A duality in classical and quantum mechanics:
General results

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\textsc{Abstract:} A duality in classical and quantum mechanics is revealed. It is shown that a potential has a series of dual potentials. All potentials that are dual to each other form a dual family. In a dual family, once the solution of one potential is solved, the solutions of all other potentials can be obtained by the dual transform immediately. The general result of the duality of one-dimensional potentials and three-dimensional central potentials are provided. Some examples are given.

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1 Introduction

It is revealed that there is a duality in classical mechanics and quantum mechanics. This duality implies that every potential has a series of dual potentials, forming a dual family. Once the dynamical equation — the Newton equation in classical mechanics or the Schrödinger equation in quantum mechanics — of a potential in a dual family is solved, the solutions of all the potentials in the family are also obtained by the dual relation.

A famous special case of this duality is the Newton-Hooke duality in classical mechanics, revealed by Newton himself in his Principia [1]. The Newton-Hooke duality is the duality between the Newton gravity potential and the harmonic-oscillator potential. In this paper, we provide a general result for arbitrary potentials in classical mechanics and in quantum mechanics.

Duality is an important concept in modern physics, such as the AdS/CFT duality [2–6], the gravity/liquid duality [7–13], the gravolectric duality [14]. The modern formulation of the Newton-Hooke duality can be found in Refs. [15–17].

In sections 2 and 4, we provide general dual relations for one-dimensional potentials and for three-dimensional central potentials in classical mechanics. In sections 3 and 5, we provide general dual relations for one-dimensional potentials and for three-dimensional central potentials in quantum mechanics. Examples are provided, such as the dual potentials of the power potentials and the Poschl-Teller potential. The conclusion is given in section 6.
2 Duality in classical mechanics: one-dimensional potential

Two one-dimensional potentials, \( U(x) \) and \( V(\xi) \), are dual to each other, if they are related by
\[
x^{-2} [U(x) - E] = \xi^{-2} [V(\xi) - \mathcal{E}]
\]
with
\[
x \leftrightarrow \xi^{2/(\sigma+3)},
\]
where \( E \) and \( \mathcal{E} \) are the energies of the orbits of \( U(x) \) and \( V(\xi) \), respectively. The solution of the potential \( V(\xi) \) with the energy \( \mathcal{E} \) and the solution of its dual potential \( U(x) \) with the energy \( E \) can be obtained from each other by the replacement of the time:
\[
t \leftrightarrow \frac{2}{\sigma+3} \tau.
\]
Here \( \sigma \) is a constant chosen arbitrarily.

Note that here a potential can have an infinite number of dual potentials, since there exist infinite choices of the parameter \( \sigma \) in the dual relation.

**Proof.** The equation of the potential \( U(x) \) with the energy \( E \) is [18]
\[
\frac{dt}{dx} = \frac{1}{\sqrt{2[E - U(x)]}}.
\]
Substituting the dual transforms (2.2) and (2.3) into Eq. (2.4) gives
\[
\frac{d\tau}{d\xi} = \frac{1}{\sqrt{-2\xi^{2/(\sigma+3)} [U(\xi^{2/(\sigma+3)}) - E]}}.
\]
Eq. (2.5) is just the equation of the potential
\[
V(\xi) = \xi^{2/(\sigma+3)} [U(x) - E] + \mathcal{E}
\]
with the energy \( \mathcal{E} \):
\[
\frac{dt}{dx} = \frac{1}{\sqrt{2[\mathcal{E} - V(\xi)]}}.
\]
Eq. (2.6) is just the dual relations (2.1) and (2.2).

3 Duality in quantum mechanics: one-dimensional potential

3.1 Duality

Two one-dimensional potentials, \( U(x) \) and \( V(\xi) \), are dual to each other, if they are related by
\[
\frac{2}{\sigma+3} \left\{ x^2 [U(x) - E] + \frac{1}{4} \right\} = \frac{\sigma+3}{2} \left\{ \xi^2 [V(\xi) - \mathcal{E}] + \frac{1}{4} \right\}.
\]
with
\[
x \leftrightarrow \xi^{2/(\sigma+3)}.
\]
The eigenfunction of the potential \( V(\xi) \) and the eigenfunction of its dual potential \( U(x) \) can be obtained from each other by the replacement

\[
u(x) \leftrightarrow \xi \frac{\sigma + 1}{2(\sigma + 3)} v(\xi).
\]

(3.3)

Here \( \sigma \) is a constant chosen arbitrarily.

