Characterizations of several convergence structures on fuzzy sets

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Abstract

In this paper, we discuss the characterizations and the relations of the Γ-convergence, the endograph metric and the sendograph metric on the realm of normal and upper semi-continuous fuzzy sets on metric space. It is found that both the Γ-convergence and the endograph metric have level decomposition property on fuzzy sets with their positive α-cuts being compact. On the basis of these results, we give the characterizations of total boundedness, relative compactness and compactness in the space of compact positive α-cuts fuzzy sets equipped with the endograph metric, and in the space of compact support fuzzy sets equipped with the sendograph metric, respectively.

Keywords: Endograph metric; Γ-convergence; Sendograph metric; Hausdorff metric; Compactness; Total boundedness; Relative compactness

1. Introduction

A fuzzy set can be identified with its endograph. Also, a fuzzy set can be identified with its sendograph. So convergence structures on fuzzy sets can be defined on their endographs or sendographs. The Γ-convergence, the endograph metric and the sendograph metric are this kind of convergence structures. In this paper, we discuss the characterizations and the relations of these three convergence structures.

In [5], we presented the level decomposition properties of the Γ-convergence and the endograph metric. Based on this, we have given the characterizations of total boundedness, relative compactness and compactness of fuzzy set spaces equipped with the endograph metric.
The results in [5] are obtained on the realm of fuzzy sets on $\mathbb{R}^m$. $\mathbb{R}^m$ is a special type of metric space. It is natural and important to consider that when the realm is fuzzy sets on a general metric space, what are the characterizations of these convergence structures?

In this paper, our discussion is on the realm of normal and upper semi-continuous fuzzy sets on metric spaces. It is shown that the $\Gamma$-convergence and the endograph metric still have level decomposition properties on compact positive $\alpha$-cuts fuzzy sets. On the basis of these results, we give the characterizations of total boundedness, relative compactness and compactness in the space of compact positive $\alpha$-cuts fuzzy sets equipped with the endograph metric, and in the space of compact support fuzzy sets equipped with the sendograph metric, respectively.

The remainder of this paper is organized as follows. In Section 2, we recall basic notions of fuzzy sets and the $\Gamma$-convergence, the endograph metric and the sendograph metric. We also introduced some subclasses of fuzzy sets. In Section 3, we investigate the level characterizations of the $\Gamma$-convergence. In Section 4, we consider the level characterizations of the endograph metric convergence. In Section 5, on the basis of the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in two classes of fuzzy set spaces equipped with endograph metric and sendograph metric, respectively.

2. Fuzzy sets and convergence structures on them

In this section, we recall basic notions about fuzzy sets and the $\Gamma$-convergence, the endograph metric and the sendograph metric on normal and upper semi-continuous fuzzy sets. We also introduce some subclasses of fuzzy sets, which will be discussed in this paper. Readers can refer to [1, 10] for more contents.

Let $(X, d)$ be a metric space and let $K(X)$ and $C(X)$ denote the set of all non-empty compact subsets of $X$ and the set of all non-empty closed subsets of $X$, respectively.

We use $H$ to denote the **Hausdorff metric** on $C(X)$ induced by $d$, i.e.,

$$H(U, V) = \max\{H^*(U, V), \ H^*(V, U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U, V) = \sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in V} d(u, v).$$
The metric \( \overline{d} \) on \( X \times [0, 1] \) is defined as
\[
\overline{d}((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.
\]
If there is no confusion, we also use \( H \) to denote the Hausdorff metric on \( C(X \times [0, 1]) \) induced by \( \overline{d} \).

The Hausdorff metric has the following important properties.

**Theorem 2.1.** Let \( (X, d) \) be a metric space and let \( H \) be the Hausdorff metric induced by \( d \). Then the following statements are true.

(i) \( (X, d) \) is complete \( \iff \) \( (K(X), H) \) is complete

(ii) \( (X, d) \) is separable \( \iff \) \( (K(X), H) \) is separable

(iii) \( (X, d) \) is compact \( \iff \) \( (K(X), H) \) is compact

Let \( (X, d) \) be a metric space. We say that a sequence of sets \( \{C_n\} \) **Kuratowski converges** to \( C \subseteq X \), if
\[
C = \lim_{n \to \infty} \inf C_n = \lim_{n \to \infty} \sup C_n,
\]
where
\[
\lim_{n \to \infty} \inf C_n = \{x \in X : x = \lim_{n \to \infty} x_n, x_n \in C_n\},
\]
\[
\lim_{n \to \infty} \sup C_n = \{x \in X : x = \lim_{j \to \infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} C_m.
\]
In this case, we’ll write \( C = \lim_{n \to \infty} C_n(Kuratowski) \) or \( C = \lim_{n \to \infty} C_n(K) \) for simplicity.

