AN ASYMPTOTIC FORMULA FOR GOLDBACH’S CONJECTURE WITH MONIC POLYNOMIALS IN $\mathbb{Z}[\theta][x]$

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Abstract. We say that $D$ satisfies the property (GC) if every element of $D[x]$ of degree $d \geq 1$ can be written as the sum of two irreducibles in $D[x]$. In 1965 Hayes [3] showed that $\mathbb{Z}$ satisfies the property (GC). In 2011 Pollack showed that $D$ satisfies the property (GC), where $D$ is any integral domain. In this note, we consider $D = \mathbb{Z}[\theta]$, where

$$\theta = \begin{cases} \sqrt{-k} & \text{if } -k \not\equiv 1 \pmod{4} \\ \frac{\sqrt{-k} + 1}{2} & \text{if } -k \equiv 1 \pmod{4} \end{cases},$$

$k$ is a squarefree integer and $k \geq 2$, and we proved that the number $R(y)$ of representation of a monic polynomial $f(x) \in \mathbb{Z}[\theta][x]$ as a sum of two irreducible monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[\theta][x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in complex modulus by $y$, is asymptotic to $(4y)^{2d-2}$.

1. Introduction

Hayes [3], in 1965, showed that Goldbach’s conjecture is considerably simpler for polynomials with integer coefficients. In fact, he proved the following result:

**Theorem 1.** If $f(x)$ is a monic polynomial in $\mathbb{Z}[x]$ with $\partial f = d > 1$, then there exist irreducible monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$ with the property that $f(x) = g(x) + h(x)$.

In a recent note, Saidak [9], improving on a result of Hayes, and gave Chebyshev-type estimates for the number $R(y) = R_f(y)$ of representations of the monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $d > 1$ as a sum of two irreducible monics $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in absolute value by $y$. Here, we do not distinguish the sum $g(x) + h(x)$ from $h(x) + g(x)$, and whenever we write that a monic polynomial $p(x) \in \mathbb{Z}[x]$ is “irreducible”, we mean irreducible over $\mathbb{Q}$. Saidak’s argument with slight modifications gives that, for $y$ sufficiently large,

$$c_1 y^{d-1} < R(y) < c_2 y^{d-1},$$

where $c_1$ and $c_2$ are constants that depend of the degree and the coefficients of the polynomial $f(x)$.

More recently, Kozek [4] gave a proof that the number $R(y)$ is asymptotic to $(2y)^{d-1}$, i.e.,

$$\lim_{y \to \infty} \frac{R(y)}{(2y)^{d-1}} = 1.$$

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Your approach implies that there is a constant $c_3$ depending only on $d$ such that if $y$ is sufficiently large, then
\[ R(y) = (2y)^{d-1} + E, \quad \text{where } |E| \leq c_3y^{d-2}\ln(y). \]

In 2011, Dubickas [1] proved a more general result for the number of representations of $f$ by the sum of $r$ monic irreducible (over $\mathbb{Q}$) integer polynomials $f_1, f_2, \ldots, f_r$ of height at most $y$, i.e.,
\[ f(x) = f_1(x) + f_2(x) + \ldots + f_r(x), \]
and, for $r = 2$, Dubickas proved that
\[ R(y) = (2y)^{d-1} + O(y^{d-2}) \]
for $d \geq 4$, (1.1)
\[ y \ln(y) << (2y)^2 - R(y) << y \ln(y) \]
for $d = 3$, and
\[ \sqrt{y} << 2y - R(y) << \sqrt{y} \]
for $d = 2$. Moreover, for each $d \geq 4$, the error term in (1.1) is best possible for some $f$. Not that this results improve the error term proved by Kozek [4].

In 2013, Dubickas [2] proved a necessary and sufficient condition on the list of nonzero integers $u_1, \ldots, u_r, r \geq 2$, under which a monic polynomial $f(x) \in \mathbb{Z}[x]$ is expressible by a linear form $u_1f_1(x) + \cdots + u_rf_r(x)$ in monic polynomials $f_1(x), \ldots, f_r(x) \in \mathbb{Z}[x]$.

We say that $D$ satisfies the property (GC) if

Every element of $D[x]$ of degree $d \geq 1$

can be written as the sum of two irreducibles in $D[x]$. When $D = \mathbb{Z}$, we have the Hayes’s theorem. Pollack [8], proved the following results:

**Proposition 2.** Suppose that $D$ is an integral domain which is Noetherian and has infinitely many maximal ideals. Then $D$ has property (GC).

