Periodic and Hyperbolic Soliton Solutions of a Number of Nonlocal PT-Symmetric Nonlinear Equations

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Abstract:
For a number of nonlocal nonlinear equations such as nonlocal, nonlinear Schrödinger equation (NLSE), nonlocal Ablowitz-Ladik (AL), nonlocal, saturable discrete NLSE (DNLSE), coupled nonlocal NLSE, coupled nonlocal AL and coupled nonlocal, saturable DNLSE, we obtain periodic solutions in terms of Jacobi elliptic functions as well as the corresponding hyperbolic soliton solutions. Remarkably, in all the six cases, we find that unlike the corresponding local cases, all the nonlocal models simultaneously admit both the bright and the dark soliton solutions. Further, in all the six cases, not only \( \operatorname{dn}(x, m) \) and \( \operatorname{cn}(x, m) \) but even their linear superposition is shown to be an exact solution. Finally, we show that the coupled nonlocal NLSE not only admits solutions in terms of Lamé polynomials of order 1, but it also admits solutions in terms of Lamé polynomials of order 2, even though they are not the solutions of the uncoupled nonlocal problem. We also remark on the possible integrability in certain cases.
1 Introduction

In recent years non-Hermitian but PT-symmetric systems have attracted widespread attention \[1\]. In the context of the nonrelativistic quantum mechanics, it has been shown that these systems exhibit real spectra so long as the PT-symmetry is not spontaneously broken. On the other hand if PT-symmetry is spontaneously broken, then the spectrum is not entirely real. In the last few years, it has been realized that optics can provide an ideal ground for testing some of the consequences of such theories. This is because, the paraxial equation of diffraction is similar in structure to the Schrödinger equation. Several studies have in fact shown that PT-symmetric optics could give rise to an entirely new class of optical structures and devices with altogether new properties \[3\]. It is thus imperative to study different types of PT-invariant nonlinear systems which could have implications in optics.

In an interesting recent paper, Ablowitz and Musslimani \[4\] (hereafter, we refer to it as I) have considered nonlocal nonlinear Schrödinger equation (NLSE) which is non-Hermitian but PT-invariant and have shown that it is an integrable system. Further, Musslimani and his collaborators \[5\] (hereafter we refer to it as II) have shown that unlike the usual NLSE, the nonlocal NLSE remarkably admits both bright and dark soliton solutions. In addition, they have also noted that unlike the local case, these solitons do not admit solutions with arbitrary shift in the transverse coordinate $x$.

The purpose of this paper is to study a number of other nonlocal nonlinear equations, all of which are non-Hermitian but PT-invariant and enquire if they too simultaneously admit both bright and dark soliton solutions. In particular, we study nonlocal Ablowitz-Ladik (AL) model, nonlocal coupled AL model, nonlocal saturable discrete NLSE (DNLSE), nonlocal coupled saturable DNLSE as well as nonlocal coupled NLSE models and obtain periodic as well as hyperbolic soliton solutions in all these cases. We show that unlike the corresponding local models, all these nonlocal models, for the focusing type of nonlinearity, simultaneously admit both the bright and the dark soliton solutions. As far as we are aware of, this is the first time that one has models simultaneously admitting both the dark and the bright soliton solutions. We also show that while these nonlocal models do not admit solutions with arbitrary shift in the transverse coordinate $x$ \[5\], they do admit solutions with specific shifts. Besides, all these models not only admit periodic solutions in terms of Jacobi elliptic functions $\text{dn}(x, m)$ and $\text{cn}(x, m)$ with modulus $m$, but even
their linear superposition [5] is also an exact periodic solution in all these cases. Further, in the coupled nonlocal NLSE models, we show that they not only admit periodic and hyperbolic soliton solutions in terms of Lamé polynomials of order 1, but they also admit periodic and hyperbolic soliton solutions in terms of Lamé polynomials of order 2, even though they are not the solutions of the uncoupled problem. Besides, the coupled model also admits periodic solutions in terms of shifted Lamé polynomials of order 1 and 2. We also obtain similar periodic soliton solutions in the case of the various discrete models, i.e. nonlocal AL, coupled nonlocal AL, nonlocal saturable DNLSE and coupled nonlocal saturable DNLSE models.

The plan of the paper is as follows. In Sec. II we consider nonlocal NLSE introduced in I and II, and obtain period solutions in terms of not only $cn(x, m)$ and $dn(x, m)$ but also their linear superposition. Further, in the same focusing model we also obtain periodic solutions in terms of $sn(x, m)$ and hence the dark and the bright hyperbolic soliton solutions. Finally, we show that this model also admits solutions in terms of shifted $sn(x, m)$, $cn(x, m)$ and $dn(x, m)$. In Sec. III we consider coupled nonlocal NLSE model and obtain periodic solutions in terms of Lamé polynomials of order 1 and 2. We show that while both nonlocal Manakov and nonlocal Mikhailov-Zakharov-Schulman (MZS) models admit periodic solutions in terms of Lamé polynomials of order 1, only MZS case admits periodic solutions in terms of Lamé polynomials of order 2 while in the Manakov limit, there are no solutions in terms of Lamé polynomials of order 2. However, we also show that the shifted Lamé polynomials of order 1 and 2 do admit solutions in both the Manakov and the MZS cases. In Sec. IV we consider a nonlocal saturable DNLSE model and show that the same focusing model simultaneously admits both $cn(x, m)$ and $dn(x, m)$ as well as $sn(x, m)$ solutions and hence the corresponding hyperbolic bright and the dark soliton solutions. Further, we show that the same model, also admits shifted $sn(x, m)$, $cn(x, m)$ and $dn(x, m)$ solutions. In Sec. V we consider a coupled nonlocal saturable DNLSE and obtain solutions in terms of Lamé polynomials of order 1. In Sec. VI we study the nonlocal AL model and show that even this model admits both the dark and the bright as well as the superposed periodic solutions. Finally, this model too admits shifted $sn(x, m)$, $cn(x, m)$ and $dn(x, m)$ solutions. In Sec. VII we consider the coupled nonlocal AL model and obtain periodic solutions in terms of Lamé polynomials of order 1. In the last section we summarize our main results and speculate about the integrability of the nonlocal AL, nonlocal Manakov and nonlocal MZS models.
2 Periodic Solutions of Nonlocal NLSE

We start with the nonlocal NLSE as considered in I and II

\[ iu_z(x, z) + u_{xx}(x, z) + gu(x, z)u^*(-x, z)u(x, z) = 0. \tag{1} \]

As has been remarked in II, one can look upon the nonlinear term as a self-induced potential of the form \( V(x, z) = u(x, z)u^*(-x, z) \) which is PT-invariant in the sense that \( V(x, z) = V^*(-x, z) \). As shown in II, this is a non-Hermitian system where total power defined by

\[ P = \int_{-\infty}^{\infty} dx |u(x, z)|^2, \tag{2} \]

is not conserved but it is the quasi-power \( Q \) and the Hamiltonian \( H \) given by

\[ Q = \int_{-\infty}^{\infty} dx u(x, z)u^*(-x, z), \quad H = \int_{-\infty}^{\infty} dx [u_x(x, z)u_x^*(-x, z) - \frac{1}{2}u^2(x, z)u^{*2}(-x, z)], \tag{3} \]

which are conserved.

Before obtaining the periodic solution, notice that the nonlocal NLS Eq. (1) has a novel (plane wave) exact solution

\[ u(x, z) = Ae^{kx - i(\omega z + \delta)}, \tag{4} \]

provided

\[ gA^2 = -(\omega + k^2). \tag{5} \]

Thus solution (4) holds good irrespective of whether \( g > 0 \) or \( g < 0 \).

Let us now obtain periodic soliton solutions to this equation.

**Solution I**

It is easily checked that one of the exact periodic solution to this equation is

\[ u = A\text{dn}(\beta x, m)e^{-i(\omega z + \delta)}, \tag{6} \]

provided

\[ gA^2 = 2\beta^2, \quad \omega = -(2 - m)\beta^2. \tag{7} \]

Here \( \delta \) is an arbitrary constant. Thus for this solution \(-2 < \omega \leq -1\) as \( 0 < m \leq 1 \),
Solution II

Similarly, another exact periodic solution to the nonlocal NLS Eq. (1) is

$$u = A\sqrt{m}\, \text{cn}(\beta x, m)e^{-i(\omega z + \delta)},$$

provided

$$gA^2 = 2\beta^2, \quad \omega = -(2m - 1)\beta^2.$$ (9)

Thus for this solution $-1 \leq \omega < 1$ as $0 < m \leq 1$,

Solution III

Remarkably, even a linear superposition of the two, i.e.

$$u = \left[A\frac{1}{2}\, \text{dn}(\beta x, m) + B\frac{1}{2}\sqrt{m}\, \text{cn}(\beta x, m)\right]e^{-i(\omega z + \delta)},$$

is also an exact periodic solution to the nonlocal NLS Eq. (1) provided

$$B = \pm A, \quad gA^2 = 2\beta^2, \quad \omega = -(1/2)(1 + m)\beta^2.$$ (11)

Thus for this solution $-1 \leq \omega < -1/2$ as $0 < m \leq 1$,

Solution IV

In the limit $m = 1$, all three periodic solutions, i.e. $\text{cn}, \text{dn}$ as well as superposed solution with $B = +A$, go over to the same hyperbolic bright soliton solution

$$u = A\text{sech}(\beta x)e^{-i(\omega z + \delta)},$$

provided

$$gA^2 = 2\beta^2, \quad \omega = -\beta^2.$$ (13)

On the other hand, the superposed solution with $B = -A$ goes over to the vacuum solution $u = 0$.

