Quantum communication protocols by quantum walks with two coins

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received 22 August 2018; accepted in final form 6 December 2018
published online 10 January 2019

PACS 03.67.Ac – Quantum algorithms, protocols, and simulations
PACS 03.67.Lx – Quantum computation architectures and implementations
PACS 03.67.Hk – Quantum communication

Abstract - We introduce some new perfect state transfer and generalized long-distance teleportation schemes by quantum walks with two coins. By encoding the transferred information in the coin-1 state and alternately using two coin operators, we can perfectly recover the information in the coin-1 state at the target position by at most two flipping operations. On the basis of quantum walks with two coins on either a line or an $N$-circle, we can perfectly transfer any qubit state. In addition, using quantum walks with two coins on regular graphs, we can first implement a perfect qudit state transfer by quantum walks. Compared with existing schemes driven by one coin, more general graph structures can be used to perfectly transfer a more general state. We also study how to realize generalized teleportation over long-distance walks by the above quantum walk models.

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Introduction. – Quantum walks have been introduced as a quantum analogue of classical random walks [1,2]. One prominent application of quantum walks is the design of algorithms in quantum information processing, such as search algorithms and element distinctness [3–7]. Another application of quantum walks is a universal quantum-computation model [8–10]. Quantum walks with multiple coins on the line were first proposed by Brun et al. in 2003 [11]. In order to implement scalable quantum networks, we need to set up various building blocks of the networks such as a line, a circle, and a complete or regular graph to complete communication tasks. For quantum networks, state transfer or teleportation are the key communication protocols. We find that quantum walks with two coins on general graphs are good carriers and can finish such quantum communication tasks more conveniently [12]. In this paper, we consider quantum state transfer and long-distance quantum teleportation protocols by quantum walks with two coins.

Quantum state transfer between different sites is a significant problem for quantum networks and quantum computers. In 2003, Bose first considered this problem using spin chains as the quantum communication carrier in quantum computing [13]. Because a spin chain can be regarded as a wire in quantum networks, state transfer by spin chains has been widely studied [14–19]. Recently, different quantum state transfer schemes were proposed [20–22]. To correct state transfer, Kay developed a family of perfect quantum error correcting codes [23]. In 2004, Christandl et al. found that the time evolution of qubit state transfer through a spin chain can be interpreted as a continuous-time quantum walk [14]. It was also generalized to a discrete-time quantum walk by adding a position-dependent coin operator [24]. In the last decade, state transfer by quantum walks has become an interesting topic. By a discrete-time quantum walk search algorithm, Stefaňák et al. discussed state transfer on star graphs and complete bipartite graphs [25]. Yalcinkaya et al. [26]...
designed qubit state transfer with a discrete-time quantum walk on an $N$-circle and $N$-line by adding a recovery operator to the entire system. However, for the $N$-circle protocol, $N$ should be even number, and the transferred state can only be transmitted to the opposite site on the circle. Zhan et al. [27] designed paths to transfer a qubit state by discrete-time quantum walks on a line. However, at the same step, different coin operators are needed to perform a flipping operation for different positions. Obviously, they will incur extra complexity for operation in a real experiment.

In this paper, we consider both state transfer and generalized long-distance teleportation schemes driven by quantum walks with two coins. In the first scheme, we encode the unknown state in the coin-1 space. After a finite number of steps, by alternately using the two coin operators, we can transfer the state to any target position and recover the transferred information perfectly in the coin-1 space at the target position. Compared with the results given by Yalcinkaya [26], we can transfer information to any target position on an infinite line. As for the $N$-circle, our circle does not necessarily have an even number of vertices, and the information can be transferred from the initial position to any other position on the circle. Compared with the routing path on a line [27], for each individual step, we use a uniform coin operator at different positions. Obviously, this reduces the extra complexity for operation in a practical experiment. In our scheme, for only at most two moments, we need to use the coin-flipping operator Pauli $X$, and we can calculate the moment accurately. Most importantly, we first show that an unknown qubit state can be perfectly transferred on a regular graph by quantum walks. The second scheme is to implement generalized teleportation over long-distance walks by two coins. This is a further extension of the work in [12]. Encoding the unknown information in the coin-1 space, after a long-distance walk, we can still recover the information in the coin-2 space perfectly by local measurement and local correction. The difference between these two schemes is that for quantum state transfer, transmitted information on coin-1 space cannot be transferred to the coin-2 space. In the second scheme, by the measurement, the information is transmitted from the coin-1 space to the coin-2 space. Here we give a possible implementation platform for quantum teleportation by quantum walks with two coins. The paper is organized as follows. In the next section, we introduce models of coined quantum walks on various graphs. In the third section, various state transfer schemes based on quantum walks with two coins are given. In the fourth section, long-distance teleportations by quantum walks with two coins on various graphs are shown. In the last section, we present the conclusions.

