Fermion Masses and Mixing and CP-Violation in \( \text{SO}(10) \) Models with Family Symmetries

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Several ideas for solving the problem of fermion mass hierarchy and mixing and specific supersymmetric models that realize it are reviewed. In particular, we discuss many models based on \( \text{SO}(10) \) in four dimensions combined with a family symmetry to accommodate fermion mass hierarchy and mixing, including the case of neutrinos. These models are compared and various tests that can be used to distinguish these models are suggested. We also include a discussion of a few \( \text{SO}(10) \) models in higher space-time dimensions.

Keywords: fermion masses; grand unification; CP violation.

1. Introduction

The origin of fermion masses and mixing and CP violation is the least understood aspect of the Standard Model (SM) of particle physics. In the SM, the fermion masses and mixing angles are completely arbitrary. In order to accommodate their diverse values (see Table 1), the Yukawa couplings must range over five orders of magnitude. Another relevant aspect is that neutrinos are massless in the framework of SM, but recent experiments strongly indicate that they do have small but non-vanishing masses. Incorporating these into theory leads to an increase in the number of parameters. The flavor problem thus consists the following aspects: how to reduce the number of parameters in the Yukawa sector, how to obtain an explanation of the mass hierarchy, how to obtain small neutrino masses and large leptonic mixing angles.

As we extend the SM to the Minimal Supersymmetric Standard Model (MSSM), the particle spectrum is doubled and many more parameters are introduced into
the model. One thus expects large flavor changing neutral current (FCNC) due to the presence of squarks. The strongest constraints come from the lighter two generations due to the type of diagram given in Fig. 1:

\[
\begin{align*}
\frac{m_d^2 - m_s^2}{m_d^2} &\lesssim 6 \times 10^{-3} \left( \frac{m_{\tilde{g}}}{\text{TeV}} \right)^2 \sin \tilde{\theta}^2 \\
\frac{m_e^2 - m_\mu^2}{m_e^2} &< 10^{-1} \left( \frac{m_{\tilde{g}}}{\text{TeV}} \right)^2 \sin \tilde{\theta}'^2
\end{align*}
\]

(1)

(2)

where \( \tilde{\theta} \) and \( \tilde{\theta}' \) refer to the mixing angles in the squark and slepton sectors in the bases where the mass matrices of (d s) and (e \( \mu \)) are diagonal. Thus, in order to suppress the supersymmetric (SUSY) FCNC, a near degeneracy between the first and the second generations of squarks and sleptons is required. This is sometimes called SUSY flavor problem.

There have been many SUSY models proposed to accommodate the observed masses and mixing angles. These models can be classified according to the family symmetry implemented in the model. We also discuss other mechanisms that have been proposed to solve the problem of the fermion mass hierarchy and mixing.

In the following two sections, we introduce various tools that have been used to solve the flavor problem in quark sector and lepton sector, respectively; in Sec. 4, \( \text{SO}(10) \) is reviewed. This is then followed by a review in Sec. 5 and Sec. 6 on various models based on \( \text{SO}(10) \) in 4D, combined with mass texture ansatz and family symmetry, respectively; a comparison of these models is given at the end of Sec. 6. Sec. 7 is devoted to models based on \( \text{SO}(10) \) in higher space-time dimensions. Sec. 8 concludes this review.

Table 1. Current status of fermions masses and CKM matrix elements at \( M_Z \).

| Mass   | Value                        |
|--------|------------------------------|
| \( m_u \) | 1.88 – 2.75 MeV             |
| \( m_c \) | 616 – 733 MeV               |
| \( m_t \) | 168 – 194 MeV               |
| \( m_s/m_d \) | 17 – 25          |
| \( m_d \) | 80.4 – 105 MeV            |
| \( m_b \) | 2.89 – 3.11 GeV            |
| \( m_e \) | 0.487 MeV                  |
| \( m_\mu \) | 103 MeV                    |
| \( m_\tau \) | 1.75 GeV                  |

\[
|V_{\text{CKM,exp}}| = \begin{pmatrix}
0.9745 & -0.9757 & 0.219 & -0.224 & 0.002 & -0.005 \\
0.218 & -0.224 & 0.9736 & -0.9750 & 0.036 & -0.046 \\
0.004 & -0.014 & 0.034 & -0.046 & 0.9989 & -0.9993
\end{pmatrix}
\]
2. Quark Masses and Mixing

Because the strong interaction eigenstates (same as the mass eigenstates) and the weak interaction eigenstates do not match, flavor mixing arises. However, currently we do not have any fundamental understanding for such a mismatch between the strong eigenstates and the weak eigenstates. In the weak basis, the quark mass terms are

\[ \mathcal{L}_{\text{mass}} = -Y_u U_R Q_L H_u - Y_d D_R Q_L H_d + h.c. \]  

where \( Q \) stands for the \( SU(2) \) doublet quark; \( U \) and \( D \) are up- and down-type fermion \( SU(2) \) singlet; \( H_u \) and \( H_d \) are Higgs fields giving masses to up- and down-type quarks. The charged current interaction is given by

\[ \mathcal{L}_{\text{cc}} = g \sqrt{2} \left( W^\mu \gamma^\mu D_L \right) + h.c. \]

The Yukawa couplings \( Y_i \), (\( i = u, d, e, \nu_{LR} \)) are in general non-diagonal. They are diagonalized by the bi-unitary transformations

\[ Y^\text{diag}_u = V_u R Y_u V_u^\dagger \]
\[ Y^\text{diag}_d = V_d R Y_d V_d^\dagger \]

where \( V_R \) and \( V_L \) are the right-handed and left-handed rotations respectively, and all the eigenvalues \( y_{i}'s \) are real and non-negative, and are obtained by diagonalizing the hermitian quantity \( Y^\dagger Y \) and \( YY^\dagger \). The Cabbibo-Kobayashi-Maskawa (CKM) matrix is then given by

\[ V_{\text{CKM}} = V_{ul} V_{dl}^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \]

The unitary matrix \( V_{\text{CKM}} \) has in general 6 phases. By phase redefinition of various quark fields, one can remove 5 of the 6 phases. The remaining one phase is one of the sources for CP violation in the quark sector. There are many ways to parameterize the CKM matrix, for example,

\[ V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\ A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1 \end{pmatrix}. \]
Here the parameters $A$, $\rho$, $\eta$ are of order 1, and $\lambda$ is the sine of the Cabbibo angle which is about 0.22, and thus a good choice as an expansion parameter. A parameterization independent measure for the CP violation is the Jarlskog invariant, defined as

$$J^q_{CP} \equiv \text{Im}\{V_{ud}V_{us}^*V_{cd}V_{cs}\}. \quad (10)$$

Unitarity of $V_{CKM}$ applied to the first and the third columns leads to the following condition

$$V_{td}V_{tb}^* + V_{ud}V_{ub}^* + V_{cd}V_{cb}^* = 0. \quad (11)$$

A geometrical representation of this equation on the complex plane gives rise to the CKM “unitarity triangle” shown in Fig. 2. The three angles of the CKM unitarity triangle are

$$\alpha \equiv \text{Arg}(-V_{td}V_{tb}^*V_{ud}V_{ub}^*), \quad \beta \equiv \text{Arg}(-V_{td}V_{tb}^*V_{cd}V_{cb}^*), \quad \gamma \equiv \text{Arg}(-V_{ud}V_{ub}^*V_{cd}V_{cb}^*). \quad (12)$$

The Jarlskog invariant $J^q_{CP}$ is proportional to the area of the unitarity triangle. A non-vanishing value for $J^q_{CP}$ thus implies non-vanishing values for $(\alpha, \beta, \gamma)$ which in turn indicates CP violation.

2.1. Textures of Mass Matrices

There have been many mass textures, with different elements having zeros in the mass matrices, proposed in order to accommodate the observed fermion mass hierarchy and mixing pattern. Imposing texture ansatz on mass matrices reduces the number of parameters in the Yukawa sector; as a consequence, masses and mixing angles may be related in some simple ways. This is illustrated in a simplified two family example in the up- and down-quark sectors, in which the mass matrices of the up- and down-type quarks are assumed to be symmetric and each contains one zero entry,

$$M_U = \begin{pmatrix} 0 & a \\ a & b \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & c \\ c & d \end{pmatrix}. \quad (13)$$

*There are five other similar conditions one can write down.
Table 2. Mass Texture combinations for up- and down-type quarks with five zeros proposed by Ramond, Roberts and Ross.

|    | I          | II         | III        | IV          | V          |
|----|------------|------------|------------|-------------|------------|
| $M_u$ | \(0 \ A_u \ 0\) | \(0 \ A_u \ 0\) | \(0 \ 0 \ E_u\) | \(0 \ A_u \ 0\) | \(0 \ 0 \ E_u\) |
|     | \(A_u \ B_u \ 0\) | \(A_u \ 0 \ C_u\) | \(0 \ B_u \ 0\) | \(A_u \ B_u \ C_u\) | \(0 \ B_u \ C_u\) |
|     | \(0 \ 0 \ D_u\) | \(0 \ C_u \ D_u\) | \(E_u \ 0 \ D_u\) | \(0 \ C_u \ D_u\) | \(E_u \ C_u \ D_u\) |
| $M_d$ | \(0 \ A_d \ 0\) | \(0 \ A_d \ 0\) | \(0 \ A_d \ 0\) | \(0 \ A_d \ 0\) | \(0 \ A_d \ 0\) |
|     | \(A_d \ B_d \ C_d\) | \(A_d \ B_d \ C_d\) | \(A_d \ B_d \ C_d\) | \(A_d \ B_d \ 0\) | \(A_d \ B_d \ 0\) |
|     | \(0 \ C_d \ D_d\) | \(0 \ C_d \ D_d\) | \(0 \ C_d \ D_d\) | \(0 \ 0 \ D_d\) | \(0 \ 0 \ D_d\) |

In this case, there is a very simple relation between the Cabbibo angle and the quark masses,

$$|V_{us}| = \sqrt{\frac{m_d}{m_s}} - e^{i\alpha} \sqrt{\frac{m_u}{m_c}} \quad (14)$$

where the CP violating phase $\alpha$ arises as the relative phase that enters when combining the up- and down-quark rotation matrices. This relation is in good agreement with experiment. Note that because $m_u/m_c$ is quite small compared to $m_d/m_s$, the value of $|V_{us}|$ is not very sensitive to the complex phase $\alpha$. One can then generalize this to consider the three family case. The ultimate goal of studying texture zero ansatz is that it may help us to understand the underlying theory of flavor, if the zeros are protected by some family symmetry. In the quark sector, assuming symmetric mass matrices, the total number of texture zeros is at most six, because there are six different quark masses. Nevertheless, Ramond, Roberts and Ross found that the observed masses and mixing angles cannot be accommodated with six texture zeros. They found five combinations of five-zero texture for up- and down-type quark mass matrices which give rise to predictions that are consistent with current observations for fermion masses and mixing angles. One should note that in the context of a grand unified theory (GUT), the texture ansatz is valid only at the GUT scale. The vanishing entries in the mass matrices at the GUT scale will be filled in by radiative corrections at lower energy scales. These five solutions are summarized in Table 2.

Lop-sided textures have also been considered in model building. In section V, we classify various SO(10) models according to whether the mass textures in the models are symmetric or lop-sided. Symmetric textures arise if SO(10) breaks down to the SM group with the left-right symmetric group as the intermediate symmetry, while lop-sided textures arise if the intermediate symmetry is SU(5).

### 2.2. Froggatt-Nielsen Mechanism and Family Symmetry

A prototype scenario which produces hierarchy in the fermion mass matrices is the Froggatt-Nielsen mechanism. The idea is that the heaviest matter fields acquire their masses through tree level interactions with the Higgs fields while masses of
lighter matter fields are produced by higher dimensional interactions involving, in addition to the regular Higgs fields, exotic vector-like pairs of matter fields and the so-called flavons (flavor Higgs fields). Schematic diagrams for these interactions are shown in Fig. 3. After integrating out superheavy vector-like matter fields of mass $M$, the mass terms of the light matter fields get suppressed by a factor of $\langle \theta \rangle$, where $\langle \theta \rangle$ is the VEVs of the flavons and $M$ is the UV-cutoff of the effective theory above which the flavor symmetry is exact. When the family symmetry is exact, only the (33) entry is non-zero. When the family symmetry is spontaneously broken, the zero entries will be filled in at some order $O(\langle \theta \rangle M)$. Suppose the family symmetry allows only the (23) and (32) elements at order $O(\langle \theta \rangle M)$,\[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{SSB}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\langle \theta \rangle}{M} \\ 0 & \frac{\langle \theta \rangle}{M} & 1 \end{pmatrix} \]Then a second fermion mass is generated at order $O(\langle \theta \rangle^2 M)$ after the family symmetry is spontaneous broken. The fermion mass hierarchy thus arises.

To illustrate how the Froggatt-Nielsen mechanism works, suppose there is a vector-like pair of matter fields $(\chi \oplus \bar{\chi})$ with mass $M$ and carrying the same quantum numbers as $\psi_R$ under the vertical gauge group (e.g. SM or $SO(10)$), but different quantum numbers under the family symmetry. It is therefore possible to have a Yukawa coupling $y \chi \psi_L H$ where $H$ is the SM doublet Higgs if the family symmetry permits such a coupling. In addition, there is a gauge singlet $\theta$ which transforms non-trivially under the family symmetry. Suppose the coupling $y \psi_R \chi \theta$ is allowed by the family symmetry, we then obtain the following seesaw mass matrix, upon $H$
and $\theta$ acquiring VEV’s
\[
(\bar{\psi}_R \chi) \begin{pmatrix} 0 & y' <\theta> \\ y <H> & M \end{pmatrix} (\psi_L).
\]  

(16)

Diagonalizing this matrix gives the following mass term for $\psi$
\[
m_\psi \simeq yy' <H><\theta> M.
\]  

(17)

So the suppression factor $<\theta>M$ is due to the mixture between the light states and the heavy states. This is very similar to how the light neutrino masses are generated in the seesaw mechanism.

So what are the possible family symmetries that can be incorporated with the Froggatt-Nielsen mechanism? The kinetic terms and gauge interactions of the SM have a very large family symmetry, $[U(3)]^5$, where the $U(3)$ factors act on the right- and left-handed multiplets of quarks and leptons, respectively. If right-handed neutrinos are included, the family symmetry becomes $[U(3)]^6$. Therefore, any family symmetry group proposed to incorporate the Froggatt-Nielsen mechanism must be contained in $[U(3)]^6$. If all the particles in each family are unified into one single multiplet, as in the case of $SO(10)$, the maximal possible family symmetry group is reduced to $U(3)$. The family symmetry can either be global or gauged. However, in the case of a global symmetry, there are problems associated with the massless Goldstone bosons when the symmetry is broken, and with possibly large gravitational quantum corrections. These problems do not arise in the case of gauged symmetries. In what follows, we discuss separately the abelian group and non-abelian group as a family symmetry.

2.2.1. Abelian Case

In compactified string theories, one usually obtains abundant Abelian symmetries below the compactification scale, in additional to the SM gauge group (or a GUT gauge group). Suppose that the flavon field $\theta$ and $\theta'$ carry $+1$ and $-1$ charges under $U(1)$, and the ratios $\frac{<\theta>}{M}$ and $\frac{<\theta'>}{M}$ have approximately the value of the Cabbibo angle 0.22 (or equivalently, the parameter $\lambda$ in the Wolfenstein parameterization). By assigning different $U(1)$ charges to different family, one in general, obtains a mass matrix of the form
\[
\begin{pmatrix}
\lambda n_{11} & \lambda n_{12} & \lambda n_{13} \\
\lambda n_{21} & \lambda n_{22} & \lambda n_{23} \\
\lambda n_{31} & \lambda n_{32} & \lambda n_{33}
\end{pmatrix}.
\]  

(18)

The exponent $n_{ij}$ is given by $Q_{L,i} + Q_{R,j}$, where $Q_{L,i}$ ($Q_{R,i}$) is the $U(1)$ charge of the left-handed (right-handed) field of the $i$-th family. It is usually not difficult to find solutions for the charge assignments that are consistent with experiments, and the solutions are not unique. An interesting model for both quarks and leptons based on anomalous $U(1)_H$ was proposed by Irges, Lavigna and Ramond which
makes use of the Green-Schwartz anomaly cancellation condition to constrain the $U(1)$ charge assignments. Due to the Abelian nature, models of this type have limited predictive power, because only the powers of the small expansion parameter, $\lambda$, are determined while the relative strengths between different entries are left un-determined. As a consequence, every entry in the above mass matrix has an unknown $O(1)$ coefficient associated with it.

Non-anomalous $U(1)_Y$ has also been considered in a model constructed by Mira, Nardi and Restrepo. In this model, anomaly cancellation condition leads to a massless up-quark, thus solving the strong CP problem.

2.2.2. Non-Abelian Case Based on $SU(2)$

Models with non-Abelian family symmetry generally have more predictive power because the relative strengths between different matrix elements can be determined. The original motivation of using non-Abelian group as the family symmetry is to solve the SUSY flavor problem. $SU(2)$ was proposed by Barbieri et al as a family symmetry. It has two attractive features: (i) As we have seen previously, the constraints from the SUSY FCNC requires that

$$\frac{m_2^2 - m_1^2}{m^2} \leq \frac{10^{-3}}{\sin \phi \left( \frac{m}{300 \text{GeV}} \right)}$$

where $(m_2^2 - m_1^2) \sim m^2$ is the average scalar mass squared and $\phi$ is some relevant CP phase. $SU(2)$ gives rise to the degeneracies between 1-2 families needed to suppress the supersymmetric FCNC in the squark sector; (ii) A multi-step breaking of $SU(2)$ gives rise to the observed inter-family hierarchy naturally. Unlike models based on the $U(1)$ family symmetry, in which one has the freedom in choosing $U(1)$ charges for various matter fields, a $SU(2)$ family symmetry appears to be a much more constrained framework for constructing realistic models.

The heaviness of the top-quark and suppression of the SUSY FCNC together suggest that the three families of matter fields transform under a $SU(2)$ family symmetry as

$$\psi_a \oplus \psi_3 = 2 \oplus 1$$

where $a = 1, 2$ and the subscripts refer to family indices. In the symmetric limit, only the third family of matter fields have non-vanishing masses. This can be understood easily since the third family of matter fields have much higher masses compared to the other two families of matter fields. $SU(2)$ breaks down in two steps:

$$SU(2) \xrightarrow{\epsilon M} U(1) \xrightarrow{\epsilon' M} nothing$$

with $\epsilon' \ll \epsilon \ll 1$ and $M$ is the UV cut-off of the effective theory mentioned before. These small parameters $\epsilon$ and $\epsilon'$ are the ratios of the vacuum expectation values of the flavon fields to the cut-off scale. Note that because

$$\psi_3 \psi_3 \sim 1_S, \quad \psi_3 \psi_a \sim 2, \quad \psi_a \psi_b \sim 2 \otimes 2 = 1_A \oplus 3$$
the only relevant flavon fields are in the $1_A, 2$ and $3$ dimensional representations of $SU(2)$, namely,

$$A^{ab} \sim 1_A, \quad \phi^a \sim 2, \quad S^{ab} \sim 3.$$  \hspace{1cm} (23)

So a generic mass matrix constrained by $SU(2)$ family symmetry is of the following form:

$$
\begin{pmatrix}
< S >, < 1_A > & < \phi > \\
< \phi >^T & 1
\end{pmatrix}
$$  \hspace{1cm} (24)

To see how the vacuum alignment in the flavon sector is achieved, let us first consider the supersymmetric limit with only one conjugate pair of doublets ($\phi \oplus \bar{\phi}$), anti-symmetric singlets ($A \oplus \bar{A}$) and triplets ($S \oplus \bar{S}$) of $SU(2)$. The most general renormalizable superpotential is then given by \cite{17}

$$W_{\text{flavon}}(\phi, \bar{\phi}, S, \bar{S}) = \phi S \phi + \bar{\phi} S \bar{\phi} + X_s S \bar{S} + X_A A \bar{A}$$  \hspace{1cm} (25)

where $X_s, X_\phi$ and $X_A$ are dimensionful parameters in the superpotential. Note that the anti-symmetric singlet fields $A$ and $\bar{A}$ are decoupled from other fields. From the F-flat conditions, one obtains the following solutions,

$$< S^{ab} > = - < \phi^a > < \phi^b > / X_s$$  \hspace{1cm} (26)

$$\sum_a < \phi^a \bar{\phi}^a > = \frac{1}{2} (X_s X_\phi)$$  \hspace{1cm} (27)

$$X_A A^{ab} = 0.$$  \hspace{1cm} (28)

Thus the relative strengths of $< S >$ and $< \phi >$ are determined. For $X_A \neq 0$, the F-flat conditions imply $< A > = < \bar{A} > = 0$. Non-vanishing $< A >$ and $< \bar{A} >$ can be obtained if non-renormalizable operators are introduced \cite{17}. If all the 16 observed matter fields in one family form a single representation as in the case of $SO(10)$, the most general effective superpotential, after integrating out all the heavy Froggatt-Nielsen fields, that generates fermion masses for a $SO(10) \times SU(2)$ model has the following very simple form

$$W = H(\psi_3 \psi_3 + \psi_3 \frac{\phi^a}{M} \psi_a + \psi_a \frac{S^{ab}}{M} \psi_b + \psi_a \frac{A^{ab}}{M} \psi_b).$$  \hspace{1cm} (29)

In a specific $SU(2)$ basis,

$$\frac{\langle \phi \rangle}{M} \sim O \left( \begin{array}{c} \epsilon' \\ \epsilon \end{array} \right), \quad \frac{\langle A^{ab} \rangle}{M} \sim O \left( \begin{array}{cc} 0 & -\epsilon' \\ \epsilon' & 0 \end{array} \right), \quad \frac{\langle S^{ab} \rangle}{M} \sim O \left( \begin{array}{cc} \epsilon'^2 & \epsilon' \\ \epsilon' & \epsilon \end{array} \right).$$  \hspace{1cm} (30)

Here we have indicated the VEVs all the flavon fields could acquire for symmetry breaking in Eq. (21). The mass matrix $\widetilde{M}$ takes the following form

$$\widetilde{M} \sim O \left( \begin{array}{ccc} \epsilon'^2 & \epsilon' & \epsilon \\ -\epsilon' & \epsilon & \epsilon \\ \epsilon' & \epsilon & 1 \end{array} \right).$$  \hspace{1cm} (31)

The hierarchy is thus built into this mass matrix.
2.2.3. Non-Abelian Case Based on SU(3)

