ON THE PATHS OF STEEPEST DESCENT FOR THE NORM OF A ONE VARIABLE COMPLEX POLYNOMIAL

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ABSTRACT. We consider paths of steepest descent, in the complex plane, for the norm of a non-constant one variable polynomial $f$. We show that such paths, starting from a zero of the logarithmic derivative of $f$ and ending in a root of $f$, draw a tree in the complex plane, and we give an upper bound estimate on their lengths. In some cases, we obtain a finer estimate that depends only on the set of roots of $f$, not on their multiplicity, and we wonder if this can be done in general. We also extend this question to finite Blaschke products for the unit disk.

1. INTRODUCTION

In our study [4] of Hermite approximations to exponentials of algebraic numbers, we were lead to several results on polynomials of $\mathbb{C}[z]$ that we could not find within the vast literature about zeros of univariate polynomials, including the exposition [3] of results prior to 1949, and more recent papers like [5]. The purpose of this note is to report on these new results with a brief outline of the proofs, to present a case where the main estimate can be improved qualitatively, and to ask if such improvement can be made in general.

To state the results, we fix a monic polynomial $f(z) \in \mathbb{C}[z]$ of degree $N \geq 1$, and factor it as a product

$$f(z) = (z - \alpha_1)^{n_1} \cdots (z - \alpha_s)^{n_s},$$

where $A = \{\alpha_1, \ldots, \alpha_s\}$ is the set of its distinct complex roots and where $n_1, \ldots, n_s$ are positive integers with sum $N$. We also denote by $\mathcal{K}$ the convex hull of $A$, and by $R$ the radius of a closed disk $D$ in $\mathbb{C}$ containing $\mathcal{K}$. To avoid trivialities, we assume that $s \geq 2$. In [4] §5, we prove the following result.

**Theorem 1.1.** Any path of steepest descent for $|f|$ linking a point $\beta$ of $\mathcal{K}$ to a root $\alpha$ of $f$ is contained in $\mathcal{K}$, with length at most $\pi NR$.

By a path of steepest descent (resp. steepest ascent) for $|f|$, we mean a continuous piecewise differentiable curve $\gamma : I \to \mathbb{C}$, defined on an interval $I$ of $\mathbb{R}$, which is an integral curve for the gradient of $|f|$, and along which $|f|$ is decreasing (resp. increasing). In the present paper, we show a case where the upper bound on the length of the path depends only on the set of roots of $f$ and not on their multiplicities.

2020 Mathematics Subject Classification. Primary 30C15; Secondary 30E10.

Key words and phrases. Jordan curve theorem, roots of polynomials, steepest descent, trees.

Research partially supported by NSERC.
**Theorem 1.2.** Suppose that a path of steepest descent for $|f|$ links a point $\beta$ of $\mathcal{K}$ to a root $\alpha$ of $f$ on the boundary of $\mathcal{K}$. Then, it has length at most $2\pi sR$.

Note that the above condition on $\alpha$ is necessary fulfilled when $s \leq 3$. We wonder if a similar estimate, with an upper bound depending only on $s$ and $R$, holds in general without the condition that $\alpha$ lies on the boundary of $\mathcal{K}$.

The proofs of the above theorems are given in the next section. In section 3, we construct a tree out of the paths of steepest descent from the zeros of $f'/f$ to the zeros of $f$ and give an application. We discuss an analog problem for finite Blaschke products in section 4.

2. Paths of steepest descent

Let $c \in \mathbb{C}\setminus\{0\}$ and let $I$ be a closed subinterval of $\mathbb{R}$. Since $f : \mathbb{C} \to \mathbb{C}$ is a ramified covering of Riemann surfaces, there exists a continuous map $\gamma : I \to \mathbb{C}$ (not unique in general) such that $f(\gamma(t)) = ct$ for each $t \in I$. More precisely, let $t_0 \in I$, let $z_0 = \gamma(t_0)$, and let $\ell$ denote the order of $f(z) - f(z_0)$ at the point $z_0$. Then there is a bi-holomorphic map $h : U \to V$ from an open neighborhood $U$ of 0 to an open disk $V$ centered at 0 of radius $\epsilon > 0$, such that

$$f(z) - f(z_0) = h(z - z_0)^\ell \quad \text{whenever } z - z_0 \in U.$$  
(2.1)

