SCALAR CURVATURE, MOMENT MAPS, AND THE DELIGNE PAIRING *

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§1. Introduction.

Let $X$ be a compact complex manifold and $L \to X$ a positive holomorphic line bundle. Assume that $\text{Aut}(X,L)/\mathbb{C}^\times$ is discrete, where $\text{Aut}(X,L)$ is the group of holomorphic automorphisms of the pair $(X,L)$. Donaldson [D] has recently proved that if $X$ admits a metric $\omega \in c_1(L)$ of constant scalar curvature, then $(X,L^k)$ is Hilbert-Mumford stable for $k$ sufficiently large. Since Kähler-Einstein metrics have constant scalar curvature, this confirms in one direction the well-known conjecture of Yau [Y1-4] which asserts that the existence of a Kähler-Einstein metric is equivalent to stability in the sense of geometric invariant theory. Additional evidence for Yau’s conjecture had been provided earlier by Tian [T2-4], who showed that the existence of constant scalar curvature metrics implies $K$-stability and $CM$-stability.

Donaldson’s proof consists of showing that constant scalar curvature implies the existence of a “balanced basis” of $H^0(L^k)$, that is, a basis which imbeds $X$ into projective space and which is orthonormal with respect to the Fubini-Study metric. The existence of a balanced basis is known to imply the Hilbert-Mumford stability of the manifold $X$, due to an earlier theorem of Zhang [Z] and Luo [L]. The general theme of approximating metrics via suitable projective imbeddings had been advocated by Yau over the years.

To obtain a balanced basis of $H^0(L^k)$, Donaldson interprets the balanced condition as the simultaneous vanishing of two moment maps on a certain infinite dimensional manifold $\mathcal{H}_0$: One moment map, $\mu_\mathcal{G}$, corresponds to the action of an infinite dimensional gauge group $\mathcal{G}$, and the other, $\mu_{su}$, to the action of a finite dimensional unitary group $SU(N+1)$.

Using Lu’s formulas [Lu] for the Tian-Yau-Zelditch [T1][Ze][C] expansion of the density of states, Donaldson begins by constructing a basis $s_0$ for which $\mu_D$ is small, where $\mu_D$ is the restriction of $\mu_{su}$ to a single $SL(N+1)$ orbit. The solution to the gradient flow equation for $|\mu_D|^2$ on the manifold $\mathcal{H}_0/\mathcal{G}$ provides a continuous family of bases $s_t$, starting at $s_0$, whose limit is the desired zero of the moment map. The key step in the proof is to show that this limit exists, and this is established by proving a certain lower bound on the derivative $d\mu_D$ of $\mu_D$. Donaldson proves this lower bound by an intricate infinite-dimensional analysis, and also formulates two conjectures for its sharp forms.

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In this paper we improve on Donaldson’s estimate for the lower bound of \( d\mu_D \). A key idea is the identification of the Kähler structure defined in [D] with the curvature of a certain line bundle on the symplectic quotient \( \mathcal{H}_0//\mathcal{G} \), namely the Deligne pairing \( \mathcal{M} = \langle \pi^*O(1), \cdots, \pi^*O(1) \rangle_{\mathcal{X}/\mathcal{Z}} \) (c.f. (4.9) below). The Deligne pairing \( \mathcal{M} \) had been introduced by Zhang [Z] in his study of heights of semistable varieties. Its curvature is given by an explicit formula due to Deligne [De]. Using this formula and estimates for the \( \bar{\partial} \) operator, we avoid the infinite-dimensional gauge group and obtain the desired sharp bounds for the derivative of the moment map. There are indications (see Remark 2 in §5) that our bounds are optimal since that is the case when \( X = \mathbb{CP}^1 \), for vector fields inside \( \text{Aut}(X,O(k)) \subseteq su(k+1) \).

The paper is organized as follows: In §2 and §3 we recall some basic facts about Deligne pairings and moment maps. In §4 we show that Donaldson’s Kähler form is the curvature of the Deligne pairing \( \mathcal{M} \) (Theorem 1) and in §5 we use the curvature formula of [De] to get the desired improvement over Donaldson’s bound (Theorem 2).

### §2. The Deligne pairing

We recall some of the basic definitions and properties in [De] and [Z]: Let \( \pi : \mathcal{X} \rightarrow S \) be a flat projective morphism of integral schemes of relative dimension \( n \). Thus for every \( s \in S \), the fiber \( \mathcal{X}_s \) is a projective variety in \( \mathbb{P}^N \) of dimension \( n \). Let \( \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n \) be line bundles on \( \mathcal{X} \). The Deligne pairing is a line bundle on \( S \), denoted \( \langle \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_{\mathcal{X}/S} \), and defined as follows: Let \( U \subseteq S \) be a small open set and let \( l_i \) be a rational section of \( \mathcal{L}_i \) over \( \pi^{-1}U \). Assume that the \( l_i \) are chosen in “general position”: This means \( \cap_i \text{div}(l_i) = \emptyset \) and for each \( s \) and \( i \) with \( s \in U \) and \( 0 \leq i \leq N \), the fiber \( \mathcal{X}_s \) is not contained in \( \text{div}(l_i) \). Then for every \( k \), the map \( (\cap_{i \neq k} \text{div}(l_i)) \rightarrow S \) is finite: For every \( s \), \( (\cap_{i \neq k} \text{div}(l_i)) \cap \mathcal{X}_s \subseteq \mathcal{X}_s \) is a zero cycle \( \sum n(s)P(s) \). This means the \( n(s) \) are integers and the \( P(s) \) are a finite set of points in \( \mathcal{X}_s \).

Now we define \( \langle \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle_{\mathcal{X}/S} \). Over a small \( U \subseteq S \), this line bundle is trivial and generated by the symbol \( l_0, \ldots, l_n \) where the \( l_i \) are chosen to be in general position. If \( l_i' \) is another set of rational sections in general position, then \( \langle l_0', \ldots, l_n' \rangle = \psi(s)\langle l_0, \ldots, l_n \rangle \) for some nowhere vanishing function \( \psi \) on \( U \) which we must specify. We do this one section at a time: Assume that \( l_i = l_i' \) for all \( i \neq k \). Assume as well that the rational function \( f_k = l_i'/l_k \) is well defined and non-zero on \( (\cap_{i \neq k} \text{div}(l_i)) \). Then \( \psi(s) = \prod f_k(P(s))^{n(s)} \).

Let \( \pi : \mathcal{X} \rightarrow S \) and \( \mathcal{L}_0, \ldots, \mathcal{L}_n \) as above. Let \( l \) be a rational section of \( \mathcal{L}_n \). Assume all components of \( \text{div}(l) \) are flat over \( S \). Then we have the following induction formula:

\[
\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle_{\mathcal{X}/S} = \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1} \rangle_{\text{div}(l)/S} \quad (2.1)
\]

