Abstract

In the conventional $p$-brane theory, a gauge $(p+1)$-form field $\phi^{(p+1)}$ mediates the interaction between $p$-branes, which is gauge invariant in the sense $\phi^{(p+1)} \rightarrow \phi^{(p+1)} + d\Lambda^{(p)}$ with $\Lambda^{(p)}$, an arbitrary $p$-form and $d$, a boundary operator. We, on the contrary, propose to introduce a new gauge field $\phi^{(p+2)}$ mediating the interaction, and we have a new type of gauge transformation: $\phi^{(p+2)} \rightarrow \phi^{(p+2)} + \delta\Lambda^{(p+3)}$, with $\delta$, a coboundary operator.
Starting with the study of a one-dimensional object (string), one concerns oneself with an arbitrary-dimensional extended object (p-brane). With the p-brane theory such an ambitious problem as unifying all fundamental interactions, which, as a result, clarifies the relation between p-brane interaction and supergravity, is now going to be attacked.

We, in the present report, want to go along this line and to deal with the interaction between p-branes. The gauge field of rank \((p+1)\) couples to p-branes, whose way of coupling is invariant under the conventional gauge transformation. The interaction Lagrangian of p-branes and gauge fields, to say noting of the kinetic Lagrangian, is made to be invariant under the conventional gauge transformation \(\phi^{(p+1)} \rightarrow \phi^{(p+1)} + d\Lambda^{(p)}\), with an arbitrary p-form \(\Lambda^{(p)}\) (\(d: a \text{ boundary operator}\)). This interaction to the p-branes of an arbitrary dimension has been described in great detail in Refs.\[2\][3].

However, the question arises as to whether there exists an alternative interaction or not which also describes the p-brane motion. The answer is affirmative and we will introduce a new type of gauge transformation \(\phi^{(p+2)} \rightarrow \phi^{(p+2)} + \delta \Lambda^{(p+3)}\), with \(\Lambda^{(p+3)}\), an arbitrary \((p+3)\)-form, and \(\delta\), a coboundary operator dual to \(d\). The gauge transformation of this kind appears in the theory with magnetic monopoles\[4\].

Feynman and Wheeler succeeded in formulating a theory equivalent to the Maxwell electromagnetic theory through the action-at-a-distance (AD) force\[1\]. They obtained, with this AD force, the electromagnetic field (gauge field) mediating the charged particle interaction. Ishikawa et al. applied the method to the p-brane theory and wrote down the interaction Lagrangian with a \((p+1)\)-form gauge field mediating the p-brane interaction\[2\][3]. There the gauge transformation was defined \(\text{à la} \) Feynman and Wheeler, i.e., in the conventional form in electromagnetism. Now, in this report, we will formulate an essentially new type of gauge transformation.

Following the reference\[4\], we consider two point particles \(a\) and \(b\), interacting with each other through the AD force in \(D\)-dimensional space-time with metric \((g^{\mu\nu}) = \text{diag}(−,+,\cdots,+)\). The action of this system is given by

\[
S = S^{\text{free}}_a + S^{\text{free}}_b + \frac{g_ag_b}{2} \int d\tau_a d\tau_b R_{ab}(X_a, X_b, \dot{X}_a, \dot{X}_b),
\]

where \(S^{\text{free}}_a\) (or \(S^{\text{free}}_b\)) is the action of the free particle \(a\) (or \(b\)), \(g_a\) (or \(g_b\)) is a coupling constant, and \(\tau_a\) (or \(\tau_b\)) is a proper time of \(a\) (or \(b\)) (\(\dot{X}_a \equiv dX_a/d\tau_a\), etc.). The existence of the functional \(R_{ab}\) just represents that of the AD force. It satisfies \(R_{ab} = R_{ba}\) for symmetry requirement. Imposing reparametrization invariance on the action, we adopt the following form:

\[
R_{ab} = M^{\mu\nu}_a M_{b\mu\nu} G((X_a - X_b)^2),
\]

\[
M_{\mu\nu} \equiv \dot{X}_{a\mu}\partial_{a\nu} - X_{a\mu}\partial_{a\nu},
\]

where \(G((X_a - X_b)^2)\) is a Green’s function (with a mass parameter \(m\)) in the \(D\)-dimensional space-time. By assuming that the action \[1\] be stationary under the variation

\[
X_{a\mu} \rightarrow X_{a\mu}^\mu + \delta X_{a\mu}^\mu,
\]
we obtain the equations of motion for the particle $a$, 
\[ \ddot{X}_a^\mu = g_a \partial_{a\rho} \left( \partial_a^{\nu} g_b \int d\tau_b M_b^{\nu\rho} G((X_a - X_b)^2) - \partial_a^{\rho} g_b \int d\tau_b M_b^{\mu\rho} G((X_a - X_b)^2) \right) \dot{X}_{av}. \] (5)

