Singular magnetic properties of effective QED action

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Employing the Bogoliubov coefficient summation method and allowing for arbitrary value of gyromagnetic ratio \( g \neq 2 \) we derive an explicit functional form of \( 3mV_{EHS}^g \), the imaginary part of Euler-Heisenberg-Schwinger (EHS) type effective action. We show that \( 3mV_{EHS}^g \) is periodic in \( g \) for any field configuration, and equal to the imaginary part obtained using a periodic in \( g \) Ramanujan integrand in the proper time representation of \( V_{EHS}^g \). This validates the Ramanujan representation of \( V_{EHS}^g \) for both real and imaginary parts and allows writing the effective action in modified Schwinger proper time format. As a function of the ratio \( b/a \) between \( E \rightarrow b \) and \( E \rightarrow a \) covariant generalizations of EM fields, we explore the singular properties of \( 3mV_{EHS}^g \) at \( g = 2 \pm 4k, k = 0, \pm 1, \pm 2 \ldots \) involving the pseudoscalar \( a/b \equiv \vec{E} \cdot \vec{B} \) in perturbative and nonperturbative behavior. We study the \( e^{-e^{+}} \)-decay vacuum instability; incorporating the physical value of \( g = 2 \) vertex diagrams when summing infinite irreducible loops. We obtain an effective expansion parameter \( \chi_b = ab/2a (\alpha = e^2/4\pi) \), characterizing the onset of nonperturbative in \( g = 2 \) suppression of vacuum instability. We demonstrate the \( \chi_b \) domains for which perturbative expansion in \( \alpha \) breaks down: The EM vacuum subject to critical electric field strength is stabilized in magnetic-dominated ‘magnetar’ environments. Considering separately the case of \( E \) and \( B \) fields, we generalize to all \( g \) the temperature representation of the \( V_{EHS}^g \) effective action.

I. INTRODUCTION

The response by virtual electron-positron \( e^{-e^{+}} \)-pairs to the action of an externally applied quasi-constant electro-magnetic (EM) field \( \vec{E}, \vec{B} \) has been explored in the seminal work by Euler-Heisenberg-Schwinger (EHS) [1–3], for further details see review [4]. The imaginary part \( 3mV_{EHS}^g \) of the effective QED action relates to the probability of the field filled vacuum state to decay into \( e^{-e^{+}} \)-pairs. The perturbative two-loop radiative corrections to this result were also explored [5–9].

The tacit assumption in any perturbative extension of \( V_{EHS}^g \) is that a perturbative expansion is meaningful near to the physical values of the relevant particle properties: mass \( m \), charge \( e \), and magnetic moment \( \mu \). The last presents challenges that will be illuminated in this work: In the following we describe the magnetic moment in terms of the gyromagnetic ratio \( g \) with \( gs = \mu/\mu_B \). Here \( \mu_B = e\hbar/2mc \) is normalizing Bohr magneton and \( s = 1/2 \); this work is only concerned with spin-1/2 particles. Where we write \( g/2 \), the reader should remember this is short for

\[
\frac{g}{2} \to gs = \frac{\mu}{\mu_B}.
\]

There has been extensive effort based on the Schwinger proper time formulation to account for the anomalous magnetic moment \( g \neq 2 \) in \( V_{EHS}^g \) creating \( V_{EHS}^g \). The form of \( V_{EHS}^g \) achieves resummation of an infinite class of Feynman diagrams. This inspires effort to find a proper time integrand which creates a convergent result for point particles such as the electron where the value \( |g| > 2 \) matters.

Formally the proper time method seems to apply to any value of \( g \). However, a closer look at the actual result reveals that the usual proper time representation converges only for \( |g| \leq 2 \) [14]. A form was discovered [19] using Weisskopf’s method [2] to obtain \( V_{EHS}^g \) valid for pure magnetic \( B \) field configuration. This result was reconfirmed in the case of pure electric \( E \) field backgrounds [16]. Ref. [13] also conjectured, based on analytical properties of the proper time integrand, the form of \( V_{EHS}^g \) for arbitrary \( g \) and arbitrary EM field configurations. Here we are able to prove this conjecture and explore some its consequences focusing on the magnetic regime of fields.

We obtain \( V_{EHS}^g \) for any value of \( g \) in any quasi-constant EM field configuration, using a constructive Bogoliubov coefficient method [17–18]. Our result allows exploration of the dependence on the pseudoscalar \( \vec{E} \cdot \vec{B} \). For pure \( B \) or \( E \) fields we recover the features from prior work [15]: Periodicity as a function of \( g \), and singular (cusp) behavior at \( g = \pm 2 \) based on the perturbative in EM field expressions. Further singular properties are uncovered considering \( V_{EHS}^g \) to all orders in \( \vec{E} \cdot \vec{B} \).

To obtain a \( g \)-dependent generalization of the EHS result \( V_{EHS}^g \), we incorporate \( g \neq 2 \) solving the relativistic quantum wave equations, following the approach seen in the work of Heisenberg and Euler [1]. Each virtual particle excitation is thus prescribed its own anomalous magnetic moment.

In the context of QED diagrams \( V_{EHS}^g \) is described by the top frame in Fig. 1. Transition to \( V_{EHS}^g \) amounts to introduction of internal photon line vertex corrections to the standard EHS diagrams, see bottom frame in Fig. 1. The resulting class of diagrams are summed to infinite irreducible loop order when we evaluate proper time integral for \( V_{EHS}^g \). Presenting a series expansion in \( g = 2 \), we identify the contribution of individual diagrams seen in bottom of Fig. 1.

Inspecting Fig. 1 further, one notices that our approach does not fully account for all possible perturbative
corrections, as we miss diagrams where an internal photon line crosses the Fermion loop isolating at least two external photon lines to the right and left. Such contributions arise in second and higher orders in the external EM field from self-energy corrections to the Fermion propagator \([19, 28]\), producing field-dependent corrections to mass and \(g\) \([29, 30]\). In closed form they produce the leading two-loop and higher order \([31, 34]\) corrections to the EHS action.

In the context of a systematic perturbative expansion we recall the Ritus-Narozhny conjecture \([19, 20]\), where the parameter \(\alpha \chi^{2/3} (\chi = (e/m^3)\sqrt{(-F_{\mu\nu}p_\nu)} \to eE/m^2)_{\text{rest frame}}\) is considered to govern the breakdown of perturbative QED, spurring exploration of convergence of higher order radiative QED corrections \([34, 36]\). This effort is based on perturbative expansion at \(g = \pm 2\). By convention the Dirac electron is negatively charged and since the magnetic moment by convention follows in the EHS action.

Our approach aims to explore the complementary magnetically dominated regime of EM fields – in covariant generalization called \(B \to b\) and \(E \to a\). Thus we seek to understand the regime in which \(b/a \gg 0\) and \(ba \neq 0\). A strong electric field can decay into electron-positron pairs, a phenomenon called ‘Schwinger’ pair production initially described by Heisenberg and Euler \([1]\). \(E\)-field decay into \(e^- e^+\)-pairs is a spontaneous process present in a quasi-constant field. Since matter is created in empty field filled space one can view this situation as spontaneous ‘Vacuum’ production of \(e^- e^+\)-pairs.

The usual EHS result shows that in presence of a strong magnetic field amplify the effect seen in pure \(E\)-field. However, we show that the opposite is to be expected allowing for the physical value of electron magnetic moment. This clarifies the objective of this work: In the presence of a strong and dominant magnetic field environment, it is natural to seek to incorporate more precisely into effective action the effects related to magnetic moment, creating \(V_g^\text{EHS}\), and to show key modifications that arise.

We aim to describe these effects nonperturbatively: The effective summation of the infinite sum in \(g\) allows us to explore any nonperturbative and singular properties of \(V_g^\text{EHS}\) related to the magnetic moment and to recognize a singularity at \(g = \pm 2\). It is in this magnetically dominated environment that the anomalous magnetic moment adds an extra nonperturbative effect that becomes important when the smallness parameter \(\chi_b = ab/2a > 1\). It is notable that the loop expansion is governed by series in \(a^n\), while the magnetic moment expansion is predominantly a series in \(\chi_b^2\).

Our presentation is organized as follows: In section [III] we briefly summarize prior \(V_g^\text{EHS}\) work valid for \(|g| \leq 2\) \([13]\). This approach is based on the Schwinger proper time formulation. In section [III] we derive \(V_g^\text{EHS}\) for any \(g\). In order to obtain an ab-initio result valid in the domain \(|g| > 2\) we:

a) In section [III A] we solve the second order Klein-Gordon-Pauli equation with a spin \(g\)-factor \(g \neq 2\). This generalizes the \(g = \pm 2\) solution to the Dirac equation used by Heisenberg and Euler \([1]\) and allows for \(E \cdot B \neq 0\) field configurations.

b) In section [III B] we apply the Bogolubov coefficient summation method, developed by Nikishov \([17]\) and recently elaborated by Kim, Lee and Yoon \([18]\), to compute the imaginary part \(3mV_g^\text{EHS}\) in specific field configurations.

c) In section [III C] we describe the Ramanujan periodic summation method, developed by Nikishov \([17]\) and recently elaborated by Kim, Lee and Yoon \([18]\), to compute the imaginary part \(3mV_g^\text{EHS}\) in specific field configurations.

