Scattering cross-section resonance originating from a spectral singularity

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Abstract
Using techniques of supersymmetric quantum mechanics, scattering properties of Hermitian Hamiltonians, which are related to non-Hermitian ones by similarity transformations, are studied. We have found that the scattering matrix of the Hermitian Hamiltonian coincides with the phase factor of the non-unitary scattering matrix of the non-Hermitian Hamiltonian. The possible presence of a spectral singularity in a non-Hermitian Hamiltonian translates into a pronounced resonance in the scattering cross section of its Hermitian counterpart. This opens a way for detecting spectral singularities in scattering experiments; although a singular point is inaccessible for the Hermitian Hamiltonian, the Hamiltonian ‘feels’ the presence of the singularity if it is ‘close enough’. We also show that cross sections of the non-Hermitian Hamiltonian do not exhibit any resonance behavior and explain the resonance behavior of the Hermitian Hamiltonian cross section by the fact that the corresponding scattering matrix, up to a background scattering matrix, is a square root of the Breit–Wigner scattering matrix.

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1. Introduction
Since the pioneering paper of Bender and Boettcher [1], non-Hermitian Hamiltonians with a purely real spectrum have attracted more and more attention from both theoretical and experimental viewpoints. Many interesting properties of these Hamiltonians have been described in the literature (for a review, see [2]). If a non-Hermitian Hamiltonian with a real spectrum can never be equivalent to a Hermitian one, it probably exhibits the most unusual properties. These are the Hamiltonians that possess exceptional points in the discrete part of the spectrum or that have spectral singularities in its continuous part. In particular, the property of a super fast evolution, called ‘faster than Hermitian evolution’, has been discovered [3].
This was explained by noting the presence of an exceptional point in the spectrum of a matrix Hamiltonian [4] and analyzed from the point of view of the Naimark extension [5]. It has recently been shown that the presence of a spectral singularity leads to infinite reflection and transmission coefficients at a real energy of a complex scattering potential, and it has been conjectured that this effect is similar to a zero width resonance that may be observed in resonating waveguides [6]. A similar effect was described for a non-Hermitian Friedrichs–Fano–Anderson model describing the decay of a discrete state coupled to a continuum of modes and in a Bragg scattering in complex crystals [7].

As far as we know, these unusual properties are mainly related to non-Hermitian Hamiltonians. Nevertheless, despite the fact that these Hamiltonians have purely real spectra, from the point of view of the standard quantum mechanics, they may not be fundamental since they cannot be directly associated with quantum mechanical observables. For this reason, there is no known way to use quantum mechanical experiments to observe these properties. On the other hand, any non-Hermitian diagonalizable Hamiltonian $H$ with a real and discrete spectrum possesses a Hermitian counterpart $h = h^\dagger$ that is related to $H$ by a similarity transformation.

For simplicity, we will consider the case when both $H$ and $h$ have purely continuous spectra and depend on a complex parameter $a$, $H = H(a)$, $h = h(a)$. At $a = a_0$, the Hamiltonian $H(a_0)$ has a spectral singularity. In this case, $H(a)$ has no exceptional points, and we will analyze the scattering properties of $h(a)$ for $a \approx a_0$. We study this case because $h(a)$, $a \neq a_0$, being a usual quantum mechanical Hamiltonian, exhibits a number of unusual properties. These properties have not been described in the literature so far, because until now no systematic approach for finding the similarity transformation in an explicit form exists. In the majority of cases, the transformation can only be calculated using an approximation scheme, notably perturbation theory [8]. On the other hand, as shown in [9, 10], supersymmetric quantum mechanics (SUSY QM) may be extremely useful for studying different properties of non-Hermitian Hamiltonians. In this communication, we will use the techniques of SUSY QM for constructing the family $H(a)$ and $h(a)$. We shall show that the scattering matrix of $h(a)$ coincides with the phase factor of the scattering matrix of $H(a)$ and for $a \approx a_0$ the scattering cross section for $h(a)$ exhibits a pronounced resonance. Moreover, we shall also show that up to a background scattering matrix, the scattering matrix of $h(a)$ is a square root of the Breit–Wigner scattering matrix. Note that the square root form of the scattering matrix appears in some models in atomic [11] and nuclear physics [12]. Finally, we illustrate our findings using the simplest but realistic toy model.

