SIMPLICITY OF LEAVITT PATH ALGEBRAS VIA GRADED RING THEORY

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Abstract. Suppose that $R$ is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. Utilizing results from graded ring theory we show, that the associated Leavitt path algebra $L_R(E)$ is simple if and only if $R$ is simple, $E^0$ has no nontrivial hereditary and saturated subset, and every cycle in $E$ has an exit. We also give a complete description of the center of a simple Leavitt path algebra.

1. Introduction

The Leavitt path algebra of a row-finite graph, over a field, was introduced in [2,5] and has since then been successively generalized (see e.g [3,20]). The Leavitt path algebra of an arbitrary directed graph, over a unital ring, was introduced in [12]. For an account of the development of the field of Leavitt path algebras, we refer the reader to [1]. Here is our first main result.

Theorem 1. Suppose that $R$ is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. The Leavitt path algebra $L_R(E)$ is simple if and only if $R$ is simple, $E^0$ has no nontrivial hereditary and saturated subset, and every cycle in $E$ has an exit.

Characterizations of simple Leavitt path algebras over fields have previously been established in e.g. [19, Thm. 6.18], [3, Thm. 3.1] and [11, Thm. 3.5]. Theorem 1 generalizes all of those results, and also partially...
generalizes [20] Thm. 7.20]. Our second main result, stated below, completely describes the center of a simple Leavitt path algebra. It generalizes [6] Thm. 4.2] from the case where $R$ is a field and $E$ is a row-finite graph.

**Theorem 2.** Suppose that $R$ is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. Furthermore, suppose that $L_R(E)$ is a simple Leavitt path algebra. The following assertions hold:

(a) If $L_R(E)$ is not unital, then $Z(L_R(E)) = \{0\}$.
(b) If $L_R(E)$ is unital, then $Z(L_R(E)) = Z(R) \cdot 1_{L_R(E)}$.

Whereas earlier proofs of Theorems 1 and 2 (when $R$ is a field) utilize ad hoc arguments, specifically designed for graph algebras, we use the general theory of graded rings to obtain our results. This makes our proofs shorter and, we believe, clearer. Indeed, we show that $L_R(E)$ is graded simple if and only if $R$ is simple and $E^0$ has no nontrivial hereditary and saturated subset (see Proposition 12). We also show that every cycle in $E$ has an exit if and only if the center of each corner subring of $L_R(E)$ at a vertex has degree zero (see Proposition 21).

We point out that there are various generalizations of Leavitt path algebras in the literature (see e.g. [1 Sec. 5] and [9, 18]). A simplicity result for Steinberg algebras was obtained in [8], and when translated to Leavitt path algebras one recovers Theorem 1 in the special case where $R$ is a commutative unital ring. Note that [6 Thm. 4.2] was generalized to Kumjian-Pask algebras in [7], and in [10], Steinberg algebra techniques were used to give a complete description of the center of a general (not necessarily simple) Leavitt path algebra $L_R(E)$, where $R$ is a commutative unital ring.

2. Simple $\mathbb{Z}$-graded rings

Let $\mathbb{Z}$ denote the rational integers and write $\mathbb{N} := \{1, 2, 3, \ldots\}$. Suppose that $S$ is a ring. By this we mean that $S$ is associative but not necessarily unital. If $S$ is unital, then we let $1_S$ denote the multiplicative identity of $S$. Furthermore, we let $Z(S)$ denote the center of $S$, that is the set of all $s \in S$ satisfying $st = ts$ for every $t \in S$. Recall that $S$ is said to be $\mathbb{Z}$-graded if, for each $n \in \mathbb{Z}$, there is an additive subgroup $S_n$ of $S$ such that $S = \oplus_{n \in \mathbb{Z}} S_n$, and $S_n S_m \subseteq S_{n+m}$, for all $n, m \in \mathbb{Z}$. In that case, each element $s \in S$ may be written as $s = \sum_{n \in \mathbb{Z}} s_n$, where $s_n \in S_n$ is zero for all but finitely many $n \in \mathbb{Z}$. The support of $s$ is defined as the finite set $\text{Supp}(s) := \{n \in \mathbb{Z} \mid s_n \neq \{0\}\}$. An ideal $I$ of $\mathbb{Z}$-graded ring $S$ is said to be graded, if $I = \oplus_{n \in \mathbb{Z}} (I \cap S_n)$. If $\{0\}$ and $S$ are the only graded ideals of $S$, then $S$ is said to be graded simple.

