On the Complexity of Submodular Function Minimisation on Diamonds

Fredrik Kuivinen

April 21, 2009

Abstract

Let \((L; \sqcap, \sqcup)\) be a finite lattice and let \(n\) be a positive integer. A function \(f : L^n \to \mathbb{R}\) is said to be submodular if \(f(a \sqcap b) + f(a \sqcup b) \leq f(a) + f(b)\) for all \(a, b \in L^n\). In this paper we study submodular functions when \(L\) is a diamond. Given oracle access to \(f\) we are interested in finding \(x \in L^n\) such that \(f(x) = \min_{y \in L^n} f(y)\) as efficiently as possible. We establish

- a min–max theorem, which states that the minimum of the submodular function is equal to the maximum of a certain function defined over a certain polyhedron; and
- a good characterisation of the minimisation problem, i.e., we show that given an oracle for computing a submodular \(f : L^n \to \mathbb{Z}\) and an integer \(m\) such that \(\min_{x \in L^n} f(x) = m\), there is a proof of this fact which can be verified in time polynomial in \(n\) and \(\max_{t \in L^n} \log |f(t)|\); and
- a pseudo-polynomial time algorithm for the minimisation problem, i.e., given an oracle for computing a submodular \(f : L^n \to \mathbb{Z}\) one can find \(\min_{t \in L^n} f(t)\) in time bounded by a polynomial in \(n\) and \(\max_{t \in L^n} |f(t)|\).

1 Introduction

Let \(V\) be a finite set and let \(f\) be a function from \(2^V\) to \(\mathbb{R}\). The function \(f\) is said to be submodular if \(f(A \cup B) + f(A \cap B) \leq f(A) + f(B)\) for all \(A, B \subseteq V\). In the sequel we will call such functions submodular set functions. Submodular set functions shows up in various fields including combinatorial optimisation, graph theory [7], game theory [30], information theory [13] and statistical physics [1]. Examples include the cut function of graphs and the rank function of matroids. There is also a connection between submodular function minimisation and convex optimisation. In particular, submodularity can be seen...
as a discrete analog of convexity [8, 23]. We refer the reader to [9, 18, 24] for a general background on submodular set functions.

Given a submodular set function \( f : 2^V \rightarrow \mathbb{R} \) there are several algorithms for finding minimisers of \( f \), i.e., finding a subset \( X \subseteq V \) such that \( f(X) = \min_{Y \subseteq V} f(Y) \), in time polynomial in \(|V|\). The first algorithm for finding such minimisers in polynomial time is due to Grötschel et al. [11]. However, this algorithm is based on the Ellipsoid algorithm and hence its usefulness in practice is limited. Almost two decades later two combinatorial algorithms were found independently by Schrijver [28] and Iwata et al. [19]. More recently the running times have been improved. The currently fastest strongly polynomial time algorithm is due to Orlin [25] and the fastest weakly polynomial time algorithm is due to Iwata [17]. In these algorithms the submodular set function is given by a value-giving oracle for \( f \) (i.e., presented with a subset \( X \subseteq V \) the oracle computes \( f(X) \)).

In this paper we investigate a more general notion of submodularity. Recall that a lattice is a partially ordered set in which each pair of elements have a least upper bound (join, \( \sqcup \)) and a greatest lower bound (meet, \( \sqcap \)). Given a finite lattice \( L \) (all lattices in this paper are finite) and a positive integer \( n \) we can construct the product lattice \( L^n \). Meet and join for \( L^n \) are then defined coordinate-wise by meet and join in \( L \). We say that a function \( h : L^n \rightarrow \mathbb{R} \) is submodular if \( h(a \sqcap b) + h(a \sqcup b) \leq h(a) + h(b) \) for all \( a, b \in L^n \). Note that the subsets of \( V \) can be seen as a lattice with union as join and intersection as meet (this lattice is a product of the two element lattice). Hence, this notion of submodularity is a generalisation of submodular set functions. For a fixed finite lattice \( L \) we are interested in the submodular function minimisation (SFM) problem:

**INSTANCE:** An integer \( n \geq 1 \) and a submodular function \( f \) on \( L^n \).

**GOAL:** Find \( x \in L^n \) such that \( f(x) = \min_{y \in L^n} f(y) \).

Following [22] we denote this problem by SFM(\( L \)). SFM(\( L \)) is said to be oracle-tractable if the problem can be solved in time polynomial in \( n \) (provided that we have access to a value-giving oracle for \( f \) and that we can assume that \( f \) is submodular, i.e., it is a promise problem). This definition naturally leads to the following question: is SFM(\( L \)) oracle-tractable for all finite lattices \( L \)? (This question was, as far as we know, first asked by Cohen et al. [3].)

Schrijver [28] showed that given a sublattice \( S \) of \( 2^V \) (i.e., \( S \subseteq 2^V \) and for any \( X, Y \in S \) we have \( X \cap Y, X \cup Y \in S \)) and submodular function \( f : S \rightarrow \mathbb{R} \) a minimiser of \( f \) can be found in time polynomial in \( n \). In particular, this implies that for any distributive lattice \( L \) the problem SFM(\( L \)) is oracle-tractable. Krokhin and Larose [22] showed that certain constructions on lattices preserve oracle-tractability of SFM. In particular, they showed that if \( X \) is a class of lattices such that SFM(\( L \)) is oracle-tractable for every \( L \in X \), then so is SFM(\( L' \)) where \( L' \) is a homomorphic image of some lattice in \( X \), a direct product of some lattices in \( X \), or contained in the Mal’tsev product \( X \circ X \). We will not define these constructions here and refer the reader to [22] instead.

A lattice \( L \) is a diamond if the elements of the lattice form a disjoint union.
of \{0_\mathcal{L}, 1_\mathcal{L}\} and \mathcal{A}, for some finite set \mathcal{A} such that |\mathcal{A}| \geq 3. Here 0_\mathcal{L} is the bottom element of \mathcal{L}, and 1_\mathcal{L} is the top element of \mathcal{L}, and all elements in \mathcal{A} (called the \textit{atoms}) are incomparable to each other. See Figure 1 for a diagram of the five element diamond. We want to emphasise that diamonds have a different structure compared to the lattices defined by union and intersection. In particular, diamonds are not \textit{distributive}, that is they do \textit{not} satisfy \(x \cap (y \cup z) = (x \cap y) \cup (x \cap z)\) for all \(x, y, z \in \mathcal{L}\). We will denote the diamond with \(k\) atoms by \(\mathcal{M}_k\). In Sections 4, 5 and 6 the complexity of \(\text{SFM}(\mathcal{M}_k)\) is investigated. In the approach taken in this paper the difficult case is \(k = 3\)—the proofs for the \(k = 3\) case generalises straightforwardly to an arbitrary \(k\). We note that none of the diamonds are captured by the combination of the results found in \cite{22, 28} (a proof of this fact can be found in \cite{22}).

\textbf{Results and techniques.} The first main result in this paper is a min–max theorem for \(\text{SFM}(\mathcal{M}_k)\) which is stated as Theorem 4.3. This result looks quite similar to Edmonds’ min–max theorem for submodular set functions \cite{6} (we present Edmonds’ result in Section 2). The key step in the proof of this result is the definition of a certain polyhedron, which depends on \(f\).

The second main result is a \textit{good characterisation} of \(\text{SFM}(\mathcal{M}_k)\) (Theorem 5.5). That is, we prove that given a submodular \(f: \mathcal{M}_k^n \to \mathbb{Z}\) and integer \(m\) such that \(\min_{x \in \mathcal{L}^n} f(x) = m\), there is a proof of this fact which can be verified in time polynomial in \(n\) and \(\max_{y \in \mathcal{L}^n} \log |f(y)|\) (under the assumption that \(f\) is submodular). This can be seen as placing \(\text{SFM}(\mathcal{M}_k)\) in the appropriately modified variant of \(\text{NP} \cap \text{coNP}\) (the differences from our setting to an ordinary optimisation problem is that we are given oracle access to the function to be minimised and we assume that the given function is submodular). The proof of this result makes use of Carathéodory’s theorem and of the known polynomial-time algorithms for minimising submodular set functions. We also need our min–max theorem.

The third result is a pseudo-polynomial time algorithm for \(\text{SFM}(\mathcal{M}_k)\) (see Section 6). We show that \(\text{SFM}(\mathcal{M}_k)\) can be solved in time polynomial in \(n\) and \(\max_{t \in \mathcal{M}_k^n} |f(t)|\). The main part of the algorithm consists of a nested application of the Ellipsoid algorithm. We also need to prove that the polyhedrons we associate with submodular functions are \(1/2\)-integral. An interesting and challenging open problem is to construct an algorithm with running time polynomial
1 INTRODUCTION

Figure 2: A lattice which can be shown to admit a pseudo-polynomial time algorithm for the submodular function minimisation problem. This lattice is a Mal’tsev product of a diamond and the two element lattice. By the results in this paper and the constructions in [22] this lattice gives a new tractable constraint language for Max CSP.

in \( n \) and \( \max_{t \in M} |f(t)| \).

Our results applies to diamonds, however, as mentioned above, in [22] two constructions on lattices (Mal’tsev products and homomorphic images) are shown to preserve tractability results for SFM. By combining these constructions with the results in this paper one gets tractability results for a much larger class of lattices than just diamonds. In particular, by the results in this paper there is a pseudo-polynomial time algorithm for minimising submodular functions over products of the lattice in Figure 2.

Connections to other problems. Minimising submodular functions on certain modular non-distributive (the diamonds are modular and non-distributive) lattices has applications to canonical forms of partitioned matrices [15, 20]. Finding a polynomial time algorithm for minimising submodular functions on these lattices was mentioned as an open problem in [19].

The notion of submodular functions over arbitrary finite lattices plays an important role in the complexity of the maximum constraint satisfaction problem (Max CSP). This connection was first observed in [3] and in later papers the connection was explored further [5, 21]. The connection between submodular function minimisation and Max CSP is that by proving oracle-tractability for new lattices for the SFM problem implies tractability results (solvability in polynomial time) for certain restrictions (so called constraint language restrictions) of Max CSP. By constructing algorithms for SFM with running times bounded by a polynomial in \( n \) and \( \max_{t \in M} |f(t)| \), as we do in Section 6 one gets solvability in polynomial time for the unweighted variant of Max CSP (with appropriate restrictions). Providing good characterisations of SFM(\( L \)), as we do in Section 5 implies coNP containment results for Max CSP (with appropriate restriction). As Max CSP is trivially in NP we get containment in NP \( \cap \) coNP for these restrictions. We refer the reader to [3, 22] for further details regarding the connection between SFM and Max CSP.

\(^1\)In [22] these constructions are shown to preserve oracle-tractability and not solvability in pseudo-polynomial time. However, it is straightforward to adapt the proofs to the pseudo-polynomial case.
In [22] it is shown that the restrictions of Max CSP which one gets from the diamonds can be solved in polynomial time. This means that the results for the diamonds in this paper do not directly imply new tractability results for Max CSP. However, as mentioned in the previous section one can combine the results in this paper with the lattice constructions in [22] to get tractability results for a larger class of lattices which implies tractability results for new constraint language restrictions of Max CSP. (We again refer to Figure 2 for an example of such a lattice.)

There is also a connection between SFM over lattices to the Valued Constraint Satisfaction Problem (VCSP). See [4] for more information on VCSP. The connection is very similar to the connection to Max CSP, proving tractability results for new lattices for SFM implies new tractable restrictions of VCSP.

For VCSP there was, before the results in this paper, no known non-trivial algorithms for the restrictions one obtains from the diamonds. We note that Raghavendra [26] recently proved almost optimal results for the approximability of Max CSP for constraint language restrictions, assuming that the unique games conjecture (UGC) holds. However, for the cases which are solvable to optimality the results in [26] gives us polynomial-time approximation schemes. This should be compared to the connection described above which gives polynomial time algorithms for some of these cases.

Organisation. This paper is organised as follows, in Section 2 we give a short background on submodular set functions, in Section 3 we introduce the notation we use, in Section 4 we prove our first main result—the min–max theorem for submodular functions over diamonds. The good characterisation is given in Section 5. In Section 6 where we give the pseudo-polynomial time algorithm for the minimisation problem. Finally, in Section 7 we give some conclusions and open problems.

