Opportunities for use of exact statistical equations  
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Abstract. Exact structure function equations are an efficient means of obtaining asymptotic laws such as inertial range laws, as well as all measurable effects of inhomogeneity and anisotropy that cause deviations from such laws. “Exact” means that the equations are obtained from the Navier-Stokes equation or other hydrodynamic equations without any approximation. A pragmatic definition of local homogeneity lies within the exact equations because terms that explicitly depend on the rate of change of measurement location appear within the exact equations; an analogous statement is true for local stationarity. An exact definition of averaging operations is required for the exact equations. Careful derivations of several inertial range laws have appeared in the literature recently in the form of theorems. These theorems give the relationships of the energy dissipation rate to the structure function of acceleration increment multiplied by velocity increment and to both the trace of and the components of the third-order velocity structure functions. These laws are efficiently derived from the exact velocity structure function equations. In some respects, the results obtained herein differ from the previous theorems. The acceleration-velocity structure function is useful for obtaining the energy dissipation rate in particle tracking experiments provided that the effects of inhomogeneity are estimated by means of displacing the measurement location.

1 Introduction

Energy dissipation rate $\varepsilon$ is used for scaling the turbulence acceleration statistics that are measured by particle tracking at Cornell University. The inertial range of the second-order velocity structure function is measured in the Cornell apparatus to determine $\varepsilon$ using the empirical inertial-range relationship that requires local isotropy. In the Cornell experiments the flow between counter rotating blades produces high Reynolds numbers but inhomogeneous, anisotropic turbulence. A motivation for the present study was to use the exact velocity structure-function equation to relate $\varepsilon$ to measurable quantities without making any assumptions about homogeneity or isotropy of the flow and hence, to obtain a relationship that applies exactly to the need to determine $\varepsilon$ in the Cornell experiment. By “exact” we mean that the equations follow from the Navier-Stokes equation and the incompressibility condition with no additional approximations. Because particle tracking is used at Cornell University to measure acceleration and velocity, it is natural to relate the acceleration-velocity structure function (i.e., the structure function of the product of acceleration and velocity increments) to both $\varepsilon$ and the third-order velocity structure function.
On the basis of local homogeneity, but without use of local isotropy, Mann, Ott and Andersen[3, 4] were the first to obtain an inertial-range relationship of the acceleration-velocity structure function to $\varepsilon$. By specializing our exact equation for the acceleration-velocity structure function to the locally homogeneous case, we efficiently obtain their inertial-range result in Sec. 4, but an extraneous derivative moment in their relationship is removed here. The exact structure function equation method given here not only obtains that asymptotic law, but also shows all quantities that must be evaluated to account for the effects of inhomogeneity and anisotropy. Such evaluation requires the displacement of the measurement volume that is defined by the Cornell laser beam and imaging system.

The theorems of Nie and Tanveer[5] and of Duchon and Robert[6] establish the inertial-range 4/3 law that relates $\varepsilon$ to the trace of the third-order velocity structure function. Instead of invoking local homogeneity, Nie and Tanveer[5] perform a large volume average, and perform a long time average instead of invoking local stationarity, and they use an average over orientations of the two measurement points instead of invoking local isotropy. Results of Nie and Tanveer[5] are efficiently obtained in Sec. 5 by use of the exact structure function equation. For the case of spatially periodic direct numerical simulation, similar results were obtained from the exact structure function equations in [2].

The theorem of Duchon and Robert[6] differs from that of Nie and Tanveer[5] in an essential manner. The space-time averaging required by Duchon and Robert is of arbitrary extent and the viscosity is zero. In Sec. 6, using space-time averaging and the limit of very large Reynolds number, the 4/3 law is obtained from the exact statistical equations. The present derivation requires conditions on the space-time average as stated in Sec. 6. All terms that describe the inhomogeneity and anisotropy of flows are retained. That result shows all quantities that should be evaluated in direct numerical simulation (DNS) of flows to determine causes of deviation from the 4/3 law.

The inertial-range 4/3 relationship of the trace of the third-order velocity structure function is easily obtained relative to the inertial-range 4/5 and 4/15 laws for the longitudinal and transverse components of the third-order velocity structure function, respectively. It is remarkable that the 4/5 and 4/15 laws were obtained by Nie and Tanveer[5] on the same basis as described above, and remarkable that the 4/5 and 4/15 laws were also proven by Eyink[7] on a basis similar to that of Duchon and Robert[6], that is, for space-time averaging of arbitrary extent in the limit of vanishing viscosity. Those 4/5 and 4/15 laws are not derived here. Eyink[7] finds that tests of the inertial-range power laws for arbitrary extent of averaging would be difficult. On the other hand, evaluation of all terms in an exact statistical equation by means of DNS data would show causes of deviations from those laws as well as the approach toward those laws.

Of course, real experiments do not provide the opportunity to perform the above mentioned averaging used in the derivation of the theorems for the inertial-range laws. Deviations from 4/5 and 4/15 laws are observed because of
turbulence inhomogeneity, anisotropy, and finite Reynolds number. Recently, there has been much work \[8, 9, 10, 11, 12, 13\] on quantifying terms in the velocity structure function equations to learn how the various terms affect the balance of the equation as a function of scale and at what scales the effects of inhomogeneity and anisotropy become pronounced and how asymptotic laws are approached. The exact structure function equations are useful in this regard because they retain all effects of inhomogeneity and anisotropy in a clearly organized manner. The exact equations obviate the need for many derivations that obtain some limited aspect of inhomogeneity. Those developments are discussed in Sec. 7. The definition of local homogeneity that directly simplifies exact structure function equations is that the rate of change of statistics with respect to where they are measured is negligible. That pragmatic definition is explained and contrasted with previous definitions in Sec. 2.

