Construction of Boundary Conditions for Hyperbolic Relaxation Approximations
II: Jin-Xin Relaxation Model

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March 9, 2022

Abstract

This is our second work in the series about constructing boundary conditions for hyperbolic relaxation approximations. The present work is concerned with the one-dimensional linearized Jin-Xin relaxation model, a convenient approximation of hyperbolic conservation laws, with non-characteristic boundaries. Assume that proper boundary conditions are given for the conservation laws. We construct boundary conditions for the relaxation model with the expectation that the resultant initial-boundary-value problems are approximations to the given conservation laws with the boundary conditions. The constructed boundary conditions are highly non-unique. Their satisfaction of the generalized Kreiss condition is analyzed. The compatibility with initial data is studied. Furthermore, by resorting to a formal asymptotic expansion, we prove the effectiveness of the approximations.

Keywords: Hyperbolic relaxation systems; Boundary conditions; Kreiss condition; Compatibility of initial and boundary data; Energy estimate.

1 Introduction

This is our second work in the series about constructing boundary conditions (BCs) for hyperbolic relaxation systems, which are an important class of partial differential equations. They describe a large number of various non-equilibrium phenomena. Important examples arise in chemically reactive flows [9], the kinetic theory [6, 10, 15, 19], compressible viscoelastic flows [8, 28], traffic flows [2, 23], thermal non-equilibrium flows [21] and so on.

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On the other hand, relaxation systems also arise as convenient approximations of hyperbolic conservation laws \[1, 5, 14\], say
\[
\partial_t u + \partial_x f(u) = 0,
\]
where \( u = u(x, t) \in \mathbb{R}^n \). A typical example is the Jin-Xin relaxation model \[14\]
\[
\begin{align*}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + a \partial_x u &= - \frac{1}{\epsilon} (v - f(u)),
\end{align*}
\]
where \( a \) is a positive constant and \( \epsilon \) is a small positive parameter called the relaxation rate. This model can provide novel numerical schemes to simulate shock waves without using Riemann solvers.

When the conservation laws (1.1) are given in a spatial domain with boundaries, say \( x > 0 \), proper BCs are needed at the boundaries for the relaxation model (1.2) to play its role. To be precise, we notice that the coefficient matrix of the relaxation model has \( n \) positive eigenvalues. According to the classical theory for hyperbolic equations \[3\], \( n \) BCs are needed at the boundary \( x = 0 \). On the other hand, the number of the given BCs for the hyperbolic conservation laws is equal to the number of positive eigenvalues of \( \frac{\partial f(u)}{\partial u} \), which is less than \( n \) in general. Thus, new BCs are required. This obvious question has not been resolved for a long time. The present paper attempts to answer this question for the one-dimensional relaxation model (1.2).

Like the first paper in this series \[29\], this work assumes that the BCs for the conservation laws are given and satisfies the Kreiss condition \[13\]. Such an assumption is reasonable for the conservation laws are classical and many mathematically correct and physically-based BCs thereof are available. In addition, for simplicity we only consider the case where the spatial domain is the half-space \( x > 0 \). According to \[17\], such a domain is representative. Furthermore, we assume that the conservation laws are linear and the boundary \( x = 0 \) is non-characteristic for the conservation laws. The corresponding nonlinear and/or multi-dimensional problems with or without characteristic boundaries are more challenging. They are our on-going project.

Under the above circumstance, the goal of this paper is to construct proper BCs for the relaxation model so that, as the relaxation rate is small, the resultant initial-boundary-value problems are good approximations to the conservation laws with the given BCs. For this purpose, we firstly resort to asymptotic expansions to obtain certain algebraic relations. The construction is based on these relations and the BC theory developed in \[26\]. It is highly non-unique. In order to show the effectiveness of the constructed BCs, we partly show that they fulfill the generalized Kreiss condition (GKC) \[26\], which is essentially necessary for the convergence when \( \epsilon \) goes to zero. Furthermore, we prove the convergence for initial data compatible with the constructed BCs.

At this point, we mention that a difficulty in this work is to verify the GKC. The first article \[29\] in this series does not directly verify the GKC but proves the strict dissipativeness of the constructed BCs. Other related works in literature all assume that the BCs are prescribed for the hyperbolic relaxation systems \[1, 7, 16, 20, 22, 24, 25\]. By contrast, the BCs in the present work are not given.

This paper is organized as follows. Section 2 contains some preliminaries and a formal asymptotic expansion. The detailed construction of BCs or the main result of this paper is presented in Section 3. In Section 4 the GKC is reviewed, while its verification is given in
Section 5. Section 6 is devoted to the compatibility of the constructed BCs with initial data. In Section 7, the formal asymptotic solution is constructed. The effectiveness is showed with error estimates by the energy method and Laplace transformation in Section 8. A special case is discussed in the appendix.

2 Preliminaries

We start with the exact equations to be studied in this paper. The one-dimensional linear hyperbolic system of conservation laws reads as

$$\partial_t u + F \partial_x u = 0, \quad x > 0, t > 0.$$  (2.1)

The hyperbolicity means that the coefficient matrix $F$ can be real diagonalized, that is, there is an invertible matrix $T$ such that

$$T^{-1} FT = \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n).$$

We assume that $F$ is invertible, which corresponds to the assumption that the boundary $x = 0$ is non-characteristic. Let $\lambda_1, \cdots, \lambda_l > 0, \lambda_{l+1}, \cdots, \lambda_n < 0$. According to the classical theory [3], $l$ BCs of the form

$$\hat{B} u(0, t) = \hat{b}(t)$$  (2.2)

are prescribed at $x = 0$. Here $\hat{B}$ is an $l \times n$-matrix such that $\hat{B} R_1^U$ is invertible, where $R_1^U$ consists of the first $l$ columns of $T$.

The corresponding Jin-Xin relaxation model is

$$\partial_t u + \partial_x v = 0,$$

$$\partial_t v + \hat{A} \partial_x u = -\frac{1}{\epsilon} (v - Fu).$$

This is more general than that in [12] for the matrix $\hat{A}$ has the form

$$\hat{A} = T \text{diag}(a_1, \cdots, a_n) T^{-1}$$

with $a_j > 0$, which includes the case $\hat{A} = a I_n$ and implies $F \hat{A} = \hat{A} F$. Let $p = v - Fu$. The relaxation model above can be rewritten as

$$\begin{pmatrix} u \\ p \end{pmatrix}_t + \begin{pmatrix} F & I_n \\ \hat{A} - F^2 & -F \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}_x = \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}. $$  (2.3)

Here and below, $I_k$ is the identity matrix of order $k$.

For this small parameter problem, we seek the following formal asymptotic solutions

$$\begin{pmatrix} u \epsilon \\ p \epsilon \end{pmatrix}(x, t) = \begin{pmatrix} \bar{u} \\ \bar{p} \end{pmatrix}(x, t; \epsilon) + \begin{pmatrix} \mu \\ \nu \end{pmatrix}(x/\epsilon, t; \epsilon).$$  (2.4)

Here the first term is the outer solution

$$\begin{pmatrix} \bar{u} \\ \bar{p} \end{pmatrix}(x, t; \epsilon) = \begin{pmatrix} \bar{u}_0 \\ \bar{p}_0 \end{pmatrix}(x, t) + \epsilon \begin{pmatrix} \bar{u}_1 \\ \bar{p}_1 \end{pmatrix}(x, t),$$  (2.5)
while the second term is the boundary-layer correction
\[
\left( \begin{array}{c}
\mu \\
\nu
\end{array} \right)(\xi, t; \epsilon) = \left( \begin{array}{c}
\mu_0 \\
\nu_0
\end{array} \right)(\xi, t) + \epsilon \left( \begin{array}{c}
\mu_1 \\
\nu_1
\end{array} \right)(\xi, t)
\]
with \( \xi = x/\epsilon \). As the boundary-layer corrections, they satisfy the matching conditions
\[
\mu_j(\infty, t) = \nu_j(\infty, t) = 0, \quad j = 0, 1.
\]

The outer solution asymptotically satisfies the relaxation system (2.3). We substitute the expansion (2.5) into the equations in (2.3) and equate the coefficients of \( \epsilon^k \) to obtain
\[
\bar{p}_0 = 0,
\]
\[
\bar{u}_{0t} + F\bar{u}_{0x} = 0,
\]
\[
\bar{p}_1 = -(\bar{A} - F^2)\bar{u}_{0x},
\]
\[
\bar{u}_{1t} + F\bar{u}_{1x} = -\bar{p}_{1x}.
\]
Similarly, substituting the corrections (2.6) into the equations in (2.3) and equating the coefficients of \( \epsilon^k \) for \( k = \{-1, 0\} \), we get
\[
F\mu_0 + \nu_0 = 0,
\]
\[
(\bar{A} - F^2)\mu_0 - F\nu_0 = -\nu_0,
\]
\[
\mu_{0t} + F\mu_{1x} + \nu_{1x} = 0,
\]
\[
\nu_{0t} + (\bar{A} - F^2)\mu_{1x} - F\nu_{1x} = -\nu_1.
\]

Since \( \mu_0(\infty, t) = \nu_0(\infty, t) = 0 \) in (2.7), we integrate the equation in (2.12) from \( \xi \) to \( \infty \) to obtain
\[
\mu_0 = -F^{-1}\nu_0.
\]
Substituting this into (2.13) and integrating from \( \xi \) to \( \infty \), we get the equation for \( \nu_0 = \nu_0(\xi, t) \):
\[
\nu_0 = F\bar{A}^{-1}\nu_0.
\]
Similarly, we integrate the equation in (2.14) from \( \xi \) to \( \infty \) to get
\[
\mu_1 = -F^{-1}\nu_1 + F^{-1} \int_{\xi}^{\infty} \mu_{0t}(s, t) ds.
\]
Finally, with (2.14) and (2.15) we deduce that \( \nu_1 = \nu_1(\xi, t) \) satisfies
\[
\nu_{1x} = F\bar{A}^{-1}\nu_1 + F^{-1}\nu_{0t}.
\]
Consequently, we derive the equations for the expansion coefficients \( \bar{u}_0, \bar{u}_1, \nu_0 \) and \( \nu_1 \). To determine them and thereby the expansion, proper boundary and initial conditions are needed. This will be discussed in Section 7.
3 Construction of Boundary Conditions

In this section, we construct BCs of the form

\[ B \begin{pmatrix} u \\ p \end{pmatrix}(0, t) = b_\epsilon(t) \]  

(3.1)

for the relaxation system (2.3), where \( B = (B_u, B_p) \) is a constant matrix and

\[ b_\epsilon(t) = b_0(t) + \epsilon b_1(t) + \epsilon^2 b_2(t). \]

Notice that coefficient matrix

\[ A := \begin{pmatrix} F & I_n \\ \bar{A} - F^2 & -F \end{pmatrix} \]

for the relaxation system has \( n \) positive eigenvalues \( \sqrt{\alpha_j} \) (\( j = 1, 2, \cdots, n \)). According to the classical theory [3], \( n \) BCs should be given for the relaxation system. Therefore, the boundary matrix \( B \) should be a full-rank \( n \times 2n \)-matrix. In what follows, by a boundary matrix we always mean that it is full-rank.

Our construction bases on the expectation that the formal asymptotic solution (2.4)-(2.6) satisfies the BCs in (3.1) with \( \epsilon = 0 \):

\[ B \begin{pmatrix} \bar{u}_0(0, t) + \mu_0(0, t) \\ \bar{p}_0(0, t) + \nu_0(0, t) \end{pmatrix} = b_0(t). \]

(3.2)

From (2.8) and (2.16) it follows that

\[ (B_u, B_p) \begin{pmatrix} \bar{u}_0(0, t) - F^{-1}\nu_0(0, t) \\ \nu_0(0, t) \end{pmatrix} = b_0(t). \]

(3.3)

In addition, it is expected that \( \bar{u}_0(x, t) \) is the solution to the conservation laws (2.1) with the BC (2.2) and certain initial data. Therefore, we require

\[ \hat{B}\bar{u}_0(0, t) = \hat{b}(t). \]

(3.4)

With (3.2) and (3.3), we construct the boundary matrix \( B \) and \( b_0(t) \), while \( b_1(t) \) and \( b_2(t) \) will be constructed in Section 6 for compatibility of boundary and initial data.

