COHERENCE OF DIRECT IMAGES OF THE DE RHAM COMPLEX

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Dedicated to the memory of Egbert Brieskorn (7.7.1937-19.7.2013)

Abstract. We show the coherence of the direct images of the relative De Rham complex relative to a flat holomorphic map with suitable boundary conditions.

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1. Introduction

In the present paper, we show the following theorem.

Main Theorem. Let \( \Phi : Z \to S \) be a flat holomorphic map between complex manifolds. Assume that there exists an open subset \( Z' \subset Z \) with smooth boundary satisfying i) \( Z' \) contains the critical set \( C_\Phi \) of \( \Phi \), ii) the closure \( \overline{Z}' \) in \( Z \) is proper over \( S \), and iii) \( Z' \) is a deformation retract of \( Z \) along the fibers of \( \Phi \). Then, the direct images \( R\Phi_* (\Omega^\bullet_{Z/S}, \Omega^\bullet_S) \) (including the higher terms) of the relative De Rham complex \( \Omega^\bullet_{Z/S} := \Omega^\bullet_Z / \Phi^{-1}(\Omega^\bullet_S) \wedge \Omega^\bullet_Z \) on \( Z \) over \( S \) are \( \mathcal{O}_S \)-coherent modules.

Main Theorem is well-known for any proper and/or projective morphism \( \Phi \) where we may choose \( Z' \) equal to \( Z \), since the \( E_1 \)-term of the spectral sequence defining the direct image, so called Hodge to De Rham spectral sequence \( \mathcal{E}_1 \), is already \( \mathcal{O}_S \)-coherent due to the proper mapping theorem of Grauert and/or Grothendieck and its differentials are \( \mathcal{O}_S \)-homomorphisms (see [12] [11]). Therefore, our main interest is the study of the case when \( \Phi \) is a non-proper morphism between open manifolds.

If the range \( S \) of \( \Phi \) is one-dimensional, i.e. \( \Phi \) is a function, and \( Z \) is a suitably small neighborhood of an isolated critical point of \( \Phi \), then Theorem was shown by Brieskorn [1]. Namely, then, \( \Phi \) is locally equivalent to a polynomial map, and then one proves the coherence by extending \( \Phi \) to a projective morphism and by applying Grothendieck’s coherence theorem for projective morphisms. Brieskorn’s result was generalized by the author to the complete intersection case in [15], where, in the proof, he did not use the algebraic method but used a complex analytic method developed by Forster and Knorr [6] whose motivation was to give a new proof of the Grauert proper mapping theorem [7]. Recently, jointly with Changzheng Li and Si Li, the author studied in [14] the morphisms \( \Phi \) which may not be defined locally in a neighborhood of an isolated critical point but may have multiple critical points as in Main Theorem. Then, \( \Phi \) may not be equivalent to a polynomial map and the algebraic method no longer seems to be applicable. However the analytic method in [15] can be generalized for this new setting, as will be presented in this paper.

The proof of Main Theorem is divided into the following 4 steps.

\footnote{This work was supported by World Premier International Research Center Initiative (WPI), MEXT, Japan, and partially by JSPS Grant-in-Aid for Scientific Research (A) No. 25247004.}
\footnote{We assume that a complex manifold is paracompact, Hausdorff and, hence, metrizable.}
Step 1. We show that the restriction along the inclusion of $Z'$ into $Z$ given in Main Theorem induces an isomorphism: $\mathbb{R} \Phi_*(\Omega^*_{Z/S}, d_{Z/S}) \simeq \mathbb{R} \Phi_*(\Omega^*_{Z'/S}, d_{Z'/S}).$

Step 2. We construct an atlas of $Z'(r)$ by relative charts in the sense of Forster-Knorr, which satisfy an additional complete intersection condition.

Step 3. We introduce the Koszul-De Rham algebra $\mathcal{K}_{D(r)} \times S^*/S^*(U)$ on each relative chart $D(r) \times S^*$, which gives an $\mathcal{O}_{D(r)} \times S^*$-free resolution of the complex $(\Omega^*_{Z'/S}, d_{Z'/S}).$

Step 4. Using the data from Steps 2 and 3, we calculate the Hodge to De Rham spectral sequence by Čech-cohomology, to which we apply Forster-Knorr’s result \[6\].

Remark 1. A flat map is an open map and $\Phi$ defines a family of Knorr, which satisfy an additional complete intersection condition.

Fact 4. We consider cohomologies of three kinds: 1. De Rham complex, 2. Koszul complex, and 3. Koszul-De Rham algebra $\mathcal{K}^*_{D(r)} \times S^*/S^*(U)$ at (4.4), (4.5) in Step 3. Both of them are elementarily constructed but are essential for our purpose to give analytic character of the subject.

Notation. We consider cohomologies of three kinds: 1. De Rham complex, 2. Koszul-De Rham algebra $\mathcal{K}^*_{D(r)} \times S^*/S^*(U)$ at (4.4), (4.5) in Step 3. Both of them are elementarily constructed but are essential for our purpose to give an analytic character of the subject.

Acknowledgment The author expresses his gratitude to Changzheng Li, Si Li, Alexander Voronov and Michael Kapranov for several interesting discussions and suggestions to improve the present paper. He expresses also his gratitude to Simeon Hellerman and Scott Carnahan for their careful reading of the manuscript.

2. Step 1: Hodge to De Rham spectral sequence

Recall that the direct image is given by the hypercohomology $\mathbb{R} \Phi_*(\Omega^*_{Z/S}, d_{Z/S})$ and is described by the limit of the following two spectral sequences:

\begin{equation}
\begin{aligned}
E_2^{p,q} &:= H^p(R^q \Phi_*(\Omega^*_{Z/S}, d_{Z/S})) \\
E_2^{p,q} &:= R^q \Phi_*(H^p(\Omega^*_{Z/S}, d_{Z/S})).
\end{aligned}
\end{equation}

The $E_1$ term $E_1^{p,q} = R^q \Phi_*(\Omega^*_{Z/S})$ of the first sequence is sometimes called Hodge to De Rham (or, Frölicher) spectral sequence for the De Rham cohomology of $\Phi$.

Let us consider the second spectral sequence $E_2^{p,q}$, which we shall denote also by $E_2^{p,q}(Z)$ in order to stress its dependence on the cohomology of the domain $Z$. We first remark that $\text{Supp}(H^p(\Omega^*_{Z/S}, d_{Z/S})) \subset C_\Phi$ for $p > 0$, since the Poincaré complex $(\Omega^*_{Z/S}, d_{Z/S})$ relative to $\Phi$ is exact outside the critical set of $\Phi$. On the other hand, we have $H^p(\Omega^*_{Z/S}, d_{Z/S}) \simeq \Phi^{-1} \mathcal{O}_S$ (since $n > 0$). That is, $H^p(\Omega^*_{Z/S}, d_{Z/S})$ is constant along fibers of $\Phi$. Therefore, we observe:

Fact 1. Let $Z'$ be an open subset of $Z$ satisfying 1. $C_\Phi \subset Z'$ and 2. $Z'$ is a deformation retract of $Z$ along fibers of $\Phi$. Then, the inclusion map $Z' \rightarrow Z$ induces bijection $E_2^{p,q}(Z) \simeq E_2^{p,q}(Z')$ and, hence, of the hypercohomology groups $\mathbb{R} \Phi_*(\Omega^*_{Z/S}, d_{Z/S}) \simeq \mathbb{R} \Phi_*(\Omega^*_{Z'/S}, d_{Z'/S})$ as $\mathcal{O}_S$-module.
3. Step 2: Atlas of complete intersection relative charts

In Step 4, we shall calculate the hypercohomology group \( \mathbb{R} \Phi_* (\Omega^\bullet_{Z(r)/S} \cdot dZ(r)/S) \) by using a generalization of Čech cohomology for relative charts of \( Z \) for \( \Phi \) (Forster-Knorr [6]), where \( Z(r) \) is a family of open subsets of \( Z \) parametrized by a real parameter \( r \) and \( S^\star \) is a Stein open subset of \( S \). Therefore, this section is devoted to a construction of an atlas of \( Z(r) \) of relative charts with suitable properties.

We start with the definition of the relative chart, and then introduce an additional condition, complete intersection, on them to adjust to our problem.

**Definition.** (6) A relative chart for a flat family \( \Phi \) is a closed embedding

\[
j : U \longrightarrow D(r) \times S_U
\]

where \( U \) is an open subset of \( Z \), \( S_U \) is an open subset of \( S \) with \( \Phi(U) \subset S_U \) and \( D(r) \) is a polycylinder of the radius \( r \) in some \( \mathbb{C}^m \) \((m \in \mathbb{Z}_{\geq 0})\) such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{j} & D(r) \times S_U \\
\Phi |_{U} & \searrow & \downarrow \text{pr}_{S_U} \\
& S_U & \\
\end{array}
\]

commutes. We sometimes call the embedding \( j \) a relative chart, for simplicity.

