CONFLICT-FREE INCIDENCE COLORING AND TWO-WAY RADIO NETWORKS

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ABSTRACT
In this paper, we introduce the conflict-free incidence coloring of graphs to model a problem of designing two-way radio networks efficiently and economically. Specifically, we call the vertex-edge pair \((v, e)\) an incidence of a graph. A conflict-free incidence coloring of a graph is a coloring of the incidences in such a way that two incidences \((u, e)\) and \((v, f)\) get distinct colors if and only if they conflict each other, i.e., (i) \(u = v\), (ii) \(uv\) is \(e\) or \(f\), or (iii) there is a vertex \(w\) such that \(uw = e\) and \(vw = f\). The minimum number of colors used among all conflict-free incidence colorings of a graph is the conflict-free incidence chromatic number. For a simple graph with maximum degree \(\Delta\), we claim that its conflict-free incidence chromatic number is either \(2\Delta\), \(2\Delta + 1\), or \(2\Delta + 2\), and each of them can be attained by infinite many graphs. We also show that the conflict-free incidence chromatic number of an outer-1-planar graph with maximum degree \(\Delta\) is either \(2\Delta\) or \(2\Delta + 1\), and moreover, characterize all outer-1-planar graphs whose conflict-free incidence chromatic numbers are exactly \(2\Delta\) or \(2\Delta + 1\).

Keywords: two-way radio network; channel assignment problem; incidence coloring; outer-1-planar graph.

1 Introduction
For groups of geographically separated people who need to keep in continuous voice communication, such as aircraft pilots and air traffic controllers, two-way radios are widely used [16]. This motivates us to investigate how to design a two-way radio network efficiently and economically.

In a two-way radio network, each node represents a two-way radio that can both transmit and receive radio waves and there is a link between two nodes if and only if they may contact each other. Waves can transmit between

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two linked two-way radios in two different directions simultaneously. For a link \( L \) connecting two nodes \( N_i \) and \( N_j \) in two-way radio network, it is usually assigned with two channels \( C(N_i, N_j) \) and \( C(N_j, N_i) \). The former one is used to transmit waves from \( N_i \) to \( N_j \) and the later one is used to transmit waves from \( N_j \) to \( N_i \). The associated channel box \( B(N_i) \) of a node \( N_i \) in two-way radio network is a multiset of channels \( C(N_i, N_j) \) and \( C(N_j, N_i) \) such that \( N_i \) is linked to \( N_j \). An efficient way to avoid possible interference is to assign channels to links so that every radio receives a rainbow associated channel box (in other words, every two channels in \( B(N_i) \) for every node \( N_i \) in the network are apart). For the sake of economy, while assigning channels to a two-way radio network, the fewer channels are used, the better. This can be modeled by the conflict-free incidence coloring of graphs.

From now on, we use the language of graph theory and then define conflict-free incidence coloring. We consider finite graphs and use \( V(G) \) and \( E(G) \) to denote the vertex set and the edge set of a graph \( G \). The degree \( d_G(v) \) of a vertex \( v \) in a graph \( G \) is the number of edges incident with \( v \) in \( G \). We use \( d(v) \) instead of \( d_G(v) \) whenever the graph \( G \) is clear from the content. We call \( \Delta(G) = \max\{d_G(v) \mid v \in V(G)\} \) and \( \delta(G) = \min\{d_G(v) \mid v \in V(G)\} \) the maximum degree and the minimum degree of a graph \( G \). Other undefined notation is referred to [3].

Let \( v \) be a vertex of \( G \) and \( e \) be an edge incident with \( v \). We call the vertex-edge pair \( (v,e) \) an incidence of \( G \). For an edge \( e = uv \in E(G) \), let \( \text{Inc}(e) = \{(u,v), (v,e)\} \), and for a vertex \( v \in V(G) \), let \( \text{Inc}(v) = \cup_{e \ni v} \text{Inc}(e) \). For a subset \( U \subseteq E(G) \), let \( \text{Inc}(U) = \{\text{Inc}(e) \mid e \in U\} \). Two incidences \( (u,e) \) and \( (v,f) \) are conflicting if (i) \( u = v \), (ii) \( uw \) is \( e \) or \( f \), or (iii) there is a vertex \( w \) such that \( uw = e \) and \( vw = f \). In other words, two incidences are conflicting if and only if there is a vertex \( w \) such that both of them belong to \( \text{Inc}(w) \).

A conflict-free incidence \( k \)-coloring of a graph \( G \) is a coloring of the incidences using \( k \) colors in such a way that every two conflicting incidences get distinct colors. The minimum integer \( k \) such that \( G \) has a conflict-free incidence \( k \)-colorable is the conflict-free incidence chromatic number of \( G \), denoted by \( \chi'_c(G) \). For a conflict-free incidence coloring \( \varphi \) of a graph \( G \) and an edge \( e = uv \in E(G) \), we use \( \varphi(\text{Inc}(e)) \) to denote the set \( \{\varphi(u,e), \varphi(v,e)\} \). For a subset \( U \subseteq E(G) \), let \( \varphi(\text{Inc}(U)) = \{\varphi(\text{Inc}(e)) \mid e \in U\} \).

