Extended Extremes of Information Combining

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Abstract—Extremes of information combining inequalities play an important role in the analysis of sparse-graph codes under message-passing decoding. We introduce new tools for the derivation of such inequalities, and show by means of a concrete example how they can be applied to solve some optimization problems in the analysis of low-density parity-check codes.

I. Setting

In order to understand iterative decoding of low-density-parity-check codes (LDPC), two operations need to be studied. These operations are the variable node convolution ⊠ and the check node convolution ⊙. They correspond to the merging of information respectively by variable nodes and by check nodes in the iterative decoding process. The reader is assumed to be familiar with LDPC codes as well as the formalism of modeling channels by densities. A very complete introduction to this topic is [1].

The notion of extremes of information combining (EIC) was introduced by I. Land, P. Hoeher, S. Huettlinger, and J. B. Huber in [3], and further extended by I. Sutskover, S. Shamai, and J. Ziv, see [2] or [6]. The idea of EIC is to associate to densities certain functionals, e.g. the entropy functional, and to see how these functionals behave under the combining of information, i.e. the two kinds of convolutions. The purpose of this work is to solve optimizing problems that arise in this setting. We will focus solely on the check convolution ⊕ although many statements can be proven in the same way for the variable node convolution.

A. Notations

There are several representations for a binary memoryless and symmetric-output channel (BMS). As is done for instance in [1], we see a BMS as a convex combination of binary symmetric channels (BSC), given by a weight distribution w. Then we have (by definition)

Example 1 (Binary Symmetric Channel BSC(ε)).

\[ w_{\text{BSC}(\epsilon)} = \delta_\epsilon \]

Example 2 (Binary Erasure Channel BEC(ε)).

\[ w_{\text{BEC}(\epsilon)} = \epsilon \delta_0 + (1-\epsilon) \delta_1 \]

The functions of interest in this domain are

\[ E(a) = \int_0^1 dw_a(\epsilon) \epsilon \]
\[ H(a) = \int_0^1 dw_a(\epsilon) h_2(\epsilon) \]
\[ B(a) = \int_0^1 dw_a(\epsilon) 2\sqrt{\epsilon(1-\epsilon)} \]

which we call respectively the error probability, the entropy and the Battacharyya functional. These can all be thought of as measures of the channel quality. They are equal to 0 for the perfect channel and equal to 1 for a useless channel. Applying these functionals to the check convolution of two densities corresponds to

\[ E(a \otimes b) = \frac{1 - (1 - 2E(a))(1 - 2E(b))}{2} \]
\[ H(a \otimes b) = \int dw_a(\epsilon) dw_b(\epsilon') h_2 \frac{1 - (1 - 2\epsilon)(1 - 2\epsilon')}{2} \]
\[ B(a \otimes b) = \int dw_a(\epsilon) dw_b(\epsilon') \sqrt{1 - ((1 - 2\epsilon)(1 - 2\epsilon'))^2} \]

In the sequel we will frequently refer to the following two functions, \( f_H \) and \( f_B \):

\[ f_H : X \in [0; 1] \mapsto h_2 \left( \frac{1 - X}{2} \right) \]
\[ f_B : X \in [0; 1] \mapsto \sqrt{1 - X^2} \]

B. Motivation

A classical result in EIC, shown in [2] and [3], is the following.

Theorem I.1. Let \( b_0 \) be any BMS channel. Amongst channels \( a \), with fixed entropy \( H \), \( H(a \otimes b_0) \) is

- minimized by the BSC(\( h_2^{-1}(h) \))
- maximized by the BEC(h)

A quick and useful application of [1] is to give bounds on the thresholds of LDPC codes. The same statement can be done with the Battacharyya functional \( B \). We will derive an alternate (calculus free) proof of the second item in [1].

Sometimes one might need to deal with non-linear expressions such as \( H(a^{0.5}) - H(a^{1.2}) \). Let us sketch very loosely, following [4], how such expressions can appear. Apart from the Shannon threshold another threshold called the Area Threshold can be defined. The Area threshold depends on
the code and channel family under consideration. In the case of a code taken from the \((d_l, d_r)\) regular ensemble, one can compute this threshold \(h^A\).

Consider a code taken from the \((d_l, d_r)\) regular ensemble, and transmission over a “gentle” channel family \(\{c_\sigma\}_\sigma\), that is a family that is smooth, ordered, and complete\(^3\). Ordered means that the bigger the channel parameter \(\sigma\) the worst the channel is, in other words, all the functionals introduced above increase with \(\sigma\), ”smooth” means we can derive \(\sigma\) in the integrals.

