We present a unified view of three different problems; rank reduction of matrices, matrix balancing, and mean-field approximation, using information geometry. Our key idea is to treat each matrix as a probability distribution represented by a log-linear model on a partially ordered set (poset), which enables us to formulate rank reduction and balancing of a matrix as projection onto a statistical submanifold, which corresponds to the set of low-rank matrices or that of balanced matrices. Moreover, the process of rank-1 reduction coincides with the mean-field approximation in the sense that the expectation parameters can be decomposed into products, where the mean-field equation holds. Our observation leads to a new convex optimization formulation of rank reduction, which applies to any nonnegative matrices, while the Nyström method, one of the most popular rank reduction methods, is applicable to only kernel positive semidefinite matrices. We empirically show that our rank reduction method achieves better approximation of matrices produced by real-world data compared to Nyström method.

1 Introduction

Rank reduction is a principal technique for matrices to efficiently store and treat them with limiting the loss of information. As increasing the amount of data to be handled, rank reductions are becoming more important and used in many applications including image processing (Friedland et al., 2011) and recommender systems (Lee et al., 2013). Since the computational complexity of the classic rank reduction method, singular value decomposition (SVD), is $O(n^3)$ for an $n \times n$ matrix, resulting in the difficulty of its application to large scale matrices, more efficient techniques using QR decomposition (Stewart, 1998), nonnegative matrix factorization (Lee and Seung, 1999), and the Nyström method (Williams and Seeger, 2001) have been developed. The Nyström method is widely used for symmetric positive semi-definite kernel matrices, as the accuracy of the approximation is theoretically guaranteed (Zhang et al. 2008) and its computation is efficient. Rank reduction is not only reducing the amount of information of matrices, but accelerating the subsequent analysis on matrices (Williams and Seeger 2001). For example, Altschuler et al. (2019) showed that matrix balancing, which is used to compute the entropy regularized optimal transport (Cuturi, 2013), can be efficiently performed on the rank reduced matrix by combining with the Nyström method.

To date, rank reduction has been widely studied as one of the main topics in linear algebra. However, despite its widespread use, it is still challenging to understand rank reduction as a learning problem and analyze its relationship to other statistical models. We challenge this problem using information geometry (Amari 2016), which enables us to treat different problems across linear algebra and machine learning in a differential geometric manner.
To geometrically analyze the problem of rank reduction on matrices, we use the log-linear model on a partially ordered set (poset) (Sugiyama et al., 2016). This model has been originally introduced to treat higher-order interactions between variables, and Sugiyama et al. (2017) showed the connection between this model and optimization on matrices. Interestingly, the problem of matrix balancing can be fully understood as projection in a statistical manifold if one models matrices by the log-linear model. Based on this connection between matrices and information geometry via the log-linear model, we address geometric understanding of rank reduction and matrix balancing, which also leads to a new rank reduction algorithm, called Legendre rank reduction.

Specifically, in this paper, we point out geometrical relationship between matrix balancing and rank-1 reduction. The matrix that achieves both conditions of matrix balancing and rank-1 reduction is realized as the unique cross point between two planes of canonical and expectation parameters.

Moreover, we extend our geometric analysis to the mean-field approximation, which has been originally introduced to discuss the phase transitions in ferromagnets (Weiss, 1907). The technique of mean-field approximation made an impact across many areas despite the simple way of reducing the many-body problem to the unity problem. This method has been widely used in not only physics but also statistics (Peterson, 1987), information theory (Bhattacharyya and Kerdn, 2000), neural networks (Mei et al., 2019) and even in game theory (Caines et al., 2006; Lions and Lasry, 2007). In the context of machine learning, the mean-field equation derived as a result of the mean-field approximation can largely simplify the computation of expected values of Boltzmann machines (Ackley et al., 1985) avoiding the combinatorial explosion of the computational cost (Anderson and Peterson, 1987). Tanaka (1999) has discussed the mean-field theory in terms of information geometry and pointed out that the problem of computing the expected values from the model parameters can be solved as a projection onto a special submanifold where the transformation of dual parameters (θ, ψ) is possible in constant time. By capturing its information geometric structure, we found unexpected theoretical relationship between rank reduction and mean-field approximation. In particular, we show that rank-1 reduction of matrices can be interpreted as mean-field approximation from the viewpoints of (1) independence of distributions, and (2) independence of expected values. Our theoretical observation leads to mean-field equation on matrices, which also point out the analogy with the typical mean-field approximation.

