THE SQUARING OPERATION FOR COMMUTATIVE DG RINGS

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Abstract. Let $A \to B$ be a homomorphism of commutative rings. The squaring operation is a functor $\text{Sq}_{B/A}$ from the derived category $\text{D}(B)$ of complexes of $B$-modules into itself. The squaring operation is needed for the definition of rigid complexes (in the sense of Van den Bergh), that in turn leads to a new approach to Grothendieck duality for rings, schemes and even DM stacks.

In our paper with J.J. Zhang from 2008 we introduced the squaring operation, and explored some of its properties. Unfortunately some of the proofs in that paper had severe gaps in them.

In the present paper we reproduce the construction of the squaring operation. This is done in a somewhat more general context than in the first paper: here we consider a homomorphism $A \to B$ of commutative DG rings. Our first main result is that the square $\text{Sq}_{B/A}(M)$ of a DG $B$-module $M$ is independent of the resolutions used to present it. Our second main results is on the trace functoriality of the squaring operation. We give precise statements and complete correct proofs.

In a subsequent paper we will reproduce the remaining parts of the 2008 paper that require fixing. This will allow us to proceed with the other papers, mentioned in the bibliography, on the rigid approach to Grothendieck duality.

The proofs of the main results require a substantial amount of foundational work on commutative and noncommutative DG rings, including a study of semi-free DG rings, their lifting properties, and their homotopies. This part of the paper could be of independent interest.

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Date: 05 Jan 2015.

Key words and phrases. DG rings, DG modules, derived categories, derived functors, resolutions.

Mathematics Subject Classification 2010. Primary: 16E45. Secondary: 16E35, 18G10, 13D03, 18E30.
0. Introduction

0.1. Background: Rigid Dualizing Complexes. The concept of rigid dualizing complex was introduced by M. Van den Bergh in his paper [VdB] from 1997. This was done for a noncommutative ring \( A \) over a base field \( K \). Let us recall the definition; but to simplify matters (and because here we are interested only in the commutative situation), we shall state it for a commutative ring \( A \).

So let \( K \) be a field and \( A \) a commutative \( K \)-ring. We denote by \( \mathcal{D}(A) \) the derived category of \( A \)-modules. Given a complex \( M \in \mathcal{D}(A) \), its square is the complex

\[
\text{Sq}_{A/K}(M) := \mathbb{R}
\text{Hom}_A \otimes_K A(A, M \otimes_K M) \in \mathcal{D}(A).
\]

The \( A \)-module structure on \( \text{Sq}_{A/K}(M) \) comes from the first argument of \( \mathbb{R}
\text{Hom} \).

Note that if \( M \) is a single module, then

\[
\text{HH}^q(\text{Sq}_{A/K}(M)) = \text{HH}^q(A, M \otimes_K M),
\]

the \( q \)-th Hochschild cohomology of the \( A \)-bimodule \( M \otimes_K M \).

Now assume \( A \) is finitely generated over \( K \) (and hence it is noetherian). A rigid dualizing complex over \( A \) relative to \( K \) is a pair \( (R, \rho) \), where \( R \in \mathcal{D}(A) \) is a dualizing complex (in the sense of [RD]), and

\[
\rho : R \cong \text{Sq}_{A/K}(R)
\]

is an isomorphism in \( \mathcal{D}(A) \). Van den Bergh proved that a rigid dualizing complex \( (R, \rho) \) exists, and it is unique up to isomorphism. Further work in this direction was done by J.J. Zhang and the author in a series of papers; see [YZ1] and its references. (These papers dealt with the noncommutative situation, which is significantly more complicated.)

In the current paper we are interested in commutative rings, but in a relative situation: instead of a base field \( K \), we have a homomorphism \( A \to B \) of commutative rings, and we want to produce a useful theory of squaring and rigidity.

If the homomorphism \( A \to B \) is flat, then there is a pretty easy way to generalize (0.1.1), as follows. Given a complex \( M \in \mathcal{D}(B) \), we may define

\[
\text{Sq}_{B/A}(M) := \mathbb{R}
\text{Hom}_{B \otimes_A B}(B, M \otimes_A^L M) \in \mathcal{D}(B).
\]

However, when \( A \to B \) is not flat, formula (0.1.3) is meaningless, since there is no way to interpret \( M \otimes_A^L M \) as an object of \( \mathcal{D}(B \otimes_A B) \)!

In our paper [YZ3], with Zhang, we proposed to solve the flatness problem by replacing the ring \( B \) with a flat DG ring resolution \( \tilde{B} \to B \) over \( A \). Unfortunately, there were serious flaws in the proofs in [YZ3], as explained in Subsection 0.7 of the Introduction. In the present paper we provide a comprehensive and correct treatment of the construction of the square, using the DG ring method. See Subsection 0.5 for applications in algebraic geometry of our work.

0.2. DG Rings and Extended Resolutions. Let \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) be a DG ring, where as usual “DG” is short for “differential graded”. (Most other texts would call \( A \) a unital associative DG algebra over \( \mathbb{Z} \).) We say that \( A \) is nonpositive if \( A^i = 0 \) for \( i > 0 \). The DG ring \( A \) is called strongly commutative if \( a \cdot b = (-1)^{ij} b \cdot a \) for \( a \in A^i \) and \( b \in A^j \), and \( a \cdot a = 0 \) if \( i \) is odd.

Notation 0.2.1. For convenience we use the term commutative DG ring as an abbreviation for “nonpositive strongly commutative DG ring.”
This is the term appearing in the title of the paper. Note that the category $\text{Ring}_c$ of commutative rings is a full subcategory of the category $\text{DGR}^{\leq 0}_{sc}$ of commutative DG rings (since a ring can be seen as a DG ring concentrated in degree 0).

To a commutative DG ring $A$ we associate the category of DG modules $M(A)$, and its derived category $D(A)$. There is an additive functor $Q : M(A) \rightarrow D(A)$, which is the identity on objects, and sends quasi-isomorphisms to isomorphisms. In Sections 1 and 2 we recall some facts about DG modules, their resolutions, and related derived functors.

Let $A \xrightarrow{u} B$ be a homomorphism of commutative DG rings. A $K$-flat commutative DG ring resolution of $A \xrightarrow{u} B$ is a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{v} & & \downarrow{w} \\
A & \xrightarrow{u} & B
\end{array}
\]

\[\xrightarrow{\text{K-flat}}\]

\[\xrightarrow{\text{qu-iso}}\]

in $\text{DGR}^{\leq 0}_{sc}$, such that $A \xrightarrow{u} A$ is a quasi-isomorphism, $B \xrightarrow{w} B$ is a surjective quasi-isomorphism, and $\hat{A} \xrightarrow{\alpha} \hat{B}$ is $K$-flat (i.e. $\hat{B}$ is $K$-flat as a DG $\hat{A}$-module). We refer to this by saying that $B/\hat{A}$ is a $K$-flat resolution of $B/A$ in $\text{DGR}^{\leq 0}_{sc}$. The set of all $K$-flat resolutions of $B/A$ in $\text{DGR}^{\leq 0}_{sc}$ is denoted by $\text{Res}_c(B/A)$. It is known (see Theorem 3.21) that $\text{Res}_c(B/A)$ is nonempty. We can even find resolutions where $\hat{A} = A$; and such a resolution $\hat{B}/\hat{A}$ is called a strict $K$-flat resolution of $B/A$. A special (yet important) case is when $A \xrightarrow{u} B$ is a homomorphism in $\text{Ring}_c$, and $\hat{B}$ consists of flat $A$-modules. (The reason we want to allow non-strict resolutions is explained in Subsection 0.5 of the Introduction.)

Let $A \xrightarrow{u} B$ be a homomorphism of commutative DG rings, and let $M \in D(B)$. Suppose we are given some resolution $\hat{B}/\hat{A} \in \text{Res}_c(B/A)$. An extended resolution of $M$ over $\hat{B}/\hat{A}$ consists of a quasi-isomorphism $\alpha : \hat{M} \rightarrow M$ in $M(\hat{B})$, where $\hat{M}$ is $K$-flat over $\hat{A}$; and a quasi-isomorphism $\beta : \hat{M} \otimes_{\hat{A}} M \rightarrow I$ in $M(\hat{B} \otimes_{\hat{A}} \hat{M})$, where $I$ is $K$-injective. Such extended resolutions exist, since we can always choose $\alpha : \hat{M} \rightarrow M$ to be a $K$-flat resolution in $M(\hat{B})$.

Again $A \xrightarrow{u} B$ is a homomorphism in $\text{DGR}^{\leq 0}_{sc}$, and $M \in D(B)$. An extended commutative resolution of $M$ is data $M = (\hat{B}/\hat{A}; \hat{M}, I)$, where $\hat{B}/\hat{A} \in \text{Res}_c(B/A)$, and $(\hat{M}, I)$ is an extended resolution of $M$ over $\hat{B}/\hat{A}$, as defined above. The set of all commutative extended resolutions of $M$ is denoted by $\text{Res}_c(B/A; M)$. We know this is a nonempty set.

Suppose we are given resolutions $\hat{B}/\hat{A}$ and $\hat{B}'/\hat{A}'$ in $\text{Res}_c(B/A)$. A morphism of resolutions $\hat{w}/\hat{v} : \hat{B}'/\hat{A}' \rightarrow \hat{B}/\hat{A}$ is the data of homomorphisms $\hat{v} : \hat{A}' \rightarrow \hat{A}$ and $\hat{w} : B' \rightarrow B$ in $\text{DGR}^{\leq 0}_{sc}$, such that the diagram
in $\text{DGR}_{w}^{\leq 0}$ is commutative. These morphisms make $\text{Res}_{c}(B/A)$ into a category.

Next let $M = (\tilde{B}/\tilde{A}; \tilde{M}, I)$ and $M' = (\tilde{B}'/\tilde{A}'; \tilde{M}', I')$ be extended resolutions in $\text{Res}_{c}(B/A; M)$. A morphism of extended resolutions $\phi : M' \to M$ is data $\phi = (\tilde{w}/\tilde{v}; \phi, \psi)$, where $\tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A}$ is a morphism in $\text{Res}_{c}(B/A)$, $\phi : \tilde{M}' \to \tilde{M}$ is a homomorphism in $\text{M}(\tilde{B}')$, and $\psi : I \to I'$ is a homomorphism in $\text{M}(\tilde{B}' \otimes_{\tilde{A}'} \tilde{B}')$.

The condition is that the diagrams

\[
\begin{array}{ccc}
\tilde{M}' & \xrightarrow{Q(\phi)} & \tilde{M} \\
\downarrow{Q(\alpha')} & & \downarrow{Q(\alpha)} \\
M & & \tilde{M}' \otimes_{\tilde{A}'} \tilde{M}' \xrightarrow{Q(\phi \otimes \phi)} \tilde{M} \otimes_{\tilde{A}'} \tilde{M} \\
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{I}' & \xleftarrow{Q(\psi)} & I \\
\downarrow{Q(\beta')} & & \downarrow{Q(\beta)} \\
\tilde{M}' \otimes_{\tilde{A}'} \tilde{M}' & & \tilde{M} \otimes_{\tilde{A}'} \tilde{M} \\
\end{array}
\]

in $\text{D}(\tilde{B}')$ and $\text{D}(\tilde{B}' \otimes_{\tilde{A}'} \tilde{B}')$ respectively are commutative. In this way the set $\text{Res}_{c}(B/A; M)$ becomes a category. The composition is

$$\phi' \circ \phi := (\tilde{w}' \circ \tilde{w} / (\tilde{v}' \circ \tilde{v}); \phi' \circ \phi, \psi \circ \psi').$$

Proposition 8.7 says that the category $\text{Res}_{c}(B/A; M)$ is nonempty and connected.

0.3. The Squaring Operation. Let $A \to B$ be a homomorphism of commutative DG rings, and let $M$ be a DG $B$-module. For any extended resolution $M = (\tilde{B}/\tilde{A}; \tilde{M}, I) \in \text{Res}_{c}(B/A; M)$ we write

\[
\text{Sq}_{B/A}^{M}(M) := \text{Hom}_{B \otimes_{A} B}(B, I) \in M(B).
\]

Observe that when $A \to B$ is a flat ring homomorphism, and we take $\tilde{B}/\tilde{A} := B/A$, then $\text{Sq}_{B/A}^{M}(M)$ is a concrete realization of $\text{(0.1.3)}$.

Given a morphism $\phi = (\tilde{w}/\tilde{v}; \phi, \psi) : M' \to M$ in $\text{Res}_{c}(B/A; M)$, let

\[
\text{Sq}_{B/A}^{\phi}(1_{M}) := \text{Hom}_{B \otimes_{A} B}(1_{B}, \psi) : \text{Sq}_{B/A}^{M}(M) \to \text{Sq}_{B/A}^{M'}(M),
\]

which is a homomorphism in $M(B)$. Thus we obtain a functor

\[
\text{Sq} : \text{Res}_{c}(B/A; M)_{\text{op}} \to M(B),
\]

called the square. According to Proposition 8.9 for any morphism $\phi$ in $\text{Res}_{c}(B/A; M)$, the homomorphism $\text{Sq}_{B/A}^{\phi}(1_{M})$ is a quasi-isomorphism.

The difficult problem is this: to what extent is $\text{Sq}_{B/A}^{M}(M)$, as an object of $\text{D}(B)$, independent of the extended resolution $M$?
The first main result of our present paper, Theorem 0.3.4 below, answers this question in a very precise way.

**Theorem 0.3.4 (Existence).** Let $A \to B$ be a homomorphism of commutative DG rings, and let $M \in D(B)$. There is an object

$$\text{Sq}_{B/A}(M) \in D(B),$$

with an isomorphism

$$\text{Sq}^M_{B/A}(1_M) : \text{Sq}_{B/A}(M) \xrightarrow{\sim} \text{Sq}_{B/A}^M(M)$$

in $D(B)$ for every commutative extended resolution $M \in \text{Res}_c(B/A; M)$, satisfying the following condition.

(*) For every morphism of extended resolutions $\phi : M' \to M$ in $\text{Res}_c(B/A; M)$, the diagram

$$\begin{array}{ccc}
\text{Sq}_{B/A}(M) & \xrightarrow{\text{Sq}^M_{B/A}(1_M)} & \text{Sq}_{B/A}^M(M) \\
\downarrow & & \downarrow \\
\text{Sq}_{B/A}^M(M) & \xrightarrow{Q(\text{Sq}^\phi_{B/A}(1_M))} & \text{Sq}_{B/A}^{M'}(M)
\end{array}$$

in $D(B)$ is commutative.

Moreover, the object $\text{Sq}_{B/A}(M)$ is unique, up to a unique isomorphism in $D(B)$.

The theorem is repeated as Theorem 8.11 in the body of the paper, and proved there. Note that the uniqueness of $\text{Sq}_{B/A}(M)$ is easy to show, since we know that the set $\text{Res}_c(B/A; M)$ is nonempty (this is Proposition 8.7).

As we already mentioned, this assertion appeared as [YZ3, Theorem 2.2]; but the proof there had a large gap in it. For a discussion, and a comparison to [AILN, Theorem 3.2], see Subsection 0.7 below.

**Definition 0.3.5.** The object $\text{Sq}_{B/A}(M)$ from Theorem 0.3.4 is called the square of $M$ over $B$ relative to $A$.

Now that the object $\text{Sq}_{B/A}(M) \in D(B)$ is well-defined, we can write

$$R\text{Hom}_{B \otimes^L_A B}(B, M \otimes^L_A M) := \text{Sq}_{B/A}(M).$$

This new expression is useful as a heuristic. Yet the reader should be warned that – at least in our “semi-classical” setting of DG rings and derived categories – the left hand side of formula (0.3.6) does not have an intrinsic meaning. Possibly such an intrinsic meaning does exist in the “avant-garde” setting of derived algebraic geometry.

**0.4. Trace Functoriality.** Let us move to the second main result of the present paper. We consider homomorphisms of commutative DG rings $A \xrightarrow{u} B \xrightarrow{v} C$. A $K$-flat commutative DG ring resolution of $A \xrightarrow{u} B \xrightarrow{v} C$ is a commutative diagram

$$\begin{array}{ccc}
\tilde{A} & \xrightarrow{\tilde{u}} & \tilde{B} & \xrightarrow{\tilde{v}} & \tilde{C} \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{u} & B & \xrightarrow{v} & C
\end{array}$$

This diagram is a commutative diagram in the category of DG rings.
in $\text{DGR}_{sc}^{\leq 0}$, such that the homomorphism $w_A$ is a quasi-isomorphism, the homomorphisms $w_B$ and $w_C$ are surjective quasi-isomorphisms, and the homomorphisms $\tilde{u}$ and $\tilde{v} \circ \tilde{u}$ are K-flat. We refer to the data in (0.4.1) succinctly as $\tilde{C}/\tilde{B}/\tilde{A}$, and say that it is a K-flat resolution of $C/B/A$ in $\text{DGR}_{sc}^{\leq 0}$. Such resolutions exist, by Theorem 3.21. Note that $\tilde{B}/\tilde{A}$ is a K-flat resolution of $B/A$, and $\tilde{C}/\tilde{A}$ is a K-flat resolution of $C/A$, as defined in Subsection 0.2.

Suppose we are given homomorphisms $A \to B \to C$ in $\text{DGR}_{sc}^{\leq 0}$, DG modules $M \in \text{D}(B)$ and $N \in \text{D}(C)$, and a morphism $\theta : N \to M$ in $\text{D}(B)$. This situation is denoted by $(C/B/A; N/M)$, or simply by $N/M$, if the DG rings are clear from the context. An extended K-flat resolution of $N/M$ consists of these data:

- A K-flat resolution $\tilde{C}/\tilde{B}/\tilde{A}$ of $C/B/A$ in $\text{DGR}_{sc}^{\leq 0}$.
- An extended resolution $(\tilde{M}, I)$ of $M$ over $\tilde{B}/\tilde{A}$, and an extended resolution $(\tilde{N}, J)$ of $N$ over $\tilde{C}/\tilde{A}$.
- A homomorphism $\tilde{\theta} : \tilde{N} \to \tilde{M}$ in $\text{M}(\tilde{B})$, and a homomorphism $\eta : J \to I$ in $\text{M}(\tilde{B} \otimes_{\tilde{A}} \tilde{B})$.

The conditions on $\tilde{\theta}$ and $\eta$ are that the diagrams

\[
\begin{array}{ccc}
M & \xleftarrow{Q(\alpha_M)} & \tilde{M} \\
\downarrow{Q(\alpha_N)} & & \downarrow{Q(\tilde{\theta})} \\
N & \xleftarrow{\theta} & \tilde{N}
\end{array}
\quad
\begin{array}{ccc}
I & \xleftarrow{Q(\eta)} & J \\
\downarrow{Q(\beta_M)} & & \downarrow{Q(\beta_N)} \\
\tilde{M} \otimes_{\tilde{A}} \tilde{M} & \xleftarrow{\tilde{\theta} \otimes \theta} & \tilde{N} \otimes_{\tilde{A}} \tilde{N}
\end{array}
\]

in the categories $\text{D}(\tilde{B})$ and $\text{D}(\tilde{B} \otimes_{\tilde{A}} \tilde{B})$ respectively, are commutative. We denote this extended resolution by $N/M = (\tilde{C}/\tilde{B}/\tilde{A}; \tilde{N}/\tilde{M}, J/I)$.

The set of all such extended resolutions is denoted by $\text{Res}_{sc}(N/M)$. According to Proposition 0.3, this set is nonempty.

Given an extended resolution $N/M \in \text{Res}_{sc}(N/M)$, we define a homomorphism

\[
\text{Sq}_{C/B/A}^{N/M}(\theta) := \text{Hom}_{\text{Res}_{sc}}(v, \eta) : \text{Sq}_{C/B/A}^{N}(M) \to \text{Sq}_{C/B/A}^{M}(N)
\]

in $\text{M}(B)$. Here $v : B \to C$ is the original DG ring homomorphism, and the homomorphisms $\tilde{v} : \tilde{B} \to \tilde{C}$ and $\tilde{\eta} : J \to I$ are part of the resolution $N/M$.

Here is the second main result of our paper.

**Theorem 0.4.3** (Trace Functoriality). Let $A \to B \to C$ be homomorphisms of commutative DG rings, let $M \in \text{D}(B)$ and $N \in \text{D}(C)$, and let $\theta : N \to M$ be a morphism in $\text{D}(B)$. There is a unique morphism

\[
\text{Sq}_{C/B/A}^{N/M}(\theta) : \text{Sq}_{C/B/A}^{N}(M) \to \text{Sq}_{C/B/A}^{M}(N)
\]

in $\text{D}(B)$ satisfying the following condition.

(†) For every extended resolution $N/M \in \text{Res}_{sc}(N/M)$, the diagram

\[
\begin{array}{ccc}
\text{Sq}_{B/A}^{M}(M) & \xleftarrow{\text{Sq}_{C/B/A}^{N/M}(\theta)} & \text{Sq}_{C/B/A}^{N}(M) \\
\downarrow{\text{Sq}_{B/A}^{M}(\theta)} & & \downarrow{\text{Sq}_{C/B/A}^{N/M}(\theta)} \\
\text{Sq}_{B/A}^{N}(M) & \xleftarrow{\text{Q(\text{Sq}_{C/B/A}^{N/M}(\theta))}} & \text{Sq}_{C/B/A}^{N}(N)
\end{array}
\]

is commutative.
in $\mathcal{D}(B)$ is commutative.

This is repeated as Theorem 9.5 in the body of the paper, and proved there.

When $C = B$ in Theorem 0.4.3 and the homomorphism $B \to B$ is the identity automorphism $1_B$, we write

\[(0.4.4) \quad \text{Sq}_{B/A}(\theta) := \text{Sq}_{B/B/A}(\theta).\]

**Corollary 0.4.5.** Let $A \to B_0 \to B_1 \to B_2$ be homomorphisms of commutative DG rings, let $M_i \in \mathcal{D}(B_i)$, and let $\theta_i : M_i \to M_{i-1}$ be morphisms in $\mathcal{D}(B_{i-1})$. Then

\[\text{Sq}_{B_i/B_0/A}(\theta_1) \circ \text{Sq}_{B_2/B_1/A}(\theta_2) = \text{Sq}_{B_2/B_1/A}(\theta_1 \circ \theta_2),\]

as morphisms $\text{Sq}_{B_2/A}(M_2) \to \text{Sq}_{B_0/A}(M_0)$ in $\mathcal{D}(B_0)$.

Moreover, for the identity automorphism $1_{M_0} : M_0 \to M_0$, there is equality

\[\text{Sq}_{B_0/A}(1_{M_0}) = 1_{\text{Sq}_{B_0/A}(M_0)}\]

of endomorphisms of the object $\text{Sq}_{B_0/A}(M_0)$ in $\mathcal{D}(B_0)$.

This is repeated as Corollary 9.10 in the body of the paper. An immediate consequence is:

**Corollary 0.4.6.** Let $A \to B$ be a homomorphism of commutative DG rings. The assignments $M \mapsto \text{Sq}_{B/A}(M)$ and $\theta \mapsto \text{Sq}_{B/A}(\theta)$ are a functor

\[\text{Sq}_{B/A} : \mathcal{D}(B) \to \mathcal{D}(B).\]

The functor $\text{Sq}_{B/A}$ is not linear; in fact it is a quadratic functor, in the sense of the next result (which is repeated as Corollary 9.12).

**Corollary 0.4.7.** Let $A \to B$ be a homomorphism of commutative DG rings. Given a morphism $\theta : N \to M$ in $\mathcal{D}(B)$ and an element $b \in B_0$, we have

\[\text{Sq}_{B/A}(b \cdot \theta) = b^2 \cdot \text{Sq}_{B/A}(\theta),\]

as morphisms $\text{Sq}_{B/A}(N) \to \text{Sq}_{B/A}(M)$ in $\mathcal{D}(B)$.

