Gap theorems in Yang-Mills theory for complete four-dimensional manifolds with a weighted Poincaré inequality

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Abstract

In this paper we prove gap theorems in Yang-Mills theory for complete four-dimensional manifolds with a weighted Poincaré inequality. We apply the theorems to many examples of manifolds. We also prove a uniqueness theorem for the basic instanton.

1 Introduction

Suppose we have a Riemannian vector bundle with structure group $G \subset O(N)$ on a Riemannian manifold $X$ and a metric connection $A$ on the bundle with curvature $F$. The connection $A$ is called a Yang-Mills connection if the divergence of the curvature $F$ is zero, which is the Euler-Lagrange equation of the Yang-Mills functional $YM(A) = \int_X |F|^2$. When the manifold is four-dimensional, the connection $A$ is called an instanton if the self-dual curvature $F^+$ is zero or the anti-self-dual curvature $F^-$ is zero, where $F^\pm = 1/2 (F \pm *F)$. Any instanton is a Yang-Mills connection. In [2] and [3] Atiyah-Drinfeld-Hitchin-Manin and Belavin-Polyakov-Schwartz-Tyupkin found an explicit $SU(2)$ instanton, known as the basic instanton. On the other hand, there are Yang-Mills connections that are not instantons. In [27] Sibner-Sibner-Uhlenbeck proved the existence of $SU(2)$ Yang-Mills connections that are not instantons (see also Bor [5], Parker [23] and Sadun-Segert [25]). For larger structure groups such as $SU(4)$ we can easily find Yang-Mills connections that are not instantons (see for example Section 2 in [28]).

By a gap theorem for the self-dual curvature $F^+$ of a Yang-Mills connection we mean the following: under certain assumptions on the four-dimensional manifold, if the self-dual curvature $F^+$ of a Yang-Mills connection has a sufficiently small norm (for example the $L^2$ norm or the $L^\infty$ norm), then the self-dual curvature $F^+$ is zero. In [6] and [7] Bourguignon-Lawson-Simons proved a $L^\infty$ gap theorem for the self-dual curvature $F^+$ of a Yang-Mills connection on compact four-dimensional manifolds with a certain positive curvature (for the extension to complete manifolds see Shen [26]). In [21] MinOo proved a $L^2$ gap theorem for the self-dual curvature $F^+$ of a Yang-Mills connection on these manifolds.
(see also Parker [22], and for the extension to complete manifolds see Dodziuk-MinOo [10]). In [12]Feehan proved a $L^2$ gap theorem for the self-dual curvature $F^+$ of a Yang-Mills connection on compact four-dimensional manifolds with a good metric. In [10]Gursky-Kelleher-Streets proved a $L^2$ gap theorem for the self-dual curvature $F^+$ of a Yang-Mills connection on compact four-dimensional manifolds with positive Yamabe invariant. A result of this type for general compact four-dimensional manifolds is still an open problem. In [30]we extended the gap theorem of Gursky-Kelleher-Streets to complete manifolds and we also described the equality in the gap theorem in terms of the basic instanton, which is interesting even for compact manifolds. By a gap theorem for the full curvature $F$ of a Yang-Mills connection we mean the following: under certain assumptions on the $n$-dimensional manifold, if the curvature $F$ of a Yang-Mills connection has a sufficiently small norm (for example the $L^{n/2}$ norm or $L^\infty$ norm), then the curvature $F$ is zero. See for example Price [24], Gerhardt [15], Zhou [31] and [32], Feehan [13] and [14], and the above references. When the manifold is four-dimensional, gap theorems for the self-dual curvature $F^+$ and the anti-self-dual curvature $F^-$ are more important than gap theorems for the full curvature $F$. In this paper we extend the gap theorem of Bourguignon-Lawson-Simons to complete four-dimensional manifolds with a weighted Poincaré inequality.

Suppose we have a continuous function $q$ on a complete Riemannian manifold $X$. The manifold $X$ is said to satisfy a weighted Poincaré inequality with a weight function $q$ if the following inequality is valid:

$$\int_X q \phi^2 \leq \int_X |\nabla \phi|^2 \tag{1}$$

for all smooth functions $\phi$ on the manifold $X$ with compact support. In [8] and [20] Carron and Li-Wang applied weighted Poincaré inequalities to Riemannian manifolds and submanifolds. In Section 2 we give many examples of manifolds with a weighted Poincaré inequality (the Euclidean space, the hyperbolic space, the complex hyperbolic space, manifolds with positive spectrum, stable minimal hypersurfaces in a general Riemannian manifold and minimal hypersurfaces in the Euclidean space) and we also give more references on this subject.