In section [IV] we explore nonperturbative behavior of \(V_g^\text{EHS}\) as a function of \(g\), focusing on a magnetically dominated environment:

a) In section [IVA] we demonstrate the singular properties at \(g = \pm 2\), in particular how the sharpness of the cusp singularity in \(3mV_g^\text{EHS}\) depends on EM fields in a nonperturbative manner. In magnetic dominated fields with nonvanishing \(E \cdot B\), \(3mV_g^\text{EHS}\) is sharply peaked as a function of \(g\) at \(g = \pm 2\), and strongly suppressed for \(g \neq \pm 2\).

b) In section [IVB] we identify expansion parameter \(\chi_b = ab/2a\), which characterizes the onset of significant suppression of the EHS pair production result. We demonstrate that perturbative expansion in radiative order \(a\) corrections to \(g = \pm 2\) break down in the \(\chi_b > 1\) domain. We also evaluate the effect of \(\chi_b > 1\) in the EM fields of magnetars. The resultant stabilizing effect dominates the otherwise monotonic enhancement of particle production by \(E\) fields when \(g = \pm 2\) is exactly.

c) In section [IVC] we explore the domains of \(g\) (relatively far from \(g = 2\)) in which asymptotic freedom arises \([15]\).
We show that in these $g$ domains, $3\text{m}V_{g}^{EHS}$ is essentially vanishing for magnetic dominated fields: in asymptotic freedom environment the QED vacuum state considered too one loop is practically stable. This parallels the recent finding \[42\] in the non-Abelian QCD context where in the asymptotic free regime the vacuum stability in (chromo) magnetic dominated fields associated with the Savvidy model of the vacuum \[43\] was recognized.

\[ \text{d)} \text{ In section IV D we apply our results to extend prior work, relating } V_{g}^{EHS} \text{ to the temperature representation.} \]

The temperature representation of $V^{EHS}$ for electric fields \[44, 45\] exhibits an inversion of spin statistics: the $g = \pm 2$ spin-$1/2$ ($g = 0$ spin 0) action takes on a Bose (Fermi) distribution. This result was extended to \[|g| \leq 2\] \[14\], establishing a connection with the Unruh thermal background \[16\] experienced by an accelerating observer. We extend this result to \[|g| > 2\], and consider the magnetic and electric field effect separately.

In section V we review our main results and discuss their implications and potential for additional study of asymptotic behavior of $V_{g}^{EHS}$ for strong fields incorporating $E \cdot B$, options for further summation of contributing diagrams, incorporation of $E > B$ in the temperature representation. We further make remarks how the singular effects we uncovered confirm the long known need to further explore the question of convergence of perturbative QED in strong field environments.

\section*{II. EHS Effective Action for \(|g| \leq 2\)}

We summarize the proper time formulation of EHS effective action with $g \neq 2$, which turns out to be limited to the domain \(|g| \leq 2\). Schwinger \[3\] in his manifestly covariant and gauge invariant approach employed the ‘squared’ Dirac equation, the product of the Dirac equation with its negative mass counterpart. To incorporate anomalous magnetic moment in this approach Kruglov extended the second order wave equation to $g \neq 2$, referred to as the Klein-Gordon-Pauli (KGP) formulation \[47, 48\]:

\[
\left( i\partial_{\mu} - eA_{\mu} \right)^{2} - m^{2} - \frac{g}{2} \epsilon^{\mu\nu} F_{\mu\nu} \right) \Psi = 0, \tag{1}
\]

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]$, $F_{\mu\nu}$ denotes the EM tensor, and

\[
\frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} = i\gamma^{5} \tilde{\Sigma} \cdot \mathcal{E} - \tilde{\Sigma} \cdot \mathcal{B}, \tag{2}
\]

with Pauli-Dirac matrices $\tilde{\Sigma} = \gamma^{5} \gamma^{0} \gamma^{\nu}$, Eq. (1) generalizes Schwinger’s proper time evolution operator Eq. (2.33) in \[3\], with ‘Hamiltonian’

\[
H = \Pi^{2} - \frac{g}{2} \epsilon \sigma_{\mu\nu} F^{\mu\nu}, \tag{3}
\]

where $\Pi_{\mu} = p_{\mu} - eA_{\mu}$. The resulting spin $1/2$ action with $g \neq 2$

\[
V_{g}^{EHS} = \frac{1}{32\pi^{2}} \int_{0}^{\infty} \frac{du}{u^{3}} e^{-im^{2}u} e^{2u^{2}ab \times \text{tr}(g/2) \sigma F/2} \frac{\sinh(eau) \sinh(ebu)}{\sinh(eau) \sinh(ebu)}, \tag{4}
\]

where the electromagnetic field invariants

\[
a^{2} - b^{2} = \mathcal{E}^{2} - \mathcal{B}^{2} \equiv 2S, \quad a^{2}b^{2} = (\mathcal{E} \cdot \mathcal{B})^{2} \equiv P^{2}. \tag{5}
\]

Eigenvalue $a$ is ‘electric-like’, following $a \to |\mathcal{E}|$ in the limit $b \to 0$. Similarly the ‘magnetic-like’ value $b$ follows $b \to |\mathcal{B}|$ for $a \to 0$.

Evaluation of Eq. (4) is straightforward since only the spin-dependent trace term is affected by $g \neq 2$. The resulting Kruglov \[13\] action is

\[
V_{g}^{EHS} = \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{du}{u^{3}} e^{-i(m^{2} - ic)u} F(eau, ebu, g \frac{2}{2}), \tag{6}
\]

\[
F(x, y, g \frac{2}{2}) = x \cosh \left( \frac{g}{2} y \right) \frac{y \cos \left( \frac{g}{2} y \right)}{\sinh(x) - \sin(y)} - 1, \quad \left| g \frac{2}{2} \right| \leq 1.
\]

However, Eq. (6) is convergent only for $|g| \leq 2$. When $|g| > 2$, the proper time integration diverges due to the cosh expression (numerator) outgrowing the sinh contribution (denominator) for large $u$. Therefore when we refer to Kruglov action we now will write $V_{g}^{EHS}$, the equal sign recreates the original EHS effective action, which for numerical expediency is often presented after path of integration is rotated $u \to -iu$.

We observe further that this divergence cannot be alleviated by renormalization; the subtraction $-1$ in Eq. (6) removes the zero point energy. A second subtraction removes the charge renormalizing logarithmically divergent contribution, which amounts to including in the integrand of Eq. (6) the term

\[
F \to F(x, y, g \frac{2}{2}) - \frac{x^{2} - y^{2}}{6} \left( 3 \left( \frac{g}{2} \right)^{2} - 1 \right), \quad \left| g \frac{2}{2} \right| \leq 1; \tag{7}
\]

for further detail on the $g$-dependent renormalization see \[49\].

The proper time method as introduced by Schwinger was in explicit terms argued for as valid because it is convergent. Thus the now recognized lack of convergence near to the physical application domain $|g| > 2$ is a mortal defect of the Schwinger proper time formulation. A completely new approach needs to be identified allowing identification of the form of the effective QED action for $|g| > 2$. A few helpful properties of effective action formally only valid for $|g| \leq 2$ will guide our future analysis.

The integrand of Eq. (6) contains a deformation of the integration contour described by shifting poles of integrand \[1\], \[3\], \[13\] to $\mathcal{E}$, \[\mathcal{B}\] near to the physical application domain.

\[
3\text{m}V_{g, 2}^{EHS} = \frac{a^{2}b^{2}}{8\pi^{2}} \sum_{l \geq 1} \frac{(-1)^{l} e^{-l\pi m^{2}/ea}}{l} \cos \left( \frac{g}{2} l\pi b/a \right) \frac{\cosh \left( \frac{g}{2} l\pi b/a \right)}{\sinh(l\pi b/a)}, \quad \left| g \frac{2}{2} \right| \leq 1. \tag{8}
\]
Finally, the effective action $V^{\text{EHS}}_g$ can be reformatted into the temperature representation. We summarize the $|g| \leq 2$ extension [14] of the original $g = \pm 2$ result obtained in [44].

To obtain the temperature form for pure electric fields ($b \rightarrow 0$), Eq. (6) becomes, after rotating the integration contour $u \rightarrow -i s$,

$$V^{\text{EHS}}_{g \leq 2} \rightarrow -\frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left( \frac{\cos(g_e \cos(eas))}{\sin(eas)} - 1 \right).$$

We apply meromorphic expansion [44, 50, 51],

$$\frac{\cos(g_e \cos(eas))}{\sin(eas)} - 1 = -\frac{e^2 a^2 s^2}{6} \left( 3 \left( \frac{g}{2} \right)^2 - 1 \right) + 2e^2 a^2 s^4 \sum_{n=1}^\infty \frac{(-1)^n \cos(\frac{n\pi}{2})}{n^2\pi^2(s^2 - n^2\pi^2/e^2a^2)},$$

and remove the first term on the RHS, which is absorbed by charge renormalization. Plugging Eq. (10) into the proper time expression Eq. (6) and exchanging summation with integration,

$$V^{\text{EHS}}_{g \leq 2} = \frac{e^2 a^2}{4\pi^2} \sum_{n=1}^\infty \int_0^\infty ds s e^{-m^2 s} \frac{(-1)^n \cos(\frac{g_n\pi}{2})}{n^2\pi^2(s^2 - n^2\pi^2/e^2a^2)}.$$  

Substituting $s \rightarrow n\pi s/ea$ and exchanging summation with integration once more,

$$V^{\text{EHS}}_{g \leq 2} = -\frac{e^2 a^2}{8\pi^2} \int_0^\infty ds \left( \frac{1}{s - 1 + i\varepsilon} + \frac{1}{s + 1} \right) \times \sum_{n=1}^\infty e^{-n\pi m^2 s/ea} \frac{(-1)^n \cos(\frac{g_n\pi}{2})}{n^2\pi^2}.$$  

We introduced the path prescription, see ‘iε’ equivalent to the negative imaginary part $m^2 \rightarrow m^2 - i\varepsilon$ inherent to proper time integration; setting $\Im[1/(s - 1 + i\varepsilon)] = -\pi i\delta(s - 1)$ we find Eq. (8).

