Missing final state puzzle in the monopole-fermion scattering

Ryuichiro Kitano\textsuperscript{1,2} and Ryutaro Matsudo\textsuperscript{1}

\textsuperscript{1}KEK Theory Center, Tsukuba 305-0801, Japan
\textsuperscript{2}Graduate University for Advanced Studies (SOKENDAI), Tsukuba 305-0801, Japan

E-mail: ryuichiro.kitano@kek.jp, ryutaro.matsudo@kek.jp

Abstract

It has been known that when a charged fermion scatters off a monopole, the fermion in the $s$-wave component must flip its chirality, i.e., fermion number violation must happen. This fact has led to a puzzle; if there are two or more flavors of massless fermions, any superposition of the fermion states cannot be the final state of the $s$-wave scattering as it is forbidden by conservation of the electric and flavor charges. The unitary evolution of the state vector, on the other hand, requires some interpretation of the final states. We solve the puzzle by finding new particle excitations in the monopole background, where multi-fermion operators exhibit condensation. The particles are described as excitations of closed-string configurations of the operators.
1 Introduction

It has been known that an 't Hooft-Polyakov monopole [1, 2] can catalyze proton decays in grand unified theories (GUTs) at the rate of the typical strong interaction [3–5]. The origin of this effect is that, when a single charged fermion collides with a monopole, the helicity of the lowest partial wave of the fermion has to flip [6]. This forces the $s$-wave component of the incoming proton to interact with the monopole core where the GUT gauge interaction is not suppressed. The higher partial waves are kept away from the monopole core according to the analysis of the Dirac equation in the monopole background [7]. The process of the fermion-monopole scattering was analysed in the $s$-wave approximation where the fields are projected into the spherically symmetric component [3–5]. In this approximation, the theory reduces to a Schwinger model, and thus when the fermion is massless, the $s$-wave theory can be solved exactly. In the massive case, the bosonization of the theory is useful, where the fermions are described as kink solitons. A numerical study of the scattering problem has been performed by solving the equation of motion in the bosonized theory [8].

The soliton picture of the scattering can also be adopted when the effect of confinement is included by regarding a proton as a skyrmion [9]. In this framework, the skyrmion decays into a positron if the boundary condition at the core of the monopole violates the baryon number conservation.
1 Introduction

Recently, scattering amplitudes including magnetically charged particles are reconsidered using the on-shell method [10]. A multi-particle state including a magnetically charged particle and an electrically charged one simultaneously cannot be written as a direct product of one-particle state because the state has an additional quantum number called the pairwise helicity, which is the “cross product” of the electric and magnetic charges. The helicity flip of the lowest partial wave can also be derived using this technique.

Although the ability of monopole to catalyze proton decay was proposed a long time ago, there remains a puzzle; when the incoming particle is a massless fermion in a theory with two or more flavors of massless Dirac fermions, there seems to be no final state consistent with all the conservation laws [11–14]. In the s-wave calculation, the final state consists of objects with fractional fermion numbers, which makes it difficult to interpret what it actually is.

In this paper, we consider an SU(2) gauge theory with massless fermions and ’t Hooft-Polyakov monopoles. We resolve the puzzle by finding new fermionic states in the system. In the monopole background, one can identify the final states as soliton excitations that are “new” fermions with opposite helicity to the original fermions. In this picture, the fermion condensation around monopole plays a crucial role. Both new and original fermions can be described as the solitons of the multi-fermion operators that exhibit condensation without adopting the s-wave approximation, i.e., in four space-time dimensions. They are walls bounded by strings around which the phases of the multi-fermion operators wind. We call them “pancakes.” The helicity flip in the scattering is understood by following the classical time evolution of the pancakes.

For \(N_f \geq 2\), a pancake with the opposite helicity cannot be interpreted as one of the original fermions because its flavor charge does not match. This means that when there is a monopole, there are one-particle states that are not the original fermions. When we add masses to the fermions, the soliton that comes from the monopole after the scattering can only be considered as an intermediate state since it is no longer stable. The soliton further deforms into one of the original fermions.

The pancake solitons we consider in this paper are similar to the \(\eta'\) strings, which are proposed to describe spin-\(N_c/2\) baryons in large \(N_c\) QCD [15]. In the case of \(\eta'\), which represents the phase of the quark bilinear, the pancake is shown to support the Chern-Simons theory which has a dynamical mode on the edge. The quantization of the edge mode provides baryon states with appropriate quantum numbers [15, 16]. Our discussion is parallel to this. The phases of the multi-fermion operators around the monopole can have pancake-like topologically non-trivial configurations. The pancakes interact with the photon via the Chern-Simons coupling, which implies the existence of the chiral edge modes on the boundary. The excitations of the edge modes describe the fermions in the theory. A closely related discussion has been given in
QED coupled with an axion. The axion string in the theory has similar properties as discussed in Ref. [17].

In the following section, we review the puzzle in the monopole-fermion system in the massless limit. The new states to account for the final states of the scattering are identified in Section 3, and the scattering processes are considered in the soliton picture in Section 3.3. We summarize the discussion in Section 5.

