Distance rationalization of anonymous and homogeneous voting rules

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Abstract
The concept of distance rationalizability of voting rules has been explored in recent years by several authors. Roughly speaking, we first choose a consensus set of elections (defined via preferences of voters over candidates) for which the result is specified a priori (intuitively, these are elections on which all voters can easily agree on the result). We also choose a measure of distance between elections. The result of an election outside the consensus set is defined to be the result of the closest consensual election under the distance measure. Most previous work has dealt with a definition in terms of preference profiles. However, most voting rules in common use are anonymous and homogeneous. In this case there is a much more succinct representation (using the voting simplex) of the inputs to the rule. This representation has been widely used in the voting literature, but rarely in the context of distance rationalizability. We show exactly how to connect distance rationalizability on profiles for anonymous and homogeneous rules to geometry in the simplex. We develop the connection for the important special case of votewise distances, recently introduced and studied by Elkind, Faliszewski and Slinko in several papers. This yields a direct interpretation in terms of well-developed mathematical concepts not seen before in the voting literature, namely Kantorovich (also called Wasserstein) distances and the geometry of Minkowski spaces. As an application of this approach, we prove some positive and some negative results about the decisiveness of distance rationalizable anonymous and homogeneous rules. The positive results connect with the recent theory of hyperplane rules, while the negative ones deal with distances that are not metrics, controversial notions of consensus, and the fact that the $\ell^1$-norm is not strictly convex. We expect that the above novel geometric interpretation will aid the analysis of rules defined by votewise distances, and the discovery of new rules with desirable properties.

Keywords Social choice theory · Collective decision-making · Rankings · Homogeneity · Anonymity · Simplex · Wasserstein metric · Kantorovich distance · Earth Mover distance

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1 Introduction

We are interested in the relation between two ways of describing voting rules (interpreted broadly), each of which has a geometric flavour.

The class of anonymous and homogeneous voting rules includes all rules used in practice, and most rules appearing in the research literature (Dodgson’s rule is a notable exception). For such rules there is an obvious concise way to describe an input profile of preferences, using the vote simplex. This approach goes back at least as far as Young (1975) and was extensively developed and popularized by Saari (1994). By allowing us to use geometric intuition, it aids in the analysis of many properties of anonymous and homogeneous voting rules.

The framework of distance rationalizability is a useful way to organize the huge number of voting rules that have been introduced. By decomposing a rule into a consensus and a notion of distance to that consensus, the framework allows systematic derivation of axiomatic properties of the rule from those of its components. This kind of analysis has been carried out recently by Elkind et al. (2012, 2015) and the present authors Hadjibeyli and Wilson (2016), following early work by Nitzan (1981), Lerer and Nitzan (1985), Campbell and Nitzan (1986) and Meskanen and Nurmi (2008).

Until now, the two approaches described above have not been explicitly connected. Specific distance-based rules have indeed been studied in the simplex or permutahedron, notably by Zwicker (2008a,b) and Cervone et al. (2012). However, a more general approach is lacking. As shown in Hadjibeyli and Wilson (2016), the theory can be developed simultaneously for social choice rules and social welfare rules, and for very general distances and consensus notions, in a way that clarifies the relationship between the profile-based and simplex-based representations.

1.1 Outline of paper and our contribution

In Sect. 2 we cover the basic notation and terminology. A rule defined directly on the simplex is automatically anonymous and homogeneous. Conversely, every anonymous and homogeneous rule can instead be defined on the simplex. In Sect. 3 we show how the usual distance rationalizability approach on profiles connects with the geometric approach on the vote simplex.

The methods of Zwicker and Saari, starting with the simplex or related geometric representations of the space of preference profiles and using usual Euclidean geometry, have led both to the discovery of some new anonymous and homogeneous rules and to improved analysis of some old ones. We give an example in Proposition 3.11 of some new rules defined in an intuitive way. However, as it seems that in order to find more interesting rules we must dig deeper and use less well-known consensuses or distances, we focus instead on the analysis of rules defined on profile space, using our geometric machinery.

Starting with an anonymous and homogeneous rule defined via distance rationalization on profile space, we consider the corresponding rule on the simplex and interpret it geometrically. Abstractly, this is straightforward if we use a quotient distance, a general construction which is nontrivial to compute in general. However, in the situation
of the present paper, we can give simple explicit formulae (Proposition 3.15). We focus in Sect. 3.3 on the special case of $\ell^p$-votewise distances, which have been shown by Elkind, Faliszewski and Slinko (and the present authors) to have many desirable properties. We sharpen further the description above to show that in this case, the quotient distance on the simplex is a Wasserstein (also called Kantorovich) distance, a concept widely used in probability theory and its applications in computer science. In particular when $p = 1$ (the most natural case for voting) each such distance is induced by a norm, and we can interpret everything in terms of the geometry of finite-dimensional normed spaces (also called Minkowski spaces). This provides a new perspective to the voting literature and suggests not only new voting rules defined using geometric intuition, but also a new geometric tool for the analysis of existing rules.

In particular, in Sect. 4 we use the simplex representation to explore the decisiveness (how often it gives a unique winner) of a distance rationalizable rule. On the positive side, we give a sufficient condition in Corollary 6.9 for a rule to be a hyperplane rule and thus admit a vanishingly small fraction of profiles where ties occur. For example, any rule defined using an $\ell^p$-votewise distance and the strong or weak unanimity consensus satisfies this condition. On the negative side, we show in Proposition 5.1 that ties can occur in a large fraction of profile space if we use $\ell^1$-votewise metrics, unless the notion of consensus is very well chosen. This sheds light on some common consensus notions and casts some doubt on that of Condorcet. In Sect. 7 we make some recommendations for desirable properties of consensus sets and distances.

The approach adopted here and in Hadjibeyli and Wilson (2016), following Elkind et al. (2015), allows for systematic exploration of the space of aggregation rules and the construction of rules with guaranteed axiomatic properties. We expect that more insight into distance-based voting rules will be obtained by exploiting the deeper geometric connections developed here.

2 Basic definitions

We use standard concepts of social choice theory. Not all of these concepts have completely standardized names.

Definition 2.1 We fix a finite set $C = \{c_1, c_2, \ldots, c_m\}$ of candidates and an infinite set $V^* = \{v_1, v_2, \ldots, \} \subset V^*$ of potential voters. For each $s$ with $1 \leq s \leq m$, an $s$-ranking is a strict linear order of $s$ elements chosen from $C$. The set of all $s$-rankings is denoted $L_s(C)$. When $s = m$ we write simply $L(C)$. When $s = 1$ we identify $L_1(C)$ with $C$ in the natural way.

Definition 2.2 A profile is a function $\pi : V \to L(C)$ where $V \subset V^*$ is finite. We denote the set of all profiles with fixed $C$ and $V$ by $\mathcal{P}(C, V)$ and the set of all profiles by $\mathcal{P}(C)$. An election is a triple $(C, V, \pi)$ with $\pi \in \mathcal{P}(C, V)$. We denote the set of all elections with fixed $C$ and $V$ by $\mathcal{E}(C, V)$, and the class of all elections by $\mathcal{E}$.

Definition 2.3 A social rule of size $s$ is a function $R$ that takes each election $E = (C, V, \pi)$ to a nonempty subset of $L_s(C)$. When there is a unique $s$-ranking chosen, the word “rule” becomes “function”. When $s = 1$, we have the usual social choice function, and when $s = |C|$ the usual social welfare function.
For each subset \( D \) of \( \mathcal{E} \) we can consider a **partial social rule with domain** \( D \) to be defined as above, but with domain restricted to \( D \).

### 2.1 Consensus

Intuitively, a consensus is simply a socially agreed unique outcome on some set of elections. We now define it formally.

**Definition 2.4** An \( s \)-**consensus** is a partial social function \( K \) of size \( s \). The domain \( D(K) \) of \( K \) is called an \( s \)-**consensus set** and is partitioned into the inverse images \( K_r := K^{-1}(\{r\}) \).

Several specific consensuses have been described in the literature. We list a few important ones. Some have been discussed by previous authors only in the case \( s = 1 \) but the definitions extend naturally.

**Definition 2.5** We use the following consensuses in this article.

- We denote by \( S^s \) the consensus \( K \) for which \( K_r \) is the election in which all voters agree that \( r \) is the ranking of the top \( s \) candidates. When \( s = |C| \), we simply write \( S \) (called the **strong unanimity consensus**), whereas when \( s = 1 \), for consistency with previous authors we denote it \( W \), the **weak unanimity consensus**.
- The \( 1 \)-**Condorcet** consensus \( C \) has domain consisting of all elections for which a Condorcet winner exists. That is, there is a candidate \( c \) (the Condorcet winner) such that for every other candidate \( b \) a fraction strictly greater than \( 1/2 \) of voters rank \( c \) above \( b \).

### 2.2 Distances

We require a notion of distance on elections. We aim to be as general as possible.

