Some remarks on invariant subspaces in real Banach spaces (revised version)

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Abstract.

It is proved that a commutative algebra $A$ of operators on a reflexive real Banach space has an invariant subspace if each operator $T \in A$ satisfies the condition

$$\|1 - \varepsilon T^2\|_e \leq 1 + o(\varepsilon) \quad \text{when} \quad \varepsilon \searrow 0,$$

where $\| \cdot \|_e$ is the essential norm. This implies the existence of an invariant subspace for every commutative family of essentially selfadjoint operators on a real Hilbert space.

1 Introduction

Notations: for a complex or real Banach space $X$ we denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on $X$; its ideals of all compact and all finite rank operators are denoted by $\mathcal{K}(X)$ and $\mathcal{F}(X)$ respectively. The quotient $\mathcal{C}(X) = \mathcal{B}(X)/\mathcal{K}(X)$ is called the Calkin algebra; the standard epimorphism of $\mathcal{B}(X)$ onto $\mathcal{C}(X)$ is denoted by $\pi$. The essential norm $\|T\|_e$ of $T \in \mathcal{B}(X)$ is the norm of $\pi(T)$ in $\mathcal{C}(X)$. In other words,

$$\|T\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}.$$

One of the most known unsolved problems in the invariant subspace theory in Hilbert spaces is the existence of a (non-trivial, closed) invariant subspace for an operator $T$ with compact imaginary part; such operators are characterized by the condition $T^* - T \in \mathcal{K}(X)$ and usually called essentially selfadjoint. It is difficult to list all the papers devoted to this subject; we only mention that the answer is affirmative if $T - T^*$ belongs to some Shatten - von Neumann class $\mathcal{S}_p$ (Livshits [11] for $p = 1$, Sahnovich [23] for $p = 2$, Gohberg and Krein [6], Macaev [18], Schwartz [24] — for arbitrary $p$), or, more generally, to the Macaev ideal (Macaev [19]). But the general problem is still open. See the review [17, Section 3] for more information.

It was proved in [14] that every essentially self-adjoint operator on a complex Hilbert space has an invariant real subspace. Then in [13] the following general theorem of Burnside type was proved:

**Theorem 1.1** Let an algebra $A$ of operators on a real or complex Banach space $X$ be non dense in $\mathcal{B}(X)$ with respect to the weak operator topology (WOT). Then there are non-zero $x \in X^{**}, y \in X^*$, such that

$$|(x, T^* y)| \leq \|T\|_e, \quad \text{for all} \quad T \in A.$$  

(1.1)
Using this result and developing a special variational techniques, Simonic [25] has obtained a significant progress in the topic: he proved that each essentially selfadjoint operator on a real Hilbert space has invariant subspace. Deep results based on Theorem 1.1 and variational methods were established then in papers of Atzmon [1], Atzmon, Godefroy [2], Atzmon, Godefroy, Kalton [3], Grivaux [8] and other mathematicians. Here we will show that every commutative family of essentially selfadjoint operators on a real Hilbert space has an invariant subspace, and consider some analogs of this result for operators on Banach spaces. Our proof is very simple and short — modulo Theorem 1.1. More precisely, we use an improvement of Theorem 1.1 proved in [15] for the case of complex Banach spaces. To formulate it, we introduce some notation.

For a pair of non-zero vectors \( x \in X^\ast \), \( y \in X^\ast \), let us denote by \( \omega_{x,y} \) the "vector functional" on \( B(X) \), acting by the rule

\[
\omega_{x,y}(T) = (x, T^* y).
\]

We say that a vector functional \( \omega_{x,y} \) is resolving for an algebra \( A \subset B(X) \), if

\[
|(x, T^* y)| \leq \|T\|_e(x, y), \text{ for all } T \in A.
\]  

(1.2)

**Theorem 1.2** Let a Banach algebra \( A \subset B(X) \) on a Banach space \( X \) over \( K \in \{\mathbb{R}, \mathbb{C}\} \) is not WOT-dense in \( B(X) \). Then at least one of the following two conditions holds:

a) \( A \) has a resolving vector functional,

b) \( K = \mathbb{R} \) and \( A \) contains an idempotent of finite rank.