2 Background on Submodular Set Functions

In this section we will give a short background on Edmonds’ min–max theorem for submodular set functions. This result was first proved by Edmonds in [6], but see also the surveys [18, 24]. Let $V$ be a finite set. For a vector $x \in \mathbb{R}^V$ (i.e., $x$ is a function from $V$ into $\mathbb{R}$) and a subset $Y \subseteq V$ define $x(Y) = \sum_{y \in Y} x(y)$. We write $x \leq 0$ if $x(v) \leq 0$ for all $v \in V$ and $x^-$ for the vector in which coordinate $v$ has the value min\{$0, x(v)$\}. Let $f$ be a submodular set function $f : 2^V \to \mathbb{R}$ such that $f(\emptyset) = 0$ (this is not really a restriction, given a submodular function $g$ we can define a new function $g'(X) = g(X) - g(\emptyset)$, $g'$ satisfies $g'(\emptyset) = 0$ and is submodular). The submodular polyhedron and the base polyhedron defined by

\[
P(f) = \{x \in \mathbb{R}^V \mid \forall Y \subseteq V, x(Y) \leq f(Y)\}, \quad \text{and}
\]

\[
B(f) = \{x \in \mathbb{R}^V \mid x \in P(f), x(V) = f(V)\}
\]
often play an important role in results related to submodular set functions. Edmonds proved the following min–max theorem

\[
\min_{X \subseteq V} f(X) = \max\{x(V) \mid x \in P(f), x \leq 0\} = \max\{x^-(V) \mid x \in B(f)\}.
\]

(1)

In Section we give an analog to (1) for submodular functions over diamonds.

3 Preliminaries

For a positive integer \( n \), \([n]\) is the set \( \{1, 2, \ldots , n\} \). Given a lattice \( (L, \cap, \cup) \) and \( x, y \in L \) we write \( x \subseteq y \) if and only if \( x \cap y = x \) (and hence \( x \cap y = y \)). We write \( x \sqsubset y \) if \( x \subseteq y \) and \( x \neq y \). As mentioned in the introduction, given a positive integer \( n \), we can construct the product lattice \( L^n \) from \( L \). The top and bottom elements of \( L^n \) are denoted by \( 1_{L^n} \) and \( 0_{L^n} \), respectively. We write \( x \prec y \) if \( x \) is covered by \( y \) (that is, if \( x \sqsubset y \), and there is no \( z \in L \) such that \( x \sqsubseteq z \sqsubset y \)).

Recall that the diamonds are modular lattices (the rank function \( \rho \) is defined by \( \rho(0_M) = 0, \rho(a) = 1 \) for all \( a \in A \) and \( \rho(1_M) = 2 \)). As direct products of modular lattices also are modular lattices it follows that direct products of diamonds are modular lattices.

For a set \( X \) we let \( \mathbb{R}^{[n] \times X} \) be the set of functions mapping \([n] \times X\) into \( \mathbb{R} \). Such functions will be called vectors and can be seen as vectors indexed by pairs from \([n] \times X\). For \( x, y \in \mathbb{R}^{[n] \times X} \) and \( \alpha \in \mathbb{R} \) we define \( \alpha x, x + y, x^- \in \mathbb{R}^{[n] \times X} \) as \( (\alpha x)(i, x) = \alpha x(i, x), (x + y)(i, x) = x(i, x) + y(i, x), \) and \( x^-(i, x) = \min\{0, x(i, x)\} \) for all \( i \in [n] \) and \( x \in X \), respectively. If \( x(i, x) \leq 0 \) for all \( i \in [n] \) and \( x \in X \) we write \( x \leq 0 \). For \( i \in [n] \) we use \( x(i) \) to denote the function \( x'(i) \in \mathbb{R}^X \) such that \( x'(i, x) = x'(x) \) for all \( x \in X \).

For \( i \in [n] \) and \( a \in A \) let \( \chi_{i,a} \in \mathbb{R}^{[n] \times A} \) be the vector such that \( \chi_{i,a}(i, a) = 1 \) and \( \chi_{i,a}(i', a') = 0 \) for \( (i', a') \neq (i, a) \). (So \( \chi_{i,a} \) is the unit vector for the coordinate \( (i, a) \).) Similarly, we use \( \chi_i \) to denote the vector \( \sum_{a \in A} \chi_{i,a} \). For a vector \( x \in \mathbb{R}^{[n] \times A} \) and tuple \( y \in M^n \) we define

\[
x(y) = \sum_{i=1}^{n} g(x(i), y(i))
\]

where the function \( g : \mathbb{R}^A \times M \to \mathbb{R} \) is defined by

\[
g(x, y) = \begin{cases} 
0 & \text{if } y = 0_M, \\
x(y) & \text{if } y \in A, \text{ and } \\
\max_{a, a' \in A, a \neq a'} x(a) + x(a') & \text{otherwise (if } y = 1_M). 
\end{cases}
\]

(This should be compared to how applying a vector to a subset is defined for submodular set functions, see [6].) For \( x, x' \in \mathbb{R}^{[n] \times A} \) we denote the usual scalar product by \( \langle x, x' \rangle \), so

\[
\langle x, x' \rangle = \sum_{i=1}^{n} \sum_{x \in A} x(i, x)x'(i, x).
\]
Let $f$ be a submodular function on $\mathcal{M}^n$ such that $f(0_M^n) \geq 0$. We define $P_M(f)$ and $B_M(f)$ as follows,

$$P_M(f) = \left\{ x \in \mathbb{R}^{[n] \times A} \mid \forall y \in \mathcal{M}^n, x(y) \leq f(y) \right\},$$

and

$$B_M(f) = \left\{ x \in \mathbb{R}^{[n] \times A} \mid x \in P_M(f), x(1_M^n) = f(1_M^n) \right\}.$$  

Due to the definition of $g$ it is not hard to see that $P_M(f)$ is a polyhedron. Note that if $t$ contains at least one $1_M$, then $t$ induce more than one linear inequality. If $t$ contains no $1_M$, then $t$ only induce one linear inequality. In general, a tuple with $m$ occurrences of $1_M$ induces $|A|^2 m$ linear inequalities. We use $I(t)$ to denote the set of all vectors $e \in \mathbb{R}^{[n] \times A}$ such that $e$ represents an inequality induced by $t$ (that is, an inequality of the form $\langle e, x \rangle \leq f(t)$, where $e \in I(t)$). Given a vector $x \in P_M(f)$ we say that a tuple $t \in \mathcal{M}^n$ such that $x(t) = f(t)$ is $x$-tight.

We will also need the following definition.

**Definition 3.1 (Unified Vector for Diamonds).** A vector $x \in \mathbb{R}^A$ is unified if there is an atom $p \in A$ such that

- if $x, y \in A, x, y \neq p$, then $x(x) = x(y)$; and
- if $a \in A$, then $x(p) \geq x(a)$.

We extend the definition of unified vectors to the vectors in $\mathbb{R}^{[n] \times A}$ by saying that $x \in \mathbb{R}^{[n] \times A}$ is unified if $x \mapsto x(i, x)$ is unified for each $i \in [n]$.

If the submodular inequality is strict for all incomparable pair of elements then we say that the function is *strictly submodular*.

## 4 A Min–Max Theorem

The main results in this section are Theorem 4.3 and Theorem 4.5. We start by a lemma which shows that $B_M(f)$ is non-empty for any submodular function which maps the bottom of the lattice to a non-negative value.

**Lemma 4.1.** Let $f : \mathcal{M}^n \rightarrow \mathbb{R}$ be submodular such that $f(0) \geq 0$. There is a vector $x \in \mathbb{R}^{[n] \times A}$ such that

- $x$ is unified; and
- $x(v_i) = f(v_i)$ for all $i \in [n]$; and
- $x(v_i[i+1 = p_i+1]) = f(v_i[i+1 = p_i+1])$ for all $i \in \{0, 1, \ldots, n-1\}$, where for $i \in [n], p_i$ is the atom in Definition 3.1 for the vector $x \mapsto x(i, x)$.

Furthermore, if $f$ is integer-valued, then $x$ can be chosen to be integer-valued.
Proof. Given a submodular \( f : \mathcal{M}^n \to \mathbb{R} \) we will construct a vector \( \mathbf{x} \) which satisfies the requirements in the lemma. To do this we define a sequence of atoms \( p_i \) for \( i \in [n] \) inductively. To start the inductive definition let \( p_1 \in \max_{a \in A} f(v_0[1 = a]) \) and set \( \mathbf{x}(1, p_1) = f(v_0[1 = p_1]) \). For the general case, choose \( p_i \in A \) so that

\[
p_i = \max_{a \in A} f(v_i[i + 1 = a]).
\]

For \( i \in [n] \) set

\[
\mathbf{x}(i + 1, p_{i+1}) = f(v_i[i + 1 = p_{i+1}]) - f(v_i),
\]

and for \( a \in A, a \neq p_{i+1} \) set

\[
\mathbf{x}(i + 1, a) = f(v_{i+1}) - f(v_i[i + 1 = p_{i+1}]).
\]

**Claim A.** If \( a \in A \), then \( \mathbf{x}(i + 1, p_{i+1}) \geq \mathbf{x}(i + 1, a) \).

Assume, without loss of generality, that \( a \neq p_{i+1} \). We now get

\[
\begin{align*}
f(v_{i+1}) + f(v_i) & \leq f(v_i[i + 1 = p_{i+1}]) + f(v_i[i + 1 = a]) \\
2f(v_i[i + 1 = p_{i+1}]) & \leq f(v_{i+1})
\end{align*}
\]

where the first inequality holds due to the submodularity of \( f \) and the second inequality follows from our choice of \( p_{i+1} \). This is equivalent to

\[
f(v_{i+1}) - f(v_i[i + 1 = p_{i+1}]) \leq f(v_i[i + 1 = p_{i+1}]) - f(v_i)
\]

which is what we wanted to prove. \( \square \)

For \( i \in [n] \) and \( j \in \{1, 2\} \) we define \( c_{i,j} \) as \( c_{i,1} = p_i \) and \( c_{i,2} = 1_\mathcal{M} \).

**Claim B.** \( \mathbf{x}(v_i[i + 1 = c_{i,j}]) = f(v_i[i + 1 = c_{i,j}]) \) for all \( (i, j) \in \{0,1 \ldots , n - 1\} \times \{1, 2\} \).

We prove this by induction over the pairs \((i, j)\) ordered lexicographically (so \((i, j) \leq (i', j')\) if and only if \(i < i'\) or \((i = i'\) and \(j \leq j')\)). With the pair \((i, j)\) we associate the tuple \( v_i[i + 1 = c_{i,j}] \). Note that \((i, j) \leq (i', j')\) if and only if \(v_i[i + 1 = c_{i,j}] \subseteq v_{i'}[i' + 1 = c_{i',j'}] \). As \( p_1 \in \max_{a \in A} f(v_0[1 = a]) \) the claim clearly holds for \((i, j) = (0, 1)\). Now assume that it holds for all pairs \((i', j')\) such that \((i', j') \leq (i, j)\). If \( j = 1 \) then the next pair is \((i, 2)\) and we get

\[
\begin{align*}
\mathbf{x}(v_i[i + 1 = c_{i,2}]) &= \mathbf{x}(v_i[i + 1 = p_{i+1}]) + \mathbf{x}(i + 1, a) \\
&= f(v_i[i + 1 = p_{i+1}]) + f(v_{i+1}) - f(v_i[i + 1 = p_{i+1}]) \\
&= f(v_{i+1}).
\end{align*}
\]

Here the first inequality follows from the definition of \( \mathbf{x}(\cdot) \) and Claim A. The second equality follows from the induction hypothesis and \( \mathbf{x}(\cdot) \). If \( j = 2 \) the next pair is \((i + 1, 1)\) and we get

\[
\begin{align*}
\mathbf{x}(v_{i+1}[i + 2 = c_{i+2,1}]) &= \mathbf{x}(v_{i+1}) + \mathbf{x}(i + 2, p_{i+2}) \\
&= f(v_{i+1}) + f(v_{i+1}[i + 2 = p_{i+2}]) - f(v_{i+1}) \\
&= f(v_{i+1}[i + 2 = p_{i+2}]).
\end{align*}
\]
As above the first equality follows from the definition of $x(\cdot)$ and Claim A. The second equality follows from the induction hypothesis and (8). By Claim A it follows that $x$ is unified. By Claim B $x$ satisfies the second condition in the statement of the lemma. It is easy to see that if $f$ is integer-valued, then so is $x$. □

**Lemma 4.2.** Let $f : \mathcal{M}^n \to \mathbb{R}$ be submodular such that $f(0) \geq 0$. Let $x$ be a vector in $\mathbb{R}^{[n] \times A}$. If for each $i \in [n]$ there is an atom $p_i$ such that

- for all $i \in [n]$ we have $x(v_i) = f(v_i)$, and
- for all $i \in \{0, 1, \ldots, n-1\}$ we have $x(v_i[i+1 = p_{i+1}]) = f(v_i[i+1 = p_{i+1}])$,

then $x \in P_{\mathcal{M}}(f)$.

**Proof.** For $i \in [n]$ and $j \in \{1, 2\}$ we define $c_{i,j}$ as follows $c_{i,1} = p_i$ and $c_{i,2} = 1_{\mathcal{M}}$. We will prove by induction that $x(y) \leq f(y)$ for all $y \in \mathcal{M}^n$. As in the proof of Claim B in Lemma 4.1 the induction will be over the pairs $\{0, 1, \ldots, n-1\} \times \{1, 2\}$ ordered lexicographically. With the pair $(i, j)$ we associate the tuples $y$ such that $y \sqsubseteq v_i[i+1 = p_{i,j}]$.