When \( l = n \), the coefficient matrix \( F \) in (2.1) has \( n \) positive eigenvalues and the boundary matrix \( \hat{B} \) in (2.2) is invertible, say \( \hat{B} = I_n \). The coefficient matrix \( F\bar{A}^{-1} \) in (2.17) has only positive eigenvalues and therefore \( \nu_0 = 0 \) is the unique bounded solution thereof. Thus, it is natural to choose

\[ B_u = \hat{B} = I_n, \quad B_p \text{ arbitrary } \quad \text{and} \quad b_0(t) = \hat{b}(t). \]

(3.4)

For \( l < n \), we recall that

\[ T^{-1}FT = \Lambda \triangleq \begin{pmatrix} \Lambda^+ \\ \Lambda^- \end{pmatrix} \]

with

\[ \Lambda^+ = \text{diag}(\lambda_1, \cdots, \lambda_l), \quad \Lambda^- = \text{diag}(\lambda_{l+1}, \cdots, \lambda_n). \]
Referring to this partition, we set $T = (R_1^U, R_1^S)$ and $T^{-1} \bar{u}_0(0, t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$. Then we have
\[
\bar{u}_0(0, t) = T \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = (R_1^U, R_1^S) \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = R_1^U \alpha_+(t) + R_1^S \alpha_-(t). 
\tag{3.5}
\]

Since $\hat{B}R_1^U$ is assumed to be invertible, the BCs with the relation can be rewritten as
\[
\alpha_+(t) = H \alpha_-(t) + J(t), 
\tag{3.6}
\]
where
\[
H = -(\hat{B}R_1^U)^{-1} \hat{B}R_1^S, \quad J(t) = (\hat{B}R_1^U)^{-1} b(t).
\]

Moreover, the relation becomes
\[
(B_u R_1^U, B_p - B_u F^{-1}) \begin{pmatrix} \alpha_+(t) \\ \nu_0(0, t) \end{pmatrix} = b_0(t) - B_u R_1^S \alpha_-(t). 
\tag{3.7}
\]

In addition, recall the equation for $\nu_0$ with $F \bar{A}^{-1} = T \text{diag}(\frac{\lambda_1}{a_1}, \cdots, \frac{\lambda_n}{a_n}) T^{-1}$. For its solution $\nu_0(\xi, t)$ to be bounded, the initial value $\nu_0(0, t)$ must fulfill
\[
L_1^U \nu_0(0, t) = 0, 
\tag{3.8}
\]
where $L_1^U$ consists of the first $l$ rows of $T^{-1} = \begin{pmatrix} L_1^U \\ L_1^S \end{pmatrix}$. Combining this with (3.7), we have
\[
\begin{pmatrix} B_u R_1^U \\ 0 \end{pmatrix} \begin{pmatrix} B_\nu - B_u F^{-1} \\ L_1^U \end{pmatrix} \begin{pmatrix} \alpha_+(t) \\ \nu_0(0, t) \end{pmatrix} = \begin{pmatrix} b_0(t) - B_u R_1^S \alpha_-(t) \\ 0 \end{pmatrix}. 
\]

Referring to Lemma 3.4 in [26], we know that the matrix
\[
\begin{pmatrix} B_u R_1^U \\ 0 \end{pmatrix} \begin{pmatrix} B_p - B_u F^{-1} \\ L_1^U \end{pmatrix}
\]
is invertible provided that the boundary matrix in (3.1) fulfills the generalized Kreiss condition (GKC) proposed in [26]. Then we can obtain $\nu_0(0, t)$ by solving the above algebraic equations. Particularly, $\nu_0(0, t)$ can be expressed as
\[
\nu_0(0, t) = C \alpha_-(t) + D(t) 
\tag{3.9}
\]
with $C$ an $n \times (n - l)$ parameter matrix and $D(t)$ a function of $t$, satisfying
\[
L_1^U C = L_1^U D(t) = 0 
\tag{3.10}
\]
due to (3.8).

With (3.9) and (3.6), the relation (3.7) becomes
\[
(B_u R_1^U, B_p - B_u F^{-1}) \begin{pmatrix} H \alpha_-(t) + J(t) \\ C \alpha_-(t) + D(t) \end{pmatrix} = b_0(t) - B_u R_1^S \alpha_-(t). 
\]
This holds for any $\alpha^{-}(t)$ determined by initial data, leading to

$$b_0(t) = B_u R_1^U J(t) + (B_p - B_u F^{-1}) D(t)$$  \hfill (3.11)

and

$$(B_u, B_p) \begin{pmatrix} R_1^U H + R_1^S - F^{-1} C \\ C \end{pmatrix} = 0.$$  \hfill (3.12)

Notice that $C \in \text{span}\{R_1^S\}$ due to (3.11). Let $C = R_1^S \tilde{C}$ with $\tilde{C}$ an arbitrary $(n - l) \times (n - l)$ square matrix. Then (3.12) can be rewritten as

$$\hat{B} Z \equiv (B_u, B_p R_1^S) \begin{pmatrix} -R_1^U (\hat{B} R_1^U)^{-1} \hat{B} R_1^S + R_1^S - F^{-1} R_1^S \tilde{C} \\ \tilde{C} \end{pmatrix} = 0,$$  \hfill (3.13)

where $\hat{B} = (B_u, B_p R_1^S)$ and $Z$ is the $(2n - l) \times (n - l)$-matrix.

Note that $Z$ is a full-rank matrix for any $\tilde{C}$. To see this, let $Zx = 0$ with $x \in \mathbb{R}^{n-l}$. Then we have $\tilde{C} x = 0$ and

$$[I_n - R_1^U (\hat{B} R_1^U)^{-1} \hat{B}]R_1^S x = 0 \quad \text{or} \quad R_1^S x = R_1^U (\hat{B} R_1^U)^{-1} \hat{B} R_1^S x.$$

The last equation means that the left-hand side belongs to $\text{span}\{R_1^S\}$, while the right-hand side is in $\text{span}\{R_1^U\}$. Since $\text{span}\{R_1^S\} \cap \text{span}\{R_1^U\} = \emptyset$, it must be

$$x = 0.$$  

Therefore, $Z$ is full-rank.

Thus, the column full-rank matrix $Z$ can be extended as a base of $\mathbb{R}^{2n-l} = \text{span}(Z, \tilde{Z})$ with $\tilde{Z}$ a full-rank $(2n - l) \times n$-matrix. Hence, there exists an invertible matrix $\begin{pmatrix} B & \tilde{B} \end{pmatrix}$ such that

$$\begin{pmatrix} B \\ \tilde{B} \end{pmatrix} (Z, \tilde{Z}) = I_{2n-l}.$$  

Particularly, we have $\tilde{B} Z = 0$. In this way, we obtain $\tilde{B} = (B_u, B_p R_1^S)$ and thereby $B = (B_u, B_p)$.

Obviously, the boundary matrix $B = (B_u, B_p)$ thus constructed is not unique. The non-uniqueness has other two sources. The first one is that the matrix $Z$ in (3.13) depends on the free matrix $\tilde{C}$. On the other hand, once $Z$ is given, there are infinitely many $\tilde{B}$ satisfying $\tilde{B} Z = 0$. However, such $\tilde{B}$ is unique up to an invertible $n \times n$-matrix multiplying $\tilde{B}$ from right.

To see this, let $\tilde{B}_1$ and $\tilde{B}_2$ satisfy $\tilde{B}_1 Z = \tilde{B}_2 Z = 0$. We can show that the matrices $(Z, \tilde{B}_1^*)$ and $(Z, \tilde{B}_2^*)$ are both invertible, where the superscript $*$ denotes the conjugate transpose. In fact, if

$$(Z, \tilde{B}_1^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = Z \alpha + \tilde{B}_1^* \beta = 0,$$

we multiply with $\tilde{B}_1$ from right to obtain

$$\tilde{B}_1 \tilde{B}_1^* \beta = 0.$$
Because $\bar{B}_1$ is full-rank, $\bar{B}_1\bar{B}_1^*$ is invertible and thereby $\beta = 0$. Moreover, since the matrix $Z$ is full-rank, we have $\alpha = 0$. Hence, the matrix $(Z, \bar{B}_1^*)$ is invertible. Consequently, $\bar{B}_2^*$ can be expressed as

$$\bar{B}_2^* = (Z, \bar{B}_1^*) \begin{pmatrix} \gamma \\ \chi \end{pmatrix} = Z\gamma + \bar{B}_1^*\chi.$$  

Multiplying with $Z^*$ from right we get $0 = Z^*Z\gamma + 0$, which implies $\gamma = 0$. This indicates that $\bar{B}_2 = \chi^*\bar{B}_1$. Furthermore, $\chi^*$ is invertible for both $\bar{B}_1$ and $\bar{B}_2$ are full-rank.

In summary, for $l < n$ we have constructed the boundary matrix $B = (B_u, B_p)$ satisfying (3.13) with the freedoms above. Once $B$ is chosen, the right-hand side $b_0(t)$ is determined completely with (3.11). For $l = n$, the construction is given in (3.4). These are the main results of this paper.

4 Generalized Kreiss Condition

According to [26], the boundary matrix $B$ constructed in Section 3 should satisfy the generalized Kreiss condition (GKC), which is essentially necessary to have a well-behaved limit when $\epsilon$ goes to zero. Thus, we review the GKC in this section. To do this, we recall the definition of a right-stable matrix for a square matrix.

**Definition 4.1.** Let $N \times N$-matrix $E$ have precisely $k$ ($0 \leq k \leq N$) stable eigenvalues. A full-rank $N \times k$-matrix $R^S_E$ is called a right-stable matrix of $E$ if

$$ER^S_E = R^S_E S_-$$

with $S_-$ a $k \times k$ stable matrix.

Notice that the coefficient matrix $A$ in (2.3) is invertible. We define

$$M(\eta, \xi_0) = A^{-1}(\eta S - \xi_0 I_{2n})$$

for parameters $\eta \geq 0$ and $\xi_0 \in \mathbb{C}$ with Re$\xi_0 > 0$. Here $S = \text{diag}(0, -I_n)$. Referring to Lemma 2.3 in [26], we know that $M(\eta, \xi_0)$ has $n$ stable eigenvalues under the sub-characteristic condition

$$a_j \geq \lambda^2_j.$$  

Here we give a direct proof of this fact. For this purpose, we recall the coefficient matrix

$$A = \begin{pmatrix} F \\ \bar{A} - F^2 \\ -F \end{pmatrix}$$

and use $F\bar{A} = \bar{A}F$ to rewrite

$$M = M(\eta, \xi_0) = A^{-1}(\eta S - \xi_0 I_{2n})$$

as

$$= \begin{pmatrix} \bar{A}^{-1} \\ 0 \\ \bar{A}^{-1} \end{pmatrix} \begin{pmatrix} F \\ \bar{A} - F^2 \\ -F \end{pmatrix} \begin{pmatrix} -\xi_0 I_n \\ 0 \\ -(\eta + \xi_0)I_n \end{pmatrix}$$

$$= \begin{pmatrix} -\xi_0\bar{A}^{-1}F \\ -(\eta + \xi_0)\bar{A}^{-1} \\ -\xi_0(I_n - \bar{A}^{-1}F^2)(\eta + \xi_0)\bar{A}^{-1}F \end{pmatrix}.$$
Moreover, with \( F = T \Lambda T^{-1} \) we have
\[
\begin{pmatrix}
T^{-1} & 0 \\
0 & T^{-1}
\end{pmatrix}
M
\begin{pmatrix}
T & 0 \\
0 & T
\end{pmatrix} =
\begin{pmatrix}
-\xi_0(\bar{\Lambda})^{-1} \Lambda & -(\eta + \xi_0)(\bar{\Lambda})^{-1} \\
-\xi_0(I_n - (\bar{\Lambda})^{-1} \Lambda^2) & (\eta + \xi_0)(\bar{\Lambda})^{-1} \Lambda
\end{pmatrix},
\]
(4.2)
where \( \bar{\Lambda} = \text{diag}(a_1, \ldots, a_n) \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \). It is known that the eigenvalues of matrix \( M \) are equal to the eigenvalues of the following matrix:
\[
\frac{1}{a_j}
\begin{pmatrix}
-\xi_0 \lambda_j & -(\eta + \xi_0) \\
\xi_0 (\lambda_j^2 - a_j) & \lambda_j(\eta + \xi_0)
\end{pmatrix}, \quad j = 1, 2 \cdots, n.
\]
For this \( 2 \times 2 \)-matrix, the corresponding characteristic polynomial is
\[
\lambda^2 - \frac{\eta \lambda_j}{a_j} \lambda - \frac{(\eta + \xi_0) \xi_0}{a_j} = 0.
\]

Denote by \( (\kappa_j)^\pm \) the two solutions of the last equation. We have

**Lemma 4.1.** Under the sub-characteristic condition, it holds that \( \text{Re}(\kappa_j)^+ \text{Re}(\kappa_j)^{-} < 0 \) for each \( j \).

