THE LOCAL WELL-POSEDNESS FOR
GROSS-PITAEVSKII HIERARCHIES

ZEQIAN CHEN

ABSTRACT. We consider the Cauchy problem for the Gross-Pitaevskii
infinite linear hierarchy of equations on $\mathbb{R}^n$. By introducing a quasi-norm
in certain Sobolev type spaces of sequences of marginal density matrices,
we establish local existence, uniqueness and stability of solutions. This
quasi-norm is compatible with the usual Sobolev space norm when the
initial data is factorized. Explicit space-time type estimates for the
solutions are obtained. The results hold without the assumption of
factorized initial conditions.

1. INTRODUCTION

We consider the Cauchy problem (initial value problem) for the Gross-
Pitaevskii infinite linear hierarchy of equations on $\mathbb{R}^n$, of the form

\begin{align}
\left\{ \begin{array}{l}
(i\partial_t + \triangle^{(k)}_{\pm}) \gamma_t^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \mu \left[ B^{(k)}(\gamma_t^{(k+1)}) \right](\mathbf{x}_k; \mathbf{x}'_k), \\
\gamma_{t=0}^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \gamma_0^{(k)}(\mathbf{x}_k; \mathbf{x}'_k), \quad k = 1, 2, \ldots,
\end{array} \right.
\end{align}

where $t \in \mathbb{R}, \mathbf{x}_k = (x_1, x_2, \ldots, x_k), \mathbf{x}'_k = (x'_1, x'_2, \ldots, x'_k) \in \mathbb{R}^{kn}, \mu = \pm 1$, and
the sequence of functions $\Gamma(t) = \{ \gamma_t^{(k)} \}_{k \geq 1}$ is referred as a Gross-Pitaevskii
hierarchy (GP hierarchy), which are symmetric, in the sense that

$$\gamma_t^{(k)}(\mathbf{x}_k, \mathbf{x}'_k) = \gamma_t^{(k)}(\mathbf{x}'_k, \mathbf{x}_k)$$

and

$$\gamma_t^{(k)}(x_1, \ldots, x_k; x'_1, \ldots, x'_k) = \gamma_t^{(k)}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x'_{\sigma(1)}, \ldots, x'_{\sigma(k)})$$

for any $\sigma \in \Pi_k$ ($\Pi_k$ denotes the set of permutations on $k$ elements). Here,

$$\Delta^{(k)}_{\pm} = \sum_{j=1}^{k} (\Delta_{x_j} - \Delta_{x'_j}) \quad \text{and} \quad B^{(k)} = \sum_{j=1}^{k} B_{j,k},$$

2010 Mathematics Subject Classification: 35Q55, 81V70.

Key words: Gross-Pitaevskii hierarchy, nonlinear Schrodinger equation, Cauchy problem, local well-posedness.
where $\Delta_{x_j}$ refers to the usual Laplace operator with respect to the variables $x_j \in \mathbb{R}^n$ and the operators $B_{j,k} = B_{j,k}^1 - B_{j,k}^2$ are defined according to

$$[B_{j,k}^1(\gamma^{(k+1)})](x_k, x'_k) = \int dx_{k+1} dx'_{k+1} \delta(x_{k+1} - x'_{k+1}) \delta(x_j - x_{k+1}) \gamma^{(k+1)}(x_{k+1}, x'_{k+1}),$$

and

$$[B_{j,k}^2(\gamma^{(k+1)})](x_k, x'_k) = \int dx_{k+1} dx'_{k+1} \delta(x_{k+1} - x'_{k+1}) \delta(x'_j - x_{k+1}) \gamma^{(k+1)}(x_{k+1}, x'_{k+1}).$$

We refer to the works of Erdős, Schlein and Yau in [7, 8, 9, 10] and references therein for the background of the GP hierarchy (1.1) and some recent fundamental results in this area.

Let $\varphi \in H^1(\mathbb{R}^n)$, then one can easily verify that a particular solution to (1.1) with factorized initial datum

$$\gamma^{(k)}_{t=0}(x_k; x'_k) = \prod_{j=1}^k \varphi(x_j) \overline{\varphi(x'_j)}, \quad k = 1, 2, \ldots,$$

is given by

$$\gamma^{(k)}_{t}(x_k; x'_k) = \prod_{j=1}^k \varphi_t(x_j) \overline{\varphi_t(x'_j)}, \quad k = 1, 2, \ldots,$$

where $\varphi_t$ satisfies the cubic non-linear Schrödinger equation

$$i \partial_t \varphi_t = -\Delta \varphi_t + \mu |\varphi_t|^2 \varphi_t, \quad \varphi_{t=0} = \varphi,$$

which is defocusing if $\mu = 1$, and focusing if $\mu = -1$. We refer to [1] and references therein for the nonlinear Schrödinger equation.

Recently, T.Chen and N.Pavlović [2] started to investigate the Cauchy problem for the GP hierarchy (1.1) without the assumption of factorized initial conditions. By introducing certain Sobolev type spaces $\mathcal{H}_\xi^\alpha$ of sequences of marginal density matrices with the parameter $0 < \xi < 1$ (see Remark 2.3 below), they prove local existence and uniqueness of solutions. However, there appear two different parameters in their results, that is, for the initial data in $\mathcal{H}_\xi^\alpha$ the solution lies in $\mathcal{H}_{\xi_2}^\alpha$ for some $0 < \xi_2 < \xi_1$. Another drawback of $\mathcal{H}_\xi^\alpha$ is that the norm $\| \cdot \|_{\mathcal{H}_\xi^\alpha}$ is not compatible with the Sobolev norm of $H^\alpha$ for factorized hierarchies $\Gamma = \{\gamma^{(k)}\}_{k\geq 1}$ with

$$\gamma^{(k)}(x_k; x'_k) = \prod_{j=1}^k \varphi(x_j) \overline{\varphi(x'_j)}, \quad k = 1, 2, \ldots, \quad \varphi \in H^\alpha(\mathbb{R}^n).$$

This means that the Cauchy problem (1.1) in $\mathcal{H}_\xi^\alpha$ is not equivalent to the one (1.2) in $H^\alpha$ when the initial condition is factorized.
In this paper we will eliminate these two undesirable issues. As a crucial ingredient of our arguments, we will define a quasi-norm in certain Sobolev type spaces of sequences of marginal density matrices, i.e., we introduce a quasi-Banach space $H^\alpha$ of all hierarchies $\Gamma = \{\gamma(k)\}_{k \geq 1} \in \bigotimes_{k=1}^\infty H^\alpha_k$ such that
\[
\sum_{k=1}^\infty \frac{1}{\lambda^k} \|\gamma(k)\|_{H^\alpha_k} < \infty \quad \text{for some } \lambda > 0,
\]
where $H^\alpha_k = H^\alpha(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$, equipped with the quasi-norm
\[
\|\Gamma\|_{H^\alpha} := \frac{1}{2} \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty \frac{1}{\lambda^k} \|\gamma(k)\|_{H^\alpha_k} \leq 1 \right\}.
\]

Our main result in this paper is that the Cauchy problem for the GP hierarchy (1.1) is locally well posed in $H^\alpha$ for $\alpha > \max\{1/2, (n-1)/2\}$. On the other hand, one can easily verify that $\|\Gamma\|_{H^\alpha} = \|\varphi\|_{H^\alpha}$ for all factorized hierarchies $\Gamma$ of the form (1.3), i.e., the quasi-norm $\|\cdot\|_{H^\alpha}$ is compatible with $\|\cdot\|_{H^\alpha}$ for any factorized hierarchy and then the Cauchy problem (1.1) in $H^\alpha$ is equivalent to the one (1.2) in $H^\alpha$ whenever the initial condition is factorized. This explains the reason why we introduce the parameter-free space $H^\alpha$ in place of $H^\alpha_\xi$ mentioned above.

As in [2], we refer to (1.1) as the cubic GP hierarchy. We note that the cubic GP hierarchy accounts for 2-body interactions between the Bose particles (see [6] for details). For $\mu = 1$ or $\mu = -1$ we refer to the corresponding GP hierarchies as being defocusing or focusing, respectively.

