FIXED POINTS AND LIMITS OF CONVOLUTION POWERS OF
CONTRACTIVE QUANTUM MEASURES

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Abstract. We study fixed points of contractive convolution operators associated to contractive quantum measures on locally compact quantum groups. We characterise the existence of non-zero fixed points respectively on $L^\infty(G)$ and on $C_0(G)$, and exploit these results to obtain for example the structure of the fixed points on the non-commutative $L^p$-spaces. Some consequences for the fixed points of classical convolution operators and Herz-Schur multipliers are also indicated.

Convolution operators associated to measures on a locally compact group $G$ form a rich and interesting class of transformations, acting on various function spaces associated with $G$, and playing a key role in the study of probabilistic, geometric and harmonic-analytic phenomena related to $G$. In particular the random walk interpretations provide motivation to analyse the limit behaviour of iterations of convolution operators (as nicely described in [Grn] and presented from the point of view of quantum generalisations in [Sal]), and thus in particular the nature of idempotent probability measures. The latter are very well understood, but, perhaps surprisingly, dropping the positivity requirement makes the problem of characterising the class of idempotent measures very difficult, and it is only fully solved for contractive measures [Gre] and for abelian groups [Coh]. More generally, one can ask about the structure of the space of fixed points of a given convolution operator (see [ChL], of which more will be said below), which at least in the positive case can be interpreted as the collection of harmonic functions for a given measure – or as the space of functions on the corresponding Poisson boundary (see for example [Kai] and references therein).

If the group $G$ in question is abelian, the convolution operators can be also viewed as Fourier multipliers. This perspective makes it natural to study also fixed points of such operators even when $G$ is non-abelian. This is to a large extent the point of view taken in the lecture notes [ChL], where a lot of care is devoted to studying for example the space of fixed points of contractive Herz–Schur multipliers acting on the group von Neumann algebra $VN(G)$.

The theory of locally compact quantum groups, as formulated by Kustermans and Vaes in [KV], provides a framework which allows asking these types of questions from a unified perspective, at the same time vastly generalising the class of objects studied. The concept of quantum convolution operators, as investigated for example in [JNR] and [Daw] has now been reasonably well understood, and in particular several questions related to the nature of idempotent states ([SaS1] and references therein) or the structure of the fixed points of positive quantum convolution operators ([KNR1], [KNR2]) have found satisfactory answers. In our previous work, [NSSS], we looked at the idempotent contractive quantum measures and characterised their form, generalising the results of Greenleaf mentioned above (see also [Kas]). Here we investigate the structure of the fixed point spaces, so in other words ‘generalised harmonic elements’, associated to arbitrary quantum contractive measures.

The quantum context necessitates a distinction between the ‘universal’ and ‘reduced’ quantum measures (respectively understood as bounded functionals on the universal and reduced algebra of continuous functions on $G$ vanishing at infinity); both of these induce convolution operators on the algebra $L^\infty(G)$ (as well as for example on $C_0(G)$). We work primarily in the more general...
setting of universal measures. It is natural to expect that the space of fixed points of a given operator is related to a limit of its Cesáro averages. To be able to exploit this fully, we introduce yet another class of generalised measures, i.e. the dual of the space of ‘universal right uniformly continuous functions’. With this in hand, we are able to show our first main result, Theorem 2.3, characterising the existence of non-zero fixed points for the convolution operator on \( L^\infty(G) \). The results on the structure of the fixed point space, often showing that it is in fact one-dimensional (if the measure in question is non-degenerate) require assuming more about the nature of a potential fixed point. The central theorems of this type are Theorem 2.7 and Proposition 4.3. These results, although at first glance quite technical in nature, turn out to have several interesting consequences; both in the general quantum group context, where they for example allow us to characterise fixed points in non-commutative \( L_p \)-spaces, but also in the classical framework and its dual, where they lead to generalisations of several earlier theorems (e.g. of [ChL]). In several instances we are able to connect the fixed points of the convolution by a quantum contractive measure \( \omega \) to the fixed points of the convolution by the absolute value of \( \omega \), which opens the way to exploiting the state results already available in the literature.

Finally we would like to recall that the fixed points of a positive (quantum) convolution operator admit a canonical structure of a von Neumann algebra equipped with the action of the (quantum) group in question; this is nothing but a (quantum) Poisson boundary, as mentioned above. Once we consider general contractive (quantum) convolution operators, the fixed point space admits a canonical TRO structure, and the corresponding (quantum) group action. The study of such actions was initiated in [SaS2]; it is however fair to say that it remains in a relatively early stage and deeper harmonic analysis applications are only to be developed.

The detailed plan of the paper is as follows: in Section 1 we recall basic facts and notations related to locally compact quantum groups and introduce the algebra \( RUC^\infty(G)^* \) and associated quantum convolution operators, which play a useful technical role in what follows. In Section 2 non-zero fixed points in \( L^\infty(G) \) are studied, and their existence characterised. Here also we get the first results on the structure of the fixed point space if the ‘universal’ convolution operator admits a non-zero fixed point in \( LUC^\infty(G) \). In Section 3 we turn our attention to the preduals of \( L^\infty(G) \)-fixed points and use them to characterise certain properties of the quantum group in question. The fourth section concerns fixed points in \( C_0(G) \); the results obtained there are used in the fifth section to discuss the existence of non-zero fixed points in \( L_p(G) \) (for tracial Haar weights). In Section 6 the general results are specialised to the context of classical locally compact groups, and appropriately strengthened. Finally in Section 7 we discuss the ‘dual to classical’ case.

1. Preliminaries

1.1. Locally compact quantum groups. We follow here the von Neumann algebraic approach to locally compact quantum groups due to Kustermans and Vaes [KV], see also [KNR1] and [KNR2] for more background. A locally compact quantum group \( G \), effectively a virtual object, is studied via the von Neumann algebra \( L^\infty(G) \), playing the role of the algebra of essentially bounded functions on \( G \), equipped with a comultiplication \( \Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \), which is a unital normal coassociative \(*\)-homomorphism. A locally compact quantum group \( G \) is by definition assumed to admit a left Haar weight \( \phi \) and a right Haar weight \( \psi \) – these are faithful, normal semifinite weights on \( L^\infty(G) \) satisfying suitable invariance conditions. We can associate to \( G \) also its algebra of continuous functions vanishing at infinity, \( C_0(G) \). It is a \( C^* \)-subalgebra of \( L^\infty(G) \) and the comultiplication restricts to a map from \( C_0(G) \) to the multiplier algebra of \( C_0(G) \otimes C_0(G) \). We say that \( G \) is compact if \( C_0(G) \) is unital, in which case we denote it simply \( C(G) \). The GNS representation space for the left Haar weight will be denoted by \( L^2(G) \). We may in fact assume that \( C_0(G) \) is a non-degenerate subalgebra of \( B(L^2(G)) \). Each locally compact quantum group \( G \) admits the dual locally compact quantum group \( \hat{G} \); in fact the algebra \( L^\infty(\hat{G}) \) acts naturally on \( L^2(G) \). We call \( G \) discrete, if \( \hat{G} \) is compact. If \( G = \hat{G} \) happens to be a locally compact group, then \( L^\infty(G) = VN(G) \). Finally note that by analogy with the classical situation we denote the predual of \( L^\infty(G) \) by \( L^1(G) \).
To each locally compact quantum group $\mathbb{G}$, we may also associate a universal $C^*$-algebra $C_0(\mathbb{G})$ with comultiplication $\Delta_u$; see [Kus]. There is a surjective $*$-homomorphism $\Lambda : C_0(\mathbb{G}) \to C_0(\mathbb{G})$ such that $(\Lambda \otimes \Lambda) \circ \Delta_u = \Delta \circ \Delta$. We call $\Lambda$ the reducing morphism of $\mathbb{G}$. The reducing morphism of the dual $\hat{\mathbb{G}}$ is denoted by $\hat{\Lambda} : C_0^u(\hat{\mathbb{G}}) \to C_0(\hat{\mathbb{G}})$.

The comultiplication $\Delta$ of $C_0(\mathbb{G})$ is implemented by the multiplicative unitary $W \in M(C_0(\mathbb{G}) \otimes C_0(\hat{\mathbb{G}}))$:

$$\Delta(x) = W^*(1 \otimes x)W, \quad x \in C_0(\mathbb{G}).$$

The multiplicative unitary $W$ admits a universal lift $\bar{W} \in M(C_0^u(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$ such that $(\Lambda \otimes \hat{\Lambda})(\bar{W}) = W$. We shall also need half-universal versions of the bicharacter and the comultiplication: define

$$W = (\text{id} \otimes \hat{\Lambda})\bar{W}.$$

The unitary antipode of $\mathbb{G}$ will be denoted by $R$. It is a $*$-antiautomorphism of $C_0(\mathbb{G})$ (and in fact of $B(L^2(\mathbb{G}))$) of order 2. Its universal version, which is a $*$-antiautomorphism of $C_0^u(\mathbb{G})$ of order 2, will be denoted by $R_u$. We have $\Lambda \circ R_u = R \circ \Lambda$.

We shall often write $M^u(\mathbb{G}) := C_0(\mathbb{G})^*$ and $M(\mathbb{G}) := C_0(\mathbb{G})^*$, and denote their unit balls by $M^u(\mathbb{G})_1$ and $M(\mathbb{G})_1$, respectively. Note that $M^u(\mathbb{G})$ is a completely contractive Banach algebra with respect to convolution $*$, defined as the adjoint of comultiplication $\Delta_u$. Similarly also $M(\mathbb{G})$ is a completely contractive Banach algebra.

1.2. Convolution operators. Fix a locally compact quantum group $\mathbb{G}$ and define

$$\Delta_s(x) = W^*(1 \otimes x)W, \quad x \in L^\infty(\mathbb{G}).$$

Note that $W$ as well as $\Delta_s(x)$ are in $M(C_0(\mathbb{G}) \otimes K(L^2(\mathbb{G})))$ [Kus]. Moreover, by equation (6.1) of [Kus] (applied to $W$), we have that

$$W^*(1 \otimes \tilde{\Lambda}(y))W = (\text{id} \otimes \hat{\Lambda})\Delta_u(y), \quad y \in C_0^u(\mathbb{G})^*,$$

where $\tilde{\Lambda} : C_0^u(\mathbb{G})^* \to L^\infty(\mathbb{G})$ is the normal extension of the reducing morphism $C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$.

For every $\omega \in M^u(\mathbb{G})$ and $\phi \in L^1(\mathbb{G})$, there is a unique $\psi \in L^1(\mathbb{G})$ such that

$$(\omega \otimes \phi \Lambda)\Delta_u(a) = (\psi \Lambda)(a), \quad a \in C_0^u(\mathbb{G}).$$

We will write $\omega \ast \phi := \psi$. Similarly $\phi \ast \omega \in L^1(\mathbb{G})$ is the unique $\psi' \in L^1(\mathbb{G})$ such that

$$(\phi \Lambda \otimes \omega)\Delta_u(a) = (\psi' \Lambda)(a), \quad a \in C_0^u(\mathbb{G}).$$

Hence we consider $L^1(\mathbb{G})$ an ideal in $M^u(\mathbb{G})$. Note that for $x \in L^\infty(\mathbb{G})$

$$\omega \ast \phi(x) = \omega((\text{id} \otimes \phi)\Delta_s(x)).$$

We will also need the following formula:

$$(\omega \ast \phi) \circ R = (\phi \circ R) \ast (\omega \circ R_u), \quad \omega \in M^u(\mathbb{G}), \phi \in L^1(\mathbb{G}).$$

It follows from the above characterisations of the respective convolutions and the equality $\chi(R_u \otimes R_u) \circ \Delta_u = \Delta_u \circ R_u$ established in [Kus].

Let

$$RUC^u(\mathbb{G}) = \overline{\text{span}}\{ (\text{id} \otimes \phi)\Delta_s(x) : \phi \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}) \}.$$

This is the universal version of $RUC(\mathbb{G})$ introduced and studied in [Run]; we have $\Lambda(RUC^u(\mathbb{G})) = RUC(\mathbb{G})$.

Lemma 1.1. (i) $C_0^0(\mathbb{G}) = \overline{\text{span}}\{ (\text{id} \otimes \phi)\Delta_s(a) : \phi \in L^1(\mathbb{G}), a \in C_0(\mathbb{G}) \}$

(ii) $RUC^u(\mathbb{G})$ is an an operator system (i.e. a unital and self-adjoint linear closed subspace) in $M(C_0^u(\mathbb{G}))$.

Proof. (i) The elements of the form $(\text{id} \otimes \sigma)W$, with $\sigma \in B(L^2(\mathbb{G})),\ast$, are dense in $C_0(\mathbb{G})$, and similarly, by equation (5.2) of [Kus], the elements of the form $(\text{id} \otimes \sigma)W$, with $\sigma \in B(L^2(\mathbb{G})),\ast$, are dense in $C_0^0(\mathbb{G})$. By (1.1) and Proposition 6.1 of [Kus], we have

$$(\Delta \otimes \text{id})(W) = (\text{id} \otimes \Lambda \otimes \text{id})(\Delta_u \otimes \text{id})(W) = W_{13}W_{23}. $$
For $\phi \in L^1(\mathbb{G})$ and $\sigma \in B(L^2(\mathbb{G}))_+$, we then have

$$(\mathrm{id} \otimes \phi)(\Delta_s((\mathrm{id} \otimes \sigma)\mathbb{W})) = (\mathrm{id} \otimes \sigma)(\mathbb{W}(\phi \otimes \mathrm{id})(\mathbb{W})).$$

Since the elements of the form $(\phi \otimes \mathrm{id})(\mathbb{W})$ are dense in $C_0(\mathbb{G})$ and $C_0(\mathbb{G})$ is non-degenerate on $L^2(\mathbb{G})$, statement (i) follows from the previous equation and the discussion above.

(ii) Since $\Delta_s(x)$ is in $M(C^*_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G})))$, it follows that $\mathcal{R}^u(\mathbb{G}) \subseteq M(C^*_0(\mathbb{G}))$. That $\mathcal{R}^u(\mathbb{G})$ is an operator system is obvious. \hfill $\square$

**Remark 1.2.** In the case when the multiplicative unitary $W$ is semi-regular, one can show that $\mathcal{R}^u(\mathbb{G})$ is in fact a $C^*$-algebra: the argument given in [HNR2, Theorem 5.6] works also in this case (with some obvious modifications that require, for example, equation (1.3)).

Similarly, we can define $\mathcal{L}^u(\mathbb{G})$ starting from the right multiplicative unitary $V \in M(C^*_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ and its half-universal lift $\mathcal{V} \in M(C_0(\mathbb{G}) \otimes C^*_0(\mathbb{G}))$. That is,

$$\mathcal{L}^u(\mathbb{G}) = \overline{\text{span}}\{ (\phi \otimes \mathrm{id})\Delta_r(x) : \phi \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}) \}.$$

where

$$\Delta_r(x) = \mathcal{V}(x \otimes 1)\mathcal{V}^*$$

(note that the right multiplicative unitary $V$ implements the comultiplication of $\mathbb{G}$ via $\Delta(a) = V(a \otimes 1)V^*$). The results concerning $\mathcal{R}^u(\mathbb{G})$ and its dual have obvious analogues for $\mathcal{L}^u(\mathbb{G})$.

We may view $\mathcal{M}^u(\mathbb{G})$ as a subspace of $\mathcal{R}^u(\mathbb{G})^*$, via strict extension.

Every $\rho \in \mathcal{R}^u(\mathbb{G})^*$ defines a map $L_\rho : \mathcal{L}^\infty(\mathbb{G}) \to \mathcal{L}^\infty(\mathbb{G})$ by

$$\phi(L_\rho(x)) = \rho((\mathrm{id} \otimes \phi)\Delta_r(x)) \quad (\phi \in L^1(\mathbb{G})).$$

The convolution operators on the universal level are defined in the following proposition.

**Proposition 1.3.**

(i) For $\omega \in \mathcal{M}^u(\mathbb{G})$, the map $R_\omega^u : M(C^*_0(\mathbb{G})) \to M(C^*_0(\mathbb{G}))$,

$$R_\omega^u(x) = (\mathrm{id} \otimes \omega)\Delta_u(x), \quad x \in M(C^*_0(\mathbb{G})),$$

maps $\mathcal{R}^u(\mathbb{G})$ to $\mathcal{R}^u(\mathbb{G})$.

