AN EXPERIMENTAL INVESTIGATION OF NEUMANN’S CONJECTURE

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Abstract. We use a large census of hyperbolic 3-manifolds to experimentally investigate a conjecture of Neumann regarding the Bloch Group. We present an augmented census including, for feasible invariant trace fields, explicit manifolds (associated to that field) that appear to generate the Bloch group of that field. We also make use of Ptolemy coordinates to compute “exotic volumes” of representations, and attempt to realize these volumes as linear combinations of generator volumes. We thus present a large body of empirical support for Neumann’s conjecture.

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1. Introduction

Our primary result, described in section 3, is a large census of hyperbolic 3-manifolds, organized by invariant trace field, together with tables expressing volumes as linear combinations of other manifolds’ volumes. This census is available at [http://www.curve.unhyperbolic.org/linComb]. Examining this census has allowed us, in many cases, to explicitly list manifolds which give rise to elements in the Bloch groups of certain invariant trace fields that span all the elements of that Bloch group we observed, supporting a conjecture of Neumann.

1.1. Neumann’s conjecture. For a hyperbolic 3-manifold $M$, we denote the element it induces in the Bloch Group (see section 2.3) as $[M]$. The conjecture we primarily studied was given in [Neu11], and stated in this form by [GTZ15]:

**Conjecture 1.1 (Neumann).** Let $F \subset \mathbb{C}$ be a number field not contained in $\mathbb{R}$, and let $\mathcal{M}$ be the set of manifolds with invariant trace fields contained in $F$,

$$\mathcal{M} := \{ M \cong \mathbb{H}^3/\Gamma \mid \mathbb{Q}(\text{tr } \Gamma(2)) \subseteq F \}.$$ 

Let $\mathcal{N} = \{ [M] \mid M \in \mathcal{M} \}$ be elements of the Bloch group determined by manifolds in $\mathcal{M}$ (see section 2.3). Then integral combinations of elements of $\mathcal{N}$ generate $B(F)$.

1.2. Neumann’s weaker conjecture. Neumann has also proposed a weaker conjecture which our results also support.

**Conjecture 1.2 ([Neu11] Conjecture 1).** Every non-real concrete number field $k$ arises as the invariant trace field of some hyperbolic manifold.

In full form, this conjecture also addresses quaternion algebras, but we only examine invariant trace fields. Relevant results are discussed in section 5.

1.3. A stronger, false conjecture. We also considered whether conjecture 1.1 could be strengthened somewhat to the following form, which directly regards volumes of manifolds rather than Bloch group elements.

**Conjecture 1.3.** Let $F \subset \mathbb{C}$ be a number field not contained in $\mathbb{R}$.

Let $S = \{ \text{vol}(M) \mid M \text{ a hyperbolic 3-manifold with invariant trace field } F \}$. The lattice generated by $S$ is linearly spanned by some $\{ \text{vol}(N_1), \ldots, \text{vol}(N_{r_2}) \} \subset S$, where $r_2$ is the number of complex places of $F$ (and each $N_i$ is a hyperbolic 3-manifold).

In section 4 we provide a counterexample to conjecture 1.3.

1.4. Exotic volumes. Recall that for a closed hyperbolic 3-manifold and a representation $\rho : \pi_1(M) \to \text{SL}(n, \mathbb{C})$, Cheeger-Chern-Simons invariant $\hat{\text{c}}(\rho)$ is given by

$$\hat{\text{c}}(\rho) = \frac{1}{2} \int_M s^* \left( \text{tr} \left( A \wedge \text{d}A + \frac{2}{3} A \wedge A \wedge A \right) \right) \in \mathbb{C}/4\pi^2\mathbb{Z},$$

with $E_\rho$ the flat $\text{SL}(n, \mathbb{C})$-bundle with holonomy $\rho$, with $A$ the flat connection in $E_\rho$, and with $s$ a section of $E_\rho$.

**Definition 1.4.** For a representation $\rho : \pi_1(M) \to \text{SL}(n, \mathbb{C})$, the **complex volume** $\text{Vol}_C(\rho)$ of $\rho$ is

$$\text{Vol}_C(\rho) = i \hat{\text{c}}(\rho).$$
If $\rho$ is $\rho_{\text{geo}}$, the geometric representation of a hyperbolic 3-manifold $M$, then

$$\text{Vol}_C(\rho_{\text{geo}}) = \text{vol}(M) + i\text{CS}(M),$$

for $\text{CS}(M)$ the Chern-Simons invariant of $M$.

When $\rho$ is not $\rho_{\text{geo}}$, we call the real part of $\text{Vol}_C(\rho)$ an exotic volume of $M$, to distinguish it from the geometric volume.

By means of the Ptolemy coordinates of [GTZ15], we were able to compute certain exotic volumes of manifolds to at least 50 decimal places of precision, and to express them as linear combinations of geometric volumes of manifolds associated to the same invariant trace field.

