The existence of Hall polynomials for posets of finite prinjective type

Justyna Kosakowska

Faculty of Mathematics and Computer Science,
Nicolaus Copernicus University,
ul. Chopina 12/18, 87-100 Toruń, Poland,
e-mail justus@mat.uni.torun.pl

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Dedicated to Professor Daniel Simson
on the occasion of his 65th birthday

Abstract

We prove the existence of Hall polynomials for prinjective representations of finite partially ordered sets of finite prinjective type. In Section 4 we shortly discuss consequences of the existence of Hall polynomials, in particular, we are able to define a generic Ringel-Hall algebra for prinjective representations of posets of finite prinjective type.

1 Introduction

Let \( K \) be a finite field and let \( A \) be a finite dimensional associative, basic \( K \)-algebra. All modules considered in the present paper are right, finite dimensional \( A \)-modules. Given \( A \)-modules \( X, Y, Z \), denote by \( F_{Z,X}^Y \) the number

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of submodules $U \subseteq Y$ such that $U \simeq X$ and $Y/U \simeq Z$. Moreover denote by $\Gamma_A$ the Auslander-Reiten quiver of the algebra $A$. The reader is referred to [3], [2] and to [13] for the definitions and the introduction to the theory of representations of algebras.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a directed Auslander-Reiten quiver, with the set of vertices $\Gamma_0$ and set of arrows $\Gamma_1$. Recall that for any field $K$ and any $K$-algebra $A$ such that $\Gamma_A = \Gamma$, we may identify a function $a : \Gamma_0 \to \mathbb{N}$ with the corresponding $A$-module $M(A, a) = M(a)$ (see [16]). It was proved by C. M. Ringel (in [16]) that for any directed Auslander-Reiten quiver $\Gamma$ and all functions $a, b, c : \Gamma_0 \to \mathbb{N}$, there exist polynomials $\varphi_{ca}^b \in \mathbb{Z}[[T]]$ with the following property: if $K$ is a finite field, and $A$ a $K$-algebra with $\Gamma_A = \Gamma$ and symmetrization index $r$, then $\varphi_{M(A, a), M(A, b)}^{M(A, c)} = \varphi_{ca}^b(|K|^r)$. The polynomials $\varphi_{ca}^b$ are called Hall polynomials. Moreover, in [17], C. M. Ringel conjectured the existence of Hall polynomials for every representation finite algebra. In [11] it was proved that there exist Hall polynomials for representation-finite trivial extension algebras. The existence of Hall polynomials for cyclic symmetric algebras was proved in [4].

Now we present consequences of the existence of Hall polynomials. We restrict our considerations to hereditary algebras. Let $\Delta$ be a Dynkin quiver, $A = K\Delta$ – path algebra of $\Delta$ and $q \in \mathbb{C}$. Following [15] we define $\mathcal{H}_q(\Delta)$ to be the free abelian group with basis $(u_M)_{[M]}$, indexed by the set of isomorphism classes of finite dimensional right $A$-modules. $\mathcal{H}_q(\Delta)$ is an associative ring with identity $u_0$, where the multiplication is defined by the formula

$$u_{X_1}u_{X_2} = \sum_{[X]} \varphi_{X_1,X_2}^X(q)u_X,$$

and sum runs over all isomorphism classes of $A$-modules. We call $\mathcal{H}_q(\Delta)$ the Ringel-Hall algebra of $A$.

The motivation for the study of Hall polynomials and Hall algebras comes from their connection with generic extensions, Lie algebras and quantum groups (see [15], [16], [17], [12]). It is known that $\mathcal{H}_1(\Delta) \otimes \mathbb{C}$ is isomorphic with the universal enveloping algebra $U(\mathfrak{n}_+)$ of $\mathfrak{n}_+$, where $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a triangular decomposition of the semisimple complex Lie algebra $\mathfrak{g}$ of type $\Delta$ (see [15]).

In the present paper we are interested in an analogous problem of the existence of Hall polynomials for prinjective modules over incidence algebras of posets of finite prinjective type (see Section 2 for definition). We define also
(Section 4) prinjective Ringel-Hall algebras for such posets. The paper is organised as follows. In Section 2 we prove some results concerning injective and surjective homomorphisms between prinjective modules and we recall main definitions and results concerning prinjective modules. In Section 3 the existence of Hall polynomials for prinjective representations of posets of finite prinjective type is proved. Section 4 contains consequences of the existence of Hall polynomials. In particular we give there a definition of prinjective Ringel-Hall algebra. Concluding remarks are also presented in Section 4.

The motivations for the study of prinjective $KI$-modules is the fact that many of the representation theory problems can be reduced to the corresponding problems for poset representations and prinjective modules (see [1], [14], [19], [20], [21]). Prinjective $KI$-modules play an important role in the representation theory of finite dimensional algebras (see [14], [19, Chapter 17]) and lattices over orders (see [19, Chapter 13], [20], [21], [22]). Moreover the study of prinjective modules is equivalent to the study of a class of bimodule matrix problems in the sense of Drozd (see [10], [19, Chapter 17]).

