QUANTUM INTEGRABILITY IN TWO–DIMENSIONAL SYSTEMS WITH BOUNDARY

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Abstract

In this paper we consider affine Toda systems defined on the half–plane and study the issue of integrability, i.e. the construction of higher–spin conserved currents in the presence of a boundary perturbation. First at the classical level we formulate the problem within a Lax pair approach which allows to determine the general structure of the boundary perturbation compatible with integrability. Then we analyze the situation at the quantum level and compute corrections to the classical conservation laws in specific examples. We find that, except for the sinh–Gordon model, the existence of quantum conserved currents requires a finite renormalization of the boundary potential.
1 Introduction

An integrable field theory possesses higher-spin integrals of motion which allow to determine exactly the on-shell properties of the system [1, 2]. For models defined on the whole two-dimensional plane the existence of these conserved currents is in one-to-one correspondence with the existence of a Lax pair formulation of the equations of motion [3, 4]. The knowledge of the Lax connection leads then to a standard, recursive construction of the whole set of conservation laws. The analysis is more complicated when the two-dimensional theory is defined on a manifold with boundary, typically the upper-half plane. In the presence of the boundary the existence of the conserved currents in the "bulk" region does not guarantee integrability unless special boundary conditions are specified appropriately [5, 6].

In this paper we consider affine Toda-like theories defined on the half plane, perturbed by a boundary potential. We study the construction of classically conserved higher-spin currents starting from the general setting of a Lax pair approach. In section 2 we introduce two gauge fields which are defined in the upper-half plane and contain a non-trivial boundary term. We show that the compatibility conditions for the corresponding Lax pairs (i.e. curvature of the gauge fields equal to zero) provide the standard Toda equations of motion in the bulk region, whereas the boundary conditions on the fields follow from the requirement that the two curvatures coincide at the boundary. This condition also fixes the most general structure of the boundary perturbation compatible with a Lax pair formulation. Then we discuss how to define a Wilson loop operator on the whole plane which is time independent and provides an infinite number of conserved quantities. However, the presence of a boundary does not guarantee the locality of these currents. In section 3 we show that in general the currents obtained from the Lax pair construction consist in the sum of a bulk term and a boundary term. A local solution can be found only if "integrable" boundary conditions are satisfied.

Finally we address the issue of boundary integrability at the quantum level. In section 4 we explicitly compute the first relevant exact quantum currents for the boundary sinh–Gordon theory and the $a_2^{(1)}$ Toda model. While in the sinh–Gordon theory quantum integrability is maintained by a finite renormalization of the currents, in the $a_2^{(1)}$ case the spin–3 current survives the quantization only if a nonperturbative renormalization of the boundary potential is performed. Section 5 contains our conclusions.

2 Lax pair for systems with boundary

In the upper-half plane we consider a Toda-like system defined by the euclidean action

$$S = \frac{1}{\beta^2} \int_{-\infty}^{+\infty} dx_0 \int_{0}^{+\infty} dx_1 \left[ \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} + V \right] - \frac{1}{\beta^2} \int_{-\infty}^{+\infty} dx_0 \ B$$

(2.1)
where $B$ is a generic boundary perturbation, function of the fields but not their derivatives, and $V$ is the affine Toda potential

$$V = \sum_{j=0}^{N} q_j e^{\vec{\alpha}_j \cdot \vec{\phi}}$$ (2.2)

The Toda theory under consideration is based on a Lie algebra $G$ of rank $N$, with simple roots $\alpha_j$, ($j = 1, \cdots, N$), $\alpha_0 = -\sum_{j=1}^{N} q_j \alpha_j$, $q_j$ being the Kac labels ($q_0 = 1$). The action in (2.1) can be rewritten as an integral on the whole $\mathbb{R}^2$ plane

$$S = \frac{1}{\beta^2} \int d^2 x \left\{ \theta(x_1) \left[ \frac{1}{2} \partial_{\mu} \vec{\phi} \cdot \partial_{\mu} \vec{\phi} + V \right] - \delta(x_1) B \right\}$$ (2.3)

In standard way one then obtains the Toda equations of motion in the bulk region

$$\square \vec{\phi} = \sum_{j=0}^{N} q_j \vec{\alpha}_j e^{\vec{\alpha}_j \cdot \vec{\phi}}$$ (2.4)

supplemented by the boundary condition

$$\left. \frac{\partial \phi_a}{\partial x_1} \right|_{x_1=0} = -\frac{\partial B}{\partial \phi_a}$$ (2.5)

Affine Toda systems without boundary are known to be classically integrable [3, 4], namely there exists an infinite number of spin $\pm n$ currents which satisfy the conservation laws

$$\partial J^{(n)} + \partial \Theta^{(n)} = 0 \quad \partial \bar{J}^{(n)} + \partial \bar{\Theta}^{(n)} = 0$$ (2.6)

where we have introduced complex coordinates

$$x = \frac{x_0 + ix_1}{\sqrt{2}} \quad \bar{x} = \frac{x_0 - ix_1}{\sqrt{2}}$$ (2.7)

and derivatives

$$\partial \equiv \partial_x = \frac{1}{\sqrt{2}} (\partial_0 - i \partial_1) \quad \bar{\partial} \equiv \partial_{\bar{x}} = \frac{1}{\sqrt{2}} (\partial_0 + i \partial_1) \quad \square = 2 \partial \bar{\partial}$$ (2.8)

The corresponding conserved charges are

$$q^{(n-1)} = \int dx_1 \left( J^{(n)} + \Theta^{(n)} \right) \quad \bar{q}^{(n-1)} = \int dx_1 \left( \bar{J}^{(n)} + \bar{\Theta}^{(n)} \right)$$ (2.9)

The integrability of these systems is a consequence of the fact that the equations of motion (2.4) are the compatibility conditions for a Lax pair

$$(\partial + A) \chi = 0 \quad (\bar{\partial} + \bar{A}) \chi = 0$$ (2.10)

or equivalently

$$(\partial + \bar{A}) \chi = 0 \quad (\bar{\partial} + A) \bar{\chi} = 0$$ (2.11)
where

\[ A(\lambda) = \partial \vec{\phi} \cdot \vec{h} + \lambda \sum_{j=0}^{N} e_j^+ \]

\[ \tilde{A}(\lambda) = \partial \vec{\phi} \cdot \vec{h} + \lambda \sum_{j=0}^{N} e_j^+ \]

\[ \tilde{A}(\lambda) = \tilde{\tilde{A}}(\lambda) = \frac{1}{2\lambda} \sum_{j=0}^{N} q_j e_j^+ \vec{\alpha} \cdot \vec{\phi} e_j^- \] (2.12)

Here \( \{ \vec{h}, e^+, e^- \} \) are a set of Cartan–Weyl generators for the Lie algebra \( \mathcal{G} \) and \( \lambda \) is the spectral parameter. The field equations (2.4) are then given by

\[ F \equiv [\partial + A, \bar{\partial} + \bar{A}] = 0 \] (2.13)

or equivalently by

\[ \tilde{F} \equiv [\partial + \tilde{A}, \bar{\partial} + \tilde{A}] = 0 \] (2.14)

which are the zero curvature conditions for the “gauge” fields \( A, \tilde{A}, \tilde{\tilde{A}}, \tilde{\tilde{A}} \) respectively. It follows that by a gauge transformation we can always set \( \tilde{A} = \tilde{\tilde{A}} = 0 \) on the whole plane so that the path–ordered Wilson loop

\[ W(\lambda) = \mathcal{P} \mathcal{E} \oint_{\gamma} A(\lambda) \] (2.15)

does not depend on the time variable. In the same way we can define a corresponding Wilson loop in terms of \( \tilde{A}, \tilde{\tilde{A}} \). Finally expanding \( W(\lambda) \) and \( \tilde{W}(\lambda) \) as a series in \( \lambda \) one obtains the infinite set of conserved quantities in (2.6).

