Global Convergence of Adaptive Gradient Methods for An Over-parameterized Neural Network

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Abstract

Adaptive gradient methods like AdaGrad are widely used in optimizing neural networks. Yet, existing convergence guarantees for adaptive gradient methods require either convexity or smoothness, and, in the smooth setting, only guarantee convergence to a stationary point. We propose an adaptive gradient method and show that for two-layer over-parameterized neural networks – if the width is sufficiently large (polynomially) – then the proposed method converges to the global minimum in polynomial time, and convergence is robust, without the need to fine-tune hyper-parameters such as the step-size schedule and with the level of over-parametrization independent of the training error. Our analysis indicates in particular that over-parametrization is crucial for the harnessing the full potential of adaptive gradient methods in the setting of neural networks.

1 Introduction

Gradient-based methods are widely used in optimizing neural networks. One crucial component in gradient methods is the learning rate (a.k.a. step size) hyper-parameter, which determines the convergence speed of the optimization procedure. A large learning rate can speed up the convergence but if it is larger than a threshold, the optimization algorithm cannot converge. This is by now well-understood for convex problems; excellent works on this topic include [Nash and Nocedal 1991, Bertsekas 1999, Nesterov 2005, Haykin et al. 2005, Bubeck et al. 2015], and the recent review for large-scale stochastic optimization to Bottou et al. [2018]. However, there is still limited work on the convergence analysis for nonsmooth and nonconvex problems, which includes over-parameterized neural networks.

Recently, a series of breakthrough papers showed that (stochastic) gradient descent can provably converge to the global minima for over-parameterized neural networks [Du et al. 2019, 2018, Li and Liang 2018, Allen-Zhu et al. 2018, Zou et al. 2018a]. However, these papers all require the step size to be sufficiently small to guarantee the global convergence. In practice, these optimization algorithms can use a much larger learning rate while still converging to the global minimum. This leads to the following question:

What is the optimal learning rate in optimizing neural networks?
While finding the optimal step size is important theoretically for identifying the optimal convergence rate, the optimal learning rate often depends on certain unknown parameters of the problem. For example, for a convex and $L$-smooth objective function, the optimal learning rate is $O(1/L)$ where $L$ is often unknown to practitioners. To solve this problem, adaptive methods [Duchi et al., 2011, McMahan and Streeter, 2010] are proposed so that they can change the learning rate on-the-fly according to gradient information received along the way. Though these methods often introduce additional hyper-parameters, compared to gradient descent methods with well-tuned stepsize, the adaptive methods are often robust to their hyper-parameters in the sense that these methods can still converge modulo (slightly) slower convergence rate. For this reason, adaptive gradient methods are widely used by practitioners in neural network optimization.

On the other hand, the theoretical investigation in adaptive methods in optimizing neural networks is limited. Existing analyses only deal with general (non)-convex and smooth functions, and thus, only concern convergence to first-order stationary points. However, a neural network is *neither smooth nor convex*. And yet, adaptive gradient methods are widely used in this setting as they converge without requiring a fine-tuned learning rate schedule. This leads to the following question:

*What is the convergence rate of adaptive gradient methods in over-parameterized networks?*

In this paper, we make progress on these two problems for the two-layer over-parameterized ReLU-activated neural networks setting.

**Our Results**

- First, we show the learning rate of gradient descent can be improved to $O(1/\|H^\infty\|)$ where $H^\infty$ is a Gram matrix that only depends on the data. Note that this upper bound is independent of the number of parameters. As a result, using this stepsize, we show gradient descent enjoys a faster convergence rate. This choice of stepsize directly leads to an improved convergence rate compared to Du et al. [2019].

- We develop an adaptive gradient method, which can be viewed as a variant of the “norm” version of AdaGrad. We prove this adaptive gradient method converges to the global minimum in polynomial time and does so robustly, in the sense that for any choice of hyper-parameters used in this method, our method is guaranteed to converge to the global minimum in polynomial time. The choice of hyper-parameters only affect the rate but not the convergence. To our knowledge, this is the first polynomial time global convergence result for an adaptive gradient method in the non-convex setting.

**Challenges and Our Techniques** To verify the improved learning rate of gradient descent, we use a more subtle analysis of the dynamics of predictions considered in [Du et al. 2019]. Our analysis shows that the dynamics are close to a linear one. This observation allows us to choose the improved learning rate.

For the adaptive method, there are two big challenges. First, because the learning rate (induced by the hyper-parameters and the dynamics) is changing at every iteration, we need to lower and upper bound the learning rate. The lower bound is required to guarantee the algorithm will converge in polynomial time and the upper bound is required to guarantee the algorithm will not diverge. The second challenge is that if at the beginning the learning rate is too large, the loss may increase...
at the beginning. The proof of Du et al. [2019] for gradient descent with well-tuned stepsize highly
depends on the fact that the loss is decreasing geometrically at each iteration, so that proof cannot
be adapted to our setting.

In this paper, we use induction with a carefully constructed hypothesis which implies both the
upper and the lower bounds of the learning rate. Furthermore, utilizing the particular property
induced by our proposed adaptive algorithm, the learning rate learns from feedback from previous
iterations and thus perseveres the distance of the updated weight matrix and its initialization
(Lemma 4.2) while does not vanishes to zero (Lemma 4.1). This property, together with the effect
of over-parameterization, we show that the loss may only increase by a bounded amount and then
decreases to zero eventually. Resolving these issues, we are able to prove the first global convergence
result for an adaptive gradient method in optimizing neural networks.

1.1 Related Work

Global Convergence of Neural Networks  Recently, a series of papers showed that gradient
based methods can provably reduce the training error to 0 for over-parameterized neural
networks [Du et al., 2019, 2018, Li and Liang, 2018, Allen-Zhu et al., 2018, Zou et al., 2018a].
In this paper we study the same setting considered in Du et al. [2019] which showed that for
learning rate \( \eta = O(\lambda_{\min}(H^\infty)/n^2) \), gradient descent finds an \( \varepsilon \)-suboptimal global minimum in
\( O\left(\frac{1}{\eta \lambda_{\min}(H^\infty)} \log(\frac{1}{\varepsilon})\right) \) iterations for the two-layer over-parameterized ReLU-activated neural
network. As a by-product of the analysis in this paper, we show that the learning rate can be improved
to \( \eta = O(1/\|H^\infty\|) \) which results in faster convergence. We believe that the proof techniques devel-
oped in this paper can be extended to deep neural networks, following the recent works [Du et al.,
2018, Allen-Zhu et al., 2018, Zou et al., 2018a].

Adaptive Gradient Methods  Adaptive Gradient (AdaGrad) Methods, first introduced inde-
pendently by Duchi et al. [2011] and McMahan and Streeter [2010], are now widely used in practice
for online learning due in part to their robustness to the choice of stepsize. The first convergence
guarantees, proved in Duchi et al. [2011], were for the setting of online convex optimization where
the loss function may change from iteration to iteration. Later convergence results for the variants
of AdaGrad were proved in Levy [2017] and Mukkamala and Hein [2017] for offline convex and
strongly convex settings. In the general non-convex and smooth setting, Ward et al. [2018] and
Li and Orabona [2018] prove that the same “norm” version of AdaGrad converges to a stationary
point at rate \( O(1/\varepsilon^2) \) for stochastic gradient descent and at rate \( O(1/\varepsilon) \) for batch gradient descent.

Many modifications to AdaGrad have been proposed, namely, RMSprop [Hinton et al., 2012],
AdaDelta [Zeiler, 2012], Adam [Kingma and Ba, 2014], AdaFTRL [Orabona and Pal, 2015], SGD-
BB [Tan et al., 2016], AdaBatch [Désorces and Bach, 2017], signSGD [Bernstein et al., 2018], SC-
AdaGrad [Mukkamala and Hein, 2017], Shah et al. 2018, WNGrad [Wu et al., 2018], AcceleGrad
Levy et al. 2018, Yogi [Zaheer et al., 2018a], Padam [Chen and Gu, 2018], to name a few. More
recently, accelerated adaptive gradient methods have also been proved to converge to station-
ary points [Barakat and Bianchi, 2018, Chen et al., 2019, Zaheer et al., 2018b, Zhou et al., 2018,
Zou et al., 2018b].

Our work is inspired by the analysis of Ward et al. 2018 and Wu et al. 2018, which quantifies
the auto-tuning property in the learning rate in AdaGrad. We propose a new adaptive algorithm
for the stepsize in the setting of over-parameterized neural networks and show global polynomial
convergence guarantee.

2 Problem Setup

Notations Throughout, $\| \cdot \|$ denotes the Euclidean norm if it applies to a vector and the maximum eigenvalue if it applies to a matrix. We use $N(\mathbf{0}, \mathbf{I})$ to denote a standard Gaussian distribution where $\mathbf{I}$ denotes the identity matrix and $U(S)$ to denote the uniform distribution over a set $S$. We use the notation $\mathbb{[}n\mathbb{]} := \{0, 1, 2, \ldots, n\}$.

Problem Setup In this paper we consider the same setup as [Du et al, 2019]. We are given $n$ data points, $(x_i, y_i)_{i=1}^n$. Following [Du et al, 2019], to simplify the analysis, we make the following assumption on the training data.

Assumption 2.1. For $i \in \mathbb{[}n\mathbb{]}$, $\|x_i\| = 1$ and $|y_i| = O(1)$.

The assumption on the input is only for the ease of presentation and analysis. See discussions in [Du et al, 2019]. The second assumption on labels is satisfied in most real world datasets.

We predict labels using a two-layer neural network of the following form

$$f(W, a, x) = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} a_r \sigma(\langle w_r, x \rangle) \quad (1)$$

where $x \in \mathbb{R}^d$ is the input, for $r \in [m]$, $w_r \in \mathbb{R}^d$ the weight vector of the first layer and $a_r \in \mathbb{R}$ is the output weight and $\sigma(\cdot)$ is ReLU activation function. For $r \in [m]$, we initialize the first layer vector $w_r(0) \sim N(\mathbf{0}, \mathbf{I})$ and output weight $a_r \sim U([-1, +1])$. We fix the second layer and train the first layer with the quadratic loss

$$L(W) = \sum_{i=1}^{n} \frac{1}{2} (f(W, a, x_i) - y_i)^2. \quad (2)$$

We will use iterative gradient-based algorithms to train $W$. The gradient of each weight vector has the following form:

$$\frac{\partial L(W)}{\partial w_r} = \frac{a_r}{\sqrt{m}} \sum_{i=1}^{n} (f(W, a, x_i) - y_i) x_i I\{w_r^T x_i \geq 0\} \quad (3)$$

We use $W(k)$ to denote the parameters at the $k$-th iteration.

