POSITIVE AND NEGATIVE DEFINITE SUBMATRICES IN AN
HERMITIAN LEAST RANK SOLUTION OF THE MATRIX
EQUATION $AXA^* = B$

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Abstract. This work is devoted to establish the extremal inertias of the two
submatrices $X_1$ and $X_4$ in a Hermitian least rank solution $X$ of the matrix
equation $AXA^* = B$. From these formulas, necessary and sufficient condi-
tions for these submatrices to be positive (nonpositive, negative, nonnegative)
definite are achieved.

1. Introduction. Throughout this note, $\mathbb{C}^{m \times n}$ and $\mathbb{C}_H^m$ stand for the sets of all
$m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices respectively,
the symbols, $A^*$, $r(A)$, $R(A)$, stand for the conjugate transpose, the rank, and the
range of $A$ respectively. $I_m$ denotes the identity matrix of order $m$. The Moore-
Penrose generalized inverse of a matrix $A \in \mathbb{C}^{m \times n}$, denoted by $A^+$, is defined to be
the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the following four matrix equations:

1. $AXA = A$, 2. $XAX = X$, 3. $(AX)^* = AX$, 4. $(XA)^* = XA$.

Many results and studies on the generalized inverse and the Moore-Penrose gen-
eralized inverse in particular, can be found in [1], [2].

We denote $E_A = I - AA^+$, $F_A = I - A^+A$ the two orthogonal projectors induced
by $A \in \mathbb{C}^{m \times n}$ such that $r(E_A) = m - r(A)$, $r(F_A) = n - r(A)$. The inertia
of $A \in \mathbb{C}_H^m$ is the set $In(A) = \{i_+(A), i_-(A), i_0(A)\}$, where $i_+(A)$, $i_-(A)$
and $i_0(A)$ are the number of positive, negative and zero eigenvalues of $A$ counted
with multiplicities respectively. For a matrix $A \in \mathbb{C}_H^m$, we know that $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = n - r(A)$.

We consider the linear matrix equation

$$AXA^* = B$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}_H^m$, are given and $X \in \mathbb{C}_H^n$ is unknown matrix.

Equation (1) is one of the best known matrix equations in matrix theory and
applications, which is a special case of the equation $AXB = C$ in which $A$, $B$
and $C$ are given and $X$ is an unknown matrix. A large amount of the equation
(1) has been derived throughout many authors, in which, they were interested in

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solving ranks and inertias minimization problems associated with matrix equations and their solutions. For example see \cite{4}, \cite{5}, such that in the first one, the authors studied the extremal ranks of submatrices in an Hermitian solution to the matrix equation $AXA^* = B$, in the second work, the authors studied the ranks of Hermitian and skew Hermitian solutions to (1). Also in \cite{10} Y. Tian gave the necessary and sufficient conditions for the least squares solutions and least rank solutions of (1) to coincide, in \cite{3} the present author and S. Guedjiba gave the necessary and sufficient conditions for the pair of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ to have a common Hermitian positive definite, negative definite, nonpositive definite, nonnegative definite least rank solution.

The concept of least-rank solutions of matrix equations was proposed in \cite{7} and in \cite{11} by Y. Tian in studying the minimal rank of the linear matrix function $A - BXC$.

In \cite{10} the Hermitian least rank solution of (1) is the matrix $X$ which minimizes the rank of the difference $B - AXA^*$ or equivalently

$$r (B - AXA^*) = \min$$ \hspace{1cm} (2)

The Hermitian least-rank solution of (1) is the solution of the consistent equation

$$E_{T_1} \left( X + TM^+T^* \right) E_{T_1} = 0$$ \hspace{1cm} (3)

Equation (3) is called the normal equation associated with (2). Hence the general expression of the Hermitian least rank solution of (1) can be written by

$$X = -TM^+T^* + T_1U + U^*T_1^*.$$ \hspace{1cm} (4)

where $M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}$, $T = \begin{bmatrix} 0 & I_n \end{bmatrix}$, $T_1 = TF_M$, and $U \in \mathbb{C}^{(m+n) \times n}$ is arbitrary. We need the following Lemmas.

