On infinitely divisible distributions with polynomially decaying characteristic functions

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Abstract

We provide necessary and sufficient conditions on the characteristics of an infinitely divisible distribution under which its characteristic function $\varphi$ decays polynomially. Under a mild regularity condition this polynomial decay is equivalent to $1/\varphi$ being a Fourier multiplier on Besov spaces.

Keywords: Deconvolution operator, Fourier multiplier theorem, Lévy process, regular density, self-decomposable distribution.

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1 Introduction

Any infinitely divisible distribution (IDD) $\mu$ is uniquely determined by its characteristic triplet $(\sigma^2, \gamma, \nu)$ where $\sigma^2 > 0$ is the diffusion coefficient, $\gamma \in \mathbb{R}$ is a drift parameter and $\nu$ is the so-called Lévy measure. By the Lévy-Khintchine formula the characteristic function of $\mu$ is given by

$$\varphi(u) := \mathcal{F}_\mu(u) := \int e^{iux} \mu(dx) = \exp \left( -\frac{\sigma^2}{2} u^2 + i\gamma u + \int (e^{iux} - 1 - iux\mathbb{1}_{[-1,1]}(x)) \nu(dx) \right), \quad u \in \mathbb{R}. $$

The key question of this article is under which conditions on the characteristic triplet $|\varphi(u)|$ decays polynomially for $|u| \to \infty$. Note that, first, $|\varphi(u)|$ does not depend on $\gamma$ and, second, if $\sigma^2 > 0$ then $|\varphi(u)|$ is of the order $e^{-\sigma^2 u^2/2}$. Hence, we basically have to study the interplay between the Lévy measure $\nu$ and the behavior of $|\varphi(u)|$ as $|u| \to \infty$.

Due to the connection between the decay of a characteristic function $\varphi$ and the regularity of the transition density of the corresponding Lévy processes established by Hartman and Wintner (1942), upper bounds for $|\varphi|$ have attracted a certain interest in the literature. Orey (1968); Kallenberg (1981); Knopova and Schilling (2013) study necessary and sufficient conditions for an exponential decay of $\varphi$ and thus for the existence of infinitely smooth transition densities. The less regular case of polynomially decaying characteristic functions is essentially studied for self-decomposable distributions only. Here, the existence of polynomial upper bounds is in detail analyzed, see Sato (1999, Chap. 28) and references therein.

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While upper bounds for $|\varphi|$ are more interesting from the probabilistic perspective, lower bounds are highly relevant from a statistical point of view. Surprisingly, polynomially lower bounds are only known for several explicit parametric classes of IDD, for instance, the family of Gamma distributions. Trabs (2014b) has established a polynomial lower bound for a class of Lévy processes which is slightly larger than self-decomposable processes.

Inspired by the results on self-decomposable distributions, we will show that under a mild regularity assumption on $\nu$ in a neighborhood of the origin, $\varphi$ decays polynomially if and only if there is no diffusion component and the Lebesgue density of $x\nu(dx)$ (which we assume to exist at zero) is bounded. From the degree of the polynomial decay we conclude how many continuous derivatives the probability density admits and we will show that this number is sharp in the sense that there cannot be more derivatives, generalizing the result on self-decomposable distributions by Wolfe (1971).

Let us illustrate the statistical interest in lower bounds on $|\varphi|$ in prototypical deconvolution model. We observe an i.i.d. sample

$$Y_i = X_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

where the target population $X_1, \ldots, X_n$ is corrupted by independent additive noise $\varepsilon_i$. In many applications the error $\varepsilon_i$ can be understood as an aggregation of many small independent influences. Therefore, IDD build a natural class of error distributions since they can be characterized as the class of limit distributions of the sum of independent, uniformly asymptotically negligible random variables. Indeed, popular examples like the normal or the Laplace distribution are IDD. As shown by Fan (1991) nonparametric convergence rates for estimating the distribution of $X_i$ depend on the decay of the characteristic function of $\varepsilon_i$. In particular, a polynomial decay corresponds to mildly ill-posed estimation problems allowing polynomial convergence rates which is much faster than the logarithmic rates in the case of an exponential decay. Due to the auto-deconvolution structure of discretely observed Lévy processes reported by Belomestny and Reiß (2006), estimating the characteristic triplet of a Lévy process depends on the decay of the characteristic function of the marginal IDD, too.