The training algorithm will be specified in Section 3 and 4. Define $u_i = f(W, a, x_i)$, the prediction of the $i$-th example and $u = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$. We also let $y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$. Then we can write the loss function as

$$L(W) = \frac{1}{2} \| u - y \|^2. \quad (4)$$

In this paper, we will study the dynamics of $u(k)$. Here we use $k$ for indexing because $u(k)$ is induced by $W(k)$. According to [Du et al, 2019], the matrix below determines the convergence rate of the randomly initialized gradient descent.
Definition 2.1. The matrix $H^\infty \in \mathbb{R}^{n \times n}$ is defined as follows. For $(i, j) \in [n] \times [n]$.

$$H^\infty_{ij} = \mathbb{E}_{w \sim N(0, I)} \left[ x_i^\top x_j I\{w^\top x_i \geq 0, w^\top x_j \geq 0\}\right] = x_i^\top x_j \frac{- \arccos (x_i^\top x_j)}{2\pi} \tag{4}$$

This matrix represents the kernel matrix induced by Gaussian initialization and ReLU activation function. We make the following assumption on $H^\infty$.

Assumption 2.2. The matrix $H^\infty \in \mathbb{R}^{n \times n}$ in Definition 2.1 satisfies $\lambda_{\min}(H^\infty) \triangleq \lambda_0 > 0$. Du et al. [2019] showed that this condition holds as long as the training data is not degenerate.

We also define the following empirical version of this Gram matrix, which will be used in our analysis. For $(i, j) \in [n] \times [n]$:

$$H_{ij} = \frac{1}{m} \sum_{r=1}^{m} x_i^\top x_j I\{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0\}. \tag{5}$$

3 Warm up: Improved Learning Rate for Gradient Descent

Before presenting our adaptive method, we first revisit the gradient descent algorithm. At each iteration $k = 0, 1, \ldots$, we update the weight matrix according to

$$W(k+1) = W(k) - \eta \frac{\partial L(W(k))}{\partial W} \tag{6}$$

where $\eta > 0$ is the learning rate. Du et al. [2019] showed if $\eta = O(\lambda_0/n^2)$, then gradient descent achieves 0 training loss at a linear rate. We improve the upper bound of learning rate used in Du et al. [2019]. This improved analysis also gives tighter bound for the adaptive method we will discuss in the next section. Our main result for gradient descent is the following theorem.

Theorem 3.1 (Convergence Rate of Gradient Descent with Improved Learning Rate). Under Assumption 2.1 and 2.2 if the number of hidden nodes $m = \Omega\left(\frac{n^6}{\lambda_0^3}\right)$ and we set the stepsize to be

$$\eta = \Theta\left(\frac{1}{\|H^\infty\|}\right),$$

then with probability at least $1 - \delta$ over the random initialization, after

$$T = \tilde{O}\left(\frac{\|H^\infty\|}{\lambda_0^2} \log\left(\frac{1}{\varepsilon}\right)\right)$$

iterations, we have $L(W(T)) \leq \varepsilon$.

Comparing with Du et al. [2019], we improve the maximum allowable learning rate from $O(\lambda_0/n^2)$ to $O(1/\|H^\infty\|)$. Note since $\|H^\infty\| \leq n$, Theorem 3.1 gives an $O(\lambda_0/n)$ improvement. The improved learning also gives a tighter iteration complexity bound $O\left(\frac{\|H^\infty\|}{\lambda_0^2} \log\left(\frac{n}{\varepsilon}\right)\right)$ comparing to the $O\left(\frac{n^2}{\lambda_0^2} \log\left(\frac{n}{\varepsilon}\right)\right)$ bound in Du et al. [2019]. Empirically, we found that if the data matrix is

\footnote{\(\tilde{O}\) and \(\tilde{\Omega}\) hide \(\log(n), \log(1/\lambda_0), \log(1/\delta)\) terms.}
approximately orthogonal, then $\|H^\infty\| = O(1)$ (see Figure 1 in Appendix B). Therefore, in certain scenarios, the iteration complexity of gradient descent is independent of $n$.

Note even though gradient descent gives fast convergence, one needs to set the learning rate $\eta$ appropriately to achieve the fast convergence rate. In practice, $\|H^\infty\|$ is unknown to users so it would be better if the learning rate can be automatically adjusted. We address this problem in the next section.

**Proof Sketch of Theorem 3.1** Our main observation is the following recursion formula.

$$
\|y - u(k+1)\|^2 \approx \|y - u(k)\|^2 - 2\eta (y - u(k))^\top (I - \eta H^\infty) H^\infty (y - u(k))
\leq \|y - u(k)\|^2 - 2\eta \lambda_0 (1 - \eta \|H^\infty\|) \|y - u(k)\|^2
\leq (1 - \eta \lambda_0) \|y - u(k)\|^2.
$$

The first approximation we used over-parameterization ($m$ is large enough) for which the width $m$ becomes larger the approximation becomes more accurate. In Section 4, we will give precise perturbation analysis. The first inequality we used the fact that $\eta = O(1/\|H^\infty\|)$ and the two symmetric matrices $(I - \eta H^\infty)$ and $H^\infty$ share same eigenvectors. The second inequality we used $\eta = O(1/\|H^\infty\|)$ again. Note this recursion formula shows the loss converges to 0 at a linear rate and if we plug in $\eta = \Theta(1/\|H^\infty\|)$ we prove theorem. The details are in Section B.

**4 An Adaptive Method for Over-parameterized Neural Networks**

In this section we present our new adaptive gradient algorithm for optimizing over-parameterized neural networks. At the high level, we use the same paradigm as existing adaptive methods [Duchi et al., 2011]. There are three positive hyper-parameters, $b_0, \eta, \alpha$ in the algorithm. $\eta$ is to ensure the homogeneity and that the units match. $b_0$ is the initialization of a monotonically increasing sequence $\{b_k\}_{k=1}^\infty$ such that $b_k$ is updated at $k$-th iteration. To control the rate of this update, we use the parameter $\alpha$. Note $\alpha$ is not the learning rate to update the parameter $W$. At $k$-th iteration, we first use $\alpha$ and the information received to obtain $b_{k+1}$, then use $\eta/b_{k+1}$ to update the parameters. Here $\eta/b_{k+1}$ is the effective learning rate at the $k$-th iteration.

In practice, we would like an adaptive method that is robust to the choices of hyper-parameters. That is, we want this method guaranteed to converge in polynomial time for any choice of hyper-parameters. The key challenge for the adaptive method is how to design an appropriate update rule for $\{b_k\}$ to achieve the goal. Our algorithm uses the following update rule:

$$
b^2_{k+1} \leftarrow b^2_k + \alpha^2 \sqrt{n} \|y - u(k)\|^2.
$$

Here one can just view $\alpha$ and $n$ together as one constant. Using $\alpha^2$ is for matching the scale of $\eta$ and using $\sqrt{n}$ is for the ease of comparison with other adaptive gradient methods that we further discuss in Section B. The key for this update is $\|y - u(k)\|$ instead of its square. Note this is sharp contrast to Duchi et al. [2011] where the scheme to update the effective learning rate can be equivalently written as $\|y - u(k)\|^2$. The main reason is that our convergence analysis requires analyzing both over-parameterization and the dynamics of the adaptive stepsize at the same time. See Section B for more discussions. We list pseudo codes in Algorithm 1. The following theorem

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2 The convergence rate will, of course, depend on the choices of the hyper-parameters. The convergence of the ideal adaptive algorithm only depends polynomially on the these hyper-parameters.
Algorithm 1 Adaptive Loss (AdaLoss)

**Input:** Tolerance $\varepsilon > 0$, initialization $W(0)$, positive constants $b_0, \eta$ and $\alpha > 0$.

Set $k = 0$.

repeat

1. $b_{k+1}^2 \leftarrow b_k^2 + \alpha^2 \sqrt{n} \|y - u(k)\|

2. $W(k+1) = W(k) - \frac{\eta}{b_{k+1}} \frac{\partial L(W(k))}{\partial W}$

until $\|y - u(k)\|^2 / 2 \leq \varepsilon$

characterizes the convergence rate of our proposed algorithm.

**Theorem 4.1 (Convergence Rate of AdaLoss).** Under Assumption 2.1 and 2.2, if the width satisfies

$$m = \Omega\left(\frac{n^6}{\lambda_0^4 \delta^3} + \frac{\eta^4 n^4 \|H^\infty\|^4}{\alpha^2 \lambda_0^3 \delta^2}\right).$$

Then Algorithm 1 admits the following convergence results.

- If the hyper-parameter satisfies $\frac{b_0}{\eta} \geq C \|H^\infty\|$, then with probability $1 - \delta$ over the random initialization $\min_{t \in [T]} \|y - u(t)\|^2 \leq \varepsilon$ after

  $$T = \tilde{O}\left(\left(\frac{b_0}{\eta \lambda_0} + \frac{\alpha^2 n}{\eta^2 \lambda_0^2 \sqrt{\delta}}\right) \log \left(\frac{1}{\varepsilon}\right)\right).$$

- If the hyper-parameter satisfies $0 < \frac{b_0}{\eta} \leq C \|H^\infty\|$, then with probability $1 - \delta$ over the random initialization $\min_{t \in [T]} \|y - u(t)\|^2 \leq \varepsilon$ after

  $$T = \tilde{O}\left(\left(\frac{\eta C \|H^\infty\|^2}{\alpha^2 \sqrt{n} \varepsilon} - \frac{b_0^2}{\alpha^2} + \frac{\alpha^2 n}{\eta^2 \lambda_0^2 \sqrt{\delta}} + \left(\frac{\|H^\infty\|}{\lambda_0}\right)^{3/2}\right) \log \left(\frac{1}{\varepsilon}\right)\right).$$

To our knowledge, this is first global convergence guarantee for the adaptive gradient method. Now we unpack the statements of Theorem 4.1. Our theorem applies to two cases. In the first case, the effective learning rate at the beginning $\eta/b_0$ is smaller than the threshold $1/(C\|H^\infty\|)$ that guarantees the global convergence of gradient descent (c.f. Theorem 3.1). In this case, the convergence has two terms, the first term $\frac{b_0}{\eta \lambda_0} \log \left(\frac{1}{\varepsilon}\right)$ is standard gradient descent rate if we use $\eta/b_0$ as the learning rate. Note this term is the same as Theorem 3.1 if $\eta/b_0 = \Theta(1/\|H^\infty\|)$. The second term comes from the upper bound of $b_T$ in the effective learning rate $\eta/b_T$ (c.f. Lemma 4.1). This case shows that if $\alpha$ is relatively small that the second term is smaller than the first term, then we have the same rate as gradient descent. See Remark 4.1 for more discussion.