0.5. **Motivation.** Let us now discuss the roles of Theorems 0.3.4 and 0.4.3 in our research. Here we consider commutative rings only (not commutative DG rings as above). Let $A$ be a noetherian ring, and let $A \to B$ be an essentially finite type ring homomorphism. Generalizing formulas (0.1.1) and (0.1.2), we define a rigid complex over $B$ relative to $A$ to be a pair $(M, \rho)$, where $M \in \mathcal{D}^f(B)$ and has finite flat dimension over $A$, and

\[(0.5.1) \quad \rho : M \to \text{Sq}_{B/A}(M)\]

is an isomorphism in $\mathcal{D}(B)$, called a rigidifying isomorphism. In the paper Ye5 we repeat (with more precise statements and with correct proofs) the last sections of YZ3. In particular, we prove that if $(M, \rho)$ is a rigid complex over $B/A$, $f^* : B \to C$ is a finite ring homomorphism, and the complex $f^!(M) := \text{RHom}_B(C, M)$ has finite flat dimension over $A$, then $f^!(M)$ has an induced rigidifying isomorphism $f^!(\rho)$ over $C/A$. If $f^* : B \to C$ is an essentially smooth ring homomorphism of relative dimension $n$, then the complex $f^!(M) := \Omega^f_{C/B}[n] \otimes_B M$ has an induced rigidifying isomorphism $f^!(\rho)$ over $C/A$. For this we need the results of Ye2 on Cohen-Macaulay DG modules (to clear up the construction of the cup product for the squaring operation).
A new result in [Ye5] – not present in [YZ3] – is the étale functoriality of the squaring operation. It says that when $B \to B'$ is an essentially étale ring homomorphism, and $M \in D(B)$, there is an isomorphism

$$\text{Sq}_{B'/A}(B' \otimes_B M) \cong B' \otimes_B \text{Sq}_{B/A}(M)$$

in $D(B')$, functorial in $M$ and $B'$. Observe that we may want to work over a base ring $K$ (see below). In case $A \to B$ is a homomorphism of $K$-rings, and $A$ is not flat over $K$, it is advantageous to use $K$-flat DG ring resolutions $\tilde{B}/\tilde{A}$ of $B/A$, such that $\tilde{A}$ is $K$-flat over $K$. This is one reason we consider non-strict DG ring resolutions. Another reason is that in the situation described in diagram (0.4.1), we sometimes need the homomorphism $\tilde{B} \to \tilde{C}$ to be a resolution of $B \to C$. For instance, this happens in [Ye5], when we establish the cup product for the squaring operation.

In the paper [Ye6] we use the results of the paper [YZ4] (with minor improvements) and [Ye5] to study rigid residue complexes on essentially finite type $K$-schemes, where $K$ is a regular noetherian base ring. This work was already outlined in the preprint [YZ2] and the survey [Ye1]. In [Ye6] we finally give full details and complete proofs.

The last paper (so far) in this series is [Ye7]. There we look at finite type Deligne-Mumford stacks over a regular base ring $K$, and establish a theory of residues and duality for them. We define the rigid residue complex $K_X$ of such a stack $X$, and prove its existence and uniqueness. To a map $f : X \to Y$ between stacks there is a trace homomorphism $\text{Tr}_f : f_* (K_Y) \to K_X$, which is a homomorphism of graded quasi-coherent sheaves. We prove the Residue Theorem, which says that when $f$ is proper, the homomorphism $\text{Tr}_f$ commutes with the differentials. And we prove the Duality Theorem, which says that when $f$ is proper and tame, $\text{Tr}_f$ induces a global duality. Both results require a mild technical condition: that $f$ is “coarsely schematic”. See the lecture notes [YZ3] for some more details. The results of [Ye7] are totally new.

0.6. Noncommutative Resolutions and the Rectangle Operation. The squaring operation described in Subsection 0.3 above is a special case of the rectangle operation. Most of the hard work in the paper goes into the proof of Theorem 7.1 that establishes the rectangle operation; and this is done using noncommutative semi-free DG ring resolutions. Let us explain this aspect of the paper briefly.

Fix a commutative DG ring $A$. A homomorphism of DG rings $u : A \to B$ is said to be central if $u(A) \subset B^{ce}$, where the latter is the center of $B$ (in the graded sense). We work in the category $\text{DGR}^{\leq 0}/_{ce} A$ of nonpositive DG rings $B$ that are central over $A$. In Section 3 we study two kinds of semi-free DG rings over $A$: the commutative kind (that were already introduced in [YZ3]) and the noncommutative kind. Theorems 3.21, 3.22, and 4.3 say that any $B \in \text{DGR}^{\leq 0}/_{ce} A$ admits a noncommutative semi-free DG ring resolution $\tilde{B}$, and this resolution is unique up to a quasi-isomorphism $w : \tilde{B}' \to \tilde{B}$, that is itself unique up to homotopy. The notion of homotopy between homomorphisms $w_0, w_1 : \tilde{B}' \to \tilde{B}$ in $\text{DGR}^{\leq 0}/_{ce} A$ that we use is due to Keller (see [Ke2]), and can be stated in terms of the cylinder DG ring.
Our noncommutative semi-free DG rings behave like cofibrant objects in a Quillen model structure. However, we do not know if the category $\text{DGR}^{\leq 0}_{/ce} A$ admits a model structure (except when $A$ is a ring). See Remark 4.6 where this issue is discussed.

In Sections 5-7 we develop noncommutative extended resolutions. These are noncommutative variants of the extended resolutions mentioned in Subsection 0.2. The highlight of this part of the paper is Theorem 7.1, and the subsequent Corollary 7.11. Here is what they say: let $B \in \text{DGR}^{\leq 0}_{/ce} A$, and let $(M_1, M_2)$ be a pair of (left, right) DG $B$-modules. There is an object

$$\text{Rect}_{B/A}(M_1, M_2) \in \text{D}(B)$$

called the rectangle of $(M_1, M_2)$, which is uniquely characterized using noncommutative extended resolutions, similarly to what is done in Theorem 0.3.4.

In case $B$ itself is commutative, we can be more explicit in our outline. Take any DG ring resolution $\hat{B}/\hat{A} \in \text{Res}_c(B/A)$, as in Subsection 0.2, and choose K-flat resolutions $\hat{M}_i \to M_i$ in $\text{M}(\hat{B})$. Then there is a canonical isomorphism

$$\text{Rect}_{B/A}(M_1, M_2) \cong \text{RHom}_{\hat{B} \otimes_{\hat{A}} \hat{B}}(B, \hat{M}_1 \otimes_{\hat{A}} \hat{M}_2)$$

in $\text{D}(B)$. The relation to the squaring is obvious:

$$\text{Sq}_{B/A}(M) = \text{Rect}_{B/A}(M, M).$$

There are two reasons we consider the bifunctorial operation $\text{Rect}_{B/A}(-,-)$, and work in the noncommutative setting. First is the importance of the bifunctorial operation (that’s explained in Remark 7.13). The second reason is this: a single-argument commutative variant of Theorem 7.1 would require almost as much work as the noncommutative two-argument variant, because we would have to use noncommutative resolutions anyhow.

After proving Theorem 7.1 we forget all about noncommutative DG rings, and stick exclusively to commutative DG rings, and to the single argument operation $\text{Sq}_{B/A}(-)$.

0.7. Discussion of Related Papers. Early versions of Theorems 0.3.4 and 0.4.3 had already appeared in our paper \cite{YZ3} with Zhang, as Theorems 2.2 and 2.3 respectively. However, to our great embarrassment, the proofs of these results in \cite{YZ3} had severe gaps in them. The gaps are well-hidden in the text; but one clear error is this: the homomorphism $\begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix}$ in the middle of page 3225 (in the proof of Theorem 2.2) does not make any sense (unless $\phi_0 = \phi_1$). In order to give a correct treatment it is necessary to work with noncommutative semi-free DG ring resolutions, as explained in Subsection 0.6 above.

The mistake in the proof of \cite{YZ3} Theorem 2.2 was discovered by Avramov, Iyengar, Lipman and Nayak \cite{AILN}. They also found a way to fix it, and their \cite{AILN} Theorem 3.2 is a generalization of our \cite{YZ3} Theorem 2.2. Indeed, \cite{AILN} Theorem 3.2 establishes the rectangle operation (0.6.1) in the case when $A$ and $B$ are rings (but not DG rings). The proof of \cite{AILN} Theorem 3.2 relies on the Quillen model structure on the category $\text{DGR}^{\leq 0}_{/ce} A$ over a commutative ring $A$, following \cite{BP}.

Our goals are different from those of \cite{AILN}, and hence we adopt a different strategy. For us the relative situation of a ring homomorphism $A \to B$ is crucial, and we must consider composed homomorphisms $A \to B \to C$. This forces us
to concentrate on commutative DG ring resolutions, and their relative properties (cf. diagram (0.4.1)). Furthermore, as already mentioned in Subsection 0.6, we do not know whether $DGR_{\leq 0}/ce A$ even admits a cofibrantly generated Quillen model structure when the commutative DG ring $A$ is not a ring.

Acknowledgments. I wish to thank James Zhang, Bernhard Keller, Vladimir Hinich, Liran Shaul, Rishi Vyas, Asaf Yekutieli and Sharon Hollander for their help in writing this paper.

1. Some Facts on DG Rings and Modules

In this section we review some known facts about DG rings and modules. We give a rather detailed account of the shift of a DG module. Finally, we introduce the cylinder construction for DG rings and modules.

A differential graded ring, or DG ring for short, is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with an additive endomorphism $d_A$ of degree $1$ called the differential. The differential $d_A$ satisfies $d_A \circ d_A = 0$, and it is a graded derivation of $A$, namely it satisfies the graded Leibniz rule

\[(1.1) \quad d_A(a \cdot b) = d_A(a) \cdot b + (-1)^i \cdot a \cdot d_A(b)\]

for $a \in A^i$ and $b \in A^j$. (Most other texts would call such $A$ an associative unital DG algebra.)

We denote by $DGR$ the category of all DG rings, where the morphisms are the graded ring homomorphisms $u : A \to B$ that commute with the differentials. We consider rings as DG rings concentrated in degree $0$.

Definition 1.2. Let $A$ be a DG ring. A DG ring over $A$, or a $DG A$-ring, is a pair $(B,u)$, where $B$ is a DG ring and $u : A \to B$ is a DG ring homomorphism.

Suppose $(B,u)$ and $(B',u')$ are DG rings over $A$. A homomorphism of DG rings over $A$, or a $DG A$-ring homomorphism, is a homomorphism of DG rings $v : B \to B'$ such that $u' = v \circ u$.

We denote by $DGR/A$ the category of DG rings over $A$.

Thus $DGR/A$ is the usual “coslice category”, except that we say “over” instead of “under”. There is an obvious forgetful functor $DGR/A \to DGR$. A morphism $v$ in $DGR/A$ is shown in this commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{u} & B' \\
\downarrow{u} & & \downarrow{u'} \\
A & & \\
\end{array}
\]

Let $1_A : A \to A$ be the identity automorphism. Then $(A,1_A)$ is the initial object of $DGR/A$.

Consider a graded ring $A$. For homogeneous elements $a \in A^i$ and $b \in A^j$ we define the graded commutator to be

\[(1.3) \quad \text{ad}(a)(b) := a \cdot b - (-1)^{ij} \cdot b \cdot a.
\]

This extends to all $a, b \in A$ using bilinearity. Namely, if $a = \sum_k a_k$ and $b = \sum_l b_l$, with $a_k \in A^{i_k}$ and $b_l \in A^{j_l}$, then

\[
\text{ad}(a)(b) := \sum_{k,l} \text{ad}(a_k)(b_l).
\]
Definition 1.4. Let $A$ be a DG ring.

(1) The center of $A$ is the DG subring
$$A^c := \{ a \in A \mid \text{ad}(a)(b) = 0 \text{ for all } b \in A \}.$$ 

(2) The DG ring $A$ is called weakly commutative if $A^c = A$. In other words, if
$$a \cdot b = (-1)^{ij} \cdot b \cdot a$$
for all $a \in A^i$ and $b \in A^j$.

(3) A homomorphism $u : A \to B$ in $\text{DGR}$ is called central if $u(A^c) \subset B^c$. In this case, the resulting DG ring homomorphism $A^c \to B^c$ is denoted by $u^c$.

(4) We denote by $\text{DGR} / A^c$ to be the the full subcategory of $\text{DGR} / A$ consisting of pairs $(B, u)$ such that $u : A \to B$ is central.

Implicit in item (1) of the definition above is the (easy to check) fact that $A^c$ is a DG subring of $A$. The DG ring $A^c$ itself is weakly commutative. If $u : A \to B$ and $v : B \to C$ are central homomorphisms in $\text{DGR}$, then $v \circ u : A \to C$ is also central. As for item (4): a morphism $v : (B, u) \to (B', u')$ in the category $\text{DGR} / A^c$ is not required to be central (the condition is only on $u$ and $u'$).

Let $A$ be a DG ring. Recall that a left DG $A$-module is a left graded $A$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with differential $d_M$ of degree 1 that satisfies $d_M \circ d_M = 0$ and the graded Leibniz rule (equation (1.1), but with $m \in M^j$ instead of $b$). Right DG $A$-modules are defined similarly. By default all DG modules in this paper are left DG modules.

Let $A$ be a DG ring, let $M$ and $N$ be left DG $A$-modules, and let $\phi : M \to N$ be a degree $k$ additive (i.e. $\mathbb{Z}$-linear) homomorphism. We say that $\phi$ is $A$-linear if
\[
\phi(a \cdot m) = (-1)^{ik} \cdot a \cdot \phi(m)
\]
for all $a \in A^i$ and $m \in M^j$. (This is the sign convention of $\text{[ML]}$.) The abelian group of $A$-linear homomorphisms of degree $k$ is denoted by $\text{Hom}_A(M, N)^k$, and we let
\[
\text{Hom}_A(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_A(M, N)^k.
\]

The graded abelian group $\text{Hom}_A(M, N)$ has a differential $d$ given by
\[
d(\phi) := d_N \circ \phi - (-1)^k \cdot \phi \circ d_M
\]
for $\phi \in \text{Hom}_A(M, N)^k$. In particular, for $N = M$ we get a DG ring
\[
\text{End}_A(M) := \text{Hom}_A(M, M).
\]

Suppose $A$ is a DG ring and $M$ a DG abelian group (i.e. a DG $\mathbb{Z}$-module). From $\text{[13]}$ we see that a DG $A$-module structure on $M$ is the same as a DG ring homomorphism $\lambda_M : A \to \text{End}_\mathbb{Z}(M)$.

If $M$ is a right DG $A$-module and $N$ a left DG $A$-module, then the usual tensor product $M \otimes_A N$ is a graded abelian group:
\[
M \otimes_A N = \bigoplus_{k \in \mathbb{Z}} (M \otimes_A N)^k,
\]
where $(M \otimes_A N)^k$ is the subgroup generated by the tensors $m \otimes n$ such that $m \in M^i$ and $n \in N^{k-i}$. The graded abelian group $M \otimes_A N$ has a differential $d$ satisfying
\[
d(m \otimes n) := d_M(m) \otimes n + (-1)^i \cdot m \otimes d_N(n)
\]
for \(m, n\) as above.

We denote by \(\text{DGMod} A\) the category of DG \(A\)-modules, where the morphisms are the \(A\)-linear homomorphisms \(\phi : M \to N\) of degree 0 that commute with the differentials. This is an abelian category. The set \(\text{DGMod} A\) also has a DG category structure on it, in which the set of morphisms \(M \to N\) is the DG abelian group \(\text{Hom}_A(M, N)\). The relation between these structures is

\[
\text{Hom}_{\text{DGMod} A}(M, N) = Z^0(\text{Hom}_A(M, N)),
\]

the group of 0-cocycles.

Let \(A\) and \(B\) be DG rings. The tensor product \(A \otimes \mathbb{Z} B\) is a DG ring, with multiplication

\[
(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{i_1 j_2} \cdot (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)
\]

for \(a_i \in A^{i_k}\) and \(b_i \in B^{j_k}\), and with differential like in (1.9).

Let \(B\) be a DG ring. Its opposite DG ring \(B^{\text{op}}\) is the same DG abelian group, but we denote its elements by \(b^{\text{op}}\), \(b \in B\). The multiplication in \(B^{\text{op}}\) is

\[
b_1^{\text{op}} \cdot b_2^{\text{op}} := (-1)^{j_2 i_1} \cdot (b_2 \cdot b_1)^{\text{op}}
\]

for \(b_k \in B^{j_k}\). Note that the centers satisfy \((B^{\text{op}})^{\text{ce}} = (B^{\text{ce}})^{\text{op}}\), and \(B^{\text{op}} = B\) iff \(B\) is weakly commutative.

If \(M\) is a right DG \(B\)-module, then it can be seen as a left DG \(B^{\text{op}}\)-module, with action

\[
b^{\text{op}} \cdot m := (-1)^{k_1} \cdot m \cdot b
\]

for \(b \in B^{j_1}\) and \(m \in M^{k_1}\). Because of this observation, working only with left DG modules is not a limitation. We shall switch between notations according to convenience.

Now consider a DG \(A\)-\(B\)-bimodule \(M\), namely \(M\) has a left \(A\) action and a right \(B\) action, and they commute, i.e. \(a \cdot (m \cdot b) = (a \cdot m) \cdot b\). Then \(M\) can be viewed as a left \(A \otimes \mathbb{Z} B^{\text{op}}\)-module with action

\[
(a \otimes b^{\text{op}}) \cdot m := (-1)^{k_1} \cdot a \cdot m \cdot b
\]

for \(a \in A^{i_1}\), \(b \in B^{j_1}\) and \(m \in M^{k_1}\). Thus a DG \(A\)-\(B\)-bimodule structure on a DG abelian group \(M\) is the same as a homomorphism of DG rings

\[
A \otimes \mathbb{Z} B^{\text{op}} \to \text{End}\mathbb{Z}(M).
\]

Let \(A\) be a weakly commutative DG ring. Any \(B \in \text{DGR} /_{\text{ce}} A\) is a DG \(A\)-bimodule. If \(B, C \in \text{DGR} /_{\text{ce}} A\), then the DG \(A\)-module \(B \otimes_A C\) has an obvious DG ring structure, and moreover \(B \otimes_A C \in \text{DGR} /_{\text{ce}} A\). Given \(M \in \text{M}(B)\) and \(N \in \text{M}(C)\), the tensor product \(M \otimes_A N\) belongs to \(\text{M}(B \otimes_A C)\), with action

\[
(b \otimes c) \cdot (m \otimes n) := (-1)^{i j} \cdot (b \cdot m) \otimes (c \cdot n)
\]

for \(c \in C^{i}\) and \(m \in M^{j}\).

Let \(M = \bigoplus_{i \in \mathbb{Z}} M^i\) be a graded abelian group. Recall that the shift (or twist, or translation, or suspension) of \(M\) is the graded abelian group \(T(M) = \bigoplus_{i \in \mathbb{Z}} T(M)^i\) defined as follows. The graded component of degree \(i\) of \(T(M)\) is \(T(M)^i := M^{i+1}\). Note that as (ungraded) abelian groups we have \(T(M) = M\), but the gradings are different. Thus the identity automorphism of \(M\) becomes a degree \(-1\) invertible homomorphism of graded abelian groups \(t : M \to T(M)\). We shall usually represent elements of \(T(M)^i\) as \(t(m)\), with \(m \in M^{i+1}\).
If $M$ is a DG $A$-module, then $T(M)$ has the following structure of DG $A$-module. The differential $d_{T(M)}$ is

\begin{equation}
  d_{T(M)}(t(m)) := -t(d_{M}(m))
\end{equation}

for $m \in M^k$. The action of $A$ is

\begin{equation}
  a \cdot t(m) := (-1)^i \cdot t(a \cdot m)
\end{equation}

for $a \in A^i$. If $M$ is a DG right module, or a DG bimodule, then the structure of $T(M)$ is determined by $\text{(1.11)}$, $\text{(1.12)}$ and $\text{(1.10)}$. Concretely,

\begin{equation}
  a \cdot t(m) \cdot b = (-1)^i \cdot t(a \cdot m \cdot b)
\end{equation}

for $a \in A^i$, $m \in M^k$ and $b \in A^j$.

Given a homomorphism $\phi : M \to N$ in $\text{DGMod} \ A$, there is an induced homomorphism $T(\phi) : T(M) \to T(N)$ in $\text{DGMod} \ A$, defined by

\begin{equation}
  T(\phi)(t(m)) := t(\phi(m))
\end{equation}

for $m \in M$. In this way $T$ becomes an automorphism of the abelian category $\text{DGMod} \ A$.

More generally, suppose $M$ and $N$ are graded abelian groups, and $\phi : M \to N$ is a degree $j$ homomorphism. We define a degree $j$ homomorphism

\begin{equation}
  T(\phi) : T(M) \to T(N), \quad T(\phi) := (-1)^j \cdot t \circ \phi \circ t^{-1}.
\end{equation}

Thus, for any element $t(m) \in T(M)^i$, represented by an element $m \in M^{i+1}$, we have

\begin{equation}
  T(\phi)(t(m)) = (-1)^j \cdot (t \circ \phi \circ t^{-1})(t(m))
\end{equation}

\begin{equation}
  = (-1)^j \cdot t(\phi(m)) \in T(N)^{i+j}.
\end{equation}

Taking the direct sum on all $j$, we obtain an isomorphism of graded abelian groups

\begin{equation}
  T : \text{Hom}_{\mathbb{Z}}(M, N) \to \text{Hom}_{\mathbb{Z}}(T(M), T(N)).
\end{equation}

Now let $L, M, N$ be graded abelian groups. It is not hard to see that for any $\phi \in \text{Hom}_{\mathbb{Z}}(L, M)^i$ and $\psi \in \text{Hom}_{\mathbb{Z}}(M, N)^j$ we have

\begin{equation}
  T(\psi) \circ T(\phi) = T(\psi \circ \phi),
\end{equation}

as elements of $\text{Hom}_{\mathbb{Z}}(T(L), T(N))^{i+j}$.

Formula $\text{(1.17)}$ explains formulas $\text{(1.11)}$ and $\text{(1.12)}$ for a DG $A$-module $M$. Indeed, viewing the differential $d_{M}$ as an element of $\text{End}_{\mathbb{Z}}(M)^i$, we have

\begin{equation}
  d_{T(M)} = T(d_{M}) \in \text{End}_{\mathbb{Z}}(T(M))^{i},
\end{equation}

and $T(d_{M}) \circ T(d_{M}) = 0$. Next, consider the DG ring homomorphism $\lambda_{M} : A \to \text{End}_{\mathbb{Z}}(M)$. Take an element $a \in A^i$. Then $\lambda_{M}(a) \in \text{End}_{\mathbb{Z}}(M)^i$, and therefore we get an element

\begin{equation}
  \lambda_{T(M)}(a) := T(\lambda_{M}(a)) \in \text{End}_{\mathbb{Z}}(T(M))^i.
\end{equation}

The equality $\text{(1.17)}$ says that the resulting homomorphism

\begin{equation}
  \lambda_{T(M)} : A \to \text{End}_{\mathbb{Z}}(T(M)),
\end{equation}

gotten by summing on all $i$, is a DG ring homomorphism. On an element $m \in M$ we get, using $\text{(1.15)}$:

\begin{equation}
  \lambda_{T(M)}(a)(t(m)) = T(\lambda_{M}(a))(t(m)) = (-1)^i \cdot t(a \cdot m),
\end{equation}
precisely as in (1.12). Formula (1.17) also implies that there is an isomorphism of DG abelian groups

\[(1.18) \quad T : \text{Hom}_A(M, N) \xrightarrow{\sim} \text{Hom}_A(T(M), T(N))\]

that respects composition. Thus T is an automorphism of the DG category structure of DGMod A. When \(N = M\), equation (1.18) becomes as isomorphism of DG rings

\[(1.19) \quad T : \text{End}_A(M) \xrightarrow{\sim} \text{End}_A(T(M)).\]

For any \(k \in \mathbb{Z}\), the \(k\)-th power of T is denoted by \(T^k\). There is a corresponding degree \(-k\) homomorphism of graded abelian groups \(T^k : M \rightarrow T^k(M)\), which is the identity on the underlying ungraded abelian group. Note that \(T^i(T^k(M)) = T^{k+i}(M)\).

We now recall the cone construction. Let \(A\) be a DG ring \(A\) and let \(\phi : M \rightarrow N\) be a homomorphism in DGMod \(A\). The cone of \(\phi\) is the DG \(A\)-module \(\text{Cone}(\phi) := N \oplus T(M)\), whose differential, when we express this module as the column \([T(M)]\), is left multiplication by the matrix of degree \(1\) abelian group homomorphisms \([d_N \phi \otimes t^{-1}]\).

We learned the concept of cylinder DG ring, defined below, from Keller. It appears (without this name) in his [Kc2].

**Definition 1.20.** Let \(M\) be a graded abelian group. The cylinder of \(M\) is the graded abelian group of \(2 \times 2\) matrices

\[
\text{Cyl}(M) := \begin{bmatrix} M & T^{-1}(M) \\ 0 & M \end{bmatrix} = \left\{ \begin{bmatrix} m_0 & t^{-1}(n) \\ 0 & m_1 \end{bmatrix} \mid m_0, m_1, n \in M \right\}.
\]

There are graded abelian group homomorphisms \(M \xrightarrow{\eta_i} \text{Cyl}(M) \xrightarrow{\rho_i} M\), for \(i = 0, 1\), with formulas

\[(1.21) \quad \epsilon(m) := \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \eta_i \left( \begin{bmatrix} m_0 & t^{-1}(n) \\ 0 & m_1 \end{bmatrix} \right) := m_i.
\]

Note that \(\eta_0 \circ \epsilon = 1_M\), the identity automorphism of \(M\).

