PERIODIC MODULES AND QUANTUM GROUPS

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Abstract. We prove that the elements $A_\leq$ defined by Lusztig in a completion of the periodic module actually live in the periodic module (in the type $A$ case). In order to prove this, we compare, using the Schur duality, these elements with Kashiwara canonical basis of an integrable module.

0. Introduction

Fix two positive integers $d$ and $p$. Let $V' \supset R \supset R^+ \supset I = \{\alpha_1, \ldots, \alpha_{d-1}\}$ be a $\mathbb{R}$-vector space of dimension $d-1$, a root system of type $A_{d-1}$, a system of positive roots and the corresponding set of simple roots. Fix a partition $c = (c_1, \ldots, c_\ell)$ of $d$ and set $I_c = \{\alpha \in R^+ \cap I, \forall \alpha \notin \{c_1 + \cdots + c_\ell\} \}$. Let $F$ be the set of all reflection hyperplanes $F_{\alpha^\vee, n} = \{v \in V' | (\alpha^\vee : v) = np\}$, where $\alpha \in R^+$ and $n \in \mathbb{Z}$. Set $F_c = \{F_{\alpha^\vee, n} | \alpha \in I_c \} \subset F$. We call alcoves the connected components of $V' \cup \bigcup_{F \in F_c} F$. Let $A'_c$ be the only alcove contained in the dominant Weyl chamber having 0 in its closure. Let $A_c$ be the set of alcoves contained in the connected component of $V' \cup \bigcup_{F \in F_c} F$ which contains $A'_c$. Finally, let $M_c$ be the $\mathbb{Z}[q, q^{-1}]$-span of the alcoves in $A_c$. It is a $H' \times R'_c$-module, where $H'$ is the affine Hecke algebra associated to $R$, and $R'_c$ is a Laurent polynomial algebra over $\ell$ variables. In [L5, 9.17(a)] Lusztig defines, for any $A \in A_c$, some element $A_\leq$ in a completion of $M_c$ and he conjectures in [L5, 12.7] that $A_\leq \in M_c$ (the conjecture is stated for all types). If $c = (1^d)$, the conjecture follows from results in [L1]. In this paper we prove it for all $c$ in Theorem 5.5.

In [L5, 17.3] two conjectural multiplicity formulas for modules of a simple Lie algebra over an algebraically closed field of characteristic $p$ are given. They are formulated via some element $A_\geq$ and it involves polynomials $\pi_{AB}$, the coefficients of the expansion of $A_\geq$ in the basis of the alcoves, [L5, 9.17(c),(d)]. Using Theorem 5.5 and [L5, Proposition 12.9] we get that for all $A$, there exist only finitely many non-zero polynomials $\pi_{AB}$.

The main tools of the proof are the Schur duality (in its quantum affine version) and a theorem of Kashiwara yielding a canonical basis for some integrable module $N$ of the quantum affine loop algebra of $\mathfrak{gl}_p$ (here $p > d$). More precisely we write down a $H' \times R'_c$-isomorphism which sends a $H' \times R'_c$-cyclic submodule of $M_c$ into a weight space of $N$ and takes $\{A_\leq\}$ into Kashiwara’s canonical basis.

This work is motivated by [VV3]. In loc.cit. the authors give a geometric construction, via quiver varieties, of the (signed) canonical basis of the module $N(\lambda_c)$. This construction is very similar to Lusztig’s conjectural construction of

2000 Mathematics Subject Classification. Primary 17B37, Secondary 17B10.
the (signed) canonical basis in the K-theory of the Springer fiber (see [L4, L5], where it is also explained the link between this K-theory object and the periodic module). It is known that, for type $A$, the Springer fibers are quiver varieties. Probably it is possible to check that both signed bases coincide via their geometric characterization. In this paper we give an elementary algebraic proof.

Acknowledgements. The idea to compare the elements $A_{\leq}$ with the Kashiwara’s canonical basis is due to E. Vasserot. I thank him to share it with me. This work started during my stay at the M.S.R.I. in March 2002. I thank the organizers of the program ”Infinite-dimensional algebras and Mathematical Physics” for inviting me.

1. Notations

1.1. Let $T \subset \text{GL}_d$ be a maximal torus. Let $X = X(T)$ be the character group of $T$, and let $R(T)$ be its representation ring. The dual group of one parameter subgroups of $T$ is denoted by $X^\vee$. Set $(\cdot)$ equal to the canonical pairing $X \times X^\vee \rightarrow \mathbb{Z}$. Set $R \subset X$ equal to the set of root of $\text{gl}_d$ with respect to $T$. Choose a positive system $R^+ \subset R$. Let $I \subset R^+$ be the set of simple roots and let $Q \subset X$ be the root lattice. We fix basis elements $\varepsilon_1, \ldots, \varepsilon_d \in X$ such that $I = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, d-1\}$. For $i = 1, \ldots, d$ set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \in X$ with $i = 1, \ldots, d$. Let $\{\omega_i^\vee\}, \{\varepsilon_i^\vee\} \subset X^\vee$ be the dual bases. Put $\gamma^\vee = \alpha_1^\vee + \cdots + \alpha_{d-1}^\vee$, and $\alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee$ with $i = 1, 2, \ldots, d-1$.

Let $\leq$ be the partial order on $X$ such that $\mu \leq \nu$ if and only if $\nu - \mu \in \sum_{i=1}^{d-1} \mathbb{N} \alpha_i$.

1.2. Let $W^f$ be the Weyl group of $\text{gl}_d$. Let $W = W^f \ltimes X$, $W' = W^f \ltimes Q$, be the extended affine Weyl group and the affine Weyl group. For any $w \in W^f$, $\lambda \in X$ we use the following notations : $w = (w, 0)$, $\tau_\lambda = (0, \lambda)$.

A composition of $d$ is a $\ell$-uple of non-negative integers $f = (f_1, f_2, \ldots, f_\ell)$ such that $\sum_i f_i = d$, for some $\ell$. It is a partition if and only if $f_1 \geq f_2 \geq \cdots \geq f_\ell \geq 0$.

For any composition $f$ of $d$, set $I_f = \{\alpha_i \in I \mid i \neq f_1 + f_2 + \cdots + f_a, \forall a\}$. Let $W_f$ be the corresponding parabolic subgroup of $W^f$, i.e. $W_f = \langle s_i \mid \alpha_i \in I_f \rangle$. Put $W_f^d = W_f \ltimes \mathbb{Z}I_f$. Let $W_f^d$ (resp. $W_f^d \subset W^f$ be the set of all elements of $W^f$ the minimal length in the cost $wW_f$ (resp. $W_f w$). Let $w_d, w_f$ be the longest elements of $W^f$, $W_f$. Set $v_d = \ell(w_d)$, $v_f = \ell(w_f)$.

Let $s_1, s_2, \ldots, s_d$ be the simple affine reflections in $W'$ such that $s_i = s_{\alpha_i^\vee}$ if $i \neq d$ and $s_d = \tau_\theta s_{\theta \varepsilon_d}$. Where for any root $\alpha \in R$, $s_{\alpha^\vee}$ is the corresponding reflection. Let $\pi$ denote the element $\tau_{\omega_1} s_1 s_2 \cdots s_{d-1}$. The cyclic group of infinite order $\langle \pi \rangle$ is isomorphic to $X/Q$ and $W$ is isomorphic to a semi-direct product $\langle \pi \rangle \ltimes W'$.

1.3. Fix a positive number, say $p$. Set $V = X \otimes_{\mathbb{Z}} \mathbb{R}$,

$$A_+ = \{ \gamma \in V \mid p > (\gamma : \alpha^\vee) > 0, \forall \alpha \in R^+ \},$$

$$X_p = \{ \gamma \in X \mid p \geq (\gamma : \varepsilon_i^\vee) > 0, \forall i = 1, ... d \}.$$  

The group $W$ acts on $X$ as follows

$$\gamma \cdot s_i = \gamma - (\gamma : \alpha_i^\vee) \alpha_i \quad \text{if } i = 1, 2, ..., d - 1,$$

$$\gamma \cdot \tau_\nu = \gamma - p\nu \quad \text{if } \nu \in X.$$  

Note that

- the set $A_+ \cap X_p$ is a fundamental domain for the right action of $W$ on $X$ ($\tilde{A}_+$ is the closure of $A_+$ in $V$),
- for all $\mu \in \tilde{A}_+ \cap X_p$, $W_\mu$ is the isotropy group of $\mu$, if $e$ is associated to $\mu$ as in (1.4.1).
1.4. Conventions.

