Regular representations of affine Kac-Moody algebras

B. Feigin, S. Parkhomenko

§1. Introduction

In this paper we investigate one Wakimoto-type construction of affine Kac-Moody algebras. Our aim is to obtain a version of the regular representation of the current algebras.

Let us discuss first what is the regular representation for finite-dimensional complex semisimple Lie groups. Let \( G \) be such a group and \( C(G) \) be the space of algebraic functions on \( G \). It is clear that \( C(G) \) is a \( G \times G \)-module where \( G \) acts from the left and from the right; the formula for the action is: \((g_1, g_2)u = g_1 \cdot u \cdot g_2^{-1}, g_1, g_2, u \in G\). It is well-known that \( C(G) \) as a \( G \times G \)-module is \( \oplus V \otimes V^* \) where the sum goes over all irreducible finite-dimensional representations of \( G \). Usually \( C(G) \) is called the regular representation. Now let us fix some Borel subgroup \( B \subset G \) and denote by \( C_B(G) \) the space of distributions on \( G \) with support on \( B \) and which are smooth along \( B \). In other words \( C_B(G) \) is isomorphic to the space of the local cohomologies of the sheaf of functions on \( G \) with support on \( B \), they are non-trivial only in dimension \( \dim G - \dim B \). The group \( G \times G \) does not act on \( C_B(G) \), but the Lie algebra \( \mathfrak{g} \oplus \mathfrak{g} \) of the group \( G \times G \) does. Let \( b \) be the Lie algebra of the Borel subgroup \( B \). It is evident that the restriction of the \( \mathfrak{g} \oplus \mathfrak{g} \)-module \( C_B(G) \) to \( b \oplus b \) can be integrated to the representation of \( B \times B \). It means that \( C_B(G) \) belongs to the “category of representations with highest weight”. In \( C_B(G) \) we have the distinguished subspace \( \delta(B) \), which is invariant with respect to \( B \times B \) and isomorphic to the space of sections of the line bundle \( \xi \) on \( B \), \( \xi \) is \( \Lambda^N T \), where \( T \) is the normal bundle to \( B \) in \( G \) and \( N \) is the dimension of \( T \). It is easy to find all \( n \oplus n \)-invariant vectors in \( \delta(B) \), where \( n \) is...
the maximal nilpotent subalgebra in \( b \). They are labeled by the elements from the weight lattice \( \Gamma \) and the weight of such vector \( v(\chi), \chi \in \Gamma \) with respect to the sum \( \mathfrak{h} \oplus \mathfrak{h} \) of two Cartan sub-algebras is \( (\chi, -2\rho - \chi) \), where \( 2\rho \) is a sum of all positive roots of \( \mathfrak{g} \).

This construction can be generalized by the following way. Let us consider the formal vicinity \( \tilde{B} \) of the submanifold \( B \) in \( G \). The fundamental group \( \pi_1(\tilde{B}) \cong \pi_1(B) \cong \pi_1(H) \), where \( H \) is maximal torus in \( G \). Therefore, each element \( \lambda \in \mathfrak{h}^* \) defines a one-dimensional bundle \( \nu_\lambda \) on \( \tilde{B} \) with flat connection and this connection is uniquely determined by the restriction of \( \nu_\lambda \) on \( H \). Lie algebra \( \mathfrak{g} \oplus \mathfrak{g} \) acts in the space of sections of \( \nu_\lambda \) and also on the space of local cohomologies of \( \nu_\lambda \) with the support on \( B \). We denote this space by \( C_\lambda^\lambda(B)(G) \).

Again in \( C_\lambda^\lambda(B)(G) \) it is possible to construct the set of vacuum vectors \( V(\chi), \chi \in \Gamma \) such that the character of \( \mathfrak{h} \oplus \mathfrak{h} \) is a pair \( (\chi + \lambda, -\lambda - \chi - 2\rho) \). It gives us for generic \( \lambda \in \mathfrak{h}^* \) a \( \mathfrak{g} \oplus \mathfrak{g} \) module \( C_\lambda^\lambda(B)(G) \) which is isomorphic to the sum \( M_{\chi+\lambda} \otimes M_{-\chi-\lambda-2\rho}, \chi \in \Gamma, M_u \) is a Verma representation with highest weight \( u \). So this is an analog of the decomposition \( C(G) = \oplus V \otimes V^* \). Note, that the algebra \( C(G) \) acts in \( C_\lambda^\lambda(B)(G) \), each function \( f \in C(G) \) defines an operator \( C_\lambda^\lambda(B)(G) \rightarrow C_\lambda^\lambda(B)(G) \) which is just the multiplication on \( f \). Therefore we obtain some “vertex operator”: \( (V \otimes V^*) \otimes \left( \bigoplus_{\chi} M_{\chi+\lambda} \otimes M_{-\chi-\lambda-2\rho} \right) \rightarrow \bigoplus_{\chi} M_{\chi+\lambda} \otimes M_{-\chi-\lambda-2\rho} \), \( V \) is a finite-dimensional representation of \( \mathfrak{g} \). The important thing is that this algebra of vertex operators is commutative.

