The six-vertex model at roots of unity and some highest weight representations of the \( \mathfrak{sl}_2 \) loop algebra

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Abstract

We discuss irreducible highest weight representations of the \( \mathfrak{sl}_2 \) loop algebra and reducible indecomposable ones in association with the \( \mathfrak{sl}_2 \) loop algebra symmetry of the six-vertex model at roots of unity. We formulate an elementary proof that every highest weight representation with distinct evaluation parameters is irreducible. We present a general criteria for a highest weight representation to be irreducible. We also give an example of a reducible indecomposable highest weight representation and discuss its dimensionality.

1 Introduction

Roots of unity representations of the quantum groups have many subtle and interesting properties, as was first discussed by Roche and Arnaudon [19, 17]. They should have significant implications in integrable systems. In fact, the roots of unity representation theory has been utilized to generalize the R matrices of the chiral Potts model [6]. There have been several approaches to connect roots of unity representations to the six-vertex model and the XXZ spin chain [1, 18]. These studies could be fundamental and should contain useful suggestions for future researches, since the six-vertex model is one of the most important exactly solvable models in statistical mechanics. Recently, it was shown that the six-vertex model at roots of unity has many spectral degeneracies which are associated with the \( \mathfrak{sl}_2 \) loop algebra [10]. Furthermore, it has been found that the \( \mathfrak{sl}_2 \) loop algebra symmetry is closely related to roots of unity representations that are discussed by Lusztig [16, 4].

Let us introduce the \( \mathfrak{sl}_2 \) loop algebra symmetry of the six-vertex model at roots of unity. It was explicitly shown that when \( q \) is a root of unity the transfer matrix of the six-vertex model commutes with the generators of the \( \mathfrak{sl}_2 \) loop algebra [10]. Let \( q_0 \) be a

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primitive root of unity satisfying $q_0^N = 1$ for an integer $N$. We introduce operators $S^{\pm(N)}$ as follows
\[
S^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q_0^j \sigma^Z \circ \cdots \circ q_0^j \sigma^Z \circ \sigma^Z_{j_1} \circ q_0^{(N-2)} \sigma^Z \circ \cdots \circ q_0^{(N-2)} \sigma^Z.
\]  
(1)

The operators $S^{\pm(N)}$ are derived from the $N$th power of the generators $S^\pm$ of the quantum group $U_q(sl_2)$ or $U_q(sl_2)$. We also define $T^{\pm}$ by the complex conjugates of $S^{\pm(N)}$, i.e. $T^{\pm(N)} = (S^{\pm(N)})^*$. The operators, $S^{\pm(N)}$ and $T^{\pm(N)}$, generate the $sl_2$ loop algebra, $U(L(sl_2))$, in the sector $S^Z \equiv 0 \pmod{N}$.

(2)

Here the value of the total spin $S^Z$ is given by an integral multiple of $N$. In the sector $\{2\}$, the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{6V}(v)$.

The loop algebra symmetry is also in common with the XXZ spin chain, which is one of the most significant quantum integrable systems. We note that the logarithmic derivative of the transfer matrix of the six-vertex model gives the Hamiltonian of the XXZ spin chain under the periodic boundary conditions:

\[
H_{XXZ} = \frac{1}{2} \sum_{j=1}^L \left( \sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right).
\]  
(3)

Here the XXZ anisotropic coupling $\Delta$ is related to the $q$ parameter by $\Delta = (q + q^{-1})/2$. In the sector $\{2\}$, the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the Hamiltonian of the XXZ spin chain: $[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0$.

(4)

Let us now discuss the spectral degeneracy of the $sl_2$ loop algebra. In the sector $\{2\}$ every Bethe ansatz eigenvector $|B\rangle$ may have the following degenerate eigenvectors

\[
S^{-(N)}|B\rangle, \quad T^{-(N)}|B\rangle, \quad (S^{-(N)})^2|B\rangle, \quad T^{-(N)}S^{+(N)}T^{-(N)}|B\rangle, \ldots
\]

However, it is nontrivial how many of them are linearly independent. The number should explain the degree of the spectral degeneracy. Thus, we want to know the dimensions of representations generated by Bethe ansatz eigenvectors.

