The equivariant de Rham complex on a simplicial $G_\ast$-manifold

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Abstract

We show that when a simplicial Lie group acts on a simplicial manifold $\{X_\ast\}$, we can construct a bisimplicial manifold and the de Rham complex on it. This complex is quasi-isomorphic to the equivariant simplicial de Rham complex on $\{X_\ast\}$ and its cohomology group is isomorphic to the cohomology group of the fat realization of the bisimplicial manifold. We also exhibit a cocycle in the equivariant simplicial de Rham complex.

1 Introduction

Simplicial manifold is a sequence of manifolds together with face and degeneracy operators satisfying some relations. There is a well-known way to construct the de Rham complex on a simplicial manifold (see [2][5][9], for instance). In [8], Meinrenken introduced the equivariant version of the de Rham complex on a simplicial manifold. That is a double complex whose components are equivariant differential forms which is called the Cartan model [1]. This complex is a generalization of Weinstein’s one in [14]. In this paper, we show that when a simplicial Lie group acts on a simplicial manifold $\{X_\ast\}$, we can construct a bisimplicial manifold and explain that the de Rham complex on it is quasi-isomorphic to the equivariant de Rham complex on $\{X_\ast\}$. We explain also that its cohomology group is isomorphic to the cohomology
group of the fat realization of the bisimplicial manifold. At the last section, we exhibit a cocycle in the equivariant de Rham complex on a simplicial manifold $\text{NSO}(4)$.

2 Review of the simplicial de Rham complex

2.1 Simplicial manifold

**Definition 2.1 (10).** Simplicial manifold is a sequence of manifolds $X = \{X_q\}, (q = 0, 1, 2 \cdots)$ together with face operators $\varepsilon_i : X_q \rightarrow X_{q-1} (i = 0, 1, 2 \cdots q)$ and degeneracy operator $\eta_i : X_q \rightarrow X_{q+1} (i = 0, 1, 2 \cdots q)$ which are all smooth maps and satisfy the following identities:

\[
\varepsilon_i \varepsilon_j = \varepsilon_{j-1} \varepsilon_i \quad i < j \\
\eta_i \eta_j = \eta_{j+1} \eta_i \quad i \leq j \\
\varepsilon_i \eta_j = \begin{cases} 
\eta_{j-1} \varepsilon_i & i < j \\
id & i = j, \ i = j + 1 \\
\eta_j \varepsilon_{i-1} & i > j + 1.
\end{cases}
\]

Simplicial Lie group $\{G_n\}$ is a simplicial manifold such that all $G_n$ are Lie groups and all face and degeneracy operators are group homomorphisms.

For any Lie group $G$, we have simplicial manifolds $NG$, $PG$ and simplicial $G$-bundle $\gamma : PG \rightarrow NG$ as follows:

\[
\text{NG}(q) = \underbrace{G \times \cdots \times G}_{q\text{-times}} \ni (g_1, \cdots, g_q) : \\
\text{face operators} \quad \varepsilon_i : \text{NG}(q) \rightarrow \text{NG}(q-1) \\
\varepsilon_i(g_1, \cdots, g_q) = \begin{cases} 
(g_2, \cdots, g_q) & i = 0 \\
(g_1, \cdots, g_i g_{i+1}, \cdots, g_q) & i = 1, \cdots, q - 1 \\
(g_1, \cdots, g_{q-1}) & i = q
\end{cases}
\]

\[
\text{PG}(q) = \underbrace{G \times \cdots \times G}_{q+1\text{-times}} \ni (\bar{g}_1, \cdots, \bar{g}_{q+1}) : 
\]

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face operators $\bar{\varepsilon}_i : PG(q) \to PG(q - 1)$

$$\bar{\varepsilon}_i(\bar{g}_1, \cdots, \bar{g}_{q+1}) = (\bar{g}_1, \cdots, \bar{g}_i, \bar{g}_{i+2}, \cdots, \bar{g}_{q+1}) \quad i = 0, 1, \cdots, q$$

Degeneracy operators are also defined but we do not need them here.

We define $\gamma : PG \to NG$ as $\gamma(\bar{g}_1, \cdots, \bar{g}_{q+1}) = (\bar{g}_1\bar{g}_2^{-1}, \cdots, \bar{g}_q\bar{g}_{q+1}^{-1})$.