(ii) For $\rho \in \mathcal{R}^u(\mathbb{G})^*$, the equation

$$(1.4) \quad \omega(L_\rho^u(x)) = \rho(R_\omega^u(x)), \quad \omega \in \mathcal{M}^u(\mathbb{G}), x \in \mathcal{R}^u(\mathbb{G}),$$

defines an operator $L^u_\rho : \mathcal{R}^u(\mathbb{G}) \to \mathcal{R}^u(\mathbb{G})$. In the case when $\rho \in \mathcal{M}^u(\mathbb{G})$, we have $L^u_\rho(x) = (\rho \otimes \mathrm{id})\Delta_u(x)$ and $L^u_\rho$ maps $\mathcal{C}^u_0(\mathbb{G})$ to $\mathcal{C}^u_0(\mathbb{G})$.

**Proof.** (i) For $\phi \in L^1(\mathbb{G})$ and $y \in L^\infty(\mathbb{G})$, we have

$$R_\phi((\mathrm{id} \otimes \phi)\Delta_u(y)) = (\mathrm{id} \otimes \omega \otimes \phi)((\mathrm{id} \otimes \Delta_u \otimes \mathrm{id})\Delta_s(y)) = (\mathrm{id} \otimes \omega \otimes \phi)(\mathrm{id} \otimes \Delta_s)\Delta_s(y) = (\mathrm{id} \otimes \omega \otimes \phi)\Delta_s(y).$$

Therefore $R_\phi^u$ maps $\mathcal{R}^u(\mathbb{G})$ to $\mathcal{R}^u(\mathbb{G})$.

(ii) The equation (1.4) defines an element $L_\rho^u(x) \in \mathcal{C}_0^u(\mathbb{G})^{**}$, so it is enough to check that in fact $L^u_\rho(x) \in \mathcal{R}^u(\mathbb{G})$. To this end, let $x = (\mathrm{id} \otimes \phi)\Delta_u(y)$ for $y \in L^\infty(\mathbb{G})$ and $\phi \in L^1(\mathbb{G})$. Similarly as above,

$$\omega(L^u_\rho(x)) = \rho(R^u_\omega(x)) = \rho(R^u_\omega((\mathrm{id} \otimes \phi)\Delta_u(y))) = \rho((\mathrm{id} \otimes \omega \otimes \phi)\Delta_s(y)) = \omega(\rho \otimes \phi(L_\rho(y))) = (\omega \otimes \phi)L_s(L_\rho(y)) = \omega((\mathrm{id} \otimes \phi)\Delta_s(L_\rho(y))).$$

It follows that $L^u_\rho$ maps $\mathcal{R}^u(\mathbb{G})$ to itself. The second statement is obvious. \hfill $\square$

Properties of $\mathcal{R}^u(\mathbb{G})^*$ and the properties of the convolution operators associated with elements in $\mathcal{R}^u(\mathbb{G})^*$ are collected in the following proposition.

**Proposition 1.4.**

(i) $\mathcal{R}^u(\mathbb{G})^*$ is a Banach algebra under the convolution product

$$\rho \ast \nu = \nu \circ L_\rho^u, \quad \rho, \nu \in \mathcal{R}^u(\mathbb{G})^*.$$  

(ii) $\mathcal{M}^u(\mathbb{G})$ is a closed subalgebra of $\mathcal{R}^u(\mathbb{G})^*$. 


(iii) For every $\omega \in M^u(G)$ and $\rho \in RUC^u(G)^*$,
\[ \rho \ast \omega = \rho \circ R^u_\omega. \]

(iv) For every $\omega, \nu \in M^u(G)$ and $\rho, \eta \in RUC^u(G)^*$,
\[ R^u_\omega \circ L^u_\rho = L^u_\rho \circ R^u_\omega, \quad R^u_\eta \circ R^u_\nu = R^u_\nu \circ R^u_\eta, \quad L^u_\eta \circ L^u_\rho = L^u_{\rho \ast \eta}. \]

(v) For every $\rho \in RUC^u(G)^*$,
\[ \Lambda \circ L^u_\rho(x) = L^u_\rho \circ \Lambda(x), \quad x \in RUC^u(G). \]

(vi) For every $\rho, \eta \in RUC^u(G)^*$,
\[ L^u_{\rho \ast \eta} = L^u_\eta \circ L^u_\rho. \]

(vii) For $\rho \in RUC^u(G)^*$, the map $L^u_\rho : L^\infty(G) \rightarrow L^\infty(G)$ is normal if and only if $\rho \in M^u(G)$.

(viii) Each of the maps $\omega \mapsto L_\omega$, $\omega \mapsto L^u_\omega$ and $\omega \mapsto R^u_\omega$ (the first two defined on $RUC^u(G)^*$, the last one on $M^u(G)$) are injective.

Proof. (iv) Here we check the first identity; the second identity is easy to verify and the third identity is proved in the next paragraph (because it needs the multiplication on $RUC^u(G)^*$). For $\omega, \nu \in M^u(G)$, $\rho \in RUC^u(G)$ and $x \in RUC^u(G)$, we have
\[ \nu(L^u_\rho \circ R^u_\omega(x)) = \rho(R^u_\nu \circ L^u_\omega(x)) = \nu(R^u_\nu \circ L^u_\omega(x)). \]

(i) It follows from Proposition 1.3 (ii) that $\rho \ast \eta = \eta \circ L^\mu_\rho$ defines an element of $RUC^u(G)^*$ (note that $RUC^u(G) \subset M(C^\omega_0(G)) \subset C^\omega_0(G)^{**}$). For every $\rho, \eta, \nu \in RUC^u(G)$, we have by definition
\[ (\rho \ast \eta) \ast \nu = \nu \circ L^u_{\rho \ast \eta}. \]

Now for every $\omega \in M^u(G)$ and $x \in RUC^u(G)$,
\[ \omega(L^u_{\rho \ast \eta}(x)) = \rho \ast \eta(R^u_\omega(x)) = \eta(L^u_\rho \circ R^u_\omega(x)) = \eta(R^u_\rho \circ L^u_\omega(x)) = \omega(L^u_\rho \circ L^u_\omega(x)). \]

Hence $L^u_{\rho \ast \eta} = L^u_\eta \circ L^u_\rho$ and
\[ (\rho \ast \eta) \ast \nu = \nu \circ L^u_{\rho \ast \eta} = (\eta \ast \nu) \circ L^u_\rho = \rho \ast (\eta \ast \nu). \]

Therefore the convolution product on $RUC^u(G)^*$ is associative, and clearly the norm is submultiplicative, so $RUC^u(G)^*$ is a Banach algebra.

(ii) is obvious and (iii) is immediate from Proposition 1.3 (ii). Also (v) is immediate and (vi) is similar to the analogous statement regarding $L^u_\rho$ in (iv). Condition (viii) is easy to check directly from the definitions.

(vii) If $\rho \in M^u(G)$, then for every $\phi \in L^1(G)$ and $x \in L^\infty(G)$,
\[ \phi(L^u_\rho(x)) = \rho((\text{id} \ast \phi)\Delta_s(x)) = \rho \ast \phi(x). \]

As $\rho \ast \phi \in L^1(G)$, it follows that $L^u_\rho$ is normal.

It remains to show that $L^u_\rho$ cannot be normal if $\rho \in C^\omega_0(G)^{**} \setminus \{0\}$. Now for every $a \in C^\omega_0(G)$, we have $L^u_\rho(\Lambda(a)) = \Lambda(L^u_\rho(a))$ and as noted in the proof of Corollary 1.6, $L^u_\rho(a) = 0$. Hence $L^u_\rho$ is normal, then $L^u_\rho = 0$ because $C_0(G)$ is weak*-dense in $L^\infty(G)$, and consequently $\rho = 0$ by (viii). \hfill \Box

Sometimes we need to work on $C^\omega_0(G)^{**}$ in which case we may consider normal maps $\widetilde{R}^u_\omega : C^\omega_0(G)^{**} \rightarrow C^\omega_0(G)^{**}$ and $\widetilde{L}^u_\omega : C^\omega_0(G)^{**} \rightarrow C^\omega_0(G)^{**}$, given $\omega \in M^u(G)$. These are defined by
\[ \widetilde{R}^u_\omega(x) = (\text{id} \ast \omega)\Delta_u(x) \quad \text{and} \quad \widetilde{L}^u_\omega(x) = (\omega \ast \text{id})\Delta_u(x). \]

Note that on $M(C^\omega_0(G))$ the maps $\widetilde{R}^u_\omega$ and $\widetilde{L}^u_\omega$ agree with the maps $R^u_\omega$ and $L^u_\omega$, respectively.

We will need one more result, essentially contained in Proposition 2.1 of [BMS]; here we provide a different proof.

Proposition 1.5. Let $A$ be a $C^*$-algebra. Then
\[ M(A)^* = A^* \oplus_1 A^\perp \]
where as usual $A^* \subset M(A)^*$ is defined using strict extensions of functionals in $A^*$. 

Proof. For every $\nu \in A^*$, denote its strict extension in $M(A)^\ast$ by $\tilde{\nu}$. For every $\mu \in M(A)^\ast$ define $\mu_0 = \mu|_A$ and $\mu_1 = \mu - \mu_0$. Then

$$\mu = \tilde{\mu}_0 + \mu_1$$

will be the required decomposition.

Suppose first that $\mu$ is a positive functional. Then clearly also $\tilde{\mu}_0$ is positive. We claim that $\tilde{\mu}_0 \leq \mu$ so that also $\mu_1$ is positive. Let $x \geq 0$ in $M(A)$. If $(\epsilon_i)_{i \in \mathcal{I}}$ is an increasing, positive contractive approximate identity in $A$, then $(x^{1/2}\epsilon_i x^{1/2})_{i \in \mathcal{I}}$ is an increasing net converging strictly to $x$. Hence

$$\tilde{\mu}_0(x) = \lim_{i \in \mathcal{I}} \mu_0(x^{1/2}\epsilon_i x^{1/2}) = \lim_{i \in \mathcal{I}} \mu(x^{1/2}\epsilon_i x^{1/2}) \leq \mu(x).$$

So $\mu = \tilde{\mu}_0 + \mu_1$ is a decomposition of $\mu$ into positive components and so

$$\|\mu\| = \mu(1) = \tilde{\mu}_0(1) + \mu_1(1) = \|\tilde{\mu}_0\| + \|\mu_1\|.$$ 

Therefore the claim holds for positive $\mu$.

Let now $\mu \in M(A)^\ast$ be arbitrary. Then $\mu$ has a (right) polar decomposition

$$\mu = \nu u$$

where $\nu \in M(A)^\ast$ is positive and $u \in M(A)^{**}$ is a partial isometry (the polar decomposition follows by considering $M(A)^\ast$ as the predual of the von Neumann algebra $M(A)^{**}$).

We claim that $\tilde{\nu}_0 u = \tilde{\mu}_0$. These functionals coincide on $A$ so it is enough to show that $\tilde{\nu}_0 u$ is strictly continuous. As there are two different extensions at play (from $A$ to $M(A)$ and from $M(A)$ to $M(A)^{**}$), it is clearer to use angle brackets to denote the duality of $M(A)^\ast$ and $M(A)^{**}$. Let $\nu_0 = a_\eta$ be a factorisation of $\tilde{\nu}_0$ on $A$ ($\eta \in A^*$ and $a \in A$). The functional $\tilde{\nu}_0$ is defined by

$$\tilde{\nu}_0(y) = \eta(ya), \quad y \in M(A),$$

so that by a weak*-approximation we have also $\tilde{\nu}_0(z) = \tilde{\eta}(za)$ for all $z \in M(A)^{**}$, where $\tilde{\eta}$ denotes the relevant weak*-extension. Let $(x_i)_{i \in \mathcal{I}} \subset M(A)$ converge strictly to $x$ in $M(A)$. Now

$$\langle \tilde{\nu}_0, uz \rangle = \langle \tilde{\eta}, uxa \rangle = \lim_{i \in \mathcal{I}} \langle \tilde{\eta}, u(x_i)a \rangle = \lim_{i \in \mathcal{I}} \langle \tilde{\eta}, u(x_i) \rangle,$$

which shows that $\tilde{\nu}_0 u$ is strictly continuous, as required.

As $\tilde{\nu}_0 u = \tilde{\mu}_0$, it follows from the uniqueness of the decomposition that $\nu u = \mu_1$. Then we have

$$\|\mu\| \leq \|\tilde{\mu}_0\| + \|\mu_1\| = \|\tilde{\nu}_0 u\| + \|\nu u\| \leq \|\tilde{\nu}_0\| + \|\nu_1\| = \|\nu\| = \|\mu\|,$$

where we used the first part of the proof (applied to the positive functional $\nu$) and the fact that $\|\mu\| = \|\nu\|$.

\[\square\]

**Corollary 1.6.** We have a decomposition

$$RUC^u(G)^\ast = M^u(G) \oplus_1 C^u_0(G)^\perp$$

where $M^u(G)$ is a subalgebra of $RUC^u(G)^\ast$ and $C^u_0(G)^\perp$ is a weak*closed ideal in $RUC^u(G)^\ast$.

**Proof.** It remains to show that $C^u_0(G)^\perp$ is an ideal. Suppose first that $\mu, \nu \in RUC^u(G)^\ast$ with $\mu \in C^u_0(G)^\perp$. For every $\omega \in RUC^u(G)^\ast$ and $a \in C^u_0(G)$, we have

$$\omega(L^u_\mu(a)) = \mu(R^u_\mu(a)) = 0$$

as $R^u_\mu(a) \in C^u_0(G)$, and so $L^u_\mu(a) = 0$. It follows that $\mu \ast \nu = \nu \circ L^u_\mu \in C^u_0(G)^\perp$.

It remains to check the case when $\mu \in RUC^u(G)^\ast$ and $\nu \in C^u_0(G)^\perp$. But now $L^u_\mu(a) \in C^u_0(G)$ whenever $a \in C^u_0(G)$, and so $\mu \ast \nu \in C^u_0(G)^\perp$. \[\square\]

**Lemma 1.7.** If $\omega \in RUC^u(G)^\ast$ is a contractive idempotent, then either $\omega \in M^u(G)$ or $\omega \in C^u_0(G)^\perp$.
Proof. Write \( \omega = \omega_0 + \omega_1 \) where \( \omega_0 \in M^u(\mathcal{G}) \) and \( \omega_1 \in C^u_0(\mathcal{G})^\perp \). Then
\[
\omega_0 + \omega_1 = \omega = \omega \ast \omega = \omega_0 \ast \omega_0 + \omega_0 \ast \omega_1 + \omega_1 \ast \omega_0 + \omega_1 \ast \omega_1.
\]
Since \( C^u_0(\mathcal{G})^\perp \) is an ideal, it follows that \( \omega_0 \ast \omega_1 = 0 \). Now \( \|\omega\| = \|\omega_0\| + \|\omega_1\| \) by Corollary 1.6, so \( \omega_0 \) is a contractive idempotent. Hence \( \|\omega_0\| \) is either 0 or 1. If \( \|\omega_0\| = 0 \), then \( \omega = \omega_1 \in C^u_0(\mathcal{G})^\perp \). If \( \|\omega_0\| = 1 \), then \( \|\omega_1\| = 0 \) and \( \omega = \omega_0 \in M^u(\mathcal{G}) \).

2. Fixed points in \( L^\infty(\mathcal{G}) \)

Let \( \mathcal{G} \) be a locally compact quantum group. For a map \( L : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{G}) \), write
\[
\text{Fix } L = \{ x \in L^\infty(\mathcal{G}) : L(x) = x \}.
\]
We are mostly interested in Fix \( L_\omega \) with \( \omega \in M^u(\mathcal{G})_1 \), in which case Fix \( L_\omega \) is weak*-closed. Note that if \( \|\omega\| < 1 \), then Fix \( L_\omega = \{0\} \), so we may concentrate on the case \( \|\omega\| = 1 \).

Given \( \omega \in M^u(\mathcal{G})_1 \) and \( n \in \mathbb{N} \) we write
\[
S_n(\omega) = \frac{1}{n} \sum_{k=1}^{n} \omega^k M \in M^u(\mathcal{G})_1.
\]
Fix a free ultrafilter \( p \) on \( \mathbb{N} \) and take the weak* limit of the sequence \( S_n(\omega) \) along \( p \) in \( RUC^u(\mathcal{G})^* \). We will denote this limit by \( \tilde{\omega}_p \) (later we will simply write \( \tilde{\omega} \) once \( p \) is fixed).