(1) In the future we will use both weights of \(\mathfrak{gl}_p\) and \(\mathfrak{gl}_d\). The fundamental weights and the simple roots of \(\mathfrak{gl}_p\) are denoted by \(\Lambda_i, i = 1, \ldots, p\) and \(\beta_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, p - 1\), respectively. The affine simple roots of \(\mathfrak{gl}_p\) are \(\beta_1, \ldots, \beta_p\). Set
\[
\delta = \beta_1 + \cdots + \beta_p, \quad \tilde{X} = \sum_{a=1}^{p} \mathbb{Z} \epsilon_a, \quad \tilde{X}^a = \mathbb{Z} \delta \oplus \tilde{X}.
\]

(2) From now on, \(\tilde{\lambda} = \sum_{a=1}^{p} d_a \epsilon_a\) will be a fixed (dominant) weight with \(d_1 \geq d_2 \geq \cdots \geq d_p \geq 0, \sum_{a=1}^{p} d_a = d\). Set \(d = (d_p, \ldots, d_1)\) and let \(c = (c_1, c_2, \ldots, c_l)\) be the partition dual to \(d\), that is \(c_i = |\{j \mid d_j \geq i\}|\). Then \(c = (p^p \ldots 2^2 1^1)\), for some positive integers \(l_1, \ldots, l_p\) such that \(\ell = \sum_{a=1}^{p} l_a\). We have
\[
\tilde{\lambda} = \sum_{a=1}^{p} l_a \Lambda_a = \sum_{a=1}^{p} \Lambda_{c_a}.
\]

(3) Given \(\mu = \sum_{i=1}^{d} b_i \varepsilon_i \in X_p\), we set
\[
\tilde{\mu} = \sum_{a=1}^{p} e_a \epsilon_a \in \tilde{X} \quad \text{with} \quad e_a = |\{i \mid b_i = a\}|, \quad e = (e_p, \ldots, e_2, e_1).
\]

Set
\[
\Omega = \{\sum_{a=1}^{p} e_a \epsilon_a \in \tilde{X} \mid e_a \geq 0, d = \sum_{a} e_a\}, \quad \Omega_{sm} = \{\sum_{a=1}^{p} e_a \epsilon_a \in \Omega \mid e \text{ is small}\}.
\]

where \(e\) is as in (1.4.1). The composition \(e\) is said small if \(e_a = 0, 1\) for all \(a\). The map \(\mu \mapsto \tilde{\mu}\) restricts to a bijection \(A_+ \cap X_p \to \Omega\) which takes \(A_+ \cap X_p\) onto \(\Omega_{sm}\). From now on if \(\tilde{\mu} \in \Omega\) (resp. \(\Omega_{sm}\)), then \(\mu\) denotes the unique preimage in \(A_+ \cap X_p\) (resp. \(A_+ \cap X_p\)).

2. \(q\)-Notations

2.1. Put \(A = \mathbb{Q}[q, q^{-1}], K = \mathbb{Q}(q), A_0 = \{f \in K \mid f \text{ is without pole at } q = 0\}\). Set
\[
R = \mathbb{Z}[X] \otimes A, \quad R_c = \mathbb{Z}[X/\mathbb{Z}I_c] \otimes A, \quad R' = \mathbb{Z}[Q] \otimes A, \quad R'_c = \mathbb{Z}[Q/\mathbb{Z}I_c] \otimes A.
\]

We identify \(R\) with the polynomial ring \(K[x_1^\pm 1, x_2^\pm 1, \ldots, x_d^\pm 1]\), via the \(A\)-algebra isomorphism which takes \(\varepsilon_i \otimes 1 \to x_i\). For all \(\gamma = \sum_i m_i \varepsilon_i \in X\), put \(x_\gamma = \prod_i x_i^{m_i}\). Then \(R_c\) is identified with the quotient of \(R\) by the ideal generated by the relations \(x_{\alpha_i} = 1\), for all \(\alpha_i \in I_c\), and \(R'\) with the sub-\(A\)-algebra of \(R\) generated by \(\{x_\alpha^\pm 1 \mid \alpha_i \in I\}\). We denote by \(x_{\gamma+\mathbb{Z}I_c}\) the image of \(x_\gamma\) in \(R_c\). Set \(\alpha(c_i) = \sum_{k=1}^{c_i} (c_i + 1 - 2k)\varepsilon_{c_1 + \cdots + c_{i-1} + k}\), and \(\alpha_c = \sum_{i=1}^{\ell} \alpha(c_i) \in Q\). Define an \(A\)-algebra homomorphism \(\psi : R \to R_c\) by the formula \(\psi(x_\gamma) = q^{\gamma \cdot \alpha_c} x_{\gamma+\mathbb{Z}I_c}\). The map \(\psi\) restricts to \(\psi : R' \to R'_c\), cf. [L5, 8.3]. We have
\[
\psi(x_{\alpha_i}) = q^{2} \quad \text{for all } \alpha_i \in I_c.
\]

2.2. The quantum loop algebra of \(\mathfrak{gl}_p\) is the \(K\)-algebra \(U_K\) generated by elements \(e_a, f_a, l_a^\pm, a = 1, 2, \ldots, p\), modulo the following defining relations
\[
l_a^{-1} l_a^{-1} = 1 = l_a^{-1} l_a, \quad l_a l_b = l_bl_a, \quad l_b e_a l_b^{-1} = q^{\delta(a=b) - \delta(a+1=b)} e_a, \quad l_b f_a l_b^{-1} = q^{\delta(a+1=b) - \delta(a=b)} f_a,
\]
\[ [e_a, f_b] = \delta(a, b) \frac{k_a - k_a^{-1}}{q - q^{-1}}, \]
\[ \sum_{p=0}^{m} (-1)^p \left[ \begin{array}{c} m \\ p \end{array} \right] e_a^p e_b e_a^{m-p} = \sum_{p=0}^{m} (-1)^p \left[ \begin{array}{c} m \\ p \end{array} \right] f_a^p f_b f_a^{m-p} = 0. \]

In the last identity \( a \neq b \), \( m = 2 \) if \( a - b = \pm 1 \) and \( 1 \) else. We have set \([n] = q^n - q^{n-1} + \ldots + q - 1\) if \( n \geq 0 \), \([n]! = [n][n-1]!\ldots[2]!\), and
\[ \left[ \begin{array}{c} m \\ p \end{array} \right] = \frac{[m]!}{[p]![m-p]!}. \]

We have also set \( k_a = l_a l_a^{-1} \), for all \( a = 1, 2, \ldots, p - 1 \) and \( k_p = l_p l_1^{-1} \). Finally \( a \equiv b \) means \( a - b \in p\mathbb{Z} \). Let \( \Delta \) be the coproduct of \( \mathbb{U}_x \), defined as follows
\[ \Delta(e_a) = 1 \otimes e_a + e_a \otimes k_a^{-1}, \quad \Delta(f_a) = k_a \otimes f_a + f_a \otimes 1, \quad \Delta(l_a) = l_a \otimes l_a. \]

The algebra \( \mathbb{U}_x \) has also a presentation in terms of Drinfeld generators \( x_i^{\pm}, h_i^{\pm}, l_i^{\pm} \), where \( r \in \mathbb{Z}, s \in \mathbb{Z} \setminus \{0\}, i = 1, 2, \ldots, p - 1, a = 1, 2, \ldots, p \). We normalize these generators as in [B] (with \( T_i = T_i^{(r)} \) in the notations of [L2]). The presentation depends on a choice of a function \( o : \{1, \ldots, p - 1\} \to \{\pm 1\} \) such that \( o(i \pm 1) = -o(i) \).

Let \( \mathbb{U}_x \) be the modified algebra of \( \mathbb{U}_x \). It is a \( \mathbb{K} \)-algebra without unity generated by \( \pi e_a, \pi f_a, \pi_1, i \in \mathbb{N}_p, a = 1, 2, \ldots, p \), with \( \pi_i \pi_j = \delta_{ij} \pi_1 \), see [L2, 23.1] for a precise definition. Let \( \mathbb{U}, \tilde{\mathbb{U}} \) be the \( \Lambda \)-forms of \( \mathbb{U}_x \), \( \mathbb{U}_x \). Then \( \mathbb{U} \) is the \( \Lambda \)-subalgebra of \( \mathbb{U}_x \) generated by \( e_a^{(n)}, f_a^{(n)}, l_a^{\pm} \), and \( \tilde{\mathbb{U}} \) is the \( \Lambda \)-subalgebra of \( \tilde{\mathbb{U}}_x \) generated by \( \pi_1 e_a^{(n)}, \pi_1 f_a^{(n)} \).

Let \( x \mapsto \overline{x} \) be the ring homomorphism of \( \mathbb{U} \) such that \( \overline{q} = q^{-1}, \overline{e}_a = e_a, \overline{f}_a = f_a, \overline{l}_a = l_a^{-1} \). We will still denote by \( x \mapsto \overline{x} \) the involution on \( \mathbb{U} \) which is also characterized by \( \overline{\pi_1} = \pi_1 \).