Note that these equations of motion are expressed in the particle coordinates and their derivatives, \textit{without gauge fields}. Following Feynman and Wheeler, we define the gauge fields:
\[ \phi_{b\mu\nu}(X) \equiv g_b \int d\tau_b M_{b\mu\nu} G((X - X_b)^2), \] (6)
and then the equations of motion reduce to
\[ \ddot{X}_a^\mu = g_a F_b^{\mu\nu}(X_a) \dot{X}_{av}, \] (7)
with the field strength
\[ F_b^{\mu\nu}(X) \equiv \partial_\rho (\partial^\mu \phi_{b}^{\nu\rho}(X) - \partial^\nu \phi_{b}^{\mu\rho}(X)). \] (8)

In terms of differential forms, Eq.(8) is rewritten as
\[ F_b^{(2)} = d\delta \phi_b^{(2)} \] (9)
with self-evident definition of $\phi^{(2)}$ and $F^{(2)}$ (e.g., $F^{(2)} = 1/2 F_{\mu\nu} dX^\mu \wedge dX^\nu$) as well as a boundary operator $d$ and a coboundary operator $\delta$. Then we immediately find that the equations of motion (7) are invariant under the transformation of gauge 2-form field,
\[ \phi^{(2)} \rightarrow \phi^{(2)} + \delta \Lambda^{(3)}, \] (10)
\[ \Lambda^{(3)} : \text{an arbitrary 3-form}. \]

In components, Eq.(10) is written as
\[ \phi_{b\mu\nu} \rightarrow \phi_{b\mu\nu} + \partial^\alpha (\Lambda_{\alpha\mu\nu} - \Lambda_{\alpha\nu\mu}), \] (11)
\[ \Lambda_{\alpha\mu\nu} : \text{an arbitrary antisymmetric tensor of rank 3}. \]

We note the conventional gauge transformation is
\[ A^{(1)} \rightarrow A^{(1)} + d\Lambda^{(0)}, \] (12)
\[ \Lambda^{(0)} : \text{an arbitrary 0-form}. \]

Here a boundary operator $d$ is dual to a coboundary one $\delta$ in Eq.(10). The total action is given, omitting the indices $a$ and $b$ indicating the particular particles, by
\[ S = - \int d\tau \left( -\dot{X}^\mu \dot{X}_\mu \right)^{1/2} + g \int d\tau \dot{X}^\mu \partial^\nu \phi_{\mu\nu} - \frac{1}{4} \int d^D X F^{\mu\nu} F_{\mu\nu} \] (13)
The mass terms of gauge fields do not appear because of gauge invariance.

The extension of the above method to $p$-branes is now straightforward. The action of interacting $p$-branes through the AD force is

$$
S = S^\text{free}_a + S^\text{free}_b + \frac{g_a g_b}{p+2} \int d^{p+1}\xi_a d^{p+1}\xi_b R_{ab}(X_a, X_b, X_{a,i}, X_{b,i}).
$$

The notation here is self-evident; the difference (and complication) arises only from the fact that our $p$-brane traces the $(p+1)$-dimensional worldvolume in the $D$-dimensional target Minkowski space-time. Adding a few more formulation to make doubly sure, we have

$$
S^\text{free} = -\int d^{p+1}\xi (-\sigma \cdot \sigma)^\frac{1}{2}.
$$

$$
\sigma \cdot \sigma \equiv \sigma^{i_0 \mu_1 \cdots \mu_p} \sigma_{i_0 \mu_1 \cdots \mu_p},
$$

$$
\sigma^{i_0 \mu_1 \cdots \mu_p} \equiv \frac{\partial (X_{\mu_0}, X_{\mu_1}, \ldots, X_{\mu_p})}{\partial (\xi^0, \xi^1, \ldots, \xi^p)},
$$

$$
d^{p+1}\xi \equiv d\xi^0 d\xi^1 \ldots d\xi^p,
$$

$$
X_{\mu_i} \equiv \frac{\partial X^\mu}{\partial \xi^i},
$$

where $\xi^i (i=0,1,\ldots,p)$ are parameters of the worldvolume. As $R_{ab}$ we adopt the following form which is also an extension of Eq.(2):

$$
R_{ab} = M_{a i_0 \mu_1 \cdots \mu_{p+1}} M_{b i_0 \mu_1 \cdots \mu_{p+1}} G((X_a - X_b)^2),
$$

where

$$
M_{a i_0 \mu_1 \cdots \mu_{p+1}} \equiv \sum_{i=0}^{p+1} (-1)^{(p+1)-i} \sigma_{a i_0 \mu_1 \cdots \mu_{p+1}} \sigma_{a i_0 \mu_1 \cdots \mu_{p+1}} \partial_{a i_1 \cdots i_{p+1}}.
$$

