A NOTE ON PERIODS

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Abstract. We construct a period regulator for motivic cohomology of an algebraic scheme over a subfield of the complex numbers. For the field of algebraic numbers we formulate a period conjecture, generalising Grothendieck period conjecture, by saying that this period regulator is surjective. By proving that a suitable Betti–de Rham realization of 1-motives is fully faithful we can verify the period conjecture in several cases.

Introduction

Let $X$ be a scheme which is separated and of finite type over a subfield $K$ of the complex numbers, for a chosen embedding $\sigma : K \rightarrow \mathbb{C}$. Consider the $q$-twisted singular cohomology $H^p(X_{an}, \mathbb{Z}_{an}(q))$ of the analytic space $X_{an}$ obtained by the $\mathbb{C}$-points of $X$ and the $p$th de Rham cohomology $H^p_{dR}(X)$, which is an algebraically defined $K$-vector space. We have the following natural $\mathbb{C}$-linear isomorphism

$$\varpi^p_X : H^p(X_{an}, \mathbb{Z}_{an}(q)) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^p_{dR}(X) \otimes_K \mathbb{C}$$

providing a comparison between these cohomology theories. As Grothendieck originally remarked, for $X$ defined over the field of algebraic numbers $K = \overline{\mathbb{Q}}$ or a number field, the position of the whole $H^p_{dR}(X)$ with respect to $H^p(X_{an}, \mathbb{Z}_{an}(q))$ under $\varpi^p_X$ "yields an interesting arithmetic invariant, generalizing the "periods" of regular differential forms" (see [20, p. 101 & footnotes (9) and (10)], cf. [3, §7.5 & Chap. 23], [35], [15], [16] and [22, Chap. 5 & 13]).

The main goal of this paper is to describe this arithmetic invariant, at least for $p = 1$ and all twists, notably, $q = 1$ and $q = 0$. In more details, we first reconstruct $\varpi^p_X$ (in Definition 1.2.4) by making use of Ayoub’s period isomorphism (see Lemma 1.2.2) in Voevodsky’s triangulated category $\text{DM}^{eff}_{et}$ of motivic complexes for the étale topology. Denote by $H^p_{\omega}(X)$ the named arithmetic invariant, i.e., the subgroup of those cohomology classes in $H^p(X_{an}, \mathbb{Z}_{an}(q))$ which are landing in $H^p_{dR}(X)$ via $\varpi^p_X$. We then show the existence of a regulator map (see Corollary 1.2.6 and Definition 1.2.7)

$$r^p,q_{\omega} : H^p,q_{\omega}(X) \rightarrow H^p,q_{\omega}(X)$$

from étale motivic cohomology groups $H^p,q_{\omega}(X)$. We here regard motivic cohomology canonically identified with $H^p_{\text{\acute{e}t}}(X, \mathbb{Z}(q))$ where $\mathbb{Z}(q)$ is the Suslin-Voevodsky motivic complex (see [27, Def. 3.1]), as a complex of sheaves for the \acute{e}t-topology (introduced in [12, §10.2]). Note that here we are mostly interested in the case of $q = 0,1$ so that $\mathbb{Z}(0) \cong \mathbb{Z}[0]$ and $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ by a theorem of Voevodsky (see [27, Thm. 4.1]).

Following Grothendieck’s idea, we conjecture that the period regulator $r^p,q_{\omega}$ is surjective over $\overline{\mathbb{Q}}$ and we actually show some evidence. We easily see that $H^0_{\omega}(X) = 0$ for $q \neq 0$ and $r^0_{\omega}$ is an isomorphism; therefore, the first non-trivial case is for $p = 1$. Moreover, by making use of Suslin-Voevodsky rigidity theorem we can show that $r^p,q_{\omega}$ is surjective on

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torsion (see Lemma 1.4.2). We can also show: if the vanishing $H^{p,q}(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ holds true then the surjectivity of $r_{\omega}^{p,q}$ is equivalent to the vanishing $H^{p}_{\text{dR}}(X) \cap H^{p}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(q)) = 0$. The divisibility properties of motivic cohomology (see [23]) imply that our conjecture is a neat generalization of the classical period conjecture for smooth and proper schemes (see Proposition 1.4.4).

In order to study the case of $p = 1$ we can make use of the description of $H^{1}$ via the Albanese 1-motive $L_{1}\text{Alb}(X)$. Recall the existence of the homological motivic Albanese complex $L_{\text{Alb}}(X)$, a complex of 1-motives whose $p$th homology $L_{p}\text{Alb}(X)$ is a 1-motive with cotorsion (see [12] for details). We can regard complexes of 1-motives as objects of $\text{DM}^{\text{eff}}$ and by the adjunction properties of $\text{LAlb}$ (proven in [12, Thm. 6.2.1]) we have a natural map

$$\text{Ext}^{p}(L_{\text{Alb}}(X), \mathbb{Z}(1)) \to H^{p,1}(X) \cong H^{p-1}_{\text{ch}}(X, \mathbb{G}_{m})$$

which is an isomorphism, rationally, for all $p$ (see the motivic Albanese map displayed in (3.2) and (3.3) below). We can also describe periods for 1-motives (see Definition 2.2.1) in such a way that we obtain suitable Betti-de Rham realizations in period categories (see Definitions 2.4.4 and 2.4.1): a key point is that these realizations are fully faithful over $\overline{\mathbb{Q}}$ (see Theorem 2.6.1). The main ingredient in the proof of the fullness is a theorem due to Waldschmidt [32, Thm. 5.2.1] in transcendence theory generalizing the classical Schneider-Lang theorem (see also [15, Thm. 4.2]). An alternative proof can be given using a theorem of Wüstholz [34].

Actually, we show that the regulator $r_{\omega}^{1,1}$ can be revisited by making use of 1-motives (see Lemmas 3.2.1 - 3.2.3 and Proposition 3.2.7). As a byproduct, all this promptly applies to show the surjectivity of $r_{\omega}^{1,1} : H^{0}_{\text{ch}}(X, \mathbb{G}_{m}) \to H^{1}_{\text{ch}}(X)$ via $\text{Ext}(L_{\text{Alb}}(X), \mathbb{Z}(1))$ verifying the conjecture for $p = 1$ and $q = 1$ (see Theorem 3.2.4). Moreover, considering the 1-motive $R^{1}\text{Pic}(X) = [L_{1} \xrightarrow{u_{1}} G_{1}]$ which is the Cartier dual of $L_{1}\text{Alb}(X) = [L_{1} \xrightarrow{u_{1}} G_{1}]$ we have a canonical isomorphism

$$\text{Ker} u_{1}^{*} \cong H^{1}_{\text{dR}}(X) \cap H^{1}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(1)) = H^{1}_{\omega}(X).$$

In particular, we obtain that $H^{1}_{\text{dR}}(X) \cap H^{1}(X_{\text{an}}, \mathbb{Z}(1)) = 0$ if $X$ is proper. With some more efforts, making now use of the motivic complex $L_{p}\text{Alb}(X)$ along with its adjunction property (as stated in [12, §5.4]), we get a map

$$\text{Ext}^{p}(L_{p}\text{Alb}(X), \mathbb{Z}) \to H^{p,0}(X) \cong H^{p}_{\text{ch}}(X, \mathbb{Z}).$$

Analysing the composition of this map for $p = 1$ with $r_{\omega}^{1,0}$ we see that

$$r_{\omega}^{1,0} : H^{1}_{\text{ch}}(X, \mathbb{Z}) \cong H^{1,1}_{\omega}(X) = H^{1}_{\text{ch}}(X) \cap H^{1}(X_{\text{an}}, \mathbb{Z}_{\text{an}})$$

is an isomorphism (see Theorem 3.3.1), which is yielding the case $p = 1$ and $q = 0$ of our conjecture. In particular, $H^{1}_{\text{dR}}(X) \cap H^{1}(X_{\text{an}}, \mathbb{Z}_{\text{an}}) = 0$ for $X$ normal.

For $p = 1$ and $q \neq 0, 1$ we have that $H^{1,q}_{\omega}(X) = 0$ (see Corollary 3.4.2) so that the period conjecture is trivially verified.

Remarkably, the description of Grothendieck arithmetic invariants $H^{p,q}_{\omega}(X)$ appears strongly related to the geometric properties encoded by motivic cohomology. These properties are almost hidden for smooth schemes, since the divisibility properties of motivic cohomology of $X$ smooth yields that for $p \notin [q, 2q]$ the surjectivity of $r_{\omega}^{p,q}$ is equivalent to the vanishing $H^{p}_{\text{dR}}(X) \cap H^{p}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(q)) = 0$. However, for $X$ smooth with a smooth compactification $\overline{X}$ and normal crossing boundary $Y$, we have that

$$\text{Ker} (\text{Div}_{Y}(\overline{X}) \xrightarrow{u_{1}} \text{Pic}_{\overline{X}/\mathbb{Q}}^{0}) \cong H^{1}_{\text{dR}}(X) \cap H^{1}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(1))$$
where $u_1^*$ is the canonical mapping sending a divisor $D$ supported on $Y$ to $\mathcal{O}_X(D)$. In fact, here $\mathbb{R}^1\mathrm{Pic}(X)$ is Cartier dual of $\mathrm{L}_1\mathrm{Alb}(X) = [0 \to \mathcal{A}^0_{X/Q}]$, the Serre-Albanese semi-abelian variety (see [12, Chap. 9]). Note that there exist smooth schemes $X$ such that $H^{1,1}_{\overline{\omega}}(X)$ can be non-zero and the vanishings in [16, Thm. 4.1 & 4.2] are particular instances of our descriptions.

With similar techniques one can make use of the Borel-Moore Albanese complex $\mathrm{L}\mathrm{Alb}^c(X)$ (see [12, Def. 8.7.1]) to describe the compactly supported variant $H^{1,1}_{\overline{\omega}}(X)$, for any twist $q$.

Finally, the cohomological Albanese complex $\mathrm{L}\mathrm{Alb}^c(X)$ (see [12, Def. 8.6.2]) shall be providing a description of $H^{2d-j-q}_{\overline{\omega}}(X)$ for $d = \dim(X)$, at least for $j = 0, 1$ and $q$ an arbitrary twist. An homological version of period regulators is also feasible and shall be discussed in a future work.

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1. Periods

Let $\mathrm{DM}^{\text{eff}}_{\text{et}}$ be the effective (unbounded) triangulated category of Voevodsky motivic complexes of $\tau$-sheaves over a field $K$ of zero characteristic, i.e., the full triangulated subcategory of $D(\mathrm{Shv}_{\tau}^b(\mathrm{Sm}_K))$ given by $\mathbb{A}^1$-local complexes (e.g. see [6, §4.1] and, for complexes bounded above, see also [27, Lect. 14]). Let $Z(q)$ for $q \geq 0$ be the Suslin-Voevodsky motivic complex regarded as a complex of étale sheaves with transfers. More precisely we consider a change of topology tensor functor

$$\alpha : \mathrm{DM}^{\text{eff}}_{\mathrm{Nis}} \to \mathrm{DM}^{\text{eff}}_{\text{et}}$$

and $Z(q) = \alpha Z_{\mathrm{Nis}}(q)$ (see [12, Cor. 1.8.5 & Def. 1.8.6]) where $Z_{\mathrm{Nis}}(q)$ is the usual complex for the Nisnevich topology (see also [29, Def. 3.1]). We have the following canonical quasi-isomorphims $Z(0) \cong Z[0]$, $Z(1) \cong \mathbb{G}_m[-1]$ and $Z(q) \otimes Z(q') \cong Z(q + q')$ for any $q, q' \geq 0$ (see [29, Lemma 3.2]). For any object $M \in \mathrm{DM}^{\text{eff}}_{\text{et}}$ we here denote $M := M \otimes Z(q)$. Recall that by inverting the Tate twist $M \sim M(1)$ we obtain $\mathrm{DM}^{\tau}$ (where every compact object is isomorphic to $M(-n)$ for some $n \geq 0$ and $M$ compact and effective). For $M \in \mathrm{DM}^{\text{eff}}_{\text{et}}$ we shall denote

$$H^{p,q}(M) := \operatorname{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{et}}}(M, Z(q)[p]).$$

For any algebraic scheme $X$ we have the Voevodsky étale motive $M(X) = \alpha C_* Z_{\text{et}}(X) \in \mathrm{DM}^{\text{eff}}_{\text{et}}$ where $C_*$ is the Suslin complex and $Z_{\text{et}}(X)$ is the representable Nisnevich sheaf with transfers (see [27, Def. 2.8, 2.14 & Properties 14.5] and compare with [12, Lemma 1.8.7 & Sect. 8.1]). Étale motivic cohomology is

$$H^{p,q}(X) := \operatorname{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{et}}}(M(X), Z(q)[p]) \cong H^{p}_{\text{et}}(X, Z(q))$$

where this cohomology is, in general, computed by the $\overline{\text{et}}$-topology (see [12, §10.2] and cf. [6] and [29, Prop. 1.8 & Def. 3.1]). In particular, if $X$ is smooth $H^{p}_{\overline{\text{et}}}(X, Z(q)) \cong H^{p}_{\overline{\text{et}}}(X, Z(q))$.

Note that we also have the triangulated category of motivic complexes without transfers $\mathrm{DA}^{\text{eff}}_{\text{et}}$ and if we are interested in rational coefficients we may forget transfers or keep the Nisnevich topology as we have equivalences

$$\mathrm{DA}^{\text{eff}}_{\text{et}, \mathbb{Q}} \cong \mathrm{DM}^{\text{eff}}_{\text{et}, \mathbb{Q}} \cong \mathrm{DM}^{\text{eff}}_{\mathrm{Nis}, \mathbb{Q}}$$

(see [6], [7, Cor. B.14] and [27, Thm. 14.30]).
1.1. de Rham regulator. Denote by $\Omega$ the object of $\text{DM}^\text{eff}_{\text{et}}$ which represents de Rham cohomology. More precisely we here denote $\Omega := \alpha \Omega_{\text{Nis}}$ where $\Omega_{\text{Nis}}$ is the corresponding object for the Nisnevich topology (see [26, §2.1] and cf. [7, §2.3] without transfers). This latter $\Omega_{\text{Nis}}$ is given by the complex of presheaves with transfers that associates to $X \in \text{Sm}_K$ the global section $\Gamma(X, \Omega^\bullet_{X/K})$ of the usual algebraic de Rham complex.

For $M \in \text{DM}^\text{eff}_{\text{et}}$ we shall denote (cf. [25, §6] and [26, Def. 2.1.1 & Lemma 2.1.2])

$$H^p_{\text{dr}}(M) := \text{Hom}_{\text{DM}^\text{eff}_{\text{et}}}(M, \Omega[p]).$$

For any algebraic scheme $X$ and $M = M(X)$ we here may also consider the sheafification of $\Omega$ for the $\acute{e}$tale topology. Actually, we set

$$H^p_{\text{dr}}(X) := \text{Hom}_{\text{DM}^\text{eff}_{\text{et}}}(M(X), \Omega[p]) \cong H^p_{\text{et}}(X, \Omega)$$

(see [12, Prop. 10.2.3]). Note that for $q = 0$ we have a canonical map $r^0 : \mathbb{Z}(0) \to \Omega$ yielding a map

$$H^{p,0}(X) \cong H^p_{\text{et}}(X, \mathbb{Z}) \to H^p_{\text{dr}}(X) \cong H^p_{\text{et}}(X, \Omega).$$

For $q = 1$ we have $r^1 := d \log : \mathbb{Z}(1) \to \Omega$ in $\text{DM}^\text{eff}_{\text{et}}$ (see [26, Lemme 2.1.3] for the Nisnevich topology and apply $\alpha$) yielding a map

$$H^{p,1}(X) \cong H^p_{\text{et}}(X, \mathbb{G}_m) \to H^p_{\text{dr}}(X) \cong H^p_{\text{et}}(X, \Omega).$$

Following [26, (2.1.5)] an internal de Rham regulator $r^q$ in $\text{DM}^\text{eff}_{\text{et}}$ for $q \geq 2$ is then obtained as the composition of

$$r^q : \mathbb{Z}(q) \cong \mathbb{Z}(q-1) \otimes \mathbb{Z}(1) \cong \mathbb{Z}(1)^{\otimes q} \xrightarrow{d \log^{\otimes q}} \Omega^{\otimes q} \to \Omega.$$

For $M \in \text{DM}^\text{eff}_{\text{et}}$, composing a map $M \to \mathbb{Z}(q)[p]$ with $r^q[p]$ we then get an external de Rham regulator map

$$r^{p,q}_{\text{dr}} : H^{p,q}(M) \to H^p_{\text{dr}}(M)$$

and in particular for $M = M(X)$ we get

$$r^{p,q}_{\text{dr}} : H^{p,q}(X) \cong H^p_{\text{et}}(X, \mathbb{Z}(q)) \to H^p_{\text{dr}}(X) \cong H^p_{\text{et}}(X, \Omega).$$

Note that if $X$ is smooth then $H^p_{\text{et}}(X, \Omega) \cong H^p_{\text{et}}(X, \Omega) \cong H^p_{\text{zar}}(X, \Omega_X)$ coincides with the classical algebraic de Rham cohomology (again, see [12, Prop. 10.2.3] and cf. [22, Prop. 3.2.4]) and we thus obtain $r^{p,q}_{\text{dr}} : H^{p,q}_{\text{et}}(X, \mathbb{Z}(q)) \to H^p_{\text{zar}}(X, \Omega_X)$ in this case.