**Definition 2.6** (distance) A **distance** (or **hemimetric**) on \( \mathcal{E} \) is a function \( d : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}^+ \cup \{\infty\} \) that satisfies for all \( x, y, z \in \mathcal{E} \)

- \( d(x, x) = 0 \),
- \( d(x, z) \leq d(x, y) + d(y, z) \).

A **pseudometric** is a distance that also satisfies \( d(x, y) = d(y, x) \). A **quasimetric** is a distance that also satisfies \( d(x, y) = 0 \Rightarrow x = y \). A **metric** is a distance that is both a quasimetric and a pseudometric. We call a distance **standard** if \( d(E, E') = \infty \) whenever \( E \) and \( E' \) have different sets of voters or candidates.

One commonly used class of distances consists of the **votewise** distances (Elkind et al. 2015) (Definition 2.11 below). First we require some preliminary definitions.

**Example 2.7** (commonly used distances on \( L(C) \)) We discuss the following distances on \( L(C) \) in this article.
• The **discrete metric** $d_H$, defined by
$$d_H(\rho, \rho') = \begin{cases} 
1 & \text{if } \rho = \rho' \\
0 & \text{otherwise.}
\end{cases}$$

• The **inversion metric** $d_K$ (also called the swap, bubblesort or Kendall-$\tau$ metric),
where $d_K(\rho, \sigma)$ is the minimum number of swaps of adjacent elements needed to change $\rho$ into $\sigma$.
• **Spearman’s footrule** $d_S$, defined by
$$d_S(\rho, \rho') := \sum_{c \in C} |rk(\rho, c) - rk(\rho', c)|.$$  
Here $rk(\rho, c)$ denotes the rank of $c$ in the preference order $\rho$.

**Definition 2.8** A **seminorm** on a real vector space $X$ is a real-valued function $N$ satisfying the identities

- $N(x + y) \leq N(x) + N(y)$
- $N(\lambda x) = |\lambda|N(x)$

for all $x, y \in X$ and all $\lambda \in \mathbb{R}$. Note that this implies that $N(0) = 0$ and $N(x) \geq 0$ for all $x \in X$.

A **norm** is a seminorm that also satisfies
- $N(x) = 0 \Rightarrow x = 0$.

**Remark 2.9** Every seminorm induces a pseudometric via $d(x, y) = ||x - y||$. This is a metric if and only if the seminorm is a norm.

**Example 2.10** Consider an $n$-dimensional space $X$ with fixed basis $e_1, \ldots, e_n$ and corresponding coefficients $x_i$ for each element $x \in X$. Fix $p$ with $1 \leq p < \infty$ and define the $\ell^p$-norm on $X$ by
$$||x||_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$  
When $p = \infty$ we define the $\ell^\infty$ norm by
$$||x||_\infty = \max_{1 \leq i \leq n} |x_i|.$$

**Definition 2.11** (votewise distances) Fix a candidate set $C$ and voter set $V$, and a distance $d$ on $L(C)$. Choose a family $\{N_n\}_{n \geq 1}$ of seminorms, where $N_n$ is defined on $\mathbb{R}^n$. Extend $d$ to a function on $\mathcal{P}(C, V)$ by taking $n = |V|$ and defining for $\sigma, \pi \in \mathcal{P}(C, V)$
$$d^{N_n}(\pi, \sigma) := N_n(d(\pi_1, \sigma_1), \ldots, d(\pi_n, \sigma_n)).$$
This yields a distance on elections having the same set of voters and candidates. We complete the definition of the extended distance (which we denote by $d^N$) on $E$ by declaring it to be standard.

We use the abbreviation $d^p$ for $d^{lp}$, and sometimes we even use just $d$ for $d^N$ if the meaning is clear.

**Remark 2.12** Note that if $d$ is a metric and $N$ is a norm, then $d^N$ is a metric.

**Example 2.13** (famous votewise distances) The distances $d_H^1$ and $d_K^1$ are called respectively the **Hamming metric** and **Kemeny metric**. The Hamming metric measures the minimum number of voters whose preferences must be changed in order to convert one profile to another, and as such has an interpretation in terms of unit cost bribery (Faliszewski et al. 2009). The Kemeny metric measures the minimum number of swaps of adjacent candidates required to convert one profile to another, and is related to models of voter error (Young 1995).

**Example 2.14** (tournament distances) Given an election $E = (C, V, \pi)$, we form the pairwise tournament digraph $\Gamma(E)$ with nodes indexed by the candidates, where the arc from $a$ to $b$ has weight equal to the net support for $a$ over $b$ in a pairwise contest. Formally, there is an arc from $a$ to $b$ whose weight equals $n_{ab} - n_{ba}$, where $n_{ab}$ denotes the number of rankings in $\pi$ in which $a$ is above $b$.

Let $M(E)$ be the weighted adjacency matrix of $\Gamma(E)$ (with respect to an arbitrarily chosen fixed ordering of $C$). Given a seminorm $N$ on the space of all $|C| \times |C|$ real matrices, we define the $N$-tournament distance by

$$d^N(E, E') = N(M(E) - M(E')).$$

A closely related distance is defined in a similar way, but where each element of the adjacency matrix is replaced by its sign ($1$, $0$, or $-1$). We call this the $N$-reduced tournament distance. We denote the special cases where $N$ is the $\ell^1$ norm on matrices by $d^T$ and $d^{RT}$ respectively. Every (reduced) tournament distance is a pseudometric.

### 2.3 Combining consensus and distance

In order for a rule to be definable via the DR construction, it is necessary that the following property holds, and we shall assume this from now on.

**Definition 2.15** Let $d$ be a distance and $\mathcal{K}$ a consensus. We say that $(\mathcal{K}, d)$ distinguishes consensus choices if whenever $x \in \mathcal{K}_r$, $y \in \mathcal{K}_{r'}$ and $r \neq r'$, then $d(x, y) > 0$.

We use a distance to extend a consensus to a social rule in the natural way. The choice at a given election $E$ consists of all $s$-rankings $r$ whose consensus set $\mathcal{K}_r$ minimizes the distance to $E$. We introduce the idea of a score in order to use our intuition about positional scoring rules.
**Definition 2.16** (DR scores and rules) Suppose that \( K \) is an \( s \)-consensus and \( d \) a distance on \( E \). Fix an election \( E \in \mathcal{E} \). For each \( r \in L_s(C) \), the \((K, d, E)\)-score of \( r \) is defined by

\[
|r| := d(E, K_r) := \inf_{E' \in K_r} d(E, E').
\]

The rule \( R := \mathcal{R}(K, d) \) is defined by

\[
R(E) = \arg \min_{r \in L_s(C)} |r|.
\]  

(1)

We say that \( R \) is **distance rationalizable** (DR) with respect to \((K, d)\).

**Example 2.17** (scoring rules) The **positional scoring rule** defined by a vector \( w \) of weights with \( w_1 \geq \cdots \geq w_m \) and \( w_1 > w_m \) elects all candidates with maximum score, where the score of a in the profile \( \pi \) is defined as \( \sum_{\nu \in V} w_{rk}(\pi(\nu), a) \).

The **plurality** \((w = (1, 0, 0, \ldots, 0))\) and **Borda** \((w = (m-1, m-2, \ldots, 0))\) rules are well-known special cases.

The positional scoring rule defined by \( w \) has the form \( R(\mathcal{W}, d^1_S(w)) \) where \( d^1_S(w) \) is the generalization of \( d_S \) given by

\[
d^1_S(w)(\rho, \rho') = \sum_{c \in C} |w_{rk}(\rho, c) - w_{rk}(\rho', c)|
\]

Elkind et al. (2015, Proposition 8). Thus the Borda rule can be expressed as \( R(\mathcal{W}, d^1_S) \).

Also, the positional scoring rule defined by \( w \) has the form \( R(\mathcal{W}, d^1_K(w)) \) where \( d^1_K(w) \) is a generalized swap distance on rankings (Lerer and Nitzan 1985). Borda’s rule has the form \( R(\mathcal{W}, d^1_K) \).

**Remark 2.18** Note that \( d_w \) is a metric on \( L_s(C) \) if and only if \( w_1, \ldots, w_s \) are all distinct. The score of \( r \) under the rule defined by \( w \) is the difference \( nw_1 - |r| \). For example, for Borda with \( m \) candidates the maximum possible score of a candidate \( c \) is \((m - 1)n\), achieved only for those elections in \( \mathcal{W}_c \). Note that as far as the distance to \( \mathcal{W} \) or \( C \) is concerned, \( d^1_S \) and \( d^1_K \) are proportional, but they are not proportional in general Meskanen and Nurmi (2008, pp. 298–299). The score of \( c \) under Borda is exactly \( n(m - 1) - K \) where \( K \) is the total number of swaps of adjacent candidates needed to move \( c \) to the top of all preference orders in \( \pi(E) \).

**Example 2.19** (Copeland’s rule) **Copeland’s rule** can be represented as \( R(\mathcal{C}, d_{RT}) \). Indeed, in an election \( E \), the Copeland score of a candidate \( c \) (the number of points it scores in pairwise contests with other candidates) equals \( n - 1 - s \), where \( s \) is the minimum number of pairwise results that must be changed for \( E \) to change to an election that belongs to \( \mathcal{C}_c \).
3 Simplex rules

Although it is far from the general case, most rules used in practice are in fact anonymous and homogeneous. In this case there is an appealing geometric interpretation. The frequency distribution of votes is sufficient information to determine the output of the rule, and so profile space can be substantially compressed.

