Complexity of full counting statistics of free quantum particles in entangled states

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We study the computational complexity of quantum-mechanical expectation values of single-particle operators in bosonic and fermionic multi-particle product states. Such expectation values appear, in particular, in full-counting-statistics problems. Depending on the initial multi-product state, the expectation values may be either easy to compute (the required number of operations scales polynomially with the particle number) or hard to compute (at least as hard as a permanent of a matrix). However, if we only consider full counting statistics in a finite number of final single-particle states, then the full-counting-statistics generating function becomes easy to compute in all the analyzed cases. We prove the latter statement for the general case of the fermionic product state and for the single-boson product state (the same as used in the boson-sampling proposal). This result may be relevant for using multi-particle product states as a resource for quantum computing.

I. INTRODUCTION

Future quantum computers are predicted to efficiently solve certain problems difficult for classical ones \([1]\). One indication of this “quantum supremacy” is the computational complexity of quantum amplitudes: computationally simple quantum states and operators may generate expectation values of higher complexity. This consideration lead to a quantum-computing proposal named “Boson sampling” \([2]\), where bosonic multi-particle amplitudes are given by (presumably) computationally difficult permanents \([3]\). In the boson-sampling proposal, the origin of the computational complexity of the corresponding non-interacting multi-particle amplitudes may be traced down to the quantum nature of the initial single-boson state. A similar construction with fermions would require suitably entangled fermionic states, in order to generate scattering amplitudes of the same complexity level \([4, 5]\).

Those examples suggest that we may benefit from a more systematic study of the complexity of expectation values for various classes of quantum states and operators. To some extent, this approach was already developed in the context of quantum optics \([6]\), but we find it instructive to discuss the bosonic and fermionic cases on equal grounds. Specifically, we restrict our study to the computational complexity of matrix elements \(\langle \Phi_1 | \hat{U} | \Phi_2 \rangle\), where \(\{|\Phi_1\rangle\}\) and \(\{|\Phi_2\rangle\}\) are multiparticle bosonic or fermionic states constructed as direct products of \(N \gg 1\) identical states and \(\hat{U}\) is a non-interacting multiparticle operator (e.g., a non-interacting evolution operator or a similar operator without the unitarity condition). In this formulation, the states \(\{|\Phi_i\rangle\}\) only require a finite number of parameters for their description (which is automatic in the fermionic case and implies an extra assumption for bosons) and the operator \(\hat{U}\) is defined by the underlying single-particle operator and is thus parametrized by \(O(N^2)\) parameters. We are interested in a criterion for the matrix element \(\langle \Phi_1 | \hat{U} | \Phi_2 \rangle\) to be computable in a polynomial in \(N\) time.

We do not have a full answer to this question, but in this paper we collect a few known examples: some of them where a polynomial in \(N\) algorithm exists and others (specifically, the boson-sampling and entangled-fermion examples) that are (at least) as complex as a permanent (and therefore are believed to belong to a higher complexity class non-computable in polynomial time).

After this overview of the known results for the general \(\hat{U}\), we consider a variation of the problem where \(\hat{U}\) is generated by a single-particle operator \(1 + V\) with \(V\) having a small rank. For the bosonic version of the problem with the single-product boson state (as in the boson-sampling construction), we find a polynomial algorithm thus proving Lemma 67 of Ref. \([2]\) presented there without proof [about the polynomial computability of the permanent \(\text{Per}(1 + V)\)]. A similar statement for the fermionic case is also formulated and proven.

The motivation for the above formulations comes partly from the full-counting-statistics (FCS) problems, where the generating function for the probability distribution of noninteracting particles has the described structure \([6]\). In particular, the results for the operators \(1 + \hat{V}\) with a finite-rank matrix \(V\) correspond to the computational complexity of the “minimal” FCS in a finite number of states (tracing over the remaining states). We elaborate on this interpretation in the corresponding section of the paper.