Our aim now is to provide the same construction for infinite-dimensional Lie algebras. To do it let us recall some main ideas of constructing of Wakimoto representations. Let \( M \) be some (may be infinite-dimensional) manifold and Lie \( (M) \) be the Lie algebra of vector fields on \( M \). Natural representations of Lie \( (M) \) are realized in different spaces of distributions on \( M \). Let us denote by \( C_N(M) \) the space of distributions on \( M \) with support on \( N \) and which are smooth along \( N \). Another name for this space – local cohomologies of the sheaf of functions with support on \( N \). If \( M \) is the finite dimensional manifold then \( C_N(M) \) is a representation of Lie \( (M) \), but in the infinite-dimensional case the situation is more complicated. The machinery of local cohomologies does not work in the infinite dimensional case, so we have to construct \( C_N(M) \) by hands and then again by hands we have to verify the functorial properties of \( C_N(M) \). In other words the construction of \( C_N(M) \) depends on the choice of the coordinate system in the small neighbourhood of \( N \).
and we need to know what it will be if we change the coordinates. Infinitesimally we want to determine the action of \( \text{Lie}(M) \) on \( C_N(M) \).

The construction of \( C_N(M) \) is the following. For simplicity we suppose that \( N \) is isomorphic to the affine space and let us fix the coordinate system: \( \{x_1, x_2, \cdots, y_1, y_2, \cdots\} \) in the neighbourhood of \( N \) such that \( \{y_1, y_2, \cdots\} \) are the coordinate in the “normal direction to \( N \)”, it means that all \( y_j \) are zero on \( N \) and they constitute the coordinate system on the transversal to \( N \) submanifold. The functions \( \{x_i\} \) form coordinates on \( N \). Now let \( D \) be an algebra of differential operators with generators: \( \{x_i, y_j, \partial/\partial x_i, \partial/\partial x_j\} \). We define \( C_N(M) \) as an irreducible representation of \( D \) with the vacuum vector \( \text{vac} \), such that \( y_j\text{vac} = 0, \ j = 1, 2, \cdots \) and \( \partial/\partial x_i\text{vac} = 0, \ i = 1, 2, \cdots \). In \([2,4]\) it is shown that on \( C_N(M) \) the central extension of \( \text{Lie}(M) \) by \( \text{Lie}(M) \)-module \( C(M) \) acts, where \( C(M) \) is an algebra of functions on \( M \). This extension is nontrivial if \( \dim(N) = \text{codim}(N) = \infty \). This means that the infinitesimal group of symmetries of local cohomologies is the Lie algebra of twisted differential operators on the manifold of the order \( \leq 1 \). The general theory of such cohomologies should exist, as in finite-dimensional case, with suitable modifications.

We are interested in the following particular case of the main construction. Let \( A \) be a semisimple complex Lie group, \( \mathfrak{A} \)-Lie algebra of \( A \), \( X \) be a homogeneous space of \( A \), \( LA \) be the loop group of \( A \), \( LX \) be the space of maps \( S^1 \to X \). It is clear that we have the map \( LA \to \text{Lie}(LX) \), where \( LA \) is the Lie algebra of the group \( LA \). Choose the submanifold \( N \subset LX \), which consists of boundary values of the analytic maps from the disk \( |z| \leq 1 \) into \( X \). In the space \( C_N(LX) \) the central extension of the Lie algebra \( LA \) by the space of functions \( C(LX) \) acts. In some cases this extension can be transformed (by the adding of a coboundary) into the extension with values only in the constants \( \mathbb{C} \subset C(LX) \). Therefore in this cases we obtain a representation of the affine Lie algebra of \( LA \) in the space \( C_N(LX) \).

We know at least two situations where it is possible to make the reduction of the extension to the constants. The first one is the case \( X = A/B, B \) is the Borel subgroup in \( A \), \( X \) is the flag manifold for \( A \). The affine algebra \( \hat{\mathfrak{A}} \) acts in \( C_N(LX) \) with level \(-g\), where \( g \) is the dual Coxeter number for \( A \). The slight variation of this construction gives us Wakimoto modules of arbitrary level \([2]\). The second example is when \( A = G \times G \) and
$X = A/G_\Delta$, where $G_\Delta$ is a diagonal subgroup in $G \otimes G$. In other words, $LX$ is the loop group $LG$, where $LA = LG \oplus LG$ acts – one $LG$ by left shifts and the other by the right shifts. So we want to define the “regular” representation in the space of distributions on $LG$. It is possible to prove that we obtain the $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$-module $C_N(LX)$, where the right and the left algebras $\hat{\mathfrak{g}}$ act with central charge $-g$. Note, that the space $C_N(LX) = C_N(LG)$ formally is very close to the space $C_B(G)$ which we discussed in the first part of the introduction. Actually, $N$ is a parabolic subgroup in $LG$, but we certainly can replace $N$ by the Borel subgroup $\bar{B}$ in the loop group $LG$. The corresponding space $C_{\bar{B}}(LG)$ is also easy to define and the values of the central charges will be the same. It is also possible to construct the family of representations $C_{\lambda, \bar{B}}(LG)$, where $\lambda$ is the character of the Cartan sub-algebra of $\mathfrak{g}$. But if we want to change the values of the central charges we need a more subtle construction.

