Truthful Approximations to Range Voting*

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Abstract

We consider the fundamental mechanism design problem of approximate social welfare maximization under general cardinal preferences on a finite number of alternatives and without money. The well-known range voting scheme can be thought of as a non-truthful mechanism for exact social welfare maximization in this setting. With $m$ being the number of alternatives, we exhibit a randomized truthful-in-expectation ordinal mechanism implementing an outcome whose expected social welfare is at least an $\Omega(m^{-3/4})$ fraction of the social welfare of the socially optimal alternative. On the other hand, we show that for sufficiently many agents and any truthful-in-expectation ordinal mechanism, there is a valuation profile where the mechanism achieves at most an $O(m^{-2/3})$ fraction of the optimal social welfare in expectation. We get tighter bounds for the natural special case of $m = 3$, and in that case furthermore obtain separation results concerning the approximation ratios achievable by natural restricted classes of truthful-in-expectation mechanisms. In particular, we show that for $m = 3$ and a sufficiently large number of agents, the best mechanism that is ordinal as well as mixed-unilateral has an approximation ratio between 0.610 and 0.611, the best ordinal mechanism has an approximation ratio between 0.616 and 0.641, while the best mixed-unilateral mechanism has an approximation ratio bigger than 0.660. In particular, the best mixed-unilateral non-ordinal (i.e., cardinal) mechanism strictly outperforms all ordinal ones, even the non-mixed-unilateral ordinal ones.

1 Introduction

We consider the fundamental mechanism design problem of approximate social welfare maximization under general cardinal preferences and without money. In this setting, there is a finite set of agents (or voters) $N = \{1, \ldots, n\}$ and a finite set of alternatives (or candidates) $M = \{1, \ldots, m\}$. Each voter $i$ has a private valuation function $u_i : M \to \mathbb{R}$ that can be arbitrary, except that we require that it is injective, i.e., we insist that it induces a total order on candidates. Standardly, the function $u_i$ is considered well-defined only up to positive affine transformations. That is, we consider $x \rightarrow au_i(x) + b$, for $a > 0$ and any $b$, to be a different representation of $u_i$. Given this, we fix the representative $u_i$ that maps the least preferred candidate of voter $i$ to 0 and the most preferred candidate to 1 as the canonical representation of $u_i$ and we shall assume that all $u_i$ are thus canonically represented throughout this paper. In particular, we shall let $V_m$ denote the set of all such functions.

We shall be interested in direct revelation mechanisms without money that elicit the valuation profile $u = (u_1, u_2, \ldots, u_n)$ from the voters and based on this elect a candidate $J(u) \in M$. We shall allow mechanisms to be randomized and $J(u)$ is therefore in general a random map. In fact, we shall define a mechanism simply to be a random map $J : V_m^n \rightarrow M$. We prefer mechanisms that are truthful-in-expectation, by

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1We make this requirement primarily for convenience; to avoid having to qualify in technically annoying ways a number of definitions and statements of this paper as well as definitions and statements of previous ones.
which we mean that the following condition is satisfied: For each voter \( i \), and all \( u = (u_i, u_{-i}) \in V_m^n \) and \( \tilde{u}_i \in V_m \), we have \( E[u_i(J(u_i, u_{-i}))] \geq E[u_i(J(\tilde{u}_i, u_{-i}))] \). That is, if voters are assumed to be expected utility maximizers, the optimal behavior of each voter is always to reveal their true valuation function to the mechanism. As truthfulness-in-expectation is the only notion of truthfulness of interest to us in this paper, we shall use “truthful” as a synonym for “truthful-in-expectation” from now on. Furthermore, we are interested in mechanisms for which the expected social welfare, i.e., \( E[\sum_{i=1}^n u_i(J(u))] \), is as high as possible, and we shall in particular be interested in the approximation ratio \( \text{ratio}(J) \) of the mechanism, defined by

\[
\text{ratio}(J) = \inf_{u \in V_m^n} \frac{E[\sum_{i=1}^n u_i(J(u))] - E[\sum_{i=1}^n u_i(\tilde{J}(u))]}{\max_{j \in M} \sum_{i=1}^n u_i(j)}
\]

trying to achieve mechanisms with as high an approximation ratio as possible. Note that for \( m = 2 \), the problem is easy; a majority vote is a truthful mechanism that achieves optimal social welfare, i.e., it has approximation ratio 1, so we only consider the problem for \( m \geq 3 \).

A mechanism without money for general cardinal preferences can be naturally interpreted as a cardinal voting scheme in which each voter provides a ballot giving each candidate \( j \in M \) a numerical score between 0 and 1. A winning candidate is then determined based on the set of ballots. With this interpretation, the well-known range voting scheme is simply the deterministic mechanism that elects the socially optimal candidate \( \arg\max_{j \in M} \sum_{i=1}^n u_i(j) \), or, more precisely, elects this candidate if ballots are reflecting the true valuation functions \( u_i \). In particular, range voting has by construction an approximation ratio of 1. However, range voting is not a truthful mechanism.

Before stating our results, we mention for comparison the approximation ratio of some simple truthful mechanisms. Let random-candidate be the mechanism that elects a candidate uniformly at random, without looking at the ballots. Let random-favorite be the mechanism that picks a voter uniformly at random and elects his favorite candidate; i.e., the (unique) candidate to which he assigns valuation 1. Let random-majority be the mechanism that picks two candidates uniformly at random and elects one of them by a majority vote. It is not difficult to see that as a function of \( m \) and assuming that \( n \) is sufficiently large, random-candidate as well as random-favorite have approximation ratios \( \Theta(m^{-1}) \), so this is the trivial bound we want to beat. Interestingly, random-majority performs even worse, with an approximation ratio of \( \Theta(m^{-2}) \).

As our first main result, we exhibit a randomized truthful mechanism with an approximation ratio of \( 0.37m^{-3/4} \). The mechanism is the following very simple one: With probability \( 3/4 \), pick a candidate uniformly at random. With probability \( 1/4 \), pick a random voter, and pick a candidate uniformly at random from his \( \lfloor m^{1/2} \rfloor \) most preferred candidates. Note that this mechanism is ordinal: Its behavior depends only on the rankings of the candidates on the ballots, not on their numerical scores. We know no asymptotically better truthful mechanism, even if we allow general (cardinal) mechanisms, i.e., mechanisms that can depend on the numerical scores in other ways. We also show a negative result: For sufficiently many voters and any truthful ordinal mechanism, there is a valuation profile where the mechanism achieves at most an \( O(m^{-2/3}) \) fraction of the optimal social welfare in expectation. The negative result also holds for non-ordinal mechanisms that are mixed-unilateral, by which we mean mechanisms that elect a candidate based on the ballot of a single randomly chosen voter.

We get tighter bounds for the natural case of \( m = 3 \) candidates and for this case, we also obtain separation results concerning the approximation ratios achievable by natural restricted classes of truthful mechanisms. Again, we first state the performance of the simple mechanisms defined above for comparison: For the case of \( m = 3 \), random-favorite and random-majority both have approximation ratios \( 1/2 + o(n) \) while random-candidate has an approximation ratio of \( 1/3 \). We show that for \( m = 3 \) and large \( n \), the best mechanism that is ordinal as well as mixed-unilateral has an approximation ratio between 0.610 and 0.611. The best ordinal mechanism has an approximation ratio between 0.616 and 0.641. Finally, the best mixed-unilateral mechanism has an approximation ratio larger than 0.660. In particular, the best mixed-unilateral mechanism strictly outperforms all ordinal ones, even the non-unilateral ordinal ones. The mixed-unilateral mechanism that establishes this is a convex combination of quadratic-lottery, a mechanism of Feige and Tennenholtz, and random-favorite, that was defined above.
1.1 Background, related research and discussion

Characterizing strategy-proof social choice functions (a.k.a., truthful direct revelation mechanisms without money) under general preferences is a classical topic of mechanism design and social choice theory. The celebrated Gibbard-Satterthwaite theorem [8, 15] states that when the number \( m \) of candidates is at least 3, any deterministic and onto truthful mechanism must be a dictatorship, i.e., it is a function of the ballot of a single distinguished voter only, and outputs the favorite (i.e., top ranking) candidate of that voter. Gibbard [9] extended the Gibbard-Satterthwaite theorem to the case of randomized ordinal mechanisms, and we shall heavily use his theorem when proving our negative results on ordinal mechanisms:

**Theorem 1.** [9] The ordinal mechanisms without money that are truthful under general cardinal preferences are exactly the convex combinations of truthful unilateral ordinal mechanisms and truthful duple mechanisms.

Here, a unilateral mechanism is a randomized mechanism whose (random) output depends on the ballot of a single distinguished voter \( i^* \) only. Note that a unilateral truthful mechanism does not have to be a dictatorship. For instance, the mechanism that elects with probability \( \frac{1}{2} \) each of the two top candidates according to the ballot of voter \( i^* \) is a unilateral truthful mechanism. A duple mechanism is an ordinal mechanism for which there are two distinguished candidates so that all other candidates are elected with probability 0, for all valuation profiles.