Since the observations $Y_i$ are distributed according to the convolution of the distributions of $X_i$ and $\varepsilon_i$, we have to divide in the Fourier domain the (estimated) characteristic function of $Y_i$ by the characteristic function of $\varepsilon_i$ to assess the distribution of $X_i$. This spectral approach gives raise to the map $f \mapsto \mathcal{F}^{-1}[\mathcal{F}f/\varphi]$. Slightly abusing notation, we consequently denote $\mathcal{F}^{-1}[1/\varphi]$ as the deconvolution operator which has a prominent role in the statistical analysis of the deconvolution model and related models. To analyze its mapping properties, the Fourier multiplier approach by Nickl and Reiß (2012) is extremely useful. Studying a Lévy process model, they have shown under certain sufficient assumptions that $1/\varphi$ is a Fourier multiplier on Besov spaces. We refer to Triebel (2010) for definitions and properties of the Besov spaces $B^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}, p, q \in [1, \infty]$. We say $1/\varphi$ satisfies the Fourier multiplier property if there exists some $\alpha > 0$ such that for all $s \in \mathbb{R}, 1 \leq p, q \leq \infty$ the linear map

$$B^{s+\alpha}_{p,q}(\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1}\left[\frac{\mathcal{F}f}{\varphi}\right] \in B^{s}_{p,q}(\mathbb{R})$$

is bounded. In the deconvolution model the Fourier multiplier approach and the closely related pseudo-differential operators were exploited in a series of recent papers Söhl and Trabs (2012); Schmidt–Hieber et al. (2013); Dattner et al. (2014); as well as Trabs (2014a) for an overview. Minimal conditions on the IDD which imply the Fourier multiplier property are desirable to be able to apply this approach in a wide area of models. It turns out that this mapping property is very natural in the
context of IDDs: We show that a polynomial decay of the characteristic function is necessary and (under a mild regularity condition) sufficient to conclude that $1/\varphi$ is a Fourier multiplier on Besov spaces.

2 Polynomial decay of the characteristic function

Before we precisely state our results, let us introduce some notation. The space of finite signed Borel measures on the real line will be denoted by $\mathcal{M}(\mathbb{R})$. For any $\mu \in \mathcal{M}(\mathbb{R})$ there are two positive finite measures $\mu^+, \mu^-$ such that $\mu(A) = \mu^+(A) - \mu^-(A)$ for any $A \in \mathcal{B}(\mathbb{R})$. Using this so-called Jordan-decomposition, the total variation norm on $\mathcal{M}(\mathbb{R})$ is defined by

$$\|\mu\|_{TV} := \mu^+(\mathbb{R}) + \mu^-(\mathbb{R}).$$

If a function $f: \mathbb{R} \to \mathbb{R}$ is locally integrable, it defines a distribution $T_f(\psi) = \int \psi f$ on the test function space $\mathcal{D}(\mathbb{R})$ of infinitely smooth functions with bounded support. If the distributional derivative of $T_f$ is a finite signed measure $\rho \in \mathcal{M}(\mathbb{R})$, the function $f$ is called (weakly) differentiable with derivative $Df := \rho$. The space of functions of bounded variation is then defined by

$$BV(\mathbb{R}) := \{f: \mathbb{R} \to \mathbb{R} \text{ locally integrable, } Df \in \mathcal{M}(\mathbb{R})\}$$

with bounded variation norm $\|f\|_{BV} := \|Df\|_{TV}$ for $f \in BV(\mathbb{R})$. The measure $Df$ satisfies $Df((a,b)) = f(b+) - f(a+)$ for $-\infty < a < b < \infty$. We will write $\lesssim, \gtrsim$ for inequalities up to constants.

Let $\mu$ be an IDD with characteristic triplet $(\sigma^2, \gamma, \nu)$. Defining the symmetrized jump measure by

$$\nu_s(A) := \nu(A) + \nu(-A)$$

for any Borel set $A \in \mathcal{B}(\mathbb{R}_+)$, the absolute value of the characteristic function is given by

$$|\varphi(u)| = \exp\left(-\frac{\sigma^2 u^2}{2} + \int (\cos(ux) - 1)\nu(\mathbb{R})\right)$$

The following proposition is inspired by the behavior of self-decomposable distributions. To prove it, we shall generalize Lemma 2.1 by Trabs (2014) and its counterpart Lemma 53.9 in Sato (1999). It turns out that a polynomial decay of the characteristic function holds true for a class of IDDs which is much larger than self-decomposable distributions.

