Dirac Monopole from Lorentz Symmetry in 
N-Dimensions: II. The Generalized Monopole

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Abstract

In a previous paper, we found an extension of the \(N\)-dimensional Lorentz generators that partially restores the closed operator algebra in the presence of a Maxwell field, and is conserved under system evolution. Generalizing the construction found by Bérard, Grandati, Lages and Mohrbach for the angular momentum operators in the \(O(3)\)-invariant nonrelativistic case, we showed that the construction can be maximally satisfied in a three dimensional subspace of the full Minkowski space; this subspace can be chosen to describe either the \(O(3)\)-invariant space sector, or an \(O(2,1)\)-invariant restriction of spacetime. When the \(O(3)\)-invariant subspace is selected, the field solution reduces to the Dirac monopole field found in the nonrelativistic case. For the \(O(2,1)\)-invariant subspace, the Maxwell field can be associated with a Coulomb-like potential of the type \(A^\mu(x) = n^\mu/\rho\), where \(\rho = (x^\mu x_\mu)^{1/2}\), similar to that used by Horwitz and Arshansky to obtain a covariant generalization of the hydrogen-like bound state. In this paper we elaborate on the generalization of the Dirac monopole to \(N\)-dimensions.

1 Introduction

The Lorentz covariance of electrodynamics can be expressed through the validity of the canonical commutation relations

\[
[x^\mu, x^\nu] = 0 \quad [p^\mu, p^\nu] = 0 \quad [x^\mu, p^\nu] = -i\hbar g^{\mu\nu}
\]

\[
[x^\mu, L^{\rho\lambda}] = i\hbar (x^\lambda g^{\mu\rho} - x^\rho g^{\mu\lambda}) \quad [p^\mu, L^{\rho\lambda}] = i\hbar (g^{\mu\rho}p^\lambda - g^{\mu\lambda}p^\rho)
\]
\[ [L^\mu\nu, L^\lambda\rho] = i\hbar (g^{\mu\lambda}L^\nu\rho - g^{\mu\rho}L^\nu\lambda - g^{\nu\lambda}L^\mu\rho + g^{\nu\rho}L^\mu\lambda) \]  

(3)

when the Lorentz generator is taken as

\[ L^\mu\nu = x^\mu p^\nu - x^\nu p^\mu \]  

(4)

and the electromagnetic field is introduced through minimal coupling of a gauge potential to the momentum

\[ p^\mu = m\dot{x}^\mu + eA^\mu. \]  

(5)

Using (5) to transform to the coordinate-velocity basis, the relations (11) become

\[ [x^\mu, x^\nu] = 0 [x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu} \]  

(6)

and

\[ [\dot{x}^\mu, \dot{x}^\nu] = \frac{1}{m^2} [p^\mu - eA^\mu, p^\nu - eA^\nu] = \frac{i\hbar e}{m^2} (\partial^\mu A^\nu - \partial^\nu A^\mu), \]  

(7)

a form that prompted Feynman [1] to seek a “derivation” of Maxwell’s equations assuming only the commutation relations (6) and noncommutivity of the velocities

\[ [\dot{x}^\mu, \dot{x}^\nu] = -\frac{i\hbar}{m^2} W^\mu\nu (x), \]  

(8)

without explicit recourse to canonical momentum, gauge potential, action, or variational principle. It was later shown [2] that assumptions (6) are sufficiently strong to establish the self-adjointness of the differential equations

\[ m\ddot{x}^\mu = F^\mu (\tau, x, \dot{x}), \]  

(9)

and it follows that this system is equivalent to a unique Lagrangian mechanics [3] from which the full canonical system follows in the standard manner. However, continuing in the spirit of Feynman’s “naive” inquiry, we construct operators

\[ M^{\mu\nu} = m (x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu) \]  

