An exact one-loop momentum-dependent Wilsonian renormalization group

Moritz Helias
Institute of Neuroscience and Medicine (INM-6) and Institute for Advanced Simulation (IAS-6) and JARA BRAIN Institute I, Jülich Research Centre, Jülich, Germany
Department of Physics, Faculty 1, RWTH Aachen University, Aachen, Germany

Abstract. Wilson’s original formulation of the renormalization group is perturbative in nature. Here, we derive a diagrammatic infinitesimal Wilsonian momentum shell RG that is exact, of one-loop structure, and complimentary to the Wegner and Houghton, the Morris-Wetterich, and the Pokhinsky scheme. A momentum scale expansion can be applied, yielding an intuitive diagrammatic framework to construct flow equations. We exemplify the method on the scalar $\phi^4$-theory, computing analytically the Wilson-Fisher fixed point, its anomalous dimension $\eta$ and the critical exponent $\nu$ non-perturbatively in $d \in [3, 4]$ dimensions. The results are in reasonable agreement with the known values, despite the simplicity of the method.
1. Introduction

The renormalization group (RG) is a standard tool to study phase transitions in statistical physics as well as to investigate renormalizable field theories in high energy physics. In its field-theoretical formulation, it is applicable only to renormalizable theories; those, in which all physical quantities can be expressed in terms of a few renormalized parameters. One here exploits the arbitrariness of the chosen scale at which the renormalized parameters are fixed to derive a differential equation in this scale parameter [1].

Conceptually quite different is the idea of Wilson’s renormalization group, which studies how the action that describes a theory changes as short-ranged degrees of freedom are marginalized out. The interpretation is thus more intuitive. Initially, this scheme has been presented as an iterative method, where in each step a fixed fraction of the the momentum space is integrated out [2, 3]. A hard cutoff separates the marginalized, short-distance modes from the long-distance modes that remain after the marginalization step. One studies how the action evolves as this cutoff is lowered. This approach has the merit of a clear interpretation as a systematic coarse-graining. Also, the procedure is applicable to theories that originally possess a high momentum cutoff. Those appear, for example, in condensed matter problems, where the lattice spacing limits the magnitude of all momenta. Thus, the Wilsonian RG is more versatile as it is applicable also to non-renormalizable theories.

The classical computation in the Wilsonian RG, however, relies on a diagrammatic perturbation expansion. For this reason it is tied to the computation of properties of systems close to their upper critical dimension $d_c$, the dimensionality above which non-Gaussian terms can be neglected. In the vicinity below $d_c$, interaction vertices are small, presenting a natural expansion parameter. The $\epsilon$-expansion, operating in $d_c - \epsilon$ dimensions, exploits this fact. A drawback of this method is that quantities must be computed from typically divergent series in $\epsilon$ that require appropriate resummation. Also, computations become complicated beyond one loop order due to the presence of the cutoff. In practice, one therefore often resorts to the field-theoretical formulation, even in the context of condensed matter problems.

Early on, Wegner and Houghton [4] presented a continuous version of the Wilsonian RG, which is of one-loop structure by employing a sharp cutoff. But the simplicity of the diagrammatic formulation by Wilson was lost to some extent due to the employed projector formalism. The authors noted difficulties of a sharp cutoff: it generates a non-analytical momentum dependence of the vertices at vanishing momenta, which correspond to long-range interactions [2, p. 153]. This general problem of treating the momentum dependence within the Wilsonian RG has been noted early on [2, cf. Fig 11.1 and surrounding text]. A sharp cutoff is, moreover, needed ensure that the action maintains its rescaling invariance, the independence of physical results under the transform $\varphi \to z \varphi$ [5]. This property must be kept in order to compute the anomalous dimension.

Morris [5] showed that there is in fact no fundamental problem of using a hard cutoff, if combined with the flow equation for the effective action [6], the Legendre transformed Helmholtz free energy. A momentum-scale expansion can be made for the flow equation, where the momentum scale $|k|$ is used as the expansion parameter. However, the author pointed out a conceptual problem: despite being an expansion for small external momenta, vertices evaluated at momenta as large as the cutoff are required to close the equations. Also, critical exponents turned out to be not as
accurate as obtained with methods of comparable complexity. Kopietz [7] showed that
the exact Wetterich flow equation for the effective action [6] indeed can be used to
systematically derive results beyond one-loop order. A drawback of this method is the
requirement to compute diagrams with more than a single loop.

The generality of the Wilsonian RG, working for renormalizable as well as for
non-renormalizable theories, as well as its intuitive interpretation in terms of coarse-
graining and its efficient diagrammatic formulation are desirable features. However,
it has been pointed out that a momentum-scale expansion was incompatible with
the closely related Wegner and Houghton RG equation, because the latter contained
tree-diagrams besides one-particle irreducible components. The reason is that tree
diagrams vanish only for zero external momenta, as the momentum on the connecting
propagator is constrained to reside above the cutoff.

We here show that in fact a continuous Wilsonian RG equation with sharp cutoff
can be derived quite naturally without the projector formalism and be interpreted
diagrammatically. We find that again tree diagrams appear, but their treatment is
actually quite simple: one can systematically keep only those tree diagrams that are
needed to integrate the remainder of the flow equation. These tree diagrams disappear
from the final effective long-range theory, as one evaluates all quantities on the long-
distance scale of small momenta.

Our derivation shows that there is no approximation implied by going to only a
single loop. Instead, we employ two approximations to arrive at closed-form results:
First, a truncation in the power of the field, only keeping a limited number of
vertices for which we compute the flow. This approximation is naturally guided
by the relevance of each term as indicated by its engineering dimension. Second,
the momentum-dependence of the vertices is approximated in the momentum-scale
approximation, as introduced by Morris [5] and employed in the context of the
functional RG [8, 9].

We exemplify this method by computing a non-zero approximation of the
anomalous dimension \( \eta \) of the scalar \( \varphi^4 \)-theory. The computation requires only
elementary spherically symmetric and uniaxial one-loop integrals. We show that the
flow equations can be evaluated far off the upper critical dimension \( d_c = 4 \), directly
obtaining results for any dimension \( d \in [3, 4] \). This demonstrates the non-perturbative
nature of the equation. We find that the approximation for the critical exponent \( \eta \)
is in between the values for the \( \epsilon \)-expansion of orders \( \epsilon^2 \) and \( \epsilon^3 \) and slightly better
than the result obtained by Morris [5]. It is, moreover, far better than the second
order \( \epsilon \)-expansion, despite only requiring one-loop integrals. In particular, there is no
divergent series to be resumed to obtain estimates for critical exponents.

The results in Section 2 are presented in a self-contained manner: The initial
sections set up the notation and present a coherent exposure of the basic idea of the
renormalization group: In Section 2.1 we introduce the notation for the field theory
and the \( \varphi^4 \)-theory as a particular example. Section 2.2 defines the
procedure of coarse-graining in Fourier domain. Section 2.3 introduces the specific use
of diagrams to carry out the decimation step. These sections can be skipped by readers
familiar with the concepts of Wilson's formulation of the renormalization group [2, 3].

The new method is developed in the subsequent sections: Section 2.4 presents
the infinitesimal momentum shell diagrammatic formulation of the Wilsonian RG and
Section 2.5 shows that two distinct classes of graphs contribute. The main result,
an exact diagrammatic one-loop RG equation, is prepared in Section 2.6. Rescaling
of momenta and fields, required to obtain fixed points, is the topic of Section 2.7.
Section 2.8 discusses how the momentum dependence of effective vertices causes different decimation steps to be connected diagrammatically in the flow equation. The method is then illustrated on the example of the $\phi^4$-theory, deriving the flow equations for the momentum-dependent interaction in Section 2.9 and of the self-energy in Section 2.10. Section 2.11 determines the fixed points for dimensionality $d \in [3, 4]$ of the flow equations and computes the critical exponents $\nu$ and $\eta$. Section 3 summarizes the results in the light of the literature and exposes the relation of this interpretation of the Wilsonian RG to the $\epsilon$-expansion [2, 3] and to the seminal work by Wegner & Houghton [4].

2. Results

We here chose a notation that should be easy to transfer to other problems. For concreteness, we here choose the language of classical statistical field theory, in particular to bosonic fields. In the following Section 2.1 we set up the language and define elementary quantities.

2.1. Model system

We assume a form of the action

$$S[\varphi] = -\frac{1}{2} \varphi^T G^{-1} \varphi, \tag{1}$$

$$+ \sum_{n=3}^{\infty} \frac{1}{n!} \int_{k_1} \cdots \int_{k_n} S^{(n)}(k_1, \ldots, k_n) \varphi(k_1) \cdots \varphi(k_n),$$

of a translation-invariant system, so that the quadratic part $G^{-1}$ and the bare interaction vertices $S^{(n)}$ all conserve momenta

$$G^{-1}(k_1, k_2) \propto \delta(k_1 + k_2),$$

$$S^{(n)}(k_1, \ldots, k_n) \propto \delta(k_1 + \ldots + k_n).$$

A particular example, we study the $\phi^4$-theory

$$S[\varphi] = -\frac{1}{2} \int_k \varphi(-k) \left( r^{(0)}(k^2) + r^{(2)} k^2 \right) \varphi(k) \tag{2}$$

$$- u^{(0)} \int_{k_1} \int_{k_2} \int_{k_3} \varphi(k_1) \varphi(k_2) \varphi(k_3) \varphi(-k_1 - k_2 - k_3),$$

as a prototypical system, for which the Gaussian part $G^{-1}$ is given by $r^{(0)} + r^{(2)} k^2$ and there is only a single bare interaction vertex $u^{(0)} \delta(k_1 + \ldots + k_4)$. The superscripts here refer to the order in $k$ of the momentum dependence. A configuration of the field $\varphi(k)$ is realized with probability of Boltzmann form

$$p[\varphi] \propto \exp \left( S[\varphi] \right),$$

and the system is described by the partition function

$$Z = \int D\varphi \exp \left( S[\varphi] \right). \tag{3}$$
2.2. Definition of coarse-graining

We assume the system lives on a space \( r \in \mathbb{R}^d \) and the elementary entities (e.g. spins) have a lattice spacing \( a \). Correspondingly, we have a high-momentum cutoff \( \Lambda = \frac{\pi}{a} \) and \(|k| < \Lambda\).

We use the notation \( \varphi_{<} \) for the coarse-grained field that is defined in terms of the long-ranged degrees of freedom

\[
\varphi_{<}(r) : = \int_{0 \leq |q| < \Lambda} \varphi(q) e^{iqr},
\]

where we only take the Fourier integral up to the new cutoff \( \Lambda \ell^{-1} \) with an \( \ell > 1 \). Here and in the following we use the short hand \( \int_{0 \leq |q| < \Lambda} = \int_{0 \leq |q| < \Lambda} \frac{d^dq}{(2\pi)^d} \).

Analogously we define the short-ranged degrees of freedom as

\[
\varphi_{>}(r) : = \int_{\Lambda \ell^{-1} < |k| < \Lambda} \varphi(k) e^{ikr}.
\]

Here and in the following we will denote momenta as \( q \) if they are long-ranged, \( 0 < |q| < \ell^{-1} \Lambda \), and those as \( k \) which short ranged, \( \Lambda \ell^{-1} < |k| < \Lambda \).

The Fourier transform decomposes the fluctuations into long-ranged \( \varphi_{<} \) and short-ranged \( \varphi_{>} \) ones; with the linearity of the Fourier transform and the orthogonality of Fourier modes, it yields a decomposition of the entire field into the direct sum

\[ \varphi = \varphi_{<} + \varphi_{>} . \]

We may ask which action \( S_\ell \) effectively controls the coarse-grained modes \( \varphi_{<} \). This question reduces to a marginalization of the fast modes and we define the action for the slow modes as

\[
\exp(S_\ell[\varphi_{<}]) : = \int D\varphi_> \exp(S[\varphi_{<} + \varphi_>]),
\]

where the notation \( \int D\varphi_> \) should be read as the integral of all Fourier modes with \( \ell^{-1} \Lambda < |k| < \Lambda \). We call this step decimation. By this definition and the orthogonality of Fourier modes we have

\[
\int D\varphi = \int D\varphi_< \int D\varphi_>
\]

so that the partition function (3) can be written in terms of the coarse-grained action alone

\[
Z = \int D\varphi_< \exp(S_\ell[\varphi_<]).
\]

Performing the marginalization over a finite momentum shell, the modes with Fourier coefficients \( k \in [\Lambda \ell^{-1}, \Lambda] \) are integrated out. This avoids any infrared divergences that would plague an ordinary loopwise expansion. After this step, the range of momenta is limited to \( 0 < |q| < \ell^{-1} \Lambda \).

To find a fixed point we need to bring the action into a comparable form as the initial action, thus rescaling the length scale by \( \ell \) and the momentum scale correspondingly as

\[
\ell q =: k_\ell,
\]

so that there are two consecutive steps
$$S[\varphi(k)] \quad \text{decimation} \quad \rightarrow \quad S_\ell[\varphi_<(q)] \quad \text{rescaling} \quad \rightarrow \quad S_\ell[\varphi_<(k_\ell)] \quad |k_\ell| = |\ell q| \in [0, \Lambda] \ .$$

\[ (7) \]

2.3. Diagrammatics of the decimation step

This section reviews the diagrammatic notation to efficiently organize the marginalization over the fast modes in (5). To obtain a flow equation, consider the difference between the coarse-grained action (5) and the original action

$$S_\ell[\varphi_<] - S[\varphi_<] = \ln \left( \int D\varphi_> \exp(S[\varphi_< + \varphi_>]) \right) - S[\varphi_<] \ .$$

We assume an action of the form (1) and write the interaction part for short as

$$\sum_{n=3}^{\infty} \frac{1}{n!} \int_{k_1} \cdots \int_{k_n} S^{(n)}(k_1, \ldots, k_n) \varphi(k_1) \cdots \varphi(k_n) =: \sum_{n=3}^{\infty} S^{(n)}_n \varphi^n \ .$$

By assumption the quadratic part is diagonal in momentum, so that slow and fast modes are not coupled by the Gaussian term. Therefore (8) takes the form

$$S_\ell[\varphi_<] - S[\varphi_<] = \ln \int D\varphi_> \exp \left( -\frac{1}{2} \varphi_>^T G^{-1} \varphi_> - \sum_{n=3}^{\infty} \frac{S^{(n)}_n}{n!} (\varphi_< + \varphi_>)^n \right)$$

$$- \sum_{n=3}^{\infty} \frac{S^{(n)}_n}{n!} \varphi_<^n \ ,$$

where we have canceled the quadratic term $-\frac{1}{2} \varphi_>^T G^{-1} \varphi_<$ contained in $S_\ell$ (cf. Section 5.1, eq. (81) for details) with the identical term that is subtracted with $S[\varphi_<]$.