Let \( F(X) \) denote the set of all fuzzy sets on \( X \). A fuzzy set \( u \in F(X) \) can be seen as a function \( u : X \to [0, 1] \). In this sense, a subset \( S \) of \( X \) can be seen as a fuzzy set
\[
\widehat{S}(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}
\]

For \( u \in F(X) \), let \( [u]_\alpha \) denote the \( \alpha \)-cut of \( u \), i.e.
\[
[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \text{supp } u = \{u > 0\}, & \alpha = 0. \end{cases}
\]

For \( u \in F(X) \), define
\[
\text{end } u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\},
\]
send $u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\} \cap ([u]_0 \times [0, 1])$.

end $u$ and send $u$ are called the endograph and the sendograph of $u$, respectively.

Let $F_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$  

We suppose that all fuzzy sets mentioned in the sequel are in $F_{USC}(X)$.

Rojas-Medar and Román-Flores have introduced the $\Gamma$-convergence on $F_{USC}(X)$:

Let $u, u_n, n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. Then $u_n \overset{\Gamma}{\longrightarrow} u$ to $u$ ($u_n \overset{\Gamma}{\longrightarrow} u$) if

$$\text{end } u = \lim_{n \to \infty} \text{end } u_n (K).$$

The endograph metric $H_{\text{end}}$ and the sendograph metric $H_{\text{send}}$ can be defined on $F_{USC}(X)$ as usual. For $u, v \in F_{USC}(X)$,

$$H_{\text{end}}(u, v) := H(\text{end } u, \text{end } v),$$

$$H_{\text{send}}(u, v) := H(\text{send } u, \text{send } v).$$

The readers can see [1, 5, 7, 10] for related contents.

**Definition 2.2.** Define two subsets of $F_{USC}(X)$ as follows.

$$F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X)\},$$

$$F_{USCG}(X) := \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \text{ when } \alpha > 0\}.$$

$F_{USCB}(X)$ consists of compact support fuzzy sets in $F_{USC}(X)$. $F_{USCG}(X)$ consists of compact positive $\alpha$-cuts fuzzy sets in $F_{USC}(X)$.

Clearly,

$$F_{USCB}(X) \subset F_{USCG}(X) \subset F_{USC}(X).$$

In this paper, we discuss the properties and the relations of the $\Gamma$-convergence, the endograph metric $H_{\text{end}}$ and the sendograph metric $H_{\text{send}}$ on $F_{USCB}(X), F_{USCG}(X)$ and $F_{USC}(X)$.

The set of (compact) fuzzy numbers are denoted by $E^m$. It is defined as

$$E^m := \{u \in F_{USCB}(\mathbb{R}^m) : [u]_\alpha \text{ is a convex subset of } \mathbb{R}^m \text{ for } \alpha \in [0, 1]\}.$$  

Fuzzy numbers have attracted much attention from theoretical research and practical applications.
3. Level characterizations of $\Gamma$-convergence

In this section, we investigate the level characterizations of the $\Gamma$-convergence. It is found that the $\Gamma$-convergence has the level decomposition property on $F_{USCG}(X)$, fuzzy sets in which has compact positive $\alpha$-cuts. It is pointed out that the $\Gamma$-convergence need not have the level decomposition property on $F_{USC}(X)$.

Rojas-Medar and Román-Flores [8] have introduced the following useful property of $\Gamma$-convergence.

**Theorem 3.1.** [8] Suppose that $u, u_n, n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$. Then $u_n \rightharpoonup u$ iff for all $\alpha \in (0, 1]$,
\[
\{u > \alpha\} \subseteq \liminf_{n \to \infty} [u_n]_\alpha \subseteq \limsup_{n \to \infty} [u_n]_\alpha \subseteq [u]_\alpha.
\] (1)

**Remark 3.2.** Rojas-Medar and Román-Flores (Proposition 3.5 in [8]) presented the statement in Theorem 3.1 when $u, u_n, n = 1, 2, \ldots$, are fuzzy sets in $E^m$. It can be checked that this conclusion also holds when $u, u_n, n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$.

**Theorem 3.3.** [3] Let $(X, d)$ be a metric space and let $\{C_n\}$ be a sequence of sets in $X$. Then $\liminf_{n \to \infty} C_n$ and $\limsup_{n \to \infty} C_n$ are closed sets.