**Lemma 1.1.** \cite{6}, Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$, $D \in \mathbb{C}^{l \times k}$. Then,

$$r \left[ \begin{array}{c} A \\ B \end{array} \right] = r (A) + r (E_AB) = r (B) + r (E_BA),$$ \hspace{1cm} (5)

$$r \left[ \begin{array}{c} A \\ C \end{array} \right] = r (A) + r (CF_A) = r (C) + r (AF_C),$$ \hspace{1cm} (6)

$$r \left[ \begin{array}{c} A \\ B \\ C \end{array} \right] = r (B) + r (C) + r (E_BAF_C).$$ \hspace{1cm} (7)

**Lemma 1.2.** \cite{8} Let $A \in \mathbb{C}^{m \times n}_H$, $B \in \mathbb{C}^{m \times n}$. Then,

$$i_\pm \left[ \begin{array}{cc} A & BF_p \\ F_pB^* & 0 \end{array} \right] = i_\pm \left[ \begin{array}{ccc} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{array} \right] - r (P).$$ \hspace{1cm} (8)

**Lemma 1.3.** \cite{8} Let $A \in \mathbb{C}^{m \times n}_H$, $B \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}^{n \times m}_H$ be given, Then,

$$i_\pm \left[ \begin{array}{cc} A & B \\ B^* & D \end{array} \right] = i_\pm \left[ \begin{array}{cc} A & -B \\ -B^* & D \end{array} \right] = i_\mp \left[ \begin{array}{cc} -A & B \\ B^* & -D \end{array} \right].$$ \hspace{1cm} (9)

**Lemma 1.4.** \cite{9} Let $S$ be a set consisting of matrices over $\mathbb{C}^{m \times n}$, and let $h$ be a set consisting of Hermitian matrices over $\mathbb{C}^{m \times m}$. Then,

a) $h$ has a matrix $X > 0$, $(X < 0)$ if and only if $\max_{X \in h} i_+ (X) = m$, $(\max_{X \in h} i_- (X) = m)$,

b) All $X \in h$ satisfy $X > 0$, $(X < 0)$ if and only if $\min_{X \in h} i_+ (X) = m$, $(\min_{X \in h} i_- (X) = m)$,
c) ℏ has a matrix \( X \geq 0 \), \((X \leq 0)\) if and only if \( \min_{X \in ℏ} i_-(X) = 0 \), \((\min_{X \in ℏ} i_+(X) = 0)\),
d) All \( X \in ℏ \) satisfy \( X \geq 0 \), \((X \leq 0)\) if and only if \( \max_{X \in ℏ} i_-(X) = 0 \), \((\max_{X \in ℏ} i_+(X) = 0)\).

Following the work of Y. Tian in [9], in which the author derived necessary and sufficient conditions for the Hermitian least squares solution of the matrix equation (1) to have a submatrices positive or negative definites, in this work we derive these results on the other solution of this equation which is the Hermitian least rank solution, also to have a submatrices positive (negative, nonpositive, nonnegative) definites.

2. Positive and negative definite submatrices in an Hermitian solution of \( AXA^* = B \).

For convenience of representation, the following notation for the collection of Hermitian least rank solutions of equation (1) is adopted

\[
S = \{ X | \min r(B - AXA^*) = \min \}. 
\]

The general least rank solutions of (1) are given by

\[
X = -TM^*T + T_1U + U^*T_1^*. 
\]

where \( M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & I_n \end{bmatrix}, T_1 = TF_M, \) and \( U \in \mathbb{C}^{m+n} \times n \) is arbitrary.

We note that, if \( B^* = B \), then \( M^* = M \) is Hermitian also. One of the fundamental concepts in matrix theory is the partition of matrix, many properties of a matrix can be derived from the submatrices in its partition. In order to show more properties of Hermitian least rank solution of Eq (1), the Hermitian least rank solution \( X \in S \) in Eq (1) is partitioned into \( 2 \times 2 \) block form

\[
X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. 
\]

So, Eq (1) can be written as

\[
\begin{bmatrix} A_1 & A_2 \\ A_1^* & A_2^* \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \end{bmatrix} = B, 
\]

where \( A_1 \in \mathbb{C}^{m \times n_1}, A_2 \in \mathbb{C}^{m \times n_2}, X_1 = X_1^* \in \mathbb{C}^{n_1 \times n_1}, X_2 \in \mathbb{C}^{n_1 \times n_2}, X_3 = X_2^* \in \mathbb{C}^{n_2 \times n_1}, X_4 = X_4^* \in \mathbb{C}^{n_2 \times n_2}, \) with \( n_1 + n_2 = n \).

