The Willmore flow with prescribed isoperimetric ratio

Fabian Rupp
Faculty of Mathematics, University of Vienna, Vienna, Austria

ABSTRACT
We introduce a non-local $L^2$-gradient flow for the Willmore energy of immersed surfaces which preserves the isoperimetric ratio. For spherical initial data with energy below an explicit threshold, we show long-time existence and convergence to a Helfrich immersion. This is in sharp contrast to the locally constrained flow, where finite time singularities occur.

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1. Introduction and main results
Finding the shape which encloses the maximal volume among surfaces of prescribed area is certainly one of the oldest and yet most prominent problems in mathematics and goes back to the legend of the foundation of Carthage. Since then generations of mathematicians have been studying isoperimetric problems, aiming to find the best possible shape in all kinds of settings. It turns out that—by the isoperimetric inequality—the optimal configuration in Euclidean space is given by a round sphere.

Likewise, the round spheres are the absolute minimizers for the Willmore energy, a functional measuring the bending of an immersed surface with various applications also beyond geometry, for instance in the study of biological membranes [1, 2], general relativity [3], nonlinear elasticity [4] and image restoration [5].

Note that the round spheres describe the optimal shape in both situations. In this article, we will study their relation using a gradient flow approach.

For an immersion $f : \Sigma \to \mathbb{R}^3$ of a closed oriented surface $\Sigma$, its Willmore energy is defined by

$$\mathcal{W}(f) := \frac{1}{4} \int_{\Sigma} |H|^2 \, d\mu.$$  

Here $\mu = \mu_f$ denotes the area measure induced by the pull-back of the Euclidean metric $g_f := f^* \langle \cdot, \cdot \rangle$, and $H = H_f := \langle \tilde{H}_f, \nu_f \rangle$ denotes the (scalar) mean curvature with respect to $\mu_f$. 

CONTACT Fabian Rupp fabian.rupp@univie.ac.at Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria.

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The unique unit normal along $f$ induced by the chosen orientation on $\Sigma$, see (2.1). A related quantity is the *umbilic Willmore energy*, given by

$$W_0(f) := \int_{\Sigma} |A^0|^2 \, d\mu,$$

where $A^0$ denotes the trace-free part of the second fundamental form. As a consequence of the Gauss–Bonnet theorem, these two energies are equivalent from a variational point of view, since for a surface with fixed genus $g$, we have

$$W_0(f) = 2W(f) - 8\pi + 8\pi g. \quad (1.1)$$

Both energies are not only *geometric*, i.e. invariant under diffeomorphisms on $\Sigma$, but—remarkably—also *conformally invariant*, i.e. invariant with respect to smooth Möbius transformations of $\mathbb{R}^3$. By [6, Theorem 7.2.2], we have $W(f) \geq 4\pi$ with equality if and only if $\Sigma = \mathbb{S}^2$ and $f: \mathbb{S}^2 \to \mathbb{R}^3$ parameterizes a round sphere.

The *isoperimetric ratio* of an immersion $f: \Sigma \to \mathbb{R}^3$ is defined as the quotient

$$I(f) := \frac{V(f)^2}{A(f)^3}, \quad \text{where}$$

$$A(f) := \int_{\Sigma} d\mu \quad \text{and} \quad V(f) := -\frac{1}{3} \int_{\Sigma} \langle f, \nu \rangle \, d\mu \quad (1.2)$$

denote the area and the algebraic volume enclosed by $f(\Sigma)$, respectively. Here, the normalizing constant is chosen such that by the isoperimetric inequality we always have $I(f) =: \sigma \in [0, 1]$ with $\sigma = 1$ if and only if $\Sigma = \mathbb{S}^2$ and $f: \mathbb{S}^2 \to \mathbb{R}^3$ parameterizes a round sphere. Critical points of the isoperimetric ratio—or equivalently, critical points of the volume functional with prescribed area—are precisely the CMC-surfaces, i.e. the surfaces with constant mean curvature, which form an important generalization of minimal surfaces and naturally arise in the modeling of soap bubbles.

The problem of minimizing the Willmore energy among all immersions of a genus $g$ surface $\Sigma_g$ with prescribed isoperimetric ratio, i.e. the minimization problem

$$\beta_g(\sigma) := \inf \left\{ W(f) \mid f: \Sigma_g \to \mathbb{R}^3 \text{ immersion with } I(f) = \sigma \right\}, \quad (1.3)$$

naturally arises in mathematical biology in the *Canham–Helfrich model* [1, 2] with zero spontaneous curvature and has already been studied mathematically in [7–9]. While the genus zero case was solved in [7], the results in [8, 9] combined with recent findings in [10] and [11] show that the infimum in (1.3) is always attained for any $g \in \mathbb{N}_0$ and $\sigma \in (0, 1)$; and satisfies $\beta_g(\sigma) < 8\pi$. The energy threshold $8\pi$ also plays an important role in the analysis of the Willmore energy, since by the famous Li–Yau inequality [12], any immersion $f$ of a compact surface with $W(f) < 8\pi$ has to be embedded.

A sufficiently smooth minimizer in (1.3) is a *Helfrich immersion*, i.e. a solution to the Euler–Lagrange equation

$$\Delta H + |A^0|^2 H - \lambda_1 H - \lambda_2 = 0 \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}, \quad (1.4)$$

where $\Delta = \Delta_{g_f}$ denotes the Laplace–Beltrami operator on $(\Sigma, g_f)$. In [13], solutions to (1.4) with small umbilic Willmore energy have been classified, depending on the sign of the *Lagrange-multipiers* $\lambda_1$ and $\lambda_2$. We observe that for $\lambda_1, \lambda_2 \in \mathbb{R}$ fixed, (1.4) is also the Euler–Lagrange equation of the *Helfrich energy* given by

$$\mathcal{H}_{\lambda_1, \lambda_2}(f) := W_0(f) + \lambda_1 A(f) + \lambda_2 V(f), \quad (1.5)$$
where the energy either penalizes or favors large area or volume, depending on the sign of $\lambda_1$ and $\lambda_2$, respectively.

The $L^2$-gradient flow of the Willmore energy was introduced and studied by Kuwert and Schätzle in their seminal works [14–16].

Their methods are very robust and allow to handle also other situations, such as the surface diffusion flow [17, 18] and the Willmore flow of tori of revolution [19]. The locally constrained Helfrich flow, i.e. the $L^2$-gradient flow for the energy (1.5), and its asymptotic behavior have been studied in [20, 21], where it was shown that fine time singularities must occur below a certain energy threshold. However, this flow does not preserve the isoperimetric ratio.

The goal of this article is to discuss a dynamic version of the minimization problem (1.3). To this end, we introduce the Willmore flow with prescribed isoperimetric ratio, which decreases $W$ as fast as possible while keeping $\mathcal{I}(f) = \mathcal{I}(f_0) = \sigma$ fixed. This yields the evolution equation

$$
\partial_t f = \left[ -\Delta H - |A^0|^2 H + \lambda \left( \frac{3}{\mathcal{A}(f)} H - \frac{2}{V(f)} \right) \right] \nu,
$$

where the Lagrange multiplier $\lambda := \lambda(t) := \lambda(f_t)$ depends on $f_t := f(t, \cdot)$ and is given by

$$
\lambda(f) := \frac{\int (\Delta H + |A^0|^2 H) \left( \frac{3}{\mathcal{A}(f)} H - \frac{2}{V(f)} \right) \, d\mu}{\int \frac{3}{\mathcal{A}(f)} H - \frac{2}{V(f)} \, d\mu}.
$$

(1.7)

In (2.9) we will justify the particular choice of $\lambda$, which yields that $\mathcal{I}$ is actually preserved along a solution of (1.6)–(1.7).

**Definition 1.1.** Let $\sigma \in (0,1)$, $T > 0$ and let $\Sigma_g$ denote a connected, oriented and closed surface with genus $g \in \mathbb{N}_0$. A smooth family of immersions $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ satisfying (1.6) with $\lambda$ as in (1.7) and $\mathcal{I}(f) \equiv \sigma$ is called a $\sigma$-isoperimetric Willmore flow with initial datum $f_0 := f(0, \cdot)$.

Stationary solutions of the flow (1.6)–(1.7) are solutions to the Helfrich equation (1.4) for $\lambda_1 = \frac{3}{\mathcal{A}(f)} \lambda$ and $\lambda_2 = -\frac{2}{V(f)} \lambda$. Conversely, any Helfrich immersion is also a stationary solution to (1.6)–(1.7), see Lemma 2.6.

However, as the Lagrange multiplier $\lambda$ defined in (1.7) depends on the solution, the isoperimetric flow (1.6) substantially differs from the $L^2$-gradient flow of the Helfrich energy (1.5), where the parameters $\lambda_1$ and $\lambda_2$ are fixed numbers and chosen a priori. On the analytic side, the integral nature of the Lagrange multiplier makes the evolution equation (1.6) a nonlocal, quasilinear, degenerate parabolic PDE of 4th order. Also geometrically, the constraint $\mathcal{I}(f) \equiv \sigma$ causes new difficulties, as we cannot control the area and the volume independently along the flow (as in [21], for instance), but only the isoperimetric ratio $\mathcal{I}$.

The Willmore flow with a constraint on either the area or the enclosed volume has been studied in [22] and a recent article by the author [23]. However, the situation here is fundamentally different and several new challenges arise.

First, if only the area or the volume is prescribed (and nonzero), constrained critical points of the corresponding variational problem are in fact Willmore immersions, i.e. solutions of (1.4) with $\lambda_1 = \lambda_2 = 0$, due to the scaling invariance of the Willmore energy. Although still an active field of research, the classification of these Willmore immersions is much better understood than that of general solutions of (1.4) and a crucial ingredient in classifying the
blow-ups in [23]. Second, in [23] the different scaling of the energy and constraint has been used to represent the Lagrange multiplier in a way that allows for good \textit{a priori estimates}. This neat trick is clearly not available for the flow (1.6)–(1.7). Third, unlike in [23], the Lagrange multiplier has a much more complicated algebraic structure and cannot be treated as a lower order term.

These obstructions are the reason for a new energy threshold in the following main result on global existence and convergence.

**Theorem 1.2.** Let $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ be a smooth immersion with $\mathcal{I}(f_0) = \sigma \in (0, 1)$ and such that $\mathcal{W}(f_0) \leq \min \{ \frac{4\pi}{\sigma}, 8\pi \}$. Then there exists a unique $\sigma$-isoperimetric Willmore flow with initial datum $f_0$. This flow exists for all times and, as $t \to \infty$, it converges smoothly after reparametrization to a Helfrich immersion $f_\infty$ with $\mathcal{I}(f_\infty) = \sigma$ solving (1.4) with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

This shows a fundamentally different behavior of the isoperimetric Willmore flow and the Helfrich flow, where finite time singularities occur, cf. [20, 21]. Consequently, despite its new analytic challenges, the introduction of the non-local Lagrange multiplier has a regularizing effect on the gradient flow, see also [24] for a related result for the mean curvature flow.

The $\frac{4\pi}{\sigma}$-threshold in Theorem 1.2 is motivated by the following simple application of the triangle inequality in $L^2(\text{d}\mu)$. With $\mathcal{I}(f) = \sigma$ and (1.2), we have

$$\int_\Sigma \left( \frac{3}{\mathcal{A}(f)} H - \frac{2}{\mathcal{V}(f)} \right)^2 \text{d}\mu \geq \frac{36}{\mathcal{A}(f)^2} \left( \sqrt{\frac{4\pi}{\sigma}} - \sqrt{\mathcal{W}(f_0)} \right)^2.$$ 

This estimate bounds the denominator in (1.7) from below if $\mathcal{W}(f_0) < \frac{4\pi}{\sigma}$. Moreover, it allows to control the Lagrange multiplier in the crucial estimates by essentially lower order quantities, see Sections 4.

We highlight that the assumption in Theorem 1.2 is not an implicit smallness of the initial energy, cf. [15, 18], but the threshold is explicitly given, although very little is known about minimizers and critical points of (1.3). Moreover, as $\sigma \nearrow 1$, the interval of admissible initial energies in Theorem 1.2 becomes arbitrarily small. This seems plausible, since if $\sigma = 1$, $f_0$ is a round sphere and the denominator in (1.7) vanishes. Thus, it is a priori unclear whether there exists an admissible immersion $f_0$ in Theorem 1.2 if $\sigma \in (\frac{1}{2}, 1)$ — in fact, this is equivalent to the condition $\beta_0(\sigma) \leq \frac{4\pi}{\sigma}$. In Theorem 7.1, we will prove $\beta_0(\sigma) < \frac{4\pi}{\sigma}$ for $\sigma \in (0, 1)$, which is asymptotically sharp as $\sigma \nearrow 1$, and consequently the existence of a suitable $f_0$ follows. We also point out that it is unknown if the energy threshold in Theorem 1.2 is optimal as it is for the classical Willmore flow [19, 25].

The proof of Theorem 1.2 is based on the methods developed by Kuwert–Schätzle for the Willmore flow [14–16]. Under a non-concentration assumption on the curvature, we use localized energy estimates to control the evolution, see Sections 3. However, as in [23], these estimates depend on certain $L^p$-type bounds on $\lambda$. The key ingredient of this paper is that for locally small curvature and if the initial energy is below the threshold of Theorem 1.2, the Lagrange multiplier can be absorbed in the estimates, see Sections 4, in particular Lemmas 4.1 and 4.2. This is an essential observation, which we can use to prove a lower bound on the lifespan and to construct a blow-up limit in the spirit of [15], see Sections 5. Using the control over the Lagrange multiplier in the energy regime of Theorem 1.2, we deduce a crucial \textit{rigidity result}: either the blowup is a \textit{compact} Helfrich immersion or a Willmore immersion,
see Proposition 5.4. In the first case, we conclude global existence and convergence by an argument based on the Łojasiewicz–Simon inequality in the spirit of [26], combined with recent progress on this inequality in the presence of constraints [27]. Due to the rigidity of the blow-up, we can follow the inversion strategy in [16] relying on the classification of compact Willmore spheres [28] to exclude the second case.

This last step is also where we crucially make use of the assumption $\Sigma_g = S^2$. In the case of higher genus, a classification result for Willmore surfaces as in [28] is currently lacking. Even if such a classification were available, a precise comprehension of the behavior under inversion would be indispensable to extend the argument beyond the spherical case. However, since the blow-up analysis is also available if $g \geq 1$, we establish the following remarkable dichotomy result.

**Corollary 1.3.** Let $\sigma \in (0, 1)$, let $\Sigma$ be a closed, oriented and connected surface and suppose that $f : [0, T) \times \Sigma \to \mathbb{R}^3$ is a maximal $\sigma$-isoperimetric Willmore flow such that $W(f_0) < \frac{4\pi}{\sigma}$. Then there exist $\hat{c} \in (0, 1)$, $(t_j)_{j \in \mathbb{N}} \subset [0, T)$, $t_j \nearrow T$, $(r_j)_{j \in \mathbb{N}} \subset (0, \infty)$ and $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^3$ such that the sequence of immersions

$$\hat{f}_j := r^{-1}_j \left( f(t_j + r^4_j \hat{c}, \cdot) - x_j \right)$$

converges, as $j \to \infty$, smoothly on compact subsets of $\mathbb{R}^3$ after reparametrization to a proper Helfrich immersion $\hat{f} : \hat{\Sigma} \to \mathbb{R}^3$ where $\hat{\Sigma} \neq \emptyset$ is a complete surface without boundary. Moreover

(a) if $\hat{\Sigma}$ is compact, then $T = \infty$ and, as $t \to \infty$, the flow $f$ converges smoothly after reparametrization to a Helfrich immersion $f_\infty$ as $t \to \infty$.

(b) if $\hat{\Sigma}$ is not compact, then $\hat{f}$ is a Willmore immersion.

Hence, under the above assumptions, in the singular case (b) the influence of the (non-local) constraint vanishes after rescaling as $t \to \infty$ and the purely local term in (1.6), coming from the Willmore functional, dominates.

We now outline the structure of this article. After a brief review of the most relevant analytic and geometric background in Sections 2, we start our analysis by carefully computing and estimating a localized version of the energy decay in Sections 3. In Sections 4, we control the Lagrange multiplier in the energy regime of Theorem 1.2 which then enables us to construct a blow-up limit in Sections 5. Finally, in Sections 6 we prove our convergence result, Theorem 1.2, and Corollary 1.3 before we show Theorem 7.1 in Sections 7, yielding that the set of admissible initial data in Theorem 1.2 is always non-empty.

2. Preliminaries

In this section, we will briefly review the geometric and analytic background and prove some first properties of the flow (1.6), see also [29] for a more detailed discussion.

2.1. Geometric and analytic background

In the following, $\Sigma_g$ always denotes an abstract compact, connected and oriented surface of genus $g \in \mathbb{N}_0$ without boundary.
An immersion \( f : \Sigma_g \to \mathbb{R}^3 \) induces the pullback metric \( g_f = f^* \langle \cdot, \cdot \rangle \) on \( \Sigma_g \), which in local coordinates is given by

\[
g_{ij} := \langle \partial_i f, \partial_j f \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean metric. The chosen orientation on \( \Sigma_g \) determines a unique smooth unit normal field \( \nu : \Sigma_g \to S^2 \) along \( f \), which in local coordinates in the orientation is given by

\[
u = \frac{\partial_1 f \times \partial_2 f}{|\partial_1 f \times \partial_2 f|}.
\] (2.1)

We will always work with this unit normal vector field.

The (scalar) second fundamental form of \( f \) is then given by \( A_{ij} := \langle \partial_i \partial_j f, \nu \rangle \) and the mean curvature and the tracefree part of the second fundamental form are defined as

\[
H := g^{ij} A_{ij} \quad \text{and} \quad A^0_{ij} := A_{ij} - \frac{1}{2} H g_{ij},
\]

where \( g^{ij} := (g_{ij})^{-1} \). Important relations are

\[
|A|^2 = |A^0|^2 + \frac{1}{2} H^2 = 2 |A^0|^2 + 2 K,
\] (2.2)

where \( K \) denotes the Gauss curvature. Consequently, using (1.1), we find

\[
\int_{\Sigma} |A|^2 \, d\mu = \mathcal{W}_0(f) + 2 \mathcal{W}(f) = 4 \mathcal{W}(f) - 8\pi + 8\pi g.
\] (2.3)

The Levi-Civita connection \( \nabla = \nabla_f \) induced by the metric \( g_f \) extends uniquely to a connection on tensors, which we also denote by \( \nabla \). For an orthonormal basis \( \{e_1, e_2\} \) of the tangent space, the Codazzi–Mainardi equations yield

\[
\nabla_i H = (\nabla_j A^0)(e_i, e_j) = 2(\nabla_j A^0)(e_i, e_j),
\] (2.4)

cf. [15, (5)].