An optimistic interpretation of Gibbard’s 1977 result as opposed to his 1973 result that was suggested, e.g., by Barbera [2], is that the class of randomized truthful mechanisms is quite rich and contains many arguably “reasonable” mechanisms in contrast to dictatorships, which are clearly “unreasonable”. However, we are not aware of any suggestions in the social choice literature of any well-defined quality measures that would enable us to rigorously compare these mechanisms and in particular find the best. Fortunately, one of the main conceptual contributions from computer science to mechanism design in general is the suggestion of one such measure, namely the notion of worst case approximation ratio relative to some objective function. Indeed, a large part of the computer science literature on mechanism design (with or without money) is the construction and analysis of approximation mechanisms, following the agenda set by the seminal papers by Nisan and Ronen [12] for the case of mechanisms with money and Procaccia and Tennenholtz [14] for the case of mechanisms without money (i.e., social choice functions). Following this research program, and using Gibbard’s characterization, Procaccia [13] gave in a paper conceptually very closely related to the present one, upper and lower bounds on the approximation ratio achievable by ordinal mechanisms for various objective functions under general preferences. However, he only considered objective functions that can be defined ordinally (such as, e.g., Borda count), and did in particular not consider approximating the optimal social welfare, as we do in the present paper.

The (approximate) optimization of social welfare (i.e. sum of valuations) is indeed a very standard objective in mechanism design. In particular, in the setting of mechanisms with money and agents with quasi-linear utilities, the celebrated class of Vickrey-Clarke-Groves (VCG) mechanisms exactly optimize social welfare, while classical negative results such as Roberts’ theorem, state that under general cardinal preferences (and subject to some qualifications), weighted social welfare is the only objective one can maximize exactly truthfully, even with money (see Nisan [11] for an exposition of all these results). It therefore seems to us extremely natural to try to understand how well one can approximate this objective truthfully without money under general cardinal preferences. One possible reason that the problem was not considered previously to this paper (to the best of our knowledge) is that, arguably, social welfare is a somewhat less natural objective function without the assumption of quasi-linearity of utilities made in the setting of mechanisms with money. Indeed, assuming quasi-linearity essentially means forcing the valuations of all agents to be in the unit of dollars, making it natural to subsequently add them up. On the other hand, in the setting of social choice theory, the valuation functions are to be interpreted as von Neumann-Morgenstern utilities (i.e., they are meant to encode orderings on lotteries), and in particular are only well-defined up to affine transformations.\(^2\)

\(^2\)Even though the theorem is usually stated for ordinal mechanisms, it is easy to see that it holds even without assuming that the mechanism is ordinal.

\(^3\)Without ties, i.e., valuation functions must be injective, as we require throughout this paper. If ties were allowed, the characterization would be much more complicated.
In this setting, the social welfare has to be defined as above, as the result of adding up the valuations of all players, after these are normalized by scaling to, say, the interval \([0,1]\). While this is arguably ad hoc, we note again that optimizing social welfare in this sense is in fact the intended (hoping for truthful ballots) outcome of the well known range voting scheme ([http://en.wikipedia.org/wiki/Range_voting](http://en.wikipedia.org/wiki/Range_voting)) which is a good piece of evidence for its naturalness.

As already noted, social welfare is a cardinal objective, i.e., it depends on the actual numerical valuations of the voters; not just their rankings of the candidates. While it makes perfect sense to measure how well ordinal mechanisms can approximate a cardinal objective, such as social welfare, it certainly also makes sense to see if improvements to the approximation of the optimal social welfare can be made by mechanisms that actually look at the numerical scores on the ballots and not just the rankings, i.e., cardinal mechanisms. The limitations of ordinal mechanisms were considered recently by Boutilier et al. ([5]) in a very interesting paper closely related to the present one, but crucially, their work did not consider incentives, i.e., they did not require truthfulness of the mechanisms in their investigations. On the other hand, truthfulness is the pivotal property in our approach and our results indicate that ordinal mechanisms are even more limited under this additional constraint. The characterization of truthful mechanisms of Theorem 1 does not apply to cardinal mechanisms. But noting that the definition of “unilateral” is not restricted to ordinal mechanisms, one might naturally suspect that a similar characterization would also apply to cardinal mechanisms. In a followup paper, Gibbard ([10]), indeed proved a theorem along those lines, but interestingly, his result does not apply to truthful direct revelation mechanisms (i.e., strategy-proof social choice functions), which is the topic of the present paper, but only to indirect revelation mechanisms with finite strategy space. Also, the restriction to finite strategy space (which is in direct contradiction to direct revelation) is crucial for the proof. Somewhat surprisingly, to this date, a characterization for the cardinal case is still an open problem! For a discussion of this situation and for interesting counterexamples to tempting characterization attempts similar to the characterization for the ordinal case of Gibbard ([9]), see [4,8], the bottomline being that we at the moment do not have a good understanding of what can be done with cardinal truthful mechanisms for general preferences. Concrete examples of cardinal mechanisms for general preferences were given in a number of a papers in the economics and social choice literature ([18, 7, 4]) and the computer science literature ([6]). It is interesting that while the social choice literature gives examples suggesting that the space of cardinal mechanisms is rich and even examples of instances where a cardinal mechanism for voting can yield a (Pareto) better result than all ordinal mechanisms ([7]), there was apparently no systematic investigation into constructing “good” cardinal mechanisms for unrestricted preferences. Here, as in the ordinal case, we suggest that the notion of approximation ratio provides a meaningful measure of quality that makes such investigations possible, and indeed, our present paper is meant to start such investigations. Our investigations are very much helped by the work of Feige and Tennenholtz ([6]) who considered and characterized the strongly truthful, continuous, unilateral cardinal mechanisms. While their agenda was mechanisms for which the objective is information elicitation itself rather than mechanisms for approximate optimization of an objective function, the mechanisms they suggest still turn out to be useful for social welfare optimization. In particular, our construction establishing the gap between the approximation ratios for cardinal and ordinal mechanisms for three candidates is based on their quadratic lottery.

1.2 Organization of paper

In Section 2 we give formal definitions of the concepts informally discussed above, and state and prove some useful lemmas. In Section 3 we present our results for an arbitrary number of candidates \(m\). In Section 4 we present our results for \(m = 3\). We conclude with a discussion of open problems in Section 5.

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4 It was pointed out to us that it is not clear that it is not completely clear that it is part of the range voting scheme that voters are asked to calibrate their scores so that 0 is the score of their least preferred candidate and 1 is the score of their most preferred candidate. However, without some calibration instructions, the statement “Score the candidates on a scale from 0 to 1” simply does not make sense and we believe that the present calibration instructions are the most natural ones imaginable.
2 Preliminaries

We let $V_m$ denote the set of canonically represented valuation functions on $M = \{1, 2, \ldots, m\}$. That is, $V_m$ is the set of injective functions $u : M \to [0, 1]$ with the property that 0 as well as 1 are contained in the image of $u$. It will sometimes be useful to restrict the set of valuation functions to a certain finite domain $R_{m,k}$ for an integer parameter $k \geq m$. Specifically, we define:

$$R_{m,k} = \left\{ u \in V_m \mid \exists(u) \subseteq \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\} \right\}$$

where $\exists(u)$ denotes the image of $u$. Given a valuation function $u \in R_{m,k}$, we define its \textit{alternation number} $a(u)$ as

$$a(u) = \#\{j \in \{0, \ldots, k-1\} \mid \left[ \frac{j}{k} \in \exists(u) \right] \oplus \left[ \frac{j+1}{k} \in \exists(u) \right] \},$$

where $\oplus$ denotes exclusive-or. That is, the alternation number of $u$ is the number of indices $j$ for which exactly one of $j/k$ and $(j+1)/k$ is in the image of $u$. Since $k \geq m$ and $\{0,1\} \subseteq \exists(u)$, we have that the alternation number of $u$ is at least 2. We shall be interested in the class of valuation functions $C_{m,k}$ with minimal alternation number. Specifically, we define:

$$C_{m,k} = \{ u \in R_{m,k} \mid a(u) = 2 \}$$

and shall refer to such valuation functions as \textit{quasi-combinatorial valuation functions}. Informally, the quasi-combinatorial valuation functions have all valuations as close to 0 or 1 as possible.

We let $\text{Mech}_{m,n}$ denote the set of truthful mechanisms for $n$ voters and $m$ candidates. That is, $\text{Mech}_{m,n}$ is the set of random maps $J : V_m^n \to M$ with the property that for voter $i \in \{1, \ldots, n\}$, and all $u = (u_i, u_{-i}) \in V_m^n$ and $\tilde{u}_i \in V_m$, we have $E[u_i(J(u_i, u_{-i}))] \geq E[u_i(J(\tilde{u}_i, u_{-i}))]$. Alternatively, instead of viewing a mechanism as a random map, we can view it as a map from $V_m^n$ to $\Delta_m$, the set of probability density functions on $\{1, \ldots, m\}$. With this interpretation, note that $\text{Mech}_{m,n}$ is a convex subset of the vector space of all maps from $V_m^n$ to $\mathbb{R}^m$.

We shall be interested in certain special classes of mechanisms. In the following definitions, we throughout view a mechanism $J$ as a map from $V_m^n$ to $\Delta_m$.