**Remark.** Kashiwara’s canonical bases are defined for the quantum loop algebra of \( \mathfrak{sl}_p \). In this paper we will use the \( \mathfrak{gl}_p \)-version. It is irrelevant since an integrable \( \mathbb{U} \)-module is a module for the quantum loop algebra of \( \mathfrak{sl}_p \) together with an extra grading coming from the action of the \( l_i \)'s.

### 2.3. Let \( H \) be the affine Hecke algebra of \( \text{GL}_d \). Recall that \( H \) is the unital associative \( \Lambda \)-algebra generated by elements \( t_1, \ldots, t_{d-1}, x_1^{\pm}, \ldots, x_d^{\pm} \), modulo the relations
\[
(t_i + q^{-1})(t_i - q) = 0, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad |i - j| \geq 2 \Rightarrow t_i t_j = t_j t_i, \\
x_i x_i^{-1} = 1 = x_i^{-1} x_i, \quad x_i x_j = x_j x_i, \quad t_i x_j t_i = x_{i+j}, \quad j \neq i, i + 1 \Rightarrow t_i x_j = x_j t_i.
\]

For any \( w \in \mathbb{W} \) let \( t_w \) be the corresponding element in \( H \). To simplify we set \( \pi = t_\pi (= x_1 t_1 t_2 \cdots t_{d-1}) \). Set also \( t_d = \pi t_{d-1} \pi^{-1} \). Let \( H', H' \subset H \) be the \( \Lambda \)-subalgebras generated by \( t_1, \ldots, t_d \) and \( t_1, \ldots, t_{d-1} \) respectively. We will identify, in the obvious way, \( \mathbb{R} \) with the sub-\( \Lambda \)-algebra of \( H \) generated by the \( x_i^{\pm} \)'s. Note that \( H' \) is the sub-\( \Lambda \)-algebra of \( H \) generated by \( t_1, \ldots, t_{d-1} \) and by \( \mathbb{R}' \). For future references, we point out that \( x_\alpha \) is equal to the element \( \theta_\alpha \) in Lusztig’s papers.

Let \( t \mapsto \bar{t} \) be the ring homomorphism of \( H \) such that \( \overline{q} = q^{-1}, \overline{t_i} = t_i^{-1}, \overline{\pi} = \pi, \overline{i} = 1, \ldots, d. \)
2.4. For any composition \( f \) of \( d \), let \( \mathbf{H}_f \subseteq \mathbf{H}^f \) be the subalgebra generated by the elements \( t_i \) with \( \alpha_i \in I_f \). Similarly, let \( \mathbf{H}_f' \subseteq \mathbf{H}' \) be the subalgebra generated by \( \mathbf{H}_f \) and the elements \( x_{\alpha_i}^\pm \) with \( i \in I_f \). Set also

\[
\rho_f = \sum_{w \in W_f} q^{\ell(w)} t_w, \quad m_f = \sum_{w \in W_f} q^{2\ell(w)}.
\]

We get \( \rho_f^2 = m_f \rho_f \) and \( \overline{m_f} = q^{-2\ell} \rho_f \).

2.5. Set \( \mathbb{A}^- \) equal to the unique left representation of \( \mathbf{H}' \) on \( \mathbb{A} \) taking the elements \( t_i \) to \(-q^{-1}I_d\), for all \( i = 1, \ldots, d \). Note that, in particular,

\[(2.5.1)\]

\[x_{\alpha_i} \cdot 1 = q^2 \cdot 1 \quad \text{for all } \alpha_i \in I\]

2.6. For any pair of compositions \( f, f' \) of \( d \) with at most \( p \) parts, let \( \mathbf{H}_{f,f'} \) be the \( \mathbb{A} \)-linear span of the elements \( t_m = \sum_{m \in W_f \setminus W/W_f} q^{\ell(w)} t_w \), where \( m \in W_f \setminus W/W_f \). Let \( S_d = \bigoplus_{f,f'} \mathbf{H}_{f,f'} \) the \( q \)-affine Schur algebra. The product in \( S_d \) is defined by setting \( t_m * t_n = \delta(f',g)m_f^{-1} t_m t_n \) for all \( m \in W_f \setminus W/W_f, \) \( n \in W_g \setminus W/W_g \). Consider the algebra homomorphism \( \chi_d : \mathbf{U} \to S_d \), defined in [SV, Proposition 2.4]. In particular it takes \( \pi_1 \) to \( \rho_d \), if \( f \) is a composition of \( d \), and \( \pi_1 \mapsto 0 \) otherwise. Here \( f^2 = (i_1, i_2, \ldots) \) and \( \ell_r = |i^{-1}(r)| \). Let \( \chi_{f,f'} : \mathbf{U} \to \mathbf{H}_{f,f'} \) be the composition of \( \chi_d \) with the canonical projection \( S_d \to \mathbf{H}_{f,f'} \). The \( \mathbb{A} \)-module \( S_d \) has a basis, \( \mathcal{B} = \{ b_s \mid \mathbf{s} \in A_d \} \), where \( A_d \) is the following subset of the set of \( \mathbb{Z} \times \mathbb{Z} \) matrices with entries in \( \mathbb{N} \),

\[A_d = \{ \mathbf{s} = (s_{ij})_{i,j \in \mathbb{Z}} \mid s_{i+p,j+p} = s_{ij}, \sum_{i \in \mathbb{Z}} \sum_{j=1}^p s_{ij} = d \}, \]

see [SV, 3.2]. Then [L6, Theorem 8.2] states that \( B_{op} = \{ b_s \mid \mathbf{s} \in A_{op}^d \} \) is a \( \mathbb{A} \)-basis of \( \text{Im} (\chi_d) \), with

\[A_{op}^d = \{ \mathbf{s} \in A_d \mid \forall j \in \mathbb{Z} \setminus \{0\}, \exists i \in \mathbb{Z} \text{ and } s_{i,i+j} = 0 \}.\]

For a future use let us mention the following.

Claim. If \( p > d \) and \( f \) is small, the map \( \chi_{f,f'} \) is surjective for all \( f' \). If \( p = d \) and \( f, f' \) are small, then \( \mathcal{B} \cap \mathbf{H}_{f,f'} = (B_{op} \cap \mathbf{H}_{f,f'}) \cup \{ \rho_f \pi_r \mid r \in \mathbb{Z} \} \).

Proof. Set \( f = (f_1, \ldots, f_p), f' = (f'_1, \ldots, f'_p) \). Then

\[b_s \in \mathbf{H}_{f,f'} \iff \mathbf{s} \in A_{f,f'} = \{ \mathbf{s} \in A_d \mid \sum_{i \in \mathbb{Z}} s_{ij} = f_i, \sum_{i \in \mathbb{Z}} s_{ij} = f'_j, \forall i, j \in \{1, 2, \ldots, p\} \} \].

Suppose that \( p \geq d \). If \( f \) is small, a direct computation gives

\[A_{f,f'} \cap A_{op}^d \neq A_{f,f'} \iff f' \text{ is small and } p = d \].

This proves the first part of the claim. If \( p = d \) and \( f, f' \) are both small, we have

\[A_{f,f'} \setminus (A_{f,f'} \cap A_{op}^d) = \{ s(r) \mid r \in \mathbb{Z} \setminus \{0\} \}, \]

with \( s(r) = (s_{ij})_{i,j \in \mathbb{Z}} \) such that \( s_{ij} \neq 0 \iff j - i = r \). Moreover \( b_{s(r)} = \rho_f \pi_r \), see [SV, 3.2]. \( \square \)
3. Kashiwara modules

3.1. Let $V$ be any integrable $U$-module and let $z$ be a formal variable. Set $U[z^\pm] = U \otimes_A \mathbb{A}[z^\pm]$. We denote by $V\{z\}$ be the representation of $U$ on the space $V \otimes \mathbb{A}[z^\pm]$ such that $x^\pm_{ir}$ acts as $x^\pm_{ir} \otimes z$ and $k^\pm_{ir}$ as $k^\pm_{ir} \otimes z^r$ (in particular $e_a$ acts as $e_a \otimes z^{a \cdot r}$ and $f_a$ as $f_a \otimes z^{-a \cdot r}$). Define

$$V^{\otimes d}\{z_1, z_2, ..., z_d\} = V\{z_1\} \otimes_A V\{z_2\} \otimes_A \cdots \otimes_A V\{z_d\}.$$ 

The element $\Lambda \in \tilde{X}$ is a weight of $V$ if the $\Lambda$-weight subspace

$$V^\Lambda = \{v \in V \mid l_a \cdot v = q^{(\Lambda: \varepsilon_a)}v, \forall a = 1, ..., p\}$$

is non-zero. If $v \in V^\Lambda$ we set $\text{wt}(v) = \Lambda$.

