Ideas Behind Kolmogorov Complexity and Related Kolmogorov’s Ideas about The Physical Impossibility of Large Integers: How Their Formalization Can Help in Foundations of Physics

Isaak A. Kunin\textsuperscript{1}, Vladik Kreinovich\textsuperscript{2}, and Yuri A. Kuperin\textsuperscript{3}

\textsuperscript{1}Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA, email kunin@uh.edu
\textsuperscript{2}Department of Computer Science, University of Texas at El Paso, El Paso, TX 79968, USA, email vladik@cs.utep.edu
\textsuperscript{3}Physics Research Institute, St. Petersburg State University, St. Petersburg, Russia, email kuperin@JK1454.spb.edu

Abstract

In addition to the equations, physicists use the following additional difficult-to-formalize property: that the initial conditions and the value of the parameters must not be abnormal. We will describe a natural formalization of this property, and show that this formalization in good accordance with theoretical physics. At present, this formalization has been mainly applied to the foundations of physics. However, potentially, more practical applications are possible.

1 Main Idea: In Short

Traditional mathematical approach to the analysis of physical systems implicitly assumed that all mathematically possible integers are physically possible as well, and all mathematically possible trajectories are physically possible. Traditionally, this approach has worked well in physics, but it does not lead to a very good understanding of chaotic systems, which, as is now known, are extremely important in the study of real-world phenomena ranging from weather...
to biological systems.

Kolmogorov was among the first who started, in the 1960s, analyzing the discrepancy between the physical and the mathematical possibility. He pinpointed two main reasons why a mathematical correct solution to the corresponding system of differential or difference equation can be not physically possible:

- First, there is a difference in understanding the term “random” in mathematics and in physics. For example, in statistical physics, it is possible (probability is positive) that a kettle, when placed on a cold stove, will start boiling by itself. From the viewpoint of a working physicist, however, this is absolutely impossible. Similarly, a trajectory which requires a highly unprobable combination of initial conditions may be mathematically correct, but, from the physical viewpoint, it is impossible.

- Second, the traditional mathematical analysis tacitly assumes that all integers and all real numbers, no matter how large or how small, are physically possible. From the physical viewpoint, however, a number like $10^{10^{10}}$ is not physically possible at all, because it exceeds the number of particles in the Universe. In particular, solutions to the corresponding systems of differential equations which lead to some numbers may be mathematically correct, but they are physically meaningless.

Attempts to formalizing these restrictions have been started by Kolmogorov himself. These attempts are at present, mainly undertaken by researchers in theoretical computer science who face a similar problem of distinguishing between theoretically possible “algorithms” and feasible practical algorithms which can provide the results of their computations in reasonable time.

The goal of the present research is to use the experience of formalizing these notions in theoretical computer science to enhance the formalization of similar constraints in working physics.

This research is mainly concentrated around the notion of Kolmogorov complexity. This notion was introduced independently by several people: Kolmogorov in Russia and Solomonoff and Chaitin in the US. Kolmogorov used it to formalize the notion of a random sequence. Probability theory describes most of the physicist intuition in precise mathematical terms, but it does not allow us to tell whether a given finite sequence of 0’s and 1’s is random or not. Kolmogorov defined a complexity $K(x)$ of a binary sequence $x$ as the shortest length of a program which produces this sequence. Thus, a sequence consisting of all 0’s or a sequence 010101... have a very short Kolmogorov complexity because these sequences can be generated by simple programs, while for a sequence of results of tossing a coin, probably the shortest program is to write `print(0101...)` and then reproduce the entire sequence. Thus, when $K(x)$ is approximately equal to the length $\text{len}(x)$ of a sequence, this sequence is random, otherwise it is not. The best source for Kolmogorov complexity is a book [60].
The definition of $K(x)$ only takes into consideration the length $\text{len}(p)$ of a program $p$. From the physical viewpoint, it is also important to take into consideration its running time $t(p)$, because if it exceeds the lifetime of the Universe, this algorithm makes no practical sense. This development is in line with Kolmogorov’s original idea that some natural numbers which are mathematically possible (like $10^{10^{10}}$) are not feasible and thus, should not considered as feasible. Corresponding modifications are also described in the above book. We plan to use the corresponding ideas in physics.

2 Main Idea: Brief Philosophical Analysis

One of the main objectives of science is to provide guaranteed estimates for physical quantities. In order to find out how estimates can be guaranteed, let us recall how quantities are estimated in physics:

- First, we must find a physical law that describes the phenomena that we are analyzing. For some phenomena, we already know the corresponding laws: we know Maxwell’s equation for electrodynamics, Einstein’s equation for gravity, Schroedinger’s equations for quantum mechanics, etc. (these laws can be usually deduced from symmetry conditions \[43, 23, 28\]). However, in many other cases, we must determine the equations from the general theoretical ideas and from the experimental data. Can we guarantee that these equations are correct? If yes, how?

There is an extra problem here. In some case, we know the equations, but we are not sure about the values of the parameters of these equations. If the theory predicts, e.g., that a dimensionless parameter is 1, and the experiments confirm it with an accuracy of 0.001, should we then use exactly 1 or $1 \pm 0.001$ for a guaranteed estimate? If the accuracy is good enough, then the physicists usually use 1. We may want to use $1 \pm 0.001$ to be on the safe side, but then, for other parameters of a more general theory (that in this particular theory are equal to 0) should we also use their experimental bounds instead of the exact 0 value? There are often many possible generalizations, and if we take all of them into consideration, we may end up with a very wide interval. This is a particular case of the same problem: when (and how) can we guarantee that these are the right equations, with the correct values of the parameters?

- Suppose now that we know the correct equations. Then, we need to describe how we will actually predict the value of the desired quantity. For example, we can get partial differential equations that describe how exactly the initial values $\phi(x, t_0)$ of all the fields change in time. Then, to predict the values of the physical quantity at a later moment of time $t$, we must do the following:
• Determine the values $\phi(x, t_0)$ from the measurement results.
• Use these values $\phi(x, t_0)$ to predict the desired value.

The problem with this idea is that reconstructing the actual values $\phi(x, t_0)$ from the results of measurements and observations is an ill-posed problem in the sense that two essentially different functions $\phi(x, t_0)$ are consistent with the same observations. For example, since all the measurement devices are inertial and thus suppress the high frequencies, the functions $\phi(x, t_0)$ and $\phi(x, t_0) + A \cdot \sin(\omega x)$, where $\omega$ is sufficiently big, lead to almost similar values of observations.

A typical example of an ill-posed problem is a problem of reconstructing the actual brightness distribution of a celestial object from its observed image.

Thus, strictly speaking, if we do not have any additional restrictions on $\phi(x, t_0)$, then for every $x$, the set of possible values of $\phi(x, t_0)$ is the entire real line. So, to get a guaranteed interval for $\phi(x, t_0)$ (and hence, for the desired physical quantity), we need to use some additional information. The process of using this additional information to get non-trivial estimates for the solution of the inverse problem is called a regularization. There are several situations where this additional information is available:

• If we are analyzing familiar processes, then we usually know (more or less) how the desired function $\phi(x, t_0)$ looks like. For example, we may know that $\phi(x, t_0)$ is a linear function $C_1 + C_2 \cdot x_1$, or a sine function $C_1 \cdot \sin(C_2 x_1 + C_3)$, etc. In mathematical terms, we know that $\phi(x, t_0) = f(x, C_1, \ldots, C_k)$, where $f$ is a known expression, and the only problem is to determine the coefficients $C_i$. This is how, for example, the orbits of planets, satellites, comets, etc., are computed: the general shape of an orbit is known from Newton’s theory, so we only have to estimate the parameters of a specific orbit. In such cases, the existence of several other functions $\phi(x, t_0)$ that are consistent with the same observations, is not a big problem, because we choose only the functions $x(t)$ that are expressed by the formula $f(t, C_1, \ldots, C_k)$. This is not, however, a frequent situation in physics, because one of the main objectives (and the main challenges) of physics is to analyze new phenomena, new effects, qualitatively new processes, and in these cases no prior expression $f$ is known.

• In some cases, we know the statistical characteristics of the reconstructed quantity $\phi(x, t_0)$ and statistical characteristics of the measurement errors. In these cases, we can formulate the problem of
choosing the maximally probable \( \phi(x, t_0) \), and end up with one of the methods of statistical regularization, or filtering (Wiener filter is one of the examples of this approach).

- If we do not have this statistical information, but we know, e.g., that the average rate of change of \( x(t) \) is smaller than some constant \( \Delta \) (i.e., \( \sqrt{\int \dot{x}(t)^2 \, dt} \leq \Delta \)), then we can apply regularization methods proposed by A. N. Tikhonov and others [84, 30, 58].

- In many cases, we do not have the desired statistical information. However, we may have some expert knowledge. For example, if we want to know how the temperature on a planet changes with time \( t \), then the experts can tell that most likely, \( x(t) \) is limited by some value \( M \), and that the rate \( \dot{x}(t) \) with which the temperature changes, is typically (or “most likely,” etc) limited by some value \( \Delta \), etc. We can also have some expert knowledge about the error, with which we perform our measurements, so the resulting expert’s knowledge about the value of measured quantity \( y \) looks like “the difference between the measured value \( \hat{y} \) and the actual value \( y \) is most likely, not bigger than \( \delta \)” (where \( \delta \) is a positive real number given by an expert). The importance of this information is stressed in [3] and in Chapter 5 of [18]. The methods of using this information and their application to testing airplane and spaceship engines is described in [17, 15, 14, 16].

- In many cases, we do not have any quantitative expert information like the one we described. In these cases, it is usually recommended to use some heuristic (or semi-heuristic) regularization techniques [84, 30, 38, 37, 38, 58, 17]. These methods often lead to reasonable results, but they do not give any guaranteed estimate for the reconstructed value \( \phi(x, t_0) \).

- Suppose that we have the equations, and that we have chosen an appropriate regularization for these equations. Then, in principle, we can have the guaranteed estimate. The problem is that the numerical methods that the physicists currently use do not give us these guaranteed estimates. For example, we may have an iterative procedure for solving the equation, and in this procedure, we stop if the next iteration is close to the previous one. The fact that iterations are close may mean that we are close to the actual solution, but how close are we? In other words, how to get a guaranteed estimate for the solution that is obtained by a heuristic method? For some equations, such methods are known [18, 19, 20], but these methods are far from being general.

There are several successful applications of interval methods (i.e., computational methods which provide guaranteed estimates) to physics:
• Stability of solar system and likewise systems: there exists a K.A.M. method (Kolmogorov-Arnold-Moser) that proves that for sufficiently small values of some parameter, the Solar system is stable. The upper bound is much lower than the actual value [68]. In [71, 72, X, 65], interval computations are used to find upper bounds for stability that for some systems, cover up to 90% of the actual stability zone.

• Relativistic stability of matter [23, 64]: for relativistic version of Schroedinger equation, it is proved that for sufficiently small charges, the energy spectrum is non-negative (i.e., for \( N \to \infty \), the system does not collapse). The estimate for the charge is close to the one for which the collapse actually occurs.

• Asymptotic energy of atoms [80, 81, 15] (based on Th. Fermi equation). All these applications and the corresponding methods are domain-specific. What can we do to get guaranteed estimates in the general case?

There are two possible approaches to this problem:

• A pessimistic approach: that we will never be able to get guaranteed estimates. This approach is typical in statistics. For example, a well-known statistician R. A. Fisher says that a “hypothesis is never proved or established, but is possibly disproved, in the course of experimentation” ([29], p. 16; for a modern description of this viewpoint, see, e.g., [35]). Strictly speaking, from this viewpoint, we cannot even say that a theory is disproved with a guarantee. Indeed, if, e.g., a theory predicts 1, and the measurement has led to 2, then, no matter how small the standard deviation of the measurement error can be, the probability that the difference is caused by the measurement error is non-zero, and so, it is possible that the theory is still correct.

• An optimistic approach, that most physicists hold, is that we can make guaranteed conclusions from the experiments. A disproved theory is wrong, and the chance that the measurement error has caused it is as large as having the cards in order after thorough shuffling, or a possibility to win the lottery every time by guessing the outcome: it is impossible.