Integrating Eq. (12) by parts,

$$V^{\text{EHS}}_{g \leq 2} = -\int_0^\infty ds \ln(s^2 - 1 + i\varepsilon) \sum_{n=1}^\infty e^{-n\beta s} \frac{(-1)^n \cos(\frac{g_n\pi}{2})}{n\pi},$$

where

$$\beta = \frac{\pi m^2}{ea}, \quad v = \frac{m^4}{8\pi^2\beta}. \quad (14)$$

Summing Eq. (13) over $n$ we obtain

$$V^{\text{EHS}}_{g \leq 2} = \int_0^\infty ds \ln(s^2 - 1 + i\varepsilon) \frac{1}{2} \sum_{\pm} \ln \left( 1 + e^{-\beta s} e^{i\pi \pm \varepsilon} \right)$$

$$= \int_0^\infty ds \ln(s^2 - 1 + i\varepsilon) \frac{1}{2} \ln \left( 1 + 2e^{-\beta s} \cos(g_e \cos(\frac{\pi}{2})) + e^{-2\beta s} \right) \quad (15)$$

compare Eq. (13) in [14], and for $g = \pm 2$ Eq. (7) in [44].

The spectral function characterizing the density of virtual particle excitations is $\ln(s^2 - 1 + i\varepsilon)$. The temperature representation has Bose gas character for $g = \pm 2$, and Fermion character for the spinless EHS result, here limit at $g = 0$, see table 1 of [14]. Interestingly, $g = 1$ case yields the Unruh temperature [46] as noted in Ref. [14]: For $g = 1$ in Eq. (15) the cos-term disappears, hence we can redefine in the final exponential $2\beta \rightarrow \beta'$, introducing the Unruh temperature in the context of a Fermi function format.

III. EULER-HEISENBERG ACTION FOR ARBITRARY $g$, AND PSEUDOSCALAR $\vec{E} \cdot \vec{B}$

Properties of the effective action $V^{\text{EHS}}_g$ which are not accessible to the proper time formulation must be explored using alternate methods. A convergent $V^{\text{EHS}}_g$ was extended to $|g| > 2$ for the case of a pure magnetic field in [15], employing the Weisskopf method of summing Landau energy eigen values. The result was a periodic in $g$ action. More recently, we obtained a convergent action for the pure electric field case [16], employing Nikishov’s method of summing Bogoliubov coefficients [17].

Kim, Lee and Yoon used the second order KGP equation to produce a convenient single expression accounting for both $g = \pm 2$ and $g = 0$ solutions [18].

This prior work on $V^{\text{EHS}}_g$ motivates our pursuit of the general case for arbitrary $E$ and $B$ fields. It was postulated [19] that the effective action when both $E$ and $B$ are present exhibits similar behavior as the pure magnetic result. That is, the periodic $g$-dependence of the magnetic result applies also to the configurations with nonzero $\vec{E} \cdot \vec{B}$, producing a convergent action. However, such a behavior implies a pseudoscalar $\vec{E} \cdot \vec{B}$-dependent cusp at $g = \pm 2$. Due to its singular nature, such a feature cannot be proven by analytical continuation of the previous results which consider the $E$ and $B$ cases separately. Thus it is necessary to obtain $V^{\text{EHS}}_g$ with $\vec{E} \cdot \vec{B}$ independently, which we present below.

A. Klein-Gordon-Pauli solution

In order to use the Bogoliubov coefficient summation procedure to obtain the spin $1/2$ $g \neq 2$ effective action we generalize the solution of the ‘squared Dirac equation’ – the Klein-Gordon-Pauli equation – to arbitrary $g$, using the Weyl representation of Dirac matrices [47]:

$$\left( -(i\partial_\mu - eA_\mu)^2 + m^2 - g\sigma(\mathcal{B} - i\lambda\mathcal{E}) \right) \Psi = 0, \quad (16)$$

where $g$ in Eq. (16) can take arbitrary values and

$$\sigma = \pm 1/2, \quad \lambda = \pm 1. \quad (17)$$

$\sigma$ is the spin projection along the axis parallel to $\mathcal{E}$ and $\mathcal{B}$, and $\lambda$ accounts for the reduction in degrees of freedom from the 4-spinor Dirac representation to the 2-spinor Weyl form. We write $\mathcal{E} = |\vec{E}|$ and $\mathcal{B} = |\vec{B}|$: we work in the reference frame where both fields are parallel. Without
Introducing only by $\psi$ as:

$$\Psi = \psi X , \quad X = \frac{1}{\sqrt{2}} \left( 1 \pm 1 \right) . \quad (18)$$

For static homogenous $E$ and $B$ pointing in the $x$-direction, the 4-potential is given by

$$A^a = (0, -\mathcal{E}t, 0, \mathcal{B}y) , \quad (19)$$

using the temporal gauge description of the electric field component. The KGP equation becomes

$$\left\{ \partial_t^2 + (p_x - e\mathcal{E}t)^2 - \partial_y^2 + (p_z + e\mathcal{B}y)^2 \\
+ m^2 - g e \sigma (B - i\lambda \mathcal{E}) \right\} \psi = 0 . \quad (20)$$

Eq. (20) allows us to separate variables in the solution $\psi$ as:

$$\psi = e^{i(p_z x + p_z z)} u(y) f(t) . \quad (21)$$

We first solve for the $u$ component of $\psi$ that is influenced only by $B$. We rewrite Eq. (20) as

$$\left( \partial_t^2 + (p_x - e\mathcal{E}t)^2 + i g e \sigma \lambda \mathcal{E} + \hat{K}^2 \right) \psi = 0 , \quad (22)$$

where operator

$$\hat{K}^2 = -\partial_y^2 + (p_z + e\mathcal{B}y)^2 + m^2 - g e \sigma B . \quad (23)$$

Introducing

$$y = \frac{\eta}{\sqrt{e\mathcal{B}}} \frac{p_z}{e\mathcal{B}} , \quad (24)$$

we recognize that $\hat{K}^2$ provides a harmonic oscillator equation satisfying

$$\hat{K}^2 u = K_n^2 u , \quad (25)$$

solved by

$$u = H_n[\eta] \left( e\mathcal{B} \right)^{1/4} e^{-\eta^2/2} \sqrt{2^n n! \sqrt{\pi}} . \quad (26)$$

$H_n$ is the Hermite polynomial describing the $n$th Landau level, and eigenvalues

$$K_n^2(g, \sigma) = m^2 + e\mathcal{B}(2n + 1 - g\sigma) , \quad n = 0, \pm 1, \pm 2, \ldots . \quad (27)$$

$n$ can have both positive and negative values. In the next section we will determine which $n$ states span the physical Hilbert space.

We return to find $f$, the $\mathcal{E}$-dependent contribution to the wavefunction $\psi$ defined in Eq. (21). Plugging Eq. (25) and Eq. (27) into Eq. (22), the KGP expression reduces to the equation for an inverted harmonic oscillator potential:

$$\left( \partial_t^2 + (p_x - e\mathcal{E}t)^2 + i g e \sigma \lambda \mathcal{E} + K_n^2 \right) f = 0 . \quad (28)$$

We translate Eq. (28) into a parabolic cylinder differential equation by introducing the following variables:

$$Z = \sqrt{\frac{2}{e\mathcal{E}}} e^{i\pi/4} (p_x - e\mathcal{E}t) , \quad (29)$$

and

$$\rho(g, \sigma, \lambda) = -\frac{1}{2} + \frac{g}{2} \sigma \lambda - i \frac{K_n^2}{2 e \mathcal{E}} , \quad (30)$$

which obeys the relation

$$\rho^*(g, \sigma, \lambda) = -\rho(g, \sigma, -\lambda) - 1 . \quad (31)$$

Plugging Eq. (29) and Eq. (30) into Eq. (28), we obtain

$$\left( \partial_t^2 - \frac{ie\mathcal{E}}{2} Z^2 + 2ie\mathcal{E} \left( \rho + \frac{1}{2} \right) \right) f = 0 , \quad (32)$$

which is solved by

$$f = D_\rho(Z) , \quad (33)$$

where parabolic cylinder function $D$ has index $\rho$ given by Eq. (30).