2 Legendre Rank Reduction

In this section, we introduce a rank reduction algorithm of matrices, called Legendre rank reduction, based on information geometry. In the following, we assume that an input to the algorithm is a nonnegative matrix \( A \in \mathbb{R}_{\geq 0}^{(n+1)\times(n+1)} \), where an element index begins from 0 to simplify our discussion. We denote by \([n] = \{0, 1, \ldots, n\}\). The rank reduction problem is to find a matrix \( A' \in \mathbb{R}_{\geq 0}^{(n+1)\times(n+1)}\) with the rank \( r < n \) that approximates \( A \).

2.1 Reminders on Log-Linear Model on Poset

In this section, we remind the log-linear model on a poset (Sugiyama et al., 2017). This model is known to be a generalization of Boltzmann machines, where we can flexibly design interaction between variables using partial orders. A poset \((S, \leq)\) is a set of elements \( S \) associated with a partial order \( \leq \) on \( S \), where the relation “\( \leq \)” satisfies the following three properties: For all \( x, y, z \in S \), (1) \( x \leq x \), (2) \( x \leq y, y \leq x \Rightarrow x = y \), and (3) \( x \leq y, y \leq z \Rightarrow x \leq z \). We consider a discrete probability distribution \( p \) on a poset \((S, \leq)\), which is treated as a mapping \( p : S \rightarrow (0, 1) \) such that \( \sum_{x \in S} p(x) = 1 \). Each entry \( p(x) \) is assumed to be strictly larger than zero. We assume that the domain \( S \) has the least element \( \perp \); that is, \( \perp \leq x \) for all \( x \in S \). A distribution \( p \) on \((S, \leq)\) is treated by the log-linear model defined as \( \log p(x) = \sum_{s \leq x} \theta(s) \), where \( \theta(\perp) \) corresponds to the normalizing factor (partition function). The convex quantity defined as the sign inverse \( \psi(\theta) = -\theta(\perp) \) is called the Helmholtz free energy of \( p \) whose parameter is \( \theta \).

The canonical parameter \( \theta \) of the log-linear model uniquely identifies the distribution \( p(x) \). Using \( \theta \) as a coordinate system in the set of distributions, which is a typical approach in information geometry (Amari, 2016), we can draw the following geometric picture: each point in the \( \theta \)-coordinate system corresponds to a distribution, and the resulting space is called the canonical space. Moreover, because the log-linear model belongs to the exponential family, we can also identify a distribution by...
We do not consider bingos on the diagonal direction as they do not have direct connection to matrix ranks. We prove the relationship between vertical and horizontal bingos and ranks of matrices.

Theorem 1 (Bingo rule and matrix rank)

Theorem 1. Let \( \mathbf{A} \) be a nonnegative matrix, and let \( \theta \) be its canonical parameter representation. We define the bingo rule as follows:

1. A vertical bingo occurs if \( \theta_i = 1 \) for some \( i \).
2. A horizontal bingo occurs if \( \theta_j = 1 \) for some \( j \).

Let \( \eta \) be the expectation parameter defined as \( \eta(x) = \sum_{s \geq x} p(s) \). Each expectation parameter \( \eta(x) \) is literally consistent with the expected value \( \mathbb{E}[F_x(s)] \) for the function \( F_x(s) \) such that \( F_x(s) = 1 \) if \( x \leq s \) and 0 otherwise (Sugiyama et al., 2016). Thus we can also identify each point using the \( \eta \)-coordinate system in the expectation space.

In addition, the \( \theta \)-coordinate and the \( \eta \)-coordinate are orthogonal with each other, which guarantees that we can combine these coordinates together as a mixture coordinate and a point specified by the mixture coordinate also identifies a distribution uniquely (Amari, 2016). As is clear from the definition, \( \eta(\perp) = 1 \) always holds. In the following, we write a distribution as \( p(\theta) \) to emphasize that it is determined by the canonical parameter \( \theta \). Similarly, we write \( p(\eta) \) if \( p \) is determined by the expectation parameter \( \eta \).

2.2 Bingo Rule

To treat matrices by the log-linear model, we follow the approach proposed by Sugiyama et al. (2017). Let us fix the domain \( S = [n]^2 \), which is the set of indices of \( (n+1) \times (n+1) \) matrices. We introduce a partial order as \( (i, j) \leq (k, l) \) if \( i \leq k \) and \( j \leq l \) for all \( (i, j), (k, l) \in S \), which allows us to make a one-to-one mapping between a nonnegative matrix \( \mathbf{A} \) and its corresponding distribution \( p \) as \( p((i, j)) = a_{ij}/\sum_{i,j} a_{ij} \). We use the notation as \( p_{ij} = p((i, j)), \eta_{ij} = \eta((i, j)) \) to simplify mathematical descriptions. The log-linear model for matrices is given as

\[
\log p_{ij} = \sum_{i' \leq i, j' \leq j} \theta_{i'j'}, \quad \text{leading to: } \theta_{ij} = \log p_{ij} - \log p_{i-1,j} - \log p_{i,j-1} + \log p_{i-1,j-1}. \tag{1}
\]

The value \( \theta_{00} = \log p_{00} = \log a_{00} \) works as a normalizing factor of the model. It is known that this particular formulation can treat the problem of matrix balancing as optimization on the log-linear model (Sugiyama et al., 2017), and it provides statistical foundation to investigate the dynamics of matrices and develop efficient algorithms from a geometric viewpoint. Note that, although the log-linear formulation cannot treat zero probability, this problem can be solved by removing the corresponding element from the domain \( S \) (Sugiyama et al., 2017).