As $x(v_0) = x(0_{\mathcal{M}^n}) = 0$ and $f(0_{\mathcal{M}^n}) \geq 0$ and

$$x(v_0[1 = p_1]) = f(v_0[1 = p_1])$$

the statement holds for the pair $(0, 1)$ (which corresponds to $y \sqsubseteq 0_{\mathcal{M}^n}[1 = p_1]$). Let $i \in \{0, 1, \ldots, n-1\}$, $j \in \{1, 2\}$, and $y \subseteq \mathcal{M}^n$, $y \sqsubseteq v_i[i+1 = c_{i+1} \sqcup c_j]$ and assume that the inequality holds for all $y' \subseteq \mathcal{M}^n$ such that $y' \sqsubseteq v_i[i' + 1 = c_{i+1, j'}]$ where $(i', j')$ is the predecessor to the pair $(i, j)$. We will prove that the inequality holds for all $y \sqsubseteq v_i[i+1 = c_{i+1, j}]$.

To simplify the notation a bit we let $k = i + 1$ and $y = y(k)$. If $y = 0_{\mathcal{M}}$ we are already done, so assume that $y \neq 0_{\mathcal{M}}$. If $y = p_k$ let $c = 0_{\mathcal{M}}$, if $y \in A$, $y \neq p_k$ let $c = p_k$ and otherwise, if $y = 1_{\mathcal{M}}$ let $c = p_k$. Now,

$$x(y) \leq x(v_i[k = y \sqcup c]) - x(v_i[k = c]) + x(y[k = y \sqcap c]) \leq x(v_i[k = y \sqcup c]) - x(v_i[k = c]) + f(y[k = y \sqcup c]) \leq f(v_i[k = y \sqcup c]) - f(v_i[k = c]) + f(y[k = y \sqcap c]) \leq f(y).$$

The first inequality follows from the supermodularity of $x$. The second inequality follows from the induction hypothesis and the fact that $y \sqcap c \sqsubseteq y$ and $y \sqcap c \in \{0_{\mathcal{M}}, p_k\}$. The third inequality follows from $y \sqcup c, c \in \{0_{\mathcal{M}}, p_k, 1_{\mathcal{M}}\}$ and the assumptions in the statement of the lemma. Finally, the last inequality follows from the submodularity of $f$. □

In the proof of Lemma 4.2, the vector $x \in \mathbb{R}^{[n] \times A}$ is constructed with a greedy approach—we order the coordinates of the vector, $[n] \times A$, in a certain way and then set each component to its maximum value subject to the constraints given.
in the definition of $B_M(f)$. The greedy algorithm does not solve the optimisation problem for $P_M(f)$. As an example, let $M_3 = \{(0,1, a, b, c, \cap, \cup)\}$ be a diamond and let $f : M_3 \rightarrow \mathbb{R}$ be defined as $f(0_M) = 0$, $f(a) = f(b) = f(c) = f(1_M) = 1$. The function $f$ is submodular. Now let $c \in \mathbb{R}^{[1] \times A}$ and $c(1, a) = 1, a = 1, c = 1, A$. From the greedy algorithm we will get a vector $\mathbf{x} \in \mathbb{R}^{[1] \times A}$ such that $\mathbf{x}(1, a) = 1$ and $\mathbf{x}(1, b) = \mathbf{x}(1, c) = 0$ (or some permutation of this vector). However, the solution to max $\langle \mathbf{c}, \mathbf{y} \rangle \mathbf{y} \in P_M(f)$ is $\mathbf{y}(1, a) = \mathbf{y}(1, b) = \mathbf{y}(1, c) = 1/2$ and $3/2 = \langle \mathbf{c}, \mathbf{x} \rangle = 1$. This example also shows that the vertices of $P_M(f)$ are not necessarily integer valued. This should be compared to submodular set functions, where the corresponding optimisation problem is solved by the greedy algorithm.

Given an algorithm which solves the optimisation problem over $P_M(f)$ in time polynomial in $n$ we can use the equivalence of optimisation and separation given by the Ellipsoid algorithm to solve the separation problem for $P_M(f)$ in polynomial time. With such an algorithm we can decide if $0 \in P_M(f)$ or not and by a binary search we can find a minimiser of $f$ in polynomial time. So a polynomial time algorithm for the optimisation problem over $P_M(f)$ would be desirable. (The approach outlined above can be used to minimise submodular set functions, see [11] or, e.g., [12].) We present a pseudo-polynomial algorithm for the optimisation problem in Section 6 which uses this technique.

We are now ready to state the two main theorems of this section.

**Theorem 4.3.** Let $f : M^n \rightarrow \mathbb{R}$ be a submodular function such that $f(0_{M^n}) = 0$, then

$$\min_{\mathbf{x} \in M^n} f(\mathbf{x}) = \max \{z(1_{M^n}) \mid z \in P_M(f), z \leq 0, z \text{ is unified} \}.$$ 

Moreover, if $f$ is integer-valued then there is an integer-valued vector $\mathbf{z}$ which maximises the right hand side.

**Proof.** If $\mathbf{z} \in P_M(f)$ and $z \leq 0$ then

$$z(1_{M^n}) \leq \mathbf{z} \leq f(\mathbf{y})$$

for any $\mathbf{y} \in M^n$. Hence, LHS $\geq$ RHS holds. Consider the function $f' : M^n \rightarrow \mathbb{R}$ defined by

$$f'(\mathbf{x}) = \min_{\mathbf{y} \subseteq \mathbf{x}} f(\mathbf{y}).$$

Then $P_M(f') \subseteq P_M(f)$.

**Claim A.** $f'$ is submodular.

Let $\mathbf{x}', \mathbf{y}' \in M^n$ and let $\mathbf{x} \subseteq \mathbf{x}', \mathbf{y} \subseteq \mathbf{y}'$ be tuples such that $f'(\mathbf{x}') = f(\mathbf{x})$ and $f'(\mathbf{y}') = f(\mathbf{y})$. Now,

$$f'(\mathbf{x}') + f'(\mathbf{y}') = f(\mathbf{x}) + f(\mathbf{y}) \geq f(\mathbf{x} \cap \mathbf{y}) + f(\mathbf{x} \cup \mathbf{y}) \geq f'(\mathbf{x}' \cap \mathbf{y}') + f'(\mathbf{x}' \cup \mathbf{y}')$$

where the first equality follows from the definition of $f'$, $\mathbf{x}$ and $\mathbf{y}$, the first inequality follows from the submodularity of $f$ and the second inequality from the definition of $f'$ and $\mathbf{x} \cap \mathbf{y} \subseteq \mathbf{x}' \cap \mathbf{y}'$ and $\mathbf{x} \cup \mathbf{y} \subseteq \mathbf{x}' \cup \mathbf{y}'$. □
Claim B. For any \( z \in P_M(f') \) we have \( z \leq 0 \).

As \( f(0_M^n) = 0 \) we have \( f'(x) \leq 0 \) for any \( x \in M^n \). For \( i \in [n] \) and \( a \in A \) define \( t_{i,a} \in M^n \) such that \( t_{i,a}(j) = 0_M \) for \( j \in [n], j \neq i \) and \( t_{i,a}(i) = a \). It follows from \( z \in P_M(f') \) that we have \( z(t_{i,a}) = z(i,a) \leq f'(t_{i,a}) \leq 0 \) for any \( a \in A \) and \( i \in [n] \).

Claim C. Any \( z \in B_M(f') \subseteq P_M(f') \) satisfies \( z(1_M^n) = f'(1_M^n) \).

Follows from the definition of \( B_M(f') \).

Finally, \( f'(1_M^n) = \min_{x \in M^n} f(x) \) which follows from the definition of \( f' \). From Lemma 4.1 and Lemma 4.2 it now follows that LHS \( \leq \) RHS holds. To prove the existence of a integer valued vector, note that the vector from Lemma 4.1 is integer valued if \( f' \) is integer valued and \( f' \) is integer valued if \( f \) is integer valued.

We can reformulate Theorem 4.3 to relate the minimum of a submodular function \( f \) to the maximum of a certain function defined over the polyhedron \( \{ x \in P_M(f) \mid x \leq 0 \} \). To do this we define a function \( S : \mathbb{R}^{[n] \times A} \rightarrow \mathbb{R} \) as follows

\[
S(x) = \sum_{i=1}^{n} \min_{a \in A} x(i,a) + \max_{a \in A} x(i,a).
\]

We then get the following corollary.

**Corollary 4.4.**

\[
\min_{y \in M^n} f(y) = \max \left\{ S(z) \mid z \in P_M(f), z \leq 0 \right\}.
\]

**Proof.** Follows from Theorem 4.3 by two observations. If \( z \) is unified, then \( z(1_M^n) = S(z) \). Furthermore, any vector \( z \) can be turned into a unified vector \( z' \) such that \( z' \leq z \) and \( S(z) = z'(1_M^n) \). (To construct \( z' \) from \( z \), for each \( i \in [n] \), choose some \( p_i = \max_{a \in A} z(i,a) \) and let \( z'(i, p_i) = z(i, p_i) \) and for \( a \in A, a \neq p_i \), let \( z'(i, a) = \min_{a \in A} z(i, a) \).)

One might ask if there is any reason to believe that the min–max characterisation given by Theorem 4.3 is the “right” way to look at this problem. That is, can this min–max relation give insight into the complexity of minimising submodular functions over diamonds? Theorem 4.3 is used in Section 5 to get a good characterisation of submodular function minimisation over diamonds, so it certainly gets us somewhere. In Section 6 we present a pseudo-polynomial time algorithm which uses \( P_M(f) \), but it does not use Theorem 4.3. Additionally, Theorem 4.3 is in some sense fairly similar to 1. In particular, in both cases the vectors are functions from the atoms of the lattices to the real numbers and when a vector is applied to a tuple (or a subset) it is computed as a sum over the coordinates of the vector and the tuple. Furthermore, in this sum the bottom of the lattice \( (0_M \) in the diamond case and \( \emptyset \) in the set case) do not contribute to the sum. There are of course differences as well. The most obvious one is, perhaps, that there is no element in the set case analogous to \( 1_M \) in the diamond case. Considering that, as far as we know, all combinatorial algorithms for submodular set function minimisation is based on 1 and the
similarity between Theorem 4.3 and (1) one could hope that Theorem 4.3 could be the basis for a polynomial time combinatorial algorithm for SFM(\(M\)).

The following theorem is an analog to the second equality in Edmonds’ min–max theorem for submodular set functions (1).

**Theorem 4.5.** Let \(f : M \rightarrow \mathbb{R}\) be a submodular function such that \(f(0M) = 0\), then

\[
\min_{x \in M} f(x) = \max \left\{ z(1M) \mid z \in P_M(f), z \leq 0, z \text{ is unified} \right\} = \max \left\{ x^-(1M) \mid x \in B_M(f), x^- \text{ is unified} \right\}.
\]

**Proof.** We prove that

\[
\max \left\{ z(1M) \mid z \in P_M(f), z \leq 0, z \text{ is unified} \right\} = \max \left\{ x^-(1M) \mid x \in B_M(f), x \text{ is unified} \right\}.
\]

The result then follows from Theorem 4.3.

Let \(x\) be a vector which maximises the right hand side. It is clear that \(x^- \in P_M(f), x^- \leq 0\), and that \(x^-\) is unified. It follows that LHS \(\geq\) RHS.

Conversely, let \(z\) be a vector which maximises the left hand side. We will define a sequence of vectors \(x_0, x_1, \ldots\). We start with \(x_0 = z\) and for \(j \geq 0\) we define \(x_{j+1}\) from \(x_j\) according to the construction below.

1. If there is some \(i \in [n]\) and \(p \in \max_{a \in A} x(i, a)\) such that \(\alpha' > 0\) where

\[
\alpha' = \max \{ \alpha \in \mathbb{R} \mid x + \alpha \chi_{i,p} \in P_M(f) \},
\]

then let \(x_{j+1} = x_j + \alpha' \cdot \chi_{i,p}\).

2. Otherwise, if there is some \(i \in [n]\) and \(p \in \max_{a \in A} x(i, a)\) such that \(\alpha' > 0\) where

\[
\alpha' = \max \{ \alpha \in \mathbb{R} \mid x_j + \alpha \cdot (\chi_i - \chi_{i,p}) \in P_M(f) \},
\]

then let \(a\) be some atom distinct from \(p\), let \(m = \min \{ \alpha', x(i, p) - x(i, a) \}\), and let \(x_{j+1} = x_j + m \cdot (\chi_i - \chi_{i,p})\).

We make four observations of this construction.