An \textit{ordinal} mechanism $J$ is a mechanism with the following property: $J(u_i, u_{-i}) = J(u_i', u_{-i})$, for any voter $i$, any preference profile $u = (u_i, u_{-i})$, and any valuation function $u_i'$ with the property that for all pairs of candidates $j, j'$, it is the case that $u_i(j) < u_i(j')$ if and only if $u_i'(j) < u_i'(j')$. Informally, the behavior of an ordinal mechanism only depends on the ranking of candidates on each ballot; not on the numerical valuations. We let $\text{Mech}_m^{\text{ord}}$ denote those mechanisms in $\text{Mech}_{m,n}$ that are ordinal.

Following Barbera [2], we define an \textit{anonymous} mechanism $J$ as one that does not depend on the names of voters. Formally, given any permutation $\pi$ on $N$, and any $u \in (V_m)^n$, we have $J(\pi \cdot u) = J(\pi \cdot u)$, where $\pi \cdot u$ denotes the vector $(u_{\pi(i)})_{i=1}^n$.

Similarly following Barbera [2], we define a \textit{neutral} mechanism $J$ as one that does not depend on the names of candidates. Formally, given any permutation $\sigma$ on $M$, any $u \in (V_m)^n$, and any candidate $j$, we have $J(\sigma \cdot u) = J(u \circ \sigma)$.

Following [3] [4], a \textit{unilateral} mechanism is a mechanism for which there exists a single voter $i^*$ so that for all valuation profiles $(u_{i^*}, u_{-i^*})$ and any alternative valuation profile $u_{i^*}'$, for the voters except $i^*$, we have $J(u_{i^*}, u_{-i^*}) = J(u_{i^*}', u_{-i^*}')$. Note that $i^*$ is not allowed to be chosen at random in the definition of a unilateral mechanism. In this paper, we shall say that a mechanism is \textit{mixed-unilateral} if it is a convex combination of unilateral truthful mechanisms. Mixed-unilateral mechanisms are quite attractive seen through the “computer science lens”: They are mechanisms of low query complexity, consulting only a single randomly chosen voter, and therefore deserve special attention in their own right. We let $\text{Mech}_{m,n}^{\text{mon}}$ denote those mechanisms in $\text{Mech}_{m,n}$ that are mixed-unilateral. Also, we let $\text{Mech}_{m,n}^{\text{ord}}$ denote those mechanisms in $\text{Mech}_{m,n}$ that are ordinal as well as mixed-unilateral.
Following Gibbard [9], a duple mechanism $J$ is an ordinal mechanism for which there exist two candidates $j_1^*$ and $j_2^*$ so that for all valuation profiles, $J$ elects all other candidates with probability 0.

We next give names to some specific important mechanisms. We let $U_{m,n}^q \in \text{Mech}_{m,n}^O$ be the mechanism for $m$ candidates and $n$ voters that picks a voter uniformly at random, and elects uniformly at random a candidate among his $q$ most preferred candidates. We let random-favorite be a nickname for $U_{m,n}^1$ and random-candidate be a nickname for $U_{m,n}^m$. We let $D_{m,n}^q \in \text{Mech}_{m,n}^O$, for $\lfloor n/2 \rfloor + 1 \leq q \leq n + 1$, be the mechanism for $m$ candidates and $n$ voters that picks two candidates uniformly at random and eliminates all other candidates. It then checks for each voter which of the two candidates he prefers and gives that candidate a “vote”. If a candidate gets at least $q$ votes, she is elected. Otherwise, a coin is flipped to decide which of the two candidates is elected. We let random-majority be a nickname for $D_{m,n}^{\lceil n/2 \rceil + 1}$. Note also that $D_{m,n}^{n+1}$ is just another name for random-candidate. Finally, we shall be interested in the following mechanism $Q_n$ for three candidates shown to be in $\text{Mech}_{3,n}^O$ by Feige and Tennenholtz [6]: Select a voter uniformly at random, and let $\alpha$ be the valuation of his second most preferred candidate. Elect his most preferred candidate with probability $(4 - \alpha^2)/6$, his second most preferred candidate with probability $(1 + 2\alpha)/6$ and his least preferred candidate with probability $(1 - 2\alpha + \alpha^2)/6$. We let quadratic-lottery be a nickname for $Q_n$. Note that quadratic-lottery is not ordinal. Feige and Tennenholtz [6] in fact presented several explicitly given non-ordinal one-voter truthful mechanisms, but quadratic-lottery is particularly amenable to an approximation ratio analysis due to the fact that the election probabilities are quadratic polynomials.

We let $\text{ratio}(J)$ denote the approximation ratio of a mechanism $J \in \text{Mech}_{m,n}$, when the objective is social welfare. That is,

$$\text{ratio}(J) = \inf_{u \in V_m^n} \frac{E(\sum_{i=1}^n u_i(J(u)))}{\max_{j \in M} \sum_{i=1}^n u_i(j)}.$$ 

We let $r_{m,n}$ denote the best possible approximation ratio when there are $n$ voters and $m$ candidates. That is, $r_{m,n} = \sup_{J \in \text{Mech}_{m,n}} \text{ratio}(J)$. Similarly, we let $r_{m,\infty}^C = \sup_{J \in \text{Mech}_{m,\infty}^C} \text{ratio}(J)$, for $C$ being either $O$, $U$ or $OU$. We let $r_m$ denote the asymptotically best possible approximation ratio when the number of voters approaches infinity. That is, $r_m = \lim_{n \to \infty} r_{m,n}$, and we also extend this notation to the restricted classes of mechanisms with the obvious notation $r_{m,\infty}^C$, $r_m^U$ and $r_m^{OU}$.

The importance of neutral and anonymous mechanisms is apparent from the following simple lemma:

**Lemma 1.** For all $J \in \text{Mech}_{m,n}$, there is a $J' \in \text{Mech}_{m,n}$ so that $J'$ is anonymous and neutral and so that $\text{ratio}(J') \geq \text{ratio}(J)$. Similarly, for all $J \in \text{Mech}_{m,n}^C$, there is $J' \in \text{Mech}_{m,n}^C$ so that $J'$ is anonymous and neutral and so that $\text{ratio}(J') \geq \text{ratio}(J)$, for $C$ being either $O$, $U$ or $OU$.

**Proof.** Given any mechanism $J$, we can “anonymize” and “neutralize” $J$ by applying a uniformly chosen random permutation to the set of candidates and an independent uniformly chosen random permutation to the set of voters before applying $J$. This yields an anonymous and neutral mechanism $J'$ with at least a good an approximation ratio as $J$. Also, if $J$ is ordinal and/or mixed-unilateral, then so is $J'$.

**Lemma 1** makes the characterizations of the following theorem very useful.

**Theorem 2.** The set of anonymous and neutral mechanisms in $\text{Mech}_{m,n}^{OU}$ is equal to the set of convex combinations of the mechanisms $U_{m,n}^q$, for $q \in \{1, \ldots, m\}$. Also, the set of anonymous and neutral mechanisms in $\text{Mech}_{m,n}$ that can be obtained as convex combinations of duple mechanisms is equal to the set of convex combinations of the mechanisms $D_{m,n}^q$, for $q \in \{\lfloor n/2 \rfloor + 1, \lceil n/2 \rceil + 2, \ldots, n, n+1\}$.

**Proof.** A very closely related statement was shown by Barbera [1]. We sketch how to derive the theorem from that statement.

Barbera (in [1], as summarized in the proof of Theorem 1 in [2]) showed that the anonymous, neutral mechanisms in $\text{Mech}_{m,n}^{OU}$ are exactly the point voting schemes and that the anonymous, neutral mechanism that are convex combinations of duple mechanism are exactly supporting size schemes. A point voting scheme

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5 Barbera et al. [4] gave a much more general definition of duple mechanism; their duple mechanisms are not restricted to be ordinal. In this paper, “duple” refers exclusively to Gibbard’s original notion.
is given by $m$ real numbers $(a_j)_{j=1}^m$ summing to 1, with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. It picks a voter uniformly at random, and elects the candidate he ranks $k$th with probability $a_k$, for $k = 1, \ldots, m$. It is easy to see that the point voting schemes are exactly the convex combinations of $U_{m,n}^q$, for $q \in \{1, \ldots, m\}$. A supporting size scheme is given by $n + 1$ real numbers $(b_i)_{i=0}^n$ with $b_0 = b_{n-1} = 0$, and $b_{i} - b_{i-1} = 1$ for $i < n/2$. It picks two different candidates $j_1, j_2$ uniformly at random and elects candidate $j_k, k = 1, 2$ with probability $b_{s_k}$ where $s_k$ is the number of voters than rank $j_k$ higher than $j_3-k$. It is easy to see that the supporting size schemes are exactly the convex combinations of $D_{m,n}^q$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n\}$. 

The following corollary is immediate from Theorem 1 and Theorem 2.

**Corollary 1.** The ordinal, anonymous and neutral mechanisms in Mech$_{m,n}$ are exactly the convex combinations of the mechanisms $U_{m,n}^q$, for $q \in \{1, \ldots, m\}$ and $D_{m,n}^q$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n\}$.

We next present some lemmas that allow us to understand the asymptotic behavior of $r_{m,n}^C$ for fixed $m$ and large $n$, for $C$ being either $O$, $U$, or $OU$.

**Lemma 2.** For any positive integers $n, m, k$, we have $r_{m,kn}^O \leq r_{m,n}^O$, for $C$ being either $O$, $U$ or $OU$.