Assume now that \( \mathcal{X}, S \) are defined over \( \mathbb{C} \), that \( \mathcal{X} \) is a smooth variety and that \( \mathcal{L}_i \) is endowed with a smooth hermitian metric. We now define by induction a hermitian metric on \( \langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle_{\mathcal{X}/S} \): Let \( \mathcal{c}_i = -\frac{1}{2\pi} \bar{\partial} \partial \log ||l||^2 \) be the normalized curvature form of
\( L_i \), where \( l \) is an invertible section of \( L_i \). When \( n = 0 \), \( \langle L \rangle_s = \otimes_{p \in \pi^{-1}(s)} L_p \) so we define
\[
\|\langle l_0 \rangle\|_s = \prod_{p \in \pi^{-1}(s)} \|l_0(p)\| \tag{2.2}
\]
In general, we define
\[
\log \|\langle l_0, ..., l_n \rangle\| = \log \|\langle l_0, ..., l_{n-1} \rangle(\text{div}(l_n)/S)\| + \int_{X/S} \log \|l_n\| |\Lambda_{i=0}^{n-1} c_i'(L_i) \tag{2.3}
\]
where the integral is the fiber integral over \( S \). If we combine the induction formula with the definition of the metric, we immediately get the following isometry:
\[
\langle L_0, ..., L_n \rangle(X/S) = \langle L_0, ..., L_{n-1} \rangle(\text{div}(l)/S) \otimes \mathcal{O} \left( -\int_{X/S} \log \|l_n\| |\Lambda_{i=0}^{n-1} c_i'(L_i) \right) \tag{2.4}
\]
where \( \mathcal{O}(f) \) denotes the trivial line bundle with metric \( \|1\| = \exp(-f) \). In particular,
\[
\langle L_0, ..., L_{n-1}, L_n \otimes \mathcal{O}(\phi) \rangle(X/S) = \langle L_0, ..., L_n \rangle(X/S) \otimes \mathcal{O}(E) \tag{2.5}
\]
where
\[
E = \int_{X/S} \phi \cdot \prod_{k<n} c_1'(L_k) \tag{2.6}
\]
Using induction we get the following change of metric formula:
\[
\langle L_0 \otimes \mathcal{O}(\phi_0), ..., L_n \otimes \mathcal{O}(\phi_n) \rangle(X/S) = \langle L_0, ..., L_n \rangle(X/S) \otimes \mathcal{O}(E) \tag{2.7}
\]
where
\[
E = \int_X \sum_{j=0}^{n} \phi_j \cdot \prod_{k<j} c_1'(L_k \otimes \mathcal{O}(\phi_k)) \cdot \prod_{k>j} c_1'(L_k) \tag{2.8}
\]
Finally, we recall a formula for the curvature which is given by Proposition 8.5 of [De]:
\[
c_1'((L_0, ..., L_N)(X/S)) = \int_{X/S} \Lambda_{i=0}^{n} c_i'(L_i) \tag{2.9}
\]
A general result of this type has also been obtained by Tian [T4].

\section{The Moment Map}

The basic facts which we require from the theory of moment maps are the following (see e.g. [DK]). Let \( K \) be a compact Lie group acting on a Kähler manifold \((V, \omega, I)\). Then the complexified group \( K^c \) satisfies \( \text{Lie}(K^c) = \text{Lie}(K) \otimes \mathbb{C} \), and \( K^c \) acts on \((V, I)\), preserving the complex structure, but not the Kähler form or the metric.
A moment map for the action of $K$ is a smooth function $\nu : V \to \text{Lie}(K)$ which satisfies the identity

$$d\langle \nu, \xi \rangle_{\text{Lie}(K)} = \iota_{X_\xi}\omega$$  \hfill (3.1)

where, for $\xi \in \text{Lie}(K)$, $X_\xi = \sigma(\xi)$ is the vector field on $V$ generating the infinitesimal action of $\xi$, and $\langle , \rangle_{\text{Lie}(K)}$ is an invariant Euclidean metric on $\text{Lie}(K)$. We say that $\nu$ is equivariant if it intertwines the action of $K$ on $V$ with the adjoint action of $K$ on $\text{Lie}(K)$. Moment maps need not always exist, but if $K$ is semi-simple, then there is a unique equivariant moment map $\nu : V \to \text{Lie}(K)$.

Let $(V, \omega, I)$ be a Kähler manifold. We say that $(V, \omega, I)$ has a “line bundle in the background” if there is a triple $(L, h, A)$ where $L$ is a holomorphic line bundle on $V$, $h$ is a hermitian metric on $L$ and $A$ is a unitary connection on $L$ whose curvature, $F_A$, satisfies: $F_A = -i\omega$. Such a structure exists if and only if the form $\frac{1}{\pi}\omega$ represents an integral cohomology class.

Now let $(V, \omega, I)$ be a Kähler manifold and assume $(L, h, A)$ is a line bundle in the background. Then there exists an equivariant moment map $\nu : V \to \text{Lie}(K)$ if and only if the action of $K$ on $(V, \omega, I)$ can be lifted to an action of $K$ on $(L, h, A)$. For example, if we are given $\nu$ then the associated action of $K$ is given infinitesimally by the formula:

$$\hat{\sigma}(\xi) = \overline{\sigma(\xi)} + \nu(\xi)\mathfrak{t}$$  \hfill (3.2)

where $\mathfrak{t}$ is the infinitesimal action of $U(1)$ and, for $Y$ a vector field on $V$, $\tilde{Y}$ is the horizontal lift to $L$ given by the connection. We thus get as well an action of $K^c$ on the holomorphic bundle $(L, I)$, given infinitesimally by:

$$\hat{\sigma}(\Xi) = \left[\sigma(\xi_1) + i\sigma(\xi_2)\right] + [\nu(\xi_1) + i\nu(\xi_2)]\mathfrak{t}$$  \hfill (3.3)

where $\xi_1, \xi_2 \in \text{Lie}(K)$ and $\Xi = \xi_1 + i\xi_2 \in \text{Lie}(K^c)$.

Now let $\tilde{\Gamma} \subseteq L$ be a fixed orbit for $K^c$ acting on $L$. Then $\tilde{\Gamma}$ is a smooth manifold which lies over an orbit $\Gamma \subseteq V$ (also a smooth manifold). Define $h : \tilde{\Gamma} \to \mathbb{R}$ by $h(\gamma) = -\log|\gamma|^2$. Let $Q = K^c/K$ and fix $\gamma_0 \in \tilde{\Gamma}$. Define $H : Q \to \mathbb{R}$ by $H(g) = h(g \cdot \gamma_0)$. The derivatives of $H_\xi(t)$ are given by the following basic formulas (see [DK], §6.5.2):

**Proposition.**

1. If $\gamma \in \tilde{\Gamma}$, then $\gamma$ is a critical point of $h$ if and only if $\nu(\pi(\gamma)) = 0$.
2. For $\xi \in \text{Lie}(K)$ let $H_\xi(t) = H(\exp(it\xi))$ and $x = x(t) = \exp(it\xi) \cdot \gamma_0$. Then

$$H'_\xi(t) = 2\langle \nu(\exp(it\xi) \cdot x_0), \xi \rangle$$  \hfill (3.4)

$$H''_\xi(t) = 2\langle \sigma_x(\xi), \sigma_x(\xi) \rangle = 2\omega(\sigma_x(\xi), \overline{\sigma_x(\xi)})$$  \hfill (3.5)

We give a proof for the convenience of the reader: If $\Xi \in \text{Lie}(K^c)$ then $\hat{\sigma}(\Xi)$ is a smooth vector field on $\tilde{\Gamma}$. We claim that the Lie derivative $\mathcal{L}_{\hat{\sigma}(\Xi)}h$ is given by

$$(\mathcal{L}_{\hat{\sigma}(\Xi)}h)(\gamma) = 2\langle \nu(x), \xi_2 \rangle$$  \hfill (3.6)
where $x = \pi(\gamma) \in V$ (here $\pi : L \to V$). Indeed, clearly $\mathcal{L}_\xi h = 0$ for any vector field $X$ on $V$ (since $|\gamma|^2$ is infinitesimally constant in the horizontal direction). Thus, (3.3) implies

$$(\mathcal{L}_\xi h)(\gamma) = -\frac{d}{dt} \log \left| \exp(it[\nu_1 + i\nu_2]) \right|^2 = -\frac{d}{dt} \log \exp(-2t\nu_2)$$

which yields (3.6). Taking $\Xi = i\xi$ in (3.6) we get (3.4). Differentiating one more time we get

$$H''(t) = 2\langle d\nu(\sigma(i\xi)), \xi \rangle = 2\langle \sigma^*\sigma(\xi), \xi \rangle = 2\langle \sigma(\xi), \sigma(\xi) \rangle$$

and this proves (3.5). Finally, to prove statement 1 in the Proposition, observe that (3.6) implies that $(\mathcal{L}_\xi h)(\gamma)|_{t=0} = 0$ for all $\Xi$ if and only if $\langle \nu(x_0), \xi_2 \rangle = 0$ for all $\xi_2 \in \text{Lie}(K)$, that is, if and only if $\nu(x_0) = 0$.

