THE RUNNING GAUGE COUPLING IN THE
EXACT RENORMALIZATION GROUP APPROACH

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Abstract

We discuss the perturbative running Yang-Mills coupling constant in the Wilsonian exact
renormalization group approach, and compare it to the running coupling in the more conven-
tional $\overline{MS}$ scheme. The exact renormalization group approach corresponds to a particular renor-
malization scheme, and we relate explicitly the corresponding $\Lambda$ parameters. The unambiguous
definition of an exact renormalization group scheme requires, however, the use of a one-loop
improved high energy effective action.

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1 Introduction

Some time ago the exact renormalization group equations (ERGEs) of Wegner and Wilson \cite{1} have been formulated in continuum quantum field theory by Polchinski \cite{2}. The original motivation was the simplification of the proof of renormalizability, and a better understanding of its meaning in cutoff field theories. Since then it has become clear that the ERGEs also provide a computational tool for the determination of a “low energy” effective action in terms of a “bare” (high energy) action: they describe the continuous evolution of effective Lagrangians (or effective actions) with a scale (or infrared cutoff) \( k \), which can be integrated from a cutoff scale \( \Lambda \) down to \( k = 0 \). In contrast to standard renormalization group equations the ERGEs describe this evolution including all irrelevant couplings, or higher dimensional operators, and are exact in spite of the appearance of one-loop diagrams only. They correspond, however, to an infinite system of coupled differential equations, which has to be approximated for practical purposes. The advantage of the method is, on the other hand, its nonperturbative nature and its large flexibility, which allows for many different approximation schemes (which may vary with the scale \( k \)).

One can, e.g., expand the effective action in powers of momenta or derivatives, keeping all powers of the involved fields (see \cite{3} for some early literature). This approach shares some features with the formulation of the theory on a finite size lattice, since on a lattice of size \( N \) only derivatives of the order \( N - 1 \) can be defined. Alternatively, one can expand the effective action in powers of fields, keeping all powers of the momenta \cite{4}. This kind of expansion is familiar from the truncation of the infinite series of Schwinger-Dyson equations.

The application of ERGEs to gauge theories has to surmount the problem that the intermediate infrared cutoff generally breaks the gauge invariance \cite{5–9}. One way to solve this problem is the use of modified Slavnov-Taylor identities \cite{7–9}, which impose “fine tuning conditions” on those couplings in the effective action at scales \( k \neq 0 \), which break gauge or BRST invariance. These “fine tuning conditions” ensure the BRST invariance of the full effective action for \( k \to 0 \).

Recently the first steps in the application of the ERGE approach to QCD were undertaken \cite{9}. In \cite{9} the ERGEs for the gluon and ghost propagators in pure SU(3) Yang-Mills theory were integrated within an approximation, which was based on an expansion of the ERGEs in powers of fields and, at the same time, imposed the modified Slavnov-Taylor identities on the effective action. A certain combination of the gluon and ghost propagator functions can be identified.
with the part of the heavy quark potential (in the quenched approximation), which is induced by “dressed” one gluon exchange (with a dressed gluon propagator and a dressed quark-gluon vertex; this combination is invariant under gluon field redefinitions).

The result of the numerical integration of the ERGEs was a form of the heavy quark potential $V(q^2)$, which has the perturbative one-loop form for $q^2 \to \infty$, but which shows a confining behaviour (like $\frac{1}{q^4}$) for small $q^2$. Actually the approximation employed in [9] ceased to be reliable for $q^2 \to 0$, but already within the trustworthy range of $q^2$ the confining behaviour is evident, and it happens to be quite well described by a form proposed by Richardson [10].

From this result, one can define a dimensionful non-perturbative quantity like the slope of the potential in ordinary space, or a string tension $\sigma$. On the other hand, the only input of the calculation was a “bare” scale invariant action; the fact, that this input action plays the role of an effective action at a “large” scale $\tilde{k}$, is only implicit in the choice of a small bare gauge coupling constant. Clearly, in pure Yang Mills theory one can calculate only dimensionless quantities like the ratio $\sigma/\tilde{k}$, as a function of the “bare” input gauge coupling constant.

It is fairly easy to see (by reintroducing the Planck constant $\hbar$) that the iterative solution of the ERGEs for the effective action [11] reproduces the perturbative series. Thus, for asymptotically free theories, the “bare” gauge coupling constant runs with the scale $\tilde{k}$, for large enough $\tilde{k}$, as in perturbation theory. Eventually one would like to relate it to the running gauge coupling within other perturbative schemes as the $\overline{MS}$ scheme. This would then allow to relate the corresponding scales and, finally, to express dimensionful nonperturbative quantities like $\sigma$ in terms of $\Lambda_{\overline{MS}}$.

The aim of the present paper is the derivation of the relation between the running coupling in the ERGE approach and the $\overline{MS}$ scheme or, equivalently, the relation between the corresponding $\Lambda$ parameters used to parametrize the respective scale dependence. This task is thus very similar to the one of Celmaster and Gonsalves [12], who related the running couplings in minimal and momentum substraction schemes. At the same time we have to clarify, of course, the meaning of the running coupling constant in context of the Wilsonian exact renormalization group.

