THE RANK OF THE 2-CLASS GROUP OF SOME FIELDS WITH LARGE DEGREE

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Abstract. Let $n \geq 3$ be an integer and $d$ an odd square-free integer. We shall compute the rank of the 2-class group of $L_{n,d} := \mathbb{Q}(\zeta_{2^n}, \sqrt{d})$, when all the prime divisors of $d$ are congruent to $\pm 3 \,(\text{mod} \, 8)$ or $9 \,(\text{mod} \, 16)$.

1. Introduction

The explicit computation of the rank of the 2-class group of a given number field $K$ is one of the difficult problems of algebraic number theory, especially for fields with large degree. For many years ago, several authors studied this problem for number fields of degree 2 or 4 (cf. [5, 9, 8]). The methods used therein are not enough to deal with the same problem for number fields with large degree, although recently some papers studied this question for some number fields of degree $2^n$ (cf. [6, 2]). Using the cyclotomic units and some results of the theory of the cyclotomic $\mathbb{Z}_2$-extension, we extend these methods to compute the rank of the 2-class group of some fields of degree $2^n$ of the form $L_{n,d} := \mathbb{Q}(\zeta_{2^n}, \sqrt{d})$, where $d$ is an odd square-free integer and $n \geq 3$ is a positive integer.

Let $k := \mathbb{Q}(\sqrt{d}, \sqrt{-1}), \mathbb{Q}(\sqrt{-2}, \sqrt{d})$ or $\mathbb{Q}(\sqrt{-2}, \sqrt{-d})$. Then the cyclotomic $\mathbb{Z}_2$-extension of $k$ is

$$k(\sqrt{2}) \subset k(\sqrt{2+\sqrt{2}}) \subset k(\sqrt{2+\sqrt{2+\sqrt{2}}}) \subset \ldots$$

which coincides with the tower $L_{3,d} \subset L_{4,d} \subset \ldots \subset L_{n,d} \subset \ldots$

The present work is a continuation of our previous work [2], in which we computed the rank of the 2-class group of $L_{n,d}$ when the prime divisors of $d$ are congruent to 3 or 5 (mod 8). Thus, we compute the rank of the 2-class group of $L_{n,d}$ when the prime divisors of $d$ are congruent to $\pm 3 \,(\text{mod} \, 8)$ or $9 \,(\text{mod} \, 16)$. Furthermore, we give the rank of the 2-class group of $L_{n,d}$ in terms of that of $L_{4,d}$, when the prime divisors of $d$ are congruent to $\pm 3 \,(\text{mod} \, 8)$ or $\pm 7 \,(\text{mod} \, 16)$.

Notations

The next notations will be used for the rest of this article:

- $d$: An odd square-free integer.

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The preliminariy results

Let us first collect some results that will be used in what follows.

**Lemma 1** ([3]). Let $K/k$ be a $\mathbb{Z}_2$-extension, $k_n$ its $n$-th layer and $n_0$ an integer such that any prime of $K$ which is ramified in $K/k$ is totally ramified in $K/k_{n_0}$. If there exists an integer $n \geq n_0$ such that \(\text{rank}_2(\text{Cl}(k_n)) = \text{rank}_2(\text{Cl}(k_{n+1}))\), then \(\text{rank}_2(\text{Cl}(k_m)) = \text{rank}_2(\text{Cl}(k_n))\) for all $m \geq n$.

**Lemma 2** ([10, Lemma 8.1, Corollary 4.13]).

1. The cyclotomic units of $K_n^+$ are generated by $-1$ and

\[
\xi_{k,n} = \zeta_{2^n}^{(1-k)/2} \frac{1 - \zeta_{2^n}^k}{1 - \zeta_{2^n}},
\]

where $k$ is an odd integer such that $1 < k < 2^{n-1}$.

2. The cyclotomic units of $K_n$ are generated by $\zeta_{2^n}$ and the cyclotomic units of $K_n^+$.

3. The Hasse’s index $Q$ of $K_n$ equals 1.

**Lemma 3** ([4]). Let $K/k$ be a quadratic extension. If the class number of $k$ is odd, then the rank of the 2-class group of $K$ is given by

\[
\text{rank}_2(\text{Cl}(K)) = t - 1 - e,
\]

where $t$ is the number of ramified primes (finite or infinite) in the extension $K/k$ and $e$ is defined by \(2^e = [E_k : E_k \cap N_K/k(K^*)]\).