Ultimately, the SU(2) family symmetry can be embedded into SU(3), under which the three families form a triplet. A model based on SU(3) is presented in Ref. 18. The three families form a SU(3) triplet before the symmetry is broken. The symmetry breaking takes place at two steps

$$SU(3) \rightarrow SU(2) \rightarrow nothing.$$ 

The SU(3) anti-triplet flavon fields, $\phi_3$ and $\phi_{23}$, acquire VEV’s along the following directions, breaking the SU(3) symmetry

$$<\phi_3> = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}, \quad <\phi_{23}> = \begin{pmatrix} 0 \\ b \\ \cdot \end{pmatrix}$$  \hspace{1cm} (32)$$

where $<\phi_3>$ triggers the first stage of breaking, and $<\phi_{23}>$ triggers the second stage of breaking. The leading order Yukawa couplings of the matter fields are

$$H\left( \frac{1}{M_3^2} \psi_i \phi_3 \psi_j \phi_3 + \frac{1}{M_{23}^2} \psi_i \phi_{23} \psi_j \phi_{23} \right).$$  \hspace{1cm} (33)$$

This gives rise to a mass matrix of the following form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{b^2}{M_{23}^2} & 0 \\ 0 & \frac{b^2}{M_{23}^2} & \frac{a^2}{M_{23}^2} + \frac{b^2}{M_{21}^2} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & \epsilon^2 & 1 \end{pmatrix}.$$  \hspace{1cm} (34)$$

Assuming $M_3 \simeq a_3 \gg M_{23} \gg b$ and $\epsilon = b/M_{23}$, one thus obtains a hierarchical structure in the (2, 3) block. There are also operators that mix $\phi_3$ with $\phi_{23}$,

$$\frac{\epsilon^2}{M_{23}M_3} H(\psi_i \phi_{23} \psi_j \phi_3 + \psi_i \phi_3 \psi_j \phi_{23}).$$  \hspace{1cm} (35)$$

These operators give rise to contributions to the (23) and (32) matrix elements of order $O(\epsilon)$, which is larger than the (22) element of order $O(\epsilon^2)$, leading to a wrong prediction for $V_{cb}$. One way to suppress these operators is to impose a discrete $Z_2$ symmetry, under which $\phi_3$ and $\phi_{23}$ have opposite parity. Thus the operators given in Eq. (38) are allowed by the $Z_2$ symmetry while the operators given in Eq. (35) only arise at higher order with a suppression factor $\epsilon^2$. To fill in the first row and column, one has to introduce additional SU(3) triplet flavon fields, $\phi_3$ and $\phi_{23}$, which acquire VEV’s along the following directions

$$<\phi_3> = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}, \quad <\phi_{23}> = \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix}$$  \hspace{1cm} (36)$$

and consider the following operators

$$\frac{\epsilon^2}{M_{23}^2} (\epsilon^{ijk} \psi_i \phi_{23,j} \psi_k^c) H$$  \hspace{1cm} (37)$$

$$\frac{\epsilon^6}{M_3^4 M_{23}^2} (\epsilon^{ijk} \psi_i \phi_{23,j} \phi_{23,k}) (\epsilon^{lmn} \psi_l \phi_{23,m} \phi_{23,n}) H.$$  \hspace{1cm} (38)$$
The operators given in Eq. (37) generate the entries $(12) = (13) = -(21) = (31)$ of order $O(\epsilon^3)$. These entries are anti-symmetric due to the presence of the anti-symmetric tensor $\epsilon^{ijk}$ in the couplings. The operators given in Eq. (38) generate the $(11)$ matrix element of order $O(\epsilon^8)$. The suppression in this operator is due to the presence of the $Z_2$ symmetry discussed above and an additional $R$-symmetry; these symmetries also forbid all the operators which could lead to un-realistic predictions. The operators given in Eq. (33), (35), (37) and (38) together give rise to a Yukawa matrix of the form, in the leading order of the expansion parameter, $\epsilon$,

$$
\begin{pmatrix}
O(\epsilon^8) & \lambda \epsilon^3 & \lambda \epsilon^3 \\
-\lambda \epsilon^3 & \epsilon^2 & \epsilon^2 \\
-\lambda \epsilon^3 & \epsilon^2 & 1
\end{pmatrix}
$$

(39)

which can accommodate realistic fermion masses and mixing angles. The vacuum alignment leading to Eq. (32) and (36) is discussed in detail in Ref. 18.

We comment that the absolute mass scale of the family symmetry, $M$, cannot be determined in the Froggatt-Nielsen type of scenario, because it is the ratio, $(\epsilon, \epsilon')$, rather than the absolute mass scale, $M$, that is phenomenologically relevant. Some attempts have been made by having the SUSY breaking messenger fields play also the role of Froggatt-Nielsen fields such that the family symmetry scale is linked to the SUSY breaking scale. Though attractive, these models have difficulties getting low SUSY breaking scale.

2.3. Ideas from Extra Dimensions

2.3.1. Split Fermion Scenario in Factorizable Geometry

It has been proposed that the existence of extra compact spatial dimensions could account for the large hierarchy between the Planck scale and the electroweak scale. Arkani-Hamed, Dimopoulos and Dvali\cite{5} showed that extra dimensions of size $\sim 1/TeV$ may provides a solution to the gauge hierarchy problem.

Based on this framework, Arkani-Hamed and Schmaltz\cite{6} proposed a non-supersymmetric model in which the mass hierarchy is generated by localizing zero modes of the weak doublet and singlet fermions at different locations. Note that this mechanism works despite of the size of the extra dimensions. Consider a chiral fermion $\Psi$ in 5D. The action for $\Psi$ coupled to a background scalar field $\Phi$ is given by

$$
S = \int d^4 dy \bar{\Psi} \left[ i \gamma^\mu \partial_\mu + i \gamma^5 \partial_5 + \Phi(y) \right] \Psi.
$$

(40)

The chiral fermion $\Psi(x, y)$ can be expanded in the product basis

$$
\Psi(x, y) = \sum_n <y|L, n > P_L \psi_n(x) + <y|R, n > P_R \psi_n(x).
$$

(41)

Here $|L, n >$ and $|R, n >$ satisfy the following equations

$$
aa^\dagger |L, n > = (\partial_5^2 + \Phi^2 + \hat{\Phi}) |L, n >= \mu_n^2 |L, n >
$$

(42)
where $\hat{\Phi} \equiv \partial^5 \Phi$ and
\begin{equation}
 a = \partial^5 + \Phi(y), \quad a^\dagger = -\partial^5 + \Phi(y).
\end{equation}

For a special choice of linearized background field, $\Phi(y) = 2\mu^2 y$ where $\mu$ is a constant of mass dimension one, the operator $a$ and $a^\dagger$ become the usual creation and annihilation operators of a simple harmonic oscillator. In what follows, we will concentrate on this special case. Expanding the 5D action in terms of $|L,n\rangle$ and $|R,n\rangle$, and then integrating out $y$, we obtain the 4D effective action
\begin{equation}
 S = \int d^4x \left[ \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \sum_{n=1}^\infty \bar{\psi}_n (i\gamma^\mu \partial_\mu + \mu_n) \psi_n \right],
\end{equation}

where the first two terms are kinetic terms of the chiral zero modes whose wave functions are Gaussian centered at $y = 0$,
\begin{equation}
 <y|L,0> = \frac{\mu^{1/2}}{(\pi/2)^{1/4}} e^{-\mu^2 y^2}.
\end{equation}

In general, there is a bulk mass term, $m \bar{\Psi} \Psi$, in the 5D action, Eq. (40), because mass terms for 5-dimensional field are allowed by all symmetries. In this case, instead of being centered at $y = 0$, the Gaussian wave function is centered at $y = m/2\mu$. Thus by having different bulk mass term, $m$, different bulk fields can be localized at different 4D slices in the 5D bulk.

For simplicity, the wave function of the Higgs doublet is assumed to have constant spread along the fifth dimension. In this case, the Yukawa coupling is
\begin{equation}
 S_{yuk} = y_{ij} \int d^4x H(x) f^*_i(x) \tilde{f}^*_j(x) \int dy \phi_{f_i}(y) \phi_{\tilde{f}_j}(y),
\end{equation}

where $\phi_{f_i}(y)$ and $\phi_{\tilde{f}_j}(y)$ are the zero mode wave functions of the $SU(2)$ doublet, $f_i$, and singlet, $\tilde{f}_j$, respectively. Integrating out the $y$ coordinate, the effective Yukawa coupling is then given by
\begin{equation}
 y_{ij} \int dy \phi_{f_i}(y) \phi_{\tilde{f}_j}(y) = y_{ij} \frac{\sqrt{2\mu}}{\sqrt{\pi}} \int dy e^{-\mu^2 (y-r_{f_i})^2} e^{-\mu^2 (y-r_{\tilde{f}_j})^2} = e^{-\mu^2 r_{f_i}^2/2},
\end{equation}

where $r_{f_i} = m_{f_i}/2\mu$ and $r_{\tilde{f}_j} = m_{\tilde{f}_j}/2\mu$ are the locations at which the Gaussian wave functions $\phi_{f_i}$ and $\phi_{\tilde{f}_j}(y)$ are centered, and $r_{ij} = |r_{f_i} - r_{\tilde{f}_j}|$ is the “distance” between the peaks of these two wave functions. Thus the large hierarchy among fermion masses can be generated by having different $m_i$.

Variations based on this mechanism have been investigated in Ref. 22, 23, 24.
2.3.2. Non-Factorizable Geometry

Randall and Sundrum\cite{25} proposed an alternative based on non-factorizable geometry from which the gauge hierarchy can be derived. In their original proposal, only gravity propagates in the bulk. If the SM Higgs doublet is confined to the TeV brane, which is required if the gauge hierarchy is assumed to arise from the warped geometry, while all other SM particles are allowed to propagate in the bulk, it is possible to understand the fermion mass hierarchy in this setup. The equations of motion for various bulk fields are given in the following compact form\cite{26,27},

\[
\left( e^{2\sigma}\eta^{\mu\nu}\partial_\mu\partial_\nu + e^{s\sigma}\partial_5(e^{-s\sigma}\partial_5) - M_\Phi^2 \right)\Phi(x^\mu, y) = 0, 
\]

where for \( \Phi = (\phi, e^{-2\sigma}\Psi_{L,R}) \) we have

\[
M_\Phi^2 = \left( a k^2 + b \sigma''(y), C(C \pm 1) k^2 \pm C \sigma''(y) \right)
\]

and

\[
s = (4, 1), \quad a, C \text{ are bulk mass terms of the scalar and fermionic fields, and } b \text{ is a boundary mass term for the scalar field.}
\]

The field \( \Phi(x^\mu, y) \) is decomposed into an infinite sum of Kaluza-Kline (KK) modes as follows,

\[
\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n \Phi(n)(x^\mu)f_n(y).
\]

The profile of the \( n \)-th mode, \( f_n(y) \), satisfies

\[
( -e^{s\sigma}\partial_5(e^{-s\sigma}\partial_5) + \hat{M}_\Phi^2 )f_n(y) = e^{2\sigma}m_n^2f_n(y),
\]

where \( \hat{M}_\Phi^2 = (ak^2, C(C \pm 1)k^2) \). \( m_n \) is the mass of the \( n \)-th KK mode. The solution for the zero modes of the bulk spin-1/2 fields are found to be\cite{26,27}

\[
f_0(y) = \frac{1}{N_0} e^{-c\sigma},
\]

where \( c = C \) for left-handed fermions and \( c = -C \) for right-handed fermions. The normalization constant \( 1/N_0 \) is given by

\[
\frac{1}{N_0^2} = \frac{(1 - 2c)\pi kR}{e^{(1-2c)\pi kR} - 1}.
\]

Thus the bulk fermion can be decomposed into

\[
\Psi_L(x^\mu, y) = e^{2\sigma}\Phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R} N_0} e^{(2-c)\sigma}\Phi_L(0)(x^\mu) + \cdots.
\]

The Yukawa coupling for the charged fermions to the Higgs doublet reads

\[
\hat{Y}_{ij} = \frac{Y_{ij}}{M_{pl}} \int d^4x \int dy \sqrt{-g} \Psi_{R,i}(x, y)\Psi_{L,j}(x, y)H(x)\delta(y - \pi R),
\]

where \( Y_{ij} \) are dimensionless \( O(1) \) coefficients. The effective Yukawa coupling in four dimensions is obtained after integrating out the fifth coordinate, \( y \):

\[
\hat{Y}_{ij} = \frac{Y_{ij}}{M_{pl}} \frac{1}{2\pi R} \int_{\pi R}^{-\pi R} dy \sqrt{-g} e^{(2-c_{R,i})\sigma} e^{(2-c_{L,j})\sigma} \delta(y - \pi R)
\]
Thus by choosing \( c_{l,i} \) and \( c_{R,i} \) all of \( \mathcal{O}(1) \), we can reproduce the observed mass hierarchy and mixing. We note that this mechanism is not predictive in the sense that it does not reduce the number of parameters; the virtue of this mechanism is that the large hierarchy observed in fermion masses arises from parameters all of \( \mathcal{O}(1) \). A configuration that reproduce the observed mass hierarchy has been found by Huber and Shafi.\(^28\)

3. Lepton Masses and Mixing

If neutrinos are massive, the mixing arises in the leptonic charged current interaction,

\[
\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} U_{LM}^\dagger (W_\mu^T Y L^\mu)_E L + h.c. .
\]  

(57)

The leptonic mixing (LM) matrix,\(^29,30,31,32\) \( U_{LM} \),\(^b\) is obtained by diagonalizing the Yukawa matrix of the charged leptons and the effective neutrino mass matrix, assuming neutrinos are Majorana particles,\(^c\)

\[
Y_e^\text{diag} = V_{eR} Y_e^\dagger V_{eL} = \text{diag}(y_e, y_\mu, y_\tau)
\]  

(58)

\[
M_{\nu LL}^\text{diag} = V_{\nu LL} M_{\nu LL}^\text{eff} V_{\nu LL}^T = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) ,
\]  

(59)

where \( V_{\nu LL} \) is an orthogonal matrix, and it can be parameterized as a product of a CKM-like mixing matrix, which has three mixing angles and one CP violating phase, with a diagonal phase matrix,

\[
U_{LM} \equiv V_{eL} V_{\nu LL}^\dagger
\approx \left( \begin{array}{ccc} 
  c_{12} c_{13} & s_{12} c_{13} e^{i \delta} & s_{13} e^{-i \delta} \\
  -s_{12} c_{13} - c_{12} s_{13} e^{i \delta} & c_{12} s_{13} e^{i \delta} & -s_{13} c_{23} e^{i \delta} \\
  s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta} & c_{12} s_{23} s_{13} e^{i \delta} & c_{23} c_{13} \\
\end{array} \right) \cdot \left( \begin{array}{c} 1 \\
 e^{i \frac{\pi}{2}} \\
 e^{-i \frac{\pi}{2}} \\
\end{array} \right) ,
\]  

(60)

which relates the neutrino mass eigen states to the flavor eigenstates by

\[
|\nu_e > = U_{ee} |\nu_1 > + U_{e2} |\nu_2 > + U_{e3} |\nu_3 >
\]  

(61)

\[
|\nu_\mu > = U_{\mu e} |\nu_1 > + U_{\mu 2} |\nu_2 > + U_{\mu 3} |\nu_3 >
\]  

(62)

\[
|\nu_\tau > = U_{\tau e} |\nu_1 > + U_{\tau 2} |\nu_2 > + U_{\tau 3} |\nu_3 >
\]  

(63)

\(^b\)The LM matrix is sometimes called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) or Maki-Nakagawa-Sakata (MNS) matrix. It was first discussed, in a two flavor case, by Pontecorvo\(^29\) and by Maki, Nakagawa and Sakata\(^30\). The mixing matrix with 3 flavors was first discussed by Lee, Pakvasa, Shrock and Sugawara\(^31,32\).

\(^c\)If neutrinos are Dirac particles, Eq. \(^56\) becomes

\[
M_{\nu}^\text{diag} = V_{\nu LL} M_{\nu LL}^\text{eff} V_{\nu LL}^T = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) .
\]
Note that the Majorana condition,
\[ C(\nu_j)^T = \nu_j, \]  
(64)
where $C$ is the charge conjugate operator, forbids the rephasing of the Majorana fields. Therefore, we can only remove 3 of the 6 phases present in the unitary matrix $U_{LM}$ by redefining the charged lepton fields and are left with three CP violating phases in the leptonic sector, if neutrinos are Majorana particles.33,34,35

Thus $U_{LM}$ can be parameterized as a product of a unitary matrix, analogous to the CKM matrix which has one phase (the so-called universal phase), $\delta_l$, and a diagonal phase matrix which contains two phases (the so-called Majorana phases), $\alpha_{21}$ and $\alpha_{31}$. The leptonic analog of the Jarlskog invariant, which measures the CP violation due to the universal phase, is given by
\[ J_{CP}^L \equiv \text{Im}\{U_{\mu\nu_2}U_{e\nu_3}U_{\mu\nu_3}^*U_{e\nu_2}^*\}. \]  
(65)

For the Majorana phases, the rephasing invariant CP violation measures are36
\[ S_1 \equiv \text{Im}\{U_{e\nu_1}^*U_{e\nu_3}\}, \quad S_2 \equiv \text{Im}\{U_{e\nu_2}^*U_{e\nu_3}\}. \]  
(66)

From $S_1$ and $S_2$, one can then determine the Majorana phases
\[ \cos \alpha_{31} = 1 - \frac{S_1^2}{|U_{e\nu_1}|^2|U_{e\nu_3}|^2}, \]  
(67)
\[ \cos(\alpha_{31} - \alpha_{21}) = 1 - \frac{S_2^2}{|U_{e\nu_2}|^2|U_{e\nu_3}|^2}. \]  
(68)

The recently reported measurements from KamLAND reactor experiment37 confirm the large mixing angle (LMA) solution to be the unique oscillation solution to the solar neutrino problem at 4.7 $\sigma$ level38,39,40. The global analysis including Solar + KamLAND + CHOOZ41 indicate the following allowed region at 3$\sigma$42
\[ 5.1 \times 10^{-5} < \Delta m^2_{21} < 9.7 \times 10^{-5} eV^2 \]  
(69)
\[ 0.29 \leq \tan^2 \theta_{12} \leq 0.86. \]  
(70)

The allowed regions at 3$\sigma$ level based on a global fit including SK42 + Solar + CHOOZ for the atmospheric parameters and the CHOOZ angle are43
\[ 1.4 \times 10^{-3} < \Delta m^2_{32} < 6.0 \times 10^{-3} eV^2 \]  
(71)
\[ 0.4 \leq \tan^2 \theta_{23} \leq 3.0 \]  
(72)
\[ \sin^2 \theta_{13} < 0.06. \]  
(73)

And the magnitudes of $U_{LM}$ elements at 1$\sigma$ (3$\sigma$) are given by44

| $U_{LM}$ | \( (0.73)0.79 - 0.86(0.88) (0.47)0.50 - 0.61(0.67) 0 - 0.16(0.23) \) | \( (0.17)0.24 - 0.52(0.57) (0.37)0.44 - 0.69(0.73) (0.56)0.63 - 0.79(0.84) \) | \( (0.20)0.26 - 0.52(0.58) (0.40)0.47 - 0.71(0.75) (0.54)0.60 - 0.77(0.82) \) |
To see what a bi-large mixing means, let us assume, to a good approximation, that $\theta_{13} = \eta \ll 1$. In this case, the LM matrix reads,

$$U_{LM} = \begin{pmatrix}
\frac{c_{12}}{\sqrt{2}} & \frac{s_{12}}{\sqrt{2}} & \eta \sqrt{2} \\
\frac{s_{12}}{\sqrt{2}} & \frac{-c_{12}}{\sqrt{2}} & \frac{-\eta s_{12}}{\sqrt{2}} \\
\frac{-\eta c_{12}}{\sqrt{2}} & \frac{-s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}}
\end{pmatrix},$$

(75)

which in turn implies the neutrino mass eigenstates given in terms of flavor eigenstates as

$$|\nu_1 > = c_{12}|\nu_e> - \frac{1}{\sqrt{2}} s_{12}(|\nu_\mu> - |\nu_\tau>) - \eta \frac{s_{12}}{\sqrt{2}} (|\nu_\mu> + |\nu_\tau>)$$

(76)

$$|\nu_2 > = s_{12}|\nu_e> + \frac{1}{\sqrt{2}} c_{12}(|\nu_\mu> - |\nu_\tau>) - \eta \frac{c_{12}}{\sqrt{2}} (|\nu_\mu> + |\nu_\tau>)$$

(77)

$$|\nu_3 > = \frac{1}{\sqrt{2}}(|\nu_\mu> + |\nu_\tau>) + \eta |\nu_e>.$$  

(78)

3.1. Generation of Small Neutrino Masses

3.1.1. Small Neutrino Masses from See-saw Mechanism

The observation of neutrino oscillations provides the first indication of beyond the Standard Model physics. It has two implications: neutrinos have non-zero masses, and lepton family numbers are violated. In the SM, neutrinos are massless because there are no $SU(2)$ singlet neutrinos nor are there $SU(2)$ triplet Higgs. Adding one $SU(2)$ singlet neutrino for each family is the simplest way to introduce neutrino masses. Because the right-handed neutrinos are SM singlets, the Majorana mass terms for the $SU(2)$ singlet neutrinos are not forbidden by the symmetry. In the Lagrangian, there are Dirac mass term for the neutrinos and the right-handed Majorana mass terms,

$$\mathcal{L}_{\text{see-saw}} = -M_{\nu L R} \overline{\nu}_R \nu_L - \frac{1}{2} M_{\nu R R} \overline{\nu}_R^T \nu_R + h.c. = -\frac{1}{2} \begin{pmatrix} \nu_L \overline{\nu}_R \end{pmatrix} \begin{pmatrix} 0 & M_{\nu L R}^T \\ M_{\nu L R} & M_{\nu R R} \end{pmatrix} \begin{pmatrix} \nu_L \\ \overline{\nu}_R \end{pmatrix}. $$

(79)

(In general, there could be a non-vanishing mass term for $(\nu_L, \nu_L)$; this is the Type II see-saw mechanism. Such a mass term can be obtained in $SO(10)$ from the Yukawa coupling given in Eq. (175)). After integrating out the heavy right-handed neutrinos,

$$-\frac{\partial \mathcal{L}}{\partial \overline{\nu}_R} = M_{\nu L R} \nu_L + M_{\nu R R} \overline{\nu}_R^T$$

$$\overline{\nu}_R = -\nu^T M_{\nu L R}^{-1} M_{\nu R R}^{-1}.$$  

(80)

one then obtains the effective light neutrino Majorana mass matrix

$$M_{LL} = M_{\nu L R}^T M_{\nu R R}^{-1} M_{\nu L R}.$$  

(81)
For $M_{LR}$ of the order of the weak scale as the mass scale of other charged fermions, the Majorana mass term $M_{RR}$ must be around $10^{12-14} GeV$ to give rise to neutrino masses of the order of 0.1 eV. As we will see later, most grand unified theories predict the existence of the right-handed neutrinos. Furthermore, the GUT scale provides an understanding why $M_{RR}$ is large.

