COMPLETE \textit{k-}CURVATURE HOMOGENEOUS
PSEUDO-RIEMANNIAN MANIFOLDS \textit{0-MODELED ON AN}
INDECOMPOSIBLE SYMMETRIC SPACE

P. GILKEY AND S. NIKEVIČ

Abstract. For \( k \geq 2 \), we exhibit complete \( k \)-curvature homogeneous neutral signature pseudo-Riemannian manifolds which are not locally affine homogeneous (and hence not locally homogeneous). The curvature tensor of these manifolds is modeled on that of an indecomposable symmetric space. All the local scalar Weyl curvature invariants of these manifolds vanish.

Dedicated to Professor Sekigawa on his 60th birthday

1. Introduction

1.1. Affine manifolds. Let \( \mathcal{A} := (M, \nabla) \) be an affine manifold where \( \nabla \) is a torsion free connection on a smooth manifold \( M \). Let \( \mathcal{R}_A \) be the associated curvature operator:
\[
\mathcal{R}_A(\xi_1, \xi_2)\xi_3 := (\nabla_{\xi_1} \nabla_{\xi_2} - \nabla_{\xi_2} \nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]} ) \xi_3.
\]

Let \( \nabla_i \mathcal{R}_A \) be the \( i \)th covariant derivative of the curvature operator. If \( P \in \mathcal{M} \), let \( \nabla_i \mathcal{R}_A \) be the restriction of \( \nabla_i \mathcal{R}_A \) to \( TP \). Consider the following algebraic structure which encodes the covariant derivatives of the curvature operator up to order \( k \):
\[
\mathfrak{A}^k(\mathcal{A}, \mathcal{P}) := (TP, \mathcal{R}_A, \mathcal{P}, \ldots, \mathcal{R}_A^k).
\]

We say that \( \phi : \mathfrak{A}^k(\mathcal{A}_1, \mathcal{P}_1) \to \mathfrak{A}^k(\mathcal{A}_2, \mathcal{P}_2) \) is an affine isomorphism if \( \phi \) is a linear map from \( TP_1 \) to \( TP_2 \) satisfying
\[
\phi^*(\nabla_i \mathcal{R}_A, \mathcal{P}_2) = \nabla_i \mathcal{R}_A, \mathcal{P}_1 \quad \text{for} \quad 0 \leq i \leq k.
\]

1.2. Pseudo-Riemannian manifolds. If \( \mathcal{M} := (M, g) \) is a pseudo-Riemannian manifold of signature \( (p, q) \) and of dimension \( m = p + q \), let \( \nabla \) be the Levi-Civita connection, let \( \mathcal{A}(\mathcal{M}) := (M, \nabla) \) be the underlying affine structure, and let
\[
R_M(\xi_1, \xi_2, \xi_3, \xi_4) := g(\mathcal{R}_M(\xi_1, \xi_2)\xi_3, \xi_4)
\]
be the curvature tensor; \( R_M \in \otimes^4 T^*M \). Similarly, let \( \nabla_i R_M \) be the \( i \)th covariant derivative of the curvature tensor. Let
\[
\mathfrak{M}^k(\mathcal{M}, \mathcal{P}) := (TP, g, R_M, \mathcal{P}, \ldots, \mathcal{R}_M^k).
\]

One says that \( \phi : \mathfrak{M}^k(\mathcal{M}_1, \mathcal{P}_1) \to \mathfrak{M}^k(\mathcal{M}_2, \mathcal{P}_2) \) is an isomorphism if \( \phi \) is a linear isomorphism from \( TP_1 \) to \( TP_2 \) so that
\[
\phi^* (g_2, \mathcal{P}_2) = g_1, \mathcal{P}_1 \quad \text{and} \quad \phi^* (\nabla_i R_M, \mathcal{P}_2) = \nabla_i R_M, \mathcal{P}_1 \quad \text{for} \quad 0 \leq i \leq k.
\]

In this situation, the metric permits one to raise indices and conclude as well that
\[
\phi^* (\nabla_i R_M, \mathcal{P}_2) = \nabla_i R_M, \mathcal{P}_1 \quad \text{for} \quad 0 \leq i \leq k.
\]
Thus \( \phi \) is also an isomorphism from \( \mathfrak{A}^k(\mathcal{A}(\mathcal{M}_1), \mathcal{P}_1) \) to \( \mathfrak{A}^k(\mathcal{A}(\mathcal{M}_2), \mathcal{P}_2) \) of the underlying affine structure.

Key words and phrases. Affine \( k \)-curvature homogeneous, \( k \)-curvature homogeneous, homogeneous space, symmetric space, Weyl invariants, vanishing scalar invariants
2000 Mathematics Subject Classification. 53B20.
We shall frequently simplify the notation by setting $R = R_A$ or $R = R_M$ when no confusion is likely to result.

1.3. Various notions of homogeneity. One is often interested in manifolds with a great deal of geometric symmetry. Sometimes this symmetry arises from a transitive group action; such manifolds are called homogeneous.

**Definition 1.1.**

1. An affine manifold $A = (M, \nabla)$ is said to be locally affine homogeneous if given $P, Q \in M$, there is a diffeomorphism $\Phi_{P,Q}$ from a neighborhood of $P$ to a neighborhood of $Q$ so $\Phi_{P,Q}^* \nabla = \nabla$ and so $\Phi(P) = Q$.

2. A pseudo-Riemannian manifold $M = (M, g)$ is said to be locally homogeneous if given $P, Q \in M$, there is a diffeomorphism $\Phi_{P,Q}$ from a neighborhood of $P$ to a neighborhood of $Q$ so $\Phi_{P,Q}^* g = g$ and so $\Phi(P) = Q$.

There are, however, other less restrictive notions of symmetry arising from the curvature operator and curvature tensor:

**Definition 1.2.**

1. One says that an affine manifold $A$ is affine $k$-curvature homogeneous if $\mathfrak{R}^k(A, P)$ and $\mathfrak{R}^k(A, Q)$ are isomorphic for any $P, Q \in M$.

2. One says that a pseudo-Riemannian manifold $M$ is $k$-curvature homogeneous if $\mathfrak{M}^k(M, P)$ and $\mathfrak{M}^k(M, Q)$ are isomorphic for any $P, Q \in M$.

One is interested finding manifolds which are affine $k$-curvature homogeneous but not locally affine homogeneous or which are $k$-curvature homogeneous but not locally homogeneous.

1.4. Previous results. There are 2-curvature homogeneous affine manifolds which are not locally affine homogeneous [9, 14, 15, 16, 21]. In the Riemannian setting ($p = 0$), Takagi [29] constructed 0-curvature homogeneous complete non-compact manifolds which are not locally homogeneous; compact examples were exhibited subsequently by Ferus, Karcher, and Münzer [8]. Many other examples are known [2, 17, 18, 19, 30, 31, 33, 34]. There are no known Riemannian manifolds which are 1-curvature homogeneous but not locally homogeneous. In the Lorentzian setting ($p = 1$) 0-curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen et. al. [6]; 1-curvature homogeneous manifolds which are not locally homogeneous were constructed by Bueken and Djorčić [4] and by Bueken and Vanhecke [5].