(10)

that, in light of (5), are not generally equivalent to the Lorentz generators (11). The resulting commutation relations among \( x^\mu, \dot{x}^\mu, \) and \( M^{\mu\nu} \) contain terms that depend on the field strength \( W^{\mu\nu} (x) \), breaking the Lie algebra for the Lorentz group.
In a previous paper [4] we studied the generators (10) in \( N \) dimensions, and found an extension

\[
\tilde{M}^{\mu\nu} = M^{\mu\nu} + Q^{\mu\nu} = m \left( x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu \right) + x^\mu x_\sigma W^{\sigma\nu} - x^\nu x_\sigma W^{\sigma\mu} - x^\sigma x_\sigma W^{\mu\nu}
\]

(11)

that partially restores the closed algebra in the coordinate-velocity basis, without explicit reference to a gauge potential \( A^\mu (x) \). It was shown that the relations

\[
\left[ x^\mu, \tilde{M}^\rho_\lambda \right] = i\hbar \left( x^\lambda g^{\mu\rho} - x^{\rho} g^{\mu\lambda} \right) \quad \left[ \dot{x}^\mu, \tilde{M}^\rho_\lambda \right] = i\hbar \left( g^{\mu\rho} \dot{x}^\lambda - g^{\mu\lambda} \dot{x}^\rho \right)
\]

(12)

hold when the field is given by

\[
W^{\mu\nu} (x) = \frac{1}{(N - 3)!} \epsilon^{\mu\nu\lambda_1 \cdots \lambda_{N-3}} \frac{x_{\lambda_0}}{(x^2)^{3/2}} U_{\lambda_1 \cdots \lambda_{N-3}}
\]

(13)

where \( U_{\lambda_1 \cdots \lambda_{N-3}} \) is totally antisymmetric, and the dynamical evolution is restricted to the subspace

\[
x (\tau) \in x^U = \left\{ x \mid x^{\lambda_1} U_{\lambda_1 \cdots \lambda_{N-3}} = 0 \right\}
\]

(14)

The algebra of the generators was found to be

\[
\left[ \tilde{M}^{\mu\nu}, \tilde{M}^{\rho_\lambda} \right] = i\hbar \left\{ g^{\mu\lambda} \tilde{M}^{\nu\rho} - g^{\mu\rho} \tilde{M}^{\nu_\lambda} - g^{\nu_\lambda} \tilde{M}^{\mu\rho} + g^{\nu\rho} \tilde{M}^{\mu_\lambda} \right\} + \Delta^{\mu\nu\rho}_{2}^{\lambda}
\]

(15)

with

\[
\Delta^{\mu\nu\rho}_{2}^{\lambda} = \frac{1}{(x^2)^{1/2}} \frac{1}{(N - 3)!} \epsilon^{\mu\nu\rho\lambda_2 \cdots \lambda_{N-3} \lambda} x^{\zeta} g^{\nu\lambda_1} U_{\lambda_1 \lambda_2 \cdots \lambda_{N-3}}
\]

(16)

so that the symmetry-breaking term \( \Delta^{\mu\nu\rho}_{2}^{\lambda} \) vanishes for the three generators of O(3) or O(2,1) that leave the subspace \( x^U \) invariant. The meaning of the electromagnetic field can be understood in four dimensions, by taking the vector \( U = \hat{t} \) along the time axis, so that the field \( W^{\mu\nu} \) describes a Dirac monopole, of the type previously found in a nonrelativistic analysis by Bérard, Grandati, Lages and Mohrbach [5]. Although the dynamical system that follows from the commutation relations must be consistent with the gauge theory posed in (1) to (4), the field \( W^{\mu\nu} \), found without reference to a gauge potential, describes a monopole solution that does not follow from a standard gauge potential in a straightforward manner. A related solution is found by taking the vector \( U = \hat{z} \) along the \( z \)-axis, for which the field \( W^{\mu\nu} \) describes an O(2,1)-invariant solution associated with a potential of the type

\[
V (x) \sim \frac{1}{\sqrt{-t^2 + x^2}}
\]

(17)
which may be seen as a relativistic generalization of the nonrelativistic Coulomb force. A description of the relativistic bound state problem for the scalar hydrogen atom was found in the context of the Horwitz-Piron formalism, using a potential of this form. In this paper, we explore the higher dimensional Dirac monopole described in expression (13). The connection with the O(2,1)-invariant generalization of the Coulomb force will be discussed in a subsequent paper.

