Perturbative Euler-Heisenberg Lagrangian in a parity-violating Abelian gauge theory

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Abstract

In this paper, we examine perturbatively the behavior of the Euler-Heisenberg effective action in the presence of a novel axial coupling among the gauge field and the fermionic matter. This axial coupling is responsible to induce a parity-violating term in the extended form of the Euler-Heisenberg effective action. In order to perform our analysis, we make use of a parametrization of the vector and axial coupling constants, $g_v$ and $g_a$, in terms of a new coupling $\beta$. This parametrization allows us to explore a hidden symmetry in the effective Lagrangian under $g_v \leftrightarrow g_a$, which is verified for a class of diagrams, in particular with $n = 4$ photon external legs. This symmetry is explicitly observed in the loop calculations, where we determine the $\lambda_i$ coefficients of $L_{EH}^{ext} = \lambda_1 F^2 + \lambda_2 G^2 + \lambda_3 F G$, in particular the coefficient $\lambda_3$ related with the parity violation due to the axial coupling. As a phenomenological application of the results, we compute the relevant cross section for the light by light scattering through the extended Euler-Heisenberg effective action.

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1 Introduction

Perhaps one of the most amazing examples of the distinct behavior of a classical and quantum theory is the electrodynamics. In one hand, we have Maxwell’s electrodynamics with its linear field equations and phenomenological predictions, such as the electromagnetic waves, serving as a keystone for the proposal of new theories, for instance Einstein’s special relativity. On the other hand, we have an important class of electrodynamics named generically as nonlinear electrodynamics (NLED), due to the presence of nonlinear terms in the field equations by virtue of quantum effects in the context of (quantum) effective field theories [1], resulting in the phenomena of (quantum) self-coupling of electromagnetic waves in the vacuum.

Almost 80 years have passed since the early proposals of the light-by-light scattering in QED [2–6], until the conceptual proposal of light-by-light scattering in ultraperipheral heavy-ion collisions [8] and its experimental verification by ATLAS Collaboration [9, 10]. Naturally, phenomenological analysis used these data to restrict a region of the parameter space constraining nonlinear corrections [11, 12].

Interestingly, such is the importance of this kind of experimental verification that the light-by-light scattering is now serving as a laboratory to investigate physics beyond the standard model scenarios, which predicts new particles that couple predominantly to photons, for instance, the search for axion-like particles [13, 14]. Other interests in nonlinear corrections to Maxwell’s electrodynamics worth mentioning are low-energy experiments, such as PVLAS [15] and BMV [16], built to detect the presence of vacuum magnetic birefringence.

On the theoretical side, we have seen in recent years great interest in the study of further nonlinear phenomena involving physics of beyond the standard model of gauge bosons, exploring new possibilities and novel (quantum) behavior of the electromagnetic waves [17–25]. Within this context, one could naturally wish to consider some generalization of these previous studies and consider the implications of parity violating effects into the photon dynamics.

It is well known that parity breaking is a richer scenario to introduce important phenomena, even inducing massive modes, regarding the behavior of the electromagnetic field in three dimensions [26]. For instance, one can cite to the quantum Hall effect as an important physical application where the topological effects of electromagnetism are the framework used to describe this physical phenomena [27]. Hence, one should expect that the violation of parity would be as interesting to the nonlinear corrections to the electrodynamics in four-dimensions.

In the light of considering parity-violating effects, we can draw some considerations: if the vacuum is invariant by C, P, and T transformations, the $\alpha^2$-order nonlinear corrections are described in terms of the Lorentz and gauge invariant quantities $F^2$ and $G^2$ to the Maxwell’s Lagrangian [see Eq.(3.2) for definitions]. However, if we allow CP violation, or simply parity violation, the term $FG$ must be also added. Hence, our main interest in this paper is to determine the coefficients of the Euler-Heisenberg effective action in the presence of parity violation, in summary determine the $\lambda_i$ coefficients of $L_{\text{ext.}}^{\text{EH}} = \lambda_1 F^2 + \lambda_2 G^2 + \lambda_3 FG$.

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1 Or even the Born and Infeld proposal of the presence of an upper limit on the strength of the electromagnetic field by means of a different nonlinear modification of the Lagrangian of QED [7].

2 Where $\alpha = e^2/4\pi$ is the fine-structure constant in natural units.
We start Sec. 2 by establishing the main aspects of the considered model, with a gauge field that possesses both vector and axial couplings, and its related symmetries. Furthermore, we present some definitions regarding the effective action formalism. In Sec. 3, we discuss some details involving the (low-energy) Euler-Heisenberg (EH) Lagrangian. In particular, we focus in developing the four-photon scattering matrix, and how to consider the different parity preserving and parity violating contributions to the amplitude. Our main analysis is presented in Sec. 4, where the evaluation of the lowest-order contribution to the box diagram (with four photon legs) is fully considered. It is important to emphasize that, since this amplitude needs to be regularized, we follow the 't Hooft-Veltman rule to perform algebraic manipulations with $\gamma_5$ within the dimensional regularization method. Moreover, in Sec. 5 we analyze our results related with the $\lambda_i$ coefficients of the (parity-violating) Euler-Heisenberg effective action and discuss some particular issues. In Sec. 6, as a phenomenological application of our results, we compute the relevant cross section for the light by light scattering through the extended Euler-Heisenberg effective action. Finally, we present our conclusions and final remarks in Sec. 7.

2 The model and main features

In this section, we introduce the model and fix our notation. The parity-violating extension of the QED minimal coupling for the Dirac fermions in the presence of an external gauge field is given by the following action

$$S_\psi = \int d^4x \bar{\psi}(x) \left[ \gamma^\mu (i\partial_\mu - (g_v + g_a \gamma_5)A_\mu) - m \right] \psi(x),$$

(2.1)

where the interacting Lagrangian density reads

$$\mathcal{L}_{int} = -\bar{\psi}\gamma^\mu(g_v + g_a \gamma_5)A_\mu \psi,$$

(2.2)

where $g_v$ and $g_a$ refer to the coupling of the external gauge field to the vector and axial vector current, respectively. We observe that the axial part of the Lagrangian (2.2) violates the parity and charge conjugation symmetry. Hence, unlike the usual QED, this model does not respect the parity (P) and charge (C) conjugation symmetry. An important consequence of the C-violation is that the Furry’s theorem is not satisfied by this model, as we expect.