Preliminary.

Quantum walks on a line. A one-dimensional quantum walk [28] takes place in a compound Hilbert space comprising the position and coin spaces, defined as

$$\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C,$$

where $\mathcal{H}_P = \text{span}\{|n\}; n \in \mathbb{Z}\}$, and $|n\}$ corresponds to a position walker localized on the line, and $\mathcal{H}_C = \text{span}\{|0\}, |1\}\}$. The walker walks on a one-dimensional line. Each step of the quantum walk is described by the equation $W(t) = S \cdot (I \otimes C)$, where $C$ is called a coin operator acting on the coin space, and the conditional shift operator $S$ is expressed as

$$S = \sum_x (|x+1\rangle \langle x| \otimes |0\rangle \langle 0| + |x-1\rangle \langle x| \otimes |1\rangle \langle 1|).$$

The conditional shift operator simulates the classical manner of random walks. The walker moves either one step to the left or right depending on the state of the coin.

Quantum walks on graphs. The definition of coined quantum walks on graphs was given by Aharonov [29]. Let $G(V,E)$ be a graph, where $V$ and $E$ are vertex and edge sets, respectively. And let $\mathcal{H}_v$ be the Hilbert space spanned by states $|v\rangle$, where the vertex $v \in V$. At each vertex $j$, there are several directed edges with labels pointing to the other vertices. The coin space $\mathcal{H}_c$ is spanned by the states $|a\rangle$, where $a \in \{0,\ldots,d-1\}$. They are the labels of the directed edges. The conditional shift operation between $\mathcal{H}_v$ and $\mathcal{H}_c$ is

$$S_j = \sum_{a,b} (|a\rangle \langle b| \otimes |a\rangle \langle b|),$$

where the label $a$ directs edge $j$ to $k$.

Quantum walks driven by many coins. In the above two quantum walks, there is only one coin space. Now, we introduce quantum walks driven by many coins [11]. We take quantum walks on a line with many coins as an example; quantum walks on graphs with many coins can be obtained in a similar manner. This walk can be seen in fig. 1, where there are $M$ coins in total. The unitary transformation that results from flipping the $m$-th coin is given by $W_m = (S \otimes P_{0m} + S^\dagger \otimes P_{1m})(I \otimes C_m)$. Here, $C_m$ is a coin operator acting on the $m$-th coin, $S = \sum_n |n+1\rangle \langle n| \otimes |0\rangle \langle 0|$, and $P_{0m} = |0\rangle \langle 0|_m$, and $P_{1m} = |1\rangle \langle 1|_m$. If we cycle among the coins, performing a total of $t$ flips ($t/M$ with each coin), then the state will be $|\Psi(t)\rangle = (W_m \cdots W_1)^{t/M}|\Psi_0\rangle$. Here, we consider the situation where $t = M$. The number of coins is the same as the steps of the walks. There are $M$ unitary transformations $\{W_m\}$, and they all commute with each other: $[W_m, W_n] = 0$ for $m \neq n$.

Perfect state transfer by quantum walks with two coins.

Qubit state transfer on a line. In the following, we will introduce a scheme to perfectly transfer a qubit (coin state) through quantum walks on a one-dimensional line with two coins. The initial state of the walker-coin-coin system is

$$|\phi\rangle^0 = |0\rangle \otimes (|a\rangle|0\rangle + |b\rangle|1\rangle) \otimes |0\rangle,$$

where $|a|^2 + |b|^2 = 1$. Let $|\phi\rangle^i$ denote the state describing the system after the $i$-th step.

In this scheme, the unknown state of coin 1 can be transferred to an arbitrary position $x$ after $n$ steps of a quantum walk. The number $n$ is correlated to $x$. The entire process
is expressed as

$$|\phi\rangle^{\text{steps}} = |x\rangle \otimes U_1(a|0\rangle + b|1\rangle) \otimes U_2|0\rangle,$$  \hspace{1cm} (3)

where $U_1$ and $U_2$ are the recovery unitary operators.

It should be noted that we alternately use two coin operators $I$ and $X$. If the coin operators are independent of step $i$, the state of $|\phi\rangle^i$ is given by the following. For even-numbered $i$, $|\phi\rangle^i = W_1^{i/2}W_2^{i/2}|\phi\rangle^0$. For odd-numbered $i$, $|\phi\rangle^i = W_1^{i/2}W_2^{i/2}W_2^{i/2}W_2^{i/2}|\phi\rangle^0$. In this scheme, the coin operators are related to step $i$. Here, we list the selected coin operators at each step. At the end of the steps, we can choose suitable unitary operators $U_1$ and $U_2$ on the coin-1 and coin-2 spaces to recover the state.