**Definition.** A relative chart is called a complete intersection if the \( j \)-image of \( U \) is a complete intersection subvariety in \( D(r) \times S_U \). That is, there exists a sequence \( f_1, \cdots, f_l \) of holomorphic functions on \( D(r) \times S_U \) with \( f = m - n \) such that \( j(U) = \{ f_1 = \cdots = f_l = 0 \} \).

**Lemma 3.1.** Let \( j_k : U_k \rightarrow D_k(r) \times S_k \) \((k \in K)\) be a finite system of relative charts. Suppose \( U_K := \cap_{k \in K} U_k \) is non-empty. Then the fiber product

\[
j_k : U_K \rightarrow D_K(r) \times S_K
\]

of the morphisms \( j_k \) \((k \in K)\) over \( S_K := \cap_{k \in K} S_k \), where we set \( D_K(r) := \prod_{k \in K} D_k(r) \), is a relative chart.

**Proof.** The morphism \( j_K \) is obviously a local embedding. We need to show that its image is closed. Suppose there is a sequence \( z_i \in U_K \) \((i = 1, 2, \cdots)\) such that the sequence \( j_K(z_i) \) converges to a point in \( D_K(r) \times S_K \). Then the projection sequence \( j_K(z_i) \) also converges in \( D_K \times S_K \), implying that the sequence \( z_i \) converges in \( U_K \) for all \( k \in K \). Then \( \lim_{i \rightarrow 1} \{ z_i \} \) belongs to \( \cap_{k \in K} U_k =: U_K \) (c.f. 6 Cor. 3.2).

**Definition.** We shall call \( j_K \) \((3.3)\) the intersection of relative charts \( j_k \) \((k \in K)\).

**Remark 1.** In general, \( j_K(U_K) \) has codimension equal to \( l_K := \sum_{k \in K} m_k - n = \sum_{k \in K} k + (\#K - 1)n \). Even if all \( j_k \) \((k \in K)\) are complete intersections, their intersection \( j_K \) may not necessarily be a complete intersection. Therefore, the following lemma needs some considerations.

**Lemma 3.2.** Let \( \Phi : Z \rightarrow S \) be any flat holomorphic map. Then there exists a function \( r : Z \rightarrow \mathbb{R}_{>0} \) and a relative chart \( j_z : U_z(r) \rightarrow D_z(r) \times S_z \) for all \( z \in Z \) and \( 0 < r < r(z) \) such that 1) \( j_z(z) \) is independent of \( r \) and 2) \( p_1 \circ j_z : U_z(r) \rightarrow D_z(r) \) is a bijection, mapping \( z \) to the center of the polycylinder of radius \( r \). Further more, any finite intersection of these relative charts is a complete intersection.

**Proof.** We first prepare a lemma of a quite general nature.

**Lemma 3.3.** Any complex manifold \( M \) of dimension \( N \) admits an atlas (= a collection of open charts covering \( M \)) such that, for any point of \( M \), the union of charts containing the point is holomorphically embeddable into an open set in \( \mathbb{C}^N \).

\[\text{A polycylinder of radius } r \text{ is a domain in } \mathbb{C}^m \text{ for some } m \in \mathbb{Z}_{\geq 0} \text{ of the form } \{ (z_1, \cdots, z_m) \in \mathbb{C}^m \mid |z_i - a_i| < r \ (i = 1, \cdots, m) \} \text{ where } (a_1, \cdots, a_m) \in \mathbb{C}^m \text{ is called the center of the cylinder.} \]
Proof. Let $d$ be a metric on $M$, and let $B(p, r) := \{ q \in M \mid d(p, q) < r \}$ be the ball neighborhood of a point $p$ of radius $r$ for any $p \in M$ and $r \in \mathbb{R}_{\geq 0}$. Define $R(p) := \sup \{ r \in \mathbb{R}_{\geq 0} \mid B(p, r) \text{ is holomorphically embeddable in a domain in } \mathbb{C}^N \}$ (actually, $R$ is a positive valued continuous function on $M$). For any fixed real number $b$ with $0 < b < 1/3$, we have the atlas $\{ (B(p, R(p)b), \varphi_p) \}_{p \in M}$, where $\varphi_p$ is a holomorphic embedding of $B(p, R(p)b)$ into $\mathbb{C}^N$, has the desired property.  

Proof: Suppose $p \in M$ belongs to the chart $B(q, R(q)b)$ centered at $q \in M$. That means $d(p, q) < R(q)b$ and then $B(p, R(q)(1-b) \subset B(q, R(q)(1-b+b'))$ where $b' := d(p, q)/R(q) < b'$ so that $1-b+b' < 1$. Hence, the ball $B(q, R(q)(1-b+b'))$ is embeddable in $\mathbb{C}^N$, and so is $B(p, R(q)(1-b))$. This implies $R(p) \geq R(q)(1-b)$. On the other hand, for any small $\varepsilon > 0$, $B(q, R(p)-d(p, q)-\varepsilon) \subset B(p, R(p)-\varepsilon)$ is embeddable in $\mathbb{C}^N$, one gets $R(q) \geq \lim_{\varepsilon \to 0} (R(p)-d(p, q)-\varepsilon) = R(p)-d(p, q)$ and, hence, $(1+b)R(q) > (1+b')R(q) = R(p)+d(p, q) \geq R(p)$. Note that the chart $B(q, R(q)b)$ is contained in the ball $B(p, R(q)2b)$ of radius $R(q)2b = R(q)(1-b) - R(q)(1-3b)$. Recalling $1-3b > 0$ and inequalities $R(q)(1-b) \leq R(p)$, $R(q) > R(p)/(1+b)$, the radius is less than $(R(p)1-R(p)(1-3b)/(1+b)) = R(p)(1-b)/(1+b)$ which is a constant $(< R(p))$ independent of the point $q$. That is, all charts containing $p$ are covered by the same ball $B(p, R(p)(1-(1-3b)/(1+b))$ which is embeddable in $\mathbb{C}^N$. \[\Box\]

We return to a proof of Lemma 3.2. Let $\{(B(z, R(z)b), \varphi_z) \}_{z \in Z}$ be the atlas of $Z$ described in Lemma 3.3 (with an additional assumption of Footnote 3). For any point $z \in Z$, let $S_z$ be a local coordinate neighborhood of $\Phi(z)$ in $S$. Then, one finds easily a positive real number $r(z)$ such that for any real $r$ with $0 < r < r(z)$, the polycylinder $D(r)$ of radius $r$ centered at $z$ is contained in the domain $\varphi_z^{-1}(B(z, R(z)b)) \subset \mathbb{C}^N$ and $\Phi(\varphi_z^{-1}(D(r))) \subset S_z$. Then, 

$$j_z : U_z(r) := \varphi_z^{-1}(D(r)) \rightarrow D(r) \times S_z \rightarrow ((\varphi_z\epsilon), \Phi(z))$$

gives a family (a parametrized by $r$) of relative chart centered at $z$. The codimension $l$ of the image $j_z(U_z(r))$ in $D(r) \times S_z$ is equal to $m-n = N-n = \dim_C S$. Actually, the image is determined by a system $\{ t_i - \Phi_i \circ \varphi_z^{-1} = 0 \}_{i=1}^{\dim_C S}$ of equations, where $(t_1, \cdots, t_\dim_C S)$ is a local coordinate system of $S_z$ and $\Phi_i$ is the $i$th coordinate component of the morphism $\Phi$. Thus $j_z$ is a complete intersection.

Let us show that, for any finite set $K$ of pairs $(z, r_z)$ of $z \in Z$ and $0 < r_z < r(z)$ such that $U_z := \cap_{(z, r_z) \in K} U_z(r_z)$ (and, hence, $S_z := \cap_{(z, r_z) \in K} S_z$) is non-empty, the intersection relative chart $j_K : U_K \rightarrow D_K(r) \times S_K$ is a complete intersection. Recall that $j_K$ is given by the fiber product morphism:

$$j_K : z' \in U_K \mapsto ((\varphi_z\epsilon), \Phi(z')) \in \bigcup_{(z, r_z) \in K} D_z(r_z) \times S_z,$$

where the codimension of $j_K(U_K)$ is equal to $l_K = \#K \cdot \dim_C S + (\#K - 1)n$.

On the other hand, the existence of a point $z_0 \in U_K$ implies the inclusion:

$$U_z(r_z) \subset \bigcup_{(z, r_z) \in K} B(z, R(z)b) \subset B(z, R(z))(1-\varepsilon)$$

for $z := (K - 3b)/(1+b)$ (Lemma 3.2). Let $z_1, \cdots, z_N$ be the coordinates of $\mathbb{C}^N$ where the ball $B(z_0, R(z_0))(1-\varepsilon)$ is embeddable by extending the domain of $\varphi_z$. We also denote by $\varphi_z^{-1}(z \in K)$ the composition map: $D_K \times S_K \rightarrow D_z \rightarrow U_z \subset Z$.

