The Okubo-Weiss Criteria in Two-Dimensional Hydrodynamic and Magnetohydrodynamic Flows

B. K. Shivamoggi* and G. J. F. van Heijst
J. M. Burgers Centre and Fluid Dynamics Laboratory
Department of Physics
Eindhoven University of Technology
NL-5600MB Eindhoven, The Netherlands

Abstract
The “slow-variation” restriction on the straining flow-velocity gradient field used in the Okubo [1]-Weiss [2] criterion is quantified via the Beltrami condition with the divorticity framework in 2D hydrodynamic flows. This turns out to provide interesting interpretations of the Okubo-Weiss criterion in terms of the topological properties of the underlying vorticity manifold. These developments are then extended to 2D quasi-geostrophic flows (via the potential divorticity framework) and magnetohydrodynamic flows and the Okubo-Weiss criteria for these cases are considered.

*Permanent Address: University of Central Florida, Orlando, FL 32816-1364, USA
1. Introduction

A central question in the problem of transport in two-dimensional (2D) turbulent flows is how to divide a vorticity field into hyperbolic (cascading turbulence) and elliptic (coherent vortex) regions because the topology of 2D turbulence is parameterized in terms of the relative dominance of flow deformation or flow rotation. Okubo [1] and Weiss [2] gave a kinematic criterion to serve as a diagnostic tool towards this goal which has been widely used in numerical simulations (Brachet et al. [3], Ohkitani [4], Babiano and Provenzale [5]) and laboratory experiments (Ouelette and Gollub [6]) of 2D hydrodynamic flows. A key assumption underlying the Okubo-Weiss criterion is that the vorticity gradient field evolves quasi-adiabatically with respect to the underlying straining flow-velocity gradient field. This issue was explored by Basdevant and Philipovitch [9] who tried to improve on it by invoking the topological properties of the pressure field, while Hua and Klein [10] tried to include the strain-rate time evolution explicitly. The purpose of this paper is to seek to quantify the “slow-variation” restriction on the straining flow-velocity gradient field used in the Okubo-Weiss criterion via the Beltrami condition with the divorticity framework in 2D hydrodynamic flows (Shivamoggi et al. [11]). This also turns out to provide interesting interpretations of the Okubo-Weiss criterion in terms of the topological properties of the underlying vorticity manifold. These developments are then extended to 2D quasi-geostrophic flows (via the potential divorticity framework) and magnetohydrodynamic flows and the Okubo-Weiss criteria for these cases are considered.

2. Beltrami Condition Interpretation of the Okubo-Weiss Criterion

The vorticity dynamics in 2D hydrodynamic flows is governed by the following equation (Kida [12], Kuznetsov et al. [13])

\[
\frac{\partial \mathcal{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathcal{B}) \tag{1a}
\]

or

\[
\frac{D\mathcal{B}}{Dt} \equiv \frac{\partial \mathcal{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathcal{B} = (\mathcal{B} \cdot \nabla) \mathbf{v} \tag{1b}
\]

where \( \mathbf{v} = \langle u, v \rangle \) is the flow velocity, \( \mathbf{\omega} \) is the vorticity,

\[
\mathbf{\omega} \equiv \nabla \times \mathbf{v} \tag{2a}
\]

and \( \mathcal{B} \) is the divorticity,

\[
\mathcal{B} \equiv \nabla \times \mathbf{\omega}. \tag{2b}
\]

Equation (1b) may be rewritten as

\[
\frac{D\mathcal{B}}{Dt} = \mathbf{A} \cdot \mathcal{B} \tag{1c}
\]

---

\(^{1}\)The Okubo-Weiss parameter describing the local strain-vorticity balance in the horizontal flow field of a shallow fluid layer turns out also to quantify the deviations from two-dimensionality of this flow (Cieslik et al. [7]). More specifically, the Okubo-Weiss parameter turns out to be the source turn in the Poisson equation for the pressure (Kamp [8]).
where $\mathbf{A}$ is the velocity gradient matrix,

$$
\mathbf{A} \equiv \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
s_1 & s_2 - \omega \\
s_2 + \omega & -s_1
\end{bmatrix}
$$

$$
s_1 \equiv -2\frac{\partial v}{\partial y}, \quad s_2 \equiv \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
$$

(3)

If the straining flow velocity gradient tensor $\nabla \mathbf{v}$ is assumed, following Okubo [1] and Weiss [2], to temporally evolve slowly so the divorticity field evolves quasi-adiabatically with respect to the straining flow-velocity gradient field, equation (1c) may be locally approximated by an eigenvalue problem with eigenvalues given by

$$
\lambda^2 = s_1^2 + s_2^2 - \omega^2 \equiv Q.
$$

(4)

The Okubo-Weiss parameter $Q$ is a measure of the relative importance of flow strain ($Q > 0$, hyperbolic) and vorticity ($Q < 0$, elliptic). Numerical simulations (Brachet et al. [3], Ohkitani [4], Babiano and Provenzale [5]) and laboratory experiments (Ouellette and Gollub [6]) of 2D hydrodynamic flows confirmed that coherent vortices are indeed located in elliptic regions while divorticity sheets are located in hyperbolic regions.