2 Pragmatic definition of local homogeneity

Exact statistical hydrodynamics involves the derivation of equations relating statistics without the use of approximations. The Navier-Stokes equation has been used to derive exact equations relating velocity structure functions of velocity increments and other statistics[2]. Such exact equations have been obtained for all orders of velocity structure functions in [14]. Structure function equations of all orders have also been given for the asymptotic case of local isotropy in [14] and [15]. The exact equation for the scalar structure function has been obtained from the continuity equation as well[16]. All such exact equations have optimal organization when the independent variables are chosen to be the spatial separation \( r \equiv x - x' \) of the two points of measurement, i.e., \( x \) and \( x' \), and the midpoint \( X \equiv (x + x') / 2 \). The derivative with respect to \( X \), i.e., \( \partial_X \), acts on some statistics within the structure function equations. Local homogeneity is the approximation that the rate of change of statistics with respect to the location of measurement may be neglected. Since that location is \( X \), the result of \( \partial_X \) acting on a statistic is neglected relative to other terms in the structure function equations. This is a truly local definition; it makes no mention of a spatial domain. This definition of local homogeneity was exploited in the formulation of exact structure function equations in [2, 14], and has been used in studies [10, 11, 12, 13] of the effects of inhomogeneity on the balance of structure function equations; it was introduced in [17].

Local homogeneity has been given various definitions by different authors. Kolmogorov[18] introduced a space-time domain that is small compared to \( L \) and \( T = (L/U) \), where \( L \) and \( U \) are “typical length and velocity for the flow in the whole.” Kolmogorov considers the two-point differences of the velocities at spatial points in the domain; one point is common to all the differences. Kolmogorov defines local homogeneity as follows: the joint probability distribution of the velocity differences is independent of the one common spatial point, and of the velocity at the one common point, and of time. There are data [19, 20, 21] that contradict the statistical independence of velocity differ-
ence and the velocity at either end point, and also contradict the statistical independence of velocity difference and the velocity at the midpoint. The exception is isotropic turbulence [21], for which case local homogeneity is assured. An alternative possibility that is particularly relevant here is that the two-point velocity sum, \( u_n + u'_n \), might be statistically independent of velocity difference, but statements in [20, 21] contradict that statistical independence as well. Therefore, Kolmogorov’s [18] definition should not be used because experimental data contradict that statistical independence [19, 20, 22], as do theoretical considerations [22]. Recently, Frisch et al. [23] have considered the inconsistency of Kolmogorov’s definition of local homogeneity.

Monin and Yaglom [24] define local homogeneity to mean that the joint probability distribution of the two-spatial-point velocity differences is unaffected by any translation of the spatial points. They do not impose a restriction on the translations to a spatial domain. It follows that statistics composed entirely of the velocity differences obey the same relationships that they do for homogeneous turbulence (namely, they are independent of where they are measured), and that the mean velocity depends linearly on position [24]. In practice, statistics of velocity differences and of derivatives do depend on where they are measured except in the ideal case of homogeneous turbulence. Frisch [25] gives a definition that is equivalent to that of Monin and Yaglom [24], except that the translations are restricted to a domain the size of the spatial scale characteristic of the production of turbulent energy (which he calls the “integral scale”). Two-point structure function equations of all orders contain a statistic that is the product of not only factors of the velocity difference but also one factor of the sum of the two velocities, i.e., \( u_n + u'_n \) [2, 14]. Because the definitions of local homogeneity by Monin and Yaglom [24] and Frisch [25] involve only the joint probability distribution of two-point differences, but do not involve \( u_n + u'_n \), it follows that those definitions are not sufficient to simplify structure function equations to the same level of simplification as does homogeneity.

3 Definitions and notation

3.1 Definition of two-point, two-time quantities

Here, we extend the results in [2] to two times and to the structure function composed of the product of acceleration difference and velocity difference. Four independent variables are considered; \( \mathbf{x} \) and \( \mathbf{x}' \) are spatial points; \( t \) and \( t' \) are times. Denote velocities by \( u_i = u_i(\mathbf{x}, t) \), \( u'_i = u_i(\mathbf{x}', t') \), and accelerations by \( a_i = a_i(\mathbf{x}, t) \), \( a'_i = a_i(\mathbf{x}', t') \), etc. Quantities that will appear when we use the
Navier-Stokes equation are
\[
\begin{align*}
  d_{ij} & \equiv (u_i - u_i') (u_j - u_j') \\
  d_{ijn} & \equiv (u_i - u_i') (u_j - u_j') (u_n - u_n') \\
  F_{ijn} & \equiv (u_i - u_i') (u_j - u_j') \frac{u_n + u_n'}{2} \\
  \tau_{ij} & \equiv (\partial_x p - \partial_x' p') (u_j - u_j') + (\partial_x p - \partial_x' p') (u_i - u_i') \\
  e_{ij} & \equiv (\partial_x, u_i) (\partial_x, u_j) + (\partial_x', u_i') (\partial_x', u_j') \\
  A_{ij} & \equiv (a_i - a_i') (u_j - u_j') + (a_j - a_j') (u_i - u_i'),
\end{align*}
\]
where \( \partial_x p \) is pressure gradient. Change of variables:

\[
X \equiv (x + x')/2 \quad \text{and} \quad r \equiv x - x'; \quad T = (t + t')/2 \quad \text{and} \quad \tau \equiv t - t'. \tag{1}
\]

