Abstract: Let $p$ be an odd prime number. In this paper, we show that the genome $\Gamma(P)$ of a finite $p$-group $P$, defined as the direct product of the genotypes of all rational irreducible representations of $P$, can be recovered from the first group of $K$-theory $K_1(\mathbb{Q}P)$. It follows that the assignment $P \mapsto \Gamma(P)$ is a $p$-biset functor. We give an explicit formula for the action of bisets on $\Gamma$, in terms of generalized transfers associated to left free bisets. Finally, we show that $\Gamma$ is a rational $p$-biset functor, i.e. that $\Gamma$ factors through the Roquette category of finite $p$-groups.

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1. Introduction

Let $p$ be a prime number. This article originates in a joint work with Nadia Romero ([3]), when we started considering the possible applications of genetic bases to the computation of Whitehead groups of finite $p$-groups. Indeed, after the comprehensive book of B. Oliver ([7]), it became clear to N. Romero that these questions have close links to rational representations of $p$-groups. So the idea emerged that possibly genetic bases would be a natural tool in this context, and a first use of this is made in [8].

In particular, when trying to compute various groups related to the Whitehead group of a finite $p$-group $P$ (for odd $p$), a specific product appears, defined in terms of the fields of endomorphisms of the irreducible $\mathbb{Q}P$-modules. After some non trivial reformulation using genetic bases, this product can be viewed as

$$\Gamma(P) = \prod_{S \in \mathcal{B}} (N_P(S)/S)$$

where $\mathcal{B}$ is a genetic basis of $P$. As the groups $N_P(S)/S$ are called the types or genotypes of the irreducible $\mathbb{Q}P$-modules, we call $\Gamma(P)$ the genome of $P$. It is the main subject of this paper.

The connection of $\Gamma(P)$ with Whitehead groups and $K$-theory is established in Theorem [4,3] the genome of $P$ can be recovered as the $p$-torsion
part of $K_1(QP)$. This induces a structure of $p$-biset functor on the correspondence $P \mapsto \Gamma(P)$, which we try to make explicit in Section 5 by giving formulae to compute the action of a $(Q,P)$-biset on $\Gamma(P)$ (Theorem 5.9). Finally, we show that $\Gamma$ is a rational $p$-biset functor, hence it factors through the Roquette category of finite $p$-groups introduced in [3].

2. Review of $K_1$

2.1. Let $A$ be a ring (with 1). Let $GL(A)$ denote the colimit of the linear groups $GL_n(A)$, for $n \in \mathbb{N}_{>0}$, where the inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$ is

$$M \in GL_n(A) \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A).$$

The group $K_1(A)$ is defined as the abelianization of $GL(A)$, namely

$$K_1(A) = GL(A)^{ab} = GL(A)/[GL(A), GL(A)].$$

2.2. Remark: In particular there is a canonical group homomorphism from the group $A^\times = GL_1(A)$ of invertible elements of $A$ to $K_1(A)$, which factors as

$$A^\times \longrightarrow A^\times/[A^\times, A^\times] \longrightarrow K_1(A)$$

2.3. There is an alternative definition of $K_1(A)$: let $\mathcal{P}(A)$ denote the category of pairs $(P,a)$ of a finitely generated projective (left) $A$-module $P$, and an automorphism $a$ of $P$. A morphism $(P,a) \rightarrow (Q,b)$ in $\mathcal{P}(A)$ is a morphism of $A$-modules $f : P \rightarrow Q$ such that $b \circ f = f \circ a$.

Let $[P,a]$ denote the isomorphism class of $(P,a)$ in $\mathcal{P}(A)$, and let $K_{det}(A)$ denote the Grothendieck group with generators the set of these equivalence classes, and the relations of the following two forms

- $[P,a \circ a'] = [P,a] + [P,a']$, for any $a, a' \in \text{Aut}_A(P),$
- $[Q,b] = [P,a] + [R,c]$ whenever there are morphisms $f : [P,a] \rightarrow [Q,b]$ and $g : [Q,b] \rightarrow [R,c]$ in $\mathcal{P}(A)$ such that the sequence

$$0 \rightarrow P \xrightarrow{f} Q \xrightarrow{g} R \rightarrow 0$$

is an exact sequence of $A$-modules (in particular, since $R$ is projective, this sequence splits).
If \( n \in \mathbb{N}_{>0} \) and \( m \in GL_n(A) \), one can view \( m \) as an automorphism of the free module \( A^n \). Let \( \lambda(m) = [A^n, m] \in K_{\text{det}}(A) \).

**2.4. Theorem:** The assignment \( m \mapsto \lambda(m) \) induces a group isomorphism \( K_1(A) \cong K_{\text{det}}(A) \).

**Proof:** See [5] Theorem 40.6. \( \square \)

**2.5.** Let now \( A \) and \( B \) be two rings, and let \( L \) be a \((B,A)\)-bimodule which is finitely generated and projective as a left \( B \)-module. If \( P \) is a finitely generated projective \( A \)-module, then \( P \) is a direct summand of some free \( A \)-module \( A^n \), and then \( L \otimes_A P \) is a direct summand of \( L \otimes_A A^n \cong L^n \) as a left \( B \)-module. Hence \( L \otimes_A P \) is a finitely generated projective left \( B \)-module. Then the functor \( P \mapsto L \otimes_A P \) induces a functor \( T_L : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \) such that

\[
T_L((P, a)) = (L \otimes_A P, L \otimes_A a) .
\]

One checks easily that the defining relations of \( K_{\text{det}}(A) \) are preserved by this functor, hence there is a well-defined induced group homomorphism

\[
t_L : K_{\text{det}}(A) \rightarrow K_{\text{det}}(B)
\]

sending the class \([P, a]\) to the class \([L \otimes_A P, L \otimes_A a]\). This group homomorphism is called the (generalized) transfer associated to the bimodule \( L \).

The properties of the tensor product of bimodules now translate to properties of this transfer homomorphism:

**2.6. Proposition:** Let \( A, B, C \) be rings. In the following assertions, assume that the bimodules involved are finitely generated and projective as left modules. Then:

1. if \( L \cong L' \) as \((B,A)\)-bimodules, then \( t_L = t_{L'} \).
2. if \( L \) is the \((A,A)\)-bimodule \( A \), then \( t_L = \text{Id}_{K_{\text{det}}(A)} \).
3. if \( L \cong L_1 \oplus L_2 \) as \((B,A)\)-bimodules, then \( t_L = t_{L_1} + t_{L_2} \).
4. if \( L \) is a \((B,A)\)-bimodule and \( M \) is a \((C,B)\)-bimodule, then

\[
t_M \circ t_L = t_{M \otimes_B L} .
\]

It follows in particular from (2) and (4) that if \( L \) is a \((B,A)\)-bimodule inducing a Morita equivalence from \( A \) to \( B \), then \( t_L \) is an isomorphism (more precisely, if \( M \) is an \((A,B)\)-bimodule such that \( M \otimes_B L \cong A \) and \( L \otimes_A M \cong B \) as bimodules, then \( t_L \) and \( t_M \) are inverse to one another).
The group $K_1(A)$ has been determined for a number of rings $A$. In particular:

2.8. Theorem:

1. Let $D$ be a division ring. Then $K_1(D) \cong D^\times/[D^\times,D^\times]$.
2. Let $F$ be a field. Then the determinant homomorphism

$$m \in GL_n(F) \to \det(m) \in F^\times$$

induces an isomorphism $K_1(F) \cong F^\times$.

Proof: See [5] Theorem 38.32.