**Case 1.1:** The position $x$ is a positive and even number (table 1).

| x-th step | 1 | 2 | ... | x | x+1 | x+2 | ... | 2x-1 | 2x |
|:---------|---|---|-----|---|----|----|-----|----|----|
| $C_1$ | I | I | ... | I | I | I | ... | I | I |
| $C_2$ | I | I | ... | I | I | I | ... | I | I |

In this case, only at the $(x+1)$-th step of a quantum walk, we choose the coin operator $C_1$ on the coin-1 space to be the Pauli operator $X$. At the other steps, the coin operators are all identity operators. After we carry out 2$x$ steps of a quantum walk, the state $a|0\rangle + b|1\rangle$ can be perfectly transferred. The detailed process is given as follows:

- If $i = 1$, $|\phi\rangle^i = |1\rangle \otimes a|0\rangle + |1\rangle \otimes b|10\rangle$.
- If $i = 2$, $|\phi\rangle^2 = |2\rangle \otimes a|00\rangle + |0\rangle \otimes b|10\rangle$.
- If $i = x - 1$, $|\phi\rangle^{x-1} = |x-1\rangle \otimes a|00\rangle + |1\rangle \otimes b|10\rangle$.
- If $i = x$, $|\phi\rangle^x = |x\rangle \otimes a|0\rangle + |0\rangle \otimes b|10\rangle$.

Here, we can check the situation in which 2$t$ steps of a quantum walk are carried out when $C_1 = C_2 = I$. The walker will move 2$t$ steps forward at the coin state $|0\rangle$, 2$t$ steps backward at the coin state $|11\rangle$, or stand still at the coin state $|10\rangle (|01\rangle)$.

After $x$ (even) steps of a quantum walk in total, the information of $a$ flows to the position $x$. Moreover, the information of $b$ rests at position 0. In the rest of the steps of a quantum walk, we should make the information of $b$ flow to the position $x$ and keep the information of $a$ at the end step of the quantum walk. Thus, we use the Pauli operator $X$ to entirely flip the coin states in the coin-1 space (i.e., $|0\rangle \rightarrow |1\rangle$, $|1\rangle \rightarrow |0\rangle$). Then, the coin states in the two coin spaces change from $(|0\rangle \otimes |0\rangle)$ to $(|10\rangle \otimes |0\rangle)$ and from $(|10\rangle \otimes |0\rangle)$ to $(|0\rangle \otimes |0\rangle)$. This will bring the result we need.

- If $i = x + 1$, $|\phi\rangle^{x+1} = |x-1\rangle \otimes a|10\rangle + |1\rangle \otimes b|00\rangle$.
- If $i = x + 2$, $|\phi\rangle^{x+2} = |x\rangle \otimes a|10\rangle + |2\rangle \otimes b|00\rangle$.
- If $i = 2x - 1$, $|\phi\rangle^{2x-1} = |x-1\rangle \otimes a|00\rangle + |x-2\rangle \otimes b|00\rangle$.
- If $i = 2x$, $|\phi\rangle^{2x} = |x\rangle \otimes a|0\rangle + |x\rangle \otimes b|00\rangle$.

The final state $|\phi\rangle^{2x}$ is equal to $|x\rangle \otimes (a|1\rangle + b|0\rangle) \otimes 0$. We can choose the recovery operator $U_1$ to be $X$. Then, the state is successfully transferred to the position $x$.

We note that perfect state transfer can be also achieved at the $(2x-1)$-th step of a quantum walk, and the position $x-1$ is odd. The positive and odd case is as follows.

**Case 1.2:** The position $x$ is a positive and odd number (table 2).

| x-th step | 1 | 2 | ... | x | x+1 | x+2 | ... | 2x-1 | 2x |
|:---------|---|---|-----|---|----|----|-----|----|----|
| $C_1$ | I | I | ... | I | X | I | ... | I | I |
| $C_2$ | I | I | ... | I | I | I | ... | I | I |

In this case, only at the $(x+2)$-th step, we choose the coin operator $C_1$ on the coin-1 space to be Pauli operator $X$. At the other steps, the coin operators are all an identity. We can perfectly transfer the unknown state with 2$x+1$ steps. $U_1 = X$, and $U_2 = I$. The proof is similar to that for the even case.