1.5. Curvature homogeneity and homogeneity. It is clear that local homogeneity implies $k$-curvature homogeneity for any $k$. The following result, due to Singer [26] in the Riemannian setting and to F. Podesta and A. Spiro [23] in the general context, provides a partial converse:

**Theorem 1.3** (Singer, Podesta-Spiro). There exists an integer $k_{p,q}$ so that if $M$ is a complete simply connected pseudo-Riemannian manifold of signature $(p, q)$ which is $k_{p,q}$-curvature homogeneous, then $(M, g)$ is homogeneous.

These constants were first studied in the Riemannian setting. Singer [26] showed $k_{0,m} < \frac{1}{2}m(m - 1)$; subsequently Yamato [35] and Gromov [13] established the bounds $3m - 5$ and $\frac{2}{2}m - 1$ for $k_{0,m}$, respectively. Sekigawa, Suga, and Vanhecke [24, 25] showed any 1-curvature homogeneous complete simply connected Riemannian manifold of dimension $m < 5$ is homogeneous; thus $k_{0,2} = k_{0,3} = k_{0,4} = 1$. We refer to the discussion in Boeckx, Vanhecke, and Kowalski [2] for further details concerning $k$-curvature homogeneous manifolds in the Riemannian setting; Opozda [22] has established an analogue of Theorem 1.3 in the affine setting. Observe that our definition of $k_{p,q}$ differs slightly from that given elsewhere by certain authors.
We constructed complete metrics of neutral signature \( (p + 3, p + 3) \) on \( \mathbb{R}^{2p+6} \) for any \( p \geq 0 \) which are \( p + 2 \)-curvature homogeneous but not affine \( p + 3 \)-curvature homogeneous \([11]\). The discussion there shows \( k_{p,q} \geq \min\{p,q\} \).

1.6. Scalar invariants. One can use the metric to contract indices in pairs and form scalar Weyl invariants. Adopt the Einstein convention and sum over repeated indices. Let \( R_{i_1i_2i_3i_4} \) denote the components of the curvature tensor. The scalar curvature \( \tau \) and the norm of the Ricci tensor \( |\rho|^2 \) are given respectively by:

\[
\tau = g^{i_1i_2}g^{j_1j_2}R_{i_1j_1i_2j_2} \quad \text{and} \quad |\rho|^2 = g^{i_1j_1}g^{j_2j_2}g^{i_3j_3}g^{i_4j_4}R_{i_1j_1i_2j_2}R_{i_3j_3i_4j_4}.
\]

There is a related result concerning scalar invariants:

**Theorem 1.4** (Prüfer, Tricerri, and Vanhecke [24]). If all local scalar Weyl invariants up to order \( \frac{1}{2} m (m - 1) \) are constant on a Riemannian manifold \( \mathcal{M} \), then \( \mathcal{M} \) is locally homogeneous and \( \mathcal{M} \) is determined up to local isometry by these invariants.

This result fails in the pseudo-Riemannian setting; Koutras and McIntosh [20] gave examples of non-flat manifolds all of whose scalar Weyl invariants vanish; see also related examples by Pravda, Pravdová, Coley, and Milson [25].

1.7. Riemannian manifolds modeled on homogeneous spaces. One says that \( \mathcal{M} \) is \( k \)-modeled on a homogeneous pseudo-Riemannian manifold \( \mathcal{N} \) if \( \mathbb{M}^k(\mathcal{M}, P) \) and \( \mathbb{M}^k(\mathcal{N}, Q) \) are isomorphic for any \( P \in \mathcal{M} \) and \( Q \in \mathcal{N} \); the precise \( Q \in \mathcal{N} \) being irrelevant as \( \mathcal{N} \) is homogeneous. One has the following results in the Riemannian and Lorentzian settings:

**Theorem 1.5.**

1. (Tricerri and Vanhecke [22]) If a Riemannian manifold \( \mathcal{M} \) is \( 0 \)-modeled on an irreducible symmetric space \( \mathcal{S} \), then \( \mathcal{M} \) is locally isometric to \( \mathcal{S} \).

2. (Cahen et al. [6]) If a Lorentzian manifold \( \mathcal{M} \) is \( 0 \)-modelled on an irreducible symmetric space, then \( \mathcal{M} \) has constant sectional curvature.

There is a bit of technical fuss here. Recall that a pseudo-Riemannian manifold \( \mathcal{M} \) is said to be irreducible if the holonomy representation is irreducible, i.e. if \( T_PM \) does not have any proper non-trivial subspace which is invariant under the holonomy representation for any (and hence for all) \( P \in \mathcal{M} \); \( \mathcal{M} \) is said to be indecomposable if there does not exist a non-trivial decomposition of \( T_PM \) which is invariant under the holonomy representation.

These two notions are equivalent in the Riemannian setting but are not equivalent in the higher signature setting. It is known that there are 1-curvature homogeneous 3-dimensional Lorentzian manifolds which are modeled on an indecomposable symmetric space (which is not irreducible) but which are not locally homogeneous; see \([22]\) for further details.

In this paper, we turn to the question of constructing pseudo-Riemannian manifolds which are 0-curvature modeled on an indecomposable symmetric space and which are \( k \)-curvature homogeneous for arbitrarily large \( k \); our construction is motivated by the examples described in \([11]\). We shall be defining several tensors. To simplify the discussion, we only give the non-zero entries in these tensors up to the usual symmetries.

1.8. The pseudo-Riemannian manifolds \( \mathcal{M}_{6+4p,f} \). For \( p \geq 1 \), let

\[
(x, z_0, ..., z_p, \bar{z}_0, ..., \bar{z}_p, x^*, z_0^*, ..., z_p^*, \bar{z}_0^*, ..., \bar{z}_p^*)
\]

be coordinates on \( \mathbb{R}^{6+4p} \). If \( f \) is a smooth function on \( \mathbb{R}^{p+1} \), a generalized plane wave manifold \( \mathcal{M}_{6+4p,f} := (\mathbb{R}^{6+4p}, g_{6+4p,f}) \) of neutral signature \((3 + 2p, 3 + 2p)\) may
be defined by setting:

\[ g_{6+4p,f}(\partial_x, \partial_x) = -2 \{f(z_0, ..., z_p) + z_0 \tilde{z}_0 + ... + z_p \tilde{z}_p \}, \quad \text{and} \]
\[ g_{6+4p,f}(\partial_x, \partial_{x^*}) = g_{6+4p,f}(\partial_{z_i}, \partial_{z_i^*}) = g_{6+4p,f}(\partial_{z_i}, \partial_{z_i^*}) = 1. \]

A word on notation. The dual variables \( \{x^*, z_i^*, \tilde{z}_i^*\} \) enter only rather trivially; Theorem 1.6 below will imply that \( \nabla^k R(\cdot) \) vanishes if any entry belongs to the span of \( \{\partial_x, \partial_{z_i}, \partial_{z_i^*}\} \). Thus \( M_{6+4p,f} \) has a parallel totally isotropic distribution of maximal dimension. The dependence of the metric on the variables \( \{z_0, ..., \tilde{z}_p\} \) is fixed and ensures that the 0-modeled space is an indecomposable symmetric space. The crucial variables are \( \{x, z_0, ..., z_p\} \).