**Proposition 1.** If $\sigma^2 = 0$ and $x\nu_s(dx)$ admits on an interval $[0, \delta]$, for some $\delta > 0$, a Lebesgue density $k_s \in BV([0, \delta])$, then for any $\varepsilon > 0$

$$(1 + |u|)^{-\alpha - \varepsilon} \lesssim |\varphi(u)| \lesssim (1 + |u|)^{-\alpha + \varepsilon} \quad \text{with} \quad \alpha := k_s(0+).$$

If, moreover, $\int_0^\delta \log(y)(Dk_s)^+(dy) < \infty$, then $|\varphi(u)| \gtrsim (1 + |u|)^{-\alpha}$.

**Proof.** Using

$$|\varphi(u)| = \exp\left(\int_0^\infty \frac{\cos(ux) - 1}{x}k_s(x)dx\right)$$

and the symmetry of the cosine function, we can assume $u > 0$ without loss of generality. We denote $\overline{\rho} = Dk_s$ and let $\tau \in (0, \delta \wedge 1)$. Since $\|k_s 1_{[0, \delta]}\|_\infty \lesssim \|k_s 1_{[0, \delta]}\|_{BV} < \infty$,
∞, we estimate for $u \leq 1/\tau$

$$1 \geq |\varphi(u)| = \exp\left(\int_0^\delta \frac{\cos(ux) - 1}{x} k_s(x)dx + \int_\delta^\infty (\cos(ux) - 1) \nu_s(dx)\right)$$

$$\geq \exp\left(\|k_s\|_{L^\infty([0,\delta])} \int_0^u \cos x - \frac{1}{x}dx - 2 \int_\delta^\infty \nu_s(dx)\right)$$

$$\geq \exp\left(-\|k_s\|_{L^\infty([0,\delta])} \sup_{v \in (0,\delta/\tau]} \int_0^v \frac{1 - \cos x}{x}dx - 2 \int_\delta^\infty \nu_s(dx)\right),$$

where the last line is a positive constant independent of $u$. It remains to consider the tail behavior of $|\varphi(u)|$ for $u > 1/\tau$.

To show the upper bound of $|\varphi(u)|$ for $u > 1/\tau$. We use Fubini’s theorem and the finite constant $c_1 := \sup_{v \geq 1} \int_1^v \cos x \, dx > 0$ to estimate

$$|\varphi(u)| \leq \exp\left(\int_{1/u}^\tau \frac{\cos(ux) - 1}{x} k_s(x)dx\right)$$

$$= \exp\left(\int_{1/u}^\tau \frac{\cos(ux) - 1}{x} \int_{1/u}^x \tilde{p}(dy)dx + k_s(\frac{1}{u}+)\int_{1/u}^\tau \frac{\cos x - 1}{x}dx\right)$$

$$= \exp\left(\int_{1/u}^\tau \frac{\cos(ux) - 1}{x} \int_{1/u}^x \tilde{p}(dy)dx + k_s(\frac{1}{u}+)\int_{1/u}^\tau \frac{\cos x - 1}{x}dx - \log(\tau u)\right)$$

$$\leq \exp\left(2 \int_{1/u}^\tau \left(\log \tau + \log(\frac{u}{\tau})\right)\tilde{p}^-(dy) - (\log u)k_s(\frac{1}{u}+)\right.$$

$$\left.+(c_1 + \log(\frac{1}{u}))k_s(\frac{1}{u}+)\right)$$

$$\leq \exp\left(- (\log u)k_s(\frac{1}{u}+) - 2 \int_{1/u}^\tau \tilde{p}^-(dy) + (c_1 + \log(\frac{1}{u}))k_s(\frac{1}{u}+)\right). \quad (3)$$

Hence, for any $\varepsilon > 0$ we find some $\tau \in (0,\delta)$ which is sufficiently small such that $|\varphi(u)| \leq u^{-(\alpha - \varepsilon)}$ for all $u > 1/\tau$.