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2. Background

2.1. Cross-ratios. It is a feature of hyperbolic geometry that the volume of a complete 3-manifold (if finite) is actually a topological invariant (see e.g. [Ben92, p. 83] for a proof). For an ideal hyperbolic 3-simplex $\Delta = (a,b,c,d)$ defined by four points in $\partial \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$, $\text{vol}(\Delta)$ can be determined knowing just the cross-ratio $[a:b:c:d] = \frac{(c-a)(d-b)}{(d-a)(c-b)}$. Since $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2,\mathbb{C})$ is 3-transitive, for the purposes of volume calculation we may assume $\Delta$ to be of the form $(\infty, 0, 1, z)$, which conveniently has cross-ratio $z$. The volume of $\Delta$ is given by

\[(2.0.1) \quad D(\Delta) = \text{Im} \left( \int_0^1 \frac{\log(1-tz)}{t} \, dt \right) + \arg(1-z) \log |z|.\]

Given a triangulation of a hyperbolic 3-manifold, one may compute cross-ratios using gluing equations 2.6.1 and 2.7.1. Using cross-ratios, one may compute volumes for each simplex in the triangulation; we refer to section 2.4 for more details. Summing those volumes gives the volume of the manifold: for $\{\Delta_i\}$ the simplices in a triangulation of $M$,

\[(2.0.2) \quad \text{vol}(M) = \sum_i D(\Delta_i).\]

In our work, we used experimental data to examine relations between the volumes of hyperbolic 3-manifolds which share the same invariant trace field. We begin by briefly reviewing the basics of the algebraic invariants of hyperbolic 3-manifolds relevant to our study.

2.2. The invariant trace field.

Definition 2.1. For a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ defined by a discrete subgroup $\Gamma$ of $\text{PSL}(2,\mathbb{C})$, the trace field of $\Gamma$, denoted $\mathbb{Q}(\text{tr}\Gamma)$, is

$$\mathbb{Q}(\{\text{tr}\gamma : \pi(\gamma) \in \Gamma\})$$

where $\pi : \text{SL}(2,\mathbb{C}) \to \text{PSL}(2,\mathbb{C})$ is the standard projection map.

Since for any matrices $a,b \in \text{SL}(2,\mathbb{C})$ we have $\text{tr}(aba^{-1}) = \text{tr}(b)$, the trace field of $\Gamma$ is a conjugation invariant of $\Gamma$. Unfortunately, it is not a commensurability invariant, since it is possible to create finite degree extensions of $\Gamma$ which extend the
trace field: we refer to [MR03, p. 116] for an explicit example. A slight modification smooths over this difficulty.

**Definition 2.2.** For a non-elementary subgroup \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \), the invariant trace field of \( \Gamma \) is the trace field \( \mathbb{Q}(\text{tr} \Gamma^{(2)}) \), where 

\[
\Gamma^{(2)} := \langle \gamma^2 \mid \gamma \in \Gamma \rangle.
\]

2.3. The Bloch group.

**Definition 2.3.** For any field \( F \), the Pre-Bloch group on \( F \), denoted \( \mathcal{P}(F) \), is defined as

\[
\mathcal{P}(F) = \langle \{ z \in F \mid z \neq 0_F, 1_F \} \rangle \cdot \frac{[z_1] - [z_2] + \frac{z_2}{z_1} - \frac{1-z_2}{1-z_1} + \frac{1-z_2^{-1}}{1-z_1^{-1}}}{[z_1] - [z_2] + \frac{z_2}{z_1} - \frac{1-z_2}{1-z_1} + \frac{1-z_2^{-1}}{1-z_1^{-1}}}.
\]

The relation in the denominator is known as the **Five Term Relation**. The variables \( z_1 \) and \( z_2 \) are to be understood as cross-ratios of hyperbolic simplices. The relation encodes that the union of two simplices with volumes \([z_1] \) and \([z_2] \) is also the union of three simplices with volumes \([\frac{z_2}{z_1}] \), \([\frac{1-z_2}{1-z_1}] \), and \([\frac{1-z_2^{-1}}{1-z_1^{-1}}] \). This equivalence is called the Pachner 2-3 move (see figure 1).

![Figure 1. The Pachner 2-3 move.](image)

**Definition 2.4.** For any field \( F \), the **Bloch group** on \( F \), denoted \( \mathcal{B}(F) \), is the kernel of the map \( d : \mathcal{P}(F) \to F^* \wedge \mathbb{Z} \wedge F^* \) defined by

\[
d : [z] \mapsto z \wedge (1 - z).
\]

The map \( d \) is an analogue of the Dehn invariant map [Neu98], which (for \( F = \mathbb{C} \)) fits into an exact sequence of scissors congruence described in e.g. [Dup01]. Thus, understanding \( \mathcal{B}(F) = \ker d \) provides insight into scissors congruence groups and generalizations of Hilbert’s third problem.

2.4. Gluing Equations. Given a hyperbolic 3-manifold \( M \) with cusps, the standard procedure for determining \( \text{vol}(M) \) is to first triangulate \( M \) into a collection of simplices \( \{\Delta_i\} \) (together with face pairings), next to compute the cross ratio \( z_i \) of each \( \Delta_i \), and finally to compute \( \text{vol}(M) \) by equation 2.0.2. The first step is well-understood: the manifolds included in the censuses of SnapPy include triangulations. The second step is a matter of numerical approximation, but the second step deserves more explanation. We present an overview of the **gluing equations** here, and refer to [NZ85] for a more detailed description, including a treatment of gluing equations for closed manifolds obtained by Dehn surgery.
Given a triangulation \( \{ \Delta_i \} \) of \( M \), the standard method of finding \( \{ z_i \} \) is to exploit the torus boundary components and local Euclidean nature of the manifold to produce and solve a system of equations in \( \{ z_i \} \). Equations 2.6.1, which arise from torus boundary components, are the **cusp equations**, and equations 2.7.1, which arise from local Euclideanness, are the **edge equations**. Together, they are the **gluing equations**.