We recall some properties of graded rings:

**Lemma 3.** Suppose that $S$ is a unital $\mathbb{Z}$-graded ring.

(a) The ring $Z(S)$ is $\mathbb{Z}$-graded with respect to the grading defined by $Z(S)_n := Z(S) \cap S_n$, for $n \in \mathbb{Z}$.
(b) If $S$ is a field, then $S = S_0$.

**Proof.** (a) is [15 p. 15, Exer. 8] and (b) is [15 Rem. 1.3.10].
Proposition 4. Suppose that $S$ is a unital $\mathbb{Z}$-graded ring. Then the following assertions are equivalent:

(i) $S$ is simple;
(ii) $S$ is graded simple and $Z(S)$ is a field;
(iii) $S$ is graded simple and $Z(S) \subseteq S_0$.

Proof. (i)⇒(ii) is clear, and (ii)⇒(iii) follows from Lemma 3. Now, we show that (iii)⇒(i). Suppose that $S$ is graded simple and that $Z(S) \subseteq S_0$. Let $I$ be a nonzero ideal of $S$. We wish to show that $1_S \in I$. Amongst all nonzero elements of $I$, choose $s$ such that $|\text{Supp}(s)|$ is minimal. Take $m \in \text{Supp}(s)$. Since $S$ is graded simple, there are $n \in \mathbb{N}$ and homogeneous elements $p_1, \ldots, p_n, q_1, \ldots, q_n \in S$, such that $\sum_{i=1}^{n} p_i s q_i = 1_S$, and $p_i s m q_i \in S_0 \setminus \{0\}$ for every $i \in \{1, \ldots, n\}$. Write $t := \sum_{i=1}^{n} p_i s q_i$. Note that $t \in I$, $t_0 = 1_S$ and $|\text{Supp}(t)| \leq |\text{Supp}(s)|$. Take $z \in \mathbb{Z}$ and $x \in S_z$. Then, $tx - xt \in I$ and, since $t_0 = 1_S$, it follows that $|\text{Supp}(tx - xt)| < |\text{Supp}(t)|$. By the assumptions on $s$ we get $|\text{Supp}(tx - xt)| = 0$ and hence that $xt = tx$. Thus, $t \in Z(S) \subseteq S_0$. We conclude that $1_S = t_0 = t \in I$. \hfill $\Box$

Let $S$ be a ring. Recall from [4] (see also [17]) that a set $U$ of idempotents in $S$ is called a set of local units for $S$, if for every $n \in \mathbb{N}$ and all $s_1, \ldots, s_n \in S$ there is some $e \in U$ such that $e s_i = s_i e = s_i$, for every $i \in \{1, \ldots, n\}$.

Remark 5. Suppose that $S$ is a $\mathbb{Z}$-graded ring. If $e \in S_0$ is an idempotent, then the corner subring $eS e$ inherits a natural $\mathbb{Z}$-grading defined by $(eSe)_n := eS ne$, for $n \in \mathbb{Z}$.

For future reference, we recall the following two results:

Proposition 6. Suppose that $S$ is a $\mathbb{Z}$-graded ring equipped with a set of local units $U \subseteq S_0$. Then, $S$ is (graded) simple if and only if, for every $f \in U$, the ring $fS f$ is (graded) simple.

Proof. First we show the “only if” statement. Suppose that $S$ is (graded) simple and that $f \in U$. Let $J$ be a nonzero (graded) ideal of $fS f$. By (graded) simplicity of $S$, it follows that $SJS = S$. Thus, $fS f = fSJS f = (fS f)J(fS f) \subseteq J$ and hence $J = fS f$. Now, we show the “if” statement. Suppose that $fS f$ is (graded) simple for every $f \in U$. Let $I$ be a nonzero (graded) ideal of $S$. Take a nonzero (homogeneous) $x \in S$. Take a nonzero (homogeneous) $y \in I$ and $f \in U$ with $fx = xf = x$ and $fy = yf = y$. By (graded) simplicity of $fS f$ it follows that $I \ni fSyS f = fS f \ni x$. Thus, $I = S$. \hfill $\Box$

Proposition 7. Suppose that $S$ is a $\mathbb{Z}$-graded ring equipped with a set of local units and that $f \in S_0$ is a nonzero idempotent. If $S$ is graded simple and $fS f$ is simple, then $S$ is simple.
Proof. Suppose that $S$ is graded simple and that $fSf$ is simple. Let $I$ be a nonzero ideal of $S$. Take a nonzero $s \in I$ and write $s = \sum_{n \in \text{Supp}(s)} s_n$. Fix $m \in \text{Supp}(s)$ and define $J := S_{m}S$. Then, $J$ is a nonzero graded ideal of $S$. By graded simplicity of $S$, it follows that $J = S$ and, in particular, that $f \in J$. Note that $f \in fJf$. Using that $f \neq 0$, it follows that there exist nonzero homogeneous $y, z \in S$ such that $fys_{m}zf$ is nonzero and $\deg(y) + \deg(z) = -m$. Now, define $s' := fyszf$. By the construction of $s'$, it follows that $s' \in I \cap fSf$ and that $s'$ is nonzero. In particular, $I \cap fSf \neq \{0\}$. Hence, by simplicity of $fSf$, we get that $I \cap fSf = fSf$. Thus, $f \in I$. Note that $SfS$ is a nonzero graded ideal of $S$. Hence, by graded simplicity of $S$, we get that $I \supseteq SfS = S$. This shows that $I = S$. \hfill \Box