2 Gauge Theory from Commutation Relations

2.1 Stueckelberg-Lorentz force law

According to Dyson’s 1991 account, Feynman observed that posing commutation relations of the form

\[ [x^i, x^j] = 0, \quad m [x^i, \dot{x}^j] = i\hbar \delta^{ij}, \quad \text{(18)} \]

among the quantum operators for Euclidean position and velocity, where \( \dot{x}^i = dx^i/dt \) and \( i, j = 1, 2, 3 \), restricts the admissible forces in the classical Newton’s second law

\[ m \ddot{x}^i = F^i(t, x, \dot{x}) \quad \text{(19)} \]

to the form

\[ m \ddot{x}^i = E^i(t, x) + \epsilon^{ijk} \dot{x}_j H_k(t, x) \quad \text{(20)} \]

with fields that must satisfy

\[ \nabla \cdot H = 0, \quad \nabla \times E + \frac{\partial}{\partial t} H = 0. \quad \text{(21)} \]

The velocity-dependent term in (20) follows from the “naive” assumption that the velocities have non-zero commutation relations,

\[ m^2 [\dot{x}^i, \dot{x}^j] = -i\hbar F^{ij}(t, x) = -i\hbar \epsilon^{ijk} H_k(t, x), \quad \text{(22)} \]

so that Feynman’s “derivation” apparently proceeds without explicit reference to canonical momentum, gauge potential, action, or variational principle. It was later shown that the assumptions are sufficiently strong to establish the self-adjointness of the differential equations, from which it follows that this system is equivalent to a unique nonrelativistic
Lagrangian mechanics \[3\] with canonical momenta whose relationship to the velocities leads directly to \((22)\). Several authors observed \[8\] that supposing Lorentz covariance in \((19)\) conflicts with the Euclidean assumptions in \((18)\), and so \((20)\) cannot be interpreted as the Lorentz force in Maxwell theory.

These results were generalized to the relativistic case \[9, 10\] in curved \(N\)-dimensional space-time by taking

\[
[x^\mu, x^\nu] = 0 \quad m[x^\mu, \dot{x}^\nu] = -ihg^{\mu\nu}(x) \quad [\dot{x}^\mu, \dot{x}^\nu] = \left(-\frac{ih}{m^2}\right)f^{\mu\nu}(\tau, x)
\]

and

\[
m\ddot{x}^\mu = F^\mu(\tau, x, \dot{x}).
\]

where \(\mu, \nu = 0, 1, \cdots, N-1\) and \(x^\mu(\tau)\) and its derivatives are functions of the Poincaré-invariant evolution parameter \(\tau\). Introducing the electric charge \(e\) and a dimensional constant \(\lambda\) required for consistency with Maxwell theory, the resulting system

\[
m[\ddot{x}^\mu + \dot{x}^\rho \Gamma^\mu_{\rho\nu} \dot{x}^\nu] = \lambda e [\epsilon^\mu(\tau, x) + f^{\mu\nu}(\tau, x) \dot{x}^\nu],
\]

with usual affine connection

\[
\Gamma_{\mu\nu\rho} = \frac{1}{2} (\partial^\rho g_{\mu\nu} + \partial^\nu g_{\mu\rho} - \partial^\mu g_{\nu\rho})
\]

and fields satisfying

\[
\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \quad \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu + \frac{\partial}{\partial \tau} f_{\mu\nu} = 0,
\]

