Estimating the Copula of a class of Time-Changed Brownian Motions: A non-parametric Approach

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Abstract
Within a high-frequency framework, we propose a non-parametric approach to estimate a family of copulas associated to a time-changed Brownian motion. We show that our estimator is consistent and asymptotically mixed-Gaussian. Furthermore, we test its finite-sample accuracy via Monte Carlo.

1 Introduction
One of the most fundamental results in probability theory is the so-called Sklar’s Theorem. It states that for every random vector \( \mathbf{X} = (X_1, \ldots, X_n) \) there exists a copula \( C \) (see Section 2) such that
\[
H(x) = C(F(x)), \quad \forall x \in \mathbb{R}^n,
\]
where \( H : \mathbb{R}^n \to [0, 1] \) is the cumulative distribution function (cdf for short) of \( \mathbf{X} \) and
\[
F(x) := (F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \quad x \in \mathbb{R}^n, \tag{1.1}
\]
and \( F_i \) is the cdf of \( X_i \), for \( i \) in \( \{1, 2, \ldots, n\} \).

Since then, copulas have been applied in a large number of sciences, for instance in hydrology \((11)\), engineering \((15)\), and, perhaps most noticeably, in finance. In finance, their primary use is in risk management and portfolio

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allocation. Specifically, copulas are used to model the joint distribution of financial assets. For a detailed account see [3] and [5]. Note that this way of using copulas can be considered as a spatial way of modeling dependence, i.e. describing the dependence between two or several stochastic processes at distinct times. Copulas have also found their use in temporal modelling as well. [4] characterized Markov processes by means of their copulas. [14] also define so-called copula processes. Furthermore, [4] derived the copula of the bivariate distributions of a Brownian motion. In addition, [13], argued that the copula of a Brownian motion could be used to construct similar processes with arbitrary marginal distributions.

In the spatial set-up, parametrical and non-parametrical inference for copulas is well documented. See for instance [12] and [6]. However, in the temporal case very little statistical analysis has been done, see for instance [2]. This paper aims at developing some results in that direction.

In the present work, we concentrate on the statistical inference for a family of copulas associated with the finite-dimensional distributions of a class of time-changed Brownian motions. More precisely, we propose a non-parametric estimator for a family of conditional copulas linked to a time-changed Brownian motion. We show consistency and asymptotic (mixed) normality under the assumption that the process is observed in a high-frequency set-up.

The paper is structured as follows. Section 2 introduces the notations used through the paper and discusses some essential preliminaries. In Section 3 we present our results and we show the performance of the estimator in finite-samples. In the last section the proofs of our main results are presented.

2 Background

In this section we recall several definitions and properties required to present our main results. Throughout this paper \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})\) will denote a filtered probability space satisfying the usual assumptions of right-continuity and completeness. If \(\mathcal{X}\) is a set, then we will denote the \(n\)-fold cartesian product of \(\mathcal{X}\) with itself as \(\mathcal{X}^n\). Similarly, if \((\mathcal{X}, d)\) is a metric space, then the product metric defined on \(\mathcal{X}^n\) is denoted \(d^n\).

\[ \begin{align*}
C(u) = 0, & \quad u \in [0, 1]^n : \exists i \in \{1, 2, \ldots, n\} : u_i = 0 \\
C(1, 1, \ldots, u, \ldots, 1) = u, & \quad u \in [0, 1] \\
\int_{[a,b]} dC \geq 0, & \quad \forall a, b \in [0, 1]^n : a \leq b.
\end{align*} \]
In this paper we focus on the time-changed Brownian motion

$$X_t = W_{T_t}, \quad t \geq 0,$$

where \((T_t)_{t \geq 0}\) is a continuous random time change independent of \(W\), that is, it is a non-decreasing process taking values in \([0, \infty)\) such that \(T_t\) is a \((\mathcal{F}_t)_{t \geq 0}\)-stopping time for all \(t \geq 0\). Within this framework, for all \(0 \leq t_0 < t_1 < t_2 < \cdots < t_n\), the copula associated to \((X_{t_0}, X_{t_1}, \ldots, X_{t_n})\), is completely determined by the law of \(T_n := (T_{t_0}, \ldots, T_{t_n})\) and the random fields

