CONVERGENCE OF DOUBLE COSINE SERIES

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ABSTRACT. In this paper we consider double cosine series whose coefficients form a null sequence of bounded variation of order \((p, 0)\), \((0, p)\) and \((p, p)\) with the weight \((jk)^{p-1}\) for some \(p > 1\). We study pointwise convergence, uniform convergence and convergence in \(L^r\)-norm of the series under consideration. In a certain sense our results extend the results of Young [7], Kolmogorov [3] and Móricz [4, 5].

1. Introduction

Consider the double cosine series

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \cos ky,
\]

on positive quadrant \(T = [0, \pi] \times [0, \pi]\) of the two dimensional torus where \(\lambda_0 = \frac{1}{2}\) and \(\lambda_j = 1\) for \(j = 1, 2, 3, \ldots\).

The rectangular partial sums \(S_{mn}(x, y)\) and the Cesàro means \(\sigma_{mn}(x, y)\) of the series (1.1) are defined as

\[
S_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky,
\]

\[
\sigma_{mn}(x, y) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} S_{jk}(x, y), \quad m, n > 0,
\]

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and for \( \lambda > 1 \), the truncated Cesáro means are defined by
\[
V_{mn}^{\lambda}(x, y) = \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor - m} \sum_{k=n+1}^{\lfloor \lambda n \rfloor - n} S_{jk}(x, y).
\]

Now assuming the coefficients \( \{a_{jk} : j, k \geq 0\} \) in (1.1) be a double sequence of real numbers which satisfy the following conditions for some positive integer \( p \):
\[
|a_{jk}|(jk)^{p-1} \to 0 \text{ as } \max\{j, k\} \to \infty,
\]
(1.2)
\[
\lim_{k \to \infty} \sum_{j=0}^{\infty} |\triangle_{0p}a_{jk}|(jk)^{p-1} = 0,
\]
(1.3)
\[
\lim_{j \to \infty} \sum_{k=0}^{\infty} |\triangle_{0p}a_{jk}|(jk)^{p-1} = 0,
\]
(1.4)
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{pp}a_{jk}|(jk)^{p-1} < \infty.
\]
(1.5)

The finite order differences \( \triangle_{pq}a_{jk} \) are defined by
\[
\begin{align*}
\triangle_{00}a_{jk} &= a_{jk}, \\
\triangle_{pq}a_{jk} &= \triangle_{p-1, q}a_{jk} - \triangle_{p-1, q-1}a_{j+1, k+1}, & p \geq 1, q \geq 0, \\
\triangle_{0q}a_{jk} &= \triangle_{p, q}a_{jk} - \triangle_{p, q-1}a_{j, k+1}, & p \geq 0, q \geq 1.
\end{align*}
\]

Also a double induction argument gives
\[
\begin{align*}
\triangle_{pq}a_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} a_{j+s, k+t}.
\end{align*}
\]

We can call the above mentioned conditions (1.2)-(1.5) as conditions of bounded variation of order \((p, 0), (0, p)\) and \((p, p)\) respectively with the weight \((jk)^{p-1}\). Obviously these conditions generalise the concept of monotone sequences. Also any sequence satisfying (1.5) with \( p = 2 \) is called a quasi-convex sequence [3, 5]. Clearly the conditions (1.3) and (1.4) can be derived from (1.2) and (1.5) for \( p = 1 \) and moreover for \( p = 1 \), the conditions (1.2) and (1.5) reduce to \(|a_{jk}| \to 0 \text{ as } \max\{j, k\} \to \infty\) and
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{11}a_{jk}| < \infty.
\]

Generally the pointwise convergence of the series (1.1) is defined in Pringsheim’s sense ([8], Vol. 2, Ch. 17) which means that the rectangular partial sums of the type
\[
S_{mn}(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{m} \lambda_j \lambda_k a_{jk} \cos jx \cos ky, \quad m, n \geq 0,
\]
are formed and then by taking both \( m, n \) tend to \( \infty \) (independently of one another) the limit \( f(x, y) \) (provided it exists) is assigned to the series (1.1) as its sum.
Also let \( \|f\|_r \) denotes the \( L^r(T^2) \)-norm, i.e,
\[
\|f\|_r = \left( \int_0^\pi \int_0^\pi |f(x,y)|^r \, dx \, dy \right)^{1/r}, \quad 1 \leq r < \infty
\]
and \( \|f\| \) denotes \( L^1(T^2) \)-norm, i.e,
\[
\|f\| = \int_0^\pi \int_0^\pi |f(x,y)| \, dx \, dy.
\]
In this paper, we will investigate the validity of the following statements:

(a) \( S_{mn}(x,y) \) converges pointwise to \( f(x,y) \) for every \( (x,y) \in T^2 \);

(b) \( S_{mn}(x,y) \) converges uniformly to \( f(x,y) \) on \( T^2 \);

(c) \( \|S_{mn}(x,y) - f(x,y)\|_r = o(1) \) as \( \min\{m,n\} \to \infty \), \( 1 \leq r < \infty \).