In this paper we prove two $L^\infty$ type gap theorems for the self-dual curvature $F^+$ of a Yang-Mills connection on complete four-dimensional manifolds with a weighted Poincaré inequality. The theorems have different assumptions and different applications.

We denote the distance function by $r(\cdot) = \text{dist}(\cdot, x_0)$. We denote the open ball by $B_R = \{r < R\}$. We denote by $\lambda_{\text{max}}(W^+)$ the largest eigenvalue of the self-dual Weyl curvature $W^+ = 1/2 (W + *W)$. We denote by $\gamma$ a constant depending only on the structure group $G$ (see Lemma [11]).

In the next result, assuming that the manifold has volume growth $\text{vol}(B_R) = O(R^p)$ and the self-dual curvature $F^+$ has growth $|F^+| = O(r^{2-p})$, we show a $L^\infty$ type gap theorem for the self-dual curvature $F^+$ without any assumption on its integral. This is very different from previous papers.
**Theorem 1.** Suppose a complete four-dimensional Riemannian manifold $X$, with scalar curvature $S$ and Weyl curvature $W$, satisfies the weighted Poincaré inequality (1) with a weight function $q$ and has volume growth $\text{vol}(B_R) = O(R^p)$ for some constant $p > 0$. Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G \subset O(N)$ ($N \geq 3$) on the manifold $X$, suppose the self-dual curvature $F^+$ has growth $|F^+| = O(r^{2-p})$ and satisfies the pointwise inequality

$$|F^+| \leq (1/\gamma) \left( 2q + (1/3) S - 2\lambda_{\text{max}}(W^+) \right).$$

Then either inequality (3) is an equality (everywhere) or the self-dual curvature $F^+$ is identically zero.

Of course we also assume that the right hand side of inequality (3) is non-negative and not identically zero.

In the next result, assuming that the weight function $q$ has growth $q = O(r^2)$ and the self-dual curvature $F^+$ is in the space $L^{2p}$, we show a $L^\infty$ type gap theorem for the self-dual curvature $F^+$ without any assumption on the volume of the manifold.

**Theorem 2.** Suppose a complete four-dimensional Riemannian manifold $X$, with scalar curvature $S$ and Weyl curvature $W$, satisfies the weighted Poincaré inequality (1) with a weight function $q$ that has growth $q = O(r^2)$. Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G \subset O(N)$ ($N \geq 3$) on the manifold $X$, suppose the self-dual curvature $F^+$ is in the space $L^{2p}$ for some constant $p > 1/4$ and satisfies the pointwise inequality

$$|F^+| \leq (1/\gamma) \left( (4p - 1)/(2p^2) q + (1/3) S - 2\lambda_{\text{max}}(W^+) \right).$$

Then either inequality (3) is an equality (everywhere) or the self-dual curvature $F^+$ is identically zero.

Of course we also assume that the right hand side of inequality (3) is non-negative and not identically zero.

The above theorems are valid by changing “self-dual”, $F^+$ and $W^+$ to “antiself-dual”, $F^-$ and $W^-$, respectively.

In [7] Bourguignon-Lawson proved a $L^\infty$ gap theorem for $F^\pm$ on the sphere $S^4$. We improve this result by a factor $\sqrt{2}$ for the structure group $G = SO(3)$. They also proved a $L^\infty$ gap theorem for $F^-$ on the complex projective space $CP^2$ (Theorem 5.26 in [7]). We improve this result by a factor $\sqrt{2}$ (resp. 2) for the structure group $G \subset SO(N)$ ($N \geq 4$) (resp. $G = SO(3)$). In [10] Dodziuk-MinOo proved a $L^2$ gap theorem for $F^\pm$ on the Euclidean space $R^4$. We show a $L^\infty$ type gap theorem for $F^\pm$ on the Euclidean space $R^3$ without any assumption on its integral. In [24] Price proved a $L^2$ type gap theorem for the full curvature $F$ on the hyperbolic space $H^n$ of dimension $n \geq 5$. In [32] Zhou proved a $L^{n/2}$ type gap theorem for the full curvature $F$ on the hyperbolic space $H^n$ of dimension $n \geq 16$. We show a $L^\infty$ type gap theorem for $F^\pm$ on the hyperbolic space $H^4$. We also show a $L^\infty$ type gap theorem for $F^-$ on the complex hyperbolic space $CH^2$ and a $L^\infty$ gap theorem for $F^\pm$ on the
cylinder $S^3 \times R$ without any assumption on its integral. Applying Theorem 1 to the manifolds with nonnegative curvature, and using the weighted Poincaré inequalities of Section 2, we get:

**Corollary 3.** Fix a Lie group $G \subset O(N)$ ($N \geq 3$). Then:

(i) Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G$ on the sphere $S^4$ (with sectional curvature 1), if $F^\pm$ satisfies the inequality $|F^\pm|_{L^\infty} \leq 4/\gamma$, then either $|F^\pm| = 4/\gamma$ (everywhere) or $F^\pm$ is identically zero.