**Remark 1.** Note that a unit $u$ of $K_n$ is a norm in $L_{n,d}/K_n$ if and only if \(\left(\frac{u,d}{p}\right) = 1\), for all prime ideal $p$ of $K_n$ ramified in $L_{n,d}$. 

Next, we need to characterize ideals of $K_n = \mathbb{Q}(\zeta_{2^n})$ that ramify in $L_{n,d} = K_n(\sqrt{d})$. Let $n \geq 3$ and $d$ be an odd square-free integer, then $d$ is congruent to 1 or 3 (mod 4). So 2 is unramified in either $\mathbb{Q}(\sqrt{d})$ or $\mathbb{Q}(\sqrt{-d})$. Thus, the ramification index of 2 in $L_{n,d}$ is strictly inferior to $2^n$. As 2 is totally ramified in $K_n := \mathbb{Q}(\zeta_{2^n})$, so the prime ideal of $K_n$ lying over 2 is unramified in $L_{n,d}$, as otherwise the ramification index of 2 in $L_{n,d}$ will be $2^n$, which is absurd. Hence we prove the following result:

**Lemma 4.** Let $d$ be an odd square-free integer. Then a prime ideal $p$ of $K_n$ is ramified in $L_{n,d}/K_n$ if and only if it divides $d$.

**Proposition 1.** Let $n \geq 5$ and $p$ be a rational prime. Then $p$ decomposes into 4 primes of $K_n$ if and only if $p \equiv 7$ or 9 (mod 16).

**Proof.** Let $K_\infty$ denote the cyclotomic $\mathbb{Z}_2$-extension of $K$. Note that $\text{Gal}(K_\infty : K) \cong \mathbb{Z}_2$. Hence the decomposition field of a prime above $p$ must be some $K_n$. Therefore if a prime $p$ of $K_n$ is inert in $K_{n+1}$, then $p$ is inert in $K_\infty/K_n$. Then Proposition 1 is from the easily verified fact that a prime $p$ decomposes into 4 primes in $K_4 = \mathbb{Q}(\zeta_{16})$ and $K_5 = \mathbb{Q}(\zeta_{32})$ if and only if $p \equiv 7, 9$ (mod 16). \qed

**Remark 2.** Let $p \equiv 7$ or 9 (mod 16) be a prime. We have

1. If $p \equiv 9$ (mod 16), then $p$ decomposes into product of four prime ideals in $K_3$ and $K_4$.

2. If $p \equiv 7$ (mod 16), then $p$ decomposes into product of two prime ideals in $K_3$ and into product of four prime ideals in $K_4$.

**Proposition 2 ([2]).** Let $m \geq 4$ and $p$ be a rational prime. Then $p$ decomposes into the product of 2 primes of $K_m$ if and only if $p \equiv 3$ or 5 (mod 8).

Now we shall do some computations:

**Lemma 5 ([2]).** Let $n \geq 3$ be a positive integer and $p$ be a prime number. Let $p_{K_n}$ denote a prime ideal of $K_n$ above $p$. We have

1. If $p \equiv 5$ (mod 8). Then
   \[
   \left( \frac{\zeta_{2^n}, p}{p_{K_n}} \right) = -1 \quad \text{and} \quad \left( \frac{\xi_{k,n}, p}{p_{K_n}} \right) = 1.
   \]

2. If $p \equiv 3$ (mod 8). Then
   \[
   \left( \frac{\zeta_{2^n}, p}{p_{K_n}} \right) = -1 \quad \text{and} \quad \left( \frac{\xi_{k,n}, p}{p_{K_n}} \right) = \begin{cases} 
   -1, & \text{if } k \equiv \pm 3 \pmod{8} \\
   1, & \text{elsewhere.}
   \end{cases}
   \]

**Lemma 6.** Let $n \geq 3$ be an integer and $p$ a prime congruent to 9 (mod 16), $p_{K_n}$ a prime ideal of $K_n$ dividing $p$.