If the theory is defined on the semi–infinite plane \( x_1 \geq 0 \), it is easy to show that from the local conservation laws (2.6) valid now in the upper–half plane one can still define conserved charges if the following boundary conditions are satisfied

\[ J_1^{(n)} \bigg|_{x_1 = 0} \equiv i \left( J^{(n)} - \tilde{J}^{(n)} - \Theta^{(n)} + \tilde{\Theta}^{(n)} \right) \bigg|_{x_1 = 0} = -\partial_0 \Sigma_0^{(n)} \] (2.16)

with \( \Sigma_0 \) any local function of the the fields at \( x_1 = 0 \). The corresponding conserved charge is given by

\[ q^{(n-1)} = \int_0^{+\infty} dx_1 J_0^{(n)} + \Sigma_0^{(n)} \] (2.17)

where \( J_0^{(n)} = J^{(n)} + \tilde{J}^{(n)} + \Theta^{(n)} + \tilde{\Theta}^{(n)} \). It has been shown that the condition (2.16) restricts the class of boundary perturbations \( B \). In the sine–Gordon case the classical integrability of the system is maintained for \( B = \gamma \cos \frac{\phi - \phi_0}{2} \) where \( \gamma \) and \( \phi_0 \) are arbitrary constants, whereas in the \( a_n^{(1)} \), \( n > 1 \) case the integrable perturbation is \( B = \sum_{j=0}^{N} d_j e^{\frac{1}{2} \vec{\alpha}_j \cdot \vec{\phi}} \) where the coefficients \( d_j \) must satisfy \( d_j^2 = 4 \).

Our aim now is to recast the above results in a Lax pair framework suitable for the description of two–dimensional systems in the presence of a boundary. We consider a theory described by the action in (2.3) with \( V \) given in (2.2) and a generic boundary perturbation of the form

\[ B = \sum_{j=0}^{N} f_j \] where \( f_j \) is a function of the \( j \)–th root of the algebra. The underlying idea is to
introduce appropriate gauge connections defined in the bulk and at the boundary, such that in the interior and at the border the field equations in (2.4) and (2.5) correspond to a zero curvature condition of the gauge fields. To this end we start again from a “chiral” curvature $F$ and its “antichiral” counterpart $\tilde{F}$ defined in the upper–half plane and at the border as

$$F = \partial \tilde{A} - \bar{\partial} A + [A, \tilde{A}]$$

$$\tilde{F} = \partial \tilde{A} - \bar{\partial} \tilde{A} - [\tilde{A}, \tilde{A}]$$

(2.18)

where now the gauge fields are chosen of the form

$$A = \theta(x_1) \left[ \partial \tilde{\phi} \cdot \tilde{h} + \lambda \sum_{j=0}^{N} e_j^+ \right]$$

$$\tilde{A} = \theta(x_1) \left[ \tilde{\partial} \phi \cdot \tilde{h} + \frac{N}{2} \sum_{j=0}^{N} f_j e_j^- \right]$$

(2.19)

With this choice (cfr. eq. (2.12)) the correct Toda bulk equations of motion (2.4) are given by $F = 0, \tilde{F} = 0$ in the interior region. At the boundary $x_1 = 0$, we impose the condition $F = \tilde{F}$ so that the curvature $\tilde{F}$ can be interpreted as the analytic continuation of $F$ in the lower–half plane. By computing explicitly $(F - \tilde{F})|_{x_1=0}$ we obtain

$$(F - \tilde{F})|_{x_1=0} = \left[ -i \frac{\partial - \bar{\partial}}{\sqrt{2}} \phi - \frac{1}{2} \sum_{j=0}^{N} f_j \tilde{\alpha}_j \right] \cdot \tilde{h}$$

$$- \frac{1}{2\lambda} \sum_{j=0}^{N} \left[ \frac{\partial f_j}{\partial \phi} - \frac{1}{2} f_j \tilde{\alpha}_j \right] \cdot (\partial + \bar{\partial}) \phi e_j^-$$

(2.20)

where we have used the relations $\partial \theta(x_1) = -\tilde{\partial} \phi(x_1) = -\frac{i}{\sqrt{2}} \delta(x_1)$ and $\theta(x_1) \delta(x_1) = \frac{1}{2} \delta(x_1)$.

Therefore $(F - \tilde{F})|_{x_1=0} = 0$ if

$$\frac{\partial f_j}{\partial \phi} = \frac{1}{2} f_j \tilde{\alpha}_j$$

(2.21)

and

$$\frac{\partial \phi}{\partial x_1} \bigg|_{x_1=0} = -\frac{1}{2} \sum_{j=0}^{N} f_j \tilde{\alpha}_j$$

(2.22)

The condition in eq. (2.21) implies that the boundary perturbation must be of the form

$$B = \sum_{j=0}^{N} d_j e^{\frac{i}{2} \tilde{\alpha}_j \tilde{\phi}}$$

(2.23)

for arbitrary coefficients $d_j$, and the relation (2.22) gives the boundary equation (2.5). We emphasize that this result holds for all Toda systems and generalizes what obtained in Ref. [6]. At this stage no further restriction needs be imposed on the boundary perturbation $B$. 

4
The bulk zero–curvature conditions $F = 0$, $\tilde{F} = 0$ and the requirement $F = \tilde{F}$ at the boundary allow now to proceed and construct Wilson loop operators. Thus we define

$$
W(\lambda) = P e^{\int_C A} = e^{\int_D F}
$$

$$
\tilde{W}(\lambda) = P e^{\int_C \tilde{A}} = e^{\int_D \tilde{F}}
$$

(2.24)

where $C$ is any close contour in the upper–half plane enclosing the region $D$. Due to the bulk conditions $F = 0$, $\tilde{F} = 0$ these Wilson operators are equal to 1 in the upper–half plane and do not depend on the choice of $C$. Moreover since $\tilde{F}$ is by definition the analytic continuation of $F$ in the lower–half plane, $\tilde{W}$ can be rewritten as

$$
\tilde{W}(\lambda) = e^{\int_{\tilde{D}} \tilde{F}}
$$

(2.25)

where $\tilde{D}$ is a region enclosed by a contour $\tilde{C}$ in the lower half plane. Therefore the operator $W(\lambda)\tilde{W}(\lambda)$, with $D$ and $\tilde{D}$ as shown in Fig. 1, can be defined on the whole circle and it is there equal to 1 except for possible boundary contributions. In fact the condition $(F - \tilde{F})|_{x_1=0} = 0$ guarantees that boundary effects are not present and we are left with a Wilson operator which is well defined on the whole plane and does not depend on time. We can then expand it in a power series in $\lambda$: if the coefficients are local in the fields they provide the conserved charges of the theory. As we will show in the next section, in the presence of a boundary the requirement of locality is not automatically satisfied and it may impose additional restrictions on the boundary perturbation $B$.