We look back into the channel assignment problem of two-way radio networks and explain why the conflict-free incidence coloring of graphs can model it. Let \( G \) be the graph representing the two-way radio network and let \( L = N_iN_j \) be an arbitrary link, i.e, \( L \in E(G) \). Assigning two channels \( C(N_i, N_j) \) and \( C(N_j, N_i) \) to \( L \) is now equivalent to coloring the incidences \( (N_i, L) \) and \( (N_j, L) \). The goal of assigning every radio \( N_i \) a rainbow associated channel box is translated to coloring the incidences of \( G \) so that every two incidences in \( \text{Inc}(N_i) \) receive distinct colors. This is exactly what we shall do while constructing a conflict-free incidence coloring of \( G \).

From a theoretical point of view, one may be interested in a fact that the conflict-free incidence coloring relates to the \( b \)-fold edge-coloring, which is an assignment of sets of size \( b \) to edges of a graph so that adjacent edges receive disjoint sets. An \( (a : b) \)-edge-coloring is a \( b \)-fold edge coloring out of \( a \) available colors. The \( b \)-fold chromatic index \( \chi'_b(G) \) is the least integer \( a \) such that an \( (a : b) \)-edge-coloring of \( G \) exists. It is not hard to check that \( \chi'_c(G) = \chi'_2(G) \) for every graph \( G \). However, there are hard problems related to \( \chi'_2(G) \), among which the most famous one is the Berge-Fulkerson conjecture [8], which states that every bridgeless cubic graph has a collection of six perfect matchings that together cover every edge exactly twice. This is equivalent to conjecture
that every bridgeless cubic graph $G$ has a $(6:2)$-edge-coloring, i.e., $\chi'_2(G) \leq 6$. This conjecture is still widely open \([7, 9, 11, 12]\) and was generalized by Seymour \([14]\) to $\gamma$-graphs.

The structure of this paper organizes as follows. In Section 2, we establish certain fundamental results for the conflict-free incidence chromatic number of graphs. In Section 3, we investigate the conflict-free incidence coloring of outer-1-planar graphs by showing that $2\Delta \leq \chi'_i(G) \leq 2\Delta + 1$ for outer-1-planar graphs $G$ with maximum degree $\Delta$, and moreover, characterizing outer-1-planar graphs $G$ with $\chi'_i(G)$ equal to $2\Delta$ or $2\Delta + 1$.

2 Fundamental results

Let $\chi'(G)$ be the chromatic index of $G$, the minimum integer $k$ such that $G$ admits an edge $k$-coloring so that adjacent edges receive distinct colors. The following is an interesting relationship between $\chi'_i(G)$ and $\chi'(G)$.

**Proposition 1.** $2\Delta(G) \leq \chi'_i(G) \leq 2\chi'(G)$.

**Proof.** Since $|\text{Inc}(v)| = 2\Delta(G)$ for a vertex $v$ with maximum degree, $\chi'_i(G) \geq 2\Delta(G)$ for every graph $G$. If $\varphi$ is a proper edge coloring of $G$ using the colors $\{1, 2, \ldots, \chi'(G)\}$, then one can construct a conflict-free incidence $2\chi'(G)$-coloring of $G$ such that $\varphi(\text{Inc}(e)) = \{\varphi(e), \varphi(e) + \chi'(G)\}$ for every edge $e \in E(G)$. It follows that $\chi'_i(G) \leq 2\chi'(G)$. 

The well-known Vizing’s theorem (see \([3]\) p128]) states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every simple graph $G$. This divides simple graphs into two classes. A simple graph $G$ belongs to class one if $\chi'(G) = \Delta(G)$, and belongs to class two if $\chi'(G) = \Delta(G) + 1$. The following are immediate corollaries of Proposition 1.

**Proposition 2.** If $G$ is a class one graph, then $\chi'_i(G) = 2\Delta(G)$.

**Proposition 3.** If $G$ is simple graph, then $\chi'_i(G) \in \{2\Delta(G), 2\Delta(G) + 1, 2\Delta(G) + 2\}$.

The well-known Kőnig’s theorem (see \([3]\) p127]) states that every bipartite graph is of class 1. So the following is immediate by Proposition 2.

**Theorem 2.1.** If $G$ is a bipartite graph, then $\chi'_i(G) = 2\Delta(G)$.

Now that we have Proposition 2, it would be worth determining the conflict-free incidence chromatic number of a certain class of graphs of class two. For example, odd cycles are of class two, and their conflict-free incidence chromatic numbers are 5, twice of its maximum degree plus one, except $C_3$, whose conflict-free incidence chromatic number is 6. Actually, given a cycle $C_{2n+1}$ ($n \geq 2$) with vertices $v_1, v_2, \ldots, v_{2n+1}$ in this order, we can construct a conflict-free incidence 5-coloring $\varphi$ in such a way that $\varphi(\text{Inc}(v_i v_{i+1})) = \{1, 2\}$ if $i \leq 2n - 3$ is odd, $\varphi(\text{Inc}(v_i v_{i+1})) = \{3, 4\}$ if $i \leq 2n - 2$ is even, $\varphi(\text{Inc}(v_{2n-1} v_{2n})) = \{1, 5\}$, $\varphi(\text{Inc}(v_{2n} v_{2n+1})) = \{2, 3\}$, and $\varphi(\text{Inc}(v_{2n+1} v_1)) = \{4, 5\}$. On the other hand, one can easily see that $C_{2n+1}$ ($n \geq 2$) does not admit a conflict-free incidence 4-coloring. Consequently, the following is immediate (note that even cycles are of class one).
Theorem 2.2.

