The technique of inverse Mellin transform for processes occurring in a background magnetic field

Guey-Lin Lin* 

*Institute of Physics, National Chiao-Tung University
Hsinchu 300, Taiwan

We develop the technique of inverse Mellin transform for processes occurring in a background magnetic field. We show by analyticity that the energy (momentum) derivatives of a field theory amplitude at the zero energy (momentum) is equal to the Mellin transform of the absorptive part of the amplitude. By inverting the transform, the absorptive part of the amplitude can be easily calculated. We apply this technique to calculate the photon polarization function in a background magnetic field.

1. INTRODUCTION

The analytic properties of scattering amplitudes in quantum field theories are well known. They are described by the so called cutting rules [1, 2]. With the cutting rules, the absorptive part of a scattering amplitude can be calculated, while the dispersive part of the amplitude is obtained from the former by the Kramers-Kronig relation. The above procedures can be easily implemented for any quantum field theory with a trivial vacuum. For such a vacuum, the energy-momentum relation of an asymptotic state is simply $E^2 = p^2 + m^2$ with $m$ the mass of the asymptotic state. On the other hand, for processes occurring in a background magnetic field, the energy-momentum relations of asymptotic states receive corrections from the magnetic-field effects. For instance, the energy-momentum relation of an electron (positron) in the background magnetic field is given by

$$E_{n,s_z}^2 = m_e^2 + p_z^2 + eB(2n + 1 + 2s_z), \tag{1}$$

where the background magnetic field is taken to be along the $+z$ direction, $s_z$ is the electron spin projection, and $n$ labels the Landau levels. Due to the above energy quantization caused by a background magnetic field, one expects the absorptive part of a physical amplitude contains an infinite number of contributions distinguished by their corresponding Landau levels. Hence the calculation of the absorptive part is becoming rather involved, and certainly the calculation of the dispersive part is even more intricate.

In this talk, we present a new approach for computing the absorptive part of a physical amplitude in the background magnetic field. The idea is based upon the analyticity. For illustration, we take the one-loop photon polarization function as an example, with a background magnetic field along the $+z$ direction. Since the energy of the internal electron in the photon polarization function is given by Eq. (1), the value of the photon longitudinal momentum $q_{\parallel}^2 \equiv q_{\parallel 0}^2 - q_{\parallel}^2$ determines the threshold for the absorptive part. Therefore the photon polarization function is an analytic function of the photon longitudinal momentum $q_{\parallel}^2$ except on the positive real axis. To obtain the photon polarization function for an arbitrary $q_{\parallel}^2$, it suffices to know the function’s power-series expansion in $q_{\parallel}^2$ at $q_{\parallel}^2 = 0$, because the analytic continuation can map the function from the neighborhood of $q_{\parallel}^2 = 0$ to any value of $q_{\parallel}^2$. A powerful way to perform this analytic continuation is through the inverse Mellin transform [3, 4]. With the inverse Mellin transform, one calculates the absorptive part of the photon polarization function with the knowledge of the above-
mentioned power-series expansion. We note that it is straightforward to obtain such a power series because the neighborhood of $q^2 = 0$ is free of resonant singularities caused by the creation of electron-positron pairs. Knowing the absorptive part, the dispersive part of the photon polarization at any $q^2$ is calculable by the Kramers-Kronig relation.

We organize this presentation as follows: In Section 2, we illustrate the technique of inverse Mellin transform using the vacuum QED as an example. In Section 3, we apply this technique to the photon polarization function in a background magnetic field. A short conclusion is given in Section 4.