## 2 The exact renormalization group approach

The ERGE approach in continuum quantum field theory has already been introduced and discussed in quite many papers [2] [11], so we will only repeat its essential features here (restricting ourselves, to this end, to the case of a single scalar field).
To start with, one considers the Euclidean partition function of a theory in the presence of an infrared cutoff $k$, which is implemented through a modification of the term quadratic in the field in the bare action $S_0(\varphi)$. The generating functional of connected Green functions $G_k(J)$ is thus of the form

$$e^{-G_k(J)} = N \int \mathcal{D}\varphi \text{Reg} e^{-S_0(\varphi) - \Delta S_k(\varphi) + (J,\varphi)}$$

where $(J,\varphi)$ is a short-hand notation for

$$(J,\varphi) \equiv \int \frac{d^4p}{(2\pi)^4} J(p) \varphi(-p) .$$

The index “Reg” to the functional integration measure indicates some ultraviolet regularization (see below), and

$$\Delta S_k(\varphi) = \frac{1}{2} \left( \varphi, R_k(p^2) \varphi \right)$$

serves as the announced infrared cutoff. In the case of a massless field a convenient choice for the IR cutoff function $R_k(p^2)$ is

$$R_k(p^2) = p^2 \frac{e^{-\frac{k^2}{p^2}}}{1 - e^{-\frac{k^2}{p^2}}}$$

such that the full propagator

$$\left(p^2 + R_k(p^2)\right)^{-1} = \frac{1 - e^{-\frac{k^2}{p^2}}}{p^2}$$

is finite for $p^2 \to 0$, and its derivative with respect to $k^2$ vanishes exponentially for large $p^2$.

The ERGE for $G_k(J)$ is obtained by differentiating eq. (2.1) with respect to $k$, and replacing $\varphi$ under the path integral by variations with respect to the sources. Then one can switch to the effective action $\Gamma_k(\varphi)$ by a Legendre transform,

$$\Gamma_k(\varphi) = G_k(J) + (J,\varphi) \equiv \hat{\Gamma}_k(\varphi) + \Delta S_k .$$

(The present field $\varphi$ is defined, as usual, by the negative of the first derivative of $G_k(J)$ with respect to $J$). The ERGE for $\Gamma_k(\varphi)$ finally assumes the form

$$\partial_k \hat{\Gamma}_k(\varphi) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \partial_k R_k(q^2) \cdot \left( \frac{\delta^2 \hat{\Gamma}_k(\varphi)}{\delta \varphi(q) \delta \varphi(-q)} + R_k(q^2) \right)^{-1} .$$
In the case of several fields, as vector fields and ghosts in the case of Yang-Mills theories, the integral $d^4q$ has simply to be extended by a (super-)trace over all degrees of freedom, and the inverse on the r.h.s. of (2.7) has to be replaced by the inverse matrix of second derivatives of $\Gamma_k + \Delta S_k$ with respect to all fields [8].

The physical meaning of the effective action $\Gamma_k(\varphi)$, as defined by eq. (2.6), is the following: the coefficients of an expansion of $\Gamma_k(\varphi)$ in powers of $\varphi$ will be called vertex functions, which will in general depend on the external momenta. These vertex functions include all quantum effects, or all one-particle irreducible Feynman diagrams, where the internal propagators are supplemented with an infrared cutoff $k$ as in eq. (2.5). For large $k$, and for asymptotically free theories, $\Gamma_k(\varphi)$ approaches the classical bare action of the theory.

The full quantum effective action of the theory is obtained from $\Gamma_k(\varphi)$ in the opposite limit $k \to 0$. Within the ERGE approach it is constructed by integrating the ERGE (2.7) from some large value of $k$, $k = \hat{k}$, down to $k = 0$. The starting point within the ERGE approach is thus an ansatz for the “high energy” effective action $\Gamma_k(\varphi)$ in the case of asymptotic freedom.

Universality now tells us that $\Gamma_k(\varphi)$ for $k \to 0$ is independent from minor changes of $\Gamma_k(\varphi)$ (again for $k$ large enough); therefore, at first sight, we may identify $\Gamma_k(\varphi)$ with the classical bare action for some large but finite value of $k$. (On the other hand, in order to shorten the domain of $k$ over which eq. (2.7) has to be integrated in the case of practical applications, it may be useful to use for $\Gamma_k(\varphi)$ a “one-loop improved” high energy effective action, see ref. [9]).

Since, in any case, the starting point within the ERGE approach is a finite ansatz for a high energy effective action $\Gamma_k(\varphi)$, problems related to ultraviolet divergences or the need to regularize the theory in the ultraviolet never arise: by definition $\Gamma_k(\varphi)$ includes already all quantum effects involving momenta $q^2 \gtrsim k^2$, but one never has to construct $\Gamma_k(\varphi)$ explicitly in terms of divergent Feynman diagrams and divergent “bare” parameters. The construction of $\Gamma_0(\varphi)$ in terms of $\Gamma_k(\varphi)$ by integrating the ERGE (2.7) involves only momenta $q^2$ with $0 \lesssim q^2 \lesssim k^2$.

