On Wilson’s theorem about domains of attraction and tubular neighborhoods

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Abstract
In this paper, we show that the domain of attraction of a compact asymptotically stable submanifold of a finite-dimensional smooth manifold of an autonomous system is homeomorphic to its tubular neighborhood. The compactness of the attractor is crucial, without which this result is false; two counterexamples are provided to demonstrate this.

Keywords: domain of attraction, compact manifold, asymptotic stability, autonomous system

1 Introduction
The domain of attraction of an attractor of a continuous dynamical system has been widely studied. An attractor is a closed invariant set of which there exists an open neighborhood such that every trajectory of the dynamical system starting within the neighborhood eventually converges to the attractor, in the sense that the distance between the trajectory and the attractor converges to zero; namely, the attractor is attractive. And the set of all initial conditions rendering the corresponding trajectories to converge to the attractor is called the domain of attraction of the attractor [6, 3].

Generally, it is difficult or sometimes impossible to find analytically the domain of attraction of an attractor. Since an attractor is attractive, if additionally it is Lyapunov stable [6, Chapter 4], then it is called an asymptotically stable attractor; sometimes Lyapunov functions can be utilized to estimate its domain of attraction, but the estimate can be conservative [6, Chapters 4 and 8].

Partly due to the difficulty of calculating the domain of attraction of an attractor, some studies in the literature instead investigate the “shapes” or “sizes” of domains of attraction in the topological sense [10, 11, 5, 4, 2, 12]. In particular, in the simplest case where the attractor is an asymptotically stable equilibrium point, it has been shown in [10, Theorem 21] that the domain of attraction is contractible. This result characterizes the “shape” of the domain of attraction, and it also implies the “size” of the domain of attraction. Namely, it leads to the topological obstruction that if the state space of the system is not contractible, then an equilibrium point cannot be stabilized globally [10, Corollary 5.9.3]. Another topological obstruction is shown in [2], which states that the domain of attraction of an asymptotically stable equilibrium point cannot be the whole state space (i.e., global...
asymptotic stability of an equilibrium is impossible) if the state space of the continuous dynamical system has the structure of a vector bundle over a compact manifold. Some studies partly generalize these results to asymptotically stable attractors that are not necessarily equilibrium points. In [8], it is proved that a compact, asymptotically stable attractor defined on a manifold (or more generally, on a locally compact metric space) is a weak deformation retract of its domain of attraction. The conclusion is further developed in [1], which shows that if the considered manifold is the Euclidean space $\mathbb{R}^n$, then the compact asymptotically stable attractor is a strong deformation retract of its domain of attraction.

Assuming that the asymptotically stable attractors are compact submanifolds of some ambient finite-dimensional smooth manifolds, stronger conclusions can be made about the domains of attraction. For example, it is proved in [3] Chapter V, Lemma 3.2 that the intersection of an $\epsilon$-neighborhood of the attractor and some sublevel set of a corresponding Lyapunov function (of which the existence is automatically guaranteed [11]) is a deformation retract of the domain of attraction of the attractor. This result is refined in [8, 13], which conclude that the attractor itself is a strong deformation retract of its domain of attraction. Therefore, the attractor and its domain of attraction are homotopy equivalent. This result has practical significance. For example, it facilitates the analysis regarding the existence of singular points and the possibility of global convergence of trajectories to desired paths in the vector-field guided path-following problem for robotic control systems [13]. Note that the results discussed in this paragraph are strengthened for the case where the attractor is an embedded submanifold, where Theorem 3.4 in [12] claims that the domain of attraction is diffeomorphic to a tubular neighborhood of the attractor, which can be either a compact or non-compact submanifold. However, in this paper, we will show that the compactness of the attractor is crucial, without which such a claim becomes false. In addition, the proof of Theorem 3.4 in [12] is very brief, only indicating the method without giving sufficient detail. In this paper, we will detail the proof for a corrected version of this theorem, where the attractor is required to be compact.

Contributions: Throughout the paper, manifolds or submanifolds are without boundaries, and they are second countable and paracompact. We assume that the attractor is compact, asymptotically stable and it is a submanifold of some finite-dimensional smooth manifold. We show that the compactness of the attractor is crucial for Theorem 3.4 in [12] by providing counterexamples where Theorem 3.4 in [12] no longer holds if the attractor is not compact. Taking the compactness of the attractor into account, Theorem 3.4 in [12] is corrected as below:

**Theorem 1** (Corrected version of Theorem 3.4 in [12]). The domain of attraction of a compact asymptotically stable submanifold of a finite-dimensional smooth manifold of an autonomous system is homeomorphic to its tubular neighborhood.

In this paper, we will give a complete and detailed proof of Theorem 1 along with some auxiliary results to gain more insight into the theorem.

The remainder of the paper is organized as follows. Section 2 provides some preparatory results for the convenience of proving Theorem 1. Then the detailed proof of Theorem 1 is elaborated in Section 3. To justify the importance of the compactness of the attractor in this theorem, we provide two counterexamples where the attractor is not compact and hence Theorem 1 fails to hold in Section 4. Finally, Section 5 concludes the paper.
2 Preparatory results

In this section, we go through some basic notions and facts that will be used in the sequel. Let \( \mathcal{M} \) and \( \mathcal{N} \) be smooth manifolds, and \( \mathcal{S} \) be a submanifold of \( \mathcal{M} \). Note that in this section, the submanifold \( \mathcal{S} \) can be compact or non-compact unless its compactness is specified explicitly. The notation \( := \) means “defined to be”. The map \( \text{id} \) is the identity map where the domain and codomain are clear from the context.

First, we recall the definitions of topological and smooth embeddings.

**Definition 2** (Topological and smooth embeddings, [7, p. 85]). A (topological) embedding is an injective continuous map that is a homeomorphism onto its image (with the subspace topology). A smooth embedding is a smooth immersion that is also a (topological) embedding.

If \( f : \mathcal{M} \to \mathcal{N} \) is an embedding, the image \( f(\mathcal{M}) \) can be regarded as a homeomorphic copy of \( \mathcal{M} \) inside \( \mathcal{N} \). If \( f : \mathcal{M} \to \mathcal{N} \) is a smooth embedding, then it is both a topological embedding and a smooth immersion.

For each \( p \in \mathcal{M} \), denote by \( T_p\mathcal{M} \) and \( T_p\mathcal{S} \) the tangent spaces respectively of \( \mathcal{M} \) and \( \mathcal{S} \) at \( p \), and by \( T\mathcal{M} \) and \( T\mathcal{S} \) the tangent bundles. Note that \( T\mathcal{S} \) can be regarded as a subbundle of \( T\mathcal{M} \) in a natural way.

**Definition 3** (Normal bundle). The normal bundle \( N\mathcal{S} \) of \( \mathcal{S} \) in \( \mathcal{M} \) is the quotient bundle \( T\mathcal{S} \mathcal{M} / T\mathcal{S} := \bigsqcup_{p \in \mathcal{S}} (T_p\mathcal{M} / T_p\mathcal{S}) \), where \( \bigsqcup \) denotes the disjoint union.