### 3 Classical conserved currents

Now we want to show that the procedure outlined in the previous section provides a consistent way to determine conserved currents for systems with boundary. In general, given the gauge connections as in eq. (2.19), we will find quantities that consist in the sum of a bulk and a boundary contribution

$$
J' = \theta(x_1)J + \delta(x_1)J_B \quad \Theta' = \theta(x_1)\Theta + \delta(x_1)\Theta_B
$$

$$
\tilde{J}' = \theta(x_1)\tilde{J} + \delta(x_1)\tilde{J}_B \quad \tilde{\Theta}' = \theta(x_1)\tilde{\Theta} + \delta(x_1)\tilde{\Theta}_B
$$

(3.1)

Here $J$, $\Theta$ and $\tilde{J}$, $\tilde{\Theta}$ are the currents which satisfy standard conservation laws in the bulk region ( cfr. eq. (2.6) )

$$
\partial_0 (J + \Theta) + i\partial_1 (J - \Theta) = 0 \quad \partial_0 (\tilde{J} + \tilde{\Theta}) - i\partial_1 (\tilde{J} - \tilde{\Theta}) = 0
$$

(3.2)

whereas $J_B$, $\Theta_B$, $\tilde{J}_B$ and $\tilde{\Theta}_B$ are the boundary terms. The Lax pair approach leads to generalized currents defined on the whole plane, with components

$$
J'_0 \equiv J' + \tilde{J}' + \Theta' + \tilde{\Theta}' = \theta(x_1)J_0 + \delta(x_1)\Sigma_0
$$

$$
J'_1 \equiv i(J' - \tilde{J}' - \Theta' + \tilde{\Theta}') = \theta(x_1)J_1 + \delta(x_1)\Sigma_1
$$

(3.3)
where we have defined
\[
J_0 \equiv J + \tilde{J} + \Theta + \tilde{\Theta} \quad \Sigma_0 \equiv J_B + \tilde{J}_B + \Theta_B + \tilde{\Theta}_B
\]
\[
J_1 \equiv i(J - \tilde{J} - \Theta + \tilde{\Theta}) \quad \Sigma_1 \equiv i(J_B - \tilde{J}_B - \Theta_B + \tilde{\Theta}_B)
\] (3.4)

It is easy to check that the conservation law
\[
\partial_0 J'_0 + \partial_1 J'_1 = 0
\] (3.5)
which can be rewritten as
\[
\partial_0(J' + \Theta') + i\partial_1(J' - \Theta') = -\partial_0(\tilde{J}' + \tilde{\Theta}') + i\partial_1(\tilde{J}' - \tilde{\Theta}')
\] (3.6)
holds in the bulk region whenever (3.2) are valid, whereas at the boundary it gives
\[
J_1 \mid_{x_1=0} = \lim_{x_1 \to 0} i(J - \tilde{J} - \Theta + \tilde{\Theta}) \equiv -\partial_0 \Sigma_0
\] (3.7)
Thus it is clear that one can construct local boundary terms in the currents only if the boundary condition in eq. (2.5) allows to express \(J_1 \mid_{x_1=0}\) as a time derivative of a functional of the fields.

If this is the case, from the conservation equation in (3.5) one obtains the corresponding charge
\[
q = \int_{-\infty}^{+\infty} dx_1 J'_0
\] (3.8)
which in fact coincides with the one defined in (2.17).

We consider now as a specific example the sinh–Gordon theory defined on the half plane. It is the simplest affine Toda system which contains a single scalar field. The corresponding action is obtained from eq. (2.1), setting \(N = 1, \alpha_1 = 1, \alpha_0 = -1, \) and \(q_0 = q_1 = 1\) in (2.2) and (2.23) so that
\[
V = e^{-\phi} + e^\phi \quad B = d_0 e^{-\frac{\phi}{2}} + d_1 e^{\frac{\phi}{2}}
\] (3.9)
For the gauge connections in equation (2.19) we choose the following realization of the \(SU(2)\) algebra: \(h = \frac{1}{2}\sigma_x, e_0^- = e_1^+ = -\frac{1}{2\sqrt{2}}(\sigma_z - i\sigma_y)\) and \(e_0^+ = e_1^- = -\frac{1}{2\sqrt{2}}(\sigma_z + i\sigma_y)\) where \(\{\sigma_x, \sigma_y, \sigma_z\}\) are the Pauli matrices.

In the interior region \(x_1 > 0\) the determination of the currents proceeds in standard manner. One writes the Lax equations
\[
(\partial + A)\chi = 0 \quad (\bar{\partial} + \bar{A})\chi = 0
\]
\[
(\partial + \bar{A})\tilde{\chi} = 0 \quad (\bar{\partial} + A)\tilde{\chi} = 0
\] (3.10)
where \(\chi = (\chi_1, \chi_2)\) and \(\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2)\) are two components vectors. Then one introduces new variables \(V = \chi_2/\chi_1, U = \log \chi_1,\) so that (3.10) give
\[
\partial V + \sqrt{2}\lambda \theta(x_1)V - \frac{1}{2} \theta(x_1)\partial \phi V^2 + \frac{1}{2} \theta(x_1)\partial \phi = 0
\]
\[
\bar{\partial} V + \frac{1}{\sqrt{2}\lambda} \theta(x_1) \left[ \cosh \phi V + \frac{1}{2} \sinh \phi (V^2 + 1) \right] = 0
\] (3.11)
and

\[
\partial U + \frac{1}{2} \theta(x_1) \partial \phi \ V - \frac{\lambda}{\sqrt{2}} \theta(x_1) = 0
\]

\[
\partial U - \frac{1}{2 \sqrt{2} \lambda} \theta(x_1) [\sinh \phi \ V + \cosh \phi] = 0
\]  \hspace{1cm} (3.12)

and analogous equations for \( \tilde{V} = \tilde{\chi}_2 / \tilde{\chi}_1 \) and \( \tilde{U} = \log \tilde{\chi}_1 \) exchanging \( \partial \) with \( \tilde{\partial} \) in (3.11), (3.12).

Thus, as a consequence of the two conditions \( F \chi = 0 \) and \( F \tilde{\chi} = 0 \) valid away from the boundary, from (3.12) and the corresponding ones for \( \tilde{U}, \tilde{V} \), using \( \partial \partial U = \tilde{\partial} \tilde{\partial} U \) and \( \partial \partial \tilde{U} = \tilde{\partial} \tilde{\partial} \tilde{U} \) one obtains two conservation equations in the bulk region (cfr. (3.2))

\[
\tilde{\partial} \left( \frac{1}{2} \theta(x_1) \partial \phi \ V - \frac{\lambda}{\sqrt{2}} \theta(x_1) \right) = - \partial \left( \frac{1}{2 \sqrt{2} \lambda} \theta(x_1) [\sinh \phi \ V + \cosh \phi] \right)
\]

\[
\tilde{\partial} \left( \frac{1}{2} \theta(x_1) \tilde{\partial} \phi \ \tilde{V} - \frac{\lambda}{\sqrt{2}} \theta(x_1) \right) = - \tilde{\partial} \left( \frac{1}{2 \sqrt{2} \lambda} \theta(x_1) [\sinh \phi \ \tilde{V} + \cosh \phi] \right)
\]  \hspace{1cm} (3.13)

As in the case of Toda systems without boundary one then expands

\[
V = \sum_{n=1}^{\infty} \frac{a_n}{(\sqrt{2} \lambda)^n} \quad \tilde{V} = \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{(\sqrt{2} \lambda)^n}
\]  \hspace{1cm} (3.14)

and from (3.13) an infinite number of conservation laws is generated. The coefficients \( a_n \) are determined recursively from the first equation in (3.11)

\[
a_1 = -\frac{1}{2} \partial \phi
\]

\[
a_n = -\partial a_{n-1} + \frac{1}{2} \partial \phi \sum_{j=1}^{n-1} a_j a_{n-1-j} \quad , \quad n > 1
\]  \hspace{1cm} (3.15)

with similar expressions for \( \tilde{a}_n \).