Fabricius and McCoy has conjectured that Bethe ansatz eigenvectors should be highest weight vectors with respect to the $sl_2$ loop algebra symmetry, and also that they have Drinfeld polynomials. In fact, it has been shown in the sector $\{2\}$ that a regular Bethe ansatz eigenvector is highest weight $[8]$. Furthermore, it has been shown that if the evaluation parameters of highest weight vector $\Omega$ are distinct, the highest weight representation generated by $\Omega$ is irreducible $[5]$. Here we note that highest weight vectors and evaluation parameters are defined in §3.1. Thus, in the sector $\{2\}$, if the evaluation parameters of
a regular Bethe state are distinct, the highest weight representation is irreducible and it has the Drinfeld polynomial by which we can determine the dimensions. However, if the evaluation parameters are not distinct, it is not trivial whether the highest weight representation is irreducible or not. We emphasize that a finite-dimensional highest weight representation is not necessarily irreducible.

In the paper, we discuss a different proof of the theorem that every finite-dimensional highest weight representation with distinct evaluation parameters is irreducible. We introduce an elementary computational scheme, and construct explicitly the basis of such a representation with distinct evaluation parameters. Furthermore, we show a general criteria for a finite-dimensional highest weight representation to be irreducible. We then discuss briefly how to apply them to the spectral degeneracy of the six-vertex model at roots of unity. Finally, we give an example of a reducible highest weight representation.

2 Loop algebra generators with parameters

We consider the classical analogue of the Drinfeld realization of the quantum $sl_2$ loop algebra, $U_q(L(sl_2))$. The classical analogues of the Drinfeld generators, $x^\pm_k$ and $h_k$ ($k \in \mathbb{Z}$), satisfy the defining relations:

$$[h_j, x^\pm_k] = \pm2x^\pm_{j+k}, \quad [x^+_j, x^-_k] = h_{j+k}, \quad \text{for } j, k \in \mathbb{Z}. \quad (5)$$

Here $[h_j, h_k] = 0$ and $[x^\pm_j, x^\pm_k] = 0$ for $j, k \in \mathbb{Z}$.

Let $A$ be a set of parameters such as $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. We define generators with $m$ parameters $x^\pm_m(A)$ and $h_m(A)$ as follows

$$x^\pm_m(A) = \sum_{k=0}^{m} (-1)^k x^\pm_{m-k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k},$$

$$h_m(A) = \sum_{k=0}^{m} (-1)^k h^\pm_{m-k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}. \quad (6)$$

In terms of generators with parameters we generalize the defining relations of the $sl_2$ loop algebra. Let $A$ and $B$ are arbitrary sets of $m$ and $n$ parameters, respectively. The operators with parameters satisfy the following:

$$[x^+_m(A), x^-_n(B)] = h_{m+n}(A \cup B), \quad [h_m(A), x^n(B)] = \pm2x^\pm_{m+n}(A \cup B). \quad (7)$$

They generalize the one-parameter operators introduced in Ref. [5]. By using the relations (7), it is straightforward to show the following useful relations:

$$[x^+_\ell(A), (x^-_m(B))^{(n)}] = (x^-_m(B))^{(n-1)}h_{\ell+m}(A \cup B) - x^+_{\ell+2m}(A \cup B \cup B)(x^-_m(B))^{(n-2)},$$

$$[h_\ell(A), (x^+_m(B))^{(n)}] = \pm2(x^+_m(B))^{(n-1)}x^\pm_{\ell+m}(A \cup B). \quad (8)$$

Here the symbol $(X)^{(n)}$ denotes the $n$th power of operator $X$ divided by the $n$ factorial, i.e. $(X)^{(n)} = X/n!$. 

3
Let the symbol \( \alpha \) denote a set of \( m \) parameters, \( \alpha_j \) for \( j = 1, 2, \ldots, m \). We denote by \( A_j \) the set of all the parameters except for \( \alpha_j \), i.e. \( A_j = \alpha \setminus \{ \alpha_j \} = \{ \alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m \} \).

We introduce the following symbol:

\[
\rho_j(\alpha) = x_{m-1}^{-1}(A_j) \quad \text{for} \quad j = 1, 2, \ldots, m.
\]  (9)

The generators \( x_j^- \) for \( j = 0, 1, \ldots, m-1 \), are expressed as linear combinations of \( \rho_j(\alpha) \).

Let us introduce \( \alpha_{kj} = \alpha_k - \alpha_j \). We show the following:

\[
\sum_{j=1}^{n} \frac{\rho_j(\alpha)}{\prod_{k=1; k \neq j}^{m} \alpha_{kj}} = x_{m-n}^{-1}(\{ \alpha_{n+1}, \ldots, \alpha_m \}) \quad (1 \leq n \leq m).
\]  (10)

It thus follows inductively on \( n \) that \( x_k^- \) \((0 \leq k \leq m-1)\) are expressed in terms of linear combinations of \( \rho_j(\alpha) \) with \( 1 \leq j \leq m \).