For any simplicial manifold $\{X_*\}$, we can associate a topological space $\|X_*\|$ called the fat realization defined as follows:

$$\|X_*\| := \coprod_n \Delta_n \times X_n / (\varepsilon_i t, x) \sim (t, \varepsilon_i x).$$

Here $\Delta_n$ is the standard $n$-simplex and $\varepsilon^i$ is a face map of it. It is well-known that $\|\gamma\| : PG \to NG$ is the universal bundle $EG \to BG$ (see [5] [9], for instance).

### 2.2 The double complex on a simplicial manifold

**Definition 2.2.** For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we have a double complex $\Omega^{p,q}(X_*) := \Omega^q(X_p)$ with derivatives defined as follows:

$$d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon^*_i, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

For any simplicial manifold the following holds.

**Theorem 2.1** ([2] [5] [9]). There exist a ring isomorphism

$$H^*(\Omega^*(X_*)) \cong H^*(\|X_*\|).$$

Here $\Omega^*(X_*)$ means the total complex.

□
3 Simplicial $G_\ast$-manifold

Let $\{X_\ast\}$ be a simplicial manifold and $\{G_\ast\}$ be a simplicial Lie group which acts on $\{X_\ast\}$ by left, i.e. $G_n$ acts on $X_n$ by left and this action is commutative with face and degeneracy operators of $\{X_\ast\}$. We call $\{X_\ast\}$ a simplicial $G_\ast$-manifold.

A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy operators which commute with each other.

Given a simplicial $G_\ast$-manifold $\{X_\ast\}$, we can construct a bisimplicial manifold $\{X_\ast \rtimes NG_\ast(\ast)\}$ in the following way:

$$X_p \rtimes NG_p(q) := X_p \times \underbrace{G_p \times \cdots \times G_p}_{\text{q-times}}.$$

Horizontal face operators $\varepsilon_{i}^{H_0} : X_p \rtimes NG_p(q) \to X_{p-1} \rtimes NG_{p-1}(q)$ are the same as the face operators of $X_p$ and $G_p$. Vertical face operators $\varepsilon_{i}^{V_\ast} : X_p \rtimes NG_p(q) \to X_{p} \rtimes NG_p(q - 1)$ are

$$\varepsilon_{i}^{V_\ast} (x, g_1, \cdots, g_q) = \begin{cases} (x, g_2, \cdots, g_q) & i = 0 \\ (x, g_1, \cdots, g_1g_{i+1}, \cdots, g_q) & i = 1, \cdots, q - 1 \\ (g_qx, g_1, \cdots, g_{q-1}) & i = q. \end{cases}$$

Example 3.1. Suppose $G_n = H$ is a compact subgroup of $G$ and $H$ acts on $NG(n)$ as follows:

$$h \cdot (g_1, g_2, \cdots, g_n) = (hg_1h^{-1}, hg_2h^{-1}, \cdots, hg_nh^{-1}).$$

Then $X_n = NG(n)$ is a simplicial $H$-manifold and $\|NG(\ast) \rtimes NH(\ast)\|$ is $B(G \rtimes H)$ (11).

Example 3.2. $PG(n)$ acts on $PG(n)$ itself by left as follows:

$$(\bar{k}_1, \cdots, \bar{k}_{n+1}) \cdot (\bar{g}_1, \cdots, \bar{g}_{n+1}) = (\bar{k}_1\bar{g}_1\bar{k}_1^{-1}, \cdots, \bar{k}_{n+1}\bar{g}_{n+1}\bar{k}_{n+1}^{-1}).$$

So $PG(\ast)$ is a simplicial $PG(\ast)$-manifold (8). If $G$ is compact, $\|PG(\ast) \rtimes N(PG(\ast))(\ast)\|$ is a fat realization of a simplicial space $PG(n) \times_{PG(n)} EPG(n)$.