3.1.2. Small Neutrino Masses from Large Extra Dimension

An interesting way to generate small Dirac neutrino masses arises in models with large extra dimensions of size $\sim (1/TeV)$. Consider the case in which only gravity can propagate in the bulk, while the SM particles and interactions are confined to the brane. Because the right-handed neutrinos are SM singlets, they are the only particles that can propagate in the bulk. Its Yukawa coupling to the charged lepton $L(x)$, and the Higgs doublet, $H(x)$, which are confined to the brane, is given by

$$S = y \int d^4 x \int dy L(x) H(x) \nu_R(x, y) \delta(y). \quad (82)$$

Compactified on a circle $S^1$, $\nu_R(x, y)$ can be decomposed into

$$\nu_R(x, y) = \frac{1}{\sqrt{2\pi R M_{pl}^{5D}}} \sum_{-\infty}^{\infty} \nu_R^{(n)}(x) e^{iny/R}. \quad (83)$$

Below the compactification scale, we thus obtain a Dirac neutrino mass

$$m_\nu = \frac{y < H >}{\sqrt{2\pi R M_{pl}^{5D}}} = \frac{y < H > M_{pl}^{4D}}{M_{pl}^{5D}}, \quad (84)$$

where the last steps follows from the relation between the 4D Planck scale and 5D Planck scale, $2\pi R M_{pl}^{5D} = (M_{pl}^{4D}/M_{pl}^{5D})^2$. With $M_{pl}^{5D} \sim (1-10)TeV$ which could avoid the gauge hierarchy problem, one obtains a highly suppressed $m_\nu$ which is consistent with the experimental observations.

3.1.3. Small Neutrino Masses from Warped Geometry

Small neutrino masses of the Dirac type are possible if right-handed neutrinos are localized toward the Planck brane while the lepton doublets are localized toward the TeV brane. This results in a small overlap between the zero mode profiles of the lepton doublets and the right-handed neutrinos based on the formulation given in Sec. II. Models of this type have been constructed in Ref. 26, 28, 43.
3.2. Bi-Large Neutrino Mixing Angles

To obtain the bi-large mixing pattern for the neutrinos, in addition to having the hierarchical mass pattern, let us consider, for example, the following mass texture,

$$
\begin{pmatrix}
0 & 0 & t \\
0 & 1 & 1 + t^n \\
t & 1 + t^n & 1
\end{pmatrix},
$$

(85)

with $t < 1$ which is a special case of the following texture

$$
\begin{pmatrix}
0 & 0 & * \\
0 & * & * \\
* & * & *
\end{pmatrix}
$$

(86)

first proposed in Ref. 49 in which the elements in (2,3) block are taken to have equal strengths to accommodate near bi-maximal mixing. The modification of adding the term $t^n$ in the (23) and (32) entries is needed in order to accommodate a large, but non-maximal solar angle in the so-called “light side” region ($0 < \theta < \pi/4$).\(^5\) It is possible to obtain the LMA solution at $3\sigma$ level with $n$ ranging from 1 to 2.

An interesting alternative in which a $3 \times 2$ neutrino Dirac mass matrix is considered was proposed recently by Kuchimanchi and Mohapatra.\(^{51,52}\) A $3 \times 2$ neutrino Dirac mass matrix arises if there are only two right-handed neutrinos, instead of three. The existence of two right-handed neutrinos is required by the cancellation of Witten anomaly, if a global leptonic $SU(2)$ family symmetry is imposed.\(^{51,52}\)

Along this line, Frampton, Glashow and Yanagida proposed a model, which has the following Lagrangian,\(^53\)

$$
L = \frac{1}{2} \left( N_1 N_2 \right) \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \left( N_1 N_2 \right) + \left( N_1 N_2 \right) \begin{pmatrix} a a' & 0 \\ 0 & b b' \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} H + h.c.
$$

(87)

The effective neutrino mass matrix due to this Lagrangian is obtained, using the see-saw formula,

$$
\begin{pmatrix}
\frac{a^2}{M_1} & \frac{a a'}{M_1} & 0 \\
\frac{a a'}{M_1} & \frac{a'^2}{M_1} + \frac{b^2}{M_2} & \frac{b b'}{M_2} \\
0 & \frac{b b'}{M_2} & \frac{b'^2}{M_2}
\end{pmatrix},
$$

(88)

where $a, b, b'$ are real and $a' = |a'| e^{i\delta}$. By taking all of them to be real, with the choice $a' = \sqrt{2}a$ and $b = b'$, and assuming $a^2/M_1 \ll b^2/M_2$, the effective neutrino masses and mixing matrix are obtained

$$
m_{\nu_1} = 0, \quad m_{\nu_2} = \frac{2a^2}{M_1}, \quad m_{\nu_3} = \frac{2b^2}{M_2}
$$

(89)

$$
U = \begin{pmatrix}
1/\sqrt{3} & 1/\sqrt{3} & 0 \\
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2}
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix},
$$

(90)
where $\theta \simeq m_{\nu_2}/\sqrt{2} m_{\nu_1}$, and the observed bi-large mixing angles and $\Delta m^2_{\text{atm}}$ and $\Delta m^2_{\odot}$ can be accommodated. An interesting feature of this model is that the sign of the baryon number asymmetry ($B \propto \xi_B = Y^2 a^2 b^2 \sin 2\delta$) is related to the sign of the CP violation in neutrino oscillation ($\xi_{\text{osc}}$) in the following way

$$\xi_{\text{osc}} = -\frac{a^4 b^4}{M_1^2 M_2^2} (2 + Y^2) \xi_B \propto -B$$

assuming the baryon number asymmetry is resulting from leptogenesis due to the decay of the lighter one of the two heavy neutrinos, $N_1$, whose mass is of $O(10^{10} \text{ GeV})$. A $SO(10)$ model which gives rise to the neutrino mass ansatz, Eq. (87), has been constructed\[54\]. A more detailed discussion on this model is given in Sec. 6.

Other phenomenologically viable textures for neutrino mass matrix are analyzed in Ref. 55.

3.2.1. $SO(10)$ GUT realization

In $SO(10)$ models in 4D, the bi-large mixing in the leptonic sector arises in two ways (A detailed classification according to how the maximal $\nu_\mu - \nu_\tau$ mixing arises is given by Barr and Dorsner\[56\]):

(i) Symmetric mass textures for the charged fermions:
This scenario is realized in symmetric textures arising from left-right symmetric breaking chain of $SO(10)$. In this case, both the large solar mixing angle and the maximal atmospheric mixing angle come from the diagonalization of the effective neutrino mass matrix. A characteristic of this class of models is that the predicted value for $|U_{e\nu_3}|$ element tends to be larger than the value predicted by models in class (ii) below.

(ii) Lop-sided mass textures for charged fermions:
In this scenario, the large atmospheric mixing angle comes from charged lepton mixing matrix. This scenario is realized in models with $SU(5)$ as the intermediate symmetry which gives rise to the so-called “lop-sided” mass textures, due to the $SU(5)$ relation,

$$M_e = M_d^T.$$  

Due to the lop-sided nature of $M_e$ and $M_d$, the large atmospheric neutrino mixing is related to the large mixing in the (23) sector of the RH charged lepton diagonalization matrix, instead of $V_{cb}$. It thus provides an explanation why the small value of $V_{cb}$ and the large value of $U_{\mu\nu_3}$ exist simultaneously. The large solar mixing angles come from the diagonalization matrix for the neutrino mass matrix. Because the two large mixing angles come from different sources, the constraint on $U_{e\nu_3}$ is not as strong as in class (i). In fact, the prediction for $U_{e\nu_3}$ in this class of models tend to
be quite small. On the other hand, this mechanism also predicts an enhanced decay rate for the flavor-violating process, $\mu \rightarrow e \gamma$, which is close to current experimental limit.

We will discuss these two classes of models in detail in Sec. 3.

3.2.2. Large Mixing from Renormalization Group Evolution

It has been shown that it is possible to obtain large neutrino mixing angles through the renormalization group evolution. Recently, Mohapatra, Parida and Rajasekaran observed in Ref. 62 that bi-large mixing angles can be driven by the renormalization group evolution, assuming that the CKM matrix and the LM matrix are identical at the GUT scale, which is a natural consequence of quark-lepton unification. The only requirement for this mechanism to work is that the masses of the three neutrinos are nearly degenerate of the form $m_3 \gtrsim m_2 \gtrsim m_1$ and have same CP parity. The one-loop RGE of the effective left-handed Majorana neutrino mass operator is given by

$$\frac{dm_\nu}{dt} = -\{\kappa_\nu m_\nu + m_\nu P + P^T m_\nu\}, \quad (93)$$

where $t \equiv \ln \mu$. In the MSSM, $P$ and $\kappa_\nu$ are given by,

$$P = -\frac{1}{32\pi^2} \frac{Y_e^\dagger Y_e}{\cos^2 \beta} \simeq -\frac{1}{32\pi^2} \frac{h_\tau^2}{\cos^2 \beta} \text{diag}(0, 0, 1) \equiv \text{diag}(0, 0, P_\tau) \quad (94)$$

$$\kappa_\nu = \frac{1}{16\pi^2} \left[\frac{6}{5} g_1^2 + 6 g_2^2 - \frac{6}{5} \frac{\text{Tr}(Y_e^\dagger Y_u)}{\sin^2 \beta}\right] \simeq \frac{1}{16\pi^2} \left[\frac{6}{5} g_1^2 + 6 g_2^2 - \frac{h_\tau^2}{\sin^2 \beta}\right], \quad (95)$$

where $g_1^2 = \frac{4}{3} g_2^2$ is the $U(1)$ gauge coupling constant, $Y_u$ and $Y_e$ are the $3 \times 3$ Yukawa coupling matrices for the up-quarks and charged leptons respectively, and $h_\tau$ and $h_\tau$ are the $t$- and $\tau$-Yukawa couplings. One can then follow the “diagonal-and-run” procedure, and obtain the RGE’s at scales between $M_R \geq \mu \geq M_{\text{SUSY}}$ for the mass eigenvalues and the three mixing angles, assuming CP violating phases vanish,

$$\frac{dm_\nu}{dt} = -4P_i m_i U_{\tau_i}^2 - m_i \kappa_\nu, \quad (i = 1, 2, 3) \quad (96)$$

$$\frac{ds_{23}}{dt} = -2P_\tau c_{23}(s_{12} U_{\tau_1} \nabla_{31} + c_{12} U_{\tau_2} \nabla_{32}) \quad (97)$$

$$\frac{ds_{13}}{dt} = -2P_\tau c_{23} c_{13}(c_{12} U_{\tau_1} \nabla_{31} + s_{12} U_{\tau_2} \nabla_{32}) \quad (98)$$

$$\frac{ds_{12}}{dt} = -2P_\tau c_{12}(c_{23} s_{13} s_{12} U_{\tau_1} \nabla_{31} - c_{23} s_{13} c_{12} U_{\tau_2} \nabla_{32} + U_{\tau_1} U_{\tau_2} \nabla_{21}), \quad (99)$$

where $\nabla_{ij} \equiv (m_i + m_j)/(m_i - m_j)$. Because the LM matrix is identical to the CKM matrix, we have, at the GUT scale, the following initial conditions, $s_{12}^0 \simeq \lambda$, $s_{23}^0 \simeq \mathcal{O}(\lambda^2)$, $s_{13}^0 \simeq \mathcal{O}(\lambda^3)$, where $\lambda$ is the Wolfenstein parameter. (Note that the RG

$^d$Note that some of the earlier results were not entirely correct; re-derivation of these results has been done in Ref. 65, 66.
evolution has negligible effect on the Wolfenstein parameter, see Eq. (138)). When the masses $m_i$ and $m_j$ are nearly degenerate, $\nabla_{ij}$ approaches infinity. Thus it drives the mixing angles to become large. Starting with the values of $(m_1^0, m_2^0, m_3^0) = (0.2983, 0.2997, 0.3383) \text{ eV}$ at the GUT scale, the solutions at the weak scale for the masses are $(m_1, m_2, m_3) = (0.2410, 0.2411, 0.2435) \text{ eV}$, which correspond to $\Delta m^2_{\text{atm}} = 1.1 \times 10^{-5} \text{ eV}^2$ and $\Delta m^2_{\odot} = 4.8 \times 10^{-5} \text{ eV}^2$. The mixing angles predicted at the weak scale are $\sin \theta_{23} = 0.68$, $\sin \theta_{12} = 0.568$ and $\sin \theta_{13} = 0.08$. Because the masses are larger than 0.1 eV, they are testable at the present searches for the neutrinoless double beta decay.

Models based on GUT and horizontal symmetry often suffer from fine-tuning or the difficulty of constructing a viable scalar potential that gives rise to the required vacua. Along the line discussed in the above paragraph, some attempts have been made to show that the maximal mixing angle and nearly degenerate neutrino masses are manifestations of infrared fixed points (IRFP) of the RGEs given above in Eq.(96)-(99), under certain assumptions.

3.2.3. Bi-large Mixing and $b - \tau$ Unification

In the minimal SO(10) model utilizing Type II see-saw mechanism with one 10 and one $\mathbf{126}$, we have the following relations:

$$M_u = f < 10 > + h < \mathbf{126}^+ >$$
$$M_d = f < 10 > + h < \mathbf{126}^- >$$
$$M_e = f < 10 > - 3h < \mathbf{126}^- >$$
$$M_{\nu,LR} = f < 10 > - 3h < \mathbf{126}^+ >$$

where $f$ and $h$ are Yukawa matrices; the mass terms $M_{\nu,LL}$ and $M_{\nu,RR}$ are both due to the coupling to $\mathbf{126}$,

$$M_{\nu,LL} = h < \mathbf{126}^+ >$$
$$M_{\nu,RR} = h < \mathbf{126}^0 >$$

where the superscripts $+/−/0$ refer to the sign of the hypercharge $Y$ (see Table 3). The small neutrino masses are explained by the Type II see-saw mechanism with the assumption that the LH Majorana mass term dominates over the usual Type I see-saw term, thus it is proportional to the Yukawa matrix $h$, which can be determined by calculating the difference between $M_d$ and $M_e$. Using down-type quark masses, charged lepton masses, and CKM matrix elements, which have roughly the form

$$M_{b,\tau} \sim \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} m_{b,\tau},$$

predictions for neutrino masses and LM matrix elements have been made.\(^{78}^{79}\)

The large atmospheric mixing results from a small deviation of $O(\lambda^2)$ from $b - \tau$ unification.
Predictions from a detailed numerical study made in Ref. 79 are $\sin^2 2\theta_{23} < 0.9$ and $\sin^2 2\theta_{12} > 0.9$, which are experimentally allowed only at $3\sigma$ level; these unique predictions can thus be used to test this type of models. Note that the best fit value of $\sin^2 2\theta_{23}$ cannot be accommodated in these models. The prediction for $U_{e3}$ is about 0.16, very close to the sensitivity of current experiments. We also note that as this type of models do not address the origin of the flavor structure, they are not as predictive as SO(10) models combined with family symmetry, in which as non-minimal Higgs content is usually present (see Sec. 6).

3.3. CP violation in Leptonic Sector

As we mentioned previously, if neutrinos are Dirac particles, there is only one phase in the LM matrix. On the other hand, if neutrinos are Majorana particles, which is the case if see-saw mechanism is implemented to give small neutrino masses, there are two additional phases. These two types of phases have very different impacts on phenomenology. The universal phase, $\delta_i$, affects the transition probability in the neutrino oscillation

$$P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i>j} \text{Re}\{U_{\alpha i}U_{\beta j}U_{\alpha j}^*U_{\beta i}^*\} \sin^2(\Delta m_{ij}^2 L / 4E)$$

$$+ 2 \sum_{i>j} \text{Im}\{U_{\alpha i}U_{\beta j}U_{\alpha j}^*U_{\beta i}^*\} \sin^2(\Delta m_{ij}^2 L / 2E).$$

(107)

Note that the only chance that one might observe CP violation in neutrino oscillation is to have LMA solution in the solar sector, and to have large value for $\theta_{13}$. The Majorana phases affect the matrix element for the neutrinoless double beta ($\beta\beta_0\nu$) decay, $|<m>|$, given in terms of the rephasing invariant quantities by

$$|<m>|^2 = m_1^2|U_{e1}|^4 + m_2^2|U_{e2}|^4 + m_3^2|U_{e3}|^4$$

$$+ 2m_1m_2|U_{e1}|^2|U_{e2}|^2 \cos \alpha_{21}$$

$$+ 2m_1m_3|U_{e1}|^2|U_{e3}|^2 \cos \alpha_{31}$$

$$+ 2m_2m_3|U_{e2}|^2|U_{e3}|^2 \cos(\alpha_{31} - \alpha_{21}).$$

(108)

The current bound on $|<m>|$ from Heidelberg-Moscow experiment is 0.11 − 0.56 eV at 95% confidence level. 

4. Grand Unified Theories Based on SO(10)

The smallest GUT group $SU(5)$ in its minimal form is very strongly constrained due to the non-observation of proton decay. The next candidate is the rank-5 SO(10), which is a very attractive candidate as a GUT group for many reasons: First of all, all of its irreducible representations are free of anomaly, unlike in the case of $SU(5)$ where the representations of the matter fields, $\bar{5} \oplus 10$, are carefully chosen to cancel the anomaly. It unifies all the 15 known fermions with the right-handed neutrino for each family into one 16-dimensional spinor representation. The seesaw
mechanism then arises very naturally, and the small but non-zero neutrino masses can thus be explained, as evidenced by recent atmospheric neutrino oscillation data from Super-Kamiokande indicating small non-zero neutrino masses. Because a complete quark-lepton symmetry is achieved, it has the promise for explaining the pattern of fermion masses and mixing. In some \(SO(10)\) models, \(R\)-parity is conserved automatically at all energy scales. This is to be contrasted with MSSM and SUSY \(SU(5)\) where \(R\)-parity must be imposed by hand. Because \((B - L)\) is a gauge symmetry contained in \(SO(10)\), it has the promises of baryogenesis. In what follows, we briefly review the structure of \(SO(10)\) models. A detail discussion can be found in Ref. 84.

4.1. The Algebra of \(SO(2n)\)

It is convenient to discuss the \(SO(2n)\) algebra in the \(SU(n)\) basis. Consider a set of \(n\) operators \(\xi_i\) (\(i = 1, ..., n\)), and their hermitian conjugates, \(\xi^\dagger_i\), satisfying

\[
\{\xi_i, \xi_j\} = \delta_{ij}, \quad \{\xi_i, \xi^\dagger_j\} = 0,
\]

where \(\{ , \}\) denotes an anti-commutator and \([ , \]\) denotes a commutator. The operators \(K^i_j\) defined as

\[
K^i_j \equiv \xi^\dagger_i \xi_j
\]

satisfy the algebra of the \(U(n)\) group

\[
[K^i_j, K^m_n] = \delta^m_j K^i_n - \delta^i_n K^m_j.
\]

We can then define the following \(2n\) operators, \(\Gamma^\mu\) (\(\mu = 1, ..., 2n\))

\[
\Gamma_{2j-1} = -i (\xi_j - \xi^\dagger_j), \quad \Gamma_{2j} = (\xi_j + \xi^\dagger_j), \quad j = 1, ..., n.
\]

The \(\Gamma^\mu\) form the Clifford algebra of rank \(2n\)

\[
\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^\mu\nu
\]

and they can then be used to construct the generators of \(SO(2n)\) as follows:

\[
\Sigma_{\mu\nu} = \frac{1}{2i} [\Gamma^\mu, \Gamma^\nu].
\]

The dimensionality of the spinor representation of \(SO(2n)\) is \(2^n\). In terms of the \(SU(n)\) basis, the spinor representation of \(SO(2n)\) can then be constructed by,

\[ |0 > \sim 1 \]
\[ \xi^\dagger_i |0 > \sim n \]
\[ \xi^\dagger_i \xi^\dagger_j |0 > \sim \frac{n(n-1)}{2} \]
\[ \xi^\dagger_i \xi^\dagger_j \xi^\dagger_l |0 > \sim \frac{n(n-1)(n-2)}{6} \]

......

\[ \xi^\dagger_1 ... \xi^\dagger_n |0 > \sim n \]
where $|0\rangle$ is the $SU(n)$ invariant vacuum state. The spinor representation can then be split into two $2^{n-1}$-dimensional representations by a chiral projection operator.

Let us define

$$\Gamma_0 \equiv i^n \Gamma_1 \Gamma_2 \ldots \Gamma_{2n}$$

and the number operator

$$N_i \equiv \xi_i^\dagger \xi_i.$$  

$\Gamma_0$ then can be written as

$$\Gamma_0 = [\xi_1, \xi_1^\dagger] [\xi_2, \xi_2^\dagger] \ldots [\xi_n, \xi_n^\dagger]$$

$$= \prod_{i-1}^n (1 - 2n_i)$$

$$= (-1)^n.$$  

To arrive at the last step, we have used the property of the number operator $n_i^2 = n_i$ to get $1 - 2n_i = (-1)^{n_i}$ and $n = \sum_i n_i$. One can then check that

$$[\Sigma_{\mu\nu}, \Gamma_0] = 0.$$  

The chirality projection operator is therefore defined by

$$\frac{1}{2} (1 \pm \Gamma_0).$$

Consider the case $n = 5$ and define a column vector $|\psi\rangle$:

$$|\psi\rangle = |0\rangle \psi_0 + \xi_j^\dagger |0\rangle \psi_j + \frac{1}{2} \xi_j^\dagger \xi_k^\dagger |0\rangle \psi_{jk} + \frac{1}{12} \epsilon^{ijklm} \xi_k^\dagger \xi_l^\dagger \xi_m^\dagger |0\rangle \psi_{ijklm}$$

$$+ \frac{1}{24} \epsilon^{ijklm} \xi_k^\dagger \xi_l^\dagger \xi_m^\dagger \xi_n^\dagger |0\rangle \psi_{ijklmn}$$

where $\overline{\psi}$ is not the complex conjugate of $\psi$ but an independent vector. This can be generalized to any $n$ if we write

$$\psi = \left( \psi_0 \psi_i \psi_{ij} \overline{\rho_i} \overline{\rho_{ij}} \right)^T.$$  

The spinor representation is then split under the chirality projection operator as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

where

$$\psi_\pm = \frac{1}{2} (1 \pm \Gamma_0) \psi$$

and

$$\psi_+ = \begin{pmatrix} \psi_0 \\ \psi_j \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \overline{\psi}_0 \\ \overline{\psi}_{ij} \end{pmatrix}.$$
In the case of \( n = 5 \), \( \psi_i \) and \( \psi_{ij} \) are 5 and 10-dimensional representations of \( SU(5) \) and \( \psi_0 \) is the singlet. All the SM fermions are assigned to \( \psi_+ \). The electric charge formula for \( SO(10) \) is given by

\[
Q = \frac{1}{2} \Sigma_{78} - \frac{1}{6} (\Sigma_{12} + \Sigma_{34} + \Sigma_{56}).
\]

(131)

The dimensionality of the adjoint representation of \( SO(2n) \) is \( (2n)(2n-1)/2 \). For \( SO(10) \), it is 45-dimensional. Under the decomposition with respect to \( SU(3) \times SU(2)_L \times SU(2)_R \) these 45 gauge bosons are:

\[
45 = (8, 1, 1) + (1, 3, 1) + (1, 1, 1) + (1, 1, 3) + ([3, 2, 2] + [3, 2, 2] + [3, 1, 1] + [\bar{3}, 1, 1]).
\]

(132)

In this basis, the 12 Standard Model gauge fields are in the \((8, 1, 1),(1, 3, 1)\) and \((1, 1, 1)\) multiplets. The rest are 33 new gauge bosons which could potentially mediate proton decay.