$$C(t_{k-1}, t_k; u, v) := \psi(T_{k-1}, T_k; u, v), \quad u, v \in [0, 1], \quad k = 1, \ldots, n,$$

where

$$\psi(s, t; u, v) := \begin{cases} \int_0^w \Phi \left( \sqrt{\frac{s \Phi^{-1}(u) - \sqrt{s \Phi^{-1}(w)}}{t - s}} \right) \, dw, & \text{if } |t - s| > 0; \\ u \wedge v & \text{if } t = s > 0; \\ uv & \text{otherwise}, \end{cases}$$

in which \(\Phi\) denotes the cdf of the standard normal distribution. For more details on the previous statements we refer the reader to [3] and [4]. It is not difficult to see that the mapping \((u, v) \mapsto C(t, s; u, v)\) satisfies (2.1) almost surely for all \(t \geq s\). Moreover, it holds that

$$C(s, t; F_s(x | T_t, T_s), F_t(y | T_t, T_s)) = \mathbb{P}(X_s \leq x, X_t \leq y | T_t, T_s),$$

where \(F_t(x | T_t, T_s)\) denotes the cdf of \(X_t\) given \((T_t, T_s)\). Motivated by (2.5) and the terminology used in [9], we will refer to \(C(t, s; \cdot, \cdot)\) as the conditional copula associated to \(X_t\).

2.2 Limit Theorems and Convergence

The notations \(\xrightarrow{P}\) and \(\xrightarrow{d}\) stand, respectively, for convergence in probability and in distribution of random vectors. As usual the space of càdlàg fields will be denoted by \(\mathbb{D}([0, \mathcal{T}]^n; \mathbb{R}^d)\). If \(X\) and \(X^n\) are two càdlàg processes we write \(X^n \xrightarrow{\mathcal{D}} P X\), whenever

$$\lim_{n \to \infty} \mathbb{P}(\sup_{0 \leq t \leq \mathcal{T}} \|X^n_t - X_t\| \geq \varepsilon) = 0, \quad \forall \mathcal{T}, \varepsilon > 0.$$

A sequence of random vectors \((\xi_n)_{n \geq 1}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is said to converge stably in law towards \(\xi\) (in symbols \(\xi_n \xrightarrow{s.d.} \xi\)), which is defined on an extension of \((\Omega, \mathcal{F}, \mathbb{P})\), say \((\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})\), if for every continuous and bounded function \(f\) and any bounded random variable \(\chi\) it holds that

$$\mathbb{E}(f(\xi_n)\chi) \to \overline{\mathbb{E}}(f(\xi)\chi),$$

where \(\overline{\mathbb{E}}\) denoted expectation w.r.t. \(\overline{\mathbb{P}}\). For a concise exposition of stable convergence see [7]. Given a stochastic process \(Z = (Z_t)_{t \geq 0}\), we will use the notation
\( \Delta_i^n Z := Z_{i/n} - Z_{(i-1)/n}, \) \( i \in \mathbb{N}. \) The \textit{realized variation} of a process \( Z = (Z_t)_{t \geq 0} \) is defined and denoted as the process

\[
[Z]_t^n = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} (\Delta_i^n Z)^2, \quad t \geq 0.
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \in \mathbb{R}. \) It is well known that if \( Z \) is a continuous semimartingale, then \( [Z]_n \) u.c.p \( \Rightarrow [Z]. \)

### 3 Estimating the conditional copula of \( X \)

As discussed in Section 2, the copula associated to \((X_{t_0}, X_{t_2}, \ldots, X_{t_m})\), for \( 0 < t_1 < t_2 < \cdots < t_m \), is completely determined by the law of \((T_{t_0}, \ldots, T_{t_m})\) as well as the family of conditional copulas \( \{C(s, t; u, v) : 0 \leq s, t \leq T, u, v \in [0, 1]\} \), where \( C \) as in (2.3). For the rest of this section, we propose a non-parametric approach for estimating the later. Our sample scheme is as follows:

The process \( X \) is observed on a fixed interval \([0, T]\), \( T > 0 \), at times \( t_i = i/n, \) for \( i = 0, 1, \ldots, \lfloor nT \rfloor \). Thus, motivated by (2.3) and the fact that \([X]_n \) u.c.p \( \Rightarrow [X] = T, \) as \( n \to \infty \), we propose to estimate \( C \) via

\[
C^n(s, t; u, v) := \psi([X]_n^{s}, [X]_n^{t}; u, v), \quad u, v \in [0, 1], 0 \leq s, t.
\]

Our first result shows that \( C^n \) is indeed a consistent estimator for \( C \).

**Theorem 1.** Let \( t_0 \geq 0 \), such that \( \mathbb{P}(T_{t_0} > 0) = 1. \) Then for all \( u, v \in [0, 1], \) \( T > t_0, \) and \( \varepsilon > 0 \), it holds

\[
\mathbb{P} \left( \sup_{t_0 \leq s, t \leq T} |C(s, t; u, v) - C^n(s, t; u, v)| \geq \varepsilon \right) \to 0, \quad n \to \infty.
\]

Now we proceed to derive second-order asymptotics for \( C^n. \) In order to do this, we require stronger assumptions on the structure of \( X. \) Specifically, we are going to assume that there is a \((G_t)_{t \geq 0}\)-Brownian motion \( B \) such that

\[
X_t = \int_0^t \sigma_s dB_s, \quad t \geq 0,
\]

where \( \sigma \) is a cádlág process. Observe that by Knight’s Theorem, \( X \) admits the representation

\[
X_t = W_{T_t}, \quad t \geq 0,
\]

where \( T_t = \int_0^t \sigma_r^2 dr, \) \( t \geq 0. \) Thus, if \( \sigma \) is assumed to be independent of \( B, \) we can further choose \( W \) to be independent of \( \sigma. \) This can be seen easily in the case when \( \int_0^\infty \sigma_r^2 dr = \infty. \) Indeed, in that situation it is well known (see for instance [1]) that (3.2) holds with

\[
W_t = X_{A_t}, \quad t \geq 0,
\]
where \( A_t = \inf\{s \geq 0 : T_s > t\} \), which easily implies that \( W \) is independent of \( \sigma \). The general case can be analysed in a similar way. Assuming that \( X \) admits the representation (3.1) we obtain the following Central Limit Theorem for \( C^n \).

**Theorem 2.** Let \( X \) be given by (3.1) and assume that \( \mathbb{P}(\sigma^2_t > 0) = 1 \) for all \( t \geq 0 \). Fix \( u, v \in (0, 1) \) and denote by \( \nabla \psi(s, t; u, v) = (\partial_t \psi(s, t; u, v), \partial_s \psi(s, t; u, v)) \). Then for \( 0 < s < t \), as \( n \to \infty \)

\[
\sqrt{n} \left[ C^n(s, t; u, v) - C(s, t; u, v) \right] \xrightarrow{s.d.} \sqrt{V_{s,t}} N(0, 1),
\]

where \( N(0, 1) \) is a standard normal random variable independent of \( F \) and

\[
V_{s,t} = 2 \nabla \psi(T_s, T_t; u, v) \left[ \frac{Q_t}{Q_s} \frac{Q_s}{Q_s} \right] \nabla \psi(T_s, T_t; u, v)' ,
\]

in which \( Q_t := \int_0^t \sigma^4_r dr, t \geq 0 \).

A simple way to estimate \( V_{s,t} \) is by using power variations: If \( \sigma \) is càdlàg (see for instance [8]), then as \( n \to \infty \)

\[
Q^n_t := \frac{n}{3} \sum_{i=1}^{[n]} |\Delta^n_i Z|^4 \Rightarrow Q_t.
\]

Thus, a feasible estimator for \( V_{s,t} \) is

\[
V^n_{s,t} := 2 \nabla \psi([X]_s^n, [X]_t^n; u, v) \left[ \frac{Q^n_t}{Q^n_s} \frac{Q^n_s}{Q^n_s} \right] \nabla \psi([X]_s^n, [X]_t^n; u, v)' \Rightarrow V_{s,t}.
\]

