The Hermite-Hadamard inequality revisited: Some new proofs and applications

Ilham A. Aliev · Mehmet E. Tamar · Cagla Sekin

Received: date / Accepted: date

Abstract New proofs of the classical Hermite-Hadamard inequality are presented and several applications are given, including Hadamard-type inequalities for the functions, whose derivatives have inflection points or whose derivatives are convex. Moreover, some estimates from below and above for the first moments of functions $f : [a, b] \to \mathbb{R}$ about the center point $c = (a + b)/2$ are obtained and the reverse Hardy inequality for convex functions $f : [0, \infty) \to (0, \infty)$ is established.

Keywords Convex functions · Hermite-Hadamard inequality · Jensen inequality · Fejer inequality.

Mathematics Subject Classification (2010) 26D15 · 26A51 · 26D10

1 Introduction

The famous Hermite-Hadamard inequality (HH) asserts that the integral mean value of a convex function $f : [a, b] \to \mathbb{R}$ can be estimated above and below by its values at the points $a, b$ and $(a + b)/2$. More precisely,

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$ (HH)
Equality holds only for functions of the form $f(x) = cx + d$. According to the notations by Niculescu and Persson [20], the right and left parts of (HH) we denote by (RHH) and (LHH), respectively. Some authors also refer to the (RHH) as Hadamard’s inequality.

(HH) has many generalizations, extensions, refinements and there is a large number of papers and book’s chapters in this area; see, e.g. books by Niculescu and Persson [21]; Mitrinovic, Pecaric and Fink [19]; Dragomir and Pearce [9] and papers [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27], which are a small part of the relevant references.

The plan of this article is as follows.

In section 2 we give two new proofs of (HH), one of which is extremely short (so to speak, "without pulling the pen out of paper"), and the other is based on Riemann’s integral sums. As an application, we give an estimation from below and above of the integral of the convex function $f : [0, \infty) \to (0, \infty)$ via the series $\sum_{k=1}^{\infty} f(k)$ and $\sum_{k=1}^{\infty} f(k-\frac{1}{2})$.

In section 3, we give some new inequalities arising as a combination of (HH) with the Hardy inequality and some variant of Hölder’s inequality. For example, as a consequence we prove that, if $f : [0, \infty) \to (0, \infty)$ is convex and $f \in L^p(0, \infty)$, $\forall p > 1$, then

$$\lim_{p \to \infty} \frac{\|\frac{1}{p} \int_0^x f\|_p}{\|f\|_p} = 1.$$ 

Section 4 is devoted to Hadamard’s type inequality, i.e. (RHH) for the functions whose first derivatives have an inflection point. As a particular case, we show that if $f'$ is concave on $[a, \frac{a+b}{2}]$ and convex on $[\frac{a+b}{2}, b]$, then

$$f(a) + f(b) \leq 2 \int_a^b f(x) dx \leq \frac{b-a}{12} (f'(b) - f'(a)).$$

In the last 5th section we prove various inequalities for functions having convex first or second order derivatives. As far as we know, in the literature on this theme there are some inequalities for functions whose absolute values of the derivatives are convex, see, e.g. [20, 24]. In our Theorem [7, 8, and 9] in section 5, the convexity condition is imposed on the derivatives themselves, but not on their absolute values. One of the interesting particular results obtained in this section is as follows.

Given $f : [a, b] \to \mathbb{R}$, let $f'$ be convex. Then

$$\int_{\frac{a+b}{2}}^b f(x) dx - \int_a^{\frac{a+b}{2}} f(x) dx \leq \frac{b-a}{4} (f(b) - f(a)).$$

Another result in this section is the estimation from above and below of the first moment of a function $f : [a, b] \to \mathbb{R}$ about the center point $c = (a + b)/2$, i.e. the integral $M_f = \int_a^b (x - \frac{a+b}{2}) f(x) dx$, when $f'$ is convex.
2 Two new proof of (HH) and some applications

At first, we give an auxiliary inequality that is satisfied by convex functions.

**Lemma 1** (cf. Lemma 1.3 in [18]) Let \( f \) be a convex function on \([a,b]\). Then
\[
f(a) + f(b) \geq f(a+b-x) + f(x), \quad (\forall x \in [a,b]).
\] (1)

**Proof** Let \( x \in [a,b] \). There exists a \( t \in [0,1] \) such that \( x = ta + (1-t)b \). Then \( a + b - x = (1-t)a + tb \) and therefore,
\[
f(a + b - x) = f((1-t)a + tb) \leq (1-t)f(a) + tf(b)
\]
\[
= f(a) + f(b) - [tf(a) + (1-t)f(b)]
\]
\[
\leq f(a) + f(b) - f(ta + (1-t)b)
\]
\[
= f(a) + f(b) - f(x).
\]

By making use of (1) we give here a short proof of the (HH), ”without pulling the pen out of paper”.