Solution V

Remarkably, unlike the NLSE case, the same (focusing) nonlocal NLS Eq. (1) also admits the periodic $\text{sn}$ solution

$$u = A\sqrt{m}\, \text{sn}(\beta x, m)e^{-i(\omega z + \delta)},$$

provided

$$gA^2 = 2\beta^2, \quad \omega = -\beta^2.$$ (14)

On the other hand, the superposed solution with $B = -A$ goes over to the vacuum solution $u = 0$. 
provided
\[ gA^2 = 2\beta^2, \quad \omega = (1 + m)\beta^2. \] (15)

Thus for this solution \(1 < \omega \leq 2\) as \(0 < m \leq 1\),

At this point, it is worth asking the following question: just as there are periodic solutions \(cn\) and \(dn\) with period \(4K(m)\) and \(2K(m)\) respectively both of which go over to the same pulse solution (sech) in the \(m = 1\) limit, why one only has sn solution with period \(4K(m)\) but no period \(2K(m)\) solution with both going to the same dark soliton solution in the \(m = 1\) limit? Such a solution is in fact possible. In particular, it is easily checked that
\[ u = Am \frac{cn(\beta x, m)sn(\beta x, m)}{dn(\beta, x)} e^{-i(\omega z + \delta)}, \] (16)
is an exact solution (with period \(2K(m)\)) to Eq. (1) provided
\[ gA^2 = 2\beta^2, \quad \omega = 2(2 - m)\beta^2. \] (17)

Thus for this solution \(2 \leq \omega < 4\) as \(0 < m \leq 1\), Note that at \(m = 1\), both solutions (14) and (16) go over to the same kink solution. Perhaps the reason why this solution is not considered in the literature is because it is related to the sn solution by Landen transformation
\[ \frac{1 + \sqrt{1 - m}}{\text{sn}(x, m)\text{cn}(x, m)} = \text{sn} \left( 1 + \sqrt{1 - m} \right) x, \left( \frac{1 - \sqrt{1 - m}}{1 + \sqrt{1 - m}} \right)^2. \] (18)

**Solution VI**

In the limit \(m = 1\), the periodic solution \(V\) (and the one given by Eq. (16) go over to the same hyperbolic dark soliton solution
\[ u = A \text{tanh}(\beta x) e^{-i(\omega t + \delta)}, \] (19)
provided
\[ gA^2 = 2\beta^2, \quad \omega = 2\beta^2. \] (20)

It is worth pointing out that while the defocusing NLSE \((g < 0)\) admits sn solution, defocusing, nonlocal NLSE (i.e. Eq. (1) with \(g < 0\)) does not admit a periodic solution in terms of either sn or sncn/dn (or cn or dn) or the hyperbolic kink or pulse solutions. However, as we now show, in the defocusing case, there are solutions in terms of shifted sn and cn.
2.1  **Shifted sn, cn, dn Solutions**

If we look at all the periodic as well as the hyperbolic soliton solutions of the nonlocal equations, we find that unlike the local NLSE case, the solutions of the nonlocal NLSE are *not invariant* with respect to shifts in the transverse coordinate $x$. For example, while $A\text{dn}[\beta(x + x_0), m]e^{-i\omega t}$ is an exact solution of the local NLSE, no matter what $x_0$ is, as remarked in II, for arbitrary $x_0$, it is not an exact solution of our nonlocal NLSE. However, we now show that for special values of $x_0$, sn, cn, dn are still the solutions of the nonlocal NLSE. In particular, we show that when $x_0 = K(m)$, there are exact solutions of the nonlocal NLSE as given by Eq. (1) for both the focusing ($g > 0$) and the defocusing ($g < 0$) cases. This is because,

$$
\begin{align*}
\text{dn}[x + K(m), m] &= \frac{\sqrt{1 - m}}{\text{dn}(x, m)}, \\
\text{sn}[x + K(m), m] &= \frac{\text{cn}(x, m)}{\text{dn}(x, m)}, \\
\text{cn}[x + K(m), m] &= -\sqrt{1 - m} \frac{\text{sn}(x, m)}{\text{dn}(x, m)},
\end{align*}
$$

(21)

and as we show now, these are indeed the solutions of the nonlocal NLSE Eq. (1).

**Solution VII**

It is easily checked that an exact periodic solution to the nonlocal NLSE (1) is

$$
\begin{align*}
u &= A\sqrt{1 - m} e^{-i(\omega z + \delta)},
\end{align*}
$$

(22)

provided conditions as given by Eq. (7) are satisfied, which are precisely the conditions under which dn is an exact solution.

**Solution VIII**

Yet another periodic solution to Eq. (1) is

$$
\begin{align*}
u &= A\sqrt{m(1 - m)} \frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega z + \delta)},
\end{align*}
$$

(23)

provided

$$
g A^2 = -2\beta^2, \quad \omega = -(2m - 1)\beta^2.
$$

(24)

**Solution IX**

Yet another exact periodic solution to Eq. (1) is

$$
\begin{align*}
u &= A\sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega z + \delta)},
\end{align*}
$$

(25)
provided

\[(1 - m)gA^2 = -2(1 - m)\beta^2, \quad \omega = (1 + m)\beta^2. \quad (26)\]

Note that while \(cn\) and \(sn\) are the exact solution in the focusing case, i.e. \(g > 0\), the above two solutions (23) and (25) (which are just the shifted \(cn\) and \(sn\) solutions respectively), are however only valid in the defocusing case, i.e. \(g < 0\). We find this to be rather interesting. Further, while \(dn\), \(cn\) and \(sn\) are the exact solutions of our nonlocal NLSE for all nonzero values of \(m\) including \(m = 1\) (i.e. for hyperbolic solitons), the above three shifted periodic solutions are only valid in case \(0 < m < 1\).

It may be noted here that there are solutions to the nonlocal NLSE Eq. (1) even when \(x_0 = iK(1 - m)\) or when \(x_0 = K(m) + iK(1 - m)\) but these are singular solutions and hence we do not consider them in this paper.

Summarizing, we find that for the nonlocal NLSE with focusing type \((g > 0)\) of nonlinearity, while \(cn\), \(dn\), \(sn\) and \(1/dn\) are the exact periodic solutions, for the defocusing case (i.e. \(g < 0\)), \(cn/dn\) and \(sn/dn\) are the exact periodic solutions.

### 3 Nonlocal Coupled NLSE

Let us consider the following coupled nonlocal NLS field equations

\[
iu_z(x, z) + u_{xx}(x, z) + [au(x, z)u^*(-x, z) + bv(x, z)v^*(-x, z)]u(x, z) = 0,\]

\[iv_z(x, z) + v_{xx}(x, z) + [fu(x, z)u^*(-x, z) + ev(x, z)v^*(-x, z)]v(x, z) = 0, \quad (27)\]

where \(u\) and \(v\) are the two coupled nonlocal NLS fields and \(a, b, f, e\) are arbitrary real numbers. This is also a non-Hermitian PT-invariant system in the sense that the self-induced potential \(V(x, z) = u(x, z)u^*(-x, z) + v(x, z)v^*(-x, z)\) satisfies \(V(x, z) = V^*(-x, z)\). In this model, whereas total power, defined by

\[
P = \int_{-\infty}^{\infty} dx \left( |u(x, z)|^2 + |v(x, z)|^2 \right), \quad (28)\]

is not conserved, but the quasi-powers given by

\[Q_1 = \int_{-\infty}^{\infty} dx \left[ u(x, z)u^*(-x, z) \right], \]

\[Q_2 = \int_{-\infty}^{\infty} dx \left[ v(x, z)v^*(-x, z) \right], \quad (29)\]
are separately conserved. In the special case when \( a = f, b = e \), the so called mixed quasi-power given by

\[
Q_3 = \int_{-\infty}^{\infty} dx \left[ u(x, z)v^*(-x, z) + v(x, z)u^*(-x, z) \right],
\]

is also conserved. Further, there is another conserved quantity

\[
C_1 = u_x(x, z)u_x^*(-x, z) + v_x(x, z)v_x^*(-x, z)
\]

\[
- \frac{1}{2} \left[ au^2(x, z)(u^*(-x, z))^2 + ev^2(x, z)(v^*(-x, z))^2 - 2gu(x, z)u^*(-x, z)v(x, z)v^*(-x, z) \right].
\]

provided \( g = b = f \). In particular, in the so called Manakov limit (i.e. \( a = b = f = e \)), we have the conserved quantity \( C_1 \).

It is worth pointing out that these conserved quantities are similar to the corresponding conserved quantities in the local coupled NLSE, in particular, the conserved quantities for the nonlocal case are obtained from the corresponding local model by simply replacing \( u^*(x, z) \) and \( v^*(x, z) \) by \( u^*(-x, z) \) and \( v^*(-x, z) \), respectively.

We now show that these coupled equations admit periodic solutions in terms of Lamé polynomials of order 1 and 2. In the special case when \( a = f = b = e \) this system reduces to the nonlocal Manakov system. In this context, it is worth remembering that the corresponding local Manakov system is a well known integrable system [7]. Remarkably, even when \( a = f = -b = -e \), the local case corresponds to the integrable MZS system [8, 9, 10]. Hence we shall call the above coupled equations as nonlocal MZS system in case \( a = f = -b = -e \). We shall however discuss the exact periodic solutions of this coupled nonlinear system when the coefficients \( a, b, f, e \) are arbitrary but real.

Before we discuss the periodic solutions to the coupled Eqs. (27), we remark that the coupled system has a novel exact solution given by

\[
u(x, z) = A e^{k_1x-i(\omega_1 z + \delta_1)}, \quad v(x, z) = B e^{k_2x-i(\omega_2 z + \delta_2)},
\]

provided

\[
\omega_1 + k_1^2 + aA^2 + bB^2 = 0, \quad \omega_2 + k_2^2 + fA^2 + eB^2 = 0.
\]

Solutions in terms of Lamé Polynomial of Order 1
We now show that these coupled equations admit 7 different solutions in terms of Lamé polynomials of order 1 in the $u$ and the $v$ fields, and 3 solutions in the corresponding hyperbolic limit. Actually, there are 10 distinct solutions but since both $u$ and $v$ fields are the nonlocal NLS fields, the truly distinct solutions are only 7.