**Fact 1** ([9, Sections 6.1 and 7.1]). Let \( g \) be any Riemannian metric on \( \mathcal{M} \). For each \( p \in \mathcal{M} \), let \( N_p \) be the orthogonal complement of \( T_p\mathcal{S} \) in \( T_p\mathcal{M} \) with respect to \( g \). Then \( \bigsqcup_{p \in \mathcal{S}} N_p \) is a subbundle of \( T\mathcal{S} \mathcal{M} \) and it is isomorphic to \( T\mathcal{S} \mathcal{M} / T\mathcal{S} \). This gives another way of defining the normal bundle of \( \mathcal{S} \) in \( \mathcal{M} \).

**Fact 2** ([9, Section 5.1]). For any vector bundle \( \mathcal{E} \) over \( \mathcal{S} \), (the image of) the zero section of \( \mathcal{E} \) can be canonically identified with \( \mathcal{S} \) via

\[
\iota : 0_S \subseteq \mathcal{E} \to \mathcal{S}
\]

\[
0_x \mapsto x
\]

where \( 0_S \subseteq \mathcal{E} \) denotes (the image of) the zero section of \( \mathcal{E} \), and \( 0_x \) denotes the zero vector in the vector space \( \mathcal{E}_x \) for \( x \in \mathcal{S} \). Therefore, \( \iota \) is a diffeomorphism from \( 0_S \) to \( \mathcal{S} \). Note that viewing \( \mathcal{S} \) as a submanifold of \( \mathcal{M} \), \( \iota \) can also be regarded as an embedding of \( 0_S \) into \( \mathcal{M} \).

**Definition 4** (Tubular neighborhood). A tubular neighborhood of \( \mathcal{S} \) is an open embedding \( \tau : \mathcal{E} \to \mathcal{M} \) from some vector bundle \( \mathcal{E} \) over \( \mathcal{S} \) to \( \mathcal{M} \) satisfying

\[
\tau|_{0_S} = \iota_S.
\]

More loosely, we often call the open set \( \mathcal{W} := \tau(\mathcal{E}) \) a tubular neighborhood of \( \mathcal{S} \).

Whether we refer to a tubular neighborhood as an embedding or an open set should be clear from the context.

**Theorem 5** (Existence of tubular neighborhood, [9, Proposition 7.1.3]). Suppose that \( \mathcal{S} \) is a submanifold of \( \mathcal{M} \). Then there exists an embedding \( \tau : N\mathcal{S} \to \mathcal{M} \) from the normal bundle \( N\mathcal{S} \) of \( \mathcal{S} \) into \( \mathcal{M} \) such that \( \tau \) keeps the zero section of \( N\mathcal{S} \) (i.e., \( \tau(0_x) = x \) for all \( x \in \mathcal{S} \), or \( \tau|_{0_S} = \iota_S \)).
Remark 6. This means that $\tau : N_S \to \mathcal{M}$ is a tubular neighborhood of $S$, and $\tau$ is a diffeomorphism between $N_S$ and $\tau(N_S)$. 

Before presenting the uniqueness result of tubular neighborhoods, we first recall the definitions of isotopy and diffeotopy.

**Definition 7** (Isotopy and diffeotopy, [5, pp. 177-178]). An isotopy from $M$ to $N$ is a map $F : M \times I \to N$, where $I \subseteq \mathbb{R}$ is an interval, such that for each $t \in I$, the map $F_t : M \to N$ defined by $x \mapsto F(x, t)$ is an embedding. We also say $F$ is an isotopy from $F_0$ to $F_1$, and $F_0$ and $F_1$ are called isotopic. If each $F_t$ is a smooth embedding, then $F$ is a smooth isotopy from $M$ to $N$. If each $F_t$ is a diffeomorphism, then $F$ is called a diffeotopy.

Throughout the paper, whenever we mention an isotopy, we mean a smooth isotopy. Now we show the uniqueness result of the tubular neighborhood as follows.

**Theorem 8** (Uniqueness of tubular neighborhood I, [9, Theorem 7.4.4]). Suppose that $f_i : E_i \to M$, $i = 0, 1$, are tubular neighborhoods of $S$. Then there exists a bundle map $\lambda : E_0 \to E_1$ such that $f_0$ and $f_1 \circ \lambda$ are isotopic (see Fig. 1).

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (E0) at (0,0) {$\mathcal{E}_0$};
  \node (E1) at (2,0) {$\mathcal{E}_1$};
  \node (M) at (1,-1) {$\mathcal{M}$};
  \node (F0) at (0,-1) {$f_0$};
  \node (F1) at (2,-1) {$f_1$};
  \draw[->] (E0) -- node[above] {$\lambda$} (E1);
  \draw[->] (E0) -- (M);
  \draw[->] (M) -- (F0);
  \draw[->] (M) -- node[left] {} (F1);
\end{tikzpicture}
\caption{Relations in Theorem 8}
\end{figure}

If $f_0$ and $f_1$ are two tubular neighborhoods, then there exists a bundle map $\lambda : E_0 \to E_1$ such that $f_0$ and $f_1 \circ \lambda$ are isotopic.

Due to Theorems 5 and 8, Theorem 1 implies the following result.

**Proposition 9.** The domain of attraction of a compact asymptotically stable submanifold $S$ of a finite-dimensional smooth manifold $M$ of an autonomous system is homeomorphic to its normal bundle $N_S$.

**Proof.** Combine Theorems 1, 5 and 8.

Denote by $G : E_0 \times (-\delta, 1 + \delta) \to \mathcal{M}$ the isotopy from $f_0$ to $f_1 \circ \lambda$. Then Theorem 5 implies that $G_t : \mathcal{E}_0 \to \mathcal{M}$ is a tubular neighborhood for any $t \in (-\delta, 1 + \delta)$. Now let

$$h(x,t) = G(f_0^{-1}(x), t)$$

for $(x,t) \in f_0(\mathcal{E}_0) \times (-\delta, 1 + \delta)$. We have the following corollary.

**Corollary 10** (Uniqueness of tubular neighborhood II, [9, Theorem 7.4.4]). Suppose that $S$ is a submanifold of $M$, and $W_0$ and $W_1$ are two tubular neighborhoods (as open sets) of $S$ in $M$, then there exists an isotopy $h : W_0 \times (-\delta, 1 + \delta) \to M$ such that

$$h_0 = j_{W_0}, \quad h_1(W_0) = W_1, \quad h_t|_S = j_S$$

More specifically, the bundle map $\gamma$ is a bundle isomorphism. This is because $f_0$ and $f_1$ are embeddings and their images are open sets in $M$; therefore, $\mathcal{E}_0$ and $\mathcal{E}_1$ are vector bundles which, as manifolds, have the same dimensions as $M$ does.
for every \( t \in (-\delta, 1 + \delta) \), where \( h_t := h(\cdot, t) \), \( j_{W_0} \) and \( j_S \) are the inclusions of \( W_0 \) and \( S \) into \( \mathcal{M} \) respectively.