**Proof of (HH)** Integrating the inequality \( f(a) + f(b) \geq f(a+b-x) + f(x) \) over \([a,b]\) and using
\[
\int_a^b f(a+b-x) \, dx = \int_a^b f(x) \, dx
\]
we have
\[
(b-a)(f(a) + f(b)) \geq 2 \int_a^b f(x) \, dx = \int_a^b (f(a+b-x) + f(x)) \, dx
\]
\[
= 2 \int_a^b f(a+b-x) + f(x) \, dx \geq 2 \int_a^b f \left( \frac{a+b-x+x}{2} \right) \, dx
\]
\[
= 2(b-a)f \left( \frac{a+b}{2} \right).
\]

**Remark 1** Although the (HH) has several proofs, as far as we know the first simple proof was given by Azbetta [3]; (see, also Niculescu and Persson [20], p. 664). Another simple proof and refinement was given by El Farissi [11].

**Remark 2** The inequality (1) enables one also to give a short proof of ”Hadamard part” of the Fejer inequality:

If \( g \geq 0 \) is integrable and symmetric with respect to \( \frac{a+b}{2} \), i.e. \( g(a+b-x) = g(x), \quad (x \in [a,b]) \), we have
\[
\int_a^b f(x)g(x) \, dx = \frac{1}{2} \left[ \int_a^b f(x)g(x) \, dx + \int_a^b f(a+b-x)g(a+b-x) \, dx \right]
\]
\[
= \frac{1}{2} \int_a^b (f(x) + f(a+b-x))g(x) \, dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) \, dx.
\]
The following theorem is the "Riemann integral's sums version" of (HH).

**Theorem 1** If \( f : [a, b] \to \mathbb{R} \) is convex and \( x_k = a + k \frac{b-a}{n}, (k = 1, 2, \ldots, n) \) then for any \( n \in \mathbb{N} \) we have

\[
f \left( \frac{(1 - \frac{1}{n}) a + (1 + \frac{1}{n}) b}{2} \right) \leq \frac{1}{n} \sum_{k=1}^{n} f(x_k) \leq \frac{1}{2} \left[ f(a) \left(1 - \frac{1}{n}\right) + f(b) \left(1 + \frac{1}{n}\right) \right]. \tag{2}
\]

**Proof** Let \( x_k = a + k \frac{b-a}{n} \), \( (k = 1, 2, \ldots, n) \). Then writing \( x_k \) as

\[
x_k = \frac{b-x_k}{b-a} a + \frac{x_k-a}{b-a} b
\]

and using

\[
f(x_k) \leq \frac{b-x_k}{b-a} f(a) + \frac{x_k-a}{b-a} f(b),
\]

one has

\[
\sum_{k=1}^{n} f(x_k) \leq \frac{f(a)}{b-a} \sum_{k=1}^{n} (b-x_k) + \frac{f(b)}{b-a} \sum_{k=1}^{n} (x_k - a)
\]

\[
= \frac{1}{2} \left[ f(a)(n-1) + f(b)(n+1) \right],
\]

and therefore,

\[
\frac{1}{n} \sum_{k=1}^{n} f(x_k) \leq \frac{1}{2} \left[ f(a) \left(1 - \frac{1}{n}\right) + f(b) \left(1 + \frac{1}{n}\right) \right]. \tag{3}
\]

On the other hand, the Jensen inequality yields

\[
\frac{1}{n} \sum_{k=1}^{n} f(x_k) \geq f \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right) = f \left( \frac{(1 - \frac{1}{n}) a + (1 + \frac{1}{n}) b}{2} \right). \tag{4}
\]

By combining (3) and (4) we obtain (2).

**Corollary 1** After taking limit as \( n \to \infty \) in (2) and using the fact that the convex function is continuous, we obtain (HH).

The following two theorems are the simple consequences of (HH).

**Theorem 2 (a refinement of (RHH))** Let \( f : [a, b] \to \mathbb{R} \) be convex. Then

\[
\frac{1}{b-a} \int_{a}^{b} f(x) dx \leq \frac{1}{b-a} \int_{a}^{b} f(x) \left[ \ln \frac{(b-a)^2}{(b-x)(x-a)} - 1 \right] dx
\]

\[
\leq \frac{f(a) + f(b)}{2}. \tag{5}
\]
Proof For any \( x \in (a, b] \) one has
\[
f\left(\frac{a + x}{2}\right) \leq \frac{1}{x - a} \int_a^x f(t) \, dt \leq \frac{f(a) + f(x)}{2}.
\]
Integrating over \((a, b)\) we have
\[
\int_a^b f\left(\frac{a + x}{2}\right) \, dx \leq \int_a^b \frac{1}{x - a} \left( \int_a^x f(t) \, dt \right) \, dx \leq \int_a^b \frac{f(a) + f(x)}{2} \, dx. \tag{6}
\]
After simple calculations, (6) leads to
\[
\int_a^b f\left(\frac{x + b}{2}\right) \, dx \leq \int_a^b f(x) \ln \frac{b - a}{x - a} \, dx \leq \frac{1}{2} \left[ f(a)(b - a) + \int_a^b f(x) \, dx \right]. \tag{7}
\]
Similarly, integrating the inequality
\[
f\left(\frac{x + b}{2}\right) \leq \frac{1}{b - x} \int_a^b f(t) \, dt \leq \frac{f(x) + f(b)}{2}
\]
over \((a, b)\) we get
\[
\int_a^b f\left(\frac{x + b}{2}\right) \, dx \leq \int_a^b \frac{1}{b - x} \left( \int_a^b f(t) \, dt \right) \, dx \leq \int_a^b \frac{f(x) + f(b)}{2} \, dx
\]
which leads to
\[
2 \int_a^{a+b} f(x) \, dx \leq \int_a^b f(x) \ln \frac{b - a}{b - x} \, dx \leq \frac{1}{2} \left[ f(b)(b - a) + \int_a^b f(x) \, dx \right]. \tag{8}
\]
After summing up (7) and (8) we obtain (5).