Then one can define a GEXIT curve in the following manner: Take a FP \((c_\sigma, x)\) and define \(y = x^\otimes d_r - 1\). Then plot,

\[
(H(c_\sigma), G(c_\sigma, y^\otimes d_l)).
\]

Here \(G(c_\sigma, \cdot) = \frac{H(d_{\sigma} + d_l)}{H(d_r)}\). In the case of the BEC, changing the channel parameter \(\sigma\), corresponds to revealing certain bits, and the kernel \(G(c_\sigma, \cdot)\) represents the probability that this bit was not previously known from the observation of the value of other neighboring bits\(^4\). In general everything has the same meaning but with soft information.

The kernel models how much more (compared to if we use only extrinsic observations) information is known about a channel, for which what BP does, is solving a system of equations iteratively solving equations where all variables are known but one. Even if the system is full rank there might still be large portions that remain unknown to BP.

where \(x\) is "the" BP fixed point with entropy \(h\) for the channel family under consideration.

The value of \(h^A\), turns out to be the right bound of the domain where the following holds

\[
-h - (d_l - 1 - \frac{d_l}{d_r})H(x^\otimes d_r) + (d_l - 1)H(x^\otimes d_r - 1) \geq 0.
\]

Here, \(x\) is "the" density evolution fixed point with entropy \(h\), using belief propagation (BP) decoding. In \([4]\) it is shown that indeed \((2)\) holds, universally over all BMS channels \(x\) with entropy lower or equal to \(\frac{d_l}{d_r}\), in the asymptotic regime \(d_l, d_r \to \infty\) with \(\frac{d_l}{d_r}\) fixed. This implies the Area threshold universally approaches the Shannon threshold. We will derive another proof of this fact in Section V (see Proposition V.2).

In \([4]\) it is then shown that a class of spatially coupled codes achieve the Area threshold, under BP decoding. This combined with the fact above, gives a new way to achieve capacity.

II. RESULTS

Our results fit in a slightly more general framework than that of Theorem I.1 we will consider expressions of the type \(\Phi(\rho(a))\) where \(\rho\) is a polynomial, and \(\Phi\) is either \(H\) or \(B\). We use the following notation

**Notation.** Let \(\rho(X) = \sum c_i X^i\) be any polynomial s.t. \(\rho(0) = 0\) i.e. \(c_0 = 0\) and \(\Phi\) be one of the functionals above. We will use the convention

\[
\Phi(\rho(a)) \overset{\text{def}}{=} \sum_i c_i \Phi(a^\otimes i).
\]

The following two statements are our main results. We prove them in the next section.

**Proposition II.1.** Let \(\rho\) be any polynomial s.t. \(\rho(0) = 0\), and \(\Phi\) be \(H\) or \(B\). Consider the following problem

\[
\begin{align*}
\text{MAX } & \Phi(\rho(a)) \\
\text{s.t. } & \Phi(a) = \phi_0 
\end{align*}
\]

Then, if \(\rho\) is \(\cup\)-convex over \([0; f_\Phi^{-1}(\phi_0)^2]\), the BEC solves this problem.

**Proposition II.2.** Let \(\rho\) be any polynomial s.t. \(\rho(0) = 0\), and \(\Phi\) be \(H\) or \(B\). Consider the following problem

\[
\begin{align*}
\text{OPT } & \Phi(\rho(a)) \\
\text{s.t. } & E(a) = \epsilon
\end{align*}
\]

Then, if \(\rho\) is increasing over \([0; 1 - 2\epsilon]\),

- the BEC minimizes this problem.
- the BSC maximizes this problem.

**Discussion.** The hypotheses for these propositions are probably not tight, they just ease the proofs. The reader should not pay too much attention to the obscure terms \(f_\Phi^{-1}(\phi_0)^2\).

\(^1\) The definitions of these terms can be found for instance in \([1]\). Examples of such families include amongst many others the \(\{\text{BEC}(h)\}_h\) and the \(\{\text{BSC}(h)\}_h\), as well as combinations of these two, and other classical families like the \(\{\text{BAWGNC}(c)\}_c\).

\(^2\) Neighbors is to be understood in the sense of the Tanner graph as usual.

