ABSTRACT. Generalizing the definition of Cartier, we introduce $p^n$-typical formal group laws over $\mathbb{Z}(p)$-algebras. An oriented cohomology theory in the sense of Levine-Morel is called $p^n$-typical if its corresponding formal group law is $p^n$-typical. The main result of the paper is the construction of 'Chern classes' from the algebraic $n$-th Morava K-theory to every $p^n$-typical oriented cohomology theory.

If the coefficient ring of a $p^n$-typical theory is a free $\mathbb{Z}(p)$-module we also prove that these Chern classes freely generate all operations to it. Examples of such theories are algebraic $mn$-th Morava K-theories $K(nm)^*$ for all $m \in \mathbb{N}$ and $\text{CH}^* \otimes \mathbb{Z}(p)$ (operations to Chow groups have been studied in a previous paper). The universal $p^n$-typical oriented theory is $BP(n)^*$ and its coefficient ring $BP/(v_j, j \nmid n)$ is also a free $\mathbb{Z}(p)$-module.

Chern classes from the $n$-th algebraic Morava K-theory $K(n)^*$ to itself allow us to introduce the gamma filtration on $K(n)^*$. This is the best approximation to the topological filtration obtained by values of operations and it satisfies properties similar to that of the classical gamma filtration on $K_0$. The major difference from the classical case is that Chern classes from the graded factors $gr^i\gamma K(n)^* \to \text{CH}^i \otimes \mathbb{Z}(p)$ are surjective for $i \leq p^n$. For some projective homogeneous varieties this allows to estimate $p$-torsion in Chow groups of codimension up to $p^n$.

INTRODUCTION

In algebraic geometry a cohomology theory can be said to be orientable if it can be endowed with a suitable notion of push-forward maps for projective morphisms. For example, push-forward maps in Chow groups can be defined using direct image of cycles, in K-theory of smooth varieties one can define push-forward maps using the right derived functor of the push-forward of coherent sheaves, and in Weil-type cohomology theories, e.g. étale cohomology, push-forward maps can be described using the Poincare duality.

It was observed by Panin and Smirnov [PS00] that the orientation can also be given by the structure of Chern classes of line bundles satisfying certain axioms. As the projective bundle theorem is usually contained in the axioms of these cohomology theories, one can define higher Chern classes due to the classical method of Grothendieck. One could say that an orientable theory is a cohomology theory which can be endowed with a suitable notion of Chern classes. Moreover, one can show that for many cohomology theories (e.g. for free theories, see below) the ring of operations from $K_0$ to them is freely generated by Chern classes which provides the K-theory with a distinguished role with respect to orientation. The scope of this paper is a search for a different notion of orientability where the distinguished role is played by Morava K-theories instead of $K_0$.

There exist the universal oriented cohomology theory $\Omega^*$ which is called algebraic cobordism of Levine-Morel [LM07]. The universality has the following meaning here: for every oriented theory $A^*$ there exist a unique morphism of presheaves of rings from $\Omega^*$ to $A^*$ which respects both push-forward and pullback maps.

Another universal property of algebraic cobordism permits to introduce many new cohomology theories. The ring of coefficients of $\Omega^*$, at least in zero characteristic, is isomorphic to the Lazarid ring classifying formal group laws. For any formal group law $F_R$ over a commutative ring $R$ the presheaf of rings $\Omega^* \otimes L, R$ is an oriented theory, called free theory, where the map $L \to R$ corresponding to $F_R$ and the identification $\Omega^*(pt) \cong L$ are hidden in the notation. Well-known examples of free theories are Chow groups and K-theory obtained via the additive and the multiplicative formal group laws over $\mathbb{Z}$, respectively.

Note that the notions of orientation due to Levine-Morel and Panin-Smirnov differ in some technical details.
If one takes $R$ to be $Z_{(p)}$ where $p$ is a prime number, and chooses a formal group law over $Z_{(p)}$ with the logarithm of the form $\log_{K(n)}(x) = x + \frac{a_1}{p}x^{p^n} + \frac{a_2}{p^2}x^{p^{2n}} + \ldots$, where $a_1 \in Z^\times_{(p)}$, then the corresponding free theory $K(n)^*$ is called an $n$-th algebraic Morava K-theory. In a special case $n = 1$ and $\log_{K(1)}(x) = \sum_{i \geq 0} \frac{x^i}{p^i}$ free theory $K(1)^*$ is isomorphic to $K_0 \otimes Z_{(p)}$ as presheaf of rings. It seems that this was the main motivation in topology to call (similarly constructed) cohomology theories ordinary or Morava K-theories.

The definition given above is quite ad hoc and demands some explanations which we adopt from topology. Algebraic cobordism of a smooth variety together with Landweber-Novikov operations on it defines a quasi-coherent sheaf on the moduli stack of formal groups (see e.g. [S18b] Section 1.4). This stack has a particularly nice description after base change to $F_p$: there exist a decreasing filtration on it by closed substacks $M_{fg}^\geq n$ which classify formal group laws of height not less than $n$, and $M_{fg}^{\geq n+1}$ is in some sense a divisor in $M_{fg}^\geq n$. Moreover, $M_{fg}^{\geq n+1}$ has an essentially unique geometric point. Formal group laws yield free theories as explained above, and isomorphic formal group laws yield multiplicatively isomorphic free theories. Thus, a point in the stack of formal groups defines a free theory as a presheaf of rings (and the orientation on it can be chosen differently). Morava K-theories $K(n)^*/p$ are those free theories which correspond to some of the $F_p$-points of $M_{fg}^\geq n \setminus M_{fg}^{\geq n+1}$, and they all become multiplicatively isomorphic with $F_p$-coefficients. Thus, in some sense, Morava K-theories exhaust all free theories with $F_p$-coefficients except for the geometric point in the intersection of all substacks $M_{fg}^\geq n$, $n \geq 1$, which corresponds to Chow groups modulo $p$.

Recall that the first Morava K-theory is almost the same as the K-theory, and, thus, the chromatic picture of the stack $M_{fg}$ places $K(n)^*$ into an intermediate position between $K_0$ and $CH^*$. Based on this it may reasonable to expect that Morava K-theories should behave simpler than Chow groups, and yet more complicated than $K_0$. One statement that supports it is the following: if $X$ is a projective homogeneous variety such that $M_{K(n)}(X)$ is a Tate motive (here $M_{K(n)}$ is a functor to the category of pure motives corresponding to an oriented theory $K(n)^*$), then $M_{K(n)}(X)$ is a Tate motive for $m < n$ ([SeSe18, Cor. 7.11]). This suggests a linear strategy for the study of algebraic cobordism of projective homogeneous varieties which starts from $K(1)^*$ (aka $K_0$) to $K(n)^*$, $n > 1$, and finishes with $CH^* \otimes Z_{(p)}$ for every prime $p$.

Even though the results about Morava K-theories $K(n)^*$ might be interesting by themselves, one also hopes to obtain some information about the more classical theories such as Chow groups with the help of it. The first step in this direction was made in [S18a] where so-called Chern classes, operations from $K(n)^*$ to $CH^* \otimes Z_{(p)}$ were constructed. These Chern classes are similar to the classical Chern classes from $K_0$ to $CH^*$: they are free generators of all operations and satisfy a Cartan-type formula. This led us to ask the question about the existence of the notion of Morava-orientable theories ([op.cit., Introduction]). The goal of this paper is to at least partially answer this question providing many oriented theories for which Chern classes from $K(n)^*$ can be defined. Among them, perhaps, most important is $K(n)^*$ itself which provides a tool to calculate Chern classes from $K(n)^*$ to $CH^* \otimes Z_{(p)}$ for certain varieties.

We start with the description of the results of the paper with the following definition generalizing Cartier’s $p$-typical formal group laws.

**Definition** (for the full definition encompassing torsion-case see Section 3.1). Let $A$ be a torsion-free $Z_{(p)}$-algebra. A formal group law $F$ over $A$ is called $p^n$-typical if its logarithm is of the form 

$$\log_F(x) = \sum_{i \geq 0} l_i x^{p^i}$$

where $l_i \in A \otimes_Z \mathbb{Q}$.

A free theory is called $p^n$-typical if the corresponding formal group law is $p^n$-typical.

The main result of this paper is the following.

**Theorem** (Theorem 5.14 if $A^* = CH^* \otimes Z_{(p)}$ this is [S18a Th. 4.2.1]). For every Morava K-theory $K(n)^*$ and every $p^n$-typical theory $A^*$ there exist a series of operations $c_j : K(n)^* \to A^*$ for $j \geq 1$ satisfying the following conditions.

1. Operation $c_j$ takes values in $\tau^j A^*$, where $\tau^j$ is the topological filtration on $A^*$.
Denote by $c_{\text{tot}} = \sum_{i \geq 1} c_i t^i$ the total Chern class in a formal variable $t$. Then the Cartan’s formula holds universally:

$$c_{\text{tot}}(x + y) = F_{K(n)}(c_{\text{tot}}(x), c_{\text{tot}}(y)),$$

where $x, y \in K(n)^* (X)$ for a smooth variety $X$ and the identity takes place in $A^*(X)[[t]]$.

ii) If $A = A^*(\text{Spec } k)$ is a free $\mathbb{Z}_p$-module, then all operations from $K(n)^*$ to $A^*$ are uniquely expressible as series in Chern classes:

$$[\tau^1 K(n)^*, A^*] = A[[c_1, \ldots, c_i, \ldots]],$$

where $K(n)^* = \mathbb{Z}_p \oplus \tau^1 K(n)^*$ as presheaves of abelian groups, and $[F, G]$ denotes the set of natural transformations of presheaves of sets $F, G$ on the category of smooth varieties.

Examples of $p^n$-typical theories include Morava K-theories $K(mn)^*$ for $m \geq 1$, Chow groups $\text{CH}^* \otimes \mathbb{Z}_p$ and the universal $p^n$-typical theory $BP[n]^*$ whose ring of coefficients is a polynomial algebra $\mathbb{Z}_p[v_n, v_{2n}, \ldots]$. Following an analogy with K-theory, $p^n$-typical theories might be called Morava-orientable even though we do not give a definition of Morava orientation.

We also provide some evidence for if the right notion of Morava-orientability exists, then the uniqueness of Morava K-theories is not an issue in topology, where there exist a unique typical theory $\mathbb{Z}_p$-module, then all operations from $K(n)^*$ to $A^*$ are uniquely expressible as series in Chern classes:

$$[\tau^1 K(n)^*, A^*] = A[[c_1, \ldots, c_i, \ldots]],$$

where $K(n)^* = \mathbb{Z}_p \oplus \tau^1 K(n)^*$ as presheaves of abelian groups, and $[F, G]$ denotes the set of natural transformations of presheaves of sets $F, G$ on the category of smooth varieties.

Examples of $p^n$-typical theories include Morava K-theories $K(mn)^*$ for $m \geq 1$, Chow groups $\text{CH}^* \otimes \mathbb{Z}_p$ and the universal $p^n$-typical theory $BP[n]^*$ whose ring of coefficients is a polynomial algebra $\mathbb{Z}_p[v_n, v_{2n}, \ldots]$. Following an analogy with K-theory, $p^n$-typical theories might be called Morava-orientable even though we do not give a definition of Morava orientation.

We also provide some evidence for if the right notion of Morava-orientability exists, then the intersection of Morava-orientable theories with oriented theories is exactly the set of $p^n$-typical theories (see Appendix A.1). Nevertheless, questions of the existence of Morava-orientable but non-orientable theories and of finding a definition of Morava-oriented theories remain open.

We should have mentioned that for each prime number $p$ and each number $n \in \mathbb{N}$ the definition of Morava K-theory above yields infinitely many oriented theories $K(n)^*$. It is not true that all of them are multiplicatively isomorphic (Appendix A.2), however we can prove the following.

**Theorem (Theorem 5.3).** Let $K(n)^*, \overline{K(n)^*}$ be two $n$-th Morava K-theories over $\mathbb{Z}_p$.

Then there exist an isomorphism of presheaves of abelian groups $K(n)^* \cong \overline{K(n)^*}$.

The uniqueness of Morava K-theories is not an issue in topology, where there exist a unique spectrum with $\mathbb{F}_p$-coefficients representing the $n$-th topological Morava K-theory. The arguments used there cannot be applied in the algebraic situation, and, moreover, we do not know whether any kind of uniqueness statement for spectra representing Morava K-theories with $\mathbb{Z}_p$-coefficients is known even in topology.

The main application of the existence of Chern classes from $K(n)^*$ in this paper is the construction of the gamma filtration on $K(n)^*$.

**Definition.** Let $c_i : K(n)^* \to K(n)^*$ be Chern classes constructed in the theorem above. Define the gamma filtration on $K(n)^*(X)$ for a smooth variety $X$ by the following formula

$$\gamma^m K(n)^*(X) := \langle c_i(\alpha_1) \cdots c_k(\alpha_k) \rangle \sum_j i_j \geq m, \alpha_j \in K(n)^*(X) \rangle, \quad m \geq 1.$$

The gamma filtration satisfies properties similar to those of the classical gamma filtration on $K_0$ which we summarize below. The main difference is that Chern classes from corresponding graded factors on $K(n)^*$ map surjectively to Chow groups $\text{CH}^* \otimes \mathbb{Z}_p$ of codimension up to $p^n$ (in contrast to up to $p$ in the case of $p$-local K-theory).

**Theorem (Prop. 6.2).** Denote by $c_i^{\text{CH}}$ the Chern classes from $K(n)^*$ to $\text{CH}^* \otimes \mathbb{Z}_p$.

The gamma filtration and the topological filtration on $K(n)^*$ satisfy the following properties:

i) $\gamma^* K(n)^* \subset \tau^1 K(n)^*$, where $\tau^* \text{ is the topological filtration}$;

ii) $\gamma^* \otimes \mathbb{Q} = \tau^* \otimes \mathbb{Q}$;

iii) $c_i^{\text{CH}} |_{\tau^i K(n)^*} = 0$;

iv) the operation $c_i^{\text{CH}}$ is additive when restricted to $\tau^i K(n)^*$, and the following maps are isomorphisms of abelian groups:

$$\text{gr}^i_1 K(n)^* \otimes \mathbb{Q} \to \text{gr}^{i+1}_1 K(n)^* \otimes \mathbb{Q} \xrightarrow{c_i^{\text{CH}}} \text{CH}^i \otimes \mathbb{Q};$$

v) $c_i^{\text{CH}} : \text{gr}^i_1 K(n)^* \to \text{CH}^i \otimes \mathbb{Z}_p$ is an isomorphism for $i: 1 \leq i \leq p^n$;

vi) $c_i^{\text{CH}} : \text{gr}^i_1 K(n)^* \to \text{CH}^i \otimes \mathbb{Z}_p$ is surjective for $i: 1 \leq i \leq p^n$. 
The property \( V_{\mathbb{K}} \) is a new tool for obtaining estimates on the \( p \)-torsion in Chow groups of codimension up to \( p^3 \). Even though the problem of calculating \( gr^i K(n)^* (X) \) is not an easy one, we describe now a situation when it can be solved. If \( X \) is a geometrically cellular variety, such that the map \( K(n)(X) \to K(n)(\overline{X}) \) is an isomorphism (where \( \overline{X} \) is the base change of \( X \) to a base field where it becomes cellular, e.g. to the separable closure of the base field), then the graded pieces of the gamma filtration can be calculated over the algebraic closure and thus depend only on the ‘combinatorial’ cellular structure of the variety \( \overline{X} \). First example of such variety was essentially found by Voevodsky and it is a Pfister quadric of dimension \( 2^{n+2} - 2 \). Other projective homogeneous varieties satisfying this condition were found by Semenov, and in a joint paper \( \text{SoSc} \) we investigate them. In particular, there we obtain new bounds on torsion in \( \text{CH}^{\leq 2n} \) of smooth projective quadrics whose class in the Witt ring lies in the \((n + 2)\)-th power of the fundamental ideal essentially using the tools of this paper.

Having summarized the results, let us briefly describe the methods of the paper. Note that algebraic cobordism admit a geometric description \( \text{L} \text{P} \text{19} \): the generators of the abelian group \( \Omega^k(X), k \in \mathbb{Z} \), for a smooth variety \( X \) are classes of isomorphisms of projective morphisms \( f : Y \to X \) of codimension \( k \) with \( Y \) being a smooth variety, and the relations are generated by an analogue of the classical cobordance relation and, also, by so-called double-point relations. However, it is rather unclear whether one could write explicit geometric relations between generators \( f : Y \to X \) in free theories. Despite this fact there exist tools to calculate operations between free theories in an efficient way. Namely, the problem of constructing operations between free theories was reduced to a purely algebraic question by Vishik in \( V_{\mathbb{I}19} V_{\mathbb{I}14} \). More precisely, Vishik’s result says that an operation \( \phi \) from \( A^* \) to \( B^* \) can be uniquely reconstructed by the data of action of \( \phi \) on products of projective spaces which commutes with pullbacks along several types of morphisms between them. For additive operations this produces a linear system of equations which depend only on formal group laws of theories involved. For general operations the system of equations can be considered as non-linear but still depends only on the formal group laws over \( A \) and \( B \).

The case of classification of operations from \( K(n)^* \) to \( \text{CH}^* \otimes \mathbb{Z} (\mathbb{P}) \) which we treated in \( S_{\mathbb{I}88} \) turns out to be the simplest case, mainly because the topological filtration is split on Chow groups by the graded components. The general case can be reduced to it by the following construction.

First, we prove that an operation \( \phi : A^* \to B^* \) takes values in the \( i \)-th part of the topological filtration \( \tau^i B^* \) if and only if it does so on products of projective spaces (Prop. 2.2). Second, we construct an injective map of \( B \)-modules (Section 2.2):

\[
tr_i : [A^*, \tau^j B^*] / [A^*, \tau^{i+1} B^*] \hookrightarrow [A^*, \text{CH}^i \otimes B], \quad i \geq 0,
\]

where \([A^*, \tau^j B^*]\) has a natural structure of an abelian group since \( \tau^j B^* \) is a presheaf of abelian groups, and we call \( tr_i \) the truncation map. For an operation \( \phi : A^* \to \tau^j B^* \) the operation \( tr_i \phi : A^* \to \text{CH}^i \otimes B \) can be characterized by the equation:

\[
\rho_B \circ tr_i \phi \equiv \phi \mod \tau^{i+1} B^*
\]

where \( \rho_B : \text{CH}^i \otimes B \to gr^i B^* \) is the canonical morphism of theories. In other words, \( tr_i \phi \) is the lift of the operation \( \phi \mod \tau^{i+1} B^* \) to Chow groups along the map \( \rho_B \).

Finally, we show that when \( A^* = K(n)^* \) and \( B^* \) is a \( p^j \)-typical theory, the truncation map is an isomorphism, and we reduce the classification of operations to the already known case where \( B^* \) is Chow groups. We should emphasize that these results are also based on the Vishik’s theorem which, thus, constitutes the main tool of the paper.

Outline. Section 1 is a recap of oriented cohomology theories, Vishik’s results on operations between them and the properties of the topological filtration on free theories.

Section 2 contains the construction of the truncation map. We classify operations between free theories which target a part of the topological filtration \( 2.1 \), provide an algebraic description of the truncation map and discuss its properties \( 2.2 \). We also consider a special refinement of the truncation map which we will need later \( 2.3 \). Finally, we investigate the general properties of modules of operations when all the truncation maps are isomorphisms \( 2.4 \).
Section 3 contains definitions of $p^n$-typical formal group laws (3.1), Morava K-theories (3.2) as well as their properties and the statement of the main theorem on the classification of operations from $K(n)^*$ to $p^n$-typical theories (3.4).

Section 4 contains the proof of the main theorem for which we need also to classify additive operations from Morava K-theories to $p^n$-typical theories (4.1).

In Section 5 we prove the uniqueness of $n$-th Morava K-theory as a presheaf of abelian groups.

Section 6 contains the definition and properties of the gamma filtration on Morava K-theories (6.1), the proof of its uniqueness (6.3) and calculation of some computational constants related to it (6.4).

Appendix A contains results which are complementary to the rest of the paper. In A.1 we show that there could not exist Chern classes from the $n$-th Morava K-theory to the $m$-th Morava K-theory except for the cases treated in the main theorem. Section A.2 contains examples of $n$-th Morava K-theories which are not multiplicatively isomorphic. Finally, Section A.3 gives an inductive formula for the image of Chern classes from $gr^i_K(n)^*$ in $\text{CH}^i \otimes \mathbb{Z}(p)$.

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1. Preliminaries

Fix a field $k$ with char $k = 0$. All varieties over $k$ are assumed to be quasi-projective.

In this section we recall basic definitions of oriented theories and describe the key input results of Vishik on the classification of operations.

1.1. Oriented theories and formal group laws. The following definition is a version of [LM07, Def. 1.1.2] (cf. also [PS00, 2.0.1]) with an additional axiom (LOC).

Definition 1.1 (V19, 2.1]). An oriented cohomology theory (or just an oriented theory) is a presheaf $A^*$ of commutative rings on a category of smooth quasi-projective varieties over $k$ supplied with the data of push-forward maps for projective morphisms. Namely, for each projective morphism of smooth varieties $f : X \to Y$, morphisms of abelian groups $f_* : A^*(X) \to A^*(Y)$ are defined.

The structure of push-forwards has to satisfy the following axioms (for precise statements see ibid): functoriality for compositions (A1), base change for transversal morphisms (A2), the projection formula, the projective bundle theorem (PB), $A^1$-homotopy invariance (EH) and localization axiom (LOC).

Recall that a pair of morphisms between smooth varieties $f : X \to Z$, $g : Y \to Z$ is called transversal if $\text{Tor}_q^O(f_*O_Y, g_*O_Y) = 0$ for all $q > 0$, and $X \times_Z Y$ is a smooth variety ([LM07, Def. 1.1.1]). The axiom (A2) says that if $f$ is a projective morphism, then $g^*f_* = f'^*g'_*$ where base change morphisms are denoted according to the following diagram:

$$
\begin{array}{ccc}
X \times_Z Y & \overset{f}{\longrightarrow} & Y \\
\downarrow g' & & \downarrow g \\
X & \overset{f}{\longrightarrow} & Z
\end{array}
$$

Note that in the definition above we do not assume that $A^*$ is a presheaf of graded rings, and it will often be the case that $A^*$ has either no grading (as in the case of $K_0$) or has $\mathbb{Z}/m$-grading for some $m \in \mathbb{N}$ (as in the case of Morava K-theories $K(n)^*$, $n \in \mathbb{N}$). If $A^*$ is a graded theory, then
the push-forward maps shift the grading by codimension. Even if \( A^\ast \) is non-graded we still keep the \* sign to distinguish the presheaf from its ring of coefficients \( A := \text{A}^\ast(\text{Spec} \, k) \).

Each oriented theory \( A^\ast \) can be endowed with Chern classes of vector bundles \( c_i^A \) using the classical method due to Grothendieck. First Chern class allows to associate the formal group law \( F_A \in A[[x, y]] \) to the theory \( A^\ast \) ([LM07, Lemma 1.1.3]) so that it satisfies the following equation for every pair of line bundles \( L_1, L_2 \) over a smooth variety \( X \):

\[
c_i^A(L_1 \otimes L_2) = F_A(c_i^A(L_1), c_i^A(L_2)).
\]

Recall that the formal group laws (FGLs) over a ring \( A \) are in 1-to-1 correspondence with the ring morphisms from the Lazard ring \( L \) to \( A \). In particular, the construction above yields a morphism of rings \( L \rightarrow A^\ast(\text{Spec} \, k) \) for all oriented theories \( A^\ast \).

**Theorem 1.2** (Levine-Morel, [LM07, 1.2.6]). There exists the universal oriented theory \( \Omega^\ast \) called algebraic cobordism, i.e. for any oriented theory \( A^\ast \) there exists a unique morphism of presheaves of rings \( p_A : \Omega^\ast \rightarrow A^\ast \) which respects the structure of push-forwards.

The associated formal group law of algebraic cobordisms is the universal formal group law, and \( \Omega^\ast(\text{Spec} \, k) \) is canonically isomorphic to the Lazard ring \( L \).

One can construct an oriented theory with a prescribed formal group law in the following way.

**Definition 1.3** (Levine-Morel, [LM07, Rem. 2.4.14]). Let \( R \) be a ring, let \( L \rightarrow R \) be a ring morphism corresponding to a formal group law \( F_R \) over \( R \).

Then \( \Omega^\ast \otimes_L R \) is an oriented theory which is called a **free theory** (the morphism \( L \rightarrow R \) and the identification \( \Omega^\ast(\text{Spec} \, k) \cong L \) are hidden in the notation). The ring of coefficients of \( \Omega^\ast \otimes_L R \) is \( R \), and its associated FGL is \( F_R \).

Chow groups \( \text{CH}^\ast \) and \( K \)-theory are the most well-known examples of free theories (see op.cit.) which also have alternative definitions. However, Morava K-theory which will be defined in Section 3 as a free theory has no other known description.

### 1.2. Operations and poly-operations.

**Definition 1.4.** Let \( A^\ast, B^\ast \) be presheaves of sets on the category of smooth varieties over a field.

An **operation** \( \phi : A^\ast \rightarrow B^\ast \) is a morphism of presheaves of sets. The set of all operations from \( A^\ast \) to \( B^\ast \) is denoted by \([A^\ast, B^\ast]\).

If \( A^\ast, B^\ast \) are presheaves of abelian groups, then an **additive operation** \( \phi : A^\ast \rightarrow B^\ast \) is a morphism of presheaves of abelian groups. The set of all additive operations from \( A^\ast \) to \( B^\ast \) is denoted by \([A^\ast, B^\ast]^{\text{add}}\).

If \( B^\ast \) is a presheaf of rings, the sets \([A^\ast, B^\ast], [A^\ast, B^\ast]^{\text{add}}\) have the natural ring structure given by the multiplication and addition on the target theory.

When working with non-additive operations one naturally starts to study poly-operations as well (e.g. see Section 1.3). There are two types of them: external and internal ones.

**Definition 1.5** ([Vi14, Def. 4.2]). Let \( A^\ast, B^\ast \) be presheaves of sets (or abelian groups, or rings) on the category of smooth varieties over a field \( k \).

An **external \( r \)-ary poly-operation** from \( A^\ast \) to \( B^\ast \) is a morphism of presheaves of sets on the \( r \)-product category of smooth varieties over a field \( k \) from \((A^\ast)^{\times r}\) to \( B^\ast \circ \prod^r \).

Explicitly, for smooth varieties \( X_1, \ldots, X_r \) an external poly-operation yields a map of sets

\[
A^\ast(X_1) \times A^\ast(X_2) \times \ldots \times A^\ast(X_r) \rightarrow B^\ast(X_1 \times X_2 \times \ldots \times X_r)
\]

in a functorial way.

An **internal \( r \)-ary poly-operation** from \( A^\ast \) to \( B^\ast \) is a morphism of presheaves of sets on the category of smooth varieties over a field \( k \) from \((A^\ast)^{\times r}\) to \( B^\ast \).

Explicitly, for a smooth variety \( X \) an internal poly-operation yields a map of sets

\[
A^\ast(X) \times A^\ast(X) \times \ldots \times A^\ast(X) \rightarrow B^\ast(X)
\]

in a functorial way.

It is not hard to see that there is a 1-to-1 correspondence between these two notions ([Vi14, Section 4]). The set of all (internal or external) \( r \)-ary poly-operations is denoted by \([(A^\ast)^{\times r}, B^\ast \circ \prod^r]\).
1.3. Derivatives and products of poly-operations. There are two straight-forward ways to produce some poly-operations from operations, or in other words to increase the arity of operations.

First, if \( \phi_1, \phi_2 \) are external \( r_1 \)-ary and \( r_2 \)-ary poly-operations, respectively, from a presheaf of sets to a presheaf of rings, then we can define an \((r_1 + r_2)\)-ary poly-operation \( \phi_1 \circ \phi_2 \) as their external product:

\[
(\phi_1 \circ \phi_2)(x_1, x_2, \ldots, x_{r_1}, y_1, y_2, \ldots, y_{r_2}) = \pi_1^* \phi_1(x_1, x_2, \ldots, x_{r_1}) \cdot \pi_2^* \phi_2(y_1, y_2, \ldots, y_{r_2}),
\]

where \( x_i \in A^*(X_i) \) for \( i: 1 \leq i \leq r_1 \), \( y_j \in A^*(Y_j) \) for \( j: 1 \leq j \leq r_2 \), and \( \pi_1: \prod_i X_i \times \prod_j Y_j \to \prod_i X_i \), \( \pi_2: \prod_i X_i \times \prod_j Y_j \to \prod_j Y_j \) are projections. Similarly, one can define products of internal poly-operations which we also denote as \( \phi_1 \circ \phi_2 \).