By modeling matrices by the log-linear model on a poset, we can treat conditions on matrices through the pair of parameters \( (\theta, \eta) \). Using this property, we introduce our key idea: the condition of low rank can be given as constraints on \( \theta \), which we call the bingo rule.

Definition 1 (Bingo rule). Given a nonnegative matrix \( \mathbf{A} \in \mathbb{R}_{\geq 0}^{(n+1) \times (n+1)} \) and its canonical parameter representation \( \theta \) given in Equation (1). For \( j \in \{1, 2, \cdots, n\} \), we say that \( j \)-th column has a vertical bingo if \( \theta_j = 1 \), and \( j \)-th row has a horizontal bingo if \( \theta_{1j} = \theta_{2j} = \cdots = \theta_{nj} = 0 \).

We do not consider bingos on the diagonal direction as they do not have direct connection to matrix ranks. In addition, the bingo on 0-th column or 0-th row does not have any effect to the matrix rank. We prove the relationship between vertical and horizontal bingos and ranks of matrices.

Theorem 1 (Bingo rule and matrix rank). For a nonnegative matrix \( \mathbf{A} \in \mathbb{R}_{\geq 0}^{(n+1) \times (n+1)} \), when \( \mathbf{A} \) has \( v \) vertical bingos and \( w \) horizontal bingos, \( \text{rank}(\mathbf{A}) = n + 1 - \max(v, w) \).
Proof. We show that a bingo on \( j \)-th column means that \( j \)-th column of a given matrix is multiplied by a constant of \((j-1)\)-th column for \( j \in \{1, 2, \ldots, n\} \). By definition of \( p \),

\[
\frac{p_{i,j}}{p_{i,j-1}} = \frac{\exp\left(\sum_{i' \leq i} \sum_{j' \leq j} \theta_{i',j'}\right)}{\exp\left(\sum_{i' \leq i} \sum_{j' \leq j-1} \theta_{i',j'}\right)} = \exp\left(\sum_{i' \leq i} \theta_{i',j}\right).
\]

(2)

If \( j \)-th column has a vertical bingo, the right-hand side of (2) becomes \( \exp(\sum_{i' \leq i} \theta_{i',j}) = \exp(\theta_{0j}) \), which is a constant and does not depend on \( i \), that is, \( j \)-th column of a given matrix is multiplied by a constant \( \exp(\theta_{0j}) \) of \((j-1)\)-th column. In the same way, for \( i \in \{1, 2, \ldots, n\} \), you can check that horizontal bingo on \( i \)-th row means that \( i \)-th row of given matrix is multiplied by a constant \( \exp(\theta_{ij}) \) of \((i-1)\)-th row. Recalling the definition of the matrix rank, you can see \( \text{rank}(A) = n + 1 - \max(w, v) \) with \( v \) vertical bingos and \( w \) horizontal bingos. □

We illustrate examples of bingos and corresponding ranks in Figure 1.

Although there are many possibilities to create rank \( r \) matrices using the bingo rule, it is in general hard to find the optimal bingo columns or rows as it costs \( \Theta(2^r) \). To avoid this issue, in this paper we always fix the bingo columns from \( r \)-th to \( n \)-th columns; that is,

\[
\theta_{1j} = \theta_{2j} = \cdots = \theta_{nj} = 0 \quad \text{if} \quad j \geq r.
\]

(3)

2.2.1 Rank Reduction as \( e \)-projection

To achieve rank reduction in the submanifold of canonical parameters \( \theta \), first we prepare the set of distributions that corresponds to the set of rank-\( r \) matrices using the Bingo rule. From Theorem [1] and the condition in Equation (3), let us define the model submanifold \( \mathcal{P}_r = \{ p(\theta) \mid \theta \text{ satisfies Equation (3)} \} \), which is defined by the linear constrained on \( \theta \) and therefore is convex with respect to \( \theta \). Due to the simplicity of our Bingo rule, we can immediately construct the model submanifold, where every distribution represents a rank-\( r \) matrix. In contrast, the \( \eta \)-coordinate allows us to formulate another submanifold from a given matrix \( A \). Let \( \eta^A \) be the \( \eta \) representation of \( A \). The data submanifold is defined as \( \mathcal{P}_A = \{ p(\eta) \mid \eta_{ij} = \eta^A_{ij} \text{ if } i = 0, j = 0, \text{ or } j < r \} \). We can therefore achieve rank reduction of a given matrix as a projection of some initial distribution in the model submanifold \( \mathcal{P}_r \) onto the data submanifold \( \mathcal{P}_A \). This operation is known as \( e \)-projection in information geometry, which is convex optimization and ensures that the global optimum is always closest the input matrix in the sense of KL divergence.