To verify the lower bound, we decompose for $\tau \in (0,\delta)$ and $u > 1/\tau$

$$|\varphi(u)| = \exp\left(\int_0^{1/u} + \int_{1/u}^\tau \frac{\cos(ux) - 1}{x} k_s(x)dx + \int_\tau^\infty (\cos(ux) - 1) \nu_s(dx)\right), \quad (4)$$

where the three integrals will be bounded separately from below. Using $\|k_s\|_{[0,\delta]} \|_\infty < \infty$, we estimate the first integral by

$$\int_0^{1/u} \frac{\cos(ux) - 1}{x} k_s(x)dx \geq \|k_s\|_{[0,\delta]} \|_\infty \int_0^{1/u} \cos x - \frac{1}{x}dx,$$

where the integral on the right-hand side is a negative finite constant independent of $u$. The third integral in (4) can be bounded by

$$\int_\tau^\infty (\cos(ux) - 1) \nu_s(dx) \geq -2 \int_\tau^\infty \nu_s(dx).$$

It remains to estimate the second integral, where we proceed similarly to the upper
bound. We obtain with the finite constant $c_2 := \inf_{v \geq 1} \int_1^v \frac{\cos x}{x} dx < 0$

\[
\int_{1/u}^\tau \frac{\cos (ux) - 1}{x} k_s(x) dx \\
= \int_{1/u}^\tau \int_y^\tau \frac{\cos (ux) - 1}{x} dx \tilde{\nu}(dy) + k_s(\frac{1}{u} +) \left( \int_1^{\tau u} \frac{\cos x}{x} dx - \log(\tau u) \right) \\
\geq -2 \int_{1/u}^\tau (\log \tau + \log(y^{-1})) \tilde{\nu}^+(dy) - k_s(\frac{1}{u} +) \log u + c_2 k_s(\frac{1}{u} +) \\
= -(\log u) \left( k_s(\frac{1}{u} +) + 2 \int_{1/u}^\tau \tilde{\nu}^+(dy) \right) + c_2 k_s(\frac{1}{u} +).
\]

This yields $|\varphi(u)| \gtrsim u^{-(\alpha + \varepsilon)}$ for any $\varepsilon > 0$ and for all $u > 1/\tau$ with $\tau$ sufficiently small.

The addendum follows from the previous estimates, the additional assumption and

\[
\int_{1/u}^\tau \frac{\cos (ux) - 1}{x} k_s(x) dx \\
= \int_0^\tau \int_y^\tau \frac{\cos (ux) - 1}{x} dx \tilde{\nu}(dy) + k_s(0+) \left( \int_1^{\tau u} \frac{\cos x}{x} dx - \log(\tau u) \right) \\
\geq -2 \int_0^\tau \log(y^{-1}) \tilde{\nu}^+(dy) - \alpha \log u + c_2 \alpha.
\]

\[\square\]

**Example.** If $\mu$ is a self-decomposable distribution, then the Lévy measure admits always a Lebesgue density of the form $\nu(dx) = \frac{k(x)}{|x|} dx$ where the so-called $k$-function $k$ is increasing on the negative half line and decreasing on the positive half line. Hence, $k_s, \mathbb{1}_{(0, \delta]} \in BV(\mathbb{R})$ is equivalent to $\|k\|_{\infty} < \infty$. The condition $\int_0^\tau \log(y^{-1})(Dk_s)^+(dy) < \infty$ is always satisfied for self-decomposable distributions owing to $(Dk_s)^+ = 0$. As indicated by Trabs (2014b) it is more generally sufficient if the quotient $(k_s(x+ - \alpha)/x$ is bounded from above uniformly in $x \in (0, \varepsilon]$ for some $\varepsilon > 0$, meaning that the largest slope of $k_s$ near zero is bounded, to obtain the sharp bound of the decay rate.

With Proposition 3 at hand we can characterize IDDs with polynomially decaying characteristic functions under the following regularity assumption on $\nu$ near the origin.

**Assumption 2.** Assume that $x \nu_s(dx)$ admits on a small interval $(0, \delta]$, for some $\delta \in (0, 1)$, a Lebesgue density $k_s$ with $k_s \in BV([\varepsilon, \delta])$ for all $\varepsilon \in (0, \delta)$. Suppose that $\|(Dk_s)^+\|_{TV([\varepsilon, \delta])} = \lim_{\varepsilon \downarrow 0} \|(Dk_s)^+\|_{TV([\varepsilon, \delta])} < \infty$.

Assumption 2 basically excludes Lévy measures which oscillate at zero or have additional singularities in any neighborhood of the origin. Both possibilities are not natural in applications, for instance, for the deconvolution setting or the modeling of stochastic processes via Lévy processes.