Due to the logarithm and by the linked cluster theorem (see e.g. [1] or [10, Appendix A.3]) the integral in the penultimate line only produces connected diagrams with propagator $G$ and interaction vertices $S^{(n)}_n/(\varphi_< + \varphi_>)^n$. Since the field of integration is $\varphi_>$, $G$ only contracts pairs of such fields; it leaves untouched the fields $\varphi_<$. Representing the interaction vertices diagrammatically, at each leg there is a sum of $\varphi_<$ and $\varphi_>$, so each interaction vertex produces a binomial factor

$$\sum_{n=3}^{\infty} \frac{1}{n!} S^{(n)}_n (\varphi_< + \varphi_>)^n \Rightarrow \sum_{n=3}^{\infty} \frac{1}{n!} S^{(n)}_n \sum_{l=0}^{n} \binom{n}{l} (\varphi_<)^l (\varphi_>)^{n-l} .$$

Among these terms, for $l = n$ there is one component $\sum_{n=3}^{\infty} S^{(n)}_n/n! \varphi_<^n$, which exactly cancels the last term in (10). For the remaining terms it is easiest to consider the graphical representation in terms of Feynman diagrams.

For concreteness, consider as an example the four point vertex of the action (2)

$$\frac{S^{(4)}}{4!} (\varphi_< + \varphi_>)^4 = \varphi_< + \varphi_> \ .$$

(12)
The combinatorial factor \( \binom{n}{l} \) can be interpreted graphically as the number of combinations to assign a field \( \phi_\prec \) to \( l \) of the \( n \) end-points of the vertex.

What remains are therefore two sorts of diagrams: First, vacuum diagrams with \( l = 0 \) in (11), where we assign a field \( \phi_\succ \) to all end points of the vertex (12); these are contracted in pairs by the Gaussian propagator \( G \). These diagrams contribute an overall constant, because they do not contain any dependence on the fields \( \phi_\prec \), so they affect the absolute value of the partition function and hence the free energy. If we are interested in the correlation functions of the system, we may hence ignore them here. In the case of the four point vertex, to lowest order in the number of loops one obtains the diagram

\[
3 \cdot \begin{tikzpicture}
  \draw (0,0) circle (1); \draw (1,0) circle (1);
  \draw (0,0) -- (1,0);
\end{tikzpicture}.
\]

The second class of diagrams has a field \( \phi_\prec \) at some external legs, whereas the remaining legs have fields \( \phi_\succ \), that are pairwise contracted by propagators \( G \); the case \( 0 < l < n \) in (11). These are non-trivial contributions to the effective interaction vertices for low momenta \( q \). An example in the case of a four-point interaction is

\[
\phi_{\prec}(q) = \phi_{\prec}(q) \phi_{\prec}(-q) \frac{4 \cdot 3}{2} \int_{\Lambda^{-1} < |k| < \Lambda} \frac{S^{(4)}(-q,q,k,-k)}{4!} G(k),
\]

where \( \phi_\prec \) denotes an amputated leg, as leg without any propagator attached. Here \( S^{(4)}(-q,q,k,-k) \) denotes the frequency dependence of the four-point vertex. We make two observations:

(i) The diagram has two (amputated) legs \( \phi_\prec \), so it contributes to the self-energy \( \Sigma \) of low momenta: Taking the second derivative of the coarse-grained action, the two fields are stripped off and the above diagram therefore changes this quadratic coefficients in the fields; it therefore, by Dyson’s equation, affects the propagator for the fields \( \phi_\prec \).

(ii) Due to momentum conservation at the vertex, the resulting diagram also conserves momentum, hence it yields again a diagonal contribution to the inverse propagator; therefore, iteration of the procedure does not generate corrections to the Gaussian part that are non-diagonal in momentum space: the separation of the trivial Gaussian part for the low \( \phi_\prec \)-degrees of freedom, as done in (10), is therefore always possible. This property is general and directly follows from momentum conservation at the vertices and propagators.

2.4. Decimating an infinitesimal momentum shell

The aim of this section is to derive a differential form for the Wilson RG. To this end, we consider the difference between the coarse-grained action \( S_{\ell, (1 + \delta)} \), defined as (5), and the action \( S_\ell \); the cutoff \( \Lambda_\ell \) is thus lowered by an infinitesimal amount controlled by \( \delta > 0 \). Assume all modes \( k > \Lambda_\ell \) have been marginalized and in (10) we integrate over the next lower interval

\[
\frac{\Lambda}{\ell \cdot (1 + \delta)} = \frac{\Lambda_\ell}{1 + \delta} < |k| < \Lambda_\ell.
\]
An exact one-loop momentum-dependent Wilsonian renormalization group

Choosing \( \delta > 0 \) infinitesimal and assuming sufficiently smooth momentum dependence of the propagator, the integral over the absolute value of the momentum is proportional to \( \Lambda_{\ell} - \Lambda_{\ell} (1 + \delta)^{-1} = \Lambda_{\ell} - \Lambda_{\ell} (1 - \delta + \mathcal{O}(\delta^2)) \), so it is proportional to the thickness of the momentum shell \( \delta \Lambda_{\ell} \). The remaining angular integral is over a \( d-1 \) dimensional hypersphere. For this marginalization step an analogous equation as (10) applies for the difference \( S_{\ell(1+\delta)}[\varphi_\ell] - S_{\ell}[\varphi_\ell] \), only that on the right hand side vertices \( S^{(n)}_{\ell} \) and propagators \( G_{\ell} \) of the coarse-grained action \( S_{\ell} \) at scale \( \ell \) appear.

All contributions with more than a single loop, \( n_L > 1 \), are proportional to \( \delta^n \), because each loop constitutes an independent integration variable. As we are after an infinitesimal flow equation, these can consequently be dropped: The one-loop calculation becomes exact.

Deriving a differential equation for \( S_{\ell} \) in \( \ell \) allows us to remove the \( |k| \)-integral altogether: A one-loop integral across the momentum shell (14) is of the form

\[
I_\delta := \int_{\Lambda_{\ell(1+\delta)} < |k| < \Lambda_{\ell}} f(k)
\]

\[
= \frac{1}{(2\pi)^d} \int_{\Lambda_{\ell(1+\delta)}}^{\Lambda_{\ell}} dk k^{d-1} \int d\Omega f(\Omega \cdot k),
\]

where \( d\Omega \) is the angular integral. Now taking the limit

\[
\lim_{\delta \to 0} \frac{S_{\ell(1+\delta)}[\varphi_\ell] - S_{\ell}[\varphi_\ell]}{\delta} \equiv \ell \frac{\partial}{\partial \ell} S_{\ell}[\varphi_\ell]
\]

we hence may expand the integral (15) in \( \delta \) to compute the limit \( \delta \to 0 \)

\[
\lim_{\delta \to 0} \frac{I_\delta}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta} \left( \delta \ell \frac{d\Lambda_{\ell}}{d\ell} \frac{d}{d\Lambda_a} \int_{\Lambda_a}^{\Lambda_{\ell}} dk k^{d-1} \int d\Omega f(\Omega \cdot k) \bigg|_{\Lambda_a=\Lambda_t} + \mathcal{O}(\delta^2) \right),
\]

\[
= \frac{\Lambda_{\ell}^d}{(2\pi)^d} \int d\Omega f(\Omega \cdot \Lambda_{\ell}).
\]

In the following we write for short

\[
\int d\Omega f(\Omega \cdot \Lambda_{\ell}) \equiv \int_{|k|=\Lambda_{\ell}} f(k).
\]

Applied, as an example, to eq. (13), we get the contribution to the self-energy

\[
\ell \frac{d}{d\ell} \varphi_\ell(q) = \varphi_\ell(q) \varphi_\ell(-q) \cdot \frac{\Lambda_{\ell}^d}{(2\pi)^d} \int_{|k|=\Lambda_{\ell}} \frac{S^{(4)}_{\ell}}{4!} (-q, q, k, -k) G_{\ell}(k).
\]

2.5. Two classes of graphs

We will now show that there are graphs of two distinct topologies that contribute to the flow equation:
(i) Diagrams that have a single propagator loop that, on either end, connects to the same vertex.

(ii) Diagrams, where a propagator connects two otherwise unconnected parts; we call such diagrams reducible.

These two contributions are qualitatively different with regard to the number of decimation steps that contribute to their flow: In the first case, all shells contribute, whereas in the latter case, only precisely a single shell contributes. This property will allow us then to directly insert the latter contributions into the former to obtain a one-loop equation. The details are explained in the following.

**Reducible contributions** The decimation step (10) produces connected diagrams that are, for example, of the form

\[
\begin{align*}
&\varphi_<(q_1) \quad \varphi_<(q_2) \quad \cdots \quad \varphi_<(q_3) \\
&= 4 \cdot 4 \cdot \frac{1}{2!} \left( \frac{S^{(4)}}{4!} \right)^2 G(q_1 + q_2 + q_3),
\end{align*}
\] (18)

where two components of a graph are connected by a propagator line. The momentum dependence of the left interaction vertex is here

\[S^{(4)}(q_1, q_2, q_3, -q_1 - q_2 - q_3).\]

Any line that appears in a diagram from the decimation step must necessarily carry a momentum that lies within the decimation shell, because these are the only Gaussian contractions computed. This holds in particular for the line that connects the two parts of the diagram (18). As a result of momentum conservation the momenta that enter the left part of the diagram need to sum up to a momentum that lies within the decimation shell. If we consider an \(n\)-point vertex at the end of such a diagram, and its \(n - 1\) amputated legs which we assume to carry momenta \(q_1, \ldots, q_{n-1}\), these must satisfy

\[\sum_{i=1}^{n-1} q_i = k_i, \quad (19)\]

where in the \(i\)-th step of the decimation we have

\[\Lambda_{\ell + 1} < |k_i| < \Lambda_\ell, \quad \ell_i = 1 + i \cdot \delta. \quad (20)\]

In what follows we write this relation for short as

\[\Lambda_{\ell + 1} < |k| < \Lambda_\ell \quad \leftrightarrow \quad k \in i.\]

In the discrete form of the flow equation considered in Section 2.4, decimation shells have a thickness controlled by \(\delta > 0\). The decimation process is thus a sequence of integrations, where the \(i\)-th step and the \(i+1\)-st step extend over adjacent but disjoint momentum shells defined by (20). The set of \(k\)-vectors \(k_i\) and \(k_j\) are therefore disjoint for \(i \neq j\). As a consequence the condition (20) describes disjoint triples of \(k\)-vectors: Each integration step contributes to a different part in momentum space of the six-point vertex (18). Stated differently, if we fix the external momenta \(q_1, \ldots, q_{n-1}\) of this vertex, the momentum \(k_i\) on the line is uniquely fixed. So there is only exactly one momentum shell, namely decimation step \(i\) for which (20) holds, that contributes to the value of the effective six-point vertex at the point in momentum space that is given by the external momenta.
An exact one-loop momentum-dependent Wilsonian renormalization group

The consequence is that the integration of the flow equation for such a reducible diagram is trivial; it simply has the value given by eq. (18) whenever the momentum on the connecting line is above the current cutoff \( \Lambda_{\ell+1} \), while it is zero if the momentum on the line is below; in this latter case, the corresponding shell has not been integrated out yet. In integrating the flow equation for the loop diagrams we can therefore replace any sub-diagram of reducible form by this rule.

Loop contributions Loop diagrams have a non-trivial flow equation, in the sense that multiple decimation steps, defined as (20), contribute to a vertex at a given and fixed set of external momenta. To see this explicitly, we take as an example a portion of the one-loop integral

\[
\varphi_c(q_1) \frac{q_1 + q_2 + k}{-k} \varphi_c(q_2),
\]

Now consider, for example, the point \( q_1 = -q_2 \) in momentum space. By momentum conservation at the vertex, any value of \( k \) is allowed. So all decimation steps will contribute to the flow of the vertex function at this set of external momenta.

In general, consider an \( n \)-point vertex that is part of a one-loop diagram, where the loop is closed by a single contraction from a decimation step. Such a vertex necessarily has two lines that each carry a momentum within the shell. Let the momenta of the remaining \( n-2 \) amputated legs be \( q_1, \ldots, q_{n-2} \). The momentum of the two legs that belong to the loop are, generalizing (21), \( \sum_{i=1}^{n-2} q_i + k \) and \( -k \). So in the \( i \)-th decimation step we have the two conditions

\[
\sum_{i=1}^{n-2} q_i + k \in i, \quad k \in i.
\]

The latter of these conditions shows that \( k \) lies on a spherical shell of inner radius \( \Lambda_{\ell+1} \) and outer radius \( \Lambda_{\ell} \) that is centered at 0. The former is a condition that \( k \) lies on such a spherical shell, too, but that is centered at \( -\sum_i q_i \). For both conditions to be simultaneously true, we hence must have

\[
\sum_{i=1}^{n-2} q_i \simeq 0,
\]

where the latter condition must be assumed with an accuracy corresponding to the thickness of the momentum shell.

The conclusion from this observation is that in the case of non-vanishing momentum transfer \( \sum_{i=1}^{n-2} q_i \neq 0 \), in each decimation step we only need to consider one-loop diagrams with a single propagator on the shell. If the remainder of the diagram is an effective vertex that is produced by additional contracting lines, these lines necessarily must belong to decimation steps that have already been integrated out. This observation serves us in the following to derive a one-loop flow equation that is valid also for non-zero momentum transfer.
An exact one-loop momentum-dependent Wilsonian renormalization group

\[ \Lambda_\ell = \frac{\Lambda}{\ell} \quad \ell_i = 1 + i\delta \]

**Figure 1. Flow of momentum dependent vertex from the decimation step.** An effective six-point vertex of reducible tree shape (top) is produced within decimation step \( i \). The shell \( i \) in which the six-point vertex is produced is determined by the momentum \( q_1 + q_2 + q_3 \in i \) on the propagator line \( G_\ell \).