- If we reach the second step, then \(\max_{a \in A} x(i, a)\) is a one element set.
- For every \(j\) the vector \(x_j\) is unified.
- For every \(j\), \(x_{j+1} \geq x_j\).
- For every \(j\), \(x_j \in P(f)\).
These observations all follow directly from the construction above. It is not hard to convince oneself that there is an integer $m$ such that $x_m = x_{m+1}$ (and thus all vectors constructed after $m$ are equal). To see this, note that for a fixed $i \in [n]$ if some atom $a$ is increased in step 1, then this atom will not be increased again at coordinate $i$. Let $y$ denote the vector $x_m$. Note that $x_{j+1} \sim (1_{M^n}) = x_j \sim (1_{M^n})$ for all $j$. Hence in particular $y \sim (1_{M^n}) \geq z(1_{M^n})$. As we have already proved that LHS $\geq$ RHS it now remains to prove that $y \in B_M(f)$. As we already know that $y \in P_M(f)$ this reduces to proving $y(1_{M^n}) = f(1_{M^n})$.

Let $p$ be a tuple such that for $i \in [n]$ we have $p(i) = \max_{a \in A} y(i, a)$. As $y = x_m = x_{m+1}$, it follows that for each $k \in [n]$ there is an atom $a \in A$, $a \neq p(k)$, and tuples $t_k, t'_k \in M^n, p(k) \subseteq t_k(k), a \subseteq t'_k(k)$ such that $t_k$ and $t'_k$ are $y$-tight. Now let,

\[ t = \bigcup_{k \in [n]} t_k \sqcup t'_k. \]

As $y \in P_M(f)$ it follows from Lemma 5.1 that $y(t) = f(t)$. Note that for each $k \in [n]$ we have $(t_k \sqcup t'_k)(k) = 1_{M^n}$, it follows that $t = 1_{M^n}$ and hence $y \in B_M(f)$. We conclude that LHS $\leq$ RHS. 

5 A Good Characterisation

In this section we show that there are membership proofs for $P_M(f)$ which can be checked in time polynomial in $n$. By using Theorem 4.3 this will lead to the existence of proofs that can be checked in time polynomial in $n$ of the fact that a certain tuple minimises a submodular function. The following lemma states that if $a$ and $b$ are $x$-tight, then so are $a \sqcup b$ and $a \sqcap b$. This simple result will be used repeatedly in the subsequent parts of the paper.

**Lemma 5.1.** Let $f : M^n \rightarrow \mathbb{R}$ be a submodular function. Let $x \in P_M(f)$ be a vector and let $a, b \in M^n$ be $x$-tight tuples. Then, $a \sqcup b$ and $a \sqcap b$ are $x$-tight.

**Proof.**

\[ x(a \sqcup b) + x(a \sqcap b) \leq f(a \sqcup b) + f(a \sqcap b) \leq f(a) + f(b) = x(a) + x(b) \]

The first inequality follows from $x \in P_M(f)$, the second follows from the submodularity of $f$. The equality follows from the assumptions in the lemma. Note that $x(a) + x(b) \leq x(a \sqcup b) + x(a \sqcap b)$. Since $x(a \sqcup b) \leq f(a \sqcup b)$ and $x(a \sqcap b) \leq f(a \sqcap b)$, it follows that $x(a \sqcup b) = f(a \sqcup b)$ and $x(a \sqcap b) = f(a \sqcap b)$. 

The following lemma is an important part of the main result in this section.

**Lemma 5.2.** Let $c \in \mathbb{R}^{[n] \times A}$ and assume that $x$ maximises $\langle x, c \rangle$ over $P_M(f)$. Furthermore, assume that $a, b \in M^n$, $a \subseteq b$ are $x$-tight and for all $t \in M^n$ such that $a \sqcap t \sqsubseteq b$ the tuple $t$ is not $x$-tight. Then, there is at most one coordinate $i \in [n]$ such that $a(i) = 0_M$ and $b(i) = 1_M$. 

13
5 A GOOD CHARACTERISATION

Proof. Assume that there is another coordinate $j \in [n], j \neq i$ such that $a(j) = 0_M$ and $b(j) = 1_M$. We can assume, without loss of generality, that

$$\sum_{x \in A} c(i, x) > \sum_{x \in A} c(j, x).$$

Let $\delta > 0$ and let $x' = x + \delta \chi_i - \delta \chi_j$. We cannot have $x' \in P_M(f)$ for any $\delta > 0$, because then $x$ is not optimal. As $x' \notin P_M(f)$ there is some $x$-tight tuple $t \in M^n$ such that $(t(i) \in A$ and $t(j) = 0_M)$ or $(t(i) = 1_M$ and $t(j) \in \{0_M\} \cup A)$. In either case, it follows from Lemma 5.1 that $t' = (b \cap t) \cup a$ is $x$-tight, which is a contradiction as $a \nsubseteq t' \nsubseteq b$. \hfill \Box

The key lemma of this section is the following result. We will use this lemma together with Lemma 5.2 in the proof of the main result of this section (Theorem 5.8).

Lemma 5.3. Let $n$ be a positive integer and let $f : M^n \to \mathbb{R}$ be submodular which is provided to us by a value-giving oracle. Let $x \in \mathbb{R}^{[n] \times A}$ and $a, b \in M^n$ such that $a \subseteq b$, $a$ is $x$-tight, and there are at most $k$ coordinates $i \in [n]$ such that $a(i) = 0_M$ and $b(i) = 1_M$. Under the assumption that for all $t \subseteq a$ we have $x(t) \leq f(t)$ it can be verified in time $O(n^{k+e})$ that $x(y) \leq f(y)$ holds for all $y \subseteq b$, for some fixed constant $c$.

Proof. Let $I \subseteq [n]$ be the set of coordinates such that $i \in I$ if and only if $a(i) = 0_M$ and $b(i) = 1_M$ and let $J = \{j \in [n] \mid a(j) \neq b(j), j \notin I\}$. Let $Z = \{z \in M^n \mid \forall i \notin I : z(i) = 0_M\}$. For a subset $Y = \{y_1, y_2, \ldots, y_m\}$ of $J$ and $z \in Z$ define $g_z : 2^J \to \mathbb{R}$ as

$$g_z(Y) = f(a[y_1 = b(y_1), \ldots, y_m = b(y_m)] \cup z).$$

We claim that $g_z$ is a submodular set function. Let $z \in Z$ and let $C = \{c_1, c_2, \ldots, c_k\}$ and $D = \{d_1, d_2, \ldots, d_l\}$ be two arbitrary subsets of $J$. Define $c, d \in M^n$ as $a[c_1 = b(c_1), \ldots, c_k = b(c_k)] \cup z$ and $a[d_1 = b(d_1), \ldots, d_l = b(d_l)] \cup z$, respectively. We now get

$$g_z(C) + g_z(D) = f(c) + f(d) \geq f(c \cap d) + f(c \cup d) = g_z(C \cap D) + g_z(C \cup D).$$

Hence $g_z$ is submodular for each $z \in Z$. For a subset $Y = \{y_1, y_2, \ldots, y_m\}$ of $J$ define $h_z : 2^J \to \mathbb{R}$ as

$$h_z(Y) = x[a[y_1 = b(y_1), \ldots, y_m = b(y_m)] \cup z].$$

We claim that $-h_z$ is a submodular set function for each $z \in Z$. As above, let $C = \{c_1, c_2, \ldots, c_k\}$ and $D = \{d_1, d_2, \ldots, d_l\}$ be two arbitrary subsets of $J$ and let $c = a[c_1 = b(c_1), \ldots, c_k = b(c_k)] \cup z$ and $d = a[d_1 = b(d_1), \ldots, d_l = b(d_l)] \cup z$, then

$$h_z(C) + h_z(D) = x(c) + x(d) \leq x(c \cap d) + x(c \cup d) = h_z(C \cap D) + h_z(C \cup D).$$
Hence, \(-h_z\) is submodular. Let \(Y = \{y_1, y_2, \ldots, y_m\}\) be an arbitrary subset of \(J\) and let \(z \in Z\). For a fixed \(k\) the inequalities
\[
\begin{align*}
\quad x(a|y_1) &= b(y_1), \ldots, y_m = b(y_m)] \cup z \leq \\
\quad f(a|y_1) &= b(y_1), \ldots, y_m = b(y_m)] \cup z \\
        \iff \\
\quad 0 &\leq g_z(Y) - h_z(Y) \quad (4)
\end{align*}
\]
can be verified to hold for every \(Y \subseteq J\) and \(z \in Z\) in time polynomial in \(n\) as, for each \(x \in X\), the RHS of (4) is a submodular set function in \(Y\). Conversely, if (4) does not hold for some \(Y \subseteq J\) and \(x \in X\), then there is a tuple \(t \subseteq b\) such that \(x(t) \not\leq f(t)\). To verify that (4) holds for all \(Y \subseteq J\) and \(z \in Z\) find the minimum value of the RHS of (4) for each \(z \in Z\) and compare it to 0 (note that \(|Z|\) only depends on \(k\) and \(|A|\)). This can be done in time polynomial in \(n\) by one of the polynomial time algorithms for submodular function minimisation (see, e.g., [12, 19, 28] for descriptions of these algorithms).

Let \(y \in M^n\) be a tuple such that \(y \not\subseteq b\). Note that if \(a \not\subseteq y\), then it follows from (4) that \(x(y) \leq f(y)\). For the sake of contradiction, assume that \(x(y) \not\leq f(y)\). By the submodularity of \(f\) we get
\[
\quad f(a \sqcup y) + f(a \sqcap y) \leq f(a) + f(y). \quad (5)
\]
As \(a \not\subseteq a \sqcup y \not\subseteq b\) it follows from (4) that \(x(a \sqcup y) \leq f(a \sqcup y)\). Furthermore, \(y \sqcap a \subseteq a\) so by the assumptions in the lemma \(x(a \sqcap y) \leq f(a \sqcap y)\). By the choice of \(a\) and \(y\) we get \(x(a) = f(a)\) and \(f(y) < x(y)\). It follows that
\[
\quad x(a \sqcup y) + x(a \sqcap y) \leq f(a \sqcup y) + f(a \sqcap y) \quad (6)
\]
and
\[
\quad f(a) + f(y) < x(a) + x(y). \quad (7)
\]
But
\[
\quad x(a) + x(y) \leq x(a \sqcup y) + x(a \sqcap y) \quad (8)
\]
so we get a contradiction by combining (6), (5), (7), and (8).

□

Before we prove the main result of this section we need a few basic facts about polyhedrons. Let \(P \subseteq \mathbb{R}^n\) be a polyhedron. The lineality space of \(P\), denoted by lin.space \(P\), is the set of vectors \(x\) such that there is a vector \(y \in P\) and \(\lambda x + y \in P\) for all \(\lambda \in \mathbb{R}\). The characteristic cone of \(P\), denoted by char.cone \(P\), is the set of vectors \(x \in \mathbb{R}^n\) such that for all \(y \in P\) and \(\lambda \geq 0\) we have \(\lambda x + y \in P\).

Given a submodular function \(f\), it is not hard to see that the characteristic cone of \(P_M(f)\) are the vectors \(x \in \mathbb{R}^{|n| \times A}\) such that \(x \leq 0\). Furthermore, the lineality space of \(P_M(f)\) is \(\{0\}\). Given a polyhedron \(P\) such that lin.space \(P = \{0\}\), it is well-known (see, e.g., [27 Chapter 8]) that any \(x \in P\) can be represented
as \( x = \sum_{i=1}^{n+1} \lambda_i y_i + c \) where \( y_1, \ldots, y_{n+1} \) are vertices of \( P \), \( c \in \text{char.cone} P \), \( \sum_{i=1}^{n+1} \lambda_i = 1 \), and \( \lambda_i \geq 0 \) for all \( i \). (That is, \( x \) is the sum of a convex combination of some of the vertices of \( P \) and a vector in the characteristic cone of \( P \).) The fact that \( n + 1 \) vertices suffice is also well-known and is a corollary to Carathéodory’s Theorem \(^2\) (see \(^{27, Chapter 7.7}\) for a proof of the theorem and the corollary). We state this result adapted to our setting as the following theorem.

**Theorem 5.4.** Let \( f : \mathcal{M}^n \to \mathbb{R} \) be submodular and let \( x \in P_M(f) \). Let \( N = n \cdot |A| \). There are vertices \( y_1, \ldots, y_{N+1} \) of \( P_M(f) \), coefficients \( \lambda_1, \ldots, \lambda_{N+1} \in \mathbb{R} \), and a vector \( c \in \mathbb{R}^{[n] \times A} \) such that

\[
x = \sum_{i=1}^{n+1} \lambda_i y_i + c,
\]

\( c \leq 0 \), \( \sum_{i=1}^{N+1} \lambda_i = 1 \), and \( \lambda_i \geq 0 \) for each \( i \in [N + 1] \).