**Theorem 3.4.** Suppose that $u, u_n, n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$. Then $u_n \rightharpoonup u$ iff for all $\alpha \in (0, 1]$,
\[
\{u > \alpha\} \subseteq \liminf_{n \to \infty} [u_n]_\alpha \subseteq \limsup_{n \to \infty} [u_n]_\alpha \subseteq [u]_\alpha.
\]

**Proof.** The desired result follows immediately from Theorems 3.1 and 3.3.

Suppose that $u$ and $u_n, n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$.

- A number $\alpha$ in $(0, 1)$ is called a platform point of $u$ if $\{u > \alpha\} \subsetneq [u]_\alpha$.
  The set of all platform points of $u$ is denoted by $P(u)$.

- $P_0(u) := \{\alpha \in (0, 1) : \lim_{\beta \to \alpha} H([u]_\beta, [u]_\alpha) \neq 0\}$.

Clearly, $P(u) \subseteq P_0(u)$. 5
Lemma 3.5. Given \( u \in F_{USCG}(X) \). Then \( P_0(u) = P(u) \) and \( P(u) \) is at most countable.

Theorem 3.6. Suppose that \( u, u_n, n = 1, 2, \ldots, \) are fuzzy sets in \( F_{USC}(X) \). Then the following statements are true.

(i) If \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for \( \alpha \in P \), where \( P \) is a dense set in \((0, 1)\), then \( u_n \xrightarrow{\Gamma} u \).

(ii) If \( u_n \xrightarrow{\Gamma} u \), then \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for all \( \alpha \in (0, 1) \setminus P(u) \).

The following Theorem states the level decomposition property of \( \Gamma \)-convergence on compact positive \( \alpha \)-cuts fuzzy sets in \( F_{USC}(X) \).

Theorem 3.7. Suppose that \( u, u_n, n = 1, 2, \ldots, \) are fuzzy sets in \( F_{USCG}(X) \). Then the following statements are true.

(i) \( u_n \xrightarrow{\Gamma} u \)

(ii) \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for a.e. \( \alpha \in (0, 1) \)

(iii) \([u]_\alpha = \lim_{n \to \infty} [u_n]_\alpha (K)\) for all \( \alpha \in (0, 1) \setminus P(u) \)

(iv) \( \lim_{n \to \infty} [u_n]_\alpha (K) = [u]_\alpha \) holds when \( \alpha \in P \), where \( P \) is a dense subset of \((0, 1) \setminus P(u) \).

(v) \( \lim_{n \to \infty} [u_n]_\alpha (K) = [u]_\alpha \) holds when \( \alpha \in P \), where \( P \) is a countable dense subset of \((0, 1) \setminus P(u) \).

Proof. The desired results follow immediately from Lemma 3.5 and Theorem 3.6.

Remark 3.8. It can be checked that the converse of the implications in statement (i) and statement (ii) of Theorem 3.6 do not hold. So the level decomposition property of \( \Gamma \)-convergence need not hold on \( F_{USC}(X) \).
4. Level characterizations of endograph metric convergence

In this section, we discuss the level characterizations of endograph metric convergence. It is shown that the endograph metric has the level decomposition property on \( F_{\text{USCG}}(X) \). It is pointed out that the endograph metric need not have the level decomposition property on \( F_{\text{USC}}(X) \).

The following are two elementary conclusions which are useful in this paper.

**Proposition 4.1.** Suppose that \( C, C_n \) are sets in \( C(X) \), \( n = 1, 2, \ldots \). Then \( H(C_n, C) \to 0 \) implies that \( \lim_{n \to \infty} C_n(K) = C \)

**Proof.** This is an already known result. Its proof is similar to the that of Theorem 4.1 in [5]. \( \square \)

**Proposition 4.2.** Given \( u, u_n, n = 1, 2, \ldots, \) in \( F_{\text{USC}}(X) \). Then

(i) \( H_{\text{send}}(u_n, u) \to 0 \) is equivalent to \( H_{\text{end}}(u_n, u) \to 0 \) and \( H([u_n], [u_0]) \to 0 \)

(ii) \( \lim_{n \to \infty} \text{send} u_n(K) = \text{send} u \) is equivalent to \( u_n \xrightarrow{\Gamma} u \) and \( \lim_{n \to \infty} [u_n](0) = [u_0] \)

Now we consider the level characterizations of \( H_{\text{end}} \) convergence.