Such type of problems have been studied by Young [7] and Kolmogorov [3] for one-dimensional case (single trigonometric series especially cosine series) and by Móricz [4, 5] and K. Kaur, Bhatia and Ram [2] for double trigonometric series. In [5], Móricz studied both double cosine series and double sine series as far as their integrability and convergence in \( L^1 \)-norm is concerned where as in [4] he studied double trigonometric series of the form
\[
\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{jk} e^{i(jx+ky)},
\]
under coefficients of bounded variation. All of them discussed the case for \( p = 1 \) or \( p = 2 \) only. Our aim in this paper is to extend the above results from \( p = 1 \) to general cases for double cosine series.

In the results, \( C_p \) and \( C_{pr} \) denote constants which may not be the same at each occurrence. Also we write \( \lambda_n = \lfloor \lambda n \rfloor \) where \( n \) is a positive integer, \( \lambda > 1 \) is a real number and \( \lfloor \cdot \rfloor \) means greatest integral part.

The first main result reads as follows.

**Theorem 1.1.** Assume that conditions (1.2)–(1.5) are satisfied for some \( p \geq 1 \), then

(i) \( S_{mn}(x,y) \) converges pointwise to \( f(x,y) \) for every \( (x,y) \in T^2 \) such that \( x, y > 0 \);

(ii) \( \|S_{mn}(x,y) - f(x,y)\|_r = o(1) \) as \( \min\{m,n\} \to \infty \), \( 1 \leq r < \infty \).

The above theorem has been proved by Móricz [4, 5] for \( p = 1 \) and \( p = 2 \) using suitable estimates for Dirichlet’s kernel \( D_j(x) \) and Fejér kernel \( K_j(x) \). In the case of a single series for \( p = 2 \), the results regarding convergence have been proved by Kolmogorov [3].

Obviously, condition (1.5) implies any of the following conditions:

\[
(1.6) \quad \lim_{\lambda_n \to 1} \lim_{n \to \infty} \sum_{j=0}^{\infty} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}|(jk)^{p-1} = 0,
\]
We introduce the following three sums for \( m, n \geq 0 \) and \( \lambda > 1 \):

\[
\sum_{10}^{\lambda}(m, n, x, y) = \sum_{j=m+1}^{\lambda m} \sum_{k=0}^{n} \frac{\lambda_{m} - j + 1}{\lambda_{m} - m} a_{jk} \cos jx \cos ky,
\]

\[
\sum_{01}^{\lambda}(m, n, x, y) = \sum_{j=0}^{\lambda n} \sum_{k=n+1}^{\lambda n} \frac{\lambda_{n} - k + 1}{\lambda_{n} - n} a_{jk} \cos jx \cos ky,
\]

\[
\sum_{11}^{\lambda}(m, n, x, y) = \sum_{j=m+1}^{\lambda m} \sum_{k=n+1}^{\lambda n} \frac{\lambda_{m} - j + 1}{\lambda_{m} - m} \frac{\lambda_{n} - k + 1}{\lambda_{n} - n} a_{jk} \cos jx \cos ky
\]

and we have

\[
\sum_{11}^{\lambda}(m, n; x, y) = \frac{1}{(\lambda_{m} - m)} \sum_{u=m+1}^{\lambda m} \left( \sum_{01}^{\lambda}(u, n; x, y) - \sum_{01}^{\lambda}(m, n; x, y) \right),
\]

\[
\sum_{11}^{\lambda}(m, n; x, y) = \frac{1}{(\lambda_{n} - n)} \sum_{v=n+1}^{\lambda n} \left( \sum_{10}^{\lambda}(m, v; x, y) - \sum_{10}^{\lambda}(m, n; x, y) \right).
\]

This implies

\[
\sum_{11}^{\lambda}(m, n; x, y) \leq \left\{ \begin{array}{l}
2 \sup_{m \leq u \leq \lambda_{m}} \left( |\sum_{01}^{\lambda}(u, n; x, y)| \right) \\
2 \sup_{n \leq v \leq \lambda_{n}} \left( |\sum_{10}^{\lambda}(m, v; x, y)| \right)
\end{array} \right\}.
\]

The second result of this paper is the following theorem.