**Solution I**

It is easily checked that

\[ u(x, t) = A \operatorname{dn}(\beta x, m)e^{-i(\omega_1 z + \delta_1)} , \]  
(34)

and

\[ v(x, t) = B \sqrt{m} \operatorname{sn}(\beta x, m)e^{-i(\omega_2 z + \delta_2)} , \]  
(35)

is an exact solution to the coupled field Eqs. (27) provided

\[ aA^2 + bB^2 = fA^2 + eB^2 = 2\beta^2 , \]  
(36)

\[ \omega_1 = m\beta^2 - aA^2, \quad \omega_2 = (1 + m)\beta^2 - fA^2 . \]  
(37)

On solving Eqs. (36) we find that so long as $bf \neq ae$, $A, B$ are given by

\[ A^2 = \frac{2\beta^2(e - b)}{ae - bf}, \quad B^2 = \frac{2\beta^2(a - f)}{ae - bf} . \]  
(38)

Few remarks are in order at this stage. Most of these remarks apply to all the solutions obtained below (in terms of Lamé polynomials of order 1).

1. It turns out that all the solutions in terms of Lamé polynomials of order 1 are only valid if relations (36) are satisfied. In case $ae = bf$, then along with Eqs. (36) this also implies that $b = e, a = f$. In that case instead of the relations (38), we only have the constraint $aA^2 + bB^2 = 2\beta^2$.

2. In the Manakov case (i.e. when $a = b = e = f$) the constraint (36) becomes $a(A^2 + B^2) = 2\beta^2$. On the other hand in the MZS case, (i.e. when $a = f = -e = -b$), the constraint becomes $a(A^2 - B^2) = 2\beta^2$.

**Solution II**

It is easily checked that

\[ u(x, t) = A \sqrt{m} \operatorname{cn}(\beta x, m)e^{-i(\omega_1 z + \delta_1)} , \]  
(39)
and

\[ v(x, t) = B\sqrt{m} \text{sn}(\beta x, m)e^{-i(\omega_2 z + \delta_2)}, \tag{40} \]

is an exact solution to the coupled field Eqs. (27) provided Eq. (36) is satisfied and further

\[ \omega_1 = \beta^2 - maA^2, \quad \omega_2 = (1 + m)\beta^2 - mfA^2. \tag{41} \]

**Solution III**

It is easily checked that

\[ u(x, t) = A\sqrt{m} \text{sn}(\beta x, m)e^{-i(\omega_1 z + \delta_1)}, \tag{42} \]

and

\[ v(x, t) = B\sqrt{m} \text{sn}(\beta x, m)e^{-i(\omega_2 z + \delta_2)}, \tag{43} \]

is an exact solution to the coupled field Eqs. (27) provided Eq. (36) is satisfied and further

\[ \omega_1 = \omega_2 = (1 + m)\beta^2. \tag{44} \]

**Solution IV**

It is easily checked that

\[ u(x, t) = A\sqrt{m} \text{cn}(\beta x, m)e^{-i(\omega_1 z + \delta_1)}, \tag{45} \]

and

\[ v(x, t) = B\sqrt{m} \text{cn}(\beta x, m)e^{-i(\omega_2 z + \delta_2)}, \tag{46} \]

is an exact solution to the coupled field Eqs. (27) provided Eq. (36) is satisfied and further

\[ \omega_1 = \omega_2 = -(2m - 1)\beta^2. \tag{47} \]

**Solution V**

It is easily checked that

\[ u(x, t) = A\text{dn}(\beta x, m)e^{-i(\omega_1 z + \delta_1)}, \tag{48} \]

and

\[ v(x, t) = B\text{dn}(\beta x)e^{-i(\omega_2 z + \delta_2)}, \tag{49} \]
is an exact solution to the coupled field Eqs. (27) provided Eq. (36) is satisfied and further

$$\omega_1 = \omega_2 = -(2 - m)\beta^2. \quad (50)$$

**Solution VI**

It is easily checked that

$$u(x, t) = A \text{dn}(\beta x, m) e^{-i(\omega_1 z + \delta_1)}, \quad (51)$$

and

$$v(x, t) = B \sqrt{m} \text{cn}(\beta x, m) e^{-i(\omega_2 z + \delta_2)}, \quad (52)$$

is an exact solution to the coupled field Eqs. (27) provided Eq. (36) is satisfied and further

$$\omega_1 = -(4 - 3m)\beta^2 - (1 - m)aA^2, \quad \omega_2 = -(2m - 1)\beta^2 - (1 - m)fA^2. \quad (53)$$

**Solution VII**

Remarkably, it turns out that a linear superposition of dn and cn is also a solution to the coupled Eqs. (27). In particular,

$$u(x, t) = \frac{1}{2} \left[ A \text{dn}(\beta x, m) + D \sqrt{m} \text{cn}(\beta x, m) \right] e^{-i(\omega_1 z + \delta_1)}, \quad (54)$$

and

$$v(x, t) = \frac{1}{2} \left[ B \text{dn}(\beta x, m) + \sqrt{m} E \text{cn}(\beta x, m) \right] e^{-i(\omega_2 z + \delta_2)}, \quad (55)$$

is an exact solution to the coupled field equations (27) provided Eqs. (36) is satisfied and further

$$D = \pm A, \quad E = \pm B, \quad \omega_1 = \omega_2 = -\frac{1}{2}(1 + m)\beta^2. \quad (56)$$

Note that the signs of $D = \pm A$ and $E = \pm B$ are correlated.

**Hyperbolic Limit**

In the limit $m = 1$, all the solutions mentioned above go over to the hyperbolic soliton solutions which we mention one by one.

**Solution VIII**

It is easily checked that in the limit $m = 1$, the Solutions I and II go over to the mixed hyperbolic (bright-dark) soliton solution

$$u(x, t) = A \text{sech}(\beta x) e^{-i(\omega_1 z + \delta)}, \quad (57)$$
and
\[ v(x, t) = B \tanh(\beta x) e^{-i(\omega_2 z + \delta)}, \] (58)
provided Eq. (36) is satisfied and further
\[ \omega_1 = \beta^2 - aA^2, \quad \omega_2 = 2\beta^2 - fA^2. \] (59)

**Solution IX**

In the limit \( m = 1 \) the Solution III goes over to the dark-dark soliton solution
\[ u(x, t) = A \tanh(\beta x) e^{-i(\omega_1 z + \delta)}, \] (60)
and
\[ v(x, t) = B \tanh(\beta x) e^{-i(\omega_2 z + \delta)}, \] (61)
provided Eq. (36) is satisfied and further
\[ \omega_1 = \omega_2 = 2\beta^2. \] (62)

**Solution X**

Finally, in the limit \( m = 1 \), Solutions IV to VII go over to the bright-bright soliton solution
\[ u(x, t) = A \text{sech}(\beta x) e^{-i(\omega_1 z + \delta)}, \] (63)
and
\[ v(x, t) = B \text{sech}(\beta x) e^{-i(\omega_2 z + \delta)}, \] (64)
provided Eq. (36) is satisfied and further
\[ \omega_1 = \omega_2 = -\beta^2. \] (65)

**Solutions in Terms of Lamé Polynomials of Order 2**

We now show that remarkably, the coupled Eqs. (27) admit 8 distinct periodic solutions in terms of Lamé polynomials of order 2, and 3 solutions in the corresponding hyperbolic limit even though neither of them are the solutions of the uncoupled, nonlocal NLS equation. Actually, there are 17 distinct solutions...
but since both $u$ and $v$ fields are the nonlocal NLS fields, hence the truly distinct possible solutions are only 11 but it turns out that out of these, only 8 solutions actually exist. Remarkably, it turns out that all the solutions in terms of Lamé polynomials of order two are only valid for the MZS case and no solution exists for the Manakov case. However, all the solutions also exist in the more general case of $a = f, b = e$.

**Solution XI**

It is easily checked that

$$u(x, t) = A \sqrt{m} \, \text{dn}(\beta x, m) \, \text{cn}(\beta x, m) \, e^{-i(\omega_1 z + \delta_1)},$$

(66)

and

$$v(x, t) = B \sqrt{m} \, \text{sn}(\beta x, m) \, \text{dn}(\beta x, m) \, e^{-i(\omega_2 z + \delta_2)},$$

(67)

is an exact solution to the coupled field equations (27) provided

$$a = f > 0, \quad b = e < 0, \quad m a A^2 = -m b B^2 = 6 \beta^2,$$

(68)

$$\omega_1 = -(5 - m) \beta^2, \quad \omega_2 = -(5 - 4 m) \beta^2.$$

(69)

Thus while this solution will hold good in the MZS case, it will not hold good in the Manakov case.

**Solution XII**

It is easily checked that

$$u(x, t) = A \sqrt{m} \, \text{dn}(\beta x, m) \, \text{cn}(\beta x, m) \, e^{-i(\omega_1 z + \delta_1)},$$

(70)

and

$$v(x, t) = B m \, \text{sn}(\beta x, m) \, \text{dn}(\beta x, m) \, e^{-i(\omega_2 z + \delta_2)},$$

(71)

is an exact solution to the coupled field equations (27) provided Eq. (68) is satisfied and further

$$\omega_1 = -(5 m - 1) \beta^2, \quad \omega_2 = -(5 - 4 m) \beta^2.$$

(72)

**Solution XIII**

It is easily checked that

$$u(x, t) = A m \, \text{cn}(\beta x, m) \, \text{sn}(\beta x, m) \, e^{-i(\omega_1 z + \delta_1)},$$

(73)
and
\[ v(x, t) = B \sqrt{m} \text{sn}(\beta x, m) \text{dn}(\beta x, m) e^{-i(\omega_2 z + \delta_2)}, \]

is an exact solution to the coupled field equations (27) provided
\[ a = f < 0, \quad b = e > 0, \quad -(1 - m)aA^2 = (1 - m)bB^2 = 6\beta^2, \]
and further
\[ \omega_1 = (4 + m)\beta^2, \quad \omega_2 = (1 + 4m)\beta^2. \]

Notice that this solution while it exists for 0 < m < 1, it does not hold good in the hyperbolic limit m = 1.

