Guaranteed cost synchronization for second-order discrete-time multi-node networks with switching transmission topologies and cost limitations

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Abstract
The guaranteed cost synchronization control for discrete-time second-order leader-following multi-node networks with varying transmission topologies and cost limitations is investigated. First, the control law and its cost function are proposed to balance the tradeoff between the synchronization regulation and the control cost. Then, we derive the sufficient conditions without cost limitations to achieve guaranteed cost synchronization with varying transmission topologies, where the dimensions of matrix variables do not rely on the number of nodes. In addition, we calculate an upper bound on the cost consumption index, which is a function of the initial conditions. Moreover, guaranteed cost synchronization criteria are formulated for control with cost limitations, which are involved by establishing the relation between the cost limitation and the upper bound on the cost consumption index. Numerical simulation experiments were conducted to validate the theoretical results.

Keywords
Multi-node network, discrete-time system, guaranteed cost synchronization, cost limitation

Introduction
As a basic problem in multi-node networks (MNNs), synchronization has received increasing attention mostly by its potential to handle problems such as multi-vehicle formation control,\textsuperscript{1–3} design of sensor networks,\textsuperscript{4–7} and rendezvous problems.\textsuperscript{8–10} Various node synchronization problems in MNNs have been addressed considering first- and second-order dynamics, where the transmission topology can be either leaderless or leader-following. In leader-following MNNs, the leader guides the motion of the whole network, and the followers should converge to the states of the leader by sharing information among nodes to realize synchronization control. To this end, a control law should be devised for all the followers to agree with the leader in some states as time proceeds.

Various studies have considered continuous-time nodes with first- and second-order dynamics.\textsuperscript{11–14} In Qin et al.,\textsuperscript{13} a sampled-data synchronization strategy was adopted for nodes modeled by double-integrator dynamics with both fixed and dynamic transmission topologies. In Xiao et al.,\textsuperscript{14} partial state synchronization problems were addressed for second-order MNNs, and an asynchronous distributed synchronization protocol with intermittent information transmission was proposed. Compared with the continuous-time approaches, the discrete control results may apply to more practical conditions; thus, the discrete control method should be studied further. In Qin et al.,\textsuperscript{15} You and Xie\textsuperscript{16} and Hengster-Movric et al.,\textsuperscript{17} synchronization strategies for a number of nodes equipped with second-order discrete-time dynamics were introduced. In Li et al.,\textsuperscript{18} synchronization control for discrete-time
leader-following MNNs was addressed considering a directed fixed topology and a limited transmission rate. Likewise, synchronization control problems of discrete-time MNNs were discussed considering the influence of time delays. In MNNs, the transmission topology usually varies due to link failures or other reasons. Therefore, switching transmission topologies should be handled for discrete-time WNN synchronization to optimize the performance of the whole network.

Most existing studies focus on synchronization problems for MNNs. However, the limited resources in real implementations should be considered while optimizing synchronization in MNNs. In Liu et al., an optimal synchronization strategy was proposed for continuous- and discrete-time models with single-integrator dynamics. In Wang et al., an optimal control algorithm based on minimization of an individual cost function by local states was proposed. In Guan et al., Cheng and Ugrinovskii, and Yu et al., guaranteed performance synchronization for MNNs was addressed, but the control consumption cost was disregarded. Moreover, the synchronization performance index was not jointly considered with the control consumption cost for the synchronization strategies. Although guaranteed cost control problems were addressed for discrete-time isolated systems, these approaches cannot be directly used for guaranteed cost synchronization in MNNs. This problem was addressed in Xu et al., where robust discrete-time MNNs synthesis was considered with the control consumption cost being disregarded. In Ren and Zhang, although control consumption cost was considered for MNNs containing time-varying delays, the synchronization regulation performance of MNNs with parameter uncertainties was investigated in the same study. Although control consumption cost and synchronization regulation performance were considered, these studies neglected the overall cost limitations in MNNs.