\[ \chi_c^c(C_n) = \begin{cases} 
4 & \text{if } n \text{ is even,} \\
5 & \text{if } n \geq 5 \text{ is odd,} \\
6 & \text{if } n = 3.
\end{cases} \]

We now pay attention to complete graphs. The famous result of Fiorini and Wilson [5] states that complete graphs on even number of vertices are of class 1. Hence Proposition 2 directly imply the following.

**Proposition 4.** \( \chi_c^c(K_{2n}) = 2\Delta(K_{2n}) = 4n - 2. \)

Fiorini and Wilson [5] also showed that complete graphs on odd number of vertices are of class 2, and thus Proposition 2 cannot be applied to those graphs. However, we can determine the conflict-free incidence chromatic numbers of them from another view of point.

**Proposition 5.** Let \( G \) be the graph obtained from \( K_{2n+1} \) by removing less than \( n/2 \) edges, then \( \chi_c^c(G) = 2\Delta(G) + 2 = 4n + 2. \)

**Proof.** We first show that \( \chi_c^c(G) \geq 4n + 2. \) Suppose for a contradiction that \( \varphi \) is a conflict-free incidence \((4n + 1)\)-coloring of \( G \). Since \( G \) totally has more than \( 4n^2 + 2n - n = (4n + 1)n \) incidences, there is a color of \( \varphi \), say 1, that has been used at least \( n + 1 \) times. Since every two strong incidences of a vertex are differently colored, there are \( n + 1 \) vertices of \( G \), say \( v_1, v_2, \ldots, v_{n+1} \), such that for each \( 1 \leq i \leq n + 1 \), \( \varphi(v_i, v_i u_i) = 1 \), where \( u_i \) is one neighbor of \( v_i \). Since every two weak incidences of a vertex are also differently colored, each \( u_i \) is different from every \( u_j \) with \( j \neq i \). If \( u_i \) coincides with some \( v_j \) with \( j \neq i \), then \( \varphi(v_i, v_i u_i) = \varphi(u_i, u_i u_j) \), a contradiction as \( (v_i, v_i u_i) \) conflicts \( (u_i, u_i u_j) \). Hence each \( u_i \) is different from every \( v_j \) with \( j \neq i \). It follows that \( V(G) \supseteq \bigcup_{i=1}^{n+1} \{u_i, v_i\} \) and thus \( |V(G)| \geq 2n + 2 \), a contradiction. To show the equality, we apply proposition 4 to \( G \). It follows that \( \chi_c^c(G) \leq 2\Delta(G) + 2 = 4n + 2 \), as desired.

Combining Propositions 4 and 5 together, we conclude the following.

**Theorem 2.3.**

\[ \chi_c^c(K_n) = \begin{cases} 
2n - 2 & \text{if } n \text{ is even,} \\
2n & \text{if } n \text{ is odd,}
\end{cases} \]

3 The conflict-free incidence chromatic number of outer-1-planar graphs

In this section we determine the conflict-free incidence chromatic numbers of outer-1-planar graphs, a subclass of planar partial 3-tree [1]. Formally speaking, a graph is outer-1-planar if it can be drawn in the plane so that vertices are on the outer-boundary and each edge is crossed at most once. The concept of outer-1-planarity was first introduced by Eggleton [4] and outer-1-planar graphs were also known as outerplanar graphs with edge crossing number one [4] and pseudo-outerplanar graphs [15, 18, 20].

The most popular result on the edge coloring of planar graphs is that planar graphs with maximum degree at least 7 is of class one [13, 17]. Since there exist class two planar graphs with maximum degree \( \Delta \) for each \( \Delta \leq 5 \),
the remaining problem is to determine whether every planar graph with maximum degree 6 is of class one, and this is still quite open (see survey [2]). Therefore, investigating the edge coloring of subclasses of planar graphs seems to be natural and interesting. Fiorini [6] showed that every outerplanar graph is of class one if and only if it is not an odd cycle, and this conclusion had been generalized to the class of series-parallel graphs by Juvan, Mohar, and Thomas [10]. Zhang, Liu, and Wu [20] showed that outer-1-planar graphs with maximum degree at least 4 is of class one. The chromatic indexes of outer-1-planar graphs with maximum degree at most 3 was completely determined by Zhang [19].

We restate Zhang’s definition [19] as follows. Let $G_2, G_4, G_8$ and $H_t$ be configurations defined by Figure 1. For any solid vertex $v$ of a configuration and any graph $G$ containing such a configuration, the degree of $v$ in $G$ is exactly the number of edges that are incident with $v$ in the picture.