**Theorem 1.2.**

(i) Let \( E \subset T^2 \). Assume that the following conditions are satisfied:

\[
\lim_{\lambda_{m} \downarrow 1} \lim_{m, n \to \infty} \left( \sup_{(x,y) \in E} |\sum_{10}^{\lambda}(m, n; x, y)| \right) = 0,
\]

\[
\lim_{\lambda_{n} \downarrow 1} \lim_{m, n \to \infty} \left( \sup_{(x,y) \in E} |\sum_{01}^{\lambda}(m, n; x, y)| \right) = 0.
\]

If \( V_{mn}^{\lambda}(x, y) \) converges uniformly to \( f(x, y) \) on \( E \subset T^2 \) as \( \min\{m, n\} \to \infty \) (that is, in the unrestricted sense), then so does \( S_{mn} \).

(ii) Assume that the following conditions are satisfied for some \( r \geq 1 \):

\[
\lim_{\lambda_{m} \downarrow 1} \lim_{m, n \to \infty} \left( \|\sum_{10}^{\lambda}(m, n; x, y)\|_{r} \right) = 0,
\]

\[
\lim_{\lambda_{n} \downarrow 1} \lim_{m, n \to \infty} \left( \|\sum_{01}^{\lambda}(m, n; x, y)\|_{r} \right) = 0.
\]

If \( \|V_{mn}^{\lambda} - f\|_{r} \to 0 \) unrestrictedly then \( \|S_{mn} - f\|_{r} \to 0 \) as \( \min\{m, n\} \to \infty \).

We will also prove the following theorem.
Theorem 1.3. Assume that the conditions (1.2)–(1.4) and (1.6)–(1.7) are satisfied for some \( p \geq 1 \), then

(i) if \( V_{mn}^\lambda(x, y) \) converges uniformly to \( f(x, y) \) as \( \min\{m, n\} \to \infty \), then so does \( S_{mn} \);

(ii) if \( \|V_{mn}^\lambda - f\|_r \to 0 \) unrestrictedly for some \( r \) with \( 1 \leq r < \infty \), then

\[ \|S_{mn} - f\|_r \to 0 \] as \( \min\{m, n\} \to \infty \).

2. Notation and Formulas

We define for every \( \alpha = 0, 1, 2, \ldots \) the sequence \( S_0^n, S_1^n, S_2^n, \ldots \) by the conditions

\[ S_0^n = S_n, \quad S_\alpha^n = \sum_{u=0}^{n} S_{\alpha-1}^u, \quad \alpha \geq 1 \]

and

\[ A_0^n = 1, \quad A_\alpha^n = \sum_{u=0}^{n} A_{\alpha-1}^u, \quad \alpha \geq 1, \]

denotes binomial coefficients. Also

\[ A_\alpha^n = \binom{n + \alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha + 1)}, \quad \alpha \neq -1, -2, -3, \ldots. \]

The Cesàro means \( T_\alpha^n \) of order \( \alpha \) of \( \sum a_n \) will be defined by

\[ T_\alpha^n = S_\alpha^n A_\alpha^n \] and also it is known [8] that \( \int_0^\pi |T_\alpha^n(x)| dx, \alpha > 0 \), is bounded for all \( n \).

3. Lemmas

We require the following lemmas for the proof of our results.

Lemma 3.1. For \( m, n \geq 0 \) and \( p > 1 \), the following representation holds:

\[ S_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky \]

\[ = \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{p\lambda a_{j+1,k+1}} S_{j-1}^{p-1}(x) S_{k-1}^{p-1}(y) + \sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{p\lambda a_{j+1,n+1}} S_{j-1}^{p-1}(x) S_{t}^{p-1}(y) \]

\[ + \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{s\lambda a_{m+1,k+1}} S_{m+1,n}^{s-1}(x) S_{t}^{s-1}(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{s\lambda a_{n+1,m+1}} S_{m+1,n+1}^{s-1}(x) S_{t}^{s-1}(y). \]

Lemma 3.2 ([1]). For \( m, n \geq 0 \) and \( \lambda > 1 \), the following representation holds:

\[ S_{mn} - \sigma_{mn} = \lambda_m + 1 \frac{\lambda_n + 1}{\lambda_m - m} (\sigma_{\lambda_m,n} - \sigma_{\lambda_m,n} - \sigma_{m,\lambda_n} + \sigma_{mn}) \]

\[ + \lambda_m + 1 \frac{\lambda_n + 1}{\lambda_m - m} (\sigma_{\lambda_m,n} - \sigma_{mn}) + \lambda_n + 1 \frac{\lambda_m + 1}{\lambda_n - n} (\sigma_{m,\lambda_n} - \sigma_{mn}) \]

\[ - \sum_{i=1}^{\lambda} (m, n, x, y) - \sum_{i=0}^{\lambda} (m, n, x, y) - \sum_{i=0}^{\lambda} (m, n, x, y). \]
Lemma 3.3. For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:

$$V_{mn}^\lambda - S_{mn} = \sum_{11}^\lambda (m, n, x, y) + \sum_{10}^\lambda (m, n, x, y) + \sum_{01}^\lambda (m, n, x, y).$$

