Canonical Quantization of the Belinskiĭ-Zakharov One-Soliton Solutions

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Abstract

We apply the algebraic quantization programme proposed by Ashtekar to the analysis of the Belinskii-Zakharov classical spacetimes, obtained from the Kasner metrics by means of a generalized soliton transformation. When the solitonic parameters associated with this transformation are frozen, the resulting Belinskii-Zakharov metrics provide the set of classical solutions to a gravitational minisuperspace model whose Einstein equations reduce to the dynamical equations generated by a homogeneous Hamiltonian constraint and to a couple of second-class constraints. The reduced phase space of such a model has the symplectic structure of the cotangent bundle over $\mathbb{R}^+ \times \mathbb{R}^+$. In this reduced phase space, we find a complete set of real observables which form a Lie algebra under Poisson brackets. The quantization of the gravitational model is then carried out by constructing an irreducible unitary representation of that algebra of observables. Finally, we show that the quantum theory obtained in this way is unitarily equivalent to that which describes the quantum dynamics of the Kasner model.

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1. Introduction

In recent years, a special attention has been paid in General Relativity to the search for exact solutions possessing two commuting Killing vectors.\textsuperscript{1,2} Spacetimes of this kind provide exact gravitational solutions in a variety of interesting physical problems, eg, in models with plane, cylindrical or stationary axial symmetry.\textsuperscript{1}

The Einstein field equations for spacetimes with a two-dimensional Abelian group of isometries which act orthogonally and transitively on non-null orbits are non-linear partial differential equations in two variables.\textsuperscript{3} A nice property of such field equations is that they admit symmetry transformations which leave invariant the set of all classical solutions. Using this result, a series of solution-generating techniques have been developed to construct new exact solutions from known ones in models with an Abelian two-parameter group of isometries.\textsuperscript{2} Among these techniques, probably the most powerful generating method is the inverse-scattering transformation of Belinskii and Zakharov,\textsuperscript{4} both because of its simplicity and generality.

The Belinskii-Zakharov (BZ) technique, or soliton transformation, is a generalization of the inverse-scattering method which has proved to be so fruitful in analysing non-linear partial differential equations in two dimensions exhibiting soliton solutions.\textsuperscript{5} This generalization consists essentially in substituting the fixed poles of the inverse-scattering transformation by pole trajectories. In particular, Belinskii and Zakharov applied this generalized soliton transformation to the Kasner metric\textsuperscript{3} (ie, to the diagonal Bianchi type I metric) to obtain a new exact solution of the Einstein equations in vacuum which is now known as the BZ one-soliton solution,\textsuperscript{4} and which can be described by the non-diagonal inhomogeneous metric

\begin{equation}
 ds^2 = f(t, z) \left( -dt^2 + dz^2 \right) + g_{ab}(t, z) \, dy^a dy^b, \quad (a, b = 1, 2), \tag{1.1}
\end{equation}

\begin{equation}
 f(t, z) = C \, t^{2p^2-1} \frac{\cosh (pr + D)}{| \sinh (\frac{r}{2}) |}, \tag{1.2}
\end{equation}
Here, $t$, $z$ and $y^a$ ($a = 1, 2$) are a set of spacetime coordinates, $A$, $B$ and $C$ are positive constants, $p$ and $D$ are real parameters, and $r$ is defined as

$$r = \ln \left( \frac{(\mu / t)^2}{(\mu / t)^2} \right), \quad \mu = (z_0 - z) - \sqrt{(z_0 - z)^2 - t^2},$$

(1.6)

$\mu$ being the pole trajectory of the BZ transformation\(^1\), and $z_0$ a real constant that can be interpreted as the origin of the $z$ coordinate.

The BZ metric (1.1-6) is defined, in principle, only for $(z - z_0)^2 \geq t^2$, so that $\mu$ in (1.6) be real. Nevertheless, the above solution can be extended to the region $(z - z_0)^2 < t^2$ for some particular values of the parameters appearing in expressions (1.2-5), like, for instance, for $p = \frac{1}{2}$.\(^6\)

Our aim in this work is the construction of a consistent quantum description for the BZ one-soliton model. The motivation is twofold. On the one hand, the analysis to be presented can be understood as a previous step before dealing with the quantization of inhomogeneous gravitational models admitting two commuting spacelike Killing fields and whose physical degrees of freedom depend not only on time, but also on one spatial coordinate. On the other hand, we want to discuss whether the relation between different families of classical spacetimes which is provided by the soliton transformation has any quantum mechanical counterpart. The quantization of the BZ model will supply an explicit example for the study of this issue, because the Kasner metric (the seed metric that leads to the BZ solutions through the soliton transformation\(^1,4\)) has already been quantized in the literature successfully.\(^7,8\)

To quantize the BZ one-soliton solutions, we will apply the extension of Dirac’s canonical quantization programme\(^9\) proposed by Ashtekar et al.\(^10,11\) We will first
identify the physical degrees of freedom of the considered model and determine the symplectic structure of the associated reduced phase space. Using this structure, we will find a complete set of real classical observables that form a Lie algebra under Poisson brackets. We will then represent these observables by quantum operators acting on a particular vector space, each element of this space representing a physical quantum state. Finally, we will fix the inner product in the space of physical states by imposing a set of reality conditions,\textsuperscript{11–13} that is, by promoting to adjointness requirements on quantum operators the complex conjugation relations that exist between the classical observables of the system.