In the second case, the initial effective learning $\eta/b_0$ is greater than the threshold that guarantees the convergence of gradient descent. Our algorithm will guarantee either of the followings happens after $T$ iterations. (1) The loss is already small, so we can stop training. This corresponds the first term $\frac{\eta C \|H^\infty\|^2 - b_0^2}{\alpha^2 \sqrt{n} \varepsilon}$. (2) The loss is still large, which will make the effective stepsize $\eta/b_k$ decrease

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3The notation $C$ is well-defined, please check Table 1 in Appendix E.
with a good rate. That is, if (2) keeps happening, the step size will decrease till \( \eta/b_k \leq 1/(C\|H^\infty\|) \) and we are in the first case. Note the first term is the same as the second term of the first case. The third term \( \left(\frac{\|H^\infty\|}{\lambda_0}\right)^{3/2} \log \left(\frac{1}{\varepsilon}\right) \) is slightly worse than the rate in the gradient descent. The reason is the loss may increase due to the large learning rate at the beginning. (c.f. Lemma 3.1).

To summarize, these two cases together show that our algorithm is robust to hyper-parameter choices. The bad choices of hyper-parameters will only hurt the constant in the convergence rate but the global polynomial time convergence is still guaranteed.

**Remark 4.1.** It is difficult to set the parameters with optimal values due to the fact that the maximum and minimum eigenvalues of the matrix \( H^\infty \) are computational costly and so generally unknown. According to Theorem 3.1, since \( n \) is an upper bound of \( \|H^\infty\| \), one may use gradient descent by setting \( \eta = \Theta \left(\frac{1}{\sqrt{n}}\right) \) and have the convergence rate of \( T_1 = \tilde{O} \left(\left(\frac{1}{\varepsilon}\right) \log \left(\frac{1}{\varepsilon}\right)\right) \).

However, this choice of step size is not optimal when \( \|H^\infty\| \) is much smaller than \( n \). Using adaptive gradient algorithm with the small initialization on the effective learning rate would results in better complexity. Indeed, for instance, let the target training error be \( \tilde{O} \left(\sqrt{n} \alpha \right) \). Now in the scenario that \( \|H^\infty\| = \Theta(1) \) and \( \frac{1}{\lambda_0} = \Theta \left(\frac{1}{\sqrt{n}}\right) \), the convergence rate of our adaptive method is \( T_2 = \tilde{O}(n^{5/4}) \) comparing to the convergence rate of gradient descent which is \( T_1 = \tilde{O}(n^{5/2}) \).

### 4.1 Proof Sketch of Theorem 4.1

We prove by induction. Our induction hypothesis is the following.

**Condition 4.1.** At the \( k' \)-th iteration, there exists a constant \( C_1 \) such that 5

\[
\|y - u(k')\|^2 \leq \left(1 - \frac{\eta\lambda_0 C_1}{b_k} \left(1 - \frac{\eta C\|H^\infty\|}{b_k}\right)\right) \|y - u(k' - 1)\|^2. \tag{8}
\]

Recall the key Gram matrix \( H(k') \) at \( k' \)-th iteration

\[
H_{ij}(k') = \frac{1}{m} \sum_{r=1}^{m} x_i^\top x_j \left\{w_r(k')^\top x_i \geq 0, w_r(k')^\top x_j \geq 0\right\}. \tag{9}
\]

We prove two cases \( b_0/\eta \geq C\|H^\infty\| \) and \( b_0/\eta \leq C\|H^\infty\| \) separately.

**Case (1):** \( b_0/\eta \geq C\|H^\infty\| \) The base case \( k' = 0 \) holds by the definition. Now suppose for \( k' = 0, \ldots, k \), Condition 4.1 holds and we want to show Condition 4.1 holds for \( k' = k + 1 \). Because \( \frac{b_0}{\eta} \geq C\|H^\infty\| \), by Lemma 4.1 we have

\[
\|w_r(k) - w_r(0)\| \leq \frac{4\sqrt{n}}{\sqrt{m\lambda_0 C_1}} \|y - u(0)\|. \tag{10}
\]

Next, plugging in \( m = \Omega \left(\frac{n^6}{\lambda_0^3}\right) \), we have \( \|w_r(k) - w_r(0)\| \leq \frac{\eta\lambda_0^5}{n^2} \). Then by Lemma 3.1 and 3.3 the matrix \( H(k) \) is positive such that the smallest eigenvalue of \( H(k) \) is greater than \( \frac{1}{\sqrt{n}} \). Consequently, we have Condition 4.1 holds for \( k' = k + 1 \).

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4 For the convenience of induction proof, we define \( \|y - u(-1)\|^2 = \|y - u(0)\|^2 / \left(1 - \frac{\eta\lambda_0 C_1}{b_0} \left(1 - \frac{n C\|H^\infty\|}{b_0}\right)\right) \).

5 See Table I in Appendix E for the expressions.
Now we have proved the induction part. Using Condition 4.1 for any \( T \in \mathbb{Z}^+ \), we have

\[
\| \mathbf{y} - \mathbf{u}(T) \|^2 \leq \prod_{i=0}^{T-1} \left( 1 - \frac{\eta \lambda_0 C_1}{2b_i} \right) \| \mathbf{y} - \mathbf{u}(0) \|^2
\]

\[
\leq \exp \left( -T \frac{\eta \lambda_0 C_1}{2b_{\infty}} \right) \| \mathbf{y} - \mathbf{u}(0) \|^2
\]

where \( b_{\infty} = b_0 + \frac{4\alpha^2 \sqrt{n}}{\eta \lambda_0 C_1} \| \mathbf{y} - \mathbf{u}(0) \| = O(b_0 + \frac{\alpha^2 n}{\eta^2 \lambda_0^2 \lambda}) \) (c.f. Lemma 4.1). This implies the convergence rate of Case (1).

**Case (2):** \( b_0/\eta \leq C\| \mathbf{H}^\infty \| \) We define

\[
\hat{T} = \arg\min_k \frac{b_k}{\eta} \geq C\| \mathbf{H}^\infty \|.
\]

Note this represents the number of iterations to make Case (2) reduce to Case (1). We first give an upper bound \( T_0 \) of \( \hat{T} \). If

\[
T_0 = \left\lceil \frac{(\eta C\| \mathbf{H}^\infty \|)^2 - b_0^2}{\alpha^2 \sqrt{n} \varepsilon} \right\rceil + 1
\]

applying Lemma 4.1 with parameters \( \gamma = \alpha^2 \sqrt{n} \), \( a_j = \| \mathbf{y} - \mathbf{u}(k) \| \) and \( L = (\eta C\| \mathbf{H}^\infty \|)^2 \) we have after \( T_0 \) step,

\[
\text{either } \min_{k \in [T_0]} \| \mathbf{y} - \mathbf{u}(k) \|^2 \leq \varepsilon, \quad \text{or} \quad b_{T_0} \geq \eta C\| \mathbf{H}^\infty \|.
\]

If \( \min_{k \in [T_0]} \| \mathbf{y} - \mathbf{u}(k) \|^2 \leq \varepsilon \), we are done. Note this bound \( T_0 \) incurs the first term of iteration complexity of the Case (2) in Theorem 4.1.

Similar to Case (1), we use induction for the proof. Again the base case \( k' = 0 \) holds by the definition. Now suppose for \( k' = 0, \ldots, k \), Condition 4.1 holds and we will show it also holds for \( k' = k + 1 \). There are two scenarios.

For \( k \leq T_0 - 1 \), Lemma 4.2 implies that \( \| \mathbf{w}_r(k) - \mathbf{w}_r(0) \| \) is upper bounded. Now plugging in our choice on \( m \) and using Lemma B.1 and B.3, we know \( \lambda_{\min}(\mathbf{H}(k)) \geq \lambda_0/2 \) and \( \| \mathbf{H}(k) \| \leq C\| \mathbf{H}^\infty \| \). These two bounds on \( \mathbf{H}(k) \) imply Condition 4.1.

When \( k \geq T_0 \), we have contraction bound as in Case (1) and then same argument follows but with the different initial values \( \mathbf{W}(T_0 - 1) \) and \( \| \mathbf{y} - \mathbf{u}(T_0 - 1) \| \). We first analyze \( \mathbf{W}(T_0 - 1) \) and \( \| \mathbf{y} - \mathbf{u}(T_0 - 1) \| \). By Lemma C.1, we know \( \| \mathbf{y} - \mathbf{u}(T_0 - 1) \| \) only increases an additive \( O \left( (\eta C\| \mathbf{H}^\infty \|)^{3/2} \right) \) factor from \( \| \mathbf{y} - \mathbf{u}(0) \| \). Furthermore, by Lemma 4.2 we know for \( r \in [m] \)

\[
\| \mathbf{w}_r(T_0 - 1) - \mathbf{w}_r(0) \| \leq \frac{4\eta^2 C\| \mathbf{H}^\infty \|}{\alpha^2 \sqrt{m}}.
\]

Now we consider \( k \)-th iteration. Applying Lemma 4.1 we have

\[
\| \mathbf{w}_r(k) - \mathbf{w}_r(0) \| \leq \| \mathbf{w}_r(k) - \mathbf{w}_r(T_0 - 1) \| + \| \mathbf{w}_r(T_0 - 1) - \mathbf{w}_r(0) \|
\]

\[
\leq \frac{4\sqrt{n}}{\sqrt{m} \lambda_0 C_1} \left( \| \mathbf{y} - \mathbf{u}(T_0 - 1) \| + \hat{R} \right)
\]

\[
\leq c \lambda_0 \delta / n^2
\]
where the last inequality we have used our choice of \( m \). Using Lemma \[B.1\] and \[B.3\] again, we can show \( \lambda_{\text{min}}(\mathbf{H}(k)) \geq \lambda_0/2 \) and \( \|\mathbf{H}(k)\| \leq C\|\mathbf{H}^\infty\| \). These two bounds on \( \mathbf{H}(k) \) imply Condition \ref{eq:4.1.1}. Now we have proved the induction. The last step is to use Condition \ref{eq:4.1.1} to prove the convergence rate. Observe that for any \( T \geq T_0 \), we have

\[
\|\mathbf{y} - \mathbf{u}(T)\|^2 \leq \exp\left(-(T - T_0 + 1)\frac{\eta \lambda_0 C_1}{2b_\infty}\right) \|\mathbf{y} - \mathbf{u}(T_0 - 1)\|^2
\]

where we have used Lemma \[4.1\] and Lemma \[C.1\] to derive

\[
b_\infty = \eta C\|\mathbf{H}^\infty\| + \frac{4\lambda^2 \sqrt{n}}{\eta \lambda_0 C_1} \|\mathbf{y} - \mathbf{u}(0)\| + \frac{2\lambda^2 \sqrt{n} (C\|\mathbf{H}^\infty\|)^{3/2}}{\alpha^2 \sqrt{n}}.
\]

With some algebra, one can show this bound corresponds to the second and the third term of iteration complexity of the Case (2) in Theorem \[4.1\].