The paper is organized as follows. In Section II we give more formal definitions and introduce notation. In Section III we list a few known examples for the case of the general noninteracting operator \(\hat{U}\). In Section IV we address the case of the operator \(\hat{U}\) based on a finite-rank single-particle operator. The interpretation of our results in the language of full counting statistics is presented in Section V. Finally, in Section VI we summarize our results and propose questions for further studies.
II. DEFINITIONS AND NOTATION

A. Fermionic and bosonic states

We consider the fermionic and bosonic multi-particle spaces (Fock spaces) generated by a large number of single-particle levels, and we are interested in the computational complexity of matrix elements of a certain class of operators as a function of this number (whether it is polynomial or higher, e.g., exponential). More specifically, we restrict our analysis to product states: tensor products of states built on a small number (one or a few) of single-particle levels. The following states will appear in our examples:

- **single-boson product state:**

\[ |BN=1 \rangle^{N} = |BN=1 \rangle \otimes \ldots \otimes |BN=1 \rangle \quad (N \text{ times}), \quad (1) \]

where \(|BN=1 \rangle = b^\dagger (|*)_{B,1} \) is a single-boson state (\(b^\dagger \) here and below denotes the boson creation operator and \(|* \rangle_{B,1} \) is the bosonic vacuum with one empty single-particle level). This is the state used in the boson-sampling proposal [2].

- **coherent-boson product state:**

\[ |BC=\lambda \rangle^{N} = |BC=\lambda \rangle \otimes \ldots \otimes |BC=\lambda \rangle \quad (N \text{ times}), \quad (2) \]

where \(|BC=\lambda \rangle = \exp(\lambda b^\dagger - \lambda^2/2)|* \rangle_{B,1} \) is a coherent boson state.

- **Fermi-sea product state:** a class of states constructed as

\[ |FS \rangle^{N} = |FS \rangle \otimes \ldots \otimes |FS \rangle \quad (N \text{ times}), \quad (3) \]

with \(|FS \rangle = \psi_1^\dagger \ldots \psi_k^\dagger |* \rangle_{F,n} \), where \(|* \rangle_{F,n} \) is a fermionic vacuum with \(n \) single-particle states and \(\psi^\dagger \) are creation operators for some \(k \leq n \) (mutually orthogonal, for the sake of normalization) linear combinations of those states. \(n \) and \(k \) are fixed small numbers (unrelated to \(N \)). The product state \(|FS\rangle^{N} \) then belongs to the multi-particle space (Fock space) generated by \(Nn \) single-particle levels. Two particular cases of such a state is the vacuum state \((k = 0)\) and the fully occupied state \((k = n)\).

- **entangled-quadruplet product state:**

\[ |\Psi_4 \rangle^{N} = |\Psi_4 \rangle \otimes \ldots \otimes |\Psi_4 \rangle \quad (N \text{ times}), \quad (4) \]

where \(|\Psi_4 \rangle = (1/\sqrt{2})(f_1^\dagger f_2^\dagger + f_1^\dagger f_2^\dagger)|* \rangle_{F,4} \). This state was used in Ref. [3]. It involves \(2N \) fermions in \(4N \) single-particle states.

B. Non-interacting operators

Every single-particle operator \(U \) generates a “multiplicative” multi-particle operator \(\hat{U} \) in the multi-particle Fock space. A “physical definition” of this construction is sometimes written as

\[ \hat{U} = \exp(\sum_{ij} a_i^\dagger (\ln U)_{ij} a_j), \quad (5) \]

where \(a_i^\dagger \) and \(a_j \) are either fermionic or bosonic creation and annihilation operators. However, this definition formally fails when \(U \) has zero eigenvalues (non-invertible). For our purpose, we extend this definition to non-invertible matrices \(U \), which can be done either by continuity or with a more explicit alternative definition

\[ \hat{U} a_i^\dagger \ldots a_k^\dagger |* \rangle = \sum_{i_1 \ldots i_k} U_{i_1,j_1} \ldots U_{i_k,j_k} a_{i_1}^\dagger \ldots a_{i_k}^\dagger |* \rangle, \quad (6) \]

which describes the action of \(\hat{U} \) on each of the basis vectors of the Fock space.

With this definition, we have a set of the “non-interacting operators” \(\hat{U} \) defined as those obtainable from single-particle matrices \(U \). This is a representation of the monoid of matrices \(U \) with respect to multiplication (i.e., \(U_1U_2 = \hat{U}_1\hat{U}_2 \)). In particular, this set is closed with respect to multiplication. An example of such an operator is a quantum evolution operator for a non-interacting system of particles (given by [2] with \(\ln U \) playing the role of the Hamiltonian).