In this way, apart from being of interest by the reasons explained above, the quantization of the BZ solutions will provide a new example to be added to the now relatively large list of minisuperspace gravitational models in which it has been possible to check the consistency and applicability of the non-perturbative canonical quantization programme elaborated by Ashtekar.\textsuperscript{7,8,14}

The remainder of this paper is organized as follows. We briefly discuss the general form of the Einstein field equations for spacetimes with two commuting spacelike Killing vectors in Sec. 2, where we also review some results on the soliton transformation. In Sec. 3 we re-examine the quantization of the diagonal Bianchi type I, following as close as possible the quantization methods that are to be employed in the study of the BZ one-soliton metrics. In Sec. 4 we prove that the Einstein equations for the family of BZ one-soliton solutions can be interpreted as the dynamical equations generated by a homogeneous gravitational Hamiltonian, supplemented with a set of second-class constraints. The corresponding global structure of the reduced phase space of the BZ model is determined in Sec. 5. In that section, we also carry out to completion the quantization of the studied gravitational system, and compare the obtained quantum theory with that constructed in Sec. 3 for the Kasner model. Finally, we present our conclusions in Sec. 6.
2. The BZ One-Soliton Transformation

For spacetimes possessing two commuting spacelike Killing vectors, the four-dimensional metric can always be expressed in the generic form (1.1). Defining then

\[ |g| = \det \{g_{ab}(t, z)\}, \quad (2.1) \]

one can show that the Einstein equations imply that \(|g|^{\frac{1}{2}}\) must satisfy the following wave equation in two dimensions

\[ \partial_t^2 |g|^{\frac{1}{2}} - \partial_z^2 |g|^{\frac{1}{2}} = 0. \quad (2.2) \]

As a consequence, the time coordinate can always be chosen proportional to \(|g|^{\frac{1}{2}}\),

\[ |g|^{\frac{1}{2}} \propto t. \quad (2.3) \]

As long as \(|g|\) depends only on time, the \((t, t)\) and \((t, z)\) components of the Einstein equations for the metrics (1.1) can be seen to adopt the respective expressions

\[ G^t_t = \frac{1}{4f} \left( \partial_t \ln f \partial_t \ln |g| + \frac{\partial^2_t |g|}{2|g|} - \frac{1}{2} g^{ab}(\partial^2_t g_{ab} + \partial^2_z g_{ab}) \right) = 0, \quad (2.4) \]

\[ G^z_t = -\frac{1}{4f} \left( \partial_z \ln f \partial_t \ln |g| - g^{ab} \partial_z \partial_t g_{ab} \right) = 0, \quad (2.5) \]

where \(g^{ab}\) is the inverse of the metric \(g_{ab}\) and the lower case Latin letters from the beginning of the alphabet denote spatial indices, with values equal to 1 or 2.

The rest of non-vanishing components of the Einstein equations turn out to be equivalent to the integrability conditions for the system (2.4,5), and can be written in the compact form

\[ \partial_t \left( A^b_a \right) - \partial_z \left( B^b_a \right) = 0, \quad (2.6) \]

with

\[ A^b_a = t \partial_t g_{ac} g^{cb} \quad \text{and} \quad B^b_a = t \partial_z g_{ac} g^{cb}. \quad (2.7) \]

The BZ generating technique exploits the fact that the non-linear system (2.6) can be regarded as well as the integrability conditions associated with the linear
eigenvalue problem  

\begin{align}
\left( \partial_t - \frac{2\lambda t}{\lambda^2 - t^2} \partial_\lambda \right) \Psi_{ab} &= - \frac{t A_c^a + \lambda B_c^a}{\lambda^2 - t^2} \Psi_{cb}, \\
\left( \partial_z - \frac{2\lambda^2}{\lambda^2 - t^2} \partial_\lambda \right) \Psi_{ab} &= - \frac{\lambda A_c^a + t B_c^a}{\lambda^2 - t^2} \Psi_{cb},
\end{align}

\lambda being a complex variable and \( \Psi(t, z, \lambda) \) a 2 \times 2 matrix. If we know a particular solution \( (f^{(0)}(t, z), g^{(0)}(t, z)) \) of the system of equations (2.2) and (2.4-6) (what we call a seed metric), the resolution of the eigenvalue problem (2.8,9), for \( A^b_a \) and \( B^b_a \) evaluated at \( g^{(0)} \) and with boundary condition

\[
\lim_{\lambda \to 0} \Psi_{ab}(t, z, \lambda) = g^{(0)}_{ab}(t, z),
\]

allows us to obtain a new solution to the Einstein equations in the following way. We first define the pole trajectory \( \mu \) as on the right hand side of Eq. (1.6). Then, the one-soliton transform of the seed metric \( (f^{(0)}(t, z), g^{(0)}(t, z)) \) can be calculated by the formulæ

\[
f = N f^{(0)} \frac{\mu^2 Q}{\sqrt{|t| (\mu^2 - t^2)}}, \quad g_{ab} = \left| \frac{\mu}{t} \right| \left( g^{(0)}_{ab} - \frac{(\mu^2 - t^2)}{\mu^2 Q} L_a L_b \right),
\]

with

\[
L_a = m_c \left( \Psi^{-1} \right)_{\lambda=\mu}^{cb} g^{(0)}_{ba}, \quad Q = L_a g^{(0)ab} L_b.
\]

Here, \( (m_1, m_2) \) are two complex parameters, \( N \) is a constant and \( (\Psi^{-1})^{cb}_{\lambda=\mu} \) is the inverse of the matrix \( \Psi_{ab} \) [solution to Eqs. (2.8-10)] evaluated at \( \lambda = \mu \).

In particular, one can take the Kasner metrics as the seed for the above transformation, since, with an appropriate choice of the time gauge, these metrics can be written in the form (1.1), with metric functions \( f \) and \( g \) given by

\[
f = \tilde{C} t^{2p^2 - \frac{1}{2}}, \quad g_{11} = AB t^{1+2p},
\]

\[
g_{22} = \frac{A}{B} t^{1-2p}, \quad g_{12} = 0.
\]
A, B and \( \tilde{C} \) are three positive constants and \( p \in \mathbb{R} \). For real solitonic parameters \((m_1, m_2)\), the new exact vacuum solutions to the Einstein equations that one reaches in this way are precisely the BZ inhomogeneous metrics displayed in Eqs. (1.1-6).