If \( B^* \) is a presheaf of rings, this construction defines a morphism of algebras

\[
[(A^*)^{\times r_1}, B^* \circ \prod_{i=1}^{r_1}] \oplus_B [(A^*)^{\times r_2}, B^* \circ \prod_{i=1}^{r_2}] \to [(A^*)^{\times (r_1 + r_2)}, B^* \circ \prod_{i=1}^{r_1 + r_2}].
\]

If this morphism of algebras is an isomorphism for particular \( A^* \) and \( B^* \), perhaps, one may it as some kind of Künneth-type property. When this property is satisfied for all \( r_1, r_2 \) (e.g. Th. L.12), we will write \([(A^*)^{\times r}, B^* \circ \prod_i^r] = [(A^*)^{\times r}]^\otimes r \).

Second, if \( \phi \) is an external \( r \)-ary poly-operation, then one can define an \((r + 1)\)-ary external poly-operation \( \partial^r \phi \) as its derivative with respect to the \( i \)-th component (Vi18 Def. 3.1, end of Section 4). Denote by \( Z_{<i} = (z_1, \ldots, z_i-1), Z_{>i} = (z_{i+1}, \ldots, z_r) \), then

\[
\partial^r \phi(Z_{<i}, x, y, Z_{>i}) := \phi(Z_{<i}, \pi_1^*(x) + \pi_2^*(y), Z_{>i}) - \phi(Z_{<i}, \pi_1^*(x), Z_{>i}) - \phi(Z_{<i}, \pi_2^*(y), Z_{>i}),
\]

where \( z_j \in A^*(X_j) \) for \( j: 1 \leq j \leq r, j \neq i \), and \( x \in A^*(X), y \in A^*(Y) \), \( \pi_1, \pi_2 \) are projections from \( X \times Y \) to \( X \) and \( Y \) respectively. Similarly, one can define internal derivatives \( \partial_i \phi \) of internal poly-operations.

If \( r = 1 \), i.e. \( \phi \) is an operation, we will omit the subscript and write \( \partial \phi \) and \( \partial^r \phi \) to mean its derivatives. Iterating the procedure one can easily define \( \partial^r \phi = \partial_{r-1} \circ \partial_{r-2} \circ \cdots \circ \partial_1 \). However, it is easy to see that all \( s \)-derivatives of an operation are symmetric and thus derivatives do not depend on the order of derivation. We will write \( \partial^r \phi \) to denote any of them. Similarly, we define \( \partial^s \phi \) to be the \( s \)-th internal derivative of \( \phi \).

By definition of the derivative of \( \phi \) one can express values of \( \phi \) on the sum of two elements as the sum of values of \( \phi \) and \( \partial^r \phi \). It is useful for computations to have analogous formulas for the values on the sum of any number of elements.

Proposition 1.6 (Discrete Taylor Expansion, [Vi18 Prop. 3.2]). Let \( f: A \to B \) be a map between abelian groups. Denote by \( \partial f: A^{\times 1} \to B \) its derivatives.

For any set \( \{a_i\}_{i \in I} \) of elements in \( A \) the following equality holds:

\[
f(\sum_{i \in I} a_i) = \sum_{\emptyset \neq J \subset I} \partial^{\#J-1} f(a_j) | j \in J).
\]

1.4. Vishik’s classification of operations from theories of rational type. Theories of rational type were introduced by Vishik in [Vi19] and are those oriented theory which satisfy an additional axiom (CONST) and whose values on varieties can be reconstructed by induction on the dimension.

Definition 1.7 ([LM07] Def 4.4.1). An oriented theory \( A^* \) satisfies the axiom (CONST) if for every smooth irreducible variety \( X \) the value of the theory in its generic point

\[
A^*(k(X)) := \lim_{U \subset X} A^*(U),
\]

is canonically isomorphic to the ring of coefficients of the theory \( A := A^*(\text{Spec}(k)) \). The limit in the above formula is taken over all non-empty open subsets of \( X \).

In particular, this allows to split \( A^* \) as presheaf of abelian groups into two summands: \( A^* = A^* \oplus A \), where \( A \) is a constant presheaf and \( A^* \) is an ideal subsheaf of elements which are trivial in generic points.
Algebraic cobordism satisfy this axiom ([LM07, Cor. 4.4.3]), and therefore every free theory does as well. As we have already mentioned theories of rational type satisfy other properties except for (CONST), which are, however, very technical. Fortunately, the following theorem allows us to skip the definition of theories of rational type.

**Theorem 1.8** (Vishik, [Vi19 Prop. 4.9]). *Theories of rational type are precisely free theories.*

Throughout the paper we will use the name ‘free theories’ for theories of rational type, which should not confuse the reader as we are not using Vishik’s definition directly.

The main tool that we use in this paper is Vishik’s theorem classifying operations from free theories to oriented theories.

**Theorem 1.9** (Vishik, [Vi19 Th. 5.1], [Vi14 Th. 5.1]). Let $A^*$ be a free theory and let $B^*$ be an oriented theory. Then the set of operations preserving zero $[A^*, B^*]$ from $A^*$ to $B^*$ is in 1-to-1 correspondence with the data of pointed maps of sets (where the values of theories are pointed by zero) $A^*((P^\infty)^{\times l}) \to B^*((P^\infty)^{\times l})$ for $l \geq 0$ (which are restrictions of an operation), which commute with the pull-backs for:

1. The permutation action of symmetric groups $\Sigma_l$;
2. The partial diagonals;
3. The partial Segre embeddings;
4. The partial point embeddings;
5. The partial projections.

See also [Vi14 Th. 5.2] for the classification of external poly-operations.

**Remark 1.10.** If the target theory is graded, then the theorem allows one to compute poly-operations to each of the components of the target.

To see this note that grading on $B^*$ yields (additive) projectors $p_m : B^* \to B^n$ and an operation to a component $B^n$ is just an operation which is zero when composed with $p_m$, $m \neq n$. As follows from the theorem, this property may be checked on products of projective spaces.

**Remark 1.11.** A similar result was known in topology due to Kashiwbara (see [Ka94 Th. 4.2] where it is formulated for spectra representing cohomology theories). The difference with the Vishik’s result is that Kashiwara’s theorem demands certain additional conditions for spectra to be satisfied. It is not always clear in particular cases, e.g. for Morava K-theories, if these conditions are fulfilled.

1.5. **Chern classes as free generators of operations from $K_0$.** The following is an application of Vishik’s classification of operations. It shows that Chern classes freely generate all operations from $K_0$ to an oriented theory $A^*$. The main result of this paper will be concerned with replacing $K_0$ in this statement by the $n$-th Morava K-theory $K_n^*$, restricting the class of oriented theories $A^*$ to a class of so-called $p^n$-typical oriented theories and defining Chern classes to them.

**Theorem 1.12** (Vishik, for the proof see [S18a Th. 2.1]). Let $A^*$ be an oriented theory. Then the ring of $r$-ary poly-operations from a presheaf $\tilde{K}_0$ to $A^*$ is freely generated over $A$ by products of Chern classes.

**Remark 1.13.** Note that there is no issue of convergence of a series of Chern classes for any particular element of $K_0$, since Chern classes $c_i^A$ are nilpotent on each variety. (Indeed, it is enough to show this for the universal oriented theory $\Omega^*$, and in this case the claim follows because $\Omega(X) = 0$ for $i > \dim X$.)

Using notations from section [1.3] we may write $[(\tilde{K}_0)^{\times r}, A^* \circ \prod'] = A[[c_1^A, \ldots, c_i^A, \ldots]]^{\otimes r}$.

---

2For a rigorous explanation of the notion of an infinite-dimensional projective space $P^\infty$ see the beginning of Section [1.6]
1.6. Notation and continuity of operations. Recall that by the projective bundle theorem for every oriented theory $A^*$ its value on the product of projective spaces is a quotient of a polynomial ring

$$A^*([\mathbb{P}^n]) = \mathbb{Z}[A] = \mathbb{Z}[c_1^A,\ldots,c_n^A]/(c_1^A)^n$$

where the pull-back of the canonical line bundle from the $i$-th factor.

One can form an ind-variety $\mathbb{P}^\infty$ as a formal colimit of linear inclusions of projective spaces. The value of cohomology theories on it can be formally defined as a limit of the values on projective spaces, and for an oriented theory $A^*$ we obtain $A^*([\mathbb{P}^\infty]) = \mathbb{Z}[c_1^A,\ldots,c_n^A]$. Throughout the article we will use the notation $z_i^A := c_i^A(\mathbb{O}(1))$, or just $z_i$ if theory $A^*$ is clear from the context. We should point out that working with infinite dimensional projective space is a handy but totally formal convention.

The restriction of every operation to products of projective spaces satisfies the property of continuity which we now explain following [VI, Proposition 5.3].

Let $G : A^* \to B^*$ be an operation from a free theory $A^*$ to an oriented theory $B^*$ which preserves 0. By Theorem [VI, Proposition 5.3] the operation $G$ is determined by maps of pointed sets $G(z) : A[[z_1,\ldots,z_i^A]] \to B[[z_1^B,\ldots,z_i^B]]$ for all $i \geq 0$. As $G(z)$'s have to commute with pull-backs along partial projections the following diagram is commutative for any $l \geq 0$:

$$
\begin{array}{ccc}
A[[z_1^A,\ldots,z_i^A]] & \xrightarrow{G(z)} & B[[z_1^B,\ldots,z_i^B]] \\
\downarrow & & \downarrow \\
A[[z_1^A,\ldots,z_i^A]] & \xrightarrow{G(z)} & B[[z_1^B,\ldots,z_i^B]]
\end{array}
$$

This allows to use only one transform, the inductive limit of maps $G(z)$:

$$G : A[[z_i^A]] \to B[[z_i^B]]$$

where $A[[z_i^A]] := \cup_{i \geq 0} A[[z_1^A,\ldots,z_i^A]]$, and similarly, for $B[[z_i^B]]$. The map $G$ uniquely determines $G(z)$ for any $l$. Since $G$ preserves 0 we have $G(0) = 0$.

Denote by $F^k_A$ an ideal in $A[[z_i^A]]$ of series of degree $\geq k$ ($F^k_B$ is defined analogously).

**Proposition 1.14** (Vishik, [VI, Proposition 5.3]).

Let $P, P' \in A[[z_i^A]]$ be s.t. $P \equiv P' \mod F^k_A$. Then

$$G(P) \equiv G(P) \mod F^k_B.$$

This allows to calculate approximation of $G(P)$ approximating $P$. In particular, the operation $G$ is determined by its restriction to the products of finite-dimensional projective spaces, or equivalently by the maps

$$G_{r,n} : A^*([\mathbb{P}^n]) = A[[z_1^A,\ldots,z_i^A]]/F^{r+1}_A \to B^*([\mathbb{P}^n]) = B[[z_1,\ldots,z_i]]/F^{r+1}_B.$$

In other words, maps $G(z)$ or map $G$ are determined by their restriction to the polynomial rings $A[z_1^A,\ldots,z_i^A] \subset A[[z_1^A,\ldots,z_i^A]]$.

Analogous statements are true for poly-operations as well.

1.7. General Riemann-Roch theorem. Let $A^*$ be a free theory, let $B^*$ be an oriented theory, and let $\phi : A^* \to B^*$ be an operation.

Let $X$ be a smooth variety and let $i : Z \to X$ be its closed smooth subvariety. Denote by $G^*_Z$ the composition

$$A^*(Z) \xrightarrow{\phi} B^*(Z \times (\mathbb{P}^\infty)^c) \xrightarrow{\phi} B^*(Z \times (\mathbb{P}^\infty)^c) \equiv B^*(Z)$$

If $\mu_i \in B^*(Z)$ for $i : 1 \leq i \leq c$ are nilpotent elements, we will denote by $G^*_Z|_{z_i^A = \mu_i}$ the composition of $G^*_Z$ with the $B^*(Z)$-linear map $B^*(Z)\langle z_1^B,\ldots,z_c^B \rangle \to B^*(Z)$ which sends $z_i^B$ to $\mu_i$.

For a formal group law $F_B$ denote by $\omega_B \in B[[t]]dt$ the unique invariant differential s.t. $\omega_B(0) = dt$. 

\[\text{CHERN CLASSES FROM MORAVA K-THEORIES TO } p^*\text{-TYPICAL ORIENTED THEORIES} \]
The following result is a general form of Riemann-Roch-type theorems for non-additive operations.

**Theorem 1.15** (Vishik, [Vil14 Th. 5.19]). Let \( \alpha \in A^*(Z) \), denote by \( \mu_1, \ldots, \mu_c \) the \( B \)-roots of the normal bundle \( N_Z/X \).

Let \( L_i \) be line bundles over \( Z \) for \( 1 \leq i \leq k \), and denote by \( x_i = c^1_i(L_i) \), \( y_i = c^1_i(L_i) \) their first Chern classes.

Then

\[
\phi \left( i_* ( \prod_{i=1}^k x_i ) \right) = i_* \text{Res}_{t=0} \frac{G^{-k}(\alpha)_{z_\leq=t+n \mu_1, 1 \leq i \leq c; z_{j+k}=y_i, c+1 \leq j \leq c+k}}{t \cdot \prod_{i=1}^c (t + \mu_i)} \omega_B.
\]

### 1.8. Topological filtration on oriented theories.

**Definition 1.16.** Let \( A^* \) be a presheaf of abelian groups on the category of smooth varieties over a field \( k \). Define the **topological** (or sometimes called support codimensional) filtration \( \tau^i \) on the values of \( A^* \) on a variety \( X \) by the formula:

\[
\tau^i A^*(X) := \cup \{ \text{codim}_X U \geq i \} \text{Ker} (A^*(X) \to A^*(U)),
\]

where the union goes over all open subvarieties \( U \) in \( X \) with the complement of codimension at least \( i \).

Note that, if \( A^* \) is an oriented theory satisfying the (CONST) axiom (e.g. free theory), then the zero-th graded quotient of the topological filtration is split: \( \tau^0 A^* = A^*, \tau^0 A^*/\tau^1 A^* = A, A^* = A^* \oplus A \) (see Section 1.4).

We summarize well-known properties of the topological filtration in the following proposition.

**Proposition 1.17.** Let \( A^* \) be a free theory.

1. The canonical map \( (\tau^i \Omega^* \otimes_L A) \to \tau^i A^* \) is surjective;
2. \( \tau^i \Omega^* = \sum_{m \leq n-i} \mathbb{L}^m \Omega^{n-m} \);
3. if \( f : X \to Y \) is a closed embedding of codimension \( c \), then \( f_* \tau^i A^*(X) \subset \tau^{i+c} A^*(Y) \);
4. Chern class \( c^1_i \) takes values in \( \tau^i A^* \);
5. the topological filtration on \( A^* \) is multiplicative, i.e. \( (\tau^i A^*) (\tau^j A^*) \subset \tau^{i+j} A^* \);
6. there exist a canonical surjective map of \( A \)-modules \( \rho_A : CH^* \otimes A \to \mathbb{P}^*_A \);
7. if \( X \) is a smooth variety, \( E \) is a vector bundle on \( X \) of rank \( r \), \( n \geq 0, i \geq 0 \), then

\[
\tau^i A^*(\mathbb{P}(E)) = \sum_{n=0}^{r-1} \tau^{i-m} A^*(X) : \xi^m,
\]

where \( \xi = s^* s_1 \mathbb{P}(E) \) is the first Chern class of the canonical line bundle \( \mathcal{O}(1) \) on \( \mathbb{P}(E) \), \( s \) is the zero section of this line bundle.

In particular, we have

\[
\tau^i A^*(X \times (\mathbb{P}^\infty)^{\times n}) = \sum_{m \geq 0} \tau^{i-m} A^*(X) \otimes A[z_1^A, \ldots, z_n^A]^{\deg \geq m},
\]

where \( z_i^A \) is the first Chern class of the line bundle \( \mathcal{O}(1)_i \) (see Section 1.5).

**Proof.**

1. Let \( \alpha \in A^*(X) \) be supported on a closed subvariety \( Z \), \( j : Z \hookrightarrow X \), of codimension \( i \) (i.e. \( \alpha \in \tau^i A^*(X) \)). If \( Z \) is smooth, then \( A^*(Z) = \Omega^*(Z) \otimes_L A \), and there exist \( \beta_k \in \Omega^*(Z), a_k \in A, k \in J \), s.t. \( \sum_{k \in J} (j_* \beta_k) \otimes a_k \) is a lift of \( \alpha \) to \( (\tau^i \Omega^*(X)) \otimes_L A \). However, if \( Z \) is not smooth, we need to consider Borel-Moore oriented cohomology theories to make the argument work, see [LM07] Section 2.1. More precisely, one extends free theories to presheaves of abelian groups on all quasi-projective varieties [LM07] Remark 2.1.4 and proves the (LOC) axiom for them. We leave details to the reader.

2. is [LM07] Th. 4.5.7. Note, however, that they use homological grading for algebraic cobordism.

3. follows from definitions.
satisfy the localization axiom.

\[ 3 \]

In the case \( A^* = \Omega^* \) the map is constructed in [LM07 Cor. 4.5.8]. From (1) it follows, that there is a canonical surjective map \((gr^*_x \Omega^*) \otimes_L A \to gr^*_x A^*\). We define \( \rho_A \) as the composition of this map with \( \rho_\tau \otimes_L A \colon CH^* \otimes A \to (gr^*_x \Omega^*) \otimes_L A \).

\[ 7 \]

Thus, we need to prove that the graded algebraic cobordism \( \langle A \rangle \), \( (A_2) \) and the projection formula for \( \tau_i \) follow from the construction of Chern classes and from the fact that \( c^i(L) \in \tau^i A^* \) for a line bundle \( L \). The latter property can be checked using the (CONST) axiom, and the statement then follows from the explicit description of the topological filtration in (2).

\[ 5 \]

The following should be well known, but we were not able to find a reference.

Lemma 1.18. Let \( B^* \) be a free theory. Then \( gr^*_x B^* := \oplus_{c \geq 0} gr^*_x B := \oplus_{c \geq 0} \tau^c B^*/\tau^{c+1} B^* \) is an oriented theory in the sense of Levine-Morel ([LM07 Def. 1.1.2]), i.e. it satisfies Definition 1.1 except for the (LOC) axiom.

Moreover, the map \( \rho_B : CH^* \otimes B \to gr^*_x B^* \) is a morphism of oriented theories.

Proof. To prove that \( gr^*_x B^* \) is an oriented theory we need to show the following:

1. The topological filtration on \( B^* \) is respected by pullbacks;
2. If \( f : X \to Y \) is a projective morphism of codimension \( c \), then \( f_! \tau^c B^*(X) \subset \tau^{c+1} B^*(Y) \) for all \( c \geq 0 \);
3. \( gr^*_x B^* \) satisfies the projective bundle theorem (PB);
4. If \( p : E \to X \) is a vector bundle on \( X \), then \( p^* \) strictly preserves the topological filtration.

1 and (2) allow to define the structure of push-forward and pullbacks on \( gr^*_x B^* \). Properties (A1), (A2) and the projection formula for \( gr^*_x B^* \) follow directly from the corresponding properties of \( B^* \). (PB) and (EH) follow from (3) and (4) above, respectively.

To show properties (1), (2), (4) it suffices to treat the case of \( B^* = \Omega^* \) due to Prop. 1.17 (1). And (3) follows from Prop. 1.17 (7). Thus, we need to prove that the graded algebraic cobordism is an oriented theory.

To show (1) note that the topological filtration on \( \Omega^* \) is defined by the structure of \( L \)-module on it (Prop. 1.17 (2)), and pullbacks are morphisms of \( L \)-modules.

To show (2) consider a projective morphism \( f : X \to Y \) of codimension \( c \) between smooth varieties. If \( \alpha \in \tau^i \Omega^c(X) \), \( i \geq k \), then without loss of generality we may assume that \( \alpha = \lambda \cdot x \) where \( \lambda \in \mathbb{L}^{i-k} \), \( x \in \Omega^c(X) \), push-forward maps are morphisms of \( L \)-modules (e.g. by the projection formula), and, thus, \( f_! (\alpha) = \lambda \cdot f_! (x) \) where \( f_! (x) \in \Omega^{i+c}(Y) \). Since \( \Omega^{i+c}(Y) = \tau^{i+c} \Omega^{i+c}(Y) \) we obtain the claim.

(4) follows since \( p^* : \Omega^* (X) \to \Omega^* (E) \) is an isomorphism of \( L \)-modules, and the topological filtration is defined by the structure of \( L \)-module.

We need also to show that the map \( \rho_B \) commutes with pullbacks and push-forwards. Again it is enough to treat the case \( B^* = \Omega^* \). However, the map \( (\rho_\tau)^{deg} : CH^* \to \tau^i \Omega^*/\tau^{i+1} \Omega^* = \Omega^*/\tau^1 \Omega^* \) is the inverse of the isomorphism of theories \( \Omega^* \otimes \mathbb{L} \cong CH^* \), and commutes with push-forwards and pullbacks. The whole map \( \rho_B \) is just the \( L \)-linearisation of this map, and since pullbacks and push-forwards are morphisms of \( L \)-modules we obtain the claim. Similarly, it follows that \( \rho \) is multiplicative. \( \Box \)

Remark 1.19. Even though \( gr^*_x \Omega^* \) is an oriented theory in the sense of Levine-Morel, it does not satisfy the localization axiom\(^3\) and therefore is not an oriented theory (Def. 1.1). In particular, Vishik’s results on classification of operations (Th. 1.9) can not be applied to operations with the target theory \( gr^*_x B^* \) where \( B^* \) is a free theory.

\(^3\)Vishik kindly provided a counter-example to this statement via personal communication.
1.9. The gamma-filtration on $K_0$. Chern classes $c_i^{K_0}$ from $K_0$ to $K_0$ are closely related to more persistent in the literature $\lambda$-operations (Gr27, Exp. 0). In particular, Theorem 1.12 actually can be restated as saying that all endo-operations of $K_0$ are freely generated by $\lambda$-operations. Nevertheless we prefer to work with Chern classes. The gamma filtration is defined by the following formula:

$$(1) \quad \gamma^i K_0(X) := \langle c_{i_1}^{K_0}(a_1) \cdots c_{i_k}^{K_0}(a_k) \rangle \int_{j} \geq m, a_j \in K_0(X) > .$$

We summarize well-known properties of the gamma filtration below, so that thereafter we could compare them with analogous statement about the gamma filtration on Morava K-theories. We do not provide proofs, as we will not use these results, however, proofs can be obtained in the same manner as it will be done for Chern classes from $K(n)^*$ in Section 6.

The gamma filtration has a description which in some sense does not use the existence or particular properties of Chern classes, namely, that in some sense the gamma filtration is the best 'operational' approximation of the topological filtration.

**Proposition 1.20.** For any $m \geq 0$

$$\gamma^m K_0(X) := \langle \phi(a_1, \ldots, a_k) \rangle \int (K_0)^{\times k}, m \gamma K_0 \circ \prod_i, a_i \in K_0(X) > ,$$

i.e. the $m$-th part of the gamma filtration is generated by the image of all internal poly-operations targeting in the $m$-th part of the topological filtration on $K_0$.

**Proposition 1.21.**

1. $\gamma^i K_0 \subset \tau^i K_0$ for $i \geq 0$;
2. $\gamma^1 = \tau^1, \gamma^2 = \tau^2$;
3. $\gamma^i \otimes Q = \tau^i \otimes Q$ as filtrations on $K_0 \otimes Q$;
4. The $i$-th Chern class $c_i^{CH} : K_0 \to CH^i$ induces additive morphisms
   $$c_i^{CH} : gr^i K_0 \to CH^i, \quad c_i^{CH} : gr^i K_0 \to CH^i;$$
5. $c_i^{CH} \otimes id_Q$ yields an isomorphism between $gr^i K_0 \otimes Q$ (or $gr^i K_0 \otimes Q$) and $CH^i \otimes Q$;
6. $c_i^{CH} : gr^i K_0 \to CH^i$ is an isomorphism, and $c_i^{CH} : gr^i K_0 \to CH^i$ is surjective.
7. $c_i^{CH} : gr^i K_0 \otimes Z(p) \to CH^i \otimes Z(p)$ is surjective for $i \leq p$.

2. Truncation of operations and the topological filtration

The Chern class $c_i^B$ considered as an operation from $K_0$ to some oriented theory $B^*$ is an example of such a construction which takes its values in $\tau^i B^*$, the $i$-th part of the topological filtration (Prop. 1.17, [7]). There is also a surjective additive map $\rho_B : CH^i \otimes B \to \tau^i B^*/\tau^{i+1} B^*$ (Prop. 1.17, [7]), and one can easily check that $\rho_B \circ c_i^{CH} = c_i^B$ as operations to $\tau^i B^*/\tau^{i+1} B^*$. Thus, $c_i^{CH}$ is a lift of $c_i^B$ mod $\tau^{i+1}$ along the map $\rho_B$.

The goal of this section is to provide a construction of such a lift for all operations. Namely, for an operation $\phi : A^* \to \tau^i B^*$ from a free theory to an oriented theory we construct its truncation

$$tr_i \phi : A^* \to CH^i \otimes B \text{ s.t. } \rho_B \circ (tr_i \phi) = \phi$$

as operations to $\tau^i B^*/\tau^{i+1} B^*$. Moreover, the truncation map gives an inclusion of $B$-modules $[A^*, \tau^i B^*]/[A^*, \tau^{i+1} B^*] \subset [A^*, CH^i \otimes B]$, and, thus, the problem of calculating all operations from $A^*$ to $B^*$ may be solved in two steps: 1. calculating all operations from $A^*$ to $CH^i \otimes B$, 2. calculating images of truncation maps.

In Section 2.1 we characterize operations from $A^*$ to $B^*$ which take values in $\tau^i B^*$ in terms of their action on products of projective spaces. In Section 2.2 we explain the construction of the truncation map and the action of the truncation of an operation on products of projective spaces. In short, series $G_{[i]} \subset B[[z_1, \ldots, z_i]]$ which determine the operation (see Section 1.6) are truncated to polynomials of degrees $i$ with respect to variables $z_j$ by forgetting the part of these series of higher degree. Section 2.3 contains a variation of the truncation construction for operations between $p$-local theories. All this is based on the Vishik’s Theorem 1.9. In Section 2.4 we show that if truncation maps are isomorphisms, and the $B$-module $[A^*, CH^i \otimes B]$ is free, then lifts of (either $B$-module or $B$-algebra) generators of the latter are generators of $[A^*, B^*]$.
The results about truncations are one of the main technical tools for the rest of the paper. In Section [4] we will apply them to the case when \( A^* \) is the \( n \)-th Morava K-theory \( K(n)^* \), and \( B^* \) is a \( p^a \)-typical oriented theory. Note that the problem of classifying all operations from \( K(n)^* \) to \( \text{CH}^* \otimes \mathbb{Z}_{(p)} \) was already solved in [S18a].

2.1. Operations which target the \( j \)-th part of the topological filtration.

**Proposition 2.1.** Let \( A^* \), \( B^* \) be oriented theories, and assume that \( A^* \) satisfies the (CONST) axiom (see Def. [7.7]).

Then there are natural isomorphisms between sets of operations:

\[
[A^*, B^*] = \prod_A [\tilde{A}^*, B^*], \quad [A^*, B^*]^{add} = \text{Hom}(A, B) \oplus [\tilde{A}^*, B^*].
\]

**Proof.** As \( A^* = A \oplus \tilde{A}^* \) by the (CONST) property, there is a natural map \( \prod_A [\tilde{A}^*, B^*] \rightarrow [A^*, B^*] \) which sends a tuple of operations \( (\phi_a)_{a \in A} \) to an operation \( (a, x) \rightarrow \phi_a(x) \). The inverse map is just the restriction of an operation to subspaces \( a \oplus \tilde{A}^* \) for all \( a \in A \).

The case of additive operations can be treated similarly. \( \square \)

For an oriented theory \( A^* \) satisfying (CONST) axiom the first piece of the topological filtration \( \tau^1 A^* \) equals to \( \tilde{A}^* \). Thus, the proposition above simplifies a description of operations from \( A^* \). Note that since any operation which preserves zero respects topological filtration, we see that classifying operations from \( A^* \) to \( \tau^1 B^*, j \geq 1 \), is equivalent to classifying operations from \( \tau^1 A^* = \tilde{A}^* \) to \( \tau^j B^* \).

Thus, dealing with \( \tilde{A}^* \) instead of \( A^* \) in the following proposition does not reduce the generality of the statement.

**Proposition 2.2.** Let \( A^* \) be a free theory, \( B^* \) be an oriented theory and let \( \phi : \tilde{A}^* \rightarrow B^* \) be an operation. Then \( \phi \) takes values in \( \tau^i B^* \) for some \( i \geq 0 \) if and only if its restriction to products of projective spaces does.