2. Real scattering potentials

It is well known (see e.g. [13]) that if the scattering particles interact by a potential that depends on the magnitude of the distance between the particles, the scattering problem is reduced to solving the radial Schrödinger equation with the so-called scattering (asymptotic) condition at infinity. Therefore, we start with a Hermitian scattering Hamiltonian $h_0$ defined by a real-valued potential $v_0(x) = v_0^\ast(x)$, 

$$h_0 = h_0^\dagger = -\frac{d^2}{dx^2} + v_0(x), \quad x \in [0, \infty),$$

on a proper domain $D_{h_0}$ in the space $L^2$ of functions that are square integrable on the positive semi-axis. We will assume that the potential $v_0$ corresponds to the zero value of the angular momentum $\ell = 0$ in the three-dimensional scattering process. In general, $v_0(x)$ may not only
be finite at the origin but also singular. In the usual radial Schrödinger equation, this singularity corresponds to the centrifugal term \( \ell (\ell + 1)x^{-2} \). Nevertheless, while modeling real scattering experiments, in many cases one needs potentials with an \( \ell \)-independent singularity at \( x = 0 \) (for a discussion, see e.g. [14]). For this reason, we will assume that
\[
\nu_0(x) \to \nu(\nu + 1)x^{-2}, \quad x \to 0,
\]
where the parameter \( \nu = 0, 1, 2, \ldots \) is called the singularity strength (see, e.g., [14]). The functions \( \psi \in D_0 \) are assumed to be smooth enough and to satisfy the Dirichlet boundary condition at the origin, \( \psi(0) = 0 \). For simplicity, we also assume that \( h_0 \) has a purely continuous and non-negative spectrum \( E = k^2 \geq 0 \) and that the eigenfunctions \( \psi_k(x) \),
\[
h_0 \psi_k = k^2 \psi_k, \quad \psi_k(0) = 0, \quad k \geq 0,
\]
satisfy the asymptotic condition at infinity (see, e.g., [13], chapter XVII),
\[
\psi_k(x) \propto A(k) e^{i k x} + B(k) e^{-i k x}, \quad x \to \infty.
\]
From here, one finds the \( S \)-matrix and phase shift \( \delta \),
\[
S(k) = \exp [2i \delta(k)] = \frac{A(k)}{B(k)},
\]
which define the scattering amplitude
\[
f(k) = \frac{1}{2ik} (S(k) - 1),
\]
extective range function
\[
g(k) = k \cot \delta(k) = i k \frac{S(k) + 1}{S(k) - 1},
\]
and cross section
\[
\sigma(k) = 4\pi |f(k)|^2 = \frac{\pi}{k^2} |S(k) - 1|^2.
\]
We assume here that all these quantities correspond to the zero value of the orbital momentum, i.e. to the \( s \)-wave which may dominate other partial waves in the case of a low-energy scattering [13].

For the Hermitian Hamiltonian \( h_0 \), the \( S \)-matrix is unitary, \( |S(k)| = 1 \), and the functions \( \psi_k \) form an orthonormal (in the sense of distributions) and complete basis in the space \( L^2 \) (see, e.g., [15]),
\[
\langle \psi_k | \psi_{k'} \rangle = \delta(k - k'), \quad \int_0^{\infty} dk |\psi_k\rangle \langle \psi_k| = 1.
\]

Below we will find the asymptotic behavior of the scattering state for a complex SUSY partner \( H \) of the Hamiltonian \( h_0 \). Therefore, the coefficients \( A(k), B(k), \) scattering matrix \( S(k) \), phase shift and cross section for the Hamiltonian \( h_0 \) will be denoted as \( A_0(k), B_0(k), S_0(k), \delta_0(k) \) and \( \sigma_0 \), respectively.