Proof. We have

$$V_{mn}^\lambda (x, y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} S_{jk}(x, y).$$

Performing double summation by parts, we have

$$V_{mn}^\lambda = \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda_n} - \frac{\lambda_m + 1}{\lambda_m - m} \frac{n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda_n}$$

$$- \frac{m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{m, \lambda_n} + \frac{m + 1}{\lambda_m - m} \frac{n + 1}{\lambda_n - n} \sigma_{mn}$$

$$= \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \left( \sigma_{\lambda_m, \lambda_n} - \sigma_{\lambda_m, \lambda_n} - \sigma_{m, \lambda_n} + \sigma_{mn} \right)$$

$$+ \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, \lambda_n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \sigma_{mn}.$$

The use of Lemma 3.2, gives

$$V_{mn}^\lambda - S_{mn} = \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky$$

$$+ \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky$$

$$+ \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \cos jx \cos ky. \quad \square$$

Lemma 3.4. For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:

$$\sum_{10}^\lambda (m, n; x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \cos jx \cos ky$$

$$= \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \Delta_{pp} a_{jk} S_{j}^{p-1}(x) S_{k}^{p-1}(y)$$

$$+ \sum_{j=m+1}^{\lambda_m} \sum_{t=0}^{p-1} \frac{\lambda_m - j + 1}{\lambda_m - m} \Delta_{pt} a_{j, n+1} S_{j}^{p-1}(x) S_{n}^{t}(y)$$

$$+ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{k=0}^{n} \Delta_{sp} a_{j+1, k} S_{j}^{s}(x) S_{k}^{p-1}(y)$$

$$+ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{st} a_{j+1, n+1} S_{j}^{s}(x) S_{n}^{t}(y).$$
\begin{equation}
- \sum_{s=0}^{p-1} \sum_{k=0}^{n} \Delta_{sp} a_{m+1,k} S^s_m(x) S^{p-1}_k(y) \\
- \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{st} a_{m+1,n+1} S^s_m(x) S^t_n(y).
\end{equation}

**Proof.** We have by summation by parts,
\[
\sum_{k=0}^{n} \cos ky \left( \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \cos jx \right) \\
= \sum_{k=0}^{n} \cos ky \left( \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} \Delta_{p0} a_{jk} S^{p-1}_j(x) \\
+ \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{p=0}^{n-1} \sum_{k=0}^{p-1} \Delta_{sp} a_{j+1,k} S^s_j(x) - \sum_{s=0}^{p-1} \Delta_{s0} a_{m+1,k} S^s_m(x) \right) \\
= \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} S^{p-1}_j(x) \left( \sum_{k=0}^{n} \Delta_{p0} a_{jk} \cos ky \right) \\
+ \sum_{j=m+1}^{\lambda_m} \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{p=0}^{n-1} \left( \sum_{k=0}^{p-1} \Delta_{sp} a_{j+1,k} \cos ky \right) S^s_j(x) \\
- \sum_{s=0}^{p-1} \left( \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \Delta_{sp} a_{m+1,k} \cos ky \right) S^s_m(x).
\]

Similarly we can have representation for \( \sum_{01}^{\lambda}(m, n; x, y). \) \( \square \)

4. **PROOF OF THEOREMS**

**Proof of Theorem 1.1.** For \( m, n \geq 0 \) and \( p > 1 \), we have from Lemma 3.1,
\[
S_{mn}(x, y) = \sum_{j=0}^{m} \sum_{s=0}^{n} \Delta_{sp} a_{j,k} S^{p-1}_j(x) S^{p-1}_k(y) + \sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{pt} a_{j,n+1} S^t_n(y) \\
+ \sum_{k=0}^{p-1} \sum_{s=0}^{p-1} \Delta_{sp} a_{m+1,k} S^s_m(x) S^{p-1}_k(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{st} a_{m+1,n+1} S^s_m(x) S^t_n(y).
\]
= \sum_1 + \sum_2 + \sum_3 + \sum_4.