Lemma 2.1. Let \( \omega \in M^u(\mathcal{G})_1 \) and let \( p \) be a free ultrafilter. Then
\[
\tilde{\omega}_p = \omega \ast \tilde{\omega}_p = \tilde{\omega}_p \ast \omega = \tilde{\omega}_p \ast \tilde{\omega}_p.
\]
In particular \( \tilde{\omega}_p \) is either 0 or a contractive idempotent. Finally, the following conditions are also equivalent:
- (i) \( \tilde{\omega}_p \) is a state for some free ultrafilter \( p \);
- (ii) \( \tilde{\omega}_p \) is a state for every free ultrafilter \( p \);
- (iii) \( \omega \) is a state.

Proof. The first statement follows from the identity
\[
S_n(\omega) \ast \omega = \omega \ast S_n(\omega) = S_n(\omega) - \frac{1}{n} \omega + \frac{1}{n} \omega^{n+1}.
\]
The implications (iii) \( \implies \) (ii) \( \implies \) (i) are obvious. Suppose that \( \tilde{\omega}_p \) is a state. Then
\[
1 = \tilde{\omega}_p(1) = \omega \ast \tilde{\omega}_p(1) = \omega(1) \tilde{\omega}_p(1) = \omega(1)
\]
so (iii) follows.

Lemma 2.2. Let \( \omega \in M^u(\mathcal{G})_1 \) and let \( p \) be a free ultrafilter. The map \( L_{\tilde{\omega}_p} \) is a projection onto Fix \( L_\omega = \text{Fix } L_{\tilde{\omega}_p} \).

Proof. To see that \( L_{\tilde{\omega}_p} \) and \( L_\omega \) have the same fixed points, note first that for all \( \phi \in L^1(\mathcal{G}) \) and \( x \in L^\infty(\mathcal{G}) \)
\[
\phi(L_{\tilde{\omega}_p}(x)) = \tilde{\omega}_p((\text{id} \otimes \phi)\Delta_s(x)) = p\text{-lim } S_n(\omega)((\text{id} \otimes \phi)\Delta_s(x)).
\]
It is immediate that Fix \( L_\omega \subset \text{Fix } L_{\tilde{\omega}_p} \). On the other hand if \( x \) is a fixed point of \( L_{\tilde{\omega}_p} \), then it follows from the above identity that
\[
\phi(L_{\omega}(x)) = \phi(L_{\omega}(L_{\tilde{\omega}_p}(x))) = \phi(L_{\tilde{\omega}_p \ast \omega}(x)) = \tilde{\omega}_p \ast \omega((\text{id} \otimes \phi)\Delta_s(x))
\]
\[
= p\text{-lim } S_n(\omega)((\text{id} \otimes (\omega \ast \phi))\Delta_s(x)) = p\text{-lim } (S_n(\omega) \ast \omega)((\text{id} \otimes \phi)\Delta_s(x))
\]
\[
= \phi(L_{\tilde{\omega}_p}(x)) = \phi(x)
\]
by (2.1).

Since \( \tilde{\omega}_p \) is either a non-zero contractive idempotent or 0, it follows that \( L_{\tilde{\omega}_p} \) is an idempotent whose image is Fix \( L_{\tilde{\omega}_p} \).

\[\Box\]
The following theorem characterises the case when there are no non-zero fixed points. Note that convolution operators determined by states always have fixed points (namely, the constants) but this need not be the case with contractive functionals.

**Theorem 2.3.** Let $\omega \in M^n(G)_1$. The following are equivalent:

(i) $L_\omega$ has no non-zero fixed points in $L^\infty(G)$;

(ii) $L_\omega$ has no non-zero fixed points in $\text{RUC}(G)$;

(iii) $L^n_\omega$ has no non-zero fixed points in $\text{RUC}^n(G)$;

(iv) $S_n(\omega) \to 0$ weak* in $\text{RUC}^n(G)^*$;

(v) $\tilde{\omega}_p = 0$ for all free ultrafilters $p$;

(vi) $\tilde{\omega}_p = 0$ for some free ultrafilter $p$.

**Proof.** (iii) $\implies$ (i): Let $x$ be a non-zero fixed point of $L_\omega$ in $L^\infty(G)$ and pick $\phi \in L^1(G)$ such that $\hat{x} := (\text{id} \otimes \phi)\Delta_s(x) \neq 0$. Then $\hat{x} \in \text{RUC}^n(G)$ and for every $\mu \in \text{RUC}^n(G)^*$

$$
\mu(L^n_\omega(\hat{x})) = \omega \ast \mu((\text{id} \otimes \phi)\Delta_s(x)) = \phi(L_\omega \ast \mu(x)) = \phi(L_\mu \circ L_\omega(x)) = \phi(L_\mu(x)) = \mu(\hat{x}).
$$

So $\hat{x}$ is a non-zero fixed point of $L^n_\omega$ in $\text{RUC}^n(G)$.

(ii) $\implies$ (i) is similar to (iii) $\implies$ (i). The converse implication (i) $\implies$ (ii) is trivial.

The implications (iv) $\iff$ (v) $\iff$ (vi) are obvious.

(i) $\implies$ (v): If $\tilde{\omega}_p \neq 0$, there is $x \in L^\infty(G)$ such that $L_{\tilde{\omega}_p}(x) \neq 0$. Moreover, $L_\omega(L_{\tilde{\omega}_p}(x)) = L_{\tilde{\omega}_p}(x)$.

(vi) $\implies$ (iii): Suppose that there exist a non-zero fixed point $x$ of $L^n_\omega$ in $\text{RUC}^n(G)$ and pick $\nu \in M^n(G)$ such that $\nu(x) \neq 0$ (recall that $\text{RUC}^n(G) \subset M(C^n_0(G)) \subset C^n_0(G)^{**}$). Then using Proposition 1.4, we have

$$
\nu(L^n_{\tilde{\omega}_p}(x)) = \tilde{\omega}_p \ast \nu(x) = \tilde{\omega}_p(R^n_\nu(x)) = p\text{-}\lim S_n(\omega)(R^n_\nu(x))
$$

$$
= p\text{-}\lim \frac{1}{n} \sum_{k=1}^{n} \nu(L^n_{\omega \ast k}(x)) = \nu(x).
$$

It follows that $\tilde{\omega}_p \neq 0$. $\square$

The right absolute value of $\omega \in M^n(G)$ is a positive functional $|\omega|$ in $M^n(G)$ defined through the polar decomposition $\omega = \omega|e(u)$ where $u \in C^n_0(G)^{**}$ is a partial isometry satisfying some extra properties; see Definition III.4.3 of [Tak]. Then the left absolute value $|\omega|_e$ is defined by $\omega = |\omega|_e(u^*\cdot)$.

**Lemma 2.4.** Let $M$ be a von Neumann algebra and $\nu \in M_+$ with $\|\nu\| = 1$. Suppose that $x \in M$ is such that $\|x\| = 1$ and $\nu(x^*) = 1$. Then $\nu = x|\nu|\phi$ and $|\nu| = \phi^*\nu$.

**Proof.** By the proof of Theorem III.4.2 of [Tak], $\nu = u|\nu|$ where $x = u|x|$ is the polar decomposition of $x$. Note that $|\nu|(\|x\|) = \nu(x^*) = 1$. By the Cauchy–Schwarz inequality

$$
|\nu|(\|x\|)^2 \leq |\nu|(\|x\|^2)|\nu|(1) \leq 1 = |\nu|(\|x\|)^2.
$$

Hence $|x|$ is in the multiplicative domain of $|\nu|$, and so for every $y \in M$

$$
x.|\nu|(|x|) = |\nu|(yu|x|) = |\nu|(yu)|\nu|(|x|) = \nu(y).
$$

As $|\nu| = u^*\cdot\nu$, we also have

$$
x^*|\nu|(|x|) = \nu(y|x|u^*) = |\nu|(y|x|) = |\nu|(y).
$$

$\square$

The above general lemma allows us to say something about fixed points of contractive convolution operators.
Lemma 2.5. Suppose that $\omega \in M^n(\mathbb{G})$ is contractive and $v \in C^0_u(\mathbb{G})^{**}$ is such that $v^*$ is a fixed point of $L^n_\omega$, $\|v\| = 1$ and $\omega(v^*) = 1$. Then

$$\Delta_u^-(k)(v) - v^{\odot (k+1)} \in N_{\|\omega\|^{\odot (k+1)}}$$

and $\omega^k = v.\|\omega\|^k$

for every $k \in \mathbb{N}$. Moreover, $|\omega|^k(au^*v) = |\omega|^k(v^*va) = |\omega|^k(a)$ for every $a \in C^0_u(\mathbb{G})^{**}$ and $k \in \mathbb{N}$.

Proof. As $\|v\| = 1$ and $\omega(v^*) = 1$, it follows from Lemma 2.4 that $\omega = v.\|\omega\|$. Moreover, $|\omega|(v^*v) = 1$.

Fix $k \in \mathbb{N}$. Since $v^*$ is a fixed point of $L^n_\omega$, we have

$$1 = \omega(v^*) = \omega^{\odot (k+1)}(\Delta_u^-(k)) = |\omega|^{\odot (k+1)}(\tilde{\Delta}_u^-(k)v^{\odot (k+1)}).$$

Therefore

$$|\omega|^{\odot (k+1)}((\tilde{\Delta}_u^-(k) - v^{\odot (k+1)})(\Delta_u^-(k)) - |\omega|^{\odot (k+1)}((\tilde{\Delta}_u^-(k)(v^*v) - v^{\odot (k+1)}(\Delta_u^-(k)) + (v^*)^{\odot (k+1)}(\tilde{\Delta}_u^-(k))v + (v^*v)^{\odot (k+1)})))$$

$$\tilde{\Delta}_u^-(k)(v) - v^{\odot (k+1)} \in N_{\|\omega\|^{\odot (k+1)}},$$

and it is then immediate that $\omega^k = v.\|\omega\|^k$.

As a by-product, we get that $|\omega|^{\odot k}(v^*v) = 1$, and so by the Cauchy–Schwarz inequality

$$1 = |\omega|^{\odot k}(v^*v)^2 \leq |\omega|^{\odot k}(v^*v) \leq \|v^*v\| \leq 1.$$ 

By Choi’s theorem on multiplicative domains, $v^*v$ is in the multiplicative domain of $|\omega|^k$, and so $|\omega|^k(au^*v) = |\omega|^k(v^*va) = |\omega|^k(a)$.

In the case when $\omega$ is a state, the next lemma trivialises (one can take $v = 1$).

Lemma 2.6. Let $\omega \in M^n(\mathbb{G})_1$. If $L_\omega$ has a non-zero fixed point in $L^\infty(\mathbb{G})$, then there exists $v \in C^0_u(\mathbb{G})^{**}$ such that

$$\tilde{\Delta}_u^-(k)(v) - v^{\odot k+1} \in N_{\|\omega\|^{\odot k+1}}$$

and $\omega^k = v.\|\omega\|^k$

for every $k \in \mathbb{N}$. Moreover, $|\omega|^k(au^*v) = |\omega|^k(v^*va) = |\omega|^k(a)$ for every $a \in C^0_u(\mathbb{G})$ and $k \in \mathbb{N}$.

Proof. Let $x \in L^\infty(\mathbb{G})$ be a non-zero fixed point of $L_\omega$. Pick a sequence $(\phi_n)_{n \in \mathbb{N}} \subseteq \text{ball}(L^1(\mathbb{G}))$ such that $\phi_n(x) > \|x\| - 1/n$ for each $n \in \mathbb{N}$. Define

$$v_n = \frac{(\text{id} \otimes \phi_n^*)\Delta(x^*)}{\phi_n^*(x^*)} \in RUC^u(\mathbb{G})$$

and let $v$ be a weak* cluster point of $(v_n)$ in $C^0_u(\mathbb{G})^{**}$. Since $L_\omega(x) = x$, it follows that $\omega(v_n^*) = 1$ for every $n \in \mathbb{N}$ and so $\omega(v^*) = 1$. Moreover, since $\phi_n(x) \to \|x\|$, it follows that $\|v\| \leq 1$, and so $\|v\| = 1$. Finally, we note that $v^*$ is a fixed point of $L^n_\omega$: indeed, it is easy to calculate that $L^n_\omega(v_n^*) = v_n^*$ and the rest follows as $\tilde{\Delta}_u^-$ is normal when $\omega \in M^n(\mathbb{G})$. Now we may apply Lemma 2.5 to obtain the result.

In the case when $L^n_\omega$ has a fixed point in $LUC^u(\mathbb{G})$ we obtain a tighter result. Note that by Lemma 2.3 a fixed point in $L^\infty(\mathbb{G})$ implies a fixed point in $RUC^u(\mathbb{G})$ but not necessarily in $LUC^u(\mathbb{G})$.

We call a contractive functional $\omega \in M^n(\mathbb{G})$ non-degenerate if both $|\omega|$ and $|\omega|^l$ are non-degenerate in the sense of [KNR], i.e. for each non-zero $x \in C^0_u(\mathbb{G})_+$ there exists $n \in \mathbb{N}$ such that $|\omega|^n(x) > 0$ and $|\omega|^l(x) > 0$. We use the analogous notion of degeneracy for $\omega \in M(\mathbb{G})$.
Theorem 2.7. Suppose that $\omega \in M^u(G)$ is non-degenerate and contractive. If $L^u_\omega$ has a non-zero fixed point in $LUC^u(G)$, then there is a unitary $v \in LUC^u(G)$ (even $UC^u(G)$) such that

$$\Delta_u(v) = v \otimes v,$$

and $\text{Fix } L^u_\omega = (\text{Fix } L^u_\omega)v^*$. Taking $u = \Lambda(v)$ we have $\text{Fix } L_\omega = (\text{Fix } L_\omega)u^*$.

Proof. Let $x \in LUC^u(G)$ be a fixed point of $L^u_\omega$ such that $\|x\| = 1$. Fix $\mu \in LUC^u(G)^*$ such that $\|\mu\| = 1$ and $\mu(x) = 1$. Define $v = R^u_\mu(x)^* \in LUC^u(G)$ (where $R^u_\mu$ is defined analogously to $L^u_\omega$, introduced in Proposition 1.4). Then

$$\omega(v^*) = \omega(R^u_\mu(x)) = \mu(L^u_\omega(x)) = \mu(x) = 1,$$

and in particular $\|v\| = 1$. Moreover, $v^*$ is a fixed point of $L^u_\omega$ as $L^u_\omega$ commutes with $R^u_\mu$. Now we may apply Lemma 2.5.

We claim that $\Delta_u(v) = v \otimes v$. For every $k, l \in \mathbb{N}$ we have

$$|\omega|^{*k} \otimes |\omega|^{*l}((\Delta_u(v) - v \otimes v)^*(\Delta_u(v) - v \otimes v))$$

$$= |\omega|^{*k} \otimes |\omega|^{*l}((\Delta_u(v^*v) - \Delta_u(v^*)v \otimes v) - (v^* \otimes v^*)\Delta_u(v) + v^*v \otimes v^*)$$

$$= |\omega|^{*(k+l)}(v^*v - \omega^{*(k+l)}(v^*)) - |\omega|^{*(k+l)}(v^*) + |\omega|^{*k}(v^*v)|\omega|^{*l}(v^*) = 0.$$

Now $X = (\Delta_u(v) - v \otimes v)^*(\Delta_u(v) - v \otimes v) \in M(C^u_0(G) \otimes C^u_0(G))$ is such that $X \geq 0$ and $|\omega|^{*k} \otimes |\omega|^{*l}(X) = 0$. Since for every $l \in \mathbb{N}$

$$|\omega|^{*l}((|\omega|^{*k} \otimes \text{id})(X)) = 0$$

it follows by non-degeneracy that $((|\omega|^{*k} \otimes \text{id})(X)) = 0$. Then, for every $\sigma \geq 0$ in $C^u_0(G)^*$, we have $(\text{id} \otimes \sigma)(X) = 0$ and $|\omega|^{*k}((\text{id} \otimes \sigma)(X)) = 0$ for every $k \in \mathbb{N}$. Therefore $(\text{id} \otimes \sigma)(X) = 0$ for every $\sigma \geq 0$ and so $X = 0$. Consequently $\Delta_u(v) = v \otimes v$ as required.