### 4.1.1 Ingredients of Proof

As we have seen in the proof sketch. Lemma \[4.1\] and Lemma \[4.2\] are most important lemmas in the proof of Theorem \[4.1\]. Here we state and prove these two lemmas.

**Lemma 4.1.** Suppose Condition \ref{eq:4.1.1} holds for \( k' = 0, \ldots, k \) and \( b_k \) is updated by Algorithm 1. Let \( T_0 \geq 1 \) be the first index such that \( b_{T_0} \geq \eta C\|\mathbf{H}^\infty\| \). Then for every \( r \in [m] \) and \( k = 0, 1, \ldots, \)

\[
b_k \leq b_{T_0-1} + \frac{4\sqrt{n}}{\eta \lambda_0 C_1} \|\mathbf{y} - \mathbf{u}(T_0 - 1)\|;
\]

\[
\|\mathbf{w}_r(k + T_0) - \mathbf{w}_r(T_0 - 1)\| \leq \frac{4\sqrt{n}}{\sqrt{m \lambda_0 C_1}} \|\mathbf{y} - \mathbf{u}(T_0 - 1)\| \triangleq \bar{R}.
\]

**Proof of Lemma 4.1** When \( b_{T_0}/\eta \geq C\|\mathbf{H}^\infty\| \) at some \( T_0 \geq 1 \), thanks to the key fact that Condition \ref{eq:4.1.1} holds \( k' = 0, \ldots, k \), we have

\[
\|\mathbf{y} - \mathbf{u}(k + T_0)\| \leq \sqrt{\left(1 - \frac{\eta \lambda_0 C_1}{2b_{k+T_0}}\right)} \|\mathbf{y} - \mathbf{u}(k + T_0 - 1)\|
\]

\[
\leq \left(1 - \frac{\eta \lambda_0 C_1}{4b_{k+T_0}}\right) \|\mathbf{y} - \mathbf{u}(k + T_0 - 1)\|
\]

\[
\leq \|\mathbf{y} - \mathbf{u}(T_0 - 1)\| - \sum_{t=0}^{k} \frac{\eta \lambda_0 C_1}{4b_{t+T_0}} \|\mathbf{y} - \mathbf{u}(t + T_0 - 1)\|
\]

\[
\Rightarrow \sum_{t=0}^{k} \frac{\|\mathbf{y} - \mathbf{u}(t + T_0 - 1)\|}{b_{t+T_0}} \leq \frac{4\|\mathbf{y} - \mathbf{u}(T_0 - 1)\|}{\eta \lambda_0 C_1}.
\]

Thus, the upper bound for \( b_k \),

\[
b_{k+T_0} \leq b_{k+T_0-1} + \frac{\alpha^2 \sqrt{n}}{b_{k+T_0}} \|\mathbf{y} - \mathbf{u}(k + T_0 - 1)\|
\]

\[
\leq b_{T_0 - 1} + \sum_{t=0}^{k} \frac{\alpha^2 \sqrt{n}}{b_{t+T_0}} \|\mathbf{y} - \mathbf{u}(t + T_0 - 1)\|
\]

\[
\leq b_{T_0 - 1} + \frac{4\alpha^2 \sqrt{n}}{\eta \lambda_0 C_1} \|\mathbf{y} - \mathbf{u}(T_0 - 1)\|.
\]
As for the upper bound of \( \| \mathbf{w}_r(k + T_0) - \mathbf{w}_r(T_0 - 1) \| \),

\[
\| \mathbf{w}_r(k + T_0) - \mathbf{w}_r(T_0 - 1) \| \leq \sum_{t=0}^{k} \frac{\eta}{b_{t+T_0}} \left\| \frac{\partial L(\mathbf{W}(t + T_0 - 1))}{\partial \mathbf{w}_r} \right\|
\]

\[
\leq \frac{4\sqrt{n}}{\sqrt{m} \lambda_0 C_1} \| \mathbf{y} - \mathbf{u}(T_0 - 1) \|.
\]

**Lemma 4.2.** Let \( T_0 \geq 1 \) be the first index such that \( b_{T_0} \geq \eta C \| \mathbf{H}^\infty \| \). Then for every \( r \in [m] \), we have for \( k = 0, 1, \ldots, T_0 - 1 \),

\[
\| \mathbf{w}_r(k) - \mathbf{w}_r(0) \| \leq \frac{4\eta^2 C \| \mathbf{H}^\infty \|}{\alpha^2 \sqrt{m}} \triangleq \hat{R}.
\]

**Proof of Lemma 4.2** For the upper bound of \( \| \mathbf{w}_r(k + 1) - \mathbf{w}_r(0) \| \) when \( b_t/\eta < C \| \mathbf{H}^\infty \| \),

\[
t = 0, 1, \ldots, k \) and \) \( k \leq T_0 - 2 \), we first observe that

\[
\sum_{t=0}^{k} \frac{\| \mathbf{y} - \mathbf{u}(t) \|}{b_{t+1}} \leq \frac{1}{\alpha^2 \sqrt{n}} \sum_{t=0}^{k} \frac{\alpha^2 \sqrt{n} \| \mathbf{y} - \mathbf{u}(t) \|}{\alpha^2 \sqrt{n} \sum_{\ell=0}^{t} \| \mathbf{y} - \mathbf{u}(\ell) \| + b_0^2}
\]

\[
\leq \frac{2}{\alpha^2 \sqrt{n}} \left( \alpha^2 \sqrt{n} \sum_{\ell=0}^{k} \| \mathbf{y} - \mathbf{u}(\ell) \| + b_0^2 \right)
\]

\[
\leq \frac{2b_{T_0-1}}{\alpha^2 \sqrt{n}}
\]

where the second inequality use Lemma 4.2 and the third inequality is due to the fact that \( b_k \leq b_{T_0-1} \leq \eta C \| \mathbf{H}^\infty \| \) for all \( k \leq T_0 - 2 \). Thus,

\[
\| \mathbf{w}_r(k + 1) - \mathbf{w}_r(0) \| \leq \sum_{t=0}^{k} \frac{\eta}{b_{t+1}} \left\| \frac{\partial L(\mathbf{W}(t))}{\partial \mathbf{w}_r} \right\| \leq \frac{\eta \sqrt{n}}{\sqrt{m}} \sum_{t=1}^{k} \frac{\| \mathbf{y} - \mathbf{u}(t) \|}{b_{t+1}} \leq \frac{2C\eta^2 \| \mathbf{H}^\infty \|}{\alpha^2 \sqrt{m}}.
\]

5 Discussion on Variants of AdaGrad

In this section we compare our proposed algorithm AdaLoss with existing adaptive methods. Algorithm 1 can be viewed as a variant of the standard AdaGrad algorithm proposed by Duchi et al. [2011], where the norm version of the update is

\[
b_{k+1}^2 = b_k^2 + \sqrt{m} \max_{r \in [m]} \left\| \frac{\partial L(\mathbf{W}(k))}{\partial \mathbf{w}_r} \right\|^2.
\]

Our algorithm AdaLoss is similar to AdaGrad, but is distinctly different from AdaGrad: we update \( b_{k+1}^2 \) using the norm of the loss instead of the squared norm of the gradient. We considered the AdaLoss update instead of AdaGrad because, in the setting considered here, the modifications allowed for dramatically better theoretical convergence rate.
Why the Loss instead of the Gradient? Indeed, our update of $b_{k+1}^2$ is not too different from the following update rule using the gradient

$$ b_{k+1}^2 = b_k^2 + \alpha^2 \sqrt{m} \max_{r \in [m]} \left\| \frac{\partial L(W(k))}{\partial w_r} \right\| . $$

(11)

The AdaLoss update can be upper and lower bounded by $t_{k}^2$ and the norm of the gradient, i.e.,

$$ b_{k+1}^2 + \alpha^2 \sqrt{m} \max_{r \in [m]} \left\| \frac{\partial L(W(k))}{\partial w_r} \right\| \leq b_{k+1}^2 \leq b_{k}^2 + \alpha^2 \sqrt{m} \max_{r \in [m]} \left\| \frac{\partial L(W(k))}{\partial w_r} \right\| , $$

where the first and second inequalities are respectively due to Proposition E.1 and Proposition 5.1.

Proposition 5.1. If $\lambda_{\min}(H) \geq \frac{\lambda_0}{2}$, then $\| y - u \| \leq \frac{\sqrt{2m}}{\sqrt{\lambda_0}} \max_{r \in [m]} \left\| \frac{\partial L(W)}{\partial w_r} \right\| . \square$

However, we use $\sqrt{n} \| y - u(k) \|$ instead of using the gradient to update $b_k$ because our convergence analysis requires lower and upper bounding the dynamics $b_1, \ldots, b_k$, in terms of $\| y - u(k) \|$. If $b_k$ were instead updated using (11), then

$$ b_{k+1}^2 \geq b_k^2 + \frac{\alpha^2}{2\lambda_0} \max_{r \in [m]} \left\| \frac{\partial L(W(k))}{\partial w_r} \right\| . $$

The above lower bound of $b_k$ results in a larger $T$ in Case (2) by a factor of $\sqrt{n/\lambda_0}$. Using the loss instead of the gradient to update $b_k$ is independently useful as reusing the already computed loss information for each iteration can save some computation cost and thus make the update more efficient.