As it was noted above, the "complex part" of Theorem 1.2 was proved in [15]. In fact a part of the proof in [15] worked only for reflexive \( X \); in the process of our proof we correct this inaccuracy, applying the Bishop-Phelps theorem.

Here Theorem 1.2 will be deduced from the following statement:

**Theorem 1.3** Let \( X \) be a real or complex Banach space. If a norm-closed algebra \( A \subset B(X) \) does not contain non-zero finite rank operators, then it has a resolving vector functional.

2 Preliminary results

**Lemma 2.1** Let \( A \) be a commutative unital algebra of operators on a real Banach space \( X \). Suppose that there are non-zero \( x_0 \in X \), \( y_0 \in X^\ast \) such that

\[
(T^2 x_0, y_0) \geq 0, \text{ for all } T \in A.
\]

Then \( A \) has a (closed non-trivial) invariant subspace.

**Proof.** Let \( K \) be the closed convex hull of the set

\[
M = \{T^2 x_0 : T \in A\} \subset X.
\]

Since \( M \) is invariant under all operators \( T^2 \), \( T \in A \), the same holds for \( K \). Furthermore \( K \neq X \), because \( (z, y_0) \geq 0 \), for all \( z \in K \). Clearly \( K \neq \{0\} \) since \( A \) is unital. If \( \partial K \) is a singleton \( \{w_0\} \) then \( w_0 = 0 \) because otherwise \( tw_0 \in \partial K \), for \( t > 0 \). Let \( w \) be a non-zero element of \( K \) and let
$q \in X$ be non-proportional to $w$ with $(q, y_0) \neq 0$. Then the line $w + Rq$ intersects $K$ in a point that does not belong to $\partial K$, a contradiction.

So we may assume that $\partial K$ is not a singleton. By the Bishop-Phelps theorem [5], $K$ has non-zero support points. So there is a non-zero functional $y_1 \in X^*$ attaining its minimum on $K$ at some non-zero point $x_1 \in K$:

$$(x_1, y_1) \leq (w, y_1), \text{ for all } w \in K.$$ 

Since $T^2x_1 \in K$, for each $T \in A$, we get that

$$(x_1, y_1) \leq (T^2x_1, y_1), \text{ for all } T \in A.$$ 

Replacing $T$ by $1 + \alpha T$, $\alpha \in \mathbb{R}$, we get that

$$2\alpha(Tx_1, y_1) + \alpha^2(T^2x_1, y_1) \geq 0, \text{ for all } \alpha \in \mathbb{R}.$$ 

This means that $(Tx_1, y_1) = 0$, for $T \in A$, so the cyclic subspace $Ax_1$ is a non-trivial invariant subspace for $A$.

**Corollary 2.2** Let $T_1, T_2, \ldots, T_n$ be commuting operators on a real Banach space $X$. If there is a positive measure $\mu$ on $\mathbb{R}^n$ and vectors $x_0 \in X, y_0 \in X^*$, such that

$$(T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x_0, y_0) = \int_{\mathbb{R}^n} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}d\mu, \text{ for all } m_i \in \mathbb{N}, \quad (2.1)$$

then operators $T_1, T_2, \ldots, T_n$ have a common invariant subspace.

**Proof.** Clearly

$$(P(T_1, \ldots, T_n)x_0, y_0) = \int_{\mathbb{R}^n} P(x_1, \ldots, x_n)d\mu, \text{ for each polynomial } P \text{ on } \mathbb{R}^n.$$ 

It follows that $(P(T_1, \ldots, T_n)x_0, y_0) \geq 0$, for each $P$, so the algebra $A$ generated by $T_1, \ldots, T_n$ satisfies the assumptions of Lemma 2.1 and, therefore, has invariant subspaces.

**Lemma 2.3** Let $T$ be an operator on a real or complex Banach space $X$. If $T$ has an eigenvalue $\lambda > \rho_e(T)$ then the norm-closed algebra $A = A(T)$ generated by $T$ contains a non-zero finite rank operator.

**Proof.** Let us firstly show that the statement holds for complex spaces. Indeed by the Puncture Neighborhood Theorem (see e.g. [20] Theorem 19.4) $\lambda$ is an isolated point of $\sigma(T)$, so the Riesz projection $P$ corresponding to $\{\lambda\}$ belongs to $A$. Thus the finite rank operator $P$ is the limit of a sequence of polynomials in $T$.