### B. Bogoliubov coefficient summation

The vacuum-to-vacuum amplitude in a constant applied field [17]

$$\langle 0_{t=-\infty} | 0_{t=+\infty} \rangle = e^{iL^3 T V_g^{\text{EHS}}} , \quad (34)$$

where $L^3 T = \text{volume} \times \text{time}$. Eq. (34) can be written as the product of the probabilities for each negative (positive) energy state ($n$) at $t = -\infty$ to remain a negative (positive) energy state at $t = +\infty$:

$$\langle 0_{t=-\infty} | 0_{t=+\infty} \rangle = \prod_k \langle 0_k, t=-\infty | 0_k, t=+\infty \rangle , \quad (35)$$

where $k$ includes all spin and momentum states. Comparing Eq. (34) and Eq. (35) we write the action as

$$V_g^{\text{EHS}} = \frac{i}{L^3 T} \sum_k \ln c_k^* , \quad (36)$$

labeling the Bogoliubov coefficient according to notation in [17]:

$$c_k^{-1} = \langle 0_k, t=-\infty | 0_k, t=+\infty \rangle . \quad (37)$$

We obtain $c_k$ from the $t \to \pm\infty$ limits of the KGP solution Eq. (33). At $t \to -\infty$, Eq. (33) takes on the form

$$D_\rho(Z)_{t\to-\infty} = e^{-Z^2/4} Z^\rho , \quad (38)$$
and at $t \to +\infty$

$$D_{\rho}(Z)_{t\to+\infty} = e^{-i\pi \rho} D_{\rho}(-Z)_{t\to-\infty} \quad (39)$$

$$+ \sqrt{\frac{2\pi}{\Gamma(-\rho)}} e^{-i\pi(\rho+1)/2} D_{-\rho-1}(-Z)_{t\to-\infty} .$$

The coefficient of the first term in Eq. (39) corresponds to the tunneling amplitude through the mass gap, while the second term gives Bogoliubov coefficient

$$c_k = \sqrt{\frac{2\pi}{\Gamma(-\rho)}} e^{-i\pi(\rho+1)/2} . \quad (40)$$

Plugging Eq. (40) into Eq. (36) we obtain the imaginary part of effective action

$$2\Im V_{g}^{EHS} = \frac{1}{L^3 T} \sum_{n,\sigma,\lambda} \left[ \sqrt{\frac{2\pi}{\Gamma(-\rho)}} e^{-i\pi(\rho+1)/2} \right]^2 , \quad (41)$$

with summation

$$\sum_{k} = \frac{1}{2} \sum_{n,\sigma,\lambda} L^2 \int \frac{dp_x dp_z}{(2\pi)^2} = L^3 T \frac{2\xi E B}{8\pi^2} \sum_{n,\sigma,\lambda} , \quad (42)$$

where factor 1/2 factor averages the $\lambda = \pm 1$ contributions, and $\int dp_z = eBL$ and $\int dp_x = eET$, as Eq. (3.7) of [17]. The imaginary action per unit time and volume Eq. (41) becomes

$$\Im V_{g}^{EHS} = \frac{e^2 B\xi}{16\pi^2} \sum_{n,\sigma,\lambda} \ln \left\{ \left[ \frac{\sqrt{2\pi}}{\Gamma(-\rho)} \right] e^{-i\pi(\rho+1)/2} \right\} . \quad (43)$$

Carrying out summation over $\lambda$ first, we have

$$\Im V_{g}^{EHS} = \frac{e^2 B\xi}{16\pi^2} \sum_{n,\sigma} \ln \left\{ \frac{2\pi e^{-i\pi(\rho(g,\sigma,1) + \rho(g,\sigma,-1))/2}}{(\rho(g,\sigma,1))\Gamma(-\rho(g,\sigma,1))} \right\}
\times \frac{2\pi e^{i\pi(\rho(g,\sigma,1) + \rho(g,\sigma,-1))/2}}{\Gamma(-\rho^*(g,\sigma,1))\Gamma(-\rho^*(g,\sigma,-1))} \right\} . \quad (44)$$

The complex conjugated terms in Eq. (44) can be rewritten using the relation Eq. (51):

$$\Im V_{g}^{EHS} = \frac{e^2 B\xi}{16\pi^2} \sum_{n,\sigma} \ln \left\{ \frac{2\pi e^{-i\pi\rho(g,\sigma,1)}}{(\rho(g,\sigma,1))\Gamma(1 + \rho(g,\sigma,1))} \right\}
\times \frac{2\pi e^{-i\pi\rho(g,\sigma,-1)}}{\Gamma(-\rho(g,\sigma,-1))\Gamma(1 + \rho(g,\sigma,-1))} \right\} , \quad (45)$$

which then allows for use of the Euler reflection formula e.g.

$$\frac{1}{\Gamma(-\rho(g,\sigma,\pm 1))\Gamma(1 + \rho(g,\sigma,\pm 1))} = -\frac{\sin(\rho(g,\sigma,\pm 1))}{\pi} . \quad (46)$$

This allows us to rewrite Eq. (45) as

$$\Im V_{g}^{EHS} = \frac{e^2 B\xi}{16\pi^2} \sum_{n,\sigma} \left\{ \ln[1 - e^{-2i\pi\rho(g,\sigma,1)}] + \ln[1 - e^{-2i\pi\rho(g,\sigma,-1)}] \right\} . \quad (47)$$

Using the series representation of the logarithms

$$\ln[1 - e^{-2i\pi\rho(g,\sigma,1)}] = \sum_{l=1}^{\infty} \frac{e^{-2il\pi\rho(g,\sigma,1)}}{l} , \quad (48)$$

and applying definition of $\rho$ in Eq. (30), Eq. (47) becomes

$$\Im V_{g}^{EHS} = \frac{e^2 B\xi}{8\pi^2} \sum_{l=1}^{\infty} \left( -1 \right)^l \frac{e^{2il\pi\xi m^2/E}}{l} \cos\left(\frac{g}{2l}\pi\right) \times \sum_{n,\sigma} e^{-i\pi(K_n^2(g,\sigma) - m^2)/eE} , \quad (49)$$

where $K_n$ is given by Eq. (27), and

$$\frac{K_n^2 - m^2}{eE} = \frac{B}{E}(2n + 1 - g\sigma) , \quad n = 0, \pm 1, \pm 2 \ldots \quad (50)$$

In Fig. 2, we plot the exponential terms $e^{-i\pi(\xi m^2/E)}$, Eq. (49), for different $n$ levels. We identify which $n$ comprise the physical spectrum, that satisfy the correct boundary conditions and preserve unitarity. These are the levels for which $|e^{-i\pi(K_n^2 - m^2)/eE}| < 1$ in Fig. 2 corresponding to the states $K_n^2 \geq m^2$. For these states, the imaginary part of $V_{EHS}^{EHS}$ vanishes as $E \to 0$, ensuring a stable QED vacuum in absence of external fields. The unphysical, non-normalizable $n$ values appear in Fig. 2 as the cases where $|e^{-i\pi(K_n^2 - m^2)/eE}| > 1$, with $K_n^2 < m^2$. These contributions would cause, for sufficiently strong $B$ fields, the imaginary part of action to be nonzero even in the limit $E \to 0$. Requiring a stable QED vacuum in constant magnetic fields, we omit such states.

We count the states in Fig. 2 that make up the physical spectrum. For the domain $-2 \leq g \leq 2$ including the well known $g = 2$ case, we admit $n \geq 0$. The situation changes for $|g| > 2$, where a shift in $g$ by multiples of 4 produces the corresponding change in $n$:

$$e^{-i\pi(K_n^2(g + 4k\sigma) - m^2)/eE} = e^{-i\pi(K_n^2 - 2nk(g\sigma) - m^2)/eE} , \quad k = 0, \pm 1, \pm 2, \ldots \quad (51)$$

The periodic values of $g = 2 + 4k$ are crossing points, at which one state disappears from the spectrum, while another state with opposite spin projection joins the physical spectrum. Thus with each shift in $g \to g + 4k$ there
The affect the condition that is convergent in the proper time formulation e.g applying the physical spectrum for be physical. The spectrum in table I agrees with the re-

ble I using Fig. 2 for the specific example of $g$. We note that while we determined the states in ta-

ble I, is essential for recovering the correct continuum limit of states (as $B \to 0$) which enter the Bogoliubov summation for electric Sauter step action. We also recover the known $g = \pm 2$ and $g = 0$ limits of the imaginary part of action in table II, compare Nik-

ishov’s [17] Eq. (3.11) and Eq. (3.8), noting that the $g = 0$ case differs from the spin-0 result by a factor $-2$ accounting for spin multiplicity and an extra sign accompanying loop Fermionic corrections.

| $-6 < g < -2$ | $\sigma = +1/2$ | $n = -1, 0, 1, \ldots$ |
| $-2 < g < 2$ | $\sigma = -1/2$ | $n = 1, 2, 3, \ldots$ |
| $2 < g < 6$ | $\sigma = +1/2$ | $n = 0, 1, 2, \ldots$ |
| $-2+4k < g < 2+4k$ | $\sigma = -1/2$ | $n = 0, 1, 0, \ldots$ |

| n states for which $|e^{-l\pi(K^2_n-g^2)/\epsilon}| < 1$ in Fig. 2 Eq. (50). |

TABLE I. n is a duplication of the physical spectrum. Which $n$ levels are admitted depends on the branch $-2+4k < g < 2+4k$ in which $g$ resides, table I.

We note that while we determined the states in table I for the specific example of $l = 1$ and $B/\epsilon = 1$, the argument can be generalized to arbitrary field strengths. The relative strengths of $B$ and $E$ do not affect the condition $K^2_n > m^2$ in order for the states to be physical. The spectrum in table I agrees with the result from prior work on the magnetic Weisskopf action for $|g| > 2$ [15].