**Proof.** Performing the dual transforms (3.2) and (3.3) to the one-dimensional stationary Schrödinger equation of the potential \( U(x) \),

\[
\frac{d^2 u(x)}{dx^2} + [E - U(x)] u(x) = 0,
\]

(3.4)

gives

\[
\frac{d^2 v(\xi)}{d\xi^2} + \left(\frac{2}{\sigma + 3}\right)^2 \left\{ \frac{1 - \left(\frac{\sigma + 3}{4}\right)^2}{4\xi^2} - \xi^{-2}\left(\frac{\sigma + 1}{\sigma + 3}\right) \left[U\left(\xi^{2/(\sigma + 3)}\right) - E\right] \right\} v(\xi) = 0.
\]

(3.5)

This is also a stationary Schrödinger equation

\[
\frac{d^2 v(\xi)}{d\xi^2} + [\mathcal{E} - V(\xi)] v(\xi) = 0
\]

(3.6)

with

\[
V(\xi) = \left(\frac{2}{\sigma + 3}\right)^2 \left\{ \frac{1 - \left(\frac{\sigma + 3}{4}\right)^2}{4\xi^2} + \xi^{-2}\left(\frac{\sigma + 1}{\sigma + 3}\right) \left[U\left(\xi^{2/(\sigma + 3)}\right) - E\right] \right\} + \mathcal{E}.
\]

(3.7)

This is just the dual relations (3.1) and (3.2). ■

### 3.2 Poschl-Teller potential: example

Take the Poschl-Teller potential as an example. For the Poschl-Teller potential

\[
U(x) = \alpha \sech^2 x,
\]

(3.8)

the stationary Schrödinger equation

\[
\frac{d^2 u(x)}{dx^2} + (E - \alpha \sech^2 x) u(x) = 0
\]

(3.9)

has the following solution:

\[
u(x) = P_l^{\sqrt{E}} \left(\frac{\theta}{\sqrt{1 - 4\alpha - 1}}\right)^{1/2} (\tanh x)
\]

(3.10)

with \( P_l^m(z) \) the associated Legendre polynomial.

The dual relations (3.1) and (3.2) give the dual potential of the Poschl-Teller potential:

\[
V(\xi) = \frac{1}{4} \left[ \left(\frac{2}{\sigma + 3}\right)^2 - 1 \right] \frac{1}{\xi^2} + \left(\frac{2}{\sigma + 3}\right)^2 \xi^{4/(\sigma + 3) - 2} \left[\alpha \sech^2 \xi^{2/(\sigma + 3)} - E\right] + \mathcal{E}.
\]

(3.11)

A potential has an infinite number of dual potentials: different choice of \( \sigma \) give different dual potentials.
Choosing $\sigma = -1$ gives the Poschl-Teller potential itself. Concretely, $\sigma = -1$ gives $V(\xi) = \alpha \operatorname{sech}^2 \xi - E + \mathcal{E}$. Taking $E = \mathcal{E}$ recovers the Poschl-Teller potential.

Choosing $\sigma = 1$ gives

$$V(\xi) = -\frac{3}{16\xi^2} + \frac{1}{4\xi} \left( \alpha \operatorname{sech}^2 \sqrt{\xi} + \alpha \xi - E \right) - \frac{\alpha}{4} + \mathcal{E}.$$  \hspace{1cm} (3.12)

Taking $\mathcal{E} = \alpha/4$, i.e., letting the coupling constant in the Poschl-Teller potential, $\alpha$, to be the energy eigenvalue of its dual potential,

$$V(\xi) = -\frac{3}{16\xi^2} + \frac{1}{4\xi} \left( \alpha \operatorname{sech}^2 \sqrt{\xi} + \alpha \xi - E \right).$$  \hspace{1cm} (3.13)

The eigenfunction of the potential $V(\xi)$ with the eigenvalue $\frac{\alpha}{4}$ then reads

$$v(\xi) = \xi^{1/4} P^{\sqrt{E}}_{\sqrt{4\xi^2 - 1}} \left( \tanh \sqrt{\xi} \right).$$  \hspace{1cm} (3.14)

Choosing $\sigma = 3$ gives

$$V(\xi) = -\frac{2}{9\xi^2} + \frac{1}{9\xi^{4/3}} \left( \alpha \operatorname{sech}^{2/3} \xi^{1/3} - \frac{2\alpha}{3} \xi^{4/3} - E \right) + \frac{2\alpha}{27} + \mathcal{E}.$$  \hspace{1cm} (3.15)