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2 Counting surjective homomorphisms

Let $I = (I, \preceq)$ be a finite poset (i.e. partially ordered set) with the partial order $\preceq$. Let $\text{max } I$ denote the set of all maximal elements of $I$ and $I^- = I \setminus \text{max } I$. Given a field $K$ we denote by $KI$ the incidence $K$-algebra of the poset $I$, that is,

$$KI = \{ (\lambda_{ij}) \in M_I(K) ; \lambda_{ij} = 0 \text{ if } i \not\preceq j \text{ in } I \} \subseteq M_I(K)$$

(see [19], [20]). The reader is referred to [19], [20], [21], [22] for a discussion of incidence algebras and their applications to the integral representation theory. A $KI$-module $X$ may be identified with the representation $(X_i, \varphi_{ij})_{i \preceq j \in I}$
of the poset $I$ (i.e. $X_i$ is a $K$-vector space for any $i \in I$ and, for all relations $i \preceq j$ in $I$, $\varphi_{ij}: X_i \to X_j$ are linear maps satisfying $\varphi_{jk} \varphi_{ij} = \varphi_{ik}$ if $i \preceq j \preceq k$). Recall that the dimension vector $\dim X \in \mathbb{Z}^I$ of $X$ is defined by $\dim(i) = \dim_K X_i$ for all $i \in I$. Denote by $P(i)$ the projective $KI$-module corresponding to the vertex $i$. Without loss of generality we may assume that $I \subseteq \mathbb{N}$ and that the order $\preceq$ in $I$ is such that $i \preceq j$ in $I$ implies $i \leq j$ in the natural order. In this case the algebra $KI$ has the following bipartition

\begin{equation}
KI = \begin{bmatrix}
KI^- & M \\
0 & K(\max I)
\end{bmatrix},
\end{equation}

where $M$ is a $KI^--K(\max I)$-bimodule.

It is well-known (see [18], [3, III.2]) that a finitely generated $KI$-module $X$ may be also identified with the triple

$$X = (X', X'', \varphi : X' \otimes_{KI^-} M \to X''),$$

where $X'$ is a $KI^-$-module, $X''$ is a $K(\max I)$-module and $\varphi$ is a $K(\max I)$-module homomorphism. A homomorphism $f : X \to Y = (Y', Y'', \psi)$ of $KI$-modules is identified with a pair $(f', f'')$, where $f' : X' \to Y'$ is a $KI^-$-module homomorphism, $f'' : X'' \to Y''$ is a $K(\max I)$-module homomorphism and $f'' \varphi = \psi(f' \otimes \text{id})$. Equivalently, we may identify $X$ with the triple

$$X = (X', X'', \overline{\varphi} : X' \to \text{Hom}_{K(\max I)}(M, X'')),$$

where $X'$ is a $KI$-module, $X''$ is a $KI^-$-module and $\overline{\varphi}$ is the $KI^-$-module homomorphism adjoint to $\varphi$. A homomorphism $f : X \to Y = (Y', Y'', \psi)$ of $KI$-modules, in this case, is identified with a pair $(f', f'')$, where $f' : X' \to Y'$ is a $KI^-$-module homomorphism, $f'' : X'' \to Y''$ is a $K(\max I)$-module homomorphism and $\overline{\psi} f' = \text{Hom}_{B}(M, f'')\overline{\varphi}$. In the present paper we use and need these three presentations of a $KI$-module $X$.

Let $\text{mod}(KI)$ denotes the category of all finite dimensional right $KI$-modules.

A $KI$-module $X$ is said to be **prinjective** if the $KI^-$-module $X'$ is projective. Let us denote by $\text{prin}(KI)$ the full subcategory of $\text{mod}(KI)$ consisting of prinjective $KI$-modules. Note that any projective $KI$-module is prinjective. The algebra $KI$ is said to be of **finite prinjective type** if the category...
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prin$(KI)$ is of finite representation type, i.e. there exist only finitely many isomorphism classes of indecomposable prinjective $KI$-modules.

**Remark.** If the poset $I$ is of finite prinjective type, the $K$-algebra $KI$ may be of infinite representation type (even wild). Moreover the category of prinjective modules is not closed under submodules. Therefore the problem of the existence of Hall polynomials for prinjective modules does not reduce to the corresponding one for representation directed algebras and Ringel's arguments given in [16] does not apply directly in our case. In this section we present a reduction which allows us, in Section 3, to develop Ringel’s arguments in our case.

Let us denote by mod$_{sp}(KI)$ the full subcategory of mod$(KI)$ consisting of socle projective modules, i.e. modules $X$ which have projective socle soc$(X)$. Following [18] we define the functor

$$\Theta : \text{prin}(KI) \to \text{mod}_{sp}(KI)$$

by

$$(X', X'', \varphi) \mapsto (\text{Im} \overline{\varphi}, X'', j_\varphi) = (\Theta(X'), \Theta(X''), j_\varphi),$$

where $j_\varphi$ is the adjoint map to the inclusion $j_\varphi : \text{Im} \overline{\varphi} \hookrightarrow \text{Hom}_{K(\max I)}(M, X'')$.

Let us collect some properties of these categories and functor.

**Lemma 2.2.** (a) A $KI$-module $X = (X', X'', \varphi)$ belongs to the category mod$_{sp}(KI)$ if and only if soc$(X)$ has the form $(0, Y, 0)$, where $Y$ is a $K(\max I)$-module.

(b) The functor $\Theta$ is full and dense with Ker $\Theta = \{(P, 0, 0) : P$ projective $KI$-module$\}$. Moreover $\Theta$ establishes a bijection between indecomposable modules which are not in Ker $\Theta$ and indecomposable modules in mod$_{sp}(KI)$.

**Proof.** See [18] and [10].

Now we prove some facts about surjective and injective homomorphisms of $KI$-modules. These facts are essentially used in Section 3.