Then, the image $j_K(U_K)$ is determined by the following two types of equations:

1) System equations for identifying polycylinders $D_z(r_z)$ ($z \in K$) each other. That is, for each fixed $j$ with $1 \leq j \leq N$, all $z_j \circ \varphi_z^{-1}$ ($z \in K$) are equal to each other. There are $(\#K - 1)N = (\#K - 1)(n + \dim_C S)$ number of equations.

2) System equations for the graph of $\Phi$ on each polycylinder $D_z(r_z)$ ($z \in K$). That is, for each fixed $i$ with $1 \leq i \leq \dim_C S$, $t_i = \Phi_i \circ \varphi_z^{-1}$ for all $z \in K$. There are $\#K \cdot \dim_C S$ number of equations. However, after the identifications in 1), we do
not need all equations but only for one point $z \in K$: $t_i = \Phi_i \circ \varphi_z^{-1}$ ($1 \leq i \leq \dim_\mathbb{C} S$), that is, the number of necessary equation is equal to $\dim_\mathbb{C} S$.

Thus the total number of necessary equations is $(\#K - 1)(n + \dim_\mathbb{C} S) + \dim_\mathbb{C} S = \#K \cdot \dim_\mathbb{C} S_K + (\#K - 1)n = \dim_\mathbb{C} (D_K \times S_K) - \dim_\mathbb{C} U_K$, showing that the image $j_K(U_K)$ is a complete intersection subvariety of $D_K \times S_K$. $\square$

This completes a proof of Lemma 3.2.

Recall the domain $Z' \subset Z$ in Main Theorem in §1 Introduction. We assume that $\partial Z'$ in Z is smooth and transversal to all fibers $\Phi^{-1}(t)$ for all $t \in S$.

**Fact 2.** For any point $t$ of $S$, there exist its Stein open neighborhood $S^*$, a finite number of relative charts over $S^*$

\[
\text{(3.4)} \quad j_k : U_k \rightarrow D_k(1) \times S^*, \quad 0 \leq k \leq k^*
\]

and a real number $0 < r^* < 1$ with the properties: for all $r$ with $r^* < r \leq 1$, set

\[
\text{(3.5)} \quad U_k(r) := j_k^{-1}(D_k(r) \times S^*) \quad \text{and} \quad Z'(r) := \cup_{k=0}^{k^*} U_k(r).
\]

Then, we have the following.

1. One has the inclusions: $Z^*_S := \Phi^{-1}(S^*) \supset Z'(r) \supset Z^*_S := \Phi^{-1}(S^*) \cap Z'$.

2. $Z'(r)$ is retractable to $Z^*_S$ along fibers of $\Phi$.

3. For any $K \subset \{0, \cdots, k^*\}$, the relative chart $j_K$ is a complete intersection.

**Corollary.** For $r$ with $r^* \leq r \leq 1$, we have $\Omega_{S^*}$-isomorphism

\[
\mathbb{R} \Phi_k(\Omega^*_{Z^*_S/S^*}, d_{Z^*_S/S^*}) \cong \mathbb{R} \Phi_k(\Omega^*_{Z^*_S(r)/S}, d_{Z^*_S(r)/S}).
\]

**Proof.** For each point $z \in Z' \cap \Phi^{-1}(t)$, we consider a relative chart $j_z : U_z(r) \rightarrow D_z(r) \times S_z$ of Lemma 3.2. We consider two cases.

Case 1. $z \in Z'$: Choose any real $r$ such that $0 < r < r(z)$ and $U_z(r) \subset Z'$.

Case 2. $z \in \partial Z'$: Choose any real $r$ such that $0 < r < r(z)$ and $U_z(r)$ (as a manifold with corners) is transversal to $\Phi^{-1}(t)$ for all real $r'$ with $0 < r' \leq r$.

Since $Z' \cap \Phi^{-1}(t)$ is compact, we can find a finite number of relative charts $\tilde{j}_k : \tilde{U}_k \rightarrow D_k(r_k) \times \tilde{S}_k (0 \leq k \leq k^*)$ centered at points $\tilde{z}_0, \cdots, \tilde{z}_{k^*}$ on $Z' \cap \Phi^{-1}(t)$ so that the union $\cup_{k=0}^{k^*} U_k$ contains the compact closure $\bar{Z'} \cap \Phi^{-1}(t)$. Then, we can find a Stein open neighborhood $S^*$ of $t$ such that 1) its compact closure $S^*$ is contained in $\cap_{k=0}^{k^*} S_k$, 2) $Z' \cap \Phi^{-1}(S^*)$ is contained in $\cup_{k=0}^{k^*} U_k(r)$, and 3) all fibers $\Phi^{-1}(t')$ for $t' \in S^*$ and $U_k(r_k) (0 < r_k \leq r_k)$ for the chart $\tilde{j}_k$ whose central point $z_k$ is on the boundary $\partial Z'$. By a suitable rescaling of the coordinate system of charts, we may assume that all radiiues $r_k (0 \leq k \leq k^*)$ are equal to 1. Then, due the compactness of $S^*$, there exists a real number $r^*$ with $0 < r^* < 1$ such that $Z' \cap \Phi^{-1}(S^*)$ is contained in $\cup_{k=0}^{k^*} U_k(r_k)$ for all $r' \cap Z'(r)$ as in (3.5). Then, 1. is trivial by definition, 2. is not trivial, however, is more or less a routine work, for instance due to R. Thom, and 3. is true since the system of relative charts $\{j_k\}_{k=0}^{k^*}$ has already this property (Lemma 3.2).

Note that $U(r) := \{(U_k(r), \varphi_k)\}_{k=0}^{k^*}$ form an atlas (a coordinate covering) of $Z'(r)$. We shall call $U(r) := \{j_k(r) := j_k|_{U_k(r)}\}_{k=0}^{k^*}$ the lifting of the atlas $U(r)$.

4. **Step 3: Koszul-De Rham algebras**

In this section, we introduce Koszul-De Rham algebra $(\mathcal{K}^*_{W/S}(U), d_{DR}, \partial_K)$ associated with a relative chart $j : U \rightarrow W$ over $S$ [11], and apply it for the construction of an $\mathcal{O}_W$-free resolution of the relative De Rham complex $\mathcal{O}^*_{U/S}$.

More precisely, the Koszul-De Rham algebra $\mathcal{K}^*_{W/S}(U)$ is a sheaf on $W$ of graded $\mathcal{O}_{W/S}$-algebra, equipped with 1) the double complex structure: De Rham operator $d_{DR}$ and Koszul operator $\partial_K$, and 2) a natural epimorphism: $(\mathcal{K}^*_{W/S}(U), d_{DR}, \partial_K)$

\[\text{To be precise, one need to show that any point in } D_K \times S_K \text{ satisfying the relations 1) and 2) is in the image of } j_K. \text{ But this can be shown by a routine work so that we omit it.}\]
a relative chart. We assume further that the defining ideal \( \mathcal{I}_U \) of the image subvariety \( j(U) \) in \( W \) (i.e. \( \mathcal{I}_U := \ker(j_*j^*|_{\mathcal{O}_W}) \)) has a finite presentation. That is, there is an \( \mathcal{O}_W \)-free exact sequence

\[
\oplus \mathcal{O}_W^{i_0} \longrightarrow \oplus \mathcal{O}_W^{i_0} \longrightarrow \mathcal{I}_U \longrightarrow 0.
\]

Explicitly, let \( f_1, \ldots, f_{l_0} \in \Gamma(W, \mathcal{O}_W) \) be a system generators of \( \mathcal{I}_U \) (i.e. the image of the basis of \( \oplus \mathcal{O}_W^{i_0} \)) and let \( (g^1_{\xi_0}, \ldots, g^{l_0}_{\xi_0}) \in \Gamma(W, \mathcal{O}_W^{i_0}) \) be a generating system of relations \( g^1_{\xi_0} f_1 + \cdots + g^{l_0}_{\xi_0} f_{l_0} = 0 \) (i.e. the image of the basis of \( \oplus \mathcal{O}_W^{i_0} \)).

For \( p \in \mathbb{Z}_{\geq 0} \), there is a natural epimorphism \( \pi = j_*j^*|_{\mathcal{O}_W^{p}} \)

\[
(\Omega^p_W, d|_{W/S}) \longrightarrow j_*(\Omega^p_U, d|_{U/S}) \longrightarrow 0,
\]

between the Kähler differentials, whose kernel, depending only on \( \mathcal{I}_U \), is given by

\[
\sum_{i=1}^{l_0} f_i \cdot \Omega^p_W + \sum_{i=1}^{l_0} df_i \wedge \Omega^{p-1}_W.
\]

We want to construct \( \mathcal{O}_W \)-free resolution of this ideal in the dg-algebra \( (\Omega^*_{W/S}, d|_{W/S}) \) generated by \( f_i \) (1 \( \leq i \leq l_0 \)) and by \( df_i \) (1 \( \leq i \leq l_0 \)). This motivates the following introduction of the Koszul-De Rham-algebra \( (\Omega^*_{W/S}(U), d_{DR}, \partial_K) \).