In this paper, we will describe a formalization of the optimistic approach.

3 Main Idea in Detail: The Notion of “Not Abnormal” and How To Formalize It
3.1 Physicists Assume That Initial Conditions And Values Of Parameters Are Not Abnormal

To a mathematician, the main contents of a physical theory is the equations. The fact that the theory is formulated in terms of well-defined mathematical equations means that the actual field must satisfy these equations. However, this fact does not mean that every solution of these equations has a physical sense. Let us give two examples:

- At any temperature greater than absolute zero, particles are randomly moving. It is theoretically possible that all the particles start moving in one direction, and, as a result, the chair that I am sitting on starts lifting up into the air. The probability of this event is small (but positive), so, from the purely mathematical viewpoint, we can say that this event is possible but highly unprobable. However, the physicists say plainly that such an abnormal event is impossible (see, e.g., [24]).

- Another example from statistical physics: Suppose that we have a two-chamber camera. The left chamber if empty, the right one has gas in it. If we open the door between the chambers, then the gas would spread evenly between the two chambers. It is theoretically possible (under appropriately chosen initial conditions) that the gas that was initially evenly distributed would concentrate in one camera, but physicists believe this abnormal event to be impossible. This is a general example of what physicists call irreversible processes: on the atomic level, all equations are invariant with respect to changing the order of time flow $t \rightarrow -t)$. So, if we have a process that goes from state $A$ to state $B$, then, if at $B$, we revert all the velocities of all the atoms, we will get a process that goes from $B$ to $A$. However, in real life, many processes are clearly irreversible: an explosion can shatter a statue, but it is hard to imagine an inverse process: an implosion that glues together shattered pieces into a statue. Boltzmann himself, the 19 century author of statistical physics, explicitly stated that such inverse processes “may be regarded as impossible, even though from the viewpoint of probability theory that outcome is only extremely improbable, not impossible.” [8] (for this similar citations from other founding fathers of statistical physics, see [49]).

- If we flip a coin 100 times in a row, and get heads all the time, then a person who is knowledgeable in probability would say that it is possible, while a physicist (and any person who uses common sense reasoning) would say that the coin is not fair, because it is was a fair coin, then this abnormal event would be impossible. To illustrate this point, G. Polya in [76] cites the following anecdote from the treatise of J. Bertrand on probabilities:

  One day in Naples the reverend Galiani saw a man from the Basilicate who, shaking three dice in a cup, wagered to three sixes; and, in fact,
he got three sixes right away. Such luck is possible, you say. Yet the man succeeded a second time, and the bet was repeated. He put back the dice in the cup, three, four, five times, and each time he produced three sixes. “Sangue di Bacco”, exclaimed the reverend, “the dice are loaded!” And they were.

- In all the above cases, we knew something about probability. However, there are examples of this type of reasoning in which probability does not enter into picture at all. For example, in general relativity, it is known that for almost all initial conditions (in some reasonable sense) the solution has a singularity point. Form this, physicists conclude that the solution that corresponds to the geometry of the actual world has a singularity (see, e.g., [67]): the reason is that the initial conditions that lead to a non-singularity solution are abnormal (atypical), and the actual initial conditions must be not abnormal.

In all these cases, the physicists (implicitly or explicitly) require that the actual values of the fields must not satisfy the equations, but they must also satisfy the additional condition: that the initial conditions should not be abnormal.

### 3.2 The Notion of “Not Abnormal” Is Difficult to Formalize

At first glance, it looks like in the probabilistic case, this property has a natural formalization: if a probability of an event is small enough (say, ≤ \( p_0 \) for some very small \( p_0 \)), then this event cannot happen. For example, the probability that a fair coin falls heads 100 times in a row is \( 2^{-100} \), so, if we choose \( p_0 \geq 2^{-100} \), then we will be able to conclude that such an event is impossible. The problem with this approach is that every sequence of heads and details has exactly the same probability. So, if we choose \( p_0 \geq 2^{-100} \), we will thus exclude all possible sequences of heads and tails as physically impossible. However, anyone can toss a coin 100 times, and this prove that some sequences are physically possible.

Historical comment. This problem was first noticed by Kyburg under the name of Lottery paradox [54]: in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is so small, then a reasonable person should not expect it. However, some people do win big prizes (for a recent discussion of this paradox, see, e.g., [14, 21, 75]).

### 3.3 How to Formalize The Notion of “Not Abnormal”:

**Idea**

“Abnormal” means something unusual, rarely happening: if something is rare enough, it is not typical (“abnormal”). Let us describe what, e.g., an abnormal height may mean. If a person’s height is \( \geq 6 \) ft, it is still normal (although it
may be considered abnormal in some parts of the world). Now, if instead of 6 pt, we consider 6 ft 1 in, 6 ft 2 in, etc, then sooner or later we will end up with a height \( h \) such that everyone who is higher than \( h \) will be definitely called a person of abnormal height. We may not be sure what exactly value \( h \) experts will call “abnormal”, but we are sure that such a value exists.

Let us express this idea is general terms. We have a Universe of discourse, i.e., a set \( U \) of all objects that we will consider. Some of the elements of the set \( U \) are abnormal (in some sense), and some are not. Let us denote the set of all elements that are typical (not abnormal) by \( T \). On this set, we have a decreasing sequence of sets \( A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \) with the property that \( \cap A_n = \emptyset \). In the above example, \( U \) is the set of all people, \( A_1 \) is the set of all people whose height is \( \geq 6 \) ft, \( A_2 \) is the set of all people whose height is \( \geq 6 \) ft 1 in, \( A_2 \) is the set of all people whose height is \( \geq 6 \) ft 2 in, etc. We know that if we take a sufficiently large \( n \), then all elements of \( A_n \) are abnormal (i.e., none of them belongs to the set \( T \) of not abnormal elements). In mathematical terms, this means that for some \( n \), we have \( A_n \cap T = \emptyset \).

In case of a coin: \( U \) is the set of all infinite sequences of results of flipping a coin; \( A_n \) is the set of all sequences that start with \( n \) heads but have some tail afterwards. Here, \( \cup A_n = \emptyset \). Therefore, we can conclude that there exists an \( n \) for which all elements of \( A_n \) are abnormal. So, if we assume that in our world, only not abnormal initial conditions can happen, we can conclude that for some \( n \), the actual sequence of results of flipping a coin cannot belong to \( A_n \). The set \( A_n \) consists of all elements that start with \( n \) heads and a tail after that. So, the fact that the actual sequence does not belong to \( A_n \) means that if an actual sequence has \( n \) heads, then it will consist of all heads. In plain words, if we have flipped a coin \( n \) times, and the results are \( n \) heads, then this coin is biased: it will always fall on heads.

Let us describe this idea in mathematical terms.

### 3.4 Formal Definition [27]

To make formal definitions, we must fix a formal theory: e.g., the set theory ZF (the definitions and results will not depend on what exactly theory we choose).

**Definition 1.** We say that a set \( S \) is definable if in ZF, there exists a formula \( P(x) \) with one free variable \( x \) such that \( P(x) \) iff \( x \in S \).

**Comment.** Crudely speaking, a set is definable if we can define it in ZF. The set of all real numbers, the set of all solutions of a well-defined equations, every set that we can describe in mathematical terms is definable. This does not means, however, that every set is definable: indeed, every definable set is uniquely determined by formula \( P(x) \), i.e., by a text in the language of set theory. There are only denumerably many words and therefore, there are only denumerably many definable sets. Since, e.g., there are more than denumerably many set of
integers, some of them are this not definable.

**Definition 2.** We say that a sequence of sets $A_1, \ldots, A_n, \ldots$ is definable if in ZF, there exists a formula $P(n, x)$ such that $x \in A_n$ iff $P(n, x)$.

**Definition 3.**
- Let a set $U$ be given. We will call it a universal set.
- A non-empty set $T \subseteq U$ is called a set of typical (not abnormal) elements if for every definable sequence of sets $A_n$ for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, there exists an $N$ for which $A_N \cap T = \emptyset$.
- If $u \in T$, we will say that $u$ is not abnormal.
- For every property $P$, we say that “normally, for all $u$, $P(u)$” if $P(u)$ is true for all $u \in T$.

### 3.5 Existence Theorems

The trivial existence result is: for every set $U$, there is a set of typical elements $T$ that satisfies Definition 3: indeed, we can take a one-element set $T = \{u\} \in U$.

A more interesting existence result appears if we take into consideration the fact that our definition did not completely capture the following property of the notion of “abnormal”: that exceptions (i.e., abnormal elements) should be rare. “Rare” usually means that the probability of an element being abnormal should be small enough (i.e., $\leq \varepsilon$ for some given $\varepsilon > 0$). We may not know the exact probabilities, so we may want to choose the set $T$ in such a way that exceptions will be rare no matter what probability measure we choose. To describe this situation, we thus need to fix a real number $\varepsilon > 0$, and a finite sequence of probability measures $p_1, \ldots, p_m$. The only problem with this idea is that definable sets may be not measurable. Therefore, in order to apply it, we will modify Definition 3 so that it will allow only sequence $A_n$ whose elements are measurable w.r.t. given measures.

**Definition 3’.
- Let a set $U$ be given. We will call it a universal set. Let $p_1, \ldots, p_m$ be probability measures on $U$.
- A non-empty set $T \subseteq U$ is called a set of $(p_1, \ldots, p_m)$-typical elements if for every definable sequence of sets $A_n$ for which $A_n \supseteq A_{n+1}$, $\cap A_n = \emptyset$, and all elements $A_n$ are measurable w.r.t. each measure $p_i$, there exists an $N$ for which $A_N \cap T = \emptyset$.

**Proposition 1.** Assume that we have a set $U$, $m$ probability measures $p_1, \ldots, p_m$ on $U$, and a real number $\varepsilon > 0$. Then, there exists a set $T$ of $(p_1, \ldots, p_m)$-typical elements for which $p_i(T) \geq 1 - \varepsilon$ for all $i$. 

10
Comment. For reader’s convenience, all the proofs are given in the special section at the end of the paper.

In other words, it is possible to define abnormal elements in such a way that for each of $m$ measures, the probability of an element to be abnormal is $\leq \varepsilon$.

Now, that we have the definition, let us show that this notion can indeed help to give guaranteed estimates.

4 Based On Finitely Many Experiments, We Can Guarantee That The Theory Is Correct

4.1 General Result

Let us show first that if we assume that the results of experiments are required not to be abnormal, then we can (potentially) guarantee that the theory is correct after only finitely many experiments.

From the viewpoint of an experimenter, a physical theory can be viewed as a statement about the results of physical experiments. If we had an infinite sequence of experimental results $r_1, r_2, \ldots$, then we will be able to tell whether the theory is correct or not. So, a theory can be defined as a set of sequences $r_1, r_2, \ldots$ that are consistent with its equations, inequalities, etc. In real life, we only have finitely many results $r_1, \ldots, r_n$, so, we can only tell whether the theory is consistent with these results or not, i.e., whether there is an infinite sequence $r_1, \ldots, r_n, \ldots$ that starts with the given results that satisfies the theory.

It is natural to require that the theory be physically meaningful in the following sense: if all experiments confirm the theory, then this theory should be correct. An example of a theory that is not physically meaningful is easy to give: assume that a theory describes the results of tossing a coin, and it predicts that at least once, there should be a tail. In other words, this theory consists of all sequences that contain at least one tail. Let us assume that actually, the coin is so biased that we always have heads. Then, this infinite sequence does not satisfy the given theory. However, for every $n$, the sequence of the first $n$ results (i.e., the sequence of $n$ heads) is perfectly consistent with the theory, because we can add a tail to it and get an infinite sequence that belongs to the set $\mathcal{T}$.

Let us describe this idea in formal terms.
Definition 4.

- Let a definable set \( R \) be given. Its elements will be called possible results of experiments. By \( S \), we will denote the set of all possible sequences \( r_1, \ldots, r_n, \ldots \), each element \( r_i \) of which is a result of an experiment (i.e., \( r_i \in R \)).