1.2. Periods. As soon as we have an embedding $\sigma : K \hookrightarrow \mathbb{C}$ we may consider a Betti realization (e.g. see [26, §3.3] or [9, Def. 2.1]) in the derived category of abelian groups $D(\mathbb{Z})$ as a triangulated functor

$$\beta_\sigma : \text{DM}^\text{eff}_{\text{et}} \to D(\mathbb{Z})$$

such that $\beta_\sigma(\mathbb{Z}(q)) \cong \mathbb{Z}_{\text{an}}(q) := (2\pi i)^q \mathbb{Z}$. Actually, following Ayoub (see also [7, §2.1.2] and [8, §1.1.2]) if we consider the analogue of the Voevodsky motivic category $\text{DM}^\text{eff}_{\text{an}}$ obtained as the full subcategory of $D(\text{Shv}^\text{md}_{\text{an}}(\text{An}_C))$ given by $\mathbb{A}^1$-local complexes, where we here replace smooth schemes $\text{Sm}_K$ by the category $\text{An}_C$ of complex analytic manifolds, we get an equivalence

$$\beta : \text{DM}^\text{eff}_{\text{an}} \xrightarrow{\sim} D(\mathbb{Z})$$

such that $M_{\text{an}}(X) \sim Sing_*(X)$ is sent to the singular chain complex of $X \in \text{An}_C$. Moreover there is a natural triangulated functor

$$\sigma : \text{DM}^\text{eff}_{\text{et}} \to \text{DM}^\text{eff}_{\text{an}}$$
such that \( M(X) \cong M_{an}(X_{an}) \) where the analytic space \( X_{an} \) is given by the \( \mathbb{C} \)-points of the base change \( X_{\mathbb{C}} \) of any algebraic scheme \( X \). We then set \( \beta_\sigma := \beta \circ \sigma \). Thus it is clear that 
\[
\beta_\sigma(\mathbb{Z}[0]) = \beta_\sigma(M(\text{Spec}(K))) = \mathbb{Z}[0].
\]
Since a \( K \)-rational point of \( X \) yields \( M(X) = \mathbb{Z} \oplus M(X) \) we also see that 
\[
\beta_\sigma(\mathbb{Z}(1)[1]) \cong \beta_\sigma(M(\mathbb{G}_m)) \cong \beta(\tilde{M}(\mathbb{G}_m)) \cong \mathbb{Z}_{an}(1)[1] \text{ and then, } \beta_\sigma(\mathbb{Z}(q)) \cong \mathbb{Z}_{an}(q) \text{ in general, as it follows from the compatibility of } \beta_\sigma \text{ with the tensor structures, i.e.,}
\]
we here use the fact that \( \beta_\sigma \) is unital and monoidal. For \( \beta \in \text{DM}_{\text{eff}}(\mathbb{Z}) \), we denote
\[
H^{p,q}_{\text{an}}(M) := \text{Hom}_{\text{DM}(\mathbb{Z})}(\beta_\sigma M, \mathbb{Z}_{an}(q)[p])
\]
and we have a Betti regulator map induced by \( \beta_\sigma \)
\[
r^{p,q}_{\text{an}} : H^{p,q}_{\text{an}}(M) \to H^{p,q}_{\text{an}}(M).
\]
In particular, for \( M = M(X) \) we obtain:

1.2.1. Lemma. For any algebraic \( K \)-scheme \( X \) and any field homomorphism \( \sigma : K \to \mathbb{C} \) we have
\[
H^{p,q}_{\text{an}}(X) := \text{Hom}_{\text{DM}(\mathbb{Z})}(\beta_\sigma M(X), \mathbb{Z}_{an}(q)[p]) \cong H^p(X_{an}, \mathbb{Z}_{an}(q))
\]
and a Betti regulator map
\[
r^{p,q}_{\text{an}} : H^{p,q}_{\text{an}}(X) \to H^p(X_{an}, \mathbb{Z}_{an}(q)).
\]
Proof. This directly follows from Ayoub’s construction (see also [26, Prop. 4.2.7]). □

Recall that the functor \( \beta_\sigma \) admits a right adjoint \( \beta^\sigma : \text{DM}(\mathbb{Z}) \to \text{DM}_{\text{eff}}(\mathbb{Z}) \) (see [8, Def. 1.7]). Note that the Betti regulator (1.4) is just given by composition with the unit
\[
r^\sigma : \mathbb{Z}(q) \to \beta^\sigma_\beta(\mathbb{Z}(q))
\]
of the adjunction. Actually, by making use of the classical Poincaré Lemma and Grothendieck comparison theorem ([20, Thm. 1']) we get:

1.2.2. Lemma (Ayoub). There is a canonical quasi-isomorphism
\[
\varpi^q : \beta^\sigma_\beta(\mathbb{Z}(q)) \otimes_{\mathbb{Z}_K} \Omega \to \mathbb{C}
\]
whose composition with \( r^\sigma_\beta \) in (1.5) is the regulator \( r^\sigma \) in (1.1) after tensoring with \( \mathbb{C} \).

Proof. See [7, Cor. 2.89 & Prop. 2.92] and also [5, §3.5]. □

1.2.3. Remark. Note that applying \( \beta_\sigma \) to \( \varpi^q \) we obtain a q.i. \( \beta_\sigma(\varpi^q) \) such that
\[
\mathbb{C} \cong \beta_\sigma(\mathbb{Z}(q)) \otimes_{\mathbb{Z}_K} \mathbb{C} \xrightarrow{\beta_\sigma(r^\sigma_\beta)} \beta_\sigma^g(\beta_\sigma(\mathbb{Z}(q)) \otimes_{\mathbb{Z}_K} \mathbb{C}) \xrightarrow{\beta_\sigma(\varpi^q)} \beta_\sigma(\Omega) \otimes_{\mathbb{C}_K} \mathbb{C}
\]
where \( \beta_\sigma(r^\sigma_\beta) \) is a split injection but it is not a q.i. (cf. [26, §4.1]).

For \( \beta \in \text{DM}_{\text{eff}}(\mathbb{Z}) \), by composition with \( \varpi^q \) we get a period isomorphism
\[
\varpi^{p,q}_M : H^{p,q}_{\text{an}}(M) \otimes_{\mathbb{Z}_K} \mathbb{C} \xrightarrow{\sim} H^{p,q}_{\text{DR}}(M) \otimes_{\mathbb{C}_K} \mathbb{C}.
\]

1.2.4. Definition. For any scheme \( X \) we shall call period isomorphism the \( \mathbb{C} \)-isomorphism
\[
\varpi^{p,q}_X : H^p(X_{an}, \mathbb{Z}_{an}(q)) \otimes_{\mathbb{Z}_K} \mathbb{C} \xrightarrow{\sim} H^p_{\text{DR}}(X) \otimes_{\mathbb{C}_K} \mathbb{C}
\]
obtained by setting \( \varpi^{p,q}_X := \varpi^{p,q}_M(X) \) as above. We shall denote \( r^{p,q}_X := (\varpi^{p,q}_X)^{-1} \) the inverse of the period isomorphism.

We also get the following compatibility.
1.2.5. Proposition. For \( M \in \text{DM}_\text{eff}^{\text{dR}} \) along with a fixed embedding \( \sigma : K \hookrightarrow \mathbb{C} \) the period isomorphism \( \varpi_{M}^{p,q} \) above induces a commutative diagram

\[
\begin{array}{cccc}
H_{\text{dR}}^p(M) & \xrightarrow{r_{\text{dR}}^{p,q}} & H_{\text{dR}}^{p,q}(M) & \xrightarrow{r_{\text{an}}^{p,q}} & H_{\text{an}}^q(M) \\
\downarrow{r_{\text{dR}}^{p,q}} & & \downarrow{r_{\text{an}}^{p,q}} & & \\
H_{\text{dR}}^{p,q}(M) & \xrightarrow{\varpi_{\text{dR}}^{p,q}} & H_{\text{dR}}^{p,q}(M) \otimes_K \mathbb{C} \cong H_{\text{an}}^{p,q}(M) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}
\]

Proof. This easily follows from Lemma 1.2.2. In fact, by construction, the claimed commutative diagram can be translated into the following commutative square:

\[
\begin{array}{cccc}
\text{Hom}_{\text{DM}}(M, \mathcal{Z}(q)[p]) & \xrightarrow{r_{\text{an}}^{p,q} \circ r_{\text{dR}}^{p,q}} & \text{Hom}_{\text{DM}}(M, \beta^p \beta_q \mathcal{Z}(q)[p])_{\mathbb{C}} & \xrightarrow{\varpi_{\text{dR}}^{p,q} \circ} \text{Hom}_{\text{DM}}(M, \Omega[p]) \xrightarrow{r_{\text{an}}^{p,q}} \text{Hom}_{\text{DM}}(M, \Omega[p])_{\mathbb{C}}
\end{array}
\]

\[\square\]

1.2.6. Corollary. Let \( X \) be an algebraic \( K \)-scheme along with a fixed embedding \( \sigma : K \hookrightarrow \mathbb{C} \). The period isomorphism \( \varpi_{X}^{p,q} \) above induces a commutative square

\[
\begin{array}{cccc}
H_{\text{dR}}^{p,q}(X) & \xrightarrow{r_{\text{dR}}^{p,q}} & H^{p} \left( X_{\text{an}}, \mathbb{Z}_{\text{an}}(q) \right) & \\
\downarrow{r_{\text{dR}}^{p,q}} & & \downarrow{r_{\text{an}}^{p,q}} & \\
H_{\text{dR}}^p(X) & \xrightarrow{\varpi_{\text{dR}}^{p,q}} & H^p(X_{\text{an}}, \mathbb{C}) \cong H^{p} \left( X_{\text{an}}, \mathbb{C} \right)
\end{array}
\]

Note that from the Corollary 1.2.6 we get a refinement of the Betti regulator.

1.2.7. Definition. Define the algebraic singular cohomology classes as the elements of the subgroup \( H_{\text{alg}}^{p,q}(X) := \text{Im} \ r_{\text{an}}^{p,q} \subseteq H^p \left( X_{\text{an}}, \mathbb{Z}_{\text{an}}(q) \right) \) given by the image of the motivic cohomology under the Betti regulator \( r_{\text{an}}^{p,q} \).

Define the \( \varpi \)-algebraic singular cohomology classes by the subgroup

\[
H_{\varpi}^{p,q}(X) := H_{\text{dR}}^p(X) \cap H^p \left( X_{\text{an}}, \mathbb{Z}_{\text{an}}(q) \right) \subseteq H^p \left( X_{\text{an}}, \mathbb{Z}_{\text{an}}(q) \right)
\]

where \( \cap \) means that we take elements in \( H^p \left( X_{\text{an}}, \mathbb{Z}_{\text{an}}(q) \right) \) which are given by the inverse image (under \( r_{\text{an}}^{p,q} \)) of elements in \( H_{\text{dR}}^p(X) \) regarded (under \( r_{\text{an}}^{p,q} \)) inside \( H^p \left( X_{\text{an}}, \mathbb{C} \right) \) via the isomorphism \( \varpi_{X}^{p,q} \) above.

The groups \( H_{\varpi}^{p,q}(X) \) shall be called period cohomology groups and

\[
r_{\varpi}^{p,q} : H^{p,q}(X) \rightarrow H_{\varpi}^{p,q}(X)
\]

induced by \( r_{\text{an}}^{p,q} \) and \( r_{\text{an}}^{p,q} \) shall be called the period regulator.

We get that:

1.2.8. Corollary. \( H_{\text{alg}}^{p,q}(X) \subseteq H_{\varpi}^{p,q}(X) \).

For example, all torsion cohomology classes are \( \varpi \)-algebraic: we shall see in Lemma 1.4.2 that are also algebraic.
In particular, if \( H^p(X_{\an}, \mathbb{Z}_{\an}(q)) \) is all algebraic, i.e., the Betti regulator \( r_{\text{B}}^{p,q} \) is surjective, then the canonical embedding \( \iota_{\text{an}}^{p,q} \) of singular cohomology \( H^p(X_{\an}, \mathbb{Q}_{\an}(q)) \) in the \( \mathbb{C} \)-vector space \( H^p(X_{\an}, \mathbb{C}) \) factors through an embedding \( H^p(X_{\an}, \mathbb{Q}_{\an}(q)) \) into the \( K \)-vector space \( H^p_{\text{DR}}(X) \). If \( K = \overline{\mathbb{Q}} \) this rarely happens. For example, if \( p = 0 \) it happens only if \( q = 0 \) and in this case \( r_{\text{an}}^{0,q} \) is always surjective (as \( H^0_{\text{an}}(X) = 0 \) for \( q \neq 0 \)).

1.3. Period Conjecture. Over \( K = \overline{\mathbb{Q}} \) it seems reasonable to make the conjecture that all \( \varpi \)-algebraic classes are algebraic, i.e., to conjecture that the period regulator \( r_{\varpi}^{p,q} \) is surjective. In other words we may say that the period conjecture holds for \( X \), in degree \( p \) and twist \( q \) if

\[
H^p_{\text{alg}}(X) = H^p_{\varpi}(X).
\]

Over a number field we may expect that this holds rationally. If (1.6) holds we also have that \( H^p(X_{\an}, \mathbb{Z}_{\an}(q)) \) modulo torsion embeds into \( H^p_{\text{DR}}(X) \) if and only if \( H^p(X_{\an}, \mathbb{Z}_{\an}(q)) \) is all algebraic. Note that using Proposition 1.2.5 we can define \( H^p_{\varpi}(M) \) providing a version of the period conjecture for any object \( M \in \text{DM}^{\text{eff}}_{\text{et}} \).

1.3.1. Proposition. For any \( q \geq 0 \) the period conjecture (1.6) holds true for \( X \), in degree \( p \) and twist \( r \) if and only if it holds true for \( M(X)(q) \), in degree \( p \) and twist \( q + r \).

Proof. By Voevodsky cancellation theorem [30] we have that twisting by \( q \) in motivic cohomology \( H^{p+r}(X) \xrightarrow{\varpi^r} H^{p+q+r}(M(X)(q)) \) is an isomorphism of groups. If \( M = M(X)(q) \) with \( q \geq 0 \) we also get \( H^p_{\text{DR}}(X) \xrightarrow{\varpi^r} H^p_{\varpi}(M(X)(q)) \) canonically by twisting. In fact, we have a diagram induced by twisting

\[
\begin{array}{ccc}
H^p(X_{\an}, \mathbb{Z}_{\an}(r)) \otimes \mathbb{C} & \xrightarrow{q_c} & H^p_{\text{an}}(M(X)(q)) \otimes_{\mathbb{Z}} \mathbb{C} \\
\varpi^p \downarrow & & \varpi_{M(X)(q)} \downarrow \\
H^p_{\text{DR}}(\varpi) \otimes_K \mathbb{C} & \xrightarrow{q_{dr}} & H^p_{\text{DR}}(M(X)(q)) \otimes_K \mathbb{C}
\end{array}
\]

where \( q_c := q \otimes \mathbb{C} \) is the \( \mathbb{C} \)-isomorphism given by the canonical integrally defined mapping \( q: H^p(X_{\an}, \mathbb{Z}_{\an}(r)) \xrightarrow{\varpi^r} H^p_{\text{an}}(M(X)(q)) \) which is sending a \( p \)-th cohomology class regarded as a map \( \beta_{q}M(X) = \text{Sing}_{q}(X_{\an}) \rightarrow \mathbb{Z}_{\an}(r)[p] \) in \( D(\mathbb{Z}) \) to the \( q \)-twist \( \beta_{q}M(X)(q) = \text{Sing}_{q}(X_{\an})(q) \rightarrow \mathbb{Z}_{\an}(q+r)[p] \). Similarly, the \( \mathbb{C} \)-isomorphism \( q_{dr} \) is induced by twisting. Note that when we twist the period q.i. \( \varpi^r \) by \( q \) we get the period q.i. \( \varpi^{q+r} \) as the composition of \( \beta^q \beta_{q}(\mathbb{Z}(q + r)) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \Omega(q) \otimes_{\mathbb{C}} \mathbb{C} \rightarrow \Omega \otimes_{\mathbb{C}} \mathbb{C} \) where \( \Omega(q) \xrightarrow{q_{\Omega}} \Omega \) is a canonical isomorphism in \( \text{DM}^{\text{eff}}_{\text{et}} \). The claim follows by diagram chase. \[\square\]

For \( X \) smooth we have that \( H^p_{\varpi}(X) \cong H^p_{\text{et}}(X, \mathbb{Z}(q)) \) and with rational coefficients we have that \( H^p_{\text{et}}(X, \mathbb{Q}(q)) \cong CH^q(X, 2q - p)_{\mathbb{Q}} \). In particular, if \( X \) is smooth and \( p = 2q \) we get that \( r_{\varpi}^{2q,q} \) is the modern refinement of the classical cycle map with rational coefficients

\[
r_{\varpi}^{2q,q} = c_{\varpi}^{2q, q} : CH^q(X)_{\mathbb{Q}} \rightarrow H^q_{\varpi}(X)_{\mathbb{Q}}
\]

for codimension \( q \) cycles on \( X \) considered in [16]. In this case, the period conjecture (1.6) with rational coefficients coincides with the classical Grothendieck period conjecture.

1.3.2. Remark. For \( K = \mathbb{C} \) we may also think to refine the Hodge conjecture as previously hinted by Beilinson, conjecturing the surjectivity of

\[
r_{\text{Hodge}}^{p,q} : H^p_{\varpi}(X)_{\mathbb{Q}} \rightarrow \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^p(X)(q))
\]

However, such a generalization of the Hodge conjecture doesn’t hold, in general, e.g. see [19].
1.4. Torsion cohomology classes are algebraic. Consider $\mathbb{Z}/n(q) := \mathbb{Z}(q) \otimes \mathbb{Z}/n$. By Suslin-Voevodsky rigidity we have a quasi-isomorphism of complexes of étale sheaves $\mu_n^{\otimes q} \to \mathbb{Z}/n(q)$ yielding $H^p(X, \mathbb{Z}/n(q)) \cong H^p_{\text{ét}}(X, \mu_n^{\otimes q}) \cong H^p_{\text{dR}}(X, \mu_n^{\otimes q})$. For a proof of this key result see [27, Thm. 10.2 & Prop. 10.7] for $X$ smooth and make use of [12, Prop. 12.1.1] to get it in general.

1.4.1. Lemma. For any algebraic scheme $X$ over $K = \overline{K} \hookrightarrow \mathbb{C}$ we have $H^p(X, \mathbb{Z}/n(q)) \cong H^p(X_{\text{an}}, \mathbb{Z}/n)$.

Proof. As étale cohomology of $\mu_n^{\otimes q}$ is invariant under the extension $\sigma : K \hookrightarrow \mathbb{C}$ of algebraically closed fields we obtain the claimed comparison from the classical comparison result after choosing a root of unity. \hfill \Box

We then have (cf. [28, Prop. 3.1]):

1.4.2. Lemma. The regulator $r_{\text{tor}}^{\beta} \mid \text{tor}: H^{p,q}(X)_{\text{tor}} \to H^{p,q}(X)_{\text{tor}}$ is surjective on torsion and $r_{\text{tor}}^{\beta} \otimes \mathbb{Q}/\mathbb{Z} : H^{p,q}(X) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow H^{p,q}(X) \otimes \mathbb{Q}/\mathbb{Z}$ is injective.

Proof. By construction, for any positive integer $n$, comparing the usual universal coefficient exact sequences, we have the following commutative diagram with exact rows

\[
\begin{array}{c}
0 \to H^p(X, \mathbb{Z}(q))/n \longrightarrow H^p(X, \mathbb{Z}/n(q)) \longrightarrow nH^{p+1}(X, \mathbb{Z}(q)) \to 0 \\
\text{1.4.1} \\
0 \to H^p(X_{\text{an}}, \mathbb{Z}_{\text{an}})/n \longrightarrow H^p(X_{\text{an}}, \mathbb{Z}/n) \longrightarrow nH^{p+1}(X_{\text{an}}, \mathbb{Z}) \to 0
\end{array}
\]

Passing to the direct limit on $n$ we easily get the claim. In fact, $nH^{p,q}(X) = nH^p(X_{\text{an}}, \mathbb{Z})$ and $r_{\text{tor}}^{\beta}/n$ factors through $r_{\text{tor}}^{\beta}$. \hfill \Box

1.4.3. Lemma. We have that $r_{\text{tor}}^{\beta} \otimes \mathbb{Q}$ is surjective if and only if $r_{\text{tor}}^{\beta}$ is surjective; moreover, if this is the case $r_{\text{tor}}^{\beta} \otimes \mathbb{Q}/\mathbb{Z}$ is an isomorphism.