2 Semiton puzzle

We consider the \( SU(2) \) gauge theory coupled with an adjoint scalar \( \Phi \) and \( 2N_f \) Weyl fermions \( \chi_j \) in the fundamental representation, whose Lagrangian is

\[
\mathcal{L} = -\frac{1}{4g^2} \text{tr}(f*f) + i \sum_{j=1}^{2N_f} \bar{\chi}_j \sigma^\mu D_\mu \chi_j + \frac{1}{2} 2 \text{tr}(D\Phi*D\Phi) - \frac{\lambda}{4} (2 \text{tr} \Phi^2 - v^2)^2,
\]

\[
f = da - i a^2 = f^a \sigma^a / 2, \quad \Phi = \Phi^a \sigma^a / 2, \quad D\Phi = d\Phi - i[a, \Phi], \quad D\chi_j = (d - ia)\chi_j,
\]

where \( a \) and \( f \) are the \( SU(2) \) gauge field and its field strength respectively. We set the potential for the adjoint scalar so that it has a non-zero expectation value. The gauge group \( SU(2) \) is spontaneously broken down to \( U(1) \). Note that this theory has \( SU(2N_f) \) flavor symmetry rather than \( SU(N_f) \times SU(N_f) \) because of the pseudo-reality of the \( SU(2) \) gauge group. The symmetry plays an important role in the following discussion. We denote the components of the left-handed Weyl fermions, \( \chi_j \), as \( a_j \) and \( b_j \), whose charges under the unbroken \( U(1) \) group are 1 and \( -1/2 \), respectively. The fermions \( a_j \) and \( b_j \) have \( 2N_f \) components which form the fundamental representation under the \( SU(2N_f) \) flavor group. At low energy, the theory reduces to the \( U(1) \) gauge theory as

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4g^2} f*f + i\bar{a}_j \sigma^\mu D_\mu a_j + i\bar{b}_j \sigma^\mu D_\mu b_j,
\]

\[
Da_j = (d - ia)a_j, \quad Db_j = (d + ia)b_j,
\]

where \( a \) without index and \( f \) are the \( U(1) \) gauge field and its field strength respectively. Let us consider what happens when a particle with a unit charge collides with an ’t Hooft-Polyakov monopole. Even if the particles are at rest, the \( U(1) \) gauge field (that is the electromagnetic field) carries an angular momentum

\[
\vec{J}_{\text{EM}} = \frac{1}{4\pi} \int d^3 x \vec{r} \times (\vec{E} \times \vec{B}) = -\frac{1}{2} \vec{r}_0,
\]

where \( \vec{r}_0 \) is the unit vector pointing from the magnetic monopole to the charge. Therefore, if the particle has a helicity \(-1/2\) (left-handed) and the impact parameter is zero, the total angular
momentum is zero. After the collision, the angular momentum from the $U(1)$ gauge field has the opposite direction to the particle momentum. See Fig. 1. This means that, in order to maintain the total angular momentum to be zero, the helicity of the particle must flip [3–6]. This causes a problem when $N_f \geq 2$. The helicity, the $U(1)$ charge and the representation of $SU(2 \times N_f)$ of the fermions are

\[ a : (L, +1, □), \quad b : (L, -1, □), \quad α : (R, -1, □), \quad b : (R, +1, □). \] (4)

However, if the incoming particle is $a$, there are no corresponding outgoing particles because the quantum numbers of the outgoing particle must be $(R, +1, □)$ for the conservation of the total angular momentum, the $U(1)$ charge, and the $SU(2 \times N_f)$ charge.

When $N_f = 2$ and the incoming state is the $s$-wave of $a_1$, the conservation of the charge and the flavor quantum number implies that the final state is something like

\[ \frac{b_1}{2} + \frac{\bar{b}_2}{2} + \frac{\bar{b}_3}{2} + \frac{\bar{b}_4}{2}, \] (5)

where $b_j/2$ ($\bar{b}_j/2$) denotes the state whose quantum numbers are halves of those of $b_j$ ($\bar{b}_j$). The state $b_j/2$ is sometimes called as a “semiton” [13].

In Ref. [12, 14], the semiton state is interpreted as follows. If we regard the theory as an approximation of the $SU(5)$ GUT with a monopole, the correspondence of the particles is

\[ a_1 = e^+_R, \quad a_2 = d^3_L, \quad a_3 = u^1_L, \quad a_4 = u^2_L, \]
\[ b_1 = d^3_L, \quad b_2 = e^-_R, \quad b_3 = u^2_L, \quad b_4 = u^1_L. \] (6)

The final state (5) is interpreted as the superposition of $e^+_R$ and $u^1_L u^2_R d^3_L$ with equal weights. However, this is problematic because the $SU(2 N_f)$ charge is not conserved. Note that $e^+_R = \bar{b}_2$ is in the antifundamental representation of $SU(2 N_f)$.

We argue that the state (5) should be interpreted as a “new particle”, whose quantum number is $(R, +1, □)$. The new particle is not a single excitation from the vacuum. It is rather a part of a two-particle state: a monopole and a new particle, but the new particle can be arbitrarily far away from the monopole. We will be able to find such a state as a soliton of the multi-fermion operators around the monopole.
The puzzle disappears when we add fermion masses since the flavor symmetry is reduced. An appropriate final state can always be found as (a combination of) the original fermions. Therefore, one can understand the Callan-Rubakov effects as a simple scattering process in the massive case. The final states are particles. The discussion of the massless limit should recover this result when masses are added. We will discuss how the massless limit and the massive case are connected in Section 3.3.

It was confirmed that the final state has the fractional fermion numbers by applying the s-wave projection [8,12,14,18]. The s-wave projection is performed by substituting the spherically symmetric fields, whose total angular momentum is zero. At low energy, the theory reduces to the $2N_f$-flavor QED in two dimensions with the space-dependent coupling. In the bosonized theory [19, 20], the scalar fields correspond to the phases of the fermion bilinears, and the fermions are described as a kink soliton made of the phases. By solving the equation of motion, we see that the final state corresponds to a kink with fractional fermion numbers [8,14]. This theory for $N_f = 2$ in $g \to 0$ limit was analysed in Ref. [18], and it was found that the theory contains new fermions with the opposite $U(1)_A$ charge as the original fermion in the action as a collective mode of the fermion fields.