3. Simple Leavitt path algebras

Let $R$ be an associative unital ring and let $E = (E^0, E^1, r, s)$ be a directed graph. Recall that $r$ (range) and $s$ (source) are maps $E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ are called edges. The elements of $E^1$ are called real edges, while for $f \in E^1$ we call $f^*$ a ghost edge. The set $\{f^* \mid f \in E^1\}$ will be denoted by $(E^1)^*$. A path $\mu$ in $E$ is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, \ldots, n-1\}$. In that case, $s(\mu) := s(\mu_1)$ is the source of $\mu$, $r(\mu) := r(\mu_n)$ is the range of $\mu$, and $|\mu| := n$ is the length of $\mu$. If $\mu = \mu_1 \ldots \mu_n$ is a (real) path in $E$, then we let $\mu^* := \mu_n^* \ldots \mu_1^*$ denote the corresponding ghost path. For any vertex $v \in E^0$ we put $s(v) := v$ and $r(v) := v$. We let $r(f^*)$ denote $s(f)$, and we let $s(f^*)$ denote $r(f)$. For $n \geq 2$, we define $E^n$ to be the set of paths of length $n$, and $E^* := \bigcup_{n \geq 0} E^n$ is the set of all finite paths.

Following Hazrat [12] we make the following definition.

Definition 8. The Leavitt path algebra of $E$ with coefficients in $R$, denoted by $L_R(E)$, is the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{f \mid f \in E^1\}$ and $\{f^* \mid f \in E^1\}$ with the coefficients in $R$, subject to the relations:

1. $uv = \delta_{u,v}v$ for all $u, v \in E^0$;
2. $s(f)f = fr(f) = f$ and $r(f)f^* = f^*s(f) = f^*$, for all $f \in E^1$;
3. $f^*f^* = \delta_{f,f^*}r(f)$, for all $f, f^* \in E^1$;
4. $\sum_{f \in E^1, s(f) = y} ff^* = v$, for every $v \in E^0$ for which $s^{-1}(v)$ is nonempty and finite.

Here elements of the ring $R$ commute with the generators.

Remark 9. (a) The Leavitt path algebra $L_R(E)$ carries a natural $\mathbb{Z}$-gradation. Indeed, put $\deg(v) := 0$ for each $v \in E^0$. For each $f \in E^1$ we put $\deg(f) := 1$ and $\deg(f^*) := -1$. By assigning degrees to the generators in this way, we obtain a $\mathbb{Z}$-gradation on the free algebra $F_R(E) = R[v, f, f^* \mid v \in E^0, f \in E^1]$. Moreover, the ideal coming from relations (1)-(4) in Definition 8 is graded. Using this it is easy to see that the natural $\mathbb{Z}$-gradation on $F_R(E)$ carries over to a $\mathbb{Z}$-gradation on the quotient algebra $L_R(E)$. 

(b) The set \( \{ \sum_{v \in F} v \mid F \text{ is a finite subset of } E^0 \} \) is a set of local units for \( L_R(E) \). If \( E^0 \) is finite, then \( L_R(E) \) is unital, and \( 1_{L_R(E)} = \sum_{v \in E^0} v \).

(c) Motivated by Definition \( 8 \), for \( u \in E^0 \), we write \( u^* := u \).

**Definition 10.** Let \( E = (E^0, E^1, r, s) \) be a directed graph. A subset \( H \subseteq E^0 \) is said to be **hereditary** if, for any \( f \in E^1 \), we have that \( s(f) \in H \) implies \( r(f) \in H \). A hereditary subset \( H \subseteq E^0 \) is called **saturated** if, whenever \( v \in E^0 \) satisfies \( 0 < |s^{-1}(v)| < \infty \), we have that \( \{ r(f) \in H \mid f \in E^1 \text{ and } s(f) = v \} \subseteq H \) implies \( v \in H \).

**Remark 11.** Note that \( \emptyset \) and \( E^0 \) are always hereditary and saturated subsets of \( E^0 \). They are referred to as **trivial**.