**Definition.** We call the isomorphism class of the sheaf of algebras \( \mathcal{K}_{W/S}(U) \) on \( W \) equipped with \( \partial_K \), \( d_{DR} \) and with bi-degrees, which we describe below (tentatively depending on the presentation \( \mathbf{4.2} \), c.f. Lemma 4.1), the Koszul-De Rham-algebra associated with the relative chart \( \mathbf{4.1} \).

Consider a graded commutative algebra over the differential graded algebra \( \Omega^*_{W/S} \)

\[
\mathcal{K}_{W/S}(U) := \Omega^*_{W/S}[^1, \ldots, ^{i_0}, \eta_1, \ldots, \eta_{l_0}] / I
\]

generated by \( \xi_1, \ldots, \xi_{i_0}, \eta_1, \ldots, \eta_{l_0} \) where \( \xi_i \)'s are considered as odd variables and \( \eta_i \)'s are considered as even variables and \( I \) is the both sided ideal generated by

\[
g^1_{\xi_0_1} \xi_1 + \cdots + g^{l_0}_{\xi_0} \xi_{l_0} \quad (j = 1, \ldots, l_1)
\]

\[
g^1_{\eta_1} \eta_1 + \cdots + g^{l_0}_{\eta_1} \eta_{l_0} + dg^1_{\xi_1} \xi_1 + \cdots + dg^l_{\xi_l} \xi_l \quad (j = 1, \ldots, l_1).
\]

Here, "graded commutative" means that

i) \( \eta_i \)'s and even degree differential forms on \( W \) commute with all variables,

ii) \( \xi_i \)'s and odd degree differentials forms on \( W \) anti-commute with each other, i.e.

\[
\xi_i \xi_j + \xi_j \xi_i = 0 \quad \text{and} \quad \xi_i dh + dh \xi_i = 0 \quad \text{for} \quad 1 \leq i, j \leq l_0 \text{ and } h \in \mathcal{O}_W.
\]

Let us equip three structures on the algebra \( \mathcal{K}_{W/S}(U) \).

1. **Koszul structure:** We define a boundary operators \( \partial_K \) on \( \mathcal{K}_{W/S}(U) \) as an \( \Omega^*_{W/S} \)-endomorphisms of the algebra satisfying the relations

\[
\partial_K \xi_i = f_i \quad \text{and} \quad \partial_K \eta_i = df_i.
\]

They automatically satisfies the relation: \( \partial_K^2 = 0 \).

5. Usually, Koszul resolution is defined only for even elements \( f_i \)'s but here we construct a resolution for odd elements \( df_i \)'s together. The interpretation to regard it as the Koszul resolution for the odd elements \( df_i \)'s was pointed out by M. Kapranov, to whom the author is grateful.
Proof. The endomorphism is well defined on the free algebra generated by $\xi_i$'s and $\eta_i$'s before dividing by the ideal $I$. Then, one check directly that the endomorphism preserves the ideal $I$, and, hence, induces an action on the quotient. 

2. De Rham structure: We regard $K_{W/S}(U)$ as De Rham complex of the Grassmann algebra $O_W[\xi_1, \cdots, \xi_n]/(I \cap O_W[\xi_1, \cdots, \xi_n])$ where $I \cap O_W[\xi_1, \cdots, \xi_n]$ is the ideal generated by the first half of (4.5). Then the classical De Rham differential operator $d_{W/S}$ acting on $\Omega_{W/S}^*$ is extended to $K_{W/S}(U)$ as

$$d_{DR} = d_{W/S} + \sum_{j=1}^{l_0} \eta_j \partial \xi_i.$$

(One first defines the operator as an endomorphism of the free algebra before dividing by the ideal $I$, but then one check directly that the endomorphism preserves the ideal $I$). The second term switches odd variables $\xi_i$ to even variables $\eta_i$, and satisfying the condition $(d_{DR})^2 = 0$. The condition is equivalent to

$$d_{DR}(\xi_i) = \eta_i, \ d_{DR}(\eta_i) = 0 \quad \text{and} \quad d_{DR}(\omega_S) = 0.$$

De Rham differential and Koszul differentials are anti-commuting each other

$$\partial_K d_{DR} + d_{DR} \partial_K = 0$$

so that the pair $(d_{DR}, \partial_K)$ form a double complex structure on $K_{W/S}(U)$.

3. Bi-degree structure: According to the double complex structure on $K_{W/S}(U)$, we introduce De Rham degree and Koszul degree on $K_{W/S}(U)$. Namely, any element of the algebra before dividing by the ideal $I$ is a linear combination of the elements of the form $\omega \Xi E$ where $\omega \in \Omega_{W/S}^p$ ($p \in \mathbb{Z}_{\geq 0}$) and $\Xi, E$ are monomials in $\xi_1, \cdots, \xi_l$ and $\eta_1, \cdots, \eta_l$. Then we define

$$\begin{align*}
\text{deg}_{DR}(\omega \Xi E) &:= \text{the total degree as a differential form} = p + \text{deg}(E) \\
\text{deg}_K(\omega \Xi E) &:= \text{the total degree of the monomial } \Xi E = \text{deg}(\Xi) + \text{deg}(E)
\end{align*}$$

which are additive with respect to the product of the algebra. Then the bi-degree is induced on the quotient $K_{W/S}(U)$, since the ideal $I$ is bi-homogeneous.

Let $K_{W/S}^{p,s}(U)$ be the linear span of those elements $\omega \Xi E$ such that $\text{deg}_{DR}(\omega \Xi E) = p$ and $\text{deg}_K(\omega \Xi E) = s$ for $p, s \in \mathbb{Z}_{\geq 0}$. We put $K_{W/S}^{p,*}(U) := \bigoplus_{s=0}^{\infty} K_{W/S}^{p,s}(U)$ and $K_{W/S}(U) = K_{W/S}^{*,*}(U) := \bigoplus_{p=0}^{\infty} K_{W/S}^{p,*}(U)$. With respect to the bi-degree, Koszul operator and De Rham operator behaves:

$$\begin{align*}
\partial_K & : K_{W/S}^{p,s}(U) \to K_{W/S}^{p,s-1}(U) \quad \text{and} \quad d_{DR} : K_{W/S}^{p,s}(U) \to K_{W/S}^{p+1,s}(U).
\end{align*}$$

This completes the definition of Koszul-De Rham algebra. In the following, we describe four basic properties A), B), C) and D) of Koszul-De Rham algebras.

**A) Independence of the definition from the presentation (4.2).**

**Lemma 4.1.** The isomorphism class of the Koszul-De Rham-algebra $K_{W/S}(U)$ is independent of the choice of the presentation (4.2), but depends only on the ideal $I_U$ in $O_W$.

**Proof.** To show this is a type of routine work so that we give only a sketch of a proof. For instance, it is sufficient to show that the algebra is invariant (i.e. canonically isomorphic) under the following operations:

1) to change the basis of generators in the presentation (4.2),

2) to change the basis of relations in the presentation (4.2),

3) to add a redundant generator to the presentation (4.2),

4) to add a redundant relation to the presentation (4.2).