Proof. This follows from a simple diagram chase. \hfill \Box

In the situation that $H^{p,q}(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ the period conjecture (1.6) is then equivalent to

\[H^p_{\text{dR}}(X) \cap H^p(X_{\text{an}}, \mathbb{Q}(q)) = 0.\] (1.8)

In particular:

1.4.4. Proposition. If $X$ is smooth then (1.6) for $p \notin [q, 2q]$ is equivalent to (1.8). If $X$ is smooth and proper then (1.6) is equivalent to the surjectivity of $c^{\text{dR}}_p$ in (1.7) for $p = 2q$ and to the vanishing (1.8) for $p \neq 2q$.

Proof. In fact, by [23, Thm. 1.3] we have that for $p \notin [q, 2q]$ the group $H^{p,q}(X)$ is an extension of torsion by divisible groups so that $H^{p,q}(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$. If $X$ is proper the latter vanishing holds true for all $p \neq 2q$. \hfill \Box

The Proposition 1.4.4 explains some weight properties related to the Grothendieck period conjecture, weight arguments which are also considered in [16].

1.4.5. Remark. For $K = \mathbb{C}$ we have that $r_{\beta}^{\text{tor}} \mid \text{tor}: H^{p,q}(X)_{\text{tor}} \to H^p(X_{\text{an}}, \mathbb{Z}_{\text{an}}(q))_{\text{tor}}$ is surjective (as also remarked in [28] for $X$ smooth projective): torsion motivic cohomology classes supply the defect of algebraic cycles providing the missing torsion algebraic cycles. In fact, from the well known Atiyah-Hirzebruch-Totaro counterexamples to the integral Hodge conjecture we know that $c^{\text{dR}}_p : CH^p(X) \to H^{2p}(X_{\text{an}}, \mathbb{Z}(p))$ cannot be surjective on torsion for $p \geq 2$ in general.
2. Periods of 1-motives

Let \( \mathcal{M}_1(K) \) be the abelian category of 1-motives with torsion over \( K \) (see [12, App. C]). We shall drop the reference to \( K \) if it is clear from the context. We shall denote
\[
M_K = [u_K : L_K \to G_K] \in \mathcal{M}_1(K)
\]
a 1-motive with torsion with \( L_K \) in degree 0 and \( G_K \) in degree 1; for brevity, we shall write \( M_K = L_K[0] \) if \( G_K = 0 \) and \( M_K = G_K[-1] \) if \( L_K = 0 \) and we omit the reference to \( K \) if unnecessary. Let \( M_{\text{tor}} := [L_{\text{tor}} \cap \text{Ker}(u) \to 0] \) be the torsion part of \( M_K \), let \( M_{\text{fr}} := [L/L_{\text{tor}} \to G/u(L_{\text{tor}})] \) be the free part of \( M_K \), and let \( M_{\text{tf}} := [L/L_{\text{tor}} \cap \text{Ker}(u) \to G] \) be the torsion free part of \( M_K \). There are short exact sequences of complexes
\[
0 \to M_{\text{tor}} \to M_K \to M_{\text{tf}} \to 0
\]
and
\[
0 \to [F = F] \to M_{\text{tf}} \to M_{\text{fr}} \to 0,
\]
where \( F = L_{\text{tor}}/L_{\text{tor}} \cap \text{Ker}(u) \). Let \( M_{\text{ab}} \) denote the 1-motive with torsion \( [L \to G/T] \) where \( T \) is the maximal subtorus of \( G \). Recall (see [12, Prop. C.7.1]) that the canonical functor \( \mathcal{M}_1 \to \mathcal{M}_1 \) from Deligne 1-motives admits a left adjoint/left inverse given by \( M \rightsquigarrow M_{\text{fr}} \).

We have that \( D^b(M_1) \cong D^b(M_1) \) (see [12, Thm. 1.11.1]) and that there is a canonical embedding (see [12, Def 2.7.1])
\[
\text{Tot} : D^b(M_1) \hookrightarrow D^b_{\text{eff}}
\]
so that we can also regard 1-motives as motivic complexes of étale sheaves. The restriction of the Betti realization \( \beta_\tau \) in (1.3) can be described explicitly for 1-motives via Deligne’s Hodge realization (see [12, Thm. 15.4.1]). Similarly, the restriction of the de Rham realization in [26] can be described via Deligne’s de Rham realization as follows.

2.1. de Rham realization. Let \( K \) be a field of zero characteristic and let \( M_K = [u_K : L_K \to G_K] \in \mathcal{M}_1(K) \) be a 1-motive with torsion over \( K \). Note that for \( M_K^2 := [u_K^2 : L_K \to G_K^2] \) the universal \( G_\alpha \)-extension of \( M_K \) we have
\[
0 \to \mathbb{V}(M) \to M_K^2 \xrightarrow{\rho^M} M_K \to 0
\]
where \( \mathbb{V}(M) := \text{Ext}(M_K, G_\alpha)^{\vee} \). The existence of universal extensions is well-known when \( L_K \) is torsion-free; for the general case see [11, Proposition 2.2.1]. Recall (see [18, §10.1.7]) the following

2.1.1. Definition. The de Rham realization of \( M_K \) is
\[
T_{\text{dR}}(M_K) := \text{Lie}(G_K^2)
\]
as a \( K \)-vector space.

2.1.2. Remark. Note that \( \rho^M = (id_L, \rho_G) \) where \( \rho_G : G_K^2 \to G_K \) is a quotient and \( \text{Ker } \rho_G = \mathbb{V}(M) \) so that \( G_K \) is the semiabelian quotient of \( G_K^2 \) and \( u_K^2 \) is a canonical lifting of \( u_K \), i.e., \( u_K = \rho_G \circ u_K^2 \). Further \( \mathbb{V}(M) \subseteq T_{\text{dR}}(M_K) \) is also the kernel of the morphism
\[
d\rho_G : \text{Lie}(G_K^2) \to \text{Lie}(G_K)
\]
induced by \( \rho_G \), so that \( T_{\text{dR}}(M_K) \) together with the \( K \)-subspace \( \mathbb{V}(M) \) can be regarded as a filtered \( K \)-vector space. This datum is called the Hodge filtration of \( T_{\text{dR}}(M_K) \).
2.1.3. Lemma. For $K \subset K'$ we have a natural isomorphism

$$(M_{K'}^2)_{K'} \cong (M_K)^2.$$ 

2.2. Base change to $\mathbb{C}$ and periods. Consider $K$ a subfield of $\mathbb{C}$ and let $M_{\mathbb{C}} = [u_{\mathbb{C}} : L_{\mathbb{C}} \rightarrow G_{\mathbb{C}}]$ be the base change of $M_K$ to $\mathbb{C}$. Let $T_{\mathbb{C}}(M_{\mathbb{C}})$ be the finitely generated abelian group in the usual Deligne-Hodge realization of $M_{\mathbb{C}}$ (see [18, 10.1.3] and [13, §1]) given by the pull-back

$$
\begin{array}{cccccc}
0 & \rightarrow & H_1(G_{\mathbb{C}}) & \rightarrow & \text{Lie}(G_{\mathbb{C}}) & \exp & G_{\mathbb{C}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T_{\mathbb{C}}(G_{\mathbb{C}}) & \rightarrow & T_{\mathbb{C}}(M_{\mathbb{C}}) & \exp & L_{\mathbb{C}} & \rightarrow & 0 \\
\end{array}
$$

After base change to $\mathbb{C}$ and Lemma 2.1.3 we then get $(M_{K'}^2)_{\mathbb{C}} \cong (M_{\mathbb{C}})^2$ hence an isomorphism

$$\iota : T_{\text{dr}}(M_{\mathbb{C}}) \sim T_{\text{dr}}(M_K) \otimes_K \mathbb{C}$$

and a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & H_1(G_{\mathbb{C}}) & \rightarrow & \text{Lie}(G_{\mathbb{C}}) & \exp & G_{\mathbb{C}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_1(G_{\mathbb{C}}) & \rightarrow & \text{Lie}(G_{\mathbb{C}}) & \exp & G_{\mathbb{C}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T_{\mathbb{C}}(G_{\mathbb{C}}) & \rightarrow & T_{\mathbb{C}}(M_{\mathbb{C}}) & \exp & L_{\mathbb{C}} & \rightarrow & 0 \\
\end{array}
$$

where all the right-hand squares are cartesian and the sequence on the bottom is equivalently obtained by pull–back of the upper sequence via $u_{\mathbb{C}}$ or of the sequence in the middle via $u_{\mathbb{C}}^\iota$. Further $T_{\mathbb{C}}(G_{\mathbb{C}}) \cong H_1(G_{\mathbb{C}}) \cong H_1(G_{\mathbb{C}}^\iota)$ is identified with the kernel of both exponential maps.

2.2.1. Definition. The homomorphism of periods is the unique homomorphism

$$\varpi_{M,Z} : T_{\mathbb{C}}(M_{\mathbb{C}}) \rightarrow T_{\text{dr}}(M_K) \otimes_K \mathbb{C}$$

such that yields $d\rho_{\mathbb{C}} \circ \varpi_{M,Z} = u_{\mathbb{C}}^\iota$ and $\exp \circ \varpi_{M,Z} = u_{\mathbb{C}}^\iota \circ \exp$ under the identification given by the isomorphism $\iota$ in (2.4).

Note that $\tilde{u}_{\mathbb{C}}$ is the pull-back of $u_{\mathbb{C}}$ along $\exp$ and for $x \in L_{\mathbb{C}}$ we may pick $\log(x) \in T_{\mathbb{C}}(M_{\mathbb{C}})$, i.e., such that $\exp(\log(x)) = x$. We then get

$$u_{\mathbb{C}}(x) = \exp(\tilde{u}_{\mathbb{C}}(\log(x))) = \exp(d\rho_{\mathbb{C}}(\varpi_{M,Z}(\log(x))))$$

2.2.2. Theorem. The induced $\mathbb{C}$-linear mapping

$$\varpi_{M,\mathbb{C}} : T_{\mathbb{C}}(M_{\mathbb{C}}) := T_{\mathbb{C}}(M_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \sim T_{\text{dr}}(M_K) \otimes_K \mathbb{C}$$

is invertible.

Proof. Making use of the identification in (2.4) we are left to see that it holds true for $K = \mathbb{C}$. The case of $L$ without torsion is treated by Deligne [18, 10.1.8]. Actually, an easy proof can be given by devissage to the case of lattices, tori and abelian varieties. For the general case note that by (2.1) $\varpi_{M,\mathbb{C}} = \varpi_{M,\text{tor}}$. Indeed $T_{\text{dr}}(M_{\text{tor}}) = 0$ and the kernel of the canonical
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morphism $T_Z(M_C) \to T_Z(M_{tfr,C})$ is torsion. Further by (2.2) the map $T_Z(M_{tfr,C}) \to T_Z(M_{fr,C})$ is an isomorphism and we have an exact sequence

$$0 \to [F = F] \to M^t_{fr} \to M^t_{fr} \to 0$$

so that the canonical morphism $T_{dR}(M_{fr}) \to T_{dR}(M_{fr})$ is an isomorphism too. Hence $\omega_{M_{fr,C}} = \omega_{M_{fr,C}}$. We conclude that $\omega_{M,C} = \omega_{M_{fr,C}}$ and the latter is an isomorphism since $M_{fr}$ is a Deligne 1-motive. \qed

2.2.3. Examples. If $M_K = [0 \to \mathbb{G}_m]$, then $T_Z(M_C) = Z$ and the first and second rows in (2.5) are given by $0 \to \mathbb{Z} \cdot \frac{2\pi i}{2} \mathbb{C} \to \mathbb{C}^* \to 0$. Hence $\omega_{M,K} = \hat{u}_C : Z \to \mathbb{C}, x \mapsto 2\pi ix$ and $\omega_{M,C} : C \to C, z \mapsto 2\pi iz$.

If $M_K = [L_K \to 0]$, then $T_Z(M_C) = L_C$ and $T_{dR}(M_C) = L_C \otimes_{\mathbb{Z}} \mathbb{C}$ and the map $\omega_{M,Z}$ is the homomorphism $L_C \to L_C \otimes_{\mathbb{Z}} \mathbb{C}, x \mapsto x \otimes 1$ and $\omega_{M,C}$ is the identity map.

Note that over $\mathbb{C}$ the Hodge filtration $\mathcal{V}(M) \subseteq T_{dR}(M_K)$ of Remark 2.1.2 is obtained from the Hodge filtration of $T_C(M_C)$ via $\omega_{M,C}$.

2.2.4. Remark. Assume $K = \overline{\mathbb{Q}}$. If $0 \neq x \in \text{Lie}(G_K)$, then $\exp(x) \in G_\mathbb{C}(\mathbb{C})$ is transcendental over $K$. The assertion is clear if $G_K = \mathbb{C}^{d,K}$. For $G_K$ an abelian variety, see [24, Theorem 2]. The general case follows by devissage. In particular Lie$(G_K) \cap \text{Ker}(\exp) = \{0\}$ in Lie$(G_\mathbb{C})$. See also [16, Corollary 3.4].

2.3. Period categories. For a fixed $\sigma : K \hookrightarrow C$ we consider a homological category for Betti-de Rham realizations as follows. Let $\text{Mod}_{Z,K}$ be the following category: (i) objects are triples $(H_Z, H_K, \omega)$ where $H_Z$ is a finitely generated abelian group, $H_K$ is a finite dimensional $K$-vector space, and $\omega : H_Z \to H_K \otimes K \mathbb{C}$ is a homomorphism of groups; (ii) morphisms $\varphi : (H_Z, H_K, \omega) \to (H'_Z, H'_K, \omega')$ are pairs $\varphi := (\varphi_Z, \varphi_K)$ where $\varphi_Z : H_Z \to H'_Z$ is a group homomorphism, $\varphi_K : H_K \to H'_K$ is a $K$-linear homomorphism and $\varphi$ is compatible with $\omega$ and $\omega'$, i.e., the following square

$$
\begin{array}{ccc}
H_Z & \xrightarrow{\omega} & H_K \otimes K \mathbb{C} \\
\varphi_Z & \downarrow & \varphi_K \otimes 1_c \\
H'_Z & \xrightarrow{\omega'} & H'_K \otimes K \mathbb{C}
\end{array}
$$

commutes. For $H = (H_Z, H_K, \omega)$ in $\text{Mod}_{Z,K}$ let

$$
\omega_C : H_Z \otimes_{\mathbb{Z}} \mathbb{C} \to H_K \otimes K \mathbb{C}
$$

be the induced $\mathbb{C}$-linear mapping and denote $\text{Mod}^C_{Z,K}$ the full subcategory of $\text{Mod}_{Z,K}$ given by those objects such that $\omega_C$ is a $\mathbb{C}$-isomorphism.

There is a $\mathbb{Q}$-linear variant $\text{Mod}_{Q,K}$ of this category where objects are $(H_Q, H_K, \omega)$ as above but $H_Q$ is a finite dimensional $\mathbb{Q}$-vector space. Note that $\text{Mod}_{Q,K} \cong \text{Mod}_{Z,K} \otimes \mathbb{Q}$ is the category $\text{Mod}_{Z,K}$ modulo torsion objects (see [12, B.3] for this notion).

2.3.1. Definition. We shall call $\text{Mod}^C_{Z,K}$ (resp. $\text{Mod}^Q_{Z,K}$) the homological periods category (resp. $\mathbb{Q}$-linear category).

Let $\text{Mod}^t_{Z,K}$ (resp. $\text{Mod}^{fr}_{Z,K}$) be the full subcategory of $\text{Mod}^C_{Z,K}$ given by those objects $H$ such that $H_Z$ is free (resp. is torsion). For any $r \in \mathbb{Z}$ we shall denote

$$\mathbb{Z}(r) := (\mathbb{Z}, K, (2\pi i)^r) \in \text{Mod}^t_{Z,K}$$
For $H = (H_Z, H_K, \omega)$ and $H' = (H'_Z, H'_K, \omega')$ we can define
\begin{equation}
H \otimes H' := (H_Z \otimes Z H'_Z, H_K \otimes K H'_K, \omega \otimes \omega')
\end{equation}
and set $H(r) := H \otimes Z(r)$ the Tate twist. For $H \in \text{Mod}^{fr}_{Z,K}$, say that $H = (H_Z, H_K, \omega)$ with $H_Z$ free, we have duals $H^\vee \in \text{Mod}^{fr}_{Z,K}$ given by
\begin{equation}
(H_Z, H_K, \omega)^\vee := (H'_Z, H'_K, \omega')
\end{equation}
where $H'_Z = \text{Hom}(H_Z, Z)$ is the dual abelian group, $H'_K = \text{Hom}(H_K, K)$ is the dual $K$-vector space, and
\[
\omega^\vee : H'_Z \to H'_K \otimes K C
\]
is the composition of the canonical mapping $H'_Z \to H'_Z \otimes Z C$ with the $C$-isomorphism $H'_Z \otimes Z C \cong H'^\vee \otimes K C$ given by the inverse of the $C$-dual of $\omega_C$ in (2.8), i.e., $\omega^\vee(f) = (f \otimes Z id_C) \circ \omega_C^{-1}$ for any $f : H_Z \to Z$, up to the canonical isomorphism $H'^\vee \otimes K C \cong (H_K \otimes K C)^\vee$. We clearly get that $(H^\vee)^\vee = H$ and $(\cdot)^\vee : \text{Mod}^{fr}_{Z,K} \to \text{Mod}^{fr}_{Z,K}$ is a dualizing functor. Note that $Z(r)^\vee = Z(-r)$ so that $H(r)^\vee = H^\vee(-r)$ for $r \in Z$.

Similar constructions can be done for the $Q$-linear variant $\text{Mod}^{\underline{Q}}_{Z,K}$. Note that $\text{Mod}^{\underline{Q}}_{Z,K}$ (resp. $\text{Mod}^{\underline{Q}}_{Z,K}$) admits an internal $\text{Hom}$ defined via the internal $\text{Hom}$ of the category of finite dimensional $Q$-vector spaces (resp. lattices). Furthermore these categories do have an identity object: $1 = Z(0) \in \text{Mod}^{\underline{Z}}_{Z,K}$ and $1 = Q(0) \in \text{Mod}^{\underline{Q}}_{Z,K}$, respectively. For any object $H$ of $\text{Mod}^{\underline{Q}}_{Z,K}$ we have $H^\vee = \underline{\text{Hom}}(H, 1)$ and $\text{End}(1) = Q$. Hence all objects of $\text{Mod}^{\underline{Q}}_{Z,K}$ are reflexive. Similarly, for $\text{Mod}^{\underline{Z}}_{Z,K}$.