The paper is organized as follows. In Section 2, some notations and the main results are presented. The main results are then proved in Sections 3 and 4. Our proof involves some natural and subtle techniques beyond the usual Picard-type fixed point argument. Finally, in Section 5, we will extend the result obtained for the cubic GP hierarchy to the so-called quintic one.

2. Preliminaries and statement of the main result

In order to state our main results, we require some more notation. We usually denote by $x$ a general variable in $\mathbb{R}^n$ and by $x_k = (x_1, \ldots, x_k)$ a point in $\mathbb{R}^{kn}$. We always use $\gamma(k), \rho(k)$ for denoting symmetric functions in $\mathbb{R}^{kn} \times \mathbb{R}^{kn}$. For $k \geq 1$ and $\alpha \in \mathbb{R}$, we denote by $H^\alpha_k = H^\alpha(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$ the space of measurable functions $\gamma(k) = \gamma(k)(x_k, x'_k)$ in $L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$, which are symmetric, such that
\[
\|\gamma(k)\|_{H^\alpha_k} := \|S^{(k,\alpha)}\gamma(k)\|_{L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})} < \infty,
\]
where
\[
S^{(k,\alpha)} := \prod_{j=1}^k [(1 - \Delta_{x_j})^{\frac{\alpha}{2}} (1 - \Delta_{x'_j})^{\frac{\alpha}{2}}].
\]
Evidently, $H_k^\alpha$ is a Hilbert space with the inner product
\[ \langle \gamma^{(k)}, \rho^{(k)} \rangle = \langle S^{(k,\alpha)} \gamma^{(k)}, S^{(k,\alpha)} \rho^{(k)} \rangle_{L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})}. \]

Moreover, the norm $\| \cdot \|_{H_k^\alpha}$ is invariance under the action of $e^{i t \Delta^{(k)}}$, i.e.,
\[ \| e^{i t \Delta^{(k)}} \gamma^{(k)} \|_{H_k^\alpha} = \| \gamma^{(k)} \|_{H_k^\alpha} \]
because $e^{i t \Delta^{(k)}}$ commutates with $\Delta_{i,j}$ for any $j$.

**Definition 2.1.** For $\alpha \in \mathbb{R}$ we define
\[ H^\alpha = \left\{ \{ \gamma^{(k)} \}_{k \geq 1} \in \bigotimes_{k=1}^\infty H_k^\alpha : \sum_{k=1}^\infty \frac{1}{\lambda^k} \| \gamma^{(k)} \|_{H_k^\alpha} < \infty \text{ for some } \lambda > 0 \right\}, \]

equipped with the quasi-norm
\[ \left\| \{ \gamma^{(k)} \}_{k \geq 1} \right\|_{H^\alpha} := \frac{1}{2} \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty \frac{1}{\lambda^k} \| \gamma^{(k)} \|_{H_k^\alpha} \leq 1 \right\}. \]

**Remark 2.1.** It is easy to check that $\| \cdot \|_{H^\alpha}$ satisfies the triangle inequality and $H^\alpha$ is only a quasi-Banach space because $\| \lambda \Gamma \|_{H^\alpha} = |\lambda| \| \Gamma \|_{H^\alpha}$ does not hold in general.

We will prove the local existence, uniqueness, and stability of solutions to the GP hierarchy (1.1) in $H^\alpha$ for $\alpha > \max\{1/2, (n-1)/2\}$. In integral formulation, (1.1) can be written as
\[ \gamma^{(k)}_t = e^{i t \Delta^{(k)}} \gamma^{(k)}_0 + \int_0^t ds \, e^{i (t-s) \Delta^{(k)}} \hat{B}^{(k)} \gamma^{(k+1)}_{s}, \quad k = 1, 2, \ldots, \]
whereafter $\hat{B}^{(k)} = -i \mu B^{(k)}$. As noted in [2], such a solution can be obtained by solving the following infinite linear hierarchy of integral equations
\[ \hat{B}^{(k)} \gamma^{(k+1)}_t = \hat{B}^{(k)} e^{i t \Delta^{(k+1)}} \gamma^{(k+1)}_0 + \int_0^t ds \, \hat{B}^{(k)} e^{i (t-s) \Delta^{(k+1)}} \hat{B}^{(k+1)} \gamma^{(k+2)}_{s}, \]
for any $k \geq 1$. If we write
\[ \hat{\Delta} \Gamma := \{ \Delta^{(k)} \gamma^{(k)} \}_{k \geq 1} \quad \text{and} \quad \hat{\Gamma} := \{ \hat{B}^{(k)} \gamma^{(k+1)} \}_{k \geq 1}, \]
then (2.2) and (2.3) can be written as
\[ \Gamma(t) = e^{i t \hat{\Delta} \Gamma_0} + \int_0^t ds \, e^{i (t-s) \hat{\Delta} \hat{\Gamma}}(s) \]
and
\[ \hat{\Gamma}(t) = \hat{B} e^{i t \hat{\Delta} \Gamma_0} + \int_0^t ds \, \hat{B} e^{i (t-s) \hat{\Delta} \hat{\Gamma}}(s), \]
respectively.

Let us make the notion of solution more precise.
\textbf{Definition 2.2.} For $T > 0$, $\Gamma(t) = \{\gamma^{(k)}_t\}_{k \geq 1} \in C([0, T], H^\alpha)$ is said to be a local (strong) solution to the Gross-Pitaevskii hierarchy (1.1) if for every $k = 1, 2, \ldots$,

\begin{equation}
\gamma^{(k)}_t = e^{it\Delta^{(k)}} \gamma^{(k)}_0 + \int_0^t ds \ e^{i(t-s)\Delta^{(k)}} B^{(k)} \gamma^{(k+1)}_s, \quad \forall t \in [0, T],
\end{equation}

holds in $H^\alpha_k$.

\textbf{Remark 2.2.} Let $\varphi \in H^\alpha(\mathbb{R}^n)$ and set for $k \geq 1$,

$$\gamma^{(k)}(x_k; x'_k) = \prod_{j=1}^k \varphi(x_j) \varphi(x'_j).$$

An immediate computation yields that

$$\|\{\gamma^{(k)}\}_{k \geq 1}\|_{H^\alpha} = \|\varphi\|^2_{H^\alpha(\mathbb{R}^n)}.$$

Thus, for $T > 0$, $\varphi_t \in C([0, T], H^\alpha)$ is a solution to (1.2) with the initial value $\varphi_{t=0} = \varphi$ if and only if

$$\Gamma(t) = \{\gamma^{(k)}_t\}_{k \geq 1} \text{ with } \gamma^{(k)}_t(x_k; x'_k) = \prod_{j=1}^k \varphi_t(x_j) \varphi_t(x'_j)$$

is a solution to (1.1) in $C([0, T], H^\alpha)$ with $\Gamma(t)|_{t=0} = \{\gamma^{(k)}\}_{k \geq 1}$. This yields that the Cauchy problem (1.1) in $H^\alpha$ is equivalent to the one (1.2) in $H^\alpha$ whenever initial conditions are factorized.

The following is one of the main results in this paper.