That this construction of De Rham structure is the universal construction of $dg$-structure on $K_{W/S}(U)$ extending that on $\Omega_{W/S}^*$ was pointed out by A. Voronov, to whom the author is grateful.
For instance, 1) can be verified as follows. Let \( f'_1, \ldots, f'_{l_0} \) be another generator system and let \( K_W/S(U)' := \Omega^{*}_{W/S}[^{k}x_1, \ldots, \xi'_1, \eta'_1, \ldots, \eta'_{l_0}]/I' \) be the associated Koszul-De Rham-algebra. This means that there exists an invertible matrix 
\( h = (h^k_i)_{i=1}^{l_0} \) with coefficients in \( \Gamma(W, \mathcal{O}_W) \) such that the generators are transformed as 
\( f_i = \sum_k h^k_i f'_k \) and the relations are transformed as \( g^j_k = \sum_i g^j_i h^k_i \). Then, define an algebra homomorphism \( h^* : K_W/S(U) \to K_W/S(U)' \) by extending the change of variables:

\[
(4.6) \quad \xi_i \mapsto \sum_k h^k_i \xi'_k \quad \text{and} \quad \eta_i \mapsto \sum_k h^k_i \eta'_k + \sum_k dh^k_i \xi'_k.
\]

Actually, this change preserves the parities of the variables and matches with the degree countings by \( \deg_{\text{DR}} \) and \( \deg_K \) in both hand sides. The fact \( h^*(I) \subset I' \) follows from the new relations \( g^j_k = \sum_i g^j_i h^k_i \). Hence, the correspondence extends to a graded algebra homomorphism. The commutativity of \( h^* \) with \( \partial_K \) follows from their definitions directly:
\[
\sum_k h^k_i \partial_K(\xi_i) = \partial_K(\sum_k h^k_i \xi'_k) = \partial_K h^* \xi_i \quad \text{and} \quad \sum_k h^k_i \partial_K(\eta_i) = \partial_K(\sum_k h^k_i \eta'_k + \sum_k dh^k_i \xi'_k) = \partial_K h^* \eta_i.
\]

The commutativity of \( h^* \) with \( d_{\text{DR}} \) can be shown similarly:
\[
\sum_k h^k_i d_{\text{DR}}(\xi_i) = \partial_K h^* \eta_i = d_{\text{DR}}(\sum_k h^k_i \eta'_k + \sum_k dh^k_i \xi'_k) = d_{\text{DR}} h^*(\eta_i).
\]

Since the matrix \( h \) is invertible, using the inverse matrix, one similarly constructs a morphism \( (h^{-1})^* : K_W/S(U)' \to K_W/S(U) \). That the morphisms \( h^* \) and \( (h^{-1})^* \) are inverse to each other follows immediately from their defining formula:
\[
(h^{-1})^* \circ h^*(\xi_i) = (h^{-1})^*(\sum_k h^k_i \xi'_k) = \sum_k h^k_i \sum_l (h^{-1})^l_k \xi_l = \xi_i \quad \text{and}
\]
\[
(h^{-1})^* \circ h^*(\eta_i) = (h^{-1})^*(\sum_k h^k_i \eta'_k + \sum_k dh^k_i \xi'_k) = \sum_k h^k_i (\sum_l (h^{-1})^l_k \eta_l + \sum_l d(h^{-1})^l_k \xi_l)
+ \sum_k dh^k_i \sum_l (h^{-1})^l_k \xi_l = \eta_i.
\]

Other cases 2), 3) and 4) can be also verified elementarily. \( \square \)

B) Functoriality.

Next, let us show the functoriality of Koszul-De Rham algebras. Actually, the functoriality shall play an essential role in Step 4 when we want to construct a lifting of the Čech coboundary operators to the relative charts.

**Lemma 4.2.** Let \( j \) and \( j' \) be two relative charts over \( S \) such that there are holomorphic maps \( u : U \to U' \) and \( w : W \to W' \) over \( S \) with the commutativity \( j' \circ u = w \circ j \). Then, there is a natural Koszul-De Rham-algebra homomorphism \( w^* : w^{-1}K_W/S(U) \to K_W/S(U) \) such that \( (w_1 \circ w_2)^* = w_2^* \circ w_1^* \).

**Proof.** By the condition, we have \( w(j(U)) \subset j(U') \). In other words, the pullback by \( w^* \) of the ideal \( I_U \) is contained in the ideal \( I_{U'} \). Let \( f_1, \ldots, f_{l_0} \) and \( f'_1, \ldots, f'_{l_0} \) be some generator systems of the ideals \( I_U \) and \( I_{U'} \), so that we have a relation \( f'_i \circ w = \sum_k w^k_i f_k \) for a matrix \( (w^k_i)_{i=1}^{l_0} \) with coefficients in \( \Gamma(W, \mathcal{O}_W) \). Then, since any relation \( \sum_k g^j_k f'_i \) on \( f'_i \)'s induces a relation \( \sum_k g^j_k w^k_i f_k = 0 \) on \( f_k \)'s, combining together with the standard pull-back morphism of differential forms \( w^* : w^{-1}\Omega^{*}_{W/S} \to \Omega^{*}_{W/S} \), we can define an algebra homomorphism \( w^* : w^{-1}K_W/S(U) \to K_W/S(U) \) by setting

\[
w^*(\xi'_i) = \sum_k w^k_i \xi_k \quad \text{and} \quad w^*(\eta'_i) = \sum_k w^k_i \eta_k + \sum_k dw^k_i \xi_k.
\]

The fact that the morphism \( w^* \) does not depend on the choices of the presentations of the ideals \( I_U \) and \( I_{U'} \) can be shown similarly as the proof of Lemma 4.1. \( \square \)

C) Comparison with the De Rham complex \( (\Omega^{*}_{U/S}, d_{U/S}) \).

Finally, we want to compare the dg-algebras \( (\Omega^{*}_{U/S}, d_{U/S}) \) and \( (\mathcal{K}^{*}_{W/S}(U), d_{\text{DR}}, \partial_K) \).

We first note the following two facts follow from the definitions of \( \deg_K \) and \( \partial_K \).
1) The $\deg_K = 0$ part $(K^{\bullet,0}_{W/S}(U), d_{DR})$ of the algebra coincides with $(\Omega^{\bullet}_{W/S}, d_{W/S})$.

2) The image $\partial_K(K^{\bullet,1}_{W/S}(U))$ in $K^{\bullet}_{W/S}(U)$ of $\deg_K = 1$ part of the algebra is equal to the ideal generated by $f_1, \ldots, f_l$ and $df_1, \ldots, df_l$ in $(\Omega^{\bullet}_{W/S}, d_{W/S})$.

As a consequence, for each $p \in \mathbb{Z}_{\geq 0}$, we obtain a complex (recall (1.3))
\[
(4.7) \quad \cdots \rightarrow K^{\bullet,3}_{W/S}(U) \xrightarrow{\partial_K} K^{\bullet,2}_{W/S}(U) \xrightarrow{\partial_K} K^{\bullet,1}_{W/S}(U) \xrightarrow{\partial_K} K^{\bullet,0}_{W/S}(U) \xrightarrow{\pi} \Omega^{\bullet}_{U/S} \rightarrow 0
\]
which is exact at the stage 0. The complex is left-bounded due to the following fact.

**Fact 3.** The set $\{(p, s) \in \mathbb{Z}^2 \mid K^{p,s}_{W/S}(U) \neq 0\}$ is contained in the strip
\[
(4.8) \quad \{(p, s) \in \mathbb{Z}^2 \mid -l_0 \leq p - s \leq \dim_W S\}
\]

**Proof.** That $K^{p,s}_{W/S}(U) \ni \sum \omega E \neq 0$ means $p - s = \deg(\omega) - \deg(\Xi)$ (recall a notation of the bidegree structure). Since $0 \leq \deg(\omega) \leq \dim W/S$ and $0 \leq \deg(\Xi) \leq l_0$, this value is bounded as in the formula. \hfill \Box

Since one has the commutativity $d_{U/S} \circ \pi = \pi \circ d_{DR}$, the morphism $\pi$ in (4.7) may be regarded as a morphism from the double complex $(K^{\bullet,\bullet}_{W/S}(U), d_{DR}, \partial_K)$ to the De Rham complex $(\Omega^{\bullet}_{U/S}, d_{U/S})$, inducing the morphism from the simplified complex to the De Rham complex
\[
(4.9) \quad (K^{\bullet}_{W/S}(U), d_{DR} + \partial_K) \xrightarrow{\pi} (\Omega^{\bullet}_{U/S}, d_{U/S}),
\]
where LHS is given by $K^{\bullet}_{W/S}(U) := \bigoplus_{p-s=0} K^{p,s}_{W/S}(U)$ is a bounded complex, again due to Fact 3. However each term of the simplified complex is an infinite sum due to the fact that $\eta_i$’s are even variables and the multiplication of any high power of them are non-vanishing and increases simultaneously the degrees $p$ and $s$. Nevertheless, such simple repetition of terms (in stable ari) seems harmless as we see in Step 4.

It is of interest to investigate whether, up to which degree, under which conditions, this morphism is quasi isomorphic? But, in order to answer to the question, it should require more careful studies of the singularities of the chart $U$, $W$ and $S$. At present early stage of the studies, we will restrict ourselves to rather specific, however still sufficiently general for our applications, cases as follows.