Suppose \(B\) is a DG ring. We define a degree 1 operator \(d_{\text{cyl}}\) on the graded abelian group \(\text{Cyl}(B)\) as follows:

\[(1.22) \quad d_{\text{cyl}} \left( \begin{bmatrix} b_0 & t^{-1}(c) \\ 0 & b_1 \end{bmatrix} \right) := \begin{bmatrix} d_B(b_0) & t^{-1}(-b_0 + b_1 - d_B(c)) \\ 0 & d_B(b_1) \end{bmatrix}
\]

for elements \(b_0, b_1, c \in B\). We also define a multiplication on \(\text{Cyl}(B)\) :

\[(1.23) \quad \left[ b_0 \quad t^{-1}(c) \\ 0 \quad b_1 \right] \cdot \left[ b_0' \quad t^{-1}(c') \\ 0 \quad b_1' \right] := \left[ b_0 \cdot b_0' \quad t^{-1}((-1)^{k_0}b_0 \cdot b_0' \cdot c' + c \cdot b_1') \quad 0 \quad b_1 \cdot b_1' \right]
\]

where \(b_0', b_1', c' \in B\) and \(b_0\) is homogeneous of degree \(k_0\).

**Proposition 1.24.** Let \(B\) be a DG ring.

1. The operator \(d_{\text{cyl}}\) and the multiplication from (1.23) make \(\text{Cyl}(B)\) into a DG ring.
2. The homomorphisms \(\epsilon\) and \(\eta_i\) are DG ring quasi-isomorphisms.
3. The homomorphisms \(\epsilon\) and \(\eta_i\) are central, and moreover the homomorphisms \(\epsilon \circ \epsilon : B^{ce} \rightarrow \text{Cyl}(B)^{ce}\) and \(\eta_i \circ \epsilon : \text{Cyl}(B)^{ce} ightarrow B^{ce}\) are bijective.

**Proof.** This is a direct (but somewhat tedious) calculation. \(\Box\)
Proposition 1.25. Let $B$ be a DG ring, and let $\text{inc}: B^{ce} \to B$ be the inclusion of the center. Then
\[
\begin{array}{ccc}
B^{ce} & \xrightarrow{\epsilon} & \text{Cyl}(B^{ce}) \\
\downarrow\text{inc} & & \downarrow\eta_i \\
B & \xrightarrow{\epsilon} & \text{Cyl}(B) \\
\end{array}
\]
is a commutative diagram of central homomorphisms in $\text{DGR}/B^{ce}$.

Proof. That the diagram is commutative is clear. Therefore there is a unique homomorphism from $B^{ce}$ to all other DG rings in the diagram, such that it becomes a diagram in $\text{DGR}/B^{ce}$.

Trivially $\text{inc}: B^{ce} \to B$ is a central homomorphism, and by Proposition 1.24(3) the homomorphisms $\epsilon, \eta_i$ are central (on both rows). Proposition 1.24(3) also tells us that $\epsilon: B^{ce} \to \text{Cyl}(B^{ce})$ and $\epsilon: B^{ce} \to \text{Cyl}(B)^{ce}$ are bijective. Therefore $\text{Cyl}(\text{inc})$ is central (and moreover $\text{Cyl}(\text{inc})^{ce}$ is bijective). Since the composition of central homomorphisms is central, we conclude that all homomorphisms in the diagram are central. Therefore this is a diagram in $\text{DGR}/B^{ce}$. □

If $M$ is a DG $B$-module, then formula (1.22), but with elements $b_0, b_1, c \in M$, defines a degree 1 operator on the graded abelian group $\text{Cyl}(M)$. Similarly, formula (1.23), with elements $b'_0, b'_1, c' \in M$, defines a multiplication $\text{Cyl}(B) \times \text{Cyl}(M) \to \text{Cyl}(M)$. If $N$ is a DG $B^{op}$-module, and if we take $b_0, b_1, c \in N$, then this formula defines a multiplication $\text{Cyl}(N) \times \text{Cyl}(B) \to \text{Cyl}(N)$. These operations are used in the next proposition.

Proposition 1.26. Let $B$ be a DG ring, let $M$ be a DG $B$-module, and let $N$ be a DG $B^{op}$-module.

1. The operations above make $\text{Cyl}(M)$ into a DG $\text{Cyl}(B)$-module, and they make $\text{Cyl}(N)$ into a DG $\text{Cyl}(B)^{op}$-module.
2. If $M = N$ is a DG $B$-bimodule, then $\text{Cyl}(M) = \text{Cyl}(N)$ is a DG $\text{Cyl}(B)$-bimodule.
3. The homomorphisms $\epsilon : M \to \text{Cyl}(M)$ and $\eta_i : \text{Cyl}(M) \to M$ are DG $\text{Cyl}(B)$-module quasi-isomorphisms. Likewise for right modules and bimodules.

In item (3), when considering the homomorphism $\epsilon : M \to \text{Cyl}(M)$, we view $\text{Cyl}(M)$ as a DG $B$-module via the DG ring homomorphism $\epsilon : B \to \text{Cyl}(B)$. Likewise, when considering the homomorphism $\eta_i : \text{Cyl}(M) \to M$, we view $M$ as a DG $\text{Cyl}(B)$-module via the DG ring homomorphism $\eta_i : \text{Cyl}(B) \to B$.

Proof. Again, a direct calculation. □

The cylinder operation (on DG rings and modules) has the following functoriality. If $u : B \to C$ is a DG ring homomorphism, then there is an induced DG ring homomorphism $\text{Cyl}(u) : \text{Cyl}(B) \to \text{Cyl}(C)$, with formula
\[
\text{Cyl}(u)\left(\begin{bmatrix} b_0 & t^{-1}(c) \\ 0 & b_1 \end{bmatrix}\right) := \begin{bmatrix} u(b_0) & t^{-1}(u(c)) \\ 0 & u(b_1) \end{bmatrix}.
\]
Likewise for modules.
Remark 1.28. In the abstract categorical sense, the DG ring \( \text{Cyl}(B) \) plays a role dual to that of a cylinder; so perhaps we should have called it “the path object of \( B \)”. Indeed, this is precisely the path object described on page 503 of \([\text{ScSh}]\).

We decided to adhere to the name “cylinder” for two reasons. First, this is the name used in \([\text{YZ3}]\). The second reason is that we are dealing with rings here, and so (as happens in algebraic geometry) arrows tend to be reversed. The same reversal occurs in Definition 1.2: we talk about “DG rings over \( A \)”, rather than about “DG rings under \( A \)”, as the categorical convention would dictate.

2. Resolutions of DG Modules

Let \( A \) be a DG ring. We already mentioned that \( \text{DGMod} A \) is a DG category, equipped with a translation automorphism \( T \). The homotopy category of \( \text{DGMod} A \) is \( \tilde{K}(\text{DGMod} A) \), where by definition

\[
\text{Hom}_{\tilde{K}(\text{DGMod} A)}(M, N) := H^0(\text{Hom}_A(M, N)).
\]

This is a triangulated category. The derived category \( \tilde{D}(\text{DGMod} A) \) is gotten from \( \tilde{K}(\text{DGMod} A) \) by inverting the quasi-isomorphisms. The localization functor (identity on objects) is

\[
Q : \text{DGMod} A \to \tilde{D}(\text{DGMod} A).
\]

If \( A \) is a ring, i.e. \( A = A^0 \), with module category \( \text{Mod} A \), then \( \text{DGMod} A = \mathcal{C}(\text{Mod} A) \), \( \tilde{K}(\text{DGMod} A) = \tilde{K}(\text{Mod} A) \) and \( \tilde{D}(\text{DGMod} A) = \tilde{D}(\text{Mod} A) \). Here are some references: derived categories of DG algebras are considered in \([\text{BL}, \text{Section 10}]\). See also \([\text{Ke1}, \text{Section 2}]\), in which the more general case of DG categories is considered. Another reference is our course notes \([\text{Ye4}]\), but there (in Sections 4 and 9) the DG category \( \mathcal{C}(M) \), associated to an abelian category \( M \), should be replaced by the DG category \( \text{DGMod} A \).

Notation 2.2. We shall use the following abbreviations for the categories related to a DG ring \( A \): \( M(A) := \text{DGMod} A \), \( K(A) := \tilde{K}(\text{DGMod} A) \) and \( D(A) := \tilde{D}(\text{DGMod} A) \). The \( i \)-th shift automorphism of these categories will sometimes be denoted by \( M[i] := T^i(M) \).

Let us recall a few facts about resolutions of DG \( A \)-modules. A DG module \( N \) is called acyclic if \( H(N) = 0 \). A DG \( A \)-module \( P \) (resp. \( I \)) is called \( K \)-projective (resp. \( K \)-injective) if for any acyclic DG \( A \)-module \( N \), the DG \( \mathbb{Z} \)-module \( \text{Hom}_A(P, N) \) (resp. \( \text{Hom}_A(N, I) \)) is also acyclic. The DG module \( P \) is called \( K \)-flat if for every acyclic DG \( A^{\text{op}} \)-module \( N \), the \( \mathbb{Z} \)-module \( N \otimes_A P \) is acyclic. As in the “classical” situation, \( K \)-projective implies \( K \)-flat. These definitions were introduced in \([\text{Sp}]\). In \([\text{Ke1}, \text{Section 3}]\) it is shown that “\( K \)-projective” is the same as “having property (P)”, and “\( K \)-injective” is the same as “having property (I)”. Every \( M \in M(A) \) admits \( K \)-projective resolutions \( P \to M \) and \( K \)-injective resolutions \( M \to I \). See also \([\text{BL}, \text{BN}]\) and \([\text{AFH}]\).

Recall that a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi_0} & N \\
\downarrow{\phi_1} & & \\
& N &
\end{array}
\]
in $\mathcal{M}(A)$ is called *commutative up to homotopy* if the homomorphisms $\phi_0$ and $\phi_1$ are homotopic. Or in other words, if the corresponding diagram in $\mathcal{K}(A)$ is commutative.

The next proposition is essentially a standard fact. We state it here because it will play a key role, especially in Section 5.

**Proposition 2.3.** Let $P, I \in \mathcal{M}(A)$. Assume either $P$ is $K$-projective or $I$ is $K$-injective.

1. The additive homomorphism
   $$Q : \text{Hom}_{\mathcal{M}(A)}(P, I) \to \text{Hom}_{\mathcal{D}(A)}(P, I)$$
   is surjective, and its kernel is the group $B^0(\text{Hom}_A(P, I))$ of null-homotopic homomorphisms.

2. Let $\psi : P \to I$ be a morphism in $\mathcal{D}(A)$. There exists a homomorphism $\phi : P \to I$ in $\mathcal{M}(A)$, unique up to homotopy, such that $Q(\phi) = \psi$.

3. Let $\phi_0, \phi_1 : P \to I$ be homomorphisms in $\mathcal{M}(A)$. The homomorphisms $\phi_0$ and $\phi_1$ are homotopic iff $Q(\phi_0) = Q(\phi_1)$ in $\mathcal{D}(A)$.

In other words, item (2) says that given a morphism $\psi$ in $\mathcal{D}(A)$, as in the first diagram below, there exists a homomorphism $\phi$ in $\mathcal{M}(A)$ making that diagram commutative; and such $\phi$ is unique up to homotopy. Item (3) says that the second diagram below, in $\mathcal{M}(A)$, is commutative up to homotopy, iff the third diagram below, in $\mathcal{D}(A)$, is commutative.

![Diagram](image)

*Proof.* (1) According to [Sp], Propositions 1.4 and 1.5], the additive homomorphism

$$\text{Hom}_{\mathcal{K}(A)}(P, I) = H^0(\text{Hom}_A(P, I)) \to \text{Hom}_{\mathcal{D}(A)}(P, I)$$

is bijective. Note that in [Sp], $A$ is a ring; but the definitions and proofs pass without change to the case of a DG ring.

(2, 3) These are immediate from (1). □

A DG ring homomorphism $u : A \to B$ induces the restriction of scalars functor $\text{rest}_u : \mathcal{M}(B) \to \mathcal{M}(A)$, which is exact, and hence gives rise to a triangulated functor $\text{rest}_u : \mathcal{D}(B) \to \mathcal{D}(A)$. We usually do not mention the functor $\text{rest}_u$ explicitly, unless it is important for the discussion at hand.

If $u : A \to B$ is a quasi-isomorphism then $\text{rest}_u : \mathcal{D}(B) \to \mathcal{D}(A)$ is an equivalence, with quasi-inverse $M \mapsto B \otimes_A^L M$. For any $M_0 \in \mathcal{D}(A^{\text{op}})$ and $M_1, M_2 \in \mathcal{D}(A)$ there are bi-functorial isomorphisms $M_0 \otimes_A^L M_1 \cong M_0 \otimes_B^L M_1$ and $\text{RHom}_A(M_1, M_2) \cong \text{RHom}_B(M_1, M_2)$ in $\mathcal{D}(\mathbb{Z})$; cf. [YZ3, Proposition 1.4].

Here is a result that will be used a lot in our paper. The restriction functors are suppressed in it. The easy proof is left to the reader.

**Proposition 2.4.** Let $A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} A_2$ and $B_0 \xrightarrow{v_1} B_1 \xrightarrow{v_2} B_2$ be homomorphisms in $\mathcal{DGR}$. For $k = 0, 1, 2$ let $M_k \in \mathcal{M}(B_k \otimes \mathbb{Z} A_k)$ and $N_k \in \mathcal{M}(B_k)$; and for $k = 1, 2$ let $\zeta_k : M_{k-1} \to M_k$ and $\theta_k : N_k \to N_{k-1}$ be homomorphisms in $\mathcal{M}(B_{k-1} \otimes \mathbb{Z} A_{k-1})$ and $\mathcal{M}(B_{k-1})$ respectively.
Proposition 2.5. Let \( \text{Hom}_{B_0}(M_0, N_0) \) be a DG \( A_0^{\text{op}} \)-module, with action
\[
(\phi \cdot a)(m) := \phi(a \cdot m)
\]
for \( \phi \in \text{Hom}_{B_0}(M_0, N_0) \), \( a \in A_0 \) and \( m \in M_0 \).

(2) The function
\[
\text{Hom}_{\nu_1}(\zeta_1, \theta_1) : \text{Hom}_{B_0}(M_1, N_1) \to \text{Hom}_{B_0}(M_0, N_0),
\]
with formula
\[
\text{Hom}_{\nu_1}(\zeta_1, \theta_1)(\phi) := \theta_1 \circ \phi \circ \zeta_1
\]
for \( \phi \in \text{Hom}_{B_0}(M_1, N_1) \), is a homomorphism in \( M(A_0^{\text{op}}) \).

(3) There is equality
\[
\text{Hom}_{\nu_1}(\zeta_1, \theta_1) \circ \text{Hom}_{\nu_2}(\zeta_2, \theta_2) = \text{Hom}_{\nu_2 \circ \nu_1}(\zeta_2 \circ \zeta_1, \theta_1 \circ \theta_2).
\]

(4) Say \( \zeta_1' : M_0 \to M_1 \) and \( \theta_1' : N_1 \to N_0 \) are also homomorphisms in \( M(B_0 \otimes Z A_0) \) and \( M(B_0) \) respectively, and there are homotopies \( \zeta_1 \Rightarrow \zeta_1' \) and \( \theta_1 \Rightarrow \theta_1' \) in \( M(B_0 \otimes Z A_0) \) and \( M(B_0) \) respectively. Then there is a homotopy
\[
\text{Hom}_{\nu_1}(\zeta_1, \theta_1) \Rightarrow \text{Hom}_{\nu_1}(\zeta_1', \theta_1')
\]
in \( M(A_0^{\text{op}}) \).

Item (2) of the proposition is illustrated here:

\[
A_0 \otimes Z B_0 \xrightarrow{u_1 \otimes v_1} A_1 \otimes Z B_1
\]

\[
\begin{array}{ccc}
M_0 & \xrightarrow{\zeta_1} & M_1 \\
\text{Hom}_{\nu_1}(\zeta_1, \theta_1)(\phi) \downarrow & & \downarrow \phi \\
N_0 & \xrightarrow{\theta_1} & N_1
\end{array}
\]

Proposition 2.5. Let \( v : A' \to A \) be a quasi-isomorphism between DG rings. Let \( Q, I \in M(A), P \in M(A^{\text{op}}), Q', I' \in M(A') \) and \( P' \in M(A'^{\text{op}}) \). Let \( \phi : P' \to P \) be a quasi-isomorphism in \( M(A'^{\text{op}}) \), and let \( \psi : Q' \to Q \) and \( \chi : I \to I' \) be quasi-isomorphisms in \( M(A') \).

(1) If \( ((P \text{ is K-flat over } A^{\text{op}} \text{ or } Q \text{ is K-flat over } A) \) and \( (P' \text{ is K-flat over } A'^{\text{op}} \text{ or } Q' \text{ is K-flat over } A')) \) then
\[
\phi \otimes v, \psi : P' \otimes_{A'} Q' \to P \otimes_A Q
\]
is a quasi-isomorphism.

(2) If \( ((Q \text{ is K-projective or } I \text{ is K-injective over } A) \) and \( (Q' \text{ is K-projective or } I' \text{ is K-injective over } A')) \) then
\[
\text{Hom}_{\nu}(\psi, \chi) : \text{Hom}_A(Q, I) \to \text{Hom}_{A'}(Q', I')
\]
is a quasi-isomorphism.

Proof. This is implicit in the proof of [YZ3 Proposition 1.4]. We will only prove (1), since (2) is proved similarly, and we leave it to the reader.

By the symmetry of the situation we can assume that either \( P' \) and \( P \) are K-flat (over the respective DG rings), or that \( Q' \) and \( P \) are K-flat. We begin by reducing
the second case to the first case. Choose a K-flat resolution \( \mu : \tilde{P}' \to P' \). Then 
\[ \tilde{P}' \otimes_{A'} Q' \to P' \otimes_{A'} Q' \]
is a quasi-isomorphism, and it suffices to prove that 
\[ (\phi \circ \mu) \otimes_{v} \psi : \tilde{P}' \otimes_{A'} Q' \to P \otimes_{A} Q \]
is a quasi-isomorphism, which is the first case.

Now we assume that \( P' \) and \( P \) are K-flat. Consider the homomorphism \( \phi \cdot 1_A : P' \otimes_{A'} A \to P, (\phi \cdot 1_A)(p' \otimes a) := \phi(p') \cdot a \). We claim this is a quasi-isomorphism. To see why, let us look at the first commutative diagram below. The homomorphism \( 1_{P'} \otimes v \) is a quasi-isomorphism since \( v \) is a quasi-isomorphism and \( P' \) is K-flat; the homomorphism \( 1_{P'} \cdot 1_A \) is an isomorphism; and \( \phi \) is a quasi-isomorphism; therefore \( \phi \cdot 1_A \) is a quasi-isomorphism.

\[
\begin{array}{ccc}
P' \otimes_{A'} A' & \xrightarrow{1_{P'} \cdot 1_A} & P' \\
\downarrow 1_{P'} \otimes v & & \downarrow \phi \\
P' \otimes_{A'} A & \xrightarrow{\phi \cdot 1_A} & P
\end{array}
\]

Next look at the second commutative diagram above. The homomorphism \( \lambda, \) defined by \( \lambda(p' \otimes q) := p' \otimes 1 \otimes q, \) is an isomorphism. The homomorphism \( 1_{P'} \otimes \psi \) is a quasi-isomorphism because \( \psi \) is a quasi-isomorphism and \( P' \) is K-flat; and \( (\phi \cdot 1_A) \otimes 1_Q \) is a quasi-isomorphism because \( \phi \cdot 1_A \) a quasi-isomorphism between K-flat DG \( A^{op} \)-modules. Therefore \( \phi \otimes_v \psi \) is a quasi-isomorphism. \( \square \)

3. Resolutions of DG Rings

In this section we introduce several special kinds of DG rings and homomorphisms. Recall that \( \text{DGR} \) is the category of DG rings.

**Definition 3.1.** A DG ring \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) is called **nonpositive** if \( A^i = 0 \) for all \( i > 0 \). The full subcategory of \( \text{DGR} \) consisting of nonpositive DG rings is denoted by \( \text{DGR}^{\leq 0} \).

**Definition 3.2.** A DG ring \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) is called **strongly commutative** if it satisfies these two conditions:

(i) \( A \) is weakly commutative (Definition 1.4), namely \( b \cdot a = (-1)^{ij} \cdot a \cdot b \) for all \( a \in A^i \) and \( b \in A^j \).

(ii) \( a \cdot a = 0 \) for all \( a \in A^i \) and \( i \) odd.

We denote by \( \text{DGR}_{sc} \) the full subcategory of \( \text{DGR} \) on the strictly commutative DG rings.

We will be mostly interested in DG rings that are both nonpositive and strongly commutative, namely in objects of \( \text{DGR}^{\leq 0}_{sc} := \text{DGR}^{\leq 0} \cap \text{DGR}_{sc} \).

Therefore we introduce the next notational convention (which occurs in the title of the paper, and is the same as Notation 0.2.1 in the Introduction).

**Notation 3.4.** The expression **commutative DG ring** shall be used throughout the paper as an abbreviation for "nonpositive strongly commutative DG ring".

**Example 3.5.** A commutative DG ring \( A \) concentrated in degree 0 (i.e. \( A = A^0 \)) is just a commutative ring.
Remark 3.6. The name “strongly commutative DG ring” here was “strictly commutative DG ring” in an earlier version of this paper, and it was “super-commutative DG algebra” in the paper [YZ3]. We hope that the terminology will stabilize, and will become standard.

Of course when the number 2 is invertible in \( A^0 \) (e.g. if \( A^0 \) contains \( \mathbb{Q} \)), then condition (i) of Definition 3.2 implies condition (ii), so that the distinction between weak and strong commutativity disappears. See also Remark 3.11.

By graded set we mean a set \( X \) with a partition \( X = \bigsqcup_{i \in \mathbb{Z}} X^i \); the elements of \( X^i \) are said to have degree \( i \). We call \( X \) nonpositive if \( X^i = \emptyset \) for all \( i > 0 \). Given a graded set \( X \), we can form the noncommutative polynomial ring \( \mathbb{Z}[X] \) over \( \mathbb{Z} \), which is the free \( \mathbb{Z} \)-module on the set of monomials \( x_1 \cdots x_m \) in the elements of \( X \), and with the obvious multiplication and grading.

The next definition is not standard. Its first occurrence seems to have been in [YZ3].

**Definition 3.7.** Let \( X = \bigsqcup_{i \in \mathbb{Z}} X^i \) be a graded set. The strongly commutative polynomial ring \( \mathbb{Z}[X] \) is the quotient of \( \mathbb{Z}[X] \) by the two-sided ideal generated by the elements \( y \cdot x - (-1)^{ij} \cdot x \cdot y \) and \( z^2 \), for all \( x \in X^i \), \( y \in X^j \) and \( z \in X^k \), with \( i,j,k \in \mathbb{Z} \) and \( k \) odd.

For \( A \in \text{DGR} \) we denote by \( A^\natural \) the graded ring gotten by forgetting the differential. Central homomorphisms were introduced in Definition 1.4.

**Definition 3.8.** Consider a commutative DG ring \( A \), and a central homomorphism \( u : A \to B \) in \( \text{DGR}^{\leq 0} \).

1. We say that \( B \) is a commutative semi-free DG ring over \( A \), and that \( u \) is a commutative semi-free DG ring homomorphism, if there is an isomorphism of graded \( A^\natural \)-rings \( B^\natural \cong A^\natural \otimes_{\mathbb{Z}} \mathbb{Z}[X] \), where \( X \) is a nonpositive graded set, and \( \mathbb{Z}[X] \) is the strongly commutative polynomial ring. Such a graded set \( X \) is called a set of commutative semi-free ring generators of \( B \) over \( A \).

2. We say that \( B \) is a noncommutative semi-free DG ring over \( A \), and that \( u \) is a noncommutative semi-free DG ring homomorphism, if there is an isomorphism of graded \( A^\natural \)-rings \( B^\natural \cong A^\natural \otimes_{\mathbb{Z}} \mathbb{Z}[X] \), where \( X \) is a nonpositive graded set, and \( \mathbb{Z}[X] \) is the noncommutative polynomial ring. Such a graded set \( X \) is called a set of noncommutative semi-free ring generators of \( B \) over \( A \).

3. We say that \( B \) is a \( K \)-projective (resp. \( K \)-flat) DG ring over \( A \), and that \( u \) is a \( K \)-projective (resp. \( K \)-flat) DG ring homomorphism, if \( B \) is \( K \)-projective (resp. \( K \)-flat) as a DG \( A \)-module (left or right – it does not matter).