### 3 Highest weight representations

#### 3.1 Evaluation parameters of a highest weight vector

In a representation of \( U(L(sl_2)) \), we call a vector \( \Omega \) a highest weight vector if \( \Omega \) is annihilated by generators \( x_k^+ \) for all integers \( k \) and such that \( \Omega \) is a simultaneous eigenvector of every generator of the Cartan subalgebra, \( h_k \ (k \in \mathbb{Z}) \):

\[
x_k^+ \Omega = 0, \quad \text{for} \quad k \in \mathbb{Z},
\]

\[
h_k \Omega = d_k^+ \Omega, \quad h_{-k} \Omega = d_{-k}^- \Omega, \quad \text{for} \quad k \in \mathbb{Z}_{\geq 0}.
\]  (12)

The representation generated by a highest weight vector is called a highest weight representation.

Hereafter in the paper, we assume that \( \Omega \) is a highest weight vector with highest weight \( d_0^\pm \) and it generates a finite-dimensional representation. We denote by \( V_\Omega \) the finite-dimensional highest weight representation.

Let us consider the \( sl_2 \)-subalgebra generated by \( x_{-k}^+, x_k^- \) and \( h_0 \) for an integer \( k \). We denote the subalgebra by \( \mathcal{U}_k \). Let \( r \) be the eigenvalue of \( h_0 \) for \( \Omega \), i.e. \( h_0 \Omega = r \Omega \). It is easy to show the following:

**Lemma 1.** The \( \mathcal{U}_k \)-subrepresentation of \( V_\Omega \) is given by the \((r+1)\)-dimensional irreducible representation of \( sl_2 \), where \( r = d_0^\pm \).

For a given integer \( n \), we define the sector of \( h_0 \Omega = r \Omega \) by the vector subspace consisting of vectors \( v_n \in V \) such that \( h_0 v_n = (r - 2n) v_n \).

**Lemma 2.** The representation \( V_\Omega \) is given by the direct sum of sectors with respect to eigenvalues of \( h_0 \). Furthermore, every vector \( v_n \) in the sector of \( h_0 \Omega = r \Omega \) is expressed as a linear combination of monomial vectors such as \( x_{j_1}^- \cdots x_{j_n}^- \Omega \) with coefficients \( C_{j_1, \ldots, j_n} \):

\[
v_n = \sum_{j_1, \ldots, j_n} C_{j_1, \ldots, j_n} \prod_{t=1}^{n} x_{j_t}^- \Omega
\]  (13)
Proof. It is clear from the PBW theorem.

Let us now introduce evaluation parameters for $\Omega$. Using commutation relations of the loop algebra, we show that $\Omega$ is a simultaneous eigenvector of operators $(x_0^+)^n(x_1^-)^n$:

$$ (x_0^+)^j(x_1^-)^j\Omega = \lambda_j \Omega, \quad \text{for} \ j = 1, 2, \ldots, r. \quad (14) $$

In terms of the eigenvalues $\lambda_k$, we define a polynomial $P_\Omega(u)$ by the following relation:

$$ P_\Omega(u) = \sum_{k=0}^r \lambda_k (-u)^k. \quad (15) $$

Making use of proposition 1, we show that the roots of $P_\Omega(u)$ are nonzero and finite, and the degree of $P_\Omega(u)$ is given by $r$. We note that the author learned the expression of the Drinfeld polynomial (15) from Jimbo (See also [12, 14]).

Let us factorize $P_\Omega(u)$ as follows

$$ P_\Omega(u) = \prod_{k=1}^s (1 - a_k u)^{m_k}, \quad (16) $$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. Then, we call $a_j$ the evaluation parameters of highest weight vector $\Omega$. We denote by $\{ a \}$ the set of $s$ parameters, $a_1, a_2, \ldots, a_s$. Here we note that $r$ is given by the sum: $r = m_1 + \cdots + m_s$. We define parameters $\hat{a}_i$ for $i = 1, 2, \ldots, r$, as follows.

$$ \hat{a}_i = a_k \quad \text{if} \ m_1 + m_2 + \cdots + m_{k-1} < i \leq m_1 + \cdots + m_{k-1} + m_k. \quad (17) $$

Then, the set $\{ \hat{a}_j \ | \ j = 1, 2, \ldots, r \}$ corresponds to the set of evaluation parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$. We denote it by $\hat{A}_\phi$.