In order to perform our perturbative analysis, we introduce a parametrization to consider the of the parity-conserving (vector) and the parity-violating (axial-vector) coupling constants in a simpler fashion

$$g_v + g_a \gamma_5 = \beta e^{i\gamma_5},$$

(2.3)

so that it is readily obtained

$$g_v = \beta \cosh \alpha, \quad g_a = \beta \sinh \alpha, \quad \beta^2 = g_v^2 - g_a^2.$$  

(2.4)

It is important to remark that in this new notation, the presence of a new symmetry $g_v \leftrightarrow g_a$ is verified in the $n$-point function of the gauge field, see comment below.
In terms of this parametrization, the Lagrangian density (2.1) is rewritten as
\[ \mathcal{L}_\psi = \bar{\psi}(x) \left[ \gamma^\mu (i\partial_\mu - \beta e^{\alpha\gamma_5} A_\mu) - m \right] \psi(x). \] (2.5)

Hence, the Feynman rule for the fermion-photon interaction is simply given by \(-i\beta \gamma^\mu e^{\alpha\gamma_5}\), having the same structure as the usual QED with the additional factor \(e^{\alpha\gamma_5}\).

In regard to the symmetries of the Lagrangian (2.5), we observe that, with massless fermions, it is invariant under the following gauge transformation
\[ \psi(x) \rightarrow e^{i\beta e^{\alpha\gamma_5} \theta(x)} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{i\beta e^{\alpha\gamma_5} \theta(x)}, \quad A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \theta(x). \] (2.6)

Moreover, we notice that in the case of \(g_a \rightarrow 0\) and \(g_v \rightarrow 0\), the usual vector gauge transformation, \(U_V(1)\), and the axial gauge transformation, \(U_A(1)\), are restored, respectively.

The one-loop effective action \(\Gamma\) for the gauge field \(A_\mu\) is defined as follows
\[ e^{i\Gamma[A]} = \int D\bar{\psi} D\psi e^{iS_\psi}. \] (2.7)

Applying the relevant techniques in the path-integral formalism, we find
\[ \Gamma[A] = -i \text{Tr} \ln \left( i\partial - \beta e^{\alpha\gamma_5} A - m \right), \] (2.8)
or equivalently in a convenient form for the perturbative analysis
\[ \Gamma[A] = \sum_{n=1}^{\infty} \int d^4x_1 \ldots \int d^4x_n \Gamma^{\mu_1 \ldots \mu_n}(x_1, \ldots, x_n) A_{\mu_1}(x_1) \ldots A_{\mu_n}(x_n). \] (2.9)

Here, \(\Gamma^{\mu_1 \ldots \mu_n}\) is the \(n\)-point function of the gauge field which is defined as
\[ \Gamma^{\mu_1 \ldots \mu_n}(x_1, \ldots, x_n) = -\frac{\beta^n}{n} \int \prod_{i=1}^{n} \frac{d^4p_i}{(2\pi)^4} \delta \left( \sum_{i=1}^{n} p_i \right) e^{i \sum_{i=1}^{n} p_i \cdot x_i} \Xi^{\mu_1 \ldots \mu_n}(p_1, \ldots, p_n), \] (2.10)
where the overall minus sign comes from the fermionic loop and \(\Xi^{\mu_1 \ldots \mu_n}\) indicates the amplitude of a graph with \(n\) external photon legs.

As a final remark, we realize from eq. (2.10) that \(\Gamma^{\mu_1 \ldots \mu_n}\) is proportional to the following power of the new (parametrized) coupling constant \(\beta^n = (g_v^2 - g_a^2)^n\). However, the dependence of \(\Gamma^{\mu_1 \ldots \mu_n}\) on the coupling constants \(g_v\) and \(g_a\) does not arise only from \(\beta^n\) since we have a factor \(e^{\alpha\gamma_5}\) for each vertex, included in \(\Xi^{\mu_1 \ldots \mu_n}\), that depends on \(g_v\) and \(g_a\). As we shall see in Sec. 4, in the case of interest \(n = 4\), we have the terms proportional to \(\beta^4 e^{4\alpha\gamma_5}\) and \(\beta^4 e^{2\alpha\gamma_5}\) within eq. (2.10) as
\[ \beta^4 e^{4\alpha\gamma_5} = (g_v^4 + g_a^4 + 6g_v^2 g_a^2) + 4(g_v^3 g_a + g_v g_a^3)\gamma^5, \] (2.11)
\[ \beta^4 e^{2\alpha\gamma_5} = (g_v^4 - g_a^4) + 2(g_v^3 g_a - g_v g_a^3)\gamma^5. \] (2.12)

We notice that \(\beta^4 e^{4\alpha\gamma_5}\) and \(\beta^4 e^{2\alpha\gamma_5}\) are completely symmetric and anti-symmetric under the exchange of \(g_v \leftrightarrow g_a\), respectively. Nevertheless, only the terms involving the four external momenta with \(\beta^4 e^{4\alpha\gamma_5}\) will contribute to the extended Euler-Heisenberg Lagrangian in Sec. 4; although finite, the remaining parts are not relevant to the Euler-Heisenberg one. Hence, this result allows us to conclude that the effective Lagrangian is symmetric under \(g_v \leftrightarrow g_a\).
3 Construction of the extended Euler-Heisenberg Lagrangian

In order to construct the one-loop effective action for the description of the light by light scattering, some comments are in order. There are strong constraints such as gauge and Lorentz invariance that should be considered in this construction. Accordingly, a gauge invariant expression to describe four-photon interaction can be built from the field strength tensor $F_{\mu \nu}$ with proper contractions, leading to a Lorentz scalar Lagrangian. Moreover, since our interaction term in (2.2) violates parity, we have an additional term in the extended E-H Lagrangian, which is absent in its usual version.

Based on these comments, there are three kinds of terms with mass dimension 8 that respect both gauge and Lorentz invariance, as well as the parity violation, as below

\[ \mathcal{F}^2, \quad \mathcal{G}^2, \quad \mathcal{FG}, \]  

with the definitions

\[ \mathcal{F} = F_{\mu \nu}F^{\mu \nu} = -2(E^2 - B^2), \]
\[ \mathcal{G} = G_{\mu \nu}F^{\mu \nu} = 4(E.B), \]

and $G_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \rho \sigma \nu} F^{\rho \sigma}$ is the dual of the field strength tensor $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Taking into account these considerations, the full structure of the extended E-H Lagrangian can be expressed as the following

\[ \mathcal{L}_{\text{E-H}}^{\text{ext}} = \lambda_1 \mathcal{F}^2 + \lambda_2 \mathcal{G}^2 + \lambda_3 \mathcal{FG}, \]

where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the effective coupling constants with mass dimension $-4$ in a four dimensional space-time. Under parity, the quantities $\mathcal{F}$ and $\mathcal{G}$ transform as scalar and pseudo-scalar, respectively. Hence, $\mathcal{F}^2$ and $\mathcal{G}^2$ are parity-even, whereas $\mathcal{FG}$ is parity-odd. This parity-odd term is not present in the ordinary E-H action. Furthermore, since under charge conjugation the field strength changes as $F_{\mu \nu} \to -F_{\mu \nu}$, the full Lagrangian (3.3) is charge-conjugation invariant.