Suppose, the Lie group $H$ is acting on its homogeneous space $Y; H_0$ is stationary subgroup of the point, $C(Y)$- the space of functions on $Y, h$ and $h_0$ are the Lie algebras of $H$ and $H_0$. The first way to deform the action of $h$ on $C(Y)$ is the following. Fix a 1-cocycle $\omega : h \to C(Y)$ and add this cocycle to the action of $h$. It means that the new action of $u \in h$ is given by the formula $u_\omega(f) = u(f) + \omega(u) \cdot f$. Because of the cocycle condition it gives us a representation $h \to \text{End}(C(Y))$. The similar thing is true for the group action. Now suppose that $\nu : A^2(h) \to \mathbb{C}$ is 2-cocycle and $\mu : h \to C(Y)$ is 1-cochain such that $d\mu(u_1, u_2) \in \mathbb{C} \cdot 1 \subset C(Y)$ $u_1, u_2 \in h$ so $d\mu$ determine 2-cochain with values in $\mathbb{C}$ and we suppose that $d\mu = \nu$. In this case the formula $u_\mu(f) = u(f) + \mu(u) \cdot f, u \in h$ gives us a projective representation of $h$ and the corresponding 2-cocycle is $\nu$. If we work with the action of groups, then $H^2(H, C(Y)) \cong H^2(H_0, \mathbb{C})$, so if we fix a class $\bar{\nu} \in H^2(H, \mathbb{C})$ such that the image of $\bar{\nu}$ in a map $H^2(H, \mathbb{C}) \to H^2(H_0, \mathbb{C})$ is zero, then we can construct the projective action of $H$ in the $C(Y)$ such that the corresponding cocycle in $\bar{\nu}$. Infinitesimal version of it is also true, if we replace $Y$ by its contractible subspace $\bar{Y}$.

Now let us apply these arguments to the case, when $H = LG \times LG$ and $Y = LG$. Let $\bar{Y}$ be some contractible open set in $Y$. Using the arguments with cocycles it is possible to show that on $C(\bar{Y})$ there exists the projective representation of $L\mathfrak{g} \oplus L\mathfrak{g}$ by the differential operators of degree $\leq 1$ such that the corresponding 2-cocycles are $(m\omega, -m\omega), m \in \mathbb{C}$.
and $\omega$ is the standard 2-cocycle of $L\mathfrak{g}$. In the space of distribution $C_N(Y)$ we get the representation of $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ and the levels are $(m - g, -m - g)$. The diagonal subalgebra $\hat{\mathfrak{g}}_{\Delta} \subset \hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ has level $-2g$. This is the case when we can add ghosts and compute semi-infinite homology which are the candidate for the space of fields in this version of topological field theory.

The paper is arranged as follows.

In Sect. 2 we briefly discuss the construction of regular representations in the simplest $G = Sl(2, \mathbb{C})$ and $G = Sl(3, \mathbb{C})$ cases. In these examples the submanifolds $N$ are chosen by using the Gauss decompositions of $G$. In Sect. 3 we give the loop versions of constructions of Section 2. As the space of distributions $D(N, LG, \xi_m)$ we consider a Fock module generated from a vacuum vector by a spin $(1, 0)$ conjugated bosonic fields and the submanifold $N$ is a set of boundary values of analytic maps from the unit disk into the open subset of $G$, which is defined by the Gauss decomposition of $G$. In this situation $m$ has arbitrary value. Sect. 4 is devoted to the generalization on $\hat{sl}(n + 1, \mathbb{C})$ case. We think that our representations can be used as an ingredient of $G/G$ topological field theory.
§2. Regular representation in the finite-dimensional case

Let us briefly discuss the regular representations of $sl(2, \mathbb{C})$ and $sl(3, \mathbb{C})$ Lie algebras.

Using the Gauss decomposition let us to introduce coordinate systems in the open subsets $Sl(2, \mathbb{C})$ and $Sl(3, \mathbb{C})$ of $Sl(2, \mathbb{C})$ and $Sl(3, \mathbb{C})$: $x \in Sl(2, \mathbb{C})$ if $x$ can be represented as a product

$$x = \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix} \begin{bmatrix} \exp(-y) & 0 \\ 0 & \exp(y) \end{bmatrix} \begin{bmatrix} 1 & x^1 \\ 0 & 1 \end{bmatrix};$$

(similarity $x \in Sl(3, \mathbb{C})$ if

$$x = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_3 & x_2 & 1 \end{bmatrix} \begin{bmatrix} \exp(-y_1) & 0 & 0 \\ 0 & \exp(y_1 - y_2) & 0 \\ 0 & 0 & \exp(y_2) \end{bmatrix} \begin{bmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{bmatrix}.$$  

We shall call them Gauss coordinate systems. Let $E, H, F$ be the standard generators of $sl(2, \mathbb{C})$ and $E_i, H_i, F_i, i = 1, 2$ be the standard generators of $sl(3, \mathbb{C})$. The following formulas give us the left and right actions of $sl(2, \mathbb{C})$ and $sl(3, \mathbb{C})$ in the Gauss coordinates.