3.2. Let $W(\Lambda_0)$, $V(\Lambda_a)$ be the $A$-form of the fundamental $U_A$-module and of Kashiwara’s maximal integrable $U_A$-module, respectively, with highest weight $\Lambda_a$, see [K1], [K2]. Fix highest weight vectors $w_{\Lambda_a} \in W(\Lambda_a)$, $v_{\Lambda_a} \in V(\Lambda_a)$. By [K2, Theorem 5.15(vii)] there exists a $U$-automorphism $z$ of $V(\Lambda_a)$ and an isomorphism of $U[z^\pm]$-modules

$$(3.2.1) \quad V(\Lambda_a) \sim W(\Lambda_a)[z]$$

such that $v_{\Lambda_a} \mapsto w_{\Lambda_a} \otimes 1$. In particular we have the $U[z_1^\pm, ..., z_d^\pm]$-isomorphism

$$V(\Lambda_1)^{\otimes d} \simeq W(\Lambda_1)^{\otimes d}\{z_1, z_2, ..., z_d\}.$$ 

Both modules $V(\Lambda_a)$, $W(\Lambda_a)$ have canonical bases $B(\Lambda_a)$, $B^f(\Lambda_a)$, see [K1, Proposition 8.2.2] and [K2, Theorem 5.15(ii)]. There is a unique $U$-semilinear (with respect to $x \mapsto \overline{x}$) and $\mathbb{Q}[\varepsilon]^\pm$-linear involution on $V(\Lambda_a)$ which fixes $v_{\Lambda_a}$. The elements in $B(\Lambda_a)$ are fixed by this involution. Via (3.2.1), we have

$$(3.2.2) \quad B(\Lambda_a) = \bigsqcup_{n \in \mathbb{Z}} B^f(\Lambda_a) \otimes z^n.$$ 

Let $L(\Lambda_a)$ be the $A_0$-lattice spanned by $B(\Lambda_a)$.

3.3. Let $V$ be the vectorial representation of $U$, i.e. $V$ is the $U[z^\pm]$-module with $A$-basis $\{u_m \mid m \in \mathbb{Z}\}$ such that $u_{m-p} = u_m \cdot z^p$ for all $m, p \in \mathbb{Z}$, and such that $U$ acts as follows ($a = 1, 2, ..., p$)

$$e_a(u_m) = \delta(m \equiv a + 1)u_{m-1}, \quad f_a(u_m) = \delta(m \equiv a)u_{m+1}, \quad l_a(u_m) = q^\delta(m \equiv a)u_m.$$ 

Here again we write $a \equiv b$ for $a - b \in p\mathbb{Z}$. The $U$-module $V$ has a $\tilde{X}^a$-gradation such that $\text{wt}^a(u_{m-p}) = \epsilon_m + l\delta$, for all $m = 1, 2, ..., p$.

**Lemma.**

1. In $V$ we have $h_{1,1}(u_1) = o(1)(-1)^{1-p}q^{-p}u_1 \cdot z$.
2. There is a unique isomorphism of $U[z^\pm]$-modules $V(\Lambda_1) \rightarrow V$ such that $\Lambda_{\Lambda_1} \mapsto u_1$.
3. $B(\Lambda_1) = \{u_m \mid m \in \mathbb{Z}\}$. 


Proof. For Claim 1, recall that \( h_{1,1} = k t^{-1}[x_{1,1}^+, f_1] \). Then use a computation as in [VV1, Theorem 3.3] (but note that we use a different coproduct and different Drinfeld generators). Claim (2) follows from Claim (1) and from [N, Proposition 3.1]. Claim (3) follows from (3.2.2) and standard properties of canonical bases. \( \square \)

From now on we identify \( V(\Lambda_1) \) with \( V \). For \( v = \otimes_i v_i \in V^{\otimes d} \) and \( \gamma = \sum_i m_i e_i \in X \), put \( v \cdot z_{\gamma} = \otimes_i (v_i \cdot z^{m_i}) \). Let \( \{ u_{\gamma} \mid \gamma \in X \} \) be the unique \( A \)-basis of \( V^{\otimes d} \) such that

\[
u_{\gamma} = (\otimes_{i=1}^d u_{m_i}) \cdot z_{\nu}, \quad \gamma, \nu \in X.
\]

Note that, if \( \gamma \in X_p, \nu \in X \), then \( \text{wt}(u_{\gamma-\nu}) = \bar{\gamma} \). The set of weights of \( V^{\otimes d} \) is exactly \( \Omega \). We have a \( \bar{X}^a \)-grading on \( V^{\otimes d} \) such that \( \text{wt}^a(u_{\gamma-\nu}) = \bar{\gamma} + \sum_{i=1}^d n_i \delta \) if \( \nu = \sum_{i=1}^d n_i e_i \).

3.4. Consider the right representation of the \( A \)-algebra \( H \) on \( V^{\otimes d} \) such that for all \( i = 1, 2, \ldots, d-1, \gamma \in X_p, \kappa, \nu \in X \),

\[
(3.4.1) \quad u_{\gamma} \circ t_i = \begin{cases} 
qu_{\gamma} & \text{if } \gamma \cdot s_i = \gamma \\ u_{\gamma-s_i} & \text{if } \gamma \cdot s_i < \gamma \\ u_{\gamma \cdot s_i} + (q - q^{-1})u_{\gamma} & \text{if } \gamma \cdot s_i > \gamma \\ u_{\nu} \circ x_{\kappa} = u_{\nu \cdot x_{\kappa}}. 
\end{cases}
\]

This action commutes with the \( U \)-action, see [VV2] for instance. Set \( T_d = \bigoplus g \rho_g H \) where \( g \) runs the set of the compositions of \( d \) with at most \( p \) parts. It is a left \( S_d \)-module such that \( t \cdot h = \delta(f, g) m_g^{-1} th \in \rho_f H \) for all \( t \in H_{f,f} \) and \( h \in \rho_g H \). Define an involution \( \iota_S \) on \( S_d \) by setting \( \iota_S(a) = q^{2w(a)} \bar{a} \), for all \( a \in H_{f,f} \).

**Lemma.**

1. There is a unique isomorphism of right \( H \)-modules

\[
\phi: V^{\otimes d} \xrightarrow{\sim} T_d, \quad u_{\mu} \mapsto \rho_e, \quad \forall \mu \in \bar{A} \cap X_p.
\]

In particular

\[
(3.4.2) \quad (V^{\otimes d})_\mu \xrightarrow{\sim} \rho_e H.
\]

2. There exists a unique \( A \)-algebra homomorphism \( \Phi_d: \bar{U} \to S_d \) such that \( \phi(uv) = \Phi_d(u) \ast \phi(v) \), for all \( u \in \bar{U}, v \in V^{\otimes d} \). We have \( \iota_S(\Phi_d(u)) = \Phi_d(\mu) \).

3. \( u_{\mu} \circ t_w = u_{\mu \cdot w} \) for all \( \mu \in \bar{A} \cap X \) and \( w \in eW \).

Proof. For Claims (1) and (2) use [VV2, Lemmas 8.3 and 8.4]. Note that the coproduct in loc. cit. is different from our.

Claim (3) is standard, we prove it by induction on the length of \( w \). Set \( w = s_i \).

Since \( \mu \in \bar{A}_+ \), we have \( (\mu : \alpha_{i}^\vee) \geq 0 \), and the equality holds if and only if \( s_i \in W_\alpha \). Then \( \mu \cdot s_i < \mu \) and \( u_{\mu} \circ t_i = u_{\mu \cdot s_i} \). Suppose now that \( \ell(w) = k > 1 \) and write \( w = w's_i \) with \( \ell(w') = k - 1 \). Then \( w' \in eW \). Otherwise there exists \( \sigma \in W_\alpha \) such that \( \ell(\sigma w') < \ell(w') \) and then \( \ell(w) < \ell(\sigma w) = \ell(\sigma w') \pm 1 \leq \ell(w') \), which is impossible, because \( w \in eW \). So, by induction, \( u_{\mu} \circ t_w = u_{\mu \cdot w'} \circ t_i \).