**Proof.** Note that we do not claim that $r_{m,kn} \leq r_{m,n}$, in fact we do not know if this is true. However, let us examine a natural idea for proving this: Suppose we are given any mechanism $J$ in Mech$_{m,kn}$ with approximation ratio $\alpha$. We will try to convert it to a mechanism $J'$ in Mech$_{m,n}$ with the same approximation ratio, hence proving $r_{m,kn} \leq r_{m,n}$. The natural idea is to let $J'$ simulate $J$ on the profile where we simply make $k$ copies of each of the $n$ ballots. However, this proof idea fails in the case of unrestricted mechanisms, because we would not be able to prove that $J'$ is truthful merely from the assumption that $J$ is truthful, as each voter now controls more than one ballot (we would need $J$ to be group-strategyproof, rather than merely strategyproof).

Now for the real proof, first of all, we note that the case of $r_{m,kn}^U$. It follows the same line of reasoning as in proof attempt, but the restricted setting allows us to successfully conclude that $J'$ is truthful in expectation. Let $J \in$ Mech$_{m,kn}$. By Lemma 4, we can assume that $J$ is anonymous. In particular, $J$ must be a convex combination of unilateral mechanisms $J = \sum_{i=1}^{kn} \frac{1}{mn} U_i$ where $U_i$ is the result of applying a fixed truthful one-voter mechanism $U$ to the valuation function of voter $i$. Then, we let $J'$ be the mechanism that on input $u \in V_m^n$ applies $J$ to a profile $u' \in V_m^n$ where each ballot to $u$ is copied $k$ times, we see that $J'$ is also a convex combination of applying the mechanism $U$ to a randomly chosen voter. Since $J'$ is therefore a convex combination of truthful mechanisms, it is itself truthful, as desired.

The proof for the case of $r_{m,kn}^O$ is completely analogous; we just observe that $U$ in that case is ordinal.

For the case of $r_{m,kn}^C$, we can again assume without loss of generality that $J \in$ Mech$_{m,kn}^O$ is anonymous and neutral. We then apply Corollary 1 and write $J$ as a convex combination of the mechanisms of the form $U_{m,kn}$ and $D_{m,kn}^q$ for various $q$. For each of these, it is easy to check that if we on input $u \in V_m^n$ apply any of these to a profile $u' \in V_m^n$ where each ballot to $u$ is copied $k$ times, we get a mechanism of the form $U_{m,n}^q$ or $D_{m,n}^q$ for some $q'$. Since each of these are truthful, their convex combination $J'$ is also truthful.

**Lemma 3.** For any $n, m$ and $k < n$, we have $r_{m,n} \geq r_{m,n-k} - \frac{kn}{n}$. Also, $r_{m,n} \geq r_{m,n-k} - \frac{kn}{n}$, for $C$ being either $O$, $U$, or $OU$.

**Proof.** We construct a mechanism $J'$ in Mech$_{m,n}$ from a mechanism $J$ in Mech$_{m,n-k}$. The mechanism $J'$ simply simulates $J$ after removing $k$ voters, chosen uniformly at random and randomly mapping the remaining voters to $\{1, \ldots, n\}$. In particular, if $J$ is ordinal (or mixed-unilateral, or both) then so is $J'$. Suppose $J$ has approximation ratio $\alpha$. Consider running $J'$ on any profile where the socially optimal candidate has social welfare $w^*$. Note that $w^* \geq n/m$, since each voter assigns valuation 1 to some candidate. Ignoring $k$ voters reduces the social welfare of any candidate by at most $k$, so $J'$ is guaranteed to return a candidate with expected social welfare at least $\alpha(w^* - k)$. This is at least a $\alpha((1 - k)/w^*) \geq \alpha - \frac{kn}{n}$ fraction of $w^*$. Since the profile was arbitrary, we are done.
We conclude that \( r^C_{m,n} \) is “asymptotically decreasing” in \( n \), for \( C \) being either \( O \), \( U \), or \( OU \), in the following sense:

**Lemma 4.** For any \( m,n \geq 2, \epsilon > 0 \) and all \( n' \geq (n-1)m/\epsilon \), we have \( r^C_{m,n'} \leq r^C_{m,n} + \epsilon \), for \( C \) being either \( O \), \( U \), or \( OU \).

**Proof.** If \( n \) divides \( n' \), the statement follows from Lemma 2. Otherwise, let \( n^* \) be the smallest number larger than \( n' \) divisible by \( m \); we have \( n^* < n' + n \). By Lemma 2, we have \( r^C_{m,n^*} = r^C_{m,n} \). By Lemma 3 we have \( r^C_{m,n} \geq r^C_{m,n'} - \frac{(n-1)m}{n^*} \). Therefore, \( r^C_{m,n'} \leq r^C_{m,n} + \frac{(n-1)m}{n^*} \leq r^C_{m,n} + \frac{(n-1)m}{n'} \).

In particular, Lemma 4 implies that \( r^C_{m,n} \) converges to a limit as \( n \to \infty \).

### 3 Mechanisms and negative results for the case of many candidates

The following lemma will be very useful, also in later sections. It states that in order to analyse the approximation ratio of an ordinal and neutral mechanism, it is sufficient to understand its performance on quasi-combinatorial valuation profiles.

**Lemma 5.** Let \( J \in \text{Mech}_{m,n} \) be ordinal and neutral. Then

\[
\text{ratio}(J) = \liminf_{k \to \infty} \min_{u \in (C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J(u))] - \min_{u \in (C_{m,k})^n} \sum_{i=1}^n u_i(1)}{\sum_{i=1}^n u_i(1)}. \]

**Proof.** For a valuation profile \( u = (u_i) \), define \( g(u) = \frac{E[\sum_{i=1}^n u_i(J(u))] - \min_{u \in (C_{m,k})^n} \sum_{i=1}^n u_i(1)}{\sum_{i=1}^n u_i(1)} \). We show the following equations:

\[
\text{ratio}(J) = \inf_{u \in (V_m)^n} E[\sum_{i=1}^n u_i(J(u))] = \inf_{u \in (V_m)^n} g(u) = \liminf_{k \to \infty} \min_{u \in (R_{m,k})^n} g(u) \leq \min_{k \to \infty} \text{ratio}(J) \leq \frac{1}{2}.
\]

Equation (2) follows from the fact that since \( J \) is neutral, it is invariant over permutations of the set of candidates, so there is a worst case instance (with respect to approximation ratio) where the socially optimal candidate is candidate 1. Equation (3) follows from the facts that (a) each \( u \in (V_m)^n \) can be written as \( u = \lim_{k \to \infty} v_k \) where \( (v_k) \) is a sequence so that \( v_k \in (R_{m,k})^n \) and where the limit is with respect to the usual Euclidean topology (with the set of valuation functions being considered as a subset of a finite-dimensional Euclidean space), and (b) the map \( g \) is continuous in this topology (to see this, observe that the denominator in the formula for \( g \) is bounded away from 0). Finally, equation (4) follows from the following claim:

\[
\forall u \in (R_{m,k})^n \exists (C_{m,k})^n : g(u') \leq g(u).
\]

With \( u = (u_1, \ldots, u_n) \), we shall prove this claim by induction in \( \sum a(u_i) \) (recall that \( a(u_i) \) is the alternation number of \( u_i \)).

For the induction basis, the smallest possible value of \( \sum a(u_i) \) is 2\( n \), corresponding to all \( u_i \) being quasi-combinatorial. For this case, we let \( u' = u \).

For the induction step, consider a valuation profile \( u \) with \( \sum a(u_i) > 2n \). Then, there must be an \( i \) so that the alternation number \( a(u_i) \) of \( u_i \) is strictly larger than 2 (and therefore at least 4, since alternation numbers are easily seen to be even numbers). Then, there must be \( m, n \in \{2, 3, \ldots, k-2\} \), so that \( r \leq s \), \( \frac{r+1}{k} \notin \mathbb{Z}(u_i), \{ \frac{r+1}{k}, \frac{r+1}{k}, \ldots, \frac{r+1}{k} \} \subseteq \mathbb{Z}(u_i) \) and \( \frac{r+1}{k} \notin \mathbb{Z}(u_i) \). Let \( \tilde{r} \) be the largest number strictly smaller
\[ u = \left( \text{we let } \tilde{u} \text{ function} \right) \]

\[ \text{in the profile} \]

\[ g \]

it is enough to bound from below \( h \).

Now consider the function \( u \).

Note that all functions of \( m, n \) works. This completes the proof.

Let \( \text{Theorem 3.} \)

\[ \text{Let } m \geq 2, m \geq 3. \text{ Let } J = \frac{3}{2} U_{m,n} + \frac{1}{3} U_{m,n}^{1/2}. \text{ Then, } \text{ratio}(J) \geq 0.37m^{-3/4}. \]

**Proof.** For a valuation profile \( u = (u_i) \), we define \( g(u) = \frac{E[\sum_{i=1}^{n} u_i(J(u))]}{\sum_{i=1}^{n} u_i(J(u))} \). By Lemma 5 since \( J \) is ordinal, it is enough to bound from below \( g(u) \) for all \( u \in (C_{m,k})^n \) with \( k \geq 1000(nm)^2 \). Let \( \epsilon = 1/k \). Let \( \delta = m \epsilon \).

Note that all functions of \( u \) map each alternative either to a valuation smaller than \( \delta \) or a valuation larger than \( 1 - \delta \).