This yields $c(t - t_0) = h(\gamma(t) - \gamma(t_0))^\ell$ for each $t \in I$ with $|t - t_0| < \ell/|c|$. If $\ell = 1$, then $\gamma(t) = \gamma(t_0) + h^{-1}(c(t - t_0))$ is an analytic function of $t - t_0$ for those $t$. Otherwise, it is represented by a convergent series in $(t_0 - t)^{1/\ell}$ to the left of $t_0$ and by a convergent series in $(t - t_0)^{1/\ell}$ to the right of $t_0$. In practice $\gamma$ is obtained by pasting such local maps (as in [2, Theorem 4.14]) and can be extended to a continuous map $\gamma : \mathbb{R} \to \mathbb{C}$ satisfying $f(\gamma(t)) = ct$ for each $t \in \mathbb{R}$.

Since $f$ is conformal, except at the zeros of $f'$, since it maps level curves of $|f|$ to circles centered at 0, and since the line $z = ct$ ($t \in \mathbb{R}$) is perpendicular to these circles, the image of the above map $\gamma$ is a curve that is perpendicular to the level curves of $|f|$ at each point $z_0 = \gamma(t_0)$ with $f'(z_0) \neq 0$. From this observation, we derive the following statement which shows a connection with the phase plot of $f$, namely the graph of $f'/|f|$ (see [3]).

**Lemma 2.1.** For any $\beta \in \mathbb{C}$, the continuous functions $\gamma : [0, 1] \to \mathbb{C}$ such that

$$\gamma(1) = \beta \quad \text{and} \quad f(\gamma(t)) = tf(\beta) \quad \text{for each } t \in [0, 1],$$  
(2.2)

parameterize (in reverse direction) the curves of steepest descent for $|f|$ from $\beta$ to a root $\alpha = \gamma(0)$ of $f$.

Let $B = \{\beta_1, \ldots, \beta_p\}$ denote the set of distinct zeros of $f'$ not in $A$ (the zeros of the logarithmic derivative $f'/f$). By a theorem of Gauss-Lucas, these zeros belong to the relative interior of $\mathcal{K}$ [3, Theorem 6.1]. In view of (1.1), we find

$$f'(z) = N(z - \alpha_1)^{n_1 - 1} \cdots (z - \alpha_s)^{n_s - 1}(z - \beta_1)^{m_1} \cdots (z - \beta_p)^{m_p},$$  
(2.3)
for positive integers $m_1, \ldots, m_p$ with sum $s - 1$. For a path $\gamma: [0, 1] \to \mathbb{C}$ as in Lemma 2.1 we find that

$$
\gamma'(t) = \frac{f(\beta)}{f'(\gamma(t))} = \frac{f(\gamma(t))}{tf'(\gamma(t))} = \frac{1}{t} \left( \sum_{j=1}^{s} \frac{n_j}{\gamma(t) - \alpha_j} \right)^{-1},
$$

for each $t \in (0, 1)$ with $\gamma(t) \notin B$. In particular the above formula holds for each $t \in (0, 1)$ such that $\gamma(t) \notin B$. In the latter situation, consider the smallest sector with vertex at $\gamma(t)$ containing $K$. Then it follows from (2.4) that $\gamma'(t)$ points in the opposite sector, away from $K$, as illustrated in Figure 1, and so from that point, the path $\gamma$ does not come back to $K$. This means that, if $\gamma(1)$ is a point $\beta$ of $K$, the image of $\gamma$ is fully contained in $K$, as asserted in Theorem 1.1.