Since, in the triangulation of \( M \), a link component appears at the vertices of each simplex, a torus can be determined by truncating the simplices, then making edge identifications to match face identifications, as in figure 2.

**Figure 2.** Piecing together a torus from a triangulation.

Since the meridian and the longitude of a torus are homotopically circles, it should be possible to require that any path following a meridian or longitude winds exactly once. In other words, it should be possible to construct a system of equations, with variables representing various angles of each \( \Delta_i \), such that certain sums are \( 2\pi \).

By considering each \( \Delta_i \) as ideal, each segment of such a path can be viewed as complex multiplication (recall that all cross-ratios are in \( \mathbb{C} \cup \{ \infty \} \)). For example, in figure 3, the rotation that takes a point from start to end of a bold arrow is the same rotation that takes 1 to \( z \): multiplication by \( z \). By re-ordering the vertices of \( \Delta_i \), other paths may be expressed as re-ordered cross-ratios.

**Figure 3.** Identifying torus traversal with multiplication in \( \mathbb{C} \).

**Definition 2.5.** Let a triangulation be fixed. To each simplex associate a variable \( z_i \). For each edge \( e_{ij} \) in the simplex, transforming any point by rotating around
$e_{ij}$ is equivalent to multiplication of that point by either $z_i$, $\frac{z_{i-1}}{z_i}$, or $\frac{1}{1-z_i}$ (fix one angle as $z_i$, determine the rest by cross-ratio rearrangement).

Let $p$ be a closed path in a triangulated surface, transverse to all edges. For every consecutive pair of edges that $p$ crosses, those edges must meet at an angle, which is associated to some $z_i$, $\frac{z_{i-1}}{z_i}$, or $\frac{1}{1-z_i}$. The path monodromy of $p$, denoted $\mathcal{M}(p)$, is the product of all these variables.

**Definition 2.6.** Let a triangulation of a manifold $M$ be fixed. For each torus boundary component $T_i$ of $M$, choose meridian and longitude paths in the torus (transverse to all edges) $m_i$ and $\ell_i$ respectively. The cusp equations are

(2.6.1) $\mathcal{M}(m_i) = 1$ and $\mathcal{M}(\ell_i) = 1$

There are two such equations for each torus boundary component.

One more piece of geometric information will be useful. For each edge in the triangulation, consider all face identifications that contain that edge as an axis. Such identifications identify the edges of neighboring faces, and these faces join to form two polygons, one around each vertex of the edge. For example, see figure 4, in which the shaded region is a polygon for the edge shown in bold. We arbitrarily consider the polygon to be the one about the terminus point of the edge.

![Figure 4. Piecing together a polygon from a triangulation.](image)

**Definition 2.7.** Let a triangulation of a manifold $M$ be fixed. For each distinct edge $e_i$ in the triangulation, let $P_i$ be the polygon described above associated to that edge. Let $p_i$ be a path in $P_i$ circling the central point. The edge equations are

(2.7.1) $\mathcal{M}(p_i) = 1$

There are as many edge equations as there are distinct edges in the chosen triangulation.

If two triangulations of $M$ are both composed of non-degenerate, geometrically viable simplices, by Mostow rigidity they must yield the same volume. Therefore, given a triangulation $\{\Delta_i\}$ of $M$, if the cross-ratios $z_i$ all have positive imaginary part, then computing $\sum_i D(\Delta_i)$ will yield $\text{vol}(M)$. It is possible, however, for a triangulation not to yield solutions that give a geometric volume. In software implementations, this is usually dealt with by re-triangulating using different parameters.

As presented, the system of cusp and edge equations can be hard to solve by hand, because their degree is unbounded. For example, the complement of the knot $6_1$ (see figure 5), known as $m032$, has the triangulation given in figure 6 and admits the boundary torus given in figure 7.
A path through the torus from left to right (for example, from the single-arrow edge of $2_2$ to the corresponding edge of $1_0$) would have to pass through at least eleven faces, inducing a cusp equation with eleven terms. Computer programs such as SnapPy [CDW] can solve these equations (numerically) quite efficiently.

2.5. Ptolemy coordinates. The Ptolemy coordinates of Garoufalidis-Thurston-Zickert offer an alternative representation of the information encoded in the gluing and cusp equations. For our purposes, their principal use is encoding information about arbitrary representations $\rho : \pi_1(M) \to \text{PSL}(n, \mathbb{C})$. We refer the reader to [GTZ15], [GGZ15], and present a brief overview here, specialized for the particular case of hyperbolic 3-manifolds.

**Definition 2.8.** For a fixed triangulation of a manifold $M$ (with vertices labeled $0, 1, 2, 3$), a **Ptolemy assignment** is an association of variables in $\mathbb{C}^\times$ to each edge of each simplex satisfying equations 2.8.1 below. We denote the variable of the $i$th simplex on the edge between vertices $j$ and $k$ as $c_{jk}^i$.