### 4.2. Symmetry Breaking

Because \( SO(10) \) is a rank-5 group while SM is a rank-4 group, there exist several intermediate symmetries through which \( SO(10) \) can descend to \( SU(3) \times U(1)_Y \). There are four maximal subgroups of \( SO(10) \): \( SU(4) \times SU(2)_L \times SU(2)_R \), \( SU(5) \times U(1) \), \( SO(9) \), and \( SO(7) \times SU(2) \). Only through the breaking chains of \( SU(4) \times SU(2)_L \times SU(2)_R \) and \( SU(5) \times U(1) \) one can obtain the correct quantum numbers for the SM particle content. Due to the presence of these intermediate scales, the predictions for proton lifetime and \( \sin^2 \theta_w \) are much less certain, compared to the case of \( SU(5) \). Details of breaking chains giving the SM are as follows:

(i) The left-right symmetry breaking chain is

\[
SO(10) \quad \frac{<54>}{\rightarrow} \quad SU(4) \times SU(2)_L \times SU(2)_R
\]

\[
\frac{<45>}{\rightarrow} \quad SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}
\]

\[
\frac{<126>}{\rightarrow} \quad SU(3) \times SU(2)_L \times U(1)_Y
\]

\[
\frac{<10>}{\rightarrow} \quad SU(3) \times U(1)_{EM}.
\]

(133)

In this case, the hypercharge \( Y \) is given by \( Y = 2T_{3R} + (B - L) \). The first step of breaking to the left-right symmetry group is achieved by a symmetric two-index tensor, \(<54>\). \( SU(4) \) in the left-right symmetry group is then broken to \( SU(3) \times U(1)_{B-L} \) by the adjoint \(<45>\). The subsequent breaking to the SM gauge group is achieved by the anti-symmetric 5-index tensor, \(<126>\) and \(<126>\); the electroweak symmetry breaking is then achieved by \(<10>\). In realistic models, the two Higgs doublets in MSSM are linear combinations of the \( SU(2) \) doublet components from different \( SO(10) \) representations of Higgses, for example, 10 and \( \overline{126} \). Thus all fields in the linear combinations contribute to electroweak symmetry breaking.
(ii) For the $SU(5)$ breaking chain,
\[
SO(10) \rightarrow SU(5) \times U(1)_x \\
\rightarrow SU(3) \times SU(2) \times U(1)_x \times U(1)_x \\
\rightarrow SU(3) \times SU(2) \times U(1)_Y \\
\rightarrow SU(3) \times U(1)_{EM}
\]
the hypercharge $Y$ is given by $Y = \frac{1}{2}(\alpha z + \beta x)$ where $z$ and $x$ are the charges under $U(1)_z$ and $U(1)_x$ respectively. There are two possible ways to embed the SM under this route: $(\alpha, \beta) = (1/6, 0)$ or $(-1/15, -2/5)$. In the case of $(\alpha, \beta) = (1/6, 0)$, we obtain $Y = \frac{1}{6}(2z)$. This corresponds to the $SU(5)$ breaking chain
\[
SO(10) \overset{<16 \oplus 16>}{\rightarrow} SU(5) \\
\overset{<45>}{\rightarrow} SU(3) \times SU(2) \times U(1)_Y \\
\overset{<10>}{\rightarrow} SU(3) \times U(1)_{EM}.
\]
In this case, the spinors $<16 \oplus 16>$ break the symmetry down to $SU(5)$; $<45>$ then breaks $SU(5)$ down to the SM. In the case of $(\alpha, \beta) = (-1/15, -2/5)$, we have $Y = \frac{1}{15}(z + 6x)$. This corresponds to flipped $SU(5)$ (that is, $SU(5) \times U(1)$) breaking chain
\[
SO(10) \overset{<45>}{\rightarrow} SU(5) \times U(1)_x \\
\overset{<16 \oplus 16>}{\rightarrow} SU(3) \times SU(2) \times U(1)_Y \\
\overset{<10>}{\rightarrow} SU(3) \times U(1)_{EM}.
\]
For this breaking to occur, again we need $<16 \oplus 16>$ and $<45>$. Each breaking chain has its characteristic set of Higgs fields and symmetry breaking superpotential. The electroweak symmetry breaking is then achieved by $<10>$. In realistic models, the two Higgs doublets in MSSM are linear combinations of the $SU(2)$ doublet components from different $SO(10)$ representations of Higgses, for example, 10 and 16. As we will see below, different symmetry breaking chains give rise to different mass relations between various sectors.

### 4.3. Renormalization Group Equation and the Georgi-Jarlskog (GJ) relations

Before describing the Yukawa sector of the $SO(10)$ models, we discuss how to relate the weak scale observables to the GUT scale parameters. We use the expressions derived from 1-loop RGEs given by [SO/ST/MSSM9]
\[
m_u = Y^0_u R_u \eta_u B^4 v_u, \quad m_c = Y^0_c R_u \eta_u B^3 v_u, \quad m_t = Y^0_c R_u B^6 v_u \\
m_d = Y^0_d R_d \eta_d v_d, \quad m_s = Y^0_s R_d \eta_d v_d, \quad m_b = Y^0_s R_d B^6 v_d \\
m_e = Y^0_e R_e v_d, \quad m_\mu = Y^0_e R_e v_d, \quad m_\tau = Y^0_e R_e v_d
\]
and

\[
V_{ij} = \begin{cases} 
V_{ij}^0, & ij = ud, us, cd, cs, tb \\
V_{ij}^0 B_t^{-1}, & ij = ub, cb, td, ts
\end{cases}
\]

(138)

where \( V_{ij} \) are CKM matrix elements; quantities with superscript 0 are evaluated at GUT scale, and all the \( m_f \) and \( V_{ij} \) are the experimental values at \( M_Z \). \( Y_f^0 \) are Yukawa couplings, and \( v_u \) and \( v_d \) are VEV of the Higgs fields \( H_u \) and \( H_d \). The SM Higgs VEV is

\[
v = \sqrt{v_u^2 + v_d^2} = 246/\sqrt{2} \text{ GeV}.
\]

The running factor \( \eta_f \) includes QCD + QED contributions: For \( f = b, c \), \( \eta_f \) is for the range \( m_f \) to \( m_t \), and for \( f = u, d, s \), \( \eta_f \) is for the range \( 1 \text{ GeV} \) to \( m_t \);

\[
\begin{align*}
\eta_u &= \eta_d = \eta_s = 2.38^{+0.24}_{-0.19} \\
\eta_c &= 2.05^{+0.13}_{-0.11} \\
\eta_b &= 1.53^{+0.03}_{-0.04}.
\end{align*}
\]

(139)

\( R_{u,d,e} \) are contributions of the gauge-coupling constants running from weak scale \( M_Z \) to the SUSY breaking scale, taken to be \( m_t \), with the SM spectrum, and from \( m_t \) to the GUT scale with MSSM spectrum;

\[
R_u = 3.53^{+0.06}_{-0.07}, \quad R_d = 3.43^{+0.07}_{-0.06}, \quad R_e = 1.50.
\]

(140)

\( B_t \) is the running induced by large top-quark Yukawa coupling defined by

\[
B_t = \exp \left[ -\frac{1}{16\pi^2} \int_{\ln M_{GUT}}^{\ln M_{SUSY}} Y_t^2(\mu) d(\ln \mu) \right]
\]

(141)

which varies from 0.7 to 0.9 corresponding to the perturbative limit \( Y_t^0 \approx 3 \) and the lower limit \( Y_t^0 \approx 0.5 \) imposed by the top-pole mass.

Naively, one would expect that masses of the down type quarks are identical to masses of the charged leptons, because of the quark-lepton unification. This turns out to be not true. Taking the experimentally measured values for masses of the down-type quarks and those of the charged leptons, evolving these values using the RGE's from the weak scale to the GUT scale, however, one finds the following approximate relations

\[
m_d \simeq 3m_e, \quad m_s \simeq \frac{1}{3}m_\mu, \quad m_b \simeq m_\tau.
\]

(142)

These are known as the Georgi-Jarlskog relations. As it will become apparent later, one way to satisfy these relations is by having a relative factor of \(-3\) in the (22) entry of the charged lepton mass matrix with respect to that of the down-type quarks, and all other elements are identical in these two mass matrices. An example suggested by Georgia and Jarlskog is:

\[
M_u = \begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & C \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 & E & 0 \\ E & 0 & F \\ 0 & F & 0 \end{pmatrix}, \quad M_e = \begin{pmatrix} 0 & E & 0 \\ E & 0 & -3F \\ 0 & 0 & G \end{pmatrix}.
\]

(143)

This factor arises naturally as a Clebsch-Gordon (CG) coefficient in some models of \( SO(10) \).
4.4. Yukawa sector in SO(10)

In SO(10), at the renormalizable level, only three types of Higgs fields can couple to fermions,

\[ 16 \otimes 16 = 10_S \oplus 120_A \oplus 126_S \]

namely, 10, 120, and 126, where the subscripts S and A refer to the symmetry property under interchanging two family indices in the Yukawa couplings \( Y_{ab} \). That is,

\[ Y_{10}^{10} = Y_{10}^{10}, \quad Y_{10}^{120} = -Y_{10}^{120}, \quad Y_{10}^{126} = Y_{10}^{126}. \]

The gauge invariant Yukawa couplings are then given by

\[ Y_{10}^{10}(16)_a(16)_b(10) = Y_{ab}^{10} \psi^T_a BC^{-1} \Gamma_\alpha \Psi_b H_\alpha \]

\[ Y_{10}^{120}(16)_a(16)_b(120) = Y_{ab}^{120} \psi^T_a BC^{-1} \Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Psi_b \Lambda_\alpha \beta \gamma \]

\[ Y_{10}^{126}(16)_a(16)_b(126) = Y_{ab}^{126} \psi^T_a BC^{-1} \Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Gamma_\delta \Gamma_\xi \Psi_b \Delta_\alpha \beta \gamma \delta \xi \]

where \( \psi \) denotes the matter fields; \( H_\alpha \), \( \Lambda_\alpha \beta \gamma \) and \( \Delta_\alpha \beta \gamma \delta \xi \) denote the 10-, 120- and 126-dim Higgs fields, respectively. \( C \) is the usual Dirac charge conjugate operator and \( B \) is the charge conjugate operator for SO(10) defined by

\[ B^{-1} \Gamma^\mu B = -\Gamma_\mu \]

and we can choose

\[ B = \prod_{\mu=\text{odd}} \Gamma_\mu. \]

Under this charge conjugation

\[ B \begin{pmatrix} \psi_0 \\ \bar{\psi}_{ij} \\ \bar{\psi}_i \\ \bar{\psi}_0 \\ \bar{\psi}_{ij} \end{pmatrix} = \begin{pmatrix} \bar{\psi}_0 \\ -\bar{\psi}_{ij} \\ -\bar{\psi}_i \\ \psi_{ij} \\ \psi_0 \end{pmatrix}. \]

Note that SO(10) can break down to SM through many different breaking chains. Different breaking chains give rise to different mass relations among the up-quark, down-quark, charged lepton and neutrino sectors. In what follows, we discuss the two symmetry breaking separately.

(i) SU(5) breaking chain:

Under the SU(5) decomposition, we have

\[ 16 = 1 + \overline{5} + 10 \]

\[ 10 = 5 + \overline{5} \]
where the $SU(5)$ component $5$ and $45$ contain the $Y = +1$ $SU(2)_L$ Higgs doublet, and the $SU(5)$ component $\overline{5}$ and $\overline{45}$ contain the $Y = -1$ $SU(2)_L$ Higgs doublet. The $SU(5)$ singlet contained in $\overline{126}$ gives masses to the RH neutrinos through the coupling $(16,3)(16,1)(126_H)$. (As we will discuss later, some models utilize non-renormalizable operators $(16,3)(16,1)(16_H)(16_H)$ to generate RH neutrino masses. This can be achieved because $16_H$ also contains a $SU(5)$ singlet component.) The $\overline{15}$ of $SU(5)$ contained in $\overline{126}$ has a $(1,3,2)$ component under $SU(3) \times SU(2)_L \times U(1)_Y$ which couples to two lepton doublets as $(1,2,-1)(1,2,-1)(1,3,2)_H$ under $G_{SM}$ and gives the LH neutrino Majorana masses in the Type II see-saw mechanism.

As the neutral components in these $SU(2)_L$ doublets acquire VEV’s of the electroweak scale, the following mass matrices are obtained

$$M_u = \mathcal{Y}^{10}_{ab} \langle 5(10) \rangle + \mathcal{Y}^{120}_{ab} \langle 45(120) \rangle + \mathcal{Y}^{\overline{126}}_{ab} \langle (\overline{5}(\overline{126}) \rangle \equiv Y_u v_u$$

$$M_d = \mathcal{Y}^{10}_{ab} \langle 5(10) \rangle + \mathcal{Y}^{120}_{ab} \langle (5(120)) \rangle + \mathcal{Y}^{\overline{126}}_{ab} \langle (45(120)) \rangle \equiv Y_d v_d$$

$$M_e = \mathcal{Y}^{\overline{5}}_{ab} \langle 5(10) \rangle + \mathcal{Y}^{120}_{ab} \langle (\overline{5}(120)) \rangle - 3 \mathcal{Y}^{\overline{126}}_{ab} \langle (45(126)) \rangle \equiv Y_e v_d$$

$$M_{\nu_{LR}} = \mathcal{Y}^{10}_{ab} \langle 5(10) \rangle + \mathcal{Y}^{120}_{ab} \langle 5(120) \rangle - 3 \mathcal{Y}^{\overline{126}}_{ab} \langle (5(\overline{126}) \rangle \equiv Y_{\nu_{LR}} v_u$$

where we denote the $m$-dim $SU(5)$ component of the $n$-dim representation of $SO(10)$ by $m(n); v_u$ and $v_d$ are the vacuum expectation values of the two Higgs doublets in MSSM. A Clebsch-Gordon coefficient $(-3)$ is generated in the lepton sectors when the $SU(5)$ component $\overline{15}$ is involved in the Yukawa couplings. This factor of $(-3)$ is very crucial for obtaining the Georgi-Jarlskog relations as we have seen in the previous section. The neutrino Majorana mass matrices are given by

$$M_{\nu_{RR}} = \mathcal{Y}^{\overline{126}}_{ab} \langle 1(\overline{126}) \rangle$$

$$M_{\nu_{LL}} = \mathcal{Y}^{\overline{126}}_{ab} \langle 15(\overline{126}) \rangle.$$

To see how these CG coefficients $(-3)$ come about, let us decompose the following representations under $SU(5)$. The $120$ and $\overline{126}$-dimensional representation of $SO(10)$ both contain a component which transforms as $\overline{45}$ under $SU(5)$. The Yukawa interactions then can be written as

$$(16)(16)(120), (16)(16)(\overline{126}) \supset (10)(\overline{5})(\overline{45}) = \psi^{\alpha\beta} \psi_\gamma < H^{\gamma}_{\alpha\beta} >. \quad (162)$$

We then write out all the terms in the summation

$$\psi^{\alpha\beta} \psi_\gamma < H^{\gamma}_{\alpha\beta} > \supset (\psi^{45} \psi_4 < H^{45}_{45} > + \psi^{a5} \psi_a < H^{a5}_{a5} >) = (-3e^+ e^- + dd^c) ^\prime. \quad (163)$$

Here the index $a = 1, 2, 3$. Note that $H^{\gamma}_{\alpha\beta}$ is anti-symmetric under inter-changing $\alpha \leftrightarrow \beta$ and it is traceless

$$H^{\gamma}_{\alpha\beta} = -H^{\gamma}_{\beta\alpha}, \quad \sum_\beta H^{\beta}_{\beta5} = 0. \quad (164)$$
This implies that
\[ 3H_{a5}^4 + H_{45}^4 = 0, \quad <H_{45}^4> = -3 <H_{a5}^a> = -3v'. \] (165)

Essentially, the CG factor of \((-3)\) is related to the fact that there are three colors.

We note that if 10 and \(\overline{126}\) are the only fields utilized in the Yukawa sector, we have the up-quark mass matrix related to the Dirac neutrino mass matrix, and the down-quark mass matrix related to the charged lepton mass matrix. When 120 is introduced, the relation between the up-quark sector and the Dirac neutrino sector is lost because these two sectors receive contributions from different components of 120.

(ii) \(SU(4) \times SU(2)_L \times SU(2)_R\) breaking chain:

Under \(SU(4) \times SU(2)_L \times SU(2)_R\), the relevant \(SO(10)\) representations have the following decomposition
\[
16 = (4, 2, 1) + (\overline{4}, 1, 2) \quad (166)
\]
\[
10 = (6, 1, 1) + (1, 2, 2) \quad (167)
\]
\[
120 = (15, 2, 2) + (6, 3, 1) + (6, 1, 3) + (1, 2, 2) + (10, 1, 1) + (\overline{10}, 1, 1) \quad (168)
\]
\[
126 = (10, 1, 3) + (\overline{10}, 3, 1) + (15, 2, 2) + (6, 1, 1) \quad (169)
\]

where the components \((15, 2, 2)\) and \((1, 2, 2)\) both contain a pair of the \(Y = \pm 1\) \(SU(2)_L\) Higgs doublets, whose neutral components give masses to the fermions. The component \((10, 1, 3)\) contained in \(\overline{126}\) gives masses to the RH neutrinos through the coupling \((16_i)(16_j)(\overline{126})\). The LH neutrino Majorana masses are generated due to the \((10, 3, 1)\) component of \(\overline{126}\). As these \(SU(2)_L\) doublets acquire VEV's, the following mass matrices are generated,

\[ M_u = Y_{ab}^{10} \langle 10^+ \rangle + Y_{ab}^{120} (\langle 120^+ \rangle + \frac{1}{3} \langle 120'^+ \rangle) + \frac{1}{3} Y_{ab}^{\overline{126}} \langle \overline{126}^+ \rangle \equiv Y_u^u v_u \] (170)

\[ M_d = Y_{ab}^{10} \langle 10^- \rangle + Y_{ab}^{120} (-\langle 120^- \rangle + \frac{1}{3} \langle 120'^- \rangle) - \frac{1}{3} Y_{ab}^{\overline{126}} \langle \overline{126}^- \rangle \equiv Y_d v_d \] (171)

\[ M_e = Y_{ab}^{10} \langle 10^+ \rangle + Y_{ab}^{120} (-\langle 120^- \rangle - \langle 120'^- \rangle) + Y_{ab}^{\overline{126}} \langle \overline{126}^- \rangle \equiv Y_e v_d \] (172)

\[ M_{\nu,L,R} = Y_{ab}^{10} \langle 10^+ \rangle + Y_{ab}^{120} (\langle 120^+ \rangle - \langle 120'^+ \rangle) + Y_{ab}^{\overline{126}} \langle \overline{126}^+ \rangle \equiv Y_{\nu,L,R} v_u. \] (173)

Note that a Clebsch-Gordon coefficient \((-3)\) is generated in the lepton sectors when the \(SU(4) \times SU(2)_L \times SU(2)_R\) components \((15, 2, 2)\) are involved in the Yukawa couplings. The neutrino Majorana mass matrices are given by

\[ M_{\nu,RR} = Y_{ab}^{\overline{126}} \langle \overline{126}^0 \rangle \] (174)

\[ M_{\nu,LL} = Y_{ab}^{\overline{126}} \langle \overline{126}^+ \rangle \] (175)
This contraction arises by integrating out a pair of 16's and inside the second parenthesis, the two 16's contract to form a 5 of SO(5) singlet direction which breaks the electroweak symmetry SO(5) down to SU(2)×U(1), we obtain
\[
(5)_{i}(10)< 1_{H_{1}} > < 5_{H_{2}} > .
\]

Inside the first parenthesis in Eq.(176), the two 16's contract to from a 5 of SU(5), while inside the second parenthesis, the two 16's contract to form a 5 of SU(5). This contraction arises by integrating out a pair of 5 and \( \mathbf{\bar{5}} \) of SU(5) from the 10's of SO(10), as shown in Fig.11. Because the 5 contains the SU(2) lepton doublet and the singlet down-type quarks, the resulting mass terms
\[
\lambda(d_{i,j}^e d_{L,j} + e_{L,i} e_{L,j})v_d
\]
are related by \( M_d = M^T \) and the lop-sided mass texture arises,
\[
(\mathbf{\bar{d}_{R,2}} \mathbf{\bar{d}_{R,3}}) \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_{L,2} \\ d_{L,3} \end{pmatrix} v_d + (\mathbf{\bar{u}_{R,2}} \mathbf{\bar{u}_{R,3}}) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} e_{L,2} \\ e_{L,3} \end{pmatrix} v_d,
\]
where various VEVs are those of the neutral components of SO(10) representations as indicated in Table 3.
if \((i,j)\) is chosen to be \((2,3)\). The \((33)\) entry which is expected to be of order \(O(1)\) is generated by tree level diagram involving a 10. The \((23)\) and \((32)\) entries of \(M_d\) and \(M_e\) also receive contributions from other non-renormalizable operators, for example, \(16_i16_j45H10_H\). If \(\lambda\) is of order \(O(1)\) and other contributions to the \((23)\) and \((32)\) entries are much smaller than one, a large mixing angle is then generated in the right-handed down quark sector, and the left-handed charged lepton sector, while the corresponding mixing angle in the left-handed down quark sector is small. This thus provides a way to explain the large mixing angle in atmospheric neutrinos while the quark mixing \(V_{cb}\) is small.

As we have seen above, when additional matter fields are introduced into the model and non-renormalizable operators that generate fermion masses are taken into account, as in the case of the Froggatt-Nielsen mechanism, other Higgs representations can play a role in the mass generation. An interesting case is the 45-dimensional Higgs representation which has Yukawa couplings to a 16- and a 10-dim matter fields. These non-renormalizable operators can be expressed generically by \(^{95}\)

\[
\mathcal{O}_{ij} = 16_i \cdot \mathcal{O}_n \cdot 10 \cdot \mathcal{O}_m \cdot 16_j
\]  

(180)

where the operator \(\mathcal{O}_n\) is given by

\[
\mathcal{O}_n = \frac{M_G^{p} \cdot 45_{p+1} \ldots 45_{q}}{M_{Pl}^{p} \cdot 45_{X}^{n-q}}, \quad n, p, q = \text{integer}.
\]  

(181)

Here \(M_G\) and \(M_{Pl}\) refer to the GUT scale and the Planck scale respectively. The 45-dimensional representation Higgs can acquire VEV along the following four directions: \(X, Y, B - L, T_{3R}\), where \(X\) and \(Y\) are defined as

\[
X = -(\Sigma_{12} + \Sigma_{34} + \Sigma_{56} + \Sigma_{78} + \Sigma_{910})
\]  

(182)

\[
Y = y \otimes \zeta
\]

where

\[
\zeta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad y = \text{diag}(1/3, 1/3, 1/3, -1/2, -1/2).
\]  

(183)

A systematical way to search for these effective operators is discussed in Ref. 95. The coefficient \(-3\) is obtained in this case when the 45-dimensional Higgs acquires
a VEV along the $B - L$ direction. We summarize in Table. all possible CG coefficients for the four possible directions of $< 45 >$.