More precisely, let \( p(t=0) \) be the initial distribution in \( \mathcal{P}_r \), for example, the uniform distribution such that \( p_{ij}(t=0) = 1/(n+1)^2 \). First we calculate the expectation parameter \( \eta^A \) of a given matrix \( A \). Legendre rank reduction performs \( e \)-projection that reduces the rank of \( A \) to \( r \) by the standard gradient descent approach. The update rule in each iteration is given in the following:

\[
\begin{align*}
\theta_{ij} &= \theta_{ij} - \epsilon(\eta_{ij} - \eta^A_{ij}), \quad \theta_{i0} = \theta_{i0} - \epsilon(\eta_{i0} - \eta^A_{i0}), \quad \theta_{0i} = \theta_{0i} - \epsilon(\eta_{0i} - \eta^A_{0i}),
\end{align*}
\]

(4)

for every \( (i, j) \in \{1, \cdots, n\} \times \{1, \cdots, r\} \), where \( \epsilon \) is a learning rate, \( \eta^A_{ij} = \sum_{i' \geq i} \sum_{j' \geq j} p^A_{i'j'} \), and \( p^A \) is the normalized distribution obtained from \( A \) such that \( p^A_{ij} = a_{ij} / \sum a_{ij} \). During these iterations, \( \theta_{ij} \) keeps to be 0 for all \( (i, j) \in \{1, \cdots, n\} \times \{r, \cdots, n\} \), which means that this algorithm searches the matrix that minimizes the KL divergence in the model submanifold \( \mathcal{P}_r \).

Let us summarize the above discussion to assemble our Legendre rank reduction algorithm. In this algorithm, instead of directly updating a matrix, we update two parameters \( \theta \) and \( \eta \) defined as

\[
\log p_{ij} = \sum_{i' \leq i} \sum_{j' \leq j} \theta_{i',j'} \quad \eta_{ij} = \sum_{i' \geq i} \sum_{j' \geq j} p_{i'j'} \quad \text{for each } i, j \in [n].
\]

(5)

We update \( \theta \) by gradient decent at each iteration \( t = 1, 2, \cdots \) using Equation (4) while fixing \( \theta_{ij} \) for \( (i, j) \in \{1, \cdots, n\} \times \{r, \cdots, n\} \). After updating from \( \theta_{ij}(t) \) to \( \theta_{ij}(t+1) \) at the \( t \)-th iteration, we compute \( \eta_{ij}(t+1) \) and \( p_{ij}(t+1) \) by Equation (5). Since this update does not ensure \( \sum p_{ij}(t+1) = 1 \), we again normalize it as \( p_{ij}(t+1) = p_{ij}(t+1) / \sum p_{ij}(t+1) \) and recompute \( \theta_{ij}(t+1) \) and \( \eta_{ij}(t+1) \).
By iterating the above update process until convergence, \( A' = (a'_{ij}) \) with \( a'_{ij} = C' p'_{ij} \) becomes a rank-1 matrix which approximates an input matrix \( A \), where \( C = \sum_{ij} a_{ij} \). From the convexity of \( P_r \) and \( P_A \), it is theoretically supported that we can always find the global optimal solution by the gradient decent method.

**Analysis of Computational Complexity.** In each iteration of gradient descent, we need to update two quantities, (1) \( p_{ij} = \exp(\theta_{ij} + p_{i,j-1} + p_{i-1,j} - p_{i-1,j-1}) \) and (2) \( \eta_{ij} = p_{ij} + \eta_{i,j+1} + \eta_{i+1,j} - \eta_{i+1,j+1} \) for every entry \((i, j) \in [n]^2\). These equations are equivalent to \( \xi \). Hence, in the general case, the time complexity is \( O(n^2) \) in each iteration.