\(^3\) Think of the BEC for which what BP does, is solving a system of equations by iteratively solving equations where all variables are known but one. Even if the system is full rank there might still be large portions that remain unknown to BP.

\(^4\) Instead of considering polynomials which vanish at 0, we could use a convention like \(a^{200} = \text{"Perfect Channel"}\).
The maximizing part in the previous result follow as a special case of Proposition III.1 with \( \rho = X^d \). Our improvement, technically speaking, is dealing with other polynomials than \( X^d \).

Proposition III.1 only addresses half of the question. We suspect that in most cases the minimizer is the BSC, and pose this as an interesting open question. Dealing with the problem requires that we have a lower bound. This is the purpose of the following lemma

**Lemma II.3.** Suppose \( \rho \) is increasing over \([0; f_\Phi^{-1}(\phi_0)^2]\). Then, for all channels \( a \) with \( \Phi(a) = \phi_0 \),

\[
\Phi(\rho(a)) \geq \rho(1) - \rho(f_\Phi^{-1}(\phi_0)^2)
\]

(4)

### III. PROOFS

Before we start the proof, a few preliminary observations are needed.

**A. Preliminary observations**

Let \( \Phi \) be either \( H \), the entropy or \( B \), the Battatcharyya functional. In both cases the "kernel" \( f_\Phi \) can be expanded in power series,

\[
f_\Phi(X) = 1 - \sum_{i=1}^{\infty} a_{\Phi,n} X^{2n}
\]

where equality still holds for \( X = 1 \). The crucial property of \( (a_{\Phi,n})_n \) is that all the terms are positive and furthermore

\[
\sum_{n \geq 0} a_{\Phi,n} = 1
\]

The explicit formulas are

\[
a_{H,n} = \frac{1}{2 \log(2)n(2n - 1)}
\]

\[
a_{B,n} = \frac{1}{(2n - 1)4^n}
\]

This expansion can be plugged in the definition of \( \Phi(a) \) to yield

\[
1 - \Phi(a) = 1 - \int dw_a(\epsilon) f_\Phi(1 - 2\epsilon)
\]

\[
= \sum_{n \geq 0} a_{\Phi,n} \int dw_a(\epsilon)(1 - 2\epsilon)^{2n}
\]

and we can proceed in a similar fashion for \( \Phi(a \boxtimes b) \) or more complicated expressions.

**Definition III.1** (moments). For a channel \( a \), its \( n \)-th moment is defined by

\[
\gamma_{a,n} = \int dw_a(\epsilon) (1 - 2\epsilon)^{2n}.
\]

We call the \( \gamma_{a,n} \)s moments even if, strictly speaking, they are not. Note that in terms of moments, the BEC is characterized by having all its moments equal, and the BSC by having moments that decrease geometrically.

**Example 3.** Fix \( \Phi = \phi_0 \), where \( \Phi \) is either the Battatcharyya functional or the entropy. Consider the BEC and the BSC s.t. there \( \Phi(.) \) is equal to \( \phi_0 \). Then,

\[
\gamma_{\text{BEC},n} = 1 - \phi_0
\]

\[
\gamma_{\text{BSC},n} = f_\Phi^{-1}(\phi_0)^2
\]

With this definition

\[
1 - \Phi(a) = \sum_{n \geq 1} a_{\Phi,n} \gamma_{a,n}
\]

(5)

Note also that if \( \Phi = H \), then \( 1 - \Phi \) is no other than \( C \), the capacity functional. Also, using Fubini, we see that

\[
\int dw_a(\epsilon) dw_b(\epsilon')(1 - 2\epsilon)^{2n}(1 - 2\epsilon')^{2n} = \gamma_{a,n} \gamma_{b,n}
\]

and it follows that

\[
1 - \Phi(a \boxtimes b) = \sum_{n \geq 1} a_{\Phi,n} \gamma_{a,n} \gamma_{b,n}
\]

(6)

and this yields straightforwardly

\[
1 - \Phi(a \boxtimes b) = \sum_{n \geq 1} a_{\Phi,n} \gamma_i^n
\]

(7)

More generally, if \( \rho = \sum_{i \geq 1} c_i X^i \) is a polynomial

\[
\Phi(\rho(a)) \equiv \sum_i c_i \Phi(a(\rho_i^2))
\]

\[
\equiv \sum_i c_i \left( 1 - \sum_{n \geq 1} a_{\Phi,n} \gamma_{a,n}^i \right)
\]

\[
\equiv \sum_i c_i \gamma_{a,n} - \sum_{n \geq 1} a_{\Phi,n} \sum_{i} c_i \gamma_{a,n}^i
\]

which can be rewritten as

\[
\Phi(\rho(a)) = \rho(1) - \sum_{n \geq 1} a_{\Phi,n} \rho(\gamma_{a,n})
\]

(8)

Although very simple, the expansion above gives an efficient way to derive numerous bounds. All the proofs presented here rely heavily on it.