**Lemma 2.3.** (a) Let $X = (X', X'', \overline{\varphi})$, $Y = (Y', Y'', \overline{\psi})$ be modules in prin$(KI)$ and let $f = (f', f'') : X \to Y$ be an injective (resp. surjective) $KI$-homomorphism. Then $\Theta(f)$ is an injective (resp. surjective) $KI$-homomorphism.
(b) Let \( X = (X', X'', \overline{\varphi}) \), \( Y = (Y', Y'', \overline{\psi}) \) be modules in \( \text{prin}(KI) \) and let \( f : X \to Y \) be a \( KI \)-homomorphism such that \( \Theta(f) = (g', g'') : \Theta(X) \to \Theta(Y) \) is surjective. If \( Y \) has no direct summand of the form \((P, 0, 0)\), where \( P \) is a projective \( KI^- \)-module, then \( f \) is surjective.

**Proof.** (a) Let \( f : X \to Y \) be a homomorphism and
\[
g = (g', g'') = \Theta(f) = (\text{Hom}_{KI}(\max I)(M, f'')|_{\text{Im}\overline{\varphi}}, f'').
\]
Assume that \( f \) is injective. Then the morphisms \( f' \) and \( f'' = g'' \) are injective. Note that \( g' \) is injective, because \( f'' \) is injective and the functor \( \text{Hom}_{KI}(\max I) \) is left exact.

Now let \( f \) be surjective. Then \( f', f'' = g'' \) are surjective. We have to show that \( g' : \text{Im}\overline{\varphi} \to \text{Im}\overline{\psi} \) is surjective. Note that
\[
g'(\text{Im}\overline{\varphi}) = \text{Hom}_{KI}(\max I)(M, f'')\text{Im}\overline{\varphi} = \text{Hom}_{KI}(\max I)(M, f'')\overline{\varphi}(X') = \overline{\psi}f'(X').
\]
Since \( f' \) is surjective we have
\[
\overline{\psi}f'(X') = \overline{\psi}(Y') = \text{Im}\overline{\psi}.
\]
Therefore \( g' \) and \( g'' \) are surjective. This finishes the proof of (a).

(b) Let \( X = (X', X'', \overline{\varphi}) \), \( Y = (Y', Y'', \overline{\psi}) \) be modules in \( \text{prin}(KI) \) and let \( \Theta(f) = (\text{Hom}_{KI}(\max I)(M, f'')|_{\text{Im}\overline{\varphi}}, f'') = (g', g'') : \Theta(X) \to \Theta(Y) \) be surjective. It follows that \( g'\overline{\varphi} : X' \to \Theta(Y)' \) and \( \overline{\psi}f' = g'\overline{\varphi} : X' \to \Theta(Y)' \) are surjective. Moreover, let \( Y \) has no direct summand of the form \((P, 0, 0)\), where \( P \) is a projective \( KI^- \)-module. By \([10, \text{Lemma 3.3}]\), \( \overline{\psi} : Y' \to \Theta(Y)' = \text{Im}\overline{\psi} \) is the projective cover of \( \Theta(Y)' \) in \( \text{mod}(KI^-) \). Since \( \overline{\psi}f' \) is surjective and \( \overline{\psi} \) is the projective cover, the morphism \( \overline{\psi} \) is essential, and therefore \( f' \) is surjective and we are done. \( \square \)

Let \( |X| \) denotes the cardinality of a finite set \( X \). Moreover, given \( KI \)-modules \( X, Y \), let \( \text{Epi}_{KI}(X, Y) \) be the set of all surjective \( KI \)-homomorphisms \( f : X \to Y \) and \( \text{Ker} \Theta(X, Y) \) be the set of all homomorphisms \( f : X \to Y \) which are in \( \text{Ker} \Theta \) (in the case \( X, Y \) are prinjective).

**Corollary 2.4.** Let \( K \) be a finite field and \( X = (X', X'', \overline{\varphi}) \), \( Y = (Y', Y'', \overline{\psi}) \) be modules in \( \text{prin}(KI) \). If \( Y \) has no direct summand of the form \((P, 0, 0)\), then
\[
|\text{Epi}_{KI}(X, Y)| = |\text{Epi}_{KI}(\Theta(X), \Theta(Y))| \cdot |\text{Ker} \Theta(X, Y)|.
\]
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Proof. By Lemma 2.2(b) and Lemma 2.3 the functor $\Theta$ induces the surjective $K$-linear map

$$\Theta : \text{Epi}_{KI}(X, Y) \to \text{Epi}_{KI}(\Theta(X), \Theta(Y))$$

by attaching to any surjective homomorphism $f : X \to Y$ the surjective homomorphism $\Theta(f)$. Lemma 2.3(a) finishes the proof. □

Lemma 2.5. Let $K$ be a finite field and $X = (X', X'', \varphi), Y = (Y', Y'', \overline{\varphi}), Z = (Z', 0, 0)$ be modules in prin($KI$). Assume that $Y$ has no direct summand of the form $(P, 0, 0)$, where $P$ is a projective $KI$-module.

(a) If there exists a surjective homomorphism $f : X \to Y$, then there exists the unique (up to isomorphism) projective $KI$-module $U'$ such that

$$\dim U' = \dim X' - \dim Y'.$$

(b) If there is no surjective homomorphism $f : X \to Y$, then there is no surjective homomorphism $g : X \to Y \oplus Z$.

(c) Let $U'$ be the module defined in (a) if there is a surjective homomorphism $f : X \to Y$ and $U' = 0$ otherwise. Then

$$|\text{Epi}_{KI}(X, Y \oplus Z)| = |\text{Epi}_{KI}(X, Y)| \cdot |\text{Epi}_{KI-}(U', Z')| \cdot |\text{Hom}_{KI-}(Y', Z')|.$$

Proof. (a) Let $f : X \to Y$ be a surjective homomorphism and consider $U = \text{Ker} f = (U', U'', \phi)$. Since the $KI$-modules $X', Y'$ are projective, the $KI$-module $U'$ is projective. Moreover $\dim U' = \dim X' - \dim Y'$ and $U'$ is uniquely determined by its dimension vector (see [14, pp 77]).