Symbols $L$ and $R$ will be used for the left right actions, respectively.

$$L_E = \frac{\partial}{\partial x_1}; \quad R_E = \exp(2y) \frac{\partial}{\partial x_1} - x^1 \frac{\partial}{\partial y} - (x^1)^2 \frac{\partial}{\partial x^1}$$

$$L_H = -2x_1 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y}; \quad R_H = \frac{\partial}{\partial y} + 2x_1 \frac{\partial}{\partial x^1}$$

$$L_F = -(x_1)^2 \frac{\partial}{\partial x^1} - x_1 \frac{\partial}{\partial y} + \exp(2y) \frac{\partial}{\partial x^1}; \quad R_F = \frac{\partial}{\partial x^1}$$

$$L_{E_1} = \frac{\partial}{\partial x_1}; \quad R_{E_1} = \exp(2y_1 - y_2) \frac{\partial}{\partial x_1} + \exp(2y_1 - y_2) x_2 \frac{\partial}{\partial x_3}$$

$$-x^3 \frac{\partial}{\partial y_1} - x^1 x^3 \frac{\partial}{\partial x^3} + (x^1 x^2 - x^3) \frac{\partial}{\partial x^2} - (x^1)^2 \frac{\partial}{\partial x^1}$$

$$L_{E_2} = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}; \quad R_{E_2} = \exp(-y_1 + 2y_2) \frac{\partial}{\partial x_2} - x^2 \frac{\partial}{\partial y_2} - (x^2)^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^1}$$

$$L_{H_1} = -2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1};$$

$$R_{H_1} = \frac{\partial}{\partial y_1} + x^3 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^2} + 2x^1 \frac{\partial}{\partial x^1}$$

$$L_{H_2} = x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_2}; \quad R_{H_2} = \frac{\partial}{\partial y_2} + x^3 \frac{\partial}{\partial x^3} + 2x^2 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^1}$$

$$L_{F_2} = x_3 \frac{\partial}{\partial x_1} - (x_2)^2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_1} + \exp(-y_1 + 2y_2) \frac{\partial}{\partial x_2}; \quad R_{F_2} = x^1 \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^2}.$$
\[ L_{F_1} = - \left( x_1 \right)^2 \frac{\partial}{\partial x_1} + (x_1 x_2 - x_3) \frac{\partial}{\partial x_2} - x_1 x_3 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial y_1} \]

\[ + \exp(2y_1 - y_2)x^2 \frac{\partial}{\partial x^2} + \exp(2y_1 - y_2) \frac{\partial}{\partial x^1}; \quad R_{F_1} = \frac{\partial}{\partial x^1} \]

The formulas (3), (4) give embeddings of the Lie algebras \( sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \) and \( sl(3, \mathbb{C}) \oplus sl(3, \mathbb{C}) \) into the Lie algebras of vector fields on \( \mathbb{S}l(2, \mathbb{C}) \) and \( \mathbb{S}l(3, \mathbb{C}) \), and equip the spaces of the distributions \( D(\mathbb{S}l(2, \mathbb{C}), Sl(2, \mathbb{C})) \) and \( D(\mathbb{S}l(3, \mathbb{C}), Sl(3, \mathbb{C})) \) with structures of modules over those Lie algebras. The embeddings (3), (4) will be called the regular representations. Regular representations of other finite dimensional Lie algebras may be derived in a similar manner.

§3. Regular representations of \( \hat{sl}(2, \mathbb{C}) \) and \( \hat{sl}(3, \mathbb{C}) \) affine Kac-Moody algebras

In this section we state affine analogues of the formulas (3), (4). Let us consider the simplest case of \( \hat{sl}(2, \mathbb{C}) \) algebra. Let \( a_1(z), \hat{a}_1(z), a^1(z), \hat{a}^1(z), b(z), \hat{b}(z) \) be three conjugate pairs of bosonic fields of spin \( (1,0) \) with the usual operator expansions:

\[
a_1(z)\hat{a}_1(w) = a^1(z)\hat{a}^1(w) = b(z)\hat{b}(w) = (z - w)^{-1} + \cdots \\
a_1(z) = \sum_{n \in \mathbb{Z}} a_1(n) z^{-n-1}; \quad a^1(z) = \sum_{n \in \mathbb{Z}} a^1(n) z^{-n-1}; \quad b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n-1} \quad (5) \\
\hat{a}_1(z) = \sum_{n \in \mathbb{Z}} a_1(n) z^{-n}; \quad \hat{a}^1(z) = \sum_{n \in \mathbb{Z}} a^1(n) z^{-n}; \quad \hat{b}(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n} \\
\]