It is possible to define running coupling constants from the $k$ dependent vertex functions at, e.g., vanishing external momenta. ($k$ regularizes infrared divergences also for exceptional external momenta). From the iterative solution of the ERGE (2.7) one can obtain the $\beta$ functions for the running coupling constants, which describe their dependence on $k$. For classically scale invariant theories (without dimensionful parameters in the bare action) the one and two-loop coefficients of the $\beta$ functions are scheme independent. The present method defines an “ERGE scheme”. (The precise definition will depend on the vertex function used to define the coupling constant,
and the form of the infrared cutoff function \( R_k(q^2) \) in (2.4).

Even if the first two coefficients of \( \beta \) functions in different schemes coincide, the running coupling constants - at the same “renormalization point” \( \mu \sim k \), but in different renormalization schemes - are generally different. If the renormalization group equation for a running coupling constant is solved explicitly, the description of its scale dependence necessitates the introduction of an (invariant) parameter \( \Lambda \). The difference between running coupling constants in different renormalization schemes can then be described in terms of different \( \Lambda \) parameters.

In the case of a perturbative treatment of gauge theories, much use has been made of a scheme based on dimensional regularization and a modified form of minimal subtraction, called the \( \overline{\text{MS}} \) scheme [13]. Since it has become clear, how coupling constants or \( \Lambda \) parameters of different renormalization schemes can be related explicitly [12], the \( \Lambda \) parameters of most schemes have been expressed in terms of \( \Lambda_{\overline{\text{MS}}} \). The purpose of the present paper is the derivation of the relation between the \( \Lambda \) parameter within an ERGE scheme (defined more precisely below) and \( \Lambda_{\overline{\text{MS}}} \). We will proceed as follows: first, we briefly repeat the relation between the finite part of the counter terms with different renormalization schemes, and the different \( \Lambda \) parameters, following ref. [12]. Second, we will clarify, that a particular definition of the running gauge coupling, within the ERGE approach, corresponds - implicitly - to a particular choice of the finite parts of the counter terms. In the next chapter we will compute, within dimensional regularization and for a particularly simple definition of the running gauge coupling, these finite parts of the counter terms explicitly, which will allow us to relate a \( \Lambda \) parameter denoted by \( \Lambda_{\text{ERGE}} \) to \( \Lambda_{\overline{\text{MS}}} \).

Let us now briefly review the role of the finite part of the counter terms. We will denote by \( g_{\text{bare}} \) the bare coupling constant, by \( g_a \) the renormalized coupling constant in the renormalization scheme \( a \), and by \( Z_i^a \) the renormalization constants in the scheme \( a \). Generally, within dimensional regularization and either a mass independent renormalization scheme or after choosing a renormalization point \( M = \mu \), the renormalization constants are of the form

\[
Z_i^a = \left[ 1 + g_a^2 \left( \frac{b_i}{\varepsilon} + c_{a,i} \right) + \mathcal{O}(\varepsilon) + \mathcal{O}(g_a^4) \right]
\]

with \( \varepsilon = 4 - d \), \( b_i \) and \( c_{a,i} \) are finite numerical coefficients; note that the \( b_i \) are scheme independent. (In gauge theories, both \( b_i \) and \( c_{a,i} \) may, however, depend on the gauge parameter \( \alpha \)).

The relation between the bare and the renormalized coupling is generally of the form
\[ g_{\text{bare}} = \mu^2 g_a \prod_i (Z_i^a)^{p_i} \]  

(2.9)

where the powers \( p_i \) depend on the vertex used to define the renormalized coupling, and \( \mu \) is the scale introduced by the dimensionful gauge coupling constant in \( d = 4 - \varepsilon \) dimensions.

Within a different renormalization scheme \( b \), both eqs. (2.8) and (2.9) hold with the indices \( a \) replaced by indices \( b \). Then, one can use the scheme independence of \( g_{\text{bare}} \) in order to derive a relation between the different renormalized couplings:

\[ g_b = g_a \prod_i \left( \frac{Z_i^a}{Z_i^b} \right)^{p_i}. \]  

(2.10)

To lowest nontrivial order within an expansion in powers of the renormalized coupling, one can neglect the difference between \( g_a \) and \( g_b \) within the renormalization constants \( Z_i^a \) and \( Z_i^b \), and one obtains a relation between \( g_a \) and \( g_b \) of the form

\[ g_b = g_a \left( 1 + g_a^2 \sum_i p_i (c_{a,i} - c_{b,i}) + \mathcal{O}(g_a^4) \right). \]  

(2.11)

This relation can be translated into a relation among the different \( \Lambda \) parameters. First, from the \( \mu \) independence of \( g_{\text{bare}} \) in eq. (2.9) one obtains the renormalization group equation

\[ \mu^2 \frac{dg_a^2}{d\mu^2} = -\beta_0 g_a^4 - \beta_1 g_a^6 + \mathcal{O}(g_a^8). \]  