Interestingly, by fixing the bingo rule as given in Equation (3), we can reduce the time complexity from \( O(n^2) \) to \( O(nr) \) as shown in the following. Let us consider computation of \( p \) from \( \theta \). First we calculate \( p_{ij} \) for each \((i, j) \in [n] \times [r-1]\), which takes \( O(nr) \). Then \( p_{i,j+1} \) for each column \( j \in \{ r, \cdots, n \} \) can be calculated in \( O(1) \) by \( \exp(\theta_{ij})p_{ij} \). Thereby the time complexity to compute \( p \) from \( \theta \) is \( O(nr) \). Next we consider computation of \( \eta \) from \( p \), where we need three steps to avoid unnecessary calculations: (i) We calculate the sum of columns \( g_j = \sum_{i=0}^n p_{ij} \) for \( j \in [r-1] \) and \( g_x = g_{j-1} \exp(\theta_{0j}) \) for \( j \in \{ r, \cdots, n \} \), resulting in \( O(nr) \) computation. (ii) Similarly, the sum of rows \( h_i = \sum_{j=0}^n p_{ij} \) is obtained as \( h_0 = \sum_{j=r-1}^n p_{0j} \) and \( h_i = h_{i-1} \exp(\sum_{j=0}^{i-1} \theta_{ij}) \) for \( i \in \{ 1, \cdots, n \} \), which also takes \( O(nr) \). (iii) Finally, we get \( \eta_{ij} = p_{ij} + \eta_{i,j+1} + \eta_{i+1,j} - \eta_{i+1,j+1} \) for \((i, j) \in [r-1] \times \{ 1, \cdots, n \} \) in \( O(nr) \). Thus all the necessary \( \eta \) to apply Equation (4) for update are computed in \( O(nr) \).

**Rank-1 Reduction and Maximum Singular Value.** In the case of rank-1 reduction, the possible bingo columns and rows are unique; that is, \( \theta_{ij} = 0 \) if \( i > 0 \) and \( j > 0 \) (see the rightmost matrix in Figure[1]). Because gradient decent can find the global minimum, the approximation archives theoretical limit and the output \( A' \) of our algorithm satisfies \( \| A - A' \|_F = \min_{A'\text{ rank}(A')=1} \| A - A'' \|_F \), which is a consequence of Eckart-Young theorem (Eckart and Young[1936]). In the case of rank-1 reduction, we can say that the algorithm finds the maximum singular value \( \sigma_{\max} \) of an input matrix \( A \) because, after convergence, the square root of the difference between the Frobenius norm of \( A \) and that of \( A' \) coincides with \( \sigma_{\max} \); that is,

\[
\sigma_{\max} = \sqrt{\| A \|_F^2 - \| A - A' \|_F^2}.
\]

You can directly derive the above relationship by Eckart-Young theorem and using the fact that \( \| A \|_F \) equals to the sum of singular values of \( A \).

**3 Rank Reduction, Matrix Balancing, and Mean-Field Approximation**

Here we discuss the relationship between rank reduction, matrix balancing, and mean-field approximation from information geometric perspective.

**3.1 Rank-1 Reduction and Mean-Field Theory**

In this section, we point out that rank-1 reduction for matrices can be interpreted as *mean-field approximation*, and we also derive the mean-field equation which seeks expected values of a distribution that approximates a given distribution. Mean-field approximation approximates a given probability distribution with independent distributions. In the typical application of Boltzmann machines defined as \( p(x) = \exp(\sum_i b_i x_i + \sum_{ij} w_{ij} x_ix_j) \) for bias \( b = (b)_i \in \mathbb{R}^n \), interaction parameters \( W = (w_{ij}) \in \mathbb{R}^{n \times n} \) and binary random variable vector \( x \in \{ 0, 1 \}^n \), the mean-field equation is given as \( \eta_i = \sigma(b_i + \sum_j w_{ij} \eta_j) \), which can efficiently approximate the expected value \( E_p[x_i] \) of the distribution \( p \) with avoiding exponential computational cost \( O(2^n) \) (Anderson and Peterson[1987]).

We show that we can derive an analogous self-consistent equation in rank-1 reduction to support our claim that rank-1 reduction can be captured as mean-field approximation.

Theoretical properties of mean-field approximation have been analyzed in information geometry, where it can be understood as a projection onto a submanifold described by the pair \((\theta, \eta)\) of canonical and expectation parameters. This submanifold has a special property that \( \eta \) can be easily computed from \( \theta \) as discussed in typical Boltzmann machines (Tanaka[1999]). We point out the analogy that rank-1 reduction also can be captured as projection onto a submanifold in which we can obviously know expectation parameters \( \eta \) from canonical parameters \( \theta \).
Let us consider the rank 1 submanifold $\mathcal{P}_1$. From its definition, the canonical parameter $\bar{\eta}$ of a distribution $\bar{p} = p(\bar{\eta}) \in \mathcal{P}_1$ always satisfies the condition $\bar{\eta}_{ij} = 0$ if $i \neq 0$ and $j \neq 0$. Therefore each probability can be directly expressed as $p(\bar{\eta}) = \exp(\sum_{k=1}^{n} \bar{\eta}_{k0}) \exp(\sum_{j=1}^{n} \bar{\eta}_{0j})$, which leads to the following two theorems. We show that we can decompose any distribution $\bar{p} \in \mathcal{P}_1$ using these two independent normalized distributions.