2.9. Proposition: Let $F$ be a field and $G$ be a finite group of order prime to the characteristic of $F$. Let $\text{Irr}_F(G)$ denote a set of representatives of isomorphism classes of irreducible $FG$-modules, and for $V \in \text{Irr}_F(G)$, let $D_V = \text{End}_{FG}(V)$ denote the skew field of endomorphisms of $V$.

Then $V$ is an $(FG,D_V^{\text{op}})$-bimodule, where the action of $g \in G$ and $f \in D_V$ on $v \in V$ is given by $g \cdot v \cdot f = gf(v) = f(gv)$. Let $V^*$ denote the $F$-dual of $V$, considered as a $(D_V^{\text{op}},FG)$-bimodule.

Then the map

$$\tau : K_1(FG) \overset{\prod t_V^*}{\longrightarrow} \prod_{V \in \text{Irr}_F(G)} K_1(D_V^{\text{op}})$$

is a well defined isomorphism of abelian groups, with inverse

$$\tau' : \prod_{V \in \text{Irr}_F(G)} K_1(D_V^{\text{op}}) \overset{\prod t_V}{\longrightarrow} K_1(FG).$$

Proof: As $|G|$ is invertible in $F$, the group algebra $FG$ is semisimple. Moreover for each $V \in \text{Irr}_F(V)$, the skew field $D_V^{\text{op}}$ is also a semisimple $F$-algebra. This shows that $V$ is projective and finitely generated as an $FG$-module, and that $V^*$ is projective and finitely generated as a $D_V^{\text{op}}$-module (that is $V^*$ is a finite dimensional $D_V$-vector space). Hence the generalized transfer maps $t_V : K_1(FG) \to K_1(D_V^{\text{op}})$ and $t_{V^*} : K_1(D_V^{\text{op}}) \to K_1(FG)$ are well defined.

Now for any two finitely generated $FG$-modules $V$ and $W$, the map

$$\alpha \otimes w \mapsto \langle v \in V \mapsto \alpha(v)w \in W \rangle$$
extends to an isomorphism (see e.g. [6] (2.32))

\[ V^* \otimes_{FG} W \to \text{Hom}_{FG}(V, W) \]

of \(((\text{End}_{FG}V)^{op}, (\text{End}_{FG}W)^{op}))\)-bimodules, where the bimodule structure on the right hand side is given by

\[ \forall h \in (\text{End}_{FG}V)^{op}, \forall \psi \in \text{Hom}_{FG}(V, W), \forall k \in (\text{End}_{FG}W)^{op}, h \cdot \psi \cdot k = k \circ \psi \circ h. \]

In case \( V, W \in \text{Irr}_F(G) \) and \( V \neq W \), this yields \( V^* \otimes_{FG} W = 0 \). And if \( V = W \), we have an isomorphism \( V^* \otimes_{FG} V \cong D^D_{V^*} \) of \((D^D_{V^*}, D^D_{V^*})\)-bimodules. Then by Assertions (2) and (4) of Proposition 2.6

\[ t_{V^*} \circ t_W = \begin{cases} 0 & \text{if } V \neq W \\ \text{Id}_{K_1(D^D_{V^*})} & \text{if } V = W. \end{cases} \]

In other words \( \tau \circ \tau' \) is the identity map of \( \prod_{V \in \text{Irr}_F(G)} K_1(D^D_{V^*}). \) Conversely

\[ \tau' \circ \tau = \sum_{V \in \text{Irr}_F(G)} t_V \circ t_{V^*} = t_L, \]

where \( L \) is the \((FG, FG)\)-bimodule \( \bigoplus_{V \in \text{Irr}_F(G)} (V \otimes_{D^D_{V^*}} V^*) \). For each \( V \in \text{Irr}_F(G) \), the bimodule \( V \otimes_{D^D_{V^*}} V^* \cong \text{End}_{D^D_{V^*}}(V) \) is isomorphic to the Wedderburn component of \( FG \) corresponding to the simple module \( V \), and the semisimple algebra \( FG \) is equal to the direct sum of its Wedderburn components. Thus \( L \cong FG \), and \( t_L \) is equal to the identity map of \( K_1(FG) \). \( \square \)

2.10. Corollary : Under the assumptions of Proposition 2.9, there is a group isomorphism

\[ K_1(FG) \cong \prod_{V \in \text{Irr}_F(G)} D^X_{V} / [D^X_{V}, D^X_{V}] . \]

Proof : This follows from Proposition 2.9 and Theorem 2.8 since \( x \mapsto x^{-1} \) is a group isomorphism \( D^X \to (D^{op})^X \), for any skew field \( D \). \( \square \)

2.11. Recall ([2] Chapter 3) that the biset category \( \mathcal{C} \) of finite groups has all finite groups as objects, the set of morphisms in \( \mathcal{C} \) from a group \( G \) to a group \( H \) being the Grothendieck group of (finite) \((H, G)\)-bisets, i.e. the Burnside group \( B(H, G) \). The composition of morphisms in \( \mathcal{C} \) is the linear extension of the product \( (V, U) \mapsto V \times_H U \), for a \((K, H)\)-biset \( V \) and an \((H, G)\)-biset \( U \). 5
A biset functor is an additive functor from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups.

For a prime number $p$, a $p$-biset functor is an additive functor from the full subcategory $\mathcal{C}_p$ of $\mathcal{C}$ consisting of $p$-groups to $\mathcal{A}b$.

Let $1\mathcal{C}$ denote the (non full) subcategory of $\mathcal{C}$ with the same objects, but where the set of morphisms from a group $G$ to a group $H$ is the Grothendieck group $1B(H,G)$ of left free $(H,G)$-bisets. A deflation biset functor is an additive functor from $1\mathcal{C}$ to $\mathcal{A}b$.

\begin{itemize}
  \item[2.12. Proposition:]\end{itemize}
  \begin{itemize}
  \item[1.] Let $R$ be a commutative ring. The assignment $G \mapsto K_1(RG)$ is a deflation functor.
  \item[2.] The assignment $G \mapsto K_1(QG)$ is a biset functor.
  \end{itemize}

\textbf{Proof}: For Assertion 1, if $G$ and $H$ are finite groups, and if $U$ is a finite left free $(H,G)$-biset, then the corresponding permutation $(RH, RG)$-bimodule $RU$ is free and finitely generated as a left $RH$-module. Hence the transfer $t_{RU} : K_1(RG) \to K_1(RH)$ is well defined. If $U'$ is an $(H, G)$-biset isomorphic to $U$, then $RU' \cong RU$ as bimodules, hence $t_{RU'} = t_{RU}$. And if $U$ is the disjoint unions of two $(H, G)$-bisets $U_1$ and $U_2$, then $RU \cong RU_1 \oplus RU_2$, thus $t_{RU} = t_{RU_1} + t_{RU_2}$. This shows that one can extend linearly this transfer construction $U \mapsto t_{RU}$ to a group homomorphism $u \in 1B(H, G) \mapsto K_1(u) \in \text{Hom}_{\mathcal{A}b}(K_1(RG), K_1(RH))$.

Moreover, if $K$ is a third group, and $V$ is a finite left free $(K, H)$-biset, then $t_{RU} \circ t_{RV} = t_{R(U \times H V)}$ since the bimodules $RV \otimes_{RH} RU$ and $R(V \times_H U)$ are isomorphic. Finally, if $U$ is the identity biset at $G$, namely the set $G$ acted on by left and right multiplication, then $RU \cong RG$ as $(RG, RG)$-bimodule, thus $t_{RU} = \text{Id}_{K_1(RG)}$. This completes the proof of Assertion (1).