1.9. The geometry of the manifolds \( M_{6+4p,f} \).

**Theorem 1.6.**

1. All geodesics in \( M_{6+4p,f} \) extend for infinite time.
2. \( \exp_{P,M_{6+4p,f}} : T_P \mathbb{R}^{6+4p} \to \mathbb{R}^{6+4p} \) is a diffeomorphism for all \( P \in \mathbb{R}^{6+4p} \).
3. The non-zero components of \( \nabla^k R \) are:

\[ \nabla^k R(\partial_{x_i}, \partial_{z_i}, \partial_{z_i}, \partial_{z_i^*}, ..., \partial_{z_{k+2}}) = -\frac{1}{2} (\partial_{x_i} \cdot \partial_{x_j} \cdot \partial_{x_j} \cdot \partial_{x_j+2} \cdot g_{6+4p,f}(\partial_{x_i}, \partial_{x_j}) \]

for \( \xi_i \in \{z_0, ..., z_p, \tilde{z}_0, ..., \tilde{z}_p\} \).
4. All scalar Weyl invariants of \( M_{6+4p,f} \) vanish.
5. \( M_{6+4p,f} \) is a symmetric space if and only if \( f \) is at most quadratic.

1.10. The symmetric space \( S_{6+4p} \).

**Theorem 1.7.** Let \( S_{6+4p} := M_{6+4p,0} \) be defined by \( f = 0 \). Then:

1. \( S_{6+4p} \) is an indecomposable symmetric space.
2. \( M_{6+4p,f} \) is 0-modeled on \( S_{6+4p} \) for any \( f = f(z_0, ..., z_p) \).

1.11. The homogeneous spaces \( H_{6+4p,k} \). Theorems 1.6 and 1.7 show that Theorems 1.4 and 1.5 fail in the higher signature context. There are other interesting properties that this family of manifolds has. Construct a sequence of pseudo-Riemannian manifolds \( H_{6+4p,k} := M_{6+4p,k} \) by defining:

\[ f_k(z_0, ..., z_p) := z_1 z_0^{k-1} + ... + z_k z_0^{k+1} \quad \text{if} \quad 1 \leq k \leq p, \]

and as exceptional cases

\[ f_{p+1}(z_0, ..., z_p) := z_1 z_0^{p+1} + ... + z_p z_0^{p+1} + z_p^{p+1}, \quad \text{and} \]
\[ f_{p+2}(z_0, ..., z_p) := z_1 z_0^{p+1} + ... + z_p z_0^{p+1} + e z_0. \]

The following result shows that the local isometry type of a homogeneous space need not be determined by the first few covariant derivatives of the curvature tensor:

**Theorem 1.8.** Let \( 1 \leq k \leq p + 2 \). Then:

1. \( H_{6+4p,k} \) is 0-modeled on the indecomposable symmetric space \( S_{6+4p} \).
2. If \( j < k \), then
   (a) \( H_{6+4p,k} \) is \( j \)-modeled on \( H_{6+4p,j} \).
   (b) \( H_{6+4p,j} \) is not \( k \)-modeled on \( H_{6+4p,k} \).
3. \( H_{6+4p,k} \) is a homogeneous space which is not symmetric.
1.12. The manifolds $\mathcal{N}_{6+4p,\psi}$. Let $\psi \in C^\infty(\mathbb{R})$ satisfy

\[ \psi^{(p+3)}(z_0) > 0 \quad \text{and} \quad \psi^{(p+4)}(z_0) > 0 \quad \text{for all} \quad z_0 \in \mathbb{R} . \]

Let $\mathcal{N}_{6+4p,\psi} := \mathcal{M}_{6+4p,\psi}$, where

\[ f_\psi := z_1 z_0^2 + \ldots + z_p z_0^{p+1} + \psi(z_0) . \]

The following Theorem shows that

\[ \alpha^{(6+4p,\psi)}_0(P) := \psi^{(k+p+3)}(\{\psi^{(p+3)}\})^{k-1}(\psi^{(p+4)})^{-k}(P) \quad \text{for} \quad k \geq 2 \]

forms a collection of affine invariants which determines the isometry types of these manifolds; these invariants are not of Weyl type. Again, this does not happen in the Riemannian setting.

**Theorem 1.9.** Suppose that $\psi_i$ are real analytic for $i = 1, 2$ and that $\psi_i^{(p+3)}$ and $\psi_i^{(p+4)}$ are positive. The following assertions are equivalent:

1. There exists a local diffeomorphism $\phi$ from $\mathcal{N}_{6+4p,\psi_1}$ to $\mathcal{N}_{6+4p,\psi_2}$ with $\phi(P_1) = P_2$ and $\phi^* \nabla_{\mathcal{N}_{6+4p,\psi_2}} = \nabla_{\mathcal{N}_{6+4p,\psi_1}}$.
2. We have $\alpha^{(6+4p,\psi_1)}_0(P_1) = \alpha^{(6+4p,\psi_2)}_0(P_2)$ for $k \geq 2$.
3. There exists an isometry $\phi : \mathcal{N}_{6+4p,\psi_1} \rightarrow \mathcal{N}_{6+4p,\psi_2}$ with $\phi(P_1) = P_2$.

1.13. Curvature and affine homogeneity. One has the following Theorem:

**Theorem 1.10.** Assume that $\psi^{(p+3)}$ and $\psi^{(p+4)}$ are positive. Then:

1. $\mathcal{N}_{6+4p,\psi}$ is $0$-modeled on the indecomposable symmetric space $\mathcal{S}_{6+4p}$.
2. $\mathcal{N}_{6+4p,\psi}$ is $j$-modeled on the homogeneous space $\mathcal{H}_{6+4p,j}$ for $1 \leq j \leq p + 2$.
3. $\mathcal{N}_{6+4p,\psi}$ is $(p + 2)$-curvature homogeneous.
4. The following conditions are equivalent:
   a. $\mathcal{N}_{6+4p,\psi}$ is homogeneous.
   b. $\mathcal{N}_{6+4p,\psi}$ is affine $(p + 3)$-curvature homogeneous.
   c. $\alpha^{(6+4p,\psi)}_0$ is constant.
   d. $\psi^{(p+3)} = a e^{b z_0}$ for some $a,b \neq 0$.

Taking $\psi = e^{z_0} + e^{2z_0}$ constructs a manifold which is $(p + 2)$-modeled on the homogeneous space $\mathcal{N}_{6+4p,e^{z_0}}$, which is curvature $0$-modeled on the indecomposable symmetric space $\mathcal{S}_{6+4p}$, and which is not affine $(p + 3)$-curvature homogeneous and hence not affine homogeneous.