- By a theory, we mean a definable subset \( T \) of the set of all infinite sequences \( S \). If \( r \in T \), we say that an infinite sequence \( r \) satisfies the theory \( T \), or, that for this sequence \( r \), the theory \( T \) is correct.

- We say that a finite sequence \( (r_1, \ldots, r_n) \) is consistent with the theory \( T \) if there exists an infinite sequence \( r \in T \) that starts with \( r_1, \ldots, r_n \) and that satisfies the theory. In this case, we will also say that the first \( n \) experiments confirm the theory.

- We say that a theory \( T \) is physically meaningful if the following is true:

  Let \( r \) be a sequence \( r \in S \) such that for every \( n \), the results of first \( n \) experiments from \( r \) conform the theory \( T \). Then, the theory \( T \) is correct for \( r \).

In this case, the universal set consists of all possible infinite sequence of experimental results, i.e., \( U = S \).

**Proposition 2.** For every physically meaningful theory \( T \), there exists an integer \( N \) such that normally, if the first \( N \) experiment confirm the theory \( T \), then this theory \( T \) is correct.

In other words, if the sequence of results \( r \) is not abnormal, and the results of first \( N \) experiments are consistent with the theory, then the theory is correct. This result shows that we can confirm the theory based on finitely many observations.

Philosophical comment: physical induction and its paradoxes. The derivation of a general theory from finitely many experiments is called physical induction (as opposed to mathematical induction). These have been many formalizations of different ideas that physicists use, and these formalizations has lead to successful programs that can find a general dependency from the cases (see, e.g., [3]). However, in spite of the success in describing several underlying ideas, the general physical induction is difficult to formalize, to the extent that a prominent philosopher C. D. Broad has called the unsolved problems concerning induction a scandal of philosophy [11]. We can say that the notion of “not abnormal” justifies physical induction (and thus resolves the corresponding scandal).

Philosophical comment: Ockham’s Razor justified. Ockham Razor is a principle according to which, one should not unnecessarily multiply the number of entities. It is usually understood as follows: if we have two properties \( A \) and \( B \), and if
in all experiments, these two properties coincide, then we should assume that these two properties are identical. The above result justifies Ockham’s razor principle: namely, if, as a theory $T$, we consider a statement that $A$ and $B$ are always identical, then from Proposition 2, we can conclude that normally, if $A$ and $B$ are identical for first $N$ experimental results, then these two properties do always coincide.

4.1.1 Abnormal Theories and Related Paradoxes

The necessary number of experiments differ from a theory to a theory. In principle, we can formulate a theory that predicts the same results as our normal physics until some arbitrarily chosen year $Y$, and then predicts something else. For this theory, $N$ must be so large as to stretch to that year $Y$. So, the larger $Y$, the larger $N$. Such theories are artificial and abnormal. It turns out that if we restrict ourselves by not abnormal theories, then we will have a universal bound on $N$:

Namely, let us assume that on the set $U$ of all pairs $(T, r)$, where $T$ is a physically meaningful theory, and $r$ is a sequence of experimental results, we have selected a subset of typical (not abnormal) pairs $T$. Then, the following result is true:

**Proposition 3.** There exists an integer $N$ such that normally, if a physically meaningful theory $T$ is confirmed by the first $N$ experiments, then this theory is correct.

In other words, if a pair $(T, r)$ is not abnormal, and first $N$ experiments from $r$ confirm the theory $T$, then $T$ is correct on $r$.

*Philosophical comment: Goodman’s paradox explained.* Formalization of physical induction is a difficult task, known to lead to paradoxes. For example, Nelson Goodman [31, 32] has proposed the following paradox. We have observed emeralds many times, we know that they are green, so we conclude that emeralds are always green. Instead of the theory “emeralds are green”, let us consider an alternative theory “emeralds are grue”, where grue stands for “green before the year 2010, and blue after the year 2010”. Then, all the evidence that we have used to conclude that emeralds are green, also confirms that they are grue. However, to conclude that emeralds are grue is strange.

There have been several idea on how to solve this paradox (see, e.g., [1, 8, 10, 22, 41]). However, as it is remarked in Chapter 14 of [31], these ideas has not yet lead to a consistent formalization, and hence, Goodman himself did not consider this paradox solved. From the physicist’s viewpoint (that Goodman himself explained in his papers and that other researchers tried to formalize), Goodman’s paradox is not a paradox at all: green is a natural property, while grue is an abnormal one. The problem is to formalize this distinction. The above formalization provides exactly this answer to the paradox: Indeed, if a not abnormal theory $T$ is confirmed by $N$ experiments and is, therefore, correct,
and if another theory $T'$ is confirmed by these same experiments, but leads to different predictions, then due to Proposition 3, it means that $T'$ is an abnormal theory.

Of course, this is not a complete solution of the related set of problems, because we still need to find out how to distinguish between normal and abnormal theories.

4.2 How To Guarantee The Exact Values of Parameters

In the following text, by $\|x\|$, we will mean a Euclidean norm of the vector $x$.

**PROPOSITION 4.** Let $d$ be an integer, and let $a$ be a definable point from $R^d$. Then, there exists an $\varepsilon > 0$ such that if $x$ is not abnormal, and $\|x-a\| \leq \varepsilon$, then $x = a$.

*Comment.* This result is actually correct for an arbitrary definable metric space $X$ with a metric $d$.

This means that for every set of typical elements $T \subseteq R^d$, there exists an $\varepsilon > 0$ such that if $x$ is not abnormal (i.e., $x \in T$), and $x$ is $\varepsilon$–close to $a$, then $x = a$.

In other words, if we make an assumption (that physicists usually make) that the actual values of parameters are not abnormal, then it is not true that we can test a theory with better and better accuracy and never be 100% guaranteed that the theory is correct: there exists an $\varepsilon > 0$ such that if we have confirmed the theory with the accuracy $\varepsilon$, then this theory is true.

From this result, we can conclude that coincidences are not accidental: Indeed, suppose that we know a constant $c$. Now, if in some other (not clearly related) physical experiments we get a constant that is very close to $c$, then we can conclude that this is the same constant. This type of argument is very frequent in physics: e.g., the discovery that light consists of electromagnetic waves was prompted by the fact that the computed velocity of these waves turned out to be very close to the measured speed of light.

*Comment.* Arguments of the type “This is too improbable to be a mere coincidence. There must be some reason.” are also used in mathematics, to deduce hypotheses from the observed facts (see, e.g., [76], Chapter XIV, Section 16).

5 Restriction To “Not Abnormal” Solutions Leads To Regularization Of Ill-Posed Problems

The material described in this section follows [13].
5.1 The Main Result

An ill-posed problem arises when we want to reconstruct the state $s$ from the measurement results $r$. Usually, all physical dependencies are continuous, so, small changes of the state $s$ result in small changes in $r$. In other words, a mapping $f : S \to R$ from the set of all states to the set of all observations is continuous (in some natural topology). We consider the case when the measurement results are (in principle) sufficient to reconstruct $s$, i.e., the case when the mapping $f$ is 1-1. That the problem is ill-posed means that small changes in $r$ can lead to huge changes in $s$, i.e., that the inverse mapping $f^{-1} : R \to S$ is not continuous.

We will show that if we restrict ourselves to states $S$ that are not abnormal, then the restriction of $f^{-1}$ will be continuous, and the problem will become well-posed.

**Definition 5.** A definable metric space $(X, d)$ is called definably separable if there exists a definable everywhere dense sequence $x_n \in X$.

**Proposition 5.** Let $S$ be a definably separable definable metric space, $T$ be a set of all not abnormal elements of $S$, and $f : S \to R$ be a continuous 1-1 function. Then, the inverse mapping $f^{-1} : R \to S$ is continuous for every $r \in f(T)$.

In other words, if we know that we have observed a not abnormal state (i.e., that $r = f(s)$ for some $s \in T$), then the reconstruction problem becomes well-posed. So, if the observations are accurate enough, we get as small guaranteed intervals for the reconstructed state $s$ as we want.

**Mathematical comment.** This Proposition uses the following Lemma that may be of independent interest:

**Lemma 1.** If $X$ is a definably separable definable metric space, and $T$ is a set of all not abnormal elements of $X$, then the closure $\overline{T}$ is a compact set.

5.2 How Can We Actually Use This Result to Get Guaranteed Estimates [42]

To actually use this result, we need an expert who will tell us what is abnormal. We will show that if we use such an expert, then for every computable function $f : X \to Y$, if we know that $x \in T$, then sufficiently accurate knowledge of $f(x)$ will enable us to reconstruct $x$ with any given accuracy.
Definition 6.

- By an expert, we mean a mapping $E : \{A_n\} \to Z$ that transforms a definable decreasing sequence with an empty intersection into an integer $N$ for which $A_N \cap T = \emptyset$ (i.e., for which all elements from $A_N$ are abnormal).

- We say that an output is computable with an expert if it is computable on a computer that can consult an expert (i.e., that sends an expert a formula defining $\{A_n\}$ and gets $N$).

The following definitions are standard in constructive analysis [5, 6, 70].

Definition 7.

- We say that an algorithm $\mathcal{U}$ computes a real number $x$ if for every natural number $k$, it generates a rational number $r_k$ such that $|r_k - x| \leq 2^{-k}$. We say that we have a computable real number if we have an algorithm $\mathcal{U}$ that computes it.

- By a computable separable metric space, we understand a separable metric space $(X, d)$ with an everywhere dense sequence $\{x_n\}$ for which there exists an algorithm, that transforms a pair of positive integers $n, m$ into a computable real number $d(x_n, x_m)$.

- By a computable element of a computable space we understand a pair consisting of an element $x \in X$ and an algorithm that given $n$, returns an integer $m(n)$ for which $x_{m(n)}$ is a $2^{-n}$-approximation to $x$.

- Let $X$ and $Y$ be computable separable metric spaces. We say that an algorithm $\mathcal{V}$ computes a function $f : X \to Y$ if $\mathcal{V}$ includes calls to an (unspecified) algorithm $\mathcal{U}$ so that when we take as $\mathcal{U}$ an algorithm that computes an element $x \in X$, then $\mathcal{V}$ will compute an element $f(x) \in Y$.

- We say that a computable function $f$ is constructively continuous on a set $S$ if there exists an algorithm, that for every $\epsilon > 0$, generates $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq \epsilon$.
Definition 8. Assume that we are given the following information:

- computable separable metric spaces $X$ and $Y$;
- a computably continuous computable 1-1 function $f : X \to Y$ (1-1 means that if $x \neq x'$, then $f(x) \neq f(x')$);
- an algorithm that, given an integer $k$, returns a $2^{-k}$-approximation to $f(x)$, where $x \in T$ is an (unknown) typical element;
- a positive integer $l$.

We say that algorithm solves the inverse problem if, given the above information, this algorithm returns a $2^{-l}$-approximation to $y$. If such an algorithm exists, then we will say that an inverse problem is computable.

PROPOSITION 6. The inverse problem is computable with an expert.

Comment.

- This general algorithm can be applied to different numerical problems: to solving a system of non-linear equations (when $X$ and $Y$ are $\mathbb{R}^k$ for some $k$), to solving integral equations (when $X$ and $Y$ are sets of functions), etc.

- If we do not restrict ourselves to not abnormal elements $x$, then in many cases, it will be impossible to have an algorithm for solving the inverse problem: indeed, if such an algorithm is possible, then the inverse function $f^{-1}$ is continuous, but, as we have already mentioned, for some continuous 1-1 mappings $f$ from a non-compact set, the inverse is not continuous [84].

- The algorithm described in the proof is general and therefore (as many general algorithms), when applied to simple problems, it may require unnecessarily many computation steps. There are cases when simpler methods are possible: e.g., if the signal that we are trying to reconstruct is a smooth function, then we can ask an expert what is the upper bound for the signal’s energy $\int (x'(t))^2 dt$, and then use known regularization techniques [84].