**Case 1.3:** The position $x$ is a negative and even number (table 3).

| x-th step | 1 | 2 | ... | |x|-1 | |x|+1 | |x|+2 | ... | 2|x|+1 |
|:---------|---|---|-----|---|----|----|-----|----|----|
| $C_1$ | I | I | ... | |x|-1 | |x|+1 | |x|+2 | ... | 2|x|+1 |
| $C_2$ | X | I | ... | I | I | I | ... | I | I |

In this case, we choose the coin operator $C_2$ to be $X$ at the second step of a quantum walk and $C_1$ to be $X$ at the $(|x|+1)$-th step. At the other steps, the coin operators are the identity operators. After we carry out 2|x| steps of a quantum walk, where $x$ is a negative number, the state $a|0\rangle + b|1\rangle$ can be perfectly transferred. The recovery operators are given by $U_1 = X$ and $U_2 = X$. The negative and odd case is as follows.

**Case 1.4:** The position $x$ is a negative and odd number (table 4).

| x-th step | 1 | 2 | ... | |x|-1 | |x|+1 | |x|+2 | ... | 2|x|+1 |
|:---------|---|---|-----|---|----|----|-----|----|----|
| $C_1$ | I | I | ... | |x|-1 | |x|+1 | |x|+2 | ... | 2|x|+1 |
| $C_2$ | X | I | ... | I | I | I | ... | I | I |
In this case, we choose the coin operator \( C_2 \) to be \( X \) at the second step of a quantum walk and \( C_1 \) to be \( X \) at the \((|x|+2)-\text{th} \) step. At the other steps, the coin operators are all an identity. After we carry out \( 2|x|+1 \) steps of a quantum walk, the state \( a(0) + b(1) \) can be perfectly transferred. The recovery operators are given by \( U_1 = X \) and \( U_2 = X \). The proof is similar to that of the even case.

Moreover, the scheme can be used to perfectly transfer the unknown coin state (qubit) from the initial position to any target position. Let \( l \) be the initial position and \( |l\rangle \otimes (a(0) + b(1)) \otimes |0\rangle \) be the initial state of a quantum walk. The first two schemes can be used to transfer the state to the right by \(|x|\) steps, i.e., from an initial position \( l \) to \( l + |x| \). The last two schemes can be used to transfer the state to the left by \(|x|\) steps, i.e., from \( l \) to \( l - |x| \). Thus, the schemes can be used for efficient quantum routing.

**Qubit state transfer on circles.** In this subsection, we can use quantum walks on graphs [29]. Choose a special graph, i.e., a circle with \( d \) vertices, where there are two edges at each vertex. Then, the coin space is spanned by \(|0\rangle, |1\rangle\). The conditional shift operator between the position and coin spaces is defined as

\[
S_1 = \sum_{k=0}^{d-1} \left[ |(k+1) \mod d\rangle \langle k| \otimes |0\rangle \langle 0| + |(k-1) \mod d\rangle \langle k| \otimes |1\rangle \langle 1| \right].
\]

(4)

In [26], perfect state transfer can be achieved by quantum walks on \( N \)-circles with one coin space, where there is an even number of vertices on the circle. The unknown state is transferred from the initial position to the opposite position, i.e., \( 0 \rightarrow N/2 \), where \( N \) is an even number. However, in our schemes, by introducing a new coin space, perfect state transfer can be achieved on circles with any vertices and from the initial position to any target position.

Here, we can check where the two steps of a quantum walk are carried out when \( C_1 = C_2 = I \). The walker on the circle will rotate \( 2 \) steps clockwise at the coin state \(|00\rangle\), 2 steps anticlockwise at the coin state \(|11\rangle\), or stand still at the coin state \(|10\rangle \) \((|01\rangle\)) Thus, to transfer the state from 0 to \( x \), there are four methods to design the scheme:

1) The information of \( a \) and \( b \) flows from 0 to \( x \) clockwise.

2) The information of \( a \) and \( b \) flows from 0 to \( x \) anticlockwise.

3) The information of \( a \) \((b)\) flows from 0 to \( x \) clockwise \((\text{anticlockwise})\).

4) The information of \( a \) \((b)\) flows from 0 to \( x \) anticlockwise \((\text{clockwise})\).

Now, we introduce the first method. The number of vertices \( d \) can be odd or even. If vertex \( x \) is even, we use the coin operators listed in table 1. If vertex \( x \) is odd, we use the coin operators listed in table 2.

If the vertex \( x \) is closer to 0 in the anticlockwise direction than in the clockwise direction \((i.e., \ x > d/2)\), we can use the second method for state transfer. The total number of steps of a quantum walk will be less than that of the first method. In addition, the number of vertices can be even or odd. The total numbers of steps of a quantum walk are \( 2d-2x \) (even case) or \( 2d-2x+1 \) (odd case).