Therefore, any two tubular neighborhoods \( W_0 \) and \( W_1 \) are homeomorphic.

**Definition 11** (Closed tubular neighborhood). Fix a Euclidean metric \( g \) on the vector bundle \( \mathcal{E} \) over \( S \), and for any \( r > 0 \), let

\[
\mathcal{B} \mathcal{E}_r = \{ v \in \mathcal{E} : g(v, v) \leq r^2 \}.
\]

A closed tubular neighborhood \( \mathcal{K} \) of \( S \) is a closed neighborhood of \( S \) in \( \mathcal{M} \) such that there is an embedding \( \phi : \mathcal{B} \mathcal{E}_r \to \mathcal{M} \) satisfying

\[
\phi(\mathcal{B} \mathcal{E}_r) = \mathcal{K}, \quad \phi|_{0_S} = \iota_S.
\]

**Remark 12.** If \( S \) is compact, then \( \mathcal{B} \mathcal{E}_r \) is by definition a closed tubular neighborhood of \( 0_S \) in \( \mathcal{E} \) and that it is compact. Since \( \mathcal{E}_r = \{ v \in \mathcal{E} : g(v, v) < r^2 \} \) can homeomorphically map to \( \mathcal{E} \) while keeping the zero section, it is an (open) tubular neighborhood of \( 0_S \) in \( \mathcal{E} \). In particular, \( \mathcal{E} \) itself is a tubular neighborhood of \( 0_S \) in \( \mathcal{E} \). \(\triangleq\)

Due to Remark 12, the following proposition holds.

**Proposition 13.** If \( S \) is compact, then there exists some tubular neighborhood \( W \) of \( S \) such that its closure \( \overline{W} \) is a closed tubular neighborhood which is also compact.

We will use a technique which relies on the following results to prove Theorem 1 later.

**Lemma 14** ([5, Chapter 8, Theorem 1.4]). Suppose that \( \mathcal{U} \) is an open set of the manifold \( \mathcal{N} \) and that \( S \) is a compact subset of \( \mathcal{N} \) contained in \( \mathcal{U} \). Suppose that \( h : \mathcal{U} \times (-\delta, 1 + \delta) \to \mathcal{N} \) is an isotopy with \( h_0 : \mathcal{U} \to \mathcal{N} \) being the inclusion. Then for any \( \delta' \in (0, \delta) \), there exists a diffeotopy \( H : \mathcal{N} \times (-\delta', 1 + \delta') \to \mathcal{N} \) with some open neighborhood \( \mathcal{U}_0 \) of \( S \) in \( \mathcal{U} \) such that

\[
H|_{\mathcal{U}_0 \times (-\delta', 1 + \delta')} = h|_{\mathcal{U}_0 \times (-\delta', 1 + \delta')}.
\]

**Remark 15.** Let \( \hat{h} \) be the level preserving map

\[
\hat{h} : \mathcal{U} \times (-\delta, 1 + \delta) \to \mathcal{N} \times (-\delta, 1 + \delta)
\]

\[
(p, t) \mapsto \left(h_t(p), t\right).
\]

Note that Theorem 1.4 in Chapter 8 of [5] requires \( \hat{h}(\mathcal{U} \times (-\delta, 1 + \delta)) \) to be open in \( \mathcal{N} \times (-\delta, 1 + \delta) \). However, this requirement is unnecessary at least in our case, since it can be easily checked that \( \hat{h} \) is a submersion and hence an open map. \(\triangleq\)

**Corollary 16.** Suppose that \( \mathcal{U} \) is an open set of the manifold \( \mathcal{N} \) and that \( S \) is a compact subset of \( \mathcal{N} \) contained in \( \mathcal{U} \). Suppose that \( h' : \mathcal{U} \times (-\delta, 1 + \delta) \to \mathcal{N} \) is an isotopy, and there exists a diffeomorphism \( f_0 : \mathcal{N} \to \mathcal{N} \) that agrees with \( h'_0 \) on \( \mathcal{U} \); i.e.,

\[
f_0|_{\mathcal{U}} = h'_0.
\]

\(\text{Note that } \mathcal{B} \mathcal{E}_r \text{ is a submanifold of } \mathcal{E} \text{ with boundary } \partial(\mathcal{B} \mathcal{E}_r) = \{ v \in \mathcal{E} : g(v, v) = r^2 \}.\)

\(\text{The map } \hat{h} \text{ is called the track of } h \text{ [p. 111].}\)

\(\text{This is because } \hat{h} \text{ is an immersion and the dimensions of } \mathcal{U} \text{ and } \mathcal{N} \text{ are the same.}\)
Then for any $\delta' \in (0, \delta)$, there is a diffeotopy $F: \mathcal{N} \times (-\delta', 1 + \delta') \rightarrow \mathcal{N}$ with some open neighborhood $U_0$ of $\mathcal{S}$ in $U$ such that

$$F|_{U_0 \times (-\delta', 1 + \delta')} = h'|_{U_0 \times (-\delta', 1 + \delta')}$$

**Proof.** Let $h = f_0^{-1} \circ h'$. Therefore, from [11], we have $h_0 = f_0^{-1} \circ h'_0 = j_{\mathcal{U}}$, where $j_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{N}$ is the inclusion map from $\mathcal{U}$ to $\mathcal{N}$. According to Lemma [14], there is a diffeotopy $H$ such that $H|_{U_0 \times (-\delta', 1 + \delta')} = h|_{U_0 \times (-\delta', 1 + \delta')}$. Then let $F = f_0 \circ H$. \qed

**Remark 17.** Note that the open set $\mathcal{U}$ in the theorems above may be $\mathcal{N}$ itself, which is the case in Lemma [13] to be discussed later.

Now we prove a lemma concerning tubular neighborhoods of the submanifold $\mathcal{S}$ of $\mathcal{M}$. This lemma greatly facilitates the arguments in Section 3.

Note that $(N_S, \pi, \mathcal{S})$, where $\pi: N_S \rightarrow 0_S$ defined by $p \mapsto 0_x$ for any $p \in N_x$ and $x \in \mathcal{S}$, is a vector bundle over $0_S$. Though this might be trivial since $0_S$ is identical to $\mathcal{S}$ in a canonical way, we still point it out as follows for the sake of clarity from the set-theoretic perspective.

![Diagram](image)

**Figure 2: Proof of Lemma 18**

**Lemma 18 (Extension of tubular neighborhoods).** Suppose that $j : N_S \rightarrow N_S$ is a tubular neighborhood of $0_S$; i.e., $j$ is an embedding and $j(0_x) = 0_x$ for all $x \in \mathcal{S}$. Then for any compact set $K$ in $N_S$, there is a diffeomorphism $\beta$ on $N_S$ such that $\beta$ agrees with $j$ on some neighborhood of $K$.

