Interpretation of Weil’s Theorem

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Abstract. This is an introductory paper based on general knowledge such as simplicial complex, nerves and their homotopy equivalents in the topology field. It focuses on the theoretical part of one of the four statements on nerve theorem of paper variations on the nerve theorem. One of many ideologies throughout this theorem is to conduct an approximating process on mathematical spaces with restrictions, which is the bridge from a theory to an application of Weil’s theorem and many others. Besides, work done by Kathryn Heal in her paper variations on the nerve theorem has proved the correctness and feasibility about Weil’s theorem. This paper aims to clarify her proof of Weil’s theorem and make it more easier for people to understand.

1. Introduction
Facing with finitely many points sampled from the space of a suitably nice manifold, people may have a lot of ideas about approximating this unknown surface $E$ [7]. Nonetheless, the complexity of the properties of the surface that we need to study usually makes it hard for us to obtain the exactly algebraic descriptions of one particular smooth manifold. To fix this problem, it requires generally to identify the topological type of $E$. A currently studied method provides us with a brand new perspective. The main idea is to replace $E$ with a covering of convex, closed sets. By studying the relationship of the homotopy types of the space $E$, a given open cover $U$, and the nerve $N(U)$ of that cover, we can approximate the surface more accurately. Mathematicians have already done their researches on different types of spaces. In this case, this paper aims to represent and re-illustrate the proof of Weil’s theorem brought up in 1952. By supplementing and clarifying the proof of Weil’s theorem, later readers are enabled to step into this topology field far more easier.

2. Analysis

2.1. Preliminaries
Definition 1. Given any open cover $\{U_i\}_{i \in I}$ of a space, if there is a partition $\{\varphi_i\}_{i \in I}$ indexed over the same index set $I = \{1, 2, 3, ..., m\}$ such that supp($\varphi_i$) $\subseteq U_i$, such a partition is said to be subordinated to the open cover $\{U_i\}_{i \in I}$.

Definition 2. A topological space $X$ is said to be normal if every two disjoint closed sets of $X$ can be contained in their own disjoint open neighborhoods.

Definition 3. A cover $X = \{X_i\}$ of $E$ is said to be locally finite if for every point of $E$, there is a neighborhood which intersects only finitely many $X_i$. If $E$ is locally compact, every compact subset of $X$ intersects only finitely many $X_i$. 
Definition 4. Set \( N \), which consists of the nonempty subsets \( J \subseteq I \) (index set) such that \( X_J \) is not empty, is called the nerve of the family \( X \).

Definition 5. If for every closed subset \( A \) of a normal space \( X \), every continuous mapping from \( A \) to \( B \) can be extended to a continuous mapping \( X \rightarrow B \), then the space \( B \) is said to have the extension property.

Definition 6. Let \( N(U) \) be the nerve of \( U \), which is a locally finite covering of a space \( E \) by open subsets \( (U_i) \). \( U \) is called topologically simple, if \( \forall \ I \in N(U) \), the set \( U_i \) has the extension property.

Definition 7. A space \( B \) has the homotopy extension property if, for every closed subset \( A \) of a normal space \( X \), given any homotopy \( G : I \times A \rightarrow B \), and given any \( f_0 : X \rightarrow B \) with \( f_0|_A = G|_{I \times A} \), there exists an extension \( F : I \times X \rightarrow B \) such that \( F|_{I \times A} = G \) and \( F|_{I \times X} = f_0 \). \[2\]

![Figure 1. The homotopy extension property](image)

Definition 8. Let \( S \) be a set. A partition of \( S \) is a collection \( S_1, S_2, \ldots, S_m \) of subsets of \( S \) such that each element of \( S \) is in exactly one of those subsets: \( S = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_m \), \( S \cap S_j = \{i = j\} \). Thus, the sets \( S_1, S_2, \ldots, S_m \) are pairwise disjoint sets, and their union is \( S \). The subsets \( S_1, S_2, \ldots, S_m \) are called the parts of the partition.

2.2. Lemma and corollary

Lemma 1. Let \( E \) be a space such that \( E \times E \times [0,1] \) is normal. Let \( (X_i)_{i \in I} \) be a locally finite family of closed subsets of \( E \). Let \( N \) be its nerve; for \( J \in N \), let \( X_J := \cap_{i \in J} X_i \). Let \( (U_j)_{j \in J} \) be a family of parts of \( E \) such that for all \( J \in N \), \( U_J \) has the extension property and contains \( X_J \), and that we have \( U_J \subset U_J' \) whenever \( J \supset J', J \in N, J' \in N \). Then there is a continuous mapping \( F(x, y, t) \) of \( \cup_{i \in J} (X_i \times X_i \times [0,1]) \) to \( E \) such that for all \( J \in N \), \( x \in X_J \) and \( y \in X_J \) we have \( F(x, x, t) = x \) for all \( t \), \( F(x, y, 0) = x \), and \( F(x, y, 1) = y \).

