New parameterized solution with application to bounding secondary variables in FE models of structures

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Abstract
In this work we propose a new kind of parameterized outer estimate of the united solution set to an interval parametric linear system. The new method has several advantages compared to the methods obtaining parameterized solutions considered so far. Some properties of the new parameterized solution, compared to the parameterized solution considered so far, and a new application direction are presented and demonstrated by numerical examples. The new parameterized solution is a basis of a new approach for obtaining sharp bounds for derived quantities (e.g., forces or stresses) which are functions of the displacements (primary variables) in interval finite element models (IFEM) of mechanical structures.

Keywords: linear algebraic equations, interval parameters, solution set, parameterized outer estimate, secondary variables.

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1. Introduction

Denote by $\mathbb{R}^{m \times n}$ the set of real $m \times n$ matrices. Vectors are considered as one-column matrices. A real compact interval is $a = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} | \underline{a} \leq a \leq \overline{a}\}$ and $\mathbb{I}^{m \times n}$ denotes the set of interval $m \times n$ matrices. We consider systems of linear algebraic equations having affine-linear uncertainty.

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structure

\[ A(p)x = a(p), \quad p \in \mathbb{P} \subseteq \mathbb{R}^K, \]

\[ A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, \quad a(p) := a_0 + \sum_{k=1}^{K} p_k a_k, \quad (1) \]

where \( A_k \in \mathbb{R}^{n \times n}, \ a_k \in \mathbb{R}^n, \ k = 0, \ldots, K \) and the parameters \( p = (p_1, \ldots, p_K)^\top \) are considered to be uncertain and varying within given non-degenerate intervals \( \mathbb{P} = (p_1, \ldots, p_K)^\top \). The so-called united parametric solution set of the system (1) is defined by

\[ \Sigma_{\text{uni}}^p = \Sigma_{\text{uni}}(A(p), a(p), \mathbb{P}) := \{ x \in \mathbb{R}^n \mid (\exists p \in \mathbb{P})(A(p)x = a(p)) \}. \]

Usually, bounding a solution set interval methods generate numerical interval vectors that contain the solution set. A new type of solution, \( x(p, l) \), called parameterized or p-solution, providing outer estimate of the united parametric solution set is proposed in [1]. This solution is in form of an affine-linear function of interval-valued parameters

\[ x(p, l) = \bar{x} + Up + l, \quad p \in \mathbb{P}, \ l \in \mathbb{I}, \]

where \( \bar{x} \in \mathbb{R}^n, \ U \in \mathbb{R}^{n \times K} \) and \( \mathbb{I} \) is an \( n \)-dimensional interval vector. The parameterized solution has the property \( \Sigma_{\text{uni}}^p \subseteq x(p, l) \), where \( x(p, l) \) is the interval hull of \( x(p, l) \) over \( p \in \mathbb{P}, \ l \in \mathbb{I} \). For a nonempty and bounded set \( \Sigma \subseteq \mathbb{R}^n \), its interval hull \( \square \Sigma \) is defined by

\[ \square \Sigma := \bigcap \{ x \in \mathbb{I}\mathbb{R}^n \mid \Sigma \subseteq x \}. \]

Since \( x(p, l) \) is a linear function of interval parameters,

\[ x(p, l) = \square \{ x(p, l) \mid p \in \mathbb{P}, l \in \mathbb{I} \} = \bar{x} + Up + l. \]

Parameterized forms of solution enclosures are proposed in relation to different numerical methods, the latter yielding interval boxes (vectors) containing the solution set, see, e.g., [1]–[4]. Parameterized enclosure of parametric AE-solution sets is developed in [5]. The potential of the parameterized solution for solving some global optimization problems where the parametric

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1 An interval \( \mathbf{a} = [\underline{a}, \overline{a}] \) is degenerate if \( \underline{a} = \overline{a} \).
linear system (1) is involved as equality constraint is shown in [3]. All parameterized solutions considered so far are functions of the initial parameters \( p \) of the system and of \( n \) (\( n \) is the dimension of the system) additional interval parameters \( l \). Therefore, and in order to distinguish the newly proposed parameterized solution, we will call all parameterized solutions considered so far Kolev-style parameterized solutions, shortly \( p, l \)-solutions instead of \( p \)-solutions.

In this work we propose a new parameterized outer estimate of the united solution set to an interval parametric linear system. Basing on a recently proposed framework for interval enclosure of the united parametric solution set, which has a broader scope of applicability [6], the new parameterized method has, respectively, a broader scope of applicability than most of the methods obtaining parameterized solutions considered so far. For parametric systems involving rank one uncertainty structure, the new parameterized solution depends only on the initial parameters of the system.

The structure of the paper is as follows. Section 2 introduces notation and known results about the parameterized \( x(p, l) \) solution. The new parameterized solution and its interval enclosure property are derived in Section 3. Some geometric properties of the parameterized solutions and theoretical comparison between the two kinds of parameterized solutions are presented in Section 4 along with numerical illustrative examples. Section 5 presents a new application direction for the newly proposed parameterized solution — a new simpler approach providing sharp bounds for derived variables in interval finite element models (IFEM) of mechanical structures. The new parameterized approach is illustrated by some example problems, which demonstrate its ability to deliver sharp bounds to derived variables with the same quality as those of the primary variables with less effort. In these examples we compare the interval enclosures obtained by the two kinds of parameterized solutions and by the direct interval approach, as well as by other approaches considered so far. The paper ends by some conclusions.

2. Preliminaries

For \( \mathbf{a} = [\underline{a}, \overline{a}] \), define its mid-point \( \mathbf{\hat{a}} := (\underline{a} + \overline{a})/2 \), the radius \( \mathbf{\hat{a}} := (\overline{a} - \underline{a})/2 \) and the magnitude \( |\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\} \). These functions are applied to interval vectors and matrices componentwise. Inequalities are understood componentwise. The spectral radius of a matrix \( A \in \mathbb{R}^{n \times n} \) is denoted by \( \varrho(A) \). The identity matrix of appropriate dimension is denoted by \( I \). For
\( A_k \in \mathbb{R}^{n \times m}, 1 \leq k \leq t, (A_1, \ldots, A_t) \in \mathbb{R}^{n \times tm} \) denotes the matrix obtained by stacking the columns of the matrices \( A_k \). Denote the \( i \)-th column of \( A \in \mathbb{R}^{n \times m} \) by \( A \bullet_i \) and its \( i \)-th row by \( A_i \).

