On completeness of the Moutard transformations

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9 June 1996

The Moutard equation

\[ u_{xy} = M(x, y) u, \quad u = u(x, y), \tag{1} \]

appeared initially in the classical differential geometry in the second half of the XIX century (see [1, 2]). It has now numerous applications in the theory of (2 + 1)-dimensional integrable systems of partial differential equations. The other its form \( u_{tt} - u_{zz} = M(x, y) u \) and the analogous elliptic equation

\[ u_{xx} + u_{yy} = M(x, y) u \tag{2} \]

(the 2-dimensional Schrödinger equation) show the role of (1) in mathematical physics. In the framework of the classical differential geometry (1) played the key role in the central problems of that epoch — the theory of surface isometries, the theory of congruences and conjugate nets. In the modern soliton theory (1) was used to obtain solutions for the Kadomtsev-Petviashvili, Novikov-Veselov equations and others ([3, 5]). The main instrument in applications of (1) is the Moutard transformation which, provided we are given two different solutions \( u = R(x, y) \) and \( u = \varphi(x, y) \) of (1) with a given "potential" \( M = M_0(x, y) \), gives us (via a quadrature) a solution \( \vartheta(x, y) \) of (1) with transformed potential \( M_1(x, y) = M_0 - 2(\ln R)_{xy} \). The corresponding transformation formulas

\[ M_1 = M_0 - 2(\ln R)_{xy} = -M_0 + \frac{2R_x R_y}{R^2} = R \left( \frac{1}{R} \right)_{xy} \tag{3} \]

\[
\begin{align*}
(R\vartheta)_x &= -R^2 \left( \frac{\varphi}{R} \right)_x, \\
(R\vartheta)_y &= R^2 \left( \frac{\varphi}{R} \right)_y,
\end{align*}
\tag{4}
\]

(and their analogues for (2), see [2, 3]) establish a (multivalued) correspondence between the solution of the Moutard equation \( (M_0) \) (i.e. (1) with the potential \( M_0(x, y) \)) and \( (M_1) \) (i.e. (1))

\*The research described in this contribution was supported in part by grant RFBR-DFG No 96-01-00050
with the potential $M_1(x, y)$. Note that $R_1 = \frac{1}{R}$ is also a solution of $(M_1)$. Let us suppose that we can find the complete solution of (I) for some given potential $M_0$; such a solution has the initial data given for example by the Goursat problem (2 functions of 1 variable), from (I) one may obtain (via a quadrature) the general solution of $(M_1)$. For example for the case $M_0 = 0$ the general solution is given by $u = \varphi(x) + \psi(y)$, the Moutard transformation yields

$$
\begin{align*}
  u &= \left(\alpha(x)\psi(y) - \beta(y)\varphi(x) + \int(\psi\beta - \beta\psi)\,dy + \\
  &\quad + \int(\varphi\alpha - \alpha\varphi)\,dx\right) \frac{1}{\alpha(x) + \beta(y)}
\end{align*}
$$

for the potential $M_1 = -2[\ln(\alpha(x) + \beta(y))]_{xy} = \frac{2\alpha_x\beta_y}{(\alpha + \beta)^2}$ (i.e. $R = \alpha(x) + \beta(y)$).

If we will build the following chain of Moutard transformations $(M_0) \rightarrow (M_1) \rightarrow (M_2) \rightarrow \ldots (M_k) \rightarrow \ldots$ we will see that (apriori) the $k$-th potential depends on the choice of $2k$ functions of 1 variable — the initial data of the solutions $R_s(x, y)$ of $(M_s)$, $s = 0, 1, \ldots, k - 1$.

In [8] a method was found that gives us the possibility to express the potential $M_k$ and all solutions of $(M_k)$ via $2k$ solutions of the initial equation $(M_0)$ (the ”pfaffian formula”, analogous to the ”wronskian formula” for the case of the Darboux transformations for $(1 + 1)$-dimensional integrable equations [5, 6]).

Nevertheless the fundamental question how ”wide” is the obtained class of potentials $M_k$ (in the space of all smooth functions of two variable) was open. In the theory of $(1 + 1)$-dimensional integrable equations the question of density of the finite gap solutions of the Korteweg-de Vries equation in the class of all quasiperiodic functions was answered positively. In this paper we show that the set of the potentials $M_k$ obtainable from any fixed $M_0$ is ”locally dense” in the space of all smooth functions in a sense to be detailed below. Hence we prove that the found in [3, 4] families of solutions of the corresponding $(2 + 1)$-dimensional integrable equations give locally ”almost every” solution.