**Proof.** The only if part is straightforward. For the converse let \( i > 0 \), and assume that \( \phi \) takes values in \( \tau^i B^* \) on products of projective spaces. Note that in particular this means that \( \phi \) is zero over a part, i.e. \( \phi \) preserves zero.

There exists a free theory \( \tilde{\Omega}^* \otimes_{L,F_n} B \) and the canonical morphism of theories \( \tilde{\Omega}^* \otimes_{L,F_n} B \rightarrow B^* \) which preserves the topological filtration. Since an operation from \( A^* \) to \( B^* \) lifts to the free theory \( \tilde{\Omega}^* \otimes_{L,F_n} B \) by Vishik’s Theorem [1.3] we may assume without loss of generality that \( B^* \) is also a free theory.

We follow Vishik’s proof of the Th. [1.10] ([Y14 5.4]) in which the operation is reconstructed on arbitrary smooth variety by induction on its dimension. The key ingredient in this argument is the Riemann-Roch theorem (Th. [1.15]) for closed smooth subvarieties.

The idea of the construction of an operation by induction is the following: if \( X \) is a variety of dimension \( d + 1 \), and \( \alpha \in \tilde{A}^*(X) \) is supported on a smooth divisor \( i : D \hookrightarrow X \), then \( \phi(i_*(\alpha)z_1 \ldots z_n) \in B^*(X \times (\mathbb{P}^\infty)^n) \) can be calculated by the Riemann-Roch formula using the value of \( \phi \) on a product of projective spaces with \( D \), while the latter is a variety of dimension \( d \).

There are, however, two technical issues which make the following proof a little bit cumbersome.

First, the divisor \( D \) does not have to be smooth, and if not the Riemann-Roch formula cannot be applied. Second, the fact that \( \phi \) sends additive generators of \( A^*(X \times (\mathbb{P}^\infty)^n) \) to the \( j \)-th part of the topological filtration does not directly imply that \( \phi \) sends every element to this part of the filtration as the operation does not have to be additive. To deal with the first problem one uses resolutions of singularities, and derivatives of the operation help to deal with the second.

**Induction assumption.** For an integer \( d \geq 0 \), every smooth variety \( X \) of dimension not greater than \( d \) and every number \( n \geq 0 \) the map

\[
\phi : A^*(X \times (\mathbb{P}^\infty)^n) \rightarrow B^*(X \times (\mathbb{P}^\infty)^n)
\]

takes values in \( \tau^i B^*(X \times (\mathbb{P}^\infty)^n) \).

**Base of induction** \((d = 0)\) is the assertion of the Proposition for the base field, and the case of finite extension of the base field clearly reduces to the assertion with the help of the (CONST) and the (PB) properties.
Induction step \((d \to d + 1)\). Let \(X\) be a smooth variety of dimension \(d + 1\).

Every element of \(\mathcal{A}'(X \times (\mathbb{P}^\infty)^{\times n})\) can be represented as a sum of two elements \(\alpha, \beta\) where \(\alpha\) lies in the subgroup \(\mathcal{A}'(X)[[z_1, \ldots, z_n]]\) and \(\beta\) lies in \(\mathcal{A}'[z_1, \ldots, z_n]^{\text{deg} \geq 1}\). Here \(\mathcal{A}'[[z_1, \ldots, z_n]]\) is considered as a subgroup of \(\mathcal{A}'(X \times (\mathbb{P}^\infty)^{\times n})\) via the map \(p^*_X,\) where \(p_X : X \times (\mathbb{P}^\infty)^{\times n} \to (\mathbb{P}^\infty)^{\times n}\) is the projection morphism.

By the definition of a derivative we have \(\phi(a + \alpha) = \phi(a) + \phi(\alpha) + \partial^1 \phi(\alpha, \alpha)\), and we are going to show that each of the latter three summands lies in \(\tau^1B^*(X \times (\mathbb{P}^\infty)^{\times n})\).

1. The value of \(\phi)\) on \(a \in \mathcal{A}'[[z_1, \ldots, z_n]]\) is equal to \(p^*_X \pi(\phi \sigma (a))\). By the base induction assumption \(\phi(a) \in \mathcal{A}'[z_1, \ldots, z_n]^{\text{deg} \geq 1}\) and therefore \(p^*_X \pi(\phi(a)) \in \mathcal{A}'(X \times (\mathbb{P}^\infty)^{\times n})\).

2. For \(\alpha \in \mathcal{A}'(X)[[z_1, \ldots, z_n]]\), there exist a divisor \(D\) in \(X\), s.t. \(\alpha\) restricts to \(0\) over \(X \setminus D \times (\mathbb{P}^\infty)^{\times n}\). By the result of Hironaka [Hir] there exist a resolution of singularities of \(D\) inside \(X\), i.e. a birational morphism \(p : X \to X\), which is an isomorphism outside of \(D\) and the preimage of \(D\) is a divisor \(E\) with strict normal crossings.

Let us show that \(\phi(p^*(\alpha))\) lies in the \(i\)-th part of the topological filtration of \(B^*(\tilde{X} \times (\mathbb{P}^\infty)^{\times n})\).

Indeed, \(p^*(\alpha)\) restricts to zero in the complement to \(\tau^{-1}(D) \times (\mathbb{P}^\infty)^{\times n}\), and therefore there exists an element \(\beta \in \mathcal{A}'(E \times (\mathbb{P}^\infty)^{\times n})\) s.t. \(p^*(\alpha) = f_* \beta\) where \(f : E \times (\mathbb{P}^\infty)^{\times n} \to \tilde{X} \times (\mathbb{P}^\infty)^{\times n}\) and \(E \subset X\) is a divisor with strict normal crossings. Denote by \(J\) the set of irreducible components of \(E\), i.e. for \(r \in J\) the divisor \(E_r \subset E\) is smooth, and for \(S \subset J\) denote by \(E_S\) the intersection of components \(E_r\), where \(s \in S\). Note that by the definition of a divisor with simple normal crossing \(d_S : E_S \hookrightarrow X\) is a closed smooth subvariety of codimension \(|S|\).

There is a well-defined push-forward map from the values on irreducible components of \(E\) to the value on \(E\): \(\oplus_r \mathcal{A}'(E_r \times (\mathbb{P}^\infty)^{\times n})) \to \mathcal{A}'(E \times (\mathbb{P}^\infty)^{\times n}))\) which is a surjection (see e.g. [VII] Section 2.2), and, thus, we may assume that \(\partial^* \alpha = \sum r \in J (d_r)_\beta r\) where \(\beta_r \in \mathcal{A}'(E_r \times (\mathbb{P}^\infty)^{\times n}))\).

Lemma 2.3 ([VII Formula (10)]).

\[
\phi(p^*\alpha) = \sum_{S \subset J} (d_S)_r \text{Res}_{t=0} \frac{\partial^{|S|-1} \phi(z_1^{r_1} \cdots z_n^{r_n} \beta_r, r \in S)}{t \cdot \prod_{r \in S} (t + p^*_r \lambda_r)} \omega_r^B,
\]

where \(\lambda_r = \alpha^B((O(E_r)) \in B^*(E_r)\) and \(j_r : E_S \hookrightarrow E_r\) is a closed embedding.

We provide the proof of this Lemma for the sake of completeness. Note that the statement differs from [VII] only in notation and that there it is used to construct an operation, while we are dealing with an already existing operation \(\phi\).

Proof of the Lemma. Using Discrete Taylor Expansion (Prop. 1.19) we have

\[
\phi(p^*\alpha) = \sum_{S \subset J} \partial^{|J|-1} \phi((d_r)_\beta r, r \in S).
\]

The internal derivative \(\partial^{|J|-1} \phi\) on \(X\) is equal to \(\Delta_X \partial^{|J|-1} \phi\) on \(X \times |J|\) where \(\Delta_X : X \hookrightarrow X \times |J|\) is the diagonal. On the other hand by the Riemann-Roch formula for external poly-operations applied to each of the ‘variables’ separately we obtain

\[
\partial^{|J|-1} \phi((d_r)_\beta r, r \in S) = (\times d_r, r \in S), \text{Res}_{t=0} \frac{\partial^{|J|-1} \phi(\beta_r A_r, r \in S)}{t \cdot \prod_{r \in S} (t + B \lambda_r)} \omega_r^B \in B^*(X \times |J|^{-1}).
\]

In order to get rid of external derivatives note that the following square is a transversal cartesian square (see Section 1.11 for the definition):

\[
\begin{array}{ccc}
E_S & \xrightarrow{d_S} & X \\
\bigcup_{r \in S} E_r \times d_r, r \in S & \xrightarrow{\Delta_X} & X \times |J|-1
\end{array}
\]

and, thus, by axiom (A2) for the theory \(B^*\) we have \(\Delta_X^* (x, r \in S) = (d_S)_r (\Delta E_S)^*\). Applying this equality to the expression of \(\phi(p^*\alpha)\) in terms of external derivatives on products of divisors
we obtain that
\[ \phi(p^*\alpha) = \sum_{S,C,J} (d_S)_j^* j_S^* \text{Res}_{t=0} \frac{\partial|S|-1|1| \phi(z^A_r, r \in S)|_{z^A_t = t + n\lambda} - \omega^B_t}{t \cdot \prod_{r \in S}(t + B \lambda_r)}. \]

Poly-operation commute with pullbacks, thus factoring \( j_S^* \Delta_{E_S}^* \phi \times j_J^* \) we see that the following holds: \( j_S^* \Delta_{E_S}^* \phi(z^A_r, r \in S) = \Delta_{E_J}^* \Delta_{E_S}^* \phi(z^A_r, r \in S) \), and the latter is actually an internal derivative \( \phi(z^A_r, r \in S) \). This finishes the proof of the Lemma. \( \square \)

To conclude that \( \phi(p^*\alpha) \) lies in the \( i \)-th part of the topological filtration we need to deal with fractions in the RHS of the equation (2). This is done in the following Lemma using the induction assumption.

**Lemma 2.4.** Let \( \phi \) be an operation from \( A^* \) to \( B^* \), let \( i \geq 1 \), \( d \geq 0 \), and assume that for each variety of dimension less than \( d + 1 \) we have \( \phi(A^* \times \langle (\mathbb{P}^\infty)^x \rangle) \subset \tau^i B^* \times \langle (\mathbb{P}^\infty)^x \rangle \) for each \( n \geq 0 \).

Let \( r \geq 1 \) and let \( D \) be a smooth variety of dimension less than \( d + 1 \), let \( \beta_i \in A^*(D), \lambda_i \in B^*(D) \) for \( i: 1 \leq i \leq r \). Then
\[ \text{Res}_{t=0} \frac{\partial|S|-1|1| \phi(z^A_r, r \in S)|_{z^A_t = t + n\lambda} - \omega^B_t}{t \cdot \prod_{r \in S}(t + B \lambda_r)} \in \tau^{i-|S|} B^*(D). \]

**Proof of the Lemma.** We need to show that \( \phi(z^A_r, r \in S) \) is divisible by \( z^B_r \). If we show that \( \phi(z^A_r, r \in S) \) is divisible by \( z^B_r \), then the expression in question will be equal to
\[ \text{Res}_{t=0} \frac{\partial|S|-1|1| \phi(z^A_r, r \in S)|_{z^A_t = t + n\lambda} - \omega^B_t}{t \cdot \prod_{r \in S}(t + B \lambda_r)} \in \tau^{i-|S|} B^*(D). \]

We know that \( \phi(z^A_r, r \in S) \) is divisible by \( z^B_r \) which is an instance of continuity of operations (see Section 1.6). On the other hand by definition the derivative \( \partial|S|-1|1| \phi(z^A_r, r \in S) \) is equal to \( \sum_{l \in S} \phi(z^A_r, r \in S) \) and the only summand of it which can be divisible by a product of variables \( z^B_r \) is \( \phi(z^A_r, r \in S) \), because others do not contain all variables \( z^B_r \) together and therefore are cancelled.

By the inductive assumptions we have \( \phi(z^A_r, r \in S) \in \tau^{i-|S|} B^*(D) \) and from Prop. 1.14 it follows that those summands of \( \phi(z^A_r, r \in S) \) which are divisible by \( z^B_r \) have coefficients in \( \tau^{i-|S|} B^*(D) \). \( \square \)

Thus, we have proved \( p^*\phi(\alpha) \in \tau^i B^*(\tilde{X}) \). If we show that \( p^* \) strictly respects the topological filtration, then \( \phi(\alpha) \in \tau^i B^*(\tilde{X}) \).

**Lemma 2.5.** If \( p: \tilde{X} \rightarrow X \) is a birational morphism, \( B^* \) is a free theory, then \( p^* : B^*(X) \rightarrow B^*(\tilde{X}) \) strictly preserves the topological filtration.

**Proof of the Lemma.** Note that \( p_1\tilde{X} \) is an invertible element in \( B^*(X) \), since by the generalized degree formula \( p_1\tilde{X} = 1 \) lies in \( \tau^1 B^*(X) \) and hence nilpotent ([LAM74, Cor. 4.4.8 (2)]). By the projection formula we have \( p_* p^* \phi(\alpha) = \phi(\alpha) \cdot p_1\tilde{X} \). By the multiplicativity of the topological filtration (Prop. 1.14) and invertibility of \( p_1\tilde{X} \) we have that \( \phi(\alpha) \) lies in the same part of the topological filtration as the element \( p_* p^* \phi(\alpha) \).

Let \( f: X \rightarrow Y \) be a projective morphism of smooth varieties of the same dimension and let \( B^* \) be a free theory. Then \( f_* \tau^j B^*(X) \subset \tau^j B^*(Y) \) for every \( j \geq 0 \) by Prop. 1.14.

If \( \alpha \in \tau^1 B^*(X) \), then by the generalized degree formula ([LAM74, Th. 4.4.7]) \( \alpha \) can be represented as \( \sum_k b_k [Z_k \rightarrow X] \) where \( b_k \in B \), images of \( Z_k \) in \( X \) are closed subvarieties of codimension no less than \( j \), and \( Z_k \) is smooth and projective over \( Z_k \). The direct image of \( \alpha \) is the element \( \sum_k b_k [Z_k \rightarrow Y] \), and the codimension of images of \( Z_k \) is no less than that in \( X \), i.e. no less than \( j \). The claim now follows. \( \square \)

3. For \( \alpha \in A[[z_1, \ldots, z_n]], \alpha \in \tilde{A}^*(X)[[z_1, \ldots, z_n]] \) we need to show that \( \partial \phi(\alpha, \alpha) \) lies in \( \tau^1 B^*(X \times \langle (\mathbb{P}^\infty)^x \rangle \). The proof is very similar to that of 2, so we only provide a sketch.
Denote by $\Delta : X \times (\mathbb{P}^\infty)^\times n \to X \times (\mathbb{P}^\infty)^\times 2n$ the morphism $\text{id}_X \times \Delta_{(\mathbb{P}^\infty)^\times n}$ where $\Delta_{(\mathbb{P}^\infty)^\times n}$ is the diagonal morphism. Then $\partial \phi(a, \alpha) = \Delta^* \phi(a, \alpha)$, and if $\tilde{\phi}(a, \alpha) \in \tau^c B^*(X \times (\mathbb{P}^\infty)^\times 2n)$ the claim follows.

To study $\partial \phi(a, \alpha)$ we can apply the Riemann-Roch formula, and acting as in 2 above we see that everything reduces to the study of $\partial^S_I \phi(a, z^B_r \beta_r, r \in S)$ for some $\beta_r \in A^*(D)$, $r \in S$ and smooth variety $D$ of dimension less than $d + 1$. Similar to Lemma 2.3 one then shows that $\partial^S_I \phi(a, z^B_r \beta_r, r \in S)$ equals to $F(a, \beta_r)z^B_1 \cdots z^B_{|S|}$ where $F(a, \beta_r)$ lies in $\tau^{|S|} B^*((\mathbb{P}^\infty)^\times n) \times D \times (\mathbb{P}^\infty)^\times |S|$). (Indeed, one has to consider only $\phi(a + \sum \beta_r z^B_r)$ and the coefficient of $z^B_1 \cdots z^B_{|S|}$ lies in $\tau^{|S|}$ by induction assumptions.)

This finishes the inductive step and the proof of the Proposition.

\[\square\]

**Remark 2.6.** A similar statement with a similar proof holds for poly-operations.

\[\angle\]

**2.2. Construction of the truncation of an operation.** Let $\mathbb{P}_n := (\mathbb{P}^\infty)^\times n$ be a product of $n$ copies of an infinite-dimensional projective space. Let $B^*$ be an oriented theory. Denote by $\pi_n$ the morphism of $B$-algebras $B^*(\mathbb{P}_n) \to (CH^* \otimes B)(\mathbb{P}_n)$ which sends $z^B_i$ to $z^\text{CH}_i$ (see Section 1.3 for the notation). Denote by $\pi_n^c$ the composition of $\pi_n$ with the projection to $(CH^c \otimes B)(\mathbb{P}_n) = B[z^\text{CH}_1, \ldots, z^\text{CH}_n]_{\text{deg}=c}$.

**Proposition 2.7.** Let $A^*$ be a free theory and let $B^*$ be an oriented theory. Let $\phi : \tilde{A}^* \to \tau^c B^*$ be an operation for some $c \geq 1$.

Then there exist an operation $\text{tr}_c \phi : \tilde{A}^* \to CH^c \otimes B$, s.t. for any $n \geq 0$ and any $\alpha \in \tilde{A}^*(\mathbb{P}_n)$

\[\text{tr}_c \phi(\alpha) = \pi_n^c(\phi(\alpha)).\]

This defines a map between groups of operations $\text{tr}_c : [\tilde{A}^*, \tau^c B^*] \to [\tilde{A}^*, CH^c \otimes B]$ and induces an inclusion

\[gr^c_\alpha [\tilde{A}^*, B^*] = [\tilde{A}^*, \tau^c B^*]/[\tilde{A}^*, \tau^{c+1} B^*] \xrightarrow{\text{tr}_c} [\tilde{A}^*, CH^c \otimes B].\]

**Proof.** Due to the Vishik’s classification of operations (Th. 1.3.19) the formula (3) defines an operation from $\tilde{A}^*$ to $CH^c \otimes B$ if these maps respect pull-backs along several types of morphisms between $\mathbb{P}_n$‘s. This follows from the next Lemma.

**Lemma 2.8.** Let $f : \mathbb{P}_n \to \mathbb{P}_m$ be one of the morphisms between products of projective spaces appearing in the list of Th. 1.5. Then for any $c \geq 1$ the following diagram is commutative:

\[
\begin{array}{ccc}
\tau^c B^*(\mathbb{P}_m) & \xrightarrow{\pi_n} & (CH^c \otimes B)(\mathbb{P}_m) \\
\downarrow f^* \circ \pi_n & & \downarrow f^* \circ \pi_n \\
\tau^c B^*(\mathbb{P}_n) & \xrightarrow{\pi_n^c} & (CH^c \otimes B)(\mathbb{P}_n).
\end{array}
\]

**Proof.** We consider only the case when $f : \mathbb{P}_n \to \mathbb{P}_{n+1}$ is a partial Segre embedding acting on the last two components, other cases can be treated similarly.

The pull-backs along Segre maps: $f_B^*$ sends $z^B_{n+1}$ to $F_B(z^B_n, z^B_{n+1}); f^*_{CH^c \otimes B}$ sends $z_n$ to $z_n + z_{n+1}$.

We can check the commutativity of the diagram on a monomial $b(z^B_1)^r_1 \cdots (z^B_n)^r_n$ where $b \in B, r_j \geq 0, \sum r_j = c$:

\[
\begin{array}{ccc}
\downarrow \ & & \downarrow \\
b(z^B_1)^{r_1} \cdots (z^B_n)^{r_n} & \to & b(z^\text{CH}_1)^{r_1} \cdots (z^\text{CH}_n)^{r_n} \\
\downarrow \ & & \downarrow \\
b(z^B_1)^{r_1} \cdots (F_B(z^B_n, z^B_{n+1}))^{r_n} & \to & b(z^\text{CH}_1)^{r_1} \cdots ((z^\text{CH} + z^\text{CH}_{n+1})^{r_n}
\end{array}
\]

As $F_B(z^B_n, z^B_{n+1}) \equiv z^B_n + z^B_{n+1} \mod (z^B_n, z^B_{n+1})$ the claim is checked by a straight-forward computation.

\[\square\]

The topological filtration on the oriented theory $B^*$ induces a decreasing filtration on the set of all operations from the theory $A^*$ to $B^*$. We denote the graded factors of it by $gr^c_\alpha [A^*, B^*]$. The truncation map $tr_c$ sends an operation to zero whenever the values of operation on the products
of projective spaces lie in $\tau^{c+1}B^*$, which is the same as the operation taking values in $\tau^{c+1}B^*$ for all varieties by Prop. 2.2. It proves the last claim of the Proposition.

\textbf{Example 2.9.} It is easy to see that $c_i^A$ takes values in $\tau^iA^*$, and $tr_i c_i^A = c_i \otimes id : K_0 \rightarrow CH^i \otimes A$.

One can check this using the construction of Chern classes in an arbitrary oriented theory, or by calculating the action of Chern classes on products of projective spaces. Namely, it is enough to show that series $c_i^A(z_1 \cdots z_j)$ have degree at least $i$ for any $j \leq i$, and that its $i$-th degree summand does not depend on $A^*$ (i.e. on the formal group law of $A^*$). We leave this to the reader.

\textbf{Proposition 2.10.} Let $\phi : \hat{A}^* \rightarrow \tau^cB^*$ be an operation between free theories.

Then the following diagram of presheaves is commutative:

$$
\begin{array}{ccc}
\hat{A}^* & \xrightarrow{\phi} & \tau^cB^* \\
tr_c, \phi & \downarrow & \downarrow \\
CH^c \otimes B & \xrightarrow{\partial^c} & \tau^cB^*/\tau^{c+1}B^*
\end{array}
$$

where the map $\rho_B$ is defined in Prop. 1.17 (6).

\textbf{Proof.} The proof goes by induction procedure which is analogous to the one in the proof of Prop. 2.2.

\textbf{Induction assumption (for $d \geq 0$).} The diagram (4) commutes for values on $X \times (\mathbb{P}^\infty)^n$ for every smooth variety $X$ of dimension not greater than $d$, and for every $n \geq 0$.

Base of induction follows by the construction of the truncation of an operation and the property \textit{(LOC)}

\textbf{Induction step.} Let $a \in \hat{A}^*(X)$. Then using Hironaka’s resolution of singularities there exist a smooth variety $\hat{X}$ and a birational morphism $p : \hat{X} \rightarrow X$ s.t. $p^*(a)$ is supported on a divisor $E$ with smooth normal crossings. Since $p^*$ is injective for every free theory, and by Lemma 2.3 it strictly preserves the topological filtration (so that the $p^*$ is also injective for $\tau^cB^*/\tau^{c+1}B^*$) it suffices to prove the commutativity of (4) for values on $\hat{X}$.

Denote by $J$ the set of irreducible components of $E$, i.e. for $r \in J$ the divisor $E_r \subset E$ is smooth, and for $S \subset J$ denote by $E_S$ the intersection of components $E_s$ where $s \in S$. Denote by $d_S : E_S \hookrightarrow X$ the inclusion of a closed smooth subvariety of codimension $|S|$. Then $p^*(a) = \sum_r (d_r)_*(\beta_r)$, and it is enough to check the commutativity of (4) for values on $p^*(a)$. For a subset $S \subset J$ denote by $\Theta(\; ; p^*(a); S)$ the element $\text{Res}_{t=0} \frac{\partial^{\vert S \vert-1}(z^{A^*}_t j^*_r \beta_r, r \in S)}{\prod_{r \in S} (t \beta_r \lambda_r)} \omega|E_r^B|$, and similarly $\Theta(tr_c; \; ; p^*(a); S)$.

Then by Lemma 2.3 we have

$$
\phi(p^*a) = \sum_{S \subset J} (d_S)_B^B \Theta(\; ; p^*(a); S), \quad (tr_c \phi)(p^*a) = \sum_{S \subset J} (d_S)_B^C \Theta(tr_c; \; ; p^*(a); S).
$$

Let us show that $\Theta(\; ; p^*(a); S) \mod \tau^{c+1-|S|} \equiv \rho_B(\Theta(tr_c; \; ; p^*(a); S))$. Since $\rho_B$ commutes with push-forwards by Lemma 1.18 and $(d_S)_B^B$ increases the topological filtration by $|S|$, it would follow that $(d_S)_B^B \Theta(\; ; p^*(a); S) \mod \tau^{c+1} \equiv \rho_B(d_S)_B^C(\Theta(tr_c; \; ; p^*(a); S))$. Thus, dealing with the each summand of $\phi(p^*a)$ in the formula (5) we will prove the commutativity of (4) on the element $p^*(a)$.

Recall that internal derivatives are defined using values of the operation (Section 1.3), and therefore by induction assumption we have the following relation in $\tau^cB^*(E_S \times (\mathbb{P}^\infty)^{|S|})/\tau^{c+1}$:

$$
\partial^{\vert S \vert-1}(z^{A^*}_r j^*_r \beta_r, r \in S) \mod \tau^{c+1} = \rho_B \left( \partial^{\vert S \vert-1}(tr_c \phi)(z^{A^*}_t j^*_r \beta_r, r \in S) \right).
$$

Note that by Prop. 1.14 (7) we have

$$
\tau^cB^*(E_S \times (\mathbb{P}^\infty)^{|S|})/\tau^{c+1} = \sum_{i=0}^c \tau^iB^*(E_S)/\tau^{i+1}B^*(E_S)[[z^B_r, r \in S]]^{\deg=c-i}.
$$
Moreover, by continuity of operations \( \partial_{[S]}^{-1} \phi(z^A j^r_! \beta_r, r \in S) \) is divisible by \( \prod_{r \in S} z_r^B \), and similarly for \( \text{tr}_r \phi \). Thus, dividing both expressions in (\ref{eq:commutative}) by a product of \( z_r \) we obtain an equality modulo \( \tau^{c+1-|S|} \).

Note, however, that

\[
\Theta(\phi; p^*(a) ; S) = \left( \frac{\partial_{[S]}^{-1} \phi(z^A j^r_! \beta_r, r \in S)}{\prod_{r \in S} z_r^B} \right)_{z_r = j^r_! \mu_r^B} \]

because \( \omega^B_t(0) = dt \) and positive powers of \( t \) do not intervene with the residue. Since \( \rho_B \) sends \( j^r_! \mu_r^B \) to \( \mu_r^{\text{CH}} \) we obtain that

\[
\Theta(\phi; p^*(a) ; S) \mod \tau^{c+1-|S|} \equiv \rho_B(\Theta(\text{tr}_r \phi; p^*(a) ; S)) \]

which is enough for the claim as explained above. \( \square \)

If a presheaf \( \oplus_c \tau^c B^*/\tau^{c+1} B^* \) were an oriented theory, then it would be enough to check the claim of Prop. \ref{prop:oriented} on products of projective spaces using Vishik’s classification theorem. However, this is not true, see Remark \ref{rem:non-oriented}.

Remark 2.11. One may wonder what motivic homotopical interpretation of the truncation map could be. One of the speculative reasons could be the following.

Let \( \mathcal{A}_t \) be a motivic space representing the presheaf \( A^t \) in the unstable motivic homotopy category \( \mathcal{H}_*(k) \), i.e. for any smooth variety \( X \) there is a natural isomorphism \( [X, \mathcal{A}_t] = A^t(X) \). Let \( \mathcal{B} \) be a spectrum representing the theory \( B^* \) in the stable motivic homotopy category. Then an unstable operation from \( A^t \) to \( B^* \) can be represented by a map \( \mathcal{A}_t \rightarrow \Omega^\infty \Omega^{-2j-1} B \) in \( \mathcal{H}_*(k) \), which by adjointness is the same as the map \( \Sigma^\infty \mathcal{A}_t \rightarrow \Omega^{-2j-1} B \) in \( \mathcal{SH}(k) \). The latter map represents an element in \( B^*(\mathcal{A}_t) \).

Assume that for the theory \( B^* \) we have got a spectral sequence (of Atiyah-Hirzebruch type) with the second page \( E_2^{p,q} = H^{p,q}(\mathcal{A}_t, B) \) which converges to \( B(\mathcal{A}_t) \). Assume also that the corresponding filtration on \( B^{2n,*}(\mathcal{A}_t) = B^*(\mathcal{A}_t) \) is induced by the topological filtration on \( B^* \) and assume that there are non-trivial arrows in this spectral sequence which target the place where Chow groups of \( \mathcal{A}_t \) stand: \( H^{2n,0}(\mathcal{A}_t, B) = \text{CH}^n(\mathcal{A}_t) \otimes B = [A^t, \text{CH}^n \otimes B] \). Then the spectral sequence would yield a map from a subgroup of \( [A^t, \text{CH}^n \otimes B] \), i.e. of unstable operations from \( A^t \) to \( \text{CH}^n \otimes B \), to the graded factor of unstable operations from \( A^t \) to \( B^* \) with respect to the topological filtration on \( B^* \). We expect that this is an inverse to the truncation map.