The propagator \( G_\ell \) and the four-point vertices \( S^{(4)}_\ell \) take the values of the corresponding RG step \( i \). Later in the flow, at \( j > i \), two legs of the six-point vertex are contracted to form a contribution to the four-point vertex \( S^{(4)}_\ell \). For given external momenta \( q_1, q_2 \), the integration over the loop momentum \( q_3 \) thus involves six point vertices produced in different shells \( i \).

**Composed contributions** The two classes of diagrams can form joint diagrams. In the example of the \( \varphi^4 \) theory, we obtain a contribution to the four point vertex from a one-loop integral containing a six-point vertex of the form (18) by contracting two of its legs. This situation is illustrated in **Figure 1** with the result

\[ \varphi^2(q_1) \quad k \quad \varphi^2(q_2) \]

\[ \varphi^2(-q_1-q_2-q_3) \quad k \quad \varphi^2(-q_1-q_2-q_3) \]

where the momentum \( k_j \in j \) is within the shell of the \( j \)-th decimation step and the momentum \( k_i \) belongs to the reducible six point diagram (18). The latter has been obtained by the decimation step \( i \); for this vertex to be present it hence must hold that \( |k_i| > \Lambda_{\ell_{i+1}} \) or \( i < j \).

The conclusions from these observations are the following:

- Contributions from tree diagrams to the coarse-grained action can be integrated trivially. They have a non-vanishing value whenever all propagators that connect the vertices of the tree are above the current cutoff \( \Lambda_\ell \).

- Only one-loop diagrams contribute non-trivially to the flow of the effective vertices; we may replace all reducible subgraphs in such diagrams by their value determined by the previous point. As a consequence, in each such loop diagram there is only precisely one single-scale propagator, a propagator whose momentum is constrained to the shell of the current decimation step.
The flow equation is of the same structure as in the Wetterich equation \[6\]: only one propagator in a graph is a single-scale propagator with a momentum \(k_j\) in the shell, while all remaining lines must carry momenta \(k_i\) above the current cutoff – in the Wetterich equation this property is enforced by the condition of the regulator that freezes all fluctuations below the current value of \(\Lambda_\ell\) – so such lines cannot appear in the diagrams of the flow equation.

### 2.6. An exact one-loop equation for the Wilson effective action (not quite)

The conclusion from the last two points is that we obtain a one-loop equation for the coarse-grained action \(S_\ell[\varphi_<]\) of the form

\[
\ell \frac{\partial}{\partial \ell} S_\ell[\varphi_<] = \sum_{1 \text{ loop}} \text{amputated connected graphs}(G_\ell, \{ \frac{1}{n!} S_\ell^{(n)}(\varphi_< + \varphi_>)^n \}),
\]

where only one single-scale propagator appears in each loop, whose momentum is constrained to an infinitesimally thin shell with \(|q_j| = \Lambda_\ell\) and all other connecting lines \(G_\ell\) must carry momenta \(k > \Lambda_\ell\); the interaction vertices are \(\{ \frac{1}{n!} S_\ell^{(n)}(\varphi_< + \varphi_>)^n \}_n\).

The example of the \(\varphi^4\)-theory yields a correction to the interaction vertex \(S^{(4)}\) of the form

\[
\ell \frac{d}{d\ell} \left( 3 \cdot 3 \cdot \frac{1}{2} \right) \cdot \int_{|k|=\Lambda_\ell} \frac{1}{6!} S_\ell^{(6)}(q_1, q_2, q_3, -q_1 - q_2 - q_3, k, -k) G_\ell(k).
\]

The effective six-point vertex \(S_\ell^{(6)}\) that appears here is the vertex that has been produced in a previous (or the same) decimation step by a diagram of the form (18). The shell in which this vertex has been produced, by the above arguments, depends on the momentum carried by the line appearing in (18). For the example above, this is shell \(j\) at scale \(\Lambda_j\).

\[
k + q_1 + q_2 \in j.
\]

If we insert the subdiagram (18) into (24), the two four-point vertices hence need to carry the index \(\ell_j\) – these vertices must have the value they had in the step when the six-point vertex has been produced; the same holds for the propagator. The following sections will address this issue to obtain a closed form one-loop equation.

The factor \(3 \cdot 3 \cdot \frac{1}{2}\) denotes the number of possibilities to choose a pair of amputated legs to contract, where one leg comes from the left interaction vertex in (18) and one from right. Contracting a pair of legs from the same vertex results in a contribution where the momentum on the connecting propagator in (18) is constrained to the long-ranged sector; thus it does not contribute to the flow.

The problem that different momentum shells contribute to the flow of a vertex only arises if one considers the dependence on external, non-zero momenta. As long
as the corrections are computed at vanishing external momenta, all propagator lines in a one-loop diagram necessarily lie on the same shell. This special case is treated in most text books and in the original review by Wilson [3].

2.7. Rescaling of momenta and wavefunction renormalization

To solve the problem exposed in the last section, we need to consider the rescaling of momenta. To find fixed points of the RG equations the length scale and hence the momentum scale are rescaled so that the ranges of momenta are the same before and after a coarse-graining step; otherwise there cannot possibly be any fixed points. It is common to define the new momenta $k_\ell$ as (6). As a consequence these now span again the same space as did the $k$ prior to any decimation

$$0 < |k_\ell| < \Lambda.$$ 

To express the coarse-grained action in the corresponding new field variables, one defines

$$\varphi_\ell(k_\ell) := \ell^{1-\frac{d}{2} - \eta} \varphi(q),$$

where the relation between $k_\ell$ and $q$ is fixed by (6) and we introduced the wavefunction renormalization factor $\ell^{1-\frac{d}{2} - \eta}$ and the anomalous dimension $\eta$ as usual to keep the $k^2$-dependent coefficient $r(2)$ in (2) invariant under rescaling [3]: the factor $\ell^{1-\frac{d}{2}}$ is chosen such that the Gaussian theory alone would maintain scale-invariance, as shown in the following:

Our aim is to bring the coarse-grained action into the same form as the original Gaussian action at the expense of modified parameters. For the Gaussian part of the action (see also Section 5.1 for details), expressing $\varphi_<$ by the rescaled field $\varphi_\ell$ (25)
yields, with the substitution

$$S_\ell[\varphi_<] = \ln Z_> - \frac{1}{2} \int_{0 \leq |q| < \Lambda} \varphi_<(q) \left(r(0) + r(2) q^2\right) \varphi<(q)$$

$$= \ln Z_> - \frac{1}{2} \ell^{2-d-\eta} \ell^{-d} \int_{0 \leq |k_\ell| < \Lambda} \varphi_<(-k_\ell) \left(r(0) + r(2) \ell^{-2} k^2_\ell\right) \varphi_<(k_\ell).$$

So the coefficient of the $k^2_\ell$-dependent term must be chosen as

$$r^{(2)}_\ell := r(2) \ell^{-\eta},$$

for the action to maintain the same form. Analogously the parameter $r^{(0)}_\ell$, the coefficient of $\varphi^2$, is read off from (26) as

$$r^{(0)}_\ell := r(0) \ell^{2-\eta}.$$ 

Performed infinitesimally, the rescaling step thus contributes

$$\ell \frac{dr^{(0)}_\ell}{d\ell} = (2 - \eta) r^{(0)}_\ell + \ldots,$$ 

$$\ell \frac{dr^{(2)}_\ell}{d\ell} = - \eta r^{(2)}_\ell + \ldots.$$ 

Here the ellipses are the terms from the decimation step.

Correspondingly, the interaction terms (9) transform under the rescaling
An exact one-loop momentum-dependent Wilsonian renormalization group

\begin{equation}
\int_{q_1} \cdots \int_{q_{n-1}} S^{(n)}(\{q\}) \varphi^{(n)}_<(q) = \ell^n \left(1 + \frac{d + \eta}{2}\right) \int_{k_{\ell,1}} \ell^{-d} \cdots \int_{k_{\ell,n-1}} \ell^{-d} S^{(n)}(\ell^{-1}\{k_\ell\}) \varphi^{(n)}_<(k_\ell),
\end{equation}

so that the coarse-grained coefficients are defined as

\begin{equation}
S^{(n)}_{\ell}(\{k_\ell\}) := \ell^n \left(1 + \frac{d + \eta}{2}\right)^{-(n-1)d} S^{(n)}(\ell^{-1}\{k_\ell\}) = \ell^n \left(1 - \frac{d + \eta}{2}\right)^{d + \eta} S^{(n)}(\ell^{-1}\{k_\ell\}).
\end{equation}

We would like to express the rescaling now for an infinitesimal change of scale, as we did for (29). We obtain from (31)

\begin{equation}
\ell \frac{dS^{(n)}_{\ell}(\{k_\ell\})}{d\ell} = \left(n \left(1 - \frac{d + \eta}{2}\right) + d\right) S^{(n)}_{\ell}(\{k_\ell\}) - k_\ell \frac{dS^{(n)}_{\ell}(\{k_\ell\})}{dk_\ell},
\end{equation}

where we used the chain rule to obtain the second term.

2.8. Momentum dependence

At the end of Section 2.6 the momentum dependence of the diagram was identified as the reason why reducible sub-diagrams from a different momentum shell are required to close the flow equation in a given shell. This section treats this momentum dependence to obtain a closed form flow equation with one-loop structure.

Decimation

We here investigate the flow for momentum-dependent vertices on the example of the four point vertex. This is illustrated in Figure 1, where the flow parameter \( \ell = 1, \ldots, \infty \) runs from top to bottom. Each decimation step performs the marginalization of the modes \( \varphi(k) \) that belong to the shell \( k \in i \).

Above, we have distinguished two classes of graphs. The tree-shaped graphs are simple: They contain a single chain of propagators. The momentum \( k \) of the contracted pair of fields belongs to precisely one decimation step, the step for which \( k \in i \). In Figure 1, the \( i \)-th step contributes a tree shaped diagram of two four point vertices; it thus yields a contribution to the six-point vertex, because there are six uncontracted fields \( \varphi_< \) left. The value of the propagator is the one that belongs to the corresponding shell, because it results from the marginalization of the corresponding modes. For spherically symmetric propagators, its value is \( G_{\ell_i}(\Lambda_{\ell_i}) \).

The entire contribution to the coarse-grained action (23) is therefore

\begin{equation}
S^{(6)}(q_1, \ldots, q_6) = 4 \cdot 4 \cdot \frac{1}{2!} \left( \frac{S^{(4)}_{\ell_i}}{4!} \right)^2 G_{\ell_i}(q_1 + q_2 + q_3).
\end{equation}

Correspondingly, it holds that

\begin{equation}
q_1 + q_2 + q_3 \in i.
\end{equation}

The combinatorial factor \( 4 \cdot 4 \) caters for the number of possible legs to choose from to be contracted; the factor \( 1/2! \) stems from the appearance of two vertices.
Now consider a contribution to the four-point vertex at non-vanishing external momenta \( q_1 + q_2 \neq 0 \). The contribution is composed of the six-point vertex with two legs contracted, as shown in eq. (33). It must necessarily be that \( j \geq i \), for otherwise the six-point vertex would not exist, so

\[ q_3 \in j. \tag{35} \]

The contribution to the flow of the four point vertex therefore reads

\[
\ell \frac{1}{4!} \frac{dS^{(4)}_{\ell}}{d\ell} = 4 \cdot 4 \cdot 3 \cdot 3 \cdot \frac{1}{2} \cdot \frac{1}{2!} \int_{|q_3| = \Lambda_j} \left( \frac{S^{(4)}_{\ell}}{4!} \right)^2 G_{\ell_i}(q_1 + q_2 + q_3) G_{\ell_j}(q_3), \tag{36}
\]

where the combinatorial factor \( 3 \cdot 3 \cdot \frac{1}{2} \) represents the combinations of selecting a pair of the legs of the six point vertex from the two sets of three legs each. The factor \( 1/2 \) has the same reason as in an \( n \) choose \( 2 \) factor: the order by which we pick the pair does not matter, so we need to correct for this factor, since we otherwise double-count contractions. The combinatorial factor in total is \( 4 \cdot 4 \cdot 3 \cdot 3 \cdot \frac{1}{2} = 72 \), is identical to that of the usual one loop diagram (24).

The combination of vertices that belong to different momentum shells, at first sight, seems to be in contradiction to the required locality property of a flow equation, which is of general form

\[
\ell \frac{dK_\ell}{d\ell} = \frac{\partial T}{\partial \ell}(K_\ell, \ell) \bigg|_{\ell = 1},
\]

where \( K_\ell \) is a coupling and \( T \) a general coarse-graining operator [11]: the right hand side should only depend on the couplings \( K_\ell \) at the current scale \( \ell \), not on the couplings of shells in the RG past \( \ell' < \ell \); this property ensures that the semi-group composition law holds. This contradiction, however, is only apparent because the value of a reducible diagram does not change except for in the one particular shell \( i \) which its momentum belongs to; so it has the same value at the later coarse-graining step. Its value is constant for all \( |k| < \Lambda_{\ell_i} \).

If we kept track of the tree diagrams separately, they would “automatically” take the right value if combined into a loop diagram. This would mean, however, that we follow the flow of the action as a functional of \( \varphi(k) \) for all values of \( k \).

Rescaling  To obtain the scaling form of the flow equation, we need to perform the rescaling of fields and momenta, as described in Section 2.7.

The six point vertex is generated in the step \( i < j \). In the rescaled units, the momenta in step \( i \) and step \( j \) obey, with (6), the relation

\[ k = \ell_j q. \]

The six point vertex, expressed in terms of the quantities at the scale \( \ell_j \), with (31) is

\[
S^{(n)}_{\ell_j}(\{k\}) = \ell_j^n \left( 1 - \frac{d+n}{2} \right) + d S^{(n)}(\ell_j^{-1}\{k\}).
\]

Conversely, we need to express \( S^{(n)} \) in terms of the rescaled coefficient at scale \( i \), for which the same relation holds, but with \( i \leftrightarrow j \), so that we get in total

\[
S^{(n)}_{\ell_j}(\{k\}) = \left( \frac{\ell_j}{\ell_i} \right)^n \left( 1 - \frac{d+n}{2} \right) + d S^{(n)}_{\ell_i}(\ell_j^{-1}\{k\}).
\]
Figure 2. Rescaling of momenta. Momenta $q$ before rescaling. Here $q_1$ and $q_2$ are external momenta and $q_3 = q'$ is the loop momentum in (36). Momenta $k = \ell_j q$ after rescaling. The rescaled loop momentum $k' = \ell_j q'$ is located at the cutoff of the decimation step, $|k'| = \Lambda$.