(ii) Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G$ on the Euclidean space $R^4$, if $F^-$ satisfies the inequality $|F^-|_{L^\infty} \leq 8/\gamma$, then either $|F^-| = 8/\gamma$ (everywhere) or $F^-$ is identically zero.

(iii) Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G$ on the complex projective space $CP^2$ (with holomorphic sectional curvature 4), if $F^\pm$ satisfies the inequality $|F^\pm|_{L^\infty} \leq 2/\gamma$, then either $|F^\pm| = 2/\gamma$ (everywhere) or $F^\pm$ is identically zero.

(iv) Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G$ on the complex hyperbolic space $H^4$ (with sectional curvature $-1$), if $F^\pm$ is in the space $L^{2p}$ for some constant $3/8 \leq p \leq 3/4$ and satisfies the pointwise inequality

$$|F^\pm| \leq (1/\gamma) \left\{ (4p - 1) / (2p^2) \right\} \left( 9/4 + (1/4) r^{-2} + (3/4) (\sinh r)^{-2} \right) - 4,$$

then $F^\pm$ is identically zero.

(v) Given a Yang-Mills connection $A$ with curvature $F$ and structure group $G$ on the complex hyperbolic space $CH^2$ (with holomorphic sectional curvature $-4$), if $F^-$ is in the space $L^1$ and satisfies the pointwise inequality

$$|F^-| \leq (2/\gamma) \left( (1/4) r^{-2} + (\sinh r)^{-2} - (\sinh (2r))^{-2} \right),$$

then $F^-$ is identically zero.

In [7] Bourguignon-Lawson proved that given a non-flat Yang-Mills connection on the sphere $S^4$ with structure group $G \subset SO(4)$, if its curvature $F$ satisfies the inequality $|F|_{L^\infty} \leq 4/\gamma$, then the bundle is one of the four-dimensional bundles of tangent spinors with the canonical Riemannian connection. Note that $SU(2) \subset SO(4)$. When the structure group is exactly $G = SU(2)$, we show that this inequality characterizes the basic instanton.

**Theorem 4.** Given a non-flat $SU(2)$ Yang-Mills connection $A$ on the sphere $S^4$ (with sectional curvature 1), if its curvature $F$ satisfies the inequality $|F|_{L^\infty(S^4)} \leq 4/\gamma$, then the connection $A$ corresponds to the basic instanton on the Euclidean space $R^4$ under a stereographic projection.
In particular, we can characterize the basic instanton as the unique corresponding \( SU(2) \) instanton on the sphere \( S^4 \) whose curvature \( F \) satisfies the equation \( |F|_{L^\infty(S^4)} = 4/\gamma \). On the other hand, we cannot characterize the basic instanton as the unique corresponding \( SU(2) \) instanton on the sphere \( S^4 \) whose curvature \( F \) satisfies the equation \( |F|_{L^2(S^4)} = (4/\gamma) \text{vol}(S^4)^{1/2} \). In fact this equation is satisfied by the curvature of any connection in the five-parameter family of \( SU(2) \) instantons on the sphere \( S^4 \) with charge 1 (see for example the proof of Theorem\( \ref{thm:charge} \)). In other words, we cannot characterize the basic instanton in terms of the \( L^2 \) norm, but surprisingly we can characterize it in terms of the \( L^\infty \) norm.

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2 Weighted Poincaré inequalities

First we discuss the weighted Poincaré inequalities that are important for Corollary\( \ref{cor:weighted} \).

The weighted Poincaré inequality of the next lemma is known as Hardy’s inequality. Substituting the weight function \( q \) of the next lemma into Theorem\( \ref{thm:poincare} \), we get the result for the Euclidean space \( \mathbb{R}^4 \) in Corollary\( \ref{cor:weighted} \).

**Lemma 5.** The Euclidean space \( \mathbb{R}^n \) satisfies a weighted Poincaré inequality with the weight function \( q = ((n-2)/2)^2 r^{-2} \), where \( r(x) = |x| \).