We carry out the summation over $n$ and $\sigma$ in Eq. (49), applying the physical spectrum $n$ in table I and Eq. (51):

$$
\sum_{\sigma=\pm 1/2} \sum_{n=2\pi k} e^{-l\pi(K^2_n(g,\sigma)-m^2)/\epsilon} = \sum_{\sigma=\pm 1/2} \sum_{n=0}^{\infty} e^{-l\pi(K^2_n(g+4k,\sigma)-m^2)/\epsilon},
$$

(52)

periodic in $g$ by shifts in $4k$. For arbitrary $|g| > 2$, we choose $k$ such that we can define a periodically reset $g_k$ which lies in the principal domain

$$-2 \leq g_k = g + 4k \leq 2. $$

(53)

Thus any summation Eq. (52) carried out for a $g$ value in the domain $|g| > 2$ has an equivalent summation using $|g_k| \leq 2$. This allows us to convert the effective action with $|g| > 2$ to an equivalent expression with $|g_k| \leq 2$ that is convergent in the proper time formulation e.g the perturbative electron $g$-factor resets according to

$$g_{\text{electron}} = 2 + \alpha/\pi \rightarrow g_k = g_{\text{electron}} - 4 = -2 + \alpha/\pi. $$

(54)

The $g_{\text{electron}} > 2$ can now be applied to the proper time formulation of $\mathfrak{Im} V^\text{EHS}_g$ Eq. (8) previously limited to $|g| \leq 2$, by resetting $g_{\text{electron}}$ to the convergent $|g_k| \leq 2$ domain.

We can now sum Eq. (52) using the series

$$
\sum_{n=0}^{\infty} e^{-l\pi B(2n+1)/\epsilon} = \frac{1}{2 \sinh(l\pi B/\epsilon)},
$$

(55)

to obtain

$$
\sum_{\sigma=\pm 1/2} \sum_{n=0}^{\infty} e^{-l\pi(K^2_n(g+4k,\sigma)-m^2)/\epsilon} = \frac{\sinh(\frac{g_k}{2}l\pi B/\epsilon)}{\sinh(l\pi B/\epsilon)}. $$

(56)

We evaluate the imaginary part of $V^\text{EHS}_g$ by plugging Eq. (56) into Eq. (49) and recognizing periodicity of the remaining $g$-dependent term

$$
\cos(g/2) = \cos(g/2 + 2\pi k) = \cos(g + 4k/2)l\pi, $$

(57)

to obtain

$$
\mathfrak{Im} V^\text{EHS}_g = \frac{e^2 ab}{8\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sinh\left(\frac{g_k l\pi b/a}{2}\right) \cosh\left(\frac{g_k l\pi b/a}{2}\right). $$

(58)

In consideration of Lorentz invariance we wrote $E$ and $B$ in terms of the EM field tensor eigenvalues $E \rightarrow a$ and $B \rightarrow b$, Eq. (58) is equivalent to Eq. (59) in the $|g| \leq 2$ domain. However, now instead of $g$ due to the periodic behavior the reset value $g_k$ appears in Eq. (53).

As a verification of our approach we consider the $B \to 0$ limit: Eq. (58) becomes

$$
\mathfrak{Im} V^\text{EHS}_g \rightarrow \frac{e^2 \pi^2}{8\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^2 \pi} \cos\left(\frac{g_k l\pi b/a}{2}\right) e^{-l\pi m^2/\epsilon}, $$

(59)

recovering the periodic in $g$ quasi-constant limit of the electric Sauter step action [16]. We note that the identification of the (discrete) physical set of Landau $n$ states, table I, is essential for recovering the correct continuum domain of states (as $B \to 0$) which enter the Bogoliubov summation for electric Sauter step action.

We also recover the known $g = \pm 2$ and $g = 0$ limits of the imaginary part of action in table I, compare Nik-

ishov’s [17] Eq. (3.11) and Eq. (3.8), noting that the $g = 0$ case differs from the spin-0 result by a factor $-2$ accounting for spin multiplicity and an extra sign accompanying loop Fermionic corrections.

| $g$ | $\cos(g/2)l\pi$ | $\cosh(g/2)l\pi B/\epsilon$ |
|-----|----------------|----------------------------|
| 0   | 1              | 1                          |
| \pm 2| $(-1)^l$       | $\cosh(l\pi B/\epsilon)$  |

TABLE II. The $g \to 0$ and $g \to \pm 2$ limits of the $g$-dependent terms in Eq. (59).

C. Ramanujan equation

Our next objective is to uniquely determine the full effective action $V^\text{EHS}_g$, including the real part. For analytical functions this only requires that we know the
imaginary part. Since we already know that our result produces a singular function of $g$ with cusps we proceed with care to show that there is a unique proper time integral representation of $V_{g}^{\text{EHS}}$, in which the integrand in proper time representation has the pole structure required to produce the imaginary part Eq. [58], thereby determining $V_{g}^{\text{EHS}}$.

Our results will arise from a $g$-dependent generalization of the Ramanujan expression for meromorphic expansion of $V_{g}^{\text{EHS}}$ as was proposed in Ref. [15]. The difference to prior work is that we have obtained $\Im mV_{g}^{\text{EHS}}$, the imaginary part of Euler-Heisenberg-Schwinger (EHS) type effective action by explicit evaluation in section III B.

The meromorphic expansion of the poles of the proper time integrand is well known for $g = \pm 2$ [11, 50]. We obtain an extension of the $g = \pm 2$ expression Eq. (6) of the work by Cho and Pak [50]:

$$\Im mV_{g}^{\text{EHS}} = \frac{e^{2}a^{2}}{8\pi^{3}} \sum_{n=1}^{\infty} \frac{(-1)^{n} \cos \left(\frac{2n}{g} \pi \right)}{n^{2}} \left( e^{-n \pi^{2}/ea} + 2(ebm) \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \cos \left(\frac{2\ell}{g} \pi \right) e^{-\ell \pi^{2}/ea}}{e^{2}a^{2}n^{2} + e^{2}b^{2}\ell^{2}} \right).$$

Swapping indices $n \leftrightarrow \ell$ in the second term we obtain

$$\Im mV_{g}^{\text{EHS}} = \frac{e^{2}a^{2}}{8\pi^{3}} \sum_{n=1}^{\infty} \frac{(-1)^{n} e^{-n \pi^{2}/ea}}{n^{2}} \cos \left(\frac{2\ell}{g} \pi \right) \left( 1 + 2 \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \cos \left(\frac{2\ell}{g} \pi \right)}{1 + a^{2}\ell^{2}/b^{2}n^{2}} \right).$$

---

\[ F(x, y, \frac{g}{2}) = \frac{x \cosh(\frac{g}{2}x) y \cos(\frac{g}{2}y)}{\sinh(x)} - 1 = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n} \cos(\frac{2n}{g} \pi \pi)}{n^{2} \pi^{2}} \left( x^{2} - y^{2} + \frac{y^{4}}{y^{2} - n^{2} \pi^{2}} - \frac{x^{4}}{x^{2} + n^{2} \pi^{2}} \right) \]

We apply the following Fourier series which is defined for $|g| \leq 2$ and periodic otherwise when $|g| > 2$:

$$\cosh\left(\frac{g}{2} \pi \right) = a_{0} + 2 \sum_{\ell=1}^{\infty} a_{\ell} \cos\left(\frac{g}{2} \ell \pi \right), \quad a_{\ell} = \frac{1}{2} \int_{0}^{2} dg \cosh\left(\frac{g}{2} \pi \right) \cos\left(\frac{g}{2} \ell \pi \right) = \frac{(-1)^{\ell}}{\pi \ell(1 + \ell^{2}/z^{2})}. $$

We plug Eq. (61) into Eq. (60) to obtain

$$F(x, y, \frac{g}{2}) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n} \cos \left(\frac{2n}{g} \pi \right)}{n^{2} \pi^{2}} \left( x^{2} - y^{2} + \frac{y^{4}}{y^{2} - n^{2} \pi^{2}} - \frac{x^{4}}{x^{2} + n^{2} \pi^{2}} \right) \left( y^{4} x^{2}/(y^{2} - n^{2} \pi^{2})/(n^{2} x^{2} + \ell^{2} y^{2}) - \frac{x^{4} y^{2}}{(x^{2} + n^{2} \pi^{2})/(n^{2} y^{2} + \ell^{2} x^{2})} \right).$$

In the last term in parenthesis of the second line Eq. (62) we exchange indices $n \leftrightarrow m$ to obtain

$$\Im mV_{g}^{\text{EHS}} = \frac{e^{2}a^{2}}{8\pi^{3}} \sum_{n=1}^{\infty} \frac{(-1)^{n} \cos \left(\frac{2n}{g} \pi \right)}{n^{2}} \left( n \right) \left( e^{-n \pi^{2}/ea} + 2(ebm) \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} \cos \left(\frac{2\ell}{g} \pi \right) e^{-\ell \pi^{2}/ea}}{e^{2}a^{2}n^{2} + e^{2}b^{2}\ell^{2}} \right).$$

We verify that the expression $F(eau, ebu, \frac{gk}{2})$, $k = 0, \pm 1, \pm 2, \ldots$

$$F(eau, ebu, \frac{gk}{2}) = F(eau, ebu, \frac{gk}{2}), \quad k = 0, \pm 1, \pm 2, \ldots,$$

where we convert $F$ to its equivalent expression in terms of $|g| \leq 2$, Eq. (63). We thus have shown that Eq. (64) can be used in the proper time integration, where any $|g| > 2$ is expressed in terms of the proper time integrand within the convergent $|g| \leq 2$ domain.

We verify that the expression $F(eau, ebu, \frac{gk}{2})$ for arbitrary $g$ contains the correct pole structure to produce
Recognizing the Fourier transform Eq. (60) in the last line, we recover Eq. (58).

Having verified the periodic in $g$ proper time integral representation Eq. (64), we obtain the $|g| > 2$ extension of Eq. (6):

$$V_{g}^{\text{EHS}} = \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{du}{u^{3}} e^{-i(m^{2}-u^{2})a} F(eau, ebu, \frac{gk}{2}), \quad (67)$$

$$F(eau, ebu, \frac{gk}{2}) = eau \cosh(\frac{gk}{2}eau) ebu \cosh(\frac{gk}{2}ebu) \frac{\sinh(eau)}{\sinh(ebu)} - 1.$$  

Eq. (67) ensures that $|g| > 2$ action has an equivalent form using $|g_k| \leq 2$ in Eq. (53) producing a convergent result with the proper time integration.

