ON STRUCTURE OF CLUSTER ALGEBRAS OF GEOMETRIC TYPE I:
IN VIEW OF SUB-SEEDS AND SEED HOMOMORPHISMS

MIN HUANG    FANG LI    YICHAO YANG

Abstract. Our motivation is to build a systematic method in order to investigate the structure of cluster algebras of geometric type. The method is given through the notion of mixing-type sub-seeds, the theory of seed homomorphisms and the viewpoint of gluing of seeds. As an application, for (rooted) cluster algebras, we completely classify rooted cluster subalgebras and characterize rooted cluster quotient algebras in detail. Also, we build the relationship between the categorification of a rooted cluster algebra and that of its rooted cluster subalgebras.

Note that cluster algebras of geometric type studied here are of the sign-skew-symmetric case.

Contents

1. Introduction and preliminaries 1
2. Seed homomorphisms and some elementary properties 5
3. Rooted cluster morphisms and the relationship with seed homomorphisms 10
4. Sub-rooted cluster algebras and rooted cluster subalgebras 16
   4.1. Sub-rooted cluster algebras and two special cases 16
   4.2. Rooted cluster subalgebras as sub-class of sub-rooted cluster algebras 18
5. On enumeration and monoidal categorification 21
   5.1. The number of rooted cluster subalgebras of the form $A(\Sigma_{I_0, I_1})$ 21
   5.2. Monoidal sub-categorification of a rooted cluster algebra 22
6. Rooted cluster quotient algebras 25
   6.1. Rooted cluster quotient algebras via pure sub-cluster algebras 25
   6.2. Rooted cluster quotient algebras via gluing method 29
References 40

1. Introduction and preliminaries

Cluster algebras are commutative algebras that were introduced by Fomin and Zelevinsky [9] in order to give a combinatorial characterization of total positivity and canonical bases in algebraic groups. The theory of cluster algebras is related to numerous other fields. Since its introduction, the study on cluster algebras mainly involves intersection with Lie theory, representation theory of algebras, its combinatorial method (e.g. the periodicity issue) and categorification and the sub-class constructed from Riemannian surfaces and its topological setting, including the Teichmüller theory.

The algebraic structure and properties of cluster algebras were originally studied in a series of articles [9] [10] [2] [11] involving bases and the positivity conjecture. The positive conjecture has been
proved by Lee and Schiffler in [20] in the skew-symmetric case and moreover, was claimed to be true by Kontsevich etc. in [13] in the skew-symmetrizable case.

In this work, we characterize cluster algebras through their internal structure. The categorical framework for cluster algebras is provided in [1]. More precisely, the so-called \textit{rooted cluster morphism} is introduced to characterize the relations among \textit{rooted cluster algebras} with “rooted” meaning fixed initial seeds. In particular, injective morphisms and surjective morphisms and isomorphisms are investigated for rooted cluster algebras in some special cases, including those from Riemanian surfaces in [1].

In [1], the structure of a cluster algebras is discussed through its rooted cluster subalgebras and quotient algebras. However, as shown in the sequel, we find that its structure is determined in general by its sub-seeds, correspondingly by the so-called \textit{sub-rooted cluster algebras}. Thus, our view is different from that in [1].

In this article, we mainly focus to study the structure of rooted cluster algebras, including all rooted cluster subalgebras and rooted cluster quotient algebras, via sub-seeds and seed homomorphisms. It partly was studied in [1][9] in some special cases. For this aim, we propose a systematic method to characterize rooted cluster subalgebras and rooted cluster quotient algebras. In addition to the methods of sub-seeds and seed homomorphisms, the method of gluing of seeds for rooted cluster quotient algebras is also an important topic in our discussion. As an incidental result, the partial answer of one problem on the rooted cluster morphism $\sigma_{x,1}$ in [1] is given in our way.

As a new idea in this article, we introduce the so-called \textit{(mixing-type) sub-seeds} and \textit{seed homomorphisms}.

The concept of mixing-type subseeds is basic for us to discuss the structure of cluster algebras. Our motivation is to unify two extremes: freeze exchange variables or delete exchange/frozen variables, into a concept. It supports us a possibility to characterize the structure of a rooted cluster algebra. In fact, we have proved in (page 18, Thm 4.4) that any rooted cluster algebras can be expressed as such form.

In [1] and [21], seed (anti-)isomorphisms and $\sigma$-similarity of seeds are defined, that are indeed consistent with isomorphisms of seeds, which are special cases of seed homomorphisms. We set up the corresponding structure of a sub-seed in a cluster algebra, which is called a \textit{sub-rooted cluster algebra}. The interesting fact is that seed homomorphisms are compatible with graph homomorphisms in graph theory, which gives a possibility to establish a connection between the cluster algebras theory and the graph theory, because a seed can be presented as a cluster quiver when the exchange matrix is skew-symmetric.

Our original motivation for introducing the concept of seed homomorphisms is to understand the structure of totally sign-skew-symmetric cluster algebras. In the sign-skew-symmetric case, many problems, e.g. positivity conjecture and $F$-polynomials, will become very difficult. So, such research in our paper is necessary. This new concept will be used in our further work [17] on the positivity and $F$-polynomials for sign-skew-symmetric cluster algebras, building on [20] and this present work.

As a preliminary application of our conclusions, we give a relation between the finite type/finite mutation type of a rooted cluster algebra and that of its sub-rooted cluster algebras and establish a connection between rooted cluster sub-algebras and their monoidal categorification.

In summary, we list our main contributions in this article as follows:

- Build the theory of seed homomorphisms (Definition 2.1) for its importance as a tool in this work.
- Introduce the notion of mixing-type sub-rooted cluster algebras via mixing-type sub-seeds and using it as the main tool, for a given rooted cluster algebra, we give the characterization of rooted
cluster subalgebras (Theorem 4.4) and that of pure sub-cluster algebras as a class of cluster quotient algebras in the acyclic case obtained via specialisation (Theorem 6.7).

- The method of gluing frozen variables is used effectively to characterize general surjective rooted cluster morphisms. Concretely, any surjective rooted cluster morphism determines uniquely a non-contractible surjective rooted cluster morphism, which can be written as a composition of a rooted cluster isomorphism and a series of surjective canonical morphisms via gluing pairs of frozen variables step-by-step (Proposition 6.9, Theorem 6.21).

- As another application of the method of mixing-type sub-seeds, we build the relationship between the categorification of a rooted cluster algebra and that of its rooted cluster subalgebra (Theorem 5.7).

The organization of this article still contains the following further contents.

In the next part of this section, we explain the notions and notations about cluster algebras. It is proved in Section 3 that a rooted cluster isomorphism is equivalent to an initial seed isomorphism (Proposition 3.8).

In Section 4, the characterization of rooted cluster subalgebras (Theorem 4.4) means that all rooted cluster subalgebras of a rooted cluster algebra is a subclass of its mixing-type sub-rooted cluster algebras. It is worth to mention that this result has been independently found in [6] in the other form of characterization. In this section, we also show that a pure cluster subalgebra is always a rooted cluster subalgebra (Proposition 4.3) and then give the characterization of proper rooted cluster subalgebras (Corollary 4.8).

As an application of Section 4, in Section 5, we calculate the number of non-trivial proper rooted cluster subalgebras in a rooted cluster algebra.

In Section 6, as a corollary of Theorem 6.7, it is proved that a sub-rooted cluster algebra of a rooted cluster algebra of finite type (respectively, finite mutation type) is also of finite type (respectively, finite mutation type) (Corollary 6.8).

The successive works [16] and [17] are based on this paper.

In this paper, we always consider totally sign-skew-symmetric cluster algebras of geometric type introduced in [9][10], as mentioned as follows.

The original definition of cluster algebra given in [11] is in terms of exchange pattern. We recall the equivalent definition in terms of seed mutation in [10]; for more details, refer to [13][9][10].

An $n \times n$ integer matrix $A = (a_{ij})$ is called sign-skew-symmetric if either $a_{ij} = a_{ji} = 0$ or $a_{ij}a_{ji} < 0$ for any $1 \leq i, j \leq n$.

An $n \times n$ integer matrix $A = (a_{ij})$ is called skew-symmetric if $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq n$.

An $n \times n$ integer matrix $A = (a_{ij})$ is called $D$-skew-symmetrizable if $d_{i}a_{ij} = -d_{j}a_{ji}$ for all $1 \leq i, j \leq n$, where $D=\text{diag}(d_{i})$ is a diagonal matrix with all $d_{i} \in \mathbb{Z}_{\geq 1}$.

Let $\tilde{A}$ be an $n \times (n+m)$ integer matrix whose principal part, denoted as $A$, is the $n \times n$ submatrix formed by the first $n$-rows and the first $n$-columns. The entries of $\tilde{A}$ are written by $a_{xy}$, $x \in X$ and $y \in \tilde{X}$. We say $\tilde{A}$ to be sign-skew-symmetric (respectively, skew-symmetric, $D$-skew-symmetrizable) whenever $A$ possesses this property.

For two $n \times (n+m)$ integer matrices $A = (a_{ij})$ and $A' = (a'_{ij})$, we say that $A'$ is obtained from $A$ by a matrix mutation $\mu_{i}$ in direction $i, 1 \leq i \leq n$, represented as $A' = \mu_{i}(A)$, or say that $A$ and
$A'$ are **mutation equivalent**, represented as $A \simeq A'$, if

$$a_{jk}' = \begin{cases} \ -a_{jk}, & \text{if } j = i \text{ or } k = i; \\ a_{jk} + \frac{|a_{ji}|a_{ik} + |a_{ij}|a_{jk}|}{\mu_i - \mu_j}, & \text{otherwise.} \end{cases}$$

(1)

It is easy to verify that $\mu_i(\mu_j(A)) = A$. The skew-symmetric/symmetrizable property of matrices is invariant under mutations. However, the sign-skew-symmetric property is not so. For this reason, a sign-skew-symmetric matrix $A$ is called **totally sign-skew-symmetric** if any matrix, that is mutation equivalent to $A$, is sign-skew-symmetric.

Give a field $\mathbb{F}$ as an extension of the rational number field $\mathbb{Q}$, assume that $u_1, \ldots, u_n, x_{n+1}, \ldots, x_{n+m} \in \mathbb{F}$ are $n + m$ algebraically independent over $\mathbb{Q}$ for a positive integer $n$ and a non-negative integer $m$ such that $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n, x_{n+1}, \ldots, x_{n+m})$, the field of rational functions in the set $\tilde{X} = \{u_1, \ldots, u_n, x_{n+1}, \ldots, x_{n+m}\}$ with coefficients in $\mathbb{Q}$.

A seed in $\mathbb{F}$ is a triple $\Sigma = (X, X_{fr}, \tilde{B})$, where

(a) $X = \{x_1, \ldots, x_n\}$ is a transcendence basis of $\mathbb{F}$ over the fraction field of $\mathbb{Z}[x_{n+1}, \ldots, x_{n+m}]$, which is called a **cluster**, whose each $x \in X$ is called a **cluster variable** (see [10]);

(b) $X_{fr} = \{x_{n+1}, \ldots, x_{n+m}\}$ are called the **frozen cluster** or, say, the **frozen part** of $\Sigma$ in $\mathbb{F}$, where all $x \in X_{fr}$ are called **stable (cluster) variables** or **frozen (cluster) variables**;

(c) $\tilde{X} = X \cup X_{fr}$ is called a **extended cluster**;

(d) $\tilde{B} = (b_{xy})_{x,y \in \tilde{X}} = (B, B_1)$ is a $n \times (n + m)$ matrix over $\mathbb{Z}$ with rows and columns indexed by $X$ and $\tilde{X}$, which is totally sign-skew-symmetric. The $n \times n$ matrix $B$ is called the **exchange matrix** and $\tilde{B}$ the **extended exchange matrix** corresponding to the seed $\Sigma$.

In a seed $\Sigma = (X, X_{fr}, \tilde{B})$, if $X = \emptyset$, that is, $\tilde{X} = X_{fr}$, we call the seed a **trivial seed**.

Given a seed $\Sigma = (X, X_{fr}, \tilde{B})$ and $x, y \in \tilde{X}$, we say $(x, y)$ is a **connected pair** if $b_{xy} \neq 0$ or $b_{yx} \neq 0$ with $(x, y) \cap X \neq \emptyset$. A seed $\Sigma$ is defined to be **connected** if for any $x, y \in \tilde{X}$, there exists a sequence of variables $(x = z_0, z_1, \ldots, z_s = y) \subseteq \tilde{X}$ such that $(z_i, z_{i+1})$ are connected pairs for all $0 \leq i \leq s - 1$.

Let $\Sigma = (X, X_{fr}, \tilde{B})$ be a seed in $\mathbb{F}$ with $x \in X$, the **mutation** $\mu_x$ of $\Sigma$ at $x$ is defined satisfying $\mu_x(\Sigma) = (\mu_x(X), X_{fr}, \mu_x(B))$ such that

(a) The **adjacent cluster** $\mu_x(X) = \{\mu_x(y) \mid y \in X\}$, where $\mu_x(y)$ is given by the exchange relation

$$\mu_x(y) = \begin{cases} \prod_{x \in \tilde{X}, b_{yx} > 0} t^{b_{yx}} + \prod_{x \in \tilde{X}, b_{yx} < 0} t^{-b_{yx}} \end{cases}, \quad \text{if } y = x; \quad \text{if } y \neq x.$$  

This new variable $\mu_x(x)$ is also called a **new cluster variable**.

(b) $\mu_x(B)$ is obtained from $B$ by applying the matrix mutation in direction $x$ and then relabeling one row and one column by replacing $x$ with $\mu_x(x)$.

It is easy to see that the mutation $\mu_x$ is an involution, i.e., $\mu_x(\mu_x(\Sigma)) = \Sigma$.

Two seeds $\Sigma'$ and $\Sigma''$ in $\mathbb{F}$ are called **mutation equivalent** if there exists a sequence of mutations $\mu_{y_1}, \ldots, \mu_{y_s}$, such that $\Sigma'' = \mu_{y_s} \cdots \mu_{y_1}(\Sigma')$. Trivially, the mutation equivalence gives an equivalence relation on the set of seeds in $\mathbb{F}$.

Let $\Sigma$ be a seed in $\mathbb{F}$. Denote by $S$ the set of all seeds mutation equivalent to $\Sigma$. In particular, $\Sigma \in S$. For any $\tilde{\Sigma} \in S$, we have $\tilde{\Sigma} = (\tilde{X}, X_{fr}, \tilde{B})$. Denote $\mathcal{X} = \cup_{\tilde{\Sigma} \in S} \tilde{X}$.

**Definition 1.1.** Let $\Sigma$ be a seed in $\mathbb{F}$. The **cluster algebra** $\mathcal{A} = \mathcal{A}(\Sigma)$, associated with $\Sigma$, is defined to be the $\mathbb{Z}[x_{n+1}, \ldots, x_{n+m}]$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$. $\Sigma$ is called the **initial seed** of $\mathcal{A}$. 

This notion of cluster algebra was given in [9][10], where it is called the cluster algebra of geometric type as a special case of general cluster algebras.

Such kind of cluster algebras is considered as the most important one, with respect to additive categorification, etc., in the theory of cluster algebras based on the views of references [10][3][13][12]. It accords with many important examples, such as those given in [9][13][18][25], including those constructed from the coordinate rings of many varieties, e.g., from Grassmannians [13] and algebraic groups [9]. They supply with a close connection between the theory of cluster algebras and representation theory, see [3][18][12], etc.

2. Seed homomorphisms and some elementary properties

By definition, cluster algebras are determined by original seeds and their mutations. We find in the next section that the relations between cluster algebras can be restricted to their seeds so as to obtain the relations between them. Motivated by this fact and our discussion in the sequel, we now introduce the so-called seed homomorphism.

For the initial seed \( \Sigma = (X, X_f, \tilde{B}) \) of a cluster algebra \( \mathcal{A} \) and two pairs \((x, y)\) and \((z, w)\) with \(x, z \in X\) and \(y, w \in \tilde{X}\), we say that \((x, y)\) and \((z, w)\) are adjacent pairs if \(b_{xz} \neq 0\) or \(x = z\).

**Definition 2.1.** Let \( \Sigma = (X, X_f, \tilde{B}) \) and \( \Sigma' = (X', X_f', \tilde{B}') \) be two seeds.

(i) A map \( f \) from \( \tilde{X} \) to \( \tilde{X}' \) is called a seed homomorphism from the seed \( \Sigma \) to the seed \( \Sigma' \) if

\[
(b_{f(x)y}b_{xy})(b_{f(z)w}b_{zw}) \geq 0 \quad \text{and} \quad |b'_{f(x)y}| \geq |b_{xy}|
\]

(ii) A seed homomorphism \( f : \Sigma \to \Sigma' \) is called a positive seed homomorphism if \( |b'_{f(x)y}|b_{xy} \geq 0 \) for all \( x \in X \) and \( y \in \tilde{X} \). In contrast, a seed homomorphism \( f \) is called a negative seed homomorphism if \( b'_{f(x)y}b_{xy} \leq 0 \) for all \( x \in X \) and \( y \in \tilde{X} \).

For seed homomorphisms \( f: \Sigma \to \Sigma' \) and \( g: \Sigma' \to \Sigma'' \), define their composition \( gf : \Sigma \to \Sigma'' \) satisfying that \( gf(x) = g(f(x)) \) for all \( x \in \tilde{X} \). Then we can define the seed category, denoted as \( \text{Seed} \), to be the category whose objects are all seeds and whose morphisms are all seed homomorphisms with composition defined as above.

A quiver \( \Gamma \) is called a cluster quiver if it is a finite quiver with no loops and no cycles of lengths 2 (see [8][21][22]). The meaning of this class of quivers follows the fact that cluster quivers can be corresponding one-to-one with the skew-symmetric integer square matrices, generating cluster algebras without frozen variables.

In fact, our idea of seed homomorphisms is original from homomorphisms of direct graphs.

As given in [15], recall that for two quivers (said as digraphs in graph theory) \( Q \) and \( P \) with the vertex sets \( Q_0 \) and \( P_0 \), a quiver homomorphism \( f \) from \( Q \) to \( P \), written as \( f : Q \to P \), is a mapping \( f : Q_0 \to P_0 \) such that there is an arrow from \( f(u) \) to \( f(v) \) in \( P \) whenever there is an arrow from \( u \) to \( v \) in \( Q \) for \( u, v \in Q_0 \).

Following this quiver homomorphism, a homomorphism \( f \) of quivers from \( Q \) to \( P \) is called a cluster quiver homomorphism if \( f(Q_0,ex) \subseteq P_0,ex \).

In the case for skew-symmetric seeds, we know the one-to-one correspondence between seeds and cluster quivers. For two connected cluster quivers \( Q \) and \( Q' \) and their seeds \( \Sigma = \Sigma(Q) \) and \( \Sigma' = \Sigma(Q') \), a positive (respectively, negative) seed homomorphism \( f_s : \Sigma \to \Sigma' \) corresponds to a cluster quiver homomorphism (respectively, anti-homomorphism) \( f_c : Q \to Q' \).

In fact, the condition (b) of the definition of seed homomorphism means that \( b'_{f_s(x)f_s(y)}b_{xy} \geq 0 \) for all \( x \in X \) and \( y \in \tilde{X} \) or \( b'_{f_s(x)f_s(y)}b_{xy} \leq 0 \) for all \( x \in X \) and \( y \in \tilde{X} \). Also, from (b), \( |b'_{f_s(x)f_s(y)}| \geq |b_{xy}| \).
which follows that we can define a cluster quiver homomorphism or anti-homomorphism \( f_c \) from the corresponding cluster quiver \( Q \) to the other one \( Q' \) with \( f_c(x) = f_s(x) \) for any vertex \( x \) in \( Q \) such that for any vertices \( x, y \) of \( Q \) if there is an arrow from \( x \) to \( y \), then there is an arrow from \( f_c(x) \) to \( f_c(y) \) for all \( x \in X \) and \( y \in \tilde{X} \) or from \( f_s(y) \) to \( f_c(x) \) for all \( x \in X \) and \( y \in \tilde{X} \). It implies that \( f_c \) is a cluster quiver homomorphism or anti-homomorphism.

According to the fact above, we can see seed homomorphisms as an improvement of quiver homomorphisms in the theory of graphs, in order to be useful for skew-symmetrizable seeds or more generally, totally sign-skew-symmetrizable seeds. In this view, the homomorphism method in graph theory will have a natural influence on our study in this article and more further work.

**Definition 2.2.** Let \( \Sigma = (X, X, f, B) \) be a seed with \( B \) an \( n \times (n + m) \) totally sign-skew-symmetric integer matrix. Assume \( I_0 \) is a subset of \( X \), \( I_1 \) a subset of \( \tilde{X} \) with \( I_0 \cap I_1 = \emptyset \) and \( \tilde{X} = I_1 \cup I'_1 \) for \( I'_1 = X \cap I_1 \) and \( I''_1 = X \cap \tilde{X} \). Denoting \( X' = X \setminus (I_0 \cup I'_1) \), \( X'' = \tilde{X} \setminus I_1 \) and \( B' \) as a \( \frac{1}{2}X' \times \frac{1}{2}X'' \)-matrix with \( b'_{xy} = b_{xy} \) for any \( x \in X' \) and \( y \in X'' \), one can define the new seed \( \Sigma' = (X', X', f', B') \), which is called a mixing-type sub-seed or, say, \((I_0, I_1)\)-type sub-seed, of the seed \( \Sigma = (X, X, f, B) \).

**Fact 2.3.** An \((I'_0, I'_1)\)-type sub-seed of an \((I_0, I_1)\)-type sub-seed of a seed \( \Sigma \) is a mixing-type sub-seed of \( \Sigma \). That is, for a seed \( \Sigma \), \((\Sigma_{I_0, I_1})_{I'_0, I'_1} = \Sigma_{I'_0 \cup (I_0 \setminus I'_1), I'_1 \cap I'_1} \).

Furthermore, we have \( X'_f = I_0 \cup (X_f \setminus I''_1) \).

To obtain an \((I_0, I_1)\)-sub-seed from the initial seed is equivalent to saying freeze the cluster variables in \( I_0 \) and delete the cluster variables in \( I_1 \).

First, we discuss two special cases of mixing-type sub-seeds of a seed \( \Sigma \).

**Case 1:** \( I_1 = \emptyset \). That is, we only freeze the variables in \( I_0 \) that are original mutable and do not delete any variables.

In this case, we have the sub-seed \( \Sigma_{I_0, \emptyset} = (X', X, f, I_0, \tilde{B}_0) \) with cluster \( X' = X \setminus I_0 \) consisting of mutable variables and extended cluster \( \tilde{X}' = \tilde{X} \). Since \( \tilde{B}_0 \) is sign-skew-symmetric, it follows that \( \tilde{B}_0 \) is also sign-skew-symmetric. The frozen variables of the sub-seed \( \Sigma_{I_0, \emptyset} \) form the set \( X'_f = \tilde{X} \setminus X' = (X \setminus X') \cap X_f = I_0 \cup X_f \. \)

We call this sub-seed \( \Sigma_{I_0, \emptyset} \) a pure sub-seed of the seed \( \Sigma = (X, X, f, \tilde{B}) \).

**Case 2:** \( I_0 = \emptyset \). That is, we only delete the variables in \( I_1 \) while the remaining variables remain unchanged and do not freeze any exchangeable variables.

In this case, we have the sub-seed \( \Sigma_{\emptyset, I_1} = (X''', X''', I_f, \tilde{B}_1) \) with \( X'' = X \setminus I_1 \), \( X'''_f = X_f \setminus I_1 \) and \( \tilde{X}'' = X'' \cap X'''_f \). \( \tilde{B}_1 \) is sign-skew-symmetric since \( \tilde{B} \) is so.

We call this sub-seed \( \Sigma_{\emptyset, I_1} \) a partial sub-seed of the seed \( \Sigma = (X, X, f, \tilde{B}) \).

For two seeds \( \Sigma_1 = (X_1, (X'_1)_f, \tilde{B}_1) \) and \( \Sigma_2 = (X_2, (X'_2)_f, \tilde{B}_2) \), if there exists (possibly empty) \( \Delta_1 \subseteq (X'_1)_f \) and \( \Delta_2 \subseteq (X'_2)_f \) such that \(|\Delta_1| = |\Delta_2|\), then \( \Sigma_1 \) and \( \Sigma_2 \) are said to be glueable along \( \Delta_1 \) and \( \Delta_2 \). Let \( \Delta \) be a family of undeterminates in bijection with \( \Delta_1 \) and \( \Delta_2 \).

Recall in [1] that the amalgamated sum of \( \Sigma_1 \) and \( \Sigma_2 \) along \( \Delta_1 \) and \( \Delta_2 \) is defined as \( \Sigma = (X, X_f, B) \), where \( X = (X_1 \setminus \Delta_1) \cup (X_2 \setminus \Delta_2) \cup \Delta \), \( X = X_1 \cup X_2 \) and the matrix \( B \) is defined as:

\[
(4) \quad \tilde{B} = \begin{pmatrix}
B_{11}^1 & B_{12}^1 & 0 & B_{13}^1 \\
0 & B_{22}^1 & 0 & B_{23}^1 \\
B_{31}^1 & 0 & B_{33}^1 & B_{33}^1 \\
0 & B_{22}^2 & 0 & B_{23}^2 \\
B_{31}^2 & 0 & B_{33}^2 & B_{33}^2 \\
0 & B_{22}^3 & 0 & B_{23}^3 \\
B_{31}^3 & 0 & B_{33}^3 & B_{33}^3
\end{pmatrix}
\]

We denote the amalgamated sum as the notations \( \Sigma = \Sigma_1 \cup \Delta_1, \Delta_2, \Sigma_2 \).

In particular, when \( \Delta_1 \) and \( \Delta_2 \) are empty sets, we call this amalgamated sum \( \Sigma = \Sigma_1 \cup \Delta_1, \Delta_2, \Sigma_2 \) the union seed of \( \Sigma_1 \) and \( \Sigma_2 \), denoted as \( \Sigma = \Sigma_1 \cup \Sigma_2 \).
For a given seed $\Sigma$, if $\Sigma_1$ and $\Sigma_2$ are partial sub-seeds of type $(\emptyset, I_1)$ of $\Sigma$ such that $X_1 \cap X_2 = \emptyset$, we replace $\Sigma_1 \cap (X_1)_{fr} \cap (X_2)_{fr} \cup \Sigma_2$ by the notation $\Sigma_1 \cap (X_1)_{fr} \cup \Sigma_2$. For subseeds $\Sigma_1, \Sigma_2$, and $\Sigma_3$ of type $(\emptyset, I_1)$ of $\Sigma$, we have $\Sigma_1 \cap (X_1)_{fr} \cup \Sigma_2 \cap (X_2)_{fr} \cup \Sigma_3 = \Sigma_1 \cap (X_1)_{fr} \cup \Sigma_2 \cap (X_2)_{fr} \cup \Sigma_3$, that is, the associative law holds for the amalgamated sum.