It is easy to see that \( X_1, X_2, X_3 \) and \( X_4 \) can be written as

\[
X_1 = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} X \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = P_1XH_1^*, 
\]

\[
X_2 = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} X \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} = P_1XH_2^*, 
\]

\[
X_3 = \begin{bmatrix} 0 & I_{n_2} \end{bmatrix} X \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = P_2XH_1^* = X_2^* 
\]

\[
X_4 = \begin{bmatrix} 0 & I_{n_2} \end{bmatrix} X \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} = P_2XH_2^*. 
\]

Substituting its Hermitian least rank solution into above formulas yields the general expressions of \( X_1, X_2, X_3 \) and \( X_4 \).
where \( U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \). In this work we want to drive the extremal inertias of the Hermitian two submatrices \( X_1 \) and \( X_4 \), so we adopt the following notations for the collections of submatrices \( X_1, X_4 \) in (11)

\[
S_i = \left\{ X_i \mid \begin{bmatrix} A_1 & A_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} A_i^* \\ A_2^* \end{bmatrix} = B \right\}, \quad i = 1, 4. \tag{20}
\]

**Lemma 2.1.** \cite{9} Let \( p(X) = A - BX - (BX)^* \). Where \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m} \) are given and \( X \in \mathbb{C}^{n \times m} \) is a variable matrix, and let \( N = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \). Then,

\[
\max_{X \in \mathbb{C}^{n \times m}} i_{\pm} [p(X)] = i_{\pm} (N), \quad \tag{21}
\]

\[
\min_{X \in \mathbb{C}^{n \times m}} i_{\pm} [p(X)] = i_{\pm} (N) - r(B). \tag{22}
\]

**Theorem 2.2.** Assume that the matrix equation in (11) is consistent, and let \( S_i \) be as given in (20). We denote,

\[
K_1 = \begin{bmatrix} B & A_2 \\ A^* & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} B & A_1 \\ A^* & 0 \end{bmatrix}.
\]

Then,

\[
\max_{X_1 \in S_1} i_{\pm} (X_1) = i_{\pm} \begin{bmatrix} M & K_1 \\ K_1^* & 0 \end{bmatrix} + n_1 - r(M), \tag{23}
\]

\[
\min_{X_1 \in S_1} i_{\pm} (X_1) = i_{\pm} \begin{bmatrix} M & K_1 \\ K_1^* & 0 \end{bmatrix} - r(K_1), \tag{24}
\]

\[
\max_{X_4 \in S_4} i_{\pm} (X_4) = i_{\pm} \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} + n_2 - r(M), \tag{25}
\]

\[
\min_{X_4 \in S_4} i_{\pm} (X_4) = i_{\pm} \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} - r(K_2). \tag{26}
\]

**Proof.** From (16) we can write

\[
X_1 = G + H_1 U_1 + (H_1 U_1)^* \tag{27}
\]

where \( G = -P_1 T M^+ T^* P_1^*, \) \( H_1 = P_1 T_1 \).

Applying the formulas (21) and (22) at (27), we obtain

\[
\max_{X_1 \in S_1} i_{\pm} (X_1) = \max_{U_1} (G + H_1 U_1 + (H_1 U_1)^*) = i_{\pm} \begin{bmatrix} G & H_1 \\ H_1^* & 0 \end{bmatrix}, \tag{28}
\]

\[
\min_{X_1 \in S_1} i_{\pm} (X_1) = \min_{U_1} (G + H_1 U_1 + (H_1 U_1)^*)
\]

\[
= i_{\pm} \begin{bmatrix} G \quad H_1 \\ H_1^* \quad 0 \end{bmatrix} - r(H_1). \tag{29}
\]
Applying (8) to the matrix \[
\begin{bmatrix}
G & H_1 \\
H_1^* & 0
\end{bmatrix},
\]
and simplifying by the formulas (9) and \(M^* = M\), and three types of elementary block congruence matrix operations we obtain,

\[
\begin{align*}
i_\pm \begin{bmatrix}
G & H_1 \\
H_1^* & 0
\end{bmatrix} &= \begin{bmatrix}
-P_i T M^+ T^* P_i^* & P_i T \\
T_i^* P_i^* & 0
\end{bmatrix} \\
&= \begin{bmatrix}
-P_i T M^+ T^* P_i^* & P_i T \\
T_i^* P_i^* & 0
\end{bmatrix} \\
&= \begin{bmatrix}
0 & P_i T \\
M & 0
\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
0 & P_i T \\
M & 0
\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
0 & P_i T \\
M & 0
\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
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&= \begin{bmatrix}
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\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
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M & 0
\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
0 & P_i T \\
M & 0
\end{bmatrix} - r(M) \\
&= \begin{bmatrix}
0 & P_i T \\
M & 0
\end{bmatrix} - r(M)
\end{align*}
\]
\[
= n_1 + i\pm \begin{bmatrix}
B & A & -B & -A_2 \\
A^* & 0 & -A^* & 0 \\
-B & -A & 0 & 0 \\
-A_2^* & 0 & 0 & 0
\end{bmatrix} - r(M)
\]