A graph belongs to the class $\mathcal{P}$, if it is isomorphic to $K_4^+$ (equal to $K_4$ with one edge subdivided) or derives from a graph $G \in \mathcal{P}$ by one of the following operations:

$G \sqcup_z G_t$ with $t = 2, 4, 8$ remove a vertex $z$ of degree two from $G$, and then paste a copy of $G_2$, or $G_4$, or $G_8$ on the current graph accordingly, by identifying $x$ and $y$ with $z_1$ and $z_2$, respectively, where $z_1$ and $z_2$ are the neighbors of $z$ (see Figure 2 for an example);

$G \vee_{z_1 z_2} H_t$ with $t \geq 1$ remove an edge $z_1 z_2$ from $G$, and then paste a copy of $H_t$ on the current graph by identifying $x_t$ and $y_t$ with $z_1$ and $z_2$, respectively (see Figure 3 for an example).

Let $\mathcal{P}^+$ be the class of connected outer-1-planar graphs with maximum degree 3 that contains some graph in $\mathcal{P}$ as a subgraph. Now we summarize the result of Zhang [19] and Zhang, Liu, and Wu [20] as follows.
Figure 2: The graph on the left shows $G$ and the one on the right shows $G \sqcup z G_2$

Figure 3: The graph on the left shows $G$ and the one on the right shows $G \vee_{z_1,z_2} H_1$

Theorem 3.1.

\[
\chi'(G) = \begin{cases} 
\Delta(G) & \text{if } G \not\in P^+ \text{ and } G \text{ is not an odd cycle,} \\
\Delta(G) + 1 & \text{otherwise,}
\end{cases}
\]

if $G$ is a connected outer-1-planar graph.

\[\square\]

**Remark on Theorem 3.1:** Zhang [19] claimed that every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \not\in P$. However, this statement is incorrect. Indeed, Zhang showed that every graph in $P$ is of class two. This further implies that every outer-1-planar graph with maximum degree 3 that contains some graph in $P$ is of class two. In other words, every graph in $P^+$ is of class two. Using the same proof of Theorem 3.3 in [19], one can show that if $G$ is a connected outer-1-planar graph with maximum degree 3 not in $P^+$ then it is of class one (note that the minimal counterexample to this statement is 2-connected and thus Zhang’s original proof works now). Conclusively, every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \not\in P^+$. Combining this with the result of Zhang, Liu, and Wu [20] that every outer-1-planar graph with maximum degree at least 4 is of class one, we have Theorem 3.1.

The following is an immediate corollary of Theorem 3.1 and Proposition 1.

**Theorem 3.2.** If $G$ is a connected outer-1-planar graph such that $G \not\in P^+$ and $G$ is not an odd cycle, then $\chi^c_i(G) = 2\Delta(G)$.

\[\square\]

The next goal of this section is to prove $\chi^c_i(G) = 2\Delta(G) + 1$ if $G \in P^+$ or $G$ is an odd cycle unless $G \cong C_3$. Theorem 2.2 supposes this conclusion while $G$ is an odd cycle of length at least 5. Hence in the following we assume that $G \in P^+$. Note that $K^+_4$ is the smallest graph (in terms of the order) in $P^+$. Now we prove $\chi^c_i(G) = 7$ for every graph $G \in P^+$ by a series of lemmas.

**Lemma 3.3.** $\chi^c_i(K^+_4) = 7$. 

6
Proof. The first picture in the first row of Figure 4 shows a conflict-free incidence 7-colorable of $K^+_4$, so it is sufficient to show contradictions for another two cases. Without loss of generality, we assume $\varphi(x_1, x_1'x_1') = \varphi(x_2, x_2'x_2') = \varphi(x_3, x_3'x_3') = 1$. If $x_i = x_j$ or $x'_i = x'_j$ or $x_i = x'_j$ for some $1 \leq i < j \leq 3$, then $(x_i, x_i'x_i')$ and $(x_j, x_j'x_j')$ are conflicting and thus they cannot in a same color. Hence $|\{x_1, x_2, x_3, x_1', x_2', x_3'\}| = 6$, contradicting the fact that $|K^+_4| = 5$.

Figure 4: A conflict-free incidence 7-colorable of $K^+_4$

Suppose for a contradiction that $\varphi$ is a conflict-free incidence 6-coloring of $K^+_4$. Since $K^+_4$ has 7 edges and 14 incidences, there is a color, say 1, such that $\varphi(x_1, x_1'x_1') = \varphi(x_2, x_2'x_2') = \varphi(x_3, x_3'x_3') = 1$. If $x_i = x_j$ or $x'_i = x'_j$ or $x_i = x'_j$ for some $1 \leq i < j \leq 3$, then $(x_i, x_i'x_i')$ and $(x_j, x_j'x_j')$ are conflicting and thus they cannot in a same color. Hence $|\{x_1, x_2, x_3, x_1', x_2', x_3'\}| = 6$, contradicting the fact that $|K^+_4| = 5$.

From now on, if we say coloring a graph or a configuration we mean coloring its incidences so that every two conflicting ones receive distinct colors.