**Theorem 4.3.** Let \( u, u_n, n = 1, 2, \ldots, \) be fuzzy sets in \( F_{\text{USC}}(X) \) and let \( P \) be a dense set in \([0, 1]\). Suppose that \( H([u_n], [u_0]) \to 0 \) for each \( \alpha \in P \). Then \( H_{\text{end}}(u_n, u) \to 0 \).

**Proof.** We proceed by contradiction. \( \square \)

**Remark 4.4.** Fan (Lemma 1 in [3]) proved a result of Theorem 4.3 type.

**Theorem 4.5.** Let \( u, u_n, n = 1, 2, \ldots, \) be fuzzy sets in \( F_{\text{USC}}(X) \). Suppose that \( H_{\text{end}}(u_n, u) \to 0 \). Then \( H([u_n], [u_0]) \to 0 \) for each \( \alpha \in (0, 1) \setminus P_0(u) \).

The following Theorem gives the level decomposition property of \( H_{\text{end}} \) on \( F_{\text{USCG}}(X) \).

**Theorem 4.6.** Suppose that \( u, u_n, n = 1, 2, \ldots, \) are fuzzy sets in \( F_{\text{USCG}}(X) \). Then the following statements are equivalent.

(i) \( H_{\text{end}}(u_n, u) \to 0 \)
(ii) $H([u_n], [u]) \xrightarrow{a.e.} 0$

(iii) $H([u_n], [u]) \to 0$ for all $\alpha \in (0, 1) \setminus P(u)$

(iv) $H([u_n], [u]) \to 0$ when $\alpha \in P$, where $P$ is a dense subset of $(0, 1) \setminus P(u)$

(v) $H([u_n], [u]) \to 0$ when $\alpha \in P$, where $P$ is a countable dense subset of $(0, 1) \setminus P(u)$

Proof. The desired result follows from Lemma 3.5 and Theorems 4.3 and 4.5.

Remark 4.7. It can be checked that the converse of the implications in Theorems 4.3 and 4.5 do not hold. So the level decomposition property of $H_{end}$ convergence need not hold on $F_{USC}(X)$.

5. Characterizations of compactness in $(F_{USCB}(X), H_{end})$ and $(F_{USCG}(X), H_{end})$

Based on the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCB}(X), H_{end})$ and $(F_{USCG}(X), H_{end})$.

5.1. Characterizations of compactness in $(K(X), H)$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(K(X), H)$. The results in this subsection are basis for contents in the sequel.

Theorem 5.1. Suppose that $X$ is complete and that $\{C_n\}$ is a Cauchy sequence in $(K(X), H)$. Let $D_n = \bigcup_{l=1}^{n} C_l$ and $D = \bigcup_{l=1}^{+\infty} C_l$. Then $D \in K(X)$ and $H(D_n, D) \to 0$

Theorem 5.2. Suppose that $(X, d)$ is a metric space. Then $D$ is totally bounded in $(K(X), H)$ is equivalent to $D = \bigcup\{C : C \in D\}$ is totally bounded in $X$

Theorem 5.3. Suppose that $(X, d)$ is a metric space. Then $D$ is relatively compact in $(K(X), H)$ is equivalent to $D = \bigcup\{C : C \in D\}$ is relatively compact in $X$

Lemma 5.4. Suppose that $(X, d)$ is a metric space. If $D$ is compact in $(K(X), H)$, Then $D = \bigcup\{C : C \in D\}$ is compact in $(X, d)$
Remark 5.5. The converse of the implication in Lemma 5.4 does not hold. Let \((X, d) = \mathbb{R}\) and let \(D = \{[0, x] : x \in (0.3, 1]\} \subset K(\mathbb{R}).\) Then \(D = [0, 1] \in K(\mathbb{R}).\) But \(D\) is not compact in \((K(\mathbb{R}), H)\).

Theorem 5.6. Suppose that \((X, d)\) is a metric space. Then \(D\) is compact in \((K(X), H)\) if and only if \(D = \bigcup \{C : C \in \mathcal{D}\}\) is compact in \((X, d)\) and \(D\) is closed.

Proof. The desired result follows from Theorem 5.3 and Lemma 5.4. \(\square\)

Remark 5.7. We are not sure whether Theorems 5.2, 5.3 and 5.6 are already existing results, so we give our proofs here.

5.2. Characterizations of compactness in \((F_{USCG}(X), H_{end})\)

In this subsection, we discuss the characterizations of totally bounded sets, relatively compact sets and compact sets in \((F_{USCG}(X), H_{end})\).