3. The Diagonal Bianchi Type I Model

Although the canonical quantization of the diagonal Bianchi type I model has already been completed,\(^7\)\(^8\) we want to present here a slightly different version of the quantization procedure which will prove specially suited to discuss the relation between the quantum theories that respectively describe the BZ one-soliton metrics and the Kasner solutions. We will begin by analysing the classical dynamics of the Kasner model in Subsection 3.a, and attain the desired quantization in Subsection 3.b.

3.a. Classical Analysis

The Kasner metrics (2.13,14) admit the generic expression

\[
\begin{align*}
  f &= e^{2Z(t)}, \\
  g_{11} &= e^{2X(t)+2Y(t)}, \\
  g_{22} &= e^{2X(t)-2Y(t)}, \\
  g_{12} &= 0.
\end{align*}
\]

From now on, we will regard these equations as the definition of a homogeneous minisuperspace model whose degrees of freedom are just the functions \( X, Y \) and \( Z \). The determinant of the metric \( g_{ab} \) depends thus only on time: \(|g| = \exp\{4X(t)\}\).

The \((t, z)\) component of the Einstein equations, given by formula (2.5), is identically zero for metrics (3.1,2). From Eq. (2.4), on the other hand, the \((t, t)\) component of the Einstein equations, which can be interpreted as the gravitational Hamiltonian constraint at each point of the spacetime, reduces in this case to

\[
\mathcal{H} \equiv \mathcal{G}_i = \frac{1}{f} \left[ 2 \frac{dX}{dt} \frac{dZ}{dt} + \left( \frac{dX}{dt} \right)^2 - \left( \frac{dY}{dt} \right)^2 \right] = 0. \tag{3.3}
\]
The integration of this Hamiltonian constraint over each surface of constant time gives the total Hamiltonian, \( H \), that generates the dynamics of the studied gravitational model. To simplify calculations, it is convenient to choose the following time gauge:

\[
\frac{dt}{|g|^{1/2}} = e^{2X(T)} dT.
\]  

(3.4)

The Hamiltonian \( H \) that dictates the evolution in the new time coordinate \( T \) can then be defined as

\[
H = \frac{1}{V_\Omega} \int_\Omega \sqrt{|g(4)|} \mathcal{H} = [2\dot{X}\dot{Z} + \dot{X}^2 - \dot{Y}^2].
\]  

(3.5)

Here, \( \int_\Omega \) denotes integration over a constant-\( T \) section of the spacetime, \( V_\Omega \) is the volume of that section (\( V_\Omega = \int_\Omega 1 \)), \( g(4) \) is the determinant of the 4-metric, and the dots represent the derivative with respect to \( T \). The right hand side of formula (3.5) has been obtained by using Eqs. (3.3,4) and the fact that, for the set of coordinates \( (T, z, y_1, y_2) \),

\[
\sqrt{|g(4)|} = f e^{4X}.
\]  

(3.6)

It is worth noticing that the Hamiltonian \( H \), as given by Eq. (3.5), is well-defined in the limit \( V_\Omega \to \infty \).

From the above Hamiltonian, it is possible to deduce the expressions of the momenta canonically conjugated to \( X, Y \) and \( Z \) by assuming the implicit dependence of the first time derivatives \( \dot{X}, \dot{Y} \) and \( \dot{Z} \) on such momenta. Let us first introduce the notation \( u_i \equiv \{X, Y, Z\} \ (i = 1, 2, 3) \). Differentiating then \( H \) with respect to \( \dot{u}_i \), applying the chain rule and substituting the Hamiltonian equations \( \partial_{p_j} H = \dot{u}_i \), we get

\[
\partial_{u_i} H = \partial_{u_i} p_j \partial_{p_j} H = \partial_{u_i} p_j \dot{u}_j,
\]  

(3.7)

where \( p_j \) is the momentum conjugate to \( u_j \). Making use of formulae (3.5) and (3.7), it is now a simple exercise to show that

\[
p_X = 2(\dot{X} + \dot{Z}), \quad p_Y = -2\dot{Y}, \quad p_Z = 2\dot{X},
\]  

(3.8)
up to irrelevant additive constant factors. Therefore, the Hamiltonian (3.5) can be rewritten in terms of the canonical momenta as

$$H = \frac{1}{4}[2p_Xp_Z - p_Y^2 - p_Z^2]. \quad (3.9)$$

The dynamical equations that follow from this Hamiltonian state that $p_X$, $p_Y$ and $p_Z$ are constants in the evolution, and that $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ are given by Eqs. (3.8). In particular, recalling that $|g|^\frac{1}{2} = \exp (2X)$ and Eq. (3.4), we have

$$\frac{d|g|^\frac{1}{2}}{dt} = 2\dot{X} = p_Z = \text{const}. \quad (3.10)$$

Thus, $|g|^\frac{1}{2}$ turns out to be linear in the time coordinate $t$, and the Einstein equation (2.2) is straightforwardly satisfied. Moreover, it is not difficult to check that, for the metrics (3.1,2), the system of non-linear equations (2.6) is already contained in the equations of motion implied by the Hamiltonian (3.9). This is not surprising because, being the $(t,z)$ component of the Einstein tensor identically vanishing in our case, Eqs. (2.6) simply provide the integrability conditions for the constraint (3.3) and, hence, for the Hamiltonian (3.9). So, the Einstein equations for the minisuperspace model analysed here are equivalent to the dynamics generated by the homogeneous Hamiltonian (3.9) and to the first-class constraint $H = 0$.