\[
p(\bar{\eta}_{01}, \ldots, \bar{\eta}_{0n})_{ij} = \begin{cases} 
\frac{\sum_{k=1}^{n} \exp(\bar{\eta}_{0k})}{1 + \sum_{k=1}^{n} \exp(\sum_{m=1}^{n} \bar{\eta}_{m0})} & \text{if } i = 0, \\
0 & \text{otherwise}
\end{cases}
\]  

(7)

\[
p(\bar{\eta}_{10}, \ldots, \bar{\eta}_{n0})_{ij} = \begin{cases} 
\frac{\sum_{k=1}^{n} \exp(\bar{\eta}_{k0})}{1 + \sum_{k=1}^{n} \exp(\sum_{m=1}^{n} \bar{\eta}_{m0})} & \text{if } j = 0, \\
0 & \text{otherwise},
\end{cases}
\]  

(8)

where $p(\bar{\eta}_{01}, \ldots, \bar{\eta}_{0n})$ and $p(\bar{\eta}_{10}, \ldots, \bar{\eta}_{n0})$ denote distributions depending only on parameters $\bar{\eta}_{01}, \ldots, \bar{\eta}_{0n}$ and $\bar{\eta}_{10}, \ldots, \bar{\eta}_{n0}$, respectively.

**Theorem 2.** Any distribution $\bar{p} = p(\bar{\eta}) \in \mathcal{P}_1$ corresponding to a rank-1 matrix can be decomposed as $p(\bar{\eta}) = p(\bar{\eta}_{01}, \ldots, \bar{\eta}_{0n})p(\bar{\eta}_{10}, \ldots, \bar{\eta}_{n0})$.

Our result means that, if a matrix is reduced to rank 1, then it can be always decomposed into the product of independent distributions. Hence rank-1 reduction can be viewed as mean-field approximation. In addition to that, we can immediately check that the obtained matrix corresponding to the distribution $\bar{p} \in \mathcal{P}_1$ is LU decomposed. We can also decompose any expected value $\bar{\eta}$ in $\mathcal{P}_1$.

**Theorem 3.** For every distribution $\bar{p} \in \mathcal{P}_1$, its expectation parameter $\bar{\eta}$ can be decomposed as $\bar{\eta}_{ij} = \bar{\eta}_{i0} \bar{\eta}_{j0}$ (Proof is in supplementary material).

Theorem 3 tells us that the expectation parameter $\bar{\eta}$ of a probability distribution $p$ becomes decomposable if it is projected onto $\mathcal{P}_1$. In the typical Boltzmann machine, mean-field approximation simplifies the calculation of expected values in its learning by decomposing them using parameter independence (Anderson and Peterson, [1987]; Kappen and Ortiz, [1998]). This is why Theorem 3 derives the connection to mean-field approximation.

Moreover, our information geometric analysis of matrices enables us to derive the mean-field equation to know the parameters of target distribution $p(\bar{\theta}) \in \mathcal{P}_1$, which is the destination of projection from a distribution $p$ which corresponds to an input matrix with any rank.

**Theorem 4.** Let $p(\bar{\theta})$ be the distribution obtained by $e$-projection of $p(\theta)$ onto $\mathcal{P}_1$. The expected values $\bar{\eta}$ of $p(\bar{\theta})$ satisfies the following self consistent equations:

\[
\bar{\eta}_{m,0} = \eta_{m+1,0} \sigma \left( - \sum_{j=0}^{n} \theta_{m,j} \eta_{0,j} \right) + \bar{\eta}_{m-1,0} \sigma \left( \sum_{j=0}^{n} \theta_{m,j} \eta_{0,j} \right),
\]

\[
\bar{\eta}_{0,m} = \eta_{0,m+1} \sigma \left( - \sum_{j=0}^{n} \theta_{j,m} \eta_{j,0} \right) + \bar{\eta}_{0,m-1} \sigma \left( \sum_{j=0}^{n} \theta_{j,m} \eta_{j,0} \right),
\]

where $\sigma$ is the sigmoid function and $m \in [n]$ (Proof is in supplementary material).