**Proof.** The idea is to use Corollary [16]. To this end, we seek an isotopy $h : N_S \times (-\delta, 1 + \delta) \rightarrow N_S$ such that $h_1 = j$ and $h_0$ is a diffeomorphism on $N_S$.

Note that both $\text{id}_{N_S} : N_S \rightarrow N_S$ and $j : N_S \rightarrow N_S$ are tubular neighborhoods of $0_S$ in $N_S$. Hence, according to Theorem [5] there exists a bundle isomorphism $\lambda : N_S \rightarrow N_S$ such that there exists an isotopy $h$ from $\text{id}_{N_S} \circ \lambda$ to $j$ (see Fig. 2). Since $\text{id}_{N_S} \circ \lambda$ is a diffeomorphism, according to Corollary [10] there exists a diffeotopy $H : N_S \times (-\delta, 1 + \delta) \rightarrow N_S$ such that $H$ agrees with $h$ on some neighborhood of $K$. Let $\beta = H_1$ and then it is a diffeomorphism and agrees with $h_1 = j$ on such a neighborhood. \qed

### 3 Proof of Theorem 1

The proof of Theorem 1 is based on [12, Lemma 3.3]. For clarity, we decompose the proof into several propositions. Denote by $\mathcal{M}$ the state space with a vector field $X$. Denote by $\varphi$ the flow of $X$ and assume that $\mathcal{S}$ is a compact boundaryless submanifold of $\mathcal{M}$ and is an asymptotic stable attractor of $\varphi$. Denote by $D_A$ the domain of attraction of $\mathcal{S}$.

We start by fixing a precompact tubular neighborhood

$$f_0 : N_S \rightarrow W$$
of $S$ in $\mathcal{D}_A$, where $W := f_0(N_S)$. The existence of $f_0$ is guaranteed by Proposition 13.

**Proposition 19.** For each compact set $K$ in the domain of attraction $\mathcal{D}_A$, there exists some $T_K > 0$, such that $\varphi^T(K) \subseteq W$ for any $T > T_K$. Consequently, $K \subseteq \varphi^{-T}(W)$ for any $T > T_K$.

**Proof.** Due to the asymptotic stability of $S$, there is some neighborhood $U$ of $S$ in $W$ such that $\varphi^{(0, \infty)}(U) \subseteq W$. For any $x \in K$, there is some $T_x > 0$ with some neighborhood $B_x$ of $x$ such that $\varphi^{T_x}(B_x) \subseteq U$. Since $K$ is compact, there is $\{B_x\}_{i=1, \ldots, k}$, where $k < \infty$, such that $\bigcup_i B_x_i \supseteq K$. Let $T_K := \max_i = 1, \ldots, k T_x_i$ and the proof is completed. \qed

Note that $S$ is invariant under $\varphi$, and hence $S \subseteq \varphi^{-T}(W)$ for any $T \in \mathbb{R}$. Since $\varphi^{-T} : W \rightarrow W_T := \varphi^{-T}(W)$ is a diffeomorphism and $W$ is a tubular neighborhood of $S$, it is natural to conjecture that $W_T$ should also be a tubular neighborhood of $S$. This is indeed true as shown in the next proposition, but it is not straightforward. According to Definition 4, we still need to find a diffeomorphism $T$ from $N_S$ to $W_T$ such that $f_T|_0 = \iota_S$. Although $\varphi^{-T} \circ f_0$ is a diffeomorphism from $N_S$ to $W_T$, we have $f|_0 = \varphi^{-T} \circ \iota_S$, which is not necessarily equal to $\iota_S$, and hence $f : N_S \rightarrow W_T$ is not necessarily a tubular neighborhood. Yet $f|_0 = \varphi^{-T} \circ \iota_S$ and $\iota_S$ are isotopic as maps from $0_S$ to $W_T$ while $f$ and $\varphi^{-T}$ are both diffeomorphisms. This makes it possible to use Lemma 14.

**Proposition 20.** For any $T > 0$, $W_T := \varphi^{-T}(W)$ is a tubular neighborhood of $S$ in $\mathcal{D}_A$. That is, there exists a diffeomorphism $f_T : N_S \rightarrow W_T$ such that $f_T|_0 = \iota_S$.

**Proof.** Obviously $f = \varphi^{-T} \circ f_0$ is a diffeomorphism from $N_S$ to $W_T$ with $0_x \in 0_S \rightarrow \varphi^{-T}(x)$. Now we need to “rectify” the map. Denote by $f_S$ the restriction of $f$ on $0_S$. Then $j_1 = f^{-1} \circ \varphi^T \circ f_S$ is a map mapping $0_S$ diffeomorphically to $0_S$. Let $j_s = f^{-1} \circ \varphi^T \circ f_S$ for $s \in (-\delta, 1 + \delta)$ and then $j : 0_S \times (-\delta, 1 + \delta) \rightarrow N_S$ is an isotopy such that $j_0$ is the inclusion map, and $f \circ j_1 = \iota_S$ on $0_S$.

Note that $g = \varphi \circ f$ with $g(x, t) = \varphi^t \circ f(x)$ is a smooth map from $N_S \times \mathbb{R}$ to $\mathcal{D}_A$. Since $\varphi^{[-\delta, 1 + \delta]} \circ f(0_S) = S \subseteq W_T$ and $[-\delta, 1 + \delta] \cdot T$ is compact, there exists an open neighborhood $U$ of $0_S$ in $N_S$ such that $\varphi^{[-\delta, 1 + \delta]} \circ f(U) \subseteq W_T$. Moreover, for any fixed $s \in [-\delta, 1 + \delta]$, $\varphi^s \circ f(\cdot)$ is an injective submersion, and hence a smooth embedding. Define

$$h : U \times (-\delta, 1 + \delta) \rightarrow N_S$$

$$h(x, s) = f^{-1} \circ \varphi^s \circ f(x),$$

which is an isotopy with $h_0$ being the inclusion map of $U$ into $N_S$ and $h_s|_{0_x} = j_s$. Then by Lemma 14 there exists a diffeotopy $H : N_S \times (-\delta', 1 + \delta') \rightarrow N_S$ for $\delta' \in (0, \delta)$ such that $H$ agrees with $h$ on $U_0 \times (-\delta', 1 + \delta')$ for some open neighborhood $U_0$ of $S$.