**Theorem 3** Let \( f : [0, \infty) \to (0, \infty) \) be a strictly convex function and \( \sum_{k=1}^{\infty} f(x_k) < \infty \). Then
\[
\sum_{k=1}^{\infty} f\left(k - \frac{1}{2}\right) + \sum_{k=1}^{\infty} f(k) \leq f(0) + \sum_{k=1}^{\infty} f(k). \tag{9}
\]

**Proof** Denote \( x_0 = a \) and \( x_k = a + (k - 1) \frac{b - a}{n}, \quad (k = 1, 2, \ldots, n) \). Since \( f \) is strictly convex, we have
\[
f\left(x_{k-1} + x_k \right) < \frac{1}{x_k - x_{k-1}} \int_{x_{k-1}}^{x_k} f(x) \, dx < \frac{f(x_{k-1}) + f(x_k)}{2}, \quad (k = 1, 2, \ldots, n).
\]
Taking into account the formulas
\[
x_k - x_{k-1} = \frac{b - a}{n}, \quad x_{k-1} + x_k = a + \left(k - \frac{1}{2}\right) \frac{b - a}{n}.
\]
and summing the inequalities above we obtain
\[
\sum_{k=1}^{n} \frac{1}{n} f\left(a + \left(k - \frac{1}{2}\right) \frac{b - a}{n}\right) < \frac{1}{b - a} \int_{a}^{b} f(x) dx < \frac{1}{n} \left[ \frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} f\left(a + \frac{k(b - a)}{n}\right) \right].
\]

Further, setting \(a = 0, b = n\) we have
\[
\sum_{k=1}^{n} f\left(k - \frac{1}{2}\right) < \int_{0}^{b} f(x) dx < \frac{f(0) + f(n)}{2} + \sum_{k=1}^{n-1} f(k).
\]

Taking limit as \(n \to \infty\) and using \(\lim_{n \to \infty} f(n) = 0\) we obtain the desired formula (9).

**Remark 3** Since \(f : [0, \infty) \to (0, \infty)\) is convex and \(\lim_{n \to \infty} f(n)\) is finite (actually, zero), then \(f\) is monotonically decreasing and therefore the comparison of the areas under graphics gives
\[
\sum_{k=1}^{\infty} f(k) < \int_{0}^{\infty} f(x) dx < f(0) + \sum_{k=1}^{\infty} f(k).
\](10)

It is clear that, the inequalities (9) are better than (10).

**Example 1** If \(f(x) = e^{-x}\) then from (9) we have
\[
\frac{\sqrt{e}}{e - 1} < 1 < \frac{1}{2} + \frac{1}{e - 1} \Leftrightarrow \sqrt{e} < e - 1 < \frac{1}{2}(e + 1),
\]
whereas the formula (10) gives the rougher estimate \(1 < e - 1 < e\).

### 3 Some inequalities arising as a combination of the Hermite-Hadamard inequality with the other ones

**Theorem 4** Let \(1 < p < \infty\) and \(\alpha p > 1\). Let further, \(f : [0, \infty) \to (0, \infty)\) be convex and such that
\[
\|x^{1-\alpha} f(x)\|_{p} = \left( \int_{0}^{\infty} (x^{1-\alpha} f(x))^{p} dx \right)^{1/p} < \infty.
\]

Then
\[
2^{1-\alpha + \frac{1}{p}} \leq \|x^{-\alpha} \int_{0}^{x} f(t) dt\|_{p} \leq \frac{1}{\alpha - 1/p}.
\](11)
Corollary 2 (a) If $\alpha = 1$, then

$$2^{\frac{1}{p}} \leq \frac{\| \frac{1}{x} \int_0^x f(t) dt \|_p}{\| f \|_p} \leq \frac{1}{1 - \frac{1}{p}}.$$ \hspace{1cm} (12)

(b) Let, in addition, $f \in L_p(0, \infty)$, ($\forall p > 1$). Then by taking the limit in (12) as $p \to \infty$ one has

$$\lim_{p \to \infty} \frac{\| \frac{1}{x} \int_0^x f(t) dt \|_p}{\| f \|_p} = 1.$$ \hspace{1cm} (13)

Proof We will use the classical weighted Hardy inequality, which asserts that

$$\left( \int_0^\infty \left| x^{-\alpha} \int_0^x f(t) dt \right|^p dx \right)^{1/p} \leq c \left( \int_0^\infty |x^{1-\alpha} f(x)|^p dx \right)^{1/p},$$ \hspace{1cm} (14)

where $c = \frac{p}{\alpha p - 1}$, $1 < p < \infty$, $\alpha p > 1$.