(2.12)

This equation can be solved in powers of \( [\ln(\mu^2/\Lambda_a^2)]^{-1} \), where \( \Lambda_a \) is a \( \mu \) independent dimensionful parameter:

\[ g_a^2 = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_a^2)} - \frac{\beta_1 \ln \ln(\mu^2/\Lambda_a^2)}{\beta_0^2 \ln^2(\mu^2/\Lambda_a^2)} + \mathcal{O}(\ln^{-3}(\mu^2/\Lambda_a^2)). \]  

(2.13)

From (2.13) and eq. (2.11) one then obtains

\[ \ln \left( \frac{\Lambda_b^2}{\Lambda_a^2} \right) = \frac{2}{\beta_0} \sum_i p_i (c_{a,i} - c_{b,i}) + \mathcal{O}(g_a^2). \]  

(2.14)

The leading term on the r.h.s. of (2.14), which does not vanish for \( \mu \to \infty \) \( (g_a \to 0) \), can thus be found from the finite parts of the renormalization constants (2.8) evaluated to one-loop order.

The next step consists in identifying the renormalization condition, or the choice of the finite part of the renormalization constants, which is implicit in the ERGE approach. To this end we
consider an iterative solution of the ERGE (2.7), to first order, expressed in terms of a bare \( k \) independent action \( \Gamma_{\text{bare}}(\varphi) \) and a dimensionally regularized momentum integration:

\[
\Gamma_{k}(\varphi) = \left[ \Gamma_{\text{bare}}(\varphi) + \frac{\mu^\varepsilon}{2} \int \frac{d^{4-\varepsilon} q}{(2\pi)^{4-\varepsilon}} \ell n \left( \frac{\delta^2 \Gamma_{\text{bare}}(\varphi)}{\delta \varphi(q) \delta \varphi(-q)} + R_{k}(q^2) \right) \right]_{\varepsilon \to 0} .
\] (2.15)

Note that \( \Gamma_{\text{bare}}(\varphi) \) has to contain \( \varepsilon \) dependent counter terms in order to render the r.h.s. of eq. (2.15) finite for \( \varepsilon \to 0 \), but that the \( k \) derivative of the second term on the r.h.s. of eq. (2.15) contains an ultraviolet finite momentum integration, which allows the limit \( \varepsilon \to 0 \) of \( \partial_k \Gamma_{k} \).

The finite parts of the counter terms in \( \Gamma_{\text{bare}}(\varphi) \) can be chosen such that \( \Gamma_{k}(\varphi) \) satisfies a desired boundary condition. This boundary condition has to coincide with the starting point of the integration of the ERGEs, namely a "high energy" effective action \( \Gamma_{k}(\varphi) \) for some large scale \( k \). Using eq. (2.13) we can express \( \Gamma_{\text{bare}}(\varphi) \) explicitly in terms of \( \Gamma_{k}(\varphi) \) since, to first order, we can replace the action in the argument of the logarithm by \( \Gamma_{k}(\varphi) \):

\[
\Gamma_{\text{bare}}(\varphi) = \Gamma_{k}(\varphi) - \frac{\mu^\varepsilon}{2} \int \frac{d^{4-\varepsilon} q}{(2\pi)^{4-\varepsilon}} \ell n \left( \frac{\delta^2 \Gamma_{k}(\varphi)}{\delta \varphi(q) \delta \varphi(-q)} + R_{k}(q^2) \right) .
\] (2.16)

Given an ansatz for \( \Gamma_{k}(\varphi) \), and expanding both sides of (2.16) in powers of fields, we can now construct the renormalization constants \( Z_{\text{ERGE}}^i \) explicitly. A priori they will depend on the scale \( k \), appearing as infrared cutoff on the r.h. side on (2.16). Only after identifying \( k \) with \( \mu \) the renormalization constants will be of the form (2.8). Then we can read off the finite coefficients \( c_{\text{ERGE},i} \). Since the corresponding coefficients in the \( \overline{\text{MS}} \) scheme \( c_{\overline{\text{MS}},i} \) are already known \cite{12}, we are subsequently able to relate the parameters \( \Lambda_{\text{ERGE}} \) and \( \Lambda_{\overline{\text{MS}}} \) using eq. (2.14).