The following proposition explains how to calculate any operation between free theories on the graded pieces of the topological filtration via classes of closed (non-needlessly-smooth) subvarieties.

Proposition 2.12. Let \( \phi : A^* \rightarrow \tau^c B^* \) be an operation between free theories. Then the composition of \( \phi \) restricted to \( \tau^c A^* \) with the projection \( \tau^c B^*/\tau^{c+1} B^* \) factors through \( \tau^{c+1} A^* \) yielding an operation which fits in the following commutative diagram

\[
\begin{array}{ccc}
\text{CH}^r \otimes A & \xrightarrow{\phi^c} & \tau^c A^*/\tau^{c+1} A^* \\
\downarrow & & \downarrow \\
\text{CH}^r \otimes B & \xrightarrow{\phi^c} & \tau^c B^*/\tau^{c+1} B^* \\
\end{array}
\]

where \( \phi^c : \text{CH}^r \otimes A \rightarrow \text{CH}^r \otimes B \) is the composition of the map \( \text{tr}_r \phi : \tau^c A^*/\tau^{c+1} A^* \rightarrow \text{CH}^r \otimes B \) with the map \( \rho_A \).

Proof. Let \( x \in A^t(X), y \in \tau^{c+1} A^t(X) \) for some smooth variety \( X \). Then there exist an open subvariety \( U \subset X \), s.t. \( y|_U = 0 \), and codimension of \( X \setminus U \) in \( X \) is greater or equal to \( c + 1 \). The element \( \phi(x + y) - \phi(x) \) is zero when restricted to \( U \) since \( \phi \) commutes with pullbacks, i.e. \( \phi(x + y) \equiv \phi(x) \mod \tau^c B^*(X) \) and \( \phi \) factors through \( \tau^{c+1} B^* \).

The commutativity of each face of the following diagram follows either by definition or from Prop. \ref{prop:oriented}.
Proof. According to Th. 1.9 one needs to check that the maps $(\tau^c \circ f_k)$ operation to the classification of operations to the Chow groups. This truncation is described here purely algebraically with the use of the Vishik’s classification theorem (Th. 1.9) and is not provided with a geometric interpretation.

Proposition 2.13. Let $p$ be a prime. Let $A^*$ and $B^*$ be free theories defined over torsion-free $\mathbb{Z}_{(p)}$-algebras $A, B$, respectively. Let $\phi : A^* \to B^* \otimes \mathbb{Q}$ be an operation, let $k \geq 1$.

Assume that on products of projective spaces $\phi$ acts integrally modulo the $k+1$-th part of the topological filtration and the action is zero modulo the $k$-th part of the topological filtration and modulo $p$, i.e. \( \forall n \geq 1 \):

\[
G_n : A^*[[z_1^A, \ldots, z_n^A]] \to \sum_{s=1}^{k-1} pB(z_1^B, \ldots, z_n^B)^s + B(z_1^B, \ldots, z_n^B)^k + B \otimes \mathbb{Q}(z_1^B, \ldots, z_n^B)^1.
\]

Then maps \((\pi_k \circ G_n) \mod p : A^*[[z_1^A, \ldots, z_n^A]] \to B/p[z_1^B, \ldots, z_n^B]^{\deg=k}, n \geq 1\), define an operation $\text{tr}_k \mod p \phi : A^* \to \text{CH}^k \otimes B/p$. If $\phi$ is additive, so is the operation $\text{tr}_k \mod p \phi$.

Proof. According to Th. 1.9 one needs to check that the maps $(\pi_k \circ G_n) \mod p$ respect pullbacks along certain class of morphisms between products of projective spaces.

As in the proof of Prop. 2.7 it is enough to check that the following diagram commutes:

\[
\begin{array}{ccc}
\sum_{s=1}^{k-1} pB(z_1^B, \ldots, z_n^B)^s + B(z_1^B, \ldots, z_n^B)^k + B \otimes \mathbb{Q}(z_1^B, \ldots, z_n^B)^1 \to B/p[z_1^B, \ldots, z_n^B]^{\deg=k} & \xrightarrow{f_n} & B/p[z_1^B, \ldots, z_n^B]^{\deg=k} \\
\sum_{s=1}^{k-1} pB(z_1^B, \ldots, z_n^B)^s + B(z_1^B, \ldots, z_n^B)^k + B \otimes \mathbb{Q}(z_1^B, \ldots, z_n^B)^1 \to B/p[z_1^B, \ldots, z_n^B]^{\deg=k} & \xrightarrow{f_{\text{CH}^k \otimes B/p}} & B/p[z_1^B, \ldots, z_n^B]^{\deg=k}
\end{array}
\]

This is done via direct computation as in Lemma 2.8.

2.3. Truncation modulo ideal. In the construction of Chern classes from the $n$-th Morava K-theory we will need a modification of the truncation process which takes an operation from $A^*$ to $B^* \otimes \mathbb{Q}$ satisfying certain conditions and produces an operation from $A^*$ to $\text{CH}^k \otimes B/p$. This truncation is described here purely algebraically with the use of the Vishik’s classification theorem (Th. 1.9) and is not provided with a geometric interpretation.

Proposition 2.14. Suppose that $|A^*, \text{CH}^i \otimes B|$ is a free $B$-module of finite rank for each $i \geq 0$, and assume that the map $\text{tr}_i : \text{gr}_i^p[A^*, B^*] \to [A^*, \text{CH}^i \otimes B]$ is an isomorphism for all $i \geq 0$. Denote...
by \( \psi_j^{(i)} \in [A^*, \tau^i B^*], j \in J_i \), a finite set of operations for each \( i \geq 0 \), s.t. their \( i \)-th truncations are a basis of the corresponding free module.

Then operations \( \psi_j^{(i)} \) freely generate the \( B \)-module of all operations from \([A^*, B^*]\).

The same is true if one considers only additive operations instead of all operations.

Proof. Let \( \phi \) be an operation, and let us construct its representation as \( \sum_{k=0}^{\infty} \sum_{j \in J_k} b_j^{(k)} \psi_j^{(k)} \). In order to do it we need the operation \( \phi - \sum_{k=0}^{i} \sum_{j \in J_k} b_j^{(k)} \psi_j^{(k)} \) to take values in \( \tau^{i+1} B^* \), since the residual takes values in \( \tau^i B^* \). Let us find coefficients \( b_j^{(k)} \) by induction on \( k \).

If \( \chi_i := \phi - \sum_{k=0}^{i} \sum_{j \in J_k} b_j^{(k)} \psi_j^{(k)} \) takes values in \( \tau^{i+1} B^* \), choose coefficients \( b_j^{(i+1)} \) as coefficients of the representation of \( tr_{i+1} \chi_i \) in the basis \( tr_{i+1} \psi_j^{(i+1)}, j \in J_{i+1} \). Thus, the \( i + 1 \)-th truncation of the operation \( \chi_{i+1} = \phi - \sum_{k=0}^{i+1} \sum_{j \in J_k} b_j^{(k)} \psi_j^{(k)} \) is zero, and therefore the operation \( \chi_{i+1} \) takes values in \( \tau^{i+1} B^* \). The process converges due to the fact that infinite sums of operations \( \psi_j^{(i)} \) are defined. Thus, operations \( \psi_j^{(k)} \) generate \( B \)-module of operations \([A^*, B^*]\).

Let us also check that there are no relations between infinite sums of operations \( \psi_j^{(k)} \). Assume that \( \phi = \sum_{k=0}^{\infty} \sum_{j \in J_k} b_j^{(k)} \psi_j^{(k)} \) is a zero operation, and suppose that \( b_j^{(k)} = 0 \) for \( k < i, j \in J_k \). Then \( tr_i \phi = \sum_{j \in J_i} b_j^{(i)} \psi_j^{(i)} \) is zero, and therefore \( b_j^{(i)} = 0 \) for all \( j \in J_i \).

Lemma 2.15. Let \( A^*, B^* \) be free theories. The multiplication of operations from \( A^* \) to \( B^* \) defined by the ring structure of \( B^* \) yields a structure of \( B \)-algebra on the set \( \oplus_i gr_i^B[A^*, B^*] \).

Moreover, the map \( tr : \oplus_i gr_i^B[A^*, B^*] \to \oplus_i [A^*, CH^i \otimes B] \) is a map of rings.

Proof. The first claim follows from the multiplicativity of the topological filtration (Prop. 1.17 (5)).

Let \( \phi : A^* \to \tau^i B^*, \psi : A^* \to \tau^j B^* \) be operations, and we need to check that \( tr_{i+j} (\phi \psi) = tr_i (\phi) tr_j (\psi) \). By Vishik’s classification theorem (Th. 1.9) it is enough to prove the equality of values on products of projective spaces. According to the construction of the truncation the claim is equivalent to the following for each \( n \geq 0 \): if \( f \in B(z_1^p, \ldots, z_n^p), g \in B(z_1^p, \ldots, z_n^p) \), then \( \pi_{i+j} (f \cdot g) = \pi_i (f) \cdot \pi_j (g) \) (see Section 2.2 for the notation). This is straight-forward.

Proposition 2.16. Let \( A^* \) be a free theory, and let \( B^* \) be an oriented theory.

Assume that the truncation maps \( tr_i : gr_i^B[A^*, B^*] \to [A^*, CH^i \otimes B^*] \) are isomorphisms for each \( i \geq 1 \). Assume that the ring \([A^*, CH^i \otimes B]^B \) is freely generated as \( B \)-algebra by operations \( t_i : A^* \to CH^i \otimes B, i \in I \), and assume that \([A^*, CH^i \otimes B]^B \) is a finite rank (free) \( B \)-module.

Denote by \( t_i \in [A^*, \tau^i B^*] \) a lift of the operation \( t_i \) with respect to the truncation map.

Then \([A^*, B^*]^B \) is freely generated as \( B \)-algebra by operations \( t_i \).

Proof. It follows from Prop. 2.14 that \( t_i \) generate the ring of all operations, i.e. any operation can be represented as a \( B \)-series in \( t_i \). Let us prove that there are no algebraic relations between these operations.

Assume that \( P \in B[[t_1, \ldots, t_i, \ldots]] \) defines a zero operation, \( P \neq 0 \). Define degree of a monomial \( \prod_k m_k r_k^{e_k} \) to be \( \sum_k m_k r_k^{e_k} \), i.e. this monomial defines an operation which takes values in the degree-th part of the topological filtration on the target. Let \( j \) be the minimal degree of monomial summands of \( P \), i.e. \( j \) is finite.

By Lemma 2.16 the operation \( tr_j P \) is equal to the polynomial in operations \( t_j^{(i)} \) obtained by truncating \( P \) as a series. As there are no polynomial relations between operations \( t_j^{(i)} \) by the assumption, we come to a contradiction.

3. Morava K-theories and operations to \( p^n \)-typical theories

Fix a prime number \( p \). In what follows all rings will be \( \mathbb{Z}(p) \)-algebras if not specified otherwise.

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4 Here ‘free’ actually means ‘topologically free’ with respect to the adic topology defined by the topological filtration on \( B^* \), i.e. infinite sums of operations whose target space has growing index of the topological filtrations are allowed.
3.1. $p^n$-typical formal group laws.

Definition 3.1. A series $\gamma \in A[[x]]$ is called $p^n$-gradable (w.r.t. to $x$) if it is of the form
$$\sum_{j \geq 0} a_j x^{j+ip^n-1}.$$  
A series $\eta \in A[[x_1, \ldots, x_n]]$ is called $p^n$-gradable if it is $p^n$-gradable w.r.t. to any variable $x_i$.

We will also say that a series $\gamma$ (or $\eta$) is $p^n$-gradable up to degree $k$ if it is of the form
$$\sum_{j=0}^{k} a_j x^{j+ip^n-1} \pmod{(x^{k+1})}$$
for each $i: 1 \leq i \leq n$.

Note that a $p^n$-gradable series over $A$ can be made a homogenous series of degree 1 over the ring $A[v_n]$ with $\deg v_n = 1 - p^n$, and $\deg x = 1$.

Proposition 3.2. 
(1) Whenever composition of a $p^n$-gradable series is defined, it is $p^n$-gradable.
(2) If a series is $p^n$-gradable and invertible w.r.t. composition, then its inverse is also $p^n$-gradable.
(3) If a series if $p^{kn}$-gradable, then it is also $p^n$-gradable.

Proof. Straight-forward.

Formal group laws (FGLs) are formal power series in two variables which model a multiplication of a formal neighbourhood of the identity of a one-dimensional algebraic group. We have already mentioned them in Section [L1] where they appeared in the connection to oriented cohomology theories. For general facts about formal group laws we refer the reader to [Ha78].

Recall that there exist a notion of a $p$-typical FGL due to Cartier, which over a torsion-free ring is simplified to the following: a FGL is $p$-typical iff its logarithm has the form $\sum_{i=1}^{\infty} l_i x^{p^i}$ ([Ca67, Th. 4]).

There exist the universal $p$-typical FGL $F_{BP}$ over the ring $BP$, and the corresponding free theory is called the Brown-Peterson cohomology. The ring $BP$ is non-canonically isomorphic to the graded ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ where the variable $v_i$ has degree $1 - p^i$, and one of the choice of these variables is known as Araki generators. It is common to include $v_0 = p$ as a ‘variable’ in notations. If we denote the coefficients of the logarithm of $F_{BP}$ as $l_i$ so that $\log_{BP} = \sum l_i x^{p^i}$, then $pl_n = \sum_{i=0}^{n} l_i v_i^{p^i}$ ([Ar73, I.4 (6.12)]). For these generators we also have $p \cdot_{BP} x = \sum_{i \geq 0} v_i^{p^i}$ ([op.cit., Th. 6.5]).

Now we are going to generalize the definition of a $p$-typical formal group law to that of a $p^n$-typical one, but beforehand let us warn the reader of the handicaps of the foregoing definition.

The definition of a $p$-typical formal group law can be fully justified by a theorem of Cartier claiming that any formal group law over any $\mathbb{Z}_{(p)}$-algebra is canonically and strictly isomorphic to a $p$-typical one ([Ca67]). In oriented cohomology theories this allows, for example, to go from $\Omega^*$ to $BP^*$ whenever the situation is $p$-local. For the definition that we are giving now no analogous statement is known at the moment, and the only thing that justifies it is the main result of this paper (Theorem 3.10) which is of a nature of oriented theories and not of formal group laws.

Definition 3.3. Formal group law $F$ over a $\mathbb{Z}_{(p)}$-algebra is called $p^n$-typical, if it is $p$-typical and $p \cdot F$ is a $p^n$-gradable series.

An oriented theory is called $p^n$-typical, if the corresponding FGL is $p^n$-typical.

Proposition 3.4. There exist a graded ring $BP\{n\}$ classifying $p^n$-typical FGL’s.

The ring $BP\{n\}$ can be naturally identified with a factor ring of $BP \cong \mathbb{Z}_{(p)}[v_n, v_{2n}, \ldots]$ where Araki generators $v_i$ are sent to zero for $i \nmid n$ and $v_{mn}$ is sent to $v_{mn}$ for $m \geq 1$.

In particular, $BP\{1\}$ is $BP$, and $BP\{kn\}$ is a natural factor-ring of $BP\{n\}$ for any $k, n \in \mathbb{N}$.

Proof. It is clear that the ring $BP\{n\}$ exists and it can be identified with a factor of the ring $BP$ over the ideal generated by those coefficients of the series $p \cdot_{F_{BP}} x$ which stand to ‘non-$p^n$-gradable’ monomials. Denote by $F_{BP\{n\}}$ the universal $p^n$-typical FGL.

Denote by the ring map $\phi: BP \to BP\{n\}$ the canonical map classifying the universal $p^n$-typical over $BP\{n\}$ which is also $p$-typical. Our goal now is to show that $\phi$ sends the Araki generator
Corollary 3.6. If $v_i$ to zero for $i \mid n$. Suppose that $i_0 = \min\{j > 0 : φ(v_j) \neq 0, j \mid n\}$ is finite. Recall that $p^i BP = \sum_i^\infty v_i x^{p^i}$, and therefore $p^i BP(n)_x = \sum_i^{F_p(n)} φ(v_i)x^{p^i}$. By our assumption

$$p^i BP \phi = \sum_{j > 0} φ(v_{j_n})x^{p^j} + BP(n) φ(v_n)x^{p^{n_0}} + BP(n) \text{ higher degree terms.}$$

Note that $log_{BP(n)}$ is $p^n$-gradable up to degree $p^n - 1$ as follows from the assumptions, and therefore $F_{BP(n)}$ is $p^n$-gradable up to degree $p^n - 1$ as well. It follows that the sum of two first summands of the RHS of (7) is $p^n$-gradable up to degree $p^n - 1$.

The rightmost summands of (7) have degree in $x$ strictly bigger than $p^n$, and therefore in $p^i BP(n)_x$ there appears the monomial $φ(v_{i_n})x^{p^{n_0}}$. Thus, this series is not $p^n$-gradable and we get a contradiction with the finiteness of $i_0$.

Conversely, to prove that the logarithm of a $p$-typical FGL over the ring $BP/(v_i, i \mid n)$ defined by the canonical map from $BP$ is $p^n$-typical. As the series $p^i x$ is a homogeneous series of degree $1$ (where $deg x = 1$), and the degrees of all of the generators of $BP/(v_i, i \mid n)$ are divisible by $p^n - 1$, it follows that this series is $p^n$-gradable.

The converse is straightforward: if $F$ is $p^n$-gradable, then $[k] \cdot x$ is $p^n$-gradable for any $k \in \mathbb{Z}$.

Corollary 3.5. A formal group law $F$ over a ring $R$ is $p^n$-typical if it is $p^n$-typical and $F = \sum a_i x^i y^i$ is $p^n$-gradable.

Proof. The universal $p^n$-typical formal group law $F_{BP(n)}$ is homogeneous of degree $1$ (with $deg x = deg y = 1$) and since the degrees of elements in $BP(n)$ are always divisible by $p^n - 1$ it follows that $F_{BP(n)}$ is $p^n$-gradable. Since a $p^n$-typical formal group law $F_A$ over a ring $A$ can be obtained as $φ(F_{BP(n)})$ where $φ : BP(n) \rightarrow A$ is a ring map, the power series $F_A$ is also $p^n$-gradable.

Conversely, to prove that the logarithm of a $p^n$-typical FGL is of the form $\sum_{i=0}^\infty l_i x^{p^{n_i}}$ it suffices to check that for the universal $p^n$-typical FGL over the ring $BP(n)$. To do this recall the formulas linking coefficients of the logarithm of $BP$ in terms of Araki generators $v_i$: $pl_{m_i} = \sum_{i=0}^\infty l_i v_{m_i - i}$, where $v_0 = p$.

Denote by $l_j \in \mathbb{Q}[v_n, v_{2n}, \ldots]$ the coefficients of the logarithm of $F_{BP(n)}$, i.e. $log_{BP(n)} = \sum_{i=0}^\infty l_j x^{p^j}$. Let $i_0 := \min\{j > 0 : l_j \neq 0, n \mid j\}$, and assume that it is finite.

Then by the definition of Araki generators we have an equality $pl_{i_0} = \sum_{i=0}^{m-1} l_i v_{m_i} + l_0 v^{p^{n_0}}$ in $BP$, and applying the map $φ : BP \rightarrow BP(n)$ to it we see that $pl_{i_0} = p^{n_0} l_{i_0}$. Indeed, if $n \mid i$ then $φ(l_i)$ is zero for $i < i_0$, and if $n \mid i$, then $n \mid (i_0 - i)$ and $φ(v_{i_0 - i})$ is zero. Thus, $l_{i_0} = 0$ which is a contradiction.

Definition 3.7. Denote by $BP(n)_+^\ast$ a free theory with the ring of coefficients $BP(n)$ and the corresponding formal group law is the universal $p^n$-typical FGL.

Proposition 3.8. Free theory $BP(n)_+^\ast$ is the universal $p^n$-typical oriented theory, i.e. for any $p^n$-typical oriented theory $A^\ast$ there exist a unique morphism of oriented theories $BP(n)_+^\ast \rightarrow A^\ast$.

Proof. Follows from the universality of algebraic cobordism (Th. 1.2).
3.2. Definition of Morava K-theories.

Definition 3.9 ([S18a Def. 4.1.1]). Let \( n \geq 1 \). A \( p^n \)-typical free theory \( K(n)^* := \Omega^* \otimes_{L} \mathbb{Z}(p) \) is called an algebraic \( n \)-th Morava K-theory \( K(n)^* \) if the formal group law \( F_{K(n)} \) mod \( p \) over \( \mathbb{F}_p \) has height \( n \).

Remark 3.10. Formal group laws which can be associated with oriented spectra in topology (and motivic topology) are defined over a graded ring of coefficients and are graded themselves, i.e. formal power series of the FGL is homogeneous of degree 1 where variables have degree 1. However, it is not clear to us which of these lifts can be isomorphic to a first Morava K-theory as a presheaf of rings.

Definition 3.9 (see e.g. [Ra04, Section 4.2]). Any such formal group law is liftable to \( \mathbb{Z}(p) \) such that \( K(n)^* = \mathbb{GK}(n)^*/(v_n - 1) \). It is \( \mathbb{GK}(n)^* \) which may be a part of a cohomology theory represented by a motivic spectrum. We prefer to work with the theory \( K(n)^* \), but all the results on operations can be reformulated for \( \mathbb{GK}(n)^* \).

Remark 3.11. In topology the \( n \)-th Morava K-theory usually is a unique spectra which homotopy groups (i.e. values of the corresponding cohomology theory on a point) is the graded field \( \mathbb{F}_p[v_n, v_n^{-1}] \) where \( \deg v_n = 1 - p^n \), and there exist an orientation of it such that the corresponding graded formal group law has height \( n \) (see e.g. [Ra04, Section 4.2]). Any such formal group law is liftable to \( \mathbb{Z}(p)[v_n, v_n^{-1}] \), and if localized at \( v_n = a \in \mathbb{Z}_p^* \) this will be a formal group of a \( n \)-th Morava K-theory as defined above. However, it is not clear to us which of these lifts can be performed to yield a (motivic) spectra representing a cohomology theory, and whether the lift is in any sense unique.

For a uniqueness statement concerning algebraic Morava K-theories \( K(n)^* \) above see Section 5.

3.3. Grading on Morava K-theories.

Proposition 3.12 ([S18a Prop. 4.3.2]). Let \( F \) be a FGL of an \( n \)-th Morava K-theory. Then its logarithm has the form

\[
\log_{K(n)}(x) = x + \frac{a_1}{p}x^{p^n} + \frac{a_2}{p^2}x^{p^{2n}} + \ldots
\]

where \( a_i \in \mathbb{Z}_{(p)}^* \) for all \( i \geq 1 \). Moreover, \( a_k \equiv (a_1)^k \mod p \).

One can easily check that the first terms of the formal group law of any Morava K-theory \( K(n)^* \) look like this:

\[
F_{K(n)}(x, y) = x + y - \frac{1}{p} \sum_{i=1}^{p^n-1} \left( \frac{p^n}{i} \right) x^i y^{p^n-i} + \text{higher degree terms.}
\]

Remark 3.13. The Artin-Hasse exponential establishes an isomorphism between formal group laws \( F_m = x + y + xy \) and a \( p \)-typical FGL over \( \mathbb{Z}(p) \) of height 1. This implies that \( K_0 \otimes \mathbb{Z}(p) \) is isomorphic to a first Morava K-theory as a presheaf of rings.

It is not true, however, that every two \( n \)-th Morava K-theories as defined above are multiplicatively isomorphic (see Appendix A.2).

Proposition 3.14 ([S18a Prop. 4.1.5]).

(1) Morava K-theories \( K(n)^* \) are \( \mathbb{Z}/(p^n - 1) \)-graded.

(2) The grading on \( K(n)^* \) is respected by Adams operations.

(3) The grading is compatible with push-forwards, i.e. for a proper morphism \( f : X \rightarrow Y \) of codimension \( c \) push-forward maps increase grading by \( c \), \( f_* : K(n)^*(X) \rightarrow K(n)^{c+}(Y) \).

In particular, \( e_1^{K(n)}(L) \in K(n)^1(X) \) for any line bundle \( L \) over a smooth variety \( X \).

---

We will drop 'algebraic' from the name as topological Morava K-theories will not be considered in this paper except in a few comparison discussions.
We denote the graded components of $n$-th Morava K-theories as $K(n)^1$, $K(n)^2$, $\ldots$, $K(n)^{p^n-1}$, and freely use the following expressions $K(n)^i$, $K(n)^i \mod p^n - 1$, $K(n)^{i+r(p^n-1)}$ to denote the component $K(n)^j$ where $j \equiv i \mod p^n - 1$, $1 \leq j \leq p^n - 1$. The reason we denote the component $K(n)^0$ as $K(n)^{p^n-1}$ is mainly because we will work with $\tilde{K}(n)^*$ instead of $K(n)^*$, $\tilde{K}(n)^{p^n-1}$ contains classes of codimensions $p^n - 1 + r(p^n-1)$ for all $r \geq 0$.

The grading on the $n$-th Morava K-theory splits the topological filtration "modulo a period of $p^n - 1$ steps" as explained in the next proposition.

**Proposition 3.15.** The topological filtration on each graded component of the $n$-th Morava K-theory changes only every $p^n - 1$ steps, i.e.,

$$\tau_{j+(p^n-1)+1} \tilde{K}(n)^j = \tau_{j+(p^n-1)+1} (\tilde{K}(n)^j) = \ldots = \tau_{j+(s+1)(p^n-1)} (\tilde{K}(n)^j),$$

where $j \in [1, p^n - 1]$, $s \geq 0$.

In particular, $gr_j^* \tilde{K}(n)^* = \tilde{K}(n)^j / \tau_{j+p^n-1} \tilde{K}(n)^j$ for $j: 1 \leq j \leq p^n - 1$, and $gr_j^* \tilde{K}(n)^* = gr_j^* (\tilde{K}(n)^j)$ for every $j$.

**Proof.** The topological filtration on algebraic cobordism has a description in terms of the structure of $L$-module by [LM07, Th. 4.5.7]: $\tau^* \Omega^* = \bigcup_{n \leq j} \Omega^n \bigotimes_p \Omega^{n-m}$, and the morphism of theories $\Omega^* \to K(n)^*$ yields a surjection $\tau^* \Omega^* \bigotimes_k K(n)^* \to \tau^* K(n)^*$ by Proposition 1.17, 1).

It follows from Proposition 3.11, 3) that $\Omega^k$ maps to $K(n)^k \mod p^n - 1$ for all $k$, i.e., $K(n)^k = \bigoplus_{s \leq j} \tilde{\tau}^i \Omega^{j+s(p^n-1)} \bigotimes_p K(n)$. Thus, for $j \in [1, p^n - 1]$ the group $\tau^j K(n)^j$ is the image of $\bigoplus_{s \leq j} \tilde{\tau}^i \Omega^{j+s(p^n-1)}$ which is equal to $\bigoplus_{s \leq j} \bigcup_{m \leq j+s(p^n-1)-1} \tilde{L}^m \Omega^{j+s(p^n-1)-m}$.

Since the formal group law of $K(n)^*$ is $p^n$-typical, the map $L \to K(n)$ can be factorized as $L \xrightarrow{\phi} Z(p)[v_n, v_n^{-1}] \xrightarrow{v_n=1} Z(p) = K(n)$ where $\phi$ is a graded map, $\deg v_n = 1 - p^n$ (cf. Remark 3.10). Therefore $L^n$ maps to zero in $K(n)$ if $j \neq 0 \mod p^n - 1$.

Thus, we see that $\tilde{L}^m \Omega^{j+s(p^n-1)-m}$ maps to zero in $K(n)^j$ if $m \neq 0 \mod p^n - 1$, and therefore the image of the group $\bigoplus_{s \leq j} \bigcup_{m \leq j+s(p^n-1)-1} \tilde{L}^m \Omega^{j+s(p^n-1)-m}$ in $K(n)^j$ changes only every $p^n - 1$ steps with the change of $i$. □

3.4. Chern classes: statement of the main theorem. In [S18a, Th. 4.2.1] we have shown that for every $n$-th Morava K-theory $K(n)^*$ there exist operations from $K(n)^*$ to $\mathcal{H}^* \bigotimes Z(p)$ which we called Chern classes. Even though these operations were not unique, let us fix and denote by $c_i^{CH}$ any choice of them. Now we are going to generalize this result to a wider class of theories in place of Chow groups, namely, to the $p^n$-typical oriented theories.