**Solution XIV**

It is easily checked that
\[ u(x, t) = [A \text{dn}^2(\beta x, m) + D] e^{-i(\omega_1 z + \delta_1)}, \]
and
\[ v(x, t) = B \sqrt{m} \text{cn}(\beta x, m) \text{dn}(\beta x, m) e^{-i(\omega_2 z + \delta_2)}, \]
is an exact solution to the coupled field equations (27) provided
\[ a = f, \quad b = e, \quad aA^2 = -bB^2, \quad (1 - m + 2z)aA^2 = 6\beta^2, \quad z = \frac{D}{A}, \]
\[ z = -(2 - m) \pm \sqrt{1 - m + m^2}, \]
\[ \omega_1 - k^2 = -[2(2 - m) + 3z] \beta^2 - aA^2 z^2, \]
\[ \omega_2 - k^2 = -(5 - m) \beta^2 - aA^2 z^2. \]

**Solution XV**

It is easily checked that
\[ u(x, t) = [A \text{dn}^2(\beta x, m) + D] e^{-i(\omega_1 z + \delta_1)}, \]
and
\[ v(x, t) = B \sqrt{m} \text{sn}(\beta x, m) \text{dn}(\beta x, m) e^{-i(\omega_2 z + \delta_2)}, \]
is an exact solution to the coupled field equations \([27]\) provided

\[
a = f, \quad b = e, \quad aA^2 = -bB^2, \quad (1 + 2z)aA^2 = 6\beta^2, \quad z = \frac{D}{A},
\]

\[
z = \frac{-(2 - m) \pm \sqrt{1 - m + m^2}}{2},
\]

\[
\omega_1 - k^2 = -\left[2(2 - m) + 3z\right] \beta^2 - aA^2z^2,
\]

\[
\omega_2 - k^2 = -(5 - 4m)\beta^2 - aA^2z^2.
\]

**Solution XVI**

It is easily checked that

\[
u(x, t) = [2\text{dn}^2(\beta x, m) + D] e^{-i(\omega_1 z + \delta_1)},
\]

and

\[
v(x, t) = Bm \text{cn}(\beta x, m) \text{sn}(\beta x, m) e^{-i(\omega_2 z + \delta_2)},
\]

is an exact solution to the coupled field equations \([27]\) provided

\[
a = f, \quad b = e, \quad aA^2 = -bB^2, \quad (2 - m + 2z)aA^2 = 6\beta^2, \quad z = \frac{D}{A},
\]

\[
z = \frac{-(2 - m) \pm \sqrt{1 - m + m^2}}{2},
\]

\[
\omega_1 - k^2 = -\left[2(2 - m) + 3z\right] \beta^2 - aA^2z^2 - (1 - m),
\]

\[
\omega_2 - k^2 = -(2 - m)\beta^2 - aA^2z^2 - (1 - m).
\]

**Solution XVII**

It is easily checked that

\[
u(x, t) = [2\text{dn}^2(\beta x, m) + D] e^{-i(\omega_1 z + \delta_1)},
\]

and

\[
v(x, t) = [2\text{dn}^2(\beta x, m) + E] e^{-i(\omega_2 z + \delta_2)},
\]

is an exact solution to the coupled field equations \([27]\) provided

\[
a = f, \quad b = e, \quad aA^2 = -bB^2, \quad (z_+ - y_+)aA^2 = 6\beta^2, \quad z = \frac{D}{A}, \quad y = \frac{E}{B},
\]

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\[ z_\pm = y_\pm = \frac{- (2 - m) \pm \sqrt{1 - m + m^2}}{3}, \]
\[ \omega_1 - k^2 = \mp 2 \sqrt{1 - m + m^2} \beta^2, \quad \omega_2 - k^2 = \pm 2 \sqrt{1 - m + m^2} \beta^2. \] (92)

From the relation (91) it follows that this solution exists only if \( y \) and \( z \) are unequal. In addition, depending on if we choose \( z_+, y_- \) or \( z_-, y_+ \), the corresponding \( \omega_1, \omega_2 \) are as given by Eq. (92).

**Solution XIII**

Remarkably, it turns out that even a superposition of \( \text{dn}^2 \) and \( \text{cn} \beta \text{dn} \) is an exact solution to the coupled Eqs. (27). In particular, it is easily checked that
\[ u(x, t) = \left[ \frac{A}{2} \text{dn}^2(\beta x, m) + D + \frac{G}{2} \sqrt{m} \text{cn}(\beta x, m) \text{dn}(\beta x, m) \right] e^{-i(\omega_1 z + \delta_1)}, \] (93)
and
\[ v(x, t) = \left[ \frac{B}{2} \text{dn}^2(\beta x, m) + E + \frac{H}{2} \sqrt{m} \text{cn}(\beta x, m) \text{dn}(\beta x, m) \right] e^{-i(\omega_2 z + \delta_2)}, \] (94)
is an exact solution to the coupled field equations (27) provided
\[ G = \pm A, \quad H = \pm B, \quad a = f, \quad b = e, \quad aA^2 = - bB^2, \]
\[ (z_\pm - y_\pm) aA^2 = 6 \beta^2, \quad z = \frac{D}{A}, \quad y = \frac{E}{B}, \] (95)
\[ z_\pm = y_\pm = \frac{- (5 - m) \pm \sqrt{1 + 14m + m^2}}{12}, \]
\[ \omega_1 - k^2 = \mp \sqrt{1 + 14m + m^2} \beta^2, \quad \omega_2 - k^2 = \pm \sqrt{1 - m + m^2} \beta^2. \] (96)

Thus akin to the last solution, this solution exists only if \( y \) and \( z \) are unequal. Further, depending on if we choose \( z_+, y_- \) or \( z_-, y_+ \), the corresponding \( \omega_1, \omega_2 \) are as given by Eq. (96).

**Hyperbolic Solitons of Order 2**

In the limit \( m = 1 \), the above 8 Lamé polynomial solutions of order 2 reduce to the hyperbolic soliton solutions, which we discuss one by one.

**Solution XIX**

In the limit \( m = 1 \), the solutions XI and XII go over to the hyperbolic bright-dark soliton solution. In particular
\[ u(x, t) = A \text{sech}^2(\beta x) e^{-i(\omega_1 z + \delta_1)}, \] (97)
and
\[
v(x, t) = B \tanh(\beta x) \sech(\beta x) e^{-i(\omega_2 z + \delta_2)},
\]
is an exact solution to the coupled field equations (27) provided
\[
a = f > 0, \quad b = e < 0, \quad aA^2 = -bB^2 = 6\beta^2,
\]
\[
\omega_1 = -4\beta^2, \quad \omega_2 = -\beta^2.
\]

Solution XX

In the limit \( m = 1 \), the solution XIV goes over to the bright-bright hyperbolic soliton solution
\[
u(x, t) = [A \sech^2(\beta x) + D] e^{-i(\omega_1 z + \delta_1)},
\]
and
\[
v(x, t) = B \sech^2(\beta x) e^{-i(\omega_2 z + \delta_2)},
\]
provided
\[
a = f < 0, \quad b = e > 0, \quad aA^2 = -bB^2, \quad aA^2 = -\frac{9}{2} \beta^2,
\]
\[
z = \frac{D}{A} = -\frac{2}{3}, \quad \omega_1 = 2\beta^2 \quad \omega_2 = -2\beta^2.
\]

Solution XXI

In the limit \( m = 1 \), the solutions XV and XVI go over to the bright-dark hyperbolic soliton solution
\[
u(x, t) = [A \sech^2(\beta x) + D] e^{-i(\omega_1 z + \delta_1)},
\]
and
\[
v(x, t) = B \tanh(\beta x) \sech(\beta x) e^{(-i(\omega_2 z + \delta_2))},
\]
provided
\[
a = f < 0, \quad b = e > 0, \quad aA^2 = -bB^2 = -18\beta^2,
\]
\[
z = \frac{D}{A} = -\frac{2}{3}, \quad \omega_1 = 8\beta^2 \quad \omega_2 = 7\beta^2.
\]

Finally, note that in the limit \( m = 1 \), the solutions XVII and XIII either go over to the solution XX or solution XX with \( u \) and \( v \) interchanged. Further as remarked earlier, solution XIII does not exist in the \( m = 1 \) limit.
3.1 Shifted Solutions in terms of Lamé Polynomials of Order 1 and 2

We now show that this model also admits shifted solutions in terms of Lamé polynomials of order 1 and 2.

**Shifted Solutions in terms of Lamé Polynomials of order 1**

**Solution XXII**

It is easily checked that

\[
u(x, t) = A \sqrt{m} \frac{cn(\beta x, m)}{dn(\beta x, m)} e^{-i(\omega_1 z + \delta_1)} ,
\]

and

\[
v(x, t) = B \sqrt{m(1 - m)} \frac{sn(\beta x, m)}{dn(\beta x, m)} e^{-i(\omega_2 z + \delta_2)} ,
\]

is an exact solution to the coupled field Eqs. (27) provided

\[
aA^2 + bB^2 = fA^2 + cB^2 = -2\beta^2 ,
\]

\[0 < m < 1 , \quad \omega_1 = (1 - m)\beta^2 - maA^2 , \quad \omega_2 = -(2m - 1)\beta^2 - mfA^2 .
\]

On solving Eqs. (36) we find that so long as \(bf \neq ae\), \(A, B\) are given by

\[
A^2 = \frac{2\beta^2(b - e)}{ae - bf} , \quad B^2 = \frac{2\beta^2(f - a)}{ae - bf} .
\]

Few remarks are in order at this stage. These remarks also apply to the next two solutions and hence we will not repeat them while discussing these two solutions.

1. It turns out that the solutions XXII, XXIII and XXIV in terms of Lamé polynomial of order 1 are only valid if relations (111) are satisfied. In case \(ae = bf\), then along with Eqs. (111) this also implies that \(b = e, a = f\). In that case instead of the relations (113), we only have the constraint

\[
aA^2 + bB^2 = -2\beta^2 .
\]

2. In the Manakov case (i.e. when \(a = b = e = f\)) the constraint (36) becomes \(a(A^2 + B^2) = -2\beta^2\).

On the other hand in the MZS case, (i.e. when \(a = f = -e = -b\)), the constraint becomes \(a(A^2 - B^2) = -2\beta^2\).