Now turning to the case of real $X$, let $Z$ be the complexification of $X$, i.e. $Z = X \oplus X$ supplied with the structure of complex space by the formula

$$(\alpha + i\beta)z = \alpha z + \beta Jz,$$

where $J$ is the operator on $Z$ acting by the rule

$$J(x \oplus y) = (-y) \oplus x.$$ 

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In matrix form
\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

For each operator \( K \) on \( X \), the operator \( K \oplus K \) on \( Z \) is complex-linear. The operator \( T \oplus T \) has the same spectrum and essential spectrum as \( T \), and \( \lambda \) is its eigenvalue. By the above, there is a sequence of polynomials with complex coefficients
\[ p_n(t) = \sum_{k=1}^{N_n} (\alpha_{nk} + i\beta_{nk}) t^k \]
such that the sequence \( p_n(T \oplus T) \) tends to a non-zero finite rank operator \( W \). So
\[ p_n(T \oplus T) = \sum_{k=1}^{N_n} (\alpha_{nk}1 + \beta_{nk}J) \begin{pmatrix} T^k & 0 \\ 0 & T^k \end{pmatrix} = \sum_{k=1}^{N_n} \begin{pmatrix} \alpha_{nk}T^k & -\beta_{nk}T^k \\ \beta_{nk}T^k & \alpha_{nk}T^k \end{pmatrix} = \begin{pmatrix} q_n(T) & r_n(T) \\ -r_n(T) & q_n(T) \end{pmatrix}, \]
where \( q_n \) and \( r_n \) are polynomials with real coefficients. It follows that
\[ W = \begin{pmatrix} W_1 & -W_2 \\ W_2 & W_1 \end{pmatrix} \]
and \( q_n(T) \to W_1, r_n(T) \to W_2 \). Since at least one of operators \( W_1, W_2 \) is non-zero, \( A(T) \) contains a non-zero finite rank operator.

\textbf{The proof of Theorem 1.3} Set
\[ F = \{ T^* \in A^* : \| T \|_e < 1 \} \]
and fix \( \varepsilon \in (0, \frac{1}{10}) \). Suppose firstly that \( Fy \) is dense in \( X^* \), for each non-zero \( y \in X^* \). Then the same is true for \( \varepsilon Fy \). Choose \( y_0 \in X^* \) with \( \| y_0 \| = 3 \) and denote by \( S \) the ball \( \{ y : \| y - y_0 \| \leq 2 \} \).

So for every \( y \in S \), there is \( T^*_y \in \varepsilon F \) with \( \| T^*_y y - y_0 \| < 1 \). By the definition of \( F \), \( T^*_y = R^*_y + K^*_y \) where \( \| R^*_y \| < \varepsilon, K^*_y \in \mathcal{K}(X) \). Thus \( T^*_y = R^*_y + K^*_y \) and
\[ \| K^*_y y - y_0 \| \leq \| T^*_y y - y_0 \| + \| R^*_y y \| < 1 + \varepsilon \| y \| \leq 1 + 5\varepsilon. \]

Now the compactness of \( K^*_y \) implies that there is a neighborhood \( V^*_y \) of \( y \) in the (relative) weak* topology \( \tau \) of \( S \), such that \( \| K^*_y w - y_0 \| < 1 + 5\varepsilon \), for all \( w \in V^*_y \). Therefore \( \| T^*_y w - y_0 \| < 1 + 5\varepsilon + 5\varepsilon < 2 \). In other words \( T^*_y \) maps \( V^*_y \) to \( S \).

The sets \( V^*_y, y \in S \), form a covering of \( S \); since \( S \) is \( \tau \)-compact there is a finite subcovering \( \{ V^*_y : 1 \leq i \leq n \} \). Let \( \{ \varphi_i : 1 \leq i \leq n \} \) be a unity partition related to the covering \( \{ V^*_y : 1 \leq i \leq n \} \). We define a map \( \Phi : S \to S \) by
\[ \Phi(y) = \sum_{i=1}^{n} \varphi_i(y) T^*_y (y). \]
By Tichonov’s Theorem, \( \Phi \) has a fixed point \( z \in S \). Thus \( Mz = z \), where \( M = \sum_{i=1}^{n} \varphi_i(z) T^*_y \).