![Figure 1. A path of steepest ascent that leaves $K$ does not come back.](image)

To estimate the length $L(\gamma)$ of a path $\gamma: [0, 1] \to \mathbb{C}$ satisfying (2.2) with $\beta = \gamma(1) \in K \setminus A$, we proceed as in [1]. For each angle $\theta \in [0, 2\pi]$ and each $r \in \mathbb{R}$, we form the line

$$
D_{r,\theta} = \{(r + iu)e^{i\theta} : u \in \mathbb{R}\}
$$

and denote by $N(r, \theta)$ the number of points of intersection of the image of $\gamma$ with this line. Since $\gamma$ is continuous and piecewise differentiable, the Cauchy-Crofton formula gives

$$
L(\gamma) = \frac{1}{4} \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} N(r, \theta) dr
$$

(see the elegant proof of [1]). We note that $f(z)/f(\beta) = t \in \mathbb{R}$ for each point $z = \gamma(t)$ with $t \in [0, 1]$, so $N(r, \theta)$ is at most equal to the number of roots $u \in \mathbb{R}$ of the polynomial

$$
g_{r,\theta}(u) = \text{Im} \left( \frac{f((r + iu)e^{i\theta})}{f(\beta)} \right) \in \mathbb{R}[u],
$$

where $\text{Im}(z)$ stands for the imaginary part of $z$. Since the coefficient of $u^N$ in this polynomial depends only on $\theta$ and does not vanish except for at most $N$ values of $\theta$ in $[0, 2\pi)$, it follows that, aside from these, we have $N(r, \theta) \leq N$ for all $r \in \mathbb{R}$. Since the image of $\gamma$ is contained in the disk $D$ with radius $R$, we also have $N(r, \theta) = 0$ outside of an interval of length $2R$ for $r$. We deduce that $\left| \int_{\mathbb{R}} N(r, \theta) dr \right| \leq 2NR$ for all but a finite number of $\theta \in [0, 2\pi]$, and so $L(\gamma) \leq \pi NR$. 

Proof of Theorem 1.2. Suppose now that \( \alpha := \gamma(0) \) lies on the boundary of \( \mathcal{K} \). Our goal is to show that \( \mathcal{L}(\gamma) \leq 2\pi sR \). To this end, we may assume that \( \mathcal{K} \) has non-empty interior because otherwise \( \mathcal{K} \) is a line segment of length at most \( 2R \) and we are done. Under that hypothesis, the point \( \beta \) belongs to that interior as well as \( \gamma(t) \) for each \( t \in (0, 1] \).

We first extend \( \gamma \) to a continuous function \( \gamma : \mathbb{R} \to \mathbb{C} \) satisfying \( f(\gamma(t)) = tf(\beta) \) for each \( t \in \mathbb{R} \). Then there exists a unique point \( c > 1 \) for which \( \gamma(c) \) lies on the boundary of \( \mathcal{K} \). For \( 0 < t < c \), the point \( \gamma(t) \) belongs to the interior of \( \mathcal{K} \) and, for \( t > c \), it is outside of \( \mathcal{K} \) (the proof uses (2.4) as above). Then, we get a simple closed curve \( \Gamma \) by following \( \gamma \) on \([0, c]\) and coming back to \( \alpha = \gamma(0) \) along the polygonal boundary of \( \mathcal{K} \). By a theorem of Jordan, \( \Gamma \) divides \( \mathbb{C} \) into two connected components \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) having \( \Gamma \) as their common boundary.

Now, consider an angle \( \theta \in [0, 2\pi] \). We claim that there are finitely many real numbers \( r \) for which \( D_{r, \theta} \) meets \( A \cup B \) or is tangent to \( \Gamma \) at a point \( \gamma(t) \in \mathcal{K} \setminus B \) with \( t \in [0, c] \). This is because, for each \( t_0 \in \mathbb{R} \), there exists \( \epsilon > 0 \) such that the restriction of \( \gamma(t) \) to \( (t_0 - \epsilon, t_0) \) and \( (t_0, t_0 + \epsilon) \) is given by convergent power series in \( |t - t_0|^{1/\ell} \) where \( \ell - 1 \) is the order of \( f' \) at \( \gamma(t_0) \). Then, the function

\[
(2.8) \quad h(t) = \text{Re}(\gamma'(t)e^{-it\theta}),
\]

