ON GROUPS IN WHICH SUBNORMAL SUBGROUPS OF INFINITE RANK ARE COMMENSURABLE WITH SOME NORMAL SUBGROUP

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Abstract. We study soluble groups $G$ in which each subnormal subgroup $H$ with infinite rank is commensurable with a normal subgroup, i.e. there exists a normal subgroup $N$ such that $H \cap N$ has finite index in both $H$ and $N$. We show that if such a $G$ is periodic, then all subnormal subgroups are commensurable with a normal subgroup, provided either the Hirsch-Plotkin radical of $G$ has infinite rank or $G$ is nilpotent-by-abelian (and has infinite rank).

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1. Introduction and statement of results

A group $G$ is said to be a $T$-group if normality in $G$ is a transitive relation, i.e. if all subnormal subgroups are normal. The structure of soluble $T$-groups was well described in the 1960s by Gaschütz, Zacher and Robinson (see [14]). Then, taking these results as a model, several authors have studied soluble groups in which subnormal subgroups have some embedding property which “approximates” normality. In particular, Casolo [2] considered $T_\ast$-groups, that is groups in which any subnormal subgroup $H$ has the property $nn$ (nearly normal), i.e. the index $|H^G : H|$ is finite. Then Franciosi, de Giovanni and Newell [11] considered $T_\ast$-groups, that is groups in which any subnormal subgroup $H$ has the property $cf$ (core-finite, normal-by-finite), i.e. the index $|H : H_G|$ is finite. Here, as usual, $H^G$ (resp. $H_G$) denotes the smallest (resp. largest) normal subgroup of $G$ containing (resp. contained in) $H$.

Recently in [4], in order to put those results in a common framework, we considered $T[\ast]$-groups, that is groups in which each subnormal subgroup $H$ is $cn$, i.e. commensurable with a normal subgroup of $G$. Recall that two subgroups $H$ and $K$ are called commensurable if $H \cap K$ has finite index in both $H$ and $K$, hence both $nn$ and $cf$ imply $cn$. Clearly all the above results rely on corresponding previous results on groups in which all subgroups are $nn$, $cf$, $cn$ resp. (see [13], [1], [3] resp.). A similar approach was adopted in [7] where finitely generated groups in which subnormal subgroups are inert have been considered, where the term inert refers to a different generalization of both $nn$, $cf$ (namely, an inert subgroup is a subgroup which is commensurable with each of its conjugates).

In the last decade, several authors have studied the influence on a soluble group of the behavior of its subgroups of infinite rank (see for instance [5], [10] or the bibliography in [9]). Recall that a group $G$ is said to have finite rank $r$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with such property and infinite rank is there is no such $r$. For example, in [8] it was proved that if $G$ is a periodic soluble group of infinite rank in which every subnormal subgroup of infinite rank is normal, then $G$ is a $T$-group indeed. Then in
[9], authors have considered groups of infinite rank with properties $T_+$ ($T^+$, resp.), that is groups in which the condition of being $nn$ (resp. $cf$) is imposed only to subnormal subgroups with infinite rank. In fact, it has been shown that a periodic soluble group of infinite rank $G$ with property $T_+$ ($T^+$, resp.) has the full $T_*$ ($T^*$, resp.) property, provided one of the following holds:

(A) the Hirsch-Plotkin radical of $G$ has infinite rank,
(B) the commutator subgroup $G'$ is nilpotent.

In this paper we show that a similar statement is true also for the property $cn$. Moreover, by a corollary, we give some further information about the property $cf$ as well. Let us call $T^+[\ast]$-group a group in which each subnormal subgroup of infinite rank is a $cn$-subgroup.

**Theorem A** Let $G$ be a periodic soluble $T^+[\ast]$-group whose Hirsch-Plotkin radical has infinite rank. Then $G$ is a $T^*\ast$-group.

**Corollary** Let $G$ be a periodic soluble $T^+[\ast]$-group (resp. $T_\ast\ast$-group) of infinite rank such that $\pi(G')$ is finite. Then all subgroups of $G$ are $cn$ (resp. $cf$).