The proof of Assertion (2) is the same, except that the transfer $t_{QU} : K_1(QG) \to K_1(QH)$ is well defined for an arbitrary finite $(H, G)$-biset $U$: indeed $QU$ is always finitely generated and projective as a $QH$-module.

\section*{3. Review of genetic subgroups}

3.1. Let $p$ be a prime number. A finite $p$-group is called a Roquette $p$-group if it has normal rank 1, i.e. if all its normal abelian subgroups are cyclic. The Roquette $p$-groups (see [3]) are the cyclic groups $C_{p^n}$, for $n \in \mathbb{N}$, if $p$ is odd. The Roquette 2-groups are the cyclic groups $C_{2^n}$, for $n \in \mathbb{N}$, the generalized
quaternion groups $Q_{2^n}$ for $n \geq 3$, the dihedral groups $D_{2^n}$ for $n \geq 4$, and the semidihedral groups $SD_{2^n}$ for $n \geq 4$.

If $P$ is a Roquette $p$-group, then $P$ admits a unique faithful irreducible rational representation $\Phi_P$ ([2] Proposition 9.3.5).

3.2. If $S$ is a subgroup of a finite $p$-group $P$, denote by $Z_P(S)$ the subgroup of $N_P(S)$ defined by $Z_P(S)/S = Z(N_P(S)/S)$. The subgroup $S$ is called genetic if it fulfills the following two conditions:

1. if $x \in P$, then $S^x \cap Z_P(S) \leq S$ if and only if $S^x = S$.
2. the group $N_P(S)/S$ is a Roquette $p$-group.

When $S$ is a genetic subgroup of $P$, let $V(S) = \text{Indinf}^P_{N_P(S)/S}\Phi_{N_P(S)/S}$ denote the $\mathbb{Q}P$-module obtained by inflation of $\Phi_{N_P(S)/S}$ to $N_P(S)$ followed by induction to $P$.

Two genetic subgroups $S$ and $T$ of $P$ are said to be linked modulo $P$ (notation $S \Leftrightarrow_P T$) if there exists an element $x \in P$ such that $S^x \cap Z_P(T) \leq T$ and $xT \cap Z_P(S) \leq S$ (where as usual $S^x = x^{-1}Sx$ and $xT = xTx^{-1}$).

3.3. Theorem: Let $p$ be a prime number and $P$ be a finite $p$-group.

1. If $V$ is a simple $\mathbb{Q}P$-module, then there exists a genetic subgroup $S$ of $P$ such that $V \cong V(S)$.
2. If $S$ is a genetic subgroup of $P$, then there is an isomorphism of $\mathbb{Q}$-algebras

$$\text{End}_{\mathbb{Q}P}(S) \cong \text{End}_{\mathbb{Q}N_P(S)/S}\Phi_{N_P(S)/S}$$

induced by the induction-inflation functor from $\mathbb{Q}N_P(S)/S$-modules to $\mathbb{Q}P$-modules.

3. If $S$ and $T$ are genetic subgroups of $P$, then $V(S) \cong V(T)$ if and only if $S \Leftrightarrow_P T$. In this case, the groups $N_P(S)/S$ and $N_P(T)/T$ are isomorphic.

Proof: See Theorem 9.4.1, Lemma 9.4.3, Definition 9.4.4, Corollary 9.4.5, Theorem 9.5.6 and Theorem 9.6.1 of [2].

It follows in particular that the relation $\Leftrightarrow_P$ is an equivalence relation on the set of genetic subgroups of $P$. A genetic basis of $P$ is by definition a set of representatives of genetic subgroups of $P$ for this equivalence.

It also follows that if $V$ is a simple $\mathbb{Q}P$-module, and if $S$ is a genetic subgroup of $P$ such that $V \cong V(S)$, then the group $N_P(S)/S$ does not depend on the choice of such a genetic subgroup $S$. This factor group is called the type of $V$ ([2] Definition 9.6.8). Laurence Barker ([1]) has introduced the word genotype instead of type, and we will follow this terminology.
3.4. Definition: Let $p$ be a prime number and $P$ be a finite $p$-group. The genome $\Gamma(P)$ of $P$ is the product group

$$\Gamma(P) = \prod_{S \in B} (N_P(S)/S) ,$$

where $B$ is a genetic basis of $P$. It is well defined up to isomorphism.

More precisely, suppose that $B$ and $B'$ are genetic bases of a $p$-group $P$. Then for $S \in B$, there exists a unique $S' \in B'$ such that there exists some $x \in P$ with

$$S^x \cap Z_P(S') \leq S'$$

and the correspondence $S \mapsto S'$ is a bijection from $B$ to $B'$. Moreover, for each $S \in B$ corresponding to $S' \in B'$, the set $D$ of elements $x$ satisfying (3.5) is a single $(N_P(S), N_P(S'))$-double coset in $P$ ([4], Proposition 9.6.9).

Let $x \in D$. Then for each $n \in N_P(S)/S$, there is a unique element $n' \in N_P(S')/S'$ such that $nSx = xS'n'$, and the map $n \mapsto n'$ is a group isomorphism $N_P(S)/S \to N_P(S')/S'$, which only depends on $x$ up to interior automorphism of $N_P(S)/S$. In particular, when $p$ is odd, the group $N_P(S)/S$ is cyclic, so this group isomorphism does not depend on $x$.

Thus for odd $p$, this yields a canonical group isomorphism

$$\prod_{S \in B} (N_P(S)/S) \xrightarrow{\gamma_{B,B'}} \prod_{S' \in B'} (N_P(S')/S') .$$

3.7. Remark: Let $p$ be a prime number, and $P$ be a finite $p$-group. Since the Roquette $p$-groups are all indecomposable (that is, they cannot be written as a direct product of two non-trivial of their subgroups), the genotypes of the simple $Q_P$-modules are determined by the group $\Gamma(P)$: by the Krull-Remak-Schmidt theorem, the group $\Gamma(P)$ can be written as a direct product of indecomposable groups $\Gamma_1, \ldots, \Gamma_r$, and such a decomposition is unique (up to permutation and isomorphism of the factors). Then $\Gamma_1, \ldots, \Gamma_r$ are the genotypes of the simple $Q_P$-modules.

In terms of the Roquette category $\mathcal{R}_p$ (see Section 7 or [3]), this means that two finite $p$-groups $P$ and $Q$ become isomorphic in $\mathcal{R}_p$ if and only if their genomes $\Gamma(P)$ and $\Gamma(Q)$ are isomorphic (as groups) (see [3] Proposition 5.14).
4. $K$-theory and genome

4.1. Lemma : Let $p$ be a prime, and $C$ be a cyclic $p$-group. Recall that $\Phi_C$ is the unique faithful irreducible rational representation of $C$, up to isomorphism.

1. If $C = 1$, then $\Phi_C = \mathbb{Q}$.
2. If $C \neq 1$, let $Z$ be the unique subgroup of order $p$ of $C$. Then there is an exact sequence
   \[(4.2) \quad 0 \to \Phi_C \to \mathbb{Q}C \to \mathbb{Q}(C/Z) \to 0,
   \]
of $(\mathbb{Q}C, \mathbb{Q}C)$-bimodules, where $\mathbb{Q}C \to \mathbb{Q}(C/Z)$ is the canonical surjection.
3. If $C$ has order $p^n$, then the algebra $\text{End}_{\mathbb{Q}C}(\Phi_C)$ is isomorphic to the cyclotomic field $\mathbb{Q}(\zeta_{p^n})$, and if $p > 2$, the map sending $c \in C$ to the endomorphism $\varphi \mapsto \varphi c$ of $\Phi_C$ is a group isomorphism from $C$ to the $p$-torsion part $\mathbb{Q}(\zeta_{p^n})^\times$ of the multiplicative group $\mathbb{Q}(\zeta_{p^n})^\times$.