**Proposition 3.9.** Let \( A \) be a commutative DG ring, and let \( u : A \to B \) be a central homomorphism in \( \text{DGR}^{\leq 0} \). Assume that \( B \) is either a commutative semi-free or a noncommutative semi-free DG ring over \( A \). Then \( B \) is a semi-free DG \( A \)-module, and therefore it is also \( K \)-projective and \( K \)-flat as a DG \( A \)-module.

**Proof.** In the noncommutative case, let \( X \) be a set of noncommutative semi-free ring generators of \( B \) over \( A \). Then the monomials \( x_1 \cdots x_m \), with \( x_1, \ldots, x_m \in X \), are a semi-basis for the DG \( A \)-module \( B \).

In the commutative case, let \( X \) be a set of commutative semi-free ring generators of \( B \) over \( A \). Choose some ordering of the set \( X \). Then the monomials \( x_1 \cdots x_m \), with \( x_1 \leq \cdots \leq x_m \) in \( X \), are a semi-basis for the DG \( A \)-module \( B \). □
Example 3.10. Let \( A \) be a commutative ring, and let \( a = (a_1, \ldots, a_n) \) be a sequence of elements in \( A \). The Koszul complex \( B := K(A; a) \) associated to this sequence is a commutative semi-free DG ring over \( A \). It has a set of commutative semi-free ring generators \( X = \{x_1, \ldots, x_n\} \) of degree \(-1\), such that \( d_B(x_i) = a_i \).

Remark 3.11. The strongly commutative polynomial ring \( \mathbb{Z}[X] \) is a free graded \( \mathbb{Z} \)-module. This is what makes Proposition 3.9 work.

The “weakly commutative polynomial ring”, which is the quotient of \( \mathbb{Z}(X) \) by the two-sided ideal generated by the elements \( y \cdot x - (-1)^{ij} \cdot x \cdot y \) for all \( x \in X^i \) and \( y \in X^j \), is not flat over \( \mathbb{Z} \). Indeed, for any odd element \( z \in X \) we have \( z^2 \neq 0 \) but \( 2 \cdot z^2 = 0 \).

Definition 3.12. Let \( A \) be a commutative DG ring, and let \( u : A \to B \) be a central homomorphism in \( \text{DGR}^{\leq 0} \). A DG ring resolution of \( u \), or a resolution of \( u \) in \( \text{DGR}^{\leq 0} \), consists of homomorphisms \( \bar{u} : \bar{A} \to \bar{B}, v : \bar{A} \to A \) and \( w : \bar{B} \to B \) in \( \text{DGR}^{\leq 0} \), such that \( \bar{A} \) is commutative, \( \bar{u} \) is central, \( v \) is a quasi-isomorphism, \( w \) is a surjective quasi-isomorphism, and \( w \circ \bar{u} = u \circ v \).

We refer to this resolution as \( (\bar{A} \xrightarrow{\bar{u}} \bar{B}, v, w) \), or simply as \( \bar{A} \to \bar{B} \) or \( \bar{B}/\bar{A} \) if the rest of the data is clear.

The resolution is called strict if \( \bar{A} = A \) and \( v = 1_A \), the identity automorphism of \( A \).

Definition 3.12 is illustrated in the next commutative diagram.

\[
\begin{array}{ccc}
\bar{A} & \xrightarrow{\bar{u}} & \bar{B} \\
v \downarrow & & \downarrow w \\
A & \xrightarrow{u} & B \\
\end{array}
\]

Remark 3.14. In \([YZ3]\) we did not require the quasi-isomorphism \( w \) to be surjective. That omission was not significant there, because we were mostly concerned with the case when \( B \) is a ring (cf. \([YZ3\), Proposition 1.8\]). When \( B \) has nonzero negative components, the surjectivity is needed (cf. proof of Theorem 3.21 below).

Definition 3.15. Let \( A \) be a commutative DG ring, and let \( u : A \to B \) be a central homomorphism in \( \text{DGR}^{\leq 0} \). Suppose \( (\bar{A} \xrightarrow{\bar{u}} \bar{B}, v, w) \) and \( (\bar{A}' \xrightarrow{\bar{u}'} \bar{B}', v', w') \) are DG ring resolutions of \( u \). A morphism of resolutions
\[
(\bar{v}, \bar{w}) : (\bar{A}' \xrightarrow{\bar{u}'} \bar{B}', v', w') \to (\bar{A} \xrightarrow{\bar{u}} \bar{B}, v, w)
\]
consists of homomorphisms \( \bar{v} : \bar{A}' \to \bar{A} \) and \( \bar{w} : \bar{B}' \to \bar{B} \) in \( \text{DGR} \) such that \( v \circ \bar{v} = v' \), \( w \circ \bar{w} = w' \) and \( \bar{u} \circ \bar{v} = \bar{w} \circ \bar{u}' \).

This is shown in the commutative diagram \([0.2.3]\). Note that the DG rings \( A, \bar{A}, \bar{A}' \) are commutative, the homomorphisms \( u, \bar{u}, \bar{u}' \) are central, and the homomorphisms \( w, w' \) are surjective quasi-isomorphisms.

Definition 3.16. Let \( A \) be a commutative DG ring, and let \( u : A \to B \) be a central homomorphism in \( \text{DGR}^{\leq 0} \). A commutative semi-free (resp. noncommutative
Lemma 3.19. Let $K$ be a Koszul complex. Then $\text{Ker}(\tilde{u} : \tilde{A} \to \tilde{B}, v, w)$ as in Definition 3.12 such that $\tilde{u} : \tilde{A} \to \tilde{B}$ is a commutative semi-free (resp. noncommutative semi-free, K-projective, K-flat) homomorphism, as in Definition 3.3.

Example 3.18. Let $u : A \to B$ be a surjective homomorphism of commutative rings, and let $a = (a_1, \ldots, a_n)$ be a sequence of elements of $A$ that generates the ideal $\text{Ker}(u)$. Assume that the sequence $a$ is regular. Define $\tilde{B} := K(A; a)$, the Koszul complex. Then $A \to \tilde{B}$ is a strict commutative semi-free DG ring resolution of $A \to B$.

The next results in this section are enhanced versions of [YZ3, Propositions 1.7 and 1.8]. The category $\text{DGR}_{\text{ce}}^{\leq 0}$ was introduced in Definition 1.4(4).

Lemma 3.19. Let $A$ be a commutative DG ring, and let $B, B' \in \text{DGR}^{\leq 0}_{\text{ce}} A$. Assume either of these two conditions holds:

1. $B$ is commutative, and $B'$ is commutative semi-free over $A$, with commutative semi-free ring generating set $X$.
2. $B'$ is noncommutative semi-free over $A$, with noncommutative semi-free ring generating set $X$.

Let $w : X \to B$ be a degree 0 function. Then:

1. The function $w$ extends uniquely to a graded $A^k$-ring homomorphism $w : B^k \to B$.
2. The graded ring homomorphism $w : B' \to B$ extends to a DG ring homomorphism $w : B' \to B$ iff $d(w(x)) = w(d(x))$ for all $x \in X$.

Proof. This is an easy exercise. □

Lemma 3.20. Let $X$ be a nonpositive graded set, and let $A$ be a commutative DG ring. Consider the graded ring $B := A^k \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ or $B := A^k \otimes_{\mathbb{Z}} \mathbb{Z}(X)$.

1. Let $d_X : X \to B$ be a degree 1 function. The function $d_X$ extends uniquely to a degree 1 derivation $d_B : B \to B$, such that $d_B|_A = d_A$, the differential of $A$.
2. The derivation $d_B$ is a differential on $B$, i.e. $d_B \circ d_B = 0$, iff $(d_B \circ d_B)(x) = 0$ for all $x \in X$.

Proof. (1) Take elements $x_j \in X^n_j$ for $j = 1, \ldots, m$, and consider the monomial $x_1 \cdots x_m \in \mathbb{Z}[X]$ (resp. in $\mathbb{Z}(X)$). We let

$$d_B(x_1 \cdots x_m) := \sum_{j=1}^m (-1)^{n_1 + \cdots + n_{j-1}} \cdot x_1 \cdots x_{j-1} \cdot d_X(x_j) \cdot x_{j+1} \cdots x_m \in B.$$

This is well-defined in the commutative case (and there is nothing to check in the noncommutative case). Extending it additively we obtain a degree $-1$ $\mathbb{Z}$-linear homomorphism $\mathbb{Z}[X] \to B$ (resp. $\mathbb{Z}(X) \to B$). This extends to a degree 1 $A$-linear homomorphism $d_B : B \to B$ that satisfies the graded Leibniz rule.

(2) This is an easy exercise. □
For a DG ring $A$ we denote by $Z(A) = \bigoplus_i Z^i(A)$ the set of cocycles, and by $B(A) = \bigoplus_i B^i(A)$ the set of coboundaries (i.e. the kernel and image of $d_i$, respectively). So $Z(A)$ a DG subring of $A$ (with zero differential of course), and $B(A)$ is a two-sided graded ideal of $Z(A)$.

**Theorem 3.21.** Let $A$ be a commutative DG ring, and let $u : A \to B$ be a central homomorphism in $\text{DGR}^{\leq 0}$.

1. If $B$ is commutative, then there exists a strict commutative semi-free DG ring resolution $A \xrightarrow{\tilde{u}} \tilde{B}$ of $A \xrightarrow{u} B$.

2. There exists a strict noncommutative semi-free DG ring resolution $A \xrightarrow{\tilde{u}} \tilde{B}$ of $A \xrightarrow{u} B$.

These strict DG ring resolutions are depicted in the commutative diagram in $\text{DGR}^{\leq 0}$ shown below.

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow & & \downarrow \tilde{u} \\
\tilde{B} & \xrightarrow{v} & B
\end{array}
\]

**Proof.** (1) We shall construct an increasing sequence of commutative DG $A$-rings $F_0(\tilde{B}) \subset F_1(\tilde{B}) \subset \cdots$. The differential of $F_i(\tilde{B})$ will be denoted by $d_i$. At the same time we shall construct an increasing sequence of graded sets $F_0(X) \subset F_1(X) \subset \cdots$, with a compatible sequence of isomorphisms $F_i(\tilde{B})^i \cong A^i \otimes_{\mathbb{Z}} Z[F_i(X)]$. We shall also construct a compatible sequence of DG ring homomorphisms $v_i : F_i(\tilde{B}) \to B$. The DG $A$-ring $\tilde{B} := \bigcup_i F_i(\tilde{B})$ and the homomorphism $v := \lim_{i \to} v_i$ will have the desired properties.

The construction is by recursion on $i \in \mathbb{N}$. Moreover, for every $i$ the following conditions will hold:

1. The homomorphisms $v_i : F_i(\tilde{B}) \to B$, $B(v_i) : B(F_i(\tilde{B})) \to B(B)$ and $H(v_i) : H(F_i(\tilde{B})) \to H(B)$ are surjective in degrees $\geq -i$.

2. The homomorphism $H(v_i) : H(F_i(\tilde{B})) \to H(B)$ is bijective in degrees $\geq -i + 1$.

We start by choosing a set $Z'_0''$ of degree $-1$ elements, and a function $v_0 : Z'_0'' \to B^{-1}$, such that the set $d_B(v_0(Z'_0''))$ generates $B^0(B)$ as an $A^0$-module. Let $Y''_0$ be a set of degree 0 elements, with a bijection $d_0 : Z''_0 \to Y''_0$. Let $v_0 : Y''_0 \to B^0$ be the unique function such that $v_0(d_0(z)) = d_B(v_0(z))$ for all $z \in Z''_0$.

Choose a set $Y'_0$ of degree 0 elements, and a function $v_0 : Y'_0 \to B^0$, such that the set $v_0(Y'_0 \cup Y''_0)$ generates $B^0$ as an $A^0$-ring.

Define the graded set $F_0(X) := Y'_0 \cup Y''_0 \cup Z'_0''$, and the graded ring $F_0(\tilde{B})^2 := A^2 \otimes_{\mathbb{Z}} Z[F_0(X)]$. Setting $d_0(y) := 0$ for $y \in Y'_0 \cup Y''_0$ gives a degree 1 function $d_0 : F_0(X) \to F_0(\tilde{B})^2$. According to Lemma 3.20 we get a differential $d_0$ on the graded ring $F_0(\tilde{B})^2$, and it becomes the DG $A$-ring $F_0(\tilde{B})$. Lemma 3.19 says that the function $v_0 : F_0(X) \to B$ extends uniquely to a DG $A$-ring homomorphism $v_0 : F_0(\tilde{B}) \to B$. Condition (i) holds, and condition (ii) holds trivially, for $i = 0$.

Now take any $i \geq 0$, and assume that $v_i : F_i(\tilde{B}) \to B$ is already defined, and it satisfies conditions (i)-(ii). Choose a graded set $Y'_{i+1}$ of degree $-i - 1$ elements, and a function $v_{i+1} : Y'_{i+1} \to Z^{-i-1}(B)$, such that the cohomology classes of the
elements of $v_{i+1}(Y'_{i+1})$ generate $H^{-i-1}(B)$ as $H^0(A)$-module. For $y \in Y'_{i+1}$ let $d_{i+1}(y) := 0 \in F_i(\tilde{B})$.

Consider the $A^0$-module
\[ J_{i+1} := \{ b \in Z^{-i}(F_i(\tilde{B})) \mid H^{-i}(v_i)(b) = 0 \}. \]

Choose a graded set $Y''_{i+1}$ of degree $-i - 1$ elements, and a function $d_{i+1} : Y''_{i+1} \to J_{i+1}$, such that the cohomology classes of the elements of $d_{i+1}(Y''_{i+1})$ generate the $H^0(A)$-module
\[ \ker(H^{-i}(v_i) : H^{-i}(F_i(\tilde{B})) \to H^{-i}(B)). \]

For any $y \in Y''_{i+1}$ there exists some $b \in B^{-i-1}$ such that $v_i(d_{i+1}(y)) = d(b)$, and we let $v_{i+1}(y) := b$.

Choose a graded set $Z''_{i+1}$ of degree $-i - 2$ elements, and a function $v_{i+1} : Z''_{i+1} \to B^{-i-2}$, such that the set $d_B(v_{i+1}(Z''_{i+1}))$ generates $B^{-i-1}(B)$ as $A^0$-module. Let $Y''_{i+1}$ be a set of degree $-i - 1$ elements, with a bijection $d_{i+1} : Z''_{i+1} \to Y''_{i+1}$. Let $v_{i+1} : Y''_{i+1} \to B^{-i-1}$ be the unique function such that $v_{i+1}(d_{i+1}(z)) = d_B(v_{i+1}(z))$ for all $z \in Z''_{i+1}$. For $y \in Y''_{i+1}$ we define $d_{i+1}(y) := 0$.

Lastly choose a graded set $Y_{i+1}$ of degree $-i - 1$ elements, and a function $v_{i+1} : Y_{i+1} \to B^{-i-1}$, such that, letting
\[ Y_{i+1} := Y'_{i+1} \cup Y''_{i+1} \cup Y_{i+1}, \]
the set $v_{i+1}(Y_{i+1})$ generates $B^{-i-1}$ as an $A^0$-module. Since $v_i : B^{-i}(F_i(\tilde{B})) \to B^{-i}(B)$ is surjective, for any $y \in Y'_{i+1}$ there exists $b \in F_i(\tilde{B})^{-i}$ such that $v_i(b) = d_B(v_{i+1}(y))$, and we define $d_{i+1}(y) := b$.

Define the graded set
\[ F_{i+1}(X) := F_i(X) \cup Y_{i+1} \cup Z''_{i+1} \]
and the commutative graded ring
\[ F_{i+1}(\tilde{B})^2 := A^2 \otimes_{Z[F_i(X)]} Z[F_{i+1}(X)]. \]

There is a degree 1 function $d_{i+1} : F_{i+1}(X) \to F_{i+1}(\tilde{B})^2$ such that $d_{i+1}|_{F_i(X)} = d_i$. According to Lemma 3.20 the function $d_{i+1}$ induces a differential $d_{i+1}$ on $F_{i+1}(\tilde{B})^2$; so we get a DG ring $F_{i+1}(\tilde{B})$. Likewise we have a degree 0 function $v_{i+1} : F_{i+1}(X) \to B$ such that $v_{i+1}|_{F_i(X)} = v_i|_{F_i(X)}$. By Lemma 3.19 there is an induced DG $A$-ring homomorphism $v_{i+1} : F_{i+1}(\tilde{B}) \to B$, and it satisfies $v_{i+1}|_{F_i(\tilde{B})} = v_i$.

(2) The proof here is the same, except that we replace $Z[F_i(X)]$ with $Z[F_{i+1}(X)]$ everywhere.

**Theorem 3.22.** Let $A$ be a commutative DG ring, and let $u : A \to B$ be a central homomorphism in $\text{DGR}^{\leq 0}$. Suppose we are given two factorizations $A \xrightarrow{\tilde{u}} \tilde{B} \xrightarrow{\nu} B$ and $A \xrightarrow{\tilde{u}'} \tilde{B}' \xrightarrow{\nu'} B$ of $u$ in $\text{DGR}^{\leq 0}/\text{ce} A$, such that $\nu$ is a surjective quasi-isomorphism, and either of these two conditions holds:

(i) The DG ring $\tilde{B}$ is commutative, and the DG ring $\tilde{B}'$ is commutative semi-free over $A$.

(ii) The DG ring $\tilde{B}'$ is noncommutative semi-free over $A$.

Then there exists a homomorphism $w : \tilde{B}' \to \tilde{B}$ in $\text{DGR}^{\leq 0}/\text{ce} A$ such that $w \circ \tilde{u}' = \tilde{u}$ and $v \circ w = v'$.
The proof is similar to that of [YZ3, Proposition 1.8]. Let \( X = \coprod_{i \leq 0} X^i \) be a set of commutative (resp. noncommutative) semi-free DG \( A \)-ring generators of \( \tilde{B} \) over \( A \). Define \( F_i(X) := \coprod_{-i \leq j \leq 0} X^i \), and let \( F_i(\tilde{B}) \) be the \( A \)-subring of \( \tilde{B} \) generated by the set \( F_i(X) \). So \( F_i(\tilde{B}) \) is a DG \( A \)-ring, \( F_i(\tilde{B}) \cong A^j \otimes_{\mathbb{Z}} \mathbb{Z}[F_i(X)] \) (resp. \( F_i(\tilde{B}) \cong A^j \otimes_{\mathbb{Z}} \mathbb{Z}[F_i(X)] \) ), and \( \tilde{B} = \bigcup F_i(\tilde{B}) \). We will construct a consistent sequence of homomorphisms \( w_i : F_i(\tilde{B}) \to \tilde{B} \) in \( \text{DGR}_{\leq 0}^{/ce} A \), satisfying \( w_i \circ \tilde{u} = \tilde{u} \) and \( v \circ v_i = v' \). The construction is by recursion on \( i \in \mathbb{N} \). Then \( w := \lim_{i \to \infty} w_i \) will have the required properties.

We start with \( i = 0 \). Take any \( x \in X^0 \). Since \( v \) is surjective, there exists \( b \in \tilde{B}^0 \) such that \( v(b) = v'(x) \) in \( \tilde{B}^0 \). We let \( w_0(x) := b \). The resulting function \( w_0 : X^0 \to \tilde{B} \) extends uniquely to a DG \( A \)-ring homomorphism \( w_0 : F_i(\tilde{B}) \to \tilde{B} \), by Lemma 3.19.

Next consider any \( i \in \mathbb{N} \), and assume a DG \( A \)-ring homomorphism \( w_i : F_i(\tilde{B}) \to \tilde{B} \) is defined, satisfying the required conditions. Take any element \( x \in X^{-i-1} \). Since \( v \) is surjective, there exists \( b \in \tilde{B}^{-i-1} \) such that \( v(b) = v'(x) \) in \( \tilde{B}^{-i-1} \). Now \( d(x) \in F_i(\tilde{B})^{-i} \) is a cocycle in \( F_i(\tilde{B}) \), so \( w_i(d(x)) \in \tilde{B}^{-i} \) is a cocycle. Let \( c := w_i(d(x)) - d(b) \in \tilde{B}^{-i} \), which is also a cocycle. We have
\[
v(c) = v(w_i(d(x))) - v(d(b)) = v'(d(x)) - v(d(b)) = d(v'(x) - v(b)) = 0.
\]
Thus the cohomology class \([c]\) satisfies \( H(v)([c]) = [v(c)] = 0 \). Because \( H(v) \) is bijective we conclude that \( [c] = 0 \) in \( H^{-i}(\tilde{B}) \). Hence there is some \( b' \in \tilde{B}^{-i} \) such that \( d(b') = c \). We define \( w_{i+1}(x) := b + b' \). Then
\[
d(w_{i+1}(x)) = d(b + b') = d(b) + c = w_i(d(x)).
\]
In this way we obtain a function \( w_{i+1} : F_{i+1}(X) \to \tilde{B} \) that extends \( w_i \). According to Lemma 3.19(1), the function \( w_{i+1} \) extends uniquely to a homomorphism of graded \( A^j \)-rings \( F_{i+1}(w) : F_{i+1}(\tilde{B})^j \to \tilde{B}^j \). Equation (3.24) and Lemma 3.19(2) imply that \( w_{i+1} : F_{i+1}(\tilde{B}) \to \tilde{B} \) is in fact a homomorphism of DG rings. \( \square \)

4. DG Ring Homotopies

Many of the ideas in this section were communicated to us privately by B. Keller (but of course the responsibility for correctness is ours).

Recall that \( \text{DGR}_{\leq 0}^{/ce} \) is the category of commutative DG rings (Notation 3.4). Central homomorphisms were defined in Definition 3.4(3).

**Definition 4.1.** Suppose \( u_0, u_1 : A \to B \) are homomorphisms of DG rings.
(1) An additive homomorphism $\gamma : A \to B$ of degree $-1$ is called a $u_0$-$u_1$-derivation if it satisfies this twisted graded Leibniz formula
\[
\gamma(a_0 \cdot a_1) = \gamma(a_0) \cdot u_1(a_1) + (-1)^{k_0} \cdot u_0(a_0) \cdot \gamma(a_1)
\]
for all $a_i \in A^{k_i}$.

(2) A DG ring homotopy $\gamma : u_0 \Rightarrow u_1$ is $u_0$-$u_1$-derivation $\gamma : A \to B$ of degree $-1$, satisfying the homotopy formula
\[
d_B \circ \gamma + \gamma \circ d_A = u_1 - u_0.
\]

Here are homotopy results, that are only valid for noncommutative semi-free DG rings (Definition [3.8]).

**Lemma 4.2.** Let $A \in \text{DGR}_{sc}^{\leq 0}$, and let $w_0, w_1 : \hat{B}' \to \hat{B}$ be homomorphisms in $\text{DGR}^{\leq 0}_{/ce} A$. Assume that $\hat{B}'$ is noncommutative semi-free over $A$, with set of noncommutative semi-free ring generators $X$. Let $\gamma : X \to \hat{B}$ be a degree $-1$ function.

1. There is a unique $A$-linear $w_0$-$w_1$-derivation $\gamma : \hat{B}' \to \hat{B}$ of degree $-1$ that extends $\gamma : X \to \hat{B}$.

2. The homomorphism $\gamma : \hat{B}' \to \hat{B}$ is a DG ring homotopy $w_0 \Rightarrow w_1$ iff
\[
(d \circ \gamma + \gamma \circ d)(x) = (w_1 - w_0)(x)
\]
for all $x \in X$.

**Proof.** (1) Define an additive homomorphism $\gamma : \mathbb{Z}(X) \to \hat{B}$ of degree $-1$ by letting
\[
\gamma(x_1 \cdots x_m) = \sum_{i=1}^{m} (-1)^{k_1 + \cdots + k_{i-1}} \cdot w_0(x_1) \cdots w_0(x_{i-1}) \cdot \gamma(x_i) \cdot w_1(x_{i+1}) \cdots w_1(x_m)
\]
for elements $x_i \in X^{k_i}$. (Here is where we need $\hat{B}'$ to be noncommutative semi-free – this won’t work for $\mathbb{Z}(X)$ unless $w_0 = w_1$.) Then extend $\gamma$ $A$-linearly to all elements of $\hat{B}'$, using the isomorphism $\hat{B}'^\mathbb{Z} \cong A^{\mathbb{Z}} \otimes \mathbb{Z}(X)$.

(2) An easy calculation, using induction on $m$, shows that
\[
(d \circ \gamma + \gamma \circ d)(a \cdot x_1 \cdots x_m) = (w_1 - w_0)(a \cdot x_1 \cdots x_m)
\]
for all $a \in A$ and $x_i \in X$.