The analysis in the s-wave projection is robust for the following reason. We can safely take $g \to 0$ limit to consider the puzzle because the helicity flip and the charge conservation is maintained. In the limit, the fermions freely propagate when it does not reach the core of the monopole. From the analysis of the Dirac equation, we see that the higher partial waves cannot reach the core due to the centrifugal barrier term [4,7], and thus no mixing between the s-wave and the higher partial wave can occur. Therefore, the analysis in the s-wave projection is exact in the $g \to 0$ limit. The boundary condition that is consistent with the flavor and electric charge conservation is uniquely determined in the s-wave theory, which gives the final state of the scattering with the fractional fermion numbers. The only remaining problem is what the interpretation of the final state is. We give an answer to this problem by giving a four dimensional picture that reproduces the two dimensional result when we restrict ourselves to the spherically symmetric system.

3 Fermions as solitons

Around the monopoles, there are condensations of multi-fermion operators which break the anomalous $U(1)$ symmetry while the $SU(2N_f)$ global symmetry is left unbroken [3,21]. We will describe in this section how the configurations of the multi-fermion operator can be identified with the fermions. We will find that there are configurations which have the same quantum numbers as the original fermions, $a_j$ and $b_j$, as well as new fermion states which are to be
3 Fermions as solitons

We consider the configuration where there is a static ’t Hooft-Polyakov monopole, which we approximate as a Dirac monopole for the $U(1)$ gauge field. Around the monopole, operators made out of the fermion fields have nonzero expectation values. The possible condensates are [14,22]

$$
\langle (a_{i_1} b_{i_2}) \cdots (a_{i_{2N_f-1}} b_{i_{2N_f}}) \rangle = \frac{1}{r^{3N_f}} c \epsilon_{i_1 \cdots i_{2N_f}},
$$

(7)

where $r$ is the distance from the monopole, and $c$ is a constant. We can consider string configurations around which the phases of these operators wind. As we will explain below, the strings can have nonzero electric charges, spin half and $SU(2N_f)$ charges, which enable us to regard the strings as fermions. This string configurations are similar to the axion strings, which can also have nonzero charge and spin half [23]. In general, such a string configuration exists when the field that is transformed under $U(1)_A$ has a nonzero expectation value.

It is possible that the condensate (7) affects the asymptotic states even though the value goes to zero at spatial infinity. To see what happens, let us first consider the s-wave theory. In the s-wave theory, there is a collective mode of the fermionic field corresponding to the semiton state, which is described as a kink in the bosonized picture. In our view, this kink solution should be lifted up to four dimensions, i.e., to a large wall surrounding the monopole. This is one of the examples of what happens in the $r \to \infty$ limit, meaning that the kinetic energy of the scalar (i.e. the phase of the operator) is preserved by expanding the size of the wall even if the value of the condensate is reduced. This large wall should be regarded as a spherically symmetric wave function of the fermion, given its correspondence to a kink in the s-wave theory. Next, let us consider the case of a pancake, a wall bounded by a string. Since we want to consider a pancake as a particle, it should be possible to make it smaller at any point in space. In the case of a pancake, there is another way to maintain its kinetic energy than to increase its size. That is to become thinner as it moves away from the monopole. Although we did not try to find an explicit pancake solution as we do not have a complete effective theory of condensates, the interpretation of the pancake as a particle is at least consistent.

3.1 Monopole bags and the Witten effect

To see the effect of the condensates (7), we try to determine the effective Lagrangian that is obtained by “integrating in” their phases. A phase shift of the operators is induced by a combination of phase shifts of the fermions, which gives a shift of the Lagrangian due to the chiral anomaly. Below, we consider an effective theory where the phases of the fermions are the dynamical degrees of freedom. Let $\alpha_j$ and $\beta_j$ be the phases of $a_j$ and $b_j$, respectively.
Note that there are configurations of $\alpha_j$ and $\beta_j$ that represent an identical configuration of the phases of the multi-fermion operators, which means there is a redundancy of description. This is because, under $SU(2N_f)$ transformations and $U(1)$ gauge transformations, $\alpha_j$ and $\beta_j$ change but the multi-fermion operators do not.

We can consider the domain wall configuration of $\alpha_j$ and $\beta_j$ because they are $2\pi$ periodic. When the wall wraps the monopole, the $U(1)$ charge and the $SU(2N_f)$ charge are induced due to the Witten effect [24]. We call such an object as a monopole bag [17]. To determine the charges of a monopole bag, we consider the background gauge field for the $U(1)$ subgroups of $SU(2N_f)$. The covariant derivative of $a_j$ is

$$\left( d - ia - iA_l[H_l]_{jj} \right)a_j,$$

where $A_l$ is the gauge field corresponding to a Cartan generator $H_l$ in $su(2N_f)$, and $[M]_{ij}$ denotes the $ij$-component of the matrix $M$. The phase rotations give the spacetime dependent “$\theta$ terms,”

$$\frac{1}{8\pi^2} \sum_j (\alpha_j(f + F_l[H_l]_{jj})^2 + \beta_j(-f + F_l[H_l]_{jj})^2).$$