The statement (b) is clear.

(c) If there is no surjective homomorphism $g : X \to Y$, then by (b) the formula given in (c) is clear.

Let $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} : X \to Y \oplus Z$ be a surjective homomorphism such that $g_1 : X \to Y, g_2 : X \to Z$ and let $U = \text{Ker} g_1$. It follows that $g_1, g_2$ are surjective. Note that $X'$ may be identified with $U' \oplus Y'$, because $X', Y'$ are projective $KI$-modules and $g'_1 : X' \to Y'$ is surjective with kernel isomorphic to $U'$. Therefore the condition $\dim U' = \dim X' - \dim Y'$ is satisfied. By [6, Lemma 2.3] there is an isomorphism of $K$-vector spaces
\[ \text{Hom}_{K^I}(V, Z) \simeq \text{Hom}_{K^I^{-}}(V', Z') \] for any \( K^I \)-module \( V \). This isomorphism is given by \( (f', f'') \mapsto f' \) and is based on the observation that \( f'' = 0 \) if \( Z = (Z', 0, 0) \). Therefore \( g_2 \) may be identified with \( g_2 = [g_{21}, g_{22}] : U' \oplus Y' \to Z' \), where \( g_{21} : U' \to Z' \), \( g_{22} : Y' \to Z' \). Consider the following commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \to & U & \to & X & \xrightarrow{g_1} & Y & \to & 0 \\
\downarrow{g_{21}} & & \downarrow{[g_1 \ g_2]} & & \downarrow{id} & & & & \\
0 & \to & Z & \to & Y \oplus Z & \xrightarrow{[0 \ 1]} & Y & \to & 0.
\end{array}
\]

Since \( g \) is surjective, by the Snake Lemma \( g_{21} \) is surjective. So, with any surjective \( K^I \)-homomorphism \( g : X \to Y \oplus Z \) we associate two surjective \( K^I \)-homomorphisms \( g_1 : X \to Y \), \( g_{21} : U \to Z \) (identified with the surjective \( K^I^{-} \)-homomorphism \( g_{21} : U' \to Z' \)) and a \( K^I^{-} \)-homomorphism \( g_{22} : Y' \to Z' \).

Conversely, let \( g_1 : X \to Y \) be a surjective \( K^I \)-homomorphism and \( U = \mathrm{Ker} \, g_1 \). Note that \( X' \simeq U' \oplus Y' \), because \( U' \), \( X' \) and \( Y' \) are projective \( K^I^{-} \)-modules. Let \( g_{21} : U' \to Z' \) be a surjective \( K^I^{-} \)-homomorphism and \( g_{22} : Y' \to Z' \) any \( K^I^{-} \)-homomorphism. Then \( g_2 = [g_{21}, g_{22}] : X \to Z \) is surjective (identified with \( g_2 : U' \oplus Y' \to Z' \)). Finally we get a surjective \( K^I \)-homomorphism \( g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} : X \to Y \oplus Z \). Indeed, let \((y, z) \in Y \oplus Z\).

Let us fix the decomposition of \( X \simeq U' \oplus Y' \oplus X'' \) as a \( K \)-linear space. Since \( g_1 \) is surjective and \( g_1(U) = 0 \), there exists \( x_1 = (0, x_1', x_1'') \) such that \( g_1(x_1) = y \). Moreover \( g_{21} \) is surjective, then there exists \( x_2 \in U' \subseteq X \) such that \( g_{21}(x_2) = z - g_{22}(x_1') \). Let \( x = (x_2, x_1', x_1'') \), therefore \( g(x) = (g_1(x_1), z - g_{22}(x_1') + g_{22}(x_1')) = (y, z) \) and lemma follows.

**Lemma 2.6.** Let \( I \) be an arbitrary finite poset, and \( K^I \) - its incidence \( K \)-algebra. Let \( P = \bigoplus_{i \in I} P(i)^{m_i}, \ n_i \geq 0, \ Q = \bigoplus_{i \in I} P(i)^{m_i}, \ m_i \geq 0 \) be projective \( K^I \)-modules. Then \( \dim_K \text{Hom}_{K^I}(P, Q) = \sum_{i \in I} (\sum_{j \prec i} n_i m_j) \). In particular \( \dim_K \text{Hom}_{K^I}(P, Q) \) is independent on the base field \( K \).

**Proof.** Let us recall that \( \dim_K \text{Hom}_{K^I}(P(i), X) = \dim_K X_i \) (see [14] pp 68]). Moreover \( P(i)_j \simeq K \) if \( i \preceq j \) in \( I \) and \( P(i)_j = 0 \) otherwise. Therefore lemma follows easily. \( \square \)
3 Hall polynomials for posets of finite prinjective type

Let $I$ be a poset of finite prinjective type and let $KI$ be its incidence $K$-algebra. In this section we prove the existence of Hall polynomials for prinjective $KI$-modules. Given finite dimensional $KI$-modules $X, Y, Z$ we define $F^X_{Z,Y}$ to be the number of modules $U \subseteq Y$ such that $U \cong X$ and $Y/U \cong Z$.

