) besides \(col(j^*_1)\) and \(col(m + 1)\). By Lemma 4.8 Part (ii), there must exist a different column with nonzero entries in row \((m - 1)\) and row \((m)\), which contradicts that being at most 2 entries in row \((m - 1)\).

In the general case, we assume \(i^*_1 \leq m - 2\), and we will define \(i^*_2 = i^*_1 + 2\). Since \(j^*_1 = j^*_1\), \(col(j^*_1)\) and \(col(m + 1)\) already have two nonzero entries. Since \(i^*_2 > 2\), these entries cannot be in either \(col(1)\) or \(col(l^*_1 + 1)\) since they only have entries in the first two rows. This implies that the columns which correspond to nonzero entries in row \((i^*_2)\) must have exactly two nonzero entries in some columns \(col(j^*_2)\) and \(col(j^*_2)\) such that \(j^*_2, j^*_2 \not\in \{j^*_1, m + 1\}\).

Since row \((i^*_1)\) has two nonzero entries in \(col(j^*_1)\) and \(col(m + 1)\), the other nonzero entries in \(col(j^*_1)\) and \(col(j^*_1)\) must be in some row \((i^*_3)\), where \(i^*_3 - i^*_2 \in \{-1, 1, 2\}\).

- If \(i^*_3 - i^*_2 = -1\), we have \(i^*_2 = m\) and \(i^*_3 = m - 1\). Here, we reach the same contradiction as when \(i^*_1 = m\) and \(i^*_1 = m - 1\).
- If \(i^*_3 - i^*_2 = 1\), then \(i^*_2 = m - 1\) and \(i^*_3 = m\). This gives the same contradiction as when \(i^*_1 = m\) and \(i^*_1 = m - 1\).
- If \(i^*_3 - i^*_2 = 2\) with \(i^*_2 = m - 2\) and \(i^*_3 = m\), then our argument is the same as when \(i^*_1 = m\) and \(i^*_1 = m - 2\).

Finally, we consider when \(i^*_3 - i^*_2 = 2\) and \(i^*_3 \neq m\). In this case, we can continue to iterate the same algorithm until we reach the same contradictions.
5 Further Directions

The primary approach of this paper utilizes the invertibility of the operator $RT_r$ on the cable space $C_{p,q}$ as well as a polynomial bound on its operator norm. The same methodology could apply in the context for the operator $RT_r$ for other cable spaces.

Although the technique may apply in the case when the cable space has positive simplicial volume, a more natural approach would be to generalize our argument to other cable spaces with simplicial volume zero. For example, we may consider the manifold defined as follows. Let $N = \Sigma_{g,2} \times S^1$ where $\Sigma_{g,2}$ is a orientable compact genus $g$ surface with 2 boundary components. Now let $\{x_i\}_{i=1}^m \subset \Sigma_{g,2}$ such that $\{x_i\}_{i=1}^m \times S^1$ is a collection of $m$ vertical fibers in $N$. We define the Seifert cable space $C(s_1, \ldots, s_m)$ where $s_i = \frac{p_i}{q_i} \in \mathbb{Q}$ to be the manifold obtained by performing $s_i$-Dehn surgery along the $i$-th vertical fiber in $N$.

If an analogous result to Theorem 1.7 holds for the Seifert cable space $C(s_1, \ldots, s_m)$, the corresponding Theorem 1.2 will also follow as well as its applications to Conjecture 1.4. Similar to the constraint of Theorem 1.7 where $r$ and $q$ must be coprime, the analogous result for the Seifert cable space may require a related caveat. This leads to the following concluding question.

**Question 5.1.** Is $RT_r(C(s_1, \ldots, s_m))$ invertible when $r$ is sufficiently large and coprime to every $q_i$?

References

[1] G. Belletti. The maximum volume of hyperbolic polyhedra. *Trans. Amer. Math. Soc.*, 374(2):1125–1153, 2021.

[2] G. Belletti, R. Detcherry, E. Kalfagianni, and T. Yang. Growth of quantum 6j-symbols and applications to the volume conjecture. *J. Differential Geom.*, 120(2):199–229, 2022.

[3] R. Benedetti and C. Petronio. On Roberts’ proof of the Turaev-Walker theorem. *J. Knot Theory Ramifications*, 5(4):427–439, 1996.

[4] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[5] Q. Chen and T. Yang. Volume conjectures for the Reshetikhin-Turaev and the Turaev-Viro invariants. *Quantum Topol.*, 9(3):419–460, 2018.

[6] R. Detcherry. Growth of Turaev-Viro invariants and cabling. *J. Knot Theory Ramifications*, 28(14):1950041, 8, 2019.
[7] R. Detcherry and E. Kalfagianni. Gromov norm and Turaev-Viro invariants of 3-manifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1363–1391, 2020.

[8] R. Detcherry, E. Kalfagianni, and T. Yang. Turaev-Viro invariants, colored Jones polynomials, and volume. *Quantum Topol.*, 9(4):775–813, 2018.

[9] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, 56:5–99 (1983), 1982.

[10] W. H. Jaco and P. B. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.*, 21(220):viii+192, 1979.

[11] K. Johannson. *Homotopy equivalences of 3-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[12] R. M. Kashaev. The hyperbolic volume of knots from the quantum dilogarithm. *Lett. Math. Phys.*, 39(3):269–275, 1997.

[13] S. Kumar. Fundamental shadow links realized as links in $S^3$. *Algebraic & Geometric Topology*, 21(6):3153–3198, 2021.

[14] S. Kumar and J. M. Melby. Asymptotic additivity of the Turaev-Viro invariants for a family of 3-manifolds. *Journal of the London Mathematical Society*, to appear, 2022.

[15] H. R. Morton. The coloured Jones function and Alexander polynomial for torus knots. *Math. Proc. Cambridge Philos. Soc.*, 117(1):129–135, 1995.

[16] H. Murakami and J. Murakami. The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.*, 186(1):85–104, 2001.

[17] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv preprint at arXiv:math/0211159*, 2002.

[18] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *arXiv preprint at arXiv:math/0307245*, 2003.

[19] G. Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint at arXiv:math/0303109*, 2003.

[20] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

[21] J. Roberts. Skein theory and Turaev-Viro invariants. *Topology*, 34(4):771–787, 1995.

[22] W. P. Thurston. *The geometry and topology of three-manifolds*. Princeton University Math Department Notes, 1979.
[23] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[24] V. G. Turaev and O. Y. Viro. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31(4):865–902, 1992.

[25] K. H. Wong. Asymptotics of some quantum invariants of the Whitehead chains. *arXiv preprint at arXiv:1912.10638*, 2019.

[26] K. H. Wong. Volume conjecture, geometric decomposition and deformation of hyperbolic structures. *arXiv preprint at arXiv:1912.11779*, 2020.

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