Since the set \( \varepsilon F \) is convex, \( M \in \varepsilon F \).

It follows that \( 1 \) is an eigenvalue of \( M \) exceeding \( \| M \|_e \); by Lemma 2.3, \( A^* \) contains a non-zero finite-rank operator. So the same is true for \( A \).
It remains to consider the case that \( Fz \) is not dense in \( X^* \), for some \( z \in X^* \). Clearly we may assume that \( Fz \neq \{0\} \), because in this case \( A^*z = \{0\} \) whence \( \omega_{x,z} \) is a resolving functional, for each \( x \in X^* \).

Let \( V = Fz \). It is a closed convex proper subset of \( X^* \), so by the Bishop - Phelps Theorem \([5]\), there are \( 0 \neq y \in V \) and \( 0 \neq x \in X^* \) with \( (x,y) = \sup\{ (x,w) : w \in V \} \) (in the case of complex scalars it is important that \( e^{it} V = V \), for all \( t \in \mathbb{R} \), see \([21]\)). Since \( F^2 \subset F \), we have that \( Fy \subset V \), so \( (x,T^*y) \leq (x,y) \), for \( T^* \in F \). Therefore \( (x,T^*y) \leq \|T\|_{\text{op}} (x,y) \), for all \( T \in A \).  ■

**The proof of Theorem 1.2** Suppose that a) does not hold: \( A \) has no resolving functionals. Then, by Theorem 1.3 \( A \) contains a non-zero finite rank operator \( T_0 \). If \( A \) had an invariant subspace \( L \) then for any \( x \in L \),\( y \in L^\perp \), the functional \( \omega_{x,y} \) would be resolving for \( A \). So \( A \) is transitive. If \( \mathbb{K} = \mathbb{C} \) this contradicts to the assumption that \( A \) is not dense in \( B(H) \) (see \([4]\), or a more general result in \([12]\)). Thus \( \mathbb{K} = \mathbb{R} \).

Let \( Y = T_0 X \) and let \( B \) be the restriction of the algebra \( T_0 A T_0 \) to \( Y \). Then \( B \) is an irreducible algebra of operators on an \( n \)-dimensional real space. There is a well known classification of such algebras (see e.g. \([9]\) which implies that each of them is isomorphic to either \( M_n(\mathbb{R}) \) or \( M_{n/2}(\mathbb{R}) \otimes \mathbb{C} \), or \( M_{n/4}(\mathbb{R}) \otimes \mathbb{H} \) where \( \mathbb{C} \) and \( \mathbb{H} \) are respectively the real algebras of complex numbers or of quaternions. It follows that \( B \) contains idempotents, so the same is true for \( T_0 A T_0 \) and therefore for \( A \).

\[ \square \]

**Corollary 2.4** Let \( X \) be a complex Banach space. The only WOT-closed subalgebra of \( B(X) \) that has no resolving functionals is \( B(X) \) itself.

In the case of real spaces the situation is different. It was proved in the recent work of E.Kissin, V.S.Shulman and Yu.V.Turovskii \([9]\) that in a real separable Hilbert space \( H \) there is a continuum of pairwise non-similar weakly closed transitive algebras containing non-zero finite rank operators.

All these algebras have no resolving functionals: if \( f = \omega_{x,y} \) is a resolving functional for a transitive algebra \( A \subset B(H) \), then, by (1.2), \( (Tx,y) = 0 \) for all \( T \in A \cap F(H) \). So the ideal \( J = A \cap F(H) \) of \( A \) has invariant subspaces and therefore the same is true for \( A \) — a contradiction.

### 3 Main results

In this section \( X \) is a real Banach space (complex spaces are considered as real ones).

Let us say that an element \( a \) of a unital real normed algebra is **positive**, if

\[ \|1 - \varepsilon a\| \leq 1 + o(\varepsilon), \text{ for } \varepsilon \searrow 0. \]

Furthermore \( a \) is **real**, if \( a^2 \) is positive.