At the boundary \( x_1 = 0 \) we impose the additional condition \( F \chi = F \tilde{\chi} \). It is easy to show that this amounts to the following equations

\[
\tilde{\partial} \left[ \sqrt{2} \lambda \theta(x_1) V - \frac{1}{2} \theta(x_1) \partial \phi \ V^2 + \frac{1}{2} \theta(x_1) \partial \phi \right] + \partial \left[ \sqrt{2} \lambda \theta(x_1) \tilde{V} - \frac{1}{2} \theta(x_1) \tilde{\partial} \phi \ \tilde{V}^2 + \frac{1}{2} \theta(x_1) \tilde{\partial} \phi \right]
\]

\[
-\partial \left[ \frac{1}{\sqrt{2} \lambda} \theta(x_1) (\cosh \phi \ V + \frac{1}{2} \sinh \phi \ (V^2 + 1)) \right]
\]

\[
-\tilde{\partial} \left[ \frac{1}{\sqrt{2} \lambda} \theta(x_1) (\cosh \phi \ \tilde{V} + \frac{1}{2} \sinh \phi \ (\tilde{V}^2 + 1)) \right]
\]

\[
-\frac{1}{4 \lambda} \partial \left\{ \delta(x_1) \left[ (d_0 e^{-\tilde{\phi}} + d_1 e^{\tilde{\phi}})(V + \tilde{V}) + \frac{1}{2} (d_1 e^{\tilde{\phi}} - d_0 e^{-\tilde{\phi}})(V^2 + \tilde{V}^2 + 2) \right] \right\} = 0
\]  \hspace{1cm} (3.16)

and

\[
\tilde{\partial} \left( \frac{1}{2} \theta(x_1) \partial \phi \ V \right) + \partial \left( \frac{1}{2} \theta(x_1) \tilde{\partial} \phi \ \tilde{V} \right)
\]

\[
+ \partial \left[ \frac{1}{2 \sqrt{2} \lambda} \theta(x_1) (\sinh \phi \ V + \cosh \phi) \right] + \tilde{\partial} \left[ \frac{1}{2 \sqrt{2} \lambda} \theta(x_1) (\sinh \phi \ \tilde{V} + \cosh \phi) \right]
\]

\[
+ \frac{1}{4 \lambda} \partial \left\{ \delta(x_1) \left[ \frac{1}{2} (d_1 e^{\tilde{\phi}} - d_0 e^{-\tilde{\phi}})(V + \tilde{V}) + (d_0 e^{-\tilde{\phi}} + d_1 e^{\tilde{\phi}}) \right] \right\} = 0
\]  \hspace{1cm} (3.17)
which are obviously well defined on the whole plane. In particular we note that (3.17) expresses the conservation equation in the same form as in (3.6). Now in order to account for the boundary contributions correctly, in the expansion (3.14) we have to replace $a_n \to \theta(x_1)a_n + \delta(x_1)b_n$ and similarly for $\tilde{a}_n$, with the bulk coefficients given in (3.15) and the boundary terms $b_n$, $\tilde{b}_n$ to be determined by the equations (3.16), (3.17).

For example one can easily show that for the stress–energy tensor and the spin–3 current (which is a total derivative), the boundary conditions (3.16) and (3.17) are automatically satisfied by $b_1 = \tilde{b}_1 = 0$ and $b_2 = \tilde{b}_2 = 0$. The first nontrivial case is at spin–4. The bulk contributions $a_3, \tilde{a}_3$ are determined using (3.15). At the boundary, the equations (3.16), (3.17) to second and third order in $\frac{1}{\lambda}$ respectively, give

$$
\partial_0 (b_3 + \tilde{b}_3) = -\frac{1}{2} \partial_0 [(\partial^2 \phi + \partial^2 \bar{\phi})(d_0 e^{-\frac{\phi}{2}} + d_1 e^{\frac{\phi}{2}})] \bigg|_{x_1=0} + \left(\partial^3 \phi - \partial^3 \bar{\phi} + \cosh \phi (\partial \phi - \bar{\partial} \phi)\right) \bigg|_{x_1=0} \tag{3.18}
$$

and

$$
\partial_0 (\partial \phi b_3 + \partial \bar{\phi} \tilde{b}_3) = -\frac{1}{2} \partial_0 [(\partial^2 \phi + \partial^2 \bar{\phi})(d_1 e^{\frac{\phi}{2}} - d_0 e^{-\frac{\phi}{2}})] \bigg|_{x_1=0} - \left(\frac{1}{4}((\partial \phi)^4 - (\partial \bar{\phi})^4) - (\partial \phi \partial^2 \phi - \partial \bar{\phi} \partial^2 \bar{\phi}) + \sinh \phi (\partial^2 \phi - \partial^2 \bar{\phi})\right) \bigg|_{x_1=0} \tag{3.19}
$$

These equations define a spin–4 conserved current with the correct behavior at the boundary if they admit a local solution, i.e. if the quantities in curly brackets are time derivatives of local expressions. While in equation (3.18) this is always true being the curly brackets equal to the $1/\lambda^2$ coefficient of $\partial_0 (V + \tilde{V})$, the expression in curly brackets in equation (3.19) is exactly the bulk component $J_1$ of the spin–4 current evaluated at the boundary. Thus the locality condition requires that $J_1|_{x_1=0}$ be the time derivative of a given function of the fields evaluated at the boundary. It is easy to show that the same pattern repeats itself at any spin level, namely for the spin–$n$ current the boundary conditions have the general form

$$
\partial_0 (b_{n-1} + \tilde{b}_{n-1}) = \partial_0 X \\
\partial_0 (\partial \phi b_{n-1} + \partial \bar{\phi} \tilde{b}_{n-1}) = -J_1^{(n)} \bigg|_{x_1=0} + \partial_0 Y \tag{3.20}
$$

with $X$ and $Y$ local functional of the fields. Notably in the sinh–Gordon case this condition is automatically satisfied by the $B$ potential in (3.9).

The previous arguments can be generalized to the $a_n^{(1)}$ Toda systems. We have checked explicitly for the spin–3 current of the $n = 2$ case that locality of the $b$ and $\tilde{b}$ coefficients requires once again $J_1|_{x_1=0}$ to be a time derivative. As shown in Ref. [6] this condition is not automatically satisfied by a perturbation of the form (2.23) but one has to choose $d_j^2 = 4$, $j = 0, 1, 2$. 

8
4 Quantum conserved currents

In this section we investigate the problem of quantum integrability of systems with boundary by studying the renormalization of the classical conservation laws. As described above in the case of systems defined on the half plane classical conserved currents give rise to physical conserved charges if the extra condition (2.16) is satisfied at the boundary. However at the quantum level the conservation laws (2.6) and the boundary condition (2.16) might be affected by anomalies. We compute these potential anomalous terms and show that they can be reabsorbed in a quantum redefinition of the currents. In addition a finite renormalization of the boundary potential is in general necessary in order to maintain integrability.