We have previously explored in detail the connection with distance rationalization (Hadjibeyli and Wilson 2016). We first recall the construction for anonymous rules. Recall that a multiset of weight \( n \) on an underlying set \( S \) of size \( M \) is “a set of \( n \) elements of \( S \) with repetitions”. Formally, there is a function \( f : S \rightarrow \mathbb{N} \) where \( f(s) \) gives the multiplicity of \( s \) in the multiset.

**Definition 3.1** Let \( E = (C, V, \pi) \in \mathcal{E} \). The vote number map \( \mathcal{N} \) is the map that associates \( E \) with the multiset \( \mathcal{N}(E) \) on \( L(C) \) of weight \( |V| \), in which the multiplicity of \( p \in L(C) \) is the number of voters in \( V \) having that preference order.

A rule \( R \) is anonymous if \( R(E) = R(E') \) whenever \( \mathcal{N}(E) = \mathcal{N}(E') \). We denote the quotient space by \( \mathcal{V} \) and call it the set of anonymous profiles.

**Remark 3.2** \( \mathcal{N}(E) \) simply keeps track of the numbers of votes of each type in \( \pi \), ignoring the identities of voters. A rule is anonymous if this information is enough to determine the output. The more usual (and equivalent) definition of anonymity is given in Remark 3.5 below.

**Definition 3.3** The vote distribution associated to \( E \) is the relative frequency distribution on \( L(C) \) corresponding to the multiset \( \mathcal{N}(E) \), which we denote \( \mathcal{D}(E) \). Explicitly, \( \mathcal{D} \) is a function from \( \mathcal{E} \) to \([0, 1]^{L(C)}\), which gives for each ranking the proportion of voters having this ranking as preference. The vote distribution map defines an equivalence relation \( \sim \) on \( \mathcal{E} \) in the usual way: \( E \sim E' \) if and only if \( \mathcal{D}(E) = \mathcal{D}(E') \).

An anonymous rule \( R \) is homogeneous if \( R(E) = R(E') \) whenever \( \mathcal{D}(E) = \mathcal{D}(E') \). We denote the quotient space by \( \mathcal{F} \), and call it the set of anonymous and homogeneous profiles.

**Remark 3.4** Note that if \( E = (C, V, \pi) \), \( E' = (C, V', \pi') \in \mathcal{E} \) and \( \mathcal{N}(E) = \mathcal{N}(E') \), then \( |V| = |V'| \). Thus there is a well-defined map \( f : \mathcal{V} \rightarrow \mathcal{F} \) (“divide by the number of voters”), and \( \mathcal{D}(E) \) simply lists each preference order according to its relative frequency in \( \pi \).

**Remark 3.5** Anonymous and homogeneous rules were already defined in the literature, but in different ways.

Let \( g \) be a permutation of \( V^* \). For each \( E = (C, V, \pi) \), define \( g(E) = (C, g(V), g(\pi)) \) where \( g(\pi)(v) = \pi(g(v)) \). A rule \( R \) is anonymous if and only if \( R(g(E)) = R(E) \) for all \( E \in \mathcal{E} \) and all \( g \).

For \( k \geq 1 \), define \( kE \) to be an election \( (C, kV, k\pi) \) where \( kV \) consists of \( k \) copies of each voter in \( V \) and \( k\pi \) the corresponding copies of their preference orders (the exact order of the voters is irrelevant since we deal only with anonymous rules). An anonymous rule \( R \) is homogeneous if and only if \( R(kE) = R(E) \) for all \( E \in \mathcal{E} \) and all \( k \).
We call anonymous and homogeneous rules **simplex rules** for short, and now explain why. So far the discussion has been coordinate-free, but it is often useful to introduce coordinates. Given any linear ordering \(\rho_1, \rho_2, \ldots, \rho_M\) on \(L(C)\), where \(|C| = m\) and \(M = m!\), we can introduce coordinates \(x_i\) such that \(x_i\) denotes the relative frequency associated to \(\rho_i\). The set of frequency distributions \(\Delta_q(L(C))\) is then coordinatized by the rational points of the standard simplex.

**Definition 3.6** The **standard simplex** in \(\mathbb{R}^M\) is the set

\[
\Delta_M := \left\{ x \in \mathbb{R}^M \mid \sum_i x_i = 1, x_i \geq 0 \text{ for } 1 \leq i \leq M \right\}.
\]

We let \(\Delta_q^M := q^M \cap \Delta_M\) denote the rational points of \(\Delta_M\).

**Remark 3.7** For simplicity we sometimes write \(x_t\) for the component of \(x \in \Delta_M\) corresponding to \(t \in L(C)\) (instead of \(x_i\) where \(i\) is the number of \(t\) in some linear ordering on \(L(C)\)).

**Example 3.8** An election on candidates \(C = \{a, b, c\}\) having 7 voters of whom 3 have preference \(a \succ b \succ c\), 2 have preference \(b \succ a \succ c\) and 2 have preference \(c \succ b \succ a\) corresponds (under the lexicographic order on \(L(C)\)) to the anonymous profile \((3, 0, 2, 0, 0, 2)\) and hence to the point \((3/7, 0, 2/7, 0, 0, 2/7)\) in \(\Delta_6\).

**Remark 3.9** We always consider \(\Delta_M\) as embedded in \(\mathbb{R}^M\). It is contained in a unique hyperplane \(H_M\), which is given by the single linear equation \(\sum_{i=1}^{M} x_i = 1\).

We can interpret each anonymous and homogeneous rule as being defined on \(\Delta_q^M\). If the rule is also **continuous** then we can in fact define it on \(\Delta_M\), which allows us to use our usual geometric intuition. For example, the domain of \(S\) consists of the corners of the simplex, while the domain of \(W\) also lies on the boundary of the simplex (for example, \(W_a\) contains all points of the simplex for whom all coordinates corresponding to rankings with \(a\) not at the top are zero).

The general approach of the last paragraph has been adopted by many previous authors. For example, Saari (1995) simply uses the terminology “profiles” to refer to vote distributions, and all rules he considers are continuous simplex rules by definition.

### 3.1 Distance rationalization in the simplex

We shall see in Sect. 3.2 how anonymous and homogeneous distance rationalizable rules defined using profiles can be interpreted using the simplex. The converse idea is to define distance rationalizable rules directly on the simplex rather than on profile space. We make the obvious definitions by analogy with those for profiles.

**Definition 3.10** Given a fixed candidate set \(C\) of size \(m\) and a distance on \(\Delta_q^M\) where \(M = m!\), a **partial social rule** on \(\Delta_q^M\) of size \(s\) with domain \(D \subseteq \Delta_q^M\) is a mapping taking each element of \(D\) to a nonempty subset of \(L_s(C)\). A **consensus** on \(\Delta_q^M\) is a...
partial social rule that is single-valued at every point (a partial social function). Given
a consensus $K$ and distance $\delta$ on $\Delta_M^Q$, the rule $R(K, \delta)$ is defined by

$$R(x) = \arg \min_{r \in L_s(C)} \delta(x, K_r). \quad (2)$$

The most obvious distances mathematically are surely the $\ell^p$ metrics. The interpretation in
terms of social choice is less compelling for $p > 1$, since we are measuring the amount of effort needed to change one election into another by transferring vote mass under a nonlinear penalty. The case $p = 1$ is by far the most commonly studied,
and also arises directly from votewise distances, unlike the case $p > 1$.

To our knowledge, several fairly obvious rules of this type have not yet been studied
in detail. Here is an example using the unanimity consensus.

**Proposition 3.11** Fix $p$ with $1 \leq p \leq \infty$ and consider the social choice rule $R(S^s, \ell^p)$
defined on $\Delta_M^Q$. This rule chooses precisely the initial $s$-ranking from each of the most
frequent ranking(s) from the input profile.

**Proof** Let $\delta$ be the $\ell^p$ distance on $\mathbb{R}^M$ and let $e_t$ denote the basis vector in $\mathbb{R}^M$
corresponding to $t \in L(C)$, a corner of the simplex. Then for $x \in \Delta_M$ and $t, t' \in L(C)$,

$$\delta^p(x, e_t) - \delta^p(x, e_{t'}) = (1 - x_t)^p + \sum_{k \neq t} x_k^p - (1 - x_{t'})^p - \sum_{k \neq t'} x_k^p$$

$$= (1 - x_t)^p - (1 - x_{t'})^p + x_{t'}^p - x_t^p.$$

Choosing $x_{t^*}$ to be the maximum value among the entries of $x$ and setting $t = t^*$
shows the right side of the above expression to be nonpositive for all $t' \neq t^*$. Hence $t^*$
is a minimizer (the same argument works for $p = \infty$ with a different computation).