Since each voter assigns valuation 1 to at least one candidate, and since \( J \) with probability 3/4 picks a candidate uniformly at random from the set of all candidates, we have \( E[\sum_{i=1}^{n} u_i(J(u))] \geq 3n/(4m) \).
Suppose $\sum_{i=1}^{n} u_i(1) \leq 2m^{-1/4}n$. Then $g(u) \geq \frac{3}{8}m^{-3/4}$, and we are done. So we shall assume from now on that
\begin{equation}
\sum_{i=1}^{n} u_i(1) > 2m^{-1/4}n. \tag{5}
\end{equation}

Obviously, $\sum_{i=1}^{n} u_i(1) \leq n$. Since $J$ with probability $3/4$ picks a candidate uniformly at random from the set of all candidates, we have that $E[\sum_{i=1}^{n} u_i(J(u))] \geq \frac{3}{4m} \sum_{i,j} u_i(j)$. So if $\sum_{i,j} u_i(j) \geq \frac{1}{3}nm^{1/4}$, we have $g(u) \geq \frac{3}{8}m^{-3/4}$, and we are done. So we shall assume from now on that
\begin{equation}
\sum_{i,j} u_i(j) < \frac{1}{2}nm^{1/4}. \tag{6}
\end{equation}

Still looking at the fixed quasi-combinatorial $u$, let a voter $i$ be called *generous* if his $|m^{1/2}| + 1$ most preferred candidates are all assigned valuation greater than $1 - \delta$. Also, let a voter $i$ be called *friendly* if he has candidate 1 among his $|m^{1/2}|$ most preferred candidates. Note that if a voter is neither generous nor friendly, he assigns to candidate 1 valuation at most $1/2$. This means that the total contribution to $\sum_{i=1}^{n} u_i(1)$ from such voters is less than $n\delta < 0.001/m$. Therefore, by equation (6), the union of friendly and generous voters must be a set of size at least $1.99m^{-1/4}n$.

If we let $g$ denote the number of generous voters, we have $\sum_{i,j} u_i(j) \geq gm^{1/2}(1-\delta) \geq 0.999gm^{1/2}$, so by equation (6), we have that $0.999gm^{1/2} < \frac{1}{2}nm^{1/4}$. In particular $g < 0.51m^{-1/4}n$. So since the union of friendly and generous voters must be a set of size at least a $1.99m^{-1/4}n$ voters, we conclude that there are at least $1.48m^{-1/4}$ friendly voters, i.e. the friendly voters is at least a $1.48m^{-1/4}$ fraction of the set of all voters. But this ensures that $U_{m,n}^{[m^{1/2}]}$ elects candidate 1 with probability at least $1.48m^{-1/4}/m^{1/2} \geq 1.48m^{-3/4}$. Then, $J$ elects candidate 1 with probability at least $0.37m^{-3/4}$ which means that $g(u) \geq 0.37m^{-3/4}$, as desired. This completes the proof. \qed

We next show our negative result. We show that any convex combination of (not necessarily ordinal) unilateral and duple mechanisms performs poorly.

**Theorem 4.** Let $m \geq 20$ and let $n = m - 1 + kg$ where $k = \lfloor m^{1/3} \rfloor$ and $q = \lfloor m^{2/3} \rfloor$. For any mechanism $J$ that is a convex combination of unilateral and duple mechanisms in Mech$_{m,n}$, we have $\text{ratio}(J) \leq 5m^{-2/3}$.

**Proof.** By applying the same proof technique as in the proof of Lemma 1 we can assume that $J$ can be decomposed into a convex combination of mechanisms $J_t$, with each $J_t$ being anonymous as well as neutral, and each $J_t$ either being a mechanism of the form $D_{m,n}^{q}$ for some $q$ (by Theorem 2), or a mechanism that applies a truthful one-voter neutral mechanism $U$ to a voter chosen uniformly at random.

We now describe a single profile for which any such mechanism $J_t$ performs badly. Let $M_1, ..., M_g$ be a partition of $\{1, ..., kg\}$ with $k$ candidates in each set. The bad profile has the following voters:

- For each $i \in \{1, ..., m - 1\}$ a voter that assigns 1 to candidate $i$, 0 to candidate $m$ and valuations smaller than $1/m^2$ to the rest.
- For each $j \in \{1, ..., g\}$ a vote that assigns valuations strictly bigger than $1 - 1/m^2$ to members of $M_j$, valuation $1 - 1/m^2$ to $m$, and valuations smaller than $1/m^2$ to the rest.

Note that the social welfare of candidate $m$ is $(1 - 1/m^2)g$ while the social welfare of the other candidates are all smaller than $2 + 1/m$. Thus, the conditional expected approximation ratio given that the mechanism does not elect $m$ is at most $(2 + 1/m)/(1 - 1/m^2)g \leq 3m^{-2/3}$. We therefore only need to estimate the probability that candidate $m$ is elected. For a mechanism of the form $D_{m,n}^{q}$, candidate $m$ is chosen with probability at most $2/m$, since such a mechanism first eliminates all candidates but two and these two are chosen uniformly at random.

For a mechanism that picks a voter uniformly at random and applies a truthful one-agent neutral mechanism $U$ to the ballot of this voter, we make the following claim: Conditioned on a particular voter $i^*$
being picked, the conditional probability that \( m \) is chosen is at most \( 1/(r+1) \), where \( r \) is the number of candidates that outranks \( m \) on the ballot of voter \( i \). Indeed, if candidate \( m \) were chosen with conditional probability strictly bigger than \( 1/(r+1) \), she would be chosen with strictly higher probability than some other candidate \( j^* \) who outranks \( m \) on the ballot of voter \( i^* \). But if so, since \( U \) is neutral, voter \( i \) would increase his utility by switching \( j^* \) and \( m \) on his ballot, as this would switch the election probabilities of \( j^* \) and \( m \) while leaving all other election probabilities the same. This contradicts that \( U \) is truthful. Therefore, our claim is correct. This means that candidate \( m \) is chosen with probability at most 
\[
\frac{1}{m} + \left( \frac{g}{m} \right) \cdot \left( \frac{1}{k} \right) \leq \frac{1}{m} + m^{2/3} / (m^{1/3} - 1) \]
\[
\leq 2m^{-2/3}, \text{ since } m \geq 20.
\]
We conclude that on the bad profile, the expected approximation ratio of any mechanism \( J_\ell \) in the decomposition is at most \( 3m^{-2/3} + 2m^{-2/3} = 5m^{-2/3} \). Therefore, the expected approximation ratio of \( J \) on the bad profile is also at most \( 5m^{-2/3} / 2m > 2 \).

**Corollary 2.** For all \( m \), and all sufficiently large \( n \) compared to \( m \), any mechanism \( J \) in \( \text{Mech}_m^O \cup \text{Mech}_m^U \) has approximation ratio \( O(m^{-2/3}) \).

**Proof.** Combine Theorem 1, Lemma 4 and Theorem 4.

---

### 4 Mechanisms and negative results for the case of three candidates

In this section, we consider the special case of three candidates \( m = 3 \). To improve readability, we shall denote the three candidates by \( A, B \) and \( C \), rather than by \( 1, 2 \) and \( 3 \).

When the number of candidates \( m \) as well as the number of voters \( n \) are small constants, the exact values of \( r_{m,n}^O \) and \( r_{m,n}^U \) can be determined. We first give a clean example, and then describe a general method.

**Proposition 1.** For all \( J \in \text{Mech}_3^O \), we have \( \text{ratio}(J) \leq 2/3 \).

**Proof.** By Lemma 1, we can assume that \( J \) is anonymous and neutral. Let \( A >_1 B \) denote the fact that voter \( i \) ranks candidate \( A \) higher than \( B \) in his ballot. Let a Condorcet profile be any valuation profile with \( A >_1 B >_1 C, \, B >_2 C >_2 A \) and \( C >_3 A >_3 B \). Since \( J \) is neutral and anonymous, by symmetry, \( J \) elects each candidate with probability \( 1/3 \). Now, for some small \( \epsilon > 0 \), consider the Condorcet profile where \( u_1(B) = \epsilon, \, u_2(C) = \epsilon \) and \( u_3(A) = 1 - \epsilon \). The socially optimal choice is candidate \( A \) with social welfare \( 2 - \epsilon \), while the other candidates have social welfare \( 1 + \epsilon \). Since each candidate elected with probability \( 1/3 \), the expected social welfare is \( (4 + \epsilon)/3 \). By making \( \epsilon \) sufficiently small, the approximation ratio on the profile is arbitrarily close to \( 2/3 \).
are greater than all elements of $M_0$ in this ordering. Let the type of a quasi-combinatorial valuation function be the partition and the total ordering $(M_0, M_1, <)$. Then, a quasi-combinatorial valuation function is given by its type and the value of $k$. For instance, if $m = 3$, one possible type is \( \{ \{B\}, \{A, C\}, \{B < A < C\} \} \), and the quasi-combinatorial valuation function $u$ corresponding to this type for $k = 1000$ is $u(A) = 0.999$, $u(B) = 0$, $u(C) = 1$. We see that for any fixed value of $m$, there is a finite set $T_m$ of possible types. In particular, we have $|T_3| = 12$. Let $\eta : T_m \times \mathbb{N} \rightarrow C_{m,k}$ be the map that maps a type and an integer $k$ into the corresponding quasi-combinatorial valuation function.