Since $p_X$, $p_Y$ and $p_Z$ are constants, Eqs. (3.8) can be immediately integrated to give the classical solutions of the model. By choosing the origin of the time coordinate $T$ in such a way that

$$X(T = 0) = 0, \quad (3.11)$$

we get

$$X(T) = \frac{p_Z}{2} T, \quad Y(T) = -\frac{p_Y}{2} T + Y_0, \quad Z(T) = \frac{p_X - p_Z}{2} T + Z_0, \quad (3.12)$$

where $Y_0$ and $Z_0$ are two integration constants, and $(p_X,p_Y,p_Z)$ must satisfy the constraint $H = 0$. 

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If we want to consider only different classical 4-geometries, it is necessary to restrict the allowed range of the parameters on which the classical solutions depend. We notice that the 4-geometries obtained from Eqs. (3.1,2), (3.4) and (3.12) are invariant under time reversal if one also flips the signs of the canonical momenta $(p_X, p_Y, p_Z)$. In order to eliminate this redundancy, we can restrict to the sector of solutions with

$$ p_Z = 2\dot{X}(T) \in \mathbb{R}^+ $$

(3.13)

which contains all possible 4-geometries. On the other hand, if we assume that the coordinates $y^1$ and $y^2$ that appear in Eq. (1.1) are physically indistinguishable, the metrics $g_{ab}$ related by an interchange of the indices 1 and 2 describe the same 4-geometry. From Eqs. (3.1,2) and (3.12), we must then identify those classical solutions which differ just in the sign of the parameters $p_Y$ and $Y_0$. To take into account each physical solution only once, we will restrict $p_Y$ to be a negative constant from now on. Thus, we will impose the following ranges for the parameters present in Eq. (3.12):

$$ Y_0, Z_0 \in \mathbb{R}, \quad p_Y \in \mathbb{R}^-, \quad p_Z \in \mathbb{R}^+, \quad (3.14) $$

whereas $p_X$ is fixed by $p_Y$ and $p_Z$ through the constraint $H = 0$:

$$ p_X = \frac{p_Z}{2} + \frac{p_Y^2}{2p_Z}. \quad (3.15) $$

Note that, being $\dot{X}$ a positive constant, we can explicitly define the time coordinate $t$ in Eq. (3.4) as

$$ t = \int_{-\infty}^{T} e^{2X(T)} dT, \quad (3.16) $$

so that, on the classical solutions (3.12), the relation between the two time gauges employed in our calculations is given by

$$ e^{pzT} = pz t. \quad (3.17) $$

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If one introduces now the redefinitions

\[ p_z = A, \quad p_y = -2pA, \quad y_0 = \frac{1}{2} \ln B - p \ln A, \tag{3.18} \]

\[ Z_0 = \frac{1}{2} \ln \tilde{C} + \left( \frac{1}{4} - p^2 \right) \ln A, \tag{3.19} \]

where, from conditions (3.14),

\[ A, B, \tilde{C}, p \in \mathbb{R}^+, \tag{3.20} \]

a trivial computation shows that the classical metrics determined by Eqs. (3.1,2), (3.12) and (3.15) are just the Kasner metrics (2.13,14), with the parameter \( p \) restricted to be positive to take into account the above discussed symmetry under interchange of coordinates.

### 3.b. Quantum Analysis

We can now proceed to the quantization of the diagonal Bianchi type I model. Our first step will consist in determining the symplectic structure of the reduced phase space of this gravitational system. This structure can be obtained as the pull-back to the constraint surface \( H = 0 \) of the symplectic form of the unreduced phase space of the model,

\[ \Gamma = dX \wedge dp_X + dY \wedge dp_Y + dZ \wedge dp_Z \big|_{H=0}, \tag{3.21} \]

where we have used the fact that \((X, Y, Z, p_X, p_Y, p_Z)\) form a canonical set of phase space variables.

The symplectic form (3.21) can be proved to be time-independent. Therefore, we can evaluate it at any constant-\( T \) section of the spacetime, the final result being insensitive to the specific section selected. Choosing the \( T = 0 \) surface and recalling condition (3.11), we get

\[ \Gamma = dY_0 \wedge dp_Y + dZ_0 \wedge dp_Z, \tag{3.22} \]
with $Y_0$ and $Z_0$ the initial values of $Y$ and $Z$ [see Eq. (3.12)]. The change of variables (3.18,19) then leads to
\[
\Gamma = dA \wedge dP_A + dp \wedge dP_p,
\]
(3.23)
where
\[
P_A = p \ln B - \frac{1}{2} \ln \tilde{C}, \quad P_p = A (\ln B - 2p),
\]
(3.24)
or, equivalently,
\[
B = e^{2p + \frac{P_p}{4}}, \quad \tilde{C} = e^{4p^2 + 2 \frac{P_p}{4} - 2P_A}.
\]
(3.25)
Equations (3.20) and (3.24) imply that
\[
A, p \in \mathbb{R}^+ \quad \text{and} \quad P_A, P_p \in \mathbb{R},
\]
(3.26)
so that we can interpret the symplectic structure obtained for the reduced phase space of the diagonal Bianchi type I model as that corresponding to the cotangent bundle over $\mathbb{R}^+ \times \mathbb{R}^+$.