Thus if $\rho^*$ denotes the initial $s$-ranking corresponding to $t^*$ (written $t^*_s = \rho^*$), then
for each $s$-ranking $\rho$, we have

$$\delta(x, S^s_{\rho^*}) = \min_{\{t: t_s = \rho\}} \delta(x, e_t) \geq \delta(x, e_{t^*}) = \delta(x, S^s_{\rho^*}).$$

\[\square\]

**Remark 3.12** For $s = m$ this rule is the modal ranking rule (Caragiannis et al. 2014)
(the term plurality ranking rule may seem more logical, but it is important not to
be confused with the ranking induced by plurality scores of candidates in the case
where these scores all differ). For $s = 1$ it is the social choice variant of the modal
ranking rule, choosing the highest ranked candidate from each modal ranking. Note
that the latter rule differs from plurality rule, which is what we would get if we used
the simplex in dimension $m$, as for example done by Saari (1995).

In order to find more interesting rules defined naturally on $\Delta_M^Q$, we may need
to use less common distances. For example, so-called statistical distances such as
the (asymmetric) Kullback–Leibler divergence (or relative entropy) and the Hellinger

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Distance rationalization of anonymous and homogeneous... are heavily used in many application areas (the simpler total variation distance arises in Example 3.22 below). We do not present a detailed study here, deferring it to future work. Instead, we now move on to explore distances on $\Delta^Q_M$ induced by distances on $E$.

### 3.2 Quotient distances

We can define a simplex rule by first starting with profiles and passing to the quotient space, provided the rule in question is anonymous and homogeneous (the case of Dodgson’s rule $R(C, d_k^L)$ which is anonymous but not homogeneous shows that care must be taken).

Since votewise distances are very natural and the $\ell^1$ norm is the most obvious choice for a votewise distance (because it just adds the distance from each voter), we obtain several rules with an $\ell^1$ flavour in this way. For example, the Hamming metric yields a constant multiple of $\ell^1$ via the Wasserstein construction as described in Example 3.22 below, while the Kemeny metric and $\ell^1$ lead to rules such as Borda and Kemeny’s rule. We discuss $\ell^1$-votewise rules in more detail in Sect. 5.

**Definition 3.13** A distance is **anonymous** if $d(g(E), g(E')) = d(E, E')$ for all $E, E' \in E$ and all permutations $g$. An anonymous distance is **homogeneous** if $d(kE, kE') = d(E, E')$ for all $E, E' \in E$ and all $k \geq 1$.

Note that votewise distances need to be normalized before becoming homogeneous, as we explain in Remark 3.20.

Every anonymous and homogeneous rule $R$ corresponds to a simplex rule $\overline{R}$. When $R = R(\mathcal{K}, d)$ where both $\mathcal{K}$ and $d$ are anonymous and homogeneous, we can express $\overline{R}$ as $R(\mathcal{K}, \delta)$ where $\mathcal{K}, \delta$ depend nicely on $\mathcal{K}, d$. The mapping from profiles to the simplex yields the obvious consensus $K = \overline{K}$. The distance $\delta$ is a little more involved. The obvious idea is to use a quotient distance (Deza and Deza 2009). This concept is standard but relatively little-known.

**Definition 3.14** We define $\overline{d} : \Delta_M \times \Delta_M \to \mathbb{R}_+$ to be the **quotient distance** induced by $\sim$. That is,

$$\overline{d}(x, y) = \inf_{E_1, E_2} \sum_{i=1}^k d(E_i, E'_i)$$

where the infimum is taken over all admissible paths, namely paths such that $E_i \sim E_{i+1}$ for $1 \leq i \leq k - 1$, $E = E_1$, $E' = E'_k$, $E$ projects to $x$ and $E'$ to $y$.

In general, quotient distances are tricky to work with, owing to the complicated definition. In our setup, it turns out that they are reasonably tractable.

**Proposition 3.15** Let $d$ be an anonymous and homogeneous standard distance on $E$. Then $\overline{d}$ is given by

$$\overline{d}(x, y) = \inf_{E'} d(E, E')$$
where $E, E'$ range over all elections having an equal number of voters, such that $D(E) = x, D(E') = y$.

**Proof** Let $x, y \in \mathcal{F}$ and consider an admissible path and its corresponding sum
\[
\sum_{i=1}^{k} d(E_i, E'_i)
\]
where $k > 1$. We show that we can reduce the value of $k$. By homogeneity of $d$, we can change the size of the voter sets so that $E_k$ and $E'_k$ have the same sized voter set. Then we can choose a permutation $g$ of $V^*$ taking $E_k$ to $E'_k$. Since $d$ is anonymous, using the triangle inequality we obtain
\[
d(E_{k-1}, E'_{k-1}) + d(E_k, E'_k) = d(E_{k-1}, E'_k) + d(g(E_k), E'_k) = d(E_{k-1}, E'_k) + d(E'_k, g(E_k)) \geq d(E_{k-1}, g(E_k)).
\]
This gives an admissible path with $k$ replaced by $k - 1$.

Thus without loss of generality, in computing $\overline{d}$ we need only deal with the case $k = 1$. Finally, if the minimum is reached for some $E = E_\ast$, then for each $E$ we can write $g(E_\ast) = E$ for some $g$. Thus by anonymity $d(E, E') = d(E_\ast, g^{-1}(E'))$. Thus the minimum can be taken only over $E'$, since $g^{-1}(E')$ ranges over all elections as $g$ ranges over all permutations.

**Remark 3.16** Proposition 3.15 shows that we can replace $\inf$ by $\min$, since by anonymity, once we fix the number of voters there are only a finite number of possible distances $d(E, E')$ to consider.

**Proposition 3.17** Let $d$ be an anonymous and homogeneous standard distance on $\mathcal{E}$, and $K$ be an anonymous and homogeneous consensus. If $R = \mathcal{R}(K, \delta)$ is anonymous and homogeneous, then $\overline{R} = \mathcal{R}(\overline{K}, \overline{d})$.

**Proof** By Definition 2.16, $R(E) = \arg\min_{r \in L_s(C)} d(E, K_r)$. By Proposition 3.15,
\[
\overline{d}(x, y) = \inf_{E'} d(E, E'),
\]
such that $D(E) = x, \overline{D}(E') = y$. In particular, $\overline{d}(D(E), \overline{K}_r) = \inf_{E'} d(E, E')$ with $E' \in K_r$, so $\overline{d}(D(E), \overline{K}_r) = d(E, K_r)$. Thus, $\overline{R}(D(E)) = R(E) = \arg\min_{r \in L_s(C)} \overline{d}(D(E), \overline{K}_r)$ and $\overline{R} = \mathcal{R}(\overline{K}, \overline{d})$.

### 3.3 The $\ell^p$-votewise case: Wasserstein distance

In the special case of $\ell^p$-votewise distances, we can describe $\overline{d}$ in more detail using a well-known construction from probability theory, the Wasserstein distance, which we now recall.

Let $S$ be a finite set of size $M$, and let $\Delta(S)$ denote the set of probability distributions on $S$. For a distance $d$ defined on $S$, the function $d^p_w : \Delta(S) \times \Delta(S) \to \mathbb{R}$ is defined by

\[
d^p_w(p, q) = \left( \int_{S \times S} d(x, y)^p \cdot p(x) \cdot q(y) \, dx \, dy \right)^{\frac{1}{p}}.
\]
\[ d_W^p(x, y)^p = \inf_{\mathcal{A}} \sum_{r, r' \in S} A_{r, r'} d(r, r')^p, \]

where the infimum is taken over all couplings of \( x \) and \( y \), defined as nonnegative square matrices of size \( M \) whose marginals are \( x \) and \( y \) respectively (i.e. \( \forall r, \sum_{r'} A_{rr'} = x_r \) and \( \forall r', \sum_r A_{rr'} = y_{r'} \)). Basically, it represents the minimum cost to move from one configuration to another, where the underlying distance \( d \) defines the cost of each movement. Indeed, this construction leads to a new distance.

**Proposition 3.18** If \( d \) is a distance on \( S \), then \( d_W^p \) is a distance on \( \Delta(S) \). If \( d \) is a metric, then so is \( d_W^p \).

**Proof** See Villani (2008, Ch. 6). \( \square \)

**Remark 3.19** The function \( d_W^p \) goes by several names, some common ones being the \( l^p \)-transportation distance, the Kantorovich \( p \)-distance, the \( p \)-Wasserstein distance. When \( p = 1 \), it is also called the Earth Mover’s distance or first Mallows metric, and is used heavily in several areas of computer science, particularly image retrieval and pattern recognition.

Now we are able to make the link with the votewise metrics, by applying the construction of \( d_W^p \) in the case \( S = L(C) \). Scaling a votewise distance based on the \( \ell^p \) norm gives a homogeneous distance with a special formula, and this turns out to be exactly the \( p \)-Wasserstein distance.