is equivalent to the \((N+1)\)-dimensional gauge theory associated with Stueckelberg’s relativistic mechanics \[11, 12\]. The commutation relations \[23\] are defined with respect to a covariant Heisenberg picture, in which operators evolve according to a Poincaré-invariant parameter \(\tau\). Thus, the position and velocity operators \(x^\mu(\tau)\) and \(\dot{x}^\mu(\tau)\), for \(\mu, \nu = 0, 1, \cdots, N-1\), are the quantum analog of the classical quantities introduced by Stueckelberg \[11\] to construct his covariant classical mechanics, in which the spacetime event \(x^\mu(\tau)\) traces out a general particle worldline as the parameter proceeds monotonically from \(\tau = -\infty\) to \(\tau = \infty\).

### 2.2 Pre-Maxwell field equations

Formally extending the indices \(\alpha, \beta, \gamma\) to \((N+1)\)-dimensions, so that

\[
\mu, \nu, \rho = 0, 1, \cdots, N-1 \quad \alpha, \beta, \gamma = 0, \cdots, N
\]

5
\[ x^N = \tau \quad \partial_\tau = \partial_N \quad f_{\mu N} = -f_{N \mu} = \epsilon_\mu \]  

(29)
equations (25) and (27) become

\[ m \left[ \ddot{x}^\mu + \Gamma^{\mu \rho \nu} \dot{x}_\rho \dot{x}_\nu \right] = \lambda e \ f^{\beta \gamma} (\tau, x) \dot{x}_\beta \]

(30)
and

\[ \partial_\alpha f_{\beta \gamma} + \partial_\beta f_{\gamma \alpha} + \partial_\gamma f_{\alpha \beta} = 0. \]

(31)
The \( N \) equations (30) imply that

\[ \frac{d}{d\tau} \left( -\frac{1}{2} m \ddot{x}_\mu \right) = \lambda e \ f_{N \alpha} (\tau, x) \dot{x}_\alpha, \]

(32)
permitting the fields and particles to exchange mass. This classical electrodynamics is equivalent \[10\] to the Lagrangian system defined by

\[ L = \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \lambda e a_\mu (\tau, x) \dot{x}_\mu + \lambda e a_\alpha (\tau, x) = \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \lambda e a_\alpha (\tau, x) \dot{x}_\alpha \]

(33)
with field strength related to the gauge potential through

\[ f_{\alpha \beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha. \]

(34)
Introducing in the action a kinetic term \[12\] for the \( \tau \)-dependent fields \( f_{\alpha \beta} (\tau, x) \)

\[ S = \int d\tau \ \left\{ \frac{1}{2} M \dot{x}_\mu \dot{x}_\mu + \lambda e a_\alpha (\tau, x) \dot{x}_\alpha \right\} - \frac{\lambda}{4} \int d^{N-1} x \ d\tau \ f^{\alpha \beta} (\tau, x) f_{\alpha \beta} (\tau, x) \]

(35)
leads to the inhomogeneous source equation

\[ \partial_\beta f^{\alpha \beta} (\tau, x) = j^\alpha (\tau, x) \]

(36)
with classical current of the form

\[ j^\alpha (\tau, x) = e \dot{z}^\alpha (\tau) \ \delta^{N-1} (x - z). \]

(37)
The associated quantum theory

\[ \left[ i \partial_\tau + \lambda e a_\alpha \right] \psi(x, \tau) = \frac{1}{2M} \left[ p^\mu - \lambda e a^\mu \right] \left[ p_\mu - \lambda e a_\mu \right] \psi(x, \tau), \]

(38)
proposed by Sa’ad, Horwitz and Arshansky \[12\], is invariant under local gauge transformations of the type

\[ \psi(x, \tau) \rightarrow e^{i \lambda e A(x, \tau)} \psi(x, \tau) \quad a_\alpha (x, \tau) \rightarrow a_\alpha (x, \tau) + \partial_\alpha A(x, \tau). \]