The value of the coupling constants $\lambda_i$ is determined from the one-loop quantum corrections, including the effects of the generalized interaction (2.2). Actually, our main goal is to determine the expressions of these effective couplings through a perturbative analysis.

To achieve this aim, we determine the total amplitude of the four-photon scattering as a function of the three coupling constants $\lambda_1$, $\lambda_2$ and $\lambda_3$. First, we will establish the tensor structure of the total amplitude of the process $\gamma \gamma \to \gamma \gamma$ through the Lagrangian (3.3); second, we will evaluate the low-energy limit ($p^2 \ll m^2$) of the Feynman amplitudes related with the process $\gamma \gamma \to \gamma \gamma$ by considering the interaction term (2.2). At last, we will compare and match the obtained results in both methods, allowing us to fix the values of the coupling constants $\lambda_i$.

3.1 Tensor structure of the total amplitude

Let us to decompose the full effective Lagrangian (3.3) into three pieces $\mathcal{L}_{\text{ext}}^{\text{E-H}} = \sum_{i=1}^{3} \mathcal{L}_i$, where we have defined

\[ \mathcal{L}_1 = \lambda_1 \mathcal{F}^2, \quad \mathcal{L}_2 = \lambda_2 \mathcal{G}^2, \quad \mathcal{L}_3 = \lambda_3 \mathcal{FG}. \]
The contribution of the parity-conserving (P.C) parts, \( \mathcal{L}_1 = \lambda_1 \mathcal{F}^2 \) and \( \mathcal{L}_2 = \lambda_2 \mathcal{G}^2 \), in the amplitude for four-photon scattering is given by [20]

\[
\mathcal{M}_1 + \mathcal{M}_2 = (\Xi_{(1)}^{\mu\nu\rho\sigma} + \Xi_{(2)}^{\mu\nu\rho\sigma}) (p_1, p_2, p_3, p_4) \varepsilon_{1\mu}\varepsilon_{3\nu}\varepsilon_{4\rho}\varepsilon_{2\sigma}.
\]

where \( \varepsilon_i \equiv \varepsilon(p_i) \) are the polarization vector corresponding to the \( p_i \) photon four-momenta. In addition, in order to obtain the correct amplitude, we shall sum over all simultaneous permutations of \( (p_1, p_2, p_3, p_4) \) and \( (\mu, \sigma, \nu, \rho) \). As a matter of fact, this corresponds to 24 permutations of the external photon legs which are included in the amplitudes as

\[
\Xi_{(1)}^{\mu\nu\rho\sigma} = \sum_{i=1}^{24} \Xi^{\mu\nu\rho\sigma}_{(1,i)} (p_1, p_2, p_3, p_4), \quad \Xi_{(2)}^{\mu\nu\rho\sigma} = \sum_{i=1}^{24} \Xi^{\mu\nu\rho\sigma}_{(2,i)} (p_1, p_2, p_3, p_4),
\]

where the first contributions are straightforwardly obtained [20]

\[
\Xi_{(1,1)}^{\mu\nu\rho\sigma} = 4\lambda_1 \left[ -(p_1 \cdot p_3)(p_2 \cdot p_4)g_{\mu\nu}g_{\rho\sigma} + 2(p_1 \cdot p_3)g_{\mu\nu}p_{4\sigma}p_{2\rho} - p_{1\nu}p_{3\mu}p_{4\sigma}p_{2\rho} \right],
\]

\[
\Xi_{(2,1)}^{\mu\nu\rho\sigma} = 8\lambda_2 \left[ (p_1 \cdot p_3)(p_4 \cdot p_2)p_{\mu\nu}g_{\rho\sigma} + 2(p_1 \cdot p_3)(p_{4\sigma}p_{2\rho} + p_{1\nu}p_{3\mu}p_{4\sigma}p_{2\rho}) + (p_1 \cdot p_3)g_{\nu\rho}p_{4\sigma}p_{2\mu}
+ (p_4 \cdot p_2)p_{1\nu}p_{3\rho}g_{\mu\sigma} + (p_3 \cdot p_4)p_{1\nu}p_{2\rho}g_{\mu\sigma} + (p_1 \cdot p_3)p_{4\nu}p_{2\rho}g_{\mu\sigma} - p_{1\nu}p_{3\rho}p_{4\sigma}p_{2\mu}
- (p_1 \cdot p_3)(p_4 \cdot p_2)g_{\mu\nu}g_{\rho\sigma} - (p_1 \cdot p_3)p_{4\nu}p_{2\rho}g_{\mu\sigma} - (p_3 \cdot p_4)p_{1\nu}p_{2\rho}g_{\mu\sigma} \right].
\]

After performing the complete permutations in (3.6) and simplifying it, we find that the complete parity-preserving amplitude reads

\[
\left(\Xi^{\mu\nu\rho\sigma}\right)_{P.C} = 32(\lambda_1 - \lambda_2) \left[ p_1^\mu p_3^\rho p_4^\nu p_2^\sigma + p_1^\mu p_4^\rho p_3^\nu p_2^\sigma - p_1^\nu p_3^\mu p_4^\rho p_2^\sigma + p_1^\nu p_4^\mu p_3^\rho p_2^\sigma + p_1^\rho p_4^\mu p_3^\nu p_2^\sigma + p_1^\rho p_4^\nu p_3^\mu p_2^\sigma \right]
- 32\lambda_2 \left[ p_1^\rho p_3^\nu p_4^\mu p_2^\sigma + p_1^\nu p_3^\mu p_4^\rho p_2^\sigma - p_1^\nu p_3^\mu p_4^\rho p_2^\sigma - p_1^\rho p_3^\nu p_4^\mu p_2^\sigma - p_1^\rho p_3^\nu p_4^\rho p_2^\sigma - p_1^\rho p_3^\nu p_4^\rho p_2^\sigma \right]
\]

(3.9)

where \( \left(\Xi^{\mu\nu\rho\sigma}\right)_{P.C} \equiv \Xi_{(1)}^{\mu\nu\rho\sigma} + \Xi_{(2)}^{\mu\nu\rho\sigma} \). It is worth mentioning that in eq (3.9) only terms without the metric are present, since the rest of the terms canceled by applying the energy-momentum conservation.