**Theorem 4.3.** Let $A$ be a commutative DG ring, and let $u : A \to B$ be a central homomorphism in $\text{DGR}^{\leq 0}$. Suppose we are given two factorizations $A \xrightarrow{\tilde{u}} \hat{B} \xrightarrow{\gamma} B$ and $A \xrightarrow{\tilde{u}'} \hat{B}' \xrightarrow{u'} B$ of $u$ in $\text{DGR}^{\leq 0}_{/ce} A$, such that $v$ is a surjective quasi-isomorphism, and $\tilde{u}'$ is noncommutative semi-free. Let $w_0, w_1 : \hat{B}' \to B$ be homomorphisms in $\text{DGR}^{\leq 0}_{/ce} A$ that satisfy $w_i \circ \tilde{u}' = \tilde{u}$ and $v \circ w_i = u'$. Then there is an $A$-linear DG ring homotopy $\gamma : w_0 \Rightarrow w_1$, that satisfies $v \circ \gamma = 0$.

The situation is depicted in the commutative diagrams below, where $i = 0, 1$. 

(4.4)
Proof. Choose a set of noncommutative semi-free ring generators $X = \bigsqcup_{i \leq 0} X^i$ for $\bar{B}'$; so $\bar{B}'^2 \cong A^3 \otimes_{Z} \mathbb{Z}(X)$ as graded rings. For any $k \geq 0$ let $F_k(X) := \bigcup_{-k \leq i \leq 0} X^i$, and let $F_k(\bar{B}')$ be the $A$-subring of $\bar{B}'$ generated by $F_k(X)$. Thus $F_k(\bar{B}')$ is in $\text{DGR}^0_{\text{ce}} A$, and $F_k(\bar{B}')^2 \cong A^3 \otimes_{Z} \mathbb{Z}(F_k(X))$. We will define an $A$-linear homomorphism $\gamma_k : F_k(\bar{B}') \to \bar{B}$ of degree $-1$, which is a DG ring homotopy $w_0|_{F_k(\bar{B}')} \Rightarrow w_1|_{F_k(\bar{B}')}$, recursively on $k$, such that $\gamma_k|_{F_{k-1}(\bar{B}')} = \gamma_{k-1}$ if $k \geq 1$, and $v \circ \gamma_k = 0$. Then we take $\gamma := \lim_{k \to \infty} \gamma_k$.

Since $v \circ w_i = v'$ it follows that $H(v) \circ H(w_i) = H(v')$. Because $H(v)$ is an isomorphism of graded rings, we see that $H(w_0) = H(w_1)$.

Let us construct $\gamma_0$. We will use the fact that all elements of $\bar{B}'^0$, $\bar{B}'^0$ and $B^0$ are cocycles. Take any $x \in X$, and define $c := w_0(x) - w_1(x) \in \bar{B}'^0$. Since $v(c) = 0$, so is its cohomology class $[v(c)] \in H^0(\bar{B})$. Because $H(v)$ is an isomorphism, we see that $[c] = 0$ in $H^0(\bar{B})$. Therefore $c = d(b)$ for some $b \in B^{-1}$. Consider the element $v(b) \in B^{-1}$. We have $d(v(b)) = v(c) = 0$, namely $v(b)$ is a cocycle. Again using the fact that $H(v)$ is bijective, we can find a cocycle $b' \in \bar{B}^{-1}$ such that $H(v)(b') = [v(b)]$ in $H^{-1}(B)$. But then $v(b - b') = 0$ in $H^{-1}(B)$, so there exists some $a \in B^{-2}$ such that $d(a) = v(b - b')$. The surjectivity of $v$ says that there is some $a' \in \bar{B}^{-2}$ with $v(a') = a$. We define

$$\gamma_0(x) := b - b' - d(a') \in \bar{B}^{-1}.$$ 

Then

$$d(\gamma_0(x)) = d(b) = c = w_0(x) - w_1(x),$$

and

$$v(\gamma_0(x)) = v(b - b') - d(a) = 0.$$ 

In this way we get a function $\gamma_0 : X \to \bar{B}^{-1}$. Lemma 4.2 shows that this function extends uniquely to a homomorphism $\gamma_0 : F_0(\bar{B}') \to \bar{B}$, which is a DG ring homotopy $w_0|_{F_0(\bar{B}')} \Rightarrow w_1|_{F_0(\bar{B}')}$, and $v \circ \gamma_0 = 0$.

Now consider $k \geq 1$, and assume we already have $\gamma_{k-1}$. Take any $x \in X^{-k}$. Note that $d(x) \in F_{k-1}(\bar{B}')$. We claim that the element

$$c := w_0(x) - w_1(x) - \gamma_{k-1}(d(x)) \in \bar{B}^{-k}$$

is a cocycle. Indeed, since $\gamma_{k-1}$ is a DG ring homotopy $w_0|_{F_{k-1}(\bar{B}')} \Rightarrow w_1|_{F_{k-1}(\bar{B}')}$, we have

$$(d \circ \gamma_{k-1} + \gamma_{k-1} \circ d)(d(x)) = (w_0 - w_1)(d(x)).$$

But $d(d(x)) = 0$ and $w_i(d(x)) = d(w_i(x))$, so $d(c) = 0$ as claimed.

From here the proof is just like in the case $k = 0$. Because $v \circ w_i = v'$ and $v \circ \gamma_{k-1} = 0$, we see that $v(c) = 0$. Therefore the cohomology class $[v(c)] \in H^{-k}(B)$ is 0. Using the fact that $H(v)$ is an isomorphism, we conclude that $[c] = 0$ in $H^{-k}(\bar{B})$. Hence there is some $b \in \bar{B}^{-k-1}$ such that $d(b) = c$. Now $d(v(b)) = v(d(b)) = v(c) = 0$, so $v(b) \in B^{-k-1}$ is a cocycle. Again using the fact that $H(v)$ is bijective, we can find a cocycle $b' \in \bar{B}^{-k-1}$ such that $H(v)(b') = [v(b)]$ in $H^{-k-1}(\bar{B})$. But then $v(b - b') = 0$ in $H^{-k-1}(B)$, so there exists some $a \in B^{-k-2}$ such that $d(a) = v(b - b')$. The surjectivity of $v$ says that there is some $a' \in \bar{B}^{-k-2}$ with $v(a') = a$. We define

$$\gamma_k(x) := b - b' - d(a') \in \bar{B}^{-k-1}.$$
Then

\[(d \circ \gamma_k + \gamma_{k-1} \circ d)(x) = c + \gamma_{k-1}(d(x)) = w_0(x) - w_1(x),\]
and \(v(\gamma_k(x)) = 0\).

In this way we get a function \(\gamma_k : X^{-k} \rightarrow \tilde{B}^{-k-1}\). We extend it to a function \(\gamma_k : F_k(X) \rightarrow \tilde{B}\) by defining \(\gamma_k(x) := \gamma_{k-1}(x)\) for \(x \in F_{k-1}(X)\). Lemma 4.2 shows that this function extends uniquely to an \(A\)-linear homomorphism \(\gamma_k : F_k(\tilde{B}') \rightarrow \tilde{B}\), which is a DG ring homotopy \(w_k|_{F_k(\tilde{B}')} \Rightarrow w_1|_{F_k(\tilde{B}')}\), and \(v \circ \gamma_k = 0\). □

DG ring homotopies can be expressed using the cylinder construction (see Proposition 1.24). The next result is similar to [Ke3, Theorem 4.3(c)].

Proposition 4.5. Let \(u_0, u_1 : A \rightarrow B\) be DG ring homomorphisms, and let \(\gamma : A \rightarrow B\) be a \(\mathbb{Z}\)-linear homomorphism of degree \(-1\). The following conditions are equivalent.

(i) The homomorphism \(\gamma\) is a DG ring homotopy \(\gamma : u_0 \Rightarrow u_1\).

(ii) The \(\mathbb{Z}\)-linear homomorphism \(u_c : A \rightarrow \text{Cyl}(B)\) with formula

\[
u_c(a) := \begin{bmatrix} u_0(a) & t^{-1}(\gamma(a)) \\ 0 & u_1(a) \end{bmatrix}
\]

is a DG ring homomorphism.

Proof. This is a straightforward calculation. □

Remark 4.6. Let \(A\) be a commutative DG ring. If \(A\) is a ring (i.e. \(A = A^0\)), then the category \(\text{DGR}_{/A}\) admits a Quillen model structure, in which the quasi-isomorphisms are the weak equivalences, and the noncommutative semi-free DG rings are cofibrant. This is proved in [BP, Theorem A.3.1]. We think it is quite plausible that the same is true even when \(A\) is not a ring (i.e. it has a nonzero negative part). Evidence for this is provided by Theorems 3.21, 3.22 and 4.3.

On the other hand, we do not know whether the category \(\text{DGR}_{sc/A}\) admits a similar Quillen model structure. A negative indication is that Theorem 4.3 does not seem to hold for commutative semi-free DG rings. Another negative indication is that even when \(A\) is a ring, this is not known (except when \(A\) contains \(\mathbb{Q}\); see [Hi]).

Recall [AILN, Theorem 3.2], that was discussed in Subsection 0.7 of the Introduction. It deals with a commutative base ring \(A\). The proof of this theorem hinges on the Quillen model structure on \(\text{DGR}_{/A}\) that was produced in [BP]. If such a model structure does exist in the more general case of a commutative DG ring \(A\) (as we predict above), then it would most likely imply Theorem 0.3.4 (by a proof similar to that in [AILN]).

However, even if we had model structures on the categories \(\text{DGR}_{/A}\) and \(\text{DGR}_{sc/A}\), that would probably not be sufficient to imply Theorem 0.4.3. This is because the proof of Theorem 0.4.3 requires a very delicate treatment of commutative semi-free resolutions, something that a Quillen model structure (being a rather coarse structure) does not appear able to provide.
5. Extended Resolutions and the Rectangle Operation

Sections 5, 6 and 7 deal with the rectangle operation. This operation is more general than the squaring operation (to be introduced in Section 8). However, the price to pay for this extra generality, in terms of work, is not high (because we must deal with noncommutative resolutions anyhow). In this section we give definitions and some initial results.

Recall that $\text{DGR}^{\leq 0}$ is the category of nonpositive DG rings, and $\text{DGR}^{\leq 0}_{\text{ce}}$ is the category of commutative DG rings (Notation 3.4). These are full subcategories of DGR. For a fixed $A \in \text{DGR}^{\leq 0}$, we have the category $\text{DGR}^{\leq 0}_{/\text{ce}} A$ of nonpositive DG rings central over $A$ (Definition 3.4).

Let $A$ be a DG ring. The category of left DG $A$-modules is $\mathbb{M}(A)$, and its derived category is $\text{D}(A)$; see Notation 2.2. There is a localization functor $Q : \mathbb{M}(A) \to \text{D}(A)$, which is the identity on objects, and inverts quasi-isomorphisms.

Throughout this section we work in the following setup:

**Setup 5.1.** Let $A$ be a commutative DG ring, and let $u : A \to B$ be a central homomorphism in $\text{DGR}^{\leq 0}$. We are given $M^l \in \mathbb{M}(B)$ and $M^r \in \mathbb{M}(B^{\text{op}})$; namely $M^l$ is a left DG $B$-module and $M^r$ is a right DG $B$-module.

**Definition 5.2.** Consider Setup 5.1. We denote by $\text{Res}(B/A)$ the category of K-flat resolutions of $A \xrightarrow{\tilde{u}} B$ in $\text{DGR}^{\leq 0}$. An object of $\text{Res}(B/A)$ is a K-flat resolution $(\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}, v, w)$ of $A \xrightarrow{u} B$, in the sense of Definition 3.16. Given another object $(\tilde{A}' \xrightarrow{\tilde{u}'} \tilde{B}', v', w')$ of $\text{Res}(B/A)$, a morphism between them is a morphism of resolutions

$$(\tilde{v}, \tilde{w}) : (\tilde{A}' \xrightarrow{\tilde{u}'} \tilde{B}', v', w') \to (\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}, v, w)$$

as in Definition 3.15. The composition of morphisms is the obvious one.

The commutative diagram (0.2.3) is an illustration of a morphism in $\text{Res}(B/A)$.

**Notation 5.3.** For convenience we often write $\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}$, or just $\tilde{B}/\tilde{A}$, instead of $(\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}, v, w)$, for an object of $\text{Res}(B/A)$. Similarly, morphisms in $\text{Res}(B/A)$ will usually be denoted by $\tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A}$.

Recall that by Definitions 3.16 and 3.12 any DG ring resolution $\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}$ is a central homomorphism in $\text{DGR}^{\leq 0}$, and the DG ring $\tilde{A}$ is commutative (Notation 3.4). The DG ring $\tilde{B}$ is allowed to be noncommutative.

**Notation 5.4.** Let $\tilde{B}/\tilde{A}$ be an object of $\text{Res}(B/A)$. We write

$$\tilde{B}^{\text{en}} := \tilde{B} \otimes_{\tilde{A}} \tilde{B}^{\text{op}} \in \text{DGR}^{\leq 0}_{/\text{ce}} \tilde{A},$$

and refer to $\tilde{B}^{\text{en}}$ as the enveloping DG ring of $\tilde{B}$ over $\tilde{A}$. If $\tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A}$ is a morphism in $\text{Res}(B/A)$, then there is an induced DG ring homomorphism

$$\tilde{u}^{\text{en}} := \tilde{w} \otimes_{\tilde{v}} \tilde{w}^{\text{op}} : \tilde{B}'^{\text{en}} \to \tilde{B}^{\text{en}}$$

in $\text{DGR}^{\leq 0}_{/\text{ce}} \tilde{A}'$.

**Lemma 5.5.** The DG ring homomorphism $\tilde{u}^{\text{en}}$ is a quasi-isomorphism.

**Proof.** We can forget that $\tilde{B}, \tilde{B}^{\text{op}}$ and $\tilde{B}', \tilde{B}'^{\text{op}}$ are DG rings, and just view them as DG modules over $\tilde{A}$ and $\tilde{A}'$ respectively. Then we can use Proposition 2.5(1). □
Given a homomorphism $\tilde{\phi} : \tilde{M}^1 \to \tilde{N}^1$ in $\text{M}(\tilde{B})$, and a homomorphism $\phi^r : \check{M}^r \to \check{N}^r$ in $\text{M}(\check{B}^{\text{op}})$, we write

$$\phi^{en} := \phi^1 \otimes \phi^r : \check{M}^{en} \to \check{N}^{en}.$$ 

This is a homomorphism in $\text{M}(\check{B}^{en})$.

**Definition 5.7.** Consider Setup 5.1 and suppose $\tilde{B}/\tilde{A} \in \text{Res}(B/A)$. An extended resolution of the pair $(M^1, M^r)$ over $\tilde{B}/\tilde{A}$ consists of the following data:

(a) A quasi-isomorphism $\alpha^1 : \tilde{M}^1 \to M^1$ in $\text{M}(\tilde{B})$, where $\tilde{M}^1$ is a DG $\tilde{B}$-module which is K-flat over $\tilde{A}$.

(b) A quasi-isomorphism $\alpha^r : \tilde{M}^r \to M^r$ in $\text{M}(\tilde{B}^{\text{op}})$, where $\tilde{M}^r$ is a DG $\tilde{B}^{\text{op}}$-module which is K-flat over $\tilde{A}$.

(c) A K-injective resolution $\beta : \check{M}^{en} \to I$ of $\check{M}^{en}$ in $\text{M}(\check{B}^{en})$.

We denote this extended resolution by $(\tilde{M}^1, \tilde{M}^r, I)$, leaving the homomorphisms $\alpha^1, \alpha^r, \beta$ implicit.

In item (c) of the definition we use Notation 5.6.

**Example 5.8.** Assume the DG ring $A$ in Setup 5.1 is a field. Then $B/A$ is a K-flat resolution of itself; in fact it is a terminal object of $\text{Res}(B/A)$. The DG modules $\tilde{M}^1 := M^1$ and $\tilde{M}^r := M^r$ are resolutions as in Definition 5.7(a,b). According to Notation 5.6 we get $\tilde{B}^{en} = B \otimes_A B$ and $\check{M}^{en} = M^1 \otimes_A M^r$. Let $\tilde{M}^{en} \to I$ be any K-injective resolution over $\tilde{B}^{en}$. Then $M := (B/A; M^1, M^r, I)$ is an extended resolution of $(M^1, M^r)$.

**Definition 5.9.** Consider Setup 5.1. Let $\tilde{w}/\tilde{v} : \tilde{B}/\tilde{A} \to \tilde{B}/\tilde{A}$ be a morphism in $\text{Res}(B/A)$. Suppose $(M^1, M^r, I)$ and $(\check{M}^1, \check{M}^r, I')$ are extended resolutions of the pair $(M^1, M^r)$ over $\tilde{B}/\tilde{A}$ and $\tilde{B}/\tilde{A}$ respectively. A morphism of extended resolutions

$$(\phi^1, \phi^r, \psi) : (\tilde{M}^1, \tilde{M}^r, I) \to (\check{M}^1, \check{M}^r, I)$$

over $\tilde{w}/\tilde{v}$ consists of:

(a) A homomorphism $\phi^1 : M^1 \to \text{rest}_{\tilde{w}}(\tilde{M}^1)$ in $\text{M}(\tilde{B})$, such that $Q(\alpha^1) = Q(\alpha^1) \circ Q(\phi^1)$ in $\text{D}(\tilde{B})$.

(b) A homomorphism $\phi^r : M^r \to \text{rest}_{\tilde{w}op}(\tilde{M}^r)$ in $\text{M}(\tilde{B}^{op})$, such that $Q(\alpha^r) = Q(\alpha^r) \circ Q(\phi^r)$ in $\text{D}(\tilde{B}^{op})$.

(c) A homomorphism $\psi : \text{rest}_{\tilde{v}}(I) \to I'$ in $\text{M}(\tilde{B}^{en})$, such that $Q(\beta') = Q(\psi) \circ Q(\beta) \circ Q(\phi^{en})$ in $\text{D}(\tilde{B}^{en})$.

To clarify, the homomorphisms $\alpha^1, \alpha^r$ and $\beta'$ in Definition 5.9 are part of the extended resolution $(\tilde{M}^1, \tilde{M}^r, I')$.

A morphism of extended resolutions $(\phi^1, \phi^r, \psi)$ is shown in the diagrams [5.10] below. These diagrams are in the categories $\text{M}(\tilde{B})$, $\text{M}(\tilde{B}^{op})$ and $\text{M}(\tilde{B}^{en})$ respectively, but they are not required required to be commutative.
Definition 5.11. The homomorphisms \( \phi^l, \phi^r, \phi^{en} \) and \( \psi \) in Definition 5.9 are quasi-isomorphisms.

Proof. By Definition 5.7, both \( \alpha^l \) and \( \alpha^r \) are quasi-isomorphisms. Since \( Q(\alpha^l) = Q(\phi^l) \circ Q(\alpha^l) \) in \( D(B') \), we see that \( Q(\phi^l) \) is an isomorphism, and hence \( \phi^l \) is a quasi-isomorphism. Likewise we see that \( \phi^r \) is a quasi-isomorphism. According to Proposition 2.5(1), the homomorphism \( \phi^{en} \) is a quasi-isomorphism. Thus \( Q(\phi^{en}) \) is an isomorphism in \( D(B'^{en}) \).

Next, by Definition 5.7 both \( Q(\beta) \) and \( Q(\beta') \) are isomorphisms. Since \( Q(\psi) = Q(\beta') \circ Q(\phi^{en})^{-1} \circ Q(\beta)^{-1} \) in \( D(B'^{en}) \), this is an isomorphism. Therefore \( \psi \) is a quasi-isomorphism. \( \square \)

Definition 5.12. Consider Setup 5.1

1. An extended resolution of the pair \( (M^l, M^r) \) is data

\[
M = (\tilde{B}/\tilde{A}; \tilde{M}^l, \tilde{M}^r, I),
\]

where \( \tilde{B}/\tilde{A} \in \text{Res}(B/A) \), and \( (\tilde{M}^l, \tilde{M}^r, I) \) is an extended resolution of \( (M^l, M^r) \) over \( \tilde{B}/\tilde{A} \), in the sense of Definition 5.7

2. Suppose

\[
M' = (\tilde{B}'/\tilde{A}'; \tilde{M}'^l, \tilde{M}'^r, I')
\]

is another extended resolution of \( (M^l, M^r) \). A morphism of extended resolutions

\[
\phi : M' \to M
\]

is data

\[
\phi = (\tilde{w}/\tilde{v}; \phi^l, \phi^r, \psi),
\]

where \( \tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A} \) is a morphism in \( \text{Res}(B/A) \), and

\[
(\phi^l, \phi^r, \psi) : (\tilde{M}'^l, \tilde{M}'^r, I') \to (\tilde{M}^l, \tilde{M}^r, I)
\]

is a morphism of extended resolutions over \( \tilde{w}/\tilde{v} \), in the sense of Definition 5.9.

The resulting category is denoted by \( \text{Res}(B/A; M^l, M^r) \).
Definition 5.13. We say that an extended resolution $M = (\tilde{B}/\tilde{A}; \tilde{M}^1, \tilde{M}^r, I)$ is K-projective if the DG modules $\tilde{M}^1$ and $\tilde{M}^r$ are K-projective over the DG rings $\tilde{B}$ and $\tilde{B}^{\text{op}}$ respectively.

Lemma 5.14. Let $\tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A}$ be a morphism in $\text{Res}(B/A)$, and let $M$ and $M'$ be extended resolutions of $(M^1, M^r)$ over $B/\tilde{A}$ and $B'/\tilde{A}'$ respectively. If $M'$ is K-projective, then a morphism $\phi : M' \to M$ over $\tilde{w}/\tilde{v}$ exists.

Proof. We use the notation of Definitions 5.7, 5.9 and 5.12 for the ingredients of $M$ and $M'$. Since $\tilde{M}^1$ is K-projective over $\tilde{B}'$, by Proposition 2.3 there exists a quasi-isomorphism $\phi^1 : \tilde{M}^1 \to \text{rest}_\tilde{w}(\tilde{M}^1)$ that commutes up to homotopy with the quasi-isomorphisms to $\tilde{M}^1$. Similarly there is a quasi-isomorphism $\phi^r : \tilde{M}^r \to \text{rest}_\tilde{w}(\tilde{M}^r)$ that commutes up to homotopy with the quasi-isomorphisms to $\tilde{M}^r$. According to Lemma 5.11, the homomorphism $\phi^m : \tilde{M}^{en} \to \text{rest}_\tilde{w}(\tilde{M}^{en})$ is a quasi-isomorphism. Since $\tilde{I}'$ is K-injective over $\tilde{B}'^{en}$, there exists a homomorphism $\psi : \text{rest}_\tilde{w}(\tilde{I}) \to \tilde{I}'$ as in Definition 5.9(c). Then $\phi := (\tilde{w}/\tilde{v}; \phi^1, \phi^r, \psi)$ is the morphism we want. \qed

There is a forgetful functor
\begin{equation}
\text{Res}(B/A; M^1, M^r) \to \text{Res}(B/A).
\end{equation}
Given any resolution $\tilde{B}/\tilde{A} \in \text{Res}(B/A)$, there exist K-projective extended resolutions $M$ over $\tilde{B}/\tilde{A}$. Thus the forgetful functor \eqref{5.15} is surjective on objects.

Recall that a category $\mathcal{C}$ is nonempty (resp. connected) if it so as a graph (forgetting the orientations of the arrows).

Proposition 5.16. In the situation of Setup 5.1, the category $\text{Res}(B/A; M^1, M^r)$ is nonempty and connected.

Proof. Fix a commutative semi-free resolution $Z \to \tilde{A}'$ of $Z \to A$, and then fix a noncommutative semi-free resolution $\tilde{A}' \to B'$ of $\tilde{A}' \to B$. These exist by Theorem 3.21. We get $\tilde{B}'/\tilde{A}' \in \text{Res}(B/A)$. Choose K-projective resolutions $M^1' \to M^1$ and $M^r' \to M^r$ in $\text{M}(\tilde{B}')$ and $\text{M}(\tilde{B}'^{op})$ respectively. Since $\tilde{B}'$ is K-flat over $\tilde{A}'$, it follows that $\tilde{M}^1$ and $\tilde{M}^r$ are both K-flat over $\tilde{A}'$. Choose a K-injective resolution $M^{en} \to I'$ in $\text{M}(\tilde{B}'^{en})$. We now have a K-projective extended resolution
\[
M' = (\tilde{B}'/\tilde{A}'; \tilde{M}^1', \tilde{M}^r', I') \in \text{Res}(B/A; M^1, M^r).
\]

Let $M$ be any object of $\text{Res}(B/A; M^1, M^r)$, with notation as in Definition 5.12(1). According to Theorem 3.22 there is a morphism $\tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A}$ in $\text{Res}(B/A)$. By Lemma 5.14 there exists a morphism $\phi : M' \to M$ in $\text{Res}(B/A; M^1, M^r)$ over $\tilde{w}/\tilde{v}$. \qed

The center $B^{ce}$ of the DG ring $B$ was defined in Definition 1.4(1). The homomorphism $w : \tilde{B} \to B$ in $\text{DGR}_{/ce} A$ makes $B$ into a DG $\tilde{B}$-bimodule, namely a DG module over $\tilde{B}^{en}$. Because the left and right actions of $\tilde{B}$ on $B$ commute (in the graded sense) with the action of the center $B^{ce}$, it follows that $B$ is a DG module over $B^{ce} \otimes_{\tilde{A}} \tilde{B}^{en}$. In the next definition we Proposition 2.4(1,2).