Why the norm and not the squared-norm? For ease of comparison with Algorithm 1, we switch from gradient information to loss and compare with two close variants:

$$ b_{k+1}^2 = b_k^2 + \alpha^2 \sqrt{n} \| y - u(k) \|^2 $$

(12)

$$ b_{k+1} = b_k + \alpha \sqrt{n} \| y - u(k) \|. $$

(13)

Equation (12) using the “square” rule update is the standard AdaGrad proposed by Duchi et al. [2011] and has been widely recognized as important optimizer in deep learning – especially for training sparse datasets. For our over-parameterized models, this update rule does give a better convergence result in Case 1 when $b_0/\eta \geq C\|H^\infty\|^2$. However, when the initialization $b_0/\eta \leq C\|H^\infty\|$, we were only able to prove convergence in case the level of over-parameterization (i.e., $m$) depends on the training error $1/\varepsilon$, the bottleneck resulting from the attempting to prove the analog of Lemma 1.2 (see Proposition 5.2 below).

Proposition 5.2. Let $T_0 \geq 1$ be the first index such that $b_{T_0} \geq \eta C\|H^\infty\|$. Consider the update of $b_k$ in (12). Then for every $r \in [m]$, we have for $k = 0, 1, \ldots, T_0 - 1$,

$$ \| w_r(k + 1) - w_r(0) \|_2 \leq \frac{\eta \sqrt{2(k + 1)}}{\alpha^2 \sqrt{m}} \sqrt{1 + 2 \log \left( \frac{C\|H^\infty\|}{b_0} \right).} \tag{12} $$

Footnotes:

6 Proof is given in Appendix D

7 The convergence proof is straightforward and similar to the first case in Theorem 4.1.
On the other hand, the update rule in (13) can resolve the problem because the growth of $b_k$ is larger than (12) such that the upper bound of $\|w_r(k) - w_r(0)\|_2$ $k = 0, 1, \ldots, T_0 - 1$, is better than that in Proposition 5.2 and even Lemma 4.2 if $c < b_0 < \eta C \|H_\infty\|$ for some small $c$. However, the growth of $b_k$ remains too fast once the critical value of $\eta C \|H_\infty\|$ has been reached – the upper bound $b_\infty$ we were able to show is exponential in $1/\lambda_0$ and also in the hyper-parameters $b_0, \eta, \alpha$ and $n$, resulting in an extremely large $T$ compared to Case (2) in Theorem 4.1.

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A Experiments

We first plot the eigenvalues of the matrices \( \{H(k)\}_{k'=0}^k \) and then provide the details.

We use two simulated Gaussian data sets: i.i.d. Gaussian (the red curves) and multivariate Gaussian (the blue curves). Observe the red curves in Figure 1 that the largest maximum eigenvalue is around 2.8 and minimum eigenvalues is around 0.19 within 100 iterations, while the maximum and minimum eigenvalues for the blue curves are around 291 and 0.033 respectively. To some extend, i.i.d. Gaussian data illustrates the case where the data points are pairwise uncorrelated such that \( \|H^\infty\| = O(1) \), while correlated Gaussian data set implies the situation when the samples are highly correlated with each other \( \|H^\infty\| = O(n) \).

In the experiments, we simulate Gaussian data with training sample \( n = 1000 \) and the dimension \( d = 200 \). Figure 1 plots the histogram of the eigenvalues of the co-variances for each dataset. Note that the eigenvalues are different from the eigenvalues in the top plots. We use the two-layer neural networks \( m = 5000 \). Although \( m \) here is far smaller than what Theorem 3.1 requires, we found it sufficient for our purpose to just illustrate the maximum and minimum eigenvalues of \( H(k) \) for iteration \( k = 0, 1, \ldots, 100 \). Set the learning rate \( \eta = 5 \times 10^{-4} \) for i.i.d. Gaussian and \( \eta = 5 \times 10^{-5} \) for correlated Gaussian. The training error is also given in Figure 1.

\[ \lambda_{\min}(H(k)) \]
\[ \lambda_{\max}(H(k)) \]
\[ \lambda_{\min}(H(k)) \]
\[ \lambda_{\max}(H(k)) \]

**Figure 1:** Top plots: y-axis is maximum or minimum eigenvalue of the matrix \( H(k) \), x-axis is the iteration. Bottom plots (left and middle): y-axis is the probability, x-axis is the eigenvalue of co-variance matrix induced by Gaussian data. Bottom plots (right): y-axis is the training error in logarithm scale, x-axis is the iteration. The distributions of eigenvalues for the co-variances matrix (\( d \times d \) dimension) of the data are plotted on the left for i.i.d. Gaussian and in the middle for correlated Gaussian. The bottom right plot is the training error for the two-layer neural network \( m = 5000 \) using the two Gaussian datasets.
B  Proof for Theorem 3.1

We prove Theorem 3.1 by induction. Our induction hypothesis is the following convergence rate of empirical loss.

Condition B.1. At the $k$-th iteration, we have for $m = \Omega \left( \frac{n^6}{\lambda_0^4 \delta^2} \right)$ such that with probability $1 - \delta$,

$$
\|y - u(k)\|^2 \leq (1 - \frac{\eta \lambda_0}{2})^k \|y - u(0)\|^2.
$$

Now we show Condition B.1 for every $k = 0, 1, \ldots$. For the base case $k = 0$, by definition Condition B.1 holds. Suppose for $k' = 0, \ldots, k$, Condition B.1 holds and we want to show Condition B.1 holds for $k' + 1$. We first prove the order of $m$ and then the contraction of $\|u(k + 1) - y\|$.

B.1  The order of $m$ at iteration $k + 1$

Note that the contraction for $\|u(k + 1) - y\|$ is mainly controlled by the smallest eigenvalue of the sequence of matrices $\{H(k')\}_{k'=0}^k$. It requires that the minimum eigenvalues of matrix $H(k')$, $k' = 0, 1, \ldots, k$ are strictly positive, which is equivalent to ask that the update of $w_r(k')$ is not far away from initialization $w_r(0)$ for $r \in [m]$. This requirement can be fulfilled by the large hidden nodes $m$.

The first lemma (Lemma B.1) gives smallest $m$ in order to have $\lambda_{\min}(H(0)) > 0$. The next two lemmas concludes the order of $m$ so that $\lambda_{\min}(H(k')) > 0$ for $k' = 0, 1, \ldots, k$. Specifically, if $R' < R$, then the conditions in Lemma B.3 hold for all $0 \leq k' \leq k$. We refer the proofs of these lemmas to Du et al. [2019].

Lemma B.1. If $m = \Omega \left( \frac{n^2 \log^2 \left( \frac{n}{\delta} \right)}{\lambda_0^2} \right)$, we have with probability at least $1 - \delta$ that $\|H(0) - H\| \leq \frac{\lambda_0}{4}$.

Lemma B.2. If Condition B.1 holds for $k' = 0, \ldots, k$, then we have for every $r \in [m]$

$$
\|w_r(k' + 1) - w_r(0)\| \leq \frac{4 \sqrt{n} \|y - u(0)\|}{\sqrt{m} \lambda_0} \triangleq R'.
$$

Lemma B.3. Suppose for $r \in [m]$, $\|w_r - w_r(0)\| \leq \frac{c \lambda_0 \delta}{n} \triangleq R$ for some small positive constant $c$. Then we have with probability $1 - \delta$ over initialization, $\|H - H(0)\| \leq \frac{\lambda_0}{4}$ where $H$ is defined in Definition 2.1.

Thus it is sufficient to show $R' < R$. Since $\|y - u(0)\|^2 = O \left( \frac{n}{\lambda} \right)$ derived from Proposition B.2 $R' < R$ implies that

$$
m = \Omega \left( \frac{n^5 \|y - u(0)\|^2}{\lambda_0^4} \right) = \Omega \left( \frac{n^6}{\lambda_0^4 \delta} \right).
$$

Note that we use the same structure as in Du et al. [2019]. For the sake of completeness in the proof, we will use most of their lemmas, of which the proofs can be found in technical section or otherwise in their paper.
B.2 The contraction of $\|u(k+1) - y\|$  

Define the event 

$$A_{ir} = \{ \exists w : \|w - w_r(0)\| \leq R, I\{x_i^\top w_r(0) \geq 0\} \neq I\{x_i^\top w \geq 0\} \}$$

for some small positive constant $c$. We let $S_i = \{ r \in [m] : I\{A_{ir}\} = 0 \}$ and $S_i^\perp = [m] \setminus S_i$. The following lemma bounds the sum of sizes of $S_i^\perp$.

**Lemma B.4.** With probability at least $1 - \delta$ over initialization, we have $\sum_{i=1}^n |S_i^\perp| \leq \frac{C_m n R}{\delta}$ for some positive constant $C_2$.

Next, we calculate the difference of predictions between two consecutive iterations,

$$u_i(k+1) - u_i(k) = \frac{1}{\sqrt{m}} \sum_{r=1}^m a_r \left( \sigma \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i \right) - \sigma \left( w_r(k)^\top x_i \right) \right).$$

Here we divide the right hand side into two parts. $I_1^i$ accounts for terms that the pattern does not change and $I_2^i$ accounts for terms that pattern may change.

Because $R' < R$, we know $I\{w_r(k+1)^\top x_i \geq 0\} \cap S_i = I\{w_r(k)^\top x_i \geq 0\} \cap S_i$.

$$I_1^i \triangleq \frac{1}{\sqrt{m}} \sum_{r \in S_i} a_r \left( \sigma \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i \right) - \sigma \left( w_r(k)^\top x_i \right) \right)$$

$$= -\frac{1}{\sqrt{m}} \sum_{r \in S_i} a_r \eta \left( \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i$$

$$= -\frac{\eta}{m} \sum_{j=1}^n x_j^\top x_j (u_j - y_j) \sum_{r \in S_i} I\{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0\}$$

$$= -\eta \sum_{j=1}^n (u_j - y_j)(H_{ij}(k) - H_{ij}^\perp(k))$$

where $H_{ij}^\perp(k) = \frac{1}{m} \sum_{r \in S_i^\perp} x_i x_j^\top I\{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0\}$ is a perturbation matrix. Let $H^\perp(k)$ be the $n \times n$ matrix with $(i,j)$-th entry being $H_{ij}^\perp(k)$. Using Lemma B.4, we obtain with probability at least $1 - \delta$,

$$\|H^\perp(k)\| \leq \sum_{(i,j)=(1,1)}^{(n,n)} \left| H_{ij}^\perp(k) \right| \leq \frac{n \sum_{i=1}^n |S_i^\perp|}{m} \leq \frac{n^2 m R}{\delta m} \leq \frac{n^2 R}{\delta}. \quad (15)$$

We view $I_2^i$ as a perturbation and bound its magnitude. Because ReLU is a 1-Lipschitz function and $|a_r| = 1$, we have

$$I_2^i \triangleq \frac{1}{\sqrt{m}} \sum_{r \in S_i^\perp} a_r \left( \sigma \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i \right) - \sigma \left( w_r(k)^\top x_i \right) \right)$$

$$\leq \frac{\eta}{\sqrt{m}} \sum_{r \in S_i^\perp} \| \frac{\partial L(W(k))}{\partial w_r(k)} \|$$

$$\Rightarrow \ |I_2^i| \leq \frac{\eta |S_i^\perp| \sqrt{n} \|u(k) - y\|}{m} \leq \frac{\eta m^{3/2} R}{\delta} \|u(k) - y\|. \quad (16)$$
Observe the maximum eigenvalue of matrix $H(k)$ upto iteration $k$ is bounded because

$$
\|H(k) - H(0)\| \leq \frac{4n^2R}{\sqrt{2\pi} \delta} \quad \text{with probability } 1 - \delta \quad \Rightarrow \quad \|H(k)\| \leq \|H(0)\| + \frac{4n^2R}{\sqrt{2\pi} \delta}
$$

Further, Lemma B.1\(^9\) implies that

$$
\left| \|H^\infty\| - \|H(0)\| \right| = O\left( \frac{n^2 \log(n/\delta)}{m} \right).
$$

That is, we could almost ignore the distance between $\|H^\infty\|$ and $\|H(0)\|$ for $m = \Omega\left( \frac{n^8}{\lambda_0^2 \delta^5} \right)$.