2. VACUUM QED

The vacuum polarization tensor in QED is written as

$$i\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^{\mu} q^{\nu})i\Pi(q^2).$$  \hfill (2)

We are interested in calculating the one-loop finite part $\Pi(q^2) = \Pi(q^2) - \Pi(0)$. We observe that $\Pi(q^2)$ satisfies the sum rule [3]:

$$\frac{1}{n!} \left( \frac{d^n}{dq^{2n}} \Pi(q^2) \right)_{q^2 = 0} = \frac{1}{\pi} \int_{M^2}^{\infty} dq^2 \text{Im}\Pi(q^2)(q^2)^{-(n+1)},$$  \hfill (3)

where $M^2$ is the threshold energy for the absorptive part $\text{Im}\Pi(q^2)$. Although the value for $M^2$ is not specified at this moment, the procedure of analytic continuation will automatically generate it. Let us rewrite the sum rule in dimensionless variables:

$$\frac{1}{n!} \left( \frac{d^n}{dt^n} \Pi(t) \right)_{t = 0} = \frac{1}{\pi} \int_{0}^{1} du \text{Im}\Pi(u)u^{n-1},$$  \hfill (4)

where $t = q^2/M^2$ and $u = 1/t$. It is easily seen that the derivatives of $\Pi$ at $t = 0$ is proportional to the Mellin transform of $\text{Im}\Pi$. Hence $\text{Im}\Pi$ at any value of $t$ is obtainable by the inverse Mellin transform:

$$\text{Im}\Pi(u) = \frac{1}{2\pi} \int_{C} ds \ a(s) u^{-s},$$  \hfill (5)

where $a(s)$ is the analytic continuation of $a(n)$ which appears in the power series expansion

$$\Pi(t) = \sum_{n=0}^{\infty} a(n) t^n.$$  \hfill (6)

For the current case, $a(n)$ is given by

$$a(n) = -\frac{2\alpha}{\pi} \frac{\Gamma^2(n+2)}{n\Gamma(2n+4)} \left( \frac{M^2}{m_c^2} \right)^n .$$  \hfill (7)

Applying Eq. (3), we obtain

$$\text{Im}\Pi(u) = -\frac{\alpha}{3} \sqrt{1 - \frac{4m_c^2}{M^2} u \left( 1 + \frac{2m_c^2}{M^2} u \right)} \times \Theta \left( 1 - \frac{4m_c^2}{M^2} u \right).$$  \hfill (8)

We like to remark that, with $u = 1/t = M^2/q^2$, $\text{Im}\Pi(u)$ is independent of the undetermined threshold scale $M^2$, and it agrees with the known result obtained by applying the cutting rules. Having obtained $\text{Im}\Pi(u)$, one can calculate $\text{Re}\Pi(u)$ by the Kramers-Kronig relation.

3. QED IN A BACKGROUND MAGNETIC FIELD

The photon polarization function in a background magnetic field is given by the following proper time representation [3]:

$$\Pi_{\mu\nu}(q) = -\frac{e^2 B}{(4\pi)^2} \int_{0}^{\infty} ds \int_{0}^{s+1} dv \times \{ e^{-is\Phi_0} [ T_{\mu\nu} N_0 - T_\perp,\mu\nu N_\perp ] - e^{-is\Phi_0} (1 - v^2) T_{\mu\nu} \},$$  \hfill (9)

where the photon momentum has been decomposed into $q_\mu^\parallel = (\omega, 0, 0, q_z)$ and $q_\mu^\perp = (0, q_x, q_y, 0)$; while $T_{\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu), T_{\parallel,\mu\nu} = (q_{\parallel}^2 g_{\mu\nu} - q_{\parallel\mu} q_{\parallel\nu}),$ and $T_{\perp,\mu\nu} = (q_{\perp}^2 g_{\mu\nu} - q_{\perp\mu} q_{\perp\nu})$. The phase $\Phi_0$ and the functions $N_0$, $N_\parallel$ and $N_\perp$ are given by

$$\Phi_0 = m_c^2 - \frac{1 - v^2}{4} q_\|^2 - \frac{\cos(zv) - \cos(z)}{2z\sin(z)} q_\perp^2,$$  \hfill (10)

with $z = eBs$, and

$$N_0 = \frac{\cos(zv) - v\cot(z) \sin(z)}{\sin(z)}.$$
The two independent eigenmodes of the above polarization tensor are $\epsilon_1^0$ and $\epsilon_2^0$ which are respectively parallel and perpendicular to the plane spanned by the photon momentum $q$ and the magnetic field $B$. They obey the dispersion equations $q^2 + \Pi_\parallel = 0$ and $q^2 + \Pi_\perp = 0$ respectively with $\Pi_{\perp,\parallel} = \epsilon_{\perp,\parallel}^\mu \epsilon_{\perp,\parallel}^\nu$. It turns out that $\Pi_\parallel$ and $\Pi_\perp$ are proportional to $N_\parallel$ and $N_\perp$ respectively. We shall not discuss the contribution by $N_0$ since it does not correspond to an independent eigenmode.