Similarly, we can express the contribution of the parity-violating (P.V) term, \( \mathcal{L}_3 = \lambda_3 \mathcal{F} \mathcal{G} \), for the amplitude as the following

\[
\mathcal{M}_3 = \Xi_{(3)}^{\mu\nu\rho\sigma} (p_1, p_2, p_3, p_4)\varepsilon_{1\mu}\varepsilon_{3\nu}\varepsilon_{4\rho}\varepsilon_{2\sigma},
\]

with the definition

\[
\left(\Xi^{\mu\nu\rho\sigma}\right)_{P.V} = \Xi_{(3)}^{\mu\nu\rho\sigma} = \sum_{i=1}^{24} \Xi_{(3,i)}^{\mu\nu\rho\sigma} (p_1, p_2, p_3, p_4),
\]

(3.11)

accounting for all possible permutation of the external photon legs. The first contribution is found as

\[
\Xi_{(3,1)}^{\mu\nu\rho\sigma} = 8\lambda_3 \left( p_1^\nu p_3^\rho - g^{\mu\nu}(p_1 \cdot p_3) \right) e^{p_2 p_4 p^\rho p^\sigma},
\]

(3.12)
where we have defined the condensed notation \( \epsilon^{\rho_2 \rho_4 \rho_3 \sigma} \equiv p_{24} \epsilon^{\xi \eta \rho \sigma} \).

Hence, inserting (3.12) back into (3.11) and summing over all other permutation, we get

\[
\left( \Xi^{\mu \nu \rho \sigma} \right)_{P,V} = 32 \lambda^3 \left[ p_1^{\sigma} p_3^{\rho} \epsilon^{\mu \nu \rho_3 \rho_4} + p_1^{\sigma} p_4^{\rho} \epsilon^{\mu \nu \rho_3 \rho_4} - p_1^{\sigma} p_4^{\rho} \epsilon^{\mu \nu \rho_4 \rho_3} + p_1^{\sigma} p_4^{\rho} \epsilon^{\mu \nu \rho_3 \rho_4} \\
+ p_1^{\rho} p_3^{\sigma} \epsilon^{\mu \nu \rho_3 \rho_4} - p_1^{\rho} p_4^{\sigma} \epsilon^{\mu \nu \rho_3 \rho_4} + p_1^{\rho} p_4^{\sigma} \epsilon^{\mu \nu \rho_4 \rho_3} - p_1^{\rho} p_4^{\sigma} \epsilon^{\mu \nu \rho_3 \rho_4} \\
+ p_3^{\nu} p_4^{\mu} \epsilon^{\rho \rho_3 \rho_4} + p_3^{\nu} p_4^{\mu} \epsilon^{\rho \rho_3 \rho_4} + p_3^{\nu} p_4^{\mu} \epsilon^{\rho \rho_3 \rho_4} + p_3^{\nu} p_4^{\mu} \epsilon^{\rho \rho_3 \rho_4} \right].
\]

(3.13)

Therefore, the total amplitude for the light by light scattering can be cast as

\[
\mathcal{M}_{\text{total}} = \left( \Xi^{\mu \nu \rho \sigma} \right)_{P,C} \epsilon_{1 \mu \nu \rho \sigma} \Xi_{4 \rho \sigma} + \left( \Xi^{\mu \nu \rho \sigma} \right)_{P,V} \epsilon_{1 \mu \nu \rho \sigma} \Xi_{4 \rho \sigma}. \quad (3.14)
\]

We can see in eq. (3.14) that the first term corresponds to the parity-preserving piece of the QED, while the second piece is the novel part corresponding to the parity-violating effects. In the next section, we will compute explicitly the total amplitude of the photon-photon scattering by considering the generalized interacting Lagrangian (2.2).

### 4 Perturbative analysis

In this section, we study the lowest order contribution to the four-photon interaction by performing the one-loop analysis through the underlying theory at the low-energy limit \((p^2 \ll m^2)\). First, we consider the term \(n = 4\) of the perturbative series (2.9), corresponding to the so-called box diagram, depicted in Fig. 1. Using the aforementioned Feynman rules, the amplitude of this box diagram is given by

\[
i^{\Lambda}_{(1)} = -\beta^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ \frac{(k^2 + m)\gamma^\mu \gamma^\nu (k^4 + m)\gamma^\rho \gamma^\sigma (k_3^2 + m) (k_3^4 + m) \gamma^\sigma \gamma^\rho (k_4^2 + m) (k_4^4 + m)}{(k^4 - m^2) (k_3^4 - m^2) (k_4^4 - m^2)} \right].
\]

(4.1)

where \((p_1^\mu, p_2^\nu, p_3^\rho, p_4^\sigma)\) are the momenta of the external legs, satisfying the energy-momentum conservation \(p_1 = p_2 + p_3 + p_4\). Moreover, we have introduced the condensed notation \(k_i \equiv k - p_i\) and \(k_{ij} \equiv k - p_i - p_j\).

As we discussed above, in order to find the total amplitude for the 4-point function, we have to consider all of 24 permutations namely,

\[
\Lambda^{\mu \nu \rho \sigma}_{\text{total}} = \sum_{i=1}^{24} \Lambda^{\mu \nu \rho \sigma}_{(i)}.
\]

(4.2)

To compute (4.1), first we apply Feynman parametrization which yields us

\[
i^{\Lambda}_{(1)} = -\beta^4 \Gamma(4) \int dX \int \frac{d^4 \ell}{(2\pi)^4} \frac{N^{\mu \nu \rho \sigma}}{(\ell^2 - \Delta)^4}.
\]

(4.3)

Here we have used \(\ell = k - u\) with \(u = xp_1 + (y + z)p_3 + yp_4\) and \(\int dX \equiv \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\). Moreover, the quantity \(\Delta\) is defined by

\[
\Delta = x(x - 1)p_1^2 + (y + z)((y + z) - 1)p_3^2 + y(y - 1)p_4^2 + 2[y(y - 1) + yz](p_3 p_4) \\
+ 2x(z + y)(p_3 p_1) + 2xy(p_4 p_1) + m^2.
\]

(4.4)
and also
\[ N_{\mu \nu \rho \sigma} = \text{tr} \left[ (\ell + \gamma_1 + m)\gamma^\mu e^{\alpha \gamma^5} (\ell + \gamma_1 + m)\gamma^\nu e^{\alpha \gamma^5} (\ell + \gamma_3 + m)\gamma^\rho e^{\alpha \gamma^5} (\ell + \gamma_{34} + m)\gamma^\sigma e^{\alpha \gamma^5} \right], \quad (4.5) \]

with the definitions \( u_i = u - p_i \) and \( u_{ij} = u - p_i - p_j \).

By definiteness, an important remark about the evaluation of (4.3) is in order because the integral should be regularized (logarithmically divergent). Since we have chosen to use the dimensional regularization and the amplitude involves a \( \gamma_5 \) matrix, it is necessary to use the 't Hooft-Veltman rule in order to correctly define the \( \gamma_5 \) within the dimensional regularization [28, 29].