Thus, we have an easy consequence of the previous theorem:

**Corollary 1.** Under the assumptions of the previous theorem we have that

\[
\sqrt{n} \frac{V^n_{s,t}}{V_{s,t}} \left[ C^n(s, t; u, v) - C(s, t; u, v) \right] \xrightarrow{s.d.} N(0, 1).
\]

**3.1 Simulation study**

In this part we study the finite-sample behavior of our proposed estimator. We use here Monte Carlo simulations to investigate the sensitivity of \( C^n \) to the variation of \( s, t, u, v \) as well as the sample size. Our set-up is as follows: The volatility term \( \sigma^2 \) is simulated according to the so-called Cox-Ingersoll-Ross process, i.e. \( \sigma^2 \) satisfies the stochastic differential equation

\[
d\sigma^2_t = \kappa(\theta - \sigma^2_t)dt + \nu \sqrt{\sigma^2_t} dW_t, \quad \sigma^2_0 = s_0.
\]

The parameters are set \((\kappa, \theta, \nu, s_0) = (0.5, 1.5, 1, 1.5)\) in such a way that the Feller condition \( 2\kappa \theta > \nu^2 \) is satisfied. Based on this, we sample over the interval \([0, 1]\) equidistant discretizations \((X_{i/n})_{i=1}^n\), where \( X \) is given as in (3.2).
In Figure 1 we have plotted the level sets of $C^n(s, t; u, v)$ and $C(s, t; u, v)$ together with the 95% confidence contours provided by Theorem 2. The confidence contours behave as we would imagine; near the line $u = v$ we see that they allow for the largest deviations. As $(u, v)$ approach the boundary of $[0, 1]^2$, the intervals diminish as expected: $C^n(s, t; u, v)$ and $C(s, t; u, v)$ coincide on the boundary. Furthermore, as $n$ increases we see that the confidence contours become increasingly narrow as expected.

![Figure 1](image-url)

(a) Level curves with estimated confidence intervals for $n = 100$ points per path. (b) Level curves with estimated confidence intervals for $n = 10000$ points per path.

Figure 1: Level curves for the estimated copula with confidence intervals ($s = 0.3$ and $t = 0.7$).

In Figure 2 we report the finite-sample distribution of our standardized error against the standard normal distribution. We can see that the accuracy of our statistic is sensitive to the boundary points where $s$ and $t$ are close. Again, this behaviour is not unexpected; the temporal gradient $\nabla \psi$, as in Theorem 2, is given by

$$
\int_0^u \varphi \left( \frac{\sqrt{t} \Phi^{-1}(v) - \sqrt{s} \Phi^{-1}(w)}{\sqrt{t - s}} \right) \left( \frac{\Phi^{-1}(v) - \sqrt{t} \Phi^{-1}(v) - \sqrt{s} \Phi^{-1}(w)}{2 \sqrt{t - s}} \right) dw 
$$

$$
\int_0^u \varphi \left( \frac{\sqrt{t} \Phi^{-1}(v) - \sqrt{s} \Phi^{-1}(w)}{\sqrt{t - s}} \right) \left( \frac{\sqrt{t} \Phi^{-1}(v) - \sqrt{s} \Phi^{-1}(w)}{2 \sqrt{t - s}} \right) dw 
$$

Here, $\varphi$ is the density function of a standard Gaussian. We see, that terms proportional to $1/\sqrt{t - s}$ appear. However, recall that as $(s, t) \to (t_0, 0)$ for $t_0 > 0$ the copula reduces to $C(t_0, 0; u, v) = u \wedge v$. Furthermore, it is also very likely that numerical errors influence the result here, due to terms such as $1/\sqrt{t - s}$.

We conclude this section by investigating whether the convergence in Theorem 1 can be extended to uniform convergence over $(u, v)$. Specifically, we investigate, via Monte Carlo simulations, the asymptotic behavior of the statistic

$$
\rho(C^n, C) := \sup_{(u, v) \in [0, 1]^2} \sup_{\tau \leq s, t \leq \tau} \left| C^n(s, t; u, v) - C(s, t; u, v) \right|
$$


Figure 2: QQ plot for $\sqrt{n/V_{s,t}} \cdot (C^n(s, t; u, v) - C(s, t; u, v))$, $(n = 10000, (u, v) = (0.7, 0.3))$.