Example 3.3. If the action of $\{G_\ast\}$ on $\{X_\ast\}$ is trivial, $\|X_\ast \rtimes NG_\ast(\ast)\|$ is $\|X_\ast\| \times \|BG_\ast\|$.
Example 3.4. Let $\Gamma_1 \xrightarrow{\simeq} \Gamma_0$ be a $G$-groupoid, i.e. both $\Gamma_1$ and $\Gamma_0$ are $G$-manifolds and all structure maps are $G$-equivariant. We define a simplicial manifold $N\Gamma$ as follows:

$$N\Gamma(p) := \{ (x_1, \cdots, x_p) \in \Gamma_1 \times \cdots \times \Gamma_1 \mid t(x_j) = s(x_{j+1}) \quad j = 1, \cdots, p-1 \}$$

face operators $\varepsilon_i : N\Gamma(p) \to N\Gamma(p-1)$

$$\varepsilon_i(x_1, \cdots, x_p) = \begin{cases} 
(x_2, \cdots, x_p) & i = 0 \\
(x_1, \cdots, m(x_i, x_{i+1}), \cdots, x_p) & i = 1, \cdots, p-1 \\
(x_1, \cdots, x_{p-1}) & i = p.
\end{cases}$$

Here $s, t, m$ mean the source and target maps, and the multiplication (13). Then $N\Gamma(*)$ is a simplicial $G$-manifold.

4 The equivariant simplicial de Rham complex

4.1 The triple complex

Definition 4.1. For a bisimplicial manifold $\{X_{*,*}\}$, we can construct a triple complex on it in the following way:

$$\Omega^{p,q,r}(X_{*,*}) := \Omega^r(X_{p,q})$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1} (-1)^i (\varepsilon^H_i)^*, \quad d'' := \sum_{i=0}^{q+1} (-1)^i (\varepsilon^V_i)^* \times (-1)^p$$

$$d''' := (-1)^{p+q} \times \text{the exterior differential on } \Omega^r(X_{p,q}).$$

Repeating the same argument in [11], we obtain the following theorem.

Theorem 4.1. There exists an isomorphism

$$H(\Omega^r(X_{*,*} \rtimes NG_*(\ast))) \cong H^r(\| X_* \rtimes NG_*(\ast) \|).$$

Here $\Omega^r(X_{*,*} \rtimes NG_*(\ast))$ means the total complex.
4.2 The equivariant simplicial de Rham complex

When a compact Lie group \(G\) acts on a manifold \(M\), there is the complex of equivariant differential forms \(\Omega^*_G(M) := (\Omega^*(M) \otimes S(G^*))^G\) with the differential \(d_G\) defined by \((d_G \alpha)(X) := (d - i_{X_M}) \alpha(X))\) ([1][3]). Here \(G\) is the Lie algebra of \(G\), \(S(G^*)\) is the algebra of polynomial functions on \(G\), \(\alpha \in \Omega^*_G(M)\), \(X \in G\) and \(X_M\) denote the vector field on \(M\) generated by \(X\). This is called the Cartan Model. We can define the double complex \(\Omega^*_G(X_*)\) in the same way as in Definition 2.2. This double complex is originally introduced by Meinrenken in [8].

Again, repeating the same argument in [11], we obtain the following theorem.

**Theorem 4.2.** If every \(G_n\) is compact, there exists an isomorphism

\[
H(\Omega^*_G(X_*)) \cong H(\Omega^*(X_* \times N_G(\ast))).
\]

Here \(\Omega^*_G(X_*)\) means the total complex. \(\square\)

**Remark 4.1.** In the case that \(G_n\) is not compact, we need to use “the Getzler model” of the equivariant cohomology in [6].

4.3 Cocycle in the equivariant simplicial de Rham complex

In this section we take \(G = SO(4)\) and construct a cocycle in \(\Omega^4_{SO(4)}(NSO(4))\), whose cohomology is isomorphic to \(H^*(B(SO(4) \times SO(4)))\).

Recall that there is a cocycle in \(\Omega^4(NSO(4))\) described in the following way.