The boundary condition at the core of the monopole is

$$\theta_A := \sum_j (\alpha_j + \beta_j) = 0,$$

$$\theta_{jk} := \alpha_j - \beta_j - \alpha_k + \beta_k = 0, \quad \forall j, k. \quad (10)$$

This condition is determined so that there is no divergence of the action other than that of the kinetic term $-f^*f/4$ corresponding to the mass of the monopole. The divergence should vanish for any value of the background gauge fields. This boundary condition ensures the conservation of the electric and flavor charges. Since the phases of the multi-fermion operators in Eq. (7) can be expressed as linear combinations of $\theta_A$ and $\theta_{jk}$, the condition (10) means that the phases of the condensates has to be fixed. From this expression, we can read off the $U(1)$ current $J$ and the currents $J_l$ for the maximal torus of $SU(2N_f)$. If $A_l = 0$, they are

$$\ast J = \frac{1}{4\pi^2} d\theta_A f,$$

$$\ast J_l = \sum_j [H_l]_{jj} \frac{1}{4\pi^2} d(\alpha_j - \beta_j). \quad (11)$$

1The right-hand side of the conditions (10) is not integer multiples of $2\pi$ but zero. When we replace the monopole by a dyon, the right-hand side changes to $2\pi$ times a corresponding integer so that the charge of the dyon is canceled by the charge coming from the Witten effect.
Therefore, the monopole bag corresponding to the positive $2\pi$ shift of $\alpha_j$, i.e., $2\pi$ shifts of $\theta_A$ and $\theta_{jk}$, has the $U(1)$ charge $Q = 1$ and the same $SU(2N_f)$ charges as $a_j$. Also, the monopole bag correspond to the $2\pi$ shift of $\beta_j$ has the same charges as $b_j$. By inserting a thin wall where $\alpha_j$ changes from 0 to $2\pi$ to the topological term (9), we obtain the Chern-Simons coupling with the level unity on the wall

$$\frac{1}{4\pi} \int_{M^3} (a + A_l[H_{l}][jj])(f + F_l[H_{l}][jj]),$$

where $M^3$ is the world volume of the wall.

3.2 Pancake soliton

Next, let us consider the string configuration around which $\alpha_j$ winds. Such an object exists because at the core of the string, the values of the multi-fermion fields can be zero so that the configuration is smooth everywhere. There is a domain wall bounded by the string, which means that the object is considered as the monopole bag with a hole. As we mentioned above, there is the Chern-Simons coupling (12) on the wall. Thus there should be a chiral edge mode on the string because otherwise the gauge symmetry is broken. Let us consider the case where the domain wall is a two-dimensional disc $D^2$. The theory on the wall bounded by the string is [25]

$$\frac{1}{4\pi} \int_{\mathbb{R} \times D^2} (a + A_l[H_{l}][jj])(f + F_l[H_{l}][jj])$$

$$+ \frac{1}{4\pi} \int_{\mathbb{R} \times \partial D^2} (D_x \phi (D_t \phi + v D_x \phi) dx dt - \phi (f + F_l[H_{l}][jj])),$$

where $x$ is a coordinate of $\partial D^2$ that is periodic and $\phi$ is a $2\pi$-periodic scalar on $\partial D^2$, that transforms as $\phi \to \phi + \lambda + \lambda_t[H_{l}][jj]$ under the gauge transformations $a \to a + d \lambda, A_t \to A_t + d \lambda_t$. The covariant derivative is defined as $D \phi := d \phi - a - A_l[H_{l}][jj]$ and $v$ is a constant. The $U(1)$ charge $Q$ and $SU(2N_f)$ charges $Q_l$ can be written as

$$Q = \frac{1}{2\pi} \int_{\partial D^2} (d \phi - a - A_l[H_{l}][jj]) + \frac{1}{2\pi} \int_{D^2} (f + F_l[H_{l}][jj]),$$

$$Q_l = \frac{1}{2\pi} \int_{\partial D^2} (d \phi - a - A_k[H_{k}][jj]) + \frac{1}{2\pi} \int_{D^2} (f + F_k[H_{k}][jj])[H_{l}][jj].$$

\(^2\)In two dimension, the definition of the current $j$ depends on the “position” of $a$ in the integral when taking the variation,

$$\int \ast ja = - \int a \ast j = \int a \ast j', \Rightarrow j = -j'.$$

In this paper, we define $j$ by putting $a$ to the right. We should note this if one combines the charges from the edge and the bulk.
Fermions as solitons

Figure 2: The direction of the spin of the pancake. The red arrow denotes the direction of the spin.

If we take the gauge where the Dirac string from the monopole does not penetrate the wall, there is no contribution from the gauge fields \( a \) or \( A_l \) due to the cancellation between the bulk and edge contributions. As a result, the charge of the object is identified with the winding number of \( \phi \). Note that, as the monopole becomes closer to the wall, the electric charge density moves from the edge to the bulk because the contribution from \( a \) partially cancels that from \( \phi \) on the edge, and \( f \) contributes to the charge in the bulk. The local operator that creates the state with charge \( \pm 1 \) is

\[
e^{\mp i \phi}, \tag{16}\]

which is confirmed by its commutator with the charge operator \( \int \partial_x \phi \, dx / (2\pi) \) for \( A = a = 0 \). The spin of the state can be calculated as the eigenvalue of \( LP_x / (2\pi) \), where \( L \) is the circumference of \( D^2 \) and \( P_x := \int (\partial_x \phi)^2 \, dx / (4\pi) \) is the generator of the translation along \( x \). The component perpendicular to \( D^2 \) of the spin of the state (16) is found to be \( 1/2 \). The direction of the spin depends only on the orientation of the domain wall, irrespective of the sign of the charge. See Fig. 2.