where \( \text{Re}(z) \) stands for the real part of \( z \), is given by convergent power series in \( |t - t_0|^{1/\ell} \) on \( (t_0 - \epsilon, t_0) \) and on \( (t_0, t_0 + \epsilon) \). On such an interval \( I \), either \( h \) is identically zero or it has finitely many zeros. Thus, \( \text{Re}(\gamma(t)e^{-it\theta}) \) takes finitely many values on the set of zeros of \( h \) on \( I \) and these are precisely the values \( r \in \mathbb{R} \) for which \( D_{r, \theta} \) is tangent to the open arc \( \gamma(I) \). The claim follows since \([0, c]\) is compact and so it can be covered by finitely many intervals \( (t_0 - \epsilon, t_0 + \epsilon) \) with those properties.

Suppose that \((ie^{it\theta})^N \notin \mathbb{R} \), so that the polynomial \( g_{r, \theta} \) given by (2.4) has degree \( N \) for each \( r \in \mathbb{R} \). Then \( D_{r, \theta} \) meets \( \gamma([0, c]) \) in at most \( N \) points

\[
\gamma_j = \gamma(t_j) = (r + iu_{j})e^{it\theta} \quad (1 \leq j \leq n := N(r, \theta)),
\]

which we order so that \( u_1 < u_2 < \cdots < u_n \). Moreover, aside from finitely many values of \( r \), the line \( D_{r, \theta} \) avoids \( A \cup B \) and meets \( \Gamma \) transversally at those points: the function \( h \) given by (2.8) is defined and non-zero at \( t_j \). For each \( j \) with \( 1 \leq j \leq n \), the open line segment \( (z_j, z_{j+1}) \) on \( D_{r, \theta} \) does not meet \( \Gamma \) and so is contained in one of the connected regions \( \mathcal{R}_1 \) or \( \mathcal{R}_2 \). Geometrically, this means that \( \gamma \) crosses \( D_{r, \theta} \) in opposite directions at \( z_j \) and at \( z_{j+1} \) so to have this line segment on the same side at both points, as illustrated on Figure 2. Thus \( h(t_j) \) and \( h(t_{j+1}) \) have opposite signs. Using (2.4), we find that \( t\gamma'(t) = P(\gamma(t))/Q(\gamma(t)) \) where

\[
P(z) = (z - \alpha_1) \cdots (z - \alpha_s) \quad \text{and} \quad Q(z) = N(z - \beta_1)^{m_1} \cdots (z - \beta_p)^{m_p}
\]

have degree \( s \) and \( s - 1 \) respectively. Since each \( t_j \) is positive, it follows that, for \( j = 1, \ldots, n \), the number \( h(t_j) \) has the same sign as \( \tilde{g}_{r, \theta}(u_j) \) where

\[
\tilde{g}_{r, \theta}(u) = \text{Re}(e^{-it\theta}P((r + iu)e^{it\theta})\overline{Q((r - iu)e^{-it\theta})})
\]

Thus the polynomial \( \tilde{g}_{r, \theta} \) alternates sign at \( u_1, \ldots, u_n \). Since it has degree at most \( 2s - 1 \), we conclude that \( N(r, \theta) = n \leq 2s \). This implies that \( \int_{\mathbb{R}} N(r, \theta) dr \leq 4sR \) and so the Cauchy-Crofton formula (2.6) yields an upper bound of \( 2\pi sR \) for the length of \( \gamma([0, c]) \). \( \square \)
In general, any continuous map $\gamma: [0, 1] \to K$ satisfying $f(\gamma(t)) = t f(\beta)$ for each $t \in [0, 1]$ and some $\beta \in K$ extends to a continuous map $\gamma: [a, c] \to K$ satisfying the same condition on a maximal interval $[a, c]$ containing $[0, 1]$, and we obtain a simple closed curve $\Gamma$ dividing $\mathbb{C}$ in two connected components by following $\gamma$ on $[a, c]$ and coming back to $\gamma(a)$ along the boundary of $K$. Then most of the argument goes through, except that, if $\gamma(0)$ is not on the boundary of $K$, then $a$ is negative and, for points $t_j$ with $a < t_j < 0$ (if any), the sign of $h(t_j)$ is opposite to that of $\tilde{g}_{r, \theta}(u_j)$. It may even happen that all $\tilde{g}_{r, \theta}(u_j)$ have the same sign if $\gamma$ winds several times around $\gamma(0)$ on $[a, 0]$ and unwinds an equal number of times on $[0, c]$ so that $t_j$ and $t_{j+1}$ have opposite signs for $j = 1, \ldots, n - 1$.

