Hydrodynamics and Hydrostatics for a Class of Asymmetric Particle Systems with Open Boundaries

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Abstract: We consider attractive particle systems in $\mathbb{Z}^d$ with product invariant measures. We prove that when particles are restricted to a subset of $\mathbb{Z}^d$, with birth and death dynamics at the boundaries, the hydrodynamic limit is given by the unique entropy solution of a conservation law, with boundary conditions in the sense of Bardos et al. (Comm Part Diff Equ 4:1017–1034, 1979). For the hydrostatic limit between parallel hyperplanes, we prove a multidimensional version of the phase diagram conjectured in Popkov and Schütz (Europhys Lett 48:257–263, 1999), and show that it is robust with respect to perturbations of the boundaries.

1. Introduction

Stochastic lattice gases in contact with reservoirs are tractable and thus widely studied examples of nonequilibrium stationary states. The derivation of the stationary macroscopic profile (hydrostatic limit) is a natural question in this context. For diffusive systems, robust methods have been developed (see e.g. [16,17,23,27]). The hydrostatic profile is the stationary solution to the hydrodynamic equation (a possibly nonlinear diffusion equation) with Dirichlet boundary conditions imposed by reservoir densities. For instance, the symmetric simple exclusion process exhibits a linear profile connecting these densities.

For driven lattice gases, the picture is different. The asymmetric simple exclusion process with open boundaries was first introduced in [29,30] as an intermediate tool for studying the process on $\mathbb{Z}^d$. Its hydrostatic profile was determined by [12] in the one-dimensional nearest-neighbor case. It consists of three phases with uniform bulk density: low-density (LD) and high-density (HD) phases, where the bulk density is given by one of the boundaries, and a maximum current (MC) phase, where the bulk density is $1/2$. Unlike in the diffusive setting, these profiles cannot satisfy both Dirichlet conditions if the reservoir densities are different. LD and HD phases are separated by a coexistence line, where the bulk state is a randomly located shock connecting the reservoir densities.
For more general models and currents, as well as higher space dimensions, mathematical results are missing. In one space dimension, the number and nature of phases is expected to depend on the current-density function through the following variational formula for the uniform bulk density ([38]):

$$\begin{align*}
\text{argmin}_{[\lambda_a, \lambda_b]} f & \text{ if } \lambda_a < \lambda_b, \\
\text{argmax}_{[\lambda_b, \lambda_a]} f & \text{ if } \lambda_b < \lambda_a,
\end{align*}$$

(1)

where $\lambda_a$, $\lambda_b$ are the left and right reservoir densities, and $f(\rho)$ is the current-density function. For the asymmetric exclusion process, $f(\rho) = \rho(1 - \rho)$, and (1) yields the three phases of [12]. For the KLS model ([24]), one obtains a seven-phase diagram, with two LD, two HD, two MC and a minimum current (mC) phase. One outcome of this paper is to prove a multidimensional version of (1), for a wide class of models including simple exclusion, with arbitrarily many phases. The boundaries are basically parallel hyperplanes, but results are somewhat robust with respect to perturbations of this geometry. The approach introduced here should be effective to treat more complex boundary-driven phase transitions, induced either by the domain geometry, or by two-species model like [19]. These will be considered in future works.

The key to our approach is to determine relevant boundary conditions in the scaling limit for asymmetric systems. The hydrodynamic behavior of particle systems with open boundaries is so far understood only in diffusive regimes ([5,17,18,27,34]), where Dirichlet boundary conditions are relevant. The celebrated result of [39] shows that, under Euler time scaling, the hydrodynamic limit of attractive particle systems on $\mathbb{Z}^d$ with product invariant measures, is given by the entropy solution to a scalar conservation law. We extend this result to systems living in an open subset of $\mathbb{R}^d$. We prove that the hydrodynamic limit is the unique entropy solution to an initial-boundary problem with BLN boundary conditions introduced in [7]. Instead of fixing the boundary value like Dirichlet conditions, these conditions impose a set of possible boundary values depending on the boundary datum ([15]). Our proof uses a generalized formulation ([46]) of the BLN boundary conditions that does not explicitly involve a trace for the solution. Producing this formulation from the microscopic boundary dynamics involves adequate coupling of open systems.

We next derive the hydrostatic profile and local equilibrium in a domain lying between two parallel hyperplanes coupled to uniform reservoirs. The result is a $d$-dimensional version of [38]: we show that the bulk density is given by the variational formula of [38] applied to the normal projection of the flux. More generally, if the boundaries are in some sense perturbations of hyperplanes, we show that, away from the perturbation, the bulk density is the same as for hyperplanes, regardless of the precise shape of the boundaries. The hydrostatic limit follows from a uniqueness theorem that we establish for measure-valued stationary entropy solutions with boundary conditions. Such a result implies (and is actually equivalent to) asymptotic stability for entropy solutions with boundary conditions, a question studied so far only for convex ([31,32]) or bell-shaped ([33]) flux functions. We prove here such a result for general fluxes.

We more generally expect BLN boundary conditions to arise in other models where convergence to the entropy solution is established on the whole space. However, a proper microscopic treatment of the boundary remains to be found for models (e.g. [24,41]) that are not attractive, or do not have explicit invariant measures. In the latter case, the “natural” definition of the boundary mechanism given here does not apply. However, it is conjectured in [20] that the macroscopic behavior of the boundary only depends on microscopic details of the boundary dynamics through an effective density.

The paper is organized as follows. In Sect. 2 we define the framework and state the main results. In Sect. 3, we establish the hydrodynamic (Theorem 2.2) and hydrostatic...