**Theorem 1** Let an initial potential $M_0(x, y) \in C^\infty$ be given in a neighborhood of $(0, 0)$. Then for any $N = 0, 1, 2, \ldots$ one may find some $K$ such that for any given numbers $P_{x_1 \ldots x_k}$, $0 \leq k \leq N$, $x_s \in \{x, y\}$, the corresponding derivatives of $M_K$ (in the chain of the Moutard transformations $(M_0) \rightarrow (M_1) \rightarrow (M_2) \rightarrow \ldots (M_n) \rightarrow \ldots$) at $(0, 0)$ coincide with $P_{x_1 \ldots x_k}$:

$$
\partial_{x_1}\partial_{x_2} \ldots \partial_{x_k} M_K(0, 0) = P_{x_1 \ldots x_k}, \quad \partial_z = \frac{\partial}{\partial z}, \quad k \leq N.
$$

(5)

**Proof** will be given by induction. For $N = 0$ we set $K = 1$, from (3)

$$
M_1(0, 0) = -M_0(0, 0) + 2\frac{R_x R_y}{R^2} \bigg|_{(0, 0)}
$$

where $R$ is a solution of $(M_0)$. As one can prove (see for example [3]), the values $\varphi(x) = R(x, 0)$, $\psi(y) = R(0, y)$ ($\varphi(0) = \psi(0)$) may be chosen as the initial data for (I). So the quantities $R(0, 0)$, $R_x(0, 0)$, $R_{xx}(0, 0)$, $\ldots$, $R_y(0, 0)$, $R_{yy}(0, 0)$, $\ldots$ are independent. Set $R(0, 0) = 1$, $R_y(0, 0) = 1/2$. Changing $R_x(0, 0)$ one can make $M_1(0, 0)$ equal to the given number $P$ which
proves Theorem for the case \( N = 0 \). In addition if we will suppose that all the other nonmixed derivatives \( \partial^k_x R, \partial^k_y R, k > 1 \), are equal to 0 then the higher derivatives \( \partial^m_y \partial^n_y M_1 \) are not arbitrary and are uniquely determined by \( P = M_1(0,0) \) (for the given \( M_0(x,y) \)). Therefore we have proved for the case \( N = N_0 = 0 \) the following main inductive proposition:

for any \( N = N_0 \) one can find \( K \) such that the corresponding derivatives \( \partial^m_x \partial^n_y M_K \) of an appropriately constructed \( M_K \) at \((0,0)\) coincide with the given numbers \( P_{\underline{x} \ldots \underline{x} y \ldots y} \) for \( m+n \leq N_0 \), and the higher derivatives \( \partial^m_x \partial^n_y M_K, m + n > N_0 \), depend (for the fixed \( M_0(x,y) \)) only upon \( P_{\underline{x} \ldots \underline{x} y \ldots y} = \partial^p_x \partial^q_y M_K(0,0) \) for \( p + q \leq N_0, p \leq m \).

The step of induction. Provided the main inductive proposition is proved for all derivatives of orders \( \leq N = N_0 \) we will prove it for \( N = N_0 + 1 \). Let \( K_0 \) be the corresponding to \( N = N_0 \) number of the potential \( M_{K_0} \) for which we have proved the inductive proposition. Performing another \( N_0 + 2 \) Moutard transformations we get \( M_P \), \( P = K_0 + N_0 + 2 \); let us check the validity of the inductive proposition for this function. From (3) we get

\[
M_P = (-1)^{N_0} \left( M_{K_0} - \frac{2 R_x^0 R_y^0}{(R^0)^2} + \frac{2 R_x^0 R_y^1}{(R^1)^2} - \cdots \pm \frac{2 R_x^{N_0+1} R_y^{N_0+1}}{(R^{N_0+1})^2} \right),
\]

where \( R^s \) are solutions of \( (M_{K_0+s})_0 \), \( s = 0, \ldots, N_0 + 1 \). Let us call the derivative \( \partial^1_x R^s \) at the origin the principal derivative of \( R^s \) and \( \partial^{N_0+2-s} R^s \) its auxiliary derivative. We suppose further that at \((0,0)\) the values of all \( R^s \) are equal to 1 and the values of their auxiliary derivatives are equal to 1/2, the principal derivatives are (so far) undefined, all the other nonmixed derivatives \( R^{(s)}_{x,\ldots,x}, R^{(s)}_{y,\ldots,y} \) of all orders are set to 0 at the origin. Consider now the derivative

\[
\frac{\partial^{N_0+1} M_P}{\partial y^{N_0+1}} = (-1)^{N_0} \left( \frac{\partial^{N_0+1} M_{K_0}}{\partial y^{N_0+1}} - \frac{2 R_x^0 \partial^{N_0+2} R_y^0}{(R^0)^2} + \cdots \pm \frac{2 R_x^{N_0+1} \partial^{N_0+2} R_y^{N_0+1}}{(R^{N_0+1})^2} \right) + F(R^s, R^s_x, R^s_y, M_{K_0+s}, \partial_y M_{K_0+s}), \tag{6}
\]