So only the ratio $\ell_i/\ell_j$ of the coarse-graining scale parameter appears. This ratio, in turn, is related to the momenta by (34) and (35) with $q := q_1 + q_2$ and $q' := q_3$ as $|q + q'| = \frac{\Lambda}{\ell_i}$ and $|q'| = \frac{\Lambda}{\ell_j}$, so that we have

$$
\frac{\ell_j}{\ell_i} = \frac{|q' + q|}{|q'|}.
$$

(37)

Here the momenta $q$ and $q'$ are measured at the absolute scale, as shown in Figure 1 — they are the arguments of $\varphi_{\epsilon_i}(q)$. But we want to know how the four point vertex depends on the momenta of the fields $\varphi_{\ell_j}(k)$. We thus need to express the momentum $q$ that appears in the latter factor by the $k$ at scale $\ell_j$, as illustrated in Figure 2. The coarse-graining scale $\ell_j$ satisfies $\ell_j |q'| = \ell_j |q_3| = \Lambda$. At this scale, the momenta $q$ take the value $\ell_j q =: k$, so

$$
q = \frac{1}{\ell_j} k = \frac{|q'|}{\Lambda} k,
$$

$$
q' = \frac{1}{\ell_j} k' = \frac{|q'|}{\Lambda} k'.
$$

We thus have

$$
\left( \frac{\ell_j}{\ell_i} \right)^2 = \frac{(q' + q)^2}{q'^2} = \frac{(k' + k)^2}{\Lambda^2}.
$$

(38)

Expressed in rescaled variables, the correction (36) to the four-point vertex hence takes the form

$$
3 \cdot 3 \cdot \frac{1}{2} \cdot \frac{1}{(2\pi)^d} \int_{|k'|=\Lambda} \theta(\ell_i - \ell_j +) \left( \frac{\ell_j}{\ell_i} \right)^6 \left( 1 - \frac{d + \eta}{2} \right) S_{\epsilon_i}^{(6)} \left( \frac{\ell_j}{\ell_i} \{ k \} \right) \frac{6!}{6!} G_{\ell_j}(k')
$$

(39)

$$
= \frac{36}{4 \cdot 4 \cdot 3 \cdot 3} \cdot \frac{1}{(2\pi)^d} \int_{|k'|=\Lambda} \theta(\ell_i - \ell_j +) \left( \frac{\ell_j}{\ell_i} \right)^6 \left( 1 - \frac{d + \eta}{2} \right) S_{\epsilon_i}^{(4)} \left( \frac{\ell_j}{\ell_i} \{ k \} \right) \frac{4!}{4!} G_{\ell_i}(\Lambda) G_{\ell_j}(\Lambda)
$$
\[
\simeq 36 \cdot \frac{1}{(2\pi)^d} \left( \frac{s^{(4)}_s}{4!} \right)^2 \int_{|k'|=\Lambda} 1 + \theta(k' \cdot k^+) \left( 6 - 2d - 3\eta \right) \times
\]
\[
\times \left( \frac{(k' \cdot k)}{\Lambda^2} + \frac{k^2}{2\Lambda^2} + \frac{(k' \cdot k)^2}{\Lambda^4} \left( 2 - d - \frac{3}{2} \eta \right) \right) G_{\ell_i}(\Lambda) G_{\ell_j}(\Lambda). \tag{40}
\]

From the second to the third line we expanded the term \( \left( \frac{\ell}{\ell} \right) \) up to second order in \( k' \) by using (38) (see Section 5.2 for details). The notation \( \theta(x+) \) is to be read as \( \lim_{\epsilon \searrow 0} \theta(x+\epsilon) \).

The four point vertices \( S^{(4)}_{\ell_i} \) and the propagator \( G_{\ell_i} \) that appear here are those expressed in terms of the rescaled variables at scale \( \ell_i \), because they take the values they had in the \( i \)-th decimation step. The momentum of each propagator in rescaled units is always equal to the original cutoff. In the last step we assumed that the flow will be close to the fixed point. As a result, we may replace the vertices and the propagators by their values at the fixed point, denoted by a star. We also assumed that we may neglect the momentum dependence of \( S^{(4)} \) here and only treat the momentum dependence produced by the rescaling prefactor.

If we assumed \( d = 4 \) and \( \eta = 0 \), the last line in (40) would simplify to

\[
36 \cdot \frac{1}{(2\pi)^4} \int_{|k'|=\Lambda} \frac{\theta(k' \cdot k^+) \Lambda^2}{(k' + k)^2} \left( \frac{s^{(4)}_s}{4!} \right)^2 G_s(\Lambda) G_s(\Lambda). \tag{41}
\]

This approximation is justified if one wants to compute the correction only in \( d = 4 - \epsilon \) dimensions, in which case \( \eta \propto \epsilon^2 \), as will be seen in the following. In the following we do not make this approximation, but rather keep the full dependence on \( d \) and \( \eta \) in (41).

So far we have assumed that \( |q_1 + q_2 + q_3| > |q_s| \) which is to say that or \( |q + q'| > |q'| \) and thus \( q \) and \( q' \) have a parallel component, giving rise to \( \theta(k' \cdot k^+) \) in the last line of (39). Of course, as we are integrating over a closed momentum shell \( k' \), within this integral we also have the case \( |q + q'| < |q'| \). We are after the contribution of the decimation step \( q' \in j \) to the four point vertex at fixed external momenta, so the contribution (39) vanishes whenever \( |q + q'| < |q'| \), because the corresponding six point vertex \( S^{(6)} \), given by (33), has not yet been produced at that point of the flow; it would only be produced further down the flow at scale \( q + q' < \Lambda_j \). This explains the appearance of the Heaviside function \( \theta(\ell_i - \ell_j^+) \) and the term \( \theta(k' \cdot k^+) \), respectively; here \( \theta(x+) \) appears, because for vanishing external momentum, both propagator lines lie on the same shell and thus contribute. In the limit of small \( q \), from the spherical integral \( \int_{|k'|=\Lambda} \) approximately only half the shell contributes for which \( |q + q'| > |q'| \).

2.9. Flow of the momentum dependent interaction vertex

We are interested in the long-distance behavior for which we want to get a flow equation. This requires the momentum dependence of (39) for small \( k \).

We here study the case that the dimensionality of the system is not close to the upper critical dimension, thus \( d \neq 4 - \epsilon \). We then cannot make the approximation in (41), but rather need to use the full expression (40). The computation is, however, not
more complicated than the approximation, because for up to order \( O(k^2) \) the same integrals appear. These are (details are given in the Appendix Section 5.4, (88), (89))

\[
\begin{align*}
\int_{|k'|=\Lambda} \theta(k' \cdot k^+) (k' \cdot k) &= S_{d-1} \Lambda \|k\| \frac{1}{d-1}, \\
\int_{|k'|=\Lambda} \theta(k' \cdot k^+) k^2 &= \frac{S_d}{2} \Lambda^2, \\
\int_{|k'|=\Lambda} \theta(k' \cdot k^+) (k' \cdot k)^2 &= \frac{S_d}{2d} \Lambda^2 k^2. 
\end{align*}
\]

We thus obtain from the decimation step and (40)

\[
\ell \frac{d}{d\ell} \frac{1}{4!} S^{(4)}_\ell(k_1, k_2, k_3, k_4)
= 36 \Lambda^d \frac{(u_\ell^{(0)})^2 G_\ell^2(\Lambda)}{(2\pi)^d} \left[ S_d + \left( 6 - 2d - 3\eta \right) \times \left( S_{d-1} \|k_1 + k_2\| \Lambda + \frac{S_d}{4d} \left( 4 - 3\eta \right) (k_1 + k_2)^2 \right) \right].
\]

We observe that the term \( \propto (k_1 + k_2)^2 \) vanishes for \( d = 4 \) and \( \eta = 0 \).

So the renormalized action requires a momentum-dependent \( S^{(4)} \)-interaction of the form

\[
-\frac{1}{4!} S^{(4)}(k_1, k_2, k_3, k_4) = u_\ell^{(0)}(0) + u_\ell^{(1)}(1) \|k_1 + k_2\| + u_\ell^{(2)}(2) (k_1 + k_2)^2 \quad (43)
\]

Defining the rescaled parameters with subscript \( \ell \), according to (31) with \( n = 4 \) yields

\[
-\frac{1}{4!} S^{(4)}_{\ell}(k_{\ell1}, k_{\ell2}) =: u_\ell^{(0)} + u_\ell^{(1)}(1) \|k_{\ell1} + k_{\ell2}\| + u_\ell^{(2)}(2) (k_{\ell1} + k_{\ell2})^2 \quad (44)
\]

In differential form the contribution from rescaling is thus

\[
\begin{align*}
\ell \frac{d}{d\ell} u_\ell^{(0)} &= (4 - d - 2\eta) u_\ell^{(0)} + \ldots, \\
\ell \frac{d}{d\ell} u_\ell^{(1)} &= (3 - d - 2\eta) u_\ell^{(1)} + \ldots, \\
\ell \frac{d}{d\ell} u_\ell^{(2)} &= (2 - d - 2\eta) u_\ell^{(2)} + \ldots,
\end{align*}
\]

where the momentum-dependent term \( k_\ell \frac{dS^{(n)}(k_\ell)}{dk_\ell} \) in (32) is taken care of by explicitly rescaling the momenta appearing in (44).

The complete flow equations of the parameters \( u_\ell^{(0)}, u_\ell^{(1)}, \) and \( u_\ell^{(2)} \) then follow by including the decimation step (42) as

\[
\ell \frac{du_\ell^{(0)}}{d\ell} = (4 - d - 2\eta) u_\ell^{(0)} \quad (48)
\]
An exact one-loop momentum-dependent Wilsonian renormalization group

\[-36 S_d \frac{\Lambda^d}{(2\pi)^d} \frac{(u^{(0)}_\ell)^2}{(r^{(0)}_\ell + r^{(2)}_\ell \Lambda^2)^2},\]

\[\ell \frac{d u^{(1)}_\ell}{d \ell} = (3 - d - 2\eta) u^{(1)}_\ell\]

\[-36 \frac{6 - 2d - 3\eta}{d - 1} \frac{S_{d-1} \Lambda^d}{(2\pi)^d} \frac{(u^{(0)}_\ell)^2}{(r^{(0)}_\ell + r^{(2)}_\ell \Lambda^2)^2} \frac{1}{\Lambda},\]

\[\ell \frac{d u^{(2)}_\ell}{d \ell} = (2 - d - 2\eta) u^{(2)}_\ell\]

\[-36 \frac{6 - 2d - 3\eta}{d - 1} \frac{S_d \Lambda^d}{4d(2\pi)^d} \frac{(u^{(0)}_\ell)^2}{(r^{(0)}_\ell + r^{(2)}_\ell \Lambda^2)^2} \frac{1}{\Lambda^2}.\]

The flow equation for \( u^{(2)}_\ell \) has the interesting property that for \( d = 3 \) the first term becomes \( \propto -3\eta \) and for \( d = 4 \), the second term becomes \( -\frac{3\eta}{4d} \). The flow equation for \( u^{(1)}_\ell \) has a first factor that is \( -3\eta \) for \( d = 3 \).

In combining the decimation step with the rescaling step, we use that the marginalization is over an infinitesimally thin shell \( \ell = 1 + \delta \). The contribution of the decimation step is thus of the order \( \delta^2 \). A subsequent rescaling therefore only produces corrections of order \( \delta^2 \), which thus do not affect \( dS/d\ell \) as given by (23).

So the rescaling step only produces additional contributions to \( dS/d\ell \) that arise from the rescaling of the action prior to decimation, given by (45) - (47). Writing the iteration as a differential equation, we need to express propagator and the vertices in the correction terms by the ones at the initial scale, as given by (32). This also replaces \( \Lambda_\ell \rightarrow \Lambda \), because the rescaling after each decimation step restores the original range of momenta and thus keeps the position of the cutoff constant.

2.10. Flow of the momentum-dependent self-energy

Inserting the momentum-dependent interaction term (43) into the self-energy corrections, this interaction vertex induces a dependence on \( k \). We have two possibilities of inserting the interaction into the self-energy diagram (13), which leads to the second and third line of the following equation

\[\ell \frac{d}{d \ell} \frac{1}{2} S^{(2)}_\ell (q, -q) := -2 \cdot 6 \cdot \frac{\varphi_\ell (q)}{\varphi_\ell (-q)}\]

\[= -2 \cdot 2 \cdot \frac{\Lambda_\ell^d}{(2\pi)^d} \int_{|k|=\Lambda_\ell} \frac{1}{4!} S^{(4)}_\ell (q, -q, k, -k) G_\ell (k)\]

\[= -2 \cdot 4 \cdot \frac{\Lambda_\ell^d}{(2\pi)^d} \int_{|k|=\Lambda_\ell} \frac{1}{4!} S^{(4)}_\ell (q, k, -k, -q) G_\ell (k).\]

The penultimate line corresponds to the 2 ways of choosing either the left pair or the right pair of amputated legs of the interaction vertex \( S^{(4)} \) to be the amputated lines of the self-energy diagram – in both cases we have \( q_1 = q \) and \( q_2 = -q \), so that
An exact one-loop momentum-dependent Wilsonian renormalization group

the momentum dependence of the four point vertex $S^{(4)}(q, -q, k, -k)$, according to eq. (43), drops out. In the latter row the four point vertex is inserted such that one external amputated leg connects to left side of $S^{(4)}$ (2 possibilities) and one to the right side (another 2 possibilities), so we get a term

$$\frac{1}{4!} S^{(4)}_\ell(q, k, -q) = u^{(0)}_\ell + u^{(1)}_\ell \sqrt{(q + k)^2} + u^{(2)}_\ell (q + k)^2,$$

which therefore is dependent on the momentum $q$. Dropping terms of order $O(q^3)$ and higher and sorting the result into terms $\propto q^0$ and $\propto q^2$ according to $-S^{(2)} \propto r^{(0)}_\ell + r^{(2)}_\ell q^2 + O(q^3)$ we get the pair of flow equations (see Section 5.5 for details)

$$\ell \frac{d}{d\ell} r^{(0)}_\ell = (2 - \eta) r^{(0)}_\ell + 2 \cdot \frac{S_d A^d}{(2\pi)^d} \cdot \frac{6 u^{(0)}_\ell + 4 u^{(1)}_\ell \Lambda + 4 u^{(2)}_\ell \Lambda^2}{r^{(0)}_\ell + r^{(2)}_\ell \Lambda^2},$$

$$\ell \frac{d}{d\ell} r^{(2)}_\ell = - \eta r^{(2)}_\ell + 8 \cdot \frac{S_d A^d}{(2\pi)^d} \cdot \frac{u^{(1)}_\ell \left( 1 - \frac{1}{3} \right) + u^{(2)}_\ell}{r^{(0)}_\ell + r^{(2)}_\ell \Lambda^2},$$

where the first line in each flow equation comes from the rescaling given by (29) and (30).