**Theorem B** Let $G$ be a periodic $T^+[\ast]$-group of infinite rank with nilpotent commutator subgroup. Then $G$ is a $T^*\ast$-group.

Note that if $G = A \rtimes B$ is the holomorph group of the additive group $A$ of the rational numbers by the multiplicative group $B$ of positive rationals (acting by usual multiplication), then, as noticed in [8], the only subnormal non-normal subgroups of $G$ are those contained in $A$ (which has rank 1) so that $G$ is $T^+[\ast]$. However all proper non-trivial subgroups of $A$ are not $cn$, since if they were $cn$ then they were $cf$ (see [6], Proposition 1) contradicting the fact that $A$ is minimal normal in $G$.

Our notation and terminology is standard and can be found in [15, 16]

2. Proofs

By a standard argument one checks easily that if $H_1$ and $H_2$ are $cn$- (resp. $cf$-) subgroups of $G$, then $H_1 \cap H_2$ is likewise $cn$ (resp. $cf$). The same holds for $H_1H_2$, provided this is a set subgroup.

**Lemma 1.** Let $G$ be a $T^+[\ast]$-group and let $A$ be a subnormal subgroup of $G$. If $A$ is the direct product of infinitely many non-trivial cyclic subgroups, then any subgroup of $A$ is a $cn$-subgroup of $G$.

**Proof.** Let $X$ be any subgroup of $A$, then $X$ is a subnormal subgroup of $G$ and $X$ is likewise a direct product of cyclic groups (see [16], 4.3.16). In order to prove that $X$ is a $cn$-subgroup of $G$ we may assume that $X$ has finite rank. Then there exist subgroups $A_1, A_2, A_3$ of $A$ with infinite rank such that $X \leq A_3$ and $A = A_1 \times A_2 \times A_3$. Thus $XA_1$ and $XA_2$ are subnormal subgroups of infinite rank, so that they are both $cn$-subgroups of $G$. Therefore $X = XA_1 \cap XA_2$ is likewise $cn$ in $G$. \qed

**Lemma 2.** Let $G$ be a periodic $T^+[\ast]$-group. If $G$ contains an abelian subnormal subgroup of infinite rank $A$, then $G$ is a $T^*\ast$-group.

**Proof.** By hypothesis there exists a normal subgroup $N$ of $G$ which is commensurable with $A$. Then $A \cap N$ has finite index in $AN$ and hence $N$ is an abelian-by-finite group of infinite rank. In particular, $N$ contains a characteristic subgroup $N$, of finite index which is an abelian group of infinite rank; hence replacing $A$ by $N$, it can be supposed that $A$ is a normal subgroup. Since $G$ is periodic and $A$ has infinite rank, it follows
that the socle $S$ of $A$ is a normal subgroup of $G$ which is the direct product of infinitely many non-trivial cyclic subgroups. Application of Lemma 1 yields that all subgroups of $S$ are $cn$-subgroups of $G$ and hence by Lemma 2.8 of [6], there exist $G$-invariant subgroups $S_0 \leq S_1$ of $S$ such that $S_0$ and $S/S_1$ are finite and all subgroups of $S$ lying between $S_0$ and $S_1$ are normal in $G$.

Let $X$ be any subnormal subgroup of finite rank of $G$. Then $X \cap S_1$ is finite, hence $S_2 = S_0(X \cap S_1)$ is likewise finite. Since $S_2X$ is commensurable with $X$, we may assume $S_2 = \{1\}$. Clearly there exist subgroups $S_3$ and $S_4$ with infinite rank such that $S_1 = S_3 \times S_4$. Since both $S_3$ and $S_4$ are normal subgroups of $G$, we have that both $XS_3$ and $XS_4$ are subnormal subgroups of infinite rank of $G$ and hence they are both $cn$. Thus $X = XS_3 \cap XS_4$ is likewise a $cn$-subgroup of $G$. □

Recall that any primary locally nilpotent group of finite rank is a Chernikov group (see [15] Part 2, p.38).