It follows from [20], [5] that the Auslander-Reiten quiver $\Gamma_I = \Gamma(\text{prin}(KI))$ (resp. $\Gamma_{I-\text{sp}} = \Gamma(\text{mod}_{\text{sp}}(KI))$) of the category prin$(KI)$ (resp. mod$_{\text{sp}}(KI)$) is directed and coincides with its preprojective component. Moreover $\Gamma_I$ and $\Gamma_{I-\text{sp}}$ do not depend on the base field $K$ (see [19, Chapter 11]). Let us recall that, by the definition, the vertices of Auslander-Reiten quiver corresponds bijectively to the isomorphism classes of indecomposable modules. For a given vertex $x \in (\Gamma_I)_0$ (resp. $x \in (\Gamma_{I-\text{sp}})_0$) we denote by $M(K,x)$ (resp. $M_{\text{sp}}(K,x)$) the corresponding indecomposable prinjective (resp. socle projective) $KI$-module. Moreover for any function $a : (\Gamma_I)_0 \to \mathbb{N}$ (resp. $a : (\Gamma_{I-\text{sp}})_0 \to \mathbb{N}$) let $M(K,a) = \bigoplus_{x \in (\Gamma_I)_0} M(K,a(x))$ (resp. $M_{\text{sp}}(K,a) = \bigoplus_{x \in (\Gamma_{I-\text{sp}})_0} M_{\text{sp}}(K,a(x))$) (see [10] for details). Moreover given a function $a \in \mathcal{B}$ we denote by $\Theta(a) \in \mathcal{B}_{\text{sp}}$ the function corresponding to the socle projective $KI$-module $\Theta(M(a))$. It follows from [19], [20], [5] and [10] that the dimension vectors $\text{dim}M(K,a)$ and $\text{dim}M_{\text{sp}}(K,a)$ depend only on the Auslander-Reiten quiver, so they do not depend on $K$. For the sake of simplicity we write $M(a)$ (resp. $M_{\text{sp}}(a)$) instead of $M(K,a)$ (resp. $M_{\text{sp}}(K,a)$) if the base field $K$ is known from the context. Denote by $\mathcal{B}$ (resp. $\mathcal{B}_{\text{sp}}$) the set of all functions $a : (\Gamma_I)_0 \to \mathbb{N}$ (resp. $a : (\Gamma_{I-\text{sp}})_0 \to \mathbb{N}$). It is clear that $\mathcal{B}$ (resp. $\mathcal{B}_{\text{sp}}$) can be identified with the set of all finite dimensional prinjective (resp. socle projective) $KI$-modules. Given an arbitrary $KI$-module $M$ we denote by $\mathcal{S}(M)$ the set of all $KI$-modules $N$ such that $\text{dim}N < \text{dim}M$ (i.e. $\text{dim}N \neq \text{dim}M$ and $(\text{dim}N)(i) \leq (\text{dim}M)(i)$ for all $i \in I$).

**Lemma 3.1.** Let $I$ be a poset of finite prinjective type. For any $a, b \in \mathcal{B}$ (resp. $\overline{a}, \overline{b} \in \mathcal{B}_{\text{sp}}$) the natural number $h(a,b) = \text{dim}_K \text{Hom}_{KI}(M(a),M(b))$ (resp. $h(\overline{a},\overline{b}) = \text{dim}_K \text{Hom}_{KI}(M_{\text{sp}}(\overline{a}),M_{\text{sp}}(\overline{b}))$) does not depend on the field $K$.

**Proof.** Since the Auslander-Reiten quivers $\Gamma_I$ and $\Gamma_{I-\text{sp}}$ are directed,
the arguments given in [16] prove our lemma. □

For $a, b \in \mathcal{B}$ (resp. $\overline{a}, \overline{b} \in \mathcal{B}_{sp}$) we define polynomial $\gamma_{ab} = T^{h(a, b)} \in \mathbb{Z}[T]$
(resp. $\gamma_{\overline{a} \overline{b}} = T^{h(\overline{a}, \overline{b})} \in \mathbb{Z}[T]$). Note that $\gamma_{ab}(|K|) = |\text{Hom}_K(M(a), M(b))|$
(resp. $\gamma_{\overline{a} \overline{b}}(|K|) = |\text{Hom}_K(M_{sp}(\overline{a}), M_{sp}(\overline{b}))|$).

**Lemma 3.2.** Let $a, b \in \mathcal{B}$ and let $\overline{a}, \overline{b} \in \mathcal{B}_{sp}$ be such that $\Theta(M(a)) = M(\overline{a})$
and $\Theta(M(b)) = M(\overline{b})$.

(a) $|\text{Ker} \Theta(M(a), M(b))| = |K|^{h(a, b) - h(\overline{a}, \overline{b})}$.

(b) There exists a polynomial $\omega_{ab} \in \mathbb{Z}[T]$ such that for any finite field $K$
we have $\omega_{ab}(|K|) = |\text{Ker} \Theta(M(a), M(b))|$. 

**Proof.** (a) By Lemma 3.1 the natural numbers $h(a, b)$ and $h(\overline{a}, \overline{b})$ are independent on the base field $K$. So let us fix a finite field $K$. By Lemma 2.2(b) we have

$$|\text{Hom}_K(M(a), M(b))| = |\text{Hom}_K(M(\overline{a}), M(\overline{b}))| \cdot |\text{Ker} \Theta(M(a), M(b))|.$$ 

To finish the prove of (a) we have only to observe that $|\text{Hom}_K(M(a), M(b))| = |K|^{h(a, b)}$ and $|\text{Hom}_K(M(\overline{a}), M(\overline{b}))| = |K|^{h(\overline{a}, \overline{b})}$.