§4. Donaldson’s Kähler structure and the Deligne pairing

In this section we show how one can recover Donaldson’s moment map construction using the Deligne pairing. More precisely, we show that the Kähler structure which Donaldson defines has a line bundle $\mathcal{M}$ in the background, where $\mathcal{M}$ is given by a certain Deligne pairing. Thus Donaldson’s Kähler structure is given by the curvature of $\mathcal{M}$, which can be computed using the general curvature formula (2.9) of Deligne.

Let $X$ be a compact complex manifold, $p : L \to X$ a positive holomorphic line bundle. If $k$ is sufficiently large, then any ordered basis $\underline{s} = (s_0, ..., s_N)$ of $H^0(L^k)$ defines an imbedding $\iota_{\underline{s}} : X \to \mathbb{P}^N$ by $\iota_{\underline{s}}(x) = (s_0(x), \cdots, s_N(x))$.

An ordered basis $\underline{s}$ also defines a canonical isomorphism of holomorphic bundles

$$\psi_{\underline{s}} : L^k \to \iota_{\underline{s}}^*O(1)$$

as follows: Let $O(1)$ be the hyperplane line bundle on $\mathbb{P}^N$, $\ell \in L^k$ and let $x = p(\ell)$. Choose $j$ such that $s_j(x) \neq 0$. Then there is a unique complex number $a_j$ such that $\ell = a_j s_j(x)$. Then $\psi_{\underline{s}}(\ell)$ is a linear map sending the line spanned by $(s_0(x), ..., s_N(x))$ to $\mathbb{C}$. It is defined by the formula

$$\begin{pmatrix} s_0(x) \\ s_1(x) \\ \vdots \\ s_N(x) \end{pmatrix} \to a_j$$

Note that this definition is independent of the choice of $j$, and $\psi_{\underline{s}}$ has the property: $\psi_{\underline{s}}^*\iota_{\underline{s}}^*\pi_i = s_i$, where $\pi_i \in H^0(O(1))$ is projection onto the $i^{th}$ component.

Let $\underline{z} = (z_0, ..., z_N)$ be an ordered basis of $H^0(L^k)$, and let $h_{FS}$ be the Fubini-Study metric on $O(1)$. Thus, if $(z_0, ..., z_N) \in \mathbb{C}^{N+1}$ represents a point in $\mathbb{P}^N$, and if $\pi_i \in H^0(O(1))$ is the projection onto the $i^{th}$ component, then $||\pi_i||^2_{h_{FS}} = \sum |z_i|^2$. Define

$$h_{\underline{z}} = \psi_{\underline{z}}^*\iota_{\underline{z}}^*h_{FS}$$

(4.3)
Then $h_s$ is a metric on $L^k$ which depends on $s$. One way to characterize $h_s$ is as follows:

$$\| \cdot \|_{h_s}^2 = \frac{\| \cdot \|_{h_0}^2}{\sum_j |s_j|_{h_0}^2}$$  \hspace{1cm} (4.4)$$

where $h_0$ is any fixed metric on $L^k$.

Now fix $k$, a large positive integer and define

$$\tilde{Z} = \{ s = (s_0, ..., s_N) : \{ s_0, ..., s_N \} \text{ is a basis of } H^0(L^k) \}/\mathbb{C}^\times$$  \hspace{1cm} (4.5)$$

and

$$Z = \tilde{Z}/(PAut(X, L^k))$$  \hspace{1cm} (4.6)$$

where $PAut(X, L^k) = Aut(X, L^k)/\mathbb{C}^\times$ and the map $Aut(X, L^k) \hookrightarrow GL(N + 1)$ is given by the action on global sections.

Assume now that $PAut(X, L^k)$ is discrete. Then the construction (4.5-4.6) provides $Z$ with a complex structure. Next Donaldson defines a map $\mu_D : Z \to su(N + 1)$ by

$$\mu_D(s) = P_{su(N+1)}[i(s_\alpha, s_\beta)_{h_s}]$$  \hspace{1cm} (4.7)$$

where $P_{su(N+1)} : u(N + 1) \to su(N + 1)$ is the projection onto the trace free subspace: $P_{su(N+1)}(C_{\alpha\beta}) = C_{\alpha\beta} - \frac{1}{N+1} \text{tr}(C) \cdot \delta_{\alpha\beta}$ (the Lie algebra $su(N + 1)$ is identified with its dual using the invariant Hilbert-Schmidt pairing). A key property of $\mu_D$ is that it is the moment map for a certain symplectic structure $\Omega_D$ on $Z$ which Donaldson constructs as follows.

Fix a hermitian metric $h_0$ on $L$ and let $I_0$ be the holomorphic structure on $X$. Let $A$ be the unitary connection on $L$ which is compatible with $I_0$. Let $J_{int}$ be the set of all integrable holomorphic structures on $X$ (so that in particular, $I_0 \in J_{int}$) and define $H_0 \subseteq \Gamma(L^k) \times \cdots \times \Gamma(L^k) \times J_{int}$ to be the set of pairs $(s, I)$ where $s = (s_0, ..., s_N)$ is an ordered basis of $H^0((L, I)^k)$. Here $(L, I)$ is the holomorphic line bundle determined by $I$ and the connection $A$. Let $G$ be the group of $C^\infty$ automorphisms of the triple $(L, h_0, A)$, so that $Lie(G) = C^\infty(X)/\mathbb{R}$. Then $G$ acts on $H_0$ and it turns out that there is a moment map $\mu_G : H_0 \to C^\infty(X)$ for the action of $G$ (it is not obvious that such a moment map should exist since $G$ is infinite dimensional). Although the complexification $G^c$ of $G$ does not exist, one can, for $w \in H_0$, still consider the “orbit” $G^c \cdot w \subseteq H_0$. Fix any real number $a > 0$. One shows that every orbit meets $\mu_G = a$ uniquely, up to the action of $G$. In this way, one obtains, by symplectic reduction, a Kähler structure on $H_0//G = H_0/G^c$, and a corresponding moment map which is the restriction to the set $\{ \mu_G = a \}$ of the map $\mu_{su}$ given by $\mu_{su}(s) = P_{su(N+1)}[i(s_\alpha, s_\beta)_{h_0}]$. The natural map $\beta : Z \to H_0//G$ is an imbedding, and provides $Z$ with the desired Kähler structure $\Omega_D$. Moreover, the moment map $\mu_D$ satisfies the relation: $\mu_D = \mu_{su} \circ \beta$. 