Taking $\mathcal{E} = -\frac{2\alpha}{27}$ gives the dual potential

$$V(\xi) = -\frac{2}{9\xi^2} + \frac{1}{9\xi^{4/3}} \left( \alpha \operatorname{sech}^{2/3} \xi^{1/3} - \frac{2\alpha}{3} \xi^{4/3} - E \right).$$  \hspace{1cm} (3.16)

The eigenfunction of the potential $V(\xi)$ with the eigenvalue $-\frac{2\alpha}{27}$ then reads

$$v(\xi) = \xi^{1/3} P^{\sqrt{E}}_{\sqrt{4\xi^2 - 1}/3} \left( \tanh \xi^{1/3} \right).$$  \hspace{1cm} (3.17)

4 Duality in classical mechanics: three-dimensional central potential

4.1 Duality

Two central potentials, $U(r)$ and $V(\rho)$, are dual to each other, if they are related by

$$r^2 [U(r) - E] = \rho^2 [V(\rho) - \mathcal{E}]$$  \hspace{1cm} (4.1)

with

$$r \leftrightarrow \rho^{2/(\sigma + 3)}.$$  \hspace{1cm} (4.2)

The orbit of the potential $U(r)$ with the energy $E$ and the orbit of its dual potential $V(\rho)$ with the energy $\mathcal{E}$ can be obtained from each other by the replacement

$$\theta \leftrightarrow \frac{2}{\sigma + 3} \phi.$$  \hspace{1cm} (4.3)

Here $\sigma$ is a constant chosen arbitrarily.
Proof. The orbit equation of the potential \( U (r) \) with the energy \( E \) is

\[
\frac{d\theta}{dr} = \frac{L/r^2}{\sqrt{2[E - L^2/(2r^2) - U(r)]}} \tag{4.4}
\]

Substituting the replacements (4.2) and (4.3) into Eq. (4.4) gives

\[
\frac{d\phi}{d\rho} = \frac{L/\rho^2}{\sqrt{2 \left\{ -L^2/(2\rho^2) - \rho^{-2(\frac{a+1}{a+3})} \left[ U \left( \rho^{2/(\sigma+3)} \right) - E \right] \right\}}} \tag{4.5}
\]

Eq. (4.5) is the equation of the potential

\[
V (\rho) = \rho^{-2(\frac{a+1}{a+3})} \left[ U \left( \rho^{2/(\sigma+3)} \right) - E \right] + \mathcal{E} \tag{4.6}
\]

with the energy \( \mathcal{E} \):

\[
\frac{d\phi}{d\rho} = \frac{L/\rho^2}{\sqrt{2 \left[ \mathcal{E} - L^2/(2\rho^2) - V(\rho) \right]}} \tag{4.7}
\]

With the replacement (4.2), Eq. (4.6) gives the duality relations (4.1) and (4.2). ■

4.2 Power potential: example

The dual potential of a power potential, generally speaking, is no longer a power potential. However, if we require that the dual potential of a power potential is still a power potential, we achieve the following result.

The dual potential of the power potential

\[
U (r) = \xi r^a, \tag{4.8}
\]

by the duality relations (4.1) and (4.2), is

\[
V (\rho) = \xi \rho^{-2(\frac{a+1}{a+3})} - E \rho^{-2(\frac{a+1}{a+3})} + \mathcal{E}. \tag{4.9}
\]

If requiring that the dual potential \( V (\rho) \) is still a power potential, we can choose \( \rho^{-2(\frac{a+1}{a+3})} = 1 \), which requires \( \sigma = a - 1 \). Thus \( V (\rho) \) becomes

\[
V (\rho) = -E \rho^{-\frac{2a}{a+2}} + \xi + \mathcal{E}. \tag{4.10}
\]

Choosing \( \mathcal{E} = -\xi \) and \( \eta = -E \) gives a power potential

\[
V (\rho) = \eta r^A \tag{4.11}
\]

with \( \eta \) the coupling constant,

\[
a + 2 \frac{2}{2} = \frac{2}{A + 2}, \tag{4.12}
\]

and

\[
r \leftrightarrow \rho^{\frac{2}{a+2}}, \tag{4.13}
\]

\[
\theta \leftrightarrow \frac{2}{a+2} \phi. \tag{4.14}
\]

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It should be noted here that there is still another choice: \( \rho^{-2(\frac{\sigma+1}{\sigma+3})} = 1 \), which gives \( \sigma = -1 \). In this case, the duality transforms (4.2) and (4.3) are identity transforms.