**Solution XXIII**
It is easily checked that
\[ u(x, t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \quad (114) \]
and
\[ v(x, t) = B \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_2 z + \delta_2)}, \quad (115) \]
is an exact solution to the coupled field Eqs. (27) provided relations (111) are satisfied and further
\[ \omega_1 = \omega_2 = (1 + m) \beta^2. \quad (116) \]

**Solution XXIV**

It is easily checked that
\[ u(x, t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \quad (117) \]
and
\[ v(x, t) = B \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_2 z + \delta_2)}, \quad (118) \]
is an exact solution to the coupled field Eqs. (27) provided relations (111) are satisfied and further
\[ \omega_1 = \omega_2 = -(2m - 1) \beta^2. \quad (119) \]

**Solution XXV**

It is easily checked that
\[ u(x, t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \quad (120) \]
and
\[ v(x, t) = B \sqrt{m} \frac{1}{\text{dn}(\beta x, m)} e^{-i(\omega_2 z + \delta_2)}, \quad (121) \]
is an exact solution to the coupled field Eqs. (27) provided
\[ bB^2 - aA^2 = eB^2 - fA^2 = 2\beta^2, \quad (122) \]
\[ 0 < m < 1, \quad \omega_1 = -(1 - m) \beta^2 - aA^2, \quad \omega_2 = -(2 - m) \beta^2 - fA^2. \quad (123) \]

On solving Eqs. (122) we find that so long as \( bf \neq ae \), \( A, B \) are given by
\[ A^2 = \frac{2\beta^2(b - e)}{ae - bf}, \quad B^2 = \frac{2\beta^2(a - f)}{ae - bf}. \quad (124) \]
However, in case $ae = bf$ then $a = f, b = e$ and then instead of the relations (124), we only have the constraint $2\beta^2 = bB^2 - aA^2$. In the Manakov limit (i.e. $a = f = b = e$), we have the constraint $a(A^2 - B^2) = -2\beta^2$ while in the MZS case (i.e. $a = f = -b = -e$) we have the constraint $a(A^2 + B^2) = -2\beta^2$.

**Solution XXVI**

It is easily checked that

$$u(x, t) = A \sqrt{m(1-m)} \frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)} e^{-i(\omega_1 z + \delta_1)},$$

and

$$v(x, t) = B \sqrt{1-m} \frac{1}{\text{dn}(\beta x, m)} e^{-i(\omega_2 z + \delta_2)},$$

is an exact solution to the coupled field Eqs. (27) provided relations (122) are satisfied and further

$$0 < m < 1, \quad \omega_1 = -\beta^2 - (1-m)aA^2, \quad \omega_2 = -(2-m)\beta^2 - (1-m)fA^2.$$  

(127)

Thus all the remarks made after the previous solution (i.e. solution XXV) are also valid in this case.

**Solution XXVII**

It is easily checked that

$$u(x, t) = A \sqrt{1-m} \frac{1}{\text{dn}(\beta x, m)} e^{-i(\omega_1 z + \delta_1)},$$

and

$$v(x, t) = B \sqrt{1-m} \frac{1}{\text{dn}(\beta x, m)} e^{-i(\omega_2 z + \delta_2)},$$

is an exact solution to the coupled field Eqs. (27) provided

$$aA^2 + bB^2 = fA^2 + eB^2 = 2\beta^2,$$

(130)

$$0 < m < 1, \quad \omega_1 = \omega_2 = -(2-m)\beta^2.$$  

(131)

On solving Eqs. (130) we find that so long as $bf \neq ae$, $A, B$ are given by

$$A^2 = \frac{2\beta^2(e-b)}{ae-bf}, \quad B^2 = \frac{2\beta^2(a-f)}{ae-bf}.$$  

(132)
However, in case $ae = bf$ then $a = f, b = e$ and then instead of the relations (132), we only have the constraint $2\beta^2 = aA^2 + bB^2$. In the Manakov limit (i.e. $a = f = b = e$), we have the constraint $a(A^2 + B^2) = 2\beta^2$ while in the MZS case (i.e. $a = f = -b = -e$) we have the constraint $a(A^2 - B^2) = 2\beta^2$.

Summarizing, we have presented 6 shifted solutions in terms of Lamé polynomials of order 1, all of which are only valid in case $0 < m < 1$ but not at $m = 1$.

**Shifted Solutions in Terms of Lamé Polynomials of order 2**

We now present 7 solutions of the coupled Eqs. (27) in terms of shifted Lamé polynomials of order 2, even though none of them is a solution of the corresponding uncoupled problem.

**Solution XXVIII**

It is easily checked that

$$u(x,t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)},$$

and

$$v(x,t) = B \sqrt{m(1-m)} \frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_2 z + \delta_2)},$$

is an exact solution to the coupled field Eqs. (27) provided

\begin{align*}
a &= f > 0, \quad b = e < 0, \quad aA^2 = -bB^2, \quad amA^2 = 6(1-m)\beta^2, \quad (135) \\
0 &< m < 1, \quad \omega_1 = -(5 - 4m)\beta^2, \quad \omega_2 = -(5 - m)\beta^2. \quad (136)
\end{align*}

Thus this solution can only be valid in the MZS case but not in the Manakov case.

**Solution XXIX**

It is easily checked that

$$u(x,t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)},$$

and

$$v(x,t) = Bm \frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_2 z + \delta_2)},$$

is an exact solution to the coupled field Eqs. (27) provided

\begin{align*}
a &= f < 0, \quad b = e < 0, \quad aA^2 = bB^2, \quad aA^2 = -6\beta^2, \quad (139) \\
0 &< m < 1, \quad \omega_1 = (4m + 1)\beta^2, \quad \omega_2 = (4 + m)\beta^2. \quad (140)
\end{align*}
Thus this solution can only be valid in the Manakov case but not in the MZS case.

**Solution XXX**

It is easily checked that

\[ u(x, t) = A \sqrt{m(1 - m)} \frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \tag{141} \]

and

\[ v(x, t) = B m \frac{\text{cn}(\beta x, m) \text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_2 z + \delta_2)}, \tag{142} \]

is an exact solution to the coupled field Eqs. (27) provided

\[ a = f < 0, \quad b = e < 0, \quad aA^2 = bB^2, \quad aA^2 = -6(1 - m)\beta^2, \tag{143} \]

\[ 0 < m < 1, \quad \omega_1 = -(5m - 1)\beta^2, \quad \omega_2 = -(5m - 4)\beta^2. \tag{144} \]

Thus this solution can only be valid in the Manakov case but not in the MZS case.

**Solution XXXI**

It is easily checked that

\[ u(x, t) = A \sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \tag{145} \]

and

\[ v(x, t) = B m \frac{\text{cn}(\beta x, m) \text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_2 z + \delta_2)}, \tag{146} \]

is an exact solution to the coupled field Eqs. (27) provided

\[ a = f, \quad b = e, \quad aA^2 = bB^2, \quad aA^2[1 + 2(1 - m)y] = 6(1 - m)\beta^2, \tag{147} \]

\[ 0 < m < 1, \quad \omega_1 = -(5m - 4)\beta^2 - (1 - m)aA^2 y^2, \quad \omega_2 = -2[2(2 - m) + 3(1 - m)x]\beta^2 - (1 - m)aA^2 y^2, \tag{148} \]

where

\[ y = \frac{D}{B} = -\frac{(2 - m)}{3(1 - m)} \pm \frac{\sqrt{1 - m + m^2}}{3(1 - m)}. \tag{149} \]

Thus this solution can only be valid in the Manakov case but not in the MZS case.

**Solution XXXII**

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It is easily checked that
\[ u(x, t) = A \sqrt{m(1 - m)} \frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \] (150)

and
\[ v(x, t) = \left[ \frac{B}{\text{dn}^2(\beta x, m)} + D \right] e^{-i(\omega_2 z + \delta_2)}, \] (151)

is an exact solution to the coupled field Eqs. (27) provided Eq. (147) is satisfied with \( y \) given by Eq. (149) and further
\[ 0 < m < 1, \quad \omega_1 = -(5 - m)\beta^2 - (1 - m)aA^2y^2, \quad \omega_2 = -2[2(2 - m) + 3(1 - m)y]\beta^2 - (1 - m)aA^2y^2. \] (152)

Thus this solution can only be valid in the Manakov case but not in the MZS case.

**Solution XXXIII**

It is easily checked that
\[ u(x, t) = Am \frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} e^{-i(\omega_1 z + \delta_1)}, \] (153)

and
\[ v(x, t) = \left[ \frac{B}{\text{dn}^2(\beta x, m)} + D \right] e^{-i(\omega_2 z + \delta_2)}, \] (154)

is an exact solution to the coupled field Eqs. (27) provided
\[ a = f, \quad b = e, \quad aA^2 = -bB^2, \quad [2 - m + 2(1 - m)y]aA^2 = -6(1 - m)\beta^2, \] (155)

\[ 0 < m < 1, \quad \omega_1 = -(2 - m)\beta^2 - [1 - (1 - m)y^2]aA^2, \quad \omega_2 = -2[2(2 - m) + 3(1 - m)y]\beta^2 - [1 - (1 - m)y^2]aA^2, \] (156)

with \( y \) given by Eq. (149).

Thus this solution can only be valid in the MZS case but not in the Manakov case.

**Solution XXXIV**

Finally, it is easily checked that
\[ u(x, t) = \left[ \frac{A}{\text{dn}^2(\beta x, m)} + E \right] e^{-i(\omega_1 z + \delta_1)}, \] (157)
and
\[ v(x,t) = \left[ \frac{B}{\text{dn}^2(\beta x, m)} + D \right] e^{-i(\omega_2 z + \delta_2)}, \] (158)
is an exact solution to the coupled field Eqs. (27) provided
\[ a = f, \quad b = e, \quad aA^2 = -bB^2, \quad y \neq z, \quad (y_\pm - z_\pm) aA^2 = 3 \beta^2, \] (159)
with \( y \neq z \) and both given by Eq. (149).