For fixed $m, n$, consider the following matrices $G$ and $H$. The matrix $G$ has a row for each of the mechanisms $U_{m,n}^q$ for $q = 1, \ldots, m$, while the matrix $H$ has a row for each of the mechanisms $U_{m,n}^q$ for $q = 1, \ldots, m$ as well as for each of the mechanisms $D_{m,n}^q$, for $q = |n/2| + 1, |n/2| + 2, \ldots, n$. Both matrices have a column for each element of $(T_m)^n$. The entries of the matrices are as follows: Each entry is indexed by a mechanism $J \in \text{Mech}_{m,n}$ (the row index) and by a type profile $\mathbf{t} \in (T_m)^n$ (the column index). We let that entry be

$$c_{J,\mathbf{t}} = \lim_{k \to \infty} \frac{E[\sum_{i=1}^n u_i(J(u^k))]\eta(t_i,k)}{\max_{j \in \mathcal{M}} \sum_{i=1}^n u_i^k(j)},$$

where $u_i^k = \eta(t_i,k)$. Informally, we let the entry be the approximation ratio of the mechanism on the quasi-combinatorial profile of the type profile indicated in the column and with $1/k$ being “infinitesimally small”. Note that for the mechanisms at hand, despite the fact that the entries are defined as a limit, it is straightforward to compute the entries symbolically, and they are rational numbers.

We now have

**Lemma 6.** The value of $G$, viewed as a matrix game with the row player being the maximizer, is equal to $r_{m,n}^O$. The value of $H$ is equal to $r_{m,n}^O$. Also, the optimal strategies for the row players in the two matrices, viewed as convex combinations of the mechanisms corresponding to the rows, achieve those ratios.

**Proof.** We only show the statement for $r_{m,n}^O$, the other proof being analogous. For fixed $k$, consider the matrix $H^k$ defined similarly to $H$, but with entries $c_{J,\mathbf{t}} = E[\sum_{i=1}^n u_i(J(u^k))]\eta(t_i,k)$, where $u_i^k = \eta(t_i,k)$. Viewing $H^k$ as a matrix game, a mixed strategy of the row player can be interpreted as a convex combination of the mechanisms corresponding to the rows, and the expected payoff when the column player responds with a particular column $\mathbf{t}$ is equal to the approximation ratio of $J$ on the valuation profile $(\eta(t_i,k))_i$. Therefore, the value of the game is the worst case approximation ratio of the best convex combination, among profiles of the form $(\eta(t_i,k))_i$ for a type profile $\mathbf{t}$. By Lemma 1, $r_{m,n}^O$ is determined by the best available anonymous and neutral ordinal mechanism. By Corollary 1, the anonymous and neutral ordinal mechanisms are exactly the convex combinations of the $U_{m,n}^q$ and the $D_{m,n}^q$ mechanisms for various $q$. Given any particular convex combination yielding a mechanism $K$, by Lemma 3 its worst case approximation ratio is given by $\liminf_{k \to \infty} \min_{u \in (C_{m,k})^n} E[\sum_{i=1}^n u_i(K(u))]\frac{E[\sum_{i=1}^n u_i(J(u^k))]}{\sum_{i=1}^n u_i(A)}$, which is equal to $\liminf_{k \to \infty} \min_{u \in (C_{m,k})^n} E[\sum_{i=1}^n u_i(K(u^k))]/\sum_{i=1}^n u_i^k(j)$, since $K$ is neutral. This means that no mechanism can have an approximation ratio better than the limit of the values of the games $H^k$ as $k$ approaches infinity. By continuity of the value of a matrix game as a function of its entries, this is equal to the value of $H$. Therefore, $r_{m,n}^O$ is at most the value of $H$. Now consider the mechanism $J$ defined by the optimal strategy for the row player in the matrix game $H$. As the entries of $H_k$ converge to the entries of $H$ as $k \to \infty$, we have that for any $\epsilon > 0$, and sufficiently large $k$, the strategy is also an $\epsilon$-optimal strategy for $H_k$. Since $\epsilon$ is arbitrary, we have that $\text{ratio}(J)$ is at least the value of $H$, completing the proof.

When applying Lemma 6 for concrete values of $m, n$, one can take advantage of the fact that all mechanisms corresponding to rows are anonymous and neutral. This means that two different columns will have identical entries if they correspond to two type profiles that can be obtained from one another by permuting voters and/or candidates. This makes it possible to reduce the number of columns drastically. After such reduction, we have applied the theorem to $m = 3$ and $n = 2, 3, 4$ and 5, computing the corresponding optimal approximation ratios and optimal mechanisms. The ratios are given in Table 1. The mechanisms achieving
Table 1: Approximation ratios for $n$ voters.

| $n$ | $r_{3,n}^O$ | $r_{3,n}^{OU}$ |
|-----|-------------|----------------|
| 2   | 2/3         | 2/3            |
| 3   | 2/3         | 105/171        |
| 4   | 2/3         | 5/8            |
| 5   | 6407/9899   | 34/55          |

Table 2: Mixed-unilateral ordinal mechanisms for $n$ voters.

| $n$ | Mechanism  | $U_{3,n}^1$ | $U_{3,n}^2$ | $U_{3,n}^3$ |
|-----|------------|-------------|-------------|-------------|
| 2   | $1/3$      | $2/3$       | 0           |
| 3   | $9/19$     | $10/19$     | 0           |
| 4   | $1/2$      | $1/2$       | 0           |
| 5   | $5/11$     | $6/11$      | 0           |

Table 3: Ordinal mechanisms for $n$ voters.

| $n$ | Mechanism  | $U_{3,n}^1$ | $U_{3,n}^2$ | $U_{3,n}^3$ | $D_{n,3}^{[n/2]+1}$ | $D_{n,3}^{[n/2]+2}$ | $D_{n,3}^{[n/2]+3}$ |
|-----|------------|-------------|-------------|-------------|---------------------|---------------------|---------------------|
| 2   | $4/100$    | $8/100$     | $0$         | $88/100$    | —                   | —                   | —                   |
| 3   | $47/100$   | $0$         | $0$         | $53/100$    | $0$                 | —                   | —                   |
| 4   | $0$        | $0$         | $0$         | $1$         | $0$                 | —                   | —                   |
| 5   | $3035/9899$| $0$         | $0$         | $3552/9899$ | $3312/9899$         | $0$                 | —                   |
the ratios are shown in Table 3 and Table 2. These mechanisms are in general not unique. Note in particular that a different approximation-optimal mechanism than random-majority found in Mech_{2,3}^O.

We now turn our attention to the case of three candidates and arbitrarily many voters. In particular, we shall be interested in \( r_3^O = \lim_{n \to \infty} r_{3,n}^O \) and \( r_3^{OU} = \lim_{n \to \infty} r_{3,n}^{OU} \). By Lemma 4, we in fact have \( r_3^O = \lim_{n \to \infty} r_{3,n}^O \) and \( r_3^{OU} = \lim_{n \to \infty} r_{3,n}^{OU} \).

We present a family of ordinal and mixed-unilateral mechanisms \( J_n \) with ratio\( (J_n) > 0.610 \). In particular, \( r_3^{OU} > 0.610 \). The coefficients \( c_1 \) and \( c_2 \) were found by trial-and-error; we present more information about how later.

**Theorem 5.** Let \( c_1 = \frac{77066611}{1377377959} \approx 0.489 \) and \( c_2 = \frac{80671148}{1377377959} \approx 0.511 \). Let \( J_n = c_1 \cdot U_{m,n}^1 + c_2 \cdot U_{m,n}^2 \). For all \( n \), we have ratio\( (J_n) > 0.610 \).

**Proof.** By Lemma 5, we have that \( \text{ratio}(J_n) = \lim_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)} \). Recall the definition of the set of types \( T_3 \) of quasi-combinatorial valuation functions on three candidates and the definition of \( \eta \) preceding the proof of Lemma 4. From that discussion, we have \( \lim_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)} = \min_{\mathbf{t} \in (T_3)^n} \lim_{k \to \infty} \frac{\sum_{i=1}^n u_i(J_n(\mathbf{u}))}{\sum_{i=1}^n u_i(A)} \), where \( u_i = \eta(t_i,k) \). Also recall that \( |T_3| = 12 \). Since \( J_n \) is anonymous, to determine the approximation ratio of \( J_n \) on \( \mathbf{u} \in (C_{m,k})^n \), we observe that we only need to know the value of \( k \) and the fraction of voters of each of the possible 12 types. In particular, fixing a type profile \( \mathbf{t} \in (C_{m,k})^n \), for each type \( k \in T_3 \), let \( x_k \) be the fraction of voters in \( \mathbf{u} \) of type \( k \). For convenience of notation, we identify \( T_3 \) with \{1, 2, \ldots, 12\} using the scheme depicted in Table 4. Let \( w_i = \lim_{k \to \infty} \sum_{i=1}^n u_i(t_i) \), where \( u_i = \eta(t_i,k) \), and let \( p_j = \lim_{k \to \infty} \Pr[E_j] \), where \( E_j \) is the event that candidate \( j \) is elected by \( J_n \) in an election with valuation profile \( \mathbf{u} \) where \( u_i = \eta(t_i,k) \). We then have \( \lim_{k \to \infty} \frac{\sum_{i=1}^n u_i(J_n(\mathbf{u}))}{\sum_{i=1}^n u_i(A)} = \left( p_A \cdot w_A + p_B \cdot w_B + p_C \cdot w_C \right) / w_A \). Also, from Table 4 and the definition of \( J_n \), we see:

\[
\begin{align*}
 w_A & = n(x_1 + x_2 + x_3 + x_4 + x_5 + x_9) \\
 w_B & = n(x_1 + x_5 + x_6 + x_7 + x_8 + x_{11}) \\
 w_C & = n(x_4 + x_7 + x_9 + x_{10} + x_{11} + x_{12}) \\
 p_A & = (c_1 + c_2/2)(x_1 + x_2 + x_3 + x_4) + (c_2/2)(x_5 + x_6 + x_9 + x_{10}) \\
 p_B & = (c_1 + c_2/2)(x_5 + x_6 + x_7 + x_8) + (c_2/2)(x_1 + x_2 + x_{11} + x_{12}) \\
 p_C & = (c_1 + c_2/2)(x_9 + x_{10} + x_{11} + x_{12}) + (c_2/2)(x_3 + x_4 + x_7 + x_8)
\end{align*}
\]