For simplicity, we shall only focus on the calculation of $\Pi_\parallel$. In particular, we only illustrate the procedure of calculating the partial contribution $\Pi^A_\parallel$ given by

$$
\Pi^A_\parallel = - \frac{\alpha \omega^2 \sin^2 \theta}{4\pi} \int_0^\infty \frac{dz}{\sin(z)} \left( 1 - v^2 + \frac{v \sin(2z)}{\sin(z)} \right) - \frac{\cos(2z)}{\sin(z)},
$$

where the integrand $\cos(2z)/\sin(z)$ is taken from the second term of $N_\parallel$, and $\theta$ is the angle between $q$ and $B$.

The scalar function $\Pi^A_\parallel$ is easy to calculate only for $q_\perp^2 < 4m_e^2$, since, in this momentum region, the contour rotation $s \to -is$ is allowed. The oscillating trigonometric functions are then rotated into hyperbolic functions. Similar to the previous section, one writes $\Pi^A_\parallel$ as a power series in $q_\perp^2$:

$$
\Pi^A_\parallel = - \frac{\alpha \omega^2 \sin^2 \theta}{4\pi} \sum_{n=0}^\infty a(n, q_\perp^2) \left( \frac{q_\perp^2}{4m_e^2} \right)^n.
$$

Following Eq. (6), we calculate $\text{Im}\Pi^A_\parallel$ by

$$
\text{Im}\Pi^A_\parallel(v) = \frac{i\alpha \omega^2 \sin^2 \theta}{8\pi} \int_C ds a(s, q_\perp^2) v^{-s},
$$

with $a(s, q_\perp^2)$ the analytic continuation of $a(n, q_\perp^2)$, and $v = 4m_e^2/q_\perp^2$. We arrive at

$$
\text{Im}\Pi^A_\parallel = \frac{2\alpha \omega B^2 \sin^2 \theta}{q_\parallel^3} \sum_{l_1=1, l_2=0}^{\infty} T_{l_1, l_2}(q_\perp^2) \times \Theta(1 - \frac{4\lambda eB}{q_\parallel^2} + \frac{4\rho^2 e^2 B^2}{q_\parallel^4}),
$$

where $\lambda = l_1 + l_2 + m_e^2/eB$, $\rho = l_1 - l_2$, and the step function indicates the threshold for creating an electron-positron pair occupying the $l_1$-th and $l_2$-th Landau levels. The detailed form of $T_{l_1, l_2}(q_\perp^2)$ is given in Ref. [4]. With $\text{Im}\Pi^A_\parallel$ determined, $\text{Re}\Pi^A_\parallel$ can once more be calculated by the Kramers-Kronig relation.

4. CONCLUSION

In this talk, we have demonstrated the usefulness of analyticity for computing field theory amplitudes. We observed that a field theory amplitude in the neighborhood of zero energy (momentum) determines the amplitude at the arbitrary energy (momentum). The latter can be calculated from the former by the inverse Mellin transform and the Kramers-Kronig relation. We also showed that such an approach is very appropriate for a process occurring in a background magnetic field, where an infinite number of thresholds occur in the amplitude of the process.

REFERENCES

1. R. E. Cutkosky, J. Math. Phys. 1 (1960) 429.
2. M. Veltman, Physica 29 (1963) 186.
3. W.-F. Kao, G.-L. Lin, and J.-J. Tseng, Phys. Lett. B495 (2000) 105.
4. W.-F. Kao, G.-L. Lin, and J.-J. Tseng, Phys. Lett. B522 (2001) 257; Erratum-ibid. B541 (2002) 411.
5. J. Schwinger, Phys. Rev. 82 (1951) 664.
6. W.-y. Tsai and T. Erber, Phys. Rev. D 10 (1974) 492.