New Trial Wave Functions for Quantum Hall States at Half Filling

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New trial wave functions corresponding to half filling quantum Hall states are proposed. These wave functions are constructed by first pairing up the quasielectrons of the 1/3 Laughlin quantum Hall state, with the same relative angular momentum for each pair, and then making the paired quasielectrons condense into a 1/4 Laughlin state. The quasiparticle excitations of the proposed wave functions carry ±1/4 of electron charge, and obey Abelian fractional statistics. In the spherical geometry, the total flux quanta \( N_\phi \) is shown to be related to the number of electrons \( N \) by \( N_\phi = 2N - (5 - q) \) with \( q \) being the relative angular momentum between the quasielectrons in each pair which takes values of non-negative even integers. The overlaps are calculated between the proposed trial wave functions, including the ground state, quasielectron states, and quasihole states, and the exact states of the finite size systems at \( N_\phi = 2N - 3 \). The near unity overlaps are obtained in the lowest Landau level, while the moderate overlaps are obtained in the second Landau level. The relevance of the wave functions to the yet to be discovered fractional quantum Hall effect in the lowest Landau level, as well as the 5/2 quantum Hall effect is also discussed.

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The observed fractional quantum Hall effect (FQHE) at \( \nu = 5/2 \)\(^2\), which deviates from the odd denominator filling factor rule, calls for a new mechanism that is different from its odd denominator filling factor counterparts. The nature of the 5/2 FQHE has been under intensive study recently, in part due to the exotic non-Abelian statistics of the quasiparticle (QP) excitations of the Moore-Read state\(^2\), which is considered to be the leading candidate for the ground state. Physically, this new state can be regarded as a p-wave BCS pairing state, apart from a Jastrow factor that attaches two flux to each of the electrons\(^2\)\(^3\). It has been shown through finite size studies that the MR wave functions have only moderate overlaps with the exact 2/5 ground state\(^3\), in particular with the low energy states\(^3\), in the second Landau level (SLL), although the overlaps can be improved by adjusting the pseudopotentials\(^4\) or fine tuning the finite thickness of the layer that confined the two dimensional (2D) electrons\(^3\).

On the other hand, the MR wave function has a rather poor overlap with the exact state in the lowest Landau level (LLL). While it is generally believed that a 2D electron system at half filling forms a compressible Fermi liquid type state\(^8\) in the LLL, it does not rule out a possibility that under certain conditions when the short range interactions are softened, an incompressible state may emerge. In fact, recent studies have shown such a possibility by including the finite width of the layer confining the 2D electrons\(^3\).

The moderate overlaps of the MR states with the exact states in the SLL, and the poor overlaps with the exact states in the LLL, in contrast with the fact that the Laughlin wave functions find near unity overlaps with their corresponding exact states, justify a continuing pursuit of alternatives to the MR states. It is noted that the MR’s particle-hole conjugate state and the superposition of both have also been studied recently\(^10\)\(^11\)\(^12\).

In this paper, new trial wave functions corresponding to half filling quantum Hall effect are proposed. These wave functions are constructed by first pairing up the quasielectrons of the 1/3 Laughlin quantum Hall state, requiring the same relative angular momentum for each pair, and then making the paired quasielectrons condense into a 1/4 Laughlin state. Hereafter, we will use the notation LQE refering to the quasielectrons of the Laughlin state, and PQE to the paired quasielectrons of the Laughlin state. It is shown that the proposed wave functions support their own QP excitations, which carry ±1/4 of electron charge, and obey ±π/4 Abelian fractional statistics. The new wave functions are explicitly constructed in the spherical geometry, with the total number of flux quanta \( N_\phi \) and the number of electrons \( N \) related by \( N_\phi = 2N - (5 - q) \), where \( q \) is the relative angular momentum between the LQEs in each pair, and takes values of zero and positive even integers. The wave functions of the low energy states at half filling (quasielectron states) as well as away from half filling (quasiparticle states), are also constructed explicitly. Their overlaps with the exact states of finite systems with \( N_\phi = 2N - 3 \) using the exact diagonalization techniques are calculated. The near unity overlaps in the lowest Landau level (LLL) are obtained. Their overlaps with the exact states in the SLL are found to be moderate. This is in contrast with the MR wave function, where it finds a larger overlap with the exact state in the SLL over the LLL.

It should be pointed out that the idea of pairing up the LQEs and forming their own Laughlin state at half filling was proposed before by the author, Su and Su\(^13\), and also discussed by others\(^14\). However, the exact mechanism to pair up the LQEs, the dependence of the
$N_\phi$-$N$ relationship on the relative angular momentum of the LQEs in each pair, the explicit construction of the ground state wave functions, the quasiexciton wave functions, and the 1/4 charged quasiparticle wave functions, as well as their overlaps with the exact states of the finite size systems are all new contributions in this paper.

