The two-capacitor problem revisited: a mechanical harmonic oscillator model approach

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Abstract
The well-known two-capacitor problem, in which exactly half the stored energy disappears when a charged capacitor is connected to an identical capacitor, is discussed based on the mechanical harmonic oscillator model approach. In the mechanical harmonic oscillator model, it is shown first that exactly half the work done by a constant applied force is dissipated irrespective of the form of dissipation mechanism when the system comes to a new equilibrium after a constant force is abruptly applied. This model is then applied to the energy loss mechanism in the capacitor charging problem or the two-capacitor problem. This approach allows a simple explanation of the energy dissipation mechanism in these problems and shows that the dissipated energy should always be exactly half the supplied energy whether that is caused by the Joule heat or by the radiation. This paper, which provides a simple treatment of the energy dissipation mechanism in the two-capacitor problem, is suitable for all undergraduate levels.

1. Introduction
When a charged capacitor is connected to another identical one, exactly half the stored energy disappears after the charge transfer is completed. Such situation also occurs when a capacitor is charged by a battery, where only half the supplied energy is left as the stored energy in the capacitor after the charging process is completed. This problem has a long history and it is an interesting problem pedagogically [1–4].

The question of 'what is the mechanism of such energy loss, and why exactly half the supplied energy is lost' has been the point of interest in this problem. It was noted in earlier works that if the presence of nonzero resistance connecting the two capacitors is assumed, the amount of Joule heat loss in the resistor becomes exactly the same as the 'missing' energy [1]. Such an explanation shows that resistance is an indispensable part in any electrical circuit. (One can imagine using the superconducting wire for the connection, but it is
obvious that the increasing current will quickly reach the critical limit which will destroy
the superconducting state. Therefore, this example shows that a critical field limit must exist
in the superconducting phenomena.

Although this problem was resolved by this simple model, an \( RC \) circuit model is of
course not realistic. Therefore, Powell has considered a more realistic model where an
inductor element is included, since any closed circuit has to have a self-inductance effect
[2]. It was shown that in all three cases of overdamped, underdamped and critically damped
cases, Joule heat loss by the resistor becomes the same, accounting for the lost energy exactly.
Powell noted that there are radiation effects which may be important. Therefore, although
the missing energy problem was resolved, it was not clearly understood why the simple \( RC \)
model was as satisfactory as the more realistic \( RLC \) model, and why discussions which do
not incorporate the radiation effect could also give perfect explanations.

We discuss this problem based on the simple harmonic oscillator model approach in this
work. From this approach, it is shown that the missing energy should always be exactly
half whether the dissipation mechanism is the Joule heat loss or the radiation effect of the
combination of them. For this purpose, motion of an object in a harmonic potential under the
influence of a damping force which is velocity and acceleration dependent is considered. Our
analysis does not require the exact solution of differential equations which may be difficult or
impossible to obtain for such cases.

2. Classical harmonic oscillator model

Let us begin our discussion by considering a simple harmonic oscillator where a frictional force
of the type \( f(v, \frac{dv}{dt}) \) which depends on the velocity \( v \) and acceleration \( \frac{dv}{dt} \) is present. If
a constant external force \( F_0 \) is applied to an object of mass \( m \) in a harmonic potential of the
form \( (1/2)kx^2 \), the equation of motion can be written as follows:
\[
m\frac{dv}{dt} = F_0 - f(v, \frac{dv}{dt}) - kx.
\]  

(1)

An analytic solution of this equation cannot be obtained in general except for some simple
special cases. Let us assume that the constant force \( F_0 \) is applied at time \( t = 0 \), and the object
is at rest at the equilibrium position at that instant. The object will come to a complete stop
at the new equilibrium position eventually; therefore, the initial and final velocity are both 0,
which means the resulting change of the kinetic energy becomes 0.

Let us first consider the energy dissipation aspect of this system when the damping force
term does not depend on acceleration and is simply proportional to the velocity such that
\( f(v, \frac{dv}{dt}) = bv \), in which \( b \) is a constant.