Thus this solution can only be valid in the MZS case but not in the Manakov case.

Summarizing, there are 7 periodic solutions in terms of shifted Lamé polynomials of order two in case \( 0 < m < 1 \) out of which 4 solutions can only exist in the Manakov case (but not in the MZS case) while 3 solutions can only exist in the MZS case (but not in the Manakov case).

4 Nonlocal, Discrete, Saturable NLSE

Motivated by the nonlocal NLSE discussed in I and II, we now introduce a nonlocal saturable DNLSE given by
\[
 i \frac{du_n}{dz}(z) + [u_{n+1}(z) + u_{n-1}(z) - 2u_n(z)] + \frac{gu_n(z)u_{n}^*(z)}{1 + u_n(z)u_{n}^*(z)} u_n(z) = 0, \] (161)
which describes a non-Hermitian but PT-invariant system. In this case the power as given by
\[
P = \sum_{i=-\infty}^{\infty} |u_i|^2, \] (162)
is not conserved while the quasi-power
\[
Q = \sum_{i=-\infty}^{\infty} u_n(z)u_{n}^*(z), \] (163)
is conserved. The nonlocal, discrete, saturable NLSE given by Eq. (161) is a non-Hermitian but PT-invariant nonlinear system in the sense that \( V_n(z) = V_n^*(z) \) where \( V_n(z) = u_n(z)u_n^*(z) \).

Before we obtain the periodic solutions to the discrete Eq. (161), we remark that Eq. (161) has a novel solution
\[ u_n(z) = A e^{kn-i\omega(z+\delta)} \] (164)
provided

\[ A^2 = \frac{4 \sin^2\left(\frac{k}{2}\right) - \omega}{g + \omega - 4 \sin^2\left(\frac{k}{2}\right)} . \]  

(165)

Let us now obtain discrete periodic soliton solutions to this equation. In order to obtain results in this and the next section, we have used a number of (not so well known) local identities for Jacobi elliptic functions [14]

Solution I

It is easily checked that one of the exact periodic solution to this equation is

\[ u_n = A \text{dn}(\beta_n, m) e^{-i(\omega z + \delta)} , \]  

(166)

provided

\[ A^2 \cos^2(\beta, m) = 1 , \quad \omega = (2 - \omega) = 2 \left[ 1 - \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right] . \]  

(167)

Solution II

Another exact periodic solution to the discrete, nonlocal, saturable NLS Eq. (161) is

\[ u_n = A \sqrt{m} \text{cn}(\beta_n, m) e^{-i(\omega z + \delta)} , \]  

(168)

provided

\[ A^2 \sin^2(\beta, m) = 1 , \quad \omega = (2 - \omega) = 2 \left[ 1 - \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right] . \]  

(169)

Solution III

Remarkably, even a linear superposition of the two, i.e.

\[ u_n = \left[ \frac{A}{2} \text{dn}(\beta_n, m) + \frac{B}{2} \sqrt{m} \text{cn}(\beta_n, m) \right] e^{-i(\omega z + \delta)} , \]  

(170)

is also an exact periodic solution to this equation provided

\[ B = \pm A , \quad A^2 \left[ \cos(\beta, m) + \text{ds}(\beta, m) \right]^2 = 4 , \quad \omega = (2 - \omega) = 2 \left[ 1 - \frac{2}{\text{cn}(\beta, m) + \text{dn}(\beta, m)} \right] . \]  

(171)

Here

\[ \cos(\beta, m) = \frac{\text{cn}(\beta, m)}{\text{sn}(\beta, m)} , \quad \text{ds}(\beta, m) = \frac{\text{dn}(\beta, m)}{\text{sn}(\beta, m)} , \quad \text{ns}(\beta, m) = \frac{1}{\text{sn}(\beta, m)} . \]  

(172)

Solution IV
In the limit $m = 1$, all three periodic solutions, i.e. $\text{cn}, \text{dn}$ as well as the superposed solution with $B = +A$ go over to the same hyperbolic bright soliton solution

$$u_n = A \text{sech}(\beta n) e^{-i(\omega z + \delta)},$$  \hspace{1cm} (173)

provided

$$A^2 = \text{sech}^2(\beta, m), \quad \omega = 2 - g = -2[\cosh(\beta, m) - 1],$$  \hspace{1cm} (174)

while the solution with $B = -A$ goes over to the vacuum solution.

**Solution V**

Remarkably, unlike the local saturable discrete NLSE [11], the nonlocal, saturable, discrete NLSE (161) (which admits both $\text{dn}$ and $\text{cn}$ solutions) also admits the periodic $\text{sn}$ solution

$$u_n = A \sqrt{m} \text{sn}(\beta n, m) e^{-i(\omega z + \delta)},$$  \hspace{1cm} (175)

provided

$$A^2 \text{sn}^2(\beta, m) = 1, \quad \omega = (2 - g) = 2[1 - \text{cn}(\beta, m)\text{dn}(\beta, m)].$$  \hspace{1cm} (176)

**Solution VI**

In the limit $m = 1$, this periodic solution goes over to the hyperbolic dark soliton solution

$$u_n = A \tanh(\beta n) e^{-i(\omega z + \delta)},$$  \hspace{1cm} (177)

provided

$$A^2 = \tanh^2(\beta), \quad \omega = (2 - g) = 2 \tanh^2(\beta).$$  \hspace{1cm} (178)

4.1 **Shifted Periodic Solutions**

We now show that this model also admits shifted periodic solutions which we now discuss one by one.

**Solution VII**

It is easily checked that

$$u_n = \frac{A}{\text{dn}(n\beta, m)} e^{-i(\omega z + \delta)},$$  \hspace{1cm} (179)
is an exact solution to Eq. (161) provided
\[ A^2 = \frac{[\text{cs}(\beta, m)\text{cn}^2(\beta, m) - 2\text{cs}(2\beta, m)\text{dn}(\beta, m)]}{\text{cn}^2(\beta, m)}, \quad \omega = 2 - g = -\frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}. \tag{180} \]

**Solution VIII**

It is easily checked that
\[ u_n = A\sqrt{m} \frac{\text{cn}(n\beta, m)}{\text{dn}(n\beta, m)} e^{-i(\omega z + \delta)}, \tag{181} \]

is an exact solution to Eq. (161) provided
\[ g = (\omega - 2)\text{cs}^2(\beta, m) = -2(1 + A^2) \text{ns}(\beta, m) [\text{cs}(2\beta, m) + \text{ds}(2\beta, m)], \]
\[ (\omega - 2)\text{cs}(\beta, m)\text{cs}(2\beta, m)\text{ds}(\beta, m) = -\text{cs}^2(\beta, m) [\text{ds}(2\beta, m) + \text{cs}(2\beta, m)] \]
\[ -A^2 \text{ds}^2(\beta, m) [\text{cs}(2\beta, m) + \text{ds}(2\beta, m)]. \tag{182} \]

On solving these two equations, one can obtain \( \omega \) and \( gA^2 \).

**Solution IX**

It is easily checked that
\[ u_n = A\sqrt{m} \frac{\text{sn}(n\beta, m)}{\text{dn}(n\beta, m)} e^{-i(\omega z + \delta)}, \tag{183} \]

is an exact solution to Eq. (161) provided
\[ g = (\omega - 2)\text{cs}^2(\beta, m) = -2(1 + A^2) \text{ds}(\beta, m) [\text{cs}(2\beta, m) + \text{ns}(2\beta, m)], \]
\[ (\omega - 2)\text{cs}(\beta, m)\text{cs}(2\beta, m)\text{ns}(\beta, m) = -\text{cs}^2(\beta, m) [\text{ns}(2\beta, m) + \text{cs}(2\beta, m)] \]
\[ -A^2 \text{ns}^2(\beta, m) [\text{cs}(2\beta, m) + \text{ds}(2\beta, m)]. \tag{184} \]

On solving these two equations, one can obtain \( \omega \) and \( gA^2 \).

### 5 Coupled Nonlocal Saturable DNLSE

We now consider a coupled nonlocal saturable DNLSE which is similar but more general than the one considered by us previously \[15\]
\[ i\frac{d u_n}{dz}(z) + [u_{n+1}(z) + u_{n-1}(z) - 2u_n(z)] + g_1 \frac{au_n(z)u_n^*(z) + bv_n(z)v_n^*(z)}{1 + au_n(z)u_n^*(z) + bv_n(z)v_n^*(z)} u_n(z) = 0, \tag{185} \]
\[ \frac{dv_n}{dz}(z) + [v_{n+1}(z) + v_{n-1}(z) - 2v_n(z)] + g_2 \frac{dv_n(z)u^*_{n}(z) + ev_n(z)v^*_{n}(z)}{1 + du_n(z)u^*_{n}(z) + ev_n(z)v^*_{n}(z)}v_n(z) = 0. \] (186)

It is easy to check that these coupled nonlocal equations will have several solutions. As an illustration, we now present a few such solutions.