It will be convenient in the sequel to know the range the moments can achieve. They are decreasing and positive. So the biggest moment is the first \( \gamma_{a,1} \). The next lemma states what channel \( a \) maximizes \( \gamma_{a,1} \).

**Lemma III.2.** Amongst all channels \( a \), s.t. \( \Phi(a) = \phi_0 \), the BSC maximizes \( \gamma_{a,1} \).

**Proof:** The function \( x \mapsto x^n \) is \( \cup \)-convex. Using Jensen’s inequality

\[
\gamma_{a,n} = \int dw_a(\epsilon)(1 - 2\epsilon)^{2n} \geq \left( \int dw_a(\epsilon)(1 - 2\epsilon)^2 \right)^n = \gamma_{a,1}^n
\]

Then notice

\[
1 - \phi_0 = \sum a_{\Phi,n} \gamma_{a,n} \geq \sum a_{\Phi,n} \gamma_{a,1}^n = 1 - f_\Phi(\sqrt{\gamma_{a,1}})
\]

Inverting this inequality - \( f_\Phi^{-1} \) is decreasing because \( f_\Phi \) is - gives

\[
\gamma_{a,1} \leq (f_\Phi^{-1}(\phi_0))^2
\]
Lemma III.3.

C. Proof of II.2

B. Proof of II.1

To conclude notice that given by the weights the bound is attained by and only by the BSC, for which 

$$\gamma_{\text{BSC},1} = f_{\Phi^{-1}}(\phi_0)^2$$ (9)

Notation. We may write $\gamma_1$ instead of $\gamma_{\text{BSC},1}$.

Bounds can be used at two different levels. Either we bound the moments themselves - like in the derivation of III.3 - that would be the first level. Or we can look at the expressions from one step further and see $\sum a_{\phi,n} \gamma_{\alpha,n}$ as an expectation $E(\gamma)$. Here the expectation is taken w.r.t to a discrete measure given by the weights $(a_{\phi,n})$. In this second setup, we can then use classical inequalities, like the Jensen inequality. That is the idea of the proof of II.1

D. Proof of III.3

Proof: We simply use $\gamma_{\alpha,n} \leq \gamma_{\alpha,1}$ and the monotonicity of $\rho$ to get

$$\forall n \rho(\gamma_{\alpha,n}) \leq \rho(\gamma_{\alpha,1})$$

Then

$$\Phi(\rho(a)) \overset{\S}{=} \rho(1) - \sum_n a_{\phi,n} \rho(\gamma_{\alpha,n}) \geq \rho(1) - \rho(\gamma_{\alpha,1}) \sum_1^n a_{\phi,n},$$

and using Lemma III.3 and again the monotonicity of $\rho$

$$\rho(\gamma_{\alpha,1}) \leq \rho(\gamma_{\text{BSC},1}) = \rho(f_{\Phi^{-1}}(\phi_0)^2)$$

(4) follows.

IV. OTHER INEQUALITIES

Here we give other inequalities that can be derived using the power series expansion, just as in the proofs of III.1 and III.2. We will only prove (10) along with the equality case which is the second part of (11). Remember that $\Phi$ stands for either $H$ or $B$. The reals $\alpha$ and $\beta$ sum to 1.

$$1 - \Phi(a \boxtimes b) \geq (1 - \Phi(a))(1 - \Phi(b))$$ (10)

$$\Phi(a^{\boxtimes d}) \leq \Phi(a) \left(\Phi(a) - \Phi(a^{\boxtimes 2}) - \cdots - \Phi(a^{\boxtimes (d-1)})\right)$$ (11)

$$1 - \Phi(a) \leq 1 - \Phi(a \boxtimes a)$$ (12)

$$1 - \Phi(a \boxtimes b) \leq 1 - \Phi(a \boxtimes a) \sqrt{1 - \Phi(b \boxtimes b)}$$ (13)