**Theorem 4.3 ([12]).** The cocycle which represents the Euler class of \(ESO(4) \to BSO(4)\) in \(\Omega^4(NSO(4))\) is the sum of the following \(E_{1,3}\) and \(E_{2,2}\):

\[
\begin{array}{c}
E_{1,3} \in \Omega^2(SO(4)) \xrightarrow{d'} \Omega^3(SO(4) \times SO(4)) \\
\uparrow d \\
E_{2,2} \in \Omega^2(SO(4) \times SO(4)) \xrightarrow{d} 0
\end{array}
\]
\[ E_{1,3} = \frac{1}{192\pi^2} \sum_{\tau \in \mathcal{S}_4} \text{sgn}(\tau) \left( (h^{-1}dh)_{\tau(1)}(h^{-1}dh)_{\tau(2)}^2 + (h^{-1}dh)_{\tau(3)}(h^{-1}dh)_{\tau(1)}^2 \right) \]
\[ E_{2,2} = -\frac{1}{64\pi^2} \sum_{\tau \in \mathcal{S}_4} \text{sgn}(\tau) \left( (h^{-1}dh_1)_{\tau(1)}(dh_2h_2^{-1})_{\tau(3)} + (h^{-1}dh_1)_{\tau(3)}(dh_2h_2^{-1})_{\tau(1)} \right) \]

**Errata 1.** In [12], there are some mistakes. Some numbers of propositions and theorems are wrong. For example, “Proposition 3.1” in P.38 should be modified as “Proposition 2.1”. The cocycle in Theorem 2.2 should be written as above. Also, \([\partial^2_{y_1, y_2} b(\gamma_1, \gamma_2)]_{y_i=0}\) and \(\alpha(\xi_1, \xi_2)\) should be written as follows.

\[ \left[ \frac{\partial^2}{\partial y_1 \partial y_2} b(\gamma_1, \gamma_2) \right]_{y_i=0} = -\frac{1}{64\pi^2} \sum_{\tau \in \mathcal{S}_4} \text{sgn}(\tau) \int_0^1 \left( \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)} \xi_2(\theta)_{\tau(3)} \xi_2(\theta)_{\tau(4)} + \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(3)} \xi_2(\theta)_{\tau(1)} \right) d\theta. \]

\[ \alpha(\xi_1, \xi_2) := -\frac{1}{64\pi^2} \sum_{\tau \in \mathcal{S}_4} \left( \text{sgn}(\tau) \cdot \int_0^1 \left( \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)} \xi_2(\theta)_{\tau(3)} + \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(3)} \xi_2(\theta)_{\tau(1)} \right) d\theta \right). \]

Now following Jeffrey and Weinstein’s idea, we construct a cocycle in \(\Omega^4_{SO(4)}(NSO(4))\).

We take a cochain \(\mu \in (\Omega^1(G) \otimes \mathcal{G}^*)^G\) as follows:

\[ \mu(X) = -\frac{1}{64\pi^2} \sum_{\tau \in \mathcal{S}_4} \text{sgn}(\tau) \left( \left( X \right)_{\tau(1)}(h^{-1}dh)_{\tau(3)}(h^{-1}dh)_{\tau(4)} + \left( X \right)_{\tau(3)}(h^{-1}dh)_{\tau(1)}(h^{-1}dh)_{\tau(2)} \right) \]

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Lemma 4.1. $i_{X} E_{1,3} = d\mu(X)$

Proof. Since $i_{X}(g^{-1}dg) = i_{X}(dgg^{-1}) = X$, the following equation holds.

$$i_{X} E_{1,3} = i_{X-\pi} E_{1,3} = i_{X} E_{1,3} - i_{X} E_{1,3}$$

$$= \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(h^{-1}dh)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h^{-1}dh)_{\tau(1)\tau(2)} $$

$$- \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(dh_{\tau}(h^{-1}dh_{\tau}))_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(dh_{\tau}(h^{-1}dh_{\tau}))_{\tau(1)\tau(2)} $$

$$= d\mu(X).$$

Lemma 4.2. $i_{XG} E_{2,2} = (\varepsilon^*_0 - \varepsilon^*_1 + \varepsilon^*_2) \mu(X)$