**Corollary 3.** If $S$ is any integral domain, then $D = S[x]$ has property (GC).

Following the results of Hayes and Pollack with $D = \mathbb{Z}[\theta]$, where $\theta$ satisfies the properties described in the Lemma [1] below, and following the ideas of Kozek [1] we will prove that the number $R(y)$ of representation of a monic polynomial $f(x) \in \mathbb{Z}[\theta][x]$ as a sum of two irreducible monic polynomials $g(x)$ and $h(x)$ in $\mathbb{Z}[\theta][x]$, with the coefficients of $g(x)$ and $h(x)$ bounded in complex modulus by $y$, is asymptotic to $(4y)^{2d-2}$.

## 2. Preliminary Results

Some well known facts are below. Let $f(x) = \sum_{i=0}^{d} f_i x^i$ be a polynomial in $\mathbb{C}[x]$, define
\[ H(f) = \max_{0 \leq i \leq d} |f_i| \quad \text{and} \quad M(f) = \exp \left( \int_0^1 \ln |f(e^{2\pi it})| dt \right). \]
These expressions \( H(f) \) and \( M(f) \) will be known as \textit{height} and \textit{Mahler’s measure} (see [6, 7]). Mahler showed that for \( 0 \leq i \leq n \), \( |f_i| \leq \binom{n}{i} M(f) \). Mahler also noted that \( M(f) \) is multiplicative. An another results de Mahler is that

\[
2.1 \quad \frac{M(f)}{\sqrt{d+1}} \leq H(f) \leq 2^{d-1} M(f).
\]

An important property of Mahler measure was proved by Landau in [5]. Landau showed that

\[
2.2 \quad 1 \leq M(f) \leq \left( \sum_{i=0}^{d} |f_i|^2 \right)^{\frac{1}{2}}.
\]

Now we consider \( f(x) = a(x)b(x) \) such that \( \partial a = d_1 \) and \( \partial b = d_2 \), this is, \( d = d_1 + d_2 \). A direct application of Jensen’s formula (see [6]) results in \( H(f) = H(ab) \leq (1 + d_1)H(a)H(b) \), if \( d_1 \leq d_2 \). To do this, let

\[
f(x) = \sum_{i=0}^{d} f_i x^i, \quad a(x) = \sum_{k=0}^{d_1} a_k x^k \quad \text{and} \quad b(x) = \sum_{t=0}^{d_2} b_t x^t.
\]

Then

\[
|f_i| = |\sum_{j=0}^{d_1} a_j b_{i-j}| \leq (1 + d_1) \max |a_k| \max |b_t| = (1 + d_1) H(a) H(b).
\]

Moreover,

\[
2.3 \quad H(ab) \geq \frac{H(a)H(b)}{\sqrt{d+1}}.
\]

The last inequality follows from the fact that \( M(f) \) is multiplicative and from the inequality (2.1) for \( a(x) \) and \( b(x) \), this is, \( H(a) \leq 2^{d_1-1} M(a), H(b) \leq 2^{d_2-1} M(b) \) and \( \sqrt{d+1} M(ab) \leq H(ab) \).

Using what has been discussed above we prove the following result.

\textbf{Lemma 1.} Let \( f(x) = a(x)b(x) \) be a polynomial of degree \( d \) in \( \mathbb{Z}[\theta][x] \), where \( a(x), b(x) \in \mathbb{Z}[\theta][x] \) and

\[
2.4 \quad \theta = \begin{cases} \sqrt{-k} & \text{if } -k \not\equiv 1 \pmod{4} \\ \sqrt{-k+1} & \text{if } -k \equiv 1 \pmod{4} \end{cases},
\]

where \( k \) is a squarefree integer and \( k \geq 2 \). Let \( a(x) \) and \( f(x) \) take the form

\[
a(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad f(x) = \sum_{i=0}^{d} f_i x^i.
\]

Then for \( 0 \leq l \leq m \), \( a_l \) satisfies

\[
|a_l| \leq 2^{d-2} \sqrt{d+1} \left( \sum_{i=0}^{d} |f_i|^2 \right)^{\frac{1}{2}}.
\]
Proof. First, we observe that $|r + s\theta| \geq \frac{1}{2}$, for any $\theta$ above and $r, s \in \mathbb{Z}$. Actually,

$$|r + s\theta| = \begin{cases} \sqrt{r^2 + s^2} & \text{if } -k \not\equiv 1 \pmod{4} \\ \frac{1}{2}\sqrt{r^2 + s^2} & \text{if } -k \equiv 1 \pmod{4} \end{cases},$$

where $r_1 = 2r + s$. But, since $r$ and $s$ are integers and $k > 1$ we have that $\sqrt{r^2 + s^2} \geq 1$ and $\frac{1}{2}\sqrt{r^2 + s^2} \geq 1$.