\textbf{Theorem 2.1.} Assume that $n \geq 1$ and $\alpha > n/2$. The Cauchy problem (1.1) is locally well posed. More precisely, there exists a constant $C_{n, \alpha} > 0$ depending only on $n$ and $\alpha$ such that

(i) For every $\Gamma_0 = \{\gamma^{(k)}_0\}_{k \geq 1} \in H^\alpha$ with $T = 1/[4C_{n, \alpha}\|\Gamma_0\|_{H^\alpha}]$, there exists a solution $\Gamma(t) = \{\gamma^{(k)}_t\}_{k \geq 1} \in C([0, T], H^\alpha)$ to the Gross-Pitaevskii hierarchy (1.1) with the initial data $\Gamma_0$ satisfying

\begin{equation}
\|\Gamma(t)\|_{C([0, T], H^\alpha)} \leq 2\|\Gamma_0\|_{H^\alpha}.
\end{equation}

(ii) Given $T_0 > 0$, if $\Gamma(t)$ and $\Gamma'(t)$ in $C([0, T_0], H^\alpha)$ are two solutions to (1.1) with initial conditions $\Gamma_{t=0} = \Gamma_0$ and $\Gamma'_{t=0} = \Gamma'_0$ in $H^\alpha$ respectively, then

\begin{equation}
\|\Gamma(t) - \Gamma'(t)\|_{C([0, T_0], H^\alpha)} \leq 2\|\Gamma_0 - \Gamma'_0\|_{H^\alpha},
\end{equation}

for $T = 1/[4C_{n, \alpha}\|\Gamma(t) - \Gamma'(t)\|_{C([0, T_0], H^\alpha)}].$

To state another main result, we introduce
Definition 2.3. For $T > 0$, we define $L^1_{t \in [0, T]} \mathcal{H}^\alpha$ to be the space of all strongly measurable functions $\Gamma(t) = \{\gamma_t^{(k)}\}_{k \geq 1}$ on $[0, T]$ with values in $\mathcal{H}^\alpha$ such that
\[
\sum_{k=1}^\infty \frac{1}{\lambda_k} \int_0^T \|\gamma_t^{(k)}\|_{\mathcal{H}_k^\alpha} dt < \infty \quad \text{for some } \lambda > 0,
\]
equipped with the quasi-norm
\[
\|\Gamma(t)\|_{L^1_{t \in [0, T]} \mathcal{H}^\alpha} := \frac{1}{2} \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty \frac{1}{\lambda_k} \int_0^T \|\gamma_t^{(k)}\|_{\mathcal{H}_k^\alpha} dt \leq 1 \right\}.
\]

Evidently, $L^1_{t \in [0, T]} \mathcal{H}^\alpha$ with $\| \cdot \|_{L^1_{t \in [0, T]} \mathcal{H}^\alpha}$ is a quasi-Banach space.

Theorem 2.2. Assume that $n \geq 2$ and $\alpha > (n - 1)/2$. Then, the Cauchy problem for the Gross-Pitaevskii hierarchy (1.1) is locally well posed in $\mathcal{H}^\alpha$. More precisely, there exist an absolute constant $A > 2$ and a constant $C = C_{n, \alpha} > 0$ depending only on $n$ and $\alpha$ such that

1. For every $\Gamma_0 = \{\gamma_0^{(k)}\}_{k \geq 1} \in \mathcal{H}^\alpha$ with $T = 1/[4C_{n, \alpha}A^2\|\Gamma_0\|_{\mathcal{H}^\alpha}^2]$, there exists a solution $\Gamma(t) = \{\gamma_t^{(k)}\}_{k \geq 1} \in C([0, T], \mathcal{H}^\alpha)$ to (1.1) with the initial data $\Gamma(0) = \Gamma_0$ satisfying
\[
\|\hat{B}\Gamma(t)\|_{L^1_{t \in [0, T]} \mathcal{H}^\alpha} \leq 4A\|\Gamma_0\|_{\mathcal{H}^\alpha}.
\]

2. Given $T_0 > 0$, if $\Gamma(t) \in C([0, T_0], \mathcal{H}^\alpha)$, then $\hat{B}\Gamma(t) \in L^1_{t \in [0, T_0]} \mathcal{H}^\alpha$ is a solution to (1.1) with the initial data $\Gamma(0) = \Gamma_0$, then (2.10) holds true provided $T = 1/[4C_{n, \alpha}A^2\max\{\|\hat{B}\Gamma(t)\|_{L^1_{t \in [0, T_0]} \mathcal{H}^\alpha}, \|\Gamma_0\|_{\mathcal{H}^\alpha}^2\}]$.

3. Given $T_0 > 0$, if $\Gamma(t)$ and $\Gamma'(t)$ in $C([0, T_0], \mathcal{H}^\alpha)$ with $\hat{B}\Gamma(t), \hat{B}\Gamma'(t) \in L^1_{t \in [0, T_0]} \mathcal{H}^\alpha$ are two solutions to (1.1) with initial conditions $\Gamma(0) = \Gamma_0$ and $\Gamma'(0) = \Gamma'_0$ in $\mathcal{H}^\alpha$ respectively, then
\[
\|\Gamma(t) - \Gamma'(t)\|_{C([0, T_0], \mathcal{H}^\alpha)} \leq (1 + 4A)\|\Gamma_0 - \Gamma'_0\|_{\mathcal{H}^\alpha},
\]

for $T = 1/[4C_{n, \alpha}A^2\max\{\|\hat{B}[\Gamma(t) - \Gamma'(t)]\|_{L^1_{t \in [0, T_0]} \mathcal{H}^\alpha}, \|\Gamma_0 - \Gamma'_0\|_{\mathcal{H}^\alpha}^2\}]$.

Remark 2.3. Let $0 < \xi < 1$. We set
\[
\mathcal{H}_\xi^\alpha = \left\{ \Gamma = \{\gamma_t^{(k)}\}_{k \geq 1} \in \bigotimes_{k=1}^\infty \mathcal{H}_k^\alpha : \|\Gamma\|_{\mathcal{H}_\xi^\alpha} := \sum_{k=1}^\infty \xi^k \|\gamma_t^{(k)}\|_{\mathcal{H}_k^\alpha} < \infty \right\}.
\]

Equipped with the norm $\| \cdot \|_{\mathcal{H}_\xi^\alpha}$, $\mathcal{H}_\xi^\alpha$ is a Banach space. This space is defined in [2] for studying the initial problem of (1.1). The local well-posedness obtained there states that for initial data $\Gamma_0 \in \mathcal{H}_\xi^\alpha$ with $\xi_1 > 0$, there exists a unique solution $\Gamma(t) \in C([0, T], \mathcal{H}_\xi^\alpha)$ for some $0 < \xi_2 < \xi_1$ and $T > 0$ (see also [5] for some improvements). That is, there are two different parameters $\xi_1, \xi_2$ in their result. On the other hand, the norm $\|\Gamma\|_{\mathcal{H}_\xi^\alpha}$ is not compatible with $\|\varphi\|_{\mathcal{H}^\alpha}$ for factorized hierarchies $\Gamma$ of the form (1.3) and hence the
Cauchy problem (1.1) in \( \mathcal{H}_x^\alpha \) is not equivalent to the one (1.2) in \( \mathcal{H}^\alpha \) when the initial condition is factorized.

Evidently, Theorems 2.1 and 2.2 eliminate these two undesirable issues. This shows that the space \( \mathcal{H}^\alpha \) with the quantity (2.1) seems more suitable for studying the Cauchy problem of the GP hierarchy (1.1).

3. Proof of Theorem 2.1

We begin with the following lemma.

**Lemma 3.1.** Suppose that \( n \geq 1 \) and \( \alpha > \frac{n}{2} \). Then, there exists a constant \( C_{n,\alpha} > 0 \) depending only on \( n \) and \( \alpha \) such that, for any \( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n}) \),

\[
\| B_{j,k} \gamma^{(k+1)} \|_{\mathcal{H}_x^\alpha} \leq C_{n,\alpha} \| \gamma^{(k+1)} \|_{\mathcal{H}_x^{\alpha+1}},
\]

for all \( k \geq 1 \), where \( j = 1, \ldots, k \).

Consequently, \( B^{(k)} \) can be extended to a bounded operator from \( \mathcal{H}_x^\alpha \) to \( \mathcal{H}_x^\alpha \), still denoted by \( B^{(k)} \), satisfying

\[
\| B^{(k)} \gamma^{(k+1)} \|_{\mathcal{H}_x^\alpha} \leq C_{n,\alpha} \| \gamma^{(k+1)} \|_{\mathcal{H}_x^{\alpha+1}},
\]

for all \( \gamma^{(k+1)} \in \mathcal{H}_x^{\alpha+1} \).

This result can be found in [2] and [5]. The proof of Theorem 2.1 is divided into two parts as follows.