6 If We Impose The Condition That The Actual State Is Not Abnormal, Then We Can Get Guaranteed Estimates Even For Heuristic Numerical Methods
6.1 How To Check Whether A Numerical Method Always Works

To check the numerical method, we can run it on several tests. Usually, the first tests are simple computer-generated ones. If a method behaves nicely on these simple tests, then it is tried on realistic or real-life examples, where the input data come from real experiments. So, we have a (potentially infinite) sequence of tests. Let us assume that testing is performed in such a way that on some stage, every part and every aspect of the method is tested. Mathematically, let us assume that the potentially infinite sequence of test cases has the following completeness property: if a method works correctly for all (infinitely many) test cases, then this method is always correct.

**Definition 9.**

- Let a definable set \( t \) be given. Its elements will be called tests. By a testing method \( S \), we mean a set of infinite sequences \( t_1, t_2, \ldots, t_n, \ldots \) of tests.
- By a numerical method, we mean a definable subset \( M \) of the set of all tests \( t \). If \( t_i \in M \), we say that a method \( M \) passed the test \( t_i \); else, that the method \( M \) failed the test \( M \).
- We say that a method is correct if it passes all tests from \( t \).
- Let a class of methods \( M \) be fixed. We say that a testing method \( S \) is complete for methods from the class \( M \) if for every sequence \( (t_1, \ldots, t_n, \ldots) \in S \), and for every method \( M \in M \), if the method \( M \) passes all the tests \( t_1, \ldots, t_n, \ldots \), then this method is correct.

**Proposition 7.** Let a testing method \( S \) be complete for a class of numerical methods \( M \). Then, for every method \( M \in M \), there exists an integer \( N \) such that if a sequence \( (t_1, \ldots, t_n, \ldots) \in S \) is not abnormal, and \( M \) passes the first \( N \) tests of this sequence, then this method \( M \) is correct.

This Proposition justifies the usual testing of a method, in which we make a conclusion about its correctness after only finitely many tests. The crucial assumption here is that we assume that the testing sequence is taken from the real-life examples, and these examples are not abnormal.

6.2 How To Get A Guaranteed Estimate For The Result

In many practical cases, we know the process \( x_k \) that is proven to converge to the desired solution \( x \), but we do not know when to stop in order to guarantee the given accuracy \( \varepsilon \) (i.e., to guarantee that \( d(x, x_k) \leq \varepsilon \)).

For example, we may use an iterative method \( x_{k+1} = F(x_k) \) to solve the equation \( F(x) = x \).
In these cases, heuristic methods are used. There are two main groups of heuristic methods:

- Usually, in iterative methods, if \( x_k = x_{k+1} \), then \( x_n \) is the required solution. Therefore, if \( x_k \) and \( x_{k+1} \) are close, we can conclude that we are close to the solution. Hence, we stop when the consequent values \( x_k \) become close enough, i.e., when \( d(x_k, x_{k+1}) \leq \delta \) for some \( \delta > 0 \). This method is often used in physics, if, e.g., we have the expression of \( x \) as a sum of the infinite series (e.g., Taylor series in perturbation methods). Then, if, e.g., second order terms are negligibly small, we neglect quadratic and higher order terms, and use the linear expression as an approximation to the desired solution (see, e.g., [25]).

- If we are solving the equation \( f(x) = y \), then we stop when \( f(x_k) \) becomes small enough (i.e., when \( d(f(x), y) \leq \delta \) for some \( \delta > 0 \)).

These stopping criteria can be described by the following general definition:

**Definition 10.** Let \( X \) be a definable metric space, and let \( S \) be a definable set of convergent sequences of \( X \).

- Let \( \{x_k\} \in S \), \( k \) be an integer, and \( \varepsilon > 0 \) a real number. We say that \( x_k \) is \( \varepsilon \)-accurate if \( d(x_k, \lim x_p) \leq \varepsilon \).

- Let \( d \geq 1 \) be an integer. By a stopping criterion, we mean a function \( c: X^d \to R_0^+ = \{x \in R \mid x \geq 0\} \) that satisfies the following two properties:
  - If \( \{x_k\} \in S \), then \( c(x_k, \ldots, x_{k+d-1}) \to 0 \).
  - If for some \( \{x_k\} \in S \) and for some \( n \), \( c(x_k, \ldots, x_{k+d-1}) = 0 \), then \( x_k = \ldots = x_{k+d-1} = \lim x_p \).

The two above-described criteria correspond to \( c(x, x') = d(x, x') \) and \( c(x) = d(f(x), y) \).

**Proposition 8.** Let \( c \) be a stopping criterion. Then, for every \( \varepsilon \), there exists a \( \delta > 0 \) such that if a sequence \( \{x_k\} \) is not abnormal, and \( c(x_k, \ldots, x_{k+d-1}) \leq \delta \), then \( x_k \) is \( \varepsilon \)-accurate.

So, if we restrict ourselves to not abnormal sequences only (i.e., sequence that stem from not abnormal, physical observations), then \( c(x_k, \ldots, x_{k+d-1}) \leq \delta \) guarantees that we are \( \varepsilon \)-close to the desired solution. In particular, \( d(x_k, x_{k+1}) \leq \delta \) and \( d(f(x), y) \leq \delta \) guarantee that \( d(x_n, x) \leq \varepsilon \). In case we are summing a numerical series \( x_k = a_1 + \ldots + a_k \), we have \( d(x_k, x_{k+1}) = |a_{k+1}| \), so, this stopping criterion means that means if the next term is negligible (\( |a_{k+1}| \leq \delta \)), then we are \( \varepsilon \)-close to the sum: \( |x_k - x| \leq \varepsilon \).
6.3 When Will The Algorithm Stop?

In practice, it is not sufficient to claim that an algorithm generates a guaranteed estimate. We would like to know when to expect the result. A computer can go wrong, so if the computations take too long, we would like to know whether it is just taking long, or there has been a computer error, and we better start anew.

Theoretically, arbitrarily long computations are possible. However, as we will see, computations do not take too long.

**Definition 11.**

- Let a set $U$ be given. Its elements are called computations.
- Let a function $t : U \to \mathbb{R} \cup \{\infty\}$ be given. The value $t(u)$ will be called the run time of the computation $u$.
- We say that computation terminates if $t(u) \neq \infty$.

**Proposition 9.** There exists a number $T_0 > 0$ such that if a computation $u$ is not abnormal, and it terminates, then its run time is $\leq T_0$.

6.4 If a Numerical Method is Polynomial-Time and Not Abnormal, Then It Is Truly Feasible

It is well known that not all algorithms are realistic (see, e.g., [59], Section 7.1). If an algorithm requires, say, $2^{2\text{len}(x)}$ computational steps for an input $x$ of length $\text{len}(x)$, then for realistic lengths (e.g., for $\text{len}(x) = 100$) this number of steps will exceed the lifetime of the Universe (according to modern cosmology). So if we are interested in separating purely theoretical algorithms from the ones that can be actually run on the computers (existing or future ones), we must somehow formalize the notion of feasibility.

The most widely used formalization of this notion is that feasible algorithms are exactly the ones that are time-polynomial, i.e., the ones for which the running time is limited by some polynomial $P(\text{len}(x))$ of the input length $\text{len}(x)$ (see, e.g., [59], Section 7.4; [66], Ch. 23). There exist formal systems of reasonable axioms that justify this choice (see, e.g., [79]).

However, the majority of the researchers agree that this is not the precise description of a feasible algorithm, because some time-polynomial algorithms are evidently not feasible. For example, an algorithm that takes $10^{10^{10^{10^{\text{len}(x)}}}}$ time to compute is time-polynomial (even linear-time) but it can hardly be called feasible: even for $\text{len}(x) = 1$, it requires the computation time that is exponentially bigger than the lifetime of the Universe.

There are two possible approaches to this situation:
We can view this situation as a problem, and try to come up with a new definition of feasibility that will really describe only physically feasible algorithms. Such a formalization is proposed, e.g., in [71].

In this section, we will pursue another approach: we will show that normally, the values of the coefficient and the exponent cannot grow indefinitely: namely, there exist $C$ and $K$ such that if we exclude abnormal methods, then running time $t_U(x)$ for all not abnormal time-polynomial algorithms $U$ is limited by $C \cdot (\text{len}(x))^K$.

**PROPOSITION 10.** There exists $C > 0$ and $K > 0$ such that if a polynomial-time algorithm $U$ is not abnormal, then its running time $t_U(n)$ is bounded by $C \cdot n^K$.

In other words, if we take the set of all polynomial-time algorithms as $U$, and denote by $T$ the set of all typical elements of $U$, then there exists $K$ and $C$ such that the running time of every algorithm from $T$ is bounded by $C \cdot n^K$.

This proposition can be confirmed by the fact that for every problem, for which a time-polynomial algorithm has been known for some time (for a few years), in a few years, a new algorithm is discovered for which the running time is limited by a cube of $n$, i.e., by $C \cdot n^3$ (see, e.g., [16]; the time during which this happens is jokingly called “incubation period”).

7 The Notion Of “Not Abnormal” Is Also Helpful For Foundations of Physics

In the previous sections, we described how the notion of “not abnormal” can lead us to guaranteed interval (and, in general, error) estimates. In other words, we showed that this notion is helpful in computational physics. In this section, we will show that this notion can also help in the problems related to foundations of physics. These results will show that our formalization is in good accordance with the modern theoretical physics.

Some of these results have been presented in [40, 27].

7.1 Every Physical Quantity is Bounded

**PROPOSITION 11.** If $U$ is a definable set, and $f : U \rightarrow \mathbb{R}$ is a definable function, then there exists a number $C$ such that if $u \in U$ is not abnormal, then $|f(u)| \leq C$.

If we use the physicists’ idea that abnormal initial conditions and/or abnormal values of parameters are impossible, then we can make the following conclusions:
Special relativity. If as $U$, we take the set of all the particles, and as $f$, we take velocity, then we can conclude that the velocities of all (not abnormal) particles is bounded by some constant $C$. This is exactly what special relativity says, with the speed of light as $C$.

Cosmology. If we take the same state $U$, and as $f$, take the distance from the a particle $u$ to some fixed point in the Universe, then we can conclude that the distances between particles in the Universe are bounded by a constant $C$. So, the Universe is finite. Similarly, if we take a time interval between the events as $f$, we can conclude that the Universe has a finite lifetime.

Why particles with large masses do not exist. If we take mass of the particle as $f$, then we can conclude that the masses of all particles are bounded by some constant $C$. This result explains the following problem:

- Several existing particle classification schemes allow particles with arbitrarily large masses. E.g., in Regge trajectory scheme, particles form families with masses $m_n = m_0 + n \cdot d$ for some constants $m_0$ and $d$. When $n \to \infty$, we have $m_n \to \infty$.

- Only particles with relatively small masses have been experimentally observed (see, e.g., [32]).

These particles with large masses, that are difficult to weed out using equations only, can be easily weeded out if use the notion of “not abnormal”.

Dimensionless constants are usually small. This is the reason why physicists can safely estimate and neglect, e.g., quadratic (or, in general, higher order terms) in asymptotic expansions, even though no accurate estimates on the coefficients on these terms is known. In particular, such methods are used in quantum field theory, where we add up several first Feynman diagrams; in celestial mechanics, etc.

Comment: Consequences for philosophy of mathematics. Physically meaningful numbers are bounded. Hence, it seems reasonable to place only physically meaningful integers in the foundations of mathematics. In the corresponding formalisms, there will be finitely only finitely many integers. The ideas of such formalisms were originally developed by Van Dantzig, Esseneine-Volpine, and Kolmogorov, and have been later transformed into a useful formalism by Parikh (see [72] and references therein).
7.2 Quantization

**Proposition 12.** Let $U$ be a definable set, and $f : U \to \mathbb{R}$ be a definable function. Then, there exists a number $\varepsilon > 0$ such that if $u$ is not abnormal and $f(u) \neq 0$, then $|f(u)| \geq \varepsilon$.

Together with the physicists’ idea that abnormal situations are impossible, we can conclude that a physical quantity cannot have arbitrarily small positive values: there must be the smallest value that is indivisible. This explains, e.g., why the electric charge cannot take any value we want: there is the charge quantum (1/3 of an electron’s charge).

