New scaling solutions in cubic Horndeski theories

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We propose a viable dark energy scenario in the presence of cubic Horndeski interactions and a standard scalar-field kinetic term with two exponential potentials. We show the existence of new scaling solutions along which the cubic coupling $G_3$ provides an important contribution to the field density that scales in the same way as the background fluid density. The solutions finally exit to the epoch of cosmic acceleration driven by a scalar-field dominated fixed point arising from the second exponential potential. We clarify the viable parameter space in which all the theoretically consistent conditions including those for the absence of ghost and Laplacian instabilities are satisfied on scaling and scalar-field dominated critical points. In comparison to Quintessence with the same scalar potential, we find that the cubic coupling gives rise to some novel features: (i) the allowed model parameter space is wider in that a steeper potential can drive the cosmic acceleration; (ii) the dark energy equation of state $w_\phi$ today can be closer to $-1$ relative to Quintessence; (iii) even if the density associated with the cubic coupling dominates over the standard field density in the scaling era, the former contribution tends to be suppressed at low redshifts. We also compute quantities associated with the growth of matter perturbations and weak lensing potentials under the quasi-static approximation in the sub-horizon limit and show that the cubic coupling leads to the modified evolution of perturbations which can be distinguished from Quintessence.

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I. INTRODUCTION

The late-time cosmic acceleration has been confirmed by many cosmological surveys, but a satisfactory theoretical explanation for this phenomenon is still lacking. Despite the overall success of the Λ-cold-dark-matter model in fitting the cosmological data [1–7], there are still some shortcomings such as the cosmological constant and coincidence problems [8–10]. An alternative explanation to the cosmic acceleration is to introduce extra fields which modify the gravitational interaction at large distances [11–15].

Inclusion of a scalar field $\phi$ (or multiple scalar fields) in the description of the cosmological dynamics sometimes gives rise to so-called scaling solutions [16–33]. The scaling solution is featured by a constant ratio between the energy density of matter components and that of the scalar field, in which case there is a possibility for alleviating the coincidence problem. Since the field density is not negligibly small compared to the background density even in the early cosmological epoch, the model can be compatible with the energy scale associated with particle physics. Moreover, the scaling solution attracts background trajectories with different initial conditions. After the solutions enter the scaling regime, the cosmological dynamics is completely fixed by theoretical parameters.

In Quintessence described by the Lagrangian $G_2 = X - V(\phi)$, where $X = -\partial_\mu \partial^\mu \phi / 2$ is the standard kinetic term and $V(\phi)$ is the scalar potential, there exists a scaling solution for the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$ ($V_0$ and $\lambda$ are constants) [16,19,21,24,26,34]. In K-essence given by the Lagrangian $G_2 = G_2(\phi, X)$, it was shown in Refs. [27,28] that the condition for the existence of scaling solutions restricts the Lagrangian to the form $G_2 = X g(Y)$, where $g$ is an arbitrary function of $Y = X e^{\lambda \phi}$. For example, this includes the diatonic ghost condensate model $G_2 = -X + c e^{\lambda \phi} X^2$ proposed in Ref. [27] ($c$ is a constant), which corresponds to the choice $g(Y) = -1 + c Y$.

A further generalization of the Lagrangian including a cubic term $G_3(\phi, X) \Box \phi$ can still give rise to scaling solutions for the function $G_3 = a_1 Y + a_2 Y^2$ (with $a_1, a_2$ constants), along with an exponential potential [32] and a direct coupling between matter and scalar fields. Finally, another option of having scaling solutions is to consider a conformal coupling to the Ricci scalar. In Ref. [29], the authors classified the possible couplings and selected three forms: exponential, polynomial, and exponential of polynomial functions. All of them depend on the coefficient characterizing the scaling.

The aim of this work is to explore the possibility for obtaining new scaling solutions from cubic Horndeski theories [35–38]. For this purpose, we employ the action

$$ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{16\pi G_N} + G_2(\phi, X) - G_3(\phi, X) \Box \phi \right], \tag{1.1} $$

where $g$ is the determinant of metric tensor $g_{\mu\nu}$, $R$ is the Ricci scalar, and $G_N$ is the Newton gravitational constant. This model is usually denoted as a kinetic gravity braiding model [39]. The name follows from the presence of a braiding term characterizing the mixing of kinetic terms between the scalar field and metric. In particular, it arises from the dependence of $G_3$ with respect to $X$,
i.e., $G_{3,X} \equiv \partial G_3/\partial X \neq 0$.

The phenomenology exhibited by the scalar field through the braiding makes the model attractive as a dark energy candidate. Indeed, the speed of propagation for the scalar mode gets modified as well as the kinetic term of scalar perturbations \[39\]. The braiding term affects the growth of perturbations, modifies the shape of the matter power spectrum, and the low-$\ell$ tail in the observed Cosmic Microwave Background (CMB) spectrum, thus showing detectable signatures \[40–42\]. In particular, the observational bound of $c_{49}$, which the theories are consistent with the observational background is equivalent to that of light \[37, 43–49\]. Hence the theories are consistent with the observational bound of $c_{49}$.

We note that, for some specific choices of cubic Horndeski interactions like covariant Galileons \[52, 53\], their dominance over other energy densities in the late Universe is not favored by the observational data of galaxy and Integrated-Sachs-Wolfe (ISW) correlations \[41, 54\]. However, it is expected that the observational constraints arising from the galaxy-ISW correlation generally depend on the form of cubic Horndeski interactions. In particular, unlike covariant Galileons, there should exist models in which the density associated with the cubic Horndeski term is subdominant to that of the standard field density.

In this paper, we construct a dark energy model in which the density of cubic Horndeski coupling gives important contributions to the field density in scaling radiation and matter eras, but it starts to be subdominant relative to the density arising from the standard field Lagrangian $G_2 = X - V(\phi)$. For this purpose, we take into account the sum of two exponential potentials $V(\phi) = V_1 e^{-\beta_1 \phi} + V_2 e^{-\beta_2 \phi}$ with $\beta_1 \gg \mathcal{O}(1)$ and $\beta_2 \lesssim \mathcal{O}(1) \ [24]$, besides the cubic Horndeski term $G_3 = A \ln Y$ ($A$ is constant) allowing for the scaling behavior. While the exponential potential $V_1 e^{-\beta_1 \phi}$ contributes to the field density in the early epoch together with the cubic Horndeski term, the potential $V_2 e^{-\beta_2 \phi}$ dominates over other densities at late time. We leave the detailed analysis about the compatibility of this model with observational data for a future work, but we show that the quantities $\mu$ and $\Sigma$ associated with Newtonian and weak lensing gravitational potentials \[55, 56\] can be consistent with the conjecture made in Ref. \[57, 58\].

This paper is organized as follows. In Sec. II, we present a suitable choice of dimensionless variables serving to study the dynamical system of background equations of motion in cubic Horndeski theories. We also discuss theoretically consistent conditions constrained from the background and the stability of perturbations. In Sec. III, we choose a specific form of the cubic coupling $G_3$ allowing for scaling solutions and study the corresponding critical points and their stability. In Sec. IV, we show a practical example of realizing scaling radiation/matter eras followed by the epoch of cosmic acceleration and study the background cosmological dynamics in detail. In Sec. V, we discuss the evolution of quantities relevant to Newtonian and weak lensing gravitational potentials. Finally, we conclude in Sec. VI.

II. DYNAMICAL SYSTEM AND STABILITY

We derive the background equations of motion in cubic Horndeski theories on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and construct the dynamical system by choosing an exponential potential for the scalar field. We also apply conditions for the absence of ghost and Laplacian instabilities derived in Ref. \[43\] to our cubic Horndeski theories.

The first step of our analysis is to write the corresponding background field equations as an autonomous system of first-order differential equations. Afterwards, the cosmological dynamics is determined by investigating the evolution around critical points. The stability of the critical points is known by linearizing the autonomous equations around each point and computing eigenvalues of the Jacobian matrix associated to the system. A critical point is stable when all the eigenvalues are negative and the point is said to be a stable node or attractor; it is unstable when the eigenvalues are all positive and the point is said to be an unstable node; finally, when at least one eigenvalue is positive and one negative, the point is a saddle. We refer the reader to Refs. \[59, 60\] for details on the procedure and to Refs. \[7, 20, 61–71\] for applications to alternative theories of gravity.