Table 4. CG coefficients for the four possible directions of $< 45 >$.

|     | u   |  |  | d   |  |  | e  |  |  | ν  |  |  |  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| B-L | 1   | -1  | 1   | -1  | -3  | 3   | -3  | 3   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $T_{3R}$ | 0 | -1/2 | 0 | 1/2 | 0 | 1/2 | 0 | -1/2 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| X   | 1   | 1   | 1   | -3  | -3  | 1   | -3  | 5   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| Y   | 1/3 | -4/3 | 1/3 | 2/3 | -1  | 2   | -1  | 0   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |

4.5. Automatic R-parity conservation and $\underline{126}$ v.s. $(16 \times 16)$ for neutrino masses

One of the salient features of $SO(10)$ is that in some classes of models, R-parity is conserved automatically. Group theoretically, the congruence number is defined as follows: For an irreducible representation of $SO(2n)$ whose Dynkin index reads $(a_1 \, a_2 \ldots a_n)$, this representation has congruence number

\[(c_1, c_2) \equiv (a_{n-1} + a_n, 2a_1 + 2a_3 + \ldots + 2a_{n-2} + (n-2)a_{n-1} + na_n)\]

mod $(2, 4)$ for $n =$ odd.

(184)

There are four possible classes of $(c_1, c_2)$: $(0, 0), (0, 2), (1, 1), (1, 3)$. The first two classes are tensor-like while the later two classes are spinor-like which is troublesome. To see how $(c_1, c_2)$ relates to R-parity, alternatively we can define, in $SO(10)$,

\[c = 3(B - L) \mod 4\]

(185)

and $c = \text{Max}(c_1, c_2)$. It has been shown that if all the Higgs representations that break $SO(10)$ down to the SM are chosen to have congruence number $c = 0$ or 2, then R-parity is preserved at all energies.\cite{96, 97} Representations having $c = 0$ are: 45, 54, 210, ...; those having $c = 2$ are: 10, 126, $\underline{126}$, ... Note that the spinor representations 16 and $\overline{16}$ have $c = 1$ and 3 respectively.

Some models avoid the use of $\underline{126}$-dim Higgses by introducing non-renormalizable operators of the form $\bar{\psi}_a \psi_b (\overline{16}_H)(\overline{16}_H)$ instead of a renormalizable $\bar{\psi}_a \psi_b \underline{126}_H$. Models utilizing the spinor representation 16 to construct the neutrino mass operators generally have R-parity broken at some high energy scale. Such models also appear to be less constrained due to the inclusion of non-renormalizable operators. Also, a discrete symmetry, the R-parity symmetry, must be imposed by hand to avoid dangerous dim-4 baryon number violating operators in the effective potential at low energies which otherwise could lead to fast proton decay rate. Utilizing $\underline{126}$-dim representation of Higgses has the advantage that R-parity symmetry is automatic.\cite{98, 99, 100, 101, 49, 102, 48, 103, 104} It is to be noted that the contribution of the
126-dimensional representation to the \(\beta\)-function makes the model nonperturbative (with the onset of the Landau pole) above the unification scale \(M_{\text{GUT}}\). One could view these models as effective theories valid below this scale where coupling constants are perturbative. One may also argue against the use of 126 with the fact that it is not possible to obtain such a large representation from string theory. It has been shown that in heterotic string theory it is not possible to get 126 of \(SO(10)\) up to Kac-Moody level-5. Nevertheless, no such constraints have been found in other types of string theories.

### 4.6. Some Related Issues

#### 4.6.1. Proton Decay

One of the signatures of any GUT model is baryon number violating processes. These processes include proton decay \((\Delta(B-L) = 0)\), \(N-\bar{N}\) oscillation \((\Delta(B-L) \neq 0)\), etc. The theoretical prediction for oscillation time in \(N-\bar{N}\) oscillation can naturally satisfy the experimental lower limit \(\tau_{N-\bar{N}} \geq 0.86 \times 10^8\) sec if a high \(B-L\) breaking scale is assumed. On the other hand, the non-observation of proton decay has put many GUT models under siege. There are three kinds of operators leading to proton decays in SUSY GUT’s:

(i) Dimension-6 operators:

As we have mentioned previously, the extra gauge bosons in GUT models, the \(X\) and \(Y\) gauge bosons, can lead to proton decay. The terms in the Lagrangian containing the \(X\) and \(Y\) gauge bosons are

\[
\mathcal{L}_{X,Y} = \frac{ig_X}{\sqrt{2}} X_{\mu,i} (\epsilon_{ijk} \bar{u}^i_k \gamma_{\mu} u^j_L + \bar{d}^i_{\gamma_{\mu}} e^+) + \frac{ig_Y}{\sqrt{2}} Y_{\mu,i} (\epsilon_{ijk} \bar{u}^i_k \gamma_{\mu} d^j_L - \bar{u}^i_{\gamma_{\mu}} e^+ + \bar{d}^i_{R} \gamma^R e^+) + \text{h.c.} \tag{186}
\]

and the following vertices are allowed

\[
\bar{X} \to uu, \bar{d}e^+, \quad \bar{Y} \to ud, \bar{u}e^+. \tag{187}
\]

These can thus lead to proton decay via the dim-6 operators. Note that these type of operators exist in both non-SUSY and SUSY GUT’s. The dominant decay mode is \(p \to e^+\pi^0\), and the decay amplitude associated with this mode is

\[
\mathcal{M}_{p \to e^+\pi^0} \simeq 4\pi \alpha_{\text{GUT}} / M_{\text{GUT}}^2 \tag{188}
\]

leading to a life-time of

\[
\tau_p \simeq \frac{1}{M_{p \to e^+\pi^0}^2 m_p^2} \simeq 4.5 \times 10^{29} \times 0.7 (\frac{M_{\text{GUT}}}{2.1 \times 10^{14} \text{GeV}})^4 \tag{189}
\]

where \(m_p\) is the mass of the proton. For \(M_{\text{GUT}} \simeq 2 \times 10^{16} \text{GeV}\), we get \(\tau_p \sim 4.5 \times 10^{37} \pm 0.7\) years which is far above the current capability of SuperKamiokande.
experiments whose limit is $\sim 10^{34}$ years.

(ii) Dimension-5 operators
In SUSY GUT's, a new channel for proton decay is possible via the dim-5 operator through the exchange of the color triplet Higgsinos where $Q Q H$ and $Q L H$ via $H H$ mixing generate an effective operator

$$QQQL/M_H.$$  

(190)

In order to suppress this operator, the masses of the color triplet Higgsinos must have superheavy masses. From the point of view of unification, we would like to have the spectrum of MSSM below $M_{GUT}$. This requires that all the Higgs fields, including the color triplet Higgsinos, to be very heavy, with masses of the order of $M_{GUT}$, except the pair of $SU(2)_L$ doublet Higgses remaining light which are then identified as the pair of Higgs doublets in MSSM. How to achieve such a mass splitting is referred to as the doublet-triplet splitting (DTS) and doublet-doublet splitting (DDS) problems. The dominant decay mode of this operator is $\tau(p \rightarrow K^+ \tilde{\nu}) \sim m_{\tilde{h}}^2 \sim M_{GUT}^2$. Its decay amplitude is

$$M_{p \rightarrow K^+ \tilde{\nu}} \simeq \frac{h_u h_d m_{gaugino} g_{GUT}^2}{16 \pi^2 M_{GUT}^2}.$$  

(191)

Dimopolous and Wilczek proposed a mechanism [109] to achieve such mass splittings using $<45_H>$ along the $(B-L)$ direction

$$<45_H> = i\tau_2 \otimes diag(a, a, a, 0, 0).$$  

(192)

Chacko and Mohapatra [110] found that with a complimentary VEV pattern to that of the Dimopoulos-Wilczek type, that is,

$$<45_H> = i\tau_2 \otimes diag(0, 0, 0, b, b)$$  

(193)

the same goals can be achieved. A solution utilizing $126_H$ instead of $45_H$ to achieve the DTS and DDS was proposed by Lee and Mohapatra [108]. Detailed calculations have shown, nevertheless, that even with these mechanisms in place, in order for the prediction of $\tau_p$ to be consistent with the experimental limit, the effective $M_H$ must be larger than $M_{GUT}$ by at least a factor of 10. This in turn requires some couplings to be much larger than 1 which is somewhat unnatural.

(iii) Dimension-4 operators:
As we have seen, the dim-4 operators are forbidden if there is R-parity in the model. In SUSY SU(5) one has to impose R-parity by hand, while in some SO(10) models, R-parity is automatically conserved if certain type of Higgs fields are chosen to construct the model, as we have discussed in the previous section.
4.6.2. Baryogenesis

The three Sakharov conditions\cite{112} (i) baryon number \((B)\) violating processes, (ii) \(C\) and \(CP\) violation, and (iii) the deviation from thermal equilibrium, for the generation of the cosmological matter anti-matter asymmetry can be naturally satisfied in the \(SO(10)\) model. Due to the presence of sphaleron effects, the only chance for GUT baryogenesis to work is to produce an asymmetry in \(B - L\) at a high scale. To see this, let us first write the baryon number \(B\) as

\[
B = \frac{1}{2}(B + L) + \frac{1}{2}(B - L). \tag{194}
\]

The electroweak sphaleron transitions rapidly erase the asymmetry \(B + L\) as soon as the temperature drops down to about \(10^{12} \text{ GeV}\). Therefore, to have a non-vanishing baryonic asymmetry requires a non-vanishing \(B - L\) asymmetry. The crucial point to note is that even though the \(B + L\) asymmetry is erased by the sphaleron transitions, the orthogonal combination \(B - L\) is left untouched, and it opens up the possibility that the baryonic asymmetry can be generated through leptogenesis.\cite{113,114,115}

The basic idea is that, since \(B + L\) must vanish at all times due to the sphaleron transitions, the asymmetry in the lepton number will consequently be converted into the asymmetry in the baryon number (with an opposite sign). The primordial leptonic asymmetry is generated by the out-of-equilibrium decay of the heavy right-handed Majorana neutrinos and their scalar partners in the supersymmetric case. The relevant superpotential is

\[
W_{\text{leptogenesis}} = (Y_e)_{ij} E^c_i L_j H_1 + (Y_{\nu LR})_{ij} N^c_i L_j H_2 + \frac{1}{2} (M_{RR})_{ij} N^c_i N^c_j. \tag{195}
\]

The heavy right-handed neutrinos and their scalar partners can decay through the following four decay modes:

\[
N_1 \rightarrow \tilde{l} + \tilde{\nu}^c \tag{196}
\]

\[
N_1 \rightarrow l + H_2 \tag{197}
\]

\[
\tilde{N}_1 \rightarrow \tilde{l} + H_2 \tag{198}
\]

\[
\tilde{N}_1 \rightarrow l + \tilde{\nu}^c. \tag{199}
\]

The interference between the tree-level and one-loop diagrams thus gives rise to the CP asymmetry.

In the basis where both charged lepton Yukawa couplings and the right-handed neutrino mass matrix are diagonal, the amount of CP asymmetry due to the interference between the tree level and one-loop diagrams for each decay mode is given by\cite{115}

\[
\epsilon_1 = -\frac{1}{8\pi} \frac{1}{(h_{\nu} h_{\nu'})_{11}} \sum_{i=2,3} Im\{(h_{\nu} h_{\nu'})_{i1}\} f\left( \frac{M_i^2}{M_1^2} \right) \tag{200}
\]

where

\[
f(x) = \sqrt{x} \left[ \ln\left( \frac{1+x}{x} \right) + \frac{2}{x-1} \right], \tag{201}
\]
and $h_\nu$ is the Dirac neutrino Yukawa matrix in the new basis. The right-handed Majorana neutrino mass matrix is diagonalized by

$$P_M O_R M_{RR} O_R^T P_M = \text{diag}(M_1, M_2, M_3)$$  \hspace{1cm} (202)

where $M_i$'s are real and non-negative, and $P_M$ is the diagonal Majorana phase matrix. In terms of the diagonalization matrices and the original Dirac neutrino Yukawa coupling, $(h_\nu h_\nu^\dagger)$ can then be rewritten as

$$h_\nu h_\nu^\dagger = P_{RR} O_R Y_{\nu LR} U_L^T U_e^T Y_e^\dagger P_{RR}^{-1} = P_{RR} O_R Y_{\nu LR} Y_{\nu LR}^\dagger O_R^{-1} P_{RR}^{-1}.$$  \hspace{1cm} (203)

We see that the phases in the right-handed neutrino mass matrix $M_{RR}$ and the Dirac neutrino mass matrix $Y_{\nu LR}$ are responsible for the CP asymmetry needed for the leptogenesis. For a hierarchical heavy right-handed neutrino mass spectrum, $M_3 \gg M_2 \gg M_1$, the argument of the function $f(x)$, $x \equiv \frac{M_2}{M_1}$, is much greater than 1. We can then approximate $f(x)$ as

$$f(x) = \sqrt{x} \left( \frac{1}{x} - \frac{1}{2x^2} + \ldots \right) + \frac{2}{x-1} \simeq \frac{3}{\sqrt{x}} = \frac{3M_1}{M_1}.$$  \hspace{1cm} (204)

The asymmetry $\epsilon_1$ can thus be rewritten as

$$\epsilon_1 \simeq -\frac{1}{4\pi} \frac{1}{(h_\nu h_\nu^\dagger)^{11}_{11}} \sum_{i=2,3} \text{Im} \left\{ (h_\nu h_\nu^\dagger)^{2}_{1i} \right\} \left( \frac{3M_1}{M_1} \right)$$  \hspace{1cm} (205)

where the factor 4 accounts for the fact that there are four decay modes. Using the fact that the mixing in $Y_{\nu LR}$ is small and that $(Y_{\nu LR})_{33}$ dominates other elements, we can further approximate

$$(h_\nu h_\nu^\dagger)^{2}_{1i} \simeq (P_{1i} P_{i1}^{-1}) |y_{\nu_e}|^2 (O_R)^{13}_{13} (O_R^*)^{i3}_{13}$$  \hspace{1cm} (206)

and

$$Im \left\{ (h_\nu h_\nu^\dagger)^{2}_{1i} \right\}_{11} \simeq |y_{\nu_e}|^2 Im \left\{ (P_{1i} P_{i1}^{-1})^{2}_{13} (O_R)^{13}_{13} (O_R^*)^{i3}_{13} \right\}. \hspace{1cm} (207)$$

To have a large amount of CP asymmetry, $\epsilon_1$, thus requires that the hierarchy among the three right-handed neutrino masses cannot be too large (that is, $\frac{M_1}{M_{2,3}}$ cannot be too small), and that the imaginary part of $\{ (P_{1i} P_{i1}^{-1})^{2}_{13} (O_R)^{13}_{13} (O_R^*)^{i3}_{13} \}$ together with the neutrino Dirac Yukawa couplings cannot be too small.

The amount of the lepton asymmetry generated is given by

$$Y_L \equiv \frac{n_L - \bar{n}_L}{s} = \frac{\epsilon_1}{g_*},$$  \hspace{1cm} (209)

Here $g_*$ is the number of relativistic degrees of freedom. For MSSM, it is $g_* = 228.75$. The out-of-equilibrium decay of the heavy Majorana neutrinos requires the decay width of the lightest neutrino, $\Gamma_1$, smaller than the Hubble constant at the temperature of the decay. That is,

$$r \equiv \frac{\Gamma_1}{H|_{T=M_1}} = \frac{M_{pl}}{(1.7)(32\pi)\sqrt{g_*}} \left( \frac{(h_\nu h_\nu^\dagger)^{11}_{11}}{M_1} \right) < 1,$$  \hspace{1cm} (210)
where $M_{Pl}$ is the Planck scale taken to be $1.2 \times 10^{19} \text{ GeV}$. In general, one can still have a sizable CP asymmetry remains even for $1 < r < 10$. The wash-out effects due to inverse decays and lepton number violating scattering processes together with the time evolution of the system is then accounted for by the factor $\kappa$. It is obtained by solving the Boltzmann equation for the system. An approximation is given by\textsuperscript{116}

$$
10^6 \leq r : \quad \kappa = (0.1 r)^{1/2} e^{- (4)(0.1 r)^{1/4}} \quad ( < 10^{-7})
$$

(211)

$$
10 \leq r \leq 10^6 : \quad \kappa = \frac{0.3}{r (\ln r)^{10/9}} \quad (10^{-2} - 10^{-7})
$$

(212)

$$
0 \leq r \leq 10 : \quad \kappa = \frac{1}{2 \sqrt{r + 9}} \quad (10^{-1} - 10^{-2})
$$

(213)

where inside the parentheses we give the order of magnitude of $\kappa$ for each corresponding $r$. We note that in order to have a small dilution factor, the lightest right-handed neutrino cannot be too light. The electroweak sphaleron effect will convert the lepton asymmetry $Y_L$ into baryon asymmetry $Y_B$, and they are related by

$$
Y_B \equiv \frac{n_B - \bar{n}_B}{s} = c Y_{B-L} = \frac{c}{c-1} Y_L
$$

(214)

with

$$
c = \frac{8N_F + 4N_H}{22N_F + 13N_H}
$$

(215)

where $N_F$ is the number of families and $N_H$ is the number of $SU(2)$ Higgs doublets. For the MSSM spectrum, $(N_F, N_H) = (3, 2)$, we have the conversion factor $(\frac{c}{c-1}) \approx -0.53$.

Models with only two RH neutrinos have been constructed by Frampton \textit{et al.}\textsuperscript{53} and by Raby\textsuperscript{54} (see Sec. 3 and 6), which give rise to bi-large mixing pattern and a correlation between the sign of the baryon number asymmetry and the sign of the CP violation in neutrino oscillation. General analyses on the consistency between constraints from simplest $SO(10)$ models and leptogenesis can be found in Ref. 117, 118.

4.7. \textbf{SUSY Breaking}

SUSY breaking can be incorporated into models by including explicitly the soft SUSY breaking terms. Since the RGE’s for the Yukawa coupling constants and gauge coupling constants do not have any dependence on the soft breaking parameters up to two-loop level, the presence of these soft SUSY breaking terms does not affect the predictions for fermion masses and mixing angles. On the other hand, since the evolutions of soft SUSY breaking parameters does depend on the Yukawa coupling constants and gauge coupling constants, whether the EW symmetry is broken (that is, the mass-square of the light Higgs doublet is driven to be negative) may depend on the Yukawa sector, which can be used as a test of the validity of the model.
Even though SUSY breaking does not affect the running of the gauge coupling constants and that of the Yukawa coupling constants, it could have a large contribution to the threshold corrections, which is the subject of the next section.

4.8. Threshold Corrections

When the RGE analysis is performed, we usually consider $\beta$ function coefficient as a constant for each coupling constant between the two relevant scales. This is done under the assumption that all the heavy modes decouple at the same scale, the symmetry breaking scale. Threshold corrections are the corrections due to the differences between the symmetry breaking scale and the masses of the heavy particles decoupled from the spectrum after symmetry breaking takes place. If all the heavy particles acquire masses exactly the same as the symmetry breaking scale, there are no threshold corrections. In practice, this is not the case. There are two possible sources of threshold corrections:

(i) GUT scale threshold corrections: Due to the presence of many large Higgs representations in $SO(10)$, the GUT scale threshold corrections could be large.

(ii) SUSY threshold corrections: Large threshold corrections to $m_b$, $\delta m_b/m_b \simeq -(0.15 \sim 0.2)$, are needed in most $SO(10)$ models in order to have a prediction for $m_b$ consistent with experiment. The dominant contributions are from the diagrams shown in Fig. 5. They give a correction

$$\delta m_b / m_b \simeq (\tan \beta/50)I$$

and $I$ is given by:

$$I \simeq \frac{50}{16\pi^2} \frac{\alpha_G}{\alpha_2} \frac{\mu m_{QY}}{m_{\tilde{g}} m_{\tilde{\chi}}^2} \left[ \frac{8}{3} \frac{\alpha_3}{\alpha_G} \frac{g_3^2 f(m_{\tilde{g}}^2)}{m_{\tilde{\chi}}^2} - 2 \lambda_t^2 f\left(\frac{\mu^2}{m_{\tilde{\chi}}^2}\right) \right]$$

where $f(x) = (1 - x + x \ln x)/(1 - x)^2$; $m_{\tilde{g}}$ and $\mu$ are the gluino mass and the $\mu$ term evaluated at the weak scale; $m_{\tilde{\chi}}^2 \equiv \frac{1}{2}(m_b^2 + m_Q^2)$ is the average of the squared masses of the $SU(2)$-singlet bottom squark and the $SU(2)$-doublet third generation squarks. A large soft SUSY breaking parameter space can give rise to such a
correction. With the typical values $\alpha_G = 0.75$ and $(\alpha_2, y_{33}) \simeq (0.124, 0.034, 1)$ at $M_{\text{weak}}$, $\delta_b$ can then be approximated as, with the assumption $\frac{2\alpha_G}{\alpha_w} m_{\tilde{G}}$,

$$I = (0.315) x t (0.69 f(t^2) - f(x^2))$$  \hspace{1cm} (218)

where $t = (\mu/m_0)$ and $x = m_{1/2}/m_0$ with $m_{1/2}$ and $m_0$ being the gluino mass and the common scalar mass respectively in the constrained MSSM (CMSSM). With $(t, x) = (5.2, 2)$ which are typical values in CMSSM, a correction $(\delta m_b/m_b) = -0.15$ for $\tan\beta = 10$ is obtained.

5. SO(10) Models with Texture Assumptions

In what follows we discuss a few selected SO(10) models combined with some texture ansatz for the mass matrices.