**Theorem 1** ([7], Theorem 4.4). Let \( A = [I - \Delta, I + \Delta] \in \mathbb{R}^{n \times n} \) with \( \Delta \in \mathbb{R}^{n \times n}, \varrho(\Delta) < 1 \). Then the inverse interval matrix

\[
A^{-1} := \left[ \min\{ A^{-1} \mid A \in A \}, \max\{ A^{-1} \mid A \in A \} \right] = [H, H]
\]

is given by

\[
H = (h_{ij}) = (I - \Delta)^{-1},
\]

\[
H = (h_{ij}), \quad h_{ij} = \begin{cases} -h_{ij} & \text{if } i \neq j \\ \frac{h_{ij}}{2h_{jj} - 1} & \text{if } i = j. \end{cases}
\]

Next we recall the simplest single step method for obtaining the \( p, l \)-solution to an united parametric solution set \( \Sigma_{\text{uni}}^p \). With the notation \( \check{A} = A_0 + \sum_{i=1}^{K} \check{p}_i A_i, \check{a} = a_0 + \sum_{i=1}^{K} \check{p}_i a_i, \) system (1) is equivalent to the interval parametric system

\[
\left( \check{A} + \sum_{i=1}^{K} p_i A_i \right) x = \check{a} + \sum_{i=1}^{K} p_i a_i, \quad p \in [-\check{p}, \check{p}] \in \mathbb{I}^{K}. \tag{2}
\]

The following theorem is modified from [2, Theorem 1] for the system (2).

**Theorem 2** ([2, Theorem 1]). Let \( \check{A} \) in (2) be nonsingular. Denote \( \check{x} = \check{A}^{-1} \check{a}, \ F = (a_1, \ldots, a_K), \ G = (A_1 \check{x}, \ldots, A_K \check{x}), \ B^0 = \check{A}^{-1} (F - G) \). Assume that

\[
\varrho\left( \sum_{i=1}^{K} |\check{A}^{-1} A_i| \check{p}_i \right) < 1. \tag{3}
\]

Then

(i) \( A(p) \) in (2) is regular for each \( p \in [-\check{p}, \check{p}] \in \mathbb{I}^{K}; \)

(ii) the united \( p, l \)-solution \( x(p, l) \) of the system (2) exists and is determined by

\[
x(p, l) = \check{x} + U p + l, \quad p \in [-\check{p}, \check{p}] \in \mathbb{I}^{K}, \quad l \in [-\check{l}, \check{l}] \in \mathbb{I}^{n}, \tag{4}
\]
where $U = \hat{H}B^0$, $\hat{t} = \hat{H}|B^0|\hat{p}$, and $\hat{H}$, $\hat{L}$, are the midpoint and radius matrices, respectively, of the inverse interval matrix $H = [\overline{H}, \overline{L}]$ obtained by Theorem 1 for $\Delta = \sum_{i=1}^{K} |A^{-1} \Delta_i| \hat{p}_i$.

Most of the $p, l$-solutions considered so far require or check the condition (3), which determines the scope of applicability of the Kolev-style parameterized solutions.

3. New parameterized solution for $\Sigma_{\text{uni}}$.

Let $K = \{1, \ldots, K\}$ and $\pi', \pi''$ be two subsets of $K$ such that $\pi' \cap \pi'' = \emptyset$, $\pi' \cup \pi'' = K$. For $\pi \subseteq K$, $\text{Card}(\pi) = K_1$, denote $p_\pi = (p_{\pi_1}, \ldots, p_{\pi_{K_1}})$. Denote by $D_{p_\pi}$ a diagonal matrix with diagonal vector $p_\pi$.

In order to obtain a new parameterized solution to the united parametric solution set of (1) we consider the following equivalent form of the parametric system

$$
(A_0 + LD_{g(p_{\pi'})} R) x = a_0 + LD_{g(p_{\pi'})} t + F_{p_{\pi'}} t, \quad p \in \mathbf{p} \tag{5}
$$

with particular $\pi', \pi'' \subseteq K$ and suitable numerical matrices $L, R$, numerical vector $t$, and a parameter vector $g(p_{\pi'})$, which provide equivalent optimal rank one representation (cf. [6] or Definition 1) of either $A(p_{\pi'}) - A_0$, or of $A^\top (p_{\pi'}) - A_0^\top$, and $\sum_{k \in \pi'} p_k a_k = LD_{g(p_{\pi'})} t$. Next definition is summarized from [6].

**Definition 1.** For a parametric matrix $A(p_{\pi'}) = A_0 + \sum_{k \in \pi'} p_k A_k$, $\text{Card}(\pi') = K_1$, the following representation (called also LDR-representation)

$$
A_0 + LD_{g(p_{\pi'})} R, \tag{6}
$$

where $g(p_{\pi'}) \in \mathbb{R}^s$, $s = \sum_{k=1}^{K_1} s_k$, $g(p_{\pi'}) = (g_1^\top (p_{\pi'_1}), \ldots, g_{K_1}^\top (p_{\pi'_{K_1}}))^\top$, $L = (L_1, \ldots, L_{K_1}) \in \mathbb{R}^{n \times s}$, $R = (R_1^\top, \ldots, R_{K_1}^\top)^\top \in \mathbb{R}^{s \times n}$ and for $1 \leq k \leq K_1$, $g_k(p_{\pi'_k}) = (p_{\pi'_1}, \ldots, p_{\pi'_k})^\top \in \mathbb{R}^{s_k}$, $p_{\pi'_k} A_{\pi'_k} = L_k D_{g_k(p_{\pi'_k})} R_k$, is an equivalent optimal rank one representation of $A(p_{\pi'})$ if

(i) (6) restores $A(p_{\pi'})$ exactly, that is

$$
A(p_{\pi'}) = A_0 + LD_{g(p_{\pi'})} R = A_0 + \sum_{k=1}^{K_1} L_k D_{g_k(p_{\pi'_k})} R_k;
$$

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(ii) for each parameter $g_i$, $1 \leq i \leq s$, $g_i \in g_i(p_{\pi'})$, its coefficient matrix $A_i$ has rank one, that is $A_i = L_i R_i$.

(iii) for each $1 \leq k \leq K_1$, the dimension $s_k$ of the diagonal vector $g_k(p_{\pi'k})$ is equal to the rank of $A_k$.

There are various ways to obtain the representation (5), cf. [8], [9]. In what follows, in the representation (5) we will not distinguish between the equivalent representations and between the representations originated from $A(p_{\pi'})$ or from $A^\top(p_{\pi'})$; the difference is essential for the applications, cf. [9, Example 8]. The following theorem presents a method (proposed in [6]) for computing numerical interval enclosure of a parametric united solution set.