Let us consider the action (1.1) in the presence of matter perfect fluids described by the action $S_\gamma$. To study the background equations of motion, we use the unit $8\pi G_N = 1$. On the flat FLRW background given by the line element $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j$, where $a(t)$ is a time-dependent scale factor, the modified Friedmann equations are

$$3H^2 = \rho_\phi + \rho_\gamma, \quad (2.1)$$
$$2H + 3H^2 = -p_\phi - p_\gamma, \quad (2.2)$$

where $H = \dot{a}/a$ is the Hubble expansion rate, $X = \dot{\phi}^2/2$, and

$$\rho_\phi = 2XG_{2,X} - G_2 + 6X\ddot{\phi}HG_{3,X} - 2XG_{3,\phi}, \quad (2.3)$$
$$p_\phi = G_2 - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}). \quad (2.4)$$

The quantities $\rho_\phi$ and $p_\phi$ correspond to the density and pressure arising from the scalar field, respectively, whereas $\rho_\gamma$ and $p_\gamma$ are those of perfect fluids. The fluid equation of state is given by $\gamma - 1 = p_\gamma/\rho_\gamma$, where $\gamma$ is a constant barotropic coefficient in the range $0 < \gamma < 2$. The dust corresponds to the choice $\gamma = 1$ and radiation to $\gamma = 4/3$.

Variation of the action with respect to $\phi$ leads to

$$\frac{1}{a^3} \frac{d}{dt} (a^3 \dot{J}) = P, \quad (2.5)$$
where
\[ J = \dot{\phi}G_{2,\phi} + 6HXG_{3,\phi} - 2\dot{\phi}G_{3,\phi}, \]
\[ P = G_{2,\phi} - 2X \left( G_{3,\phi\phi} + \dot{\phi}G_{3,\phi\phi} \right). \]

In the following, we consider the quadratic Lagrangian \( G_2(\phi, X) \) of a standard canonical scalar field, i.e.,
\[ G_2(\phi, X) = X - V(\phi), \]
with an exponential potential
\[ V(\phi) = V_0 e^{-\beta\phi}, \]
where \( V_0 \) and \( \beta \) are constants.

In the search for scaling solutions, let us now write \( G_3 \) in the form
\[ G_3(\phi, X) = g(Y), \quad Y = X e^{\lambda \phi}, \]
where \( g \) is an arbitrary function of \( Y \), and \( \lambda \) is a constant. With this definition, the derivatives of \( G_3 \) with respect to \( \phi \) and \( X \) can be written as
\[ G_{3,\phi} = \lambda Y g_Y, \]
\[ G_{3,X} = Y g_Y, \]
\[ G_{3,XX} = \frac{Y^2 g_{YY}}{X}, \]
\[ G_{3,\phi\phi} = \left( g_Y + Y g_{YY} \right) \lambda^2 Y, \]
\[ G_{3,\phi X} = G_{3,\phi X} = (g_Y + Y g_{YY}) \lambda Y \frac{Y}{X}. \]

To study the background cosmological dynamics, we introduce the dimensionless variables:
\[ x = \frac{\phi}{\sqrt{6}H}, \quad y = \frac{\sqrt{V}}{\sqrt{3}H}, \quad \Omega_\gamma = \frac{\rho_\gamma}{3H^2}. \]

On using Eqs. (2.1)- (2.6) and the relations (2.10) - (2.14), we obtain the following autonomous system of first-order differential equations:
\[ x' = \frac{1}{\sqrt{6}} f(x, y) - \frac{\dot{H}}{H^2} x, \]
\[ y' = -\frac{3}{2} \frac{\beta xy}{H} - \frac{\dot{H}}{H^2} y, \]
where a prime represents the derivative with respect to \( N = \ln a \), and
\[ f(x, y) \equiv \frac{\phi}{H^2} \left\{ -3\sqrt{6}x + 3\beta y^2 - 6\lambda x (Y g_Y + Y^2 g_{YY}) \right\} \]
\[ +3xY g_Y (2\lambda x - \sqrt{6}) \right\}, \]
with
\[ s(x, y)^{-1} = 1 + 2 \left( Y g_Y + Y^2 g_{YY} \right) \left( \frac{\sqrt{6}}{x} - \lambda \right) \]
\[ +2Y g_Y (3Y g_Y - \lambda), \]
and
\[ \frac{\dot{H}}{H^2} = Y g_Y \frac{3}{2} \gamma \Omega_\gamma + 3xY g_Y \left( 2\lambda x - \sqrt{6} \right) - 3x^2. \]
The critical points \((x_c, y_c)\) of the above dynamical system can be derived by setting \( x' = 0 \) and \( y' = 0 \) in Eqs. (2.16) - (2.17).

The constraint equation (2.2) can be expressed as
\[ \Omega_\gamma = 1 - \Omega_\phi, \]
where \( \Omega_\phi \) is the field density parameter defined by
\[ \Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = x^2 + y^2 + 2xyY g_Y \left( \sqrt{6} - \lambda x \right). \]

For a positive fluid density \( (\Omega_\gamma \geq 0) \), \( \Omega_\phi \) has an upper bound, \( \Omega_\phi \leq 1 \). This condition will be exploited when exploring the region in which the critical points exist. We also introduce the density parameter associated with the cubic coupling \( G_3 \), as
\[ \Omega_{G_3} = 2xY g_Y \left( \sqrt{6} - \lambda x \right). \]
The scalar-field equation of state \( w_\phi \) and the effective equation of state \( w_{\text{eff}} \), are defined, respectively, by
\[ \frac{w_\phi}{\rho_\phi} = \frac{x^2 - y^2 - 2Y g_Y (f/3 + \lambda x^2)}{x^2 + y^2 + 2xyY g_Y (\sqrt{6} - \lambda x^2)}, \]
\[ w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2}. \]

The Universe exhibits the accelerated expansion for \( w_{\text{eff}} < -1/3 \). In general, the condition \( w_\phi < -1/3 \) is not sufficient for realizing the cosmic acceleration. If the energy density of the Universe is dominated by the scalar field, then the cosmic acceleration occurs under the condition \( w_\phi < -1/3 \).

So far, we have maintained the form of \( G_3 = g(Y) \) completely open. In order to close the system, however, we have to make a choice for this function. If we choose \( \beta = \lambda \), as done in Refs. [27, 28], the quantity \( Y \) can be expressed as \( Y = (x^2/y^2)V_0 \) and hence Eqs. (2.16) - (2.17) are immediately closed. In the following, we will not make this assumption and explore another possibility to close the dynamical system. An explicit model is presented in Sec. III.

To guarantee the viability of our model at the background level, we impose the following conditions:

- \textbf{Existence condition:} The critical points \((x_c, y_c)\) must be real.
• **Stability of critical points**: We need to identify critical points responsible for radiation/matter eras and for the late-time cosmic acceleration. From a cosmological point of view, the fixed points during the radiation and matter eras need to be either an unstable node or a saddle point. The system finally has to approach an attractor/stable point with the cosmic acceleration. For the late-time scalar-field dominated solution, we demand the condition \( w_\phi < -1/3 \).

• **Phase-space constraint**: We impose that the field density parameter is in the range \( \Omega_\phi \leq 1 \).

In Horndeski theories, there are two tensor and one scalar degrees of freedom. In cubic Horndeski theories, the second-order action of tensor perturbations is the same as that in General Relativity (GR) \(^{(22)}^{(23)}\), so there are neither ghost nor Laplacian instabilities in the tensor sector. In particular, the speed of gravitational waves is equivalent to that of light. On the other hand, the no-ghost condition and the sound speed \( c_s \) of scalar perturbations get modified by cubic Horndeski terms compared to a canonical scalar field with the Lagrangian \( G_2 = X - V(\phi) \).