A complete set of elementary variables in this reduced phase space (ie, a complete set of observables) is provided by $A, p$ and their generalized momenta$^8$
\[
L_A = AP_A, \quad L_p = pP_p.
\]
(3.27)
Conditions (3.26) ensure that all these observables are real. On the other hand, since their only non-vanishing Poisson brackets are
\[
\{A, L_A\} = A, \quad \{p, L_p\} = p,
\]
(3.28)
they form the Lie algebra $L(T^*GL(1, \mathbb{R}) \times T^*GL(1, \mathbb{R}))$, $T^*GL(1, \mathbb{R})$ being the semidirect product of $\mathbb{R}$ and $\mathbb{R}^+$.

In order to quantize the model, we will represent the observables $(A, p, L_A, L_p)$ as operators acting on the vector space of complex functions $\Psi(A, p)$ over $\mathbb{R}^+ \times \mathbb{R}^+$. Each complex function $\Psi$ will represent in this way a physical quantum state of the
diagonal Bianchi type I model, because we have already got rid of all the constraints of the system. The action of the operators \( \hat{A}, \hat{p}, \hat{L}_A, \hat{L}_p \) can be explicitly defined as

\[
\begin{align*}
\hat{A} \Psi(A, p) &= A \Psi(A, p), \\
\hat{p} \Psi(A, p) &= p \Psi(A, p), \\
\hat{L}_A \Psi(A, p) &= -iA \partial_A \Psi(A, p), \\
\hat{L}_p \Psi(A, p) &= -ip \partial_p \Psi(A, p),
\end{align*}
\]

where we have taken \( \hbar = 1 \). It is easy to check that the above operators form a closed algebra under commutators which coincides with the Lie algebra of Poisson brackets (3.28).

According to Ashtekar,\(^{11}\) the inner product in the space of physical states can be determined by imposing a set of reality conditions.\(^{12}\) For the elementary observables that we have chosen, the reality conditions demand that the operators (3.29,30) be self-adjoint, because they all represent real classical observables. These hermiticity requirements select the physical inner product

\[
< \Phi, \Psi > = \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{dA dp}{A p} \Phi^*(A, p) \Psi(A, p),
\]

with \( \Phi^* \) the complex conjugate to \( \Phi \). So, the Hilbert space of physical states is \( L^2(\mathbb{R}^+ \times \mathbb{R}^+, dAdp(Ap)^{-1}) \). Actually, what we have obtained by implementing Ashtekar’s programme is just an irreducible unitary representation of the algebra of observables \( L(T^*GL(1, \mathbb{R}) \times T^*GL(1, \mathbb{R})) \). We notice in this sense that the measure that appears in Eq. (3.31) is precisely that corresponding to the product of two copies of the group \( T^*GL(1, \mathbb{R}) \).\(^{17}\) The results presented in this section are in complete agreement with those reached in Ref. 7 for the quantization of the Bianchi type I once the diagonal reduction to the Kasner model is taken into account.

4. The BZ One-Soliton Model

We turn now to the analysis of the BZ one-soliton solutions, described in Eqs. (1.1-6). In the following, we will restrict our attention exclusively to families of BZ
metrics with fixed values of the parameters $z_0$ and $D$. This restriction can be proved to guarantee that the BZ solutions possess exactly the same number of degrees of freedom as the seed metrics from which they can be obtained by means of a soliton transformation,\textsuperscript{15} that is, as the diagonal Bianchi type I model. In fact, one can show that fixing the constants $z_0$ and $D$ results in freezing all the solitonic parameters involved in the BZ transformation.\textsuperscript{15}

From Eq. (1.6), it is obvious that the value of $z_0$ can always be absorbed by shifting the origin of the $z$ coordinate:

$$\tilde{z} = z - z_0. \tag{4.1}$$

Restricting $z_0$ to be a given constant is thus equivalent to consider a unique pole trajectory, determined by

$$\mu = -\tilde{z} - \sqrt{\tilde{z}^2 - t^2}. \tag{4.2}$$

Using this equation and the definiton $r = \ln [(\mu/t)^2]$, one can show that the variable $r$ changes its sign under the transformation $\tilde{z} \to -\tilde{z}$. On the other hand, by applying this transformation to the metrics (1.1-5) and flipping the sign of the spatial coordinate $y^2$, it is straightforward to conclude that the BZ solutions that differ only in the sign of the constant $D$ turn out to describe the same classical four-dimensional spacetime geometry. With the aim at keeping this symmetry in our model while fixing the value of $D$, we will restrict from now on this parameter to vanish:

$$D = 0. \tag{4.3}$$

The family of BZ solutions to be studied can then be represented in the form

$$f = \frac{\cosh (\beta(t)r)}{|\sinh (\frac{r}{2})|} e^{2Z(t)}, \tag{4.4}$$

$$g_{11} = \frac{\cosh \left( \left\{ \frac{1}{2} + \beta(t) \right\} r \right)}{\cosh (\beta(t)r)} e^{2X(t)+2Y(t)}, \tag{4.5}$$
\[ g_{22} = \frac{\cosh \left( \frac{1}{2} - \beta(t) \right) r}{\cosh (\beta(t)r)} e^{2X(t)-2Y(t)}, \quad (4.6) \]
\[ g_{12} = -\frac{\sinh \left( \frac{r}{2} \right)}{\cosh (\beta(t)r)} e^{2X(t)}. \quad (4.7) \]

These equations can be regarded as the definition of a gravitational minisuperspace model whose degrees of freedom are the functions \( X, Y, Z \) and \( \beta \), which depend only on time.