Note that, although we encountered divergent expressions (for \( \varepsilon \to 0 \)) in the form of eqs. (2.15) and (2.16), ultraviolet divergences are absent if we express \( \Gamma_{k}(\varphi) \) in terms of \( \Gamma_{k}(\varphi) \). Since this all that matters in the ERGE approach, there is, in principle, no need to formulate the ERGEs in \( d = 4 - \varepsilon \) dimensions. It was only our desire to relate \( \Gamma_{k}(\varphi) \) to some bare action within dimensional regularization, in order to make the relation with the \( \overline{\text{MS}} \) scheme explicit, which forced us away from four dimensions. In the next chapter we will proceed towards the explicit calculation of the renormalization constants \( Z_{\text{ERGE}}^i \) in Yang-Mills theories.
3 Renormalization constants for Yang-Mills theories in the ERGE scheme

Let us first present our conventions for the classical action of a $SU(N)$ Yang-Mills theory in four-dimensional Euclidean space-time, with the usual gauge fixing and ghost parts included, and with external sources $K_\mu^a$, $L^a$ and $\bar{L}^a$ coupled to the BRST variations of the fields $A_\mu^a$, $c^a$ and $\bar{c}^a$ respectively:

\[
S = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\alpha} \partial_\mu A_\mu^a \partial_\nu A_\nu^a + + \partial_\mu c^a \left( \partial_\mu c^a + gf_{bc} A_\mu^b A_\mu^c \right) 
- K_\mu^a \left( \partial_\mu c^a + gf_{bc} A_\mu^b A_\mu^c \right) - \frac{1}{2} g L^a f_{bc}^a c^b c^c + \frac{1}{\alpha} \bar{L}^a \partial_\mu A_\mu^a \right\} \tag{3.1}
\]

with

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{bc} A_\mu^b A_\nu^c . \tag{3.2}
\]

The ERGE approach demands the presence of infrared cutoff terms, cf. eqs. (2.1) and (2.3), for the gluon and ghost propagators:

\[
\Delta S_k = \frac{1}{2} \left( A_\mu^a, R_{k,\mu\nu}(p^2) A_\nu^a \right) + \left( c^a, \bar{R}_k(p^2) c^a \right) \tag{3.3}
\]

Explicit expressions for $R_{k,\mu\nu}$ and $\bar{R}_k$ will be given below. The fact that the presence of $\Delta S_k$ in eq. (2.1) breaks gauge or BRST invariance can be translated into a modification of the Slavnov-Taylor identities, which have to be satisfied by the effective action $\Gamma_k(A_\mu^a, c^a, \bar{c}^a)$ constructed along eqs. (2.1) and (2.6) \cite{8,9}. They read

\[
\int \frac{d^4p}{(2\pi)^4} \left\{ \frac{\delta \hat{\Gamma}_k}{\delta K_\mu^a(-p)} \frac{\delta \hat{\Gamma}_k}{\delta A_\mu^a(p)} - \frac{\delta \hat{\Gamma}_k}{\delta L^a(-p)} \frac{\delta \hat{\Gamma}_k}{\delta c^a(p)} - \frac{\delta \hat{\Gamma}_k}{\delta \bar{L}^a(-p)} \frac{\delta \hat{\Gamma}_k}{\delta \bar{c}^a(p)} \right\} \bigg|_{L=0}
- \int \frac{d^4p}{(2\pi)^4} \sum_B \left[ R_{k,\mu\nu}(p^2) \frac{\delta^2 \hat{\Gamma}_k}{\delta K_\mu^a(-p) \delta \varphi_B} \left( \Gamma_k^{(2)} \right)^{-1}_{\varphi_B, A_\mu^a(p)} 
- \bar{R}_k(p^2) \frac{\delta^2 \hat{\Gamma}_k}{\delta L^a(-p) \delta \varphi_B} \left( \Gamma_k^{(2)} \right)^{-1}_{\varphi_B, c^a(p)} \right] \bigg|_{L=0} \tag{3.4}
\]

where the sum over $B$ runs over the fields $\varphi_B = \{ A_\mu^a(p), c^a(p), \bar{c}^a(p) \}$ and $\bar{\varphi}_B = \{ A_\mu^a(-p), -\bar{c}^a(-p), c^a(-p) \}$. $\Gamma_k$ and $\hat{\Gamma}_k$ are related as in eq. (2.6). $\left( \Gamma_k^{(2)} \right)^{-1}_{\varphi_B, \varphi_C}$ denotes the $(\varphi_B, \varphi_C)$
component of the inverse of the matrix $\Gamma_k^{(2)} = \delta^2 \Gamma_k / \delta \bar{\varphi} \delta \varphi$ of second derivatives of $\Gamma_k$ with respect to the field $\varphi^B$ and $\bar{\varphi}^B$.

The analog of the ERGE (2.7) reads now

$$
\partial_k \hat{\Gamma}_k = \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{1}{2} \partial_k R_{k,\mu\nu}(p^2) \left( \Gamma_k^{(2)} \right)^{-1} A^a_{\nu}(-p), A^a_{\mu}(p) - \partial_k \tilde{R}_k(p^2) \left( \Gamma_k^{(2)} \right)^{-1} c^{a(-p)}, c^{a(p)} \right\} .
$$

(3.5)

In [8] it has been shown that once (3.4) is satisfied for some scale $k = \hat{k}$, it will be satisfied by any $\hat{\Gamma}_k$ provided $\hat{\Gamma}_k$ is obtained from $\hat{\Gamma}_{-k}$ by integrating the ERGE (3.5). In particular $\hat{\Gamma}_{k=0}$ will satisfy the standard Slavnov-Taylor identity, eq. (3.4) with a vanishing right-hand side, if the infrared cutoff functions $R_{k,\mu\nu}$ and $\tilde{R}_k$ vanish for $k \to 0$.