**Case 2.1:** The number \( d - x \) is even (table 5).

**Table 5:** Case 2.1 on a circle.

| 1st step | 2nd step | 3rd step | 4th step |
|---------|---------|---------|---------|
| \( C_1 \) | \( I \) | \( X \) | \( I \) |
| \( C_2 \) | \( X \) | \( I \) | \( I \) |

**Case 2.2:** The number \( d - x \) is odd (table 6).

**Table 6:** Case 2.2 on a circle.

| 1st step | 2nd step | 3rd step | 4th step |
|---------|---------|---------|---------|
| \( C_1 \) | \( I \) | \( I \) | \( I \) |
| \( C_2 \) | \( X \) | \( I \) | \( I \) |

The last two methods can also be used to perfectly transfer a state. The total number of steps of a quantum walk is independent of \( x \), whereas the number of vertices should be even. We take the third method as an example and then list the coin operators at each step and the recovery unitary operators.

**Case 3.1:** The position \( x \) is even, and the total number of vertices \( d \) is even (table 7).

**Table 7:** Case 3.1 on a circle.

| 1st step | 2nd step | 3rd step | 4th step |
|---------|---------|---------|---------|
| \( C_1 \) | \( I \) | \( I \) | \( I \) |
| \( C_2 \) | \( X \) | \( I \) | \( I \) |

In this case, we let \( C_2 = X \) only at the \((x+2)\)-th step. At the other steps, the coin operators are all an identity. The calculation process is as follows:

- If \( i = 1 \), \(|\phi\rangle^1 = |1\rangle \otimes a(00) + |d-1\rangle \otimes b(10)\).
- If \( i = 2 \), \(|\phi\rangle^2 = |2\rangle \otimes a(00) + |0\rangle \otimes b(10)\).
- If \( i = x - 1 \), \(|\phi\rangle^{x-1} = |x-1\rangle \otimes a(00) + |d-1\rangle \otimes b(10)\).
- If \( i = x \), \(|\phi\rangle^x = |x\rangle \otimes a(00) + |0\rangle \otimes b(10)\).

After \( x \) steps of a quantum walk in total, the information of a flows to the position \( x \). Moreover, the information of \( b \) stays at position 0. In the rest of the steps of a quantum walk, we should make the information of \( b \) flow to the position \( x \) and keep the information of \( a \) at the last step.

- If \( i = x + 1 \), \(|\phi\rangle^{x+1} = |x+1\rangle \otimes a(00) + |d-1\rangle \otimes b(10)\).
- If \( i = x + 2 \), \(|\phi\rangle^{x+2} = |x\rangle \otimes a(01) + |d-2\rangle \otimes b(11)\).
- If \( i = x + (d-x-1) \), \(|\phi\rangle^{d-1} = |x+1\rangle \otimes a(01) + |x+1\rangle \otimes b(11)\).
- If \( i = d \), \(|\phi\rangle^d = |x\rangle \otimes a(01) + |x\rangle \otimes b(11)\).
The final state $|\phi|^d$ is equal to $|x\rangle \otimes (a|0\rangle + b|1\rangle) \otimes |1\rangle$, which means that the state is successfully transferred to the position $x$. Let $U_2 = X$.

The positive and odd case is as follows.

**Case 3.2**: The position $x$ is odd, and the total number of vertices $d$ is even (table 8).

Table 8: Case 3.2 on a circle.

| Step | $C_1$ | $C_2$ |
|------|-------|-------|
| 1    | $X$   | $I$   |
| 2    | $x+1$ | $I$   |
| 3    | $x+1$ | $I$   |
| 4    | $d-3$ | $d-2$ |
| $d$  | $d$   | $d$   |

In this case, let $C_1 = X$ at the 1st step and $C_2 = X$ at the $(x + 1)$-th step. At the other steps, the coin operators are all an identity. The proof is similar to that for the even case.

**Qudit state transfer on a $d$-regular graph.** Now, we introduce a scheme to perfectly transfer a qudit (coin state) through a quantum walk on a $d$-regular graph with two coins. Denote the number of vertices by $n$. The qudit is $\sum_{k=0}^{d-1} a_k |k\rangle$, where $|a_k|^2 = 1$. Each coin space is spanned by $|0\rangle, \ldots, |d-1\rangle$. The conditional shift operator between the position and coin spaces is defined as

$$S = \sum_{k=0}^{n-1} \sum_{j=0}^{d-1} ((k + j) \mod n) |k\rangle \langle j| \langle j|.$$  

This operator simulates the shift regulations between the vertices for the different tossed coin states.