$$1 - \Phi(a \boxtimes b) \leq 1 - \Phi(a \boxtimes a \boxtimes b) \sqrt{1 - \Phi(b \boxtimes b)}$$ (14)

$$\Phi\left((aa + \beta b)^{\boxtimes d}\right) \geq \alpha \Phi(a^{\boxtimes d}) + \beta \Phi(b^{\boxtimes d})$$ (15)

$$\sqrt{1 - \Phi((aa + \beta b)^{\boxtimes d})} \leq \sqrt{1 - \Phi(a^{\boxtimes d}) + \beta \sqrt{1 - \Phi(b^{\boxtimes d})}}$$ (16)

$$\Phi(a) \leq f_a(1 - 2E(a))$$ (17)

$$1 - \Phi(a \boxtimes b) \leq (1 - \Phi(a))(1 - 2E(b))$$ (18)

Proof: (10): We do the same as in III-B except using another inequality than Jensen. Recall from (6) that

$$1 - \Phi(a \boxtimes b) = \sum a_{\phi,n} \gamma_{\alpha,n} \gamma_{\beta,n}.$$ We use the following corollary of FKG inequality

$$\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g),$$

whenever $f,g$ have the same monotonicity. Equality case is when $f$ or $g$ is constant a.e. Here $f : n \mapsto \gamma_{\alpha,n}, g : n \mapsto \gamma_{\beta,n}$ and $\mathbb{E}(f) = \sum a_{\phi,n} f_n$.

So, since the moments are decreasing, we get

$$1 - \Phi(a \boxtimes b) = \sum a_{\phi,n} \gamma_{\alpha,n} \gamma_{\beta,n} \geq \sum a_{\phi,n} \gamma_{\alpha,n} \sum a_{\phi,n} \gamma_{\beta,n} = (1 - \Phi(a))(1 - \Phi(b))$$

with equality when $a$ or $b$ is from the BEC family.
V. AN APPLICATION : STUDYING THE AREA THRESHOLD

Remember our initial problem which was to study when \( (19) \) holds. Fix \( c_0 > 0 \), we would like to know first, when
\[
-A = -h - (d_l - 1 - \frac{d_l}{d_r})H(a^\otimes d_r) + (d_l - 1)H(a^\otimes d_r^{-1}) \geq c_0
\]
holds. We are going to show

Lemma V.1. If the following two conditions are fulfilled then \( (19) \) holds.
(i) \( (1 - 2h_2^{-1}(h))^2 \leq \left( \frac{c_0}{d_l - 1} \right)^{d_l-1} \)
(ii) \( h \leq \frac{d_l}{d_r} - 2c_0 \)

Proof: Define
\[
d = d_r, \quad \kappa = \frac{d_l - 1 - \frac{d_l}{d_r}}{d_l - 1} \quad \rho = X^{d_l-1} - \kappa X^d
\]
We are going to use the bound from Lemma II.3. The condition for \( \rho \) to be increasing over the range of interest is \( (1 - 2h_2^{-1}(h))^2 \leq \frac{d_l - 1}{d_l - \frac{d_l}{d_r}} \), which is always true when \( \kappa \) is given the value \( \kappa = \frac{d_l - 1 - \frac{d_l}{d_r}}{d_l - 1} \). So by Lemma II.3
\[
H(\rho(a)) \geq 1 - \kappa - (1 - 2h_2^{-1}(h))^{2(d_l-1)} + \kappa (1 - 2h_2^{-1}(h))^{2d}
\]
and then,
\[
h - (d_l - 1 - \frac{d_l}{d_r})H(a^\otimes d_r) + (d_l - 1)H(a^\otimes d_r^{-1})
\]
\[
= -h + (d_l - 1)H(\rho(a))
\]
\[
\geq -h + (d_l - 1) \left[ 1 - \kappa - \rho((1 - 2h_2^{-1}(h))^2) \right]
\]
Also \( (d_l - 1)(1 - \kappa) = \frac{d_l}{d_r} \). So for \( (19) \) to hold it is enough that
\[
h + (d_l - 1) \left[ 1 - \kappa - \rho((1 - 2h_2^{-1}(h))^2) \right] \geq c_0
\]
\[
\Leftrightarrow \frac{d_l}{d_r} - h - (d_l - 1)\rho((1 - 2h_2^{-1}(h))^2) \geq c_0
\]
which can be rewritten
\[
d_l \frac{d_r}{d_r} - h \geq (d_l - 1)\rho((1 - 2h_2^{-1}(h))^2) + c_0
\]
(i) is s.t. \( \xi(h) \leq c_0 \), and then (ii) makes \( (19) \) true. Indeed,
\[
\xi(h) = (d_l - 1)(1 - 2h_2^{-1}(h))^{2d_r-2} - (d_l - 1 - \frac{d_l}{d_r})(1 - 2h_2^{-1}(h))^{2d_r}
\]
\[
\leq (d_l - 1)(1 - 2h_2^{-1}(h))^{2d_r-2}
\]
\[
\leq (d_l - 1) \left( \frac{c_0}{d_l - 1} \right)^{\frac{d_l-1}{d_l - \frac{d_l}{d_r}}} = c_0
\]
If we are interested only in the sign of \( A(h) \) and not how far it is from 0, we can let \( c_0 = f(d_l, d_r) \) to increase the range of valid \( h \). For instance, taking \( c_0 = (d_l - 1)\exp(-\sqrt{d_r} - 1) \),
\[
(i) \Leftrightarrow h_2^{-1}(h) \geq \frac{K}{\sqrt{d_r}}(1 + o(1))
\]
Where \( K \) is some constant. Asymptotically this can be taken (changing the constant) to be simply
\[
h_2^{-1}(h) \geq \frac{K}{\sqrt{d_r}}
\]
In the end, we are left with