(at \((0,0)\)), where \( F \) comprises all the terms except the given in parentheses. The mixed derivatives \( \partial^m_x \partial^n_y R^s \) are eliminated using (3). As one can easily see the only principal derivative \( R_x^{(0)} \) does not appear in \( F \).

We will use an additional induction over \( k \) to prove that the values of \( M_{K_0+s} \) and their derivatives w.r.t \( y \) included in \( F \) (they have the orders \( \leq N_0 \)) at the origin are determined uniquely by the equations \( \partial^k_y M_P = P_{\underline{y} \ldots \underline{y} y}, k = 0,1, \ldots, N_0 \). Indeed for \( k = k_0 = 0 \)

\[
P = M_P\big|_{(0,0)} = (-1)^{N_0} \left( M_{K_0} - \frac{2 R_x^0 R_y^0}{(R^0)^2} + \cdots \pm \frac{2 R_x^{N_0+1} R_y^{N_0+1}}{(R^{N_0+1})^2} \right)\big|_{(0,0)},
\]

where all the derivatives of \( R^s \) except \( R_x^0 \) are not principal i.e. fixed. Since the coefficient of the principal derivative \( R_x^{(0)} \) is \( R_y^0 = 0 \), we can find \( M_{K_0}(0,0) \). The values \( M_{K_0+s} \) are found
from
\[ M_{K_0+s} = (-1)^s \left( M_{K_0} - \frac{2 R_y^{(0)} R_y^{(s)}}{(R^{(0)})^2} + \cdots + \frac{2 R_y^{(s-1)} R_y^{(s-1)}}{(R^{(s-1)})^2} \right). \tag{7} \]

For \( k = k_0 + 1 \leq N_0 \) (the step of the additional induction)

\[ P_{y \ldots y} = \partial_y^{k_0+1} M_P = (-1)^{N_0} \left( \partial_y^{k_0+1} M_{K_0} - \sum_{s=0}^{N_0+1} (-1)^s \frac{2 R_y^{(s)} \partial_y^{k_0+1} R_y^{(s)}}{(R^{(s)})^2} \right) + \]

\[ + F(R^{(q)}, R_x^{(q)}, \partial_y R^{(q)}, M_{K_0+q}, \partial_y m M_{K_0+q}), \]

where again the only principal derivative \( R_x^{(0)} \) has everywhere zero coefficients (since \( k_0 + 1 \leq N_0 \)) and the derivatives \( \partial_y^m M_{K_0+q} \) have the orders \( m \leq k_0 \) i.e. are already found. The derivatives \( \partial_y^{k_0+1} M_{K_0+1} \) are found differentiating \( \partial_y \). The additional induction is finished.

Since according to the main inductive proposition we can actually choose \( \partial_y^m M_{K_0}, \, 0 \leq m \leq N_0 \), arbitrarily, we set them in \( \partial_y \) to be equal to the values found above. Among the still indefinite quantities in \( \partial_y \) only the principal derivative \( R_y^{(0)} \) is present. Its coefficient is equal to \( \frac{2 \partial_y^{N_0+2} R_y^{(0)}}{(R_y^{(0)})^2} = 1 \). Choosing appropriately the value of \( R_y^{(0)} \) we will obtain the desired equality \( P_{y \ldots y} = \partial_y^{N_0+1} M_P \) for arbitrary \( \partial_y^{k_0+1} M_{K_0} \). Since due to the inductive proposition \( \partial_y^{N_0+1} M_{K_0} \) depends only upon \( \partial_y^m M_{K_0}, \, m \leq N_0 \), the chosen \( R_y^{(0)} \) depends only on them and

\[ P_{y \ldots y} = \partial_y^{N_0+1} M_P. \]

Consequently calculating all the higher derivatives \( \partial_y^m M_P \) of the orders \( m > N_0 + 1 \), we conclude that they depend only on the quantities \( P_{y \ldots y}, \, m \leq N_0 + 1 \).