Just as \((x, x', t, t')\) are independent variables, so are \((X, r, T, \tau)\). The significance of variable \(X\) is that it is the location of measurement. Nonzero values of derivatives of statistics with respect to \(X\), evaluated at position \(X\), are the result of local inhomogeneity of the flow. Likewise, nonzero values of derivatives of statistics with respect to \(T\) are the result of nonstationarity of the flow. The relationships between the spatial derivatives are

\[
\partial_x = \partial_x + \frac{1}{2} \partial X, \quad \partial_x' = -\partial_x + \frac{1}{2} \partial X, \quad \partial X_i = \partial_x + \partial_x', \quad \partial_r = \frac{1}{2} (\partial_x - \partial_x'),
\]
which give the useful properties

\[
\partial_x, [f(x, t) \pm g(x', t')] = \partial X, [f(x, t) \mp g(x', t')] / 2. \tag{2}
\]

Similarly,

\[
\partial_T = \partial_t + \partial \tau, \quad \partial_t \equiv \frac{1}{2} (\partial_x - \partial_x').
\]

Use of the derivative formulas on \(\tau_{ij}\) gives

\[
\tau_{ij} = -2 (p - p') (s_{ij} - s_{ij}') + \partial X_i [(p - p') (u_j - u_j')] + \partial X_j [(p - p') (u_i - u_i')],
\]
where \( s_{ij} \equiv (\partial_x, u_j + \partial_x, u_i) / 2 \) is the rate of strain. The trace of \(\tau_{ij}\) is

\[
\tau_{ii} = 2 \partial X_i [(p - p') (u_i - u_i')]. \tag{3}
\]

Use of derivative formulas on \(e_{ij}\) and taking the trace and use of Poisson’s equation, i.e., \(\partial_x, \partial_{x_i} p = -\partial_x, u_j \partial_{x_j} u_i\), gives

\[
e_{ii} = \nu^{-1} (\varepsilon + \varepsilon') + \partial X_n \partial X_n (p + p'),
\]
where \(\varepsilon \equiv 2 \nu s_{ij} s_{ij}\) is the energy dissipation rate.
3.2 Use of the Navier-Stokes equation

The Navier-Stokes equation at \((x, t)\) is

\[
a_i = \partial_t u_i + u_n \partial_{x_n} u_i = -\partial_{x_i} p + \nu \partial_{x_n} \partial_{x_n} u_i , \quad \text{and} \quad \partial_{x_n} u_n = 0.
\]

Subtracting the Navier-Stokes equations at \((x', t')\), multiplying by \((u_j - u'_j)\), and adding the equation needed to produce symmetry under interchange of \(i\) and \(j\) gives

\[
A_{ij} = \partial_t d_{ij} + \partial_{X_i} F_{ijn} + \partial_{r_n} d_{ijn} = -\tau_{ij} + 2\nu \left( \partial_{r_n} \partial_{r_n} d_{ij} + \frac{1}{4} \partial_{X_n} \partial_{X_n} d_{ij} - e_{ij} \right).
\] (5)

The trace gives

\[
A_{ii} = \partial_t d_{ii} + \partial_{X_i} F_{iin} + \partial_{r_n} d_{iin} = 2\nu \partial_{r_n} \partial_{r_n} d_{ii} - 2(\varepsilon + \varepsilon') + W,
\] (6)

where

\[
W \equiv -2\partial_{X_i} [(p - p') (u_i - u'_i)] + \frac{\nu}{2} \partial_{X_n} \partial_{X_n} d_{ii} - 2\nu \partial_{X_n} \partial_{X_n} (p + p').
\] (7)

No average has been used yet.

3.3 The r-sphere and orientation averages

Local isotropy has been used in the past to remove the divergence from \(\partial_{r_n} d_{iin}\) and to proceed to Kolmogorov’s equation. Without the assumption of local isotropy, an average over an \(r\)-sphere removes the divergence from \(\partial_{r_n} d_{iin}\). The \(r\)-sphere average also mitigates anisotropy\([26, 2]\). Energy dissipation rate \(\varepsilon\) averaged over a sphere in \(r\)-space, \(\langle \varepsilon \rangle_{r\text{-sphere}}\), was introduced by Obukhov\([27]\) and Kolmogorov\([28]\) in 1962; it is a recurrent theme in small-scale similarity theories. We have produced exact dynamical equations containing the sphere-averaged energy dissipation rate\([2]\). The volume average over an \(r\)-space sphere of radius \(r_S\) of a quantity \(Q\) is defined by

\[
\langle Q \rangle_{r\text{-sphere}} (X, r_S, T, \tau) \equiv \frac{4\pi r_S^3}{3} \int \int \int_{|r| \leq r_S} Q (X, r, T, \tau) \, dr.
\] (8)

The orientation average over the surface of the \(r\)-space sphere of radius \(r_S\) of the outward normal component of any vector \(q_n\) is defined by

\[
\frac{1}{\Omega} \int_{\Omega} q_n \, d\Omega = \frac{4\pi r_S^2}{3} \int \int_{|r| = r_S} \frac{r_n}{r} q_n (X, r, t) \, ds
\] (9)

where \(ds\) is the differential of surface area and \(d\Omega\) is the differential of solid angle. In \(\langle Q \rangle\), the inner product \(\frac{r_n}{r} q_n\) produces the so-called longitudinal component.
of \( q_n \). The above averages can be performed on data. The divergence theorem is
\[
\langle \partial_r q_n \rangle_{\text{r-sphere}} = (3/rS) \oint_{r_n} q_n.
\] (10)

