Abstract for JSC

This paper reports on a new algorithm to compute the asymptotic solutions of a linear differential system. A feature of the algorithm is the ability to accommodate periodic coefficients.
A new algorithm for computing the asymptotic solutions of a class of linear differential systems

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1. Introduction

In this paper we discuss a new algorithm for estimating and improving error terms in the asymptotic solution of linear differential systems. We consider systems of the form

\[ Z'(x) = \rho(x)\{D + R(x)\}Z(x) \quad (a \leq x < \infty), \]  

where \( Z \) is an \( n \)-component vector, \( \rho \) is a real or complex scalar factor, \( D \) is a constant \( n \times n \) diagonal matrix,

\[ D = dg(d_1, \ldots, d_n) \]  

with distinct \( d_k \), and \( R \) is also an \( n \times n \) matrix whose entries tend to zero as \( x \to \infty \), that is, \( R(x) \to 0 \) as \( x \to \infty \).

If it is the case that \( \rho(x)R(x) \) is \( L(a, \infty) \), the Levinson asymptotic theorem (Eastham 1989, section 1.3); (Levinson 1948) states that there are solutions \( Z_k \) \((1 \leq k \leq n)\) of (1.1) such that

\[ Z_k(x) = \{e_k + \eta_k(x)\} \exp(d_k \int_a^x \rho(t)dt), \]  

where \( e_k \) is the unit coordinate vector in the \( k \)-direction and \( \eta_k(x) \to 0 \) as \( x \to \infty \). The size of the error term \( \eta_k \) is related to the size of \( R(x) \) as \( x \to \infty \), and therefore the accuracy of (1.3) can be improved if the perturbation \( R(x) \) can be improved — that is, made smaller in magnitude — as \( x \to \infty \). Under suitable conditions on \( \rho \) and \( R \), this improvement can be effected by applying a sequence of transformations to the solution vector \( Z \) in (1.1). Our algorithm is concerned with the implementation of this sequence of transformations.

In order to introduce the ideas involved, we consider the transformation

\[ Z = (I + P)W, \]  

where \( I \) is the identity matrix.
where $I$ is the $n \times n$ identity matrix, $dgP = 0$, and the off-diagonal entries of $P$ are defined by

$$PD - DP = R - dgR.$$  (1.5)

Thus, in terms of the $(i, j)$ entries in the matrices,

$$p_{ij} = r_{ij}/(d_j - d_i) \quad (i \neq j).$$  (1.6)

On substituting (1.4) into (1.1) and using (1.5), we have

$$W' = \rho\{\hat{D} + (I + P)^{-1}(RP - PdgR - \rho^{-1}P')\}W,$$  (1.7)

where

$$\hat{D} = D + dgR.$$  (1.8)

By (1.6), $P(x) \to 0$ as $x \to \infty$ and therefore it is clear that there are circumstances, to be detailed later, in which the perturbation term in the $W$-system has a smaller order of magnitude for large $x$ than the original perturbation $R$. Repetition of the process successively improves the perturbation term. It is to the final system in the process that the Levinson theorem (1.3) is applied, when a prescribed accuracy in the error term has been achieved.

The transformation back from the final system to the original system (1.1) yields an improvement of (1.3) in which the factor $e_k + \eta_k(x)$ is replaced by

$$\{I + P_0(x)\}{e_k + \eta_k(x)}$$  (1.9)

with a new $\eta_k$ which has the prescribed accuracy, and the matrix $P_0$ is generated explicitly by our algorithm. The terms in $P_0$ tend to zero as $x \to \infty$ and $\eta_k = o(P_0)$. Thus (1.9) provides explicit sub-dominant terms for the asymptotic solution of (1.1). In sections 2 and 3, we discuss the sequence of transformations and, in sections 4 and 5, we discuss the algorithm for the generation of the terms in $P_0$.

In a recent paper (Brown, Eastham, Evans & McCormack 1996), we consider a particular system of the form (1.1) which arises from the $n$-th order differential equation

$$y^{(n)}(x) - Q(x)y(x) = 0,$$  (1.10)

and we formulated an algorithm which implements a sequence of transformations of the type (1.4)–(1.6). The main emphasis in (Brown et al. 1996), however, is on the analytic and asymptotic implications of the transformations for the solution of (1.10) and for applications to spectral theory. Here, on the other hand, we wish to develop our algorithm from the point of view of symbolic algebra in the context of the general system (1.1). We also demonstrate the versatility of our procedure by applying it to other situations than the one covered in (Brown et al. 1996).