C. Multi-particle complexity of a quantum state (or of a pair of states)

Now we are ready to define the main object of our study. We define the multi-particle complexity of a pair of states \(|\Phi_1 \rangle, |\Phi_2 \rangle \) (or of a single state \(|\Phi_0 \rangle \)) as the maximal computational complexity of the matrix element

\[ \langle \Phi_1 | \hat{U} | \Phi_2 \rangle \quad \text{or} \quad \langle \Phi_0 | \hat{U} | \Phi_0 \rangle, \quad \text{respectively} \quad (7) \]

(with the maximum taken over all non-interacting operators \(\hat{U} \)). The computational complexity is understood as scaling of the required number of operations as a function of \(N \) (see more explanations in Section III below). The operator \(\hat{U} \) is parametrized by its single-particle counterpart \(U \), which requires \(N^2 \) parameters. The quantum states \(|\Phi_i \rangle \), in their full generality, use an exponential number of amplitudes, therefore the definition is only meaningful if we restrict it to a subclass of states parametrized by at most a polynomial in \(N \) set of parameters. One possible restriction of this sort is to consider product states (as defined in Section IIIA), where each of the factors involves only a finite number of parameters (we do not formalize this restriction further).
In Section [I] below, we also consider a modification of this definition where \( \hat{U} \) is further restricted to be generated by a matrix \( U = 1 + V \), where \( V \) is a matrix of a finite rank. We will call this a finite-rank complexity of a state (or of a pair of states).

### D. Computational complexity for real and complex functions

Defining computational complexity for functions with continuous variables is sometimes a subtle issue \([8]\), and we do not want to go deeply into this topic here. Instead, since the expectation values of interest are all polynomials of the matrix elements of \( U \) and of the wave function components, we define the computational complexity as the scaling of the number of required arithmetic operations with \( N \) (with the exception of the coherent-boson case, which involves the exponentiation operator, see more details in Section III B). To simplify our notation, we only distinguish two levels of complexity: “easy” (computable in a polynomial in \( N \) number of operations) and “hard” (at least as difficult as computing a matrix permanent).

There is a general belief that computing a permanent requires a higher than polynomial number of operations, which implies \( P \neq \text{NP} \). We also need this assumption in order for our classification to be meaningful. However otherwise we never make use of it.

### III. COMPLEXITY IN CASE OF GENERAL \( \hat{U} \)

We do not have a general criterion for product states to be “easy” or “hard”, but we can give a few examples of states of each of them:

- Single-boson product state is “hard”.
- Coherent-boson product state is “easy”.
- Fermi-sea product state is “easy”.
- Entangled-quadruplet product state is “hard”.

#### A. Single-boson product state is “hard”

The corresponding expectation value is a permanent,

\[
\langle BN=1|N \hat{U}|BN=1 \rangle^N = \text{Per} \, U ,
\]

so it is “hard” by definition. This high complexity was used in Ref. [2] to conjecture the “quantum supremacy” of Boson Sampling.

### B. Coherent-boson product state is “easy”

Since non-interacting operators \( \hat{U} \) act within the space of coherent states (and this action can be written in single-particle terms), one can easily calculate the matrix element of \( \hat{U} \) between any two coherent states. In particular,

\[
\langle BC=\lambda|N \hat{U}|BC=\lambda \rangle^N = \exp \left[ \lambda^2 \left( \sum_{ij} U_{ij} - N \right) \right]. \tag{9}
\]

In this example, unlike in all the others, we use a sloppy definition of complexity: instead of the wave-function components (there are infinitely many of them), we use the parameter \( \lambda \) of the coherent state and are allowed one exponentiation at the end of the calculation.

### C. Fermi-sea product state is “easy”

The product of Fermi seas \([3]\) is also a Fermi sea with \( Nn \) fermions. For this large Fermi sea, one easily finds

\[
\langle FS|N \hat{U}|FS \rangle^N = \det \langle \psi_{i,j}|U|\psi_{i,j} \rangle , \tag{10}
\]

where the determinant is of the \( Nn \)-dimensional matrix of the single-particle matrix elements between the states generating the large Fermi sea. This proves that this matrix element is computable in polynomial time. Note that this argument equally applies to products of non-identical Fermi seas.