\textbf{2.3.2. Lemma.} The categories $\text{Mod}^{\underline{Z}}_{Z,K}$ and $\text{Mod}^{\underline{Z}}_{Z,K}$ are abelian tensor categories. The category $\text{Mod}^{\underline{Q}}_{Z,K}$ is a neutral Tannakian category with fibre functor the forgetful functor to $Q$-vector spaces.

Note that there is a cohomological version of $\text{Mod}^{\underline{Z}}_{Z,K}$ and $\text{Mod}^{\underline{Q}}_{Z,K}$, which is called the de Rham–Betti category in the existing literature (cf. [3, 7.5]).

\textbf{2.3.3. Definition.} Let $\text{Mod}^{\underline{Z}}_{K,Z}$ be the category whose objects are triples $(H_K, H_Z, \eta)$ where $H_K$ is a finite dimensional $K$-vector space, $H_Z$ is a finitely generated abelian group and
\[
\eta : H_K \otimes K C \cong H_Z \otimes Z C
\]
is an isomorphism of $C$-vector spaces. We shall call $\text{Mod}^{\underline{Z}}_{K,Z}$ and its $Q$-linear variant $\text{Mod}^{\underline{Q}}_{K,Z}$ the \textit{cohomological period} categories.

The category $\text{Mod}^{\underline{Q}}_{Z,K}$ is denoted $\text{C}_{\text{DRB}}$ in [16, §2.1] and in [15, §5.3]. The $Q$-linear variant $\text{Mod}^{\underline{Q}}_{K,Q}$ is denoted $(K, Q)$-\text{Vec} in [22, Chap. 5]. For these categories we have an analogue of Lemma 2.3.2; in particular, a dualizing functor exists.

\textbf{2.3.4. Lemma.} There is canonical equivalence given by the functor
\[
\varsigma : \text{Mod}^{\underline{Z}}_{Z,K} \to \text{Mod}^{\underline{Z}}_{K,Z} \quad \varsigma(H, H_K, \omega) := (H_K, H_Z, \omega^{-1})
\]
which induces an equivalence between the tensor subcategories $\text{Mod}^{\underline{Z}}_{Z,K} \text{Mod}^{\underline{Q}}_{K,Z}$.

We set
\[
Z(r) := \varsigma(Z(r)) \in \text{Mod}^{\underline{Z}}_{K,Z}.
\]
Note that, for $H \in \text{Mod}^{\underline{Z}}_{Z,K}$ we may consider $H^\circ \in \text{Mod}^{\underline{Z}}_{K,Z}$ setting
\begin{equation}
(H_Z, H_K, \omega)^\circ := (H'_Z, H'_K, \omega^{-1}) = \varsigma(H^\vee) = \varsigma(H)^\vee
\end{equation}
where \( \omega : H^*_K \otimes_K \mathbb{C} \xrightarrow{\sim} H^*_Z \otimes_{\mathbb{Z}} \mathbb{C} \) is just given by the \( \mathbb{C} \)-dual of \( \omega_C \) in (2.8). We then have \( \mathbb{Z}(r)^\partial = \mathbb{Z}(-r) \in \text{Mod}^{\mathbb{Z}}_{K,Z} \) so that \( H(r)^\partial = H^0(-r) \) for all \( r \in \mathbb{Z} \).

The functor \( ( )^\partial \) is an anti-equivalence and there is an induced equivalence \( \text{Mod}^{\mathbb{Z}}_{Q,K} \cong (\text{Mod}^{\mathbb{Z}}_{K,Q})^{\text{op}} \) of neutral Tannakian categories.

### 2.4. Betti–de Rham realization and Cartier duality.

Now recall the period mapping \( \varpi_{M,Z} : T^*_Z(M_C) \to T^*_dR(M_K) \otimes_K \mathbb{C} \) provided by Definition 2.2.1. According to Theorem 2.2.2 we have that \( \varpi_{M,C} \) is a \( \mathbb{C} \)-isomorphism.

#### 2.4.1. Definition.

For \( K \) a subfield of \( \mathbb{C} \), \( M_K \in \mathcal{M}_1(K) \) and \( \varpi_{M,Z} \) we set

\[
T_{\text{BdR}}(M_K) := (T^*_Z(M_C), T^*_dR(M_K), \varpi_{M,Z}) \in \text{Mod}^{\mathbb{Z}}_{Z,K}
\]

and the \( \mathbb{Q} \)-linear variant

\[
T^*_Q_{\text{BdR}}(M_K) := (T_Q(M_C), T^*_dR(M_K), \varpi_{M,Q}) \in \text{Mod}^{\mathbb{Z}}_{Q,K}
\]

where \( T_Q(M_C) := T^*_Z(M_C) \otimes_{\mathbb{Z}} \mathbb{Q} \). Call these realizations the Betti–de Rham realizations.

Since the period mapping \( \varpi_{M,Z} \) in \( T^*_dR(M_K) \) is covariantly functorial, by the constructions in (2.5) and (2.4), the Betti–de Rham realization yields a functor

\[
T_{\text{BdR}} : \mathcal{M}_1(K) \to \text{Mod}^{\mathbb{Z}}_{Z,K}
\]

in the homological category \( \text{Mod}^{\mathbb{Z}}_{Z,K} \). Similarly, with rational coefficients, we get a functor from 1-motives up to isogenies \( \mathcal{M}_1^Q(K) \cong \mathcal{M}_1^Q(K) \) to \( \text{Mod}^{\mathbb{Z}}_{Z,K} \). By Examples 2.2.3 we have \( T_{\text{BdR}}(\mathbb{Z}[0]) = \mathbb{Z}(0) \) and \( T_{\text{BdR}}(\mathbb{G}_m[-1]) = \mathbb{Z}(1) \).

#### 2.4.2. Definition.

For \( H = (H_Z, H_K, \omega) \in \text{Mod}^{\mathbb{Z}}_{Z,K} \) define the Cartier dual

\[
H^* := (H^*_Z, H^*_K, 2\pi i \omega^\vee) = H^\vee(1) = H(-1)^\vee = \text{Hom}(H, \mathbb{Z}(1)) \in \text{Mod}^{\mathbb{Z}}_{Z,K}.
\]

Note that this construction is reflexive.

#### 2.4.3. Theorem.

For \( M_K \in \mathcal{M}_1(K) \) free with Cartier dual \( M^*_K \) we have that

\[
T_{\text{BdR}}(M_K)^* \cong T_{\text{BdR}}(M^*_K)
\]

**Proof.** It suffices to prove that the Poincaré biextension of \( M_K \) provides a natural morphism \( T(M_K) \otimes T(M^*_K) \to \mathbb{Z}(1) \) which induces the usual dualities \( \langle , \rangle_Z \) on \( T_Z \)'s and \( \langle , \rangle_{dR} \) on \( T_{dR} \)'s constructed in [18, §10.2.3 & §10.2.7]. This is proved in [18, Prop. 10.2.8].

Note that we also have a de Rham–Betti contravariant realization in the cohomological category \( \text{Mod}^{\mathbb{Z}}_{K,Z} \). Recall from [12, §1.13] that we also have the category of 1-motives with cotorsion \( \mathcal{M}_1 \). Cartier duality

\[
( )^* : \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_1
\]

is an anti-equivalence of abelian categories.

#### 2.4.4. Definition.

For \( M \in \mathcal{M}_1 \) denote

\[
T^*_dR(B)(M) := \langle T^*_dR(M^*) \rangle = (T^*_dR(M^*), T_Z(M^*), \eta_{M^*}) \in \text{Mod}^{\mathbb{Z}}_{K,Z}
\]

where \( \eta_{M^*} := \varpi_{M^*,C}^{-1} \) is the inverse of the \( \mathbb{C} \)-linear period isomorphism \( \varpi_{M^*,C} \) of the Cartier dual \( M^* \in \mathcal{M}_1 \) (see Theorem 2.2.2). Call this realization (and its \( \mathbb{Q} \)-linear variant) the de Rham–Betti realization.
With this definition we get a functor
\[(2.14) \quad T_{dRB} : \iota \mathcal{M}_1^{op} \to \text{Mod}_{K,z}^\wedge.\]
Now we have \(T_{dRB}(\mathbb{Z}[0]) = \mathbb{Z}(1)\) and \(T_{dRB}(\mathbb{G}_m[-1]) = \mathbb{Z}(0)\). With the notation adopted in (2.11), we also have
\[
T_{\text{BdR}}(M)^\circ(1) = (T_{\text{dR}}(M)^\vee, T_{\text{dR}}(M)^\vee, (2\pi i)^{-1} \omega_M^\circ) .
\]

2.4.5. Lemma. We have a natural isomorphism of functors \(T_{dRB}(k) \cong T_{BdR}(k)^\circ(1)\).

Proof. For \(M \in \mathcal{M}_1\) and its Cartier dual \(M^*\) we have that \(T_{\text{BdR}}(M)^* \cong T_{\text{BdR}}(M^*) \in \text{Mod}_{K,z}^\phi\)
by Theorem 2.4.3. Thus the period isomorphism of the Cartier dual \(\omega_{M^*, z} = 2\pi i \omega_M^\circ\) and its \(\mathbb{C}\)-inverse \(\omega_{M*, \mathbb{C}}^{-1} = (2\pi i)^{-1} \omega_M^\circ\).

2.5. Weight and Hodge filtrations. We may consider \(\text{FMod}_{\mathbb{Z}, K}\) given by objects in \(\text{Mod}_{\mathbb{Z}, K}\) endowed with finite and exhaustive filtrations and morphisms that respects the filtrations.

More precisely, an object of \(\text{FMod}_{\mathbb{Z}, K}\) is an abelian group \(H_Z\) endowed with a (weight) filtration \(W_i H_Z\) and a \(K\)-vector space \(H_K\) endowed with two filtrations \(W_i H_K, F_i H_K\), along with the corresponding compatibilities of the \(\omega\)'s on weight filtrations.

Since the Betti-de Rham realization (2.12) is functorial and compatible with the canonical weight filtration on \(T_Z(M_C)\) given by sub-\(1\)-motives and \(T_{dR}(M_K)\) is filtered by \(V(M)\), the Hodge filtration as in Remark 2.1.2, we also get a realization functor
\[(2.15) \quad F T_{\text{BdR}} : \iota \mathcal{M}_1(K) \to \text{FMod}_{\mathbb{Z}, K}.\]

We have
\[(T_Z(M_C), T_{dR}(M_K), \omega_{M, z}) \supseteq (T_Z(G_C), T_{dR}(G_K), \omega_{G, z}) \supseteq (T_Z(T_C), T_{dR}(T_K), \omega_{T, z}).\]

Note that:

2.5.1. Lemma. Let \(K = \overline{\mathbb{Q}}\). Let \(M_K\) and \(N_K\) be two free \(1\)-motives over \(K\). Then any morphism \(\varphi : T_{\text{BdR}}(M_K) \to T_{\text{BdR}}(N_K)\) in \(\text{Mod}_{K,z}^\phi\) preserves the weight filtrations.

Proof. Let \(\varphi = (\varphi_\mathbb{Z}, \varphi_K) : (T_Z(M_C), T_{dR}(M_K), \omega_{M, z}) \to (T_Z(N_C), T_{dR}(N_K), \omega_{N, z})\) for \(M_K\) and \(N_K\) one of the following pure \(1\)-motives: \([Z_K \to 0]\), \([0 \to \mathbb{G}_{m,K}\]) and \([0 \to A_K]\) if \(A_K\) is an abelian variety. We show that \(\varphi = 0\) for different weights, in all cases. As \(K\) is algebraically closed this implies that \(\varphi = 0\) for all pure \(1\)-motives of different weights and this easily yields the claimed compatibility.

For \(M_K = [Z_K \to 0]\) and \(N_K = [0 \to \mathbb{G}_{m,K}\] \(\) respectively \(M_K = [0 \to \mathbb{G}_{m,K}\) and \(N_K = [Z_K \to 0]\) we have \(T_{\text{BdR}}([Z_K \to 0]) = \mathbb{Z}(0), T_{\text{BdR}}([0 \to \mathbb{G}_{m,K}\]) = \mathbb{Z}(1)\) and \(\varphi = 0\) as \(\varphi_K : K \to K\) is given by the multiplication by an algebraic number but the compatibility (2.7) forces such algebraic number to be \(n2\pi i\) (respectively \(n/2\pi i\)) for some \(n \in \mathbb{Z}\).

Similarly, for \(M_K = [Z_K \to 0]\) and \(N_K = [0 \to A_K]\) we have that \(\omega_{Z, \mathbb{N}} \circ \varphi_Z(1) = \varphi_K(1)\) if, and only if, \(\varphi = 0\). Indeed, the preceding equality implies that \(d\rho_A \circ \omega_{Z, \mathbb{N}} \circ \varphi_Z(1) = d\rho_A \circ \varphi_K(1)\).

Now, the right-hand term is in \(\text{Lie}(A_K)\) while by Remark 2.2.4 the left-hand term would give a transcendental point of \(\text{Lie}(A_K)\) if \(\varphi_Z(1) \neq 0\).

Dually, for \(M_K = [0 \to A_K]\) and \(N_K = [0 \to \mathbb{G}_{m,K}\] \(\) by making use of Theorem 2.4.3 we then get \(\varphi^* = 0\) thus \(\varphi = 0\).

Finally, for \(M_K = [0 \to \mathbb{G}_{m,K}\) and \(N_K = [0 \to A_K]\) we can apply [16, Thm. 3.1] to the pair \((\varphi_K, \varphi_Z)\) so that, dually, making use of Theorem 2.4.3, the same holds for \(M_K = [0 \to A_K]\) and \(N_K = [Z_K \to 0]\).
Let $M_K = [u_K: L_K \to G_K]$ and $N_K = [v_K: F_K \to H_K]$ be free and let $\varphi: T_{\text{Bdr}}(M_K) \to T_{\text{Bdr}}(N_K)$ be a morphism in $\text{Mod}_{\mathbb{Z},K}^\text{gr}$. Then we have a $K$-linear mapping $\varphi_K: T_{\text{dr}}(M_K) \to T_{\text{dr}}(N_K)$ and an homomorphism $\varphi_Z: T_Z(M_K) \to T_Z(N_K)$ which is compatible with the weight filtrations, by Lemma 2.5.1. Moreover, $\varphi_Z$ and $\varphi_K$ are compatible with the $\omega$'s as in (2.7). We have that $\varphi_Z$ restricts to a homomorphism

$$(2.16) \quad W_{-1} \varphi_Z: W_{-1} T_Z(M_K) := T_Z(G_C) \cong H_1(G_C^\natural) \to W_{-1} T_Z(N_K) := T_Z(H_C) \cong H_1(H_C^\natural)$$

and we get an induced map on $\text{gr}_0^W$ as follows

$$(2.17) \quad \varphi_{Z,0}: \text{gr}_0^W T_Z(M_K) = T_Z(M_K)/T_Z(G_C) = L_C \to \text{gr}_0^W T_Z(N_K) = T_Z(N_K)/T_Z(H_C) = F_C$$

Note that $\varphi_{Z,0}$ is indeed defined over $\overline{K}$.

2.5.2. Lemma. Let $K = \overline{K}$. Let $M_K$ and $N_K$ be two free $1$-motives over $K$. Then any morphism $\varphi: T_{\text{Bdr}}(M_K) \to T_{\text{Bdr}}(N_K)$ in $\text{Mod}_{\mathbb{Z},K}^\text{gr}$ preserves the Hodge filtrations.

Proof. Let $M_K = [u_K: L_K \to G_K]$ and $N_K = [v_K: F_K \to H_K]$ be free and let $\varphi: T_{\text{Bdr}}(M_K) \to T_{\text{Bdr}}(N_K)$ be a morphism in $\text{Mod}_{\mathbb{Z},K}^\text{gr}$. We have to show that $\varphi_K(\mathbb{V}(M)) \subseteq \mathbb{V}(N)$ where $\mathbb{V}(M)$ is the additive part of $G_K^\natural$ and $\mathbb{V}(N)$ is that of $H_K^\natural$; see Remark 2.1.2. Recall the commutative diagram

$$(2.18) \quad \begin{array}{ccc}
T_Z(M_C) & \xrightarrow{\varphi_Z} & T_Z(N_C) \\
\text{gr}_0^W & \downarrow{\omega_{M,K}} & \downarrow{\omega_{N,K}} \\
T_{\text{dr}}(M_C) & \xrightarrow{\varphi_K \otimes \text{id}_C} & T_{\text{dr}}(N_C)
\end{array}$$

By (2.16) and (2.5) there exists an analytic morphism $h_C: G_C^\natural \to H_C^\natural$ with $dh_C = \varphi_K \otimes \text{id}_C$. It is sufficient to prove that $h_C$ is algebraic and defined over $\overline{K}$ to conclude by the structure theorem of algebraic $K$-groups that $h_K(\mathbb{V}(M)) \subseteq \mathbb{V}(N)$ and hence that $\varphi_K = dh_K$ preserves the Hodge filtrations.

If $L_K = 0$, by Lemma 2.5.1, $\varphi$ factors through $W_{-1} T_{\text{Bdr}}(N_K) = T_{\text{Bdr}}(H_K)$. Hence we may assume $F_K = 0$ as well. It follows then from [16, Thm. 3.1] applied to $G^\natural$ and $H^\natural$ that the above morphism $h_C$ is indeed algebraic and defined over $K$. Hence $\varphi_K$ preserves the Hodge filtrations.

Now let $L_K \neq 0$ and set $L_K^\natural = L_K \otimes G_{\text{a},K}$. Since $\varphi$ preserves the weights by Lemma 2.5.1 we get $W_{-1} \varphi: T_{\text{Bdr}}(G_K) \to T_{\text{Bdr}}(H_K)$. By the previous step $W_{-1} \varphi_K(\mathbb{V}(G)) \subseteq \mathbb{V}(H) \subseteq \mathbb{V}(N)$. We thus obtain the following commutative diagram

$$(2.19) \quad \begin{array}{ccc}
T_Z(M_C) & \xrightarrow{\omega_{M,Z}} & T_{\text{dr}}(M)/\mathbb{V}(G) \otimes \mathbb{C} \xrightarrow{\text{exp}} G_C^\natural/\mathbb{V}(G)_C \cong G_C \oplus L_K^\natural \\
\downarrow{\varphi_Z} & & \downarrow{\gamma} \\
T_Z(N_C) & \xrightarrow{\omega_{N,Z}} & T_{\text{dr}}(N)/\mathbb{V}(N) \otimes \mathbb{C} \xrightarrow{\text{exp}} H_C^\natural/\mathbb{V}(N)_C \cong H_C
\end{array}$$

where the mapping $\gamma$ is induced by $\varphi_K$, we have the canonical identification of $G_K^\natural/\mathbb{V}(G) = G_K \oplus L_K^\natural$ and $\delta = g_C + \beta$ with $g_C = g_K \otimes \text{id}_C$ and $g_K: G_K \to H_K$ induced by $W_{-1} \varphi_K$. We are left to show that $\beta: L_C^\natural \to H_C$ is zero. Since the composition of the upper arrows in the
previous diagram maps \( T_\mathbb{Z}(G_\mathbb{C}) \) to \( 0 \oplus 0 \), we obtain a commutative square

\[
\begin{array}{ccc}
L_\mathbb{C} & \overset{(u,1)}{\longrightarrow} & G_\mathbb{C} \oplus L^\mathbb{C} \\
\varphi_{Z,0} \downarrow & & \delta \\
F_\mathbb{C} & \overset{\psi}{\longrightarrow} & H_\mathbb{C}
\end{array}
\]

where \( \varphi_{Z,0} \) is the induced map as in (2.17). In particular, for \( x \in L_K(K) \) we have \( \beta(x \otimes 1) = \psi(\varphi_{Z,0}(x)) - g_K(u(x)) = \gamma - dg_K \otimes \text{id}_\mathbb{C} \) is in \( H_K(K) \). On the other hand \( \beta(x \otimes 1) = \exp d\beta(x \otimes 1) \).