2. Completeness

**Proof of Theorem 1.10.** To simplify the notation a bit, we introduce the variables

\[ s = (s_1, \ldots, s_{2+2p}) := (z_0, \ldots, z_p, \tilde{z}_0, \ldots, \tilde{z}_p) , \quad \text{and} \quad s^* = (s^*_1, \ldots, s^*_{2+2p}) := (z^*_0, \ldots, z^*_p, \tilde{z}^*_0, \ldots, \tilde{z}^*_p) . \]

Let $1 \leq i \leq 2 + 2p$. The metric then takes the form

\[ g_{6+4p,f}(\partial_{x_i}, \partial_x) = -2F(s) \quad \text{and} \quad g_{6+4p,f}(\partial_{x_i}, \partial_{x^*}) = g_{6+4p,f}(\partial_{s_i}, \partial_{s^*}) = 1 . \]

For $F := f(z_0, \ldots, z_p) + z_0 \tilde{z}_0 + z_1 \tilde{z}_1 + \ldots + z_p \tilde{z}_p$. We compute the non-zero Christoffel symbols of the first and second kinds:

\[ g_{6+4p,f}(\nabla_{\partial_x} \partial_x, \partial_s) = \partial_s F , \quad g_{6+4p,f}(\nabla_{\partial_s} \partial_{s^*}, \partial_x) = g_{6+4p,f}(\nabla_{\partial_s} \partial_{x^*}, \partial_x) = -\partial_x F , \quad \nabla_{\partial_x} \partial_x = \sum_i \partial_{s_i} F \cdot \partial_{s^*_i} , \quad \text{and} \quad \nabla_{\partial_x} \partial_{s_i} = \nabla_{\partial_{s^*_i}} \partial_x = -\partial_x F \cdot \partial_{x^*_i} . \]

The curve $\gamma(t) = (x(t), s(t), x^*(t), s^*(t))$ is a geodesic if and only if

\[ 0 = \ddot{x} , \quad 0 = \ddot{s}_i , \quad 0 = \ddot{s}^*_i - 2\dot{x} \sum_i \dot{s}_i \partial_{s_i} F , \quad \text{and} \quad 0 = \ddot{s}^*_i + \dot{x} \dot{s}^*_i \partial_{s_i} F . \]
We solve the geodesic equation with initial conditions \( \gamma(0) = (\alpha, \xi, \alpha^*, \xi^*) \) and \( \dot{\gamma}(0) = (\beta, \eta, \beta^*, \eta^*) \) by setting:

\[
\begin{align*}
    x(t) &= \alpha + \beta t, \\
x^*(t) &= \alpha^* + \beta^* t + 2 \beta \int_0^t \left( \sum_i \eta_i \partial_i F(\xi + t \eta) \right) d\sigma d\tau, \\
    s_i(t) &= \xi_i + t \eta_i,
\end{align*}
\]

The solution exists for all time. Furthermore, there exists a unique geodesic with \( \gamma(0) = P \) and \( \gamma(1) = Q \); this establishes Assertions (1) and (2).

Since \( \nabla \partial_{x^*} = \nabla \partial_{s^*} = 0 \), Assertion (3) follows as the quadratic terms in the Christoffel symbols play no role in the covariant derivatives. Let

\[
\nabla \partial_{x^*} = D_{x^*} = \sum_i \partial_i X_i...
\]

3. A 0-model for \( \mathcal{M}_{6+4p,f} \)

It is convenient to work in the purely algebraic setting. Let \( V \) be an \( m \)-dimensional vector space. Let

\[
\mathfrak{M}^k := (V, \langle \cdot, \cdot \rangle, A^0, \ldots, A^k)
\]

where \( \langle \cdot, \cdot \rangle \) is a non-degenerate inner product on \( V \) and where \( A^i \in \otimes^{k+i} V^* \) satisfies the appropriate symmetries of the covariant derivatives of the curvature tensor; if \( k = \infty \), then the sequence is infinite. We say that \( \mathfrak{M} \) is a \( k \)-model for \( \mathcal{M} = (M, g) \) if for each point \( P \in M \), there is an isomorphism \( \phi : T_P M \to V \) so that

\[
\phi^* \langle \cdot, \cdot \rangle = g_P \quad \text{and} \quad \phi^* A^i = \nabla^i R_P \quad \text{for} \quad 0 \leq i \leq k.
\]

Clearly \( \mathcal{M} \) is \( k \)-curvature homogeneous if and only if it admits a \( k \)-model as one could take \( \mathfrak{M}^k := \mathfrak{M}(\mathcal{M}, P) \) for any \( P \in \mathcal{M} \).

3.1. Models for the manifolds \( \mathcal{M}_{6+4p,f} \)

Let

\[
\{ X, Z_0, \ldots, Z_p, \tilde{Z}_0, \ldots, \tilde{Z}_p, X^*, Z_0^*, \ldots, Z_p^*, \tilde{Z}_0^*, \ldots, \tilde{Z}_p^* \}
\]

be a basis for \( \mathbb{R}^{6+4p} \). Define a hyperbolic inner product on \( \mathbb{R}^{6+4p} \) by pairing ordinary variables with the corresponding dual variables:

\[
\langle X, X^* \rangle = \langle Z_i, Z_i^* \rangle = \langle \tilde{Z}_i, \tilde{Z}_i^* \rangle = 1 \quad \text{for} \quad 0 \leq i \leq p.
\]

Define an algebraic curvature tensor \( A^0 \) supported on \( \text{Span}\{X, Z_i, \tilde{Z}_i\} \) by:

\[
A^0(\langle X, Z_i, \tilde{Z}_i, X \rangle) = 1 \quad \text{for} \quad 0 \leq i \leq p.
\]

Define higher order covariant derivative curvature tensors \( A^i \) for \( 1 \leq i \leq p \) by:

\[
\begin{align*}
    A^i(X, Z_0, Z_i, X; Z_0, \ldots, Z_0) &= 1, \\
    A^i(X, Z_0, Z_i, X; Z_i, Z_0, \ldots, Z_0) &= 1, \ldots, \\
    A^i(X, Z_0, Z_i, X; Z_0, \ldots, Z_0, Z_i) &= 1.
\end{align*}
\]

The vectors \( \{Z_i, \tilde{Z}_i\} \) for \( 0 \leq i \leq p \) are linked by \( A^0 \); the vectors \( Z_0 \) and \( Z_i \) are linked by \( A^i \) for \( 1 \leq i \leq p \). Set

\[
\begin{align*}
    \mathfrak{M}^{p+1}_{6+4p} &: (X, Z_0, Z_i, X; Z_0, \ldots, Z_0) = 1, \quad \text{and} \\
    \mathfrak{M}^{p+2} &: (X, Z_0, Z_i, X; Z_0, \ldots, Z_0) = 1.
\end{align*}
\]

For \( 0 \leq k \leq p + 2 \), we define models:

\[
\mathfrak{M}^k_{6+4p} := (\mathbb{R}^{6+4p}, \langle \cdot, \cdot \rangle, A^0, \ldots, A^k).
\]
Proof of Theorem 1.7. Let $0 \leq i, j \leq p$. By Theorem 1.6,

$$R(\partial_x, \partial_{z_i}, \partial_{z_j}, \partial_{z_k}) = 1$$

and

$$R(\partial_x, \partial_{z_i}, \partial_{z_j}, \partial_x) = \partial_{z_i} \partial_{z_j} F$$

where $F = f(z_0, \ldots, z_p) + z_0 \bar{z}_0 + \ldots + z_p \bar{z}_p$. We set

$$X := \partial_x + F \partial_x^*, \quad X^* := \partial_x^*$$

(3.a)

$$Z_i := \partial_{z_i} - \frac{1}{2} \sum_j \partial_{z_j} f \cdot \partial_{z_j}, \quad Z_i^* := \partial_{z_i}^*$$

$$\tilde{Z}_i := \partial_{z_i}, \quad \tilde{Z}_i^* := \partial_{z_i}^* + \frac{1}{2} \sum_j \partial_{z_j} f \cdot \partial_{z_j}^*.$$