Prepare the initial state of the walker-coin-coin system to be

$$|\phi\rangle^0 = |0\rangle \otimes \left(\sum_{k=0}^{d-1} a_k |k\rangle\right) \otimes |0\rangle.$$  

Let $|\phi\rangle^i$ denote the state describing the system after the $i$-th step.

It is noted that the coin operations are qudit operations. Define a permutation operation as follows:

$$X_d = \sum_{j=0}^{d-1} |(j + 1) \mod d\rangle \langle j|.$$  

Here, we list the coin operators at each step (table 9).

Table 9: Case on a $d$-regular graph.

| Step | $C_1$ | $C_2$ |
|------|-------|-------|
| 1    | $I$   | $I$   |
| 2    | $2n-2x$ | $2n-2x+1$ |
| 3    | $2n-2x+2$ | $2n-2x+2$ |
| 4    | $2n-1$ | $2n$ |
| $d$  | $X_d$ | $I$   |

Let $C_1 = C_2 = I$, if we carry out two steps of a quantum walk, the position will shift from 0 to $k$ at the coin state $|k\rangle|0\rangle$. The detailed process is as follows:

- If $i = 2n - 2x$,

$$|\phi\rangle^{2n-2x} = \sum_{k=0}^{d-1} a_k ((n-x)k \mod n) |k\rangle |0\rangle.$$  

- If $i = 2n - 2x + 1$,

$$|\phi\rangle^{2n-2x+1} = \sum_{k=0}^{d-1} a_k ((n-x)(k+1) \mod n) |k\rangle |0\rangle.$$  

- If $i = 2n - 2x + 2$,

$$|\phi\rangle^{2n-2x+2} = \sum_{k=0}^{d-1} a_k ((n-x)(k+1) \mod n) |k\rangle |1\rangle.$$  

- If $i = 2n$,

$$|\phi\rangle^{2n} = \sum_{k=0}^{d-1} a_k ((n-x)(k+x \mod n) |k\rangle |1\rangle.$$  

$$= |x\rangle \otimes \left(\sum_{k=0}^{d-1} a_k |k\rangle\right) \otimes |1\rangle.$$  

Let $U_2 = X_d^{-1}$, the qudit is perfectly transferred from 0 to $x$.

**Generalized teleportation over long-distance walks by quantum walks with two coins.** If a measurement is used as a tool, information can also be transferred between coin spaces. Recently, teleportation schemes by quantum walks with two coins were introduced [12]. By two steps of a quantum walk, we can transmit an unknown qubit/qudit state from the coin-1 space to the coin-2 space. However, teleportation based on quantum walks with two coins by a long-distance walk remains unclear. In this section, we discuss the implementation conditions to conduct generalized teleportation by quantum walks with two coins for various models. Unlike the above state transfer schemes, by introducing quantum measurement, we realize communication between two coin spaces.

**Qubit teleportation on a one-dimensional line over long-distance walks.** Suppose that there are two coins in total. The initial state of the walker-coin-coin system is

$$|\phi\rangle^0 = |0\rangle \otimes (a|0\rangle + b|1\rangle) \otimes |+\rangle.$$  

Our goal is to revive the state $a|0\rangle + b|1\rangle$ in the coin-2 space at some position.

In this scheme, local measurements in the position and coin-1 spaces are allowed, and $S$ is chosen to be the evolution between the position and coin spaces. Let the coin operators be an identity, i.e., $C_1 = C_2 = I$.

After the first step of $W_1$, the state evolves to

$$|\phi\rangle^1 = (W_1^1 W_2^1)|\phi\rangle^0 = |1\rangle \otimes (a|0\rangle \otimes |+\rangle + |1\rangle \otimes b|1\rangle \otimes |+\rangle.$$  

After the second step of $W_2$, the state evolves to

$$|\phi\rangle^2 = (W_1^2 W_2^2)|\phi\rangle^0 = |0\rangle \otimes (a|0\rangle + b|1\rangle) \otimes (|1\rangle \otimes b|1\rangle \otimes |+\rangle + |-1\rangle \otimes b|1\rangle \otimes |+\rangle.$$  

$$+ |2\rangle \otimes (a|0\rangle + b|1\rangle) \otimes (|1\rangle \otimes b|1\rangle \otimes |+\rangle).$$
Let \( n \) be an even number. At time \( t = n \), the state evolves to
\[
|\phi\rangle^t = (W_1^{n/2}W_2^{n/2})|\phi\rangle^0
= |0\rangle \otimes (a(00\rangle + b(10\rangle) + \sqrt{2} \\
+ (|1-n\rangle \otimes b(11\rangle + |n\rangle \otimes a(00\rangle)/\sqrt{2}.
\]
(14)

Then, we take a measurement with the basis \{\ket{0}, \ket{n}, \ket{-n}, \ldots\} in the position space and use the measurement basis \{\ket{+}, \ket{-}\} in the coin-1 space, namely, |\pm n\rangle = \frac{1}{\sqrt{2}} (|1-n\rangle \pm |n\rangle)/\sqrt{2}. The state in the coin-2 space will recover to the target state \( a(0) + b(1) \) if some local unitary operations are carried out according to the measurement results.