\[
\left( \frac{\zeta_{2^n}, p}{p_{K_n}} \right) = -1 \quad \text{and} \quad \left( \frac{\xi_{k,n}, p}{p_{K_n}} \right) = \begin{cases} 
   -\left( \frac{2}{p} \right)_4, & \text{if } k \equiv \pm 3 \pmod{8} \\
   1, & \text{elsewhere.}
   \end{cases}
\]
Proof. For all $n \geq 3$, the prime $p$ decomposes into product of four prime ideals of $K_n$ (see Proposition [1]), denote by $p_{K_n}$ one of them (such that $p_{K_{n-1}} \subset p_{K_n}$). We have $\zeta_{2^n} = \zeta_{2^{n-1}}$, so the minimal polynomial of $\zeta_{2^n}$ over $K_{n-1}$ is $X^2 - \zeta_{n-1}$ and $N_{K_n/K_{n-1}}(\zeta_{2^n}) = -\zeta_{2^{n-1}}$. Thus

$$
\left( \frac{\zeta_{2^n} \cdot p}{p_{K_n}} \right) = \left( \frac{-\zeta_{2^{n-1}} \cdot p}{p_{K_{n-1}}} \right) = \left( \frac{-\zeta_{2^{n-1}} \cdot p}{p_{K_{n-1}}} \right) = \ldots = \left( \frac{\zeta_n \cdot p}{p_{K_3}} \right) = (-1)^{\frac{q-1}{8}} = -1 \, (\text{see [1]}).
$$

$$
\left( \frac{\zeta_{k,n} \cdot p}{p_{K_n}} \right) = \left( \frac{\zeta_{2^n}^{(1-k)/2} \cdot p}{p_{K_n}} \right) \left( \frac{1 - \zeta_k \cdot p}{p_{K_3}} \right) \left( \frac{1 - \zeta_{n-1} \cdot p}{p_{K_3}} \right) \left( \frac{1 - \zeta_{n} \cdot p}{p_{K_3}} \right) \left( \frac{1 - \zeta_{p} \cdot p}{p_{K_3}} \right)
$$

\begin{align*}
&= (-1)^{(1-k)/2} \left( \frac{1+\zeta_k \cdot p}{p_{K_3}} \right) \left( \frac{1-\zeta_k \cdot p}{p_{K_3}} \right), \quad \text{if } k \equiv 3 \pmod{8} \\
&= (-1)^{(1-k)/2} \left( \frac{-1-\zeta_k \cdot p}{p_{K_3}} \right) \left( \frac{1-\zeta_{n-1} \cdot p}{p_{K_3}} \right), \quad \text{if } k \equiv 5 \pmod{8} \\
&= (-1)^{(3-k)/2} \left( \frac{-1-\zeta_k \cdot p}{p_{K_3}} \right) \left( \frac{1-\zeta_{n} \cdot p}{p_{K_3}} \right) \left( \frac{1-\zeta_{p} \cdot p}{p_{K_3}} \right), \quad \text{if } k \equiv 7 \pmod{8} \\
&= \left( \frac{2}{p} \right) \frac{-1}{p_{K_3}}, \quad \text{if } k \equiv 1 \pmod{8} \\
&= \left( \frac{2}{p} \right) \frac{-1}{p_{K_3}} - \left( \frac{2}{p} \right) \frac{\sqrt{2}}{p_{K_3}} - \left( \frac{2}{p} \right) \frac{\sqrt{2}}{p_{K_3}} - \left( \frac{2}{p} \right) \frac{\sqrt{2}}{p_{K_3}} = 1, \quad \text{if } k \equiv 1 \pmod{8}.
\end{align*}

Using the proof of [1] Lemma 3.4 we get

$$
\left( \frac{1-i \cdot p}{p_{K_3}} \right) = \left( \frac{1+i \cdot p}{p_{K_3}} \right) = \left( \frac{1+i \cdot p}{p_{K_3}} \right) = \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) \left( \sqrt{2} \cdot p_{K_3} \right) = \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) \left( \frac{-1}{p} \frac{\sqrt{2}}{p_{K_3}} \right) \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) = \left( \frac{-1}{p} \frac{\sqrt{2}}{p_{K_3}} \right) \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) = -1.
$$

Similarly we have

$$
\left( \frac{2-\sqrt{2} \cdot p}{p_{K_3}} \right) = \left( \frac{2+\sqrt{2} \cdot p}{p_{K_3}} \right) = \left( \frac{\sqrt{2} \cdot p}{p_{K_3}} \right) = \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) = \left( \frac{-1}{p} \frac{\sqrt{2}}{p_{K_3}} \right) \left( \frac{\zeta_{2} \cdot p}{p_{K_3}} \right) = -1. \quad \text{Which achieves the proof.}
$$