• **Physical viability conditions**: The conditions for the absence of ghost and Laplacian instabilities in the small-scale limit are given, respectively, by \(^{(13)}\)

\[
Q_s = \frac{4w_3 + 9w_2^2}{3w_2^2} > 0 , \quad \text{(2.26)}
\]

\[
c_s^2 = \frac{2(Hw_2 - \dot{w}_2 - \gamma \rho_\gamma) - w_2^2}{w_2^2 Q_s} > 0 , \quad \text{(2.27)}
\]

where

\[
w_2 = 2H \left( 1 - Y g_{,Y} \sqrt{6} x \right) ,
\]

\[
w_3 = 9H^2 \left[ x^2 \left( 1 - 2\lambda Y^2 g_{,YY} - 4\lambda Y g_{,Y} \right) + 2\sqrt{6}x \left( 2Y g_{,Y} + Y^2 g_{,YY} \right) - 1 \right] . \quad \text{(2.28)}
\]

For a given model, the viable parameter space is constrained to satisfy all the conditions mentioned above.

### III. MODEL WITH SCALING SOLUTIONS

From Eqs. (2.16)–(2.17) with Eqs. (2.18)–(2.20), the dynamical system is closed for \( g(Y) \) satisfying the conditions that both \( Y g_{,Y} \) and \( Y^2 g_{,YY} \) are constants. Let us consider the cubic coupling given by

\[
G_3(\phi, X) = g(Y) = A \ln Y , \quad \text{(3.1)}
\]

where \( A \) is a constant. In this case, we have

\[
Y g_{,Y} = -Y^2 g_{,YY} = A = \text{constant} . \quad \text{(3.2)}
\]

In the following, we derive the critical points and discuss the stability of them for the cubic coupling (3.1).

In Table I, we show the critical points and their corresponding values of \( \Omega_\phi \) and \( w_\phi \). The fluid density parameter is known from the relation \( \Omega_\gamma = 1 - \Omega_\phi \). For the model (3.1), the physical viability conditions (2.26)–(2.27) reduce, respectively, to

\[
Q_s = \frac{3x^2 \left( 1 - 2A\lambda + 6A^2 \right)}{\left( 1 - \sqrt{6}Ax \right)^2} > 0 , \quad \text{(3.3)}
\]

\[
c_s^2 = \frac{3x \left( 1 - 2A\lambda - 2A^2 \right) + 4\sqrt{6}A}{3x \left( 1 - 2A + 6A^2 \right)} > 0 . \quad \text{(3.4)}
\]

In the limit \( A \to 0 \), we have \( Q_s \to 3x^2 \) and \( c_s^2 \to 1 \), so the conditions \( Q_s > 0 \) and \( c_s^2 > 0 \) are automatically satisfied. In presence of the cubic coupling (3.1), the parameters \( A \) and \( \lambda \) are constrained to satisfy the conditions (3.3) and (3.4). In Table I, we present concrete values of \( Q_s \) and \( c_s^2 \) for each fixed point.

In what follows, we will illustrate the main characteristics of the critical points and discuss their stability following the criteria mentioned in Sec. II. In Appendix A, we explain the detail for the stability of the fixed points by explicitly computing eigenvalues of the Jacobian matrix associated with homogeneous perturbations around each point. In total, there are five fixed points presented below and in Table I.

• **Point (a)**: This point is characterized by

\[
x_c = \sqrt{\frac{3}{2} \beta} , \quad y_c = \sqrt{\frac{3\left( 2 - \gamma \right)}{2\beta^2} \left[ \gamma + 2A \left( \beta - \gamma \lambda \right) \right]} , \quad \text{(3.5)}
\]

with \( w_\phi = w_{\text{eff}} = \gamma - 1 \) and

\[
\Omega_\phi = \frac{3}{\beta^2} \left[ \gamma + A \left( \gamma (\beta - 2\lambda) + 2\beta \right) \right] . \quad \text{(3.6)}
\]

This corresponds to the scaling solution along which the ratio of energy densities between \( \phi \) and the matter fluid are constant \( (\Omega_\phi/\Omega_\gamma = \text{constant}) \) with \( w_\phi \) equivalent to the matter equation of state \( (w_\phi = \gamma - 1) \). The dark energy density scales as the fluid density regardless of the value of \( \gamma \). The scaling ratio \( \Omega_\phi/\Omega_\gamma \) depends on the parameters \( \beta, \gamma, A, \lambda \). In the limit \( A \to 0 \), the values of \( \Omega_\phi \) and \( y_c \) given above recover those derived for a canonical scalar field with the Lagrangian \( G_2 = X - V(\phi) \).

\[
\Omega_{G_3} = \frac{3}{\beta^2} A \gamma (2\beta - \gamma \lambda) , \quad \text{(3.7)}
\]

\footnote{We do not use the more appropriate wording “physical stability conditions” in order to avoid confusion with the “stability conditions” of critical points which determine the dynamics of the system.}
The cube coupling $G_3$ leads to the modification to the standard value $\Omega_{\phi}^{(r)} = 4/\beta^2$ derived for a canonical scalar field with the exponential potential $V(\phi) = V_0 e^{-\beta \phi}$. Whether $\Omega_{\phi}^{(r)}$ is increased or decreased by $G_3$ depends on the sign of $A(5 \beta - 4 \lambda)$.

In 2015, the Planck team [23] put a more stringent bound on the field density parameter from CMB measurements: $\Omega_{\phi} < 0.02$ (95% C.L.) at the redshift $z = 1/a - 1 \approx 50$. If the solution is in the scaling regime during the matter era, the field density parameter $\Omega_{\phi}^{(m)} \equiv \Omega_{\phi}(\gamma = 1)$ is constrained to be

$$\Omega_{\phi}^{(m)} = \frac{3}{\beta^2} [1 + A (3 \beta - 2 \lambda)] < 0.02.$$  \hfill (3.12)

In Sec. [IV] we will present a model with the early-time scaling solution followed by the late-time cosmic acceleration. We show that it is possible to find the parameter space consistent with all the bounds derived above.

- Point (b): This point corresponds to

$$x_c = \frac{\beta - 6 A}{\sqrt{6 [1 + A (\beta - 2 \lambda)]}},$$  \hfill (3.13)

$$y_c = \sqrt{\frac{[1 + 2 A (3 \lambda - A)] [6 - 3 \beta^2 + 12 A (\beta - \lambda)]}{6 [1 + A (\beta - 2 \lambda)]^2}}.$$ \hfill (3.14)

with $\Omega_{\phi} = 1$, and

$$w_{\phi} = w_{\text{eff}} = -1 + \frac{\beta (\beta - 6 A)}{3 [1 + A (\beta - 2 \lambda)]}.$$ \hfill (3.15)

This is the scalar-field dominated point which can be used for the late-time dark energy. In this case, the cosmic acceleration occurs for $w_{\text{eff}} < -1/3$, i.e.,

$$\frac{\beta (\beta - 6 A)}{3 [1 + A (\beta - 2 \lambda)]} < 2.$$ \hfill (3.16)

From the values of $Q_\phi$ and $c_2^\phi$ shown in Table I there are neither ghost nor Laplacian instabilities under the conditions:

$$1 + 2 A (3 \lambda - A) > 0,$$ \hfill (3.17)