(b) Put $\omega_{ab} = T^{h(a, b) - h(\overline{a}, \overline{b})}$. Then (b) follows from (a). □

**Theorem 3.3.** Let $I$ be a poset of finite prinjective type and let $a \in \mathcal{B}$
(resp. $\overline{a} \in \mathcal{B}_{sp}$). There exists a monic polynomial $\alpha_a \in \mathbb{Z}[T]$ (resp. $\alpha_{\overline{a}} \in \mathbb{Z}[T]$)
such that for any finite field $K$

$$|\text{Aut}_K(M(a))| = \alpha_a(|K|), \quad (\text{resp. } |\text{Aut}_K(M_{sp}(\overline{a}))| = \alpha_{\overline{a}}(|K|)).$$

**Proof.** We may follow the proof given in [16]. This theorem also follows from [11] Proposition 2.1. □

Given functions $x, y, z \in \mathcal{B} \cup \mathcal{B}_{sp}$, etc., for the sake of simplicity, we
denote by capital letters $X, Y, Z$, etc. the $KI$-modules $M(K, x), M_{sp}(K, x), M(K, y), M(K, z)$, respectively. However we should remember that $KI$-modules are identified with functions from the sets $\mathcal{B}, \mathcal{B}_{sp}$ and depend on the base field $K$. Moreover given a function $x \in \mathcal{B}$ we denote by $\Theta(x)$ the
function in $\mathcal{B}_{sp}$ corresponding to the module $\Theta(X)$.

**Lemma 3.4.** Let $I$ be a poset of finite prinjective type. Let $x, y \in \mathcal{B}_{sp}$.
There exist polynomials $\sigma_x^y, \eta_x^y, \mu_x^y, \epsilon_x^y \in \mathbb{Z}[T]$ such that for any finite field $K$:
\( \sigma^y_x(|K|) \) equals the number of submodules \( U \subseteq Y \), such that \( U \cong X \),
\( \eta^y_x(|K|) \) equals the number of submodules \( U \subseteq Y \), such that \( Y/U \cong X \),
\( \mu^y_x(|K|) \) equals the number of injective homomorphisms \( X \to Y \),
\( \varepsilon^y_x(|K|) \) equals the number of surjective homomorphisms \( Y \to X \).

**Proof.** One can prove this lemma by developing Ringel’s arguments given in [16]. For the convenience of the reader we outline the proof.

If \( \dim X \nless \dim Y \), we set \( \sigma^y_x = 0 = \eta^y_x \).

Let \( \dim X \less \dim Y \). We apply induction on \( \dim Y \). If \( \dim Y = 0 \), then \( X = 0 = Y \) and \( \sigma^y_x = 1 = \eta^y_x \). Let \( Y \neq 0 \) and we start with induction on \( \dim X \). Define two polynomials \( \mu^y_x = \gamma_{xy} - \sum_{U \in \text{S}(X)} \eta^y_u \sigma^x_u \), \( \varepsilon^y_x = \gamma_{yx} - \sum_{U \in \text{S}(X)} \eta^y_u \sigma^y_x \). Since the category \( \text{mod}_{\text{sp}}(KI) \) is closed under submodules, we may assume that \( U \) arising in these sums is socle projective, because otherwise \( \sigma^y_u = 0 = \sigma^x_u \). Moreover these sums are finite, because the poset \( I \) is of finite prinjective type. All summands on the right side are defined by induction hypothesis.

We claim that \( \eta^y_u \sigma^y_x(|K|) \) equals the number of morphisms \( f : X \to Y \) such that \( \text{Im} f \cong U \). Indeed, for a given submodule \( V \subseteq X \) such that \( X/V \cong U \) we fix a surjective homomorphism \( g_V : X \to U \) with \( \text{Ker} g_V = V \). Similarly, if \( W \subseteq Y \) is a submodule such that \( W \approx U \), we fix an injective homomorphism \( h_W : U \to Y \) with \( \text{Im} h_W = W \). Homomorphisms \( X \to Y \) with kernel \( V \) and image \( W \) correspond bijectively to automorphisms of \( U \). This bijection is given by attaching to any automorphism \( f : U \to U \) the following homomorphism \( X \to Y \):

\[
X @>g_V>> U @>f>> U @>h_W>> Y.
\]

A homomorphism \( X \to Y \) is injective if and only if its image is not isomorphic to any \( U \) with \( \dim U < \dim X \). Therefore \( \mu^y_x(|K|) \) is the number of injective homomorphisms \( X \to Y \). Dually, \( \varepsilon^y_x(|K|) \) is the number of surjective homomorphisms \( Y \to X \).

Note that for all finite fields \( K \), \( \mu^y_x(|K|) (\alpha_x(|K|))^{-1} \) equals the number of submodules \( U \subseteq Y \) with \( U \cong X \) and therefore it is an integer. By [16] page 441] the polynomial \( \alpha_x \) divides \( \mu^y_x \) in \( \mathbb{Z}[T] \). Similarly, \( \alpha_x \) divides \( \varepsilon^y_x \) in \( \mathbb{Z}[T] \). We put \( \sigma^y_x = \mu^y_x (\alpha_x)^{-1} \) and \( \eta^y_x = \varepsilon^y_x (\alpha_x)^{-1} \). This finishes the proof.

**Lemma 3.5.** Let \( I \) be an arbitrary poset and let \( X, Y \) be projective \( KI \)-modules there exist polynomials \( \eta^y_x, \varepsilon^y_x \subseteq \mathbb{Z}[T] \) such that for any finite field \( K \):
\[ \eta^y_x(|K|) \] equals the number of submodules \( U \subseteq Y \) such that \( Y/U \cong X \),
\[ \varepsilon^y_x(|K|) \] equals the number of surjective homomorphisms \( Y \to X \).