**Infinites in quantum field theory disappear.** This result can be justified by the joint use of quantization and boundedness: indeed, in quantum field theory, infinites are caused by the fact that we have to integrate over all momenta $p$ of all particles. Infinites happen because we have to integrate over $p \to \infty$ and $p \to 0$ (see, e.g., [25]). If we apply boundedness and quantization results to momentum, we conclude that $p$ is bounded from above and from below. Therefore, all the integrals should be finite (another interval-related argument that makes infinites disappear is given in [39]).

**Schroedinger’s cat paradox stops being paradoxical.** According to traditional quantum mechanics, states are described by a vectors from a Hilbert space $L^2$, and all vectors have a physical meaning. E. Schroedinger has shown that this assumption leads to the following paradox (for description and discussions, see, e.g., [4]): Suppose that we place a cat into a box with a gun aimed at it. A gun is controlled by the switch, which can be triggered by a left-polarized photon. If we send a photon in a left-polarized state $s_1$, the gun fires, and the cat is dead. If we send a photon in a right-polarized state $s_2$, the gun does not fire, and the cat is alive. Suppose now that we send a photon in a state $s$ that is a superposition of $s_1$ and $s_2$ (i.e., in mathematical terms, a linear combination). Equations of quantum mechanics are linear, so, as a result, we get the state that is a superposition of dead and alive. Such a superposition is difficult to imagine, because in real life, an animal is either dead, or alive.

This paradox is based on the assumption that all vectors from a Hilbert space are physically meaningful states. If we impose the additional condition that only not abnormal states are physically possible, then we can exclude some states as being abnormal. Indeed, from the quantization result, it follows that there exists an $\varepsilon$ such that if a physically meaningful state is $\varepsilon$-close to the “dead” state, then it is the “dead” state. Paradoxical continuous transition between dead and alive thus disappears.

7.3 Chaos

**The origin of chaos.** Restriction to not abnormal also explains the origin of chaotic behavior of physical systems. In mathematical terms, chaos means that
after some time, the states of the system form a so-called \textit{strange attractor}, i.e., in topological terms, a \textit{completely disconnected set} in the following precise sense:

\textbf{Definition 12.} A set $S$ in a metric space $X$ is called \textit{completely disconnected} if for every $s_1, s_2 \in S$, there exist open sets $S_1$ and $S_2$ such that $s_1 \in S_1$, $s_2 \in S_2$, $S_1 \cap S_2 = \emptyset$, and $S \subseteq S_1 \cup S_2$.

In other words, every two points belong to different topological components of the set $S$. The relationship between this definition and typical elements is given by the following result:

\textbf{PROPOSITION 13.} In a definable separable metric space, the set of typical elements is completely disconnected.

So, if we assume (as physicists do) that abnormal states are impossible, then we immediately arrive at the chaotic dynamics.

\textbf{Spontaneous symmetry violations.} Equations of physics have lots of symmetries. If an equation is, e.g., invariant w.r.t. rotations, and the initial condition is rotation-invariant, then the solution stays rotation-invariant for all moments of time. From the mathematical viewpoint, symmetric solutions are quite possible. However, in real life, we only observe \textit{approximately symmetric solutions}. E.g., in cosmology, from the observations of the 3K radiation, we can conclude that the initial state of the Universe was highly isotropic and homogeneous (see, e.g., [67]). However, the observed Universe is not. This means that the initial conditions were only \textit{approximately} isotropic and homogeneous.

In each particular case, we may have specific physical reasons for symmetry violation. Restriction to “not abnormal” leads to a general explanation [27]: namely, with this restriction, the theory consists not only of the set of equations, but also of the set $T$ of physically possible (not abnormal) initial conditions. We are going to show that even if the equations are invariant, the set $T$ is not.

\textbf{Definition 13.} Let $X$ be a topological space.

- By a \textit{continuous transformation group}, we mean a connected continuous group $G$ with a continuous mapping $a : G \times X \to X$ such that $a_{g_1g_2}(x) = a_{g_1}(a_{g_2}(x))$.
- A set $S$ is called \textit{invariant} w.r.t. $G$ if $a_g(S) = S$ for all $g \in G$ (where $a_g(S) = \{a_g(s) \mid s \in S\}$).
- We say that the continuous transformation group $G$ is \textit{non-trivial} if $a_g(x) \neq x$ for some $g \in G$ and for some not abnormal $x \in X$.

\textbf{PROPOSITION 14.} Let $X$ be a definable separable metric space, $T$ be the set of typical elements of $X$, and let $G$ be a non-trivial continuous transformation group. Then, the set $T$ is not invariant w.r.t. $G$. 

24
7.4 All The Processes In The World Are Connected

Let us assume that we are studying two physically unrelated processes. Let $X_1$ denote the set of all possible states (initial conditions) of the first process, and $X_2$ denote the set of all initial conditions of the second process. Then, for each process, we can formalize the physicists’ idea that initial conditions cannot be abnormal by saying that $x_1 \in T_1$ and $x_2 \in T_2$, where $T_i \subseteq X_i$ are the corresponding sets of typical elements.

We could also analyze the two processes as a whole. The state of a pair of processes can be characterized by a pair of states $(x_1, x_2)$. Since the processes are unrelated, the state $x_1$ of the first process cannot influence the state of the second one. So, for every $x_1$, the set of all possible states $x_2$ of the second process is $X_2$. Therefore, the set of all possible pairs $(x_1, x_2)$ is equal to the set $X = X_1 \times X_2$ of all the pairs $(x_1, x_2), x_i \in X_i$. We can now formulate our “not abnormal” idea by saying that $(x_1, x_2) \in T$, where $T$ is the set of all typical pairs.

Since we assumed that the processes are physically unrelated, it seems like the choice of the state of the first process should not change whatever states are possible for the second one. Therefore, we would expect that the set of physically possible (not abnormal) pairs coincides with the Cartesian product $T = T_1 \times T_2$.

In principle, it is possible to find such a $T$: e.g., we can take a one-point set $T$. However, as we will show, if we take into consideration that typical states must form a majority in some reasonable sense, then such $T = T_1 \times T_2$ is no longer possible. In other words, every two processes in the world are related, even if the equations that describe these processes are independent.

The strongest result occurs if we consider two identical processes, for which it is natural to assume that if $(x_1, x_2) \in T$, then $(x_2, x_1) \in T$:

**Definition 14.**

- Let $X$ be an arbitrary set. By a permutation, we mean a mapping $X \times X \to X \times X$ defined as $(x_1, x_2) \to (x_2, x_1)$.
- For arbitrary sets $X_1 \times X_2$, we say that a set $T \subseteq X_1 \times X_2$ is factorizable if $T = T_1 \times T_2$ for some $T_i \subseteq X_i$.
- Let $p$ be a probability measure on $X$ that is non-atomic (i.e., $p(\{x\}) = 0$ for all $x \in X$). We say that a set $T_i \subseteq X$ is a majority set if $p(T_i) > 1/2$. We say that a set $T \subseteq X_1 \times X_2$ is a majority set if $(p \times p)(T) \geq 1/2$.

In physical terms, factorizable sets correspond to truly independent processes:
Definition 15.

- Let $X_1$ and $X_2$ be sets. Elements of $X_1$ will be called states of the first process. Elements of $X_2$ will be called states of the second process. A pair $(x_1, x_2) \in X_1 \times X_2$ is called a joint state.

- Let $T \subseteq X_1 \times X_2$ be a set of joint states. States from $T$ will be called physically possible.

- We say that a state $x_1 \in X_1$ of the first process is physically possible if $(x_1, x_2) \in T$ for some $x_2 \in X_2$.

- We say that a state $x_2 \in X_2$ of the second process is physically possible if $(x_1, x_2) \in T$ for some $x_1 \in X_1$.

- Let a state $x_1 \in X_1$ be given. We say that given a state $x_1$, the state $x_2$ is possible for the second process if $(x_1, x_2) \in T$. The set of all states of the second process that are possible for a given $x_1$ will be denoted by $P_2(x_1)$.

- Similarly, we can define $P_1(x_2)$.

- We say that the processes are truly independent if the set of possible states of the second process does not depend on the state of the first process (and vice versa), i.e., if $x_1$ and $x'_1$ are both physically possible, then $P_2(x_1) = P_2(x'_1)$.

The following result is easy to prove:

**Proposition 15.** Two processes are truly independent iff the set $T$ is factorizable.

**Proposition 16.** Let $X$ be a definable separable metric space, and let $T$ be an infinite set of typical elements of $X \times X$ that is invariant w.r.t. permutation. Then, $T$ is not factorizable (i.e., two processes are not truly independent).

The proof of this Proposition is based on the following Lemma that may be of independent interest:

**Lemma 2.** If $X$ and $Y$ are definable sets, $f : X \to Y$ is a definable mapping, and $T$ is a set of typical elements of $X$, then $f(X)$ is a set of typical elements of $Y$.

In other words, this Lemma says that if an element $x \in X$ is not abnormal, then its image is also not abnormal.

In the general case, we can usually assume that $X_1 = X_2$ (e.g., in quantum mechanics, the set of all possible states of any system is a Hilbert space $L^2$).

**Proposition 17.** Let $X$ be a definable separable metric space, and let $T$ be a majority set of typical elements of $X_1 \times X_2$. Then, $T$ is not factorizable (i.e., two processes are not truly independent).
PROPOSITION 18. Let $X$ be a definable separable metric space, and let $T_1$ and $T_2$ be majority sets of typical elements of $X$. Then, $T = T_1 \times T_2$ is not a set of typical elements of $X \times X$ (i.e., the processes are not truly independent).

The proof of these statements is based on the following Lemma:

LEMMA 3. If $T$ is a set of typical elements of $X$, then every non-empty subset of $T$ is also a set of typical elements of $X$.

If we have $s > 2$ systems, then we can prove an even stronger statement:

PROPOSITION 19. Let $X$ be a definable separable metric space, and let $T_1, \ldots, T_s$ be sets of typical elements of $X$ for which $p(T_i) > 1/s$ for all $i$. Then, $T = T_1 \times \ldots \times T_s$ is not a set of typical elements of $X \times \ldots \times X$.

We can generalize the above definition of true independence to $s > 2$ processes, and claim that under the conditions of Proposition 19, these $s$ processes are not truly independent.

This proposition take into consideration the fact that “typical” does not necessarily mean “belonging to the majority”: e.g., a “typical professor” may combine several features, each of which may be typical for a majority, but when combined, may be rather rare.

Let us describe how these results are related to theoretical physics.

EPR paradox. Analyzing quantum mechanics, Einstein, Podolsky and Rosen came up with the conclusion that in quantum mechanics, it is potentially possible to have correlation between the states of spatially separated particles at the same moment of time. This conclusion clearly contradicts special relativity, according to which immediate communication between spatially separated events is impossible. Because of this contradiction, this conclusion was called a paradox (named EPR by first letters of their names). For a detailed description and references, see, e.g., [74]. The above results show that if we take into consideration the fact that a theory is not only equations, but also initial conditions, then connection even between spatially separated events is possible, so EPR paradox is not a paradox anymore (other solutions to this paradox are presented in [74]).

Interaction between parallel worlds. Modern physics is formulated in terms of probabilities. Because of that, even if we measure everything accurately, we cannot uniquely predict the results of future experiments. One way to describe it is to say that instead of a single world history, there are several possible world histories. Usually, only one history is considered real, all others are viewed as purely mathematical objects. However, starting from Wheeler and Everett, some researchers started to consider the possibility that all world histories are real: one of them describes our world, in which we live. Others describe other worlds (these other worlds do not intersect with ours are are therefore called parallel worlds [74]).
If we assume that parallel worlds do not influence our world, then whether we call them real or not is a question of semantics: no experiments in this world are influenced by anything that happens in these parallel worlds. However, some theorists suggested that a small interaction is possible. For example, in 1972, A. Sakharov have suggested that the space-times of the worlds do have intersections; these intersections are elementary particles, and observable properties of the particles can be interpreted as topological characteristics of the intersection (for details and references, see, e.g., [67]). From the described viewpoint, this is a quite natural idea: as soon as we adopted the model with parallel worlds, then automatically we adopted the postulate that what’s going on in all these worlds is typical with respect to this theory. Hence, due to the above propositions, what is happening in one of the worlds can influence the others.