Since \( w'(\alpha_i) \in R^+ \), we have \( (\mu : w': \alpha_i^\vee) = (\mu : w'(\alpha_i^\vee)) \geq 0 \). If \( (\mu : w': \alpha_i^\vee) = 0 \), then \( s_{w'(\alpha_i)} \in W_\alpha \). Thus \( w' = s_{w'(\alpha_i)} w \) is an element of \( W_\alpha w \) such that \( \ell(w') \leq \ell(w) \) which is impossible by assumption. \( \square \)

Note that the map \( \Phi_d \) defined in the lemma above is a renormalization of the map \( \Lambda_d \) in 2.6. It is easy to check that the Claim in section 2.6 still hold for \( \Phi_{f,f'} \), the composition of \( \Phi_d \) with the canonical projection \( S_d \to H_{f,f'} \).
3.5. Taking $d = 1$ in (3.4.1) we get $v_{\Lambda_1} \circ x_1 = v_{\Lambda_1} \cdot z$, because $v_{\Lambda_1} = u_1$. So $x_1$ acts as Kashiwara’s operator $z$ on $V$. The module $W(\Lambda_1)$ is identified with the subspace of $V$ spanned by $u_1, \ldots, u_p$. Then, formulas (3.4.1) endow $W(\Lambda_1)$ with a right $H^J$-action, such that, if $p > d$, $W(\Lambda_1)^\otimes_d \otimes H^J \mathbb{A}^{-}$ is a simple module of the quantum enveloping algebra of $\mathfrak{gl}_p$ with highest weight $\Lambda_d$. Set
\[
V^{[d]} = W(\Lambda_1)^\otimes_{(q)} \{(-q)^{-d}z_1, (-q)^{3-d}, \ldots, (-q)^{d-1}z_d\}.
\]
Take $c = (d)$, then $R_c = \mathbb{A}[\psi(x_1)]$. We can endow $V^{[d]} \otimes H^J \mathbb{A}^{-}$ with a (right)-$R_c$-action by setting
\[
(v \otimes 1) \cdot \psi(x_1) = (v \circ x_1) \otimes 1,
\]
for all $v \in V^{[d]}$. It is well defined by (2.1.1), (2.5.1).

**Proposition.** Assume that $p > d$ and let $\tilde{\lambda} = \Lambda_d$. Then $c = (d)$, $\rho_d = 1$. Set $x = \psi((-q)^{-d}x_1)$.

1. There is a unique isomorphism of $U$-modules
\[
(3.5.1) \quad V(\Lambda_d) \xrightarrow{\sim} V^{[d]} \otimes H^J \mathbb{A}^{-} \quad \text{s.t.} \quad v_{\Lambda_d} \cdot z^i \mapsto (u_d \otimes \cdots u_2 \otimes u_1 \otimes 1) \cdot x_i, \quad \forall i \in \mathbb{Z}.
\]

2. The set of the weights of $V(\Lambda_d)$ is $\Omega_{sm}$. Fix $\tilde{\mu} \in \Omega_{sm}$. The map (3.5.1) takes $B(\Lambda_d)_{\tilde{\mu}}$ to $\{ (u_\mu \otimes 1) \cdot x^i \mid i \in \mathbb{Z} \}$.

**Proof.** Claim (1). There exists $\alpha \in \mathbb{A}^\times$ such that $(W(\Lambda_1)^\otimes_d \otimes H^J \mathbb{A}^{-}) \{\alpha z\} \simeq V(\Lambda_d)$, because $W(\Lambda_1)^\otimes_d \otimes H^J \mathbb{A}^{-}$ and $W(\Lambda_d)$ are both integral $\mathbb{A}$-forms of simple $U_\mathfrak{g}$-modules which are simple and of highest weight $\Lambda_d$ as modules for the quantum enveloping algebra of $\mathfrak{gl}_p$. By (2.1.1) we have $R_c = \mathbb{A}[x^{\pm 1}]$. Consider the $U$-module $W(\Lambda_1)^\otimes_d \{ \alpha z_1, q^a \alpha z_2, \ldots q^{2(d-1)} \alpha z_d \} \otimes H^J \mathbb{A}^{-}$. The element $f_p \in U$ acts as
\[
\sum_{i=1}^d k_i^{-1} \otimes (f_p \circ x_i a q^{2(i-1)} \otimes 1^{d-i}) = \sum_{i=1}^d k_i^{-1} \otimes (f_p \circ \alpha x_1) \otimes 1^{d-i},
\]
by formula (2.5.1). A similar formula holds for $e_p$. We need to compute $\alpha$. Set $A = o(d)(-1)^{p-1}q^{-p}$. Using a computation as in [VV1, Theorem 3.3] we get
\[
h_{d,1}(u_d \otimes \cdots \otimes u_1 \otimes 1) = (-q)^{d-1} A u_d \otimes \cdots \otimes u_1 \otimes x_1 \otimes 1.
\]
Then use [N, Proposition 3.1] to get $\alpha = (-q)^{-d}$.

Claim (2). If $\mu \in (A_+ \setminus A_+) \cap X_p$, then the element $u_\mu \otimes 1 \in V^{[d]} \otimes H^J \mathbb{A}^{-}$ is zero since $qu_\mu \otimes 1 = u_\mu \otimes t_i \otimes 1 = u_\mu \otimes (t_i \cdot 1) = -q^{-1} u_\mu \otimes 1$, for all $i \in \{1, 2, \ldots, d-1\}$ such that $\mu_i = \mu_{i+1}$. Thus $\Omega_{sm}$ is the set of weights of $V(\Lambda_d)$. Since $f_2^2$ acts trivially on $V(\Lambda_d)$, Kashiwara’s operator $f_\alpha$ coincide with $f_a$, for all $a$. Hence $B(\Lambda_d) + qL(\Lambda_d) \ni f_{a_1} \cdots f_{a_r}(v_{\Lambda_d}) = u_\mu$ or $0$ with $\mu \in A_+ \cap X_p$, for any $a_1, \ldots, a_r \neq p$. We are done by (3.2.2).

3.6. Set
\[
V_c = \bigotimes_{i=1}^\ell V(\Lambda_{c_i}) \simeq \bigotimes_{i=1}^\ell W(\Lambda_{c_i})\{z_i\}, \quad v_c = \otimes_{i=1}^\ell v_{\Lambda_{c_i}} \in V_c.
\]
It is convenient to identify the ring \( A[z_1, \ldots, z_{\ell}] \) with the quotient of the polynomial algebra \( A[z_1, \ldots, z_{\ell}] \) by the ideal generated by the relations \( z_{\alpha_i} = 1 \), for all \( \alpha_i \in I_c \). Let \( z_{\gamma+zi_I} \) be the image of \( z_{\gamma} \) in this quotient. Using Proposition 3.5 we get a morphism of \( U \)-modules

(3.6.1) \[
\bigotimes_{i=1}^{\ell} V^{[c_i]} \rightarrow V_c, \quad \text{such that} \quad \otimes_{b=\mu}(\otimes_{a=b} u_a) \otimes e_b \mapsto v_c.
\]

Define a right action of the \( A \)-algebra \( R_c \) on \( \bigotimes_{i=1}^{\ell} V^{[c_i]} \otimes_{H_c} A^- \) by setting

\[
(v \otimes 1) \cdot \psi(x_{\gamma}) = (v \circ x_{\gamma}) \otimes 1,
\]

for all \( v \in \bigotimes_{i=1}^{\ell} V^{[c_i]}, \gamma \in X \). It is well defined by (2.1.1), (2.5.1). Via (3.4.2), we get also, for all \( e \), a right action of \( R_c \) on \( \rho_c H \otimes_{H_c} A^- \).

**Corollary.** Assume that \( p > d \).

1. The map (3.6.1) induces an isomorphism of \( U \)-modules

(3.6.2) \[
V_c \cong \bigotimes_{i=1}^{\ell} V^{[c_i]} \otimes_{H_c} A^-,
\]

such that \( v_c \cdot z_{\gamma+zi_I} \mapsto (\otimes_{b=\mu}(\otimes_{a=b} u_a) \otimes e_b) \cdot \psi((-q)^{-\gamma_{\lambda}} x_{\gamma}) \).

2. By composing (3.6.2) with (3.4.2) we get, for any \( \tilde{\mu} \), a unique isomorphism of \( R_c \)-modules

\[s_{\tilde{\mu}} : V_{c,\tilde{\mu}} \cong \rho_c H \otimes_{H_c} A^-\]

such that \( s_{\tilde{\mu}}(uv) = \Phi_{e,\tilde{\mu}}(u) \cdot s_{\tilde{\mu}}(v) \) if \( v \in V_{c,\tilde{\mu}}, u \in \tilde{U}, uv \in V_{c,\tilde{\mu}}, \) and \( f \) is the partition associated to \( v \) as in (1.4.1).

3.7. Let \( N(\tilde{\lambda}) \subseteq V_c \) be the smallest \( U \times R_c \)-submodule containing the element \( v_c \). For any composition \( f \) of \( d \), let \( f W^c \) be the set of elements \( w \in W^c \) having a maximal length in \( W_T w \). The set \( d W^c \) contains only one element, denoted by \( \sigma_c \), see \([L_5, 13.11(a)]\).