**IV. SINGULAR AND NONPERTURBATIVE PROPERTIES**

This section describes several properties of $V_{g}^{\text{EHS}}$ that rely on the nonperturbative treatment of the magnetic moment. Without doubt other interesting properties will appear in the future; this should be considered an initial consideration of the opportunities opened up by applying a nonperturbative method to achieve resummation of diagrams, see Fig. 1. The highly singular outcome of the magnetic moment anomaly should be also a warning not to draw quantitative conclusions too soon about physical processes based on a relatively small subset of all diagrams the original EHS effective action $V_{g}^{\text{EHS}}|_{g=\pm 2}$ represents.

**A. Singular properties of $V_{g}^{\text{EHS}}$ as a function of $g$**

As a consequence of the periodicity in $g$, the effective action $\Im V_{g}^{\text{EHS}}$ peaks at the singular points $g = \pm 2$. This is particularly well visible in considering the imaginary part: Normalizing Eq. (58) to the $g = \pm 2$ EHS value

$$R(g) = \frac{\Im V_{g}^{\text{EHS}}}{\Im V_{\text{EHS}}} = \sum_{l=1}^{\infty} \frac{(-1)^{l} e^{-l\pi m^{2}/ea}}{l} \frac{\cosh(\frac{gk}{2}l\pi b/a) \cosh((g_k/2)l\pi b/a)}{\sinh(l\pi b/a)}, \quad (68)$$

where we use Eq. (58) both for $g = 2$ in denominator and as a general expression in the numerator. Considering the leading terms in both denominator and numerator we note that $\frac{\cosh(\frac{gk}{2}l\pi b/a)}{\sinh(l\pi b/a)}$ in the numerator is always equal or smaller than $\frac{\cosh(\pi b/a)}{\sinh(\pi b/a)} = \coth(\pi b/a)$ in the denominator, since we converted $\Im V_{g}^{\text{EHS}}$ for arbitrary $|g| > 2$ to an equivalent expression in terms of periodically reset $|g_k| \leq 2$, Eq. (53). As a result, $\Im V_{g}^{\text{EHS}}$ is at normalized maximum at $g_k = 2$, corresponding to values $g = 2 + 4k, k = 0, \pm 1, \pm 2, \ldots$. It is of considerable interest to understand analytically how $g \neq 2$ suppresses $\Im V_{g}^{\text{EHS}}$. Tewit we use the cosh addition theorem

$$\frac{\cosh(\frac{gk}{2}l\pi b/a)}{\sinh(l\pi b/a)} = \cosh((1 - |g_k|/2)l\pi b/a) \coth(l\pi b/a) - \sinh((1 - |g_k|/2)l\pi b/a), \quad (69)$$

where $(1 - |g_k|/2) \geq 0$. Using in Eq. (68) relation Eq. (69) with $\coth(l\pi b/a) \to \coth(l\pi b/a) + 1 - 1$ we obtain after some algebra

$$R(g) = \sum_{l=1}^{\infty} \frac{(-1)^{l} e^{-l\pi m^{2}/ea}}{l} \frac{\cosh(\frac{gk}{2}l\pi b/a)}{\sinh(l\pi b/a)} \frac{e^{-(1-|g_k|/2)l\pi b/a} + e^{-l\pi b/a} \cosh((1 - |g_k|/2)l\pi b/a)}{\coth(l\pi b/a)}, \quad (70)$$

Remembering $1 + e^{-x} / \sinh x = \coth x$ we recover $R \to 1$ for any value of $b$ for $g_k \to \pm 2$.

We find suppression of the imaginary part of action in the presence of $(1 - |g_k|/2)l\pi b/a \gg 1$ for all field strengths and $g$ values. The larger $(1 - |g_k|/2)l\pi b/a$, i.e. the more magnetic-interaction dominated is the particle-EM field configuration, the more pronounced is the resultant suppression of the imaginary part of the action due to the first term.
To demonstrate that the cusp points, here as example \( g = 2 \), in Fig.~3 are truly singular for \( b \neq 0 \) (and not only a sharply peaked function that is smooth at \( g = 2 \)), we compute the discontinuity in the derivative of \( 3mV_e^{\text{EHS}} \) with respect to \( g \). Differentiating Eq. (58) with respect to \( g \) at opposite sides of \( g = 2 \) and taking the difference we obtain

\[
\begin{align*}
\frac{\partial^2 mV_e^{\text{EHS}}}{\partial g^2} \bigg|_{g=2+} - \frac{\partial^2 mV_e^{\text{EHS}}}{\partial g^2} \bigg|_{g=2-} &= e^2 ab \sum_{l=1}^{\infty} \left( -1 \right)^l e^{-l \pi m^2/ea} l \cos\left( \frac{bk}{2} \right) \bigg|_{g=2} \\
& \times \frac{\partial}{\partial g} \left( \frac{\cosh\left( \frac{\pi l b}{a} \right) - \cosh\left( \frac{\pi b}{2 a} \right)}{\sinh\left( \frac{l \pi b}{a} \right)} \right) \bigg|_{g=2} \\
&= \frac{e^2 b^2}{8\pi} \sum_{l=1}^{\infty} e^{-l \pi m^2/ea} \frac{e^{2b^2}}{8\pi (e^{\pi m^2/ea} - 1)} ,
\end{align*}
\]

where \( g = 2+ \) and \( g = 2- \) denote \( g \) approaching 2 from opposite sides of the cusp. 2+ is already smaller than 2 and thus requires no \( g \)-reset, while 2− is barely above 2 thus requiring \( g \)-reset 2+ → 2− + 4 according to Eq. (53).

We see that only at exactly \( b = 0 \) the discontinuity vanishes.

### B. Relevance of non-perturbative treatment of electron magnetic moment

When \( g \) is not exactly equal to 2, Fig.~3 demonstrates a stabilizing effect on the vacuum. In magnetic dominated fields \( b/a \rightarrow \infty \), the width of the peak in \( 3mV_e^{\text{EHS}} \) shrinks until the particle production can only occur at points \( g = 2 \pm 4k \), \( k = 0, \pm 1, \pm 2, \ldots \). Thus even a small deviation from \( g = \pm 2 \) can cause \( 3mV_e^{\text{EHS}} \) to fall off the peak (and tend to zero), suppressing particle production.

The above indicates that even though the deviation of the magnetic moment of electrons from the Dirac value is relatively small, with the gyromagnetic ratio \( g_{\text{electron}} = -2 - \alpha/\pi + \ldots \) we cannot assume that nonperturbative evaluation of electron-positron pair production allowing for this small correction is not required. We now clarify EM field regimes for which the non-perturbative treatment of electron magnetic moment for pair production matters, which turns out to be the magnetically dominated environment.

Using the reset Eq. (54) we recognize as noted below Eq. (70) the relevant parameter

\[
\frac{g_{\text{electron}}}{2} = -1 - \frac{\alpha}{2\pi} \rightarrow g_{2b} = 1 - \frac{\alpha}{2\pi} ,
\]

\[
\chi_b \equiv (1 - |g_{2b}|) \pi b/a = \alpha b/2a .
\]

We now restate \( R \), Eq. (70), using Eq. (76) and \( \cos(\pi l - x) = (-1)^l \cos x \).
This is the final exact nonperturbative in $O(\alpha)$ analytical result describing instability of the QED vacuum with regard to $e^{-e^+}$-pair production, using as reference the EHS result with $g = \pm 2$. The perturbative expansion in $\alpha$ of the nominator term in Eq. (77) creates the coefficient function $A_n$ shown to sixth order below

\begin{align}
A_1 & = -\frac{lb/a}{2}, \\
A_2 & = \frac{l^2}{8} \left( (b/a)^2 - 1 \right) \coth(l\pi b/a), \\
A_3 & = \frac{l^3}{48} \left( 3(b/a) - (b/a)^3 \right), \\
A_4 & = \frac{l^4}{384} \left( 1 - 6(b/a)^2 + (b/a)^4 \right) \coth(l\pi b/a), \\
A_5 & = \frac{l^5}{3840} \left( -5(b/a) + 10(b/a)^3 - (b/a)^5 \right), \\
A_6 & = \frac{l^6}{46080} \left( -1 + 15l^2(b/a)^2 - 15(b/a)^4 + (b/a)^6 \right) \times \coth(l\pi b/a). 
\end{align}

The series in powers of $\alpha$ represents the individual contributions of the diagrams in Fig. [1].

The reference decay rate is seen in the denominator in Eq. (77) and requires inclusion of the canceling common factor $ab/2\pi$, compare Eq. (5). Note that when the normalized electric field $\tilde{a}$ is sufficiently small only $l = 1$ terms contribute in the $\alpha$ series of the nominator. In Fig. [4] we present results for $\tilde{a} = 1$, and $\tilde{a} = 1/10$ representative of weak but still functional $e^-e^+$-pair production.