**Theorem 3.** Let the system (7) have equivalent representation (5) with optimal rank one representation of $A(p)$ and let the matrix $A(\bar{p})$ be nonsingular. Denote $C = A^{-1}(\bar{p})$ and $\bar{x} = Ca(\bar{p})$. If

$$g \left(\left|(RCL)Dg(\bar{p}_{\pi'\pi''}(-p_{\pi'}))\right|\right) < 1,$$

then

(i) $\Sigma_{uni}(A(p), a(p), p)$ and the united solution set $\Sigma_{uni}(8)$ of the interval parametric system

$$(I - RCLDg(\bar{p}_{\pi'})) y = R\bar{x} - RCFp_{\pi''} - RCLDg(\bar{p}_{\pi'}) \cdot t, \quad p \in [\hat{p}, \hat{p}]$$

are bounded,

(ii) $y \supseteq \Sigma_{uni}(8)$ is computable by methods that require (3),

(iii) every $x \in \Sigma_{uni}(A(p), a(p), p)$ satisfies

$$x \in \bar{x} - (CF)[\hat{p}_{\pi''}, \hat{p}_{\pi''}] + (CL) \left(Dg([[\hat{p}_{\pi''}, \hat{p}_{\pi''}]) |y - t\right).$$

**Proof.** If (7) holds, by [6, Theorem 3.3] (see also [9, Theorem 7]), the parametric matrix $A(p)$, is regular in $p \in p$ and the interval parametric matrix $I - (RCL)Dg(\bar{p}_{\pi'})$, $g(p_{\pi'}) \in g([\hat{p}_{\pi''}, \hat{p}_{\pi''}])$, in (8) is also regular. This implies (i), while (ii) follows because the latter interval parametric matrix satisfies also the condition (3). For more details see [6].
The solution \( x \) of the interval parametric linear system

\[
(I - CLD_{g(p_{\pi'})}R)x = \tilde{x} - (CF)p_{\pi''} + (CL)D_{g(p_{\pi'})}(y - t),
\]

\( p \in [-\hat{p}, \hat{p}] \) \hspace{1cm} (10)

which is equivalent to that of (1) (respectively, that of (5)), and the solution \( y \) of system (8) are related via \( y = Rx \). Hence, each solution \( \tilde{x} \in \Sigma_{uni}(A(p), a(p), p) \) satisfies

\[
\tilde{x} = \tilde{x} - (CF)(\tilde{p}_{\pi''}) + (CL)D_{g(\tilde{p}_{\pi'})}(y - t)
\]

for some \( \tilde{p} \in [-\hat{p}, \hat{p}] \), and some \( y \in \Sigma_{uni}(8) \). Then, the inclusion isotonicity of interval operations gives (9). Since \( g([-\hat{p}_{\pi''}, \hat{p}_{\pi''}]) \) is a symmetric interval vector, the range of \( D_{g(p_{\pi'})}(y - t) \) for \( p_{\pi'} \in [-\hat{p}_{\pi'}, \hat{p}_{\pi'}] \), \( y \in \Sigma_{uni}(8) \) is equal to \( D_{g([-\hat{p}_{\pi''}, \hat{p}_{\pi''}])}(y - t) \).

**Theorem 4.** Let the system (11) have equivalent representation (5) with optimal rank one representation of \( A(p) \) and let the matrix \( A(\hat{p}) \) be nonsingular. Denote \( C = A^{-1}(\hat{p}) \) and \( \bar{x} = Ca(\hat{p}) \). If (7) holds true, then

i) there exists an united parameterized solution of the system (11), respectively the system (5),

\[
x(p_{\pi''}, g) = \tilde{x} - (CF)p_{\pi''} + (CL)D_{g(p_{\pi'})}(y - t),
\]

\( p_{\pi''} \in [-\hat{p}_{\pi''}, \hat{p}_{\pi''}], g \in g([-\hat{p}_{\pi'}, \hat{p}_{\pi'}]), \) \hspace{1cm} (12)

where \( y \supseteq \Sigma_{uni}(8) \),

ii) with the same \( y \) used in (9) and in (12), interval vector \( x([-\hat{p}_{\pi''}, \hat{p}_{\pi''}], g([-\hat{p}_{\pi'}, \hat{p}_{\pi'}])) \) is equal to the interval vector obtained by Theorem 3.

**Proof.** The assumptions of the theorem imply that Theorem 3 holds true. By \([10, \text{Theorem 5.6}], \Sigma_{uni}(8) \subseteq \Sigma_{uni}(13) \),

\[
(I - RCLD_{g}) y = R\tilde{x} - RCFp_{\pi''} - RCLD_{g(p_{\pi'})}t,
\]

\( p \in [-\hat{p}, \hat{p}], g \in g([-\hat{p}_{\pi'}, \hat{p}_{\pi'}]) \). \hspace{1cm} (13)

\( a = [a, \overline{a}] \in \mathbb{R} \) is called symmetric if \( a = -\overline{a} \).
It follows from relation (11) that each solution \( \tilde{x} \in \Sigma_{uni}(A(p), a(p), p) \) satisfies also the relation

\[
\tilde{x} = \tilde{x} - (CF)\tilde{p}_{\pi''} + (CL)D_{\tilde{g}}(y - t)
\]  

for some \( \tilde{p}_{\pi''} \in p_{\pi''} \), some \( \tilde{g} \in g([-\hat{p}_{\pi''}, \hat{p}_{\pi''}]) \), and some \( y \in y \supseteq \Sigma_{uni}(13) \).

We consider the expression in the right-hand side of (14) as a function of interval parameters \( p_{\pi''} \) and \( g \). To this end we rearrange this expression equivalently as

\[
\tilde{x} - (CF)\tilde{p}_{\pi''} + (CL)D_{\tilde{g}}(y - t) = \tilde{x} - (CF)\tilde{p}_{\pi''} + (CLD_{y-t})\tilde{g}.
\]

The latter implies (12) and (ii). Thus, the existence of (12) and (ii) follows from Theorem 3.

It is clear from (12) that the newly proposed parameterized solution \( x(p_{\pi''}, g) \) is a linear function of \( \text{Card}(\pi'') + s \) interval parameters \( p_{\pi''}, g \). More precisely, this parameterized solution is a function of \( K + (s - K_1) \) interval parameters \( p, g' \), where the vector \( g \) involves \( s - K_1 \) auxiliary interval parameters \( g' \).

Parametric linear systems involving rank one interval parameters are widely spread in various application domains. Examples of such systems originating from models of electrical circuits, in biology and structural mechanics are presented in [11].

Corollary 1. Let the system (1) have equivalent representation (5) with optimal rank one representation of \( A(p) \) and each \( A_k \) have rank one. Let the matrix \( A(\tilde{p}) \) be nonsingular. Denote \( C = A^{-1}(\tilde{p}) \) and \( \tilde{x} = Ca(\tilde{p}) \). If (7) holds true, then

i) there exists a united parameterized solution of the system (1), respectively the system (5),

\[
x(p) = \tilde{x} - (CF)p_{\pi''} + (CLD_{|y-t|})p_{\pi''}, \quad p \in [-\hat{p}, \hat{p}], \quad (15)
\]

where \( y \supseteq \Sigma_{uni}(8) \),

ii) with the same \( y \) used in (9) and in (15), interval vector \( x([-\hat{p}, \hat{p}]) \) is equal to the interval vector obtained by Theorem 3.
4. Properties and Comparison

In this section we present some properties of the parameterized solutions and compare the two kinds of these solutions.