We show that $\mathfrak{M}_{6+4p}$ is a 0-model for $\mathcal{M}_{6+4p,f}$ by noting that the non-zero components of $g_{6+4p,f}$ and $R$ are then given by

$$g_{6+4p,f}(X, X^*) = g_{6+4p,f}(Z_i, Z_i^*) = g_{6+4p,f}(\tilde{Z}_i, \tilde{Z}_i^*) = 1, \quad \text{and}$$

$$R(X, Z_i, \tilde{Z}_i, X) = 1 \quad \text{for} \quad 0 \leq i \leq p.$$

By Theorem 1.6, $\mathcal{S}_{6+4p}$ is a symmetric space. As $\mathfrak{M}_{6+4p}^0$ is a 0-model for $\mathcal{S}_{6+4p}$ and $\mathfrak{M}_{6+4p}^0$ is a 0-model for $\mathcal{M}_{6+4p,f}$, $\mathfrak{M}_{6+4p}$ is a 0-model for $\mathcal{M}_{6+4p,f}$. To show complete the proof, we must only show $\mathfrak{M}_{6+4p}^0$ is indecomposable.

Suppose we have a non-trivial decomposition $\mathbb{R}^{6+4p} = V_1 \oplus V_2$ such that

$$A^0 = A_1^0 \oplus A_2^0 \quad \text{and} \quad \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2.$$

We argue for a contradiction. Denote the natural projections induced by this decomposition by $\pi_i : \mathbb{R}^{6+4p} \to V_i$. Since

$$1 = \langle X, X^* \rangle = \langle \pi_1 X, X^* \rangle + \langle \pi_2 X, X^* \rangle$$

we may assume without loss of generality $\langle \pi_1 X, X^* \rangle \neq 0$. Set $\alpha := \pi_1(X)$. Let $\beta \in (X^*)^+ \cap V_2$. Then $A^0(\alpha, \cdot, \beta, \alpha) = 0$ as $\alpha \in V_1$ and $\beta \in V_2$. Since $\beta$ doesn’t involve $X$,

$$0 = A^0(\alpha, Z_i, \beta, \alpha) = \langle \alpha, X^* \rangle^2 \langle \beta, \tilde{Z}_i^* \rangle, \quad \text{and}$$

$$0 = A^0(\alpha, \tilde{Z}_i, \beta, \alpha) = \langle \alpha, X^* \rangle^2 \langle \beta, Z_i^* \rangle.$$

Consequently $\langle \beta, X^* \rangle = 0$, $\langle \beta, Z_i^* \rangle = 0$, and $\langle \beta, \tilde{Z}_i^* \rangle = 0$. Thus

$$\beta \in \text{Span}\{X^*, Z_1^*, \ldots, Z_p^*, \tilde{Z}_1^*, \ldots, \tilde{Z}_p^*\}$$

so $(X^*)^+ \cap V_2$ is totally isotropic. Since the restriction of $\langle \cdot, \cdot \rangle$ to $V_2$ is non-degenerate and since

$$\dim\{(X^*)^+ \cap V_2\} \geq \dim\{V_2\} - 1,$$

we conclude that $\dim\{V_2\} = 2$. Furthermore there must exist an element of $V_2$ not in $(X^*)^+$. We can therefore interchange the roles of $V_1$ and $V_2$ to see that $\dim\{V_1\} = 2$. Consequently $6 + 4p = \dim\{V_1\} + \dim\{V_2\} = 4$ which provides the desired contradiction.

\[ \square \]

Theorem 1.8 (1) and Theorem 1.10 (1) are specials cases of Theorem 1.7 (2). Theorem 1.8 (2b) follows since $\nabla^j R_{6+4p,k} = 0$ if $j > k$ whereas $\nabla^j R_{6+4p,k} \neq 0$. Theorem 1.8 (2a) and Theorem 1.10 (2, 3) will follow from the following result.

Lemma 3.1.

1. If $f = \psi(z_0) + z_1 z_0^2 + \ldots + z_k z_0^{k+1}$ for $1 \leq k \leq p$, then $\mathfrak{M}_{6+4p}^k$ is a k-model for $\mathcal{M}_{6+4p,f}$.

2. If $f = \psi(z_0) + z_1 z_0^2 + \ldots + z_p z_0^{p+1}$ and if $\psi^{(p+3)}$ is positive on $\mathbb{R}$, then $\mathfrak{M}_{6+4p}^{p+1}$ is a $(p+1)$-model for $\mathcal{M}_{6+4p,f}$.

3. If $f = \psi(z_0) + z_1 z_0^2 + \ldots + z_p z_0^{p+1}$ and if $\psi^{(p+3)}$ and $\psi^{(p+4)}$ are positive on $\mathbb{R}$, then $\mathfrak{M}_{6+4p}^{p+2}$ is a $(p+2)$-model for $\mathcal{M}_{6+4p,f}$.
Proof. We adopt the notation of Equation (3.4). The normalizations of Equation (3.1) are then satisfied. Suppose \( f = \psi(z_0) + z_1 z_0^2 + \ldots + z_k z_0^{k+1} \). If \( 1 \leq i \leq k \) and \( 1 \leq j \leq p \),
\[
\nabla^i R(X, Z_0, Z_0, X; Z_0, \ldots, Z_0) = \varepsilon_i,
\]
\[
\nabla^i R(X, Z_0, Z_j, X; Z_0, \ldots, Z_0) = \varepsilon_{j,i},
\]
\[
\nabla^i R(X, Z_0, Z_0, X; Z_0, \ldots, Z_j, \ldots, Z_0) = \varepsilon_{j,i}
\]
where \( \varepsilon_i = (\partial z_0)^i f \) and \( \varepsilon_{j,i} = (\partial z_0)^{i+j} \partial z_i f \). Note that
\[
\varepsilon_{i,i} \neq 0 \quad \text{for} \quad 1 \leq i \leq k \quad \text{and} \quad \varepsilon_{i,i} = 0 \quad \text{for} \quad 1 \leq j < i \leq k.
\]
To prove Assertion (1), we must define a new frame \( \{ 1 X, 1 X^*, 1 Z_1, 1 Z^*_1, 1 Z^*_1 \} \) so that in addition to the relations of Equation (3.8), the only non-zero components of \( \nabla^i R \) are given by
\[
\nabla^i R(1 X, 1 Z_0, 1 Z_1, 1 X; 1 Z_0, \ldots, 1 Z_0) = \ldots = \nabla^i R(1 X, 1 Z_0, 1 Z_1, 1 X, 1 Z_0, \ldots, 1 Z_0, 1 Z_1) = 1.
\]
Set
\[
1 X := X \quad \text{and} \quad 1 Z_0 := Z_0 + a_1 Z_1 + \ldots + a_k Z_k.
\]
To ensure \( \nabla^i R(1 X, 1 Z_0, 1 Z_1, 1 X, 1 Z_0, \ldots, 1 Z_0) = 0 \) for \( 1 \leq \ell \leq k \), we must have:
\[
0 = \varepsilon_k + (k+2)\varepsilon_{k,k} a_k,
\]
\[
0 = \varepsilon_{k-1} + (k+1)\{\varepsilon_{k,k-1} a_k + \varepsilon_{k-1,k-1} a_{k-1}\},
\]
\[
0 = \varepsilon_1 + 3\{\varepsilon_{k,1} a_k + \ldots + \varepsilon_{1,1} a_1\}.
\]
Because \( \varepsilon_{i,i} \neq 0 \) for \( 1 \leq i \leq k \), this upper triangular system of equations is recursively solvable for \( a_k, \ldots, a_1 \).