Let $f_T = f \circ H_1$ and this is a diffeomorphism between $N_S$ and $W_T$. Moreover, restricted on $S$, $f_T = f \circ h_1 = f \circ j_1 = \iota_S$. Hence, $f_T : N_S \rightarrow W_T$ is a tubular neighborhood. \qed

Since the domain of attraction $\mathcal{D}_A$ is a smooth manifold with the second countability, there exists an ascending chain of compact subsets $K_0 \subseteq K_1 \subseteq \cdots$ such that $\bigcup_{i \in \mathbb{N}} K_i = \mathcal{D}_A$. Choose $0 < T_0 < T_1 < \cdots$ such that

$$W_i := \varphi^{-T_i}(W)$$

contains $K_i$ for each $i$ and that $W_i \subseteq W_{i+1}$. This is possible due to the precompactness of $W$. By Proposition 20 there exist tubular neighborhoods $f_i : N_S \rightarrow W_i$ for all $i \in \mathbb{N}$. The strategy to prove Theorem 1 is to construct by induction an ascending chain of compact subsets $C_0 \subseteq C_1 \subseteq \cdots$ with
tubular neighborhoods \( g_i : N_S \to W_i \) “rectified” from \( f_i \) such that \( g_i(C_i) \supseteq K_i, g_{i+1} \) agrees with \( g_i \) on \( C_i \) and \( \bigcup_i C_i = N_S \). Then the theorem follows by defining a map \( g : N_S \to D_A \) with \( g = g_i \) on \( C_i \).

**Theorem 21.** There exists a diffeomorphism \( g : N_S \to D_A \) such that \( g|_{\bar{0}_S} = \iota_S \).

**Proof.** Let \( \mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots \) be an ascending chain of compact subsets such that \( \bigcup_{i \in \mathbb{N}} \mathcal{K}_i = D_A \) and \( \mathcal{K}_0 \supseteq S \). Since \( W \) is precompact in \( D_A \), \( \varphi^{-T}(W) \) is precompact for any \( T > 0 \) in \( D_A \). Then by Proposition 19 we can choose inductively \( 0 < T_0 < T_1 < \cdots \) such that \( W_i \cup \mathcal{K}_{i+1} \subseteq W_{i+1} \). According to Proposition 20, for each \( i \in \mathbb{N} \), there is a diffeomorphism \( f_i : N_S \to W_i \) such that \( f_i(0_S) = x \) for all \( x \in S \). Now we construct \( \{(g_i, C_i) : i \in \mathbb{N}\} \) with \( C_i \) being compact sets in \( N_S \) and \( g_i : N_S \to W_i \) being tubular neighborhoods such that

1. \( C_i \subseteq \text{int} C_{i+1} \);
2. \( g_i(C_i) \supseteq \mathcal{K}_i \);  
3. \( g_{i+1} \mid_{C_i} = g_i \mid_{C_i} \);
4. \( \bigcup_{i \in \mathbb{N}} C_i = N_S \).

Take \( g_0 = f_0 \) and \( C_0 = B_{E_{r_0}} \) with \( r_0 \) large enough such that \( B_{E_{r_0}} \supseteq g_0^{-1}(\mathcal{K}_0) \). Let \( j_1 = f_1^{-1} \circ g_0 \). Then \( j_1 : N_S \to N_S \) is a tubular neighborhood of \( 0_S \) in \( N_S \) and \( f_1 \circ j_1 = g_0 \). According to Lemma 18, there is a bundle isomorphism \( \beta_1 : N_S \to N_S \) such that \( \beta_1 \) agrees with \( j_1 \) on \( C_0 \). Let \( g_1 = f_1 \circ \beta_1 \). Then \( g_1 : N_S \to W_1 \) is a diffeomorphism and \( g_1 = g_0 \) on \( C_0 \). Take \( r_1 \) large enough such that \( r_1 > 2r_0 \) and \( C_1 = B_{E_{r_1}} \) contains \( g_1^{-1}(\mathcal{K}_1) \).

Suppose that for \( n \in \mathbb{N} \), \( A_n = \{(g_i, C_i) : 0 \leq i \leq n\} \) such that (i) (ii) (iii) are satisfied and \( C_n = B_{E_{r_n}} \) with \( r_n > 2^n r_0 \). Let \( j_{n+1} = f_{n+1}^{-1} \circ g_n \). Then \( j_{n+1} : N_S \to N_S \) is a tubular neighborhood of \( 0_S \) in \( N_S \). Again, according to Lemma 18 there exists a diffeomorphism \( \beta_{n+1} \) on \( N_S \) such that \( \beta_{n+1} = j_{n+1} \) on \( C_n \). Set \( g_{n+1} = f_{n+1} \circ \beta_{n+1} \), and then \( g_{n+1} = g_n \) on \( C_n \). Pick a positive number \( r_{n+1} \) such that \( r_{n+1} > 2r_n \) and \( C_{n+1} = B_{E_{r_{n+1}}} \supseteq g_{n+1}^{-1}(\mathcal{K}_{n+1}) \). Then \( A_{n+1} = A_n \cup \{(g_{n+1}, C_{n+1})\} \) again satisfies (i) (ii) (iii) with \( r_{n+1} > 2^{n+1} r_0 \). By induction we have \( \{(g_k, C_k) : k \in \mathbb{N}\} \) satisfying (i) (ii) (iii) with \( r_k > 2^k r_0 \) for all \( k \in \mathbb{N} \).

Define \( g : N_S \to D_A \) with \( g|_{C_i} = g_i \mid_{C_i} \) for all \( i \in \mathbb{N} \). Then \( g \) is well defined and \( \text{Im} \ g = D_A \) due to (iii) and (ii) respectively. Moreover, since \( g_i \)’s are diffeomorphisms and \( \bigcup_i \text{int} C_i = N_S \), \( g \) is a local diffeomorphism. It is also obvious that (iv) is satisfied. For any \( p, q \in N_S \), there exists \( i \) such that \( C_i \) contains \( p \) and \( q \). Then \( g(p) = g(q) \implies g_i(p) = g_i(q) \implies p = q \). Hence \( g \) is also injective. Therefore, \( g \) is a diffeomorphism from \( N_S \) onto \( D_A \). Moreover, since \( \mathcal{K}_0 \) is chosen to contain \( S \) in the beginning and \( g_0 \) keeps the zero section \( (i.e., g(0_S) = x \) for all \( x \in S \), \( S \subseteq 0_S \). Therefore \( g|_{0_S} = g_0|_{0_S} = \iota_S \), which concludes the proof. \( \square \)

### 4 Two counterexamples

In this section, we illustrate two counterexamples to invalidate the original claim in [12] Theorem 3.4.1. The first counterexample in Section 4.1 discusses how [12] Theorem 3.4 fails to hold when the attractor is a noncompact manifold. The idea of constructing the counterexample is straightforward, but it usually involves an **incomplete** Riemannian manifold as the ambient space. Nevertheless, another counterexample in Section 4.2 involves a **complete** Riemannian manifold as the ambient
space. The idea of the counterexample is to present two topologically equivalent dynamical systems, where the domains of attraction of the noncompact attractors are not homotopy equivalent. As a result, the domain of attraction of the noncompact attractor of either of the system is of a different homotopy type from its tubular neighbourhood, contradicting \[12, \text{Theorem 3.4}\]. Note that all the vector fields of the dynamical systems in this section are complete; i.e., solutions exist for all \( t \in \mathbb{R} \).