Finally in this introduction, we note that the origins of the transformation (1.3)–(1.6) lie in the work of Harris and Lutz (Harris & Lutz 1974) with more recent developments of these ideas by Eastham (Eastham 1986) (Eastham 1989, section 1.7). The nature of the matrix $I + P$ is that it is an explicit approximation to the matrix whose columns are
eigenvectors of $D + R$, these eigenvectors in general only being explicit in terms of $D$ and
$R$ when $n = 2$.

2. The sequence of transformations

We now define a sequence of transformations

$$Z_m = (I + P_m)Z_{m+1} \quad (m = 1, 2, \ldots) \quad (2.1)$$

of the type introduced in (1.4) – (1.9), the purpose of which is to produce differential
systems for the $Z_m$, similar to (1.1), but with the perturbation term successively improved.
The definition is almost the same as that given in (Brown et al. 1996, section 4) for the
particular system (1.1) which arises from (1.10), and so we shall be brief in this part of
the paper. A typical system in the process is

$$Z_m' = \rho(D_m + R_m)Z_m, \quad (2.2)$$

where $D_m$ is diagonal, with (1.1) being the case $m = 1$. The process ends at $m = M$ when
the perturbation $R_M$ has a pre-assigned accuracy in terms of its order of magnitude as
$x \to \infty$.

As already indicated by (1.7), $R_m$ will contain terms of different orders of magnitude as
$x \to \infty$, and it is the essence of our algorithm to identify and collate these terms according
to their size. Hence we write

$$R_m = V_m + E_m = V_{1m} + V_{2m} + \ldots + V_{\mu m} + E_m, \quad (2.3)$$

where

$$V_{km} = o(V_{jm}) \quad (x \to \infty, k > j) \quad (2.4)$$

and

$$E_m = o(V_{\mu m}) \quad (x \to \infty). \quad (2.5)$$

Here $E_m$ represents terms which are already of the pre-assigned accuracy, and they take
little part in the transformation process (2.1). The $V_{jm}$ represents terms which are not of
that accuracy, and they are successively replaced by smaller-order terms as we go through
the process. Also as indicated by (1.8), we take any diagonal terms in $V_{1m}$ over to $D_m$ in
(2.3). Thus we arrange that

$$\text{deg}V_{1m} = 0 \quad (2.6)$$

and we write

$$D_m = D + \Delta_m. \quad (2.7)$$

We substitute (2.1) into (2.2) to eliminate the dominant term $V_{1m}$ in (2.3) and to define
the resulting terms $V_{j,m+1}$ constructively in terms of the $V_{jm}$. Corresponding to (1.3), we
define $P_m$ by

$$P_mD - DP_m = V_{1m} \quad (2.8)$$
with \( \text{dg}P_m = 0 \). Then it is easily checked that (2.1) and (2.2) give

\[
Z_{m+1}' = \rho \{D_m + (I + P_m)^{-1}( - \rho^{-1} P'_m + T_m + V_{1m}P_m \\
+ (R_m - V_{1m})(I + P_m))\} Z_{m+1},
\]

where

\[
T_m = \Delta_m P_m - P_m \Delta_m.
\]

As in (Brown et al. 1996, section 4), we show that (2.9) can be expressed as

\[
Z_{m+1}' = \rho (D_{m+1} + R_{m+1}) Z_{m+1}
\]

where \( R_{m+1} \) has the form

\[
R_{m+1} = V_{1m+1} + E_{m+1} = V_{1,m+1} + \ldots + V_{\mu,m+1} + E_{m+1}
\]

as in (2.3)-(2.5), but with a different \( \mu \). To do this, we let \( U \) denote any of the terms on which \((I + P_m)^{-1}\) acts in (2.9). Then we write

\[
(I + P_m)^{-1} = I - P_m + P^2_m - \ldots + (-1)^\nu P^{\nu}_m + (-1)^{\nu+1}(I + P_m)^{-1}P^{\nu+1}_m
\]

where, for each \( U, \nu \) is chosen so that the product

\[
(I + P_m)^{-1}P^{\nu+1}_m U
\]

has a sufficiently small order of magnitude to be included with \( E_m \) and form part of \( E_{m+1} \).