**Proposition 12.** The Leavitt path algebra \( L_R(E) \) is graded simple if and only if \( R \) is simple and \( E^0 \) has no nontrivial hereditary and saturated subset.

**Proof.** First we show the “if” statement. Suppose that \( R \) is simple and that \( E^0 \) has no nontrivial hereditary and saturated subset. Let \( I \) be a nonzero graded ideal of \( L_R(E) \). Consider the set \( H_I := \{ v \in E^0 \mid kv \in I \text{ for some nonzero } k \in R \} \). By the same argument as in [20, Lem. 5.1], \( H_I \) is nonempty. Furthermore, since \( R \) is simple, it follows that \( H_I = \{ v \in E^0 \mid v \in I \} \). We wish to show that \( H_I \) is hereditary and saturated. To this end, take \( v \in H_I \). Suppose that \( e \in E^1 \) with \( s(e) = v \). Then, \( r(e) = e^*e = e^*ve \in I \).

Thus, \( H_I \) is hereditary. Now, take \( v \in E^0 \) such that \( 0 < |s^{-1}(v)| < \infty \), and suppose that \( r(s^{-1}(v)) \subseteq H_I \). For each \( e \in s^{-1}(v) \) we have \( r(e) \in H_I \) and hence \( ee^* = er(e)e^* \in I \). Thus, \( v = \sum_{e \in s^{-1}(v)} ee^* \in I \) and \( v \in H_I \).

Therefore, \( H_I \) is saturated. By assumption, we get that \( H_I = E^0 \). This shows that \( I \) must contain all the local units of \( L_R(E) \) and thus \( I = L_R(E) \). Hence, \( L_R(E) \) is graded simple.

Now, we show the “only if” statement. Suppose that \( L_R(E) \) is graded simple. Let \( J \) be a nonzero ideal of \( R \). Note that \( J \cdot L_R(E) \) is a nonzero graded ideal of \( L_R(E) \). Thus, \( J \cdot L_R(E) = L_R(E) \) and we conclude that \( J = R \). This shows that \( R \) is simple.

Let \( H \) be a proper hereditary and saturated subset of \( E^0 \). Following [2, 3], we let \( F := (F^0, F^1, r, s) \) be the graph consisting of all vertices not in \( H \) and all edges whose range is not in \( H \). For \( v \in E^0 \), define \( \Psi(v) := v \) if \( v \in F^0 \), and \( \Psi(v) := 0 \), otherwise. For \( e \in E^1 \), define \( \Psi(e) := e \) if \( e \in F^1 \), and \( \Psi(e) := 0 \), otherwise. Furthermore, define \( \Psi(e^*) := e^* \), if \( e^* \in (F^1)^* \), and \( \Psi(e^*) := 0 \), otherwise. The argument in loc. cit. shows that this yields a well-defined ring homomorphism \( \Psi : L_R(E) \to L_R(F) \). Clearly, \( \Psi \) is graded. Thus, the ideal \( I := \ker(\Psi) \) of \( L_R(E) \) is graded. Note that \( F^0 \) is nonempty, because \( H \) is proper, and hence \( I \neq L_R(E) \). By assumption, we get that \( I = \{ 0 \} \). By the construction of \( \Psi \) it follows that \( H \subseteq I \). Thus, \( H = \emptyset \).

**Definition 13.** Define an additive map \( \mathcal{L} : L_R(E) \to L_R(E) \) by requiring that \( \mathcal{L}(\lambda \alpha \beta^*) = \lambda \beta \alpha^* \), for all \( \lambda \in R \), and \( \alpha, \beta \in E^* \).
Remark 14. The map \( \mathcal{L} \) is an isomorphism of additive groups such that 
\( \mathcal{L}((L_R(E))_N) = (L_R(E))_{-N} \) for every \( N \in \mathbb{Z} \).

Lemma 15. Suppose that \( u \in E^0 \). The map \( \mathcal{L} \) restricts to an isomorphism 
of additive groups \( \mathcal{L}|_{Z(uL_R(E)u)} : Z(uL_R(E)u) \to Z(uL_R(E)u) \). In particular, the equality 
\( \mathcal{L}|_{Z(uL_R(E)u)} = (Z(uL_R(E)u))_{-N} \) holds for every \( N \in \mathbb{Z} \).