**D) Complete Intersection Case.**

As in Step 2, let us call a chart $j : U \rightarrow W$ is a **complete intersection** if $j(U)$ is a complete intersection subvariety of $W$. In the rest of the present paper, we restrict our attention only to that case. By choosing minimal number, say $l = l_0$, of global generators $f_1, \ldots, f_l$ of the defining ideal $\mathcal{I}_U$ of $j(U)$, the Koszul-De Rham-algebra \[14\] is defined without relations $\mathcal{I} = 0$. Fixing one such generator system, we have the following explicit description of the degree decomposition of the Koszul-De Rham algebra\[8\]
\[
(4.10) \quad K^{p,s}_{W/S}(U) = \bigoplus_{a=0}^{\infty} \Xi \text{ is a monomial in } \xi_i \text{'s of } \deg(\Xi) = a. \quad \mathcal{E} \text{ is a monomial in } \eta_i \text{'s of } \deg(\mathcal{E}) = s-a.
\]

Then, the Koszul derivation $\partial_K$ splits into a direct sum $\partial + \tilde{\partial}$, where each of the factors $\partial$ and $\tilde{\partial}$ is defined as $\Omega^{\bullet}_{W/S}$-linear endomorphisms\[8\] such that
\[
\partial \xi_i = f_i, \quad \partial \eta_i = 0 \quad \text{and} \quad \tilde{\partial} \eta_i = df_i, \quad \tilde{\partial} \xi_i = 0.
\]

We see immediately the relations $\partial, \tilde{\partial} : K^{p,s}_{W/S} \rightarrow K^{p,s-1}_{W/S}$ for all $p, s \in \mathbb{Z}$ and $\partial^2 = \tilde{\partial}^2 = \partial \tilde{\partial} + \tilde{\partial} \partial = 0$. That is, for each fixed $p \in \mathbb{Z}$, the subcomplex $(K^{p,s}_{W/S}(U), \partial_K)$

---

7One should note that the decomposition (4.10) depends on the choice of the generators $f_1, \ldots, f_l$, since a change of the generators (except for $\mathbb{C}$-constant linear changes) induces change of the variables $\eta_i$’s to elements where $\xi_i$’s and $\eta_i$’s are mixed (see (1.0)).

8We stress that non of $\partial$ or $\tilde{\partial}$ is well-defined for a non-complete intersection chart, since they do not preserve the defining ideal $\mathcal{I}$ (except for $\mathbb{C}$-linear relations).
can be regarded as a unified complex of a double complex \((K_{W/S}^{p, *}(U), \partial, \ddot{\partial})\). More precisely, let us denote by \(\Omega_{W/S}^p e \eta^c\) the space spanned by those elements of the form \(\omega \Xi \varepsilon\) with \(\omega \in \Omega_{W/S}^a\), and \(\Xi\) and \(\varepsilon\) are monomials of \(\xi_j\)'s and \(\eta_j\)'s of degree \(\deg(\Xi) = b\) and \(\deg(\varepsilon) = c\), respectively. Then, we have \(\partial : \Omega_{W/S}^p e \eta^c \rightarrow \Omega_{W/S}^{p-1} e \eta^c\) and \(\ddot{\partial} : \Omega_{W/S}^p e \eta^c \rightarrow \Omega_{W/S}^{p-1} e \eta^{c-1}\), so that the subcomplex \((K_{W/S}^{p}, \partial, \ddot{\partial})\) is given by the unification of the following double complex.

\[
\begin{array}{cccccccc}
\Omega_{W/S}^p & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^{p-1} e \eta^1 & \overset{\ddot{\partial}}{\rightarrow} & \ldots & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^1 e \eta^p & \overset{\ddot{\partial}}{\rightarrow} & \mathcal{O}_W e \eta^p & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \ldots & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
\Omega_{W/S}^p e \xi^1 & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^{p-1} e \xi^1 \eta^1 & \overset{\ddot{\partial}}{\rightarrow} & \ldots & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^1 e \xi^p \eta^p & \overset{\ddot{\partial}}{\rightarrow} & \mathcal{O}_W e \xi^p & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \ldots & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
\Omega_{W/S}^p e \xi^{l-1} & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^{p-1} e \xi^{l-1} \eta^1 & \overset{\ddot{\partial}}{\rightarrow} & \ldots & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^1 e \xi^{l-1} \eta^p & \overset{\ddot{\partial}}{\rightarrow} & \mathcal{O}_W e \xi^{l-1} & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \ldots & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
\Omega_{W/S}^p e \xi^l & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^{p-1} e \xi^l \eta^1 & \overset{\ddot{\partial}}{\rightarrow} & \ldots & \overset{\ddot{\partial}}{\rightarrow} & \Omega_{W/S}^1 e \xi^l \eta^p & \overset{\ddot{\partial}}{\rightarrow} & \mathcal{O}_W e \xi^l & \leftarrow 0 \\
\uparrow \partial & & \uparrow \partial & & \ldots & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial \\
0 & & 0 & & \ldots & & 0 & & 0 & & 0 \\
\end{array}
\]

Here, the vertical direction is the classical Koszul resolution for the regular sequence of functions \(f_1, \ldots, f_t\), and horizontal directions are the resolution for the system of forms \(df_1, \ldots, df_t\). In particular, if \(W/S\) and \(U\) are smooth, due to De Rham Lemma ([5][17]) and Koszul lemma ([7]), all horizontal and vertical sequences are exact except at the end of each sequence. Thus the unified complex for the differential \(\partial_K = \partial + \ddot{\partial}\) gives a resolution of the \(\mathcal{O}_S\)-module \(j_*(\Omega_{W/S}^p | U)\). That is, \[\text{Fact 4. Suppose } W/S \text{ and } U \text{ are smooth. Then } \pi \text{ in } [4.7] \text{ is quasi-isomorphic.}\]

5. Step 4: Lifting of Čech cohomology groups

Let \(\Phi : Z \rightarrow S\) be a flat holomorphic map and \(Z'\) be an open subset of \(Z'\) given in Main Theorem in the introduction. At the end of Step 2, we have seen that, for any point \(t \in S\), there exists a Stein open neighborhood \(S^* \subset S\) of \(t\) and a system of relative charts \(\mathcal{U} := \{j_k : U_k \rightarrow D_k(1) \times S^*\}_{k=0}^{\kappa} \) and a real number \(0 < r^* < 1\) such that, for any \(r\) with \(r^* \leq r \leq 1\) and any Stein open subset \(S' \subset S^*\), we have \(Z \cap \Phi^{-1}(S') \supset Z(r, S') := \cup_{k=0}^{\kappa} U_k(r, S') \supset Z \cap \Phi^{-1}(S)\) where we set \(U_k(r, S') := j_k^{-1}(D_k(r) \times S')\), and some further properties (see Fact 2). We called \(\mathcal{U} := \mathcal{U}(r, S') := \{j_k : U_k(r, S') \rightarrow D_k(r) \times S^*\}_{k=0}^{\kappa} \) the lifting of the atlas \(\mathcal{U} := \mathcal{U}(r, S') := \{U_k(r, S'), \varphi_k\}_{k=0}^{\kappa}\) of \(Z'(r, S')\) (for simplicity, we shall often omit \(r\) and \(S'\) from the notation \(\mathcal{U}(r, S')\) and \(\mathcal{U}(r, S')\)).

In this section, we express the Hodge to De Rham spectral sequence of the hypercohomology group \(\mathbb{R}\Phi_*(\Omega_{Z(r, S')}^{*, S'} | d_{Z(r, S')} | S')\) by the Čech cohomology group with respect to the atlas \(\mathcal{U}\) of \(Z(r, S')\). Then, using the functoriality, the Čech complex is lifted to a “Čech complex” on relative charts with coefficients in Koszul-De Rham algebra with respect to the lifted atlas \(\mathcal{U}\). Showing certain finiteness property for the lifted Čech complex, we can apply Forster-Knorr’s result to the complex and show that \(\mathbb{R}\Phi_*(\Omega_{Z(S)}^{*, S} | d_{Z(S)} | S)\) is \(\mathcal{O}_S\)-coherent in a neighborhood of \(t \in S\).
We first remark that \( \mathcal{U} := \{U_k(r, S')\}_{k=0}^{k'} \) is a Stein covering of \( Z(r, S') \), since the intersection \( U_k(r, S') := \cap_{k \in K} U_k(r, S') \) for any subset \( K \subset \{0, \cdots, k'\} \) is isomorphic to a closed submanifold of \( D_K(r) \times S' \) and, hence, is Stein. Then, due to Leray’s lemma, the cohomology group \( H^q(\check{C}(\mathcal{U}, \Omega_{Z(r, S')/S}')^p) \) of the Čech-complex \((\check{C}(\mathcal{U}, \Omega_{Z(r, S')/S}'), \bar{\partial}) \) of the covering \( \mathcal{U} \) coincides with the sheaf cohomology group \( H^q(Z(r, S'), \Omega_{Z(r, S')/S}^p) \) \((p, q \in \mathbb{Z}_{\geq 0}) \). That is, the first page \( E^1_p(q)(Z(r, S')) \) of the Hodge to De Rham spectral sequence is describe by the Čech-cohomology group.