From the inequalities (2.1) and (2.3) we have

$$H(a)H(b) \leq 2^{2d-3}\sqrt{d} + 1M(f).$$

Now, using the inequality (2.2) follows that

$$\frac{1}{2}|a_i| \leq \frac{1}{2}H(a) \leq H(a)H(b) \leq 2^{2d-3}\sqrt{d} + 1 \left(\sum_{i=0}^{d} |f_i|^2\right)^{\frac{1}{2}}.$$

Consequently

$$|a_i| \leq 2^{2d-2}\sqrt{d} + 1 \left(\sum_{i=0}^{d} |f_i|^2\right)^{\frac{1}{2}}.$$

Lemma 2. Let $d > 1$ be an integer and $g_{d-1} \in \mathbb{Z}[\theta]$ is fixed, $\theta$ as in Lemma [1]. For each $y \geq 2$, let $r_y$ denote the number of $d$-tuples $(g_{d-1}, g_{d-2}, \ldots, g_1, g_0)$ of elements in $\mathbb{Z}[\theta]$ satisfying $|g_i| \leq y$ for $i \in \{0, 1, \ldots, d - 1\}$ such that the polynomial

$$\sum_{i=0}^{d-1} g_ix^i + x^d$$

is reducible. Then $r_y = O_d(y^{2d-4}\ln y)$. In particular, $r_y = 0$ if $y < |g_{d-1}|$.

Proof. Let $g(x) \in \mathbb{Z}[\theta][x]$ be a reducible, monic polynomials of degree $d > 1$ such that all of its coefficients are $\leq y$ in complex modulus and $g_{d-1}$ is fixed as in the lemma. Then exists two monic polynomials $a(x)$ and $b(x) \in \mathbb{Z}[\theta][x]$ of degree $\geq 1$ such that $g(x) = a(x)b(x)$. Let us further take

$$\deg(a) = m \geq n = \deg(b),$$

where $m + n = d$. We write $a(x)$ and $b(x)$ in the following forms:

$$a(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1x + a_0$$

$$b(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0.$$

The number of monic polynomials we are considering with $g_0 = 0$ is $O_{d,k}(y^{2d-4})$. Indeed, denoting by $g_j = g_{j,1} + ig_{j,2}$ and as $|g_i| \leq y$ we have that $|g_j| = \sqrt{g_{j,1}^2 + k g_{j,2}^2} \leq y$. Therefore, the number of possibilities for $g_j$ is bounded by

$$|g_j| \leq y \sqrt{1 + \frac{1}{k}} - 2\sqrt{\frac{2y - 1}{k}} - \ldots - 2\sqrt{\frac{y^2 - 1}{k}},$$

with the sum having $y$ terms, this is, the term (2.5) is $O_{d,k}(y^2)$ and the assertion follows. It is sufficient to show that the number of $d$-tuples

$$(a_{m-1}, a_{m-2}, \ldots, a_1, a_0, b_{n-1}, b_{n-2}, \ldots, b_1, b_0)$$

as above with $a_0b_0 \neq 0$ is equal to $O_{d,k}(y^{2d-4}\ln y)$.
We consider $a(x)$ which has degree $m \leq d - 1$. A similar argument applies to $b(x)$. For $1 \leq l \leq m - 1$, Lemma \[\text{H}\] implies
\[
|a_l| \leq 2^{2d-2}\sqrt{d+1} \left( \sum_{i=0}^{d} |g_i|^2 \right)^{\frac{1}{2}} \leq 2^{2d-2}\sqrt{d+1} ((d+1)y^2)^{\frac{1}{2}} = C_d y,
\]
where $C_d$ depends only $d$. Thus, the number of $(d-4)$-tuples
\[
(a_{m-2}, \ldots, a_1, b_{n-2}, \ldots, b_1)
\]
is $O_d(k(y^{2m-4}y^{2n-4}) = O_d(k(y^{2d-8})$.

Observe that when we multiply $a(x)$ and $b(x)$, since they are both monic polynomials, the value the coefficient $g_{d-1}$ is the sum $a_{m-1} + b_{n-1}$. Also, recall that $g_{d-1}$ is fixed, so determining $a_{m-1}$ also determines $b_{n-1}$. Hence, the number of 2-tuples $(a_{m-1}, b_{n-1})$ is $O_k(y^2)$.