**Theorem 3.** Let $\mu$ be an infinitely divisible distribution satisfying Assumption 2. Then the following are equivalent:

(i) $\sigma^2 = 0$ and $\|k_s\|_{BV([\varepsilon, \delta])} < \infty$,

(ii) $\sigma^2 = 0$ and $\lim_{x \to 0} k_s(x) < \infty$,
(iii) there is some $\alpha > 0$ and some constant $c > 0$ such that $|\varphi(u)| \geq c(1 + |u|)^{-\alpha}$ for all $u \in \mathbb{R}$.

**Proof.** Denoting $\tilde{\rho} := Dk$, we define the monotone decreasing functions $k^+_s(x) := \int_x^\delta \tilde{\rho}^+(dy)$ and $k^-_s(x) := \int_x^\delta \tilde{\rho}^-(dy)$ for $x \in (0, \delta)$. To verify equivalence of (i) and (ii), we conclude from $k_s(\delta+) - k_s(x+) = k^+_s(x) - k^-_s(x)$ that

$$\|k_s\|_{BV([0,\delta])} - \|\tilde{\rho}^+\|_{TV([0,\delta])} = \|\tilde{\rho}^-\|_{TV([0,\delta])} = \sup_{x \in (0,\delta]} k^-_s(x) = \sup_{x \in (0,\delta]} \{k_s(x+) + k^+_s(x) - k^-_s(x)\}.$$

Hence, $\|k_s\|_{BV([0,\delta])} < \infty$ if and only if $k_s(0+) = \infty$, owing to $0 \leq k^+_s(0+) = \|\tilde{\rho}^+\|_{TV([0,\delta])}$.

The inclusion (i)$\Rightarrow$(iii) immediately follows from the lower bound in Proposition 1.

To show (iii)$\Rightarrow$(i), we first note that if $\sigma^2 > 0$ then $|\varphi(u)| \lesssim e^{-c\sigma^2}$, $u \in \mathbb{R}$, for some constant $c > 0$ which contradicts (iii). So let $\sigma^2 = 0$ and $u > 0$ without loss of generality. We deduce similarly to (2) and for any $\tau \in (0,\delta)$ and any $u > 1/\tau$

$$|\varphi(u)| \leq \exp\left( \int_{1/\tau}^\tau \int_y^\tau \frac{1 - \cos(ux)}{x} dx \tilde{\rho}^-(dy) + k_s(\frac{1}{\delta} +) \int_1^{\tau u} \frac{1}{x} \cos(x) - 1 dx \right)$$

$$\leq \exp\left( - \int_1^{\tau u} \frac{1 - \cos(x)}{x} dx \left(k_s\left(\frac{1}{\delta}\right) - \int_1^{1/\tau} \tilde{\rho}^-(dy)\right) \right)$$

$$= \exp\left( - \int_1^{\tau u} \frac{1 - \cos(x)}{x} dx \left(k_s(\delta+) - k^+_s\left(\frac{1}{\delta}\right) + k^-_s(\tau)\right)\right). \quad (5)$$

Using

$$\int_1^{\infty} \frac{1 - \cos(x)}{x} dx = \log y + \int_y^{\infty} \frac{\cos x}{x} dx + c_1 \quad \text{with} \quad c_1 := - \int_1^{\infty} \frac{\cos x}{x} dx > 0$$

and $\lim_{y \to \infty} \int_y^{\infty} \frac{\cos x}{x} dx = 0$, we find a function $f: [1, \infty) \to \mathbb{R}$ such that for some $T > 1$

$$\int_1^{2T} \frac{1 - \cos(x)}{x} dx = f(y) \log y \quad \text{and} \quad f(y) \in (\frac{1}{2}, 2) \quad \text{for all} \quad y \geq T.$$

Combining the lower bound on $|\varphi(u)|$ in Condition (iii) and the upper bound (5), we conclude

$$\log c - \alpha \log 2 - \alpha \log u \leq \log |\varphi(u)| \leq -f(\tau u) \log(\tau u) \left(k_s(\delta+) - k^+_s\left(\frac{1}{\delta}\right) + k^-_s(\tau)\right)$$

and thus for $c_2 := \alpha \log 2 - \log c$

$$\log u \left(f(\tau u) \left(k_s(\delta+) - k^+_s\left(\frac{1}{\delta}\right) + k^-_s(\tau)\right) - \alpha\right) \leq c_2 + \log(\tau^{-1}) f(\tau u) \left(k_s(\delta+) - k^+_s\left(\frac{1}{\delta}\right) + k^-_s(\tau)\right).$$