$$(\beta + 2 A) (\beta - 6 A) > 0.$$ \hfill (3.18)
where Eq. (3.17) is the same as Eq. (3.9). Since \( y_c \) must be real, we require that
\[
6 - \beta^2 + 12A(\beta - \lambda) \geq 0, \quad (3.19)
\]
where we used the condition (3.17). If we demand that point (b) is the late-time attractor, the two eigenvalues \( \mu_1 \) and \( \mu_2 \) given in Appendix A need to be negative, so that
\[
\begin{align*}
\mu_1 &= \frac{\beta(\beta - 6A)}{1 + A(\beta - 2\lambda)} - 3 < 0, \quad (3.20) \\
\mu_2 &= \frac{\beta^2 - 6 + 12A(\lambda - \beta)}{2[1 + A(\beta - 2\lambda)]} < 0, \quad (3.21)
\end{align*}
\]
where we have chosen the value \( \gamma = 1 \) in Eq. (3.20). In the limit \( A \to 0 \), the conditions (3.17), (3.18) are automatically satisfied, while the other conditions (3.16), (3.19), (3.20), and (3.21) are satisfied for \( \beta^2 < 2 \). This upper bound of \( \beta \) is modified by the nonvanishing coupling \( A \). From Eq. (3.15), we observe that it is possible to realize \( w_\phi \approx -1 \) for the coupling \( A \) close to \( \beta / 6 \). We need to caution that the bound (3.18), which arises from the condition \( c_s^2 > 0 \), places the upper limit on the amplitude of \( A \). For \( \beta > 0 \), this bound translates to
\[
-\frac{\beta}{2} < A < \frac{\beta}{6}, \quad (3.22)
\]
and hence \( A \) cannot be larger than \( \beta / 6 \). For point (b), the density parameter \( \Omega_{c_2} \) is associated with the cubic coupling given by
\[
\Omega_{c_2} = \frac{A(\beta - 6A)[6 + 6A(\beta - \lambda) - \beta \lambda]}{3[1 + A(\beta - 2\lambda)]^2}, \quad (3.23)
\]
which vanishes for \( A = \beta / 6 \). For \( A \) close to \( \beta / 6 \), the cubic coupling slowdowns the evolution of \( \phi \), so that \( x_\phi \approx 0 \) in Eq. (3.13). In this case, the dominant contribution to \( \Omega_{c_2} \) comes from the potential energy, i.e., \( \Omega_{c_2} \approx y_c^2 = 1 \).

For the model presented in Sec. IV, the late-time cosmic acceleration is driven by point (b). There exists a viable parameter space in which all the conditions (3.16), (3.21) are consistently satisfied.

**Point (c):** This is a kinetic scaling solution which exists for \( A \neq 0 \). Since \( y_c = 0 \), the field potential does not play any role. The nonvanishing field kinetic energy \( x_\phi = \sqrt{6A}/(2A - 1) \) leads to the constant density parameter \( \Omega_{c_2} = 6A^2/(2A - 1) \). Since the scalar propagation speed squared is negative \( c_s^2 = -1/3 \), the physical viability condition \( 2.27 \) is not satisfied for \( A \neq 0 \). In the limit that \( A \to 0 \), this fixed point corresponds to a fluid-dominated solution (\( \Omega_{c_2} = 1 \)) with \( c_s^2 = 1 \), see Eq. (3.4). For \( A = 0 \), the eigenvalues \( \mu_1, \mu_2 \) given in Appendix A are in the ranges \( \mu_1 < 0 \) and \( \mu_2 > 0 \), so the fluid-dominated solution corresponds to a saddle point which can be used for the radiation or matter era.

We stress that, for \( A \neq 0 \), point (c) is excluded by the negative \( c_s^2 \), so it cannot play the role of scaling radiation or matter eras.

**Points (d1) and (d2):** These fixed points are kinetically dominated scalar field solutions where the kinetic energy of \( \phi \) is the dominant component to the total energy density. One of the eigenvalues \( \mu_1 = 3(2 - \gamma) \) is positive, so they are either unstable or saddle points. However, since \( w_{eff} = w_\phi = 1 \) and \( \Omega_\phi = 1 \) on these points, they are responsible for neither radiation nor matter eras. Moreover, it cannot be used for the late-time cosmic acceleration.

In summary, we showed that neither points (c) nor (d1,d2) are suitable to describe a viable cosmic expansion history after the onset of the radiation-dominated epoch. The point (a) can be responsible for scaling radiation and matter eras, but the solution does not exit from the scaling regime to the epoch of cosmic acceleration. This comes from the property that the scaling solution (a) is stable for \( \Omega_\phi < 1 \) with the other theoretical consistent conditions (3.8) and (3.9). From Eqs. (3.11) and (3.12), we generally require that the value of \( |\beta| \) be larger than order unity. In this case, the exponential potential is so steep that it is also difficult to satisfy all the conditions required for the existence and stability of point (b) with the cosmic acceleration. This situation changes for the scalar potential in which the slope \( \beta = -V_{\phi}/V \) decreases in time. In Sec. IV we will study a modified version of the present model by including a second potential term. As discussed in Ref. [24] for standard Quintessence, this allows the possibility for realizing scaling radiation/matter eras followed by the late-time accelerated expansion.

In Ref. [22], the authors showed that there exist scaling solutions for the cubic coupling \( g_3(Y) = c_1Y + c_2Y^2 \) by considering a field-dependent coupling \( Q(\phi) \) between the scalar field and nonrelativistic matter. In addition to the fact that a specific form of the coupling \( Q(\phi) = (b_1\phi + b_2)^{-1} \) was chosen, they made some additional assumptions for deriving the solution to \( g_3(Y) \). Moreover, their results are not directly applicable to the case \( Q = 0 \). As we explicitly showed above, scaling solutions are present even for the function \( g_3(Y) = A\ln Y \). Indeed, as we will show in a separate publication, there are scaling solutions even for general arbitrary functions \( g_3(Y) \).

While the cubic coupling \( g_4(Y) = A\ln Y \) is chosen in this paper due to its simplicity, it can accommodate most of the important properties of scaling solutions. Moreover, this is an explicit and simple example of showing the existence of scaling solutions other than the coupling \( g_3(Y) = c_1Y + c_2Y^2 \). In Sec. IV we will consider a double exponential potential for realizing an exit from the scaling matter era. In any scaling solution relevant to
radiation/matter eras, we need a mechanism of transition from the scaling matter era to the epoch of cosmic acceleration. Our choice of the double exponential potential does not lose the generality for describing such a transition.

IV. COSMOLOGICAL EVOLUTION FOR A CONCRETE DARK ENERGY MODEL

The model discussed in the previous section does not allow for viable cosmological evolution with the scaling radiation/matter era followed by the late-time dark energy attractor. Since the critical point (a) is always stable for \( \Omega_\phi < 1 \), the scaling solution does not exit to the epoch of cosmic acceleration driven by the fixed point (b). On the contrary, if the parameters are chosen such that the field has to approach point (b) at late time, the scaling behavior at early time is lost.

In this work, we would like to maintain the scaling behaviour in the early cosmological epoch as this property allows for a natural large value of the energy density of the field in the past despite of its small value at late time. To realize the proper cosmic expansion history with an early-time scaling period and a late-time dark energy attractor, we construct a model in which the two features associated with the critical points (a) and (b) discussed in Sec. III are present by adding a second exponential potential term similar to that used for Quintessence in Ref. [24]. More precisely, the model has the same \( G_3 \) function as Eq. (3.1) but with two exponential potentials of the form:

\[
V = V_1 e^{-\beta_1 \phi} + V_2 e^{-\beta_2 \phi}, \tag{4.1}
\]

where \( V_1, V_2, \beta_1, \beta_2 \) are positive constants with \( \beta_1 \gg O(1) \) and \( \beta_2 \lesssim O(1) \). The first potential \( V_1 e^{-\beta_1 \phi} \) gives rise to the scaling fixed point (a) with \( \beta = \beta_1 \), whereas the second potential \( V_2 e^{-\beta_2 \phi} \) leads to the scalar-field dominated point (b) with \( \beta = \beta_2 \). The expansion history of this model is not the same as standard Quintessence (hereafter QE) with two potentials because it is modified by the \( G_3 \) term. We call this model G3.