\end{proof}

3. The main results

The authors of [2] computed the rank of the 2-class group of $L_{n,d}$, when the prime divisors of $d$ are congruent to 3 or 5 (mod 8). In this section we shall compute the rank of the 2-class group of $L_{n,d}$, when the prime divisors of $d$ are congruent to $\pm 3$ (mod 8) or 9 (mod 16).
Theorem 1. Let $n \geq 3$ and $d > 2$ be an odd composite square-free integer of prime divisors congruent to $\pm 3 \pmod{8}$ or $9 \pmod{16}$. Let $r$ denote the number of prime divisors of $d$ which are congruent to $3$ or $5 \pmod{8}$ and $q$ the number of those which are congruent to $9 \pmod{16}$. Set $t = 4q + 2r$. We have

1. If there are two primes $p_1$ and $p_2$ dividing $d$ such that $p_1 \equiv -p_2 \equiv 5 \pmod{8}$, then $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 3$.

2. If $d$ is divisible by a prime congruent to 3 (mod 8) and none of the primes $p | d$ is congruent to 3 (mod 8), then $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 2$ or $t - 3$. More precisely, $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 3$ if and only if there is a prime $p \equiv 1 \pmod{8}$ dividing $d$ such that $\left(\frac{2}{p}\right)_4 = -1$.

3. If $d$ is divisible by a prime congruent to 5 (mod 8) and none of the primes $p | d$ is congruent to 3 (mod 8), then $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 2$ or $t - 3$. More precisely, $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 3$ if and only if there is a prime $p \equiv 1 \pmod{8}$ dividing $d$ such that $\left(\frac{2}{p}\right)_4 = 1$.

4. If all the primes $p | d$ are congruent to 9 (mod 16), then $\text{rank}_2(\text{Cl}(L_{n,d})) = 4q - 2$ or $4q - 3$. More precisely, $\text{rank}_2(\text{Cl}(L_{n,d})) = 4q - 3$ if and only if there are two prime divisors $p_1$ and $p_2$ of $d$ such that $\left(\frac{2}{p_1}\right)_4 = 1$ and $\left(\frac{2}{p_2}\right)_4 = -1$.

Proof. We shall firstly prove the items of the above theorem assuming that $n \in \{3, 4, 5\}$. We have $h(K_n^+) = 1$ (see [7]). So by [10, Theorem 8.2], the unit group of $K_n^+$ is generated by $-1$ and the cyclotomic units $\xi_{k,n}$, for odd integers $k$ such that $1 < k < 2^{n-1}$. Thus, by Lemma 2 we have

$$E_{K_n} = \langle \zeta_{2^n}, \xi_{k,n} \rangle,$$

with $k$ is an odd integer such that $1 < k < 2^{n-1}$.

By Lemma 3, Proposition 1 and Lemma 4 we have $\text{rank}_2(\text{Cl}(L_{n,d})) = t - 1 - e_{n,d}$, where $e_{n,d}$ is defined by $(E_{K_n} : E_{K_n} \cap \mathcal{N}(L_{n,d})) = 2^{e_{n,d}}$. We shall determine the classes representing $E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d}))$. Let $\alpha \in E_{K_n}$. $\overline{\alpha}$ denotes the class of $\alpha$ in $E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d}))$. Let $p$ be a prime dividing $d$ and $p_{K_n}$ be a prime ideal of $K_n$ lying over $p$. We have

$$\left(\frac{\alpha, d}{p_{K_n}}\right)_q = \prod_{q | d} \left(\frac{\alpha, d}{p_{K_n}}\right) = \left(\frac{\alpha, p}{p_{K_n}}\right) \prod_{q \neq p} \left(\frac{\alpha, q}{p_{K_n}}\right) = \left(\frac{\alpha, p}{p_{K_n}}\right) \prod_{q | d \text{ and } q \neq p} \left(\frac{q}{p_{K_n}}\right)^0 = \left(\frac{\alpha, p}{p_{K_n}}\right).$$

Note that the units $\xi_{k,n}$ for $k \equiv \pm 1 \pmod{8}$ are norms in $L_{n,d}/K_n$ so we will disregard them. Let $k$ and $k'$ denote any two positive integers such that $k, k' \equiv \pm 3 \pmod{8}$. 