The calculation is most easily performed using massless perturbation theory. For the action in (2.1) the massless propagator is defined by the equations

$$\frac{1}{\beta^2} \Box G_{ij}(x, x') = -\delta_{ij} \delta^{(2)}(x - x') \quad \frac{\partial G_{ij}(x, x')}{\partial x_1} \bigg|_{x_1=0} = 0 \quad (4.1)$$

Thus using the relation $$\bar{\partial} \left( \frac{1}{x} \right) = 2\pi \delta^{(2)}(x)$$ one finds

$$G_{ij}(x, x') = -\frac{\beta^2}{4\pi} \delta_{ij} \left[ \log 2|x - x'|^2 + \log 2|x - \bar{x}'|^2 \right] \quad (4.2)$$

Then one treats $$S_{int} = \frac{1}{\beta^2} \int^{+\infty}_{-\infty} dx_0 \int^{+\infty}_{0} dx_1 V$$, with $$V$$ the affine Toda potential (2.2) and $$S_{int}^B = -\frac{1}{\beta^2} \int^{+\infty}_{0} dx_0 B$$, with $$B$$ the boundary perturbation, as interaction terms. In $$V$$ and $$B$$ the exponentials are normal ordered so that perturbative calculations are free from ultraviolet divergences.

At the quantum level the conservation equations become

$$\bar{\partial} \left\langle J^{(n)}(x, \bar{x}) \right\rangle \equiv \bar{\partial} \left\langle J^{(n)}(x, \bar{x}) e^{-S_{int}} \right\rangle_0 = \partial \left\langle \Theta^{(n)} \right\rangle \quad (4.3)$$

for $$x_1 > 0$$ and

$$\left\langle J^{(n)}_1(x, \bar{x}) \right\rangle \bigg|_{x_1=0} \equiv \left\langle J^{(n)}_1(x, \bar{x}) e^{-S_{int}^B} \right\rangle_0 \bigg|_{x_1=0} = \partial_0 - \text{derivative} \quad (4.4)$$

at $$x_1 = 0$$. Anomalous contributions would correspond to local terms obtained by Wick contracting the currents with the exponentials in (1.3) and (4.4). Mixing between bulk and boundary interactions have not been included since they would always produce non–local expressions.

First we consider $$\bar{\partial} \left\langle J^{(n)} \right\rangle$$ in (4.3). Being interested only in local contributions it is sufficient to expand the exponential to first order in $$S_{int}$$. Indeed performing the Wick contractions with the propagator (4.2), we obtain terms of the form

$$\bar{\partial} \int d^2 w \ M(x, \bar{x}) \left[ \frac{1}{(x - w)^k} + \frac{1}{(x - \bar{w})^k} \right] N(w, \bar{w}) \quad (4.5)$$

where $$M, N$$ are products of the fields and their $$\partial$$–derivatives and the integration is performed in the upper–half plane. Now local expressions arise using in the half plane the relation

$$\bar{\partial} \frac{1}{(x - w)^k} = \frac{2\pi}{(k - 1)!} \partial^{k-1} \delta^{(2)}(x - w) \quad (4.6)$$
The contributions obtained in this way either can be rewritten as total $\partial$-derivatives and then give corrections to the quantum trace in (4.3), or must be reabsorbed in a renormalization of the classical current $J^{(n)}$. Following this procedure, which is exact to all-loop orders, one determines the quantum current $J^{(n)}$ and its corresponding trace $\Theta^{(n)}$ defined for $x_1 > 0$.

Then one has to study the condition (4.4) at the boundary, using the exact quantum expression of $J^{(n)}_1 = i(J^{(n)} - \tilde{J}^{(n)} - \Theta^{(n)} + \tilde{\Theta}^{(n)})$. Again we are looking for potential anomalies, namely for local terms which are not expressible as total $\partial_0$-derivatives. In this case one needs consider Wick contractions with higher-order terms in the expansion of the boundary potential. Typically, expanding the exponential in (4.4) to first order, we produce terms with the following general structure

$$\lim_{x_1 \to 0} \int_{-\infty}^{+\infty} dw_0 \left[ \mathcal{P}(x, \bar{x}) \left( \frac{1}{(x - w)^k} + \frac{1}{(\bar{x} - w)^k} \right) - \mathcal{P}(x, \bar{x}) \left( \frac{1}{(\bar{x} - w)^k} + \frac{1}{(x - w)^k} \right) \right] Q(w_0)$$

Here $\mathcal{P}$ is a function of $\partial^p \phi$ and $\tilde{\mathcal{P}}$ correspondingly of $\bar{\partial}^p \phi$. Since $w = \bar{w}$ ($w_1 = 0$) (4.7) gives

$$2(\sqrt{2})^k \lim_{x_1 \to 0} \int_{-\infty}^{+\infty} dw_0 \left[ \mathcal{P}(x, \bar{x}) \frac{1}{(x_0 - w_0 + ix_1)^k} - \tilde{\mathcal{P}}(x, \bar{x}) \frac{1}{(x_0 - w_0 - ix_1)^k} \right] Q(w_0)$$

Then in order to identify local boundary contributions we isolate in $\mathcal{P}$ and $\tilde{\mathcal{P}}$ terms which are identical and use the relation

$$\lim_{x_1 \to 0^+} \left( \frac{1}{(x_0 - w_0 - ix_1)^k} - \frac{1}{(x_0 - w_0 + ix_1)^k} \right) = \frac{2\pi i}{(k-1)!} \partial_{w_0}^{k-1} \delta(x_0 - w_0)$$

In a similar manner contractions with higher-order factors in the expansion of the boundary interaction give rise to local terms whenever the number of delta functions produced in the limit $x_1 \to 0$ equals the number of integrations.

At this stage one has to analyze the local contributions which cannot be written as $\partial_0$-derivatives of suitable expressions, and understand whether they correspond to real boundary anomalies or if it is possible to eliminate them by coupling-constant dependent modifications of the boundary potential. It is easy to show that for any Toda theory defined on the half plane the quantum stress-energy tensor satisfies the condition (2.16) without any new condition on the boundary perturbation $B$. In general restrictions arise when the construction of quantum higher-spin conservation laws is attempted. We have performed the explicit calculation for two cases, namely the spin–4 current of the sinh–Gordon system and the spin–3 current of the $a_2^{(1)}$ Toda model. We illustrate our results in the next two subsections.

### 4.1 The sinh–Gordon model

The first nontrivial classical conserved current for the sinh–Gordon model with boundary is the spin–4 current. In order to determine its quantum version we consider the general expression

$$J^{(4)} = \frac{A}{4}(\partial \phi)^4 + \frac{D}{2}(\partial^2 \phi)^2$$

(4.10)
and study \( \tilde{\partial} \langle J^{(4)} \rangle \) following the procedure outlined above. Thus we evaluate

\[
\tilde{\partial} \langle (\partial^2 \phi)^2 \rangle \sim -\frac{1}{\beta^2} \int d^2 w \left\{ 2 \partial^2 \phi \frac{\beta^2}{4\pi} \left[ \tilde{\partial} \frac{1}{(x-w)^2} + \tilde{\partial} \frac{1}{(x-\bar{w})^2} \right] \frac{\partial V}{\partial \phi} \right. \\
+ \left. \left( \frac{\beta^2}{4\pi} \right)^2 \left[ \tilde{\partial} \frac{1}{(x-w)^4} + \tilde{\partial} \frac{1}{(x-\bar{w})^4} \right] \frac{\partial^2 V}{\partial \phi^2} \right\} \tag{4.11}
\]

where the integrations are performed in the upper–half plane. Using the relation (4.6) we obtain

\[
\tilde{\partial} \langle (\partial^2 \phi)^2 \rangle \sim -\int d^2 w \partial^2 \phi \left[ (\partial \delta^{(2)}(x-w)) + (\partial \delta^{(2)}(x-\bar{w})) \right] \frac{\partial V}{\partial \phi} \\
- \frac{\alpha}{24} \int d^2 w \left[ (\partial^3 \delta^{(2)}(x-w)) + (\partial^3 \delta^{(2)}(x-\bar{w})) \right] \frac{\partial^2 V}{\partial \phi^2} \tag{4.12}
\]

having defined \( \alpha \equiv \frac{\beta^2}{2\pi} \). In the upper–half plane only one of the two delta functions contributes and we find

\[
\tilde{\partial} \langle (\partial^2 \phi)^2 \rangle \sim \partial \left( \frac{\partial V}{\partial \phi} \right) \partial^2 \phi + \frac{\alpha}{24} \partial \left( \frac{\partial^2 V}{\partial \phi^2} \right) \tag{4.13}
\]