Clearly, potential singularities for the flow (1.6) occur if \( \mathcal{W}(f) \) becomes zero or if the denominator in (1.7) vanishes. Note that in the latter case \( H \equiv \text{const} \), thus \( f \) is a constant mean curvature immersion.

**Lemma 2.1.** Let \( \sigma \in (0,1) \) and let \( f : \Sigma_g \to \mathbb{R}^3 \) be an immersion with \( \mathcal{I}(f) = \sigma \). Then

(i) \( \mathcal{W}(f) \neq 0 \);

(ii) if \( g = 0 \), i.e. \( \Sigma_g = S^2 \), or if \( f \) is an embedding, then \( H \not\equiv \text{const} \). In particular, the denominator in (1.7) is nonzero.

**Proof** The first statement follows immediately from the definition of \( \mathcal{I} \). For (ii), we assume by contradiction that \( H \equiv \text{const} \), so \( f : \Sigma_g \to \mathbb{R}^3 \) is an immersion with constant mean curvature. If \( \Sigma_g = S^2 \), then \( f \) has to parameterize a round sphere by a result of Hopf [30, Theorem 2.1, Chapter VI]. In the second case, \( f \) has to parameterize a round sphere by the famous theorem of Aleksandrov [31]. In both cases this contradicts \( \sigma \neq 1 \). □

Despite its geometric degeneracy, (1.6) is still a parabolic equation. Thus, starting with a smooth nonsingular initial datum, it is possible to prove the following short-time existence
result in similar fashion as it is outlined in [32, Chapter 4, Proposition 2.1], after observing that we can integrate by parts in (1.7) so that the numerator of the Lagrange-multiplier contains no second order derivatives of \( A \) any more.

**Proposition 2.2.** Let \( f_0 : \Sigma \rightarrow \mathbb{R}^3 \) be a smooth immersion with \( H_{f_0} \neq \text{const} \) and \( I(f_0) = \sigma \in (0, 1) \). Then there exist \( T \in (0, \infty) \) and a unique, non-extendable \( \sigma \)-isoperimetric Willmore flow \( f : [0, T) \times \Sigma \rightarrow \mathbb{R}^3 \) with initial datum \( f(0) := f(0, \cdot) = f_0 \).

If \( \Sigma = \mathbb{S}^2 \), assumption \( H_{f_0} \neq \text{const} \) in Proposition 2.2 follows from \( \sigma \in (0, 1) \) by Lemma 2.1 (ii).

### 2.2. Evolution of geometric quantities

In this subsection, we will briefly review the variations of the relevant geometric quantities and energies.

**Lemma 2.3.** [23, Lemma 2.3] Let \( f : [0, T) \times \Sigma \rightarrow \mathbb{R}^3 \) be a smooth family of immersions with normal velocity \( \partial_t f = \xi \nu \). For an orthonormal basis \( \{ e_1, e_2 \} \) of the tangent space, the geometric quantities induced by \( f \) satisfy

\[
\begin{align*}
\partial_t (d\mu) &= -H \xi \, d\mu, \\
\partial_t H &= \Delta \xi + |A|^2 \xi, \\
(\partial_t A)(e_i, e_j) &= \nabla^2_{ij} \xi - A_{ik} A_{kj} \xi.
\end{align*}
\]

As a consequence, we have the following first variation identities, cf. [23, Lemma 2.4].

**Proposition 2.4.** Let \( f : \Sigma \rightarrow \mathbb{R}^3 \) be an immersion and let \( \varphi \in C^\infty(\Sigma; \mathbb{R}^3) \). Then we have

\[
\begin{align*}
W_0'(f)\varphi &= \langle \nabla W_0(f), \varphi \rangle_{L^2(d\mu)} = \int \langle (\Delta H + |A|^2 H) \nu, \varphi \rangle \, d\mu, \\
A'(f)\varphi &= \langle \nabla A(f), \varphi \rangle_{L^2(d\mu)} = -\int \langle Hv, \varphi \rangle \, d\mu, \\
V'(f)\varphi &= \langle \nabla V(f), \varphi \rangle_{L^2(d\mu)} = -\int \langle v, \varphi \rangle \, d\mu.
\end{align*}
\]

Moreover, if \( I(f) > 0 \), we have

\[
\mathcal{I}'(f)\varphi = \langle \nabla \mathcal{I}(f), \varphi \rangle_{L^2(d\mu)} = \sigma \int \left\langle \frac{3}{A(f)} Hv - \frac{2}{V(f)} v, \varphi \right\rangle \, d\mu.
\]

**Proof** Since \( \mathcal{W}_0, A, \) and \( V \) are invariant under orientation-preserving diffeomorphisms of \( \Sigma \), we only need to consider normal variations, as any tangential variation corresponds to a suitable orientation-preserving family of reparametrizations (see for instance [33, Theorem 17.8]), which leaves the quantities unchanged.

The variation of \( A \) then follows immediately from (2.5). For \( \mathcal{W}_0 \) and \( V \) consider [23, Lemma 2.4], for instance. The variation of \( \mathcal{I} \) then follows. \( \Box \)

The scaling behavior of the energies yields the following important identities.
Lemma 2.5. Let \( f : \Sigma_g \to \mathbb{R}^3 \) be an immersion. Then we have
\[
\int \langle (\Delta H + |A^0|^2 H) \nu, f \rangle \, d\mu = 0, \quad - \int \langle Hv, f \rangle \, d\mu = 2A(f), \quad - \int \langle v, f \rangle \, d\mu = 3 \mathcal{V}(f).
\]

Proof By the scaling invariance of the Willmore energy, we find
\[
\langle \nabla W_0(f), f \rangle_{L^2(d\mu)} = \left. \frac{d}{d\alpha} \right|_{\alpha=1} W(\alpha f) = 0,
\]
so Proposition 2.4 yields the claim. For \( A \) and \( \mathcal{V} \) we may proceed similarly, using the scaling behavior
\[
A(\alpha f) = \alpha^2 A(f), \quad \mathcal{V}(\alpha f) = \alpha^3 \mathcal{V}(f)
\]
for all \( f : \Sigma_g \to \mathbb{R}^3, \alpha > 0. \)

This yields that Helfrich immersions are precisely the stationary solutions of (1.6)–(1.7).

Lemma 2.6. Let \( f : \Sigma_g \to \mathbb{R}^3 \) be an immersion with \( \mathcal{I}(f) = \sigma \in (0, 1) \) and \( H_f \neq \text{const} \). Then \( f \) is a Helfrich immersion if and only if it is a stationary solution to the \( \sigma \)-isoperimetric Willmore flow.

Proof The “if” part of the statement is immediate. Suppose \( f \) is a Helfrich immersion. We multiply (1.4) with \( \langle f, \nu \rangle \), integrate and use Lemma 2.5 to conclude
\[
2\lambda_1 A(f) + 3\lambda_2 \mathcal{V}(f) = 0.
\]
By Lemma 2.1(i) we have \( \mathcal{V}(f) \neq 0 \). Hence, with \( \lambda := -\frac{\lambda_1 \mathcal{V}(f)}{2} \), Equation (1.4) reads
\[
\Delta H + |A^0|^2 H - \lambda \left( \frac{3}{\mathcal{A}(f)} H - \frac{2}{\mathcal{V}(f)} \right) = 0. \tag{2.8}
\]
We have \( \nabla \mathcal{I}(f) \neq 0 \) by Proposition 2.4, so by testing (2.8) with \( \nabla \mathcal{I}(f) \nu \) and integrating it follows that \( \lambda \) is given as in (1.7), so \( f \) is indeed stationary.  

It is not difficult to see that along a solution of (1.6) with \( \mathcal{I}(f) > 0 \), the isoperimetric ratio is indeed preserved, since by Proposition 2.4, (1.6) and (1.7) we have
\[
\frac{d}{dt} \mathcal{I}(f) = \int \langle \nabla \mathcal{I}(f), \partial_t f \rangle \, d\mu
\]
\[
= \int \left( \nabla \mathcal{I}(f), -\nabla W_0(f) + \frac{\lambda}{\mathcal{I}(f)} \nabla \mathcal{I}(f) \right) \, d\mu
\]
\[
= - \int \langle \nabla \mathcal{I}(f), \nabla W_0(f) \rangle \, d\mu + \frac{\lambda}{\mathcal{I}(f)} \int |\nabla \mathcal{I}(f)|^2 \, d\mu = 0. \tag{2.9}
\]
On the other hand, the Willmore energy decreases since by (2.9)
\[
\frac{d}{dt} W_0(f) = \int \langle (\Delta H + |A^0|^2 H) v, \partial_t f \rangle \, d\mu = - \int |\partial_t f|^2 \, d\mu \leq 0. \tag{2.10}
\]
Equations (2.9) and (2.10) are the key features in studying the flow (1.6) and of vital importance for our further analysis. We highlight two immediate consequences.

Remark 2.7. (i) The computation in (2.10) implies that \( W_0 \) is a strict Lyapunov function along the flow (1.6), i.e. \( W_0 \) is strictly decreasing unless \( \partial_t f = 0 \), so \( f \) is stationary (by uniqueness of the solution). By (1.1), this also holds for \( \mathcal{W} \).
(ii) Since $\mathcal{W}$ is monotone, the limit $\lim_{t \to T} \mathcal{W}(f(t, \cdot)) \in [\beta_g(\sigma), \mathcal{W}(f_0)]$ exists.

As (1.6) is a (degenerate) parabolic equation, the scaling behavior in time and space is central in understanding the problem. Therefore, we gather the scaling behavior of some important quantities in the following lemma. The powers appearing in the time integrals below will naturally appear later in our energy estimates, see Proposition 3.3.

**Lemma 2.8.** Let $\sigma \in (0, 1), f : [0, T] \times \Sigma_g \to \mathbb{R}^3$ be a $\sigma$-isoperimetric Willmore flow and let $r > 0$. Let $\tilde{f} : [0, r^{-4}T] \times \Sigma_g \to \mathbb{R}^3, \tilde{f}(t, p) := r^{-1}f(r^4t, p)$. Then

(i) $\tilde{f}$ is a $\sigma$-isoperimetric Willmore flow;
(ii) the Lagrange multiplier $\tilde{\lambda}$ of $\tilde{f}$ satisfies $\tilde{\lambda}(t) = \lambda(r^4t)$;
(iii) $\int_0^T \frac{\lambda^2}{A(f)^2} \, dt = \int_0^{r^{-4}T} \frac{\tilde{\lambda}^2}{A(\tilde{f})^2} \, dt$ and $\int_0^T \frac{\lambda^4}{|V(f)|^4} \, dt = \int_0^{r^{-4}T} \frac{\tilde{\lambda}^4}{|V(\tilde{f})|^4} \, dt$.

**Proof** Follows from the scaling behavior of the geometric quantities and a direct calculation. \hfill \qed

### 3. Localized energy estimates

As in [15, Section 3] and [23, Sections 2.3 and 3], we will start our analysis by localizing the energy decay (2.10). The main goal of this section is to show that all derivatives of $A$ can be bounded along the flow, if the energy concentration and a suitable time integral involving the Lagrange multiplier are controlled. Note that at this stage, we do not yet need to assume $\Sigma_g = S^2$ or any restriction on the initial energy.

**Lemma 3.1.** Let $\sigma \in (0, 1)$ and let $f : [0, T] \times \Sigma_g \to \mathbb{R}^3$ be a $\sigma$-isoperimetric Willmore flow. Let $\tilde{\eta} \in C^\infty_c(\mathbb{R}^3)$ and define $\eta := \tilde{\eta} \circ f$. Then we have

$$
\partial_t \left( \frac{1}{2} H^2 \eta \, d\mu + \int |\nabla \mathcal{W}_0(f)|^2 \eta \, d\mu \right) = \frac{3\lambda}{A(f)} \int \Delta H \eta \, d\mu + \int \left( \frac{3\lambda}{A(f)} H \frac{2\lambda}{V(f)} \right) |A^0|^2 H \eta \, d\mu - 2 \int \langle \nabla \mathcal{W}_0(f), v \rangle \langle \nabla H, \nabla \eta \rangle_g \, d\mu - \int \langle \nabla \mathcal{W}_0(f), v \rangle H \Delta \eta \, d\mu + \int \frac{1}{2} H^2 \partial_t \eta \, d\mu.
$$

and

$$
\partial_t \left( |A^0|^2 \eta \, d\mu + \int |\nabla \mathcal{W}_0(f)|^2 \eta \, d\mu \right) = \frac{6\lambda}{A(f)} \int \langle \nabla^2 H, A^0 \rangle_g \eta \, d\mu + \int \left( \frac{3\lambda}{A(f)} H \frac{2\lambda}{V(f)} \right) |A^0|^2 H \eta \, d\mu - 2 \int \langle \nabla \mathcal{W}_0(f), v \rangle \left( \langle \nabla H, \nabla \eta \rangle_g + \langle A^0, \nabla^2 \eta \rangle_g \right) \, d\mu + \int |A^0|^2 \partial_t \eta \, d\mu.
$$

**Proof** This computation is very similar to [32, Chapter 4, Lemma 2.8] (see also [15, Section 3]) if one replaces $\lambda \nu$ with $\left( \frac{3\lambda}{A(f)} - \frac{2\lambda}{V(f)} \right) \nu$, so we will focus on the differences. We will use a
local orthonormal frame \( \{ e_i(t) \}_{i=1,2} \) for our computations and find
\[
\partial_t \int \frac{1}{2} H^2 \eta \, d\mu + \int |\nabla \mathcal{W}_0(f)|^2 \eta \, d\mu \\
= \int \left( \frac{3\lambda}{A(f)} H - \frac{2\lambda}{V(f)} \right) \Delta H \eta \, d\mu + \int \left( \frac{3\lambda}{A(f)} H - \frac{2\lambda}{V(f)} \right) |A_0^1|^2 H \eta \, d\mu \\
+ \int (2\xi \nabla \nu \nabla i \eta + H \xi \Delta \eta) \, d\mu + \int \frac{1}{2} H^2 \partial_i \eta \, d\mu,
\]
writing \( \partial_t f = \xi \nu \). Moreover, we have
\[
\int (2\xi \nabla \nu \nabla i \eta + H \xi \Delta \eta) \, d\mu \\
= -2 \int \langle \nabla \mathcal{W}_0(f), \nu \rangle \nabla \nu \nabla i \eta \, d\mu + \frac{6\lambda}{A(f)} \int \nabla \nabla \nu \nabla i \eta \, d\mu - \frac{4\lambda}{V(f)} \int \nabla \nabla \nu \nabla i \eta \, d\mu \\
- \int \langle \nabla \mathcal{W}_0(f), \nu \rangle H \Delta \eta \, d\mu + \frac{3\lambda}{A(f)} \int H^2 \Delta \eta \, d\mu - \frac{2\lambda}{V(f)} \int H \Delta \eta \, d\mu.
\]
If we carefully combine the terms with \( \lambda \) in (3.1) and (3.2), the claim follows after integrating by parts, where the terms involving derivatives of \( H \) and the factor \( \frac{2\lambda}{V(f)} \) cancel.

For the second identity, arguing similarly as in [32, Chapter 4, Lemma 2.8] we have
\[
\partial_t (|A_0^1|^2 \, d\mu) = 2\nabla \nu \nabla \xi A_0^0(e_i, e_j) \, d\mu - \nabla \nu \nabla \xi H \, d\mu + \langle \nabla \mathcal{W}_0(f), \xi \nu \rangle \, d\mu \\
= 2\nabla \nu \nabla \xi A_0^0(e_i, e_j) \, d\mu - \nabla \nu \nabla \xi H \, d\mu - |\nabla \mathcal{W}_0(f)|^2 \, d\mu \\
+ \lambda (\Delta H + |A_0^1|^2 H) \left( \frac{3}{A(f)} H - \frac{2}{V(f)} \right) \, d\mu.
\]
Integrating by parts and using (2.4) we conclude
\[
\partial_t \int \frac{1}{2} |A_0^1|^2 \eta \, d\mu + \int |\nabla \mathcal{W}_0(f)|^2 \eta \, d\mu \\
= \int \left[ -2\nabla \nu \nabla \xi \nabla \nu \nabla i \eta + \nabla \nu \nabla \xi H \nabla \nu \nabla i \eta + \lambda (\Delta H + |A_0^1|^2 H) \left( \frac{3}{A(f)} H - \frac{2}{V(f)} \right) \eta \right] \, d\mu + \int |A_0^1|^2 \partial_i \eta \, d\mu \\
= -2 \int \langle \nabla \mathcal{W}_0(f), \nu \rangle A_0^0 \nabla \nu \nabla i \eta \, d\mu + 2\lambda \int \left( \frac{3}{A(f)} H - \frac{2}{V(f)} \right) A_0^1 \nabla \nu \nabla i \eta \, d\mu \\
- 2 \int \langle \nabla \mathcal{W}_0(f), \nu \rangle \nabla \nu \nabla i \eta \nabla \delta \, d\mu + 2\lambda \int \left( \frac{3}{A(f)} H - \frac{2}{V(f)} \right) \nabla \nu \nabla i \eta \nabla \delta \, d\mu \\
+ \lambda \int (\Delta H + |A_0^1|^2 H) \left( \frac{3}{A(f)} H - \frac{2}{V(f)} \right) \eta \, d\mu + \int |A_0^1|^2 \partial_i \eta \, d\mu.
\]
Now, using integration by parts and (2.4) once again, we have
\[
\int \left( \frac{3\lambda}{A(f)} H - \frac{2\lambda}{V(f)} \right) \left( 2\nabla \nu \nabla i \eta + 2A_0^1 \nabla \nu \nabla i \eta + \Delta H \eta \right) \, d\mu = \frac{6\lambda}{A(f)} \int \langle \nabla^2 H, A_0^0 \rangle \eta \, d\mu.
\]
The claim follows.