The bottom of the spectrum of the Laplacian of the hyperbolic space \( H^n \) is \( \lambda_1(\Delta_{H^n}) = ((n-1)/2)^2 \). Substituting the weight function \( q = \lambda_1(\Delta_{H^n}) \) into Theorem\( \ref{thm:poincare} \), we can get a result for the hyperbolic space \( H^4 \). Here we use a weight function that improves the bottom of the spectrum of the Laplacian and leads to a slightly better result (note that the weight function \( q \) of the next lemma satisfies \( q > \lambda_1(\Delta_{H^n}) \)). In \[1\] Akutagawa-Kumura proved the weighted Poincaré inequality of the next lemma, and in \[4\] Berchio-Ganguly-Grillo proved that the inequality is sharp. Substituting the weight function \( q \) of the next lemma into Theorem\( \ref{thm:poincare} \) using the fact that the hyperbolic space \( H^4 \) has scalar curvature \( S = -12 \) and Weyl curvature \( W = 0 \), and using the fact that for \( 3/8 \leq p \leq 3/4 \) we have \( ((4p-1)/(2p^2)) q - 4 > 0 \), we get the result for the hyperbolic space \( H^4 \) in Corollary\( \ref{cor:weighted} \).

**Lemma 6.** The hyperbolic space \( H^n \) (with sectional curvature \(-1\)) satisfies a weighted Poincaré inequality with the weight function

\[
q = ((n-1)/2)^2 + (1/4) r^{-2} + \{(n-1)(n-3)/4\} (\sinh r)^{-2},
\]

where \( r(\cdot) = \text{dist}_{H^n}(\cdot, x_0) \).

The bottom of the spectrum of the Laplacian of the complex hyperbolic space \( CH^n \) is \( \lambda_1(\Delta_{CH^n}) = n^2 \) (see for example Li-Wang \[19\]). Substituting the weight function \( q = \lambda_1(\Delta_{CH^n}) \) into Theorem\( \ref{thm:poincare} \), we cannot get a result for the
complex hyperbolic space $\mathbb{CH}^2$. To overcome this difficulty, using a method of Akutagawa-Kumura [1], we find a weighted Poincaré inequality that improves the bottom of the spectrum of the Laplacian (note that the weight function $q$ of the next lemma satisfies $q > \lambda_1(\Delta_{\mathbb{CH}^n})$). The method in [1] led to a sharp weighted Poincaré inequality in the hyperbolic space (see the discussion above Lemma [3]), so we would like to know if this method also leads to a sharp weighted Poincaré inequality in the complex hyperbolic space. Substituting the weight function $q$ of the next lemma and $p = 1/2$ into Theorem [2] and using the fact that the complex hyperbolic space $\mathbb{CH}^2$ has scalar curvature $S = -24$ and Weyl curvature $W^{-} = 0$, we get the result for the complex hyperbolic space $\mathbb{CH}^2$ in Corollary [3].

**Lemma 7.** The complex hyperbolic space $\mathbb{CH}^n$ (with holomorphic sectional curvature $-4$) satisfies a weighted Poincaré inequality with the weight function

$$q = n^2 + (1/4) r^{-2} + (n - 1)^2 (\sinh r)^{-2} - (\sinh (2r))^{-2},$$

where $r(\cdot) = \text{dist}_{\mathbb{CH}^n}(\cdot, x_0)$.

**Proof.** In [1] Akutagawa-Kumura proved that any complete Riemannian manifold with a pole $x_0$ satisfies a weighted Poincaré inequality with the weight function

$$q = (1/4) r^{-2} + (1/4) (\Delta r)^2 - (1/2) |\nabla r|^2 - (1/2) \text{Ric}(\nabla r, \nabla r),$$

where $r(\cdot) = \text{dist}(\cdot, x_0)$. Substituting the distance function $r$ into Bochner’s formula and using the fact that $|\nabla r|^2 = 1$, we see that the last two terms on the right hand side are equal to $(1/2) \langle \nabla \Delta r, \nabla r \rangle$, so

$$q = (1/4) r^{-2} + (1/4) (\Delta r)^2 + (1/2) \langle \nabla \Delta r, \nabla r \rangle.$$

In the special case of the complex hyperbolic space $\mathbb{CH}^n$ the Laplacian of the distance function $r$ is (see for example Theorem 1.6 in Li-Wang [19])

$$\Delta r = 2(n - 1) \coth r + 2 \coth (2r),$$

so

$$(1/4) (\Delta r)^2 = n^2 + n(n - 1) (\sinh r)^{-2} + (\sinh (2r))^{-2},$$

$$(1/2) \langle \nabla \Delta r, \nabla r \rangle = (1/2) \frac{d}{dr} (\Delta r) = - (n - 1) (\sinh r)^{-2} - 2 (\sinh (2r))^{-2}.$$ Substituting these two terms into the equation of the weight function $q$, we get the result.