We derive the following two formulae in the middle of the proof of Theorem 4:

\[
\bar{\eta}_{m,0} = \frac{\sum_{k=m}^{n} \exp(\sum_{i=1}^{k} \bar{\eta}_{i,0})}{1 + \sum_{k=1}^{n} \exp(\sum_{i=1}^{k} \bar{\eta}_{i,0})}, \quad \bar{\eta}_{0,m} = \frac{\sum_{k=m}^{n} \exp(\sum_{j=1}^{k} \bar{\eta}_{0,j})}{1 + \sum_{k=1}^{n} \exp(\sum_{j=1}^{k} \bar{\eta}_{0,j})},
\]

(9)

We can evaluate the expectation parameters $\bar{\eta}$ from parameters $\theta$ by Equations (9). This problem has been introduced in [Tanaka, 1999, Problem 1], which supports the fact that rank-1 reduction can be treated as one of the mean-field approximation, which is defined as an $e$-projection to a submanifold in which the solution has the closed form.

### 3.2 Connecting Rank Reduction And Matrix Balancing

In this section, we explain the geometric relationship between rank-1 reduction and matrix balancing. We also give an analytical solution if $n = 2$ and illustrate geometric situation on 3D plots.
We theoretically pointed out that both algorithms of rank-r reduction and matrix balancing can be captured as projection onto special submanifold that combines distribution decomposability, expected values decomposability, and mean-field equation. That is, rank-1 reduction can be captured as projection onto special submanifold where the transform of dual parameters \( \theta, \eta \) is possible in constant time.

To get the intuition of geometric structure across conditions on rank-1 reduction, matrix balancing, and mean-field approximation, we illustrate a simple case of \( n = 1 \) as 3D plots in Figure 2. Let us

Figure 2: Balancing submanifold \( \mathcal{P}_B \) (blue) and rank-1 submanifold (orange) \( \mathcal{P}_1 \) in \( \theta \) (top) and \( \eta \) space (bottom) in case of \( n = 1 \) in each parameters (a) \( s = t = (0.4, 0.6) \) (b) \( s = t = (0.5, 0.5) \) (c) \( s = (0.8, 0.2), t = (0.3, 0.7) \).
We empirically evaluate the performance of Legendre rank reduction. To measure the quality of obtained low-rank matrices, we use the Frobenius norm and the KL (Kullback–Leibler) divergence from a given matrix to the approximated low rank matrix. KL divergence from a distribution \( p \) to \( q \) is defined as 
\[
D(p||q) = \sum_{ij=0}^{n} p_{ij} \log(p_{ij}/q_{ij}).
\]
As a baseline of rank reduction, we use Nyström method, one of the most popular rank reduction methods (Williams and Seeger, 2001). Both methods were implemented in Python with numpy library (Oliphant, 2006). Experiments were conducted on CentOS 6.10 with a single core of 2.2 GHz Intel Xeon CPU E7-8880 v4 and 3TB of memory. We use Nyström method implemented in scikit-learn (Pedregosa et al., 2011). We run our algorithm until \( \|q - q^A\|_2 \) is below the threshold 10^{-4}. Since the Nyström method can treat only positive semidefinite matrices, in our experiments we consistently use kernel matrices given as 
\[
A_{ij} = \exp(-\lambda \|x_i - x_j\|_2)
\]
as inputs to the methods, where \( \lambda \) is a hyper parameter, which is constructed from a d-dimensional real-valued dataset \( X = \{x_0, \cdots, x_n\} \in \mathbb{R}^{d \times (n+1)} \). We used the following synthetic and real-world kernel matrices: (a) randomly sampled \( X \) generated by the multidimensional uniform distribution. (b) the radial basis function kernel matrix generated from Stone Flakes Open Data Set (Ferreira et al., 2016). In the Stone Flakes Open Data Set, we removed five instances which include missing values. In experiments using the dataset (a), the dimensionality \( n \) of an input matrix is fixed to 50 and \( d = 3, \lambda = 5.0 \), and the learning rate \( \epsilon = 0.01 \). In experiments on the dataset (b), \( n = 75, \lambda = 0.01, \) and \( \epsilon = 0.01 \).

Results are shown in Figure 3. These results show that the approximation error of our method is stable in terms of both the KL divergence and the Frobenius norm. In the Nyström method, we have to chose partial matrix randomly in advance and the randomness makes the performance unstable especially when the ranks are small, while our convergence condition guarantees the performance in any rank in our algorithm.
5 Conclusion

In this paper, we have proposed a new rank reduction method for matrices. Our key idea is to realize the low rank condition on not matrices directly but the canonical parameter space of the log-linear model on posets, where each matrix is treated as a discrete probability distribution. Our theoretical contribution is that we have firstly shown the relationship between the matrix balancing algorithm, the rank reduction algorithm, and mean-field approximation using information geometry. Our geometric analysis uncovered a tight connection: rank-1 reduction coincides with mean-field approximation. Moreover, we have shown the effectiveness of our rank reduction method. Our work will become a basis of further investigation between linear algebraic matrix operations, statistics, and information theory via information geometry.