With these estimates at hand, we are ready to prove the induction hypothesis.

$$
\|y - u(k+1)\|^2 = \|y - (u(k) + I_1 + I_2)\|^2
\leq \|y - u(k)\|^2 - 2\langle y - u(k), (I_1 + I_2) + 2\|I_1\|^2 + 2\|I_2\|^2
\leq \|y - u(k)\|^2 - 2\eta\langle y - u(k), (H(k) - H(k)^\perp) (y - u(k)) - 2(y - u(k))^\top I
\leq \|y - u(k)\|^2 - 2\eta^2\langle y - u(k), (H(k) - H(k)^\perp)^2 (y - u(k)) + 2\|I\|^2
\leq \|y - u(k)\|^2 - 2\eta\langle y - u(k), (I - \eta H(k)) H(k) (y - u(k))
\leq \|y - u(k)\|^2 - 2\eta^2\langle y - u(k), H(k) H(k)^\perp (y - u(k))
\leq \frac{1}{\|H(k)\|^2}, \text{ we have}

Term1 \geq (1 - \eta\|H(k)\|) \lambda_{\min}(H(k)) \|y - u(k)\|^2 \geq \frac{\lambda_0}{2} (1 - \eta\|H(k)\|) \|y - u(k)\|^2 > 0
$$

Due to (15), we could bound Term2

$$
Term2 \leq \left( \eta\|H^\perp(k)\|^2 + \|H^\perp(k)\| \right) \|y - u(k)\|^2 \leq \left( \frac{\eta n^2 R}{\delta} + 1 \right) \frac{n^2 R}{\delta} \|y - u(k)\|^2
$$

Due to (16), we could bound Term3

$$
Term3 \leq (\|I\| + \|y - u(k)\|) \|I\| \leq \left( \frac{\eta n^{3/2} R}{\delta} + 1 \right) \frac{\eta n^{3/2} R}{\delta} \|u(k) - y\|^2
$$

Finally, for Term4

$$
Term4 \leq \|H(k)\| \|H^\perp(k)\| \|u(k) - y\|^2 \leq \frac{n^2 R}{\delta} \|H(k)\| \|u(k) - y\|^2
$$

\(^9\)For more details, please see Lemma 3.1 in [Du et al., 2019]
Let $C_1 = 1 - 2 \left( \frac{1}{\sqrt{n}} + 1 \right) \frac{n^2 R}{\lambda_0 \sigma}$ and $C = \frac{\left( 1 + \frac{2n^2 R}{\lambda_0} \right) \left( \|H(0)\|_2 + \frac{2n^2 R}{\sqrt{2n} \delta} \right) + 2\left( \frac{1}{\sqrt{n}} \right) \frac{n^2 R^2}{\lambda_0 \sigma^2}}{c \|H\|_\infty}$. Putting Term1, Term2, Term3 and Term4 back to inequality (12), we have with probability $1 - \delta$

$$\|y - u(k + 1)\|^2 - \|y - u(k)\|^2 \leq - \lambda_0 \eta \left( 1 - \frac{2n^{3/2} \sqrt{n} + 1}{\lambda_0 \delta} \right) - \eta \left( \frac{(\lambda_0 \delta + 2n^2 R) \|H(k)\| + 2(n + 1)n^2 R^2}{\lambda_0 \delta} \right) \|y - u(k)\|^2$$

$$\leq - \eta \lambda_0 C_1 (1 - \eta C \|H\|_\infty) \|y - u(k)\|^2$$

(20)

where in the last inequality the upper bound of $H(k)$ is $\|H\|_\infty + \frac{4n^2 R}{\sqrt{2n} \delta}$ due to (17) and (18). Recall that $R = \frac{c_0 \sqrt{\delta}}{n^2}$ for very small constant $c$. We have contractions for $\|y - u(k + 1)\|^2$, if the stepsize satisfy

$$\eta \leq \frac{C_1}{C \|H\|_\infty} \Rightarrow \eta \leq \frac{1}{\|H(k)\|}.$$ 

We could pick $\eta = \frac{C_1}{2C \|H\|_\infty} = \Theta \left( \frac{1}{\|H\|_\infty} \right)$ for large $m$ such that

$$\|y - u(k + 1)\|^2 \leq \left( 1 - \frac{\lambda_0 C_1}{4C \|H\|_\infty} \right) \|y - u(k)\|^2.$$ 

Therefore Condition 3.1 holds for $k' = k + 1$. Now by induction, we prove Theorem 3.1.

C Proof for Theorem 4.1

In this section, we give the detailed proof for Theorem 4.1. The proof are organized into two parts. Part I in Subsection C.1 is to prove the convergence for the initialization $b_0/\eta \geq C \|H\|_\infty$. Part II in Subsection C.2 is to prove the convergence for the initialization $b_0/\eta < C \|H\|_\infty$. Several key lemmas will be stated and used during the proof, and the proof of these lemmas will be deferred to subsection C.3.

C.1 Part I

In this part, we prove the following condition by induction: show Condition 4.1 for every $k = 0, 1, 2, \ldots$. Based on this condition, we then obtain the upper bound of $b_k$ and so the convergence result.

Condition 4.1 At the $k$-th iteration,

$$\|y - u(k)\|^2 \leq \left( 1 - \frac{\eta \lambda_0}{b_k} \left( 1 - \frac{\eta C \|H\|_\infty}{b_k} \right) \right) \|y - u(k - 1)\|^2$$

(21)

where we define $\|y - u(-1)\|^2 = \|y - u(0)\|^2 / \left( 1 - \frac{\eta \lambda_0}{b_0} \left( 1 - \frac{\eta C \|H\|_\infty}{b_0} \right) \right)$, $C_1$ and $C$ are some constant of order 1 (see Table 4 in Appendix E for details).

For the base case $k' = 0$, by definition Condition 4.1 holds. Suppose for $k' = 0, \ldots, k$, Condition 4.1 holds and we want to show it still holds for $k' = k + 1$. We first prove the order of $m$ and then the contraction of $\|u(k + 1) - y\|$.
For the order of \(m\) that controls the strict positiveness of \(\{H(k')\}_{k'=0}^k\) and so the contraction of \(\|u(k+1) - y\|\), we first have that \(m = \Omega\left(\frac{n^2 \log^2(n)}{\lambda_0^2}\right)\) from Lemma 3.1. Further, by Lemma 4.1 with \(T_0 = 1\) for the adaptive stepsize \(b_k > \eta \lambda_0 \|H\|\infty\), we can upper bound \(\|w_{r}(k+1) - w_r(0)\|\) by \(R\). By Lemma 3.3, we ask for \(R \leq R\) for \(k' = 0, 1, \cdots, k + 1\), which results in \(m = \Omega(\frac{n^2}{\lambda_0^2 \delta^2})\).

**Lemma 4.1** Suppose Condition 4.1 holds for \(k' = 1, 2, \cdots, k\), and we have

\[b_k \leq b_{T_0} + \frac{4 \alpha_2 \sqrt{n}}{\eta \lambda_0 C_1} \|y - u(T_0 - 1)\|;
\]

\[\|w_r(k + T_0) - w_r(T_0 - 1)\| \leq \frac{4 \sqrt{n}}{\sqrt{m \lambda_0 C_1}} \|y - u(T_0 - 1)\| \triangleq \tilde{R}.
\]

Now given the strictly positiveness of \(\{H(k')\}_{k'=0}^k\) such that \(\lambda_{\min}(H(k')) > \frac{\eta}{\lambda_0}\), we will prove (21) at iteration \(k + 1\). We follow the same argument as Subsection B.2, and straightforwardly modify the constant learning rate \(\eta\) for the adaptive learning rate \(\eta / b_{k+1}\). Observe the key inequality (20) in Subsection B.2 that expresses the gradient descent with constant learning rate \(\eta\) in Theorem 3.1 and have

\[\|y - u(k + 1)\|^2 \leq \left(1 - \frac{\eta \lambda_0 C_1}{b_{k+1}} \left(1 - \frac{\eta}{b_{k+1}} C_{\|H_0\|}\right)\right) \|y - u(k)\|^2 \tag{22}
\]

which is Condition 4.1 for \(k' = k + 1\).

Now that we have proved Condition 4.1 for all \(k = 0, 1, 2, \cdots\), when \(b_0 / \eta \geq C \|H\|\infty\). We use Lemma 4.1 again to bound \(b_k\) denoted by \(b_\infty\):

\[b_\infty \leq b_0 + \frac{4 \alpha_2 \sqrt{n}}{\eta \lambda_0 C_1} \|y - u(0)\| = O\left(b_0 + \frac{\alpha^2 n}{\eta \lambda_0}\right)
\]

where the equality is from Proposition 2.2. Thus, iteratively substituting \(\|y - u(t)\|^2, t = k - 1, k - 1, \cdots, 0\) in inequality (22), we have

\[\|y - u(T)\|^2 \leq \prod_{t=0}^{T-1} \left(1 - \frac{\eta \lambda_0 C_1}{2 b_t}\right) \|y - u(0)\|^2 \leq \exp\left(-T \frac{\eta \lambda_0 C_1}{2 b_\infty}\right) \|y - u(0)\|^2.
\]

For tolerance error \(\varepsilon\) such that \(\|y - u(T)\|^2 \leq \varepsilon\), we get the maximum step by plugging the upper bound \(b_\infty\) into above inequality.