1. By Lemmas 5 and 6, we have \( \zeta_{2n} \) and \( \xi_{k,n} \) are not norms in \( L_{n,d}/K_n \). Furthermore

\[
\begin{cases}
\left( \frac{\zeta_{2n}\xi_{k,n,p}}{p_{Kn}} \right) = -1, & \text{if } p \equiv 5 \pmod{8}, \\
\left( \frac{\xi_{k,n}\xi_{k',n,p}}{p_{Kn}} \right) = 1, & \text{for all prime } p \text{ dividing } d.
\end{cases}
\]

Thus \( \bar{\zeta}_{2n} \neq \bar{\xi}_{k,n} \) and \( \bar{\xi}_{k,n} = \bar{\xi}_{k',n} \). Hence \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}, \bar{\xi}_{k,n}, \bar{\zeta}_{2n}\bar{\xi}_{k,n}\} \). It follows that \( \text{rank}_2(\text{Cl}(L_{n,d})) = t - 3 \).

2. By Lemmas 5 and 6 we have \( \zeta_{2n} \) and \( \xi_{k,n} \) are not norms in \( L_{n,d}/K_n \). Furthermore

\[
\begin{cases}
\left( \frac{\zeta_{2n}\xi_{k,n,p}}{p_{Kn}} \right) = \left( \frac{2}{p} \right), & \text{if } p \equiv 3 \pmod{8}, \\
\left( \frac{\xi_{k,n}\xi_{k',n,p}}{p_{Kn}} \right) = 1, & \text{for all prime } p \text{ dividing } d.
\end{cases}
\]

Then \( \bar{\xi}_{k,n} = \bar{\xi}_{k',n} \) and \( \bar{\zeta}_{2n} \neq \bar{\xi}_{k,n} \) if and only if \( \left( \frac{2}{p} \right) = -1 \). Thus, \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}\} \) if \( \left( \frac{2}{p} \right) = 1 \) and \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}, \bar{\xi}_{k,n}, \bar{\zeta}_{2n}\bar{\xi}_{k,n}\} \) if not. So the second item.

3. By Lemmas 5 and 6 we have \( \zeta_{2n} \) are not norm in \( L_{n,d}/K_n \) and \( \xi_{k,n} \) is not norm in \( L_{n,d}/K_n \) is and only if \( \left( \frac{2}{p} \right) = 1 \). So with similar discussion as above, one shows that \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}\} \) if \( \left( \frac{2}{p} \right) = -1 \) and \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}, \bar{\xi}_{k,n}, \bar{\zeta}_{2n}\bar{\xi}_{k,n}\} \) if not. So the results.

4. Assume that for all prime \( p \) dividing \( d \) we have \( \left( \frac{2}{p} \right) = 1 \), then \( \left( \frac{\zeta_{2n}\xi_{k,n,p}}{p_{Kn}} \right) = 1 \). It follows that \( \bar{\zeta}_{2n} = \bar{\xi}_{k,n} \). So \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}\} \) and \( e_{n,d} = 1 \).

- If for all the primes \( p \) dividing \( d \) we have \( \left( \frac{2}{p} \right) = -1 \), then \( \xi_{k,n} \) is norm in \( L_{n,d}/K_n \). So \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}\} \) and \( e_{n,d} = 1 \).

- Suppose now there are two primes \( p_1 \) and \( p_2 \) dividing \( d \) such that \( \left( \frac{2}{p_1} \right) = 1 \) and \( \left( \frac{2}{p_2} \right) = -1 \). We have \( \left( \frac{\xi_{k,n,p_1}}{p_{Kn}} \right) = \left( \frac{\zeta_{2n}\xi_{k,n,p_2}}{p_{Kn}} \right) = -1 \). Thus \( \bar{\zeta}_{2n} \neq \bar{\xi}_{k,n} \) and \( \bar{\xi}_{k,n} \neq \bar{\zeta}_{2n} \). We infer that \( E_{K_n}/(E_{K_n} \cap \mathcal{N}(L_{n,d})) = \{1, \bar{\zeta}_{2n}, \bar{\xi}_{k,n}, \bar{\zeta}_{2n}\bar{\xi}_{k,n}\} \) and \( e_{n,d} = 2 \). So the third item.