For any \( \mu \in A_+ \cap X_p \), consider the morphism of \( H \times R_c \)-modules

\[a_{\tilde{\mu}} : \rho_c H \otimes_{H_c} A^- \rightarrow \left( \bigotimes_{i=1}^{\ell} V^{[c_i]} \otimes_{H_c} A^- \right)_{\tilde{\mu}}\]

such that \( (\rho_c H \otimes 1) \cdot \psi(x_{\gamma}) \mapsto (u_{\mu} \circ \tilde{h} \otimes 1) \cdot \psi(x_{\gamma}) \).

**Proposition.** Assume that \( p > d \).

1. For any weight \( \tilde{\mu} \) we have the following commutative square of \( R_c \)-modules

\[
\begin{array}{ccc}
H_{c,d} & \cong & R_c \otimes 1 \\
\downarrow^{a_{\tilde{\mu}}} & & \downarrow^{a_{\tilde{\mu}}} \\
N(\tilde{\lambda})_{\tilde{\mu}} & \cong & V_{c,\tilde{\mu}}
\end{array}
\]

The vertical maps are \( a_{\tilde{\mu}} \) and its restriction. If \( \tilde{\mu} = \tilde{\lambda} \) then

\[a_{\tilde{\lambda}}(q^{v_a} \rho_d t_{\sigma_c} \otimes 1) = v_c.\]

2. If \( \tilde{\mu} \in \Omega_{sm} \), the left vertical map is an isomorphism.
4.1. Set $\rho$ such that $V$ is simply transitive. Let $A'$ of $W$. We denote by $V$ the connected components of the open set $A$ chamber having 0 in its closure. Thus, $S$.

Proof. Claim (1). Take $\hat{\mu} = \hat{\lambda}$, then $\bigotimes_{a=p}^1 u_a^{\otimes d_a} = u_\lambda$. Set $\sigma_c = w_d \sigma$, where $\sigma \in ^dW$. Then $t_{\sigma_c} = t_{w_d} t_\sigma$. Using (3.4.1) and Lemma 3.4(3) we get

\[
a_{\lambda}^{-1}(q^{a_d} \rho_d t_{\sigma_c} \otimes 1) = u_\lambda \cdot (q^{-a_d} t_{\sigma_c} \otimes 1) = u_\lambda \cdot t_\sigma \otimes 1 = \bigotimes_{b=p}^1 (\bigotimes_{a=b}^{\otimes d_b}) \otimes t_b \otimes 1.
\]

Moreover

\[
N(\hat{\lambda})_{\hat{\mu}} = \big((U \otimes R_c)v_\lambda\big)_{\hat{\mu}} \overset{s_{p}}{\rightarrow} \Phi_{e,d}(\hat{U}) \ast s_{\lambda}(v_c)R_c.
\]

If $\hat{\mu}$ is small, we have $\Phi_{e,d}(\hat{U}) = H_{e,d} = H \rho_d$, by Claim 2.6 and the equality $\rho_e = 1$. Claim (2) easily follows. $\square$

4. Periodic modules

4.1. Set $V' := Q \otimes Z R$, $F_{c^{\alpha}} = \{ \gamma \in V' | (\alpha : \gamma) = np \}$ for all $\alpha \in R^+$, $n \in Z$. The connected components of the open set $V' \bigcup_{\alpha \in R^+, n \in Z} F_{c^{\alpha}}$ are called alcoves. The connected components of the open set $V' \bigcup_{\alpha \in R^+, n \in Z} F_{c^{\alpha}}$ are called boxes. We denote by $A$ the set of alcoves. Consider the restriction to $V'$ of the right action of $W' \subset W$ on $V$. It induces a right $W'$-action on $A$, denoted by $A \mapsto A \cdot w$, which is simply transitive. Let $A'_+$ be the unique alcove contained in the dominant Weyl chamber having 0 in its closure. Thus, $A'_+ = A_+ \cap V'$. We define a left $W'$-action on $A$ by

\[
w \cdot (A'_+ \cdot v) = A'_+ \cdot wv, \quad \forall v, w \in W'.
\]

We denote by $\leq$ the partial order on $A$ defined in [L1, 1.5].

4.2. We now consider the parabolic case. Fix $c$ as in Convention 1.4. The connected components of the open set $V' \bigcup_{\alpha \in R^+ \cap L_c, n \in Z} F_{c^{\alpha}}$ are called $I_c$-alcoves. Let $A_c$ be the unique $I_c$-alcove containing $A'_+$. Set $A_c \subseteq A$ equal to the set of alcoves contained in $S_c$.

Let $M_c$ be the free $A$-module on $A_c$. There is a unique left action of $H'$ on $M_c$ such that

\[
t_i \cdot A = \begin{cases} -q^{-1}A & \text{if } s_i \cdot A \notin A_c \\ s_i \cdot A & \text{if } s_i \cdot A > A, s_i \cdot A \in A_c \\ s_i \cdot A + (q - q^{-1})A & \text{if } s_i \cdot A < A, s_i \cdot A \in A_c, \end{cases}
\]

for all $A \in A_c$, $i = 1, 2, \ldots, d$, see [L5, 9.3(b)].

By [L5, Lemma 9.6(b)] we have $A'_+ \cdot w = t_w \cdot A'_+$, for all $w \in W_c$. For any $\gamma \in Q$ there is a unique element $w_\gamma \in W_c$ such that $S_c - p\gamma = S_c - w_\gamma$, because $W_c$ acts simply transitively on the set of $I_c$-alcoves. Thus the map $A \mapsto (A - p\gamma) \cdot w_\gamma^{-1}$ permutes the alcoves in $A_c$. Let $g_\gamma : M_c \rightarrow M_c$ be corresponding $A$-linear map. Then

\[
g_\gamma(A'_+) = (-q)^{-(\gamma : c_{\alpha_c})} x_\gamma \cdot A'_+, \quad g_\gamma(h' A'_+) = h' g_\gamma(A'_+), \quad \forall h' \in H',
\]

see [L5, 9.2 and Lemma 9.5]. Consider the unique right $R_c'$-action on $M_c$ such that

\[
A : x_{\gamma + ZI_c} = (-1)^{\gamma : c_{\alpha_c}} g_\gamma(A), \quad \forall A \in A_c, \forall \gamma \in Q
\]

see [L5, 9.7].
4.3. The element
\[ m_c = q^{-\nu_d} \sum_{w \in W_d} q^{\ell(w)} A'_+ \cdot w \sigma_c, \]
belongs to \( M_c \) because \( W_d \sigma_c \subset W_c \), see [L5, Lemma 13.9.(b)]. Let \( M'_c \subset M_c \) be the \( H' \otimes R'_c \)-submodule generated by \( m_c \). Since, for all \( w \in W_d \), \( \ell(\sigma_c) = \ell(w \sigma_c) + \ell(w) \) we get
\[ m_c = q^{-\nu_d} \sum_{w \in W_d} q^{\ell(w)} \bar{t}_{w \sigma_c} A'_+ = q^{-\nu_d} \rho_d \bar{t}_{\sigma_c} : A'_+. \]
The right \( R_c \)-action on \( H \otimes H_c \ A^- \) defined in 3.6 restricts to a right \( R'_c \)-action on \( H' \otimes H'_c \ A^- \).

**Lemma.**

1. \( \{g_\gamma(A'_+ \cdot w) | \gamma \in Q, w \in W_c \} = A_c. \)
2. The linear map
\[ b : M_c \to H' \otimes H'_c \ A^- \quad (A'_+ \cdot w) \cdot x_{\gamma + z I_c} \mapsto (\bar{t}_w \otimes 1) \cdot \psi(q^{-(\gamma \cdot \alpha_c)} x_\gamma), \]
where \( w \in W_c, \gamma \in Q \), is a morphism of \( H' \times R'_c \)-modules such that \( b(m_c) = q^{-\nu_d} \rho_d \bar{t}_{\sigma_c} \otimes 1. \)
3. The map \( b \) yields a commutative square of \( H' \times R'_c \)-modules
\[
\begin{array}{ccc}
H' \rho_d \bar{t}_{\sigma_c} R'_c \otimes 1 & \leftrightarrow & H' \otimes H'_c \ A^-\\
M'_c & \leftrightarrow & M_c
\end{array}
\]
with invertible vertical maps.

**Proof.** Claim (1) follows from [L3, 2.12(f), 4.9(c)] and the identity \( w_d W_c w_c = W_c \).

As for Claims (2) and (3), use [L5, 9.7(b) and Proposition 8.5, Lemma 10.2].