Thus we can establish that \( \text{ratio}(J_n) > 0.610 \) for all \( n \), by showing that the quadratic program “Minimize \( p_A \cdot w_A + p_B \cdot w_B + p_C \cdot w_C \) - 0.610w_A subject to \( x_1 + x_2 + \cdots + x_{12} = 1, x_1, x_2, \ldots, x_{12} \geq 0 \)” , where \( w_A, w_B, w_C, p_A, p_B, p_C \) have been replaced with the above formulae using the variables \( x_i \), has a strictly positive minimum (note that the parameter \( n \) appears as a multiplicative constant in the objective function and can be removed, so there is only one program, not one for each \( n \)). This was established rigorously by solving the program symbolically in Maple by a facet enumeration approach (the program being non-convex), which is easily feasible for quadratic programs of this relatively small size.

We next present a family of ordinal mechanisms \( J_n' \) with ratio\( (J_n') > 0.616 \). In particular, \( r_3^O > 0.616 \). The coefficients defining the mechanism \( c_1 \) and \( c_2 \) were again found by trial-and-error; we present more information about how later.

**Theorem 6.** Let \( c_1' = 0.476, c_2' = 0.467 \) and \( d = 0.057 \) and let \( J_n = c_1' \cdot U_{m,n}^1 + c_2' \cdot U_{m,n}^2 + d \cdot D_{m,n}^{n/2} \). Then \( \text{ratio}(J_n) > 0.616 \) for all \( n \).

**Proof.** The proof idea is the same as in the proof of Theorem 5. In particular, we want to reduce proving the theorem to solving quadratic programs. The fact that we have to deal with the \( D_{m,n}^{n/2} \), i.e., random-majority, makes this task slightly more involved. In particular, we have to solve many programs rather than just one. We only provide a sketch, showing how to modify the proof of Theorem 5.
As in the proof of Theorem 5 we let \( w_j = \lim_{k \to \infty} \sum_{i=1}^{n} u_i(i) \), where \( u_i = \eta(t_i, k) \). The expressions for \( w_A \) as functions of the variables \( x_i \) remain the same as in that proof. Also, we let \( p_j = \lim_{k \to \infty} \Pr[E_j] \), where \( E_j \) is the event that candidate \( j \) is elected by \( J'_n \) in an election with valuation profile \( u \) where \( u_i = \eta(t_i, k) \). We then have

\[
\begin{align*}
p_A &= (\epsilon'_1 + \epsilon'_2/2)(x_1 + x_2 + x_3 + x_4) + (\epsilon'_2/2)(x_5 + x_6 + x_9 + x_{10}) + d \cdot q_A(t) \\
p_B &= (\epsilon'_1 + \epsilon'_2/2)(x_5 + x_6 + x_7 + x_8) + (\epsilon'_2/2)(x_1 + x_2 + x_{11} + x_{12}) + d \cdot q_B(t) \\
p_C &= (\epsilon'_1 + \epsilon'_2/2)(x_9 + x_{10} + x_{11} + x_{12}) + (\epsilon'_2/2)(x_3 + x_4 + x_7 + x_8) + d \cdot q_C(t)
\end{align*}
\]

where \( q_j(t) \) is the probability that random-majority elects candidate \( j \) when the type profile is \( t \). Unfortunately, this quantity is not a linear combination of the \( x_i \) variables, so we do not immediately arrive at a quadratic program.

However, we can observe that the values of \( q_j(t) \), \( j = A, B, C \) depend only on the outcome of the three pairwise majority votes between \( A, B \) and \( C \), where the majority vote between, say, \( A \) and \( B \) has three possible outcomes: \( A \) wins, \( B \) wins, or there is a tie. In particular, there are 27 possible outcomes of the three pairwise majority votes. To show that \( \min_{t \in (T_3)^n} \lim_{k \to \infty} E \left[ \eta \left( \sum_{i=1}^{n} u_i(J'_n(A)) \right) \right] > 0.616 \), where \( u_i = \eta(t_i, k) \), we partition \( (T_3)^n \) into 27 sets according to the outcomes of the three majority votes of an election with type profile \( t \) and show that the inequality holds on all 27 sets in the partition. We claim that on each of the 27 sets, the inequality is equivalent to a quadratic program. Indeed, each \( q_A(t) \) is now a constant, and the constraint that the outcome is as specified can be expressed as a linear constraint in the \( x_i \)'s and added to the program. For instance, the condition that \( A \) beats \( B \) in a majority vote can be expressed as \( x_1 + x_2 + x_3 + x_4 + x_9 + x_{10} > 1/2 \) while \( A \) ties \( C \) can be expressed as \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1/2 \). Except for the fact that these constraints are added, the program is now constructed exactly as in the proof of Theorem 5. Solving the programs confirms the statement of the theorem.

We next show that \( r^{OU}_{3,n^*} \leq 0.611 \) and \( r^{O}_{3} \leq 0.641 \). By Lemma 4 it is enough to show that \( r^{OU}_{3,n^*} \leq 0.611 \) and \( r^{O}_{3,n^*} \leq 0.641 \) for some fixed \( n^* \). Therefore, the statements follow from the following theorem.

**Theorem 7.** \( r^{OU}_{3,23000} \leq 0.611 \) and \( r^{O}_{3,23000} \leq 0.641 \).

**Proof.** Lemma 6 states that the two upper bounds can be proven by showing that the values of two certain matrix games \( G \) and \( H \) are smaller than the stated figures. While the two games have a reasonable number of rows, the columns of the games is astronomical, so we cannot solve the games exactly. However, we can prove upper bounds on the values of the games by restricting the strategy space of the column player. Note that this corresponds to selecting a number of bad type profiles. We have constructed a catalogue of just 5 type profiles, each with 23000 voters. Using the “fraction encoding” of profiles suggested in the proof of Theorem 5 the profiles are:

- \( x_2 = 14398/23000, x_3 = 2185/23000, x_{11} = 6417/23000 \).
- \( x_2 = 6000/23000, x_3 = 8000/23000, x_{12} = 9000/23000 \).
- \( x_1 = 11500/23000, x_{11} = 11500/23000 \).

6To make the program amenable to standard facet enumeration methods of quadratic programming, we changed the sharp inequalities \( > \) expressed the majority vote constraints into weak inequalities \( \geq \). Note that this cannot decrease the cost of the optimal solution.
Lemma 7. Let reason that we focus on this particular cardinal mechanism is given by the following lemma.

Proposition 2. Suppose \( u_1(A) = u_2(A) = 1 - \epsilon, u_1(B) = u_2(C) = 0, \) and \( u_1(C) = u_2(B) = 1. \) As in the proof of Theorem 4, by neutrality, we must have that the probability of \( u \) being elected is at most \( \frac{1}{2}. \) The statement follows by considering a sufficiently small \( \epsilon. \)

The lower bound follows from an analysis of the quadratic-lottery of Feige and Tennenholtz \([6]\). The main reason that we focus on this particular cardinal mechanism is given by the following lemma.