**Theorem 3.16.** For every Morava K-theory $K(n)^*$ and every $p^n$-typical theory $A^*$ there exist a series of operations $c_j : K(n)^* \to A^*$ for $j \geq 1$ satisfying the following conditions.

i) Operation $c_j$ restricted to the graded components of $K(n)^*$ is zero except for $K(n)^j \mod (p^n-1)$.

ii) Operation $c_j$ takes values in $\tau^j A^*$, and $\tau^j c_j$ is the Chern class $c_j^{CH} \bigotimes \text{id}_A$.

iii) Denote by $c_{tot} = \sum_{i \geq 1} c_i^t$ the total Chern class in a formal variable $t$. Then the Cartan’s formula holds universally:

$$c_{tot}(x+y) = F_{K(n)}(c_{tot}(x), c_{tot}(y)),$$

where $x, y \in K(n)^*(X)$ for any smooth variety $X$ and the identity takes place in $A^*(X)[[t]]$.

iv) If $A$ is a free $Z(p)$-module, then all operations from $K(n)^*$ to $A^*$ are uniquely expressible as series in Chern classes:

$$[K(n)^*, A^*] = A[[c_1, \ldots, c_i, \ldots]].$$

Moreover, the analogous statement is true for poly-operations.

We call operations $c_j$ described in this Theorem Chern classes, even though it may lead to confusion especially if the usual Chern classes are involved. For the clarity we suggest to use the notation $c_j^{K(n)^* \to A^*}$, $c_j^{K(n) \to A^*}$ to distinguish the notions in writing when needed.

**Remark 3.17.** Petrov and Semenov introduced operations $c_1, c_2, \ldots, c_{p^n}$ from a specific Morava K-theory $K(n)^*$ to Chow groups modulo $p$-torsion in [PS14]. One may choose lifts of their operations $c_1, c_2, \ldots, c_{p^n-1}$ to $\mathcal{H}^* \bigotimes Z(p)$ as the starting point of the construction of operations
in [S18a]. However, the natural choice of the operation \( c_p^n \) then will differ by a sign to the choice of Petrov-Semenov.

**Remark 3.18.** No uniqueness of Chern classes is claimed in the theorem even when the Chern classes to Chow groups are fixed. Moreover, one can check through the proof of Theorem 3.16 that having constructed operations \( c_1, \ldots, c_l \) one can define \( c_i^{\text{new}} = c_i + \phi_i \) where \( \phi_i \) is any additive operation to \( \tau_i^{+1}A^* \), and then construct operations \( c_j^{\text{new}} \) for \( j > i \) so that all properties of the theorem are satisfied.

Even though constructed operations are not unique, one can define the gamma filtration on the \( n \)-th Morava K-theory using them and it does not depend on any choices (Section 4).

It should also be noted that one can substitute the formal group law of \( K(n)^* \) in the Cartan’s formula by any \( p^n \)-typical FGL over \( \mathbb{Z}(p) \) of height \( n \), i.e. to any formal group law defining \( n \)-th Morava K-theory. At the moment it seems that there is no advantage of using one or another FGL for the Cartan’s formula for Chern classes from \( K(n)^* \), however, using the same FGL as that of the orientation of \( K(n)^* \) make it look similar to the classical case of \( K_0 \).

**Remark 3.19.** If \( A \) is not a free \( \mathbb{Z}(p) \)-module, then it is not true that all operations from \( K(n)^* \) to \( A^* \) are expressible in terms of Chern classes for \( A \). In particular, we have shown in [S18a] End of Sec. 4.5] that the ring of operations from \( K(n)^* \) to \( \text{CH}^+/p \) generated by Chern classes is not stable under the action of the Steenrod algebra.

### 4. Proof of Theorem 3.16

In this section an \( n \)-th Morava K-theory \( K(n)^* \) is fixed, its FGL is denoted by \( F_{K(n)} \) and its logarithm is \( \log_{K(n)}(x) = x + \sum_{i=1}^{\infty} \frac{a_i}{p^i}x^{p^i} \) for some \( a_i \in \mathbb{Z}(p) \) (see Prop. 3.12).

First, we classify additive operations from \( K(n)^* \) to \( B\mathbb{P}\{n\}^* \) in Section 4.1 and use them to construct Chern classes with values in \( B\mathbb{P}\{n\}^* \) in Section 4.2. Then for any \( p^n \)-typical theory \( A^* \) Chern classes from \( K(n)^* \) are defined as compositions of constructed operations \( c_i : K(n)^* \to B\mathbb{P}\{n\}^* \) with the unique morphism of oriented theories \( B\mathbb{P}\{n\}^* \to A^* \). Finally, in Section 4.3 based on the results about the truncation of operations from Section 2 we prove that if \( A \) is a free \( \mathbb{Z}(p) \)-module then Chern classes generate all operations to \( A^* \).

#### 4.1. Additive operations from Morava K-theory to \( B\mathbb{P}\{n\}^* \)

We will need to work with additive operations from \( K(n)^* \) to \( B\mathbb{P}\{n\}^*/p \), however, their classification is more difficult than of integral operations. We restrict ourselves to so-called gradable operations which are much more amenable to investigations (cf. [S18a] Section 4.5).

**Definition 4.1 (cf. [S18a] Def. 4.5.2]).** An additive operation \( \phi : K(n)^* \to A^* \) to an oriented theory \( A^* \) is called *gradable* if series \( G_l \in A[[z_1, \ldots, z_l]] \) defining it (see Section 1.6) are \( p^n \)-gradable for all \( l \geq 1 \).

**Proposition 4.2 (cf. [S18a] Prop. 4.5.4]).** Let \( A^* \) be a \( p^n \)-typical oriented theory s.t. \( A \) is a torsion-free \( \mathbb{Z}(p) \)-module. Then all additive operations from \( K(n)^* \) to \( A^* \) are gradable.

**Proof.** Let \( \phi : K(n)^* \to A^* \) be an additive operation. Its composition with the Chern character \( ch_A : A^* \to CH_0^* \otimes A \) is an additive operation to \( CH_0^* \otimes A \). However, as a presheaf of abelian groups \( CH_0^* \otimes A \) is a direct sum of presheaves \( CH_0^* \), and by [S18a] Prop. 4.5.4] any additive operation to this presheaf is gradable. Thus, \( ch_A \circ \phi \) is gradable as well.

Note that the operation \( (ch_A)^{-1} : CH_0^* \otimes A \to A^* \otimes \mathbb{Q} \) is gradable. Indeed, it is a multiplicative operation, and therefore \( G_l(1) = \prod_{i=1}^{l} G_l(1)_{\text{ch}_n} \otimes \tau_i^i \), so it is enough to check that \( G_l \) is gradable. However, \( G_l(1) = \log_{A}(z_{\text{CH}}) \) which is gradable by definition of \( p^n \)-typical theory and Cor. 3.3.6.

Thus, the operation \( \phi \otimes id_Q = (ch_A)^{-1} \circ (ch_A \circ \phi) \) from \( K(n)^* \) to \( A^* \otimes \mathbb{Q} \) is gradable, and as \( A \) is a torsion-free ring therefore so is \( \phi \).

**Proposition 4.3.** The truncation map (see Prop. 2.7)

\[ tr_i : gr_i^{+}[K(n)^*, \tau_i^i BP\{n\}^*]_{\text{add}} \hookrightarrow [K(n)^*, CH^i \otimes BP\{n\}]_{\text{add}} \]

defines an isomorphism of free \( BP\{n\} \)-modules of rank 1 for each \( i \geq 0 \).
Moreover, for each $i \geq 0$ there exists an additive operation $\psi_i$ supported on $K(n)^i$ and taking values in $\tau^i BP\{n\}^i$ which generates the module above.

**Proof.** The group $[K(n)^*, CH^i \otimes \mathbb{Z}_p]^{\text{add}}$ is a free $\mathbb{Z}_p$-module of rank 1 by [SINa] Prop. 3.6. Since $BP\{n\}$ is a free $\mathbb{Z}_p$-module, the presheaf of abelian groups $CH^i \otimes BP\{n\}$ is isomorphic to a direct sum of presheaves $CH^i \otimes \mathbb{Z}_p$. Combining these two facts together we obtain that the group $[K(n)^*, CH^i \otimes BP\{n\}]^{\text{add}}$ is a free $BP\{n\}$-module of rank 1.

Thus, to prove the proposition it suffices to find an additive operation from $K(n)^i$ to $BP\{n\}^i = \tau^i BP\{n\}^i$ such that its $i$-th truncation comes from a generator of the $\mathbb{Z}_p$-module $[K(n)^*, CH^i \otimes \mathbb{Z}_p]^{\text{add}}$ which is a submodule of the $BP\{n\}$-module $[K(n)^*, CH^i \otimes BP\{n\}]^{\text{add}}$. Let $p_i : BP\{n\}^i \to CH^i \otimes \mathbb{Z}_p$ be the $i$-th component of the unique morphism of theories from $BP\{n\}^*$ to $CH^i$. Direct computation on products of projective spaces shows that that for an operation $\phi : K(n)^* \to BP\{n\}^i$ we have $p_i \circ \phi = tr_i(\phi)$.

In what follows we will use Vishik’s Theorem [17] without mentioning the identities of the operations from $K(n)^*$ to $CH^i \otimes \mathbb{Z}_p$ (or to $BP\{n\}^*$) as subsets of those operations to $CH^i \otimes \mathbb{Q}$ (or to $BP\{n\}^* \otimes \mathbb{Q}$, respectively) which act on the products of projective spaces integrally.

Let us first construct an additive operation $\psi_j : K(n)^j \to BP\{n\}^j_\mathbb{Q} = \tau^j BP\{n\}^j_\mathbb{Q}$ for any $j \geq 1$ s.t. its truncation $tr \psi_j : K(n)^* \to CH^j \otimes \mathbb{Q}$ is a generator of additive operations to $CH^j \otimes \mathbb{Z}_p$. We have an isomorphism $[K(n)^*, BP\{n\}^j_\mathbb{Q}]^{\text{add}} \cong [K(n)^*, BP\{n\}^j_\mathbb{Z}_p]^{\text{add}}$, and there exist the Chern character isomorphism $BP\{n\}^j_\mathbb{Q} \xrightarrow{\tau^j} CH^j \otimes BP\{n\}$ which identifies $\tau^j BP\{n\}^j_\mathbb{Q}$ with $\otimes_{s \geq k}(CH_s^j \otimes BP)^{\text{deg} = j}$. It follows that the truncation map

$$tr_j : gr^j_{\mathbb{Q}}(K(n)^*), BP\{n\}^j_\mathbb{Q}^{\text{add}} \to [K(n)^*, CH^j_\mathbb{Q}]^{\text{add}}$$

is an isomorphism. Thus, a generator of additive operations from $K(n)^*$ to $CH^j \otimes \mathbb{Q}$ which is supported on $K(n)^j$ (SINa Prop. 4.1.6) can be lifted to an additive operation $\psi_j$ from $K(n)^j$ to $BP\{n\}^j_\mathbb{Q}$.

Second, let us define the additive operation $\phi_1$ as an infinite $BP\{n\} \otimes \mathbb{Q}$-linear combination of operations $\psi_j$, $j \geq i$, $j \equiv i \mod (p^m - 1)$. Such an operation will be supported on $K(n)^i$. The linear combination is chosen by the following inductive procedure.

On the $k$-th step of the induction we construct an additive operation $\phi_k^i : K(n)^i \to BP\{n\}^i \otimes \mathbb{Q}$, s.t. $tr \phi_k^i$ is a generator of additive operations to $CH^i \otimes \mathbb{Q}_p$ and $\phi_k^i$ acts integrally on products of projective spaces modulo the $(i + k(p^m - 1) + 1)$-th piece of the topological filtration. The latter condition also could also be written as series $G_l$ defining the operation for all $l \geq 1$ satisfy

$$G_l \mod (z_1, \ldots, z_l)^{i+k(p^m - 1) + 1} \in BP\{n\}[z_1, \ldots, z_l]/(z_1, \ldots, z_l)^{i+k(p^m - 1) + 1}.$$ 

We should also note that if a gradable operation $\phi$ on $K(n)^i$ is integral modulo the $(i + k(p^n - 1) + 1)$-th piece of the topological filtration, then it is also integral modulo the $(i + (k + 1)(p^n - 1))$-th piece of the topological filtration. Indeed, the series $G_k$ defining the operations has non-trivial summands of degrees $k + r(p^n - 1)$ only, where $r \geq 0$. However, for the operation supported on $K(n)^i$ we have $G_k = 0$ if $k \not\equiv i \mod p^n - 1$.

**Base of induction** ($k = 0$). Take $\phi_0^i$ to be equal $\psi_i$. This operation takes values in $\tau^i$, and its $i$-th truncation is integral, which means that it acts integrally modulo $(i + 1)$-th piece of the topological filtration on products of projective spaces.

**Induction step** ($k \to k + 1$). The operation $\phi_k^i$ acts integrally on the projective spaces modulo $(i + k(p^n - 1) + 1)$-th part of the topological filtration. Since series $G_l$ defining the operation $\phi_k^i$ are gradable for each $l \geq 1$, they are integral modulo $(i + (k + 1)(p^n - 1))$-th part of the topological filtration.

Denote by $m$ the maximal $p$-power in the denominators of degree $i + (k + 1)(p^n - 1)$ summands in series $G_l$ for all $l \geq 1$. In other words, $m$ is the minimal number $M$, s.t. $p^M \phi_k^i$ acts integrally on projective spaces modulo $(i + (k + 1)(p^n - 1) + 1)$-th part of the topological filtration. If $m = 0$, then one can define $\phi_{k+1}^i = \phi_i$.

Assume now that $m > 0$. The operation $p^m \phi_k^i$ satisfies the assumptions of Proposition 21.3 and therefore there is a well-defined additive operation $tr_{i+(k+1)(p^n-1)+1}(p^m \phi_k^i)$ from $K(n)^i$
to $\text{CH}^{i+k} \otimes BP\{n\}/p$. This operation acts on the products of projective spaces via the series 
$(p^nG_i \mod (z_1, \ldots, z_l)^{(i+k)(p^n-1)+1}) \mod p$, $l \in \mathbb{N}$, which are in fact polynomials of degree 
$i + (k + 1)(p^n - 1)$ in variables $z_i$.

Clearly, this operation is also gradable. Since the module of gradable additive operations to 
$\text{CH}^{i+k(k+1)(p^n-1)}/p$ is of rank 1 by [S18a Cor. 4.5.12], the operation $tr_{i+k(k+1)(p^n-1)+1}(p^n\phi^i_k)$ is 
propotional to the truncation of the operation $\psi_{i+k(k+1)(p^n-1)}$. Therefore there exist 
b $\in BP\{n\}^{i-k-1}$ s.t. $p^n\phi^k - b\psi_{i+k(k+1)(p^n-1)}$ has denominators at most $p^n-1$ in the 
i $(k+1)(p^n-1)$-th part of filtration. This induction process reduces $m$ to zero in finitely 
many steps, and one defines $\phi_i$ as $\phi^{k,i}$ for some $x \in BP\{n\}^{i-k-1} \otimes \mathbb{Q}$. Thus, 
$tr_i\phi^{k,i} = tr_i\phi^k_i$ is a generator of additive operations.

Note that since the operation $\psi_{i+k(p^n-1)+1}$ takes values in $\tau^{i+k(p^n-1)+1}$, infinite linear 
combinations of these operations are well-defined and the induction process on $k$ converges.

Thus, the induction process yields the additive operation $\phi_i : K(n)^* \rightarrow \tau^i BP\{n\}^*$ such that its 
i-th truncation is a generator of the free $BP\{n\}$-module $[\{K(n)^*, \text{CH}^{i} \otimes BP\{n\}]^{add}$ of rank one. $\blacksquare$

Let us fix the notation $\phi_i : K(n)^* \rightarrow \tau^i BP\{n\}^*$ for the operation constructed in the proof above 
throughout this section.

**Corollary 4.4.** Let $A^*$ be a $p^n$-typical oriented theory. Denote by $\phi_A^i$ the composition of the additive operation $\phi_i$ with the unique morphism of theories $p : BP\{n\}^* \rightarrow A^*$.

1. If $A$ is a $F_p$-algebra, then the $A$-module of gradable additive operations from $K(n)^*$ to $A^*$ is 
   freely generated by $\phi_A^i$, i.e.
   
   $$[K(n)^*, A^*]^{add,grad} = \{ \sum_{i=1}^{\infty} a_i \phi_A^i | a_i \in A \}.$$

2. If $A$ is a free $\mathbb{Z}_p$-module, then the $A$-module of all additive operations from $K(n)^*$ to $A^*$ is 
   freely generated by $\phi_A^i$, i.e.
   
   $$[K(n)^*, A^*]^{add} = \{ \sum_{i=1}^{\infty} a_i (\phi_A^i \mod p) | a_i \in A \}.$$ 

**Proof.** It follows from the construction of the truncation, that $tr_i(\phi_A^i) = tr_i(\phi_i) \otimes \text{id}_A$.

If $A$ is a $F_p$-algebra, then $\text{CH}^* \otimes A$ as a presheaf of abelian groups is isomorphic to a direct sum 
of presheaves $\text{CH}^*/p$. In this case $tr_i(\phi_A^i)$ is an additive generator of gradable additive operations 
to $\text{CH}^*/p$ ([S18a Cor. 4.5.12]), and therefore is a generator of the $A$-module of gradable additive operations 
to $\text{CH}^* \otimes A$. The first part of this Proposition is proved then exactly in the same way as 
Proposition 2.14 where one considers only gradable additive operations instead of all additive operations.

If $A$ is a free $\mathbb{Z}_p$-module, then $\text{CH}^* \otimes A$ as a presheaf of abelian groups is isomorphic to a direct sum 
of $\text{CH}^* \otimes \mathbb{Z}_p$. By construction $tr_i(\phi_A^i)$ is a generator of the $A$-module of all additive operations 
to $\text{CH}^* \otimes A$. The claim follows from Proposition 2.14. $\blacksquare$

**Remark 4.5.** A particular case of Corollary 4.4 is a classification of additive endo-operations of 
Morava $K$-theories $K(n)^*$. Note that additive endo-operations of $K_0$ were classified by Vishik 
([V19 Th. 6.8]) and are infinite linear combinations of special finite sums of Adams operations. 
In other words one could say that additive endo-operations of $K_0$ (and by essentially the same 
argument of $K(1)^*$) are generated by Adams operations.

Rather surprisingly, this is not the case for $K(n)^*$, $n > 1$. More precisely, one can show that for 
a sufficiently big $i$ the operation $\phi_i^{K(n)}$ constructed above can not be equal to a ‘convergent’ 
infinite $\mathbb{Z}_p$-linear combination of Adams operations. In Topology a similar fact about topological 
Morava $K$-theories with $F_p$-coefficients was obtained by Yagita in [Ya80, Remark on p. 437].

4.2. The construction of Chern classes.

**Proposition 4.6.** There exist operations $c_i : K(n)^i \rightarrow BP\{n\}^i$ for any $i \geq 1$ s.t.
the operation defined by the polynomial
\[ P \sum \text{op.cit.} \]
copying the argument of \( \tau \)

operation \( tr_i c_i \) is equal to \( c_i^{\text{CH}} \otimes id_{BP[n]} \).

**Proof.** We will construct operations \( c_i \) by induction, in the same way as it was done for operations to Chow groups in [S18a, Sec. 4.4]. This is possible as the Cartan formula restricted to the coefficient of the monomial \( t^i \) describes non-additivity of the operation \( c_i \) in terms of operations \( c_j \) where \( j < i \).

**Base of induction.** Note that operations \( c_i^{\text{CH}}, i: 1 \leq i \leq p^n - 1 \), are chosen in [S18a] to be generators of modules \( [K(n)^*], \text{CH}^i \otimes \mathbb{Z}(p) ]^{\text{prod}} \). Thus, by Prop. 4.3 there exist numbers \( a_i \in \mathbb{Z}_p^* \) for \( i: 1 \leq i \leq p^n - 1 \) s.t. \( tr_i(a_i \phi) = c_i^{\text{CH}} \).

**Induction step.** Assume that operations \( c_1, \ldots, c_{i-1} \) are constructed and satisfy properties of the proposition.

Similar to the case of Chern classes with values in Chow groups we define the operation \( c_i \) as a sum of a polynomial in \( c_1, \ldots, c_{i-1} \) and an additive operation. In the current situation this additive operation is taking values in \( \tau^* BP[n]^* \).

Denote by \( P_i \in \mathbb{Q}[c_1, \ldots, c_{i-1}] \) the coefficient of \( t^i \) in the formal power series \( \mu_i - (\log \mu_i)^{\text{cotor}} \) in \( t \). Let \( \mu_i = \max(0, -\nu_P(P_i)) \) where \( \nu_P \) is the minimal \( p \)-valuation of coefficients of \( P_i \).

Note that the topological filtration is multiplicative on \( BP[n]^* \) (Prop. 4.4.5). By definition of the operation defined by the polynomial \( P_i \) takes values in \( \tau^* BP[n]^* \).

It follows from Lemma 2.13 that \( tr_i P_i(c_1, \ldots, c_{i-1}) = c_i^{\text{CH}} \otimes \mathbb{Z}(p) \).

By Prop. 4.3 there exist \( a_i \in \mathbb{Z}_p^* \) such that \( tr_i(a_i \phi_i) = c_i^{\text{CH}} \).

**Claim.** There exist an additive operation \( \psi_i^{(r)} : K(n)^* \rightarrow \tau^* BP[n]^* \) such that 1) the operation \( tr_i \psi_i^{(r)} \) is a generator of additive operations to \( \text{CH}^i \otimes BP[n] \), 2) the operation \( p^{\mu_i - r} P_i(c_1, \ldots, c_{i-1}) + \psi_i^{(r)} \) acts integrally on products of projective spaces.

**Base of induction** (\( r = 0 \)). By definition of \( \mu_i \), the polynomial \( p^{\mu_i} P_i \) is integral, and one can choose \( \psi^{(0)} \) to be the operation \( \phi_i \) of Prop. 4.3

**Induction step** \( (r \rightarrow r + 1) \). Let \( p^{\mu_i + r} P_i(c_1, \ldots, c_{i-1}) + \psi_i^{(r)} \) be an operation acting integrally on products of projective spaces.

Note that the derivative of this operation is equal to \( p^r \partial tr_i P_i \), and \( \partial P_i \) is an integral polynomial in \( c_1, \ldots, c_i \) by [loc. cit.]. Thus, if \( r > 0 \), then this operation modulo \( p \) is an additive operation which takes values in \( \tau^* BP[n]^*/p \). Moreover, it is gradable as follows by copying the argument of [op.cit., Prop. 4.5.6]. Thus, by Corollary 4.3 (1) it is equal to a sum \( \sum_{s \geq 1}(b_s \phi_s \mod p) \) modulo \( p \), and one can choose \( \psi_i^{(r+1)} := \psi_i^{(r)} - p^r \sum_{s \geq 1} b_s \phi_s \). Note that \( tr_i \psi_i^{(r+1)} = tr_i \psi_i^{(r)} - p^r b_i tr_i \phi_i \), and if \( r > 0 \), then by induction \( tr_i \psi_i^{(r+1)} \) is a generator of additive operations. In the induction step \( r = 0 \rightarrow r = 1 \) similar arguments apply as in [S18a, p. 33, Lemma 4.4.1] follows from Lemma 4.4.2] to show that \( tr_i \psi_i^{(1)} \) is a generator of additive operations.

Having finished the induction we define \( c_i \) as \( P_i + \psi_i^{(r)} \) which satisfied demanded properties.

Thus, we have constructed Chern classes from \( K(n)^* \) to the universal \( p^n \)-typical theory \( BP[n]^* \). As already mentioned we define then Chern classes \( c_i^A \) from \( K(n)^* \) to any \( p^n \)-typical theory \( A^* \) as compositions of \( c_i \) \( BP[n]^* \) with the unique morphism of theories \( \pi_A : BP[n]^* \rightarrow A^* \). Operations \( c_i^A \) satisfy the Cartan-type formula since any morphism of theories is a multiplicative map. To finish the construction we need to check that \( tr_i c_i^A = c_i^{\text{CH}} \).
**Proposition 4.7.** Operations $c_i^A$ take values in $\tau^i A^*$, and $tr_i c_i^A = c_i^\mathbb{H} \otimes id_A$.

**Proof.** Every morphism of theories preserves the topological filtration. In particular, $\pi_A$ maps $\tau^i BP[n]^*$ to $\tau^i A^*$.

On products of projective spaces $\pi_A$ sends $b z_1^{BP(n)} \cdots z_i^{BP(n)}$ to $\pi_A(b) z_1^A \cdots z_i^A$, and thus by the construction of the truncation map on products of projective spaces we have $tr_i(\pi_A \circ c_i^{BP(n)}) = \pi_A(tr_i c_i^{BP(n)})$. However, $tr_i c_i^{BP(n)}$ takes values in $CH^i \otimes \mathbb{Z}(p) \subset CH^i \otimes BP(n)$, i.e. coefficients of the corresponding polynomials are integral. Being a multiplicative map, $\pi_A$ acts on these coefficients canonically (if not to say trivially).

4.3. **Chern classes generate all operations.** The following proposition finishes the proof of Theorem 4.16.

**Proposition 4.8.** Let $A^*$ be a $p^n$-typical theory s.t. $A$ is a free $\mathbb{Z}(p)$-module. Then all operations from $K(n)^*$ to $A^*$ are uniquely expressible as series in Chern classes: $[K(n)^*, A^*] = A[[c_1, \ldots, c_i, \ldots]]$.

Moreover, the analogous statement is true for poly-operations.

**Proof.** If $A$ is a free $\mathbb{Z}(p)$-module, then presheaf of abelian groups $CH^* \otimes A$ is isomorphic to a direct sum of $CH^* \otimes \mathbb{Z}(p)$. Thus, $A$-module of operations $[K(n)^*, CH^* \otimes A]$ is isomorphic to $A \otimes (K(n)^*, CH^* \otimes \mathbb{Z}(p))$.

The $\mathbb{Z}(p)$-algebra $[K(n)^*, CH^* \otimes \mathbb{Z}(p)]$ is freely generated by Chern classes $c_i^CH$ as was shown in [S18] Th. 4.2.1, and in Section 4.2 the lifts of these operations with respect to the truncation map were constructed. By Proposition 4.16 the claim now follows.

A similar proof works for poly-operations based on the same statement [S18] Th. 4.2.1 for the classification of poly-operations to Chow groups.

5. **On the uniqueness of Morava K-theories**

In this section we construct an additive isomorphism between every two $n$-th Morava K-theories. In order to do this recall the classification of additive endooperations in $K(n)^*$ from Corollary 4.4.

There exist additive operations $\phi_i^{K(n)} : K(n)^i \rightarrow \tau^i K(n)^i$ for each $i \geq 0$, such that every additive endo-operations in $\tilde{K}(n)^*$ can be uniquely represented as an infinite sum of operations $\phi_i$ with $\mathbb{Z}(p)$-coefficients. We can take $\phi_0$ to be a projection to the canonical summand $\mathbb{Z}(p)$ of $K(n)^*$ (in other words, $\phi_0$ is an analogue of virtual rank for K-theory).

**Proposition 5.1.** An additive endo-operation $\phi := \sum_{i \geq 0} a_i \phi_i^{K(n)}$ is an isomorphism if and only if $a_i \in \mathbb{Z}(p)$ for $0 \leq i \leq p^n - 1$.

**Proof.** Since any additive operation preserves the topological filtration, operations $\phi_i$, $i \geq 1$ are supported on $\tilde{K}(n)^* = \tau^i K(n)^*$. It is clear then, that the operation $\phi$ is an isomorphism if and only if the operation $\sum_{i \geq 1} a_i \phi_i^{K(n)}$ is an isomorphism on $\tilde{K}(n)^*$ and $a_0 \in \mathbb{Z}(p)$.

To prove the proposition it is useful to understand the 'coordinates' of operations $(\phi_i^{K(n)})^2$ in the 'basis' $\phi_i^{K(n)}$.

**Lemma 5.2.** For each $i \geq 1$ there exist numbers $\beta_i \in \mathbb{Z}(p)$ s.t. for some $b^i_k \in \mathbb{Z}(p), k > i$ the following equation holds:

$$\phi_i^{K(n)} c^2 = \beta_i \phi_i^{K(n)} + \sum_{k > i} b^i_k \phi_k^{K(n)}.$$  

Moreover, $\beta_i \in \mathbb{Z}(p)$ for $i < p^n$, and $\beta_i \in p\mathbb{Z}(p) \setminus \{0\}$ for $i \geq p^n$.