**Theorem 5.** Geometrically, the two kinds of parameterized solutions, Kolev-style \((p, l)\)-solutions and the newly proposed \((p, g)\)-solution, are bounded convex polytopes.

**Proof.** From the representations (12) and (4), it is obvious that the two kinds of parameterized solutions are convex polytopes as affine images of the interval boxes \((-\hat{p}_{\pi''}, \hat{p}_{\pi''})^\top, g([-\hat{p}_{\pi''}, \hat{p}_{\pi''})^\top)\) for \(x(p_{\pi''}, g)\) and \((-\hat{p}, \hat{p})^\top, [-\hat{l}, \hat{l})^\top)\) for \(x(p, l)\). The convex polytopes are bounded due to the regularity conditions (7) and (3), respectively. \(\square\)

The first difference between the two kinds of parameterized solutions follows from the conditions (3), (7) for their existence, which imply their scope of applicability. It is proven in [6, Theorem 3.2] that condition (7) is more general than (3) and more powerful for large class of problems. Therefore, the newly proposed parameterized solution \(x(p_{\pi''}, q)\) is applicable to a wider class of parametric interval linear systems. The expanded scope of applicability is demonstrated in [9, Examples 5 and 8], as well as in [12]. In what follows we will not consider examples for which Kolev-style \((p, l)\)-solutions cannot be found. The focus will be on comparing the two kinds of parameterized solutions when both exist.

**Theorem 6.** For a system (1) involving only rank one interval parameters\(^3\) in the matrix and for which both (3), (7) hold true, the convex polytope representing a Kolev-style \((p, l)\)-solution, \(l \neq 0\), contains the convex polytope representing the newly proposed \((p, g)\)-solution.

**Proof.** Since all vertices of the box \(p\) are vertices of the box \((p^\top, I^\top)^\top\), the proof follows from the properties of affine transformations, from Corollary 1 and from \(x(p, I) \supseteq x(p)\). \(\square\)

Our first example demonstrates Theorem 6 on a parametric system for which the interval enclosures of the united parametric solution set, obtained by the corresponding numerical methods, are the same.

\(^3\text{rank}(A_k) = 1, k = 1, \ldots, K_1\)
Example 1. Consider the interval parametric linear system

\[
\begin{pmatrix}
-1 + \frac{1}{2}p_2 & -1 - \frac{1}{2}p_2 \\
-1 - p_2 & -1 + p_2
\end{pmatrix}
x = \begin{pmatrix}
2 + p_2 \\
-2p_2 + 3p_1
\end{pmatrix}, \quad p_1 \in \left[-\frac{1}{4}, 1\right], p_2 \in \left[\frac{1}{2}, \frac{3}{2}\right].
\]  

For this system both conditions (3) and (7) are satisfied. Also, the two numerical methods (Theorem 2 and Theorem 3) yield the same interval vector

\[
x = \begin{pmatrix}
-\frac{17}{12} & 55 \\
-\frac{27}{8} & -\frac{11}{12}
\end{pmatrix}^\top
\]  

containing the united parametric solution set.

The \(p, l\)-solution, obtained by Theorem 2, is

\[
x'(p, l) = \begin{pmatrix}
\frac{7}{16} & 0 \\
-\frac{9}{16} & \frac{11}{16}
\end{pmatrix} + \begin{pmatrix}
-\frac{27}{16}p_1 - \frac{21}{64}p_2 \\
\frac{61}{16}p_1 + \frac{13}{192}l_2
\end{pmatrix},
\]

\[p_1 \in \left[-\frac{5}{8}, \frac{5}{8}\right], p_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right], l_1, l_2 \in [-1, 1].\]  

Its numerical evaluation \(x'(p, l)\) gives (17).

In order to obtain the newly proposed parameterized solution we first obtain the equivalent form (5) of the parametric system (16)

\[
(\check{A} + L.D_{p_2}R)x = \check{a} + F.(p_1) + L.D_{p_2}t, \quad p_1 \in \left[-\frac{5}{8}, \frac{5}{8}\right], p_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right],
\]

where

\[
\check{A} = \check{p}_2 \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-1 & 1
\end{pmatrix}, \quad L = \begin{pmatrix}
\frac{1}{2} \\
-1
\end{pmatrix}, \quad R = (1, -1), \quad D_{p_2} = (p_2),
\]

\[
\check{a} = \begin{pmatrix}
2 + \check{p}_2 \\
-2\check{p}_2 + 3\check{p}_1
\end{pmatrix}, \quad F = \begin{pmatrix}
0 \\
3
\end{pmatrix}, \quad t = (2).
\]

The coefficient matrix of the parameter \(p_2\) in (16) has rank one. The interval parametric equation (8) has the form

\[
(1 - p_2)y = \frac{31}{12} + 2p_1 - 2p_2, \quad p_1 \in \left[-\frac{5}{8}, \frac{5}{8}\right], p_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right].
\]

An interval enclosure of the solution set of the last equation is

\[y = \left[-\frac{1}{2}, \frac{17}{3}\right].\]
Then, by Corollary 1, the parameterized solution is

\[ x''(p) = \left( \frac{7}{16} \frac{1}{48} \right) + \left( \frac{3}{2} p_1 + \frac{11}{6} p_2 \right), \quad p_1 \in [-\frac{5}{8}, \frac{5}{8}], p_2 \in [-\frac{1}{2}, \frac{1}{2}]. \]  (19)

Its interval evaluation \( x''(p) \) gives also (17). However, (19) is a 2-polytope (in particular skew-box), with a much smaller volume than the polytope of the \( p, l \)-solution (18), both presented in Figure 1.