Proof : Assertion 1 is trivial. Assertion 2 follows e.g. from [2], Proposition 9.3.5. A different proof consists in observing that if $C$ has order $p^n$, then the algebra $\mathbb{Q}C$ is isomorphic to $\mathbb{Q}[X]/(X^{p^n} - 1)$, and the projection map $\mathbb{Q}C \to \mathbb{Q}(C/Z)$ becomes the canonical map
   \[\mathbb{Q}[X]/(X^{p^n} - 1) \to \mathbb{Q}[X]/(X^{p^{n-1}} - 1).\]
The kernel of this map is now clearly isomorphic to $\mathbb{Q}[X]/(\gamma_{p^n})$, where $\gamma_{p^n}$ is the $p^n$-th cyclotomic polynomial, that is, the $p^n$-th cyclotomic field, which is clearly a simple faithful module for the cyclic group generated by $X$ in the algebra $\mathbb{Q}[X]/(X^{p^n} - 1)$. Observe moreover that the exact sequence $(4.2)$ is indeed a sequence of $(\mathbb{Q}C, \mathbb{Q}C)$-bimodules.

The first part of Assertion 3 follows easily. For the last part, let $\zeta_{p^n}$ be a primitive $p^n$-th root of unity. Observe that a $p$-torsion element in $\mathbb{Q}(\zeta_{p^n})^\times$ is a $p^n$-th root of unity. Hence the $p$-torsion part of $\mathbb{Q}(\zeta_{p^n})^\times$ is cyclic of order $p^n$, generated by $\zeta_{p^n}$. \hfill $\Box$

4.3. Theorem : Let $p$ be an odd prime, and $P$ be a finite $p$-group, and $\mathcal{B}$ be a genetic basis of $P$. If $S$ is a genetic subgroup of $P$, and $a \in N_P(S)/S$, view $a$ as an automorphism of $\Phi_{N_P(S)/S}$, and let $\hat{a}$ denote the corresponding automorphism of $V(S) = \text{Ind}^P_{N_P(S)} \Phi_{N_P(S)/S}$.
1. The group homomorphism

\[ \Gamma(P) = \prod_{S \in \mathcal{B}} \left( N_P(S)/S \right)^{\nu_B} K_1(QP) \]

sending \( a \in N_P(S)/S \), for \( s \in \mathcal{B} \), to the class \([V(S), \tilde{a}]\) in \( K_1(QP) \) is an isomorphism of the genome \( \Gamma(P) \) onto the \( p \)-torsion part \( pK_1(QP) \) of \( K_1(QP) \).

2. If \( \mathcal{B}' \) is another genetic basis of \( P \), and \( \gamma_{\mathcal{B}', \mathcal{B}} \) is the canonical isomorphism defined in 3.6, then

\[ \nu_{\mathcal{B}'} \circ \gamma_{\mathcal{B}', \mathcal{B}} = \nu_{\mathcal{B}}. \]

**Proof:** Since \( p \) is odd, the Roquette \( p \)-groups are the cyclic \( p \)-groups. Assertion 1 now follows from Proposition 2.9, Theorem 3.3, and Lemma 4.1.

For Assertion 2, let \( S \in \mathcal{B} \) and let \( S' \) be the unique element of \( \mathcal{B}' \) such that \( S' \prec_p S \). Let \( \varphi : N_P(S)/S \to N_P(S')/S' \) be the restriction of \( \gamma_{\mathcal{B}', \mathcal{B}} \) to \( N_P(S)/S \). If \( a \in N_P(S)/S \), let \( a' = \varphi(a) \). Then \( \varphi \) induces an isomorphism of \( QP \)-modules \( \tilde{\varphi} : V(S) \to V(S') \) such that the diagram

\[
\begin{array}{ccc}
V(S) & \xrightarrow{\tilde{\varphi}} & V(S) \\
\varphi \downarrow & & \downarrow \tilde{\varphi} \\
V(S') & \xrightarrow{\tilde{a}'} & V(S')
\end{array}
\]

is commutative. Hence \( (V(S), \tilde{a}) \cong (V(S'), \tilde{a}') \) in \( \mathcal{P}(QP) \), thus \([V(S), \tilde{a}] = [V(S'), \tilde{a}'] \) in \( K_1(QP) \), as was to be shown.

**4.4. Remark:** The elements of odd order of \( \mathbb{Q}(\zeta_p^n) \times \) are the \( p^n \)-th roots of unity. So \( \Gamma(P) \) is also the odd-torsion part of \( K_1(QP) \).

**4.5. Corollary:** Let \( p \) be an odd prime. Then the correspondence sending a finite \( p \)-group \( P \) to its genome \( \Gamma(P) \) is a \( p \)-biset functor.

**Proof:** Indeed by Proposition 2.12, the assignment \( P \mapsto K_1(QP) \) is a \( p \)-biset functor. So its \( p \)-torsion part is also a \( p \)-biset functor.

5. Explicit transfer maps

We begin with a slight generalization of the transfer homomorphism, associated to a left-free biset:
5.1. Lemma and Definition: Let $G$ and $H$ be finite groups, and let $\Omega$ be a left free $(H, G)$-biset. Let $[H \setminus \Omega]$ be a set of representatives of $H$-orbits on $\Omega$. For $g \in G$, and $x \in \Omega$, let $h_{g,x} \in H$ and $\sigma_g(x) \in [H \setminus \Omega]$ be the elements defined by $xg = h_{g,x}\sigma_g(x)$.

1. The map $g \in G \mapsto \prod_{x \in [H \setminus \Omega]} h_{g,x}$ (in any order) induces a well defined group homomorphism

$$\text{Ver}_\Omega : G/[G, G] \to H/[H, H]$$

called the (generalized) transfer associated to $\Omega$.

2. If $\Omega' \cong \Omega$ as $(H, G)$-bisets, then $\text{Ver}_{\Omega'} = \text{Ver}_{\Omega}$.

3. If $\Omega = \Omega_1 \sqcup \Omega_2$ as $(H, G)$-bisets, then $\text{Ver}_\Omega = \text{Ver}_{\Omega_1} + \text{Ver}_{\Omega_2}$.

4. If $K$ is another finite group, and $\Omega'$ is a finite left free $(K, H)$-biset, then $\Omega' \times_H \Omega$ is a finite left free $K$-set, and

$$\text{Ver}_{\Omega'} \circ \text{Ver}_\Omega = \text{Ver}_{\Omega' \times_H \Omega}.$$

The notation and terminology comes from the classical transfer from $G/[G, G]$ to $H/[H, H]$, when $H$ is a subgroup of $G$: the corresponding biset $\Omega$ is the set $G$ itself, in this case.

**Proof:** Changing the set of representatives $[H \setminus \Omega]$ amounts to replacing each $x \in [H \setminus \Omega]$ by $\eta_x x$, for some $\eta_x \in H$. This changes the element $h_{g,x}$ in $h_{g',x} = \eta_x h_{g,x} \eta_{\sigma_g(x)}^{-1}$, so the product over $x \in [H \setminus \Omega]$ of the elements $h_{g',x}$ is equal to the product of the elements $h_{g,x}$ in the abelianization $H/[H, H]$. Hence $\text{Ver}_\Omega$ does not depend on the choice of a set of representatives.