3. Complex SUSY partners of real scattering potentials

Assuming that we know a solution \( u(x) \) of the differential equation
\[
h_0 u(x) = \alpha u(x), \quad \alpha \in \mathbb{C},
\]
we can construct an exactly solvable non-Hermitian Hamiltonian
\[
H = -\frac{d^2}{dx^2} + V(x), \quad V(x) \neq V^*(x),
\]

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with a real spectrum using the approach of SUSY QM [16]. In this approach, the operators \( h_0 \), \( H \) and \( H^\dagger = -\partial_x^2 + V^*(x) \) are related via the intertwining relations

\[
L h_0 = H L, \quad h_0 L^\dagger = L^\dagger H^\dagger,
\]

where \( L \) is assumed to be a differential operator. In the simplest case, that we will consider below, the operator \( L \) is a first-order differential operator,

\[
L = -\frac{d}{dx} + w, \quad w = w(x) := [\log u(x)]^\dagger.
\]

Here and in what follows the prime denotes the derivative with respect to \( x \). The function \( u(x) \) is known as a superpotential (complex valued in the current case) defined with the help of a complex-valued solution \( \psi \) with a real spectrum using the approach of SUSY QM [16]. In this approach, the operators \( J. \text{Phys. A: Math. Theor.} \), \( H \) and \( H^\dagger \) are related via the intertwining relations

\[
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where \( L \) is assumed to be a differential operator. In the simplest case, that we will consider below, the operator \( L \) is a first-order differential operator,

\[
L = -\frac{d}{dx} + w, \quad w = w(x) := [\log u(x)]^\dagger.
\]

The potential \( V(x) \) is expressed via the superpotential in the usual way,

\[
V(x) = v_0(x) - 2w'(x).
\]

Any solution \( \psi_k \) of the differential equation

\[
H \psi_k = k^2 \psi_k, \quad k \in \mathbb{C}, \quad k^2 \neq \alpha.
\]

may be obtained by applying the operator \( L \) to a solution to equation (1),

\[
\psi_k = L \psi_k.
\]

For \( k^2 = \alpha \), a solution to this equation is the function \( \psi_{\sqrt{\alpha}} = 1/u \) (see e.g [16, 17]). Another solution to equation (13) corresponding to \( k^2 = \alpha \), \( \psi_{\sqrt{\alpha}}(x) \), which is linearly independent with \( \psi_{\sqrt{\alpha}}(x) \), may be obtained from the property

\[
\psi_{\sqrt{\alpha}}(x) \frac{\partial}{\partial \sqrt{\alpha}}(x) - \psi_{\sqrt{\alpha}}(x) \frac{\partial}{\partial \sqrt{\alpha}}(x) = 1.
\]

To select ‘physical’ solutions to equation (13), we will solve it with the Dirichlet boundary condition at the origin, \( \psi_k(0) = 0 \), and the asymptotic condition at infinity (2).

In general, the transformation operator \( L \) violates boundary conditions. The spectral problem for the Hamiltonian \( H \) is simplified if \( L \) transforms eigenfunctions of the Hamiltonian \( h_0 \) to eigenfunctions of the Hamiltonian \( H \). Such transformations are known in the literature as conservative [18]. For simplicity, we will consider below only conservative transformations.

We thus assume that \( L \) preserves the Dirichlet boundary condition at the origin. In this case, all spectral points of the Hamiltonian \( h_0 \) (note that they are real!) are also the spectral points of the Hamiltonian \( H \). Therefore, the Hamiltonian \( H \) has a real spectrum provided the complex factorization constant \( \alpha \) does not belong to its spectrum.