**Qubit teleportation on circles over long-distance walks.**

Suppose that there are two coins in total. The initial state of the walker-coin-coin system is
\[
|\phi\rangle^0 = |0\rangle \otimes (a(0) + b(1)) \otimes |+\rangle.
\]
(15)

Let the coin operators be an identity and independent with time, i.e., \( C_1 = C_2 = I \). After the first step of \( W_1 \), the state evolves to
\[
|\phi\rangle^1 = (W_1^{|n/2|}W_2^{|n/2|})|\phi\rangle^0
= |1\rangle \otimes a(00\rangle + |d-1\rangle \otimes b(1) \otimes |+\rangle.
\]
(16)

After the second step of \( W_2 \), the state evolves to
\[
|\phi\rangle^2 = (W_1^{d/2}W_2^{d/2})|\phi\rangle^0
= |0\rangle \otimes (a(01\rangle + b(10\rangle) + \sqrt{2} \\
+ (|d-2\rangle \otimes b(11\rangle + |2\rangle \otimes a(00\rangle)/\sqrt{2}.
\]
(17)

Let the number of vertices \( d \) be an even number. At time \( t = d/2 \), the state evolves to
\[
|\phi\rangle^t = (W_1^{d/4}W_2^{d/4})|\phi\rangle^0
= |0\rangle \otimes (a(01\rangle + b(10\rangle) + \sqrt{2} \\
+ |d/2\rangle \otimes (a(00\rangle + b(11\rangle)/\sqrt{2}.
\]
(18)

Then, we take a measurement with the basis \{\ket{0}, \ldots, \ket{d-1}\} in the position space and use the measurement basis \{\ket{+}, \ket{-}\} in the coin-1 space. The state in the coin-2 space will recover to the target state \( a(0) + b(1) \) after some local unitary operations.

**Qudit teleportation on a d-regular graph over long-distance walks.** We can also select a d-regular graph with \( n \) vertices to teleport an unknown qudit, where \( n \geq 2d-1 \). Suppose that the number of vertices is in the range of \( \{d+1, \ldots, 2d-2\} \). The qudit cannot be teleported with a probability of 1. Let \( n/2d-1 \) be the integer part, where \( t \in \{1, \ldots, [n/2d-1] \} \). The teleportation scheme will occur at 2t-th steps. When \( t > [n/2d-1] \), the existing state in the position space exceeds the maximum number of labels of the vertices.

The first step of the walk is given by \( W_1 = E_1 \cdot (I_0 \otimes C_1 \otimes I_2) \), where \( E_1 = \sum_{j=0}^{d-1} \sum_{k=0}^{n-1} (|k+j\rangle |k\rangle \otimes |j\rangle \otimes I_2) \). Similarly, the coin operator \( C_1 \) can be any qudit operation for successful teleportation. Here, let \( C_1 = I \). The second step of the walk is given by \( W_2 = E_2 \cdot (I_0 \otimes I_1 \otimes I_2) \), where \( E_2 = \sum_{j=0}^{d-1} \sum_{k=0}^{n-1} (|k+j\rangle |k\rangle \otimes |j\rangle \otimes I_2) \). The initial state is
\[
|\phi\rangle^0 \otimes \sum_{m=0}^{d-1} a_m |m\rangle \otimes \sum_{k=0}^{d-1} |k\rangle/\sqrt{d}.
\]
(19)

After two steps of a quantum walk, we have the state given by
\[
\sum_{m=0}^{d-1} \sum_{k=0}^{d-1} a_m |m+k\rangle |m\rangle |k\rangle/\sqrt{d}.
\]
(20)

After 2t steps of a walk, the final state is
\[
\sum_{m=0}^{d-1} \sum_{k=0}^{d-1} a_m |t(m+k) \mod n\rangle |m\rangle |k\rangle/\sqrt{d}.
\]
(21)

The set of all integers that is mutually prime with \( n \) and no more than \( n \) is denoted by \( A(n) \). Let \( \phi(n) \) represent the size of \( A(n) \). That is,
\[
A(n) = \{x|1 \leq x \leq n, x \in \mathbb{Z}, (x, n) = 1 \}
= \{A_1, A_2, \ldots, A_{\phi(n)} \}
= \{1, 2, \ldots, \phi(n) \}.
\]
(22)