In fact, $(\Sigma_1 \cap (X_1)_{fr} \cup \Sigma_2 \cap (X_2)_{fr})$ has the set of exchangeable cluster variables $(X_1 \cup X_2) \cup X_3$ and the set of frozen cluster variables $((X_1)_{fr} \cup (X_2)_{fr})$ and $\Sigma_1 \cap (X_2)_{fr} \cup \Sigma_3$ has the set of exchangeable cluster variables $X_1 \cup (X_2 \cup X_3)$ and the set of frozen cluster variables $(X_1)_{fr} \cup ((X_2)_{fr} \cup (X_3)_{fr})$. Following the associative law of the sets, we get the associative law of the amalgamated sum, since subseeds are uniquely determined by their cluster variables.

Example 2.4. Let $Q : x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4$, $Q_1 : x_1 \rightarrow x_3 \rightarrow x_4$, and $Q_2 : x_2 \rightarrow x_3$ be quivers with $x_1, x_3, x_4$ exchangeable variables, $x_2$ frozen. Since $\{x_1\} \cap \{x_3\} = \emptyset$, we get $\Sigma(Q_1) \cap \Sigma(Q_2) = \Sigma(Q')$, where $Q' : x_1 \rightarrow x_2 \rightarrow x_3$.

Definition 2.5. A seed $\Sigma = (X, X_{fr}, B)$ is called indecomposable if it is connected and for any decomposition $\Sigma_1 \cap (X_1)_{fr} \cup \Sigma_2$, either $\Sigma_1$ or $\Sigma_2$ is a trivial seed, equivalently, either $\Sigma = \Sigma_2$ or $\Sigma = \Sigma_1$.

Note that if $X_{fr} = \emptyset$, then the meaning of indecomposable and connected coincide.

Example 2.6. Let $Q : x_1 \rightarrow x_2 \rightarrow x_3$, $Q_1 : x_1 \rightarrow x_2 \rightarrow x_3$, and $Q_2 : x_2 \rightarrow x_3$ be quivers with $x_1, x_3$ exchangeable variables, $x_2$ frozen. Then according to the definition, $\Sigma(Q)$ is connected, but it is not indecomposable, since $\Sigma(Q) = \Sigma(Q_1) \cap \Sigma(Q_2)$.

Remark 2.7. In case $\Sigma$ is skew-symmetrizable, the indecomposability of $\Sigma$ is defined in [1] in terms of valued ice quiver.

Due to the definitions of indecomposability and positive seed homomorphism, we have the following lemmas, which are easy to see.

Lemma 2.8. A seed $\Sigma = (X, X_{fr}, B)$ is indecomposable if and only if for any $x, y \in \tilde{X}$ with $x \neq y$, there exists a sequence of exchangeable cluster variables $(x_1, \cdots, x_s)$ in $X$ such that $b_{x_i, x_i} \neq 0$, $b_{x_i, x_{i+1}} \neq 0$, and $b_{x_i, x_j} = 0$ for $i = 1, \cdots, s$.

Proof. “Only if”: Otherwise, there exist $x, y \in \tilde{X}$, $x \neq y$, which do not satisfy the condition. Set $I_1 = \{x\} \cup \{z \in \tilde{X} : (x_1, \cdots, x_s) \subseteq X \text{ such that } b_{x_i, x_{i+1}} \neq 0, b_{x_i, x_j} \neq 0 \text{ and } b_{x_i, x_j} = 0 \text{ for } i = 1, \cdots, s\}$ and $I_1' = \tilde{X} \setminus I_1$. Then $x \in I_1$ and $y \in I_1'$, that is, $\Sigma_I, I_1, \Sigma_I, I_1'$ are non-trivial.

Since $\Sigma$ is connected, we have $\Sigma = \Sigma_{I, I_1} \cap \Sigma_{I, I_1'}$, which contradicts to the indecomposability of $\Sigma$.

“If”: Clearly, $\Sigma$ is connected. If $Q$ is decomposable, then we have $\Sigma = \Sigma_1 \cap \Sigma_2$, with non-trivial $\Sigma_1 = (X_1, (X_1)_{fr}, \tilde{B}^1)$ and $\Sigma_2 = (X_2, (X_2)_{fr}, \tilde{B}^2)$. Then we can find $x \in X_1, y \in X_2$ such that there exists a sequence of exchangeable cluster variables $(x_1, \cdots, x_s)$ in $X = X_1 \cup X_2$ satisfying that $b_{x_i, x_{i+1}} \neq 0$, $b_{x_i, x_{i+1}} \neq 0$, and $b_{x_i, x_j} = 0$ for $i = 1, \cdots, s$, which is impossible according to [1], the form of $\tilde{B}$.

Lemma 2.9. If a non-trivial seed $\Sigma = (X, X_{fr}, B)$ is indecomposable, then any seed homomorphism $f : \Sigma \rightarrow \Sigma'$ is either positive or negative.

Proof. Assume there exist $x \in X$ and $w \in \tilde{X}$ such that $b'_{f(z)f(w)} b_{zw} > 0$. For any $x \in X$ and $y \in \tilde{X}$ with $b_{xy} \neq 0$, by Lemma 2.8, there exists a sequence $(z = z_0, z_1, \cdots, z_{s-1}, z_s = x)$ in $X$ such that $b_{xz_{i+1}} = 0$ for $0 \leq k \leq s - 1$. Set $z_{s+1} = y$. Since $f$ is a seed homomorphism, we have $b'_{f(z)f(w)} b_{zw} b_{z_{s+1}} \geq 0$, and $|b'_{f(z)f(w)}| \geq |b_{zw}| > 0, |b'_{f(z)f(w)}| \geq |b_{z_{s+1}}| > 0$.
Thus, $b'_{f(z_0)f(z_1)}b_{z_0z_1} > 0$. Similarly, for $0 \leq i \leq s - 1$, we have
\[
(b'_{f(z_i)f(z_{i+1})}b_{z_{i}z_{i+1}})(b'_{f(z_{i+1})f(z_{i+2})}b_{z_{i+1}z_{i+2}}) \geq 0
\]
and
\[
|b'_{f(z_i)f(z_{i+1})}| \geq |b_{z_{i}z_{i+1}}| > 0, |b'_{f(z_{i+1})f(z_{i+2})}| \geq |b_{z_{i+1}z_{i+2}}| > 0.
\]
Therefore, using induction, we have
\[
b'_{f(z_i)f(z_{i+1})}b_{z_{i}z_{i+1}} > 0 \text{ for } 0 \leq i \leq s - 1.
\]
In particular, for $i = s$, it follows that for any $x \in X$ and $y \in \bar{X}$,
\[
b'_{f(z_s)f(y)}b_{xy} = b'_{f(z_s)f(z_{s+1})}b_{z_sz_{s+1}} > 0,
\]
which means that $f$ is a positive seed homomorphism.

Similarly, if there exist $z \in X$ and $w \in \bar{X}$ with $b'_{f(z)f(w)}b_{zw} < 0$, then $f$ is negative. \hfill \Box

**Lemma 2.10.** Let $\Sigma = (X, X_{fr}, \bar{B})$ be a connected seed of a cluster algebra $\mathcal{A}$, then there uniquely exist indecomposable subseeds $\Sigma_1 = (X_1, (X_1)_{fr}, \bar{B}_1), \ldots, \Sigma_t = (X_t, (X_t)_{fr}, \bar{B}_t)$ of type $(0, I_1)$ of $\Sigma$ for some integer $t$ such that
\begin{enumerate}
\item $(X_i \cap X_j) = \emptyset$ if $i \neq j$ and
\item $\Sigma = \bigcup_{i=1}^{t} \Sigma_i$ is the decomposition of the amalgamated sum.
\end{enumerate}

**Proof.** In the $n \times m$ totally sign-skew-symmetric matrix $\bar{B}$, by an appropriate permutation of the $n$ row indices and the first $n$ column indices of $\bar{B}$ simultaneously, the principal part $B$ of $\bar{B}$ can be decomposed into a block diagonal matrix $\text{diag}(B_1, B_2, \ldots, B_t)$. Then $\bar{B}$ can be written through an appropriate permutation of row indices and column indices as follows:
\[
\begin{pmatrix}
B_1 & 0 & \cdots & 0 & B'_1 \\
0 & B_2 & \cdots & 0 & B'_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_t & B'_t
\end{pmatrix},
\]
satisfying that (i) all $B_i$ are indecomposable (i.e. $B_i$ can not be decomposed as block diagonal matrices with smaller ranks via the above operation), (ii) all $B_i$ are totally sign-skew-symmetric.

For $1 \leq i \leq t$, let $X_i$ be the subset of the exchangeable variables of $X$ corresponding to the row indexes of $B_i$. By the uniqueness (up to the permutation of $\{B_i\}_i$) of the block diagonal decomposition of the principal part $B$, we have $X_i \cap X_j = \emptyset$ for $i \neq j$.

The set $(X_i)_{fr}$ of frozen variables adjacent to $X_i$ is just the subset of $X_{fr}$ corresponding to the column indexes of $B'_i$ which are non-zero. Then $X_{fr} = \bigcup_{i=1}^{t} (X_i)_{fr}$.

Let $I_i = \bar{X} \setminus (X_i \cup (X_i)_{fr})$ and $\Sigma_i = \Sigma_{\emptyset, I_i}$ for $1 \leq i \leq t$. By the definition of decomposition of amalgamated sum and comparing with the form of matrix in \ref{Lemma 2.10}, we have $\Sigma = \bigcup_{i=1}^{t} \Sigma_i$. \hfill \Box

**Definition 2.11.** Let $\Sigma$ and $\Sigma'$ be two seeds and $f : \Sigma \to \Sigma'$ be a seed homomorphism.
\begin{enumerate}
\item $f$ is called a seed isomorphism if $f$ induces bijections $X \to X'$ and $\bar{X} \to \bar{X}'$ and $|b_{xy}| = |b'_{f(x)f(y)}|$ for all $x \in X$ and $y \in \bar{X}$;
\item A seed isomorphism $f$ is called positive (respectively, negative) if $f$ is positive (respectively, negative) as a seed homomorphism.
\end{enumerate}

Trivially, we have the following lemmas by the definitions of (positive) seed homomorphisms and seed isomorphisms:

**Lemma 2.12.** A seed homomorphism $f : \Sigma \to \Sigma'$ is an isomorphism if and only if there exists a unique seed homomorphism $f^{-1} : \Sigma' \to \Sigma$ such that $f^{-1}f = \text{id}_\Sigma$ and $ff^{-1} = \text{id}_{\Sigma'}$. 
Lemma 2.13. A seed homomorphism \( f \) is a positive (respectively, negative) seed isomorphism if and only if \( f \) is isomorphic and \( b_{xy} = b'_{f(x)f(y)} \) (respectively, \( b_{xy} = -b'_{f(x)f(y)} \)) for all \( x \in X \) and \( y \in \bar{X} \).

Proof. “Only If”: The condition (b) in Definition 2.21 means that \( b'_{f(x)f(y)}b_{xy} \geq 0 \) for all \( x \in X \) and \( y \in \bar{X} \) or \( b'_{f(x)f(y)}b_{xy} \leq 0 \) for all \( x \in X \) and \( y \in \bar{X} \). It always holds that \( |b'_{f(x)f(y)}| \geq |b_{xy}| \). Since \( f \) is positive, we have \( b'_{f(x)f(y)}b_{xy} \leq 0 \). Then it follows \( b_{xy} = b'_{f(x)f(y)} \) for all \( x \in X \) and \( y \in \bar{X} \).

“If”: It follows immediately by Definition 2.11. \( \square \)

Remark 2.14. In \([\text{1}]\), two seeds \( \Sigma = (X, X_f, \bar{B}) \) and \( \Sigma' = (X', X'_f, \bar{B}') \) are called isomorphic (respectively, anti-isomorphic) if there is a bijection \( \varphi : \bar{X} \rightarrow \bar{X}' \), including a bijection \( \varphi : X \rightarrow X' \), such that \( b'_{\varphi(x)\varphi(y)} = b_{xy} \) (respectively, \( b'_{\varphi(x)\varphi(y)} = -b_{xy} \)) for \( x \in X \) and \( y \in \bar{X} \). Obviously, their isomorphism (respectively, anti-isomorphism) given in \([\text{1}]\) is just our positive (respectively, negative) seed isomorphism defined here, which are both only the special cases of seed isomorphisms given by us. For example, let \( Q : x_1 \rightarrow x_2 \rightarrow x_3 \) and \( Q' : x_1 \rightarrow x_2 \rightarrow x_3 \). Then we have a seed isomorphism \( f : \Sigma(Q) \rightarrow \Sigma(Q') \), i.e., \( \varphi(x_1) = x_2 \), \( \varphi(x_2) = x_3 \), which is neither positive nor negative.

Proposition 2.15. Applying the notations in Definition 2.21, for a seed \( \Sigma \) and its mixing-type subseeds \( \Sigma_i \) and \( \Sigma_j \), if \( \Sigma_i \cong \Sigma_j \) in Seed, then it holds that \#(\Sigma_i) = \#(\Sigma_j) = \#(\varnothing) = 1.

Proof. Since \( \Sigma_i \cong \Sigma_j \), we have \( \#(\Sigma_i) = \#(\varnothing) \). Obviously, their mixing-type subseeds \( \Sigma_i \) and \( \Sigma_j \) are called isomorphic (respectively, anti-isomorphic) if there is a bijection \( \varphi : \bar{X} \rightarrow \bar{X}' \), including a bijection \( \varphi : X \rightarrow X' \), such that \( b'_{\varphi(x)\varphi(y)} = b_{xy} \) (respectively, \( b'_{\varphi(x)\varphi(y)} = -b_{xy} \)) for \( x \in X \) and \( y \in \bar{X} \). As \( I_1, J_1 \subseteq \bar{X} \), \( I_0 \cup J_1 \subseteq X \) and \( J_0 \cup J_1 \subseteq X \), therefore, we get \#(\Sigma_i) = \#(\Sigma_j) = \#(\varnothing).

Corollary 2.16. Following Proposition 2.15, it holds that

(i) if \( \Sigma_i \cong \Sigma_j \) in Seed and \( I'_1 = J'_1 = \varnothing \), then \#(\Sigma_i) = \#(\Sigma_j) = 1.

(ii) if \( \Sigma_i \cong \Sigma_j \) in Seed, then \#(\Sigma_i) = \#(\Sigma_j).

(iii) if \( \Sigma_i \cong \Sigma_j \) in Seed, then \#(\Sigma_i) = 1.

Note that

(1) The converse of the above proposition is not true. For example, for a cluster quiver \( Q : x_1 \rightarrow x_2 \rightarrow x_3 \) with the exchangeable variables 1 and 3 and frozen variable \( x_2 \), it is clear that \( \Sigma(Q) \cong \Sigma(Q') \).

(2) In general, \#(\Sigma_i) = \#(\Sigma_j) = 1.

Proposition 2.17. Let \( \Sigma = (X, X_f, \bar{B}) \) be a seed and \( \Sigma_i \) be a mixing-type subseed of \( \Sigma \). Then there is a positive seed isomorphism \( \mu_{\Sigma}(\Sigma_i) \cong (\mu_{\Sigma}(\Sigma))_i \) for any \( x \in X \setminus (I_0 \cup I_1) \).

Proof. Denote \( \mu_{\Sigma}(\Sigma_i) \) by \( (X', X'_f, \bar{B}') \) and \( (\mu_{\Sigma}(\Sigma))_i \) by \( (X'', X''_f, \bar{B}'') \). By definition, we have

\[
X' = X \setminus (I_0 \cup I_1 \cup \{x\}) \cup \{\Sigma_i(x)\}, \quad \bar{X}' = \bar{X} \setminus (I_1 \cup \{x\}) \cup \{\Sigma_i(x)\},
\]

\[
X'' = X \setminus (I_0 \cup I_1 \cup \{x\}) \cup \{\mu_{\Sigma}(x)\}, \quad \bar{X}'' = \bar{X} \setminus (I_1 \cup \{x\}) \cup \{\mu_{\Sigma}(x)\},
\]

To compare the set \( \bar{X}' \) with \( \bar{X}'' \), their elements can be correspondent one-to-one with the identity map but the correspondence between \( \mu_{\Sigma}(\Sigma_i) \) and \( \mu_{\Sigma}(\Sigma) \).

According to the definition of mutation of matrices, for all \( y \in X' \) and \( z \in \bar{X}' \), we have

\[
b'_{yz} = \begin{cases} \frac{b_{yz} + |b_{xz} + b_{zx}|}{2}, & \text{if } y \neq \mu_{\Sigma}(\Sigma_i)(x), z \neq \mu_{\Sigma}(\Sigma_i)(x); \\ -b_{yz}, & \text{otherwise}, \end{cases}
\]
and for all \( y \in X'' \) and \( z \in \widetilde{X}'' \), we have
\[
b_{yz}'' = \begin{cases} 
    b_{yz} + \frac{|b_{xz}| + |b_{zy}|}{2}, & \text{if } y \neq \mu^\Sigma_x(x), z \neq \mu^\Sigma_z(x); \\
    -b_{yz}, & \text{otherwise.}
\end{cases}
\]

After comparing \( b_{yz}'' \) with \( b_{yz}'' \), it is obvious that \( \widetilde{B} = \widetilde{B}'' \). Therefore, the result holds.

**Definition 2.18.** Let \( f : \Sigma \rightarrow \Sigma' \) be a seed homomorphism. The **image seed** of \( \Sigma \) under \( f \) is defined to be \( f(\Sigma) = (f(X), f(\widetilde{X}) \setminus f(X), B'') \), where \( B'' \) is a \( \#(f(X)) \times \#(f(\widetilde{X})) \)-matrix with \( b''_{xy} = b'_{xy} \) for any \( x \in f(X) \) and \( y \in f(\widetilde{X}) \).

It is easy to see that for \( I'_1 = \widetilde{X}' \setminus (\widetilde{X}' \cap f(X)) \) and \( I'_0 = X' \setminus (f(X) \cup I'_1) \), we have
\[
f(\Sigma) = \Sigma'_{I'_0, I'_1};
\]

Using the image seed \( f(\Sigma) \) and by \([1]\), we can introduce the notions of injective/surjective seed homomorphisms as follows.

**Definition 2.19.**
(i) A seed homomorphism \( f : \Sigma \rightarrow \Sigma' \) is called **injective** if \( \Sigma \cong f(\Sigma) \) in \Seed.
(ii) A seed homomorphism \( f : \Sigma \rightarrow \Sigma' \) is called **surjective** if \( f(\Sigma) = \Sigma' \).

Following this definition and that of seed isomorphism, trivially, we have the following.

**Proposition 2.20.** A seed homomorphism \( f : \Sigma \rightarrow \Sigma' \) is isomorphic if and only if \( f \) is injective and surjective.

## 3. Rooted cluster morphisms and the relationship with seed homomorphisms

In \([1]\), a **rooted cluster algebra** is defined as a cluster algebra \( A(\Sigma) \) together with its initial seed \( \Sigma \), denoted by \( A(\Sigma) \). Moreover, given a rooted cluster algebra \( A(\Sigma) \), a sequence of cluster variables \( (y_1, y_2, \cdots, y_l) \) is called **\( \Sigma \)-admissible** if \( y_1 \) is exchangeable in \( \Sigma \) and \( y_i \) is exchangeable in \( \mu_{y_{i-1}} \cdots \mu_{y_1}(\Sigma) \) for every \( i \geq 2 \). Let \( A(\Sigma') \) be another rooted cluster algebra and \( f : F(\Sigma) \rightarrow F(\Sigma') \) a map as sets. A sequence of cluster variables \( \{y_1, y_2, \cdots, y_l\} \subseteq A(\Sigma) \) is called **(\( f, \Sigma, \Sigma' \))-biadmissible** if it is \( \Sigma \)-admissible and \( (f(y_1), f(y_2), \cdots, f(y_l)) \) is \( \Sigma' \)-admissible.

**Definition 3.1.** (Definition 2.2, \([1]\)) A **rooted cluster morphism** \( f \) from \( A(\Sigma) \) to \( A(\Sigma') \) is a ring morphism which sends 1 to 1 satisfying:
\begin{align*}
CM1. & \quad f(\widetilde{X}) \subseteq \widetilde{X}' \sqcup \mathbb{Z}; \\
CM2. & \quad f(X) \subseteq X' \sqcup \mathbb{Z}; \\
CM3. & \quad \text{For every } (f, \Sigma, \Sigma')-\text{biadmissible sequence } (y_1, y_2, \cdots, y_s) \text{ and for any } y \in \widetilde{X}, \text{ we have} \\
& \quad f(\mu_{y_s} \cdots \mu_{y_1}(y)) = \mu_{f(y_s)} \cdots \mu_{f(y_1)}(f(y)).
\end{align*}

By Definition 3.1, a rooted cluster morphism is first a ring morphism. Following this, we say a rooted cluster morphism \( f \) to be **surjective** (respectively, **injective**) if \( f \) is surjective (respectively, injective) as a ring morphism. A rooted cluster morphism \( f : A(\Sigma) \rightarrow A(\Sigma') \) is called an **isomorphism** in \Clus if it is both injective and also surjective and is written as \( A(\Sigma) \cong f A(\Sigma') \).

If there is an injective rooted cluster morphism \( f \) from \( A(\Sigma) \) to \( A(\Sigma') \), then the rooted cluster algebra \( A(\Sigma) \) is called a **rooted cluster subalgebra** of \( A(\Sigma') \) (see \([1]\)).

Dually, if there is a surjective rooted cluster morphism \( f \) from \( A(\Sigma) \) to \( A(\Sigma') \), then the rooted cluster algebra \( A(\Sigma') \) is called a **rooted cluster quotient algebra** of \( A(\Sigma) \). Note that, in the category \Clus, the surjective morphism and epimorphism are not coincide. See \([1]\).
The category of rooted cluster algebras is defined as the category Clus whose objects are all rooted cluster algebras and whose morphisms between two rooted cluster algebras are all rooted cluster morphisms.

It is easy to see that the conditions CM1 and CM2 in Definition 3.1 give the restriction of the rooted cluster morphism \( f \) on the seed \( \Sigma \), which shows the relation between the seeds \( \Sigma \) and \( \Sigma' \). Motivated by this fact and our discussion, we have introduced the theory of seed homomorphisms in the last section, which will be used to unify and understand the research on rooted cluster algebras via the relations among seeds.

**Definition 3.2.** Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be a rooted cluster morphism and

\[
I_1 = \{ x \in \tilde{X} | f(x) \in \mathbb{Z} \}.
\]

From \( f \), define a new seed \( \Sigma(f) = (X(f), X'_f, \tilde{B}(f)) \) satisfying that:

(I). \( X(f) = X \setminus I_1 \) with \( \{ x \in X | f(x) \notin \mathbb{Z} \} \);

(II). \( \tilde{X}(f) = \tilde{X} \setminus I_1 \) with \( \{ x \in \tilde{X} | f(x) \notin \mathbb{Z} \} \);

(III). \( B(f) = (b_{x,y}(f)) \) is a \( \#(X(f)) \times \#(X(f)) \) matrix with

\[
b_{x,y}(f) = \begin{cases} b_{x,y}, & \text{if } f(z) \neq 0 \forall z \in I_1 \text{ adjacent to } x \text{ or } y; \\ 0, & \text{otherwise.} \end{cases}
\]

We call this seed \( \Sigma(f) = (X(f), X'_f, \tilde{B}(f)) \) the contraction of \( \Sigma \) under \( f \).

For a rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \), if \( I_1 = \{ x \in \tilde{X} | f(x) \in \mathbb{Z} \} = \emptyset \), then \( \Sigma(f) = \Sigma \); in this case, we say \( f \) to be a noncontractible morphism and the seed \( \Sigma \) to be a noncontractible seed under \( f \).

**Remark 3.3.** Using the definitions of \( I_1 \) in (4) and of the new seed \( \Sigma(f) \), we have \( \Sigma(f) = \Sigma_{\emptyset, I_1} \) if \( f(x) \neq 0 \) for any \( x \in \tilde{X} \), since \( b_{x,y} = b_{x,y} \) in (III) for any \( x, y \) in this case.

**Proposition 3.4.** A rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) determines uniquely a seed homomorphism \( f^S, \Sigma(f), \Sigma' \) from \( \Sigma(f) \) to \( \Sigma' \) via \( f^S(x) = f(x) \) for \( x \in \tilde{X}(f) \).

**Proof.** By the definition of \( \Sigma(f), \varphi \) satisfies the condition (a) of Definition 2.1 For any two adjacent pairs \( (x, y) \) and \( (z, w) \) with \( x, z \in X(f) \) and \( y, w \in \tilde{X}(f) \), if either \( b_{x,y} = 0 \) or \( b_{z,w} = 0 \), then it always holds that \( (b_{x,y})^T (b_{y,z}) = 0 \). So, now we assume \( b_{x,y} \neq 0 \) and \( b_{z,w} \neq 0 \).

Under this condition, by the definition of \( B(f) \), we have \( b_{x,y} = b_{x,y} \) and \( b_{z,w} = b_{z,w} \). Then it follows that \( f(u) \neq 0 \) for all \( u \) adjacent to \( x \) or \( z \).

The following discussion is only made in the case where \( b_{x,y}, b_{z,w} \geq 0 \). For the other cases, the same conclusion can be obtained in a similar way.

By CM3, we have \( f(\mu(x)) = \mu(f(x)) \), which means that

\[
\frac{f(\prod_{b_{x,y} \geq 0, u \neq y} u_{x}^{b_{x,u}} + \prod_{b_{x,u} \leq 0} u_{x}^{-b_{x,u}})}{f(x)} = \prod_{b_{x,y} \geq 0, u \in \tilde{X}'} u_{x}^{b_{x,y}'} + \prod_{b_{x,u} \leq 0, u \in \tilde{X}'} u_{x}^{-b_{x,u}'}.
\]

By the algebraic independence of variables in \( \tilde{X}' \), we have

\[
f(\prod_{b_{x,y} \geq 0, u \neq y} u_{x}^{b_{x,u}}) = \prod_{b_{x,y} \geq 0, u \in \tilde{X}'} u_{x}^{b_{x,y}'} \quad \text{and} \quad f(\prod_{b_{x,u} \leq 0} u_{x}^{-b_{x,u}}) = \prod_{b_{x,u} \leq 0, u \in \tilde{X}'} u_{x}^{-b_{x,u}'}.
\]
or
\[ (9) \quad f(y^{b_{xy}} \prod_{b_{zu} \geq 0, u \neq y} u^{b_{zu}} = \prod_{b_{f(x)=u} \leq 0, v \in \mathcal{X}} v^{-b_{f(x)v}} \quad \text{and} \quad f(\prod_{b_{zu} \leq 0} u^{-b_{zu}}) = \prod_{b_{f(x)=u} \geq 0, v \in \mathcal{X}} v^{b_{f(x)v}}. \]

**Case 1:** Assume \( x = z \).

If \( \mathfrak{S} \) holds, then we get \( f(y)|_{b_{f(x)=u} \geq 0, v \in \mathcal{X}} = v^{-b_{f(x)v}}. \) For the pair \((x, y)\), comparing the exponents of \( f(y) \) in the two sides of the first expression in \( \mathfrak{S} \), we have 
\[ b_{f(x)f(y)} = \sum_{f(w) = f(y), b_{zu} > 0} b_{zu} > 0; \]
similarly, for the pair \((z, w)\), we have 
\[ b_{f(z)f(w)} > 0. \]
If \( x = z \), then it follows 
\[ (b_{xy}b_{f(x)f(y)})(b_{zw}b_{f(z)f(w)}) = (b_{xy}b_{f(x)f(y)})(b_{zw}b_{f(z)f(w)}) > 0. \]
If \( \mathfrak{S} \) holds, then we get 
\[ b_{f(x)f(y)} = \sum_{f(w) = f(y), b_{zu} > 0} (-b_{zu}) < 0, b_{f(z)f(w)} < 0, \] and similarly, \( \mathfrak{S} \) also follows.