Definition 5.17. Consider Setup 5.1

1. Given an extended resolution
\[
M = (\tilde{B}/\tilde{A}; \tilde{M}^1, \tilde{M}^r, I) \in \text{Res}(B/A; M^1, M^r),
\]
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let
\[ \text{Rect}(M) := \text{Hom}_{B^\text{en}}(B, I) \in M(B^\text{en}). \]

(2) Given another extended resolution \( M' \), and a morphism
\[ \phi = (\bar{w}/\bar{v}; \phi^l, \phi^r, \psi) : M' \rightarrow M \]
in \( \text{Res}(B/A; M^1, M^r) \), let
\[ \text{Rect}(\phi) : \text{Rect}(M) \rightarrow \text{Rect}(M') \]
be the homomorphism
\[ \text{Rect}(\phi) := \text{Hom}_{\bar{B}^\text{en}}(1_B, \psi) : \text{Hom}_{\bar{B}^\text{en}}(B, I) \rightarrow \text{Hom}_{\bar{B}'^\text{en}}(B, I') \]
in \( M(B^\text{en}) \).

In this way we obtain a functor
\[ \text{Rect} : \text{Res}(B/A; M^1, M^r)^{\text{op}} \rightarrow M(B^\text{en}) \]
called the rectangle functor.

Lemma 5.18. The homomorphism \( \text{Rect}(\phi) \) in Definition 5.17 is a quasi-isomorphism.

Proof. The DG modules \( I \) and \( I' \) are K-injective over \( \bar{B}^\text{en} \) and \( \bar{B}'^\text{en} \) respectively. According to Lemmas 5.5 and 5.11 the homomorphisms \( \bar{w}^\text{en} : \bar{B}'^\text{en} \rightarrow \bar{B}^\text{en} \) and \( \psi : I \rightarrow I' \) respectively are quasi-isomorphisms. And of course \( 1_B : B \rightarrow B \) is a quasi-isomorphism. Therefore, Proposition 2.5(2) says that \( \text{Rect}(\phi) \) is a quasi-isomorphism. \( \square \)

Lemma 5.19. In the setup of Notation 5.6 let \( \phi^1_0, \phi^1_1 : \bar{M}^1 \rightarrow \bar{N}^1 \) and \( \phi^r_0, \phi^r_1 : \bar{M}^r \rightarrow \bar{N}^r \) be pairs of homotopic homomorphisms. Define \( \phi^en_1 := \phi^1_1 \otimes \phi^r_1 \). Then the homomorphisms \( \phi^en_0, \phi^en_1 : \bar{M}^\text{en} \rightarrow \bar{N}^\text{en} \) in \( M(\bar{B}^\text{en}) \) are homotopic.

We omit the easy proof.

6. THE RECTANGLE OPERATION: HOMOTOPY INVARIANCE

In this section we continue with the material of Section 5. There are many lemmas here, leading up to Lemma 6.25 which shall be used in the proof of Theorem 7.1. The tool we use here is the cylinder DG ring and related constructions.

The category of extended resolutions \( \text{Res}(B/A; M^1, M^r) \) of a pair of DG modules was introduced in Definition 5.12. The rectangle functor was introduced in Definition 5.17.

Throughout this section we work in the following setup:

Setup 6.1. Let \( A \) be a commutative DG ring, let \( u : A \rightarrow B \) be a central homomorphism in \( \text{DGR}^{\leq 0} \), let \( M^1 \in M(B) \) and let \( M^r \in M(B^{\text{op}}) \). We are given extended resolutions
\[ M = (\bar{B}/\bar{A}; \bar{M}^1, \bar{M}^r, I), \quad M' = (\bar{B}'/\bar{A}'; \bar{M}'^1, \bar{M}'^r, I') \in \text{Res}(B/A; M^1, M^r) \]
and morphisms
\[ \phi_i = (\bar{w}_i/\bar{v}_i; \phi^l_i, \phi^r_i, \psi_i) : M' \rightarrow M \]
in \( \text{Res}(B/A; M^1, M^r) \), for \( i = 0, 1 \). These satisfy the following conditions:

(i) The homomorphism \( \bar{u}' : \bar{A}' \rightarrow \bar{B}' \) is NC semi-free.
(ii) The homomorphisms \( \bar{v}_0, \bar{v}_1 : \bar{A}' \rightarrow \bar{A} \) are equal.
(iii) The extended resolution \( M' \) is K-projective.

In condition (i), \((\tilde{A}' \rightarrow \tilde{B}') = (\tilde{B}'/\tilde{A}')\) is the DG ring ingredient of the extended resolution \( M' \); cf. Definitions 5.12 and 5.2. See Definition 5.13 regarding the K-projective extended resolution mentioned in condition (iii). We use the notation of Definitions 5.2, 5.7 and 5.12 for the ingredients of \( M \) and \( M' \).

In the constructions and proofs in this section we use Proposition 2.3 several times, usually without explicit mention.

From here on we adopt the following notational convention. This abbreviation will simplify the text significantly, yet hopefully will not cause confusion.

**Notation 6.2.** We avoid writing the restriction functors explicitly, whenever they can be understood from the context. Thus, if \( f : C \rightarrow D \) is a given DG ring homomorphism, and no other DG ring homomorphism \( C \rightarrow D \) is mentioned, we allow ourselves to suppress the restriction functors \( \text{rest}_f : \text{M}(D) \rightarrow \text{M}(C) \) and \( \text{rest}_f : \text{D}(D) \rightarrow \text{D}(C) \).

Recall the cylinder DG ring \( \text{Cyl}(\tilde{B}) \) from Proposition 1.24(1). Note that it is neither nonpositive nor weakly commutative (even if \( \tilde{B} \) happens to be a commutative DG ring). There are DG ring quasi-isomorphisms \( \epsilon : \tilde{B} \rightarrow \text{Cyl}(\tilde{B}) \) and \( \eta_i : \text{Cyl}(\tilde{B}) \rightarrow \tilde{B}, \) where \( i = 0, 1 \). The homomorphisms \( \epsilon, \eta_i \) are central according to Proposition 1.24(3). By definition the homomorphism \( \tilde{u} : \tilde{A} \rightarrow \tilde{B} \) is central, and DG ring \( \tilde{A} \) is commutative. Because \( \tilde{u} \) and \( \epsilon \) are central, so is their composition \( \epsilon \circ \tilde{u} : \tilde{A} \rightarrow \text{Cyl}(\tilde{B}) \).

Let us write

\[
\epsilon^n : \tilde{B}^n \rightarrow \text{Cyl}(\tilde{B})^n \quad \eta^n_i : \text{Cyl}(\tilde{B})^n \rightarrow \tilde{B}^n \quad \text{in DGR}_\text{ce} \tilde{A}.
\]

There are homomorphisms \( \epsilon^n : \tilde{B}^n \rightarrow \text{Cyl}(\tilde{B})^n \) and \( \eta^n_i : \text{Cyl}(\tilde{B})^n \rightarrow \tilde{B}^n \) in DGR\(\text{ce} \tilde{A} \). Since \( \tilde{B} \) and \( \text{Cyl}(\tilde{B}) \) are K-flat over \( \tilde{A} \), it follows that \( \epsilon^n \) and \( \eta^n_i \) are quasi-isomorphisms.

Recall that \( \text{B}^\text{ce} \) is the center of \( B \). Since \( v : \tilde{A} \rightarrow A \) and \( u : A \rightarrow B \) are central DG ring homomorphisms, the composition \( u \circ v = w \circ \tilde{u} : A \rightarrow B \) is central. Because \( \tilde{A} \) is commutative, there is an induced DG ring homomorphism \( \tilde{A} \rightarrow \text{B}^\text{ce} \).

**Lemma 6.4.**

1. The DG ring homomorphisms \( \epsilon : B \rightarrow \text{Cyl}(B) \) and \( \text{Cyl}(w) : \text{Cyl}(\tilde{B}) \rightarrow \text{Cyl}(B) \) make \( \text{Cyl}(B) \) into a DG module over \( \text{B}^\text{ce} \otimes \tilde{A} \text{Cyl}(\tilde{B})^\text{en} \).

2. The homomorphisms \( \eta_i : \text{Cyl}(B) \rightarrow B \) are linear over \( \text{B}^\text{ce} \otimes \tilde{A} \text{Cyl}(\tilde{B})^\text{en} \).

**Proof.**

1. Because \( \epsilon : B \rightarrow \text{Cyl}(B) \) is a central homomorphism (Proposition 1.24(3)), it follows that the action of \( \text{B}^\text{ce} \) on \( \text{Cyl}(B) \) commutes (in the graded sense) with the left and right actions of \( \text{Cyl}(\tilde{B}) \) on \( \text{Cyl}(B) \).

2. By Proposition 1.26(3), the homomorphisms \( \eta_i : \text{Cyl}(B) \rightarrow B \) are \( \text{Cyl}(\tilde{B})^\text{en} \)-linear. By Proposition 1.24(3) we know that the homomorphisms \( B \rightarrow \text{Cyl}(B) \) are central. \( \square \)

Matrix multiplication makes \( \text{Cyl}(\tilde{M}') \) and \( \text{Cyl}(\tilde{M}') \) into left and right DG modules over \( \text{Cyl}(\tilde{B}) \), respectively; see Proposition 1.26. These are K-flat DG modules over \( \tilde{A} \).

Define

\[
\text{Cyl}(\tilde{M})^\text{en} := \text{Cyl}(\tilde{M}) \otimes \tilde{A} \text{Cyl}(\tilde{M}') \in \text{M}(\text{Cyl}(\tilde{B})^\text{en}).
\]
Choose a K-injective resolution

\[(6.6) \quad \beta_{\text{cyl}} : \text{Cyl}({\tilde{M}})^{\text{en}} \to I_{\text{cyl}}\]

over \(\text{Cyl}({\tilde{B}})^{\text{en}}\).

For \(i = 0, 1\) there is a quasi-isomorphism \(\eta_i : \text{Cyl}({\tilde{M}}) \to {\tilde{M}}\) in \(\mathbf{M}(\text{Cyl}({\tilde{B}}))\), and a quasi-isomorphism \(\eta_i : \text{Cyl}(M^i) \to M^i\) in \(\mathbf{M}(\text{Cyl}(B)^{\text{en}})\). By tensoring them we get a quasi-isomorphism \(\eta_i^{\text{en}} : \text{Cyl}(M)^{\text{en}} \to M^{\text{en}}\) in \(\mathbf{M}(\text{Cyl}(B)^{\text{en}})\). Likewise there is a quasi-isomorphism \(\epsilon^{\text{en}} : M^{\text{en}} \to \text{Cyl}(M)^{\text{en}}\) in \(\mathbf{M}(B)^{\text{en}}\). These homomorphisms satisfy \(\eta_i^{\text{en}} \circ \epsilon^{\text{en}} = 1\).

Recall that \(\beta : {\tilde{M}}^{\text{en}} \to I\) is part of the extended resolution \(\mathbf{M}\). Since \(I_{\text{cyl}}\) is K-injective in the category \(\mathbf{M}(\text{Cyl}(B)^{\text{en}})\), according to Proposition 2.3 there is (for \(i = 0, 1\)) a quasi-isomorphism

\[(6.7) \quad \tilde{\psi}_i : I \to I_{\text{cyl}}\]

in \(\mathbf{M}(\text{Cyl}(B)^{\text{en}})\), unique up to homotopy, such that

\[(6.8) \quad Q(\tilde{\psi}_i) = Q(\beta_{\text{cyl}}) \circ Q(\eta_i^{\text{en}})^{-1} \circ Q(\beta)^{-1}\]

in \(\mathbf{D}(\text{Cyl}(B)^{\text{en}})\). Cf. diagram (6.13) below.

Lemma 6.4 tells us that \(\text{Hom}_{\text{Cyl}(B)^{\text{en}}}(\text{Cyl}(B), I_{\text{cyl}})\) is a DG \(B^{\text{ce}}\)-module. We also have, for \(i = 0, 1\), a homomorphism

\[(6.9) \quad \chi_i : \text{Hom}_{\text{Cyl}(B)^{\text{en}}}(B, I_{\text{cyl}}) \to \text{Hom}_{\text{Cyl}(B)^{\text{en}}}(\text{Cyl}(B), I_{\text{cyl}}), \quad \chi_i := \text{Hom}_{\eta_i^{\text{en}}}(\eta_i, \tilde{\psi}_i)\]

in \(\mathbf{M}(B^{\text{en}})\).

**Lemma 6.10.** The homomorphisms \(\chi_0\) and \(\chi_1\) from equation (6.9) satisfy \(Q(\chi_0) = Q(\chi_1)\) in \(\mathbf{D}(B^{\text{ce}})\).

**Proof.** Choose a quasi-isomorphism

\[(6.11) \quad \tilde{\psi} : I_{\text{cyl}} \to I\]

in \(\mathbf{M}(\tilde{B}^{\text{en}})\) such that

\[(6.12) \quad Q(\tilde{\psi}) = Q(\beta) \circ Q(\epsilon^{\text{en}})^{-1} \circ Q(\beta_{\text{cyl}})^{-1}\]

in \(\mathbf{D}(\tilde{B}^{\text{en}})\). We obtain the following diagram (6.13) of isomorphisms in \(\mathbf{D}(\tilde{B}^{\text{en}})\), for \(i = 0, 1\). The two squares in this diagram are commutative by the properties of \(\tilde{\psi}_i\) and \(\tilde{\psi}\), see equations (6.8) and (6.12). The top half-moon is commutative since \(\eta_i^{\text{en}} \circ \epsilon^{\text{en}} = 1\). The outer path is trivially commutative. It follows that the bottom half-moon is commutative. We conclude that \(Q(\tilde{\psi} \circ \tilde{\psi}_i) = 1_I\) in \(\mathbf{D}(\tilde{B}^{\text{en}})\). But \(I\) is K-injective over \(\tilde{B}^{\text{en}}\), and thus \(\tilde{\psi} \circ \tilde{\psi}_i\) and \(1_I\) are homotopic in \(\mathbf{M}(\tilde{B}^{\text{en}})\).
Now look at diagram (6.14) in $\mathcal{M}(B^{ce})$, for $i = 0, 1$. The triangle is commutative because $\eta_i \circ \epsilon = 1$. By the previous paragraph and Proposition 2.4(4), the half-moon is commutative up to homotopy. Therefore, writing $\chi := \text{Hom}_{en}(\epsilon, \tilde{\psi})$, it follows that $Q(\chi) \circ Q(\chi_i) = 1$ in $D(B^{ce})$. Because $I_{cly}$ and $I$ are K-injective over the DG rings $\text{Cyl}(\tilde{B})^{en}$ and $\tilde{B}^{en}$ respectively, and because $\epsilon^{en}$, $\epsilon$ and $\tilde{\psi}$ are quasi-isomorphisms, Proposition 2.5(2) says that $\chi$ is a quasi-isomorphism. The conclusion is that $Q(\chi)$ is an isomorphism in $D(B^{ce})$, and therefore $Q(\chi_0) = Q(\chi_1)$.

In view of condition (ii) in Setup 6.1, we may write $\tilde{\nu} := \tilde{v}_0 = \tilde{v}_1$; this is a quasi-isomorphism $\tilde{\nu} : \tilde{A}' \to \tilde{A}$ in $\text{DGR}_{\leq 0}^{<c}$. The DG ring input of the setup can be summarized as the following commutative diagrams in the category $\text{DGR}_{\leq 0}^{<c}$, for $i = 0, 1$:

Cf. diagram (0.2.3).

**Lemma 6.15.** There is a homomorphism

\[ \tilde{\nu}_{cly} : \tilde{B}' \to \text{Cyl}(\tilde{B}) \]

in $\text{DGR}_{\leq 0}^{<c} \tilde{A}'$, such that diagram (6.16) is commutative for $i = 0, 1$. 
Proof. We know that $\tilde{w}'$ is NC semi-free and $w$ is a surjective quasi-isomorphism. So Theorem 4.3 applies. We obtain an $A'$-linear DG ring homotopy $\gamma: \tilde{w}_0 \Rightarrow \tilde{w}_1$ such that $w \circ \gamma = 0$. By Proposition 4.5 we deduce the existence of a DG ring homomorphism $\tilde{w}_{cyl}: \tilde{B}' \rightarrow \text{Cyl}(\tilde{B})$. The formula, for $b \in \tilde{B}'$, is

$$\tilde{w}_{cyl}(b) = \tilde{w}_0(b) t^{-1}(\gamma(b)) 0 \tilde{w}_1(b).$$

This shows the commutativity of the half-moon in the diagram. Next, we have

$$\text{Cyl}(w)(\tilde{w}_{cyl}(b)) = \begin{bmatrix} w(\tilde{w}_0(b)) & w(t^{-1}(\gamma(b))) \\ 0 & w(\tilde{w}_1(b)) \end{bmatrix} = \begin{bmatrix} w'(b) & 0 \\ 0 & w'(b) \end{bmatrix}.$$ 

We deduce the commutativity of the first square in the diagram. The second square is trivially commutative. □

By applying the enveloping operation (see Notation 5.4 and formula (6.3)) to the top row of diagram (6.16), we obtain the top row of diagram (6.17) below. Define a DG bimodule homomorphism $w' \cdot w'^{op}: \tilde{B}'^{en} \rightarrow B$ by the formula

$$(w' \cdot w'^{op})(b_0 \otimes b_1) := w'(b_0) \cdot w'^{op}(b_1)$$

for $b_i \in \tilde{B}'$. Note that this is not a DG ring homomorphism, unless $B$ is weakly commutative. Similarly define DG bimodule homomorphisms $w \cdot w'^{op}$ and $\text{Cyl}(w) \cdot \text{Cyl}(w'^{op})$. We get the commutative diagram (6.17), in which the objects belong to $\text{DGR}/_{ce} \tilde{A}'$, the horizontal arrows are in the category $\text{DGR}/_{ce} \tilde{A}'$, but the vertical arrows are only bimodule homomorphisms.

For the next constuctions we use the abbreviated Notation 6.2 with respect to the DG ring homomorphisms $\tilde{w}_{cyl}: \tilde{B}' \rightarrow \text{Cyl}(\tilde{B})$ and $\tilde{w}_{cyl}^{en}: \tilde{B}'^{en} \rightarrow \text{Cyl}(\tilde{B})^{en}$. The commutativity of diagrams (6.16) and (6.17) will be used implicitly.

Recall the quasi-isomorphism $a^1: M^1 \rightarrow M^1$ in $M(\tilde{B})$, which is is part of the extended resolution $M$; cf. Definition 5.7(a). Its cylinder is a quasi-isomorphism.
Cyl($\alpha^1$) : Cyl($\tilde{M}^1$) → Cyl($M^1$) in $\mathcal{M}$($\text{Cyl}(\tilde{B})$). Because $\tilde{M}^1$ is K-projective in $\mathcal{M}(\tilde{B}')$, there exists a quasi-isomorphism $\tilde{\phi}^1 : M^1 \rightarrow \text{Cyl}(\tilde{M}^1)$ in $\mathcal{M}(\tilde{B}')$, such that $Q(\tilde{\phi}^1) = Q(\text{Cyl}(\alpha^1))^{-1} \circ Q(\epsilon) \circ Q(\alpha^1)$ in $\mathcal{D}(\tilde{B}')$. The homomorphism $\tilde{\phi}^1$ makes the left square in diagram (6.18) below commutative up to homotopy in $\mathcal{M}(\tilde{B}')$.

\[
\begin{array}{ccc}
\tilde{M}^1 & \xrightarrow{\phi^1} & \text{Cyl}(\tilde{M}^1) \\
\downarrow{\alpha^1} & & \downarrow{\eta^1} \\
M^1 & \xrightarrow{\epsilon} & \text{Cyl}(M^1) \\
\end{array}
\]

Similarly, there exists a quasi-isomorphism $\tilde{\phi}^r : \tilde{M}^r \rightarrow \text{Cyl}(\tilde{M}^r)$ in $\mathcal{M}(\tilde{B}'^{op})$, such that $Q(\tilde{\phi}^r) = Q(\text{Cyl}(\alpha^r))^{-1} \circ Q(\epsilon) \circ Q(\alpha^r)$ in $\mathcal{D}(\tilde{B}'^{op})$. There is a diagram similar to (6.18). Tensoring $\tilde{\phi}^1$ and $\tilde{\phi}^r$ we obtain a quasi-isomorphism

\[
\tilde{\phi}^{en} := \tilde{\phi}^1 \otimes \tilde{\phi}^r : \tilde{M}^{en} \rightarrow \text{Cyl}(\tilde{M})^{en}
\]

in $\mathcal{M}(\tilde{B}'^{en})$.

**Lemma 6.20.** Let $i$ be either 0 or 1. The quasi-isomorphisms

\[
\tilde{\phi}_i^{en} : \eta_i^{en} \circ \tilde{\phi}^{en} : \tilde{M}^{en} \rightarrow \tilde{M}^{en}
\]

in $\mathcal{M}(\tilde{B}'^{en})$ are homotopic to each other.

**Proof.** Consider diagram (6.18). It is a diagram in the category $\mathcal{M}(\tilde{B}')$. By our choice of $\tilde{\phi}^1$, the left square is commutative up to homotopy. The right square and the bottom half-moon are trivially commutative. The outer path is commutative up to homotopy, by the condition on $\tilde{\phi}^1$ in Definition 5.9(a). Therefore the whole diagram becomes a commutative diagram of isomorphisms in $\mathcal{D}(\tilde{B}')$. Because $\tilde{M}^1$ is K-projective, we conclude that the top half-moon is commutative up to homotopy in $\mathcal{M}(\tilde{B}')$. In other words, the homomorphisms $\tilde{\phi}^1_1, \eta_i \circ \tilde{\phi}^1 : \tilde{M}^1 \rightarrow \tilde{M}^1$ in $\mathcal{M}(\tilde{B}')$ are homotopic.

Similarly, the homomorphisms $\tilde{\phi}^r_1, \eta_i \circ \tilde{\phi}^r : \tilde{M}^r \rightarrow \tilde{M}^r$ in $\mathcal{M}(\tilde{B}'^{op})$ are homotopic. Therefore, according to Lemma 5.19, the homomorphisms $\tilde{\phi}^{en}_i = \tilde{\phi}^1_i \otimes \phi^r_i$ and $\eta_i^{en} \circ \tilde{\phi}^{en} = (\eta_i \circ \tilde{\phi}^1) \otimes (\eta_i \circ \tilde{\phi}^r)$ are homotopic.

Recall the quasi-isomorphisms $\tilde{\psi}_i : I \rightarrow I_{\text{cyl}}$ that we chose in (6.7), for $i = 0, 1$. These satisfy formula (6.8). Because $I_{\text{cyl}}$ is K-injective in $\mathcal{M}(\text{Cyl}(\tilde{B})^{en})$, we see that the right square in diagram (6.22) below is commutative up to homotopy in $\mathcal{M}(\text{Cyl}(\tilde{B})^{en})$. (Compare to the right square in diagram (6.13).)
Because $I'$ is K-injective in $\mathcal{M}(\tilde{B}'^{\text{en}})$, there is a quasi-isomorphism
\[(6.21) \tilde{\psi}': I_{\text{cyl}} \to I' \]
in $\mathcal{M}(\tilde{B}'^{\text{en}})$ such that
\[Q(\tilde{\psi}') = Q(\beta') \circ Q(\tilde{\psi}^{\text{en}})^{-1} \circ Q(\beta_{\text{cyl}})^{-1} \]
in $D(\tilde{B}'^{\text{en}})$. Thus the left square in diagram (6.22) is commutative up to homotopy in $\mathcal{M}(\tilde{B}'^{\text{en}})$.

\[(6.22) \begin{array}{ccc}
\tilde{M}'^{\text{en}} & \xrightarrow{\tilde{\psi}^{\text{en}}} & \text{Cyl}(\tilde{M})^{\text{en}} \\
\beta' & \downarrow & \beta_{\text{cyl}} \\
I' & \xrightarrow{\tilde{\psi}'} & I_{\text{cyl}} \\
\psi_i & \downarrow & \psi_i \\
I & \xleftarrow{\psi_i} & I
\end{array} \]

For $i = 0,1$ define
\[(6.23) \chi'_i := \text{Hom}_{\tilde{\omega}^{\text{en}}}(1_B, \tilde{\psi}' \circ \tilde{\psi}_i) : \text{Hom}_{\tilde{B}^{\text{en}}}(B, I) \to \text{Hom}_{\tilde{B}'^{\text{en}}}(B, I'). \]

These are homomorphisms in $\mathcal{M}(B^{\text{ce}})$.