That implies for $u \geq T/\tau > 1$

$$\log u \left(\frac{1}{\delta} k_s(\delta+) - \frac{2}{\delta} \|\tilde{\rho}^+\|_{TV([0,\delta])} + \frac{1}{2} k^-_s(\tau)\right) - \alpha \leq c_2 + 2 \log(\tau^{-1}) \left(k_s(\delta+) + k^-_s(\tau)\right).$$

Since the right-hand side is independent of $u$ and $\log u \to \infty$ as $u \to \infty$, we obtain

$$\frac{1}{\delta} k_s(\delta+) - \frac{2}{\delta} \|\tilde{\rho}^+\|_{TV([0,\delta])} + \frac{1}{2} k^-_s(\tau) - \alpha < 0$$

and therefore

$$\|\tilde{\rho}^-\|_{TV([0,\delta])} = \sup_{\tau \in (0, \delta)} k^-_s(\tau) \leq 2\alpha + 4 \|\tilde{\rho}^+\|_{TV([0,\delta])} < \infty,$$

implying $\|k_s\|_{BV([0,\delta])} < \infty$. \qed
We conclude the following corollary on the regularity of the probability densities of IDDs. Later we will show that there cannot exist more than \( \alpha - 1 \) derivatives.

**Corollary 4.** Let \( \mu \) be an infinitely divisible distribution satisfying Assumption \( 3 \) with \( k_2 \in BV([0,\delta]) \) and suppose that \( xv \) is absolutely continuous on \( \mathbb{R} \). Let \( \alpha := k_2(0+) > 1 \). Then \( \mu \) admits a Lebesgue-density \( f \in C^n(\mathbb{R}) \) for any integer \( 0 \leq n < \alpha - 1 \).

**Proof.** Due to the absolute continuity of \( \nu \) and infinite activity \( \int_{\mathbb{R}} \nu(dx) = \infty \) by \( \alpha > 0 \), the measure \( \mu \) is absolutely continuous (Sato, 1999, Thm. 27.7). Proposition 1 yield \( |\varphi(u)| \lesssim (1 + |u|)^{-\alpha+\varepsilon} \) for any \( \varepsilon > 0 \). Therefore, \( \int_{\mathbb{R}} |\varphi(u)||u|^n du < \infty \) for any \( n < \alpha - 1 \). By standard Fourier analysis we obtain \( f := \mathcal{F}^{-1}\varphi \in C^n(\mathbb{R}) \).

\[ \square \]

### 3 A Fourier multiplier theorem

To motivate Fourier multiplier theorem and to explain its statistical relevance, let us dive a little deeper into the statistical analysis of the deconvolution model (1).

Assume the laws of \( X_1 \) and \( \varepsilon_1 \) have Lebesgue densities \( f \) and \( f_\varepsilon \), respectively. Consequently, \( Y_1 \) is distributed according to \( f_{Y} = f * f_{\varepsilon} \). Denoting the characteristic functions of \( Y_1 \) and \( \varepsilon_1 \) by \( \varphi_Y \) and \( \varphi_{\varepsilon} \), respectively, we obtain the Fourier inversion formula

\[ f = \mathcal{F}^{-1}\left[ \frac{\varphi_Y}{\varphi_\varepsilon} \right] = \frac{1}{\varphi_\varepsilon} * f_Y, \]

where the second equality has to be understood in distributional sense. Hence, the deconvolution map is given by \( g \mapsto \mathcal{F}^{-1}[\mathcal{F}g/\varphi_\varepsilon] \). Slightly abusing notation, we denote it by \( \mathcal{F}^{-1}[1/\varphi_\varepsilon] \). Replacing \( \varphi_Y \) by the empirical characteristic function of \( (Y_j) \) and regularizing with a band-limited kernel \( K \) with bandwidth \( h > 0 \), the previous display gives us immediately the natural kernel density estimator, which was proposed by Fan (1991)

\[ \widehat{f}_h := \mathcal{F}^{-1}\left[ \frac{\mathcal{F}K(h,)\varphi}{\varphi_\varepsilon} \right] * \mu_n \]