We start with showing that the vertices of \( P_M(f) \) can be encoded in not too many bits. This is needed in the proof of Theorem 5.8.

Let \( m \) be a positive integer. Given a set of vector \( X \subseteq \mathbb{R}^m \) we use \( \text{conv}(X) \) to denote the convex hull of \( X \) and \( \text{cone}(X) \) to denote

\[
\{ \lambda_1 x_1 + \ldots + \lambda_t x_t \mid t \in \mathbb{N}, x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0 \}.
\]

**Definition 5.5** (Facet- and vertex-complexity \(^{12}\)). Let \( P \subseteq \mathbb{R}^m \) be a polyhedron and let \( \phi \) and \( \nu \) be positive integers.

- \( P \) has facet-complexity at most \( \phi \) if there exists a system of linear inequalities with rational coefficients that has solution set \( P \) and such that any equation can be encoded with at most \( \phi \) bits. If \( P = \mathbb{R}^m \) we require that \( \phi \geq m + 1 \).

- \( P \) has vertex-complexity at most \( \nu \) if there exist finite sets \( V \) and \( E \) of rational vectors such that \( P = \text{conv}(V) + \text{cone}(E) \) and such that each vector in \( V \) and \( E \) can be encoded with at most \( \nu \) bits. If \( P = \emptyset \) we require that \( \nu \geq m \).

**Lemma 5.6** (Part of Lemma 6.2.4 in \(^{12}\)). Let \( P \subseteq \mathbb{R}^m \) be a polyhedron. If \( P \) has facet-complexity at most \( \phi \), then \( P \) has vertex-complexity at most \( 4m^2 \phi \).

**Lemma 5.7.** There is a constant \( c \) such that for any submodular \( f : \mathcal{M}^n \to \mathbb{Z} \), the polyhedron \( P_M(f) \) has vertex-complexity at most

\[
c \cdot |A| n^3 \cdot \log \max(|f|).
\]

**Proof.** From the definition of \( P_M(f) \) it follows that \( P_M(f) \) has facet-complexity at most \( c \cdot |A| n \cdot \log \max(|f|) \). for some constant \( c \). The lemma now follows from Lemma 5.6. \( \square \)
Lemma 5.7 tells us that the vertices of $P_M(f)$ can be encoded with not too many bits (that is, the size is bounded by a polynomial in $n$ and $\log \max(|f|)$).

We are now ready to prove the main theorem in this section, that $\mathcal{M}$ is well-characterised.

**Theorem 5.8.** For every $k \geq 3$ the lattice $\mathcal{M}_k$ is well-characterised.

As usual we let $\mathcal{M}$ denote an arbitrary diamond. The idea in the proof is that any point in $P_M(f)$ can be represented as a convex combination of at most $n|A| + 1$ vertices of $P_M(f)$ (this is Carathéodory’s theorem). Furthermore, by Lemma 5.2 and an iterated use of Lemma 5.3 there are membership proofs for the vertices of $P_M(f)$ which can be checked in polynomial time. Hence, we get membership proofs for all of $P_M(f)$ which can be checked efficiently and by Theorem 4.3 we obtain the result.

**Proof.** Let $f : \mathcal{M}^n \to \mathbb{Z}$ be a submodular function and let $m$ be some integer. We will show that if $\min_{t \in \mathcal{M}^n} f(t) = m$, then there is a proof of this fact which can be checked in time polynomial in $n$.

We can assume that $f(0_{\mathcal{M}^n}) = 0$ as $t \mapsto f(t) - f(0_{\mathcal{M}^n})$ is submodular. Let $N = n \cdot |A|$. The proof consists of a tuple $m \in \mathcal{M}^n$, $N + 1$ vectors $x_1, \ldots, x_{N+1} \in \mathbb{R}[n] \times A$, for each $i \in [N+1]$ a sequence $t_i^1, \ldots, t_i^{2n} \in \mathcal{M}^n$ of tuples, and finally an integer-valued vector $c \in \mathbb{R}[n] \times A$. To verify the proof we first find $\lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and $y \in \mathbb{R}[n] \times A$ such that

$$y \leq 0, \quad \sum_{i=1}^{N+1} \lambda_i x_i + y = c, \quad \lambda \geq 0, \quad \text{and} \quad \sum_{i=1}^{N+1} \lambda_i = 1. \quad (9)$$

This can be done in time polynomial in $n$. Reject the proof if there are no solutions to (9). We proceed by checking that for each $i$

- $0_{\mathcal{M}^n} = t_i^1 \subseteq t_i^2 \subseteq \ldots \subseteq t_i^{2n-1} \subseteq t_i^{2n} = 1_{\mathcal{M}^n}$, and
- $t_i^1, \ldots, t_i^{2n}$ are $x_i$-tight, and
- for any $j \in [2n-1]$ there it at most one coordinate $l \in [n]$ such that $t_i^j(l) = 0_M$ and $t_i^{j+1}(l) = 1_M$.

Reject the proof if any of these checks fail. We now want to verify that $x_i \in P_M(f)$, this can be done by using Lemma 5.3 repeatedly. For $j = 1, 2, \ldots, 2n-1$ we use the algorithm in Lemma 5.3 with $a = t_i^j$ and $b = t_i^{j+1}$. If all invocations of the algorithm succeeds we can conclude that $x_i \in P_M(f)$, otherwise the proof is rejected. Finally, compute

$$c = \sum_{i=1}^{N+1} \lambda_i x_i + y$$

and accept the proof if $c \leq 0$, $c$ is unified, and $c(1_{\mathcal{M}^n}) = f(m) = m$.

We now prove that this proof system is sound and complete.
Lemma 5.9. Let $k$ be some fixed positive integer. Let $f: \mathcal{M}^n \to \mathbb{Z}$ be submodular and let $\mathbf{x}$ be a vector in $P_M(f)$. Let $\mathbf{t}_1, \ldots, \mathbf{t}_m \in \mathcal{M}^n$ be $\mathbf{x}$-tight tuples such that $0_{\mathcal{M}^n} = \mathbf{t}_1 \subseteq \ldots \subseteq \mathbf{t}_m = 1_{\mathcal{M}^n}$ and for each $j \in [m-1]$ there is at most $k$ distinct $i_1, i_2, \ldots, i_k \in [n]$ such that $t_j(i_1) = t_j(i_2) = \ldots = t_j(i_k) = 0_M$ and $t_{j+1}(i_1) = t_{j+1}(i_2) = \ldots = t_{j+1}(i_k) = 1_M$.

For $i \in [m]$ let $E_i \subseteq I(t_i)$ such that $\mathbf{e} \in E_i$ if and only if $\langle \mathbf{e}, \mathbf{x} \rangle = f(t_i)$. Given $\mathbf{c} \in \mathbb{Q}[n]^{\times A}$, $\mathbf{x}$, and $\mathbf{t}_1, \ldots, \mathbf{t}_m$ it is possible to compute $\max(\mathbf{c}, \mathbf{y})$ subject to $\mathbf{y} \in P_M(f)$ and $\langle \mathbf{e}, \mathbf{y} \rangle = f(t_i)$ for all $i \in [m]$ and $\mathbf{e} \in E_i$ in time polynomial in $n$, $\log \max(\{|f|\})$ and the encoding length of $\mathbf{c}$.

Note that we do not require that the running time depend polynomially on $k$.

Proof. We construct a separation algorithm for the polyhedron

$$\{ \mathbf{y} \in P_M(f) \mid \forall i \in [m], \mathbf{e} \in E_i : \langle \mathbf{e}, \mathbf{y} \rangle = f(t_i) \}.$$ (10)
Given a vector \( y \in \mathbb{R}^{[n] \times A} \) we first test that for all \( i \in [m] \) and \( e \in E_i \) we have \( \langle e, y \rangle = f(t_i) \). For each \( i \in [m] \) we do this as follows: For each \( j \in [n] \) such that \( t_i(j) = 1_M \) the set of pairs of atoms \( a, b \in A \) such that \( x(j, a) + x(j, b) \) is maximised must be a subset of the set of pairs of atoms \( a', b' \in A \) such that \( y(j, a') + y(j, b') \) is maximised. (Otherwise there is some \( e \in E_i \) such that \( \langle x, e \rangle = f(t_i) \neq \langle y, e \rangle \).) If this is the case then \( \langle y, e \rangle = f(t_i) \) for all \( e \in E_i \); if and only if \( \langle y, e \rangle = f(t_i) \) for some \( e \in E_i \).

Note that this test can be done in polynomial time in \( n \) and \( \log \max(|f|) \) as \( m \leq |A| \cdot n \). We can then use the algorithm in Lemma 5.3 to test if \( y \in P_M(f) \). By combining these two tests we have a separation oracle for \((10)\) and hence the lemma follows.

### 6 Finding the Minimum Value

In this section we will show that there is an algorithm which finds the minimum value of a submodular function \( f : \mathcal{M}^n \to \mathbb{Z} \) in time polynomial in \( n \) and \( \max(|f|) \). Note that from an algorithm which computes \( \min_{t \in \mathcal{M}^n} f(t) \) one can construct an algorithm to find a minimiser of \( f \), i.e., find a tuple \( y \in \mathcal{M}^n \) such that \( f(y) = \min_{t \in \mathcal{M}^n} f(t) \). This can be done by for each \( x \in \mathcal{M} \) minimising \( f_x : \mathcal{M}^{n-1} \to \mathbb{R} \) defined by \( f_x(t) = f(x, t) \). If \( \min_{t \in \mathcal{M}^{n-1}} f_x(t) = \min_{t \in \mathcal{M}^n} f(t) \), then there is a minimiser \( y \in \mathcal{M}^n \) to \( f \) such that \( y(1) = x \). By iterating this procedure \( n \) times one finds a minimiser of \( f \).

We start with a high level description of the algorithm. The starting point is the separation problem for \( P_M(f) \) and the observation that \( 0 \in P_M(f) \) if and only if \( \min_{t \in \mathcal{M}^n} f(t) \geq 0 \). Hence, given an algorithm for deciding if \( 0 \) is contained in \( P_M(f) \) we can apply a binary search strategy to find a minimiser of \( f \). (Note that for any \( c \in \mathbb{R} \) the function \( f + c \) is submodular if \( f \) is submodular.)

In each iteration \( i \) of the algorithm we maintain an upper bound \( u_i \) and lower bound \( l_i \) on \( \min_{t \in \mathcal{M}^n} f(t) \). If \( 0 \in P_M(f - (u_i - l_i)/2) \) (note that \( f - (u_i - l_i)/2 \), i.e., the function \( f' : \mathcal{M}^n \to \mathbb{R} \) defined by \( f'(t) = f(t) - (u_i - l_i)/2 \), is submodular if \( f \) is submodular), we iterate the algorithm with \( u_{i+1} = u_i \) and \( l_{i+1} = (u_i - l_i)/2 \). Otherwise, if \( 0 \not\in P_M(f) \), we set \( u_{i+1} = (u_i - l_i)/2 \) and \( l_{i+1} = l_i \). For an initial upper bound we can use \( u_1 = f(0_{\mathcal{M}^n}) \). To find a lower bound \( l_1 \) we can use Theorem 4.3 together with the greedy algorithm in Lemma 4.4. The running time of this algorithm is \( O(S \log \max(|f|) + n) \), where \( S \) is the time taken to decide if \( 0 \in P_M(f) \).

By the equivalence of separation and optimisation given by the Ellipsoid algorithm it is sufficient to solve the optimisation problem for \( P_M(f) \). (The results we will need which are related to the Ellipsoid algorithm are given in Subsection 6.1. We refer the reader to [12] for an in-depth treatment of the theory related to this topic.) In the optimisation problem we are given \( c \in \mathbb{Q}^{[n] \times A} \) and are supposed to solve \( \max\langle c, y \rangle, y \in P_M(f) \). To get the running time we are aiming for we must do this in time polynomial in \( n \), \( \max(|f|) \) and \( 19 \).
the encoding length of $c$.

To solve this problem our algorithm starts with a vertex of $P_M(f)$ and either finds an adjacent vertex with a strictly better measure or concludes that no such vertex exists. (This technique is called the primal-dual method, see Section 12.1.) The initial vertex is found by the greedy algorithm in Lemma 4.1. To make this approach run in pseudo-polynomial time two parts are needed. The first one is that the existence of a pseudo-polynomial algorithm to go from vertex to a better one or conclude that no such vertex exists. We present an algorithm for this in Section 6.4. The other part is that we must ensure that the algorithm makes enough progress in each iteration so that we get the bound on the running time we are aiming for. To this end, we prove in Section 6.3 that the vertices of $P_M(f)$ are half-integral.