**Proof.** (i) Let \( \alpha > n/2 \). Given \( \Gamma_0 = \{ \gamma^{(k)}_0 \}_{k \geq 1} \in \mathcal{H}_x^\alpha \). For \( m \geq 1 \), set

\[
\gamma^{(k)}_{m,t} = e^{it \triangle \gamma^{(k)}_0} B^{(k)} \gamma^{(k+1)}_{m-1,s}, \quad t > 0, k \geq 1,
\]

with the convention \( \gamma^{(k)}_{0,t} = \gamma^{(k)}_0 \). By expansion, for every \( m \geq 1 \) one has

\[
\gamma^{(k)}_{m,t} = e^{it \triangle \gamma^{(k)}_0} \gamma^{(k)}_0 + \sum_{j=1}^{m-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_j-1} dt_j e^{i(t-t_1) \triangle \gamma^{(k)}_0} B^{(k)} \gamma^{(k)}_{m-1,s} + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_{m} e^{i(t-t_1) \triangle \gamma^{(k)}_0} B^{(k)} \gamma^{(k)}_{m-1,s} \]

\[
\vdots \]

\[
\Delta \sum_{j=0}^{m} \gamma^{(k)}_{j,t} \]
with the convention \( t_0 = 0 \). Then, for \( j = 1, \cdots, m - 1 \), by Lemma 3.1 we have

\[
\| \Xi_{j,t}^{(k)} \|_{H_0^\alpha} \leq \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{j-1}} dt_j \left\| e^{i(t-t_1)\frac{\triangle^\alpha}{\gamma_0}} B^{(k)} \cdots \right. \\
\times e^{i(t_{j-1}-t_j)\frac{\triangle^\alpha}{\gamma_0}} B^{(k+j-1)} e^{i t_j \gamma_0^\alpha (k+j)} \left. \right\|_{H_0^\alpha} \\
\leq \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{j-1}} dt_j k \cdots \\
\times (k + j - 1) (C_{n,\alpha})^j \| e^{i t_j \gamma_0^\alpha (k+j)} \|_{H_{k+j}^\alpha} \\
\leq \frac{t^j}{j!} k \cdots (k + j - 1) (C_{n,\alpha})^j \| \gamma_0^\alpha (k+j) \|_{H_{k+j}^\alpha} \\
= \left( \begin{array}{c} k + j - 1 \\ j \end{array} \right) (C_{n,\alpha} t)^j \| \gamma_0^\alpha (k+j) \|_{H_{k+j}^\alpha},
\]

and

\[
\| \Xi_{m,t}^{(k)} \|_{H_0^\alpha} \leq \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{m-1}} dt_m \left\| e^{i(t-t_1)\frac{\triangle^\alpha}{\gamma_0}} B^{(k)} \cdots \right. \\
\times e^{i(t_{m-1}-t_m)\frac{\triangle^\alpha}{\gamma_0}} B^{(k+m-1)} \gamma_0^\alpha (k+m) \left. \right\|_{H_0^\alpha} \\
\leq \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{m-1}} dt_m \left. k \cdots \right. \\
\times (k + m - 1) (C_{n,\alpha})^m \| \gamma_0^\alpha (k+m) \|_{H_{k+m}^\alpha} \\
\leq \frac{t^m}{m!} k \cdots (k + m - 1) (C_{n,\alpha})^m \| \gamma_0^\alpha (k+m) \|_{H_{k+m}^\alpha} \\
= \left( \begin{array}{c} k + m - 1 \\ m \end{array} \right) (C_{n,\alpha} t)^m \| \gamma_0^\alpha (k+m) \|_{H_{k+m}^\alpha}. 
\]

Then, for \( T > 0 \) (\( T \) will be fixed in the sequel) we obtain

\[
\| \gamma_{m,T}^{(k)} \|_{C([0,T],H_0^\alpha)} \leq \sum_{j=0}^{m} \| \Xi_{j,T}^{(k)} \|_{C([0,T],H_0^\alpha)} \\
\leq \sum_{j=0}^{m} \left( \begin{array}{c} k + j - 1 \\ j \end{array} \right) (C_{n,\alpha} T)^j \| \gamma_0^\alpha (k+j) \|_{H_{k+j}^\alpha} 
\]
Hence, for \( \lambda > 0 \) one has
\[
\sum_{k=1}^{\infty} \frac{1}{\lambda^k} \| \gamma^{(k)}_{m,t} \|_{C([0,T],H^\omega_k)} \\
\leq \sum_{j=0}^{m} \sum_{k=1}^{\infty} \frac{1}{\lambda^k} \binom{k + j - 1}{j} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \\
\leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda^k} \binom{k + j - 1}{j} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}}.
\]

By the direct computation, one has
\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda^k} \binom{k + j - 1}{j} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \\
= \sum_{j=0}^{\infty} \sum_{\ell=j+1}^{\infty} \frac{1}{\lambda^{\ell-j}} \binom{\ell - 1}{j} (C_{n,\alpha} T)^\ell \| \gamma^{(\ell)}_0 \|_{H^\omega_{\ell}} \\
= \sum_{\ell=1}^{\infty} \sum_{j=0}^{\ell-1} \binom{\ell - 1}{j} (C_{n,\alpha} T \lambda)^j \frac{1}{\lambda^{\ell}} \| \gamma^{(\ell)}_0 \|_{H^\omega_{\ell}} \\
= \sum_{\ell=1}^{\infty} (1 + C_{n,\alpha} T \lambda)^{\ell-1} \frac{1}{\lambda^{\ell}} \| \gamma^{(\ell)}_0 \|_{H^\omega_{\ell}} \\
\leq \sum_{\ell=1}^{\infty} \left( 1 + C_{n,\alpha} T \right)^{\ell-1} \frac{1}{\lambda^{\ell}} \| \gamma^{(\ell)}_0 \|_{H^\omega_{\ell}}.
\]

Set \( \Gamma_m(t) = \{ \gamma^{(k)}_{m,t} \} \). Let \( T = 1/[4C_{n,\alpha} \| \Gamma_0 \|_{H^\omega}] \). Then, by choosing \( \lambda = 4\| \Gamma_0 \|_{H^\omega} \) we conclude from (3.4) that
\[
\| \Gamma_m(t) \|_{C([0,T],H^\omega)} \leq 2 \| \Gamma_0 \|_{H^\omega}.
\]

Now, fix \( k \geq 1 \), by the above estimates for \( \Xi^{(k)}_{j,t} \) we have for any \( m, n \) with \( n > m \gg k \)
\[
\| \gamma^{(k)}_{m,t} - \gamma^{(k)}_{n,t} \|_{C([0,T],H^\omega_k)} \leq \sum_{j=m}^{n} \binom{k + j - 1}{j} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \\
\leq 2 \sum_{j=m}^{n} \binom{2j - 1}{j} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \\
\leq 2 \sum_{j=m}^{n} \frac{4^j}{\sqrt{j}} (C_{n,\alpha} T)^j \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \\
\leq 2 \frac{\| \Gamma_0 \|_{H^\omega}}{\sqrt{m}} \sum_{j=m}^{n} \frac{1}{\| \Gamma_0 \|_{H^\omega}} \| \gamma^{(k+j)}_0 \|_{H^\omega_{k+j}} \to 0,
\]
as \( m \to \infty \), where we have used Stirling’s formula \( m! \approx m^{m+1/2}e^{-m} \) by which we have \( (2m^{-1}) \approx \frac{4^m}{\sqrt{m}} \). This concludes that for every \( k \geq 1 \), \( \gamma_{m,t}^{(k)} \) converges in \( C([0,T], H^2_k) \), whose limitation is denoted by \( \gamma_t^{(k)} \).

Set \( \Gamma(t) = \{ \gamma_t^{(k)} \}_{k \geq 1}. \) By Lemma 3.1 one has

\[
\| \int_0^t ds e^{i(t-s)\Delta^{(k)}_{\pm}} \tilde{B}(k) [\gamma_{m-1,s}^{(k+1)} - \gamma_{n-1,s}^{(k+1)}] \|_{H^g_k} \\
\leq \int_0^T ds \| \tilde{B}(k) [\gamma_{m-1,s}^{(k+1)} - \gamma_{n-1,s}^{(k+1)}] \|_{H^a_k} \\
\leq kC_{n,a}T \| \gamma_{m-1,t}^{(k+1)} - \gamma_{n-1,t}^{(k+1)} \|_{C([0,T], H^a_{k+1})}.
\]

Then, taking \( m \to \infty \) in (3.2) we conclude that \( \Gamma(t) \) is a solution to (1.1). Moreover, taking \( m \to \infty \) in (3.5) we obtain (2.7).