**Lemma 6.24.** For $i = 0,1$ there is equality
\[Q(\chi'_i) = Q(\text{Rect}(\phi_i)) \]
of morphisms in $D(B^{\text{ce}})$.

**Proof.** We fix $i$, and examine diagram (6.22). This is a diagram in the category $\mathcal{M}(\tilde{B}'^{\text{en}})$, via the forgetful functors $\text{rest}_{\tilde{\omega}^{\text{en}}}$ and $\text{rest}_{\tilde{\omega}^{\text{en}}}$.

Let us apply the functor $Q$ to this diagram, namely we look at the image of this diagram in $D(\tilde{B}'^{\text{en}})$. According to Lemma 6.20 the upper half-moon is commutative in $D(\tilde{B}'^{\text{en}})$. By the choice of $\tilde{\psi}'$, the left square is commutative in $D(\tilde{B}'^{\text{en}})$. The choice of $\tilde{\psi}_i$ (see (6.8)) says that the right square is commutative in $D(\tilde{B}'^{\text{en}})$. The outer boundary is commutative by the condition on $\psi_i$ in Definition 5.9(c). We conclude that the whole diagram is commutative in $D(\tilde{B}'^{\text{en}})$.

Now we restrict attention to the bottom half-moon in (6.22). Since $I'$ is K-injective, this part of the diagram is commutative up to homotopy in $\mathcal{M}(\tilde{B}'^{\text{en}})$. In other words, the homomorphisms $\psi_i$, $\tilde{\psi}' \circ \tilde{\psi}_i : I \to I'$ in $\mathcal{M}(\tilde{B}'^{\text{en}})$ are homotopic. Comparing the formula for $\chi'_i$ in (6.23) to the formula for $\text{Rect}(\phi_i)$ in Definition 5.17(2), we see that they are homotopic in $\mathcal{M}(B^{\text{ce}})$. Thus they are equal in $D(B^{\text{ce}})$.

**Lemma 6.25.** In the situation of Setup 6.1 there is equality
\[Q(\text{Rect}(\phi_0)) = Q(\text{Rect}(\phi_1)) \]
of morphisms $\text{Rect}(M) \xrightarrow{\sim} \text{Rect}(M')$ in $D(B)$. 

\[\blacksquare\]
Proof. Recall the homomorphisms $\chi_i$ from equation (6.9), for $i = 0, 1$. Let’s write $\chi' := \text{Hom}_{\tilde{w}}(\epsilon, \tilde{\psi}')$.

Then diagram (6.26) below (for $i = 0, 1$), of homomorphisms in the category $M(B^{ce})$, is commutative.

\[
\begin{array}{ccc}
\text{Hom}_{\text{Cyl}(B)^{en}}(\text{Cyl}(B), I_{\text{cyl}}) & \xleftarrow{\chi' = \text{Hom}_{\tilde{w}}(\epsilon, \tilde{\psi}')} & \text{Hom}_{\tilde{w}}(B, I) \\
\text{Hom}_{B^{en}}(B, I') & \xleftarrow{\chi_i = \text{Hom}_{\tilde{w}}(1, \tilde{\psi}' \circ \tilde{\psi}_i)} & \text{Hom}_{\tilde{w}}(B, I)
\end{array}
\]

We see that
\[
Q(\chi') \circ Q(\chi_i) = Q(\chi'_i)
\]
in $D(B^{ce})$, for $i = 0, 1$.

Now the homomorphisms $\tilde{w}_{\text{cyl}} : \tilde{B}^{en} \to \text{Cyl}(\tilde{B})^{en}$, $\epsilon : B \to \text{Cyl}(B)$ and $\tilde{\psi}' : I_{\text{cyl}} \to I'$ are quasi-isomorphisms, and the DG modules $I_{\text{cyl}}$ and $I'$ are K-injective over the DG rings $\text{Cyl}(\tilde{B})^{en}$ and $\tilde{B}^{en}$ respectively. According to Proposition 2.5(2), the homomorphism $\chi'$ is a quasi-isomorphism. We see that $Q(\chi')$ is an isomorphism in $D(B^{ce})$. On the other hand, according to Lemma 6.10 we have $Q(\chi_0) = Q(\chi_1)$. Plugging this into equation (6.27) we deduce that $Q(\chi'_0) = Q(\chi'_1)$. Finally, by Lemma 6.24 we know that $Q(\chi'_i) = Q(\text{Rect}(\phi_i))$. We conclude that $Q(\text{Rect}(\phi_0)) = Q(\text{Rect}(\phi_1))$. □

7. The Rectangle Operation: Independence of Resolutions

This is the third section dealing with the rectangle operation. The main result here is Theorem 7.1.

Recall that $\text{DGR}^{\leq 0}$ is the category of nonpositive DG rings, and $\text{DGR}^{\leq 0}_{\text{sc}}$ is the category of commutative DG rings (Notation 3.4). These are full subcategories of $\text{DGR}$. For any $A \in \text{DGR}^{\leq 0}_{\text{sc}}$, we have the category $\text{DGR}^{\leq 0}_{\text{ce}} A$ of nonpositive DG rings central over $A$ (Definition 1.4(4)). An object of $\text{DGR}^{\leq 0}_{\text{ce}} A$ is a nonpositive DG ring $B$, together with a DG ring homomorphism $u : A \to B$, such that $u(A)$ is contained in $B^{ce}$, the center of $B$.

For a DG ring $B$, its category of left DG modules is $M(B)$, and the derived category is $D(B)$; see Notation 2.2. There is a localization functor $Q : M(B) \to D(B)$, which is the identity on objects, and inverts quasi-isomorphisms.

Suppose we are given a commutative DG ring $A$, a DG ring $B \in \text{DGR}^{\leq 0}_{\text{ce}} A$, and DG modules $M^1 \in M(B)$ and $M^* \in M(B^{op})$. To this data we attach the category of extended resolutions $\text{Res}(B/A; M^1, M^*)$. See Definition 5.12. The rectangle functor

$\text{Rect} : \text{Res}(B/A; M^1, M^*)^{op} \to M(B^{ce})$

was introduced in Definition 5.17.
Theorem 7.1. Let $A$ be a commutative DG ring, let $u : A \to B$ be a central homomorphism in $\text{DGR}^{\leq 0}$, let $M^1 \in \text{M}(B)$, and let $M^\circ \in \text{M}(B^\circ)$. Suppose 

$$\phi_0, \phi_1 : M^\circ \to M$$

are morphisms in $\text{Res}(B/A; M^1, M^\circ)$. Then the morphisms 

$$Q(\text{Rect}(\phi_i)) : \text{Rect}(M) \to \text{Rect}(M^\circ)$$

in $\text{D}(B^\circ)$ are isomorphisms, and furthermore 

$$Q(\text{Rect}(\phi_0)) = Q(\text{Rect}(\phi_1)).$$

Proof. The assertion that $Q(\text{Rect}(\phi_i))$ are isomorphisms is an immediate consequence of Lemma 5.18. The proof that they are equal requires much more work.

The DG ring input is shown in diagram (7.2), for $i = 0, 1$. This is a commutative diagram in the category $\text{DGR}^{\leq 0}_{/ \text{ce} \tilde{A}}$. Recall that the DG rings $A, \tilde{A}, \tilde{A}^\prime$ are commutative, and the homomorphisms $u, \tilde{u}, \tilde{u}^\prime$ are central.

Choose the following DG ring resolutions: a strict K-flat resolution $(A \xrightarrow{u^\circ} B^\circ, 1, x)$ of $A \xrightarrow{u} B$; a strict NC semi-free resolution $(\tilde{A} \xrightarrow{\tilde{u}^\circ} \tilde{B}^\circ, 1, \tilde{x})$ of $\tilde{A} \xrightarrow{\tilde{u}} \tilde{B}$; and a strict NC semi-free resolution $(\tilde{A}^\prime \xrightarrow{\tilde{u}^\prime\circ} \tilde{B}^\prime, 1, \tilde{x}^\prime)$ of $\tilde{A}^\prime \xrightarrow{\tilde{u}^\prime} \tilde{B}^\prime$. These resolutions are shown as the horizontal arrows in diagram (7.3). According to Theorem 3.22 there exist homomorphisms $\tilde{w}^\circ : \tilde{B}^\circ \to B^\circ$ and $\tilde{w}^\circ_i : \tilde{B}^\circ \to B^\circ$ in $\text{DGR}^{\leq 0}$, for $i = 0, 1$, that make diagram (7.3) commutative.

Diagram (7.3), for $i = 0, 1$, can be interpreted as a commutative diagram in the category $\text{Res}(B/A)$; this is diagram (7.4) below.
Now choose an extended resolution $N$ of $(M^i, M^r)$ over $B^\oplus/A$, and K-projective extended resolutions $M^\psi$ and $M^{\psi'}$ over $B^\oplus/\tilde{A}$ and $B^{\psi'}/\tilde{A}'$ respectively. According to Lemma 5.14 there are morphisms $\phi^\psi_i$, $\phi'$, $\phi$ and $\psi$ in the category $\text{Res}(B/A; M^i, M^r)$, lying above the morphisms $\tilde{w}_i^\psi/v_i$, $\tilde{x}'/1$, $\tilde{x}/1$ and $w^\psi/v$ respectively. They fit into diagram (7.5) below (for $i = 0, 1$). Of course there is no reason for this diagram to be commutative.

\begin{equation}
(7.5)
\begin{array}{ccc}
B^\oplus/\tilde{A} & \xrightarrow{\tilde{x}/1} & \tilde{B}^\oplus/\tilde{A}' \\
\downarrow \tilde{w}_i^\psi/v_i & & \downarrow \tilde{w}_i/v_i \\
B^\oplus/\tilde{A} & \xrightarrow{\tilde{x}'/1} & \tilde{B}^\oplus/\tilde{A}' \\
\downarrow (w^\psi \circ \tilde{w}_i^\psi)/\psi' & & \downarrow \tilde{w}_i/v_i \\
B^\oplus/A & & \\
\end{array}
\end{equation}

Fix $i = 0, 1$. The pair of morphisms

$$
\phi_i \circ \phi', \phi \circ \phi^\psi_i : M_i^\psi \to M
$$

(i.e. the square in diagram (7.5)) satisfy the conditions of Setup 6.1. Indeed, the DG ring resolution $\tilde{B}^{\psi'}/\tilde{A}'$ is NC semi-free; the DG ring homomorphisms $\tilde{A}' \to \tilde{A}$ are equal (they are both $\tilde{v}_i$); and the extended resolution $M^{\psi'}$ is K-projective. (Actually the two DG ring homomorphisms $\tilde{B}^{\psi'}/\tilde{A}'$ are equal here, so we do not need the full force of Setup 6.1) Therefore, by Lemma 6.25, we have

$$
(7.6)
Q(\text{Rect}(\phi')) \circ Q(\text{Rect}(\phi_i)) = Q(\text{Rect}(\phi_i \circ \phi'))
$$

$$
= Q(\text{Rect}(\phi \circ \phi^\psi_i)) = Q(\text{Rect}(\phi^\psi_i)) \circ Q(\text{Rect}(\phi))
$$

in $D(B^{\psi'})$. Note that all the morphisms in (7.6) are isomorphisms; this is by item (1) of the theorem.

Equation (7.6) implies that it suffices to prove that

$$
(7.7)
Q(\text{Rect}(\phi^\psi_i)) = Q(\text{Rect}(\phi^\psi_i)).
$$

Because

$$
Q(\text{Rect}(\phi^\psi_i)) \circ Q(\text{Rect}(\psi)) = Q(\text{Rect}(\psi \circ \phi^\psi_i))
$$
for $i = 0, 1$, it suffices to prove that
\[(7.8) \quad Q(\text{Rect}(\psi \circ \phi_i)) = Q(\text{Rect}(\psi \circ \phi_1)) .\]
Now the pair of morphisms
\[
\psi \circ \phi_0, \psi \circ \phi_1 : \mathcal{M}^\otimes \to \mathcal{N}
\]
also satisfy the conditions of Setup 6.1: the DG ring resolution \(\tilde{B}^\otimes / \tilde{A}^\otimes\) is NC semi-free; the DG ring homomorphisms \(\tilde{A}^\otimes \to \tilde{A}^\otimes\) are equal (they are both \(v \circ v_0 = v \circ v_1 = v')\); and the extended resolution \(\mathcal{M}^\otimes\) is K-projective. Therefore, by Lemma 6.25, equation (7.8) is true. □

Here we must introduce a general concept in category theory, that does not seem to have a proper name in the literature.

**Definition 7.9.** Let \(F : C \to D\) be a functor between categories. We say that \(F\) is an **isoconstant functor** if \(F\) is isomorphic to a constant functor.

The proof of the next proposition is left to the reader. (It is merely an exercise in category theory.)

**Proposition 7.10.** Let \(F : C \to D\) be a functor between categories. Assume the source category \(C\) is nonempty and connected. The following four conditions are equivalent.

(i) \(F\) is isoconstant.

(ii) For any pair of morphisms \(f_0, f_1 : C \to C'\) in \(C\), the morphisms \(F(f_0), F(f_1) : F(C) \to F(C')\) in \(D\) are equal, and they are isomorphisms.

(iii) The limit of \(F\) exists, and for any object \(C \in C\) the canonical morphism \(\lim(F) \to F(C)\) in \(D\) is an isomorphism.

(iv) The colimit of \(F\) exists, and for any object \(C \in C\) the canonical morphism \(F(C) \to \text{colim}(F)\) in \(D\) is an isomorphism.

Now we return to the rectangle operation.

**Corollary 7.11.** In the situation of Theorem 7.1, the functor \(Q \circ \text{Rect} : \text{Res}(B/A; M^l, M^r)^\text{op} \to \text{D}(B^\text{ce})\) is isoconstant, and the source category is nonempty connected.

**Proof.** Proposition 5.16 says that \(\text{Res}(B/A; M^l, M^r)\) is a nonempty connected category. Hence so is the opposite category. Theorem 7.1 says that the functor \(Q \circ \text{Rect}\) satisfies condition (ii) of Proposition 7.10. □

In view of the corollary and Proposition 7.10, we can make the following definition.

**Definition 7.12.** Let \(A\) be a commutative DG ring, let \(u : A \to B\) be a central homomorphism in \(\text{DGR}^{\leq 0}\), let \(M^l \in \text{M}(B)\), and let \(M^r \in \text{M}(B^{\text{op}})\). We define the **rectangle of \((M^l, M^r)\) over \(B\) relative to \(A\)** to be the object
\[
\text{Rect}_{B/A}(M^l, M^r) := \lim(Q \circ \text{Rect}) \in \text{D}(B^\text{ce}),
\]
the limit of the functor
\[
Q \circ \text{Rect} : \text{Res}(B/A; M^l, M^r)^\text{op} \to \text{D}(B^\text{ce}).
\]
Remark 7.13. Let \( \mathbb{K} \) be a regular finite dimensional noetherian commutative ring, and let \( A \) be a cohomologically noetherian commutative DG ring (see [Ye2] Section 3). Suppose \( \mathbb{K} \to A \) is a DG ring homomorphism, such that \( \mathbb{K} \to H^0(A) \) is essentially finite type. Let \( D^+_A(A) \) be the full subcategory of \( D(A) \) consisting of DG modules whose cohomology is bounded below, and whose cohomology modules are finite over \( H^0(A) \).

For \( M, N \in D^+_A(A) \) let us write
\[ M \otimes^1_{A/\mathbb{K}} N := \text{Rect}_A/M(N) \in D(A). \]

In [Ga], D. Gaitsgory states that when \( \mathbb{K} \) is a field of characteristic 0, this operation makes \( D^+_A(A) \) into a monoidal category. No proof of this statement is given in [Ga].

Now assume \( A \) is a ring. According to [YZ4] and its corrections in [Ye5], there is a rigid dualizing complex \( R_{A/\mathbb{K}} \) over \( A \) relative to \( \mathbb{K} \). The functor \( D := \text{RHom}_A(-, R_{A/\mathbb{K}}) \) is a duality of the category \( D_A(A) \), and it exchanges \( D^+_A(A) \) with \( D^-_A(A) \). Recently L. Shaul [Sh1] showed that there is a bifunctorial isomorphism
\[ (7.14) \quad M \otimes^1_{A/\mathbb{K}} N \cong D(D(M) \otimes^L_A D(N)) \]
for \( M, N \in D^+_A(A) \). The proof relies on the reduction formula for Hochschild cohomology [AILN, Theorem 1], [Sh2]. The isomorphism \( (7.14) \) directly implies that the operation \( - \otimes^1_{A/\mathbb{K}} - \) is a monoidal structure on \( D^+_A(A) \), with unit object \( R_{A/\mathbb{K}} \).

8. The Squaring Operation

In this section all DG rings are commutative by default (Notation 3.4). Recall that for a DG ring \( A \), its category of DG modules is \( M(A) \), and the derived category is \( D(A) \); see Notation 2.2. There is an additive functor \( Q : M(A) \to D(A) \), which is the identity on objects, and inverts quasi-isomorphisms.

The \( K \)-flat resolutions of a DG ring homomorphism \( A \to B \) form a category, that we denote by \( \text{Res}(B/A) \); see Definitions 3.12, 3.15 and 5.2 and Notation 5.3.

Definition 8.1. Let \( A \to B \) be a homomorphism of commutative DG rings. A resolution \( \tilde{B}/\tilde{A} \) in \( \text{Res}(B/A) \) is called a \textit{\( K \)-flat commutative DG ring resolution} if the DG ring \( \tilde{B} \) is commutative. We denote by \( \text{Res}_c(B/A) \) the full subcategory of \( \text{Res}(B/A) \) on the commutative resolutions.

A morphism in \( \text{Res}_c(B/A) \) is shown in diagram (0.2.3).

Example 8.2. If the homomorphism \( A \to B \) is \( K \)-flat, then \( B/A \) is the terminal object of the category \( \text{Res}_c(B/A) \).

Here is the commutative analogue of Notation 5.6.

Notation 8.3. Let \( \tilde{B}/\tilde{A} \in \text{Res}_c(B/A) \), and let \( M \in M(\tilde{B}) \). We write \( \tilde{B}^{en} := \tilde{B} \otimes \tilde{A} \tilde{B} \) and \( \tilde{M}^{en} := \tilde{M} \otimes \tilde{A} \tilde{M} \). So \( \tilde{B}^{en} \in \text{DGR}^{\leq 0}_{\tilde{A} / \tilde{A}} \tilde{B} \) and \( \tilde{M}^{en} \in M(\tilde{B}^{en}) \). Given a homomorphism \( \tilde{\theta} : \tilde{N} \to \tilde{M} \) in \( M(\tilde{B}) \), we write \( \tilde{\theta}^{en} := \tilde{\theta} \otimes \tilde{\theta} : \tilde{N}^{en} \to \tilde{M}^{en} \).

Definition 8.4. Let \( A \to B \) be a homomorphism of commutative DG rings, and let \( M \in M(B) \). A \textit{commutative extended resolution} of \( M \) consists of these data:

(a) A resolution \( \tilde{B}/\tilde{A} \in \text{Res}_c(B/A) \).
(b) A quasi-isomorphism \( \alpha : \tilde{M} \to M \) in \( \mathcal{M}(\tilde{B}) \), where \( \tilde{M} \) is a DG \( \tilde{B} \)-module which is K-flat over \( \tilde{A} \).

(c) A K-injective resolution \( \beta : \tilde{M}^\text{en} \to I \) in \( \mathcal{M}(\tilde{B}^\text{en}) \).

We denote this commutative extended resolution by \( M = (\tilde{B}/\tilde{A}; \tilde{M}, I) \). The set of all commutative extended resolutions of \( M \) is denoted by \( \text{Res}_c(B/A; M) \).

In item (c) of the definition we use Notation 8.3. Such commutative extended resolutions exist.

**Definition 8.5.** A resolution \( M = (\tilde{B}/\tilde{A}; \tilde{M}, I) \) in \( \text{Res}_c(B/A; M) \) is called K-projective if \( \tilde{M} \) is K-projective in \( \mathcal{M}(\tilde{B}) \).

**Definition 8.6.** Let \( A \to B \) be a homomorphism between commutative DG rings, and let \( M \in \mathcal{M}(B) \). Suppose \( M = (\tilde{B}/\tilde{A}; \tilde{M}, I) \) and \( M' = (\tilde{B}'/\tilde{A}', \tilde{M}', I') \) are commutative extended resolutions of \( M \). A morphism

\[
\phi = (\tilde{w}/\tilde{v}; \phi, \psi) : M' \to M
\]

consists of:

(a) A morphism \( \tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A} \) in \( \text{Res}_c(B/A) \).

(b) A homomorphism \( \phi : M' \to M \) in \( \text{M}(\tilde{B}') \), such that \( Q(\phi') = Q(\alpha) \circ Q(\phi) \).

(c) A homomorphism \( \psi : I \to I' \) in \( \text{M}(\tilde{B}'^\text{en}) \), such that \( Q(\psi') = Q(\psi) \circ Q(\beta) \circ Q(\phi'^\text{en}) \).

In this way the set \( \text{Res}_c(B/A; M) \) becomes a category.

Homomorphisms \( Q(\phi) \) and \( Q(\psi) \) as in the definition are shown in the two commutative diagrams [0.2.4], that are in the categories in \( \text{D}(\tilde{B}') \) and \( \text{D}(\tilde{B}'^\text{en}) \) respectively. There is a forgetful functor\n
\[
\text{Res}_c(B/A; M) \to \text{Res}_c(B/A).
\]

**Proposition 8.7.** The category \( \text{Res}_c(B/A; M) \) is nonempty and connected.

**Proof.** Fix a commutative semi-free resolution \( Z \to \tilde{A}' \) of \( Z \to A \), and then find a commutative semi-free resolution \( \tilde{A}' \to \tilde{B}' \) of \( \tilde{A}' \to B \). These exist by Theorem [3.21].1. Next fix a K-projective resolution \( \tilde{M}' \to M \) in \( \mathcal{M}(\tilde{B}) \), and a K-injective resolution \( \tilde{M}'^\text{en} \to I' \) in \( \mathcal{M}(\tilde{B}'^\text{en}) \). We get a commutative extended resolution \( M' = (\tilde{B}'/\tilde{A}'; \tilde{M}', I') \) of \( M \), i.e. an object of \( \text{Res}_c(B/A; M) \).

Let \( M = (\tilde{B}/\tilde{A}; M, I) \) be any object of \( \text{Res}_c(B/A; M) \). According to Theorem [3.22]1 there is a morphism \( \tilde{w}/\tilde{v} : \tilde{B}'/\tilde{A}' \to \tilde{B}/\tilde{A} \) in \( \text{Res}_c(B/A) \). As in the proof of Lemma [5.14]1, there is a morphism \( (\phi, \psi) \) over \( \tilde{w}/\tilde{v} \). Thus \( \phi := (\tilde{w}/\tilde{v}; \phi, \psi) \) is a morphism \( M' \to M \) in \( \text{Res}_c(B/A; M) \). \( \square \)

**Definition 8.8.** Let \( A \to B \) be a homomorphism in \( \text{DGR}_\text{sc}^\text{\text{en}} \), and let \( M \in \mathcal{M}(B) \). For any object \( M = (\tilde{B}/\tilde{A}; \tilde{M}, I) \) in \( \text{Res}_c(B/A; M) \) define the DG module\n
\[
\text{Sq}^M_{B/A}(M) := \text{Hom}_{B^\text{en}}(B, I) \in \mathcal{M}(B).
\]

For any morphism \( \phi := (\tilde{w}/\tilde{v}; \phi, \psi) : M' \to M \) in \( \text{Res}_c(B/A; M) \) define the homomorphism\n
\[
\text{Sq}^\phi_{B/A}(1_M) := \text{Hom}_{\tilde{B}^\text{en}}(1_B, \psi) : \text{Sq}^M_{B/A}(M) \to \text{Sq}^{M'}_{B/A}(M)
\]

in \( \mathcal{M}(B) \); cf. Proposition [2.4].
Proposition 8.9. For any morphism $\phi : M' \to M$ in $\text{Res}_c(B/A; M)$, the homomorphism $\text{Sq}^\phi_{B/A}(1_M)$ is a quasi-isomorphism.

Proof. We can view $\phi$ as a morphism in $\text{Res}(B/A; M, M)$. Then $\text{Sq}^M_{B/A}(M) = \text{Rect}(iM)$, $\text{Sq}^{M'}_{B/A}(M) = \text{Rect}(M')$, and $\text{Sq}^\phi_{B/A}(1_M) = \text{Rect}(\phi)$. According to Theorem 7.1, the morphism $Q(\text{Rect}(\phi))$ is an isomorphism in $D(B^{ce}) = D(B)$. □

Remark 8.10. Actually, it is not hard to show that $\text{Sq}^M_{B/A}(M)$ and $\text{Sq}^{M'}_{B/A}(M)$ are K-injective in $M(B)$. This implies that $\text{Sq}^\phi_{B/A}(1_M)$ is a homotopy equivalence. (We will not need this fact.)