The theory on the wall bounded by the string for \( \beta_j \) is similar to that for \( \alpha_j \) with the exception of the sign in front of \( A_l \):

\[
\frac{1}{4\pi} \int_{\mathbb{R} \times D^2} (a - A_l[H_l]_{jj})(f - F_l[H_l]_{jj}) + \frac{1}{4\pi} \int_{\mathbb{R} \times \partial D^2} (D_x \tilde{\phi} (D_t \tilde{\phi} + v_D x \tilde{\phi}) \, dx dt - \phi (f - F_l[H_l]_{jj})). \tag{17}\]

The covariant derivative is defined as \( D \tilde{\phi} = (d - ia + iA_l[H_l]_{jj} ) \tilde{\phi} \).

The pancakes have unit charge, unit flavor charge and spin \( 1/2 \). Therefore we can regard them as fermions. However, the original fermions \( a_j \) and \( \bar{b}_j \) correspond to the pancakes with an appropriate direction of motion, and those moving in the opposite direction should be regarded as new particle states. See Fig. 3. We will see in the next subsection that one of them is the final state of the monopole scattering when the initial state is one of \( a_j \) and \( \bar{a}_j \).

The boundary of the pancake cannot just shrink because of the conserved quantum numbers. However, when the wall surrounds the monopole it gets possible, and thus the pancake can
Figure 3: Pancakes corresponding to the original fermions (upper panel) and the new fermions (bottom panel). The double arrows denote the direction of the motion. The red arrows denote the direction of the spin.

continuously deform to a monopole bag. When the pancake wraps the monopole, the charge at the edge moves toward the bulk as we can see in Eq. (15), then finally there are no charge at the edge and it can shrink.

Since we find a stable configuration of the phases, it just exists. Although it sounds surprising to find such a new state, the existence has already been admitted in the analyses of the s-wave reduced two-dimensional theory. In the bosonized picture, there are solitons which can be the final state of the scattering. We argue that the soliton state in the two-dimensional analyses is simply promoted to be a particle state in four dimensions as it should be.

Because the (original and new) fermions are massless, the pancake has to be a massless soliton, which has different features than a usual massive soliton. Since a massless soliton cannot be static by definition, we cannot define the tension, which is defined using the energy density of the static solution. Correspondingly, the potential term should vanish. For example, let us consider the bosonized version of the free fermion theory in two dimension. In the theory, a fermion is described as a sine-Gordon kink, where the fermion mass term can be regarded as a potential for the scalar field. In the massless limit of fermions, the potential disappears while the massless fermions are still described as kinks. The kink solution can no longer be static, reflecting the fact that fermions move at the speed of light. Then we see that a potential
3 Fermions as solitons

By identifying the fermions $a_j$ and $b_j$ as solitons, one can discuss the scattering with the monopole as classical time evolutions of the solitons. We will see how the initial state deforms into the new fermion states. We also discuss what happens when fermion masses are added.

In this section, we set $N_f = 2$ and consider the situation where a pancake hits the monopole. For concreteness, we take the initial state as the charge-one excited pancake made of $\alpha_1$, i.e., $a_1$. The corresponding situation for the axion strings is considered in Ref. [17].

We depict the evolution of the phases $\alpha_i$ and $\beta_i$ during the scattering in Fig. 4. We explain in the following the five evolution steps in Fig. 4.

1. The wall where the phase $\alpha_1$ gradually changes from $\alpha_1 = 0$ to $\alpha_1 = 2\pi$ goes towards the monopole. The solid line represents the domain wall and its boundary is the string. The monopole is in the region where $\alpha_1 = 0$.

2. When the wall reaches the location of the monopole, $\beta_i$’s change to satisfy the boundary

3. is needed for the existence of the static soliton solution, which cannot be massless, while the potential should vanish for the existence of a massless soliton, which cannot be static. In our case, the pancake describes the massless fermion, then the potential should vanish. [The interaction with photon gives the potential corresponding to the Coulomb interaction, but it can be neglected when we consider an isolated fermion.]

3.3 Soliton picture of the scattering

Figure 4: The soliton picture of the monopole scattering. The black solid line shows the wall where $\alpha_1$ changes.
conditions in Eq. (10) as
\[ \beta_1 = \frac{\alpha_1}{2}, \quad \beta_2 = \beta_3 = \beta_4 = -\frac{\alpha_1}{2}. \] (18)

At the location of the monopole, \( \alpha_1 \) changes from 0 to \( 2\pi \) and thus \( \beta_1 \) changes from 0 to \( \pi \) while \( \beta_{2,3,4} \) change from 0 to \(-\pi\). Due to this change of \( \beta_j \) near the monopole, there appears two types of walls that are connected: the wall where the value of \( \beta_j \) changes with \( \alpha_1 \) fixed to be \( 2\pi \) (black line) and the wall where both \( \alpha_1 \) and \( \beta_j \) change (red line). The latter one is in fact transparent because the values of the condensates do not change on the wall. Here the term “transparent” means that there is no object actually.

(3) The string goes away from the monopole, and a monopole bag with charge 1 is left. Note that the charge is completely transferred from the string to the bag. The neutral string is no longer stable and thus it disappears after the scattering.

(4) The wall can break by creating a string around which \( \alpha_1 \) changes from 0 to \( 2\pi \). There also appears the transparent wall on which \( \alpha_1 \) and \( \beta_1,2,3,4 \) change while satisfying the boundary condition.