2.11. Fixed points and critical exponents for $3 \leq d \leq 4$

We may now compute the Wilson-Fisher fixed point for arbitrary dimensions between 3 and 4. We determine $\eta$ such that $r^{(2)} \equiv 1$ is a fixed point of (54). Throughout we approximate the mass to be small compared to the cutoff, $r^{(0)}_\ell \ll \Lambda^2$. The details can be found in Section 5.6.

**Interaction** The fixed point for the momentum-independent four point coupling obeys

$$\Lambda^{d-4} u^{(0)}_\ast \simeq (2\pi)^d \frac{4 - d - 2\eta}{36 S_d}.$$  (55)

In the limits $d \to 3$ and $d \to 4$ we get

$$\Lambda^{d-4} u^{(0)}_\ast \simeq \begin{cases} \pi^2 \frac{1 - 2\eta}{18} \simeq \frac{\pi^2}{18} & d = 3 \\ \pi^2 \frac{2}{5} \left( \epsilon - 2\eta \right) & d = 4 - \epsilon \end{cases}.$$  (56)

As explained by Wilson [3, section V], all fixed point values can be related back to the value of the interaction $u^{(0)}_\ast$, the single marginal coupling of the Gell-Mann-Low theory. The first-order momentum dependence depends quadratically on $u^{(0)}_\ast$

$$\Lambda^{5-d} \left( \frac{u^{(1)}_\ast}{u^{(0)}_\ast} \right)^2 \simeq \frac{36}{(2\pi)^d} \frac{2(3 - d - 3\eta_S^{d-1})}{3 - d - 2\eta} \frac{S_d}{d - 1}.$$  (57)

In the limits $d \to 3$ and $d \to 4$ we get

$$\Lambda^{d-3} u^{(1)}_\ast \simeq \begin{cases} \frac{\pi^2}{4\epsilon} \left( 1 - 2\eta \right)^2 \simeq \frac{\pi^2}{4\epsilon} & d = 3 \\ \frac{2(1 + \epsilon - 3\eta)}{5} \left( \epsilon - 2\eta \right)^2 \frac{\pi}{(3 - \epsilon)} \simeq \frac{2\pi}{5} \epsilon^2 + O(\epsilon^3) & d = 4 - \epsilon \end{cases}.$$  (58)
The quadratic momentum dependence of the interaction is
\[ \Lambda^{6-d} \frac{u_s^{(2)}}{(u_s^{(0)})^2} \simeq 36 \left( \frac{S_d}{(2\pi)^d} \right) \frac{2(3-d) - 3\eta}{2 - d - 2\eta} \frac{4 - d - 3\eta}{4d}. \] (59)

As expected from the \( \epsilon \)-expansion, the fixed-point value depends quadratically on \( u_s^{(0)} \). For \( d \to 3 \) and \( d \to 4 \) we get
\[ \Lambda^{d-2} u_s^{(2)} \simeq \begin{cases} \frac{\pi^2}{72} \frac{\eta}{1 + 2\eta} (1 - 3\eta)(1 - 2\eta + 4\eta^2) \simeq \frac{(2\pi)^3}{72} \eta + O(\eta^2) & d = 3 \frac{2\pi^2}{2} \frac{2(1 - \epsilon + 3\eta)(\epsilon - 3\eta)(\epsilon - 2\eta)^2}{4(4 - \epsilon)} \simeq \frac{\pi^2}{72} \epsilon^3 + O(\epsilon^4), & d = 4 - \epsilon (60) \end{cases} \]

**Mass term** To check if the approximation \( r_s^{(0)} \ll \Lambda^2 \) is justified, one observes that the fixed point value for \( r_s^{(0)} \), in this approximation, is determined as
\[ \Lambda^{-2} r_s^{(0)} \simeq - \frac{6 S_d}{(2\pi)^d} \Lambda^{d-4} u_s^{(0)} \] (61)

which shows the approximate linear relationship between \( r_s^{(0)} \) and \( u_s^{(0)} \) and explains why the mass term stays small as long as \( u_s^{(0)} \) is small. In this case, it is justified to approximate \( \Lambda^2 + r_s^{(0)} \to r_s^{(0)} \). The ratio between mass and coupling term is
\[ \Lambda^{2-d} \frac{r_s^{(0)}}{u_s^{(0)}} \simeq - \frac{6 S_d}{(2\pi)^d} \frac{4}{3 - d - 2\eta} \left( \frac{S_{d-1}}{d - 1} + \frac{4 - d - 3\eta}{4d} S_d \right). \] (62)

The correction to the linear relation in the second line is \( \propto \eta \) for \( d = 4 \), but becomes notable in for \( d < 3 \), as shown in Figure 3c. Inserting the expression for \( u_s^{(0)} \) from (55) yields \( r_s^{(0)} \) as a function of \( d \), shown in Figure 3a.

**Critical exponent \( \eta \)** We get by demanding stationarity of (54) and \( r_s^{(2)} = 1 \) to get
\[ \eta = 8 \cdot \frac{S_d}{(2\pi)^d} \left( \frac{1}{2} \Lambda^{d-3} u_s^{(1)} \left( 1 - \frac{1}{d} \right) + \Lambda^{d-2} u_s^{(2)} \right). \] (63)

Expressed in terms of \( u_s^{(0)} \)
\[ \Lambda^{8-2d} \frac{\eta}{(u_s^{(0)})^2} \simeq 8 \left( \frac{S_d}{(2\pi)^d} \right) \frac{1}{2} \left( 1 - \frac{1}{d} \right) 36 \frac{2(3-d) - 3\eta}{3 - d - 2\eta} \frac{S_{d-1}}{d - 1} \] \[ \quad + 36 \frac{2(3-d) - 3\eta}{2 - d - 2\eta} \frac{4 - d - 3\eta}{4d} S_d. \] (64)

This expression has a factor \( \Lambda^{2d-8} (u_s^{(0)})^2 \) appearing in both terms. So, as expected, expressing \( u_s^{(0)} \) by (55) one obtains a universal result that does not depend on \( \Lambda \), the
microscopic details of the system, reorganized as a cubic equation of which we need to determine the roots

$$0 \simeq \frac{8}{36} \frac{1}{S_d} \frac{2(3-d) - 3\eta}{d} \left(\frac{S_{d-1}}{2} (2 - d - 2\eta) + \frac{S_d}{4} (3 - d - 2\eta)(4 - d - 3\eta) \right) \left(4 - d - 2\eta \right)^2 - \eta (3 - d - 2\eta)(2 - d - 2\eta).$$

(65)

For $3 < d \leq 4$, we solve the cubic equation for $\eta$ numerically. The result is shown in Figure 3b.

Dropping from this cubic equation in $\eta$ all terms that are $O(\eta^2)$, for $d = 4$ we get a linear equation in $\eta$, relating it to $(u_\ast^{(0)})^2$ as

$$\eta \simeq \frac{1}{16\pi^8} \frac{9S_4S_3}{2} + \frac{1}{64\pi} (27S_4^2 - 90S_4S_3) (u_\ast^{(0)})^2 \quad (u_\ast^{(0)})^2 + O((u_\ast^{(0)})^4)$$

$$= \frac{9}{4\pi^8} (u_\ast^{(0)})^2 + O((u_\ast^{(0)})^4)$$

$$\simeq 0.0074 (u_\ast^{(0)})^2 + O((u_\ast^{(0)})^4).$$

With $u_\ast^{(0)} \simeq \pi^2 \frac{2}{9} \epsilon + O(\epsilon^2)$ (cf. (56)) one sees that this result is consistent with the limiting expressions for $d = 4 - \epsilon$ for $u_\ast^{(1)} \simeq \frac{8\pi}{27} \epsilon^2 + O(\epsilon^3)$ (cf. (58)) and $u_\ast^{(2)} = \frac{72}{72} \epsilon^3 + O(\epsilon^4)$ inserted into (63) to find

$$\eta \simeq \frac{1}{9\pi} \epsilon^2 + O(\epsilon^3).$$

(67)

The latter consideration shows that the $||k||$-dependence of the interaction $u (\propto u_\ast^{(1)})$ causes the anomalous dimension close to four dimensions and the $k^2$-dependence of the interaction $\propto u_\ast^{(2)}$ only plays a role at higher orders $O(\epsilon^3)$.

The value for $\eta$ (67) is by a factor $6/\pi \simeq 1.9$ larger than the two-loop result from the $\epsilon$ expansion to second order [1, p. 626]

$$\eta \simeq \frac{1}{54} \epsilon^2$$

$$\simeq \frac{9}{24\pi^4} (u_\ast^{(0)})^2$$

$$\simeq 0.0038 (u_\ast^{(0)})^2.$$

For $d = 3$ we obtain a quadratic equation from (65)

$$0 \simeq \eta^2 + \frac{17S_3 + 2S_2}{43S_3 + 4S_2} \eta - \frac{S_2}{43S_3 + 4S_2}$$

$$0 \simeq \eta^2 + \frac{72}{180} \eta - \frac{1}{90}$$

so

$$\eta^{(d=3)} \pm = \frac{36}{180} \pm \sqrt{\left(\frac{36}{180}\right)^2 + \frac{1}{90}}$$

$$= \left\{ \begin{array}{l}
0.0261 + \\
-0.426 -
\end{array} \right.$$
Critical exponent \( \nu \)  
The critical exponent \( \nu \) is determined by linearizing the flow about the fixed point and by determining the eigenvalue \( \lambda_r \) in the direction of the parameter \( r \). The critical exponent then takes the value

\[
\nu = \lambda_r^{-1} = \frac{1}{2 - \eta - \frac{1}{16} (4 - d - 2\eta)(2(3 - d) - 3\eta)} \left( \frac{6}{2(3-d)-3\eta} + \frac{4}{d-1} \frac{4-d-2\eta}{3-d-2\eta} \frac{S_d - 1}{8d} + \frac{1}{d} \frac{(4-d-3\eta)(4-d-2\eta)}{2-2d-2\eta} \right).
\]

(70)

So for \( d = 3 \) and \( d = 4 - \epsilon \) we obtain

\[
\nu = \begin{cases} 
\frac{1}{2-\eta+\frac{1}{3}(1-2\eta)} & d = 3 \\
\frac{1}{2-\frac{1}{4} \epsilon - \frac{1}{24} \epsilon^2 + \mathcal{O}(\epsilon^3)} & d = 4 - \epsilon
\end{cases}
\]

(71)

For the approximation for \( d = 4 - \epsilon \) we neglected \( \eta \propto \epsilon^2 \) throughout.

3. Summary and discussion

Summary of quantitative results for the \( \phi^4 \)-theory

A summary of the derived expressions for the fixed point values of the mass \( v^{(0)} \), and the momentum-dependent interaction \( \{u^{(0)}, u^{(1)}, u^{(2)}\} \) is shown in Figure 3a for \( d \in [3, 4] \).

Critical exponents \( \nu \) and \( \eta \) as functions of the dimension are shown in Figure 3b. For \( d = 5 \), the obtained critical exponent \( \nu \simeq 0.632 \) compares well to the result of an order five \( \epsilon \)-expansion \( \nu \simeq 0.631(2) \) [1, p. 635]. It is better that the result from the momentum-scale expansion for the effective action (\( \nu = 0.532 \) [5]) and to the second order \( \epsilon \)-expansion (\( \nu = 0.627 \)). The increase in accuracy is despite the observation that the quadratic momentum dependence \( \mathcal{O}(k^2) \) of the interaction vertex remains small for all dimensions \( d \in [3, 4] \) (Figure 3c).

Also the critical exponent \( \eta \simeq 0.026 \) is not too far from the result of the order five \( \epsilon \)-expansion of \( \eta = 0.035(3) \), estimated from the divergent series in powers of \( \epsilon \). The here obtained value is closer to the true value than the two loop result of \( \eta \simeq 1/54 \simeq 0.0185 \) [1, p. 625 eq. 28.7]. This is so despite the simplicity of the computation presented here, which only requires elementary one-loop integrals. Compared to the momentum-scale expansion of the effective action [5, p. 494] (\( \eta = 0.0225 \)), the improvement in accuracy is only small.

The fixed point values in a Gell-Mann-Low theory, a theory with a single marginal variable, can entirely be expressed in terms of this single marginal variable [3, Section V]. This is the basis of a renormalizable theory, where only a single renormalized coupling constant must be fixed. Correspondingly, the resulting expressions for the mass and for the momentum-dependent part of the interaction can be expressed in terms of the momentum-independent part of the interaction, \( u^{(0)} \). Mass and the

| exponent | \( \epsilon \)-expansion \( \mathcal{O}(\epsilon) \) | \( \epsilon \)-expansion \( \mathcal{O}(\epsilon^2) \) | this method | best approx. |
|----------|--------------------------------|--------------------------------|-------------|-------------|
| \( \nu \) | 0.583 | 0.627 | 0.632 | 0.631(2) |
| \( \eta \) | 0 | 0.0185 | 0.026 | 0.035(3) |

Table 1. Critical exponents in \( d = 3 \) dimensions. Comparison of this method to known approximations. Best reference approximation from [1, p. 635] and [1, p. 626].
Figure 3. Wilson-Fisher fixed point in $\phi^4$-theory as function of dimension $d \in [3, 4]$. 

**a** Fixed point as function of dimension $d$: i) Mass $r^{(0)}$ (red solid curve, (61)); red dashed curve: linear approximation in $\epsilon = 4 - d$; only first line of (61). ii) Momentum-independent interaction $u^{(0)}$ (blue curve, (55)); limits $d \to 3, 4$ (56) blue dot and blue dashed line). iii) First-order momentum-dependence $u^{(1)}$ (orange curve (57) and (55); orange dot: limit $d \to 3$ (58)). iv) Second-order momentum-dependence $u^{(2)}$ (green curve, (59) and (55); green dot: limit $d \to 3$ (60)).