Let \( t = xn + Ai, \) where \( x = 0, 1, 2, \ldots, \) and \( i = 1, 2, \ldots, \phi(n) \). Thus, the final state is
\[
|\phi\rangle^{2t} = \sum_{m=0}^{d-1} \sum_{k=0}^{d-1} a_m |A_i(m+k) \mod n\rangle |m\rangle |k\rangle/\sqrt{d}.
\]
(23)

Obviously, \( (A_i(m+k) \mod n) \in \{0, 1, \ldots, n-1\} \). For arbitrary different values of \( (m+k) \), they correspond to different \( A_i(m+k) \mod n \). There are \( (2d-1) \) different states in total in the position space. For convenience, we list the position states following the order of \( (m+k) \) from small to large:
\[
|A_1(0)\rangle, |A_2(0)\rangle, \ldots, |(2d-1)(0)\rangle.
\]

Thus, Alice measures the position state with the basis
\[
\{(|k\rangle \pm |d+k\rangle)/\sqrt{2}, |(d-2)(0)\rangle, \ldots, |(n-1)(0)\rangle \}
\]
for \( k = 0, \ldots, d-2 \). Denote the result as \( k, k+d, \) or \( d-1 \) if the state degenerates to \( (|k\rangle + |d+k\rangle)/\sqrt{2}, \) \( (|k\rangle - |d+k\rangle)/\sqrt{2}, \) or \( |(d-1)(i)\rangle \), respectively.

In addition, Alice measures the coin-1 state with the basis \( \{|i\rangle : k = 0, \ldots, d-1 \} \) obtained from a Fourier transform. For example, when the result in the position space is \( k \), the states of coin 1 and coin 2 are
\[
\sum_{m=0}^{k} a_m |m\rangle \otimes |k-m \rangle + \sum_{m=k+1}^{d-1} a_m |m\rangle \otimes |k+d-m \rangle.
\]
(24)

Expressing the state in the coin-1 space with the basis obtained from the Fourier transform, this state is equal to
\[
\sum_{m=0}^{d-1} \sum_{k=0}^{d-1} a_m e^{-2\pi i m/k} |\hat{F}_k \rangle \otimes |k-m \rangle
+ \sum_{m=k+1}^{d-1} a_m e^{-2\pi i m/k} |\hat{F}_k \rangle \otimes |k+d-m \rangle.
\]
(25)
If the result of the coin-1 state is \( \hat{t} \), the state of coin 2 is
\[
\sum_{m=0}^{k} a_m e^{-2\pi i m/d} |k - m\rangle
+ \sum_{m=k+1}^{d-1} a_m e^{-2\pi i m/d} |k + d - m\rangle.
\]
In order to recover the state \( \sum_{m=0}^{d-1} a_m |m\rangle \), Bob performs the local unitary operation expressed by
\[
U_{k\hat{t}} = \sum_{m=0}^{k} e^{2\pi i m/d} |m\rangle\langle k - m| + \sum_{m=k+1}^{d-1} e^{2\pi i m/d} |m\rangle\langle k + d - m|.
\]
If the results of the position and coin-1 states are \( d + k \) and \( \hat{t} \), Bob performs the local unitary operation given by
\[
U_{(d+k),\hat{t}} = \sum_{m=0}^{k} e^{2\pi i m/d} |m\rangle\langle k - m| - \sum_{m=k+1}^{d-1} e^{2\pi i m/d} |m\rangle\langle k + d - m|.
\]
If the results of the position and coin-1 states are \( d - 1 \) and \( \hat{t} \), respectively, Bob performs the following local unitary operation:
\[
U_{(d-1),\hat{t}} = \sum_{m=0}^{d-1} e^{2\pi i m/d} |m\rangle\langle d - 1 - m|.
\]

Conclusions. – In this paper, we study quantum communication protocols for various building blocks for quantum networks by quantum walks with two coins. By alternately using two coins, we find that perfect qubit state transfer can be implemented on either a line or an \( N \)-circle. In particular, perfect qubit state transfer by quantum walks can be first realized on \( d \)-regular graphs, which cannot be obtained by quantum walks with one coin. Furthermore, we study how to carry out generalized teleportation [12] by a quantum walk with two coins after long-distance walks on these graphs. These works explore some new application for quantum walks with multiple coins. Because quantum walks are a universal quantum computation model, they may provide a universal platform for the implementation of scalable quantum networks.

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This work was partially supported by the National Key Research and Development Program of China under grant No. 2016YFB1000902, the National Natural Science Foundation of China (grant No. 61472142, 61872352), and the Program for Creative Research Group of the National Natural Science Foundation of China (grant No. 61621003).

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