### 3.4 The X-space and X-surface averages

Let the spatial average be over a region \( \mathcal{R} \) in \( \mathbf{X} \)-space. The spatial average of any quantity \( Q \) is denoted by \( \langle Q \rangle_{\mathcal{R}} \), which has argument list \((\mathcal{R}, r, T, t)\); that is,
\[
\langle Q \rangle_{\mathcal{R}} (\mathcal{R}, r, T, t) = \frac{1}{V} \int \int \int_{\mathcal{R}} Q(X, r, T, t) \, dX,
\]
where \( V \) is the volume of the space region \( \mathcal{R} \). Given any vector \( q_n \), the divergence theorem relates the volume average of \( \partial_{X_n} q_n \) to the surface average; that is,
\[
\langle \partial_{X_n} q_n \rangle_{\mathcal{R}} = \frac{1}{V} \int \int \int_{\mathcal{R}} \partial_{X_n} q_n \, dX = \frac{S}{V} \left( \frac{1}{S} \int \int_{\mathcal{N}_n} \nabla_n q_n \, dS \right) = \frac{S}{V} \oint_{X_n} q_n,
\] (11)
where \( S \) is the surface area bounding \( \mathcal{R} \), \( dS \) is the differential of surface area, and \( \hat{N}_n \) is the unit vector oriented outward and normal to the surface. As seen on the right-hand side of (11), we adopt, for brevity, the integral-operator notation
\[
\oint_{X_n} \equiv \frac{1}{S} \int \int \hat{N}_n \, dS.
\]

### 3.5 Time average

Consider the term \( \partial_T d_{ij} \) in (5). The time average of a quantity \( Q \) from an initial time \( T_0 \) and over a duration \( T \) is defined by
\[
\langle Q \rangle_T (X, r, T_0, T, t) = \frac{1}{T} \int_{T_0}^{T_0+T} Q(X, r, T, t) \, dT.
\] (12)
Of course, \( \partial_T \) does not commute with the integral operator (12); it follows that
\[
\langle \partial_T d_{ij} \rangle_T = \frac{1}{T} \int_{T_0}^{T_0+T} (\partial_T d_{ij}) \, dT = \left[ d_{ij}(X, r, T_0 + T, t) - d_{ij}(X, r, T_0, t) \right] / T.
\] (13)
This shows that it is easy to evaluate \( \langle \partial_T d_{ij} \rangle_T \) using experimental data because only the data at times \( T_0 \) and \( T_0 + T \) are used. One can make \( \langle \partial_T d_{ij} \rangle_T \) as small as one desires by allowing \( T \) to be very large, provided that \( d_{ij}(X, r, T_0 + T, t) \) does not differ greatly from \( d_{ij}(X, r, T_0, t) \).
3.6 All averages commute

The above averages are integrations with respect to independent variables $X$, $r$, and $T$; also, an ensemble average is a sum over realizations. All those averages commute with one another. They commute with derivatives with the following exceptions: Volume and surface averages $\langle Q \rangle_R$ and $\oint_X q$ do not commute with the derivative $\partial_X$, nor do $\langle Q \rangle_r$ and $\oint_r q$ commute with $\partial_r$, nor does $\langle Q \rangle_T$ commute with $\partial_T$.

4 Theorem of Mann, Ott, and Andersen

Mann et al.\textsuperscript{[3]} and Ott and Mann\textsuperscript{[4]} studied turbulent dispersion of particles by means of particle tracking. They therefore used an ensemble average over particle trajectory events. The ensemble average is formed from observations of many particle-pair trajectories and $r$ values. It is also possible to use single trajectories; then two times, $t$ and $t'$, are needed. From the data, averages are stored in bins of the direction and length of $r$. Denote the ensemble average by $\langle \rangle_E$. The ensemble average is a sum and therefore commutes with the spatial and temporal derivatives. Mann et al.\textsuperscript{[3]} assumed local homogeneity. From the present definition of local homogeneity, we have

$$\langle \partial_X \delta \rangle_E = \partial_X \langle \delta \rangle_E = 0,$$

and similarly $\langle W \rangle_E = 0$. Consequently, the ensemble average of (6) can be written as

$$\langle A_{ii} \rangle_E = \partial_T \langle d_{ii} \rangle_E + \partial_r \langle d_{ii} \rangle_E = 2\nu \partial_r \partial_r \langle d_{ii} \rangle_E - 2\langle \varepsilon + \varepsilon' \rangle_E. \quad (14)$$

For $r$ much larger than dissipation-range scales, Mann et al.\textsuperscript{[3]} neglect the term $2\nu \partial_r \partial_r \langle d_{ii} \rangle_E$. Doing so, and using the definition of $A_{ij}$, we have

$$\langle (a_i - a'_i) (u_i - u'_i) \rangle_E = -\langle \varepsilon + \varepsilon' \rangle_E. \quad (15)$$

With two exceptions, this is the result of \textsuperscript{[8]} in their equation (90). First, their result contains an extraneous derivative moment that is absent above. Second, $\langle \varepsilon + \varepsilon' \rangle_E$ appears above, whereas $2\langle \varepsilon \rangle_E$ is in their result. In an experimental apparatus, there can be a systematic difference between the energy dissipation rates at $(x, t)$ versus $(x', t')$, even if $t = t'$, such that $\langle \varepsilon \rangle_E \neq \langle \varepsilon' \rangle_E$.