Now for every $a \in C^u_0(G)$ we have

$$|\omega|^{*k}((av^*v - a^*v^*v - a^*a^*) = |\omega|^{*k}(v^*va^*v - v^*va^*a - a^*a^*v + a^*a^*) = 0.$$

Non-degeneracy implies that $av^*v = a$, and then it follows that $v^*v = 1$. Similarly $vv^* = 1$ follows by applying non-degeneracy of the left-hand sided absolute value $|\omega|$. If $y \in \text{Fix } L^u_\omega$, then

$$L^u_\omega(yv^*) = (\omega \otimes \text{id})\Delta_u(yv^*) = (\omega \otimes \text{id})(\Delta_u(y)(v^* \otimes v^*)) = (|\omega| \otimes \text{id})(\Delta_u(y))^*v^* = yv^*.$$

Hence Fix $L^u_\omega v^* \subseteq \text{Fix } L^u_\omega$. Conversely, if $z \in \text{Fix } L^u_\omega$, then

$$L^u_\omega(\omega(zv^*)) = \omega(\Delta_u(\omega(zv^*)) = \omega(\Delta_u(z)(v^* \otimes v^*)) = \omega(\Delta_u(z))v^* = zv^*.$$

Therefore Fix $L^u_\omega = (\text{Fix } L^u_\omega)v^*$ and Fix $L^u_\omega = (\text{Fix } L^u_\omega)v$.

Finally, $u = \Lambda(v)$ is a group-like unitary in $M(C^u_0(G))$ and $u^*$ is a fixed point of $L_\omega$. Similarly as above Fix $L_\omega = (\text{Fix } L_\omega)u^*$ and Fix $L_\omega = (\text{Fix } L_\omega)u$. \qed

We say that a locally compact quantum group $G$ is universally SIN if $LUC^u(G) = RUC^u(G)$. Note that all discrete and compact quantum groups are universally SIN as are duals of locally compact groups. Recall that Hu, Neufang and Ruan called a locally compact quantum group SIN (from ‘having small invariant neighbourhoods’) if $LUC(G) = RUC(G)$. If $G$ is universally SIN, it is SIN, and the two notions trivially coincide for coamenable $G$. We do not know if they coincide in general.

A weak*-closed subspace of a von Neumann algebra is called a $W^*$-sub-TRO if it is closed under the ternary product $(x, y, z) \mapsto xy^*z$.

Corollary 2.8. Suppose that $G$ is a locally compact quantum group that is universally SIN, and let $\omega \in M^u(G)_1$ be non-degenerate. If Fix $L_\omega$ is a $W^*$-sub-TRO of $L^\infty(G)$, then $\text{dim}(\text{Fix } L_\omega) \leq 1$ (moreover if Fix $L_\omega$ is non-zero, it contains a unitary).
Moreover, if (iv) holds then \( \omega \). We then first use Theorem 2.3 and the assumption that \( G \) is universally SIN to deduce that the assumptions of Theorem 2.7 hold. Thus there is a group-like unitary \( u \) such that \( \text{Fix} \, L_\omega = (\text{Fix} \, L_\omega)u^* \). If \( \text{Fix} \, L_\omega \) is a \( W^* \)-sub-TRO, then \( \text{Fix} \, L_{|\omega|} \) is a von Neumann subalgebra. Hence \( \text{Fix} \, L_{|\omega|} = \mathbb{C} \) by [KNR₁, Theorem 3.6]. \( \square \)

We do not know if the universally SIN assumption of the previous result is actually necessary; our methods seem however to be limited to this case.

3. Annihilators of \( \text{Fix} \, L_\omega \) in \( L^1(G) \)

In this short section we describe properties of the pre-annihilators of fixed point spaces of the type \( \text{Fix} \, L_\omega \subseteq L^\infty(G) \), i.e. certain one-sided ideals in \( L^1(G) \).

**Definition 3.1.** Given \( \omega \in P^u(G) \), define a closed right ideal \( I_\omega \) in \( L^1(G) \) by the formula

\[
I_\omega := \text{cl} \{ f - \omega \ast f : f \in L^1(G) \}.
\]

Moreover, let \( L_0^1(G) \) denote the augmentation ideal in \( L^1(G) \), i.e. \( L_0^1(G) = \{ f \in L^1(G) : f(1) = 0 \} \).

**Lemma 3.2.** Let \( \omega \in P^u(G) \). Then the right ideal \( I_\omega \) is equal to the pre-annihilator of \( \text{Fix} \, L_\omega \). In particular \( I_\omega \subseteq L_0^1(G) \).

**Proof.** To show the first statement it suffices to consider the following chain of equivalences:

\[
x \in I_\omega \iff \forall f \in L^1(G) \, f(x) - \omega \ast f(x) = 0 \iff \forall f \in L^1(G) \, f(x) = f(L_\omega(x)) \iff x = L_\omega(x).
\]

The second statement is now obvious. \( \square \)

The following result can be proved exactly as its classical counterpart, [Wil, Theorem 1.2]. We outline the argument for completeness.

**Proposition 3.3.** Assume that \( G \) is second countable (i.e. \( C_0(G) \) is separable) and let \( I = \{ I_\omega : \omega \in P^u(G) \} \). Then each element in \( I \) is contained in a maximal element of \( I \).

**Proof.** The proof proceeds in two steps. First one shows a counterpart of Lemma 1.1 in [Wil], namely the following result: suppose \( X \subseteq L^1(G) \) is a closed subspace and \( \mathcal{F} \subseteq P^u(G) \) is a closed convex semigroup such that \( I_\omega \subseteq X \) for \( \omega \in \mathcal{F} \) and that for each finite subset \( A \subseteq X \) and \( \epsilon > 0 \) one can find \( \omega_A, \epsilon \in \mathcal{F} \) such that for each \( f \in A \) we have \( d(f, I_{\omega_A}) < \epsilon \). Then \( X = I_\omega \) for some \( \omega \in \mathcal{F} \). The proof follows line by line as in [Wil, Lemma 1.1]. Secondly, one uses this fact to prove that every chain in \( I \) has an upper bound and concludes by the Kuratowski-Zorn lemma. \( \square \)

**Theorem 3.4.** Assume that \( L^1(G) \) is separable. Consider the following list of conditions:

(i) \( G \) is coamenable;

(ii) \( G \) is amenable;

(iii) for each \( \omega \in P^u(G) \), the right ideal \( I_\omega \) admits a bounded left approximate identity;

(iv) the collection \( I := \{ I_\omega : \omega \in P^u(G) \} \) admits a unique maximal element \( I_{\text{max}} \).

Then the following implications/equivalences hold: (ii)\( \iff \) (iii) and (i)+(ii)\( \iff \) (iii)+(iv). Moreover if (iv) holds then \( I_{\text{max}} = L_0^1(G) \); in particular (iv) holds if and only if the augmentation ideal belongs to \( I \).

**Proof.** We begin by proving the last statement. Suppose then that (iv) holds and we have the largest ideal \( I_{\text{max}} \subseteq \mathcal{I} \). By Lemma 3.2 we have \( I_{\text{max}} = \{ x \in L^\infty(G) : \forall \omega \in P^u(G) \, L_\omega(x) = x \} \). Suppose then that \( x \in L^\infty(G) \) and \( L_\omega(x) = \omega(1) x \) for all \( \omega \in L^1(G) \). This is equivalent to \( \Delta(x) = 1 \otimes x \). As is well-known this holds if and only if \( x \in \mathcal{C} \) (see for example Result 5.13 of [KV]) and \( I_{\text{max}} = L_0^1(G) \).

The implication (ii)\( \iff \) (iv) follows now from the forward implication of Theorem 4.2 of [KNR₁]. Further (iv)\( \implies \) (ii) follows as in the backward implication of Theorem 4.2 of [KNR₁], although the latter was formulated only for \( \omega \in \mathcal{P}(G) \). Again, for completeness we sketch the proof. Suppose that (iv) holds, i.e. there is \( \omega \in P^u(G) \) such that \( I_\omega = L_0^1(G) \); equivalently, \( \text{Fix} \, L_\omega = \mathcal{C} \). Pick
a state $\mu \in L^1(G)$ and consider normal states $\omega_n := S_n(\omega) \ast \mu$, $n \in \mathbb{N}$. Let $\gamma \in S(L^\infty(G))$ be the weak* limit point of the sequence $(\mu_n)_{n \in \mathbb{N}}$ along some free ultrafilter. Then for every $x \in L^\infty(G)$ we have $R_\gamma(x) \in \text{Fix} L_\omega = \mathcal{C}$ so that we can define a state $\gamma'$ on $L^\infty(G)$ by the formula $\gamma'(x)1 = R_\gamma(x)$, $x \in L^\infty(G)$. One can then check that $\gamma = \gamma'$ and in fact $\gamma$ is a left invariant mean on $L^\infty(G)$.

The implication $(i) \implies (iii)$ follows as in the intro to [Wil]. Indeed, fix $\omega \in P^u(G)$. The fact that $G$ is coameasurable implies that $L^1(G)$ admits a bounded approximate identity, say $(u_\lambda)_{\lambda \in \Lambda}$. Then one can check that the double-indexed net $(u_\lambda - S_n(\omega) \ast u_\lambda)_{\lambda \in \Lambda, n \in \mathbb{N}}$ is a bounded left approximate identity in $L_\omega$.

The implication $(iii)+(iv) \implies (i)+(ii)$ follows from Proposition 16 of [HNR1]; formally the formulation in [HNR1] requires the two-sided bounded approximate identity in $L^1_\gamma(G)$, but as $L^1_\gamma(G)$ is an involutive Banach algebra (with the involution given by composing with the unitary antipode), the existence of a one-sided bounded approximate identity implies that the two-sided one also exists.

4. Fixed points in $C_0(G)$

In this section, we consider the case when there are fixed points in $C_0(G)$. This only happens in the following cases: $G$ is compact or $\omega$ is degenerate.

**Lemma 4.1.** Let $\omega \in M^u(G)_1$ and suppose that the Cesàro averages $S_n(\omega)$ do not converge to 0 weak* on $C^*_0(G)$. Then the weak* limit

$$\tilde{\omega} := \lim_{n \to \infty} S_n(\omega)$$

exists and $\tilde{\omega}$ is a non-zero contractive idempotent in $M^u(G)$.

**Proof.** Since the Cesàro sums $S_n(\omega)$ do not converge to 0 weak* on $C^*_0(G)$, there is some free ultrafilter $p$ such that the limit $\tilde{\omega}_p$ is non-zero on $C^*_0(G)$. By Lemma 1.7, $\tilde{\omega}_p$ is a contractive idempotent in $M^u(G)$. We shall show that in fact $\tilde{\omega}_p$ is the weak* limit of the sequence $(S_n(\omega))_{n=1}^\infty$ by showing that every subnet of $(S_n(\omega))_{n=1}^\infty$ has a subnet converging to $\tilde{\omega}_p$ (see [Wi, Exercise 11D.(c)]). Since $(S_n(\omega))_{n=1}^\infty$ is bounded, every subnet of $(S_n(\omega))_{n=1}^\infty$ has a subnet converging weak* to some $\tilde{\omega}_q$, where $q$ is some free ultrafilter. Let $a \in C^*_0(G)$ be arbitrary. Since $\tilde{\omega}_p \ast \omega = \tilde{\omega}_p$, it follows that $\omega^k(L^u_{\tilde{\omega}_p}(a)) = \tilde{\omega}_p(a)$. Therefore $S_n(\omega)(L^u_{\tilde{\omega}_p}(a)) = \tilde{\omega}_p(a)$ and so $\tilde{\omega}_q(L^u_{\tilde{\omega}_p}(a)) = \tilde{\omega}_p(a)$. It follows from Lemma 1.7 that $\tilde{\omega}_q$ is also a non-zero contractive idempotent in $M^u(G)$.

Since $\tilde{\omega}_q \in M^u(G)$, we have by Proposition 1.4 that

$$\tilde{\omega}_p = \tilde{\omega}_q \ast L^u_{\tilde{\omega}_p} = \tilde{\omega}_p \ast \tilde{\omega}_q = \tilde{\omega}_p \ast R^u_{\tilde{\omega}_q},$$

But

$$\tilde{\omega}_p \ast R^u_{\tilde{\omega}_q}(a) = p\lim S_n(\omega)(R^u_{\tilde{\omega}_q}(a)) = \tilde{\omega}_q(a)$$

as $\omega \ast \tilde{\omega}_q = \tilde{\omega}_q$. Consequently, $\tilde{\omega}_q = \tilde{\omega}_p$, as required.

**Remark 4.2.** The final condition in the following result may be interpreted as a weak analogue to ‘the subgroup generated by the support of $\omega$ is compact’. Note that classically the closed subsemigroup generated by some set is compact if and only if the closed subgroup generated by the set is compact (because a compact semigroup with cancellation laws is a group). See also Corollary 4.6.

The first result covers the general, possibly degenerate case.

**Proposition 4.3.** Let $\omega \in M^u(G)_1$. Then the following are equivalent:

(i) Cesàro sums $S_n(\omega)$ do not converge to 0 weak* on $C^*_0(G)$;

(ii) for some (equivalently for every) free ultrafilter $p$ the functional $\tilde{\omega}_p := p\lim S_n(\omega)$ is a non-zero contractive idempotent in $M^u(G)$;

(iii) $L^u_\omega$ has a non-zero fixed point in $C^*_0(G)$;

(iv) $L_\omega$ has a non-zero fixed point in $C_0(G)$;
(v) there is a non-zero \( \tau \in M^n(\mathbb{G}) \) such that \( \tau \ast \omega = \tau \);
(vi) \( L_\omega \) has a non-zero fixed point in \( L^\infty(\mathbb{G}) \) and there exists \( e \in C^u_0(\mathbb{G})_+ \) such that \( \omega e = e \cdot |\omega|^k e = e \cdot |\omega|^k \) and \( a e = a \in N_{|\omega|^k e} \) for every \( k = 1, 2, \ldots \) and \( a \in C^u_0(\mathbb{G}) \).

The proof of this result will be based on the following lemma. For \( \nu \in M(\mathbb{G}) \), we define
\[ R_\nu : C_0(\mathbb{G}) \to C^u_0(\mathbb{G}), \quad R_\nu(a) = (\text{id} \otimes \nu) \Delta_\nu(a). \]
This is possible because (1.1) implies that \( \Delta_\nu(a) \in M(C^u_0(\mathbb{G}) \otimes C_0(\mathbb{G})) \) for every \( a \in C_0(\mathbb{G}) \).

**Lemma 4.4.** Let \( \omega \in M^n(\mathbb{G})_1 \). If \( L_\omega \) has a non-zero fixed point in \( C_0(\mathbb{G}) \), then there exists \( v \in C_0(\mathbb{G}) \) such that \( \omega^k = v \cdot |\omega|^k \) for every \( k \in \mathbb{N} \).

**Proof.** Suppose that \( L_\omega \) has a non-zero fixed point \( x \) in \( C_0(\mathbb{G}) \). Then we may choose \( \nu \in M(\mathbb{G})_1 \) such that \( \nu(x) = \|x\| \). Then \( v = R_\nu(x^*) / \nu(x) \in C^u_0(\mathbb{G}) \) satisfies \( \|v\| \leq 1 \) and \( \omega(v^*) = 1 \). Repeating the argument of Lemma 2.6, we see that this \( v \) satisfies the statement (we may put in the proof \( \phi_n = \nu \) so that \( v_n = v \) for all \( n \in \mathbb{N} \)).

**Proof of Proposition 4.3.** (i) \( \Rightarrow \) (ii) (for every free ultrafilter) follows from Lemma 4.1.

(ii) (for a fixed free ultrafilter) \( \Rightarrow \) (iii) is clear because every element in \( L^n_\nu(\mathbb{G})_\rho \) is a fixed point of \( \text{Fix} L^n_\nu \). (Note also that the map \( \mu \mapsto L^n_\mu \) is injective.)

(iii) \( \Rightarrow \) (i): Let \( a \in C^u_0(\mathbb{G}) \) be a non-zero fixed point of \( L^n_\nu \). Consider \( \nu \in M^n(\mathbb{G}) \) such that \( \nu(a) \neq 0 \). Then we have
\[ \omega^* \nu(R_\nu(a)) = \nu(L_\omega \ast \nu(a)) = \nu(L_\omega \ast \nu(a)) = \nu(a), \]
and it follows that \( S_n(R_\nu(a)) = \nu(a) \neq 0 \) for every \( n \in \mathbb{N} \).

(ii) (for a fixed free ultrafilter) \( \Rightarrow \) (iv): If \( \tilde{\omega}_\rho \) is a non-zero contractive idempotent in \( M^n(\mathbb{G}) \), then \( L_{\tilde{\omega}_\rho}(C_0(\mathbb{G})) = \text{Fix} L_\omega \cap C_0(\mathbb{G}) \) is non-zero (see Lemma 2.2).