**Case 2:** Assume \( x \neq z \).

Applying the result in Case 1 on the adjacent pairs \((x, y)\) and \((x, z)\), we have
\[ (b_{xy}b_{f(x)f(y)})(b_{xz}b_{f(x)f(z)}) > 0. \]
On the other hand, applying the result in case 1 on the adjacent pairs \((z, x)\) and \((z, w)\), we get 
\[ (b_{xz}b_{f(z)f(x)})(b_{zw}b_{f(z)f(w)}) > 0. \]
Combining \( \mathfrak{S} \) and \( \mathfrak{T} \), therefore, we have 
\[ (b_{xy}b_{f(x)f(y)})(b_{xz}b_{f(x)f(z)}) > 0. \]
In summary, it follows 
\[ (b_{xy}b_{f(x)f(y)})(b_{xz}b_{f(x)f(z)}) > 0. \]
Moreover, no matter whether \( \mathfrak{S} \) or \( \mathfrak{T} \) holds, we have 
\[ |b_{f(x)f(y)}| = \sum_{f(w) = f(y), b_{zu} > 0} |b_{xy}| \geq |b_{xy}| = |b_{xy}|. \]
Therefore, \( f^S \) is a seed homomorphism from \( \Sigma(f) \) to \( \Sigma' \). \qed

Following this proposition, we call \( f^S \) the **restricted seed homomorphism** of the rooted cluster morphism \( f \).

Conversely, for any seed homomorphism \( g : \Sigma \to \Sigma' \), we can induce a ring homomorphism \( G : \mathbb{Q}[X_f][X^{\pm 1}] \to \mathbb{Q}[X_f'][X'^{\pm 1}] \) by defining \( G(x) = g(x) \) for \( x \in \mathbb{X} \) and \( G(x^{-1}) = g(x)^{-1} \) for all \( x \in X \).

The restriction \( G|_{\mathcal{A}(\Sigma)} \) of \( G \) is also a ring homomorphism from \( \mathcal{A}(\Sigma) \) to \( \mathbb{Q}[X_f][X^{\pm 1}] \). If \( \text{Im}(G|_{\mathcal{A}(\Sigma)}) \subseteq \mathcal{A}(\Sigma') \) and \( G|_{\mathcal{A}(\Sigma)} \) is a rooted cluster morphism from \( \mathcal{A}(\Sigma) \) to \( \mathcal{A}(\Sigma') \), we call \( G|_{\mathcal{A}(\Sigma)} \) the **induced rooted cluster morphism** of \( g \). In this case, denote \( G|_{\mathcal{A}(\Sigma)} \) as \( g^E \).

Motivated by the above discussion, the natural question one has to consider are that for a rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) and a seed homomorphism \( g : \Sigma \to \Sigma' \).

**Question** (I). When \((f^S)^E\) and \(g^E\) exist, under what conditions \( f = (f^S)^E \) and \( g = (g^E)^S \) hold?

First, we have the following easy observation:

**Lemma 3.5.** Let \( f, g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be rooted cluster morphisms. If \( f(x) = g(x) \neq 0 \) for all \( x \in \mathbb{X} \) of \( \Sigma \), then \( f = g \).

**Proof.** It suffices to prove \( f(y) = g(y) \) for all cluster variable \( y \in \mathcal{A}(\Sigma) \). By Laurent phenomenon, \( y \in \mathbb{Q}[X_f][X^{\pm 1}] \), so \( y = \frac{h}{m} \) for a monomial \( m \) and a polynomial \( h \). Thus, \( f(y) = \frac{f(h)}{f(m)} = \frac{g(h)}{g(m)} = g(y). \) \qed

In particular, this proposition is satisfied if \( f, g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) are noncontractible.

Now, we use this lemma to answer the question.

In fact, for a seed homomorphism \( g : \Sigma \to \Sigma' \), if its induced rooted cluster morphism \( g^E \) exists, then we always have \( g = (g^E)^S : \Sigma \to \Sigma' \), since \( \Sigma(g^E) = \Sigma \) and for all \( x \in \mathbb{X} \), \((g^E)^S(x) = g^E(x) = g(x). \)
Moreover, for a rooted cluster morphism \( f : A(\Sigma) \to A(\Sigma') \), if the induced rooted cluster morphism \((f^S)^E : A(\Sigma(j)) \to A(\Sigma')\) exists, then \( f = (f^S)^E \) if and only if \( f \) is noncontractible. Indeed, \( \Sigma(j) = \Sigma \) if and only if \( f \) is noncontractible, and in this case, \((f^S)^E(x) = f^S(x) = f(x)\) for all \( x \in \tilde{X} \). Then by Lemma 3.3, \( f = (f^S)^E \) if and only if \( f \) is noncontractible.

In summary, we obtain the following result as the answer of Question (I).

**Proposition 3.6.** For a rooted cluster morphism \( f : A(\Sigma) \to A(\Sigma') \) and a seed homomorphism \( g : \Sigma \to \Sigma' \), if the induced rooted cluster morphisms \((f^S)^E\) and \( g^E \) exist, then (1) \( g = (g^E)^S : \Sigma \to \Sigma' \) and (2) \( f = (f^S)^E : A(\Sigma(j)) \to A(\Sigma') \) if and only if \( f \) is noncontractible.

A further question is that: under what condition, do there exist \((f^S)^E\) and \( g^E\)? This question seems difficult for us now. Maybe we would study it in the future work.

**Proposition 3.7.** For a rooted cluster morphism \( g : A(\Sigma) \to A(\Sigma') \) and any \((g, \Sigma, \Sigma')\)-biadmissible sequence \((y_1, \ldots, y_t)\), the morphism

\[
\gamma : A(\mu_{y_1} \cdots \mu_{y_t}(\Sigma)) \to A(\mu_{g(y_1)} \cdots \mu_{g(y_t)}(\Sigma'))
\]

is still a rooted cluster morphism on the seed \( \mu_{y_1} \cdots \mu_{y_t}(\Sigma) \).

**Proof.** By induction, it suffices to prove the result for the case \( t = 1 \). First, note that for any \( x \in \tilde{X} \) and \( x \neq y_1 \), we have \( g(x) \neq g(y_1) \) since \( g(\mu_{y_1}(x)) = g(x) = \mu_{g(y_1)}(g(x)) \) by CM3 on \( A(\Sigma) \). Hence, \( g(\tilde{X} \setminus \{y_1\}) \subseteq (\tilde{X} \setminus \{g(y_1)\}) \cup \mathbb{Z} \) and \( g(X \setminus \{y_1\}) \subseteq (X \setminus \{g(y_1)\}) \cup \mathbb{Z} \). Following this, CM1 and CM2 for \( g \) on \( A(\mu_{y_1}(\Sigma)) \) are obtained directly. For any \((g, \mu_{y_1}(\Sigma), \mu_{g(y_1)}(\Sigma'))\)-biadmissible sequence \((z_2, \ldots, z_s)\), by definition, \((y_1, z_2, \ldots, z_s)\) is a \((g, \Sigma, \Sigma')\)-biadmissible sequence. Then by CM3 for \( g \) on \( A(\Sigma) \), we have \( g(\mu_{z_2} \cdots \mu_{g(y_1)}(y)) = \mu_{g(z_2)} \cdots \mu_{g(y_1)}(y) \) for \( y \in \tilde{X} \). Combining the fact that \( \mu_{g(y_1)}(\tilde{X}) \) is the extended cluster of \( \mu_{y_1}(\Sigma) \), CM3 holds for \( g \) on \( A(\mu_{y_1}(\Sigma)) \).

Note that as algebras, we have \( A(\Sigma) = A(\mu_{y_1} \cdots \mu_{y_t}(\Sigma)) \) and \( A(\Sigma') = A(\mu_{g(y_1)} \cdots \mu_{g(y_t)}(\Sigma')) \). From the fact of (13), we know that the morphism \( g \) is still of rooted cluster on the seed \( \mu_{y_1} \cdots \mu_{y_t}(\Sigma) \) for any \((g, \Sigma, \Sigma')\)-biadmissible sequence \((y_1, \ldots, y_t)\). For this reason, we say \( g \) to be a **rooted cluster morphism generated** by \((y_1, \ldots, y_t)\).

For a seed homomorphism \( g_0 : \Sigma \to \Sigma' \), assume that its induced rooted cluster morphism \( g_0^E : A(\Sigma) \to A(\Sigma') \) exists. Then by Proposition 3.6, for any \((g_0^E, \Sigma, \Sigma')\)-biadmissible sequence \((y_1, \ldots, y_t)\), the morphism \( g_0^E : A(\mu_{y_1} \cdots \mu_{y_t}(\Sigma)) \to A(\mu_{g_0(y_1)} \cdots \mu_{g_0(y_t)}(\Sigma')) \) is still a rooted cluster morphism on the initial seed \( \mu_{y_1} \cdots \mu_{y_t}(\Sigma) \). By Proposition 3.6 for \( \Sigma \), we have \( (g_0^E)^S = g_0 : \Sigma \to \Sigma' \); for this reason, for the seed \( \mu_{y_1} \cdots \mu_{y_t}(\Sigma) \) and the rooted cluster morphism \( g_0^E \) generated by \((y_1, \ldots, y_t)\), we denote

\[
\mu_{y_1} \cdots \mu_{y_t}(g_0) = (g_0^E)^S : \mu_{y_1} \cdots \mu_{y_t}(\Sigma) \to \mu_{g_0(y_1)} \cdots \mu_{g_0(y_t)}(\Sigma').
\]

where we say \( \mu_{y_1} \cdots \mu_{y_t}(g_0) \) to be obtained from \( g_0 \) by the \( t \)-mutations of seed homomorphisms at the exchangeable variables \( y_1, \ldots, y_t \) for any positive integer \( t \).

Of course, \( g_0^E \) is noncontractible; by Proposition 3.6, \( ((g_0^E)^S)^E = g_0^E \). Using (14), we obtain \( (\mu_{y_1} \cdots \mu_{y_t}(g_0))^E = g_0^E \) as algebra morphisms. Therefore, Proposition 3.6 indeed tells us that

For a seed homomorphism \( g_0 \), the operation for giving the induced rooted cluster morphisms \( g_0^E \) is invariant under mutations of seed homomorphisms.

The following result illustrates the relation between seed isomorphism and rooted cluster isomorphism.

**Proposition 3.8.** \( A(\Sigma) \cong A(\Sigma') \) in \( \text{Clus} \) if and only if \( \Sigma \cong \Sigma' \) in \( \text{Seed} \).
Proof. “Only if”: Let $f: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a rooted cluster isomorphism with inverse $g$. According to Proposition 3.4, we have $f^S$ and $g^S$ as the restricted seed homomorphisms of $f$ and $g$, respectively. As the restrictions of $f$ and $g$ are seed homomorphisms, it is clear that $g^S f^S|_{\widetilde{X}} = id_{\widetilde{X}}$ and $f^S g^S|_{\widetilde{X}'} = id_{\widetilde{X}'}$. Note that $f$ and $g$ cannot map an exchangeable variable to a frozen variable. We also have $g^S f^S|_{X} = id_{X}$ and $f^S g^S|_{X'} = id_{X'}$.

Moreover, since $f^S$ is a seed homomorphism, $|b_{xy}| \leq |b_{f^S(x)f^S(y)}|$ for all $x, y \in \widetilde{X}$. On the other hand, for $g^S$ as a seed homomorphism, we have $|b_{g^S f^S(x)g^S f^S(x)}| = |b_{xy}|$ for all $x, y \in \widetilde{X}$. Therefore, $f^S: \Sigma \to \Sigma'$ is a seed isomorphism.

“If”: For a seed isomorphism $\Sigma \cong \Sigma'$, we have its prolongation $\tilde{f}: Q[X_f][X^{\pm 1}] \to Q[X_{f'}][X'^{\pm 1}]$ as an algebra isomorphism of Laurent polynomials, which satisfies that $\tilde{f}(x) = F(x)$ for all $x \in \widetilde{X}$ and $\tilde{f}(x)^{-1} = (F(x))^{-1}$ for $x \in X$. Now we prove that $\tilde{f}$ can induce a rooted cluster isomorphism $F^E: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ for $F^E = \tilde{f}|_{\mathcal{A}(\Sigma)}$.

First, we prove that $F^E = \tilde{f}|_{\mathcal{A}(\Sigma)}$ satisfies the conditions CM1, CM2 and CM3 such that $F^E(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(\Sigma')$.

In fact, CM1 and CM2 for $F^E$ hold clearly since $F$ is a seed homomorphism. Now we prove that any $\Sigma$-admissible sequence $(z_1, \ldots, z_s)$ is $(F^E, \Sigma, \Sigma')$-biadmissible and then that CM3 holds for $F^E$.

For $s = 1$, $(z_1)$ is $(F^E, \Sigma, \Sigma')$-biadmissible trivially due to $F$ as a seed isomorphism.

For $X \ni x \neq z_1$, it is clear that $F^E(\mu_{z_1}(x)) = \mu_{F^E(z_1)}(F^E(x))$ by the definition of mutation and the injection of $F^E$.

Now consider the case for $x = z_1$. Since $F$ is a seed isomorphism, we have $b_{z_1y} = b_{F(z_1)y} = b_{F^E(z_1)y}$ for all $b_{z_1y} \neq 0$ or $b_{z_1y} = -b_{F(z_1)y} = -b_{F^E(z_1)y}$ for all $b_{z_1y} \neq 0$. In the both cases, it holds

$$
\prod_{b_{z_1y} > 0} F^E(y^{b_{z_1y}}) + \prod_{b_{z_1y} < 0} F^E(y^{-b_{z_1y}}) = \prod_{b_{F^E(z_1)y} > 0} F^E(y^{b_{F^E(z_1)y}}) + \prod_{b_{F^E(z_1)y} < 0} F^E(y^{-b_{F^E(z_1)y}}).
$$

Thus, $F^E(\mu_{z_1}(z_1)) = \mu_{F^E(z_1)}(F^E(z_1))$, since $F^E|_{\widetilde{X}} = \tilde{f}|_{\widetilde{X}}$ is a bijection.

Assume that $(z_1, \ldots, z_s)$ is $(F^E, \Sigma, \Sigma')$-biadmissible and $F^E$ satisfies CM3 for $s < t$.

Now consider the case for $s = t$. By the definition of seed isomorphisms, we have the observation:

$$
\mu_{z_t-1} \cdots \mu_{z_1}(F) : \mu_{z_t-1} \cdots \mu_{z_1}(\Sigma) \to \mu_{F(z_t-1)} \cdots \mu_{F(z_1)}(\Sigma')
$$

as a seed isomorphism. Note that $F(z_t) = F^E(z_t)$ for any $t$.

Since $(z_1, \ldots, z_t)$ is $\Sigma$-admissible, $(z_t)$ is $\Sigma$-admissible. Using the isomorphism in 15, we know that $F^E(z_t) = \mu_{F(z_t-1)} \cdots \mu_{F(z_1)}(\Sigma')$-admissible; thus, $(z_t)$ is $(F^E, \mu_{z_t-1} \cdots \mu_{z_1}(\Sigma), \mu_{F(z_t-1)} \cdots \mu_{F(z_1)}(\Sigma'))$-admissible.

Therefore, $(z_1, \ldots, z_t)$ is also $(F^E, \Sigma, \Sigma')$-biadmissible. Moreover, from 15, it follows that

$$
F^E(\mu_{z_t}(z_t)) = \mu_{F(z_t)}(F^E(z_t))
$$

for all cluster variables $z$ in the seed $\mu_{z_t-1} \cdots \mu_{z_1}(\Sigma)$, where $\mu_{z_t}$ and $\mu_{F(z_t)}$ mean the mutations at $z_t$ and $F(z_t)$ in the seeds $\mu_{z_t-1} \cdots \mu_{z_1}(\Sigma)$ and $\mu_{F(z_t-1)} \cdots \mu_{F(z_1)}(\Sigma')$, respectively. Hence, $F^E(\mu_{z_t}(z_t)) = \mu_{F(z_t)}(F^E(\mu_{z_t-1} \cdots \mu_{z_1}(z_t))) = \mu_{F(z_t)}(F^E(\mu_{z_t-1} \cdots \mu_{z_1}(z_t))) = \mu_{F(z_t)}(\mu_{F(z_t-1)} \cdots \mu_{F(z_1)}(F^E(z_t)))$ for all $z_t \in \widetilde{X}$, where the second equality is by 15 and the third one is by the induction assumption. Thus, CM3 follows.

Since $\mathcal{A}(\Sigma')$ is generated by all its cluster variables, we have $F^E(\mathcal{A}(\Sigma)) = \tilde{f}(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(\Sigma')$ due to CM1, CM2 and CM3 shown above.
Second, the above discussion on $\tilde{f}$ is also suitable for $\tilde{f}^{-1}$; hence, similarly, we have $(\tilde{f})^{-1}(A(\Sigma')) \subseteq A(\Sigma)$. It follows that $A(\Sigma') \subseteq \tilde{f}(A(\Sigma))$. Hence, $F^E(A(\Sigma)) = \tilde{f}(A(\Sigma)) = A(\Sigma')$.

Note that $F^E$ is injective. Therefore, $F^E : A(\Sigma) \to A(\Sigma')$ is a rooted cluster isomorphism. □

**Remark 3.9.** Note that Theorem 3.9 in [1] states that $A(\Sigma) \cong A(\Sigma')$ in Clus if and only if either $\Sigma \cong \Sigma'$ or $\Sigma \cong \Sigma^{op}$. In fact, this result has dealt only the rooted cluster algebras with indecomposable seeds. For example, let $\mathcal{Q} : x_1 \rightarrow x_2 \rightarrow x_3$ and $\mathcal{Q}' : x_1 \rightarrow x_2 \rightarrow x_3$, it is clear that $f : \mathcal{Q}(\Sigma) \to \mathcal{Q}'(\Sigma'), x_i \to x_i$ for $i = 1, 2, 3$, is a rooted cluster isomorphism, while $\Sigma(\mathcal{Q}) \cong \Sigma(\mathcal{Q}')$ or $\Sigma(\mathcal{Q}) \cong \Sigma(\mathcal{Q})^{op}$ in sense of [1].

For a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$, in [1], the authors defined the image seed of $f$ to be the image seed of $f^S : \Sigma(\mathcal{S}) \to \Sigma'$ by Proposition 3.8, that is, $f^S(\Sigma(\mathcal{S}))$ by Definition 2.18.

In [1], a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$ is called ideal if $A(f^S(\Sigma(\mathcal{S}))) = f(A(\Sigma))$.

**Lemma 3.10.** (Proposition 2.36(2), [6]) Let $f : A(\Sigma) \to A(\Sigma')$ be an ideal rooted cluster morphism. Then $f = \tau f_1$, with a surjective rooted cluster morphism $f_1$ and an injective rooted cluster morphism $\tau$, that is, $f : A(\Sigma) \to A(f^S(\Sigma(\mathcal{S}))) \cong A(\Sigma')$.

**Lemma 3.11.** ([1]) Any injective rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$ is ideal.

Proof. Since $f$ is injective, we know clearly $\Sigma(\mathcal{S}) = \Sigma$. Then by Proposition 3.8, $f^S : \Sigma \to \Sigma'$ is a seed homomorphism. By definition, $(f^S)_1 : \Sigma \to f^S(\Sigma)$ satisfies $(f^S)_1(x) = f^S(x)$ for all $x \in \bar{X}$. We will prove that $(f^S)_1$ is a seed isomorphism as follows.

Denote $f^S(\Sigma) = (\bar{Y}, Y_{fr}, \bar{C})$. Owing to the definition of $f^S(\Sigma)$, $(f^S)_1|_{\bar{X}}$ and $(f^S)_1|_{\bar{X}}$ are bijections by injection of $f$. For any $x \in \bar{X}$ and $y \in \bar{X}$, by CM3 for $f$, we have $f(\mu_x(x)) = \mu_{f(x)}(f(x))$; thus,

$$f(\prod_{b_{xz} > 0, z \in \bar{X}} z^{b_{xz}} + \prod_{b_{xz} < 0, z \in \bar{X}} z^{-b_{xz}}) = \prod_{b_{f(x)w} > 0, w \in \bar{X}'} w^{b_{f(x)w}} + \prod_{b_{f(x)w} < 0, w \in \bar{X}'} w^{-b_{f(x)w}}.$$

Comparing the exponent of $f(y)$ in the two sides of the above equation, we get either $b_{xy} = b_{f(x)f(y)}$ or $b_{xy} = -b_{f(x)f(y)}$. Thus, $[b_{xy}] = [b_{f(x)f(y)}] = |c_{(f^S)_1(x)f^S(\Sigma))}|$. So, $(f^S)_1$ is a seed isomorphism.

According to Proposition 3.8, $A(\Sigma) \cong A(f^S(\Sigma))$ is a rooted cluster isomorphism. By definition, clearly $((f^S)_1)^E = f$ on $A(\Sigma)$. Moreover, since $f$ is injective, we have $A(\Sigma) \cong f(A(\Sigma))$. It follows that $f(A(\Sigma)) = A(f^S(\Sigma))$.

Using Lemma 3.11 and ([1], Lemma 3.1), we have the following:

**Proposition 3.12.** (1) If $A(\Sigma)$ is a rooted cluster subalgebra of $A(\Sigma')$ with an injective rooted cluster morphism $f$, then $A(\Sigma) := A(f^S(\Sigma))$ in Clus. Moreover, $\Sigma \cong f^S(\Sigma)$ in Seed.

(2) If $A(\Sigma)$ is a rooted cluster quotient algebra of $A(\Sigma)$ with a surjective rooted cluster morphism $f$, then $f^S(\Sigma) = \Sigma'$. Moreover, $A(f^S(\Sigma)) = A(\Sigma')$ as rooted cluster algebras.

Proof. (1) Since $f$ is injective, we have $A(\Sigma) \cong f(A(\Sigma))$ in Clus. Then by Lemma 3.11 it follows that $A(\Sigma) \cong A(f^S(\Sigma))$ in Clus. By Proposition 3.8, $\Sigma \cong f^S(\Sigma)$ in Seed.

(2) By ([1], Lemma 3.1), since $f$ is surjective, we have $f(X) \supseteq X'$ and $f(X) \supseteq \bar{X}'$. Then by the definition of image seed, we obtain $f^S(\Sigma) = \Sigma'$. Therefore, $A(f^S(\Sigma)) = A(\Sigma')$.

Owing to this proposition, the corresponding injective (respectively, surjective) seed morphisms are deduced from injective (respectively, surjective) rooted cluster morphisms as the below observation:
Corollary 3.13. (1) The restricted seed morphism from $\Sigma$ to $\Sigma'$ of an injective rooted cluster morphism $f : A(\Sigma) \rightarrow A(\Sigma')$ is injective.

(2) The restricted seed morphism from $\Sigma^{(f)}$ to $\Sigma'$ of a surjective rooted cluster morphism $f : A(\Sigma) \rightarrow A(\Sigma')$ is surjective.

Proof. (i) By Proposition 3.12, $\Sigma \cong f^S(\Sigma)$, and by (6), $f^S(\Sigma') = \Sigma'_{I_0, I'_1}$. Then an injective seed homomorphism is given.

(ii) By Proposition 3.3 (ii), $\varphi : \Sigma^{(f)} \rightarrow \Sigma'$ is a seed homomorphism via $\varphi(x) = f(x)$ for all $x \in \tilde{X}^{(f)}$. By (1), Lemma 3.1, $f(X) \cong X'$ and $f(\tilde{X}) \cong \tilde{X}'$. Then, $\varphi(\tilde{X}^{(f)}) \cap \tilde{X}' = f(\tilde{X}^{(f)}) \cap \tilde{X}' = \tilde{X}'$ and $\varphi(X^{(f)}) \cap X' = f(X)^{(f)}) \cap X' = f(X) \cap X' = X'$. Thus, $\varphi(\tilde{X}^{(f)}) \cong \tilde{X}'$ and $\varphi(X^{(f)}) \cong X'$. Therefore, we have $\varphi(\Sigma^{(f)}) = \Sigma'$. It follows that $\varphi$ is a surjective seed homomorphism.

4. Sub-rooted cluster algebras and rooted cluster subalgebras

4.1. Sub-rooted cluster algebras and two special cases.

The following notion on the sub-structure of a rooted cluster algebra is a key in this research, which will be used to supply a unified view-point for the internal structure of a rooted cluster algebra.

Definition 4.1. A rooted cluster algebra $A' = A(\Sigma')$ is called a (mixing-type) sub-rooted cluster algebra of type $(I_0, I_1)$ of the rooted cluster algebra $A = A(\Sigma)$ if $A(\Sigma') \cong A(\Sigma_{I_0, I_1})$ in the category $	ext{Clus}$.

From the definition of rooted cluster algebras, we can recognize the initial seed of a rooted cluster algebra as its “root”. So, it is natural for us to say the name of sub-rooted cluster algebra in the above definition since $A$ is obtained from mixing-type sub-seed as the “sub-root”.

By Proposition 3.3, a rooted cluster algebra $A' = A(\Sigma')$ is a mixing-type sub-rooted cluster algebra of type $(I_0, I_1)$ of $A = A(\Sigma)$ if and only if $\Sigma' \cong \Sigma_{I_0, I_1}$ in $\text{Seed}$.

Now, we discuss two special cases of mixing-type sub-rooted cluster algebras of $A(\Sigma)$.

Case 1: $I_1 = \emptyset$. That is, the sub-seed is a pure sub-seed $\Sigma_{I_0, \emptyset} = (X', \tilde{B}_0)$ of the seed $\Sigma$.

Since the extended clusters of $\Sigma$ and $\Sigma_{I_0, \emptyset}$ are the same by $\tilde{X}' = \tilde{X}$, their fields of rational functions in the independent extended cluster variables are $F$ with coefficients in the rational field $Q$. But, the ground ring $P$ of $\Sigma$ becomes a sub-ring of the ground ring $P'$ of $\Sigma_{I_0, \emptyset}$ generated by all frozen variables of $\Sigma_{I_0, \emptyset}$ with unit, since the frozen variables of $\Sigma$ are only a part of the frozen variables of $\Sigma_{I_0, \emptyset}$.

By Definition 3.3, from the seed $\Sigma_{I_0, \emptyset} = (X', \tilde{B}_0)$, we get its associated cluster algebra $A' = A(\tilde{B}_0)$ over $P'$ as the $P'$-subalgebra of $F$ generated by all cluster variables in all seeds mutation equivalent to $\Sigma_{I_0, \emptyset}$. An elementary fact is the following:

Proposition 4.2. $A' = A(\Sigma_{I_0, \emptyset})$ is a subalgebra of the rooted cluster algebra $A(\Sigma)$ over $Q$ as associative algebras.

