ad-NILPOTENT IDEALS CONTAINING A FIXED NUMBER OF SIMPLE ROOT SPACES

PAOLA CELLINI
PIERLUIGI MÖSENEDER FRAJRIA
PAOLO PAPI

Abstract. We give formulas for the number of ad-nilpotent ideals of a Borel subalgebra of a Lie algebra of type B or D containing a fixed number of root spaces attached to simple roots. This result solves positively a conjecture of Panyushev [12, 3.5] and affords a complete knowledge of the above statistics for any simple Lie algebra. We also study the restriction of the above statistics to the abelian ideals of a Borel subalgebra, obtaining uniform results for any simple Lie algebra.

1. Introduction

Let \( \mathfrak{g} \) be a complex finite-dimensional simple Lie algebra. Fix a Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \), and let \( \mathfrak{n} \) be its nilradical. If \( \mathfrak{g} \) is of type \( X \), denote by \( \mathcal{I}(X) \) denote the set of ad-nilpotent ideals \( \mathfrak{b} \), i.e. the ideals of \( \mathfrak{b} \) which are contained in \( \mathfrak{n} \). Let \( \Delta^+ \), \( \Pi \) denote respectively the positive and simple systems of the root system \( \Delta \) of \( \mathfrak{g} \) corresponding to \( \mathfrak{b} \). Then \( i \in \mathcal{I}(X) \) if and only if \( i = \bigoplus_{\alpha \in \Phi_i} g\alpha \), where \( g\alpha \) is the root space attached to \( \alpha \) and \( \Phi_i \subseteq \Delta^+ \) is a dual order ideal of \( \Delta^+ \) (w.r.t. the usual order: \( \alpha < \beta \) is \( \beta - \alpha \) is a sum of positive roots). ad-nilpotent ideals have been intensively investigated in recent literature: see references in [12]. The first goal of this short paper is to solve positively conjecture 3.5 of [12]. This conjecture regards the following statistics on \( \mathcal{I}(X) \):

\[ P_X(j) = \left| \{i \in \mathcal{I}(X) : |\Pi \cap \Phi_i| = j \} \right| \]

\((0 \leq j \leq n)\). The formulas expressing \( P_X(j) \) for the classical Lie algebras are given in the following theorem. The result in type \( A \) has been proved in [12, Theorem 3.4], together with the equality \( P_{B_n} = P_{C_n} \). The formulas for types \( B, D \) are conjecture 3.5 of the same paper.
Theorem 1.1. For $0 \leq j \leq n$ we have

\[
P_{A_n}(j) = \frac{j + 1}{n + 1} \binom{2n - j}{n},
\]

\[
P_{B_n}(j) = P_{C_n}(j) = \binom{2n - j - 1}{n - 1},
\]

\[
P_{D_n}(j) = \begin{cases} 
\binom{2n - 2 - j}{n - 2} + \binom{2n - 3 - j}{n - 2} & \text{if } j = 0 \\
\binom{2n - 2 - j}{n - 2} + \binom{2n - 3 - j}{n - 2} & \text{if } 1 \leq j \leq n.
\end{cases}
\]

We remark that the numerical values of $P_X(j)$ in the exceptional cases are easily calculated from the knowledge of $P_X(0)$ using the inclusion-exclusion principle: see [12, §3]. On the other hand, the number $P_X(0)$ can be uniformly described: see Remark 2.1.

The relevance of the statistics $P_X$ is motivated by the following discussion. It is known [4] that the cardinality of $I$ is given by the generalized Catalan number $\frac{1}{n+1} \prod_{i=1}^{n} (e_i + h + 1)$ (see Remark 2.1 for undefined notation) as well as that of clusters, certain subsets of $\Delta^+ \cup -\Pi$ which play a major role in Zelevinsky’s theory of cluster algebras [7]. Panyushev noticed that $P_X(j)$ also counts the number of clusters having $j$ elements in $-\Pi$. Looking for a conceptual explanation of the interplay between $ad$-nilpotent ideals and clusters is an interesting open problem.

Theorem 1.1 is proved in the next section. The final section deals with a formula for the same statistics on the subset $I_{ab}$ of $I$ consisting of abelian ideals. Abelian ideals turn out to appear in several contexts, ranging from the structure of the exterior algebra of $g$ [9], to affine algebras [2] and to difficult problems in classical invariant theory [11].