5. **Comparing canonical bases**

5.1. Let \( M_{c, \leq} \) be the set of formal \( \lambda \)-linear combinations \( m = \sum_{A \in A_c} m_A A \) such that the set \( \{A | m_A \neq 0\} \) is bounded above under \( \leq \), i.e. there exists \( B \in A_c \) such that \( A < B \) for all \( A \) such that \( m_A \neq 0 \). The \( H' \times R'_c \)-action on \( M_c \) extends in an obvious way to \( M_{c, \leq} \).

There is a unique continuous (in the sense of [L3, 4.13]) involution \( \iota_M : M_{c, \leq} \to M_{c, \leq} \) such that, for all \( m \in M_{c, \leq} \), \( h \in H', x_{\gamma + z I_c} \in R'_c \),

(5.1.1) \[ \iota_M(m_c) = m_c, \quad \iota_M(hm) = \overline{h} \iota_M(m), \quad \iota_M(m \cdot x_{\gamma + z I_c}) = \iota_M(m) \cdot x_{\gamma + z I_c}, \]

see [L5, Lemmas 9.16.(a) and 13.12].

For all \( A \in A_c \) there is a unique element \( A_{\leq} = \sum_{B \leq A} \pi_B A B \in M_{c, \leq} \) such that
\[ \iota_M(A_{\leq}) = A_{\leq}, \quad \pi_{AA} = 1 \text{ and } \pi_{BA} \in q^{-1} Z[q^{-1}] \text{ for all } B < A, \]
see [L5,9.17(a),(b)]. We have \( m_c = (A'_+ \cdot w_d \sigma_c)_{\leq} \) by [L5, Theorem 13.13(a)]. Set \( B_M = \{A_{\leq} | A \in A_c \}. \)

**Conjecture** [L5, 12.7]. \( B_M \subset M_c \).

For a future use let us recall the following standard fact.
Lemma. Let $A^t = A + \sum_{A' \neq A} m_{A'} A' \in M_c$, with $m_{A'} \in q^{-1}Q[q^{-1}]$. If $\iota_M(A^t) = A^t$, then $A^t \in B_M$.

Proof. For all $B \in \mathcal{A}_c$ we have $B = B_\ell + \sum_{\ell' < \ell} c_{B'} B'_\ell$ in $M_{c, \leq}$, with $c_{B'} \in q^{-1}Q[q^{-1}]$. Thus $A^t = \sum_{\ell} n_B B_\ell$, with $n_B \in q^{-1}Q[q^{-1}]$ for all $B \neq A$, and $n_A \in 1 + q^{-1}Q[q^{-1}]$. Since $\iota_M(A^t) = A^t$, we get $\pi_B = n_B$ for all $B$. Thus $A^t = A_\leq$.

5.2. Following [K2, Theorem 8.5, Proposition 8.6], the $\Lambda$-module $N(\tilde{\lambda})$ is endowed with a canonical basis. More precisely, define

$$N_{c, \leq} = V_c \otimes Q[z_{\alpha_i + z_{Ic}} | \alpha_i \in I] \otimes Q[[z_{\alpha_i + z_{Ic}} | \alpha_i \in I]],$$

$$L_{c, \leq} = (\otimes_{\ell} L(\Lambda_{c_i}) \otimes Q[z_{\alpha_i + z_{Ic}} | \alpha_i \in I] \otimes Q[[z_{\alpha_i + z_{Ic}} | \alpha_i \in I]],$$

There is a unique involution $\iota_N : N_{c, \leq} \to N_{c, \leq}$ such that, for all $v \in N_{c, \leq}, u \in U, f \in Q[[z_{\alpha_i + z_{Ic}} | \alpha_i \in I] \otimes Q[z_{\gamma} | \gamma \in X]$,

$$(5.2.1) \quad \iota_N(v_c) = v_c, \quad \iota_N(uv) = \overline{u} \iota_N(v), \quad \iota_N(v \cdot f) = \iota_N(v) \cdot f.$$ For any $t \in \bigotimes_{i=1}^{\ell} B(\Lambda_{c_i})$ there is a unique element $F(t) \in N_{c, \leq}$ such that

$$(5.2.2) \quad F(t) - t \in \sum_{\ell' < \ell} qQ[q]^{t'}.$$ The $\Lambda$-module $V(\Lambda_{c_i})$ has a $\tilde{X}_c$-gradation induced by the $\tilde{X}_c$-gradation of $V^\otimes d$, using Proposition 3.5. Let $wt^a(v) \in \tilde{X}_c$ be the degree of the element $v \in V(\Lambda_{c_i})$. There is a unique partial order, $\leq_c$, on $\bigotimes_i B(\Lambda_{c_i})$ such that $t' = t'_1 \otimes \cdots \otimes t'_\ell <_c t = t_1 \otimes \cdots t_\ell$ if and only if

$$\sum_{i=1}^{\ell} \text{wt}^a(t'_i) - \sum_{i=1}^{\ell} \text{wt}^a(t_i) \in \tilde{Q}_+ \setminus \{0\}, \quad \forall m = 1, \ldots, \ell - 1,$$

where $Q_+ = \sum_{a=1}^{\rho} N_{\beta_a}$. Then

$$F(t) - t \in \sum_{\ell' < \ell} qQ[q]^{t'}.$$ The U $\times R_c$-submodule $N(\tilde{\lambda})$ of $N_{c, \leq}$ is stable by $\iota_N$. Furthermore $F(t) \in N(\tilde{\lambda})$. The family $B_N = \{F(t) | t \in \bigotimes_i B(\Lambda_{c_i})\}$ is the canonical basis of $N(\tilde{\lambda})$.

5.3. The basis $\bigotimes_{i=1}^{\ell} B(\Lambda_{c_i})$ of $V_c$ is identified with its image in $\bigotimes_{i=1}^{\ell} V[\gamma] \otimes H_c \Lambda^-$ by (3.6.2).

Claim 1. $\bigotimes_{i=1}^{\ell} B(\Lambda_{c_i}) = \{(u_{\mu} \cdot 1) \cdot z_{\gamma + z_{Ic}} | \mu \in A_+ \cap X_p, w \in W^c, \gamma \in X\}$.

Proof. Given $\mu \in A_+ \cap X_p, w \in W^f$, write $\mu \cdot w = \sum_{j=1}^{d} a_j \epsilon_j$. Then $w \in W^c$ if and only if $p > a_k - a_l > 0$ for all $c_1 + \cdots + c_i \geq k > \ell \geq c_1 + \cdots c_i - 1 + 1$. The claim follows from Proposition 3.5(2). Assume that $p > d$, then $\Omega_{sm} \neq \emptyset$. Take $\tilde{\mu} \in \Omega_{sm}$. The map $d_{\tilde{\mu}} = a_{\tilde{\mu}} \circ b$ gives an injective morphism of $H^c \times R_c$-modules

$$d_{\tilde{\mu}} : M_c \to \bigotimes_{i=\ell}^{1} V[\gamma] \otimes H_c \Lambda^-,$$

$$(A'_+ \cdot w) \cdot x_{\gamma + z_{Ic}} \mapsto (-1)^{(\gamma, \alpha_c)}(u_{\mu} \cdot t_w \otimes 1) \cdot z_{\gamma + z_{Ic}},$$

for all $w \in W^c, \gamma \in Q$. Moreover $d_{\tilde{\mu}}$ maps $M_c$ into $N(\tilde{\lambda})_{\tilde{\mu}}$ by Lemma 4.3(3) and Proposition 3.7(1).
Claim 2. $d_{\tilde{\mu}}(A_c) \subset \bigotimes_{i=1}^{\ell} B(\Lambda_{c,i})$.

Proof. By Lemma 4.3(1) any alcove in $A_c$ is of the type $A = g_{\gamma}(A'_+ \cdot w)$ for some \( \gamma \in Q \), \( w \in W^c \). Thus $d_{\mu}(A) = (u_{\mu} \circ t_w \otimes 1) \cdot z_{\gamma + zL_c}$ and Claim 2 follows from Claim 1 and Lemma 3.4(3) (remember that $\mu \in A_+ \cap X_p$). \( \square \)

5.4. Set $X' = (\tilde{A}_+ \cap X_p) \cdot W'$.

Lemma. There exists a family $B'_{N,c} \subset B_N$ such that the $\Lambda$-span of $B'_{N,c}$ is equal to the intersection of $N(\tilde{\lambda})$ with the $\Lambda$-span of \( \{ u_\gamma \otimes 1 \mid \gamma \in X' \} \).