(iv) \( \Rightarrow \) (i) is similar to “(iii) \( \Rightarrow \) (i)” when we take \( \nu \in M(\mathbb{G}) \) and \( R_\nu : C_0(\mathbb{G}) \to C^u_0(\mathbb{G}) \) is the map defined before Lemma 4.4.

(ii) (for a fixed free ultrafilter) \( \Rightarrow \) (v): Take \( \tau = \tilde{\omega}_\rho \).

(v) \( \Rightarrow \) (iii): If \( a \in C^u_0(\mathbb{G}) \) is such that \( L^n_\nu(a) \neq 0 \), then the latter is a fixed point of \( L^n_\omega \) as \( L^n_\nu \circ L^n_\omega = L^n_\tau \). \( \ast \)

(iv) \( \Rightarrow \) (vi): Suppose that \( L_\omega \) has a non-zero fixed point \( x \in C_0(\mathbb{G}) \). Then we can apply Lemma 4.4 to obtain a suitable \( v \in C^u_0(\mathbb{G}) \). As (iv) implies (i), we know that the limit \( \tilde{\omega} := w^*-\lim_{n \to \infty} S_n(\omega) \) exists by Lemma 4.1. We shall then show that \( \tilde{\omega}(a) = |\omega|^{-k} \) for every \( a \in C^u_0(\mathbb{G}) \). Note that \( \omega^k(a) = |\omega|^k(\nu \ast v) \) by Lemma 4.4, and hence
\[ \tilde{\omega}(a) = \lim_{n \to \infty} \sum_{k=1}^n \omega^k(a) = \lim_{n \to \infty} \sum_{k=1}^n |\omega|^k(\nu \ast v) = |\omega|^{-k}. \]

Recalling that \( |\omega|^k(\nu \ast v) = |\omega|^k(a) \) for all \( k \in \mathbb{N} \), it follows similarly that \( \tilde{\omega}(\nu \ast v) = |\omega|^{-k}(a) \). These identities imply that \( \|\tilde{\omega}\| = \|\omega\|^{-k} \). Moreover, by the Cauchy–Schwarz inequality,
\[ |\tilde{\omega}(a)|^2 \leq |\omega|^{-k}(aa^*) \|\nu \ast v\| \leq |\omega|^{-k}(aa^*). \]

It now follows from the uniqueness of absolute value (Proposition III.4.6 of [Tak]) that \( |\tilde{\omega}| = |\omega|^{-k} \).

Since \( x \in C_0(\mathbb{G}) \) is a fixed point of \( L_\omega \), we have \( L_\omega(x) = x \). Since \( \tilde{\omega} \) is a contractive idempotent in \( M^n(\mathbb{G}) \) it follows from Lemma 3.1 of [NSSS] (see also equation (2.5) of [Kas]) that
\[ xx^* = L_\tilde{\omega}(x)L_\tilde{\omega}(x)^* = L_{|\tilde{\omega}|}(xx^*) = L_{|\tilde{\omega}|}(xx^*). \]

Pick \( \nu \in M(\mathbb{G})_+ \) such that \( \nu(xx^*) = \|xx^*\| \). Then \( R_\nu(xx^*) \in C^u_0(\mathbb{G}) \) is a fixed point of \( L^n_\nu \) and \( |\omega|^{-k}(R_\nu(xx^*)) = \nu(xx^*) \). Put \( e = R_\nu(xx^*) / \nu(xx^*) \in C^u_0(\mathbb{G}) \). Since \( \tilde{\omega} \) is an idempotent state in \( M^n(\mathbb{G}) \) and \( e \in L^n_\nu(C^u_0(\mathbb{G})) \), \( e \) is in the multiplicative domain of \( |\tilde{\omega}| \) by Lemma 2.5 of [SaS1].

Since \( |\tilde{\omega}| = |\omega|^{-k} \), we have for any \( k \in \mathbb{N} \) that
\[ L_{|\tilde{\omega}|}(xx^*) = L_{|\tilde{\omega}|}(L_{|\tilde{\omega}|}(xx^*)) = L_{|\tilde{\omega}|}(L_{|\tilde{\omega}|}(xx^*)) = L_{|\tilde{\omega}|}(xx^*) = xx^* \]
and so \( |\omega|^k(e) = 1 \). Hence
\[
1 = |\omega|^k(e)^2 \leq |\omega|^k(e^2) \leq 1
\]
(by Cauchy–Schwarz) and it follows that \( e \) is also in the multiplicative domain of \( |\omega|^k \) for every \( k \in \mathbb{N} \). Now for every \( a \in C^*_0(\mathbb{G}) \) and \( k \in \mathbb{N} \)
\[
|\omega|^k((ae - a)^*(ae - a)) = |\omega|^k(ea^*ae) - |\omega|^k(a^*ae) - |\omega|^k(ea^*a) + |\omega|^k(a^*a) = 0
\]
because \( e \) is in the multiplicative domain of \( |\omega|^k \). Therefore \( ae - a \in N_{|\omega|^k} \).

(vi) \implies (i): Fix a free ultrafilter \( p \) and let \( \bar{\omega}_p \in RUC^n(\mathbb{G})^* \) be the weak* limit of \( S_n(\omega) \) along \( p \). Since \( L_\omega \) has a non-zero fixed point in \( L^\infty(\mathbb{G}) \), there is, by Lemma 2.3, \( x \in RUC^n(\mathbb{G})^* \) such that \( \bar{\omega}_p(x) \neq 0 \). Let \( v \in C^*_0(\mathbb{G})^{**} \) be as in Lemma 2.6. It follows from (vi) that \( |\omega|^k.e = |\omega|^k \) on \( C^*_0(\mathbb{G})^{**} \) for every \( k \in \mathbb{N} \). Therefore\[
\bar{\omega}_p(ex) = p\text{-}\lim S_n(|\omega|)(ex) = p\text{-}\lim S_n(|\omega|)(xv) = p\text{-}\lim S_n(\omega)(x) = \bar{\omega}_p(x) \neq 0.
\]
Since \( ex \in C^*_0(\mathbb{G}) \), (i) holds.

The above theorem has an immediate corollary, which can be also proved directly.

**Corollary 4.5.** If \( \omega \in M^n(\mathbb{G}) \) and \( \omega^n \to 0 \) in the weak* topology of \( M^n(\mathbb{G}) \), then \( \text{Fix} L_\omega \cap C_0(\mathbb{G}) = \{0\} \).

In the non-degenerate case, Proposition 4.3 can be used to completely characterise the fixed points in \( C_0(\mathbb{G}) \). Note that the second statement can be deduced from Theorem 2.7 and the analogous fact for \( \omega \) being positive.

**Corollary 4.6.** Suppose that \( \omega \in M^n(\mathbb{G}) \) is non-degenerate. If \( \mathbb{G} \) is not compact, then \( \text{Fix} L_\omega \cap C_0(\mathbb{G}) = \{0\} \). If \( \mathbb{G} \) is compact, then \( \text{Fix} L_\omega \) is either \( \{0\} \) or \( \mathbb{C}1 \) where \( u \) is a group-like unitary in \( C(\mathbb{G}) \).

**Proof.** For non-degenerate \( \omega \), the element \( e \) in statement (iv) of Proposition 4.3 is a positive right identity, hence an identity. Therefore \( \mathbb{G} \) is necessarily compact if \( L_\omega \) has a non-zero fixed point in \( C_0(\mathbb{G}) \).

Now suppose that \( \mathbb{G} \) is compact. By Theorem 2.7 and Lemma 2.3, if \( \text{Fix} L_\omega \neq \{0\} \) there is a group-like unitary \( u \in C(\mathbb{G}) \) such that \( \text{Fix} L_\omega = \text{Fix} L_{|\omega|}.u \). By [KNR1, Theorem 3.6], \( \text{Fix} L_{|\omega|} = \mathbb{C}1 \).

Finally we note that for discrete quantum groups and positive functionals the above corollary can be strengthened: in [Ka1, Theorem 2] it is shown that if \( \mathbb{G} \) is discrete and infinite (in other words non-compact) and \( \omega \in P^n(\mathbb{G}) \) is non-degenerate then for every \( x \in C_0(\mathbb{G}) \) we have \( L_{\omega^n}(x) \xrightarrow{n \to \infty} 0 \).

5. **Fixed points in \( L_p(\mathbb{G}) \) for tracial Haar weights**

Yau showed in [Yau] that any harmonic function \( f \in L_p(M) \) on a complete manifold \( M \) is constant, for any \( p \in (1, \infty) \). Motivated by this result, Chu [Chu] introduced and studied the space of \( L_p \)-fixed points of the convolution operator \( L_\omega \) for \( p \in [1, \infty) \). The main result of [Chu] states that if \( \omega \) is an adapted probability measure, then any such fixed point must be a constant function, see [Chu, Theorem 3.12, Corollary 3.14].

A quantum group version of Chu’s result has been obtained by Kalantar in [Ka2]. More precisely, let \( \mathbb{G} \) be a locally compact quantum group with tracial (left) Haar weight, \( p \in [1, \infty) \), and \( \omega \in P(\mathbb{G}) \) a non-degenerate quantum probability measure. Consider the space of \( \omega \)-harmonic vectors in the non-commutative \( L_p \)-space \( L_p(\mathbb{G}) \):
\[
H^\omega_p(\mathbb{G}) := \{ f \in L_p(\mathbb{G}) \mid L_\omega f = f \}.
\]
If \( \mathbb{G} \) is non-compact, then \( H^\omega_p(\mathbb{G}) = \{0\} \) [Ka2, Theorems 2.4, 2.6]. If \( \mathbb{G} \) is compact, then \( H^\omega_p(\mathbb{G}) = \mathbb{C}1 \) [Ka2, Theorem 2.8].

We will now generalise these results to our setting, extending the context from \( M(\mathbb{G}) \) to \( M^n(\mathbb{G}) \) and allowing non-positive quantum measures. We begin by outlining the construction of the action of the convolution operators on the non-commutative \( L_p \)-spaces, with \( p \in [1, \infty) \). Recall that we
denote by \( \varphi \) the tracial left Haar weight, and by \( \psi = \varphi R \) the right Haar weight, where \( R \) is the unitary antipode of \( L^\infty(G) \). We write \( L_p(G) \) and \( \hat{L}_p(G) \) for the non-commutative \( L_p \)-spaces associated with \( \varphi \) and \( \psi \), respectively. Recall that these are defined as the completions of \( \mathcal{M}_\varphi \) and \( \mathcal{M}_\psi \) under the norms \( \|x\|_p = \varphi(|x|^p)^{\frac{1}{p}} \) and \( \|x\|_p = \psi(|x|^p)^{\frac{1}{p}} \), respectively; here,

\[
\mathcal{M}_\varphi := \{ x \in L^\infty(G)^+ \mid \varphi(x) < \infty \}.
\]

For \( p, q \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), the spaces \( L_p(G) \) and \( \hat{L}_q(G) \) are the duals of each other, via

\[
\langle a, b \rangle = \varphi(aR(b)) = \psi(R(a)b) \quad (a \in L_p(G), b \in \hat{L}_q(G));
\]

cf. [Ka2, p. 3972]. Recall that we denote by \( L^1(G) \) the predual of \( L^\infty(G) \); the spaces \( L^1(G) \) and \( L_1(G) \) are isometrically isomorphic via the map \( \Phi : L_1(G) \to L^1(G) \) such that for \( x \in \mathcal{M}_\varphi \) and \( y \in L^\infty(G) \) we have \( \Phi(x)(y) = \varphi(xy) \).

The convolution action of \( M(G) \) on \( L_p(G) \) is defined by complex interpolation; for \( \omega \in P(G) \) this is discussed in in [Ka2, section 2], cf. also [BrR, p. 19, in particular Lemma 4.3]. Below we will outline what is needed to extend this to \( \omega \in M^1(G) \).

Begin by recalling the argument appearing in [Ka2]: if \( \omega \in M^1(G)_+ \), then the convolution operator \( L_\omega : L^\infty(G) \to L^\infty(G) \) preserves the weight \( \phi \) (so also the space \( \mathcal{M}_\varphi \)). Further Kalantar checks in [Ka2] that we have the following equality:

\[
(5.1) \quad \Phi \circ L_\omega(z) = (\omega \circ R) \ast \Phi(z) \quad (z \in \mathcal{M}_\varphi).
\]

This implies that in fact \( L_\omega \) yields (by continuous extension) a bounded map on \( L_1(G) \), and hence by complex interpolation on all the spaces \( L_p(G) \). Further the above formula is true (simply by linearity and continuity) for all \( \omega \in M(G) \) and \( z \in L_1(G) \). A similar procedure shows that the right convolution operators \( R_\omega \) act boundedly on \( L_q(G) \).

Further Kalantar shows in the proof of [Ka2, Theorem 2.2] that the following is true for all \( p \in (1, \infty), \omega \in P(G), f \in L_p(G) \) and \( g \in \hat{L}_q(G) \):

\[
(5.2) \quad \langle L_\omega(f), g \rangle = \langle f, R_\omega(g) \rangle.
\]

Before we extend these formulas to operators associated with \( \omega \in M^1(G) \), we need to recall that for every \( p \in (1, \infty) \) we have that each of the subspaces \( L_1(G) \cap L_p(G) \) and \( L^\infty(G) \cap L_p(G) \) is dense in \( \hat{L}_p(G) \) (with similar statements holding for \( \hat{L}_p(G) \)). Finally note one more easy property: the span of \( \{ L_\nu(z) : \nu \in L^1(G), z \in \mathcal{M}_\varphi \} \) is dense in \( L_1(G) \) (and naturally the span of \( \{ R_\nu(z) : \nu \in L^1(G), z \in \mathcal{M}_\varphi \} \) is dense in \( \hat{L}_1(G) \)). Indeed, it suffices to use the formula (5.1), the fact that \( R \) is an isometry on \( L^\infty(G) \) and finally the fact that the linear span of \( L^1(G) \ast L^1(G) \) is dense in \( L^1(G) \).

**Lemma 5.1.** Let \( \omega \in M^1(G), p \in [1, \infty) \). Then the operator \( L_\omega : L^\infty(G) \to L^\infty(G) \) defines by restriction/continuous extension a bounded operator on \( L_p(G) \) (to be denoted by the same symbol). Further we have the following equalities:

\[
(5.3) \quad \Phi \circ L_\omega(z) = (\omega \circ R^\nu) \ast \Phi(z), \quad z \in L_1(G),
\]

\[
(5.4) \quad \langle L_\omega(f), g \rangle = \langle f, R_\omega(g) \rangle, \quad f \in L_p(G), g \in \hat{L}_q(G).
\]

**Proof.** We follow the line of argument in [Ka2]. Note we can (and do) assume that \( \omega \in P^1(G) \) and then argue by linearity. The fact that \( L_\omega \) preserves the left Haar weight is Lemma 3.4 of [KNN1]. In the next step we show that the formula (5.1) holds for \( z \) in a dense subset of \( L_1(G) \). Indeed, take \( z = L_\nu(z') \), where \( z' \in \mathcal{M}_\varphi \) and \( \nu \in L^1(G) \). Then we have

\[
\Phi \circ L_\omega(z) = \Phi \circ L_\omega(L_\nu(z')) = \Phi \circ L_{\nu \ast \omega}(z') = ((\nu \ast \omega) \circ R) \ast \Phi(z') = ((\omega \circ R_\nu) \ast (\nu \circ R)) \ast \Phi(z') = (\omega \circ R_\nu) \ast (\nu \circ R) \ast \Phi(z') = (\omega \circ R_\nu) \ast \Phi \circ L_\nu(z') = (\omega \circ R_\nu) \ast \Phi(z)
\]

where in the third and the sixth equalities we used (5.1) and in the fourth one (1.2). Now, as the span of \( L_\nu(z') \) with \( z' \in \mathcal{M}_\varphi \) and \( \nu \in L^1(G) \) is dense in \( L_1(G) \), we deduce that \( L_\omega \) restricts/extends to a bounded map on the whole \( L_1(G) \) and further (5.3) holds for all \( z \in L_1(G) \). Now that we have established that \( L_\omega \) is bounded on \( L_1(G) \) (as well as on \( L^\infty(G) \)), it follows by complex...
interpolation that $L_\omega$ is bounded also on $L_p(\mathbb{G})$ for $p \in (1, \infty)$. Similarly, $R_\omega$ is bounded on all $L_q(\mathbb{G})$.