Let the final equilibrium position of the object be denoted as \( x_0 \). Since both the velocity
and acceleration of the object become 0 eventually, as long as the damping force disappears
when \( v = 0 \) and \( \frac{dv}{dt} = 0 \), which is naturally satisfied when the object comes to a complete
stop, it could be noted that the final object position can still be expressed as \( x_0 = \frac{F_0}{k} \).
Therefore the resulting change of potential energy can be written as \((1/2)kx_0^2\). If these facts
are used, the integrated result of this equation can be expressed as
\[
F_0 \int_0^\infty v \, dt = \int_0^\infty f(v, \frac{dv}{dt}) \, dt + \frac{1}{2}kx_0^2.
\]  

(2)

The left-hand side of the equation indicates the amount of work done by the force and the
first term on the right-hand side indicates the amount of dissipated energy. Although \( v(t) \) may
have a complicated form, it is important to note at this point that \( \int_0^\infty v \, dt \) can be identified
as the eventual position \( x_0 \) of the object. Since \( x_0 = \frac{F_0}{k} \), this means that the amount of
supplied work can be expressed as \( kx_0^2 \). Therefore, the following interesting relation can be obtained:

\[
kx_0^2 = \int_0^\infty bv^2 \, dt + \frac{1}{2} kx_0^2.
\] (3)

This relation shows that exactly half the work done by the applied force is eventually lost as the dissipated energy. Let us call this relation, ‘dissipation energy relation in a harmonic potential system’. This relation can also be proved using the analytic solutions (which are usually classified as overdamped, underdamped and critical-damped cases) for this particular case.

It should be noted again that this relation has been obtained without resorting to the solution of differential equations; therefore, we are allowed to have a general form of the damping force. Let us assume that the damping force function has a polynomial form of the type \( f(v) = b_1 v + b_2 v^2 + \cdots \). It is known that, in the case of viscous resistance on a body moving through a fluid, such a form has to be assumed [5]. The equation of motion in this case becomes nonlinear and it is usually impossible to obtain the exact solution of such equations, but of course, this dissipation energy relation still holds. An interesting point is that this is true even when the damping force function depends on the acceleration. This fact will be used as an essential point in the following two-capacitor problem discussion.

In some physical systems, potential function may include anharmonic terms such as \( kx^n \), where \( n \) is an integer not equal to 2, which could lead to a nonlinear equation of motion. (An anharmonic potential term is important in describing the intermolecular potential in molecular bonding.) The exact solution of differential equations for these cases is usually not available. But it can be noted that a definite relation between the supplied and dissipated energy can still be found based on our analysis, although the dissipation energy relation for these cases will become different from the simple harmonic systems.

3. Discussion of the two-capacitor problem

Let us now extend our discussion to the two-capacitor problem. For this purpose, we consider an RLC circuit with resistance \( R \), and inductance \( L \) connected to an emf source of magnitude \( \epsilon_0 \) as in figure 1(a), which has the following circuit equation:

\[
\epsilon_0 - R i - L \frac{di}{dt} - \frac{1}{C} q = 0.
\] (4)

If the current \( i \) and inductance \( L \) are considered to correspond to velocity \( v \) and mass \( m \) in the mechanical system, this equation is the exact equivalent of the damped harmonic oscillator. It could also be shown that this equation is essentially equivalent to the two-capacitor problem equation [2].

Now if the current \( i \) is multiplied to this equation, we obtain

\[
\epsilon_0 i - R i^2 - \frac{du_m}{dt} - \frac{du_e}{dt} = 0,
\] (5)

where \( U_m = \frac{1}{2} Li^2 \), and \( U_e = \frac{1}{2} C q^2 \). If this equation is integrated, then the following relation can be obtained:

\[
\epsilon_0 \int_0^\infty i \, dt = \int_0^\infty R i^2 \, dt + \Delta U_m\big|_0^\infty + \Delta U_e\big|_0^\infty.
\] (6)

Since the current \( i \) is equal to 0 at \( t = 0 \) and at \( t = \infty \), \( \Delta U_m \) part vanishes. Also since both \( i \) and \( di/dt \) vanish eventually, the amount of capacitor charge becomes \( C\epsilon_0 \) from
equation (4), and therefore the $\Delta U_e$ part becomes $(1/2)C\epsilon_0^2$ and the following expression can be obtained:

$$C\epsilon_0^2 = \int_{0}^{\infty} Ri^2 \, dt + \frac{1}{2} C\epsilon_0^2. \quad (7)$$

This shows that only half the supplied energy is left as the stored energy in the capacitor and the other half is necessarily dissipated in the resistor whenever one attempts to charge a capacitor.

Now consider the circuit in which an initially charged capacitor with charge $Q_0$ is connected to another capacitor of the same capacitance as in figure 1(b). If the charge of the connected capacitor is denoted as $q$, the circuit equation for this case can be written as

$$Q_0 - q \frac{C}{C} - Ri - L \frac{di}{dt} - \frac{1}{C}q = 0. \quad (8)$$

If we replace $\frac{1}{2}Q_0$ as $\epsilon_0$, this equation can be rewritten as

$$\epsilon_0 - Ri - L \frac{di}{dt} - \frac{2q}{C} = 0, \quad (9)$$

which is the same as equation (4), except that the capacitor has been replaced by one with a capacitance of $C/2$.