We will now carefully estimate the integrals in Lemma 3.1. To this end, we choose a particular class of test functions. Let \( \tilde{\gamma} \in C_0^\infty(\mathbb{R}^2) \) with \( 0 \leq \tilde{\gamma} \leq 1 \) and assume \( \| D \tilde{\gamma} \|_\infty \leq \Lambda, \| D^2 \tilde{\gamma} \|_\infty \leq \Lambda^2 \) for some \( \Lambda > 0 \). Then setting
\[
\gamma := \tilde{\gamma} \circ f : [0, T) \times \Sigma_g \to \mathbb{R}
\]
\[ |\nabla \gamma| \leq \Lambda \text{ and } |\nabla^2 \gamma| \leq \Lambda^2 + |A| \Lambda, \quad (3.3) \]

and note that \( \gamma(t, \cdot) \) has compact support in \( \Sigma_g \), which is compact, for all \( 0 \leq t < T \), see also [23, (3.1)].

For the rest of this article, we denote by \( C \) a universal constant with \( 0 < C < \infty \) which may change from line to line.

**Lemma 3.2.** Let \( \sigma \in (0, 1) \), let \( f : [0, T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow and let \( \gamma \) be as in \((3.3)\). Then we have

\[
\partial_t \int |A|^2 \gamma^4 \, d\mu + \frac{3}{2} \int |\nabla \mathcal{W}_0(f)|^2 \gamma^4 \, d\mu \\
\leq \frac{C|\lambda|}{\mathcal{A}(f)} \left( \int \langle \nabla^2 H, A \rangle_{\mathcal{B}} \gamma^4 \, d\mu + \int |A|^4 \gamma^4 \, d\mu + A \int |A|^3 \gamma^3 \, d\mu \right) \\
+ \frac{C|\lambda|}{|\nabla(f)|} \left( \int |A|^3 \gamma^4 \, d\mu + \Lambda \int |A|^2 \gamma^3 \, d\mu \right) + \mathcal{C}^4 \int_{|\gamma| > 0} |A|^2 \, d\mu + \mathcal{C}^2 \int |A|^4 \gamma^2 \, d\mu.
\]

**Proof.** We have \( 2\langle \nabla^2 \varphi, A \rangle_{\mathcal{B}} = 2\langle \nabla^2 \varphi, A^0 \rangle_{\mathcal{B}} + H \Delta \varphi \) for any \( \varphi \in C^\infty([0, T) \times \Sigma_g) \) by a direct computation in a local orthonormal frame. Hence, using **Lemma 3.1** and \( |A|^2 = |A^0|^2 + \frac{1}{2} H^2 \), cf. (2.2), we find

\[
\partial_t \int |A|^2 \gamma^4 \, d\mu + 2 \int |\nabla \mathcal{W}_0(f)|^2 \gamma^4 \, d\mu \\
= \frac{6\lambda}{\mathcal{A}(f)} \left( \int \langle \nabla^2 H, A \rangle_{\mathcal{B}} \gamma^4 \, d\mu + \int |A^0|^2 H^2 \gamma^4 \, d\mu \right) - \frac{4\lambda}{|\nabla(f)|} \int |A^0|^2 H \gamma^4 \, d\mu \\
- 4 \int \langle \nabla \mathcal{W}_0(f), v \rangle \langle \nabla H, \nabla \gamma^4 \rangle_{\mathcal{B}} \, d\mu + 2 \int \langle \nabla \mathcal{W}_0(f), v \rangle \langle \nabla^2 \gamma^4, A \rangle_{\mathcal{B}} \, d\mu + \int |A|^2 \partial_t \gamma^4 \, d\mu.
\]

The terms \( \int \langle \nabla \mathcal{W}_0(f), v \rangle \langle \nabla H, \nabla \gamma^4 \rangle_{\mathcal{B}} \, d\mu \) and \( \int \langle \nabla \mathcal{W}_0(f), v \rangle \langle \nabla^2 \gamma^4, A \rangle_{\mathcal{B}} \, d\mu \) can be estimated as in [15, Lemma 3.2]. Since \( \partial_t \gamma^4 = 4 \gamma^3 \nabla \gamma \mathcal{W}_f \mathcal{W}_f g \) we have by \((3.3)\)

\[
|\partial_t \gamma^4| \leq C \mathcal{A} \gamma^3 \left( |\nabla \mathcal{W}_0(f)| + \frac{|\lambda|}{\mathcal{A}(f)} |A| + \frac{|\lambda|}{|\nabla(f)|} \right).
\]

Consequently, we find

\[
\int |A|^2 \partial_t \gamma^4 \, d\mu \leq \varepsilon \int |\nabla \mathcal{W}_0(f)|^2 \gamma^4 \, d\mu + C(\varepsilon) \Lambda^2 \int |A|^4 \gamma^2 \, d\mu \\
+ \mathcal{C} \frac{|\lambda|}{\mathcal{A}(f)} \int |A|^3 \gamma^3 \, d\mu + \mathcal{C} \frac{|\lambda|}{|\nabla(f)|} \int |A|^2 \gamma^2 \, d\mu.
\]

Choosing \( \varepsilon > 0 \) small enough, the claim follows from the estimates above. \( \square \)

Note that on the right hand side of **Lemma 3.2**, terms involving the Lagrange multiplier multiplied with powers of \( A \) up to 4-th order and even second derivatives of \( H \) appear. With the energy, we can only control the \( L^2 \)-norms of \( H \) and \( A \). In the following **Proposition 3.3** we will close this gap by using higher powers of the Lagrange multiplier, the area and the volume, see also [23, Proposition 3.3]; these powers behave correctly under rescaling, cf. **Lemma 2.8**. We will combine this with the interpolation techniques from [14, 15] to get control on the local \( W^{2,2} \)-norm of \( A \), in terms of the (localized) Willmore gradient, at least if the \( L^2 \)-norm of \( A \) is locally small.
Proposition 3.3. There exist universal constants \( \varepsilon_0, c_0, C \in (0, \infty) \) with the following property: Let \( \sigma \in (0, 1) \), let \( f : [0, T) \times \Sigma \rightarrow \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow and let \( \gamma \) be as in (3.3). If we have
\[
\int_{[\gamma > 0]} |A|^2 \, d\mu < \varepsilon_0 \quad \text{for some time } t \in [0, T),
\]
then at time \( t \) we can estimate
\[
\partial_t \int |A|^2 \gamma^4 \, d\mu + c_0 \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 \, d\mu
\leq CA^4 \int_{[\gamma > 0]} |A|^2 \, d\mu + C \left( \frac{\lambda^2}{A^2} + \frac{|\lambda|^3}{|V(f)|^2} \right) \int |A|^2 \gamma^4 \, d\mu.
\]
Here \( \int_{[\gamma > 0]} |A_0|^2 \, d\mu_0 := \int_{[\gamma > 0]} |A|^2 \, d\mu |_{t=0} \).

Proof. Using the assumption and the interpolation inequality in [15, Proposition 2.6] (see also [23, Proposition 3.2]), we have at time \( t \in [0, T) \)
\[
\int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 \, d\mu \leq C \int |\nabla \mathcal{W}_0(f)|^2 \gamma^4 \, d\mu + CA^4 \int_{[\gamma > 0]} |A|^2 \, d\mu.
\]
Consequently, from Lemma 3.2, we find for some \( c_0 \in (0, \infty) \)
\[
\partial_t \int |A|^2 \gamma^4 \, d\mu + 2c_0 \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 \, d\mu
\leq \frac{C|\lambda|}{A(f)} \left( \int \langle \nabla^2 H, A \rangle_{g_f} \gamma^4 \, d\mu + \int |A|^4 \gamma^4 \, d\mu + \Lambda \int |A|^3 \gamma^3 \, d\mu \right)
+ \frac{C|\lambda|}{|V(f)|} \left( \int |A|^3 \gamma^4 \, d\mu + \Lambda \int |A|^2 \gamma^3 \, d\mu \right) + CA^2 \int |A|^4 \gamma^2 \, d\mu + CA^4 \int_{[\gamma > 0]} |A|^2 \, d\mu.
\]
(3.5)

For the first term on the right hand side of (3.5), we infer using Young’s inequality
\[
\frac{|\lambda|}{A(f)} \left( \int \langle \nabla^2 H, A \rangle_{g_f} \gamma^4 \, d\mu + \int |A|^4 \gamma^4 \, d\mu + \Lambda \int |A|^3 \gamma^3 \, d\mu \right)
\leq \varepsilon \int |\nabla^2 A|^2 \gamma^4 \, d\mu + \varepsilon \int |A|^6 \gamma^4 \, d\mu + \Lambda^2 \int |A|^4 \gamma^2 \, d\mu + C(\varepsilon) \frac{|\lambda|^2}{A(f)^2} \int |A|^2 \gamma^4 \, d\mu.
\]
(3.6)

The second term on the right hand side of (3.5) can be estimated by using Young’s inequality with \( p = 4 \) and \( q = \frac{4}{3} \) and \( \gamma \leq 1 \) to obtain
\[
\frac{C|\lambda|}{|V(f)|} \left( \int |A|^3 \gamma^4 \, d\mu + \Lambda \int |A|^2 \gamma^3 \, d\mu \right)
\leq \varepsilon \int |A|^6 \gamma^4 \, d\mu + C(\varepsilon) \Lambda^4 \int_{[\gamma > 0]} |A|^2 \, d\mu.
\]
(3.7)

Moreover, we have the estimate \( CA^2 \int |A|^4 \gamma^2 \, d\mu \leq \varepsilon \int |A|^6 \gamma^4 \, d\mu + C(\varepsilon) \Lambda^4 \int_{[\gamma > 0]} |A|^2 \, d\mu \) using Young’s inequality. Combining this with (3.5), (3.6), and (3.7) and choosing \( \varepsilon > 0 \) sufficiently small, the claim follows.
Assumption (3.4) means that the second fundamental form is small on the support of \( \gamma \). Note that this will only be satisfied locally, since by (2.3) we always have \( \int |A|^2 \, d\mu \in [8\pi, 4V(f) - 8\pi + 8\pi g] \). We will now study the situation, where (3.4) is satisfied on all balls with a certain radius, yielding a control over the concentration of the Willmore energy in \( \mathbb{R}^3 \). Following [16] we introduce the following notation.

**Definition 3.4.** For a smooth family of immersions \( f : [0, T) \times \Sigma_g \to \mathbb{R}^3, t \in [0, T), r > 0 \), we define the curvature concentration function

\[
\kappa(t, r) := \sup_{x \in \mathbb{R}^3} \int_{B_r(x)} |A|^2 \, d\mu. \tag{3.8}
\]

Here and in the rest of this article, we follow the notation of [14], i.e. the integrals over balls \( B_r(x) \subset \mathbb{R}^3 \) have to be understood over the preimages under \( f_t \).

If \( \Gamma > 1 \) denotes the minimal number of balls of radius 1/2 necessary to cover \( B_1(0) \subset \mathbb{R}^3 \), then

\[
\kappa(t, \rho) \leq \Gamma \cdot \kappa(t, \rho/2) \quad \text{for all } 0 \leq t < T. \tag{3.9}
\]

We now prove an integrated form of Proposition 3.3.

**Proposition 3.5.** Let \( \varepsilon_0 > 0 \) be as in Proposition 3.3. There exist universal constants \( \varepsilon_1 \in (0, \varepsilon_0), \varepsilon_2, C > 0 \) with the following property: Let \( \sigma \in (0, 1) \), let \( f : [0, T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow and let \( \rho > 0 \) be such that

\[
\kappa(t, \rho) < \varepsilon_1 \quad \text{for all } t \in [0, T). \tag{3.10}
\]

Then for all \( x \in \mathbb{R}^3 \) and \( t \in [0, T) \) we have

\[
\int_{B_{\rho/2}(x)} |A|^2 \, d\mu + \varepsilon_0 \int_0^t \int_{B_{\rho/2}(x)} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \, d\mu \, dt
\leq \int_{B_\rho(x)} |A_0|^2 \, d\mu_0 + \frac{C(1 + \sigma^{-2})}{\rho^4} \int_0^t \int_{B_\rho(x)} |A|^2 \, d\mu \, dt + C \int_0^t \frac{\lambda^2}{A(f)^2} \int_{B_{\rho}(x)} |A|^2 \, d\mu \, dt.
\]

**Proof** Fix \( x \in \mathbb{R}^3 \). Let \( \tilde{\gamma} \in C_c^\infty(\mathbb{R}^3) \) be a cutoff function with \( \chi_{B_{\rho/2}(x)} \leq \tilde{\gamma} \leq \chi_{B_\rho(x)} \), \( \|D\tilde{\gamma}\|_\infty \leq \frac{C}{\rho} \) and \( \|D^2\tilde{\gamma}\|_\infty \leq \frac{C}{\rho^2} \). Therefore, \( \gamma : = \tilde{\gamma} \circ f \) is as in (3.3) with \( \Lambda = \frac{C}{\rho} \). Moreover, if we take \( \varepsilon_1 > 0 \) small enough, we have the estimate

\[
\rho^2 \leq CA(f) \tag{3.11}
\]

as a consequence of Simon’s monotonicity formula [34], see also [23, Lemma 4.1]. Now, since we have \( |\mathcal{V}(f)|^\frac{1}{3} = \left( \frac{\sigma}{36\pi} \right)^{\frac{1}{3}} A(f)^2 \) by (2.9) and (1.2), we observe

\[
\frac{|\lambda|^2}{|\mathcal{V}(f)|} = \left( \frac{36\pi}{A(f)^2} \right)^{\frac{1}{3}} |\lambda|^\frac{4}{3} \sigma^{-\frac{2}{3}} \leq \frac{C(\lambda^2 + \sigma^{-2})}{A(f)^2} \leq C \left( \frac{\lambda^2}{A(f)^2} + \frac{\sigma^{-2}}{\rho^4} \right), \tag{3.12}
\]

where we used Young’s inequality (with \( p = \frac{3}{2} \) and \( q = 3 \)). The statement then immediately follows by integrating Young’s inequality in time. \( \square \)

**Remark 3.6.** If we directly integrate Proposition 3.3, we have to deal with two terms involving \( \lambda \), both of whose time integrals behave correctly under parabolic rescaling, cf. Lemma 2.8.
The estimate (3.12) above reveals that if (3.10) is satisfied, then it suffices to control merely the $\frac{\lambda^2}{\mathcal{A}(f)^2}$-term, since

$$\frac{|\lambda|^3}{|V(f)|^2} \leq C \left( \frac{\lambda^2}{\mathcal{A}(f)^2} + \frac{\sigma^{-2}}{\rho^4} \right). \quad (3.13)$$

For the blow-up construction in Sections 5, we will need the following higher order estimates for the flow in the case of non-concentrated curvature, cf. [15, Theorem 3.5], [23, Proposition 3.5].

**Proposition 3.7.** Let $\sigma \in (0,1)$ and let $f : [0,T) \times \Sigma \to \mathbb{R}^3$ be a $\sigma$-isoperimetric Willmore flow. Suppose $\rho > 0$ is chosen such that $T \leq T^* \rho^4$ for some $0 < T^* < \infty$ and

$$\kappa(t, \rho) \leq \varepsilon < \varepsilon_1 \quad \text{for all } 0 \leq t < T, \quad (3.14)$$

where $\varepsilon_1 > 0$ is as in Proposition 3.5. Moreover, assume

$$\int_0^T \frac{\lambda^2}{\mathcal{A}(f)^2} \, dt \leq \bar{L} < \infty. \quad (3.15)$$

Then for all $t \in (0,T)$, $x \in \mathbb{R}^3$ and $m \in \mathbb{N}_0$ we have the local estimates

$$\|\nabla^m A\|_{L^2(B_{\rho/8}(x))} \leq C(m, T^*, \bar{L}, \sigma) \sqrt{\varepsilon} t^{-\frac{m}{4}},$$

$$\|\nabla^m A\|_{L^\infty} \leq C(m, T^*, \bar{L}, \sigma) \sqrt{\varepsilon} t^{-\frac{m+1}{4}},$$

and the global bounds

$$\|\nabla^m A\|_{L^2(\mu)} \leq C(m, T^*, \bar{L}, \sigma) t^{-\frac{m}{4}} \left( \int |A_0|^2 \, d\mu_0 \right)^{\frac{1}{2}}. \quad (3.16)$$

In contrast to [15, Theorem 3.5] and [23, Proposition 3.5], we do not only prove local bounds, but also the global $L^2$-control (3.16). Note that the global $L^2$-norms could also be estimated by the $L^\infty$-bounds and the area. However, this is disadvantageous since the area cannot be controlled along the flow, and in fact is always expected to diverge in the blow-up process, cf. Lemma 6.1. The necessity for the finer estimates leading to (3.16) is why we give full details on the proof here, even though the argument is very similar to [15, Theorem 3.5].