8 Proofs

Proof of Proposition 1. We have already mentioned that there are denumerably many definable sequences of sets. Therefore, there are no more than denumerably many sequences of sets $A_n$ for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$. So, we can enumerate such sequences. Let us denote the elements of the $k$-th sequence by $A^k_0, A^k_1, \ldots, A^k_n, \ldots$. For every $k$ and for every $i$, from $\cap A^k_n = \emptyset$ and monotonicity, it follows that $p_i(A^k_n) \to 0$. This means that there exists an integer $N_i^k$ such that if $n \geq N_i$, then $p_i(A^k_n) \leq \varepsilon \cdot 2^{-k}$. Let us define $N(k) = \max(N_i^1, \ldots, N_i^m)$. Then, $N_i(k) \geq N_i^k$ for all $i$, and therefore, $p_i(A^k_N(k)) \leq \varepsilon \cdot 2^{-k}$. As $T$, we will take the complement to the set

$$A = \bigcup_{k=1}^{\infty} A^k_N(k).$$

It is easy to see that $A$ is a set of typical elements: indeed, for every sequence $A^k_0, A^k_1, \ldots, A^k_n, \ldots$, we have $T \cap A^k_N(k) = \emptyset$. Now, let us show that elements from $A$ are rare. Indeed, for every $i$, we have $p_i(A) \leq \sum p_i(A^k_N(k))$. For every $i$, we have $p_i(A^k_N(k)) \leq \varepsilon \cdot 2^{-k}$, and therefore, $p_i(A) \leq \sum(\varepsilon \cdot 2^{-k}) = \varepsilon$. Q.E.D.

Proof of Proposition 2. As $A_n$, let us take the set of all sequence $r \in S$ for which first $n$ experiments confirm the theory $T$, but some further experiments do not confirm $T$. Then, it is easy to show that $A_n \supseteq A_{n+1}$ for all $n$. Since the theory is physically meaningful, we have $\cap A_n = \emptyset$. Therefore, there exists $N$ for which $A_N \cap T = \emptyset$, i.e., for which all not abnormal sequences belong to the complement of $A_N$. Due to our definition of $A_N$, $r \notin A_N$ means that if first $N$ experiments confirm the theory, then this theory is correct. Q.E.D.

Proof of Proposition 3. As $A_n$, take the set of all pairs $(T, \nabla)$ for which first $n$ experiments confirm the sequence $r$ confirm the theory, but the theory is not correct on $r$. The proof is similar to the proof of Proposition 2.
Proof of Proposition 4. Let us prove this result for an arbitrary definable metric space $X$ with a metric $d$. As $A_n$, let us take the set of all $x$ for which $0 < d(x, a) \leq 2^{-n}$. This sequence is decreasing, and $\cap A_n = \emptyset$. Therefore, there exists an $N$ for which $A_N \cap T = \emptyset$. This means that none of the elements from $T$ belong to $A_N$. This, in its turn, means that for elements $x \in T$, either $d(x, a) = 0$, or $d(x, a) > 2^{-n}$. For $\varepsilon = 2^{-n}$, we get the desired result.

Proof of Lemma 1. A set $S$ in a metric space $X$ is compact iff it is closed and for every $\varepsilon$, it has a finite $\varepsilon$-net, i.e., a finite set $S(\varepsilon)$ with the property that every $s \in S$, there exists an element $s(\varepsilon) \in S(\varepsilon)$ that is $\varepsilon$-close to $s$.

The closure of $T$ is clearly closed, so, to prove that the closure of $T$ is a compact, it is sufficient to prove that it has an $\varepsilon$-set for all $\varepsilon$. For that, it is sufficient to prove that for every $\varepsilon > 0$, there exists an $\varepsilon$-net for $T$.

If a set $S$ is a $\varepsilon$-net $S(\varepsilon)$, and $\varepsilon' > \varepsilon$, then, as one can easily see, this same set $S(\varepsilon)$ is also a $\varepsilon'$-net for $S$. Therefore, it is sufficient to show that $\varepsilon$-nets for $T$ exist for $\varepsilon = 2^{-k}, k = 0, 1, 2, \ldots$

Let us fix $\varepsilon = 2^{-k}$. Since $X$ is definably separable, there exists a definable sequence $x_1, \ldots, x_i, \ldots$ that is everywhere dense in $X$. As $A_n$, we will take the complement to the union $U_n$ of $n$ closed spheres $D_\varepsilon(x_1), \ldots, D_\varepsilon(x_n)$ of radius $\varepsilon$ with centers in $x_1, \ldots, x_n$. Clearly, $A_n \supseteq A_{n+1}$. Since $x_i$ is an everywhere dense sequence, for every $x$, there exists an $n$ for which $x \in D_\varepsilon(x_i)$ and for which, therefore, $x \in U_n$ and $x \notin A_n = X \setminus A_n$. Hence, the intersection of all the sets $A_n$ is empty. Therefore, there exists an $N$ for which $A_N \cap T = \emptyset$. This means that $T \subseteq U_N$. This, in its turn, means that the elements $x_1, \ldots, x_N$ form an $\varepsilon$-net for $T$.

So, the set $T$ has an $\varepsilon$-net for $\varepsilon = 2^{-k}, k = 0, 1, 2, \ldots$. Hence, $\overline{T}$ is compact.

Q.E.D.

Proof of Proposition 5. This proof follows from the known result that if a function $f$ is continuous and 1-1 on a compact, then its inverse is also continuous (see, e.g., [84]). In our case, such a function is $f : \overline{T} \rightarrow f(\overline{T})$.

Proof of Proposition 6. Due to Proposition 5, the inverse function $f^{-1}$ is continuous on $f(T)$. In particular, for every $l$, there exists a $\delta > 0$ such that if $x, x' \in T$ and $d(f(x), f(x')) \leq \delta$, then $d(x, x') \leq 2^{-l}$. If we know $\delta$, then we can compute the desired approximation to $x$ as follows. Since $X$ is definably separable, there exists a definable sequence $x_n$ that is everywhere dense in $X$. Using this sequence, we:

- Compute $f(x)$ with accuracy $\delta/8$; the result of this computation (one of the elements of the everywhere dense sequence $y_m$) will be denoted by $\tilde{f}(x)$.

- For $n = 1, 2, \ldots$, compute $f(x_n)$ with accuracy $\delta/8$, and for the result $\tilde{f}(x_n)$ of this computation, we compute the distance $d(f(x), \tilde{f}(x_n))$ with an accuracy $\delta/8$. When this estimate $\tilde{d}$ is $\leq \delta/2$, we stop, and produce $x_n$ as the desired result.

29
Let us show that this algorithm will work.

- First, let us prove that this algorithm will stop. Indeed, since the sequence \( x_n \) is everywhere dense in \( X \), we have a subsequence \( x_{n_k} \) that tends to \( x \). Since \( f \) is continuous, we have \( f(x_{n_k}) \to f(x) \). So, there exists a \( k \) for which \( d(f(x_{n_k}), f(x)) \leq \delta/8 \). Since \( f(x) \) and \( f(x_{n_k}) \) are \( (\delta/8) \)-approximations to \( f(x) \), we can conclude that

\[
d(f(x_{n_k}), f(x)) \leq d(f(x_{n_k}), f(x_{n_k})) + d(f(x_{n_k}), f(x)) \leq \delta/8 + \delta/8 + \delta/8 = (3/8)\delta.
\]

Hence \( \delta \leq d(f(x_{n_k}), f(x)) + \delta/8 \leq \delta/2 \). So, if the algorithm did not stop before the value \( n_k \), it will stop at this point.

- Let us now show that the algorithm produces the desired value. Indeed, if \( d(f(x_{n_k}), f(x)) \leq \delta/8 \), then we return \( \delta \).

We must find \( \delta \) such that if \( d(f(x), f(x')) \leq \delta \), then \( d(x, x') \leq 2^{-l} \). To find this \( \delta \), let us choose an integer \( p \). Since \( f \) is constructively continuous, we can compute the value \( \eta \) such that if \( d(x, x') \leq \eta \), then \( d(f(x), f(x')) \leq 2^{-p} \). Let us take \( \beta = \min(\eta, 2^{-p}) \). For this choice of \( \beta \), if \( d(x, x') \leq \beta \), then \( d(x, x') \leq 2^{-p} \) and \( d(f(x), f(x')) \leq 2^{-p} \) for all \( \delta \)-approximations to \( f(x) \), we have \( d(x, x_n) \leq 2^{-l} \).

So, to complete the description of the algorithm, we must describe how to compute \( \delta \).

We must find \( \delta \) such that if \( d(f(x), f(x')) \leq \delta \), then \( d(x, x') \leq 2^{-l} \). To find this \( \delta \), let us choose an integer \( p \). Since \( f \) is constructively continuous, we can compute the value \( \eta \) such that if \( d(x, x') \leq \eta \), then \( d(f(x), f(x')) \leq 2^{-p} \). Let us take \( \beta = \min(\eta, 2^{-p}) \). For this choice of \( \beta \), if \( d(x, x') \leq \beta \), then \( d(x, x') \leq 2^{-p} \) and \( d(f(x), f(x')) \leq 2^{-p} \). Let us find a \( \beta \)-net \( x^{(1)}, \ldots, x^{(m)} \) for \( X \). This can be done similarly to the proof of Lemma 1, only instead of referring to existence of the desired \( N \), we use the expert to produce such an \( N \). For this \( \beta \)-net, we take all pairs \( x^{(i)}, x^{(j)} \) for which \( d(x^{(i)}, x^{(j)}) \geq 2^{-l} - 2\beta \), and find the smallest value \( M \) of \( d(f(x^{(i)}), f(x^{(j)})) \) for all such pairs. If \( M > 2\beta \), then we return \( \delta = M - 2\beta \). Else, we increase \( p \) by 1, and repeat the process again and again.

Let us prove that this part of the algorithm does produce the correct value of \( \delta \) (and thus, that the entire algorithm is correct). Indeed:

- Let us first show that this algorithm will stop. Indeed, due to Proposition 5, there exists a value \( \delta' > 0 \) for which if \( d(f(x), f(x')) \leq \delta' \), then \( d(x, x') \leq (1/2) \cdot 2^{-l} \). So, if \( d(f(x^{(i)}), f(x^{(j)})) > (1/2) \cdot 2^{-l} \), then

\[
d(f(x^{(i)}), f(x^{(j)})) > \delta'.
\]

Hence, if we take \( p \) so big that \( 2^{-p} < \min(\delta'/2, (1/4) \cdot 2^{-l}) \), then from \( d(x^{(i)}, x^{(j)}) > 2^{-l} - 2\beta \), and from \( \beta \leq 2^{-p} < (1/4) \cdot 2^{-l} \), we can conclude that \( d(x^{(i)}, x^{(j)}) > (1/2) \cdot 2^{-l} \), and therefore, that \( M \geq \delta' > 2 \cdot 2^{-p} \geq 2\beta \).

- Let us now show that if it did stop, then we got the desired \( \delta \). Indeed, let \( d(f(x), f(x')) \leq M - 2\beta \). Since elements \( x^{(i)} \) are a \( \beta \)-net for \( X \), there exist elements \( x^{(i)} \) and \( x^{(j)} \) that are \( \beta \)-close to \( x \) and \( x' \) correspondingly. Due to the choice of \( \beta \), we can conclude that \( d(f(x), f(x^{(i)})) \leq \beta \) and \( d(f(x'), f(x^{(j)})) \leq \beta \). Hence, \( d(f(x^{(i)}), f(x^{(j)})) \leq d(f(x), f(x')) + 2\beta \leq M \). By definition of \( M \), this means that \( d(x^{(i)}, x^{(j)}) \leq 2^{-l} - 2\beta \). Therefore, \( d(x, x') \leq d(x^{(i)}, x^{(j)}) + 2\beta \leq 2^{-l} \).
So, the second part of the algorithm produces correct $\delta$. Q.E.D.