Corollary 1. The hypotheses being those of the lemma, let \( f \) and \( f' \) be two continuous mappings of a space \( A \) into \( E \) such that, for any \( u \in A \), there is any \( i \in I \), for which \( f(u) \in X_i \) and \( f'(u) \in X_i \). Then \( f \) and \( f' \) are homotopic [1].

2.3. Main statement

Theorem 1. If \( E \) is a space for which \( E \times E \times [0,1] \) is normal, and if \( U \) is a topologically simple cover of \( E \), the nerve \( N(U) \) has the same homotopy type as \( E \) [3].

2.4. Proof of main statement

In order to show \( E \) and \( N \) are homotopy equivalent, we have to find two continuous maps \( f : E \rightarrow N \) and \( f^{-1} = g : N \rightarrow E \) for which \( f \circ g \equiv \text{id}_N \) and \( g \circ f \equiv \text{id}_E \) [3].

First, showing the existence of \( f : E \rightarrow N(U) \). As it is required to be a normal space, a closed subspace of normal space \( E \times E \times [0,1] \) is normal. Therefore, space \( E \) is normal. Let \( U = \{u\} \) be any locally finite cover of space \( E \). There is a partition \( f = \{f_i\} \) subordinate to \( U \). We define \( f(p) = f(p) \), and this map is continuous from open cover \( U \) of \( E \) to the nerve \( N(U) \) of open cover \( U \). It is clear that continuous mapping \( f \) can be extended into the whole space \( E \) for open cover \( U \) is topologically simple. Assume that \( p \in E, J \) is the set of \( i \in I \) such that \( p \in u_i, f(p) \) is in the simplex \( \Sigma_J \). That is to say that every \( p \) is in \( U_i \), \( p \) is in \( U_j \) and once the nerve is identified, then \( J \) is assigned to \( \Sigma_J \). If \( \{F\} \) is another partition
subordinate to open cover $U$, $F(p)$ and the line segment between $f(p)$ and $F(p)$ is contained in $\Sigma_i$. We can derive that the mapping $p \to (1-t)f(p) + tF(p)$ is a homotopy between $f$ and $F$.

Second, showing the existence of $g: N(U) \to E$. Now we assume that $U_I = \cap_{i \in I} U_i$. For all given $J \subseteq \mathbb{N}$, let $e_J$ be the center of gravity of $\Sigma_i$. For an increasing sequence $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots \subseteq J_m$ of elements of $N(U)$, let $\Sigma_0(J_0, J_1, J_2, \ldots, J_m)$ be the simplex of vertices $e_{J_0}, e_{J_1}, e_{J_2}, \ldots, e_{J_m}$. In this case, we can see that nerve $N(U)$ is a union of all these simplexes.

In Figure 2, the forming process [2] is illustrated. We define a mapping $g$ from $N(U)$ to $E$ such that $g(\Sigma(J_0, J_1, J_2, \ldots, J_m)) \subseteq U_{J_0}$ for every increasing sequence $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots \subseteq J_m$. In other words, this mapping sends every barycentric subdivision back to the subset of the open cover $U$.

In Figure 3, the action of $g$ on $N$ [2] is depicted. Without losing its unity, we can see that the union of all simplices divided by barycentric subdivision is the nerve $N(U)$ itself. Then we define a continuous mapping $g$ on the simplexes of barycentric subdivision of $N(U)$ of dimension $\leq m - 1$ and $\Sigma' = \Sigma^*(J_0, J_1, J_2, \ldots, J_{m-1}, J_{m+1}, \ldots, J_m)$ for $0 \leq \mu \leq m$. With the help of a picture of the union of the simplexes $\Sigma'_{\mu}$ we can see that the union of the simplexes $\Sigma'_{\mu}$ is the connection of edges of geometrically realization of the simplex $g(\Sigma^*(J_0, J_1, J_2, \ldots, J_m))$, which is obvious.

Moreover, if $g'$ is another mapping from $N(U)$ to $E$, we can construct a homotopy between $g$ and $g'$ simply by conducting the above procedure backwards.