Now we group together terms of the same order of magnitude and denote the dominant term by \( S_{m+1} \). We then obtain (2.12) (with \( S_{m+1} \) in place of \( V_{1,m+1} \)), where \( E_{m+1} \) has the pre-assigned accuracy and, by (2.8), \( S_{m+1} \) and the \( V_{j,m+1} \) are known explicitly in terms of the \( V_{jm} \). Then, finally, we obtain (2.11) and (2.12) by defining

\[
D_{m+1} = D_m + \text{dg}S_{m+1}
\]

and

\[
V_{1,m+1} = S_{m+1} - \text{dg}S_{m+1}.
\]

3. Orders of magnitude

The transformation process (2.1) is carried out for \( m = 1, 2, \ldots, M - 1 \) and, in order to express the process in terms of an algorithm which can be implemented in the symbolic algebra system Mathematica, it is necessary to specify more precisely the orders of magnitude involved. The starting point is \( m = 1 \) and, in (2.3), we suppose that

\[
V_1 = V_{11} + V_{21} + \ldots + V_{N1},
\]

where

\[
V_{j1}(x) = O(x^{-\theta_j}) \quad (1 \leq j \leq N)
\]

as \( x \to \infty \) and, corresponding to (2.4),

\[
0 < \theta_1 < \theta_2 < \ldots < \theta_N.
\]
A new algorithm for computing the asymptotic solutions of a class of linear differential systems

We assume that the $\theta_j$ in (3.2) are chosen to have their minimum possible values and, in practice, (3.2) represents the exact order of magnitude of $V_j$. We denote by $\sigma$ the set of positive numbers

$$\sigma = \{ n_1\theta_1 + n_2\theta_2 + \ldots + n_N\theta_N; \ n_1 \geq 1, \ n_2 \geq 0, \ldots, \ n_N \geq 0 \} \quad (3.3)$$

the $n_j$ being integers. It is possible that different values of the $n_j$ give the same number in $\sigma$ and, allowing for this, we denote the distinct numbers in $\sigma$ by $\sigma_1, \sigma_2, \ldots$ in increasing order.

Let us suppose that the pre-assigned accuracy represented by $E_m$ in (2.3) is expressed as

$$E_m(x) = O(x^{-K}) \quad (3.4)$$

for a given $K > 0$. Then we choose the integer $L$ so that

$$\sigma_L < K \leq \sigma_{L+1}. \quad (3.5)$$

The definition of $\sigma$ in (3.3) allows us to postulate orders of magnitude

$$V_{jm}(x) = O(x^{-\sigma_{m+j-1}}) \quad (3.6)$$

where we can allow the possibility that some of the $V_{jm}$ (even $V_1$) may be identically zero. To justify (3.4), we note first that $P_m = O(x^{-\sigma_m})$ by (2.8). Then, recalling the use of (2.13) in (2.9), we also have

$$P_m V_{jm} = O(x^{-r\sigma_m-\sigma_{m+j-1}}),$$

and again $r\sigma_m + \sigma_{m+j-1} \in \sigma$ by (3.3). Further, since the combination $r = 0$ and $j = 1$ does not occur together here, we have

$$r\sigma_m + \sigma_{m+j-1} \geq \sigma_{m+1}.$$ 

The term $T_m$ in (2.9) is treated similarly. A simple induction argument on $m$ now establishes (3.6) for all $j$ and $m$, provided only that we add a suitable hypothesis on the term $\rho^{-1}P_m'$ which appears in (2.4) but is not so far included in the argument. We therefore add the hypothesis that

$$\rho^{-1}P_m' = W_{1m} + \ldots + W_{lm} \quad (3.7)$$

where, similarly to (3.6),

$$W_{jm}(x) = O(x^{-\sigma_{m+j}}) \quad (3.8)$$

and again we allow the possibility that some $W_{jm}$ may be zero. Since $P_m'$ depends on $V_m'$ (see (2.8)), which in turn depends on the previous matrices in the process (2.1), the nature of (3.7) and (3.8) is that they are conditions on the successive derivatives of the original $V_{11}$ which occur in $R_1$ in (2.3) and (3.1). The exact form of these conditions on $V_{11}$ determines classes of matrices $R_1$ to which this theory and the consequent algorithms are applicable. Examples of such classes will be given in section 5. Thus (3.7) and (3.8) are consequences of these conditions on $R_1$ which must be established (usually by induction) in each application of the theory. It is these $W_{jm}$ which will appear in our algorithms.
We can now summarise this section by saying that, subject to (3.2), (3.7) and (3.8), we have established that

\[ V_{jm}(x) = O(x^{-\sigma_m+j-1}) \]

in (2.3)-(2.5). Also, allowing for the fact that some \( V_{jm} \) in (2.5) may be zero, we can write \( \mu = L - m + 1 \) by (3.4) and (3.5). The transformation process (2.1) ends when (2.3) reduces to

\[ R_M = E_M = O(x^{-K}), \]  \hspace{1cm} (3.9)

the pre-assigned accuracy, and it follows from (3.3) that

\[ M = L + 1, \mu = M - m. \]

4. The algorithm

In this section of the paper we show how the theory that has been developed in sections 1 through 3 may be used to obtain a computer code to calculate the asymptotic expansion of the solutions of (1.1) together with an explicit error bound at some point \( x \geq X > 0 \). A consequence of the theory that we have exhibited is that, given sufficient computational power, the quality of the asymptotics that we obtain allows us to take \( X \) to be quite small and still maintain high accuracy in the solutions.