**Proof of Proposition 7.** This proof is similar to the proof of Proposition 2. Let us fix $M$. As $A_n$, let us take the set of all testing sequences $s \in S$ for which the method $M$ passes this first $n$ tests, but fails some other test. Then, it is easy to show that $A_n \supseteq A_{n+1}$ for all $n$. Since the testing method is assumed to be complete, we have $\cap A_n = \emptyset$. Therefore, there exists $N$ for which $A_N \cap T = \emptyset$, i.e., for which all *not abnormal* testing sequences belong to the complement of $A_N$. Due to our definition of $A_N$, $s \not\in A_N$ means that if the method passes first $N$ tests, then this method is correct. Q.E.D.

**Proof of Proposition 8.** As $A_n$, we will take the set of all sequences for which for some $k$, $c(x_k, \ldots, x_{k+d-1}) \leq 2^{-n}$ and $d(x_k, x) > \varepsilon$. Clearly, $A_n \supseteq A_{n+1}$.

Let us show that the intersection $\cap A_n$ is empty. Indeed, suppose that the sequence $\{x_k\}$ belongs to this intersection. This means that for every $n$, there exists a $k(n)$ such that $c(x_k(n), \ldots, x_{k(n)+d-1}) \leq 2^{-n}$ and $d(x_k(n), x) > \varepsilon$. If some value $k$ is equal to $k(n)$ for infinitely many $n$, this means that $c(x_k, \ldots, x_{k+d-1}) \leq 2^{-n}$ for all $n$ and hence, that $c(x_k, \ldots, x_{k+d-1}) = 0$. From the definition of a stopping criterion, it then follows that $x_k = x$, so $d(x_k, x) = 0 \not\in \varepsilon$. Hence, $k(n) \rightarrow \infty$, so (since $\{x_k\}$ is convergent), $d(x_k(n), x) \rightarrow 0$ and $d(x_k(n), x) \not\in \varepsilon$. The contradiction shows that the intersection is empty.

So, there exists an $N$ for which $A_N \cap T = \emptyset$. Hence, we can take $\delta = 2^{-N}$. Q.E.D.

**Proof of Proposition 9.** As $A_n$, we take the set of all computations $u$ for which $t(u) > n$ and $t(u) \neq \infty$. Then, there exists $N$ such that $A_N \cap T = \emptyset$. Hence, we can take $N$ as the desired $T_0$.

**Proof of Proposition 10.** As $A_n$, we take the set of all polynomial-time algorithms $U$ for which $t_{ul}(x) > n \cdot (\text{len}(x))^n$ for some input $x$. Clearly, $A_n \supseteq A_{n+1}$. Let us prove that $\cap A_n = \emptyset$.

Indeed, let us take an arbitrary algorithm $U$ from $U$ and show that it does not belong to the intersection $\cap A_n$. Every algorithm from $U$ is time-polynomial, i.e., $t_{ul}(x) \leq c \cdot (\text{len}(x))^k$ for some $c$ and $k$. Therefore, for $n = \max(c, k)$, we have $U \not\in A_n$. Hence, $U \not\in \cap A_n$, and therefore, $\cap A_n = \emptyset$.

So, there exists an $N$ for which $A_N \cap T = \emptyset$. This means that if $U \in T$, then $t_{ul}(x) \leq n \cdot (\text{len}(x))^n$ for all inputs $x$. So, we can take $C = K = N$. Q.E.D.

**Proof of Proposition 11.** As $A_n$, we take $\{u | |f(u)| > n\}$. Then, $A_n \supseteq A_{n+1}$, $\cap A_n = \emptyset$, and hence, there exists an $N$ for which $A_N \cap T = \emptyset$. This means that if $u \in T$, then $|f(u)| \leq N$. Q.E.D.

**Proof of Proposition 12.** Take $A_n = \{u | f(u) \neq 0 \& |f(u)| \leq 2^{-n}\}$. Then, $A_n \supseteq A_{n+1}$, $\cap A_n = \emptyset$, and so, there exists an $N$ for which $A_N \cap T = \emptyset$. So, we can take $\varepsilon = 2^{-N}$. Q.E.D.

**Proof of Proposition 13.** Let $s_1, s_2 \in T$ and $s_1 \neq s_2$. Then, $d(s_1, s_2) > 0$. Since $X$ is definably separable, there exist a definable everywhere dense sequence
Proof of Proposition 14. Since $G$ is non-trivial, there exist an element $x \in T$ and $g \in G$ for which $a_g(x) \neq x$. For this $x \in T$, the orbit $Gx = \{a_g(x) \mid g \in G\}$ is a continuous image of the connected set $G$ and is, therefore, connected. It contains more than 2 points. Since $T$ is completeley disconnected, $T$ cannot contain a connected subset different from a single point. Hence, $Gx \not\subseteq T$. This means that there exists a $g$ for which $a_g(x) \not\in T$. So, $x \in T$, $a_g(x) \not\in T$, hence, $a_g(T) \neq T$. Q.E.D.

Proof of Lemma 2. Indeed, let $A_n$ be a sequence of subsets of $Y$ for which $A_n \supseteq A_{n+1}$, and $\cap A_n = \emptyset$. Then, for $B_n = f^{-1}(A_n)$, we have $B_n \supseteq B_{n+1}$.

If $x \in \cap B_n$, then $f(x) \in A_n$ for all $n$, so, $f(x) \in \cap A_n$, which contradicts to our assumption that $\cap A_n = \emptyset$. Hence, $\cap B_n = \emptyset$. Since $T$ is a set of typical elements, we can conclude that there exists an $N$ for which $B_N \cap T = f^{-1}(A_N) \cap T = \emptyset$.

Let us show that $A_N \cap f(T) = \emptyset$. Indeed, suppose that there exists an element $y \in A_N \cap f(T)$. Since $y \in f(T)$, there exists an $x \in T$ for which $y = f(x)$. This $x$ belongs both to $T$ and to $f^{-1}(A_N) = B_N$, which contradicts to our choice of $N$. So, such an element $y$ is impossible. Hence, $A_N \cap f(T) = \emptyset$. Q.E.D.

Proof of Proposition 16. We will prove this Proposition by reduction to a contradiction. Assume that $T = T_1 \times T_2$ is a set of typical elements of $X \times X$ that is invariant w.r.t. permutations. Since $T$ is invariant w.r.t. permutations, we have $T_1 = T_2$.

If we take $A_n = \{(x_1, x_2) \mid |d(x_1, x_2)| < 2^{-n} \& x_1 \neq x_2\}$, then we can conclude that there exists an $N$ for which $A_N \cap T = \emptyset$. This means that if $(x_1, x_2) \in T$ and $x_1 \neq x_2$, then $d(x_1, x_2) \geq 2^{-n}$. Since $T = T_1 \times T_2$ and $T_1 = T_2$, we can reformulate this condition as follows: if $x_1, x_2 \in T_1$, and $x_1 \neq x_2$, then $d(x_1, x_2) \geq 2^{-n}$. So, every two elements from $T_1$ are $\geq 2^{-n}$-different from each other.

The set $T_1$ is a projection of $T$ on $X$: $T_1 = \pi_1(T)$, where $\pi_1 : X \times X \to X$ is a definable mapping (defined as $(x_1, x_2) \to x_1$). So, due to Lemma 2, $T_1$ is a set of typical elements of $X$.

Due to Lemma 1, this set $T_1$ is pre-compact. In a compact set, there can be at most finitely many elements that are $2^{-n}$-different from each other. So, $T_1$
is finite. Hence, $T = T_1 \times T_1$ is also finite, and this contradicts to our assumption that $T$ is infinite. Q.E.D.

Proof of Lemma 3: this statement immediately follows from Definition 3.

Proof of Proposition 18. We will prove this proposition by reduction to a contradiction. Assume that $T_1 \times T_2$ is a set of typical elements of $X \times X$. Similarly to Proposition 16, we can prove that there exists a $\varepsilon > 0$ such that for all typical pairs $(x_1, x_2) \in T$, either $x_1 = x_2$, or $d(x_1, x_2) \geq \varepsilon$.

Due to Lemma 3, the intersection $T_1 \cap T_2 \subseteq T_1$ is a set of all typical elements of $X$. Similarly to the proof of Proposition 16, we can now conclude that this intersection is finite. Since $p$ is a non-atomic measure, we have $p(T_1 \cap T_2) = 0$. Hence, $p(T_1 \cup T_2) = p(T_1) + p(T_2) - p(T_1 \cap T_2) = p(T_1) + p(T_2)$. But $p(T_1) > 1/2$, so, $p(T_1 \cup T_2) > 1$, which contradicts to the fact that $p$ is a probability measure (and so, $p(T_1 \cup T_2) \leq p(X) = 1$). The contradiction proves that our assumption is wrong, and $T$ is not the set of typical elements. Q.E.D.

Proof of Proposition 17. Assume that $T$ is factorizable, i.e., $T = T_1 \times T_2$. Then, due to Lemma 2, each of the sets $T_i$ is a set of typical elements. Since $T_1 \times T_2$ is $(p \times p)$-measurable, both its projections $T_i$ must be $p$-measurable sets, and $(p \times p)(T_1 \times T_2) = p(T_1) \cdot p(T_2)$. Since $p(T_i) \leq 1$, we have $p(T_i) \geq (p \times p)(T_1 \times T_2) > 1/2$. So, both sets $T_i$ are majority sets. The result follows from Proposition 18. Q.E.D.

Proof of Proposition 19. This proof is similar to the proof of Proposition 17. Indeed, suppose that $T_1 \times \ldots \times T_s$ is a set of typical elements for $X \times \ldots \times X$. Then, similarly to that proof, we conclude that the intersection of each pair $T_i$ and $T_j$ is finite and therefore, $p(T_i \cap T_j) = 0$. Hence, $p(T_1 \cup \ldots \cup T_s) = p(T_1) + \ldots + p(T_s) > (1/s) + \ldots + (1/s) = 1$, which contradicts to the assumption that $p$ is a probability measure. Q.E.D.

Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209 and grant NCC 2-1232, by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-00-1-0365, by the Grant No. W-00016 from the U.S.-Czech Science and Technology Joint Fund, and by Grant NSF 9710940 Mexico/Conacyt.

References

[1] S. Barker and P. Achinstein, “On the new riddle of induction”, Philosophical Review, 1960, Vol. 69.
[2] J. M. Barone, *Fuzzy least squares and fuzzy entropy*. Proceedings of the 1992 International Fuzzy Systems and Intelligent Control Conference, Louisville, KY, 1992, pp. 170–181.

[3] J. Barzdin, A. Brazma, and E. Kinber, “Models of inductive syntactical synthesis”, In: J. E. Hayes, D. Michie, and E. Tyugu (eds.), *Machine Intelligence 12*, Claredon Press, Oxford, 1991, pp. 139–148.

[4] V. B. Berestetsky and E. M. Lifshits, *Relativistic quantum theory*, Pergamon Press, Oxford, N.Y., 1974.

[5] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[6] E. Bishop, D. S. Bridges, *Constructive Analysis*, Springer, N.Y., 1985.

[7] S. Blackburn, *Reason and prediction*, Cambridge University Press, Cambridge, MA, 1973, pp. 61–96.

[8] L. Boltzmann, “Bemrkungen über einige Probleme der mechanischen Wärmtetheorie”, *Wiener Ber. II*, 1877, Vol. 75, pp. 62–100.

[9] D. S. Bridges, *Constructive Functional Analysis*, Pitman, London, 1979.

[10] L. Brink and M. Henneaux, *Principles of String Theory* (Plenum Press, N.Y., 1988).

[11] C. D. Broad, *Ethics and th history of philosophy*, Routledge and Kegan Paul, London, 1952.

[12] V. A. Brumberg, *Essential relativistic celestial mechanics*, Adam Higler, Bristol, England, and Philadelphia, PA, 1991.

[13] G. A. Chebotarev, *Analytical and numerical methods of celestial mechanics*, Elsevier, N.Y., 1967.

[14] L. J. Cohen, *An introduction to the philosophy of induction and probability*, Claredon Press, Oxford, 1989.

[15] A. Córdoba, L. A. Seco, *Regular Weyl sums and atomic energy oscillations*, Revista Matematica Iberoamericana, 1994 (to appear).