In this paper, we investigate the guaranteed cost synchronization control for leader-following discrete-time MNNs with varying transmission topologies. On the basis of the relative states between nodes, the synchronization control law with the cost consumption index function and varying topologies are constructed. Using the characteristic of the orthonormal matrix, the dynamics of the state errors between the nodes are modeled. For discrete-time MNN synchronization without cost limitations, a new design approach using lemmas is proposed to design the control parameters of gain matrices, and sufficient conditions are derived. Furthermore, an upper bound on the cost consumption index is computed according to the initial state of the proposed Lyapunov function candidate. For discrete-time MNN synchronization with cost limitations, guaranteed cost synchronization sufficient criteria are derived, in which we determine the control parameters that relate the cost limitations and the upper bound on the cost function.

Compared with existing studies on MNN synchronization, we provide various contributions from our study. First, a guaranteed cost synchronization control law and a cost consumption index function are presented for discrete-time MNN with varying transmission topologies, where the tradeoff between synchronization regulation performance and control consumption cost is considered. Most existing studies have only addressed guaranteed performance control without control cost optimization for discrete-time MNNs. Second, the impacts of cost limitations on synchronization criteria are considered to redefine the sufficient conditions with the additional constraints, which means that the whole cost consumption for discrete-time MNNs can be constrained and is applicable in practice. In Zhou et al. and Wang et al., a different upper bound on the cost consumption index was reported, and the discrete-time MNNs did not consider cost limitations in those studies.

The remainder of the study is organized as follows. The second section describes the preliminaries and problem description used in this study. In the third section, we state the sufficient conditions of guaranteed cost synchronization for discrete-time MNNs without cost limitations, derive the explicit upper bound on the cost function, and finally extend the synchronization criteria to include cost limitations. In the fourth section, it reports numerical simulation examples and their results as the effectiveness evidence for the proposed methods. Finally, we draw conclusions in the fifth section.

Notations: We use the following general notations throughout this study. $\mathbb{R}^k$ is the $k$-dimensional Euclidean space of nodes, and $\mathbb{R}^{k \times g}$ is used to describe $k \times g$-dimensional matrices over real field. Value 0 represents both the scalar zero and a column vector of zeroes with appropriate dimensions. $I_m$ represents the $m$-dimensional identity matrix. $Z^T = Z > 0$ and $Z^T = Z < 0$ denote symmetric matrix $Z$ being positive and negative definite, respectively. $\text{diag}(u_1, u_2, \cdots, u_N)$ denotes a diagonal matrix with elements $u_1, u_2, \cdots, u_N$ along the diagonal. Symbol $*$ in a matrix indicates a symmetric element.

Preliminaries and problem description

For the following analysis, we first introduce the model of the topology graph and the graph theory notation. Problem descriptions are then provided.

Graph model of transmission topology

A second-order leader-following MNN consists of $N$ nodes, which can be represented as weighted graph $G = (V, E)$. $\mathcal{V}(G) = \{v_1, v_2, \cdots, v_N\}$ is a limited collection of vertices and $\mathcal{E}(G) \subseteq \mathcal{V}(G) \times \mathcal{V}(G)$ is a nonempty collection of edges in graph $G$. All vertices belong to a finite index set $\omega = \{1, 2, \cdots, N\}$. An edge of the MNN is denoted by $e_{ij} = (v_i, v_j)$, which means there is a transmission channel from $v_i$ to $v_j$, and $v_j$ is the neighbor of $v_i$. In addition, the coupling weight of edge $e_{ij}$ is denoted...
by $a_{ji}$. $N_j = \{ (v_j, v_i) \in \mathcal{E}(G) \}$ is the set of all neighbors of vertex $j$. A path in a directed graph from node $v_j$ to node $v_i$ is an ordered sequence $v_j, v_{j_1}, \ldots, v_{j_{k-1}}, v_i$ of distinct vertices, such that $(v_{j_k}, v_{j_{k-1}}) \in \mathcal{E}(G)$, $k = 1, 2, \ldots, g - 1$. The Laplacian matrix $L$ of a weighted graph $G$ is represented as $L = \{ l_{ij} \in \mathbb{R} \}^{N \times N}$, where $l_{ji} = \sum_{i \in N} a_{ij}$. $G$ has a directed spanning tree, whose root vertex is connected to all the other vertices via paths. We suppose that there exists a spanning tree in the entire topology of the second-order MNN and designate the root vertex as the leader. The transmission states of the leader are independent, whereas those of the followers are influenced by the states of the leader and some neighbors, and the local transmission topology among other neighbors is undirected. More details on related concepts of graph theory can be found in Godsil and Royle.\textsuperscript{37}

In order to prove obtained results, the lemmas stated below are crucial for us.