Proof. Trivially, any frozen variables in $A'$ are always in $A$. For any exchangeable variable $x \in X'$ of $A'$, by Proposition 3.17 there exists a positive isomorphism $\mu_x(\Sigma_{I_0, \emptyset}) \cong \mu_x(\Sigma)_{I_0, \emptyset}$. But since $\tilde{X}' = \tilde{X}$, it is easy to see that $\mu_{\Sigma_{I_0, \emptyset}}(x) = \mu_x(\tilde{X})$. Hence, $\mu_x(\Sigma_{I_0, \emptyset})$ and $\mu_x(\Sigma)_{I_0, \emptyset}$ have the same cluster variables. Therefore, the above positive isomorphism is in fact an identity, that is,

\begin{equation}
\mu_x(\Sigma_{I_0, \emptyset}) = \mu_x(\Sigma)_{I_0, \emptyset}
\end{equation}

Then by induction, any exchange cluster variable $y_s = \mu_{y_{s-1}} \cdots \mu_y(y_1)$ is in $A$ by (17).
For this reason, we say this sub-rooted cluster algebra \( A' = A(\Sigma') \) to be a pure cluster sub-algebra of \( A = A(\Sigma) \) if \( A(\Sigma') \cong A(I_{0}, \emptyset) \) in Clus; equivalently, \( \Sigma' \cong \Sigma_{I_{0}, \emptyset} \) in Seed for \( I_{0} \subseteq X \).

Obviously, the rank of \( A' \) is \( n_{1} \).

Case 2: \( I_{0} = \emptyset \). That is, the sub-seed is a partial sub-seed \( \Sigma_{\emptyset, I_{1}} = (X', X_{fr}', \tilde{B}_{1}) \) of the seed \( \Sigma \).

A sub-rooted cluster algebra \( A' = A(\Sigma') \) is called a pure sub-cluster algebra of \( A = A(\Sigma) \) if \( A(\Sigma') \cong A(I_{0}, \emptyset) \) in Clus; equivalently, \( \Sigma' \cong \Sigma_{I_{0}, \emptyset} \) in Seed for some \( I_{1} \subseteq \tilde{X} \).

We give an example from \cite{1}. For two seeds \( \Sigma_{1} = (X_{1}, (X_{1})_{fr}, \tilde{B}_{1}) \) and \( \Sigma_{2} = (X_{2}, (X_{2})_{fr}, \tilde{B}_{2}) \) and their cluster algebras \( A(\Sigma_{1}) \) and \( A(\Sigma_{2}) \), assume that there exists (possibly empty) \( \Delta_{1} \subseteq (X_{1})_{fr} \) and \( \Delta_{2} \subseteq (X_{2})_{fr} \), such that \( \Sigma_{1} \) and \( \Sigma_{2} \) are glueable along \( \Delta_{1} \) and \( \Delta_{2} \).

Recall in \cite{1} that the amalgamated sum along \( \Delta_{1} \) and \( \Delta_{2} \) is defined as the rooted cluster algebra \( A(\Sigma_{1}) \Pi_{\Delta_{1}, \Delta_{2}} A(\Sigma_{2}) = A(\Sigma) \) where \( \Sigma = \Sigma_{1} \Pi_{\Delta_{1}, \Delta_{2}} \Sigma_{2} \).

\( A(\Sigma_{i}) \) \((i = 1, 2)\) can be viewed easily as pure sub-cluster algebras of \( A(\Sigma_{1}) \Pi_{\Delta_{1}, \Delta_{2}} A(\Sigma_{2}) \).

Denote \( I_{1} = \{x_{s_{1}}, \cdots, x_{s_{n}}\} \). Then, we can get a series of sub-cluster algebras as follows:

\[
A(\Sigma_{1}) \Pi_{\Delta_{1}, \Delta_{2}} A(\Sigma_{2}) = A(\Sigma) \subseteq A(\Sigma_{1}) \Pi_{\Delta_{1}} A(\Sigma_{2}) \subseteq \cdots \subseteq A(\Sigma_{1}) \Pi_{\Delta_{1}} \cdots \Pi_{\Delta_{n_{2}}} A(\Sigma_{2}).
\]

It is known that the exchange relation (2) for the adjacent cluster of \( A \) in direction \( k \in [1, n] \) can be given equivalently using the following formula:

\[
x_{k}x'_{k} = p_{k}^{+} \prod_{1 \leq i \leq n; b_{k,i} > 0} x_{i}^{b_{k,i}} + p_{k}^{-} \prod_{1 \leq i \leq n; b_{k,i} < 0} x_{i}^{-b_{k,i}},
\]

where

\[
p_{k}^{+} = \prod_{1 \leq i \leq m; b_{k,n+i} > 0} x_{n+i}^{b_{k,n+i}}, \quad p_{k}^{-} = \prod_{1 \leq i \leq m; b_{k,n+i} < 0} x_{n+i}^{-b_{k,n+i}}.
\]

are, respectively, the products of frozen variables and their inverses.

On the other hand, the field of rational functions in \( \tilde{X}'' \), written as \( F'' \), is a sub-field of \( F \) in \( \tilde{X} \) with coefficients in the rational field \( Q \).

By the definition of cluster algebras of geometric type, from the partial sub-seed \( \Sigma_{\emptyset, I_{1}} = (X', X_{fr}', \tilde{B}_{1}) \), its associated cluster algebra \( A'' = A(\tilde{B}_{1}) \) over \( F'' \) is generated as the \( F'' \)-subalgebra of \( F'' \) generated by all cluster variables in all seeds mutation equivalent to \( \Sigma_{\emptyset, I_{1}} \).

Referring to the equivalent form (19) of the exchange relation (2), we can describe the exchange relation for the adjacent cluster of \( A'' \) in direction \( k \in [1, n] \) using the following formula:

\[
x_{ik}x'_{ik} = p_{ik}^{+} \prod_{1 \leq t \leq n_{1}; b_{ik,t} > 0} x_{i,t}^{b_{ik,t}} + p_{ik}^{-} \prod_{1 \leq t \leq n_{1}; b_{ik,t} < 0} x_{i,t}^{-b_{ik,t}},
\]

where

\[
p_{ik}^{+} = \prod_{1 \leq t \leq n_{2}; b_{ik,n+t} > 0} x_{n+t}^{b_{ik,n+t}}, \quad p_{ik}^{-} = \prod_{1 \leq t \leq n_{2}; b_{ik,n+t} < 0} x_{n+t}^{-b_{ik,n+t}}.
\]

Note that the above \( p_{ik}^{+} \) and \( p_{ik}^{-} \) are the divisors of \( p_{ik}^{+} \) and \( p_{ik}^{-} \) in (20), respectively, and in (21), the products

\[
\prod_{1 \leq t \leq n_{1}; b_{ik,t} > 0} x_{i,t}^{b_{ik,t}}, \quad \prod_{1 \leq t \leq n_{1}; b_{ik,t} < 0} x_{i,t}^{-b_{ik,t}}
\]

are, respectively, the divisors of the corresponding products in (19). Hence, in general, the \( x'_{ik} \) in the adjacent cluster of \( A'' \) is not the \( x'_{ik} \) in that of \( A \). Thus,
$A''$ is not a subalgebra of the cluster algebra $A(\Sigma)$ even over $Q$.

Obviously, the rank of $A''$ is also $n_1$, which is similar to that of $A'$.

In general, analogous to the pure sub-algebra $A''$ above, a sub-rooted cluster algebra $A(\Sigma_{I_0, I_1})$ is NOT a subalgebra of the cluster algebra $A(\Sigma)$ even over $Q$.

4.2. Rooted cluster subalgebras as sub-class of sub-rooted cluster algebras.

We present a combinatorial characterization of rooted cluster subalgebras as a sub-class of mixing-type sub-rooted cluster algebras.

Given a cluster algebra $A(\Sigma)$, we denote by

$\mathcal{A}_1(\Sigma)$: the set of all pure cluster subalgebras of $A(\Sigma),
\mathcal{A}_2(\Sigma)$: the set of all rooted cluster subalgebras of $A(\Sigma)$ and,
\mathcal{A}_3(\Sigma): the set of all mixing-type sub-rooted cluster algebras of $A(\Sigma)$.

In the following discussion, we will have the inclusion relation:

$$\mathcal{A}_1(\Sigma) \subseteq \mathcal{A}_2(\Sigma) \subseteq \mathcal{A}_3(\Sigma).$$

First, pure cluster subalgebras are special rooted cluster subalgebras in a rooted cluster algebra. In fact, since $F(\Sigma_{I_0, \emptyset}) = F(\emptyset)\mathcal{A}(\Sigma_{I_0, \emptyset})$ is a subalgebra of $A(\Sigma)$. Hence, we have the embedding $\tilde{f} = id_{\mathcal{A}(\Sigma_{I_0, \emptyset})} : A(\Sigma_{I_0, \emptyset}) \hookrightarrow A(\Sigma)$, and trivially, the conditions CM1 and CM2 hold. Using the condition $\mu_x(\Sigma_{I_0, \emptyset}) \subseteq \mu_x(\Sigma_{I_0, \emptyset})$ for any $x \in X'$ and using induction, the condition CM3 is satisfied. Hence, $\tilde{f}$ is an injective rooted cluster morphism. So, we have:

**Proposition 4.3.** For a seed $X = (X, B)$ and its pure sub-seed $\Sigma_{I_0, \emptyset} = (X', B')$ with $X' = X \setminus I_0$, the pure cluster subalgebra $A(\Sigma_{I_0, \emptyset})$ of $A(\Sigma)$ is always a rooted cluster subalgebra of $A(\Sigma)$.

Let $\Sigma = (X, B)$, where $X = (x_1, x_2)$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $X_{fr} = \emptyset$ and $\Sigma' = (X', B')$, where $X' = \emptyset$ and $X'_{fr} = \{x_1\}$. Then, $\Sigma' = \Sigma_{I_0, I_1}$ with $I_0 = \{x_1\}$ and $I_1 = \{x_2\}$, and $A(\Sigma')$ is a rooted cluster subalgebra but not a pure cluster subalgebra of $A(\Sigma)$.

Then, the first strict inclusion relation in (22) follows.

A rooted cluster subalgebra $A(\Sigma') \in \mathcal{A}_2(\Sigma)$ is called proper if it does not belong to $\mathcal{A}_1(\Sigma)$, that is, it is not a pure cluster sub-algebra.

Note that we think $\mathcal{A}(\Sigma)$ as a special pure cluster sub-algebra of itself since $\Sigma = \Sigma_{\emptyset, \emptyset}$ for $I_0 = \emptyset$ and then a proper rooted cluster subalgebra of $A(\Sigma)$ never equals to $A(\Sigma)$.

Now, we give a characterization of rooted cluster subalgebras as a sub-class of (mixing-type) sub-rooted cluster algebras in a rooted cluster algebra $A(\Sigma)$.

**Theorem 4.4.** $A(\Sigma')$ is a rooted cluster subalgebra of $A(\Sigma)$ if and only if there exists a mixing-type sub-seed $\Sigma_{I_0, I_1}$ of $\Sigma$ such that $\Sigma' \cong \Sigma_{I_0, I_1}$ satisfies $b_{xy} = 0$ for any $x \in X \setminus (I_0 \cup I_1)$ and $y \in I_1$.

**Proof.** “Only if”: Let $f : A(\Sigma') \to A(\Sigma)$ be the injective rooted cluster morphism. By Proposition 3.12, we have $A(\Sigma') \cong A(f^S(\Sigma'))$. By Proposition 3.12 (1), $f^S(\Sigma') = \Sigma_{I_0, I_1}$ for $I_1 \supset X \setminus (X' \cup f(\tilde{X}'))$ and $I_0 = X \setminus (f(X') \cup I_1)$. Hence, $\Sigma' \cong \Sigma_{I_0, I_1}$ by Proposition 3.8.

Now we show that the above sets $I_0$ and $I_1$ satisfy the condition in the theorem. Otherwise, there exists $x_0 \in X \setminus (I_0 \cup I_1)$ and $y_0 \in I_1$ such that $b_{x_0 y_0} \neq 0$. Thus, in the rooted cluster algebra $A(\Sigma)$,

$$\mu_{x_0, \Sigma}(x_0) = \frac{\prod_{y \in X, b_{xy} > 0} y^{b_{xy}} \prod_{y \in X, b_{xy} < 0} y^{-b_{xy}}}{x_0}.$$
and in its sub-rooted cluster \((I_0, I_1)\)-algebra \(A(\Sigma_{I_0, I_1})\), we have

\[
\mu_{x_0, \Sigma_{I_0, I_1}}(x_0) = \frac{\prod_{y \in \tilde{X} \setminus I_1, b_{xy} > 0} y^{b_{xy}} + \prod_{y \in \tilde{X} \setminus I_1, b_{xy} < 0} y^{-b_{xy}}}{x_0}.
\]

Owing to \(b_{xy} \neq 0\), the term \(y^{b_{xy}}\) appears in the first equality but not in the second one. Since \(\tilde{X}\) is a transcendence basis of \(F(\Sigma)\), it follows that \(\mu_{x_0, \Sigma_{I_0, I_1}}(x_0) \neq \mu_{x_0, \Sigma}(x_0)\), which contradicts the condition (CM3) for the injective rooted cluster morphism \(f\).

“\(\text{If}\)”: We know that \(b_{xy} = 0\) for any \(x \in X \setminus (I_0 \cup I_1)\) and \(y \in I_1\) for the given \(I_0\) and \(I_1\). Let \(\Delta = I_0 \cup X_{fr}\). Furthermore, let \(\Sigma(I_1 \cup \Delta)\) be the sub-seed of \(\Sigma\) generated by the cluster variables of \(I_1\) and \(\Delta\). By definition of amalgamated sum, it is easy to see that \(\Sigma_{I_0, \emptyset} = \Sigma(I_1 \cup \Delta)_{I_0, \emptyset} \bigcup_{\Delta_1, \Delta_2} \Sigma_{I_0, I_1}\), where \(\Delta_1 = \Delta_2 = \Delta\). Thus, we get \(A(\Sigma_{I_0, \emptyset}) = A(\Sigma(I_1 \cup \Delta))_{I_0, \emptyset} \bigcup_{\Delta_1, \Delta_2} A(\Sigma_{I_0, I_1})\).

By Lemma 4.13 in \(\S 4\), \(A(\Sigma_{I_0, I_1})\) is a rooted cluster subalgebra of \(A(\Sigma_{I_0, \emptyset})\). By Proposition 4.21, \(A(\Sigma_{I_0, \emptyset})\) is a rooted cluster subalgebra of \(A(\Sigma)\). As the composition of two injective rooted cluster morphisms, it follows that \(A(\Sigma_{I_0, I_1})\) is a rooted cluster subalgebra of \(A(\Sigma)\).

This theorem tells us that in a cluster algebra, all rooted cluster subalgebras form a proper sub-set of the set of rooted sub-cluster algebras, that is, the second strict inclusion relation in \(\S 2\) follows.

**Remark 4.5.** According to Theorem 4.4, the existence of rooted cluster subalgebras is dependent on the initial seed of the rooted cluster algebra via, more precisely, mixing-type subseeds of the initial seed. The following is an example to illustrate that a rooted cluster subalgebra \(A(\Sigma_{I_0, I_1})\) of a rooted cluster algebra \(A(\Sigma)\) may not be isomorphic to any rooted cluster subalgebra of \(A(\Sigma')\) anymore, for a seed \(\Sigma'\), which is mutation equivalent to \(\Sigma\).

Let \(Q\) be the quiver \(1 \to 2 \to 3\) and the seed \(\Sigma = \Sigma(Q)\). Then \(A(\Sigma(21, \emptyset))\), a rooted cluster subalgebra of \(A(\Sigma)\), has 4 cluster variables and 1 frozen variable. It is easy to see that \(A(\mu_2(\Sigma))\) has no rooted cluster subalgebra that possesses 4 cluster variables and 1 frozen variable. Hence \(A(\Sigma(21, \emptyset))\) is not isomorphic to any rooted cluster subalgebra \(A((\mu_2(\Sigma))_{I_0, I_1})\) of \(A(\mu_2(\Sigma))\).

**Remark 4.6.** For any integer \(m \geq 3\), denote by \(\Pi_m\) the \(m\)-gon whose points are labeled cyclically from 1 to \(m\). For \(m \geq 4\), let \(A(\Pi_m)\) be the cluster algebra from the fan triangulation \(T_m\) of \(\Pi_m\) in Fig. 1, which is of type \(A_{m-3}\) with coefficients associated with boundary arcs. We construct by induction a family \(\{T_m\}_{m \geq 3}\) and then obtain a family of cluster algebras \(\{A(\Pi_m)\}_{m \geq 3}\).

**Figure 1**

For \(m, m'\) satisfying \(0 < m < m'\), the inclusion of \(T_m\) in \(T_{m'}\) defines naturally the injective rooted cluster morphism \(j_{m, m'} : A(\Pi_m) \to A(\Pi_{m'})\) given in [1]. Now we can interpret \(j_{m, m'}\) by the language.
of a mixing-type sub-rooted cluster algebra about some \((I_0, I_1)\). Actually, as in Fig. 1, we denote \(I_0\) to be the set consisting of single vertex corresponding to the diagonal \(\beta_{m-2}\) and \(I_1\) the set consisting of frozen vertices corresponding to the edges \(\alpha_m, \cdots, \alpha_{m'}\) and exchangeable vertices corresponding to the diagonals \(\beta_{m-1}, \cdots, \beta_{m' - 3}\). Then, the rooted cluster subalgebra \(\mathcal{A}(\Pi_m)\) of \(\mathcal{A}(\Pi_{m'})\) is a mixing-type sub-rooted cluster subalgebra of \((I_0, I_1)\)-type.

Obviously, in this example, \((I_0, I_1)\) satisfies the condition of Theorem 4.4.

In order to use in the sequel, we introduce the so-called diagonal-unitization matrix of an extended exchange matrix from a cluster algebra.

**Definition 4.7.** For an extended exchange matrix \(\tilde{B}_{n \times (n + m)}\) of a cluster algebra, we define its related diagonal-unitization matrix \(U \tilde{B}\) to be a matrix \(U \tilde{B} = (c_{ij})_{n \times (n + m)}\) such that for any \(i, j\),

\[
c_{ij} = \begin{cases} 
  b_{ij}, & \text{if } i \neq j; \\
  1, & \text{if } i = j.
\end{cases}
\]

Since all diagonal entries of \(\tilde{B}\) are zero due to its skew-symmetrizability, we have indeed

\[
U \tilde{B} = \tilde{B} + (E_n O_{n \times m}),
\]

where \(E_n\) is an \(n \times n\) identity matrix and \(O_{n \times m}\) is a zero matrix. Note that all diagonal entries of \(U \tilde{B}\) are 1; in the sequel, one will see that \(U \tilde{B}\) is just a tool to judge when the row-index set and the column-index set of certain submatrices are disjoint.

We need to understand a special case of cluster algebras, that is, a cluster algebra that is called trivial if it has no exchangeable cluster variables except frozen cluster variables. All other cluster algebras are called non-trivial.

We will say a sub-matrix of a matrix to be an empty sub-matrix if its either row-index set or column-index set is empty. To facilitate the statement of the conclusion, we think any empty submatrices are zero matrices.

**Corollary 4.8.** Using the above notations, \(\mathcal{A}(\Sigma')\) is a proper rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\) if and only if there exist \(I' \subseteq X\) and \(\emptyset \neq I_1 \subseteq \bar{X}\) such that the \(I' \times I_1\) sub-matrix of the diagonal-unitization matrix \(U \tilde{B}\) is a zero matrix and \(\Sigma' \cong \Sigma_{I_0, I_1}\) with \(I_0 = X \setminus (I' \cup I_1)\).

In particular, (i) a proper rooted cluster subalgebra \(\mathcal{A}(\Sigma')\) of \(\mathcal{A}(\Sigma)\) is trivial if and only if it can be written as \(\mathcal{A}(\Sigma_{I_0, I_1})\) with \(I_0 = X \setminus I_1\) and \(I_1 \neq \emptyset\) and

(ii) all proper rooted cluster subalgebras of \(\mathcal{A}(\Sigma)\) are trivial if and only if all entries of \(U \tilde{B}\) are nonzero. In this case, there does not exist non-trivial proper rooted cluster subalgebras.

**Proof.** “If”: In case \(I' = \emptyset\), equivalently \(I_0 \cup I_1 \supseteq X\), it is easy to see \(\mathcal{A}(\Sigma_{I_0, I_1})\) as a trivial rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\). Furthermore, since \(I_1 \neq \emptyset\), \(\mathcal{A}(\Sigma_{I_0, I_1})\) is a proper trivial rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\).

In case \(I' \neq \emptyset\), since \(I' \subseteq X\), \(\emptyset \neq I_1 \subseteq \bar{X}\) and \(I_0 = X \setminus (I' \cup I_1)\), we have \(X = I_0 \cup I' \cup (X \cap I_1)\). Moreover, the \(I' \times I_1\) sub-matrix of the diagonal-unitization matrix \(U \tilde{B}\) is a zero matrix, which implies that \(I' \cap I_1 = \emptyset\). Thus, \(I' = X \setminus (I_0 \cup I_1)\). As \(b_{xy} = 0\) for all \(x \in I' = X \setminus (I_0 \cup I_1)\) and \(y \in I_1\), according to Theorem 4.4, \(\mathcal{A}(\Sigma_{I_0, I_1})\) is a rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\). Moreover, as \(I' \neq \emptyset\), \(\mathcal{A}(\Sigma_{I_0, I_1})\) is a proper non-trivial rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\).

“Only If”: If \(\mathcal{A}(\Sigma')\) is a proper rooted cluster subalgebra of \(\mathcal{A}(\Sigma)\), then by Theorem 4.4 there exist \(I_0 \subseteq X\) and \(I_1 \subseteq \bar{X}\) satisfying \(b_{xy} = 0\) for any \(x \in X \setminus (I_0 \cup I_1)\) and \(y \in I_1\) such that \(\mathcal{A}(\Sigma') \cong \mathcal{A}(\Sigma_{I_0, I_1})\).

Now let \(I' = X \setminus (I_0 \cup I_1)\), then \(I' \cap I_1 = \emptyset\) and the \(I' \times I_1\) sub-matrix of the diagonal-unitization matrix \(U \tilde{B}\) is a zero matrix. Since \(I_0\), \(I_1\) and \(I' \cap X\) are pairwise disjoint, we have \(I_0 = X \setminus (I' \cup (I_1 \cap X)) = X \setminus (I' \cup I_1)\). Finally, \(I_1 \neq \emptyset\) follows due to the fact that \(\mathcal{A}(\Sigma')\) is proper.
Example 4.10. Let \( A(\Sigma') \cong A(\Sigma_{I_0,I_1}) \), it is trivial if and only if \( I_1 \neq \emptyset \) and \( X \subseteq (I_0 \cup I_1) \) and, equivalently, if and only if \( I_1 \neq \emptyset \) and \( I_0 = X \setminus I_1 \) since \( I_0 \cap I_1 = \emptyset \) and \( I_0 \subseteq X \).

(ii) “If”: For any proper rooted cluster subalgebra \( A(\Sigma') = A(\Sigma_{I_0,I_1}) \), since all entries of \( U\bar{B} \) are nonzero, the \( I \times I_1 \) zero submatrix in the above must be empty. But, \( I_1 \neq \emptyset \), we have to get \( I' = \emptyset \) and \( I_0 = X \setminus I_1 \), and then \( X \subseteq (I_0 \cup I_1) \) due to \( I_0 \cap I_1 = \emptyset \), which means \( \Sigma_{I_0,I_1} \) has no exchangeable cluster variable, i.e. \( A(\Sigma') = A(\Sigma_{I_0,I_1}) \) is trivial.

“Only if”: Otherwise, there exist \( i \) and \( j \) such that \( (i,j) \)-entry \( (U\bar{B})_{i,j} = 0 \). By choosing \( I' = \{i\} \), \( I_1 = \{j\} \) and \( I_0 = X \setminus \{i\} \) and then by Theorem 4.4, \( A(\Sigma_{I_0,I_1}) \) is a rooted cluster subalgebra of \( A(\Sigma) \).

Since \( I_0 \cap I_1 = \emptyset \), we have \( j \neq i \) and thus \( I_0 \neq X \setminus I_1 \), which means that \( A(\Sigma_{I_0,I_1}) \) is non-trivial by (i) and is proper due to \( I_1 \neq \emptyset \). It contradicts with the given condition. \( \Box \)

In summary, in a rooted cluster algebra \( A(\Sigma) \), we have:

\[
\mathcal{A}_2(\Sigma) = \{\text{rooted cluster subalgebras}\}
\]

\[
= \{\text{pure cluster subalgebras}\} \cup \{\text{proper rooted cluster subalgebras}\}
\]

\[
= \{\text{pure cluster subalgebras}\} \cup \{\text{trivial proper rooted cluster subalgebras} \}
\]

\[
\emptyset \{\text{non-trivial proper rooted cluster subalgebras}\}
\]

with \{ pure cluster subalgebras\} = \{A(\Sigma_{I_0,I_1}) : I_1 = \emptyset, I_0 \subseteq X\},

\{ trivial proper rooted cluster subalgebras\} = \{A(\Sigma_{I_0,I_1}) : I_1 \neq \emptyset, X \subseteq I_0 \cup I_1\} and

\{ non-trivial proper rooted cluster subalgebras\} = \{A(\Sigma_{I_0,I_1}) : I_1 \neq \emptyset, X \not\subseteq I_0 \cup I_1\},

where \( \cup \) means the disjoint union of sets.

Remark 4.9. (i) When \( I' = \emptyset \), the zero sub-matrix \( O_{I' \times I_1} \) is indeed an empty matrix and then \( A(\Sigma') \) is trivial.

(ii) If we assume \( I_1 = \emptyset \) in this corollary, then we get indeed a pure cluster subalgebra \( A(\Sigma_{I_0,\emptyset}) \) of \( A(\Sigma) \) for \( I_0 = X \setminus I' \). Moreover, a pure cluster subalgebra \( A(\Sigma_{I_0,\emptyset}) \) is trivial if and only if \( I_0 = X \) and \( I_1 = \emptyset \). This type of algebras is a unique trivial pure cluster subalgebra and is a special case of general (proper) trivial rooted cluster subalgebras.

The conclusion of Corollary 4.8 can be stated equivalently: a rooted cluster algebra \( A(\Sigma) \) has no proper non-trivial rooted cluster subalgebras if and only if all entries of \( U\bar{B} \) are nonzero. Or say, in case all entries of \( U\bar{B} \) are nonzero, in \( \mathcal{A}_3(\Sigma) \), the rooted cluster algebra \( A(\Sigma) \) has no non-trivial rooted cluster subalgebras except pure cluster subalgebras.

Meantime, the purpose of Corollary 4.8 is to provide a detailed program to construct proper rooted cluster subalgebras from a given rooted cluster subalgebra \( A(\Sigma) \).

Example 4.10. Let \( Q \) be the quiver of type \( A_2 \), that is, \( Q : 1 \to 2 \). For the corresponding cluster algebra \( A(\Sigma(Q)) \) of \( Q \), its diagonal-unitization matrix \( U\bar{B}(Q) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \). Obviously, all the entries of \( U\bar{B}(Q) \) are non-zero. The only proper rooted cluster subalgebras of \( A(\Sigma(Q)) \) are:

\[
A(\Sigma_{\{1\},\{2\}}) \text{ and } A(\Sigma_{\{2\},\{1\}})
\]

which are both trivial.