Next we give more examples of weighted Poincaré inequalities. Using these inequalities, we could get more applications of Theorem [4] and Theorem [2].

Manifolds with positive spectrum satisfy a weighted Poincaré inequality.
Example 8. The bottom of the spectrum of the Laplacian of a Riemannian manifold $X$ is defined by $\lambda_1(\Delta) = \inf_{\phi} \int_X |\nabla \phi|^2$, where the infimum is taken over all smooth functions $\phi$ on the manifold $X$ with compact support and $\int_X \phi^2 = 1$. In particular the manifold $X$ satisfies a weighted Poincaré inequality with the weight function $q = \lambda_1(\Delta)$.

Stable minimal hypersurfaces in a general Riemannian manifold satisfy a weighted Poincaré inequality.

Example 9. A minimal hypersurface $X^n$ in a Riemannian manifold $Y^{n+1}$ is called stable if the second variation of the area functional is nonnegative for any normal variation with compact support. It is well known that the hypersurface $X^n$ is stable if and only if it satisfies a weighted Poincaré inequality with the weight function $q = |A|^2 + \text{Ric}_Y(N, N)$. Here we have the second fundamental form $A$ of the hypersurface and the Ricci curvature $\text{Ric}_Y$ of the ambient space applied to the unit normal vector $N$.

Minimal hypersurfaces in the Euclidean space satisfy a weighted Poincaré inequality.

Example 10. In [8] Carron proved that a minimal hypersurface $X^n$ in the Euclidean space $\mathbb{R}^{n+1}$ satisfies a weighted Poincaré inequality with the weight function $q = ((n-2)/2)^2 r^{-2}$, where $r(\cdot) = \text{dist}_{\mathbb{R}^{n+1}}(\cdot, x_0)|X|$. Finally we give more references on weighted Poincaré inequalities. For applications of weighted Poincaré inequalities to the geometry and topology of Riemannian manifolds and submanifolds see for example Carron [8], Li-Wang [20], Cheng-Zhou [9], Lam [18] and Vieira [29]. In [17], Corollary 1.3 with $\alpha = 1$, Hein proved a weighted Poincaré inequality in certain asymptotically conical manifolds that may lead to new applications of Theorem 1. We would like to find weighted Poincaré inequalities in interesting ALE manifolds (for example the Eguchi-Hanson manifold) and ALF manifolds (for example the Euclidean Schwarzschild manifold) leading to new applications of Theorem 1.

3 Proofs

3.1 Bochner formula

In the proof of Theorem 1 and Theorem 2 we use a Bochner type formula based on Bourguignon-Lawson [7] and Gursky-Kelleher-Streets [16] (for details see [30]). Although the constant in the term of the Weyl curvature is different, the result follows easily from the proof in [30].

Lemma 11. Suppose we have a four-dimensional Riemannian manifold $X$ with scalar curvature $S$ and Weyl curvature $W$, and a Yang-Mills connection $A$ with curvature $F$ and structure group $G \subset O(N)$ ($N \geq 3$) on the manifold $X$. 

Fix a constant $p > 0$. Then the self-dual curvature $F^+$ satisfies the pointwise inequality
\[|F^+|^p \Delta |F^+|^p \geq (1 - 1/(2p)) |\nabla |F^+|^p|^2 + (p/3) S |F^+|^{2p} - 2p \lambda_{\max}(W^+) |F^+|^{2p} - p \gamma |F^+|^{2p+1}.\]

Here we use the inner products
\[|F^+|^2 = \sum_{1 \leq i < j \leq 4} |F_{ij}|^2, \quad \langle A, B \rangle_{so(N)} = -c \cdot tr(AB), \quad c > 0,
\]
and we denote
\[\gamma = \begin{cases} 4/\sqrt{12c}, & N = 3, \\ 4/\sqrt{6c}, & N \geq 4. \end{cases}\]

Recall that $F^+ = 1/2 (F + \ast F)$ and $\lambda_{\max}(W^+)$ is the largest eigenvalue of $W^+ = 1/2 (W + \ast W)$.