Figure 3: Density estimates for $\rho(C^n, C)$ on a logarithmic scale.
as the sample size increases.

Figure 3 shows the density estimates, obtained via a Gaussian kernel density estimate, for $\rho(C^n, C)$ on a logarithmic scale focusing on the peaks. Observe that the mean is located in the lower tail, due to a substantial number of simulations resulting in $\rho(C^n, C)$ being very close to 0. Similarly, $1/\sqrt{n}$ is also added to the plot, showing that the shift in the distribution is, relatively, proportional to $1/\sqrt{n}$ on a logarithmic scale. Based on Figure 3, these simulations indicate that Theorem 1 may be extended to include the supremum over $(u, v) \in [0, 1]^2$.

4 Proofs

The following lemmas are key for the proof of our main results.

**Lemma 1.** Let $\psi$ be as in (2.4). For all $u, v \in [0, 1]$, the mapping $(s, t) \mapsto \psi(s, t; u, v)$ is continuous in $(0, \infty)^2$ and continuously differentiable in $\{ (s, t) | 0 < s < t \}$.

**Proof.** If $(u, v)$ is in the boundary of $[0, 1]^2$ the result is trivial. Suppose that $u, v \in (0, 1)$ and put

$$g(t, s, w) := \Phi\left(\frac{\sqrt{t} \vee s \Phi^{-1}(v) - \sqrt{t \wedge s} \Phi^{-1}(w)}{\sqrt{|t - s|}}\right), \quad t, s > 0, w \neq v,$$

and $g(t, s, w) \equiv 0$ when $w = v$. From Example 5.32 in [13], it follows that for almost all $w \in [0, u]$

$$\lim_{(t, s) \to (t_0, s_0)} g(t, s, w) = \begin{cases} 1_{[0, v]}(w) & \text{if } t_0 = s_0; \\ g(t_0, s_0, w) & \text{if } t_0 \neq s_0. \end{cases} \quad (4.1)$$

The continuity then follows by the Lebesgue’s Dominated Convergence Theorem. On the other hand, for $w \neq v$ we have that for $0 < s < t$

$$\begin{bmatrix} \frac{\partial g(s, t, w)}{\partial t} \\ \frac{\partial g(s, t, w)}{\partial s} \end{bmatrix} = \begin{bmatrix} \phi\left(\frac{\sqrt{t \vee s \Phi^{-1}(v)} - \sqrt{t \wedge s} \Phi^{-1}(w)}{2\sqrt{|t - s|}}\right) & \Phi^{-1}(v) - \frac{\Phi^{-1}(w)}{2\sqrt{|t - s|}} \\ \phi\left(\frac{\sqrt{t \vee s \Phi^{-1}(v)} - \sqrt{t \wedge s} \Phi^{-1}(w)}{2\sqrt{|t - s|}}\right) & \frac{\Phi^{-1}(v) - \sqrt{t \wedge s} \Phi^{-1}(w)}{2\sqrt{|t - s|}} \end{bmatrix},$$

where $\phi$ is density of a standard normal distribution. Since for any constants $c, k$ such that $k \neq 0$, it holds that $\phi(c + kx)x \to 0$ as $|x| \to \infty$, we deduce that

$$\sup_{w \in [0, 1]} \left\| \left(\frac{\partial g(s, t, w)}{\partial t}, \frac{\partial g(s, t, w)}{\partial s}\right) \right\| < \infty, \quad 0 < s < t. \quad (4.2)$$

Interchanging roles between $s$ and $t$ allow us to conclude that (4.2) is fulfilled for all $(s, t) \in \{(s, t) | 0 < s < t\}$. Another application of the Dominated Convergence Theorem concludes the proof. ■
Lemma 2. Let $g \in C((0, \infty)^n; \mathbb{R}^m)$, $m, n \in \mathbb{N}$ and $\mathcal{T} > 0$. Then the mapping

$$
\Psi : (\mathbb{F}([0, \mathcal{T}]; \mathbb{R})^n, d_{\infty}^\prime) \to (\mathbb{D}([0, \mathcal{T}]; \mathbb{R}^m), d_{\infty})
$$

$$
x \mapsto g(x_1(t_1), x_2(t_2), \ldots, x_n(t_n)), \quad \forall t = (t_1, t_2, \ldots, t_n) \in [0, \mathcal{T}]^n,
$$

is continuous, where $d_{\infty}$ denotes the supremum metric.