(iii) Given \( T_0 > 0 \). Suppose \( \Gamma(t), \Gamma'(t) \in C([0,T_0], H^a) \) are two solutions to (1.1) with the initial datum \( \Gamma_0 \) and \( \Gamma'_0 \) in \( H^a \), respectively. Since (1.1) is linear, it suffices to consider \( \Gamma(t) \) instead of \( \Gamma(t) - \Gamma'(t) \). By (2.2), for every \( m \geq 1 \) one has

\[
\gamma_t^{(k)} = e^{it\Delta_{\pm}^{(k)}} \gamma_0^{(k)} + \sum_{j=1}^{m-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j e^{it-t_1)\Delta_{\pm}^{(k)}} \tilde{B}(k) \cdots \\
\times e^{it_{j-1} - t_j)\Delta_{\pm}^{(k+j-1)}} \tilde{B}(k+j-1) e^{it_j \Delta_{\pm}^{(k+j)}} \gamma_0^{(k+j)} \\
+ \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m e^{it-t_1)\Delta_{\pm}^{(k)}} \tilde{B}(k) \cdots \\
\times e^{it_{m-1} - t_m)\Delta_{\pm}^{(k+m-1)}} \tilde{B}(k+m-1) \gamma_{t_m}^{(k+m)} \\
\triangleq \sum_{j=0}^{m-1} \Xi_{j,t}^{(k)} + \Xi_{m,t}^{(k)},
\]

with the convention \( t_0 = t \). Note that,

\[
\| \Xi_{m,t}^{(k)} \|_{H^g_k} \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \| e^{it-t_1)\Delta_{\pm}^{(k)}} \tilde{B}(k) \cdots \\
\times e^{it_{m-1} - t_m)\Delta_{\pm}^{(k+m-1)}} \tilde{B}(k+m-1) \gamma_{t_m}^{(k+m)} \|_{H^a_k} \\
\leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m k \cdots \\
\times (k + m - 1)(C_{n,a})^m \| \gamma_{t_m}^{(k+m)} \|_{H^a_{k+m}} \\
\leq m(m + 1) \cdots (2m - 1)(C_{n,a})^m \\
\times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} \| \gamma_{t_m}^{(k+m)} \|_{H^a_{k+m}} dt_m.
\]
Set $T = 1/[4C_{n,\alpha}] \|\Gamma(t)\|_{C([0,T_0];H^\omega)}$. Then, combining the above estimate and (3.3) yields
\[
\|\gamma_t^{(k)}\|_{C([0,T];H^2_{k^n})} \leq \sum_{j=0}^{m-1} \binom{k+j-1}{j} (C_{n,\alpha}T)^j \|\gamma_0^{(k+j)}\|_{H^\omega_{k+j}} + \frac{m(m+1)\cdots(2m-1)}{2} (C_{n,\alpha})^m \\
\times \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} \|\gamma_t^{(k+m)}\|_{H^\omega_{k+m}} dt_m \\
\leq \sum_{j=0}^{\infty} \binom{k+j-1}{j} \left(\frac{1}{4\|\Gamma_0\|_{H^\omega}}\right)^j \|\gamma_0^{(k+j)}\|_{H^\omega_{k+j}} + (C_{n,\alpha}T)^m \frac{2m-1}{m} \|\gamma_t^{(k+m)}\|_{C([0,T];H^\omega_{k+m})} \\
\leq \sum_{j=0}^{\infty} \binom{k+j-1}{j} \left(\frac{1}{4\|\Gamma_0\|_{H^\omega}}\right)^j \|\gamma_0^{(k+j)}\|_{H^\omega_{k+j}} + \frac{\|\Gamma(t)\|_{C([0,T];H^\omega)}}{4^m} \frac{2m-1}{m} \|\gamma_t^{(k+m)}\|_{C([0,T];H^\omega_{k+m})}.
\]
Since $\frac{2m-1}{m} \approx \frac{4^m}{\sqrt{m}}$, taking $m \to \infty$ we conclude that
\[
\|\gamma_t^{(k)}\|_{C([0,T];H^2_{k^n})} \leq \sum_{j=0}^{\infty} \binom{k+j-1}{j} \left(\frac{1}{4\|\Gamma_0\|_{H^\omega}}\right)^j \|\gamma_0^{(k+j)}\|_{H^\omega_{k+j}}.
\]
Then, taking $\lambda = 4\|\Gamma_0\|_{H^\omega}$ we have
\[
\sum_{k=1}^{\infty} \frac{1}{\lambda^k} \|\gamma_t^{(k)}\|_{C([0,T];H^2_{k^n})} \\
\leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \binom{k+j-1}{j} (C_{n,\alpha}T\lambda)^j \frac{1}{\lambda^{k+j}} \|\gamma_0^{(k+j)}\|_{H^\omega_{k+j}} \\
\leq \sum_{\ell \geq 1} \left(\frac{1}{2\|\Gamma_0\|_{H^\omega}}\right)^\ell \|\gamma_0^{(\ell)}\|_{H^\omega_{\ell}} \leq 1.
\]
This concludes that $\|\Gamma(t)\|_{C([0,T];H^\omega)} \leq \|\Gamma_0\|_{H^\omega}$. This proves (2.8). \qed

**Remark 3.1.** From the above proof we find that the assumption that functions $\gamma$ are symmetric can be dropped in Theorem 2.1.

**4. Proof of Theorem 2.2**

For simplicity, we denote by $L^2_\mathbb{R}(H^\omega_k) = L^2(\mathbb{R}, H^\omega_k)$, equipped with the norm
\[
\|f\|_{L^2_\mathbb{R}(H^\omega_k)} = \left(\int_{\mathbb{R}} \|f(t)\|_{H^\omega_k}^2 dt\right)^{\frac{1}{2}}.
\]
Evidently, \( L^2_k(H^\alpha_k) \) is a Hilbert space.

**Lemma 4.1.** Assume that \( n \geq 2 \) and \( \alpha > (n - 1)/2 \). Then there exists a constant \( C_{n,\alpha} \) depending only on \( n \) and \( \alpha \) such that, for any symmetric\( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n})\),

\[
\|B_{j,k} e^{\pm t \Delta^{(k+1)}} \gamma^{(k+1)} \|_{L^2_k(H^\alpha_k)} \leq C_{n,\alpha} \|\gamma^{(k+1)}\|_{H^\alpha_{k+1}},
\]

for all \( k \geq 1 \), where \( j = 1, \ldots, k \).

Consequently, \( B^{(k)} \) can be extended to a bounded operator from \( H^\alpha_{k+1} \) to \( H^\alpha_k \), still denoted by \( B^{(k)} \), satisfying

\[
\|B^{(k)} e^{\pm t \Delta^{(k+1)}} \gamma^{(k+1)} \|_{L^2_k(H^\alpha_k)} \leq C_{n,\alpha} k \|\gamma^{(k+1)}\|_{H^\alpha_{k+1}},
\]

for all \( \gamma^{(k+1)} \in H^\alpha_{k+1} \).

This lemma was proved in [11].

As in [11], we introduce the notation

\[
D_j^{(k)}(\Gamma)(t) := \int_{0}^{t} \cdots \int_{0}^{t_{j-1}} J_j^{(k)}(t_j) \gamma^{(k+j+1)}(t_j) dt_1 \cdots dt_j
\]

for \( k, j \geq 1 \), where

\[
J_j^{(k)}(t_j) = \prod_{i=1}^{j} e^{i(t_{i-1} - t_i) \Delta^{(k+1)}} \tilde{B}^{(k+i)}
\]

with \( t_j = (t, t_1, \ldots, t_j) \) and with the convention \( t_0 = t \).