Since for \( k = \sqrt{\alpha} \) the function \( \psi_{\sqrt{\alpha}} = 1/u \) is a solution to equation (13), to ensure that the spectrum of the operator \( H \) be real for a complex \( \alpha \), we choose the function \( u(x) \) such that

\[
\psi_{\sqrt{\alpha}}(x) = \frac{1}{u(x)} \exp(-ax) \to \infty, \quad x \to \infty,
\]

with

\[
\alpha = -a^2, \quad a = d + ib, \quad d < 0.
\]

We note that the function \( u(x) \) with the asymptotic behavior as given in (15) is the Jost solution to equation (8). (For a more detailed discussion, see [17]). Below we will need the asymptotic form \( L_\alpha \) as \( x \to \infty \) of the transformation operator (10) which follows from (15),

\[
L_\alpha \equiv -\partial_x + a \equiv -\partial_x + d + ib.
\]
We emphasize that for the first-order transformation (10) to preserve the Dirichlet boundary condition at the origin, the initial potential should be singular at the origin. In this case, the SUSY transformation decreases the singularity strength $\nu$ by one unit and at $\nu = 0$, the potential becomes finite at $x = 0$ (see, e.g., [14]). As was shown in [17] for $d < 0$, the functions

$$\phi_k = (k^2 - \alpha)^{-1/2} L \psi_k, \quad k \geq 0,$$

form a continuous biorthonormal set in $L^2$.

Let us denote $H|_{d=0} = H_0$. According to (15), if $d = 0$, the eigenfunction of $H = H_0$ at $k^2 = k_0^2 = \alpha = b^2$, $\varphi_{k_0}(x) = 1/u(x)$, has a pure exponential asymptotic behavior at infinity,

$$\varphi_{k_0}(x) \to \exp(-ibx), \quad x \to \infty.$$

We emphasize that due to the choice of the transformation function, the function $\varphi_{k_0}(x)$ satisfies both the Dirichlet boundary condition at the origin, $\varphi_{k_0}(0) = 0$, and the asymptotic condition at infinity (2) with either $A(k) = 0$ if $b > 0$ or $B(k) = 0$ if $b < 0$. For these reasons, the point $k^2 = k_0^2 = \alpha = b^2$ belongs to the continuous spectrum of $H_0$ but the resolution of the identity operator over the set of the eigenfunctions of $H_0$ becomes divergent at $k = k_0$ [19] and needs a special regularization procedure [20, 17]. The spectral point $E = k_0^2$ possessing such properties is known in the literature as the spectral singularity. (The interested reader can find a rigorous mathematical definition of the spectral singularity in [21], see also a recent discussion in [6, 20, 17] and references therein.) Moreover, if $H$ possesses a spectral singularity, there is no way to relate it to a Hermitian Hamiltonian by a similarity transformation. Therefore, below we will assume $d < 0$ since just in this case, functions (17) form a continuous biorthonormal set in $L^2$ [17] and will analyze properties of a Hermitian counterpart of $H$ when $d \to -0$ and $H \to H_0$.

4. Hermitian operator equivalent to $H$

To establish an equivalence between the non-Hermitian operator $H$ and a Hermitian operator $h$, we will use ideas formulated in [22] for quasi-Hermitian Hamiltonians and further developed in [23] for pseudo-Hermitian Hamiltonians combined with the supersymmetric approach. In this way, we will express the equivalence transformation in terms of SUSY transformation operators.

From (9), one easily deduces that

$$\eta H = H^\dagger \eta, \quad \eta := (LL^\dagger)^*.$$

The operator $\eta$, as introduced in (18) with $L$ given in (10), is positive definite, Hermitian and invertible [17] second-order differential operator

$$\eta = (-\partial_x + w^\dagger)(\partial_x + w) = \eta^\dagger.$$

It has a unique Hermitian, positive definite and invertible square root

$$\rho = \eta^{1/2} = \rho^\dagger, \quad \rho > 0.$$

Hence, from (18) one finds the Hermitian operator $h$ equivalent to $H$,

$$h = \rho H \rho^{-1} = \rho^{-1} H^\dagger \rho = h^\dagger,$$

which also has a purely continuous real and non-negative spectrum. Its eigenfunctions $\Phi_k$,

$$h \Phi_k = k^2 \Phi_k, \quad k \geq 0,$$
are obtained by applying the operator \( \rho \) to the eigenfunctions of \( H \),

\[
\Phi_k = (k^2 - \alpha)^{-1/2} \rho \psi_k = (k^2 - \alpha)^{-1} \rho L \psi_k.
\]