**Remark 3.20** A votewise distance based on \( \ell^p \) is not homogeneous, as its value depends on the number of voters. However, we may make an equivalent homogeneous version by scaling. For each real number \( p \) with \( 1 \leq p \leq \infty \), and each positive integer \( n \), we define a new norm on \( \mathbb{R}^n \) by defining for each \( x \in \mathbb{R}^n \)

\[ ||x||_p^* = \frac{1}{n} ||x||_p. \]

Denote the votewise distance corresponding to this norm by \( d^p_* \); then \( \mathcal{R}(\mathcal{K}, d^p_*) = \mathcal{R}(\mathcal{K}, d^p_W) \).

**Proposition 3.21** Let \( d \) be a finite distance on \( L(C) \). Then \( \overline{d^p_*} = d^p_W \).

**Proof** Write \( S = L(C) \). Since \( d^p_* \) is anonymous and homogeneous, Proposition 3.15 implies that \( \overline{d^p_*}(x, y) = \min_{E, E'} d^p_*(E, E') \), where \( E = x, E' = y \) and \( |V(E)| = |V(E')| \). Fix \( n \geq 1 \) and let \( E = (C, V, \pi) \) and \( E' = (C, V, \pi') \) denote elections with \( n \) voters such that \( E = x, E' = y \). Then

\[ \overline{d^p_*}(x, y)^p = \min_{\pi'} \frac{1}{n} \sum_{i} d(\pi_i, \pi'_i)^p \]

\[ \geq \frac{1}{n} \sum_{r, r' \in S} \sum_{i: \pi_i = r, \pi'_i = r'} d(r, r')^p = \sum_{r, r' \in S} \frac{1}{n} d(r, r')^p, \]
where the $a_{r,r'}$’s are nonnegative integers such that for all $r \in S$, $\sum_r \frac{a_{r,r'}}{n} = x_r$ and for all $r' \in S$, $\sum_{r'} \frac{a_{r,r'}}{n} = y_{r'}$, which corresponds to the Wasserstein distance restricted to matrices $A$ respecting the conditions and with coefficients of the form $\frac{k}{n}$ with $0 \leq k \leq n$ (in other words, a rational coupling with restricted denominators). So clearly, $d_{W, p}(x, y) \leq d_{\ast, p}(x, y)$.

Let assume that this inequality is strict. Then there is a coupling $A'$ (not all of whose entries are rational) with $\sum_{r,r'} a_{r,r'} d(r,r')^p + \epsilon/2 < \sum_{r,r'} a_{r,r'} d(r,r')^p$ for all rational couplings $A$. However since $\max_{r,r'} d(r,r') < \infty$ and we can choose $n$ as large as we want, $d_{W, p}(x, y)$ can be approximated arbitrarily closely, since we can approach all entries of $A'$ simultaneously arbitrarily closely by a rational matrix satisfying the coupling constraints. This contradiction yields the final result. \(\Box\)

**Example 3.22** Let $d_1$ be the $\ell^1$ distance between probability measures on $L(C)$ and let $d = d_H$. Then $\overline{d_1} = \frac{1}{2}d_1$ (also called the total variation distance). This was observed (without the current notation) in Mossel et al. (2013, Lemma 3.2): if $E, E'$ are elections on $(C, V)$ with $|V| = n$, then

$$d_1(E, E') \leq \frac{2}{n}d_H(E, E')$$

and given $E, E' \in \Delta_M^\mathbb{Q}$, we can choose $n$ and the preimages $E, E'$ so that equality holds.

**Example 3.23** If $x \in V$ has 2 “abc voters” (voters with preferences $a \succ b \succ c$) and 3 bac voters, while $y$ has 2 bac voters and 3 cba voters, the quotient distance corresponding to the normalized version of $d^1_H$ is 3/5. This is because we must change at least 3 of the 5 voters to convert $x$ to $y$ (switch both abc voters to cba, and one of the bac voters to cba). For the Kemeny metric $d^1_K$, the analogous quantity is 8/5, because we must make at least 8 swaps of adjacent candidates to convert $x$ to $y$ (switching abc to cba requires 3 swaps, and switching bac to cba requires 2 swaps; similarly, switching both abc votes to bac and all three bac to cba requires 8 swaps).

### 4 Tied sets and decisiveness

All social rules used in practice encounter the problem of breaking ties. However, many commonly used social rules have the property that the subset of profiles where a unique output is not determined is “small” (for example asymptotically negligible as $n \to \infty$, for fixed $m$). This is important, because if the region where ties occur is small, then ties can be ignored for many purposes, whereas if that region is asymptotically large, our rule may suffer extreme lack of decisiveness.

In the worst case, the rule may do nothing, and simply return all possible $s$-rankings at every profile, making it useless. In the DR framework, this extreme indecision cannot occur, because some consensus set must be nonempty. However, it is certainly possible to have “large” subsets of profile space on which a DR rule is not single-valued. We
investigate this question for simplex rules, giving both positive (few ties) and negative (many ties) results.

4.1 Boundaries

We formalize the concept of “tied region” in our geometric context.

**Definition 4.1** The boundary of the social rule $R$ is the set of all elections $E$ at which $R(E)$ is of size at least 2.

**Example 4.2** Suppose that $m = 2$, with alternatives $a$ and $b$, $K = C$, and $d$ is an anonymous neutral standard distance. The concepts of majority winner and Condorcet winner coincide when $m = 2$. When $n$ is odd, the boundary $R(K, d)$ is empty, whereas when $n$ is even, the elections having an equal number of $ab$ and $ba$ voters are in the boundary.

On the other hand, for the simplex rule $R(C, d)$, the boundary is the point $(1/2, 1/2)$.

Our intuition is that the boundary of a well-behaved simplex rule should be “small” in $\Delta_M$, and be geometrically “nice”. The relevant geometric theory is that of Voronoi diagrams. We now digress to review some known results, which will help develop a more refined intuition.

4.2 Geometric background

Voronoi theory is usually defined for a metric space, and most commonly for a Minkowski space (a finite-dimensional real normed vector space). All definitions below work for an arbitrary metric space. The theory can no doubt be generalized to distances, but we do not deal with maximum generality here. Our main interest is in explaining that there are several interesting reasons why DR rules defined by $\ell_p$-votewise distances may fail to be decisive.

For a fixed set of of sites (subsets of the entire space), the open Voronoi cell of each site $X$ is defined as the set of points closer to $X$ than to any other site. The boundaries of these cells are contained in the union of bisectors, where a bisector of the sites $X$ and $Y$, denoted $\beta(X, Y)$, is defined to be the set of points equidistant from those two sites.

Interpreting the sites as the consensus sets $K_r$, we see that the open Voronoi cell corresponding to $K_r$ is precisely the set on which $R(K, d)$ is single-valued with value $r$. Also, the boundary as defined above is just the union of boundaries of all Voronoi cells.

We first discuss bisectors, because if these are well-behaved, so will the boundary of the rule be. We first restrict to the nicest situation (where our intuition is strongest), namely where the space is $\mathbb{R}^n$ under the Euclidean $\ell^2$ norm. Suppose first that the sites are all distinct points. In this case for each pair of distinct sites $X$ and $Y$, the bisector $\beta(X, Y)$ is a hyperplane normal to the line joining the points. The Voronoi cells are therefore convex polyhedra that tile the entire space.
However, this situation is rather special. In fact $\beta(X, Y)$ is a hyperplane for all $X$ and $Y$ if and only if the space is indeed Euclidean (in other words the norm is $\ell^2$) (Mann 1935). Thus we should expect to see bisectors that are not hyperplanes. Of course, such bisectors may still be well-behaved, for example smooth hypersurfaces. Note that $\beta(X, Y)$ is known to be homeomorphic to a hyperplane provided the norm is strictly convex (recall that a norm is strictly convex if its unit sphere contains no line segment) and even sometimes when it is not Horvath (2000). This does not preclude nasty behaviour such as two bisectors intersecting in arbitrarily complicated ways, but for norms defined algebraically, such as $\ell^p$, that does not happen.

When the sites are not single points (in particular when they are not separated), bisectors may be poorly behaved even in $\ell^2$ (see Example 4.4). We conclude that well-behaved bisectors should not be expected in general, and we explore this in the next section.

4.3 Large boundary

The most obvious way for a rule to be rather indecisive is if the underlying distance does not distinguish points well. We say that a subset of a Minkowski space (possibly defined with a seminorm) is large if it contains an open ball, and small otherwise.

Example 4.3 (pseudometric) Consider the Copeland rule whose boundary contains all points with no unique Copeland winner. This contains in particular the set where all candidates have the same Copeland score, for example because the majority tournament contains a cycle that includes all candidates. This is a large subset of $\Delta_M$. To see this, note that in terms of coordinates, the majority cycle is described by $\binom{m}{2}$ equations of the form $\sum_{i \in S_{ab}} x_i > \sum_{i \in S_{ba}} x_i$, where $S_{ab}$ denotes the set of rankings for which $a$ is above $b$. The tied set contains a sufficiently small neighborhood of every point for which all inequalities are strict, because small changes to the proportions of voter types will not change the majority tournament.