**Proof.** Firstly, we show that \( \text{Re}(\kappa_j)^+ \text{Re}(\kappa_j)^{-} \neq 0 \). Otherwise, we may assume that \( (\kappa_j)^+ = ib \) with \( b \) a real number. It follows that
\[
(\kappa_j)^+ + (\kappa_j)^- = \frac{\eta \lambda_j}{a_j}, \quad (\kappa_j)^+ (\kappa_j)^- = -\frac{(\eta + \xi_0) \xi_0}{a_j},
\]
(4.3)
and thereby \( (\kappa_j)^- = \frac{\eta \lambda_j}{a_j} - ib \). Set \( \xi_0 = \alpha + i\beta \) with \( \alpha > 0 \). We deduce that
\[
(\kappa_j)^+ (\kappa_j)^- = b^2 + i \frac{\eta \lambda_j}{a_j} b = \frac{\beta^2 - (\eta + \alpha) \alpha}{a_j} - i \frac{(\eta + 2\alpha) \beta}{a_j}
\]
and, therefore,
\[
\frac{\eta \lambda_j}{a_j} b = -\frac{(\eta + 2\alpha) \beta}{a_j}, \quad b^2 = \frac{\beta^2 - (\eta + \alpha) \alpha}{a_j}.
\]
These lead to
\[
(\eta + 2\alpha)^2 \frac{\beta^2}{\lambda_j^2} = \eta^2 \frac{\beta^2 - (\eta + \alpha) \alpha}{a_j}.
\]
By the sub-characteristic condition (4.1), we see that
\[
(\eta + 2\alpha)^2 \beta^2 \leq \eta^2 (\beta^2 - (\eta + \alpha) \alpha) < \eta^2 \beta^2,
\]
which is impossible. Thus, we have shown that \( \text{Re}(\kappa_j)^+ \text{Re}(\kappa_j)^{-} \neq 0 \).

On the other hand, it is clear that \( \text{Re}(\kappa_j)^+ \text{Re}(\kappa_j)^{-} < 0 \) when \( \xi_0 = 1 \) and \( \eta = 0 \). Hence, by the continuity of \( (\kappa_j)^\pm \) with respect to the coefficients, we have \( \text{Re}(\kappa_j)^+ \text{Re}(\kappa_j)^{-} < 0 \) for all \( \eta \geq 0 \) and complex number \( \xi_0 \) with \( \text{Re}\xi_0 > 0 \). This completes the proof.
According to the above fact, the right-stable matrix \( R_M^S(\eta, \xi_0) \) of \( M(\eta, \xi_0) \) is a \( 2n \times n \)-matrix. Note that the boundary matrix \( B \) is an \( n \times 2n \) full-rank matrix. Then the GKC can be stated as [26]: there exists a constant \( c_K > 0 \) such that

\[
|\det\{BR_M^S(\eta, \xi_0)\}| \geq c_K \sqrt{|\det\{R_M^{S*}(\eta, \xi_0)R_M^S(\eta, \xi_0)\}|}
\]

(4.4)

for all \( \eta \geq 0 \) and \( \xi_0 \) with \( \text{Re}\xi_0 > 0 \). Here the superscript \( * \) denotes the conjugate transpose.

In order to verify the GKC for the boundary matrix \( B \) constructed in Section 3, we need a detailed expression of \( R_M^S(\eta, \xi_0) \). As above, let \( (\kappa_j)_+ \) denote unstable eigenvalues of \( M \) with \( \text{Re}(\kappa_j)_+ > 0 \) and \( (\kappa_j)_- \) stand for stable eigenvalues with \( \text{Re}(\kappa_j)_- < 0 \) \( (j = 1, 2, \cdots, n) \). Then, according to \([12]\) and \([13]\) we have

\[
\begin{pmatrix}
T^{-1} & 0 \\
0 & T^{-1}
\end{pmatrix} M \begin{pmatrix}
T & 0 \\
0 & T
\end{pmatrix} \begin{pmatrix}
e_j \\
\frac{a_j}{\eta + \xi_0}(\kappa_j)_+ - \lambda_j e_j
\end{pmatrix}
= \begin{pmatrix}
-\xi_0(\bar{\Lambda})^{-1} \Lambda & -(\eta + \xi_0)(\bar{\Lambda})^{-1} \\
-\xi_0(I_n - (\bar{\Lambda})^{-1} A^2) & (\eta + \xi_0)(\bar{\Lambda})^{-1} A
\end{pmatrix} \begin{pmatrix}
e_j \\
\frac{a_j}{\eta + \xi_0}(\kappa_j)_+ - \lambda_j e_j
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\xi_0 + \frac{\lambda_j \eta}{a_j} e_j \\
-\xi_0 - \frac{\eta \lambda_j^2}{a_j} + \lambda_j (\kappa_j)_+ e_j
\end{pmatrix}
= \begin{pmatrix}
\frac{\kappa_j}{\eta + \xi_0}(\kappa_j)_+ e_j \\
\frac{\kappa_j}{\eta + \xi_0}(\kappa_j)_+ - \lambda_j (\kappa_j)_- e_j
\end{pmatrix}
\]

\[
= (\kappa_j)_- \begin{pmatrix}
e_j \\
\frac{a_j}{\eta + \xi_0}(\kappa_j)_+ - \lambda_j e_j
\end{pmatrix}.
\]

Thus, \( R_M^S(\eta, \xi_0) \) can be taken to be

\[
R_M^S(\eta, \xi_0) = \begin{pmatrix}
T & 0 \\
0 & T
\end{pmatrix} \begin{pmatrix}
e_1 \\
\frac{a_1}{\eta + \xi_0}(\kappa_1)_+ - \lambda_1 e_1 \\
\cdots \\
\frac{a_n}{\eta + \xi_0}(\kappa_n)_+ - \lambda_n e_n
\end{pmatrix}
= \begin{pmatrix}
T & 0 \\
0 & T
\end{pmatrix} \begin{pmatrix}
I_n \\
Q
\end{pmatrix},
\]

(4.5)

where \( e_j \) represents the j-th column of the identity matrix \( I_n \) and \( Q = \text{diag}(q_1, q_2, \cdots, q_n) \) with \( q_j = \frac{a_j}{\xi_0 + \eta}(\kappa_j)_+ - \lambda_j \) \( (j = 1, 2, \cdots, n) \). Consequently, we obtain

\[
BR_M^S(\eta, \xi_0) = (B_u, B_p) \begin{pmatrix}
T & 0 \\
0 & T
\end{pmatrix} \begin{pmatrix}
I_n \\
Q
\end{pmatrix} = \tilde{B}_u + \tilde{B}_p Q.
\]
Here \( \tilde{B}_u = B_uT \) and \( \tilde{B}_p = B_pT \).

Set \( \tilde{B} = (\tilde{B}_u, \tilde{B}_p) \) and

\[
\tilde{R}_M^S(\eta, \xi_0) = \begin{pmatrix} I_n \\ Q \end{pmatrix}.
\]

Then the GKC (4.4) can be rewritten as

\[
\frac{|\det\{\tilde{B}\tilde{R}_M^S(\eta, \xi_0)\}|}{\sqrt{\det\{\tilde{R}_M^{*S}(\eta, \xi_0)\tilde{R}_M^S(\eta, \xi_0)\}}} \geq c_K > 0,
\]

where Lemma 3.3 in [26] is used. And the denominator can be calculated as

\[
\det\{\tilde{R}_M^{*S}(\eta, \xi_0)\tilde{R}_M^S(\eta, \xi_0)\} = \det\{I_n + Q^*Q\} = \prod_{j=1}^{n}(1 + |q_j|^2).
\]

Here is a uniform estimate for \( q_j = \frac{a_j}{\xi_0 + \eta}(\kappa_j)^+ - \lambda_j \) depending on the parameters \( \eta \) and \( \xi_0 \).

**Lemma 4.2.** Under the sub-characteristic condition, we have the uniform estimate

\[
|q_j(\eta, \xi_0)| = \left| \frac{a_j}{\xi_0 + \eta}(\kappa_j)^+ - \lambda_j \right| \leq (\sqrt{2} + 1)\sqrt{a_j}, \quad j = 1, 2, \ldots, n.
\]

Under the strict sub-characteristic condition \( a_j > \lambda_j^2 \), there is a positive constant \( c \) such that

\[
|q_j(\eta, \xi_0)| \geq c
\]

for \( j > l \), all \( \eta \geq 0 \) and all complex number \( \xi_0 \) with \( \text{Re}\xi_0 > 0 \).

**Proof.** For \( \eta \geq 0 \) and \( \text{Re}\xi_0 > 0 \), under the sub-characteristic condition we have

\[
|q_j| = \left| \frac{\sqrt{4a_j\xi_0^2 + 4a_j\xi_0\eta + \lambda_j^2\eta^2} - \lambda_j(\eta + 2\xi_0)}{2(\xi_0 + \eta)} \right|
\]

\[
\leq \left| \frac{\sqrt{4a_j\xi_0^2 + 4a_j\xi_0\eta + \lambda_j^2\eta^2} - \lambda_j(\eta + 2\xi_0)}{\eta + 2\xi_0} \right|
\]

\[
= \left| \sqrt{a_j + (\lambda_j^2 - a_j)(\frac{\eta}{\eta + 2\xi_0})^2} - \lambda_j \right|
\]

\[
\leq \sqrt{2a_j - \lambda_j^2} + \sqrt{a_j}
\]

\[
\leq (\sqrt{2} + 1)\sqrt{a_j}.
\]
For the lower bound, we refer to [24] and know that \( q_j = h_j - \lambda_j \) is analytic in \( \text{Re}\xi_0 = 0 \) for each \( \eta \geq 0 \) under the sub-characteristic condition, where

\[
\begin{align*}
    h_j(\xi_0) = \frac{a_j}{\xi_0 + \eta} (\kappa_j) &= \frac{\eta\lambda_j + \sqrt{4a_j\xi_0^2 + 4a_j\xi_0\eta + \lambda_j^2\eta^2}}{2(\xi_0 + \eta)}.
\end{align*}
\]

From this we can solve

\[
\xi_0 = \frac{\eta h_j(\xi_0)(h_j(\xi_0) - \lambda_j)}{a_j - h_j^2(\xi_0)}
\]

under the strict sub-characteristic condition \( \lambda_j < \sqrt{a_j} \). This indicates that \( h_j(\xi_0) \) is a univalent analytic function. According to the conformal mapping theorem, \( h_j = h_j(\xi_0) \) maps the half plane \( \text{Re}\xi_0 > 0 \) to a simply connected closed bounded domain \( \Omega \subset \mathbb{C} \) and maps the imaginary axis \( \text{Re}\xi_0 = 0 \) to the boundary of \( \Omega \). Let \( \xi_0 = \theta_1 + i\theta_2 \) with \( \theta_1 > 0 \). The boundary is parametrized as

\[
\begin{align*}
    h_j(i\theta_2) &= \frac{\eta\lambda_j + \sqrt{-4a_j\theta_2^2 + 4a_j\eta\theta_2 + \lambda_j^2\theta_2^2}}{2(i\theta_2 + \eta)},
\end{align*}
\]

which is a closed curve and intersects the real axis only at \( \theta_2 = 0 \) and \( \theta_2 = \pm\infty \):

\[
\begin{align*}
    h_j(0) &= \frac{|\lambda_j| + \lambda_j}{2}, \quad h_j(\pm\infty) = \sqrt{a_j}.
\end{align*}
\]

When \( j > l \), we have \( \lambda_j < 0 \), \( h_j(0) = 0 \) and therefore \( \lambda_j \notin \Omega \). Because \( \Omega \) is closed, there exists a positive constant \( c \) such that

\[
|q_j(\eta, \xi_0)| = |h_j(\eta, \xi_0) - \lambda_j| \geq c, \quad j = l + 1, \ldots, n.
\]

This completes the proof.

\[
\square \]

5 Verification of the Generalized Kreiss Condition

In this section, we verify the GKC for the constructed boundary matrix \( B = (B_u, B_p) \). According to Lemma 4.2, the determinant \( \det\{\hat{R}_M^s(\eta, \xi_0)\hat{R}_M^s(\eta, \xi_0)\} \) in (4.6) has a positive upper bound. Thus, it suffices to show that \( |\det\{\hat{R}_M^s(\eta, \xi_0)\}| \) has a positive lower bound.