Lemma 7. Let \( J \in \text{Mech}_{3,n} \) be a convex combination of \( Q_n \) and any ordinal and neutral mechanism. Then

\[
\text{ratio}(J) = \lim_{k\to\infty} \inf_{u \in (C_{n,k})^n} \frac{E[\sum_{i=1}^n u_i(J(u))] - E[\sum_{i=1}^n u_i(1)]}{\sum_{i=1}^n u_i(1)}. 
\]

Proof. The proof is a simple modification of the proof of Lemma 4. As in that proof, for a valuation profile \( u = (u_i) \), define \( g(u) = \frac{E[\sum_{i=1}^n u_i(J(u))] - E[\sum_{i=1}^n u_i(1)]}{\sum_{i=1}^n u_i(1)} \). We show the following equations:

\[
\text{ratio}(J) = \inf_{u \in V_3^n} g(u) \quad (7)
\]

\[
= \lim_{k\to\infty} \min_{u \in (R_{3,k})^n} g(u) \quad (8)
\]

\[
= \lim_{k\to\infty} \min_{u \in (C_{3,k})^n} g(u) \quad (9)
\]

Solving the corresponding matrix games yields the stated upper bound. \( \square \)

While the catalogue of bad type profiles of the proof of Theorem 7 suffices to prove Theorem 1, we should discuss how we arrived at this particular “magic” catalogue. This discussion also explains how we arrived at the “magic” coefficients in Theorems 5 and 6. In fact, we arrived at the catalogue and the coefficients iteratively in a joint local search process (or “co-evolution” process). To get an initial catalogue, we used the fact that we had already solved the matrix games yielding the values of \( r_{3,n}^Q \) and \( r_{3,n}^O \) for \( n = 2, 3, 5. \) By the theorem of Shapley and Snow \([10]\), these matrix games have optimal strategies for the column player with support size at most the number of rows of the matrices. One can think of these supports as a small set of bad type profiles for 2, 3 and 5 voters. Utilizing that 2, 3 and 5 all divide 1000, we scaled all these up to 1000 voters. Also, we had solved the quadratic programs of the proofs of Theorem 5 and Theorem 6 but with inferior coefficients and resulting bounds to the ones stated in this paper. The quadratic programs obtained their minima at certain type profiles. We added these entries to the catalogue, and scaled all profiles to their least common multiple, i.e. 23000. Solving the linear programs of the proof of Theorem 7 now gave not only an upper bound on the approximation ratio, but the optimal strategy of Player I in the games also their minima at certain type profiles. We added by hand some bad type profiles along the way, and iterated the procedure until no further improvement was found. In the end we pruned the catalogue into a set of five, giving the same upper bound as we had already obtained.

We finally show that \( r_{3,2}^U \) is between 0.660 and 0.750. The upper bound follows from the following proposition and Lemma 4.

**Proposition 2.** \( r_{3,2}^U \leq 0.75. \)

**Proof.** Suppose \( J \in \text{Mech}_{3,2} \) has ratio \( (J) > 0.75. \) By Lemma 4 we can assume \( J \) is neutral. For some \( \epsilon > 0, \) consider the valuation profile with \( u_1(A) = u_2(A) = 1 - \epsilon, u_1(B) = u_2(C) = 0, \) and \( u_1(C) = u_2(B) = 1. \) Then \( \text{ratio}(J) \geq \frac{1}{2}. \) The statement follows by considering a sufficiently small \( \epsilon. \)

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\]

**Proof.** The proof is a simple modification of the proof of Lemma 4. As in that proof, for a valuation profile \( u = (u_i) \), define \( g(u) = \frac{E[\sum_{i=1}^n u_i(J(u))] - E[\sum_{i=1}^n u_i(1)]}{\sum_{i=1}^n u_i(1)} \). We show the following equations:

\[
\text{ratio}(J) = \inf_{u \in V_3^n} g(u) = \lim_{k\to\infty} \min_{u \in (R_{3,k})^n} g(u) = \lim_{k\to\infty} \min_{u \in (C_{3,k})^n} g(u) \quad (7)
\]

\[
= \inf_{u \in V_3^n} \max_{j \in M} \sum_{i=1}^n u_i(j) \quad (8)
\]

\[
= \lim_{k\to\infty} \min_{u \in (R_{3,k})^n} g(u) \quad (9)
\]

\[
= \lim_{k\to\infty} \min_{u \in (C_{3,k})^n} g(u) \quad (10)
\]
Equation (3), (9) follows as in the proof of Lemma 5. Equation (10) follows from the following argument.

For any profile $u = (u_i) \in (R_{3,k})^n$, let $c_u$ denote the number of pairs $(i,j)$ with $i$ being a voter and $j$ a candidate, for which $u_i(j) - 1/k$ and $u_i(j) + 1/k$ are both in $[0,1]$ and both not in the image of $u_i$. Then, $C_{3,k}$ consists of exactly those $u$ in $R_{3,k}$ for which $c_u = 0$. To establish equation (10), we merely have to show that for any $u \in R_{3,k}$, there is a $u' \in R_{3,k}$ for which $g(u') \leq g(u)$ and $c_{u'} < c_u$. We will now construct such $u'$. Since $c_u > 0$, there is a pair $(i,j)$ so that $u_i(j) - 1/k$ and $u_i(j) + 1/k$ are both in $[0,1]$ and both not in the image of $u$. Let $\ell_-$ be the smallest integer value so that $u_i(j) - \ell/k$ is not in the image of $u_i$, for any integer $\ell \in \{\ell, \ldots, j - 1\}$. Let $\ell_+$ be the largest integer value so that $u_i(j) + \ell/k$ is not in the image of $u_i$, for any integer $\ell \in \{j + 1, \ldots, \ell_+\}$. We can define a valuation function $u^x \in V_m$ for any $x \in [-\ell_-/k; \ell_+/k]$ as follows: $u^x$ agrees with $u_i$ except on $j$, where we let $u^x(j) = u_i(j) + x$. Let $u = (u^x, u_{-i})$. Now consider the valuation $h : x \rightarrow g(u^x)$. Since $J$ is a convex combination of quadratic-lottery and a neutral ordinal mechanism, we see by inspection of the definition of the function $g$, that $h$ on the domain $[-\ell_-/k; \ell_+/k]$ is the quotient of two quadratic polynomials where the numerator has second derivative being a negative constant and the denominator is positive throughout the interval. This means that $h$ attains its minimum at either $\ell_-/k$ or at $\ell_+/k$. In the first case, we let $u' = u^{\ell_-/k}$ and in the second, we let $u' = u^{\ell_+/k}$. This completes the proof. \(\square\)

Theorem 8. The limit of the approximation ratio of $Q_n$ as $n$ approaches infinity, is exactly the golden ratio, i.e., $(\sqrt{5} - 1)/2 \approx 0.618$. Also, let $J_n$ be the mechanism for $n$ voters that selects random-favorite with probability 29/100 and quadratic-lottery with probability 71/100. Then, ratio($J_n$) > $\frac{33}{50} = 0.660$.

Proof. (sketch) Lemma 7 allows us to proceed completely as in the proof of Theorem 5 by constructing and solving appropriate quadratic programs. As the proof is a straightforward adaptation, we leave out the details. \(\square\)

5 Conclusion

Several open problems remain. For general $m$, can we close the gap between $\Omega(m^{-3/4})$ and $O(m^{-2/3})$ for ordinal mechanisms? Can the upper bound be extended to cardinal mechanisms that are not mixed-unilateral? Can we close the gaps for ordinal mechanisms and for mixed-unilateral mechanisms for $m = 3$? How well can cardinal mechanisms do for $m = 3$? Does a lemma similar to Lemma 4 hold for general cardinal mechanisms? Getting upper bounds on the performance of general cardinal mechanisms is impaired by the lack of a characterization of cardinal mechanisms a la Gibbard’s. However, by direct manipulation arguments, it is possible to obtain some upper bounds. As a proof of concept, we present the following simple one. We did not attempt to optimize and leave a thorough investigation of the approximation abilities of cardinal mechanisms for future work.

Theorem 9. All mechanisms $J \in \text{Mech}_{3,3}$ have ratio$(J) < 0.94$.

Proof. Assume for contradiction that there exists a mechanism $J \in \text{Mech}_{3,3}$, with ratio$(J) \geq 0.94$. Consider the valuation profile $u$ with three voters $\{1, 2, 3\}$, three candidates $\{A, B, C\}$, and valuations $u_1(B) = u_2(B) = u_3(C) = 1$, $u_1(C) = u_2(C) = u_3(B) = 0$, $u_1(A) = 0.7$ and $u_2(A) = u_3(A) = 0.8$. The social optimum on profile $u$ is candidate $A$, with social welfare $w_A = 2.3$, while $w_B = 2$ and $w_C = 1$. Since $J$’s expected social welfare is at least a 0.94 fraction of $w_A$, i.e. $2.162$, the probability of $A$ being elected is at least 0.54, as otherwise the expected social welfare would be smaller than $0.54 \cdot 2.3 + 0.46 \cdot 2 = 2.162$. The expected utility $\tilde{u}$ of voter 1 in that case is at most $0.54 \cdot 0.7 + 0.46 \cdot 1 = 0.883$.

Next, consider the profile $u'$ identical to $u$ except that $u'_1(A) = 0.0001$. Let $p_A, p_B, p_C$ be the probabilities of candidates $A, B$ and $C$ being elected on this profile, respectively. The social optimum is $B$ with social welfare 2. It must be that $0.7p_A + p_B \leq \tilde{u}$, otherwise on profile $u$, voter 1 would have an incentive to misreport $u_1(A)$ as 0.0001. Also, since $J$ has an approximation ratio of at least 0.94, it must be the case that $1.6001 \cdot p_A + 2 \cdot p_B + p_C \leq 1.888$. By those two inequalities, we have: $0.9001p_A + p_B + p_C \geq 1.8800 - \tilde{u} \Rightarrow 0.9001(p_A + p_B + p_C) + 0.0999p_B + 0.0999p_C \geq 1.8800 - \tilde{u} \Rightarrow 0.0999(p_B + p_C) \geq 1.0419 \Rightarrow p_B + p_C \geq 1.42$ which is not possible. Hence, it cannot be that ratio$(J) \geq 0.94$. \(\square\)
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