We take into account radiation and nonrelativistic matter whose background densities are given, respectively, by \( \rho_r \) and \( \rho_m \), so that \( \rho_\phi = \rho_r + \rho_m \). To study the background cosmological dynamics, we define the following dimensionless variables:

\[
y_1 = \sqrt{\frac{V_1 e^{-\beta_1 \phi}}{3H}}, \quad y_2 = \sqrt{\frac{V_2 e^{-\beta_2 \phi}}{3H}}, \\
\Omega_r = \frac{\rho_r}{3H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2}, \tag{4.2}
\]

with \( x = \phi / (\sqrt{6}H) \) and \( y^2 = y_1^2 + y_2^2 \). From the constraint Eq. (2.1), we obtain

\[
\Omega_m = 1 - \Omega_r - \Omega_\phi, \tag{4.3}
\]

where

\[
\Omega_\phi = \Omega_{G_2} + \Omega_{G_3}, \quad \Omega_{G_2} = x^2 + y_1^2 + y_2^2, \tag{4.4}
\]

with \( \Omega_{G_3} \) given by Eq. (2.23). We obtain the autonomous equations in the forms:

\[
x' = \frac{1}{\sqrt{6}} \tilde{f}(x, y) - \frac{\tilde{H}^2}{H^2} x, \tag{4.5}
\]

\[
y'_i = -\sqrt{\frac{3}{2}} \beta_i y_i - \frac{\tilde{H}^2}{H^2} y_i, \tag{4.6}
\]

\[
\Omega_r' = -4\Omega_r - 2 \frac{\dot{\tilde{H}}}{\tilde{H}^2} \Omega_r, \tag{4.7}
\]

where \( i = 1, 2 \). The function \( \tilde{f}(x, y) \) follows from Eq. (2.18) after the replacements \( \beta y^2 \rightarrow \beta_1 y_1^2 + \beta_2 y_2^2 \) and \( \gamma \Omega_\phi \rightarrow (4/3)\Omega_r + \Omega_m \), with Eq. (3.2). The derivative term \( \tilde{H}/H^2 \) is given by Eq. (2.20) with the correspondence \( \gamma \Omega_r \rightarrow (4/3)\Omega_r + \Omega_m \). The dark energy equation of state \( w_\phi \) follows from Eq. (2.24) after the replacements \( y^2 \rightarrow y_1^2 + y_2^2 \) and \( f \rightarrow \tilde{f} \).

The scaling radiation era corresponds to the fixed point (a1) given by

\[
(x, y_1, y_2, \Omega_m) = \left( \frac{2\sqrt{6}}{3\beta_1}, \frac{\sqrt{12 + 6A(3\beta_1 - 4\lambda)}}{3\beta_1}, 0, 0 \right), \tag{4.8}
\]

and \( \Omega_r = 1 - \Omega_\phi \) with \( \Omega_\phi = [4 + 2A(5\beta_1 - 4\lambda)]/\beta_1^2 \), whereas the scaling matter era is characterized by the critical point (a2) given by

\[
(x, y_1, y_2, \Omega_r) = \left( \frac{3}{2\beta_1}, \frac{\sqrt{6 + 12A(\beta_1 - \lambda)}}{2\beta_1}, 0, 0 \right), \tag{4.9}
\]

and \( \Omega_m = 1 - \Omega_\phi \) with \( \Omega_\phi = 3[1 + A(3\beta_1 - 2\lambda)]/\beta_1^2 \). The scalar-field dominated point (b) corresponds to

\[
x = \frac{\beta_2 - 6A}{\sqrt{6[1 + A(\beta_2 - 2\lambda)]}}, \quad y_1 = 0, \\
y_2 = \sqrt{\frac{1 + 2A(3A - \lambda)[6 - \beta_2^2 + 12A(\beta_2 - \lambda)]}{6[1 + A(\beta_2 - 2\lambda)]}}, \tag{4.10}
\]

with \( \Omega_m = \Omega_r = 0 \) and \( \Omega_\phi = 1 \).

Now, we have four parameters \{\( \beta_1, \beta_2, \lambda, A \)\} in our G3 model. We choose the two parameters \{\( \beta_1, \beta_2 \)\} and then constrain the values of \( \lambda \) and \( A \) according to the viability conditions discussed in Sec. III. The theoretically consistent conditions for points (a1) and (a2) are given by Eqs. (3.8), (3.10) with the replacement \( \beta \rightarrow \beta_1 \), where \( \gamma = 4/3 \) for (a1) and \( \gamma = 1 \) for (a2). There are also the BBN and CMB bounds (8.11) and (3.12) derived by setting \( \beta \rightarrow \beta_1 \). For point (b), we also require that the conditions (3.16), (3.21) hold with the replacement \( \beta \rightarrow \beta_2 \).

In Fig. 1, we plot the allowed parameter space in the \((\lambda, A)\) plane (light blue color) for (1) \( \beta_1 = 100, \beta_2 = 0.7 \)
show the parameter space in which the dark energy density parameter space (light blue) in the ($\lambda, \Omega$) plane for the two cases: (i) $\beta_1 = 100$, $\beta_2 = 0.7$ (left) and (ii) $\beta_1 = 100$, $\beta_2 = 2.5$ (right). Each boundary is obtained by using the conditions (3.8)-(3.10) for points (a1) and (a2) as well as the conditions (3.11)-(3.12) for point (b). The observational bounds (3.11) and (3.12) are also plotted, together with the region $\Omega_{\phi}^{(r)} < 10^{-3}$. The labels M1, M2, M3, and M4 correspond to the G3 models presented in Table II.

In case (i), the allowed parameter space is surrounded by several boundaries determined by the conditions $c_2^2(a^2) > 0$, $c_2^2(b^2) < 0$, $Q_s(a1) > 0$. They translate, respectively, to

\[
8A\beta_1 - 6A(A + \lambda) + 3 > 0, \quad (4.11)
\]
\[
-\frac{\beta_2}{2} < A < \frac{\beta_2}{6}, \quad (4.12)
\]
\[
1 + 2A (3A - \lambda) > 0, \quad (4.13)
\]

where the condition (4.12) corresponds to $-0.35 < A < 0.117$ for $\beta_2 = 0.7$. All the other theoretically consistent conditions are satisfied in the viable parameter region plotted in the left panel of Fig. 1. The observational bounds (3.11) and (3.12) exclude only a narrow region of the theoretically consistent parameter space. We also show the parameter space in which the dark energy density parameter in the scaling radiation era is in the range $\Omega_{\phi}^{(r)} < 10^{-3}$. This condition is not obligatory, but we plot such a region for the purpose of understanding the parameter space in which the primordial scaling value of $\Omega_{\phi}$ is small.

In case (ii), in the limit $A \to 0$, the slope $\beta_2 = 2.5$ is too large to satisfy the stability condition (3.20) of point (b). Moreover, for $A = 0$, the cosmic acceleration occurs for $w_{\text{eff}} = w_{\phi} = -1 + \beta_2^2/3 < -1/3$, i.e., $\beta_2^2 < 2$. On the other hand, the nonvanishing cubic coupling $A$ allows for the possibility of cosmic acceleration even for $\beta_2^2 > 2$. Indeed, the viable region for $A > 0$ is determined by the condition $w_{\text{eff}}(b) = w_{\phi}(b) < -1/3$, i.e.,

\[
\frac{\beta_2(\beta_2 - 6A)}{1 + A(\beta_2 - 2\lambda)} - 2 < 0, \quad (4.14)
\]

under which $\mu_1(b) < 0$. Note that the condition (4.14) also determines the upper bound for $A < 0$ (see the right panel of Fig. 1). The other boundaries of viable parameter space are determined by the conditions $c_2^2(a2) > 0$ and $c_2^2(b) < 0$, i.e., by Eqs. (4.11) and (4.12). The important difference from case (i) is that the observational bounds (3.11) and (3.12) restrict a broader range of theoretically consistent model parameters. Since there exists a viable parameter space even for $\beta_2^2 > 2$, the cubic coupling allows a wider allowed range of $\beta_2$ compared to QE.

In Table II we show four different models M1, M2, M3, and M4, all of which are inside the viable region depicted in Fig. 1. We also consider two QE models with $\beta_1 = 100$: QE1 ($\beta_2 = 0.7$) and QE2 ($\beta_2 = 2.5$). In the following, we study the cosmological evolution in these models by paying particular attention to the effect of the cubic coupling on the background dynamics. In doing so, we first comment on the issue of ICs. Unless otherwise stated, we select the ICs of $x$ and $y_1$ corresponding to

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2 We note that theoretically consistent regions resulting from this analysis ensures the viability at critical points not along the whole evolution of the system, which instead needs to be confirmed. We have tested for several combinations of parameters in these regions and found that the system is stable at any time.
Figure 2. (Left) Evolution of the total fluid density $\rho_m + \rho_r$ (blue, solid line) and the scalar-field density $\rho_\phi$ for M1 (orange, dashed line) and QE1 (green, dot-dashed line). The model parameters for M1 and QE1 are given in Table II. The ICs of $x, y_1$ are chosen to be close to those of critical point (a), with $y_2, \Omega_r$ realizing today’s density parameters $\Omega^{(0)}_m = 0.68$ and $\Omega^{(0)}_r = 10^{-4}$. (Right) Evolution of $\rho_m + \rho_r$ and $\rho_\phi$ for the same model parameters as those in the left, but with different ICs: $x = 0.015$, $y_1 = 0.04$ for M1 and $x = 0.105$, $y_1 = 0.1$ for QE1.