Consider now the second statement and assume first that $p = 1$. Suppose that $f = L_\omega(f')$ for some $\nu \in L^1(\mathbb{G})$ and $f' \in L_1(\mathbb{G})$ and $g \in \hat{L}_q(\mathbb{G})$. We compute:

$$
\langle L_\omega(f), g \rangle = \langle L_\omega(L_\nu(f')), g \rangle = \langle L_{\nu \ast \omega}(f'), g \rangle = \langle f', R_{\nu \ast \omega}(g) \rangle = \langle f', R_\nu(g) \rangle = \langle f, R_\omega(g) \rangle,
$$

where in the third and fifth equalities we used (5.2). By continuity, (5.4) follows (for $p = 1$).

Let then $p \in (1, \infty)$. As each duality is given by computing the trace on the respective (strong) products of measurable operators, we have by the above

$$
\langle L_\omega(f), g \rangle = \langle f, R_\omega(g) \rangle
$$

for all $f \in L_p(\mathbb{G}) \cap L_1(\mathbb{G})$, $g \in \hat{L}_q(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then the general statement follows by first approximating a general $f \in L_p(\mathbb{G})$ by $f_i \in L_p(\mathbb{G}) \cap L_1(\mathbb{G})$ and then a general $g \in \hat{L}_q(\mathbb{G})$ by $g_i \in \hat{L}_q(\mathbb{G}) \cap L^\infty(\mathbb{G})$. This uses the continuity of the operators $L_\omega$ and $R_\omega$ on respectively $L_p(\mathbb{G})$ and $\hat{L}_q(\mathbb{G})$.

We are ready for the main result of this section.

**Theorem 5.2.** Let $\mathbb{G}$ be a locally compact quantum group with tracial (left) Haar weight, let $p \in [1, \infty)$, and let $\omega \in M(\mathbb{G})_1$ be non-degenerate.

(i) Assume $\mathbb{G}$ is non-compact. Then $H^p_\omega(\mathbb{G}) = \{0\}$.

(ii) Assume $\mathbb{G}$ is compact and co-amenable. If $L_\omega$ has a non-zero fixed point in $L_p(\mathbb{G})$, then $H^p_\omega(\mathbb{G}) = C_u$ where $u$ is a group-like unitary in $C(\mathbb{G})$.

**Proof.** (i) Case 1: $p > 1$. Let $f \in H^p_\omega(\mathbb{G})$, and $g \in \hat{L}_q(\mathbb{G})$, $g = R_\nu(g')$ for some $\nu \in L^1(\mathbb{G})$ and $g' \in L_q(\mathbb{G})$. Then the element $\Omega_{f,g'} \in L_\infty(\mathbb{G})$ defined through

$$
\langle \Omega_{f,g'}, h \rangle = \langle L_h(f), g' \rangle \quad (h \in L^1(\mathbb{G}))
$$

belongs to $C_0(\mathbb{G})$; cf. [Ka2, proof of Theorem 2.4] (note that the latter argument assumes $p \in [1, 2]$, but it can be modified to work for all $p > 1$ by using the density result obtained in [Mas, Theorem 8] to approximate $f$ and $g$). Note also that $S_n(\omega) \to 0$ ($w^*$) since otherwise, by our Proposition 4.3, $L_\omega$ would have a non-zero fixed point in $C_0(\mathbb{G})$, which would contradict the non-compactness of $\mathbb{G}$, in view of our Corollary 4.6. Thus we obtain, using twice (5.4):

$$
\langle f, g \rangle = \langle L_{S_n(\omega)}(f), g \rangle = \langle f, R_{S_n(\omega)}(R_\nu(g')) \rangle = \langle f, R_{S_n(\omega) \ast \nu}(g') \rangle = \langle L_{S_n(\omega) \ast \nu}(f), g' \rangle = (S_n(\omega) \ast \nu)(\Omega_{f,g'}) = S_n(\omega)((\text{id} \otimes \nu)\Delta(\Omega_{f,g'}))^{n \to \infty} 0
$$

whence $f = 0$ (since $g$ was an arbitrary element of a dense subset of $\hat{L}_q(\mathbb{G})$).

Case 2: $p = 1$. Let $f \in H^1_\omega(\mathbb{G})$. Note that, as in Case 1, we have that $S_n(\omega) \to 0$ ($w^*$). By (5.3) we have for any $n \in \mathbb{N}$ the following equality:

$$
\Phi(f) = \Phi \circ L_{S_n(\omega)}(f) = (S_n(\omega) \circ R_n) \ast \Phi(f)
$$

Thus for all $a \in C_0(\mathbb{G})$:

$$
\Phi(f)(a) = ((S_n(\omega) \circ R_n) \ast \Phi(f))(a) = (S_n(\omega) \circ R_n)((\text{id} \otimes \Phi(f))\Delta(a)) = S_n(\omega)(R_n((\text{id} \otimes \Phi(f))\Delta(a)))^{n \to \infty} 0
$$

so that $\Phi(f) = 0$ and of course $f = 0$.

(ii) Since $\mathbb{G}$ is compact, we have $L^\infty(\mathbb{G}) \subseteq L_p(\mathbb{G}) \subseteq L_1(\mathbb{G})$. Let $f \in H^p_\omega(\mathbb{G}) \setminus \{0\}$. As $\mathbb{G}$ is co-amenable, we have a bounded approximate identity $(e_\lambda)_{\lambda \in A}$ in $L^1(\mathbb{G})$. For all $n \in \mathbb{N}$ choose $e_\lambda$ such that $\|\Phi(f) \ast e_\lambda - \Phi(f)\|_1 < \frac{1}{n}$. Since $L^\infty(\mathbb{G})$ is dense in $L_1(\mathbb{G})$, we can further find $x_n \in L^\infty(\mathbb{G})$ with $\|\Phi(x_n) - e_\lambda\|_1 < \frac{1}{n\|f\|_1}$. Hence, for all $n \in \mathbb{N}$ we obtain:

$$
\|\Phi(f) \ast \Phi(x_n) - \Phi(f)\|_1 \leq \|\Phi(f) \ast \Phi(x_n) - \Phi(f) \ast e_\lambda\|_1 + \|\Phi(f) \ast e_\lambda - \Phi(f)\|_1 < \frac{2}{n}.
$$
so \( f_n := \Phi^{-1}(\Phi(f) \ast \Phi(x_n)) \) in \( L_1(\mathbb{G}) \). We can now exploit the formula (5.1) to see that
\[
\Phi(f) \ast \Phi(x_n) = ((\Phi(f) \circ R) \circ R) \ast \Phi(x_n) = \Phi(L_{\Phi(f) \circ R}(x_n)).
\]
Thus \( f_n = L_{\Phi(f) \circ R}(x_n) \in L^1(\mathbb{G}) \ast L^\infty(\mathbb{G}) = RUC(\mathbb{G}) = C(\mathbb{G}) \). Finally
\[
L_\omega \circ L_{\Phi(f) \circ R}(x_n) = L_{\Phi(f) \circ R}(x_n)
\]
and using the (5.1) once again we check that
\[
(\Phi(f) \circ R) \ast \omega = ((\omega \circ R) \ast \Phi(f)) \circ R = \Phi(L_\omega(f)) \circ R = \Phi(f) \circ R.
\]
As \( f_n \to f \neq 0 \), we have \( f_n \neq 0 \) for \( n \) large enough, so that \( L_\omega \) has a non-zero fixed point in \( C(\mathbb{G}) \). Thus, our Corollary 4.6 implies, since \( \mathbb{G} \) is compact, that \( \text{Fix} L_\omega = \mathbb{C}u \), where \( u \) is a group-like unitary in \( C(\mathbb{G}) \). So, \( f_n \in \mathbb{C}u \) for \( n \) large enough, whence \( f \in \mathbb{C}u \). Thus, we have \( \mathcal{H}^p(\mathbb{G}) \subseteq \mathbb{C}u \).

The reverse inclusion is clear as \( u \in \text{Fix} L_\omega \subseteq L^\infty(\mathbb{G}) \subseteq L^p(\mathbb{G}) \). □

6. Classical case

In this section we discuss the classical case, i.e. the situation where \( \mathbb{G} \) is a locally compact group.

Given a probability measure \( \omega \in \text{Prob}(\mathbb{G}) \), we let \( S_\omega \) and \( G_\omega \) denote, respectively, the closed semigroup and group generated by \( \supp \omega \). We say that \( \omega \) is non-degenerate (or irreducible) if \( S_\omega = \mathbb{G} \), and adapted if \( G_\omega = \mathbb{G} \); so that non-degeneracy implies adaptedness, but the converse implication does not hold. Given a general \( \omega \in M(\mathbb{G}) \) we will call it non-degenerate (respectively, adapted) if \( |\omega| \) is non-degenerate (respectively, adapted). It is easy to see that this notion of non-degeneracy coincides with the one introduced earlier. In the non-degenerate case, Theorem 2.7 says that if \( L_\omega \) has a non-zero fixed point in \( LUC(\mathbb{G}) \), then there is \( \chi \in \hat{\mathbb{G}} \) such that \( \omega = \chi|\omega| \) and \( \text{Fix} L_\omega = (\text{Fix} L_{|\omega|})^\mathbb{T} \). The following theorem addresses this result in the degenerate situation.

Theorem 6.1. Suppose that \( \mathbb{G} \) is a locally compact group and that \( \omega \in M(\mathbb{G})_1 \). If \( L_\omega \) has a non-zero fixed point in \( LUC(\mathbb{G}) \), then there is a continuous character \( \chi : S_{|\omega|} \to \mathbb{T} \) such that \( \omega = \chi|\omega| \).

Proof. Let \( f \in LUC(\mathbb{G}) \) be a non-zero fixed point of \( L_\omega \). Let \( \tilde{f} \) denote the extension of \( f \) to the LUC-compactification \( G^{LUC} \). Then there exists \( x \in G^{LUC} \) such that \( \|f\| = |\tilde{f}(x)| \). Multiplying \( f \) by a scalar we may and shall assume that \( 1 = \tilde{f}(x) = \|f\| \).

Write \( \omega = u|\omega| \) where \( u \) is a unimodular, measurable function. With \( x \) as above, which can be viewed also as an element of \( LUC(\mathbb{G})^* \), we define a function \( x \circ f \) by
\[
(x \circ f)(s) = \langle x, \ell_s f \rangle = \tilde{f}(sx) \quad (s \in \mathbb{G})
\]
where \( sx \) is defined using the multiplication on \( G^{LUC} \). Then \( x \circ f \in LUC(\mathbb{G}) \) and for any \( n \in \mathbb{N} \), if we choose a net \( (t_i) \in \mathbb{T} \) of elements of \( G \) convergent to \( x \) inside \( G^{LUC} \), we have
\[
\langle \omega^n, x \circ f \rangle = \langle \omega^n, x, f \rangle = \lim_{i \in \mathbb{I}} \langle \omega^n \cdot t_i, f \rangle = \lim_{i \in \mathbb{I}} \langle \omega^n, r_{t_i} f \rangle = \lim_{i \in \mathbb{I}} \langle L^n_\omega f(t_i) \rangle = \langle x, L^n_\omega f \rangle = \langle x, f \rangle,
\]
where we first used the definition of the product in \( LUC(\mathbb{G})^* \) and then the fact that \( \omega^n \), as a measure, belongs to the topological centre of \( LUC(\mathbb{G})^* \) (see for example [Won]). Hence letting \( S = \supp |\omega| \) we have that
\[
\tilde{f}(x) = \int_{S \times \mathbb{N}} f(s_1 \ldots s_n x) u(s_1) \ldots u(s_n) d|\omega|^\mathbb{N}(s_1, \ldots, s_n).
\]
Since \( |\tilde{f}| \) attains its maximum at \( x \), and \( \tilde{f}(x) = 1 \),
\[
\tilde{f}(s_1 \ldots s_n x) u(s_1) \ldots u(s_n) = \tilde{f}(x) = 1
\]
for \( |\omega|^\times \) almost every \( (s_1, \ldots, s_n) \). Now
\[
\overline{\chi(s)} = \tilde{f}(sx)
\]
defines a bounded continuous function \( \chi \) on \( \mathbb{G} \). Taking \( n = 1 \) above we see that
\[
\chi(s) = u(s) \quad \text{for } |\omega|\text{-almost every } s, \text{ and hence } \chi(S) \subseteq \mathbb{T}.
\]
For arbitrary $n$ we then see that
\[ \chi(s_1 \ldots s_n) = u(s_1) \ldots u(s_n) \quad \text{for} \quad |\omega|^n \text{-almost every } (s_1, \ldots, s_n) \]
while the $n = 1$ case implies that $u(s_1) \ldots u(s_n) = \chi(s_1) \ldots \chi(s_n)$ for $|\omega|^n$-almost every $(s_1, \ldots, s_n)$. Since $\chi$ is continuous we conclude that
\[ \chi(s_1 \ldots s_n) = \chi(s_1) \ldots \chi(s_n) \quad \text{for every } n \in \mathbb{N} \text{ and } (s_1, \ldots, s_n) \in S^{\times n}. \]
We let $S^n \subseteq G$ denote the set of products from $S^{\times n}$. An obvious induction shows that $\chi$ is multiplicative on the semigroup $S^{|\omega|} = \bigcup_{n=1}^{\infty} S^n$, and hence, by continuity, $\chi$ is multiplicative on its closure $S_{|\omega|}$.

Notice that if $|\omega|$ in the last theorem is non-degenerate then the character $\chi$ with the desired properties is defined on the whole of $G$. In such a situation, if $f \in \text{Fix } L_{|\omega|}$, then it is clear that $f\tau \in \text{Fix } L_{|\omega|}$. In other cases there is no evident analogue of this fact. For any closed subsemigroup $S$ of $G$ and continuous character $\chi : S \to \mathbb{T}$ we let
\[ LUC_{\chi,S}(G) = \{ f \in LUC(G) : f(st) = \bar{\chi}(s)f(t) \text{ for } s \in S, t \in G \} \]
If $\omega = |\omega|$ as in the theorem above, then it is easy to see that
\[ LUC_{\chi,S_{|\omega|}}(G) \subseteq \text{Fix } L_{|\omega|}. \]
If $\omega = |\omega|$ is a probability measure, then in many cases – e.g. $G$ abelian ([ChD]) or $G$ is SIN and $\omega$ is non-degenerate ([Jaw]) – we have the Choquet-Deny theorem: $\text{Fix } L_{|\omega|} = LUC_{1,S_{|\omega|}}(G)$.

Suppose we are given a closed subsemigroup $S$ densely generating a locally compact group $H$. Except in the case of abelian groups, it is not evident how to extend a character $\chi_0 : S \to \mathbb{T}$ to a character on $H$. Even in the abelian case it is not clear that the extension can be assumed continuous if $\chi_0$ is continuous.

**Lemma 6.2.** Let $S$ be a closed subsemigroup of $G$, let $H$ be the smallest closed subgroup of $G$ containing $S$, and let $\chi_0 : S \to \mathbb{T}$ be a continuous character. If there exists a non-zero bounded continuous function $f$ such that $f(st) = \bar{\chi_0}(s)f(t)$ for all $s \in S, t \in G$ then $\chi_0$ extends to a continuous character $\chi$ on $H$. Furthermore $f(st) = \bar{\chi}(s)f(t)$ for all $s \in H, t \in G$.

**Proof.** Set $U_f = \{ t \in G : f(t) \neq 0 \}$. Let $s \in S$ and $t \in U_f$. We have that $\bar{\chi_0}(s)f = \ell_s f$ (left translation) and hence $\ell_{s^{-1}}f = \chi_0(s)f$ which implies that
\[ \bar{\chi_0}(s) = \frac{f(st)}{f(t)} \quad \text{and} \quad \chi_0(s) = \frac{f(s^{-1}t)}{f(t)} \quad \text{so} \quad \langle S \rangle U_f \subseteq U_f \]
where $\langle S \rangle$ is the subgroup generated (algebraically) by $S$. We fix some $t_0 \in U_f$ and define $\chi : G \to \mathbb{C}$ by
\[ \bar{\chi}(s) = \frac{f(st_0)}{f(t_0)} \]
so that $\chi$ is continuous. Then for $s, s' \in S \cup S^{-1}$ we have that
\[ \bar{\chi}(ss') = \frac{f(ss't_0)}{f(t_0)} = \frac{f(s's't_0)}{f(t_0)} = \bar{\chi}(s') \bar{\chi}(s). \]
As simple induction then shows that $\chi$ is multiplicative on $\langle S \rangle = \bigcup_{n=1}^{\infty} (S \cup S^{-1})^n$, so also on $H$. Similarly, we see that $\ell_s f = \bar{\chi}(s)f$ on $\langle S \rangle$, and hence also on $H$, which ends the proof.