As in the discussion in (Brown et al. 1996) the algorithm is formulated and implemented in three stages. All the symbolic algorithms that we shall discuss are implemented in the symbolic algebra system Mathematica. The first algorithm, which is concerned with the generation of a set of recurrence relations to compute the matrix quantities \( S_j \), assumes only that the quantities involved satisfy non-commutative multiplication. We recall the comments made after (3.8) that general classes of matrices \( R_1 \) to which the algorithm is applicable will be given in section 5. In the following, we write \( A_m = (I + P_m)^{-1} \) and we note that expressions such as \( P_m, T_m \) and \( W_{jm} \) appear in the algorithm by virtue of their orders of magnitude as indicated in section 3.

Algorithm 4.1. (a) Define \( K \) to specify the accuracy (3.4).
(b) Define \( N \) and \( \theta_1, \ldots, \theta_N \) in (3.2) and arrange the distinct numbers in the set \( \sigma \) in increasing order. This defines \( \sigma_1, \sigma_2, \ldots \) and determines \( L \) in (3.3). For a given \( K \), \( n_j \) in (3.3) satisfies

(I) \( 0 \leq n_j \leq \left[ \frac{K-\theta_1}{\theta_j} \right] \) (\( j \geq 2 \))

(II) \( 1 \leq n_1 \leq \left[ \frac{K}{\theta_1} \right] \).

(c) Start with \( D_1 \) and \( V_{j1} \) (\( 1 \leq j \leq N \)) as in (3.1)-(3.2) and put \( E_1 = 0 \).
(d) For \( m = 1 \) to \( M - 1 \),

(I) \( E_{m+1} = A_m E_m (I + P_m) \).
(II) For each \( U \in \{ W_{jm} (1 \leq j \leq l), T_m, V_{1m} P_m, V_{jm}, V_{jm} P_m \ (2 \leq j \leq M - m) \} \),
A new algorithm for computing the asymptotic solutions of a class of linear differential systems

\( i \) In (2.14) determine \( \nu \).
\( ii \) For \( r = 0 \) to \( \nu \),
\[
determine the order \( \sigma_{m+k} = r \sigma_m + (\text{order of } U) \) of \( P_m^r U \);
Update \( V_{k,m+1} = V_{k,m+1} + (-1)^r P_m^r U \).
\( iii \) Update \( E_{m+1} = E_{m+1} + (-1)^\nu A_m P_m^{\nu+1} U \).

\( III \) Output \( S_{m+1} = V_{1,m+1} \).

At each stage of the algorithm, \( S_{m+1} \) depends on the terms \( D_k, P_k, W_{jk} \) and \( V_k \) (\( 1 \leq k \leq m \)). However, because of part \( b \), the algorithm requires more precise information than its counterpart, Algorithm 6.1, in Brown et al. (1996). The set \( \sigma \) in Brown et al. (1996) has a very simple form, consisting only of numbers \( na \), where \( n \) is a positive integer and \( a \) (> 0) is a parameter. Thus \( \sigma_n = na \) in (3.3), and Algorithm 6.1 in Brown et al. (1996) can be executed without specifying the value of \( a \). We give a more general example of the same situation in Example 5.1 below. However, in the wider context of (3.3), sufficient information about the parameters \( \theta_1, ..., \theta_N \) must be provided to Algorithm 4.1 in order to generate all the necessary values of \( \sigma_n \). We therefore defer further discussion of the output of Algorithm 4.1 to Example 5.2 in the next section, where values of the parameters are specified.

**Algorithm 4.2.** Starting with the precise form of the matrices \( D_1 \) and \( V_1 \) and with \( E_1 = 0 \), the expressions \( S_2, ..., S_{M-1} \) generated by Algorithm 4.1 are evaluated in order. These are then used to evaluate the matrices \( D_{m+1} \) and \( V_{1,m+1} \).

The structure of the algorithm is similar to that of Algorithm 6.2 of Brown et al. (1996). As noted in that paper, in order to reduce the computation time a detailed assessment of the mathematical issues involved at each simplification of an expression must first be made. Thus the judicious use of the Together, Apart commands instead of the Simplify command can result in a dramatic decrease in the time needed to perform the computation. At this stage it is necessary to keep the expressions in symbolic form since the \( W_{jm} \) must be obtained explicitly. These are computed in terms of \( P_m' \), which in turn is obtained from \( S_j \) by differentiation.