In view of this example, we should require our distances to be quasimetrics. However, there are other more subtle problems that can occur. Note that in the next example, the distance is $\ell^2$ and is hence as nice as could be expected: a metric induced by a norm that is strictly convex, symmetric, and algebraically defined. The problem is that the consensus notion is wrong—intuitively, consensus sets should be separated, because otherwise how could the consensus choice be uncontroversial?

Example 4.4 (non-separated consensus) Let $m = 3$ and consider $\Delta_6$ with the usual $\ell^2$ metric (induced from $\mathbb{R}^5$ which coordinatizes $H_6$ in the usual way).

Consider the half-open line segments $L_1, L_2$ that join the center $P$ of $\Delta_6$ to the points $x_{abc} = 1$ and $x_{bca} = 1$ respectively (Fig. 1 gives some intuition in lower dimension). Each contains the endpoint on the boundary, but neither contains the center of the simplex.

Define a 1-consensus by letting $K_a = L_1$, $K_b = L_2$ and $K_c$ be the single point $x_{cab} = 1$. Let $H_1, H_2$ be the hyperplanes (in $\Delta_6$, i.e. having dimension 4) normal to $L_1, L_2$ at $P$. Let $S$ be the set of points in $\Delta_6$ that lie on the other side of $H_1$ from $L_1$ and on the other side of $H_2$ from $L_2$. Then each point of $S$ lies on the bisector of $L_1$,
Distance rationalization of anonymous and homogeneous.

Fig. 1 Illustration for Example 4.4

$L_2$ with respect to the usual $\ell^2$ metric $d$, because the closest point of $L_1$ is $P$ and this is the closest point of $L_2$. Every point of $S$ that is closer to the centre of $\Delta_6$ than to $K_c$ (this includes the point $L_1 \cap L_2$) is in the boundary of the social choice rule $R(K, d)$, which is therefore large. Note that $K$ satisfies anonymity and homogeneity, but not neutrality. Furthermore, every $K_r$ is a convex polyhedral subset of the simplex.

If instead we define $K$ symmetrically by letting $K_c$ be the line segment $L_3$ joining the point $x_{cab} = 1$ to the centre of the simplex, then although the bisectors are large on $\Delta_6$, the boundary of the rule is small. This is because points in $S$ are now closer to $L_3$ than either $L_1$ or $L_2$.

Also note that if the line segments $L_1$ and $L_2$ did not approach arbitrarily closely, the bisectors would all be small.

Thus we should require that consensus sets be separated. However, there is another common way in which bisectors can fail to behave well, which is when the underlying norm in a Minkowski space is not strictly convex. We now analyse a special case of this in some detail.

5 Analysis of $\ell^1$-votewise metrics

Votewise distances based on the $\ell^1$ norm are very commonly used. They are typically computationally easy and have a clear interpretation in terms of adding distances corresponding to each voter. In fact we are not aware of a named $\ell^p$-votewise rule that has been defined for any $p \neq 1$. However, when we consider decisiveness, there are some potential negative consequences to using the $\ell^1$ norm.

We first show in Propositions 5.1 and 5.3 that each $\ell^1$-votewise metric corresponds to a (Wasserstein) distance on the simplex that is induced by a norm that is not strictly convex.

We fix a candidate set $C$ with $|C| = m$ and let $M = m!$ as in previous sections. Let $c = (1, 1, \ldots, 1)/M \in \mathbb{R}^M$ be the center of the simplex $\Delta_M$. Now, we translate the center $c$ to the origin, and we denote by $\Delta'$ the image of the simplex under this translation. We denote by $H$ the hyperplane containing $\Delta'$. Our study of the geometry under the Wasserstein distance will be facilitated by the following observations.
Proposition 5.1 Let $d$ be an $\ell^1$-votewise distance. Then $d$ induces a norm $N$ on $\mathcal{H}$.

Proof It is a well known property of the Wasserstein 1-distance [the norm is called the Kantorovich-Rubinstein norm Villani (2008, Ch 6)]. Explicitly, one first shows that any Wasserstein distance is translation-invariant. Then one shows that the function $f : x \mapsto d_W^p(x + c, c)$ is homogeneous on $\Delta'$. Since every translation-invariant and homogeneous metric is induced by a norm, $d$ induces a norm $N$ on $\mathcal{H}$ by setting $N(x) = f(x)$ on $\Delta'$ and then extending it to $\mathcal{H}$ by requiring it to be homogeneous. \(\Box\)

Remark 5.2 This result is not true for the other Wasserstein metrics, because they do not satisfy the homogeneity property of norms. For example, let us choose $z \in \Delta'$ such that $z_1 \leq 0$ and $\forall i \neq 1$, $z_i \geq 0$. It is easy to show that the matrix $A$ reaching the minimum in the definition of $d_W^p(x, y)$ is such that $A_{rr} = \min(x_r, y_r)$ for all $r$ and then $d_W^p(x, y) = d_W^p((x - y)^+, (y - x)^+)$ where $x^+ = (\max(x_1, 0), \ldots, \max(x_n, 0))$. Then $d_W^p(z + c, c) = d_W^p((z_1, 0, \ldots, 0), (0, z_2, \ldots, z_n)) = \sum r z_r^* d(r, r')^p$, and $d_W^p(\lambda z + c, c) = |\lambda|^{\frac{1}{p}} \sum r z_r^* d(r, r')^p = |\lambda|^{\frac{1}{p}} d_W^p(z + c, c)$.

Proposition 5.3 The norm $N$ of Proposition 5.1 is not strictly convex.

Proof We need to show that the unit sphere is not strictly convex. Fix a ranking $r \in L(C)$ and consider the subset $S_r$ of all points $x \in \mathcal{H}$ where only the component corresponding to $r$ is negative. In $S_r$, we have

$$N(x) = \sum r x_r^* d(r, r').$$

Thus the equation $N(x) = 1$ of the unit sphere defines a hyperplane in $S_r$. Since $S_r$ is large, the unit ball is not strictly convex. \(\Box\)

We can now show that for any distance $d$, there are two distinct points whose bisector under the Wasserstein distance $d_W^1$ is large.

Proposition 5.4 Consider a norm $N$ induced over $\mathcal{H}$ by an $\ell^1$-votewise metric. Let $r_1, r_2$ be rankings. We denote by $d_1$ and $d_2$ the distances $d(r, r_1)$ and $d(r, r_2)$. Let $\epsilon > 0$. We define $x$ and $y$ as the two points of $\mathcal{H}$ such that $x_r = -x_{r_1} = \frac{\epsilon}{d_1}$, $y_r = -y_{r_2} = \frac{\epsilon}{d_2}$, and all other components are equal to zero. Then, any point $z \in \mathcal{H}$ such that $z_r \leq 0$ and $z_{r'} \geq 0$ for all $r' \neq r$ is equidistant from $x$ and $y$ according to $N$.

Proof Let $z$ be such a point. Then, $x - z$ and $y - z$ have only one positive component: the one corresponding to the ranking $r$. So $N(x - z) = \sum r' \neq r (x_{r'} - z_{r'}) d(r, r')$ and $N(y - z) = \sum r' \neq r (y_{r'} - z_{r'}) d(r, r')$. Since the only components (different from $r$) where $x$ and $y$ differ are $r_1$ and $r_2$, they are equidistant from $z$ if and only if $(x_{r_1} - z_{r_1}) d_1 + (x_{r_2} - z_{r_2}) d_2 = (y_{r_1} - z_{r_1}) d_1 + (y_{r_2} - z_{r_2}) d_2$, which is equivalent to $x_{r_1} d_1 + x_{r_2} d_2 = y_{r_1} d_1 + y_{r_2} d_2$, which is in turn equivalent to $x_{r_1} d_1 = y_{r_2} d_2$, which is true by definition. \(\Box\)

It follows that the behaviour of $\ell^1$-votewise distances is rather counterintuitive.
Corollary 5.5 Let $d$ be an $\ell^1$-votewise metric. Then there is a consensus $\mathcal{K}$ consisting of isolated points, such that the boundary of $\mathbb{R}(\mathcal{K}, d)$ is large.

Proof Write $x_\epsilon$ and $y_\epsilon$ for points $x$ and $y$ of the form defined in Proposition 5.4. That proposition implies that, if we set $\mathcal{K}_a = \{x_\epsilon\}$ and $\mathcal{K}_b = \{y_\epsilon\}$ and choose a sufficiently small $\epsilon$, then $\beta(\mathcal{K}_a, \mathcal{K}_b)$ will be large. Also, for any other candidate $c$, if we set $\mathcal{K}_c = \{x_\epsilon\}$ with $\epsilon < \epsilon_\epsilon$, then for any $z$ such that $z_\epsilon$ is the only negative component, $N(z, \mathcal{K}_a) = N(z, \mathcal{K}_b) < N(z, \mathcal{K}_c)$. □

Remark 5.6 Note that the consensus in the proof of Corollary 5.5 is somewhat unnatural. For example, it is not neutral and does not intersect the boundary of $\Delta_M$.

The next question is how often this kind of situation happens. For simplicity we focus on the case $d = d_H$, when the induced norm is exactly $\ell^1$. We can give an exact characterization of when two points have a large bisector. This is directly connected with the well-known integer partition problem.