Proof. First of all, $wt^a(u_{\gamma_1}) = wt^a(u_{\gamma_2})$ iff $\gamma_1 \in \gamma_2 \cdot W'$, for any $\gamma_1, \gamma_2 \in X$. Namely, write $\gamma_i = \nu_i - p\mu_i$ with $\nu_i \in X_p, \mu_i \in X$. Then $wt^a(u_{\gamma_1}) = wt^a(u_{\gamma_2})$ iff $\mu_1 - \mu_2 \in Q$ and $\nu_1 = \nu_2$ i.e. $\nu_1 = \nu_2 \cdot w$ for some $w \in W'$, hence iff $\gamma_1 = \gamma_2 \cdot w\gamma_1 - w\gamma_2$ and $\mu_1 - \mu_2 \cdot w \in Q$.

Given $\nu, \gamma \in X$ such that $u_\nu \otimes 1, u_\gamma \otimes 1 \in q^2 \bigotimes_{i=1}^{\ell} B(A_{c_i})$, we write $u_\nu \otimes 1 <_c u_\gamma \otimes 1$ if the corresponding elements in $\bigotimes_{i=1}^{\ell} B(A_{c_i})$ satisfy the same relation. Then $u_\nu \otimes 1 < u_\gamma \otimes 1$ implies $wt^a(u_\nu) = wt^a(u_\gamma)$, hence $\nu \in \gamma \cdot W'$. For any $\gamma \in X$, set $F(u_\gamma \otimes 1) = F(q^a u_\gamma \otimes 1)$ if $q^{a} u_\gamma \otimes 1 \in \bigotimes_{i=1}^{\ell} B(A_{c_i})$. We have $F(u_\gamma \otimes 1) \in \bigoplus_{w \in W'} A u_{\gamma,w} \otimes 1$ by (5.2.2). Thus, for all $\gamma$,

$$\bigoplus_{w \in W'} A u_{\gamma,w} \otimes 1 = \bigoplus_{w \in W'} A F(u_{\gamma,w} \otimes 1).$$

Since $N(\tilde{\lambda})$ is spanned by the elements $F(t)$ with $t \in \bigotimes_{i=1}^{\ell} B(A_{c_i})$ we are done because $X' \cdot W' \subseteq X'$.

Fix $p > d$ and $\tilde{\mu} \in \Omega_{sm}$. Set $B'_{N,\tilde{\mu}} = B'_{N,c} \cap N(\tilde{\lambda})_{\tilde{\mu}}$, and let $N'((\tilde{\lambda})_{\tilde{\mu}})$ be the $\Lambda$-span of $B'_{N,\tilde{\mu}}$. The $\Lambda$-span of $\{ u_\mu \mid \mu \in X' \}$ in $V^{\otimes d}$ is identified to $\bigoplus_{g} \rho_g H'$, see (3.4.1) and Lemma 3.4(1). We have $a_{\tilde{\mu}}^{-1}(N'(\tilde{\lambda})_{\tilde{\mu}}) = H' \rho_d \tilde{\tau}_c R_c \otimes 1$ since $a_{\tilde{\mu}}^{-1}(N'(\tilde{\lambda})_{\tilde{\mu}}) = (\rho_e H' \otimes H_k A^{-}) \cap (H \rho_d \tilde{\tau}_c R_c \otimes 1)$. Hence $N'(\tilde{\lambda})_{\tilde{\mu}}$ is a right $R_c'$-module by the lemma above and (3.4.1), and a left $H'$-module. We have $d_{\tilde{\mu}}(M'_{c}) = a_{\tilde{\mu}}(H' \rho_d \tilde{\tau}_c R_c' \otimes 1) = N'(\tilde{\lambda})_{\tilde{\mu}}$. Let $c_{\tilde{\mu}}$ be the inverse map $c_{\tilde{\mu}} : N'(\tilde{\lambda})_{\tilde{\mu}} \to M'_{c}$. It is an isomorphism of $H' \otimes R_c'$-modules.

5.5. We can now prove Conjecture 5.1.

Theorem. $B_M$ is a $\Lambda$-basis of $M'_{c}$ for all $p > d$.

Proof. Let $\tilde{\mu} \in \Omega_{sm}$. Set $h = q^{-v_a} \rho_d \tilde{\tau}_c$. We consider $h$ as an element of $\rho_e H \subset T_d$. Define $v_{c,\tilde{\mu}} = a_{\tilde{\mu}}(h \otimes 1)$. Then

$$v_{c,\tilde{\mu}} = u_\mu \otimes \tilde{t} \otimes 1 = u_\mu \otimes q^{-v_a} \rho_d t_c \otimes 1 = d_{\tilde{\mu}}(m_c) \in N'(\tilde{\lambda})_{\tilde{\mu}}.$$ 

The element $v_c = a_{\tilde{\mu}}(q^{v_a} \rho_d \tilde{\tau}_c \otimes 1) = u_\lambda \otimes q^{-v_a} t_c \otimes 1 \in N'(\tilde{\lambda})_{\tilde{\mu}}$ is fixed by $\iota_N$. Hence $v_N(v_c,\tilde{\mu}) = v_{c,\tilde{\mu}}$, because $v_{c,\tilde{\mu}} = \rho_d \ast v_c$ and $\iota_N(\rho_d) = \rho_d$.

For any $v \in N'(\tilde{\lambda})_{\tilde{\mu}}$ there exists $u \in U$ and $r \in R'_{c}$ such that $v = uv_{c,\tilde{\mu}} \cdot r$, since $\Phi_{e,d}(U) \ast (q^{-v_a} \rho_d t_c \otimes 1) \subseteq \Phi_{e,e}(U) \ast (h \otimes 1)$, by Claim 2.6. We want to prove that $c_{\tilde{\mu}}$ is compatible with the involutions $\iota_N, \iota_M$. By (5.1.1), (5.2.1) and the $R_c'$-linearity of $c_{\tilde{\mu}}$, we can forget $r$. We have $c_{\tilde{\mu}}(uv_{c,\tilde{\mu}}) = \Phi_{e,e}(u) \cdot m_c$. Namely, set $t = \Phi_{e,e}(u)$. Then $a_{\tilde{\mu}}(uv_{c,\tilde{\mu}}) = a_{\tilde{\mu}}(t \ast (u_\mu \otimes \tilde{t} \otimes 1)) = a_{\tilde{\mu}}(u_\mu \otimes t \tilde{h} \otimes 1) = \tilde{t} h = b(\tilde{t} \cdot m_c)$. 


Using 5.1.1, 5.2.1, and Lemma 3.4(2) we get

\[
c_{\tilde{\mu}}(\iota_N(uw_{e,\tilde{\mu}})) = \frac{c_{\tilde{\mu}}(uw_{e,\tilde{\mu}})}{\Phi_{e,e}(\pi) \cdot m_e}
\]
\[
= \frac{\iota_S(\Phi_{e,e}(u)) \cdot m_e}{\Phi_{e,e}(u) \cdot m_e}
\]
\[
= \iota_M(\Phi_{e,e}(u) \cdot m_e)
\]
\[
= \iota_M(c_{\tilde{\mu}}(uw_{e,\tilde{\mu}})).
\]

Hence any element in \( c_{\tilde{\mu}}(B'_{N,\tilde{\mu}}) \) is fixed by \( \iota_M \). Claim 2 in 5.3 gives

\[
c_{\tilde{\mu}}(N'(\lambda)_{\tilde{\mu}} \cap L_{e,\leq}) \subseteq \sum_{A \in A_e} \mathbb{Q}[q^{-1}]A.
\]

Thus, by Lemma 5.1, \( c_{\tilde{\mu}}(B'_{N,\tilde{\mu}}) \subseteq B_M \). We are done since \( c_{\tilde{\mu}} \) is invertible. \( \square \)

**Remarks.**

1. As a corollary of Theorem 5.5 and [L5, 15.13], the set \( B'_{B_e} \) in [L5, 5.11] is a signed basis of the K-theory of the Springer fiber.

2. Even in the \( A_2 \) case the two orders are not comparable via \( d_{\tilde{\mu}} \). Fix \( d = 3, p > 3, e = (1^3) \), and \( \mu = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3 \). Then \( d_{\tilde{\mu}}(A_+^\prime) = u_\mu, d_{\tilde{\mu}}(A_+^\prime \cdot w) = u_{\mu \cdot w} \), for any \( w \in W^I \).

   - If \( w = s_2 \), we have that \( A_+^\prime \cdot w < A_+^\prime \), but \( u_\mu \not< u_{\mu \cdot w} \).

   - If \( w = s_{\theta_0} \tau_{-\alpha_1-2\alpha_2} \), we have \( u_\mu < u_{\mu \cdot w} \) but \( A_+^\prime \not< A_+^\prime \cdot w \), since \( A_+^\prime \cdot w \) is not in the Weyl chamber.

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