The 't Hooft-Veltman method consists in the splitting of the \( \omega \) dimensional spacetime into two parts: a 4-dimensional (physical) and a \((\omega - 4)\)-dimensional subspace
\[ \int d^\omega \ell \rightarrow \int d^\omega Q = \int d^4 \ell \int d^{\omega-4} L. \quad (4.6) \]

In this case, the internal momentum is expressed as below
\[ Q = \ell + L = (\gamma^0 \ell_0 + \ldots + \gamma^3 \ell_3) + (\gamma^4 L_4 + \ldots + \gamma^{\omega-1} L_{\omega-1}), \quad (4.7) \]

where we have denoted the internal momentum as \( L \) for the remaining \( \omega - 4 \) components.

Within the 't Hooft-Veltman rule, the \( \gamma_5 \) algebra is written as
\[ \{ \gamma_5, \gamma^\mu \} = 0, \quad \mu = 0, 1, 2, 3 \quad (4.8) \]
\[ [\gamma_5, \gamma^\mu] = 0, \quad \mu = 4, \ldots, \omega - 1, \quad (4.9) \]
and all other familiar rules are still valid, including the algebra
\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu \nu}, \quad \mu, \nu = 0, 1, \ldots, \omega - 1, \quad (4.10) \]
with the metric tensor components \( g_{\mu \nu} = \text{diag} (+1, -1, \ldots, -1) \). We observe that all the external momenta \( p_i \) remain 4-dimensional. This implies that \( \ell^2 = -L^2 \) and \( \ell \ell + L L = 0 \).

Hence, applying the 't Hooft-Veltman rule in (4.5) i.e. \( \ell \rightarrow \ell + L \), we find
\[ N_{\mu \nu \rho \sigma} = \text{tr} \left[ (Q + \gamma_1 + m)\gamma^\mu e^{\alpha \gamma^5} (Q + \gamma_1 + m)\gamma^\nu e^{\alpha \gamma^5} (Q + \gamma_3 + m)\gamma^\rho e^{\alpha \gamma^5} (Q + \gamma_{34} + m)\gamma^\sigma e^{\alpha \gamma^5} \right]. \quad (4.11) \]
We then observe the presence of two possibly divergent terms, those proportional to $\ell^4$ and $L^4$. We shall focus on these terms in our discussion below about their cancelling and finiteness of the quantum effective action.

The numerator (4.5) can be split into two parts according to the power of $m$ as below

$$N_{\mu\nu\rho\sigma} = N_{1,\mu\nu\rho\sigma} + N_{2,\mu\nu\rho\sigma}, \quad (4.12)$$

where, $N_{1,\mu\nu\rho\sigma}$ and $N_{2,\mu\nu\rho\sigma}$ include the odd and even powers of $m$, respectively. Furthermore, we can write these terms in regard to the power of $m$ as the following

$$N_{1,\mu\nu\rho\sigma} = N_{(mu^3)} + N_{(mu^3u^2)} + N_{(mu^3u^3)} + N_{(m^3u^3)}, \quad (4.13)$$

$$N_{2,\mu\nu\rho\sigma} = N_{(\ell^4)} + N_{(\ell^2u^2)} + N_{(u^4)} + N_{(\ell^2m^2)} + N_{(m^2u^2)} + N_{(m^4)}, \quad (4.14)$$

One can observe that since every term in $N_{1,\mu\nu\rho\sigma}$ includes a trace of an odd number of gamma matrices, we conclude that $N_{1,\mu\nu\rho\sigma} = 0$. Thus, we now insert $N_{2,\mu\nu\rho\sigma}$ into (4.3) and arrange it according to the power of the internal momentum $\ell$ as the following

$$i\Lambda_{(1,a)}^{\mu\nu\rho\sigma} = -\beta^4\Gamma(4) \int dX \int \frac{d^4Q}{(2\pi)^4} \frac{N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)}}{(\ell^2 - L^2 - \Delta)^4}, \quad (4.15)$$

$$i\Lambda_{(1,b)}^{\mu\nu\rho\sigma} = -\beta^4\Gamma(4) \int dX \int \frac{d^4Q}{(2\pi)^4} \frac{N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)}}{(\ell^2 - L^2 - \Delta)^4}, \quad (4.16)$$

$$i\Lambda_{(1,c)}^{\mu\nu\rho\sigma} = -\beta^4\Gamma(4) \int dX \int \frac{d^4Q}{(2\pi)^4} \frac{N_{(\mu\nu\rho\sigma)} + N_{(\mu\nu\rho\sigma)}}{(\ell^2 - L^2 - \Delta)^4}. \quad (4.17)$$

The explicit form of these terms in $N_{2,\mu\nu\rho\sigma}$ can be found in the appendix A, eqs. (A.1)-(A.10), and also the $L$ dependent terms in $N$ are similar to those of $\ell$.

Now, we can show how the divergent contributions in (4.17) are cancelled. As mentioned above, we have the following logarithmically divergent terms

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{d^{\omega-4}L}{(2\pi)^4} \frac{1}{(\ell^2 - L^2 - \Delta)^4} \text{tr} [e^{4\alpha\gamma\xi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma], \quad (4.18)$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{d^{\omega-4}L}{(2\pi)^4} \frac{1}{(\ell^2 - L^2 - \Delta)^4} \text{tr} [L \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma], \quad (4.19)$$

we notice that both terms have similar tensor structure (due to the trace operation). Thus, the tensor and regularized structure is nearly common for the above $\ell^4$ and $L^4$ terms and we can use it to apply our considerations. Hence, we can evaluate straightforwardly the momentum integration with help of (A.1) and use of the identity $Q_\xi Q_\gamma Q_\eta Q_\rho \rightarrow \frac{Q^2}{\omega(\omega+2)} (g_{\delta\tau} g_{\xi\eta} + g_{\delta\xi} g_{\tau\eta} + g_{\delta\eta} g_{\tau\xi})$, and show that both contributions are proportional to $\Gamma \left( \frac{\epsilon}{2} \right)$, where $\epsilon = 4 - \omega \rightarrow 0^+$, and to the tensor structure

$$\tilde{N}_{\mu\nu\rho\sigma} = \frac{1}{4} \left( g_{\delta\tau} g_{\xi\eta} + g_{\delta\xi} g_{\tau\eta} + g_{\delta\eta} g_{\tau\xi} \right) \text{Tr} [e^{4\alpha\gamma\xi} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma], \quad (4.20)$$
where the $L$ contribution in eq. (4.19) is obtained from (4.20) as $\alpha = 0$. Moreover, we notice that the terms including $\cosh(4\alpha)$ and $\sinh(4\alpha)$ refer to the parity-preserving and parity-violating contributions, respectively. However, in the usual QED, there is only terms with $\alpha \to 0$, so that $\beta \to g_0$.