**Lemma 1.** For any square matrix $A \in \mathbb{R}^{m \times m}$, there exists a positive matrix $P \in \mathbb{R}^{m \times m}$ satisfying $A^T PA - P + T < 0$ if and only if there exists a matrix $X > 0$ such that
\[
\begin{bmatrix}
-X & AX \\
X^T & -X + XTX
\end{bmatrix} < 0.
\]

**Lemma 2.** Let $P$ and $H$ be real symmetric matrices with $H$ being invertible.\textsuperscript{39} Then,
\[
\begin{bmatrix}
P & Y \\
* & -H
\end{bmatrix} < 0 \iff \begin{cases}
H > 0, \\
P + YH^{-1}Y^T < 0.
\end{cases}
\]

**Problem description**

Let a second-order discrete-time MNN consist of a group of $N$ linear identical nodes, which are modeled as vertices in transmission topology graph $G$. Root node 1 is represented as a leader, and the others correspond to $N - 1$ followers. A discrete-time dynamic description of the $j$th node is given by
\[
\begin{align*}
x_j(r+1) &= x_j(r) + v_j(r), \\
v_j(r+1) &= v_j(r) + u_j(r),
\end{align*}
\]
where $x_j(r) \in \mathbb{R}$ is the queue length, $v_j(r) \in \mathbb{R}$ is the transmission rate, and $u_j(r) \in \mathbb{R}$ is the coupling input for $j = 1, 2, \ldots, N$. Update instant $r$ takes values $r = 0, 1, \ldots$. Assume that the states of the leader remain fixed and $u_1(r) = 0$. Let all possible transmission topologies be described by set $\Pi = \{ G_1, G_2, \ldots, G_M \}$ with finite index set $I_M = \{ 1, 2, \ldots, M \}$ and $\emptyset: [0, 1, \ldots] \rightarrow I_M$ denote a switching signal. In addition, assume that switching instants satisfy $i_d - i_{d-1} \geq T_d$ for switching instant $i_d (d = 1, 2, \ldots)$ and dwell time $T_d > 0$. To obtain guaranteed cost synchronization for the second-order MNN represented by (1), we construct a distributed neighbor-based synchronization law:
\[
\begin{align*}
u_j(r) &= \sum_{i \in N_{\emptyset}(j)} a_{j(i), i} (k_1 (x_i(r) - x_j(r))) \\
&\quad + k_2 (v_i(r) - v_j(r)), \\
J_\Phi &= \sum_{r=0}^{\infty} (J_u(r) + J_s(r)),
\end{align*}
\]
where $s_1, s_2 \in \mathbb{R} > 0$ and $\xi \in \mathbb{R} > 0$ are available, $k_1 > 0$ and $k_2 > 0$ are control parameters, $j = 2, 3, \ldots, N$. $N_{\emptyset(j)}$ corresponds to the neighbor set of node $j$ at instant $r$, and
\[
\begin{align*}
J_u(r) &= \sum_{j=2}^{N} \xi u_j^2(r), \\
J_s(r) &= \sum_{j=2}^{N} \sum_{i \in N_{\emptyset}(j)} a_{j(i), i} (s_1 (x_i(r) - x_j(r))^2 \\
&\quad + s_2 (v_i(r) - v_j(r))^2).
\end{align*}
\]

Note that $a_{\emptyset(j), i}$ appears in the synchronization control law given by (2), and hence data can be exchanged among nodes and local neighboring nodes. Therefore, $J_u(r)$ and $J_s(r)$ can be designed using the control inputs and relative state information, respectively. In addition, $J_u(r)$ and $J_s(r)$ can correspond to the overall consumption cost containing both the control effort consumption and synchronization regulation performance for a leader-following MNN in practice.