6
We now give a simple description of \( \Omega_D \): Let
\[
\tilde{\mathcal{X}} = \{ (x, s) : x \in \mathbb{P}^N, \ s = (s_0, ..., s_N) \text{ a basis of } H^0(L^k), \ x \in \iota_s(X) \} \tag{4.8}
\]
and let \( \mathcal{X} = \tilde{\mathcal{X}} / \text{PAut}(X, L^k) \). Then \( \mathcal{X} \rightarrow \mathcal{Z} \) is a smooth holomorphic fibration whose fibers are all isomorphic to \( X \). Let \( \pi : \mathcal{X} \rightarrow \mathbb{P}^N \) be projection onto the second factor, and let
\[
\mathcal{M} = \langle \pi^*O(1), ..., \pi^*O(1) \rangle(\mathcal{X} / \mathcal{Z}) \tag{4.9}
\]
be the Deligne pairing of \( n + 1 \) copies of the line bundle \( \pi^*O(1) \), in the sense explained in \$2. The line bundle \( \mathcal{M} \) is a hermitian line bundle on \( \mathcal{Z} \), invariant under the action of \( SU(N + 1) \). Let \( \Omega_{\mathcal{M}} \) be the curvature of \( \mathcal{M} \). Formula (2.9) says that \( \Omega_{\mathcal{M}} \), the curvature of \( \mathcal{M} \), is given by the formula
\[
\Omega_{\mathcal{M}} = \int_{\mathcal{X} / \mathcal{Z}} \omega_{FS}^{n+1} \tag{4.10}
\]
which is positive, since \( \omega_{FS} \) is positive. Since \( SU(N + 1) \) is semi-simple, there is a unique equivariant moment map
\[
\mu_{\mathcal{M}} : \mathcal{Z} \rightarrow su(N + 1) \tag{4.11}
\]
with respect to the form \( \Omega_{\mathcal{M}} \) (see, for example, \$4.9 of [CS]).

**Theorem 1.** Let \( \mathcal{M} \) be the line bundle over \( \mathcal{Z} \) given by (4.9). Then Donaldson’s symplectic form \( \Omega_D \) and moment map \( \mu_D \) are given respectively by the curvature \( \Omega_{\mathcal{M}} \) and the moment map \( \mu_{\mathcal{M}} \) of \( \mathcal{M} \):
\[
\mu_{\mathcal{M}} = \mu_D, \quad \Omega_{\mathcal{M}} = \Omega_D.
\]

**Proof.** To prove this theorem, we compute \( \mu_{\mathcal{M}} \) using (3.4): Fix \( z = [s] \in \mathcal{Z} \) and let \( \gamma \in \mathcal{M} \) be a point above \( z \). Let \( \xi \in su(N + 1) \) and define \( H(t) = -\log |\sigma_t \cdot \gamma|^2_{\mathcal{M}} \) where \( \sigma_t = \exp(it\xi) \). Then the formula (3.4) can be rewritten as
\[
H'(0) = 2\langle \mu_{\mathcal{M}}(z), \xi \rangle \tag{4.12}
\]
On the other hand, the change of metric formula (2.7) for the Deligne pairing tells us that
\[
H(t) - H(0) = E(\phi_t) = \int_X \phi_t \cdot \sum_{j=0}^n (\sigma_t^* \omega_{FS}^j \wedge \omega_{FS}^{n-j}) \tag{4.13}
\]
where
\[
\phi_t(x) = \log \frac{|\sigma_t(x)|^2}{|x|^2} \tag{4.14}
\]
The right hand side can be recognized as the familiar component \( F_\omega(\phi_t) \) of the energy functional \( F_\omega(\phi_t) \) due to Yau and Aubin. Its variational derivative is well-known and can be obtained by a straightforward computation
\[
\frac{d}{dt} E(\phi_t) = \int_X \dot{\phi_t} \sigma_t^* \omega_{FS}^0 \tag{4.15}
\]
Now $\dot{\phi}_t$ is given explicitly by
\[
\dot{\phi}_t = \frac{2x^* \exp(2i\xi t)(i\xi)x}{x^* \exp(2i\xi t)x}
\] (4.16)
for $x \in P^N$. Differentiating (4.13) and substituting in (4.15) and (4.16) produces
\[
H'(0) = 2 \text{Tr} \left( i \int_X x x^* \omega^n \right)
\] (4.17)
Comparing (4.12) and (4.17) we see that
\[
\mu_M(z) = Psu(N+1) \left( i \int_X x x^* \omega^n \right) = Psu(N+1) \left( i \langle s_\alpha, s_\beta \rangle h \right) = \mu_D(z)
\] (4.18)
In view of the defining relation (3.1) between the moment map and the symplectic form, and the fact that the symplectic forms are compatible with the complex structure, it follows that the symplectic forms $\Omega_M$ and $\Omega_D$ must coincide on each $SL(N+1)$ orbit. But then $\Omega_M = \Omega_D$ on $Z$, since $Z$ consists of a single $SL(N+1)$ orbit.

§5. Estimates for the moment map

In order to explain the statement of our theorem, we sketch briefly the steps in Donaldson’s proof [D], introducing the necessary notation along the way. One wants to show that constant scalar curvature implies the existence of a “balanced basis” of $H^0(L^k)$, that is, an ordered basis $\underline{s} = \{s_0, ..., s_N\}$ which imbeds $X$ into $P^N$ and which satisfies the following two conditions:

i) $\sum_{j=0}^N |s_j|_{h_{FS}}^2 = 1$ at each point of $X$.

ii) The matrix $\langle s_\alpha, s_\beta \rangle_{h_{FS}}$ is diagonal, where $h_{FS}$ is the Fubini-Study metric.

To find the balanced basis, [D] first uses Lu’s formula [L] for the coefficients in the Tian-Yau-Zelditch asymptotic expansion for the density of states to produce, for each $k$, an ordered basis $\underline{s}' = \{s'_0, ..., s'_N\}$ of $H^0(L^k)$ which is almost balanced in the following sense:
\[
\sum_{j=0}^N |s'_j|_{h_{FS}}^2 = 1 \text{ at each point of } X \text{ and }
\]
\[
\langle s'_\alpha, s'_\beta \rangle_{h_{FS}} = D_k + E_k
\] (5.1)
where $D_k$ is a scalar matrix with $D_k \to 1$ as $k \to \infty$, and $E_k$ is a trace-free hermitian matrix whose operator norm $\|E_k\|_{op}$ tends to zero rapidly as $k$ tends to infinity. He then interprets $E = E_k$ as the value $\mu_D(z')$ where $\mu_D$ is the moment map corresponding to $\Omega_D$ on the manifold $Z$, and $z' \in Z$ is the point determined by $\underline{s}'$. The problem now becomes one of showing that if $\mu_D(z')$ is small for some $z' \in Z$, then there is a point $z \in Z$ which is close to $z'$ with $\mu_D(z) = 0$. The standard technique for finding the zero of a moment map is to follow the gradient flow of the function $|\mu_D|^2$. But to guarantee that the flow will, after a short distance $\delta$, reach a zero of the moment map, one must prove that $d\mu_D$, the derivative of $\mu_D$, is large in a $\delta$ neighborhood of $z'$. 