**Proof of the Lemma.** An endo-operation $(\phi_i^{K(n)})^2$ takes values in $\tau^i K(n)^*$ and therefore is represented by an infinite sum of operations $\phi_k, k \geq i$, with $\mathbb{Z}(p)$-coefficients. Denote the coefficients in this sum as in (8).
Denote by \( \eta_i \in \mathbb{Z}_{(p)} \) the coefficient of the monomial \( z_1 \cdots z_i \) in the series \( G_i \) corresponding to the operation \( \phi_i \). We claim that \( \eta_i = \beta_i \). Indeed, the series \( G_i \) of the composition \( \phi_i^2 \) has the coefficient \( \eta_i^2 \). Since for operations \( \phi_j \), \( j > i \), the corresponding coefficient is equal to 0, we have \( \eta_i^2 = \beta_i \eta_i \). Therefore either \( \eta_i = 0 \) or \( \beta_i = \eta_i \).

By the construction the \( i \)-th truncation of \( \phi_i \) has the polynomial \( G_i \) equal to \( \eta_i z_1 \cdots z_i \). Therefore, in order to prove the lemma we need to show that for a generator of additive operations from \( K(n)^* \) to \( \text{CH}^i \otimes \mathbb{Z}_{(p)} \) the coefficient \( \eta_i \neq 0 \) for all \( i \geq 1 \), \( \eta_i \in \mathbb{Z}_{(p)} \) for \( i \leq p^n - 1 \), and \( \eta_i \in p\mathbb{Z}_{(p)} \) for \( i \geq p^n \).

An additive operation \( ch_i : K(n)^* \to \text{CH}^i \otimes \mathbb{Q} \) has the polynomial \( G_i \) equal to \( z_1 \cdots z_i \). However, the vector space over \( \mathbb{Q} \) of additive operations \([K(n)^*, \text{CH}^i \otimes \mathbb{Q}]^{\text{add}}\) is 1-dimensional by [S18a, Prop. 3.6], and therefore a generator of integral additive operations is proportional to \( ch_i \). In particular, \( \eta_i \neq 0 \).

One checks by a direct computation that \( ch_i \) acts integrally on products of projective spaces for \( i \leq p^n - 1 \), and therefore \( \eta_i \in \mathbb{Z}_{(p)} \) for \( i \leq p^n - 1 \).

On the other hand \( ch_i(z) = \log_{K(n)}^1(z) = z - \frac{a_i}{p} z^{p^n} + \ldots \) for some \( a_i \in \mathbb{Z}_{(p)}^* \), and the polynomial \( G_{1-p^n-1} \) of the operation \( ch_i \) is equal to \( -\frac{a_i}{p} \sum_{j=1}^{p^n-1} z_1 \cdots z^{p^n} z_1 \cdots z_{1-p^n-1} \) which is not integral. Thus, a generator of additive operations to \( \text{CH}^i \otimes \mathbb{Z}_{(p)} \) is proportional to \( ch_i \) with the coefficient in \( p\mathbb{Z}_{(p)} \), and \( \eta_i \in p\mathbb{Z}_{(p)} \).

Let \( \phi := \sum_{i=1}^{\infty} a_i \phi_i^{K(n)} \) be an additive endo-operation of \( \tilde{K}(n)^* \) which is an isomorphism. The inverse of \( \phi \) is also an operation, denote it by \( \psi \).

The series \( G_i \) of the operation \( \phi \) for \( i : 1 \leq i \leq p^n - 1 \) equals to \( a_i \beta_i z_1 \cdots z_i + \) higher degree terms, where \( \beta_i \) is defined in Lemma 5.2. Indeed, operations \( \phi_j^{K(n)} \) do not contribute to the series \( G_i \) if \( j > i \), since they take values in higher part of the topological filtration. Also, operation \( \phi_j^{K(n)} \) is supported on \( K(n)^i \mod p^n-1 \), which means that series \( G_i \) corresponding to it are zero for \( r \neq j \mod p^n - 1 \). Therefore only \( \phi_i^{K(n)} \) contributes to the starting monomial of the series \( G_i \) of the operation \( \phi \).

Suppose that the series \( G_i \) of the operation \( \psi \) is equal to \( \alpha_i z_1 \cdots z_i + \) higher degree terms for some number \( \alpha_i \in \mathbb{Z}_{(p)} \). Therefore the series \( G_i \), \( i \leq p^n - 1 \) of the composition \( \psi \circ \phi = \text{id}_{K(n)} \) starts with \( \alpha_i \beta_i z_1 \cdots z_i \), and on the other hand is equal to \( z_1 \cdots z_i \), since \( \psi \circ \phi = \text{id} \). Thus, we may conclude that \( \alpha_i \in \mathbb{Z}_{(p)}^* \), and one part of the proposition is proved.

Now assume that \( a_i \in \mathbb{Z}_{(p)}^* \) for \( i \leq p^n - 1 \) and let us construct the inverse of the operation \( \phi \).

We do this by induction approximating the inverse of \( \phi \) modulo parts of the topological filtration.

**Claim.** There exists an operation \( \psi_k : \tilde{K}(n)^* \to K(n)^* \) s.t. \( \psi_k \circ \phi \circ \text{id} \) takes values in \( \tau^k K(n)^* \).

**Base of induction** \((k = p^n)\).

Define the operation \( \psi_{p^n} := \sum_{i=1}^{p^n-1} \frac{1}{a_i} \beta_i \phi_i \). Considerations as above show that for the composition \( \phi \circ \psi_{p^n} \) the series \( G_i \) starts with \( a_i \beta_i (a_i \beta_i^2)^{-1} z_1 \cdots z_i = z_1 \cdots z_i \) for \( i \leq p^n - 1 \). On the other hand series \( G_i \) are gradable for every additive operation (Prop. 1.1), and for the operation \( \phi \circ \psi_{p^n} \circ \text{id} \) the series \( G_i \) starts with monomials of degrees higher than \( i \) for \( i \leq p^n - 1 \), and therefore by monomials of degree higher or equal to \( p^n \). Thus, by Proposition 2.2 the operation \( \phi \circ \psi_{p^n} \circ \text{id} \) takes values in \( \tau^{p^n} K(n)^* \).

**Induction step** \((k \rightarrow k + 1, \frac{1}{p} \rightarrow \frac{1}{p^2})\).

By the induction assumption we have \( \psi_k \circ \phi \circ \text{id} = \sum_{i \geq k} \alpha_i \phi_i^{K(n)} \) for some \( \alpha_i \in \mathbb{Z}_{(p)} \). Let us find \( x \in \mathbb{Z}_{(p)} \) such that the operation \( \psi_{k+1} = (\text{id} + x \phi_k^{K(n)}) \circ \psi_k \) is the next approximation of the inverse of \( \phi \), i.e. \( \psi_{k+1} \circ \phi \circ \text{id} \) takes values in \( \tau^{k+1} K(n)^* \).

Using Lemma 5.2 we have
\[
\psi_{k+1} \circ \phi = (\text{id} + x \phi_k^{K(n)}) \circ \psi_k \circ \phi = \text{id} + x \phi_k^{K(n)} + \alpha_k \phi_k^{K(n)} + x \alpha_k \beta_k \phi_k^{K(n)} + \sum_{i \geq k} \delta_i \phi_i^{K(n)},
\]
where \( \delta_i \in \mathbb{Z}_{(p)} \). Therefore setting \( x = -\frac{\alpha_i}{1 + \alpha_k \beta_k} \) the claim is obtained. Note that \( x \in \mathbb{Z}_{(p)} \) since \( \beta_k \in p\mathbb{Z}_{(p)} \) for \( k \geq p^n \) by Lemma 5.2.
The infinite induction process 'converges' as infinite linear combinations of operations \( \phi_i \) are well-defined. Thus, we have constructed the left inverse of \( \phi \), however, the same argument applies to construct the right inverse of \( \phi \), and therefore \( \phi \) is an isomorphism.

**Theorem 5.3.** Let \( K(n)^* \), \( \overline{K}(n)^* \) be two \( n \)-th Morava K-theories over \( \mathbb{Z}(p) \).

Then there exist an isomorphism of presheaves of abelian groups \( K(n)^* \stackrel{\sim}{\rightarrow} \overline{K}(n)^* \).

**Proof.** In what follows we prove the isomorphism between \( \overline{K}(n)^* \) and \( \overline{K}(n)^* \), however, we omit ~ in what follows for the clarity of reading. Clearly, this implies the statement of the theorem.

By Corollary 4.4 for \( i \geq 1 \) there exist additive operations \( \varphi_i := \phi_i^{K(n)^* \rightarrow \overline{K}(n)^*} : K(n)^i \rightarrow \overline{K}(n)^i \) and \( \psi_i := \phi_i^{\overline{K}(n)^* \rightarrow K(n)^*} : K(n)^i \rightarrow K(n)^i \) such that \( \varphi_i, \psi_i \) take values in the \( i \)-th part of the topological filtration and their truncations are generators of additive operations to \( \text{CH}^i \otimes \mathbb{Z}(p) \).

Denote \( \varphi = \sum_{i=1}^{p^n-1} \varphi_i \) and \( \psi = \sum_{i=1}^{p^n-1} \psi_i \). We will show now that compositions of these operations are isomorphisms of \( K(n)^* \) and \( K(n)^* \), respectively. It would follow that both these operations are isomorphisms. Clearly, it is enough to consider only one composition by symmetry of the argument.

Let us show that for \( i : 1 \leq i \leq p^n - 1 \) the series \( G_i \) of operations \( \varphi \) and \( \psi \) starts with \( a_i z_1 \ldots z_i \) and \( b_i z_1 \ldots z_i \), respectively, where \( a_i, b_i \in \mathbb{Z}(p) \). Let us consider only the operation \( \varphi \). Since operations \( \varphi_i \) are supported on \( K(n)^i \), the series \( G_i \) of \( \varphi \) for \( 1 \leq i \leq p^n - 1 \) equals to the series \( G_i \) of \( \varphi_i \). The truncation of \( \varphi_i \) is a generator of additive operations from \( K(n)^i \rightarrow \text{CH}^i \otimes \mathbb{Z}(p) \), and its corresponding polynomial \( G_i \) is exactly \( a_i z_1 \ldots z_i \). However, \( c_{i} \) is an integral generator of additive operations to \( \text{CH}^i \otimes \mathbb{Z}(p) \) for \( i \leq p^n - 1 \), its polynomial \( G_i \) is equal to \( z_1 \cdots z_i \). The operation \( \text{tr} \varphi_i \) has to be proportional to \( c_{i} \) by a number in \( \mathbb{Z}(p)^* \), and therefore \( a_i \in \mathbb{Z}(p)^* \).

We can represent \( \psi \circ \varphi \) as a sum \( \sum_{i=1}^{p^n-1} c_i \varphi_i \) for some \( c_i \in \mathbb{Z}(p) \). The series \( G_i \) of this operation starts with \( a_i b_i z_1 \ldots z_i \) for \( 1 \leq i \leq p^n \). However, calculating the series \( G_i \) using the composition of corresponding series of \( \psi \) and \( \phi \), one obtains that it starts with \( a_i b_i z_1 \ldots z_i \). Thus, \( a_i = b_i \in \mathbb{Z}(p) \) for \( i : 1 \leq i \leq p^n \), and it follows by Proposition 5.1 that \( \psi \circ \varphi \) is an isomorphism.

**Remark 5.4.** Not all of \( n \)-th Morava K-theories are multiplicatively isomorphic, see Appendix A.2. Note, however, that every two formal group laws over \( \mathbb{F}_p \) are isomorphic by a theorem of Lazard, and therefore there exist a multiplicative isomorphism between formal theories \( K(n)^* \otimes \mathbb{F}_p \) and \( K(n)^* \otimes \mathbb{F}_p \) for every two \( n \)-th Morava K-theories \( K(n)^*, \overline{K}(n)^* \).

**Remark 5.5.** Theorem 5.3 suggests that there is a uniquely defined motivic spectrum representing \( n \)-th Morava K-theory. Since there exist many different multiplicative structures on the \( n \)-th Morava K-theory \( K(n)^* \), it is also possible that this hypothetical unique spectrum admits several multiplicative structures, or admits none of them. These questions are open to the author's knowledge.

6. **The Gamma Filtration on Morava K-theories**

In this section we define and investigate properties of a functorial filtration on \( K(n)^*(X) \) for all \( n \geq 1 \) which we call the gamma filtration. The definition of this filtration is verbatim the definition of the gamma filtration on \( K_0 \) as defined with Chern classes.

The gamma filtration on \( K(n)^* \) has the universal property as the best approximation to the topological filtration which is defined by values of poly-operations (Prop. 5.8). Thus, even though Chern classes which we have constructed in Theorem 5.1 are not unique, the gamma filtration does not depend on their choice and is strictly compatible with additive isomorphisms between \( n \)-th Morava K-theories (Prop. 5.9). In other words, the gamma filtration is unique on the presheaf of abelian groups \( K(n)^* \). As expected the gamma filtration on \( K(1)^* \) coincides with the gamma filtration on \( K_0 \otimes \mathbb{Z}(p) \) under an additive isomorphism between them (Prop. 5.10).

An important difference between properties of gamma filtrations on \( K_0 \) and \( K(n)^* \), \( n \geq 2 \), is the fact that Chern classes \( c_i^{K(n)^* \rightarrow \text{CH}} \) are additive surjective homomorphisms from the \( i \)-th graded piece \( \gamma^i K(n)^* \gamma^{i+1} K(n)^* \rightarrow \text{CH}^i \otimes \mathbb{Z}(p) \) for \( i \leq p^n \) (in the case of \( K_0 \) we have surjectivity only for \( i \leq p \)), see Prop. 5.2. This yields a combinatorial tool for the study of \( p \)-torsion in Chow groups for
geometrically cellular varieties $X$ such that the pullback morphism $K(n)^*(X) \to K(n)^*(X \times_k \bar{k})$ is an isomorphism. Pfister quadric is a well-known example of a variety satisfying this property, and we show in Section 6.1 how the calculations work out in this case. New examples of varieties satisfying the property above are studied in [ScSe18].

6.1. Definitions and properties. In what follows we denote by $c_i, c_i^{CH}$ Chern classes from $K(n)^*$ to itself and to Chow groups, respectively, which were constructed in Theorem 3.10.

Definition 6.1. Define the gamma filtration on $K(n)^*$ of a smooth variety by the following formulas:

$$\gamma^0 K(n)^*(X) = K(n)^*(X),$$
$$\gamma^m K(n)^*(X) := \langle c_{i_1}(\alpha_1) \cdots c_{i_k}(\alpha_k) \rangle \sum_{j} i_j \geq m, \alpha_j \in K(n)^*(X) >, \quad m \geq 1,$$

where $\langle, \rangle$-brackets denote generation as $\mathbb{Z}(p)$-modules.

It is clear from the definition that $\gamma^m K(n)^*$ is an ideal subsheaf of $K(n)^*$ for all $m \geq 0$.

Proposition 6.2. The gamma and the topological filtrations on $K(n)^*$ satisfy the following properties:

i) $gr^i_i K(n)^* = gr^i_i K(n)^i \mod p^{a+1}$;

ii) $\gamma^i \subseteq \tau^i$;

iii) $c_i^{CH}$ is zero on $\tau^{i+1} K(n)^*$;

iv) the operation $c_i^{CH}$ is additive when restricted to $\tau^i K(n)^*$;

v) the additive map $c_i^{CH} : gr^j_i K(n)^* \to CH^i \otimes \mathbb{Z}(p)$

1) is an isomorphism after tensoring with $\mathbb{Q}$;

2) is an isomorphism for $1 \leq i \leq p^a$ and $p K(n)^*$ is its inverse;

vi) the additive map $c_i^{CH} : gr^j_i K(n)^* \to CH^i \otimes \mathbb{Z}(p)$

1) is an isomorphism after tensoring with $\mathbb{Q}$;

2) is an isomorphism for $i = 1$ and surjective for $1 \leq i \leq p^a$;

vii) $\gamma^i \otimes \mathbb{Q} = \tau^i \otimes \mathbb{Q}$.

Proof. \footnote{Recall that the Chern class $c_i : K(n)^* \to BP\{n\}^i$ takes values in the $i$-th graded part, and as the classifying morphism of theories from $BP\{n\}^* \to K(n)^*$ respects the grading, the $i$-th Chern class from $K(n)^*$ to itself takes values in $K(n)^i$.}
The space of all polynomials in Chern classes is split into $p^{a_1} - 1$ summands by their degree modulo $p^{a_1}$. It is clear that the filtration by the degree of polynomials jumps on each summand every $p^{a_1} - 1$ steps, and thus the same is true for the gamma filtration. In particular, $gr^j_i K(n)^{i} = 0$ if $j \neq i \mod p^{a_1} - 1$.

\footnote{\textcolor{red}{The}} The Chern class $c_i$ takes values in $\tau^i K(n)^*$ by Theorem 3.10\footnote{\textcolor{red}{2}}. Since the topological filtration is multiplicative (Prop. 1.17), this property follows.

\footnote{\textcolor{blue}{III}} Follows from the fact that any operation (which preserves 0) preserves the topological filtration, and that $\tau^{i+1} CH^i = 0$.

\footnote{\textcolor{red}{3}} By the Cartan’s formula the internal derivative of the operation $c_i$ is expressible as a polynomial in operations $c_j, j < i$. However, these operations vanish on $\tau^i$ by the property \footnote{\textcolor{blue}{III}}, and therefore $c_i$ is additive.

\footnote{\textcolor{red}{4}} We will need the following Lemmata.

Lemma 6.3. Let $i \geq 1$. Denote by $a_i \in \mathbb{Z}(p)$ the coefficient of the monomial $z_1 \cdots z_i$ of $c_i(z_1 \cdots z_i)$ in the notation of Section 6.4.

Then the following diagram commutes:

$$\begin{array}{ccc}
CH^i \otimes \mathbb{Z}(p) & \xrightarrow{\partial_i} & CH^i \otimes \mathbb{Z}(p) \\
\downarrow & & \downarrow \\
gr^j_i K(n)^* & \xrightarrow{c_i} & gr^j_i K(n)^*
\end{array}$$

Moreover, $a_i \in \mathbb{Z}^\times(p)$ for $1 \leq i \leq p^a$, and $a_i \neq 0$ for all $i \geq 1$. 

Proof of the Lemma. This is a version of Proposition 2.12 with the claim that the corresponding operation on Chow groups is the multiplication by \( a_i \). The operation \( \text{gr}^i K(n)^* \xrightarrow{\sim} \text{gr}^i K(n)^* \) is additive by \( \text{iv} \), and therefore the corresponding operation on \( \text{CH}^i \otimes \mathbb{Z}(p) \) is also additive. By Vishik’s Theorem 1.9 an additive operation \( \text{CH}^i \otimes \mathbb{Z}(p) \to \text{CH}^i \otimes \mathbb{Z}(p) \) is defined by the additive map \( G_i : \mathbb{Z}(p) \to \mathbb{Z}(p) : z_1 \cdots z_i \), i.e. it is a multiplication by the coefficient of the monomial \( z_1 \cdots z_i \) in \( G_i(1) \). A computation on products of projective spaces shows that this is the coefficient \( a_i \) as specified in the Lemma.

Note that the coefficient of the monomial \( z_1 \cdots z_i \) in \( c_i(z_1 \cdots z_i) \) is the same as the coefficient of the monomial \( z_1^{CH} \cdots z_i^{CH} \) for \( (\tau_1 c_i = c_i^{CH})(z_1 \cdots z_i) \) by the construction of the truncation. Thus, we may work with Chern classes to Chow groups instead. For \( i : 1 \leq i \leq p^n - 1 \) we have \( a_i \in \mathbb{Z}_p^\times \) because in this range Chern classes are additive and \( c_i^{CH} \) can be chosen as \( ch_i \) as an operation to \( \text{CH}^i \otimes \mathbb{Q} \) (this was already discussed in the proof of Lemma 5.2). For \( i = p^n \) the construction of Chern classes yields \( c_i^{CH} = (c_i^{CH})^p + \phi_{p^n} \) and one checks that a generator of integral additive operations \( \phi_{p^n} \) is equal to \( a_{p^n} p \cdot \text{ch}_{p^n} \) where \( a_{p^n} \in \mathbb{Z}_p^\times \) (see e.g. Appendix A.9). Since \((c_i^{CH})^p\) is zero on \( z_1 \cdots z_{p^n} \) we see that this number \( a_{p^n} \) is the coefficient we are looking for.

The last claim is that \( a_i \neq 0 \) for every \( i \geq 1 \). As above we can consider the operation \( c_i^{CH} \). By [S18a, Lemma 4.4.1] the following relation \( c_i^{CH} = P(1^{CH}, \ldots, 1^{CH}) + \sum_{j < i} \) holds for the action of operations on projective spaces where \( P_i \) is a certain polynomial, \( \phi_i \) is a generator of integral additive operations \( K(n)^* \to \text{CH}^i \otimes \mathbb{Z}(p) \). Chern classes \( c_i^{CH} \) send \( z_1 \cdots z_i \) to zero if \( j < i \) because of the continuity of operations, and therefore so does the polynomial \( P(1^{CH}, \ldots, 1^{CH}) \). Thus, it is enough to show that the operation \( \phi_i \) sends \( z_1 \cdots z_i \) to \( a_i p^{a_i} z_1 \cdots z_i \) where \( a_i \neq 0 \). However, the space of additive operations from \( K(n)^* \) to \( \text{CH}^i \otimes \mathbb{Q} \) is 1-dimensional by [S18a, Prop. 3.3.1], and therefore \( \phi_i \) is rationally proportional to \( ch_i \) with a non-zero coefficient, and \( ch_i \) sends \( z_1 \cdots z_i \) to \( a_i p^{a_i} z_1 \cdots z_i \). This proves the claim. 

Lemma 6.4. The composition \( \text{CH}^i \otimes \mathbb{Z}(p) \to \text{gr}^i K(n)^* \xrightarrow{\sim} \text{CH}^i \otimes \mathbb{Z}(p) \) is multiplication by \( a_i \) (the same number as in Lemma 6.3).

Proof of the Lemma. The map \( \text{CH}^i \otimes \mathbb{Z}(p) \to \text{gr}^i K(n)^* \) is a surjective additive operation by Lemma 1.18 and thus the composition is an additive operation. Calculating the composition on the element \( z_1 \cdots z_i \in \text{CH}^i(\mathbb{P}^\infty)^{\times i} \) we obtain the result.

\[ \text{VIII} \] \[ \text{IX} \] now follow from Lemmata 6.3 and 6.4. \[ \text{V} \] We will first prove the surjectivity statement \[ \text{VII} \]. The statement \[ \text{VI} \] will follow from \[ \text{VII} \] and \[ \text{VIII} \].

Lemma 6.5. The composition
\[
\text{CH}^i \otimes \mathbb{Z}(p) \to \text{gr}^i K(n)^* \xrightarrow{\sim} \text{gr}^i K(n)^* \xrightarrow{c_i^{CH}} \text{CH}^i \otimes \mathbb{Z}(p)
\]
is multiplication by \( a_i^2 \) (the number \( a_i \) as in Lemma 6.3).

Proof of the Lemma. The composition is an additive operation, and it is enough to consider its value on the element \( z_1^{CH} \cdots z_i^{CH} \) of \( \text{CH}^i(\mathbb{P}^\infty)^{\times i} \otimes \mathbb{Z}(p) \) (see Section 1.1 for the notation). The morphism of the theories \( \rho_{K(n)} \) sends it to \( z_1 \cdots z_i \), the operation \( c_i \) sends this elements to \( a_i z_1 \cdots z_i \) as in the proof of Lemma 6.3. Similarly, \( c_i^{CH} \) sends \( z_1 \cdots z_i \) to \( a_i z_1 \cdots z_i \), and, thus, sends \( a_i z_1 \cdots z_i \) to \( a_i^2 z_1 \cdots z_i \).

By Lemma 6.3 if \( a_i \in \mathbb{Z}_p^\times \), then the map \( \text{gr}^i K(n)^* \xrightarrow{\sim} \text{gr}^i K(n)^* \) is an isomorphism. In particular, this is true for \( i \leq p^n \) by Lemma 6.3. The image of this map lies in \( \gamma^i/\tau^i+1 K(n)^i \subset \text{gr}^i K(n)^i \), and thus we see that \( \tau^i = \gamma^i \) on \( K(n)^i \) in this range. Together with \[ \text{VII} \] this implies \[ \text{VII} \].

\[ \text{VIII} \] The inclusion \( \gamma Q \subset \tau Q \) is known by \[ \text{II} \]. Note that the map \( \text{CH}^i \otimes \mathbb{Q} \to \text{gr}^i K(n)^* \otimes \mathbb{Q} \) is an isomorphism, and it follows from Lemma 6.3 and \[ \text{II} \] that \( c_i \) acts a multiplication by a non-zero...
number \(a_i\) on \(gr_{\tau}^i K(n)^* \otimes \mathbb{Q}\). In particular, if \(x \in \tau^i K(n)^*(X)_{\mathbb{Q}}\) for a smooth variety \(X\), then \(a_i x - c_i(x) \in \tau^{i+1} K(n)^*(X)_{\mathbb{Q}}\).

For a variety \(X\) of dimension \(d\) we prove by a decreasing induction that \(\gamma^i K(n)^* \otimes \mathbb{Q} = \tau^i K(n)^* \otimes \mathbb{Q}\).

**Base of induction.** The equality holds for \(i = d + 1\), since \(\tau^{d+1} K(n)^*(X) = 0\) for dimensional reasons, and polynomials of degree \(\geq 1\) in Chern classes \(c^*_i BP[n]\) are zero because \(BP[n] \geq d+1\) is zero and therefore \(\gamma^{d+1} K(n)^*(X) = 0\).

**Induction step.** Let \(x \in \tau^i K(n)^*(X)_{\mathbb{Q}}\) and define \(y_i(x) = a_i x - c_i(x) \in \tau^{i+1} K(n)^*(X)_{\mathbb{Q}}\). The element \(x + \frac{1}{a_i} y_i(x)\) lies in \(\gamma^i K(n)^*(X)_{\mathbb{Q}}\), therefore it is enough to show that \(y_i(x) \in \gamma^i K(n)^*(X)_{\mathbb{Q}}\). However, we know that \(y_i(x) \in \tau^{i+1} K(n)^*(X)_{\mathbb{Q}}\) and by induction assumption we know that \(y_i(x) \in \gamma^{i+1} K(n)^*(X)_{\mathbb{Q}}\). Thus, \(x \in \gamma^i K(n)^*(X)_{\mathbb{Q}}\).

6.2. **An application:** \(K(n)\)-motives of Tate type. For the following corollary we need the notion of \(A^*\)-motive where \(A^*\) is an oriented theory. This is an analogue of Grothendieck pure motives with Chow groups replaced by \(A^*\). We refer the reader to \([\text{Ma68, NZ06}]\) for details. Note that we call \(A^*\)-motive \(M\) Tate if it is a direct sum of powers of the Lefschetz motive in the terminology of \([\text{NZ06}]\). For a smooth projective variety \(X\) we denote by \(M_A(X)\) its \(A^*\)-motive. Note that by \([\text{YY07}]\) there is a canonical map from the set of irreducible objects of the category of Chow motives to the set of objects of the category of \(A^*\)-motives for any oriented cohomology theory \(A^*\), i.e. there is a way to “lift” an irreducible Chow motive to \(A^*\)-motive, but it may further decompose then.

**Corollary 6.6.** Let \(X\) be a smooth projective variety such that \(\dim X \leq p^n - 2\), then

\[
\text{End } M_{K(n)}(X) = \text{End } M_{\text{CH} \otimes \mathbb{Z}(p)}(X),
\]

and therefore the decompositions of motives \(M_{K(n)}(X)\), \(M_{\text{CH}}(X)\) are the same, i.e. the canonical lift of irreducible summands of \(M_{\text{CH} \otimes \mathbb{Z}(p)}(X)\) stay irreducible in \(K(n)^*\)-motives.

Moreover, \(K(n)^*\)-motive of \(X\) is Tate if and only if \(CH^* \otimes \mathbb{Z}(p)\)-motive of \(X\) is Tate.

**Proof.** The decomposition of a \(A^*\)-motive is determined by the structure of the algebra \(A^{\dim X}(X \times X)\) where the multiplication is defined by the composition of correspondences. In particular, a \(A^*\)-motive of \(X\) is Tate if and only if the unit of this algebra is a sum of pairwise orthogonal projectors with free images of rank 1.