The final algorithm that is needed in the symbolic part of the computation is concerned with obtaining an upper bound for the norm \( \| E_m \| \) of the error term \( E_m \). The norm is computed using the sup. norm by applying the triangle and Cauchy inequalities

\[
\| AB \| \leq n \| A \| \| B \|, \quad \| A + B \| \leq \| A \| + \| B \|
\]

for \( n \times n \) matrices. As in Brown et al. (1996) a bound for the norm of the inverse matrix \( A_m = (I + P_m)^{-1} \) is given by

\[
\| A_m \| \leq 1 + \| P_m \| / (1 - n \| P_m \|)
\]

provided \( \| P_m \| < 1/n \).
Algorithm 4.3. Compute the sup. norm of each matrix in $E_m$, using (4.4) for the inverse matrices. Next apply the triangle and Cauchy inequalities to obtain an upper bound for the sup. norm of $E_m$ itself.

We note that, since Algorithm 4.1 expresses $E_m$ in terms of matrices arising at earlier stages of (2.2) and therefore ultimately in terms of the $V_{j_1}$ in (1.1) so also Algorithm 4.3 ultimately expresses the norm of $E_m$ in terms of norms derived from the original system (1.1).

Before moving on to examples of the implementation of the algorithms, we add some detail to (1.9) concerning the generation of the sub-dominant terms in the asymptotic solution of (1.1). By (3.9) and (3.5), the final system in the sequence (2.2) is

$$Z'_M = \rho(D_M + E_M)Z_M,$$

where

$$\|E_M\| \leq c_M x^{-\sigma_M}$$

and $c_M$ is a constant. We recall that the numbers $\sigma_m$ cover all orders of magnitude which occur. Algorithm 4.3 provides a definite value for $c_M$ in any particular example. As in (1.3), the asymptotic solution of (4.2) has the form

$$\{e_k + \eta_k(x)\} \exp(\int_a^x d_{kM}(t)\rho(t)dt),$$

where the $d_{kM}$ are the diagonal entries in $D_M$ and the size of $\eta_k$ can be expressed in terms of $c_M$ and $\sigma_M$ as in (Brown et al. 1996, (3.15)). What we wish to emphasise here is the transformation back from (4.2) to the original system (1.1). As indicated in (1.9), this adds to (4.4) the extra factor

$$I + P_0(x) = \prod_{m=1}^{M-1} \{I + P_m(x)\}.$$  

Now the definition of $P_m$ in (2.8) is in terms of $V_{1m}$, which is provided by Algorithms 4.1 and 4.2. Further, by (2.8) and (3.6), $P_m(x) = O(x^{-\sigma_m}) \ (1 \leq m \leq M-1)$. Thus, in terms of (1.3), our algorithms provide sub-dominant terms up to $O(x^{-\sigma_{M-1}})$ in the asymptotic solution of (1.1).

We mention one further point concerning the transformation process which leads from (2.2) to (2.9). Since the derivative $P'_m$ appears in (2.9) and since $P_m$ ultimately depends on $V_1$ and $\rho$, each step in the process requires the existence of a further derivative of $V_1$ and $\rho$. Thus the sub-dominant terms in (1.5) require the existence of $M-1$ derivatives of $V_1$ and $\rho$. If $V_1$ and $\rho$ are infinitely differentiable then, subject to convergence considerations, (4.3) would yield a full asymptotic expansion. It is hoped to deal with this matter in a future paper.

5. Examples
5.1. Example 1

Let \( \rho(x) = x^\gamma \) and \( R(x) = x^{-(1+\gamma)}C \), where \( \gamma > 0 \) and \( C \) is a constant matrix. Here we have just \( N = 1 \) in (3.1) and

\[
\sigma_m = m(1 + \gamma) \quad (m = 1, 2, \ldots)
\]

This example is basically the case considered in [Brown et al. 1996, section 5] with a special choice of \( C \) and, as in [Brown et al. 1996], the condition (3.7) is easily verified by induction on \( m \). The present code has been tested on this example and the results from Algorithm 4.1 are, up to notational differences, identical with those reported on in [Brown et al. 1996]. Further, Algorithms 4.2 and 4.3 return values of the solutions computed with 4 iterations that, at \( X = 40 \), are within \( 10^{-11} \) of those reported on in [Brown et al. 1996].