If two edges are identified, their associated variables are the same. This immediately implies, for example, that $c_{jk}^i = -c_{kj}^i$. 
The Ptolemy relations are

\[(2.8.1)\quad c_{03}c_{12}^i + c_{01}c_{23}^i = c_{02}c_{13}^i\]

**Proposition 2.9.** For a fixed triangulation of a manifold \(M\), and for any Ptolemy assignment of this triangulation,

\[(2.9.1)\quad z_i = \pm \frac{c_{03}c_{12}^i}{c_{02}c_{13}^i}\]

where the sign is determined by an obstruction class, as described in [GGZ15] and [GTZ15].

Since in our work we are interested only in \(z_i\), we may often assume without loss of generality that certain \(c_{jk}\) = 1, where the number of such variables depends on the number of cusps of the manifold.

In the case of \(m032\) (the manifold presented above), there are only three distinct edges, therefore three independent \(c_{jk}\) variables. The Ptolemy relations are homogeneous degree 2 equations in two variables, and taking one of the variables to be 1 makes this a readily solvable system.

2.6. **Lattice matching.** Given a large collection of (numerically approximated) hyperbolic 3-manifold volumes (with the same invariant trace field), we desired to construct the coarsest lattice containing the point given by each volume. We desired to reconstruct the linear combination whose sum was the volume of the manifold - we refer to the volume as the projection of the lattice point to \(\mathbb{R}\). Searching for representative manifolds of the generating basis of this lattice would then provide insight into the the relation between volumes and the Bloch group. The question we sought to answer therefore was “Given the dimension \(d\) of a lattice, as well as a set \(S\) of projections of points in that lattice to \(\mathbb{R}\), what is a \(d\)-subset \(B\) of \(S\) which, up to a minimal scaling factor, contains all other lattices generated by \(d\)-subsets of \(S\)?”

2.6.1. **An assumption.** In answering this problem, we had no exact norms: instead of \(S\), we had numerical approximations of elements of \(S\). Using only numerical approximations, we cannot prove that any two elements of \(S\) are linearly independent. However, we can be reasonably confident that two numerical representations represent linearly independent elements. In our work, we used upwards of 50 places of precision and used the LLL algorithm implemented in PARI [GRO14]. If the algorithm indicated that coefficients with magnitude greater than \(2^{12}\) was required to obtain a linear dependence, we considered the two elements linearly independent. As we very rarely observed coefficients with magnitudes greater than 10 and never observed enough linearly independent elements to contradict Borel’s result below, we consider this reasonable.

2.6.2. **The lattice dimension.** By a result of Borel [Neu98, p. 397], for an invariant trace field \(\mathbb{Q}[x]/(p)\), where \(p\) has \(r_2\) complex places, \(\mathcal{B}(F)/(\text{torsion})\) is isomorphically a lattice in \(\mathbb{R}^{r_2}\). We therefore worked under the assumption that \(d = r_2\). In some case, we observed strictly less than \(r_2\) linearly independent volumes; we believe this is due to insufficient data, and that given unlimited resources we would eventually discover another, linearly independent volume.
2.6.3. An example. Suppose \( d = 2 \), and we wish to obtain a \( B \) for the following (numerical approximations of) \( S \):

\[
\begin{align*}
V &= 2.7182818284590 \quad (\approx e) \\
W &= 5.4365636569180 \quad (\approx 2e) \\
X &= 6.3496623769612 \quad (\approx 15\pi - 15e) \\
Y &= 11.7197489640976 \quad (\approx 2\pi + 2e) \\
Z &= 21.1445269248670 \quad (\approx 5\pi + 2e).
\end{align*}
\]

The coarsest lattice generated by \( V, W, X, Y, Z \) may be generated by \( \pi \) and \( e \). We cannot obtain a 2 element subset which generates this lattice, but we would like to recover something close, ideally containing \( V \). We would also like to obtain a measure of how much this 2 element subset \( B \) fails to completely generate \( S \).

**Definition 2.10.** For a \( d \)-subset \( B \) of \( S \), as described above, the **fit ratio** of \( B \) is the smallest positive integer \( f \) such that the lattice with generators \( B \) contains \( fS \).

If no such integer exists, the fit ratio of \( B \) is \( \infty \).

We certainly do not wish to select the two smallest elements of \( S \): in the example \( \{V, W\} \) has fit ratio \( \infty \). Nor is it sufficient to simply select the two least linearly independent elements: \( \{V, X\} \) has fit ratio 15 by considering \( Y \).

A good choice would be \( B = \{V, Y\} \), with fit ratio 2. Our algorithm for detecting such \( B \) is formalized in algorithm 1, which is implemented in our experimentation software.

2.6.4. An algorithm. To find a good choice of \( B \), we first find some linearly independent basis of \( S \) (by brute force, if necessary), then express each element of \( S \) in terms of that basis, yielding vectors in \( \mathbb{Q}^d \) (if this could not be achieved, it would be a contradiction of Borel’s result). We then test all \( d \)-sized sets of vectors, and select the one with the least determinant. The ratio of this determinant to the fractional GCD of all such determinants yields the fit ratio.

In our implementation, we use PARI for the `lindep` algorithm, and as noted in 2.6.1, we use a combination of high precision and detection of large coefficients to determine linear dependence.