8
The heart of the argument in \[D\] is the proof of a lower bound estimate on \(d\mu_D\). This is equivalent to bounding \(|\sigma_Z(\xi)|\) from below, where \(\xi \in su(N+1)\) has length one and \(\sigma_Z(\xi)\) is the vector field on \(Z\) given by the infinitesimal action of \(\xi\). Since \(Z \subseteq H_0/G\), the symplectic reduction formalism tells us that \(\sigma_Z(\xi)\) is the projection of \(\sigma_{H_0}(\xi)\) (the vector field on \(H_0\) corresponding to the infinitesimal action of \(\xi\)) onto the orthogonal complement of the tangent space \(T(G^c \cdot w)\), where \(G^c \cdot w\) is a complexified orbit. This description turns out to be sufficiently explicit to allow the following estimation of \(|\sigma_Z(\xi)|\):

**Theorem.** (Donaldson) Suppose \(\text{Aut}(X, L)\) is discrete. For any \(R > 1\) there are positive constants \(C\) and \(\epsilon < \frac{1}{10}\) such that, for any \(k\), if the basis \(s_\alpha\) of \(H^0(L^k)\) has \(R\)-bounded geometry, and if \(\|E\|_{op} < \epsilon\), then

\[
\Lambda_z \leq C^2 \cdot k^4
\]

Here the basis \(s_\alpha\) is viewed as defining a metric \(\tilde{\omega}\) on \(X\), which is in the cohomology class \(k \cdot c_1(L)\). The metric \(\tilde{\omega}\) is said to have \(R\)-bounded geometry if \(\tilde{\omega} > R^{-1} \tilde{\omega}_0\) and \(\|\tilde{\omega} - \tilde{\omega}_0\|_{C^r(\tilde{\omega}_0)} < R\), where \(\tilde{\omega}_0 = k \omega_0\), \(\omega_0\) is a fixed reference metric in \(c_1(L)\), \(\|\cdot\|_{C^r(\tilde{\omega}_0)}\) is the \(C^r\) norm defined by \(\tilde{\omega}_0\), and \(r \geq 4\) is a fixed integer. The basis \(s_\alpha\) is said to have \(R\)-bounded geometry if the corresponding metric \(\tilde{\omega}\) does. The expression \(\Lambda_z^{-1}\) denotes the smallest eigenvalue of the operator

\[
Q_z = \sigma_z^* \sigma_z : su(N + 1) \longrightarrow su(N + 1),
\]

where \(\sigma : su(N + 1) \to TZ\) is the infinitesimal action and \(\sigma_z^*\) is its adjoint with respect to the metric on \(TZ\) and the invariant metric on \(su(N + 1)\). At the end of his paper, Donaldson sketches a refinement of the argument which replaces \(k^4\) by \(k^{2+\epsilon}\), and conjectures that in fact the estimate (5.2) holds with \(k^4\) replaced by \(k\). In this section, we shall prove:

**Theorem 2.** Under the same hypotheses as in Donaldson’s theorem, we have

\[
\Lambda_z \leq C^2 \cdot k^2
\]

**Proof.** The estimate (5.3) is equivalent to the estimate

\[
|\sigma_Z(\xi)|^2 \geq c_R k^{-2} ||\xi||^2
\]

for all \(\xi \in su(N + 1)\) and a positive constant \(c_R\) depending only on \(R\). To prove this, our starting point is the identity

\[
|\sigma_Z(\xi)|^2 = \int_X \langle \xi, \tilde{\omega}^{n+1} \rangle
\]

which follows from Theorem 1. Indeed, \(|\sigma_Z(\xi)|^2 = -i \Omega_D(\sigma_Z(\xi), \sigma_Z(\xi))\) by definition. On the other hand, \(\Omega_D(\sigma_Z(\xi), \sigma_Z(\xi)) = \Omega_{\mathcal{M}}(\sigma_Z(\xi), \sigma_Z(\xi))\) by Theorem 1, and this last
expression is given by the right hand side of (5.5) in view of Deligne’s formula (2.9). (Alternatively, (5.5) has also been proved directly in [PS], formula (3.5)).

Consider now the exact sequence of holomorphic vector bundles:

\[ 0 \to TX \to \iota^*TP^N \to Q \to 0 \]

where \( TX \) is the tangent bundle of \( X \), \( TP^N \) is the tangent bundle of \( P^N \), \( \iota : X \to P^N \) is the embedding, and \( Q \) is the quotient. Let \( N \subseteq \iota^*TP^N \) be the orthogonal complement of \( TX \). Then \( N \) is a smooth vector bundle and

\[ \iota^*TP^N = TX \oplus N \]

Let \( \pi_T \) and \( \pi_N \) be the projections onto the first and second components of \( \iota^*TP^N \). We observe that

\[ \int_X \iota_X \xi \bar{\xi} \omega_{FS}^{n+1} = ||\pi_N X\xi||^2 \quad (5.6) \]

where \( || \cdot || \) denotes the \( L^2 \) norm with respect to the metric \( \bar{\omega} \) on the base and the Fubini-Study metric on the fiber. The desired inequality (5.4) is a direct consequence of (5.5), (5.6) and the following inequalities:

\[ ||\xi||^2 \leq c'_R k ||X\xi||^2 \quad (5.7) \]

\[ ||X\xi||^2 = ||\pi_T X\xi||^2 + ||\pi_N X\xi||^2 \quad (5.8) \]

\[ c_R ||\pi_T X\xi||^2 \leq k ||\pi_N X\xi||^2 \quad (5.9) \]

where \( c_R, c'_R > 0 \) are constants independent of \( k \).

The identity (5.8) is an immediate consequence of the orthogonality of \( \pi_N X\xi \) and \( \pi_T X\xi \). To prove (5.7) we argue as follows: If we view points \( z \in P^N \) as a column vectors in \( \mathbb{C}^{N+1} \) (modulo the action of \( \mathbb{C}^\times \), then tangent vectors \( X \) on \( CP^N \) can be viewed as ordered pairs \((z,v)\) modulo the equivalence relation: \((z,v) \sim (z',v')\) if \( z' = \lambda z, v' - \lambda v = \mu z \) for some \( \lambda \in \mathbb{C}^\times \) and \( \mu \in \mathbb{C} \). Then the vector field \( X\xi \) is defined, for \( \xi \in su(N+1) \), by

\[ X\xi = (z,\xi z) \]

and its norm with respect to the Fubini-Study metric \( \omega_{FS} \) is given by the well-known formula

\[ ||X\xi||^2 = \frac{(z^*\xi^*z)(z^*z) - (z^*\xi z)^2}{(z^*z)^2} \]

Observe that the function \( z^*\xi z/z^*z \) on \( M \) is just \( \hat{\phi} \equiv \hat{\phi}_t \), at \( t = 0 \), where \( \phi_t = \log |\sigma_t|^2/|z|^2 \) and \( \sigma_t(z) = \exp(it\xi)z \). Integrating the above identity with respect to \( \omega_{FS} \) gives

\[ tr \left( \xi^* \xi \cdot \int_M \frac{zz^*}{z^*z} \bar{\omega}^n \right) = ||X\xi||^2 + \int_M \hat{\phi}^2 \bar{\omega}^n \quad (5.10) \]

We claim that the following Poincaré inequality holds for \( \bar{\omega} \), with uniform constants in \( \bar{\omega} \)

\[ c \int_M \hat{\phi}^2 \bar{\omega}^n \leq k \int_M \bar{\partial} \hat{\phi} \wedge \bar{\partial} \hat{\phi} \wedge \bar{\omega}^n + k^{-n} \left( \int_M \hat{\phi} \bar{\omega}^n \right)^2 \quad (5.11) \]
Indeed, for $\tilde{\omega}_0 \equiv k\omega_0$, this is just the standard Poincaré inequality for the fixed metric $\omega_0$, scaled up by a factor of $k$. To establish it for $\tilde{\omega}$, we note first that $R^{-1}\tilde{\omega}_0 < \tilde{\omega} < 2R\tilde{\omega}$ since $\tilde{\omega}$ is $R$-bounded. Writing

$$\int_M \dot{\phi} \tilde{\omega}_0^n = \int_M \dot{\phi} \tilde{\omega}^n - \int_M \dot{\phi} (\tilde{\omega}^n - \tilde{\omega}_0^n)$$