The fields \( a^1, a_1, b \) are the loop algebra versions of the operators \( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y} \) and the fields \( \hat{a}^1, \hat{a}_1, \hat{b} \) are the loop algebra versions of the operators \( x^1, x_1, y \). Let \( \Gamma \) be the Heisenberg algebra, generated by \( a_1, \hat{a}_1, a^1, \hat{a}^1, b, \hat{b} \) and \( M \) be the irreducible representation of \( \Gamma \) with the vacuum vector, annihilated by \( \hat{a}_1(n), \hat{a}^1(n), \hat{b}(n), n > 0 \), and by \( a_1(n), a^1(n), b(n), n \geq 0 \). \( M \) can be identified with some space of distributions on the manifold \( \mathbb{L}Sl(2, \mathbb{C}) \) of loops on \( \mathbb{S}l(2, \mathbb{C}) \). The regular representation of \( \hat{sl}(2, \mathbb{C}) \oplus \hat{sl}(2, \mathbb{C}) \) is given by the formulas:

\[
L_{E} = a_1; \quad R_{E} = \exp(2b_\hat{a})a_1 - \hat{a}^1 b - : (\hat{a}^1)^2 a^1 : - (k + 4) \partial \hat{a}^1 + (k + 2) \hat{a} \partial b \\
L_{H} = -2 : \hat{a}_1 a_1 : - b - (k + 2) \partial \hat{b}; \quad R_{H} = b + 2 : \hat{a}^1 a^1 : - (k + 2) \partial \hat{b} \quad (6) \\
L_{F} = - : (\hat{a}_1)^2 a_1 : + b + \exp(2b) a^1 + k \partial a_1 - (k + 2) a_1 \partial b; \quad R_{F} = a^1 
\]
The regular representation of \( \widehat{sl(3, \mathbb{C})} \) algebra is given in a similar way. Let \( a_i(z), \dot{a}_i(z), a^i(z), \dot{a}^i(z), i = 1, 2, 3, b_i(z), \dot{b}_i(z), i = 1, 2 \) be a set of spin \((1,0)\) conjugate pairs of bosonic fields with operator expansions:

\[
\begin{align*}
a_i(z)a_j(w) &= a^i(z)\dot{a}^j(w) = (z - w)^{-1}\delta_{ij} + \cdots, i, j = 1, 2, 3 \\
\dot{b}_i(z)b_j(w) &= (z - w)^{-1}\delta_{ij} + \cdots, i, j = 1, 2
\end{align*}
\]  

Then

\[
L_{E_1} = a_1
\]

\[
R_{E_1} = (a_1 + \dot{a}_2a_3)\exp(2\dot{b}_1 - \dot{b}_2) - \dot{a}^1 b_1 - : \dot{a}^1 \dot{a}^3 : + : (\dot{a}^1 \dot{a}^2 - \dot{a}^3) a^2 : - : (\dot{a}^1) a^1 : - \dot{a}^1 (-k + 3)\partial b_1 + \alpha_1 \partial b_1) - (k + 6)\partial \dot{a}^1
\]

\[
L_{E_2} = a_2 + \dot{a}_1 a_3
\]

\[
R_{E_2} = a_2 \exp(-\dot{b}_1 + 2\dot{b}_2) - \dot{a}^2 b_2 - : (\dot{a}^2) a^2 : + \dot{a}^3 a^1 - \dot{a}^2 (\alpha_2 \partial b_1 - (k + 3)\partial b_2) - (k + 5)\partial \dot{a}^2
\]

\[
L_{H_1} = -2 : \dot{a}_1 a_1 : + : \dot{a}_2 a_2 : - : \dot{a}_3 a_3 : - b_1 - (k + 3)\partial b_1 + \alpha_2 \partial b_2
\]

\[
R_{H_1} = b_1 + : \dot{a}^3 a^3 : - : \dot{a}^2 a^2 : + 2 : \dot{a}^1 a^1 : - (k + 3)\partial b_1 + \alpha_1 \partial b_1
\]

\[
L_{H_2} = : \dot{a}_1 a_1 : - 2 : \dot{a}_2 a_2 : - : \dot{a}_3 a_3 : - b_2 + \alpha_1 \dot{b}_1 + (k + 3)\partial b_2
\]

\[
R_{H_2} = b_2 + : \dot{a}^3 a^3 : + 2 : \dot{a}^2 a^2 : + : \dot{a}^1 a^1 : + \alpha_2 \partial b_1 - (k + 3)\partial b_1
\]

\[
L_{F_2} = \dot{a}_3 a_1 - : (\dot{a}_2)^2 a_2 : - \dot{a}_2 b_2 + \exp(-\dot{b}_1 + 2\dot{b}_2)a^2
\]

\[
\quad + \dot{a}_2(\alpha_1 \partial b_1 - (k + 3)\partial b_2) + (k + 1)\partial \dot{a}_2
\]

\[
R_{F_2} = \dot{a}^1 a^3 + a^2
\]

\[
L_{F_1} = - : (\dot{a}_1)^2 a_1 : + : (\dot{a}_1 \dot{a}_2 - \dot{a}_3) a_2 : - \dot{a}_1 : \dot{a}_3 a_3 : - \dot{a}_1 b_1
\]