Now, by (LHH) we have

$$f \left( \frac{x}{2} \right) < \frac{1}{x} \int_0^x f(t) dt \Rightarrow x^{1-\alpha} f \left( \frac{x}{2} \right) < x^{-\alpha} \int_0^x f(t) dt,$$

and therefore

$$\int_0^\infty \left( x^{1-\alpha} f \left( \frac{x}{2} \right) \right)^p dx \leq \int_0^\infty \left( x^{-\alpha} \int_0^x f(t) dt \right)^p dx \leq \left( \frac{p}{\alpha p - 1} \right)^p \int_0^\infty (x^{1-\alpha} f(x))^p dx. \hspace{1cm} (15)$$

Since

$$\int_0^\infty \left( x^{1-\alpha} f \left( \frac{x}{2} \right) \right)^p dx = 2^{p(1-\alpha)+1} \int_0^\infty (x^{1-\alpha} f(x))^p dx,$$

we have from (15) the desired result (11) and its consequences (12) and (13).

Remark 4 The left hand side of (11) shows that under the conditions of Theorem 4 the following reverse Hardy’s inequality is valid:

$$\| x^{-\alpha} \int_0^x f \|_p \geq 2^{1-\alpha + \frac{1}{p}} \| x^{1-\alpha} f(x) \|_p.$$ 

Example 2 Let $k > 0$ and $f(x) = e^{-kx}$. Then (13) yields

$$\lim_{p \to \infty} \left( \int_0^\infty \left( \frac{1 - e^{-kx}}{x} \right)^p dx \right)^{1/p} = k.$$
In the next theorem we will make use of a combination of Hadamard’s inequality
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]
and the inequality
\[
\left( \int_a^b \left( \prod_{k=1}^n u_k(x) \right) \, dx \right)^n \leq \prod_{k=1}^n \left( \int_a^b u_k^n(x) \, dx \right).
\tag{16}
\]
where \(u_1 \geq 0, \ldots, u_n \geq 0\).

Recall that the inequality (16) is a consequence of Hölder’s inequality and can be proved by induction.

We need also the following

**Lemma 2** If \(u : [a, b] \to (0, \infty)\) is convex, then \(u^n\) is convex as well for any \(n \in \mathbb{N}\).

A simple proof follows by induction. Namely, if the functions \(u > 0\) and \(u^n\) are convex for some \(n \geq 2\), i.e.
\[
u(\alpha x + \beta y) \leq \alpha u(x) + \beta u(y)
\]
and
\[
u^n(\alpha x + \beta y) \leq \alpha \nu^n(x) + \beta \nu^n(y), (\alpha + \beta = 1),
\]
then by multiplying these inequalities we have
\[
u^{n+1}(\alpha x + \beta y) \leq (\alpha u(x) + \beta u(y)) (\alpha \nu^n(x) + \beta \nu^n(y))
\leq \alpha \nu^{n+1}(x) + \beta \nu^{n+1}(y),
\]
where the last estimate is equivalent to the following obvious inequality:
\[
(u(x) - u(y)) (\nu^n(x) - \nu^n(y)) \geq 0.
\]

**Remark 5** The convexity of the functions \(u_1 \geq 0, u_2 \geq 0, \ldots, u_n \geq 0\) does not guarantee the convexity of their product \(u_1 u_2 \cdots u_n\). Indeed, for example, although the functions \(u_1(x) = x^2, u_2(x) = x^2, \ldots, u_{n-1}(x) = x^2\) and \(u_n(x) = (2-x)^{2n-2}, (n \geq 2)\) are convex on \([0,2]\), their product \(u(x) = x^{2n-2}(2-x)^{2n-2}\) is not convex because of \(f''(1) = 4(2n-2)(1-n) < 0\).

**Theorem 5** For given \(n \geq 2\), let the functions \(u_1 \geq 0, u_2 \geq 0, \ldots, u_n \geq 0\) be convex on \([a, b]\). Then
\[
\frac{1}{b-a} \int_a^b \left( \prod_{k=1}^n u_k(x) \right) \, dx \leq \frac{1}{2} \prod_{k=1}^n \left( u_k^n(a) + u_k^n(b) \right)^{1/2}.
\tag{17}
\]
Proof Since \( u_k, (k = 1, 2, \cdots, n) \) is convex on \([a, b]\), then \( u_k^n \) is also convex by Lemma 2. Then Hadamard’s inequality yields
\[
\frac{1}{b-a} \int_a^b u_k^n(x)dx \leq \frac{1}{2} [u_k^n(a) + u_k^n(b)], \ (k = 1, 2, \cdots, n).
\]
By multiplying these inequalities we have
\[
\frac{1}{(b-a)^n} \prod_{k=1}^n \left( \int_a^b u_k^n(x)dx \right) \leq \frac{1}{2^n} \prod_{k=1}^n (u_k^n(a) + u_k^n(b)). \tag{18}
\]
Here, by making use of the inequality \(16\), we get
\[
\frac{1}{(b-a)^n} \left( \int_a^b \left( \prod_{k=1}^n u_k(x) \right) dx \right)^n \leq \frac{1}{2^n} \prod_{k=1}^n (u_k^n(a) + u_k^n(b)),
\]
from which the inequality \(17\) follows.