Note that the topological filtration restricted to a graded component of \((n)^*\) changes every \(p^n - 1\) steps (Prop. \([3, 15]\), e.g. \(\tau^2 K(n)^1 = \tau^3 K(n)^1 = \ldots = \tau^{p^n} K(n)^1\)). Moreover, \(\tau^1 K(n)^*\) splits by the (CONST) property of free theories. Thus, for a variety \(X\) of dimension less or equal to \(p^n - 1\) we have \(gr_{\tau}^{\dim X} K(n)^*(X \times X) = K(n)^{\dim X}(X \times X)\).

The canonical morphism \(\rho_{K(n)} : CH^{\dim X}(X \times X) \otimes \mathbb{Z}(p) \to gr_{\tau}^{\dim X} K(n)^*(X \times X)\) is an isomorphism of \(\mathbb{Z}(p)\)-modules by Proposition \([6, 2, 22]\). Since \(\rho_{K(n)}\) is a morphism of theories by Lemma \([1, 18]\) i.e. commutes with pull-backs, push-forwards and multiplication, it follows that \(\rho_{K(n)}\) is an isomorphism of algebras of correspondences. Note also that \(\rho_{K(n)}\) preserves the class of the diagonal.

Let \(A^*\) be a \(Z\) or \(\mathbb{Z}/m\)-graded oriented theory. Denote by \(\pi : X \to \text{Spec } k\) the structural morphism, and by \(p_1, p_2 : X \times X \to X\) the canonical projections. The property of \(A^*\)-motive of \(X\) to be Tate is equivalent to the existence of elements \(a_i, b_i \in A^*(X)\) for a set of indices \(i \in I\), such that

1. \(a_i \cdot b_i \in A^{\dim X}(X)\) for all \(i \in I\);
2. \(\pi_*(a_i \cdot b_i) = 1\) in \(A^0(\text{Spec } k)\) for all \(i \in I\);
3. \(\pi_*(a_i \cdot b_j) = 0\) for all \(i, j \in I, i \neq j\);
4. \(\Delta_A = \sum_{i,j} (p_1^*(a_i) \cdot p_2^*(b_j))\) in \(A^{\dim X}(X \times X)\), where \(\Delta_A\) is the class of the diagonal of \(X\).

If \(\dim X \leq p^n - 2\), then \(\rho_{K(n)} : CH^i(X) \to gr_{\tau}^i K(n)^i(X) = K(n)^i(X)\) is an isomorphism for all \(i\). Moreover, this isomorphism for \(i = \dim X\) agrees with the push-forwards to the point. Assume that \((n)^*\)-motive of \(X\) is Tate, and let \(\Delta_{K(n)} = \sum_{i \in I} (p_1^*(a_i) \cdot p_2^*(b_i))\) be the
decomposition of the diagonal as above. Then set \( \alpha_i := \rho_{K(n)}^{-1}(a_i), \beta_i := \rho_{K(n)}^{-1}(b_i) \), and we have \( \Delta_{CH} = \sum_{\ell=1}^r (p_1)(\alpha_i) \cdot (p_2)(\beta_i) \) yielding the needed decomposition of the Chow motive of \( X \).

The proof in the other direction can be done in the same way, or one could use the well-known corollary of Vishik and Yangita [VY07], which says that if the Chow motive of \( X \) is Tate, then the \( A^* \)-motive of \( X \) is Tate for every oriented cohomology theory \( A^* \).

**Remark 6.7.** If one assumes that \( X \) is a projective homogeneous variety (or, more generally, that the Chow motive of \( X \times_k k \) is Tate and Rost nilpotence holds for \( M_{CH} \otimes \mathbb{Z}(p)(X) \)), then one can improve the bounds in the previous corollary. Namely, under these assumptions if \( K(n)^* \)-motive of \( X \) is Tate and \( \dim X \leq p^n \), then the Chow motive of \( X \) is Tate, see [SeSc18, Cor. 7.12].

Corollary 6.9 shows that if one looks for smooth projective varieties such that its \( K(n)^* \)-motive is Tate, but Chow motive is not Tate, then one has to consider varieties of sufficiently big dimension. The search of such varieties is motivated by possible applications of the gamma filtration discussed at the end of Section 6.4.

### 6.3. Uniqueness of the gamma filtration.

The gamma filtration turns out to have another, yet similar definition which does not mention Chern classes explicitly. It shows that the gamma filtration is the best approximation of the topological filtration defined by values of polyoperations.

Recall that all internal poly-operations from \( K(n)^* \) to \( K(n)^* \) are freely generated by products of Chern classes (Th. 3.16). It is clear that a series in Chern classes takes values in \( \tau^1 K(n)^* \) where \( v \) is the minimal degree of non-trivial monomials in the series, \( \deg c_i = i \). Together with the classification of all poly-operations this gives a description of the set of internal poly-operations which take values in \( \tau^j K(n)^* \) for any \( j \geq 0 \).

**Proposition 6.8.** For \( m \geq 0 \) denote by \( P^m \) the disjoint union of internal poly-operations from \( (K(n)^*)^\times_k \) to \( \tau^m K(n)^* \) for all \( k \geq 1 \).

For any \( m \geq 0 \) we have

\[
\gamma^m K(n)^*(X) := \bigcup_{P \in P^m} \text{Im} P_{|(K(n)^*)^\times_k(X)}.
\]

In particular, the gamma filtration does not depend on the choice of Chern classes \( c_i \) (satisfying properties of Theorem 3.16).

**Proof.** We need only to check that if \( \phi \) is an internal poly-operation which takes values in \( \tau^1 K(n)^* \), then it is expressible as a series in external products of Chern classes with the minimal degree of a summand being \( i \). Assume the contrary, i.e. without loss of generality a polynomial \( P(c_1, \ldots, c_j) \) of degree \( j < i \) in external products of Chern classes takes values in \( \tau^j K(n)^* \).

Then the following diagram commutes (poly-operation version of Prop. 2.10):

\[
\begin{array}{ccc}
(K(n)^*)^\times_r & \overset{P}{\longrightarrow} & \tau^1 K(n)^* \\
\downarrow_{P(c_1^{CH}, \ldots, c_j^{CH})} & & \downarrow \\
CH^1 \otimes \mathbb{Z}(p) & \overset{\rho_{K(n)}}{\longrightarrow} & gr^1_{K(n)^*}
\end{array}
\]

However, \( P(c_1^{CH}, \ldots, c_j^{CH}) \) takes values in \( CH^{-r} \otimes \mathbb{Z}(p) \) by assumptions on degrees. Thus, in the diagram above the left down arrow is zero, and therefore \( P \) is zero as well.

**Proposition 6.9.** Let \( K(n)^*, \overline{K(n)^*} \) be two \( n \)-th Morava K-theories, and let \( \phi : K(n)^* \to \overline{K(n)^*} \) an additive isomorphism between them (which exists by Theorem 7.5).

Then \( \phi \) is strictly compatible with the gamma filtration.

**Proof.** It is enough to show that \( \phi \) respects gamma-filtration, since by symmetry it will follow that its inverse also respects it, and therefore both \( \phi \) and \( \phi^{-1} \) are strictly compatible with it.

Let \( P \) be a series of degree \( i \) in Chern classes defining an \( r \)-ary poly-operation from \( (K(n)^*)^\times_r \) to \( \tau^m K(n)^* \). Using the description of the gamma filtration in Prop. 6.8 it is enough to show that \( \phi \circ P = P \circ \phi^r \), where \( P \) is an internal poly-operation from \( (K(n)^*)^\times_r \) to \( \tau^r K(n)^* \).
Define $\bar{P}$ as the composition $\phi \circ P \circ \psi$ where $\psi$ is an inverse additive operation to $\phi$. The operation $\bar{P}$ takes values in $\tau^*\overline{K}(n)^*$ since $P$ does and $\phi$ preserves the topological filtration.

**Proposition 6.10.** Denote by $\theta : K_0 \otimes \mathbb{Z}(p) \to K(1)$ an invertible multiplicative operation defined by the Artin-Hasse exponential which gives an isomorphism of $K_0$ and a first Morava $K$-theory with the logarithm $\log_{K(1)} = \sum_{i=0}^{\infty} \frac{x^p}{p^i}$ (cf. [S1Sa, Rem. 4.14]). Denote by $\gamma^*K_0$ the classical gamma-filtration on $K_0$.

Then $\theta(\gamma^*K_0 \otimes \mathbb{Z}(p)) = F^1\overline{K}(n)$.

**Proof.** The proof is essentially the same as of Proposition 6.9 one needs only to note that the usual Chern classes $c_i^{K_0} : K_0 \to K_0$ take values in $\tau^*K_0$, and the classification of poly-operations in $K_0$ works integrally (Theorem 1.12).

---

**6.4. Computational constants of Chern classes for the Riemann-Roch formula.**

If we have a geometrically cellular variety $X$ such that the pull-back morphism $p^* : K(n)^*(X) \to K(n)^*(\overline{X})$ is an isomorphism, then $p^*$ induces an isomorphism of abelian groups $gr_i^\gamma K(n)^*(X) \cong gr_i^\gamma K(n)^*(\overline{X})$. Thus, we can compute the graded factors of the gamma filtration of $X$ using only the cellular variety $\overline{X}$. Since $gr_i^\gamma K(n)^*(X)$ maps surjectively to $CH^i(X) \otimes \mathbb{Z}(p)$ by Proposition $6.2$ the computation of the torsion in the abelian group $gr_i^\gamma K(n)^*(\overline{X})$ provides an estimate of the $p$-torsion in $CH^i(X)$ for such variety.

The first example of a variety $X$ as above was found by Voevodsky based on the results of Rost: it is a Pfister quadric corresponding to a pure symbol of degree $n + 2$ (see [Vo95]). Other examples are provided in [S1Sa, 12].

In order to calculate the gamma filtration of a cellular algebraic variety it is necessary to be able to compute values of Chern classes on the classes of subvarieties. The only tool to do this at the moment is the Riemann-Roch formula (Th. 4.13). Let us recall how it applies.

Let $\phi : A^* \to B^*$ be an operation, let $X$ be a smooth variety, and let $i : Z \hookrightarrow X$ be its smooth closed subvariety of codimension $c$. It follows from the Riemann-Roch formula that the value $\phi(i_*1_Z)$ is equal to $b \cdot 1_Z$ modulo $(c + 1)$-th part of the topological filtration, where $b \in B$ is the coefficient of $z_1^B \cdots z_c^B$ in the series $\phi(z_1^A \cdots z_c^A)$ (for the notation see Section 1.6). We compute these coefficients for some of the Chern classes in the following propositions. Partial calculations of the gamma filtration on split quadrics based on these numbers are performed in a forthcoming paper [SaSe13].

**Proposition 6.11.** Let $c_{p^n}$ be the operation from $K(n)^1 \to \tau^{p^n} K(n)^1$ constructed in Theorem 3.16.

Denote by $e_j$, $j \geq 0$, the coefficient of the monomial $z_1^{1+j(p^n-1)}$ in the series $c_{p^n}(z_1^c \cdots z_{1+j(p^n-1)}) \in K(n)^1((\mathbb{F}_p^\infty)^{x+1} \cdot z_j^{p^n-1})$.

Then for all $j \geq 1$ we have $e_j \in \mathbb{Z}(p)$. The proof of this and of the following proposition uses the next Lemma.

**Lemma 6.12.** Let $\phi : K(n)^1 \to K(n)^1$ be an additive operation, such that in the notation of Section 1.6

$$G_{1+i(p^n-1)}(z_1, \ldots, z_{1+i(p^n-1)}) = \alpha_i z_1 \cdots z_{1+i(p^n-1)} + \beta_i \sum_{k=1}^{1+i(p^n-1)} z_1 \cdots z_k^{p^n} \cdots z_{1+i(p^n-1)}$$

$$+ \delta_i \sum_{k,s=1, k<s} z_1 \cdots z_k^{p^n} \cdots z_s^{p^n} \cdots z_{1+i(p^n-1)} + \text{other terms,} \quad i \geq 0,$$

where $\alpha_i, \beta_i, \delta_i \in \mathbb{Z}(p)$.

Denote by $v_n \in \mathbb{Z}(p)$ the coefficient in the following equation: $p^*K(n)x = px + v_n x^{p^n} \equiv x^{p^n+1}$.

---

Note, however, that the results of Voevodsky are stated a little bit differently, even though the spirit of the result is clearly the same. Note also that he worked with conjectural at that moment big Morava $K$-theories and not their small parts directly as we do. See below another proof of this property of Morava $K$-theories of Pfister quadrics which follows from [YY07].
Then \( \alpha_{i+1} = \alpha_i - \frac{p^n - 1}{p^n - 1} \beta_i \) for \( i \geq 0 \), \( \beta_{i+1} = \beta_i + \frac{p^n - 1}{p^n - 1} \delta_i \) for \( i \geq 1 \).

Proof of the Lemma. The pull-back of the Veronese map \([p]\) of degree \( p \) on \( \mathbb{P}^\infty \) induces the multiplicative map \( z \to p \cdot K(n) \cdot z \) on \( K(n)^*(\mathbb{P}^\infty) \). The action of the operation \( \phi \) on products of projective spaces has to commute with pull-backs along maps \([p] \times \text{id}^{(p^n - 1)}\) for each \( i \geq 0 \) which gives a non-trivial relation on series \( G_i \). Explicitly, comparing the value of \(([p] \times \text{id}^{(p^n - 1)} \circ \phi)\) on \( z_1 z_2 \cdots z_{1+i(p^n-1)} \) we obtain the following. The series \( G_{1+i(p^n-1)}(p \cdot K(n) \cdot z_1, z_2, \ldots, z_{1+i(p^n-1)}) \) has to be equal to \( pG_{1+i(p^n-1)}(z_1, z_2, \ldots, z_{1+i(p^n-1)}) + v_n G_{1+i(p^n-1)}(z_1^{p^n}, z_2^{p^n}, \ldots, z_{1+i(p^n-1)}) \mod (z_1^{p^n}+1) \).

Calculating the coefficient of the monomial \( z_1^{p^n} z_2 \cdots z_{1+i(p^n-1)} \) in this equation we obtain: \( v_n \alpha_{i+1} + p\beta_i = v_n \alpha_i + p\beta_i p \delta_i \). Similarly calculating the coefficient of the monomial \( z_1^{p^n} z_2^{p^n} z_3 \cdots z_{1+i(p^n-1)} \) we obtain: \( v_n \beta_{i+1} + p\beta_i = v_n \beta_i + p^{n+1} \delta_i \). Note, however, that we have assumed in the last conclusion that \( 1 + i(p^n - 1) \geq 2 \), i.e. \( i \geq 1 \).

Proof of Proposition 6.11. The operation \( c_{p^n} : K(n)^1 \to \tau^{p^n} K(n)^1 \) as defined by the construction of Section 6.2 acts on products of projective spaces as \( ch_{p^n} \otimes \psi_{p^n} \) where \( \psi_{p^n} \) is an additive endooperation which takes values in \( \tau^{p^n} K(n)^1 \) and its truncation \( tr_{p^n} \psi_{p^n} \) is a generator of additive integral operations to \( \text{CH}^{p^n} \otimes \mathbb{Z}_p \).

As \( ch_p \) sends \( z_1 \cdots z_{1+i(p^n-1)} \) to \( p^n \)-th power of a series divisible by \( z_1 \cdots z_{1+i(p^n-1)} \) for any \( i \geq 1 \), the coefficient of \( z_1 \cdots z_{1+i(p^n-1)} \) in \( ch_{p^n} \otimes \psi_{p^n} \) is zero for any \( j \). Thus, the coefficient of \( z_1 \cdots z_{1+i(p^n-1)} \) in \( c_{p^n} \otimes \psi_{p^n} \) is equal to the corresponding coefficient of the operation \( \psi_{p^n} \) divided by \( p \).

We are going to show that for the operation \( \psi_{p^n} \) we have \( \alpha_1 \in p\mathbb{Z}_p^\times \) and \( \beta_1 \in p\mathbb{Z}_p^\times \) in the notation of Lemma 6.12. This follows by induction on \( i \) from relations there that \( \beta_i \in p\mathbb{Z}_p^\times \) for any \( i \geq 1 \), and therefore also by induction on \( i \) that \( \alpha_i \in p\mathbb{Z}_p^\times \) for all \( i \geq 1 \). Thus, we would have \( \psi_{p^n} \in \mathbb{Z}_p^\times \) as needed.

To calculate the coefficient \( \alpha_1 \) we note that it coincides with the corresponding coefficient of the truncation \( tr_{p^n} \psi_{p^n} \) which is a generator of additive operations to \( \text{CH}^{p^n} \otimes \mathbb{Z}_p \). By a straightforward computation one checks that a generator can be chosen to be \( p \cdot ch_{p^n} \) where \( ch : K(n) \to \text{CH}^*_Q \) is the Chern character. Since we have \( p \cdot ch_{p^n}(z_1 \cdots z_j) = p z_1 \cdots z_j + \ldots \), it follows that \( \alpha_1 \) is proportional to \( p \).

Since \( p c_{p^n} = c_{p^n} - \psi_{p^n} \), we know that \( \psi_{p^n} \equiv c_{p^n} \mod p \), and therefore all coefficients of those monomials of the series \( c_{p^n} \) which are not \( p^n \)-powers have to be zero modulo \( p \).

In particular, \( \beta_1 \) is a coefficient of a monomial \( z_1^{p^n} \cdots z_{p^n} \), and therefore we have \( \beta_1 \in p\mathbb{Z}_p^\times \).}

For the operation \( c_{p^n} \) we have seen that the \( p \)-valuation of the constant term of the series \( \psi_{p^n}(z_1 \cdots z_{1+i(p^n-1)}) \) does not depend on the choice of the operation itself. However, this is not true for higher Chern classes.

In order to estimate the graded groups of the gamma filtration in the best way one has to choose those operations in which the corresponding coefficients have minimal \( p \)-valuation. Perhaps, this explains the importance of the following clumsy proposition which will be used in Section 6.3.

**Proposition 6.13.** Let \( j \geq 0 \), and let \( p \) be the prime number corresponding to the \( n \)-th Morava K-theory, i.e. \( K(n)^*(\text{Spec} \ k) = \mathbb{Z}_p^\times \).

There exist operations \( \chi, \psi : K(n)^1 \to \gamma z^{p^n-1} K(n)^1 \) which satisfy the following.

Denote by \( h_j, a_j \in \mathbb{Z}_p \) the coefficients of the monomial \( z_1 \cdots z_{1+j(p^n-1)} \) in the series

\[
\chi(z_1 \cdots z_{1+j(p^n-1)}) = K(n)^1(\text{Spec} \ k), \quad \psi(z_1 \cdots z_{1+j(p^n-1)}) \in K(n)^1(\mathbb{P}^\infty \times 1+j(p^n-1)),
\]

respectively. Then

1) we have \( h_j = a_j = 0 \) for \( j = 0, 1 \).

2) if \( p \neq 2 \) we have \( h_j \in p^j \mathbb{Z}_p^\times \) where \( t_j = v_p(j+1) \).

8or, for a more general discussion yielding this, see Appendix A.3.
We fix an $e$ of $\mathbb{Z}_{p}$ which finishes the proof of 2).

Since (e.g. $k$ is a generator of additive operations with $\phi$ as its $p$-th Morava $K$-theory corresponding to the prime 2 in what follows. Let $L = 0$ be coprime to $p$, we need to look at the order of $\mathbb{Z}_{p}$ of $\mathbb{Z}_{p}$ if $n$ is even.

Note that by Prop. 6.8 every operation $\chi_{k}$ sends additive generators $z_{1} \cdots z_{1+1(p-1)}$ to elements of $\mathbb{Z}_{p}$ which is non-trivial only for $j = 0, 1$ and can be checked by direct computations using the fact that $c_{p\alpha}$ takes values in $\mathbb{Z}$. Therefore, if we choose $\chi_{k}$ for specific $k$.

First, we claim that $\chi_{k}$ takes values in $\mathbb{Z}$. Using the Cartan's formula one obtains that $\partial \chi_{k}(u,v) = (\Psi_{k}-k^{p}\cdot id)\left(-\frac{1}{p} \sum_{i=1}^{p-1} c_{i}(u)c_{j}(v)^{p^{n}-1}\right)$. Hence, $\Psi_{k}-k^{p}\cdot id$ sends $z_{1} \cdots z_{1+1(p-1)}$ to a series in $\mathbb{Z}$ for $j \geq 1$ (the claim is non-trivial only for $j = 1$), and one deduces that this operation sends a product of $p^{n}$ elements in $\mathbb{K}(n)^{1}$ to an element of $\mathbb{Z}^{1}$. Thus, it is enough to check that the (non-additive) operation $\chi_{k}$ sends additive generators $z_{1} \cdots z_{1+1(p-1)}$ to elements of $\mathbb{Z}^{1}$. This claim is non-trivial only for $j = 0, 1$ and can be checked by direct computations using the fact that $c_{p\alpha}$ takes values in $\mathbb{Z}$.

Second, we claim that for specific $k$ its truncation is a generator of primitive operations to $\mathbb{C}^{p\alpha}$ and $\mathbb{Z}^{1}$, and we denote an operation $\chi_{k}$ by Proposition 6.11 we calculate $\chi_{k}(z_{1} \cdots z_{2p^{n}-1}) = c_{k}k^{p}(k^{p^{n}-1}-1)z_{1}z_{2} \cdots z_{2p^{n}-1}$ higher degree terms. Thus, it is left to calculate the $p$-adic valuation of $\mathbb{Z}$.

Let $k$ be coprime to $p$, then $\nu_{p}(k^{p\alpha}) = 0$, and $[k] \in (\mathbb{Z}/(p)^{\infty}$.

If $p \neq 2$, then $(\mathbb{Z}/(p)^{\infty} \cong \mathbb{Z}/p \times \mathbb{Z}/(p-1))$, and since $p \nmid p^{n} - 1$ we see that $p^{2} \mid k^{p^{n}-1} - 1$ if $k$ is a generator of the multiplicative group $(\mathbb{Z}/(p)^{\infty}$.

If $p = 2$, then $(\mathbb{Z}/4)^{\infty} \cong \mathbb{Z}/2$, and since $2^{n} - 1$ is odd, we similarly obtain the claim for $k \equiv 3 \mod 4$. Thus, if we choose $\chi_{k}$ to be $\chi_{k}$ for these $k$, its truncation is a primitive operation and at most the $p$-adic valuation of coefficient $z_{1} \cdots z_{1+2(p-1)}$ of $\chi_{k}(z_{1} \cdots z_{1+2(p-1)})$ can not be reduced.

Third, we can now calculate constants $h_{j}$ of the operation $\chi_{k}$. The operation $\chi_{k}$ sends $z_{1} \cdots z_{1+1(p-1)}$ to $e_{j}k^{p^{n}}(k^{1(p-1)} - 1)z_{1} \cdots z_{1+1(p-1)}$ higher degree terms. If $p \neq 2$, then $(\mathbb{Z}/(p)^{\infty} \cong \mathbb{Z}/p^{1} \times \mathbb{Z}/(p-1))$.

Thus, if $k$ is a generator of multiplicative groups $(\mathbb{Z}/(p)^{\infty}$ for all $l$ (e.g. $k = 1 + p$), we need to look at the order of $j - 1$ in $\mathbb{Z}/(p^{2}-1)$. If $l - 1 = \nu_{p}(j - 1) + 1$, then $p^{2l} \mid e_{j}k^{p^{n}}(k^{1(p-1)} - 1)$ and $h_{j} \in \mathbb{Z}/(p^{2}$ where $l_{j} = \nu_{p}(j - 1) + 1$.

If $p = 2$, then $(\mathbb{Z}/2)^{\infty} \cong \mathbb{Z}/2^{l-1} \times \mathbb{Z}/(p-1)$ for $l \geq 2$. Similarly, we need to look at the order of $j - 1$ in $\mathbb{Z}/(2^{l}-2)$. However, if $j - 1$ is odd, then $4 \nmid (j-1)(2^{l-1} - 1)$ for $k \equiv 3 \mod 4$. If $j - 1$ is even and $l - 2 = \nu_{2}(j - 1) + 1$, then $2^{2l} \mid e_{j}k^{p^{n}}(k^{1(p-1)} - 1)$ and $h_{j} \in \mathbb{Z}/(2^{l})$ where $l_{j} = \nu_{2}(j - 1) + 2$.

This finishes the proof of 2).

However, in the construction above we can also choose $k$ to be $p$, and then the $p$-adic valuation of $e_{j}k^{p^{n}}(k^{1(p-1)} - 1)$ is $p^{n}$ for every $p$. Thus, if we choose $\psi := \chi_{p}$, this shows 3.

Let us provide an example of a computation using some of the constants we have calculated. We fix an $n$-th Morava $K$-theory corresponding to the prime 2 in what follows. Let $\alpha \in K_{n+2}(k)/2$ be a pure symbol, and let $Q_{\alpha}$ be a Pfister quadric corresponding to $\alpha$. In [VY07] Vishik and Yagita show that the pull-back map $\Psi_{\alpha}(Q_{\alpha}) \rightarrow \Psi_{\alpha}(\mathbb{Q})$ is injective, and its image equals to $\bigoplus_{i=0}^{2^{n+1}-1} L \cdot H_{i}$ where $H_{i}$ is the class of a hyperplane section of the quadric, $L_{i}$ is the class of the $i$-dimensional projective linear subspace in the split quadric $\mathbb{Q}$, and $I(2,n+1) = (2,v_{1}, \ldots, v_{n})$. Since for every $n$-th Morava $K$-theory the map $L \rightarrow K(n) \cong \mathbb{Z}_{(p)}$ sends...


\[ v_n \] to an invertible element, we see that the pull-back restriction map \( \rho^* : K(n)^*(Q_\alpha) \rightarrow \mathcal{K}(n)^*(Q_\alpha) \) is an isomorphism. We will denote by \( h_i, l_i \) the generators of the Morava K-theory of the split quadric which are imaged to \( H, L_i \), respectively, i.e. \( K(n)^*(Q_\alpha) = \bigoplus_{i=0}^{\infty} \mathbb{Z}(p) h_i \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}(p) l_i \).

Our goal now is to perform the computation of the abelian group \( gr^*_h K(n)^*(Q_\alpha) \cong \mathbb{Z} gr^*_h K(n)^*(Q_\alpha) \) for \( i \leq 2^n \). We will need the following identity in \( K(n)^*(\mathcal{Q}) \):

\[ h_i \cdot l_i = l_i \cdot h_i \] can be obtained using the transversality axiom.

For the Pfister quadric \( Q_\alpha \) we have \( l_{2^n+1} \in K(n)^1(Q_\alpha) \) since \( 2^{n+1} - 1 \equiv 1 \mod 2^n - 1 \). Since \( l_{2^n+1} = i_1 \cdot p_2 \cdot l_{2^n+1} \) where \( i \) is a closed embedding, we can apply the Riemann-Roch formula to calculate \( c_2(l_{2^n+1}) \). Using Proposition 6.11 and discussion before it we obtain that \( c_2(l_{2^n+1}) = c_2 l_{2^n+1} + b_1 l_{2^n} + b_0 l_1 \) for some \( b_0, b_1 \in \mathbb{Z}(p) \), \( e_2 \in \mathbb{Z}(2) \) (others \( l_i \) do not appear as they lie in different graded components). Similarly, applying \( c_2(l) \) to \( l_{2^n} \), \( l_1 \) we obtain \( c_3 l_{2n} + b l_1 \), \( c_4 l_1 \), respectively. All the elements obtained lie in the \( 2^n \)-th part of the group (and hence topological filtration).

Using the fact that \( e_j \in \mathbb{Z}(2)^* \), \( j \geq 1 \), one obtains that \( l_{2^n+1} \) lies in the \( 2^n \)-part of the topological filtration.

The class \( h_i \) lives in \( \tau^1 K(n)^*(Q_\alpha) = \gamma^1 K(n)^*(Q_\alpha) = \mathcal{K}(n)^*(Q_\alpha) \), and multiplication by \( h_i \) 'shifts' classes \( l_i \) and increases gamma and topological filtrations. Using this one can show that \( l_i \in \gamma^{2^n+1} K(n)^*(Q_\alpha) \) for \( i < 2^{n+1} - 1 \). Thus, \( gr^*_h K(n)^*(Q_\alpha) = \mathbb{Z}(2) \cdot h^i \) for \( i < 2^n \), and \( gr^*_h K(n)^*(Q_\alpha) \) is generated by \( h^2^n \) and \( l_{2^n+1} \) where \( h^2^n \) generates a free \( \mathbb{Z}(2) \)-module and \( l_{2^n+1} \) generates a torsion subgroup.