(21)

The factor \( (k^2 - \alpha)^{-1/2} \) is introduced to guarantee both the normalization of these functions, \( \langle \Phi_k \mid \Phi_k \rangle = \delta(k - k') \), and their completeness [17]. From (21), it follows that the eigenfunctions \( \varphi_k \) of \( h_0 \) and \( \Phi_k \) of \( h \) are related by an isometry \( U \),

\[
\Phi_k = U \psi_k,
\]

(22)

where

\[
U = \rho L (h_0 - \alpha)^{-1} = L^* [(L^L)^*]^{-1/2} = (U^*)^{-1}.
\]

Another remarkable property of the operators \( \rho \) and \( \rho^* \) is that their product factorizes a polynomial of the operator \( h \),

\[
\rho (\rho^*)^2 \rho = (h - \alpha)(h - \alpha^*),
\]

which follows from (9), (11) and (20). Using the spectral decomposition of the operator \( h \), one can express \( h \) in terms of \( \rho \), \( L \), \( L^1 \) and the resolvent of \( h_0 \),

\[
h = \frac{\rho L}{\alpha - \alpha^*} [\alpha (h_0 - \alpha)^{-1} - \alpha^* (h_0 - \alpha^*)^{-1}] L^1 \rho.
\]

(23)

Note that for any \( a = d + ib \) with \( d < 0 \), the point \( \alpha = -a^2 \) does not belong to the spectrum of \( h_0 \) and operator (23) is well defined.

Although equation (21) (or (22)) formally solves the problem of finding the eigenfunctions of \( h \), it contains the non-local operator \( \rho \) (or \( (L^L)^* \)) and, therefore, in general, no explicit expression for \( \Phi_k \) exists. Fortunately, to describe the scattering properties of \( h \), one needs only the asymptotic form of these functions.

5. Scattering matrices and cross sections for \( H \) and \( h \)

According to (2) and (3), the \( S \)-matrix for a Hermitian Hamiltonian is defined by the asymptotic behavior (i.e. as \( x \to \infty \)) of the corresponding scattering state. In the case of an inelastic scattering, the non-unitary scattering matrix is defined by the same equation (3) [13] where \( A(k) \) and \( B(k) \) should be replaced by \( A_H(k) \) and \( B_H(k) \), respectively, found from the asymptotic behavior of the function \( \psi_k \) given in (14), (2) and (3) [13]. Now using (17) and (16), we express the asymptotic form \( \psi_{ka} \) of the function \( \psi_k \) in terms of the Hamiltonian \( h_0 \) scattering state asymptotics \( \psi_{ka} \),

\[
\psi_{ka} = L_a \psi_{ka} = (-\partial_k + d + ib) \psi_{ka}.
\]

(24)

Here we dropped the inessential normalization factor \( (k^2 - \alpha)^{-1/2} \), so that \( \psi_k \propto \psi_{ka} \) as \( x \to \infty \).

Using (2), where the coefficients \( A(k) \) and \( B(k) \) are replaced by \( A_0(k) \) and \( B_0(k) \), respectively, we obtain for \( \psi_{ka} \) the same expression (2) with the coefficients

\[
A(k) \equiv A_H(k) = A_0(k)(d + ib - ik),
\]

(25)

\[
B(k) \equiv B_H(k) = B_0(k)(d + ib + ik).
\]

(26)

From here it follows the scattering matrix \( S_H \) for the Hamiltonian \( H \),

\[
S_H(k) = \frac{A(k)}{B(k)} = S_0(k) \tilde{S}(k),
\]

(27)
where

\[
\tilde{S}(k) = \frac{d + ib - ik}{d + ib + ik}
\]

We note a non-unitary character of \( S_H \),

\[
|S_H(k)| = |\tilde{S}(k)| = \left[ \frac{(b - k)^2 + d^2}{(b + k)^2 + d^2} \right]^{1/2}.
\]  

(28)

As is known from the theory of inelastic collisions [13], \( S_H \) defines the elastic cross section \( \sigma_e = \frac{π}{σ} |S_H - 1|^2 \). Its absolute value \(|S_H|\) determines a reaction cross section, \( \sigma_r = \frac{π}{σ} (1 - |S_H|^2) \). The total cross section is their sum, \( \sigma_t = \sigma_e + \sigma_r \).