Applied to the example, `rational_vecs` is calculated as

\[
\left\{ (V, (1, 0)), (X, (0, 1)), (Y, (\frac{1}{4}, -\frac{2}{15})), (Z, (-7, -\frac{1}{3})) \right\},
\]

discarding \( W \) since it is a linear multiple of \( V \). Calculating determinants of potential bases gives:

\[
\begin{align*}
\{V, X\} &\to \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \\
\{V, Y\} &\to \det \begin{bmatrix} 1 & 4 \\ 0 & -\frac{2}{15} \end{bmatrix} = -\frac{2}{15} \\
\{V, Z\} &\to \det \begin{bmatrix} 1 & -\frac{7}{1} \\ 0 & -\frac{1}{3} \end{bmatrix} = -\frac{1}{3} \\
\{X, Y\} &\to \det \begin{bmatrix} 0 & 4 \\ 1 & -\frac{2}{15} \end{bmatrix} = 4 \\
\{X, Z\} &\to \det \begin{bmatrix} 0 & -\frac{7}{1} \\ 1 & -\frac{1}{3} \end{bmatrix} = 7 \\
\{Y, Z\} &\to \det \begin{bmatrix} 4 & -\frac{7}{1} \\ -\frac{2}{15} & -\frac{1}{3} \end{bmatrix} = -\frac{34}{15}.
\end{align*}
\]

Since \( -\frac{2}{15} \) is the smallest absolute value of all possible determinants, \( B = \{V, Y\} \) is returned by the algorithm. Since the fractional GCD of all possible determinants is \( \frac{1}{15} \), the fit ratio of \( B \) is 2.
Algorithm 1  Best-fit lattice matching algorithm.

\[
\text{manifolds} \leftarrow \text{SORT\_BY\_INCREASING\_VOLUME}(\text{manifolds})
\]

\[
\text{next}_i \leftarrow 2 \quad \triangleright \text{Which } e_i \text{ we seek a manifold for}
\]

\[
\text{rational\_vecs} \leftarrow \{(\text{manifolds}_1, e_1)\} \quad \triangleright \text{Tuples } (M, v), \text{ with } v \in \mathbb{Q}^n
\]

\[
\text{rational\_basis} \leftarrow \{\text{manifolds}_1\} \quad \triangleright \text{All } M \text{ associated to some } e_i
\]

\begin{algorithmic}
\ForAll {M \in \text{manifolds except } \text{manifolds}_1}
\State \text{dep} = \text{LINDEP}(\text{vol}(M), \text{rational\_basis}) \quad \triangleright \text{Via PARI; } (\text{dep}, \text{volumes}) = 0
\If {\text{dep} shows } \text{vol}(M) \text{ is not a linear combination of others}
\If {\text{dep} shows } \text{vol}(M) \text{ is a rational combination of others}
\State \text{rational\_vecs} \leftarrow \text{rational\_vecs} \cup \{(M), \text{dep}^{-1}\text{dep}_2, \ldots, n+1\}
\Else
\If {\text{next}_i > n}
\State \text{return} \text{Failure} \quad \triangleright \text{There are more than } n \text{ l.i. manifolds}
\EndIf
\State \text{rational\_basis} \leftarrow \text{rational\_basis} \cup \{M\}
\State \text{rational\_vecs} \leftarrow \text{rational\_vecs} \cup \{(M), e_{\text{next}_i}\}
\State \text{next}_i \leftarrow \text{next}_i + 1
\EndIf
\EndIf
\EndFor
\State \text{gcd} \leftarrow \infty
\State \text{best\_det} \leftarrow \infty
\State \text{best\_basis} \leftarrow \varnothing
\ForAll {B \subseteq \text{rational\_vecs} \text{ with } |B| = n}
\State \text{det} \leftarrow \text{det}\{(v \mid (M, v) \in B)\}
\If {\text{det} \neq 0}
\State \text{gcd} \leftarrow \text{fractional\_gcd}(\text{gcd, det})
\If {\text{det} < \text{best\_det}}
\State \text{best\_det} \leftarrow \text{det}
\State \text{best\_basis} \leftarrow B
\EndIf
\EndIf
\EndFor
\State \text{return} (\text{basis, fit ratio}) := (\text{best\_basis}, \frac{\text{best\_det}}{\text{gcd}})
\end{algorithmic}

3. Experimentation

Our main results are the following data.

For our work, we used the volumes for a large number of hyperbolic 3-manifolds, which we obtained by performing Dehn surgery on manifolds from a base census. We used \texttt{SnapPy} \cite{CDW}, although for invariant trace field calculations we used the older \texttt{Snap} \cite{Goo} interface.

3.1. Manifold generation. The code we used to collect manifold information can be found at \url{https://github.com/s-gilles/maps-reu-code}. We considered all orientable, cusped, hyperbolic manifolds which can be triangulated using 9 tetrahedra or less (the \texttt{OrientableCuspedCensus} within \texttt{SnapPy} at the time of work), as well as all link complements using 3 crossings or more (a subset of
We also considered additional manifolds in the LinkExteriors and HTLinkExteriors collections. Of these manifolds, we performed Dehn surgery of type \((p, q)\) over each of the \(n\) cusps, with \(p \in [0, L(n)]\) and \(q \in [-L(n), L(n)]\), (and \((p, q) = 1\)) where \(L(n)\) is given by figure 8.

| \(L(n)\) | 16 | 12 | 8 | 6 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|---|
| \(n\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | \(\geq 8\) |

Figure 8. \(L(n)\)

Of the resulting manifolds, we discarded those whose invariant trace field resulted in a polynomial with degree 9 or more (or if SnapPy or Snap were unable to triangulate the result). For each remaining manifold, we stored intermediate information about the manifold. The resulting file, available at http://www.curve.unhyperbolic.org/linComb/finalized_data/volumes/all_volumes.csv, contains over 790,000 manifolds, representing over 6,300 distinct invariant trace fields, with volumes accurate to at least 50 decimal places. The work was performed on the University of Maryland’s computation cluster.