**b** Critical exponents $\nu, \eta$ as function of dimension $d$: i) Anomalous dimension $10\eta$ (blue curve, positive root of (65); blue dots: limit $d \to 3$ (69) and $d \to 4$, $\eta = 0$). ii) Critical exponent $\nu$ (orange curve: (70); orange dots: limits $d \to 3, 4$ (71)). Lowest order approximations of $\epsilon$-expansion (dashed curves (72) for $\eta$ and $\nu = \left(2 - (4 - d)/3\right)^{-1}$ for $d = 4 - \epsilon$). c Coupling constants relative to interaction: i) Relative mass $r^{(0)}/u^{(0)}$ (red curve (62); dashed curve showing only linear dependent part on $u^{(0)}$, first line in (62)). ii) Relative interaction $u^{(1)}/(u^{(0)})^2$ (orange curve (57)) and $u^{(2)}/(u^{(0)})^2$ (green curve (57)). Literature values of order $5\epsilon$-expansion for $\eta$ (blue cross; $\eta = 0.035$ [1, p. 626] and $\nu = 0.631(2)$ [1, p. 635]). d Relative anomalous dimension $\eta/(u^{(0)})^2$ (blue curve, (64); blue dot: limit for $d \to 4$ (66); $\eta/(u^{(0)})^2 = 9/(4\pi^2) \simeq 0.0074$). Limit for $d \to 4$ of classical $\epsilon$ expansion to order $\epsilon^2$ (gray dashed line, (68); $\eta/(u^{(0)})^2 = 9/(24\pi^2) \simeq 0.0038$).

momentum-dependent part of the interaction relative to $u^{(0)}$ (for the mass) or relative to $(u^{(0)})^2$ (for both interaction terms) are shown in Figure 3c.

Relation to earlier work

Even though the numerical values for the anomalous dimension $\eta$ obtained by the here-presented method and by an $\epsilon$-expansion of second order are not so far apart, expressing the anomalous dimension relative to $(u^{(0)})^2$ shows that the two methods are in fact quite different: The $\epsilon$-expansion, to second order, yields [12, 1, p. 625 eq. 28.7]

$$\eta = \frac{1}{54} \epsilon^2 + O(\epsilon^3)$$

(72)
and thus, with the leading order behavior of \( u^{(0)} = \pi^2 \frac{2}{3} \epsilon + \mathcal{O}(\epsilon^2) \) (Figure 3a), a ratio independent of \( \epsilon \) and thus \( d \)

\[
\frac{\eta}{(u^{(0)})^2} = \frac{9}{24 \pi^4}.
\]

The here-proposed method instead yields a saturating form of \( u^{(0)} \) as the dimension is lowered from \( d = 4 \) to \( d = 3 \) (Figure 3a), while the relative quantity \( \eta/(u^{(0)})^2 \) increases as \( d \) is lowered. Thus, the anomalous dimension, expressed in relation to the renormalized coupling constant, is much larger in the here-proposed method for \( d = 3 \), as shown in Figure 3d.

Relation to the work by Wegner & Houghton [4] The method proposed here is practically identical to the exact equations derived by Wegner & Houghton [4]. In fact, their equation (2.20) is the formal analogue of the diagrammatic rules described in the present manuscript in sections Section 2.6 - Section 2.8. The first main contribution of the current manuscript is the diagrammatic formulation which yields an alternative derivation and interpretation of these equations. The second contribution is the combination of this approach with a momentum-scale expansion to demonstrate the quantitative accuracy in a non-perturbative setting far off the upper critical dimension. A closely related approach is the functional renormalization group [6, 13] for the effective action, the first Legendre transform of the free energy. Employing a hard cutoff, this method has been used previously to compute the momentum dependence of the self-energy in weakly interacting Bosons [8, 9]. Here as well, a momentum-scale expansion of the interaction leads to a non-differentiable momentum dependence \( \propto ||k|| \) of the interaction vertex that enters the computation of the anomalous dimension (cf. i.p. [9] eq. (11)) and that becomes marginal in \( d = 3 \) dimensions, as in the present manuscript (cf. eq. (49)).

It must be noted that the computation to order \( \epsilon^2 \) of the anomalous dimension performed by Wegner and Houghton yields a different result than found here. The cause of this difference is the step from eq. (3.13) to eq. (3.16) in their work: Here the authors approximately solve the set of their differential equations (3.9) - (3.12). In doing so, they insert the contribution of the tree-diagrams of the form (33) contributing to the six-point vertex \( v_6 \) given by their eq. (3.12) into the second line of their eq. (3.11). What they thus implicitly assume is that the six-point vertex \( v_6 \) is produced within the same decimation shell as the one integrated out in the flow of \( v_4 \). This, however, is only true for vanishing external momenta. Stated differently, they neglect that, for non-zero external momenta \( k \) of their four point vertex \( v_4(k) \), the two propagator lines in the composed diagram of the form (22) necessarily belong to different decimation shells. The propagator line that connects the two four point vertices in their eq. (3.12) of \( v_6 \), as a consequence, stems from a different decimation shell than the propagator in their eq. (3.11). Our calculation takes this momentum-dependence into account. This results in the factor \( \left( \frac{\ell_j}{\tau} \right)^{6-3\eta-2d} \) in front of \( S^{(0)}_{\ell_j} \left( \frac{\ell_i}{\tau} \{ k \} \right) \) in (39), which is missing in Wegner & Houghton’s approximate solution, their eq. (3.16). This additional approximation they make yields a result for the critical exponent \( \eta \) that, to order \( \epsilon^2 \), is identical to the known two-loop result. A careful interpretation of their equations in fact yields the same result as presented in the current manuscript; to this end one needs to take seriously the rescaling term
Relation to the ε-expansion An important cross check of the method is its compatibility with previous methods, foremost the established ε-expansion [12, 2, 3]. The interaction \( u^{(0)}_\epsilon \simeq \pi^2 \frac{2}{3} \epsilon + O(\epsilon^2) \), given by (56), is in fact identical to order \( O(\epsilon) \) in the two approaches. Also, the momentum dependence only arises at order \( O(\epsilon^2) \), given by (58) and (60), again in line with the ε-expansion. A qualitative difference is, though, that the leading order \( O(\epsilon^2) \) momentum-dependence of the interaction vertex is \( \propto u^{(1)} \|k_1 + k_2\| \) with \( \Lambda^{d-3} u^{(1)} \|k_1 + k_2\| = \frac{8\pi}{27} \epsilon^2 + O(\epsilon^3) \), whereas the quadratic momentum-dependence \( \propto u^{(2)}_\epsilon (k_1 + k_2)^2 \) only arises at the next order in \( \epsilon \), namely \( \Lambda^{d-2} u^{(2)}_\epsilon \propto \epsilon^3 + O(\epsilon^4) \). The correction to \( \eta \) to order \( O(\epsilon^2) \) comes from the third line of the self-energy correction (51) by inserting the momentum-dependence \( \propto u^{(1)}_\epsilon \) in the middle term of (52).

Such contribution \( \propto u^{(1)}_\epsilon \) is not present in the classical second order ε-expansion, because this method treats the momentum dependence of the interaction differently.

The leading order \( k \)-dependence of the self-energy in the ε-expansion comes from the diagram (cf. [2, Fig 5.4])

\[
\begin{align*}
\begin{array}{c}
\text{k} \\
\bigcirc \\
\text{−k}
\end{array}
\end{align*}
\]

The \( k \)-dependence of this diagram results from an expansion in the external momentum \( k \). This diagram can also be seen as arising from contracting a pair of legs of the one-loop fluctuation correction \( \Gamma^{(4)}_n \) to the four point vertex, as in eq. (51). This sub-diagram of (73) is of the form

\[
-\Gamma^{(4)}_n(q,k,-k,-q) = \begin{array}{c}
\text{–} \\
\bigcirc \\
\text{–}
\end{array}
\]

where \( q \) is the momentum of integration of the upper loop in (73). The dependence on \( k \) then results from the algebraic form of the diagram

\[
-\Gamma^{(4)}_n(q,k,-k,-q) \propto \int \frac{dk'}{(2\pi)^d} \frac{u_0^2}{(r_0 + (q + k')^2) (r_0 + (k - k')^2)},
\]

when expanding to quadratic order in \( k \): One obtains only spherically symmetric diagrams \( \propto k^2 \) and uniaxial diagrams \( \propto \int dk' (k \cdot k')^2 \propto k^2 \); to lowest order in \( k \) one thus obtains a dependence \( \propto k^2 \). In particular, there is no dependence \( \propto \|k\| \) arising in the perturbative approach, because expanding for small \( k \), terms \( \propto \int dk' (k \cdot k') \) vanish by the point-symmetry of the integrand. In contrast, in our approach we have found this very term \( \propto u^{(1)}_\epsilon \|k\| \) to produce the dominant contribution to \( \eta \).

What is the reason for this qualitative difference between the two approaches? The non-differentiable momentum dependence \( \propto u^{(1)}_\epsilon \|k\| \) of the interaction in our approach comes about by the fact that a fluctuation correction of the form (74) at scale \( \ell \) necessarily must have one of the propagator lines above the current cutoff \( \|q + k'\| > \Lambda \ell \) or \( \|k - k'\| > \Lambda \ell \). When expanded for small momenta, this leads to integrals of the form \( \propto \int dk' (k \cdot k') \theta(k \cdot k') \propto \|k\| \), where \( \theta(k \cdot k') \) constrains the integration to half the momentum shell. These contributions thus do not vanish, as opposed to the perturbative approach, where one integrates over full momentum shells.
On a more abstract level, we can summarize these observations as follows. The infinitesimal Wilsonian RG studied here leads to a functional differential equation of the form

$$\ell \frac{dS_\ell}{d\ell} = \beta\{S_\ell\},$$

(76)

where $\beta : \mathcal{F} \mapsto \mathcal{F}$ is a mapping from the space of functionals $\mathcal{F} := \{f \mid f : \mathcal{C} \mapsto \mathbb{R}\}$ into itself, where $\mathcal{C}$ is the space of functions, in our case the configurations $\varphi$ of our system. In the chosen Fourier representation of $\varphi$, the functional mapping $\beta$ may thus couple different $k$-vectors. This is in fact the case, as we illustrate in Figure 1: The flow for the Taylor coefficient $S^{(4)}_\ell(q_1, q_2, q_1', q_2')$ of $S_\ell$ depends on the Taylor coefficient $S^{(6)}_\ell(q_1, q_2, q_3, \ldots)$; here $S^{(4)}_\ell$ appears on the left hand side of (76) and $S^{(6)}_\ell$ on the right hand side. The here-followed approach takes this functional seriously; as a result, the value $S^{(6)}_\ell$ that appears on the right hand side of (76) depends on the momentum $q_1 + q_2 + q_3$ given by the arguments $q_1$ and $q_2$ of $S^{(4)}_\ell$ and the loop momentum $q_3$, because the decimation step $\ell_i(q_1 + q_2 + q_3)$ in which $S^{(6)}_\ell$ has been produced is a function of this very momentum, as explained in Section 2.8.

This is what appears to us being neglected in the perturbative approach of the $\epsilon$-expansion: Here, the diagram (75) is composed of vertices $u_0$ and propagators $(r_0 + k^2)^{-1}$, all of which take the values of the current decimation step; the perturbative calculation thus neglects the fact that each propagator line necessarily belongs to a single decimation step that integrates out the degrees of freedom at the very momentum that is carried by the propagator.

The difference between the perturbative calculation and our approach consequently shows up only in the flow of vertices at non-zero external momenta; the value for $u_0$ at vanishing momentum to order $\epsilon$ is consequently identical. In contrast, the two methods yield quantitatively different results for the anomalous dimension, because this quantity is defined in terms of the momentum dependence of the self-energy.

The estimate of the anomalous dimension by the $\epsilon$-expansion, to quadratic order $O(\epsilon^2)$, is roughly by about a factor 2 smaller than the estimate of a high temperature expansion, as already noted in Fig. 1 of [12]. In their ref 16, the authors point out that their estimate is in fact too small. Also the comment after eq. (8.29) on p. 137 of [2] points out that the result of the $\epsilon$-expansion at second order is too small by about a factor of 2. Our estimate is larger by a factor $6/\pi \simeq 1.9$ than the $\epsilon$-expansion at second order. Despite comparable complexity in the integrals to be computed, the here-presented method is thus in better agreement with the high temperature expansion and with higher orders of the $\epsilon$-expansion.

On the technical side a difference of the proposed method is that it does not require coping with a divergent series, as opposed to the $\epsilon$-expansion. The flow equations expose the desired quantities simply by their fixed point values, as demonstrated by the elementary calculations presented here. This shows the non-perturbative nature of the method. The approximation made here is not in the number of loops – we have shown above that the one-loop calculation is exact; this is simply a result of the phase space volume contributing to an infinitesimal marginalization step being proportional to the (infinitesimal) thickness of the momentum shell times the number of loops. Formally, this result is thus in line with exact functional RG equations [4, 6, 13], where also only single-loop integrals are required.

Two approximations enter our practical calculation: First, the truncation in
the number of effective vertices, here up to fourth order. Second, the momentum-
dependence of the vertex, here up to second order. For the functional RG of the
effective action [6, 13], a momentum scale expansion has been combined with the
vertex expansion to investigate critical behavior of weakly interacting Bosons and
in particular to obtain an approximation of the the momentum dependence of the
self-energy and thus the anomalous dimension [8, 9].

In the simple example shown here we did not compute the flow of the six-
point vertex. At \( d = 3 \), however, the six point vertex becomes marginal so that its
contribution can potentially become substantial. Since the \(|k|\) -momentum-dependence
of the six point vertex is less relevant by one power of \( \ell \), an approximation taking the
the momentum-independent part into account is still simple and yields potentially
better results.

3.1. Conclusion and outlook

We here presented an exact one-loop diagrammatic Wilsonian renormalization group
equation. The required diagrams can be derived from ordinary perturbation theory,
but the result is still non-perturbative. We showed that the flow equations can
successfully be closed by combining two approximations, the vertex expansion and
the momentum-scale approximation. For the scalar \( \varphi^4 \) -theory, one obtains a highly
accurate approximation for \( \nu \) and a reasonable accurate approximation for \( \eta \); even
though, the latter vanishes at one-loop order of the orthodox \( \epsilon \)-expansion.