Considering the necessity of MNN guaranteed cost synchronization, we choose positive constant parameters $s_1, s_2$, and $\xi$ to balance the tradeoff, in which the consumption cost of the nodes can be reduced, and the synchronization performance can be optimized. Moreover, considering that nodes have limited cost budgets in practice, we set the total cost limitation as a constraint when calculating the control law parameters for guaranteed cost synchronization.

Let $J_\Phi > 0$ be the total cost limitation of the MNN represented by (1). Guaranteed cost synchronization is defined as follows for this discrete-time leader-following MNN with switching transmission topologies and cost limitations.

**Definition 1.** For any given $J_\Phi > 0$, the discrete-time leader-following MNN represented by (1) achieves guaranteed cost synchronization by the control law in (2) if there exist control parameters $k_1$ and $k_2$ such that
\[
\lim_{r \to \infty} (x_j(r) - x_1(r)) = 0, \quad \lim_{r \to \infty} (v_j(r) - v_1(r)) = 0,
\]
and $J_\Phi \leq J_\Phi$ for any bounded disagreement initial conditions $x_j(0)$ and $v_j(0) (j = 2, 3, \ldots, N)$.

We focus on the effect of switching transmission topologies and cost limitations on guaranteed cost synchronization of the discrete-time MNN represented by (1).
Remark 1

In comparison with the guaranteed cost synchronization laws in Ren and Zhang30 and Xi et al.,32,33 two key aspects can be highlighted from the synchronization law in (2). First, the relative states between nodes instead of the node states are used to construct the synchronization control law. This is because the node states may not converge to zero, but it is noted that the state errors among nodes converge to zero. Moreover, the cost consumption index function upper bound is more difficult to calculate, because it involves the node state errors and discrete-time control parameters. Second, the cost limitations are considered, which means that the whole cost supply is limited and modeled as the constrained conditions. Thus, the sufficient conditions should reflect the coupling relations between the upper bound on the cost consumption function and the cost limitations, where the real consumption cost must be lower than the cost limitations.

Guaranteed cost synchronization for MNNs

We address the cases without and with cost limitations for second-order discrete-time MNNs with switching transmission topologies. Without cost limitations, we derive the sufficient condition to solve the control parameters for the guaranteed cost synchronization and calculate a cost consumption function upper bound that reflects both the control effort consumption and the synchronization performance regulation. We also provide a sufficient condition when considering the cost limitations.

To facilitate the analysis, the Laplacian matrix \( L_{\theta(t)}^T \) corresponds to the transmission topology at time \( r \) among follower nodes and the transmission weight matrix from the leader to the followers at time \( r \) is defined as \( \Lambda_{\theta(t)}^T = \text{diag}\{a_{\theta(t),21}, a_{\theta(t),31}, \ldots, a_{\theta(t),N1}\} \). We also assume the existence of spanning trees in all switching topologies and undirected local transmission topologies among followers. Then, there exists orthonormal matrix \( W_{\theta(t)} \) such that \( W_{\theta(t)}^T \left( L_{\theta(t)}^T + \Lambda_{\theta(t)}^T \right) W_{\theta(t)} = \text{diag}\{\lambda_{\theta(t),21}, \lambda_{\theta(t),31}, \ldots, \lambda_{\theta(t),N1}\} \), where \( 0 < \lambda_{\theta(t),21} \leq \lambda_{\theta(t),31} \leq \cdots \leq \lambda_{\theta(t),N} \). Let \( \lambda_2 = \min\{\lambda_{\theta(t),21}\} \), \( \lambda_N = \max\{\lambda_{\theta(t),N}\} \), \( \Delta x_j(r) = x_j(r) - x_l(r) \), \( \Delta v_j(r) = v_j(r) - v_l(r) \), and from (1)–(3), we obtain

\[
\begin{align*}
\Delta x_j(r + 1) &= \Delta x_j(r) + \Delta v_j(r), \\
\Delta v_j(r + 1) &= \Delta v_j(r) + \sum_{i \in N_{\theta(t)}(j)} a_{\theta(t),ij} (k_1 (\Delta x_i(r) - \Delta x_j(r)) \ldots + k_2 (\Delta v_i(r) - \Delta v_j(r))).
\end{align*}
\]