(5) The wall representing the new particle remains as the final state. The quantum number is \((R, +1, \square)\), which is the leftmost one in the bottom panel of Fig. 3.

This description of the scattering is only schematic, and the actual process is not determined. However, for consistency, the following two points must be satisfied. (1) The helicity has to flip. This is consequence of the fact that the direction of the spin is determined by the direction of the wall. (2) The bag configuration surrounding the monopole can deform continuously to the pancake. This deformation is the reverse of the process that the string shrinks and disappears, which is allowed when the charges are conserved under the process.

The new particle can go away from the monopole for an arbitrary distance. Note, however, that there is no distance scale in the theory when the fermions are massless. Therefore, there is no actual meaning of far or close.

3.4 The initial state with the opposite helicity

We can also consider the case where the incoming particle is \( \bar{b}_j \), which is a right-handed particle with charge +1. In this case, the total angular momentum is \( \mathbf{J} = \mathbf{n} \), where \( \mathbf{n} \) is the direction of motion of the incoming particle, and therefore the process cannot be described in the \( s \)-wave theory. Because the initial state is not \( s \)-wave, the particle cannot interact with the monopole core and thus the helicity should not flip. The soliton picture correctly reproduce this feature.
at least when the energy of the initial particle is large enough so that the string goes through the monopole. In the process, the $U(1)$ charge moves from the bulk to the edge due to the Witten effect, and then the monopole bag with charge $-1$ and the pancake with charge $+2$ remain. Because the edge state of the pancake has winding number 2, the helicity of the pancake is $+1$, and thus the total angular momentum is conserved. The flavor charge of the initial state is $[H_{ljj}]$, and after the collision, the bag and the pancake has flavor charges $[H_{ljj}]$ and $-2[H_{ljj}]$ respectively. The bag is deformed to the pancake corresponding to $b_j$, which is a left-handed particle with charge $-1$. The pancake with charge $+2$ can collapse to two $\bar{b}_j$. The final state is $b_j + 2\bar{b}_j$, which can be regarded as just a pair production. A similar process in the skyrmion-monopole scattering is discussed in Ref. [9].

### 4 Effects of fermion masses

Let us introduce the mass terms

\[ -m \varepsilon_{ab}(\chi_{1a}\chi_{2b} + \chi_{3a}\chi_{4b} + \text{h.c.}) \]  

where $\chi_{ja}$ is a Weyl fermion with the flavor index $j$ and the $SU(2)$ index $a$. For this mass term, the global symmetry reduces to $Sp(2)$ [26]. In the effective theory, the mass terms are written as

\[ -m(a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 + \text{h.c.}), \]

where the minus sign in front of $a_2b_1$ and $a_4b_3$ is needed for the $SU(2)$ gauge invariance. The $U(1) \times U(1)$ subgroup of $Sp(2)$ acts on the fermions as

\[
\begin{align*}
    a_1 &\to e^{i\gamma_1}a_1, & a_2 &\to e^{-i\gamma_1}a_2 & a_3 &\to e^{i\gamma_2}a_3, & a_4 &\to e^{-i\gamma_2}a_4, \\
    b_1 &\to e^{i\gamma_1}b_1, & b_2 &\to e^{-i\gamma_1}b_2 & b_3 &\to e^{i\gamma_2}b_3, & b_4 &\to e^{-i\gamma_2}b_4.
\end{align*}
\]

Therefore, $\bar{b}_2$ has the same flavor charge as $a_1$. This means that there is a candidate of the final state, and thus the puzzle disappears. Therefore, we expect that the semiton states disappear when the mass term is introduced. According to the numerical simulation [8], the semiton state appears only as an intermediate state, and it decays into a usual fermion. In the following, we give a four dimensional picture of this phenomenon.

In the massive case, there appears additional condensates:

\[
\begin{align*}
    \langle a_1b_2 \rangle &\simeq -\frac{\tilde{c}_\mu}{r^2}, & \langle a_2b_1 \rangle &\simeq \frac{\tilde{c}_\mu}{r^2}, & \langle a_3b_4 \rangle &\simeq -\frac{\tilde{c}_\mu}{r^2}, & \langle a_4b_3 \rangle &\simeq \frac{\tilde{c}_\mu}{r^2},
\end{align*}
\]
where $\mu$ is a renormalization scale and $\tilde{c}$ is a constant. Let $\phi_1, \phi_2, \phi_3$ and $\phi_4$ be the phases of these operators. Using $\alpha_j$ and $\beta_j$, they are expressed as

$$\phi_1 = \alpha_1 + \beta_2, \quad \phi_2 = \alpha_2 + \beta_1, \quad \phi_3 = \alpha_3 + \beta_4, \quad \phi_4 = \alpha_4 + \beta_3. \quad (23)$$

The s-wave analysis indicates that, in the effective theory of $\phi_j$, the mass term is written as

$$\frac{\mu}{4\pi r^2} \sum_{j=1}^{4} (1 - \cos \phi_j). \quad (24)$$

In the s-wave theory, a kink of $\phi_1$ describes a mixed state of $a_1$ and $\bar{b}_2$. By analogy, in four dimensions, a pancake of $\phi_1$ should describe it. Also in the massive case, the scattering of a monopole and a fermion can be described by using pancakes.