Remark 6 For \( n = 2 \), the inequality \(17\) was proved by Amrahov [1]. Another generalization of Amrahov’s result for the product of two functions was noted by D. A. Ion [16]:
\[
\text{If } u \geq 0, v \geq 0 \text{ are convex and } \frac{1}{p} + \frac{1}{q} = 1, (1 < p, q < \infty), \text{ then}
\]
\[
\frac{1}{b-a} \int_a^b u(t)v(t)dt \leq \frac{1}{2} (u^p(a) + u^p(b))^{1/p} (u^q(a) + u^q(b))^{1/q}.
\]
It should also be mentioned that, in the same paper [16] Ion gives some generalization of Amrahov’s result for the product of two functions in Orlicz spaces.

4 Hadamard’s type inequality for the functions whose derivatives have an inflection point

Theorem 6 Given \( c \in [a, b] \) and \( f : [a, b] \to \mathbb{R} \), let its derivative \( f' \) be concave on \([a, c]\) and convex on \([c, b]\). Then
\[
\left[ \frac{c-a}{b-a} f(a) + \frac{b-c}{b-a} f(b) \right] + \frac{1}{b-a} \int_a^b f(x)dx \\
\leq \frac{1}{3} \left[ (b-c)^2 f'(b) - \frac{(c-a)^2}{b-a} f'(a) + \left( \frac{a+b}{2} - c \right) f'(c) \right]. \tag{19}
\]

Corollary 3 In case of \( c = \frac{a+b}{2} \) we have
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{b-a}{12} (f'(b) - f'(a)) \tag{20}
\]
Proof Integration by parts yields
\[
\frac{c - a}{b - a} f(a) + \frac{b - c}{b - a} f(b) - \frac{1}{b - a} \int_a^b f(x)dx = \frac{1}{b - a} \int_a^b (x - c)f'(x)dx
\]
\[
= \frac{1}{b - a} \int_a^c (x - c)f'(x)dx + \frac{1}{b - a} \int_c^b (x - c)f'(x)dx
\]
\[
\equiv A + B. \quad (21)
\]

By changing variables as \(x = (1 - \lambda)a + \lambda c\), (0 < \(\lambda< 1\)) in A and \(x = (1 - \lambda)c + \lambda b\) in B and applying Jensen’s inequality, we have

\[
A \equiv \frac{1}{b - a} \int_a^c (x - c)f'(x)dx = \frac{(a - c)^2}{b - a} \int_0^1 (\lambda - 1)(1 - \lambda)f'(a + \lambda c)d\lambda
\]
\[
\leq \frac{(a - c)^2}{b - a} \int_0^1 (\lambda - 1)(1 - \lambda)(f'(a) + \lambda f'(c))d\lambda
\]
\[
= \frac{(a - c)^2}{6(b - a)}[2f'(a) + f'(c)]; \quad (22)
\]

\[
B \equiv \frac{1}{b - a} \int_c^b (x - c)f'(x)dx = \frac{(b - c)^2}{b - a} \int_0^1 \lambda(1 - \lambda)f'(a + \lambda b)d\lambda
\]
\[
\leq \frac{(b - c)^2}{(b - a)} \int_0^1 \lambda(1 - \lambda)(f'(c) + \lambda^2 f'(b))d\lambda
\]
\[
= \frac{(b - c)^2}{6(b - a)}[f'(c) + 2f'(b)]. \quad (23)
\]

It follows from (22) and (23) that

\[
A + B \leq \frac{1}{3} f'(b) \frac{(b - c)^2}{b - a} - \frac{1}{3} f'(a) \frac{(a - c)^2}{b - a} + \frac{1}{6} f'(c)(a + b - 2c),
\]

which completes the proof.

Remark 7 A simple calculation shows that the equality in (19) holds for the functions \(f(x) = k(x - c)^2 + m\), \((k, m \in \mathbb{R})\).

Remark 8 In the ”critic” cases \(c = a\) or \(c = b\), i.e. in the cases when \(f'\) is convex or concave on \([a, b]\) we have from (19)

\[
f(b) - \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{b - a}{6}[f'(a) + 2f'(b)]
\]
and

\[
f(a) - \frac{1}{b - a} \int_a^b f(x)dx \leq -\frac{b - a}{6}[2f'(a) + f'(b)],
\]
respectively.
5 Various inequalities for functions having convex first or second order derivatives

The first moment of a function $f$ about the center point $c = (a + b)/2$ is defined by $M_f = \int_a^b (x - \frac{a+b}{2}) f(x)dx$. In the following Theorem we obtain some estimation from above and below for $M_f$, when $f'$ is convex.