5.2. Example 2

A significantly different example is obtained when \( \rho(x) \) and \( R(x) \) in (1.1) contain periodic factors. Let \( \rho(x) = x^\gamma \phi(x^\beta) \) and

\[
R(x) = x^{-(1+\gamma-\beta)}F_1(x^\beta) + x^{-(1+\gamma)}F_2(x^\beta),
\]

where

\[
0 < \beta < 1 + \gamma
\]

and \( \phi(t) \), \( F_1(t) \) and \( F_2(t) \) have the same period \( \omega \) in \( t \), with \( \phi \) nowhere zero. Here we have \( N = 2 \) in (3.1) and

\[
\theta_1 = 1 + \gamma - \beta, \quad \theta_2 = 1 + \gamma
\]

in (3.1)-(3.2). Corresponding to (3.6), we make the induction hypothesis

\[
V_{jm} = x^{-\sigma_{m+j-1}}U_{jm}(x^\beta)
\]

where \( U_{jm}(t) \) has period \( \omega \) in \( t \). Then, by (2.8),

\[
P_m(x) = x^{-\sigma_m}\Pi_m(x^\beta)
\]

where \( \Pi_m(t) \) has period \( \omega \) and the entries \( \pi_{ijm} \) in \( \Pi_m \) are obtained from those in \( U_{1m} \) by the formula

\[
\pi_{ijm} = u_{ijm}/(d_j - d_i) \quad (i \neq j).
\]

Then considering (3.7), we have

\[
\rho^{-1}P_m' = x^{-(\sigma_m + \gamma + 1 - \beta)}(\Pi_m'/\phi)(x^\beta) - \sigma_m x^{-(\sigma_m + \gamma + 1)}(\Pi_m/\phi)(x^\beta) = x^{-(\sigma_m + \gamma + 1 - \beta)}W_1(x^\beta) + x^{-(\sigma_m + \gamma + 1)}W_2(x^\beta)
\]

and hence (3.7) holds with \( l = 2 \).

We note that the upper bound (5.2) placed on \( \beta \) is a restriction on the frequency of oscillations of \( \rho \) and \( R \) in this example. The same type of condition was imposed
in (Eastham 1989, Example 2.4.1) in connection with the method of repeated diagonalization. When $\beta > 1 + \gamma$, the asymptotic solution of (1.1) requires transformations of an entirely different nature from those based on (2.1) and (2.8) (Eastham 1992), (Sultanaev 1984).

In order to discuss the output of Algorithm 4.1 for this example, we have to choose specific values of $\beta$ and $\gamma$, so that part (b) can generate the list of values $\sigma_m$. We make the simple choice $\beta = \gamma = 1$, so that $\theta_1 = 1$ and $\theta_2 = 2$ in (5.3). Then, by (3.3), $\sigma_m = m$.

Also, by (5.1) and (5.4), we have

$$V_{11}(x) = x^{-1}F_1(x), \quad V_{21}(x) = x^{-2}F_2(x),$$
$$W_{1m}(x) = x^{-(m+1)}W_1(x), \quad W_{2m}(x) = x^{-(m+2)}W_2(x)$$

in the notation of section 3.

We have noted in section 4 that $S_{m+1}$ depends on $D_k$, $P_k$, $W_{jk}$ and $V_k$ ($1 \leq k \leq m$). However, the formulae for the $S_{m+1}$ can often be simplified by expressing them in terms of the earlier $S_k$. This reduces the number of terms in the formulae with a consequent reduction in the computational effort required. We now give the output for $S_2$, $S_3$ and $S_4$:

$$S_2 = V_{11}P_1 + T_1 + V_{12} - W_{11}$$
$$S_3 = -P_1S_2 + V_{12}P_1 + T_2 - W_{12} - W_{21}$$
$$S_4 = T_3 - W_{13}$$

We note that, at this stage, these expressions appear no more complex that those computed in (Brown et al. 1996). However, increased difficulties do occur in the evaluation of the entries in $T_3$, $W_{12}$, $W_{21}$ and $W_{31}$ at the next stage when Algorithm 4.2 is implemented. We discuss this point further in Example 5.3. The time needed on a Sun SPARC-station 10 to compute $S_4$ is 1.6 seconds, which is comparable with the comparative time 1.1833 seconds reported on in (Brown et al. 1996). The similar times are a reflection of the low number of terms that must be manipulated by the symbolic algebra system. As we remarked previously, the output of Algorithm 4.1 at this point consists only of a set of symbolic expressions which satisfy non-commutative multiplication.