### 3.2 Proof of Theorem

First we combine an estimate obtained from Lemma with an estimate obtained from the weighted Poincaré inequality. Using Lemma we have
\[- |F^+|^{1/2} \Delta |F^+|^{1/2} + (1/6) S |F^+| - \lambda_{\max}(W^+) |F^+| - (\gamma/2) |F^+|^2 \leq 0.\]

Recall that $r(\cdot) = dist(\cdot, x_0)$ and $B_R = \{ r < R \}$. Take the logarithmic cutoff function $\phi$ on the manifold given by
\[\phi = \begin{cases} 1 & B_R, \\ 2 - \log r \log R & B_{R^2} \setminus B_R, \\ 0 & X \setminus B_{R^2}. \end{cases}\]

This cutoff function is often used in minimal surface theory and it is important in the next part of the proof. Multiplying this inequality by the function $\phi^2$ and integrating by parts, we get
\[\int |\nabla |F^+|^{1/2}|^2 \phi^2 + 2 \int |F^+|^{1/2} \phi \langle \nabla |F^+|^{1/2}, \nabla \phi \rangle + (1/6) \int S |F^+| \phi^2 - \int \lambda_{\max}(W^+) |F^+| \phi^2 - (\gamma/2) \int |F^+|^2 \phi^2 \leq 0.
\]

On the other hand, substituting the compactly supported function $|F^+|^{1/2} \phi$ into the weighted Poincaré inequality and rearranging the terms, we have
\[- \int |\nabla |F^+|^{1/2}|^2 \phi^2 - 2 \int |F^+|^{1/2} \phi \langle \nabla |F^+|^{1/2}, \nabla \phi \rangle + \int q |F^+|^2 \phi^2 \leq \int |F^+| |\nabla \phi|^2.
\]

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Summing the above estimates, we get
\[
\int \left\{ q + (1/6) S - \lambda_{\text{max}} (W^+) - (\gamma/2) |F^+| \right\} |F^+| \phi^2 \leq \int |F^+| |\nabla \phi|^2.
\]

Next we estimate the right hand side of this inequality. We denote by \( C \) a positive constant independent of \( R \), which may change from line to line. Substituting the gradient of the function \( \phi \) (which vanishes outside the ring \( B_{R^2} \setminus B_R \)), writing the ring \( B_{R^2} \setminus B_R \) as the disjoint union \( B_{R^2} \setminus B_R = \bigcup_{i=\log R}^{2 \log R-1} B_{e^{i+1}} \setminus B_{e^i} \), and using the fact that the self-dual curvature \( F^+ \) has growth \(|F^+| = O(r^{2-p})\), we see that for \( R \) sufficiently large
\[
\int \left\{ q + (1/6) S - \lambda_{\text{max}} (W^+) - (\gamma/2) |F^+| \right\} |F^+| \phi^2 \leq C (\log R)^{-2} \sum_{i=\log R}^{2 \log R-1} \int_{B_{e^{i+1}} \setminus B_{e^i}} r^{-p}.
\]
Using the fact that the manifold has volume growth \( \text{vol} (B_R) = O(R^p) \), we have \( \int_{B_{e^{i+1}} \setminus B_{e^i}} r^{-p} = O(1) \), so for \( R \) sufficiently large
\[
\int \left\{ q + (1/6) S - \lambda_{\text{max}} (W^+) - (\gamma/2) |F^+| \right\} |F^+| \phi^2 \leq C (\log R)^{-1}.
\]

Finally we show that either inequality (2) is an equality (everywhere) or the self-dual curvature \( F^+ \) is identically zero. By assumption, the term inside the braces is nonnegative. Taking the limit as \( R \to \infty \), we get
\[
\{ q + (1/6) S - \lambda_{\text{max}} (W^+) - (\gamma/2) |F^+| \} |F^+| = 0.
\]
Suppose that inequality (2) is not an equality (everywhere). In this case the term inside the braces is positive at some point of the manifold, so by continuity the self-dual curvature \( F^+ \) vanishes in an open subset of the manifold. Using the unique continuation principle, we conclude that the self-dual curvature \( F^+ \) is identically zero.

### 3.3 Proof of Theorem 2

First, as in the proof of Theorem 1 we combine an estimate obtained from Lemma 11 with an estimate obtained from the weighted Poincaré inequality, but the calculation here is more involved. Using Lemma 11 we have
\[
- |F^+|^p \Delta |F^+|^p + (1 - 1/ (2p)) |\nabla |F^+|^p|^2 + (p/3) S |F^+|^{2p} - 2p \lambda_{\text{max}} (W^+) |F^+|^{2p} - p\gamma |F^+|^{2p+1} \leq 0.
\]
Recall that \( r(\cdot) = \text{dist} (\cdot, x_0) \) and \( B_R = \{ r < R \} \). Take the cutoff function \( \phi \) on the manifold given by
\[
\phi = \begin{cases} 
1 & B_R, \\
2 - r/R & B_{2R} \setminus B_R, \\
0 & X \setminus B_{2R}.
\end{cases}
\]
Multiplying this inequality by the function $\phi^2$, integrating by parts and writing $\tilde{p} = 2 - 1/(2p)$ (which is positive by assumption), we get