It is easy to show the reduction relations in the following [3, 8, 9]:

Lemma 3. In the representation $V_\Omega$ we have

$$ x^{\ell+1}_{\ell+r+1} \Omega = \sum_{j=1}^r (-1)^{r-j} \lambda_{r+1-j} x^{\ell+1}_{\ell+j} \Omega, \quad \text{for} \ \ell \in \mathbb{Z}. \quad (18) $$

Here we recall $r = \bar{d}^+_{\ell}$ and $\lambda_j$ denote the eigenvalues defined in (14).

Proof. The reduction relation (18) is derived from the following:

$$ (\bar{x}^+)(n)(\bar{x}^-_{\ell+1})(n+1) = \bar{x}^+_{\ell+1}(\bar{x}^-_{\ell})(n+1) + \frac{1}{2} [\bar{h}_1, (\bar{x}^+_{\ell})(n+1)(\bar{x}^-_{\ell})(n)] $$

$$ - (\bar{x}^-_{\ell})(n+1)(\bar{x}^+_{\ell+1})(n+1) \bar{x}^+_{\ell}, \quad \text{for} \ \ell \in \mathbb{Z}. \quad (19) $$

The relation (19) has been shown for the case of $U_q(L(sl(2)))$ [3].

The reduction relation (18) for $\ell = -1$ is expressed as follows

$$ x^-_{\ell} (\hat{A}_\phi) \Omega = 0. \quad (20) $$

Making use of the reduction relation (20), any monomial vector $x_{j_1}^- x_{j_2}^- \cdots x_{j_n}^- \Omega$ can be expressed as a linear combination of $\rho_{k_1}(a) \rho_{k_2}(a) \cdots \rho_{k_n}(a) \Omega$ over some sets of integers $k_1, \ldots, k_n$. Here we note the following:
Lemma 4. If \( x_n^-(A)\Omega = 0 \) for some set of parameters, \( A \), then we have \( x_n^-(A \cup B)\Omega = 0 \) for any set of parameters \( B \).

### 3.2 The case of distinct evaluation parameters

Let us discuss the case where all the evaluation parameters \( a_j \) have multiplicity 1, i.e. \( m_j = 1 \) for \( j = 1, \ldots, s \). We call it the case of distinct evaluation parameters. Here we note that \( s = r \). We therefore have

\[
x_s^-(a)\Omega = 0. \tag{21}
\]

Hereafter, we denote by \( a_j^{\otimes m} \) the set of parameter \( a_j \) with multiplicity \( m \), i.e. \( a_j^{\otimes m} = \{a_j, a_j, \ldots, a_j\} \). For instance, we have

\[
x_s^\pm(a_j^{\otimes s}) = \sum_{k=0}^s \frac{s!}{(s-k)!k!} (-a_j)^k x_{s-k}^\pm. \tag{22}
\]

In the case of \( s = 1 \), we write \( x_1^\pm(a_j^{\otimes 1}) \) simply as \( x_1^\pm(a_j) \).

Lemma 5. If all evaluation parameters \( \hat{a}_j \) are distinct (\( m_j = 1 \) for all \( j \)), we have

\[
(\rho_j(a))^2 \Omega = 0 \tag{23}
\]

Proof. First, we show

\[
x_0^\pm(\rho_j(a))^2 \Omega = 0. \tag{24}
\]

From eq. \( \[8 \] \) we have

\[
x_0^\pm(\rho_j(a))^{(2)} \Omega = x_{s-1}^- (A_j) h_{s-1}(A_j)\Omega - x_{2s-2}^- (A_j \cup A_j)\Omega
\]

In terms of \( a_{kj} = a_k - a_j \), we have \( h_{s-1}(A_j)\Omega = \prod_{k \neq j} a_{kj} \Omega \), and using eq. \( \[21 \] \) and lemma 4 we have

\[
x_{2s-2}^- (A_j \cup A_j)\Omega = \prod_{k \neq j} a_{kj} x_{s-1}^- (A_j)\Omega.
\]

We thus obtain eq. \( \[24 \] \). Secondly, we apply \( (x_0^\pm)^{(r-1)}(x_1^-(a_j))^{(r-1)} \) to \( (\rho_j(a))^2 \Omega \). The product is given by zero since it is out of the sectors of \( V_\Omega \) due to the fact that \( (r-1)+2 > r \):

\[
(x_0^\pm)^{(r-1)}(x_1^-(a_j))^{(r-1)} (\rho_j(a))^2 \Omega = 0.
\]