The use of the renormalization group to study phenomena that arise from the
interaction of processes on many different scales is not restricted to systems from the
core domains of physics. Our motivation to employ the renormalization group,
for example, arises from the wish to understand how activity organizes in neuronal
systems. Field theoretical formulations of neuronal networks have begun to be
employed in this field (see e.g. [14, 15] for reviews). One here knows the dynamical
equations on the microscopic level, namely for individual neurons, but experiments
often only yield coarse-grained signals, such as the local-field potential [16]. Moreover,
there is experimental evidence that neuronal systems may operate at critical points
[17]. Methods from statistical physics are therefore commonly employed in this field;
often from equilibrium statistical mechanics. For example the pairwise maximum-
entropy model [18], the Ising model, is fitted to binned neuronal activity data [19].
More recent developments have formulated neuronal systems as a Ginzburg-Landau
field theory, a genuine non-equilibrium formulation [20]. Such systems bear strong
similarity to the Kardar-Parisi-Zhang model [21]. For the latter model it is known
that the momentum dependence of the self-energy is a key ingredient of the flow
equations. Such models thus necessitate efficient approximations of the momentum
dependence, as investigated here. But not only the study of coarse-graining and critical
dynamics in biological neuronal networks requires renormalization group techniques;
also the transformations performed by deep artificial neuronal networks are composed
of concatenations of large numbers of relatively simple transformations by each layer.
Addressing the question how representations of data arise in these networks [22]
may therefore also be studied by renormalization group methods [23]. Applying an
established technique to a new field, often shows a method in a new light. This is
how the question of the momentum dependence in the Wilson RG, as discussed by
the current manuscript, arose in the present case.

We hope that the here presented method may become fruitful to obtain simple
and intuitive diagrammatic approximations for critical phenomena in various systems that so far required more elaborate non-perturbative methods. We also hope that the partly pedagogical presentation may be helpful for the accessibility by a broad readership.

4. Acknowledgments

We are grateful to the comments by Peter Kopietz on an earlier version of this manuscript. These comments in particular considerably improved the discussion of the presented work to the $\epsilon$-expansion and to non-perturbative techniques using the functional RG and the momentum-scale expansion. We would further like to acknowledge helpful discussions with Tobias Kuehn and Kirsten Fischer that lead to the initial question on the momentum-dependence of the infinitesimal RG.

This work was partially supported by the European Union’s Horizon 2020 research and innovation programme under grant agreement No. 785907 (Human Brain Project SGA2), the Exploratory Research Space (ERS) seed fund neuroIC002 (part of the DFG excellence initiative) of the RWTH university and the German Federal Ministry for Education and Research (BMBF Grant 01IS19077A).

5. Appendix

5.1. Gaussian part of the action

Before proceeding to the general problem to apply the coarse-graining defined in Section 2.2 to the full action, consider the quadratic term which is

$$S_0[\varphi] := -\frac{1}{2} \int_k \varphi(-k) \left( r^{(0)} + r^{(2)} k^2 \right) \varphi(k) = : -\frac{1}{2} \varphi^T G^{-1} \varphi, \quad (77)$$

where we introduce a matrix-vector notation and define

$$G^{-1}(k,k') = \delta(k + k') \left( r^{(0)} + r^{(2)} k^2 \right).$$

This matrix is diagonal in the sense that it only couples the component $\varphi(k)$ with $\varphi(-k)$.

To perform the marginalization step (5) one needs to evaluate $S_0[\varphi_+ + \varphi_-]$

$$\varphi^T G^{-1} \varphi = (\varphi_+ + \varphi_-)^T G^{-1} (\varphi_+ + \varphi_-). \quad (78)$$

As $G^{-1}$ is diagonal in frequency domain, it couples only frequencies $0 < |q| < \Lambda \ell^{-1}$ among another and frequencies $\Lambda \ell := \Lambda \ell^{-1} < |k| < \Lambda$ among another, but it does not couple $q$ and $k$; cross terms in multiplying out (78) are thus dropped to get

$$\varphi^T G^{-1} \varphi = \varphi_-^T G^{-1} \varphi_- + \varphi_+^T G^{-1} \varphi_+. \quad (79)$$

The decimation step in (5) thus only affects the term in the second line of (79), which yields

$$Z_> := \int D\varphi_> \exp(-\frac{1}{2} \varphi_>^T G^{-1} \varphi_>) \quad (80)$$

$$= \prod_{\Lambda \ell < |k| < \Lambda} \left( 2\pi G(k) \right)^{\frac{1}{2}}$$

$$\ln Z_> = \frac{1}{2} \int_{\Lambda \ell < |k| < \Lambda} \ln 2\pi G(k).$$
An exact one-loop momentum-dependent Wilsonian renormalization group

This contribution is hence a multiplicative factor $Z_>$ changing the partition function, independent of $\varphi_<$. Thus it affects the free energy. But it has no consequence on the correlation functions of the field, because it becomes an additive correction to the cumulant-generating functional $W = \ln Z$. Since we here want to calculate only the critical exponents, we will neglect this term.

The remaining part of the coarse-grained action stems from the first line in (79), which just yields

$$\exp\left(-\frac{1}{2}\varphi^T_<> G^{-1} \varphi_<\right).$$

So together the coarse-grained action is

$$S_\ell[\varphi_<] = \ln \left(Z_> \exp\left(-\frac{1}{2}\varphi^T_<> G^{-1} \varphi_<\right)\right)$$

(81)

$$= \ln Z_> - \frac{1}{2}\varphi^T_<> G^{-1} \varphi_<.$$

The last line shows that the rescaled action, apart from the inconsequential constant, is the same as the original one (77). With one exception: the momenta of the fields only extend up to $\Lambda^\ell_-1$ instead of $\Lambda$.

5.2. Expansion of momentum dependence of rescaling term

The momentum dependence of the rescaling term appearing in (39), to order $O(k^2)$, takes the form

$$\left(\ell_j \ell_i\right)^{6\left(1 - \frac{d + \eta}{2}\right)} + d = \exp\left[\left(\frac{\ell_j}{\ell_i}\right)^{6\left(1 - \frac{d + \eta}{2}\right)} + d\right] \frac{1}{2} \ln \left(\ell_j \ell_i\right)^2$$

(82)

$$= \exp\left(6\left(1 - \frac{d + \eta}{2}\right) + d\right) \frac{1}{2} \ln \left(\frac{\Lambda^2}{k^2 + 2 k' \cdot k + k^2}\right)$$

$$= \exp\left(6\left(1 - \frac{d + \eta}{2}\right) + d\right) \frac{1}{2} \ln \left(1 + \frac{2 k' \cdot k + k^2}{\Lambda^2}\right)$$

$$\simeq \exp\left(6\left(1 - \frac{d + \eta}{2}\right) + d\right) \left(\frac{2 k' \cdot k + k^2}{\Lambda^2} - \frac{1}{2} \frac{4 (k' \cdot k)^2}{\Lambda^4}\right) + O(k^3)$$

$$\simeq 1 + \left(6\left(1 - \frac{d + \eta}{2}\right) + d\right) \left(\frac{2 k' \cdot k + k^2}{2 \Lambda^2} - \frac{(k' \cdot k)^2}{\Lambda^4}\right)$$

$$+ \frac{1}{2} \left(6\left(1 - \frac{d + \eta}{2}\right) + d\right)^2 \frac{(k' \cdot k)^2}{\Lambda^4} + O(k^3)$$

$$= 1 + \frac{(k' \cdot k)}{\Lambda^2} \left(6\left(1 - \frac{d + \eta}{2}\right) + d\right)$$

$$+ \frac{k^2}{2 \Lambda^2} \left(6\left(1 - \frac{d + \eta}{2}\right) + d\right)$$
An exact one-loop momentum-dependent Wilsonian renormalization group

\[ + \frac{(k' \cdot k)^2}{\Lambda^4} \left[ \frac{1}{2} \left( 6 \left( 1 - \frac{d + \eta}{2} \right) + d \right)^2 - \left( 6 \left( 1 - \frac{d + \eta}{2} \right) + d \right) \right] \]

\[ = 1 + \left( 6 - 2d - 3\eta \right) \left( \frac{(k' \cdot k)}{\Lambda^2} + \frac{k^2}{2\Lambda^2} + \frac{(k' \cdot k)^2}{\Lambda^4} \right) \left[ 2 - d - \frac{3}{2}\eta \right]. \]

5.3. Uniaxial and spherical symmetric integrals

We write for short

\[ \int d\Omega f(\Omega \cdot \Lambda) \equiv \int_{|k| = \Lambda} f(k). \]

If the integrand is spherically symmetric, so that it does not depend on \( \Omega \), we may perform the angular integral to obtain

\[ \int d\Omega = S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \]

which yields the area of the \( d \)-dimensional unit sphere and \( \Gamma \) denotes the gamma function

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt. \]

For uniaxial integrals we have [24, p. 379]

\[ I = \int \frac{d^d q}{(2\pi)^d} f(|q|, \theta) \]

\[ = \int dq q^{d-1} \int d\Omega f(|q|, \theta) \]

\[ = \int dq q^{d-1} C \int_0^\pi d\theta (\sin \theta)^{d-2} f(|q|, \theta), \]

\[ C = \frac{S_d}{(2\pi)^d} \int_0^\pi d\theta (\sin \theta)^{d-2} = \frac{S_{d-1}}{(2\pi)^d}. \]

We need such integrals to compute the \( k \)-dependence of the fluctuation corrections. Here the integrand \( f(|q|, \theta) \) only depends on the absolute magnitude \( |q| \) and a single angle \( \theta \) between a specified direction in \( \mathbb{R}^d \) and the \( q \)-vector to integrate over.

We need in particular integrands of the form \( f(|q|) \cos^2(\theta) \) for which it holds that

\[ J = \int \frac{d^d q}{(2\pi)^d} f(|q|) \cos^2(\theta) \]

\[ = \frac{1}{(2\pi)^d} \int d\Omega \cos^2(\theta) \int dq q^{d-1} f(|q|) \]

\[ = \frac{S_d}{(2\pi)^d} \frac{1}{d} \int dq q^{d-1} f(|q|). \]

This can be seen as follows: For the angular part of (85) alone we get with (84)

\[ \int d\Omega \cos^2(\theta) = C \int_0^\pi d\theta \sin^{d-2}(\theta) \cos^2(\theta) \]

\[ = C \int_0^\pi d\theta \sin^{d-2}(\theta) (1 - \sin^2(\theta)) \]

\[ = C \int_0^\pi d\theta \sin^{d-2}(\theta) - \sin^d(\theta). \]
We may reduce the latter integral to the former for \( d > 1 \)

\[
I := \int_0^\pi d\theta (\sin \theta)^d = \int_0^\pi d\theta \sin \theta (\sin \theta)^{d-1}
\]

\[
= -\cos \theta (\sin \theta)^{d-1}\bigg|_0^\pi + \int_0^\pi d\theta \cos^2 \theta (d-1)(\sin \theta)^{d-2}
\]

\[
= (d-1) \int_0^\pi d\theta (1 - \sin^2 \theta)(\sin \theta)^{d-2}
\]

\[
= (d-1) \int_0^\pi d\theta (\sin \theta)^{d-2} - (d-1) I.
\]

So we get

\[
I = \frac{d-1}{d} \int_0^\pi d\theta (\sin \theta)^{d-2}.
\]

We can therefore combine both angular integrals (86) as

\[
\frac{1}{(2\pi)^d} \int d\Omega \cos^2(\theta) = \frac{S_d}{(2\pi)^d} (1 - \frac{d-1}{d}) = \frac{S_d - 1}{d} \frac{1}{d},
\]

from which follows (85).

5.4. Momentum-dependent integral of four-point vertex

Here we compute the integral appearing in (39) at small momenta \( k \), using (82). We hence have to compute two spherically symmetric and two uniaxial terms. The latter uniaxial term \( \propto (k' \cdot k)^2 \) in (82) yields with \( (k \cdot k')^2 = ||k||^2 \Lambda^2 \cos(\Omega_1)^2 \) and (87)

\[
\int_{|k'|=\Lambda} \theta(k' \cdot k+) (k \cdot k')^2 = \frac{1}{2} \Lambda^2 k^2 \frac{S_d}{d},
\]

where the factor 1/2 comes from the constraint that by \( \theta(k' \cdot k) \) we are only integrating over half a momentum shell. The spherically symmetric integrals (\( \propto (k')^0 \)) simply produce one factor \( S_d \); here the entire shell contributes for vanishing external momenta. The remaining integral is given with (84) by

\[
\int_{|k'|=\Lambda} \theta(k' \cdot k+) k \cdot k' \]

\[
= \Lambda ||k|| \int_{|k'|=\Lambda} \theta(\cos(\Omega_1)+) \cos(\Omega_1)
\]

\[
= \Lambda S_{d-1} ||k|| \int_0^\pi d\Omega_1 (\sin \Omega_1)^{d-2} \theta(\cos(\Omega_1)+) \cos(\Omega_1)
\]

\[
= S_{d-1} \Lambda ||k|| \int_0^{\pi/2} d\Omega_1 (\sin \Omega_1)^{d-2} \cos(\Omega_1)
\]

\[
= S_{d-1} \Lambda ||k|| \frac{1}{d-1} (\sin \Omega_1)^{d-1} \int_0^{\pi/2} = \Lambda S_{d-1} ||k|| \frac{1}{d-1}.
\]

Taken together, we get (42) in the main text.
5.5. Momentum dependence of self-energy

Approximating the dependence on the momentum \( q \) for \( q \ll k \) up to orders of \( \mathcal{O}(q^2) \), the resulting integrals appearing in (51) are again either spherically symmetric or uniaxial which follows with 
\[
\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^2)
\]
from the expansion (43)
\[
\frac{1}{4!} S^{(4)}_\ell (q, k, -k, q) \approx u^{(0)}_\ell ||k|| \left( 1 + \frac{q \cdot k}{||k||^2} + \frac{q^2}{2 ||k||^2} - \frac{1}{2} \frac{(q \cdot k)^2}{||k||^4} \right) \]
\[
+ u^{(1)}_\ell (q^2 + 2q \cdot k + k^2) + \mathcal{O}(||q||^3).\]

Taking the integral over the sphere of \( k \)-vectors in (13), the terms \( \propto 2q \cdot k \) do not contribute by their point symmetry, indicated by the underbraces. The terms \( \propto q^2 \) yield a momentum dependence of the self-energy. The uniaxial term is again of the form (85) and yields
\[
- \frac{u^{(1)}_\ell}{2} \int_{||k||=\Lambda} \frac{(q \cdot k)^2}{||k||^3} = - \frac{u^{(1)}_\ell}{2\Lambda} S_d \frac{2}{d} \frac{q^2}{d}.\]