3.2. Lattice matching. With the data stored in all_volumes.csv, we have enough data to conjecture partial lattice generators for 5,900 invariant trace fields. Of those, for 312 invariant trace fields we have enough data to conjecture not only generating volumes for the full lattice, but also representative manifolds associated to the invariant trace field which exhibit those volumes. This data can be found at http://www.curve.unhyperbolic.org/linComb/finalized_data/volume_spans.csv, and the code we used for conjecturing lattice generators is available at https://github.com/s-gilles/maps-reu-code, an implementation of algorithm 1.

Sample data can be found in section 6.

3.3. Linear combinations for exotic volumes. While we used the Ptolemy coordinates for computing volumes of hyperbolic 3-manifolds, the computations may be generalized to compute the volume of any representation \(\rho : \Gamma \to \text{PSL}(n, \mathbb{C})\).

The extended Bloch group is exactly the Bloch group up to torsion, so another method for experimentally testing conjecture 1.1 is to produce linear combinations of hyperbolic volumes matching exotic volumes, where these exotic volumes arise from generalized representations into \(\text{PSL}(n, \mathbb{C})\) where \(n\) is not necessarily 2.

Using the data of all_volumes.csv, we have been able to find a great many such combinations. This data is available at http://www.curve.unhyperbolic.org/linComb/finalized_data/linear_combinations/.

Sample data can be found in section 6.

4. A counter-example to conjecture 1.3

A secondary result of our work is a counter-example to conjecture 1.3. Assuming the conjecture holds, two of the manifold volumes catalogued would imply the existence of a manifold with an impossibly small volume.

**Theorem 4.1** ([GMM09, Corollary 1.3]). The Weeks manifold is the unique closed orientable hyperbolic 3-manifold of smallest volume: 0.9427 . . .
Counter-example 4.2. Conjecture 1.3 is false.

Proof. The polynomial $p(x) = x^3 - x^2 + 1$ has exactly one complex place. The manifold $m003(2, 1)$ has hyperbolic volume $0.9427 \ldots$, and the manifold $10^2_3(2, 3)(5, 11)$ has hyperbolic volume $1.4140 \ldots$

If conjecture 1.3 were true, the volumes of these manifolds (which have invariant trace field $\mathbb{Q}[x]/(p)$) would be expressable as a linear combination of one generator: $\text{vol}(M)$. Then

$$\text{vol}(M) \mid \frac{0.9427 \ldots}{2} = (\text{vol}(10^2_3(2, 3)(5, 11)) - \text{vol}(m003(2, 1))).$$

This would contradict theorem 4.1. □

In our entire census, we did not observe any counterexamples where the obstruction factor was less than $\frac{1}{2}$. We suspect this is related to the following theorem of Neumann.

**Theorem 4.3** ([Neu11, Theorem 2.7]). If $M$ has cusps then $|M|$ is defined in $\mathcal{B}(F)$, for $F$ the invariant trace field of $M$, while if $M$ is closed $2|M|$ is defined in $\mathcal{B}(F)$.

5. Evidence for Conjecture 1.2

Our data provides an opportunity for examining the strength of conjecture 1.2. It is known that every field with one complex place arises as the invariant trace field of a hyperbolic manifold [MR03], but an open question in general [Neu11].

5.1. Examples of missing fields. Our data is necessarily incomplete. Some interesting absences:

1. Some fields associated to “simple” polynomials such as $x^4 - 2x^2 + 4$, $x^4 + 9$, and $x^7 - 3$ do not arise. This also includes some degree 2 polynomials such as $x^2 + 13$, $x^2 + 22$, and $x^2 - x + 17$, even though it is known that all fields with one complex place must be exhibited by some manifold.

2. Some number fields, such as the two concrete number fields arising from $\mathbb{Q}[x]/(x^4 - x + 2)$, produced surprisingly little data.

For the concrete number field associated to root $-0.850 \ldots \pm 1.01 \ldots i$, we observed only two volumes: $21.531 \ldots$ and $16.383 \ldots$. These two volumes are linearly independent, but we know very little about the proposed lattice.

For the concrete number field associated to root $0.850 \ldots \pm 0.654 \ldots i$, we only observed one volume: $9.054 \ldots$, though the lattice should be of dimension 2. This is (from some perspective) the simplest example from our data which does not support conjecture 1.1.

5.2. Data analysis. By comparing the fields we observed (under some reasonable restrictions) to a proven-complete census under the same restrictions, we obtain an estimate of the probability that an arbitrary field (within these restrictions) appears in our census. If conjecture 1.2 holds, then our census samples from all possible fields. If our sample was uniformly random, we would then expect that our census contains the same percentage of fields with $r_2 = 1$ as for $r_2 \neq 1$.