(39)
Global gauge symmetry leads to an \((N + 1)\)-dimensional conserved current

\[
\partial_\alpha j^\alpha(\tau, x) = \partial_\mu j^\mu + \partial_\tau j^N = 0 \tag{40}
\]

where

\[
j^\mu(x, \tau) = \frac{-ie}{2M} \left[ \psi^*(\partial^\mu - i\lambda a^\mu)\psi - \psi(\partial^\mu + i\lambda a^\mu)\psi^* \right] \quad j^N(x, \tau) = e \left| \psi(x, \tau) \right|^2 \tag{41}
\]

so that \(\left| \psi(x, \tau) \right|^2\) is interpreted as the positive-definite probability density at \(\tau\) of finding the event at the spacetime point \(x\). Equations (31) and (36) are formally identical to the standard Maxwell equations, but differ in two significant ways — first, these are \((N + 1)\)-dimensional equations in \(N\)-dimensions, and second, the structure of the source current (37) depends directly on the instantaneous event \(x^\mu(\tau)\), while the Maxwell current has support on the entire worldline

\[
J^\mu(x) = e \int d\tau \; \dot{z}^\mu(\tau) \; \delta^{N-1}(x - z). \tag{42}
\]

Standard Maxwell theory can be recovered from the Stueckelberg theory as an equilibrium limit, defined pointwise in \(x\) as \(\tau \to \pm \infty\) by the conditions

\[
f_{N\mu}(\tau, x) = 0 \quad \text{and} \quad \partial_\tau f^{\mu\nu}(\tau, x) = 0. \tag{43}
\]

Integrating (31) and (36) over \(\tau\) concatenates events into worldlines [13], recovering

\[
\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \quad \partial_\nu F^{\mu\nu} = J^\mu \tag{44}
\]

where

\[
F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \; f^{\mu\nu}(x, \tau) \quad \text{and} \quad J^\mu(x) = \int_{-\infty}^{\infty} d\tau \; j^\mu(x, \tau). \tag{45}
\]

It follows that \(\lambda\) has dimensions of time. Thus the instantaneous pre-Maxwell field is written as \(f^{\mu\nu}(x, \tau)\), and Maxwell-like fields, obtained by concatenation or by direct solution of Maxwell equations in \(N\)-dimensions, are written as \(F^{\mu\nu}(x)\). In the remainder of this paper, we will consider only \(\tau\)-independent Maxwell-like fields.

### 3 Electromagnetic Duality and the Monopole

The standard treatment of the Dirac monopole in four dimensional spacetime is deceptively simple. By introducing a magnetic current vector, the magnetic field acquires a non-zero
divergence (source), and the Maxwell equations become symmetric under exchange of the electric and magnetic sectors, so that any electric field solution induced by an electric current can be associated with a magnetic field solution induced by a magnetic current. However, this exchange symmetry is an artifact of four dimensional spacetime. Ignoring the laboratory origins of electromagnetics and viewing the Maxwell theory abstractly, as a pair of Lorentz covariant differential equations for a second rank tensor field on $N$-dimensional spacetime, there are two alternative approaches to the introduction of a second source current — in four dimensions, these approaches become equivalent.

The first approach, in analogy to the historical route, follows from the non-covariant decomposition of the tensor field, in some particular time + space reference frame, into an electric field associated with a source and a sourceless magnetic field. In this approach, a source may be introduced for the magnetic field, but this current does not make the Maxwell theory symmetric under exchange of the electric and magnetic sectors, except in the well-known case of $N = 4$. Moreover, for $N \neq 4$, the exchange of these non-covariant sectors is not equivalent to a duality transformation, and the tensor Maxwell equations are not duality-symmetric.

The second approach generalizes the Maxwell tensor field to a Clifford number field, leading to duality symmetric field equations. This approach provides an alternative description of the magnetic monopole, based on a covariant decomposition of the field into symmetric sectors, however, only in $N = 4$ can the field sectors exchanged by this duality transformation be identified with the field sectors found by non-covariant decomposition. In this section, we will show that the generalized magnetic monopole found in \[4\] supports approach of the duality invariance.