\[
\quad + (a^1 + \dot{a}^2) a^3) \exp(2b_1 - b_2) + \dot{a}_1(- k + 3)\partial b_1 + \alpha_2 \partial b_2) + k\partial \dot{a}_1
\]

\[
R_{F_1} = a^1,
\]

where \( \alpha_1, \alpha_2 \) are complex numbers such that

\[
\alpha_1 + \alpha_2 = k + 3
\]
In contrast to the $sl(2, \mathbb{C})$ case, formulas (8) depend on one arbitrary value $\alpha = \alpha_1 - \alpha_2$. But there is the change of variables $b_1(z), b_2(z)$ preserving the relations (7) and eliminating the dependence on $\alpha$ in (8):

\[
\begin{align*}
    b_1(z) &\to b_1(z) - \alpha_2 \partial^{+} b_2(z), \\
    b_2(z) &\to b_2(z) + \alpha_2 \partial^{+} b_1(z).
\end{align*}
\]

In conclusion of this section we introduce another form of representations $\hat{sl}(2, \mathbb{C})$ and $\hat{sl}(3, \mathbb{C})$ algebras which seems important in connection with Wakimoto representations. Let us introduce a free bosonic fields by equations:

\[
\begin{align*}
    \partial \rho &= \frac{i}{\sqrt{k + 2}}((k + 2)\partial^{+} b - b); \quad \partial \lambda = \frac{i}{\sqrt{k + 2}}((k + 2)\partial^{+} b + b) \\
    \partial \rho_1 &= \frac{i}{\sqrt{k + 3}}((k + 3)\partial^{+} b_1 - \alpha_1 \partial^{+} b_2 - b_1); \\
    \partial \lambda_1 &= \frac{i}{\sqrt{k + 3}}((k + 3)\partial^{+} b_1 - \alpha_2 \partial^{+} b_2 + b_1) \\
    \partial \rho_2 &= \frac{i}{\sqrt{k + 3}}((k + 3)\partial^{+} b_2 - \alpha_2 \partial^{+} b_1 - b_2); \\
    \partial \lambda_2 &= \frac{i}{\sqrt{k + 3}}((k + 3)\partial^{+} b_2 - \alpha_1 \partial^{+} b_1 + b_2).
\end{align*}
\]

The operator product expansions of fields (11), (12) are given by:

\[
\begin{align*}
    \rho(z)\rho(w) &= -\lambda(z)\lambda(w) = 2\ln(z - w) + \cdots, \\
    \rho_i(z)\rho_j(w) &= -\lambda_i(z)\lambda_j(w) = K_{ij}\ln(z - w) + \cdots,
\end{align*}
\]

where $K_{ij}$ is the Cartan matrix of $sl(3, \mathbb{C})$. Then $\hat{sl}(2, \mathbb{C}) \oplus \hat{sl}(2, \mathbb{C})$ currents are given by

\[
\begin{align*}
    L_E &= a_1; \quad R_E = - : (\hat{a}^+ )^2 a_1 : - (k + 2) \partial \hat{a}^+ - i\sqrt{k + 2} \partial \rho + \exp[- \frac{i}{\sqrt{k + 2}}(\rho + \lambda)]a_1 \\
    L_H &= - 2 : \hat{a}^+ a_1 : + i\sqrt{k + 2} \partial \lambda; \quad R_H = 2 : \hat{a}^+ a_1 : + i\sqrt{k + 2} \partial \rho \\
    L_F &= - : (\hat{a}_1^+ )^2 a_1 : + k \partial \hat{a}_1^+ + i\sqrt{k + 2} \partial \lambda + \exp[- \frac{i}{\sqrt{k + 2}}(\rho + \lambda)]a_1; \quad R_F = a_1^+
\end{align*}
\]

**Remark.** We see that these formulas are very close to the standard Wakimoto formulas. Let us consider the Heisenberg algebra $\tilde{\Gamma}$ with generators $a_1(n), \hat{a}_1(n), a_1(n)$, $\hat{a}_1(n), \rho(n) =$
\[ \oint_0 dzz^n i \partial \rho(z), \lambda(n) = \oint_0 dzz^n i \partial \lambda(z). \]

Let \( F_{(l,r)} \) be irreducible representation of \( \hat{\Gamma} \) with vacuum vector \( \vartheta_{l,r} \) annihilated by \( \hat{a}^1(n), \hat{a}^1(n), \rho(n), \lambda(n), n > 0 \) and by \( a^1(n), a_1(n), n \geq 0 \), such that

\[
\rho(0) \vartheta_{l,r} = \frac{2r}{\sqrt{k} + 2} \vartheta_{l,r}; \quad \lambda(0) = \frac{2l}{\sqrt{k} + 2} \vartheta_{l,r}
\] (16)

The action of the generators \( R_E(n) = \oint dzz^n R_E(z), L_F(n) = \oint dzz^n L_F(z) \) on \( \vartheta_{l,r} \) is defined if

\[
2(r + l) = N(k + 2), \quad \text{where} \quad N \in \mathbb{Z}
\] (17)