5. On enumeration and monoidal categorification

5.1. The number of rooted cluster subalgebras of the form \( A(\Sigma_{I_0,I_1}) \).

Using the result in Section 4, e.g. Theorem 4.4 and the above conclusions, we numberize some sub-classes in the internal structure of a rooted cluster algebra \( A(\Sigma) \) in \( \text{Clus} \).
Therefore, we have \( n \epsilon y \) rooted cluster algebras of \( A \). 

Remark 5.2. In Proposition 5.1 (2), the corresponding rooted cluster algebras of the empty sub-

\[ \text{Proof.} \]

(1) All pure cluster sub-algebras \( A(\Sigma_{I_0}, \emptyset) \) of \( A(\Sigma) \) are determined by \( I_0 \subseteq X \); hence, the required number is the chosen number of \( I_0 \), that is, \( C_n^0 + C_n^1 + \cdots + C_n^n = 2^n \).

(2) Denoting by \( S_1 \) the set of rooted cluster sub-algebras of \( A(\Sigma) \) in the form \( A(\Sigma_{I_0}, I_1) \) and by \( S_2 \) the set of all zero submatrices of \( \tilde{U} \). Assume that \( A(\Sigma_{I_0}, I_1) \) is a rooted cluster subalgebra of \( A(\Sigma) \) by Theorem 4.4, \( B_{xy} = 0 \) for all \( y \in I_1 \) and \( x \in X \setminus (I_0 \cup I_1) \). Now we define a map \( \varphi : S_1 \to S_2 \\) with \( \varphi(A(\Sigma_{I_0}, I_1)) = U \hat{B}(\tilde{X} \setminus (I_0 \cup I_1) \times I_1) \), where \( U \hat{B}(\tilde{X} \setminus (I_0 \cup I_1) \times I_1) \) is the zero submatrix of \( \hat{U} \). Hence, we can define the map \( \phi : S_2 \to S_1 \) with \( \phi(U \hat{B}(\tilde{X} \setminus (I_0 \cup I_1) \times I_1)) = A(\Sigma_{X \setminus (I_0 \cup I_1) \times I_1}) \).

(3) The set of sub-rooted cluster algebras of \( A(\Sigma) \) in the form \( A(\Sigma_{I_0}, I_1) \) is the disjoint union of the subset of the proper ones and that of pure sub-rooted cluster algebras of \( A(\Sigma) \). Hence, this result follows directly from (1) and (2).

Remark 5.2. In Proposition 5.1 (2), the corresponding rooted cluster algebras of the empty sub-

\[ \text{of} \] 

5.2. Monoidal sub-categorification of a rooted cluster algebra.

Let us recall the definition of the monoidal categorification of a cluster algebra (refer to 19). For a field \( K \), let \( M \) be a \( K \)-linear abelian monoidal category, where \( K \)-linearity means that the tensor functor \( \otimes \) is \( K \)-linear and exact. Moreover, we assume \( M \) to satisfy that (i) any object of \( M \) is of finite length and (ii) \( K \cong \text{Hom}_M(S, S) \) for any simple object \( S \) of \( M \).

Definition 5.3. (19) Let \( \mathcal{F} = \{(M_i)_{i=1}^{n+m}, \tilde{B}\} \) be a pair of a family \( \{M_i\}_{i=1}^{n+m} \) of simple objects in \( M \) and an integer-valued \( n \times (n + m) \)-matrix \( \tilde{B} = (b_{M_i, M_j})_{i=1, \cdots, n; j=1, \cdots, n+m} \) whose principal
part is skew-symmetric. We call \( \mathcal{I} \) a monoidal seed in \( \mathcal{M} \) if (i) \( M_i \otimes M_j \cong M_j \otimes M_i \) for any \( 1 \leq i, j \leq n + m \) and (ii) \( \bigotimes_{i=1}^{n+m} M_i^{\otimes a_i} \) is simple for any \((a_i) \in \mathbb{Z}_{\geq 0}^{n+m}\).

**Definition 5.4.** ([19]) For \( 1 \leq k \leq n \), we say that a monoidal seed \( \mathcal{I} = (\{M_i\}_{i=1}^{n+m}, \tilde{B}) \) admits a mutation \( \mu_{M_k} \) in direction \( M_k \) if there exists a simple object \( M'_k \in \mathcal{M} \) such that

(i) there exist exact sequences in \( \mathcal{M} \):

\[
0 \rightarrow \bigotimes_{b_{M_k, M_i} > 0} M_i^{\otimes b_{M_k, M_i}} \rightarrow M_k \otimes M'_k \rightarrow \bigotimes_{b_{M_k, M_i} < 0} M_i^{\otimes (-b_{M_k, M_i})} \rightarrow 0,
\]

\[
0 \rightarrow \bigotimes_{b_{M_k, M_i} < 0} M_i^{\otimes (-b_{M_k, M_i})} \rightarrow M'_k \otimes M_k \rightarrow \bigotimes_{b_{M_k, M_i} > 0} M_i^{\otimes b_{M_k, M_i}} \rightarrow 0;
\]

(ii) the pair \( \mu_{M_k}(\mathcal{I}) := (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_{M_k}(\tilde{B})) \) is a monoidal seed in \( \mathcal{M} \).

In this case, we denote \( \mu_{M_k}(M_k) \triangleq M'_k \) and \( \mu_{M_k}(M_i) \triangleq M_i \) if \( i \neq k \).

Similarly, as in the case of cluster algebras, for any \( I_0 \subseteq \{M_1, \cdots, M_n\} \) and \( I_1 \subseteq \{M_1, \cdots, M_{n+m}\} \) with \( I_0 \cap I_1 = \emptyset \), we can define \( \mathcal{I}_{I_0, I_1} \) for any monoidal seed \( \mathcal{I} \). More precisely, \( \mathcal{I}_{I_0, I_1} = (\{M_i\}_{i \not\in I_1}, \tilde{B'} \) , where \( \tilde{B}' \) is the matrix obtained from \( \tilde{B} \) by deleting the \( I_0 \cup I_1 \) rows and \( I_1 \) columns.

**Definition 5.5.** ([19]) Using the notations above, \( \mathcal{M} \) is called a monoidal categorification of a cluster algebra \( A = A(\Sigma) \) if

(a) there is a ring isomorphism \( \varphi : K_0(\mathcal{M}) \cong A \), where \( K_0(\mathcal{M}) \) is the Grothendieck ring of \( \mathcal{M} \),

(b) there exists a monoidal seed \( \mathcal{I} = (\{M_i\}_{i=1}^{n+m}, \tilde{B}) \) in \( \mathcal{M} \) such that \( |\mathcal{I}| := (\{M_i\}_{i=1}^{n+m}, \tilde{B}) \) is the initial seed \( \Sigma \) of \( A \) and \( \mathcal{I} \) admits successive mutations in all directions.

**Remark 5.6.** If a rooted cluster algebra \( A(\Sigma) \) admits a monoidal categorification \( \mathcal{M} \) such that the monoidal seed \( \mathcal{I} = (\{M_i\}_{i=1}^{n+m}, \tilde{B}) \) corresponds to \( \Sigma = (\{x_i\}, \tilde{B}) \), we always assume that \( M_i \) correspond to \( x_i \) for all \( i \). According to the definition of mutation of monoidal seeds, it is easy to see that

\[
\varphi([\mu_{M_i}(M)]) = \mu_{x_i}(\varphi([M]))
\]

for all \( x_i \in X \).

Following the above preparations, we find the relation between the categorification of a rooted cluster algebra \( A(\Sigma) \) and the categorification of a rooted cluster subalgebra of \( A(\Sigma) \).

**Theorem 5.7.** Let the abelian monoidal category \( \mathcal{M} \) be a monoidal categorification of a cluster algebra \( A(\Sigma) \) and \( A(\Sigma_{I_0, I_1}) \) be a rooted cluster subalgebra of \( A(\Sigma) \). Then,

(i) \( A(\Sigma_{I_0, I_1}) \) has a monoidal categorification \( \mathcal{M}' \) that is also an abelian monoidal sub-category of \( \mathcal{M} \);

(ii) For the Grothendieck rings \( K_0(\mathcal{M}) \) and \( K_0(\mathcal{M}') \), the following diagram commutes via ring homomorphisms:

\[
\begin{array}{ccc}
K_0(\mathcal{M}) & \xrightarrow{i_1} & K_0(\mathcal{M}') \\
\downarrow \cong \varphi' & & \downarrow \cong \varphi \\
A(\Sigma_{I_0, I_1}) & \xrightarrow{i_2} & A(\Sigma)
\end{array}
\]

where \( i_1 \) and \( i_2 \) mean the injective ring homomorphisms.

**Proof.** (i) Since \( A(\Sigma_{I_0, I_1}) \) is a rooted cluster subalgebra of \( A(\Sigma) \), by ([11], Corollary 4.6(2)), the cluster variables of \( A(\Sigma_{I_0, I_1}) \) are cluster variables of \( A(\Sigma) \). Let \( \varphi : K_0(\mathcal{M}) \cong A(\Sigma) \) be the ring isomorphism by Definition 5.5 and let

\[
\mathcal{G} = \{S \in \mathcal{M} | \varphi([S]) \text{ is a cluster variable of } A(\Sigma_{I_0, I_1})\}.
\]
Denote by $\mathcal{M}'$ the fully faithful abelian monoidal subcategory of $\mathcal{M}$ generated by $\mathcal{S}$. Now we prove that $\mathcal{M}'$ is the monoidal categorification of $\mathcal{A}(\Sigma_{I_{0},I_{1}})$.

Let $W'$ be the set of cluster variables in $\mathcal{A}(\Sigma_{I_{0},I_{1}})$. We denote $[\mathcal{S}] = \{ [S] | S \in \mathcal{S} \}$. Then, $\varphi([\mathcal{S}]) = W'$.

(a) Since $\mathcal{M}'$ is the fully faithful abelian monoidal subcategory of $\mathcal{M}$ generated by $\mathcal{S}$, we have $K_0(\mathcal{M}')$ as the sub-ring of $K_0(\mathcal{M})$ generated by $[\mathcal{S}]$. Let $\varphi' = \varphi|_{K_0(\mathcal{M}')}$. Since $\mathcal{A}(\Sigma_{I_{0},I_{1}})$ is generated by $W' = \varphi([\mathcal{S}])$ as a ring, we have $\varphi'(K_0(\mathcal{M}')) = \mathcal{A}(\Sigma_{I_{0},I_{1}})$. Since $\varphi'$ is the restriction of the isomorphism $\varphi$, we get $\varphi' : K_0(\mathcal{M}') \cong \mathcal{A}(\Sigma_{I_{0},I_{1}})$, that is, Definition 5.7 (a) holds for $\mathcal{M}'$ and $\mathcal{A}(\Sigma_{I_{0},I_{1}})$, and then the commutes diagram in the proposition follows.

(b) Since $\mathcal{M}$ is the monoidal categorification of $\mathcal{A}(\Sigma)$, by definition, there exists a monoidal seed $\mathcal{S} = (\{M_i\}, \mathcal{B})$ such that $[\mathcal{S}] = \Sigma$. Set $\mathcal{T}_0 = \{ M \in \{ M_i \} | \varphi([M]) \in \mathcal{I}_0 \}$ and $\mathcal{T}_1 = \{ M \in \{ M_i \} | \varphi([M]) \in \mathcal{I}_1 \}$, so that all objects $M_i \in \mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}$ are in one-one correspondence to cluster variables in $\Sigma_{I_{0},I_{1}}$. $\mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}$ is a monoidal seed in $\mathcal{M}'$ by definition, and furthermore, $[\mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}] = \Sigma_{I_{0},I_{1}}$.

For any $M_k \in \{ M_1, M_2, \ldots, M_n \} \setminus (\mathcal{T}_0 \cup \mathcal{T}_1)$, since $\mathcal{M}$ is the monoidal categorification of $\mathcal{A}(\Sigma)$, there exist $\mu_{M_k}(M_k) \in \mathcal{M}$ and two exact sequences in $\mathcal{M}$:

\begin{align*}
(23) & \quad 0 \to \bigotimes_{b_{M_k,M_i} > 0} M_i \otimes b_{M_k,M_i} M_i \to M_k \otimes \mu_{M_k}(M_k) \to \bigotimes_{b_{M_k,M_i} < 0} M_i \otimes (-b_{M_k,M_i}) \to 0, \\
(24) & \quad 0 \to \bigotimes_{b_{M_k,M_i} < 0} M_i \otimes (-b_{M_k,M_i}) \to \mu_{M_k}(M_k) \otimes M_k \to \bigotimes_{b_{M_k,M_i} > 0} M_i \otimes b_{M_k,M_i} \to 0.
\end{align*}

Since $\mathcal{A}(\Sigma_{I_{0},I_{1}})$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$, according to Theorem 4.4, for all $M_i \in \{ M_{j_{1},j_{2},\ldots,j_{m}} \}$, if $b_{M_k,M_i} \neq 0$, we have $M_i \notin \mathcal{T}_1$; thus, $M_i \in \mathcal{M}'$ in (23) and (24).

By Remark 5.6, $\varphi([\mu_k(M_k)]) = \mu_k(\varphi([M_k]))$, which is a cluster variable in $\mathcal{A}(\Sigma_{I_{0},I_{1}})$. Hence, $\mu_k(M_k) \in \mathcal{S} \subseteq \text{Ob}(\mathcal{M}')$.

Therefore, the exact sequences (23) and (24) are also in $\mathcal{M}'$. Thus, for $\mathcal{M}'$, there is a mutation $\mu'_{M_k}$ in direct $k$ satisfying $\mu'_{M_k}(\mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}) = (\{ M_i | i \notin \mathcal{T}_1 \} \cup \{ \mu_{M_k}(M_k) \}) \cup \{ \mu_{M_k}(M_k) \}$ with $\mu'_{M_k}(M_k) = \mu_{M_k}(M_k)$ and $\mu'_{M_k}(M_i) = M_i$ for $i \neq k$ and $M_i \notin \mathcal{T}_1$. Thus, $\mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}$ admits successive mutations in all directions by induction.

From (a), (b), by Definition 5.5, $\mathcal{M}'$ is the monoidal categorification of $\mathcal{A}(\Sigma_{I_{0},I_{1}})$. So, (i) holds.

(ii) As mentioned in the proof of (i), the cluster variables of $\mathcal{A}(\Sigma_{I_{0},I_{1}})$ are also cluster variables of $\mathcal{A}(\Sigma)$. By definition, $\varphi' = \varphi|_{K_0(\mathcal{M}')}$. Hence, the commutativity of the diagram follows naturally. \(\square\)

Analog to $\Sigma_{I_{0},I_{1}}$ in Proposition 2.17, we can prove that $\mu_{M_k}(\mathcal{S}_{\mathcal{T}_0,\mathcal{T}_1}) = (\mu_{M_k}(\mathcal{S}))_{\mathcal{T}_0,\mathcal{T}_1}$.

The monoidal category $\mathcal{M}'$ is called a monoidal sub-categorification of the cluster algebra $\mathcal{A}(\Sigma)$.

By Theorem 4.4 for the rooted cluster subalgebra $\mathcal{A}(\Sigma')$, there is a mixing-type sub-seed $\Sigma_{I_{0},I_{1}}$ for $I_1 = \tilde{X} \setminus (\tilde{X} \cap f(\tilde{X}'))$, $I_0 = X \setminus (f(X') \cup I_1)$ satisfying $b_{x,y} = 0$ for any $x \in X \setminus (I_0 \cup I_1)$ and $y \in I_1$ such that $\Sigma' \cong \Sigma_{I_{0},I_{1}}$. Hence, the monoidal categorification $\mathcal{M}'$ of $\mathcal{A}(\Sigma')$ can be obtained by using Theorem 5.7.

The remaining interesting question is:

(a) What is the relationship between the monoidal categorification $\mathcal{M}'$ of an arbitrary sub-rooted cluster algebra isomorphic to $\mathcal{A}(\Sigma_{I_{0},I_{1}})$ and $\mathcal{M}$ of the (rooted) cluster algebra $\mathcal{A}(\Sigma)$?

In particular, dually, when the abelian monoidal category $\mathcal{M}'$ is a quotient category of $\mathcal{M}$ with the natural quotient functor $\pi : \mathcal{M} \to \mathcal{M}'$ preserving the tensor product, we call $\mathcal{M}'$ a monoidal
quotient categorification of the (rooted) cluster algebra $\mathcal{A}(\Sigma)$. Therefore, we have the other question as follows:

(b) For a sub-rooted cluster algebra $\mathcal{A}(\Sigma')$ isomorphic to $\mathcal{A}(\Sigma_{I_0,I_1})$, what is the condition of $\mathcal{A}(\Sigma')$ that makes its monoidal categorification $\mathcal{M}'$ to be a monoidal quotient categorification of the (rooted) cluster algebra $\mathcal{A}(\Sigma)$?

As a partial answer of (b), a necessary condition is that $\mathcal{A}(\Sigma')$ is a quotient algebra of $\mathcal{A}(\Sigma)$. Indeed, if $\mathcal{M}'$ is a monoidal quotient categorification of $\mathcal{A}(\Sigma)$, by definition, we have the algebra isomorphisms $K_0(\mathcal{M}) \cong \mathcal{A}(\Sigma)$ and $K_0(\mathcal{M}') \cong \mathcal{A}(\Sigma')$, and since the quotient functor $\pi : \mathcal{M} \to \mathcal{M}'$ preserves tensor product, we obtain a surjective algebra homomorphism $[\pi] : K_0(\mathcal{M}) \to K_0(\mathcal{M}')$ naturally, which means that $\mathcal{A}(\Sigma')$ is a quotient algebra of $\mathcal{A}(\Sigma)$.

These two questions will be discussed more in our further work.

6. Rooted cluster quotient algebras

6.1. Rooted cluster quotient algebras via pure sub-cluster algebras.

We use the same notations in [1]. Let $\Sigma \setminus \{x\} = (X \setminus \{x\}, \overline{B} \setminus \{x\})$ denote the seed obtained from $\Sigma$ by deleting $x \in \overline{X}$, where $\overline{B} \setminus \{x\}$ means the matrix obtained by deleting the row and column labeled by $x$ from $\overline{B}$ (when $x \in \overline{X}$, $\overline{B}$ has no row labeled by $x$, so $\overline{B} \setminus \{x\}$ is obtained by deleting only the column labelled by $x$ from $\overline{B}$). Denote $\sigma_{x,1}$ as the unique algebra homomorphism from $\mathcal{A}(\Sigma)$ to $F(\Sigma \setminus \{x\})$, which sends $y$ to $y$ for $y \in \overline{X} \setminus \{x\}$ and $x$ to 1. Hence, $\sigma_{x,1} : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma \setminus \{x\})$ is an algebra homomorphism if and only if $\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(\Sigma \setminus \{x\})$. We call $\sigma_{x,1}$ the simple specialisation of $\mathcal{A}(\Sigma)$ at $x$. In the sequel, we will need to consider the composition of some simple specialisations for $x$ in a certain subset $I \subseteq \overline{X}$, i.e. $\sigma_{I,1} = \prod_{x \in I} \sigma_{x,1}$, which is called the specialisation of $\mathcal{A}(\Sigma)$ at $I$.

Denote by $\sigma_{x,1}(\mu_y(\Sigma) \setminus \{x\})$ the seed $(\{\sigma_{x,1}(\mu_y(x_1)), \ldots, \sigma_{x,1}(\mu_y(x_n))\} \setminus \{1\}, \mu_y(\overline{B} \setminus \{x\}))$ in case $x \neq y \in X$.

In [1], Problem 6.10), the authors propose a problem on whether $\sigma_{x,1}$ induces a surjective ideal rooted cluster morphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma \setminus \{x\})$.

[1], Proposition 6.9) says that $\sigma_{x,1}$ is a surjective ideal rooted cluster morphism from $\mathcal{A}(\Sigma)$ to $\mathcal{A}(\Sigma \setminus \{x\})$ if and only if $\sigma_{x,1}$ is an algebra homomorphism and then if and only if $\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(\Sigma \setminus \{x\})$.

[1], Corollary 6.14 and Theorem 6.15) tells us that if the seed $\Sigma$ is either acyclic or arising from a surface with certain condition on $x$, then the homomorphism $\sigma_{x,1}$ is surjective.

More generally, ([1], Theorem 6.17) shows that if $\mathcal{A}(\Sigma)$ admits a 2-CY categorification, in particular, if the skew-symmetric initial seed $\Sigma$ is mutation equivalent to an acyclic seed or say that $\mathcal{A}(\Sigma)$ is acyclic, then $\sigma_{x,1}$ induces an algebra homomorphism between the cluster character algebras.

Associated with these, in the sequel, we obtain the fact in Lemma 6.5 that for a totally sign-symmetric seed $\Sigma$ if $\mathcal{A}(\Sigma \setminus \{x\})$ is acyclic, then $\sigma_{x,1}$ induces an algebra homomorphism between cluster algebras. This fact is used to get Theorem 6.7, the main result of this section.

According to ([1], Corollary 6.14) and [18], we have the following.

**Proposition 6.1.** For an acyclic seed $\Sigma$ and $I_1 \subset X$, the pure sub-cluster algebra $\mathcal{A}(\Sigma_{I_0,I_1})$ is a rooted cluster quotient algebra of the rooted cluster algebra $\mathcal{A}(\Sigma)$.

It is known well that a (rooted) cluster algebra $\mathcal{A}(\Sigma)$ is called acyclic if the initial seed $\Sigma$ is mutation equivalent to an acyclic seed. Hence, the above proposition means that for an acyclic rooted cluster algebra $\mathcal{A}(\Sigma)$ with an acyclic initial seed, its pure sub-cluster algebras are always...
rooted cluster quotient algebras. Now, we can discuss the general case of an acyclic rooted cluster algebra \( \mathcal{A}(\Sigma) \), that is, its acyclic seed does not need to be the initial seed \( \Sigma \).

**Lemma 6.2.** Using the foregoing notations, let \((y_1, \cdots, y_s)\) be a \( \Sigma \)-admissible sequence with \( y_i \neq x \) for all \( 1 \leq i \leq s \). Denote by \( \mu \) and \( \mu' \), respectively, the mutations in the cluster algebras \( \mathcal{A}(\mu_y, \cdots, \mu_y(\Sigma)) \) and \( \mathcal{A}(\sigma_{x,1}(\mu_y, \cdots, \mu_y(\Sigma) \setminus \{x\})) \). Then for any \( y \in \mu_{y_1} \cdots \mu_{y_s}(X) \) and \( z \in \mu_{y_1} \cdots \mu_{y_s}(X) \) with \( y, z \neq x \), it holds that

1. \( \sigma_{x,1}(\mu_{y_1}(z)) = \mu'_{x,1}(\sigma_{x,1}(z)) \);
2. \( \sigma_{x,1}(\mu_{y_1}(\Sigma) \setminus \{x\}) = \mu'_{x,1}(\sigma_{x,1}(\{y\}) \setminus \{x\}) \);
3. Any \( \sigma_{x,1}(\Sigma \setminus \{x\}) \)-admissible sequence \((y_1, \cdots, y_i)\) can be lifted to a \((\sigma_{x,1}(\Sigma), \Sigma \setminus \{x\})\)-biadmissible sequence \((w_1, \cdots, w_i)\) satisfying that \( \sigma_{x,1}(w_i) = z_i \) for \( i = 1, \cdots, t \) and

\[
\sigma_{x,1}(\mu_{w_1} \cdots \mu_{w_1}(\Sigma) \setminus \{x\}) = \mu'_{z_1} \cdots \mu'_{z_1}(\Sigma \setminus \{x\}).
\]

**Proof.** We denote \( \mu_{y_1} \cdots \mu_{y_s}(\Sigma) \) by \( \Sigma' = (X', \tilde{B}') \).

1. In the cluster algebra \( \mathcal{A}(\Sigma') \), since

\[
\mu_{y}(z) = \left\{ \begin{array}{ll}
\prod_{t \in X', \sigma_{x,1}(y) < 0} t^{s_y} + \prod_{t \in X', \sigma_{x,1}(y) > 0} t^{-s_y} & \text{if } z = y; \\
z, & \text{if } z \neq y,
\end{array} \right.
\]

we have

\[
\sigma_{x,1}(\mu_{y}(z)) = \left\{ \begin{array}{ll}
\prod_{t \in X', \sigma_{x,1}(y) > 0} \sigma_{x,1}(y)^{d_y} + \prod_{t \in X', \sigma_{x,1}(y) < 0} \sigma_{x,1}(y)^{-d_y} & \text{if } z = y; \\
\sigma_{x,1}(z), & \text{if } z \neq y.
\end{array} \right.
\]

On the other hand, in the seed \( \sigma_{x,1}(\Sigma' \setminus \{x\}) \),

\[
\mu'_{\sigma_{x,1}(y)}(\sigma_{x,1}(z)) = \left\{ \begin{array}{ll}
\prod_{t \in X', \sigma_{x,1}(y) > 0} \sigma_{x,1}(y)^{d_y} + \prod_{t \in X', \sigma_{x,1}(y) < 0} \sigma_{x,1}(y)^{-d_y} & \text{if } z = y; \\
\sigma_{x,1}(z), & \text{if } z \neq y.
\end{array} \right.
\]

Therefore, the result follows.

2. Denote

\[
(25) \quad \sigma_{x,1}(\mu_{y}(\Sigma') \setminus \{x\}) = (X'', \tilde{B}'') \quad \text{and} \quad \mu'_{\sigma_{x,1}(y)}(\sigma_{x,1}(\Sigma \setminus \{x\})) = (X'''', \tilde{B}''').
\]

Then by definition, we have

\[
X'' = \{\sigma_{x,1}(\mu_{y}(x'_i)) : i = 1, \cdots, n, \mu_{y}(x'_i) \neq x\} = (\sigma_{x,1}(X') \setminus \{x, \sigma_{x,1}(y)\}) \cup \{\sigma_{x,1}(\mu_{y}(y))\}
\]

and by the definitions of \( \sigma_{x,1} \) and the mutation \( \mu' \), we have

\[
X''' = ((\sigma_{x,1}(X') \setminus \{x\}) \setminus \sigma_{x,1}(y)) \cup \{\mu'_{\sigma_{x,1}(y)}(\sigma_{x,1}(y))\} = (\sigma_{x,1}(X') \setminus \{x, \sigma_{x,1}(y)\}) \cup \{\mu'_{\sigma_{x,1}(y)}(\sigma_{x,1}(y))\}.
\]

Thus by (i), it follows that \( X'' = X''' \).

Owing to the first formula of (25), \( \tilde{B}'' \) is the matrix obtained by applying the mutation in the direction \( y \) on \( \tilde{B}' \) and then deleting the row and column labeled \( x \); on the other hand, due to the second formula of (25), since \( y \neq x \), \( \tilde{B}''' \) is identified with the matrix by first deleting the row and column labeled \( x \) in \( \tilde{B}' \) and then applying the mutation in the direction \( y \).