To ensure that
\[
\nabla^i R(1 X, 1 Z_0, 1 Z_1, 1 X^*; 1 Z_0, \ldots, 1 Z_0) = 1, \quad \text{and}
\]
\[
\nabla^i R(1 X, 1 Z_0, 1 Z_j, 1 X^*; 1 Z_0, \ldots, 1 Z_0) = 0 \quad \text{for} \quad i \neq j,
\]
we set \( 1 Z_i = Z_i \) for \( k < i \leq p \), while for \( 1 \leq i \leq k \), we set
\[
1 Z_1 = a_{1,1} Z_1, \quad 1 Z_2 = a_{2,1} Z_1 + a_{2,2} Z_2, \quad \ldots, \quad 1 Z_k = a_{k,1} Z_1 + \ldots + a_{k,k} Z_k.
\]
To ensure that \( 1 Z_k \) is properly normalized, the following relations must hold:
\[
1 = a_{k,k} \varepsilon_{k,k},
\]
\[
0 = a_{k,k-1} \varepsilon_{k-1,k-1} + a_{k,k} \varepsilon_{k,k-1},
\]
\[
0 = a_{k,1} \varepsilon_{1,1} + \ldots + a_{k,k} \varepsilon_{1,k}.
\]
This determines \( 1 Z_k \). We continue in this fashion to determine the remaining coefficients. This ensures the proper normalizations for \( \nabla^i R \) for \( 1 \leq i \leq k \).

We now return to the relations of Equation (3.1) for \( g \) and \( R \). We regard \( R(X, \cdot, \cdot, X) \) as defining a neutral signature inner product on
\[
\text{Span}\{\partial z_0, \ldots, \partial z_p, \partial z_{1}, \ldots, \partial z_{p}\}.
\]
Since \( 1 X = \partial x + F \partial x^* \) and since \( \{1 Z_0, \ldots, 1 Z_p\} \subset \text{Span}\{Z_0, \ldots, Z_p\} \) we may choose
\[
\{1 Z_0, \ldots, 1 Z_p\} \subset \text{Span}\{\tilde{Z}_0, \ldots, \tilde{Z}_p\}
\]
so the only non-zero components of \( R \) are \( R(1 X, 1 Z_i, 1 \tilde{Z}_i, X) = 1 \). Finally, we choose a dual basis
\[
\{1 X^*, 1 Z_0^*, \ldots, 1 Z_p^*, 1 \tilde{Z}_0^*, \ldots, 1 \tilde{Z}_p^*\} \subset \text{Span}\{X^*, Z_0^*, \ldots, Z_p^*, \tilde{Z}_0^*, \ldots, \tilde{Z}_p^*\}
\]
so the non-zero components of the metric \( g \) are
\[
g(1 X, 1 X^*) = g(1 Z_i, 1 Z_i^*) = g(1 \tilde{Z}_i, 1 \tilde{Z}_i^*) = 1.
\]
Assertion (1) of the Lemma now follows.
There is a final bit of flexibility that we use in proving Assertions (2) and (3) of the Lemma. The relations of Equation (3.b) continue to hold. We rescale the basis we have constructed by setting:

\[
\begin{align*}
2X &= \varepsilon^{-1}X, & 2X^* &= \varepsilon^{-1}X^*, \\
2Z_i &= \varepsilon_i Z_i, & 2Z_i^* &= \varepsilon_i Z_i^*.
\end{align*}
\]

The non-zero components of \(g\) and of \(R\) are

\[
\begin{align*}
g(2X, 2X^*) &= g(2Z_i, 2Z_i^*) = g(2\tilde{Z}_i, 2\tilde{Z}_i^*) = 1, \\
R(2X, 2Z_i, 2\tilde{Z}_i, 2X) &= 1.
\end{align*}
\]

for \(0 \leq i \leq p\). The non-zero components of \(\nabla^i R\) for \(1 \leq i \leq p\) are

\[
\begin{align*}
\nabla^i R(2X, 2Z_0, 2\tilde{Z}_i, 2X, 2Z_0, \ldots, 2Z_0) &= \ldots \\
&= \nabla^i R(2X, 2Z_0, 2\tilde{Z}_i, 2X, 2Z_0, \ldots, 2Z_i) = \varepsilon^2 \varepsilon_i \varepsilon^{i+1}.
\end{align*}
\]

The non-zero components of \(\nabla^{p+1} R\) and \(\nabla^{p+2} R\) are:

\[
\begin{align*}
\nabla^{p+1} R(2X, 2Z_0, 2\tilde{Z}_i, 2X, 2Z_0, \ldots, 2Z_0) &= \varepsilon^2 \varepsilon_0^{p+3} \psi^{(p+3)}, \\
\nabla^{p+2} R(2X, 2Z_0, 2\tilde{Z}_i, 2X, 2Z_0, \ldots, 2Z_0) &= \varepsilon^2 \varepsilon_0^{p+4} \psi^{(p+4)}.
\end{align*}
\]

We set \(\varepsilon_i := \varepsilon^{-2}\varepsilon^{i-1}\) for \(1 \leq i \leq p\) to ensure \(\nabla^i R\) has the proper normalization for \(1 \leq i \leq p\). Suppose that \(\psi^{(p+3)}\) is positive on \(\mathbb{R}\). We normalize \(\nabla^{p+1} R\) and prove Assertion (2) of the Lemma by setting:

\[
\varepsilon_0 = 1 \quad \text{and} \quad \varepsilon = \left\{\psi^{(p+3)}\right\}^{-1/2}.
\]

If additionally \(\psi^{(p+4)}\) is positive on \(\mathbb{R}\), we may set

\[
\varepsilon_0 := \psi^{(p+3)} \left\{\psi^{(p+4)}\right\}^{-1} \quad \text{and} \quad \varepsilon = \left\{\varepsilon_0^{p+3} \psi^{(p+3)}\right\}^{-1/2}
\]

to ensure that both \(\nabla^{p+1}\) and \(\nabla^{p+2}\) are normalized appropriately. This establishes Assertion (3) of the Lemma.

4. ISOMETRIES

Let \(\mathcal{M}_1 := \left(\mathcal{M}_1, g_1\right)\) be real analytic pseudo-Riemannian manifolds for \(i = 1, 2\). Assume there exist points \(P_i \in \mathcal{M}_i\) so \(\exp_{\mathcal{M}_i, P_i} : T_{P_i} \mathcal{M}_i \rightarrow \mathcal{M}_i\) is a diffeomorphism and so there exists an isomorphism \(\Phi\) between \(\mathcal{M}_1\) and \(\mathcal{M}_1\) and \(\mathcal{M}_2\) and \(\mathcal{M}_2\). Then \(\phi := \exp_{\mathcal{M}_2, \mathcal{M}_2} \circ \Phi \circ \exp_{\mathcal{M}_1, \mathcal{M}_1}\) is an isometry from \(\mathcal{M}_1\) to \(\mathcal{M}_2\).