![Figure 1: The united parametric solution set of the system (16) (the most inner butterfly region with red boundary), its interval enclosure (17) (dashed line box), the \( p, l \)-solution (light gray polytope) and the newly proposed p-solution (dark gray polytope).](image)

**Example 2.** Consider the interval parametric linear system

\[
\begin{pmatrix}
1 + \frac{1}{2} p_2 - 2 p_3, & -1 - \frac{1}{2} p_2 \\
-1 - p_2, & -1 + p_2
\end{pmatrix} x = \begin{pmatrix}
p_2 + 3 p_1 - 1 \\
-2 p_2 + 2 p_1 + 3
\end{pmatrix},
\]

\[ p_1 \in [-\frac{1}{4}, 1], \; p_2 \in [\frac{1}{2}, \frac{3}{2}], \; p_3 \in [\frac{1}{5}, \frac{2}{3}]. \]  (20)

For this system both conditions (3) and (7) are satisfied. The method from Theorem 2 yields interval vector

\[
([-5.725, 3.975], [-9.0805 \ldots 5, 9.175])^T
\]

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and a parameterized solution enclosure

\[ x'(p, l) = \left( \frac{7}{17} \right) \left( \frac{5}{360} \right) + \left( \frac{-11881p_1}{3969p_1} + \frac{3124703p_2}{1512000} + \frac{1108559}{198000} \right) l_1, \]

\[ p_1 \in \left[ -\frac{5}{8}, \frac{5}{8} \right], \quad p_2 \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \quad l_1, l_2 \in [-1, 1]. \]

The equivalent optimal rank one representation of system (20) is obtained for

\[ L = \begin{pmatrix} 1, & \frac{1}{2} \\ 0, & -1 \end{pmatrix}, \quad R = \begin{pmatrix} -2, & 0 \\ 1, & -1 \end{pmatrix}, \quad D_p = \begin{pmatrix} p_2, & 0 \\ 0, & p_3 \end{pmatrix}, \]

\[ F = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad t = (0, 2)^\top. \]

The interval parametric equation (8) has the form

\[ \begin{pmatrix} 1 \\ -\frac{2}{7}p_3 \\ -\frac{58}{45}p_2 \end{pmatrix} y = \left( \begin{pmatrix} \frac{7}{8} - 2p_1 + 2p_2 \\ \frac{5}{8} - \frac{4}{5}p_1 - \frac{16}{45}p_2 \end{pmatrix} \right), \]

\[ p_1 \in \left[ -\frac{5}{8}, \frac{5}{8} \right], \quad p_2 \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \quad p_3 \in \left[ -\frac{7}{30}, \frac{7}{30} \right]. \]

An interval enclosure of the solution set of the last equation is

\[ y = ([-5.7, 9.2], [-10.4, 77/9])^\top. \]

Then, by Corollary 1, the parameterized solution is

\[ x''(p) = \left( \frac{7}{17} \right) \left( \frac{6}{360} \right) + \left( \frac{p_1 + 6.2p_2}{157p_1 - 2201p_2 - 92p_3} \right), \]

\[ p_1 \in \left[ -\frac{5}{8}, \frac{5}{8} \right], \quad p_2 \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \quad p_3 \in \left[ -\frac{7}{30}, \frac{7}{30} \right]. \]

The two parameterized solutions and their interval hulls are presented and compared in Figure 2.

In general, when comparing the two parameterized solutions \( x(p, l) \) and \( x(p_{\pi'}, g) \), one has to consider the two relations \( K + n \preceq K + s - K_1 \) and \( x(p, l) \simeq x(p_{\pi'}, g) \), where \( \sim \in \{\subseteq, \supseteq\} \).
Example 3. Consider the interval parametric linear system

$$
\begin{pmatrix}
\frac{1}{2} - p_2, & p_2, & 2 \\
 p_2, & -p_2, & p_1 \\
 2, & p_1, & -2 + p_1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_1 + x_2
\end{pmatrix}
= \begin{pmatrix} p_2 \\
 2 - p_1 - p_2 \\
p_1 - 1
\end{pmatrix},
$$

$$
p_1 \in \left[\frac{2}{3}, \frac{4}{3}\right], p_2 \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{21}
$$

For this system both conditions \[3\] and \[7\] are satisfied. The interval hull of the united parametric solution set, rounded outwardly and presented by 6 digits in the mantissa, is

$$
\left([-1.56997, 0.363637], [-0.727273, 0.5972697], [0.1896562, 0.4927185]\right)^T.
$$

The method from Theorem 2 yields interval vector

$$
\left([-0.782941, 0.782941], [-1.014773, 1.6814392], [0.1896551, 0.49271845]\right)^T.
$$
and a parameterized solution enclosure

\[
  x'(p, l) = \left( \begin{array}{c} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{array} \right) + \left( \begin{array}{c} -0.38474p_1 - 0.256493p_2 \\ 0.924365p_1 + 0.728287p_2 \\ -0.392728p_1 + 0.0374027p_2 \end{array} \right) + \left( \begin{array}{c} 0.526447l_1 \\ 0.67584l_2, \\ 0.101283l_3 \end{array} \right),
\]

\[ p_1 \in [-\frac{1}{3}, \frac{1}{3}], \quad p_2 \in [-\frac{1}{2}, \frac{1}{2}], \quad l_1, l_2, l_3 \in [-1, 1]. \]

The coefficient matrix of \( p_2 \) has rank one, while the coefficient matrix of \( p_1 \) has rank two. Therefore, the equivalent optimal rank one representation of system \((21)\) is obtained for

\[
D_p = \left( \begin{array}{ccc} p_1, & 0, & 0 \\ 0, & p_1, & 0 \\ 0, & 0, & p_2 \end{array} \right), \quad L = \left( \begin{array}{ccc} 0, & 0, & 1 \\ 0, & 1, & -1 \\ 1, & 1, & 0 \end{array} \right), \quad R = \left( \begin{array}{ccc} 0, & 1, & 0 \\ 0, & 0, & 1 \\ -1, & 1, & 0 \end{array} \right),
\]

\[ t = (2, -1, 1)^T. \]

By Theorem \(4\) the parameterized solution is

\[
x''(p_2, g) = \left( \begin{array}{c} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{array} \right) + (\tilde{A}^{-1}L) \left( \begin{array}{ccc} g_1, & 0, & 0 \\ 0, & g_2, & 0 \\ 0, & 0, & p_2 \end{array} \right) \left( \begin{array}{c} 1.462224 \\ 0.563382 \\ 1.9156691 \end{array} \right),
\]

\[ g_1, g_2 \in [-\hat{p}_1, \hat{p}_1] = [-\frac{1}{3}, \frac{1}{3}], \quad p_2 \in [-\frac{1}{2}, \frac{1}{2}]. \]

Its interval evaluation \( x''(p_2, g) \) is

\[
([-1.032869, 1.032869], [-0.795558, 1.462224], [-0.1032854, 0.5633813])^T.
\]

For this example \( x''(p_2, g) \) involves less number of interval parameters than \( x'(p, l) \), however \( x''(p_2, g) \supseteq x'_i(p, l) \) for \( i \in \{1, 3\} \).

5. Bounding secondary (derived) variables

In this section we present a new application direction for the parameterized solution enclosures and demonstrate the value of the newly proposed parameterized solution.