The following lemma is crucial for the proof of Theorem 2.2.

**Lemma 4.2.** Assume that \( n \geq 2 \) and \( \alpha > (n - 1)/2 \). Then there exist an absolute constant \( A > 2 \) and a constant \( C_{n,\alpha} > 0 \) depending only on \( n \) and \( \alpha \) so that the estimates below hold

1. For any \( \Gamma_0 = \{\gamma^{(k)}_0\}_{k \geq 1} \in \bigotimes_{k=1}^\infty H^\alpha_k \),

\[
\left\| \tilde{B}^{(k)} D_j^{(k)}(\Gamma_0)(t) \right\|_{L^1_{t \in [0,T]} H^\alpha_k} \leq kA^{k+j}(C_{n,\alpha}T)^{\frac{j+1}{2}} \|\gamma^{(k+j+1)}_0\|_{H^\alpha_{k+j+1}},
\]

for \( k, j \geq 1 \) and \( T > 0 \), where \( \Gamma_0(t) = \{e^{i t \Delta^{(k)}} \gamma^{(k)}_0\}_{k \geq 1} \).

2. For any \( T > 0 \) and \( \Gamma(t) = \{\gamma^{(k)}(t)\}_{k \geq 1} \) with \( \gamma^{(k)}_t \in L^1_{t \in [0,T]} H^\alpha_k \),

\[
\left\| \tilde{B}^{(k)} D_m^{(k)}(\Gamma)(t) \right\|_{L^1_{t \in [0,T]} H^\alpha_k} \leq kA^{k+m}(C_{n,\alpha}T)^{\frac{m}{2}} \|B^{(k+m+1)}(k+m+1)(t)\|_{L^1_{t \in [0,T]} H^\alpha_{k+m}},
\]

for \( k, m \geq 1 \).

**Proof.** The inequalities (4.3) and (4.4) can be proved by using the so-called “board game” argument presented in [11]. For the details see the proof of Proposition A.2 in [2].
Now we are ready to prove Theorem 2.2. To this end, we introduce the system

\[
\Gamma(t) = e^{it\hat{\Delta}_\pm} \Gamma_0 + \int_0^t ds \ e^{i(t-s)\hat{\Delta}_\pm} \Xi_s,
\]

which is formally equivalent to the system (2.4), (2.5). The proof is divided into three parts as follows.

**Proof.** (1) Let \( \alpha > (n-1)/2 \) and \( n \geq 2 \). Let \( \Gamma_0 = \{ \gamma_0^{(k)} \}_{k \geq 1} \in \mathcal{H}_\alpha \) and \( \Xi_0 = \{ \rho_0^{(k)}(t) \}_{k \geq 1} = 0 \). Given \( k \geq 1 \), for any \( m \geq 1 \) we define

\[
\rho_m^{(k)}(t) = \tilde{B}^{(k)} e^{it\Delta_{\pm}^{(k+1)}} \gamma_0^{(k+1)} + \int_0^t ds \tilde{B}^{(k)} e^{i(t-s)\Delta_{\pm}^{(k+1)}} \rho_{m-1}^{(k+1)}(s)\]

for \( t \in [0, T] \), where \( T \) will be fixed later. Set \( \Xi_m(t) = \{ \rho_m^{(k)}(t) \}_{k \geq 1} \) for every \( m \geq 1 \). By expansion, for every \( m \geq 2 \) one has

\[
\rho_m^{(k)}(t) = \sum_{j=0}^{m-1} \tilde{B}^{(k)} D_j^{(k)}(\Gamma_0)(t)
\]

with the convention \( t_0 = t \), that is,

\[
\rho_m^{(k)}(t) = \sum_{j=0}^{m-1} \tilde{B}^{(k)} D_j^{(k)}(\Gamma_0)(t)
\]

with the convenience \( D_0^{(k)}(\Gamma_0)(t) = e^{it\Delta_{\pm}^{(k+1)}} \gamma_0^{(k+1)} \). By Lemma 4.2 (1) we have

\[
\left\| \sum_{j=0}^{m-1} \tilde{B}^{(k)} D_j^{(k)}(\Gamma_0)(t) \right\|_{L_{t \in [0,T]}^1 \mathcal{H}_\alpha^0} \leq k \sum_{j=0}^{m-1} A^{k+j}(\sqrt{C_{n,\alpha} T})^{j+1}\left\| \gamma_0^{(k+j)} \right\|_{\mathcal{H}_{k+j+1}^0}.
\]

Then,

\[
\|\rho_m^{(k)}\|_{L_{t \in [0,T]}^1 \mathcal{H}_\alpha^0} \leq k \sum_{j=0}^{m-1} A^{k+j}(\sqrt{C_{n,\alpha} T})^{j+1}\left\| \gamma_0^{(k+j)} \right\|_{\mathcal{H}_{k+j+1}^0}.
\]

Set \( T := 1/[4C_{n,\alpha} A^2 \|\Gamma_0\|^2_{\mathcal{H}_\alpha}] \). For \( \lambda > 0 \) one has by (4.9)

\[
\sum_{k \geq 1} \frac{1}{\lambda^k} \|\rho_m^{(k)}\|_{L_{t \in [0,T]}^1 \mathcal{H}_{k+m}^0} \leq \sum_{k \geq 1} \left( \frac{2A}{\lambda} \right) \sum_{j=0}^{m-1} \frac{1}{(2\|\Gamma_0\|_{\mathcal{H}_\alpha})^{j+1}} \left\| \gamma_0^{(k+j)} \right\|_{\mathcal{H}_{k+j+1}^0}.
\]
Choosing $\lambda = 8A\|\Gamma_0\|_{\mathcal{H}^\alpha}$, we have

$$
\sum_{k \geq 1} \frac{1}{\lambda^k} \|\rho_m^{(k)}\|_{L_t^1[0,T] H^o_{k+m}} \leq \sum_{k \geq 1} \frac{1}{2^k} \sum_{j \geq 1} \frac{1}{(2\|\Gamma_0\|_{\mathcal{H}^\alpha})^{k+j+1}} \|\gamma_0^{(k+j+1)}\|_{H_{k+j+1}^o} \leq 1.
$$

This concludes that for every $m \geq 1$, $\Xi_m \in L_t^1[0,T] \mathcal{H}^\alpha$ and

$$
(4.10) \quad \|\Xi_m\|_{L_t^1[0,T] \mathcal{H}^\alpha} \leq 4A\|\Gamma_0\|_{\mathcal{H}^\alpha}.
$$

Now, for fixed $k \geq 1$ and any $n, m$ with $n > m$ we have

$$
\|\rho_m^{(k)} - \rho_n^{(k)}\|_{L_t^1[0,T] H^o_k}
\leq \sum_{j=m}^{n-1} (2A)^{k+j}(\sqrt{C_{n,\alpha} T}j+1)\|\gamma_0^{(k+j+1)}\|_{H_{k+j+1}^o}
\leq (4A\|\Gamma_0\|_{\mathcal{H}^\alpha})^k \sum_{j \geq m} \frac{1}{(2\|\Gamma_0\|_{\mathcal{H}^\alpha})^{k+j+1}} \|\gamma_0^{(k+j+1)}\|_{H_{k+j+1}^o}.
$$

This concludes that for each $k \geq 1$, $\rho_m^{(k)}$ converges in $L_t^1[0,T] H^o_k$ as $m \to \infty$, whose limitation is denoted by $\rho^{(k)}$.