Because of the need to satisfy eq. (3.4) for $k = \hat{k}$, we are actually not free to choose the starting point $\Gamma_k$ of the integration of the ERGEs as we wish ; in particular it cannot be identified with the classical action (3.1). Perturbatively, however, a consistent starting point $\Gamma_k$ can be constructed by solving (3.4) iteratively around the classical action (3.1), which satisfies (3.4) with a vanishing right-hand side. Alternatively, to first order in the coupling, a consistent starting point $\Gamma_k$ can be constructed from eq. (2.15) for $k = \hat{k}$, where $\Gamma_{\text{bare}}$ is of the form of the classical action (3.1) supplemented with local counter terms.

Our task is, however, the construction of local counter terms from eq. (2.16), using a (consistent) starting point action $\Gamma_k$ as input. First we emphasize a feature of the ERGE approach, which is actually well known from momentum subtraction schemes : since the infrared cutoff $\hat{k}$ breaks the BRST invariance (as, e.g., an off-shell renormalization condition), the precise definition of the renormalization scheme (the finite parts of the renormalization constants) depends on the vertex used to define the coupling constant.

Here we will proceed by using the ghost gluon vertex for this definition. The terms in the bare action, which involve the renormalization constants $Z_A$, $Z_c$, and $Z_{Ac\bar{c}}$ required to relate the renormalized coupling $g_{\text{ERGE}}$ to the bare coupling $g_{\text{bare}}$ in this case, are as follows:

\[
\begin{align*}
\Gamma_{\text{bare}} &= \frac{1}{2} \int_{p_1,p_2} A^a_{\mu}(p_1) \left\{ Z_A \left( p_1^2 \delta_{\mu\nu} - p_{1\mu}p_{1\nu} \right) + \frac{p_{1\mu}p_{1\nu}}{\alpha} \right\} A^a_{\nu}(p_2) \\
&\quad + \int_{p_1,p_2} \bar{c}^a(p_1) Z_c p_1^2 c^a(p_2) \\
&\quad + i g_{\text{ERGE}} \int_{p_1,p_2,p_3} \bar{c}^a(p_1) Z_{Ac\bar{c}} \ p_{1\mu} \ A^b_{\mu}(p_2) \ c^{c}(p_3) + \cdots
\end{align*}
\]

(3.6)

with
The relation between \( g_{\text{ERGE}} \) and \( g_{\text{bare}} \) is then

\[
g_{\text{ERGE}} = Z_{A}^{1/2} Z_{c} Z_{Ac} \cdot g_{\text{bare}}
\] (3.8)

In order to compute the renormalization constants \( Z_{A}, Z_{c} \) and \( Z_{Ac} \) from the analog of eq. (2.16) for Yang-Mills theories, we have to consider the terms of \( \mathcal{O}(A^2), \mathcal{O}(c\bar{c}) \) and \( \mathcal{O}(Ac\bar{c}) \), respectively. Now note that the corresponding terms of eq. (2.16) relate momentum dependent vertex functions of \( \Gamma_{\text{bare}} \) and \( \Gamma_{k} \) : if we require that \( \Gamma_{\text{bare}} \) is of the form of the classical action (3.1) supplemented with local counter terms (and satisfies the classical Slavnov-Taylor identity), \( \Gamma_{k} \) necessarily has to contain nontrivial momentum dependent vertex functions. These nontrivial momentum dependent vertex functions correspond to the one-loop improved starting point action employed in ref. [9].

As a result we have to decide, which values of the external momenta we use, in order to fix the definition of the gauge coupling and thus the renormalization scheme completely. The simplest choice is, of course, vanishing external momenta. (For external momenta \( p^{2} = k^{2} \), e.g., we would obtain different finite parts of the counter terms).

The diagrams which contribute to \( Z_{A}, Z_{c} \) and \( Z_{Ac} \) are shown in figs. 1, 2 and 3, respectively. Of course we have to take appropriate derivatives with respect to the external momenta before setting the external momenta to zero, in order to reproduce the tensor structures in eq. (3.6). For completeness, we took in \( Z_{A} \) the presence of \( n_{f} \) massless fermions in the fundamental representation of \( SU(N) \) into account ; their propagator is supposed to be cutoff in the infrared due to a contribution to \( \Delta S_{k} \), eq. (3.3), of the form \( \langle \bar{\psi}^{a}, \beta R^{\psi}_{k}(p^{2}) \psi^{a} \rangle \).