4.1 \( \mathcal{M} \) is an incomplete Riemannian manifold

Theorem 3.4 in \[12\] states that the domain of attraction of a uniformly asymptotically stable attractor, be it a compact or non-compact manifold, of a complete autonomous system is diffeomorphic to its tubular neighborhood. While the argument in Section 3 holds for a compact attractor \( \mathcal{S} \), it does not hold for a non-compact attractor, since Proposition 19 may be invalid when the attractor is noncompact. More specifically, when \( \mathcal{S} \) is noncompact, it is possible that none of its tubular neighborhood contains any \( \epsilon \)-neighborhood of \( \mathcal{S} \). To see this, note that if we take out one point from a submanifold, the \( \epsilon \)-neighborhood of the new submanifold will only miss one point compared to that of the original submanifold, while its tubular neighbourhood (viewed as a vector bundle) would lose the whole fiber over the missing point. Exploiting this observation, we can construct a counterexample by starting with a compact asymptotically stable attractor and then taking one fixed point out of it.

Example 22. Start with the smooth function \( \bar{f}(x) = (\text{dist}(x, S^1))^2 \) on \( \mathbb{R}^2 \) and let \( \bar{X} = -\text{grad} \bar{f} \). This system has the unit circle \( S^1 \subseteq \mathbb{R}^2 \) as the asymptotically stable attractor, and all points on \( S^1 \) are fixed points. Now consider the state space \( \mathcal{M} = \mathbb{R}^2 - \{(1,0)\} \). Let \( \mathcal{S} = S^1 - \{(1,0)\} \). It is a closed set and also a submanifold of \( \mathcal{M} \), but it is noncompact. Let \( f \) be the function on \( \mathcal{M} \) such that \( f(x) = (\text{dist}(x, S))^2 \). The function \( f \) is the restriction of \( \bar{f} \) on \( \mathcal{M} \), and hence it is smooth. The vector field \( X = -\text{grad} f \) is then the restriction of \( \bar{X} \) on \( \mathcal{M} \), and it has \( \mathcal{S} \) as an attractor, which is uniformly asymptotically stable. The domain of attraction is \( \mathcal{M} - \{(0,0)\} \), which is not contractible. However, a tubular neighborhood of \( \mathcal{S} \) is homeomorphic to \( \mathcal{S} \times \mathbb{R} \), which is contractible, contradicting Theorem 3.4 in \[12\].

4.2 \( \mathcal{M} \) is a complete Riemannian manifold

In this section we demonstrate a dynamical system \((\mathcal{M}, \varphi)\) as a counterexample to Theorem 3.4 in \[12\] where the state space \( \mathcal{M} \) is a complete Riemannian manifold and the asymptotically stable attraction \( \mathcal{S} \) is not compact. Instead of directly showing the construction of the flow map \( \varphi \) on \( \mathcal{M} \), we first construct an auxiliary system \((\mathcal{M}_0, \varphi_0)\), and then obtain \((\mathcal{M}, \varphi)\) via a topological conjugacy \[4, \text{Chapter 2}\] \( h : \mathcal{M}_0 \to \mathcal{M} \). As an extra benefit to be seen later, such a demonstration shows that uniformly asymptotic stability is rather a “geometric” concept than a “topological” one. Namely, even if two dynamical systems are topologically conjugate, properties concerning the uniform asymptotic stability of the systems may not be (fully) preserved by the conjugacy.

4.2.1 The auxiliary system \((\mathcal{M}_0, \varphi_0)\)

Let

\[
\mathcal{M}_0 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}
\]
and
\[ S_0 = \{(x, y, z) \in M_0 : x = 0, z = 1\}. \]

Endow \( M_0 \) with the Riemannian metric \( g_0 \) induced by the standard Riemannian metric \((dx)^2 + (dy)^2 + (dz)^2\) on \( \mathbb{R}^3 \). Then \((M_0, g_0)\) is a complete Riemannian manifold with the distance \( d_{M_0} \).

Let \( Y_0, Z_0 \) be the vector fields on \( M_0 \) defined by
\[ Y_0(x, y, z) = \begin{cases} e^{-\frac{1}{\epsilon} \frac{\partial}{\partial y}} (x, y, z) & y > 0 \\ 0 & y \leq 0 \end{cases} \]
and
\[ Z_0(x, y, z) = x \cdot \left( \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \bigg|_{M_0}. \]

Let \( X_0 = Y_0 + Z_0 \) and denote by \( \varphi_0 \) the flow of \( X_0 \) on \( M_0 \). Then \( S_0 \) is a uniformly asymptotically stable manifold of the dynamical system \((M_0, \varphi_0)\) with its domain of attraction being
\[ D_0 = \{(x, y, z) \in M_0 : z > -1\}. \]

The following characterization of the stability of \( S_0 \) will be needed later. Namely, given any \( a > -1 \) with \( W_{z>a} = M_0 \cap \{(x, y, z) \in M_0 : z > a\}, \) corresponding to each \( \epsilon > 0 \), there exists some \( T_\epsilon > 0 \) such that \( d_{M_0}(\varphi_0^{[T_\epsilon, +\infty)}(W_{z>a}), S_0) < \epsilon \). To see this, denote by \((x'_\epsilon, y'_\epsilon, z'_\epsilon)\) the orbit \( \varphi_0^t(p') \) for \( p' = (x', y', z') \in M_0 \). Then \((x'_\epsilon, z'_\epsilon) \leq S^1 \) is subject to the equation
\[ \frac{d}{dt}(x'_\epsilon, z'_\epsilon) = (-x'_\epsilon z'_\epsilon, x'_\epsilon^2). \]

Note that the dynamical system \( (2) \) on \( S^1 \) has the point \( q'_0 = (0, 1) \) as an asymptotically stable equilibrium with the domain of attraction \( \{(x, z) \in S^1 : z \neq -1\}. \) Hence for any \( \epsilon > 0 \), there exists \( T'_\epsilon > 0 \) such that for any \( t \geq T'_\epsilon \) and \( q' = (x', z') \in S^1 \) with \( z' \geq a \), \( \text{dist}(\phi^t(q'), q'_0) < \epsilon \), where \( \text{dist} \) is the distance on \( S^1 \) measured by lengths of minor arcs, and \( \phi \) is the flow of \( (2) \). Therefore,

\[ d_{M_0}(\varphi_0^t(x', y', z'), S_0) \leq \text{dist}(\phi^t(x', z'), q'_0) < \epsilon \]

for all \( t > T'_\epsilon \) and \((x', y', z') \in W_{z>a}'. \)

For a point \((0, y, -1) \in M_0 - D_0\) with \( y \leq 0 \), it holds that \( X_0|_{(x, y, z)} = 0 \). For any \( y > 0 \),
\[ X_0|_{(0, y, -1)} = Y_0|_{(0, y, -1)} = e^{-\frac{1}{\epsilon} \frac{\partial}{\partial y}}|_{(0, y, -1)} \]

implying
\[ \varphi_0^t(0, y, -1) = (0, \gamma(t), -1) \]
with
\[ \gamma(t) = e^{-\frac{t}{\epsilon y}} > 0. \]

Therefore, both \( \gamma(t) \) and \( \dot{\gamma}(t) \) increase strictly with respect to \( t > 0 \).
### 4.2.2 The system \((\mathcal{M}, \varphi)\)

Now we construct the dynamical system \((\mathcal{M}, \varphi)\) which will serve as a counterexample. More specifically, a vector field \(X\) on some Riemannian manifold \((\mathcal{M}, g)\) is to be constructed with a uniformly asymptotically stable submanifold \(\mathcal{S}\) of which the domain of attraction \(\mathcal{D}\) is not homotopy equivalent to \(\mathcal{S}\) itself.