It follows moreover from the definition that for $g, g' \in G$ and $x \in [H \setminus \Omega]$, we have $h_{gg',x} = h_{g,x}h_{g',\sigma_g(x)}$. Hence

$$\prod_{x \in [H \setminus \Omega]} h_{gg',x} = \prod_{x \in [H \setminus \Omega]} h_{g,x} \prod_{x \in [H \setminus \Omega]} h_{g',\sigma_g(x)} = \prod_{x \in [H \setminus \Omega]} h_{g,x} \prod_{x \in [H \setminus \Omega]} h_{g',x}$$

in $H/[H, H]$, so $\text{Ver}_\Omega$ is a group homomorphism. This proves Assertion 1.

For Assertion 2, let $f : \Omega \to \Omega'$ be an isomorphism of $(H, G)$-bisets. Then the set $f([H \setminus \Omega])$ is a set of representatives of the $H$-orbits on $\Omega'$. Moreover for $x \in [H \setminus \Omega]$ and $g \in G$,

$$f(x)g = f(xg) = f(h_{g,x}\sigma_g(x)) = h_{g,x}f(\sigma_g(x)),$$
so \( \text{Ver}_{\Omega'}(g) = \prod_{x \in \Omega} h_{g,x} = \text{Ver}_{\Omega}(g) \), which proves Assertion 2.

Assertion 3 is clear, since \([H \setminus \Omega] = [H \setminus \Omega_1] \sqcup [H \setminus \Omega_2]\).

For Assertion 4, it is straightforward to check that \( \Omega' \times_H \Omega \) is left free. Moreover, the set of pairs \((x', x) \in \Omega' \times_H \Omega\), for \(x' \in [K \setminus \Omega']\) and \(x \in [H \setminus \Omega]\), is a set of representatives of \( K \) orbits on \( \Omega' \times_H \Omega \). Then for \(x' \in [K \setminus \Omega']\) and \(x \in [H \setminus \Omega]\), and \(g \in G\)

\[
(x', x)g = (x', xg) = (x', h_{g,x} \sigma_g(x)) = (x' h_{g,x}, \sigma_g(x)) = h_{h_{g,x}, x'} \tau_{h_{g,x}}(x'), \sigma_g(x) \,,
\]

where \(k_{h,x'} \in K\) and \(\tau_{h}(x') \in [K \setminus \Omega']\) are defined by \(x'h = k_{h,x'} \tau_{h}(x')\), for \(h \in H\) and \(x' \in [K \setminus \Omega']\).

It follows that

\[
\text{Ver}_{\Omega'} \times_h \Omega(g) = \prod_{x' \in [K \setminus \Omega']} h_{x', x} = \text{Ver}_{\Omega'} \left( \prod_{x \in [H \setminus \Omega]} h_{g,x} \right) = \text{Ver}_{\Omega'} \circ \text{Ver}_{\Omega}(g) \,,
\]

which completes the proof.

\[\square\]

5.2. Corollary 4.5 shows that there exists a \( p \)-biset functor structure on the assignment \( P \mapsto \Gamma(P) \) for \( p \)-groups, when \( p \) is odd. This raises the following question: suppose that \( P \) and \( Q \) are finite \( p \)-groups, that \( \mathcal{B}_P \) is a genetic basis of \( P \), and \( \mathcal{B}_Q \) is a genetic basis of \( Q \). When \( U \) is a finite \((Q,P)\)-biset, how can we compute the map

\[
\Gamma(U) : \Gamma(P) = \prod_{S \in \mathcal{B}_P} (N_P(S)/S) \rightarrow \Gamma(Q) = \prod_{T \in \mathcal{B}_Q} (N_Q(T)/T)
\]

giving the action of the biset \( U \)?

This amounts to finding the map

\[
\Gamma(U)_{T,S} : \overline{N}_P(S) = N_P(S)/S \rightarrow \overline{N}_Q(T) = N_Q(T)/T
\]

for each pair \((T, S)\) of a genetic subgroup \( T \) of \( Q \) and a genetic subgroup \( S \) of \( P \), defined as follows: if \( a \in \overline{N}_P(S) \), then \( a \) can be viewed as an automorphism of the \( Q \overline{N}_P(S) \)-module \( \Phi_{\overline{N}_P(S)} \), viewed as an ideal of \( Q \overline{N}_P(S) \) as in 4.2. Then \( \tilde{a} = \text{Ind}_{\overline{N}_P(S)}^{Q \overline{N}_P(S)} a \) is an automorphism of \( V(S) = \text{Ind}_{\overline{N}_P(S)}^{Q \overline{N}_P(S)} \Phi_{\overline{N}_P(S)} \), hence an element \( \tilde{a} = [V(S), \tilde{a}] \) of \( K_1(QP) \). This element is mapped by \( t_{QU} \) to the element

\[
t_{QU}(\tilde{a}) = [QU \otimes_{QP} V(S), QU \otimes_{QP} \tilde{a}]
\]

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of $K_1(\mathbb{Q}Q)$. This in turn is mapped to the element $t_{V(T)} \circ t_{QU}(\hat{a})$ of the direct summand $K_1(F_T)$ of $K_1(\mathbb{Q}Q)$ corresponding to the simple $\mathbb{Q}Q$-module $V(T)$ as in Proposition 2.9, where $F_T$ is the field $D_{V(T)} = \text{End}_{\mathbb{Q}Q}V(T)$.

Thus to find $\Gamma(U)_{T,S}(a)$, we have to compute the element

$$[V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} V(S), V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} \hat{a}]$$

of $K_1(F_T) \cong F_T^*$, and identify it as an element of $\overline{N}_Q(T)$.

We set $L(U)_{T,S} = V(T)^* \otimes_{\mathbb{Q}Q} \mathbb{Q}U \otimes_{\mathbb{Q}P} V(S)$ for simplicity. First we observe that the induction-inflation functor $\text{Indinf}_{\mathbb{Q}P(S)}^{N_{P(S)}}$ is isomorphic to the functor $Q(P/S) \otimes_{N_{P}(S)}(-)$, where $Q(P/S)$ is endowed with its natural structure of $(QP, Q(N_{P}(S)))$-bimodule. Hence

$$\mathbb{Q}U \otimes_{\mathbb{Q}P} V(S) = \mathbb{Q}U \otimes_{\mathbb{Q}P} \text{Indinf}_{\mathbb{Q}P(S)}^{N_{P(S)}} \Phi_{N_{P}(S)}$$

$$\cong \mathbb{Q}U \otimes_{\mathbb{Q}P} Q(P/S) \otimes_{Q_{N_{P}(S)}} \Phi_{N_{P}(S)}$$

$$\cong \mathbb{Q}(U/S) \otimes_{Q_{N_{P}(S)}} \Phi_{N_{P}(S)}$$

where $\mathbb{Q}(U/S)$ is given its natural structure of $(QP, Q(N_{P}(S)))$-bimodule.

Tensoring on the left with $V(T)^*$, an using a similar argument, we get that

$$L(U)_{T,S} \cong \Phi_{\overline{N}_Q(T)}^* \otimes_{Q_{\overline{N}_Q(T)}} Q(T/U/S) \otimes_{Q_{N_{P}(S)}} \Phi_{N_{P}(S)}$$

where $Q(T/U/S)$ is the permutation $(Q\overline{N}_Q(T), Q\overline{N}_P(S))$-bimodule associated to the $(\overline{N}_Q(T), \overline{N}_P(S))$-biset $T/U/S$. Moreover $\Phi_{\overline{N}_Q(T)}$ is self dual, since it is the unique faithful rational irreducible representation of $\overline{N}_Q(T)$, so we can replace $\Phi_{\overline{N}_Q(T)}^*$ by $\Phi_{\overline{N}_Q(T)}$ in the right hand side of the previous isomorphism.