and $\tilde{\omega}^n - \tilde{\omega}_0^n = \partial \bar{\theta} \land \sum_{p=0}^{n-1} \tilde{\omega}_0^{n-1-p} \tilde{\omega}^p$, with $\theta$ normalized so that $\int_M \theta \tilde{\omega}_0^n = 0$, we have

$$| \int_M \dot{\phi} (\tilde{\omega}^n - \tilde{\omega}_0^n) | = | \int_M \partial \bar{\theta} \land \partial \theta \land \sum_{p=0}^{n-1} \tilde{\omega}_0^{n-1-p} \tilde{\omega}^p | \leq \sum_{p=0}^{n-1} \int_M | \partial \bar{\theta} | | \tilde{\omega}_0^n | | \tilde{\omega}^p | \frac{\tilde{\omega}^n}{\tilde{\omega}_0^n}$$

$$\leq C_1 (\int_M | \partial \bar{\theta} |^2 | \tilde{\omega}_0^n |^{1/2} (\int_X | \partial \theta |^2 | \tilde{\omega}_0^n |^{1/2})^{1/2})$$

$$\leq C_2 (\int_M | \partial \bar{\theta} \land \partial \theta \land \tilde{\omega}_0^{n-1} |^{1/2} k^{\frac{1}{2}(n+1)}$$

The last inequality follows from the uniform boundedness of $||\Delta \omega_0 \theta||_{C^\infty}$ (which holds by the R-boundedness assumption) and the inequality $||\bar{\Delta} \theta||_{\omega_0}^2 \leq Ck||\Delta \omega_0 \theta||_{\omega_0}^2$ (which holds since $k \Delta \omega_0 - c > 0$ on the space of $\theta$'s with mean value 0, for some small positive constant $c$). The inequality (5.11) now follows.

Writing $\int_M \tilde{\omega}_0^n = D_k + E_k$ and using the fact that $D_k \to 1$ as $k \to \infty$, and $||E_k||_{op} < \epsilon$, we see that $tr \left( \xi \xi \cdot \int_M \tilde{\omega}_0^n \right) \geq c ||\xi||^2$. On the other hand,

$$| \int_M \dot{\phi} \tilde{\omega}^n | = | tr(\xi E) | \leq \sqrt{N+1} ||\xi|| \cdot ||E||_{op} \leq c k^{n/2} ||\xi|| \cdot ||E||_{op}$$

Combining this with (5.10) and (5.11), and using $||E_k||_{op} < \epsilon$, we obtain, for $\epsilon$ small,

$$c ||\xi||^2 \leq ||X_{\xi}||^2 + k \int_M \partial \bar{\theta} \land \partial \theta \land \tilde{\omega}^n$$

But now observe that $\partial \bar{\theta} = \iota_{X_{\xi}} \omega_{FS}$. Restricting to $M$ we get $\partial \bar{\theta} = \iota_{\pi T X_{\xi}} \tilde{\omega}$ which implies

$$c ||\xi||^2 \leq ||X_{\xi}||^2 + k ||\pi T X_{\xi}||^2$$

and this proves (5.7).

It remains to prove (5.9). We begin by assuming that Aut$(X)$ is discrete, that is, $X$ does not admit any nonzero holomorphic vector field. This assumption will be removed later. Since there are no nonzero holomorphic vector fields on $X$, we have the standard inequality for the $\bar{\partial}$ operator:

$$c ||W||_{L^2(\omega_0)}^2 \leq ||\bar{\partial}(W)||_{L^2(\omega_0)}^2$$

for some $c > 0$, where $W$ is any smooth vector field on $X$ and $|| \cdot ||_{L^2(\omega_0)}$ is the $L^2$ norm with respect to the metric $\omega_0$. Replacing $\omega_0$ by $\tilde{\omega}_0 = k\omega_0$, we obtain the inequality
we have
\[ c_R \|W\|^2 \leq k \|\partial(W)\|^2 \]
for some \(c_R > 0\), depending on \(R\) but not on \(k\). Applying this inequality with \(W = \pi_T X_\xi\), we see that in order to establish (5.11) it suffices to prove
\[ c_R \|\partial(\pi_T V)\|^2 \leq \|\pi_N V\|^2, \] (5.14)
for all holomorphic vector fields \(V\) on \(\mathbb{P}^N\).

But \(V = \pi_T V + \pi_N V\) so \(\bar{\partial} V = 0 = \bar{\partial}(\pi_T V) + \bar{\partial}(\pi_N V)\). Thus it suffices to prove:
\[ \|\pi_N V\|^2 \geq c_R \cdot \|\bar{\partial}(\pi_N V)\|^2 = c_R \cdot \|\bar{\partial}(\pi_N)(V)\|^2 \] (5.15)

In fact, we shall prove the pointwise estimate:
\[ |\pi_N V|^2 \geq c_R \cdot |\bar{\partial}(\pi_N V)|^2 = c_R \cdot |\bar{\partial}(\pi_N)(V)|^2 \] (5.16)

Fix \(x \in X\) be a point and choose a local holomorphic frame of \(\iota^*T\mathbb{P}^N\), in a neighborhood of \(x\), of the form: \(e_1, \ldots, e_n, f_1, \ldots, f_m\) with \(m + n = N\), satisfying the following:

a) The \(e_1, \ldots, e_n, f_1, \ldots, f_m\) form an orthonormal basis of \(\iota^*T_x \mathbb{P}^N\).

b) The \(e_1, \ldots, e_n\) form a local holomorphic frame of \(TX\) near \(x\).

Then we can express \(V = \sum_i a_i e_i + \sum_j b_j f_j\), where the \(a_i\) and the \(b_j\) are holomorphic functions. Since \(\pi_N\) is a projection, \(\pi_N(\pi_N(f_j) - f_j) = 0\). Thus \(\pi_N(f_j) - f_j\) is a linear combination of the vectors \(e_i\)'s, and we can write
\[ \pi_N(f_j) = f_j - \sum_{i=1}^n \phi_{ij} e_i \] (5.17)

where the \(\phi_{ij}\)'s are smooth functions, vanishing at \(x\). Then \(\pi_N V = \sum_{j=1}^m b_j (f_j - \sum_i \phi_{ij} e_i)\), and
\[ \bar{\partial}(\pi_N V) = \sum_{j=1}^m b_j (-\sum_{i=1}^n (\bar{\partial}\phi_{ij}) e_i) \] (5.18)

To establish (5.16) we must prove:
\[ \sum_{i=1}^n \left(\sum_{j=1}^m b_j \bar{\partial}\phi_{ij}\right)^2 \leq c_R^{-1} \sum_{j=1}^m |b_j|^2 \] (5.19)

But for each \(i\), the Cauchy-Schwarz inequality for quadratic forms defined by hermitian matrices implies \(\sum_{j=1}^m b_j \bar{\partial}\phi_{ij}^2 \leq \sum_{j=1}^m |b_j|^2 \cdot \sum_{j=1}^m |\bar{\partial}\phi_{ij}|^2\), and thus it suffices to prove
\[ \sum_{i=1}^n \sum_{j=1}^m |\bar{\partial}\phi_{ij}|^2 \leq c_2 \] (5.20)
where \( c_2 = c_2(R) \) is independent of \( k \) (and depends only on \( R \)). But the matrix

\[
A^* = (\bar{\partial}\phi_{ij})
\]

is precisely the dual of the second fundamental form \( A \) of \( TX \) in \( \iota^*TP^N \). One can see this as follows: Let \( S = TX, E = \iota^*TP^N \) and \( Q = E/S \). Then we have the exact sequence of holomorphic bundles:

\[
0 \to S \to E \overset{p}{\to} Q \to 0
\]