**Solution I**

It is easy to check that

\[ u_n = Adn(n\beta, m)e^{-i(\omega_1 z + \delta)}, \] (187)

\[ v_n = B\sqrt{msn(n\beta, m)}e^{-i(\omega_2 z + \delta)}, \] (188)

is an exact solution to the coupled Eqs. (185) and (186) provided

\[ aA^2cs^2(\beta, m) + bB^2ns^2(\beta, m) = dA^2cs^2(\beta, m) + eB^2ns^2(\beta, m) = 1, \] (189)

\[ g_1 = 2 - \omega_1 = 2(1 - bB^2)\frac{dn(\beta, m)}{cn^2(\beta, m)}, \quad g_2 = 2 - \omega_2 = 2(1 - dA^2)cn(\beta, m)dn(\beta, m). \] (190)

On solving relations (189) we find that so long as \( ae \neq bd \)

\[ A^2cs^2(\beta, m) = \frac{e - b}{ae - bd}, \quad B^2ns^2(\beta) = \frac{a - d}{ae - bd}. \] (191)

However, in case \( ae = bd \), this implies that \( a = d, b = e \) and instead of relations (191), we only have the constraint \( aA^2cs^2(\beta, m) + bB^2ns^2(\beta, m) = 1. \)

**Solution II**

It is easy to check that

\[ u_n = A\sqrt{m}cn(n\beta, m)e^{-i(\omega_1 z + \delta)}, \] (192)

\[ v_n = B\sqrt{msn(n\beta, m)}e^{-i(\omega_2 z + \delta)}, \] (193)

is an exact solution to the coupled Eqs. (185) and (186) provided

\[ aA^2ds^2(\beta, m) + bB^2ns^2(\beta, m) = dA^2ds^2(\beta, m) + eB^2ns^2(\beta, m) = 1, \] (194)

\[ g_1 = 2 - \omega_1 = 2(1 - bB^2m)\frac{cn(\beta, m)}{dn^2(\beta, m)}, \quad g_2 = 2 - \omega_2 = 2(1 + dA^2m)cn(\beta, m)dn(\beta, m). \] (195)

On solving relations (194) we find that so long as \( ae \neq bd \)

\[ A^2ds^2(\beta, m) = \frac{e - b}{ae - bd}, \quad B^2ns^2(\beta) = \frac{a - d}{ae - bd}. \] (196)
However, in case $ae = bd$, this implies that $a = d, b = e$ and instead of relations (196), we only have the constraint $aA^2cs^2(\beta, m) + bB^2ns^2(\beta, m) = 1$.

**Solution III**

It is easy to check that

\[
\begin{align*}
u_n &= A\text{dn}(n\beta, m) e^{-i(\omega_1 z + \delta)}, \\
v_n &= B\text{dn}(n\beta, m) e^{-i(\omega_2 z + \delta)},
\end{align*}
\]

(197) \hspace{1cm} (198)

is an exact solution to the coupled Eqs. (185) and (186) provided

\[
(aA^2 + bB^2)cs^2(\beta, m) = (dA^2 + eB^2)cs^2(\beta, m) = 1,
\]

(199)

\[
\omega_1 = \omega_2 = 2 \left[ 1 - \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)} \right], \quad g_1 = g_2 = 2 - \omega_1.
\]

(200)

On solving relations (199) we find that so long as $ae \neq bd$

\[
A^2cs^2(\beta, m) = \frac{e - b}{ae - bd}, \quad B^2cs^2(\beta) = \frac{a - d}{ae - bd}.
\]

(201)

However, in case $ae = bd$, this implies that $a = d, b = e$ and instead of relations (201), we only have the constraint $(aA^2 + bB^2)cs^2(\beta, m) = 1$.

**Solution IV**

It is easy to check that

\[
\begin{align*}
u_n &= A\sqrt{m}\text{cn}(n\beta, m) e^{-i(\omega_1 z + \delta)}, \\
v_n &= B\sqrt{m}\text{cn}(n\beta, m) e^{-i(\omega_2 z + \delta)},
\end{align*}
\]

(202) \hspace{1cm} (203)

is an exact solution to the coupled Eqs. (185) and (186) provided

\[
(aA^2 + bB^2)ds^2(\beta, m) = (dA^2 + eB^2)ds^2(\beta, m) = 1,
\]

(204)

\[
\omega_1 = \omega_2 = 2 \left[ 1 - \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)} \right], \quad g_1 = g_2 = 2 - \omega_1.
\]

(205)

On solving relations (204) we find that so long as $ae \neq bd$

\[
A^2ds^2(\beta, m) = \frac{e - b}{ae - bd}, \quad B^2ds^2(\beta) = \frac{a - d}{ae - bd}.
\]

(206)
However, in case $ae = bd$, this implies that $a = d, b = e$ and instead of relations (206), we only have the constraint $(aA^2 + bB^2)ds^2(\beta, m) = 1$.

**Solution V**

It is easy to check that

$$u_n = A\text{dn}(n\beta, m) e^{-i(\omega_1 z + \delta)},$$

$$v_n = B\sqrt{mcn}(n\beta, m) e^{-i(\omega_2 z + \delta)},$$

is an exact solution to the coupled Eqs. (185) and (186) provided

$$aA^2cs^2(\beta, m) + bB^2ds^2(\beta, m) = dA^2cs^2(\beta, m) + eB^2ds^2(\beta, m) = 1,$$  \hspace{1cm} (209)

$$g_1 = 2 - \omega_1 = 2[1 - (1 - m)bB^2] \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}, \quad g_2 = 2 - \omega_2 = 2[1 + (1 - m)dA^2] \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}.$$  \hspace{1cm} (210)

On solving relations (209) we find that so long as $ae \neq bd$

$$A^2ds^2(\beta, m) = \frac{a - d}{ae - bd}, \quad B^2ds^2(\beta, m) = \frac{a - d}{ae - bd}.$$  \hspace{1cm} (211)

However, in case $ae = bd$, this implies that $a = d, b = e$ and instead of relations (211), we only have the constraint $(aA^2 + bB^2)ds^2(\beta, m) = 1$.

**Solution VI**

It is easy to check that

$$u_n = A\sqrt{m}\text{sn}(n\beta, m) e^{-i(\omega_1 z + \delta)},$$

$$v_n = B\sqrt{m}\text{sn}(n\beta, m) e^{-i(\omega_2 z + \delta)},$$

is an exact solution to the coupled Eqs. (185) and (186) provided

$$(aA^2 + bB^2)ns^2(\beta, m) = (dA^2 + eB^2)ns^2(\beta, m) = 1,$$  \hspace{1cm} (214)

$$\omega_1 = \omega_2 = 2[1 - \text{cn}(\beta, m)\text{dn}(\beta, m)], \quad g_1 = g_2 = 2 - \omega_1.$$  \hspace{1cm} (215)

On solving relations (214) we find that so long as $ae \neq bd$

$$A^2ns^2(\beta, m) = \frac{e - b}{ae - bd}, \quad B^2ns^2(\beta, m) = \frac{a - d}{ae - bd}.$$  \hspace{1cm} (216)
However, in case $ae = bd$, this implies that $a = d, b = e$ and instead of relations (216), we only have the constraint $(aA^2 + bB^2)ns^2(\beta, m) = 1$.

In the limit $m = 1$ these coupled solutions go over to the corresponding hyperbolic pulse or kink solutions which can be easily worked out from here.

6 Nonlocal Ablowitz-Ladik Equation

Yet another nonlocal nonlinear discrete equation that we now consider is the nonlocal AL equation

$$i \frac{du_n(z)}{dz} + u_{n+1}(z) + u_{n-1}(z) + gu_n(z)u^*_n(z)[u_{n+1}(z) + u_{n-1}(z)] = 0.$$  \hspace{1cm} (217)

Note that this is also a non-Hermitian, but PT-invariant system. In this case, the power as given by

$$P = \sum_{i=-\infty}^{\infty} \ln[1 + gu_n(z)u^*_n(z)]$$  \hspace{1cm} (218)

is not conserved while the quasi-power as given by

$$Q = \sum_{i=-\infty}^{\infty} \ln[1 + gu_n(z)u^*_n(z)]$$  \hspace{1cm} (219)

is conserved. Note that the corresponding local AL equation is a well known discrete integrable equation [12].

Before obtaining periodic solutions, we note that there is a novel exact solution to the nonlocal AL Eq. (217) given by

$$u_n(z) = Ae^{kn-i(\omega z+\delta)}$$  \hspace{1cm} (220)

provided

$$gA^2 = -\frac{\omega + 2 \cos(k)}{2 \cos(k)}.$$  \hspace{1cm} (221)

Let us now obtain discrete periodic soliton solutions to this equation.

Solution I

It is easily checked that one of the exact periodic solution to this equation is

$$u_n = Adn(\beta n, m) e^{-i(\omega z+\delta)}.$$  \hspace{1cm} (222)
provided

\[ gA^2 \text{cs}^2(\beta, m) = 1, \quad \omega = -2 \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}. \] (223)

**Solution II**

Another exact periodic solution to the nonlocal AL Eq. (217) is

\[ u_n = A \sqrt{m} \text{cn}(\beta n, m) e^{-i(\omega z + \delta)} \] (224)

provided

\[ gA^2 \text{ds}^2(\beta, m) = 1, \quad \omega = -2 \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}. \] (225)

**Solution III**

Remarkably, even a linear superposition of the two, i.e.

\[ u_n = \left[ A \text{dn}(\beta n, m) + B \sqrt{m} \text{cn}(\beta n, m) \right] e^{-i(\omega z + \delta)}, \] (226)

is also an exact periodic solution to this equation provided

\[ B = \pm A, \quad gA^2 [\text{cs}(\beta, m) + \text{ds}(\beta, m)]^2 = 4, \quad \omega = -\frac{4}{\text{cn}(\beta, m) + \text{dn}(\beta, m)}. \] (227)

**Solution IV**

In the limit \( m = 1 \), all three periodic solutions, i.e. \( \text{cn}, \text{dn} \) as well as the superposed solution with \( B = +A \) go over to the same hyperbolic pulse solution

\[ u_n = A \text{sech}(\beta n) e^{-i(\omega z + \delta)}, \] (228)

provided

\[ gA^2 = \text{sech}^2(\beta, m), \quad \omega = -2 \cosh(\beta, m); \] (229)

while the solution with \( B = -A \) goes over to the vacuum solution.

**Solution V**

Remarkably, unlike the usual focusing AL case [13], the same nonlocal, AL Eq. (217) (which admits \( \text{cn} \) and \( \text{dn} \) solutions), also admits the periodic \( \text{sn} \) solution

\[ u_n = A \sqrt{m} \text{sn}(\beta n, m) e^{-i(\omega z + \delta)}, \] (230)
provided
\[ gA^2 \text{ns}^2(\beta, m) = 1, \quad \omega = -2\text{cn}(\beta, m)\text{dn}(\beta, m). \tag{231} \]

**Solution VI**

In the limit \( m = 1 \), this periodic solution goes over to the hyperbolic kink solution
\[ u_n = A \tanh(\beta n) e^{-i(\omega z + \delta)}, \tag{232} \]
provided
\[ gA^2 = \tanh^2(\beta), \quad \omega = -2\text{sech}^2(\beta). \tag{233} \]

### 6.1 Shifted Periodic Solutions

We now show that this model also admits shifted periodic solutions which we now discuss one by one.