### 3.1 Clifford algebra formulation

The spacetime algebra formalism \[14\] provides a high degree of notational compactness by representing the usual tensorial objects of physics as index-free elements in a Clifford algebra. The formalism considerably simplifies the treatment of the magnetic monopole in higher dimensions because it eases the transition between covariant and space + time points of view, and contains a natural geometric representation of the duality transformation. In Clifford algebra, the product of two vectors separates naturally into a symmetric part and
antisymmetric part
\[ ab = \frac{1}{2} (ab + ba) + \frac{1}{2} (ab - ba) = a \cdot b + a \wedge b \] (46)

where the symmetric part is identified with the scalar inner product, and the rank 2 antisymmetric part is called a bivector. The general Clifford number is a direct sum of multivectors of rank 0, 1, \ldots, N
\[ A = A_0 + A_1 + A_2 + A_3 + \cdots + A_N \] (47)
\[ = A_0 + A_1^\alpha e_\alpha + \frac{1}{2} A_2^{\alpha \beta} e_\alpha \wedge e_\beta + \cdots + \frac{1}{N!} A_N^{\alpha_0 \alpha_2 \cdots \alpha_{N-1}} e_{\alpha_0} \wedge e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{N-1}} \] (48)
expanded on the \(2^N\)-dimensional basis
\[ \{1, e_\alpha, e_\alpha \wedge e_\beta, e_\alpha \wedge e_\beta \wedge e_\gamma, \cdots, e_\alpha \wedge e_\beta \wedge \cdots \wedge e_{N-1} \}. \] (49)

The most important algebraic rules are
\[ a A_r = a \left( a_1 \wedge a_2 \wedge \cdots \wedge a_r \right) = a \cdot A_r + a \wedge A_r \] (50)
\[ a \cdot A_r = \sum_{k=1}^{r} (-1)^{k+1} (a \cdot a_k) \left( a_1 \wedge \cdots \wedge a_{k-1} \wedge a_{k+1} \wedge \cdots \wedge a_r \right) \] (51)
\[ a \wedge A_r = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r \] (52)
\[ (a \cdot b) \cdot A_r = a \cdot (b \cdot A_r) \] (53)
\[ i = e_0 \wedge e_1 \wedge \cdots \wedge e_{N-1} = e_0 e_1 \cdots e_{N-1} \] (54)
\[ i^2 = (-1)^{N(N-1)/2} \det (g_{ij}) \] (55)
\[ i [e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}] = g_{\alpha_1 \alpha_2} \cdots g_{\alpha_r \alpha_r} \frac{1}{(N-r)!} \epsilon^{\alpha_1 \cdots \alpha_r \alpha_{r+1} \cdots \alpha_N} \left[ e_{\alpha_{r+1}} \wedge \cdots \wedge e_{\alpha_N} \right] \] (56)
\[ a \cdot (i A_r) = (-1)^{N-1} i (a \wedge A_r) \] (57)
\[ a \wedge (i A_r) = (-1)^{N-1} i (a \cdot A_r) \] (58)

We choose the flat metric
\[ g_{\alpha \beta} = e_\alpha \cdot e_\beta = \text{diag} (-1, 1, \cdots, 1) \] (59)
for which the unit pseudoscalar \(i\) satisfies
\[ i^2 = (-1)^{N(N-1)/2} \det (g_{ij}) = -(-1)^{N(N-1)/2} \] (60)