Since $a_0b_0 = g_0$, we have $1 \leq |a_0b_0| \leq y$, this is,
\[
1 \leq (a_{0,1} + k a_{0,2}) (b_{0,1} + k b_{0,2}) \leq y^2.
\]

Thus, the number of 2-tuples $(a_0, b_0)$ is bounded by
\[
16 \sum_{q \leq y^2} \sum_{\delta | q} 1 = 16 \sum_{\delta \leq y^2} \frac{1}{\delta} \sum_{q \leq y^2} 1 \leq 16 \sum_{\delta \leq y^2} \frac{y^2}{\delta} \leq 16 y^2 \sum_{\delta \leq y^2} \frac{1}{\delta} \leq 16 y^2 \left( 1 + \int_1^{y^2} \frac{1}{t} dt \right) = O(y^2 \ln y^2) = O(y^2 \ln y),
\]
where the 16 appears above since each term of $a_0$ and $b_0$ may be either positive or negative.

Finally, for an integer $d > 1$ and a fixed $g_{d-1} \in \mathbb{Z}[\theta]$, the number of $d$-tuples
\[
(a_{0}, a_{1}, \ldots, a_{m-1}, b_{0}, b_{1}, \ldots, b_{n-1}),
\]
corresponding to the coefficients of two monic polynomials $a(x)$ and $b(x)$ in $\mathbb{Z}[\theta][x]$ of degrees $m, n \geq 1$ such that $g(x) = a(x)b(x)$ and the coefficients of $g(x)$ are bounded in absolute value by $y$, as $y \to \infty$ is:
\[
r_y = O_d(k(y^{2m-4}y^{2n-4}y^2(\ln y)y^2) = O_d(k(y^{2d-4} \ln y)),
\]
as the constants depend only on $d$. \hfill \Box

Remark 4. If we substitute (2.4) by
\[
\theta = \begin{cases} 
\sqrt{k} & \text{if } k \not\equiv 1 \pmod{4} \\
\frac{\sqrt{k}+1}{2} & \text{if } k \equiv 1 \pmod{4}
\end{cases},
\]
where $k \geq 2$ is a squarefree integer and we consider $\mathbb{Z}[\theta]$ with the usual norm induced by $\mathbb{Q}$, our argument can not be applied because, for $y$ large enough, there are endless possibilities for $g_j = g_{j,1} + \sqrt{k}g_{j,2}$ with $g_{j,1}, g_{j,2} \in \mathbb{Z}$ and satisfying $|g_1| \leq y$. Therefore, we could not get the limitation (2.5).
Remark 5. If we remove the condition in Lemma 2 that \( g_{d-1} \) is fixed, then \( r_y = O_{d,k}(y^{2d-2} \ln y) \). This is a direct consequence the same arguments used to prove (2.3). If degree of \( g \) is \( d-1 \) and in this case \( g_{d-2} \) isn’t fixed, then \( r_y = O_{d,k}(y^{2d-4} \ln y) \).

Lemma 3. Let \( f(x) \) be a monic polynomial in \( \mathbb{Z}[\theta][x] \) of degree \( d > 1 \), such that \( f(x) = g(x) + h(x) \), where \( g(x) \) or \( h(x) \) is a reducible monic polynomial such that \( \partial g = d \) and \( 1 \leq \partial h \leq d - 1 \). Moreover, the coefficients of \( g(x) \) and \( h(x) \) bounded in complex modulus by \( y \). Then the number of pairs \( (g(x), h(x)) \) where at least one \( g(x) \) or \( h(x) \) is a reducible monic polynomial is \( O_{d,k}(y^{2d-4} \ln y) \).

Proof. First, we write

\[
(2.6) \quad f(x) = x^d + \sum_{j=0}^{d-1} f_j x^j, \quad g(x) = x^d + \sum_{j=0}^{d-1} g_j x^j \quad \text{and} \quad h(x) = x^n + \sum_{j=0}^{n-1} h_j x^j,
\]

where \( f_j = g_j + h_j \) and \( 1 \leq n \leq d - 1 \).