**Proof.** Let \( X, Y, Z \) be \( KI \)-modules. By [13, Section 4], the number of submodules \( U \subseteq Y \), such that \( Y/U \cong X \), equals
\[ F^Y_{X,Z} = \frac{|\text{Ext}^1_{KI}(X,Z)_Y||\text{Aut}_{KI}(Y)|}{|\text{Aut}_{KI}(Z)||\text{Aut}_{KI}(X)||\text{Hom}_{KI}(Z,X)|}. \]

where \( \text{Ext}^1_{KI}(X,Z)_Y \) is the set of all exact sequences in \( \text{Ext}^1_{KI}(X,Z) \) with the middle term \( Y \). Let us assume that \( Y \) and \( X \) are projective \( KI \)-modules. Let us fix a submodule \( Z \subseteq Y \) such that \( Y/Z \cong X \). Since the category of projective modules is closed under kernels of surjective homomorphisms, the submodules \( U \subseteq Y \) with \( Y/U \cong X \) are projective. Moreover \( U \cong Z \), because any exact sequence \( 0 \to U \to Y \to X \to 0 \) splits. Therefore \( F^Y_{X,Z} \) equals the number of submodules \( U \subseteq Y \) such that \( Y/U \cong X \). Note also that \( \text{Ext}^1_{KI}(X,Z)_Y = 0 \) and therefore \( |\text{Ext}^1_{KI}(X,Z)_Y| = 1 \). By Lemma 2.6 the number \( h(z,x) = \dim_K \text{Hom}_{KI}(Z,X) \) is independent on the base field \( K \) and the number of \( KI \)-homomorphisms \( f : Z \to X \) equals \( \gamma_{z,x}(|K|) \). We define
\[ \eta^y_x = \frac{\alpha_y}{\alpha_z \alpha_x \gamma_{z,x}}. \]

By Theorem 3.3 and (*), \( F^Y_{X,Z} = \eta^y_x(|K|) \) for any finite field \( K \). Then the number
\[ \alpha_z(|K|) \alpha_x(|K|) \gamma_{z,x}(|K|) \]
divides \( \alpha_y(|K|) \) for infinitely many finite fields \( K \). Since the polynomial \( \alpha_z \alpha_x \gamma_{z,x} \) is monic, it follows from [16, page 441] that it divides the polynomial \( \alpha_y \) in \( \mathbb{Z}[T] \) and therefore \( \eta^y_x \in \mathbb{Z}[T] \). Consequently \( \eta^y_x(|K|) \) equals the number of submodules \( U \subseteq Y \) such that \( Y/U \cong X \).

We put \( \varepsilon^y_x = \eta^y_x \alpha_x \in \mathbb{Z}[T] \). Note that \( \varepsilon^y_x(|K|) \) equals the number of surjective homomorphisms \( f : Y \to X \). This finishes the proof. \( \square \)

**Corollary 3.6.** Assume that \( I \) is of finite prinjective type and \( x, y \in \mathcal{B} \). There exists a polynomial \( \varepsilon^y_x \in \mathbb{Z}[T] \) such that for any field \( K \):
\[ \varepsilon^y_x(|K|) = \text{Epi}_{KI}(Y,X). \]

**Proof.** If there is no surjective homomorphism \( f : Y \to X \) for any field \( K \), we put \( \varepsilon^y_x = 0 \). Otherwise, let \( X = Y \oplus Z \), where \( Z = (P,0,0) \) with projective \( KI^{-} \)-module \( P \) and \( Y \) has no direct summand of the form \( (P,0,0) \).
Then \( \Theta(X) = \Theta(\overline{X}) \). In our case there exists a surjective homomorphism \( f : Y \to X \) for some field \( K \). Let \( U' \simeq Y' \overline{X} \) be the unique (up to isomorphism) projective \( KI' \)-module such that \( \dim U' = \dim Y' - \dim \overline{X}' \).

By Lemma 3.4, there exists a polynomial \( \varepsilon_{\Theta(x)}^{\Theta(y)} \in Z[T] \) such that \( \varepsilon_{\Theta(x)}^{\Theta(y)}(|K|) \) equals the number of surjective homomorphisms \( \Theta(Y) \to \Theta(X) \). By Lemma 3.5, there exists a polynomial \( \varepsilon_{\varepsilon}^{\gamma} \in Z[T] \) such that \( \varepsilon_{\varepsilon}^{\gamma}(|K|) \) equals the number of surjective homomorphisms \( U' \to Z' \).

Put

\[
\varepsilon_{\varepsilon}^{\gamma} = \varepsilon_{\Theta(x)}^{\Theta(y)} \cdot T^{h(y,x) - h(\Theta(y),\Theta(x))} \cdot T^{h(x',z')} \cdot \varepsilon_{\varepsilon}^{\gamma}.
\]

By Corollary 2.4, Lemma 2.5 and Lemma 3.2, \( \varepsilon_{\varepsilon}^{\gamma} \) is the required polynomial.

\[ \Box \]

**Corollary 3.7.** Let \( I \) be a poset of finite prinjective type and let \( x, y \in B \). There exists a polynomial \( \eta_{\varepsilon}^{x} \in Z[T] \) such that for any finite field \( K \):

\[ \eta_{\varepsilon}^{x}(|K|) \text{ equals the number of submodules } U \subseteq Y \text{, such that } Y/U \simeq X. \]

**Proof.** By Corollary 3.6, there exists a polynomial \( \varepsilon_{\varepsilon}^{y} \in Z[T] \) such that \( \varepsilon_{\varepsilon}^{y}(|K|) = Epi_{KI}(Y,X) \) for any finite field \( K \). Note that, for any finite field \( K \), the number \( \varepsilon_{\varepsilon}^{y}(|K|) \cdot \alpha_{x}^{-1}(|K|) \) is an integer, because it counts the number of submodules \( U \subseteq Y \) such that \( Y/U \simeq X \). Since \( \alpha_{x} \) is a monic polynomial, it follows from [16, page 441] that \( \alpha_{x} \) divides \( \varepsilon_{\varepsilon}^{y} \) in \( Z[T] \). Therefore \( \eta_{\varepsilon}^{x} = \varepsilon_{\varepsilon}^{y} \cdot \alpha_{x}^{-1} \in Z[T] \) is the required polynomial.