This section is organised as follows: in Subsection 6.1 state some results we will need related to the Ellipsoid algorithm. In Subsection 6.2 we prove a couple of results of the structure of the vertices of $P_M(f)$. We also show that a submodular function can be turned into a strictly submodular function such that any minimiser of the latter is also a minimiser of the former. This will be useful to us in the subsequent parts of the algorithm. In Subsection 6.3 we prove that the vertices of $P_M(f)$ are half-integral. Finally, in Subsection 6.4 we show how we can go from one vertex of $P_M(f)$ to a better one (if there is one) and how this can be used to construct an optimisation algorithm for $P_M(f)$.

### 6.1 The Ellipsoid Algorithm

In this subsection we present some definitions and results which are related to the Ellipsoid algorithm. They are all from the book [12]. As in [12] we make the general assumption that for any oracle $O$ there is an integer $c$ such that when $O$ is given input data of length $n$ the length of the output is $O(n^c)$.

**Definition 6.1** (Oracle-polynomial time). An algorithm $A$, with access to an oracle $O$, runs in oracle-polynomial time if there is an integer $c$ such that given any input of length $n$ $A$ makes $O(n^c)$ calls to $O$ and performs $O(n^c)$ additional primitive operations.

This is definition 6.2.2c in [12].

**Definition 6.2** (Well-described polyhedron). A well-described polyhedron is a triple $(P; n, \phi)$ where $P \subseteq \mathbb{R}^n$ is a polyhedron with facet-complexity at most $\phi$. The encoding length of $(P; n, \phi)$ is $\phi + n$.

This is definition 6.2.1 in [12].

**Definition 6.3** (Strong optimization problem). Given a polyhedron $P \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{Q}^n$, either

- assert that $P$ is empty, or
- find a vector $y \in P$ maximising $\langle c, x \rangle$ over $P$, or
6.2 The Structure of the Vertices of $P_M(f)$

The following lemma is stated in [29] Theorem 2.1 for the boolean lattice. Essentially the same proof works for modular lattices. We give a version of the lemma specialised to $M^n$.

**Lemma 6.6.** Let $t, u \in M^n$ such that $t \not\subseteq u$ and $u \not\subseteq t$, then

$$\rho(t)(2n-\rho(t)) + \rho(u)(2n-\rho(u)) > \rho(t \cap u)(2n-\rho(t \cap u)) + \rho(t \cup u)(2n-\rho(t \cup u))$$

\[ \rho(t)(2n-\rho(t)) + \rho(u)(2n-\rho(u)) > \rho(t \cap u)(2n-\rho(t \cap u)) + \rho(t \cup u)(2n-\rho(t \cup u)) \]

**Proof.** Let $\alpha = \rho(t \cap u), \beta = \rho(t) - \rho(t \cap u), \gamma = \rho(u) - \rho(t \cap u)$, and $\delta = 2n - \rho(t \cup u)$. Then the LHS is equal to

$$2\alpha\beta + 2\beta\gamma + \alpha\beta + \beta\gamma$$

as $\rho$ is modular (i.e., $\rho(t) + \rho(u) = \rho(t \cup u) + \rho(t \cap u)$). The RHS is equal to

$$2\alpha\delta + \alpha\gamma + \beta\delta + \alpha\beta + \gamma\delta.$$  

Since $\beta\gamma > 0$ the lemma follows. \[ \square \]

The lemma above tells us that the function $t \mapsto \rho(t)(2n-\rho(t))$ is strictly submodular. Note that if $f : M^n \rightarrow \mathbb{R}$ is submodular, then $f$ can be turned into a strictly submodular function $f' : M^n \rightarrow \mathbb{R}$ by $f'(t) = f(t) + \epsilon\rho(t)(2n-\rho(t))$. Observe that if $\epsilon > 0$ is chosen small enough then any minimiser of $f'$ is also a minimiser of $f$. Strictly submodular functions are an interesting subset of the submodular functions due to this observation and the following lemma.

**Lemma 6.7.** Let $f : M^n \rightarrow \mathbb{R}$ be strictly submodular and let $x$ be a vertex of $P_M(f)$. Then, the $x$-tight tuples form a chain.

**Proof.** Assume, for the sake of contradiction, that $t, u \in M^n$ are $x$-tight and $t \not\subseteq u$ and $u \not\subseteq t$. It follows that

$$x(t) + x(u) = f(t) + f(u) > f(t \cap u) + f(t \cup u) = x(u \cup v) + x(u \cap v).$$  

(11)
The last equality follows from the fact that the $x$-tight tuples are closed under $\sqcap$ and $\sqcup$. However, (11) contradicts the supermodularity of $x$. □

Lemma 6.7 tells us that, for strictly submodular $f$, for each vertex $x$ of $P_M(f)$ the set of $x$-tight tuples is a chain in $\mathcal{M}^n$. As the dimension of $P_M(f)$ is $|A|n$, for every vertex $x$ of $P_M(f)$ there are $|A|n$ linearly independent inequalities which are satisfied with equality by $x$. This means that for every such $x$ there is a chain $t_1 \sqsupset t_2 \sqsupset \ldots \sqsupset t_m$ in $\mathcal{M}^n$ and linearly independent vectors $e_1, e_2, \ldots, e_{|A|n}$ such that for each $i \in [|A|n]$ there is some $j(i) \in [m]$ such that $e_i \in I(t_{j(i)})$ and for all $i \in [|A|n]$ we have $(e_i, x) = f(t_{j(i)})$. Furthermore, $x$ is the only vector which satisfies $(e_i, x) = f(t_i)$ for all $i \in [|A|n]$.

For general (not necessarily strict) submodular functions the set of $x$-tight tuples is not necessarily a chain, but one can prove that for every vertex there is a chain of tuples such that some subset of the inequalities induced by the tight tuples characterises the vertex. That is, given the subset of inequalities induced by such a chain of tight tuples there is only one point in $P_M(f)$ which satisfies all the inequalities with equality. Formally we state this as the following lemma.

**Lemma 6.8.** Let $f: \mathcal{M}^n \rightarrow \mathbb{R}$ be submodular and let $x$ be a vertex of $P_M(f)$. Then there is a chain $t_1 \sqsupset t_2 \sqsupset \ldots \sqsupset t_m$ in $\mathcal{M}^n$ and linearly independent vectors $e_1, e_2, \ldots, e_{|A|n}$, which are all $x$-tight, such that for each $i \in [|A|n]$ there is some $j(i) \in [m]$ such that $e_i \in I(t_{j(i)})$.

**Proof.** Define $f'(t) = f(t) + \epsilon \rho(t)(2n - \rho(t))$ and choose $\epsilon > 0$ small. From Lemma 6.6 it follows that $f'$ is strictly submodular.

Let $c \in \mathbb{R}^{[n]\times [A]}$ such that $x$ is the unique optimum to $\max\langle c, y \rangle, y \in P_M(f)$. Let $x'$ be a vertex of $P_M(f')$ which is an optimum to $\max\langle c, y \rangle, y \in P_M(f')$. From Lemma 6.7 it follows that as $x'$ is a vertex of $P_M(f')$ there are $t_1 \sqsupset t_2 \sqsupset \ldots \sqsupset t_m$ and $x'$-tight $e_1, \ldots, e_{|A|n}$ as in the statement of the lemma. Let $e_1, \ldots, e_{|A|n}$ be the rows of the matrix $A$ and define $b = (f(t_{j(1)}), \ldots, f(t_{j(m)}))^T$ and

$$\epsilon = \epsilon \cdot \rho(t_{j(1)})(2n - \rho(t_{j(1)})), \ldots, \rho(t_{j(m)})(2n - \rho(t_{j(m)})))^T.$$

It follows that $x' = A^{-1}(b + \epsilon)$. We proceed by establishing two claims.

**Claim A.** $A^{-1}b \in P_M(f)$.

To see this assume for the sake of contradiction that $A^{-1}b \notin P_M(f)$, then there is some $t \in \mathcal{M}^n$ and $e \in I(t)$ such that $eA^{-1}b > f(t)$. However,

$$eA^{-1}b = eA^{-1}(b + \epsilon) = eA^{-1}b + eA^{-1}\epsilon \leq f(t) + \epsilon \rho(t)(2n - \rho(t)). \quad (12)$$

As $eA^{-1}b > f(t)$ we can choose some $\delta > 0$ such that $eA^{-1}b > f(t) + \delta$. By choosing $\epsilon$ so that $|\epsilon \rho(t)(2n - \rho(t)) - eA^{-1}\epsilon| < \delta$ we get

$$eA^{-1}b + eA^{-1}\epsilon > f(t) + \epsilon \rho(t)(2n - \rho(t))$$

which contradicts (12). We conclude that $A^{-1}b \in P_M(f)$. □

**Claim B.** $\langle A^{-1}b, c \rangle = \langle x, c \rangle$.
Assume, for the sake of contradiction, that \( \langle A^{-1} b, c \rangle < \langle x, c \rangle \). (The inequality ≤ follows from our choice of \( x \) and Claim A.) Note that \( x \in P_M(f) \subseteq P_M(f') \). For sufficiently small \( \epsilon \) we get
\[
\langle x', c \rangle = \langle A^{-1}(b + \epsilon), c \rangle < \langle x, c \rangle
\]
which contradicts the optimality of \( x' \).

We have shown that for every vertex \( x \in P_M(f) \) there is some vertex \( x' \in P_M(f') \) which satisfies some inequalities, given by the matrix \( A \), with equality. As \( f' \) is strictly submodular it follows from Lemma 6.7 that the \( x'-\text{tight} \) tuples form a chain. By Claim A the inequalities in \( A \) also defines a point in \( P_M(f) \). Furthermore, by Claim B this point maximises \( \langle y, c \rangle \) over \( P_M(f) \). By our choice of \( c \) it follows that \( A^{-1} b = x \). The lemma follows.

### 6.3 \( P_M(f) \) is Half-integral

In this subsection we will prove that if \( f : \mathcal{M}^n \to \mathbb{R} \) is submodular, then the vertices of \( P_M(f) \) are half-integral.

**Lemma 6.9.** Let \( f : \mathcal{M}^n \to \mathbb{R} \) be submodular and let \( x \) be a vertex of \( P_M(f) \). For each \( i \in [n] \) there are three possibilities

1. \( x(i, a) = x(i, b) \) for all \( a, b \in A \); or
2. there is exactly one atom \( a' \in A \) such that \( x(i, a') > \min_{a \in A} x(i, a) \); or
3. there is exactly one atom \( a' \in A \) such that \( x(i, a') < \max_{a \in A} x(i, a) \).

**Proof.** As \( x \) is a vertex of \( P_M(f) \) there is a \( c \in \mathbb{R}^{[n] \times A} \) such that \( x \) is the unique optimum to \( \langle c, y \rangle, y \in P_M(f) \). As the optimum exist it follows that \( c \geq 0 \). Assume, for the sake of contradiction, that there is a coordinate \( i \in [n] \) such that the statement of the lemma does not hold for \( i \). Let \( A_1 \) be the atoms \( a' \in A \) which satisfies \( x(i, a') = \max_{a \in A} x(i, a) \). Similarly, let \( A_2 \) be the atoms \( a' \in A_2 \) which satisfies \( x(i, a') = \min_{a \in A} x(i, a) \). Finally, let \( A_3 = A \setminus (A_1 \cup A_2) \). We will first prove the following claim.

**Claim.** There are distinct atoms \( b, c \in A \) and \( x \)-tight tuples \( t_1, t_2 \in \mathcal{M}^n \) such that \( t_1(i) = b \) and \( t_2(i) = c \), furthermore \( b \in A_3 \) or \( b, c \in A_2 \).

If \( a \in A_3 \) let \( x' = x + \delta \chi(i, a) \) for some small \( \delta > 0 \). As \( x \) is the unique optimum it follows that \( x' \notin P_M(f) \) and hence there is an \( x \)-tight tuple \( t \in \mathcal{M}^n \) such that \( t(i) = a \). So if \( |A_3| \geq 2 \), then the claim holds. Similarly, if \( |A_1| \geq 2 \), then any for any \( a \in A \setminus A_1 \) we get an \( x \)-tight tuple \( t \) such that \( t(i) = a \). (Again this follows from considering the vector \( x' = x + \delta \chi(i, a) \).)

So \( |A_3| \leq 1 \) and \((|A_1| = 1 \lor |A_1| \geq |A| - 1)\). If \( |A_3| = 0 \) and \((|A_1| = 1 \lor |A_1| \geq |A| - 1)\), then the statement of the lemma holds, so we must have \(|A_3| = 1\). This implies that \(|A_1| = 1\).