Theorem 1.2. The number $P_{ab}^X(j)$ of abelian ideals of $b$ in a Lie algebra $g$ of type $X$ and rank $n$ containing $j$ simple roots is given by

\[
P_{ab}^X(j) = \begin{cases} 
2^n - z(g) + 1 & \text{if } j = 0, \\
z(g) - 1 & \text{if } j = 1, \\
0 & \text{if } j > 1.
\end{cases}
\]
2. Proof of Theorem 1.1

Our approach to Panyushev’s conjecture is based on Shi’s encoding [13] of ad-nilpotent ideals for classical Lie algebras via (possibly shifted) shapes as formulated in [3]. More precisely, consider a staircase diagram \( T_X \) of shape \( (n, n - 1, \ldots, 1) \) in type \( A_n \) (respectively a shifted staircase diagram of shape \( (2n - 1, 2n - 3, \ldots, 1) \) for \( B_n \) and \( C_n \), and of shape \( (2n - 2, 2n - 4, \ldots, 2) \) for \( D_n \)). Arrange in the diagram the positive roots of \( \Delta \) according to the formulas

\[
\tau_{i,j} = \alpha_i + \cdots + \alpha_{n-j+1} \quad 1 \leq i \leq j \leq n.
\]

\[
\tau_{i,j} = \begin{cases} 
\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_n + \cdots + \alpha_{n-1}) + \alpha_n & \text{if } j \leq n - 1, \\
\alpha_i + \cdots + \alpha_{2n-j} & \text{if } n \leq j \leq 2n - i.
\end{cases}
\]

\[
\tau_{i,j} = \begin{cases} 
\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_{j+1} + \cdots + \alpha_n) & \text{if } j \leq n - 1, \\
\alpha_i + \cdots + \alpha_{2n-j} & \text{if } n \leq j \leq 2n - i.
\end{cases}
\]

\[
\tau_{i,j} = \begin{cases} 
\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_{j+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n & \text{if } j \leq n - 2, \\
\alpha_i + \cdots + \alpha_{n-2} + \alpha_n & \text{if } j = n - 1, \\
\alpha_i + \cdots + \alpha_{2n-j-1} & \text{if } n \leq j \leq 2n - 1 - i.
\end{cases}
\]

in types \( A_n, C_n, B_n, D_n \) respectively. E.g., in types \( A_4, C_3, B_3, D_4 \) we have, respectively

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \quad \alpha_1 + \alpha_2 + \alpha_3 & \quad \alpha_1 + \alpha_2 & \quad \alpha_1 \\
2\alpha_2 + \alpha_3 & \quad 2\alpha_2 + \alpha_3 & \quad \alpha_1 + \alpha_2 + \alpha_3 & \quad \alpha_1 + \alpha_2 & \quad \alpha_1 \\
\alpha_3 + \alpha_4 & \quad \alpha_2 + \alpha_3 & \quad \alpha_2 & \quad \alpha_2 \\
\alpha_4 & & \quad \alpha_3 \\
\end{align*}
\]

Then \( \mathcal{I}(X) \) is in bijection with the set \( \mathcal{S}_X \) of subdiagrams of \( T_X \) when \( X = A, B, C \) whereas in type \( D \) one has to consider also the sets of boxes of \( T_D \) which become subdiagrams of \( T_D \) upon switching columns \( n - 1, n \) (see [13] or [3]).
In turn to each subdiagram we can associate a lattice path of length $2n$, starting from the origin and never going under the $x$-axis, with step vectors $(1,1), (1,-1)$ (see [10]). The correspondence between subdiagrams and paths is best explained with an example at hand. Let $n = 9$ and consider, for type $B_n$ or $C_n$, the shifted partition $(16, 13, 11, 8, 7, 5, 3)$, see Figure 1 (here, as in Figure 3, the origin coincides with the left upper corner of the diagram, and the y-axis points downwards). Connect the point $(2n, 0)$ to the border of the subdiagram with an horizontal segment, and consider the zig-zag line formed by the horizontal segment and the right border of the subdiagram. Rotate the figure by $45^\circ$ in the positive direction and then flip it across a vertical line. After rescaling (in the obvious way) we obtain the desired lattice path. See Figure 2 for the path corresponding to the partition of Figure 1. (To make a comparison easy, the steps which correspond to thick segments in Figure 1 are also made thick in Figure 2.)