To see that all selfadjoint operators on a real Hilbert space \( H \) are real, note that the map \( T \mapsto T_e \) from \( B(H) \) to \( B(H_e) \) is \( \mathbb{R} \)-linear, involution-preserving and isometric, so if \( T \in B(H) \) is selfadjoint then

\[ \|1 - \varepsilon T^2\| = \|1 - \varepsilon (T_e)^2\| < 1, \text{ if } 0 < \varepsilon < 1/\|T\|^2. \]

It is not difficult to check that Hermitian operators in complex Banach spaces (defined by the condition \( \|\exp(itT)\| = 1 \), for \( t \in \mathbb{R} \)) are real. Indeed, in this case \( \|\cos(tT)\| = \|\mathcal{R}(\exp(itT))\| \leq 1 \) for all \( t \in \mathbb{R} \).
whence
\[ \|1 - \frac{t^2}{2} T^2\| = \|\cos tT\| - \sum_{n=2}^{\infty} (-1)^n \frac{t^{2n}}{n!} \leq 1 + o(t^2); \]
denoting \( \frac{t^2}{2} \) by \( \varepsilon \), we obtain that \( \|1 - \varepsilon T^2\| \leq 1 + o(\varepsilon) \). In passing we obtain a similar criterion of reality for operators on real Banach spaces:

if \( \|\cos(tT)\| \leq 1 \), for all \( t \in \mathbb{R} \), then \( T \) is real.

Clearly all involutions and all nilpotents of order 2 satisfy the latter condition, so these operators are also real.

An operator \( T \in B(X) \) is essentially real, if \( \pi(T) \) is a real element of the Calkin algebra (recall that by \( \pi : B(X) \to \mathcal{C}(X) \) we denote the natural epimorphism).

Thus \( T \) is essentially real if
\[ \|1 - \varepsilon T^2\|_e \leq 1 + o(\varepsilon), \quad \text{when } \varepsilon \searrow 0 \]

If \( T \) is an essentially selfadjoint operator on a real Hilbert space then it is essentially real. Indeed
\[ \|1 - \varepsilon T^2\|_e = \|1 - \varepsilon \pi(T)^2\| = \|1 - \varepsilon \pi(T)^* \pi(T)\| = \|\pi(1 - \varepsilon T^* T)\| \leq \|1 - \varepsilon T^* T\| \leq 1, \]
if \( \varepsilon < \|T\|^{-2} \).

**Theorem 3.1** If \( A \) is a commutative algebra of essentially real operators on a Banach space \( X \), then there is a closed subspace of \( X^* \), invariant for the algebra \( A^* = \{T^* : T \in A\} \).

**Proof.** Note that the set of all positive elements of a Banach algebra is a convex cone. Moreover this cone is closed. Indeed let \( a = \lim_{n \to \infty} a_n \) where all \( a_n \) are positive. If \( a \) is not positive then there is a sequence \( \varepsilon_n \to 0 \) and a number \( C > 0 \), such that \( \|1 - \varepsilon_n a\| > 1 + C\varepsilon_n \) for all \( n \). Taking \( k \) with \( \|a - a_k\| < C/2 \), we get that \( \|1 - \varepsilon_n a_k\| > 1 + C\varepsilon_n - \|a - a_k\|\varepsilon_n > 1 + (C/2)\varepsilon_n \), a contradiction to positivity of \( a_k \).

It follows that the set of real elements is closed. Applying this to \( \mathcal{C}(X) \) we see that the set of essentially real operators is closed. This allows us to assume that the algebra \( A \) is closed. Obviously we may assume also that \( A \) is unital.

If \( A \) contains a non-zero finite rank operator \( K \) then, by commutativity, the finite dimensional subspace \( KX \) is invariant for \( A \). So we may restrict by the case that \( A \) has trivial intersection with \( \mathcal{F}(X) \). By Theorem 1.3 there are vectors \( x_0 \in X^{**}, y_0 \in X^* \), such that the condition \( \mathbb{1.2} \) holds. Clearly if \( (x_0, y_0) = 0 \) then \( A^* \) has an invariant subspace (namely \( \overline{A^* y_0} \)), so we may assume that \( (x_0, y_0) > 0 \).