Using (21) and dropping once again the factor \((k^2 - \alpha)^{-1/2}\), we find the asymptotic form \( \Phi_{kw} \) of the function \( \Phi_k \),

\[
\Phi_{kw} = \rho_w \psi_{kw},
\]

where \( \rho_w \) is the asymptotic form of the operator \( \rho \).

To find the action of the operator \( \rho_w = \eta_w^{1/2} \) on function (24), we note that according to (18) and (16),

\[
\eta_w = (L_a L_a^\dagger) = (-\partial_a + d - ib)(\partial_a + d + ib)
\]

and therefore that

\[
\eta_w e^{ikx} = [(k + b)^2 + d^2] e^{ikx}.
\]

From this, one finds the action of the operator \( \rho_w = \eta_w^{1/2} \) on the exponential function

\[
\rho_w e^{ikx} = \sqrt{(k + b)^2 + d^2} e^{ikx},
\]

which finally yields the asymptotic form (2) of the Hamiltonian \( h \) scattering state \( \Phi_{kw} \) with the coefficients

\[
A(k) = A_H(k)[(k + b)^2 + d^2]^{1/2}
\]

(30)

and

\[
B(k) = B_H(k)[(k - b)^2 + d^2]^{1/2}.
\]

(31)

We would like to stress the presence of the square root branch points in these formulas which come from (29). Here, in agreement with (19), only one branch of the square root is chosen.

Using (25) and (26), one sees that coefficients (30) and (31) contain the common factor \( [(d + ib)^2 + k^2]^{1/2} \) which cancels out in the ratio \( A(k)/B(k) \). For this reason, the asymptotics of the state \( \Phi_k \) may be written in the form (2) with the coefficients

\[
A(k) \equiv A_k(k) = A_0(k)[(d - i\delta)^2 + k^2]^{1/2}
\]

(32)

and

\[
B(k) \equiv B_k(k) = B_0(k)[(d + i\delta)^2 + k^2]^{1/2}.
\]

(33)

Once the asymptotic form of the scattering state is established, one finds the scattering matrix \( S_h \) for \( h \) with the help of the asymptotic boundary condition (2),

\[
S_h(k) = S_0(k)S_R(k), \quad S_R(k) = \tilde{S}(k) \left[ \frac{(b + k)^2 + d^2}{(b - k)^2 + d^2} \right]^{1/2}.
\]

(34)

Comparing equations (27), (28) and (34), we see that the phase factor of \( S_H \) coincides with the scattering matrix of the Hermitian Hamiltonian \( h \) equivalent to \( H \).
\[ S_h = \frac{S_H}{|S_H|}. \]

We note that the scattering matrix
\[ S_{BW} = S_R^2(k) = \frac{b^2 + (d - ik)^2}{b^2 + (d + ik)^2}, \]  
leads to the Breit–Wigner resonance formula (see e.g. [24])
\[ \sigma_{BW} = \frac{16\pi d^2}{(k^2 + d^2 - b^2)^2 + 4b^2 d^2}, \]
which in the energy scale reads
\[ \sigma_{BW} = \frac{4\pi}{b^2} \frac{(\Gamma/2)^2}{(E - E_0)^2 + (\Gamma/2)^2}, \]
with \( \Gamma = 4bd \) and \( E_0 = b^2 - d^2 \). We assume that \( |d| \) is small enough so that \( b^2 > d^2 \). Near the resonance, we have \( E \approx E_0 \) and \( k \approx b \) so that equation (37) reduces to the celebrated Breit–Wigner formula (see e.g. [25]). From here we conclude that the S-matrix \( S_R \) is a square root of the Breit–Wigner S-matrix \( S_{BW} \) given in (36). Note that an electron scattering S-matrix square root plays an essential role in the theory of spin effects in photoionization developed by Stewart [11] who noted that ‘a photoionization experiment contains half of an electron scattering experiment’. A square root of a unitary S-matrix appears also in some models describing absorptive processes in high-energy reactions involving elementary particles [12].