Recall that the qth Čech co-chain module \((q \in \mathbb{Z}_{\geq 0}) \) is given by the direct sum
\[
\check{C}^q(\mathcal{U}, \Omega_{Z(r, S')/S}^p) := \bigoplus_{K=1}^{q+1} \Gamma(U_K(r, S'), \Omega_{Z(r, S')/S}^p).
\]

Inspired by this, we consider a triple complex. Namely, for \( p, q, s \in \mathbb{Z}_{\geq 0} \), we set
\[
(5.1) \quad \check{C}^{p, q, s}(\mathcal{U}(r, S')) := \bigoplus_{K=1}^{q+1} \Gamma(D_K(r) \times S', K_{D_K(r) \times S'/S}(U)).
\]

It was shown already in previous section Step 3 that the (co-)boundary operators \( d_{DR} \) and \( \partial_K \), acting on the coefficient \( K_{DR}^* \), define complexes \( \check{C}^{p, q, *}(\mathcal{U}(r, S')) \) and \( \check{C}^{p, q, *}(\mathcal{U}(r, S')) \), respectively. Let us clarify the role of the new index \( q \) by “lifting” the Čech-coboundary operator \( \bar{\partial} \) to obtain a generalized Čech complex on the relative charts \( \{j_K\} \) with respect to the atlas \( \mathcal{U} \).

**Lemma 5.1.** For all \( p, q, s \in \mathbb{Z} \), there exists an \( \mathcal{O}_S \)-homomorphisms
\[
\bar{\partial} : \check{C}^{p, q, s}(\mathcal{U}(r, S')) \to \check{C}^{p, q, s+1}(\mathcal{U}(r, S'))
\]
with the following properties.

1) They satisfy the relations
\[
\bar{\partial}^2 = 0, \quad \bar{\partial}d_{DR} + d_{DR}\bar{\partial} = 0, \quad \text{and} \quad \bar{\partial}\partial_K + \partial_K\bar{\partial} = 0.
\]

2) For each fixed \( p, q \in \mathbb{Z} \), we have a resolution
\[
\cdots \to \check{C}^{p, q, 2}(\mathcal{U}(r, S')) \xrightarrow{\partial_K} \check{C}^{p, q, 1}(\mathcal{U}(r, S')) \xrightarrow{\partial_K} \check{C}^{p, q, 0}(\mathcal{U}(r, S')) \xrightarrow{\partial_K} \check{C}^0(\mathcal{U}, \Omega^p_{Z(r, S')/S}) \to 0,
\]
where the Čech coboundary operator and the De Rham differential operator on RHS commutes with the operators \( \bar{\partial} \) and \( d_{DR} \) in LHS, respectively.

**Proof.** Recall that the qth Čech-coboundary operator \( \bar{\partial} \) on \( \check{C}^0(\mathcal{U}, \Omega^p_{Z(r, S')/S}) \) is an alternating sum \( \sum_{K \subset K'}(\pm \rho_{K, K'}^\mathcal{U}) \) of the restriction maps \( \rho_{K, K'}^\mathcal{U} : \Gamma(U_{K'}, \Omega^p_{Z(r, S')/S}) \to \Gamma(U_K, \Omega^p_{Z(r, S')/S}) \) for \( K \subset K' \) with \( \#K = q+1 \) and \( \#K' = q+2 \). Then, the lifting \( \bar{\partial} \) (denoted by the same notation) is defined by an alternating sum \( \sum_{K \subset K'}(\pm (\pi_K^p)^*) \), with the same choices of signs for the Čech-coboundary operator, of the pull-back homomorphisms \( \pi_K^p * : \Gamma(D_K(r) \times S', K^p_{DR}) \to \Gamma(D_K(r) \times S', K^p_{DR}) \) \((p, q \in \mathbb{Z}_{\geq 0}) \) induced from the projection \( D_K(r) \times S' \to D_K(r) \times S' \) functorially in Lemma 4.2.

1) To show that \( \bar{\partial}^2 = 0 \) is the same as the standard Čech-coboundary calculation. Other relations follows from the fact that the pull-back homomorphism preserves the Koszul-De Rham-structure, and commute with \( d_{DR} \) and \( \partial_K \) (Lemma 4.2).

2) The \( \check{C}^0 \) together with **Fact 4** at the end of Step 3, gives a resolution of \( \Omega^p_{U_K(r, S')/S} \) by \( K^p_{DR}(D_K(r) \times S'/S) \) as sheaf of \( \mathcal{O}_{D_K(r) \times S'/S} \)-module. Since \( D_K(r) \times S' \) is Stein, the resolution induces that of the module of sections on \( D_K(r) \times S' \).

We saw already in the last section the commutativity of \( \pi \) with Čech-coboundary operator follows from the commutativity \( \rho_{K, K'}^\mathcal{U} \circ \pi = \pi \circ (\pi_K^p)^* \).

As a consequence of Lemma 5.1, for each \( p \in \mathbb{Z} \), we see that the double complex \((\check{C}^{p, q, *}(\mathcal{U}(r, S')), \bar{\partial}, \partial_K) \) gives a resolution of the Čech-complex \((\check{C}^0(\mathcal{U}, \Omega^p_{Z(r, S')/S}'), \bar{\partial}) \).
Let us consider the simplification of the double complex by putting $\tilde{\kappa} := \kappa - \kappa$:

\[(5.2) \quad (\tilde{C}^p_{\kappa}(\mathcal{U}(r, S'))^\kappa, \delta + \partial_k) \quad \text{where} \quad \tilde{C}^p_{\kappa}(\mathcal{U}(r, S')) := \bigoplus_{\kappa \to \kappa} \tilde{C}^p_{\kappa-\kappa}(\mathcal{U}(r, S'))\]

and the morphism $(\tilde{C}^p_{\kappa}(\mathcal{U}(r, S'))^\kappa, \delta + \partial_k) \to (\mathcal{C}^p(\mathcal{U}, \Omega^p_{\mathcal{U}(r, S')^\kappa}), \delta)$.

The following finiteness of the simplified complex is making a big contrast to the case $(5.1)$, where, by putting $\tilde{\kappa}_0 := \kappa - \kappa$, we obtained a resolution $\mathcal{K}^p_{\delta k}$ of the De Rham complex $\Omega^p_{\mathcal{U}/S'}$ of infinite sequence. This finiteness is one of the most subtle point of the whole theory where Koszul-De Rham algebra works mysteriously well.

**Fact 5.** The RHS of $(5.2)$ for fixed $p$ and $\kappa$ is a finite direct sum.

**Proof.** The index $\kappa = \#K - 1$ for $K \subset \{0, \cdots, k^*\}$ is bounded. Then the condition $\#K - 1 - \kappa$ means that the range of $\kappa$ is bounded.

**6.** Recall $\tilde{C}^p_{\kappa}(\mathcal{U}(r, S')) := \bigoplus_{\kappa \to \kappa} \Gamma(D_K(r) \times S', \mathcal{K}^p_{\delta k}(r) \times S' \times S')$. If there is a non-vanishing term in RHS, then due to $(5.5)$, one has $-l_K \leq p - \kappa \leq \dim \mathcal{K}(D_K(r))$. Then, adding $\kappa = \kappa' + \kappa''$ in both hand side, we have $-l_K + \kappa' \leq \kappa'' + p \leq \dim \mathcal{K}(D_K(r)) + \kappa''$.

Since $\kappa = \#K - 1$ runs over the index $K \subset \{0, 1, \ldots, k^*\}$ such that $U_K = \bigcup_{k \in K} U_k$ is non-empty, and $\dim \mathcal{K}(D_K(r))$ takes the maximal value when $\kappa = \#K - 1$ takes the maximal value (recall Lemma’s $3.1$ and $3.2$ so that $\dim \mathcal{K}(D_K(r)) = N \cdot \#K$), we obtain $(5.3)$. \hfill \Box

Due to the finiteness **Fact 5**, for all $p, q \in \mathbb{Z}$, the morphism $\pi$ induces the isomorphisms of cohomology groups with respect to $\delta + \partial_k$ and $\tilde{\delta}$:

\[(5.4) \quad H^q(\tilde{C}^p_{\kappa}(\mathcal{U}(r, S'))^\kappa, \delta + \partial_k) \simeq H^q(\mathcal{C}^p(\mathcal{U}, \Omega^p_{\mathcal{U}(r, S')^\kappa}), \delta) \simeq H^q(Z(r, S'), \Omega^p_{\mathcal{U}(r, S')^\kappa}) \simeq \mathcal{E}^{p,q}(r, S').\]

Here we recall that $\mathcal{E}^{p,q}(r, S')$ in RHS is the first page of the Hodge to De Rham spectral sequence of $Z(r, S')^\kappa$ for $r \in \mathbb{R}$ satisfies $r^* \leq r \leq 1$, $S'$ is a Stein open subset of $S^\kappa$ and $Z(r, S') := \bigcup_{k \leq \kappa} U(r, S')$ for $U_k(r, S') := j_k(D_k(r) \times S')$.