In Haldane’s spherical geometry, the Laughlin wave function can be written as [4]:

$$\Phi_{mL} = \prod_{i<j} (u_i v_j - v_i u_j)^{m_l},$$

(1)

where $(u, v)$ are the spinor variables describing electron coordinates, and $m_L$ is an odd integer as the result of the Pauli exclusion principle. The total flux quanta of $N$ by an amount of $\frac{\pi}{\Omega}$ for the low energy states from a single LQE to multiple LQEs in each pair, the explicit construction of the ground state wave functions, the quasiexciton wave functions Eq. (2) provide a rather exact description with $(\alpha, \beta)$ being the spinor variables describing the LEQ coordinates. It has been shown numerically that the LQE wave functions Eq. (2) provide a rather exact description for the low energy states from a single LQE to multiple LQEs, and all the way to the 2/5 filling factor, at which a new hierarchy state is formed when the LQEs form their own Laughlin state [10]. This is in part due to the fact that a LQE behaves the same as a charged particle in its own LLL with the total flux quasi equal to the total number of the underlying electrons $N$. This can be best seen by expanding $S^{qe}(\alpha, \beta)$ in terms of $(\alpha, \beta)$:

$$S^{qe}(\alpha, \beta) = \prod_{j=1}^{N} (\beta^* \frac{\partial}{\partial u_j} - \alpha^* \frac{\partial}{\partial v_j}),$$

(3)

where $\psi_{\frac{\pi}{\Omega}, m}(\alpha, \beta)$ is the LLL wave function of a particle which sees the total flux quanta

$$G_{\frac{\pi}{\Omega}, m}^{qe} = (-1)\frac{N}{2} m [\frac{N!}{(\frac{N}{2} + m)! (\frac{N}{2} - m)!}]^{-1/2} \sum_{1 \leq l_1 < l_2 < \ldots \leq \frac{N}{2} + m} \frac{\partial}{\partial v_{l_1}} \frac{\partial}{\partial v_{l_2}} \ldots \frac{\partial}{\partial v_{\frac{N}{2} + m}} \prod_{j \neq l_1, l_2, \ldots, \frac{N}{2} + m} \frac{\partial}{\partial u_j}.$$ 

(5)

When applied to the Laughlin wave function, $G_{\frac{\pi}{\Omega}, m}^{qe}$ will generate a LQE with angular momentum $(L, L_z) = (\frac{N}{2}, m)$.

Now we come to the main part of this paper. Suppose we have an even number of LQEs generated:

$$(G_{\frac{\pi}{\Omega}, m_1}^{qe} G_{\frac{\pi}{\Omega}, m_2}^{qe}) (G_{\frac{\pi}{\Omega}, m_3}^{qe} G_{\frac{\pi}{\Omega}, m_4}^{qe}) \ldots (G_{\frac{\pi}{\Omega}, m_{N_{qe}-1}}^{qe} G_{\frac{\pi}{\Omega}, m_{N_{qe}}}^{qe}) \Phi_{mL}.$$ 

(6)

In the above equation, we have bracketed the $G_{\frac{\pi}{\Omega}, m}^{qe}$ in pair, as we would like to pair up $m_1$ with $m_2$, $m_3$ with $m_4$, ..., and $m_{N_{qe}-1}$ with $m_{N_{qe}}$. Since the angular momentum for each LQE is $N/2$, the total angular momentum for each pair is therefore $L_p = N - q$ with $q$ being a non-negative even integer representing a relative angular momentum of the LQE pair. The following wave function:

$$G_{L_p, p_1}^{qe} G_{L_p, p_2}^{qe} \ldots G_{L_p, p_{N_p}}^{qe} \Phi_{mL},$$

(7)

where

$$G_{L_p, p}^{qe} = \sum_{m_{1}, m_{2}} < L_p, p | \frac{N}{2} m_{1}; \frac{N}{2} m_{2} > G_{\frac{\pi}{\Omega}, m_{1}}^{qe} G_{\frac{\pi}{\Omega}, m_{2}}^{qe},$$

(8)

and $< L_p, p | \frac{N}{2} m_{1}; \frac{N}{2} m_{2} >$ being Clebsch-Gordon coefficients, describe $N_p = N_{qe}/2$ PQEs, each LQE has an angular momentum $L_p = N - q$ with it’s $z$ component $p$ taking values from $-L_p$ to $L_p$. Similar to Eq. (4), one can define a new operator corresponding to $G_{L_p, p}^{qe}$ as

$$S^{qe}(\xi, \eta) = \sum_{p = -L_p}^{L_p} \psi_{L_p, p}(\xi, \eta) G_{L_p, p}^{qe},$$

(9)

where $\psi_{L_p, p}(\xi, \eta)$ is the LLL wave function of a particle which sees the total flux quanta $2L_p = 2(N - q)$.