Proof. $(\varepsilon^*_0 - \varepsilon^*_1 + \varepsilon^*_2) \mu(X)$

$$= \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(h^{-1}dh_{2})_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h^{-1}dh_{2})_{\tau(1)\tau(2)} $$

$$- \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(dh_{2}(h^{-1}dh_{2}))_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(dh_{2}(h^{-1}dh_{2}))_{\tau(1)\tau(2)} $$

$$+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(h^{-1}h_{1}^{-1}dh_{1}h_{2} + h^{-1}h_{2}dh_{2})_{\tau(3)\tau(4)} $$

$$+ (X)_{\tau(3)\tau(4)}(h^{-1}h_{1}^{-1}dh_{1}h_{2} + h^{-1}h_{2}dh_{2})_{\tau(1)\tau(2)} $$

$$+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau) ( (X)_{\tau(1)\tau(2)}(dh_{1}(h^{-1} + h_{1}dh_{2}h^{-1}h_{1})_{\tau(3)\tau(4)} $$

$$+ (X)_{\tau(3)\tau(4)}(dh_{1}(h^{-1} + h_{1}dh_{2}h^{-1}h_{1})_{\tau(1)\tau(2)} $$

$$= (\varepsilon^*_0 - \varepsilon^*_1 + \varepsilon^*_2) \mu(X).$$
Lemma 4.3. \(-i\mathcal{X}_G\mu(X) = 0\)

Proof. 
\[-i\mathcal{X}_G\mu(X) = -i\mathcal{X}_{-\mathcal{X}}\mu(X)\]
\[= \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(1)\tau(2)})\]
\[+ \frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau)((h^{-1}Xh)_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (h^{-1}Xh)_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)})\]
\[-\frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau)((hXh^{-1})_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (hXh^{-1})_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)})\]
\[-\frac{1}{64\pi^2} \sum_{\tau \in \Theta_4} \text{sgn}(\tau)((X)_{\tau(1)\tau(2)}(X)_{\tau(3)\tau(4)} + (X)_{\tau(3)\tau(4)}(X)_{\tau(1)\tau(2)}) = 0.\]

As a result, we obtain the following theorem.

Theorem 4.4. \(E_{1,3} + E_{2,2} + \mu\) is a cocycle in \(\Omega^3_{SO(4)}(NSO(4))\).
References

[1] N. Berline, E. Getzler, and M. Vergne, Heat Kernels and Dirac Operators, Grundlehren Math. Wiss. 298, Springer-Verlag, Berlin, 1992.

[2] R. Bott, H. Shulman, J. Stasheff, On the de Rham Theory of Certain Classifying Spaces, Adv. in Math. 20 (1976), 43-56.

[3] H. Cartan, La transgression dans un groupe de Lie et dans un espace fibré principal, Colloque de Topologie, CBRM Bruxelles, 1950, pp. 57-71.

[4] J. L. Dupont, Simplicial de Rham cohomology and characteristic classes of flat bundles, Top. Vol 15(1976), 233-245, Perg Press.

[5] J. L. Dupont, Curvature and Characteristic Classes, Lecture Notes in Math. 640, Springer Verlag, 1978.

[6] E. Getzler, The equivariant Chern character for non-compact Lie groups, Adv. Math. 109(1994), no.1, 88-107.

[7] L. Jeffrey, Group cohomology construction of the cohomology of moduli spaces of flat connections on 2-manifolds, Duke Math. J. 77(1995) 407-429.

[8] E. Meinrenken, Witten’s formulas for intersection pairings on moduli spaces of flat G-bundles, Adv. in Math. 197 (2005), 140-197.

[9] M. Mostow and J. Perchick, Notes on Gel’fand-Fuks Cohomology and Characteristic Classes (Lectures by Bott). In Eleventh Holiday Symposium. New Mexico State University, December 1973.

[10] G. Segal, Classifying spaces and spectral sequences. Inst. Hautes Études Sci.Publ. Math. No.34, 1968, 105-112.

[11] N. Suzuki, The equivariant simplicial de Rham complex and the classifying space of a semi-direct product group. Math. J. Okayama Univ. 57 (2015), 123-128.

[12] N. Suzuki, The Euler class in the Simplicial de Rham Complex, International Electronic Journal of Geometry, Vol 9, No.2, (2016), pp. 36-43.
[13] M. Stiénon, Equivariant Dixmier-Douady classes. Math. Res. Lett. 17 (2010), no. 1, 127-145.

[14] A. Weinstein, The symplectic structure on moduli space. The Floer memorial volume, Progr. Math., 133, Birkhäuser, Basel, 1995, 627-635.

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