Proof. We must show that for every $x = (x_1, \ldots, x_n) \in \mathbb{F}([0, \mathcal{T}]; \mathbb{R})^n$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$
d_{\infty}^\prime(x, y) < \delta \implies d_{\infty}(\Psi(x), \Psi(y)) < \varepsilon. \quad (4.3)
$$

To this end, note first that for all $i = 1, \ldots, n$, $\inf_{t \in [0, \mathcal{T}]} x_i(t) = x_i(0) > 0$. Now, let $i = 1, \ldots, n$ and consider the set $B_{x_i} = \{ z \in \mathbb{F}([0, \mathcal{T}]; \mathbb{R}) \mid d_{\infty}(x_i, y) \leq x_i(0)/2 \}$. For $z \in B_{x_i}$, we have

$$
\sup_{t \in [0, \mathcal{T}]} |x_i(t) - z(t)| = \sup_{t \in [0, \mathcal{T}]} |x_i(t) - z(t)| < \max x_i(0)/2 =: c_{x_i}
$$

This implies $\sup_{t \in [0, \mathcal{T}]} |z(t)| \leq \sup_{t \in [0, \mathcal{T}]} |x_i(t)| + c_{x_i} < \infty$. Now set $C_{x_i} = c_{x_i} + \sup_{t \in [0, \mathcal{T}]} x_i(t)$. Then, $z(t), x_i(t) \in [c_{x_i}, C_{x_i}]$ for every $t \in [0, \mathcal{T}]$. Let $K = \prod_{i=1}^n [c_{x_i}, C_{x_i}] \subset (0, \infty)^n$. By the Heine-Cantor Theorem, it follows that the restriction of $g$ to $K$ is uniformly continuous. This means that we can find $\delta_0 > 0$ such that for all $|t - s| < \delta_0 \implies |g(t) - g(s)| < \varepsilon$. Now, put $\delta = \min(\delta_0, c_{x_1}, \ldots, c_{x_n})$. We conclude from above that if $y = (y_1, \ldots, y_n) \in \mathbb{F}([0, \mathcal{T}]; \mathbb{R})^n$ and $d_{\infty}^\prime(x, y) < \delta$ then $x(t), y(t) \in K$ for all $0 \leq t \leq \mathcal{T}$, from which (4.3) follows.

Proof of Theorem 1. Let $\mathcal{T} > t_0$. From Lemma 1 the mapping $(s, t) \mapsto \psi(s, t; u, v)$ is continuous in $(0, \infty)^2$. Thus, we deduce that $\psi$ extends (as in Lemma 2) to a continuous function $\Psi : \mathbb{F}([0, \mathcal{T} - t_0]; (0, \infty)^2) \to \mathbb{D}([0, \mathcal{T}]^2; \mathbb{R})$. Moreover

$$
C^n(\cdot, \cdot; u, v) = \Psi((S_{t_0}[X]^n, S_{t_0}[X]^n)); \quad C(\cdot, \cdot; u, v) = \Psi(S_{t_0}T, S_{t_0}T),
$$

where $S_{t_0} : \mathbb{D}([0, \mathcal{T}]; \mathbb{R}) \to \mathbb{D}([0, \mathcal{T} - t_0]; \mathbb{R})$ denotes the shift operator, i.e.

$$
x \mapsto (S_{t_0}x)(t) = x(t + t_0), \quad \forall t \in [0, \mathcal{T} - t_0].
$$