**Theorem 7** Suppose that the derivative $f'$ of the function $f : [a, b] \rightarrow \mathbb{R}$ is convex. Then the first moment of $f$ about the center point $c = (a+b)/2$ satisfies the following inequality

$$A \leq \int_a^b \left( x - \frac{a+b}{2} \right) f(x)dx \leq B,$$

where

$$A = \frac{(a-b)^2}{8}(f(b) - f(a)) - \frac{(b-a)^3}{48}(f'(a) + f'(b))$$

and

$$B = \frac{(b-a)^3}{24}(f'(a) + f'(b)).$$

**Proof** Integration by parts leads to

$$\int_a^b (x-a)(b-x)f'(x)dx = \int_a^b (x-a)(b-x)df(x)$$

$$= 2 \int_a^b \left( x - \frac{a+b}{2} \right) f(x)dx.$$

Hence,

$$\int_a^b \left( x - \frac{a+b}{2} \right) f(x)dx = \frac{1}{2} \int_a^b (x-a)(b-x)f'(x)dx$$

(set $x = (1-t)a + tb$, $(x-a)(b-x) = (b-a)^2t(1-t)$, $0 \leq t \leq 1$)

$$= \frac{1}{2}(b-a)^3 \int_0^1 t(1-t)f'((1-t)a + tb)dt$$

$$\leq \frac{1}{2}(b-a)^3 \int_0^1 t(1-t)[f'(a)(1-t) + f'(b)t]dt$$

$$= \frac{(b-a)^3}{24}(f'(a) + f'(b)).$$

This proved the right hand side of (24).

Further, again using integration by parts we have

$$\int_a^b \left( x - \frac{a+b}{2} \right)^2 f'(x)dx = \frac{(b-a)^2}{4}(f(b) - f(a)) - 2 \int_a^b \left( x - \frac{a+b}{2} \right) f(x)dx,$$
and therefore,

\[ \int_a^b \left( x - \frac{a + b}{2} \right) f(x)dx = \frac{(b-a)^2}{8}(f(b)-f(a)) - \frac{1}{2} \int_a^b \left( x - \frac{a + b}{2} \right)^2 f'(x)dx. \]  

(25)

Furthermore, setting \( x = (1-t)a + tb \), \( (x-a)^2 + b^2 = (b-a)^2 \left( t - \frac{1}{2} \right)^2 \) and \( dx = (b-a)dt, (0 \leq t \leq 1) \), we get

\[ \int_a^b \left( x - \frac{a + b}{2} \right)^2 f'(x)dx = (b-a)^3 \int_0^1 \left( t - \frac{1}{2} \right)^2 f'((1-t)a + tb)dt \]

\[ \leq (b-a)^3 \int_0^1 \left( t - \frac{1}{2} \right)^2 [(1-t)f'(a) + tf'(b)]dt \]

\[ = (b-a)^3 \left[ f'(a) \int_0^1 \left( t - \frac{1}{2} \right)^2 (1-t)dt + f'(b) \int_0^1 t \left( t - \frac{1}{2} \right)^2 dt \right] \]

\[ = \frac{(b-a)^3}{24} (f'(a) + f'(b)). \]

Taking into account this in (25) we obtain the left hand side of inequality (24).

The proof is complete.

A straightforward calculation shows that the equality in both sides of (24) is attained for \( f(x) = k(x^2 - (a + b)x) + n \), where \( k \) and \( n \) are arbitrary real numbers.

**Theorem 8** Given \( f : [a, b] \to \mathbb{R} \), let \( f'' \) be convex. Then the following inequality holds

\[ A \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq B, \]  

(26)

where

\[ A = \frac{b-a}{8}(f'(b) - f'(a)) - \frac{(b-a)^2}{48}(f''(a) + f''(b)) \]

and

\[ B = \frac{(b-a)^2}{24}(f''(a) + f''(b)). \]

**Proof** Integration by parts twice gives

\[ \int_a^b (x-a)(b-x)f''(x)dx = (b-a)(f(a) + f(b)) - 2 \int_a^b f(x)dx. \]
Hence,

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx = \frac{1}{2(b - a)} \int_a^b (x - a)(b - x)f''(x)dx
\]

(Set \( x = (1 - t)a + tb, 0 \leq t \leq 1 \))

\[
= \frac{(b - a)^2}{2} \int_0^1 t(1 - t)f''((1 - t)a + tb)dt
\]

\[
\leq \frac{(b - a)^2}{2} \int_0^1 t(1 - t)[f''(a) + tf''(b)]dt
\]

\[
= \frac{(b - a)^2}{24}(f''(a) + f''(b)).
\]

The right hand side of (26) is proved.