Set $\Xi(t) = \{\rho^{(k)}(t)\}_{k \geq 1}$. Note that for any $n, m \geq 1$,

$$
\left\| \int_0^t ds \tilde{B}^{(k)} e^{i(t-s)\Delta^{(k+1)\pm}} [\rho_{m-1}^{(k+1)}(s) - \rho_{n-1}^{(k+1)}(s)] \right\|_{L_t^1[0,T] H^o_k}
\leq \sum_{\ell=1}^k \int_0^T \int_0^T dt ds \| B_{\ell,k} e^{i(t-s)\Delta^{(k+1)\pm}} [\rho_{m-1}^{(k+1)}(s) - \rho_{n-1}^{(k+1)}(s)] \|_{H^o_k}
\leq T^{1/2} \sum_{\ell=1}^k \int_0^T ds \| B_{\ell,k} e^{i(t-s)\Delta^{(k+1)\pm}} [\rho_{m-1}^{(k+1)}(s) - \rho_{n-1}^{(k+1)}(s)] \|_{L_t^2[0,T] H^o_k}
\leq C_{n,\alpha} k T^{1/2} \left\| \rho_{m-1}^{(k+1)} - \rho_{n-1}^{(k+1)} \right\|_{L_t^1[0,T] H^o_{k+1}},
$$

where we have used the Cauchy-Schwarz inequality with respect to the integral in $t$ in the second inequality and used Lemma 4.1 in the last inequality. Thus, taking $m \to \infty$ in (4.7) we prove that $\Xi$ is a solution to (4.6). Moreover, taking $m \to \infty$ in (4.10) we obtain (2.10).

(2) Fix $T_0 > 0$. Suppose $\Gamma(t) \in C([0, T_0], \mathcal{H}^\alpha)$ is a solution to (1.1) so that $B\Gamma(t) \in L_{t \in [0,T]} \mathcal{H}^\alpha$. Given $T \in (0, T_0]$, which will be fixed later. By
(2.3), for every $m \geq 1$ one has

$$
\tilde{B}^{(k)} \gamma^{(k+1)}_{t} = \tilde{B}^{(k)} e^{it\Delta_{\pm}^{(k+1)}} \gamma_{0}^{(k+1)} + \sum_{j=1}^{m-1} \tilde{B}^{(k)} \int_{0}^{t} \ldots \int_{0}^{t_{j-1}} dt_{1} \ldots dt_{j} e^{i(t-t_{1})\Delta_{\pm}^{(k+1)}} \tilde{B}^{(k+1)}
\times \ldots e^{i(t_{j}-t_{j})\Delta_{\pm}^{(k+j)}} B^{(k+j)} e^{it_{j}\Delta_{\pm}^{(k+j+1)}} \gamma_{0}^{(k+j+1)} + \tilde{B}^{(k)} \int_{0}^{t} \int_{0}^{t_{1}} \ldots \int_{0}^{t_{m-1}} dt_{1} dt_{2} \ldots dt_{m} e^{i(t-t_{1})\Delta_{\pm}^{(k+1)}} \tilde{B}^{(k+1)}
\times \ldots e^{i(t_{m-1}-t_{m})\Delta_{\pm}^{(k+m)}} B^{(k+m)} \gamma^{(k+m+1)}(t_{m}),
$$

with the convention $t_{0} = t$, that is,

$$
\tilde{B}^{(k)} \gamma^{(k+1)}_{t} = \tilde{B}^{(k)} e^{it\Delta_{\pm}^{(k+1)}} \gamma_{0}^{(k+1)} + \sum_{j=1}^{m-1} \tilde{B}^{(k)} D_{j}^{(k)} (\Gamma_{0})(t) + \tilde{B}^{(k)} D_{m}^{(k)} (\Gamma)(t).
$$

By Lemma 4.2 we have

$$
\left\| \tilde{B}^{(k)} \gamma^{(k+1)}_{t} \right\|_{L_{t}^{1}[0,T]H_{k}^{p}} \leq \sum_{j=0}^{m-1} kA^{k+j} (\sqrt{C_{n,\alpha}T})^{j+1} \left\| \gamma^{(k+j+1)}_{0} \right\|_{H_{k+j+1}^{p}} + kA^{k+m} (\sqrt{C_{n,\alpha}T})^{m} \left\| B^{(k+m)} \gamma^{(k+m+1)}_{t} \right\|_{L_{t}^{1}[0,T]H_{k+m}^{p}}.
$$

Set $T = 1/[4C_{n,\alpha}A^{2}] \max \{ \| \tilde{B} \Gamma(t) \|_{L_{t}^{1}[0,T]H_{k}^{p}}, \| \Gamma_{0} \|_{H_{k}^{p}}^{2} \}$. Taking $m \to \infty$ we have

$$
\left\| \tilde{B}^{(k)} \gamma^{(k+1)}_{t} \right\|_{L_{t}^{1}[0,T]H_{k}^{p}} \leq (2A)^{k} \sum_{j=0}^{\infty} \frac{1}{(2\| \Gamma_{0} \|_{H_{k}^{p}})^{j+1}} \left\| \gamma^{(k+j+1)}_{0} \right\|_{H_{k+j+1}^{p}}.
$$

Then, for $\lambda > 0$ we have

$$
\sum_{k \geq 1} \frac{1}{\lambda^{k}} \left\| \tilde{B}^{(k)} \gamma^{(k+1)}_{t} \right\|_{L_{t}^{1}[0,T]H_{k}^{p}} \leq \sum_{k \geq 1} \sum_{j=0}^{\infty} \left( \frac{2A}{\lambda} \right)^{k} \frac{1}{(2\| \Gamma_{0} \|_{H_{k}^{p}})^{j+1}} \left\| \gamma^{(k+j+1)}_{0} \right\|_{H_{k+j+1}^{p}}.
$$
Choose \( \lambda = 8A\|\Gamma_0\|_{\mathcal{H}^\alpha} \). Then we have
\[
\sum_{k \geq 1} \frac{1}{\lambda^k} \left\| \tilde{B}_t \gamma_t^{(k+1)} \right\|_{L^1_t([0,T])^4_{\mathcal{H}_k}} \leq \sum_{k \geq 1} \frac{1}{2^k} \sum_{j=0}^\infty \left( 2\|\Gamma_0\|_{\mathcal{H}^\alpha} \right)^{k+j+1} \left\| \gamma_0^{(k+1)} \right\|_{\mathcal{H}_{k+j+1}} \leq 1.
\]
This completes the proof of (2).

(3) Fix \( T_0 > 0 \). Suppose \( \Gamma(t), \Gamma'(t) \in C([0,T_0], \mathcal{H}^\alpha) \) are two solutions to (1.1) such that \( \tilde{B}_t \Gamma(t), \tilde{B}_t \Gamma'(t) \in L^1_{t \in [0,T_0]} \mathcal{H}^\alpha \). Since (1.1) is linear, it suffices to consider \( \Gamma(t) \) instead of \( \Gamma(t) - \Gamma'(t) \). Choosing
\[
T = 1/[4C_{n,\alpha}A^2] \max \{ \| \tilde{B}_t \Gamma(t) \|_{L^1_{t \in [0,T_0]} \mathcal{H}^\alpha}, \| \Gamma_0 \|_{\mathcal{H}^\alpha} \},
\]
by (2) we have
\[
\| \Gamma(t) \|_{C([0,T], \mathcal{H}^\alpha)} \leq \| \Gamma_0 \|_{\mathcal{H}^\alpha} + 4A\|\Gamma_0\|_{\mathcal{H}^\alpha} = (1 + 4A)\|\Gamma_0\|_{\mathcal{H}^\alpha}.
\]
This completes the proof. \( \square \)

**Remark 4.1.** For \( n = 3 \), using Theorem 1.3 in [11] instead of Lemma 4.1 we can prove as done above that the Cauchy problem (1.1) is locally well posed in \( \mathcal{H}^1 \) in the sense of Theorem 2.2. We omit the details.

### 5. The Quintic Gross–Pitaevskii Hierarchy

In this section, we consider the so-called quintic Gross–Pitaevskii hierarchy. Recall that the quintic Gross–Pitaevskii hierarchy \( \Gamma(t) = \{ \gamma_t^{(k)} \}_{k \geq 1} \) is given by
\[
(5.1) \quad i\partial_t \gamma_t^{(k)} = \left[ -\Delta^{(k)}, \gamma_t^{(k)} \right] + \mu Q^{(k)} \gamma_t^{(k+2)}, \quad \Delta^{(k)} = \sum_{j=1}^k \Delta_{x_j}, \quad \mu = \pm 1,
\]
in \( n \) dimensions, for \( k \in \mathbb{N} \), where the operator \( Q^{(k)} \) is defined by
\[
Q^{(k)} \gamma_t^{(k+2)} = \sum_{j=1}^k \text{tr}_{k+1,k+2} \left[ \delta(x_j - x_{k+1})\delta(x_j - x_{k+2}), \gamma_t^{(k+2)} \right].
\]
It is defocusing if \( \mu = 1 \), and focusing if \( \mu = -1 \). We note that the quintic Gross–Pitaevskii hierarchy accounts for 3-body interactions between the Bose particles (see [3] and references therein for details).