Let \( A_1 = \{a\} \). If \( c(i, a) \geq \sum_{b \in A_2} c(i, b) \), then let \( x' = x + \delta \chi(i, a) - \delta \sum_{b \in A_2} \chi(i, b) \) for some small \( \delta > 0 \). It follows that \( x' \notin P_M(f) \) and hence there is an \( x \)-tight tuple \( t \) with \( t(i) = a \). In the other case, when \( c(i, a) < \)}
Let $b$ and $c$ be the atoms in the claim above and let $t_1$ and $t_2$ be the $x$-tight tuples in the claim. As $f$ is submodular we have

$$f(t_1 \cup t_2) + f(t_1 \cap t_2) \leq f(t_1) + f(t_2) = x(t_1) + x(t_2).$$

From this inequality and the fact that $x \in P_M(f)$ it follows that

$$x(t_1 \cup t_2) + x(t_1 \cap t_2) \leq x(t_1) + x(t_2).$$

We conclude that $x(t_1 \cup t_2) + x(t_1 \cap t_2) = x(t_1) + x(t_2)$. However, this leads to a contradiction:

$$x(t_1) + x(t_2) = x(i, b) + x(i, c) + x(t_1[i = 0_M]) + x(t_2[i = 0_M]) \leq x(i, b) + x(i, c) + x((t_1 \cup t_2)[i = 0_M]) + x(t_1 \cap t_2) < x(t_1 \cup t_2) + x(t_1 \cap t_2)$$

So the coordinate $i$ cannot exist.

The lemma above can be strengthened if $|A| = 3$, in this case only 1 and 2 are possible. To see this, assume that $A = \{a_1, a_2, a_3\}$ and $x(i, a_1) = x(i, a_2) > x(i, a_3)$. Let $x' = x + \delta \chi(i, a_1) - \delta \chi(i, a_2)$ (or $x' = x - \delta \chi(i, a_1) + \delta \chi(i, a_2)$ if $c(i, a_1) < c(i, a_2)$). As $x' \not\in P_M(f)$ it follows that there is some $x$-tight tuple $t$ with $t(i) = a_1$ (or $t(i) = a_2$). We can then proceed as in the proof above.

We will need the following lemma from [33] in our proof of the half-integrality of $P_M(f)$.

**Lemma 6.10.** Let $A$ be a $m \times n$ integral matrix satisfying

$$\sum_{i=1}^m |A_{ij}| \leq 2$$

for $j \in [n]$. Then, for every square non-singular submatrix $S$ of $A$, $S^{-1}$ is half-integral.

By combining Lemma 6.7, Lemma 6.9 and Lemma 6.10 we are able to obtain the following theorem which asserts the half-integrality of $P_M(f)$.

**Theorem 6.11.** Let $f : M^n \to \mathbb{Z}$ be submodular. For any vertex $x$ of $P_M(f)$ and any $i \in [n]$ and $a \in A$ we have $x(i, a) \in \{1/2 \cdot k \mid k \in \mathbb{Z}\}$.

**Proof.** Let $x$ be a vertex of $P_M(f)$. By Lemma 6.8 there is a chain of $x$-tight tuples $t_1 \sqcup \ldots \sqcup t_m$ and linearly independent vectors $e_1, \ldots, e_{|A|n}$ such that for each $i \in [|A|n]$ there is some $j(i) \in [m]$ such that $e_i \in I(t_{j(i)})$. We can
also assume that for \( i \leq i' \) we have \( j(i) \leq j(i') \). Let \( E \) be the matrix with rows \( e_1, \ldots, e_{|A|n} \), then \( x \) is the unique solution to \( Ex = b \), where

\[
b = (f(t_{j(1)}), f(t_{j(2)}), \ldots, f(t_{j(|A|n)}))^T.
\]

Let \( A = \{a_1, a_2, \ldots, a_{|A|}\} \). By Lemma 6.9 we can assume, without loss of generality, that and \( x(i, a_2) = x(i, a_3) = \ldots = x(i, a_{|A|}) \). If \( x(i, a_1) = x(i, a_2) = \ldots = x(i, a_{|A|}) \) we can identify \( x(i, a_1), \ldots, x(i, a_{|A|}) \) without changing the set of solutions to \( Ex = b \), in the other case when \( x(i, a_1) > \min_{a \in A} x(i, a) \) or \( x(i, a_1) < \max_{a \in A} x(i, a) \) we can identify \( x(i, a_2), \ldots, x(i, a_{|A|}) \) without changing the set of solutions to \( Ex = b \). After having identified these variables we get a system of linear equations, \( E'x' = b' \), which has a unique solution. Furthermore, the solution to \( E'x' = b' \) is half-integral if and only if \( E\overline{x} = b \) has a half-integral solution (that is, if and only if \( x \) is half-integral).

Let \( X \subseteq [n] \times \{1, 2\} \) such that for each \( i \in [n], (i, 1) \in X \) and \( (i, 2) \in X \) if and only if \( x(i, a_1) > \min_{a \in A} x(i, a) \) or \( x(i, a_1) < \max_{a \in A} x(i, a) \). We can describe the rows, \( e_1', e_2', \ldots, e_{|A|n}' \in \mathbb{R}^X \) of \( E' \) as follows

- if \( x(i, a_1) = x(i, a_2) = \ldots = x(i, a_{|A|}) \), then \( e_j'(i, 1) = \sum_{a \in A} e_j(i, a) \);
- otherwise (if \( x(i, a_1) > \min_{a \in A} x(i, a) \) or \( x(i, a_1) < \max_{a \in A} x(i, a) \)), then \( e_j'(i, 1) = e_j(i, a_1) \) and \( e_j'(i, 2) = \sum_{a \in A, a \neq a_1} e_j'(i, a) \).

As the solution to \( E'x' = b' \) and \( E\overline{x} = b \) are equal, modulo the identification of some of the variables, there is a subset \( R = \{r_1, r_2, \ldots, r_{|X|}\} \subseteq [|A|n] \) with \( r_1 < r_2 < \ldots < r_{|X|} \) such that the matrix \( E'' \), with rows \( \{e_i' \mid i \in R\} \), has an inverse. Furthermore, this inverse is half-integral (that is, \( E''^{-1} \) is half-integral) if and only if the solution to \( E'x' = b' \) is half-integral.

It is easy to see that the entries of \( E'' \) are contained in \( \{0, 1, 2\} \). Furthermore, if \( c \) is an arbitrary column of \( E'' \), then it is of the form \( (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2)^T \) or \( (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T \) (in these patterns, \( x, \ldots, x \) means that \( x \) occurs zero or more times). It follows that for each \( (i, k) \in X \) we have

\[
\sum_{i=1}^{|X|} |e_{r_{i+2}'}(i, k) - e_{r_1'}(i, k)| \leq 2.
\]

Following the proof of Theorem 1 in [10] we now define

\[
U = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 1
\end{pmatrix}.
\]

We can then express the inverse of \( E'' \) as \((UE'')^{-1}U\). By (13) and Lemma 6.10 it follows that \((UE'')^{-1}\) is half-integral and hence \(E''^{-1}\) is half-integral as well, which implies that \( x \) is half-integral. \(\square\)
6.4 Finding Augmentations

Let $f : \mathcal{M}^n \to \mathbb{Z}$ be submodular. In this section we will show that there is an algorithm which decides if $0 \in P_M(f)$ in time polynomial in $n$ and max$(|f|)$. The strategy of the algorithm is to use the equivalence between separation and optimisation given by the Ellipsoid algorithm and solve the optimisation problem for $P_M(f)$ instead. In the optimisation problem we are given $c \in \mathbb{Q}^{[n] \times \mathcal{A}}$ and are supposed to find max$(\langle c, y \rangle, y \in P_M(f))$. This problem is solved by iterating an augmentation step in which we are in some vertex $x$ of $P_M(f)$ and wish to find some vertex $x'$, adjacent to $x$, such that $\langle c, x' \rangle > \langle c, x \rangle$.

Let $c \in \mathbb{Q}^{[n] \times \mathcal{A}}$ and assume that we want to solve max$(\langle c, y \rangle, y \in P_M(f))$. Let $T$ be the set of all $x$-tight tuples and let $E \subseteq \bigcup_{t \in T} I(t)$ such that $e \in E$ if and only if there is some $t \in T$ with $e \in I(t)$ and $\langle e, x \rangle = f(t)$. Finding a vector $y \in \mathbb{R}^{[n] \times \mathcal{A}}$ such that there is some $\delta > 0$ which satisfies $\langle c, x \rangle < \langle c, x + \delta y \rangle$ and $x + \delta y \in P_M(f)$ or conclude that no such vector $y$ exists is equivalent to solving the linear program

$$\max \langle c, z \rangle \text{ subject to } \forall e \in E : \langle e, z \rangle \leq 0 \text{ and } \langle c, z \rangle \leq 1. \quad (14)$$

(Here $z$ contains the variables.) The optimum of this linear program is 0 if $x$ is optimal and 1 otherwise. The separation problem for this polyhedron reduces to computing

$$\max_{e \in E} \langle e, z \rangle.$$

Define $f' : \mathcal{M}^n \to \mathbb{Z}$ as $f'(t) = (n^2 + 1) \cdot f(t) + \rho(t)(2n - \rho(t))$. It is not hard to see that a minimiser of $f'$ is also a minimiser of $f$. Furthermore, by Lemma 6.6 $f'$ is strictly submodular. When minimising submodular functions we can thus assume that the function is strictly submodular. By Lemma 6.7 if $x$ is a vertex of $P_M(f')$, then $T$ (the $x$-tight tuples) is a chain. This implies that $|T| \leq 2n$.

Lemma 6.12. If $f : \mathcal{M}^n \to \mathbb{Z}$ is strictly submodular and $x$ a vertex of $P_M(f)$, then the linear program (14) can be solved in time polynomial in $n$, log max$(|f|)$ and the encoding length of $c$. (Assuming that $T$ is available to the algorithm.)

Proof. As $f$ is strictly submodular it follows from Lemma 6.7 that $|T| \leq 2n$. Hence, the separation problem for (14) can be solved in polynomial time. By the equivalence of separation and optimisation given by the Ellipsoid algorithm it follows that (14) can be solved in time polynomial in $n$, log max$(|f|)$ and the encoding length of $c$. (Note that even though $|T| \leq 2n$, the number of inequalities in $E$ may be exponential in $n$. In particular the tuple $1_{\mathcal{M}^n}$ can induce as many as $\binom{[2]}{n}$ inequalities.)

By the algorithm in Lemma 6.12 we can find an optimal solution $z$ to (14). We can use this algorithm to find adjacent vertices which are better (if there are any). We also need to find the largest $\delta > 0$ such that $x + \delta z \in P_M(f)$. We construct an algorithm for this in Lemma 6.14 but first we need a lemma.
Lemma 6.13. Let $y$ be an optimal solution to (14) which is a vertex such that $\langle c, y \rangle = 1$. Assume that there are $t_1, t_2 \in M^n$, $t_1 \sqsubset t_2$, and $e_1 \in E \cap I(t_1), e_2 \in E \cap I(t_2)$ such that $(e_1, y) = f(t_1)$ and $(e_2, y) = f(t_2)$. Furthermore, assume that there is no $u \in T$ such that $t_1 \sqsubset u \sqsubset t_2$ with any $e \in E \cap I(u)$ and $(e, y) = f(u)$. Then, there are no three distinct coordinates $i, j, k \in [n]$ such that $t_1(i) = t_1(j) = t_1(k) = 0, t_2(i) = t_2(j) = t_2(k) = 1$.

Proof. Let $E' \subseteq E$ be the vectors which define tight inequalities for $y$. As $y$ is a vertex and $\langle c, y \rangle = 1$, it follows that the polyhedron $P = \{ z \in \mathbb{R}^{n \times A} | \langle e, z \rangle = f(t), e \in E', e \in I(t) \}$ is one dimensional.

Let $\alpha, \beta \in \mathbb{R}$ be arbitrary and define $y' \in \mathbb{R}^{n \times A}$ by

$$y' = y + (\alpha + \beta)x_i - \alpha x_j - \beta x_k.$$ 

From the non-existence of any $u \in M^n$ such that $t_1 \sqsubset u \sqsubset t_2$ and $e \in E \cap I(u), (e, y) = f(u)$ it follows that $(e, y') = f(t)$ for all $e \in E', e \in I(t)$. However, this means that $y' \in P$ and as $\alpha$ and $\beta$ were arbitrary it follows that $P$ is not one-dimensional. This is a contradiction and the lemma follows. \hfill \Box

The following lemma is a crucial part of our pseudo-polynomial time algorithm for $SFM(M)$. With the algorithm in this lemma we are able to go from one vertex in $P_M(f)$ to a better one (if there is a better one).

Lemma 6.14. Let $f : M^n \rightarrow \mathbb{Z}$ be a strictly submodular function. Given $c \in \mathbb{Q}^{n \times A}$, a vertex $x$ of $P_M(f)$, and the set of $x$-tight tuples $T$, there is an algorithm which is polynomial in $n$, $\log \max(|f|)$ and the encoding length of $c$ which finds a vertex $y \in P_M(f)$ such that $\langle c, y \rangle > \langle c, x \rangle$ or concludes that no such vertex exist. If $y$ exists the set of $y$-tight tuples can be computed within the same time bound.