### D. Entangled-quadruplet product state is “hard”

This was shown in Ref. [5] (it also follows from the results on mixed discriminants in Refs. [9], [10]). Strictly speaking, in that work, the hardness of \( \langle x|U|\Psi_4 \rangle^N \) was proven, where \( |x\rangle \) is a Fermi sea with arbitrary \( 2N \) states (orthogonal, for simplicity), \( |x\rangle = \psi_1^\dagger \ldots \psi_{2N}^\dagger |\pm \rangle_{F,4N} \). However, we can easily convert this statement into one for the expectation value in the state \( |\Psi_4 \rangle^N \). Namely, consider a single-particle operator \( Y \) transforming the states \( \psi_1, \ldots, \psi_{2N} \) into the basis states \( f_1, f_2, f_3, f_6, \ldots, f_{4N-3}, f_{4N-2} \) (in arbitrary order) and zeroing out the orthogonal complement of \( \psi_1, \ldots, \psi_{2N} \). Then

\[
\langle \Psi_4 |^N Y \hat{U} |\Psi_4 \rangle^N = \langle \Psi_4 |^N |x\rangle \langle x| \hat{U} |\Psi_4 \rangle^N
\]

\[
= 2^{-N/2} \langle x| \hat{U} |\Psi_4 \rangle^N , \tag{11}
\]

which proves the hardness of the left-hand side of the above equation.

### IV. FINITE-RANK COMPLEXITY

In this section, we consider the finite-rank complexity: a modified version of the complexity definition (Sec-
where the operators \( \hat{U} \) are restricted to those generated by

\[
U = 1 + V ,
\]

where \( V \) is a matrix of a finite rank:

\[
V_{ij} = \sum_{s=1}^{k} u_i^{(s)} v_j^{(s)} .
\]

Obviously, the finite-rank complexity cannot be higher than the complexity for the general \( U \). In particular, for all the examples considered above, the finite-rank complexity is “easy” (polynomial). Moreover, we can prove that the finite-rank complexity is polynomial for a general product state in the fermionic case. Specifically, we prove the following two statements below:

- The finite-rank complexity of the single-boson product state is “easy”. We can further prove that the number of required operations scales as \( O(N^{2k+1}) \).
- The finite-rank complexity of any fermionic product state is “easy”. The number of required operations is also limited as \( O(N^{2k+1}) \).

### A. Finite-rank complexity of the single-boson product state is “easy”

The matrix element is given by the permanent

\[
(\text{BN}=1|\hat{U}|\text{BN}=1)^N = \text{Per} U = \text{Per}(1+V) .
\]

Below we show that, if \( V \) has a finite rank \( k \), the permanent \((14)\) may be expressed in terms of the coefficients of an auxiliary polynomial of degree \( 2N \) in \( 2k \) variables, which, in turn, requires only a polynomial in \( N \) number of operations.

A simple combinatorial argument expresses the permanent \((14)\) in terms of the vectors \( u^{(s)} \) and \( v^{(s)} \) from Eq. \((13)\):

\[
\text{Per}(1+V) = \sum_{X \subseteq \{1, \ldots, N\}} \prod_{s \in X} u_{s_u(X)}^{(s_u(x))} v_{s_v(x)}^{(s_v(x))} \prod_{r=1}^{k} n_r ! ,
\]

where the first sum is taken over all subsets \( X \) of \( S \) of indices \( \{1, \ldots, N\} \), the second sum is over the label sets \( s_u \) and \( s_v \) (ranging from 1 to the rank \( k \)) for elements of \( X \) such that they form identical multisets (sets with repetitions) \([s_u(X)] = [s_v(X)]\), but possibly permuted with respect to each other. Finally, in the last product \( n_r \) denotes the multiplicity of \( r \) in the multiset \([s_u(X)]\) (or, equivalently \([s_u(X)]\)). This expression may, in turn, be computed with the help of the auxiliary polynomial of \( 2k \) formal variables

\[
F(u^{(1)}_u, \ldots, a^{(k)}_u, v^{(1)}_v, \ldots, a^{(k)}_v) = \prod_{x=1}^{N} \left[ 1 + \sum_{s=1}^{k} \sum_{s'=1}^{k} a_u^{(s)} a_v^{(s')} u_x^{(s)} v_x^{(s')} \right] = \sum_{\{n_r\}} F_{n_1, \ldots, n_k, n'_1, \ldots, n'_k}
\]

\[
\times \left(a^{(1)}_u n_1 \ldots a^{(k)}_u n_k a^{(1)}_v n'_1 \ldots a^{(k)}_v n'_k\right) ,
\]

where the first equality is the definition of the polynomial \( F(u^{(1)}_u, \ldots, a^{(k)}_u, v^{(1)}_v, \ldots, a^{(k)}_v) \) and the second equality is its expansion in powers of \( a^{(s)}_u \) and \( a^{(s)}_v \) defining its coefficients. On inspection, the “diagonal” coefficients of this polynomial reproduce the terms in the sum \( (15) \), up to combinatorial coefficients, and one finds

\[
\text{Per}(1+V) = \sum_{\{n_r\}} F_{n_1, \ldots, n_k, n'_1, \ldots, n'_k} \prod_{r=1}^{k} n_r ! .
\]