Since \( d\beta = d\delta - dg_K \otimes \text{id}_\mathbb{C} \) we have that \( d\beta(x \otimes 1) \) belongs to \( \text{Lie}(H_K) \) regarded as a \( K \)-linear subspace of \( \text{Lie}(H_\mathbb{C}) \). By Remark 2.2.4 we get that \( \beta(x \otimes 1) = 0 \) and therefore that \( \beta = 0 \). \( \square \)

2.6. Full faithfulness. We are now ready to show that our previous Lemmas 2.5.1 and 2.5.2 yield the full faithfulness of Betti–de Rham and de Rham–Betti realizations.

2.6.1. Theorem. The functors \( T_{\text{BdR}} \) in (2.12) and \( T_{\text{dR}} \) in (2.14) restricted to \( M_1(K) \) are fully faithful over \( K = \overline{\mathbb{Q}} \).

Proof. Clearly, the functor \( T_{\text{BdR}} \) (resp. \( T_{\text{dR}} \)) is faithful (cf. [4, proof of Lemma 3.3.2]) and we are left to show the fullness. Making use of Lemma 2.4.5 we are left to check the fullness for \( T_{\text{BdR}} \). Let \( M_K = [u_K : L_K \rightarrow G_K] \) and \( N_K = [v_K : F_K \rightarrow H_K] \) be free and let \( \phi : T_{\text{BdR}}(M_K) \rightarrow T_{\text{BdR}}(N_K) \) be a morphism in \( \text{Mod}^\mathbb{C}_{Z,K} \).

For 0-motives, i.e., if \( G_K = H_K = 0 \), we have \( L_K \cong \mathbb{Z}^r_K \) and \( F_K \cong \mathbb{Z}^s_K \). \( \varphi_Z : T_\mathbb{Z}(M_C) \cong \mathbb{Z}^r \rightarrow T_\mathbb{Z}(N_C) \cong \mathbb{Z}^s \) provides a morphism \( f : M_K \rightarrow N_K \) such that \( T_{\text{BdR}}(f) = \phi \).

If the weight \( -1 \) parts are non-zero, by Lemma 2.5.1 \( \varphi_Z \) restricts to an homomorphism \( W_{-1}\varphi_Z \) as in (2.16) and it yields a morphism \( \varphi_Z,0 \) as in (2.17) i.e., \( \varphi_{Z,0} \) is the map induced by \( \varphi_Z \) on \( \text{gr}_W^0 \). If we set \( f_C := \varphi_{Z,0} \) the homomorphism \( f_C : L_C \rightarrow F_C \) trivially descends to a homomorphism \( f_K : L_K \rightarrow F_K \) over \( K = \overline{\mathbb{Q}} \).

Let’s now consider \( \varphi_C := \varphi_K \otimes \text{id}_C \) and translate (2.16) and (2.18), as in the proof of Lemma 2.5.2, in the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & H_1(G_\mathbb{C}) \\
\downarrow W_{-1}\varphi_Z & & \downarrow \varphi_Z & \downarrow \psi \\
0 & \longrightarrow & H_1(H_\mathbb{C})
\end{array}
\]

yielding a morphism of analytic groups \( \psi : G_\mathbb{C} \rightarrow H_\mathbb{C} \) on the quotients via the exponential mapping \( \exp \), as indicated above. Now, since by Lemma 2.5.2, we have \( \varphi_K(\mathbb{V}(M)) \subseteq \mathbb{V}(N) \), \( \psi(\mathbb{V}(M_C)) \subseteq \mathbb{V}(N_C) \), the diagram (2.19) induces a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H_1(G_\mathbb{C}) & \overset{\exp}{\longrightarrow} & G_\mathbb{C} & \longrightarrow & 0 \\
\downarrow W_{-1}\varphi_Z & & \downarrow \varphi_Z & \downarrow \psi & & \downarrow \psi' & \downarrow \psi' & \downarrow \psi' \\
0 & \longrightarrow & H_1(H_\mathbb{C}) & \overset{\exp}{\longrightarrow} & H_\mathbb{C} & \longrightarrow & 0
\end{array}
\]

As \( \varphi'_C \) is the base change of the \( K \)-linear map \( \text{Lie}(G_K) \rightarrow \text{Lie}(H_K) \) induced by \( \varphi_K \), it follows from [16, Thm. 3.1] that \( \psi' = g_C \) is the base change of the morphism \( g_K : G_K \rightarrow H_K \) over \( K = \overline{\mathbb{Q}} \) induced by \( W_{-1}\varphi_K \) (see the proof of Lemma 2.5.2).

We are left to check that \( h := (f_K,g_K) \) gives a morphism \( h : M_K \rightarrow N_K \), i.e., that \( g_K \circ u_K = v_K \circ f_K \), and to see that \( T_{\text{BdR}}(h) = \phi \). To show that \( h \) is a morphism of 1-motives we may
work after base change to $\mathbb{C}$ and, using (2.5), it suffices to prove that $\psi \circ u^\flat_C = v^\flat_C \circ f_C$. Consider the following diagram

$$
\begin{array}{ccc}
T_{\mathbb{Z}}(M_C) & \xrightarrow{\varphi_Z} & T_{\mathbb{Z}}(N_C) \\
\xrightarrow{\exp} & & \xrightarrow{\exp} \\
L_C & \xrightarrow{f_C} & F_C \\
\xrightarrow{\psi} & & \xrightarrow{\psi} \\
T_{\text{dR}}(M_K) \otimes_K \mathbb{C} & \xrightarrow{\varphi_C} & T_{\text{dR}}(N_K) \otimes_K \mathbb{C} \\
\xrightarrow{\exp} & & \xrightarrow{\exp} \\
G^2_C & \xrightarrow{\psi} & H^2_C
\end{array}
$$

All squares are commutative. Indeed, $\exp \circ \varpi_{M,Z} = u^\flat_C \circ \exp$ and $\exp \circ \varpi_{N,Z} = v^\flat_C \circ \exp$ by (2.5), $f_C \circ \exp = \exp \circ \varphi_Z$ by definition of $f_C$, $\varphi_C \circ \varpi_{M,Z} = \varpi_{N,Z} \circ \varphi_Z$ by the compatibility of $\varphi_Z$ with $\varphi_K$ as in (2.7), and finally $\psi \circ \exp = \exp \circ \varphi_C$ by (2.19). One concludes by the surjectivity of the map $\exp$ that also $\psi \circ u^\flat_C = v^\flat_C \circ f_C$.

Now consider the morphism $\alpha := T_{\text{BdR}}(h) - \varphi: T_{\text{BdR}}(M_K) \to T_{\text{BdR}}(N_K)$ in $\text{Mod}_{\mathbb{Z},K}$. By construction $W_{-1} \varphi_Z = W_{-1} T_{\mathbb{Z}}(h)$ so that $\alpha$ is vanishing on $T_{\text{BdR}}(G_K)$. Moreover we have that $\text{gr}_0^W T_{\mathbb{Z}}(h) = \varphi_{Z,0}$ so that $\alpha$ induces a morphism in $\text{Mod}_{\mathbb{Z},K}^w$ from $T_{\text{BdR}}(L_K)$ to $T_{\text{BdR}}(H_K)$ which is trivial by Lemma 2.5.1.

2.6.2. Remark. In the proof of the Theorem 2.6.1, in order to show that $(f_K, g_K): M_K \to N_K$ is a morphism we are left to check that $g_K \circ u_K = v_K \circ f_K$. Remark that this also follows from two key facts: (i) the pullbacks $\tilde{u}_C$ and $\tilde{v}_C$ of $u_C$ and $v_C$ factor through the period mappings $\varpi$’s and (ii) the mappings $\varphi_Z$ and $\varphi_K$ are compatible with the $\varpi$’s.

In fact, according with the notation adopted above, for $x \in L_C$ pick $\tilde{\log}(x) \in T_{\mathbb{Z}}(M_C)$ and note that $\varphi_Z(\log(x)) = \tilde{\log}(f(x))$. Making use of (2.6) we obtain

$$
g_C(u_C(x)) = g_C \exp d\rho \varpi_{M,Z}(\tilde{\log}(x)) = \exp d\rho H d\tilde{g}_C \varpi_{M,Z}(\tilde{\log}(x))
$$

by the functoriality of $\exp$. Now $\psi = \varphi_K \otimes_K 1_C$ and we are assuming the compatibility $(\varphi_K \otimes_K 1_C) \circ \varpi_{M,Z} = \varpi_{N,Z} \circ \varphi_Z$ so that

$$
g_C(u_C(x)) = \exp d\rho H \varpi_{N,Z} \varphi_Z(\tilde{\log}(x)) = \exp d\rho H \varpi_{N,Z}(\tilde{\log}(f(x))) = v_C(f(x))
$$

using (2.6) again, as claimed.

We notice that an alternative proof of Theorem 2.6.1 can be given using the following theorem ([34, Thm. 1]):

2.6.3. Theorem (Wüstholz). Let $W$ be a commutative connected group scheme over $\overline{\mathbb{Q}}$. Let $S$ be a subset of $\exp^{-1}(W(\overline{\mathbb{Q}}))$ and let $V \subset \text{Lie} W$ be the smallest $\overline{\mathbb{Q}}$-subvector space whose $\mathbb{C}$-span contains $S$. Then, there exists a connected algebraic subgroup $Z \subset W$ such that $\text{Lie} Z = V$

We use this theorem to deduce the following:

Alternative Proof of Theorem 2.6.1. Let $M_K = [u_K: L_K \to G_K]$ and $N_K = [v_K: F_K \to H_K]$
be free 1-motives and let $\varphi = (\varphi_Z, \varphi_K): T_{\text{BdR}}(M_K) \to T_{\text{BdR}}(N_K)$ be a morphism in $\text{Mod}^\fr_{\mathbb{Z},K}$. Let $W = G^+_K \times H^+_K$ and note that we have commutative squares

$$T := T_Z(M_C) \xrightarrow{(id,\varphi_Z)} T_Z(M_C) \times T_Z(N_C) \xrightarrow{\varphi_M \times \varphi_N} \text{Lie} W_C = \text{Lie} G^+_C \oplus \text{Lie} H^+_C$$

where the horizontal arrows are injective. Let $S$ denote the image of $T$ in $\text{Lie} W_C$; it is contained in $\exp^{-1}(W(\overline{\mathbb{Q}}))$ since the image of $L_C \times F_C$ via $u^2 \times v^2$ is contained in $W(\overline{\mathbb{Q}})$. Let $V$ denote the image of $L_C \times F_C$ in $\text{Lie} W$ via the map $id \oplus \varphi_K$. By the compatibility of $\varphi_Z$ and $\varphi_K$ over $\mathbb{C}$ via the homomorphisms of periods, $V_C$ coincides with the $\mathbb{C}$-span of $S$. It then follows from Theorem 2.6.3 that there exists an algebraic subgroup $Z \subset W$ whose Lie algebra is $V$. Now, the composition of the inclusion $Z \to W$ with the projection $W \to G^+_K$ is an isogeny, since it is an isomorphism on Lie algebras. In fact, it is an isomorphism; indeed, it is an isomorphism; indeed, the injective map $T \to V_C = \text{Lie} Z_C \subset \text{Lie} W_C$ maps $H_1(G^+_C) \subset T$ into $H_1(Z_C) \subset H_1(W_C)$ and hence the isomorphism $\text{Lie} Z_C \xrightarrow{\sim} \text{Lie} G^+_C$ restricts to an isomorphism $H_1(Z_C) \xrightarrow{\sim} H_1(G^+_C)$.

Let $\gamma: G^+_K \to H^+_K$ be the homomorphism of algebraic $K$-groups defined by composing the inverse of the isomorphism $Z \to G^+_K$ with the inclusion $Z \to W$ and the second projection $W \to H^+_K$. By construction $\gamma = \varphi_K$.

In order to see that $f := (\varphi_Z, 0, \gamma)$ is a morphism of 1-motives with $T_{\text{BdR}}(f) = \varphi$ it suffices to check that $\gamma_C \circ u^2 = v^2 \circ \varphi_Z, 0: L_C \to H^+_C$. The latter fact is equivalent to the equality $(id_{G^+_C}, \gamma) \circ u^2 = (u^2, v^2 \circ \varphi_Z, 0): L_C \to W_C$, and, by the above diagram, this is satisfied whenever $(id_{\text{Lie} G^+_C}, \text{Lie} \gamma_C) \circ \varphi_{M,Z} = (\varphi_{M,Z}, \varphi_{N,Z} \circ \varphi_Z): T \to \text{Lie} W_C$. Then we conclude by the commutativity of diagram (2.7) since $\text{Lie} \gamma_C = \varphi_K \circ id_C$. □

2.7. Descent to number fields. Let $K'/K$ be a field extension with $K' \subset \overline{\mathbb{Q}}$ and fix an embedding $\sigma: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Note the following commutative diagram of functors

$$\begin{array}{ccc}
\mathcal{M}_1(K) & \xrightarrow{T_{\text{BdR}}} & \text{Mod}^\fr_{\mathbb{Z},K} \\
\downarrow & & \downarrow \\
\mathcal{M}_1(K') & \xrightarrow{T_{\text{BdR}}} & \text{Mod}^\fr_{\mathbb{Z},K'}
\end{array}$$

where the functor on the left is the usual base-change and the vertical functor on the right maps $(H_Z, H_K, \omega)$ to $(H_Z, H_K \otimes_K K', \omega)$ using the canonical isomorphism $(H_K \otimes_K K') \otimes \mathbb{C} \cong H_K \otimes \mathbb{C}$.

2.7.1. Proposition. Let $K$ be a subfield of $\overline{\mathbb{Q}}$. The functor $T_{\text{BdR}}: \mathcal{M}_1(K) \to \text{Mod}^\fr_{\mathbb{Z},K}$ is fully faithful.

Proof. The functor $T_{\text{BdR}}$ is fully faithful over $\overline{\mathbb{Q}}$ by Theorem 2.6.1; hence it is faithful over $K$, since the left-hand vertical functor in (2.20) is faithful.

Assume now $M_K = [u: L_K \to G_K], N_K = [v: F_K \to H_K]$ are 1-motives over $K$ and let $(\varphi_Z, \varphi_K): T_{\text{BdR}}(M_K) \to T_{\text{BdR}}(N_K)$ be a morphism in $\text{Mod}^\fr_{\mathbb{Z},K}$. By Theorem 2.6.1 there exists a morphism $\psi: M_{\overline{\mathbb{Q}}} \to N_{\overline{\mathbb{Q}}}$ such that $T_{\text{BdR}}(\psi) = (\varphi_Z, \varphi_K \otimes_K \overline{\mathbb{Q}})$. Note that there exists a subfield $K' \subset \overline{\mathbb{Q}}$ with $K'/K$ finite Galois and $\psi = (f,g)$ is defined over $K'$. We may
further assume that $L_{K'}, F_{K'}$ are constant free. Hence $\text{gr}_0^W \varphi_\mathbb{Z}$ descends over $K'$ and we have a commutative square

\[
\begin{array}{c}
\text{L}_{K'} \quad \text{f} = \text{gr}_0^W (\varphi_\mathbb{Z}) \\
\downarrow \quad \downarrow \\
\text{L}_{K'} \otimes \mathbb{G}_{a,K'} \quad \text{gr}_0^W (\varphi_{K'}) \quad \text{F}_{K'} \otimes \mathbb{G}_{a,K'}
\end{array}
\]

where the vertical morphisms maps $x$ to $x \otimes 1$ (and descend the homomorphism of periods for $\text{gr}_0^W (M)$ and $\text{gr}_0^W (N)$ respectively). By diagram (2.21) $f$ descends over $K$ since $\text{gr}_0^W (\varphi_{K'}) = \text{gr}_0^W (\varphi_K) \otimes \text{id}_{K'}$ and the vertical morphisms are injective on points. In order to check that $\psi$ descends over $K$, we may then reduce to the case $L = F = 0$. By Cartier duality, we may further reduce to the case where $L = F = 0$ and $G = A, H = B$ are abelian varieties.

We have to check that for any $\tau \in \text{Gal}(K'/K)$ it is $\tau_B \circ \psi = \psi \circ \tau_A$, where $\tau_A$ is not a morphism of $K'$-schemes in general. Let $\iota_{A, \tau}: A_{K'} \to \tau^* A_{K'}$ be the canonical morphism of $K'$-schemes (i.e., $\tau_A$ is the composition of $\iota_{A, \tau}$ with the projection $\tau^* A_{K'} \to A_{K'}$). We are left to check that $\iota_{B, \tau} \circ \psi = (\tau^* \psi) \circ \iota_{A, \tau}$ as morphisms of $K'$-schemes $A_{K'} \to \tau^* B_{K'}$. By faithfulness of $T_{\text{BdR}}$, it is sufficient to check that

\[
T_{\text{BdR}} (\iota_{B, \tau}) \circ T_{\text{BdR}} (\psi) = T_{\text{BdR}} (\tau^* \psi) \circ T_{\text{BdR}} (\iota_{A, \tau}).
\]

Note that since $A_K$ is a $K$-form of $A_{K'}$, we may identify $A_K$ with $\tau^* A_K$, so that $\iota_{A, \tau}$ becomes the identity map. Further $T_{\text{dR}} (\tau^* \psi) = \tau^* (\varphi_K \otimes \text{id}_{K'})$ may be identified with $\varphi_K \otimes \text{id}_{K'}$, and $T_{\text{Z}} (\tau^* \psi)$ with $\varphi_\mathbb{Z}$. We conclude that $T_{\text{BdR}} (\tau^* \psi)$ may be identified with $(\varphi_\mathbb{Z}, \varphi_K \otimes \text{id}_{K'})$ and hence (2.22) is clear.