Proof. Let \( x = \sum_{j=1}^{m} \lambda_j \alpha_j \beta_j^* \in Z(uL_R(E)u) \), where \( \lambda_j \in R \), \( \alpha_i, \beta_j \in E^* \) 
and \( s(\alpha_j) = s(\beta_j) = u \) for \( j \in \{1, \ldots, m\} \). Take \( r \in R \). Then, \( = xru - ru = \sum_{j=1}^{m} (\lambda_j r - r \lambda_j) \alpha_j \beta_j^* \). Therefore, \( 0 = \omega(0) = \sum_{j=1}^{m} (\lambda_j r - r \lambda_j) \alpha_j \beta_j^* = \omega(x)ru - ru\omega(x) \). Thus, \( \omega(x)ru = ru\omega(x) \). Take \( \gamma, \delta \in E^* \) with \( s(\gamma) = s(\delta) = u \). Then, \( 0 = x\gamma \delta^* - \gamma \delta^* x = \sum_{j=1}^{m} \lambda_j (\alpha_j \beta_j^* \gamma \delta^* - \gamma \delta^* \alpha_j \beta_j^*) \). Therefore, \( 0 = \omega(0) = \sum_{j=1}^{m} \lambda_j \omega(\alpha_j \beta_j^* \gamma \delta^* - \gamma \delta^* \alpha_j \beta_j^*) = \omega(x)ru - ru\omega(x) \). Thus, \( \omega(x)ru = ru\omega(x) \). Finally, \( \omega(x)ru \delta^* = \omega(x)ru = \omega(x)ru \delta^* = ru\omega(x)ru \delta^* = ru\omega(x)ru \delta^* = \omega(x)ru \delta^* \omega(x) \). This shows that \( \omega(x) \in Z(uL_R(E)u) \). \( \square \)

Lemma 16. Suppose that \( u, v \in E^0 \) and that \( \alpha \in E^* \) is such that \( s(\alpha) = u \) 
and \( r(\alpha) = v \). If \( x \in Z(uL_R(E)u) \), then \( \alpha^* x \alpha \in Z(vL_R(E)v) \).

Proof. Let \( x \in Z(uL_R(E)u) \). Take \( y \in vL_R(E)v \). Since \( \alpha \gamma \alpha^* \in uL_R(E)u \), it follows that \( vy^* x \alpha \gamma \alpha^* \in \alpha^* x \alpha \gamma \alpha^* \). Thus, \( \alpha^* x \alpha \in Z(vL_R(E)v) \). \( \square \)

Definition 17 (cf. [20]). Let \( E = (E^0, E^1, r, s) \) be a directed graph. A cycle 
in \( E \) is a path \( \mu \in E^* \setminus E^0 \) such that \( s(\mu) = r(\mu) \). An edge \( f \in E^1 \) is said to 
be an exit for the cycle \( \mu = \mu_1 \ldots \mu_n \) if, for some \( i \in \{1, 2, \ldots, n\} \), we have 
\( s(f) = s(\mu_i) \) but \( f \neq \mu_i \).

Remark 18. The definition of a cycle in a directed graph varies in the literature 
on Leavitt path algebras. In contrast to the most common definition of a cycle (cf. [2] p. 320), following [20] 
we allow a cycle to “intersect” itself. In Theorem 1 the condition that “every cycle in \( E \) has an exit” appears.
That condition is commonly known as Condition (L). It is easy to see that 
Condition (L) is satisfied with the first definition of a cycle [2], if and only 
if it is satisfied with the second definition of a cycle [20].

Lemma 19. Every element in \( E^0 \cup E^1 \cup (E^1)^* \) is nonzero in \( L_R(E) \), and 
the set of real (resp. ghost) paths is linearly independent in the left \( R \)-module 
\( L_R(E) \) and in the right \( R \)-module \( L_R(E) \).

Proof. The proof of [20] Prop. 4.9 immediately carries over to the case where \( R \) is a noncommutative unital ring. The same holds for the proof of [20] 
Prop. 3.4 in case \( E^0 \) and \( E^1 \) are countable sets. Otherwise, the proof may 
be adapted by taking \( k \) to be an infinite cardinal at least as large as card \( (E^0 \cup \ E^1) \) and defining \( Z := \oplus_R R \) (with the notation of [20] Prop. 3.4)). \( \square \)

Remark 20. Let \( x \) be a nonzero element of \( L_R(E) \). It is clear from the 
definition of \( L_R(E) \) that \( x \) can be represented as a finite sum \( x = \sum_{i=1}^{n} r_i \alpha_i \beta_i^* \).
where $r_i \in R \setminus \{0\}$ and $\alpha_i, \beta_i \in E^*$. Following [20] Def. 4.8, we define the real degree (resp. ghost degree) of this representation as $\max\{\deg(\alpha_i) \mid 1 \leq i \leq n\}$ (resp. $\max\{\deg(\beta_i) \mid 1 \leq i \leq n\}$). Note that, in general, the real degree and ghost degree of $x$ depend on the particular choice of representation. If, however, $x$ has a representation in only real (resp. ghost) edges, then, by Lemma [19], the real (resp. ghost) degree is independent of the choice of representation of $x$ in real (resp. ghost) edges.