Lemma 3.4. If the configuration $G_2$ is colored with 6 colors under $\varphi$, then $\varphi(\text{Inc}(vx)) \cap \varphi(\text{Inc}(wy)) = \emptyset$.

Proof. If $\varphi$ is a conflict-free incidence 6-coloring of $G_2$, then $\varphi(u, uv), \varphi(v, uv), \varphi(u, uw), \varphi(w, uw), \varphi(v, vw)$ and $\varphi(w, vw)$ are pairwise distinct, so we assume, without loss of generality, that they are $1, 2, 3, 4, 5,$ and $6$, respectively. This forces that $\varphi(\text{Inc}(vx)) = \{3, 4\}$ and $\varphi(\text{Inc}(wy)) = \{1, 2\}$, as desired.

Lemma 3.5. If the configuration $G_4$ is colored with 6 colors under $\varphi$, then $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$.

Proof. If $\varphi$ is a conflict-free incidence 6-coloring of $G_4$, we have three cases: $\varphi(\text{Inc}(u_1x)) = \varphi(\text{Inc}(v_1y))$, or $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, or $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = 1$. If $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume $\varphi(\text{Inc}(u_1x)) = \{1, 2\}, \varphi(\text{Inc}(u_1v_0)) = \{3, 4\},$ and $\varphi(\text{Inc}(u_0u_1)) = \{5, 6\}$.

Case 1. $\varphi(\text{Inc}(u_1x)) = \varphi(\text{Inc}(v_1y))$.

$\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0u_1)) = \{1, 2, 5, 6\}$ and $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_1v_0)) = \{1, 2, 3, 4\}$ forces $\varphi(\text{Inc}(u_0v_1)) = \{3, 4\}$ and $\varphi(\text{Inc}(v_0v_1)) = \{5, 6\}$, respectively. And then $\varphi(\text{Inc}(u_0u_1, u_0v_1)) = \varphi(\text{Inc}(u_0v_0, v_0v_1)) = \{3, 4, 5, 6\}$ forces $\varphi(\text{Inc}(u_0w)) = \varphi(\text{Inc}(v_0w)) = \{1, 2\}$, which is impossible.

Case 2. $|\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y))| = 1$.

Assume, by symmetry, that $\varphi(\text{Inc}(v_1y)) = \{1, a\}$, where $a \in \{3, 4\}$. It follows that $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0u_1)) = \{1, a, 5, 6\}$, forcing $\varphi(\text{Inc}(u_0v_1)) = \{2, b\}$, $b \in \{3, 4\} \setminus \{a\}$. Now $\varphi(\text{Inc}(u_0u_1, u_0v_1)) = \{2, b, 5, 6\}$ and $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0v_1)) = \{1, 2, 3, 4\}$, which implies $\varphi(\text{Inc}(u_0w)) = \{1, a\}$ and
\[\varphi(\text{Inc}(v_0v_1)) = \{5, 6\}, \text{respectively. It follows that } \varphi(\text{Inc}(u_1v_0, v_0v_1, u_0w)) = \{1, 3, 4, 5, 6\} \text{ and thus } \text{Inc}(uw_0) \text{ have to be colored with 2, which is impossible.} \]

Lemma 3.6. If the configuration \( G_8 \) is colored with 6 colors under \( \varphi \), then \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset. \)

Proof. If \( \varphi \) is a conflict-free incidence 6-coloring of \( G_8 \), we have three cases: \( \varphi(\text{Inc}(u_2x)) = \varphi(\text{Inc}(v_1y)) \), or \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), or \( |\varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y))| = 1. \) If \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume \( \varphi(\text{Inc}(v_1y)) = \{1, 2\} \), \( \varphi(\text{Inc}(u_0v_1)) = \{3, 4\} \), and \( \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \).

Case 1. \( \varphi(\text{Inc}(u_2x)) = \varphi(\text{Inc}(v_1y)). \)

\[\varphi(\text{Inc}(u_2x)) \cup \varphi(\text{Inc}(v_0v_1)) = \{1, 2, 5, 6\} \text{ and } \varphi(\text{Inc}(u_0v_1, v_0v_1)) = \{3, 4, 5, 6\} \text{ forces } \varphi(\text{Inc}(u_2v_0)) = \{3, 4\} \text{ and } \varphi(\text{Inc}(u_0v_0)) = \{1, 2\} \text{, respectively. And then } \varphi(\text{Inc}(u_0v_0, u_0v_1)) = \varphi(\text{Inc}(u_2x)) \cup \varphi(\text{Inc}(u_2v_0)) = \{1, 2, 3, 4\} \text{ forces } \varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(u_1u_2)) = \{5, 6\}, \text{ which is impossible.} \]

Case 2. \( |\varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y))| = 1. \)

Assume, by symmetry, that \( \varphi(\text{Inc}(u_2x)) = \{1, a\} \), where \( a \in \{5, 6\} \). It follows that \( \varphi(\text{Inc}(u_0v_1)) \cup \varphi(\text{Inc}(v_0v_1)) = \{3, 4, 5, 6\} \) and \( \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \), forcing \( \varphi(\text{Inc}(u_0v_0)) = \{1, 2\} \) and \( \varphi(\text{Inc}(u_2v_0)) = \{3, 4\} \). Now \( \varphi(\text{Inc}(u_0v_0, u_0v_1)) = \{1, 2, 3, 4\} \) which implies \( \varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(u_1u_2)) = \{5, 6\} \). It follows that \( \varphi(\text{Inc}(u_0u_1, u_2x, u_2v_0)) = \{1, 3, 4, 5, 6\} \) and thus \( \text{Inc}(u_1u_2) \) have to be colored with 2, which is impossible. \)