Next we have to specify the infrared cutoff terms \( R_{k,\mu\nu}, \bar{R}_{k} \) and \( R_{k}^{\psi} \) for the gluon, ghost and fermion fields. In principle these terms could depend on the \( k \)-dependent parameters of \( \Gamma_{k} \) [9] (in order to ensure the absence of unwanted poles in the propagators), but to the presently required order a simple choice analogous to eq. (2.3) is sufficient :

\[
\bar{R}_{k}(p^{2}) = p^{2} \frac{e^{-k^{2}/2}}{1 - e^{-k^{2}/2}}
\] (3.9a)

\[
R_{k,\mu\nu}(p) = \bar{R}_{k}(p^{2}) \left\{ \delta_{\mu\nu} - \left( 1 - \frac{1}{\alpha} \right) \frac{p_{\mu}p_{\nu}}{p^{2}} \right\}
\] (3.9b)
\[ R_k^\psi(p^2) = \frac{e^{-\frac{k^2}{p^2}}}{1 - e^{-\frac{k^2}{p^2}}} \] . (3.9c)

Finally we fix the gauge ; the Landau gauge \( \alpha = 0 \) turns out to be particularly convenient, because in this gauge the diagrams in fig. 3 vanish (for vanishing external momenta).

We then obtain :

\[ Z_{\text{ERGE}}^A = 1 + \frac{g^2}{16\pi^2} \left[ \frac{2}{\varepsilon} \left( \frac{13}{6} N - \frac{2}{3} n_F \right) + N \left( \frac{13}{6} \ell n(4\pi) + \frac{229}{96} \right) - n_F \left( \frac{1}{3} \ell n(4\pi) + \frac{1}{36} \right) \right] \] (3.10a)

\[ Z_{\text{ERGE}}^c = 1 + \frac{g^2}{16\pi^2} \left[ \frac{2}{\varepsilon} \left( \frac{3}{4} N \right) + N \left( \frac{3}{4} \ell n(4\pi) + \frac{1}{4} \right) \right] \] (3.10b)

\[ Z_{\text{ERGE}}^{Ac} = 1 \] (3.10c)

The corresponding renormalization constants in the \( \overline{\text{MS}} \) scheme, one the other hand, are as follows [12] (note that \( d = 4 + \varepsilon \) in [12], whereas \( d = 4 - \varepsilon \) here) :

\[ Z_{\overline{\text{MS}}}^A = 1 + \frac{g^2}{16\pi^2} \left[ \frac{13}{6} N \left( \frac{2}{\varepsilon} + \ell n(4\pi) - \gamma_E \right) - \frac{2}{3} n_F \left( \frac{2}{\varepsilon} + \ell n(4\pi) - \gamma_E \right) \right] \] (3.11a)

\[ Z_{\overline{\text{MS}}}^c = 1 + \frac{g^2}{16\pi^2} \left[ \frac{3}{4} N \left( \frac{2}{\varepsilon} + \ell n(4\pi) - \gamma_E \right) \right] \] (3.11b)

\[ Z_{\overline{\text{MS}}}^{Ac} = 1 \] (3.11c)

where \( \gamma_E \) is Euler’s constant: \( \gamma_E = 0.577216 \ldots \)

From eq. (3.8), and the analogs of eqs. (2.8) and (2.11), we thus find

\[ g_{\text{ERGE}} = g_{\overline{\text{MS}}} \left( 1 + g_{\overline{\text{MS}}}^2 \left[ N \left( \frac{11}{6} \gamma_E + \frac{277}{192} \right) - n_F \left( \frac{1}{3} \gamma_E + \frac{1}{72} \right) \right] + \mathcal{O}(g_{\overline{\text{MS}}}^4) \right) \] (3.12)

or, from eq. (2.14) and with \( \beta_0 = \left( \frac{11}{3} N - \frac{2}{3} n_F \right) / 16\pi^2 \),

\[ \ell n \left( \frac{\Lambda_{\text{ERGE}}^2}{\Lambda_{\overline{\text{MS}}}^2} \right) = \frac{N \left( \frac{11}{3} \gamma_E + \frac{277}{96} \right) - n_F \left( \frac{2}{3} \gamma_E + \frac{1}{36} \right)}{\frac{11}{3} N - \frac{2}{3} n_F} + \mathcal{O}(g^2) \] . (3.13)
This is our main result; numerically it leads, for \( n_F = 0 \) and \( N \) arbitrary (pure Yang-Mills), to \( \Lambda_{\text{ERGE}} \sim 2 \Lambda_{\text{MS}} \).

In the remaining part of this section we will briefly discuss, how this result may be applied within the ERGE formalism. Let us note again, that the starting point in the ERGE formalism is a high energy effective action \( \Gamma_k \). The only free parameter in \( \Gamma_k \) is a coupling constant \( g_k \). Due to the one-loop improvement of \( \Gamma_k \) required to satisfy eqs. (2.15) and (2.16) with a local bare action, or, equivalently, the modified Slavnov-Taylor identity (3.4), the precise definition of \( g_k \) will depend on the vertex function used, on the external moment and the gauge. Here we confine ourselves to the ghost gluon vertex at vanishing external momenta and the Landau gauge.