The second fundamental form is the bundle map \( A : S \to Q \otimes \Omega^{1,0} \) defined by \( A = p \circ D_E \), where \( D_E \) is the Chern connection on \( E \), compatible with the metric \( h = h_{FS} \) and the holomorphic structure. Then \( A^* : Q \to S \otimes \Omega_{1,0} \) is characterized by

\[
\langle As, q \rangle_h = \langle s, A^*\tilde{q} \rangle_h = \langle s, A^*q \rangle_h
\]

where \( s \) is a section of \( S \) and \( q \) a section of \( Q \), and \( \tilde{q} \) is the section of \( S^\perp \) which corresponds to \( q \) via the canonical isomorphism of smooth bundles \( S^\perp \to Q \). We claim that

\[
A^*(\tilde{f}_j) = \sum_{i=1}^n (\bar{\partial}\phi_{ij})e_i
\]

To see this, first note that \( \langle s, \tilde{f}_j \rangle = \langle s, f_j - \sum_{i=1}^n \phi_{ij}e_i \rangle = 0 \). Differentiating both sides:

\[
\langle D_Es, f_j - \sum_{i=1}^n \phi_{ij}e_i \rangle + \langle s, D_E(f_j - \sum_{i=1}^n \phi_{ij}e_i) \rangle = 0
\]

In other words,

\[
\langle As, f_j - \sum_{i=1}^n \phi_{ij}e_i \rangle = \langle s, \sum_{i=1}^n \bar{\partial}(\phi_{ij})e_i \rangle
\]

This proves the claim. Since \( A \) is the second fundamental form, the curvature tensors of the sub-bundle and of the ambient bundle are related by (see, e.g., [GH], page 78):

\[
-A^* \wedge A = \pi_T \circ (F_{\iota^*TP^N|TX}) - F_{TX}
\]

Let us explain the meaning of this notation: The tensor \( F_{\iota^*TP^N} \) is a 2-form on \( X \) with values in \( End(\iota^*TP^N) \). Thus \( F_{\iota^*TP^N|TX} \) is a 2-form with values in \( Hom(TX, \iota^*TP^N) \) and \( \pi_T \circ (F_{\iota^*TP^N|TX}) \) is a 2-form with values in \( Hom(TX, TX) = End(TX) \) as is the tensor \( F_{TX} \). Thus

\[
\sum_{i=1}^n \sum_{j=1}^m |\bar{\partial}\phi_{ij}|^2 = C_0 Tr[(\pi_T \circ (F_{\iota^*TP^N|TX}) - F_{TX})]
\]

where

\[
Tr[(\pi_T \circ (F_{\iota^*TP^N|TX}) - F_{TX})]
\]

13
is a two form on $X$ and $C_\tilde{\omega}$ is the contraction with the metric $\tilde{\omega}$.

We now compute the terms in the equation above: First, $R = F_T P_N$ is the Riemann curvature of the Fubini-Study metric. It is well known that the Fubini-Study metric has constant bisectional curvature. In fact:

$$R(X, \bar{Y}, Z, \bar{W}) = g(X, \bar{Y})g(Z, \bar{W}) + g(X, \bar{W})g(Z, \bar{Y})$$  \hspace{1cm} (5.30)

where $g = g_{FS}$ is the Fubini-Study metric. In other words, in any system of local coordinate for $P^N$, we have $R_{ijkl} = g_{ij}g_{kl} + g_{il}g_{kj}$, and thus

$$R_{i^j_{kl}} = \delta_i^j g_{kl} + g_{il} \delta_k^j$$  \hspace{1cm} (5.31)

Now for a fixed $x \in X$, choose a local coordinate system $(x_1, \ldots, x_N)$, centered at $x$, such that $T_x X$ is the plane defined by: $x_{n+1} = \cdots = x_N = 0$. Then

$$Tr[(\pi_T \circ (F^* T P N)|_{T X})] = \sum_{k,l=1}^{n} \left( \sum_{i=1}^{n} R_{i^i_{kl}} \right) dz_k \wedge d\bar{z}_l$$

$$= n \sum_{k,l=1}^{n} g_{kl} \cdot dz_k \wedge d\bar{z}_l + \sum_{k,l=1}^{n} g_{kl} \cdot dz_k \wedge d\bar{z}_l = (n + 1)\tilde{\omega}$$  \hspace{1cm} (5.32)

Thus

$$C_\tilde{\omega} Tr[(\pi_T \circ (F^* T P N)|_{T X})] = (n + 1)$$  \hspace{1cm} (5.33)

On the other hand, $C_\tilde{\omega}(F_{TX})$ is the scalar curvature of $X$ with respect to the pullback of $\omega_{FS}$. But our assumption in Theorem 2 is that $\omega_{FS}$ has $R$-bounded geometry with respect to $k\omega_0$. This is readily seen to imply that $||\nabla^r\tilde{\omega}||_{C^0(\omega_0)} \leq C_R k^{1+\frac{2}{r}}$, and hence

$$C_\tilde{\omega} Tr[F_{TX}] \leq C_R$$  \hspace{1cm} (5.34)

with a constant depending only on $R$. This completes the proof under the assumption $Aut(X)$ is discrete.

Finally, we remove the hypothesis that $Aut(X)$ is discrete. This step relies on Lemma 12 of [D], which we now review:

Let $p : L \to X$ be a positive holomorphic vector bundle on a compact manifold $X$. Let $Aut(X)$ be the group of holomorphic automorphisms of $X$ and let $Aut(X, L)$ the group of holomorphic automorphisms of the pair $(X, L)$. Thus an element of $Aut(X, L)$ is a pair $(F, \tilde{F})$ where $F : X \to X$ is biholomorphic, $\tilde{F} : L \to L$ is biholomorphic, and the diagram commutes: $p\tilde{F} = Fp$.

We clearly have a map $Aut(X, L) \to Aut(X)$ defined by $(F, \tilde{F}) \mapsto F$. The kernel of the map is $C^\times$. We are interested in the image of this homomorphism. More precisely, we
want to characterize the infinitesimal image, that is, the image of $\text{Lie}(\text{Aut}(X,L))$ inside $\text{Lie}(\text{Aut}(X))$.

Recall that if $v$ is a vector field on $X$ then $v \in \text{Lie}(\text{Aut}(X))$ if and only if $v$ is the real part of a holomorphic vector field, i.e.:

1. $v = w + \bar{w}$ where $w = v^j \frac{\partial}{\partial z^j}$
2. $\bar{\partial}w = 0$

An element of $\text{Lie}(\text{Aut}(X,L))$ is a vector field $V$ in $\text{Aut}(L)$ (i.e., $V$ is the real part of a holomorphic vector field on $L$) which is $C^\infty$ invariant. We have a well defined map $q : \text{Lie}(\text{Aut}(X,L)) \rightarrow \text{Lie}(\text{Aut}(X))$: If $V \in \text{Lie}(\text{Aut}(X,L))$, and if $x \in X$, then $q(V)(x) \in T_xX$ is defined as follows: Let $l \in L$ be any point in $p^{-1}(x)$. Then $q(V)(x) = dp(V(l))$. This does not depend on the choice of $l$, since $V$ is $C^\infty$ invariant.