Table II. Model parameters $\beta_2, A, \lambda$ used in the numerical simulations of Figs. 2-6. For all of them, $\beta_1 = 100$. In each model, we also show today’s dark energy equation of state $w_\phi^{(0)}$, today’s effective equation of state $w_\phi^{(eq)}$, and the dark energy density parameter $\Omega_\phi$ at the redshift $z = 50$. Except for QE2, all the other models give rise to the cosmic acceleration today ($w_\phi^{(eq)} < -1/3$).

| Model | $\beta_2$ | $A$ | $\lambda$ | $w_\phi^{(0)}$ | $w_\phi^{(eq)}$ | $\Omega_\phi(z = 50)$ |
|-------|-----------|-----|-----------|---------------|----------------|---------------------|
| M1    | 0.7       | -0.3| 154       | -0.993        | -0.675        | 1.0 x 10^{-4}      |
| M2    | 0.7       | 0.09| -8        | -0.988        | -0.672        | 8.9 x 10^{-4}      |
| M3    | 0.7       | -0.28| 148.3    | -0.993        | -0.675        | 4.3 x 10^{-4}      |
| M4    | 2.5       | -1   | 150       | -0.975        | -0.663        | 3.6 x 10^{-4}      |
| QE1   | 0.7       | 0    | 0         | -0.927        | -0.630        | 3.2 x 10^{-4}      |
| QE2   | 2.5       | 0    | 0         | -0.358        | -0.167        | 3.3 x 10^{-4}      |

in the early cosmological epoch ($\rho_\phi \propto \rho_m + \rho_r$). In this case, the field density parameters corresponding to points (a1) and (a2) are given by $\Omega_\phi(a1) = 7.4 \times 10^{-3}$ and $\Omega_\phi(a2) = 1.9 \times 10^{-4}$, respectively, which are consistent with the bounds (5.11) and (5.12). They are by one order of magnitude larger than the corresponding values in QE. Indeed, this property can be confirmed in the left panel of Fig. 2.

In the right panel of Fig. 2, we show the evolution of $\rho_\phi$ and $\rho_m + \rho_r$ for M1 by changing the ICs of $x$ by 0.01% and $y_1$ by 1%. At the same time, the ICs for QE1 are changed by 0.01% in $x$ and by 10% in $y_1$. For M1, the solutions approach the scaling critical point (a1) after a few oscillations in the field density. For QE1, the larger change of $y$ relative to M1 does not induce oscillations in the field density, but its takes some time to reach the scaling regime. The important point is that, even in presence of the cubic coupling $G_3$, the first exponential potential $V_1 e^{-\beta_1 \phi}$ leads to stable scaling fixed points (a1) and (a2) with $\Omega_\phi < 1$ that always attracts solutions with different ICs.

Since the additional exponential potential $V_2 e^{-\beta_2 \phi}$ is present, the solutions finally exit from the scaling matter era to the epoch of cosmic acceleration driven by the critical point (b). From Eqs. (3.15) and (3.23), the dark energy equation of state and the density parameter arising from $G_3$ at point (b) are

$$w_\phi = -1 + \frac{\beta_2(\beta_2 - 6A)}{3[1 + A(\beta_2 - 2\lambda)]}, \quad (4.15)$$

$$\Omega_{G_3} = \frac{A(\beta_2 - 6A)(6 + 6A(\beta_2 - \lambda) - \beta_2 \lambda)}{3[1 + A(\beta_2 - 2\lambda)]^2}. \quad (4.16)$$
which give $w_\phi = -0.994$ and $\Omega_{G_3} = -5.0 \times 10^{-3}$ for M1. The density associated with the $G_3$ term is suppressed at low redshifts, so that the dominant contribution to $\Omega_\phi$ comes from the standard field density $\Omega_{G_2}$. For QE1 we have $w_\phi = -0.837$ at point (b), so the field density $\rho_\phi$ for M1 decreases more slowly relative to that for QE1 in the future ($z < 0$). This behavior can be confirmed in the numerical simulation of Fig. 2.

To ensure the absence of ghost and Laplacian instabilities, we need to confirm that $Q_\phi$ and $c_s^2$ given by Eq. (3.3) and (3.4) remain positive. In Fig. 3, we plot the evolution of those quantities for the model M1 by choosing ICs same as those used in the left panel of Fig. 2. The values of $c_s^2$ on points (a1) and (a2) can be obtained by substituting $\gamma = 4/3$ and $\gamma = 1$ with $\beta = \beta_1$ into $c_s^2$ at point (a) given in Table I, respectively, while $c_s^2 = 1+8A/(\beta_2 -6A)$ on point (b). They are in good agreement with the numerical simulation of Fig. 3. Moreover, during the transient regimes between critical points, $c_s^2$ remains positive without crossing 0. This is also the case for $Q_\phi$, so the model M1 suffers neither ghost nor Laplacian instabilities during the whole cosmological evolution. We have confirmed that such conditions are also satisfied for all the models listed in Table II.

In Fig. 3 the evolution of density parameters is plotted for the model M2, which exists in the region $\lambda < 0$ and $A > 0$ in the left panel of Fig. 1. We observe that $\Omega_{G_3}$ dominates over $\Omega_{G_2}$ during the early cosmological epoch, but the main contribution to $\Omega_\phi$ comes from $\Omega_{G_2}$ at redshifts $z \lesssim 10$. From Eq. (4.16), we have $\Omega_{G_2} = 0.012$ on point (b), which is positive. This property is different from the model M1, in which $\Omega_{G_3}$ is negative on point (b). The important point is that the cubic coupling can provide the dominant contribution to $\Omega_\phi$ in the scaling radiation and matter eras, but its effect on $\Omega_\phi$ tends to be suppressed ($|\Omega_{G_3}| \ll 1$) at low redshifts.

From Eq. (4.15), the dark energy equation of state on point (b) for M2 is given by $w_\phi = -0.985$, which is again closer to $-1$ relative to the value $-0.837$ for QE1. Indeed, it approaches the value $w_\phi = -1$ as the model shifts to the upper boundary $A = \beta_2/6$ of the viable region plotted in the left panel of Fig. 1. At the same time, the contribution of $\Omega_{G_3}$ to $\Omega_\phi$ decreases toward 0. For some specific models like covariant Galileons, the field density dominated by cubic interactions at low redshifts can give rise to the galaxy-ISW anti-correlation incompatible with current observations [41] [51], so it is anticipated that the G3 models satisfying the condition $|\Omega_{G_3}| \ll 1$ at late times may evade such constraints.

As we see in the right panel of Fig. 1 there are models...
in which all the theoretically consistent conditions are satisfied even for $\beta_2^2 > 2$. In Fig. 5, we show the evolution of $\rho_\phi$ and $\rho_m + \rho_r$ for the models M4 and QE2 (in which $\beta_2 = 2.5$). For QE2, the eigenvalue $\mu_1$ on point (b) is positive with $w_{\text{eff}} = w_\phi > -1/3$, so the Universe does not enter the stage of cosmic acceleration. Instead, the scaling matter era ($\Omega_\phi = 3/\beta_2^2 = 3 \times 10^{-4}$) is followed by the other scaling matter fixed point (a2) driven by the second exponential potential $V_2 e^{-\beta_2 \phi}$ with $\Omega_\phi = 3/\beta_2^2 = 0.48$. As we see in Table I, today’s value of $w_{\text{eff}}$ for QE2 is larger than $-1/3$, so the Universe does not exhibit the cosmic acceleration today.