Lemma 3.7. If the configuration \( H_t \) with some \( t \geq 1 \) is colored with 6 colors under \( \varphi \), then \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)). \)

Proof. We prove it by induction on \( t \). If \( \varphi \) is a conflict-free incidence 6-coloring of \( H_1 \), then we assume, without loss of generality, \( \varphi(x', x'y'), \varphi(y', x'y'), \varphi(x', x'y'), \varphi(y_0, x'y_0), \varphi(x', x'x_0), \) and \( \varphi(x_0, x'x_0) \) are 1, 2, 3, 4, 5, and 6, respectively. Since \( \varphi(\text{Inc}(x'y', x'x_0)) = \{1, 2, 5, 6\} \) and \( \varphi(\text{Inc}(x'y', x'y_0)) = \{1, 2, 3, 4\} \), we have \( \varphi(\text{Inc}(x_0'y')) = \{3, 4\} \) and \( \varphi(\text{Inc}(y_0'y_0)) = \{5, 6\} \), which imply \( \varphi(\text{Inc}(x_0x_1)) = \varphi(\text{Inc}(y_0y_1)) = \{1, 2\} \). This completes the proof of the base case. Now suppose that the lemma holds for \( H_{t-1} \) with some \( t \geq 2 \) and prove that it also holds for \( H_t \). By the induction hypothesis, \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})). \) This implies \( \varphi(\text{Inc}(x_{t-1}x_t)) = \{1, 2, 3, 4, 5, 6\} \setminus \varphi(\text{Inc}(x_{t-2}x_{t-1})) \cup \varphi(\text{Inc}(x_{t-1}y_{t-1})) \} \) and \( \varphi(\text{Inc}(y_{t-1}y_t)) = \{1, 2, 3, 4, 5, 6\} \setminus \varphi(\text{Inc}(y_{t-2}y_{t-1})) \cup \varphi(\text{Inc}(x_{t-1}y_{t-1})) \}, \) and thus \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)) \), as desired. \)

Lemma 3.8. If \( \varphi \) is a partial incidence coloring of the configuration \( G_2 \) such that \( \varphi(\text{Inc}(vx)) \cap \varphi(\text{Inc}(wy)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_2 \).

Proof. Suppose \( \varphi(\text{Inc}(vx)) = \{1, 2\} \) and \( \varphi(\text{Inc}(wy)) = \{3, 4\} \). It is easy to see that we can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_2 \) by coloring \( \text{Inc}(uw, uw, vw) \) so that \( \varphi(\text{Inc}(uv)) = \{3, 4\}, \varphi(\text{Inc}(uw)) = \{1, 2\}, \) and \( \varphi(\text{Inc}(vw)) = \{5, 6\}. \)
Lemma 3.9. If \( \varphi \) is a partial incidence coloring of the configuration \( G_4 \) such that \( \varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_4 \).

Proof. Suppose \( \varphi(\text{Inc}(u_1x)) = \{1, 2\} \) and \( \varphi(\text{Inc}(v_1y)) = \{3, 4\} \). It is easy to see that we can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_4 \) by coloring \( \text{Inc}(v_0v_1, v_0w, u_0w, u_1v_0, u_0u_1, v_0v_1) \) so that \( \varphi(\text{Inc}(v_0v_1)) = \varphi(\text{Inc}(v_0w)) = \{1, 2\}, \varphi(\text{Inc}(u_0w)) = \varphi(\text{Inc}(u_1v_0)) = \{3, 4\}, \) and \( \varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \).

\[ \square \]

Lemma 3.10. If \( \varphi \) is a partial incidence coloring of the configuration \( G_8 \) such that \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_8 \).

Proof. Suppose \( \varphi(\text{Inc}(u_2x)) = \{1, 2\} \) and \( \varphi(\text{Inc}(v_1y)) = \{3, 4\} \). It is easy to see that we can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_4 \) by coloring the incidences on \( v_0v_1, u_0u_1, u_0v_0, u_1u_2, u_0v_1, \) and \( u_2v_0 \) so that \( \varphi(\text{Inc}(v_0v_1)) = \varphi(\text{Inc}(u_0u_1)) = \{1, 2\}, \varphi(\text{Inc}(u_0v_0)) = \varphi(\text{Inc}(u_1u_2)) = \{3, 4\}, \) and \( \varphi(\text{Inc}(u_0v_1)) = \varphi(\text{Inc}(u_2v_0)) = \{5, 6\} \).

\[ \square \]

Lemma 3.11. If \( \varphi \) is a partial incidence coloring of the configuration \( H_t \) with some \( t \geq 1 \) such that \( \varphi(\text{Inc}(x_{t-1}x_1)) = \varphi(\text{Inc}(y_{t-1}y_t)) \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( H_t \).