5.1. Buchmuller and Wyler

Buchmuller and Wyler\textsuperscript{122} assume symmetric mass textures for the up- and down-type quarks

$$M_{u,d} \sim \begin{pmatrix} 0 & \epsilon^3 e^{i\phi} & 0 \\ \epsilon^3 e^{i\phi} & \rho e^2 & \eta e^2 \\ 0 & \eta e^2 & e^{i\psi} \end{pmatrix}.$$  \hspace{1cm} (219)

Here $\rho = |\rho| e^{i\alpha}$ and $\eta = |\eta| e^{i\beta}$ are complex parameters of $O(1)$. Using the SO(10) relations, they have

$$m_{\nu}^{\text{Dirac}} = m_u, \quad m_e = m_d$$  \hspace{1cm} (220)

assuming the incorrect mass relations in the lighter two generations are lifted when higher dimensional Higgs representations are introduced. The right-handed neutrino Majorana mass matrix is generated by $\nu_{\nu} H$, and is assumed to have the following form

$$M_{\nu,RR} = \begin{pmatrix} 0 & M_{12} & 0 \\ M_{12} & M_{22} & M_{23} \\ 0 & M_{23} & M_{33} \end{pmatrix}.$$  \hspace{1cm} (221)

With the relations

$$M_{12} : M_{22} : M_{33} = \epsilon^5 : \epsilon^4 : 1, \quad M_{23} \sim M_{22},$$  \hspace{1cm} (222)

the resulting effective neutrino mass matrix has the following form

$$M_{\nu}^{\text{eff}} = \begin{pmatrix} 0 & \epsilon \epsilon^{2i\phi} & 0 \\ \epsilon \epsilon^{2i\phi} & -\sigma \epsilon^{2i\phi} + 2\rho \epsilon e^{i\phi} & \eta \epsilon e^{i\phi} \\ 0 & \eta \epsilon e^{i\phi} & \epsilon^{2i\phi} \end{pmatrix} \cdot \frac{v^2}{M_3}.$$  \hspace{1cm} (223)

where $M_3$ is the heaviest eigenvalue of $M_{\nu,RR}$ which is of the same order as $M_{33}$, and $\sigma$ is defined as $\sigma \epsilon^4 = M_{22}/M_3$. The parameters $\eta$, $\rho$ and $\sigma$ are all of $O(1)$, and
η and ρ are determined using the quark masses. The effective neutrino masses form
the following pattern,

\[ m_{\nu_1} : m_{\nu_2} : m_{\nu_3} = \epsilon : \epsilon : 1 \]  
(224)

with \( \epsilon \sim 0.1 \), we have \((m_{\nu_2}^2 - m_{\nu_1}^2)/(m_{\nu_3}^2 - m_{\nu_2}^2) \sim 10^{-2}\), which is consistent with
the LMA solution. Nevertheless, it is not clear whether the predicted solar angle is
consistent with experiment or not. The prediction for \( U_{e\nu_3} \) is of order
\( O(\epsilon) \).

An interesting feature of this model is that, with \( \epsilon \sim 0.1 \), the scale of
\( M_1 \) is about \( 10^9 \) GeV. The amount of baryonic asymmetry which is very sensitive to this
scale is given by

\[ Y_B \sim -\kappa \text{sign}(\sigma) \sin(\phi - \alpha) \times 10^{-9}. \]  
(225)

The result has the promise to be consistent with the observed value, once the
parameters \( \kappa, \sigma, \phi \) and \( \alpha \) are fixed.

5.2. Matsuda, Fukuyama and Nishiura

Matsuda, Fukuyama and Nishiura\(^{123}\) proposed Hermitian textures with four zeros
in the context of \( SO(10) \)

\[
M_{u,\nu} = \begin{pmatrix}
0 & A_{u,\nu} & 0 \\
A_{u,\nu} & B_{u,\nu} & C_{u,\nu} \\
0 & C_{u,\nu} & D_{u,\nu}
\end{pmatrix},
\]  
(226)

\[
M_{d,e} = \begin{pmatrix}
0 & A_{d,e}e^{i\alpha_{12}} & 0 \\
A_{d,e}e^{-i\alpha_{12}} & B_{d,e} & C_{d,e}e^{i\alpha_{23}} \\
0 & C_{d,e}e^{-i\alpha_{23}} & D_{d,e}
\end{pmatrix}.
\]  
(227)

For a general matrix of the form

\[
M = \begin{pmatrix}
0 & A & 0 \\
A & B & C \\
0 & C & D
\end{pmatrix},
\]  
(228)

one can relate its three eigenvalues, \( m_1, m_2, \) and \( m_3, \) to the matrix elements by

\[
DA^2 = -m_1m_2m_3
\]  
(229)

\[
BD - A^2 - C^2 = m_1m_2 + m_2m_3 + m_3m_1
\]  
(230)

\[
B + D = m_1 + m_2 + m_3.
\]  
(231)

Because of the observed fermion mass hierarchy, \( |m_3| \gg |m_2| \gg |m_1| \), it is a good
approximation to write \( B = m_2 \) and \( D = m_3 - m_1 \) and express the mass matrix
in terms of its three eigenvalues as

\[
M \simeq \begin{pmatrix}
0 & \sqrt{-m_1m_2} & 0 \\
\sqrt{-m_1m_2} & m_2 & \sqrt{-m_1m_3} \\
0 & \sqrt{-m_1m_3} & m_3 + m_1
\end{pmatrix}.
\]  
(232)
To put this idea to work in the context of $SO(10)$, we first note that the set of equations given in Eq.(160)-(163) can be re-written as, assuming $<120>$ is small and contribute to $M_e$ and $M_d$ only,

$$M_u = Y^{10} < 10 > + \epsilon Y^{126} < \underline{126} >$$

$$M_d = \alpha Y^{10} < 10 > + Y^{126} < \underline{126} > + Y^{120} < 120 >$$

$$r M_e = \alpha Y^{10} < 10 > - 3 Y^{126} < \underline{126} > + \delta Y^{120} < 120 >$$

$$r' M_{\nu}^{\text{Dirac}} = Y^{10} < 10 > - 3 \epsilon Y^{126} < \underline{126} >$$

$$s M_{\nu, LL} = \beta Y^{126} < \underline{126} >$$

$$s' M_{\nu, RR} = \gamma Y^{126} < \underline{126} >$$

where parameters $\alpha, \beta, \gamma$ are ratios of SM Higgs doublet VEVs from different $SO(10)$ representations. The symmetric (anti-symmetric) matrices $Y^{10,126}$ ($Y^{120}$) can be expressed in terms of the symmetric (anti-symmetric) part of the mass matrices $M_{u,d,e,\nu}$ as

$$(1 - \alpha \epsilon) Y^{10} < 10 > = (M_u)_s - \epsilon (M_d)_s$$

$$Y^{126} < \underline{126} > = \frac{1}{4} (M_d)_s - \frac{1}{4} r (M_e)_s$$

$$Y^{120} < 120 > = (M_d)_a$$

where the subscripts $s$ and $a$ refer to symmetric part and anti-symmetric part, respectively. One can then solve for the elements in matrices $Y$’s in terms of quark masses. As the simple approximations, $B = m_2$ and $D = m_3 + m_1$, work well for quark masses and CKM matrix elements, in order to obtain viable neutrino masses and mixing angles, a deviation must be made in the charged lepton mass matrix, $B_e = m_\mu (1 + \xi)$ and $D_e = m_\tau + m_e - \xi m_\mu$. With $\xi \sim 0.01$, maximal $\nu_\mu - \nu_\tau$ mixing angle and the LMA solution can be accommodated. In this model, the allowed region for the leptonic CP violating Dirac phase can be obtained.

5.3. Bando and Obara

Bando and Obara\cite{124,125} pursue along the line of Matsuda, Fukuyama and Nishijima\cite{123} to analyze mass matrices of the type given in Eq.(226), and have a detailed analysis on all possible combinations of contributions from either $< 10_H >$ or $< 126_H >$ for each non-vanishing entry. In other words, all possible ways the CG factor $(-3)$ due to $< 126_H >$ can appear in the neutrino Dirac mass matrix

$$\begin{pmatrix}
0 & *a_\nu & 0 \\
* a_\nu & *b_\nu & *c_\nu \\
0 & *c_\nu & 1
\end{pmatrix}$$

(242)
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where $*$ is either 1 or $-3$ depending upon whether the coupling is due to $<10_H>$ or $<\overline{126}_H>$. They found the following texture has best agreement with experiments

$$M_u = \begin{pmatrix} 0 & 126 & 0 \\ 126 & 10 & 10 \\ 0 & 10 & 126 \end{pmatrix} = \begin{pmatrix} 0 & a_u & 0 \\ a_u & b_u & c_u \\ 0 & c_u & 1 \end{pmatrix}.$$ \hspace{1cm} (243)

In this case, the Dirac neutrino mass matrix is

$$M_{\nu,\text{Dirac}} = \begin{pmatrix} 0 & -3a_u & 0 \\ -3a_u & b_u & c_u \\ 0 & c_u & -3 \end{pmatrix} m_t.$$ \hspace{1cm} (244)

and the right-handed neutrino Majorana mass matrix is generated by the coupling to $\overline{126}$ Higgs representation, and is of the form

$$M_{\nu,\text{RR}} = \begin{pmatrix} 0 & r & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_R.$$ \hspace{1cm} (245)

The effective neutrino mass matrix is thus given by

$$M_{\nu,\text{eff}} = \begin{pmatrix} 0 & \frac{a}{r} & \frac{2}{r} \\ \frac{2}{r} & 2ab & c c(\frac{a}{r} + 1) \\ \frac{a}{r} & c(\frac{a}{r} + 1) & d^2 \end{pmatrix} \frac{m_t^2}{m_R}.$$ \hspace{1cm} (246)

where $a = -3a_u$, $b = b_u$, $c = -3c_u$, and $d = -3$. The typical predictions for this type of mass matrices are

$$\sin^2 \theta_{23} \sim 0.98 - 1$$ \hspace{1cm} (247)
$$\tan^2 \theta_{12} \sim 0.29 - 0.46$$ \hspace{1cm} (248)
$$|\theta_{13}| \sim 0.037 - 0.038$$ \hspace{1cm} (249)
$$|m_{\nu_3}| \sim 0.053 - 0.059 \text{eV}$$ \hspace{1cm} (250)
$$|m_{\nu_2}| \sim 0.003 - 0.008 \text{eV}$$ \hspace{1cm} (251)
$$|m_{\nu_1}| \sim 0.0006 - 0.001 \text{eV}.$$ \hspace{1cm} (252)

6. $SO(10)$ Models with Family Symmetry in 4-dimensions

A natural framework to accommodate small neutrino masses is a grand unified theory based on $SO(10)$ in which a right-handed neutrino in each family is predicted and the see-saw mechanism can be implemented naturally. Many other models based on $SO(10)$ besides those mentioned in Section 5 have been proposed to accommodate the observed fermion masses and mixing angles. Here we concentrate only on models which utilize family symmetry. We classify these models according to their intermediate symmetry below $SO(10)$ breaking scale and the family symmetry. Different symmetry breaking pattern of $SO(10)$ have different mass relations among the quark and lepton sectors, resulting in different way to generate large leptonic mixing angles.
6.1. Models with Symmetric Mass Textures

Symmetric mass textures naturally arise if the SO(10) is broken down to the SM gauge group through the left-right symmetry breaking chain. Due to its symmetric nature, this type of models tend to be more predictive compared to models with lop-sided/asymmetric mass texture.

\section*{SU(2) Family Symmetry}

6.1.1. Chen and Mahanthappa

The model proposed by Chen and Mahanthappa\cite{49,102,48} has SU(2) family symmetry. Because SO(10) breaks down through the left-right symmetry breaking chain, symmetric mass matrices arise. The field content of this model is given by, in terms of SO(10) $\times$ SU(2) quantum numbers,

\begin{itemize}
    \item \textbf{Matterfields}: $1(16, 2), 1(16, 1)$
    \item \textbf{Higgsfields}: $5(10, 1), 3(126, 1)$
    \item \textbf{Flavonfields}: $3(1, 2), 3(1, 3)$.
\end{itemize}

After the symmetry is broken, the following mass matrices are generated

\begin{equation}
    M_{\nu_{LR}} = \begin{pmatrix}
        0 & 0 & \langle 10_+^1 \rangle e' \\
        0 & \langle 10_+^1 \rangle e & \langle 10_+^2 \rangle e' \\
        \langle 10_+^2 \rangle e' & \langle 10_+^3 \rangle e & \langle 10_+^1 \rangle e
    \end{pmatrix}
    = \begin{pmatrix}
        0 & 0 & r_2 e' \\
        0 & r_4 e & \epsilon \\
        r_2 e' & \epsilon & 1
    \end{pmatrix} M_U
    \tag{253}
\end{equation}

\begin{equation}
    M_{d,e} = \begin{pmatrix}
        0 & \langle 10_+^1 \rangle e' & 0 \\
        0 & \langle 10_+^2 \rangle e' (1, -3) & \langle 126 \rangle e & 0 \\
        0 & 0 & \langle 10_+^1 \rangle e
    \end{pmatrix}
    = \begin{pmatrix}
        0 & e' & 0 \\
        0 & (1, -3) p e & 0 \\
        0 & 0 & 1
    \end{pmatrix} M_D
    \tag{254}
\end{equation}

where

\begin{equation}
    M_U \equiv \langle 10_+^1 \rangle, \quad M_D \equiv \langle 10_+^1 \rangle
\end{equation}

\begin{equation}
    r_2 \equiv \langle 10_+^2 \rangle / \langle 10_+^1 \rangle, \quad r_4 \equiv \langle 10_+^3 \rangle / \langle 10_+^1 \rangle \quad p \equiv \langle 126 \rangle / \langle 10_+^1 \rangle.
\end{equation}

$\epsilon M$ and $e' M$ are the VEV’s accompanying the flavon fields. The mass hierarchy arises due to the Froggatt-Nielsen mechanism. The right-handed neutrino masses are generated due to $126_H$, thus R-parity is preserved at all energy scales, and no additional proton decay modes are allowed, which is to be contrasted to the case when $16_H$’s are implemented to generate right-handed neutrino Majorana masses (see the model of Babu, Pati and Wilczek discussed in the next section.) The right-handed neutrino mass matrix is given by

\begin{equation}
    M_{\nu_{RR}} = \begin{pmatrix}
        0 & 0 & \langle 126_0^1 \rangle \delta_1 \\
        0 & \langle 126_0^2 \rangle \delta_2 & \langle 126_0^3 \rangle \delta_3 \\
        \langle 126_0^0 \rangle \delta_1 & \langle 126_0^0 \rangle \delta_2 & \langle 126_0^0 \rangle \delta_3
    \end{pmatrix}
    = \begin{pmatrix}
        0 & 0 & \delta_1 \\
        0 & \delta_2 & \delta_3 \\
        \delta_1 & \delta_2 & \delta_3
    \end{pmatrix} M_R
    \tag{257}
\end{equation}
with \( M_R \equiv \left( \frac{126}{1} \right)^0 \). The effective neutrino mass matrix is of the following form

\[
M_{\nu LL}^{\text{eff}} = \begin{pmatrix}
0 & 0 & t \\
0 & 1 & 1 + t^{3/2} \\
t & 1 + t^{3/2} & 1
\end{pmatrix} \frac{g^2 v^2}{M_R}
\]

where \( t < 1 \). This model can accommodate the LMA solution, in addition to LOW and “Just SO” VO solutions. Its prediction for \( U_{e3} \) is about 0.15. With 11 input parameters, this model predicts 22(+9) masses, mixing angles, and CP violating phases for quarks and leptons (and right-handed neutrinos).

We note that in this model, the mass matrices, \( M_{\nu LR}, M_{\nu RR} \) and \( M_{\nu LL}^{\text{eff}} \), have identical form. In other words, the texture considered in Eq. (258) is invariant under the see-saw mechanism. The form invariance also occurs in a model of neutrino mixing\(^\text{126}\) which uses different texture.

### 6.2. Models with Lop-sided/Asymmetric Mass Textures

Models having \( SU(5) \) as the intermediate symmetry have lop-sided Yukawa matrices. This is due to the \( SU(5) \) relation, \( M_d = M_d^T \). This opens up the possibility of large leptonic mixing angles due to the large mixing angle in the charged lepton mixing matrix.

#### U(1) family symmetry

6.2.1. Babu, Pati and Wilczek

The model proposed by Babu, Pati and Wilczek\(^\text{127}\) utilizes the Abelian \( U(1) \) as its family symmetry. Because it is based on dimension-5 operators to generate fermion masses, its field content, in terms of \( SO(10) \) quantum numbers, is somewhat simple

**Matterfields:** \( 16_1, 16_2, 16_3 \)

**Higgsfields:** \( 1(10), 1(16 + 16), 45 \).

The mass matrices generated are given by

\[
M_u = \left( \begin{array}{ccc}
0 & \epsilon' & 0 \\
-\epsilon' & 0 & \epsilon + \sigma \\
0 & -\epsilon + \sigma & 1
\end{array} \right) \cdot m_u
\]

\[
M_d = \left( \begin{array}{ccc}
0 & \epsilon' + \eta' & 0 \\
-\epsilon' + \eta' & 0 & \epsilon + \eta \\
0 & -\epsilon + \eta & 1
\end{array} \right) \cdot m_d
\]

\[
M_{\nu LL} = \left( \begin{array}{ccc}
0 & -3\epsilon' & 0 \\
3\epsilon' & 0 & -3\epsilon + \sigma \\
0 & 3\epsilon + \sigma & 1
\end{array} \right) \cdot m_u
\]

(259)

(260)

(261)
The right-handed neutrino Majorana mass matrix is generated by the effective operator, \( \frac{1}{M} 16_j 16_H \overline{16}_H \) and is given by

\[
M_{\nu,RR} = \begin{pmatrix}
x & z \\
y & 1
\end{pmatrix} \cdot M_R
\]

and the resulting effective neutrino mass matrix is given by

\[
M_{\nu}^{\text{eff}} = \begin{pmatrix}
9\epsilon'^2 (x-z^2) & 3\epsilon' y(-3\epsilon' z + (3\epsilon + \sigma)x) & 3\epsilon (xy - (3\epsilon + \sigma)(x-z^2)) \\
3\epsilon' y(-3\epsilon' z + (3\epsilon + \sigma)x) & -9\epsilon'^2 y^2 & (3\epsilon + \sigma)y(3\epsilon' z - (3\epsilon + \sigma)x) \\
3\epsilon (xy - (3\epsilon + \sigma)(x-z^2)) & (3\epsilon + \sigma)y(3\epsilon' z - (3\epsilon + \sigma)x) & (3\epsilon + \sigma)(-2xy + (3\epsilon + \sigma)(x-z^2))
\end{pmatrix} \cdot m_{\nu}^{\text{eff}}.
\]

The large atmospheric mixing comes from the effective neutrino mixing matrix, by choosing the value of parameter \( y \). As \( y \) also gives rise to small \( V_{cb} \) value, the smallness of \( V_{cb} \) and the maximality of the atmospheric mixing angle are thus related. This model can only accommodate SMA solution for the solar neutrinos. The LMA solution can be obtained if an intrinsic LH neutrino Majorana mass term arising from a dim-7 operator is included, assuming it dominates over the regular Type I seesaw term. \[128, 129] A characteristic of this model is the presence of a new prominent dim-5 proton decay mode, \( p \rightarrow \mu^+ K^0 \), in addition to the \( p \rightarrow \pi K^+ \) mode. This is a consequence of utilizing the \( 16_H \) to generate neutrino masses. \[130\]

In this model, the \( \theta_{13} \) angle is predicted to be about \( 5.5 \times 10^{-4} \). As all parameters are taken to be real, CP is conserved in this model. With 14 parameters, this model accommodates 18 (+6) masses and mixing angles for quarks and leptons (and RH neutrinos), yielding 7 predictions in accord with the data.

6.2.2. Albright, Babu and Barr

The model proposed by Albright, Babu and Barr \[94, 131, 132, 133\] has \( U(1) \times Z_4 \times Z_4 \) as family symmetry. The model has the following particle content, in terms of \( SO(10) \) representations,

Matter fields: \( 16_1, 16_2, 16_3, 2(16 \oplus \overline{16}), 2(10), 6(1) \)

Higgs fields: \( 4(10), 2(16 \oplus \overline{16}), 1(45), 5(1) \).

The mass matrices in this model are generated by the Froggatt-Nielsen mechanism, and are given by

\[
M_u = \begin{pmatrix}
\eta & 0 & 0 \\
0 & 0 & \epsilon/3 \\
0 & -\epsilon/3 & 1
\end{pmatrix} \cdot m_u, \quad M_d = \begin{pmatrix}
\eta & \delta & \delta' e^{i\phi} \\
\delta & 0 & \sigma + \epsilon/3 \\
\delta' e^{i\phi} - \epsilon/3 & 0 & 1
\end{pmatrix} \cdot m_d
\]
The right-handed neutrino Majorana mass matrix is generated by the effective operator of the type, \( \overline{16} \ell \ell \overline{16} H 16 H \), and is given by

\[
M_{\nu_{LR}} = \begin{pmatrix}
\eta & 0 & 0 \\
0 & \epsilon & 0 \\
0 & -\epsilon & 1
\end{pmatrix} \cdot m_u, \quad M_e = \begin{pmatrix}
\eta & \delta & \delta' e^{i\phi} \\
\delta & 0 & -\epsilon \\
\delta' e^{i\phi} & 1 & \sigma + \epsilon
\end{pmatrix} \cdot m_d. \tag{265}
\]

The right-handed neutrino Majorana mass matrix is generated by the effective operator of the type, \( \overline{16} \ell \ell \overline{16} H 16 H \), and is given by

\[
M_{\nu_{RR}} = \begin{pmatrix}
c^2 \eta^2 & -b \epsilon \eta & a \eta \\
b \epsilon \eta & c^2 & -\epsilon \\
a \eta & -\epsilon & 1
\end{pmatrix} \cdot \Lambda_R. \tag{266}
\]

And the effective neutrino mass matrix that accommodates the LMA solution is given by

\[
M_{\nu_{eff}} \sim \begin{pmatrix}
0 & \epsilon/(a-b) \\
\epsilon/(a-b) & -\epsilon^2/(a-b)^2 - b\epsilon/(a-b) & 0 \\
0 & -b\epsilon/(a-b) & 1
\end{pmatrix} m_u^2/\lambda_R
\]

\[
= \begin{pmatrix}
0 & -\epsilon & 0 \\
-\epsilon & 0 & 2\epsilon \\
0 & 2\epsilon & 1
\end{pmatrix} m_u^2/\lambda_R, \quad \text{(choosing } a = 1, b = c = 2). \tag{267}
\]

This model can also accommodate the SMA, and “Just So” VO solutions. An interesting property of this model is that the large mixing angle in atmospheric neutrinos is due to the lop-sided structure of \( M_e \), in which \( \epsilon \sim 0.1 \) and \( \sigma \sim 1 \), giving rise to a large left-handed mixing angle in the \((2,3)\) block of \( V_{e,L} \). The matrix \( M_e \) is related to \( M_d \) by the \( SU(5) \) relation, \( M_e = M_d^T \), thus such a lop-sided structure gives rise to a large mixing angle in the right-handed rotation matrix for the down-type quarks, \( V_{d,R} \), making it un-observable. The large solar mixing angle is due to the structure in the \((1,2)\) block of \( M_{\nu_{eff}} \),

\[
\begin{pmatrix}
0 & -\epsilon \\
-\epsilon & 0
\end{pmatrix}.
\tag{268}
\]

Because the large mixing angles in the atmospheric and solar neutrinos are due to different mass matrices, the prediction for \( U_{e\nu_3} \) can be made to be extremely small. In this model, \( |U_{e\nu_3}| \) is predicted to be 0.014. With 10 parameters, this model accommodates all the 22 masses, mixing angles, and CP violating phases at low energies.