\[ \Box \]

**Theorem 3.8.** Let \( I \) be a poset of finite prinjective type and \( x, y, z \) be functions in \( B \) (resp. \( \pi, \tau, \sigma \in B_{sp} \)). There exist polynomials \( \varphi_{x}^{y} \in Z[T] \) (resp. \( \varphi_{x}^{\tau} \in Z[T] \)) such that for any finite field \( K \):

\[ \varphi_{x}^{y}(|K|) = F_{XZ} \quad \text{(resp. } \varphi_{x}^{\tau}(|K|) = F_{XZ}). \]

**Proof.** We prove this theorem developing arguments given in [16] and facts proved in Sections 2 and 3.

If \( \dim Y \neq \dim Z + \dim X \) we put \( \varphi_{x}^{y}(x) = 0 \). Let \( \dim Y = \dim Z + \dim X \). We apply induction on \( \dim Z \). If \( \dim Z = 0 \) we put \( \varphi_{x}^{y} = 1 \) and \( \varphi_{x}^{y} = 0 \) if \( X \neq Y \).

Assume that \( Z \neq 0 \) and \( Z = U_{1} \oplus U_{2} \), where \( U_{1} \neq 0 \), \( U_{1} \simeq W^{m} \), \( W \) is indecomposable, \( W \) is not a direct summand of \( U_{2} \) and no indecomposable direct summand of \( U_{2} \) is a predecessor of \( W \) in \( \Gamma_{I} \) (resp. \( \Gamma_{I-sp} \) in the "socle projective" case).

Let us consider two cases:
Case 1. $U_2 \neq 0$. We define
\[ \varphi_{yz} = \sum_d \varphi_{dx1}^d \varphi_{du2}^d, \]
where the sum runs over all modules $D$ such that $\dim D = \dim X + \dim U_1$. Note that this sum is finite and runs over prinjective modules (resp. socle projective modules), because the category of prinjective modules (resp. socle projective modules) is closed under extensions and the poset $I$ is of finite prinjective type. Moreover the right side is already defined by induction hypothesis. One can prove that $\varphi_{yz}(|K|) = F_{XZ}^Y$ (see [16]).

Case 2. $U_2 = 0$. We define
\[ \varphi_{yz} = \eta_{xy} - \sum_{d \neq z} \varphi_{zd}, \]
where $d$ runs over all modules such that $\dim D = \dim Z$. Since the category of prinjective is closed under kernels of epimorphisms and the category of socle projective modules is closed under submodules, we may assume that the modules $D$ are prinjective (resp. have projective socle). Note that $D$ is not a direct power of indecomposable, because $Z$ is a direct power of indecomposable, $Z \not\cong D$ and $\dim Z = \dim D$ (see [3 IX.2.1]). Therefore the polynomials $\varphi_{zd}$ are defined in Case 1. The polynomials $\eta_{xy}$ are defined in Corollary 3.7 for prinjective modules and in Lemma 3.4 for socle projective modules. It is clear that $\varphi_{yz}(|K|) = F_{XZ}^Y$ and this finishes the proof. \qed

The polynomials $\varphi_{yz}$ are called **Hall polynomials**.

In the last chapter we present consequences of the existence of Hall polynomials for prinjective modules.

## 4 Prinjective Ringel-Hall algebras

We denote by $\mathcal{H}_{prin}(I)$ the free $\mathbb{Q}(T)$-module with basis $\{u_x\}_{x \in B}$, indexed by the elements of the set $B$. $\mathcal{H}_{prin}(I)$ is equipped with a multiplication defined by the formula:
\[ u_{x_1} u_{x_2} = \sum_{x \in B} \varphi_{x_1 x_2}^x u_x. \]
Note that this sum is finite, because the poset $I$ is of finite prinjective type and $\varphi_{x_1,x_2}^x \neq 0$ only if $\text{dim} X = \text{dim} X_1 + \text{dim} X_2$. By [16, Proposition 4], $\mathcal{H}_{\text{prin}}(I)$ is an associative ring and the element $u_0$ is the identity element of $\mathcal{H}_{\text{prin}}(I)$. By the results of Section 3 this ring depends only on the poset $I$. We call $\mathcal{H}_{\text{prin}}(I)$ the \textbf{prinjective generic Ringel-Hall algebra} for the poset $I$.

\textbf{Concluding remarks.} (1) In the forthcoming paper [8] description of $\mathcal{H}_{\text{prin}}(I)$ by generators and relations is given. Moreover in [8] we show connections of the prinjective Ringel-Hall algebra with Lie algebras and Kac-Moody algebras.

(2) In [7] the existence of generic extensions for prinjective modules over posets of finite prinjective type is proved. It would be interesting to find connections between the monoid of generic extensions of prinjective modules and some specialization of prinjective Ringel-Hall algebra. Such a connection, for Dynkin quivers, one ca find in [12].

(3) In the paper [8] generators of prinjective Ringel-Hall algebra are given. Most of these generators are in the kernel of the functor $\Theta$. We can’t see natural candidates for generators in the ”socle projective case”, therefore the category of prinjective modules is more convenient in our considerations.

(4) In [9] the existence of Hall polynomials for representations of finite type bisected posets is proved. However, in our case, it solves only the problem of the existence of Hall polynomials for socle projective modules over posets of finite prinjective type with exactly one maximal element.

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