(4)

Then,

\[
\begin{align*}
\tilde{W}_{\theta(t)}^T \left[ \Delta x_2(r + 1) - \Delta x_3(r), \Delta x_3(r + 1) - \Delta x_3(r), \\
&\cdots, \Delta x_N(r + 1) - \Delta x_3(r), \right] \\
= &\left[ x_2(r), x_3(r), \ldots, x_N(r) \right]^T, \\
\tilde{W}_{\theta(t)}^T \left[ \Delta v_2(r + 1) - \Delta v_3(r), \Delta v_3(r + 1) - \Delta v_3(r), \\
&\cdots, \Delta v_N(r + 1) - \Delta v_3(r) \right]^T \\
= &\left[ v_2(r), v_3(r), \ldots, v_N(r) \right]^T.
\end{align*}
\]

(5)

Let \( \chi_j(r) = \left[ \bar{x}_j(r), \bar{v}_j(r) \right]^T \). By (3)–(5), the MNN represented by (1) can be expressed as

\[
\chi_j(r + 1) = \left[ -k_1 \lambda_{\theta(t),j} 1 - k_2 \lambda_{\theta(t),j} \right] \chi_j(r),
\]

(6)

for \( j = 2, 3, \ldots, N \). Note that \( \tilde{W}_{\theta(t)} \) is an orthonormal matrix, and thus \( \Delta x_j(r) = 0 \) and \( \Delta v_j(r) = 0 \) when \( \bar{x}_j(r) = 0 \) and \( \bar{v}_j(r) = 0 \). Hence, the synchronization of a second-order discrete-time MNN represented by (1) can be achieved.

The main result of guaranteed cost synchronization is provided as the following theorem for a second-order discrete-time MNN represented by (1) with switching transmission topologies.

Theorem 1. The discrete-time leader-following MNN represented by (1) achieves guaranteed cost synchronization by the control law in (2) if there exist matrix \( W > 0 \) and matrix \( X = X^T > 0 \) such that \( \tilde{Q}_j < 0(j = 2, N) \), where

\[
\tilde{Q}_j = \begin{bmatrix}
-X & AX & -\tilde{A} & BW & 0 & 0 \\
* & -X & X & W^T & 0 \\
* & * & -\tilde{A} & 0 & 0 \\
* & * & * & -\frac{1}{\lambda_N^2} & 0 \\
\end{bmatrix},
\]

(7)

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},
\]

\[
B = [0, 1]^T,
\]

and \( Q = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}^{-1} \). In this case, \( K = [k_1, k_2] = WX^{-1} \).
Proof: Given control input $u(r)$, sufficient conditions such as $\lim_{r \to \infty} \bar{x}_j(r) = 0$ and $\lim_{r \to \infty} \bar{y}_j(r) = 0$ ($j = 2, 3, \ldots, N$) are first stated to achieve guaranteed cost synchronization. We apply a following Lyapunov-Krasovskii functional candidate approach for MNN represented by (1):

$$V_j(r) = x_j^T(r)P x_j(r),$$

where $j = 2, 3, \ldots, N$ and $P \in \mathbb{R}^{2 \times 2}$ is a positive definite and symmetric matrix. Hence, $V_j(r) \geq 0$. Moreover, under the state trajectory in (6), we obtain

$$\Delta V_j(r) = V_j(r + 1) - V_j(r)$$

$$= x_j^T(r) \left( \begin{bmatrix} 1 & -k_1 \lambda_{0\theta_0,j} \\ 1 & -k_2 \lambda_{0\theta_0,j} \end{bmatrix} - P \right) x_j(r).$$