The phase shift \( \delta_R \) corresponding to \( S_R \) is one half of \( \delta_{BW} \), \( \delta_R = \frac{1}{2} \delta_{BW} \). It leads to a cross section with a square root branch point
\[ \sigma_R(k) = 2\pi \left[ 1 + \frac{k^2 - b^2 - d^2}{\sqrt{(k^2 + d^2 - b^2)^2 + 4b^2 d^2}} \right]. \]
As was already mentioned, we choose here that sign of the square root which corresponds to the positive definite operator \( \rho \) (19). It is not difficult to see that \( \sigma_R(0) = 4\pi d^2 / (b^2 + d^2)^2 > 0 \), \( \lim_{k \to \infty} \sigma_R(k) = 0 \), \( \sigma_R(k) > 0 \) for \( 0 < k < \infty \) and \( d \sigma_R(k)/dk > 0 \) for \( b^2 > \frac{1}{2} d^2 \). These results mean that for any fixed value of \( b \) and small enough value of \( |d| \), the function \( \sigma_R(k) \) (38) has a maximum and therefore exhibits resonance behavior. This evidently is just a consequence of the fact that the scattering matrix \( S_h \) (34) is a square root of \( S_{BW} \) (36).

The factor \( S_0(k) \) produces a background phase shift \( \delta_0(k) = \frac{1}{2} \log S_0(k) \) (see e.g. [26]). In the vicinity of the resonance energy, this is usually a slowly changing function of \( k \) leading to a small shift of the resonance maximum. Therefore, it is natural to expect that the cross section \( \sigma_h \) for the Hamiltonian \( h \) will be close to the cross section \( \sigma_R \) (38). This property will be illustrated in the next section.

It is instructive to compare the scattering amplitude \( f_{BW}(k) \) for the Breit–Wigner S-matrix (36) with the scattering amplitude \( f_R(k) \) for \( S_R \) given in (34). From (4), one respectively finds
\[ \frac{1}{f_{BW}(k)} = \frac{b^2 + (d - ik)^2}{-2d}, \]
and
\[ \frac{1}{f_R(k)} = \frac{1}{f_{BW}(k)} + \Delta(k), \]
where
\[
\Delta(k) = \frac{1}{|f_{BW}|} = \frac{[(k^2 + d^2 - b^2)^2 + 4b^2d^2]^{1/2}}{-2d}
\]
may be called the interference term. From here according to (5), it follows that the effective range function \( g_R(k) \) for \( S_R(k) \) is expressed in terms of the Breit–Wigner effective range function
\[
g_{BW}(k) = \Re \frac{1}{f_{BW}(k)} = \frac{k^2 - b^2 - d^2}{2d}
\]
and the interference term as
\[
g_R(k) = g_{BW}(k) + \Delta(k).
\]
In the following section, using a simple toy model, we will show that the interference term gives an essential contribution to the cross section \( \sigma_R \) as compared to the cross section \( \sigma_{BW} \). It not only disturbs the Lorentzian behavior of the \( \sigma_{BW} \) curve but also decreases twice the value of its maximum.