So we have associated to any subdiagram of $T_{B_n}$ (or $T_{C_n}$) a lattice path of length $2n$. In a similar way we can associate to any subdiagram of $T_{D_n}$ a lattice path of length $2n - 1$. Slight modifications are needed to define a correspondence in type $A_n$. Start from the point $(n + 1, 0)$, reach and follow the right border of the diagram. End in the point...
(0, n + 1): see Figures 3, 4 for the case of the partition (5, 3, 1, 1, 1, 0, 0), relative to $A_7$.

In type $A_n$ this correspondence turns out to be a bijection between $\mathcal{I}(A_n)$ and the set of Dyck paths of length $2n + 2$, whereas in types $B_n, C_n$ one gets a bijection with the set of paths of length $2n$ not necessarily ending on the $x$-axis.

Remark that in cases $B_n, C_n$ our statistics $P_X$ translates into the one which counts the number of returns of the paths, i.e. the number of contact points of the path with the $x$-axis minus one. In type $A_n$ the statistics $P_X$ counts the number of returns minus one (so the statistics has value 0 for the path of Figure 4).

Denote by $\mathcal{B}_{n,h,j}$ the set of paths of the previous type having length $n$, ending in the point $(n, h)$ and having exactly $j$ returns. The enumeration of such objects has been known since a long time (see [3] §2 for historical details and generalizations). As usual we set $\binom{n}{m} = 0$ if $m < 0$.

**Proposition 2.1.** [3, 13, Cor. 3.2] Assume $n \equiv h \mod 2$. Then

$$|\mathcal{B}_{n,h,j}| = \binom{n - (j + 1)}{\frac{n+h}{2} - 1} - \binom{n - (j + 1)}{\frac{n+h}{2}}.$$
Note that if a path has length $n$ and ends at height $j$, then $n + j$ is even. In particular, if $n + h$ is odd then $B_{n,h,j} = \emptyset$ for any $j$. We have immediately

\[ P_{A_n}(j) = |B_{2n+2,0,j+1}| = \frac{j + 1}{n + 1} \binom{2n-j}{n}, \]

\[ P_{B_n}(j) = P_{C_n}(j) = \sum_{h=0}^{2n} |B_{2n,h,j}| = \binom{2n-j-1}{n}, \]

which are the desired formulas in cases $A_n, B_n, C_n$.

For type $D$ we argue as follows. First observe that, in the diagramatic encoding, ideals can be counted as

\[
|\mathcal{S}_{D_n}| - |\mathcal{D}_n| = 2^{2n-1} \sum_{h=3}^{2n-1} |B_{2n-1,h,j}| + |B_{2n-1,1,j-1}|
\]

\[
= \binom{2n-j-2}{n} + \binom{2n-j-1}{n-1} - \binom{2n-j-1}{n} + \binom{2n-j-2}{n-2}.
\]

We have used relation (2.1) to evaluate the left hand side of the previous expression.

Now remark that the contribution to the piece of degree $j$ of our statistics coming from $\mathcal{D}_n$ is

\[ P_{B_{n-1}}(j) - P_{A_{n-2}}(j-1) + P_{A_{n-2}}(j-2). \]

Note in fact that to any diagram in $\mathcal{D}_n$ we can associate a diagram in $T_{B_{n-1}}$ by deleting the $n$-th column. In so doing our statistics counts:

(a) all paths for type $B_{n-1}$ having $j$ returns and end point not lying on the $x$-axis;

(b) all paths for type $B_{n-1}$ having $j - 1$ returns and end point on the $x$-axis.

It is clear that paths for $B_{n-1}$ having $k$ returns and end point on the $x$-axis are the same as paths for $A_{n-2}$ with $k - 1$ returns. Hence
contribution (a) is $P_{B_{n-1}}(j) - P_{A_{n-2}}(j - 1)$, and contribution (b) is $P_{A_{n-2}}(j - 2)$. Relation (2.2) and some elementary calculations yield the last formula in the Theorem.