Coulomb potential and harmonic-oscillator potential. The duality between the Coulomb potential and the harmonic-oscillator potential is just the Newton-Hooke duality. The Coulomb potential \( U(r) = \frac{e}{r} \), in fact, has an infinite number of dual potentials corresponding to various choices of the parameter \( \sigma \). Choosing \( \sigma = -2 \), the dual relation (4.1) becomes \( r^2 \left( \frac{e}{2} - E \right) = \rho^2 \left[ V(\rho) - \mathcal{E} \right] \) with \( r \leftrightarrow \rho^2 \) and \( \theta \leftrightarrow 2\phi \). The dual potential then reads \( V(\rho) = -E\rho^2 + \xi + \mathcal{E} \). Taking \( E = -\xi \), i.e., the energy of the system becoming the coupling constant of its dual system, we arrive at a harmonic-oscillator potential \( V(\rho) = -E\rho^2 = \eta\rho^2 \). The energy of the Coulomb potential system becomes the coupling constant of its dual potential.

5 Duality in quantum mechanics: three-dimensional central potential

5.1 Duality

Two central potentials, \( U(r) \) and \( V(\rho) \), are dual to each other, if they are related by

\[
\frac{2}{\sigma + 3} \rho^2 \left[ U(r) - E \right] = \frac{\sigma + 3}{2} \rho^2 \left[ V(\rho) - \mathcal{E} \right]
\]

with

\[
r \leftrightarrow \rho^{2/(\sigma+3)}. \tag{5.2}
\]

The radial eigenfunction of the potential \( V(\rho) \) and the radial eigenfunction of its dual potential \( U(r) \) can be obtained from each other by the replacement

\[
u_l(r) \leftrightarrow \rho^{-}\frac{\sigma+1}{2(\sigma+3)} v_\ell(\rho). \tag{5.3}
\]

The relation between the angular momenta of the dual systems, then, is

\[
l + \frac{1}{2} \leftrightarrow \frac{\sigma + 3}{2} \left( \ell + \frac{1}{2} \right). \tag{5.4}
\]

Here \( \sigma \) is a constant chosen arbitrarily.

Proof. The radial equation with the potential \( U(r) \) is

\[
\frac{d^2 u_l(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - U(r) \right] u_l(r) = 0. \tag{5.5}
\]

The dual relations (5.2) and (5.3) transform Eq. (5.5) to

\[
\frac{d^2 v_\ell(\rho)}{d\rho^2} + \left\{ -\frac{2}{\sigma+3} \left( \ell + \frac{1}{2} \right) - \frac{1}{2} \right\} \frac{2}{\rho^2} \left( \ell + \frac{1}{2} \right) + \frac{1}{2} - \left( \frac{2}{\sigma+3} \right)^2 \rho^{-2(\frac{\sigma+1}{\sigma+3})} \left[ U\left( \rho^{2/(\sigma+3)} \right) - E \right] \right\} v_\ell(\rho) = 0. \tag{5.6}
\]

Eq. (5.6) is the eigenequation of the potential

\[
V(\rho) = \left( \frac{2}{\sigma + 3} \right)^2 \rho^{-2(\frac{\sigma+1}{\sigma+3})} \left[ U\left( \rho^{2/(\sigma+3)} \right) - E \right] + \mathcal{E}. \tag{5.7}
\]
with the energy $E$ and the angular momentum

$$\ell (\ell + 1) = \left[ \frac{2}{\sigma + 3} \left( l + \frac{1}{2} \right) - \frac{1}{2} \right] \left[ \frac{2}{\sigma + 3} \left( l + \frac{1}{2} \right) + \frac{1}{2} \right],$$  \hspace{1cm} (5.8)

i.e.,

$$\frac{d^2 v_\ell (\rho)}{d \rho^2} + \left[ E - \frac{\ell (\ell + 1)}{\rho^2} \right] v_\ell (\rho) = 0.$$  \hspace{1cm} (5.9)

Eq. (5.7) gives the duality relation (5.1) and Eq. (5.8) gives the duality relations (5.4) and (5.2).

It should be emphasized that the dual relation of the angular momentum, Eq. (5.4), is a result of the dual transforms (5.2) and (5.3).