Suppose that \(U\) is a subset of \(F_{USCG}(X)\) and \(\alpha \in [0, 1].\) For writing convenience, we denote

- \(U(\alpha) := \bigcup_{u \in U} [u]_\alpha,\) and
- \(U_\alpha := \{[u]_\alpha : u \in U\}.\)

Theorem 5.8. Let \(U\) be a subset of \(F_{USCG}(X)\). Then \(U\) is totally bounded in \((F_{USCG}(X), H_{end})\) if and only if \(U(\alpha)\) is totally bounded in \(X\) for each \(\alpha \in (0, 1]\)

Theorem 5.9. Let \(U\) be a subset of \(F_{USCG}(X)\). Then \(U\) is relatively compact in \((F_{USCG}(X), H_{end})\) if and only if \(U(\alpha)\) is relatively compact in \(X\) for each \(\alpha \in (0, 1]\)

Theorem 5.10. Let \(U\) be a subset of \(F_{USCG}(X)\). Then the following statements are equivalent.

(i) \(U\) is compact in \((F_{USCG}(X), H_{end})\)

(ii) \(U(\alpha)\) is relatively compact in \(X\) for each \(\alpha \in (0, 1]\) and \(U\) is closed in \((F_{USCG}(X), H_{end})\)

(iii) \(U(\alpha)\) is compact in \(X\) for each \(\alpha \in (0, 1]\) and \(U\) is closed in \((F_{USCG}(X), H_{end})\)

Theorem 5.11. Let \((X, d)\) be a metric space. Then the following statements are equivalent.

(i) \(X\) is complete

(ii) \((F_{USCG}(X), H_{end})\) is complete
5.3. Characterizations of compactness in \((F_{USCB}(X), H_{send})\)

In this subsection, we give the characterizations of totally bounded sets, relatively compact sets and compact sets in \((F_{USCB}(X), H_{send})\).

Even if \((X, d)\) is complete, \((F_{USCB}(X), H_{send})\) need not be complete. Consider

\[
P_{USCB}(X) := \{ u \in X \times [0,1] : [u]_\alpha = \bigcap_{\beta < \alpha} [u]_\beta \text{ for all } \alpha \in (0,1];
\]

\[
[u]_\alpha \in K(X) \text{ for all } \alpha \in [0,1]\};
\]

where \([u]_\alpha := \{x : (x, \alpha) \in u\} \text{ for each } u \in X \times [0,1] \text{ and } \alpha \in [0,1].

Note that \(u \in K(X \times [0,1])\) for each \(u \in P_{USCB}(X)\). We can formally define

\[
H_{send}(u, v) := H(u, v),
\]

\[
H_{end}(u, v) := H(u, v),
\]

where \(u := u \cup (X \times \{0\})\).

\(F_{USCB}(X)\) can be regarded as a subset of \(P_{USCB}(X)\) if we think of each \(u \in F_{USCB}(X)\) as its sendograph. Thus, we can see \((F_{USCB}(X), H_{send})\) as a subspace of \((P_{USCB}(X), H_{send})\).

**Theorem 5.12.** Suppose that \(U\) is a subset of \(F_{USCB}(X)\). Then \(U\) is totally bounded in \((F_{USCB}(X), H_{send})\) if and only if \(U(0)\) is totally bounded in \(X\).

**Theorem 5.13.** Let \((X, d)\) be a metric space. Then the following statements are equivalent.

(i) \(X\) is complete

(ii) \((P_{USCB}(X), H_{send})\) is complete

**Remark 5.14.** In fact, \((P_{USCB}(X), H_{send})\) is the completion of \((F_{USCB}(X), H_{send})\).

- \(U \subset F_{USCB}(X)\) is said to be equi-right-continuous at 0 if for each \(\varepsilon > 0\), there is a \(\delta > 0\) such that \(H([u]_\delta, [u]_0) < \varepsilon\) for all \(u \in U\).

**Theorem 5.15.** Suppose that \(U\) is a subset of \(F_{USCB}(X)\). Then \(U\) is relatively compact in \((F_{USCB}(X), H_{send})\) if and only if \(U(0)\) is relatively compact in \(X\) and \(U\) is equi-right-continuous at 0.
Theorem 5.16. Suppose that $U$ is a subset of $F_{USCB}(X)$. Then the following statements are equivalent.

(i) $U$ is compact in $(F_{USCB}(X), H_{send})$

(ii) $U$ is closed in $(F_{USCB}(X), H_{send})$, $U(0)$ is relatively compact in $X$ and $U$ is equi-right-continuous at 0

(iii) $U$ is closed in $(F_{USCB}(X), H_{send})$, $U(0)$ is compact in $X$ and $U$ is equi-right-continuous at 0
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