On the determination of the relation between the matrix elements of $CP_q(n)$ and covariant $q$-oscillators

Salih CELIK
Department of Mathematics, Faculty of Sciences, Mimar Sinan University
80690 Besiktas, Istanbul, TURKEY.

Abstract: In this article it is explicitly shown that the matrix elements of $CP_q(n)$ are the annihilation operators of Pusz-Woronowicz (P-W) oscillators provided they are rescaled.
1. Introduction

Recently, using the first order differential calculus and the second quantization procedure the algebra of $q$-oscillators covariant under the coaction of the unitary quantum group $SU_q(n)$ was given by Pusz and Woronowicz [1]. Wess and Zumino [2] have developed a differential calculus for the quantum hyperplane which is covariant under the action of $GL_q(n)$. In Ref. 3 it has been shown that the unitary quantum group $U_q(n)$ can be constructed in terms of $n(n-1)/2$ $q$-oscillators and $n$-commuting phases. This approach shows that the quantum projective space $CP_q(n)$ can be identified with $n$ $q$-oscillators.

A new realization of the quantum group $SU_q(2)$ has been obtained by introducing a $q$-analogue of the usual harmonic oscillator and using the Jordan-Schwinger mapping [4,5]. For early studies on the $q$-oscillator see Arik et al [6,7]. Similar deformations of the quantum harmonic oscillator algebra have attracted a lot of attention [8-17].

The purpose of this study is to show that the elements of the matrix representations of the quantum projective space $CP_q(n)$, by using the representations found in [3], are the covariant annihilation operators of Pusz-Woronowicz (P-W) oscillators.

In section 2 we introduced the quantum group $SU_q(2)$. And also we showed the matrix elements of $SU_q(2)$ can map to a Biedenharn-Macfarlane (B-M) oscillator algebra [4,5]. In section 3 we proved that the matrix elements of $CP_q(n)$ are the annihilation operators of (P-W) oscillators. The results has been discussed in section 4.

2. The quantum group $SU_q(2)$ and $q$-oscillators

The quantum group $SU_q(2)$ consists of all matrices of the form
\[
M = \begin{pmatrix} a & b \\ -q b^* & a^* \end{pmatrix}
\]
where
\[
ab = qba, \quad ab^* = qb^* a, \quad bb^* = b^* b, \quad aa^* + q^2 bb^* = 1, \quad a^* a + bb^* = 1,
\]
and $0 < q < 1 \ (q \in \mathbb{R})$.

The $^*$-operation is defined as follows:
\[
(a)^* = a^+ \quad \text{and} \quad (a^+)^* = a.
\]
From the equations in (2.2) one obtains

$$aa^* - q^2a^*a = 1 - q^2. \quad (2.3)$$

This equation can be used to define our $q$-oscillator [3] where the operator $a^*$ is the creation and its hermitean conjugate $a$ is the annihilation operator. Note that the normalization of the oscillators is chosen so that in the $q \to 1$ limit this operators commute and gives c-numbers. The spectrum of the $q$-oscillator defined by equation (2.3) is completely fixed for $0 < q < 1$ and it is shown that [3]

$$a^*a = 1 - q^{2N} = [N]_q \quad (2.4)$$

where $N$ is the number operator. From this we can get immediately

$$[N, a] = -a, \quad [N, a^*] = a^* \quad (2.5)$$

Now if $a$ is defined as follows

$$a = (1 - q^2)^{1/2} q^{N/2} a_q$$

then

$$a^* = (1 - q^2)^{1/2} a_q^+ q^{N/2}$$

is obtained. Here $N^+ = N$. Then (2.3) becomes

$$a_q a_q^+ - q a_q^+ a_q = q^{-N}. \quad (2.6)$$

It follows that the matrix elements of the two-dimensional representation of $SU_q(2)$ can be mapped to a (B-M) oscillator algebra, i.e. the algebra (2.3,5) is equivalent to (B-M) oscillator algebra. Note that although for $0 < q < 1$ equations (2.3) and (2.6) are related by just a rescaling, in the $q \to 1$ limit they are fundamentally different. In this limit the oscillator creation and annihilation operators in (2.3) are commutative so that $a$ and $a^*$ can be considered as commuting complex numbers.

3. The quantum projective space $CP_q(n)$ and covariant $q$-oscillators

Now the quantum group $SU_q(n)$ is discussed. It can be shown that any element of $SU_q(n)$ can be expressed uniquely as [3]

$$M = \prod_{k=1}^{n-1} M_{i,i+1}(a_{ik}) \prod_{i=1}^{n-1} \chi_{i,i+1}(\beta_i) \quad (3.1)$$

and

$$H = \prod_{i=1}^{n} M_{i,i+1}(a_i) \quad (3.1)$$
being identified with the matrix representatives of the n-dimensional quantum projective space \( CP_q(n) \). Here each \( M_{i,i+1}(a_i) \) is the matrix whose 2x2 diagonal block in the \( i, i+1 \) position is a \( q \)-oscillator matrix

\[
\begin{pmatrix}
a_i \\
-q(1-a_i^*a_i)^{1/2}
\end{pmatrix}
\begin{pmatrix}
(1-a_i^*a_i)^{1/2} \\
a_i^*
\end{pmatrix}
\]

and the remaining diagonal elements contain 1, and all the elements apart from these are zero. In (3.1) each \( a_i \) is a \( q \)-oscillator and their properties are determined by the commutation relations,

\[
[a_i, a_j] = 0 = [a_i^*, a_j^*], \quad i, j = 1, 2, ..., n \quad (3.2a)
\]

\[
[a_i, a_j^*] = (1-q^2)q^{2N_i}\delta_{ij}, \quad (3.2b)
\]

\[
[a_i, N_j] = \delta_{ij}a_i, \quad [a_i^*, N_j] = -\delta_{ij}a_i^*, \quad (3.2c)
\]

where \( \delta_{ij} \) denotes the Kronecker delta.