3. Some evidence

As in the previous sections, we here mean by scheme a separated scheme which is of finite type over a field $K$ that we will assume to be $\overline{\mathbb{Q}}$. In order to show the period conjecture (1.6) we are left to deal with rational coefficients. If we work with rational coefficients, we then have that motivic cohomology $H^{p,q}(X)_\mathbb{Q}$ is computed by the cdh-topology, i.e., $H^p_{\text{cdh}}(X, \mathbb{Q}(q)) \cong H^p_{\text{cdh}}(X, \mathbb{Q}(q))$, and $H^p_{\text{cdh}}(X, \mathbb{Q}(q)) \cong H^p_{\text{Zar}}(X, \mathbb{Q}(q))$ if $X$ is smooth. However, we prefer to keep the arguments integral when possible. In general, for any algebraic scheme $X$ over $K$, by making use of the period isomorphism $\varpi_X^{p,q}$ and its inverse $\eta_X^{p,q}$ in Definition 1.2.4 we set

$H^{p,q}_{\text{BdR}}(X) := (H^p(X_{\text{an}}, \mathbb{Z}_{\text{an}}(q)), H^p_{\text{dR}}(X), \varpi_X^{p,q}) \in \text{Mod}_{\mathbb{Z}_{\text{K}}}$

and

$H^{p,q}_{\text{dR}}(X) := (H^p_{\text{dR}}(X), H^p_{\text{an}}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(q)), \eta_X^{p,q}) \in \text{Mod}_{\mathbb{Z}_{\text{K}, \mathbb{Z}}}$.

Note that $\varsigma (H^{p,q}_{\text{BdR}}(X)) = H^{p,q}_{\text{dR}}(X)$. We have that $\varpi_X^{p,q} = (2\pi i)^q \varpi_X^{0,0}$ and $\eta_X^{p,q} = (2\pi i)^{-q} \eta_X^{p,0}$ where $\eta_X^{p,0}: H^p_{\text{dR}}(X) \otimes _{\mathbb{K}} \mathbb{C} \isom H^p(X_{\text{an}}, \mathbb{C})$ is the usual de Rham–Betti comparison isomorphism (up to a sign cf. [22, Def. 5.3.1] and [26, Lemma 4.1.1 & Prop. 4.1.2] for the Nisnevich topology). In particular we have that $H^{p,q}_{\text{dR}}(X) = H^{p,0}_{\text{dR}}(X)(q)$.

3.1. Period cohomology revisited. For $H \in \text{Mod}_{\mathbb{Z}_{\text{K}}}$ we set

$H_{\varpi} := \text{Hom}(\mathbb{Z}(0), H)$

where the Hom-group is taken in $\text{Mod}_{\mathbb{Z}_{\text{K}}}$. This yields a functor

$( )_{\varpi} : \text{Mod}_{\mathbb{Z}_{\text{K}}} \to \text{Mod}_{\mathbb{Z}}$
to the category of finitely generated abelian groups. Similarly, let $H_\omega := \text{Hom}(\mathbb{Z}(0),H)$ for $H \in \text{Mod}_{K,\mathbb{Z}}$ where now the Hom-group is taken in $\text{Mod}^{\text{fr}}_{K,\mathbb{Z}}$. By Lemma 2.3.4, for $H \in \text{Mod}^{\text{fr}}_{\mathbb{Z},K}$ we clearly have that

$$H_\omega = \omega(H).$$

Moreover, for $H \in \text{Mod}^{\text{fr}}_{\mathbb{Z},K}$ we have $H^* = H(-1)^\vee \in \text{Mod}^{\text{fr}}_{\mathbb{Z},K}$ so that

$$H^*_\omega = \text{Hom}(\mathbb{Z}(0),H(-1)^\vee) = \text{Hom}(H,\mathbb{Z}(1)).$$

Note that for $H \in \text{Mod}^{\text{fr}}_{\mathbb{Z},K}$ we also have $H^0(1) = H(-1)^{0} \in \text{Mod}^{\text{fr}}_{K,\mathbb{Z}}$ (see (2.11)) and we shall denote

$$H^0_\omega := \text{Hom}_{\text{Mod}_{\mathbb{Z},K}}(\mathbb{Z}(0),H(-1)^0) = \text{Hom}_{\text{Mod}^{\text{fr}}_{\mathbb{Z},K}}(H,\mathbb{Z}(1)) = H^*_\omega.$$

With rational coefficients, for $H \in \text{Mod}_{\mathbb{Z},K} \otimes \mathbb{Q}$ and the corresponding $H^\mathbb{Q} \in \text{Mod}_{\mathbb{Q},K}$ we then have $H_\omega \otimes \mathbb{Z} \mathbb{Q} \cong H^\mathbb{Q} := \text{Hom}(\mathbb{Q}(0),H^\mathbb{Q})$ and, similarly, $H^\omega \otimes \mathbb{Z} \mathbb{Q} \cong H^\omega_\mathbb{Q}$. We have (cf. [16, Def. 2.1] and [15, (5.15)]):

**3.1.1. Lemma.** For $H = (H_Z,H_K,\omega) \in \text{Mod}_{\mathbb{Z},K}$ we have $H_\omega \cong H_Z \cap H_K$ where $\cap$ is the inverse image of $H_K$ under $\omega : H_Z \to H_K \otimes \mathbb{C}$. Moreover, for $H = (H_K,H_Z,\eta) \in \text{Mod}^{\text{fr}}_{K,\mathbb{Z}}$, we have that $H_\omega = H_K \cap H_Z$ where $\cap$ is the inverse image of $H_K$ under the composition of $H_Z \to H_Z \otimes \mathbb{C} \xrightarrow{\eta^{-1}} H_K \otimes \mathbb{C}.$

**Proof.** The identifications are given by mapping $\varphi \in \text{Hom}(\mathbb{Z}(0),H) \leadsto \varphi(1) \in H_Z \cap H_K$. \qed

Similarly, for $H = (H_K,H_Q,\omega) \in \text{Mod}^{\text{fr}}_{K,\mathbb{Q}}$ we have that $H^\mathbb{Q}_\omega \cong H_K \cap H_Q$. We then clearly obtain:

**3.1.2. Corollary.** For $H = H^{\text{p,q}}_{\text{dR}}(X)$ we have that $H_\omega \cong H^{\text{p,q}}_{\text{dR}}(X)$ coincide with the period cohomology of Definition 1.2.7.

Moreover, composing the functor $H \leadsto H_\omega$ with the Betti–de Rham realization of 1-motives $\text{T}_{\text{BdR}}$ in (2.12) we obtain a functor

$$(3.1) \quad T_\omega := ( )_\omega \circ \text{T}_{\text{BdR}} : \check{\mathcal{M}}_1(K) \to \text{Mod}_{\mathbb{Z}}.$$

For a 1-motive $M \in \check{\mathcal{M}}_1(K)$ we also have $T_{\text{dR}}(M) \in \text{Mod}^{\text{fr}}_{K,\mathbb{Z}}$. Composing $H \leadsto H_\omega$ with the de Rham–Betti realization $T_{\text{dR}}$ in (2.14) now yields a functor

$$T^\omega := ( )_\omega \circ T_{\text{dR}} : \check{\mathcal{M}}_1(K)^{\text{op}} \to \text{Mod}_{\mathbb{Z}}.$$

We also note that Lemma 2.4.5 yields:

**3.1.3. Corollary.** For $M \in \check{\mathcal{M}}_1(K)$ we have that $T^\omega(M) := T_{\text{dR}}(M)_\omega \cong T_{\text{BdR}}(M)$.\omega.

Working with rational coefficients we have $T^\mathbb{Q}_{\text{BdR}} := T_{\text{BdR}} \otimes \mathbb{Q}$ (resp. $T^\mathbb{Q}_{\text{dR}} := T_{\text{dR}} \otimes \mathbb{Q}$) and we then get a functor $T^\mathbb{Q}_\omega$ (resp. a contravariant functor $T^\mathbb{Q}_\omega$) from the category of 1-motives up to isogenies $\check{\mathcal{M}}_1^{\mathbb{Q}} := \check{\mathcal{M}}_1 \otimes \mathbb{Q} \cong \check{\mathcal{M}}_1 \otimes \mathbb{Q} \cong \check{\mathcal{M}}_1 \otimes \mathbb{Q}$ to the category of finite dimensional $\mathbb{Q}$-vector spaces. Moreover, applying our Theorem 2.6.1 we have:

**3.1.4. Corollary.** For $M = [u : L \to G] \in \check{\mathcal{M}}_1(\mathbb{Q})$ with Cartier dual $M^* = [u^* : L^* \to G^*] \in \check{\mathcal{M}}_1(\mathbb{Q})$ we have that

$$T_\omega(M) \cong T_{\text{Z}}(M_C) \cap T_{\text{dR}}(M_K) \cong \text{Ker} u$$

and

$$T^\omega(M) \cong T_{\text{dR}}(M^*_K) \cap T_{\text{Z}}(M^*_C) \cong \text{Ker} u^*.$$
Proof. Note that \( Z(0) = T_{\text{BdR}}(\mathbb{Z}[0]) \) and for \( T_{\text{BdR}}(M) = (T_Z(M_{\text{C}}), T_{\text{dr}}(M_K), \varpi_{M,\mathbb{Z}}) \) we have

\[
\text{Ker} \ u \cong \text{Hom}_{M_1(K)}(\mathbb{Z}[0], M) \xrightarrow{T_{\text{BdR}}} \text{Hom}(T_{\text{BdR}}(\mathbb{Z}[0]), T_{\text{BdR}}(M)) = T_{\varpi}(M)
\]

which is an isomorphism over \( K = \overline{\mathbb{Q}} \) as proven in Theorem 2.6.1. We just apply Lemma 3.1.1. Moreover, \( T_{\text{dR}}(M) = (T_{\text{dR}}(M^*_K), T_Z(M^*_C), \eta_{M^*}) \), \( Z(0) = T_{\text{dR}}(\mathbb{G}_m[-1]) \) and we have an isomorphism

\[
\text{Hom}_{M_1(K)}(M, \mathbb{G}_m[-1]) \xrightarrow{T_{\text{dR}}} \text{Hom}(T_{\text{dR}}(\mathbb{G}_m[-1]), T_{\text{dR}}(M)) = T_{\varpi}(M)
\]

and \( \text{Hom}_{M_1(K)}(M, \mathbb{G}_m[-1]) \cong \text{Hom}_{M_1(K)}(\mathbb{Z}[0], M^*) = \text{Ker} \ u^* \) showing the claim. \( \square \)

3.2. Period conjecture for \( q = 1 \). Recall that \( \mathbb{Z}(1) \in \text{DM}^{\text{eff}}_{\text{gm}} \) is canonically identified with \( \text{Tot}([0 \to \mathbb{G}_m]) = \mathbb{G}_m[-1] \) (see \([12, \text{Lemma 1.8.7}]\)). We then have

\[
H^{p-1}(X) \cong H_{\text{eh}}^{p-1}(X, \mathbb{G}_m)
\]

for all \( p \in \mathbb{Z} \). Recall the motivic Albanese triangulated functor

\[
\text{LA} \text{lb} : \text{DM}^{\text{eff}}_{\text{gm}} \to D^b(\mathcal{M}_1)
\]

where \( \text{DM}^{\text{eff}}_{\text{gm}} \subset \text{DM}^{\text{eff}}_{\text{Nis}} \) is the subcategory of compact objects, i.e., the category of geometric motives, which has been constructed in \([12, \text{Def. 5.2.1}]\) (see also \([10, \text{Thm. 2.4.1}]\)). Rationally, \( \text{LA} \text{lb} \) yields a left adjoint to the inclusion functor given by \( \text{Tot} \) in \((2.3)\) (see \([12, \text{Thm. 6.2.1}]\)).

Applying \( \text{LA} \text{lb} \) to the motive of any algebraic scheme \( X \) we get \( \text{LA} \text{lb}(X) \in D^b(\mathcal{M}_1) \), a complex of 1-motives whose \( p \)-th homology \( L_p \text{Alb}(X) \in \mathcal{M}_1 \) is a 1-motive (with cotorsion, see \([12, \text{Def. 8.2.1}]\)). Dually, we have \( \text{RP} \text{ic}(X) \in D^b(\mathcal{M}_1) \) (see \([12, \text{§8.3}]\)). Taking the Cartier dual of \( L_p \text{Alb}(X) \) we get \( \text{RP} \text{ic}(X) \in \mathcal{M}_1 \) and conversely via \((2.13)\). Now, the motivic Albanese map

\[
M(X) \to \text{Tot} \text{LA} \text{lb}(X)
\]

in \( \text{DM}^{\text{eff}}_{\text{et}} \) (see \([12, \text{§8.2.7}]\)) yields an integrally defined map

\[
\text{Hom}^{D^b(\mathcal{M}_1)}(\text{LA} \text{lb}(X), [0 \to \mathbb{G}_m][p]) \to \text{Hom}_{\text{DM}^{\text{eff}}_{\text{et}}}(M(X), \mathbb{Z}(1)[p]) \cong H_{\text{eh}}^{p-1}(X, \mathbb{G}_m).
\]

Rationally (by adjunction), this map becomes a \( \mathbb{Q} \)-linear isomorphism

\[
H_{\text{eh}}^{p-1}(X, \mathbb{G}_m)_{\mathbb{Q}} \cong \text{Hom}_{\text{DM}^{\text{et}}_{\mathbb{Q}}}(M(X), \mathbb{Z}(1)[p]) \cong \text{Hom}_{D^b(\mathcal{M}^*_1)}(\text{LA} \text{lb}(X), \mathbb{G}_m[-1][p]).
\]

Using \((2.13)\) we set

\[
\text{Ext}^p(\mathbb{Z}, \text{RP} \text{ic}(X)) := \text{Hom}_{D^b(\mathcal{M}_1)}(\mathbb{Z}, \text{RP} \text{ic}(X)[p]) \cong \text{Hom}_{D^b(\mathcal{M}_1)}(\text{LA} \text{lb}(X), \mathbb{G}_m[-1][p])
\]

for all \( p \in \mathbb{Z} \) and we also have (cf. \([12, \text{Lemma 10.5.1}]\)):

3.2.1. Lemma. For any \( X \) over \( K = \overline{\mathbb{Q}} \) and \( p \in \mathbb{Z} \) there is an extension

\[
0 \to \text{Ext}(\mathbb{Z}, \text{RP} \text{ic}(X)) \to \text{Ext}^p(\mathbb{Z}, \text{RP} \text{ic}(X)) \xrightarrow{\epsilon} \text{Hom}(\mathbb{Z}, \text{RP} \text{ic}(X)) \to 0
\]

where the \( \text{Hom} \) and \( \text{Ext} \) are here taken in the category \( \mathcal{M}_1 \) of 1-motives with torsion. The composition of \((3.2)\) with the period regulator \( \tau_{\mathbb{Q}}^{p,1} : H_{\text{eh}}^{p-1}(X, \mathbb{G}_m) \to H_{\mathbb{Z}}^{p,1}(X) \) induces a mapping

\[
\theta_{\mathbb{Q}}^{p,1} : \text{Hom}(\mathbb{Z}, \text{RP} \text{ic}(X)) \to H_{\mathbb{Z}}^{p,1}(X).
\]
Proof. In fact, the canonical spectral sequence
\[ E^{p,q}_2 = \text{Ext}^p(\mathbb{Z}, \text{R}^q \text{Pic}(X)) \Rightarrow \text{Ext}^{p+q}(\mathbb{Z}, \text{RPic}(X)) \]
yields the claimed extension since the abelian category of 1-motives with torsion \( ^1\mathcal{M}_1(K) \) is of homological dimension 1 over the algebraically closed field \( K = \overline{\mathbb{Q}} \). Moreover, for any 1-motive \( M = \text{R}^p \text{Pic}(X) \in ^1\mathcal{M}_1(K) \) the group \( \text{Ext}(\mathbb{Z}, M) \) is divisible and the group \( \text{Hom}(\mathbb{Z}, M) \) is finitely generated (as it follows easily by making use of [12, §C.8]). The mapping in the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}(\mathbb{Z}, \text{R}^{p-1} \text{Pic}(X)) & \xrightarrow{\text{zero}} & \\
\downarrow & \downarrow & \\
\text{Ext}^p(\mathbb{Z}, \text{RPic}(X)) & \xrightarrow{\pi} & H^{p-1}_{\text{et}}(X, \mathbb{G}_m) \xrightarrow{r_{\eta}^{p,1}} H^{p,1}_{\text{et}}(X) \\
\end{array}
\]

obtained by the composition of (3.2) with the period regulator \( r_{\eta}^{p,1} \), is therefore sending \( \text{Ext}(\mathbb{Z}, M) \) to zero, since \( H^{p,1}_{\text{et}}(X) \) is finitely generated. We then get the induced mapping \( \theta_{\eta}^{p} \) as indicated in the diagram. \( \square \)

Also for the Betti realization, there is an integrally defined group homomorphism
\[ \theta_{\mathbb{Z}}^{p} : T_{\mathbb{Z}}(\text{R}^p \text{Pic}(X)_{\mathbb{C}})_{\text{fr}} \rightarrow H^{p}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(1))_{\text{fr}} \]
induced via Cartier duality, by applying the Betti realization \( \beta_{\alpha} \) in (1.3) to the motivic Albanese (3.2) in a canonical way. This is justified after the natural identification of Deligne’s \( T_{\mathbb{Z}} \) with the Betti realization \( \beta_{\alpha} \) on 1-motives (see [12, Thm. 15.4.1] and [31] for an explicit construction of the natural isomorphism \( T_{\mathbb{Z}} \cong \beta_{\alpha} \text{Tot} \)). Rationally, it yields an injection
\[ \theta_{\mathbb{Q}}^{p} : T_{\mathbb{Q}}(\text{R}^p \text{Pic}(X)_{\mathbb{C}}) \cong H^{p}_{(1)}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(1)) \subset H^{p}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(1)) \]
where the notation \( H^{p}_{(1)} \) is taken to indicate the largest 1-motivic part of \( H^{p}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(1)) \) (more precisely, this is given by the underlying \( \mathbb{Q} \)-vector space associated to the mixed Hodge structure, see [12, Cor. 15.3.1]).