Here the left-hand-side is given by

\[
\rho_j(a)^2 (x_0^\pm)^{(r-1)}(x_1^-(a_j))^{(r-1)} \Omega = \prod_{k=1; k \neq j}^r a_{kj} \times (\rho_j(a))^2 \Omega
\]

Since \( a_{kj} \neq 0 \) for \( k \neq j \), we obtain eq. \( \[23 \] \). \( \square \)
Lemma 6. In the sector $h_0 = r - 2n$ of $V_\Omega$, every vector $v_n$ is expressed as follows:

$$v_n = \sum_{1 \leq j_1 \leq \cdots \leq j_n \leq s} C_{j_1, \ldots, j_n} \prod_{t=1}^{n} \rho_{j_t}(a) \Omega$$

(25)

If $v_n$ is zero, all the coefficients $C_{j_1, \ldots, j_n}$ are given by zero.

Proof. By the definition of $\rho_{j_t}(a)$, any vector in the sector is expressed as a sum over $\rho_{j_1}(a) \cdots \rho_{j_n}(a) \Omega$. We multiply both sides of eq. (25) with $(x^{(\ell)}(a))^{(n)}$, and use the Vandermonde determinant whose $(i, \ell)$ entry is given by $(a_{j_1} \cdots a_{j_n})^\ell$ to show that if $v_n = 0$, all the coefficients $C_{j_1, \ldots, j_n}$ are zero. $\square$

We have from lemma 5 and lemma 6 the next proposition:

Proposition 1. If evaluation parameters $\hat{a}_j$ of $\Omega$ are distinct, the set of vectors $\prod_{t=1}^{n} \rho_{j_t}(a) \Omega$ for $1 \leq j_1 \leq \cdots \leq j_n \leq s$ gives a basis of the sector $h_0 = r - 2n$ of $V_\Omega$.

Theorem 1. Let $\Omega$ be a highest weight vector with distinct evaluation parameters $a_1, \ldots, a_r$. The representation generated by $\Omega$ is irreducible.

Proof. We show that every nonzero vector of $V_\Omega$ has such an element of the loop algebra that maps it to $\Omega$. Suppose that there is a nonzero vector $v_n$ in the sector $h_0 = r - 2n$ of $V_\Omega$ that has no such element. Then, we have

$$x_{k_1}^+ \cdots x_{k_n}^+ v_n = 0$$

(26)

for all monomial elements $x_{k_1}^+ \cdots x_{k_n}^+$. Let us express $v_n$ in terms of the basis vectors $\rho_{j_1}(a) \cdots \rho_{j_n}(a) \Omega$ with coefficients $C_{j_1, \ldots, j_n}$ such as shown in eq. 25. Then, by the same argument as in lemma 6 using the Vandermonde determinant, we show from eq. (26) that all the coefficients $C_{j_1, \ldots, j_n}$ vanish. However, this contradicts with the assumption that $v_n$ is nonzero. It therefore follows that $v_n$ has such an element that maps it to $\Omega$. We thus obtain the theorem. $\square$

3.3 The case of degenerate evaluation parameters

Let us discuss a general criteria for a finite-dimensional highest weight representation to be irreducible.

Theorem 2. Let $\Omega$ be a highest weight vector that has evaluation parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$. We denote by $a$ the set of evaluation parameters, i.e. $a = \{a_1, a_2, \ldots, a_s\}$, and by $V_\Omega$ the representation generated by $\Omega$. Then, $V_\Omega$ is irreducible if and only if $x_s^-(a) \Omega = 0$.

We prove it by generalizing the proof of theorem 1. However, we shall discuss the derivation in the next report. $\square$

Let us discuss an application of theorem 2 for the degenerate case. It plays an important role when we consider the spectral degeneracy of the XXZ spin chain at roots of
unity associated with the \( sl_2 \) loop algebra symmetry. It has been shown that a regular Bethe ansatz eigenvector \(| R \rangle\) in the sector (2) is a highest weight vector \([8]\). However, if the evaluation parameters of the regular Bethe state \(| R \rangle\) are not distinct, it is nontrivial whether the highest weight representation \( V_{| R \rangle} \) is irreducible or not. Suppose that we have the following relation for \(| R \rangle\):

\[
x^{-}_{a}(a) | R \rangle = 0, \tag{27}
\]

where \( a \) denotes the set of evaluation parameters \( a_1, a_2, \ldots, a_s \) of the highest weight vector \(| R \rangle\). Then, it follows from theorem 2 that the highest weight representation \( V_{| R \rangle} \) is irreducible.