for the empirical measure \( \mu_n = \sum_{j=1}^{n} \delta_{Y_j} \) with Dirac measure \( \delta_y \) in the point \( y \in \mathbb{R} \). To estimate the linear functional \( \int \zeta(x)f(x)dx \) for suitable functions \( \zeta \), say \( \zeta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), a plug-in approach yields

\[ \int \zeta(x)\widehat{f}_h(x)dx = \int \zeta(x)\left( \mathcal{F}^{-1}\left[ \frac{\mathcal{F}K(h,)\varphi}{\varphi_\varepsilon} \right] * \mu_n \right)(x)dx \]

\[ = \int \left( \mathcal{F}^{-1}\left[ \frac{\mathcal{F}K(-h,)\varphi}{\varphi_\varepsilon(-\bullet)} \right] * \zeta \right)(x)\mu_n(dx). \]

Since the regularizing \( \mathcal{F}K(h,) \) degenerates to one as \( h \downarrow 0 \), we have to study the mapping properties of the deconvolution operator \( \mathcal{F}^{-1}[1/\varphi_\varepsilon(-\bullet)] \).

Applying a Mihlin multiplier theorem, Nickl and Reiß (2012) quantified the mapping properties of the Fourier multiplier in a Besov scale under certain assumptions. However, their conditions already exclude simple examples like the Poisson process, which satisfies (2) even for \( \alpha = 0 \). The target of this section is to study necessary and sufficient conditions under which \( 1/\varphi \) satisfies the Fourier multiplier property.

Using that \( B_{p,q}^s(\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1}[(1 + |u|)\alpha f] \in B_{p,q}^{s+\alpha}(\mathbb{R}) \) is an isomorphism for any \( \alpha \in \mathbb{R} \) and that the set of Fourier multipliers on \( B_{p,q}^s(\mathbb{R}) \) does not depend on \( s,q \) and are nested in \( p \) Triebel (2010, Prop. 2.6.2 and Thm. 2.6.3), the characteristic function \( \varphi \) has to satisfy necessarily

\[ |\varphi(u)| \gtrsim (1 + |u|)^{-\alpha}, \quad u \in \mathbb{R}. \]
In the previous section we have characterized infinitely divisible distributions whose characteristic function decays only with a polynomial rate. Our main result shows that under the regularity Assumption the necessary polynomial decay of \( \varphi \) already implies that \( 1/\varphi \) is a Fourier multiplier on Besov spaces.

**Theorem 5.** Let \( \mu \) be an infinitely divisible distribution with characteristic function \( \varphi \) satisfying Assumption \( 2 \) If and only if one (and thus all) of the conditions (i) to (iii) of Theorem \( 3 \) are satisfied, \( 1/\varphi \) is a Fourier multiplier on Besov spaces: There exists some \( \alpha > 0 \) such that for all \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \) the linear map

\[
B_{p,q}^{s+\alpha} (\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1} \left[ \frac{\mathcal{F}f}{\varphi} \right] \in B_{p,q}^s (\mathbb{R})
\]

is bounded.

**Proof.** If \( 1/\varphi \) is a Fourier multiplier, then (iii) in Theorem 3 has to be fulfilled as carried out above.

Now, let the assumptions of Theorem 3 be satisfied. Using \( \sigma^2 = 0 \) and noting that \( k_\gamma \in BV([0, \delta]) \) implies boundedness of \( k_\gamma \) and thus \( \int_0^1 x \nu_s (dx) < \infty \), the characteristic function of \( \mu \) can be represented by

\[
\varphi (u) = \exp \left( i \gamma_0 u + \int (e^{itx} - 1) \nu (dx) \right) = \varphi_c(u) \varphi_p(u), \quad \text{for} \quad u \in \mathbb{R},
\]

where \( \gamma_0 = \gamma - \int_0^1 x \nu_s (dx) \in \mathbb{R} \) is a drift parameter and

\[
\varphi_c(u) := \exp \left( \int_{[-\delta, \delta]} (e^{itx} - 1) \nu (dx) \right), \quad \varphi_p(u) := \exp \left( i \gamma_0 u + \int_{[\mathbb{R} \setminus [-\delta, \delta]]} (e^{itx} - 1) \nu (dx) \right).
\]