For the de Rham realization, similarly, we have a \( K \)-linear mapping
\[ \theta_{\text{dR}}^{p} : T_{\text{dR}}(\text{R}^p \text{Pic}(X)) \rightarrow H^{p}_{\text{dR}}(X). \]
Actually, for \( M = L_{\mu} \text{Alb}(X) \) and \( M^{*} = \text{R}^p \text{Pic}(X) \), we have \( \eta_{M^{*}} \) the \( \mathbb{C} \)-inverse of the period isomorphism \( \varpi_{M^{*}, \mathbb{C}} \) in Theorem 2.2.2 and \( \eta_{X}^{p,1} \) which is the inverse of the period isomorphism in Definition 1.2.4. Together with \( \theta_{\mathbb{Z}}^{p} \) and \( \theta_{\text{dR}}^{p} \), we obtain a diagram

\[
\begin{array}{cccc}
T_{\mathbb{Z}}(\text{R}^p \text{Pic}(X)_{\mathbb{C}}) & \xrightarrow{\eta_{M^{*}} \otimes \mathbb{C}} & H^{p}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(1))_{\mathbb{C}} & \xrightarrow{2\pi i} & H^{p}(X_{\text{an}}, \mathbb{C}) \\
\downarrow & & \downarrow & \downarrow & \\
T_{\text{dR}}(\text{R}^p \text{Pic}(X)_{\mathbb{C}}) & \xrightarrow{\theta_{\text{dR}}^{p} \otimes \mathbb{C}} & H^{p}_{\text{dR}}(X) \otimes_{K} \mathbb{C} & \\
\end{array}
\]

We have that this diagram commutes, in fact:
3.2.2. **Lemma.** Let $X$ be over the field $K = \overline{\mathbb{Q}}$ and $p \in \mathbb{Z}$. There is a morphism

$$\theta^p_\text{dR} := (\theta^p_\text{dR}, \theta^p_\text{B}) : T^\text{dR}(L_p \text{Alb}(X))_{\text{fr}} \to H^{p,1}_\text{dR}(X)_{\text{fr}}$$

in the category $\text{Mod}^\text{fr}_{K,\mathbb{Z}}$. Rationally $\theta^p_\text{dR} \otimes \mathbb{Q}$ becomes injective. Moreover, $\theta^0_\text{dR}$ and $\theta^1_\text{dR}$ are integrally defined isomorphisms.

**Proof.** This is a consequence of [12, Cor. 16.3.2]. For $p = 0, 1$ it is straightforward that they are isomorphisms. \qed

3.2.3. **Lemma.** The map $\theta^p_\text{v}$ defined in Lemma 3.2.1 factors through the de Rham-Betti realization via the Cartier duals (2.13), i.e., we have the following factorization

$$\theta^p_\text{v} : \text{Hom}(K, R^p \text{Pic}(X)) \xrightarrow{\text{Hom}(L_p \text{Alb}(X), \mathbb{G}_m[-1])} T^\omega(L_p \text{Alb}(X)) \xrightarrow{T_{\text{dR}}^\omega} H^{p,1}_\text{dR}(X)$$

such that $\iota$ is given by $\theta^p_{\text{dR}}$ in Lemma 3.2.2, using Corollary 3.1.2, as follows

$$T^\omega(L_p \text{Alb}(X)) = \text{Hom}(K(0), T_{\text{dR}}(L_p \text{Alb}(X))) \to \text{Hom}(K(0), H^{p,1}_\text{dR}(X)) \cong H^{p,1}_\omega(X)$$

and the latter $\text{Hom}$ is here taken in $\text{Mod}^\omega_{K,\mathbb{Z}}$.

**Proof.** By construction $\theta^p_\text{v}$ is induced by $r^1_\omega$ on a quotient via the motivic Albanese (3.2) applying Betti and de Rham realizations so that the claimed factorization is clear. \qed

Thus, showing the period conjecture (1.6) for $q = 1$ is equivalent to seeing that $\theta^p_\text{v}$ is surjective, rationally. Recall (see [12, Prop. 10.4.2]) that for any $X$ of dimension $d = \dim(X)$ the 1-motive $L_{d+1} \text{Alb}(X)$ is a group of multiplicative type and

$$L_p \text{Alb}(X) = \begin{cases} 0 & \text{if } p < 0 \\ [\mathbb{Z}[\pi_0(X)] \to 0] & \text{if } p = 0 \\ [L_1 \xrightarrow{u_1} G_1] & \text{if } p = 1 \\ 0 & \text{if } p > \max(2, d + 1) \end{cases}$$

where $G_1$ is connected, so that $L_p \text{Alb}(X) \in \mathcal{M}_1$ is free for $p = 0, 1$ (see [12, Prop. 12.6.3 c)]). Thus $R^0 \text{Pic}(X) = [\mathbb{Z}[\pi_0(X)] \to 0]^* = [0 \to \mathbb{Z}[\pi_0(X)]^\vee \otimes \mathbb{G}_m]$ is a torus and we have that $\text{Ext}(K, R^0 \text{Pic}(X)) = \text{Hom}_K(K(0), R^0 \text{Pic}(X)) = K^* \otimes_\mathbb{Z} [\pi_0(X)]^\vee$ (see [12, Prop. C.8.3 (b)]).

3.2.4. **Theorem.** For any $X$ over $K = \overline{\mathbb{Q}}$ we have that (1.6) holds true for $p = q = 1$, i.e., the period regulator $r^1_\omega : H^1_{\text{dR}}(X, \mathbb{G}_m) \to H^{1,1}_\omega(X)$ is surjective. Moreover, considering the 1-motive $\text{R}^1 \text{Pic}(X) = [L_1^+ \xrightarrow{u_1^+} G_1^\ast]$ which is the Cartier dual of $L_1 \text{Alb}(X)$ we have a canonical isomorphism

$$\text{Ker} u_1^* \cong H^1_{\text{dR}}(X) \cap H^1(X_{\text{an}}, \mathbb{Z}_{\text{an}}(1)) = H^{1,1}_\omega(X).$$

In particular, if $X$ is proper $H^1_{\text{dR}}(X, \mathbb{G}_m) \cong K^* \otimes_\mathbb{Z} [\pi_0(X)]^\vee$ and $H^1_{\text{dR}}(X) \cap H^1(X_{\text{an}}, \mathbb{Z}(1)) = 0$.

**Proof.** In fact, $\text{R}^1 \text{Pic}(X)$ is free and therefore $\text{Hom}(L_1 \text{Alb}(X), \mathbb{G}_m[-1]) \cong \text{Hom}(K, \text{R}^1 \text{Pic}(X)) \cong \text{Ker} u_1^*$. Thus the extension in Lemma 3.2.1 is

$$0 \to K^* \otimes_\mathbb{Z} [\pi_0(X)]^\vee \to \text{Ext}^1(K, \text{R}^1 \text{Pic}(X)) \to \text{Ker} u_1^* \to 0.$$

Moreover $\theta^1_\omega : \text{Hom}(K, \text{R}^1 \text{Pic}(X)) \cong \text{Ker} u_1^* \xrightarrow{\sim} H^{1,1}_\omega(X)$ is an isomorphism, which in turn implies that $r^1_\omega$ is a surjection. Actually, see Lemma 3.2.3, $\theta^1_\omega$ factors as follows

$$\text{Hom}(K, \text{R}^1 \text{Pic}(X)) \cong_{(a)} T^\omega(L_1 \text{Alb}(X)) \cong_{(b)} \text{Hom}(K(0), H^{1,1}_{\text{dR}}(X)) \cong_{(c)} H^{1,1}_\omega(X)$$
where: (a) is the isomorphism obtained applying Corollary 3.1.4 to \( M = L_1 Alb(X) \); (b) is the \( \text{Hom}(\mathbb{Z}(0), -) \) of the isomorphism \( \theta_{\text{dR}} : T_{\text{dR}}(L_1 Alb(X)) \cong H_{\text{dR}}^{1,1}(X) \) given by \( p = 1 \) in Lemma 3.2.2; (c) is the isomorphism in Corollary 3.1.2. If \( X \) is proper then \( L_1^* = 0 \), i.e., \( L_1 Alb(X) = [L_1 \xrightarrow{u} G_1] \) with \( G_1 \) an abelian variety (see [12, Cor. 12.6.6]) in such a way that \( R^1 \text{Pic}(X) = [0 \to G_1^*] \), and \( H_{\text{dR}}^0(X, \mathbb{G}_m) \cong \mathbb{G}_m(\pi_0(X)) \) (see [12, Lemma 12.4.1]).

3.2.5. **Remark.** We may actually compute \( R^1 \text{Pic}(X) \) by using descent. For example, if \( X \) is normal let \( \overline{X} \) be a normal compactification of \( X \), \( p : \overline{X} \to X \) a smooth hypercovering and \( \overline{X}_* \), a smooth compactification with normal crossing boundary \( Y_* \), such that \( p : \overline{X}_* \to X \) is an hypercovering. Then \( p^* : \text{Pic}_X^0(\overline{X}_*) \to \text{Pic}_X^0(\overline{X}/K) \) is an abelian variety and

\[
R^1 \text{Pic}(X) = [\text{Div}_{\overline{X}_*}^0(\overline{X}_*) \xrightarrow{u_1} \text{Pic}_X^0(\overline{X}/K)]
\]

where \( \text{Div}_{\overline{X}_*}^0(\overline{X}_*) := \text{Ker}(\text{Div}_0^0(\overline{X}_0) \to \text{Div}_1^0(\overline{X}_1)) \) (see [12, Prop. 12.7.2]).

For \( X \) smooth we have that (see [12, Cor. 9.2.3])

\[
L_p Alb(X) = \begin{cases} 
[\mathbb{Z}[\pi_0(X)] \to 0] & \text{if } p = 0 \\
[0 \to \mathcal{A}_X^0] & \text{if } p = 1 \\
[0 \to \text{NS}_X^*/K] & \text{if } p = 2 \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \mathcal{A}_X^0/K \) is the Serre-Albanese semi-abelian variety and \( \text{NS}_X^*/K \) denotes the group of multiplicative type dual to the Néron-Severi group \( \text{NS}_X^*/K \). In this case, we then have

\[
R^p \text{Pic}(X) = \begin{cases} 
[0 \to \mathbb{Z}[\pi_0(X)]^*] & \text{if } p = 0 \\
[\text{Div}_1^0(\overline{X}) \xrightarrow{u_1} \text{Pic}_X^0(\overline{X}/K)] & \text{if } p = 1 \\
[\text{NS}_X/K \to 0] & \text{if } p = 2 \\
0 & \text{otherwise,}
\end{cases}
\]

for a smooth compactification \( \overline{X} \) with normal crossing boundary \( Y \). Note that, reducing to the smooth case by blow-up induction we can see that the map (3.2) is an isomorphism for \( p = 0, 1 \) (cf. [12, Lemma 12.6.4 b]]). We deduce the following:

3.2.6. **Corollary.** For any scheme \( X \) over \( K = \overline{K} \) we have a short exact sequence

\[
0 \to K^* \otimes \mathbb{Z}[\pi_0(X)]^* \to H_{\text{dR}}^0(X, \mathbb{G}_m) \xrightarrow{r_{\text{dR}}^{1,1}} H_{\text{dR}}^{1,1}(X) \to 0.
\]

In general, we also have:

3.2.7. **Proposition.** For \( K = \overline{K} \) the period regulator \( r_{\text{dR}}^{1,1} \) admits a factorization

\[
H_{\text{dR}}^{p-1}(X, \mathbb{G}_m)_Q \to T_{\text{dR}}^Q(L_p Alb(X)) \hookrightarrow H_{\text{dR}}^p(X) \cap H^p(X_{\text{an}}, Q_{\text{an}}(1)) = H_{\text{dR}}^{p,1}(X)_Q
\]

where the projection is given by Lemma 3.2.1 via \( T_{\text{dR}}^Q \) and the inclusion is given by \( \theta_{\text{dR}}^p \otimes Q \) in Lemma 3.2.2. Therefore, the conjecture (1.6) is equivalent to \( T_{\text{dR}}^Q(L_p Alb(X)) \cong H_{\text{dR}}^{p,1}(X)_Q \).
Proof. In fact, using the adjunction (3.3), the Cartier dual $\pi^*$ of $\pi$ in Lemma 3.2.1, the factorization of Lemma 3.2.3 and Theorem 2.6.1 we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{D^b(\mathcal{M}_1^0)}(\text{LAlb}(X), \mathbb{G}_m[-1][p]) & \cong & H^{p-1}_{\text{ch}}(X, \mathbb{G}_m) \\
\pi^* \otimes \mathbb{Q} & \Downarrow & H^{p-1}_{\text{ch}}(X, \mathbb{G}_m) \otimes \mathbb{Q} \\
\text{Hom}(L_p\text{Alb}(X), \mathbb{G}_m[-1]) & \cong & T_{dR}^\mathbb{Q}(L_p\text{Alb}(X))
\end{array}
\]

For $X$ smooth we further have that

\[
H^{p,1}_w(X) \cong H^{p-1}_{\text{ch}}(X, \mathbb{G}_m) \cong H^{p-1}_{\text{st}}(X, \mathbb{G}_m)
\]

and this latter is vanishing after tensoring with $\mathbb{Q}$ for all $p \neq 1, 2$ (see [21, Prop. 1.4]). Accordingly, the period conjecture (1.6) for $X$ smooth and $p \neq 1, 2$ is in fact equivalent to (1.8), i.e.,

\[
H^{p,1}_w(X) = H^{p}_{\text{deR}}(X) \cap H^p(X_{\text{an}}, \mathbb{Q}_{\text{an}}(1)) = 0 \quad p \neq 1, 2
\]

For $p = 2$ and $X$ smooth we have that $H^{2,1}(X) \cong \text{Pic}(X)$, $r^{2,1}_w = \text{cl}$ is induced by the usual cycle class map and $T_{dR}^\mathbb{Q}(L_2\text{Alb}(X)) = \text{NS}(X)_{\mathbb{Q}}$.

We here recover the results of Bost-Charles (see [15, Thm. 5.1] and [16, Cor. 3.9-3.10]) as follows. We refer to [12, Chap. 4] for the notion of biextension of 1-motives. The following is a generalization of [16, Thm 3.8 2] and of the discussion of the sign issue in [16, §3.4]:

3.2.8. Lemma. For $N, M \in \mathcal{M}_1(\mathbb{Q})$ we have that

\[
\text{Biext}(N, M; \mathbb{G}_m) \cong (T_{\text{dR}}(N))^\vee \otimes T_{\text{dR}}(M)^\vee \otimes \mathbb{Z}(1)_{\infty}
\]

and, when $N = M$, the subgroup of symmetric biextensions corresponds to alternating elements.

Proof. Recall that $\text{Biext}(-, M; \mathbb{G}_m)$ is representable by the Cartier dual $M^*$ for $M \in \mathcal{M}_1(\mathbb{Q})$ (see [12, Prop. 4.1.1]). Thus $\text{Biext}(N, M; \mathbb{G}_m) = \text{Hom}(N, M^*) \cong \text{Hom}(T_{\text{dR}}(N), T_{\text{dR}}(M)^*)$ where we here use Theorems 2.4.3 - 2.6.1. Now $T_{\text{dR}}(M)^* = T_{\text{dR}}(M)^\vee(1)$ in such a way that $\text{Hom}(T_{\text{dR}}(N), T_{\text{dR}}(M)^*) = \text{Hom}(\mathbb{Z}(0), T_{\text{dR}}(N)^\vee \otimes T_{\text{dR}}(M)^\vee \otimes \mathbb{Z}(1))$ making use of the tensor structure of the category $\text{Mod}_{\mathbb{Q}}^\infty_K$ by Lemma 2.3.2.

Assume $N = M$. Since $\text{Biext}(M, M; \mathbb{G}_m) \cong \text{Hom}(T_{\text{dR}}(M), T_{\text{dR}}(M)^*)$, any biextension $\mathcal{P}$ corresponds to a pairing $T_{\text{dR}}(M) \otimes T_{\text{dR}}(M) \rightarrow \mathbb{Z}(1)$ which induces the pairing [18, 10.2.3] on Deligne-Hodge realizations and the pairing [18, 10.2.7] on de Rham realizations; if $\mathcal{P}$ is symmetric, the pairing is alternating by [18, 10.2.5 & 10.2.8].

3.2.9. Proposition. For $X$ over $K = \overline{\mathbb{Q}}$ we have that

\[
\text{Biext}(L_1\text{Alb}(X), L_1\text{Alb}(X); \mathbb{G}_m)^\text{sym} \cong (H^{1,0}_{\text{dR}}(X) \otimes H^{1,0}_{\text{dR}}(X) \otimes \mathbb{Z}(1))_{\infty}.
\]

Proof. Applying Lemma 3.2.8 to the free 1-motive $L_1\text{Alb}(X)$ we obtain the claimed formula. In fact, recall that $T_{\mathbb{Z}}(L_1\text{Alb}(X)) \cong H_1(X_{\text{an}}, \mathbb{Z}_{\text{an}})_R$ and observe that $H^{1,0}_{\text{dR}}(X)$ is identified with $(T_{\text{dR}}(L_1\text{Alb}(X))_{\infty}^\vee$ up to inverting the period isomorphism by the same argument of Lemma 3.2.2.

This implies that the period conjecture for $p = 2$ holds true in several cases, e.g. for abelian varieties, as previously indicated by Bost (see [15, Thm. 5.1]).
3.3. The case of \( q = 0 \). Consider the case of \( \mathbb{Z}(0) \) which is canonically identified with \( \text{Tot} ([\mathbb{Z} \to 0]) = \mathbb{Z}[0] \). Note that \( H^{p,0}(X) \cong H^{p}_{\text{et}}(X, \mathbb{Z}) \). Let \( \mathcal{M}_{0}(K) \subset \mathcal{M}_{1}(K) \) be the full subcategory of 0-motives or Artin motives over \( K \). Recall that the motivic \( \pi_{0} \) (see [12, §5.4] and [10, Cor. 2.3.4]) is a triangulated functor

\[
L \pi_{0} : \text{DM}^{\text{eff}}_{\text{gm}} \to D^{b}(\mathcal{M}_{0})
\]

whence \( L \pi_{0}(X) \in D^{b}(\mathcal{M}_{0}) \), a complex in the the derived category of Artin motives, associated to the motive of \( X \). We have that \( M(X) \to \text{Tot} L \pi_{0}(X) \in \text{DM}^{\text{eff}}_{\text{et}} \) (see (2.3) for \( \text{Tot} \)) induces

\[
\text{Hom}_{D^{b}(\mathcal{M}_{0})}(L \pi_{0}(X), \mathbb{Z}[p]) \to \text{Hom}_{\text{DM}^{\text{eff}}_{\text{et}}}(M(X), \mathbb{Z}[0]) \cong H^{p}_{\text{et}}(X, \mathbb{Z}).
\]

This map is an isomorphism, integrally, for \( p = 0, 1 \) (cf. [12, Lemma 12.6.4 b]) and it becomes, by adjunction, a \( \mathbb{Q} \)-linear isomorphism, for all \( p \). Recall that for any \( M \in \text{DM}^{\text{eff}}_{\text{gm}} \) we have (see [12, Prop. 8.2.3])

\[
L \text{Alb}(M(q)) \cong \begin{cases} 
L \pi_{0}(M)(1) & \text{if } q = 1 \\
0 & \text{for } q \geq 2
\end{cases}
\]

where an Artin motive twisted by one is a 1-motive of weight \(-2\), i.e., the twist by one functor \((-)(1) : D^{b}(\mathcal{M}_{0}) \to D^{b}(\mathcal{M}_{1}) \) is induced by \( L \sim [0 \to L \otimes \mathbb{G}_{m}] \). Note that as soon as \( K = \overline{\mathbb{Q}} \) Artin motives are of homological dimension 0 and we have that

\[
\text{Hom}_{D^{b}(\mathcal{M}_{0})}(L \pi_{0}(X), \mathbb{Z}[p]) = \text{Hom}_{\mathcal{M}_{0}}(L \pi_{0}(X), \mathbb{Z}).
\]

Moreover, we have that

\[
H^{p}_{\text{et}}(X, \mathbb{Z}) \cong \text{Hom}_{\text{DM}^{\text{eff}}_{\text{et}}}(M(X)(1), \mathbb{Z}[0])
\]

by Voevodsky cancellation theorem [30].