In terms of kernel functions, the Cauchy problem for the quintic GP hierarchy can be written as follows
\[
(5.2) \quad \left\{ \begin{array}{ll}
(i\partial_t + \Delta^{(k)}_{\pm}) \gamma_t^{(k)}(x_k; x'_k) = \mu (Q^{(k)} \gamma_t^{(k+2)})(x_k; x'_k), \\
\gamma_{t=0}(x_k; x'_k) = \gamma_0^{(k)}(x_k; x'_k), \quad k \in \mathbb{N},
\end{array} \right.
\]
where, the action of $Q^{(k)}$ on $\gamma^{(k+2)}(x_{k+2}, x'_{k+2}) \in \mathcal{S}(\mathbb{R}^{(k+2)n} \times \mathbb{R}^{(k+2)n})$ is given by

$$(Q^{(k)}\gamma^{(k+2)})(x_k, x'_k) := \sum_{j=1}^{k} (Q_{j,k}\gamma^{(k+2)})(x_k, x'_k)$$

$$:= \sum_{j=1}^{k} \int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\gamma^{(k+2)}(x_k, x_{k+2}, x'_{k}, x'_{k+2})$$

$$\times \left[ \prod_{\ell=k+1}^{k+2} \delta(x_j - x_{\ell})\delta(x_j - x'_{\ell}) - \prod_{\ell=k+1}^{k+2} \delta(x'_j - x_{\ell})\delta(x'_{j} - x'_{\ell}) \right].$$

Let $\varphi \in H^1(\mathbb{R}^n)$, then one can easily verify that a particular solution to (5.2) with initial conditions

$$\gamma^{(k)}_{t=0}(x_k; x'_k) = \prod_{j=1}^{k} \varphi(x_j)\overline{\varphi(x'_j)}$$ \quad $k = 1, 2, \ldots,$

is given by

$$\gamma^{(k)}_{t}(x_k; x'_k) = \prod_{j=1}^{k} \varphi_{t}(x_j)\overline{\varphi_{t}(x'_j)}$$ \quad $k = 1, 2, \ldots,$

where $\varphi_t$ satisfies the quintic non-linear Schrödinger equation

$$(5.3) \quad \imath \partial_t \varphi_t = -\Delta \varphi + \mu |\varphi|^4 \varphi_t, \quad \varphi_{t=0} = \varphi.$$ 

The Gross-Pitaevskii hierarchy (5.2) can be written in the integral form

$$(5.4) \quad \gamma^{(k)}_{t} = e^{it\Delta^{(k)}_{\pm}} \gamma^{(k)}_{0} + \int_{0}^{t} ds \; e^{i(t-s)\Delta^{(k)}_{\pm}} \mathring{Q}^{(k)} \gamma^{(k+2)}_{s}, \quad k = 1, 2, \ldots,$$

where $\mathring{Q}^{(k)} = -\imath \mu Q^{(k)}$. Formally we can expand the solution $\gamma^{(k)}_{t}$ of (5.4) for any $m \geq 1$ as

$$\gamma^{(k)}_{t} = e^{it\Delta^{(k)}_{0}} \gamma^{(k)}_{0} + \sum_{j=1}^{m-1} \int_{0}^{t} dt_1 \int_{0}^{t_1} dt_2 \cdots \int_{0}^{t_{j-1}} dt_j e^{i(t-t_1)\Delta^{(k)}_{\pm}} \mathring{Q}^{(k)} \gamma^{(k+2)}_{0}$$

$$\times e^{i(t_1-t_j)\Delta^{(k+2)}_{\pm}} \mathring{Q}^{(k+2)} e^{i(2(j-1))t} e^{i(t-(j-1))\Delta^{(k+2)}_{\pm}} \gamma^{(k+2)}_{0}$$

$$+ \int_{0}^{t} dt_1 \int_{0}^{t_1} dt_2 \cdots \int_{0}^{t_{m-1}} dt_m e^{i(t-t_1)\Delta^{(k)}_{\pm}} \mathring{Q}^{(k)} \gamma^{(k+2)}_{0}$$

$$\times e^{i(t_1-t_m)\Delta^{(k+2(m-1))}_{\pm}} \mathring{Q}^{(k+2(m-1))} e^{i((2m-2))t}$$

$$\times e^{i((2m-2)(m-1))t} e^{i((m-2))t} e^{i(t-(m-1))\Delta^{(k+2(m-1))}_{\pm}} \gamma^{(k+2(m-1))}_{0}$$

with the convention $t_0 = t$. 

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Let $q = (p_{k+1}, p_{k+2})$ and $q' = (p'_{k+1}, p'_{k+2})$ we have
\[ (Q_{j,k} \gamma^{(k+2)})(p_k, p'_k) \]
\[ = \int dq dq' \left[ \gamma^{(k+2)}(p_1, \ldots, p_j + p_{k+1} + p_{k+2} - p'_{k+1} - p'_{k+2}, \ldots, p_k, q; p'_{k+2}) \right. \]
\[ - \left. \gamma^{(k+2)}(p_{k+2}, p'_1, \ldots, p'_j + p'_{k+1} + p'_{k+2} - p_{k+1} - p_{k+2}, \ldots, p'_k, q') \right] \]

It is proved in [3] (Theorem 4.3 there) that for $\alpha > n/2$ there exists a constant $C_{n,\alpha} > 0$ depending only on $n$ and $\alpha$ such that
\[ \|Q_{j,k} \gamma^{(k+2)}\|_{H_{\alpha}^0} \leq C_{n,\alpha} \|\gamma^{(k+2)}\|_{H_{k+2}^0}, \quad \forall j = 1, \ldots, k. \]

Then, by slightly repeating the proof of Theorem 2.1, we can obtain the following theorem.

**Theorem 5.1.** Assume that $n \geq 1$ and $\alpha > n/2$. The Cauchy problem (5.2) is locally well posed. More precisely, there exists a constant $C_{n,\alpha} > 0$ depending only on $n$ and $\alpha$ such that

(i) For every $\Gamma_0 = \{\gamma_0^{(k)}\}_{k \geq 1} \in H^\alpha$ with $T = 1/[4C_{n,\alpha} \|\Gamma_0\|_{H^\alpha}^2]$, there exists a solution $\Gamma(t) = \{\gamma_t^{(k)}\}_{k \geq 1} \in C([0,T], H^\alpha)$ to the Gross-Pitaevskii hierarchy (5.2) with the initial data $\Gamma_0$ satisfying
\[ \|\Gamma(t)\|_{C([0,T], H^\alpha)} \leq 2\|\Gamma_0\|_{H^\alpha}. \]  

(ii) Given $T_0 > 0$, if $\Gamma(t)$ and $\Gamma'(t)$ in $C([0,T_0], H^\alpha)$ are two solutions to (5.2) with initial conditions $\Gamma_{t=0} = \Gamma_0$ and $\Gamma'_{t=0} = \Gamma'_0$ in $H^\alpha$ respectively, then
\[ \|\Gamma(t) - \Gamma'(t)\|_{C([0,T_0], H^\alpha)} \leq 2\|\Gamma_0 - \Gamma'_0\|_{H^\alpha}, \]
\[ \quad \text{for } T = 1/[4C_{n,\alpha} \|\Gamma(t) - \Gamma'(t)\|_{C([0,T_0], H^\alpha)}^2]. \]

Similarly, we can obtain an analogue of Theorem 2.2 for the quintic GP hierarchy. The details are omitted.

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Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, West District 30, Xiao-Hong-Shan, Wuhan 430071, China