Here is the first main theorem of our paper (a repetition of Theorem 0.3.4 from the Introduction).

Theorem 8.11 (Existence). Let $A \to B$ be a homomorphism of commutative DG rings, and let $M \in D(B)$. There is an object

$$\text{Sq}^M_{B/A}(M) \in D(B),$$

with an isomorphism

$$\text{Sq}^{M'}_{B/A}(1_M) : \text{Sq}^M_{B/A}(M) \xrightarrow{\cong} \text{Sq}^{M'}_{B/A}(M)$$

in $D(B)$ for every extended resolution $M \in \text{Res}_c(B/A; M)$, satisfying the following condition.

(*) For every morphism of resolutions $\phi : M' \to M$ in $\text{Res}_c(B/A; M)$, there is equality

$$\text{Sq}^{M'}_{B/A}(1_M) = Q(\text{Sq}^\phi_{B/A}(1_M)) \circ \text{Sq}^M_{B/A}(1_M)$$

of isomorphisms $\text{Sq}^M_{B/A}(M) \xrightarrow{\cong} \text{Sq}^{M'}_{B/A}(M)$ in $D(B)$.

Moreover, the object $\text{Sq}^M_{B/A}(M)$ is unique, up to a unique isomorphism in $D(B)$.

Proof. The formulas $\text{Sq}(M) := \text{Sq}^M_{B/A}(M)$ and $\text{Sq}(\phi) := \text{Sq}^\phi_{B/A}(1_M)$ give rise to a functor

$$\text{Sq} : \text{Res}_c(B/A; M)^{op} \to M(B).$$

Now $\text{Res}_c(B/A; M)$ is a subcategory of $\text{Res}(B/A; M, M)$, cf. Definition 5.12. Indeed, an extended resolution $(\tilde{B}/\tilde{A}; \tilde{M}^1, \tilde{M}^2, I)$ belongs to $\text{Res}_c(B/A; M)$ if and only if $B$ is commutative and $\tilde{M}^1 = \tilde{M}^2$. A morphism $(\tilde{u}/\tilde{v}; \phi^1, \phi^2, \psi)$ belongs to $\text{Res}_c(B/A; M)$ if and only if $\phi^1 = \phi^2$. The functor $\text{Sq}$ is the restriction of $\text{Rect}$ (see Definition 5.17) to the subcategory $\text{Res}_c(B/A; M)$.

According to Proposition 8.7, the source category $\text{Res}_c(B/A; M)$ is nonempty connected. By Theorem 7.1, the functor $Q \circ \text{Sq}$ satisfies condition (ii) of Proposition 7.10. Condition (iii) of Proposition 7.10 says that the limit of the functor

$$Q \circ \text{Sq} : \text{Res}_c(B/A; M)^{op} \to D(B)$$

exists, and we define $\text{Sq}^M_{B/A}(M)$ to be this limit. For any $M \in \text{Res}_c(B/A; M)$, the canonical morphism of the limit

$$\text{Sq}^M_{B/A}(1_M) : \text{Sq}^M_{B/A}(M) \xrightarrow{\cong} \text{Sq}^M_{B/A}(M)$$

is an isomorphism, and these isomorphisms satisfy condition (*). The uniqueness statement is clear (it is the usual uniqueness of a limit of a functor). □
In view of the theorem we can make the following definition.

**Definition 8.12.** The object $\text{Sq}_{B/A}(M)$ from Theorem 8.11 is called the *square of* $M$ *over* $B$ *relative to* $A$.

## 9. Trace Functoriality

In this section again all DG rings are commutative (Notation 3.4). Recall that for a DG ring $A$, its category of DG modules is $\text{M}(A)$, and the derived category is $\text{D}(A)$; see Notation 2.2. There is an additive functor $\text{Q} : \text{M}(A) \to \text{D}(A)$, which is the identity on objects, and inverts quasi-isomorphisms.

The category of commutative DG rings is $\text{DGR}_{\text{sc}}^{<0}$. A homomorphism $A \to B$ in $\text{DGR}_{\text{sc}}^{<0}$ induces restriction functors $\text{M}(B) \to \text{M}(A)$ and $\text{D}(B) \to \text{D}(A)$. We judiciously ignore these restriction functors. See Notation 6.2.

**Definition 9.1.** Let $A \xrightarrow{u} B \xrightarrow{v} C$ be homomorphisms in $\text{DGR}_{\text{sc}}^{<0}$. We refer to this situation as $C/B/A$. A *K-flat resolution of* $C/B/A$ *in* $\text{DGR}_{\text{sc}}^{<0}$ is a commutative diagram

\[
\begin{array}{ccc}
\hat{A} & \xrightarrow{\hat{u}} & \hat{B} \xrightarrow{\hat{v}} \hat{C} \\
\downarrow{w_A} & & \downarrow{w_B} \\
A & \xrightarrow{u} & B \xrightarrow{v} C
\end{array}
\]

in $\text{DGR}_{\text{sc}}^{<0}$, such that the homomorphism $w_A$ is a quasi-isomorphism, the homomorphisms $w_B$ and $w_C$ are surjective quasi-isomorphisms, and the homomorphisms $\hat{u}$ and $\hat{v} \circ \hat{u}$ are K-flat. We refer to this resolution as $\hat{C}/\hat{B}/\hat{A}$.

The set of all K-flat resolutions of $C/B/A$ in $\text{DGR}_{\text{sc}}^{<0}$ is denoted by $\text{Res}_k(C/B/A)$.

A resolution $\hat{C}/\hat{B}/\hat{A} \in \text{Res}_k(C/B/A)$ as above can be broken up into resolutions $\hat{B}/\hat{A} \in \text{Res}_k(B/A)$ and $\hat{C}/\hat{A} \in \text{Res}_k(C/A)$, as in Definition 8.1. According to Notation 8.3, there are DG rings $\hat{B}^{\text{en}} = \hat{B} \otimes_{\hat{A}} \hat{B}$ and $\hat{C}^{\text{en}} = \hat{C} \otimes_{\hat{A}} \hat{C}$, that belong to $\text{DGR}_{\text{sc}}^{<0} A$. Likewise, for DG modules $\hat{M} \in \text{M}(\hat{B})$ and $\hat{N} \in \text{M}(\hat{C})$, there are DG modules $\hat{M}^{\text{en}} = \hat{M} \otimes_{\hat{A}} \hat{M} \in \text{M}(\hat{B}^{\text{en}})$ and $\hat{N}^{\text{en}} = \hat{N} \otimes_{\hat{A}} \hat{N} \in \text{M}(\hat{C}^{\text{en}})$.

**Definition 9.2.** Let $A \xrightarrow{u} B \xrightarrow{v} C$ be homomorphisms in $\text{DGR}_{\text{sc}}^{<0}$, let $M \in \text{D}(B)$, let $N \in \text{D}(C)$, and let $\theta : N \to M$ be a morphism in $\text{D}(B)$. We refer to this data as $N/M$ (leaving the rest of the input implicit). An *extended resolution* of $N/M$ consists of the data listed below:

(a) A resolution $\hat{C}/\hat{B}/\hat{A} \in \text{Res}_k(C/B/A)$.

(b) A commutative extended resolution $(\hat{M}, I)$ of $M$ over $\hat{B}/\hat{A}$, and a commutative extended resolution $(\hat{N}, J)$ of $N$ over $\hat{C}/\hat{A}$. So that

\[
M := (\hat{B}/\hat{A}; \hat{M}, I) \in \text{Res}_k(B/A; M)
\]

and

\[
N := (\hat{C}/\hat{A}; \hat{N}, J) \in \text{Res}_k(C/A; N),
\]

as in Definition 8.3.
(c) A homomorphism $\tilde{\theta} : \tilde{N} \to \tilde{M}$ in $M(\tilde{B})$, and a homomorphism $\eta : J \to I$ in $M(\tilde{B}^\text{en})$, such that the diagrams

$$
\begin{array}{c}
\tilde{M} \xleftarrow{Q(\tilde{\theta})} \tilde{N} \\
\downarrow & \downarrow \\
M & N
\end{array}
\quad
\begin{array}{c}
I \xleftarrow{Q(\eta)} J \\
\downarrow & \downarrow \\
\tilde{M}^\text{en} & \tilde{N}^\text{en}
\end{array}
$$

in $D(B)$ and $D(\tilde{B}^\text{en})$ respectively, are commutative.

We denote this extended resolution by $N/M$. The set of all extended resolutions of $N/M$ is denoted by $\text{Res}_\text{e}(N/M)$.

**Proposition 9.3.** In the situation of Definition 9.2, the set $\text{Res}_\text{e}(N/M)$ is nonempty.

**Proof.** Choose commutative semi-free resolutions $\mathbb{Z} \to \tilde{A}', \tilde{A}' \to \tilde{B}'$ and $\tilde{B}' \to \tilde{C}'$ of $\mathbb{Z} \to A$, $\tilde{A}' \to B$ and $\tilde{B}' \to C$ respectively (in this order). Let $(M', I')$ be a K-projective commutative extended resolution of $M$ over $\tilde{B}'/\tilde{A}'$, and let $(\tilde{N}', J')$ be a K-projective over $\tilde{B}'$; it follows that $\tilde{N}'$ is K-projective over $\tilde{B}'$. Therefore there is a homomorphism $\theta' : \tilde{N}' \to M'$ in $M(\tilde{B}')$ satisfying the condition in item (c) of Definition 9.2, i.e. $Q(\theta')$ represents $\theta$. Because $I'$ is K-injective over $\tilde{B}'^\text{en}$, there is a homomorphism $\eta' : J' \to I'$ as in Definition 9.2, i.e. $Q(\eta')$ represents $Q(\tilde{\theta}')$. We obtain an extended resolution $N'/M'$ of $N/M$. \qed

**Definition 9.4.** Consider an extended resolution $N/M$ of $N/M$ as in Definitions 9.1 and 9.2. According to Definition 8.8, we have DG modules $\text{Sq}^{M}_{C/A}(\tilde{N}) \in M(B)$ $\text{Sq}^{M}_{C/A}(\tilde{N}) \in M(C)$. Define the homomorphism

$$
\text{Sq}^{N/M}_{C/B/A}(\theta) : \text{Sq}^{N}_{C/B/A}(\tilde{N}) \to \text{Sq}^{M}_{B/A}(M)
$$

in $M(B)$ to be

$$
\text{Sq}^{N/M}_{C/B/A}(\theta) := \text{Hom}_{B^\text{en}}(v, \eta),
$$

as in Proposition 2.4(2).

To clarify, the DG ring homomorphism $v : B \to C$ is part of the input $N/M$. The DG ring homomorphism $\tilde{v} : \tilde{B} \to \tilde{C}$ and the DG module homomorphism $\eta : J \to I$ are part of the extended resolution $N/M$, and $\tilde{v}^\text{en} = \tilde{v} \otimes \tilde{v}$.

In the situation of Definition 9.4, there are isomorphisms

$$
\text{Sq}^{M}_{B/A}(1_M) : \text{Sq}^{M}_{B/A}(M) \cong \text{Sq}^{M}_{B/A}(M)
$$

and

$$
\text{Sq}^{N}_{C/A}(1_N) : \text{Sq}^{N}_{C/A}(N) \cong \text{Sq}^{N}_{C/A}(N)
$$

in the categories $D(B)$ and $D(C)$ respectively. See Theorem 8.11.

**Theorem 9.5** (Trace Functoriality). Let $A \to B \to C$ be homomorphisms of commutative DG rings, let $M \in D(B)$, let $N \in D(C)$, and let $\theta : N \to M$ be a morphism in $D(B)$. There exists a unique morphism

$$
\text{Sq}^{N/M}_{C/B/A}(\theta) : \text{Sq}^{N/M}_{C/B/A}(N) \to \text{Sq}^{M}_{B/A}(M)
$$

in $D(B)$, satisfying the following condition:
For any extended resolution $N/M \in \text{Res}_c(N/M)$, there is equality
\[
\text{Sq}_{C/B/A}(\theta) = \text{Sq}_{B/A}(1_M)^{-1} \circ Q(\text{Sq}_{C/B/A}(\theta)) \circ \text{Sq}_{C/A}(1_N)
\]
of morphisms
\[
\text{Sq}_{C/A}(N) \rightarrow \text{Sq}_{B/A}(M)
\]
in $D(B)$

**Proof.** Let $N'/M'$ be the extended resolution of $N/M$ from the proof of Proposition 9.3. Define the morphism $\text{Sq}_{C/B/A}(\theta)$ in $D(B)$ to be
\[
(9.6) \quad \text{Sq}_{C/B/A}(\theta) := \text{Sq}_{B/A}(1_M)^{-1} \circ Q(\text{Sq}_{C/B/A}(\theta)) \circ \text{Sq}_{C/A}(1_N).
\]

Suppose we are given any extended resolution $N/M \in \text{Res}_c(N/M)$. By Theorem 3.21 there are homomorphism $\tilde{\theta}_A : \tilde{A} \rightarrow \tilde{A}$, $\tilde{\theta}_B : \tilde{B} \rightarrow \tilde{B}$ and $\tilde{\theta}_C : \tilde{C} \rightarrow \tilde{C}$ that make the first diagram in (9.7) a commutative diagram in $D\mathcal{G}_{\leq 0}$ lying above $A \rightarrow B \rightarrow C$.

Because $\tilde{M}'$ is K-projective over $\tilde{B}'$, there is a homomorphism $\tilde{\phi}_M : \tilde{M}' \rightarrow \tilde{M}$ in $M(\tilde{B}')$ that commutes up to homotopy with the quasi-isomorphisms to $M$. Likewise there is a homomorphism $\tilde{\phi}_N : \tilde{N}' \rightarrow \tilde{N}$ in $M(\tilde{C}')$ that commutes up to homotopy with the quasi-isomorphisms to $N$. Now $Q(\tilde{\theta}') = \tilde{\theta} = Q(\tilde{\theta})$. Since $\tilde{N}'$ is also K-projective over $\tilde{B}'$, it follows that the second diagram below, in $M(\tilde{B}')$, is commutative up to homotopy. Therefore the third diagram in (9.7), in the category $M(\tilde{B}')$, is commutative up to homotopy.

(9.7)
\[
\begin{array}{ccccccccc}
\tilde{A}' & \overset{\tilde{\theta}'}{\longrightarrow} & \tilde{B}' & \overset{\tilde{\theta}'}{\longrightarrow} & \tilde{C}' & \overset{\tilde{\phi}_M}{\longrightarrow} & \tilde{M}' & \overset{\tilde{\phi}_N}{\longrightarrow} & \tilde{N}' & \overset{\tilde{\theta}'}{\longrightarrow} & \tilde{N}' \\
\tilde{\theta}_A & \downarrow & \tilde{\theta}_B & \downarrow & \tilde{\theta}_C & \downarrow & \phi_M & \downarrow & \phi_N & \downarrow & \tilde{\theta}_N \\
\tilde{A} & \overset{\tilde{\theta}}{\longrightarrow} & \tilde{B} & \overset{\tilde{\theta}}{\longrightarrow} & \tilde{C} & \overset{\phi_M}{\longrightarrow} & \tilde{M} & \overset{\phi_N}{\longrightarrow} & \tilde{N} & \overset{\tilde{\theta}}{\longrightarrow} & \tilde{N}
\end{array}
\]

Choose a homomorphism $\psi_M : I \rightarrow I'$ in $M(\tilde{B}')$ such that $Q(\psi_M) = Q(\tilde{\theta}_M)^{-1}$. Likewise choose $\psi_N : J \rightarrow J'$ in $M(\tilde{C}')$. We obtain a morphism $\tilde{\phi}_M : \tilde{M}' \rightarrow \tilde{M}$ in $\text{Res}_c(B/A;M)$, with components $\tilde{\phi}_M = (\tilde{\theta}_B/\tilde{\theta}_A; \phi_M, \psi_M)$; and a morphism $\tilde{\phi}_N : \tilde{N}' \rightarrow \tilde{N}$ in $\text{Res}_c(C/A;N)$, with components $\tilde{\phi}_N = (\tilde{\theta}_C/\tilde{\theta}_A; \phi_N, \psi_N)$.

Because the third diagram in (9.7) is commutative up to homotopy, we have
\[
Q(\tilde{\theta}_N) \circ Q(\tilde{\phi}_N) = Q(\tilde{\phi}_N) \circ Q(\tilde{\theta}_N)
\]
in $D(\tilde{B}')$. By the conditions in Definition 9.2(c) and 8.6(c) it follows that
\[
Q(\eta) \circ Q(\psi_N)^{-1} = Q(\psi_M)^{-1} \circ Q(\eta')
\]
in $D(\tilde{B}')$. But $I'$ is K-injective in $M(\tilde{B}')$, and therefore diagram (9.8) below, in the category $M(\tilde{B}')$, is commutative up to homotopy.

(9.8)
\[
\begin{array}{ccc}
I' & \overset{\eta'}{\longrightarrow} & J' \\
\psi_M & \downarrow & \psi_N \\
I & \overset{\eta}{\longrightarrow} & J
\end{array}
\]
Finally consider diagram (9.9) below, in the category $\mathcal{D}(B)$. The bottom square is commutative because (9.8) is commutative up to homotopy. The top square is commutative by the definition of $\text{Sq}_{C/B/A}(\theta)$; see equation (9.6). The two half moons are commutative by condition (♦) in Theorem 8.11. We conclude that the outer boundary of the diagram is commutative. This is what we had to prove.

$$
\begin{array}{c}
\text{Sq}_{B/A}(M) \xrightarrow{\text{Sq}_{C/B/A}(\theta)} \text{Sq}_{C/A}(N) \\
\text{Sq}^{M}_{B/A}(1_M) \downarrow \quad \downarrow \text{Sq}^{N'}_{C/A}(1_N) \\
\text{Sq}^{M'}_{B/A}(1_M) \xrightarrow{\text{Q(Sq}_{C/B/A}(\theta))} \text{Sq}^{N'}_{C/A}(1_N) \\
\end{array}
$$

\[\text{Corollary 9.10.}\] Let $A \to B_0 \to B_1 \to B_2$ be homomorphisms of commutative DG rings, let $M_i \in \mathcal{D}(B_i)$, and let $\theta_i : M_i \to M_{i-1}$ be morphisms in $\mathcal{D}(B_{i-1})$. Then

$$\text{Sq}_{B_1/B_0/A}(\theta_1) \circ \text{Sq}_{B_2/B_1/A}(\theta_2) = \text{Sq}_{B_2/B_0/A}(\theta_1 \circ \theta_2),$$

as morphisms $\text{Sq}_{B_2/A}(M_2) \to \text{Sq}_{B_0/A}(M_0)$ in $\mathcal{D}(B_0)$.

Moreover, for the identity automorphisms $1_{B_0} : B_0 \to B_0$ and $1_{M_0} : M_0 \to M_0$, there is equality

$$\text{Sq}_{B_0/B_0/A}(1_{M_0}) = 1_{\text{Sq}_{B_0/A}(M_0)}.$$

\textbf{Proof.} Like in the proof of Proposition 9.3 we can find extended resolutions $M_i/M_{i-1}$ of $M_i/M_{i-1}$, where $M_i = (B_i/A; M_i, I_i)$, $B_i \to B_i$ are commutative semi-free, and $\tilde{M}_i$ are $K$-projective over $B_i$. There are homomorphisms $\tilde{\theta}_1 : \tilde{M}_i \to \tilde{M}_{i-1}$ that represent $\theta_1$, and homomorphisms $\tilde{\eta}_i : I_i \to I_{i-1}$ that represent $\theta_1^{en}$.

Now the homomorphism $\tilde{\theta}_1 \circ \tilde{\theta}_2 : \tilde{M}_2 \to \tilde{M}_0$ represents $\theta_1 \circ \theta_2 : M_2 \to M_0$, and the homomorphism $\tilde{\eta}_1 \circ \tilde{\eta}_2 : I_2 \to I_1$ represents $\tilde{\theta}_1^{en} \circ \tilde{\theta}_2^{en} : M_2^{en} \to M_0^{en}$. Let us denote the resulting resolution of $M_2/M_0$ by $M_2^e/M_0$. We then have

$$\text{Sq}_{B_2/B_0/A}(\theta_1 \circ \theta_2) = \text{Sq}_{B_1/B_0/A}(\theta_1) \circ \text{Sq}_{B_2/B_1/A}(\theta_2),$$

as homomorphisms $\text{Sq}_{B_2/A}(M_2) \to \text{Sq}_{B_0/A}(M_0)$ in $\mathcal{M}(B_0)$. By condition (♦) of Theorem 9.5 we conclude the assertion about $\theta_1$ and $\theta_2$.

The assertion about $1_{B_0}$ and $1_{M_0}$ is clear. \hfill $\square$

Taking $B_1 = B_0 := B$ in Corollary 9.10 and writing $\text{Sq}_{B/A}(\theta) := \text{Sq}_{B/B/A}(\theta)$, we obtain:

\textbf{Corollary 9.11.} Let $A \to B$ be a homomorphism of commutative DG rings. The assignments $M \mapsto \text{Sq}_{B/A}(M)$ and $\theta \mapsto \text{Sq}_{B/A}(\theta)$ are a functor

$$\text{Sq}_{B/A} : \mathcal{D}(B) \to \mathcal{D}(B).$$
The functor $\text{Sq}_{B/A}$ is not linear; in fact it is a quadratic functor, in the sense of the next result.

**Corollary 9.12.** Let $A \to B$ be a homomorphism of commutative DG rings. Given a morphism $\theta : N \to M$ in $\mathcal{D}(B)$ and an element $b \in B^0$, we have

$$\text{Sq}_{B/A}(b \cdot \theta) = b^2 \cdot \text{Sq}_{B/A}(\theta),$$

as morphisms $\text{Sq}_{B/A}(N) \to \text{Sq}_{B/A}(M)$ in $\mathcal{D}(B)$.

**Proof.** Take any extended resolution $N/M$ of $\theta : N \to M$, with ingredients $\tilde{v} := 1_{\tilde{B}} : \tilde{B} \to \tilde{B}$, $\tilde{\theta} : \tilde{N} \to \tilde{M}$ in $\text{M}(\tilde{B})$ and $\eta : J \to I$ in $\text{M}(\tilde{B}^n)$. We know that $\text{Sq}_{B/A}(\theta)$ is represented by $\text{Hom}_{\mathcal{E}^m}(v, \eta)$, where $v := 1_B : B \to B$.

The rings $B^0$ and $\tilde{B}^0$ act on the derived categories $\mathcal{D}(B) \approx \mathcal{D}(\tilde{B})$ via $\text{H}^0(B) \cong \text{H}^0(\tilde{B})$. Let $\tilde{b} \in \tilde{B}^0$ be a lifting of $b \in B^0$ modulo coboundaries, i.e. they have same class in $\text{H}^0(B) \cong \text{H}^0(\tilde{B})$. Then the same DG modules in $N/M$, but with homomorphisms $\tilde{b} \cdot \tilde{\theta} : \tilde{N} \to \tilde{M}$ and $(\tilde{b} \otimes \tilde{b}) \cdot \eta : J \to I$, represent the morphism $\tilde{b} \cdot \tilde{\theta} : \tilde{N} \to \tilde{M}$. Thus $\text{Sq}_{B/A}(b \cdot \theta)$ is represented by $\text{Hom}_{\mathcal{E}^m}(v, (\tilde{b} \otimes \tilde{b}) \cdot \eta)$. Finally we calculate that

$$\text{Hom}_{\mathcal{E}^m}(v, (\tilde{b} \otimes \tilde{b}) \cdot \eta) = \text{Hom}_{\mathcal{E}^m}((\tilde{b} \otimes \tilde{b}) \cdot v, \eta) = \tilde{b}^2 \cdot \text{Hom}_{\mathcal{E}^m}(v, \eta)$$

as homomorphisms in $\text{M}(\tilde{B})$. \qed

**Question 9.13.** In the situation of Theorem 9.5, the morphism $\theta : N \to M$ in $\mathcal{D}(B)$ is called a trace morphism if the induced morphism $N \to \text{RHom}_B(C, M)$ in $\mathcal{D}(C)$ is an isomorphism. Is it true that if $\theta : N \to M$ is a trace morphism, then $\text{Sq}_{B/A}(\theta) : \text{Sq}_{C/A}(N) \to \text{Sq}_{B/A}(M)$ is also a trace morphism?

We know that this is true under some finiteness assumptions. See formula (5.6) in the proof of [YZ] Theorem 5.3.

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