Proof. Belger and Kowalski [11] note about analytic pseudo-Riemannian metrics that the “metric \(g\) is uniquely determined, up to local isometry, by the tensors \(R, \nabla R, \ldots, \nabla^k R, \ldots\) at one point.”; see also Gray [12] for related work. The desired result now follows.

Proof of Theorem 4.3. Let \(f = \psi(z_0) + z_1 z_0 + \ldots + z_p z_0^{p+1}\). We assume \(\psi^{(p+3)}\) and \(\psi^{(p+4)}\) are positive. If \(k \geq p + 1\), then the non-zero components of the curvature operator \(\nabla^k R\) are given by

\[
\begin{align*}
(\nabla_{\partial_{z_0}})^k R(\partial_{z_0}, \partial_{z_0})\partial_{z_0} &= -(\nabla_{\partial_{z_0}})^k R(\partial_{z_0}, \partial_z)\partial_{z_0} = \psi^{(k+2)}\partial_{z_0}, \quad \text{and} \\
(\nabla_{\partial_{z_0}})^k R(\partial_z, \partial_{z_0})\partial_z &= -(\nabla_{\partial_{z_0}})^k R(\partial_{z_0}, \partial_z)\partial_z = -\psi^{(k+2)}\partial_{z_0}.
\end{align*}
\]

Choose \(X, Z_0 \in T_P \mathbb{R}^{6+4p}\) and \(\Theta \in T_P^* (\mathbb{R}^{6+4p})\) so:

\[
\Theta \{ (\nabla_{Z_0})^{p+1} R(X, Z_0)X \} \neq 0.
\]
For example one could take \( \Theta = ds_0^* \), \( X = \partial_x \) and \( Z_0 = \partial_{z_0} \). Equation (4.a) is an invariant of the affine \( p+1 \)-model as it does not depend on the metric and is preserved by local affine isomorphisms. Expand

\[
X = a\partial_x + a^*\partial_x^* + \sum_i \{a_i\partial_{z_i} + \bar{a}_i\partial_{\bar{z}_i} + a_i^*\partial_{z_i^*} + \bar{a}_i^*\partial_{\bar{z}_i^*}\}, \\
Z_0 = b\partial_x + b^*\partial_x^* + \sum_i \{b_i\partial_{z_i} + \bar{b}_i\partial_{\bar{z}_i} + b_i^*\partial_{z_i^*} + \bar{b}_i^*\partial_{\bar{z}_i^*}\}.
\]

If \( k \geq p+1 \),

\[
\Theta\{(\nabla_{z_0})^k R(X, Z_0)X\} = (ba_0 - ab_0)b_0^k\Theta\{(\nabla_{z_0})^k R(\partial_x, \partial_{z_0})(a\partial_x + a_0\partial_{z_0})\} \\
= (ba_0 - ab_0)b_0^k\psi^{(k+2)}\Theta(\partial_a z_0 - a_0\partial_x^*)
\]

By hypothesis this is non-zero when \( k = p+1 \). Thus

\( a \neq 0, \ b_0 \neq 0, \ ba_0 - ab_0 \neq 0, \) and \( \Theta(a\partial_{z_0} - a_0\partial_x^*) \neq 0 \).

Set \( \gamma := \Theta(a\partial_{z_0} - a_0\partial_x^*) \). We may now compute:

\[
\Theta\{(\nabla_{z_0})^{k+p+1} R(X, Z_0)Z_0\} \Theta\{(\nabla_{z_0})^{p+1} R(X, Z_0)Z_0\}^{k-1} \\
= \frac{(ba_0 - ab_0)b_0^{k+p+1}\psi^{(k+p+3)}\gamma \cdot (ba_0 - ab_0)b_0^{p+1}\psi^{(p+3)}\gamma^{-1}}{\gamma^{k+1} \gamma^{k-1}} = a_0^{k+1} 6+4p,\psi.
\]

This shows that \( a_0^{k+1} 6+4p,\psi \) is an affine invariant. Consequently Assertion (1) implies Assertion (2) in Theorem 1.9.

We now show Assertion (2) implies Assertion (3) in Theorem 1.9; this will complete the proof as it is immediate that Assertion (3) implies Assertion (1). By Lemma 3.1 (3), we can choose a basis \( \{X, X^*, Z_i, \tilde{Z}_i, Z_i^*, \tilde{Z}_i^*\} \) which normalizes \( g_{6+4p,f} \) and \( \nabla^i R \) appropriately for \( 0 \leq i \leq p+2 \). Since

\[
\nabla^i R(X, Z_0, Z_0, X; Z_0, \ldots, Z_0) = 1 \quad \text{for} \quad i = p+1, p+2,
\]

we have

\[
\nabla^{k+p+1} R(X, Z_0, Z_0, X; Z_0, \ldots, Z_0) = a_0^{k} 6+4p,\psi \quad \text{for} \quad k \geq 2.
\]

This shows that the higher covariant derivatives are controlled by \( a_0^{k} 6+4p,\psi \). Consequently if \( a_0^{k} 6+4p,\psi_1(P_1) = a_0^{k} 6+4p,\psi_2(P_2) \) for \( k \geq 2 \), there is an isomorphism between \( \mathfrak{M}^\infty(N_{6+4p,\psi_1}, P_1) \) and \( \mathfrak{M}^\infty(N_{6+4p,\psi_2}, P_2) \) and hence by Lemma 4.1 an isometry between \( (N_{6+4p,\psi_1}, P_1) \) and \( (N_{6+4p,\psi_2}, P_2) \) as desired. \( \Box \)

**Proof of Theorem 1.8 (4).** Let \( f_k := z_1^2 z_0^2 + \ldots + z_k z_0^{k+1} \). Let \( P_i \in \mathbb{R}^{6+4p} \). By Lemma 4.1

\[
(4.b) \quad \mathfrak{M}^i(M_{6+4p,f}, P_1) \cong \mathfrak{M}^i(M_{6+4p,f}, P_2)
\]

for \( i = k \). Since \( \nabla^j R = 0 \) for \( j > k \), we may take \( i = \infty \) in Equation 1.11. Thus by Lemma 4.1 there is an isometry of \( M_{6+4p,f} \) taking \( P_1 \) to \( P_2 \). This shows \( M_{6+4p,f} \) is a homogeneous space. The argument is the same if \( f = z_1^2 z_0^2 + \ldots + z_p z_0^{p+1} + z_0^{p+3} \) where we start with \( i = p+1 \) in Equation 1.11.

If \( f = z_1^2 z_0^2 + \ldots + z_p z_0^{p+1} + a e^{b z_0} \), then

\[
\alpha_0^{k} 6+4p,\psi = b^{k+p+3} \psi^{(p+3)(k-1)} \psi^{(p+4)(-k)}
\]

is independent of the point in question. We use Theorem 1.9 (2) to see \( M_{6+4p,f} \) is a homogeneous space. \( \Box \)
Proof of Theorem 1.10 (4). By Theorem 1.9, (4a) \(\Rightarrow\) (4b) \(\Rightarrow\) (4c). Set \(h = \psi^{(p+3)}\).