A simple computation leads to the result
\[ |g|^{\frac{1}{2}} = e^{2X(t)}. \quad (4.8) \]

Choosing now the time coordinate \( t \) as in Eq. (2.3), the Einstein equation (2.2) reduces in our case to the condition
\[ \frac{dX}{dt} = \frac{1}{2t}. \quad (4.9) \]

The Einstein equations (2.4,5), on the other hand, provide the two gravitational constraints that exist in our minisuperspace model. They can be interpreted in turn as the Hamiltonian constraint and the only non-vanishing momentum constraint at each point of the spacetime. Realizing that all the \( z \)-dependence of the metrics (4.4-7) is contained in the variable \( r \), it is easy to check that the momentum constraint \( G^z_t = 0 \) and the linear combination of constraints
\[ \tilde{G}^t_t \equiv G^t_t + \frac{\partial_t r}{\partial_r} G^z_t = 0 \quad (4.10) \]
determine, respectively, the first partial derivatives \( \partial_t \ln \tilde{f} \) and \( \partial_r \ln \tilde{f} \) of the function \( \tilde{f}(t,r) \equiv f(t,z(t,r)) \) once the metric \( g_{ab} \) is known. The non-linear equations (2.6), as we have already explained, are just the integrability conditions for the two constraints of the system. These conditions can be equivalently imposed by demanding the coincidence at all points of the second partial derivatives
\[ \partial_t \partial_r \ln \tilde{f} = \partial_r \partial_t \ln \tilde{f}, \quad (4.11) \]
which can be computed from the constraints $G^z_t = 0$ and $\tilde{G}^t_t = 0$. Employing Eq. (4.9), a detailed and lengthy but trivial calculation shows that the requirement (4.11), and thus Eqs. (2.6), are satisfied if and only if the following dynamical equations hold

\[
\left( \frac{dY}{dt} \right)^2 = \left( \frac{\beta}{t} \right)^2, \quad (4.12)
\]

\[
\frac{d\beta}{dt} = 0. \quad (4.13)
\]

Finally, substituting Eqs. (4.4-7), (4.9) and (4.12,13) in expression (2.5), one can prove that there exist no classical solutions with $\beta = -t(dY/dt)$, whereas for

\[
\frac{dY}{dt} = \frac{\beta}{t} \quad (4.14)
\]

the momentum constraint $G^z_t = 0$ is trivially satisfied. We thus conclude that, for the considered minisuperspace model, the whole set of Einstein equations reduces to the constraint (4.10) and to the dynamical equations (4.9) and (4.13,14).

Note that Eqs. (4.13,14) determine the metric function $\beta$ in terms of $Y(t)$. Our task in the rest of this section will be to demonstrate that these two equations can be interpreted in fact as second-class constraints which allow us to eliminate $\beta$ as a physical degree of freedom.

Let us first adopt the same time gauge (3.4) that was used in the analysis of the diagonal Bianchi type I model. In this gauge, Eqs. (4.9) and (4.13,14) translate into

\[
\chi_1 \equiv -S(T)\dot{Y} + \beta = 0, \quad (4.15)
\]

\[
\chi_2 \equiv \dot{\beta} = 0, \quad (4.16)
\]

\[
S(T)\dot{X} = \frac{1}{2}, \quad (4.17)
\]

where the dots denote differentiation with respect to $T$ and

\[
S(T) = \frac{t(T)}{e^{2X(T)}}. \quad (4.18)
\]
Defining now
\[ \mathcal{H} \equiv \tilde{G}_t^r - \frac{1}{f e^{4X}} \left[ \frac{\sinh^2 \left( \frac{r}{2} \right)}{S^2(T) \cosh^2 (\beta r)} (S(T) \dot{Y} + \beta) \chi_1 + \tanh (\beta r) r \dot{X} \chi_2 \right], \tag{4.19} \]
a careful calculation, employing Eqs. (4.4-7), (4.10) and (2.4,5), leads to the result
\[ \mathcal{H} = \frac{1}{f e^{4X}} \left[ 2 \dot{X} \dot{Z} + \dot{X}^2 - \dot{Y}^2 + \frac{1}{4S^2(T)} - \frac{\sinh^2 \left( \frac{r}{2} \right)}{4 \cosh^2 (\beta r)} r^2 \beta^2 \right]. \tag{4.20} \]

Assuming that \( \chi_1 \) and \( \chi_2 \) form a set of second-class constraints, \( \mathcal{H} \), as given by Eqs. (4.10) and (4.19), turns out to be a linear combination of all the constraints of the system. In that case, the integration of \( \mathcal{H} \) over each surface of constant time \( T \) will supply us with a total Hamiltonian for the model.\(^9\) We will also admit at this point that Eq. (4.17) is one of the dynamical equations implied by the Hamiltonian evolution. This assumption allows us to substitute Eq. (4.17) in formulae (4.15) and (4.20) to get equivalent expressions on shell for the classical Hamiltonian and the constraint \( \chi_1 \). In particular, \( \chi_1 \) will now read
\[ \chi_1 = \beta - \frac{\dot{Y}}{2X}. \tag{4.21} \]
The consistency of the hypotheses introduced in this paragraph will be proved later on in this section.

Using then Eqs. (4.17) and (4.20), and following a procedure similar to that explained in Sec. 3 for the Kasner model, we obtain a total Hamiltonian for the BZ one-soliton model of the form
\[ H = \frac{1}{V_\Omega} \int_\Omega \sqrt{|g^{(4)}|} \mathcal{H} = 2 \ddot{X} \dot{Z} + 2 \dot{X}^2 - \dot{Y}^2 - \frac{1}{4} W(\beta) \dot{\beta}^2, \tag{4.22} \]
where
\[ W(\beta) = \frac{1}{V_\Omega} \int_\Omega \frac{\sinh^2 \left( \frac{r}{2} \right)}{\cosh^2 (\beta r)} r^2, \tag{4.23} \]
and we are employing the same notation as in Sec. 3.a.