By Theorem 1, we can conclude that

$$\Delta V_j(r) \leq -x_j^T(r) \left[ \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} \xi k_1 k_2 + \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \right] x_j(r) \leq 0,$$

and $\Delta V_j(r) = 0$ if and only if $\bar{x}_j(r) = 0$ and $\bar{y}_j(r) = 0$. Hence, the MNN represented by (1) with the control law in (2) can achieve synchronization. Next, we analyze the guaranteed cost synchronization condition. From (2), we can obtain

$$J_o = \sum_{j=0}^{\infty} \left( J_0(r) + J_0(r) \right)$$

$$= \sum_{j=0}^{\infty} \left( \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} \xi k_1 k_2 + \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \right) x_j(r)$$

$$+ 2 \lambda_{0\theta_0,j} \left[ \begin{bmatrix} s_1 \\ 0 \\ s_2 \end{bmatrix} x_j(r) \right].$$

Hence, we obtain

$$J_o = \sum_{j=0}^{\infty} \left( J_0(r) + J_0(r) \right)$$

$$= \sum_{j=0}^{\infty} \left( \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} \xi k_1 k_2 + \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \right) x_j(r)$$

$$+ 2 \lambda_{0\theta_0,j} \left[ \begin{bmatrix} s_1 \\ 0 \\ s_2 \end{bmatrix} x_j(r) \right].$$

(14)

For $V(r) = \sum_{j=0}^{\infty} V_j(r)$, we obtain

$$J_o = \sum_{j=0}^{\infty} \left( J_0(r) + J_0(r) \right)$$

$$= \sum_{j=0}^{\infty} \left( \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} \xi k_1 k_2 + \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \right) x_j(r)$$

$$+ 2 \lambda_{0\theta_0,j} \left[ \begin{bmatrix} s_1 \\ 0 \\ s_2 \end{bmatrix} x_j(r) \right].$$

(15)

where

$$\Psi_{\theta_0,j} = \left[ \begin{matrix} \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} \xi k_1 k_2 \\ \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \\ 1 - k_1 \lambda_{0\theta_0,j} \\ 1 - k_2 \lambda_{0\theta_0,j} \end{matrix} \right].$$

By Lemma 1, there exists a matrix $P > 0$ such that $\Psi_{\theta_0,j}$ is negative if and only if there exists a matrix $B = P^{-1}$ such that

$$\Psi_{\theta_0,j} = \left[ \begin{matrix} -X \\ \tilde{\Psi}_{12, \theta_0,j} \\ \tilde{\Psi}_{22, \theta_0,j} \end{matrix} \right] < 0.$$

(16)

where

$$\tilde{\Psi}_{12, \theta_0,j} = \left[ \begin{matrix} \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} s_1 \\ \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \\ 1 - k_1 \lambda_{0\theta_0,j} \end{matrix} \right] X, \left[ \begin{matrix} 1 & 1 \\ 1 - k_2 \lambda_{0\theta_0,j} \end{matrix} \right].$$

$$\tilde{\Psi}_{22, \theta_0,j} = -X + X$$

$$\times \left[ \begin{matrix} \xi \lambda_{0\theta_0,j}^2 k_1^2 + 2 \lambda_{0\theta_0,j} s_1 \\ \xi \lambda_{0\theta_0,j}^2 k_2^2 + 2 \lambda_{0\theta_0,j} s_2 \end{matrix} \right] X.$$

Then, by the Schur complement and (16), we can obtain
Let $K = [k_1, k_2]$, $W = KX$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = [0, 1]^T$, and $Q = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}^{-1}$. Then, (17) is expressed as

$$\tilde{\Omega}_{\theta(i),j} < 0,$$

where

$$\tilde{\Omega}_{\theta(i),j} = \begin{bmatrix} -X & AX - \lambda \theta(i),j BW & 0 & 0 \\ * & -X & X & W^T \\ * & * & -\frac{Q}{\lambda_{\theta(i),j}} & 0 \\ * & * & * & -\frac{1}{\lambda_{\theta(i),j}} \end{bmatrix}.$$ 

According to the convexity of linear matrix inequalities, if $\tilde{\Omega}_{\theta(i),j}$ is negative for $j = 2, N$, then $\tilde{\Omega}_{\theta(i),j}$ is negative for $j = 2, 3, \ldots, N$. Hence, by the conclusion of Theorem 1, we obtain that $\tilde{\Omega}_{\theta(i),j} < 0$. In this case,

$$J_\theta \leq V(0).$$

Thus, the MNN represented by (1) achieves discrete-time guaranteed cost synchronization. From the above analysis progress, we can obtain the main result of Theorem 1.