3.3.1. Theorem. For any \( X \) over \( K = \overline{\mathbb{Q}} \) we have that (1.6) holds true for \( p = 1 \) and \( q = 0 \). Moreover, we have

\[
H^{1}_{\text{et}}(X, \mathbb{Z}) \cong H^{1}_{\text{DR}}(X) \cap H^{1}(X_{an}, \mathbb{Z}_{an}) \cong H^{1,0}_{\text{et}}(X)
\]

which is vanishing if \( X \) is normal.

Proof. Making use of Proposition 1.3.1 we are left to show the period conjecture for \( M(X)(1) \). We have that

\[
\text{Hom}_{\text{DM}^{\text{eff}}_{\text{et}}}(M(X)(1), \mathbb{Z}[1]) \cong \text{Hom}_{\mathcal{M}_{1}}(L \text{Alb}(M(X)(1)), \mathbb{Z}[1]).
\]

We have \( L_{0} \text{Alb}(M(X)(1)) \cong L_{0} \pi_{0}(X)(1) \cong [0 \to \mathbb{Z}[\pi_{0}(X)] \otimes \mathbb{G}_{m}] \) in such a way that

\[
\text{Ext}_{\mathcal{M}_{1}}(L_{0} \text{Alb}(M(X)(1)), \mathbb{G}_{m}[-1]) = 0
\]

and (cf. (3.3) for \( M(X)(1) \)) we obtain

\[
H^{1}_{\text{et}}(X, \mathbb{Z}) \cong \text{Hom}_{\mathcal{M}_{1}}(L_{1} \text{Alb}(M(X)(1)), \mathbb{G}_{m}[-1]).
\]

Now \( T_{\text{et}}(L_{1} \text{Alb}(M(X)(1))) \cong H^{1,1}_{\text{et}}(M(X)(1)) \cong H^{1,0}_{\text{et}}(X) \) by Lemma 3.2.2 twisted by \((-1)\) and the same argument in the proof of Theorem 3.2.4 applies here. Finally, recall that \( H^{1}_{\text{et}}(X, \mathbb{Z}) \cong H^{1}_{\text{et}}(X, \mathbb{Z}) \) for any scheme \( X \) and \( H^{1}_{\text{et}}(X, \mathbb{Z}) = 0 \) if \( X \) is normal (see [12, Lemma 12.3.2 & Prop. 12.3.4]).

3.3.2. Remark. For \( X \) not normal (e.g. for the nodal curve) the group \( H^{1}_{\text{et}}(X, \mathbb{Z}) \) can be non-zero. Moreover, for any \( X \) we have a geometric interpretation \( H^{1}_{\text{et}}(X, \mathbb{Z}) \cong L \text{Pic}(X) \hookrightarrow \text{Pic}(X[t, t^{-1}]) \) by a theorem of Weibel [33, Thm. 7.6]. Note that this \( L \text{Pic}(X) \) is also a sub-quotient of the negative \( K \)-theory group \( K_{-1}(X) \) (see [33, Thm. 8.5]).
For $X$ smooth we have a quasi-isomorphism $L\pi_0(X) \cong \mathbb{Z}[\pi_0(X)][0]$ (see [12, Prop. 5.4.1]) which means that $H^{p,0}(X)_Q = 0$ for $p \neq 0$. This yields (as it also does the Proposition 1.4.4 for $X$ smooth) that the period conjecture (1.6) is equivalent to

$$H^{p,0}_\text{dR}(X) \cap H^p(X_{\text{an}}, \mathbb{Q}_{\text{an}}) = 0 \quad p \neq 0.$$  

(3.5)

3.3.3. Remark. The period conjecture (1.6) for $q = 0$ and $X$ smooth is also equivalent to the surjectivity of $f^p_w : H^{p,0}_\text{dR}(\pi_0(X))_Q \to H^{p,0}_\text{dR}(X)_Q$ induced by the canonical morphism $f : X \to \pi_0(X)$, for all $p \geq 0$. In fact, the morphism $f$ induces a map $M(X) \to M(\pi_0(X))$ and a commutative square by functoriality

$$
\begin{array}{ccc}
H^{p,0}(X)_Q & \xrightarrow{r^{p,0}_w} & H^{p,0}(X)_Q \\
\downarrow f^p & & \downarrow f^p \\
H^{p,0}(\pi_0(X))_Q & \xrightarrow{\sim} & H^{p,0}(\pi_0(X))_Q
\end{array}
$$

where $f^p : \text{Hom}_{\text{DM}_{\text{et}}}(M(\pi_0(X)), \mathbb{Z}[p])_Q \to \text{Hom}_{\text{DM}_{\text{et}}}(M(X), \mathbb{Z}[p])_Q$ is an isomorphism for $X$ smooth; since $\dim \pi_0(X) = 0$ then $r^{p,0}_w$ is clearly an isomorphism for $\pi_0(X)$. For $p = 0$ the group $H^0(X_{\text{an}}, \mathbb{Z}_{\text{an}}(0))$ has rank equal to the rank of $\mathbb{Z}[\pi_0(X)]$ and $f^0_w$ is an isomorphism; for $p \neq 0$ the surjectivity of $f^p_w$ is equivalent to the vanishing of all groups.

3.4. Arbitrary twists. We now apply Waldschmidt’s Theorem [32, Thm 5.2.1] to arbitrary twists.

3.4.1. Proposition. For $M = [L \to G]$ a free 1-motive over $K = \overline{\mathbb{Q}}$ and $q \in \mathbb{Z}$ an integer we have that

1) the group $\text{Hom}(\mathbb{Z}(q), T^{\text{BdR}}(M))$ of homomorphisms in $\text{Mod}^{\mathbb{Z}}_{\mathbb{Z},K}$ or $\text{Mod}^{\mathbb{Z}}_{\mathbb{Q},K}$ is trivial for $q \neq 0, 1$;

2) the group $\text{Hom}(\mathbb{Z}(q), T^{\text{dR}}(M))$ of homomorphisms in $\text{Mod}^{\mathbb{Z}}_{K,\mathbb{Z}}$ or $\text{Mod}^{\mathbb{Z}}_{K,\mathbb{Q}}$ is trivial for $q \neq 0, 1$.

Proof. 1) We work in $\text{Mod}^{\mathbb{Z}}_{\mathbb{Z},K}$ and leave to the reader the other case. We suppose first that $L = 0$. Consider a non trivial $\varphi \in \text{Hom}_{\text{BdR}}(\mathbb{Z}(q), T^{\text{BdR}}(M))$ and the subgroup $\Gamma = T^{\mathbb{Z}}(\mathbb{Z}(q)) = \mathbb{Z} \subset T^{\text{BdR}}(\mathbb{Z}(q))_\mathbb{C} = \mathbb{C}$. Via the non trivial map $\varphi_K \otimes \mathbb{C} : T^{\text{BdR}}(\mathbb{Z}(q))_\mathbb{C} \to T^{\text{BdR}}(G)_\mathbb{C} = \text{Lie}(G^\mathbb{C})$ we can identify $\Gamma$ with a subgroup of $\text{Lie}(G^\mathbb{C})$. This subgroup is contained in $V_\mathbb{C}$ with $V \subset \text{Lie}(G^\mathbb{C})$ defined by the image $\varphi_K(T^{\text{dR}}(\mathbb{Z}(q)))$. Via the exponential map $\text{Lie}(G^\mathbb{C}) \to G^\mathbb{C}(\mathbb{C})$ the image of $\Gamma$ is $0 \in G^\mathbb{K}(K)$ as $\varphi$ is a map in the category $\text{Mod}^{\mathbb{Z}}_{\mathbb{Z},K}$ (respectively $\text{Mod}^{\mathbb{Z}}_{\mathbb{Q},K}$). We deduce from Waldschmidt’s Theorem [32, Thm 5.2.1] that $V \subset \text{Lie}(G^\mathbb{C})$ is the Lie algebra of a 1-dimensional algebraic subgroup $H$ of $G^\mathbb{C}$. There are only two possibilities $H = \mathbb{G}_a$ and $H = \mathbb{G}_m$. In both cases the period morphism for $\mathbb{Z}(q)$ identifies $\Gamma$ with the subgroup $(2\pi i)^q \mathbb{Z} \subset \text{Lie}(H_\mathbb{C})$ that goes to $0$ via $\exp_{H_\mathbb{C}}$. For $H = \mathbb{G}_a$ the map $\exp_{H_\mathbb{C}}$ is the identity, leading to a contradiction. For $H = \mathbb{G}_m$ the kernel of $\exp_{H_\mathbb{C}}$ is $(2\pi i)^q \mathbb{Z}$ forcing $q = 1$.

Secondly we suppose that $G = 0$. Consider a non trivial $\varphi \in \text{Hom}_{\text{BdR}}(\mathbb{Z}(q), T^{\text{BdR}}(M))$. Recall that $T^{\text{dR}}(M) = L \otimes K$ and the period map is induced by the inclusion $L \subset L \otimes K$. Let $e = \varphi_K(1) \in L \otimes K$. It is a non-zero element. Using that $T^{\mathbb{Z}}(\mathbb{Z}(q))$ is identified via the period morphism for $\mathbb{Z}(q)$ with $(2\pi i)^q \mathbb{Z}$, we deduce that $\varphi_{\mathbb{Z}}(1) = (2\pi i)^q \cdot e$ should lie in $L \subset L \otimes K$. As $\pi$ is transcendental, this forces $q = 0$.

For general $M = [L \to G]$ we reduce to $G$ and $L$ to conclude the statement.
2) We prove the statement for Mod_{\overline{k}, \mathbb{Q}} using Lemma 3.3.4. The analogue for Mod_{\overline{k}, \mathbb{Q}} follows similarly. Given a 1-motive \( M \) and its Cartier dual \( M^* \) we have a natural identification
\[
\varsigma : \text{Hom}(\mathbb{Z}(q), T_{\text{BDR}}(M^*)) \cong \text{Hom}(\varsigma(\mathbb{Z}(q)), \varsigma(T_{\text{BDR}}(M^*)) = \text{Hom}(\mathbb{Z}(q), T_{\text{DRB}}(M)).
\]

The statement follows then from 1).

Denote \( H^{p,q}_{\text{dR},(1)}(X)_{\mathfrak{f}} \subset H^{p,q}_{\text{dR}}(X)_{\mathfrak{f}} \) the image of \( T_{\text{dR}}(L_p \text{Alb}(X))_{\mathfrak{f}}(q-1) \) under \( \theta_{\text{dR}}^p(q-1) \) of Lemma 3.2.2 twisted by \( q - 1 \). We have:

3.4.2. Corollary. We get that \( H^{p,q}_{\overline{\omega},(1)}(X)_{\mathfrak{f}} = 0 \) if \( q \neq 0, 1 \). For \( p = 1 \) we have \( H^{1,q}_{\overline{\omega},(1)}(X) = H^{1,q}_{\overline{\omega}}(X) \) and
\[
H^{1,q}_{\overline{\omega}}(X) = \begin{cases} 
H^1_{\text{dR}}(X, \mathbb{Z}) \text{ (see Theorem 3.3.1)} & \text{if } q = 0 \\
\text{Ker } u_1^* \text{ (see Theorem 3.2.4)} & \text{if } q = 1 \\
0 & q \neq 0, 1.
\end{cases}
\]

Proof. We apply the Proposition 3.4.1 2) to \( M = L_p \text{Alb}(X) \) to deduce that \( H^{p-1,q}_{\overline{\omega},(1)}(X)_{\mathfrak{f}} = \text{Hom}_{\text{dR}}(\mathbb{Z}(0), H^{p,1}_{\text{dR},(1)}(X)(-q)) = \text{Hom}_{\text{dR}}(\mathbb{Z}(q), T_{\text{dR}}(L_p \text{Alb}(X))_{\mathfrak{f}}) = 0 \) if \( q \neq 0, 1 \).

Thus, for the period conjecture in degree \( p = 1 \), the previous computations for the twists \( q = 0, 1 \) are the only relevant.

3.5. Higher odd degrees. Next, let \( X \) be a smooth and projective variety over \( K = \overline{\mathbb{Q}} \). Denote \( J^{2k+1}(X) \) the intermediate Jacobian: as a real analytic manifold, it is defined as the quotient of the image \( H_{\overline{\omega}}^{2k+1}(X) \) of \( H^{2k+1}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(k)) \) in \( H^{2k+1}(X_{\text{an}}, \mathbb{R}(k)) \). This defines a full lattice of \( H^{2k+1}(X_{\text{an}}, \mathbb{R}(k)) \) so that \( J^{2k+1}(X) \) is compact. It has also a natural complex analytic structure induced by the identification
\[
J^{2k+1}(X) := H^{2k+1}(X_{\text{an}}, \mathbb{C})/(F^{k+1}H^{2k+1}(X_{\text{an}}, \mathbb{C}) + (\overline{\omega} X_{\text{an}}^{2k+1,k})^{-1}(H^{2k+1}(X))).
\]

Thus \( J^{2k+1}(X) \) is a complex torus.

For integers \( n \) define \( N^n H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \subset H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \), the \( n \)-th step of the geometric coniveau filtration, as the kernel of
\[
H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \rightarrow \bigoplus_{Z \subset X} H^{2k+1}(X_{\text{an}} \setminus Z, \mathbb{Q}_{\text{an}}(k))
\]
for \( Z \subset X \) varying among the codimension \( \geq n \) closed subschemes.

3.5.1. Lemma. Assume that \( H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \) has geometric coniveau \( k \), i.e., that we have \( N^k H^{2k+1}(X_{\text{an}}, \mathbb{C}) = H^{2k+1}(X_{\text{an}}, \mathbb{C}) \). Then \( J^{2k+1}(X) \) is an abelian variety, which descends to an abelian variety \( J^{2k+1}(X)_K \) over \( K \) with
\[
T_{\text{dR}}(J^{2k+1}(X)_K) = (H^{2k+1}_{\text{dR}}(X), H^{2k+1}_{\overline{\omega}}(X), J^{2k+1}_{\overline{\omega}})_K.
\]

Proof. Under the assumption, \( H^{2k+1}_{\overline{\omega}}(X) \) is a polarized Hodge structure of type \((1, 0)\) and \((0, 1)\) so that \( J^{2k+1}(X) \) is polarizable and, hence, an abelian variety. The second statement follows from [2, Thm. A] where it is proven that there exists an abelian variety \( J \) over \( K \) and a correspondence \( \Gamma \in \text{CH}^k(X_{\text{an}}) \) over \( K \), for \( h = k + \dim J^{2k+1}(X) \), inducing an isomorphism \( \Gamma^* : H^1(J_{\text{an}}(q), \mathbb{Q}_{\text{an}}) \cong H^{2k+1}(X_{\text{an}}(q), \mathbb{Q}_{\text{an}}(k)) \) (and hence in de Rham cohomology, compatibly with the period morphisms). Then set \( J^{2k+1}(X)_K := J \).

The period conjecture (1.6) in odd degrees for \( X \) predicts that \( H^{2k+1,q}_{\overline{\omega}}(X) = H^{2k+1}_{\overline{\omega}}(X) \cap H^{2k+1}(X_{\text{an}}, \mathbb{Z}_{\text{an}}(q))_{\mathfrak{f}} = 0 \) for every \( k \in \mathbb{N} \) and every \( q \in \mathbb{Z} \).
3.5.2. **Proposition.** The period conjecture (1.6) in degree \( p = 2k + 1 \) and any twist \( q \) for \( X \) smooth and projective holds true if \( H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \) has geometric coniveau \( k \).

**Proof.** Apply Lemma 3.5.1 and we have that
\[
H^{2k+1,q}_{\text{dR}}(X) = \text{Hom}(\mathbb{Z}(0), T_{\text{dR}}(J^{2k+1}(X)_K)(q-k)) = \text{Hom}(\mathbb{Z}(k-q), T_{\text{dR}}(J^{2k+1}(X)_K)).
\]

This is trivial for \( k - q \neq 0 \) and 1 by Proposition 3.4.1. Now use Theorem 2.6.1. For \( k - q = 0 \) we get that this coincides with the homomorphisms of 1-motives from \([\mathbb{Z} \to 0]\) to \([0 \to J^{2k+1}(X)_K]\), which is 0. For \( k - q = 1 \) this coincides with the homomorphisms of 1-motives from \([0 \to \mathbb{G}_m]\) to \([0 \to J^{2k+1}(X)_K]\), which is also 0. \( \square \)

3.5.3. **Remark.** The Lemma 3.5.1 is proven more generally in [2] for the Hodge structure \( N^k H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \) defined by the \( k \)-th step of the coniveau filtration. Namely, if \( X \) is defined over a number field \( L \subset K \), there is an abelian variety \( J^{2k+1}_a(X) \) over \( L \) with \( T_{\text{dR}}(J^{2k+1}_a(X)) = (N^k H^{2k+1}_{\text{dR}}(X), N^k H^d_{Z}(X), \eta^{2k+1,k}_X) \). The proof of Proposition 3.5.2 using \( J^{2k+1}_a(X) \) gives the following weak version of the period conjecture:

\[
(3.6) \quad N^k H^{2k+1}_{\text{dR}}(X) \cap N^k H^{2k+1}(X_{\text{an}}, Z_{\text{an}}(q))_{\mathbb{Q}} = 0 \quad \text{for every } k \in \mathbb{N} \text{ and every } q \in \mathbb{Z}.
\]

The assumption in Lemma 3.5.1 amounts to say that \( N^k H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \) is equal to \( H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \). This equality holds, for example, for \( k = 1 \) for uniruled smooth projective threefolds; see [1].

The assumption implies, and under the generalized Hodge conjecture is equivalent to, the fact that the Hodge structure \( H^{2k+1}(X_{\text{an}}, \mathbb{Q}) \) has Hodge coniveau \( k \), i.e.,

\[
H^{2k+1}(X_{\text{an}}, \mathbb{Q}) = H^{k+1,k}(X) \oplus H^{k,k+1}(X).
\]

Under this weaker condition on the Hodge coniveau one can still prove that \( H^{2k+1}(X_{\text{an}}, \mathbb{Q}_{\text{an}}(k)) \) is the Hodge structure associated to the abelian variety \( J^{2k+1}_\text{alg}(X) \) over \( \mathbb{C} \), called the algebraic intermediate Jacobian in \( J^{2k+1}(X) \). Unfortunately one lacks the descent to \( K \). See the discussion in [1].

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