Generalizing now the analysis carried out for the diagonal Bianchi type I case, it is straightforward to deduce the expressions of the momenta canonically conjugate
to $X$, $Y$, $Z$ and $\beta$ from the Hamiltonian (4.22):

$$p_X = 4\dot{X} + 2\dot{Z}, \quad p_Y = -2\dot{Y}, \quad p_Z = 2\dot{X}, \quad (4.24)$$

$$p_\beta = -\frac{1}{2}W(\beta)\dot{\beta}. \quad (4.25)$$

Equations (4.16,17) and (4.21,22) can thus be rewritten

$$\chi_1 = \beta + \frac{p_Y}{2p_Z} = 0, \quad \chi_2 = p_\beta = 0, \quad (4.26)$$

$$H = \frac{1}{2}p_Xp_Z - \frac{1}{2}p_Z^2 - \frac{1}{4}p_Y^2 - \frac{p_\beta^2}{W(\beta)}, \quad (4.27)$$

$$p_Z = \frac{1}{S(T)}. \quad (4.28)$$

The Hamiltonian (4.27), on the other hand, must satisfy the constraint $H = 0$.

It is a simple exercise to check that \{\chi_1, \chi_2\} = 1 and that the Poisson brackets of $\chi_1$ and $\chi_2$ with $H$ vanish weakly (i.e., as long as $\chi_2 = 0$). Therefore, $\chi_1$ and $\chi_2$ form a set of second-class constraints for the system under consideration, as we wanted to prove. The imposition of these constraints eliminates the degrees of freedom $(\beta, p_\beta)$ and leads to a reduced model with associated Hamiltonian

$$H_R = \frac{1}{2}p_Xp_Z - \frac{1}{2}p_Z^2 - \frac{1}{4}p_Y^2. \quad (4.29)$$

The Poisson and Dirac brackets of the variables that describe this reduced model are easily seen to coincide, for $X$, $Y$, $Z$, $p_X$, $p_Y$ and $p_Z$ commute strongly with the constraint $\chi_2$.

It is worth remarking that the reduced Hamiltonian (4.29) is independent of the topology of the constant-$T$ surfaces. As a consequence, this Hamiltonian is well-defined in the limit of non-compact surfaces.

The Hamiltonian equations reached from $H_R$ reproduce the first order equations (4.24) and assure that $p_X$, $p_Y$ and $p_Z$ are constants of motion. Selecting the origin of time by condition (3.11), the classical solutions of our model turn out then to be
formally identical to those obtained in Eq. (3.12) for the Kasner metrics, except for
that now
\[ Z(T) = \frac{p_X - 2p_Z}{2} T + Z_0. \] (4.30)
Besides, from the first constraint in Eq. (4.26) we get
\[ \beta = -\frac{p_Y}{2p_Z} = \text{const.}, \] (4.31)
so that \( \chi_2 = p_\beta \propto \dot{\beta} = 0 \) is automatically satisfied.

If we want to consider only different classical 4-geometries, a discussion similar to
that presented for the diagonal Bianchi type I in Sec. 3.a leads us to conclude that,
owing to the existing symmetry under time reversal, we must fix the sign of one of
the canonical momenta of our reduced model. Hence, we will restrict, e.g., \( p_Z \) to be a
positive constant. On the other hand, it is not difficult to check that an interchange
of the spatial coordinates \( y^1 \) and \( y^2 \) in Eq. (1.1) can be reinterpreted in the BZ model
as a flip of sign in the parameters \( p_Y \) and \( Y_0 \) of the classical solutions. Therefore,
to study all possible 4-geometries, it will suffice to analyse the sector \( p_Y < 0 \). In
this way, the parameters \((Y_0, Z_0, p_Y, p_Z)\) on which the BZ solutions depend will take
on the range of values displayed in Eq. (3.14). The constant momentum \( p_X \) is
determined through the Hamiltonian constraint \( H_R = 0 \), which implies that
\[ p_X = p_Z + \frac{p_Y^2}{2p_Z}. \] (4.32)
Since the expression of the classical solutions \( X(T) \) is the same for the Kasner
and the BZ one-soliton metrics, and \( p_Z \in \mathbb{R}^+ \) in both cases, the relation (3.17)
between the time coordinates \( t \) and \( T \) is still valid in the BZ model, provided that
we define the coordinate \( t \) as in Eq. (3.16). We then have that, on the classical
solutions,
\[ e^{2X(T)} = e^{p_Z T} = p_Z t. \] (4.33)
Substituting this equality in formula (4.18) and making use of the last of the Hamil-
tonian equations in (4.24), we arrive at the result

\[ \frac{1}{S(T)} = p_Z = 2\dot{X}, \]  

(4.34)

which is precisely Eq. (4.17). So, this equation is simply a consequence of the dynamical evolution generated by the Hamiltonian (4.29), as we had anticipated. We thus conclude that the work hypotheses introduced above Eq. (4.21) are completely consistent.

Finally, adopting the redefinitions (3.18) and

\[ Z_0 = \frac{1}{2} \ln C + \left( \frac{1}{2} - p^2 \right) \ln A, \]  

(4.35)

it is not difficult to prove that the classical metrics obtained from Eqs. (4.4-7), (4.30-32) and the two first relations in Eq. (3.12) are nothing else but the family of BZ one-soliton solutions (1.2-5) with \( D = 0 \) and \( p \) restricted to be positive because of the symmetry under the interchange of spatial coordinates \( y^1 \) and \( y^2 \).