We show in Fig. [4] the infinite order vertex summation alongside its perturbative expansion. Top frame is for $\tilde{a} = 1$, while bottom for $\tilde{a} = 1/10$. The solid line labeled ‘∞’ denotes the vertex correction summation to infinite order as seen in Eq. (77). The plots ‘1, 2, 3,…’ denote the perturbative summations in Eq. (78), coefficients were shown to 6th order in $\alpha$. We see that for sufficiently large $b/a$ where $\chi_b > 1$, the even power perturbative expansion in $\alpha$ breaks down while the odd-power becomes inaccurate. The nonperturbatively summed solution is needed to describe the physical $\beta_nV_{\text{EHS}}^\gamma$ behavior when $\chi_b > 1$. In sufficiently strong $b$-fields the perturbative summation is reliable only for $\chi_b < 1$.

The results we presented are of interest in study of pair production in a magnetar environment. The magnetic fields in range $5 < \tilde{b} < 100$ are accompanied by electric fields which are at most subcritical [52], offering a suitable environment for probing the nonperturbative $\chi_b$ regime seen in Fig. [4]. The search for particle production on magnetars is ongoing and remains an open question potentially testing QED [53–55], considering both charged and neutral particles [56, 57]. Up to now studies of the electron vacuum response in magnetars have been based on evaluation of effective action at $g = \pm 2$ and in following we extend this to physical values of $g$.

However, in extreme environments of magnetars we cannot be sure that the magnetic moment of an electron is what it is in vacuum. Therefore we will consider the magnetic stabilizing effect varying the value of $g$ near $g_{\text{electron}} = -(2 + \alpha/\pi)$ (solid line) shown as a function of the (generalized) magnetic to electric field strength ratio $b/a$; top frame for $\tilde{a} = 1$, bottom frame for $\tilde{a} = 1/10$. Other lines: perturbative expansion in powers of $\alpha$ of the analytic nonperturbative exact result. The orders in $\alpha$ are labeled. Vertical lines indicate the parameter $\chi_b$ Eq. (78) characterizing the magnetic dominance.

\begin{align}
R_{\text{electron}} &= \frac{\sum_{l=1}^{\infty} \frac{e^{-l^2m^2/\alpha}}{l} \cos(l\alpha/2) \left( e^{-l\chi_b} + e^{-l\pi b/a} \coth(l\pi b/a) \right)}{\sum_{l=1}^{\infty} \frac{e^{-l^2m^2/\alpha}}{l} \coth(l\pi b/a)} \\
&= \frac{\sum_{l=1}^{\infty} \frac{e^{-l^2m^2/\alpha}}{l} \left( \coth(l\pi b/a) + \sum_{n=1}^{\infty} \alpha^n A_n(b/a;l) \right)}{\sum_{l=1}^{\infty} \frac{e^{-l^2m^2/\alpha}}{l} \coth(l\pi b/a)}. 
\end{align}
FIG. 5. $R$ from Eq. (70) plotted as a function of $g$, near to $g_{\text{electron}} \sim 2 + \alpha/\pi$ and the corresponding $\chi_b$ values from Eq. (76), labeled. We present three different configurations of $\tilde{a}$ and $\tilde{b}$, Eq. (72).

urations and values of $g$.

We demonstrate how the nonperturbative in $g$ suppression impacts this result by evaluating $R$ for different $\chi_b$ values relevant to magnetar fields. In Fig. 5, we plot $R$ from Eq. (70) for a small domain of $g$ centered on $g \sim 2 + \alpha/\pi$. We consider a subcritical electric-likie $\tilde{a} = 0.1$, and three magnetic $\tilde{b}$ values within the expected magnetar regime of 1-100 times the EHS critical field (Eq. (73)) [58, 59]. At $\pm g = 2 + \alpha/\pi$, we find for the case $\tilde{a} = 0.1, \tilde{b} = 5$ (blue, dashed, $\chi_b \sim 1/5$) that the suppression effect is relatively small (20%). The $g = \pm 2$ pair production is reduced by factor 2 for $\tilde{a} = 0.1, \tilde{b} = 25$ (red, dot-dashed, $\chi_b \sim 1$). In the case $\tilde{a} = 0.1, \tilde{b} = 100$ (black, solid, $\chi_b \sim 4$), the suppression is nearly two orders of magnitude.

To recognize this large suppression ab-initio consideration of $g \neq 2$ is required. Had we considered $g = \pm 2$ exclusively, there would be a monotonically increasing (linear) in $b$ enhancing effect on particle production in magnetic dominated fields, quite opposite of the results we demonstrated.

C. Connection to non-abelian theory

We comment on the behavior of the beta-function at $g$ values far from the singular points. Within the domain $|g| \leq 2$, the beta-function changes sign according to the charge-renormalization subtraction Eq. (7), see [49]. This result was extended to periodic domains for $|g| > 2$ [14].

\begin{equation}
\beta(e) = \mu \frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2} \left( \frac{3g_b^2}{8} - \frac{1}{2} \right).
\end{equation}

Interestingly, Eq. (79) is negative between the points where lines cross in Fig. 3, thus in domains of $g$ as follows

$$\beta(e) < 0, \quad -\sqrt{4/3} + 4k < g < \sqrt{4/3} + 4k.$$  \hspace{1cm} (80)

Asymptotic freedom thus arises in such domains of the Abelian $V_g^{\text{EHS}}$ formulation, allowing comparison with the non-Abelian Yang Mills vacuum [43].

We note that in these $g$ domains $\Im V_g^{\text{EHS}}$ is negative as seen in Fig. 3. Since both signs, $\beta$-function and pair instability, change sign in the asymptotically free domains Eq. (80), a study of QED vacuum structure may resolve the paradox of growing vacuum persistence probability that a negative imaginary part signals. Even so, recall that in the limit $b/a \rightarrow \infty$, the vacuum becomes completely stable even though pseudoscalar $\tilde{\xi} \cdot \tilde{B}$ is nonzero. This feature agrees with the recent finding that chromo-magnetic dominated fields with a nonvanishing pseudoscalar can be stable in the Savvidy vacuum state [42].

D. Temperature representation

The format of periodic in $g$ function $V_g^{\text{EHS}}$ now available can be applied to extend prior work on the temperature representation of EHS action [14, 44]. First we consider the electric-dominated action. Like the proper time integration method for inserting $g \neq 2$ in Eq. (6), the temperature representation is on first sight restricted to the domain $|g| \leq 2$, see Eq. (13) in section II, or Eq. (13) of [14].

To extend this result to arbitrary $g$, we recall the conversion between effective action with $|g| > 2$ to an equivalent form periodically reset to $|g| \leq 2$, Eq. (53). This allows for a convergent result in both the proper time and the temperature representation integrands. Thus the prior result for $|g| \leq 2$ in Eq. (13) of Ref. [14] requires only replacement $g \rightarrow g_k$ in order to describe all possible magnetic moments. Consequently, $V_g^{\text{EHS}}$ for $g = 2 + 4k$ values ($k = 0, \pm 1, \pm 2, \ldots$) corresponds to the (spin inverted) bosonic distributions, while for $g = 0 + 4k$ the representation is Fermionic.

We consider separately the case of magnetic-dominated fields. The procedure for deriving the corresponding temperature representation follows closely to the electric case summarized in section II. We start with the proper time integral form of $V_g^{\text{EHS}}$ given by Eq. (67), allowing a convergent expression for any $g$. In the vanishing electric field limit, with rotation of the integration contour $u \rightarrow -iu$, Eq. (67) becomes

$$V_g^{\text{EHS}} = -\frac{1}{8\pi^2} \int_0^\infty \frac{du}{u^3} e^{-m^2 u} \left( \frac{ebu \cosh(\frac{3}{2}ebu)}{\sinh(ebu)} - 1 \right),$$

(81)
which we write in terms of the meromorphic expansion
\[ \frac{\text{ebs}(\frac{n\pi}{2}) \text{ebs}}{\sin(\text{ebs})} - 1 = -\frac{e^2 b^2 u^2}{3} \left( \frac{3g^2}{8} - \frac{1}{2} \right) \] (82)
\[ - 2e^2 b^2 u^4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\frac{n\pi}{2} \text{ebs})}{n^2 \pi^2 (u^2 + n^2 \pi^2 / e^2 b^2)} . \]

We plug Eq. (82) into Eq. (81), remove the charge renormalization contribution, and exchange summation with integration to obtain
\[ V_g^{\text{EHS}} = \frac{e^2 b^2}{4\pi^2} \sum_{n=1}^{\infty} \int_{0}^{\infty} du u e^{-m^2 u} \frac{(-1)^n \cos(\frac{n\pi}{2} \text{ebs})}{n^2 \pi^2 (u^2 + n^2 \pi^2 / e^2 b^2)} . \]
(83)

Substituting \( u \to n\pi u / eb \) and exchanging summation with integration again,
\[ V_g^{\text{EHS}} = \frac{e^2 b^2}{8\pi^2} \int_{0}^{\infty} du \frac{2u}{u^2 + 1} \times \sum_{n=1}^{\infty} e^{-n\pi m u / eb} \frac{(-1)^n \cos(\frac{n\pi}{2} \text{ebs})}{n^2 \pi^2} . \]
(84)

Integrating Eq. (84) by parts,
\[ V_g^{\text{EHS}} = v \int_{0}^{\infty} du \ln(u^2 + 1) \sum_{n=1}^{\infty} e^{-n\beta u} \frac{(-1)^n \cos(\frac{n\pi}{2} \text{ebs})}{n\pi} , \]
where now
\[ \beta = \pi m^2 / eb , \quad v = \frac{m^4}{8\pi^2 \beta} . \]
(85)

Summing Eq. (85) over \( n \) we obtain
\[ V_g^{\text{EHS}} = -v \int_{0}^{\infty} du \ln(u^2 + 1) \left[ \frac{1}{2} \sum_{\pm} \ln(1 + e^{-\beta u} e^{\pm i\frac{n\pi}{2} \text{ebs}}) \right] . \]
(86)

Comparing the magnetic field action Eq. (87) to the electric case Eq. (13) of [14] (Eq. (15) in section [11] here), the \( g \)-dependent logarithmic terms on the RHS are identical within the domain \( |g| \leq 2 \). Thus in this \( g \) domain the two expressions obey the same statistical model representation. Given that our conversion \( g \to g_b \) is equal for electric and magnetic fields, the periodic \( |g| > 2 \) extensions of these two statistical representations are also equal. The one difference is between the spectral function terms describing density of the virtual particles: \( \ln(u^2 + 1) \) appears in the B-dominated integrand, compared to \( \ln(u^2 - 1 + i\varepsilon) \) in the E field case.