Now, we consider a pairs of monic polynomials \( (g(x), h(x)) \) where at least one of \( g(x) \) or \( h(x) \) is reducible. Once a particular \( g(x) \) or \( h(x) \) is fixed, it determines the other. Thus, we can count separately when \( g(x) \) is reducible and when \( h(x) \) is reducible. We count the ways \( g(x) \) might be reducible. Since \( f(x) = g(x) + h(x) \), by (2.6), we have that either \( g_{d-1} = f_{d-1} \) or \( g_{d-1} = f_{d-1} - 1 \). Therefore, in any case \( g_{d-1} \) is fixed. Now, the coefficients of \( g \) are bounded in complex modulus by \( y \), and thus by Lemma 2 we have that the number of monic polynomials reducibles \( g(x) \) is \( O_{d,k}(y^{2d-4} \ln y) \). Now, we count the ways \( g(x) \) might be reducible. If \( \partial h = d - 1 \), by Remark 5 we have that the number of monic polynomials reducibles \( h(x) \) is \( O_{d,k}(y^{2d-4} \ln y) \) and if \( \partial h < d - 1 \), then the amount of reducible \( h(x) \) is smaller, also by Remark 5.

\[ \square \]

With the results above we can prove the main result.

3. Main Result

Theorem 6. Let \( f(x) \) be a monic polynomial in \( \mathbb{Z}[\theta][x] \) of degree \( d > 1 \), \( \theta \) as in Lemma 1. The number \( R(y) \) of representation of \( f(x) \) as a sum of two irreducible monic \( g(x) \) and \( h(x) \) in \( \mathbb{Z}[\theta][x] \), with the coefficients of \( g(x) \) and \( h(x) \) bounded in complex modulus by \( y \), is asymptotic to \( (4y)^{2d-2} \).

Proof. Let \( f(x) \in \mathbb{Z}[\theta][x] \) be a given monic polynomial of degree \( d > 1 \) that takes the form

\[
x^d + \sum_{j=0}^{d-1} f_j x^j,
\]

where \( f_j = f_{j,1} + i\sqrt{k} f_{j,2} \). We are looking for pairs of monic polynomials \( g(x), h(x) \in \mathbb{Z}[\theta][x] \) such that \( f(x) = g(x) + h(x) \) and the coefficients of \( g(x) \) and \( h(x) \) are bounded in complex modulus by \( y \). Without loss of generality, let \( \partial(g) > \partial(h) \), and observe that \( \partial(g) = d \) and \( 1 \leq \partial(h) \leq d - 1 \). In this case,

\[
g(x) = x^d + \sum_{j=0}^{d-1} g_j x^j \quad \text{and} \quad h(x) = x^n + \sum_{j=0}^{n-1} h_j x^j,
\]

where \( g_j = g_{j,1} + i\sqrt{k} g_{j,2}, h_j = h_{j,1} + i\sqrt{k} h_{j,2}, f_j = g_j + h_j \) and \( 1 \leq n \leq d - 1 \).
If \( y \geq 1 + \{|f_0|, |f_1|, \cdots, |f_{d-1}|\} \), then the total number pairs of monic (not necessarily irreducible) polynomials \( g(x), h(x) \) is

\[
\sum_{S=0}^{d-2} \prod_{j=0}^{S} \left( 2 \lfloor y \rfloor + 1 - |f_{j,1}| \right) \left( 2 \lfloor y \rfloor + 1 - |f_{j,2}| \right) = (4y)^{2d-2} + O_f(y^{2d-4}),
\]

since \( f_{j,1} + i\sqrt{k}f_{j,2} = g_{j,1} + h_{j,1} + i\sqrt{k}(g_{j,2} + h_{j,2}) \).

By Lemma 3 almost all of these pairs of monic polynomials \( g(x), h(x) \) are irreducible. In fact, the number of pairs \((g(x),h(x))\) where at least one \(g(x)\) or \(h(x)\) is a reducible monic polynomial is \(O_{d,k}(y^{2d-4}\ln y)\). Thus,

\[
R(y) = \sum_{S=0}^{d-2} \prod_{j=0}^{S} \left( 2 \lfloor y \rfloor + 1 - |f_{j,1}| \right) \left( 2 \lfloor y \rfloor + 1 - |f_{j,2}| \right) + O_{d,k}(y^{2d-4}\ln y)
\]

\[
= ((4y)^{2d-2} + O_f(y^{2d-4}\ln y)) + O_{d,k}(y^{2d-4}\ln y)
\]

\[
= (4y)^{2d-2} + O_{d,k}(y^{2d-4}\ln y),
\]

where we have used that any constant depending only on the coefficients and degree of \( f(x) \) is small compared to \( \ln y \) when \( y \) is sufficiently large. Therefore,

\[
R(y) \sim (4y)^{2d-2}.
\]

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