[16] T. H. Cormen, C. E. Leiserson, R. L. Rivest, *Introduction to Algorithms*, MIT Press, 1990.

[17] I. Craig and J. Brown. *Inverse problems in astronomy*. Adam Hilget Ltd., Bristol, 1986.

[18] B. S. Dobronets, “Difference correction by finite elements of higher order”, In: *Mathematical models and methods of solving the problems of continuous media*, Krasnoyarsk, 1986, pp. 80–88 (in Russian).
[19] B. S. Dobronets, V. V. Shaydurov, *Two-sided Numerical Methods*, Nauka Publ. (Siberian Branch), Novosibirsk, 1990 (in Russian).

[20] B. S. Dobronets, “A posteriori error estimation and corrected solution of partial differential equation”, *Reliable Computing*, 1995, Supplement (Extended Abstracts of APIC’95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995), pp. 70–72.

[21] D. W. Etherington, S. Kraus, and D. Perlis, “Nonmonotonicity and the scope of reasoning”, *Proceedings of the Eighth National Conference on Artificial Intelligence AAAI-90*, MIT Press, Vol. 2, pp. 600–607.

[22] H. Fain, “The very thought of grue”, *Philosophical Review*, 1967, Vol. 76, pp. 61–73.

[23] C. Fefferman, R. de la Llave, *Relativistic stability of matter – I*, Revista Matemática Iberoamericana, 1986, Vol. 2, No. 1/2, pp. 119–213.

[24] R. P. Feynman, *Statistical Mechanics*, Reading, MA, 1972.

[25] R. P. Feynman, Leighton, and Sands, *The Feynman Lectures*, Addison-Wesley, 1965.

[26] A. M. Finkelstein and V. Kreinovich. “Derivation of Einstein’s, Brans-Dicke and other equations from group considerations,” *On Relativity Theory. Proceedings of the Sir Arthur Eddington Centenary Symposium, Nagpur India 1984*, Vol. 2, Choque-Bruhat, Y.; Karade, T. M. (eds), World Scientific, Singapore, 1985, pp. 138–146.

[27] A. M. Finkelstein and V. Kreinovich. “Impossibility of hardly possible events: physical consequences,” *Abstracts of the 8th International Congress on Logic, Methodology and Philosophy of Science*, Moscow, 1987, Vol. 5, Pt. 2, pp. 23–25.

[28] A. M. Finkelstein, V. Kreinovich, and R. R. Zapatrin. “Fundamental physical equations uniquely determined by their symmetry groups,” *Lecture Notes in Mathematics*, Springer-Verlag, Berlin-Heidelberg-N.Y., Vol. 1214, 1986, pp. 159–170.

[29] R. A. Fisher, *The design of experiments*, Oliver and Boyd, Edinburgh, 1947.

[30] V. B. Glasko. *Inverse problems of mathematical physics*. American Institute of Physics, N. Y., 1984.

[31] N. Goodman, “A query on confirmation”, *Journal of Philosophy*, 1946, Vol. 43.

[32] N. Goodman, *Fact, fiction, and forecast*, Harvard University Press, Cambridge, MA, 1955.
[33] D. Griffiths, *Introduction to Elementary Particles*, (Harper & Row, N.Y., 1987).

[34] M. Hesse, “Ramifications of ‘Grue’”, *British Journal for the Philosophy of Science*, 1969, Vol. 20, pp. 13–25.

[35] C. Howson and P. Urbach, *Scientific reasoning: the Bayesian approach*, Open Court, La Salle, IL, 1989.

[36] *Inverse problems*. SIAM–AMS Proceedings, Vol. 14. American Mathematical Society, Providence, RI, 1983.

[37] *Inverse problems*. Birkhauser Verlag, Basel, 1986.

[38] *Inverse problems*. Lecture Notes in Mathematics, Vol. 1225, Springer-Verlag, Berlin–Heidelberg, 1986.

[39] A. B. Korlyukov and V. Kreinovich, “Equations of physics become consistent take measurement uncertainty into consideration”, *Reliable Computing*, 1995, Supplement (Extended Abstracts of APIC’95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995), pp. 130–133.

[40] O. M. Kosheleva and V. Kreinovich. “On the Algorithmic Problems of a Measurement Process,” *Research Reports in Philosophy of Physics, University of Toronto, Ontario, Canada, Department of Philosophy*, No. 5, 1978, 63 pp.

[41] A. N. Kolmogorov, “Automata and life”, In: *Cybernetics expected and Cybernetics unexpected*, Nauka publ., Moscow, 1968, p. 24 (in Russian).

[42] V. Kozlenko, V. Kreinovich, and G. N. Solopchenko. “A method for solving ill-defined problems,” *Leningrad Center of Scientific and Technical Information*, Leningrad, Technical Report No. 1067, 1984, 2 pp. (in Russian).

[43] V. Kreinovich. “Derivation of the Schroedinger equations from scale invariance,” *Teoreticheskaya i Matematicheskaya Fizika*, 1976, Vol. 26, No. 3, pp. 414–418 (in Russian); English translation: *Theoretical and Mathematical Physics*, 1976, Vol. 8, No. 3, pp. 282–285.

[44] V. Kreinovich, Ching-Chuang Chang, L. Reznik, G. N. Solopchenko. “Inverse problems: fuzzy representation of uncertainty generates a regularization”, *Proceedings of NAFIPS’92: North American Fuzzy Information Processing Society Conference, Puerto Vallarta, Mexico, December 15–17, 1992*, NASA Johnson Space Center, Houston, TX, 1992, Vol. II, pp. 418–426.
[45] V. Kreinovich, C. Quintana, R. Lea, O. Fuentes, A. Lokshin, S. Kumar, I. Boricheva, and L. Reznik. “What non-linearity to choose? Mathematical foundations of fuzzy control,” Proceedings of the 1992 International Conference on Fuzzy Systems and Intelligent Control, Louisville, KY, 1992, pp. 349–412.

[46] V. Kreinovich, C. Quintana, L. Reznik. “Gaussian membership functions are most adequate in representing uncertainty in measurements”. Proceedings of NAFIPS’92: North American Fuzzy Information Processing Society Conference, Puerto Vallarta, Mexico, December 15–17, 1992, NASA Johnson Space Center, Houston, TX, 1992, Vol. II, pp. 618–624.

[47] V. Kreinovich and L. K. Reznik. Methods and models of formalizing prior information (on the example of processing measurements results). In: Analysis and formalization of computer experiments, Proceedings of Mendeleev Metrology Institute, Leningrad, 1986, pp. 37-41 (in Russian).

[48] R. Kruse and K. D. Meyer. Statistics with vague data. D. Reidel, Dordrecht, 1987.

[49] Th. S. Kuhn, Black-body theory and the quantum discontinuity, Oxford University Press, N.Y., 1978.

[50] H. E. Kyburg, Jr., Probability and the logic of rational belief, Wesleyan University Press, Middletown, 1961, p. 197.

[51] H. E. Kyburg, Jr., Philosophy of Science: A Formal Approach, MacMillan Co., N.Y., 1968.

[52] L. D. Landau and E. M. Lifshits, The classical theory of fields, Addison Wesley, Cambridge, MA, 1951.

[53] L. D. Landau and E. M. Lifshits, Fluid Mechanics, Pergamon Press, London; Addison Wesley, Reading, MA, 1959.

[54] L. D. Landau and E. M. Lifshits, Theory of Elasticity, Pergamon Press, London; Addison Wesley, Reading, MA, 1959.

[55] L. D. Landau and E. M. Lifshits, Electrodynamics of continuous media, Pergamon Press, N.Y., 1960.

[56] L. D. Landau and E. M. Lifshits, Quantum mechanics: non-relativistic theory, Pergamon Press, Oxford, N.Y., 1965.

[57] L. D. Landau and E. M. Lifshits, Mechanics, Pergamon Press, Oxford, N.Y., 1969.
[58] M. M. Lavrentiev, V. G. Romanov, and S. P. Shishatskii. *Ill-posed problems of mathematical physics and analysis*. American Mathematical Society, Providence, RI, 1986.

[59] H. R. Lewis, C. H. Papadimitriou, *Elements of the Theory of Computations*, Prentice-Hall, Englewood Cliffs, NJ, 1981.

[60] M. Li and P. M. B. Vitanyi, *An Introduction to Kolmogorov Complexity and its Applications*, Springer-Verlag, N.Y., 1997.

[61] R. de la Llave, D. Rana, *Accurate strategies for small divisor problems*. In: M. Mebkhout, R. Seneor (eds.), *Lecture given at the VIII International Congress on Mathematical Physics*, World Scientific, Singapore, 1987.

[62] R. de la Llave, D. Rana, *Accurate strategies for small divisor problems*, Bulletin of the American Mathematical Society, 1990, Vol. 22, No. 1, pp. 85-90.

[63] R. de la Llave, D. Rana, *Accurate strategies for small divisor problems*, Preprint, 1990.

[64] R. de la Llave, D. Rana, *Computer assisted proofs of stability of matter*. In: K. Meyer, D. Schmidt (eds.), *Computer Aided Proofs in Analysis*, Springer-Verlag, N.Y., 1991, pp. 116–126.

[65] R. de la Llave, D. Rana, *Accurate strategies for K.A.M. bounds and their implementation*. In: K. Meyer, D. Schmidt (eds.), *Computer Aided Proofs in Analysis*, Springer-Verlag, N.Y., 1991, pp. 127–146.

[66] J. C. Martin. *Introduction to Languages and the Theory of Computation*, McGraw-Hill, N.Y., 1991.

[67] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco, 1973.

[68] J. Moser, *Is the solar system stable?*, Mathematical Intelligencer, 1978, Vol. 1, pp. 65–71.

[69] O. Nachtmann, *Elementary Particle Physics: Concepts and Phenomena* (Springer-Verlag, Berlin, 1990).

[70] M. Nakamura, R. Mines, and V. Kreinovich. “Guaranteed intervals for Kolmogorov’s theorem (and their possible relation to neural networks)”*, Interval Computations*, 1993, No. 3, pp. 183–199.

[71] Hung T. Nguyen and Vladik Kreinovich, “When is an algorithm feasible? Soft computing approach”, *IEEE-FUZZ’95*, Japan, March 1995 (to appear).
[72] R. Parikh. *Existence and feasibility in arithmetic*, Journal of Symbolic Logic, 1971, Vol. 36, pp. 494–508.

[73] Particle Data Group, Phys. Lett., Vol. 170B, 1 (1986).

[74] R. Penrose, *The Emperor’s new mind*, Oxford Univ. Press, Oxford, 1989.

[75] J. L. Pollock, Oxford University Press, N.Y., Oxford, 1990.

[76] G. Polya, *Patterns of plausible inference*, Princeton University Press, Princeton, NJ, 1968.

[77] D. Rana, *Proof of accurate upper and lower bounds to stability domains in small denomination problems*, Princeton University, Department of Mathematics, Thesis, 1987.

[78] W. Salmon, “On vindicating induction”, In: H. E. Kyburg and E. Nagel (eds.), *Induction: some current issues*, Wesleyan University Press, Middletown, 1963, pp. 27–41.

[79] V. Yu. Sazonov, “A logical approach to the problem ‘P=NP?’ ”, *Lecture Notes in Computer Science*, Vol. 88, Springer-Verlag, N.Y., 1980, pp. 562–575.

[80] L. A. Seco, *Lower bounds for the ground states energy of atoms*, Ph.D. Dissertation, Department of Mathematics, Princeton University, 1989.

[81] L. A. Seco, *Computer assisted lower bounds for atomic energies*, In: K. R. Meyer and D. D. Schmidt (eds.), *Computer Aided Proofs in Analysis*, Springer-Verlag, N.Y., 1991, pp. 241–251.

[82] H. Smokler, “Goodman’s paradox and the problem of rules of acceptance”, *American Philosophical Quarterly*, 1966, Vol. 3.

[83] L. G. Taff, *Celestial mechanics: a computational guide for the practitioner*, Wiley, N.Y., 1985.

[84] A. N. Tikhonov, V. Y. Arsenin. *Solutions of ill-posed problems*. V. H. Winston & Sons, Washington, DC, 1977.