**Proof of Proposition 3.7** After parabolic rescaling, cf. Lemma 2.8, we may assume $\rho = 1$. Let $x \in \mathbb{R}^3$ and define $K(t) := \int_{B_1(x)} |A|^2 \, d\mu$ and $L(t) := \frac{\lambda^2}{\mathcal{A}(f)^2}$. Then, for all $t \in [0,T)$ using that $K \leq \varepsilon < \varepsilon_1$ by (3.14), we deduce from Proposition 3.5

$$\int_0^t \int_{B_{1/2}(x)} |\nabla^2 A|^2 \, d\mu \, dt \leq C \int_{B_1(x)} |A_0|^2 \, d\mu_0 + C(1 + \sigma^{-2}) \int_0^t \int_{B_1(x)} |A|^2 \, d\mu \, dt + C \int_0^t \int_{B_1(x)} |A|^2 \, d\mu \, dt \leq C(\sigma) \left( K(0) + \int_0^t (1 + L) \, K \, dt \right). \quad (3.17)$$
Moreover, as $K(t) \leq \varepsilon < \varepsilon_1$ by (3.14) we can interpolate by combining [15, Lemma 2.8] and [14, Lemma 4.2] to find
\[
\int_0^t \|A\|_{L^\infty(B_{1/4}(x))}^4 \, d\tau \leq C(\sigma) \varepsilon_1 \left( K(0) + \int_0^t (1 + L) K \, d\tau \right) \leq C(T^*, \tilde{L}, \sigma),
\]
where we used the assumptions (3.14), (3.15) and $T \leq T^*$ in the last step. Thus, defining $a(t) := \|A\|_{L^\infty(B_{1/4}(x))}^4$ and using $T \leq T^*$ and (3.15), we have the estimate
\[
\int_0^t (1 + L + a) \, d\tau \leq C(T^*, \tilde{L}, \sigma). \tag{3.18}
\]

Now, we pick $\tilde{y} \in C_c^\infty(\mathbb{R}^3)$ with $\chi_{B_{1/4}(x)} \leq \tilde{y} \leq \chi_{B_{1/4}(x)}$ and $\gamma := \tilde{y} \circ f$. Note that (3.3) is satisfied with a universal $\Lambda > 0$, which we do not keep track of. As in [15, Theorem 3.5], we define Lipschitz cutoff functions in time via
\[
\xi_j(t) := \begin{cases} 0 & \text{for } t \leq (j - 1) \frac{T}{m}, \\ \frac{m}{T} (t - (j - 1) \frac{T}{m}) & \text{for } (j - 1) \frac{T}{m} \leq t \leq j \frac{T}{m}, \\ 1 & \text{for } t \geq j \frac{T}{m}, \end{cases}
\]
where $m \in \mathbb{N}$ and $0 \leq j \leq m$. We also define $\xi_{-1}(t) := 0$ and $\xi_0(t) := 1$ for all $t \in \mathbb{R}$ if $m = 0$. We note that $\xi_m(T) = 1$ and
\[
0 \leq \frac{d}{dt} \xi_j \leq \frac{m}{T} \xi_{j-1} \quad \text{for all } j \in \mathbb{N}_0. \tag{3.19}
\]

Furthermore, for $0 \leq j \leq m$ we define $E_j(t) := \int |\nabla^2 A|^2 \gamma^4 \, d\mu$. Then, by Proposition B.3, using $\gamma \leq 1$ and (3.13) we have
\[
\frac{d}{dt} E_j(t) + \frac{1}{2} E_{j+1}(t) \leq C(j, m, \sigma) \left[ (1 + L(t) + a(t)) E_j(t) + (1 + L(t) + a(t)) K(t) \right].
\]
Therefore, if we define $e_j := \xi_j E_j$ this implies using (3.19) and $\xi_j \leq 1$
\[
\frac{d}{dt} e_j(t) \leq \frac{m}{T} \xi_{j-1}(t) E_j(t) + C(j, m, \sigma) (1 + L(t) + a(t)) e_j(t) + C(j, m, \sigma) (1 + L(t) + a(t)) K(t) - \frac{1}{2} \xi_j(t) E_{j+1}(t). \tag{3.20}
\]

We now claim that for all $0 \leq j \leq m$ and $t \in (0, T)$ we have
\[
e_j(t) + \frac{1}{2} \int_0^t \xi_j E_{j+1} \, ds \leq \frac{C(j, m, T^*, \tilde{L}, \sigma)}{T} \left( K(0) + K(t) + \int_0^t (1 + L + a) K \, d\tau \right). \tag{3.21}
\]

We proceed by induction on $j$. For $j = 0$ we have $\xi_0 \equiv 1$ on $(0, T)$. Therefore, we clearly have $e_0 = \int |A|^2 \gamma^4 \, d\mu \leq K$. Moreover, by (3.17) we find
\[
\int_0^t E_1(s) \, ds = \int_0^t \int |\nabla^2 A|^2 \gamma^8 \, d\mu \, ds \leq C(\sigma) \left( K(0) + \int_0^t (1 + L) K \, d\tau \right).
\]

For $j \geq 1$, integrating (3.20) on $[0, t]$ and using $e_j(0) = 0$, we find
\[
e_j(t) + \frac{1}{2} \int_0^t \xi_j E_{j+1} \, d\tau \leq C(j, m, \sigma) \int_0^t (1 + L + a) e_j \, d\tau + C(j, m, \sigma) \int_0^t (1 + L + a) K \, d\tau + \frac{m}{T} \int_0^t \xi_{j-1} E_j \, d\tau
\]
For the global exponential term by (3.18), we find by the induction hypothesis and since \( T \leq T^* \). Using Gronwall's inequality and estimating the exponential term by (3.18), we find
\[
e_j(t) \leq -\frac{1}{2} \int_0^t \xi_j e_{j+1} \, ds + \frac{C(j, m, T^*, L, \sigma)}{T_j} \left( K(0) + K(t) + \int_0^t (1 + L + a)K \, d\tau \right)
\]
where we used (2.3), (2.10), and (3.18). After renaming the desired local and global with the assumption that the initial energy is not too large. In contrast to [23], the crucial power of \( \lambda \) is not of lower order when compared to the left hand side of Proposition 3.3. Nevertheless, Lemma 4.1. Let \( \sigma \in (0, 1) \) and let \( f : [0, T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow with \( \mathcal{W}(f_0) < \frac{4\pi}{\sigma} \). Then
\[
\int \left| \frac{3}{A(f)} - \frac{2}{V(f)} \right|^2 \, d\mu \geq \frac{36}{A(f)^2} \left( \sqrt{\frac{4\pi}{\sigma}} - \sqrt{\mathcal{W}(f_0)} \right)^2.
\]
Proof This follows from the reverse triangle inequality in $L^2(d\mu)$, (2.9) and (2.10).

While the scaling techniques from [23, Lemma 4.3] are not available here, we still get the following key estimate, which gives a control over $\lambda$ by quantities which will be suitably integrable.

Lemma 4.2. Let $\sigma \in (0, 1)$ and let $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ be a $\sigma$-isoperimetric Willmore flow with $\mathcal{W}(f_0) < \frac{4\pi}{\sigma}$. Then, we have

$$|\lambda| \leq \frac{\sqrt{A(f)}}{6 \left( \sqrt{\frac{4\pi}{\sigma}} - \sqrt{\mathcal{W}(f)} \right)} \left| \int \langle \partial_t f, \nu \rangle \, d\mu + \int |A^0|^2 H \, d\mu \right|.$$

Proof We test the evolution equation (1.6) with the normal $\nu$ and integrate to obtain

$$\int \langle \partial_t f, \nu \rangle \, d\mu = - \int |A^0|^2 H \, d\mu + \left( \frac{3}{A(f)} \int H \, d\mu - \frac{2}{V(f)} A(f) \right) \lambda,$$

where we used the divergence theorem. We now estimate the prefactor of $\lambda$ by

$$\left| \frac{3}{A(f)} \int H \, d\mu - \frac{2}{V(f)} A(f) \right| \geq \frac{2}{|V(f)|} A(f) - \frac{3}{A(f)} \left| \int H \, d\mu \right| \geq \frac{6}{\sqrt{A(f)}} \left( \sqrt{\frac{4\pi}{\sigma}} - \sqrt{\mathcal{W}(f)} \right),$$

using the fact that $I(f) \equiv \sigma$ by (2.9). By the assumption and (2.10) this is strictly positive and the claim follows.

We remark that the existence of $f_0 : \Sigma_g \to \mathbb{R}^3$ with $I(f_0) = \sigma \in (0, 1)$ satisfying the assumption $\mathcal{W}(f_0) < \frac{4\pi}{\sigma}$ is not yet known and—in general—not true. For the case $g = 0$, this will follow from Theorem 7.1. However, for tori we have $\mathcal{W}(f_0) \geq 2\pi^2$ by [35], and hence $2\pi^2 \leq \mathcal{W}(f_0) < \frac{4\pi}{\sigma}$ can only hold for $\sigma < \frac{2}{\pi} < 1$. On the other hand, for $\sigma \in (0, \frac{1}{2}]$ and arbitrary genus, there exists $f_0$ with $\mathcal{W}(f_0) < \frac{4\pi}{\sigma}$ since $\beta_g(\sigma) < 8\pi$ by [11, Theorem 1.2]. We now use Lemma 4.2 to deduce the time integrability (3.15) for $\lambda$ in the case of small curvature concentration, which enables us to bound all derivatives of the second fundamental form by Proposition 3.7.

Lemma 4.3. Let $\sigma \in (0, 1)$ and let $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ be a $\sigma$-isoperimetric Willmore flow. Let $\mathcal{W}(f_0) \leq K < \frac{4\pi}{\sigma}$ and let $\rho > 0$ be such that

$$\kappa(t, \rho) \leq \varepsilon \leq \varepsilon_1 \quad \text{for all } t \in [0, T),$$

where $\varepsilon_1 > 0$ is as in Proposition 3.5. Then for all $\tau \in [0, T)$ we have

$$\int_0^\tau \frac{\lambda^2}{A(f)^2} \, dt \leq \frac{C}{\left( \sqrt{\frac{4\pi}{\sigma}} - \sqrt{K} \right)^4} \left( \mathcal{W}(f_0) - \mathcal{W}(f(\tau)) + C(\sigma, g) \left( \frac{\tau^3}{\rho^2} + \frac{\tau}{\rho^4} \right) \right).$$

Note that by the invariance of the Willmore energy and the isoperimetric ratio, this estimate is preserved under parabolic rescaling, cf. Lemma 2.8.
Proof of Lemma 4.3} By the assumption we get the local control from Proposition 3.5. As in [23, Proposition 4.2], we can sum up these local bounds to get the global estimate
\[
\int_0^\tau \int |A|^6 \, d\mu \, dt \leq C \int |A_0|^2 \, d\mu_0 + \frac{C(1 + \sigma^{-2})}{\rho^4} \int_0^\tau \int |A|^2 \, d\mu \, dt + C \int_0^\tau \frac{\lambda^2}{A(f)^2} \int |A|^2 \, d\mu \, dt.
\]
Now, by (2.3), the energy decay (2.10) and the assumption, we have
\[
\int_0^\tau \int |A|^2 \, d\mu \, dt \leq C(\sigma, g).
\] (4.1)
Thus, we obtain the estimate
\[
\int_0^\tau \int |A|^6 \, d\mu \, dt \leq C(\sigma, g) \left( 1 + \frac{\tau}{\rho^4} + \int_0^\tau \frac{\lambda^2}{A(f)^2} \, dt \right).
\] (4.2)
By (2.2) we have \(|A|^2 |H| \leq C |A|^3\). Therefore, using Lemma 4.2 we find
\[
\int_0^\tau \frac{\lambda^2}{A(f)^2} \, dt \leq \frac{C}{b^2} \left( \int_0^\tau \int |\partial_t f|^2 \, d\mu \, dt + \int_0^\tau \frac{C(\sigma, g)}{A(f)} \int |A|^4 \, d\mu \, dt \right),
\] (4.3)
by Cauchy–Schwarz and (4.1), where \(b = b(K, \sigma) := \sqrt{\frac{4\pi}{\sigma}} - \sqrt{K} > 0\). For the first term in (4.3), by (2.10) and (1.1) we have \(\int_0^\tau \int |\partial_t f|^2 \, d\mu \, dt = \mathcal{W}(f_0) - \mathcal{W}(f(\tau))\). For the second term in (4.3), we use (3.11), Cauchy–Schwarz in time and space, (4.1) and then (4.2) to find
\[
\frac{C(\sigma, g)}{b^2} \int_0^\tau \frac{1}{A(f)} \int |A|^4 \, d\mu \, dt \leq \frac{C(\sigma, g) \rho^{-2}}{b^2} \left( \int_0^\tau \int |A|^6 \, d\mu \, dt \right)^\frac{1}{2} \left( \int_0^\tau \int |A|^2 \, d\mu \, dt \right)^\frac{1}{2}
\leq \frac{C(\sigma, g)}{b^2} \frac{\tau^{\frac{1}{2}}}{\rho^2} \left( 1 + \frac{\tau^{\frac{1}{2}}}{\rho^2} + \left( \int_0^\tau \frac{\lambda^2}{A(f)^2} \, dt \right)^{\frac{1}{2}} \right)
\leq \frac{C(\delta, \sigma, g)}{b^4} \left( \frac{\tau^{\frac{1}{2}}}{\rho^2} + \frac{\tau}{\rho^4} \right) + \delta \int_0^\tau \frac{\lambda^2}{A(f)^2} \, dt,
\]
for every \(\delta > 0\) by Young’s inequality, and estimating \(b = \sqrt{\frac{4\pi}{\sigma}} - \sqrt{K} \leq C(\sigma)\) in the last step. The statement then follows from (4.3) by taking \(\delta > 0\) sufficiently small. \(\square\)

5. The blow-up and its properties

In this section, we will rescale an isoperimetric Willmore flow as we approach the maximal existence time to obtain a limit immersion. Analyzing the properties of this limit will be the keystone in proving our main result, Theorem 1.2.

5.1. A lower bound on the existence time

As in [14] and [23], the first step is to prove a lower bound on the existence time of an isoperimetric flow which respects the parabolic rescaling in Section 5.2.

To that end, we state a general lifespan result for possible future reference, where the lower bound only depends on the radius of concentration \(\rho\), the isoperimetric ratio \(\sigma\) and the behavior of the \(L^2\)-norm of \(\frac{\lambda}{A(f)}\) near \(t = 0\).
Proposition 5.1. Let $\varepsilon_1 > 0$ be as in Proposition 3.5. There exist universal constants $\tilde{\delta} > 0$ and $\tilde{\varepsilon} \in (0, \min\{8\pi, \varepsilon_1\})$ with the following property: Let $f_0 : \Sigma_g \rightarrow \mathbb{R}^3$ be an immersion with $\mathcal{I}(f_0) = \sigma \in (0, 1), H_{f_0} \neq \text{const}$ and $\mathcal{W}(f_0) < \frac{4\pi}{\sigma}$. Let $f$ be the $\sigma$-isoperimetric Willmore flow with initial datum $f_0$. Assume that

(a) $\kappa(t, \rho) \leq \varepsilon < \tilde{\varepsilon}$ for some $\rho > 0$;
(b) there exists $\rho > 0$ with the following property: For any $t_0 \in [0, \min\{T, \rho^4 \tilde{\omega}\}]$ with $\kappa(t, \rho) < \varepsilon_1$ for all $0 \leq t < t_0$, we have $\int_{t_0}^t \frac{\lambda^2}{A(t)} \, dt \leq \tilde{\delta}.$

Then the maximal existence time of the flow satisfies $T > \hat{\rho}^4$ for some $\hat{\rho} = \hat{\rho}(\sigma, \tilde{\omega}) \in (0, 1)$ and

$$\kappa(t, \rho) \leq \hat{\rho}^{-1} \varepsilon \quad \text{for all } t \in [0, \hat{\rho}^4].$$

(5.1)

Note that we always have $\lim_{t \searrow 0} \int_0^t \frac{\lambda^2}{A(t)} \, dt = 0$. The crucial insight here is that only the decay behavior of the $L^2$-norm of $\frac{\lambda}{A}$ under the assumption of small concentration allows control on the existence time in a way which transforms correctly under parabolic rescaling.

Proof of Proposition 5.1. Without loss of generality, we may assume $\rho = 1$, otherwise we rescale as in Lemma 2.8, see also [23, Proposition 3.5]. Let $T$ denote the maximal existence time of the flow and let $\tilde{\delta} > 0$ to be chosen. We define $\tilde{\varepsilon} := \min\{\frac{\varepsilon_1}{\Gamma_1}, 8\pi\}$ where $\varepsilon_1 > 0$ is as in Proposition 3.3 and $\Gamma_1 > 1$ is as in (3.9) and set $\kappa(t) := \kappa(t, 1)$ for $t \in [0, T]$. By compactness of $f([0, t] \times \Sigma_g)$ for $t < T$, the supremum in the definition of $\kappa = \kappa(\cdot, 1)$ in (3.8) is always attained and the function $\kappa : [0, T] \rightarrow \mathbb{R}$ is continuous with $\kappa(0) \leq \varepsilon < \tilde{\varepsilon}$ by (a).

For a parameter $\omega \in (0, \tilde{\omega})$, to be chosen later, we now define

$$t_0 := \sup\{0 \leq t \leq \min\{T, \omega\} \mid \kappa(\tau) \leq 3\Gamma \varepsilon \text{ for all } 0 \leq \tau < t\} \in [0, \min\{T, \omega\}].$$

(5.2)

By continuity of $t \mapsto \kappa(t)$ and (a), we have $t_0 > 0$. For $t \in [0, t_0)$, we have $\kappa(t) \leq 3\Gamma \varepsilon < \varepsilon_1$ by (5.2) and the definition of $\tilde{\varepsilon}$. Hence, by Proposition 3.5 and assumption (b) we find

$$\int_{B_{1/2}(x)} |A|^2 \, d\mu \leq \int_{B_{1}(x)} |A_0|^2 \, d\mu_0 + 3c_1(1 + \sigma^{-2}) \Gamma \varepsilon t + 3c_1 \Gamma \varepsilon \tilde{\delta},$$

(5.3)

for all $0 \leq t < t_0$ where $c_1 = C$ from Proposition 3.5. Now, if we choose $\tilde{\delta} := (6c_1 \Gamma)^{-1} > 0$ and $\omega = \omega(\sigma, \tilde{\omega}) = \min\{(6c_1(1 + \sigma^{-2}) \Gamma)^{-1}, \tilde{\omega}\} > 0$ we find from (5.3)

$$\int_{B_{1/2}(x)} |A|^2 \, d\mu \leq \int_{B_{1}(x)} |A_0|^2 \, d\mu_0 + \frac{\varepsilon}{2} \omega^{-1} t + \frac{\varepsilon}{2} \leq 2 \varepsilon \quad \text{for all } 0 \leq t < t_0.$$  

(5.4)

However, if $t_0 < \min\{T, \omega\}$, together with (3.9), this implies $\kappa(t) \leq 2\Gamma \varepsilon < \varepsilon_1$ for all $0 \leq t < t_0$ by our choice of $\tilde{\varepsilon}$. On the other hand, by (5.2) and continuity, we must have $\kappa(t_0) = 3\Gamma \varepsilon$, a contradiction.