Using the results as given in [6] that \( S_j^p(x) = O\left(\frac{1}{xp}\right) \), for all \( p \geq 2, 0 < x \leq \pi \), etc, we have for \( 0 < x, y \leq \pi \),

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)| < \infty \quad \text{(by (1.2))}
\]

and also by (1.3)–(1.5), we have

\[
\sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{pt}a_{j,n+1} \leq \sum_{t=0}^{p-1} \sum_{v=0}^{t} \left( \sum_{j=0}^{m} |\Delta_{p0}a_{j,n+1+v+1}| \right)
\]

\[
\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\Delta_{p0}a_{jk}|
\]

\[
\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\Delta_{p0}a_{jk}| \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty.
\]

Thus,

\[
\sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^{p-1}(y) \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty.
\]

And similarly

\[
\sum_{s=0}^{p-1} \sum_{k=0}^{n} \Delta_{sp}a_{m+1,k} \leq \sum_{s=0}^{p-1} \sum_{u=0}^{s} \left( \sum_{k=0}^{n} |\Delta_{0p}a_{m+1,u+k}| \right)
\]

\[
\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\Delta_{0p}a_{jk}|
\]

\[
\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\Delta_{0p}a_{jk}| \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty.
\]

Thus,

\[
\sum_{s=0}^{n} \sum_{k=0}^{p-1} \Delta_{sp}a_{m+1,k}S_m^{s}(x)S_k^{p-1}(y) \rightarrow 0,
\]

as \( \min\{m, n\} \rightarrow \infty \). Also

\[
\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{st}a_{m+1,n+1} \leq \sum_{s=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{u} \left( \sum_{t=0}^{v} |\Delta_{00}a_{m+1,u+v+1}| \right)
\]

\[
\leq \sup_{j>m,k>n} |a_{jk}| \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty.
\]

So,

\[
\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \Delta_{st}a_{m+1,n+1}S_m^{s}(x)S_n^{p}(y) \rightarrow 0 \text{ as } \min\{m, n\} \rightarrow \infty.
\]
Consequently, series (1.1) converges to the function \( f(x, y) \) where

\[
f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Delta_{p} \alpha_{jk} \mathcal{S}_{\delta}^{\va})(x) \mathcal{S}_{\delta}^{\va}(y) \quad \text{and} \quad \lim_{m,n \to \infty} S_{mn}(x, y) = f(x, y).
\]

Now we will calculate \( \| \Sigma_1 \|, \| \Sigma_2 \|, \| \Sigma_3 \| \) and \( \| \Sigma_4 \| \) in the following way:

\[
\| \Sigma_1 \| = \left\| \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{p} \alpha_{jk} \mathcal{S}_{\delta}^{\va}(x) \mathcal{S}_{\delta}^{\va}(y) \right\|
\leq \sum_{j=0}^{m} \sum_{k=0}^{n} \left| \Delta_{p} \alpha_{jk} \right| \int_{0}^{\pi} \int_{0}^{\pi} | \mathcal{S}_{\delta}^{\va}(x) \mathcal{S}_{\delta}^{\va}(y) | dx dy
\leq \sum_{j=0}^{m} \sum_{k=0}^{n} \left| \Delta_{p} \alpha_{jk} \right| A_{p}^{\va} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | \mathcal{T}_{\delta}^{\va}(x) \mathcal{T}_{\delta}^{\va}(y) | dx dy
\leq C_{p} \sum_{j=0}^{m} \sum_{k=0}^{n} \left| \Delta_{p} \alpha_{jk} \right| j^{p-1} k^{p-1},
\]

\[
\| \Sigma_2 \| = \left\| \sum_{j=0}^{m} \sum_{t=0}^{n} \Delta_{l} \alpha_{j,n+1} \mathcal{S}_{\delta}^{\va}(x) \mathcal{S}_{\delta}^{\va}(y) \right\|
\leq \sum_{j=0}^{m} \sum_{t=0}^{n} \left( \frac{t}{v} \right) \left( \sum_{j=0}^{m} \left| \Delta_{p} \alpha_{j,n+1} \right| \right) A_{p}^{\va} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | \mathcal{T}_{\delta}^{\va}(x) \mathcal{T}_{\delta}^{\va}(y) | dx dy
\leq C_{p} \sup_{n \leq k \leq n+p} \sum_{j=0}^{m} \left| \Delta_{p} \alpha_{jk} \right| j^{p-1} \left( \sum_{t=0}^{n} n^{t} \right)
\leq C_{p} \sup_{n \leq k \leq n+p} \sum_{j=0}^{m} \left| \Delta_{p} \alpha_{jk} \right| j^{p-1} k^{p-1},
\]