**C.2** **Part II**

Starting with \(b_0 / \eta < C \|H\|\infty\), we use Lemma 2.1 with \(\gamma = \alpha^2 \sqrt{n}\), \(a_1 = \|y - u(k)\|\) and \(L = (\eta C \|H\|\infty\|^2)^2\) to prove that eventually after step

\[T_0 = \left[\left(\frac{(\eta C \|H\|\infty\|^2 - b_0^2)}{\alpha^2 \sqrt{n \varepsilon}}\right)\right] + 1, \tag{23}
\]

\(\text{For more descriptions of the relationship between } m \text{ and } \|u(k + T_0) - y\|, \text{ we refer to Subsection B.1.} \text{}}
we have

\[ \text{either } \min_{t \in [T_0]} \| y - u(t) \|^2 \leq \varepsilon, \quad \text{or} \quad b_{T_0} \geq \eta C \| H^\infty \|. \]  \hfill (24)

**Lemma E.1** Fix \( \varepsilon \in (0, 1] \), \( L > 0 \), \( \gamma > 0 \). For any non-negative \( a_0, a_1, \ldots \), the dynamical system

\[ b_0 > 0; \quad b_{j+1}^2 = b_j^2 + 4a_j \]

has the property that after \( N = \lfloor \frac{L^2 - b_0^2}{\gamma \sqrt{\varepsilon}} \rfloor + 1 \) iterations, either \( \min_{k=0:N-1} a_k \leq \sqrt{\varepsilon} \), or \( b_N \geq L \).

*Now similar to Part I, we first use induction to prove Condition 4.1 with \( m \) satisfying*

\[ m = \Omega \left( \frac{4n^5 \lambda^4 \delta^2}{\lambda^4 \delta^2} \left( 2\| y - u(0) \|^2 + \frac{\eta^4 C^2 \lambda_0 \| H^\infty \|^2 \left( 4 \sqrt{C \| H^\infty \| + \lambda_0} \right)^2}{2\alpha^4 n} \right) \right). \]  \hfill (25)

*Note that since \( T_0 > 1 \), we will first prove the induction before \( k \leq T_0 - 1 \) and then \( k \geq T_0 \). Based on this condition, we then obtain the upper bound of \( b_k \) and so the convergence result.*

For \( k' = 0 \), by definition Condition 4.1 holds. Suppose for \( k' = 0, 1, \ldots, k \leq T_0 - 2 \), Condition 4.1 holds and we want to show it holds for \( k' = k + 1 \leq T_0 - 1 \). Similar to Part I, we use Lemma B.3 in order to maintain the strict positiveness of \( H(k) \) for \( k' = k + 1 \leq T_0 - 1 \). That is, we ask for \( k' = 0, 1, \ldots, k + 1 \) such that \( \| w_r(k') - w_r(0) \| \leq R = \frac{\sqrt{\lambda_0 \delta}}{n \sqrt{m}} \). From Lemma 4.1 we know that the upper bound of the distance \( \| w_r(t) - w_r(0) \| \) for \( t \leq T_0 - 1 \) grows only up to a finite number proportional to \( \eta C \| H^\infty \| \).

**Lemma 4.2** Let \( L = \eta C \| H^\infty \| \) and \( T_0 \geq 1 \) be the first index such that \( b_{T_0} \geq L \). Then for every \( r \in [m] \), we have \( k \leq T_0 - 2 \)

\[ \| w_r(k + 1) - w_r(0) \| \leq \frac{2\eta^2 C \| H^\infty \|}{\alpha^2 \sqrt{\gamma}} \triangleq \tilde{R}. \]

Thus, we have the strict positiveness of \( \{ H(k') \}_{k=0}^{k'} \), \( k \leq T_0 - 1 \) such that \( \lambda_{\min} (H(k')) > \frac{\lambda_0}{2} \), as long as \( \tilde{R} \leq \frac{\sqrt{\lambda_0 \delta}}{n \sqrt{m}} \) holds. But that is guaranteed for large \( m \) satisfying (25). Again, use the same argument as the derivation of inequality (22), we have Condition 4.1 holds for \( k' = k + 1 \). Thus we prove Condition 4.1 for \( k' = 0, 1, \ldots, T_0 - 1 \).

Now we are at \( k' = T_0 \). We have no clue at iteration \( k' = T_0 \) since there is no contraction bound for \( \| y - u(k') \|^2 \), \( k' = 0, 1, k \leq T_0 - 1 \). However, we have (24) at \( k' = T_0 \).

If we are lucky to have

\[ \min_{t \in [T_0]} \| y - u(t) \|^2 \leq \varepsilon, \]

then we are done. Otherwise, we have \( b_{T_0} \geq \eta C \| H^\infty \| \). We will continue to prove the Condition 4.1 for \( k' = T_0, T_0 + 1, \ldots \).

For the case \( k' = T_0 \), we need to prove the strictly positive \( H(T_0) \) given Condition 4.1 holds for \( k' = 0, 1, \ldots, T_0 - 1 \). At this time, the upper bound of \( \| y - u(T_0 - 1) \| \) only grows up to a factor of \( \eta C \| H^\infty \| \) as stated in following lemma,

**Lemma C.1** Let \( T_0 \geq 1 \) be the first index such that \( b_{T_0} \geq \eta C \| H^\infty \| \). Suppose Condition 4.1 holds for \( k = 0, 1, \ldots, k \). Then

\[ \| y - u(T_0 - 1) \| \leq \| y - u(0) \| + \frac{2\eta^2 \sqrt{\lambda_0} (C \| H^\infty \|)^{3/2}}{\alpha^2 \sqrt{\gamma}}. \]
Then the distance between $w_r(T_0)$ and $w_r(0)$ is
\[
\|w_r(T_0) - w_r(0)\| \leq \|w_r(T_0) - w_r(T_0 - 1)\| + \|w_r(T_0 - 1) - w_r(0)\|
\]
\[
\leq \frac{\eta}{b_{T_0}} \|\partial L(W(T_0 - 1)) / \partial w_r\| + \frac{2\eta^2 C\|H^\infty\|}{\alpha^2 \sqrt{m}}
\]
\[
\leq \frac{\sqrt{n} ||u(T_0 - 1)||}{C\|H^\infty\| \sqrt{m}} + \frac{2\eta^2 C\|H^\infty\|}{\alpha^2 \sqrt{m}}
\]
\[
\leq \frac{1}{\sqrt{m}} \left( \frac{\sqrt{n} ||u(T_0 - 1)||}{C\|H^\infty\|} + \frac{\eta^2}{\alpha^2} \left( \frac{\sqrt{\lambda_0 C\|H^\infty\|} + C\|H^\infty\|}{2} \right) \right)
\]
\[
\leq \frac{c\lambda_0 \delta}{n^2}
\]
where the last inequality is due to large $m$ satisfying equation (25). Thus Lemma B.3 implies the strict positive for $H(T_0)$ such that $\lambda_{\min}(H(T_0)) \geq \frac{b_{T_0}}{2}$ > 0 and so Condition 4.1 holds for $k = T_0$.

Now suppose for $k' = 0, \ldots, T_0 - 1, T_0, \ldots, k + T_0 - 1$, Condition 4.1 holds and we want to show it holds for $k' = k + T_0$. The bound $\|w_r(k + T_0) - w_r(0)\|$ can be obtained by using Lemma 4.1, Lemma 1.2
\[
\|w_r(k + T_0) - w_r(0)\| \leq \|w_r(k + T_0) - w_r(T_0 - 1)\| + \|w_r(T_0 - 1) - w_r(0)\|
\]
\[
\leq \frac{4\sqrt{n}}{\sqrt{m\lambda_0 C_1}} \left( ||y - u(T_0 - 1)|| + \frac{\eta^2 \lambda_0 C\|H^\infty\|}{2\alpha^2 \sqrt{n}} \right).
\]
Putting back the upper bound of $||y - u(0)||$ given in Lemma C.1 and using Lemma B.3 that ask for $\|w_r(k + T_0) - w_r(0)\| \leq R = \frac{c\lambda_0 \delta}{n^2}$, we require following
\[
\frac{c\lambda_0 \delta}{n^2} \geq \frac{4\sqrt{n}}{\sqrt{m\lambda_0 C_1}} \left( ||y - u(0)|| + \frac{\eta^2 \lambda_0 C\|H^\infty\|}{2\alpha^2 \sqrt{n}} \right).
\]
Rearranging $m$ to one side, we get (25). Given the strictly positive $\{H(k')\}_{k'=0}^{k+T_0}$, we have Condition 4.1 holds for $k' = k + T_0$ by the same argument as the derivation of inequality (22). Thus we prove Condition 4.1 for $k' = T_0, T_0 + 1, \ldots$.

Now that we prove Condition 4.1 for $k' = 0, \ldots, T_0 - 1, T_0, T_0 + 1, \ldots$, we use Lemma 4.1 and Lemma C.1 to bound $b_k$ denoted by $\bar{b}_\infty$:
\[
\bar{b}_\infty = \eta C\|H^\infty\| + \frac{4\alpha^2 \sqrt{n}}{\eta \lambda_0 C_1} \left( ||y - u(0)|| + \frac{2\eta^2 \sqrt{\lambda_0 (C\|H^\infty\|)^{3/2}}}{\alpha^2 \sqrt{n}} \right).
\]
Thus, using the fact that $b_{T_0}/\eta \geq C\|H^\infty\|$ with $T_0 = \frac{\eta C\|H^\infty\|^{2} - \bar{b}_\infty}{\alpha^2 \sqrt{n}}$ and iteratively substituting $\|y - u(t)\|^2, t = T - 1, T - 2, \ldots, T_0 - 1$ in inequality (22) gives
\[
\|y - u(T)\|^2 \leq \Pi_{t=T_0}^{T} \left( 1 - \frac{\eta \lambda_0 C_1}{2 b_t} \right) \|y - u(T_0 - 1)\|^2
\]
\[
\leq \exp \left( -(T - T_0 + 1) \frac{\eta \lambda_0 C_1}{2 \bar{b}_\infty} \right) \left( ||y - u(0)|| + \frac{2\eta^2 (C\|H^\infty\|)^{3/2}}{\alpha^2 \sqrt{\lambda_0}} \right)^2.
\]
For the tolerance $\varepsilon$, the maximum step $T$ can be derived by plugging the upper bound $\bar{b}_\infty$ into above inequality.
C.3 Proof of Lemmas