The error term

$$E_4 = -A_3W_{23} + A_4T_4 - A_4W_{14}A_4W_{24}$$
$$+ A_1P_1^2(V_{12}P_1 - W_{21})$$
$$+ A_2(-P_1(T_1 + V_{12} - W_{11}) + V_{12}P_1 - P_1V_{11}P_1 - W_{21})P_2$$
$$+ A_3V_{31}P_3 + A_4V_{41}P_4$$

is more complex than that found in (Brown et al. 1996), which consists of only 11 additive terms. This additional complexity in the $E_4$ term is reflected in the time needed to compute norms when the third stage, Algorithm 4.3, is implemented.
5.3. Example 3

Algorithms 4.2 and 4.3 require the input of specific matrices \( D_1 \) and \( V_1 \) and, in this example, we introduce a special case of Example 5.2 which arises from the \( n \)-th order differential equation

\[
y^{(n)}(x) - Q(x)y(x) = 0. \tag{5.5}
\]

Again, \( Q(x) \) contains a periodic factor of the form

\[
Q(x) = x^\alpha f(x^\beta) \tag{5.6}
\]

where \( f(t) \) is periodic in \( t \) and nowhere zero, with \( 0 < \beta < 1 + \frac{\alpha}{n} \). As in (Brown et al. 1996, section 3), we write (5.5) in the system form

\[
Z' = Q^{1/n}(D + Q'Q^{-1-1/n}C)Z, \tag{5.7}
\]

where \( Z \) has a first component \( y \), \( D \) is the diagonal matrix formed by the \( n \)-th roots of unity,

\[
D = \text{dg}(\omega_1, \ldots, \omega_n), \tag{5.8}
\]

and \( C \) is constant with

\[
dgC = -(n-1)(2n)^{-1}I. \tag{5.9}
\]

It follows from (5.6) that

\[
Q'Q^{-1-1/n} = \beta x^{-(1+\alpha/n-\beta)}(f'f^{-1-1/n})(x^\beta) + \alpha x^{-(1+\alpha/n)}f^{-1/n}(x^\beta).
\]

Hence (5.7) is the special case of (1.1) and (5.1) in which

\[
\gamma = \alpha/n, \quad \phi = f^{1/n}, \quad F_1 = \beta f'f^{-1-1/n}C, \quad F_2 = \alpha f^{-1/n}C.
\]

We now write (5.7) in the form (2.2) (with \( m = 1 \)), where \( dgV_{11} = 0 \) as in (2.6). Thus taking the diagonal terms from \( F_1 \) over to \( D \) and using (5.9), we define

\[
D_1 = D - (n-1)(2n)^{-1} \beta x^{-(1+\alpha/n-\beta)}(f'f^{-1-1/n})(x^\beta)I = D + \frac{1}{2}(n-1)pI, \tag{5.10}
\]

where

\[
p = x^{-\alpha/n}\{f^{-1/n}(x^\beta)\}'
\]

and

\[
R_1 = V_{11} + V_{21} \quad (= V_1), \tag{5.11}
\]

where

\[
V_{11} = -x^{-\alpha/n}np(C - dgC), \quad V_{21} = \alpha x^{-(1+\alpha/n)}f^{-1/n}(x^\beta)C.
\]

Thus (5.10) and (5.11) are our choice of \( D_1 \) and \( V_1 \).

As in the discussion of Example 5.2, we choose the parameter values \( \beta = \gamma = 1 \), that is, \( \alpha = n \) and \( \beta = 1 \) in (5.6). Finally, we must also choose the values of \( n \) in order to
complete the requirements for implementing Algorithms 4.2 and 4.3. We choose $n = 4$, in which case a short calculation gives

$$C = \frac{1}{8} \begin{pmatrix} -3 & 1+i & 1 & 1-i \\ 1-i & -3 & 1+i & 1 \\ 1 & 1-i & -3 & 1+i \\ 1+i & 1 & 1-i & -3 \end{pmatrix}$$

as in [Brown et al. 1996, Algorithm 6.2].

The periodic nature of the function $Q(x)$ introduces additional matrices over the case discussed in Example 5.1 and the theory expounded in [Brown et al. 1996]. As we remarked above, Algorithm 4.1 needs the specific values of $\beta$ and $\gamma$ to be available. A consequence of this is that all three algorithms must be run for each set of parameter values. However the main additional computational difficulties occur in Algorithms 4.2 and 4.3. In Algorithm 4.2 the extra matrices generated as a consequence of the periodic nature of $Q$ must have their entries evaluated, while in Algorithm 4.3 the norm of the error matrix, which is considerably more complex than that which occurs in Example 5.1, must be evaluated.