$$
\tilde{p} \int |\nabla |F^+|^p|^2 \phi^2 + 2 \int |F^+|^p \phi (\nabla |F^+|^p, \nabla \phi) + (p/3) \int S |F^+|^{2p} \phi^2
- 2p \int (p) |F^+|^{2p} \phi^2 - pr \int |F^+|^{2p+1} \phi^2 \leq 0.
$$

Fix a small $\delta > 0$. Applying the inequality $2ab \leq \delta a^2 + \delta^{-1}b^2$ to the second term on the left hand side and rearranging terms, we get (i):

$$
(\tilde{p} - \delta) \int |\nabla |F^+|^p|^2 \phi^2 + (p/3) \int S |F^+|^{2p} \phi^2 - 2p \int \lambda_{\max} (W^+) |F^+|^{2p} \phi^2
- pr \int |F^+|^{2p+1} \phi^2 \leq \delta^{-1} \int |F^+|^{2p} |\nabla \phi|^2.
$$

On the other hand, substituting the compactly supported function $|F^+|^{2p} \phi$ into the weighted Poincaré inequality and using the inequality $2ab \leq \delta a^2 + \delta^{-1}b^2$ as before, we have

$$
\int q |F^+|^{2p} \phi^2 \leq (1 + \delta) \int |\nabla |F^+|^p|^2 \phi^2 + (1 + \delta^{-1}) \int |F^+|^{2p} |\nabla \phi|^2.
$$

We denote by $C$ a positive constant independent of $R$ and $\delta$, which may change from line to line. Multiplying this inequality by $(\tilde{p} - \delta) (1 + \delta)^{-1}$, using the inequalities $(\tilde{p} - \delta) (1 + \delta)^{-1} \geq \tilde{p} - C\delta$ and $(1 + \delta^{-1}) (\tilde{p} - \delta) (1 + \delta)^{-1} \leq C\delta^{-1}$, and rearranging terms, we get (ii):

$$
- (\tilde{p} - \delta) \int |\nabla |F^+|^p|^2 \phi^2 + \tilde{p} \int q |F^+|^{2p} \phi^2
\leq C \left( \delta \int q |F^+|^{2p} \phi^2 + \delta^{-1} \int |F^+|^{2p} |\nabla \phi|^2 \right).
$$

Summing the above estimates ((i) and (ii)), we get

$$
\int \left\{ (2 - 1/(2p)) q + (p/3) S - 2p\lambda_{\max} (W^+) - pr |F^+| \right\} |F^+|^{2p} \phi^2
\leq C \left( \delta \int q |F^+|^{2p} \phi^2 + \delta^{-1} \int |F^+|^{2p} |\nabla \phi|^2 \right).
$$

Next, as in the proof of Theorem 1 we estimate the right hand side of this inequality. The choice of $\delta$ is tricky. Fix a small $\epsilon > 0$ and take $\delta = \epsilon R^{-2}$. Substituting the function $\phi$ (which vanishes outside the ball $B_{2R} \setminus B_R$) and its gradient (which vanishes outside the ring $B_{2R} \setminus B_R$), and using the fact that the weight function $q$ has growth $q = O(r^2)$ (say $q \leq Cr^2$ for $r \geq R_0$), we get

$$
\int \left\{ (2 - 1/(2p)) q + (p/3) S - 2p\lambda_{\max} (W^+) - pr |F^+| \right\} |F^+|^{2p} \phi^2
\leq C \left( \epsilon R^{-2} \int_{B_{R_1}} q |F^+|^{2p} + \epsilon \int_{B_{2R} \setminus B_{R_0}} |F^+|^{2p} + \epsilon^{-1} \int_{B_{2R} \setminus B_{R}} |F^+|^{2p} \right).
$$
Finally, as in the proof of Theorem 1, we show that either inequality (3) is an equality (everywhere) or the self-dual curvature $F^+$ is identically zero. By assumption, the term inside the braces on the left hand side is nonnegative and the self-dual curvature $F^+$ is in the space $L^{2p}$. Taking the limit as $R \to \infty$ and then taking the limit as $\epsilon \to 0$, we see that the right hand side becomes zero, so
\[
\left\{(2 - 1/(2p))q + (p/3)S - 2p\lambda_{\max}(W^+) - p\gamma |F^+|\right\} |F^+|^{2p} = 0.
\]
As in the proof of Theorem 1, if inequality (3) is not an equality (everywhere), then the self-dual curvature $F^+$ is identically zero.