It is the key assumption of this paper that the PQEs described by $S^{qe}(\xi, \eta)$ condense into their own Laughlin state to form a new state of the system, described by the following wave functions:

$$\int d\Omega \prod_{1 \leq a < b \leq N_p} \Phi_{a} \Phi_{b} \sum_{a=1}^{N_p} S^{qe}(\xi_a, \eta_a) \Phi_{mL}.$$ 

(10)

where $d\Omega = \prod_{a=b}^{N_p} d\Omega_a = \prod_{a=1}^{N_p} \sin(\partial_a) d\partial_a d\varphi_a$ if we let the spinor variables $(\xi, \eta) = (\cos(\partial/2)e^{i\varphi/2}, \sin(\partial/2)e^{-i\varphi/2})$, and $n$ is a positive integer. The filling factor of the electron system described by Eq. (10) can be derived by the the equation between $N_\phi$ and $N_p$:

$$N_\phi = m_L(N - 1) - 2N_p,$$

(11)

which describes $2N_p$ LQE created on top of the $1/m_L$ Laughlin state, and that between the total number of the flux quanta seen by the PQEs $2(N - q)$ and $N_p$:

$$2(N - q) = 2n(N_p - 1),$$

(12)

which describes the PQEs condense into their own $1/(2n)$ Laughlin state. Combining these two equations and setting $m_L = 3$ and $n = 2$, we arrive at:

$$N_\phi = 2N - (5 - q).$$

(13)
which gives 1/2 filling factor in the thermodynamic limit. The "shift" defined as \( S = 2N - N_q \) is given by \( 5 - q \), which can take values 5, 3, 1, etc., corresponding to the relative angular momentum 0, 2, 4, etc. The case of \( S = 3 \) is of particular interest for two reasons. First of all, it is widely believed to be responsible for the 5/2 FQHE observed experimentally [4]. Secondly, from the potential energy point of view, it seems more favorable to have a non-zero relative angular momentum between the LEQs in each pair to avoid them on top of each other (which corresponds to the zero relative angular momentum, i.e., the shift \( S = 5 \)). On the other hand, the relative angular momentum which corresponds to the separation of the LEQs in each pair shall not be too large in order to justify their condensation into a Laughlin state. The relative angular momentum \( q = 2 \) which corresponds to the shift \( S = 3 \) is therefore seems to be an optimal choice, although there is no solid reason to rule out other choices of the relative angular momentum.

The charge of the QPs supported by the wave functions Eq. (10) can be calculated by starting with Eq. (11) and increasing (decreasing) the number of electrons by 2 in order to keep all the LQEs paired up, and examining the total number of the shortness (excess) of the flux quanta seen by the PQEs in Eq. (12), which turns out to be 8. This corresponds to \( \pm 1/4 \) of the electron charge. The statistics is the Abelian fractional statistics \( \pm \pi/4 \).

It is well known that the Laughlin state supports a quasiexciton as its low energy excitations. By the same reasoning we propose the following quasiexciton wave functions to describe the low energy excitations at half filling:

\[
\int d\Omega s^{\text{exc}}(\mu_1, \nu_1)s^{\text{qh}}(\mu_2, \nu_2)\prod_{1 \leq a < b \leq N_p}(\xi_a \eta_b - \xi_b \eta_a)^{2n} \prod_{a=1}^{N_p} S^{\text{PQE}}(\xi_a, \eta_a)\Phi_{mL}.
\]

(14)

where \( s^{\text{exc}}(\mu, \nu) \) is the quasiexciton operator of the PQE Laughlin state, similar to Eq. (13):

\[
s^{\text{exc}}(\mu, \nu) = \prod_{a=1}^{N_p} (\nu^a \partial/\partial \xi_a - \mu^a \partial/\partial \eta_a).
\]

(15)

and \( s^{\text{qh}}(\mu, \nu) \) is the quasiholo operator of the PQE Laughlin state,

\[
s^{\text{qh}}(\mu, \nu) = \prod_{a=1}^{N_p} (\nu \xi_a - \mu \eta_a).
\]

(16)

where the spinor \((\mu, \nu)\) represents the coordinates of the quasiexciton or the quasiho of the PQE Laughlin state. The angular momentum described by Eq. (14) is from 1, 2, ..., to \( N_p \).