Finally, making use of this result and by taking into account all the 24 (tensor and momentum) permutations, and also performing the relevant traces through FeynCalc program, in both parity-preserving and parity-violating pieces, we find that

\[ \Lambda_{\text{total}}^{\mu\nu\rho\sigma} \bigg|_{\text{div.}} = \sum_{i=1}^{24} \Lambda_{(i,c)}^{\mu\nu\rho\sigma} \bigg|_{\text{div.}} = 0, \tag{4.21} \]

which is in accordance to the result of the usual QED. Hence, we are left with a finite result for the total amplitude (4.2).

After proving the UV-finiteness of the total amplitude, we are now ready to obtain the generalized expression for the Euler-Heisenberg effective Lagrangian. Thus, by making use of the standard Feynman integrals, we arrive at the following expressions for the finite part

\[ \Lambda_{(1,a)}^{\mu\nu\rho\sigma} = -\frac{\beta^4}{16\pi^2} \int dX \frac{1}{\Delta^2} \left[ \mathcal{N}_{(m^4)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2 u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(u^4)}^{\mu\nu\rho\sigma} \right], \tag{4.22} \]

\[ \Lambda_{(1,b)}^{\mu\nu\rho\sigma} = \frac{\beta^4}{16\pi^2} \int dX \frac{1}{\Delta} \left[ \mathcal{N}_{(u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(0)}^{\mu\nu\rho\sigma} \right], \tag{4.23} \]

the quantities with tilde and hat in (4.23) indicate the integrated expressions of (4.16) over the internal momenta $\ell$ and $L$.

Assuming on-shell photons, $p_i^2 = 0$, the expression $\Delta$ in (4.4) changes to

\[ \Delta = m^2(1 + \xi), \tag{4.24} \]

where

\[ \xi = 2y(y - 1 + z)(\frac{p_3.p_1}{m^2}) + 2x(z + y)(\frac{p_3.p_1}{m^2}) + 2xy(\frac{p_4.p_1}{m^2}). \tag{4.25} \]

Now, in the low-energy limit the photon energies are small compared to the fermionic mass $m$, i.e. $p_i, p_j \ll m^2$. Under these considerations, we obtain

\[ \Lambda_{(1,a)}^{\mu\nu\rho\sigma} = -\frac{\beta^4}{16\pi^2 m^4} \int dX \left[ \mathcal{N}_{(m^4)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2 u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(u^4)}^{\mu\nu\rho\sigma} \right] \left[ 1 - 2\xi + 3\xi^2 \right], \tag{4.26} \]

\[ \Lambda_{(1,b)}^{\mu\nu\rho\sigma} = \frac{\beta^4}{16\pi^2 m^2} \int dX \left[ \mathcal{N}_{(u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(u^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(m^2)}^{\mu\nu\rho\sigma} + \mathcal{N}_{(0)}^{\mu\nu\rho\sigma} \right] \left[ 1 - \xi + \xi^2 \right]. \tag{4.27} \]

As we have previously mentioned, we observe that $\Lambda_{(1,a)}^{\mu\nu\rho\sigma}$ and $\Lambda_{(1,b)}^{\mu\nu\rho\sigma}$ are UV finite.

To determine the generalized Euler-Heisenberg effective action, we should concentrate in examining on the term which includes four momenta of the external photons, i.e. $\mathcal{N}_{(u^4)}^{\mu\nu\rho\sigma}$ in (4.26), and discarding the $\xi \ll 1$ parts. Thus, using the explicit form of $\mathcal{N}_{(u^4)}^{\mu\nu\rho\sigma}$ in (A.9), we find

\[ \Lambda_{(1,a)}^{\mu\nu\rho\sigma} \big|_{u^4} = -\frac{\beta^4}{16\pi^2 m^4} \text{Tr} \left[ \left[ \cosh(4\alpha) + \sinh(4\alpha) \gamma^5 \right] \gamma^\delta \gamma^\mu \gamma^\tau \gamma^\rho \gamma^\gamma \xi^\xi \gamma^\xi \gamma^\gamma \gamma^\gamma \right] \int dX \; u_\delta u_\tau u_\xi u_\eta. \tag{4.28} \]
Now, by making use of eq. (2.4) to return to the \((g_v, g_a)\) couplings, and separating eq. (4.28) into the parity-conserving (P.C) and parity-violating (P.V) contributions, we have

\[
(\Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4})_{P.C} = -\frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{16\pi^2 m^4} Tr \left( \gamma^\delta \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\gamma \xi^0_{u^4} \right) \int dX \ u_{1\delta} u_{\tau} u_{3\xi} u_{3\eta}, \tag{4.29}
\]

\[
(\Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4})_{P.V} = -\frac{4(g_v^2g_a + g_vg_a^2)}{16\pi^2 m^4} Tr \left( \gamma^\delta \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\gamma \xi^0_{u^4} \right) \int dX \ u_{1\delta} u_{\tau} u_{3\xi} u_{3\eta}. \tag{4.30}
\]

A first comment about these expressions is that in the limit \(g_a \to 0\), the P.V contribution vanishes and the value of P.C reduces to the ordinary result. Furthermore, we observe that both contributions, P.C and P.V., are totally symmetric under the exchange of \(g_v \leftrightarrow g_a\), corroborating our arguments of the presence of this symmetry for \(n = 4\) point function. However, this symmetry property is in disagreement with the behavior of the results found in ref. [22]. But we should emphasize that our symmetry arguments are general, since the parametrization (2.4) is independent of the perturbative analysis (that reflects the generality of our arguments).

After performing the remaining traces and taking the integral over the Feynman parameters in (4.29) and (4.30) through FeynCalc program, we should apply all of the 24 permutations

\[
\left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.C} = \sum_{i=1}^{24} \left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.C}, \quad \left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.V} = \sum_{i=1}^{24} \left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.V}, \tag{4.31}
\]

which finally lead to the following expression of the total parity-preserving contribution

\[
\left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.C} = \frac{(g_v^4 + g_a^4 + 6g_v^2g_a^2)}{60\pi^2 m^4} \times \left[ \frac{7}{3} \left( p_1^\rho p_3^\rho p_4^\rho p_3^\mu + p_1^\mu p_4^\rho p_3^\rho p_3^\mu + p_3^\rho p_1^\rho p_4^\rho p_3^\mu + p_3^\mu p_1^\rho p_4^\rho p_3^\mu - p_4^\rho p_1^\rho p_3^\rho p_3^\mu - p_4^\mu p_1^\rho p_3^\rho p_3^\mu \right) \right] , \tag{4.32}
\]

and also to the total parity-violating contribution

\[
\left( \Lambda_{\mu\nu\rho\sigma}^{(1,\alpha)} \xi^0_{u^4} \right)_{P.V} = -\frac{i(g_v^3g_a + g_v^3g_a)}{24\pi^2 m^4} \left[ p_1^\rho p_3^{\mu\nu p_3 p_4} + p_1^\mu p_4^{\rho\nu p_3 p_4} - p_1^\mu p_4^{\rho\nu p_3 p_4} + p_1^\rho p_4^{\mu\nu p_3 p_4} + p_1^\rho p_4^{\mu\nu p_3 p_4} + p_1^\rho p_4^{\mu\nu p_3 p_4} - p_1^\rho p_4^{\mu\nu p_3 p_4} \right] , \tag{4.33}
\]