Proof of Lemma 4.1 and 4.2 are given in subsection 4.1.1

Proof of Lemma C.1

\[ \| y - u(T_0 - 1) \| \leq \sqrt{1 + \frac{\eta^2 \lambda_0 C \| H^\infty \|}{b_{T_0 - 1}^2}} \| y - u(T_0 - 2) \| \]

\[ \leq \| y - u(T_0 - 2) \| + \frac{\eta \sqrt{\lambda_0 C \| H^\infty \|}}{b_{T_0 - 1}} \| y - u(T_0 - 2) \| \]

\[ \leq \| y - u(0) \| + \frac{\eta \lambda_0 C \| H^\infty \|}{\alpha^2 \sqrt{n}} \sum_{t=0}^{T_0 - 2} \alpha^2 \sqrt{n} \| y - u(t) \| + b_0^2 \]

\[ \leq \| y - u(0) \| + \frac{2\eta \sqrt{\lambda_0 C \| H^\infty \|}}{\alpha^2 \sqrt{n}} \sqrt{\alpha^2 \sqrt{n} \sum_{t=0}^{T_0 - 2} \| y - u(\ell) \| + b_0^2} \]

\[ \leq \| y - u(0) \| + \frac{2\eta^2 \sqrt{\lambda_0 C \| H^\infty \|}^{3/2}}{\alpha^2 \sqrt{n}} \]

D Proof of Propositions

**Proposition 5.1** If \( \lambda_{\min}(H) \geq \frac{\lambda_0}{2} \), then \( \| y - u \| \leq \frac{\sqrt{2m}}{\sqrt{\lambda_0}} \max_{r \in [m]} \| \frac{\partial L(W)}{\partial w_r} \| \).

**Proof.** For \( a_r \sim \text{unif}(\{-1, 1\}) \), we have

\[
\max_{r \in [m]} \| \frac{\partial L(W)}{\partial w_r} \|^2 = \frac{1}{m} \max_{r \in [m]} \| \sum_{i=1}^{n} (y_i - u_i) a_r x_i \mathbb{I} \{ w_r x_i \geq 0 \} \|^2 \\
= \frac{1}{m} \max_{r \in [m]} \left( \sum_{i,j} (u_i - y_i)(u_j - y_j) \langle x_i, x_j \rangle \mathbb{I} \{ w_r x_i \geq 0, w_r x_j \geq 0 \} \right) \\
\geq \frac{1}{m} \left( \sum_{i,j} (u_i - y_i)(u_j - y_j) \langle x_i, x_j \rangle \frac{1}{m} \sum_{r=1}^{m} \mathbb{I} \{ w_r x_i \geq 0, w_r x_j \geq 0 \} \right) \\
= \frac{1}{m} (u - y)^\top H (u - y) \\
\geq \frac{\lambda_0}{2m} \| u - y \|^2
\]

where the last inequality use the condition that \( \lambda_{\min}(H) \geq \frac{\lambda_0}{2} \). \( \square \)

**Proposition 5.2** Let \( L = \eta C \| H^\infty \| \) and \( T_0 \geq 1 \) be the first index such that \( b_{T_0} \geq L \). Consider the update: \( b_{k+1}^2 = b_k^2 + \alpha^2 \sqrt{n} \| y - u(k) \|^2 \). Then for every \( r \in [m] \), we have

\[
\| w_r(k + 1) - w_r(0) \|_2 \leq \frac{\eta \sqrt{2(k + 1)}}{\alpha^2 \sqrt{m}} \sqrt{1 + 2 \log \left( \frac{C \eta \| H^\infty \|}{b_0} \right)}
\]

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Proof. For the upper bound of $\|w_r(k+1) - w_r(0)\|_2$ when $b_t/\eta < C\|H_\infty\|$, $t = 0, 1, \ldots, k$ and $k \leq T_0 - 2$, we have

$$\sum_{t=0}^k \frac{\|y - u(t)\|_2^2}{b_{t+1}^2} \leq \frac{1}{\alpha^2 n} \sum_{t=0}^k \frac{\alpha^2 \sqrt{n} \|y - u(t)\|_2^2 / b_0^2 + 1}{\alpha^2 \sqrt{n} \sum_{\ell=0}^k \|y - u(\ell)\|_2^2 / b_0^2 + 1}
$$

$$\leq \frac{1}{\alpha^2 n} \left( 1 + \log \left( \frac{\alpha^2 \sqrt{n} \sum_{t=0}^k \|y - u(t)\|_2^2 / b_0^2 + 1}{1 + 2 \log \left( \frac{C\eta\|H_\infty\|}{b_0} \right)} \right) \right)
$$

$$\leq \frac{1}{\alpha^2 n} (1 + 2 \log \left( \frac{b_{T_0-1}/b_0}{b_0} \right))$$

where the second inequality uses Lemma 6 in Ward et al. [2018]. Thus

$$\|w_r(k+1) - w_r(0)\|_2 \leq \eta \sqrt{n} \sqrt{\frac{1}{m}} \frac{\sqrt{1 + 2 \log \left( \frac{C\eta\|H_\infty\|}{b_0} \right)}}{n^2} \sqrt{1 + \log \left( \frac{b_{T_0-1}/b_0}{b_0} \right)}$$

E Technical Lemmas

Lemma E.1. Fix $\varepsilon \in (0, 1]$, $L > 0$, $\gamma > 0$. For any non-negative $a_0, a_1, \ldots$, the dynamical system

$$b_0 > 0; \quad b_{j+1}^2 = b_j^2 + \gamma a_j$$

has the property that after $N = \left\lceil \frac{T^2 - b_0^2}{\gamma \varepsilon^2} \right\rceil + 1$ iterations, either $\min_{k=0:N-1} a_k \leq \sqrt{\varepsilon}$, or $b_N \geq L$.

Lemma E.2. For any non-negative $a_1, \ldots, a_T$, such that $a_1 > 0$,

$$\sum_{\ell=1}^T \frac{a_\ell}{\sqrt{\sum_{i=1}^\ell a_i}} \leq 2 \sqrt{\sum_{i=1}^T a_i}$$

Since the above two lemmas correspond to Lemma 7 and Lemma 8 in Ward et al. [2018], we omit their proofs.

Proposition E.1. Under Assumption 2.1 and Assumption 2.2, then

$$\max_{r \in [m]} \left\| \frac{\partial L(W)}{\partial w_r} \right\|_2 \leq \frac{\sqrt{n}}{\sqrt{m}} \|y - u\|_2.$$ 

The proof is straightforward as follows

$$\max_{r \in [m]} \left\| \frac{\partial L(W)}{\partial w_r} \right\| \leq \frac{1}{\sqrt{m}} \sqrt{\sum_{i=1}^n |y_i - u_i|^2} \sqrt{\sum_{i=1}^n \|x_i\|^2} \leq \frac{\sqrt{n}}{\sqrt{m}} \|y - u\|_2$$

Observe that at initialization, we have following proposition
Proposition E.2. Under Assumption 2.1 and 2.2, with probability $1 - \delta$ over the random initialization,

$$\|\mathbf{y} - \mathbf{u}(0)\|^2 \leq \frac{n}{\delta}.$$  

We get above statement by Markov’s Inequality with following

$$\mathbb{E} \left[ \|\mathbf{y} - \mathbf{u}(0)\|^2 \right] = \sum_{i=1}^{n} (y_i^2 + 2y_i\mathbb{E}[f(\mathbf{W}(0), \mathbf{a}, \mathbf{x}_i)]) + \mathbb{E}[f^2(\mathbf{W}(0), \mathbf{a}, \mathbf{x}_i)]) = \sum_{i=1}^{n} (y_i^2 + 1) = O(n).$$

Finally, we analyze the upper bound of the maximum eigenvalues of Gram matrix that plays the most crucial role in our analysis. Observe that

$$\|\mathbf{H}^\infty\| = \sup_{\|v\|=1} \sum_{i,j} v_i v_j \langle \mathbf{x}_i, \mathbf{x}_j \rangle \frac{1}{m} \sum_{r=1}^{m} \mathbb{I}\{w_r(0)^\top \mathbf{x}_i \geq 0, w_r(0)^\top \mathbf{x}_j \geq 0\} \leq \sqrt{\sum_{i \neq j} |\langle \mathbf{x}_i, \mathbf{x}_j \rangle|^2 + 1}$$

If the data points are pairwise uncorrelated (orthogonal), i.e., $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0, i \neq j$, then the maximum eigenvalues is close to 1, i.e., $\|\mathbf{H}^\infty\| \leq 1$. In contrast, we could have $\|\mathbf{H}^\infty\| \leq n$ if data points are pairwise highly correlated (parallel), i.e., $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 1, i \neq j$.

Table 1: Some notations of parameters to facilitate understanding the proofs in Appendix B and C

| Expression | Order | First Appear |
|------------|-------|--------------|
| $c$ is a small value, say less than 0.1 | $O(1)$ | Lemma B.3 |
| $R = \frac{c\lambda_0\delta}{n^2}$ | $O\left(\frac{\lambda_0\delta}{n^2}\right)$ | Lemma B.3 |
| $R' = \frac{4\sqrt{n}\|\mathbf{y} - \mathbf{u}(0)\|}{\sqrt{m}\lambda_0}$ | $O\left(\frac{n}{\sqrt{m}\lambda_0}\right)$ | Lemma B.2 |
| $C_1 = 1 - 2 \left(\frac{1}{\sqrt{n}} + 1\right) \frac{2^2\eta^2}{\lambda_0\delta}$ | $O(1)$ | Equation (20), Condition 4.1 |
| $C = \frac{1 + 2\eta^2 R}{\sqrt{m}\lambda_0} \frac{\|\mathbf{H}(0)\| + 1}{\sqrt{m}\lambda_0} + 2(\frac{1}{\sqrt{n}} + 1) \frac{\alpha^2 R^2}{\lambda_0\delta^2}$ | $O(1)$ | Equation (20), Condition 4.1 |
| $\bar{R} = \frac{4\sqrt{n}}{\sqrt{m}\lambda_0 C_1} \|\mathbf{y} - \mathbf{u}(0)\| | O\left(\frac{n}{\sqrt{m}\lambda_0}\right)$ | Lemma 4.1 |
| $\hat{R} = \frac{2\eta^2 C^2 \|\mathbf{H}^\infty\|}{\alpha^2 \sqrt{m}} | O\left(\frac{\eta^2 \|\mathbf{H}^\infty\|}{\alpha^2 \sqrt{m}}\right)$ | Lemma 4.2 |