Consequently, $t_0 = \min\{T, \omega\}$ has to hold. Assume $t_0 = T \leq \omega$. Then, as before, from (5.4) and (3.9) we find

$$\kappa(t) \leq 2\Gamma \varepsilon < \varepsilon_1 \quad \text{for all } 0 \leq t < T = t_0,$$

by the definition of $\tilde{\varepsilon}$. As $T \leq \omega \leq \tilde{\omega}$ by assumption and $\int_0^{\tilde{\omega}} \frac{\lambda^2}{A(t)^2} \, dt \leq \tilde{\delta}$ by (b), we can apply Proposition 3.7 to conclude that for any $0 < \xi < T$ we have

$$\|\nabla^m A\|_\infty \leq C(m, \xi, \sigma, \tilde{\omega}) \quad \text{for all } m \in \mathbb{N}_0, t \in [\xi, T),$$

(5.5)
and \( \| \nabla m A \|_{L^2} \leq C(m, \xi, \mathcal{W}(f_0), \sigma, \tilde{\omega}) \). Consequently, for all \( t \in [\xi, T) \) we can estimate

\[
\left| \frac{\lambda}{\mathcal{A}(f)} \right|^2 \leq \frac{C \left( \| \Delta H \|_{L^2}^2 + \| A \|_{L^\infty}^4 \| A \|_{L^2}^2 \right)}{\mathcal{A}(f)^2 \left\| \frac{3}{\mathcal{A}(f)} H - \frac{2}{V(f)} \right\|_{L^2}^2} \leq C(\xi, \mathcal{W}(f_0), \sigma, \tilde{\omega}),
\]

(5.6)

using (1.7), Cauchy-Schwarz, (2.2) and Lemma 4.1. Similarly, we find

\[
\left| \frac{\lambda}{\mathcal{V}(f)} \right|^2 \leq \frac{C(\sigma)}{\mathcal{A}(f)} \int (|\Delta H|^2 + |A|^6) \, d\mu \leq C(\sigma) \left( \| \Delta H \|_{L^\infty}^2 + \| A \|_{L^\infty}^6 \right) \leq C(\xi, \sigma, \mathcal{W}(f_0), \tilde{\omega}),
\]

(5.7)

where we used \( \mathcal{I}(f) \equiv \sigma \) and (5.5). Exactly with the same arguments as in [14, pp. 330–332] (see also [32, Chapter 4, Proof of Theorem 1.1 after (5.8)]), we can deduce that \( f(t) \) smoothly converges to a smooth immersion \( f(T) \) as \( t \rightarrow T \). By assumption and the energy decay, we infer from Lemma 4.1 that the denominator in (1.7) is bounded away from zero for all \( t \in [0, T) \), so \( f(T) \) is not a constant mean curvature immersion. By Proposition 2.2, we can then restart the flow with initial datum \( f(T) \) which contradicts the maximality of \( T \).

Hence, \( T > \omega \) has to hold. The estimate (5.1) then follows from (5.4) and (3.9) after choosing \( \hat{\epsilon} = \hat{\epsilon}(\sigma, \tilde{\omega}) = \min\{\omega, (2\Gamma)^{-1}, 1\} > 0 \).

Together with the integral estimate for the Lagrange multiplier in Lemma 4.3, this now implies the following

**Proposition 5.2** (Lifespan bound for small energy gap). Let \( \sigma \in (0, 1) \), let \( f: [0, T) \times \Sigma_g \rightarrow \mathbb{R}^3 \) be a maximal \( \sigma \)-isoperimetric Willmore flow such that

(i) \( \mathcal{W}(f_0) \leq K < \frac{4\pi}{\sigma} \),

(ii) \( \kappa(0, \rho) \leq \epsilon < \tilde{\epsilon} \), where \( \tilde{\epsilon} > 0 \) is as in Proposition 5.1;

(iii) \( \mathcal{W}(f_0) - \lim_{t \rightarrow T} \mathcal{W}(f(t)) \leq \tilde{d} \), where \( \tilde{d} = \tilde{d}(K, \sigma, g) > 0 \).

Then the maximal existence time is bounded from below by

\[
T > \hat{\epsilon} \rho^4,
\]

where \( \hat{\epsilon} = \hat{\epsilon}(K, \sigma, g) \) and for all \( 0 \leq t \leq \hat{\epsilon} \rho^4 \) we have \( \kappa(t, \rho) \leq \hat{\epsilon}^{-1} \epsilon \).

Note that the limit in (iii) exists due to Remark 2.7 (ii).

**Proof of Proposition 5.2** We check that the assumptions in Proposition 5.1 are satisfied. Let \( \tilde{\epsilon}, \tilde{d} > 0 \) be as in Proposition 5.1. Assumption (a) of Proposition 5.1 holds true by assumption (ii). We now verify assumption (b) in Proposition 5.1. To that end, let \( \tilde{\omega} > 0 \) to be chosen and assume that for some \( t_0 \in [0, \min\{T, \rho^4 \tilde{d} \omega\}] \) we have \( \kappa(t, \rho) < \epsilon_1 \) for all \( 0 \leq t < t_0 \). By (i), we may apply Lemma 4.3 and use (iii) to find the estimate

\[
\int_0^{t_0} \frac{\lambda^2}{\mathcal{A}^2} \, dt \leq C(K, \sigma, g) \left( \tilde{d} + \tilde{\omega} + \tilde{\omega}^\frac{1}{2} \right) \leq \tilde{d},
\]

if we choose \( \tilde{d} = \tilde{d}(K, \sigma, g) > 0 \) and \( \tilde{\omega} = \tilde{\omega}(K, \sigma, g) > 0 \) small enough. The assumptions of Proposition 5.1 are thus fulfilled and the result follows with \( \hat{\epsilon} = \hat{\epsilon}(\sigma, \tilde{\omega}) = \hat{\epsilon}(K, \sigma, g) \). \qed
5.2. Existence of a blow-up

In this section, we will rescale as we approach the maximal existence time $T \in (0, \infty]$ of a $\sigma$-isoperimetric Willmore flow $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ with $\sigma \in (0, 1)$. To that end, let $(t_j)_{j \in \mathbb{N}} \subset [0, T), t_j \nearrow T, (r_j)_{j \in \mathbb{N}} \subset (0, \infty), (x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^3$ be arbitrary. By translation invariance and Lemma 2.8 for all $j \in \mathbb{N}$ the flow

$$f_j : [0, r_j^{-4}(T - t_j)) \times \Sigma_g \to \mathbb{R}^3, \quad f_j(t, p) := r_j^{-1}(f(t_j + r_j^4 t, p) - x_j)$$

is also a $\sigma$-isoperimetric Willmore flow with initial datum $f_j(0) = r_j^{-1}(f(t_j, \cdot) - x_j)$ and maximal existence time $r_j^{-4}(T - t_j)$. Throughout this section, we will denote all geometric quantities of the flow $f_j$ with a subscript $j$, such as $A_j, \lambda_j, \kappa_j, \mu_j$ for example. The next lemma guarantees the existence of suitable $t_j, r_j$ and $x_j$.

**Lemma 5.3.** Let $\sigma \in (0, 1)$ and let $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ be a maximal $\sigma$-isoperimetric Willmore flow with $\mathcal{W}(f_0) \leq K < \frac{4\pi}{\sigma}$. Let $\hat{c} = \hat{c}(K, \sigma, g) \in (0, 1)$ be as in Proposition 5.2. Then, there exist sequences $(t_j)_{j \in \mathbb{N}} \subset [0, T), t_j \nearrow T, (r_j)_{j \in \mathbb{N}} \subset (0, \infty)$ and $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^3$ such that for all $j \in \mathbb{N}$ we have

(i) $t_j + r_j^4 \hat{c} < T$;

(ii) $\kappa_j(t, 1) \leq \hat{\varepsilon}$ for all $t \in [0, \hat{c}]$, where $\varepsilon > 0$ is as in Proposition 5.1;

(iii) $\inf_{j \in \mathbb{N}} \int_{B_j(0)} |A_{j_\varepsilon}(\cdot)|^2 d\mu_{j_\varepsilon}(\cdot) > 0$.

**Proof** Given any $t \in [0, T)$, with essentially the same arguments as in [23, Lemma 6.6], one finds a radius $r_t \in (0, \infty)$ such that

$$\alpha \leq \kappa(t, r_t) \leq \hat{\varepsilon},$$

where $\alpha = \alpha(K, \sigma, g) > 0$. One then argues as in [16, p. 349] (see also [23, Proposition 6.7]), to prove the existence of $t_j \nearrow T$ and $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^3$ such that choosing $r_j := r_j$, we find that (iii) is satisfied.

Now, the flow $f_j$ satisfies $\kappa_j(0, 1) = \kappa(t_j, r_j) = \kappa(t_j, r_j) \leq \hat{\varepsilon} < \varepsilon$ by (5.9) and since $\hat{c} \in (0, 1)$. Moreover, by the invariances of the Willmore energy we have $\mathcal{W}(f_j(0)) \leq K < \frac{4\pi}{\sigma}$ for all $j \in \mathbb{N}$ and

$$\mathcal{W}(f_j(0)) - \lim_{t_j \nearrow r_j^{-4}(T - t_j)} \mathcal{W}(f_j(t)) = \mathcal{W}(f(t)) - \lim_{t_j \nearrow T} \mathcal{W}(f(t)) \to 0, \quad \text{as } j \to \infty.$$

Consequently, for $j$ sufficiently large, we can apply Proposition 5.2, to find that the maximal existence time of the flow $f_j$ is bounded from below by $r_j^{-4}(T - t_j) > \hat{c}$ which proves (i) and $\kappa_j(t, 1) \leq \hat{\varepsilon}$ for all $t \in [0, \hat{c}]$ by (5.1) which proves (ii).

**Proposition 5.4 (Existence and properties of the limit immersion).** Let $\sigma \in (0, 1)$ and suppose $f : [0, T) \times \Sigma_g \to \mathbb{R}^3$ is a maximal $\sigma$-isoperimetric Willmore flow with $\mathcal{W}(f_0) \leq K < \frac{4\pi}{\sigma}$. Let $\hat{c} \in (0, 1), t_j \nearrow T, (r_j)_{j \in \mathbb{N}} \subset (0, \infty)$ and $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^3$ be as in Lemma 5.3. Then, there exists a complete, orientable surface $\hat{\Sigma} \neq \emptyset$ without boundary and a proper immersion $\hat{f} : \hat{\Sigma} \to \mathbb{R}^3$ such that, after passing to a subsequence, $r_j \to r \in [0, \infty]$ and

(i) as $j \to \infty, \hat{f}_j := f_j(\cdot) \to \hat{f}$ smoothly on compact subsets of $\mathbb{R}^3$, after reparametrization;
(ii) we have \( \int_{B_1(0)} |\hat{A}|^2 \, d\hat{\mu} > 0 \) and \( \mathcal{W}(\hat{f}) \leq \mathcal{W}(f_0) \);

(iii) \( \hat{f} \) is a Helfrich immersion, i.e. a solution to (1.4);

(iv) if \( \mathcal{A}(\hat{f}_j) \to \infty \), then \( \hat{f} \) is a Willmore immersion.

Any Helfrich immersion \( \hat{f} : \hat{\Sigma} \to \mathbb{R}^3 \) which arises from the process described above is called a concentration limit. More precisely, we call \( \hat{f} \) a blow-up if \( r_j \to 0 \), a blow-down for \( r_j \to \infty \) and a limit under translation if \( r_j \to r \in (0, \infty) \). Note that by Lemma 5.3 (i) the last two can only occur if \( T = \infty \).

We highlight that Proposition 5.4 (iv) is particularly remarkable, since it means that under the assumption of diverging area, the constraint vanishes in the concentration limit, see also [23, Theorem 6.2] for a similar rigidity result. This will be essential in the proof of Theorem 1.2.

**Proof of Proposition 5.4** After passing to a subsequence, we may assume \( r_j \to r \) in \([0, \infty)\).

We have \( \varepsilon < \varepsilon_1 \) and \( \hat{c} \in (0, 1) \) by Proposition 5.1 and hence by Lemma 5.3 (ii) we find \( \kappa_j(t, 1) < \varepsilon_1 \) for all \( t \in [0, \hat{c}] \). We may thus use Lemma 4.3 to bound \( \int_0^t \frac{\lambda_j^2}{\mathcal{A}(f_j)} \, d\tau \leq C(K, \sigma, g) \) for all \( t \in [0, \hat{c}] \) and for all \( j \in \mathbb{N} \). Consequently, using Proposition 3.7 we conclude that for any \( j \in \mathbb{N} \) we have

\[
\|\nabla^m A_j\|_\infty \leq C(m, K, \sigma, g)t^{-\frac{m+1}{4}},
\]

\[
\|\nabla^m A_j\|_{L^2(\mu_j)} \leq C(m, K, \sigma, g)t^{-\frac{m}{4}} \quad \text{for } 0 < t \leq \hat{c}.
\]

Moreover, from Simon's monotonicity formula, cf. [34, (1.3)], for any \( R > 0 \) we find

\[
R^{-2}\mu_j(B_R(0)) \leq CK < \infty \quad \text{for all } j \in \mathbb{N}.
\]

Thus, we may apply the localized version of Langer's compactness theorem ([15, Theorem 4.2], see also [23, Appendix A]) to the sequence of immersions \( f_j := f_j(\hat{c}, \cdot) \). After passing to a subsequence, we thus find a proper limit immersion \( \hat{f} : \hat{\Sigma} \to \mathbb{R}^3 \), where \( \hat{\Sigma} \) is a complete (possibly empty) surface without boundary, diffeomorphisms \( \phi_j : \hat{\Sigma}(j) \to U_j \), where \( U_j \subset \Sigma_g \) are open sets and \( \hat{\Sigma}(j) = \{ p \in \hat{\Sigma} \mid |\hat{f}(p)| < j \} \), and functions \( u_j \in C^\infty(\hat{\Sigma}(j); \mathbb{R}^3) \) such that we have

\[ \hat{f}_j \circ \phi_j = \hat{f} + u_j \quad \text{on } \hat{\Sigma}(j) \]

as well as \( \|\nabla^m u_j\|_{L^\infty(\hat{\Sigma}(j), \hat{\mu})} \to 0 \) as \( j \to \infty \) for all \( m \in \mathbb{N}_0 \), so (i) is proven.

Moreover, sending \( j \to \infty \) in Lemma 5.3 (iii) and using the smooth convergence on compact subsets, it follows \( \int_{B_1(0)} |\hat{A}|^2 \, d\hat{\mu} > 0 \) and hence in particular \( \hat{\Sigma} \neq \emptyset \). The second statement in (ii) follows from the scaling invariance and the lower semicontinuity of the Willmore energy with respect to smooth convergence on compact subsets of \( \mathbb{R}^3 \), see [19, Appendix B] for instance.

Let \( \xi \in (0, \hat{c}) \) be arbitrary. Using (5.11) and arguing as in (5.6) and (5.7), we find

\[
\left| \frac{\lambda_j}{\mathcal{A}(f_j)} \right| + \left| \frac{\lambda_j}{\mathcal{V}(f_j)} \right| \leq C(\xi, K, \sigma, g) \quad \text{for all } t \in [\xi, \hat{c}], j \in \mathbb{N},
\]

which when combined with (1.6) and (5.10) immediately yields

\[
\|\partial_t f_j\|_\infty \leq C(\xi, K, \sigma, g) \quad \text{for all } t \in [\xi, \hat{c}], j \in \mathbb{N}.
\]
Now, as a consequence of Lemma B.1, we find
\[
\| \partial_t \nabla^m A_j \|_\infty \leq C(m, \xi, K, \sigma, g),
\]
\[
\| \partial_t \nabla^m A_j \|_{L^2(\mu_j)} \leq C(m, \xi, K, \sigma, g) \quad \text{for all } t \in [\xi, \hat{\xi}], m \in \mathbb{N}_0, j \in \mathbb{N},
\]  
(5.14)
using (5.10), (5.11), and (5.12). Similarly, using Lemma B.2 instead we obtain
\[
\| \partial_t \nabla^m H_j \|_\infty \leq C(m, \xi, K, \sigma, g),
\]
\[
\| \partial_t \nabla^m H_j \|_{L^2(\mu_j)} \leq C(m, \xi, K, \sigma, g) \quad \text{for all } t \in [\xi, \hat{\xi}], j \in \mathbb{N}, m \in \mathbb{N}_0.
\]  
(5.15)
We will now use this to bound the derivative of the Lagrange multiplier. To that end, we observe that using \( I(f_j) \equiv \sigma \) and integration by parts, we find
\[
\frac{\lambda_j}{A(f_j)} = \frac{-3 \int |\nabla H_j|^2 \, d\mu_j + 3 \int |A_j^0|^2 H_j^2 \, d\mu_j - \frac{12\sqrt{T}}{A(f_j)^{\frac{3}{2}}} \int |A_j^0|^2 H_j \, d\mu_j}{\int 3H_j - \frac{12\sqrt{T}}{A(f_j)^{\frac{3}{2}}} \, d\mu_j}.
\]
Note that by Lemma 4.1 the denominator is bounded from below by some \( C(K, \sigma) > 0 \). Using (2.5), (5.10),(5.11), (5.13), (5.14), and (5.15), by direct computation we find
\[
\left| \partial_t \frac{\lambda_j}{A(f_j)} \right| \leq C(\xi, K, \sigma, g) \quad \text{for all } t \in [0, \hat{\xi}], j \in \mathbb{N}.
\]  
(5.16)
Now, using \( I(f_j) \equiv \sigma \) and (2.5) we infer
\[
\partial_t \frac{\lambda_j}{V(f_j)} = C(\sigma) \left( \partial_t \frac{\lambda_j}{A(f_j)} A(f_j)^{-\frac{1}{2}} - \frac{1}{2} \frac{\lambda_j}{V(f_j)} A(f_j)^{-\frac{3}{2}} \partial_t A(f_j) \right)
\]
\[
= C(\sigma) \left( \partial_t \frac{\lambda_j}{A(f_j)} A(f_j)^{-\frac{1}{2}} + \frac{1}{2} \frac{\lambda_j}{A(f_j)} A(f_j)^{-\frac{3}{2}} \int H_j(\partial_t f_j, v_j) \, d\mu_j \right).
\]
Since \( \kappa(t, 1) < \epsilon \) for \( t \in [0, \hat{\xi}] \), we can apply (3.11) with \( \rho = 1 \) to obtain \( A(f_j)^{-1} \leq C \) and hence using (5.16), (5.12), (5.13), and (2.10) we have
\[
\left| \partial_t \frac{\lambda_j}{V(f_j)} \right| \leq C(\xi, K, \sigma, g) \quad \text{for all } t \in [0, \hat{\xi}], j \in \mathbb{N}.
\]  
For \( j \in \mathbb{N} \), we now define the flows \( \tilde{f}_j := f_j \circ \phi_j = f_j(\cdot, \phi_j(\cdot)) \colon (0, \hat{\xi}] \times \tilde{\Sigma}(j) \to \mathbb{R}^3 \) and observe that they satisfy the \( L^\infty \)-estimates (5.10) with \( \tilde{A}_j \) instead of \( A_j \) and the evolution equation
\[
\partial_t \tilde{f}_j = \left[ -\Delta \tilde{H}_j - |\tilde{A}_j^0|^2 \tilde{H}_j + \lambda_j \left( \frac{3}{\tilde{A}(f_j)} \tilde{H}_j - \frac{2}{\tilde{V}(f_j)} \right) \right] v_j \circ \phi_j.
\]  
(5.17)
As in [23, Proof of Theorem 6.2], the estimates for \( \tilde{f}_j \) together with the \( C^1 \)-estimates for \( \frac{\lambda_j}{\tilde{A}(f_j)} \) and \( \frac{\lambda_j}{\tilde{V}(f_j)} \) can then be used to deduce that, after passing to a subsequence, the flows \( \tilde{f}_j \) converge in \( C^1([\xi, \hat{\xi}]; C^m(P; \mathbb{R}^3)) \) for all \( m \in \mathbb{N} \) and all \( P \subset \tilde{\Sigma} \) compact to a limit flow \( f_{\lim} : [\xi, \hat{\xi}] \times \tilde{\Sigma} \to \mathbb{R}^3 \). Moreover, we may assume \( v_j \circ \phi_j \to v_{\lim} \) in \( C^1([\xi, \hat{\xi}]; C^m(P; \mathbb{R}^3)) \) for all \( m \in \mathbb{N} \) and all \( P \subset \tilde{\Sigma} \) compact, where \( v_{\lim}(t, \cdot) \) is a smooth normal vector field along \( f_{\lim}(t, \cdot) \) for all \( t \in [\xi, \hat{\xi}] \), as well as
\[
\frac{\lambda(f_j)}{A(f_j)} \to \lambda_{\lim, 1} \quad \text{and} \quad \frac{\lambda(f_j)}{V(f_j)} \to \lambda_{\lim, 2} \quad \text{in } C^0([\xi, \hat{\xi}]; \mathbb{R}) \text{ as } j \to \infty.
\]
Now, let $P \subset \hat{\Sigma}$ be a fixed compact set and let $j \in \mathbb{N}$ be large enough. Then, using (5.17), (2.9), and (2.10) we find
\[
\int_{\xi}^{\hat{\xi}} \int_{P} |\partial \tilde{f}_j|^2 \, d\mu_j \, dt \leq \int_{\xi}^{\hat{\xi}} \int_{\Sigma} \langle -\nabla W_0(f_j) + \lambda_j \sigma^{-1} \nabla I(f_j), \partial \tilde{f}_j \rangle \, d\mu_j \, dt = \int_{\xi}^{\hat{\xi}} \partial \nabla W_0(f_j) \, dt.
\]
In particular, taking $j \to \infty$ and using $\tilde{f}_j \to f_{\text{lim}}$ in $C^1([\xi, \hat{\xi}; C^m(P; \mathbb{R}^3))$ for all $m \in \mathbb{N}$, we find by Remark 2.7 (ii)
\[
\int_{\xi}^{\hat{\xi}} \int_{P} |\partial f_{\text{lim}}|^2 \, d\mu_{\text{lim}} \, dt \leq \lim_{j \to \infty} \left( W_0(f(t_j + r^4_j \xi)) - W_0(f(t_j + r^4_j \hat{\xi})) \right) = 0.
\]
Consequently, we have $f_{\text{lim}} \equiv f_{\text{lim}}(\hat{\xi}, \cdot) = \lim_{j \to \infty} f_{\tilde{j}}(\hat{\xi}, \phi_j(\cdot)) = \lim_{j \to \infty} \hat{f}_j \circ \phi_j = \hat{f}$ in $C^m(P; \mathbb{R}^3)$ for all $m \in \mathbb{N}$ and $P \subset \hat{\Sigma}$ compact.