Defining \( \mu_c := \mathcal{F}^{-1} [\varphi_c] \) and \( \mu_c := \mathcal{F}^{-1} [\varphi_p] \), this yields the decomposition \( \mu = \mu_c * \mu_p \) into a convolution of an IDD with compactly supported jump measure and distribution of compound Poisson type. The deconvolution operator decomposes into a composition \( \mathcal{F}^{-1}[1/\varphi] = \mathcal{F}^{-1}[1/\varphi^c] * \mathcal{F}^{-1}[1/\varphi^p] \), where

\[
\mathcal{F}^{-1}[1/\varphi^p] = \delta_{-\gamma_0} \ast \left( e^{\nu([\mathbb{R} \setminus [-\delta, \delta]])} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\nu([\mathbb{R} \setminus [-\delta, \delta]])^k) \right)
\]

is a finite signed measure. Since convolution with a finite signed measure is a bounded map on \( L^p(\mathbb{R}) \) for any \( 0 < p \leq \infty \), we conclude from the Littlewood–Paley representation of Besov spaces that \( B_{p,q}^s (\mathbb{R}) \ni f \mapsto \mathcal{F}^{-1}[1/\varphi^p] * f \in B_{p,q}^s (\mathbb{R}) \) is a bounded linear map. For \( \varphi_c \) we use \( \varphi_c(u) = i \mathcal{F}[x \nu_s]([-\delta, \delta]) (u) \varphi_c(u) \) and the estimate \( |\mathcal{F}[x \nu_s]([-\delta, \delta]) (u)| \lesssim (1 + |u|)^{-1} \) by the bounded variation of \( x \nu \) near the origin. The polynomial decay of \( \varphi_c \) implies then for some \( \alpha > 0 \)

\[
(1 + |u|)^{-\alpha} |\varphi_c^{-1}(u)| \lesssim 1 \quad \text{and} \quad (1 + |u|)^{-\alpha} (|\varphi_c^{-1}(u)|^\prime (u)) = (1 + |u|)^{-\alpha} |\mathcal{F}[x \nu_s]([-\delta, \delta]) (u) / \varphi_c(u)| \lesssim (1 + |u|)^{-1}.
\]

Therefore, we can apply Corollary 4.11 by Girardi and Weis (2003) to conclude that \( (1 + |u|)^{-\alpha} / \varphi_c(u) \) is a Fourier multiplier on all Besov spaces \( B_{p,q}^s (\mathbb{R}) \) for all \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \). The assertion follows because \( f \mapsto \mathcal{F}^{-1}[(1 + i u)^\alpha \mathcal{F}f] \) is an isomorphism from \( B_{p,q}^{s+\alpha} (\mathbb{R}) \) onto \( B_{p,q}^s (\mathbb{R}) \).

As a small application of Theorem 5 we will extend Corollary 4 generalizing of the result for self-decomposable distributions by Wolfe (1971) to a much larger class, but for simplicity restricted to non-integer \( \alpha = k_\gamma (0+) \).
Corollary 6. Let $\mu$ be an infinitely divisible distribution satisfying Assumption 2 with $k_s \in BV([0,\delta])$ and suppose that $x\nu$ is absolutely continuous on $\mathbb{R}$. Let $\alpha := k_s(0+) \in \mathbb{N}$ be positive. Then the Lebesgue-density of $\mu$ satisfies $f \in C^n(\mathbb{R})$ for $n \in \mathbb{N}$ if and only if $0 \leq n < \alpha - 1$.

Proof. Let $n$ satisfy $n-1 < \alpha < n+2$ and suppose $f \in C^{n+1}(\mathbb{R})$. For the function $g(x) = e^{-x} \mathbb{1}_{(0,\infty)}(x)$, a direct calculation or using $g \in B^{1,1}_{1,1}(\mathbb{R})$ shows $f \ast g \in C^{n+2}(\mathbb{R}) \subseteq B^{n+2}_{\infty,\infty}$. Then Theorem 5 applied to $f \ast g$ yields for any $\varepsilon > 0$ that $g = F^{-1} \left[ F[f \ast g]/\phi \right] \in B^{n+2}_{\infty,\infty}$. Since $n+2-\alpha > 0$ and $\varepsilon$ can be chosen small enough, the Besov embedding yields $g \in B^{n+2-\alpha-\varepsilon}_{\infty,\infty}(\mathbb{R})$ which contradicts to $g$ being discontinuous.

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