### 5.2 Power potential: example

Like that in classical mechanics, in quantum mechanics a potential can also have an arbitrary number of dual potentials. The dual potential of a power potential, generally speaking, is not a power potential. Nevertheless, if we still require that the dual potential of a power potential

$$U (r) = \xi r^a$$  \hspace{1cm} (5.10)

is also a power potential, we can choose $\sigma = a - 1$ in Eq. (5.2). Then the dual potential reads

$$V (\rho) = -\left( \frac{2}{\sigma + 3} \right)^2 E \rho^\frac{2a}{\sigma + 2} + \left( \frac{2}{\sigma + 3} \right)^2 \xi + \mathcal{E}.$$  \hspace{1cm} (5.11)

Choosing

$$\mathcal{E} = -\left( \frac{2}{\sigma + 3} \right)^2 \xi,$$  \hspace{1cm} (5.12)

$$\eta = -\left( \frac{2}{\sigma + 3} \right)^2 E,$$  \hspace{1cm} (5.13)

we arrive at a power potential

$$V (\rho) = \eta \rho^A.$$  \hspace{1cm} (5.14)

The corresponding dual relations are

$$\frac{a + 2}{2} = \frac{2}{A + 2}$$  \hspace{1cm} (5.15)

and

$$r \leftrightarrow \rho^{2/(a+2)},$$  \hspace{1cm} (5.16)

$$u_l (r) \leftrightarrow \rho^{-\frac{a}{2(a+2)}} v_\ell (\rho).$$  \hspace{1cm} (5.17)

There is another choice giving dual power potentials: $\rho^{-\frac{2\sigma+1}{\sigma+3}} = 1$, but this gives $\sigma = -1$, which leads to a trivial identity transform.
Coulomb potential and harmonic-oscillator potential. Like that in the classical mechanics, the dual potential of the Coulomb potential $U(r) = \frac{\xi}{r}$ with $\sigma = -2$ is the harmonic-oscillator potential. The dual transforms (5.15), (5.16), and (5.17) become

$$r \leftrightarrow \rho^2,$$

$$u_l (r) \leftrightarrow \rho^{1/2} v_l (\rho),$$

and

$$l + \frac{1}{2} \leftrightarrow \frac{1}{2} \left( l + \frac{1}{2} \right).$$

The dual relation (5.1) now becomes $2r^2 \left[ U(r) - E \right] = \frac{1}{2} \rho^2 \left[ V(\rho) - E \right]$. We have

$$V(\rho) = -4E\rho^2 + 4\xi + E.$$  \hspace{2cm} (5.21)

Taking $E = -4\xi$ gives

$$V(\rho) = -4E\rho^2.$$  \hspace{2cm} (5.22)

The radial eigenfunction of the Coulomb potential is

$$u_l (r) = A_l e^{-\sqrt{-E}r} \left( 2\sqrt{-E} \right)^{l+1} r^{l+1} \, _1F_1 \left( l + 1 + \frac{\xi}{2\sqrt{-E}}, 2(l + 1), 2\sqrt{-E}r \right).$$  \hspace{2cm} (5.23)

The dual transforms (5.18), (5.19), and (5.20) give the radial eigenfunction of the Coulomb potential:

$$v_l (\rho) = A_l e^{-\sqrt{-E}\rho^2} \left( 2\sqrt{-E} \right)^{\frac{l+3}{2}} \rho^{\ell+1} \, _1F_1 \left( \frac{\ell}{2} + \frac{3}{4} + \frac{\xi}{2\sqrt{-E}}, \ell + \frac{3}{2}, 2\sqrt{-E}\rho^2 \right).$$  \hspace{2cm} (5.24)

This is the solution of the radial equation of the central harmonic-oscillator potential with the eigenvalue $E = -4\xi$ and the coupling constant $\eta = -4E$:

$$\frac{d^2v_l (\rho)}{d\rho^2} + \left[ -4\xi - \frac{\ell(\ell + 1)}{\rho^2} + 4E\rho^2 \right] v_l (\rho) = 0.$$  \hspace{2cm} (5.25)

6 Conclusion

A dual relation between potentials in classical and quantum mechanics is revealed. The duality considered in the present paper is the duality in classical mechanics and in quantum mechanics. In classical mechanics the dynamical equation is the Newton equation. In quantum mechanics the dynamical equation is the Schrödinger equation. In future work, we will discuss the duality of the potential in relativistic quantum mechanics: the duality of potentials in the Klein-Gordon equation and the duality in the Dirac equation. Moreover, in the present paper we only consider the duality of conservative forces which can be described by a potential. In the subsequent work, we will take the nonconservative force, such as the magnetic force, into account.

In the subsequent work, we will concentrate on the duality family. In such a duality family, all potentials are dual to each other and the solutions of the potentials in the family can be transformed to each other by the dual relation. It is worthy to discuss the general properties of the duality family.
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