While various methods and techniques are devised for obtaining very sharp (even the exact) bounds for the unknowns (called primary variables)
of an interval parametric linear system, obtaining sharp enclosure of the so-called derived (secondary) variables is referred as a challenging problem. Secondary (derived) variables are functions of the primary variables or of both primary variables and the initial interval model parameters. Due to the dependency, the derived quantities are obtained with significant overestimation. Some special techniques are usually applied to decrease the overestimation in the secondary quantities. In [13] a new mixed formulation of interval finite element method (IFEM) is proposed, where both primary and derived quantities of interest are involved as primary variables in an expanded interval parametric linear system. In this section we propose an alternative approach based on the newly proposed parameterized solution. The new approach requires that the interval enclosure of the primary variables is obtained as a parameterized solution. Thus, interval estimation of the secondary variables reduces to range enclosure of the expressions representing secondary variables as functions of the initial interval model parameters. In formal notations the approach we propose based on the new parameterized solution of primary variables is presented in Algorithm 1.

Let $A(p)u = a(p), p \in p$, be an interval parametric linear system for the primary variables $u$ and $p \in p$ be the interval model parameters. For simplicity of the presentation we assume that the coefficient matrices of all interval parameters have rank one and the system for the primary variable can be solved by Theorem 3. Let $v = p_ib^\top u$ be a secondary (derived) variable, where $p_i$ is one of the model parameters and $b$ is a numerical vector.

**Algorithm 1.** Interval enclosure of the secondary variable $v$ obtained by the new parameterized enclosure (Corollary 7) of the primary variables $u$.

**Input:** numerical matrices $A_0 \in \mathbb{R}^{n \times n}$, $L, R^\top \in \mathbb{R}^{n \times K}$, $F \in \mathbb{R}^{n \times (K - K_1)}$ and vectors $a_0 \in \mathbb{R}^n$, $t \in \mathbb{R}^K$ providing an equivalent representation (3); vectors $b \in \mathbb{R}^n$ and $p \in \mathbb{R}^K$.

**Output:** interval $v = [v^-, v^+]$ for the unknown secondary variable.

1. Obtain the new parameterized interval enclosure of the primary variables by Theorem 3

   $$u_0(p) = u_0 + Up, \quad p \in [-\hat{p}, \hat{p}], \quad u_0 \in \mathbb{R}^n, U \in \mathbb{R}^{n \times K}.$$  

   There is a flexibility in the implementation of this step of the algorithm, which is discussed in [10].

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2. Generate $p' = [-\hat{p}, \hat{p}]$, \textcolor{red}{v'} = p_i \left( b^\top u_0 + (b^\top U)p' \right).

3. Since $v'(p)$ is a quadratic function of $p_i$, $v'$ may overestimate the true range $v'(p)$. To reduce the overestimation we may prove if the endpoints of $v'(p)$ are attained at some endpoints of $p_i$. To this end we evaluate

$$\frac{\partial v'(p)}{\partial p_i} = b^\top u_p + p_i b^\top \frac{\partial u_p}{\partial p_i} = (b^\top u_0 + (b^\top U)p') + p_i (b^\top U_{\bullet i}).$$

Evaluate $v_1 = [v_1^-, v_1^+] = b^\top u_0 + (b^\top U)p'$ and $v_2 = p_i (b^\top U_{\bullet i})$.

3.1 If $0 \in v_1^- + v_2$, then $v^- = (v')^-$
else $s_1 = \text{sign}(v_1^- + v_2) \in \{-1, 1\}$;
$\text{sign}(p'_i) = -s_1 \hat{p}_i$;
$v^- = (\hat{p}_i - s_1 \hat{p}_i)(b^\top u_0 + (b^\top U)p')$;

3.2 If $0 \in v_1^+ + v_2$, then $v^+ = (v')^+$
else $s_2 = \text{sign}(v_1^+ + v_2) \in \{-1, 1\}$;
$\text{sign}(p'_i) = s_2 \hat{p}_i$;
$v^+ = (\hat{p}_i + s_1 \hat{p}_i)(b^\top u_0 + (b^\top U)p')$;

4. Return $v = [v^-, v^+]$.

Theorem 7. Algorithm 1 provides interval enclosure of a secondary variable $v$ with quality, which is not worse than the quality of the enclosure of primary variables $u$ obtained by Theorem 3.

Proof. The proof follows from the linear transformation applied to the new parameterized enclosure obtained by Theorem 3. \hfill \Box

In what follows, the approach proposed in Algorithm 1 is demonstrated on two examples and compared to various other approaches.

5.1. Example 1

Consider a 6-bar truss structure as presented in Fig. after [14]. The structure consists of 6 elements. The crisp values of the parameters of the truss are presented in Table 1.

The traditional finite element method (FEM) for this structure leads to a linear system

$$K(E, A, L)u = f(Q),$$

where $K(E, A, L)$ is the reduced stiffness matrix depending on the structural parameters (modulus of elasticity $E$, cross sectional area $A$, length $L$) for each
element, $f(Q)$ is the load vector and $u$ is the displacement vector. Namely,

$$
K(E, A, L) = \begin{pmatrix}
\frac{E_1 A_1}{L_1} + 0.36 \frac{E_5 A_5}{L_5} & -0.48 \frac{E_5 A_5}{L_5} & -\frac{E_1 A_1}{L_1} & 0 \\
-0.48 \frac{E_5 A_5}{L_5} & \frac{E_1 A_1}{L_1} + 0.64 \frac{E_5 A_5}{L_5} & 0 & 0 \\
-\frac{E_1 A_1}{L_1} & 0 & \frac{E_1 A_1}{L_1} + 0.36 \frac{E_6 A_6}{L_6} & 0.48 \frac{E_6 A_6}{L_6} \\
0 & 0 & 0.48 \frac{E_6 A_6}{L_6} & \frac{E_4 A_4}{L_4} + 0.64 \frac{E_6 A_6}{L_6}
\end{pmatrix},
$$

$$
f(Q) = (Q, 2Q, 2.5Q, -1.5Q)^{\top}, \quad u = (ux_2, uy_2, ux_3, uy_3)^{\top}. \quad (22)
$$

Let the force parameter $Q$ be unknown-but-bounded in the interval $Q = [20, 21] kN$ and the cross sectional areas $A_5, A_6$ be also uncertain varying in the intervals $[1.008, 1.092] \times 10^{-3} \ m^2, [1.1, 1.1] \times 10^{-3} \ m^2$, respectively. The aim is to obtain interval enclosure for the displacements (as primary variables depending on interval model parameters) and for the element axial forces (as secondary variables). Axial forces are quantities of practical interest in design. For the considered example, the global force vector $F = (F_{e_1}, F_{e_3}, F_{e_4}, F_{e_5}, F_{e_6})^{\top}$ is determined by $F = D_v.T.u$, where $F_{e_i}$ are the
Table 1: Crisp values of the parameters for the 6-bar truss structure.