To this end, we recall the constraint (3.13) for the boundary matrix \( B = (B_u, B_p) \):

\[
B_p R_1^S \hat{C} + B_u (-F^{-1} R_1^S \hat{C} + R_1^S + R_1^U H) = 0
\]

with \( H = -(\hat{B} R_1^U)^{-1} \hat{B} R_1^S \). In terms of \( \hat{B}_u = B_u T \) and \( \hat{B}_p = B_p T \), this constraint can be rewritten as

\[
\hat{B}_p (T^{-1} R_1^S \hat{C}) + \hat{B}_u (-\Lambda^{-1}(T^{-1} R_1^S \hat{C}) + T^{-1} R_1^S + T^{-1} R_1^U H) = 0.
\]

Because \( I_n = T^{-1} T = T^{-1} (R_1^U, R_1^S) \), we have

\[
T^{-1} R_1^U = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{-1} R_1^S = \begin{pmatrix} 0 & 0 \\ I_{n-l} \end{pmatrix}.
\]
Consequently, the constraint becomes

\[ \tilde{B}_p \begin{pmatrix} 0 \\ \tilde{C} \end{pmatrix} + \tilde{B}_u \begin{pmatrix} H \\ I_{n-l} - \Lambda^{-1} \tilde{C} \end{pmatrix} = 0 \] 

(5.1)

with \( \Lambda = \text{diag}(\lambda_{l+1}, \ldots, \lambda_n) < 0 \) and \( \tilde{C} \) a free \((n-l) \times (n-l)\)-matrix.

Next we proceed for different \( n \) or \( l \).

### 5.1 \( n = 1 \)

In this case, the coefficient matrix \( F \) in (2.1) is a number and \( T = 1 \). When \( F > 0 \), the constructed BC in (3.4) reads as

\[ B_u = \tilde{B} = 1, \quad B_p \text{ arbitrary.} \]

(5.2)

Then

\[ \det \{ \tilde{B} \hat{R}^S_M(\eta, \xi_0) \} = \tilde{B}_u + \tilde{B}_p q = B_u T + B_p T q = 1 + B_p q, \]

where \( q = \frac{a_1}{\xi_0 + \eta} (\kappa_1)_+ - F \). When \( B_p = 0 \), the GKC obviously holds. When \( B_p \neq 0 \), referring to the proof of Lemma 4.2 we know that \( |\det \{ \tilde{B} \hat{R}^S_M(\eta, \xi_0) \}| \) has a positive lower bound if

\[ \frac{1}{B_p} \notin [F - \sqrt{a_1}, 0]. \]

This indicates that \( B_p > \frac{1}{F - \sqrt{a_1}} \).

For \( F < 0 \), the constraint (5.1) becomes

\[ \tilde{B}_p \tilde{C} + \tilde{B}_u (1 - F^{-1} \tilde{C}) = 0 \]

with number \( \tilde{C} \) to be determined. If \( \tilde{C} = F \), this constraint implies that \( \tilde{B}_p = 0 \). Then we must have \( \tilde{B}_u = 1 \) in order that the boundary matrix \( B = (B_u, B_p) \) is full-rank. Thus the determinant

\[ \det \{ \tilde{B} \hat{R}^S_M(\eta, \xi_0) \} = \tilde{B}_u + \tilde{B}_p q = 1 \]

and thereby the GKC holds.

If \( \tilde{C} \neq F \), it follows from the constraint that

\[ \tilde{B}_u = \frac{F \tilde{C}}{\tilde{C} - F} \tilde{B}_p. \]

Thus we must have \( \tilde{B}_p \neq 0 \) in order that the boundary matrix \( B = (B_u, B_p) \) is full-rank. Then

\[ \det \{ \tilde{B} \hat{R}^S_M(\eta, \xi_0) \} = \tilde{B}_u + \tilde{B}_p q = \tilde{B}_p \left( \frac{F \tilde{C}}{\tilde{C} - F} + q \right), \]

where \( q = \frac{a_1}{\xi_0 + \eta} (\kappa_1)_+ - F \). Referring to the proof of Lemma 4.2 this determinant has a positive lower bound if

\[ \frac{F \tilde{C}}{\tilde{C} - F} \notin [F - \sqrt{a_1}, F]. \]

This means that \( \tilde{C} \in (F - \frac{F^2}{\sqrt{a_1}}, F) \cup (F, +\infty) \). Combining the above discussions, we have the following conclusion.
Proposition 5.1. For \( n = 1 \), when \( F > 0 \), the GKC holds for the construction \((5.2)\) with \( B_p > \frac{1}{F - \sqrt{a_1}} \). When \( F < 0 \), the GKC holds for the full-rank matrix \( B = (B_u, B_p) \) satisfying the constraint \((5.1)\) with 
\[
\dot{C} > F - \frac{F^2}{\sqrt{a_1}}.
\]

5.2 \( n = 2 \), \( l = 1 \)

In this case, we partition \( \tilde{B}_u = (\tilde{B}_u, \tilde{B}_{u2}) \) and \( \tilde{B}_p = (\tilde{B}_{p1}, \tilde{B}_{p2}) \). Then the constraint \((5.1)\) becomes 
\[
(\tilde{B}_{p1}, \tilde{B}_{p2}) \begin{pmatrix} \frac{H}{1 - \frac{\dot{C}}{a_2}} \end{pmatrix} = 0.
\]
(5.3)

Notice that \( H \) and \( \dot{C} \) are numbers.

If \( \dot{C} = 0 \), it follows from \((5.3)\) that \( \tilde{B}_{u2} = -\tilde{B}_{u1}H \). Then we have
\[
\dot{B}\tilde{R}_M^S(\eta, \xi_0) = \dot{B}_u + \dot{B}_p \begin{pmatrix} Q^+ & 0 \\ 0 & Q^- \end{pmatrix}
\]
\[
= \left( \tilde{B}_{u1}, -\tilde{B}_{u1}H \right) + (\tilde{B}_{p1}, \tilde{B}_{p2}) \begin{pmatrix} Q^+ & 0 \\ 0 & Q^- \end{pmatrix}
\]
\[
= \left( \tilde{B}_{u1} + \tilde{B}_{p1}Q^+, -\tilde{B}_{u1}H + \tilde{B}_{p2}Q^- \right)
\]
and
\[
\text{det}\{\dot{B}\tilde{R}_M^S(\eta, \xi_0)\}
\]
\[
= \text{det}\{(\tilde{B}_{u1}, \tilde{B}_{p2})\}Q^- + \text{det}\{(\tilde{B}_{p1}, -\tilde{B}_{u1}H + \tilde{B}_{p2}Q^-)\}Q^+.
\]

Considering that \( Q^+ \) may vanish, it is necessary for the GKC to be true that \( (\tilde{B}_{u1}, \tilde{B}_{p2}) \) is invertible. Thus, it is easy to see the following conclusion.

Proposition 5.2. For \( n = 2, l = 1 \), the GKC holds for the constructed boundary matrix \( B = (B_u, B_p) \) with \( \dot{B} = (B_u T, B_p T) \equiv (\tilde{B}_{u1}, \tilde{B}_{u2}, \tilde{B}_{p1}, \tilde{B}_{p2}) \) satisfying \( \tilde{B}_{u2} = -HB_{u1}, (\tilde{B}_{u1}, \tilde{B}_{p2}) \) invertible and \( \tilde{B}_{p1} \) close to zero.

If \( \dot{C} \neq 0 \), the constraint \((5.3)\) becomes
\[
\tilde{B}_{p2} = -\tilde{B}_{u1} \frac{H}{\dot{C}} + \tilde{B}_{u2} \frac{\dot{C} - \lambda_2}{\lambda_2 \dot{C}}.
\]
(5.4)
Then we have

\[ \tilde{BR}_M^S (\eta, \xi_0) = \tilde{B}_u + \tilde{B}_p \begin{pmatrix} Q^+ & 0 \\ 0 & Q^- \end{pmatrix} \]

\[ = \left( \tilde{B}_{u1}, \tilde{B}_{u2} \right) + \left( \tilde{B}_{p1}, -\tilde{B}_{u1} \frac{H}{C} + \tilde{B}_{u2} \frac{\tilde{C} - \lambda_2}{\lambda_2 C} \right) \left( \begin{array}{cc} Q^+ & 0 \\ 0 & Q^- \end{array} \right) \]

\[ = \left( \tilde{B}_{u1} + \tilde{B}_{p1} Q^+, -\tilde{B}_{u1} \frac{H}{C} Q^- + \tilde{B}_{u2} \left( 1 + \frac{\tilde{C} - \lambda_2}{\lambda_2 C} Q^- \right) \right) \]

and

\[ \det{\tilde{BR}_M^S (\eta, \xi_0)} \]

\[ = \det{\left( \tilde{B}_{u1}, \tilde{B}_{u2} \right)} \left( 1 + \frac{\tilde{C} - \lambda_2}{\lambda_2 C} Q^- \right) + \det{\left( \tilde{B}_{p1}, -\tilde{B}_{u1} \frac{H}{C} Q^- + \tilde{B}_{u2} \left( 1 + \frac{\tilde{C} - \lambda_2}{\lambda_2 C} Q^- \right) \right)} Q^+. \]

Considering that \( Q^+ \) may vanish, it is necessary for the GKC to be true that \( (\tilde{B}_{u1}, \tilde{B}_{u2}) = B_u T \) is invertible. Having this, it is not difficult to see the following conclusion.

**Proposition 5.3.** For \( n = 2, l = 1 \), the GKC holds for the constructed boundary matrix \( B = (B_u, B_p) \) satisfying that \( B_u \) is invertible, the first column of \( B_p T \) is close to zero, and the second column is given in \((5.4)\) with

\[ \tilde{C} \in \left( \lambda_2 - \frac{\lambda_2^2}{\lambda_2}, 0 \right) \cup (0, +\infty). \]

5.3 \( l < n \)

In this general case, the constraint \((5.1)\) depends on the free \((n - l) \times (n - l)\)-matrix \( \tilde{C} \) and we will verify the GKC only with \( \tilde{C} = 0 \) or \( \tilde{C} = \Lambda_- \).