The “slow-variation” restriction on the straining flow-velocity gradient field used above may be quantified via the Beltrami condition with the divorticity framework in 2D hydrodynamic flows (Shivamoggi et al. [11]). Equations (1a-c) yield for the Beltrami state (Shivamoggi et al. [11]),

$$
\mathbf{B} = a \mathbf{v}
$$

(5)

$a$ being an arbitrary constant. Using (5), the Okubo-Weiss parameter $Q$ becomes

$$
Q = \frac{4}{a^2} \left[ \left( \frac{\partial^2 \omega}{\partial x \partial y} \right)^2 - \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 \omega}{\partial y^2} \right].
$$

(6)

(6) implies that the Okubo-Weiss parameter also characterizes the topological properties of the vorticity manifold - it is in fact the negative of the Gaussian curvature of the vorticity surface. Thus, the character of the ensuing time-dependent 2D flow behavior appears to be rooted in the local topological properties of the underlying equilibrium vorticity manifold. It may be mentioned that the above reduction was pointed out by Larcheveque [16] on the premise of replacing streamlines by isovorticity lines which lacked, as Larcheveque [16] admitted, any dynamical meaning - streamlines are actually isomorphic to divorticity lines (as implied by the Beltrami condition (5)).

---

2 It may be noted that divorticity sheets are also more likely to occur near vorticity nulls due to selective rapid viscous decay of vorticity in these layers (Shivamoggi et al. [11]), just as vortex sheets are more likely to form near velocity nulls in 3D hydrodynamic flows.

3 A similar approach was taken previously (Shivamoggi and van Heijst [14]) in the quantification of the “slow variation” restriction used in Flierl-Stern-Whitehead [15] zero angular momentum theorem for localized nonlinear structures in 2D quasi-geostrophic flows on the $\beta$-plane.
3. The Okubo-Weiss Criterion for Quasi-geostrophic Flows

Consider a 2D quasi-geostrophic flow in which the baroclinic effects are produced by the deformed free surface of the ocean. The governing equation (in appropriate units) is (Charney [17])

\[ \frac{\partial q}{\partial t} + (v \cdot \nabla) q = 0 \]  

(7)

where \( q \) is the potential vorticity vector,

\[ q \equiv \omega - k^2 \psi + f \]  

(8)

\( f \) is the Coriolis parameter, \( k \) is the inverse Rossby radius of deformation, \( k \equiv \sqrt{f_0^2 / gH} \), \( f_0 \) being the local value of \( |f| \) and \( H \) the depth of the ocean, which is taken to be uniform), and \( v \equiv -\nabla \times \psi \)

Upon taking the curl of equation (7), we obtain

\[ \frac{\partial D}{\partial t} = \nabla \times (v \times D) \]  

(10a)

where \( D \) is the potential divorticity vector (in analogy to the potential vorticity vector \( q \)),

\[ D \equiv \nabla \times q = B + k^2 v + h \]  

(11)

and

\[ h = \nabla \times f = \langle \beta, 0, 0 \rangle \]

\( \beta \) being the planetary vorticity gradient.

Equation (10a) may be rewritten as

\[ \frac{D D}{Dt} = \mathcal{A} \cdot D. \]  

(10b)

Following Okubo [1] and Weiss [2], and assuming that the potential divorticity field evolves quasi-adiabatically with respect to the straining flow velocity gradient field, equation (10b) may again be locally approximated by an eigenvalue problem with eigenvalues given by,

\[ \lambda^2 = \frac{1}{4} \left( u_y v_x + v_y^2 \right) = Q. \]  

(12)

Equation (10a) yields for the Beltrami state,

\[ D = b v \]  

(13)

\( b \) being an arbitrary constant. Using (13), the Okubo-Weiss parameter \( Q \) becomes

\[ Q \equiv \frac{1}{4b^2} \left[ \left( \frac{\partial^2 \omega}{\partial x \partial y} \right)^2 - \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 \omega}{\partial y^2} \right] \]  

(14)

which is the same as (6) for 2D hydrodynamic case. This shows that the Okubo-Weiss parameter is robust and remains intact under extension to 2D quasi-geostrophic flows (in the
β-plane approximation to the Coriolis parameter). The inclusion of the nonlinear variation in the Coriolis parameter (the so-called \( \gamma \)-effect) will, however, lead to changes in the Okubo-Weiss parameter,

\[
Q = \frac{1}{4b^2} \left[ \left( \frac{\partial^2 \omega}{\partial x \partial y} \right)^2 - \frac{\partial^2 \omega}{\partial x^2} \left( \frac{\partial^2 \omega}{\partial y^2} + \gamma \right) \right].
\]  

(15)

4. The Okubo-Weiss Criterion for Magnetohydrodynamic Flows

Consider a 2D incompressible magnetohydrodynamic (MHD) flow. The equation governing the transport of the magnetic field \( \mathbf{B} = \langle B_1, B_2 \rangle \) is (Goedbloed and Poedts [18])

\[
\frac{DB}{Dt} = (\mathbf{B} \cdot \nabla) \mathbf{v}
\]  