Let

\[
  r(y) = \begin{cases} 
  1 - e^{-\frac{y}{2}} & y > 0 \\
  1 & y \leq 0 
  \end{cases}.
\]

Let \(\mathcal{M}\) be the two-dimensional cylinder embedded in \(\mathbb{R}^3\) defined by

\[
\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r(y)\},
\]

and let

\[
\mathcal{S} = \{(x, y, z) \in \mathcal{M} : x = 0, z = \sqrt{r(y)}\}.
\]

Then \(\mathcal{S}\) is an embedded submanifold and a closed subset in \(\mathcal{M}\). Endowed with the Riemannian metric \(g_\mathcal{M}\) induced by the standard Riemannian metric \(g = (dx)^2 + (dy)^2 + (dz)^2\) on \(\mathbb{R}^3\), \(\mathcal{M}\) is a complete Riemannian manifold. Note that although the Riemannian metric \(g_\mathcal{M}\) is induced by \(g\), the corresponding distance \(d_\mathcal{M}\) on \(\mathcal{M}\) is not the restriction on \(\mathcal{M}\) of the Euclidean distance \(d\) on \(\mathbb{R}^3\).

Generally speaking, it holds that \(d_\mathcal{M}(p, q) \geq d(p, q)\) for \(p, q \in \mathcal{M}\). However, the topology \(\tau_\mathcal{M}\) induced by \(d_\mathcal{M}\) on \(\mathcal{M}\) is exactly the subspace topology inherited from \(\mathbb{R}^3\), meaning that \(\tau_\mathcal{M}\) is the same as the topology induced by (the restriction of) \(d\). Then, if a sequence \(\{p_n\}\) on \(\mathcal{M}\) is a Cauchy sequence with respect to \(d_\mathcal{M}\), it is also a Cauchy sequence with respect to \(d\). Due to the completeness of \(\mathbb{R}^3\) and the closedness of \(\mathcal{M}\) in \(\mathbb{R}^3\), there exists \(\bar{p} \in \mathcal{M}\) such that \(p_n \xrightarrow{d_\mathcal{M}} \bar{p}\) (i.e., the sequence \(\{p_n\}\) converges to \(\bar{p}\) with respect to the metric \(d\)). Since \(d_\mathcal{M}\) and \(d\) induce the same topology on \(\mathcal{M}\), this implies that \(p_n \xrightarrow{d_\mathcal{M}} \bar{p}\) (i.e., the sequence \(\{p_n\}\) converges to \(\bar{p}\) with respect to the metric \(d_\mathcal{M}\)), ensuring the completeness of \((\mathcal{M}, d_\mathcal{M})\).

The map \(h : \mathcal{M}_0 \to \mathcal{M}\) defined by

\[h(x, y, z) = (\sqrt{r(y)} \cdot x, y, \sqrt{r(y)} \cdot z)\]

is a diffeomorphism between the pairs \((\mathcal{M}_0, \mathcal{S}_0)\) and \((\mathcal{M}, \mathcal{S})\). Here, we define \(X\) to be the vector field on \(\mathcal{M}\) related to \(X_0\) by \(h\). That is, \(X = h_* (X_0)\), where \(h_* : TM_0 \to TM\) is the tangent map. Let \(\varphi\) be the flow of \(X\) on \(\mathcal{M}\). Then \(h\) is a conjugacy between the flows \(\varphi_0\) and \(\varphi\). That is, the identity \(h \circ \varphi_0 = \varphi \circ h\) holds, or equivalently,

\[\varphi^t(p'') = h \circ \varphi_0^t \circ h^{-1}(p'')\]  \hspace{1cm} (6)

for all \(p'' \in \mathcal{M}\).

Note that for a point \(p' = (x', y', z')\) on \(\mathcal{M}_0\), the distance \(d_{\mathcal{M}_0}(p', \mathcal{S}_0)\) is exactly the length of the minor arc on the circle \(\mathcal{M}_0 \cap \{(x, y, z) \in \mathbb{R}^3 : y = y'\}\) between \(p'\) and \((0, y', 1)\). Meanwhile, for a point \(p'' = (x'', y'', z'') = h(p')\) on \(\mathcal{M}\), the distance \(d_{\mathcal{M}}(p'', \mathcal{S})\) is no larger than the length of the minor arc on the circle \(\mathcal{M} \cap \{(x, y, z) \in \mathbb{R}^3 : y = y''\}\) between \(p''\) and \((0, y'', \sqrt{r(y'')})\). With \(r(y) \leq 1\), this
implies
\[ d_M(h(p'), S) \leq d_M(p', S_0) \]
for all \( p' \in S_0 \). Combined with (1), it yields the following inequality:
\[ d_M(\varphi^t(p''), S) = d_M(h \circ \varphi_0^t \circ h^{-1}(p''), S) \leq d_M(\varphi_0^t \circ h^{-1}(p''), S_0). \] (7)

Since \( h^{-1} \) maps \( \tilde{D}_0 := \{(x, y, z) \in M : z > -1\} \) diffeomorphically to \( D_0 \), it implies that as \( t \to +\infty \),
\[ d_M(\varphi^t(p), S) \to 0 \] for all \( p \in \tilde{D}_0 \). However, if \( S \) is an attractor, then the domain of attraction of \( S \)
should be
\[ D = \tilde{D}_0 \cup \{(0, y, -\sqrt{r(y)}) : y > 0\}. \]

To see this, first note that for any point \( p'' = (x'', y'', z'') \) in \( \{(0, y, -\sqrt{r(y)}) : y > 0\} \),
\[ \varphi^t(p'') = h \circ \varphi_0^t \circ h^{-1}(p'') \]
\[ = h \circ \varphi_0^t(0, y'', -1) \]
\[ = h(0, \gamma''(t), -1) \]
\[ = (0, \gamma''(t), \sqrt{r \circ \gamma''(t)}), \]
where \( \frac{d\gamma''}{dt} > 0 \). Then from (5) we can deduce that \( \gamma''(t) \) and \( \frac{d\gamma''}{dt} \) both strictly increase with respect
to \( t \). Hence \( d_M(\varphi^t(p''), S) \leq \pi \sqrt{r \circ \gamma''(t)} \to 0 \) as \( t \to +\infty \). Meanwhile, for any point \( p \in M - D \),
i.e. \( p = (0, y, -1) \) with \( y \leq 0 \), \( X|_p = h_* (X|_{\partial p}) = 0 \), and hence \( p \) stays stationary under the flow \( \varphi \).
Therefore, if \( p'' \in M \), then \( \varphi^t(p) \to S \) as \( t \to +\infty \) if and only if \( p'' \in D \). Since \( D \) contains circles in the form \( \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r(y), y > 0\} \) in \( M \), its fundamental group is non-zero and hence
is not homotopy equivalent to \( S \).