\[
\| \Sigma_3 \| = \left\| \sum_{s=0}^{m} \sum_{k=0}^{n} \Delta_{s} \alpha_{m+1,k} \mathcal{S}_{\delta}^{\va}(x) \mathcal{S}_{\delta}^{\va}(y) \right\|
\leq \sum_{s=0}^{m} \sum_{u=0}^{n} \left( \frac{s}{u} \right) \left( \sum_{k=0}^{n} \left| \Delta_{p} \alpha_{m+u+1,k} \right| \right) A_{p}^{\va} \int_{0}^{\pi} \int_{0}^{\pi} | \mathcal{T}_{\delta}^{\va}(x) \mathcal{T}_{\delta}^{\va}(y) | dx dy
\leq C_{p} \sup_{m \leq k \leq m+p} \sum_{s=0}^{n} \left| \Delta_{p} \alpha_{jk} \right| k^{p-1} \left( \sum_{s=0}^{m} m^{s} \right)
\leq C_{p} \sup_{m \leq k \leq m+p} \sum_{s=0}^{n} \left| \Delta_{p} \alpha_{jk} \right| j^{p-1} k^{p-1},
\]

\[
\| \Sigma_4 \| = \left\| \sum_{s=0}^{m} \sum_{t=0}^{n} \Delta_{s} \alpha_{m+1,n+1} \mathcal{S}_{\delta}^{\va}(x) \mathcal{S}_{\delta}^{\va}(y) \right\|
\]
\[
\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} t u \left( \frac{t}{u} \right) |\Delta_{00} \alpha_{m,u+1,n+1} | A_m^s A_n^t \int_{0}^{\pi} \int_{0}^{\pi} |T_m^s(x) T_n^t(y)| \, dx \, dy \\
\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}.
\]

Now let \( R_{mn} \) consists of all \((j, k)\) with \( j > m \) or \( k > n \), that is,

\[
\sum \sum_{(j, k) \in R_{mn}} = \sum_{j=m+1}^{\infty} \sum_{k=0}^{n} + \sum_{j=0}^{n} \sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty}.
\]

Then

\[
\|f - S_{mn}\|_r = \left( \int_{0}^{\pi} \int_{0}^{\pi} |f(x, y) - S_{mn}(x, y)|^r \, dx \, dy \right)^{1/r}, \quad 1 \leq r < \infty,
\]

\[
\leq \sum \sum_{(j, k) \in R_{mn}} \Delta_{pq} |a_{jk}| S_j^{p-1} \left( x \right) S_k^{p-1} \left( y \right)\bigg|_r \\
+ \sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{pt} |a_{j,n+1}| S_j^{p-1} \left( x \right) S_t^{p-1} \left( y \right)\bigg|_r \\
+ \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{sp} |a_{m+1,k}| S_m^{s} \left( x \right) S_k^{p-1} \left( y \right)\bigg|_r \\
+ \sum_{s=0}^{p-1} \sum_{k=0}^{p-1} \Delta_{st} |a_{m+1,k+1}| S_m^{s} \left( x \right) S_t^{p-1} \left( y \right)\bigg|_r
\]

\[
\leq C_p \left\{ \left( \sum \sum_{(j, k) \in R_{mn}} |\Delta_{pq} a_{jk}| j^{p-1} k^{p-1} \right) \right. \\
+ \left( \sup_{n < k \leq n+p} m \sum_{j=0}^{m} |\Delta_{pq} a_{jk}| j^{p-1} k^{p-1} \right) \\
+ \left( \sup_{m < j \leq m+p} n \sum_{k=0}^{n} |\Delta_{pq} a_{jk}| j^{p-1} k^{p-1} \right) \\
+ \left( \sup_{j > m,k > n} |a_{jk}| j^{p-1} k^{p-1} \right) \right. \\
\left. \rightarrow 0 \right. \text{ as } \min\{m, n\} \rightarrow \infty \quad \text{(by (1.2)-(1.5))},
\]

which proves (ii) part.

**Proof of Theorem 1.2.** Using the relation (1.8), we find that (1.9) or (1.10) implies

\[
\lim_{\lambda \uparrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \sum_{1}^{\lambda} (m, n; x, y) \right) = 0.
\]
Assume that \( V^\lambda_{mn}(x, y) \) converges uniformly on \( E \) to \( f(x, y) \). Then by Lemma 3.3, we get
\[
\lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \left| S_{mn}(x, y) - V^\lambda_{mn}(x, y) \right| \right) \leq \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \left| \sum^\lambda_{01} (m, n; x, y) \right| \right) + \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \left| \sum^\lambda_{11} (m, n; x, y) \right| \right).
\]

After taking \( \lambda \downarrow 1 \) the result follows from (1.9), (1.10) and (4.1).