(16a)

which may be rewritten as

\[
\frac{DB}{Dt} = \mathcal{A} \cdot \mathbf{B}.
\]  

(16b)

If we now assume that the magnetic field evolves quasi-adiabatically with respect to the straining flow velocity gradient field, equation (16b) may again be locally approximated by an eigenvalue problem with eigenvalues given by,

\[
\lambda^2 = \frac{1}{4} \left( u_y v_x + v_y^2 \right) \equiv Q.
\]  

(17)

Noting that the MHD Beltrami state (Shivamoggi [19]) corresponds to the so-called Alfvénic state (Hasegawa [20])

\[
\mathbf{v} = c \mathbf{B}
\]  

(18)

c being an arbitrary constant, (17) becomes

\[
Q = \frac{c^2}{4} \left( B_1 v_y + B_2 v_x \right)^2.
\]  

(19)

In terms of the magnetic vector potential \( \mathbf{A} \) given by

\[
\mathbf{B} \equiv \nabla \times \mathbf{A}, \quad \mathbf{A} = A \hat{\mathbf{i}}_z
\]  

(20)

(19) becomes

\[
Q = \frac{c^2}{4} \left[ \left( \frac{\partial^2 A}{\partial x \partial y} \right)^2 - \frac{\partial^2 A}{\partial x^2} \frac{\partial^2 A}{\partial y^2} \right].
\]  

(21)

(21) implies that the Okubo-Weiss parameter \( Q \) for the MHD case characterizes the topological properties of the magnetic flux surface - it is the negative of the Gaussian curvature of the magnetic flux surface. As with the case of 2D hydrodynamic flows, (21) can serve as a useful diagnostic tool to parameterize the magnetic field topology in 2D MHD flows.
5. Discussion

The “slow variation” restriction on the straining flow-velocity gradient field used in the Okubo-Weiss criterion may be quantified via the Beltrami condition with the divorticity framework in 2D hydrodynamic flows (Shivamoggi et al. \[11\]). This also turns out to provide interesting interpretations of the Okubo-Weiss criterion in terms of the topological properties of the underlying vorticity manifold. Extension of these considerations to 2D quasi-geostrophic flows (via the potential divorticity framework) shows the robustness of the Okubo-Weiss parameter under varying 2D hydrodynamic flow situations. Extension to 2D MHD flows, on the other hand, provides one again with a useful diagnostic tool to parameterize the magnetic field topology in 2D MHD flows.

6. Acknowledgments

The authors are thankful to Dr. Leon Kamp for helpful discussions. BKS would like to thank The Netherlands Organization for Scientific Research (NWO) for the financial support.

References

[1] A. Okubo: Deep Sea Res. 17, 445, (1970).
[2] J. Weiss: Physica D 48, 273, (1991).
[3] M. E. Brachet, M. Meneguzzi, H. Politano and P. L. Sulem: J. Fluid Mech. 194, 333, (1988).
[4] K. Ohkitani: Phys. Fluids A 3, 1598, (1991).
[5] A. Babiano and A. Provenzale: J. Fluid Mech. 574, 429, (2007).
[6] N. T. Ouelette and J. P. Gollub: Phys. Rev. Lett. 99, 194502, (2007).
[7] A. R. Cieslik, L. P. J. Kamp, H. J. H. Clercx and G. J. F. van Heijst: Europhys. Lett. 85, 54001, (2009).
[8] L. P. J. Kamp: Phys. Fluids, Submitted, (2011).
[9] C. Basdevant and T. Philipovitch: Physica D 73, 17, (1994).
[10] B. L. Hua and P. Klein: Physica D 113, 98, (1998).
[11] B. K. Shivamoggi, G. J. F. van Heijst and J. Juul Rasmussen: Phys. Lett. A 374, 2309, (2010).
[12] S. Kida: J. Phys. Soc. Japan 54, 2840, (1985).
[13] E. A. Kuznetsov, V. Naulin, A. H. Nielsen and J. Juul Rasmussen: Phys. Fluids 19, 105110, (2007).
[14] B. K. Shivamoggi and G. J. F. van Heijst: Geophys. Astrophys. Fluid Dyn. 103, 293, (2009).

[15] G. R. Flierl, M. E. Stern and J. A. Whitehead: Dyn. Atmos. Oceans 7, 233, (1983).

[16] M. Larcheveque: Theor. Comp. Fluid Dyn. 5, 215, (1993).

[17] J. G. Charney: Geophys. Publ. Kosjones. Nor. Vidensk. Akad. Oslo 17, 1, (1947).

[18] H. Goedbloed and S. Poedts: Principles of Magnetohydrodynamics, Ch. 4, Cambridge Univ. Press, (2004).

[19] B. K. Shivamoggi: Euro. Phys. J. D 64, 393, (2011).

[20] A. Hasegawa: Adv. Phys. 34, 1, (1985).