By calculation, since \( X' \ni x \neq y \in X', \) for \( s \in X' \setminus \{x\} \) and \( t \in X' \setminus \{x\} \), we have

\[
b_{st}' = b_{st}'' = \left\{ \begin{array}{ll} 
b_{st} + \frac{[b_{st}]_{k_{st} + b_{st}'} + [b_{st}']_{k_{st}}}{2}, & \text{if } s \neq y \neq t; \\
-b_{st}', & \text{if } s = y \text{ or } t = y.
\end{array} \right.
\]
Lemma 6.4. Let \( w_1 = z_1 \). As \( z_1 \neq x \), we have \( \sigma_{x,1}(w_1) = z_1 \). By (ii), \( \sigma_{x,1}(\mu_{w_1}(\Sigma) \setminus \{x\}) = \mu_{z_1}(\Sigma \setminus \{x\}) \). Assume that for \( t - 1, (z_1, \ldots, z_{t-1}) \) can be lifted to a \( (\sigma_{x,1}, \Sigma \setminus \{x\}) \)-biadmissible sequence \( (w_1, \ldots, w_{t-1}) \) and \( \sigma_{x,1}(\mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \setminus \{x\}) = \mu^*_{z_{t-1}} \cdots \mu^*_1(\Sigma \setminus \{x\}) \). Then we consider the case for \( t \). Since \( z_t \) is exchangeable in \( \mu^*_{z_{t-1}} \cdots \mu^*_1(\Sigma \setminus \{x\}) \) and \( \sigma_{x,1}(\mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \setminus \{x\}) = \mu^*_{z_{t-1}} \cdots \mu^*_1(\Sigma \setminus \{x\}) \), there exists \( w_t \) that is exchangeable in \( \mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \) such that \( \sigma_{x,1}(w_t) = z_t \). Using (ii) and \( \Sigma^* = \mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \), by the induction assumption, we have
\[
\sigma_{x,1}(\mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \setminus \{x\}) = \mu^*_{z_t} \sigma_{x,1}(\mu_{w_1} \cdots \mu_{w_{t-1}}(\Sigma) \setminus \{x\}) = \mu^*_{z_t} \mu^*_{z_{t-1}} \cdots \mu^*_1(\Sigma \setminus \{x\}).
\]

Lemma 6.3. If \( \mathcal{A}(\Sigma) \) is an acyclic rooted cluster algebra, then \( \mathcal{A}(\Sigma \setminus \{x\}) \) is acyclic for any \( x \in \bar{X} \).

Proof. Since \( \mathcal{A}(\Sigma) \) is acyclic, there exists a sequence \( (x_{i_1}, x_{i_2}, \ldots, x_{i_s}) \) with \( 1 \leq i_1 \leq n \) for \( 1 \leq i \leq s \) such that \( \mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(\Sigma) \) is acyclic. By calculation, it is obvious that
\[
\mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(\Sigma) \setminus \{ \mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(x) \} = \mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(\Sigma \setminus \{x\}),
\]
where the sequence \( (x_{j_1}, \ldots, x_{j_t}) \) is obtained by deleting \( x \) from the sequence \( (x_{i_1}, \ldots, x_{i_s}) \). If we denote \( \Sigma = (X, B) \), then
\[
\mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(\Sigma) \setminus \{ \mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(x) \} = \mu_{x_{j_1}} \cdots \mu_{x_{j_t}}(X) \setminus \{ \mu_{x_{i_1}} \cdots \mu_{x_{i_s}}(x) \},
\]
where \( \{ \mu_{x_{j_1}} \cdots \mu_{x_{j_t}}(x) \} = \mu_{x_{j_1}} \cdots \mu_{x_{j_t}}(\Sigma \setminus \{x\}) \).

It follows that
\[
\mu_{x_{j_1}} \cdots \mu_{x_{j_t}}(\Sigma \setminus \{x\}) = \mu_{x_{j_1}} \cdots \mu_{x_{j_t}}(B \setminus \{x\}).
\]

For a cluster algebra \( \mathcal{A} = \mathcal{A}(\Sigma) \), recall that the upper cluster algebra of \( \mathcal{A} \) is defined in [2] as
\[
\mathcal{U} := \bigcap_{\text{clusters } X \text{ of } \mathcal{A}} \mathbb{Z}[X^\pm],
\]
where \( \mathbb{Z}[P] \) is the coefficient ring of \( \mathcal{A} \). By this definition, \( \mathcal{U} \) is determined by all seeds of \( \mathcal{A} \). If a seed \( \Sigma \) of \( \mathcal{A} \) is given as initial, we denote the upper cluster algebra as \( \mathcal{U} = \mathcal{U}(\Sigma) \). So, for any other seed \( \Sigma^* \), we have \( \mathcal{U}(\Sigma) = \mathcal{U}(\Sigma^*) \).

Given any (initial) seed \( \Sigma = (X, B) \) of \( \mathcal{A} \), the upper bound of \( \mathcal{A}(\Sigma) \) on \( X \) is defined in [2] as
\[
\mathcal{U}_X(\Sigma) := \mathbb{Z}[P][X^\pm] \bigcap \bigcap_{x \in X} \mathbb{Z}[P][\mu_x X^\pm].
\]

By Laurent phenomenon and the definitions of upper cluster algebra and upper bound, clearly
\[
\mathcal{A} \subseteq \mathcal{U} = \bigcap_{\text{clusters } X \text{ of } \mathcal{A}} \mathcal{U}_X(\Sigma).
\]

Lemma 6.4. Let \( \xi = (X, B) \) be an initial seed, where \( x \in X \). Then \( \sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{U}(\Sigma \setminus \{x\}) \) is the upper cluster algebra of \( \mathcal{A}(\Sigma \setminus \{x\}) \).

Proof. By ([1], Proposition 6.13), \( \sigma_{x,1}(\mathcal{U}_X(\Sigma)) \subseteq \mathcal{U}_X(\Sigma \setminus \{x\}) \). Since \( \mathcal{A} \subseteq \mathcal{U}_X(\Sigma) \) and \( \mathcal{U}_X(\Sigma \setminus \{x\}) \subseteq \mathbb{Z}[P][X^\pm \setminus \{x^\pm\}] \), we have
\[
\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathbb{Z}[P][X^\pm \setminus \{x^\pm\}].
\]
For any cluster $X'$ of $\mathcal{A}(\Sigma \setminus \{x\})$, there exists a $\Sigma \setminus \{x\}$-admissible sequence $(z_1, \ldots, z_s)$ such that $\mu'_z \cdots \mu'_{z_1} (X \setminus \{x\}) = X'$. By Lemma 6.2(iii), $(z_1, \ldots, z_s)$ can be lifted to a $(\sigma_{x,1}, \Sigma \setminus \{x\})$-admissible sequence $(y_1, \ldots, y_s)$ and $\sigma_{x,1}(\mu_y \cdots \mu_{y_1}(\Sigma \setminus \{x\})) = \mu_z \cdots \mu_{z_1}(\Sigma \setminus \{x\})$. Hence, $\sigma_{x,1}(\mu_y \cdots \mu_{y_1}(X \setminus \{x\})) = \sigma_{x,1}(\mu_y \cdots \mu_{y_1}(X)) \setminus \{1\} = X'$, and then,

$$\sigma_{x,1}(\mathcal{ZP}([\mu_y \cdots \mu_{y_1}(X)]^{\pm1})) \subseteq \mathcal{ZP}[X'^{\pm1}].$$

By Laurent phenomenon,

$$\sigma_{x,1}(\mathcal{ZP}([\mu_y \cdots \mu_{y_1}(X)]^{\pm1})) \subseteq \mathcal{ZP}[X'^{\pm1}].$$

From (28) and (29), we have $\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{ZP}[X'^{\pm1}]$. Hence by the definition of upper cluster algebra and the arbitrariness of $X'$ as cluster in $\mathcal{A}(\Sigma \setminus \{x\})$, it follows that $\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{U}(\Sigma \setminus \{x\})$. \qed

**Lemma 6.5.** Given an initial seed $\Sigma = (X, \tilde{B})$, in case either $x \in X_{fr}$ or $\mathcal{A}(\Sigma \setminus \{x\})$ is acyclic, then (i) $\sigma_{x,1}(\mathcal{A}(\Sigma)) = \mathcal{A}(\Sigma \setminus \{x\})$ and (ii) $\sigma_{x,1}$ is surjective.

**Proof.** (ii) is trivial from (i). So, we only need to prove (i).

For any cluster variable $z \in \mathcal{A}(\Sigma \setminus \{x\})$, $z = \mu_{z_1} \cdots \mu_{z_t}(z_0)$ for some $\Sigma \setminus \{x\}$-admissible sequence $(z_1, \ldots, z_t)$ and $z_0 \in \tilde{X} \setminus \{x\}$. According to Lemma 6.2(iii), $(z_1, \ldots, z_t)$ can lift to a $(\sigma_{x,1}, \Sigma, \Sigma \setminus \{x\})$-admissible sequence $(y_1, \ldots, y_t)$. Thus, $\sigma_{x,1}(\mu_y \cdots \mu_{y_1}(y_0)) = \mu_{z_1} \cdots \mu_{z_t}(z_0) = z$, where $y_0 = z_0 \in \tilde{X} \setminus \{x\}$. Therefore, $\sigma_{x,1}(\mathcal{A}(\Sigma)) \supseteq \mathcal{A}(\mathcal{A}(\Sigma \setminus \{x\}))$ since $\mathcal{A}(\Sigma \setminus \{x\})$ is generated by all cluster variables.

In order to prove $\sigma_{x,1}(\mathcal{A}(\Sigma)) \subseteq \mathcal{A}(\mathcal{A}(\Sigma \setminus \{x\}))$, it is enough to claim $\sigma_{x,1}(\mu_{y_1} \cdots \mu_{y_2} \mu_{y_1}(y)) \in \mathcal{A}(\mathcal{A}(\Sigma \setminus \{x\}))$ for any composition of mutations $\mu_{y_1} \cdots \mu_{y_2} \mu_{y_1}$ and $y \in \tilde{X}$ since $\mathcal{A}(\Sigma \setminus \{x\})$ is generated by all cluster variables.

Suppose $x \in X_{fr}$. Since $x$ is frozen, $y_i \neq x$ for any $i = 1, \ldots, n$.

When $n = 1$, it is clear by Lemma 6.2(i). Using induction on $n$ and by Lemma 6.2(ii), in the case for $n \geq 2$, we have

$$\sigma_{x,1}(\mu_y \cdots \mu_{y_1}(y)) = \left\{\begin{array}{ll} \mu'_y \cdots \mu'_y \mu_{y_1}(y), & \text{if } y \neq x; \\ 1, & \text{if } y = x. \end{array}\right.$$ 

which is certainly in $\mathcal{A}(\Sigma \setminus \{x\})$. Thus, the result follows.

Suppose $\mathcal{A}(\mathcal{A}(\Sigma \setminus \{x\}))$ is acyclic. Then by (23, Theorem 2), $\mathcal{A}(\mathcal{A}(\Sigma \setminus \{x\})) = \mathcal{U}(\Sigma \setminus \{x\})$. Thus, the result follows from Lemma 6.3. \qed

This lemma improves the results in [1] in some cases mentioned before Proposition 6.1 as the partial answer of the problem in [1] that whether $\sigma_{x,1}$ is surjective.

**Remark 6.6.** (1) The case $x \in X_{fr}$ has been mentioned in [1], 2.39.

(2) Following this lemma and Lemma 6.3, $\sigma_{x,1}$ is surjective if $\mathcal{A}(\Sigma \setminus \{x\})$ is acyclic.

Since $I_1 = I'_1 \cup I''_1$ for $I'_1 \subseteq X$ and $I''_1 \subseteq X_{fr}$, using Lemma 6.3 step-by-step at $x \in I_1$ and according to ([1] Proposition 6.9), $\sigma_{x,1}$ is an ideal surjective rooted cluster morphism. Since $\sigma_{x,1}$ is an algebra homomorphism, we have the following.

**Theorem 6.7.** For a seed $\Sigma$, if the rooted cluster algebra $\mathcal{A}(\Sigma)$ is acyclic, then for any $I_1 \subseteq \tilde{X}$, the pure sub-cluster algebra $\mathcal{A}(\Sigma_{0,I_1})$ is a rooted cluster quotient algebra of $\mathcal{A}(\Sigma)$ via $\sigma_{x,1} = \prod_{x \in I_1} \sigma_{x,1}: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma_{0,I_1})$, i.e. the specialisation of $\mathcal{A}(\Sigma)$ at $I_1$.

Trivially, Proposition 6.1 is a special case of this theorem.

By Theorem 6.7, we know that a part of the set of rooted cluster quotient algebras consists of some pure sub-cluster algebras of rooted cluster algebras.
Then the following remaining question is still open:

**Question:** In the case that the cluster algebra $\mathcal{A}(\Sigma \setminus \{x\})$ is cyclic, is $\sigma_{x,1}$ surjective or, equivalently, $\text{Im} \sigma_{x,1} \subseteq \mathcal{A}(\Sigma \setminus \{x\})$?

As a corollary of Theorem 6.7, we show the finite type and finite mutation type of mixing-type sub-rooted cluster algebras of a rooted cluster algebra as follows.

A (rooted) cluster algebra $\mathcal{A} = \mathcal{A}(\Sigma)$ is said to be of **finite type** if it has finite cluster variables, and to be of **finite mutation type** if it has finite number of exchange matrices up to similar permutation of rows and columns, under all mutations (see [9][10]). It is easy to see that a cluster algebra of finite type is always of finite mutation type.

**Corollary 6.8.** All mixing-type sub-rooted cluster algebras of a rooted cluster algebra of finite type (respectively, finite mutation type) are of finite type (respectively, finite mutation type).

**Proof.** Let $\mathcal{A}(\Sigma)$ be a rooted cluster algebra of finite type. According to ([10], Theorem 1.4), since it is of finite type, the corresponding Cartan matrix of $\mathcal{A}(\Sigma)$ is of Dynkin type, which means that $\mathcal{A}(\Sigma)$ is acyclic. Then, $\mathcal{A}(\Sigma_{I_0, I_1})$ is also acyclic. Note that $\Sigma_{I_0, I_1} = (\Sigma_{I_0, I_1})_{I_0, I_1}$. By Theorem 6.7 there is a surjective morphism from $\mathcal{A}(\Sigma_{I_0, I_1})$ to $\mathcal{A}(\Sigma_{I_0, I_1})$ in $\text{Clus}$. Hence, in order to show the finite type of $\mathcal{A}(\Sigma_{I_0, I_1})$, it is enough to prove by ([1], Corollary 6.3) that $\mathcal{A}(\Sigma_{I_0, I_1})$ is of finite type. By Theorem 4.4, $\mathcal{A}(\Sigma_{I_0, I_1})$ is a rooted cluster subalgebra of $\mathcal{A}(\Sigma)$. So, by ([1], Corollary 4.6(2)), all cluster variables of $\mathcal{A}(\Sigma_{I_0, I_1})$ are also in $\mathcal{A}(\Sigma)$. Then, $\mathcal{A}(\Sigma_{I_0, I_1})$ is of finite type since $\mathcal{A}(\Sigma)$ is so.

It is easy to see that $(\mu_x(\Sigma_{I_0, I_1}))_{I_0, I_1} = \mu_x(\Sigma_{I_0, I_1})$ for any $x \in X \setminus (I_0 \cup I_1)$. Therefore, to show that $\mathcal{A}(\Sigma_{I_0, I_1})$ is of finite mutation type, it suffices to prove that $\mathcal{A}(\Sigma_{I_0, I_1})$ is of finite mutation type. By [17], all mutations of $\Sigma_{I_0, I_1}$ are sub-seeds of some mutations of $\Sigma$. Hence, $\mathcal{A}(\Sigma_{I_0, I_1})$ is of finite mutation type since $\mathcal{A}(\Sigma)$ is of finite mutation type.

Furthermore, in the next sub-section, we give the characterization of rooted cluster quotient algebras of rooted cluster algebras, which are not sub-cluster algebras, through the method of gluing frozen variables.

### 6.2. Rooted cluster quotient algebras via gluing method.

In this section, we give another class of rooted cluster quotient algebras via gluing frozen variables.

For a surjective rooted cluster morphism $g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ and $I_1 = \{x \in \tilde{X} | g(x) \in \mathbb{Z}\}$, by Definition 3.2 we can define $\Sigma'(g) = (X(g), B(g))$, the contraction of $\Sigma$ under $g$. We will show that there is a surjective rooted cluster morphism $f : \mathcal{A}(\Sigma'(g)) \to \mathcal{A}(\Sigma')$ satisfying $f(X(g)) = X'$ and $f(X'(g)) = \tilde{X}'$.

First, we have a unique algebra homomorphism

\begin{equation}
(30) \quad f : \mathbb{Q}[X_{fr}^{(g)}]|X_{fr}^{(g)\pm 1}] \to \mathbb{Q}[X_{fr}'|X_{fr}'^{\pm 1}]
\end{equation}

such that $f(x) = g(x)$ for $x \in \tilde{X}(g)$ and $f(x^{-1}) = g(x)^{-1}$ for $x \in X(g)$. By Laurent phenomenon, $\mathcal{A}(\Sigma(g)) \subseteq \mathbb{Q}[X_{fr}^{(g)}]|X_{fr}^{(g)\pm 1}]$.

**Proposition 6.9.** Let $g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ be a surjective rooted cluster morphism. Define $f$ as above in (30) and restrict $f$ to $\mathcal{A}(\Sigma'(g))$. Then $f : \mathcal{A}(\Sigma'(g)) \to \mathcal{A}(\Sigma')$ is a surjective rooted cluster morphism satisfying $f(X(g)) = X'$ and $f(X'(g)) = \tilde{X}'$.

This surjective rooted cluster morphism $f$ is called the **contraction** of $g$.

**Proof.** First, we show that $f$ is a rooted cluster morphism. Obviously, $f$ satisfies CM1 and CM2. Now we prove that CM3 holds for $f$. For this, we need the following two claims.

**Claim 1:** If $(y_1, \cdots, y_t)$ is $\Sigma'(g)$-admissible, then it is $(f, \Sigma(g), \Sigma')$-biadmissible.
Comparing the sets $\mu(32)$

Therefore, (two sides of this formula, since $30$ MIN HUANG FANG LI YICHAO YANG

There exists a positive isomorphism over $I$

Denote $(33)$

By definition, we have

Case 2.

Following this discussion, comparing (31) and (32), we get that

By the definition of $\Sigma$,

Thus, $g(\prod_1)g(A_1) + g(\prod_2)g(A_2) = B_1 + B_2$. Comparing the two sides of this formula, since $g(\prod_1)$ are monomials over $\tilde{X}'$ and $g(A_i) \in \mathbb{Z}$, by the algebraic independence of $\tilde{X}'$, we discuss the following cases:

Case 1. $g(A_1)g(A_2) = 0$. Without loss of generality, assume $g(A_1) = 0$. Then $g(\prod_2) = 1$ and $g(A_2) = B_1 + B_2 = 2$.

Case 2. $g(A_1)g(A_2) \neq 0$. Then, we have:

(1) if $g(\prod_1)g(\prod_2) \neq 1$, then $g(A_1) = g(A_2) = 1$ and

(2) if $g(\prod_1) = g(\prod_2) = 1$, then $g(A_2) + g(A_2) = B_1 + B_2 = 2$.

Both (1) and (2) mean $g(\prod_1)g(A_1) + g(\prod_2)g(A_2) = g(\prod_1) + g(\prod_2)$.

Following this discussion, comparing (31) and (32), we get that

Now we need to prove the important relation on the new seed $\Sigma^{(g)}$, which is as follows.

**Lemma 6.10.** There exists a positive isomorphism $(\mu^{\Sigma}_y(\Sigma))^{(g)} \cong \mu^{\Sigma^{(g)}}_y(\Sigma^{(g)})$, where $y \in X \setminus I_1$.

**Proof.** Denote $(\mu^{\Sigma}_y(\Sigma))$ by $(Y', C')$, $(\mu^{\Sigma^{(g)}}_y(\Sigma^{(g)}))$ by $(Y', C')'$ and $\mu^{\Sigma^{(g)}}_y(\Sigma^{(g)})$ by $(Y'', C'')$.

By definition, we have

Comparing the sets $\tilde{Y}'$ and $\tilde{Y}''$, their elements can be in one-to-one correspondence with the identity map but the correspondence between $\mu^{\Sigma}_y(y)$ and $\mu^{\Sigma^{(g)}}_y(y)$. 
According to definition, we have
\[
c_{xz} = \begin{cases} 
  b_{xz} + \frac{|b_{yw} + b_{xz}||b_{yw}|}{2}, & \text{if } x \neq \mu_y(y) \neq z; \\
  -b_{xz}, & \text{if } x = \mu_y(y) \text{ or } z = \mu_y(y). 
\end{cases}
\]
\[
c'_{xz} = \begin{cases} 
  c_{xz}, & \text{if } g(w) \neq 0 \forall w \in I_1 \text{ adjacent to } x \text{ or } z; \\
  0, & \text{otherwise.} 
\end{cases}
\]
\[
e''_{xz} = \begin{cases} 
  b_{(g)}(g) + \frac{|b_{w}(g) + b_{xz}(g) + b_{yw}(g)|}{2}, & \text{if } x \neq \mu_y(y) \neq z; \\
  -b_{(g)y} \text{ or } -b_{(g)xz}, & \text{if } x = \mu_y(y) \text{ or } z = \mu_y(y). 
\end{cases}
\]

We will show \( \tilde{C}' = \tilde{C}'' \) in three cases, which are as follows:

(i) In case \( x \in X \setminus (I_1 \cup \{y\}) \) and \( z \in X \setminus (I_1 \cup \{y\}) \).

(1) If there exists no \( w \in \tilde{X} \) adjacent to \( y \) with \( g(w) = 0 \), then
\[
c'_{xz} = c''_{xz} = b_{xz} + \frac{|b_{yw} + b_{xz}||b_{yw}|}{2}, \text{ if } g(w) \neq 0 \forall w \in I_1 \text{ adjacent to } x \text{ or } z;
\]
otherwise.

(2) If there exists a \( w \in \tilde{X} \) adjacent to \( y \) with \( g(w) = 0 \), then from the formula \( [34] \), \( b_{yw}b_{xy} < 0 \), \( b_{yu}b_{wy} > 0 \) and \( b_{yx}b_{zy} < 0 \), we have
\[
c'_{xz} = c''_{xz} = \begin{cases} 
  b_{xz}, & \text{if } g(w) \neq 0 \forall \in I_1 \text{ adjacent to } x \text{ or } z; \\
  0, & \text{otherwise.} 
\end{cases}
\]

(ii) In Case \( z \in X \setminus (I_1 \cup \{y\}) \).

(1) If there exists no \( w \in \tilde{X} \) adjacent to \( y \) with \( g(w) = 0 \), then
\[
c'_{(\mu_y(y))z} = c''_{(\mu_y(y))z} = \begin{cases} 
  -b_{yz}, & \text{if } g(w) \neq 0 \forall w \in I_1 \text{ adjacent to } z; \\
  0, & \text{otherwise.} 
\end{cases}
\]

(2) If there exists a \( w \in \tilde{X} \) adjacent to \( y \) with \( g(w) = 0 \), then \( c'_{(\mu_y(y))z} = c''_{(\mu_y(y))z} = 0 \).

(iii) In case \( x \in X \setminus (I_1 \cup \{y\}) \), we have similarly \( c'_{x(\mu_y(y))} = c''_{x(\mu_y(y))} \).

According to the above discussion on \( c'_{xy} \) and \( c''_{xy} \), under the correspondence between the sets \( \tilde{Y}' \) and \( \tilde{Y}'' \), we get that \( \tilde{C}' = \tilde{C}'' \).

Hence, \( (\mu_y(y)(\Sigma))^{(g)} \cong (\mu_{(g)}(\Sigma))^{(g)} \).

Let us return to prove the proposition.

By Lemma \[6.13\] we have \( (\mu_y(y)(\Sigma))^{(g)} \cong (\mu_{(g)}(\Sigma))^{(g)} \). Since \((y_1, \ldots, y_t) \text{ is } (\Sigma)'\)-admissible, it means \((y_2, \ldots, y_t) \text{ is } (\mu_{(g)}(\Sigma))^{(g)}\)-admissible.

By Proposition \[3.7\] \( g : \mathcal{A}(\mu_y(y)(\Sigma)) \to \mathcal{A}(\mu_{(g)}(\Sigma)) \) is a surjective rooted cluster morphism. For this \( g \), using the induction assumption, \((y_2, \ldots, y_t) \text{ is } (f, \mu_{(g)}(\Sigma))^{(g)}\)-biadmissible. Therefore, \((y_1, \ldots, y_t) \text{ is } (f, \Sigma)(g)\)-biadmissible. Hence, the claim holds.

Claim 2: For a \((f, \Sigma)(g), \Sigma')\)-biadmissible \((y_1, \ldots, y_t)\), there exists uniquely a \((g, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, \ldots, x_t)\) such that
\[
f(\mu_{(g)}(\Sigma))^{(g)} \cdot \mu_{(g)}(\Sigma)(y) = g(\mu_{(g)}(\Sigma)(y)) \text{ and } g(x_i) = f(y_i)
\]
for all \(1 \leq i \leq t\) and any \(y \in X(g) \subseteq \tilde{X}\). Conversely, for a \((g, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, \ldots, x_t)\), there exists uniquely an \((f, \Sigma(g), \Sigma')\)-biadmissible \((y_1, \ldots, y_t)\) such that the relations in \[(34)\] are satisfied.
In the case $t = 1$, let $x_1 = y_1$. Then we have $g(x_1) = f(y_1)$, and by the formula (33), $f(\mu_{y_1}^{\Sigma(y)}(z)) = g(\mu_{x_1}^{\Sigma (g)}(z)) = \mu_{f(y_1)}(f(z))$ for any $z \in \widetilde{X}^g$. So, Claim 2 holds for $t = 1$.

Assume that Claim 2 holds in the case for $t - 1$. In the case for $t$, by Proposition 3.7, $g : \mathcal{A}(\mu_{x_1}^{\Sigma(\Sigma)}) \to \mathcal{A}(\mu_{g(x_1)}^{\Sigma(g)})$ is a surjective rooted cluster morphism. By (10), $(\mu_{x_1}^{\Sigma (g)}(\Sigma)) \cong \mu_{y_1}^{\Sigma(y)}(\Sigma^{(g)})$. Hence, by induction assumption, there exits a $(g, \mu_{x_1}(\Sigma), \mu_{g(x_1)(\Sigma^{(g)})})$-biadmissible sequence $(x_2, \ldots, x_t)$ such that $g(x_i) = f(x_i)$ for $2 \leq i \leq t$ and $g(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$ for all cluster variables $y$ in $\mu_{y_1}^{\Sigma(y)}(\Sigma^{(g)})$. Therefore, $(x_1, \ldots, x_t)$ is the $(g, \Sigma, \Sigma^{(g)})$-biadmissible sequence such that $g(x_i) = f(y_i)$ for all $1 \leq i \leq k$ and $f(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$ for all $z \in \widetilde{X}^{(g)}$.

Conversely, we can prove similarly for a $(g, \Sigma, \Sigma^{(g)})$-biadmissible sequence $(x_1, \ldots, x_t)$ that there exists uniquely a $(f, \Sigma^{(g)}, \Sigma^{(g)})$-biadmissible sequence $(y_1, \ldots, y_t)$ such that the relations in (34) are satisfied.

Hence, the claim follows.