6.2.3. *Maekawa*

Maekawa\textsuperscript{[134]} has proposed a \( SO(10) \) model combined with an anomalous \( U(1)_A \) symmetry. This anomalous \( U(1) \) symmetry is important for achieving DTS via Dimopoulos-Wilczek mechanism and for generating fermion mass hierarchy. The \( U(1)_A \) charge assignments project out terms that destabilize the VEV of \( 45_H \), and thus guarantee that \(< 45_H > \) is along the \( B-L \) direction, which otherwise can only be achieved with fine-tuning and introducing additional Higgs multiplets. These
charge assignments also give rise to the fermion mass matrices of the following form:

\[
M_u = \begin{pmatrix} \lambda^6 \lambda^5 \lambda^3 \\ \lambda^5 \lambda^4 \lambda^2 \\ \lambda^3 \lambda^2 \end{pmatrix} < H_u >, \quad M_d = \begin{pmatrix} \lambda^4 \lambda^{7/2} \lambda^3 \\ \lambda^3 \lambda^{5/2} \lambda^2 \\ \lambda \lambda^{1/2} \end{pmatrix} < H_d >. \tag{269}
\]

The charged lepton mass matrix is \( M_e = M_T^d \cdot \eta \), where \( \eta \) characterizes the renormalization effects, and the Dirac neutrino mass matrix is given by

\[
M_{\nu}^{\text{Dirac}} = \begin{pmatrix} \lambda^4 \lambda^3 \lambda \\ \lambda^{7/2} \lambda^{5/2} \lambda^{1/2} \\ \lambda^3 \lambda^2 \end{pmatrix} \cdot \lambda^2 \eta < H_u >. \tag{270}
\]

The right-handed neutrino Majorana masses are generated by couplings to 16-dim Higgs representations

\[
M_{\nu,RR} = \begin{pmatrix} \lambda^6 \lambda^5 \lambda^3 \\ \lambda^5 \lambda^4 \lambda^2 \\ \lambda^3 \lambda^2 \end{pmatrix} \cdot \lambda^9 \tag{271}
\]

and the resulting effective neutrino mass matrix is

\[
M_{\nu}^{\text{eff}} = \begin{pmatrix} \lambda^2 \lambda^{3/2} \lambda \\ \lambda^{3/2} \lambda \lambda^{1/2} \\ \lambda \lambda^{1/2} \end{pmatrix} \cdot \lambda^{-5} \eta^2 < H_u^2 >. \tag{272}
\]

The LM matrix is then given by

\[
U_{LM} = \begin{pmatrix} 1 \lambda^{1/2} \lambda \\ \lambda^{1/2} \lambda^2 \lambda^{1/2} \\ \lambda \lambda^{1/2} \lambda^{1/2} \end{pmatrix}. \tag{273}
\]

The LMA solution can be accommodated in this model. The prediction for \( U_{\nu e} \) is about \( \lambda \) which is very close to the current bound from experiment. All order \( O(1) \) coefficients in the mass matrices are not specified in this model, thus the validity of this model is unclear.

6.2.4. \textit{Shafi and Tavartkiladze}

Shafi and Tavartkiladze proposed a SUSY SO(10) model combined with an anomalous \( U(1)_H \) symmetry. By extending matter content, the \( U(1)_H \) can account for the observed mass hierarchy and mixing angles in the charged fermion sector. In this model, the three light lepton families are contained in the additional matter fields, 10, rather than the three usual 16, in which quarks reside. From this point of view, the quark-lepton unification is lost. The bi-large neutrino mixing is achieved by introducing three additional SO(10) singlet states. Due to the \( U(1)_H \) charge assignment, only two of these singlets interact with the lepton doublets giving a \( 3 \times 2 \) neutrino Dirac mass matrix, and the neutrino RH Majorana mass matrix is
approximately $2 \times 2$. The LMA solution can be accommodated in this model.

**U(2) family Symmetry**

6.2.5. Blazek, Raby and Tobe

The model proposed by Blazek, Raby and Tobe\textsuperscript{136,137} has $U(2) \times U(1)^n$ as its family symmetry. It has the following field content, in terms of $SO(10) \times U(2)$:

- **Matter fields**: $1(16, 2), 1(16, 1), 1(1, 2), 1(1, 1)$
- **Higgs fields**: $1(10, 1), 1(45, 1)$
- **Flavon fields**: $2(1, 2), 1(1, 3), 2(1, 1_3)$

The mass matrices of this model have the following form

\[ Y_u = \begin{pmatrix} \kappa_1 \rho & (\epsilon' + \kappa_2 \epsilon) \rho & 0 \\ - (\epsilon' - \kappa_2 \epsilon) \rho & \epsilon \rho & \epsilon \tau_T \sigma \\ 0 & \epsilon \tau_T \sigma & 1 \end{pmatrix} \lambda \]

\[ Y_d = \begin{pmatrix} \kappa_1 \epsilon \rho & \epsilon' + \kappa_2 \epsilon & 0 \\ - (\epsilon' - \kappa_2 \epsilon) & \epsilon \rho & \epsilon \tau_T \sigma \\ 0 & \epsilon \tau_T \sigma & 1 \end{pmatrix} \lambda \]

\[ Y_{\nu_{LR}} = \begin{pmatrix} 3\kappa_1 \omega & -(\epsilon' - 3 \kappa_2 \epsilon) \omega & 0 \\ (\epsilon' + 3 \kappa_2 \epsilon) \omega & 3 \epsilon \omega & \frac{1}{2} \epsilon \tau_T \sigma \omega \\ 0 & \frac{1}{2} \epsilon \tau_T \sigma \omega & 1 \end{pmatrix} \lambda \]

\[ Y_e = \begin{pmatrix} 3\kappa_1 \epsilon & -(\epsilon' - 3 \kappa_2 \epsilon) & 0 \\ (\epsilon' + 3 \kappa_2 \epsilon) \omega & 3 \epsilon \omega & \frac{1}{2} \epsilon \tau_T \sigma \omega \\ 0 & \frac{1}{2} \epsilon \tau_T \sigma \omega & 1 \end{pmatrix} \lambda \]

where $\omega = 2\sigma/(2\sigma - 1)$ and $T_f = (\text{Baryon number} - \text{Lepton number})$ of multiplet $f$. The right-handed neutrino Majorana mass matrix is generated by the effective operator of the type, $\overline{16}_i \overline{16}_j H H$, and is given by

\[ M_{\nu_{RR}} = \begin{pmatrix} \kappa_1 S & \kappa_2 S & 0 \\ \kappa_2 S & S & \phi \\ 0 & \phi & 0 \end{pmatrix} . \]

This model can accommodate LMA solution, in addition to SMA and “Just So” VO solutions. In the case of LMA solution, its prediction for $U_{\tau\tau}$ is 0.049. It has 16 input parameters which yield 22 masses and mixing angles.

6.2.6. Raby

Raby has proposed a $SO(10)$ model combined with $SU(2) \times U(1)$ family symmetry\textsuperscript{59} in which the ansatz proposed by Frampton, Glashow and Yanagida\textsuperscript{33}
discussed in Sec. 3 naturally arises. The representations utilized in this model, in
terms of $SO(10) \times SU(2)$, are given as follows:

**Matter fields**: 
1(16, 2), 1(16, 1), 3(1, 1)

**Higgs fields**: 
(10, 1), (16, 1), (45, 1)

**Flavon fields**: 
(1, 1), (1, 1_A), (1, 2), (1, 2), (1, 3),
in addition to several vector-like Froggatt-Nielsen fields. For the purpose of discus-
sion, we denote the three $SO(10) \times SU(2)$ singlets by $N_i (i = 1, 2, 3)$ which play an
important role in the neutrino sector, the $SU(2)$ (anti-)doublet flavon fields by $(\tilde{\phi})$
$\phi$, and the $SU(2)$ singlet flavon fields by $\theta$ and $S_i (i = 1, 2)$. In the charged fermion
sector, the Yukawa matrices are given by

$$Y_u = \begin{pmatrix}
0 & \epsilon' \rho & -\epsilon \\
-\epsilon' \rho & \epsilon & \epsilon \\
\epsilon & 1 & 1
\end{pmatrix} \lambda$$

$$Y_d = \begin{pmatrix}
0 & \epsilon' & -\epsilon \\
-\epsilon' & \epsilon & -\epsilon \\
\epsilon & 1 & 1
\end{pmatrix} \lambda$$

$$Y_{\nu_{LL}} = \begin{pmatrix}
0 & -\epsilon' \omega & 3\epsilon \omega \\
-\epsilon' \omega & 3\epsilon \omega & 3\epsilon \omega \\
3\epsilon \omega & 3\epsilon \omega & 3\epsilon \omega
\end{pmatrix} \lambda$$

$$Y_e = \begin{pmatrix}
0 & -\epsilon' & 3\epsilon \\
-\epsilon' & 3\epsilon & 3\epsilon \\
-3\epsilon \sigma & -3\epsilon \sigma & 1
\end{pmatrix} \lambda.$$  

The ansatz of Frampton, Glashow and Yanagida is obtained by considering the
following superpotential in which the three $SO(10) \times SU(2)$ singlets $N_i (i = 1, 2, 3)$ which play an
important role in the neutrino sector, the $SU(2)$ (anti-)doublet flavon fields by $(\tilde{\phi})$
$\phi$, and the $SU(2)$ singlet flavon fields by $\theta$ and $S_i (i = 1, 2)$. In the charged fermion
sector, the Yukawa matrices are given by

$$W_\nu = \frac{\lambda}{16} \left( N_1 \tilde{\phi}^a 16_a + N_2 \phi^a 16_a + N_3 \theta 16_a \right) + \frac{1}{2} (S_1 N_1^2 + S_2 N_2^2).$$

The complete neutrino mass terms are thus given by

$$\nu m_{\nu \nu} + \nu V N + \frac{1}{2} N M_N N$$

where $m_{\nu}$ ($\propto Y_{\nu_{LL}}$) is the Dirac mass matrix due to the coupling between $\nu$ and
$\tilde{\nu}$, which is related to the up-quark mass matrix, and $V$ and $M_N$ are the Majorana
mass matrices due to the coupling between $N_i$ and $\tilde{\nu}$, and the self coupling of $N_i$
in Eq. (283).

$$V = \frac{\nu_{16}}{M} \begin{pmatrix}
0 & \phi^1 & 0 \\
\phi^2 & \phi^3 & 0 \\
0 & 0 & \theta
\end{pmatrix}, \quad M_N = \text{diag}(M_1, M_2, 0).$$
After integrating out the heavy SM singlet neutrinos, $\nu_i$ and $N_i$, the neutrino effective mass matrix is obtained,

$$M_{\nu}^{eff} = m_\nu (V^T)^{-1} M_N V^{-1} m_\nu^T.$$  \hspace{1cm} (286)

The key observation is that if one defines

$$D^T \equiv m_\nu (V^T)^{-1} M_N P = \begin{pmatrix} a & 0 \\ a' & b \\ 0 & b' \end{pmatrix}, \quad \text{with} \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$  \hspace{1cm} (287)

the effective neutrino mass matrix can be rewritten in the following form

$$D^T \hat{M}_N^{-1} D$$  \hspace{1cm} (288)

where

$$\hat{M}_N \equiv \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$  \hspace{1cm} (289)

which is the ansatz proposed by Frampton et al. Note that in this model, because the mixing in the charged lepton sector is small, the LM matrix is approximately the diagonalization matrix of $M_{\nu}^{eff}$, and the prediction of a bi-large mixing pattern is not affected. Even though the bi-large mixing pattern can naturally arise in this model, the connection between the CP violation in neutrino oscillation and the sign of the baryogenesis, which exists in the model of Frampton, Glashow and Yanagida, is lost, due to additional CP phases and more complicated structure in this model.

**SU(3) Family Symmetry**

6.2.7. Berezhiani and Rossi

The model proposed by Berezhiani and Rossi [89] has $SU(3)$ as family symmetry with some unspecified discrete symmetries imposed. It has the following field content, in terms of $SO(10) \times SU(3)$:

**Higgs fields**: 2(10, 1), 1(16, 1), (16, 1), 2(45, 1), 1(54, 1)

**Flavon fields**: 1(1,6), 3(1,3), 1(1,8).

The mass matrices of this model are given as follows:

$$Y_u = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \quad Y_d = \begin{pmatrix} y_u D e^{i \xi} A e^{i \sigma} & A e^{i \sigma} & C \\ A e^{i \sigma} & y_c D e^{i \xi} & B \\ \frac{1}{b} C & \frac{1}{b} B & D \end{pmatrix}$$  \hspace{1cm} (290)

$$Y_{e L R} = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & \frac{1}{b} y_t \end{pmatrix}, \quad Y_e = \begin{pmatrix} y_u D e^{i \xi} & k_3 A e^{i \sigma} & \frac{1}{b} k_2 C \\ k_3 A e^{i \sigma} & y_t D e^{i \xi} & \frac{1}{b} k_1 B \\ k_2 C & k_1 B & D \end{pmatrix}.$$  \hspace{1cm} (291)
The right-handed neutrino Majorana mass matrix is taken to be diagonal. Because both \( Y_{\nu_{LR}} \) and \( M_{\nu_{RR}} \) are diagonal, the effective neutrino mass matrix is also diagonal. Thus the leptonic mixing matrix is proportional to \( V_{e,L} \). Due to the lop-sided structure in the \((2,3)\) block of \( Y_e \) arising from the \( SU(5) \) breaking chain, the maximal mixing angle in atmospheric neutrinos is obtained. Nevertheless, the solar mixing angle in this model is very small, which is in the range of SMA solution. This model has 14 input parameters in the Yukawa sector. The value of \( \sin^2 \theta_{13} \) is predicted to be \( \mathcal{O}(10^{-2}) \).

6.2.8. Kitano and Mimura

Kitano and Mimura\(^{139} \) propose a \( SO(10) \) model combined with \( SU(3) \times U(1)_H \) family symmetry. The three families of matter fields form a triplet of \( SU(3) \). The Yukawa couplings in this model have the form
\[
\sum_{i,j=1}^{3} \left( \frac{\Phi}{M_{pl}} \right)^{x_i + x_j} \frac{\xi_i \xi_j}{M^2} \tag{292}
\]
The field \( \Phi \) is a singlet of \( SU(3) \), but it has non-vanishing \( U(1)_H \) charges. Thus its VEV \( \lambda M_{pl} \) provides the mass hierarchy, if different generations, \( i \), have different \( U(1)_H \) charge, proportional to \( -x_i \). The field \( \xi_i \)'s are \( \mathbb{T} \) representation of \( SU(3) \), whose VEV break the \( SU(3) \) family symmetry
\[
< \xi_1 > \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} M_e, \quad < \xi_2 > \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} M_s, \quad < \xi_3 > \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} M_t \tag{293}
\]
where \( M_i \) is the family symmetry breaking scale. With the \( U(1)_H \) charge assignment, the mass matrices of the up- and down-type quarks are
\[
Y_u \sim \frac{1}{M_t^2} \left( < \xi_3 > < \xi_2 > < \xi_1 > \right) \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \begin{pmatrix} < \xi_3^T > \\ < \xi_2^T > \\ < \xi_1^T > \end{pmatrix}
\]
\[
\sim \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \tag{294}
\]
\[
Y_d = Y_e^T \sim \frac{1}{M_e^2} \left( < \xi_3 > < \xi_2 > < \xi_1 > \right) \begin{pmatrix} \lambda^5 & \lambda^4 & \lambda^1 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \begin{pmatrix} < \xi_3^T > \\ < \xi_2^T > \\ < \xi_1^T > \end{pmatrix}
\]
\[
\sim \begin{pmatrix} \lambda^5 & \lambda^4 & \lambda^1 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \tag{295}
\]
These Yukawa matrices give rise to the observed hierarchical masses
\[
m_u : m_c : m_t \sim \lambda^6 : \lambda^4 : 1 \tag{296}
\]
\[
m_d : m_s : m_b = m_c : m_{\mu} : m_\tau \sim \lambda^4 : \lambda^2 : 1 \tag{297}
\]
and the CKM matrix

\[ V_{CKM} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ -\lambda^3 & -\lambda^2 & 1 \end{pmatrix}. \] (298)

The role of these \( \xi \) fields is that it gives rise to the bi-maximal mixing pattern in the neutrino sector. With the \( U(1) \) charge assignment, the neutrino mass matrix is

\[ (\langle \xi_3 \rangle < \xi_2 < \langle \xi_1 \rangle) \begin{pmatrix} \lambda^2 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda^4 \end{pmatrix} (\langle \xi_3^T \rangle < \langle \xi_2^T \rangle < \langle \xi_1^T \rangle) \sim \begin{pmatrix} \lambda^2 & \lambda & 1 \\ 1 & 1 & \lambda \\ \lambda & 1 & 1 \end{pmatrix}. \] (299)

To implement the above idea in \( SO(10) \) is not trivial, because all the three families must have the same charge assignment under \( U(1)_H \). This is overcome by introducing many Froggatt-Nielsen fields and diagrams so that the phenomenologically viable mass matrices discussed above are reproduced. The neutrino masses are generated by a complicated mechanism in which each SM singlet in the 16-dim matter field mixes with a Froggatt-Nielsen field which transforms as singlet under \( SO(10) \). Thus, instead of a \( 6 \times 6 \) neutrino mass matrix, this model has a \( 9 \times 9 \) neutrino mass matrix. The Dirac neutrino mass matrix is different from that of the up-type quark because they are due to different Froggatt-Nielsen diagrams. Large mixing angle in the atmospheric sector can be accommodated, and the LMA solution can also be accommodated.

6.2.9. Ross and Velasco-Sevilla

Ross and Velasco-Sevilla\cite{140} utilize \( SU(3) \) as the family symmetry in combination with \( SO(10) \). Based on the formulation given in Sec. 2\cite{115}, the up- and down-type quark mass matrices are of the form

\[ M = \begin{pmatrix} \lambda' \epsilon^8 & \lambda \epsilon^3 & \lambda' \epsilon^3 \\ -\lambda^3 \epsilon^2 & \lambda'' \epsilon^2 & \lambda'' \epsilon^2 \\ -\lambda \epsilon^3 \end{pmatrix} M_{33}. \] (300)

Note the expansion parameter \( \epsilon \) is different for up- and down-quark sectors. The Dirac neutrino mass matrix is of the form

\[ M_{\nu}^{Dirac} = \begin{pmatrix} \mathcal{O}(\epsilon^8) & \epsilon^3(z + \epsilon(x + y)) & \epsilon^3(x + \epsilon(x - y)) \\ \epsilon^2(z + \epsilon(x + y)) & \epsilon^2(aw + cu) & \epsilon^2(aw - cu) \\ \epsilon^3(x + \epsilon(x - y)) & \epsilon^2(aw - cu) & 1 \end{pmatrix} M_{\nu, 33}^{Dirac}. \] (301)

where \( M_{\nu, 33}^{Dirac} \) is the \((33)\) component of the Dirac neutrino mass matrix, and the real parameters \( z, a, w \) and complex parameters \( x, y, u \) are of \( \mathcal{O}(1) \). Note that the higher order terms in symmetry breaking parameters have been included in \( M_{\nu}^{Dirac} \).
given above, as they are crucial for getting near bi-maximal mixing pattern. The right-handed neutrino Majorana mass matrix is assumed to be diagonal

\[
M_{\nu,RR} = \begin{pmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 \\
\end{pmatrix}
\]

(302)

The mass eigenstates of the effective neutrino mass matrix are given by

\[
\nu_1 \simeq \frac{|r|\nu_e - |z|e^{-i\xi}\nu_h}{\sqrt{z^2 + r^2}}
\]

(303)

\[
\nu_2 \simeq \frac{|r|e^{i\xi}\nu_e + |z|\nu_h}{\sqrt{z^2 + r^2}}
\]

(304)

\[
\nu_3 \simeq \nu_a
\]

(305)

where \( \xi = \text{Arg}(z) - \text{Arg}(r) \) and \( r = \sqrt{2}(zu - a_\nu wy)/z \). Here \( \nu_a \) is the heaviest mass eigenstate given by

\[
\nu_a = \frac{(z + \epsilon x)(\nu_\tau + \nu_\mu) - \epsilon y(\nu_\tau - \nu_\mu)}{\sqrt{(z + \epsilon(x + y))^2 + (z + \epsilon(x - y))^2}}
\]

(306)

and \( \nu_h \) is orthogonal to \( \nu_a \). As one can see in \( \nu_3 \), \( \nu_\mu \) contributes equally as \( \nu_\tau \), thus the mixing angle between \( \nu_\mu \) and \( \nu_\tau \) is maximal. A large solar mixing angle is also obtained if \( \tan^2\theta_{12} \simeq |z/r|^2 \) is in the right parameter space. Solutions for these parameters that are consistent with other charged fermion masses and CKM matrix elements have been found; it can accommodate LMA, LOW and “Just So” VO solutions. The CHOOZ angle in this model is predicted to be about \( U_{e\nu_3} \simeq 0.07 \).

6.3. Comparisons of Models and Other Issues

6.3.1. Distinguishing Models using CKM Unitarity triangle

To distinguish various existing models, clearly we need more precise results from the experiments. The most sensitive test are the three angles in the CKM unitarity triangle, as suggested in Ref. 141. To illustrate this point, in Table 5 predictions for the three angles of the CKM unitarity triangle from various models are given. Clearly, the predictions from these models are very different. Many models will be ruled out as soon as one can pin down the values for these three angles more accurately.