The kernel of $q$ is $C \cdot t$, where $t$ is the vector field generated by the infinitesimal action of $U(1)$. The hypothesis which Donaldson imposes in his theorem is that the image of $q$ is trivial, that is, $\text{Lie}(\text{Aut}(X,L))/(C \cdot t)$ is trivial. This group is characterized in Lemma 12 of [D] as follows: Fix a hermitian metric $h$ on $L$ with positive curvature $\omega$ (such an $h$ exists since, by assumption, $L$ is positive). Then Lemma 12 of [D] says that $v$ is in the image of $q$ if and only if $w^i = \omega^{ij} f_j$ for some smooth complex valued function $f$ on $X$. Note that this does not depend on the choice of $h$: If $h'$ is another positive metric, then $\omega$ is replaced by $\omega' = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$ for some $\phi$. So if $\omega_{ij} w^i = \bar{\partial} f$ then $\omega'_{ij} w^i = \bar{\partial} f + \frac{\sqrt{-1}}{2\pi} \phi_{ij} w^i = \bar{\partial} f'$ where $f' = f + \frac{\sqrt{-1}}{2\pi} \phi_i w^i$ (here we are using the fact that $\bar{\partial}w = 0$). Note as well that if we replace $L$ by $L^k$, the image of $q$ does not change since if $\omega_{ij} w^i = \bar{\partial} f$ then $k \omega_{ij} w^i = \bar{\partial} (kf)$.

We now show how Lemma 12 of [D] can be used to remove the hypothesis that $\text{Aut}(X)$ is discrete: Let $L \rightarrow X$ be a hermitian line bundle on a compact complex manifold $X$. Assume that $\omega$, the curvature of $L$, is a positive $(1,1)$ form. If $X$ has no holomorphic vector fields, then there is a constant $c > 0$ such that

$$||w||_{L_1^2(\omega_0)} \leq c \cdot ||\bar{\partial}w||_{L_2^2(\omega_0)}$$

(5.35)

for all $L_1^2$ vector fields $w = w^i \frac{\partial}{\partial z^i}$ on $X$.

Now we drop the assumption that $X$ has no holomorphic vector fields. Then (5.35) no longer holds for all $w \in L_1^2$. Let $H_0 \subseteq L_1^2$ be the space of vector fields of the form $w^i = \omega^{ij} f_j$ where $f$ ranges over all smooth complex valued functions on $X$, and let $H \subseteq L_1^2$ be the completion of $H_0$. By the standard elliptic estimates for the $\bar{\partial}$ operator, the space $H$ can be identified with vector fields of the form $\omega^{ij} f_j$, where $f$ is a function in the Sobolev space $L_2^2$.

**Lemma :** Assume that $\text{Aut}(X,L)/C^\infty$ is discrete. Then (5.35) holds for all $w \in H$.

Proof. Lemma 12 of [D] says that the kernel of $\bar{\partial}|_{H_0}$ is $\text{Lie}(\text{Aut}(X,L))/C$. But by elliptic regularity, $\ker(\bar{\partial}|_H) = \ker(\bar{\partial}|_{H_0})$. Now we are assuming

$$\text{Lie}(\text{Aut}(X,L)/C^\infty) = \text{Lie}(\text{Aut}(X,L))/C = 0$$

(5.36)

15
Thus the kernel of $\bar{\partial}|_H$ is trivial. Now let $w \in H$. Then $w' = w - \theta \in \ker(\bar{\partial})^\perp$, for some unique $\theta \in \ker(\bar{\partial})$. Here the orthogonal complement $\ker(\bar{\partial})^\perp$ is taken with respect to the $L^2$ norm. Thus
\[
||w'||_{L^2(\omega_0)} \leq c \cdot ||\bar{\partial}w'||_{L^2(\omega_0)} = c \cdot ||\bar{\partial}w||_{L^2(\omega_0)} \tag{5.37}
\]
On the other hand, there is a constant $c' > 0$ such that
\[
||w||_{L^2(\omega_0)} \leq c' ||w'||_{L^2(\omega_0)} \tag{5.38}
\]
for all $w \in H$. To see this, assume that $||w'_n||_{L^2(\omega_0)} \to 0$ for some sequence $w_n \in H$ such that $||w_n||_{L^2(\omega_0)} = 1$. Since $w_n = w'_n + \theta_n$ is an orthogonal decomposition in $L^2$, we have $||\theta_n||_{L^2(\omega_0)} \leq 1$. But the $\theta_n$ are holomorphic vector fields. Thus, after passing to a subsequence, the $\theta_n$ converge in $C^\infty$ to an element $\theta \in \ker(\bar{\partial})$. Since $w'_n = w_n - \theta_n \to 0$ in $L^2(\omega_0)$, we conclude that $w_n \to \theta$ in $L^2(\omega_0)$. Since $w_n \in H$ and since $H$ is closed in $L^2(\omega_0)$, we have $\theta \in H$. But we have seen that $\ker(\bar{\partial}|_H) = 0$. Thus $\theta = 0$. On the other hand, $w_n \to \theta$ in $L^2(\omega_0)$ so 1 = $||w_n||_{L^2(\omega_0)} \to 0$, which is a contradiction. This proves (5.38), and together with (5.37), $||w||_{L^2(\omega_0)} \leq c' ||\bar{\partial}w||_{L^2(\omega_0)}$. Our claim follows, in view of the standard elliptic estimate
\[
||w||_{L^2(\omega_0)} \leq c''(||\bar{\partial}w||_{L^2(\omega_0)} + ||w||_{L^2(\omega_0)}). \tag{5.39}
\]
Now we conclude the proof of Theorem 2. The discreteness assumption of Aut$(X)$ was used earlier only at one point, namely to ensure that (5.12) holds with $W = \pi_T \nu$. The lemma shows that if we only assume that Aut$(X,L)/C^\infty$ is discrete, that this inequality continues to hold provided $W$ is in $H$. Thus it suffices to show that $\pi_T \nu \in H$: Let $\nu$ be the vector field on $\text{P}^N$ given by the infinitesimal action of some element in sl$(N+1)$. Then $\omega_{ij}^{FS} \nu^i = \bar{\partial}F$ for some $F$, smooth on $\text{P}^N$. In other words, $\bar{\partial}F(Y) = \omega^{FS}(V,Y)$ for any $Y \in T\text{P}^N$. Now suppose that $Y \in T\nu$, and let $f = F|_\nu$ and let $\omega = \omega^{FS}|_\nu$. Then
\[
\bar{\partial}_X f(Y) = \bar{\partial}_\nu F(Y) = \omega^{FS}(V,Y) = \omega^{FS}(\pi_T \nu, Y) = \omega(\pi_T \nu, Y) \tag{5.40}
\]
so
\[
(\pi_T \nu)^i = \omega^{ij} f_j \tag{5.41}
\]
Thus $\pi_T \nu \in H_0 \subset H$, which is what we wanted to prove.

**Remark 1.** Donaldson also conjectures that $\Lambda \leq C k$ holds with the Hilbert-Schmidt norm on $su(N+1)$ replaced by the operator norm. This problem is still open.

**Remark 2.** Let $X = \text{P}^1$ be imbedded in $\text{P}^k$ by $O(k)$. Consider $\xi \in su(k+1)$ given by
\[
\xi_{ii} = i^2 - \frac{1}{6} k(2k+1) - k(i - \frac{1}{2})
\]
and $\xi_{ij} = 0$ if $i \neq j$. Then $\xi$ is orthogonal to the Lie algebra of Aut$(\text{P}^1, O(k))$ and
\[
||\xi||^2 = \frac{1}{180} k^5 + O(k^4), \quad ||X_\xi||^2 = \frac{1}{30} k^4 + O(k^3) \quad ||\pi_T X_\xi||^2 = \frac{1}{30} k^4 + O(k^3)
\]
Thus $||\pi_N X_\xi||^2 = O(k^3)$, and the bound $||\xi||^2 \leq C k^2 ||\pi_N X_\xi||^2$ is sharp.

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