In the most general cases, however, it is not clear whether the condition (27) holds for all the regular Bethe eigenstates in the sector (2). In order to discuss the degenerate multiplicity of the \( sl_2 \) loop algebra symmetry of the XXZ spin chain, it is thus important to know the dimensions of reducible finite-dimensional highest weight representations. We propose the following conjecture:

**Conjecture 1.** Let \( \Omega \) be a highest weight vector with evaluation parameters \( \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r \).

If \( x^{-}_{a_j}(\hat{A}_{\phi})\Omega \neq 0 \) for all \( j \), the representation \( V_{\Omega} \) generated by \( \Omega \) is reducible and indecomposable, and it has the dimensions \( 2^r \). Here \( \hat{A}_{\phi} \) denote the set \( \hat{A}_{\phi} \setminus \{ \hat{a}_j \} \) with \( \hat{A}_{\phi} = \{ \hat{a}_j | j = 1, 2, \ldots, r \} \).

For an illustration of reducible indecomposable highest weight representations, let us consider the case of \( r = 3 \) with \( m_1 = 2 \) and \( m_2 = 1 \). In this case, the highest weight vector \( \Omega \) has evaluation parameters, \( \hat{a}_1 = a_1, \hat{a}_2 = a_1 \) and \( \hat{a}_3 = a_2 \), i.e. \( \hat{A}_{\phi} = \{ a_1, a_1, a_2 \} \). Here we recall lemma 2. The highest weight representation \( V_{\Omega} \) has four sectors of \( h_0 = 3, 1, -1, 3 \), respectively. Let us introduce some symbols.

\[
\rho_1 = x^{-}_1(a_2), \quad \rho_2 = x^{-}_1(a_1), \quad w_1 = x^{-}_2(a_1, a_2). \tag{28}
\]

Here \( x^{-}_2(a_1, a_2) \) denotes \( x^{-}(B) \) with \( B = \{ a_1, a_2 \} \). It is easy to show from the reduction relation, \( x^{-}_{a_j}(\hat{A}_{\phi})\Omega = 0 \), that \( x^{-}_n\Omega \) is expressed in terms of \( \rho_1\Omega, \rho_2\Omega, \) and \( w_1\Omega \) as follows.

\[
x^{-}_{a_j}(\hat{A}_{\phi})\Omega = \frac{a_1^n}{a_{12}} \rho_1\Omega + \frac{a_2^n}{a_{21}} \rho_2\Omega + \left( \frac{na_1^{n-1}}{a_{12}} - \frac{a_1^n - a_2^n}{a_{12}} \right) w_1\Omega \tag{29}
\]

We now have

\[
x^{-}_n w_1\Omega = 0, \quad \text{for} \quad n \in \mathbb{Z}. \tag{30}
\]

Therefore, it follows that the representation \( V_{\Omega} \) is reducible and indecomposable if \( w_1\Omega \neq 0 \). By a similar argument with lemma 5 we show

\[
\rho_1^2\Omega \neq 0, \quad \rho_1^3\Omega = 0; \quad \rho_2^2\Omega \neq 0, \quad \rho_2^3\Omega = 0; \quad w_1^2\Omega = 0. \tag{31}
\]

We also show

\[
\rho_1 w_1\Omega = 0, \tag{32}
\]
and $\rho_2 w_1 \Omega \neq 0$ if $w_1 \Omega \neq 0$. It thus follows that the four sectors of $V_\Omega$ have the following sets of basis vectors:

$$
\Omega, \quad \rho_1 \Omega, \quad \rho_2 \Omega, \quad w_1 \Omega, \quad \text{for } h_0 = 3;
$$

$$
\rho_1 \rho_2 \Omega, \quad \rho_2 \rho_1 \Omega, \quad \rho_2 w_1 \Omega, \quad \text{for } h_0 = 1;
$$

$$
\rho_1^2 \Omega, \quad \rho_2^2 \Omega, \quad \rho_2 \rho_2 \Omega, \quad \rho_2^2 w_1 \Omega, \quad \text{for } h_0 = 1.
$$

Consequently, we have the following result:

**Proposition 2.** The highest weight representation with three evaluation parameters $a_1, a_1, a_2$ is reducible, indecomposable and of $2^3$ dimensions, if and only if $w_1 \Omega \neq 0$. It is irreducible and of $6$ dimensions, if and only if $w_1 \Omega = 0$.

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