Proof. If there is such a vertex $y$, then the value of the optimum of the linear program (14) is 1. By Lemma 6.12 this optimum $y'$ can be found in polynomial time. The set of tuples $T' \subseteq T$ which are $y'$-tight can be found in polynomial time (as $|T| \leq 2n$). Furthermore, by Lemma 6.13 the gap between two successive tuples in $T'$ is not too large. It follows from Lemma 5.9 that we can find a vertex $y$ of $P_M(f)$ such that $\langle c, y \rangle > \langle c, x \rangle$ in polynomial time.

It remains to find the rest of the $y'$-tight tuples within the stated time bound. By Lemma 6.13 for any consecutive tuples $a, b$ in $T'$ there are at most two distinct coordinates $i, j \in [n]$ such that $a(i) = a(j) = 0_M$ and $b(i) = b(j) = 1_M$. We will show that for every such pair $a, b$ in $T'$ we can find the $y$-tight tuples $t$ which satisfies $a \sqsubset t \sqsubset b$. To do this, for each $p, q \in M$, we find the minimisers to the submodular function $f_{p, q}$ defined as $f_{p, q}(x) = f(x[i = p, j = q]) - y(x[i = p, j = q])$ over the set $X = \{ x \in M^n | a \sqsubset x \sqsubset b \}$. As $f$ is submodular and $y$ is supermodular it follows that $f'$ is submodular. To minimise $f_{p, q}$ over $X$ we can minimise it over at most $n^2 |A|^2$ intervals defined by $\{ x \in M^n | a^* \sqsubseteq x \sqsubseteq b_\ast \}$ where $a \prec a^*$ and $b_\ast \prec b$ (there are at most $n|A|$ choices for $a^*$ and at most $n|A|$ choices for $b_\ast$).

Note that each of these intervals is a product of the two element lattice and hence this minimisation can be done with the known algorithms for minimising...
submodular set functions. We can use this method to find all minimisers of \( f_{p,q} \) in the interval we are interested in. When we have found one minimiser \( m \) we iteratively minimise \( f_{p,q} \) over the sets \( \{ x \in M^n \mid a \subseteq x \subseteq m \} \) and \( \{ x \in M^n \mid m \subseteq x \subseteq b \} \). As the \( y \)-tight tuples is a chain in \( M^n \) there are only a polynomial number of \( y \)-tight tuples and hence this step of the algorithm runs in polynomial time. Hence the set of all \( y \)-tight tuples can be found within the stated time bound.

We are now finally ready to show the existence of a pseudo-polynomial time separation algorithm for \( P_M(f) \).

**Theorem 6.15.** Let \( f : M^n \to \mathbb{Z} \) be submodular. It is possible to decide if \( 0 \) is contained in \( P_M(f) \) or not in time polynomial in \( n \) and \( \max(|f|) \).

**Proof.** By the equivalence of separation and optimisation given by the Ellipsoid algorithm there is an algorithm which decides if \( 0 \) is contained in \( P_M(f) \) or not which makes use of an optimisation oracle for \( P_M(f) \). The number of calls to the optimisation oracle is bounded by a polynomial in \( n \) and \( \log \max(|f|) \), furthermore the objective function given to the optimisation oracle is given by a vector \( c \in \mathbb{Q}^{[n] \times A} \) such that the encoding length of \( c \) is bounded by a polynomial in \( n \) and \( \log \max(|f|) \).

To prove the lemma it is therefore sufficient to construct an algorithm such that given \( c \in \mathbb{Z}^{[n] \times A} \) (there is no loss of generality in assuming that \( c \) is integral, a simple scaling of \( c \) achieves this) it solves \( \max(y,c), y \in P_M(f) \) in time polynomial in \( n \), \( \max(|f|) \) and the size of the encoding of \( c \). Let \( f'(t) = (n^2 + 1) \cdot f(t) + \rho(t)(2n - \rho(t)) \). By Lemma 6.6 \( f' \) is strictly submodular. Furthermore, it is easy to see that any minimiser of \( f' \) is also a minimiser of \( f \). By Lemma 6.7 each vertex \( x \) of \( P_M(f') \) is “characterised” of a chain of \( x \)-tight tuples.

The algorithm consists of a number of iterations. In iteration \( j \) a current vertex \( x_j \) of \( P_M(f') \) is computed together with its associated chain \( C_j \) of \( x_j \)-tight tuples. The initial vertex \( x_0 \) and initial chain \( C_0 \) is computed by the greedy algorithm from Lemma 6.11.

In iteration \( j \), either \( x_j \) is the optimum or there is some other vertex \( x_{j+1} \) such that \( \langle x_{j+1}, c \rangle > \langle x, c \rangle \). To find such an \( x_{j+1} \) or conclude that no such vertex exists we use the algorithm from Lemma 6.14. In the case when \( x_{j+1} \) exists we also get the chain \( C_{j+1} \) of \( x_{j+1} \)-tight from the algorithm in Lemma 6.14.

By Theorem 6.11 the vertices of \( P_M(f) \) are half-integral. This implies that \( \langle x_{j+1}, c \rangle \geq \langle x_j, c \rangle + 1/2 \). So the algorithm is polynomial if we can prove that the optimum value is not too far from the starting point \( x_0 \). That is, the difference between \( \langle c, x_0 \rangle \) and \( \max\langle c, y \rangle, y \in P_M(f) \) should be bounded by a polynomial in \( n \), \( \max(|f|) \) and the encoding length of \( c \). Note that as the size of the encoding of \( c \) is bounded by a polynomial in \( n \) and \( \log \max(|f|) \) it follows that \( \max_{c \in [n], a \in A} |c(i,a)| \) is bounded by a polynomial in \( n \) and \( \max(|f|) \). Furthermore, as \( x_0 \) is obtained by the greedy algorithm it follows that for any
\[i \in [n], a \in A \text{ we have } -2 \max(|f|) \leq x_0(i, a). \text{ We now obtain the inequality}
\]
\[-2 \max(|f|) \cdot n|A| \left( \max_{i \in [n], a \in A} |c(i, a)| \right) \leq \langle c, x_0 \rangle \leq \langle c, y \rangle \leq
\]
\[\max(|f|) \cdot n|A| \left( \max_{i \in [n], a \in A} |c(i, a)| \right). \]

From this inequality and the fact that \(\max_{i \in [n], a \in A} |c(i, a)|\) is bounded by a polynomial in \(n\) and \(\max(|f|)\) it follows that the difference between \(\langle c, x_0 \rangle\) and \(\langle c, y \rangle\) is bounded by a polynomial in \(n\) and \(\max(|f|)\). As the objective function increases by at least \(1/2\) in each iteration this implies that the number of iterations is bounded by a polynomial in \(n\) and \(\max(|f|)\). \(\Box\)

From Theorem \ref{thm:main} we now get our desired result, a pseudo-polynomial time algorithm for minimising submodular functions over diamonds. The proof of this final step was given in Section \ref{sec:proof}.

7 Conclusions and Open Problems

The most obvious open problem is to find a polynomial time algorithm, as opposed to a pseudo-polynomial time algorithm established in this paper, for minimising submodular functions over diamonds. One possible approach may be to use some kind of scaling technique see, e.g., \cite{16,19}. The pseudo-polynomial algorithm as it is presented here is very inefficient: it consists of a nested application of the Ellipsoid algorithm. Usually, one layer of the Ellipsoid algorithm is considered to be too inefficient to be used in practise. It would clearly be desirable to have a simpler and more efficient minimisation algorithm.

References

[1] J.-C. Anglès-d’Auriac, F. Iglói, M. Preissmann, and Á. Sebő. Optimal cooperation and submodularity for computing potts’ partition functions with a large number of states. \textit{Journal of Physics A: Mathematical and General}, 35:6973–6983(11), 2002.

[2] C. Carathéodory. Über den variabilitätsbereich der fourierschen konstanten von positiven harmonischen funktionen. \textit{Rend. Circ. Mat. Palermo}, 32:193–217, 1911.

[3] D. Cohen, M. Cooper, P. Jeavons, and A. Krokhin. Supermodular functions and the complexity of Max CSP. \textit{Discrete Appl. Math.}, 149(1-3):53–72, 2005.

[4] D. Cohen, M. Cooper, P. Jeavons, and A. Krokhin. The complexity of soft constraint satisfaction. \textit{Artificial Intelligence}, 170(11):909–1030, 2006.

[5] V. Deineko, P. Jonsson, M. Klasson, and A. Krokhin. The approximability of Max CSP with fixed-value constraints. \textit{J. ACM}, 55(4):1–37, 2008.
[6] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In R. Guy, H. Hanani, N. Sauer, and J. Schönheim, editors, Combinatorial Structures and Their Applications. Gordon and Breach, 1970.

[7] A. Frank. Submodular functions in graph theory. Discrete Math., 111(1-3):231–243, 1993.

[8] S. Fujishige. Theory of submodular programs: A fenchel-type min-max theorem and subgradients of submodular functions. Math. Program., 29(2):142–155, 1984.

[9] S. Fujishige. Submodular Functions and Optimization, volume 58 of Ann. Discrete Math. Elsevier Science, 2 edition, 2005.

[10] D. Gijswijt and G. Pap. An algorithm for weighted fractional matroid matching. Technical report, EGRES TR-2008-11, MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Dept. of Operations Research, Eötvös University, Budapest, 2008.

[11] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. Combinatorica, 1(2):169–197, 1981.

[12] M. Grötschel, L. Lovász, and A. Schrijver. Geometric Algorithms and Combinatorial Optimization, volume 2 of Algorithms and Combinatorics. Springer, 1988.

[13] T. S. Han. The capacity region of a general multiple access channel with certain correlated sources. Inf. Control, 40:37–60, 1979.

[14] D. S. Hochbaum, N. Megiddo, J. Naor, and A. Tamir. Tight bounds and 2-approximation algorithms for integer programs with two variables per inequality. Mathematical Programming, 62:69–83, 1993.

[15] H. Ito, S. Iwata, and K. Murota. Block-triangularizations of partitioned matrices under similarity/equivalence transformations. SIAM J. Matrix Anal. Appl., 15(4):1226–1255, 1994.

[16] S. Iwata. A capacity scaling algorithm for convex cost submodular flows. In Proceedings of the seventh Annual ACM-SIAM Symposium on Discrete algorithms (SODA ’96), pages 482–489, Philadelphia, PA, USA, 1996. Society for Industrial and Applied Mathematics.

[17] S. Iwata. A faster scaling algorithm for minimizing submodular functions. In Proceedings of the 9th International IPCO Conference on Integer Programming and Combinatorial Optimization (IPCO ’02), volume 2337 of Lecture Notes in Computer Science, pages 1–8. Springer-Verlag, 2002.

[18] S. Iwata. Submodular function minimization. Math. Program., 112:45–64, 2008.
[19] S. Iwata, L. Fleischer, and S. Fujishige. A combinatorial strongly polynomial algorithm for minimizing submodular functions. *J. ACM*, 48(4):761–777, 2001.

[20] S. Iwata and K. Murota. A minimax theorem and a Dulmage–Mendelsohn type decomposition for a class of generic partitioned matrices. *SIAM J. Matrix Anal. Appl.*, 16(3):719–734, 1995.

[21] P. Jonsson, M. Klasson, and A. Krokhin. The approximability of three-valued Max CSP. *SIAM J. Comput.*, 35(6):1329–1349, 2006.

[22] A. Krokhin and B. Larose. Maximizing supermodular functions on product lattices, with application to maximum constraint satisfaction. *SIAM J. Discret. Math.*, 22(1):312–328, 2008.

[23] L. Lovász. Submodular functions and convexity. In A. Bachem, M. Grötschel, and B. Korte, editors, *Mathematical Programming—The State of the Art*, pages 235–257. Springer, 1982.

[24] S. T. McCormick. Submodular function minimization, 2007. Version 3a. Based on Chapter 7 of the *Handbook on Discrete Optimization*, Elsevier, K. Aardal, G. Nemhauser, and R. Weismantel, editors, 321–391.

[25] J. B. Orlin. A faster strongly polynomial time algorithm for submodular function minimization. In *Integer Programming and Combinatorial Optimization, 12th International IPCO Conference (IPCO ’07)*, volume 4513 of *Lecture Notes in Computer Science*, pages 240–251. Springer-Verlag, 2007.

[26] P. Raghavendra. Optimal algorithms and inapproximability results for every CSP? In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC ’08)*, pages 245–254, New York, NY, USA, 2008. ACM Press.

[27] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, Inc., New York, NY, USA, 1986.

[28] A. Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *J. Comb. Theory Ser. B*, 80(2):346–355, 2000.

[29] A. Schrijver. *Combinatorial Optimization – Polyhedra and Efficiency*. Springer, 2003.

[30] L. S. Shapley. Cores of convex games. *Internat. J. Game Theory*, 1(1):11–26, 1971.