It can be shown that the minimum number, which is 2, of 1/4 fractionally charged QHs can be created when \( N \) is an odd number and the flux quanta \( N_q \) is increased by 1 from Eq. (13). The corresponding wave function can be described by

\[
\int d\Omega s^{\text{exc}}(\mu_1, \nu_1)s^{\text{qh}}(\mu_2, \nu_2)\prod_{1 \leq a < b \leq N_p}(\xi_a \eta_b - \xi_b \eta_a)^{2n} \prod_{a=1}^{N_p} S^{\text{PQE}}(\xi_a, \eta_a)\Phi_{mL}.
\]

(17)

In the angular momentum space, it forms independent states with the total angular momentum equal to \( N_p \), \( N_p - 2 \), ..., and 1.

In order to validate the proposed wave functions, we have calculated the overlaps of the ground state wave functions Eq. (10), quasiexciton wave functions Eq. (14), and the two QH wave functions Eq. (17) with the exact states of the finite systems at \( N_q = 2N - 3 \) using the exact diagonalization technique in the spherical geometry. In Fig. 1(a), we plot the energy spectrum in an arbitrary units of a \((N_q, N) = (9, 6)\) finite system in the LLL versus angular momentum \( L \). The numbers (0.9980 and 0.9676) on top of the two energy bars are the overlaps of the ground state wave function described by Eq. (10) and the exciton wave function described by Eq. (17) with the corresponding exact states at \( L = 0 \) and \( L = 2 \), respectively. The overlaps are near unity. It should be noted, while Eq. (14) also provides angular momentum state at \( L = 3 \), there is only one \( L = 3 \) state in the \((N_q, N) = (9, 6)\) finite system where the overlap is trivially equal to 1, and is therefore not shown in the figure.

In Fig. 1(b), we plot the energy spectrum for \((N_q, N) = (12, 7)\), which corresponds to two 1/4 fractionally charged QHs in the LLL. In this case, Eq. (17) describes two states in the \( L_z = 0 \) sector with the angular momentum equal to 1 and 3. The overlaps of them with the exact states are shown on top of the corresponding energy bars. Again, exceedingly large overlaps are obtained.

We have also calculated overlaps for the finite systems shown in Fig. 1(a) and 1(b) by varying the hard-core pseudopotential component \( V_1 \), while keeping other pseudopotential components at their LLL Coulomb values. When \( V_1 \) is changed by a factor of 0.8, 0.85, 0.9, and 1.2 from its LLL Coulomb value, the overlap in Fig. 1(a) at \( L = 0 \) changes to 0.9861, 0.9861, 0.9948, and 0.9996, respectively. The overlap in Fig. 1(a) at \( L = 2 \) changes to 0.9856, 0.9878, 0.9762, and 0.9587, respectively. The overlap in Fig. 1(b) at \( L = 1 \) changes to 0.9280, 0.9916, 0.9976, and 0.9938, respectively. The overlap in Fig. 1(b) at \( L = 3 \) changes to 0.7970, 0.9833, 0.9946, and 0.9899, respectively. In general, the results show that the large overlaps have maintained until \( V_1 \) decreased below around 80% of its corresponding Coulomb value, manifesting the importance of the short range interaction to the validity of the proposed wave functions. This shows a competing requirement towards the formation of the incompressible FQHE state where a certain degree of softening of the short range interactions is needed. Therefore a delicate balance to meet both requirements
is important and that’s probably one of the reasons why the FQHE at half filling in the LLL has not yet been observed so far.

We have also calculated the overlaps in the SLL. They are 0.6730 and 0.8608 at $L = 0$ and $L = 2$ in Fig. 1(a), and 0.8074 and 0.6382 at $L = 1$ and $L = 3$ in Fig. 1(b). These rather moderate overlaps are comparable to the performance of the MR states, making them an alternative candidate to explain the observed 5/2 FQHE.

Finally, we would like to briefly discuss situations when there is one LQE left unpaired. This can happen when the total flux quanta deviates from Eq. (13) by 1 and $N$ is an even number. In this case, the wave function is simply obtained by applying the LQE operator Eq. (3) to Eq. (10). The unpaired situation can also happen when Eq. (13) is satisfied but $N$ is an odd number. In this case, there will be a LQE left unpaired. There will also be two 1/4 fractionally charged quasiholes created. The corresponding wave functions can be obtained by simply applying the LQE operator Eq. (3) to Eq. (17).

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