Therefore, for \( T \in A \) and \( \varepsilon \searrow 0 \),
\[ (x_0, (1 - \varepsilon (T^2)^*) y_0) \leq \|1 - \varepsilon T^2\|_e (x_0, y_0) \leq (1 + o(\varepsilon))(x_0, y_0), \]
whence
\[ -\varepsilon (x_0, (T^2)^* y_0) \leq o(\varepsilon)(x_0, y_0). \]

It follows that \( (x_0, (T^2)^* y_0) \geq 0 \), because \( (x_0, y_0) > 0 \). Now it remains to apply Lemma 2.1 to the algebra \( A^* \).
Corollary 3.2 A commutative algebra of essentially real operators on a reflexive real Banach space has an invariant subspace.

Since the algebra generated by a commutative family of essentially selfadjoint operators on a Hilbert space consists of essentially selfadjoint (and therefore, essentially real) operators we get the following result:

Theorem 3.3 Any commutative family of essentially selfadjoint operators on a real Hilbert space has an invariant subspace.

It seems desirable to strengthen Theorem 3.3 by replacing commuting essentially selfadjoint operators with essentially commuting essentially selfadjoint, or at least essentially commuting self-adjoint ones. But this is impossible: any two selfadjoint compact operators essentially commute but not necessarily have a common invariant subspaces.

Returning to individual criteria of non-transitivity let us denote by $E(X)$ the class of all operators $T$ on a real Banach space $X$ such that each polynomial $P(T)$ of $T$ is essentially real.

Corollary 3.4 If $T \in E(X)$ then the operator $T^*$ has an invariant subspace. As a consequence, if $X$ is reflexive then any operator $T \in E(X)$ has an invariant subspace.

Atzmon, Godefroy and Kalton [3] introduced the class $S(X)$ of all operators on $X$, satisfying the condition

$$
\|P(T)\|_e \leq \sup\{|P(t)| : t \in \Omega\},
$$

(3.1)

where $\Omega$ is a compact subset of $\mathbb{R}$. It was proved in [3] that all operators in $(S)$ have invariant subspaces if $X$ is reflexive (in general their adjoints have invariant subspaces). The usefulness of this result was cogently demonstrated by S. Grivaux [8] who applied it to the proof of non-transitivity of a wide class of tridiagonal operators in sequence spaces.

It is not difficult to see that $S(X) \subset E(X)$. Indeed if $T \in S(X)$ then

$$
\|1 - \varepsilon P(T)^2\|_e \leq \sup\{|1 - \varepsilon P(t)^2| : t \in \Omega\} \leq 1,
$$

for sufficiently small $\varepsilon$.

So the above mentioned result of [3] follows from Corollary 3.4.

4 Concluding remarks and problems

This version of the paper differs from the first one ([16]) in several ways. Among several improvements and additions we mention that Theorem 1.2 now is obtained as a consequence of Theorem 1.3 which has a more convenient formulation which does not differ the real and the complex fields of scalars. Furthermore in the proof of our main result, Theorem 3.3 we use the quadratic function instead of the exponential one (this trick has appeared in our review [17] Proof of Theorem 3.9); the same idea was proposed two years later by Godefroy [7]. Here we present it separately in Lemma 2.1.

Note also that the whole text is much more detailed and (hopefully) transparent than [16].

Now we will formulate several unsolved problems.
Q1. (the main question): can the word *real* in Theorem 3.3 be replaced by *complex*?

Q2. Does every essentially selfadjoint operator $T$ on a real Hilbert space $H$ have hyperinvariant subspace (that is a subspace invariant for all operators commuting with $T$)?

Note that the positive answer for Q2 would imply the positive answer for Q1. Indeed let $H$ be complex, $H_r$ the underlying real Hilbert space and $J$ the operator of multiplication by $i$ on $H_r$. An essentially selfadjoint operator $T$ on $H$ can be considered as an essentially selfadjoint operator $T_r$ on $H_r$ that commute with $J$. A hyperinvariant subspace $L$ of $T_r$ must be invariant for $T_r$ and $J$, so it is a subspace of $H$, invariant for $T$.

Q3. Does every essentially selfadjoint operator $T$ on a real Hilbert space $H$ have conditionally hyperinvariant subspace (that is a subspace invariant for all essentially selfadjoint operators commuting with $T$)?

Q4. Does every essentially normal operator on real Hilbert space have invariant subspace? A more weak version: the same question for a compact perturbation of a normal operator.

Q5. Does every essentially skew self-adjoint operator ($T + T^* \in \mathcal{K}(H)$) on a real Hilbert space have invariant subspace?

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