To show that this is a counterexample, it remains to prove that \( S \) is indeed a uniformly asymptotically stable manifold of the system \((M, \varphi)\). Let
\[ W := \{(x, y, z) \in M : z > 0\} \cup \{(x, y, z) \in M : y > 1\}. \]
We will first show that \( W \) contains some \( \alpha \)-neighborhood \( N_\alpha \) of \( S \) for some \( \alpha > 0 \), and then show
that for each \( \epsilon > 0 \), there exists some \( T_\epsilon > 0 \) such that \( d_M(\varphi^{[T_\epsilon, +\infty]}(W), S) < \epsilon \).

To see that \( W \) contains some \( \alpha \)-neighborhood of \( S \), we only need to show that there is a positive
distance between its complement \( W^c \) and \( S \). Note that
\[ W^c = \{(x, y, z) \in M : z \leq 0, y \leq 1\} = C \cup K \]
with
\[ C := \{(x, y, z) \in M : z \leq 0, y \leq -\pi\} \]
and
\[ K := \{(x, y, z) \in M : z \leq 0, -\pi \leq y \leq 1\}. \]
Then \( K \) is compact and \( C \) is closed in \( M \), and \( C \cap S, K \cap S \) are both empty. Since \((M, g_M)\) is a
complete Riemannian manifold with the distance \( d_M \), it holds that \( d_M(S, K) > 0 \) as a consequence of
the disjointedness of a closed subset and a compact subset. To see that \( d_M(S, C) > 0 \), note
that \( d_M(S_{y\leq0}, C) = \pi/2 \) and \( d_M(S_{y\geq0}, C) \geq \pi, \) where \( S_{y\leq0} := S \cap \{(x, y, z) \in \mathbb{R}^3 : y \leq 0\} \) and
\[ S_{y \geq 0} := \mathcal{S} \cap \{(x, y, z) \in \mathbb{R}^3 : y \geq 0\} \text{.} \] Then for any \( 0 < \alpha < \min\{d_M(\mathcal{S}, \mathcal{K}), d_M(\mathcal{S}, \mathcal{C})\} \), there holds \( \mathcal{N}_\alpha \subset \mathcal{W} \).

Now we proceed to show that for any \( \epsilon > 0 \), there exists \( T_\epsilon > 0 \). Denote by \( \mathcal{W}_{y>1} \) the set \( \{(x, y, z) \in \mathcal{M} : y > 1\} \). Then \( \mathcal{W} = \mathcal{W}_{z>0} \cup \mathcal{W}_{y>1} \). Note that for each point \( p = (x, y, z) \in \mathcal{W}_{y>1} \), \( X_p \) takes the form \( a_p \frac{\partial}{\partial x} + e^{-1} \frac{\partial}{\partial y} + c_p \frac{\partial}{\partial z} \), and therefore, \( \mathcal{W}_{y>1} \) is an invariant open set of the system \( (\mathcal{M}, \varphi) \). It holds that \( \varphi_t(p) = (x_t, y_t, z_t) \) with \( \frac{dy_t}{dt} > e^{-1} \) for any \( p \in \mathcal{W}_{y>1} \). Choose \( T'' \) to be some positive number large enough such that \( r(e^{-1} \cdot T'') < \frac{\epsilon}{\pi} \). Then for any \( t \geq T'' \) and \( p \in \mathcal{W}_{y>1} \), it holds that \( r(y_t) < r(e^{-1} \cdot T'') \) and therefore \( d_M(\varphi_t(p), \mathcal{S}) \leq \pi \sqrt{r(y_t)} < \epsilon \). To see that the points in \( \mathcal{W}_{z>0} \) converge uniformly towards \( \mathcal{S} \), first note that \( h^{-1}(\mathcal{W}_{z>0}) = \mathcal{W}_{z>0}' \cap \{(x, y, z) \in \mathbb{R}^3 : z > 0\} \). Then combined with \( 7 \), there holds \( d_M(\varphi_t^{[T', +\infty]}(\mathcal{W}_{z>0}), \mathcal{S}) < \epsilon \). Finally, one only needs to choose \( T_\epsilon \) to be \( \min\{T', T''\} \) and the whole argument is complete.

## 5 Conclusion

In this paper, we have revisited Wilson’s theorem (i.e., Theorem 3.4 in [12]) about the relation between the domain of attraction of an attractor and its tubular neighborhood. Specifically, we show with detailed and rigorous proofs that the domain of attraction of a compact asymptotically submanifold of a finite-dimensional smooth manifold of a continuous dynamical system is homeomorphic to its tubular neighborhood. We emphasize that the compactness of the attractor is crucial, without which Wilson’s theorem cannot hold. This is shown by two counterexamples where the attractor is not compact and the state space is either complete or incomplete.

### References

[1] Emmanuel Bermuau, Emmanuel Moulay, Patrick Coirault, and Qing Hui. Topological properties for compact stable attractors in \( \mathbb{R}^n \). In 2019 Proceedings of the Conference on Control and its Applications, pages 15–21. SIAM, 2019.

[2] Sanjay P Bhat and Dennis S Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. Systems & Control Letters, 39(1):63–70, 2000.

[3] Nam Parshad Bhatia and Giorgio P Szegö. Stability theory of dynamical systems. Springer Science & Business Media, 2002.

[4] Michael Brin and Garrett Stuck. Introduction to dynamical systems. Cambridge university press, 2002.

[5] Morris W Hirsch. Differential topology, volume 33. Springer Science & Business Media, 2012.

[6] H.K. Khalil. Nonlinear Systems. Prentice Hall, third edition, 2002.

[7] J.M. Lee. Introduction to Smooth Manifolds: Second Edition. Graduate texts in mathematics. Springer, 2015.

[8] Emmanuel Moulay and Sanjay P Bhat. Topological properties of asymptotically stable sets. Nonlinear Analysis: Theory, Methods & Applications, 73(4):1093–1097, 2010.
[9] Amiya Mukherjee. *Differential topology*. Springer, 2016.

[10] Eduardo D Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2013.

[11] F Wesley Wilson. Smoothing derivatives of functions and applications. *Transactions of the American Mathematical Society*, 139:413–428, 1969.

[12] F Wilson Jr. The structure of the level surfaces of a Lyapunov function. 1967.

[13] Weijia Yao, Bohuan Lin, Brian D. O. Anderson, and Ming Cao. Topological analysis of vector-field guided path following on manifolds. *IEEE Transactions on Automatic Control*, 2021. Conditionally accepted.