Proposition 21. Every cycle in $E$ has an exit if and only if for every $u \in E^0$ the inclusion $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$ holds.

Proof. First we show the “if” statement by contrapositivity. Suppose that there is a cycle $p \in E^* \setminus E^0$ without any exit. Set $u := s(p)$ and write $p^0 := u$. Take $r \in R$ and $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta) = u$ and $r(\alpha) = r(\beta)$. Since $p$ has no exit, there are $m, n \in \mathbb{N} \cup \{0\}$ and $\gamma \in E^*$ such that $\alpha = p^m\gamma$ and $\beta = p^n\gamma$. Note that $\gamma^* = u = pr^*$. We get that $p\alpha\beta^* = prp^n\gamma^*(p^*)^n = rp^{m+1}(p^*)^n$ and $r\alpha\beta^*p = rp^m\gamma^*(p^*)^np = rp^m(p^*)^np$. If $n = 0$, then $p^{m+1}(p^*)^n = p^m(p^*)^np$ and if $n > 0$, then $p^{m+1}(p^*)^n = p^m(p^*)^{n-1}p^*p = p^m(p^*)^np$. In either case, we get that $p\alpha\beta^* = r\alpha\beta^*$. Hence, $p \in Z(uL_R(E)u) \setminus (uL_R(E)u)_0$.

Now we show the “only if” statement. Suppose that every cycle in $E$ has an exit. Take $u \in E^0$. We wish to show that $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$. By Lemma [3] a) and Lemma [15] it is enough to show that $(Z(uL_R(E)u))_N = \{0\}$ for every negative integer $N$.

We now adapt parts of the proof of [3] Thm. 3.1 to our situation. Take $N < 0$. Seeking a contradiction, suppose that the set

$$M := \{(u, x) \mid u \in E^0 \text{ and } x \in (Z(uL_R(E)u))_N \setminus \{0\}\}$$

is nonempty. If $(u, x), (v, y) \in M$, then we write $(u, x) \leq (v, y)$ if $x$ has a representation in $L_R(E)$ of real degree less than or equal to all real degrees of representations of $y$ in $L_R(E)$. We write $(u, x) = (v, y)$ whenever $(u, x) \leq (v, y)$ and $(v, y) \leq (u, x)$. Clearly, clearly, $\leq$ is a total order on $M$ which therefore has a minimal element $(u, x)$. Choose a minimizing representation $x = \sum_{i=1}^n e_i a_i + b$ where $e_1, \ldots, e_n \in E^1$ are all distinct, each $a_i \in L_R(E)$ is either zero, or nonzero and representable as an element of smaller real degree than that of $x$, and $b$ is a polynomial (possibly zero) in only ghost paths whose source and range equals $u$. Take $i \in \{1, \ldots, n\}$. Write $v_i := r(e_i)$. By Lemma [16] $e_i^* x e_i \in (Z(v_i L_R(E) v_i))_N$. Since $e_i^* x e_i$ is of smaller real degree than that of $x$, it follows that $e_i^* x e_i = 0$. Using that $x \in (Z(uL_R(E)u))_N$, it follows that $e_i^* x = e_i^* e_i e_i^* x = e_i^* x e_i e_i^* = 0$. Thus, $0 = e_i^* x = a_i + e_i^* b$ and hence $a_i = -e_i^* b$.