We observe that $\hat{\nu} := \nu_{\text{lim}}(\hat{\xi}, \cdot)$ is a global and smooth normal vector field along $\hat{f}$ and hence $\hat{\Sigma}$ is orientable. Setting $\lambda_1 := \lambda_{\text{lim},1}(\hat{\xi}), \lambda_2 := \lambda_{\text{lim},2}(\hat{\xi})$ and using (5.17) we find
\[
\left( -\Delta \hat{H} - |A^0|^2 \hat{H} + 3\lambda_1 \hat{H} - 2\lambda_2 \right) \hat{\nu} = \lim_{j \to \infty} \partial_t \tilde{f}_j (\hat{\xi}, \cdot) = \partial_t f_{\text{lim}}(\hat{\xi}, \cdot) = 0 \quad \text{on } \hat{\Sigma},
\]
so $\hat{f}$ is a Helfrich immersion and (iii) is proven.

For (iv), we now assume $A(\tilde{f}_j) \to \infty$ as $j \to \infty$. By Lemma 4.2, for all $j \in \mathbb{N}$ and $\xi \in (0, \hat{\xi})$, we have by Cauchy–Schwarz
\[
\int_{\xi}^{\hat{\xi}} \left| \frac{\lambda_j}{A(f_j)} \right|^2 \, dt \leq C(K, \sigma) \int_{\xi}^{\hat{\xi}} \left( \int |\partial f_j|^2 \, d\mu_j + A(\tilde{f}_j)^{-1} \left( \int |A_j|^2 |H_j| \, d\mu_j \right)^2 \right) \, dt \leq C(K, \sigma, g) \left[ W(\tilde{f}(t_j + r^4_j \xi, \cdot)) - W(\tilde{f}(t_j + r^4_j \hat{\xi}, \cdot)) \right] + \int_{\xi}^{\hat{\xi}} A(\tilde{f}_j)^{-1} \, dt,
\]
where we estimated $\int |A_j|^2 \, d\mu_j \leq C(\xi, K, \sigma, g)$ for all $t \in [\xi, \hat{\xi}]$, using (5.10) and (2.10). Moreover, as a consequence of (2.5) and (2.10), for all $t \in [0, \hat{\xi}]$ we have
\[
|A(\tilde{f}_j(t, \cdot)) - A(f_j(\hat{\xi}, \cdot))| \leq \int_{t_1}^{\hat{\xi}} \int |H_j| \, d\mu_j \, dt \leq 2\Delta W(0) + \frac{1}{2} \int_{0}^{\hat{\xi}} |\partial f_j|^2 \, d\mu_j \, dt \leq C(K, \sigma, g),
\]
so that $A(\tilde{f}_j(t, \cdot)) \geq A(f_j(\hat{\xi})) - C(K, \sigma, g)$ for all $t \in [0, \hat{\xi}]$ and hence the last term on the right hand side of (5.18) goes to zero as $j \to \infty$. Since $t_j \not\to T$, the first term in (5.18) converges to zero for $j \to \infty$. Consequently
\[
0 = \lim_{j \to \infty} \int_{\xi}^{\hat{\xi}} \left| \frac{\lambda_j}{A(f_j)} \right|^2 \, dt = \int_{\xi}^{\hat{\xi}} |\lambda_{\text{lim},1}|^2 \, dt,
\]
so that in particular, $\hat{\lambda}_1 = \lambda_{\text{lim},1}(\hat{\xi}) = 0$. Moreover, as $I(\tilde{f}_j) \equiv \sigma$, from (1.2) we obtain
\[
\left| \frac{\lambda_j(\hat{\xi})}{\hat{V}(f_j(\hat{\xi}, \cdot))} \right| = C(\sigma) \left| \frac{\lambda_j(\hat{\xi})}{A(f_j(\hat{\xi}, \cdot))} \right| A(f_j(\hat{\xi}, \cdot))^{-\frac{1}{2}} \to 0, \quad j \to \infty,
\]
so $\hat{\lambda}_2 = 0$ and thus $\hat{f}$ is a Willmore immersion. \hfill \square
5.3. The constrained Łojasiewicz–Simon inequality

In this section, we establish a Łojasiewicz–Simon inequality \cite{36–38}. While the unconstrained Willmore energy satisfies such an inequality \cite{26}, the constraint of fixed isoperimetric ratio requires us to prove a refined estimate. To that end, we rely on the general framework of constrained or refined Łojasiewicz–Simon inequalities on submanifolds of Banach spaces \cite{27}, see also \cite[Chapter 1, Section 1.2]{32}.

**Theorem 5.5** (Constrained Łojasiewicz–Simon inequality). Let \( f : \Sigma_g \to \mathbb{R}^3 \) be a Helfrich immersion with \( \mathcal{I}(f) = \sigma \in (0,1) \) such that \( H_f \neq \text{const} \). Then, there exist \( C, r > 0 \) and \( \theta \in (0, \frac{1}{2}] \) such that for all immersions \( h \in W^{4,2}(\Sigma_g; \mathbb{R}^3) \) with \( \| h - f \|_{W^{4,2}} \leq r \) and \( \mathcal{I}(h) = \sigma \) we have

\[
|W_0(h) - W_0(f)|^{1-\theta} \leq C \| \nabla W_0(h) - \lambda(h)\sigma^{-1} \nabla \mathcal{I}(h) \|_{L^2(d\mu_h)}.
\]

The proof of this result is very similar to \cite[Section 7.1]{23}, so we will only provide full details on the differences. Throughout this section we will fix a smooth immersion \( f : \Sigma_g \to \mathbb{R}^3 \) with \( \mathcal{I}(f) = \sigma \in (0,1) \). The normal Sobolev spaces along \( f \) are

\[
W^{k,2}(\Sigma_g; \mathbb{R}^3) : = \{ \phi \in W^{k,2}(\Sigma_g; \mathbb{R}^3) | P^\perp \phi = \phi \},
\]

for \( k \in \mathbb{N}_0 \), with \( L^2(\Sigma_g; \mathbb{R}^3)^\perp : = W^{0,2}(\Sigma_g; \mathbb{R}^3)^\perp \). Here, the \( L^2 \)-inner product always has to be understood with respect to the measure \( \mu_f \) and \( P^\perp \) denotes the normal projection along \( f \), given by \( P^\perp X : = (X, \nu_f)\nu_f \) for any vector field \( X \) along \( f \).

Let \( r > 0 \) be sufficiently small and let

\[
\tilde{U} : = \{ \phi \in W^{4,2}(\Sigma_g; \mathbb{R}^3)^\perp | \phi = u\nu_f \text{ for } \| u \|_{W^{4,2}(\Sigma_g; \mathbb{R}^3)} < r \}.
\]

Consider the shifted energies, defined by

\[
\begin{align*}
W : \tilde{U} &\to \mathbb{R}, W(\phi) : = \mathcal{W}_0(f + \phi), \\
I : \tilde{U} &\to \mathbb{R}, I(\phi) : = \mathcal{I}(f + \phi).
\end{align*}
\]

Note that this is well-defined, since \( f + \phi \) is an immersion for all \( \phi \in \tilde{U} \) with \( r > 0 \) small enough, cf. \cite[Lemma 7.5 (i)]{23}. The first main ingredient toward proving Theorem 5.5 is the analyticity of the energy and the constraint.

**Lemma 5.6.** For \( r > 0 \) small enough, the following maps are analytic.

(i) the function \( \tilde{U} \to \mathbb{R}, \phi \mapsto W(\phi) \);

(ii) the function \( \tilde{U} \to L^2(\Sigma_g; \mathbb{R}^3), \phi \mapsto \nabla \mathcal{W}_0(f + \phi)\rho_{f+\phi}, \) where \( d\mu_{f+\phi} = \rho_{f+\phi} \, d\mu_f \);

(iii) the function \( \tilde{U} \to \mathbb{R}, \phi \mapsto I(\phi) \);

(iv) the function \( \tilde{U} \to L^2(\Sigma_g; \mathbb{R}^3), \phi \mapsto \nabla \mathcal{I}(f + \phi)\rho_{f+\phi} \).

**Proof** Statements (i) and (ii) are exactly as in \cite[Lemma 7.6 (ii) and (iii)]{23}. By \cite[Lemma 7.6 (i) and (iv)]{23}, the maps \( \tilde{U} \to \mathbb{R}, \phi \mapsto A(f + \phi) \) and \( \tilde{U} \to \mathbb{R}, \phi \mapsto \mathcal{V}(f + \phi) \) are analytic and hence so is \( I \) by definition of the isoperimetric ratio and since \( A(f + \phi) > 0 \) for all \( \phi \in \tilde{U} \). For statement (iv) recall from Proposition 2.4 that for \( \phi \in \tilde{U} \) we have

\[
\nabla \mathcal{I}(f + \phi) = \mathcal{I}(f + \phi) \left( \frac{3}{A(f + \phi)} H_{f+\phi} v_{f+\phi} - \frac{2}{\mathcal{V}(f + \phi)} v_{f+\phi} \right).
\]
We note that $\tilde{U} \rightarrow C^0(\Sigma_g; \mathbb{R}^3), \phi \mapsto v_{f+\phi}$ is analytic by [23, Lemma 7.5 (ii)] and $\tilde{U} \rightarrow L^2(\Sigma_g; \mathbb{R}^3), \phi \mapsto H_{f+\phi}v_{f+\phi}$ is analytic by [26, Lemma 3.2 (iv)]. We have $\mathcal{A}(f + \phi) > 0$ for all $\phi \in \tilde{U}$ and $\mathcal{V}(f + \phi) \neq 0$ by continuity for $r > 0$ sufficiently small, since $\mathcal{I}(f) = \sigma > 0$. This implies (iv).

We now compute the first and second variations of $W$ and $I$ in terms of their $H$-gradients, see [27, Section 5].

**Lemma 5.7.** Let $H : = L^2(\Sigma_g; \mathbb{R}^3)^\perp$ and let $r > 0$ be sufficiently small. For each $\phi \in \tilde{U}$, the $H$-gradients of $W$ and $I$ are given by

$$\nabla_H W(\phi) = P^\perp \nabla W_0(f + \phi) \rho_{f+\phi},$$

$$\nabla_H I(\phi) = I(\phi) \left( \frac{3}{\mathcal{A}(f+\phi)} H_{f+\phi} - \frac{2}{\mathcal{V}(f+\phi)} \right) P^\perp v_{f+\phi} \rho_{f+\phi}.$$

Moreover, the Fréchet-derivatives of the $H$-gradient maps of $W$ and $I$ at $u = 0$ satisfy

$$(\nabla_H W)'(0) : W^{4,2}(\Sigma_g; \mathbb{R}^3)^\perp \rightarrow L^2(\Sigma_g; \mathbb{R}^3)^\perp$$ is a Fredholm operator with index zero,

$$(\nabla_H I)'(0) : W^{4,2}(\Sigma_g; \mathbb{R}^3)^\perp \rightarrow L^2(\Sigma_g; \mathbb{R}^3)^\perp$$ is compact.

**Proof** For $\phi, \psi \in \tilde{U}$, we have by Proposition 2.4

$$\frac{d}{dt} \bigg|_{t=0} W(\phi + t\psi) = \int \langle \nabla W_0(f + \phi), \psi \rangle \, d\mu_{f+\phi}$$

$$= \int \left( P^\perp \nabla W_0(f + \phi) \rho_{f+\phi}, \psi \right) \, d\mu_{f+\phi} = \left( P^\perp \nabla W_0(f + \phi) \rho_{f+\phi}, \psi \right)_H.$$  

Similarly, the statement for $\nabla_H I$ can be shown. The Fredholm property of $(\nabla_H W)'(0)$ follows from (1.1) and [26, Lemma 3.3 and p. 356]. For the last statement, we observe that for all $\phi \in W^{4,2}(\Sigma_g; \mathbb{R}^3)^\perp$ we have $\left( \frac{d}{dt} \bigg|_{t=0} v_{f+\phi} \right) = \frac{1}{2} \left| \frac{d}{dt} \bigg|_{t=0} |v_{f+t\phi}|^2 = 0. \right.$ Hence, using (2.6) with $\xi = \langle v_f, \phi \rangle$ and Proposition 2.4 we find

$$(\nabla_H I)'(0) \phi = \frac{d}{dt} \bigg|_{t=0} \left( I(t\phi) - \frac{3}{\mathcal{A}(f+\phi)} H_f + \frac{2}{\mathcal{V}(f+\phi)} \right) v_f$$

$$= \langle \nabla \mathcal{I}(f), \phi \rangle L^2(d\mu_f) \left( \frac{3}{\mathcal{A}(f)} H_f - \frac{2}{\mathcal{V}(f)} \right) v_f - \frac{3\sigma}{\mathcal{A}(f)^2} \langle \nabla \mathcal{A}(f), \phi \rangle L^2(d\mu_f) H_f v_f$$

$$+ \frac{3\sigma}{\mathcal{A}(f)} \left( \Delta \langle v_f, \phi \rangle + |A_f|^2 \langle v_f, \phi \rangle \right) v_f + \frac{2\sigma}{\mathcal{V}(f)^2} \langle \nabla \mathcal{V}(f), \phi \rangle L^2(d\mu_f) v_f$$

$$- \sigma \left( \frac{3}{\mathcal{A}(f)} H_f - \frac{2}{\mathcal{V}(f)} \right) H_f \langle v_f, \phi \rangle v_f,$$

where we used $\mathcal{I}(f) = \sigma$ and $\frac{d}{dt} \bigg|_{t=0} H_{f+t\phi} = -H_f \langle v_f, \phi \rangle$ by (2.5). Since this only involves terms of order two or less in $\phi \in W^{4,2}(\Sigma_g; \mathbb{R}^3)^\perp$, the claim follows from the Rellich-Kondrachov Theorem, see for instance [39, Theorem 2.34].

**Proof of Theorem 5.5** From the assumption $H_f \neq \text{const}$, it follows that $\nabla \mathcal{I}(f) \neq 0$ and hence $\nabla_H I(0) \neq 0$. As in [23, Proposition 7.4], we can thus apply [27, Corollary 5.2] to deduce
that Theorem 5.5 is satisfied in normal directions, i.e. for the functional $W$ with the constraint $I = \sigma$. With the methods from [26, p. 357], one can then use the invariance of the energies under diffeomorphisms to conclude that Theorem 5.5 holds in all directions. 

As in [26, Lemma 4.1], the Łojasiewicz–Simon inequality yields the following asymptotic stability result, see also [23, Lemma 7.9] and [40, Theorem 2.1] for related results in the context of constrained gradient flows in Hilbert spaces.