In the matrix \( H \) given by (3.1) instead of taking the n-1 \( a_i \)'s as independent one can take the first n-1 elements of the first row as independent. Then it can be shown that these elements can be shown they are the annihilation operators of (P-W) oscillators [1]. Note that the first elements taken are

\[
a = (1-q^2)^{1/2} A
\]

\[
a^* = (1-q^2)^{1/2} A^+.
\]

In fact, if we take

\[
A_k = (1-q^2)^{-1/2} \left( \prod_{i=1}^{k-1} x_i \right) a_k, \quad \text{fork} = 2, 3, ..., n \quad (3.4)
\]

where \( x_i = (1-a_i^*a_i)^{1/2} \) and

\[
A_1 = (1-q^2)^{-1/2} a_1
\]

one has

\[
A_iA_j = qA_jA_i, \quad i < j
\]

\[
A_iA_j^+ = qA_j^+A_i, \quad i \neq j \quad (3.5)
\]

\[
A_i^+A_j^+ = qA_j^+A_i^+, \quad i > j
\]

\[
A_iA_i^+ - q^2A_i^+A_i = 1 + (q^2 - 1) \sum_{k<i} A_k^+ A_k.
\]

Now let's take

\[
a_1 = A_1.
\]
For a representation in a Hilbert space where $a_1^*$ denotes the hermitean conjugate of $a_1$, the eigenvalues of $A_1^+A_1$ are real and non-negative. Moreover, by the relation (2.2) the operator $A_1^+A_1$ is bounded and the eigenvalues of this operator are in the interval $[0,1)$. Thus one can take

$$a_2 = (1 - A_1^+A_1)^{-1/2}A_2.$$ 

We require that the relations which will be obtained, after the defining the transformations, are the same with the relations (3.2). Thereby we will take

$$ a_1 = (1 - q^2)^{1/2}A_1, \quad (3.6a) $$
$$ a_2 = (1 - q^2)^{1/2}X^{-1}A_2, \quad (3.6b) $$

where

$$X^2 = A_1A_1^+ - A_1^+A_1.$$ 

Note that

$$A_1X^2 = q^2X^2A_1, \quad A_1^+X^2 = q^{-2}X^2A_1^+, \quad A_1X^{\pm 1} = q^{\pm 1}X^{\pm 1}A_1, \quad X^{\pm 1}A_1^+ = q^{\pm 1}X^{\pm 1}A_1^+,$$ 

$$[A_2, X^2] = 0 = [A_2^+, X^2].$$ 

Now by using the commutation relations (3.5) and (3.7) it is easy to check that the relations (3.2) are invariant under the transformations (3.6). Indeed, for example,

$$a_1a_2 = (1 - q^2)A_1X^{-1}A_2 = (1 - q^2)X^{-1}A_2A_1 = a_2a_1.$$ 

Consequently, we have shown that there is a one to one correspondence between the matrix elements of the matrix representation of $CP_q(n)$ and covariant (P-W) oscillators.

In the general case, one can take

$$a_1 = (1 - q^2)^{1/2}A_1$$

and for $k = 2, 3, ..., n$

$$a_k = (1 - q^2)^{1/2} \left( \prod_{i=1}^{k-1} X_i^{-1} \right) A_k \quad (3.8)$$

where

$$X_i^2 = (A_iA_i^+ - A_i^+A_i)$$

and a straightforward computation shows that the relations (3.2) are satisfied.
4. Discussion

In this study by taking the matrix elements of the unitary quantum group as the independent $q$-oscillators we proved that the entries of the matrix representations of the quantum projective space $CP_q(n)$ are the annihilation operators of Pusz-Woronowicz oscillators.

The quantum projective space has two different $q \rightarrow 1$ limits. In the first case it reduces to the ordinary complex projective space $CP(n)$ and in the other case it reduces to the n-dimensional ordinary oscillator.

References

[1] W. Pusz and S. L. Woronowicz, Rep. Math. Phys. 27 (1989) 231.
[2] J. Wess and B. Zumino, Nucl. Phys. B (Proc.Suppl) 18 (1990) 302.
[3] M. Arik and S. Celik, Z.Phys. C 59 (1993) 99.
[4] A. J. Macfarlane, J. Phys. A : Math. Gen. 22 (1989) 4581.
[5] L. C. Biedenharn, J. Phys. A : Math. Gen. 22 (1989) L873.
[6] M. Arik, D. D. Coon and Y.Lam, J. Math. Phys. 16 (1975) 1765.
[7] M. Arik and D. D. Coon, J. Math. Phys. 17 (1976) 524.
[8] Y. J. Ng, J. Phys. A : Math. Gen. 23 (1990) 1023.
[9] M. Chaichian, P. Kulish and J. Lukierski, Phys.Lett. B 237 (1990)401.
[10] B. Fairlie and C. K. Zachos, Phys.Lett. B 256(1) (1991) 43.
[11] M. Arik, Z.Phys. C 51 (1991) 627.
[12] S. P. Vokos, J. Math. Phys. 32 (11) (1991) 2979.
[13] M. Chaichicn, P.Kulish and J.Lukierski, Phys.Lett. B 262 (1991)43.
[14] C. Daskaloyannis, J. Phys. A : Math. Gen. 23 (1990) L789.
[15] P. Kulish and E. V. Damaskinsky, J.Phys.A:Math.Gen. 23 (1990) L415.
[16] R. M. Mir-Kasimov, J.Phys. A: Math. Gen. 24 (1991) 4283.
[17] M. Arik and M. Mungan, Phys. Lett. B 282 (1992) 101.