Thus we proved the theorem for \( n \in \{3, 4, 5\} \).

Since \( \text{rank}_2(\text{Cl}(L_{3,d})) = \text{rank}_2(\text{Cl}(L_{4,d})) \), then Lemma 11 achieves the proof.

\( \square \)

We similarly get the following result:
Theorem 2. Let $n \geq 3$ be a positive integer and let $p$ denote a prime such that $p \equiv 9 \pmod{16}$. Then

$$\text{rank}_2(\text{Cl}(L_{n,p})) = 2.$$ 

4. Appendix

In this appendix, we give the rank of the 2-class group of $L_{n,d}$ according to that of $L_{4,d}$, when the prime divisors of $d$ are congruent to $\pm 3 \pmod{8}$ or $\pm 7 \pmod{16}$.

Lemma 7. Let $n \geq 4$ be an integer and $p$ a prime integer congruent to $7 \pmod{16}$. Then for all prime ideal $p_{K_n}$ of $K_n$ dividing $p$ we have

$$\left(\frac{\zeta_{2^n}, p}{p_{K_n}}\right) = \left(\frac{\zeta_{16}, p}{p_{K_4}}\right) \quad \text{and} \quad \left(\frac{\xi_{k,n}, p}{p_{K_n}}\right) = \begin{cases} \left(\frac{\xi_{5,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 3 \pmod{16} \\ \left(\frac{\xi_{9,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 5 \pmod{16} \\ \left(\frac{\xi_{7,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 7 \pmod{16} \\ 1, & \text{if } k \equiv 1 \pmod{16} \end{cases}$$

And there are $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ such that for all prime ideal $p_{K_n}$ of $K_n$ dividing $p$ we have

$$\left(\frac{\xi_{k,n}, p}{p_{K_n}}\right) = \begin{cases} \varepsilon_1 \left(\frac{\xi_{5,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 11 \pmod{16} \\ \varepsilon_2 \left(\frac{\xi_{9,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 13 \pmod{16} \\ \varepsilon_3 \left(\frac{\xi_{7,4}, p}{p_{K_4}}\right), & \text{if } k \equiv 15 \pmod{16} \\ 1, & \text{if } k \equiv 9 \pmod{16} \end{cases}$$

Proof. By Remark 2 and Proposition 1, there are four prime ideals of $K_4$ lying over $p$, and these primes are inert in $K_n$ for all $n \geq 5$. Since the minimal polynomial of $\zeta_{2^n}$ over $K_{n-1}$ is $X^2 - \zeta_{n-1}$, then $N_{K_n/K_{n-1}}(\zeta_{2^n}) = -\zeta_{2^{n-1}}$. We have

$$\left(\frac{\zeta_{2^n}, p}{p_{K_n}}\right) = \left(\frac{\zeta_{2^n}, p}{p_{K_n}}\right) = \left(\frac{\zeta_{2^n-1}, p}{p_{K_{n-1}}}ight) = \ldots = \left(\frac{\zeta_{16}, p}{p_{K_4}}\right),$$

and

$$\left(\frac{1 - \zeta_{2^n}, p}{p_{K_n}}\right) = \left(\frac{N_{K_n/K_{n-1}}(1 - \zeta_{2^n}), p}{p_{K_{n-1}}}ight) = \ldots = \left(\frac{1 - \zeta_{16}, p}{p_{K_4}}\right).$$
Theorem 3. Let $d$ be a square-free integer such that the prime divisors of $d$ are congruent to $±3 \pmod{8}$ or $±7 \pmod{16}$. Then, for all positive integer $n \geq 4$, we have

$$\text{rank}_2(\text{Cl}(L_{n,d})) = \text{rank}_2(\text{Cl}(L_{4,d})).$$
Proof. Suppose that \( n \in \{4, 5\} \). By Lemmas 5, 6 and 7 we have \( e_{n,d} = e_{4,d} \). By Propositions 1 and 2 the number of prime divisors of \( d \) in \( K_n \) is the same. Then \( \text{rank}_2(\text{Cl}(L_5,d)) = \text{rank}_2(\text{Cl}(L_4,d)) \) (see Lemma 3). Hence, Lemma 1 completes the proof. \( \Box \)

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