| Parameter                        | Value             |
|----------------------------------|-------------------|
| Modulus of elasticity for all elements | $E_i, i = 1, \ldots, 6 \ (kN/m^2)$ | $2.1 \times 10^8$ |
| Cross sectional area             | $A_1, A_2, A_3, A_4 \ (m^2)$ | $1.0 \times 10^{-3}$ |
| Cross sectional area             | $A_5, A_6 \ (m^2)$ | $1.05 \times 10^{-3}$ |
| Load $Q \ (kN)$                  |                   | $20.5$ |
| Length of the first and second element | $L_1, L_2 \ (m)$ | $0.6$ |
| Length of the third and fourth element | $L_3, L_4 \ (m)$ | $0.8$ |
| Length of the fifth and sixth element | $L_5, L_6 \ (m)$ | $1$ |

corresponding element forces and

$$v = \left( \frac{E_1 A_1}{L_1}, \frac{E_3 A_3}{L_3}, \frac{E_4 A_4}{L_4}, \frac{E_5 A_5}{L_5}, \frac{E_6 A_6}{L_6} \right)^\top, \quad T = \begin{pmatrix} -1, & 0, & 1, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ -\frac{6}{10}, & \frac{8}{10}, & 0, & 0 \\ 0, & 0, & \frac{8}{10}, & \frac{6}{10} \end{pmatrix}.$$ 

Above, the displacements $u = u(A_5, A_6, Q)$ (as primary variables), the vector $v = v(A_5, A_6)$ and the secondary variables – element axial forces $F_{e_i}, i = 1, 3, 4, 5, 6$ – are functions of the interval model parameters $A_5, A_6, Q$.

First, we find interval enclosures for the displacements as parameterized solutions to the interval parametric linear system $K(A_5, A_6)u = f(Q)$. Applying Theorem 2 we obtain

$$10^4 u'(A_5, A_6, Q, l) \approx \begin{pmatrix} 8.5846 - 2153.0 A_5 - 2134.2 A_6 + .41896Q + 10^{-3}26.693l_1 \\ 3.2669 + 491.791 A_5 - 559.409 A_6 + .15937Q + 10^{-3}6.6039l_2 \\ 8.9579 - 1876.6 A_5 - 2449.5 A_6 + .43727Q + 10^{-3}26.952l_3 \\ -3.1109 + 491.80 A_5 - 559.420 A_6 + .15176Q + 10^{-3}6.6017l_4 \end{pmatrix}.$$
It is readily seen that $u^{i}(A_{5}, A_{6}, Q, l) \subset \{[8.151, 9.018], [3.131, 3.402], [8.511, 9.405], [-3.242, -2.979]\}^{T}$.

With the crisp values from Table I the optimal equivalent rank one representation of the system (22) is $(A_{6} + L D_{y} R) u = f(Q)$, where $g = (A_{5}, A_{6})^{T}$ and

$$10^{-5} A_{0} = \begin{pmatrix} \frac{7}{2} & 0 & -\frac{7}{2} & 0 \\ 0 & \frac{7}{2} & 0 & 0 \\ -\frac{7}{2} & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & \frac{21}{8} \end{pmatrix}, \quad 10^{-5} L = \begin{pmatrix} 756 & 0 \\ -1008 & 0 \\ 0 & 756 \\ 0 & 1008 \end{pmatrix}, \quad R_{g}^{T} = \begin{pmatrix} 1 & 0 \\ -\frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}.$$

The application of Corollary 1 to the above system yields the parameterized solution

$$10^{4} u^{\prime\prime}(A_{5}, A_{6}, Q) \approx \begin{pmatrix} 8.5846 + 2306.60 A_{5} + 2285.45 A_{6} - .41876 Q \\ 3.2669 - 527.104 A_{5} + 599.307 A_{6} - .15936 Q \\ 8.9579 + 2010.11 A_{5} + 2622.56 A_{6} - .43697 Q \\ -3.1109 - 527.104 A_{5} + 599.307 A_{6} + .15175 Q \end{pmatrix},$$

where $A_{i} \in [-\hat{A}_{i}, \hat{A}_{i}], i = 5, 6, Q \in [-\hat{Q}, \hat{Q}]$. Its interval hull is

$$10^{4} u^{\prime\prime}(A_{5}, A_{6}, Q) \subset \{[8.164, 9.006], [3.135, 3.399], [8.523, 9.392], [-3.239, -2.982]\}^{T}.$$

It is readily seen that $u^{\prime\prime}(A_{5}, A_{6}, Q)$ provides sharper interval enclosure to the displacements than $u^{i}(A_{5}, A_{6}, Q, l)$. Percentage by which the latter overestimates the former is $(2.95, 2.32, 2.87, 2.39)^{T}$. This implies that the newly proposed parameterized solution $u^{\prime\prime}(A_{5}, A_{6}, Q)$ will provide a sharper enclosure of the element axial forces.

For the particular example we have

$$10^{-5} D_{v} T = 10^{-5} D_{v} T^{\prime\prime}$$

$$= 10^{-5} D_{v}^{\prime}\begin{pmatrix} \frac{7}{2} & 0 & \frac{7}{2} & 0 \\ 0 & \frac{7}{2} & 0 & 0 \\ -\frac{7}{2} & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & \frac{21}{8} \end{pmatrix}, v^{\prime\prime} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ A_{5} \end{pmatrix},$$

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which shows that the element axial forces \( F_e, \ i = 1, 3, 4, \) are linear functions
of the interval model parameters \( A_5, A_6, Q, \) while the axial forces \( F_{e5}, F_{e6} \) are
quadratic functions of the interval model parameters which shows that the element axial forces
\( F \) obtained via the two kinds of parameterized solutions.

Table 2 presents and compares interval enclosures of the element axial forces
in the 6-bar truss structure obtained via direct interval computation, and the enclosures obtained via the two kinds parameterized solutions \( u'(A_5, A_6, Q, l) \) and \( u''(A_5, A_6, Q). \)

| \( F' \) | \( F'' \) | \( F''(A_5, A_6, Q) \) |
|---|---|---|
| \( e_1 \) | \([-17.740, 43.875]\) | \([-16.843, 42.978]\) | \([11.722, 14.412]\) |
| \( e_3 \) | \([82.215, 89.298]\) | \([82.297, 89.216]\) | \([82.297, 89.216]\) |
| \( e_4 \) | \([-85.102, -78.218]\) | \([-85.020, -78.300]\) | \([-85.019, -78.300]\) |
| \( e_5 \) | \([-66.621, -45.919]\) | \([-66.388, -46.135]\) | \([-62.365, -49.848]\) |
| \( e_6 \) | \([102.13, 132.51]\) | \([102.39, 132.23]\) | \([104.86, 129.51]\) |