In this case the generators \( R_E(n), L_F(n) \) acts from \( F_{(l,r)} \) to another \( F_{(l',r')} \), such that \( l' + r' = l + r \). It is natural to consider the direct sum of Fock modules

\[
M_N = \bigoplus_{2(r+l) = N(k+2)} F_{(r,l)}
\] (18)

as a representation of \( \widehat{sl(2,\mathbb{C})} \oplus \widehat{sl(2,\mathbb{C})} \). Let \( \mathfrak{g}_b \) be subalgebra in the left \( \widehat{sl(2,\mathbb{C})} \) which consists in \( \{ L_E(n), n \in \mathbb{Z}, L_H(n), n > 0 \} \). The algebra \( \mathfrak{g}_b \) has a natural decomposition \( \mathfrak{g}_b = \mathfrak{g}_b^+ \oplus \mathfrak{g}_b^- \), \( \mathfrak{g}_b^+ = \{ L_E(n), n > 0, L_H(n), n > 0 \} \), \( \mathfrak{g}_b^- = \{ L_E(n), n \leq 0 \} \). This decomposition gives us possibility to define the semi-infinite cohomology of \( \mathfrak{g}_b \) with coefficients in \( M_N \) [6].

On the cohomology the right \( \widehat{sl(2,\mathbb{C})} \) is acting and this is given exactly by the Wakimoto formulas. Remainder of the left action is the screening operator.

The \( \widehat{sl(3,\mathbb{C})} \oplus \widehat{sl(3,\mathbb{C})} \) currents are given by

\[
L_{E_1} = a_1
\]
\[
R_{E_1} = - : \hat{a}^1 \hat{a}^1 + \hat{a}^2 \hat{a}^2 + \hat{a}^3 \hat{a}^3 : - \hat{a}^3 \hat{a}^2 - i \sqrt{k + 3} \hat{a}^1 \partial \rho_1
\]
\[\quad - (k + 6) \partial \hat{a}^1 + \exp \left[ - \frac{i}{\sqrt{k + 3}} (\rho_1 + \lambda_1) \right] (a_1 + \hat{a}_2 a_3) \]
\[
L_{E_2} = a_2 + \hat{a} a_3
\]
\[
R_{E_2} = - : \hat{a}^2 \hat{a}^2 : + \hat{a}^3 \hat{a}^1 - i \sqrt{k + 3} \hat{a}^2 \partial \rho_2 - (k + 5) \partial \hat{a}^2 + \exp \left[ - \frac{i}{\sqrt{k + 3}} (\rho_2 + \lambda_2) \right] a_2
\]
\[
L_{H_1} = - 2 : \hat{a}_1 a_1 : + : \hat{a}_2 a_2 : - : \hat{a}_3 a_3 : + i \sqrt{k + 3} \partial \lambda_1
\]
\[ R_{H_1} = 2 : \frac{1}{a} a^1 : - : \frac{i}{a} a^2 : + : \frac{1}{a} a^3 : + : i \sqrt{k+3} \partial \rho_1 \]  
(19)

\[ L_{H_2} = \frac{1}{a} a_1 : -2 : \frac{1}{a} a_2 : - : \frac{1}{a} a_3 : +i \sqrt{k+3} \partial \lambda_2 \]

\[ R_{H_2} = - : \frac{1}{a} a_1 : +2 : \frac{1}{a} a_2 : + : \frac{1}{a} a_3 : +i \sqrt{k+3} \partial \rho_2 \]

\[ L_{F_2} = - : (\frac{1}{a} a_2 )^2 a_2 : +a_3 a_1 + i \sqrt{k+3} a_2 \partial \lambda_2 + (k+1) \partial a_2 + \exp \left[ -\frac{i}{\sqrt{k+3}}(\rho_2 + \lambda_2) \right] a^2 \]

\[ R_{F_2} = a^2 + a^3 \]

\[ L_{F_1} = - : a_1 (a_1 a_1 - a_2 a_2 + a_3 a_3) : -a_3 a_2 + i \sqrt{k+3} a_1 \partial \lambda_1 \]

\[ + k \partial a_1 + \exp \left[ -\frac{i}{\sqrt{k+3}}(\rho_1 + \lambda_1) \right] (a^1 + a^2 a^3) \]

\[ R_{F_1} = a^1. \]

§4. Regular representations of \( \hat{sl}(n+1) \) Kac-Moody algebras

The generalization of (15), (19) for \( sl(n+1, \mathbb{C}) \) algebras is immediate. Let \( \alpha_1, \cdots, \alpha_n \) be the set of simple roots of \( sl(n+1, \mathbb{C}) \). Denote

\[ \overset{\dagger}{a}_{ij}(z) = \overset{\dagger}{a}_{(\alpha_i, \cdots, \alpha_j)}(z), \quad a_{ij}(z) = a_{(\alpha_i, \cdots, \alpha_j)}(z), \quad l \leq i \leq j \leq n \]

\[ \overset{\dagger}{a}(z) = \overset{\dagger}{a}_{-(\alpha_i, \cdots, \alpha_j)}(z), \quad a^j(z) = a_{(\alpha_i, \cdots, \alpha_j)}(z), \quad 1 \leq i \leq j \leq n \]  
(20)

and put:

\[ a_{ij}(z) \overset{\dagger}{a}^{nm}(w) = a^{ij}(z) \overset{\dagger}{a}^{nm} (w) = (z - w)^{-1} \delta_{in} \delta_{jm} + \cdots \]  
(21)