Now, we prove the CM3 condition for $f$. For any $(f, \Sigma^{(g)}, \Sigma^{(g)})$-biadmissible sequence $(y_1, \ldots, y_t)$, by Claim 2, there exists uniquely $(g, \Sigma, \Sigma^{(g)})$-biadmissible sequence $(x_1, \ldots, x_t)$ such that $g(x_i) = f(y_i)$ for all $1 \leq i \leq t$ and $g(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$ for all $z \in \widetilde{X}^{(g)}$. By CM3 for $g$, $g(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = \mu_{g(x_1)}(\Sigma) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$. Then, $f(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = \mu_{f(y_1)}(\Sigma) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$ for all $z \in \widetilde{X}^{(g)}$. Hence, CM3 for $f$ follows.

To show $f : \mathcal{A}(\Sigma^{(g)}) \to \mathcal{A}(\Sigma^{(g)})$ is a ring homomorphism, it suffices to prove that $f(\mathcal{A}(\Sigma^{(g)})) \subseteq \mathcal{A}(\Sigma^{(g)})$ since $g$ is a ring homomorphism. In fact, for any cluster variable $y \in \mathcal{A}(\Sigma^{(g)})$, there exists a $\Sigma^{(g)}$-admissible sequence $(y_1, \ldots, y_t)$ and $y_0 \in \widetilde{X}^{(g)}$ such that $y = \mu_{y_1}^{\Sigma(y)}(\Sigma) \cdot \cdots \cdot \mu_{y_t}^{\Sigma(y)}(\Sigma)$. By Claim 1, any $\Sigma^{(g)}$-admissible sequence is $(f, \Sigma^{(g)}, \Sigma^{(g)})$-biadmissible. Then by CM3, $f(y) = \mu_{f(y_1)}(\Sigma) \cdot \cdots \cdot \mu_{f(y_t)}(\Sigma)$. By CM1 and the definition of $f$, $f(y)$ is a cluster variable of $\mathcal{A}(\Sigma^{(g)})$. So, $f(\mathcal{A}(\Sigma^{(g)})) \subseteq \mathcal{A}(\Sigma^{(g)})$.

Now, we verify that $f$ is surjective. As $\mathcal{A}(\Sigma^{(g)})$ is generated by all cluster variables, it suffices to prove that any cluster variable $z \in \mathcal{A}(\Sigma^{(g)})$ can be lifted to a cluster variable in $\mathcal{A}(\Sigma^{(g)})$.

We have $z = \mu_1 \cdots \mu_k(z_0)$ for some $\Sigma'$-admissible sequence $(z_1, \ldots, z_k)$ and $z_0 \in \widetilde{X}$. As $g$ is surjective, by [11], Proposition 6.2), there exists a $(g, \Sigma, \Sigma')$-biadmissible sequence $(x_1, \ldots, x_t)$ and $x_0 \in \widetilde{X}$ such that $g(\mu_{x_1}^{\Sigma}(\Sigma)) = \mu_{x_1}(\Sigma) = \mu_{x_1}(z_0) = z$. It is clear to see that $x_0 \in \widetilde{X} \setminus I_1$.

According to Claim 2, there exists a $(f, \Sigma, \Sigma')$-biadmissible sequence $(y_1, \ldots, y_t)$ such that $f(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = g(\mu_{x_1}^{\Sigma(g)}(\Sigma^{(g)}))$. So, it follows that $f(\mu_{y_1}^{\Sigma(y)}(\Sigma)) = z$. It means that $f$ is surjective.

In summary, we have shown that $f : \mathcal{A}(\Sigma^{(g)}) \to \mathcal{A}(\Sigma^{(g)})$ is a surjective rooted cluster morphism.

Last, by the definition of $f$, we have $f(X^{(g)}) = X'$ and $f(\widetilde{X}^{(g)}) = \widetilde{X}'$.

In Proposition 6.9, $f = g$ if $g$ is noncontractible, i.e. $I_1 = \emptyset$ and then $\Sigma^{(g)} = \Sigma$. By definition and Proposition 6.9 it is easy to see the following.

**Lemma 6.11.** A surjective rooted cluster morphism $g : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is noncontractible if and only if $g(X) = X'$ and $g(\widetilde{X}) = \widetilde{X}'$.

Owing to this and Proposition 6.9 in the sequel, in order to characterize the quotient from a surjective rooted cluster morphism $f$, we always assume that $f$ is noncontractible, that is, the conditions $f(X) = X'$ and $f(\widetilde{X}) = \widetilde{X}'$ are satisfied.

For two sets $U$ and $V$, $\varphi$ is called a partial map from $U$ to $V$ on a subset $W$ of $U$ if $\varphi : W \to V$ is a map, denoted as $\varphi_W : U \to V$. 
Definition 6.12. Given a seed $\Sigma = (X, X_{fr}, B)$, a subset $S \subseteq \bar{X}$ and a partial injective map $\varphi_S : \bar{X} \to \bar{X}$ satisfying $\varphi_S(S \cap X_{fr}) \subseteq X_{fr}$, we define a new seed $\Sigma_{\varphi_S} = (\bar{X}, \bar{B})$ (denoted as $\Sigma_{\varphi_S(S)}$) by gluing $S$ and $\varphi_S(S)$ under $\varphi_S$ as follows:

(i) For any $s \in S$, define a new variable $\bar{s}$, called the gluing variable of $s$ and $\varphi_S(s)$ on $\varphi_S$. For any subset $T \subseteq S$, let $\bar{T} = \{ \bar{s} \mid s \in T \}$, which is called the gluing set of $T$ and $\varphi_S(T)$ on $\varphi_S$.

(ii) Define $\bar{X} = (\bar{X} \setminus (S \cup \varphi_S(S))) \cup \bar{S}$ and $\bar{X} = (X \setminus (S \cup \varphi_S(S))) \cup \bar{S} \cap \bar{X}$ and its extended exchange matrix $\bar{B}$ to be a $\#X \times \#X$ matrix satisfying:

$$
\bar{b}_{\bar{z}_1 \bar{z}_2} = \begin{cases} 
  b_{z_1z_2}, & \text{if } z_1 \in X \setminus (S \cup \varphi_S(S)), z_2 \in \bar{X} \setminus (S \cup \varphi_S(S)); \\
  b_{y_1y_2}, & \text{if } z_1 = \bar{y}_1 \in S \cap \bar{X}, z_2 = \bar{y}_2 \in \bar{S}; \\
  b_{z_1y_2}, & \text{if } z_1 \in X \setminus (S \cup \varphi_S(S)), z_2 = \bar{y}_2 \in \bar{S}, y_2 = \varphi_S(y_2); \\
  b_{z_1y_2} + b_{z_1\varphi_S(y_2)}, & \text{if } z_1 \in X \setminus (S \cup \varphi_S(S)), z_2 = \bar{y}_2 \in \bar{S}, y_2 \neq \varphi_S(y_2); \\
  b_{y_1y_2}, & \text{if } z_1 = \bar{y}_1 \in S \cap \bar{X}, z_2 \in \bar{X} \setminus (S \cup \varphi_S(S)).
\end{cases}
$$

Following this definition, in the new seed $\Sigma_{\varphi_S} = (\bar{X}, \bar{B})$, we have $X_{fr} = (X_{fr} \setminus (S \cup \varphi_S(S))) \cup \bar{S} \cap X_{fr}$. It is easy to see that $\bar{B}$ is skew-symmetrizable.

Example 6.13. Let $Q$ be the quiver: $x_1 \xrightarrow{x_2} x_3$ and let $S = \{x_1\}$. Define $\varphi_S(x_1) = x_3$. Then $\Sigma(Q)_{\varphi_S} = \Sigma(Q')$, where $Q' : x_1 \xrightarrow{x_2} x_3$.

Remark 6.14. For $y_1, y_2 \in X_{fr}$ in $\Sigma$, define $S = \{y_1\}$ and $\varphi_S : \{y_1\} \to \{y_2\}$ to get $\varphi_S(S) = \{y_2\}$. Then, we get $\Sigma_{y_1y_2} = (\bar{X}, \bar{B})$, where $\bar{X} = X, \bar{X} = \bar{X} \setminus \{y_1, y_2\} \cup \{\bar{y}\}$ and its extended exchange matrix $\bar{B}$ is of $n \times (n + m - 1)$ satisfying $\bar{b}_{\bar{y}y} = b_{xy}$ and $\bar{b}_{\bar{y}x} = b_{xy} + b_{x\bar{y}}$ for all $x \in X$ and $y \in \bar{X} \setminus \{\bar{y}\}$. In this situation, we say $\bar{y}$ is the gluing variable of $y_1$ and $y_2$.

Let $\Sigma = (X, B)$ be a seed with $y_1$ and $y_2$ as two frozen variables. We can obtain a natural ring morphism $\pi : Q[X_{fr}][X^{\pm 1}] \to E(\Sigma_{y_1y_2})$ satisfying

$$
\pi(x) = x \forall x \in \bar{X}, x \neq x_1, x \neq y_2, \text{ and } \pi(y_1) = \pi(y_2) = \bar{y}.
$$

Restricting $\pi$ on $A(\Sigma)$, such morphism $\pi_0 = \pi|_{A(\Sigma)}$ is called the canonical morphism induced by gluing $y_1, y_2 \in X_{fr}$.

To illustrate our result, we need the following lemma.

Lemma 6.15. Let $y_1, y_2 \in X_{fr}$ in seed $\Sigma$ such that $b_{x_1y_1}b_{x_2y_2} \geq 0$ for all $x \in X$. Then for any exchangeable variable $z \in X$, there is a positive isomorphism $\mu_z(\Sigma) \cong (\mu_z(\Sigma))_{y_1y_2}$.

Proof. Denote $\mu_z(\Sigma_{y_1y_2}) = (Y, C)$ and $(\mu_z(\Sigma))_{y_1y_2} = (\bar{Y}, \bar{C})$. Then by definition,

$$
Y = X \setminus \{z\} \cup \{\mu_z(S_{y_1y_2})(z)\} = X \setminus \{z\} \cup \{\mu_z(S_{y_1y_2})(z)\}, \\
\bar{Y} = \bar{X} \setminus \{z, y_1, y_2\} \cup \{\mu_z(S_{y_1y_2})(z), \bar{y}\},
$$

Comparing the sets $\bar{Y}$ and $\bar{Y}$, their elements can be one-to-one corresponded by a bijection $h$ with the identity map but the correspondence between $\mu_z(S_{y_1y_2})(z)$ and $\mu_z(S_{y_1y_2})(z)$. 

\]
Then the fact below follows.

\[ \Sigma \]

According to the above expressions of \( \Sigma \), hence, it induces a rooted cluster isomorphism

\[ \pi : \Sigma \to \Sigma \]

with the gluing variable \( \bar{y} \).

On the other hand, for all \( x \in \overline{Y} \) and \( y \in \overline{Y} \), we have the entries of \( \overline{c}_{xy} \):

\[
\overline{c}_{xy} = \begin{cases} 
  b_{xy} + \frac{b_{xy} + b_{xy} | b_{xy} + b_{xy} |}{2}, & \text{if } y \neq \overline{y}, x \neq \mu^\Sigma_{z}(z) \neq y; \\
  -b_{xy}, & \text{if } y \neq \overline{y}, x \neq \mu^\Sigma_{z}(z) \text{ or } y = \mu^\Sigma_{z}(z); \\
  \sum_{i=1}^{2} (b_{xyi} + b_{xyi} | b_{xyi} + b_{xyi} |), & \text{if } y = \overline{y}, x \neq \mu^\Sigma_{z}(z) \neq y; \\
  -b_{xy1} - b_{xy2}, & \text{if } y = \overline{y}, x = \mu^\Sigma_{z}(z) \text{ or } y = \mu^\Sigma_{z}(z).
\end{cases}
\]

According to the above expressions of \( c_{xy} \) and \( \overline{c}_{xy} \), under the correspondence between the sets \( \bar{Y} \) and \( \overline{Y} \), we get \( \bar{C} = \overline{C} \). Hence, \( \mu_{z}(\Sigma_{y_{1}y_{2}}) \cong (\mu_{z}(\Sigma))_{y_{1}y_{2}} \).

With the above preparations, another class of rooted cluster quotient algebras is given as follows.

**Proposition 6.16.** Let \( y_{1}, y_{2} \in X_{fr} \) in the seed \( \Sigma \) with the gluing variable \( \bar{y} \). Then, \( A(\Sigma_{y_{1}y_{2}}) \) is a rooted cluster quotient algebra of \( A(\Sigma) \) under the canonical morphism \( \pi_{0} = \pi | A(\Sigma) \) if and only if \( b_{xy1}^{'}, b_{xy2}^{'}, b_{xy} \geq 0 \) for any exchangeable variable \( x \) in any seed \( \Sigma' \) mutation equivalent to \( \Sigma \) and its exchange matrix \( \overline{B} \).

**Proof.** “Only if”: Let \( \Sigma' = (X', B') \) be a seed mutation equivalent to \( \Sigma \). Hence, there exists a \( \Sigma \)-admissible sequence \( (z_{1}, \ldots, z_{s}) \) such that \( \Sigma' = \mu_{z_{s}} \cdots \mu_{z_{1}}(\Sigma) \). According to Proposition 6.7, \( \pi_{0} : A(\Sigma') \to A(\mu_{\pi_{0}(z_{s})} \cdots \mu_{\pi_{0}(z_{1})}(\Sigma_{y_{1}y_{2}})) \) is a rooted cluster morphism. Moreover, using Lemma 6.15, it is easy to see that \( \mu_{\pi_{0}(z_{s})} \cdots \mu_{\pi_{0}(z_{1})}(\Sigma_{y_{1}y_{2}}) \cong \Sigma_{y_{1}y_{2}} \) by a seed isomorphism \( h \); hence, it induces a rooted cluster isomorphism

\[ h : A(\mu_{\pi_{0}(z_{s})} \cdots \mu_{\pi_{0}(z_{1})}(\Sigma_{y_{1}y_{2}})) \to A(\Sigma_{y_{1}y_{2}}). \]

Then the fact below follows.

**Fact 6.17.** \( h_{\pi_{0}} : A(\Sigma') \to A(\Sigma_{y_{1}y_{2}}) \) is the rooted cluster morphism given by gluing variables \( y_{1} \) and \( y_{2} \).

In fact, when \( s = 1 \), from the seed isomorphism defined in Lemma 6.15, for \( x \in \mu_{z_{1}}(\overline{X}) = (\overline{X} \setminus \{z_{1}\}) \cup \{\mu_{z_{1}}^{\Sigma}(z_{1})\} \), we know

\[
h_{\pi_{0}}(x) = \begin{cases} 
  h(x) = x, & \text{if } x \in \overline{X} \setminus \{z_{1}, y_{1}, y_{2}\}; \\
  h(\overline{y}) = \overline{y}, & \text{if } x = y_{1} \text{ or } y_{2}; \\
  h(\mu_{\pi_{0}(z_{1} \cdots z_{1} y_{1} y_{2})}) = \mu_{z_{1}}^{\Sigma}(z_{1}), & \text{if } x = \mu_{z_{1}}^{\Sigma}(z_{1}).
\end{cases}
\]

Hence, \( h_{\pi_{0}} \) is the rooted cluster morphism given by gluing variables \( y_{1} \) and \( y_{2} \). Using the above mutation step by step, we know that \( h_{\pi_{0}} \) is the rooted cluster morphism given by gluing variables \( y_{1} \) and \( y_{2} \) for any \( s \).
In particular, when \( \pi_0 \) is surjective, then \( h \pi_0 \) is also surjective.

Now for any \( x \in X' \), it is clear that the sequence \( (x) \) of length 1 is a \((h \pi_0, \Sigma', \Sigma_{y_1y_2})\)-biadmissible sequence. By CM3, we have \( h \pi_0(\mu_x(x)) = \mu_{h \pi_0(x)}(h \pi_0(x)) \). Equivalently, we have

\[
(35) \quad h \pi_0( \prod_{y'_{xy} > 0, y \in \tilde{X}'} y^{b_{y'_{xy}}} + \prod_{y'_{xy} < 0, y \in \tilde{X}'} y^{-b_{y'_{xy}}} = \prod_{y'_{xy} > 0, y \in \tilde{X}'} y^{v_0(x)y'_{xy} + \prod_{y'_{xy} < 0, y \in \tilde{X}'} y^{-v_0(x)y'_{xy}},}
\]

Assuming \( b'_{xy_1} b'_{xy_2} < 0 \), without loss of generality, we may assume that \( b'_{xy_1} > 0 \) and \( b'_{xy_2} < 0 \). Therefore, by the definition of \( \pi_0 \) and the construction of \( h \), (35) becomes the equality:

\[
(36) \quad \prod_{y'_{xy} > 0, y \neq \tilde{y} \in \tilde{X}'} y^{b_{y'_{xy}}} y^{b_{y'_{xy}}y_{xy_1}} + \prod_{y'_{xy} < 0, y \neq \tilde{y} \in \tilde{X}'} y^{-b_{y'_{xy}}} y^{b_{y'_{xy}}y_{xy_2}} = \prod_{y'_{xy} > 0, y \in \tilde{X}'} y^{v_{xy_1}y'_{xy}} + \prod_{y'_{xy} < 0, y \in \tilde{X}'} y^{-v_{xy_2}y'_{xy}}.
\]

By the algebraical independence of \( \tilde{X} \) and the skew symmetrizability of \( \tilde{B} \), the right hand side of (36) can not include a cluster monomial divisor, but the left hand side of (36) has \( \tilde{w}^{-\min(b'_{xy_1}, b'_{xy_2})} \) as its divisor. It is a contradiction. Hence, the assumption \( b'_{xy_1} b'_{xy_2} < 0 \) is not true and we have \( b'_{xy_1} b'_{xy_2} \geq 0 \).

“\( \Pi' \)”: By definition, trivially \( \pi_0 \) satisfies the conditions CM1 and CM2 of root cluster morphism.

Now we need to prove the CM3 for \( \pi_0 \) with \( \pi_0(A(\Sigma)) \subseteq A(\Sigma_{y_1y_2}) \). First, we have

**Claim 1:**

(a) Any \( \Sigma \)-admissible sequence \((z_1, \cdots, z_s)\) is \((\pi_0, \Sigma, \Sigma_{y_1y_2})\)-biadmissible.

(b) \( \pi_0 \) satisfies CM3, that is, for any \((\pi_0, \Sigma, \Sigma_{y_1y_2})\)-biadmissible sequence \((z_s, \cdots, z_1)\) and \( y \in \tilde{X} \),

\[
\pi_0(\mu_{z_s} \cdots \mu_{z_1}(y)) = \mu_{\pi_0(z_1) \cdots \pi_0(z_s)}(\pi_0(y)).
\]

When \( s = 1 \), since \( z_1 \in X \), \((z_1)\) is a \( \Sigma \)-admissible sequence and \( \pi_0(z_1) = z_1 \) by definition of \( \pi_0 \). Then, \((\pi_0(z_1))\) is \( \Sigma_{y_1y_2} \)-admissible, and thus, \((z_1)\) is \((\pi_0, \Sigma, \Sigma_{y_1y_2})\)-biadmissible. Moreover, as \( b_{xy_1}, b_{xy_2} \geq 0 \) for all \( x \in X \), without loss of generality assume that \( b_{z_1y_1} \geq 0, b_{z_1y_2} \geq 0 \). For any \( y \in \tilde{X} \), we have

\[
\pi_0(\mu_{z_1}(y)) = \left\{ \begin{array}{ll}
\prod_{t \in X, b_{z_1t} \geq 0} \pi_0(t^{b_{z_1t}}) + \prod_{t \in X, b_{z_1t} < 0} \pi_0(t^{-b_{z_1t}}), & \text{if } y = z_1; \\
\pi_0(y), & \text{if } y \neq z_1;
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\prod_{t \in X, b_{z_1t} \geq 0} t^{v_{z_1y}b_{z_1t} + b_{z_1y} + b_{z_1y} + b_{z_1yt} + b_{z_1yt} + b_{z_1yt} + b_{z_1yt}}, & \text{if } y = z_1; \\
\pi_0(y), & \text{if } y \neq z_1.
\end{array} \right.
\]

On the other hand, we have

\[
\mu_{\pi_0(z_1)}(\pi_0(y)) = \left\{ \begin{array}{ll}
\prod_{t \in X, b_{z_1t} > 0} \pi_0(t^{v_{z_1y}b_{z_1t}}) + \prod_{t \in X, b_{z_1t} < 0} \pi_0(t^{-v_{z_1y}b_{z_1t}}), & \text{if } \pi_0(y) = \pi_0(z_1); \\
\pi_0(y), & \text{if } \pi_0(y) \neq \pi_0(z_1);
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\prod_{t \in X, b_{z_1t} > 0} \pi_0(t^{v_{z_1y}b_{z_1t}}) + \prod_{t \in X, b_{z_1t} < 0} \pi_0(t^{-v_{z_1y}b_{z_1t}}), & \text{if } y = z_1; \\
\pi_0(y), & \text{if } y \neq z_1.
\end{array} \right.
\]

As \( \tilde{v}_{z_1y} = b_{z_1y} \) for \( y \neq \tilde{y} \) and \( \tilde{v}_{z_1y} = b_{z_1y} + b_{z_1z_2} \) for \( y = \tilde{y} \), therefore, we get \( \pi_0(\mu_{z_1}(y)) = \mu_{\pi_0(z_1)}(\pi_0(y)) \) for all \( y \in \tilde{X} \).

Assume that Claim 1 holds for \( s - 1 \). Similar to Fact 6.17, it can be proved straightforwardly that

\[
(37) \quad A(\mu_{z_1}, \Sigma) \xrightarrow{\ddot{\gamma}} F(\mu_{\pi_0(z_1)}(\Sigma_{y_1y_2})) \xrightarrow{\ddot{\gamma}} F((\mu_{z_1}, \Sigma)_{y_1y_2})
\]
is a ring homomorphism obtained by gluing variables $y_1$ and $y_2$ with $h$. The field isomorphism induced by the seed isomorphism $\mu_{\pi(z_1)}(\Sigma_{y_1y_2}) \rightarrow (\mu_{\pi(z_1)})_{y_1y_2}$ in Lemma 6.15 in particular, $A(\mu_{\pi(z_1)}(\Sigma_{y_1y_2})) \rightarrow A((\mu_{\pi(z_1)})_{y_1y_2})$ is a rooted cluster isomorphism. Therefore, by the induction assumption, $(z_2, \ldots, z_s)$ is $(h\pi_0, \mu_{\pi_0}(\Sigma), (\mu_{\pi_0(z_1)}(\Sigma))_{y_1y_2})$-biadmissible and

$$h\pi_0(\mu_{z_s} \cdots \mu_{z_2}(y)) = \mu_{\pi_0(z_s)} \cdots \mu_{\pi_0(z_2)}(h\pi_0(y))$$

for all $y \in \mu_{z_2}(X)$, and since $h$ is a seed isomorphism, $(z_2, \ldots, z_s)$ is $(\pi_0, \mu_{z_2}(\Sigma), \mu_{z_1}(\Sigma_{y_1y_2}))$-biadmissible. Then $\pi_0(\mu_{z_2} \cdots \mu_{z_1}(y)) = \mu_{\pi_0(z_s)} \cdots \mu_{\pi_0(z_1)}(\pi_0(y))$ for all $y \in \mu_{z_2}(X)$. Therefore, $(z_1, \ldots, z_s)$ is $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible and $\pi_0(\mu_{z_2} \cdots \mu_{z_1}(y)) = \mu_{\pi_0(z_s)} \cdots \mu_{\pi_0(z_1)}(\pi_0(y))$ for all $y \in X$.

In summary, we have shown $\pi_0(\Sigma) \subseteq A(\Sigma_{y_1y_2})$. For any cluster variable $z$ in $A(\Sigma)$, there exists a $\Sigma$-admissible sequence $(z_1, \ldots, z_s)$ and $z_0 \in X$ such that $\mu_{z_2} \cdots \mu_{z_1}(z_0) = z$. According to Claim 1, $(z_1, \ldots, z_s)$ is $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible, and by CM3, we have $\pi_0(\mu_{z_2} \cdots \mu_{z_1}(y)) = \mu_{\pi_0(z_s)} \cdots \mu_{\pi_0(z_1)}(\pi_0(y)) \in A(\Sigma_{y_1y_2})$. It follows that $\pi_0(z) \in A(\Sigma_{y_1y_2})$.

In summary, we have shown $\pi_0 = \pi|_{A(\Sigma)}$ to be a rooted cluster morphism.

Last, we need to prove that $\pi_0$ is surjective on $A(\Sigma_{y_1y_2})$. For this, we only need to show that $\pi_0(\Sigma) \supseteq A(\Sigma_{y_1y_2})$.

**Claim 2:** Any $\Sigma_{y_1y_2}$-admissible sequence $(w_1, \ldots, w_t)$ can be lifted to a $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible sequence $(z_1, \ldots, z_s)$.

For $t = 1$, since $w_1 \in X = X$, let $z_1 = w_1$. Then, $\pi_0(z_1) = w_1$ by definition of $\pi$ and $(z_1)$ is $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible.

Assume that Claim 2 holds for $t - 1$.

By (37), $h\pi_0 : A(\mu_{z_1}(\Sigma)) \rightarrow A((\mu_{z_1})_{y_1y_2})$ is the rooted cluster morphism given by gluing variables $y_1$ and $y_2$, where $h : A((\mu_{w_1}(\Sigma_{y_1y_2})) \rightarrow A((\mu_{z_1})_{y_1y_2})$ is the rooted cluster isomorphism induced by the seed isomorphism $h$.

Therefore, by induction assumption, $(h(w_2), \ldots, h(w_s))$ can be lifted to a $(h\pi_0, \mu_{z_1}(\Sigma), \mu_{z_1}(\Sigma_{y_1y_2}))$-biadmissible sequence $(z_2, \ldots, z_s)$, as $h$ is a rooted cluster isomorphism. Hence, we have $\pi_0(z_i) = w_i$ for $2 \leq i \leq s$. Therefore, $(w_1, \ldots, w_s)$ can be lifted to a $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible sequence $(z_1, \ldots, z_s)$. Hence, the claim follows.

For any cluster variable $w \in A(\Sigma_{y_1y_2})$, there exists a $(\Sigma_{y_1y_2})$-admissible sequence $(w_1, \ldots, w_s)$ and $w_0 \in X$ such that $w = \mu_{w_s} \cdots \mu_{w_1}(w_0)$. According to Claim 2, $(w_1, \ldots, w_s)$ can be lifted to a $(\pi_0, \Sigma, \Sigma_{y_1y_2})$-biadmissible sequence $(z_1, \ldots, z_s)$. It is clear that there exists $z_0 \in X$ such that $\pi_0(z_0) = w_0$. Thus, by CM3, we have $w = \mu_{w_s} \cdots \mu_{w_2}(w_0) = \pi_0(z_2) \cdots \pi_0(z_0)$. Hence, $\pi_0$ is surjective follows.

This proposition tells us the condition for two frozen variables $y_1$ and $y_2$ to be glued so as to make the canonical morphism $\pi_0$ to be surjective. So, we define two frozen variables $y_1$ and $y_2$ of a rooted cluster algebra $A(\Sigma)$ to be **glueable** if the condition of Proposition 6.10 is satisfied. The following Lemma 6.20 (i) can be thought as another characterization for two frozen variables $y_1$ and $y_2$ in $\Sigma$ to be glueable via noncontractible rooted cluster morphisms.