As a result of the integration of the ERGEs one can obtain dimensionful nonperturbative quantities like hadron masses or the slope \( \sigma \) of a confining potential. For dimensional reasons quantities like \( \sigma \) are necessarily given in units of the ultraviolet starting scale \( \bar{k} \), on the other hand they have to depend on \( \bar{k} \) and \( g_k \) such that their total derivative with respect to \( \bar{k} \) vanishes:

\[
\sigma^2 = \gamma k^2 e^{-\frac{\bar{k}}{\bar{g}_k}} \left( \beta_0 \frac{g_k^2}{\bar{g}_k} \right)^{-\frac{\beta_1}{\beta_0}} \left( 1 + \mathcal{O}(g_k^2) \right)^{-1}
\]

with \( \gamma \) a \( \bar{k} \) independent constant. Eq. (3.14) can and has to be checked within the ERGE formalism, for \( g_k \) small enough. Most importantly, it requires that \( g_k \) runs with \( \bar{k} \) according to the renormalization group equation (2.12) with the two-loop term \( \beta_1 \) included. Only then it becomes possible to extract from eq. (3.14) the quantity \( \gamma \) in the limit \( g_k \to 0 \).

Using eq. (2.13) with \( \mu^2 = \bar{k}^2 \) and \( \Lambda_a = \Lambda_{\text{ERGE}} \), eq. (3.14) becomes, as it should,

\[
\sigma^2 = \gamma \Lambda_{\text{ERGE}}^2 \left( 1 + \mathcal{O}(g_k^2) \right)
\]

Hence, having determined \( \gamma \) within the ERGE approach and having compared \( \sigma \) with some measured quantity, one is able to determine \( \Lambda_{\text{ERGE}} \) in, say, MeV. At this stage we can apply our result, eq. (3.13), and obtain \( \Lambda_{\text{MS}} \) in MeV. This information would certainly be highly welcome.

A first effort in this direction has been made in ref. [9]. There the ERGEs for the gluon and ghost propagators in pure SU(3) Yang-Mills theory were integrated, within a certain approximation, with the aim to study the potential between heavy quarks. The result was indeed a confining form of the potential, which allows the determination of a phenomenologically known quantity like \( \sigma \) in eqs. (3.14) and (3.15). In addition, a one-loop improvement of the action \( \Gamma_k \) at the starting point was used, which renders the definition of the coupling \( g_k \) free from the ambiguities discussed above. However, the approximations performed on the r.h. sides of the ERGEs
in \[ 9 \] (in order to turn them into a low-dimensional closed system of differential equations) were too crude to reproduce correctly the two-loop coefficient \( \beta_1 \) of the \( \beta \)-function for \( g_k \). Therefore eq. (3.14) was not satisfied to the required order, and a parameter \( \gamma \) as in eq. (3.15) could not be extracted. It should be clear that the determination of \( \Lambda_{\overline{MS}} \) in the ERGE approach allows only for approximations, which reproduce correctly the two-loop \( \beta \)-function.

4 Conclusions

In the present paper we have discussed the relation between the ERGE approach and the standard renormalization procedure. In particular we have shown, that a particular choice of the starting point action \( \Gamma_k \), within the ERGE approach, corresponds implicitly to a particular choice of the renormalization condition. Our aim was to establish the explicit relation eq. (3.13) between the \( \Lambda \) parameter of the ERGE approach (in the case of a particularly convenient definition of the coupling \( g_k^{\text{ERGE}} \), and for a given choice of the infrared regulators \( R_k(p^2) \)) and \( \Lambda_{\overline{MS}} \) for SU(N) gauge theories with \( n_F \) massless quarks.

First we had to clarify, however, that the starting point action \( \Gamma_k \) in the ERGE approach has to be of the form of a one-loop improved action, in order to render the relation between the coupling constants in the ERGE approach and the \( \overline{MS} \) scheme free from ambiguities: if one would naively choose \( \Gamma_k \) to be of the form of the classical action, the definition of the coupling constant would, of course, not depend on the vertex nor on the external momenta of the corresponding particles. The finite parts of the counterterms, which relate \( \Gamma_k \) to \( \Gamma_{\text{bare}} \), do, however, depend on these conventions and would therefore be ambiguous. From the one-loop relation between \( \Gamma_k \) and \( \Gamma_{\text{bare}} \) (which is necessary and sufficient for the present discussion) one also sees that a “classical” choice of \( \Gamma_k \) corresponds, in fact, to a nonlocal form of \( \Gamma_{\text{bare}} \) which does not coincide with the bare action implicit in the \( \overline{MS} \) scheme.

Second, as one may have suspected, a practical application of our result in order to determine \( \Lambda_{\overline{MS}} \) from the ERGE approach requires an approximation, which is general enough in order to reproduce the two-loop \( \beta \) function. This condition is certainly much harder to satisfy, in practice, than the use of a one-loop improved starting point action (which has already been used in [9]) although it would be, in principle, straightforward to generalize the approach in [9] sufficiently. In any case the present result is a prerequisite for such a program.
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Figure 1: Diagrams contributing to $Z_A$. All internal propagators are supplemented with infrared cutoffs, cf. eqs. (2.5) and (3.9).

Figure 2: Diagram contributing to $Z_c$.

Figure 3: Diagrams contributing to $Z_{Ac\bar{c}}$ (which vanish, however, in the Landau gauge at vanishing external momenta).