For our calculations, we used an online census [JR14] of abstract number fields expressed as polynomials. For the field restrictions we used, this census has been proven complete. In figures 9 and 10, $n$ refers to the degree of the polynomial $p$ to which the field is associated, and $D$ is the discriminant of the polynomial. We
only consider restrictions for which multiple values of \( r_2 \) are possible, and we ignore \( r_2 = 4 \) due to relative scarcity of data (the total fields for \( r_2 = 4 \) are so numerous that our results are scarce enough to be statistical noise).

| Restrictions | \( |D|^{1/n} \) | \( r_2 = 1 \) | \( r_2 = 2 \) | \( r_2 = 3 \) |
|--------------|----------------|----------------|----------------|----------------|
| \( n \)      | Found Total % | Found Total % | Found Total % | Found Total % |
| 4 \( \leq 8 \) | 56 137 40.8% | 76 408 18.6% |                   |                   |
| 4 \( \leq 10 \) | 65 444 14.6% | 82 1100 7.4% |                   |                   |
| 4 \( \leq 12 \) | 66 1056 6.2% | 84 2550 3.2% |                   |                   |
| 4 \( \leq 15 \) | 66 3069 2.1% | 84 6728 1.2% |                   |                   |
| 5 \( \leq 8 \) | 47 77 61.0% | 226 736 30.7% |                   |                   |
| 5 \( \leq 10 \) | 65 472 13.7% | 326 3470 9.3% |                   |                   |
| 5 \( \leq 12 \) | 73 1670 4.3% | 356 10992 3.2% |                   |                   |
| 5 \( \leq 15 \) | 76 7556 1.0% | 362 41776 .8% |                   |                   |
| 6 \( \leq 8 \) | 11 40 27.5% | 180 1222 14.7% | 166 1851 8.9% |                   |
| 6 \( \leq 10 \) | 28 405 6.9% | 374 8434 4.4% | 291 10887 2.6% |                   |
| 6 \( \leq 12 \) | 37 2335 1.5% | 514 38722 1.3% | 352 42123 .8% |                   |
| 6 \( \leq 15 \) | 48 15556 .3% | 578 204108 .2% | 380 190095 .1% |                   |
| 7 \( \leq 10 \) | 10 137 7.2% | 276 8070 3.4% | 391 38103 1.0% |                   |
| 7 \( \leq 12 \) | 14 1473 .9% | 455 57292 .7% | 637 219879 .2% |                   |
| 7 \( \leq 15 \) | 30 16759 .1% | 599 506188 .1% | 890 1612152 .0% |                   |
| 8 \( \leq 10 \) | 1 22 4.5% | 75 752 9.9% | 288 3141 9.1% |                   |
| 8 \( \leq 12 \) | 7 246 2.8% | 199 5808 3.4% | 584 16764 3.4% |                   |
| 8 \( \leq 15 \) | 11 2560 .4% | 383 50268 .7% | 976 120111 .8% |                   |

**Figure 9.** Observed concrete field percentages by restriction

| Restrictions | \( |D|^{1/n} \) | \( r_2 = 1 \) | \( r_2 = 2 \) | \( r_2 = 3 \) |
|--------------|----------------|----------------|----------------|----------------|
| \( n \)      | Found Total % | Found Total % | Found Total % | Found Total % |
| 4 \( \leq 8 \) | 56 137 40.8% | 60 204 29.4% |                   |                   |
| 4 \( \leq 10 \) | 65 444 14.6% | 66 550 12.0% |                   |                   |
| 4 \( \leq 12 \) | 66 1056 6.2% | 68 1275 5.3% |                   |                   |
| 4 \( \leq 15 \) | 66 3069 2.1% | 68 3364 2.0% |                   |                   |
| 5 \( \leq 8 \) | 47 77 61.0% | 168 368 45.6% |                   |                   |
| 5 \( \leq 10 \) | 65 472 13.7% | 264 1735 15.2% |                   |                   |
| 5 \( \leq 12 \) | 73 1670 4.3% | 294 5496 5.3% |                   |                   |
| 5 \( \leq 15 \) | 76 7556 1.0% | 300 20888 1.4% |                   |                   |
| 6 \( \leq 8 \) | 11 40 27.5% | 162 611 26.5% | 143 617 23.1% |                   |
| 6 \( \leq 10 \) | 28 405 6.9% | 350 4217 8.2% | 263 3629 7.2% |                   |
| 6 \( \leq 12 \) | 37 2335 1.5% | 489 19361 2.5% | 324 14041 2.3% |                   |
| 6 \( \leq 15 \) | 48 15556 .3% | 553 102054 .5% | 352 63665 .5% |                   |
| 7 \( \leq 10 \) | 10 137 7.2% | 271 4035 6.7% | 383 12701 3.0% |                   |
| 7 \( \leq 12 \) | 14 1473 .9% | 450 28646 1.5% | 629 73293 .8% |                   |
| 7 \( \leq 15 \) | 30 16759 .1% | 594 253094 .2% | 882 537384 .1% |                   |
| 8 \( \leq 10 \) | 1 22 4.5% | 75 376 19.9% | 286 1047 27.3% |                   |
| 8 \( \leq 12 \) | 7 246 2.8% | 199 2904 6.8% | 582 5588 10.4% |                   |
| 8 \( \leq 15 \) | 11 2560 .4% | 383 25134 1.5% | 974 40037 2.4% |                   |

**Figure 10.** Observed abstract field percentages by restriction
Figure [9] implies, in general, a negative correlation between the number of complex places of the field and the probability that it is exhibited by some manifold. Figure [10] offers hope for an explanation.