For (ii) part of theorem, we have
\[
\left\| \sum^\lambda_{11} (m, n; x, y) \right\|_r = \frac{1}{\lambda_m - m} \sum_{u=m+1}^{\lambda_m} \left( \left\| \sum^\lambda_{01} (u, n; x, y) \right\|_r + \left\| \sum^\lambda_{01} (m, n; x, y) \right\|_r \right) \leq 2 \left( \sup_{m \leq u \leq \lambda_m} \left( \left\| \sum^\lambda_{01} (u, n; x, y) \right\|_r \right) \right).
\]
Thus (1.11) implies
\[
\lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left\| \sum^\lambda_{11} (m, n; x, y) \right\|_r = 0.
\]
Thus, the result of Theorem 1.2 (ii) follows.

**Proof of Theorem 1.3.** Using the Lemma 3.4, we can write the expression for \( \sum^\lambda_{01}(m, n; x, y) \) as
\[
\sum^\lambda_{01}(m, n; x, y) = \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky
\]
\[
= \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp}a_{jk} S^{p-1}_j(x) S^{p-1}_k(y)
\]
\[
+ \frac{1}{\lambda_n - n} \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{p-1} \Delta_{sp}a_{m+1,k} S^{s}_m(x) S^{p-1}_k(y)
\]
\[
+ \frac{1}{\lambda_n - n} \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \sum_{l=0}^{p-1} \sum_{s=0}^{p-1} \Delta_{st}a_{m+1,k+1} S^{s}_m(x) S^{t}_k(y)
\]
\[
- \sum_{l=0}^{p-1} \sum_{j=0}^{m} \Delta_{pl}a_{j,n+1} S^{p-1}_j(x) S^{l}_n(y)
\]
\[ - \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S^s_m (x) S^{t}_n (y) \]
\[ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \]

Now by using (1.2)–(1.4) and (1.6) along with estimates of \( S^p_j (x) \) etc., as mentioned in [6], we have the following estimates in brief:

\[
|I_1| = \left| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{pp} a_{jk} S^p_j (x) S^{j-1}_k (y) \right| \\
\leq \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| \\
\rightarrow 0 \quad \text{as } \min\{m,n\} \rightarrow \infty.
\]

Consequently, \( \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |I_1| \right) \rightarrow 0 \) as \( \min\{m,n\} \rightarrow \infty \). Also,

\[
|I_2| = \left| \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{sp} a_{m+1,k} S^s_m (x) S^{j-1}_k (y) \right| \\
\leq \sum_{s=0}^{p-1} \sum_{u=0}^{t} \frac{\lambda_n}{\lambda_k - n} |\triangle_{sp} a_{jk}| \\
\leq \sup_{m<j \leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{sp} a_{jk}| \rightarrow 0 \quad \text{as } \min\{m,n\} \rightarrow \infty.
\]

So, \( \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |I_2| \right) \rightarrow 0 \) as \( \min\{m,n\} \rightarrow \infty \). Also,

\[
|I_3| \leq \sup_{n<k \leq \lambda_n} \sum_{t=0}^{p-1} \sum_{j=0}^{m} |\triangle_{pt} a_{j,k+1}| \\
\leq \sup_{n<k \leq \lambda_n} \sum_{t=0}^{p-1} \sum_{v=0}^{m} \left( \frac{t}{v} \right) \sum_{j=0}^{m} |\triangle_{pt} a_{j,v+1}| \\
\leq \sup_{n<k \leq \lambda_n} \sum_{p=0}^{m} |\triangle_{pt} a_{jk}| \rightarrow 0 \quad \text{as } \min\{m,n\} \rightarrow \infty,
\]

which implies \( \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |I_3| \right) \rightarrow 0 \) as \( \min\{m,n\} \rightarrow \infty \). Now,

\[
|I_4| \leq \sup_{n<k \leq \lambda_n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\triangle_{st} a_{m+1,k+1}| \\
\leq \sup_{j>m,k>n} |a_{jk}| \rightarrow 0 \quad \text{as } \min\{m,n\} \rightarrow \infty.
\]
Thus, combining all these, we have
\[ |I_5| \leq \sum_{t=0}^{p-1} \sum_{s=0}^{t} \left( \frac{t}{v} \right) \sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}| \leq \sup_{m} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \to 0 \text{ as } \min \{m,n\} \to \infty,
\]
which implies
\[ \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |I_5| \right) \to 0 \text{ as } \min \{m,n\} \to \infty. \]

Proof of (ii). We have
\[ \|S_{mn} - f\|_r \leq \|S_{mn} - V_{mn}^\lambda\|_r + \|V_{mn}^\lambda - f\|_r. \]