According to the s-wave analysis, the boundary condition at the core of the monopole is given as

$$\sum_j \partial_t \phi_j = \partial_t (\phi_1 - \phi_2) = \partial_t (\phi_3 - \phi_4) = \partial_r (\phi_1 + \phi_2 - \phi_3 - \phi_4) = 0 \quad \text{at} \quad r = 0, \quad (25)$$

where $r$ is the radial distance from the monopole. This condition is obtained so that the $U(1)$ current $J$ and the flavor currents $J_l$ are conserved at the origin, which is the same as the massless case. In contrast with the massless case, the values of the multi-fermion fields can change at the origin. When the wall of $\phi_1$ traveling nearly at the speed of light reaches the core of the monopole, the values of $\phi_j$ gradually change toward

$$\phi_1 = \pi, \quad \phi_2 = \pi, \quad \phi_3 = -\pi, \quad \phi_4 = -\pi, \quad \text{at} \quad r = 0. \quad (26)$$

This gives the maximum of the mass term (24), and therefore the region where the values of $\phi_j$ given as this should decay into the vacuum.

The scattering of the s-wave fermion and the monopole can be calculated in the s-wave theory. A classical numerical simulation was performed in the s-wave theory in Ref. [8]. We can interpret the result as follows:

(1) Let the initial particle be $a_1$, which can be described as a kink soliton where $\phi_1(r = 0) = 0$ and $\phi_1(r = \infty) = 2\pi$.

(2) In the region where $r \leq 1/m$, one can ignore the mass term, and thus the semiton state appears after the collision. In the bosonized theory, the state is described as a kink soliton where the values of the boson fields at the origin and the infinity are

$$\phi_1(r = 0) = \phi_2(r = 0) = \pi, \quad \phi_3(r = 0) = \phi_4(r = 0) = -\pi,$$

$$\phi_1(r = \infty) = 2\pi, \quad \phi_2(r = \infty) = \phi_3(r = \infty) = \phi_4(r = \infty) = 0. \quad (27)$$
(3) Due to the mass term, the region where $\phi_1 = \phi_2 = \pi$, $\phi_3 = \phi_4 = -\pi$ is no longer the vacuum, but it gives the maximum of the potential. Therefore, at some point of $r$ and $t$, the value of $\phi_j$ starts to change towards $\phi_j = 0 \bmod 2\pi$, which can be regarded as the pair creation of the semiton states $\bar{b}_1/2 + b_2/2 + b_3/2 + b_4/2$ and $a_1/2 + \bar{a}_2/2 + a_3/2 + a_4/2$. Because the s-wave states of $a_j$ and $\bar{a}_j$ can only be the initial state, $a_1/2 + \bar{a}_2/2 + a_3/2 + a_4/2$ goes back to the monopole. After the collision, it changes to $\bar{b}_2$ in the same way that $a_1$ changes to the semiton state, and thus the final state is $\bar{b}_2$. The pair of the semiton states is seen as a ripple of the boson field.

4.1 Interpretation in four dimensions

In the massive case, the field variables in the effective theory is $\phi_1, \phi_2, \phi_3$ and $\phi_4$ due to the additional condensates (22), which is different from the massless case. For the spherically symmetric configuration, the same scattering process as the s-wave approximation is expected. In the same way as the massless case, the fermions can be described as pancake solitons of $\phi_j$. On the other hand, the semiton states cannot be described as isolated pancake solitons, but instead pancakes connected with the monopole by the region where $\phi_j = \pm \pi$. As expected from the s-wave approximation, the pair creation of the semiton states occurs at some point. See Fig. 5.

5 Summary

The monopole catalyzed proton decay, called the Callan-Rubakov effect, is a somewhat counter intuitive phenomenon. Even though the scale of the baryon number violating interaction is much higher than that of the proton mass or size, the rate of the proton decay around the monopole is of order unity in the unit of the proton mass. This effects have been used to put strong constraints on the monopole abundance in the Universe by searching for neutrino flux.
from the Sun [27–30] as well as by the limits on the X-rays from neutron stars [31,32].

Although the existence of the effect is theoretically robust with massive fermions, there remains a puzzle in the interpretation of the final states of the scattering processes in the limit of massless fermions. Just by the conservation of quantum numbers, the final states cannot be constructed from the superposition of the original fermion states. Logically, in this circumstance one may conclude that the massless limit of QED with the monopole is not unitary, or something wrong in the calculation. In this paper, we find that there is no such pathology. The final states of the scattering can actually be found as a form of solitons made of the multi-fermion operators in the monopole background.

The solitonic object is a domain wall whose boundary is a loop of a string. By the quantization of the edge mode on the string, fermionic states can be found. The states indeed have the correct quantum numbers for the missing final states. Moreover, the original fermions can also be described as the solitons. By using the soliton picture, one can describe the monopole-fermion scattering as the evolution of the solitons. We can understand how the chirality flipping happens and how the original particle deforms into the new particle. The massless QED seems to have a peculiar feature. The fermion spectrum is doubled when there is a monopole somewhere!

By adding masses, $m$, to fermions, such new fermion states disappear. In the soliton picture, the small mass deforms the particle into a particle with a tail which stretches to the monopole. The object is unstable when the distance between the particle and the monopole gets further than $O(1/m)$. The tail is detached from the monopole and eventually the particle-tail system collapses into one of the original fermions. The asymptotic states of the scattering are only the original fermions in the massive case. The “new particle” is an approximate picture of an intermediate state that can only exist sufficiently near the monopole. In the massless limit, everywhere gets sufficiently near the monopole.

Throwing an electron to the monopole turns the electron into a pancake. Not only the soliton picture provides us with understanding of the process, the discussion gives us a picture of how the bosonization in two dimensions is lifted up to four dimensions.