Following the previous analysis, considering now that $\int_{0}^{\infty} i \, dt = \frac{1}{2}Q_0$ and that eventually the capacitor charge becomes $\frac{1}{2}Q_0$, we obtain the relation

$$\frac{1}{2} C\epsilon_0^2 = \int_{0}^{\infty} Ri^2 \, dt + \frac{1}{4} C\epsilon_0^2. \quad (10)$$

This shows that the total dissipated energy is equal to $\frac{1}{4} C\epsilon_0^2$, which is just half the original stored energy. In fact, the circuit equation for the capacitor charging case by a battery as is given by equation (4) and the two-capacitor circuit case as is given by equation (8) become essentially the same if differentiated, which means that two situations are basically identical.
We can therefore successfully resolve the two-capacitor problem in a much simpler way than Powell, i.e., without resorting to the solution of differential equations. In fact, what Powell did was to find solutions for the differential equation, and prove this result by evaluating the integrals directly [2]. Powell found that the overdamped case closely approximates the less realistic $RC$ situation, which may explain why the unrealistic $RC$ model can also be successful. But the interesting point is that the $RC$ model explains the missing energy exactly, not approximately. Furthermore, it was found that most practical circuits belong to the underdamped case, not the overdamped case. Therefore the reason for the success of the $RC$ model could not be understood. In the present analysis, since the $\Delta U_m = \frac{1}{2} Li^2$ part vanishes anyway, it is easy to see why the presence of an inductor in the circuit does not make any difference at all for this problem.

Although the circuit including the inductor component is more realistic, it is not truly realistic because it does not consider the radiation effect which becomes especially important in high-frequency oscillating circuits. To resolve this situation, Boykin et al have adopted the magnetic dipole model and have shown that just the radiation effect due to the magnetic dipole model can also explain the missing energy [3]. Using the so-called lumped-parameter model (in which nonlinear circuit elements excluding the capacitor radiation effect have been represented by an equivalent radiation resistance), it was shown that the presence of a resistor is not essential to resolve the two-capacitor problem.

There exists an oscillating electric field inside the capacitor which can also radiate. Such an electric dipole radiation effect from the capacitor part was also considered by Choy, who showed that the electric dipole model can also explain the missing energy [4]. It is amazing to find that all approaches from the simple $RC$ model to the more elaborate radiating dipole models can all successfully resolve the two-capacitor problem. But the reason for such an outcome cannot be understood as yet.

To understand this situation, let us now extend the above model which corresponds to the velocity-dependent damping force model to a case which includes an acceleration-dependent damping force. It is well known that the radiation effect can be incorporated as a resistive term [6]. Since the dipole radiation effects involve the acceleration term, our resistive force function is now considered to have a form of $f (v, dv/dt)$. However, we have already found that our dissipation energy relation remains effective even for such forms of resistance functions. Therefore it could be stated that regardless of the nature of the damping force, i.e., whether it is the Ohmic resistive force or radiative resistive force or a combination of both, the amount of dissipation energy should always be exactly half the supplied energy in the two-capacitor problem.

4. Conclusion

In this work, the energy loss mechanism in the two-capacitor problem has been considered using a harmonic oscillator model. We have shown, without resorting to solutions of differential equations, that the amount of energy loss in the form of Joule heat should be exactly half the supplied energy in the $RLC$ circuit. Since our analysis does not require the solution of differential equations, we could extend our analysis to damping forces which include an acceleration term, which corresponds to radiative resistance. This explains why perfect explanations of the two-capacitor problem have been possible, whether just the Joule heat loss or dipole radiation energy loss is considered, providing a comprehensive understanding of the two-capacitor problem.
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References

[1] Cuvaj C 1968 On conservation of energy in electric circuits Am. J. Phys. 36 909–10
[2] Powell R A 1979 Two capacitor problem: a more realistic view Am. J. Phys. 47 460–2
[3] Boykin T R, Hite D and Singh N 2002 The two-capacitor problem with radiation Am. J. Phys. 70 415–20
[4] Choy T C 2004 Capacitors can radiate: further results for the two-capacitor problem Am. J. Phys. 72 662–70
[5] Fowles 1975 Analytical Mechanics 2nd ed (New York: Wiley) p 783
[6] Jackson J D 1975 Classical Electrodynamics 2nd ed (New York: Wiley) p 783