One should note that the coefficient \(i\) in eq. (4.33) arises from the trace of \(\gamma^5\) with the gamma matrices. Now that we have evaluated the one-loop effective action for the generalized parity-violating coupling, we are in a position to determine the effective couplings \(\lambda_i\)’s by comparing these results to those from Section (3.1).
5 Determination of the effective couplings \((\lambda_1, \lambda_2, \lambda_3)\)

At this stage, we can determine the value of three coupling constants appearing in the effective Lagrangian (3.3)

\[
L_{\text{ext.}}^{E.H} = \lambda_1 F^2 + \lambda_2 G^2 + \lambda_3 F G.
\]  

(5.1)

This can be achieved by matching the results found in eqs. (3.9) and (4.32) for the parity-preserving piece

\[
(\Xi^{\mu\nu\rho\sigma})_{\text{P.C}} = (\Lambda^{\mu\nu\rho\sigma})^{(u^4)}_{\text{P.C}},
\]  

(5.2)

as well as eqs. (3.13) and (4.33) for the parity-violating part

\[
(\Xi^{\mu\nu\rho\sigma})_{\text{P.V}} = (\Lambda^{\mu\nu\rho\sigma})^{(u^4)}_{\text{P.V}}.
\]  

(5.3)

After some algebraic manipulations, we obtain the values of \((\lambda_1, \lambda_2)\) from eq. (5.2), while (5.3) gives us the value of \(\lambda_3\). These expressions are summarized as

\[
\begin{align*}
\lambda_1 &= \frac{1}{512\pi^2 m^4} \left[ \frac{16}{45} g_v^4 + \frac{32}{15} g_v^2 g_a^2 + \frac{16}{45} g_a^4 \right], \\
\lambda_2 &= \frac{1}{512\pi^2 m^4} \left[ \frac{28}{45} g_v^4 + \frac{56}{15} g_v^2 g_a^2 + \frac{28}{45} g_a^4 \right], \\
\lambda_3 &= -\frac{i}{768\pi^2 m^4} \left[ g_v^3 g_a + g_v g_a^3 \right].
\end{align*}
\]  

(5.4)

Since the one-loop 4-point function is symmetric under the change of \(g_v \leftrightarrow g_a\), it was expected that the effective couplings \(\lambda_i\) present a similar behavior, as verified in (5.4). Here, as we have mentioned above, we observe again that the values of the coupling constants are different from those obtained in the ref. [22], that do not display such a symmetric behavior.

To clarify this issue, let us consider two cases in the interacting Lagrangian (2.2), or equivalently in the effective couplings (5.4):

\[
\begin{align*}
\lambda_1 &= \frac{g_v^4}{1440\pi^2 m^4}, \\
\lambda_2 &= \frac{7g_v^4}{5760\pi^2 m^4}, \\
\lambda_3 &= 0,
\end{align*}
\]  

(5.5)

that correspond to the two limiting cases: \(g_a = 0, g_v \neq 0\) and \(g_v = 0, g_a \neq 0\).

Both cases, in contrast to [22], lead to the same result for the three effective coupling constants as the following which coincides exactly with effective couplings of an ordinary parity-conserving Euler-Heisenberg Lagrangian [1, 20].

Therefore, we realize that considering the pure-vector interaction (even-parity) or the pure-axial interaction (odd-parity) of the gauge field with fermionic matter leads to an identical effective action, which is parity-preserving. In the other words, the interaction term with a distinct behaviour under parity produces a parity-conserving effective action. While the case \(g_v \neq 0, g_a \neq 0\) yields us a generalized effective Euler-Heisenberg action with parity-conserving and parity-violating terms (5.4).
As the last case, we consider the V+A and V-A interactions, which correspond to $g_a = \pm g_v$, respectively,

$$L_{V+A} = -g_v \bar{\psi} \gamma^\mu (1 + \gamma^5) A_\mu \psi, \quad L_{V-A} = -g_v \bar{\psi} \gamma^\mu (1 - \gamma^5) A_\mu \psi,$$

and find the effective coupling constants as follows

$$\lambda_1 \bigg|_{g_a = \pm g_v} = \frac{g_v^4}{180\pi^2 m^4}, \quad \lambda_2 \bigg|_{g_a = \pm g_v} = \frac{7g_v^4}{720\pi^2 m^4}, \quad \lambda_3 \bigg|_{g_a = \pm g_v} = \mp \frac{ig_v^4}{384\pi^2 m^4}.$$  

(5.6)

Naturally, as we expected, the values of the constants $\lambda_i$’s (5.7) are also in disagreement with the results obtained in [22].

6 $\gamma\gamma \rightarrow \gamma\gamma$ differential cross section

In order to examine the phenomenology of the effective action (5.1), with the couplings (5.4), it is interesting to calculate the differential cross section for the light by light scattering. To this purpose, we first consider the tensorial structure of the total amplitude in (3.14) as the following

$$\Xi^{\mu\nu\rho\sigma} = (\Xi^{\mu\nu\rho\sigma})_{P.C} + (\Xi^{\mu\nu\rho\sigma})_{P.V}.$$  

(6.1)

As a first check, we verify the Ward identity by inserting eqs. (3.9) and (3.13) back into (6.1), and then contracting the resulting expression with the external momenta, which give the following

$$p_{1\mu} \Xi^{\mu\nu\rho\sigma} = p_{2\sigma} \Xi^{\mu\nu\rho\sigma} = p_{3\nu} \Xi^{\mu\nu\rho\sigma} = p_{4\rho} \Xi^{\mu\nu\rho\sigma} = 0.$$  

(6.2)

This result is expected since the effective Lagrangian (5.1) is constructed from the gauge invariant quantities $\mathcal{F}$ and $\mathcal{G}$. The unpolarized differential cross section is proportional to the average of the absolute square of the total amplitude as $\frac{1}{4} \sum |M|^2$ that includes the following sum over the photon polarization states

$$\sum_{\lambda=1}^2 \varepsilon_{\nu}^*(p,\lambda) \varepsilon_{\nu}(p,\lambda) = -g_{\mu\nu} + \mathbb{L}_{\mu\nu},$$  

(6.3)

where $\mathbb{L}_{\mu\nu}$ is a longitudinal part which contraction with $\Xi^{\mu\nu\rho\sigma}$ vanishes, due to the Ward identity (6.2). In the center of mass frame (CM), we define $\omega$ as the photon energy and $\theta$ as the departure angle of the final state photons. Hence, the unpolarized differential cross section is given by

$$\frac{d\sigma}{d\Omega} \bigg|_{CM} = \left(278g_v^8 + 3561g_v^6g_a^2 + 11014g_v^4g_a^4 + 3561g_v^2g_a^6 + 278g_a^8\right) \left(\frac{\omega^6}{m^8}\right) \left[7 + \cos(2\theta)\right]^2.$$  

(6.4)

It is worth noticing that the presence of the parity-violating term $\lambda_3 \mathcal{F} \mathcal{G}$ does not change the angular distribution of the differential cross section of the QED description. Rather, its contribution is only as a numerical factor; therefore, an enhancement in regard of the usual value measured in the related cross section could be a signal of the presence of parity violation in QED.