Proof. We prove it by induction on \( t \). If \( \varphi \) is a partial incidence coloring of the configuration \( H_1 \) such that \( \varphi(\text{Inc}(x_0x_1)) = \varphi(\text{Inc}(y_0y_1)) = \{1, 2\} \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of \( H_t \) by coloring \( \text{Inc}(x'y', x'y_0, x_0y', x'x_0, y'y_0) \) so that \( \varphi(\text{Inc}(x'y')) = \{1, 2\}, \varphi(\text{Inc}(x'y_0)) = \varphi(\text{Inc}(x_0y')) = \{3, 4\}, \) and \( \varphi(\text{Inc}(x'x_0)) = \varphi(\text{Inc}(y'y_0)) = \{5, 6\} \). This completes the proof of the base case. Now suppose that the lemma holds for \( H_{t-1} \) with some \( t \geq 2 \) and prove that it also holds for \( H_t \). Assume, without loss of generality, that \( \varphi(\text{Inc}(x_{t-1}x_1)) = \varphi(\text{Inc}(y_{t-1}y_t)) = \{1, 2\} \). We extend \( \varphi \) by coloring \( \text{Inc}(x_{t-2}x_{t-1}, y_{t-2}y_{t-1}, x_{t-1}y_{t-1}) \) so that \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})) = \{3, 4\} \) and \( \varphi(\text{Inc}(x_{t-1}y_{t-1})) = \{5, 6\} \). This constructs a partial incidence coloring of the configuration \( H_{t-1} = H_t - \{x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t\} \) such that \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})) \). Since any incidence of \( \text{Inc}(x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t) \) is conflict-free to any incidence of \( \text{Inc}(H_{t-1}) \), by the induction hypothesis, the extended \( \varphi \) can be further extended to a conflict-free incidence 6-coloring of the configuration \( H_t \).

\[ \square \]

Proposition 6. If \( G \in \mathcal{P} \), then \( \chi^c_t(G) = 7 \).

Proof. We proceed by induction on \( |G| \). Since the smallest graph in \( \mathcal{P} \) is \( K^+_4 \), and \( \chi^c_t(K^+_4) = 7 \) by Lemma 3.3, the proof of the base case has been done. Now assume \( |G| > 5 \). By the construction of \( \mathcal{P} \), we meet four cases. Here and elsewhere, once \( G \) contains a configuration as shown in Figure 1, we use the same labelling of any vertex appearing on the configuration as the one marked in the corresponding picture.
Case 1. There is a graph $G' \in \mathcal{P}$ and a degree 2 vertex $z$ of $G'$ such that $G = G' \sqcup z G_2$ (or $G = G' \sqcup z G_4$, or $G = G' \sqcup z G_8$, respectively).

By the induction hypothesis, $\chi^c_i(G') = 7$. Let $z_1, z_2$ be two neighbors of $z$ in $G'$ and let $\varphi$ be a conflict-free incidence $7$-coloring of $G'$. Clearly, $\varphi(Inc(zz_1)) \cap \varphi(Inc(zz_2)) = \emptyset$. We construct a conflict-free incidence $7$-coloring $\phi$ of $G$ as follows. Let $\phi(Inc(vx)) = \varphi(Inc(zz_1))$ and $\phi(Inc(uy)) = \varphi(Inc(zz_2))$ (or $\phi(Inc(u_1x)) = \varphi(Inc(zz_1))$ and $\phi(Inc(v_1y)) = \varphi(Inc(zz_2))$, or $\phi(Inc(u_2x)) = \varphi(Inc(zz_1))$ and $\phi(Inc(v_1y)) = \varphi(Inc(zz_2))$, respectively). This makes a partial incidence coloring of the configuration $G_2$ (or $G_4$, or $G_8$, respectively) such that $\varphi(Inc(vx)) \cap \varphi(Inc(uy)) = \emptyset$ (or $\varphi(Inc(u_1x)) \cap \varphi(Inc(v_1y)) = \emptyset$, or $\varphi(Inc(u_2x)) \cap \varphi(Inc(v_1y)) = \emptyset$, respectively). By Lemma 3.8 (or Lemma 3.9 or Lemma 3.10, respectively), $\varphi$ can be extended to a conflict-free incidence $7$-coloring of the configuration $G_2$ (or $G_4$, or $G_8$, respectively) and thus any two conflicting incidences of $I(E(G) \setminus E(G'))$ receive distinct colors. Now for every edge $e \in E(G) \cap E(G')$, let $\phi(Inc(e)) = \varphi(Inc(e))$. This completes a $7$-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence $6$-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence $6$-coloring of $G$. By Lemma 3.4 (or Lemma 3.5 or Lemma 3.6, respectively), $\phi(Inc(vx)) \cap \phi(Inc(uy)) = \emptyset$ (or $\phi(Inc(u_1x)) \cap \phi(Inc(v_1y)) = \emptyset$, or $\phi(Inc(u_2x)) \cap \phi(Inc(v_1y)) = \emptyset$, respectively). This makes us possible to construct a conflict-free incidence $6$-coloring $\varphi$ of $G'$ by setting $\varphi(Inc(zz_1)) = \phi(Inc(vx))$, $\varphi(Inc(zz_2)) = \phi(Inc(uy))$, (or $\varphi(Inc(zz_1)) = \phi(Inc(u_1x))$, $\varphi(Inc(zz_2)) = \phi(Inc(v_1y))$, or $\varphi(Inc(zz_1)) = \phi(Inc(u_2x))$, $\varphi(Inc(zz_2)) = \phi(Inc(v_1y))$, respectively) and $\varphi(Inc(e)) = \phi(Inc(e))$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction.