Now, $0 \neq x = (u - \sum_{i=1}^n e_i e_i^*) b$. Thus, $u \neq \sum_{i=1}^n e_i e_i^*$ and $b \neq 0$. This implies that there is some $f \in E^1 \setminus \{e_1, \ldots, e_n\}$ with $s(f) = u$. Furthermore, $f^* x = f^* b$, and, by Lemma [19] $f^* b \neq 0$ since it is a sum of distinct ghost paths. Write $v := r(f)$. By Lemma [16] it follows that
\[ f^*xf \in (Z(vL_R(E)v))_N. \] Using that \( 0 \neq f^*x = f^*ff^*x = f^*xf^* \), we get that \( f^*xf \neq 0 \). Note that the real degree of \( f^*xf \) is less or equal to the real degree of \( x \). Hence, by the assumption made on \((u, x)\), and possibly after replacing \((u, x)\) by \((v, f^*xf)\), we may assume that \( a_i = 0 \) for every \( i \in \{1, \ldots, n\} \). Therefore, suppose that \( x = \sum_{j=1}^m r_j \beta_j^* \) for some nonzero \( r_j \in R \) and some distinct paths \( \beta_j \in E^{-N} \) with \( s(\beta_j) = r(\beta_j) = u \). Take \( k \in \{1, \ldots, m\} \). By Lemma 16 it follows that \( r_k \beta_k^* = \beta_k^*x\beta_k \in Z(uL_R(E)u) \). By assumption, the cycle \( \beta_k \) has an exit at some \( w \in E^0 \). Thus, there are \( \gamma, \delta \in E^* \) and \( \epsilon \in E^1 \) such that \( \beta_k = \gamma \delta, r(\gamma) = s(\epsilon) \) and \( \epsilon^* \delta = 0 \). By Lemma 16 it follows that \( r_k(\delta \gamma)^* = r_k \gamma^* \delta^* \gamma^* \gamma = \gamma^* r_k \beta_k^* \gamma \in Z(vL_R(E)v) \). We now reach a contradiction, because \( 0 \neq \epsilon \epsilon^* r_k(\delta \gamma)^* = r_k(\delta \gamma)^* \epsilon \epsilon^* = 0 \). \( \square \)

Now, we prove our main result.

**Proof of Theorem 7** First we show the “only if” statement. Suppose that \( L_R(E) \) is simple. Then \( L_R(E) \) is graded simple and hence, by Proposition 12 it follows that \( R \) is simple and that \( E^0 \) has no nontrivial hereditary and saturated subset. Furthermore, Proposition 6 implies that \( uL_R(E)u \) is simple for every \( u \in E^0 \), and hence, by Proposition 4 \( Z(uL_R(E)u) \subset (uL_R(E)u)_0 \) for every \( u \in E^0 \). Thus, by Proposition 21 every cycle in \( E \) has an exit.

Now we show the “if” statement. Suppose that \( R \) is simple, \( E^0 \) has no nontrivial hereditary and saturated subset, and every cycle in \( E \) has an exit. By Proposition 12 \( L_R(E) \) is graded simple. Take \( u \in E^0 \). It follows from Proposition 21 that \( Z(uL_R(E)u) \subset (uL_R(E)u)_0 \). Furthermore, by Proposition 6 \( uL_R(E)u \) is graded simple. Thus, by Proposition 4 we get that \( uL_R(E)u \) is simple. Hence, by Proposition 7 \( L_R(E) \) is simple. \( \square \)

4. The center of a simple Leavitt path algebra

In this section we prove Theorem 2 using results from the previous sections together with some auxiliary observations.

**Remark 22.** Let \( E = (E^0, E^1, r, s) \) be a directed graph.

(a) Take \( v \in E^0 \). We write \( w \leq v \), for \( w \in E^0 \), if there is \( \mu \in E^* \) with \( s(\mu) = v \) and \( r(\mu) = w \). The set \( T(v) := \{ w \in E^0 \mid w \leq v \} \) is the smallest hereditary subset of \( E^0 \) containing \( v \).

(b) Suppose that \( X \subseteq E^0 \). Put \( T(X) := \cup_{x \in X} T(x) \). The hereditary saturated closure \( \overline{X} \) of \( X \) is defined as the smallest hereditary and saturated subset of \( E^0 \) containing \( X \). One can show (see [6], p. 626) and the references therein) that \( \overline{X} = \cup_{n=0}^\infty X_n \) where \( X_0 := T(X) \) and, for \( n \geq 1, X_n := \{ y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \) and \( r(s^{-1}(y)) \subseteq X_{n-1} \} \cup X_{n-1} \).

The following result can be proved by induction (see [14], Prop. 14.11) and [20], Lem. 5.2).

**Proposition 23.** Suppose that \( R \) is an associative unital ring and that \( E = (E^0, E^1, r, s) \) is a directed graph. If \( a \in (L_R(E))_0 \) is nonzero, then there exist \( \alpha, \beta \in E^*, v \in E^0 \) and a nonzero \( k \in R \) such that \( \alpha^* a \beta = kv \).
Now, we prove our second main result.

**Proof of Theorem 2.** Write $S := L_{R}(E)$. If $S$ is not unital, then it follows immediately from [21, 3.3] that $Z(S) = \{0\}$. This proves (a). Now, we show (b). Suppose that $S$ is unital, i.e. $E^{0}$ is finite. Take a nonzero $x \in Z(S)$. By Proposition 4, it follows that $\alpha, \beta$ are nonempty hereditary and saturated subset of $E$ for a moment, that this claim holds. Then $x = 1_{S} \cdot x = \sum_{w \in E^{0}} wx = \sum_{w \in E^{0}} kw = k \cdot \sum_{w \in E^{0}} w = k \cdot 1_{S} \in Z(R) \cdot 1_{S}$. Thus, $Z(S) \subseteq Z(R) \cdot 1_{S}$. Clearly, $Z(R) \cdot 1_{S} \subseteq Z(S)$ holds.