5.3.1 \( \tilde{C} = 0 \)

In this case, the constraint \((5.1)\) becomes

\[ \tilde{B}_u \begin{pmatrix} H \\ I_{n-l} \end{pmatrix} = 0 \]

with \( H = -(BR_1^T)^{-1}BR_1^S \). This implies that the rank of \( B_u \) is not larger than \( l \). When \( l = 0 \), we have \( \tilde{B}_u = 0 \). Then it must be that \( \tilde{B}_p = I_n \) in order that the boundary matrix \( \tilde{B} = (\tilde{B}_u, \tilde{B}_p) \) is full-rank. Thus it follows from Lemma 4.2 that

\[ |\det{\tilde{BR}_M^S (\eta, \xi_0)}| = |\det{\tilde{B}_u + \tilde{B}_p Q}| = |\det{Q}| \geq c_n > 0. \]

Therefore the GKC holds.
For \( l > 0 \), the constraint is \( \tilde{B}_{u2} = -\tilde{B}_{u1}H \) with \((\tilde{B}_{u1}, \tilde{B}_{u2}) = B_u T = B_u (R_U^l, R_S^l)\). Notice that \( \tilde{B}_{u1} \) is an \( n \times l \)-matrix. By a linear transformation, we may as well assume that \( \tilde{B}_{u1} \) has the form

\[
\tilde{B}_{u1} = \begin{pmatrix}
\tilde{B}_{u11} \\
0
\end{pmatrix}
\]

with \( \tilde{B}_{u11} \) an \( l \times l \)-matrix. Here the main result is

**Proposition 5.4.** The GKC holds for

\[
\tilde{B} = (B_u T, B_p T) = \begin{pmatrix}
\tilde{B}_{u11} & -\tilde{B}_{u11}H & \tilde{B}_{p11} & * \\
0 & 0 & 0 & \tilde{B}_{p22}
\end{pmatrix}
\]

with both \( \tilde{B}_{u11} \) and \((n-l) \times (n-l)\)-matrix \( \tilde{B}_{p22} \) invertible, and the spectral radius \( \rho(\tilde{B}_{u11}^{-1}\tilde{B}_{p11}) < \frac{1}{\max(\sqrt{2+1})^{\frac{1}{\sqrt{\sigma_j}}}} \). Here \( * \) stands for an arbitrary \( l \times (n-l) \)-matrix.

**Proof.** Set

\[
Q^+ = \text{diag}(q_1, \ldots, q_l), \quad Q^- = \text{diag}(q_{l+1}, \ldots, q_n).
\]

Then we have

\[
\tilde{B} \tilde{R}_M^S(\eta, \xi_0) = \tilde{B}_u + \tilde{B}_p \begin{pmatrix}
Q^+ & 0 \\
0 & Q^-
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\tilde{B}_{u11} & -\tilde{B}_{u11}H \\
0 & 0
\end{pmatrix}
+ \begin{pmatrix}
\tilde{B}_{p11} & * \\
0 & \tilde{B}_{p22}
\end{pmatrix}
\begin{pmatrix}
Q^+ & 0 \\
0 & Q^-
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\tilde{B}_{u11} + \tilde{B}_{p11}Q^- & -\tilde{B}_{u11}H + *Q^- \\
0 & \tilde{B}_{p22}Q^-
\end{pmatrix}
\]

and

\[
\det\{\tilde{B} \tilde{R}_M^S(\eta, \xi_0)\} = \left| \tilde{B}_{u11} \right| \left| I_l + \tilde{B}_{u11}^{-1}\tilde{B}_{p11}Q^+ \right| \left| \tilde{B}_{p22} \right| \prod_{j=l+1}^n q_j.
\]

Since \( \left| \tilde{B}_{u11} \right| \left| \tilde{B}_{p22} \right| \neq 0 \) is independent of the parameters \( \eta \) and \( \xi_0 \), \( |q_j(\eta, \xi_0)| (j > l) \) has a uniform positive lower bound due to Lemma 4.2, and \( |I_l + \tilde{B}_{u11}^{-1}\tilde{B}_{p11}Q^+| \) has a uniform positive lower bound under the given condition, \( |\det\{\tilde{B} \tilde{R}_M^S(\eta, \xi_0)\}| \) has a positive lower bound. Hence the proof is completed.

**Rem. 5.1.** For \( l = 1 \), it is not difficult to see from the above proof that the spectral radius condition can be relaxed as \( \tilde{B}_{u11}^{-1}\tilde{B}_{p11} > \frac{1}{\lambda_1 - \sqrt{a_1}} \).
5.3.2 $\tilde{C} = \Lambda$

In this case, the constraint (5.1) becomes

$$\tilde{B}_p \begin{pmatrix} 0 \\ \Lambda \end{pmatrix} + \tilde{B}_u \begin{pmatrix} H \\ 0 \end{pmatrix} = 0.$$  

When $l = 0$, it follows from this constraint that $\tilde{B}_p = 0$. Then we must have $\tilde{B}_u = I_n$ in order that the boundary matrix $\tilde{B} = (\tilde{B}_u, \tilde{B}_p)$ is full-rank. Thus the determinant

$$\det\{\tilde{B} \tilde{R}_M(\eta, \xi_0)\} = |\tilde{B}_u + \tilde{B}_p Q| = 1$$

and thereby the GKC holds.

For $l > 0$, we have the following result.

**Proposition 5.5.** The GKC holds for

$$\tilde{B} = (B_u T, B_p T) = \begin{pmatrix} \tilde{B}_{u11} & \ast & \tilde{B}_{p11} & -\tilde{B}_{u11} H \Lambda^{-1}_- \\ 0 & \tilde{B}_{u22} & 0 & 0 \end{pmatrix}$$

with both $\tilde{B}_{u11}$ and $\tilde{B}_{u22}$ invertible, and the spectral radius $\rho(\tilde{B}_{u11}^{-1} \tilde{B}_{p11}) < \frac{1}{\max(\sqrt{2}+1) \sqrt{a_j}}$. Here $\ast$ stands for an arbitrary $l \times (n-l)$-matrix.

**Proof.** Because

$$\tilde{B} \tilde{R}_M^S(\eta, \xi_0) = \tilde{B}_u + \tilde{B}_p \begin{pmatrix} Q^+ & 0 \\ 0 & Q^- \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{B}_{u11} & \ast & \tilde{B}_{p11} & -\tilde{B}_{u11} H \Lambda^{-1}_- \\ 0 & \tilde{B}_{u22} & 0 & 0 \end{pmatrix} \begin{pmatrix} Q^+ & 0 \\ 0 & Q^- \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{B}_{u11} + \tilde{B}_{p11} Q^+ & \ast - \tilde{B}_{u11} H \Lambda^{-1}_- Q^- \\ 0 & \tilde{B}_{u22} \end{pmatrix},$$

we have

$$\det\{\tilde{B} \tilde{R}_M^S(\eta, \xi_0)\} = |\tilde{B}_{u11} | I_l + \tilde{B}_{u11}^{-1} \tilde{B}_{p11} Q^+ | \tilde{B}_{u22} |.$$

Then the same argument in the proof of Proposition 5.4 leads to the conclusion. This completes the proof.

6 Compatibility

The BCs constructed in Section 3 do not guarantee that the initial-boundary-value problems (IBVPs) have smooth solutions. To clarify this point, we introduce further constraints on the boundary data $b_\epsilon(t)$ in (3.1) so that the initial and boundary data are compatible, up to a certain order, at $(x,t) = (0,0)$. In this and next sections, we only consider the case where $l < n$, while the simple case $l = n$ is studied in the appendix.
To do this, we denote by \( u_0 = u_0(x) \) the initial value for the conservation laws (2.1). Assume that this initial value is compatible, up to order 2, with the boundary data in (2.2) for the conservation laws, that is,

\[
\hat{B}(-F \partial_x)^i u_0(0) = \partial_i^* \hat{b}(0), \quad i = 0, 1, 2. \tag{6.1}
\]

Similarly, the compatibility up to order 2 for the relaxation system (2.3) reads as

\[
(B_u, B_p) \left( \begin{array}{c}
\partial_i^* u \\
\partial_i^* p
\end{array} \right) (0, 0) = \partial_i^* b_i(0), \quad i = 0, 1, 2, \tag{6.2}
\]

where

\[
b_i(t) = b_0(t) + \epsilon b_1(t) + \epsilon^2 b_2(t).
\]

In view of (2.8), we refer to (2.8) and (2.10) and choose

\[
\left( \begin{array}{c} u \\ p \end{array} \right) (x, 0) = \left( \begin{array}{c} u_0 \\ 0 \end{array} \right) + \epsilon \left( \begin{array}{c} 0 \\ -(\bar{A} - F^2) u_{0x} \end{array} \right) + \epsilon^2 \left( \begin{array}{c} 0 \\ p_{02} \end{array} \right) \tag{6.3}
\]

as initial data for the relaxation system. Here \( p_{02} \) is to be determined.

Next we deduce from the relaxation system (2.3) that

\[
\left. \left( \begin{array}{c}
\partial_t^* u \\
\partial_t^* p
\end{array} \right) \right|_{t=0} = \left. \left( \begin{array}{c}
-F \partial_x u_0 \\
0
\end{array} \right) \right|_{t=0} + \epsilon m_{11} + \epsilon^2 m_{12}
\]

with

\[
m_{11} = \left( \begin{array}{c}
(\bar{A} - F^2) \partial_{xx} u_0 \\
-F(\bar{A} - F^2) \partial_{xx} u_0 - p_{02}
\end{array} \right), \quad m_{12} = \left( \begin{array}{c}
-\partial_x p_{02} \\
F \partial_x p_{02}
\end{array} \right)
\]

and

\[
\left. \left( \begin{array}{c}
\partial_t^2 u \\
\partial_t^2 p
\end{array} \right) \right|_{t=0} = \left. \left( \begin{array}{c}
F^2 \partial_{xx} u_0 \\
p_{02} + 2F(\bar{A} - F^2) \partial_{xx} u_0
\end{array} \right) \right|_{t=0} + \epsilon m_{21} + \epsilon^2 m_{22}, \tag{6.4}
\]

where

\[
m_{21} = \left( \begin{array}{c}
\partial_x p_{02} \\
-\bar{A}(\bar{A} - F^2) \partial_{xx} u_0 - 2F \partial_x p_{02}
\end{array} \right), \quad m_{22} = \left( \begin{array}{c}
0 \\
\bar{A} \partial_{xx} p_{02}
\end{array} \right).
\]

In view of (6.4), we take

\[
p_{02} = -2F(\bar{A} - F^2) \partial_{xx} u_0. \tag{6.5}
\]

Consequently, the compatibility condition (6.2) becomes

\[
\partial_t^* b_0(0) = (B_u, B_p) \left( \begin{array}{c}
(-F \partial_x)^i u_0(0) \\
0
\end{array} \right) = B_u(-F \partial_x)^i u_0(0), \quad i = 0, 1, 2 \tag{6.6}
\]

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and 
\[ \partial_t^i b_1(0) = (B_u, B_p)m_{i1}, \quad \partial_t^i b_2(0) = (B_u, B_p)m_{i2}, \quad i = 0, 1, 2, \] (6.7)
where 
\[ m_{01} = \begin{pmatrix} 0 \\ - (\bar{A} - F^2)u_{0x} \end{pmatrix}, \quad m_{02} = \begin{pmatrix} 0 \\ p_{02} \end{pmatrix}. \]

Recall (3.11) where 
\[ b_0(t) = B_u R_1^U J(t) + (B_p - B_u F^{-1}) D(t), \]
while \( b_1(t) \) and \( b_2(t) \) have not been determined up to now. Our main result of this section is

**Theorem 6.1.** For \( l < n \), let the boundary matrix \( B = (B_u, B_p) \) be given with the constraint (3.12) and \( b_0(t) \) be given with (3.11). Then the compatibility of order 2 holds if the initial data are chosen according to (6.3) with (6.5), and the boundary data satisfy (6.7) and
\[ \partial_t^i D(0) = -(0, C) T^{-1} (-F \partial_x)^i u_0(0), \quad i = 0, 1, 2. \] (6.8)

**Proof.** With (6.7), we only need to check (6.6). To do this, it follows from (3.11) and (6.8) that
\[ \partial_t^i b_0(0) = \partial_t^i (B_u R_1^U J(t))|_{t=0} - (B_p - B_u F^{-1})(0, C) T^{-1} (-F \partial_x)^i u_0(0). \]

Furthermore, from (3.6) and (3.12) we deduce that
\[ \partial_t^i b_0(0) = B_u R_1^U \partial_t^i (\alpha^+(0) - H \partial_t^i \alpha^-(0)) \]
\[ - (B_p - B_u F^{-1})(0, C) T^{-1} (-F \partial_x)^i u_0(0) \]
\[ = B_u [R_1^U \partial_t^i \alpha^+(0) + R_1^S \partial_t^i \alpha^-(0)] + (B_p - B_u F^{-1}) \left[ C \partial_t^i \alpha^-(0) - (0, C) T^{-1} (-F \partial_x)^i u_0(0) \right]. \]

Recall (3.5) that 
\[ T \begin{pmatrix} \alpha^+(t) \\ \alpha^-(t) \end{pmatrix} = (R_1^U, R_1^S) \begin{pmatrix} \alpha^+(t) \\ \alpha^-(t) \end{pmatrix} = \mathbf{u}(0, t) \]
with \( \mathbf{u} = \mathbf{u}(x, t) \) the solution to the conservation laws (2.1). Since \( \partial_t^i \mathbf{u}(0, t)|_{t=0} = (-F \partial_x)^i u_0(0) \)
due to the conservation laws, we have
\[ \partial_t^i b_0(0) = B_u \partial_t^i \mathbf{u}(0, 0) + (B_p - B_u F^{-1}) \left[ C \partial_t^i \alpha^-(0) - (0, C) T^{-1} \partial_t^i \mathbf{u}(0, t)|_{t=0} \right] \]
\[ = B_u \partial_t^i \mathbf{u}(0, 0) + (B_p - B_u F^{-1}) \left[ C \partial_t^i \alpha^-(0) - (0, C) \left( \partial_t^i \alpha^+(0) \right) \right] \]
\[ = B_u (-F \partial_x)^i u_0(0). \]

This completes the proof.