Table 2: Interval enclosures for the element axial forces in the 6-bar truss structure obtained via the two kinds parameterized solutions \( u'(A_5, A_6, Q, l) \) and \( u''(A_5, A_6, Q). \)

Due to \( u'' \subset u' \), it is clear that \( F'' \subset F' \) and the latter overestimation is
\((2.9, 2.3, 2.4, 2.2, 1.8)^T \%\). Note that the enclosures \( F', F'' \) are so bad that the
sign of \( F_{e1} \) cannot be determined. Interval values for \( F'(A_5, A_6, Q, l) \) are not
present in Table 2 because \( F'(A_5, A_6, Q, l) = F' \). This means that Kolev-style
parameterized solution was not able to improve the bounds \( F' \). Intervals \( F'' \) overestimate intervals \( F''(A_5, A_6, Q) \) by \((95.5, 0, 0, 38.2, 17.4)^T \%\), respectively.
Since \( F_{e5}, (A_5, A_6, Q), i = 5, 6, \) are quadratic polynomials of the interval
parameters \( A_5, A_6, \) respectively, their interval values presented in Table 2 in general, may not be equal to the corresponding ranges. Evaluating partial derivatives as presented in Algorithm \( \Pi \) we prove that \( F''_{e5}(A_5, A_6, Q) \) is

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monotonic decreasing on $A_5$, while $F''(A_5, A_6, Q)$ is monotonic increasing on $A_6$. Thus $F''(A_5, A_6, Q)$ presented in Table 2 are the exact ranges of the corresponding expressions and the quality of the enclosures $F''(A_5, A_6, Q)$ is the same as the quality of the enclosures $u''$. Note that neither $u''$ nor $F''(A_5, A_6, Q)$ are the exact ranges of the corresponding unknowns. In order to demonstrate the quality of the enclosures $F''(A_5, A_6, Q)$ we give below the corresponding exact ranges rounded outwardly.

$$F \in ([11.8215, 14.3755], [82.4287, 89.1673], [-84.9499, -78.4121],$$
$$[-58.9591, -53.0358], [109.960, 123.970])^{\top}.$$  

5.2. Example 2

Consider a finite element model of a one-bay 20-floor truss cantilever presented in Fig. 4 after [15]. The structure consists of 42 nodes and 101 elements. The bay is $L = 1$m, every floor is $0.75L$, the element cross-sectional area is $A = 0.01 \text{ m}^2$, and the crisp value for the element Young modulus is $E = 2 \times 10^8 \text{kN/m}^2$. Twenty horizontal loads with nominal value $P = 10 \text{kN}$ are applied at the left nodes. The boundary conditions are determined by the supports: at $A$ the support is a pin, at $B$ the support is roller. It is assumed 10% uncertainty in the modulus of elasticity $E_k$ of each element ($\pm 5\%$ from the corresponding mean value) and 10% uncertainty in the twenty loads. The goal is to obtain bounds for the axial force ($F_{40}$) in element 40.

![Image](https://via.placeholder.com/150)
Exactly this problem is used in [13] as a benchmark problem for the applicability, computational efficiency and scalability of the approach proposed therein for structures with complex configuration and a large number of interval parameters. The aim of using this example in the present work is similar: to check these properties for the newly proposed Algorithm 1 based on the new parameterized solution. In addition, the interval result obtained by the approach proposed here will be compared to the results obtained by various other approaches considered in [13, Example 2].

Table 3 presents intervals for the axial force $F_{40}$ in element 40, which are obtained by:

- the special expanded finite element formulation, proposed in [13], ($F_{40}$);
- the newly proposed parameterized solution and step 2 of Algorithm 1, ($F'_{40}(E, P)$);
- the newly proposed parameterized solution and step 3 of Algorithm 1, ($F''_{40}(E, P)$).

| $F_{40}$ by [13] | $F'_{40}(E, P)$ | $F''_{40}(E, P)$ |
|------------------|-----------------|------------------|
| [60.652, 98.991] | [55.729, 106.03] | [61.595, 98.639] |

Table 3: Axial force $F_{40}$ (kN) in element 40 of the cantilever truss obtained by various approaches.

Interval values for the axial force $F_{40}$, obtained by Pownuk’s “gradient-free” method [16] and by the Neumaier’s enclosure $z_2(u)$ [12, Eqn. (4.13)], are presented in [13] and can be compared.

It should be mentioned that the coefficient matrices of all interval parameters in the linear system for the displacements have rank one. Therefore, there are no exceed interval parameters in the parameterized solution enclosure for the displacements. The symbolic expression of $F_{40}(E, P)$ is a quadratic function of the interval parameter $E_{40}$. Applying step 3 of Algorithm 1 we prove numerically that both the lower and the upper bounds of $F_{40}(E, P)$ are attained at the upper bound of $E_{40}$. Note that this does not mean monotonic dependence of $E_{40}$. Note also that the above proof is very easy compared to proving monotonic dependence of the displacements on the interval parameters. Step 3 in Algorithm 1 costs nothing compared to step 2.
of the algorithm. Thus, we obtain an improvement \( F''_{40}(E, P) \) of the bounds \( F'_4(\mathbf{E}, \mathbf{P}) \) in Table 3. Interval \( F''_{40}(\mathbf{E}, \mathbf{P}) \) is sharper than the interval \( F'_{40} \), obtained by the approach of \([13]\), which shows the efficiency of the newly proposed approach based on the new parameterized solution. It should be also mentioned that the interval axial force \( z_2(u) \), showed in \([13, \text{Table 4}]\) and obtained by the Neumaier’s approach \([12, \text{Eqn. (4.13)}]\), is the same as the interval \( F'_{40}(\mathbf{E}, \mathbf{P}) \) in Table 3.

6. Conclusion

We presented a new kind of parameterized solution to interval parametric linear systems. It is based on optimal rank one representation of the parameter dependencies. This representation determines the number of interval parameters in the parameterized solution, as well as, whether the new parameterized solution will have better properties than the Kolev-style parameterized solution.

The major advantage of the newly proposed parameterized solution is for interval parametric linear systems involving rank one uncertainty structure. Such systems appear often in various domain-specific models, cf., \([11]\). A general application direction is presented in this article and illustrated by some numerical examples originated from worst-case analysis of truss structures in mechanics.

While bounding secondary variables by the approach of \([13]\) requires a dedicated IFEM formulation for each particular problem and the system to be solved is expanded by the number of derived quantities, the approach based on the new parameterized solution of primary variables does not depend on the IFEM formulation, does not require solving an expanded interval parametric linear system, and provides sharp bounds for the derived quantities by a simple interval evaluation. The proposed new approach could be applied to various other problems of enclosing secondary (derived) variables.

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