**Solution VII**

It is easily checked that
\[ u_n = \frac{A}{\text{dn}(n\beta, m)} e^{-i(\omega z + \delta)}, \tag{234} \]
is an exact solution to Eq. (217) provided
\[ gA^2 = \frac{[\text{cs}(\beta, m)\text{cn}^2(\beta, m) - 2\text{cs}(2\beta, m)\text{dn}(\beta, m)]}{\text{cn}^2(\beta, m)}, \quad \omega = -\frac{2\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}. \tag{235} \]

**Solution VIII**

It is easily checked that
\[ u_n = A\sqrt{m} \text{cn}(n\beta, m) \text{dn}(n\beta, m) e^{-i(\omega z + \delta)}, \tag{236} \]
is an exact solution to Eq. (217) provided
\[ \omega \text{cs}^2(\beta, m) = -2(1 + gA^2) \text{ns}(\beta, m)[\text{cs}(2\beta, m) + \text{ds}(2\beta, m)], \]
\[ \omega \text{cs}(\beta, m)\text{cs}(2\beta, m)\text{ds}(\beta, m) = -\text{cs}^2(\beta, m)[\text{ds}(2\beta, m) + \text{cs}(2\beta, m)] \]
\[ -gA^2 \text{ds}^2(\beta, m)[\text{cs}(2\beta, m) + \text{ds}(2\beta, m)]. \tag{237} \]

On solving these two equations, one can obtain \( \omega \) and \( gA^2 \).

**Solution IX**
It is easily checked that
\[
u_n = A\sqrt{m} \frac{sn(n\beta, m)}{dn(n\beta, m)} e^{-i(\omega z + \delta)}, \tag{238}
\]
is an exact solution to Eq. \([161]\) provided
\[
\omega \text{cs}^2(\beta, m) = -2(1 + gA^2) \text{ds}(\beta, m)[\text{cs}(2\beta, m) + \text{ns}(2\beta, m)],
\]
\[
\omega \text{cs}(\beta, m)\text{cs}(2\beta, m)\text{ns}(\beta, m) = -\text{cs}^2(\beta, m)[\text{cs}(2\beta, m) + \text{ns}(2\beta, m)]
\]
\[
- gA^2 \text{ns}^2(\beta, m)[\text{cs}(2\beta, m) + \text{ds}(2\beta, m)]. \tag{239}
\]

On solving these two equations, one can obtain \(\omega\) and \(gA^2\).

7 Coupled Nonlocal Ablowitz Ladik Model

We now consider a coupled nonlocal AL model which is similar but more general than the one considered by us elsewhere \([15]\)
\[
i\frac{du_n}{dz}(z) + [u_{n+1}(z) + u_{n-1}(z)] + [au_n(z)u^*_{-n}(z) + bv_n(z)v^*_{-n}(z)][u_{n+1}(z) + u_{n-1}(z)] = 0, \tag{240}
\]
\[
i\frac{dv_n}{dz}(z) + [v_{n+1}(z) + v_{n-1}(z) - 2v_n(z)] + [du_n(z)u^*_{-n}(z) + ev_n(z)v^*_{-n}(z)][v_{n+1}(z) + v_{n-1}(z)] = 0. \tag{241}
\]
It is easy to check that these coupled nonlocal equations will have several solutions. As an illustration, we now present few such solutions.

Solution I

It is easy to check that
\[
u_n = \text{Adn}(n\beta, m) e^{-i(\omega_1 z + \delta)}, \tag{242}
\]
\[
v_n = B\sqrt{m}\text{sn}(n\beta, m) e^{-i(\omega_2 z + \delta)}, \tag{243}
\]
is an exact solution to the coupled Eqs. \([240]\) and \([241]\) provided Eqs. \([189]\) are satisfied and further
\[
\omega_1 = -2(1 - bB^2)\frac{dn(\beta, m)}{cn^2(\beta, m)}, \quad \omega_2 = -2(1 + dA^2)\text{cn}(\beta, m)\text{dn}(\beta, m). \tag{244}
\]

Solution II
It is easy to check that

$$u_n = A \sqrt{m} \text{cn}(n\beta, m) e^{-i(\omega_1 z + \delta)} ,$$  \hspace{1cm} (245)  

$$v_n = B \sqrt{m} \text{sn}(n\beta, m) e^{-i(\omega_2 z + \delta)} ,$$  \hspace{1cm} (246)  

is an exact solution to the coupled Eqs. (240) and (241) provided Eqs. (194) are satisfied and further

$$\omega_1 = -2(1 - bB^2) \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}, \quad \omega_2 = -2(1 + dA^2 m) \text{cn}(\beta, m) \text{dn}(\beta, m).$$  \hspace{1cm} (247)

**Solution III**

It is easy to check that

$$u_n = A \text{dn}(n\beta, m) e^{-i(\omega_1 z + \delta)} ,$$  \hspace{1cm} (248)  

$$v_n = B \sqrt{m} \text{cn}(n\beta, m) e^{-i(\omega_2 z + \delta)} ,$$  \hspace{1cm} (249)  

is an exact solution to the coupled Eqs. (240) and (241) provided Eqs. (199) are satisfied and further

$$\omega_1 = \omega_2 = -2 \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}. \hspace{1cm} (250)$$

**Solution IV**

It is easy to check that

$$u_n = A \text{dn}(n\beta, m) e^{-i(\omega_1 z + \delta)} ,$$  \hspace{1cm} (251)  

$$v_n = B \sqrt{m} \text{cn}(n\beta, m) e^{-i(\omega_2 z + \delta)} ,$$  \hspace{1cm} (252)  

is an exact solution to the coupled Eqs. (240) and (241) provided Eqs. (204) are satisfied and further

$$\omega_1 = \omega_2 = -2 \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}. \hspace{1cm} (253)$$

**Solution V**

It is easy to check that

$$u_n = A \text{dn}(n\beta, m) e^{-i(\omega_1 z + \delta)} ,$$  \hspace{1cm} (254)  

$$v_n = B \sqrt{m} \text{cn}(n\beta, m) e^{-i(\omega_2 z + \delta)} ,$$  \hspace{1cm} (255)  

is an exact solution to the coupled Eqs. (240) and (241) provided Eqs. (209) are satisfied and further

$$\omega_1 = -2[1 - (1 - m)bB^2] \frac{\text{dn}(\beta, m)}{\text{cn}^2(\beta, m)}, \quad \omega_2 = -2[1 + (1 - m)dA^2] \frac{\text{cn}(\beta, m)}{\text{dn}^2(\beta, m)}. \hspace{1cm} (256)$$
Solution VI

It is easy to check that

\[ u_n = A \sqrt{m} \text{sn}(n\beta, m) e^{-i(\omega_1 z + \delta)}, \]

\[ v_n = B \sqrt{m} \text{sn}(n\beta, m) e^{-i(\omega_2 z + \delta)}, \]

is an exact solution to the coupled Eqs. (240) and (241) provided Eqs. (214) are satisfied and further

\[ \omega_1 = \omega_2 = -2 cn(\beta, m) dn(\beta, m). \]

In the limit \( m = 1 \) these coupled solutions go over to the corresponding hyperbolic pulse or kink solutions which can be easily worked out from here.

8 Conclusion and Open Problems

In this paper, we have considered a number of discrete, continuum and coupled nonlocal nonlinear equations which are non-Hermitian but invariant under PT symmetry. We have shown that like the local nonlinear equations (which are Hermitian), these nonlocal nonlinear equations also admit periodic solutions in terms of \( \text{cn}(x, m) \) and \( \text{dn}(x, m) \) as well as in terms of their linear superposition (and hence the corresponding hyperbolic bright soliton solution). However, what is remarkable is that unlike the usual local models, all the six nonlocal models, which admit \( \text{dn}(x, m) \) and \( \text{cn}(x, m) \) solutions, at the same time also admit periodic \( \text{sn}(x, m) \) and hence hyperbolic dark soliton solution. This is probably for the first time that the same nonlinear model simultaneously admits both the dark and the bright soliton solutions. Further, though naively it would seem that no solutions with a shift in the transverse coordinate \( x \) are allowed in such nonlinear models, but in this paper we have shown that Jacobi elliptic function solutions with definite shifts are allowed in all the six cases. Moreover, in the coupled nonlocal NLSE case, we have shown that as in the local case, it admits solutions in terms of Lamé polynomials of order 1 and 2. What is interesting to note is that while in the local case, for both the Manakov and the MZS case, there are solutions in terms of Lamé polynomials of order 2, in the nonlocal case that we have considered, no solutions in terms of Lamé polynomials of order 2 are admitted in the Manakov case while a large number of such solutions
are allowed in the MZS case. However, periodic solutions in terms of shifted Lamé polynomials of order 2 do exist in the Manakov case.

It is worth pointing out here that in this paper we have restricted ourselves to only the nonsingular solutions. In fact several singular solutions are admitted as well in all the six cases [like \(\text{cn}(x,m)/\text{sn}(x,m)\)] but we do not consider them here.

Finally, it is worth enquiring whether, like the nonlocal NLSE, the coupled nonlocal NLSE of Manakov type or MZS type are also integrable systems? Looking at the structures of the nonlocal NLSE and the coupled nonlocal NLSE, we conjecture that indeed both nonlocal Manakov and nonlocal MZS are also likely to be integrable systems. One reason why we think so is because the conserved quantities in both the uncoupled and the coupled nonlocal NLSE models are simply obtained from the corresponding local case by replacing \(u^*(x,z)\) by \(u^*(-x,z)\). Remarkably, a similar situation also occurs for the nonlocal Ablowitz-Ladik case and hence we conjecture that the nonlocal Ablowitz-Ladik system is also likely to be integrable. It will be interesting to prove or disprove our conjectures. Further, it would be worthwhile to study the stability of the various soliton solutions of the system.

9  Acknowledgments

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