**Remark 6.1.** Recall that \( \nu_0(0, t) = C \alpha^-(t) + D(t) \). Then the condition (6.8) is equivalent to
\[ \partial_t^i \nu_0(0, 0) = 0, \quad i = 0, 1, 2. \]

In addition, \( \mu_0|_{t=0} = 0 \) for \( \mu_0 = -F^{-1} \nu_0 \).

In summary, we have introduced new constraints (6.7) and (6.8) on \( b_1(t) \) such that the initial and boundary data for the relaxation system (2.3) are compatible, up to order 2, at \((x, t) = (0, 0)\).
7 Formal Asymptotic Solutions

In order to show the effectiveness of the initial and boundary conditions constructed before, we seek a formal approximate solution to the resultant IBVP in this section. To this end, we follow Section 2 to fix the asymptotic expansion coefficients in (2.4)-(2.6). Thanks to (2.8), (2.10), (2.16) and (2.18), we only need to determine $\bar{u}_0$, $\bar{u}_1$, $\nu_0$ and $\nu_1$, solving equations (2.9), (2.11), (2.17) and (2.19), respectively.

According to (6.3), it is natural to take

$$\bar{u}_0(x, 0) = u_0(x), \quad \bar{u}_1(x, 0) = 0.$$  \hfill (7.1)

In addition, we need BCs for $\bar{u}_0$ and $\bar{u}_1$ at the boundary $x = 0$, and initial data for $\nu_0$ and $\nu_1$. This can be obtained from the expectation that the asymptotic solution (2.4)-(2.6) satisfies the BC (3.1), that is,

$$(Bu, B_p) \left( \bar{u}_0(0, t) + \mu_0(0, t) \right) = b_0(t)$$

and

$$(Bu, B_p) \left( \bar{u}_1(0, t) + \mu_1(0, t) \right) = b_1(t).$$

With (2.8), (2.10), (2.16) and (2.18), the last two relations become

$$Bu \bar{u}_0(0, t) + (B_p - Bu F^{-1}) \nu_0(0, t) = b_0(t)$$  \hfill (7.2)

and

$$Bu \bar{u}_1(0, t) + (B_p - Bu F^{-1}) \nu_1(0, t) = b_1(t) + B_p(\tilde{A} - F^2) \bar{u}_0 x(0, t) - Bu F^{-1} \int_0^\infty \mu_{0t}(s, t) ds.$$  \hfill (7.3)

For $l < n$, we refer to Theorem 3.2 in [26]. If the constructed BC satisfies the GKC, then there exists a full-rank $l \times n$-matrix $\hat{B}_1$ such that $l \times l$-matrix $\hat{B}_1 Bu R^U_1$ is invertible and $\hat{B}_1 (B_p - Bu F^{-1}) R^S_1 = 0$ with $(R^U_1, R^S_1) = T$. On the other hand, from (3.8) we know that $\nu_0(0, t) = R^S_1 \alpha$ with $\alpha$ an $(n-l)$-vector. Then we multiply (7.2) with $\hat{B}_1$ from right to obtain

$$\hat{B}_1 Bu \bar{u}_0(0, t) = \hat{B}_1 b_0(t).$$  \hfill (7.4)

This will be shown to be just the given BC (2.2). Consequently, $\bar{u}_0$ can be obtained by solving the given IBVP of the conservation laws (2.1).

To see the equivalence of (7.4) and (2.2), we recall the constraint in (3.13):

$$Bu R^U_1 H + Bu R^S_1 + (B_p - Bu F^{-1}) R^S_1 \tilde{C} = 0$$

with $H = -(\hat{B} R^U_1)^{-1} (\hat{B} R^S_1)$. Multiplying this with $\hat{B}_1$ from right we obtain

$$\hat{B}_1 Bu \left( I - R^U_1 (\hat{B} R^U_1)^{-1} \hat{B} \right) R^S_1 = -\hat{B}_1 Bu R^U_1 (\hat{B} R^U_1)^{-1} \hat{B} R^S_1 + \hat{B}_1 Bu R^S_1 = 0.$$
Thus we have
\[
\dot{B}_1 B_u T = \dot{B}_1 B_u (R_1^U, R_1^S)
\]
\[
= (\dot{B}_1 B_u R_1^U (\dot{B} R_1^U)^{-1} \dot{B} R_1^U, \dot{B}_1 B_u R_1^U (\dot{B} R_1^U)^{-1} \dot{B} R_1^S)
\]
and thereby
\[
\dot{B}_1 B_u = \dot{B}_1 B_u R_1^U (\dot{B} R_1^U)^{-1} \dot{B}.
\] (7.5)

In addition, recall the constraint in (3.11):
\[
b_0(t) = B_u R_1^U J(t) + (B_p - B_u F^{-1}) D(t)
\]
with \( J(t) = (\dot{B} R_1^U)^{-1} \dot{b}(t) \) and \( D(t) \in \text{span}\{R_1^S\} \) due to (3.10). Then we multiply it with \( \dot{B}_1 \) from right to obtain
\[
\dot{B}_1 b_0(t) = \dot{B}_1 B_u R_1^U (\dot{B} R_1^U)^{-1} \dot{b}(t).
\] (7.6)
Since \( \dot{B}_1 B_u R_1^U \) is invertible, it is easy to see from (7.5) and (7.6) that the reduced BC (7.4) is equivalent to the BC given in (2.2).

On the other hand, there exists an \((n-l) \times n\)-matrix \( \dot{B}_2 \) such that \( \begin{pmatrix} \dot{B}_1 \\ \dot{B}_2 \end{pmatrix} \) is invertible for \( \dot{B}_1 \) is full-rank. Multiplying (7.2) with \( \dot{B}_2 \) from right, we get
\[
\dot{B}_2 (B_p - B_u F^{-1}) \nu_0(0,t) = \dot{B}_2 \left( b_0(t) - B_u \bar{u}_0(0,t) \right).
\] (7.7)
According to Lemma 3.4 in [26], \( \dot{B}_2 (B_p - B_u F^{-1}) R_1^S \) is invertible. Thus, the initial value \( \nu_0(0,t) \) is uniquely determined by (7.7).

Similarly, for (2.19) to have a bounded solution \( \nu_1 = \nu_1(\xi, t) \), the initial value \( \nu_1(0,t) \) has to be in the form \( \nu_1(0,t) = R_1^S \zeta \) with \( \zeta \) an \((n-l)\)-vector. Thus we multiply (7.3) with \( \dot{B}_1 \) from right to obtain
\[
\dot{B}_1 B_u \bar{u}_1(0,t) = \dot{B}_1 \left[ b_1(t) + B_p (\bar{A} - F^2) \bar{u}_0x(0,t) - B_u F^{-1} \int_0^\infty \mu_0x(s,t) ds \right].
\]
With this and (7.1), we can get the unique solution \( \bar{u}_1 \) to the IBVP of equation (2.11). Having \( \bar{u}_1 \), we multiply (7.3) with \( \dot{B}_2 \) from right to get
\[
\dot{B}_2 (B_p - B_u F^{-1}) \nu_1(0,t) = \dot{B}_2 \left[ b_1(t) + B_p (\bar{A} - F^2) \bar{u}_0x(0,t) \right.
\]
\[
- B_u F^{-1} \int_0^\infty \mu_0x(s,t) ds - B_u \bar{u}_1(0,t) \right].
\]
From this, we get the initial value \( \nu_1(0,t) \).

**Remark 7.1.** From the last relation, (6.7), (7.1) and Remark 6.1 we deduce that \( \nu_1(0,0) = 0 \). In addition, from Remark 6.1 and (2.18) it follows that \( \mu_1|_{t=0} = 0 \).

**Remark 7.2.** The boundary and initial data of \( \bar{u}_1 \) are compatible up to order 1 if the corresponding boundary and initial data for \( \bar{u}_0 \) are compatible up to order 2.

In conclusion, we have determined all the coefficients in the expansion (2.4)-(2.6) and thus constructed a formal asymptotic solution to the constructed IBVP of the relaxation system (2.3).
8 Effectiveness

In this section, we prove the convergence by estimating the difference between the formal approximate solution and the exact solution to the constructed IBVP of the relaxation system (2.3):

$$\begin{align*}
\begin{cases}
\left( \begin{array}{c}
\u^\v \\
p^\v 
\end{array} \right)_t + \left( \begin{array}{cc}
F & I_n \\
\bar{A} - F^2 & -F
\end{array} \right) \left( \begin{array}{c}
\u^\v \\
p^\v 
\end{array} \right) = \frac{1}{\v} \left( \begin{array}{cc}
0 & 0 \\
0 & -I_n
\end{array} \right) \left( \begin{array}{c}
\u^\v \\
p^\v 
\end{array} \right), \\
B \left( \begin{array}{c}
\u^\v \\
p^\v 
\end{array} \right)(0,t) = b_0(t) + \v b_1(t) + \v^2 b_2(t), \\
\left( \begin{array}{c}
\u^\v \\
p^\v 
\end{array} \right)(x,0) = \left( \begin{array}{c}
u_0 \\
0 
\end{array} \right)(x) + \v \left( \begin{array}{c}
0 \\
-(\bar{A} - F^2)u_{0x}
\end{array} \right)(x) + \v^2 \left( \begin{array}{c}
0 \\
0
\end{array} \right)(x).
\end{cases}
\end{align*}$$

(8.1)

Recall that the formal asymptotic solution is

$$\left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right)(x,t) = \left( \begin{array}{c}
u_0 \\
p_0 
\end{array} \right)(x,t) + \v \left( \begin{array}{c}
\bar{u}_1 \\
\bar{p}_1 
\end{array} \right)(x,t) + \v \left( \begin{array}{c}
\mu_0 \\
\nu_0 
\end{array} \right)(x/t, t) + \v \left( \begin{array}{c}
\mu_1 \\
\nu_1 
\end{array} \right)(x/t, t).$$

According to Section 7 and Remark 6.1, it is not difficult to see that the formal approximate solution \((u^\v, p^\v)\) satisfies

$$\begin{align*}
\begin{cases}
\left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right)_t + \left( \begin{array}{cc}
F & I_n \\
\bar{A} - F^2 & -F
\end{array} \right) \left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right) = \frac{1}{\v} \left( \begin{array}{cc}
0 & 0 \\
0 & -I_n
\end{array} \right) \left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right) + \v \left( \begin{array}{c}
0 \\
I_n
\end{array} \right) y + \v Y, \\
B \left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right)(0,t) = b_0(t) + \v b_1(t), \\
\left( \begin{array}{c}
u^\v \\
p^\v 
\end{array} \right)(x,0) = \left( \begin{array}{c}
u_0 \\
0 
\end{array} \right)(x) + \v \left( \begin{array}{c}
0 \\
-(\bar{A} - F^2)u_{0x}
\end{array} \right)(x),
\end{cases}
\end{align*}$$

(8.2)

where

$$y = y(x,t) = \partial_t \bar{p}_1 + (\bar{A} - F^2)\partial_x \bar{u}_1 - F \partial_x \bar{p}_1, \quad Y = Y(x/t,t) = \left( \begin{array}{c}
\partial_x \mu_1 \\
\partial_t \nu_1 
\end{array} \right).$$

To show the convergence, we make the following assumptions.