Proposition 5.7 Let $M \geq 1$ and let $x, y \in \mathbb{R}^M$. We denote by $S$ the set of values $(x_i - y_i)$. Then $x$ and $y$ have a large bisector under $\ell^1$ if and only if there exists a subset $S' \subset S$ such that $\sum_{e \in S'} e = \sum_{e \notin S'} e$.

Proof By definition $\beta(x, y) = \{z | \sum_i |x_i - z_i| = \sum_i |y_i - z_i| \}$. We divide $\mathbb{R}^M$ into $4^M$ subspaces corresponding to the possible signs of the values $(x_i - z_i)$ and $(y_i - z_i)$. Let $V$ be one of these subspaces: in $V$, the equality $\sum_i |x_i - z_i| = \sum_i |y_i - z_i|$ is equivalent to $\sum_i \epsilon_i (x_i - z_i) = \sum_i \epsilon'_i (y_i - z_i)$, where $\forall i, \epsilon_i, \epsilon'_i = \pm 1$. This is equivalent to $\sum_i (\epsilon_i - \epsilon'_i) z_i = \sum_i (\epsilon_i x_i - \epsilon'_i y_i)$.

There are two cases. First, if for some $i$, $\epsilon_i \neq \epsilon'_i$, then the linear equation in $z$ is nontrivial and $z$ lies in a hyperplane, so that $V \cap \beta(x, y)$ is small. The other case is when $\epsilon_i = \epsilon'_i$ for all $i$, in which case the left side of the equation is 0. If the right side is nonzero there is no solution, and $V \cap \beta(x, y) = \emptyset$. If the right side is zero, then $V \cap \beta(x, y)$ is large (for each $i$, it contains all points for which $z_i$ is sufficiently large, for example). The right side is zero if and only if $\sum_i \epsilon_i (x_i - y_i) = 0$, which is equivalent to the fact that there exists $S' \subset S$, $\sum_{e \in S'} e = \sum_{e \notin S'} e$. □

The argument in the proof gives insight into the shape of any large bisector of two points: any ball included in the bisector is contained in cells where $(x_i - z_i)$ and $(y_i - z_i)$ are of the same sign, and thus in a subset defined by a set of equations $z_i \leq \min(x_i, y_i)$ or $z_i \geq \max(x_i, y_i)$ for all $i$. It implies, for example, that if the points are corners of the simplex, the large bisector in $\mathbb{R}^M$ intersects $\Delta_M$ in a small set. Thus, for example, large boundaries cannot occur with $S$ (which also follows from Corollary 6.9 below).

Definition 5.8 The standard decision problem PARTITION is defined as follows. Input is a vector $(z_1, \ldots, z_M)$ of natural numbers. We must decide whether there is a subset $S \subseteq \{1, \ldots, M\}$ for which $\sum_{i \in S} z_i = \sum_{i \notin S} z_i$.

Define the decision problem LARGE-BISECTOR as follows. Input is a pair $(x, y)$ of points of $\mathbb{Q}_+^M$ and we must decide whether $\beta(x, y)$ contains an open ball under the $d_H^1$ metric.
Remark 5.9  Note that $M$ is part of the input in each case. When $M$ is bounded, PARTITION can be solved trivially by exhaustive enumeration of subsets. Note that if we let $K := \sum_i x_i$, then a standard dynamic programming algorithm solves PARTITION in $O(KM)$ time.

Proposition 5.10  LARGE-BISECTOR is NP-complete.

Proof  Given an instance $(z_1, \ldots, z_M)$ of PARTITION, let $x_i = z_i, y_i = 0$. This gives an instance of LARGE-BISECTOR, which is a yes instance if and only if the original instance is a yes instance of PARTITION. Thus LARGE-BISECTOR is NP-hard. On the other hand, given a yes-instance $(x, y, M)$ of LARGE-BISECTOR, the criterion in Proposition 5.7 gives a polynomial-sized certificate checkable in polynomial time, so LARGE-BISECTOR is in NP. Thus, LARGE-BISECTOR is NP-complete.  

Remark 5.11  We suspect the analogue of LARGE-BISECTOR to be NP-hard for every $\ell^1$-votewise metric. Presumably it is in NP for “nice” distances, but of course there exist distances which cannot even be computed in polynomial time, so that an analogue of Proposition 5.7 may not exist.

The question of large bisectors is quite subtle, because large bisectors do not occur when the consensus sets are hyperplanes instead of points.

Proposition 5.12  The bisector of two distinct hyperplanes under any norm on $\mathbb{R}^M$ is contained in a union of at most two hyperplanes.

Proof  The distance from a point $x$ to a hyperplane $H$ defined by $a^T x = b$ is equal to $d(x, H) = \frac{|a^T x - b|}{||a||_*}$ where $*$ denotes the dual norm (see for example Mangasarian (1997); the exact definition is not necessary here). Now, let $H'$ be another hyperplane defined by the equation $a'^T x = b'$. We assume that $||a||_* = ||a'||_*$ (without loss of generality since multiplying by a scalar still defines the same hyperplane). The bisector of $H$ and $H'$ can be defined as the set of points $x$ satisfying $|a^T x - b| = |a'^T x - b'|$. So, we have two cases, depending on the sign of these absolute values: either $\sum_i (a_i - a'_i)x_i = b - b'$ or $\sum_i (a_i + a'_i)x_i = b + b'$. Since $H \neq H'$, each of these is the equation of a hyperplane.  

6 Small bisectors and hyperplane rules

All our results in this section show that the bisectors in question are contained in a finite union of hyperplanes. Rules which have a well-defined winner on each component of the complement in $\Delta_M$ of a finite set of hyperplanes have been studied recently. Mossel et al. (2013) call such simplex rules hyperplane rules and show their equivalence with the generalized scoring rules of Xia and Conitzer (2008). These rules can be defined axiomatically using finite local consistency (Xia and Conitzer 2009). Although originally introduced for social choice rules only, the definition extends to social welfare rules (Caragiannis et al. 2014) and it is clear that it also extends to general $s$.  

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Most rules that have ever been studied by social choice theorists are hyperplane rules. A notable exception is Copeland’s rule. In order to interpret Copeland’s rule as a hyperplane rule, Mossel et al. (2013) require that the winner be (arbitrarily) specified on the tied region. This seems to us to be stretching the definition too far – we could do the same thing for any indecisive rule.

We now give a sufficient condition for a DR rule to be a hyperplane rule.

**Definition 6.1** Let $\mathcal{K}$ be a homogeneous consensus and $d$ a homogeneous distance.

Say that $(\mathcal{K}, d)$ satisfies the **votewise minimizer property** (VMP) if the following condition is satisfied.

There is a mapping $\xi : L(C^*) \times L_s(C^*) \to L(C^*)$ such that for each $r \in L_s(C^*)$ with $\mathcal{K}_r \neq \emptyset$, and each election $E = (C, V, \pi) \in \mathcal{E}$, $m(E, r) := (C, V, \xi(\pi, r))$ minimizes the distance under $d$ from $E$ to $\mathcal{K}_r$, where $\xi(\pi, r) := (\xi(\pi_1, r), \ldots, \xi(\pi_n, r))$.

**Remark 6.2** The VMP allows us to find a minimizer by dealing with each voter separately, and if two voters have the same preference order in $E$ then the corresponding votes in $m(E, r)$ are equal. Also if $d$ is votewise based on $N$, then

$$d(E, \mathcal{K}_r) = N(d(\pi_1, \xi(\pi_1, r)), \ldots, d(\pi_n, \xi(\pi_n, r)).$$

If $N$ is also symmetric then $d(E, \mathcal{K}_r)$ depends only on the multiset of all values $d(\pi_i, \xi(\pi_i, r))$.

**Proposition 6.3** Let $d$ be $\ell^p$-votewise for some $1 \leq p < \infty$ and suppose that $(\mathcal{K}, d)$ satisfies the VMP. Then on $\Delta_M$, $\beta(\mathcal{K}_r, \mathcal{K}_{r'})$ is defined by

$$\sum_{t \in L(C)} x_t \delta(t, r)^p = \sum_{t \in L(C)} x_t \delta(t, r')^p.$$ 

**Proof** The distance between $E = (C, V, \pi)$ and the minimizer $m(E, r) = (C, V, \pi^*)$ equals $N(\Sigma)$ where $\Sigma$ is the multiset with entries $\delta(t, r)$ occurring according to their multiplicities $n_{xt}$, for all $t \in L(C)$. The specific form of $N$ then shows that $d(E, m(E, r))^p = n \left(\sum_t x_t \delta(t, r)^p\right)$. Applying the same argument for $r'$ yields the result. \qed

**Definition 6.4** Suppose that the $s$-consensus $\mathcal{K}$ satisfies the following: for each $r \in L_s(C)$, there is a subset $S_r$ of $L(C)$ such that $\mathcal{K}_r$ consists precisely of those elections for which every voter has a ranking in $S_r$. Then we call $\mathcal{K}$ a **generalized unanimity** consensus.