**Lemma 6.18.** For $y_1 \neq y_2 \in X$, if $f(y_1) = f(y_2) \in X'$ for a rooted cluster morphism $f : A(\Sigma) \rightarrow A(\Sigma')$, then $y_1, y_2 \in X_{fr}$.

**Proof.** If $y_1 \notin X_{fr}$, then $y_1 \in X$. According to CM3, we have $\mu_{f(y_1)}(f(y_2)) = f(\mu_{y_1}(y_2)) = f(y_2)$. Then, $f(y_2) = \mu_{f(y_1)}(f(y_2))$. However, it is impossible due to the definition of mutation.

**Lemma 6.19.** For a surjective noncontractible rooted cluster morphism $g : A(\Sigma) \rightarrow A(\Sigma')$,

(i) $g(X_{fr}) \subseteq X'_{fr}$;
(ii) \( \# \tilde{X} \geq \# \tilde{X}' \);

(iii) if \( \# \tilde{X} \geq \# \tilde{X}' \), then there exist \( y_1, y_2 \in X_{fr} \) with \( y_1 \neq y_2 \) such that \( g(y_1) = g(y_2) \) and

(iv) \( \# \tilde{X} = \# \tilde{X}' \) if and only if \( \# X = \# X' \) and \( X_{fr} = X'_{fr} \).

**Proof.** (i) Otherwise, there exists a \( y \in X_{fr} \) such that \( g(y) \in X' \). Then by (Lemma 3.1, [1]), we have an \( x \in X \) such that \( g(y) = g(x) \in X' \). By CM3, we have \( g(\mu_x(y)) = \mu_g(y)(g(y)) \), however, which is impossible since \( g(\mu_x(y)) = g(y) \neq \mu_g(y)(g(y)) = \mu_g(x)(g(y)) \).

(ii) Since \( g \) is noncontractible, we have \( g(\tilde{X}) \subseteq X' \). Because \( g \) is surjective, \( g(\tilde{X}) = \tilde{X}' \). So, we have \( \# \tilde{X} = \# g(\tilde{X}) = \# \tilde{X}' \).

(iii) By Lemma 6.18 the images of any two various exchangeable cluster variables under \( g \) are always various. Thus, \( \# X = \# g(X) \leq \# X' \). Moreover, since \( \# X + \# X_{fr} = \# \tilde{X} \geq \# \tilde{X}' = \# X' + \# X'_{fr} \), so we have \( \# X_{fr} \leq \# X'_{fr} \). Thus, there always exist \( y_1, y_2 \in X_{fr} \) such that \( g(y_1) = g(y_2) \).

(iv) By (Lemma 3.1, [1]), \( X' \subseteq g(X) \). Moreover, since \( g \) is noncontractible, we have \( g(X) \subseteq X' \). Thus, \( g(X) = X' \). By Lemma 6.18 \( g|_X \) is injective. Thus, \( \# X = \# X' \). \( \Box \)

**Lemma 6.20.** (i) For a seed \( \Sigma \), two frozen variables \( y_1 \) and \( y_2 \) in \( \Sigma \) are glueable if and only if there is another seed \( \Sigma' \) and a noncontractible surjective rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) such that \( f(y_1) = f(y_2) \).

(ii) In the situation of (i), the noncontractible surjective rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) can be decomposed into \( f = h_1 f_1 \) for \( f_1 = \pi_0 \), a surjective canonical morphism, and another surjective rooted cluster morphism \( h_1 : \mathcal{A}(\Sigma_{g(y_2)}) \to \mathcal{A}(\Sigma') \).

**Proof.** (i) “Only If”: By the definition of “glueable” and Proposition 6.16 it follows immediately that \( \Sigma' = \Sigma_{g(y_2)} \) and \( f = \pi_0 \).

“If”: To show \( \pi_0 \) is a surjective rooted cluster morphism, by Proposition 6.16 it suffices to prove that \( c_{xy_1} c_{xy_2} \geq 0 \) for any exchangeable variable \( x \) in any seed \( \mu_z \cdots \mu_z(\Sigma) = (Y, \tilde{C}) \) obtained through mutations.

According to Claim 1 in the proof of Proposition 6.9 \( (z_1, \cdots, z_s) \) is \( (f, \Sigma, \Sigma') \)-biadmissible. By Proposition 6.24 \( f : \mu_z \cdots \mu_z(\Sigma) \to \mu_f(z_1) \cdots \mu_f(z_1)(\Sigma') \) is a rooted cluster morphism. It is clear that \( f(Y) = \mu_f(z_1) \cdots \mu_f(z_1)(X') \) and \( f(Y) = \mu_f(z_1) \cdots \mu_f(z_1)(X') \). For any \( x \in Y \), by CM3, we have \( f(\mu_x(x)) = \mu_f(x)(f(x)) \). Equivalently,

\[
\sum_{y \in Y, c_{xy} > 0} f \left( y^{c_{xy}} + \prod_{y \in Y, c_{xy} < 0} y^{-c_{xy}} \right) = \sum_{x \in X, c_{xy} > 0} f(x) \prod_{x \in X, c_{xy} < 0} x^c_{xy} = \prod_1 + \prod_2,
\]

where \( \prod_1 \) and \( \prod_2 \) are coprime cluster monomials in \( \mu_f(z_1) \cdots \mu_f(z_1)(\Sigma') \); hence, there is no non-trivial divisor in the right hand side of (38). Thus, it can be seen that for the given \( y_1, y_2 \) in (1), we have \( c_{xy_1} c_{xy_2} \geq 0 \), since otherwise, there is a non-trivial divisor \( f(y_1)^{\text{min}(|c_{xy_1}|, |c_{xy_2}|)} \). Hence, \( \pi_0 \) is a surjective.

(ii) From (i), \( f_1 = \pi_0 \) is a surjective rooted cluster morphism.

Owing to \( f(\tilde{X}) = \tilde{X}' \), let \( h_1 : \mathcal{A}(\Sigma_{g(y_2)}) \to \mathcal{F}(\Sigma') \) be the unique ring homomorphism by defining \( h_1(x) = f(x) \) for any \( x \in \tilde{X} \setminus \{7\} \) and \( h_1(7) = f(y_1) = f(y_2) \), where \( 7 \) is the gluing variable in \( \Sigma_{g(y_2)} \). We will see below that \( h_1 : \mathcal{A}(\Sigma_{g(y_2)}) \to \mathcal{A}(\Sigma') \) is a surjective rooted cluster morphism satisfying \( h_1(\tilde{X}) = X' \) and \( h_1(\tilde{X}) = \tilde{X}' \).

It is clear that \( h_1(\tilde{X}) = X' \) and \( h_1(\tilde{X}) = \tilde{X}' \), so CM1 and CM2 hold for \( h_1 \). For any \( \Sigma_{g(y_2)} \)-admissible sequence \( (w_1, \cdots, w_s) \), by Proposition 6.19 there exists uniquely a \( (f_1, \Sigma, \Sigma_{g(y_2)}) \)-biadmissible sequence \( (z_1, \cdots, z_s) \) such that \( f_1(z_i) = w_i \) for \( i = 1, \cdots, s \).
Claim: $f(z_i) = h_1(w_i)$ for $i = 1, \ldots, s$ and $h_1(\mu_{w_2} \cdots \mu_{w_1}(w)) = f(\mu_{z_2} \cdots \mu_{z_1}(z))$ for all $w \in \overline{X}$, where $z \in \overline{X}$ such that $f_1(z) = w$.

In case $s = 1$, we have $z_1 = w_1$; hence, $f(z_1) = h_1(z_1)$ by the definition of $h_1$. Without loss of generality, we may assume that $b_{w_1y} \geq 0$. Equivalently, $b_{w_1y_1}, b_{w_1y_2} \geq 0$. Hence, we have

$$h_1(\mu_{w_1}(w)) = \begin{cases} \prod_{t \in \overline{X}, b_{w_1t} > 0} h_1(t^{w_1}) \prod_{t \in \overline{X}, b_{w_1t} < 0} h_1(t^{-w_1}), & \text{if } w = w_1; \\ h_1(w), & \text{if } w \neq w_1; \\ f(w), & \text{if } y \neq w \neq w_1; \\ f(y_1), & \text{if } y = y \neq w_1, \\ f(y), & \text{if } y = y. \\ \end{cases}$$

On the other hand,

$$f(\mu_{w_1}(z)) = \begin{cases} \prod_{t \in \overline{X}, b_{w_1t} > 0} f(t^{w_1}) \prod_{t \in \overline{X}, b_{w_1t} < 0} f(t^{-w_1}), & \text{if } z = w_1; \\ f(z), & \text{if } z \neq w_1. \\ \end{cases}$$

Thus, for all $w \in \overline{X}$ and $z \in \overline{X}$ such that $h_1(z) = w$, we have $h_1(\mu_{w_1}(w)) = f(\mu_{z_1}(z))$.

Assume that the Claim holds for $s - 1$. By Fact 6.17, $h_{f_1} : A(\mu_{w_1}(\Sigma)) \rightarrow A(\mu_{w_2}(\Sigma))_{y_1y_2}$ is the surjective rooted cluster morphism obtained by gluing $y_1$ and $y_2$, where $h : A(\mu_{z_1}(\Sigma))_{y_1y_2} \rightarrow A(\mu_{z_1}(\Sigma))_{y_1y_2}$ is the rooted cluster isomorphism induced by the corresponding seed isomorphism defined in Lemma 4.15. Similarly, it can be seen that $h_1h^{-1} : A(\mu_{z_1}(\Sigma))_{y_1y_2} \rightarrow A(\mu_{f(z_1})(\Sigma'))$ is the ring homomorphism induced by the surjective rooted cluster morphism $f : A(\mu_{z_1}(\Sigma)) \rightarrow A(\mu_{f(z_1)}(\Sigma'))$.

Since $(h(w_2), \ldots, h(w_s))$ is $(\mu_{z_1}(\Sigma))_{y_1y_2}$-admissible, it can be lifted to a $(h_{f_1}, \mu_{z_1}(\Sigma), (\mu_{z_1}(\Sigma))_{y_1y_2})$-admissible sequence $(z_2, \ldots, z_s)$. Therefore, by induction assumption, we have $f(z_i) = (h_1h^{-1})(h(w_i)) = h_1(w_i)$ for $2 \leq i \leq s$ and

$$h_1(\mu_{w_2} \cdots \mu_{w_1}(w)) = (h_1h^{-1})(\mu_{h(w_1)} \cdots \mu_{h(w_2)}(h(w))) = f(\mu_{z_2} \cdots \mu_{z_1}(z))$$

for all $w \in \overline{w_1X}$ and $z \in \overline{z_1X}$ such that $f_1(z) = w$. For $w \in \overline{X}$ and $z \in \overline{X'}$ such that $f_1(z) = w$, we have $f_1(h(w_1))(z) = \mu_{z_2}(f_1(z))$. Thus, for $1 \leq i \leq s$, $f(z_i) = h_1(w_i)$ and $h_1(\mu_{w_2} \cdots \mu_{w_1}(w)) = f(\mu_{z_2} \cdots \mu_{z_1}(z))$ for all $w \in \overline{X}$, where $z \in \overline{X}$ such that $f_1(z) = w$. Hence, claim is proved.

Let $(w_1, \ldots, w_s)$ be $\Sigma_{y_1y_2}$-admissible, so there exists $u \in \overline{X} = \overline{X'}$ such that $w_i = \mu_{w_{i-1}} \cdots \mu_{w_1}(u_i)$ for each $1 \leq i \leq s$. Let $(z_1, \ldots, z_s)$ be the $(f_1, \Sigma, \Sigma_{y_1y_2})$-biadmissible sequence such that $f_1(z_1) = w_i$. Since $f(\overline{X}) = \overline{X'}$, it is easy to see that $(z_1, \ldots, z_s)$ is $(f, \Sigma, \Sigma')$-biadmissible. According to the above claim, we have

$$h_1(w_i) = h_1(\mu_{w_{i-1}} \cdots \mu_{w_1}(u_i)) = f(\mu_{z_{i-1}} \cdots \mu_{z_1}(u_i)) = f_1(z_{i-1}) \cdots f_1(z_1)(f(u_i)).$$

As $f(u_i) \in X'$, so $f_1(w_i)$ is exchangeable in $\mu_{f(z_{i-1})} \cdots \mu_{f(z_1)}(\Sigma')$ for each $1 \leq i \leq s$. Hence, $(w_1, \ldots, w_s)$ is $(h_1, \Sigma_{y_1y_2}, \Sigma')$-biadmissible.

Now, we prove that $h_1$ satisfies CM3. From the discussion above, any $\Sigma_{y_1y_2}$-admissible sequence $(w_1, \ldots, w_s)$ is $(h_1, \Sigma_{y_1y_2}, \Sigma')$-biadmissible. For any $w \in \overline{X}$, let $z \in \overline{X}$ such that $f_1(z) = w$. By the above claim, we have

$$h_1(\mu_{w_2} \cdots \mu_{w_1}(w)) = f(\mu_{z_2} \cdots \mu_{z_1}(z)) = \mu_{f(z_2)} \cdots \mu_{f(z_1)}(f(z)) = \mu_{h_1(w_2)} \cdots \mu_{h_1(w_1)}(h_1(w)).$$

Thus, CM3 follows.
Second, \( h_1(\mathcal{A}(\Sigma_{g_{1/2}})) \subseteq \mathcal{A}(\Sigma') \) follows immediately from CM3 for \( h_1 \) and the fact that any \( \Sigma_{g_{1/2}} \)-admissible sequence is \( (h_1, \Sigma_{g_{1/2}}, \Sigma') \)-biadmissible.

Third, we show that \( h_1 \) is surjective. Any cluster variable \( v \) of \( \mathcal{A}(\Sigma') \) can be written as \( v = \mu_{v_1} \cdots \mu_{v_t}(v_0) \) for \( v_0 \in \Sigma' \). There exists a \((f, \Sigma, \Sigma')\)-biadmissible sequence \((z_1, \cdots, z_s)\) and \( z_0 \in \bar{X} \) such that \( f(z_i) = v_i \) for \( i = 0, \cdots, s \). According to Claim 1 in the proof of Proposition 6.16, \((z_1, \cdots, z_s)\) is \((f_1, \Sigma, \Sigma_{g_{1/2}})\)-biadmissible. Therefore, by the above Claim, we have
\[
\mu_{z_s} \cdots \mu_{z_1}(f_1(z_0)) = f(z).
\]

Thus, \( h_1 f_1 = f \) holds.

Following these lemmas, we can get the main conclusion.

**Theorem 6.21.** Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be a noncontractible surjective rooted cluster morphism and \( s = \#\bar{X} - \#\bar{X}' \). Then, either \( f = g_0 \) or \( f = g_s f_s \cdots f_2 f_1 \), \((s \geq 1)\) for a series of surjective rooted cluster morphisms:
\[
\mathcal{A}(\Sigma) \xrightarrow{f_1} \mathcal{A}(\Sigma_1) \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} \mathcal{A}(\Sigma_{s-1}) \xrightarrow{f_s} \mathcal{A}(\Sigma_s) \xrightarrow{g} \mathcal{A}(\Sigma'),
\]
where \( g_s \) is a rooted cluster isomorphism and each \( f_i \) is just the canonical morphism on \( \mathcal{A}(\Sigma_{i-1}) \) with \( \Sigma_i \) the seed given from \( \Sigma_{i-1} \) by gluing a pair of frozen cluster variables with the same images under \( f \) for \( i = 1, \cdots, s \) and \( \Sigma = \Sigma_0 \).

**Proof.** By Lemma 6.19 (ii), \( s = \#\bar{X} - \#\bar{X}' \geq 0 \).

If \( s = 0 \), then \( \#\bar{X} = \#\bar{X}' \). Therefore, \( f^S : \bar{X} \to \bar{X}' \) is a bijective map. For any \( x \in X \) and \( y \in \bar{X} \), by CM3, \( f(\mu_x(x)) = \mu_{f(x)}(f(x)) \), that is
\[
\left( \prod_{b_{xz} > 0, x \in X} z^{b_{xz}} + \prod_{b_{xz} < 0, x \in \bar{X}} z^{-b_{xz}} \right) f(\prod_{b'_{xz', w > 0, w \in \bar{X}'} z^{b'_{xz', w}} + \prod_{b'_{xz', w < 0, w \in \bar{X}'} z^{-b'_{xz', w}}}) = \prod_{b'_{f(x)w > 0, w \in \bar{X}'}} w^{b'_{f(x)w}} + \prod_{b'_{f(x)w < 0, w \in \bar{X}'}} w^{-b'_{f(x)w}} f(x).
\]
Comparing the exponent of \( f(y) \) in this equality, we have \( |b_{xy}| = |b'_{f(x)f(y)}| \). Hence, by Lemma 6.19 (iv) and Definition 2.11, \( f^S \) is a seed isomorphism. According to Proposition 6.8, \( g_0 = f \) is a rooted cluster isomorphism.

Fix a positive integer \( t \) and assume that the result holds for any \( s < t \). Consider the case \( s = t \). By Lemma 6.19 there exists \( y_1 \) and \( y_2 \in X_f \), with \( y_1 \neq y_2 \) such that \( f(y_1) = f(y_2) \). According to Lemma 6.20, \( h = h_1 f_1 \) for the surjective canonical morphism \( f_1 : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma_{g_{1/2}}) \) and another surjective rooted cluster morphism \( h_1 : \mathcal{A}(\Sigma_{g_{1/2}}) \to \mathcal{A}(\Sigma') \). Since \( \#\bar{X}_{g_{1/2}} = \#\bar{X} - 1 \), we have \( \#\bar{X}_{g_{1/2}} - \#\bar{X}' = t - 1 \). By the inductive assumption, \( h_1 = g_t f_t \cdots f_2 \) for a rooted cluster isomorphism \( g_t \), and for \( 2 \leq i \leq t, f_i \) are the surjective canonical morphisms on \( \mathcal{A}(\Sigma_{i-1}) \) with \( \Sigma_i \) the seed given from \( \Sigma_{i-1} \) by gluing a pair of cluster variables with the same images under \( f \) for \( i = 1, \cdots, s \) and \( \Sigma = \Sigma_0 \). Thus, the result holds.

**Corollary 6.22.** Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be a surjective rooted cluster morphism with \( 0 \notin f(\bar{X}) \) and \( \mathcal{A}(\Sigma) \) be acyclic. Denote \( I_1 = \{ x \in \bar{X} | f(x) \in \mathbb{Z} \} \). Then,

(i) A surjective rooted cluster morphism \( f' : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) can be uniquely constructed from \( f \) satisfying \( f'(x) = f(x) \) for \( x \in \bar{X} \setminus I_1 \) and \( f'(x) = 1 \) for \( x \in I_1 \), which is called the unitary morphism of \( f \) on \( I_1 \).
(ii) \( f' = f_0\sigma_{I_1,1} \) with two surjective rooted cluster morphisms \( \sigma_{I_1,1} : A(\Sigma) \rightarrow A(\Sigma_{\emptyset,I_1}) \) and \( f_0 : A(\Sigma_{\emptyset,I_1}) \rightarrow A(\Sigma') \), where \( \sigma_{I_1,1} \) is the specialisation of \( A(\Sigma) \) at \( I_1 \) and \( f_0 \) is the contraction of \( f \), which can be decomposed into a composition of a rooted cluster isomorphism \( g_* \) and a series of some surjective canonical morphisms \( f_1, \cdots, f_s \) obtained step-by-step by gluing a pair of frozen cluster variables with the same images under \( f \) for \( i = 1, \cdots, s \) as given in Theorem 6.21.

Proof. We first prove (ii). Since \( A(\Sigma) \) is acyclic, by Theorem 6.7, the specialisation \( \sigma_{I_1,1} = \prod_{x \in I_1} \sigma_x, 1 : A(\Sigma) \rightarrow A(\Sigma_{\emptyset,I_1}) \) is a surjective rooted cluster morphism. Let \( f_0 \) be the contraction of \( f \). Since \( 0 \not\in f(X) \), we have \( \Sigma(f) = \Sigma_{\emptyset,I_1} \) by Remark 3.3. By Proposition 6.9, \( f_0 : A(\Sigma_{\emptyset,I_1}) \rightarrow A(\Sigma') \) is a noncontractible surjective rooted cluster morphism. By Theorem 6.21, \( f_0 \) can be decomposed as required. Setting \( f' = f_0\sigma_{I_1,1} \), then (ii) follows immediately. We will show that this \( f' \) is just required in (i).

In fact, \( f' \) is a surjective rooted cluster morphism, since \( f_0 \) and \( \sigma_{I_1,1} \) are so. We have \( f'(x) = (f_0\sigma_{I_1,1})(x) = f_0(x) = f(x) \) for all \( x \in X \setminus I_1 \) and \( f'(x) = 1 \) for all \( x \in I_1 \) by the definitions of contraction and specialisation. The uniqueness of \( f' \) follows from Lemma 3.5.

Example 6.23. For two seeds \( \Sigma_1 \) and \( \Sigma_2 \), we have the rooted cluster algebras \( A(\Sigma_1 \sqcup \Sigma_2) \) and \( A(\Sigma_1 \coprod_{\Delta_1,\Delta_2} \Sigma_2) \) from the union seed and the amalgamated sum, respectively. Define a rooted cluster morphism \( f : A(\Sigma_1 \sqcup \Sigma_2) \rightarrow A(\Sigma_1 \coprod_{\Delta_1,\Delta_2} \Sigma_2) \) satisfying \( f(x) = x \) for all \( x \in (X_1 \sqcup X_2) \setminus (\Delta_1 \cup \Delta_2) \) and \( f(y') = f(y'') = \bar{y} \) the image variable in \( \Delta \) for a pair of corresponding variables \( y' \in \Delta_1 \) and \( y'' \in \Delta_2 \). Trivially, \( f \) is noncontractible and surjective. By Theorem 6.21, we can decompose \( f \) into \( f = g_1 f_1 \cdots f_s \sigma_1, 1 \) for surjective canonical morphisms \( f_i \) and a rooted cluster isomorphism \( g_* \). In this case, \( g_* = id_{A(\Sigma_1 \coprod_{\Delta_1,\Delta_2} \Sigma_2)} \). Assuming that all pairs of the corresponding variables from \( \Delta_1 \) and \( \Delta_2 \) are \( (y'_1, y''_1), \cdots, (y'_s, y''_s) \), then \( f \) can be obtained by gluing \( y'_i \) and \( y''_i \), i.e. \( f(y'_i) = f(y''_i) = \bar{y}_i \) for \( i = 1, \cdots, s \).

Acknowledgements: This project is supported by the National Natural Science Foundation of China (No.11271318 and No.11571173) and the Zhejiang Provincial Natural Science Foundation of China (No.LZ13A010001).

References

[1] I. Assem, G. Dupont, R. Schiffler, On a category of cluster algebras. J. Pure Appl. Alg. 218(3), 553-582, 2014.
[2] A. Berenstein, S. Fomin and A. Zelevinsky, Cluster Algebras III: Upper Bounds and Double Bruhat Cells, Duke Math J. Vol. 126, Number 1 (2005), 1-52.
[3] E. Brugallé and K. Shaw. A bit of tropical geometry. The American Mathematical Monthly, 121(7):pp. 563C589, 2014.
[4] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2) (2006) 572C618. MR2249625 (2007f:16033).
[5] P. Caldero and B. Keller, From triangulated categories to cluster algebras, Invent. Math. Vol. 172 (2008) 169-211.
[6] W. Chang and B. Zhu, On rooted cluster morphisms and cluster structures in 2-Calabi-Yau triangulated categories, arXiv:1410.5702.
[7] S. Fomin, Total positivity and cluster algebras. (English summary) Proceedings of the International Congress of Mathematicians. Volume II, 125-145, Hindustan Book Agency, New Delhi, 2010.
[8] S. Fomin, M. Shapiro, D. Thurston, Cluster algebras and triangulated surfaces. Part I: Cluster complexes. Acta Math. 201, 83-146(2008).
[9] S. Fomin, A. Zelevinsky, Cluster algebras I: Foundations. J. Amer. Math. Soc. 15, 497-529(2002).
[10] S. Fomin, A. Zelevinsky, Cluster algebras II: Finite type classification. Inven. Math. 154, 63-121 (2003).
[11] S. Fomin, A. Zelevinsky, Cluster algebras IV: Coefficients. Compositio Mathematica, 143(1): 112-164, 2007.
[12] G.Christof, B.Leclerc, J.Schroer, Rigid modules over preprojective algebras, Inventions Math. 165 (2006), 589-632.
[13] M.Gekhtman, M.Shapiro, A.Vainshtein, Cluster Algebras and Poisson Geometry. Math. Surveys and Monographs Vol.167, Amer. Math. Soc., Providence, Rhode Island (2010).
[14] M.Gross, P. Hacking, S. Keel, M. Kontsevich, Canonical bases for cluster algebras, arXiv:1411.1394v1 [math.AG], 2014.
[15] P.Hell, J.Nesetril, Graphs and Homomorphisms, Oxford Univ. Press Inc., New York, 2004.
[16] M.Huang, F.Li, On Structure of cluster algebras of geometric type, II: Green’s equivalences and paunched surfaces, Pure and Applied Mathematics Quarterly, to appear, 2016.
[17] M.Huang, F.Li, Unfolding of acyclic sign-skew-symmetric cluster algebras and applications to positivity and $F$-polynomials, preprint, 2016, see http://www.math.zju.edu.cn/teacherintroen.asp?userid=18
[18] B.Keller, Cluster algebras, quiver representations and triangulated categories. math.RT, arXiv:0807.1960.
[19] S.-J.Kang, M.Kashiwara, M.Kim, S.-J.Oh, Monoidal categorification of cluster algebras. math.RT, arXiv:1412.8106
[20] K.Y. Lee and R. Schiffler, Positivity for cluster algebras, Annals of Mathematics 182 (2015), 73-125
[21] F.Li, J.C.Liu, Y.C.Yang, Genuses of cluster quivers of finite mutation type, Pacific Journal of Mathematics, 269(1): 133-148 (2014).
[22] F.Li, J.C.Liu, Y.C.Yang, Non-planar cluster quivers from surface, Annals of Combinatorics, 18 (2014) 675C707.
[23] G.Muller, A=U for Locally acyclic Cluster algebras, Symmetry, Integrability and Geometry: Methods and Applications 10 (2014), 094.
[24] I.Salch, Exchange maps of cluster algebras. International Electronic J. of Algebra 16, 1-15 (2014).
[25] R.Schiffler, Cluster algebras and cluster categories. Lecture notes for the XVIII Latin American Algebra Colloquium, San Pedro Brazil, 2009.

Min Huang
Department of Mathematics, Zhejiang University (Yuquan Campus), Hangzhou, Zhejiang 310027, P.R.China
E-mail address: minhuang1989@hotmail.com

Fang Li
Department of Mathematics, Zhejiang University (Yuquan Campus), Hangzhou, Zhejiang 310027, P.R.China
E-mail address: fangli@zju.edu.cn

Yichao Yang
Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Quèbec, Canada
E-mail address: yichao.yang@usherbrooke.ca