There are altogether \( O(N^{2k}) \) coefficients \( F_{n_1, \ldots, n_k, n'_1, \ldots, n'_k} \), including \( O(N^k) \) diagonal coefficients (with \( n_r = n'_r \)). Their calculation involves multiplying out \( N \) terms in Eq. \((16)\), where at each multiplication the \( O(N^k) \) coefficients need to be updated. Therefore the calculation of \( \text{Per}(1+V) \) using Eqs. \((16)\) and \((17)\) can be done in \( O(N^{2k+1}) \) operations, as claimed. This proves Lemma 67 of Ref. [2] and the “finite-rank easiness” of the single-boson product state.

### B. Finite-rank complexity of any fermionic product state is “easy”

The idea of the proof is that \( \hat{U} \), in the finite-rank construction \( \{14-13\} \), acts nontrivially only in a small subspace spanned by a small number of fermionic states and therefore may be written in terms of a small number of fermionic operators. Specifically, \( \hat{U} \) may be written in terms of the creation and annihilation operators defined as

\[
\hat{u}^s = \sum_{i} u^{(s)}_i f_i^\dagger , \quad \hat{\nu}^s = \sum_{j} v^{(s)}_j f_j ,
\]

where \( f_i^\dagger \) and \( f_j \) are the fermionic creation and annihilation operators in the original basis. Then \( \hat{U} \) is a polynomial in those operators

\[
\hat{U} = 1 + \sum_{r=0}^{k} \sum_{s_1, \ldots, s_r} A_{s_1, \ldots, s_r} \hat{u}_{s_1}^\dagger \cdots \hat{u}_{s_r}^\dagger \hat{\nu}_{s_1} \cdots \hat{\nu}_{s_r} .
\]

The coefficients \( A_{s_1, \ldots, s_r} \) may be calculated, for example, iteratively, starting with \( r = 1 \), then \( r = 2 \), etc.,
by requiring that at each step the definition (19) is satisfied for r particles. Note that the degree of the polynomial (19) is bounded by k due to the fermionic statistics. The number of its coefficients is then at most \( O(2^k) \), and their calculation requires an exponentially large number of operation in \( k \), but only linearly growing with \( N \) (since the only operation depending on \( N \) is the calculation of scalar products between \( u^{(s)} \) and \( v^{(s')} \)).

Now consider the expectation value of each term of the polynomial (19) in any product state

\[
|F\rangle = |F_1\rangle \otimes \ldots \otimes |F_N\rangle,
\]

where each of the states \( |F_i\rangle \) belongs to a Fock space generated by a “small” number of single-particle states (we do not even need these states to be identical). Our states (3) and (4) are particular cases of this construction.

To calculate the expectation value of a term of degree \( r \) in the polynomial (19) in such a state, we decompose each of the operators \( \hat{u}_i^\dagger \) and \( \hat{v}_i \) into N components:

\[
\hat{u}_i^\dagger = (\hat{u}_i^\dagger)_1 \oplus \ldots \oplus (\hat{u}_i^\dagger)_N,
\]

where \( (\hat{u}_i^\dagger)_k \) acts in the \( i \)-th space (and the same for \( \hat{v}_i \)). Expanding the product of \( 2r \) operators, we obtain \( N^{2r} \) terms (and \( r \leq k \)). Each of these terms is itself a product of \( N \) components by the number of subspaces in Eq. (20), where each component can be calculated in a “small” number of operations, since it is an expectation value in a fermionic space of a small dimension. Therefore, in total, we need \( O(N^{2k+1}) \) operations, which proves our statement.