If (4c) holds, then

\[ k = \alpha^{2}_{6+4p,\psi} = h^{(2)}h^{(1)} - 2 . \]

We integrate the relation \(h^{(2)}h = kh^{(1)}h^{(1)}\) to see there exist \((a, b)\) so

\[ h(z_0) = \begin{cases} ae^{b_{z_0}} & \text{if } k = 1, \\ a(z_0 + b)^{1/(1-k)} & \text{if } k \neq 1. \end{cases} \]

Since \(a(z_0 + b)^{1/(1-k)}\) vanishes when \(z_0 = -b\), these solutions are ruled out by the assumption \(h\) is always positive and smooth. Consequently \(h(z_0) = ae^{b_{z_0}}\) and (4d) holds. By Theorem 1.9, (4d) \(\Rightarrow\) (4a).

\[ \square \]

Acknowledgments

Research of P. Gilkey partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig). Research of S. Nikčević partially supported by MM 1646 (Srbija) and by the DAAD (Germany). Both authors wish to express their thanks to the Technical University of Berlin where parts of research reported here were conducted.

References

[1] M. Belger and O. Kowalski, Riemannian metrics with the prescribed curvature tensor and all its covariant derivatives at one point, Math. Nachr. 168 (1994), 209–225.
[2] E. Boeckx, O. Kowalski, and L. Vanhecke, Riemannian manifolds of conullity two, World Scientific (1996).
[3] P. Bueken, On curvature homogeneous three-dimensional Lorentzian manifolds, J. Geom. Phys. 22 (1997), 349–362.
[4] P. Bueken and M. Djorić, Three-dimensional Lorentz metrics and curvature homogeneity of order one, Ann. Global Anal. Geom. 18 (2000), 85–103.
[5] P. Bueken and L. Vanhecke, Examples of curvature homogeneous Lorentz metrics, Classical Quantum Gravity 14 (1997), L93–L96.
[6] M. Cahen, J. Leroy, M. Parker, F. Tricerri, and L. Vanhecke, Lorentz manifolds modeled on a Lorentz symmetric space, J. Geom. Phys. 7 (1990), 571–581.
[7] G. Calvaruso, R. A. Marinochi, and D. Perrone, Three-dimensional curvature homogeneous hypersurfaces, Arch. Math. (Brno) 36 (2000), 269–278.
[8] D. Ferus, H. Karcher, and H. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, Math. Z. 177 (1981), 479–502.
[9] E. Garcia-Río, D. Kupeli, M. E. Vázquez-Abal, and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemann extensions. Differential Geom. Appl. 11 (1999), 145–153.
[10] P. Gilkey and S. Nikčević, Curvature homogeneous spacelike Jordan Osserman pseudo-Riemannian manifolds, Classical Quantum Gravity, 21 (2004), 497–507.
[11] P. Gilkey and S. Nikčević, Complete \(k\)-curvature homogeneous pseudo-Riemannian manifolds [math.DG/0409023].
[12] A. Gray, The volume of a small geodesic ball of a Riemannian manifold, Mich. Math. J. 20 (1973), 329–344.
[13] M. Gromov, Partial differential relations, Ergeb. Math. Grenzgeb 3. Folge, Band 9, Springer-Verlag (1986).
[14] O. Kowalski, B. Opozda, and Z. Vlášek, Curvature homogeneity of affine connections on two-dimensional manifolds, Colloq. Math. 81 (1999), 123–139.
[15] —, A classification of locally homogeneous affine connections with skew-symmetric Ricci tensor on 2 dimensional manifolds, Monatsh. Math. 130 (2000), 109–125.
[16] —, A classification of locally homogeneous connections on 2-dimensional manifolds via group-theoretical approach, Central European Journal of Mathematics (2004) 2, 87–102.
[17] O. Kowalski and F. Prüfer, Curvature tensors in dimension four which do not belong to any curvature homogeneous space, Arch. Math. (Brno) 30 (1994), 45–57.
[18] O. Kowalski, F. Tricerri, and L. Vanhecke, Curvature homogeneous Riemannian manifolds J. Math. Pures Appl. 71 (1992), 471–501.
[19] —, Curvature homogeneous spaces with a solvable Lie group as homogeneous model, J. Math. Soc. Japan 44 (1992), 461–484.
[20] A. Kontras and C. McIntosh, A metric with no symmetries or invariants, Classical Quantum Gravity 13 (1996), L47–L49.
[21] B. Opozda, On curvature homogeneous and locally homogeneous affine connections, *Proc. Amer. Math. Soc.* **124** (1996), 1889–1893.

[22] B. Opozda, Affine versions of Singer’s theorem on locally homogeneous spaces, *Ann. Global Anal. Geom.* **15** (1997), 187–199.

[23] F. Podesta and A. Spiro, Introduzione ai Gruppi di Trasformazioni, *Volume of the Preprint Series of the Mathematics Department "V. Volterra" of the University of Ancona, Via delle Brecce Bianche, Ancona, ITALY* (1996).

[24] F. Prüfer, F. Tricerri, and L. Vanhecke, Curvature invariants, differential operators and local homogeneity, *Trans. Am. Math. Soc.* **348** (1996), 4643–4652.

[25] V. Pravda, A. Pravdová, A. Coley, and R. Milson, All spacetimes with vanishing curvature invariants, *Classical Quantum Gravity* **19** (2002), 6213–6236.

[26] I. M. Singer, Infinitesimally homogeneous spaces, *Commun. Pure Appl. Math.* **13** (1960), 685–697.

[27] K. Sekigawa, H. Suga, and L. Vanhecke, Four-dimensional curvature homogeneous spaces, *Commentat. Math. Univ. Carol.* **33** (1992), 261–268.

[28] K. Sekigawa, H. Suga, and L. Vanhecke, Curvature homogeneity for four-dimensional manifolds, *J. Korean Math. Soc.* **32** (1995), 93–101.

[29] H. Takagi, On curvature homogeneity of Riemannian manifolds, *Tohoku Math. J.* **26** (1974), 581–585.

[30] A. Tomassini, Curvature homogeneous metrics on principal fibre bundles, *Ann. Mat. Pura Appl.* **172** (1997), 287–295.

[31] F. Tricerri, Riemannian manifolds with the same curvature as a homogeneous space, and a conjecture of Gromov, *Geometry Conference (Parma, 1988)*. *Riv. Mat. Univ. Parma* (4) **14** (1988), 91–104.

[32] F. Tricerri and L. Vanhecke, Variétés riemanniennes dont le tenseur de courbure est celui d’un espace symétrique riemannien irréductible, *C. R. Acad. Sci., Paris, Sér. I* **302** (1986), 233–235.

[33] K. Tsukada, Curvature homogeneous hypersurfaces immersed in a real space form, *Tohoku Math. J.* **40** (1988), 221–244.

[34] L. Vanhecke, Curvature homogeneity and related problems, *Proceedings of the Workshop on Recent Topics in Differential Geometry* (Puerto de la Cruz, 1990), 103–122, Informes, 32, Univ. La Laguna, La Laguna, 1991.

[35] K. Yamato, Algebraic Riemann manifolds, *Nagoya Math. J.* **115** (1989), 87–104.

---

PG: *Mathematics Department, University of Oregon, Eugene OR 97403 USA.*

Email: gilkey@darkwing.uoregon.edu

SN: *Mathematical Institute, SANU, Knez Mihailova 35, p.p. 367, 11001 Belgrade, Serbia and Montenegro.*

Email: stanan@mi.sanu.ac.yu