We consider below two cases in which the Choquet–Deny theorem always holds in the context of probability measures, and show that it also holds in the more general framework of contractive measures we consider here. For the abelian case we adapt the remarkably simple proof of [Rau], and for the weakly almost periodic case we adapt a proof from [Tem], which Tempelman attributes to Ryll-Nardzewski. Below $WAP(G)$ denotes the space of weakly almost periodic functions on $G$; recall also the notation introduced in (6.1).
Theorem 6.3. Let $G$ be a locally compact group and $\omega \in M(G)_1$.

(i) If $G$ is abelian and $L_\omega$ has a non-zero fixed point in $LUC(G)$, then there is a continuous character $\chi : G_{|\omega|} \to \mathbb{T}$ such that $\omega = \chi_{|\omega|}$ and

$$
\text{Fix } L_\omega = LUC_{X,G_{|\omega|}}(G)
$$

(ii) If $L_\omega$ has a non-zero fixed point in $WAP(G)$ then there is a continuous character $\chi : G_{|\omega|} \to \mathbb{T}$ such that $\omega = \chi_{|\omega|}$ and

$$
\text{Fix } L_\omega \cap WAP(G) = WAP_{X,G_{|\omega|}}(G)
$$

where $WAP_{X,G_{|\omega|}}(G) = WAP(G) \cap LUC_{X,G_{|\omega|}}(G)$. In particular, we have that $\chi \in WAP(G)_{|G_{\omega}|}$.

Proof. In either case above, Theorem 6.1 provides a character $\chi : S_{|\omega|} \to \mathbb{T}$ for which $\omega = \chi_{|\omega|}$. We will proceed to show for each case that the fixed points of $L_\omega$ are contained in the set $LUC_{X,S_{|\omega|}}(G)$.

Note that Lemma 6.2 shows that we may deem from the beginning $\chi$ to be a character on $G_{|\omega|}$.

(i) We let $f \in \text{Fix } L_\omega \setminus \{0\}$ and write $S = \text{supp } |\omega|$. Define for $n \in \mathbb{N}$ and $t \in G$

$$
g_n(t) = \int_{S^n} |\chi(s_1 \ldots s_n)f(s_1 \ldots s_n t) − \chi(s_1 \ldots s_{n-1})f(s_1 \ldots s_{n-1} t)|^2 |\omega|^{\times n}(s_1, \ldots, s_n)
$$

Expanding the integrand, exploiting the fact that $G$ is abelian (so that $s_1 \ldots s_n t = s_n s_1 \ldots s_{n-1} t$) and that $f \in \text{Fix } L_\omega$ we see that $g_n = L_\omega |\omega|^\times(|f|^2) − L_\omega |\omega|^\times(|f|^2)$. Hence the telescopic series $\sum_{n=1}^{\infty} g_n$ converges (as increasing and bounded), so that $\lim_{n \to \infty} g_n = 0$. On the other hand, the Cauchy-Schwarz inequality tells us that for each $t \in G$

$$
\int_{S \times (n-1)} |\chi(s_1 \ldots s_n)f(s_1 \ldots s_n t) − \chi(s_1 \ldots s_{n-1})f(s_1 \ldots s_{n-1} t)|^2 |\omega|^{\times (n-1)}(s_1, \ldots, s_{n-1})
\geq \left| \int_{S \times (n-1)} |\chi(s_1 \ldots s_n)f(s_1 \ldots s_n t) − \chi(s_1 \ldots s_{n-1})f(s_1 \ldots s_{n-1} t)| |\omega|^{\times (n-1)}(s_1, \ldots, s_{n-1}) \right|^2
= |\chi(s_n)f(s_n t) − f(t)|^2.
$$

Hence an application of the Fubini–Tonelli theorem shows that $g_n \geq g_1$. Thus $g_1 = 0$, which tells us for any $t$ that $\chi(s)f(st) = f(t)$ for $|\omega|$-a.e. $s$, and hence, by continuity, for every $s \in S$. In other words $\ell_s f = \chi(s)f$ for $s \in S$. A simple induction shows this holds for $s \in \bigcup_{n=1}^{\infty} S^n$ and hence for all $s \in S_{|\omega|}$.

(ii) Let $f \in \text{Fix } L_\omega \cap WAP(G) \setminus \{0\}$. Consider the product probability space

$$(\Omega, \mathcal{B}_\infty, P) = (G^{\infty}, \mathcal{B}_\infty, |\omega|^{\times \infty})$$

where $\mathcal{B}_\infty$ is the $\sigma$-algebra generated by the sequence of $\sigma$-algebras given for each $n \in \mathbb{N}$ by

$$(\mathcal{B}_n) = \sigma(B_1 \times \cdots \times B_n) = G \times G \times \cdots : \text{ each } B_t \text{ is a Borel set in } G.$$ 

Let $\mathcal{L}(\mathcal{B}_n) = L^1(\Omega, \mathcal{B}_n, P; WAP(G))$ for each $n = 1, 2, \ldots, \infty$. We have conditional expectations $\mathbb{E}_n : \mathcal{L}(\mathcal{B}_\infty) \to \mathcal{L}(\mathcal{B}_n)$ given by

$$
[\mathbb{E}_n(\alpha)](s_1, s_2, \ldots) = \int_{\Omega} \alpha(s_1, \ldots, s_{n-1}, s_n, s_{n+1}, \ldots) dP(s_{n+1}, s_{n+2}, \ldots).
$$

Define $\beta_n$ in $\mathcal{L}(\mathcal{B}_n)$ for $P$-a.e. $(s_1, s_2, \ldots)$ by

$$
\beta_n(s_1, s_2, \ldots) = \chi(s_1 \ldots s_n) \ell_{s_1} \ldots s_n f
$$

and observe that $\|\beta_n\|_{L^1} \leq \|f\|_{\infty}$. Also, observe that for $P$-a.e. $(s_1, s_2, \ldots)$ we have

$$
\|\chi(s_1) \ell_{s_1} \ldots s_n f - f\|_{\infty} = \|\chi(s_1 \ldots s_{n-1}) \ell_{s_n} \ldots s_n f - f\|_{\infty} = \|\beta_n - \beta_{n-1}(s_1, s_2, \ldots)\|_{\infty}
$$

We compute, for any $n \geq 2$, $t \in G$ and $(s_1, s_2, \ldots)$ as above that

$$
([\mathbb{E}_{n-1}(\beta_n)](s_1, s_2, \ldots))(t) = (\chi(s_1 \ldots s_{n-1}) \ell_{s_{n-1}} \ldots s_n f(\ell_{s_{n-1}} \ldots s_n f)(t) = (\beta_{n-1}(s_1, s_2, \ldots))(t).
$$

Hence $(\beta_n)_{n=1}^{\infty}$ is a martingale in $\mathcal{L}(\mathcal{B}_\infty)$. Moreover, the essential range of each $\beta_n$ is contained in $T \{f : g \in G\}$, and thus is relatively weakly compact as $f \in WAP(G)$. Hence by [Cha2, VIII
Theo. 2], there is in \( \mathcal{L}(\mathcal{B}) \), where \( (\Omega, \mathcal{B}, \mathcal{P}) \) is the completion of \( \mathcal{B}_\infty \), for which \( \beta = \lim_{n \to \infty} \beta_n \), \( P \)-a.e., and each \( n \in \mathbb{N} \) we have \( \beta_n = \mathbb{E}_n(\beta) \). By [Cha, Theo. 1], \( \lim_{n \to \infty} \| \beta_n - \beta \|_{L^1} = 0 \). Thus for \( P \)-a.e. \((s_1, s_2, \ldots)\) we have, using (6.2), that
\[
\int_{G} \| \chi(s_n) |\ell_s \cdot f - \int_{\Omega} \| \beta_n - \beta_{n-1} \|_{L^1} \, d\mathcal{P} = \| \beta_n - \beta_{n-1} \|_{L^1} \to 0
\]
and hence \( \int_{G} \| \chi(s) |\ell_s \cdot f - \int_{\Omega} \| \beta_n - \beta_{n-1} \|_{L^1} \, d\mathcal{P} = \| \beta_n - \beta_{n-1} \|_{L^1} \to 0 \)

and above we see that \( f \in \mathcal{WAP}_{\chi, S_{\omega}}(G) \).

Notice that by taking right translations, we may assume that our non-zero fixed point \( f \) satisfies \( f(e) \neq 0 \). Thus for \( s \) in \( G_{|\omega|} \) we must have \( f(s) = \chi(s) f(e) \), so that \( \chi \in \mathcal{WAP}(G)_{G_{|\omega|}} \). Clearly the algebra \( \mathcal{WAP}(G)_{G_{|\omega|}} \) of functions on \( G_{|\omega|} \) is self-adjoint.

We remark that the proof of (i) shows that if \( \omega = |\omega| \) for a continuous character on \( S_{|\omega|} \), then \( \ell_s h = \chi(s) h \) for \( s \) in \( S_{|\omega|} \) and every bounded Borel-measurable function \( h \).

If \( G \) is a semisimple Lie group with finite center, then it is shown in [Vee] ([Cho] for \( GL_2(\mathbb{R}) \)) that for any non-compact closed one-parameter subgroup \( R \cong \mathbb{R} \) we have \( \mathcal{WAP}(G)_{R} = C_0(\mathbb{R}) \oplus \mathcal{C}_1 \). Hence there are many measures \( \omega \) in \( M(G)_1 \) for which \( Fix L_{\omega} \cap \mathcal{WAP}(G) = \{ 0 \} \).

If we ask about the fixed points in \( C_0(G) \), also here in the classical case we obtain a more refined result than Proposition 4.3.

**Corollary 6.4.** Let \( G \) be a locally compact group and \( \omega \in M(G)_1 \). Then, the following are equivalent:

(i) \( Fix L_{\omega} \cap C_0(G) \neq \{ 0 \} \);

(ii) \( G_{|\omega|} \) is compact and there is a continuous character \( \chi : G_{|\omega|} \to \mathbb{T} \) such that \( \omega = |\omega| \);

(iii) Cesàro sums, \( S_n(\omega) \), do not converge weak* to 0.

In this case we have that \( |\chi|_{G_{|\omega|}} = w^* \cdot \lim_{n \to \infty} S_n(\omega) \), and that
\[
Fix L_{\omega} \cap C_0(G) = Fix L_{|\chi|_{G_{|\omega|}}} \cap C_0(G) = L_{|\chi|_{G_{|\omega|}}}(C_0(G)).
\]

In particular, if \( G \) is compact and \( |\omega| \) is adapted, then \( Fix L_{\omega} = \mathcal{C}_{\omega} \).

**Proof.** The equivalence of (i) and (iii) is from Proposition 4.3. It follows from Theorem 6.3 that (i) is equivalent to (ii), where the compactness of \( G_{|\omega|} \) follows form the fact that if \( f \) is a non-zero element of \( C_0(G) \cap \mathcal{WAP}_{\chi, G_{|\omega|}}(G) \) (see the terminology introduced in the last theorem) then \( |f| \) is constant on cosets of \( G_{|\omega|} \). Since \( |\chi|_{G_{|\omega|}} \) is an idempotent, a rudimentary computation shows that \( \chi_{G_{|\omega|}}(G) := \{ f \in C_0(G) : f(st) = \chi(s)f(t) \text{ for } s \in G_{|\omega|}, t \in G \} = L_{|\chi|_{G_{|\omega|}}}(C_0(G)) = Fix L_{|\chi|_{G_{|\omega|}}} \cap C_0(G) \).

Finally, by Lemma 4.1 \( \mathcal{W} = w^* \cdot \lim_{n \to \infty} S_n(\omega) \) is a non-zero contractive idempotent with \( \chi_{G_{|\omega|}}(G) = Fix L_{\mathcal{W}} \cap C_0(G) \). Hence by [Gre], \( \mathcal{W} = \theta m_K \) for some compact group \( K \) and a continuous character \( \theta : K \to \mathbb{T} \). But \( Fix L_{\theta m_K} \cap C_0(G) = L_{\theta m_K}(C_0(G)) = C_0(\theta, K) \). We conclude that \( K = G_{|\omega|} \) and \( \theta = \chi \), and hence \( \mathcal{W} = \chi_{G_{|\omega|}} \).

Note that for \( \omega \) in \( \text{Prob}(G) \), it is shown in [KI] that \( m_{G_{|\omega|}} = w^* \cdot \lim_{n \to \infty} S_n(\omega) \).

We finish with some observations generalising the results of this section to a more abstract context, and present one more application.

Let \( E \) be a Banach space. A strong operator continuous representation \( \pi : G \to \mathcal{G}(E) \) will be called essentially weakly almost periodic if there is a separating subspace \( F \subseteq E^* \) such that
\[
M_{\pi,F} = \text{span}\{ \{ f, \pi(\cdot)x \} : x \in E, f \in F \} \subseteq \mathcal{WAP}(G).
\]
i.e. the space generated by matrix coefficients of \( \pi \) with \( F \) consists of weakly almost periodic functions. Notice that the uniform boundedness principle shows that \( \pi \) must have bounded range. If \( \omega \in M(G) \), then we define the operator \( \pi(\omega) \) in terms of Bochner integrals: \( \pi(\omega)x = \int_G \pi(s)x \, d\omega(s) \) for \( x \in E \).
Corollary 6.5. Let $G$ be a locally compact group, $\omega \in M(G)_1$, and $\pi : G \to \mathcal{GL}(E)$ be an essentially weakly almost periodic representation. If $\pi(\omega)$ has a non-zero fixed point, then there is a continuous character $\chi : G_{|\omega|} \to \mathbb{T}$ such that $\omega = \chi|\omega|$ and

$$\text{Fix} \pi(\omega) = \{ x \in E : \pi(s)x = \chi(s)x \text{ for all } s \in G_{|\omega|} \}.$$ 

In particular, we have that $\bar{\chi} \in M_{\bar{\pi},F|G_{|\omega|}}$, where $F$ is as in the definition of essential weak almost periodicity of $\pi$.

Proof. Let $\bar{\pi}(s) = \pi(s^{-1})$ for $s \in G$, so $\bar{\pi}$ is an anti-homomorphism. If $x \in E$ and $f \in F$, let $\pi_{f,x} = \langle f, \pi(\cdot)x \rangle$. We also let $\omega$ denote the unique measure which satisfies $\int_G g \, d\omega = \int_G \tilde{g} \, d\omega$ where $\tilde{g}(s) = g(s^{-1})$ for $g \in \text{WAP}(G)$. Notice that $\pi_{f,x} = \langle f, \bar{\pi}(\cdot)x \rangle$. We then have for $t$ in $G$ that

$$L_\omega \pi_{f,x}(t) = \int_G \langle f, \pi(s^{-1}t)x \rangle \, d\omega(s) = \int_G \langle f, \pi(t)\pi(s)x \rangle \, d\omega(s) = \pi_{f,\pi(\omega)x}(t).$$

Since $F$ is separating, it follows that, for $x \in E$, we have $x \in \text{Fix} \pi(\omega)$ if and only if $\pi_{f,x} \in \text{Fix} L_\omega \cap \text{WAP}(G)$ for every $f \in F$. Suppose that $\text{Fix} \pi(\omega) \neq \{0\}$. By Theorem 6.3 (ii), $\bar{\omega} = \bar{\chi}|\omega| = \chi|\omega|$ for some continuous character $\chi$ on $G_{|\omega|} = G_{|\omega|}$, and hence $\omega = \chi|\omega|$. Furthermore,

$$\pi_{f,x} \in \text{WAP}_{\chi,G_{|\omega|}}(G)$$

whenever $x \in \text{Fix} \pi(\omega)$ and $f \in F$. Hence, for $x \in E$, we have that $x \in \text{Fix} \pi(\omega)$ exactly when for $f$ in $F$ and $s$ in $G_{|\omega|}$ we have

$$\langle f, \pi(s)x \rangle = \pi_{f,x}(s^{-1}) = \chi(s^{-1})\pi_{f,x}(e) = \chi(s)\pi_{f,x}(e) = \langle f, \chi(s)x \rangle$$

and, again since $F$ is separating, this happens exactly when $\chi(s)x = \pi(s)x$. In this case, a variant of the last computation also shows that $\chi(s)\pi_{f,x}(e) = \pi_{f,x}(s)$, so $\bar{\chi} \in M_{\pi,F|G_{|\omega|}}$. \hfill $\square$

Finally consider the left regular representation $\lambda_\omega : G \to \mathcal{GL}(L^p(G))$, $\lambda_\omega(s)f(t) = f(s^{-1}t)$ for each $s \in G$ and almost every $t \in G$. As before we let $H^p_\omega(G) = \text{Fix} \lambda_\omega(\omega)$. In the commutative setting (i.e. the case of locally compact groups), we gain the following improvement to Theorem 5.2.