Case 2. There is a graph $G' \in \mathcal{P}$ and an edge $z_z z_2$ of $G'$ such that $G = G' \vee z_z z_2 H_i$.

By the induction hypothesis, $\chi^c_i(G') = 7$. Let $\varphi$ be a conflict-free incidence $7$-coloring of $G'$. We construct a conflict-free incidence $7$-coloring $\phi$ of $G$ as follows. Let $\phi(Inc(x_{i-1} x_i)) = \phi(Inc(y_{i-1} y_i)) = \varphi(Inc(zz_2))$. This makes a partial incidence coloring of the configuration $H_i$ such that $\phi(Inc(x_{i-1} x_i)) = \phi(Inc(y_{i-1} y_i))$. By Lemma 3.11, $\phi$ can be extended to a conflict-free incidence $7$-coloring of the configuration $H_i$. Now for every edge $e \in E(G) \cap E(G')$, let $\phi(Inc(e)) = \varphi(Inc(e))$. This completes a $7$-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence $6$-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence $6$-coloring of $G$. By Lemma 3.7 $\phi(Inc(x_{i-1} x_i)) = \phi(Inc(y_{i-1} y_i))$. This makes us possible to construct a conflict-free incidence $6$-coloring $\varphi$ of $G'$ by setting $\varphi(Inc(zz_2)) = \phi(Inc(x_{i-1} x_i))$ and $\varphi(Inc(e)) = \phi(Inc(e))$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction. 

We are ready to prove our desired result as follows.

**Theorem 3.12.** If $G \in \mathcal{P}^+$, then $\chi^c_i(G) = 7$. 
Proof. We proceed by induction on $|G|$. Note that the base case is supported by Lemma 3.3 By the definition of $\mathcal{P}$, every graph in $\mathcal{P}$ has exactly one vertex of degree 2, besides which all vertices are of degree 3. By Proposition 6, we assume $G \in \mathcal{P}^+ \setminus \mathcal{P}$.

Suppose that $G$ contains a graph $H \in \mathcal{P}$ as a proper subgraph. Let $u$ be the unique vertex of degree 2 of $H$ and let $v$ and $w$ be the two neighbors of $u$ in $H$. Since $\Delta(G) \leq 3$ and $G$ is connected, the degree of $u$ in $G$ must be 3. Let $x$ be the third neighbor of $u$ in $G$. Since every vertex in $V(H) \setminus \{u\}$ has degree 3 in $H$ (and thus in $G$), $u$ is a cut-vertex of $G$.

Let $H'$ be the other component of $G$ containing $u$ besides $H$. Since $u$ has degree 1 in $H'$, $H'$ is not an odd cycle. Therefore, if $H' \in \mathcal{P}^+$, then $\chi_i^c(H') = 7$ by the induction hypothesis, and if $H' \notin \mathcal{P}^+$, then $\chi(H') = \Delta(H') \leq 3$ by Theorem 3.1 and thus $\chi_i^c(H') \leq 6$ by Proposition 1. In each case, there is a conflict-free incidence 7-coloring $\varphi'$ of $H'$.

Since $H \in \mathcal{P}$, there is a conflict-free incidence 7-coloring $\varphi$ of $H$ by Proposition 6. We exchange (if necessary) the colors of $\varphi$ so that $\varphi(\text{Inc}(uv)), \varphi(\text{Inc}(uw))$, and $\varphi'(\text{Inc}(ux))$ are pairwise disjoint, and then obtain a conflict-free incidence 7-coloring of $G$ by combining $\varphi$ with $\varphi'$. This implies $\chi_i^c(G) \leq 7$.

On the other hand, $\chi_i^c(G) \geq \chi_i^c(H) = 7$. Hence $\chi_i^c(G) = 7$. □

According to Theorem 2.2, Theorem 3.2, and Theorem 3.12 we can completely determine the conflict-free incidence chromatic number of connected outer-1-planar graphs (and thus all outer-1-planar graphs). Note that the proofs involved in this section make us possible to efficiently find a conflict-free incidence $\chi_i^c(G)$-coloring of every outer-1-planar graph $G$ in polynomial time.

**Theorem 3.13.**

$$
\chi_i^c(G) = \begin{cases} 
6 & \text{if } G \cong C_3, \\
2\Delta(G) & \text{if } G \notin \mathcal{P}^+ \text{ and } G \text{ is not an odd cycle,} \\
2\Delta(G) + 1 & \text{otherwise}
\end{cases}
$$

for every connected outer-1-planar graph $G$.

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