Computing in the same manner the term \( \tilde{\partial} \langle (\partial \phi)^4 \rangle \) the total result is

\[
\tilde{\partial} \langle J^{(4)} \rangle = \\
= \partial \left[ \frac{D}{4} \frac{\partial^2 V}{\partial \phi^2} (\partial \phi)^2 + \alpha \frac{D}{48} \frac{\partial^2 V}{\partial \phi^2} + \alpha^2 \frac{A}{32} \frac{\partial^4 V}{\partial \phi^4} + \alpha^3 \frac{A}{353} \frac{\partial^6 V}{\partial \phi^6} \right] \\
+ \frac{1}{2} \left[ \frac{A}{\partial \phi} + \left( -\frac{D}{2} + \alpha \frac{3A}{4} \right) \frac{\partial^3 V}{\partial \phi^3} + \alpha^2 \frac{A}{16} \frac{\partial^5 V}{\partial \phi^5} \right] (\partial \phi)^3 \tag{4.14}
\]

Therefore in the bulk region the conservation law is not anomalous if the coefficients \( A \) and \( D \) satisfy

\[
A \frac{\partial V}{\partial \phi} + \left( -\frac{D}{2} + \alpha \frac{3A}{4} \right) \frac{\partial^3 V}{\partial \phi^3} + \alpha^2 \frac{A}{16} \frac{\partial^5 V}{\partial \phi^5} = 0 \tag{4.15}
\]

Using now the explicit expression for the sinh–Gordon potential \( V = 2 \cosh \phi \) we obtain

\[
D = 2A \left( 1 + \frac{3}{4} \alpha + \frac{\alpha^2}{16} \right) \tag{4.16}
\]

We note that in the classical limit (\( \alpha = 0 \)) this coincides with the standard relation [7]. The corresponding quantum trace is given in (4.14) and can be rewritten as

\[
\Theta^{(4)} = -\frac{1}{4} \left( D + \alpha \frac{D}{12} + \alpha^2 \frac{A}{8} + \alpha^3 \frac{A}{96} \right) \frac{\partial^2 V}{\partial \phi^2} (\partial \phi)^2 - \frac{\alpha}{48} \left( D + \alpha^2 \frac{A}{8} \right) \frac{\partial V}{\partial \phi} \partial^2 \phi \tag{4.17}
\]
where the coefficients $A$ and $D$ satisfy (4.16). The currents with opposite spin $\tilde{J}^{(4)}$, $\tilde{\Theta}^{(4)}$ have similar expressions with $\partial$–derivatives substituted by $\bar{\partial}$–ones.

Now we consider the boundary condition in equation (4.4) and evaluate
\[
\left\langle i(J^{(4)} - \tilde{J}^{(4)} - \Theta^{(4)} + \tilde{\Theta}^{(4)}) e^{\frac{1}{\beta^2} \int dw B} \right\rangle |_{x_1 = 0}
\]

Local contributions from $\left\langle \Theta^{(4)} - \tilde{\Theta}^{(4)} \right\rangle$ arise only from the first order expansion in $B$. First we compute
\[
\left\langle \frac{\partial^2 V}{\partial \phi^2} (\partial \phi)^2 - \frac{\partial^2 V}{\partial \phi^2} (\bar{\partial} \phi)^2 \right\rangle |_{x_1 = 0}
\sim \lim_{x_1 \to 0} \frac{1}{\beta^2} \int dw_0 \frac{\partial B}{\partial \phi} \frac{\partial^2 V}{\partial \phi^2} \left( -\frac{\beta^2}{4\pi} \right) \left[ \frac{(\partial_0 + i\partial_1)\phi}{x_0 - w_0 + ix_1} - \frac{(\partial_0 - i\partial_1)\phi}{x_0 - w_0 - ix_1} \right]
\sim 2i \frac{\partial B}{\partial \phi} \frac{\partial^2 V}{\partial \phi^2} \partial_0 \phi
\]

where the local contribution arises from the terms proportional to $\partial_0 \phi$ by using the relation (4.9). Performing an analogous calculation for the other term in the trace finally we obtain
\[
\left\langle \Theta^{(4)} - \tilde{\Theta}^{(4)} \right\rangle = -\frac{i}{2} \left( D + \alpha \frac{D}{12} + \alpha^2 \frac{A}{8} + \alpha^3 \frac{A}{96} \right) \frac{\partial^2 V}{\partial \phi^2} \frac{\partial B}{\partial \phi} \partial_0 \phi - i \alpha \frac{D}{24} \frac{\partial V}{\partial \phi} \frac{\partial^2 B}{\partial \phi^2} \partial_0 \phi
\]

The calculation of $\left\langle J^{(4)} - \tilde{J}^{(4)} \right\rangle$ is more complicated since in this case local terms arise up to third order in the $B$ expansion. As an example of higher–order contributions we describe here the computation of $\left\langle (\partial \phi)^4 - (\bar{\partial} \phi)^4 \right\rangle$. To first order in $B$ the Wick contractions, up to total $\partial_0$–derivatives, give
\[
\lim_{x_1 \to 0} \frac{1}{\beta^2} \int dw_0 \left\{ 8\sqrt{2} \frac{\partial B}{\partial \phi} \left( -\frac{\beta^2}{4\pi} \right) \left[ \frac{(\partial \phi)^3}{x_0 - w_0 + ix_1} - \frac{(\bar{\partial} \phi)^3}{x_0 - w_0 - ix_1} \right] \right.
\left. + 48 \frac{\partial^2 B}{\partial \phi^2} \left( -\frac{\beta^2}{4\pi} \right)^2 \left[ \frac{(\partial \phi)^2}{(x_0 - w_0 + ix_1)^2} - \frac{(\bar{\partial} \phi)^2}{(x_0 - w_0 - ix_1)^2} \right] \right.
\left. + 64 \sqrt{2} \frac{\partial^3 B}{\partial \phi^3} \left( -\frac{\beta^2}{4\pi} \right)^3 \left[ \frac{\partial \phi}{(x_0 - w_0 + ix_1)^3} - \frac{\bar{\partial} \phi}{(x_0 - w_0 - ix_1)^3} \right] \right\}
\]