Corollary 6.6. Let $G$ be a locally compact group, let $p \in [1, \infty)$, and let $\omega \in M(G)_1$. Suppose that $H^p(G) \neq \{0\}$. Then $G_{|\omega|}$ is compact, $\omega = \chi|\omega|$ for some continuous character $\chi : G_{|\omega|} \to \mathbb{T}$, and

$$H^p_\omega(G) = \text{Ran} \lambda_\omega(\chi m_{G_{|\omega|}})$$

where $m_{G_{|\omega|}}$ is normalised Haar measure on $G_{|\omega|}$. In particular, if $\omega$ is adapted, then $H^p_\omega(G) = \mathbb{C}\bar{\chi}$.

Proof. Letting $F = L^p(G)$ (dual space) for $p > 1$, or $C_0(G)$ for $p = 1$, we see that $M_{\lambda_\omega,F} \subseteq C_0(G) \subseteq \text{WAP}(G)$. We then appeal directly to Corollaries 6.5 and 6.4. \hfill $\square$

7. Mukherjea condition and the dual of the classical case

We devote the last section to a discussion of possible generalizations in the ‘dual to classical’ case of the following classical result due to Mukherjea [Muk] (see also [Der]), and draw several consequences.

Theorem 7.1. Let $G$ be a locally compact group and let $\omega \in \text{Prob}(G)$. Then the following are equivalent:

(i) Cesàro sums $S_n(\omega)$ converge to 0 weak$^*$ on $C_0(G)$,

(ii) convolution iterates $\omega \ast n$ converge to 0 weak$^*$ on $C_0(G)$,

(iii) the group $G_\omega$ is not compact.

The result is due to Derriennic (Theorem 8 of [Der]) who assumed that $\omega$ is adapted. However, the form above is equivalent since extension holds, i.e. $C_0(G)_{G_\omega} = C_0(G_\omega)$. Mukherjea established this result earlier in [Muk] under the second countability assumption.

In this section we consider the cocommutative case, i.e. the case when $G$ is the dual of a locally compact group $G$. In this case, $C_0(G)^*$ is identified with the Fourier–Stieltjes algebra $B(G)$ (see
which is the linear span of $P(G)$ – the cone of continuous positive definite functions on $G$. Moreover, in this case $L^\infty(G)$ is the group von Neumann algebra $VN(G)$, the predual of which is the Fourier algebra $A(G)$ (again, see [Eym]). As we will show below in Theorem 7.5, the analogous result of Theorem 7.1 holds in the dual case, if we interpret condition (iii) in the following way: the pre-image of 1 with respect to the normalised function $\omega \in B(G)^+$ is not open.

We cannot hope to extend Theorem 7.1 to contractive functionals as the implication (ii)$\implies$(iii) can fail – there exist nilpotent contractive measures on compact groups – as can (i)$\implies$(ii) – consider simply $\omega = -\delta_e$. We will still see in Theorem 7.5 that there is a simple way to cut out such ‘pathological’ examples.

Finally note that in the classical context (iii)$\implies$(ii) holds for $\omega \in M(G)_1$, as follows from the above result and a simple convolution estimate $|\omega^n| \leq |\omega|^n$, $n \in \mathbb{N}$.

For $\omega \in B(G)$, write $Z_\omega = \{ s \in G : \omega(s) = 1 \}$.

**Theorem 7.2.** Suppose that $G$ is a locally compact group and $\omega \in B(G)_1$.

(i) If $Z_\omega$ is empty, then $\text{Fix } L_\omega = \{0\}$. This happens in particular when $\|\omega\| < 1$.

(ii) If $Z_\omega$ is not empty, then $Z_\omega = sH$ where $H$ is a closed subgroup of $G$ and $s \in G$. Moreover, $\text{Fix } L_\omega = \pi(s)VN(H)$.

In particular, $\text{Fix } L_\omega$ is always a $W^*$-sub-TRO of $VN(G)$, i.e. $\text{Fix } L_\omega$ is a weak*-closed subspace that is closed under the ternary product $(x, y, z) \mapsto xy^*z$.

**Proof.** (i) Suppose that $x \neq 0$ is a fixed point of $L_\omega$. Let $s \in \text{supp } x$ (see [Eym, Définition 4.5] for the definition of supp $x$) and pick $f \in A(G)$ such that $f(s) \neq 0$. Now

$$0 = L_f(x - L_\omega(x)) = L_{f-f_\omega}(x).$$

Therefore, since $f - f_\omega \in A(G)$, it follows from Proposition 4.4 of [Eym] that $(f - f_\omega)(s) = 0$. Since $f(s) \neq 0$, we have that $\omega(s) = 1$. So if $Z_\omega = \emptyset$, then $\text{Fix } L_\omega = \{0\}$.

The latter statement is clear because if $\|\omega\| < 1$, then $|\omega(s)| < 1$ for every $s \in G$.

(ii) Let $s \in Z_\omega$ and define $\tau(t) = \omega(st)$ for $t \in G$. The $\tau \in B(G)_1$ and $\tau(\epsilon) = 1$ so $\tau$ is a state. Let $\pi : G \to M(C^*(G))$ be the natural map. For every $t \in Z_\tau$ also $t^{-1} \in Z_\tau$ as $\tau$ is positive definite and

$$\tau(\pi(t)^*\pi(t)) = \tau(t^{-1}t) = 1 = \overline{\tau(t)}\tau(t) = \tau(\pi(t)^*\pi(t)).$$

Hence $\pi(t)$ is in the multiplicative domain of $\tau$ (considered as a state on $M(C^*(G))$) and it follows that $Z_\tau$ is closed under multiplication. Therefore $H := Z_\tau$ is a closed subgroup of $G$ and $Z_\omega = sZ_\tau = sH$.

Let $t \in G$. Then $L_\omega(\pi(t)) = \omega(t)\pi(t) = \pi(t)$ if and only if $t \in sH$. It follows that $\pi(s)VN(H) \subseteq \text{Fix } L_\omega$. Conversely, if $x \in \text{Fix } L_\omega$, then the proof of the statement (i) shows that supp $x \subseteq Z_\omega$, and so $\text{Fix } L_\omega \subseteq \pi(s)VN(H)$.

**Corollary 7.3.** Suppose that $\omega \in B(G)_1$ is non-degenerate. Then $\dim(\text{Fix } L_\omega) \leq 1$. Moreover, if $\text{Fix } L_\omega$ is non-zero, it contains a unitary.

**Proof.** We shall show that if $\omega$ is non-degenerate, then $Z_{|\omega|} \subseteq \{e\}$. Suppose otherwise and pick $s \in Z_{|\omega|} \setminus \{e\}$. Then

$$|\omega|^k((\pi(s) - \pi(e))^*(\pi(s) - \pi(e))) = 2|\omega|^k(e) - |\omega|^k(s) - |\omega|^k(s^{-1}) = 0$$

for every $k \in \mathbb{N}$, which is in contradiction with non-degeneracy. The preceding theorem implies that if $\text{Fix } L_\omega \neq \{0\}$, then $\text{Fix } L_\omega = \pi(s)\mathbb{C}$ for some $s \in G$ as $Z_{|\omega|} = H = \{e\}$ (in the notation of that theorem). Now $\pi(s)$ is the required unitary element.

Before we proceed with the main result of this section, we record below how Theorem 7.2 provides an answer to a question asked by Chu and Lau, who posed the following in [ChL, Remark 3.3.16]: Let $\omega \in B(G)_1$ be such that $Z_\omega$ is discrete and $A(G)/I_\omega$ has a bounded approximate identity (recall that $I_\omega$ denotes the pre-annihilator of $\text{Fix } L_\omega$ in $A(G)$; see Definition 3.1). Does Arens regularity of $A(G)/I_\omega$ imply finiteness of $Z_\omega$? We now show that this is indeed the case.
**Theorem 7.4.** Let $G$ be a locally compact group, and let $\omega \in B(G)_1$ be such that $Z_\omega$ discrete. Assume that the Banach algebra $A(G)/I_\omega$ has a BAI and is Arens regular. Then $Z_\omega$ is finite.

**Proof.** By assumption, we are in the situation of [ChL, Proposition 3.3.15 (ii)]. Hence, $E := I^*_\omega = \text{Fix } I_\omega$ is, as a Banach space, isomorphic to a Hilbert space, so in particular $E$ is reflexive. We can assume that $Z_\omega$ is non-empty. By Theorem 7.2 (ii), we have $Z_\omega = sH$ and $E = \lambda_s VN(H)$, where $H$ is a closed subgroup of $G$ and $s \in G$. Thus, the Banach space $VN(H)$ is isomorphic to $E$, hence reflexive. But a von Neumann algebra whose underlying Banach space is reflexive, is finite-dimensional; cf., e.g., [Li, Proposition 1.11.7]. So $H$ is finite, whence $Z_\omega = sH$ is finite as well. \hfill $\square$

We now return to the main theme of the section.

**Theorem 7.5.** If $\omega \in B(G)$ such that $\|\omega\| = 1$ and $Z_\omega \neq \emptyset$, then the following are equivalent:

(i) Cesàro sums $S_n(\omega)$ converge to 0 weak* on $C^*(G)$;

(ii) convolution iterates $\omega^*n$ converge to 0 weak* on $C^*(G)$;

(iii) the set $Z_\omega$ is not open (equivalently, $Z_\omega$ is locally null).

In particular, the above statements are equivalent when $\omega$ is a state.

**Proof.** Note first that by Theorem 7.2 the set $Z_\omega$ is a coset of a closed subgroup $H$ of $G$. Naturally $Z_\omega$ is open if and only if $H$ is open, and locally null with respect to the left Haar measure of $G$ if and only if $H$ is locally null. Now the fact that $H$ is not locally null is equivalent to $H$ being open is standard (and follows from a Steinhaus type theorem, see for example Corollary 20.17 in [HeR]). We now proceed with the proof of the main equivalences.

(ii)$\implies$(i): trivial.

(i)$\implies$(iii): assume that (iii) is not true. Since $Z_\omega$ is open, there is a compactly supported continuous function $f$ on $G$ such that $\text{supp } f \subseteq Z_\omega$ and $\int f(s) \, ds = 1$ (note that $Z_\omega \neq \emptyset$ by assumption). If $\pi : L^1(G) \to C^*(G)$ is the canonical map, then

$$\omega^*k(\pi(f)) = \int f(s)\omega(s)^k \, ds = \int f(s) \, ds = 1$$

for every $k \in \mathbb{N}$. Hence $S_n(\omega)(\pi(f)) = 1$ for every $n \in \mathbb{N}$ and (i) is not true.

(iii)$\implies$(ii): let then $\omega$ be as in the assumptions of the theorem. The proof of Theorem 7.2 shows that there exists $s \in Z_\omega$ and a state $\tau \in B(G)$ such that $\tau(t) = \omega(st)$, $t \in G$; moreover $Z_\omega = sZ_\tau$ (so that (iii) is equivalent to $Z_\tau$ being null). If $f \in L^1(G)$ then

$$\omega^*n(\pi(f)) = \int_G \omega^n(t)f(t)\, dt, \quad \tau^*n(\pi(f)) = \int_G \omega^n(st)f(t)\, dt = \int_G \omega^n(t)f(s^{-1}t)\, dt.$$ 

This implies that (ii) is equivalent to the convolution iterates $\omega^*n$ converging to 0 weak* on $C^*(G)$.

The discussion above implies that we can assume without loss of generality that $\omega$ is a state. There is then a strongly continuous unitary representation $\pi : G \to B(H)$ on some Hilbert space $H$ and a unit vector $\xi \in H$ such that $\omega(g) = \langle \xi, \pi(g)\xi \rangle, g \in G$. Write $U := \omega^{-1}(\mathbb{T})$. As for any $z \in \mathbb{T}$ and $g \in G$ we have

$$\omega(s) = z \iff z = \langle \xi, \pi(g)\xi \rangle \iff \pi(g)\xi = z\xi,$$

it is easy to see that $U$ is a closed subgroup of $G$ and $\omega|_U : U \to \mathbb{T}$ is a homomorphism with kernel equal to $Z_\omega$. Thus $\omega|_U = \chi \circ q$, where $q : U \to U/Z_\omega$ is the quotient map and $\chi \in \overline{U/Z_\omega}$. Further $U/Z_\omega \cong \omega(U) \subset \mathbb{T}$.

Let $f \in L^1(G)$. If $U$ is not open, then $\mu_U(U) = 0$, as argued in the beginning of the proof, and we have

$$\omega^*n(\pi(f)) = \int_{G\setminus U} \omega^n(t)f(t)\, dt \xrightarrow{n \to \infty} 0$$

by the dominated convergence theorem. We may then assume that $U$ is open; and then $U/Z_\omega$ cannot be a finite subgroup of $\mathbb{T}$ (as $Z_\omega$ is assumed not open). Hence $U/Z_\omega \cong \mathbb{T}$, and $\chi \neq 1$. By
the standard integration formula we have
\[
\int_U \omega^n(t)f(t)dt = \int_{U/Z_\omega} \left(\int_{Z_\omega} \omega(st)^n f(st)d\mu_{Z_\omega}(t)\right) d\mu_{U/Z_\omega}(sZ_\omega)
\]
and the latter tends to 0 by the Riemann-Lebesgue lemma – as the function \(sZ_\omega \mapsto \int_{Z_\omega} f(st)d\mu_{Z_\omega}(t)\) is integrable. This ends the proof. \(\square\)

Note that the above theorem is related to Theorem 3 of [Ka3], which is on one hand only treating the positive-definite functions \(\omega\), but on the other obtains a stronger notion of convergence in (ii), i.e. the point–norm convergence of the convolution iterates.

Furthermore, we note that the equivalence of (i) and (iii) in Theorem 7.5 above also follows from the recent, independently obtained result in [Mus, Proposition 2.5]. However, the equivalence with (ii) in Theorem 7.5 does not follow from [Mus], as the example of \(u(t) = e^{it}\) on \(G = \mathbb{R}\) shows, which is covered by our Theorem 7.5, but does not satisfy the assumption of [Mus, Proposition 4.4 (a)]; cf. the remark after the proof of the latter.

Finally we stress that the condition that \(Z_\omega\) is not open is connected to the fact that the ‘support’ of \(\omega\) is non-compact. We see this clearly in the case where \(G\) is abelian.

**Proposition 7.6.** Let \(G\) be a locally compact abelian group and let \(\mu \in M(G), \|\mu\| = 1\). Then \(Z_\mu \neq \emptyset\) if and only if \(\frac{d\mu}{d|\mu|} = \chi\) \(\mu\)-a.e. for some character \(\chi \in G\). Moreover, in that case \(G_\mu\) is compact if and only if \(Z_\mu\) is open.

**Proof.** If \(\chi \in Z_\mu\), then
\[
1 = \hat{\mu}(\chi) = \int \overline{\chi(s)} d\mu(s) = \int \overline{\chi(s)} \frac{d\mu}{d|\mu|}(s) d|\mu|(s),
\]
which implies that \(\frac{d\mu}{d|\mu|} = \chi\) \(\mu\)-a.e. (because \(\|\mu\| = 1\)). The converse is trivial.

Moreover, if \(\chi \in Z_\mu\), then
\[
\eta \in Z_\mu \iff 1 = \int \overline{\eta(s)} \chi(s) d|\mu|(s)
\]
\[
\iff \overline{\eta} \chi = 1 \text{ on } \text{supp } \mu
\]
\[
\iff \eta \chi = 1 \text{ on } G_\mu
\]
\[
\iff \eta \in \chi G_\mu^+.
\]
Then the second statement follows from the fact that \(G_\mu\) is compact if and only if \(G_\mu^+\) is open. \(\square\)

Note that the preceding two results imply Mukherjea’s result for locally compact abelian groups.

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