5.Canonical Quantization of the BZ model

We have shown that, in the BZ model, the degree of freedom \( \beta(T) \) is determined by a set of second-class constraints. Once these constraints are eliminated, the rest of degrees of freedom of the system, described by the canonical set of variables \( (X, Y, Z, p_X, p_Y, p_Z) \), are still subjected to the Hamiltonian constraint (4.29). Therefore, to obtain the symplectic structure of the reduced phase space, we must simply pull-back to the surface \( H_R = 0 \) the symplectic form

\[ dX \wedge dp_X + dY \wedge dp_Y + dZ \wedge dp_Z. \]  

(5.1)

Given that the desired pull-back is time-independent, we can evaluate it at any constant-\( T \) section of the spacetime. Selecting the \( T = 0 \) surface and recalling condition (3.11), we arrive at a symplectic form on the reduced phase space which coincides formally with that reached in Eq. (3.22) for the diagonal Bianchi type I.
Introducing then the change of variables defined by Eqs. (3.18) and (4.35), a simple calculation shows that both the symplectic form (3.23) and relations (3.24), which were deduced for the Kasner metrics in Sec. 3.b, are valid in the BZ model with the replacement of $C$ for $\tilde{C}$. Moreover, since Eqs. (3.14), and hence restrictions (3.20) still apply, the ranges of the canonical variables $(A, p, P_A, P_p)$ that appear in Eq. (3.23) turn out to be given again by expression (3.26), as in the diagonal Bianchi type I case. The symplectic structure of the reduced phase space of the BZ one-soliton model can thus be identified with that corresponding to the Kasner model, namely, the symplectic structure of the cotangent bundle over $\mathbb{R}^+ \times \mathbb{R}^+$.

The quantization of this reduced phase space was already carried out in Sec. 3.b. All the results presented there [below Eq. (3.26)] hold as well in the BZ minisuperspace model studied here.

In this way, we see in particular that the BZ one-soliton solutions with constant values of $z_0$ and $D$ have the same degrees of freedom as the Kasner metrics. This is due to the fact that, for fixed solitonic parameters, the BZ transformation that relates the two mentioned types of classical solutions preserves the physical degrees of freedom. We recall, in this sense, that fixing the constants $z_0$ and $D$ is equivalent to freezing the solitonic parameters of the BZ transformation.

We have also proved that the quantum theories constructed for the two analysed models are totally equivalent. The observables and physical Hilbert spaces of these two quantum theories can mutually be identified. Note, nevertheless, that the physical interpretation of these observables in terms of the metric differs in the two considered models, because the 4-geometries studied are not the same in each case. Whether or not the symmetry that exists between different classical spacetimes related by the BZ transformation with fixed solitonic parameters translates always into the equivalence of the quantum theories which describe those gravitational

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*Substituting $C$ for $\tilde{C}$. 

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systems is an issue which deserves further research. The analysis presented in this work can be understood in this line as the discussion of a particular example which supports the validity of this conjecture.

We want to close this section with some remarks about the kind of predictions that can be obtained from the quantum theory that we have built up for the BZ one-soliton metrics. The gravitational model employed to describe the BZ one-soliton solutions possesses as its only degrees of freedom the functions $\beta$, $X$, $Y$ and $Z$, which depend exclusively on time. Therefore, the minisuperspace model considered is in fact homogeneous. This explains why the Hamiltonian (4.29) and the second-class constraints (4.26) are independent of the spatial coordinates. As a consequence, all reachable predictions in the corresponding quantum theory refer only to homogeneous variables, like, for instance, the parameters $A$, $B$, $C$ and $p$ on which the 3-geometry depends on each constant-time surface. The quantum theory constructed does not supply us with an appropriate framework to address questions about local quantities, such as the expectation value of the Riemann tensor at each point of the spacetime, a value which would allow us to elucidate whether the singularities of the BZ geometries (1.1-6) disappear or not in the quantum evolution. To analyse this kind of local problems quantum mechanically, we should first enlarge our gravitational model to permit the dependence of the physical degrees of freedom on spatial position. The quantization of such an enlarged gravitational system would then lead us to a true quantum field theory.

6. Conclusions

Following Ashtekar’s programme, we have carried out to completion the quantization of the family of BZ one-soliton metrics (1.1-6) with vanishing parameter $D$ and a given constant value of $z_0$. This family of classical spacetimes can be obtained from the Kasner metrics by means of a generalized soliton transformation in which the
pole trajectory is fixed and all the solitonic degrees of freedom are frozen.

In order to quantize the BZ one-soliton metrics, we have shown that they can be regarded as the classical solutions to a gravitational model whose degrees of freedom solely depend on time. The Einstein equations for this minisuperspace have been proved equivalent to the dynamical equations generated by the homogeneous Hamiltonian constraint of the system when supplemented by a couple of second-class constraints. We have then imposed all these constraints on the model and eliminated the unphysical degrees of freedom. The resulting reduced phase space possesses the symplectic structure of the cotangent bundle over $\mathbb{R}^+ \times \mathbb{R}^+$. In this reduced phase space, it is possible to select a complete set of real variables (observables in this case) which, under Poisson brackets, form the Lie algebra $L(T^*GL(1, \mathbb{R}) \times T^*GL(1, \mathbb{R}))$. We have represented these observables as operators acting on the vector space of complex functions over $\mathbb{R}^+ \times \mathbb{R}^+$. The inner product has been determined by promoting the reality conditions on classical observables to hermiticity requirements on their corresponding quantum operators. In this way, we have obtained an irreducible unitary representation of the considered algebra of observables.

We have also revisited the quantization of the diagonal Bianchi type I (that is, the Kasner model), adopting as close as possible the language and methods employed in the quantization of the BZ one-soliton solutions. The quantum theories constructed for the description of these two models have been shown to be unitarily equivalent, because the Lie algebras of observables and Hilbert spaces of physical states associated with these theories are formally identical. This result can be understood as an indication supporting the conjecture that, under quantization, the symmetry between different types of classical spacetimes that underlies in the BZ transformation with frozen solitonic parameters would translate into equivalence.
among the quantum theories which describe such spacetimes.

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