Straightforward calculations show that, integration by parts twice yields

\[
\int_a^b \left(x - \frac{a + b}{2}\right)^2 f''(x)dx
\]

\[
= \left(\frac{b - a}{2}\right)^2 (f'(b) - f'(a)) - 2(b - a) \left[\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx\right].
\]

Hence,

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx
\]

\[
= \frac{b - a}{8}(f'(b) - f'(a)) - \frac{1}{2(b - a)} \int_a^b \left(x - \frac{a + b}{2}\right)^2 f''(x)dx.
\]  \hspace{1cm} (27)

Setting \( x = (1 - t)a + tb, (0 \leq t \leq 1) \) and using the convexity of \( f'' \), we have

\[
\int_a^b \left(x - \frac{a + b}{2}\right)^2 f''(x)dx = (b - a)^3 \int_0^1 \left(t - \frac{1}{2}\right)^2 f''((1 - t)a + tb)dt
\]

\[
\leq (b - a)^3 \left[f''(a) \int_0^1 \left(t - \frac{1}{2}\right)^2 (1 - t)dt + f''(b) \int_0^1 \left(t - \frac{1}{2}\right)^2 tdt\right]
\]

\[
= \frac{(b - a)^3}{24}(f''(a) + f''(b)).
\]

By making use of this in (27) we obtain the left hand side of inequality (26).

The proof is complete.

It is easy to verify that the equality in both sides of (24) is attained for the functions \( f(x) = k(2x^3 - 3(a + b)x^2) + mx + n \), with arbitrary real numbers \( k, m \) and \( n \).
Remark 9 There are several results in the literature under the condition of the convexity of $|f'|$ or $|f''|$ (see, e.g. [2, 6, 24]). As far as we know, the conditions and assertions of our theorems 7, 8 and 9 completely differ from those known in the literature.

In the following theorem we give some estimations for the mean value of a function $f$ whose first derivative is convex.

**Theorem 9** Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and its derivative $f'$ be convex. Then

(a) $$N \leq \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq M,$$

where $$N = \frac{1}{3}(f(a) + 2f(b)) - \frac{1}{6}f'(b)(b - a)$$

and $$M = \frac{1}{3}(f(b) + 2f(a)) + \frac{1}{6}f'(a)(b - a);$$

(b) $$N \leq \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq M,$$

where $$N = f(a) + 2f \left( \frac{a + b}{2} \right) - \frac{4}{b - a} \int_{a}^{\frac{a + b}{2}} f(x) dx$$

and $$M = f(b) + 2f \left( \frac{a + b}{2} \right) - \frac{4}{b - a} \int_{\frac{a + b}{2}}^{b} f(x) dx.$$

**Corollary 4**

$$\int_{\frac{a + b}{2}}^{b} f(x) dx - \int_{a}^{\frac{a + b}{2}} f(x) dx \leq \frac{1}{4}(b - a)(f(b) - f(a)).$$

**Proof** Since $f'$ is convex, (HH) leads to

$$f' \left( \frac{a + x}{2} \right) \leq \frac{1}{x - a}(f(x) - f(a)) \leq \frac{f'(a) + f'(x)}{2};$$

$$f' \left( \frac{x + b}{2} \right) \leq \frac{1}{b - x}(f(b) - f(x)) \leq \frac{f'(x) + f'(b)}{2}.$$ Multiplying the inequalities (31) by $(x - a)$ and integrating over $[a, b]$, after simple calculations we obtain

$$2(b - a)f \left( \frac{a + b}{2} \right) - 4 \int_{a}^{\frac{a + b}{2}} f(x) dx \leq \int_{a}^{b} f(x) dx - f(a)(b - a)$$
\[
\leq \frac{1}{4} f'(a)(b-a)^2 + \frac{1}{2} (b-a)f(b) - \frac{1}{2} \int_a^b f(x)dx.
\]

The above inequalities can be written as two separate inequalities:

\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{3}(2f(a) + f(b)) + \frac{1}{6} f'(a)(b-a) \quad (33)
\]

and

\[
\frac{1}{b-a} \int_a^b f(x)dx + \frac{4}{b-a} \int_a^{b-\frac{1}{2}} f(x)dx \geq f(a) + 2f \left( \frac{a+b}{2} \right). \quad (34)
\]

In a similar way, multiplying inequalities (32) by \((b-x)\) and integrating over \([a,b]\), after some calculations we have the following two inequalities:

\[
\frac{1}{b-a} \int_a^b f(x)dx \geq \frac{1}{3}(f(a) + 2f(b)) - \frac{1}{6} f'(b)(b-a) \quad (35)
\]

and

\[
\frac{1}{b-a} \int_a^b f(x)dx + \frac{4}{b-a} \int_a^{b+\frac{1}{2}} f(x)dx \leq f(b) + 2f \left( \frac{a+b}{2} \right). \quad (36)
\]

Now, the inequalities (33) and (35) yields (28) and the inequalities (34) and (36) yields (29). The Corollary follows by subtracting (34) from (36).

The proof is complete.