To determine the flow equations for \( r^{(0)} \) and \( r^{(2)} \), we need to decompose the self-energy corrections \( \ell \frac{d}{dt} S^{(2)}_\ell (q, -q) \) according to their \( q^2 \)-dependence as
\[
\ell \frac{d}{dt} r^{(0)} = - \ell \frac{d}{dt} S^{(2)}_\ell (0, 0), \quad (90)
\]
\[
\ell \frac{d}{dt} r^{(2)} = - \lim_{q \to 0} \frac{\ell \frac{d}{dt} S^{(2)}_\ell (q, -q) - \ell \frac{d}{dt} S^{(2)}_\ell (0, 0)}{q^2}.\]

It is therefore sufficient to compute (51) for small momenta \( q \). This, in turn, means we need the \( q \)-dependence of the interaction vertex in the first two entries, given by (43). The projections (90) hence yield the flow equation for the momentum-independent parameter
\[
\ell \frac{d}{dt} r^{(0)} = 2 \cdot 4 \cdot \frac{\Lambda^d}{(2\pi)^d} \int_{||k||=\Lambda} \left( u^{(0)}_\ell + u^{(1)}_\ell ||k|| + u^{(2)}_\ell k^2 \right) G_\ell(k).
\]
\[
= 2 \cdot \frac{S_d \Lambda^d}{(2\pi)^d} \cdot \frac{6 u^{(0)}_\ell + 4 u^{(1)}_\ell \Lambda + 4 u^{(2)}_\ell \Lambda^2}{r^{(0)} + r^{(2)} \Lambda^2}.
\]

The flow of the momentum-dependent term of the self-energy then follows from (51), (52), and (90) as
\[
\ell \frac{d}{dt} r^{(2)} = 2 \cdot 4 \cdot \frac{\Lambda^d}{(2\pi)^d} \int_{||k||=\Lambda} \left( \frac{u^{(1)}_\ell}{2\Lambda} \left( 1 - \frac{1}{d} \right) + u^{(2)}_\ell \right) G_\ell(k).
\]
\[
= 8 \cdot \frac{S_d \Lambda^d}{(2\pi)^d} \cdot \frac{u^{(1)}_\ell}{r^{(0)} + r^{(2)} \Lambda^2}.
\]
5.6. Detailed calculation of fixed points and critical exponents for $3 \leq d \leq 4$

We may now compute the values of the fixed point for arbitrary dimensions between 3 and 4. We again determine $\eta$ such that $r_s^{(2)} = 1$.

**Interaction $u$** The fixed point for the momentum-independent four point coupling must obey $0 = d u^{(0)}_{\ell} / d \ell$ from which follows with (48)

$$0 = (4 - d - 2\eta) u^{(0)}_{\ell} - 36 \frac{S_d \Lambda^d}{(2\pi)^d} \frac{(u_s^{(0)})^2}{(r_s^{(0)} + \Lambda^2)^2}$$

$$r_s^{(0)} \approx (4 - d - 2\eta) u^{(0)}_{\ell} - 36 \frac{S_d \Lambda^{d-4}}{(2\pi)^d} \cdot (u_s^{(0)})^2$$

$$\Lambda^{d-4} u_s^{(0)} \approx (2\pi)^d \frac{4 - d - 2\eta}{36 S_d}. \quad (91)$$

The first-order momentum dependence becomes with $0 = d u^{(1)}_{\ell} / d \ell$ and (49)

$$0 = (3 - d - 2\eta) u^{(1)}_{\ell} - 36 \left(2(3 - d) - 3\eta\right) \frac{S_{d-1} \Lambda^d}{d - 1} \frac{(u_s^{(0)})^2}{(r_s^{(0)} + \Lambda^2)^2} \frac{1}{\Lambda}$$

$$r_s^{(0)} \approx (3 - d - 2\eta) u^{(1)}_{\ell} - 36 \left(2(3 - d) - 3\eta\right) \frac{S_{d-1} \Lambda^{d-5}}{d - 1} (u_s^{(0)})^2$$

$$u_s^{(1)} \approx 36 \left(2(3 - d) - 3\eta\right) \frac{S_{d-1} \Lambda^{d-5}}{3 - d - 2\eta} \left(\frac{r_s^{(0)} + \Lambda^2}{2}\right)^2 \frac{1}{(d - 1) S_d^2}. \quad (92)$$

The quadratic momentum dependence of the interaction with $0 = d u^{(2)}_{\ell} / d \ell$ and (50)

$$0 = (2 - d - 2\eta) u^{(2)}_{\ell} - 36 \left(2(3 - d) - 3\eta\right) \frac{4 - d - 3\eta}{4d} \frac{S_d \Lambda^d}{(2\pi)^d} \frac{(u_s^{(0)})^2}{(r_s^{(0)} + \Lambda^2)^2} \frac{1}{\Lambda^2}$$

$$r_s^{(0)} \approx (2 - d - 2\eta) u^{(2)}_{\ell} - 36 \left(2(3 - d) - 3\eta\right) \frac{4 - d - 3\eta}{4d} \frac{S_d \Lambda^{d-6}}{(2\pi)^d} (u_s^{(0)})^2$$

$$u_s^{(2)} \approx 36 \left(2(3 - d) - 3\eta\right) \frac{4 - d - 3\eta}{2d - 2\eta} \frac{S_d \Lambda^{d-6}}{(2\pi)^d} (u_s^{(0)})^2$$

$$\Lambda^{d-2} u_s^{(2)} \approx \frac{2d - 2\eta}{4d} \left(2(3 - d) - 3\eta\right) \frac{(4 - d - 3\eta)(4 - d - 2\eta)^2}{4d}. \quad (93)$$

**Mass $r$** To value for $r_s^{(0)}$, in the approximation $r_s^{(0)} \ll \Lambda^2$, is determined from $0 = d r^{(0)}_{\ell} / d \ell$ and (53)

$$0 = \left(2 - \eta\right) r_s^{(0)} + 2 \cdot \frac{S_d \Lambda^d}{(2\pi)^d} \cdot \frac{6 u_s^{(0)} + 4 u_s^{(1)} \Lambda + 4 u_s^{(2)} \Lambda^2}{r_s^{(0)} + \Lambda^2}$$

$$r_s^{(0)} \ll \Lambda^2, \eta \ll 2$$

$$r_s^{(0)} \approx 2 r_s^{(0)} + 2 \cdot \Lambda^2 \frac{S_d \Lambda^{d-4}}{(2\pi)^d} \left(\frac{6 u_s^{(0)} + 4 u_s^{(1)} \Lambda + 4 u_s^{(2)} \Lambda^2}{r_s^{(0)} + \Lambda^2}\right)$$
\( \Lambda^{-2} r_s^{(0)} \simeq - \frac{S_d}{(2\pi)^d} \left( 6 A^{d-4} u_s^{(0)} + 4 A^{d-3} u_s^{(1)} + 4 A^{d-2} u_s^{(2)} \right) \)

\( \simeq - \frac{6 S_d}{(2\pi)^d} A^{d-4} u_s^{(0)} - \frac{144 S_d}{(2\pi)^{2d}} \frac{2(3 - d) - 3\eta}{3 - d - 2\eta} \left( \frac{S_{d-1}}{d-1} + \frac{4 - d - 3\eta}{d} S_d \right) A^{d-4} u_s^{(0)}^2. \) 

Critical exponent \( \eta \) Demanding \( 0 = \frac{1}{\ell} \frac{dr_s^{(2)}}{d\ell} \) the anomalous dimension \( \eta \) is determined from (54) as

\[ \eta = 8 \frac{S_d}{(2\pi)^d} \frac{u_s^{(1)}}{r_s^{(0)} + \Lambda^2} \left( \frac{1}{2} \left( 1 - \frac{1}{d} \right) A^{d-3} u_s^{(1)} + A^{d-2} u_s^{(2)} \right) \]

\[ \simeq 8 \frac{S_d}{(2\pi)^{2d}} \frac{1}{2} \left( 1 - \frac{1}{d} \right) 2(3 - d) - 3\eta \left( \frac{2}{d - 2\eta} \right) \Lambda^{2d-8} u_s^{(1)} u_s^{(2)}. \]

Critical exponent \( \nu \) To determine the critical exponent \( \nu \), we need to linearize the flow about the fixed point and determine the eigenvalue \( \lambda_r \) in the direction of the parameter \( r \). We have the general form of the flow equation

\[ \ell \frac{d}{d\ell} \begin{pmatrix} r_s^{(0)} \\ r_s^{(2)} \\ u_s^{(1)} \\ u_s^{(1)} \\ u_s^{(2)} \end{pmatrix} = \beta(r_s^{(0)}, r_s^{(2)}, u_s^{(0)}, u_s^{(1)}, u_s^{(2)}). \]

Since the influence of \( r_s^{(0)} \) on the flow equations of the \( u_s^{(n)} \) is negligible, we can also neglect all terms \( \frac{\partial \beta_s^{(n)}}{\partial r_s^{(0)}} \simeq 0 \). Linearizing the flow equation (53) with \( \delta r_s^{(0)} = r_s^{(0)} - r_s^{(0)} \), we get

\[ \ell \frac{d\delta r_s^{(0)}}{d\ell} = (2 - \eta) \delta r_s^{(0)} - 2 \frac{6 u_s^{(0)} + 4 u_s^{(1)} + 4 u_s^{(2)} \Lambda^2}{\left( r_s^{(0)} + \Lambda^2 \right)^2} \delta r_s^{(0)} \]

\( \simeq (2 - \eta) \delta r_s^{(0)} - 2 \frac{6 u_s^{(0)} + 4 u_s^{(1)} + 4 u_s^{(2)} \Lambda^2}{\left( r_s^{(0)} + \Lambda^2 \right)^2} \delta r_s^{(0)} \)

\( \approx (2 - \eta) \delta r_s^{(0)} - \frac{1}{18} (4 - d - 2\eta)(2(3 - d) - 3\eta) \left( \frac{6}{2(3 - d) - 3\eta} + \frac{4}{d - 1} \frac{4 - d - 2\eta}{S_{d-1}} \right) \)
\[
\lambda_r = \frac{(4-d-3\eta)(4-d-2\eta)}{2-d-2\eta} \delta r
\]

The critical exponent \( \nu = \lambda_r^{-1} \) thus takes the form (70), as stated in the main text.

6. Bibliography

[1] Zinn-Justin J 1996 Quantum field theory and critical phenomena (Clarendon Press, Oxford)
[2] Wilson K G and Kogut J 1974 Physics Reports 12 75 – 199 ISSN 0370-1573 URL http://www.sciencedirect.com/science/article/pii/0370157374900234
[3] Wilson K G 1975 Rev. Mod. Phys. 47(4) 773–840 URL http://link.aps.org/doi/10.1103/RevModPhys.47.773
[4] Wegner F J and Houghton A 1973 Phys. Rev. A 8(1) 401–412 URL https://link.aps.org/doi/10.1103/PhysRevA.8.401
[5] Morris T R 1996 Nuclear Physics B 458 477–503 URL https://doi.org/10.1016/0550-3213(95)00541-2
[6] Weiszloch C 1993 Physics Letters B 309 90–94 ISSN 0370-2693 URL http://www.sciencedirect.com/science/article/pii/037026939390726X
[7] Kopietz P 2001 Nuclear Physics B 595 493–518 URL https://doi.org/10.1016/s0550-3213(00)00726-X
[8] Hasselmann N, Ledowski S and Kopietz P 2004 Phys. Rev. A 70 URL https://doi.org/10.1103/physreva.70.063621
[9] Ledowski S, Hasselmann N and Kopietz P 2004 Phys. Rev. A 69 URL https://doi.org/10.1103/physreva.69.061601
[10] Kühn T and Helias M 2018 Journal of Physics A: Mathematical and Theoretical 51 375004 URL http://stacks.iop.org/1751-8121/51/i=37/a=375004
[11] Delamotte B 2012 An introduction to the nonperturbative renormalization group Renormalization groups and effective field theory approaches to many-body systems ed Janos Polonyi J S (Springer) pp 49–130
[12] Wilson K G and Fisher M E 1972 Physical Review Letters 28 240–243 URL https://doi.org/10.1103/physrevlett.28.240
[13] Morris T R 1994 International Journal of Modern Physics A 09 2411–2449 (Preprint https://doi.org/10.1142/S0217751X94000972) URL https://doi.org/10.1142/S0217751X94000972
[14] Hertz J A, Roudi Y and Sollich P 2017 Journal of Physics A: Mathematical and Theoretical 50 033001 URL http://stacks.iop.org/1751-8121/50/i=3/a=033001
[15] Helias M and Dahmen D 2019 arXiv 1901.10416 [cond-mat.dis-nn]
[16] Lindén H, Tetzlaff T, Potjans T C, Pettersen K H, Grün S, Diesmann M and Einevoll G T 2011 Neuron 72 859–872 URL https://doi.org/10.1016/j.neuron.2011.11.006
[17] Beggs J M and Plenz D 2003 J. Neurosci. 23 11167–11177
[18] Roudi Y, Aurell E and Hertz J A 2009 Front. in Comput. Neurosc. 3 1–15
[19] Mora T and Bialek W 2011 Journal of Statistical Physics 144 268–302 ISSN 1572-9613 URL https://doi.org/10.1007/s10955-011-0229-4
[20] DiSanto S, Villegas P, Burioni R and Munoz M A 2018 Proceedings of the National Academy of Sciences 115 E1356–E1365 ISSN 0027-8424 (Preprint http://www.pnas.org/content/early/2018/01/26/1712989115.full.pdf) URL http://www.pnas.org/content/early/2018/01/26/1712989115
[21] Kardar M, Parisi G and Zhang Y C 1986 Phys. Rev. Lett. 56(9) 889–892 URL https://link.aps.org/doi/10.1103/PhysRevLett.56.889
[22] Bengio Y, Courville A and Vincent P 2013 IEEE Transactions on Pattern Analysis and Machine Intelligence 35 1798–1826 URL https://doi.org/10.1109/tpami.2013.50
[23] Mehta P and Schwab D J 2014 ArXiv e-prints (Preprint 1410.3831)
[24] Goldenfeld N 1992 Lectures on phase transitions and the renormalization group (Reading, Massachusetts: Perseus books)