6.3.2. Distinguish Models Using \( \sin^2\theta_{13} \)

The two classes of models discussed in this section predict very different relations between \( U_{e\nu_3} \) and \( \Delta m^2_\odot/\Delta m^2_{\text{atm}} \), as discussed in Ref. 56, 142, 143, 48. The predictions for \( \sin^2\theta_{13} \) of various \( SO(10) \) models are summarized in Table 6. In the model of Chen and Mahanthappa, the typical value for \( \sin^2\theta_{13} \) is very close to
the sensitivity of current experiments. In this model, the value of $U_{e3} = \sin \theta_{13}$ is

Table 5. Predictions for the three angles of the CKM unitarity triangle from different models.

| Model                        | $\sin 2\alpha$ | $\sin 2\beta$ | $\gamma$  |
|------------------------------|-----------------|----------------|-----------|
| Albright-Barr132             | −0.2079         | 0.6428         | 64°       |
| Blazek-Raby-Tobe137 (LMA solution) | 0.94          | 0.39           | 47°       |
| Berezhiani-Rossi138 (Ansatz B) | 0.27           | 0.75           | 73°       |
| Chen-Mahanthappa102          | −0.8913         | 0.7416         | 34.55°    |
| Raby54                      | 0.92            | 0.50           | 71.7°     |
| Experiments                 | −0.95 ± 0.33    | 0.79 ± 0.12    | 34° ~ 82° |

Table 6. Predictions for $\sin \theta_{13}$ of various models. The upper bound from CHOOZ experiment is $\sin \theta_{13} \lesssim 0.24$. First nine models use SO(10). Last two models are not based on SO(10).

| Model                        | family symmetry | solar solution | $\sin \theta_{13}$ |
|------------------------------|-----------------|----------------|---------------------|
| Albright-Bar133              | $U(1)$          | LMA            | 0.014               |
| Babu-Pati-Wilczek127         | $U(1)$          | SMA            | $5.5 \times 10^{-4}$|
| Blazek-Raby-Tober157         | $U(2) \times U(1)^n$ | LMA     | 0.049               |
| Berezhiani-Rossi             | $SU(3)$         | SMA            | $O(10^{-2})$        |
| Chen-Mahanthappa135          | $SU(2)$         | LMA            | 0.149               |
| Kitano-Mimura139             | $SU(3) \times U(1)$ | LMA     | $\sim \lambda \sim 0.22$ |
| Maekawa134                   | $U(1)$          | LMA            | $\sim \lambda \sim 0.22$ |
| Raby54                       | $3 \times 2$ seesaw with $SU(2)_F$ | LMA | $\sim m_{\nu_2}/2m_{\nu_3} \sim O(0.1)$ |
| Ross-Velasco-Sevilla140      | $SU(3)$         | LMA            | 0.07                |
| Frampton-Glashow-Yanagida53  | $3 \times 2$ seesaw | LMA     | $\sim m_{\nu_2}/2m_{\nu_3} \sim O(0.1)$ |
| Mohapatra-Parida-Rajasekeran | RG enhancement | LMA            | 0.08 – 0.10         |
related to the ratio $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}}$ as

$$U_{e\nu_3} \sim (\Delta m^2_{\odot}/\Delta m^2_{\text{atm}})^{1/3}. \quad (307)$$

Thus as this ratio increases, the value of the angle $\theta_{13}$ increases. As the LMA solution is the most favored solution, the angle $\theta_{13}$ in this model is predicted to be very close to the current sensitivity of experiments. We note that $\theta_{13}$ of this order of magnitude leads to observable CP violation in neutrino oscillation.

In the model of Albright and Barr,[12][13] the relation between $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}}$ and $\sin \theta_{13}$ is quite different: as the ratio $\Delta m^2_{\odot}/\Delta m^2_{\text{atm}}$ increases, the prediction for $|U_{e\nu_3}|$ decreases. This is due to the fact that the large mixing angles in the atmospheric sector and solar sector have different origins. The most favored LMA solution thus implies that the value of $U_{e\nu_3}$ is extremely small; a neutrino factory is needed in this case in order to pin down its value.

6.3.3. $b - \tau$ unification

In most models, the prediction for $m_b$ at the weak scale tends to be higher than the experimental observed value, and thus a threshold correction of the size $-(15-20)\%$ is needed to bring down $m_b$. Such a large threshold corrections for $m_b$ are expected due to loop diagrams of $SU(2)$-singlet bottom squark, the $SU(2)$-doublet third generation squark, gluinos and charginos, as discussed in Sec. 4. Barr and Dorsner[14] suggested that, instead of these threshold corrections to $m_b$ being large and negative, $m_b/m_\tau$ may indeed be smaller than one at the GUT scale, and the deviation from the naive $b - \tau$ unification in $SU(5)$ is due to the large off-diagonal element of the charged lepton mixing matrix which also explain the large mixing in atmospheric neutrinos in models with lop-sided mass textures.

6.3.4. CP Violation

CP violation arises in these models from the complex phases in the VEV’s of the scalar fields, and from the complex phases of the Yukawa coupling constants. Thus they are free parameters in the models. In the quark sector, the complex phase is constrained by the masses and the mixing angles, thus definite predictions for the three angles $(\alpha, \beta, \gamma)$ in the CKM unitarity triangle can be obtained. In the leptonic sector, on the contrary, it is not possible at this moment to obtain definite predictions for the three CP violating phases. The situation will be improved once the absolute scales of neutrino masses and the three mixing angles are known to much better precision.

7. SUSY SO(10) in Higher Dimensions

The idea of orbifold (SUSY) GUT’s was first proposed by Kawamura to solve the doublet-triplet splitting problem,[15][16][17] and later developed by Altarelli and Feruglio[18] and Hall and Nomura[19]. The size of this type of extra dimensions are
small, being inverse of the GUT scale, \( R \sim 1/M_{GUT} \). To see how it works, let us consider the case with only one extra dimension, which is compactified on a \( S^1/\mathbb{Z}_2 \) orbifold. The circle \( S^1 \) has radius \( R \) and is defined by \( y = y + 2\pi R \). Under \( \mathbb{Z}_2 \), \( y \) is mapped to \(-y\). Thus the physical region can be taken as \( 0 \leq y \leq \pi R \). Various components transform under the \( \mathbb{Z}_2 \) symmetry as follows

\[
A_{\mu}(x,y) \rightarrow A_{\mu}(x,-y) = PA_{\mu}(x,y)P^{-1} \tag{308}
\]

\[
A_5(x,y) \rightarrow A_5(x,-y) = -PA_5(x,y)P^{-1} \tag{309}
\]

\[
\Phi(x,y) \rightarrow \Phi(x,-y) = \pm P\Phi(x,y) \tag{310}
\]

where \( A_{\mu}(x,y) \) and \( A_5(x,y) \) are the components of the gauge fields along the usual 4D and the 5th-dimension, respectively; \( \Phi(x,y) \) is a generic matter or Higgs field. The generators transform according to the following transformation rules,

\[
PT^aP^{-1} = T^a, \quad PT^{\bar{a}}P^{-1} = -T^{\bar{a}}. \tag{311}
\]

Here \( T^a \) are generators of the residual symmetry group while \( T^{\bar{a}} \) are the broken generators. The 5D bulk field can be decomposed into a infinite tower of KK states

\[
\phi_+(x^n, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \phi_+^{(n)} \cos \frac{ny}{R} \tag{312}
\]

\[
\phi_-(x^n, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \phi_-^{(n)} \sin \frac{ny}{R} \tag{313}
\]

The mode \( \phi_+ \) is even under the \( \mathbb{Z}_2 \) symmetry,

\[
P\phi_+(x,y) = \phi_+(x,-y) = +\phi_+(x,y). \tag{314}
\]

After compactification, it has a zero mode \( \phi_+^{(0)} \). The \( \phi_- \) is odd under \( \mathbb{Z}_2 \) transformation,

\[
P\phi_-(x,y) = \phi_-(x,-y) = -\phi_-(x,y). \tag{315}
\]

thus it does not have a zero mode, and its \( n \)-th KK mode has a GUT scale mass, \( (2n + 1)/\pi R \). For the broken generators, the corresponding gauge bosons thus acquire GUT scale masses; only those corresponding to the un-broken generators have 4D zero modes, which are then identified as the gauge fields of the little group. From the conditions given in Eq.(311), one sees that this symmetry breaking mechanism only works for non-Abelian symmetry, because the generators of an Abelian symmetry always commutes with the parity operator, \( P \). In other words, it is not possible to reduce the rank of a group by Abelian orbifolding,\(^\text{5}\) and additional \( U(1) \) symmetries survive, along with the SM gauge group, if the GUT symmetry has rank larger than 4. Breaking these \( U(1) \) symmetries can be achieved by the usual Higgs mechanism. In orbifold GUT models, because the GUT symmetry is broken by the

\(^{5}\text{In very limited cases of orbifold breaking by outer automorphism, the rank reduction may be possible.}^{150,151}\)
orbifold boundary conditions, one does avoid the task of constructing symmetry breaking scalar potential which is usually non-trivial.

One should also note that in higher space-time dimensions, supersymmetry is enlarged: In 5D, \( N = 1 \) SUSY has 8 super charges; it corresponds to \( N = 2 \) SUSY from the 4D point of view. By having different orbifold boundary conditions for different components in a “super” multiplet, one can reduce supersymmetry to 4D \( N = 1 \), similar to the case of GUT breaking. To break both SUSY and the gauge symmetry by orbifolding, a larger discrete Abelian orbifold is thus needed when building a realistic model. A \( N = 2 \) hypermultiplet can be decomposed into two \( N = 1 \) chiral multiplets; and a \( N = 2 \) vector multiplet can be decomposed into a \( N = 1 \) vector multiplet (denoted by “\( V \”) and a \( N = 1 \) chiral multiplet (denoted by “\( \Sigma \”). (See, for example, pp. 348-351 of Ref. 84.) To break \( N = 2 \) SUSY down to \( N = 1 \), we thus require \( V \) to be even under the parity, and \( \Sigma \) to be odd.

An immediate question one might ask is that how does the Georgi-Jarlskog \((-3)\) factor arise? Recall that in 4D GUT models, the GJ factor arises as the Clebsch-Gorden coefficients associated with the VEV’s of scalar fields along certain symmetry breaking directions. In orbifold GUT scenario, because the GUT symmetry is broken by orbifold boundary conditions, the GJ factor must arise in some other way.

### 7.1. Breaking SUSY \( SO(10) \) by Orbifolding

Several orbifoldings have been found to break \( SO(10) \). The number of extra dimensions that has been considered is either one or two. Some orbifoldings break \( SO(10) \) to only its maximal subgroups; others break \( SO(10) \) fully down to \( SU(3) \times SU(2)_L \times U(1)_Y \times U(1) \). Here we summarize various orbifolding that have been constructed.

#### 7.1.1. \( SUSY \) \( SO(10) \) in 5D

Dermisek and Mafi\(^{152}\) consider \( SO(10) \) in 5D and the extra dimension is compactified on \( S^1/(Z_2 \times Z_2') \) orbifold. The parities are chosen to be

\[
P = I_{5 \times 5} \otimes I_{2 \times 2}
\]

\[
P' = \text{diag}(-1, -1, -1, 1, 1) \otimes I_{2 \times 2}
\]

and the orbifold boundary conditions are given by

\[
45_v : \begin{array}{c}
V_{(15,1,1)}^{++} \\
V_{(1,3,1)}^{++} \\
V_{(1,1,3)}^{++} \\
V^{++}_{(6,2,2)} \\
\Sigma^{++}_{(15,1,1)} \\
\Sigma^{++}_{(1,3,1)} \\
\Sigma^{++}_{(1,1,3)} \\
\Sigma^{++}_{(6,2,2)}
\end{array}
\]

\[
10_H : \begin{array}{c}
H^{++}_{(1,2,2)} \\
H^{++}_{(1,1,1)} \\
H_c^{--}_{(1,2,2)} \\
H^{--}_{(6,1,1)}
\end{array}
\]

After compactification, the parity \( Z_2 \) reduces \( N = 2 \) SUSY to \( N = 1 \) SUSY in 4D, and the parity \( Z_2' \) reduces \( SO(10) \) to \( G_{PS} \). The residual symmetry below the compactification scale is the Pati-Salam group \( SU(4)_c \times SU(2)_L \times SU(2)_R \) on the
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4D “hidden” brane at fixed point \( y = \pi R/2 \) is then broken down to \( G_{SM} \) by the usually Higgs mechanism, and the symmetry on the “visible” brane at the fixed point \( y = 0 \) is \( SO(10) \).

Kim and Raby \cite{KimRaby} pursue along this line, and analyze the renormalization group evolution in this model: The 5D gauge coupling constant unification scale \( M_* \) (to be contrasted with the 4D unification scale \( M_{GUT} \)) is found to be \( \sim 3 \times 10^{17} \text{ GeV} \) and the compactification scale is found to be \( \sim 1.5 \times 10^{14} \text{ GeV} \).

Kyae and Shafi \cite{KyaeShafi} consider different parities,

\[
P = \text{diag}(I_{3\times3},I_{2\times2},-I_{3\times3},-I_{2\times2})
\]

\[
P' = \text{diag}(-I_{3\times3},I_{2\times2},I_{3\times3},-I_{2\times2})
\]

and the \( Z_2 \times Z'_2 \) charge assignments for the components of the \( SO(10) \) gauge field is

\[
45_Y : V^{++}_{(8,1,0)} V^{++}_{(1,3,0)} V^{++}_{(1,1,0)} V^{+-}_{(3,2,-5/6)} V^{+-}_{(3,2,5/6)} V^{-+}_{(3,1,-2/3)} 2V^{++}_{(3,2,1/6)} V^{-+}_{(1,1,1)} \cdot
\]

With these boundary conditions, they are able to break \( SO(10) \) down to \( SU(3) \times SU(2)_L \times U(1)_Y \times U(1)' \).

7.1.2. SUSY \( SO(10) \) in 6D

Asaka, Buchmuller and Covi et al \cite{AsakaBuchmullerCovi} consider \( SO(10) \) in 6D and the extra dimensions are compactified on a \( T^2/(Z_2 \times Z_2^{GG} \times Z_2^{PS}) \) orbifold. The idea is based on the observation that a simple extension of the SM gauge group to \( SU(3) \times SU(2)_L \times U(1)_Y \times U(1)' \) (which has the same rank as \( SO(10) \)) is the common symmetry subgroup of the Pati-Salam gauge group, \( SU(4) \times SU(2)_L \times SU(2)_R \) \((G_{PS})\) and the Georgi-Glashow gauge group, \( SU(5) \times U(1) \), \((G_{GG})\). The first parity \( P \) breaks supersymmetry down to \( N = 1 \) in 4D, upon compactification on \( T^2/Z_2 \). The other two parities break the \( SO(10) \) gauge symmetry, and can be taken to be

\[
P_{GG} = \begin{pmatrix} 
\sigma_2 & 
\sigma_2 & 
\sigma_2 \\
\sigma_2 & 
\sigma_2 & 
\sigma_2 \\
\sigma_2 & 
\sigma_2 & 
\sigma_2 
\end{pmatrix}, \quad
P_{PS} = \begin{pmatrix} 
-\sigma_0 & 
-\sigma_0 & 
-\sigma_0 \\
-\sigma_0 & 
-\sigma_0 & 
-\sigma_0 \\
-\sigma_0 & 
-\sigma_0 & 
-\sigma_0 
\end{pmatrix}
\]

in the vector representation of \( SO(10) \). At the fixed point of \( Z_2^{GG} \), SUSY \( G_{GG} \) is respected; at the fixed point of \( Z_2^{PS} \), SUSY \( G_{PS} \) is respected. The charge assignments for the gauge fields are chosen such that component fields belonging to the symmetric subgroup have positive parity and those belonging to the coset space have negative parity. At the intersection of two 5D subspaces of the 6D bulk, in which \( G_{PS} \) and \( G_{GG} \) are un-broken, respectively, the extended SM gauge group, \( SU(3) \times SU(2)_L \times [U(1)]^2 \) is realized. At this intersection (which is also one of the fixed points of the orbifold transformations), the electroweak symmetry and the additional \( U(1)' \) are broken by the usual Higgs mechanism.
7.2. Fermion Mass Hierarchy in SUSY SO(10) Models in Higher Dimensions

A few mechanisms have been proposed to solve the fermion mass hierarchy problem in orbifold GUTs. Some models address the realistic mass relations; the gauge symmetry breaking in this type of models is due to both orbifolding and the usual Higgs mechanism from which the GJ factor $-3$ needed to achieve the realistic mass relations arises as the CG coefficient associated with the VEVs of some Higgs multiplets. Other models make use of mechanisms that are purely higher dimensional, e.g. overlap wave function, to generate the mass hierarchy. No mechanisms have been found to generate the CG factor by compactification.

7.2.1. Hall et al

Hall et al\textsuperscript{157} proposed three SO(10) models in 6\textit{D}, and the two extra dimensions are compactified on $T^2 \mathbb{Z}_2$, $T^2 \mathbb{Z}_6$, and $T^2 (\mathbb{Z}_2 \times \mathbb{Z}_2)$ tori, respectively, in each of these three models. In the first model there is $N=1$ SUSY in the bulk, and the other two models have $N=2$ SUSY. These models incorporate a mechanism proposed by Hall et al\textsuperscript{149,158} to solve the fermion mass hierarchy problem in which the correct mass relations are generated by mixing the brane localized matter fields with additional matter fields that propagate along the fixed line. The generic Yukawa interactions for a brane-confined field, $\psi(x)$, and a bulk field, $\Phi(x, y)$, are given as follows:

$$L \sim \int dy \left\{ \lambda_0 \delta(y) \psi^3 + \lambda_1 \delta(y) \psi^2 \Phi(y) + \lambda_2 \delta(y) \psi \Phi(y)^2 + \lambda_3 \delta(y) \Phi(y)^3 + \lambda_4 \Phi(y)^3 \right\}.$$ (323)

The effective Lagrangian below the compactification scale is

$$\lambda_0 \psi^3 + \frac{\lambda_1}{\sqrt{1/2}} \psi^2 \Phi(0) + \frac{\lambda_2}{\sqrt{1/2}} \psi \Phi^2(0) + \frac{\lambda_3}{\sqrt{3/2}} \Phi^3(0) + \frac{\lambda_4}{\sqrt{1/2}} \Phi^3(0).$$ (324)

where $V = M_{\text{string}} \cdot R$ is the volume factor. Thus different Yukawa coupling constants in the 4\textit{D} effective Lagrangian have different volume suppression factors, depending upon how many bulk fields are involved in the interactions. Therefore by having different matter multiplets locate at different locations in the bulk, the mass hierarchy can be generated.

7.2.2. Albright and Barr

Albright and Barr\textsuperscript{159} consider SO(10) in 5\textit{D} and the fifth dimension is compactified on a $S^1 / (\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold. All quarks and leptons and most of the Higgs fields employed in Ref. 131 are confined to the SO(10) 3-brane; only the SO(10) gauge fields and a 10- and a 45-dimensional Higgs fields are placed in the 5\textit{D} bulk. As a consequence, most features of the 4\textit{D} model\textsuperscript{131} also exist here; the only exception is that because the DTS problem is solved by orbifolding, the Higgs superpotential in this case is simpler as the terms needed for DTS are absent. The parity $\mathbb{Z}_2$ breaks
N = 2 SUSY (from the 4D point of view) down to N = 1 SUSY; \( Z_2 \) breaks \( SO(10) \) down to the Pati-Salam group. The orbifold boundary conditions under \( Z_2 \times Z_2 \) are given by

\[
45v : V^{++}_{(15,1,1)} V^{++}_{(1,1,3)} V^{++}_{(6,2,2)} \Sigma^{-+}_{(15,1,1)} \Sigma^{-+}_{(1,1,3)} \Sigma^{-+}_{(6,2,2)}
\]

\[
45_H : H^{++}_{(15,1,1)} H^{++}_{(1,1,3)} H^{++}_{(6,2,2)} H^{--}_{(15,1,1)} H^{--}_{(1,1,3)} H^{--}_{(6,2,2)}
\]

\[
10_H : H^{++}_{(1,2,2)} H^{++}_{(6,1,1)} H^{--}_{(1,2,2)} H^{--}_{(6,1,1)}
\]

The VEV < 45_H > arising from the complete Higgs superpotential on the visible \( SO(10) \) brane is along the \( (B - L) \) direction, thus the GJ factor of \(-3\) remain in this model. Because all the matter fields are confined to the 4D brane, the Yukawa sector of this 5D model is essentially the same as that given in Ref. 131.

7.2.3. Kitano and Li

Kitano and Li [160] propose a supersymmetric \( SO(10) \) model in 5D; the extra dimension is compactified on \( S^1/Z_2 \) orbifold, which breaks \( N = 2 \) SUSY down to \( N = 1 \), from the 4D point of view. The gauge symmetry \( SO(10) \) can be broken either by the orbifold boundary conditions or by the usual Higgs mechanism. All three families of matter fields along with the gauge fields propagate in the bulk; a \( 45_H \), a pair of \( 16 \oplus \overline{16} \) which are needed to break the rank of the symmetry, and a \( 10_H \) are confined to the visible brane.

In this model, the fermion mass hierarchy is accommodated utilizing the overlap between zero mode profiles along the fifth dimension, as discussed in Sec. 2. In a 5D SUSY theory compactified on \( S^1/Z_2 \) orbifold, the zero mode wave function of a 5D bulk field with a bulk mass term \( m \) is localized exponentially as

\[
fo(y) \sim e^{-my}.
\]

In the \( SO(10) \) symmetric limit, all fields in one family must have the same bulk mass term, \( m_i \), resulting in unrealistic mass spectrum. When the \( U(1)_X \) subgroup of \( SO(10) \) is broken by the Higgs mechanism, the VEV of the scalar field \( < \phi > \) which triggers this breaking also contributes to the bulk mass terms of the matter fields. The resulting bulk mass terms are of the form

\[
m_i \rightarrow m_i - \sqrt{2}g_X Q_X^i < \phi > .
\]

Because different \( SU(5) \) components of \( SO(10) \), 1, \( \overline{5} \) and 10, have different \( U(1)_X \) charges, \( Q_X^i \), a realistic mass spectrum can be obtained

\[
Y_u \sim \begin{pmatrix} \lambda^6 & \lambda^3 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad Y_d = Y_e^T \sim \begin{pmatrix} \lambda^4 & \lambda^2 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda & 1 & 1 \end{pmatrix}, \quad m_{eff} \sim \begin{pmatrix} \lambda^2 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}.
\]

The lop-sidedness of \( Y_e \) thus gives the maximal atmospheric mixing angle and the LMA solution to the solar neutrino problem is accommodated.
7.2.4. Shafi and Tavartkiladze

Shafi and Tavartkiladze considered SO(10) in 5D compactified on a $S^1/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ orbifold. The flavor structure of this model arises at the fixed point which has $G_{PS}$ symmetry, thus the mechanism which generates fermion mass hierarchy and mixing angles is purely 4-dimensional. By extending the matter content of their model and by imposing $U(1)_H$ symmetry, the charged fermion masses and mixing angles can be accommodated. Bi-large neutrino mixing pattern is achieved by imposing “flavor democracy” in the neutrino sector, which has been made possible due to the extension of matter content.

8. Conclusion

SUSY GUT is one of the promising candidates for physics beyond the standard model: the hierarchy problem is solved, charge quantization is explained, gauge coupling constants unification is achieved; as a consequence, a prediction for the weak mixing angle $\sin^2 \theta_w$ is obtained. It provides a natural framework for small neutrino masses to arise, and it has the promise for baryogenesis. We have seen in this review how the fermion mass hierarchy can arise from a very constrained framework of SO(10); this is achieved by imposing family symmetries. As proton decay has not been observed, SUSY GUT’s in 4D are under siege. This situation can be alleviated if the SUSY GUT model is constructed in higher dimensions. The presence of extra dimensions also provides new ways to understand fermion mass hierarchy. On the experimental side, one hopes that more precise measurements for the masses and CKM matrix elements will enable us to distinguish these models, thus pointing out the right direction for model building. On the theory side, one hopes to obtain an understanding of the complicated symmetry breaking patterns and charge assignments that are needed in many of these models, which will then shed some light on Physics beyond the Standard Model.

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