**Proposition 6.5** Let $d$ be an $\ell^p$-votewise distance on $\mathcal{E}$ and let $\mathcal{K}$ be a generalized unanimity consensus. Then $(\mathcal{K}, d)$ satisfies the VMP.

**Proof** If $\mathcal{K}_r \neq \emptyset$, define $\xi(\pi_i)$ to be the closest element of $S_r$ to $\pi_i$ under the underlying distance on $L(C)$ (if there is more than one such element, make an arbitrary choice).

For each $E = (C, V, \pi)$, the element $E^* = (C, V, \xi(\pi))$ belongs to $\mathcal{K}_r$. If $F =$
\((C, V, \pi') \in K_r\) then \(d(E, F) \geq d(E, E^*)\) because \(d(\pi_i, \xi(\pi_i)) \leq d(\pi_i, \pi_i')\) for each \(i\), and the \(\ell^p\)-norm is increasing in each argument in the positive orthant. Thus \(E^*\) is the desired minimizer.

**Corollary 6.6** \((S^*, d^p)\) satisfies the VMP for every distance \(d\).

**Proof** We can take \(S_r\) to be the set of rankings which agree with \(r\) in their initial \(s\)-ranking, showing that \(S^*\) is a generalized unanimity consensus.

**Example 6.7** Let \(K = W\) and \(d = d_K\), and \(N = \ell^2\). For each \(E = (C, V, \pi) \in E\) and \(a \in C\), we can take \(\xi(\pi, a)\) to be the ranking derived from \(\pi\) by swapping \(a\) to the top as efficiently as possible in each \(\pi_i\). Thus \(d(E, W_a)^2 = \sum_{t \in L(C)} n(t)(\xi(t, a) - 1)^2\), where \(n(t)\) is the number of times \(t\) occurs in \(\pi\).

**Remark 6.8** We do not know of any “natural” consensus and distance which satisfy the VMP, apart from those already mentioned. We can easily create strange examples, however, by creating generalized unanimity consensuses. If for each candidate \(a\) we choose a single ranking with \(a\) at the top, this yields a generalized unanimity consensus that is extended by \(W\). Note that this consensus is not neutral. Alternatively, we could choose all rankings having \(a\) in the first or second position (in which case \(a\) is the consensus winner), or \(b\) in the first position as long as \(a\) is not in the second position (in which case \(b\) is the consensus winner), or \(c\) in the first or second position (provided \(a\) is not first or second and \(b\) is not first), in which case \(c\) is the consensus winner. Again, this is not neutral.

**Corollary 6.9** Suppose that \(d\) is \(\ell^p\)-votewise with \(1 \leq p < \infty\), \(d\) is finite and not identically zero, and \(K\) is a generalized unanimity consensus. Then \(R(K, d)\) is a hyperplane rule.

**Proof** Since \(d < \infty\) we may rearrange the formula in Proposition 6.3 to get

\[
\sum_i (\delta(t, r) - \delta(t, r')) = 0.
\]

It suffices to show that the linear function on the left side is not identically zero. That could only happen if \(\delta(t, r) = \delta(t, r')\) for all \(t\). However, note that the distance from \(x\) to \(K_r\) is attained at a point \(m(x, r)\) where \(m(x, r)_t = x_t\) for all \(t \notin S\), and \(d(x, K_r) = \sum_{t \in S} x_t \delta(t, r)^p\). If \(r \neq r'\) then by definition \(S_r \neq S_{r'}\). Thus taking \(t \notin S \cap S'\), without loss of generality \(\delta(t, r) = 0\) and \(\delta(t, r') \neq 0\).

**Corollary 6.10** Every rule of the form \(R(S^*, d^p)\), where \(1 \leq p < \infty\) and \(d\) is a distance on \(L(C)\) that is neither infinite nor identically zero, is a hyperplane rule.

**Remark 6.11** This result does not extend to general distances. For example, Copeland’s rule as we have defined it is not a hyperplane rule, yet it can be defined as \(R(W, d_{RT})\). Also note that when \(p = \infty\), we do not obtain a hyperplane rule. For example, every point \(x \in \Delta_M^0\) for which every coordinate is nonzero is equidistant from all \(S_r\), so \(R(S, d^\infty)\) is almost maximally indecisive.

**Remark 6.12** Rules of the type described in Proposition 6.3 are rather special. Since the distance to \(K_r\) is of the form \(\sum_i x_i \delta(t, r)^p\), each can be thought of as a differently weighted version of the rule with \(p = 1\).
7 Discussion and future work

We now summarize what we have learned about the boundary of a DR simplex rule.

- Using a pseudometric that is not a metric can easily lead to a large boundary.
- Large bisectors can occur even with $\ell^2$, if consensus sets are not separated.
- Large bisectors can occur with $\ell^1$-votewise rules, even for consensus sets that are isolated points, and it can be difficult to determine whether they do occur.
- Even when bisectors are large in the ambient space, using consensus sets on the boundary of the simplex often yields small bisectors on the simplex.
- Even when bisectors are large on the simplex, neutrality often makes the boundary of the rule small.

We have seen some desirable properties of consensus sets, such as homogeneity and neutrality. We argue that convexity (defined in the usual way via restriction from $\mathbb{R}^M$) of each $K_r$ is another essential condition. In the following example, it seems ridiculous that $a$ should win at the extra point.

**Example 7.1** Consider the case $m = 3$, and the consensus formed by extending $W$ so that $a$ is the winner whenever $x_{bca} = x_{cba} = 1/2$ (and similarly for $b, c$). This consensus is anonymous and homogeneous, but $K_a, K_b, K_c$ are not convex.

**Remark 7.2** In the simplex model, convexity (over $Q$) is equivalent to the notion of consistency: if we split the voter set into two parts each of which elects $r$, the original voter set should elect $r$. It rules out the above example. Note that $C$ and $S'$ are convex. In fact we do not know of a consensus that has been used in the literature that is not convex.

Based on the above results, we suggest that the following criteria be required of consensus classes in the simplex (anonymity and homogeneity come for free)

- neutrality
- convexity
- separation
- intersecting the boundary of the simplex

while distances should be required to be metrics.

Note that the separation requirement rules out $C$ as a consensus notion. This may of course be somewhat controversial. It may turn out that neutral rules based on $C$ and using metrics always have small boundary (we do not know of a counterexample, but have no proof yet). However, it seems strange to consider a situation arbitrarily close to a complete tie among all rankings (the centre of the simplex) to be an election on which a “consensus” can be formed.

We saw above that $\ell^1$ votewise distances can lead to major problems with decisiveness. However there are many natural examples of $\ell^1$ votewise distances, as we have seen. We do not know of any “natural” simplex rule satisfying the above requirements for which the boundary is large. However, not all obvious rules have been thoroughly explored.

Systematic exploration of rules $\mathcal{R}(K, d)$, where $K$ and $d$ satisfy the recommendations above, may prove fruitful in finding new rules with desirable properties. For
example, by the results in this paper and Hadjibeyli and Wilson (2016), the rules $R(S, d^p)$ where $d$ is a neutral metric on permutations, are anonymous, homogeneous, neutral, continuous, hyperplane rules. There are many neutral (also called right-invariant) metrics on permutations we have not discussed here, such as the $\ell^q$-metrics (the cases $q = 1, 2, \infty$ being called Spearman’s footrule, Spearman’s rank correlation and the maximum displacement distance), and the Lee distance (Deza and Deza 2009). Even the rules $R(W, d^p)$ and $R(S, d^p)$ have not been fully explored, to our knowledge.

Even less understood are rules of the form $R(C, \ell^p)$. For example, when $p = 1$, we obtain a homogeneous version of the recently described Voter Replacement Rule (Elkind et al. 2012). Little is known about the Voter Replacement Rule other than that it is not homogeneous (Hadjibeyli and Wilson 2016).

Beyond the realm of votewise and $\ell^p$ distances, we have already mentioned more general statistical distances. Finally, rules involving various matrix norms on the tournament matrices have not been well studied.

Distance-based aggregation of preferences is a more general procedure than we have studied here: it could be applied with many different input and output spaces [Zwic2014]. If the input consists of the tournament matrix rather than the profile, there is a natural hypercube representation of the input in $\binom{m}{2}$ dimensions. Saari and Merlin (2000) showed that the Kemeny rule can be described in this way using distance rationalization with respect to the $\ell^1$ norm and $S$. This is the same as using an element-wise norm on the weighted tournament matrix, in our framework. When using profiles as input, the simplex geometry is hard enough to visualize that some authors have used a fixed projection to the permutahedron and essentially used $S$ as a consensus. The cases $p = 2$ (mean proximity rules) Zwicker (2008b), Lahaie and Shah (2014) and $p = 1$ (mediancenter rules) Cervone et al. (2012) have received attention. These can be interpreted in our framework by changing the distance—detailed formulae might be interesting.

A question which partially motivated the present work remains unanswered. Does (a homogenization of) Dodgson’s rule have a “small and nice” boundary? What about other Condorcet rules $R(C, d)$ where $d$ is a votewise metric, or even rules based on $d_T$, such as the maximin rule?

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