For the model M4, the coupling $G_3$ allows the possibility for realizing the scaling radiation era with $\Omega_\phi = 0.02$. While $\Omega_{G_1}$ dominates over $\Omega_{G_2}$ on point (a1), the contribution $\Omega_{G_3}$ exactly vanishes on point (a2) for M4 and hence $\Omega_\phi = 3/\beta_2^2 = 3 \times 10^{-4}$. Since $\Omega_\phi$ for point (a2) is by two orders of magnitude smaller than that for point (a1), it takes some time for the solutions to move from (a1) to (a2). Indeed, the second exponential potential $V_2 e^{-\beta_2 \phi}$ starts to contribute to the field density before the solutions completely approach point (a2). This is the reason why $\rho_\phi$(M4) in Fig. 5 decreases faster than $\rho_m + \rho_r$ in the redshift range $10 \lesssim z \lesssim 1000$.

For M4, around the redshift $z \lesssim 10$, the solutions start to approach point (b) characterized by $w_\phi = -0.976$ and $\Omega_{G_3} = -0.016$, so the cosmic acceleration occurs even for $\beta_2^2 > 2$. In spite of the dominance of $\Omega_{G_3}$ in the scaling radiation era, $\Omega_{G_3}$ is suppressed relative to $\Omega_{G_2}$ at low redshifts. With this example, we showed that the allowed parameter space for $\beta_2$ is wider than that for QE.

In Fig. 6, we plot the evolution of $w_\phi$ for all the G3 models and QE1 presented in Table I. For M1 and M2, the scaling radiation era ($w_\phi \simeq 1/3$) is followed by the scaling matter epoch ($w_\phi \simeq 0$). In the models M3 and M4, the scaling matter era is practically absent by reflecting the fact that $\Omega_\phi$ at point (a2) is much smaller than that at point (a1), e.g., $\Omega_\phi(a1) = 5.6 \times 10^{-3}$ and $\Omega_\phi(a2) = 1.4 \times 10^{-5}$ for M3. In such cases, the matter-dominated epoch corresponds to the transient period between critical points (a1) and (b). For M3, the field density parameter at the redshift $z = 50$ is $\Omega_\phi = 4.3 \times 10^{-5}$, which is smallest among the G3 models studied above. The Planck bound (3.12) is satisfied for all the models listed in Table II.

The dark energy equation of state today ($w_\phi^{(0)}$) is related to the model parameters $A$ and $\beta_2$. For increasing $|A|$ from 0, we need to choose larger values of $\beta_2$ for the reason of theoretical viability. The larger $\beta_2$ results in $w_\phi^{(0)}$ deviating from $-1$. In Fig. 6, we observe that M4 gives the highest values of $w_\phi^{(0)}$ and $w_{\text{eff}}^{(0)}$ among the G3 models listed in Table II. However, all the G3 models give rise to $w_\phi^{(0)}$ closer to $-1$ than that in QE1. This behavior of G3 models is also in better agreement with the latest cosmological constraints [12, 73] compared to QE with the large deviation of $w_\phi^{(0)}$ from $-1$.

In summary, we found the following new features in our cubic Horndeski model.

- The model allows for a scaling behavior at early time and a dark energy attractor at late time;
- The viable parameter space is wider than that for QE;
- There exist G3 models in which the dark energy equation of state today ($w_\phi^{(0)}$) is close to $-1$ even for $\beta_2^2 > 2$;
- The model can be consistent with the BBN and CMB bounds (3.11) and (3.12);
- The cubic coupling can provide the dominant contribution to $\Omega_\phi$ in the early scaling epoch, but its contribution to the field density is typically suppressed at low redshifts ($|\Omega_{G_3}| \ll \Omega_{G_2}$).

V. INSIGHT ON MODIFICATIONS OF GRAVITY AT LARGE SCALES

Finally, we discuss the impact of our G3 models on the evolution of linear scalar perturbations relevant to the growth of structures. Let us consider the perturbed line element on the flat FLRW background given by

$$ds^2 = -(1 + 2\Psi) dt^2 + a^2(t) (1 - 2\Phi) \delta_{ij} dx^i dx^j,$$

where $\Psi$ and $\Phi$ are gravitational potentials in the Newtonian gauge. We define the gravitational slip parameter

$$\eta = \frac{\Phi}{\Psi},$$

which characterizes the difference between two gravitational potentials.
For the matter sector, we take into account nonrelativistic matter components with the background density \( \rho_i \) and the density contrast \( \Delta_i = \delta \rho_i / \rho_i \). In Fourier space with the coming wavenumber \( k \), we relate \( \Psi \) and the total matter density perturbation \( \rho \Delta = \sum_i \rho_i \Delta_i \), as

\[
-k^2 \Psi = 4\pi G_N a^2 \mu \rho \Delta ,
\]

which corresponds to the modified Poisson equation. If the quantity \( \mu \) deviates from 1, this leads to the modified growth of matter density contrast \( \Delta \) compared to that in GR. Along with \( \mu \) and \( \eta \), we also define a quantity \( \Sigma \) that relates the weak lensing potential \( \Psi + \Phi \) with \( \Delta \), as

\[
- k^2 (\Psi + \Phi) = 8\pi G a^2 \mu \rho \Delta ,
\]

where

\[
\Sigma = \frac{1 + \eta}{2} \mu .
\]

In GR with the field Lagrangian \( G_2(\phi, X) \) we have \( \eta = \mu = \Sigma = 1 \), so any departure from these values translates to a signature of the modification of gravity.

In the scaling radiation and matter epochs driven by points (a1) and (a2), respectively, \( \mu \) depends on \( A, \lambda, \beta \). On the scalar-field dominated point (b), \( \mu \) is affected by \( \beta_2 \) besides \( A, \lambda \).

In Fig. 7 we plot the evolution of \( \mu - 1 \) for all the G3 models listed in Table 1. Around the critical points (a1), (a2), and (b), the numerical values of \( \mu - 1 \) exhibit good agreement with the analytic results (5.9)-(5.11), e.g., in the model M1, \( \mu - 1 = 0.0054, 0.014, 0.048 \) for (a1), (a2), (b), respectively. The quantity \( \mu - 1 \) increases with time for all the cases shown in Fig. 7. The deviation of \( \mu \) from 1 today is about 2% for M1 and M3, 0.08% for M2, and 8% for M4. As expected from Eqs. (5.9)-(5.11), the largest deviation from GR arises for the model with highest value of \( A^2 \).

In Ref. [58], the authors studied phenomenological constraints on \( \mu \) and \( \Sigma \) by using a specific parametrization for time-dependent functions (Padé functions) appearing in the effective field theory of dark energy. The parameter space exists mostly in the region \( (\mu - 1)(\Sigma - 1) \geq 0 \).

\[ \eta = 1, \quad \mu = \Sigma = 1 + \frac{\alpha_B^2}{Q_s c_s^2 (1 + \alpha_B)^2} , \] (5.6)

where the braiding parameter \( \alpha_B \) is given by

\[
\alpha_B = - \frac{\phi X G_{3, X}}{H} = - \sqrt{6} A x .
\] (5.7)

On using Eqs. (3.3) and (3.4), it follows that

\[
\mu = \Sigma = 1 + \frac{6 A^2 x}{4 \sqrt{6} A - 3 x (2 A^2 + 2 A \lambda - 1)} .
\] (5.8)

From the above expressions there is no gravitational slip \( (\Psi = \Phi) \), but the cubic coupling \( G_3 \) modifies the growth of structures \( (\mu \neq 1) \) and the evolution of weak lensing potential \( \Sigma \neq 1 \). The deviation of \( \Sigma \) from 1 also gives rise to modifications to the integrated Sachs-Wolfe effect in the CMB. Moreover, we infer from Eq. (5.6) that \( \mu = \Sigma > 1 \) under the absence of scalar ghosts \((Q_s > 0)\) and Laplacian instabilities \((c_s^2 > 0)\).

For perturbations relevant to the observations of large-scale structures and weak lensing, we are interested in the modes deep inside the sound horizon \((c_s^2 k^2 > a^2 H^2)\).

Provided that the oscillating mode of scalar-field perturbations is negligible compared to the matter-induced mode, we can resort to the so-called quasi-static approximation under which the dominant contributions to the perturbation equations of motion correspond to those containing \( k^2 / a^2 \) and \( \delta \rho_i \). In our cubic Horndeski theory, the quasi-static approximation for the modes deep inside the sound horizon gives the results were derived by choosing the unitary gauge in which the field perturbation \( \delta \phi \) vanishes, but \( \eta, \mu, \Sigma \) are gauge-invariant quantities in the small-scale limit.
In particular, if the stability conditions (absence of ghost and Laplacian instabilities) and observational priors are imposed, it was shown that the region with $\mu \geq 1$ and $\Sigma \geq 1$ is most favored. Since our G3 models predict $\mu = \Sigma > 1$, they are consistent with the recent bounds on $\mu$ and $\Sigma$.