Now the biset $T/U/S$ splits as a disjoint union

$$T/U/S = \biguplus_{\omega \in N_{Q}(T)/U/N_{P}(S)} T/\omega/S$$

of transitive $(\overline{N}_Q(T), \overline{N}_P(S))$-bisets, where $N_{Q}(T)/U/N_{P}(S)$ is the set of $(N_{Q}(T), N_{P}(S))$-orbits on $U$. This yields a decomposition

$$(5.3) \quad \mathbb{Q}(T/U/S) \cong \bigoplus_{\omega \in N_{Q}(T)/U/N_{P}(S)} \mathbb{Q}(T/\omega/S)$$

as $(Q\overline{N}_Q(T), Q\overline{N}_P(T))$-bimodules.
5.4. Lemma : Let $C$ and $D$ be cyclic $p$-groups, and let $\Omega$ be a transitive $(D, C)$-biset. Then $\Phi_D \otimes_{QD} Q\Omega = 0$ unless $\Omega$ is left free, and $Q\Omega \otimes_{QC} \Phi_C = 0$ unless $\Omega$ is right free.

Proof : Suppose that the action of $C$ is not free. This means that $C$ is non-trivial, and that the unique subgroup $Z$ of order $p$ of $C$ acts trivially on $\Omega$: indeed since $\Omega$ is a transitive biset, the stabilizers in $C$ of the points of $\Omega$ are conjugate in $C$, hence equal since $C$ is abelian. So these stabilizers all contain $Z$ if one of them is non trivial. Then $\Omega$ is inflated from a $(C/Z)$-set $\Omega$, and then $Q\Omega \cong Q\Omega \otimes_{Q(C/Z)} Q(C/Z)$. But $Q(C/Z) \otimes_{QC} \Phi_C$ is the module of $Z$-coinvariants on $\Phi_C$, hence it is zero, since $\Phi_C$ is faithful. Hence $Q\Omega \otimes_{QC} \Phi_C = 0$ in this case. Similarly, if the action of $D$ is not free, then $\Phi_D \otimes_{QD} Q\Omega = 0$.

5.5. It follows from Lemma 5.4 that to compute

$$\Phi^*_{NQ(T)} \otimes_{QNQ(T)} Q(T \setminus U/S) \otimes_{QNP(S)} \Phi_{NP(S)}$$

using decomposition 5.3, we can restrict to orbits $\omega = N_Q(T)uNP(S)$, where $u \in U$, for which the $(N_Q(T), NP(S))$-biset $T \setminus \omega/S$ is left and right free. The left stabilizer of the element $TuS$ of this biset is equal to

$$\{ xT \in N_G(T) \mid \exists s \in S, xu = us \} ,$$

hence $T \setminus \omega/S$ is left free if and only if

$$uS \cap N_Q(T) \leq T ,$$

where $uS = \{ x \in Q \mid \exists s \in S, xu = us \}$ (2 Notation 2.3.16).

Similarly $T \setminus \omega/S$ is right free if and only if

$$T^u \cap NP(S) \leq S ,$$

where $T^u = \{ x \in P \mid \exists t \in T, tu = ux \}$.

Finally, the bimodule $L(U)_{T,S}$ is isomorphic to

$$\bigoplus_{\substack{u \in [N_Q(T)] \setminus U/\ NP(S) \\ uS \cap N_Q(T) \leq T \\ T^u \cap NP(S) \leq S}} \Phi_{NQ(T)} \otimes_{QNQ(T)} Q(T \setminus N_Q(T)uNP(S)/S) \otimes_{QNP(S)} \Phi_{NP(S)} ,$$

where $[N_Q(T)] \setminus U/\ NP(S)$ is a set of representatives of $(N_Q(T), NP(S))$-orbits on $U$. 

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5.7. Lemma: Let $p$ be an odd prime, and let $C$ and $D$ be cyclic $p$-groups. Let moreover $\Omega$ be a left and right free finite $(D,C)$-biset. Let $a \in C$, viewed as an automorphism of the $\mathbb{Q}C$-module $\Phi_C$. Then the image of $[\Phi_C,a]$ in $K_1(\mathbb{Q}D)$ by the transfer associated to the bimodule $L = \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega$ is equal to the image of $\text{Ver}_\Omega(a) \in D = D/[D,D]$ by the map $\alpha_{\mathbb{Q}D}$ of Remark 2.2.

Proof: By Lemma 5.1 and Proposition 2.6 we can assume that $\Omega$ is a transitive biset, of the form $(D \times C)/B$ for some subgroup $B$ of $D \times C$. Then $\Omega$ is left and right free if and only if there exists a subgroup $E$ of $C$ and an injective group homomorphism $\varphi : E \rightarrow D$ such that $B = \{(\varphi(e),e) \mid e \in E\}$. There are two cases:

- either $E = 1$: in this case $\Omega = D \times C$, so $\mathbb{Q}\Omega \cong \mathbb{Q}D \otimes_{\mathbb{Q}} \mathbb{Q}C$, and $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C \cong \Phi_D \otimes_{\mathbb{Q}} \Phi_C$. As a vector space over the cyclotomic field $F$ of endomorphisms of $\Phi_D$, it is isomorphic to $F \otimes_{\mathbb{Q}} \Phi_C$. The action of $a \in C$ on this vector space is given by the matrix of $a$ acting on $\Phi_C$.

Suppose that $a$ is a generator of $C$, of order $p^n$. Then this action is the action by multiplication of a primitive $p^n$-th root of unity $\zeta$ on the field $\mathbb{Q}(\zeta)$. As an element of $K_1(F)$, it is equal to the determinant of the matrix representing this multiplication, i.e. to the norm $N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta)$, which is equal to 1, as the $p^n$-th cyclotomic polynomial has even degree $p^{n-1}(p-1)$ and value 1 at 0. It follows that $[\Phi_C,a]$ is mapped to the identity element of $K_1(\mathbb{Q}D)$ in this case. Since this holds for a generator $a$ of $C$, the same is true for any element $a$ of $C$.

In this case also, a set of representatives of $[D\backslash \Omega]$ is the set $1 \times C$, which is invariant by right multiplication by $C$. It follows that the elements $d_{a,x} \in D$ defined for $a \in C$ and $x \in [D\backslash \Omega]$ by $xa = d_{a,x}x'$, for $x' \in [D\backslash \Omega]$, are all equal to $1$. So the transfer $\text{Ver}_\Omega$ is also the trivial homomorphism in this case.

- or $E \neq 1$: let $Z$ denote the unique subgroup of order $p$ of $C$. Tensoring over $\mathbb{Q}C$ the exact sequence of $(\mathbb{Q}C,\mathbb{Q}C)$-bimodules

$$0 \rightarrow \Phi_C \rightarrow \mathbb{Q}C \rightarrow \mathbb{Q}(C/Z) \rightarrow 0$$

with $\Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega$ gives the exact sequence

$$0 \rightarrow \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \otimes_{\mathbb{Q}C} \Phi_C \rightarrow \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}\Omega \rightarrow \Phi_D \otimes_{\mathbb{Q}D} \mathbb{Q}(\Omega/Z) \rightarrow 0 .$$

But $\Omega/Z$ is not free as a left $D$-set, since the unique subgroup $\varphi(Z)$ of order $p$ of $D$ stabilizes $BZ \in \Omega/Z$, as for $z \in Z$ and $e \in E$

$$\varphi(z)(\varphi(e),e) = (\varphi(ze),e) = (\varphi(ze),ze)z .$$
By Lemma 5.4, it follows that $\Phi_D \otimes Q \Omega \otimes QC \Phi_C \cong \Phi_D \otimes Q \Omega$.