**Remark 2.1.** It is worth recalling that the value $P_{X}(0)$ has a special geometric meaning. Indeed, $\text{ad}$-nilpotent ideals correspond to connected components in the dominant chamber of $\mathfrak{h}_{\mathbb{B}}$ ($\mathfrak{h}$ being a Cartan subalgebra of $\mathfrak{g}$) determined by the hyperplanes $(\alpha, x) = 0$, $(\alpha, x) = 1$, $\alpha \in \Delta^+$. More precisely, the open region associated to the ideal $i$ is determined by the inequalities $0 < (\alpha, x) < 1$ if $g_\alpha \not\subset i$, and $(\alpha, x) > 1$ if $g_\alpha \subset i$. Panyushev proved that an ideal in $I$ does not contain a simple root space if and only if the corresponding region is bounded (see [12, Proposition 3.7]). He also found the following remarkable formula (see [12, Proposition 3.10]):

$$P_{X}(0) = \frac{1}{|W|} \prod_{i=1}^{n}(h + e_i - 1).$$

Here $W$ is the Weyl group, $h$ the Coxeter number and $e_1, \ldots, e_n$ the exponents of $g$. $P_{X}(0)$ is also the number of positive clusters.

### 3. Proof of Theorem 1.2

**Lemma 3.1.** An abelian ideal $i \in I_{\text{ab}}$ may contain at most one simple root space.

**Proof.** Let $\alpha, \alpha' \in \Pi$ such that $g_\alpha, g_{\alpha'} \subset i$. Consider a minimal length path from $\alpha$ to $\alpha'$ in the Dynkin diagram of $g$. By Corollaire 3 in [11 VI, 1.7] the sum $\gamma$ of the simple roots in the path belongs to $\Delta^+$ as well as $\gamma - \alpha$. Moreover $\gamma > \alpha$, $\gamma - \alpha > \alpha'$. Therefore $g_\gamma \subset i$, $g_{\gamma - \alpha} \subset i$. But $[g_\alpha, g_{\gamma - \alpha}] = g_\gamma$, hence $i$ is not abelian. □

Recall that an $\text{ad}$-nilpotent ideal is nilpotent, i.e. its descending central series

$$i \supset [i, i] \supset [[i, i], i] \supset [[[i, i], i], i] \supset \cdots$$

has a finite number $n(i)$ of non zero terms. In particular, $i$ is an abelian ideal if and only if $n(i) \leq 1$. Also recall that $\text{ad}$-nilpotent ideals are in canonical bijection with antichains (i.e., subset formed by mutually non-comparable elements) in the root poset. The correspondence is given by mapping an ideal to its minimal roots w.r.t $<$, and the inverse map associates to an antichain $A$ the ideal $\bigoplus_{\beta \in A} \bigoplus_{\alpha \geq \beta} g_\alpha$.

If $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, denote by $\theta = \sum_{i=1}^{n} a_i \alpha_i$ the highest root of $\Delta$.

**Lemma 3.2.** Let $i_j = \bigoplus_{\beta \geq \alpha_j} g_\alpha$, $1 \leq j \leq n$. Then

$$n(i_j) = a_j.$$
Proof. We use the following result of Chari, Dolbin and Ridenour [5, Theorem 1]. Let \(i\) an ad-nilpotent ideal corresponding to the antichain \(A = \{\beta_1, \ldots, \beta_k\}\). Then \(n(i) = s\) if and only if \(s\) is the minimal non-negative integer such that \(\beta_1 + \ldots + \beta_{s+1} \leq \theta\) (repetitions in the \(\beta\) are allowed). The claim follows immediately, because the antichain attached to \(i_j\) consists only of \(\alpha_j\), and \(\theta - a_j\alpha_j = \sum_{i=1}^{j-1} a_i\alpha_i + \sum_{i=j+1}^{n} a_i\alpha_i\) belongs to the positive root lattice, whereas

\[
\theta - (a_j + 1)\alpha_j = \sum_{i=1}^{j-1} a_i\alpha_i - \alpha_j + \sum_{i=j+1}^{n} a_i\alpha_i
\]
does not. \(\square\)

We are ready to prove Theorem 1.2. The result follows combining (1.1) and Lemma 3.1 if we prove that \(P^X_{ab}(1) = z(g) - 1\). On the other hand Lemma 3.2 implies that \(P^X_{ab}(1)\) equals the number of indices \(i\) such that \(a_i = 1\). The latter number is known to coincide with \(z(g) - 1\) (see [1, VI, §2.3]).

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**P.C.**: Dipartimento di Scienze, Università di Chieti-Pescara, Viale Pindaro 42, 65127 Pescara, ITALY; 
cellini@sci.unich.it

**P.MF.**: Politecnico di Milano, Polo regionale di Como, Via Valleggio 11, 22100 Como, ITALY; 
pierluigi.moseneder@polimi.it

**P.P.**: Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185, Roma, ITALY; 
papi@mat.uniroma1.it