**Example 3** For \(0 < a < x < b < \infty\) and \(f(x) = \ln x\), the inequality (30) yields

\[
\frac{a^{\alpha} \cdot b^{\beta}}{\alpha^{\alpha+\beta} \cdot \beta^{\alpha+\beta}} \leq \frac{a+b}{2}. \quad (37)
\]

Since \(\alpha + \beta = 1\) for \(\alpha = \frac{3a+b}{4(a+b)}\) and \(\beta = \frac{a+3b}{4(a+b)}\), then by the generalized AM-GM inequality we have

\[
a^\alpha \cdot b^\beta < a \cdot a + \beta \cdot b = \frac{3a+b}{4(a+b)} \cdot a + \frac{a+3b}{4(a+b)} \cdot b. \quad (38)
\]

A simple calculation shows that

\[
\frac{a+b}{2} < \frac{3a+b}{4(a+b)} \cdot a + \frac{a+3b}{4(a+b)} \cdot b,
\]

and therefore, the inequality (37) is better than (38).
References

1. Amrahov, S.È.: A note on Hadamard inequalities for the product of the convex functions, International J. of Research and Reviews in Applied Sciences, 5(2), 168-170 (2010).
2. Alomari, M., Darus, M. and Dragomir, S.S.: New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, Tamkang J. of Math., 41(4), 353-359 (2010).
3. Azpetia, A.G.: Convex functions and the Hadamard inequality, Revista Colombiana Mat., 28, 7-12 (1994).
4. Bakula M.K., Pecaric, J.: Note on some Hadamard-type inequalities, J. Ineq. Pure Appl. Math. 5(3), Article 74 (2004).
5. Bakula, M.K.; Ozdemir M. E. and Pecaric, J. Hadamard-type inequalities for m-convex and (α, m)-convex functions, J. Ineq. Pure Appl. Math., 9(4), Article 96 (2008).
6. Chen, F. and Liu, X.: On Hermite-Hadamard type inequalities for functions whose second derivatives absolute values are s-convex, ISRN Applied Mathematics, volume 2014, 1-4.
7. Dahmani, Z.: On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ano. Funct. Anal., 1(1), 51-58 (2010).
8. Dragomir, S.S. and Fitzpatrick, S.: The Hadamard inequalities for s-convex functions in the second sense, Demonstr. Math., 32(4), 687-696 (1999).
9. Dragomir, S.S. and Pearce, C.E.M.: Selected topics on Hermite-Hadamard inequalities and applications, RGMA monographs, Victoria University (2000).
10. Dragomir, S.S.: On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math., 31(1)(2002).
11. El Farissi, A.: Simple proof and refinement of Hermite-Hadamard inequality. J. Math. Inequal, 4(3), 365-369 (2010).
12. Fink, A.M.: A best possible Hadamard inequality, Math. Inequal Appl., 1, 223-230 (1998).
13. Florea, A. and Niculescu, C.P.: A Hermite-Hadamard inequality for convex-concave symmetric functions, Bull. Soc. Sci. Math. Roum., 50(98): No:2, 149-156 (2007).
14. Gill, P.M., Pearce, C.E.M. and Pecaric, J.: Hadamard’s inequality for r-convex functions, J. Math. Anal. Appl. 215(2), 461-470 (1997).
15. Ion, D.A.: Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. 34, 82-87 (2007).
16. Ion, D.A.: On an inequality due to Amrahov, Annals of University of Craiova, Math. Comp. Sci. Ser. 38(1), 92-95 (2011).
17. Kirmaci, V., Bakula, M.; Ozdemir, M. E. and Pecaric, J., Hadamard-type inequalities for s-convex functions, Appl. Math. Comp. 193, 26-35 (2007).
18. Mercer, A.M.D.: A variant of Jensen’s inequality, J. Ineq. Pure and Appl. Math. 4(4), Article 73 (2005).
19. Mitroinovic, D.S.; Pecaric, J. E. and Fink, A. M., Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, (1993).
20. Niculescu, C.P. and Persson, L.-E.: Old and new on the Hermite-Hadamard inequality, Real Anal. Exchange, 29, 663-686 (2003/2004).
21. Niculescu, C.P. and Persson, L.-E.: Convex functions and their applications, A contemporary approach, CMS Books in Mathematics, V.23, Springer-Verlag, (2006).
22. Niculescu, C.P.: The Hermite-Hadamard inequality for log-convex functions, Nonlinear Analysis, 75, 662-669 (2012).
23. Sarikaya, M.Z.: Saglam, A. and Yildirim, H., On some Hadamard-type inequalities for h-convex functions, J. of Math. Inequalities, 2(3), 335-341 (2008).
24. Sarikaya, M.Z.: Saglam, A. and Yldirim, H., New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex or quasi-convex, International J. of Open Problems in Comp. Sci. and Math., 5(3), 1-14 (2012).
25. Tseng, K.-L., Hwang, S.R. and Dragomir, S.S.: On some new inequalities of Hermite-Hadamard-Fejer type involving convex functions, Demonstr. Math., 40(1), 51-64 (2007).
26. Qi, F. and Yang, Z.-L.: Generalizations and refinements of Hermite-Hadamard’s inequality, The Rocky Mountain J. of Math., 35, 235-251 (2005).
27. Wu, S.-H.: On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39, 1741-1749 (2009).