The above ensemble averages depend on $r$, which is the vector separation of points on trajectories from which data are obtained. That dependence on $r$ allows a further average over the $r$-sphere. Anisotropy of the flow might favor the occurrence of some orientations of $r$; by weighting of the occurrences, the $r$-sphere average is constructed for uniform orientation of $r$.

4.1 Applying the $r$-sphere average

The $r$-sphere average causes each point $x$ to coincide with point $x'$ when the orientation of $r$ is reversed. Consequently,

$$\langle (\varepsilon + \varepsilon') \rangle_{r\text{-sphere}} = 2\langle \varepsilon \rangle_{x\text{-sphere}} = 2\langle \varepsilon' \rangle_{x'\text{-sphere}}$$
where averages over spheres in $x$-space and $x'$-space are introduced. The particle tracking experiment is temporally stationary such that $\partial T \langle d_{ii} \rangle_E$ vanishes. Then, the $r$-sphere average of (14) gives
\[ \oint r n \langle d_{ii} \rangle_E = -\frac{4}{3} r S \langle \varepsilon \rangle_E \text{-sphere}, \]
which is an extension of the 4/3 law without use of local isotropy, and
\[ \langle(a_i - a'_i) (u_i - u'_i) \rangle_{r\text{-sphere}} = -2 \langle \varepsilon \rangle_E \text{-sphere}. \]
Mann et al.\[3\] give (16), except that their result contains an extraneous derivative moment. See equation (91) in \[3\].

4.2 Quantifying effects of inhomogeneity

The terms that are neglected on the basis of local homogeneity, such as $\partial X_n \langle f_{ii} \rangle_E$, can be measured by particle tracking. To evaluate the effects of inhomogeneity, the rate of change with respect to displacement of the averaging volume is needed, i.e., $X$ must be changed. The most troublesome of the terms describing inhomogeneity is the term $-2 \partial X_n \langle (p - p') (u_i - u'_i) \rangle$ in (7) because it requires a measurement of pressure difference. By applying the derivative relationship (2), the $r$-sphere average (8), and the divergence theorem (10), the most troublesome term can be expressed as
\[ \partial X_n \langle (p - p') (u_i - u'_i) \rangle_{r\text{-sphere}} = 6 \frac{r S}{\nu} \oint r n \langle d_{ii} \rangle_E - 2 \langle \varepsilon \rangle_R + \langle W \rangle_R, \]
where
\[ \langle W \rangle_R = \frac{S}{V} \oint X_n \left[ -2 (p - p') (u_n - u'_n) + \nu \partial X_n d_{ij} - 2 \nu \partial X_n (p + p') \right]. \]
These averages are for any arbitrary region $R$.

5 Theorem of Nie and Tanveer

The spatial average of (9) and use of the divergence theorem (11) gives
\[ \langle A_{ii} \rangle_R = \frac{S}{V} \oint X_n \langle f_{ii} \rangle_R + \partial X_n \langle d_{ii} \rangle_R = 2 \nu \partial X_n \langle d_{ii} \rangle_R - 2 \langle \varepsilon + \varepsilon' \rangle_R + \langle W \rangle_R, \]
where
\[ \langle W \rangle_R = \frac{S}{V} \oint X_n \left[ -2 (p - p') (u_n - u'_n) + \nu \partial X_n d_{ij} - 2 \nu \partial X_n (p + p') \right]. \]
Reynolds number and small $|r|$. Second, as in [2], those terms vanish for the case of periodic boundary conditions, such as are often used in DNS, provided that the spatial average is over the entire spatial period. Third, those terms vanish if the turbulence is of limited spatial extent and the volume average is over a much larger region such that the velocity and pressure fields are negligible on the surface that bounds $R$; this possibility causes the statistics to decrease as $V$ increases. Fourth, if the region $R$ is enclosed by rigid boundaries with the no-slip boundary condition on velocities, then only the pressure gradient term at far right in (18) is nonzero at points on the boundaries; a separate hypothesis that $\oint_{X_n} \partial X_n \left( p + p' \right)$ vanishes is required. Nie and Tanveer[5] eliminate the subject terms by integrating over the “entire volume”; the meaning must be of an infinite volume unless boundary conditions are specified. Neglecting the subject terms, we have

$$\langle A_{ii} \rangle_R = \partial_T \langle d_{ii} \rangle_R + \partial_{r_n} \langle d_{iin} \rangle_R = 2 \nu \partial_{r_n} \partial_r \langle d_{ii} \rangle_R - 2 \langle \varepsilon + \varepsilon' \rangle_R. \quad (19)$$

For $r$ much larger than dissipation-range scales, neglect the term $2 \nu \partial_{r_n} \partial_r \langle d_{ii} \rangle_R$. For an average over the entire flow and for $t = t'$,

$$\langle \varepsilon + \varepsilon' \rangle_R = 2 \langle \varepsilon \rangle_R = 2 \langle \varepsilon' \rangle_R.$$

If $t \neq t'$, then clearly, $\langle \varepsilon \rangle_R \neq \langle \varepsilon' \rangle_R$. The definition of $A_{ij}$ and (19) give

$$\langle (a_i - a'_i) (u_i - u'_i) \rangle_R = -2 \langle \varepsilon \rangle_R. \quad (20)$$

Apply the time average (12) to (19). Nie and Tanveer[5] take the averaging time, $T$, to be infinite such that (13) vanishes. Alternatively, one can assume sufficient stationarity that the trace of (13) is negligible. Also apply the $r$-sphere average (8). Then (19) gives, for $t = t'$,

$$\oint_{r_n} \langle \langle d_{iin} \rangle_R \rangle_T = -\frac{4}{3} r S \langle \langle \varepsilon \rangle_R \rangle_T / r\text{-sphere}. \quad (21)$$

Note that the orientation average of $d_{iin}$, as defined in (10), appears in (21) as a consequence of the divergence theorem (14). This is a theorem of Nie and Tanveer[5]. Their order of averaging is different, namely $R$ then $r$ then $T$, but the averages commute as discussed in Sec. 3.6.