Proposition V.2. For, \( d_l, d_r \) large enough, the range for which \( (20) \) holds contains an interval of the form \( [L(d_l, d_r); R(d_l, d_r)] \) where
\[
L(d_l, d_r) = h_2 \left( \frac{K}{\sqrt{d_r}} \right)
\]
\[
R(d_l, d_r) = \frac{d_l}{d_r} - o(\exp(-\sqrt{d_r}))
\]
Remark. Actually, changing \( c_0(d_l, d_r) \), we could replace any \( \sqrt{c} \) by \( (\cdot)^\alpha \) for any \( \alpha < 1 \).

VI. CONVEX OPTIMIZATION AND THE SHAPE OF EXTREMAL DENSITIES

Classical convex analysis provides powerful tools that allow - at least in the case where the target functional is linear - to drastically reduce the range of possible optimizers. Remember we represented the channels by probability measures over \([0; 1]\]. The basic principle is as follows

Theorem VI.1 (Dual Caratheodory). Take \( \Phi \) any continuous linear functional over BMS channels, like all those discussed above, and consider the following problem

\[
\text{OPT} \Phi(a) \quad \text{s.t.} \quad (\Phi_1(a), \ldots, \Phi_m(a)) = (\phi_1, \ldots, \phi_m)
\]
Then there are extremal densities \( a_+ \) and \( a_- \) with support of cardinality at most \( m + 1 \).

Discussion. The constraints are also assumed to be linear. A more extensive source on the topic is [5].

This principle sheds some light on the fact that the BSC (which has one mass point in our representation) and the BEC (which has two) appear so often as extremal densities, when we consider problems with a single constraint. Indeed, one constraint corresponds to at most two mass points.
Extensions of the Caratheodory Principle were amongst the tools used in [6] to track two channel parameters (namely $H$ and $E$) through the process of iterative decoding. As a result new bounds on iterative decoding were shown.

It seems hard to derive proofs using solely VI.1 However it can be used to do numerical experiments. One way to proceed is as follows. Consider the target functional $\Psi(\rho(a))$ where $\rho$ is of degree $d$. Introduce $d$ variables $a_1, a_2, \ldots, a_d$ and replace (for $k \leq d$

$$\Phi(a^{\boxtimes k}) \leadsto \binom{d}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq d} \Phi(a_{i_1} \boxtimes \ldots \boxtimes a_{i_k})$$

Denote $\tilde{\Phi}(a_1, \ldots, a_d)$ the expression we get. If it is maximized by a tuple where all coefficients $a_i$ are the same, then we know the initial expression has the same maximizer. To maximize $\tilde{\Phi}$, a simple tractable heuristic is optimizing coordinate after coordinate. Starting from random $a_i$s, to fix each coordinate except coordinate $i$, then find the best combination of two $\delta$'s for this coordinate. And repeat for all $i \leq d$. This gave good results for the motivational expression of (2) and led to the claim

**Claim.** The expression in (2), for any $h$ and when $(d_l, d_r) = (3, 6)$ or $(5, 10)$ (the cases we tested) is always minimized by the BSC and maximized by the BEC.

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