**Assumption 8.1.**

1. The boundary \(x = 0\) is non-characteristic for the conservation laws (2.1), that is, the coefficient matrix \(F\) is invertible.
2. The initial data \(u_0 \in H^5(\mathbb{R}^+)\) and boundary data \((\bar{b}, \hat{b}) \in H^4(0, t_*)\).
3. At \((x,t) = (0,0)\), these initial and boundary data are compatible up to order 3.

**Assumption 8.2.**
(1) The initial and boundary data in (8.1) are compatible up to order 2.

(2) \( p_{02} \in H^3(\mathbb{R}^+), b_0 \in H^4(0, t_*) \) and \( b_1, b_2 \in H^3(0, t_*) \).

Under these assumptions, we use Lemma 7.1 in [29] (see also [18]) and can obtain the following existence result, in which

\[
CH_{t_*}^s = \bigcap_{k \leq s} C^k([0, t_*]; H^{s-k}(\mathbb{R}^+))
\]

and \( H^k(\mathbb{R}^+) \) is the Sobolev space of functions on \( \mathbb{R}^+ \) with all derivatives, up to order \( k \), being square-integrable.

**Lemma 8.1.**

1. The IBVP (8.1) has a unique solution \((u', p') \in CH_{t_*}^3\).
2. There is a unique \( \bar{u}_0 \in CH_{t_*}^4 \) and an unique \((\bar{u}_1, \bar{p}_1) \in CH_{t_*}^2\). Moreover, \( \bar{u}_0(0, t) \in H^2(0, t_*), \bar{u}_1(0, t) \in H^2(0, t_*) \).
3. \( \mu_0, \nu_0 \in H^4([0, t_*] \times \mathbb{R}^+), \mu_1, \nu_1 \in H^2([0, t_*] \times \mathbb{R}^+) \).

Now we can state the main result of this section.

**Theorem 8.2.** Under the strict sub-characteristic condition, the assumptions 8.1 and 8.2, there exists a constant \( C > 0 \) such that

\[
\max_{t \in [0, t_*)} \| (u' - u_\epsilon, p' - p_\epsilon)(\cdot, t) \|_{L^2(\mathbb{R}^+)} \leq C \epsilon^{3/2}.
\]

**Proof.** Set

\[
W = \begin{pmatrix} W^I \\ W^{II} \end{pmatrix} = \begin{pmatrix} u' \\ p' \end{pmatrix} - \begin{pmatrix} u_\epsilon \\ p_\epsilon \end{pmatrix}.
\]

From (8.1) and (8.2), it follows that \( W(x, t) \) satisfies

\[
\begin{cases}
W_t + AW_x = \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W - \epsilon \begin{pmatrix} 0 \\ I_n \end{pmatrix} y - \epsilon Y, \\
BW(0, t) = \epsilon^2 b_2(t), \\
W(x, 0) = \epsilon^2 \begin{pmatrix} 0 \\ p_{02} \end{pmatrix}(x),
\end{cases}
\]

where

\[
A = \begin{pmatrix} F & I_n \\ \bar{A} - F^2 & -F \end{pmatrix}.
\]

Recall that \( F = T\bar{A}T^{-1} \) and \( \bar{A} = T\bar{A}T^{-1} \). Set

\[
A_0 = \begin{pmatrix} T^{-*}(\bar{A} - \Lambda^2)T^{-1} & 0 \\ 0 & T^{-*}T^{-1} \end{pmatrix}.
\]

It is not difficult to verify that \( A_0A \) is symmetric and \( A_0 \) is symmetric positive definite under the strict sub-characteristic condition \( a_j > \lambda_j^2 \). This indicates that the system in (8.3) satisfies
the structural stability condition [27] and is symmetrizable hyperbolic. Therefore, there exists an invertible matrix \( L \) such that
\[
LAL^{-1} = \text{diag}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}, -\sqrt{a_1}, -\sqrt{a_2}, \ldots, -\sqrt{a_n}) \equiv D.
\]

Denote by \( L_{\pm} \) the first (last) \( n \) rows of \( L \). We follow [11] and decompose the solution \( W(x,t) \) as
\[
W(x,t) = W_1(x,t) + W_2(x,t).
\]
Here \( W_1 = W_1(x,t) \) solves
\[
\begin{align*}
W_{1t} + AW_{1x} &= \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W_1 - \epsilon \begin{pmatrix} 0 \\ I_n \end{pmatrix} y - \epsilon Y, \\
L_+ W_1(0,t) &= 0, \\
W_1(x,0) &= \epsilon^2 \begin{pmatrix} 0 \\ p_{02} \end{pmatrix}(x),
\end{align*}
\]
while \( W_2 = W_2(x,t) \) satisfies
\[
\begin{align*}
W_{2t} + AW_{2x} &= \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W_2, \\
BW_2(0,t) &= -BW_1(0,t) + \epsilon^2 b_2(t), \\
W_2(x,0) &= 0.
\end{align*}
\]

It is known from [11] that the BC in (8.4) satisfies the Uniform Kreiss Condition. According to the existence theory in [31], there exists a unique solution \( W_1 \in C([0,t_*]; L^2(\mathbb{R}^+)) \). In addition, by Remark 3.2 in [26], the Uniform Kreiss Condition is implied by the GKC. Thus, the existence theory in [31] indicates that there exists a unique solution \( W_2 \in C([0,\infty); L^2(\mathbb{R}^+)) \) provided that \( W_1(0,t) \) and \( b_2(t) \) in (8.5) are replaced by their zero-extensions.

For \( W_1 \), we multiply (8.4) with \( W_1^* A_0 \) from right to get
\[
(W_1^* A_0 W_1)_t + (W_1^* A_0 A W_1)_x = \frac{2}{\epsilon} W_1^* A_0 \left( \begin{array}{cc} 0 & 0 \\ 0 & -I_n \end{array} \right) W_1 - 2 \epsilon W_1^* A_0 \left( \begin{array}{c} 0 \\ I_n \end{array} \right) y - 2 \epsilon W_1^* A_0 Y \\
\leq -c_0 \frac{|W_1^{II}|^2}{\epsilon} + C\epsilon^3|y|^2 + C|W_1|^2 + Ce^2|Y|^2,
\]
where \( W_1^{II} \) stands for the last \( n \) components of \( W_1 \), \( C > 0 \) is a generic constant and \( c_0 > 0 \) is a small constant. Integrating the last inequality over \( x \in [0,\infty) \) we have
\[
\frac{d}{dt} \int_0^\infty W_1^* A_0 W_1 dx + c_0 \int_0^\infty \frac{|W_1^{II}|^2}{\epsilon} dx - W_1^* A_0 A W_1|_{x=0} = C \left( \int_0^\infty \epsilon^3|y|^2 dx + \int_0^\infty |W_1|^2 dx + \int_0^\infty \epsilon^2|Y(x/\epsilon,t)|^2 dx \right)
\]

\[
= C \left( \int_0^\infty \epsilon^3|y|^2 dx + \int_0^\infty |W_1|^2 dx + \epsilon^3 \int_0^\infty |Y(\xi,t)|^2 d\xi \right), \tag{8.6}
\]

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For the boundary term, we notice that $L^{-*}A_0L^{-1}$ is block-diagonal and have

$$-W_1^*A_0AW_1|_{x=0} = -W_1^*(0,t)A_0L^{-1}DLW_1(0,t) = -W_1^*(0,t)L^*L^{-*}A_0L^{-1}DLW_1(0,t) \geq c_1|L_-W_1(0,t)|^2$$

with $c_1$ a positive constant. In addition, we have $C^{-1}W_1^*A_0W_1 \leq |W_1|^2 \leq CW_1^*A_0W_1$ due to the positiveness of $A_0$. Consequently, applying Gronwall’s inequality to (8.6) we obtain

$$\|W_1(\cdot, t)\|_{L^2(\mathbb{R}^+)}^2 \leq Ce^{C\epsilon^3} \left( \|W_1(\cdot, 0)\|_{L^2(\mathbb{R}^+)}^2 + \epsilon^3 \int_0^{t_*} \int_0^\infty \|y(x, t)\|^2 dx + \epsilon^3 \int_0^{t_*} \int_0^\infty |Y(\xi, t)|^2 d\xi \right).$$

(8.7)

From Assumption 8.2 and Lemma 8.1, we know that

$$p_{02} \in L^2(\mathbb{R}^+), \quad \mu_1, \nu_1 \in H^1([0, t_*] \times \mathbb{R}^+),$$

$$y = \partial_t \tilde{p} + (\tilde{A} - F^2)\partial_x \tilde{u} - F\partial_x \tilde{p} \in L^2([0, t_*] \times \mathbb{R}^+).$$

Thus, it follows that

$$\|W_1(\cdot, 0)\|_{L^2(\mathbb{R}^+)}^2 = \int_0^\infty \|p_{02}\|^2 dx \leq C,$$

$$\int_0^{t_*} \int_0^\infty \|y(x, t)\|^2 dx \leq C, \quad \int_0^{t_*} \int_0^\infty |Y(\xi, t)|^2 d\xi \leq C.$$

Combining these with (8.7), we get

$$\|W_1(\cdot, t)\|_{L^2(\mathbb{R}^+)}^2 \leq Ce^{C\epsilon^3}.$$  

Then we integrate (8.6) over $t \in [0, t_*]$ to obtain

$$\|W_1\|_{C([0, t_*]; L^2(\mathbb{R}^+))}^2 + \frac{1}{\epsilon} \|W_1\|_{L^2([0, t_*] \times \mathbb{R}^+)}^2 + \|L_-W_1|_{x=0}\|_{L^2([0, t_*])}^2 \leq Ce^{C\epsilon^3}. \quad (8.8)$$

Next we follow [11] to estimate $W_2$. Denote by $\hat{W}_2 = \hat{W}_2(x, \xi_0)$ the Laplace transform of $W_2 = W_2(x, t)$ with respect to time $t$. It follows from (8.5) that

$$\begin{cases}
\hat{W}_2x = A^{-1}(\eta S - \xi_0I_n)\hat{W}_2 \equiv M(\eta, \xi_0)\hat{W}_2, \\
B\hat{W}_2(0, \xi_0) = \epsilon^2 b_2(\xi_0) - BW_1(0, \xi_0), \\
\|\hat{W}_2(\cdot, \xi_0)\|_{L^2(\mathbb{R}^+)} < \infty, \quad \text{for a.e.} \ \xi_0,
\end{cases}$$

(8.9)

where $\eta = 1/\epsilon$ and $S = \text{diag}(0, -I_n)$.
Recall from Lemma 2.3 in [26] that, under the sub-characteristic condition, \( M = M(\eta, \xi_0) \) has \( n \) stable eigenvalues and \( n \) unstable eigenvalues for all \( \eta \geq 0 \) and all \( \xi_0 \) with \( \text{Re}\xi_0 > 0 \). By Shur's theorem, there exists an unitary matrix \( U \) such that

\[
U^*MU = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix},
\]

where the \( M_{11} \) is a stable \( n \times n \)-matrix and the \( M_{22} \) is a unstable \( n \times n \)-matrix. Set \( \varphi = U^*\hat{W}_2 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \). The equation in (8.9) becomes

\[
\partial_x \begin{pmatrix} \varphi_1(x, \xi_0) \\ \varphi_2(x, \xi_0) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} \varphi_1(x, \xi_0) \\ \varphi_2(x, \xi_0) \end{pmatrix}.
\]

The bounded solution to the last equation is \( \varphi_2 = 0 \) and

\[
\varphi_1(x, \xi_0) = e^{M_{11}x}\varphi_1(0, \xi_0).
\]

The corresponding BC becomes

\[
B\hat{W}_2(0, \xi_0) = BU_1\varphi_1(0, \xi_0) + BU_{11}\varphi_2(0, \xi_0) = e^2\hat{b}_2(\xi_0) - B\hat{W}_1(0, \xi_0),
\]

where \( U = (U_1, U_{11}) \). Thus we have

\[
BU_1\varphi_1(0, \xi_0) = e^2\hat{b}_2(\xi_0) - B\hat{W}_1(0, \xi_0).
\]