6 Theorem of Duchon and Robert

The important distinction between the theorem of Nie and Tanveer (discussed above) and that of Duchon and Robert[6] is that the space-time averaging required by Duchon and Robert is of arbitrary extent and the viscosity is zero. In this case, the averaging region $R$ is a subdomain of the entire flow. Recall that $d_{ij} = (u_i - u'_i) (u_j - u'_j)$, and write

$$F_{ijn} = \langle \tilde{u}_n \rangle_R d_{ij} + \frac{\tilde{u}_n + \tilde{u}'_n}{2} d_{ij}, \quad (22)$$
where
\[ \hat{u}_n = u_n - \langle u_n \rangle_R, \quad \hat{u}'_n = u'_n - \langle u'_n \rangle_R, \quad \text{and} \quad \hat{u}_n = \frac{u_n + u'_n}{2} \]

Recall that (17)-(18) apply for any arbitrary space averaging region \( R \). Let \( \nu = 0 \) in (17)-(18), substitute (22), and multiply by \( r_S/3 \), then

\[
\frac{r_S}{3} \langle A_{ii} \rangle_R = \frac{r_S}{3} \partial_T \langle d_{ii} \rangle_R + \frac{r_S}{3} \langle \langle \hat{u}_n \rangle_R \partial_{X_n} d_{ii} \rangle_R + \frac{1}{3} \frac{S r_S}{V} \int_{X_n} \frac{\hat{u}_n + \hat{u}'_n}{2} d_{ij} + \frac{r_S}{3} \partial_{r_n} \langle d_{inin} \rangle_R
\]

\[
= -\frac{4}{3} r_S \left\langle \frac{\varepsilon + \varepsilon'}{2} \right\rangle_R - \frac{2}{3} S r_S \int_{X_n} (p - p') (u_n - u'_n) .
\]  

(23)

Take the averaging region \( R \) to have a simple topology such that the volume to surface ratio \( V/S \) is the size of \( R \). Because \( \nu = 0 \) is effectively the limit of infinite Reynolds number, \( r_S \) can be as small as desired. Therefore, the limit \( S r_S/V \to 0 \) can now be applied. This limit means that \( r_S \) is very small compared to the size of the averaging volume \( V/S \), but there is no requirement that \( V/S \) be a length scale at which energy is generated. Thus, the size of the averaging volume is arbitrary provided that \( V/S \gg r_S \). The limit \( S r_S/V \to 0 \) applied to (23) gives

\[
\frac{r_S}{3} \langle A_{ii} \rangle_R = \frac{r_S}{3} \partial_T \langle d_{ii} \rangle_R + \frac{r_S}{3} \langle \langle \hat{u}_n \rangle_R \partial_{X_n} d_{ii} \rangle_R + \frac{1}{3} \frac{S r_S}{V} \int_{X_n} \langle \hat{u}_n \rangle_R d_{ii}
\]

\[
= -\frac{4}{3} r_S \left\langle \frac{\varepsilon + \varepsilon'}{2} \right\rangle_R - \frac{2}{3} S r_S \int_{X_n} (p - p') (u_n - u'_n) .
\]  

(24)

The two terms that explicitly describe effects of inhomogeneity are eliminated from (24). Of course, (24) contains the following advective derivative

\[
\frac{r_S}{3} \langle \langle \hat{u}_n \rangle_R \partial_{X_n} d_{ii} \rangle_R = \frac{1}{3} S r_S \int_{X_n} \langle \hat{u}_n \rangle_R d_{ii}
\]

(25)

which is seen to vanish in the limit \( S r_S/V \to 0 \). The reason for not eliminating this advective term from (24) in the previous step is to make a point about random sweeping at the end of this section.

The theorem of Duchon and Robert[6] applies to the case \( t = t' \). In the previous section where the average was over the entire flow we had \( \langle \varepsilon \rangle_R = \langle \varepsilon' \rangle_R \) if \( t = t' \). That is not true here because \( R \) is now a subdomain of the entire flow; as \( X \) varies over \( R \) with \( r \) fixed, not every spatial point occupied by \( x \) coincides with a spatial point occupied by \( x' \) and vice versa. However, including the \( r \)-sphere average does cause \( x \) and \( x' \) to equally occupy every spatial point in the double integration. That is, for \( t = t' \), \( \langle \langle \varepsilon \rangle_R \rangle_{\text{r-sphere}} = \langle \langle \varepsilon' \rangle_R \rangle_{\text{r-sphere}} \). Henceforth, consider only the case \( t = t' \). Thus,

\[
\left\langle \frac{\varepsilon + \varepsilon'}{2} \right\rangle_R = \langle \langle \varepsilon \rangle_R \rangle_{\text{r-sphere}} = \langle \langle \varepsilon' \rangle_R \rangle_{\text{r-sphere}}.
\]
Apply the $r$-sphere average \( \mathbf{8} \) and the divergence theorem \( \mathbf{10} \), and neglect the advective term in \( \mathbf{25} \), and apply the time average, then \( \mathbf{24} \) gives

\[
\frac{r_S}{3} \left[ \frac{\partial_T \left( \langle \langle d_{ii} \rangle \rangle_{r\text{-sphere}} \right)}{T} + \left( \frac{1}{r_n} \langle d_{iin} \rangle_{\mathbb{R}} \right) \right] = -\frac{4}{3} r \left( \langle \langle \varepsilon \rangle \rangle_{r\text{-sphere}} \right)_{T}. \tag{26}
\]