**Lemma 5.8.** Let $f_W: \Sigma_g \rightarrow \mathbb{R}^3$ be a Helfrich immersion with $\mathcal{I}(f_W) = \sigma \in (0, 1)$, $H_{f_W} \neq \text{const}$. Let $k \in \mathbb{N}$, $k \geq 4$, $\delta > 0$. Then there exists $\varepsilon = \varepsilon(f_W) > 0$ such that if $f: [0, T) \times \Sigma_g \rightarrow \mathbb{R}^3$ is a $\sigma$-isoperimetric Willmore flow satisfying

(i) $\|f_0 - f_W\|_{C^k, \alpha} < \varepsilon$ for some $\alpha > 0$;
(ii) $W_0(f(t)) \geq W_0(f_W)$ whenever $\|f(t) \circ \Phi(t) - f_W\|_{C^k} \leq \delta$ for diffeomorphisms $\Phi(t): \Sigma_g \rightarrow \Sigma_g$;

then the flow exists globally, i.e. we may take $T = \infty$. Moreover, as $t \rightarrow \infty$, it converges smoothly after reparametrization by some diffeomorphisms $\tilde{\Phi}(t): \Sigma_g \rightarrow \Sigma_g$ to a Helfrich immersion $f_{\infty}$, satisfying $W_0(f_{\infty}) = W_0(f_W)$ and $\|f_{\infty} - f_W\|_{C^k} \leq \delta$.

Note that by Lemma 2.6, $f_W$ above is a stationary solution to (1.6)–(1.7). Consequently, the proof of Lemma 5.8 is a straightforward adaptation of [23, Lemma 7.9], applying our Łojasiewicz–Simon inequality in Theorem 5.5 and can be safely omitted. As an important consequence one then finds the following convergence result by following the lines of [26, Section 5] (see also [23, Theorem 7.1]), which yields that in the case where $\hat{\Sigma}$ is compact, below the explicit energy threshold no blow-ups or blow-downs may occur.

**Theorem 5.9.** Let $\sigma \in (0, 1)$, let $f: [0, T) \times \Sigma_g \rightarrow \mathbb{R}^3$ be a maximal $\sigma$-isoperimetric Willmore flow with $W(f_0) < \frac{4\pi}{\sigma}$ and let $\hat{f}: \hat{\Sigma} \rightarrow \mathbb{R}^3$ be a concentration limit as in Proposition 5.4. If $\hat{\Sigma}$ has a compact component and $H_{\hat{f}} \neq \text{const}$, then $\hat{f}$ is a limit under translation. Moreover, the flow exists for all times, i.e. $T = \infty$, and, as $t \rightarrow \infty$, converges smoothly after reparametrization to a Helfrich immersion $f_{\infty}$ with $W(f_{\infty}) = W(\hat{f})$.

**Proof** Let $\hat{c} \in (0, 1)$, $t_j \searrow T$, $(r_j) \in \mathbb{N} \subset (0, \infty)$ and $(x_j) \in \mathbb{N} \subset \mathbb{R}^3$ be as in Proposition 5.4. By arguing as in [15, Lemma 4.3], we may assume $\hat{\Sigma} = \Sigma_g$ and, by Proposition 5.4 (i), we hence have $\hat{f}_j \circ \Phi_j \rightarrow \hat{f}$ smoothly on $\Sigma_g$, where $\Phi_j: \Sigma_g \rightarrow \Sigma_g$ are diffeomorphisms. Let $\varepsilon = \varepsilon(\hat{f}) > 0$ be as in Lemma 5.8. Then for some fixed $j_0 \in \mathbb{N}$ sufficiently large and any $\alpha \in (0, 1)$, we may assume $\|\hat{f}_j \circ \Phi_{j_0} - \hat{f}\|_{C^k, \alpha} < \varepsilon$. Moreover, the $\sigma$-isoperimetric Willmore flow

$$\tilde{f}_{j_0}(t, \cdot) := r_{j_0}^{-1} \left(f(t_0 + r_{j_0}^4 t, \cdot) - x_{j_0}\right) \circ \Phi_{j_0}, \quad t \in [0, r_{j_0}^{-4}(T - t_{j_0})),$$

satisfies $\tilde{f}_{j_0}(\hat{c}) = \hat{f}_{j_0} \circ \Phi_{j_0}$. Using (2.10) and the invariance of the Willmore energy, for any $t \in [0, r_{j_0}^{-4}(T - t_{j_0}))$, we find from Remark 2.7 (ii)

$$W_0(\tilde{f}_{j_0}(t)) \geq \lim_{s \rightarrow r_{j_0}^4(T - t_{j_0})} W_0(f(t_0 + r_{j_0}^4 s)) = \lim_{s \rightarrow T} W_0(f(s)) = \lim_{k \rightarrow \infty} W_0(\hat{f}_k) = W_0(\hat{f}),$$
where we used the smooth convergence \( \hat{f}_k \circ \Phi_k \to \hat{f} \) in the last step. Also note that we have \( I(\hat{f}_j \circ \Phi_{j\circ}) = I(\hat{f}) = \sigma \). Thus, by Lemma 5.8 the flow \( \hat{f}_j \) exists globally and, as \( t \to \infty \), converges smoothly after reparametrization by appropriate diffeomorphisms \( \hat{\Phi}(t) : \Sigma_g \to \Sigma_g \) to a Helfrich immersion \( f_\infty \) with \( \mathcal{W}_0(f_\infty) = \mathcal{W}_0(\hat{f}) \), so \( \mathcal{W}(f_\infty) = \mathcal{W}(\hat{f}) \) by (1.1). Consequently, \( T = \infty \) and for all \( t \geq t_{j_0} \) we have

\[
f(t, \Phi_{j_0} \circ \Phi(r_{j_0}^{-4}(t - t_{j_0}))) = r_{j_0} \tilde{f}_{j_0} \left( r_{j_0}^{-4}(t - t_{j_0}), \tilde{\Phi}(r_{j_0}^{-4}(t - t_{j_0})) \right) + x_{j_0} \to r_{j_0} f_\infty + x_{j_0},
\]
as \( t \to \infty \) smoothly on \( \Sigma_g \). It remains to prove \( r \in (0, \infty) \). To that end, we choose times \( s_k := r_{j_0}^{-4}(t_k - t_{j_0} + \hat{c}r_k) \) for \( k \in \mathbb{N} \), such that we find \( s_k \to \infty \) as \( k \to \infty \), since \( t_k \to T = \infty \). We thus obtain

\[
r_{j_0}^{-1}(f(t_k + \hat{c}r_k, \cdot) - x_{j_0}) \circ \Phi_{j_0} \circ \Phi(s_k) = \tilde{f}_{j_0}(s_k) \circ \tilde{\Phi}(s_k) \to f_\infty \quad \text{smoothly as } k \to \infty.
\]

Consequently, the diameters converge, so \( d_k := \text{diam}(f(t_k + \hat{c}r_k, \Sigma_g)) \to r_{j_0} \text{diam}(f_\infty, \Sigma_g) \) as \( k \to \infty \), whence \( \lim_{k \to \infty} d_k \in (0, \infty) \) since \( \Sigma_g \) is compact.

On the other hand, since \( \hat{f}_k \circ \Phi_k \to \hat{f} \) smoothly and \( r_k \to r \in [0, \infty) \) the limit

\[
\lim_{k \to \infty} r_k^{-1} d_k = \lim_{k \to \infty} \text{diam}(\hat{f}_k) = \text{diam}(\hat{f}(\Sigma_g)) \in (0, \infty)
\]
exists, and consequently \( r_k \to r \in (0, \infty) \) has to hold. \( \square \)

6. Convergence for spheres

The goal of this section is to prove Theorem 1.2. To that end, we want to use the fact that compactness of the concentration limit \( \hat{\Sigma} \) yields convergence of the flow by Theorem 5.9. We first note that the desired compactness follows, if the area along the sequence \( \hat{f}_j \) in Proposition 5.4 remains bounded.

**Lemma 6.1.** Let \( \sigma \in (0, 1) \), let \( f : [0, T) \times \Sigma_g \to \mathbb{R}^3 \) be a maximal \( \sigma \)-isoperimetric Willmore flow with \( \mathcal{W}(f_0) < \frac{4\pi}{\sigma} \) and let \( \hat{f}_j \) be as in Proposition 5.4. If \( \sup_{j \in \mathbb{N}} A(\hat{f}_j) < \infty \), then \( \hat{\Sigma} \) is compact.

**Proof** By Lemma 5.3 (iii), we have \( \hat{f}_j(\Sigma_g) \cap B_1(0) \neq \emptyset \) for all \( j \in \mathbb{N} \), where \( \hat{f}_j = f_j(\hat{c}, \cdot) \) with \( f_j \) as in (5.8). We now use the diameter bound [34, Lemma 1.1] to estimate \( \text{diam}(\hat{f}_j(\Sigma_g)) \leq C \sqrt{A(\hat{f}_j)} \mathcal{W}(\hat{f}_j) \), such that using the assumption, the invariances of the Willmore energy and the energy decay (2.10), we find \( \sup_{j \in \mathbb{N}} \text{diam}(\hat{f}_j(\Sigma_g)) < \infty \). Consequently, there exists \( R \in (0, \infty) \) such that \( \hat{f}_j(\Sigma_g) \subset B_R(0) \) for all \( j \in \mathbb{N} \). Letting \( j \to \infty \) and using Proposition 5.4 (i) and the definition of smooth convergence on compact subset of \( \mathbb{R}^3 \), one then easily deduces \( \hat{f}(\hat{\Sigma}) \subset B_R(0) \) and then, since \( \hat{f} \) is proper, compactness of \( \hat{\Sigma} \). \( \square \)

We will now use Lemma 6.1 and Proposition 5.4 to conclude that if the concentration limit is non-compact, then it is not only a Helfrich, but even a Willmore immersion. In the spherical case, the classification in [28] and the inversion strategy from [16] will then yield a contradiction. Combined with Theorem 5.9, this will prove our main result.
Proof of Theorem 1.2} Since $\Sigma_g = \mathbb{S}^2$ and $\sigma \in (0, 1)$, the existence of a unique, non-extendable $\sigma$-isoperimetric Willmore flow with initial datum $f_0$ follows from Proposition 2.2 and Lemma 2.1 (ii). Moreover, by Remark 2.7 (i), the Willmore energy strictly decreases unless the flow is stationary, in which case global existence and convergence to a Helfrich immersion follow trivially. Thus, we may assume $\mathcal{W}(f_0) < \min\{\frac{4\pi}{\sigma}, 8\pi\}$.

Let $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^3$ be a concentration limit as in Lemma 5.4. If $\hat{\Sigma}$ is compact, we find $\hat{\Sigma} = \mathbb{S}^2$ by [15, Lemma 4.3] and long-time existence and convergence follow from Theorem 5.9 and the fact that $H_f \neq \text{const}$ by Lemma 2.1 (ii).

For the sake of contradiction, we assume that $\hat{\Sigma}$ is not compact. Then we may assume $A(\hat{f}_j) \rightarrow \infty$ by Lemma 6.1. Consequently, by Proposition 5.4 we find that $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^3$ is a Willmore immersion with $\mathcal{W}(\hat{f}) \leq \mathcal{W}(f_0) < 8\pi$. The rest of the argument is as in [23, Proof of Theorem 1.2]: Denote by $I$ the inversion in a sphere with radius 1 centered at $x_0 \notin \hat{f}(\hat{\Sigma})$ and let $\tilde{\Sigma} := I(\hat{f}(\hat{\Sigma})) \cup \{0\}$. Then $\tilde{\Sigma}$ is compact. By [16, Lemma 5.1], $\tilde{\Sigma}$ is a smooth Willmore sphere with $\mathcal{W}(\tilde{\Sigma}) < 8\pi$ and hence, using Bryant’s classification result [28], has to be a round sphere. Thus, $\hat{f}(\hat{\Sigma})$ has to be either a plane or a sphere. Since $\tilde{\Sigma}$ is non-compact by assumption, this yields that $\hat{f}$ has to parametrize a plane, a contradiction to Proposition 5.4 (ii).

Now the limit immersion $f_\infty : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ satisfies $\mathcal{W}(f_\infty) \leq 8\pi$ and solves (1.4) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. It remains to prove $\lambda_1, \lambda_2 \neq 0$. Arguing as in the proof of Lemma 2.6, we infer
$$2\lambda_1 A(f_\infty) + 3\lambda_2 \mathcal{V}(f_\infty) = 0.$$

Now, $\mathcal{V}(f_\infty) \neq 0$ by Lemma 2.1 (i) as $\sigma \in (0, 1)$ and also $A(f_\infty) > 0$. Consequently, if one of $\lambda_1, \lambda_2$ is zero, then so is the other. In this case $f_\infty$ is a Willmore sphere with $\mathcal{W}(f_\infty) \leq 8\pi$. By Bryant’s result [28], it then has to be a round sphere, so $I(f_\infty) = 1$, a contradiction and hence $\lambda_1, \lambda_2 \neq 0$. \hfill $\Box$

Corollary 1.3 is an immediate consequence of the previous results.

Proof of Corollary 1.3} By the assumption on the initial energy, Proposition 5.4 yields the existence of a suitable blow-up sequence and a concentration limit $\hat{f}$ with the desired properties. If $\hat{f}$ has constant mean curvature $\hat{H} \equiv c$, using (2.2) Equation (1.4) reads
$$\frac{1}{2}\hat{H}^3 - 2\hat{K}\hat{H} - \lambda_1\hat{H} - \lambda_2 = 0.$$ 

If $\hat{\Sigma}$ is compact, we conclude $\hat{H} \equiv c \neq 0$ and hence $\hat{K}$ also has to be constant. But then $\hat{f}$ has to parametrize a round sphere (see for instance [30, Chapter V.1]), a contradiction to $I(\hat{f}) = \sigma \in (0, 1)$. Therefore, statement (a) follows from Theorem 5.9. If $\hat{\Sigma}$ is not compact, we may assume $A(\hat{f}_j) \rightarrow \infty$ by Lemma 6.1 after passing to a subsequence. In this case, $\hat{f}$ is a Willmore immersion by Proposition 5.4 (iv), yielding statement (b). \hfill $\Box$

7. An upper bound for $\beta_0$

In this section, we will prove an upper bound for the minimal Willmore energy of spheres with isoperimetric ratio $\sigma \in (0, 1)$.

Theorem 7.1. For every $\sigma \in (0, 1)$ we have $\beta_0(\sigma) < \frac{4\pi}{\sigma}$.
We remark that this estimate becomes sharp for $\sigma \to 1$ since $\beta_0(1) = 4\pi$. On the other hand for $\sigma \in (0, \frac{1}{2}]$, the statement follows since by [7, Lemma 1] we have $\beta_0(\sigma) < 8\pi$ for all $\sigma \in (0, 1)$. We will prove Theorem 7.1 by comparing energy and isoperimetric ratio of an ellipsoid. To that end, for $a \in (0, 1]$, we define the half-ellipse
\[ c_a(t) := (0, a \cos t, \sin t)^T, \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \]
in the $y$-$z$-plane in $\mathbb{R}^3$. By rotating the curve $c_a$ around the z-axis we obtain a particular type of ellipsoid, a prolate spheroid. More explicitly, we define
\[ f_a(t, \theta) = (a \cos t \cos \theta, a \cos t \sin \theta, \sin t)^T \quad \text{for} \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \theta \in [0, 2\pi]. \]

Fortunately, its area, volume and also its Willmore energy can be explicitly computed without the use of elliptic integrals.

**Lemma 7.2.** Let $a \in (0, 1)$. Then we have

(i) $\mathcal{V}(f_a) = \frac{4\pi}{3} a^2$;

(ii) $\mathcal{A}(f_a) = 2\pi a \left( a + \frac{\arcsin \sqrt{1-a^2}}{\sqrt{1-a^2}} \right)$;

(iii) $\mathcal{W}(f_a) = \frac{7\pi}{3} + \frac{2\pi}{3} a^2 + \frac{\pi}{a} \frac{\arcsin \sqrt{1-a^2}}{\sqrt{1-a^2}}$.

**Proof** (i) and (ii) are standard formulas, see for instance [41, Section 4.8]. For (iii), we observe that the mean curvature and the surface element of $f_a$ are given by

\[
\begin{align*}
\frac{\partial \mu_{f_a}}{\partial t} &= a \cos t \sqrt{a^2 \sin^2 t + \cos^2 t} \, dt \, d\theta, \\
H_{f_a} &= \frac{(1 + a^2) \cos^2 t + 2a^2 \sin^2 t}{a(\cos^2 t + a^2 \sin^2 t)^{\frac{3}{2}}},
\end{align*}
\]

by standard formulas for surfaces of revolution, see for instance [42, Section 3C]. In order to compute the Willmore energy, we thus have to evaluate the integral

\[
\mathcal{W}(f_a) = \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( (1 + a^2) \cos^2 t + 2a^2 \sin^2 t \right)^2 \cos t \, dt.
\]

Substituting $u = \sin t$, this integral can then be explicitly computed yielding (iii). \hfill \Box

**Proof of Theorem 7.1** Clearly, we have $\beta_0(\mathcal{I}(f_a)) \leq \mathcal{W}(f_a)$. Moreover, by Lemma 7.2 and a short computation we have

\[
\mathcal{I}(f_a) = \frac{8a}{(a + \frac{\arcsin \sqrt{1-a^2}}{\sqrt{1-a^2}})^3} \quad \text{for all} \quad a \in (0, 1).
\]

An elementary computation yields $\mathcal{I}(f_a) \to 1$ as $a \to 1$ and similarly $\mathcal{I}(f_a) \to 0$ as $a \to 0$. Consequently, we have $\{\mathcal{I}(f_a) \mid a \in (0, 1)\} = (0, 1)$ by a continuity argument.

Now, by (7.1), we find for all $a \in (0, 1)$

\[
\mathcal{W}(f_a) - \frac{4\pi}{\mathcal{I}(f_a)} = \frac{\pi}{6} \left( 14 + 4a^2 + \frac{6 \arcsin \sqrt{1-a^2}}{a \sqrt{1-a^2}} - \frac{3}{a} \left( a + \frac{\arcsin \sqrt{1-a^2}}{\sqrt{1-a^2}} \right)^3 \right) = \frac{\pi}{6} F(a),
\]

where the function $F$ is negative for $a \in (0, 1)$ by Lemma 7.3. \hfill \Box
Figure 1. The function $F(a)$ in Lemma 7.3.