In view that $S_{t_0}$ is a continuous operator from $\mathbb{D}([0, \mathcal{T}]; \mathbb{R})$ to $\mathbb{D}([0, \mathcal{T} - t_0]; \mathbb{R})$ and $[X]^n \Rightarrow T$, as $n \to \infty$, we can now apply The Continuous Mapping Theorem (see for instance [3]) to conclude that

$$
d_{\infty}(\Psi(S_{t_0}[X]^n, S_{t_0}[X]^n), \Psi(S_{t_0}T, S_{t_0}T)) \to 0,
$$

which is exactly the conclusion of the theorem. □
Proof of Theorem 3. First note that thanks to Lemma 5.3.12 in [8] we may and do assume that \(|\sigma_t| \leq C\) for some deterministic constant \(C > 0\). Now, let
\[
Z_n := ([X]_t^n, [X]_s^n); \quad Z := (T_t, T_s).
\]
Since \(P(\sigma_t^2 > 0) = 1\) for all \(t \geq 0\), we can find \(n\) large enough such that \(Z_n \in \{(s, t) : 0 < s, t, t \neq s\}\). Moreover, by Taylor’s Theorem
\[
\sqrt{n} |C^n(s, t; u, v) - C(s, t; u, v), | = \int_0^1 \nabla \psi(Z + y(Z_n - Z), u, v)dy \cdot \sqrt{n}(Z_n - Z).
\]
From Theorem 5.4.2 in [8], it follows that for any \(s \neq t\)
\[
\sqrt{n}(Z_n - Z) \xrightarrow{s.d} \sqrt{2} \left( \int_0^t \sigma_t^2 dW_t, \int_0^s \sigma_t^2 dW_t' \right),
\]
where \(W^r\) is a Brownian motion independent of \(\mathcal{F}\). Therefore, it is enough to show that
\[
\int_0^1 \nabla \psi(Z + y(Z_n - Z), u, v)dy \xrightarrow{P} \nabla \psi(T_t, T_s, u, v). \quad (4.4)
\]
In view of \(Z_n \xrightarrow{P} Z\), every subsequence \(Z_n^k\) contains a further subsequence \(Z_{n(i)}^k\) such that \(Z_{n(i)}^k \xrightarrow{a.s.} Z\). Fix \(\omega \in \Omega_{t,s} := \{\omega \in \Omega : Z_{n(i)}^k(\omega) \rightarrow Z(\omega), Z(\omega) \in \{(s, t) \mid 0 < s < t\}\}\). By using that \(Z(\omega) \in V\), we can find an open ball with center \(Z(\omega)\) and radius \(\rho(\omega) > 0\) which is totally contained in \(\{(s, t) \mid 0 < s < t\}\). Moreover, for every \(\rho > \varepsilon > 0\) there is \(n_0 \equiv n_0(\omega) \in \mathbb{N}\) such that
\[
\|Z_{n(i)}^k(\omega) - Z(\omega)\| < \varepsilon, \quad \forall n_k(i) \geq n_0.
\]
This in particular implies that for all \(0 \leq y \leq 1\) and \(n_k(i) \geq n_0\), \(Z(\omega) + y(Z_n^k(\omega) - Z(\omega))\) is contained in an open ball with center \(Z(\omega)\) and radius \(\varepsilon > 0\). Therefore, we can find a compact set \(K_\omega \subseteq \{(s, t) \mid 0 < s < t\}\) such that \(Z(\omega) + y(Z_n^k(\omega) - Z(\omega)) \in K_\omega\) for all \(0 \leq y \leq 1\) and \(n_k(i) \geq n_0\). Hence, by the continuity of \(\nabla \psi\) on \(\{(s, t) \mid 0 < s < t\}\) (see Lemma [1]) and the Dominated Convergence Theorem, we deduce that as \(n_k(i) \rightarrow \infty\)
\[
\int_0^1 \nabla \psi(Z(\omega) + y(Z_n^k(\omega) - Z(\omega)), u, v)dy \rightarrow \nabla \psi(Z(\omega); u, v), \quad \forall \omega \in \Omega_{t,s}.
\]
(4.4) follows now by Theorem 6.3.1 in [10]. \(\blacksquare\)

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