As a vector space over the cyclotomic field $F$ of endomorphisms of $\Phi_D$, this is isomorphic to $F \otimes Q [D \setminus \Omega]$. The action of $a \in C$ on this vector space is given for $x \in [D \setminus \Omega]$ and $\lambda \in F$ by

$$(\lambda \otimes x)a = \lambda \otimes xa = \lambda \otimes d_{a,x} \sigma_a(x) = \lambda d_{a,x} \otimes \sigma_a(x),$$

where $d_{a,x} \in D$ and $\sigma_a(x) \in [D \setminus \Omega]$ are defined by $xa = d_{a,x} \sigma_a(x)$. In other words, the matrix of the action of $a$ is the product of the permutation matrix of $\sigma_a$ with a diagonal matrix of coefficients $d_{a,x}$, for $x \in [D \setminus \Omega]$. In $K_1(F)$, this matrix is equal to its determinant, that is the signature of $\sigma_a$, which is equal to 1 as $\sigma_a$ is a product of cycles of odd length (equal to some power of $p$), multiplied by the product of the elements $d_{a,x}$, that is the image in $K_1(Q)$ of $\text{Ver}_\Omega(a)$, as was to be shown.

5.8. Remark : Recall that if $Q$ is a central subgroup of finite index $n$ in a group $G$, then the transfer $G/[G,G] \to Q$ is induced by the map $g \mapsto g^n$ from $G$ to $Q$ (see [10] Theorem 7.47). It follows easily that in the situation of Lemma 5.7, if $\Omega = (D \times C)/B$, where $B = \{(\varphi(e), e) \mid e \in E\}$ for a subgroup $E$ of $C$ and an injective homomorphism $\varphi : E \to D$, the transfer $\text{Ver}_\Omega : C \to D$ is given by $a \mapsto \varphi(a^{[C:E]})$. Moreover $|C : E| = |D \setminus \Omega|$.

5.9. Theorem : Let $p$ be an odd prime. Let $P$ and $Q$ be finite $p$-groups, and let $U$ be a finite $(Q, P)$-biset.

1. Let $S$ be a genetic subgroup of $P$ and $T$ be a genetic subgroup of $Q$. Let $D(U)_{T,S}$ be the set of orbits $N_Q(T)uN_P(S)$ of those $u \in U$ for which $T^u \cap N_P(S) \leq S$ and $u \cap N_Q(T) \leq T$ (see 5.6 for notation). Then for $\omega \in D(U)_{T,S}$, the set $T \setminus \omega / S$ is a left and right free $(N_Q(T)/T, N_P(S)/S)$-biset, and the map

$$\Gamma(U)_{T,S} : N_P(S)/S \to N_Q(T)/T$$

sending $a \in N_P(S)/S$ to

$$\prod_{\omega \in D(U)_{T,S}} \text{Ver}_{T \setminus \omega / S}(a)$$

is a well defined group homomorphism.
2. Let $\mathcal{B}_P$ and $\mathcal{B}_Q$ be genetic bases of $P$ and $Q$, respectively. Then the map $\Gamma(U) : \Gamma(P) \to \Gamma(Q)$ giving the biset functor structure of $\Gamma$ is the map

$$\Gamma(P) = \prod_{S \in \mathcal{B}_P} (N_P(S)/S) \to \prod_{T \in \mathcal{B}_Q} (N_Q(T)/T) = \Gamma(Q)$$

with component $(T, S)$ equal to $\Gamma(U)_{T,S}$.

**Proof:** This results from Paragraph 5.2, Lemma 5.3, Paragraph 5.5, and Lemma 5.7.

6. Examples

6.1. Proposition: Let $P$ be a finite $p$-group, for $p$ odd, and let $\mathcal{B}$ be a genetic basis of $P$. Let $N \trianglelefteq P$, and $\mathcal{P} = P/N$. Let $\mathcal{B}_N$ be the subset of $\mathcal{B}$ defined by

$$\mathcal{B}_N = \{ S \in \mathcal{B} | S \supseteq N \}.$$ 

Then:

1. The set $\overline{\mathcal{B}} = \{ \overline{S} = S/N \mid S \in \mathcal{B}_N \}$ is a genetic basis of $\mathcal{P}$.
2. Up to the identification of $N_\mathcal{P}(\overline{S})$ with $N_P(S)/S$, for $S \in \mathcal{B}_N$, the inflation morphism

$$\text{Inf}^\mathcal{P}_{\mathcal{P}/N} : \Gamma(P/N) = \prod_{S \in \mathcal{B}} (N_\mathcal{P}(\overline{S})/\overline{S}) \to \Gamma(P) = \prod_{S \in \mathcal{B}} (N_P(S)/S)$$

is the embedding in the product of the factors of $\Gamma(P)$ corresponding to genetic subgroups $S$ containing $N$.
3. Similarly, the deflation morphism

$$\text{Def}^\mathcal{P}_{\mathcal{P}/N} : \Gamma(P) = \prod_{S \in \mathcal{B}} (N_P(S)/S) \to \Gamma(P/N) = \prod_{\overline{S} \in \overline{\mathcal{B}}} (N_\mathcal{P}(\overline{S})/\overline{S})$$

is the projection onto the product of the factors of $\Gamma(P)$ corresponding to genetic subgroups $S$ containing $N$.

**Proof:** Assertion 1 is clear from the definitions: if $N \leq S \leq P$, then $S$ is
genetic in $P$ if and only if $S/N$ is genetic in $P/N$. Moreover the relation $\sim_{P/N}$ gives the relation $\sim_{P}$ by inflation.

Now the inflation morphism $\text{Inf}_{P/N}^{P}$ is defined by the $(P, \mathcal{F})$-biset $U = P/N$, with natural actions of $P$ and $\mathcal{F}$. Let $u = xN$ be an element of $U$, for some $x \in P$. Let $T \in \mathcal{B}$ and $\mathcal{S} \in \mathcal{B}$.

$$T^u \cap N_{\mathcal{F}}(\mathcal{S}) = \{yN \in N_{\mathcal{F}}(\mathcal{S}) \mid \exists t \in T, tu = u\mathcal{F}\}$$
$$= \{yN \in N_{\mathcal{F}}(\mathcal{S}) \mid \exists t \in T, txN = xyN\}$$
$$= \{yN \in N_{\mathcal{F}}(\mathcal{S}) \mid yN \in T^x N\}$$
$$= (T^x N \cap N_{P}(S))/N = (T^x \cap N_{P}(S))N/N,$$

where the last two equalities hold because $S \geq N$. Hence $T^u \cap N_{\mathcal{F}}(\mathcal{S}) \leq \mathcal{S}$ if and only if $T^x \cap N_{P}(S) \leq S$.

On the other hand

$$\mathcal{S}^u \cap N_{P}(T) = \{y \in N_{P}(T) \mid \exists s \in \mathcal{S}, yu = us\mathcal{F}\}$$
$$= \{y \in N_{P}(T) \mid \exists s \in \mathcal{S}, yxN = xNs\}$$
$$= \{y \in N_{P}(T) \mid y \in ^xS\}$$
$$= ^xS \cap N_{P}(T).$$

Hence $\mathcal{S}^u \cap N_{P}(T) \leq T$ if and only if $^xS \cap N_{P}(T) \leq T$.