Using the boundary equation of motion to zero order in perturbation theory $\frac{\partial \phi}{\partial x_1} |_{x_1 = 0} = 0$ and the identity (4.9), one obtains
\[
\left\langle (\partial \phi)^4 - (\bar{\partial} \phi)^4 \right\rangle |_{x_1 = 0} \sim i \left( 2 \frac{\partial B}{\partial \phi} + 6\alpha \frac{\partial^3 B}{\partial \phi^3} + 2\alpha^2 \frac{\partial^5 B}{\partial \phi^5} \right) (\partial_0 \phi)^3
\]
Following the same procedure one finds that local contributions from the second order expansion in the boundary perturbation vanish being proportional to $\frac{\partial \phi}{\partial x_1}|_{x_1=0}$. The third order contributions give

$$
\left( \frac{1}{\beta^2} \right)^3 \int dw_0 dw'_0 dw''_0 \left\{ 32 \frac{\partial B}{\partial \phi}(w_0) \frac{\partial B}{\partial \phi}(w'_0) \frac{\partial B}{\partial \phi}(w''_0) \left( -\frac{\beta^2}{4\pi} \right)^3 \right. \\
\left. \frac{\partial \phi}{(x-w)(x-w')(x-w'')} - \frac{\partial \phi}{(x-w)(x-w')(x-w'')} \right) \\
+96 \frac{\partial^2 B}{\partial \phi^2}(w_0) \frac{\partial B}{\partial \phi}(w'_0) \frac{\partial B}{\partial \phi}(w''_0) \left( -\frac{\beta^2}{4\pi} \right)^4 \\
\left( \frac{1}{(x-w)^2(x-w')(x-w'')} - \frac{1}{(x-w)^2(x-w')(x-w'')} \right) \right\}
$$

and the result is

$$
\langle (\partial \phi)^4 - (\bar{\partial} \phi)^4 \rangle_{x_1=0}^{(3)} \sim -2i \left( \frac{\partial B}{\partial \phi} \right)^3 + 6\alpha \left( \frac{\partial B}{\partial \phi} \right)^2 \frac{\partial^3 B}{\partial \phi^3} \partial_0 \phi
$$

Contributions from the fourth–order expansion in $B$ give total $\partial_0$–derivatives. The same procedure is applied to compute the local contributions from $\langle (\partial^2 \phi)^2 - (\bar{\partial}^2 \phi)^2 \rangle$. In this case only the first–order expansion in $B$ gives non trivial contributions and the bulk equations of motion are used to evaluate $\partial_1^2 \phi$ at the boundary.

Finally, adding all the local terms which are not total $\partial_0$–derivatives, we find the following quantum boundary condition

$$
\left. \langle i\mathcal{J}^{(4)} - \bar{\mathcal{J}}^{(4)} - \Theta^{(4)} + \bar{\Theta}^{(4)} \rangle \right|_{x_1=0} = \\
-\frac{1}{2} \left[ A \left( \frac{\partial B}{\partial \phi} + 3\alpha \frac{\partial^3 B}{\partial \phi^3} + \alpha^2 \frac{\partial^5 B}{\partial \phi^5} \right) - 2D \frac{\partial^3 B}{\partial \phi^3} \right] (\partial_0 \phi)^3 \\
+ \frac{1}{2} \left\{ - \left( D + \alpha \frac{D}{12} + \alpha^2 \frac{A}{8} + \alpha^3 \frac{A}{96} \right) \frac{\partial^2 V}{\partial \phi^2} \frac{\partial B}{\partial \phi} + \left[ 2D - \alpha \left( D + \alpha^2 \frac{A}{8} \right) \right] \frac{\partial^2 V}{\partial \phi \partial \phi^2} \partial_0 \phi \\
+ A \left( \frac{\partial B}{\partial \phi} \right)^3 + 3A \alpha \left( \frac{\partial B}{\partial \phi} \right)^2 \frac{\partial^3 B}{\partial \phi^3} \right\} \partial_0 \phi
$$

In order to cancel the potential anomalies on the right hand side of this equation, first we need impose the condition

$$
A \frac{\partial B}{\partial \phi} + \left( -2D + 3\alpha A \right) \frac{\partial^3 B}{\partial \phi^3} + \alpha^2 A \frac{\partial^5 B}{\partial \phi^5} = 0
$$

Using the expressions of the quantum coefficients in (4.16) (cfr. also (4.15)) we find that the above condition is always satisfied by a boundary perturbation of the form

$$
B = d_0 e^{-\frac{\phi}{2}} + d_1 e^{\frac{\phi}{2}}
$$
Moreover, in this case the term in (4.25) proportional to $\partial_0 \phi$ is a total $\partial_0$–derivative and no further restrictions need be imposed on the coefficients $d_j$. In conclusion, for the sinh–Gordon theory the existence of a spin–4 quantum conserved current does not require quantum corrections to the boundary perturbation $B$.

### 4.2 The $a_2^{(1)}$ model

For this Toda theory the action in (2.1) is written in terms of two independent scalar fields. With a realization of the simple roots in terms of two dimensional vectors

\[ \vec{\alpha}_1 = (\sqrt{2}, 0) \quad \vec{\alpha}_2 = \left(-\frac{1}{\sqrt{2}}, -\sqrt{3} \frac{1}{2}\right) \]  

(4.28)

the potential in (2.2) becomes

\[ V = e^{\sqrt{2} \phi_1} + e^{-\sqrt{2} \phi_1 - \sqrt{2} \phi_2} + e^{-\sqrt{2} \phi_1 + \sqrt{2} \phi_2} \]

(4.29)

This model has a spin–3 classical conserved current with the general form

\[ J^{(3)} = \frac{1}{3} A_{abc} \partial \phi_a \partial \phi_b \partial \phi_c + b_{ab} \partial^2 \phi_a \partial \phi_b \]

(4.30)

where the coefficients $A_{abc}$ and $b_{ab}$, completely symmetric and antisymmetric respectively, are determined so that the classical conservation law (2.6) is satisfied \[3\]. At the classical level, as shown in \[3\], the boundary condition (2.16) fixes the values of the coefficients, $d_j^2 = 4$, which appear in the boundary perturbation $B = \sum_{j=0}^{2} d_j e^{\frac{1}{2} \vec{\alpha}_j \cdot \vec{\phi}}$.

Now we study the conservation of the current in (4.30) at the quantum level. We start considering the conservation law in the upper–half plane and evaluate

\[ \hat{\partial} \left< J^{(3)} \left(-\frac{1}{\beta^2}\right) \int d^2 w V \right> \]

Using the massless propagator (4.2) and following a procedure similar to the one described for the sinh–Gordon model, we easily find

\[ \hat{\partial} \left< J^{(3)} \right> \sim \]

\[ \sim \partial \left[ \frac{1}{2} b_{ab} V_b \partial \phi_a + \frac{\alpha^2}{48} A_{abc} \partial V_{abc} \right] \]

\[ + \frac{1}{2} \left[ A_{abc} V_a + b_{ac} V_{ab} + b_{ab} V_{ac} + \frac{\alpha}{4} A_{abd} V_{acd} + \frac{\alpha}{4} A_{acd} V_{abd} \right] \partial \phi_b \partial \phi_c \]

(4.31)

where we have defined $V_a \equiv \frac{\partial V}{\partial \phi_a}$ and dropped all the non–local contributions. Therefore absence of quantum anomalies in the conservation of $J^{(3)}$ requires that the terms on the right–hand–side which are not total $\partial$–derivatives vanish

\[ A_{abc} V_a + b_{ac} V_{ab} + b_{ab} V_{ac} + \frac{\alpha}{4} A_{abd} V_{acd} + \frac{\alpha}{4} A_{acd} V_{abd} = 0 \]

(4.32)
We note that in the classical limit this expression reproduces the result of [6]. Introducing the explicit expression of $V$ we compute the quantum coefficients and obtain

$$J^{(3)} = (\partial \phi_1)^2 \partial \phi_2 - \frac{1}{3}(\partial \phi_2)^3 + \frac{1}{\sqrt{2}}(1 + \frac{\alpha}{2})\partial^2 \phi_2 \partial \phi_1 - \frac{1}{\sqrt{2}}(1 + \frac{\alpha}{2})\partial^2 \phi_1 \partial \phi_2$$

(4.33)

This current coincides with the one determined in Ref. [8] up to a total $\partial$–derivative. Finally from equation (4.31) we construct the quantum trace

$$\Theta^{(3)} = -\frac{1}{2} b_{ab} V_b \partial \phi_a - \frac{\alpha^2}{48} A_{abc} \partial V_{abc} =$$

$$= -\frac{1}{2\sqrt{2}} (1 + \frac{\alpha}{2}) V_2 \partial \phi_1 + \frac{1}{2\sqrt{2}} (1 + \frac{\alpha}{2}) V_1 \partial \phi_2 - \frac{\alpha^2}{16} \partial V_{112} + \frac{\alpha^2}{48} \partial V_{222}$$

(4.34)

The same procedure can be applied to compute the quantum currents $\tilde{J}^{(3)}$, $\tilde{\Theta}^{(3)}$ whose expressions are obtained from (4.33), (4.34) by exchanging holomorphic derivatives with antiholomorphic ones.