### 3.4 Proof of Corollary 3

We proved the results for Euclidean space $R^4$, the hyperbolic space $H^4$ and the complex hyperbolic space $CH^2$ above Lemma 5, Lemma 6 and Lemma 7 respectively. Substituting the weight function $q = 0$ into Theorem 1 and using the fact that the sphere $S^4$ has scalar curvature $S = 12$ and Weyl curvature $W = 0$, the complex projective space $CP^2$ has scalar curvature $S = 24$ and Weyl curvature $W^{-} = 0$, and the cylinder $S^3 \times R$ has scalar curvature $S = 6$ and Weyl curvature $W = 0$, we get the results for the sphere $S^4$, the complex projective space $CP^2$ and the cylinder $S^3 \times R$.

### 3.5 Proof of Theorem 4

Note that $SU(2) \subset SO(4)$. Taking $c = 1/2$ (without loss of generality) and $N = 4$ in the definition of $\gamma$ (see Lemma 1), we see that the inequality in the assumption of the theorem becomes the inequality $|F|_{L^\infty(S^4)} \leq \sqrt{3}$.

First we show that the connection $A$ is an instanton, the curvature $F$ satisfies the equation $|F|_{g_{S^4}} = \sqrt{3}$ (everywhere) and the charge is 1. The proof is as follows. Using the fact that $|F|^2 = |F^+|^2 + |F^-|^2$, we get $|F^+|_{L^\infty(S^4)} \leq \sqrt{3}$ and $|F^-|_{L^\infty(S^4)} \leq \sqrt{3}$. Applying Corollary 3 to the self-dual curvature $F^+$ (resp. anti-self-dual curvature $F^-$), we see that $|F^+|_{g_{S^4}} = \sqrt{3}$ everywhere or $F^+$ is identically zero (resp. $|F^-|_{g_{S^4}} = \sqrt{3}$ everywhere or $F^-$ is identically zero). Using the fact that $|F^+|_{g_{S^4}} = \sqrt{3}$ and $|F^-|_{g_{S^4}} = \sqrt{3}$ cannot happen at the same time, we see that either: (i) $|F^+|_{g_{S^4}} = \sqrt{3}$ everywhere and $F^-$ is identically zero or (ii) $F^+$ is identically zero and $|F^-|_{g_{S^4}} = \sqrt{3}$ everywhere. In both cases we conclude that the connection $A$ is an instanton and the curvature $F$ satisfies $|F|_{g_{S^4}} = \sqrt{3}$ everywhere. Using the fact that $vol(S^4) = 8\pi^2/3$, we see that the charge is $(8\pi^2)^{-1} |F|_{L^2(S^4)}^2 = 1$.

Next we discuss the corresponding instanton on the Euclidean space $R^4$. Using the fact that the connection $A$ is a $SU(2)$ instanton on the sphere $S^4$ with charge 1, we see that under the stereographic projection of $S^4 \setminus \{N\}$ the connection $A$ corresponds to a $SU(2)$ instanton on the Euclidean space $R^4$ with charge 1. Using the fact that the curvature $F$ satisfies the equation $|F|_{g_{S^4}} = \sqrt{3}$.
everywhere and writing the metric of the sphere $S^4$ in terms of the stereographic projection as $g_{S^4} = 4 \left(1 + |x|^2\right)^{-2} g_{R^4}$, we see that the curvature of the corresponding instanton (also denoted by $F$) satisfies the equation

$$|F|^2_{g_{S^4}} (x) = 48 \left(1 + |x|^2\right)^{-4}.$$  

Finally we show that the corresponding instanton is the basic instanton. It is well known that $SU(2)$ instantons on the Euclidean space $R^4$ with charge 1 are an explicit five-parameter family of connections $\{A_{x_0,a}\}$ determined by a center $x_0$ in $R^4$ and a scale $a > 0$, and the curvature $F(A_{x_0,a})$ satisfies the equation

$$|F(A_{x_0,a})|^2_{g_{R^4}} (x) = 48a^4 \left(a^2 + |x - x_0|^2\right)^{-4}.$$  

See for example Chapter 3 in [11] and Section 2 in [28]. Comparing the above equations, we see that the connection $A$ corresponds to the instanton $A_{x_0,a}$ that has center $x_0 = 0$ and scale $a = 1$, which is known as the ADHM-BPST basic instanton ([2, 3]).

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