If moreover $T^u \cap N_{\mathcal{F}}(\mathcal{S}) \leq \mathcal{S}$, i.e. $T^x \cap N_{P}(S) \leq S$, it follows that $T \sim_{P} S$, hence $T = S$ since $T$ and $S$ belong to the same genetic basis $\mathcal{B}$. Moreover $x \in N_{P}(S)$, and the induced group homomorphism $N_{\mathcal{F}}(\mathcal{S})/S \to N_{P}(T)/T$ is the canonical isomorphism $N_{\mathcal{F}}(\mathcal{S})/\mathcal{S} \to N_{P}(S)/S$. This completes the proof of Assertion 2.

For Assertion 3, we consider the deflation map $\text{Def}_{P/N}^{P} : \Gamma(P) \to \Gamma(P/N)$.

It corresponds to the biset $V = P/N$, with left action of $\mathcal{F}$ and right action of $P$. For $v = yN \in V$, for $T \in \mathcal{B}$ and $\mathcal{S} \in \mathcal{B}$, and with the same notation as above, the computation is similar: we have $T^v \cap N_{\mathcal{F}}(\mathcal{S}) \leq \mathcal{S}$ if and only if $T^u \cap N_{P}(S) \leq S$, and $\mathcal{S}^v \cap N_{P}(T) \leq T$ if and only if $^xS \cap N_{P}(T) \leq T$. These two conditions are fulfilled if and only if $S = T$ and $y \in N_{P}(S)$. This completes the proof.
6.2. Corollary: Let $p$ be an odd prime and $P$ be a finite $p$-group. Let $B$ be a genetic basis of $P$. Then the faithful part $\partial \Gamma(P)$ of $\Gamma(P)$ is equal to

$$\partial \Gamma(P) = \prod_{S \in B, S \cap Z(P) = 1} (N_P(S)/S) .$$

7. Genome and Roquette category

7.1. Let $F$ be a $p$-biset functor. It is shown in [2] Theorem 10.1.1 that if $P$ is a finite $p$-group and $B$ is a genetic basis of $P$, then the map

$$I_B = \bigoplus_{S \in B} \text{Ind}_{N_P(S)/S}^P : \bigoplus_{S \in B} \partial F(N_P(S)/S) \to F(P)$$

is always split injective. When $I_B$ is an isomorphism for one particular genetic basis $B$ of $P$, then $I_B$ is an isomorphism for any other genetic basis $B'$ of $P$.

The functors for which $I_B$ is an isomorphism for any finite $p$-group $P$ and any genetic basis $B$ of $P$ are called rational $p$-biset functors. It has been shown further ([3]) that these rational $p$-biset functors are exactly those $p$-biset functors which factorize through the Roquette category $\mathcal{R}_p$ of $p$-groups: more precisely ([3], Definition 3.3), the category $\mathcal{R}_p$ is defined as the idempotent additive completion of a specific quotient $\mathcal{R}_p^2$ of the category $\mathcal{C}_p$, so there is a canonical additive functor $\pi_p : \mathcal{C}_p \to \mathcal{R}_p$, equal to the composition of the projection functor $\mathcal{C}_p \to \mathcal{R}_p^2$ and the inclusion functor $\mathcal{R}_p^2 \to \mathcal{R}_p$. The rational $p$-biset functors are the additive functors $F : \mathcal{C}_p \to Ab$ for which there exists an additive functor $\overline{F} : \mathcal{R}_p \to Ab$ such that $F = \overline{F} \circ \pi_p$. In this case, the functor $\overline{F}$ is unique.

7.2. Proposition: Let $p$ be an odd prime. Then the genome $p$-biset functor $\Gamma$ is rational.

Proof: Let $P$ be a $p$-group, and $B$ be a genetic basis of $P$. If $S \in B$, then $Q = N_P(S)/S$ is cyclic, so the trivial subgroup $S/S$ of $Q$ is the only one intersecting trivially the center of $Q$, and it is a genetic subgroup of $Q$. By Corollary 6.2 we have that

$$\partial \Gamma(N_P(S)/S) = N_P(S)/S .$$
Now the induction-inflation map $\text{Ind}_{N_p(S)/S}^P$ is given by the $(P, N_P(S)/S)$-biset $U = P/S$. Let $T \in B$, and let $u = xS \in U$ such that

$$T^u \cap N_Q(S/S) \leq S/S \quad \text{and} \quad u(S/S) \cap N_P(T) \leq T.$$ 

The first inclusion means that

$$\{yS \in N_P(S)/S \mid \exists t \in T, txS = xSy\} = S/S.$$ 

In other words $T^x \cap N_P(S) \leq S$. The second inclusion means similarly that

$$\{y \in N_P(T) \mid \exists s \in S/S, txS = xSs\} \leq T,$$

that is $N_P(T) \cap xS \leq T$. Hence $T \preceq_p S$, thus $T = S$ since $T$ and $S$ both belong to a genetic basis of $P$. Moreover $x \in N_P(S)$, and the morphism we get from $N_P(S)/S$ to $N_P(T)/T$ is the identity map.

In other words, the map

$$\text{Ind}_{N_p(S)/S}^P : \partial \Gamma(N_P(S)/S) = N_P(S)/S \to \Gamma(P)$$

is the canonical embedding of $N_P(S)/S$ in $\Gamma(P)$. It clearly follows that the map $\mathcal{I}_B$ is an isomorphism, hence $\Gamma$ is rational. 

7.3. Corollary: Let $p$ be an odd prime. Then there exists a unique additive functor $\tilde{\Gamma}$ from the Roquette category $R_p$ to $\mathsf{Ab}$ such that $\Gamma = \tilde{\Gamma} \circ \pi_p$. Moreover $\tilde{\Gamma}(\partial P) = \partial \Gamma(P)$ for any finite $p$-group $P$, where $\partial P$ is the edge of $P$ in $R_p$. In particular $\Gamma(\partial C) = C$ for any cyclic $p$-group $C$.

Proof: This follows from the definition and properties of the category $R_p$ (for the definition of the edge $\partial P$ of a $p$-group $P$ in the Roquette category, see [3] Definition 3.7).

References

[1] L. Barker. Genotypes of irreducible representations of finite $p$-groups. Journal of Algebra, 306:655–681, 2007.

[2] S. Bouc. Biset functors for finite groups, volume 1990 of Lecture Notes in Mathematics. Springer, 2010.

[3] S. Bouc. The Roquette category of finite $p$-groups. Journal of the European Mathematical Society, 17:2843–2886, 2015.
[4] S. Bouc and N. Romero. The Whitehead group of (almost) extra-special $p$-groups with $p$-odd. Preprint, arXiv:1604.06306, 2016.

[5] C. Curtis and I. Reiner. Methods of representation theory with applications to finite groups and orders, volume II of Wiley classics library. Wiley, 1990.

[6] C. Curtis and I. Reiner. Methods of representation theory with applications to finite groups and orders, volume I of Wiley classics library. Wiley, 1990.

[7] R. Oliver. Whitehead groups of finite groups, volume 132 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1988.

[8] N. Romero. Computing Whitehead groups using genetic bases. Journal of Algebra, 450:646–666, 2016.

[9] P. Roquette. Realisierung von Darstellungen endlicher nilpotenter Gruppen. Arch. Math., 9:224–250, 1958.

[10] J. J. Rotman. An introduction to the theory of groups, volume 148 of Graduate Texts in Mathematics. Springer-Verlag, New York, fourth edition, 1995.

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