Introduce the set of free bosonic fields \( \lambda_i(z), \rho_i(z), i = 1, \cdots, n \) and put:

\[ -\lambda_i(z) \lambda_j(w) = \rho_i(z) \rho_j(w) = \ln(z - w) K_{ij}, \]  
(22)

where \( K_{ij} \) is Cartan matrix of \( sl(n+1) \). Denote \( \nu^2 = k + n + 1 \). Then

\[ L_{E_{n+1-i}} = a_{n+1-i, n+1-i} + \sum_{j=i+1}^{n} a_{n+1-j, n-i} a_{n+1-j, n+1-i} \]

\[ R_{E_{n+1-i}} = \sum_{j=1}^{i+j-1} a_{n+1-j, n+1-j} \left( \sum_{j=1}^{n+2-i, n+1-j} a^{n+2-i, n+1-j} \right) \]

\[ a_{n+1-i, n+1-j} a^{n+1-i, n+1-j} \]
\[-i\nu \dot{a} + n+1-i, n+1-i + \sum_{j=i+1}^{n} a^+_{n+1-i, n+1-j} a^{n+1-j, n-i}\]

\[-\sum_{j=1}^{i-1} a^+_{n+1-i, n+1-j} a^{n+2-i, n+1-j} : -(\nu^2 + 1 + i)\dot{a}\]

\[+ \exp\left(-\frac{i}{\nu}(\rho + \lambda)_{n+1-i}\right) \left(\sum_{j=1}^{i-1} a^+_{n+2-i, n+1-j} a^{n+1-i, n+1-j}\right)\]

\[L_{H_{n+1-i}} = -2 : \dot{a}_{n+1-i, n+1-i} a_{n+1-i, n+1-i} :\]

\[-\sum_{j=1}^{i-1} (a^+_{n+1-i, n+1-j} a^{n+1-i, n+1-j} - a_{n+1-j, n+1+i} a^{n+1-i, n+1-j} ) :\]

\[+ \dot{\lambda}_{n+1-i} \]

\[R_{H_{n+1-i}} = 2 : \dot{a} a_{n+1-i, n+1-i} a^{n+1-i, n+1-i} :\]

\[+ \sum_{j=1}^{i-1} (a^+ a^{n+1-i, n+1-j} - a^+ a^{n+1-j, n+1-i} a^{n+1-j, n+1-i} ) :\]

\[-\sum_{j=1}^{i-1} (a^+ a^{n+1-j, n+1-i} - a^+ a^{n+1-j, n+1-i} a^{n+1-j, n+1-i} ) :\]

\[+ i\nu \dot{\rho}_{n+1-i} \]

\[L_{F_{n+1-i}} = : \dot{a}_{n+1-i, n+1-i} a^{n+1-i, n+1-i} (\sum_{j=1}^{i-1} a^+ a^+_{n+2-i, n+1-j} a^{n+2-i, n+1-j} ) :\]

\[-\sum_{j=1}^{i-1} a^+_{n+1-i, n+1-j} a^+ a^{n+1-j, n+1-i} : i\nu \dot{a}_{n+1-i, n+1-i} \dot{\lambda}_{n+1-i}\]

\[+ \sum_{j=i+1}^{n} a^+_{n+1-j, n+1-i} a^+ a^{n+1-j, n+1-i} :\]

\[-\sum_{j=1}^{i-1} a^+ a^{n+2-j, n+1-j} a^{n+1-i, n+1-i} : + (\nu^2 - 1 - i) \dot{a}^+ a^{n+1-i, n+1-i} :\]

\[+ \exp\left(-\frac{i}{\nu}(\rho + \lambda)_{n+1-i}\right) \left(\sum_{j=1}^{i-1} a^+ a^{n+1-j, n+1-i} a^{n+1-j, n+1-i} :\right)\]

\[R_{F_{n+1-i}} = a^{n+1-i, n+1-i} + \sum_{j=i+1}^{n} a^+ a^{n+1-j, n+1-i} a^{n+1-j, n+1-i} :\]

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determine the structure of regular representation of the $\widehat{sl(n+1, \mathbb{C})} \oplus \widehat{sl(n+1, \mathbb{C})}$ affine Kac-Moody algebra.

Note that the structure of these formulas is the following. We have two copies of free fields: \{\(a_{ij}, \dot{a}_{ij}, \lambda_i\) and \(a^{ij}, \dot{a}^{ij}, \rho_i\). Then we write down Wakimoto formulas for the action of left and right $\widehat{sl(n+1, \mathbb{C})}$ in terms of these free fields. The next step - we add to the action of $F_i$ from the left algebra the “screening” currents for the right algebra. And we also add left “screening” currents to the action of $E_i$ from right algebra. The similar procedure can be done for arbitrary semi-simple Lie algebra.

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