Notice that \((BU_1)^{-1}\) is uniformly bounded due to the GKC. We have

\[
|\varphi_1(0, \xi_0)| = \left| (BU_1)^{-1} [e^2\hat{b}_2(\xi_0) - B\hat{W}_1(0, \xi_0)] \right| \\
\leq C \left( |e^2\hat{b}_2(\xi_0)| + |\hat{W}_1(0, \xi_0)| \right).
\]

Since \( U \) is a unitary matrix, it is easy to see that

\[
|\hat{W}_2(0, \xi_0)| \leq C \left( e^2|\hat{b}_2(\xi_0)| + |\hat{W}_1(0, \xi_0)| \right).
\]

According to the Parseval equality, the last inequality leads to

\[
\int_0^\infty e^{-2t\text{Re}\xi_0} |W_2(0, t)|^2 dt \leq C \left( \int_0^\infty e^{-2t\text{Re}\xi_0} |e^2b_2(t)|^2 dt + \int_0^\infty e^{-2t\text{Re}\xi_0} |W_1(0, t)|^2 dt \right) \\
\leq C \left( e^4 \int_0^\infty |b_2(t)|^2 dt + \int_0^\infty |W_1(0, t)|^2 dt \right).
\]

Because the right-hand side is independent of \( \text{Re}\xi_0 \), then we have

\[
\int_0^\infty |W_2(0, t)|^2 dt \leq C \left( e^4 \int_0^\infty |b_2(t)|^2 dt + \int_0^\infty |W_1(0, t)|^2 dt \right).
\]
By a standard argument in [11], the last inequality implies
\[
\int_0^{t^*} |W_2(0, t)|^2 dt \leq C \left( \epsilon^4 \|b_2(t)\|_{L^2([0,t^*])}^2 + \|W_1(0, t)\|_{L^2([0,t^*])}^2 \right)
\leq C \left( \epsilon^4 + \epsilon^3 \right) .
\] (8.10)

Here the estimate (8.8) has been used.

Finally, we multiply (8.3) with \( W_2^* A_0 \) from right to obtain
\[
(W_2^* A_0 W_2)_t + (W_2^* A_0 A W_2)_x = \frac{2}{\epsilon} W_2^* A_0 \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W_2 \leq 0.
\]

Integrating the above inequality over \( (x, t) \in [0, \infty) \times [0, t^*] \) and using (8.10), we get
\[
\max_{t \in [0, t^*]} \| W_2(\cdot, t) \|_{L^2(\mathbb{R}^+)} \leq C \int_0^{t^*} |W_2(0, t)|^2 dt \leq C \epsilon^3.
\]

This together with (8.8) completes the proof. \( \square \)

Furthermore, we have the following \( H^1 \)-estimate.

**Theorem 8.3.** Under the strict sub-characteristic condition, assumptions 8.1 and 8.2 there exists a constant \( C > 0 \) such that
\[
\max_{t \in [0, t^*]} \| (u^e - u_{\epsilon}, p^e - p_{\epsilon}) (\cdot, t) \|_{H^1(\mathbb{R}^+)} \leq C \epsilon^2.
\]

**Proof.** We firstly estimate \( W_1 \). From (8.3), it is easy to see that \( W_1 = W_1(x, t) \) satisfies
\[
\begin{cases}
(W_1)_t + A(W_1)_x = \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W_1 - \epsilon \begin{pmatrix} 0 \\ I_n \end{pmatrix} \partial_t y - \epsilon \partial_t Y, \\
BW_1(0, t) = \epsilon^2 \partial_t b_2(t), \\
W_1(x, 0) = -AW_1(x, 0) - \epsilon \begin{pmatrix} 0 \\ p_{02} \end{pmatrix} (x, 0) - \epsilon \begin{pmatrix} 0 \\ I_n \end{pmatrix} y(x, 0) - \epsilon Y(x/\epsilon, 0).
\end{cases}
\]

For the initial data of \( W_1 \), we use Assumption 8.2 and Lemma 8.1 to get
\[
\| W_1(\cdot, 0) \|_{L^2(\mathbb{R}^+)}^2 \leq C \left( \| W_1(\cdot, 0) \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^2 \| p_{02} \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^2 \| y(\cdot, 0) \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^2 \| Y(\cdot/\epsilon, 0) \|_{L^2(\mathbb{R}^+)}^2 \right)
= C \left( \epsilon^4 \| \partial_x p_{02} \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^2 \| p_{02} \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^2 \| y(\cdot, 0) \|_{L^2(\mathbb{R}^+)}^2 + \epsilon^3 \| Y(\cdot, 0) \|_{L^2(\mathbb{R}^+)} \right)
\leq C \epsilon^2.
\]

In analogue to the estimate for \( W \) in Theorem 8.2, we obtain
\[
\| W_1(\cdot, t) \|_{L^2(\mathbb{R}^+)}^2 \leq C \epsilon^4 \| \partial_t b_2(t) \|_{L^2([0, t^*])}^2 + C \epsilon^2
+ C \epsilon^{2t^*} \left( \epsilon^3 \int_0^{t^*} \int_0^\infty |y_t(x, t)|^2 dx + \epsilon^3 \int_0^{t^*} \int_0^\infty |Y_t(\xi, t)|^2 d\xi \right).
\]

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Moreover, since $b_2 \in H^1(0, t_*)$, it follows from the last inequality that

$$\|W_t(\cdot, t)\|_{L^2(\mathbb{R}^+)}^2 \leq C\epsilon^2.$$  

Next we estimate $W_x$ in terms of equation in (8.3). Because the matrix $A$ is invertible, then

$$\|W_x(\cdot, t)\|_{L^2(\mathbb{R}^+)} \leq \epsilon \left\| A^{-1} \left( -W_t(\cdot, t) + \frac{1}{\epsilon} \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} W(\cdot, t) - \epsilon \begin{pmatrix} 0 \\ I_n \end{pmatrix} y(\cdot, t) - \epsilon Y(\cdot/\epsilon, t) \right) \right\|_{L^2(\mathbb{R}^+)}$$

$$\leq C \left( \|W_t(\cdot, t)\|_{L^2(\mathbb{R}^+)} + \frac{1}{\epsilon} \|W(\cdot, t)\|_{L^2(\mathbb{R}^+)} + \epsilon \|y(\cdot, t)\|_{L^2(\mathbb{R}^+)} + \epsilon \|Y(\cdot/\epsilon, t)\|_{L^2(\mathbb{R}^+)} \right)$$

$$\leq C\epsilon^{\frac{3}{2}}.$$  

These together with the estimate of $W$ lead to

$$\max_{t \in [0, t_*)} \| (u^\epsilon - u_\epsilon, p^\epsilon - p_\epsilon)(\cdot, t) \|_{H^1(\mathbb{R}^+)} \leq C\epsilon^{\frac{3}{2}}.$$  

This completes the proof.

At the end of this section, we deduce from Theorem 8.2 that

$$\| u^\epsilon(\cdot, t) - \bar{u}_0(\cdot, t) \|_{L^2(\mathbb{R}^+)} \leq \epsilon \| u_1(\cdot, t) + \mu_0(\cdot/\epsilon, t) \|_{L^2(\mathbb{R}^+)} + C\epsilon^{\frac{3}{2}}$$

$$= \| \mu_0(\cdot/\epsilon, t) \|_{L^2(\mathbb{R}^+)} + C\epsilon^{\frac{3}{2}}$$

$$\leq C\epsilon^{\frac{3}{2}},$$

where $\bar{u}_0(x, t)$ is the solution to the IBVP of the conservation laws (2.1) and $u^\epsilon(x, t)$ is the solution to the IBVP (8.1).

**Appendix**

This appendix presents the contents of Sections 5-7 for the simple case where $l = n$. In this case, the constructed BC in (3.4) reads as

$$B_u = \hat{B} = I_n, \quad B_p \text{ arbitrary and } b_0(t) = \hat{b}(t).$$

The GKC can be easily verified under certain constraints on $B_p$.

**Proposition 8.1.** Under the strict sub-characteristic condition, the GKC holds if the spectral radius $\rho(B_p) < \frac{1}{\max_{j}(\sqrt{2}+1)\sqrt{a_j}}$ or $T^{-1} B_p T$ is a lower (upper) triangle matrix with its $j$-th diagonal element $\delta_j > \frac{1}{\sqrt{a_j}}$. 

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Proof. From (4.5) we have \( BR_X^S(\eta, \xi_0) = T + B_pTQ \). Thus the determinant \( BR_X^S(\eta, \xi_0) \) is far from zero for the spectral radius \( \rho(Q) = \max_j (\sqrt{2} + 1)\sqrt{a_j} \) due to Lemma 4.2 and the condition \( \rho(B_p) < \frac{1}{\max_j (\sqrt{2} + 1)\sqrt{a_j}} \).

Under the other condition, we may as well assume that \( T - 1 \) is upper triangle. Thus we deduce that

\[
BR_X^S(\eta, \xi_0) = T + B_pTQ
\]

and

\[
det\{BR_X^S(\eta, \xi_0)\} = |T| \prod_{j=1}^n (1 + \delta_jq_j).
\]

For \( \delta_j = 0 \) or \( \delta_j \neq 0 \) but

\[
-\frac{1}{\delta_j} \notin \text{closure}\{q_j(\eta, \xi_0) : \text{Re}\xi_0 > 0, \eta \geq 0\},
\]

it is clear that \( |1 + \delta_jq_j| \) has a positive lower bound. On the other hand, we refer to the proof of Lemma 4.2 and know that the intersection of the closure and the real axis is \([0, \sqrt{a_j} - \lambda_j]\). Thus the condition (8.11) is satisfied if \( \delta_j \) is a real number and

\[
-\frac{1}{\delta_j} \notin [0, \sqrt{a_j} - \lambda_j].
\]

The latter holds if \( \delta_j > \frac{1}{\lambda_j - \sqrt{a_j}} \). This completes the proof.

As to the compatibility, at \((x, t) = (0, 0)\), of the initial and boundary data for the relaxation system (2.3), we follow the discussion in Section 6. Thus, we only need to check the relation (6.6). When \( B_u = \hat{B} = I_n \) and \( b_0(t) = \hat{b}(t) \), (6.6) is just the assumption (6.1).

To determine the coefficients \( \bar{u}_0, \bar{u}_1, \nu_0 \) and \( \nu_1 \) of the formal asymptotic solution, we observe that the coefficient matrix \( FA^{-1} \) in Equation (2.17) has only positive eigenvalues. Then the unique bounded solution thereof is \( \nu_0 = 0 \). Similarly, we have \( \nu_1 = 0 \) due to (2.19). On the other hand, the BCs for \( \bar{u}_0 \) and \( \bar{u}_1 \) at boundary \( x = 0 \) can be obtained as follows. From the expectation that the asymptotic solution satisfies the BC (3.1):

\[
(B_u, B_p) \begin{pmatrix} \bar{u}_0(0, t) \\ \bar{p}_0(0, t) \end{pmatrix} = b_0(t), \quad (B_u, B_p) \begin{pmatrix} \bar{u}_1(0, t) \\ \bar{p}_1(0, t) \end{pmatrix} = b_1(t),
\]

it follows from (2.8) and (2.10) that

\[
B_u \bar{u}_0(0, t) = b_0(t), \quad B_u \bar{u}_1(0, t) = b_1(t) + B_p(\hat{A} - F^2)\bar{u}_0x(0, t).
\]
Since $B_u = I_n$ and $b_0(t) = \hat{b}(t)$, we obtain
\[
\bar{u}_0(0, t) = \hat{b}(t), \quad \bar{u}_1(0, t) = b_1(t) + B_p(\hat{A} - F^2)\bar{u}_{0x}(0, t).
\] (8.12)

Moreover, the choice (7.1) of initial data for $\bar{u}_0$ and $\bar{u}_1$ is still valid in this case. In this way, $\bar{u}_0$ can be uniquely obtained by solving the IBVP of the conservation laws (2.11), while $\bar{u}_1$ can be solved from the IBVP (2.11), (7.1) and (8.12).

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