VI. CONCLUSIONS

In this paper, we have proposed a viable model of cosmic acceleration in the framework of cubic-order Horndeski theories. We searched for scaling solutions to alleviate the coincidence problem of dark energy in the presence of cubic Horndeski interactions besides the standard field kinetic term with two exponential potentials. Extending the analysis of scaling solutions performed for the K-essence Lagrangian $G_2(\phi, X)$ [27,28], we assumed that the cubic coupling $G_3$ is a function of $Y = X e^{\lambda \phi}$ and found a new type of scaling solutions for the coupling $G_3(Y) = A \ln Y$.

In Sec. IV, we first performed a thorough dynamical analysis of the background cosmology for the cubic coupling $G_3(Y) = A \ln Y$ with a single exponential potential $V(\phi) = V_0 e^{-\beta \phi}$. We derived critical points of the system and studied their stability. For each critical point, we also discussed conditions for the absence of ghost and Laplacian instabilities in the small-scale limit. The scaling solution (a) and the scalar-field dominated solution (b) given in Table I are the two important critical points of our system. On using theoretically consistent conditions, we showed that point (a) is always stable for $\Omega_\phi < 1$. Hence the solutions do not exit from the scaling regime to the epoch of cosmic acceleration driven by point (b). This situation is analogous to what happens for Quintessence with the potential $V(\phi) = V_0 e^{-\beta \phi}$.

To realize scaling radiation and matter eras followed by the epoch of cosmic acceleration, we considered two exponential potentials $V(\phi) = V_1 e^{-\beta_1 \phi} + V_2 e^{-\beta_2 \phi}$ with $\beta_1 \gg \mathcal{O}(1)$ and $\beta_2 \lesssim \mathcal{O}(1)$ in Sec. IV. In this case, the first potential $V_1 e^{-\beta_1 \phi}$ gives rise to the scaling radiation and matter critical points (a1) and (a2) given by Eqs. (4.8) and (4.9), respectively. The density parameter $\Omega_{\phi 1}$ arising from the cubic coupling can provide an important contribution to the total field density parameter $\Omega_\phi$ in the scaling regime. Finally, the solutions enter the epoch of cosmic acceleration by approaching the critical point (b) arising from the second exponential potential $V_2 e^{-\beta_2 \phi}$.

For the model proposed in Sec. IV, we have found some interesting features which make the model appealing as a viable dark energy candidate. Our findings are summarized below.

1. Scaling solutions followed by the dark energy attractor: As illustrated in Fig. 2, we have numerically confirmed that the solutions first enter the scaling radiation regime and finally approach the dark energy attractor. The duration of the scaling matter era $(w_\phi \approx 0)$ depends on how similar the values of $\Omega_\phi$ on the critical points (a1) and (a2) are (see Fig. 6). Even if $\Omega_{G_3}$ dominates over the standard field density parameter $\Omega_G$ in scaling radiation/matter dominated epochs, the former tends to be suppressed at low redshifts (see Fig. 4).

2. Parameter space: We derived theoretically consistent conditions for points (a1), (a2), (b) and showed the existence of viable parameter space in Fig. 1 for some values of $\beta_1, \beta_2$. We also considered four different G3 models existing inside the viable parameter space and showed that the quantities $Q_s$ and $c_s^2$ remain positive throughout the cosmological evolution (see Fig. 3). The G3 models can be also cosmologically viable even for $\beta_2^2 > 2$ (such as the model M4 shown in Fig. 4), while this is not the case for Quintessence. Hence the cubic coupling offers the possibility for realizing the cosmic acceleration even for the steep second exponential potential $V_2 e^{-\beta_2 \phi}$ with $\beta_2^2 > 2$.

3. Dark energy equation of state today, $w_\phi^{(0)}$: For all the G3 models presented in Table II, we found that $w_\phi^{(0)}$ is closer to $-1$ in comparison to Quintessence. Then, these G3 models are in better agreement with recent observational data. Since the evolution of $w_\phi$ after the onset of the matter-dominated epoch is also different among different G3 models, it will be also possible to observationally distinguish between them with future high-precision data.

4. Early-time dark energy density: The BBN and CMB bounds (3.11) and (3.12) on the density parameter $\Omega_\phi$ put further constraints on the model parameters, but a wide range of viable parameter space is still left (see Fig. 1). There exist G3 models like M3 given in Table II where $\Omega_\phi$ around the redshift $z = 50$ is smaller than that in Quintessence by one order of magnitude.

5. Linear perturbations: We gave a hint to the expected modification of gravity on scales relevant to the growth of large-scale structures. Under the quasi-static approximation for modes deep inside the sound horizon, we showed that the parameters $\mu$ and $\Sigma$, which are related to the Newtonian and weak lensing gravitational potentials respectively, are given by Eq. (5.8). As we observe in Fig. 7, the deviation of $\mu$ from 1 induced by the cubic coupling $G_3$ tends to increase for lower redshifts. Thus, the G3 models give rise to observational signatures that can be distinguished from Quintessence.

We conclude that the proposed model reveals very interesting features for realizing the late-time acceleration and alleviating the coincidence problem. It will be of interest to further analyze its phenomenology and to put observational constraints on the model parameters. In
particular, although the density associated with cubic interactions is typically suppressed at low redshifts in our model, the galaxy-ISW correlation data may put further bounds on the allowed parameter space. The investigation of other forms of function $G_m(Y)$ could also exhibit interesting phenomenology. It would be also relevant to study the cosmology in the presence of couplings between the scalar field and matter (which are present for the original construction of scaling solutions in K-essence [27, 28]). These issues are left for future works.

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Appendix A: Eigenvalues of the critical points

The stabilities of critical points $(x_c, y_c)$ presented in Table I are known by considering homogenous perturbations $(\delta x, \delta y)$ around them, such that

$$x = x_c + \delta x, \quad y = y_c + \delta y. \quad (A1)$$

Substituting these expressions into Eqs. (2.16) and (2.17), the perturbations, at linear order, obey the differential equations

$$\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad (A2)$$

where the Jacobian matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \bigg|_{x=x_c, y=y_c}. \quad (A3)$$

The general solutions to $\delta x$ and $\delta y$ can be expressed as the linear combinations of two terms $e^{\mu_1 N}$ and $e^{\mu_2 N}$, where $\mu_1$ and $\mu_2$ are the eigenvalues of $\mathcal{M}$. As we explained in Sec. II, the stabilities of fixed points are determined by the signs of $\mu_1$ and $\mu_2$.

For the critical points presented in Table I, we obtain the following eigenvalues:

- Point (a):
  $$\mu_1 = \frac{3}{4} (\gamma - 2), \quad \mu_2 = \frac{3(2 - \gamma)}{2[1 + A(\beta - 2\lambda)]}, \quad (A4)$$

  where $\Omega_\phi$ is given by Eq. (3.6).

- Point (b):
  $$\mu_1 = -3\gamma + \frac{\beta(\beta - 6A)}{1 + A(\beta - 2\lambda)}, \quad \mu_2 = \frac{\beta^2 - 6 + 12A(\lambda - \beta)}{2[1 + A(\beta - 2\lambda)]}. \quad (A5)$$

- Point (c):
  $$\mu_1 = -\frac{3}{2} (2 - \gamma), \quad \mu_2 = \frac{3\gamma}{2} + \frac{3A\beta}{1 - 2A\lambda}. \quad (A6)$$

- Points (d1) and (d2):
  $$\mu_1 = 3(2 - \gamma), \quad \mu_2 = \frac{6 + 6A(\beta - 2\lambda) \mp \sqrt{6}\beta\sqrt{1 + 2A(3A - \lambda)}}{2(1 - 2A\lambda)}. \quad (A7)$$

where the ($-$) and ($+$) signs of $\mu_2$ correspond to the critical points (d1) and (d2), respectively.

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