From \( \mathbf{13} \), the time-derivative term in \( \mathbf{26} \) is

\[
\frac{r_S}{3} \left( \frac{\partial_T \langle \langle d_{ii} \rangle \rangle_{r\text{-sphere}}}{T} \right) = \frac{r_S}{3T} \left[ \langle \langle d_{ii} (X, r, T_0 + T, 0) \rangle \rangle_{r\text{-sphere}} - \langle \langle d_{ii} (X, r, T_0, 0) \rangle \rangle_{r\text{-sphere}} \right]. \tag{27}
\]

If \( \langle \langle d_{ii} \rangle \rangle_{r\text{-sphere}} \) decreases as \( r_S \) decreases, then there is an averaging duration \( T \) sufficiently large that the left-most term in \( \mathbf{26} \) may be neglected. The classic inertial-range case serves as an example; namely, \( \langle \langle d_{ii} \rangle \rangle_{r\text{-sphere}} \) scales with \( \epsilon^{2/3} r_S^{2/3} \), where, for brevity, \( \epsilon \equiv \langle \langle \varepsilon \rangle \rangle_{r\text{-sphere}} \). For that case, the ratio of the right-hand side of \( \mathbf{27} \) to the right-hand side of \( \mathbf{26} \) is not greater than order \( \epsilon^{-1/3} r_S^{2/3} / T \). That is, to neglect the time-derivative term the averaging duration \( T \) must be much greater than the inertial-range time scale \( \epsilon^{-1/3} r_S^{2/3} \).

Recall that \( \nu = 0 \) such that \( r_S \), and also \( \epsilon^{-1/3} r_S^{2/3} \), may be as small as desired. Hence \( T \) may be much smaller than integral time scales.

Thus, with \( \nu = 0 \) and a space average of arbitrary length scale, provided that length scale is much greater than \( r_S \), and a time average of arbitrary duration, provided that duration is much greater than the inertial-range time scale based on \( r_S \), we have

\[
\left( \frac{1}{r_n} \langle d_{iin} \rangle_{\mathbb{R}} \right)_{T} = -\frac{4}{3} r_S \left( \langle \langle \varepsilon \rangle \rangle_{r\text{-sphere}} \right)_{T}. \tag{28}
\]

This is closely related to the theorem of Duchon and Robert \( \mathbf{5} \). The order of the averaging operations in \( \mathbf{28} \) is the same as in \( \mathbf{9} \), although those averaging operations commute (see Sec. 3.6). Note that the orientation average, as defined in \( \mathbf{9} \), appears in \( \mathbf{28} \) as a consequence of the divergence theorem \( \mathbf{10} \).

### 6.1 Random Sweeping

Eyink \( \mathbf{7} \) inquires into the feasibility of numerical and experimental tests of his theorems and the theorem of Duchon and Robert. An aspect of such testing is the fact that the sum of the time derivative and advective terms in \( \mathbf{24} \) can be much smaller than either term taken separately. The time derivative and advective terms in \( \mathbf{24} \) constitute a comoving derivative. For simplicity, remove the factor \( r_S/3 \) and the \( r \)-sphere average from these two terms; then their sum can be written as

\[
\partial_T \langle d_{ii} \rangle_{\mathbb{R}} + \langle \langle \bar{u}_n \rangle \rangle_{\mathbb{R}} \partial_{X_n} d_{ii} = \partial_T \langle d_{ii} \rangle_{\mathbb{R}} + \langle \bar{u}_n \rangle_{\mathbb{R}} \langle \partial_{X_n} d_{ii} \rangle_{\mathbb{R}} = \partial_T \langle d_{ii} \rangle_{\mathbb{R}} + \frac{S}{\mathbb{V}} \langle \bar{u}_n \rangle_{\mathbb{R}} \left( \frac{1}{S} \int \int \tilde{N}_n d_{ii} dS \right),
\]
wherein two of many ways to express the sum are given. The last expression contains the product of the scalar \( d_{ii} \) and the outward normal unit vector \( \hat{N}_n \) averaged over the surface of the averaging volume followed by the inner product with \( \langle \hat{u}_n \rangle_R \). Recall that \( \hat{u}_n \equiv (u_n + u_n')/2 \); thus \( \langle \hat{u}_n \rangle_R \) constitutes the sweeping velocity of the scales larger than the averaging volume as well as any mean flow advection. The effect of changes of the advective term is a corresponding change of the time derivative \( \partial_T \langle d_{ii} \rangle_R \) such that the sum of the two terms can be much smaller than either taken separately.

### 6.2 Acceleration-velocity structure function

The relationship of the acceleration-velocity structure function to the energy dissipation rate is given in (15) and (16) on the basis of the ensemble average without and with the \( r \)-sphere average, respectively, and in (20) on the basis of an average over the entire flow. Now, (24) contains

\[
\langle (a_i - a_i') (u_i - u_i') \rangle_R = - \langle \varepsilon + \varepsilon' \rangle_R. \tag{29}
\]

This is obtained on the basis of \( \nu = 0 \) and the local space average without the time and \( r \)-sphere averages and without requiring \( t = t' \). Terms describing random and mean-flow sweeping are absent from the derivation of (29).