3. A tree of paths

We construct a graph on the set $A \cup B$ in the following way. For each $j = 1, \ldots, p$, the difference $f(z) - f(\beta_j)$ has a zero of multiplicity $\ell = m_j + 1$ at $\beta_j$ and so it is locally the $\ell$-th power of a diffeomorphism as in (2.1) (with $z_0 = \beta_j$). It follows that, in a sufficiently small neighborhood of $\beta_j$, there are exactly $\ell$ paths of steepest descent issued from $\beta_j$ and their tangents make equal angles of $2\pi/\ell$ at that point. Each of these paths can be continued uniquely until it reaches another point of $A \cup B$. Doing this for each $j = 1, \ldots, p$, we obtain a total of $\sum_{j=1}^{p} (m_j + 1) = s + p - 1$ paths from a point of $B$ to a point in $A \cup B$. These paths intersect only on their end-points because there is only one path of steepest descent through each point not in $A \cup B$. So they make the edges of a graph with $A \cup B$ as its set of vertices.

We claim that this graph has no cycle. Indeed, if there were a cycle, we would obtain a simple closed curve $\Gamma$ by composing a number of these paths. Consider the bounded connected region $\mathcal{R}$ delimited by $\Gamma$. By the maximum modulus principle, the maximum of $|f|$ on $\mathcal{R}$ is achieved at a point of $\Gamma$ and so at some $\beta_j$ on $\Gamma$. This is impossible because, in a neighborhood of $\beta_j$, the curve $\Gamma$ consists of two paths of steepest descent issued from $\beta_j$. However, in each of the two angles formed by these paths at $\beta_j$, there is at least one path of steepest ascent for $|f|$ issued from $\beta_j$ (locally, there are exactly $m_j + 1$ such paths and their tangents at $\beta_j$ are bisectors of the angles formed by the $m_j + 1$ tangents to the paths.
of steepest descent from $\beta_j$). This is impossible since one of these paths would enter in $\mathcal{R}$, and $|f|$ increases along such a path.

Since the graph has no cycle and its number of edges is $s + p - 1$, one less than the cardinality of its set of vertices $A \cup B$, we conclude that it is a tree (a connected graph with no cycle).

In [4, §8], we use the above to estimate some integrals. To present this application, fix some $\beta_j \in B$ and consider two paths of steepest descent for $|f|$ from $\beta_j$ to roots of $f$. Since the graph constructed above has no cycle, these paths have distinct end-points, say $\alpha_i$ and $\alpha_k$ with $i \neq k$. By composing these paths, we obtain a continuous map $\gamma: [0, 1] \to \mathcal{K}$ with $\gamma(0) = \alpha_i$, $\gamma(1/2) = \beta_j$ and $\gamma(1) = \alpha_k$ such that $|f(\gamma(t))|$ is maximal equal to $|f(\beta_k)|$ at $t = 1/2$. According to Theorem 1.1, the length of $\gamma$ is at most $2\pi NR$, and so we obtain

$$\int_{\alpha_i}^{\alpha_k} f(z)e^{-z}dz \leq \mathcal{L}(\gamma) \max_{\gamma} |f| \max_k |e^{-z}| \leq 2\pi NR e^R|f(\beta_j)|$$

by integrating along $\gamma$, and assuming the disk $D$ centered at 0. In view of Theorem 1.2 if $\alpha_i$ and $\alpha_k$ lie on the boundary of $\mathcal{K}$, we may replace the factor $N$ by $2s$ in this estimate. We wonder if one can replace $N$ by a function of $s$ and $R$ in general.