In order to test the performance of the set of algorithms we have chosen to take

$$f(x) = 2 + \sin x.$$  \hspace{1cm} (5.12)

However the performance of Algorithms 4.2 and 4.3 is considerably improved if we work with a generic function $f$ together with the simplification rule

$$f'' = 2 - f$$  \hspace{1cm} (5.13)

and the results that we report are based on this latter situation. A further consequence of the extra complexity in $Q$ is that, with the CPU power and memory that we have available, we can not evaluate the entries in $S_5$. Thus effectively, we can only perform 4 iterations of Algorithm 1. The time needed on a SPARC 10 workstation to compute the entries in $S_4$ is 190 seconds of CPU time. This compares with the 75 CPU seconds that was needed in the work reported on in [Brown et al. 1996]. The final algorithm, Algorithm 4.3, deals with the estimation of the sup. norm of the error matrix, in our case $E_4$. This involves first applying the Cauchy and triangle inequalities to each matrix component of $E_4$ to obtain an upper bound for $\| E_4 \|$ in terms of the norms of its components. An estimate for the norm of the inverse matrix $A_4$ is given by (4.1).

These norms are estimated by examining and evaluating each component of each matrix. In doing this we encounter terms involving $p$ and its derivatives. In order to obtain upper bounds for these terms we symbolically compute expressions for them and note that, for this example,

$$3 \geq |f(x)| \geq 1 \hspace{0.5cm} (X < x < \infty)$$

This gives the necessary bound. Again the increase in the complexity of the expressions, resulting from the more complex structure of the initial data means that the computation
time that is required is increased over (Brown et al. 1996). It takes some 1110 CPU seconds to compute the norm of \( E_4 \) compared with approximately 550 CPU seconds for \( E_4 \) reported in (Brown et al. 1996). At \( X = 40 \) the bound for the norm of this error matrix is \( 1.63099 \times 10^{-6} \).

We now compute the factor

\[
I + P_0(x)
\]

discussed in (4.5) and apply this to the asymptotic solution (4.4) to yield the asymptotic solution of (5.7). We mention that in view of the size of the expressions that are generated in (4.3), we have chosen to evaluate it at \( x = 40 \) using 30 digits of accuracy.

6. Concluding remarks

6.1. The transformations of Harris and Lutz

In the introduction, we indicated that the origins of our basic transformation (1.4)-(1.6) lie in the 1974 paper of Harris and Lutz. In a subsequent paper ((Harris & Lutz 1977, 2.4)), an extension of (1.5) is also discussed. Whereas (1.5) can be described as providing a first-order approximation to the exact diagonalization of \( D + R \) in (1.1), the extension provides a more accurate second-order approximation. These ideas are also discussed in ((Eastham 1989) pp26-8).

The question arises whether this extension accelerates the process leading to (3.9), and here we indicate why it does not achieve this objective. The essential feature of both (1.5) and the extension in (Harris & Lutz 1977, 2.4) is that they are linear algebraic equations to determine \( P \). They do not involve \( P' \). Thus, in (2.9), both the corresponding definitions of \( P_m \) yield a term \( \rho^{-1} P_m' \) which, by (3.7), contributes expressions \( W_{jm} \) satisfying (3.8).

Now, although we have allowed the possibility that \( W_{1m} \) may be zero, there is no reason to suppose that it is necessarily zero, and there is therefore nothing to be gained by departing from the simplest definition of \( P_m \) based on (1.5) and (1.6).

6.2. Other computational algorithms

Here we indicate how our algorithm has a quite different purpose as compared to the algorithms of (Dora, Crescenzo & Tournei 1982) and (Dietrich 1992), and it is convenient to refer specifically to the latter. In (Dietrich 1992), the differential system is

\[
Y'(x) = x^{-1}B(x)Y(x)
\]

where \( x \) can be a complex variable,

\[
B = \begin{pmatrix}
0 & 1 & & \\
& & \ddots & \\
& & & 0 & 1 \\
-b_n & & -b_2 & -b_1
\end{pmatrix}
\]
and each \( b \) in the last row has a Laurent series
\[
b(x) = (\text{const.})x^c(1 + a_1x^{-1} + \ldots) \quad (x \to \infty)
\]
with rational \( c \). With \( \infty \) as an irregular singular point, the solutions of the corresponding \( n \) th order differential equation have the asymptotic form
\[
f(x)\{1 + o(1)\}
\]
where the dominant term \( f(x) \) comprises the usual logarithmic and exponential factors. The algorithm developed by (Dietrich 1992) determines \( f(x) \) from a knowledge of \( B \). The algorithm of (Della Dora, Di Crescenzo and Tournier 1982) also allows sub-dominant components of \( f(x) \) to be computed. In contrast, our algorithm is concerned with improvements to the \( o(1) \) term in (6.2) and the construction of sub-dominant terms as explained above in section 1 and at the end of section 4. It is also the purpose of our paper to cover classes of coefficients which, unlike (6.1), contain periodic factors as well as powers of \( x \). This was the subject of section 5.

We are grateful to the referees for raising the issues which are covered in this section.

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