### 7 Effect of large-scale inhomogeneities at small scales

Recent research has quantified the influence of inhomogeneous terms on the balance of structure function equations. Those studies have applied to channel flow \cite{10, 11, 12}, wind tunnel grid-generated turbulence \cite{8, 9, 11, 12}, and turbulent jets \cite{13}. The Reynolds decomposition of all terms in the exact equation \( \ref{eq:5} \) is given in \( \ref{eq:29} \). That decomposition contains all possible terms contributing to the effects of inhomogeneous large-scale structures. For the ensemble average, the Reynolds decomposition of \( u_i(\mathbf{x}, t) \) is defined by

\[
u(\mathbf{x}, t) = U_i(\mathbf{x}, t) + \hat{u}_i(\mathbf{x}, t), \quad U_i(\mathbf{x}, t) = \langle u_i(\mathbf{x}, t) \rangle_E, \quad \langle \hat{u}_i(\mathbf{x}, t) \rangle_E = 0,
\]

and similarly at the point \( \mathbf{x}' \). Consider the Reynolds decomposition of the average of the term \( \partial_{X_n} F_{ijn} \) in \( \ref{eq:5} \)

\[
\partial_{X_n} \langle F_{ijn} \rangle_E = \frac{U_n + U_n'}{2} \partial_{X_n} D_{ij} + \partial_{X_n} \left( \Delta_i \hat{G}_{jn} + \Delta_j \hat{G}_{in} + \hat{G}_{ijn} \right), \tag{30}
\]

where, for brevity, we define

\[
D_{ij} \equiv \langle d_{ij} \rangle_E, \quad \Delta_i \equiv (U_i - U_i') ,
\]

\[
\hat{G}_{in} \equiv \left( (\hat{u}_i - \hat{u}_i') \frac{\hat{u}_n + \hat{u}_n'}{2} \right)_E,
\]

\[
\hat{G}_{ijn} \equiv \left( (\hat{u}_i - \hat{u}_i') \left( \hat{u}_j - \hat{u}_j' \right) \frac{\hat{u}_n + \hat{u}_n'}{2} \right)_E.
\]
Note that $D_{ij}$ is the structure function of velocity, not of velocity fluctuations. For grid-generated turbulence data, one can let subscript 1 denote the down-stream direction and approximate $U_1 = U_1'$, such that the first term on the right-hand side of (30) is $U_1 \partial_X D_{ij}$; this is the form of the inhomogeneous term derived in the above-mentioned studies.\cite{8, 9, 11, 12} One can easily show that the second term on the right-hand side of (30) is a generalization of the inhomogeneous term given in \cite{10, 11} for the case of uniform channel flow. Of course, (30) and the other Reynolds decompositions given in \cite{29} are exact for every flow, whether laminar or turbulent. Equations for structure functions of velocity fluctuations differ significantly from equations for the velocity; that topic has been treated in detail in \cite{29}.

The exact structure functions are useful in other respects. For example, from (3) and (4) it is clear that the pressure velocity term vanishes from the trace equations such as (14) and (17) on the basis of incompressibility and local homogeneity. It is not necessary to assume the more restrictive basis of local isotropy as in \cite{13}. The combination of (6) and (30) shows that both $\partial_T D_{ij}$ and $\frac{1}{2} (U_n + U_n') \partial_X D_{ij}$ must appear in the structure function equation as was correctly deduced by Lindborg \cite{8} on the basis of mean-flow Galilean invariance, but replacing $\partial_T D_{ij}$ with $\frac{1}{2} (U_n + U_n') \partial_X D_{ij}$, as was done in early derivations on the basis of Taylor's hypothesis, does not preserve that invariance.

8 Summary

The formulation of the exact statistical equations in the variables $X$ and $r$ of (1) produces the pragmatic definition of local homogeneity discussed in Sec. 2. It is the single definition that simplifies structure function equations for the case of local homogeneity. The analogous definition of local stationarity arises from use of the temporal variables $T$ and $\tau$ of (11). The resultant organization of the exact statistical equations provides immediate and improved derivation of the inertial-range relationships (14)–(16) between the acceleration-velocity structure function, the third-order velocity structure function, and the energy dissipation rate. Further, all terms that must be evaluated to include the effects of turbulence inhomogeneity and anisotropy are evident in the formulation. That makes the exact statistical equation method useful to particle tracking experiments such as the turbulence acceleration experiments at Cornell University. The recent theorems of Nie and Tanveer \cite{5} in (21) and of Duchon and Robert \cite{6} in (28) are likewise obtained easily, as are their generalizations, e.g., (17). The relationship between the acceleration-velocity structure function and energy dissipation rate is obtained in (29) on the basis of only a local spatial average for $\nu = 0$. With (30) as one example of the Reynolds decomposition of all terms in the exact statistical equations in (29), it is evident that the exact statistical equation formulation efficiently reveals all terms that must be evaluated to quantify the influence of inhomogeneity on the balance of structure function equations. It is not necessary to derive the individual terms that describe the effects of inhomogeneity that are missing from equations valid only
for homogeneous turbulence. All such terms are now known.
The above are examples of the use of exact statistical equations. Their usefulness arises from the organization of the equations.

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