V. IMPLICATIONS FOR FULL COUNTING STATISTICS

The above discussion of the complexity of the expectation values may be interpreted in the language of so-called full counting statistics (FCS): a class of problems addressing the probability distribution of a quantum observable \( \hat{F} \). Namely, our results may be reformulated in terms of complexity of FCS generating functions for non-interacting particles initially prepared in a certain state \( |\Phi_1\rangle \). Indeed, consider an initial state \( |\Phi_1\rangle \) that is subject to a non-interacting evolution \( \hat{U}_0 \). We may further count particles in the final states and define the generating function

\[
\chi(\lambda_1, \ldots, \lambda_N) = \sum_{\{n_i\}} e^{i \sum_{i=1}^{N} \lambda_i n_i} P(n_1, \ldots, n_N),
\]

where

\[
P(n_1, \ldots, n_N) = \left| \langle n_1, \ldots, n_N | \hat{U}_0 | \Phi_1 \rangle \right|^2
\]

is the probability to observe the counts \( n_i \) in the single-particle states \( i \) after the evolution \( \hat{U}_0 \). The above expression obviously has the required structure \( \langle \Phi_1 | \hat{U} | \Phi_1 \rangle \), where

\[
\hat{U} = \hat{U}_0^{-1} e^{i \sum_{i=1}^{N} \lambda_i \hat{n}_i \hat{U}_0} = \exp \left( \sum_{i=1}^{N} \lambda_i \hat{U}_0^{-1} \hat{n}_i \hat{U}_0 \right)
\]

and \( \hat{n}_i = a_i^\dagger a_i \) is the particle-number operator in the single-particle state \( i \).

Taking into account that we may choose the “counting parameters” \( \lambda_i \) to be arbitrary complex numbers and also take the limit \( \lambda_i \to i\infty \) (which projects onto the state with zero occupancy), we may generate nearly any non-interacting operator \( \hat{U} \). Therefore, we believe that our results from Section III literally translate into the computational complexity of the generating function (22), (23) for the general choice of the non-interacting evolution \( U_0 \) and for the general choice of the complex variables \( \lambda_i \).

At the same time, the finite-rank complexity studied in Section IV corresponds to the case of a finite number of non-zero variables \( \lambda_i \). In other words, in this case we only count particles in a finite number of channels. Our main results of this paper is that such generating functions are polynomially computable in the fermionic case for any initial product state \( |\Phi_1\rangle \) and in the bosonic case for the product of single-boson states.

VI. SUMMARY AND DISCUSSION

The purpose of this paper is two-fold. First, we introduce the notion of the “multi-particle complexity” of product states. This definition naturally leads to the question of formulating a criterion for a product state to be “hard”. From examples, one may conjecture that most of such states are actually “hard” except for a few special cases. One such special case are so called Gaussian states, where Wick theorem applies (see, e.g., Ref. [11] for definition in the bosonic case). Our examples of coherent-boson and Fermi-sea product states belong to this class of Gaussian states. One finds more Gaussian states among the mixed states described by a density matrix, but in this paper we restrict our discussion to pure states only. We do not know if there is any non-Gaussian state that would generate “easy” product states.

Another question in connection with this “multi-particle complexity” concept is its possible implications for quantum computing. Ref. [2] suggests that this setup (specifically, the example of Boson sampling) is insufficient for universal quantum computing, but, to our knowledge, without solid justification. In any case, it would be an interesting problem to characterize the class of problems solvable in polynomial time with this “full-counting statistics” setup: an initial preparation of a certain product state (e.g., one of our “hard” examples), then evolution with a single-particle operator (which encodes the “quantum algorithm”), and finally a measurement of a certain generating function (or of a set of generating functions) [23]. It seems plausible that all
“hard” quantum states are equivalent for this quantum-computing setup, and therefore all of them would be equivalent to Boson sampling.

We would like to remind the reader that the “hardness” of a matrix element $\langle \Phi_0 | \hat{U} | \Phi_0 \rangle$ does not imply a possibility to actually compute this quantity with a quantum system. There are two reasons for this. First, the quantum measurement implies sampling, and achieving a good precision in a typically exponentially small expectation value would require exponentially many repeated measurements. Second, in this paper we only address the question of an exact computation, while for experimental implications it may be more relevant to study approximations. Computational complexity of approximate computations of permanents, mixed discriminants and other related functions is addressed in many recent works [6, 9, 10, 12, 13].

The second goal of the paper is to report two results related to the “finite-rank” full-counting statistics. For the cases we managed to prove (any fermionic product states and the single-boson product state), we have shown that counting particles in a finite number of final states is an “easy” task (computable in polynomial time) and therefore does not present any interest from the point of view of quantum computing. It seems plausible that this statement might be extended to a wider class of bosonic states (e.g., to any bosonic product states based on states with a finite number of particles). We leave this extension for future studies.

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