Here, we mention that the obtained result (6.4) is also symmetric under the exchange of $g_a \leftrightarrow g_v$. Moreover, in the pure-vector and pure-axial coupling limits, we arrive at

$$\lim_{g_a \rightarrow 0} \frac{d\sigma}{d\Omega} \bigg|_{CM} = \lim_{g_v \rightarrow 0} \frac{d\sigma}{d\Omega} \bigg|_{CM} = \frac{139g_v^8}{33177600\pi^6} \left(\frac{\omega^6}{m^8}\right) \left[7 + \cos(2\theta)\right]^2.$$  

(6.5)
We observe that these limiting cases yield us the same result, which coincides exactly with the standard differential cross section for the photon-photon scattering in the usual QED \cite{30,31}, as expected.

7 Conclusion

In this paper, we have perturbatively examined the Euler-Heisenberg effective action in the presence of an axial coupling of the gauge field with the fermionic matter. This novel coupling effects are responsible for generating a parity-breaking term $\mathcal{FG}$ in the effective action.

One important aspect of our analysis is in regard of the regularization to the Feynman amplitude. Since the amplitude of the process $\gamma\gamma \rightarrow \gamma\gamma$ is divergent, and we have the presence of an axial coupling in terms of the $\gamma_5$ matrix, we presented a detailed analysis of the algebraic manipulations using the 't Hooft-Veltman rule. We have shown the presence of two possibly divergent terms, and discussed how their contributions are cancelled when the 24 permutations are considered.

Another point is related to a hidden symmetry of the vector and axial couplings, which was observed in terms of a proper parametrization. The piece of the 4-point function contributing to the extended Euler-Heisenberg effective Lagrangian is written in terms of the parametrized couplings $\beta^4 e^{4\alpha\gamma^5} = (g_v^4 + g_a^4 + 6g_v^2g_a^2) + 4(g_v^3g_a + g_vg_a^3)\gamma^5$, which allows us to verify that it is symmetric under the change of $g_v \leftrightarrow g_a$. Furthermore, this observation also allowed us to conclude that the usual Euler-Heisenberg effective action, which is parity-preserving, can be generated whether by the pure-vector interaction (even-parity) or the pure-axial interaction (odd-parity), see eq.(5.5) and comments below it.

It is important to remark that our results are in contrast to those of ref. \cite{22}. But we believe that, although following different approaches, a crucial point showing the correctness of our analysis is the presence of the aforementioned symmetry in the effective Lagrangian. Actually, this hidden symmetry of the $n = 4$ legs graph under $g_v \leftrightarrow g_a$ is absent in the results of ref. \cite{22}.

We believe that the results presented here are very interesting. Therefore, we proceeded to a phenomenological application of the generalized effective action to compute the differential cross section related with the photon-photon scattering. As a result, we found that the presence of the parity-violating term does not change the angular distribution of the differential cross section and changes only the related numerical coefficient. Hence, the effects of the parity-violating term could be measured in terms of the enhanced value of the cross section.

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A The explicit form of $N_{2}^{\mu\nu\rho\sigma}$

Here, we present the explicit form of the six terms appeared in the relation (4.14)

$$N_{2}^{\mu\nu\rho\sigma} = N_{(e_{1}^{2})}^{\mu\nu\rho\sigma} + N_{(e_{2}^{2})}^{\mu\nu\rho\sigma} + N_{(u_{1}^{1})}^{\mu\nu\rho\sigma} + N_{(m_{1}^{2})}^{\mu\nu\rho\sigma} + N_{(m_{2}^{2})}^{\mu\nu\rho\sigma} + N_{(m_{3}^{2})}^{\mu\nu\rho\sigma},$$

as the following:

$$N_{(e_{1}^{2})}^{\mu\nu\rho\sigma} = \ell_{\delta} \ell_{\tau} \ell_{\xi} \ell_{\eta} \text{Tr} \left[ e^{4\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}} \right], \quad (A.1)$$

$$N_{(e_{2}^{2})}^{\mu\nu\rho\sigma} = \left[ u_{35} u_{34} \ell_{\delta} \ell_{\tau} + u_{36} u_{35} \ell_{\xi} \ell_{\eta} + \ell_{\delta} \ell_{\tau} u_{34} + \ell_{\xi} \ell_{\eta} u_{35} + \ell_{\xi} \ell_{\eta} u_{36} \right] \times \text{Tr} \left[ e^{4\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}} \right], \quad (A.2)$$

$$N_{(m_{1}^{2})}^{\mu\nu\rho\sigma} = m^{2} \ell_{\delta} \ell_{\tau} \left[ \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) + \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) 
+ \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) \right], \quad (A.3)$$

$$N_{(m_{2}^{2})}^{\mu\nu\rho\sigma} = m^{2} \ell_{\delta} \ell_{\tau} \ell_{\xi} \ell_{\eta} \left[ \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) \right], \quad (A.4)$$

$$N_{(m_{1}^{2})}^{\mu\nu\rho\sigma} = m^{2} \ell_{\delta} \ell_{\tau} \ell_{\xi} \ell_{\eta} \left[ \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) \right], \quad (A.5)$$

$$N_{(m_{2}^{2})}^{\mu\nu\rho\sigma} = m^{2} \ell_{\delta} \ell_{\tau} \ell_{\xi} \ell_{\eta} \left[ \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) + \text{Tr}(\gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}) \right], \quad (A.6)$$

$$N_{(u_{1}^{1})}^{\mu\nu\rho\sigma} = \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + u_{36} u_{34} \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}), \quad (A.7)$$

$$N_{(m_{1}^{2})}^{\mu\nu\rho\sigma} = m^{2} \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}) + u_{36} \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}), \quad (A.8)$$

$$N_{(u_{1}^{1})}^{\mu\nu\rho\sigma} = u_{36} \text{Tr}(e^{2\alpha_{\gamma}^{5} \gamma^{\delta} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}}), \quad (A.9)$$

$$N_{(m_{1}^{2})}^{\mu\nu\rho\sigma} = m^{4} \text{Tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma}). \quad (A.10)$$
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