6. A toy model
To illustrate general properties of the Hamiltonian \( h \), we choose the Hamiltonian \( h_0 \) with the simplest (\( \nu = 1 \)) scattering potential singular at the origin,
\[
v_0 = 2a_1^2 \sinh^{-1}(a_1 x), \quad a_1 \in \mathbb{R}, \quad a_1 \neq 0
\]
with the scattering states of the form
\[
\psi_k(x) = \sqrt{\frac{2}{\pi (k^2 - a_1^2)}} [a_1 \coth(a_1 x) \sin(kx) - k \cos(kx)].
\]
Using the asymptotic behavior of this function
\[
\psi_k(x) \propto (ia_1 - k) \exp(-ikx) - (ia_1 + k) \exp(ikx),
\]
we find the background scattering matrix
\[
S_0(k) = \frac{a_1 - ik}{a_1 + ik}
\]
as well as the cross section
\[
\sigma_0 = 4\pi (k^2 + a_1^2)^{-1}.
\]
The transformation function with the asymptotic behavior as given in (15) has the form
\[
u(x) = \exp(\alpha x)[a_1 \coth(a_1 x) - a], \quad a = d + ib, \quad d < 0.
\]
From (12) and (10), we find the complex scattering potential
\[
V(x) = \frac{2a_1^2 (a^2 - a_1^2)}{[a_1 \cosh(a_1 x) - a \sinh(a_1 x)]^2}
\]
(40)
(previously obtained in [10]) which, after a change of parameters, can be reduced to a complex version of the well-known one-soliton potential.

Figure 1 shows the cross sections \( \sigma_c \) (the dash-dotted curve), \( \sigma_r \) (the dotted curve) and \( \sigma_t \) (the solid curve) related to the scattering matrix \( S_H \). (Everywhere the units are such that \( \hbar^2/(2m) = 1 \) and \( a_1 = 3, b = 0.5 \).) None of these curves exhibit any resonance behavior. In contrast, for \( d \) values close to zero, the scattering matrix \( S_0 \) gives rise to a pronounced maximum in the scattering cross section \( \sigma_h \). Figure 2 illustrates the phase shift \( \delta_h = \frac{1}{2i} \log S_0 \) for \( d = -1 \).
Figure 1. Scattering cross sections for the non-Hermitian Hamiltonian $H$: $\sigma_e$ (the dot-dashed curve), $\sigma_r$ (the dotted curve) and $\sigma_t$ (the solid curve).

Figure 2. Phase shifts $\delta_h$ for $d = -0.1$ (the solid curve), $d = -0.5$ (the dot-dashed curve) and $d = -1.0$ (the dotted curve).

In figure 3, we compare the behavior of different cross sections for $d = -0.1$. As already discussed, the background cross section is close to zero for the whole drawn momentum interval (see the dashed curve). The Hermitian Hamiltonian $h$ cross section $\sigma_h$ (the solid curve) is wider and less pronounced as compared to the Breit–Wigner cross section $\sigma_{BW}$ (the dot-dashed curve). This agrees with the fact that $\sigma_R$ is the square root of $\sigma_{BW}$. The difference between the curves $\sigma_R$ and $\sigma_{BW}$ is due to the interference term (39) and it illustrates an important contribution from this term to the cross section $\sigma_R$. The dotted and solid curves are close to each other. This is because of a small influence that the background scattering matrix $S_0$ exerts on the resonance scattering matrix $S_R$. 

...
7. Conclusion

We have found a physical meaning for the phase factor of the non-unitary scattering matrix \( S_H \) of the non-Hermitian Hamiltonian \( H \), that is, this phase factor is shown to be the same as the unitary scattering matrix \( S_h \) of the Hermitian operator \( h \) related to the given non-Hermitian one by a similarity transformation. We have demonstrated that the possible presence of a spectral singularity in the continuous spectrum of the non-Hermitian Hamiltonian \( H \) translates as a resonance in the scattering cross section of its Hermitian counterpart \( h \) (see figure 3). The closer in the space of parameters the Hamiltonian \( H \) to the singular point, the more pronounced the resonance in the cross section of \( h \). This means that although the singular point is inaccessible for the Hermitian Hamiltonian, it ‘feels’ the presence of the singularity if it is ‘close enough’. Based on this property, one may conjecture that it might be possible to detect spectral singularities in scattering experiments. We explain the resonance behavior of the Hermitian Hamiltonian cross section by the fact that the corresponding scattering matrix up to a background scattering matrix is a square root of the Breit–Wigner scattering matrix.

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