Now we show the claim. We will use induction to prove that for every $n \geq 0$ the implication $w \in X_{n} \Rightarrow wx = kw$ holds. From this the claim follows. Base case: $n = 0$. Suppose that $w \in X_{0}$, i.e. $w \leq v$. Then there is a path $\delta$ from $v$ to $w$. We get that $wx = \delta^{*} \delta x = \delta^{*} v \delta x = \delta^{*} v x \delta = \delta^{*} k v \delta = k^{*} v \delta = k \delta^{*} \delta = kw$. Induction step: Suppose that $wx = kw$ for every $w \in X_{n-1}$. Take $y \in X_{n} \setminus X_{n-1}$ and note that $0 < |s^{-1}(y)| < \infty$ and $r(s^{-1}(y)) \subseteq X_{n-1}$. We get that $yx = \sum_{e \in s^{-1}(y)} ee^{*} x = \sum_{e \in s^{-1}(y)} e r(e) x e^{*} = \sum_{e \in s^{-1}(y)} e k r(e) e^{*} = k \sum_{e \in s^{-1}(y)} ee^{*} = ky$. □

**References**

[1] G. Abrams, Leavitt path algebras: the first decade, *Bull. Math. Sci.* 5 (2015), no. 1, 59–120.
[2] G. Abrams, A. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* 293 (2005), 319–334.
[3] G. Abrams, A. Aranda Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* 34 (2008), no. 2, 423–442.
[4] P. N. Anh and L. Marki, Morita equivalence for rings without identity, *Tsukuba J. Math.* 11(2) (1987), 1–16.
[5] P. Ara, M. A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, *Algebras Represent. Theory* 10 (2007), no. 2, 157–178.
[6] G. Aranda Pino, K. Crow, The center of a Leavitt path algebra, *Rev. Mat. Iberoam.* 27 (2011), no. 2, 621–644.
[7] J. H. Brown, A. an Huef, Centers of algebras associated to higher-rank graphs, *Rev. Mat. Iberoam.* 30 (2014), no. 4, 1387–1396.
[8] L. O. Clark, C. Edie-Michell, Uniqueness theorems for Steinberg algebras, *Algebr. Represent. Theory* 18 (2015), no. 4, 907–916.
[9] L. O. Clark, C. Farthing, A. Sims, M. Tomforde, A groupoid generalisation of Leavitt path algebras, *Semigroup Forum* 89 (2014), no. 3, 501–517.
[10] L. O. Clark, D. Martín Barquero, C. Martín González, M. Siles Molina, Using the Steinberg algebra model to determine the center of any Leavitt path algebra, *Israel J. Math.* 230 (2019), no. 1, 23–44.
[11] D. Gonçalves, J. Öinert, D. Royer, Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics, *J. Algebra* **420** (2014), 201–216.

[12] R. Hazrat, The graded structure of Leavitt path algebras, *Israel J. Math.* **195** (2013), no. 2, 833–895.

[13] E. Jespers, Simple abelian-group graded rings, *Boll. Un. Mat. Ital. A* (7) **3** (1989), no. 1, 103–106.

[14] D. Lännström, P. Lundström, J. Öinert and S. Wagner, Prime group graded rings with applications to partial crossed products and Leavitt path algebras. arXiv:2105.09224 [math.RA]

[15] C. Năstăsescu and F. van Oystaeyen, *Methods of graded rings*, Springer Lecture Notes (2004).

[16] P. Nystedt, J. Öinert, Simple semigroup graded rings, *J. Algebra Appl.* **14** (2015), no. 7, 1550102, 10 pp.

[17] P. Nystedt, A survey of s-unital and locally unital rings. *Rev. Integr. Temas Mat.* **37** (2019), no. 2, 251–260.

[18] B. Steinberg, A groupoid approach to discrete inverse semigroup algebras, *Adv. Math.* **223** (2010), no. 2, 689–727.

[19] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318** (2007), no. 1, 270–299.

[20] M. Tomforde, Leavitt path algebras with coefficients in a commutative ring, *J. Pure Appl. Algebra* **215** (2011), no. 4, 471–484.

[21] R. Wisbauer, *Foundations of module and ring theory*, Algebra, Logic and Applications, 3. Gordon and Breach Science Publishers, Philadelphia, PA (1991).