By assumption \( \|V_{mn}^\lambda - f\|_r \to 0 \), so it is sufficient to show that
\[ \|S_{mn} - V_{mn}^\lambda\|_r \to 0 \text{ as } \min \{m,n\} \to \infty. \]

By Lemma 3.3, we have
\[ \|S_{mn} - V_{mn}^\lambda\|_r \leq \| \sum_{j=0}^{\lambda} (m,n;x,y) \|_r + \| \sum_{j=0}^{\lambda} (m,n;x,y) \|_r + \| \sum_{j=0}^{\lambda} (m,n;x,y) \|_r. \]

Now in order to estimate \( \| \sum_{j=0}^{\lambda} (m,n;x,y) \|_r \), we first find \( \|I_1\|, \|I_2\|, \|I_3\|, \|I_4\|, \|I_5\| \) and \( \|I_6\| \), so we have
\[ \|I_1\| = \left\| \sum_{j=0}^{\lambda} \sum_{k=n+1}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) \right\|
\leq \sum_{j=0}^{\lambda} \sum_{k=n+1}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp} a_{jk} A_j^{p-1} A_k^{p-1} \int_0^{\pi} \int_0^{\pi} |T_j^{p-1}(x) T_k^{p-1}(y)| \, dx \, dy. \]
Thus, we can estimate

\[
\|I_2\| \leq C_p \left( \sum_{j=0}^{m} \frac{\lambda_n}{\lambda_n - n} |\Delta_{pp} \alpha_{jk}| j^{p-1} k^{p-1}, \right.
\]

\[
\|I_3\| \leq C_p \left( \sup_{m \leq n+1} \left( \sum_{s=0}^{p} u \right) \sum_{k=n+1}^{p} \frac{\lambda_n}{\lambda_n - n} \Delta_{0p} \alpha_{m+1,k} |k^{p-1} m^s \right)
\]

\[
\|I_4\| \leq C_p \left( \sup_{m \leq n+1} \left( \sum_{s=0}^{p} u \right) \sum_{k=n+1}^{p} \frac{\lambda_n}{\lambda_n - n} \Delta_{0p} \alpha_{m+1,k} |k^{p-1} m^s \right)
\]

\[
\|I_5\| \leq C_p \left( \sup_{m \leq n+1} \left( \sum_{s=0}^{p} u \right) \sum_{k=n+1}^{p} \frac{\lambda_n}{\lambda_n - n} \Delta_{0p} \alpha_{m+1,k} |k^{p-1} m^s \right)
\]

\[
\|I_6\| \leq C_p \left( \sup_{m \leq n+1} \left( \sum_{s=0}^{p} u \right) \sum_{k=n+1}^{p} \frac{\lambda_n}{\lambda_n - n} \Delta_{0p} \alpha_{m+1,k} |k^{p-1} m^s \right)
\]

Thus, we can estimate

\[
\left\| \sum_{r=1}^{\lambda} (m, n; x, y) \right\| \leq C_{pr} \left( \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_{pp} \alpha_{jk}| j^{p-1} k^{p-1}, \right.
\]

\[
+ C_{pr} \left( \sup_{m \leq n+1} \left( \sum_{s=0}^{p} u \right) \sum_{k=n+1}^{p} \frac{\lambda_n}{\lambda_n - n} \Delta_{0p} \alpha_{m+1,k} |j^{p-1} k^{p-1} \right)
\]
\[ C_{pr} \left( \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\Delta_{p0}a_{jk}|j^{p-1}k^{p-1} \right) \]
\[ + C_{pr} \left( \sup_{j>m,k>n} |a_{jk}|j^{p-1}k^{p-1} \right) \]
\[ + C_{pr} \left( \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\Delta_{p0}a_{jk}|j^{p-1}k^{p-1} \right) \]
\[ + C_{pr} \left( \sup_{j>m,k>n} |a_{jk}|j^{p-1}k^{p-1} \right). \]

By (1.2)–(1.4) and (1.6), we conclude that
\[ \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \| \sum_{01}^{\lambda} (m, n; x, y) \|_r \right) = 0. \]

Similarly, by conditions (1.2)–(1.4) and (1.7), we get
\[ \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \| \sum_{10}^{\lambda} (m, n; x, y) \|_r \right) = 0. \]

Also, by (1.8), we have
\[ \lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \| \sum_{11}^{\lambda} (m, n; x, y) \|_r \right) = 0. \]

Thus, \( \| S_{mn} - V_{\lambda mn}^{\lambda} \|_r \to 0 \) as \( \min\{m, n\} \to \infty. \)

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