**Lemma 7.3.** The function $F: (0, 1) \to \mathbb{R}$ defined by

$$F(a) := 14 + 4a^2 + \frac{6 \arcsin \sqrt{1 - a^2}}{a \sqrt{1 - a^2}} - \frac{3 \left( a + \frac{\arcsin \sqrt{1 - a^2}}{\sqrt{1 - a^2}} \right)^3}{a}$$

satisfies $F(a) < 0$ for all $a \in (0, 1)$.

We will prove Lemma 7.3 in Sections A. A quick glimpse at the plot of $F$ in Figure 1 illustrates that the statement of Lemma 7.3 is true. However, a rigorous proof seems to be surprisingly difficult, since the function combines trigonometric functions with polynomials and its graph becomes very flat near $F(1) = 0$.

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A. Proof of Lemma 7.3

This section is devoted to proving Lemma 7.3. The idea is to make a change of variables, such that the problem is equivalently formulated in terms of a polynomial in $x, \cos x$ and $\sin x$. Then, we use the power series representation of the Cosine and Sine functions, to reduce the problem to the question if a certain polynomial has roots in a given interval. This last point can then be discussed by studying the Sturm chain of the polynomial.

**Proof of Lemma 7.3** For $x \in (0, \frac{\pi}{2})$ we consider the function $G(x) := F(\cos x) \sin^3 x \cos x$, so that expanding we find

$$G(x) = -3x^3 - 9x^2 \cos x \sin x + 6x \sin^2 x - 9x \cos^2 x \sin^2 x + 14 \cos x \sin^3 x + \cos^3 x \sin^3 x.$$ 

We observe that $F(a) < 0$ for all $a \in (0, 1)$ is equivalent to $G(x) < 0$ for all $x \in (0, \frac{\pi}{2})$.

Using the power series expansion of the Cosine and Taylor's theorem with the Lagrange form of the remainder, for any $N \in \mathbb{N}$ we infer

$$\cos x = \sum_{k=0}^N \frac{(-1)^k}{(2k)!} x^{2k} + \frac{1}{(2N + 1)!} \cos^{(2N+1)}(\xi) x^{2N+1},$$

where $\xi$ lies between $x$ and $0$. 

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for some \( \xi = \xi(N) \in (0, x) \subset (0, \frac{\pi}{2}) \). An induction argument yields \( \cos^{(2N+1)} = (-1)^{N+1} \sin \), so the remainder has a sign, depending on the parity of \( N \). Hence, denoting 
\[
T_{\cos}^N(x) := \sum_{k=0}^{N} \frac{(-1)^k}{(2k)!} x^{2k},
\]
we infer
\[
T_{\cos}^{2n+1}(x) < \cos x < T_{\cos}^{2n}(x) \quad \text{for all } x \in (0, \frac{\pi}{2}), n \in \mathbb{N}_0. \quad (A.1)
\]

By similar arguments, defining 
\[
T_{\sin}^N(x) := \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{and using } \sin^{(2N+2)} = (-1)^{N+1} \sin
\]
we find
\[
T_{\sin}^{2n+1}(x) < \sin x < T_{\sin}^{2n}(x) \quad \text{for all } x \in (0, \frac{\pi}{2}), n \in \mathbb{N}_0. \quad (A.2)
\]

We will now use (A.1) and (A.2) to estimate \( G \). For \( x \in (0, \frac{\pi}{2}) \), we have
\[
G(x) < -3x^3 - 9x^2 T_{\cos}^3(x) T_{\sin}^3(x) + 6x T_{\sin}^2(x) - 9x T_{\sin}^2(x) T_{\sin}^2(x) + 14 T_{\sin}^2(x) T_{\sin}^2(x) - T_{\sin}^2(x) T_{\sin}^2(x) =: k(x).
\]

Now, we observe that \( k(x) \) is polynomial of degree 27. Using Mathematica, we find that this can be simplified to
\[
k(x) = \frac{x^9}{5852528640000} q(x),
\]
for the degree 18 polynomial
\[
q(x) = -984711168000 + 660770611200x^2 - 209922048000x^4 + 40156646400x^6
\]
\[
- 5069859840x^8 + 43718400x^{10} - 25717120x^{12} + 994464x^{14} - 22944x^{16} + 241x^{18}.
\]

By substituting \( x^2 = z \), in order to prove \( G(x) < 0 \) for \( x \in (0, \frac{\pi}{2}) \) it thus suffices to show that
\[
p(z) = -984711168000 + 660770611200z - 209922048000z^2 + 40156646400z^3
\]
\[
- 5069859840z^4 + 43718400z^5 - 25717120z^6 + 994464z^7 - 22944z^8 + 241z^9 < 0
\]
for all \( z \in (0, \frac{\pi^2}{4}) \). To this end, one may compute the Sturm chain of the polynomial \( p \) (see [43, Theorem 8.8.15] for instance), to find that there exist no real roots of \( p \) in the interval \([0, 3] \supset [0, \frac{\pi^2}{4}] \). Consequently, since \( p(0) < 0 \), we find \( p(z) < 0 \) for all \( z \in [0, \frac{\pi^2}{4}] \), and hence the claim follows.

\[
\square
\]

**B. Higher order evolution**

In this section, we will prove a higher order version of Proposition 3.3. To this end, we follow [14, 15] and denote by \( \phi \ast \psi \) any multilinear form, depending on \( \phi \) and \( \psi \) in a universal bilinear way, where \( \phi, \psi \) are tensors on \( \Sigma g \). In particular, we have \( |\phi \ast \psi| \leq C|\phi||\psi| \) for a universal constant \( C > 0 \) and \( \nabla (\phi \ast \psi) = \nabla \phi \ast \psi + \phi \ast \nabla \psi \). Moreover, for \( m \in \mathbb{N}_0 \) and \( r \in \mathbb{N}, r \geq 2 \) we denote by \( P^m_r(A) \) any term of the type
\[
P^m_r(A) = \sum_{i_1 + \cdots + i_r = m} \nabla^{i_1} A \ast \cdots \ast \nabla^{i_r} A.
\]

In addition, for \( r = 1 \) we extend this definition by denoting by \( P^m_1(A) \) any contraction of \( \nabla^mA \) with respect to the metric \( g \).

With this notation, we observe that along an isoperimetric Willmore flow the covariant derivatives of the second fundamental form \( A \) also satisfy a 4-th order evolution equation.
Lemma B.1. Let \( \sigma \in (0,1) \) and let \( f : [0,T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow. Then for all \( m \in \mathbb{N}_0 \) we have
\[
\partial_t (\nabla^m A) + \Delta^2 (\nabla^m A) = P_{3}^{m+2}(A) + P_{5}^{m}(A) + \frac{\lambda}{A(f)}(P_{1}^{m+2}(A) + P_{3}^{m}(A)) + \frac{\lambda}{V(f)} P_{2}^{m}(A).
\]

Proof. We observe \( \partial_t f = \xi \nu \) with
\[
\xi = -\Delta H + P_{0}^{0}(A) + \frac{\lambda}{A(f)} P_{1}^{0}(A) - \frac{2\lambda}{V(f)}.
\]
For \( m = 0 \), we thus find by (2.7)
\[
\partial_t A = \nabla^2 \xi + A * A * \xi = -\Delta^2 A + P_{3}^{2}(A) + P_{5}^{0}(A) + \frac{\lambda}{A(f)} (P_{1}^{2}(A) + P_{3}^{0}(A)) + \frac{\lambda}{V(f)} P_{2}^{0}(A),
\]
where we used \( \nabla^2 \Delta H = \Delta^2 A + P_{3}^{2}(A) \) as a consequence of Simons’ identity [44]. Assume the statement is true for \( m \geq 1 \). By [14, Lemma 2.3] with \( \phi = \nabla^m A \) and the fact that we are in codimension one, we find
\[
\partial_t \nabla^{m+1} A + \Delta^2 \nabla^{m+1} A = \nabla \left( P_{3}^{m+2}(A) + P_{5}^{m}(A) + \frac{\lambda}{A(f)} (P_{1}^{m+2}(A) + P_{3}^{m}(A)) + \frac{\lambda}{V(f)} P_{2}^{m}(A) \right) + \sum_{i+j+k=3} \nabla^i A * \nabla^j A * \nabla^{k+m} A + A * \nabla \xi * \nabla^m A + \nabla A * \nabla^m A + \nabla^2 \nabla^m A
\]
using (B.1) in the last step. \( \square \)

With similar computations as above, one finds the following

Lemma B.2. Let \( \sigma \in (0,1) \) and let \( f : [0,T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow. The for all \( m \in \mathbb{N}_0 \) we have
\[
\partial_t (\nabla^m H) + \Delta^2 (\nabla^m H) = P_{3}^{m+2}(A) + P_{5}^{m}(A) + \frac{\lambda}{A(f)} (P_{1}^{m+2}(A) + P_{3}^{m}(A)) + \frac{\lambda}{V(f)} P_{2}^{m}(A).
\]

We can now prove the following higher order analogue of Proposition 3.3.

Proposition B.3. Let \( \sigma \in (0,1) \), let \( f : [0,T) \times \Sigma_g \to \mathbb{R}^3 \) be a \( \sigma \)-isoperimetric Willmore flow and let \( \gamma \) be as in (3.3). Then for all \( m \in \mathbb{N}_0, s \geq 2m + 4 \) and \( \phi = \nabla^m A \) we have
\[
\frac{\text{d}}{\text{d}t} \int |\phi|^2 \gamma^s \, \text{d}\mu + \frac{1}{2} \int |\nabla^2 \phi|^2 \gamma^s \, \text{d}\mu
\]
\[
\leq C \left( \frac{\lambda^2}{A(f)^2} + \frac{|\lambda|^s}{|V(f)|^3} + \|A\|_{L^\infty(|\gamma| > 0)} \right) \int |\phi|^2 \gamma^s \, \text{d}\mu
\]
\[
+ C \left( 1 + \frac{\lambda^2}{A(f)^2} + \frac{|\lambda|^s}{|V(f)|^3} + \|A\|_{L^\infty(|\gamma| > 0)} \right) \int_{|\gamma| > 0} |A|^2 \, \text{d}\mu,
\]
where \( C = C(s, m, \Lambda) > 0 \).
In order to prove Proposition B.3, we first recall the following

**Lemma B.4.** [14, Lemma 3.2] Let \( f : [0, T] \times \Sigma_g \to \mathbb{R}^3 \) be a normal variation, \( \partial_t f = \xi \nu \). Let \( \phi \) be a \( \left( \begin{smallmatrix} 0 & 0 \end{smallmatrix} \right) \)-tensor satisfying \( \partial_t \phi + \Delta^2 \phi = Y \). Then for any \( \gamma \in C^2([0, T] \times \Sigma_g) \) and \( s \geq 4 \) we have

\[
\frac{d}{dt} \int |\phi|^2 \gamma^s \, d\mu + \int |\nabla^2 \phi|^2 \gamma^s \, d\mu \\
\leq 2 \int \langle Y, \phi \rangle \gamma^s \, d\mu + \int A \ast \phi \ast \phi \ast \xi \gamma^s \, d\mu + \int |\phi|^2 s \gamma^{s-1} \partial_t \gamma \, d\mu \\
+ C \int |\phi|^2 \gamma^{s-4} (|\nabla \gamma|^4 + \gamma^2 |\nabla^2 \gamma|^2) \, d\mu + C \int |\phi|^2 (|\nabla A|^2 + |A|^4) \gamma^s \, d\mu,
\]

where \( C = C(s) \).

**Proof of Proposition B.3** In the following, note that the value of \( C = C(s, m, \Lambda) \) is allowed to change from line to line. We apply Lemma B.4 with \( Y = \partial_t \phi + \Delta^2 \phi, \xi = P_7^1(A) + P_5^0(A) + \frac{k}{\Lambda(f)} \frac{k}{\Lambda(f)} - \frac{2\lambda}{\Lambda(f)} \) by (B.1) and estimate the terms on the right hand side. Using \( \phi = \nabla^m A \) and Lemma B.1, we thus have

\[
2 \int \langle Y, \phi \rangle \gamma^s \, d\mu + \int A \ast \phi \ast \phi \ast \xi \gamma^s \, d\mu + C \int |\phi|^2 (|\nabla A|^2 + |A|^4) \gamma^s \, d\mu \\
= \int \left( P_3^{m+2}(A) + P_5^m(A) \right) \ast \phi \gamma^s \, d\mu \\
\quad + \frac{\lambda}{\Lambda(f)} \int \left( P_1^{m+2}(A) + P_3^m(A) \right) \ast \phi \gamma^s \, d\mu + \frac{\lambda}{\Lambda(f)} \int P_2^m(A) \ast \phi \gamma^s \, d\mu. \tag{B.2}
\]

Moreover, by (3.3) we find

\[
\int |\phi|^2 \gamma^{s-1} \partial_t \gamma \, d\mu = \int |\phi|^2 \gamma^{s-1} \langle D\gamma \circ f, \nu \rangle \left( -\Delta H - |A|^{2} H + \frac{3\lambda}{\Lambda(f)} H - \frac{2\lambda}{\Lambda(f)} \right) \, d\mu. \tag{B.3}
\]

We proceed by estimating all the terms involving \( \lambda \) in (B.2) and (B.3). For the first \( \lambda \)-term in (B.2), since \( |P_1^{m+2}(A)| \leq C |\nabla^2 \phi| \) we find for every \( \varepsilon > 0 \)

\[
\frac{\lambda}{\Lambda(f)} \int P_1^{m+2}(A) \ast \phi \gamma^s \, d\mu \leq \varepsilon \int |\nabla^2 \phi|^2 \gamma^s \, d\mu + C(\varepsilon) \frac{\lambda^2}{\Lambda(f)^2} \int |\phi|^2 \gamma^s \, d\mu.
\]

For the second term, we use [14, Corollary 5.5] with \( k = m, r = 4 \) to obtain

\[
\frac{\lambda}{\Lambda(f)} \int P_3^m(A) \ast \phi \gamma^s \, d\mu \leq C \left| \frac{\lambda}{\Lambda(f)} \right| |A|_{L^\infty((\gamma > 0))}^2 \left( \int |\phi|^2 \gamma^s \, d\mu + \int_{\gamma > 0} |A|^2 \, d\mu \right).
\]

The last \( \lambda \)-term in (B.2) can be estimated by [14, Corollary 5.5] with \( k = m \) and \( r = 3 \), yielding

\[
\frac{\lambda}{\Lambda(f)} \int P_2^m(A) \ast \phi \gamma^s \, d\mu \leq C \left| \frac{\lambda}{\Lambda(f)} \right| |A|_{L^\infty((\gamma > 0))} |A|^2 \left( \int |\phi|^2 \gamma^s \, d\mu + \int_{\gamma > 0} |A|^2 \, d\mu \right).
\]
Now for the first \( \lambda \)-term in (B.3), we use Young’s inequality twice to obtain
\[
\frac{\lambda}{A(f)} \int |\phi|^2 \gamma^{s-1} \langle D\tilde{\gamma} \circ f, \nu \rangle \, d\mu \\
\leq C \frac{\lambda^2}{A(f)^2} \int |\phi|^2 \gamma^s \, d\mu + C \int |\phi|^2 |A|^2 \gamma^{s-2} \, d\mu \\
\leq C \frac{\lambda^2}{A(f)^2} \int |\phi|^2 \gamma^s \, d\mu + C \parallel A \parallel_{\mathcal{L}^\infty([\gamma > 0])} \int |\phi|^2 \gamma^s \, d\mu + C \int |\phi|^2 \gamma^{s-4} \, d\mu.
\]

For the second \( \lambda \)-term in (B.3), we can use Young’s inequality with \( p = \frac{4}{3} \) and \( q = 4 \) to estimate
\[
\frac{\lambda}{V(f)} \int |\phi|^2 \gamma^{s-1} \langle D\tilde{\gamma} \circ f, \nu \rangle \, d\mu \leq C \frac{|\lambda|^\frac{3}{4}}{|V(f)|^\frac{1}{3}} \int |\phi|^2 \gamma^s \, d\mu + C \int |\phi|^2 \gamma^{s-4} \, d\mu.
\]

Choosing \( \varepsilon > 0 \) sufficiently small and absorbing, by Lemma B.4, (B.2) and (B.3)
\[
\frac{d}{dt} \int |\phi|^2 \gamma^s \, d\mu + \frac{3}{4} \int |\nabla^2 \phi| \gamma^s \, d\mu \\
\leq \int \left( P_3^{m+2}(A) + P_5^m(A) \right) \ast \phi \gamma^s \, d\mu + \int |\phi|^2 \gamma^{s-1} \langle D\tilde{\gamma} \circ f, \nu \rangle \left( -\Delta H - |A^0|^2 H \right) \, d\mu \\
+ C \left( \frac{\lambda^2}{A(f)^2} + \frac{|\lambda|^\frac{3}{4}}{|V(f)|^\frac{1}{3}} + \| A \|_{\mathcal{L}^\infty([\gamma > 0])}^4 \right) \left( \int |\phi|^2 \gamma^s \, d\mu + \int_{[\gamma > 0]} |A|^2 \, d\mu \right) \\
+ C \int |\phi|^2 \gamma^{s-4} + \int |\phi|^2 \gamma^{s-4} \left( |\nabla^2 \gamma|^4 + \gamma^2 |\nabla^2 \gamma|^2 \right) \, d\mu,
\]

where we used Young’s inequality to obtain the correct powers of \( \lambda \) and \( \| A \|_{\mathcal{L}^\infty([\gamma > 0])} \). Now, all the terms involving \( \lambda \) on the right hand side of (B.4) are as in the statement. For the second and the last term in (B.4), one may proceed exactly as in the proof of [14, Proposition 3.3]. This way, one creates additional terms which can be estimated by
\[
\int |\phi|^2 \gamma^{s-4} \, d\mu + \int |\nabla \phi|^2 \gamma^{s-2} \, d\mu \leq \varepsilon \int |\nabla^2 \phi|^2 \gamma^s \, d\mu + C \varepsilon \int_{[\gamma > 0]} |A|^2 \gamma^{s-4-2m} \, d\mu,
\]
for every \( \varepsilon > 0 \), using twice the interpolation inequality [14, Corollary 5.3] (which trivially also holds in the case \( k = m = 0 \)). The first term on the right hand side of (B.4) can then be estimated by means of [14, (4.15)]. After choosing \( \varepsilon > 0 \) small enough and absorbing, the claim follows since \( s \geq 2m + 4 \) and \( \gamma \leq 1 \).