**Remark 2**

When analyzing the synchronization conditions, we use Lyapunov stability theory and supporting lemmas. The variables are transformed by an orthonormal matrix $W_{\theta(i)}$. Based on the preliminary conditions of discrete-time control, it may be complex to determine the control parameters given as the irregular matrix form. Nevertheless, the introduced lemma allows to transform the irregular matrix into a regular form and then obtain the control parameters by linear matrix inequalities. Then, the synchronization conditions in terms of these inequalities can be obtained for the MNN represented by (1) with switching transmission topologies. In this case, when the MNN achieves guaranteed cost synchronization, the initial state value of the Lyapunov function candidate can serve as a specified upper bound on the cost consumption index function. It should be noted that according to the convexity of linear matrix inequalities, the sufficient conditions depend on non-zero eigenvalues $\tilde{\lambda}_2$ and $\tilde{\lambda}_N$. Hence, the dimensions of the matrix variables do not rely on the quantity of nodes. To reduce the computational load, we can compute $\tilde{\lambda}_2$ and $\tilde{\lambda}_N$ using the methods in Horn and Johnson. Moreover, we do not need to estimate all nonzero eigenvalues of the switching transmission topologies.

Considering the cost limitations, we prove the existence of control parameters in Theorem 2.

**Theorem 2.** For any given $J_\theta > 0$, the discrete-time leader-following MNN represented by (1) achieves guaranteed cost synchronization by the control law in (2) if there exist matrix $W > 0$ and matrix $X = X^T > 0$ such that

$$\sum_{j=2}^N x_j^T(0) P x_j(0) \leq J_\theta,$$

where

$$\tilde{\Omega}_j = \begin{bmatrix} -X & AX - \tilde{\lambda}_j BW & 0 & 0 \\ * & -X & X & W^T \\ * & * & -\frac{Q}{\lambda_j} & 0 \\ * & * & * & -\frac{1}{\lambda_j} \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$B = [0, 1]^T,$$

$$Q = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}^{-1},$$

and $P = X^{-1}$. In this case, $K = [k_1, k_2] = WX^{-1}$.

**Proof:** Given control input $u_i(r)$, by (7)–(9), we have

$$V(r) = \sum_{j=2}^N x_j^T(r) P x_j(r).$$

For $r = 0$, we obtain

$$V(0) = \sum_{j=2}^N x_j^T(0) P x_j(0).$$

Using the proof of Theorem 1, we obtain
with cost limitations, completing the proof. (1) allows to reach the guaranteed cost synchronization the discrete-time leader-following MNN represented by function index function, we can show that the practice. With the upper bound on the cost consumption index function can be calculated as a consumption index function upper bound value and can ensure a limited cost satisfying the consumption cost. Then, the coupling correlation between the cost can ensure that matrix $\mathcal{L}$ in Theorem 1 is negative and ensure the guaranteed cost synchronization. In addition, an explicit upper bound on the cost consumption index function can be calculated as a function of the initial state errors and control parameters. We introduce cost limitations in Theorem 2 to obtain the sufficient conditions that can be applied in practice. With the upper bound on the cost consumption index function, we can show that $J_\Phi \leqslant V(0) \leqslant J_\Phi^*$ can ensure a limited cost satisfying the consumption cost. Then, the coupling correlation between the cost consumption index function upper bound value and the cost limitations can be established. Hence, the sufficient conditions can ensure that the cost function remains below the constraint values. Moreover, the additional factors are still relevant to eigenvalues $\bar{\lambda}_2$ and $\bar{\lambda}_3$, and thus no more eigenvalues should be calculated, maintaining the computational load from the case without cost limitations.