Note that, for $r = |f(\beta_j)|$, the above points $\alpha_i$ and $\alpha_k$ belong to disjoint connected components of the set $U_r := \{z \in \mathbb{C} ; |f(z)| < r\}$ because otherwise we could construct a simple closed curve passing through $\beta_j$, contained in $\gamma([0, 1]) \cup U_r$, and that would violate the maximum modulus principle for $f$. Thus, along any continuous path linking $\alpha_i$ and $\alpha_k$, the maximum of $|f|$ is at least equal to $|f(\beta_j)|$. In general, for any $r > 0$, the number of connected components of $U_r$ is one more than the number of zeros of $f'$ outside of $U_r$, counting multiplicities [7, §2].

4. Finite Blaschke products

Many of the above observations apply as well to finite Blaschke products mapping the open unit disk $D = \{z \in \mathbb{C} ; |z| < 1\}$ to itself. These are rational functions of the form

$$f(z) = c \prod_{j=1}^{s} \left( \frac{z - \alpha_j}{1 - \alpha_j z} \right)^{n_j}$$

with $|c| = 1$, where $A = \{\alpha_1, \ldots, \alpha_s\}$ is the set of distinct zeros of $f$ in $D$, and where $n_j \geq 1$ denotes the multiplicity of $\alpha_j$ for $j = 1, \ldots, s$. Set $N = n_1 + \cdots + n_s$ and suppose for simplicity that $f$ is normalized so that $f(0) = 0$ and $f(1) = 1$. Then, its derivative $f'$ has exactly $N - 1$ zeros in $D$ counting multiplicities, and these in turn completely determine $f$ (see [6] for details and references). For the set $B$ of zeros $\beta_1, \ldots, \beta_p$ of $f'$ in $D \setminus A$ and their respective multiplicities $m_1, \ldots, m_p$, this means in particular that $m_1 + \cdots + m_p = s - 1$.

Paths of steepest descent for $|f|$ starting on a point of $D$ remain in $D$ and may be continued until they reach a zero of $f$ in $A$, since $|f|$ decreases along such a path. In particular, we may form all paths of steepest descent from an element of $B$ down to a first new element.
of $\mathcal{A} \cup \mathcal{B}$. Following the argument of section 3, this yields $p + s − 1$ curves which draw the edges of a tree on $\mathcal{A} \cup \mathcal{B}$.

Finally, fix any path of steepest descent for $|f|$ from a point $\beta \in D \setminus \mathcal{A}$ to a zero of $f$ in $\mathcal{A}$. We claim that its length is at most $2\pi N$. Since $D$ has radius 1, the Cauchy-Crofton formula reduces this to showing that, except for finitely many angles $\theta \in [0, 2\pi]$, the lines $D_{r, \theta}$ with $r \in \mathbb{R}$ given by (2.5) meet the path in at most $2N$ points. Note that, as $f(0) = 0$, we may assume that $\alpha_1 = 0$ and so, for each $z \in D$, the complex number $\tilde{f}(z)$ is a real multiple of

$$\tilde{f}(z) = cz^{n_1} \prod_{j=2}^{s} ((z - \alpha_j)(1 - \alpha_j \overline{z}))^{n_j}.$$ 

By Lemma 2.1 the ratio $f(z)/f(\beta)$ is a real number at each point $z$ along the path, and thus $\text{Im}(\tilde{f}(z)/f(\beta)) = 0$. So, for given $\theta \in [0, 2\pi]$ and $r \in \mathbb{R}$, the number of points of the path on the line $D_{r, \theta}$ is at most equal to the number of real zeros of the polynomial $\text{Im}(\tilde{f}((r + iu)e^{i\theta})/f(\beta)) \in \mathbb{R}[u]$. Since this polynomial has degree at most $2N - n_1$ and since its coefficient of $u^{2N-n_1}$ is $\text{Im}(c'e^{im_1\theta})$ with a constant $c' \neq 0$, the conclusion follows.

Again, we wonder if there is an upper bound for that length which depends only on $s$.

Acknowledgments. The author thanks the anonymous referee for his suggestions and for mentioning the interesting techniques of Ruscheweyh in [5]. He also thanks the journal editor Javad Mashreghi for the reference [8] and his suggestion to look at finite Blaschke products.

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