Third, showing that $f \circ g \cong \text{id}_{N(U)}$. Suppose that $F_i$ is the union of the images of simplexes $\Sigma(J_0, J_1, J_2, \ldots, J_m)$. Since there are only finitely many simplexes, $F_i$ is compact. Then we construct an open subset of $U_i$, $U_i'$ such that $F_i$ is contained in $U_i'$. It is easy to see that $U'$ forms a cover of $E$. Now we have an identity map $N$, which is also defined as another partition subordinate to the open cover $U$. According to step 1, we can realize that the homotopy class of $f$ is completely determined by $U$. Therefore, $f \circ g$ is homotopic to $\text{id}_E$.

Fourth, showing that $g \circ f \cong \text{id}_E$. Assume that $J$ is the set of $i$ that belongs to $I$ such that $f_i > 0$ on $F_i$, and $f_i = 0$ outside $U_i'$. Let $p \in E$ and let $I$ be the set $i \in I$ such that $\text{supp}(I)$ defined in $I$. Then we have $p \in U_i$, and thus $f(p) \in \Sigma_i$. Therefore, $f(p)$ is an element of a simplex of barycentric subdivision of $\Sigma_i$. Since we already have $\Sigma(J_0, J_1, J_2, \ldots, J_m)$ where $J_m \subseteq J$, if we pick any $i \in J_m$, we will have $g(f(p))$ in $F_i$. Hence, $g(f(p))$ and $p$ are both in $U_i'$. Now let us define $X_i$ as complementary set of $U_i'$, and let $N$ be the nerve of $X_i$. Suppose that $U_I$ have the extension property, we will see families $(X_i)_{i \in I}$ and $(U_i)_{i \in N}$ satisfy all the conditions of the following lemma 1. Therefore, we can finally assert that $g \circ f$ is homotopic to $\text{id}_E$ from corollary 1.
3. Suggestion

Will’s theorem itself provides more theoretical value, which creates a good beginning for the whole nerve theorem, provides guidance for the following theoretical development and delineates the research scope and framework. As for its application value, the theorem itself has a little more restrictions for itself to be put into use. On this issue, follow-up scholars such as McCord, Borsuk, Edelsrunner and Harer have made great contributions to this theory, making the whole theorem more and more universal. Among them, the most valuable theorem is Leray’s theorem which was adopted by Edelsbrunner and Harer:[5] If $U$ is a finite collection of closed, convex sets in Euclidean space, then the nerve of $U$ is homotopy equivalent to the union of the sets in $U$. Under this theorem, there are no conditions for the space we study, which makes the range of objects that we can study expand.

Nerve theorem often needs to integrate with other knowledge to play its great value. Topology data analysis technology is a popular technology to analyze data with the help of topology related knowledge in recent years. It can be expected that nerve theorems can be integrated with them to help people quickly and easily solve many problems that were difficult to solve in the past. For example, with the help of topological data analysis technology, nerve theorem can be applied to the study of tumor vasculature. Tumor vessel network structure can reveal the presence of potential underlying disease and how patients might respond to their treatment [4]. Further, most measurement only depicts images at a single spatial scale, while by implementing nerve theorem together with topology data analysis, images can be depicted at multiple spatial scales. After obtaining the raw images of functions of medicine and distribution and development of tumor vasculature, nerve theorem not only can make the image clearer, but also helps people to see more and more comprehensive tumor vasculature.

Another possible application of nerve theorem is based on a physical theory. At present, in the scientific community, we agree with the explanation of Adams and Lavlin, an astrophysicist. They divide the whole life of the universe into five parts. In the final stage of the universe, all is gone. If human beings are still alive, there is nothing in the universe that we can touch. We can see from one end of the universe to the other. There may be subatomic particles and photons with infinitely low capacity in the universe, but they can be ignored. We can say that the universe is dead. At this time, if we insist on adding a concept of time, then this era will continue endless [6]. If human beings at that time mastered advanced space technology and topology knowledge, they might escape from this chaotic universe and go to a new universe full of light and vitality. At that time, there may be more powerful theoretical tools than the nerve theorem. However, no matter how complex the geometry of the universe at that time, human beings can always simulate this geometry and find the exit of the universe approximately.

4. Conclusion

The proof of Heal is correct, but not concise. However, this paper is lack of application and originality. It only expands a kind of proof method, but does not give sufficient possible methods to prove the result. At the same time, the paper only makes a simple assumption on the possible application field, and has no real application. In the follow-up research process, we will explore other ways of proof, and apply the theory to specific cases to analyze specific problems.

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