Applying the operation \( \chi \) from Prop. 6.13 to \( l_{2^n+1} \) we obtain \( h_2 l_{2^n+1} + B l_{2^n} + C l_1 \in \gamma^{2^n+1} \) for some \( B, C \in \mathbb{Z}(2) \). Since \( h_2 l_{2^n+1} + B l_{2^n} + C l_1 \in \gamma^{2^n+1} \) for \( B, C \in \mathbb{Z}(2) \), \( 2^n+1 = 1 + 2(2^n - 1) \) and \( 2 \) is even, we see that \( h_2 l_{2^n+1} + B l_{2^n} + C l_1 \) is generated by \( e^\mathcal{K}_0((\rho^*)^{-1}(l_{2^n+1})) \) and \( B l_{2^n} + C l_1 \) is generated by \( e^\mathcal{K}_0((\rho^*)^{-1}(l_{2^n})) \). Since \( B l_{2^n} + C l_1 \) is generated by \( e^\mathcal{K}_0((\rho^*)^{-1}(l_{2^n+1})) \) and \( B l_{2^n} + C l_1 \) is generated by \( e^\mathcal{K}_0((\rho^*)^{-1}(l_{2^n})) \), we obtain that \( h^{2^n+1} \) and \( l_{2^n+1} \) are generated by \( \mathbb{Z}(2) \cdot h^i \) and \( \mathbb{Z}(2) \cdot l_{2^n+1} \). Using Proposition 6.12 and the fact that only torsion in Chow groups of quadrics is \( 2 \)-torsion this shows that \( CH^i(Q_\alpha) = \mathbb{Z} \) for \( i < 2^n \), and that the torsion in \( CH^{2^n}(Q_\alpha) \) is generated by \( e^\mathcal{K}_0((\rho^*)^{-1}(l_{2^n+1})) \) and \( B l_{2^n} + C l_1 \). In fact, Rost computations (which can be re-obtained using the calculation of \( \Omega^*(Q_\alpha) \) of Vishik and Yagita described above) show that the torsion in \( CH^{2^n}(Q_\alpha) \) is precisely \( \mathbb{Z}/2 \), i.e. computations with the Morava K-theory give an exact bound.

One may wonder whether other torsion elements in Chow groups of a Pfister quadric can be obtained as values of other Chern classes. This is, however, not true, since one can show that Chern classes \( c_i \) on \( gr^*_h K(n)^* \) take values in \( p CH^i \otimes \mathbb{Z}(p) \) for \( i > p^n \) (see Appendix A.3), and the only torsion in Chow groups of Pfister quadrics is \( 2 \)-torsion.

For further details and more general computations see [SeSe18].

**Appendix A.**

**A.1. Non-existence of some operations from \( K(n)^* \).** In the paper we have constructed a subset of the set of oriented cohomology theories for which Chern classes from \( K(n)^* \) exist. Perhaps, vaguely speaking these \( p^n \)-typical theories can be called Morava-orientable. It is reasonable to ask whether this subset can be expanded. We do not answer this question completely here as we have no definition of Morava-orientability, however the following results suggest that theories which are not \( p^n \)-typical (up to a change of orientation) do not admit sufficiently good Chern classes from \( K(n)^* \).

More precisely, Prop. 6.14 shows that free theories whose height is less than \( n \) can not have a good theory of Chern classes since they do not admit even additive operations from \( K(n)^* \). On the other hand Prop. 6.13 shows that there could be no lifting (with respect to the truncation, see Prop. 2.10) of Chern classes from \( K(n)^* \) to \( CH^* \otimes \mathbb{Z}(p) \) to operations with the target theory \( K(m)^* \) when \( m \leq n \).

**Proposition A.1.** Let \( A^* \) be a free theory s.t. \( A \) is an \( \mathbb{F}_p \)-algebra, and \( p \cdot A x \equiv a_k x^k \) mod \( x^{k+1} \), where \( a_k \in A \) is not a zero-divisor.

If \( k < n \), then there exist no non-trivial additive operations from \( \overline{K(n)^*}/p \to A^* \). In particular, there exist no additive operations from \( \overline{K(n)^*}/p \to BP(k)^* \) or \( K(k)^* \) for \( k: 1 \leq k < n \).
Proof. Let \( \phi \) be a non-trivial additive operation from \( \tilde{K}(n)^* \) to \( A^* \). Let us consider its action on products of projective spaces which is non-trivial by Vishik’s Theorem [3]. There exist \( i > 0 \) s.t. \( G_i := \phi(z_1^{K(n)} \cdots z_i^{K(n)}) \neq 0 \) where \( G_i \) is a symmetric series in \( z_1^1, \ldots, z_i^1 \) divisible by \( \prod_{j=1}^i z_j^A \). Let \( d \geq 1 \) be the minimal degree of \( z_1 \) in \( G_i \).

The pull-back along the \( m \)-Veronese map \([m]_\infty \) acts on the first Chern class \( z \) in an oriented theory \( B^* \) by the formula \( z \mapsto m \cdot B \cdot z \). Thus, since \( \phi \) has to commute with the pull-back along the map \([p^N] \times \id^* \) on \((\mathbb{P}, \mathbb{P}) \cdot \mathbb{Z}^i \) we have \( \phi((p^N \cdot K(n)) z_1^{K(n)} \cdots z_i^{K(n)}) = G_i \mid_{z_1=p^n \cdot A z_1^p}. \)

By the assumptions on the series \( p \cdot A \cdot z \) (note that \( p \cdot A \cdot z = p \cdot A \cdot (\cdots (p \cdot A \cdot z)) \)) we can see that the minimal degree of \( z_1 \) in the series \( G_i \mid_{z_1=p^n \cdot A z_1^p} \) equals to \( dp^{kN} \). On the other hand, the series \((p^N \cdot K(n)) z_1^{K(n)} \cdots z_i^{K(n)} \) has the minimal degree of \( z_1 \) equal to \( p^{Nn} \). By the continuity of operations the minimal degree of \( z_1 \) in \( \phi((p^N \cdot K(n)) z_1^{K(n)} \cdots z_i^{K(n)}) \) is greater or equal to \( p^{Nn} \) which is bigger than \( dp^{kN} \) for sufficiently big \( N \). Contradiction.

Every additive operation \( \phi \) from \( K(n)^* \) to \( BP[k]^* \) or \( K(k)^* \) factors through to an additive operation from \( \tilde{K}(n)^*/p \) to \( BP[k]^*/p \) or \( K(k)^*/p \), respectively. As follows from above this mod-p operation has to be zero. It follows from Vishik’s theorem that one can canonically divide \( \phi \) by \( p \) to get a new additive operation. Again, \( \phi \) has to be zero modulo \( p \). Continuing this we see that \( \phi \) is zero modulo \( p^N \) for every \( N \geq 1 \). However, the action of \( \phi \) on products of projective spaces is defined by series with \( BP[k] \)- or \( \mathbb{Z}_p \)-coefficients, and therefore they are equal to zero if they are equal to zero modulo \( p^N \) for every \( N \geq 1 \). Thus, \( \phi \) has to be a trivial operation.

\[ \square \]

Remark A.2. One can show that between any two free theories there exist a non-trivial (in most cases, non-additive) operation. For example, for all \( n, m \) there exist an operation \( c : K(n)^* \to K(m)^* \) defined as the composition of operations \( c = c_1^{K_n \to K(m)} \circ \iota \circ c_1^{K(n) \to CH} \) where \( \iota \) denotes a non-additive map \( CH^1 \cong \text{Pic} \to K_0 \).

Lemma A.3. Assume that \( m \nmid n \). Let \( \phi : (K(n)^*) \to (K(m)^*) \) be an additive operation.

Let \( \psi : (K(n)^*) \to CH^i \otimes \mathbb{Z}_p \) be any additive operation, where \( i \neq ip^m \mod p^n - 1 \).

Then the composition \( \psi \circ \phi \) is 0 modulo \( p \).

Proof. Without loss of generality we may assume that \( \psi \) is a generator \( \psi_i \) of additive operations to some component \( CH^i \otimes \mathbb{Z}_p \). The composition \( \psi \circ \phi \) has to be supported on \( K(n)^* \mod p^n - 1 = K(n)^i \) (S18p Prop. 4.1.6), and we may also assume that \( \phi \) is supported on \( K(n)^i \).

By S18p Cor. 4.3.5 there is a relation \( \psi_i^{p^m} \equiv \psi_i \mod p \), where \( \psi_i \equiv c_{i}^{K_n \to K(m)} \) is a generator of additive operations. Therefore we have \( (\psi \circ \phi)^{p^m} \equiv \psi_i \circ \phi \mod p \). However, \( \psi_i \circ \phi \) is an additive operation from \( K(n)^i \) to \( CH^i \otimes \mathbb{Z}_p \). If \( ip^m \neq i \mod (p^n - 1) \), this operation is zero by S18p Prop. 4.1.6, and therefore the composition \( \psi_i \circ \phi \) is zero modulo \( p \).

If the operation \( \psi \circ \phi \) is non-trivial modulo \( p \), then it acts non-trivially on products of projective spaces, and since the theory \( CH^i/p \) has no nilpotents in the coefficient ring the \( p^n\)-th power of this operation also acts non-trivially on products of projective spaces. Contradiction.

\[ \square \]

Proposition A.4. Let \( m \nmid n \). Then there exist \( i \) s.t. \( 1 \leq i \leq p^n - 1 \) and \( ip^m \neq i \mod (p^n - 1) \).

For such \( i \) the truncation map \( tr_i : [K(n)^*, \tau^i K(m)^*[\text{add}] \to [K(n)^*, CH^i \otimes \mathbb{Z}_p[\text{add}] \] is not surjective.

Proof. Let \( \phi_i : (K(m)^*) \to CH^i \otimes \mathbb{Z}_p \) be a generator of additive operations. Then for \( i \) in the given range the composition of an operation from \( K(n)^* \) to \( \tau^i K(m)^* \) with the operation \( \phi_i \) is the same as the truncation \( tr_i \). One applies Lemma A.3 and the claim follows.

\[ \square \]

A.2. Existence of \( n \)-th Morava K-theories which are not multiplicatively isomorphic. Recall that an \( n \)-th Morava K-theory \( K(n)^* \) is defined as \( \Omega^* \otimes \mathbb{Z}_p[\varepsilon_n, \varepsilon_n^{-1}] \) where the morphism of rings \( \theta : \Omega^*(k) \cong \mathbb{E} \to \mathbb{Z}_p \) corresponds to a \( p^n\)-typical formal group law \( F_{K(n)} \) over \( \mathbb{Z}_p \) s.t. \( F_{K(n)} \mod p \) has height \( n \). However, we remarked after introducing Definition 3.9 that if a cohomology theory is represented in the stable motivic category, then its ‘geometric’ part is a graded theory. For the case of Morava K-theories this would mean investigating a theory \( \Omega^* \otimes \mathbb{Z}_p[\varepsilon_n, \varepsilon_n^{-1}] \)
where \( L \to \mathbb{Z}(p)[v_n, v_n^{-1}] \) is a graded lift of the ring map \( \theta \) above. Let us call this theory a graded \( n \)-th Morava K-theory throughout this section and denote it by \( GK(n)^* \). Note that \( GK(n)^* \otimes_{\mathbb{Z}(p)[v_n, v_n^{-1}]} \mathbb{Z}(p) = K(n)^* \) for some choice of the ring map \( \mathbb{Z}(p)[v_n, v_n^{-1}] \to \mathbb{Z}(p) \).

Studying additive operations from or to \( n \)-th Morava K-theory and its graded version is essentially the same. Indeed, we have an isomorphism of presheaves of abelian groups between \( GK(n)^* \) and \( \bigoplus_{i \in \mathbb{Z}(p)} K(n)^* \). In particular, it follows from Th. 5.3 that any two graded \( n \)-th Morava K-theories are additively graded-isomorphic. Comparing different graded and non-graded versions of Morava K-theories as presheaves of rings is more subtle, however, this is a question about isomorphisms between formal group laws.

**Proposition A.5.** Let \( GK(n)^* \) be a graded \( n \)-th Morava K-theory, let \( \phi_1, \phi_2 : \mathbb{Z}(p)[v_n, v_n^{-1}] \to \mathbb{Z}(p) \) be two ring maps sending \( v_n \) to \( a_1, a_2 \in \mathbb{Z}(p) \), respectively. If \( a_1 \not\equiv a_2 \mod p \), then \( n \)-th Morava K-theories \( K(n)^*_1 := GK(n)^* \otimes_{\phi_i} \mathbb{Z}(p), i = 1, 2 \) are not multiplicatively isomorphic.

**Proof.** By [VII] Th. 6.9 the set of invertible multiplicative operations \( \phi : K(n)^*_1 \to K(n)^*_2 \) is in bijective correspondence with the set of series \( \gamma(x) \in \mathbb{Z}(p)[x, v_n, v_n^{-1}] \) s.t. \( F_i(\gamma(x), \gamma(y)) = \gamma F_2(x, y) \). Since we are working over a torsion-free ring, the latter equation is equivalent to \( \gamma := \log_1^{-1}(\log_2(x)) \), where \( \log_i(x) \) is the logarithm of the formal group law \( F_i \) of the theory \( K(n)^*_i \).

Using Araki relations we can write \( \log_i(x) = x + \frac{a_i}{p-x} x^p + \ldots, i = 1, 2 \), and a direct computation shows that \( \log_1^{-1}(\log_2(x)) \) is not integral under the conditions on \( a_1, a_2 \).

The situation with graded theories is slightly different, since we have non-trivial automorphisms of \( \mathbb{Z}(p)[v_n, v_n^{-1}] \).

**Proposition A.6.** There exist two \( n \)-th graded Morava K-theories which are not graded multiplicatively isomorphic.

**Proof.** Two \( n \)-th graded Morava K-theory \( GK(n)^* = BP(n)^* \otimes_{BP(n)} \mathbb{Z}(p)[v_n, v_n^{-1}], i = 1, 2 \), are defined by morphism of graded rings \( \psi_i : BP(n) \to \mathbb{Z}(p)[v_n, v_n^{-1}], i = 1, 2 \), respectively, which define two formal group laws \( F_1, F_2 \). Let us denote \( \psi_i(v_n) = a_i v_n, \psi_i(v_{2n}) = b_i v_n^{p^{n+1}} \) for some numbers \( a_i \in \mathbb{Z}(p), b_i \in \mathbb{Z}(p) \).

By [VII] Th. 6.9 a multiplicative (graded) isomorphism between these two theories consists of a graded isomorphism \( \phi : GK(n)^*[(Spec k) \to Spec k] \to GK(n)^*[(Spec k)[v_n, v_n^{-1}] \) which sends \( v_n \) to \( \alpha v_n \) for some \( \alpha \in \mathbb{Z}(p) \) and a homogeneous series \( \gamma \in \mathbb{Z}(p)[x, v_n, v_n^{-1}][[x]][x^2] \) of degree 1 s.t. \( \phi(F_i)(\gamma(x), \gamma(y)) = \gamma F_2(x, y) \). Without loss of generality we may assume that \( \gamma(x) \equiv x \mod x^2 \), since we can twist the isomorphism by an invertible Adams operation otherwise.

Thus, \( \gamma(x) \equiv x + c v_n x^{p^n} \mod x^{p^{n+1}} \).

Using the Araki generators of \( BP[n] \) (see Prop. 5.2) we can write for \( i = 1, 2 \):

\[
p \cdot F_i x \equiv \psi_i(v_n) x^{p^n} + \psi_i(v_{2n}) x^{p^{2n}} = a_i v_n x^{p^n} + b_i v_n^{p^{n+1}} x^{p^{2n}} \mod (p, x^{p^{2n+1}}).
\]

On the other hand it follows from the equation on \( \gamma \) that

\[
(9) \phi(p \cdot F_1 \gamma(x)) = \gamma(p \cdot F_2 x),
\]

where on the left hand side \( \phi \) is applied to the series \( p F_1, \) and then \( \gamma(x) \) is plugged in into it.

Rewriting this equation with given series we obtain

\[
a_1 v_n \gamma(x)^{p^n} + b_1 v_n^{p^{n+1}} + b_1 \gamma(x)^{p^{2n}} \equiv \gamma(a_2 v_n x^{p^n} + b_2 v_n^{p^{n+1}} x^{p^{2n}}) \mod (p, x^{p^{2n+1}}),
\]

\[
a_1 v_n x^{p^n} + (a_1 \gamma(x)^{p^n} + b_1 \gamma(x)^{p^{2n}}) x^{p^{2n}} \equiv a_2 v_n x^{p^n} + (b_2 + c v_n^{p^n}) v_n^{p^{n+1}} x^{p^{2n}} \mod (p, x^{p^{2n+1}}).
\]

We get two equations from which we obtain \( \alpha \equiv \frac{a_2}{a_1} \mod p, b_1 \alpha^{p^n} \equiv b_2 \mod p \). However, since we can choose \( b_1, b_2 \) as we want, e.g. \( b_1 = 0, b_2 \neq 0 \), these equations can not always be satisfied, and \( (\phi, \gamma) \) does not always exist.
A.3. **Image of Chern classes from $K(n)^*$ in Chow groups.** One of the main results on Chern classes obtained in this paper is that operations $c_i : K(n)^* \to \text{CH}^i \otimes \mathbb{Z}(p)$, $i : 1 \leq i \leq p^n$, are surjective. This is not true for Chern class $c_i$, $i > p^n$, however, the image of this operation is always a subgroup of the form $b_i \text{CH}^i \otimes \mathbb{Z}(p)$. In this section we provide an inductive way to compute numbers $b_i$.

Denote by $d_i \in p^\mathbb{Z}$ the number s.t. $d_i \cdot c_{d_i}$ acts integrally on products of projective spaces and this action is not zero modulo $p$. In other words, $d_i \cdot c_{d_i}$ can be lifted to a generator $\phi_i$ of additive operations from $K(n)^* \to \text{CH}^i \otimes \mathbb{Z}(p)$. It is clear that $d_i$ is uniquely defined by the above.

**Proposition A.7.** We have $d_1 = 1$.

- If $i \neq p^n$ for some $s \in \mathbb{N}$, then $d_i = \max_{j=1}^{i-1}(d_j, d_{i-j})$.
- If $i = p^n$, then $d_i = p \cdot \max_{j=1}^{i-1}(d_j, d_{i-j})$.

**Proof.** The operation $c_{d_1}$ acts integrally on projective spaces, it is defined by series $G_1(z) = z$, $G_j(z) = 0$ for $j \geq 2$. Thus, $d_1 = 1$.

**Lemma A.8.** The $\mathbb{Z}(p)$-module of poly-additive poly-operations of arity $r$ from $K(n)^*$ to $\text{CH}^r \otimes \mathbb{Z}(p)$ is a free $\mathbb{Z}(p)$-module generated by external products $\phi_{j_1} \circ \cdots \circ \phi_{j_r}$, where $\sum_{r=1}^r j_s = i$, $j_s : 0 \leq j_s \leq i$.

**Proof.** It follows from Vishik’s classification of poly-operations theorem that this module is free. Thus, its rank is equal to the dimension of $\mathbb{Q}$-vector space of additive polyoperations from $K(n)^* \otimes \mathbb{Q}$ to $\text{CH}^r \otimes \mathbb{Q}$. However, using a multiplicative isomorphism $c_{r} : K(n)^* \otimes \mathbb{Q} \cong \otimes \text{CH}^r \otimes \mathbb{Q}$ one reduces the problem of calculating this dimension to the dimension of the vector space of poly-operations of arity $r$ from Chow groups to Chow groups with rational coefficients. One can show that the ring of poly-operations in Chow groups is generated by multiplications of components, and, thus, the dimension is equal to the number of external products $\phi_{j_1} \circ \cdots \circ \phi_{j_r}$.

The number of poly-operations $\phi_{j_1} \circ \cdots \circ \phi_{j_r}$ coincides with the rank of the $\mathbb{Z}(p)$-module, and thus it is enough to show that there are no $\mathbb{F}_p$-relations between them. But this is precisely the claim of [ST18a Lemma 4.6.2].

Denote by $m$ the multiplication in $K(n)^*$ which is a poly-additive poly-operation of arity 2. As follows from Lemma A.8, for each $i$ the poly-operation $\phi_i \circ m$ is of the form

$$\phi_i \circ \phi_0 + \phi_0 \circ \phi_i + \sum_{j=1}^{i-1} b_j^{(i)} \phi_j \circ \phi_{i-j}$$

where $b_j^{(i)} \in \mathbb{Z}(p)$ and $\phi_0$ is the additive operation $K(n)^* \to \text{CH}^0 \otimes \mathbb{Z}(p)$ which sends 1 to 1 and is zero on $K(n)^*$.

On the other hand, since $c_{d_1}$ is a multiplicative operation, we have $c_{d_1} \circ m = \sum_{j=0}^{d_1} c_{d_1+j} \circ c_{d_1-j}$. Combining these two equations together we obtain that $\phi_i \circ m = \phi_i \circ \phi_0 + \phi_0 \circ \phi_i + \sum_{j=1}^{d_1} \frac{d_1}{d_1-j} \phi_j \circ \phi_{d_1-j}$, and therefore $b_j^{(i)} = \frac{d_1}{d_1-j}$. The number $b_j^{(i)}$ has to be integer since $\phi_i \circ m$ is an integral polyoperation, and therefore $d_i \geq \max_{j=1}^{i-1}(d_j, d_{i-j})$.

Assume that $b_j^{(i)} \in p\mathbb{Z}(p)$ for all $j : 1 \leq j \leq i - 1$, i.e. $\phi_i \circ m \equiv \phi_i \circ \phi_0 + \phi_0 \circ \phi_i \mod p$. In particular, this means that $(\phi_i \mod p)(z_1, \ldots, z_i) = 0$ for all $j \geq 2$. However, $\phi_i \mod p \neq 0$ as it is a generator of additive operations, and therefore $(\phi_i \mod p)(z)$ can not be equal to zero. Thus, $(\phi_i \mod p)(z) = a z^i$ for some $a \in \mathbb{F}_p^\times$ and we see that $\phi_i \equiv a(\phi_1)^i \mod p$.

In order for $(\phi_1)^i$ to be additive modulo $p$ we need $i$ to be a power of $p$. However, if $i \neq p^n$ for some $s \geq 0$, then $\phi_i$ and $\phi_0$ are supported on different graded components of $K(n)^*$ by [ST18a Prop. 4.1.6] (or one could argue that $(\phi_1)^i \mod p$ is not $p^n$-gradable). Thus, we see that if $i \neq p^n$, then there exist $j$ s.t. $\frac{d_1}{d_1-j} \in \mathbb{Z}(p)^\times$. The claim follows.

If $i = p^n$, then by [ST18a Cor. 4.3.5] we have the relation $\phi_i \equiv (\phi_1)^i \mod p$, and thus $\phi_i \circ m \equiv \phi_i \circ \phi_0 + \phi_0 \circ \phi_i \mod p$. We claim, however, that $\phi_i \neq (\phi_1)^i \mod p^2$. Without loss of generality we may choose $n$-th Morava K-theory to have the logarithm $\log_{K(n)}(x) = \sum_{s=0}^{\infty} \frac{x^{p^n}}{p^s}$. 

By the construction of Chern classes \[\text{Lemma 4.4.1, Lemma 4.3.3 (2p)}\] we have \[\log_{K(n)}(c_1 t + c_2 t^2 + \ldots + c_p = \frac{\phi_{p^n}}{p}).\] Multiplying this equation by \(p^n\) and taking it modulo \(p^2\) we obtain

\[p(c_1 t + \ldots + c_p = \frac{\phi_{p^n}}{p}) = \frac{\phi_{p^n}}{p^2}.\]

Using equality \(c_1 = \phi_1\) we may rewrite it as

\[pQ(c_1, \ldots, c_p) + \phi_{p^n} = \phi_{p^n} \mod p^2,

where \(Q\) is a polynomial in Chern classes. The polynomial \(Q\) contains \((c_p = \frac{\phi_{p^n}}{p})\) as a summand which comes from the term \((c_1 t + \ldots + c_p = \frac{\phi_{p^n}}{p})\) and can not be cancelled by any other summands coming from the other term. Thus, \(Q \neq 0\), and therefore it is not zero as an operation to Chow groups modulo \(p\), because Chern classes form a basis of operations and there are no relations between them modulo \(p\). Thus, we see that \(\phi_i \neq (\phi_1)^i\) \(mod p^2\), and therefore as explained above \(\phi_i \circ m \neq \phi_i \circ \phi_0 + \phi_0 \circ \phi_i \mod p\). Therefore the minimal \(p\)-valuation of coefficients \(b_i^{(i)}\) is equal to 1, and \(d_i = p \cdot \max_{j=1}^{i-1}(d_j, d_{i-j})\). \(\square\)

**Example A.9.** Prop. \[\text{A.7}\] allows to calculate \(d_2 = \cdots = d_{p^n - 1} = 1\), \(d_{p^n} = \cdots = d_{2p^n - 1} = p\), and, more generally, \(d_{kp^n} = \cdots = d_{kp^n + p^n - 1} = p^k\) for \(k < p^n\).

Define numbers \(b_i\) according to the rule \(b_i = \begin{cases} d_i, & \text{if } p^n \nmid i; \\ p^{-k}d_i, & \text{if } i = p^av^n \text{ where } p^n \mid v. \end{cases} \)

**Proposition A.10.** For each \(i \geq 0\) numbers \(b_i \in \mathbb{P}^n\).

The image of the map \(c_i^{CH} : gr^i_{K(n)^*} \to CH^i \otimes \mathbb{Z}(p)\) equals to \(b_i \mathbb{Z}(p)(CH^i \otimes \mathbb{Z}(p))\).

**Proof.** It follows from Prop. \[\text{4.12}\] that the image is equal to the image of the operation \(\Theta_i := (c_i^{CH})^{CH} : CH^i \otimes \mathbb{Z}(p) \to CH^i \otimes \mathbb{Z}(p)\) which is obtained as the composition of \(c_i\) with the canonical morphism \(\rho_{K(n)} : CH^i \to gr^i_{K(n)^*}\). We are going to prove that this operation is multiplication by \(b_i\).

The operation \(\Theta_i\) is a composition of additive operations and therefore it is additive itself. By Vishik’s classification of additive operations \[\text{[V19, Th. 6.2]}\] \(\Theta_i\) is determined by a symmetric polynomial \(G_i(z_1, \ldots, z_i)\) of degree \(i\) which is divisible by \(\prod_{j=1}^{i} z_j\), i.e. \(G_i = \alpha z_1 \cdots z_i\) for some \(\alpha \in \mathbb{Z}(p)\), and it is clear that \(\Theta_i\) is multiplication by \(\alpha\). The polynomial \(G_i\) for the operation \(c_i^{CH}\) is also equal to \(\alpha z_1 \cdots z_i\) which can be checked by comparing the action of \(\Theta_i\) and \(c_i^{CH}\) on products of projective spaces using the definition of \(\Theta_i\). Thus, our goal is to show that \(c_i^{CH}(z_1 \cdots z_i) = ab_1z_1 \cdots z_i\) for some \(a \in \mathbb{Z}(p)\).

By construction of the operation \(c_i^{CH}\) \[\text{[S18a Cor. 4.3.1]}\] we have \(c_i - P_i(c_1, \ldots, c_{i-1}) = \frac{\phi}{p^i}\) as operations to \(CH^i \otimes Q\) where \(P_i\) is a rational polynomial, \(\phi_i\) is a generator of additive operations to \(CH^i \otimes \mathbb{Z}(p)\). For operations \(c_j, j < i\) the polynomial \(G_i\) is equal to 0 by degree reasons, and therefore so it is for \(P_i\). Thus, the polynomial \(G_i\) of the operation \(c_i\) coincides with the polynomial of the operation \(\frac{\phi}{p^i}\).

The operation \(\phi_i\) is equal to \(ap^d_i \cdot ch\), where \(a \in \mathbb{Z}(p)\) by definition of numbers \(d_i\) above. Therefore \(G_i\) of the operation \(\phi_i\) is equal to \(ap^d_i \cdot z_1 \cdots z_i\), and \(\alpha = ap^{d_i - \mu_i}\). To finish the proof recall that \(\mu_i = \max(0, -\nu_p(P_i))\) \[\text{[S18a Cor. 4.3.1]}\] and it follows from \[\text{Lemma 4.3.3 (2, 2p)}\] that \(\mu_i = 0\) if \(p^n \nmid i\) and \(\mu_i = k\) if \(i = p^a v^n\) where \(p^n \nmid v, v \in \mathbb{Z}(p)\). Thus, \(\alpha = ab_1\).

Since \(\Theta_i\) is an integral operation, it follows that \(b_i\) is integral, i.e. \(b_i \in \mathbb{P}^n\), which is the first claim of the Proposition. \(\square\)

**Example A.11.** If \(i = kp^n + j < p^{2n}\) where \(j: 1 \leq j \leq p^n\), then the image \(c_i(gr^i_{K(n)^*})\) is equal to \(p^k CH^i \otimes \mathbb{Z}(p).\)
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