The possibility to obtain \( P_{x \ldots x y \ldots y} = \partial_x^m \partial_y^{N_0+1-n} M_P, \, n \leq N_0 + 1 \), will be proved by an auxiliary induction over \( n \). The case \( n = 0 \) has been just considered.

Provided we have proved for all \( n \leq n_0 \) that

a) the principal derivatives \( R_x^{(0)}, R_x^{(1)}, \ldots, R_x^{(n)}, \ldots, x \), are already chosen in such a way that

\[ P_{x \ldots x y \ldots y} = \partial_x^k \partial_y^{N_0+1-k} M_P, \, k \leq n, \text{ hold,} \]

b) in

\[ \partial_x^k \partial_y^{N_0+1-k} M_P = (-1)^{N_0} \left( \partial_x^k \partial_y^{N_0+1-k} M_{K_0} \right) \]

\[ - \sum_{s=0}^{N_0+1} (-1)^s \frac{2 \partial_x^{k+1} R_x^{(s)} \partial_y^{N_0+2-k} R_y^{(s)}}{(R_x^{(s)})^2} \]

\[ + F(R^{(q)}, \partial_x R^{(q)}, \partial_y R^{(q)}, M_{K_0+q}, \partial_x^m M_{K_0+q}), \]

for \( k \leq n \) in the terms collected in \( F \) only the principal derivatives defined during the previous steps as well as \( \partial_x^r \partial_y^m M_{K_0+q} \) with \( r \leq n, \, m + r \leq N_0, \, K_0 + q < P \), are present,
c) all the higher derivatives $\partial^m_x \partial^k_y M_P$, $m + k > N_0 + 1$, $m \leq n$, and the already defined principal derivatives $\partial^{n+1}_x R^{(s)}$, $s \leq n$, depend (according to our construct of $M_P$) only on the choice of $P_x \ldots x y \ldots y, p + q \leq N_0 + 1, p \leq m$.

we will make the inductive step $n = n_0 + 1$. Then in the rest $F$ only the already defined principal derivatives of $R^{(s)}$ and $\partial^R_x \partial^m_y M_{K_0+q}$ with $r \leq n_0 + 1, m + r \leq N_0$, will appear alongside with the still unknown $\partial^{n_0+1}_x \partial^m_y M_{K_0+q}$.

In order to determine these quantities we will make another imbedded induction over $m$ using the equations $P_x \ldots x y \ldots y = \partial^{n_0+1}_x \partial^{N-n_0-1}_y M_P, N \leq N_0$. Indeed for $m = 0$ in

$$\partial^{n_0+1}_x M_P = (-1)^{N_0} \left( \partial^{n_0+1}_x M_{K_0} - \sum_{s=0}^{N_0+1} (-1)^s \frac{2\partial^{n_0+2}_x R^{(s)} R^{(s)}_y}{(R^{(s)})^2} \right) + F$$

(9)

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$$\partial^{n_0+1}_x M_P = (-1)^{N_0} \left( \partial^{n_0+1}_x M_{K_0} - \sum_{s=0}^{N_0+1} (-1)^s \frac{2\partial^{n_0+2}_x R^{(s)} R^{(s)}_y}{(R^{(s)})^2} \right) + F$$

(9)

Differentiating (9) we find $\partial^{n_0+1}_x M_{K_0+q}|_{(0,0)}$, $K_0 + q < P$ inductively over $q$. The step of the imbedded induction we perform as earlier applying the operator $\partial^m_y$ to (9). Again according to the main inductive proposition (valid for $M_{K_0}$) $\partial^{n_0+1}_x \partial^{N-n_0}_y M_{K_0}$ depends only upon $\partial^m_x \partial^m_y M_{K_0}$, $m + n \leq N_0$, $n \leq n_0 + 1$, which due to (9) implies the same dependence of the principal derivative $\partial^{n_0+2}_x R^{(n_0+1)}$. The validity of a), b), c) is easily checked now for $n = n_0 + 1$.

The auxiliary induction over $n$ proves that we can consecutively choose definite values of $R^{(0)}_x, R^{(1)}_{xx}, \ldots, \partial^{N_0+2}_x R^{(N_0+1)}$ in such a way that (9) hold for $N = N_0 + 1$ and (8) holds also for $N \leq N_0$ after an appropriate choice of $\partial^m_x \partial^m_y M_{K_0}, m + r \leq N_0$, which is possible due to the inductive proposition. The higher derivatives $\partial^m_x \partial^m_y M_{K_0}, m + r > N_0 + 1$, depend only upon the lower order ones as indicated in the inductive proposition. Therefore the main inductive proposition as well as Theorem are proved.

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