\[
= n_1 + i\pm \begin{bmatrix}
M & K_1 \\
K_1^* & 0
\end{bmatrix} - r(M).
\]

(30)

Now, by applying (6) to the matrix \(H_1\) we find,
\[
\begin{align*}
\rho(H_1) &= \rho(P_1 T_1) = \rho(P_1 T F_M) \\
&= r \begin{bmatrix}
P_1 T & M \\
M & \end{bmatrix} - r(M) \\
&= r \begin{bmatrix}
0 & I_{n_1} & 0 \\
B & A_1 & A_2 \\
A_1^* & 0 & 0 \\
A_2^* & 0 & 0
\end{bmatrix} - r(M) \\
&= r \begin{bmatrix}
I_{n_1} & 0 & 0 \\
0 & B & A_2 \\
0 & A_1^* & 0 \\
0 & A_2^* & 0
\end{bmatrix} - r(M) \\
&= n_1 + r(K_1) - r(M).
\end{align*}
\]

(31)

Substituting (30) and (31) into (28) and (29) yields to (23) and (24), The formulas (25) and (26) can be proved similarly.

Next, we achieve at the necessary and sufficient conditions of the submatrices \(X_1\) and \(X_4\) to be positive (negative, nonpositive, nonnegative) definite, from Theorem 2.2 and Lemma 1.4.

**Theorem 2.3.** Assume that the matrix equation in (11) is consistent, and let \(S_i\) be as given in (20). Then,

a) Equation (11) has a solution of the form \(\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\) such that \(X_1 \in \mathbb{C}^n_{H}\) satisfying \(X_1 \succeq 0, (X_1 \preceq 0)\) if and only if
\[
i_- \begin{bmatrix}
M & K_1 \\
K_1^* & 0
\end{bmatrix} = r(K_1), \quad (i_+ \begin{bmatrix}
M & K_1 \\
K_1^* & 0
\end{bmatrix} = r(K_1)).
\]

b) All solutions \(\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\) of the equation (11) have \(X_1 \succeq 0, (X_1 \preceq 0)\) if and only if
\[
i_- \begin{bmatrix}
M & K_1 \\
K_1^* & 0
\end{bmatrix} = r(M) - n_1, \quad (i_+ \begin{bmatrix}
M & K_1 \\
K_1^* & 0
\end{bmatrix} = r(M) - n_1).
\]

c) Equation (11) has a solution of the form \(\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\) such that \(X_1 \in \mathbb{C}^n_{H}\) satisfying \(X_1 \succ 0, (X_1 \prec 0)\) if and only if
d) All solutions of the equation (11) have a solution of the form if and only if

\[ i_+ \begin{bmatrix} M & K_1 \\ K_1^* & 0 \end{bmatrix} = r(M), \quad (i_- \begin{bmatrix} M & K_1 \\ K_1^* & 0 \end{bmatrix} = r(M)). \]

e) Equation (11) has a solution of the form \( \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \) such that \( X_4 \geq 0 \), \( (X_4 \leq 0) \) if and only if

\[ i_- \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(K_2), \quad (i_+ \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(K_2)). \]

f) All solutions of the equation (11) have \( X_4 \geq 0 \), \( (X_4 \leq 0) \) if and only if

\[ i_- \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(M) - n_2, \quad (i_+ \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(M) - n_2). \]

g) Equation (11) has a solution of the form \( \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \) such that \( X_4 > 0 \), \( (X_4 < 0) \) if and only if

\[ i_+ \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(M), \quad (i_- \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(M)). \]

h) All solutions of the equation (11) have \( X_4 > 0 \), \( (X_4 < 0) \) if and only if

\[ i_+ \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(K_2) + n_2, \quad (i_- \begin{bmatrix} M & K_2 \\ K_2^* & 0 \end{bmatrix} = r(K_2) + n_2). \]

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