The gauge field for $p$-branes is

$$
\phi_{b i_0 \mu_1 \cdots \mu_{p+1}}(X) \equiv g_b \int d^{p+1}\xi_b M_{b i_0 \mu_1 \cdots \mu_{p+1}} G((X - X_b)^2).
$$

The field strength is given by

$$
F_{b i_0 \mu_1 \cdots \mu_{p+1}}(X) \equiv \sum_{i=0}^{p+1} (-1)^{(p+1)-i} \partial_{b i_0} \partial_{b i_1} \phi_{b i_0 \mu_1 \cdots \mu_{p+1}}.\phi_{b i_0 \mu_1 \cdots \mu_{p+1}} \mu_0 \cdots \mu_{i_1} \cdots \mu_{i_{p+1}}\nu.
$$

Based on the variational principle we have the equations of motion

$$
(p+1)D_{a 0 \mu_1 \cdots \mu_{p+1}} \left[ \frac{\sigma^{a i_0 \mu_1 \cdots \mu_{p+1}}}{(-\sigma_a \cdot \sigma_a)^\frac{1}{2}} \right] = g_a \sigma_{a i_0 \mu_1 \cdots \mu_{p+1}} F_{b i_0 \mu_1 \cdots \mu_{p+1}}\phi_{b i_0 \mu_1 \cdots \mu_{p+1}} = g_a \sigma_{a i_0 \mu_1 \cdots \mu_{p+1}} F_{b i_0 \mu_1 \cdots \mu_{p+1}}.
$$
where
\[ D_{\mu_1\mu_2...\mu_p} \equiv \sum_{i=0}^{p} K_i^{\mu_1\mu_2...\mu_p} \frac{\partial}{\partial \xi_i}, \] (21)
and
\[ K_i^{\mu_1\mu_2...\mu_p} \equiv \frac{\partial \sigma_{\alpha\mu_1...\mu_p}}{\partial X_{\alpha}^i}. \] (22)

In the definition of \( K_i \), we do not sum over the index \( \alpha \).

The equations of motion (20) are invariant under the gauge transformations for \( \phi_{\mu_0...\mu_{p+1}} \):
\[ \phi_{\mu_0\mu_1...\mu_{p+1}} \rightarrow \phi_{\mu_0\mu_1...\mu_{p+1}} + \partial^{\mu_{p+2}} \Lambda_{\mu_0\mu_1...\mu_{p+1}\mu_{p+2}}, \] (23)
\( \Lambda_{\mu_0\mu_1...\mu_{p+1}\mu_{p+2}} \) : an arbitrary tensor of rank \((p+3)\),
which, expressed in the differential form, become
\[ \phi^{(p+2)} \rightarrow \phi^{(p+2)} + \delta \Lambda^{(p+3)}, \] (24)
\( \Lambda^{(p+3)} \) : an arbitrary \((p+3)\)-form.

Now we finally have the total action for interacting \( p \)-branes
\[ S = - \int d^{p+1} \xi (-\sigma \cdot \sigma)^{1/2} + g \int d^{p+1} \xi \sigma_{\alpha\mu_1...\mu_p} \partial^{\mu_{p+1}} \phi_{\mu_0\mu_1...\mu_{p+1}} - \frac{1}{2(p+2)!} \int d^D X F_{\mu_0\mu_1...\mu_{p+1}\mu_{p+2}} F^{\mu_0\mu_1...\mu_{p+1}.} \] (25)

Now we come to the conclusion. Contrary to the conventional gauge transformation with a boundary operator \( d \), our transformation is represented in Eq.(24) with a coboundary operator \( \delta \). \( R_{ab} \), being written in the AD force of Eq.(16), we are to have a gauge invariant theory. In Ref.[2], for an open \( p \)-brane, it was emphasized that the primitive form of gauge field as in Eq.(18) should be modified to recover the gauge invariance of the action. Based on this modification, an interesting result was obtained that the massiveness does not prevent the gauge invariance, which is a conspicuous difference from the case of a closed \( p \)-brane. Our model shows, however, that no modification is necessary and the gauge invariance of the action holds in the form Eq.(18), both for a closed and open \( p \)-brane.

Of course, our choice for \( R_{ab} \) (Eq.(16)), is not unique. Our guiding principle to obtain Eq.(16) is that \( R_{ab} \) should have as simple a form as possible and that it be invariant under a new type of gauge transformation(24) with a coboundary operator \( \delta \).

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