V. SUMMARY, CONCLUSIONS AND OUTLOOK

We have studied the Euler-Heisenberg-Schwinger (EHS) effective action \( V_g^{\text{EHS}} \) generalized to, and as a function of, the gyromagnetic ratio \( g \). We have demonstrated a cusp singularity in the vicinity of \( g = \pm 2, \pm 6, \ldots \). For arbitrary quasi-constant field configurations we have shown periodicity in \( V_g^{\text{EHS}} \) as a function of \( g \) recognized before for the pure electric \( E \) [10], and pure magnetic \( B \) field cases [13].

The nonperturbative in \( g \) singular behavior of \( V_g^{\text{EHS}} \) in the presence of an anomalous magnetic moment \( g \neq 2 \) was conjectured in our prior work [15] [18]. In this work using the Bogoliubov coefficient summation method [17,18] we were able prove this singular behavior dominated by nonvanishing pseudoscalar \( \vec{E} \cdot \vec{B} \). This singularity escaped prior attention since considered for pure electric \( E \), or pure magnetic \( B \) fields, \( V_g^{\text{EHS}} \) is smooth and differentiable at \( g = \pm 2 \).

We have shown that the sharpness of the cusp in \( \Im V_g^{\text{EHS}} \) at \( g = \pm 2 \) is dependent on the EM fields in a nonperturbative fashion, and occurs for nonzero \( \vec{E} \cdot \vec{B} \), based on the nonperturbative discontinuity in \( dV_g^{\text{EHS}} / dg \) shown in Eq. (74), section [IV.A]. We believe that the importance of nonvanishing pseudoscalar \( \vec{E} \cdot \vec{B} \) relates to the relatively strong coupling of two photons to the singlet pseudoscalar 0-para-positronium and the related fast decay channel (\( \tau_{\gamma} = 0.124 \) ns, to be compared to \( \tau_{\gamma} = 142 \) ns for the triplet ortho-positronium coupling to odd number of photons).

We have explored some of the nonperturbative properties of \( V_g^{\text{EHS}} \). Most interesting is that the cusp effect for magnetically dominated fields \( \Im V_g^{\text{EHS}} \), viz. the pair production is heavily suppressed, see Fig. 3 in section [IV.A]. We presented explicit dependence of this suppression effect exploring several important values of EM fields \( E \to a, B \to b \), Fig. 3 in section [IV.A], confirming the conjectured results presented in Ref. [60]. Considering proportionality of the magnetic moment \( \mu \propto g/m \) and viewing our results as a function of \( \mu \) rather than \( g \) we conjecture equivalence of our results to an effective mass modification, Eq. (71). In this case our result is reminiscent of the analysis of higher order loop contributions to the imaginary part of EHS action carried out in [61] [62]. However this was applied to electrically dominated fields, while our present work focuses on magnetically dominated environments.

We have identified a smallness parameter \( \chi_b \), Eq. (76) describing at which value of \( b/a \) the nonperturbative \( g \)-modification of the EHS action is significant, Fig. 4 in section [IV.B]. Our result demonstrates parallels with the study of perturbative QED breakdown based on the Ritus-Narozhny conjecture [33] [41], for an extensive review see [63]. As we have shown, the nonperturbative in \( \vec{E} \cdot \vec{B} \) magnetically dominated EM fields present an entirely different environment in which we identified strong suppression of particle production.

We have presented explicit dependence of this suppression effect on the ratio \( B/E \), allowing for direct application of our results to EM fields relevant to astrophysical environments in which \( B \) field dominates near-critical
\( \mathcal{E} \) fields [53–55]. The anomalous magnetic moment can suppress the particle production rate by orders of magnitude, see Fig.3 in section [IVB] This nonperturbative \( g \)-dependence is a step towards addressing the question as to whether magnetar fields generate pair production or are pair-stable environments. Our \( g \) correction is important argument greatly influencing in magnetically dominated conditions the process of pair production.

This singular behavior at \( g = \pm 2 \) for quasi-constant fields of any strength leads us to the question: more generally, could there be higher order modifications of the conventional perturbative QED expansion which is carried out at \( g = \pm 2 \) reflecting on this singular behavior in presence of external fields? This is probably so so in specific channels in which even powers of pseudoscalar as the perturbative series for \( g \) in QED relies on the evaluation of the energy change of a particle in presence of an external EM field and this is exactly what we have done using the external field EHS method. Importantly, since the effective action dependence on \( g \) is nonperturbative for certain external EM field configurations, a perturbative series defining \( g \) has to be reviewed to allow for singular behavior; even a small deviation from the Dirac equation value \( g = \pm 2 \) can have a significant effect. Addressing this situation in the context of actual precision experimental environment is perhaps the most important open question arising from our work.

In this work we illuminated the singular behavior of the imaginary part of the effective action \( 3m\mathcal{V}_{EHS}^{g} \) as a function of \( g \) considering the pair production rate in Eq. (74). However, the singular properties of the full effective action \( \mathcal{V}_{EHS}^{g} \) require a deeper consideration beyond scope of this work: \( g \) appears associated with the magnetic field \( \hat{\mathcal{E}} \) since \( g \) acts as a spin - field coupling. Thus a singular behavior in \( g \) seen in Eq. (74) indicates also singular behavior of \( \mathcal{V}_{EHS}^{g} \) as function of \( \hat{\mathcal{E}} \). Presence of a cusp as a function of \( \hat{g}_{\mathcal{E}} \) would appear as a discontinuity in the magnetic susceptibility. We conclude that our results may be indicating presence of 2nd order phase transition in QED with magnetically dominated strong fields.

Our QED result for strong fields considered with variable magnetic moment can mimic asymptotic freedom of strong interactions and there are some parallels of our work with the those usually associated with vacuum structure in QCD. For example, \( 3m\mathcal{V}_{EHS}^{g} \) is suppressed in certain domains of \( g \) in which also asymptotic freedom arises in our Abelian theory, Eq. (80). This feature parallels recent results of Suvvidy [42] who demonstrated that the asymptotically free Yang Mills Lagrangian is stable in (chromo) magnetic dominated fields, allowing for the presence of nonvanishing (chromo) \( \hat{\mathcal{E}} \cdot \hat{\mathcal{B}} \) field configurations.

To further compare our QED result with features of QCD vacuum requires understanding what ratio \( \mathcal{B}/\mathcal{E} \) is needed to stabilize the vacuum. This condition is clearly met in the here adopted strong field diagram resummation only in the \( \mathcal{B}/\mathcal{E} \to \infty \) limit: When \( \mathcal{E} \) and \( \mathcal{B} \) are of the same order there remains an exponentially suppressed in \( b/a \), see Eq. (70) nonzero imaginary part. Further study of this interesting result may require consideration of resummation of infinitely many higher order corrections to the effective action.

We have extended the temperature representation of the \( \mathcal{V}_{EHS}^{g} \) effective action for all \( \vert g \vert \), extending prior work based on \( g = \pm 2 \) [44] and \( \vert g \vert \leq 2 \) [14]. We obtained a result for pure magnetic fields, which exhibits the same statistical representation as the electric case. Further exploration is needed to understand the role of the pseudoscalar \( \hat{\mathcal{E}} \cdot \hat{\mathcal{B}} \) contribution. An indication that same statistical form arises for nonvanishing \( \hat{\mathcal{E}} \cdot \hat{\mathcal{B}} \) can be found in section [IV A].

While the \( \mathcal{V}_{EHS}^{g} \) result considers exclusively the (irreducible) vertex contributions to the EHS action, there are also reducible diagrams recently found to be nonvanishing in constant EM fields: as effective action [64–66] and propagator [67, 68] contributions. In these works it was shown that the reducible corrections dominate the irreducible contributions, based on the strong field asymptotic behavior of \( \mathcal{S} = (\mathcal{E}^{2} – \mathcal{B}^{2})/2 \)-dominated effective action. This comparison could be further explored for EM configurations with nonperturbative in pseudoscalar \( P = \hat{\mathcal{E}} \cdot \hat{\mathcal{B}} \) dependence, in which the \( g \)-dependent singular properties can influence the strong field asymptotic.

To conclude: We have obtained the generalized EHS effective action accounting for anomalous \( g \not= 2 \) and included the effect of pseudoscalar \( \hat{\mathcal{E}} \cdot \hat{\mathcal{B}} \). Nonperturbative phenomena are uncovered in resummed expressions: radiative corrections, previously assumed to be small within perturbative QED context, are the dominant contributions for certain EM field configurations. Our result could be a step toward novel understanding of the singular properties in QED noted for example by Dyson [69] and Källén [70, 71]. Our results also provide means to identify parallels of strong field vacuum QED phenomena with the strongly interacting QCD vacuum.

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