Now we concentrate on the boundary condition (4.4). Thus we consider

$$\left< i(J^{(3)} - \tilde{J}^{(3)} - \Theta^{(3)} + \tilde{\Theta}^{(3)}) e^{\frac{i}{\hbar} \int dw_0 B} \right|_{x_1=0} \right. 0$$

(4.35)

where for the time being we leave the coefficients $A_{abc}$ and $b_{ab}$ unspecified. Local corrections to the classical condition (2.16) come from contractions of the currents with the exponential expanded up to the third order. The calculation is performed along the lines described in detail for the sinh–Gordon model. Summing all the contributions the final result is

$$\left< i(J^{(3)} - \tilde{J}^{(3)} - \Theta^{(3)} + \tilde{\Theta}^{(3)}) \right|_{x_1=0} =$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{1}{3} A_{abc} B_a B_b B_c + 2 b_{ab} V_a B_b + \frac{\alpha^2}{24} A_{abc} V_{abcd} B_d \right]$$

$$- \frac{1}{\sqrt{2}} [A_{abc} B_a + 2 b_{ab} B_a B_c + 2 b_{ac} B_a B_b + \alpha A_{abd} B_{acd} + \alpha A_{acd} B_{abd}] \partial_0 \phi_b \partial_0 \phi_c$$

(4.36)

In order to cancel the term proportional to $\partial_0 \phi_b \partial_0 \phi_c$ we require

$$A_{abc} B_a + 2 b_{ab} B_a B_c + 2 b_{ac} B_a B_b + \alpha A_{abd} B_{acd} + \alpha A_{acd} B_{abd} = 0$$

(4.37)

In the classical limit it matches the result in Ref. [3]. Comparing (4.37) with (4.32) it is easy to see that the quantum corrections in both identities are such that if (4.32) is satisfied with $V = \sum_{j=0}^2 q_j e^{\tilde{\alpha}_j \tilde{\phi}}$, then (4.37) is also satisfied with $B = \sum_{j=0}^2 d_j e^{\frac{i}{\hbar} \tilde{\alpha}_j \tilde{\phi}}$. The coefficients $d_j$ are determined and actually acquire an explicit quantum correction once the other condition

$$\frac{1}{3} A_{abc} B_a B_b B_c + 2 b_{ab} V_a B_b + \frac{\alpha^2}{24} A_{abc} V_{abcd} B_d = 0$$

(4.38)

is imposed, with $A_{abc}$ and $b_{ab}$ determined from eq. (4.32) or equivalently from (4.37). This relation follows from the requirement of complete cancellation of local anomalies in (4.36)
is nonlinear in $B$ and thus it imposes nontrivial constraints on the coefficients. We note that setting $\alpha = 0$ the classical result in Ref. [6] is reproduced, with $d^0_j = 4$, $j = 0, 1, 2$. However the presence in (4.37) and (4.38) of quantum corrections modifies this solution. We solve the equation (4.38) by introducing the following notation (see Ref. [6])

$$V = \sum_{j=0}^{2} e^2_j \quad B = \sum_{j=0}^{2} d_j e^2_j \quad (4.39)$$

where we have defined $e_i = e^{\frac{\alpha}{2}} e_i$. Moreover we define

$$A_{ijk} \equiv A_{abc}(\alpha_i)_{a}(\alpha_j)_{b}(\alpha_k)_{c} \quad b_{ij} \equiv b_{ab}(\alpha_i)_{a}(\alpha_j)_{b} \quad C_{ij} \equiv \bar{\alpha}_i \cdot \alpha_j \quad (4.40)$$

which, as a consequence of (4.32), satisfy

$$A_{ijk} + b_{ik} C_{ij} + b_{ij} C_{ik} + \frac{\alpha}{4} [A_{iij} C_{ik} + A_{iik} C_{ij}] = 0 \quad (4.41)$$

From this equation and the antisymmetry of $b_{ij}$ one easily derives

$$A_{iii} = 0 \quad A_{iij} = -A_{jjj} = -\frac{2}{1 + \frac{\alpha}{2}} b_{ij} \quad (4.42)$$

for any $i, j = 0, 1, 2$. Using the previous relations we rewrite (4.38) as

$$\sum_{i \neq j} \left( -\frac{1}{4} + \frac{1}{1 + \frac{\alpha}{2}} d^2_j + 1 \right) b_{ij} d^2_j e^2_i e_j = 0 \quad (4.43)$$

Consequently, in order to maintain at the quantum level the conservation of the $q^{(2)}$ charge the coefficients $d_j$ must be modified as

$$d^2_j = 4 \left( 1 + \frac{\alpha}{2} \right) \quad j = 0, 1, 2 \quad (4.44)$$

This result is exact to all loop orders. Therefore the conservation of the $q^{(2)}$ charge requires the addition to the action (2.1) of an infinite number of finite boundary interactions, i.e. a nonperturbative renormalization of the coefficients $d_j$.

5 Conclusions

We have studied the integrability properties of Toda theories defined in the upper–half plane presenting a Lax pair approach which allows to determine the general structure of the boundary perturbation compatible with the existence of classical higher–spin conserved charges. We have found that, given the bulk potential $V = \sum_j q_j e^{\alpha_i} \bar{\phi}$, the integrable boundary perturbation is $B = \sum_j d_j e^{\alpha_j} \bar{\phi}$. This result generalizes to all Toda models what obtained in Ref. [6] for the
Then we have illustrated the procedure for the construction of the conserved currents in the interior region supplemented by the boundary condition.

At the quantum level the conservation laws have been studied using a suitable generalization of the massless perturbation procedure which is standard for systems without boundary. The requirement of cancellation of local anomalies leads to a renormalization of the classical currents. We have studied in detail two explicit examples, the spin–4 current of the sinh–Gordon theory and the spin–3 current of the $a_2^{(1)}$ Toda model. In the first case quantum corrections induce a coupling–constant modification of the current, but no restrictions need be imposed on the $d_0$, $d_1$ coefficients of the boundary potential. In the $a_2^{(1)}$ example we have found that in order to insure the quantum conservation of the corresponding charge, in addition to a renormalization of the spin–3 current, a finite, nonperturbative renormalization of the boundary perturbation is necessary. We emphasize that these results are quantum exact. As argued in Ref. at the classical level the restriction on the coefficients of the boundary perturbation seems to be a common feature of all Toda models except for the sinh–Gordon theory, i.e. $a_1^{(1)}$. It would be interesting to investigate whether the existence of quantum higher–spin conserved charges requires a finite renormalization of the boundary potential in all Toda systems.
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Figure 1: Contours for the definition of the Wilson loop operators
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