**Acknowledgements**

RK would like to thank Take Okui, Keisuke Harigaya, and John Terning for collaborations and useful discussions in the early stage of this work. The work is supported by JSPS KAKENHI Grant No. 19H00689 (RK and RM), and MEXT KAKENHI Grant No. 18H05542 (RK).
References

[1] G. ’t Hooft, Magnetic monopoles in unified gauge theories, *Nucl. Phys. B* 79 (1974) 276.

[2] A.M. Polyakov, Particle Spectrum in the Quantum Field Theory, *JETP Lett.* 20 (1974) 194.

[3] V. Rubakov, Adler-Bell-Jackiw anomaly and fermion-number breaking in the presence of a magnetic monopole, *Nucl. Phys. B* 203 (1982) 311.

[4] C.G. Callan, Dyon-fermion dynamics, *Phys. Rev. D* 26 (1982) 2058.

[5] C.G. Callan, Monopole catalysis of baryon decay, *Nucl. Phys. B* 212 (1983) 391.

[6] Y. Kazama, C.N. Yang and A.S. Goldhaber, Scattering of a Dirac Particle with Charge Ze by a Fixed Magnetic Monopole, *Phys. Rev. D* 15 (1977) 2287.

[7] R. Jackiw and C. Rebbi, Solitons with fermion number 1/2, *Phys. Rev. D* 13 (1976) 3398.

[8] S. Dawson and A.N. Schellekens, Monopole-fermion interactions: The soliton picture, *Phys. Rev. D* 28 (1983) 3125.

[9] C.G. Callan and E. Witten, Monopole catalysis of Skyrmion decay, *Nucl. Phys. B* 239 (1984) 161.

[10] C. Csaki, S. Hong, Y. Shirman, O. Telem, J. Terning and M. Waterbury, Scattering Amplitudes for Monopoles: Pairwise Little Group and Pairwise Helicity, arXiv:2009.14213.

[11] A. Sen, Conservation laws in the monopole-induced baryon-number-violating processes, *Phys. Rev. D* 28 (1983) 876.

[12] C.G. Callan, The monopole catalysis S-matrix, *AIP Conf. Proc.* 116 (1984) 45.

[13] J. Preskill, Magnetic Monopoles, *Ann. Rev. Nucl. Part. Sci.* 34 (1984) 461.

[14] V.A. Rubakov, Monopole catalysis of proton decay, *Rep. Prog. Phys.* 51 (1988) 189.

[15] Z. Komargodski, Baryons as Quantum Hall Droplets, arXiv:1812.09253.

[16] R. Kitano and R. Matsudo, Vector mesons on the wall, *JHEP* 03 (2021) 23 [arXiv:2011.14637].
[17] I.I. Kogan, *Kaluza-Klein and axion domain walls. Induced charge and mass transmutation*, *Phys. Lett. B* 299 (1993) 16.

[18] J.M. Maldacena and A.W. Ludwig, *Majorana fermions, exact mapping between quantum impurity fixed points with four bulk fermion species, and solution of the “unitarity puzzle”*, *Nucl. Phys. B* 506 (1997) 565 [cond-mat/9502109].

[19] S. Mandelstam, *Soliton operators for the quantized sine-Gordon equation*, *Phys. Rev. D* 11 (1975) 3026.

[20] S. Coleman, *Quantum sine-Gordon equation as the massive Thirring model*, *Phys. Rev. D* 11 (1975) 2088.

[21] N. Craigie, W. Nahm and V. Rubakov, *Towards a complete QFT treatment of monopole induced baryon number violating transitions*, *Nucl. Phys. B* 241 (1984) 274.

[22] Y. Kazama, *Condensates and the Boundary Condition in Monopole-Fermion Dynamics*, *Prog. Theor. Phys.* 70 (1983) 1166.

[23] C. Callan and J. Harvey, *Anomalies and fermion zero modes on strings and domain walls*, *Nucl. Phys. B* 250 (1985) 427.

[24] E. Witten, *Dyons of charge $e\theta/2\pi$*, *Phys. Lett. B* 86 (1979) 283.

[25] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, *Nucl. Phys. B* 326 (1989) 108.

[26] N. Strodthoff, B.-J. Schaefer and L. von Smekal, *Quark-meson-diquark model for two-color QCD*, *Phys. Rev. D* 85 (2012) 074007 [arXiv:1112.5401].

[27] J. Arafune and M. Fukugita, *A limit on the solar monopole abundance*, *Phys. Lett. B* 133 (1983) 380.

[28] J. Arafune, M. Fukugita and S. Yanagita, *Monopole abundance in the Solar System and the intrinsic heat in the Jovian planets*, *Phys. Rev. D* 32 (1985) 2586.

[29] T. Kajita, K. Arisaka, M. Koshiba, M. Nakahata, Y. Oyama, A. Suzuki et al., *Search for Nucleon Decays Catalyzed by Magnetic Monopoles*, *J. Phys. Soc. Jpn.* 54 (1985) 4065.

[30] K. Ueno, K. Abe, Y. Hayato, T. Iida, K. Iyogi, J. Kameda et al., *Search for GUT monopoles at Super–Kamiokande*, *Astropart. Phys.* 36 (2012) 131 [arXiv:1203.0940].
[31] S. Dimopoulos, J. Preskill and F. Wilczek, *Catalyzed nucleon decay in neutron stars*, Phys. Lett. B 119 (1982) 320.

[32] E.W. Kolb and M.S. Turner, *Limits from the soft X-ray background on the temperature of old neutron stars and on the flux of superheavy magnetic monopoles*, Astrophys. J. 286 (1984) 702.