While figure [9] lists the observed percentage of concrete number fields, figure [10] lists the observed percentage of fields considered as $\mathbb{Q}[x]/(p)$: in particular, we treat any two concrete fields arising from different roots of the same polynomial as identical. The percentages are relatively even across rows in this case, which supports that our census was selecting from all possible abstract number fields. This partially supports conjecture [1.2] and more solidly supports the following weaker version.

**Conjecture 5.1.** Every non-real abstract number field $\mathbb{Q}[x]/(p)$ arises as abstract number field associated to the invariant trace field of some hyperbolic manifold.

### 6. Selected examples

We present a selection of lattices, together with the manifolds which may be generators. These are constructed from the volume_spans.csv and linear_combinations files of our results.

Each of the following lattices are two-dimensional: each is associated to a polynomial with two complex places. Our results contain data for three and even four complex places, but these are harder to represent visually. In the following graphs, one potential generator is arbitrarily assigned to the $x$-axis, and the other to the $y$-axis. A point at position $(x, y)$ represents a volume $v = a \text{vol}(g_1) + b \text{vol}(g_2)$, where $x = a \text{vol}(g_1)$ and $y = b \text{vol}(g_2)$.

The gray dots indicate our predicted lattice (based on the volumes of all manifolds associated to that invariant trace field), while the crosses indicate manifolds we have observed: blue + markers indicate geometric volumes, while red $\times$ markers indicate exotic volumes. The relevant volume (whether geometric or exotic) of the manifold is the projection to $\mathbb{R}$ of the point in the lattice.

Figure [11] represents a typical “good” lattice from our data. We obtained a substantial number of distinct volumes (each volume is recorded many times in our census), and we have a number of pairs of volumes which differ by exactly one of our conjectured basis elements.

Figure [12] represents a typical “almost good” lattice from our data. Our fit ratio is 2, which points out that we are unable to fit the exotic volumes of $v2489$ and $t09825$ (either of them) completely into our lattice. Nonetheless, this lattice allows us to conjecture the existence of smaller manifolds which would “fix” this lattice. For example, we might suppose that the nearby exotic volumes of $v2489$ and $t09825$ differ by a generator (or, perhaps, a multiple of a generator): this would imply the existence of a manifold with volume $\frac{\text{vol}(v3318)}{2} \approx 3.2253$ (this is by no means the only volume which would “fix” the lattice).
Figure 11. $p(x) = x^4 - 3x^2 + 4$, root $-1.322 \ldots + 0.5i$, prospective basis: $\text{vol}(v_{1859}(-1,3)), \text{vol}(t_{07828})$ with fit ratio 1.
| Manifold      | Volume   | Type      | As linear combination                                                                 |
|--------------|----------|-----------|---------------------------------------------------------------------------------------|
| $v_{3318}$   | 6.4506... | geometric | $1 \cdot \text{vol}(v_{3318}) + 0 \cdot \text{vol}(L_{14n54610})$                   |
| $L_{14n534610}$ | 15.6881... | geometric | $0 \cdot \text{vol}(v_{3318}) + 1 \cdot \text{vol}(L_{14n54610})$                  |
| $t_{09825}$  | 2.2704... | exotic    | $4 \cdot \text{vol}(v_{3318}) - 1.5 \cdot \text{vol}(L_{14n54610})$               |
| $v_{3318}$   | 0.8770... | exotic    | $5 \cdot \text{vol}(v_{3318}) - 2 \cdot \text{vol}(L_{14n54610})$                  |
| $10^3$       | 5.5736... | exotic    | $-4 \cdot \text{vol}(v_{3318}) + 2 \cdot \text{vol}(L_{14n54610})$                 |
| $L_{13n5993}$ | 1.9097... | exotic    | $-7 \cdot \text{vol}(v_{3318}) + 3 \cdot \text{vol}(L_{14n54610})$                 |
| $K_{12n809}$ | 3.6638... | exotic    | $3 \cdot \text{vol}(v_{3318}) - 1 \cdot \text{vol}(L_{14n54610})$                  |
| $L_{13n5993}$ | 9.2374... | exotic    | $-1 \cdot \text{vol}(v_{3318}) + 1 \cdot \text{vol}(L_{14n54610})$                 |
| $v_{2489}$   | 1.8319... | exotic    | $1.5 \cdot \text{vol}(v_{3318}) - 0.5 \cdot \text{vol}(L_{14n54610})$              |
| $t_{09825}$  | 5.0572... | exotic    | $2 \cdot \text{vol}(v_{3318}) - 0.5 \cdot \text{vol}(L_{14n54610})$               |
| $v_{3548}$   | 2.7868... | exotic    | $-2 \cdot \text{vol}(v_{3318}) + 1 \cdot \text{vol}(L_{14n54610})$                 |

Figure 12. $p(x) = x^4 + x^2 - 2x + 1$, root $-0.624... + 1.300...i$, prospective basis: $\text{vol}(v_{3318}), \text{vol}(L_{14n54610})$ with fit ratio 2.
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