Lemma 3. Let $G$ a $T[+]$-group of infinite rank. If $G$ is a Baer $p$-group, then $G$ is a nilpotent $T[*]$-group.

Proof. Let $X$ be any subnormal subgroup of $G$ with finite rank. Then $X$ is a Chernikov group (see [15] Part 2, p.389) and $X$ contains an abelian divisible normal subgroup $J$ of finite index. Hence $J$ is subnormal in $G$, and so $J^G$ is abelian and divisible (see [15] Part 1, Lemma 4.46). If $A$ is any abelian subnormal subgroup of $G$, the subgroup $J^G A$ is nilpotent and $[J, A] = \{1\}$ (see [15] Part 1, Lemma 3.13). Since $G$ is generated by its subnormal abelian subgroups, it follows that $J \leq Z(G)$ and so $X/X_G$ is finite. This proves that $G$ is a $T[*]$-group, hence nilpotent (see [4], Proposition 20). □

The following lemma is probably well-known but we are not able to find it in the literature, hence we write also the proof.

Lemma 4. Let $G$ be a periodic finite-by-abelian group of finite rank. Then $G/Z(G)$ is finite.

Proof. Clearly $C = C_G(G')$ is a normal subgroup of finite index of $G$ which is nilpotent and has finite rank; in particular, any primary component of $C$ is a Chernikov group. Let $\pi = \pi(G')$ be the set of all primes $p$ such that $G'$ contains some element of order $p$. Then $\pi$ is finite and so the subgroup $C_\pi$ is a Chernikov group; hence $C_\pi Z(G)/Z(G)$ is finite (see [15] Part 1, Lemma 4.3.1). On the other hand $C_\pi^*$ is abelian, and so it follows that $C/Z(C)$ is finite. Thus $G$ is both abelian-by-finite and finite-by-abelian and hence $G/Z(G)$ is finite. □

Proof of Theorem A. Assume, for a contradiction, that the statement is false and let $X$ be a subnormal subgroup $G$ which is not a $cn$-subgroup; in particular, $X$ has finite rank. Among all counterexamples choose $G$ in such a way that $X$ has the smallest possible derived length. Then the derived subgroup $Y = X'$ of $X$ is a $cn$-subgroup by the minimal choice on the derived length of $X$; on the other hand, $Y$ has finite rank and so $Y/Y_G$ is finite (see [6], Proposition 1). Then $X/Y_G$ is a finite-by-abelian group of finite rank, and hence its centre $Z/Y_G = Z(X/Y_G)$ has finite index in $X/Y_G$ by Lemma 4. Thus $Z$ is a subnormal subgroup of $G$ which has finite index in $X$, so that the index $[Z : Z_G]$ is infinite and hence $Z$ cannot be a $cn$-subgroup of $G$ (see [6], Proposition 1). Since $Y_G$ has finite rank, the Hirsch-Plotkin radical of $G/Y_G$ has infinite rank and so $G/Y_G$ is also a counterexample; thus replacing $G$ with $G/Y_G$ and $X$ with $Z/Y_G$ it can be supposed that $X$ is abelian. Hence $X$ is contained in the Hirsch-Plotkin radical $H$ of $G$. 

Let $P$ any primary component of $H$, and suppose that $P$ has infinite rank. If $F$ is the Fitting subgroup of $P$, then $F$ is nilpotent by Lemma 3. Let $A$ be a maximal abelian normal subgroup of $F$, then $A = C_P(A)$ (see [15] Part 1, Lemma 2.19.1) and so $A$ has infinite rank (see [15] Part 1, Theorem 3.29). Hence $G$ is a $T[\ast]$-group by Lemma 2. This contradiction proves that each primary component of $H$ has finite rank. In particular, as $H$ has infinite rank, there exist $H_1$ and $H_2$ subgroups of infinite rank such that $H = H_1 \times H_2$ and $\pi(H_1) \cap \pi(H_2) = \emptyset$. By the same reason, for $i \in \{1, 2\}$, two subgroups of infinite rank $H_{i,1}$ and $H_{i,2}$ can be found such that $H_i = H_{i,1} \times H_{i,2}$ and $\pi(H_{i,1}) \cap \pi(H_{i,2}) = \emptyset$. If $i, j \in \{1, 2\}$ and $i \neq j$, considered $\pi_i = \pi(H_i)$ and denoted by $X_i$ the $\pi_i$-component of $X$, the subgroups $X_iH_{j,1}$ and $X_iH_{j,2}$ are subnormal subgroups of infinite rank of $G$, so that they are both $cn$-subgroups of $G$ and hence $X_i = X_iH_{j,1} \cap X_iH_{j,2}$ is likewise a $cn$-subgroup of $G$. Therefore $X = X_1X_2$ is a $cn$-subgroup of $G$ and this final contradiction concludes the proof.