**Remark 3**

By the discrete-time control law in (2), the node dynamics can be represented using (6). By the Lyapunov stability theory and supporting lemmas, the sufficient conditions with regard to linear matrix inequalities can ensure that matrix $\mathcal{L}_i$ in Theorem 1 is negative and maintain the guaranteed cost synchronization. Moreover, the upper bound on the cost consumption index function can be calculated as a function of the initial state errors and control parameters. We introduce cost limitations in Theorem 2 to obtain the sufficient conditions that can be applied in practice. With the upper bound on the cost consumption index function, we can show that $J_\Phi \leqslant V(0) \leqslant J_\Phi^*$ can ensure a limited cost satisfying the consumption cost. Then, the coupling correlation between the cost consumption index function upper bound value and the cost limitations can be established. Hence, the sufficient conditions can ensure that the cost function remains below the constraint values. Moreover, the additional factors are still relevant to eigenvalues $\bar{\lambda}_2$ and $\bar{\lambda}_3$, and thus no more eigenvalues should be calculated, maintaining the computational load from the case without cost limitations.

**Numerical simulation**

We verify the effectiveness of the proposed synchronization control method through simulations. We consider a two-dimensional MNN with six nodes labeled from 1 to 6. We set node 1 as the leader and the others correspond to five followers. The dynamics of all nodes are determined by (1). In addition, we consider the switching transmission topologies shown in Figure 1 and the switching sequences of topologies are shown in Figure 2. We assume that all edge weights are 1 and the adjacency matrices are 0–1 matrices. Then, the minimum nonzero and maximum eigenvalues of all the corresponding Laplacian matrices are $\bar{\lambda}_2 = 0.4038$ and $\bar{\lambda}_3 = 4.2631$, respectively. Moreover, we evaluate the following initial states for the nodes: 

\[ [x_1(0), v_1(0)]^T = [4.0, 2.1]^T, [x_2(0), v_2(0)]^T = [5.6, 2.1]^T, [x_3(0), v_3(0)]^T = [3.1, 1.2]^T, [x_4(0), v_4(0)]^T = [5.9, 2.7]^T, [x_5(0), v_5(0)]^T = [4.7, 0.9]^T, [x_6(0), v_6(0)]^T = [8.0, 1.7]^T. \]

For the cost function in (2), we set constant parameters $\xi = 9.11$, $\varsigma_1 = 9.02$, and $\varsigma_2 = 9.01$. From Theorem 2, we obtain $k_1 = 0.12$ and $k_2 = 0.15$. Moreover, the cost limitation is $J_\Phi = 24000$, which is an upper bound value on the cost consumption index function in (2) and includes both the control effort consumption and the synchronization regulation performance.

**Figure 1.** Switching transmission topology set.

**Figure 2.** Switching signal $\phi(r)$.
limitations. In the existing literature, authors investigated the continuous-time guaranteed cost synchronization problem, where the control effort was evaluated and an upper bound of the cost function was presented. Distinguished from Wang et al., this study can achieve the discrete-time guaranteed cost synchronization.
considering both switching transmission topologies and cost limitations for the MNNs.

Conclusion

For a discrete-time leader-following MNN with switching transmission topologies, we proposed a guaranteed cost synchronization control law with a cost consumption index function. The tradeoff between synchronization regulation performance and control consumption cost was balanced. In addition, Lyapunov stability theory and supporting lemmas were allowed to derive the sufficient conditions without cost limitations with regard to linear matrix inequalities, where the dimensions of the matrix variables did not rely on the quantity of nodes. Moreover, we determined a specified upper bound on the cost function, which is related to the initial relative states and control parameters. Then, we extended the leader-following guaranteed cost synchronization to include cost limitations, where the coupling relation between the cost limitation and the upper bound on the cost consumption index function was considered. The resulting criteria to achieve guaranteed cost synchronization only required checking whether the matrix variations were negative. Especially, further works mainly focus on designing the synchronization control law for the discrete-time leader-following MNN with jointly connected transmission topologies. It should also be mentioned that some new synchronization control approaches are required to solve these problems.

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