Since we obviously have the vanishing of the RHS of $(5.4)$ if $p \not\in \mathbb{Z} \cap [0, \dim \mathcal{K}]$, so should be the cohomology group in LHS. This observation leads to the following truncation of the double complex.

**Facts.** Consider the truncated double complex $(\mathcal{C}^p_{\kappa}(\mathcal{U}(r, S'))^\kappa, d_{DR, \delta + \partial_k})$, where

\[
\text{Tr} \mathcal{C}^p_{\kappa}(\mathcal{U}(r, S')) := \begin{cases} 
\mathcal{C}^p_{\kappa}(\mathcal{U}(r, S')) & \text{if } 0 \leq p \leq \dim \mathcal{K} \\
0 & \text{else}
\end{cases}
\]

Then, we have the following $7$, $8$, $9$ and $10$.

**7.** The cohomology with respect to the coboundary operator $\delta + \partial_k$ of the double complex $(\mathcal{C}^p_{\kappa}(\mathcal{U}(r, S'))^\kappa, d_{DR, \delta + \partial_k})$ gives the Hodge to De Rham spectral sequence $\mathcal{E}^{p,q}(r, S')$ for the morphism $\Phi$.

**8.** The spectral sequence is bounded, and converges in finitely many steps to the hypercohomology group $\mathcal{R}\Phi_{\mathcal{U}(r, S')^\kappa, d_{Z/S}}(\mathcal{U}(r, S'))$, which is independent of $r$ for $r^* \leq r \leq 1$.

**9.** For any Stein open subset $S' \subset S^\kappa$ and $r^* \leq r \leq 1$, each $(p, q)$-entry of the double complex is a Fréchet vector space isomorphic to a finite direct sum of vector spaces of the form $\prod_{(\text{finite})} \mathcal{H}(D(r) \times S', \mathcal{C}(D(r) \times S'))$, where $D(r)$ is a polycylinder of radius $r$ of dimension $m$ depending on each factor.
10. The coboundary operator $d_{DR}$ and $\hat{\partial} + \partial_K$ of the double complex are $\mathcal{O}_S$-homomorphisms, which are continuous with respect to the Fréchet topology.

Proof. Obviously, the truncation does not affect to the $\hat{\partial} + \partial_K$-cohomology. Then, the equality (5.4) show that the $\hat{\partial} + \partial_K$-cohomology and $\mathcal{E}^p_q(r, S')$ are naturally isomorphic as vector spaces. Then, the commutativity $d_{U/S} \circ \pi = \pi \circ d_{DR}$ in (1.9) implies that the derivations as spectral sequences also coincide with each other.

8. The boundedness of the truncated complex was shown already in above Fact 6. It was shown in Step 1, that the limit cohomology group $\mathcal{H}_q^\bullet((\mathcal{O}_{Z/S}, d_{Z/S})|_S)$ does not depend on the parameter $r$.

9. The Fact 5, together with (5.8), (4.10) show that each component of the double complex $TrC^\bullet\bullet((\mathfrak{U}(r, S'))$ is a finite direct sum of modules of the form

$$\Gamma(D_K(r) \times S', \mathcal{O}_{D_K(r) \times S'}) \simeq \bigoplus_{1 \leq i_1 < \cdots < i_p \leq m_K} \Gamma(D_K(r) \times S', \mathcal{O}_{D_K(r) \times S'})$$

for some $K \subset \{0, \cdots, k^*\}$ and $p \in \mathbb{Z}$. It is well known that the holomorphic function ring (Stein algebra) $\Gamma(D_K(r) \times S', \mathcal{O}_{D_K(r) \times S'})$ carries naturally a Fréchet topology (2, 8, p266) with respect to the compact open convergence.

10. The operators $\partial$ and $\partial_K$ are $\mathcal{O}_{D_K(1) \times S'}$-homomorphisms and induce continuous morphisms on the modules. The operator $d_{DR}$ is no-longer $\mathcal{O}_{D_K(1) \times S'}$-homomorphism but is only an $\mathcal{O}_S$-homomorphism. Nevertheless, it is also well known that differentiation operators on a Stein algebra are also continuous w.r.t. the Fréchet topology.

We are now able to apply the following key Lemma due to Forster and Knorr [6] (the formulation here of the result is taken form their unpublished note which is slightly modified from the published one, however can be deduced).

**Lemma 5.2.** (Forster-Knorr) Let $m$ be a given integer, $S$ a smooth complex manifold, $0$ a point in $S$, and $D(r) = \{ z \in \mathbb{C}^n \mid |z_i| < r \ (1 \leq i \leq n) \}$ a polydisc of radius $r \in \mathbb{R}_{>0}$. Suppose $(C^\bullet(r), d)$ is a complex of $\mathcal{O}_S$-module bounded from the left such that

i) for any Stein open subset $S' \subset S$ and $q \in \mathbb{Z}$, we have an isomorphism

$$C^q(r)(S') \simeq \prod_{\text{finite}} \Gamma(D(r) \times S', \mathcal{O}_{D(r) \times S'})$$

together with the Fréchet topology.

ii) $d : C^q(r) \to C^{q+1}(r)$ is an $\mathcal{O}_S$-homomorphism, which is continuous with respect to the Fréchet topology.

iii) There exists $r_1$ and $r_2$ such that, for any $r$, $r_1 \geq r \geq r' \geq r_2 > 0$, the restriction $C^\bullet(r') \to C^\bullet(r)$ is a quasi-isomorphism.

Then, there exists a small neighborhood $S_m$ of $0$ in $S$ such that, for $q \geq m$, $H^q(C^\bullet(r))|_{S_m}$ is an $\mathcal{O}_{S_m}$-coherent module.

We apply Lemma to the truncated double complex $TrC^\bullet\bullet((\mathfrak{U}(r, S'))$. Precisely, Lemma is formulated for a complex $(C^\bullet(r), d)$ but not for a double complex, however our double complex is bounded in both horizontal and vertical directions (Fact 8) and, hence, the spectral sequence converges in finite steps. Then, to adjust Lemma to it is a question of a formalism, which we leave to the reader. Let us check that the double complex satisfies the assumptions in Lemma by putting $S = S^*$, $0$ to be $t \in S^*$ and $r_1 = 1$, $r_2 = r^*$. Namely, the finite sum expression with Fréchet topology in i) follows from Fact 9, the continuity of the coboundary operator in ii) follows from Fact 10, and the quasi isomorphy in iii) is shown by combining Fact 7 with Corollary to Fact 2 (c.f. Fact 1). Finally, choosing $m = -1$, we obtain the coherence of $\mathcal{H}_q^\bullet((\mathfrak{U}_{Z/S}, d_{Z/S})|_S)$ in a neighborhood of $t \in S^*$.

This completes a proof of Main Theorem given in Introduction. □
REFERENCES

[1] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. 2 (1970), 103–161.

[2] H. Cartan, Séminaire Cartan, Éc. norm. Supér. '51-52

[3] P. Deligne, *Théorie de Hodge. II*. Publ. Math. IHES, 1971, 40, 5-58.

[4] P. Deligne, *Théorie de Hodge. III*. Publ. Math. IHES, 1974, 44, 5-77.

[5] G. De Rham, *Sur la division de formes et de courants par une forme linéaire*, Comment. math. Helv. 28, 346-352 (1954)

[6] O. Forster and K. Knorr, *Relativ-analytische Räume und die Kohärenz von Bildgarben*. Invent. Math. 16 (1972), 113-160.

[7] H. Grauert, *Ein Theorem der Analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Publ. Math. I.H.E.S. 5, Paris 1960.

[8] H. Grauert, Th. Peternell and R. Remmert, *Several Complex variables VII*, encyclopedia of Mathematical Sciences, Volume 74

[9] G. M. Greuel, *Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. 1975, 214, 235-66.

[10] H. Haman, *Zur analytischem und algebraischen Beschreibung der Picard-Lefschetz-Monodromie*, Habilitationsschrift, Göttingen, 1974, 106pp.

[11] N. Katz, *The regularity theorem in Algebraic Geometry*, Proceedings of I.C.M., Nice, 1970.

[12] N. Katz and T. Oda, *On the differentiation of De Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. 8 (1968), 199-213.

[13] K. Knorr, *Der Grauerts Projektionssatz*, Inventiones math. 12, 118-172 (1971).

[14] C. Li, S. Li and K. Saito, *Primitive Forms via Poly-vector Fields*, preprint at arXiv: math.AG/1311.1659.

[15] K. Saito, *Calcul algébrique de la monodromie*, Société Mathématique de France, Astérisque, 7 et 8, 195-212, (1973).

[16] K. Saito, *Regularity of Gauss-Manin connection of a flat family of isolated singularities*, Quelques journée singulières, École Polytechnique, Paris (1973).

[17] K. Saito, *On a generalization of De Rham Lemma*, Ann. Inst Fourier, Grenoble 26, 2 (1976), 165-170.