Proof of Corollary. One may refine the derived series of $G'$ to a series $G_1 = G' \supseteq \cdots \supseteq G_n = \{1\}$ whose factors $A_i = G_i/G_{i+1}$ are $p$-groups (for possibly different primes). Let $C_i = C_G(A_i)$ for each $i$. If $A_i$ has finite rank, then $A_i$ is a Chernikov group and the same holds for $G_i/C_i$ as a periodic group of automorphisms of a Chernikov group (see [15] Part 1, Theorem 3.29). If $A_i$ has infinite rank, then Lemma 2 yields that each subgroup of $A_i$ is a $cn$-subgroup (resp $cf$-) of $G$. Hence, according to Proposition 14 in [3], $G/C_i$ is finite as a periodic group of power automorphisms of $p$-groups (see [14], Lemma 4.1.2). Thus if $C$ is the intersection of all $C_i$’s, then $G/C$ is a Chernikov group and therefore has finite rank. It follows that $C$ has finite rank. Om the other hand, $C$ is nilpotent by a well-known fact (see [12]). Then by Theorem A, the group $G$ has property $T[\ast]$ (resp $T_\ast$). Further, by Theorem 15 in [4], $G$ all subgroups are $cn$ (resp $cf$).

Proof of Theorem B. Assume that the statement is false. As in the first part of proof of Theorem A, there exists a counterexample $G$ containing an abelian subnormal subgroup $X$ that is not a $cn$-subgroup; in particular, $X$ has finite rank and the index $|X : X_G|$ is infinite. Then $L = XG'$ is a nilpotent normal subgroup and hence has finite rank by Theorem A. Let $p \in \pi(X)$, then $L/L_{p'}$ is a nilpotent $p$-group of finite rank and hence it is a Chernikov group; thus $G/C_G(L/L_{p'})$ is finite (see [13] Part 1, Corollary p.85) and hence $C_{G/L_{p'}}(L/L_{p'})$ is a nilpotent normal subgroup of infinite rank of $G/L_{p'}$. Thus Theorem A yields that $G/L_{p'}$ is a $T[\ast]$-group. Therefore $X_{p}L_{p'}$ is a $cn$-subgroup of $G$, and hence it is even $cf$ because it has finite rank (see [6], Proposition 1). The $p$-component of the core $(X_{p}L_{p'})_G$ of $X_{p}L_{p'}$ in $G$ is $G$-invariant, it coincides with the subgroup $X_{p} \cap (X_{p}L_{p'})_G$ and so has finite index in $X_{p}$, therefore $X_{p}$ is $cf$. In particular, the set $\pi$ of all primes $p$ in $\pi(X)$ such that $X_{p}$ is not normal in $G$ is infinite. Replacing $G$ by $G/L_{p'}$ it can be supposed that $\pi = \pi(L)$. Then there exists an infinite subset $\pi_0$ of $\pi$ such that $G/L_{\pi_0}$ contains a nilpotent normal subgroup of infinite rank (see [9], Corollary 11); hence $G/L_{\pi_0}$ is a $T[\ast]$-group by Theorem A. Therefore $X_{\pi_0}L_{\pi_0}$ is a $cn$-subgroup of $G$ and so even a $cf$-subgroup (see [6], Proposition 1); hence $X_{\pi_0}$ is $cf$ and this is a contradiction because $X_{p}$ is not normal in $G$ for each $p \in \pi_0$.

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