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A Proof of Theorem 3

Proof. To simplify the proof, we denote the decomposability of the distributions on $P_1$ as $p_{ij} = p_ip_j$. Using independence of probability and normalization factor $\sum_{0 \leq i'} \sum_{0 \leq j'} p_{i'j'} = 1$, it follows that

$$\eta_{ij} = \sum_{i \leq i'} \sum_{j \leq j'} p_{i'j'} = \left( \sum_{i \leq i'} p_{i'} \right) \left( \sum_{j \leq j'} p_{j'} \right) = \left( \sum_{i \leq i'} p_{i'} \right) \left( \sum_{j \leq j'} p_{j'} \right) \left( \sum_{j' \leq j} p_{j'} \right) = \left( \sum_{i \leq i'} p_{i'} \right) \left( \sum_{j \leq j'} p_{j'} \right) = \eta_{i0} \eta_{0j}. \Box$$

B Proof of Theorem 4

Proof. The potential function $\psi(\theta) = -\theta_00$ can be derived from normalization condition.

$$\psi(\theta) = \log \left( 1 + \sum_{k=1}^{n} \exp \left( \sum_{i=1}^{k} \theta_{i0} \right) \right) + \log \left( 1 + \sum_{k=1}^{n} \exp \left( \sum_{j=1}^{k} \theta_{0j} \right) \right)$$

(11)

By differentiating $\psi(\theta)$ by the canonical parameters $\theta$, we can get expectation parameters $\eta_{m,0}$ and $\eta_{0,m}$ for $m \in \{0, 1, \cdots, n\}$. See Theorem 2 in Sugiyama et al. [2017].

$$\eta_{m,0} = \frac{\partial \psi(\theta)}{\partial \theta_{m,0}} = \frac{\sum_{k=m}^{n} \exp \left( \sum_{i=1}^{k} \theta_{i0} \right)}{1 + \sum_{k=1}^{n} \exp \left( \sum_{i=1}^{k} \theta_{i0} \right)}$$

(12)

$$\eta_{0,m} = \frac{\partial \psi(\theta)}{\partial \theta_{0,m}} = \frac{\sum_{k=m}^{n} \exp \left( \sum_{j=1}^{k} \theta_{0j} \right)}{1 + \sum_{k=1}^{n} \exp \left( \sum_{j=1}^{k} \theta_{0j} \right)}$$

(13)

In inverse, we can solve equations (12) and (13) for $\theta$,.

$$\theta_{m,0} = \log \left( \frac{\eta_{m,0} - \eta_{m+1,0}}{\eta_{m-1,0} - \eta_{m,0}} \right), \quad \theta_{0,m} = \log \left( \frac{\eta_{0,m} - \eta_{0,m+1}}{\eta_{0,m-1} - \eta_{0,m}} \right).$$

(14)

We define $\eta_{0,n+1} = \eta_{n+1,0} = 0$. KL divergence between $p(\theta)$ and $p(\theta)$ is given as

$$D(p(\theta)\|p(\theta)) = \psi(p(\theta)) + \phi(p(\theta)) - \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} \eta_{ij}. $$

(15)

Here, $\phi(p(\theta))$ is a potential function in $\eta$ space of $P_1$, which is defined as Legendre transformation of $\psi(\theta)$. See Theorem 1 in Sugiyama et al. [2017]. We can get it by conservation law for the sum of dual potential functions (Amari [2016]).

$$\phi(p(\theta)) = \sum_{i=1}^{n} \left( \theta_{i0} \eta_{0i} + \theta_{0i} \eta_{i0} \right) - \psi(p(\theta))$$

(16)

$$\phi(p(\theta)) = \sum_{i=1}^{n} \log \left( \frac{\eta_{i0} - \eta_{i+1,0}}{\eta_{i-1,0} - \eta_{i0}} \right) \eta_{0i} \left( \frac{\eta_{0i} - \eta_{0i+1}}{\eta_{0i-1} - \eta_{0i}} \right) \eta_{i0} \right) + \log(1 - \eta_{10})(1 - \eta_{01})$$

(17)

Since Legendre rank-1 reduction minimize KL divergence (15),

$$\frac{\partial}{\partial \eta_{m,0}} D(p(\theta)\|p(\theta)) = \log \left( \frac{\eta_{m,0} - \eta_{m+1,0}}{\eta_{m-1,0} - \eta_{m,0}} \right) - \sum_{j=1}^{n} \theta_{m,j} \eta_{0,j} = 0$$

(18)

should be satisfied. We used

$$\frac{\partial \psi(p(\theta))}{\partial \eta_{m,0}} = 0.$$