In the Clifford algebra formulation, Maxwell’s equations in \(N\)-dimensions are
\[ dF = -J \] (61)
in which the electromagnetic field strength bivector (antisymmetric second rank tensor) is
\[ F = \frac{1}{2} F^{\alpha\beta} (e_\alpha \wedge e_\beta), \] (62)
and the gradient and current vectors are \( d = \partial^\alpha e_\alpha \) and \( J = J^\alpha e_\alpha \). The LHS of (61) separates into the divergence and exterior derivative
\[ d \cdot F + d \wedge F = -J \] (63)
so that associating terms of equal rank rank leads to
\[ d \cdot F = -J \] (64)
\[ d \wedge F = 0. \] (65)
Equation (64) expresses the inhomogeneous Maxwell equations, as seen from the component tensor form
\[ d \cdot F = \partial^\alpha \left( \frac{1}{2} F^{\beta\gamma} \right) e_\alpha \cdot (e_\beta \wedge e_\gamma) = \partial^\alpha F^{\beta\gamma} \frac{1}{2} (g^{\alpha\beta} e_\gamma - g^{\alpha\gamma} e_\beta) = (\partial_\beta F^{\beta\gamma}) e_\gamma. \] (66)
Similarly, using the total antisymmetry of \( e_\alpha \wedge e_\beta \wedge e_\gamma \) we expand (65) as
\[ d \wedge F = \partial^\alpha \left( \frac{1}{2} F^{\beta\gamma} \right) e_\alpha \wedge e_\beta \wedge e_\gamma = \frac{1}{3!} \left( \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} \right) e_\alpha \wedge e_\beta \wedge e_\gamma = 0, \] (67)
expressing the homogeneous Maxwell equations. The wave equation and current conservation follow from (61) as
\[ d^2 F = -dJ = -d \cdot J - d \wedge J, \] (68)
which separates by rank into
\[ d^2 F = -d \wedge J \quad d \cdot J = 0. \] (69)
We will refer to the field equations in the form (64) and (65) as the Jacobi point of view, because the exterior derivative in (67) follows from the Jacobi identity for the commutator \([\hat{x}^\alpha, [\hat{x}^\beta, \hat{x}^\gamma]]\). On the other hand, writing the dual to \( F \) as
\[ \tilde{F} = i F \] (70)
relation (57) permits equation (65) to be written
\[ (-1)^{N-1} i (d \wedge F) = d \cdot (i F) = d \cdot \tilde{F} = 0, \] (71)
and in what we will call the divergence point of view, Maxwell theory can be represented as a pair of inequivalent tensor structures, whose divergences are associated with a source (or sourcelessness) through

\[ d \cdot F = J \quad \text{and} \quad d \cdot \tilde{F} = 0. \]

In this point of view, the divergenceless of \( \tilde{F} \) expresses the asymmetry of the Maxwell theory under duality — the exchange of \( F \) and \( \tilde{F} \) — and is related to the absence of a source for the magnetic field in four dimensions. Thus, a theory describing magnetic monopoles with no electric charges would appear in the divergence point of view as

\[ d \cdot G = 0 \quad \text{and} \quad d \cdot \tilde{G} = \tilde{J}^{(m)} \]

and in the Jacobi point of view

\[ d \cdot G = 0 \quad \text{and} \quad d \wedge G = J^{(m)} \]

with trivector current \( J^{(m)} \).

### 3.2 Non-manifestly covariant field equations

The usual distinction between the electric and magnetic fields is based on the decomposition of the manifestly covariant field equations into a time + space formulation in some reference frame. Choosing a time direction \( e_0 \), the field strength separates into time and space components as

\[ F = \frac{1}{2} F^{\alpha \beta} (e_\alpha \wedge e_\beta) = F^{0i} (e_0 \wedge e_i) + \frac{1}{2} F^{ij} (e_i \wedge e_j) = e_0 \wedge E + F \]

where the vector \( E \) and the bivector \( F \), defined as

\[ E = F^{0i} e_i \quad \text{and} \quad F = \frac{1}{2} F^{ij} (e_i \wedge e_j) \quad i, j = 1, ..., N - 1, \]

have only space components. In this reference frame, the classical Lorentz force decomposes as

\[ m \frac{d^2 x}{d \tau^2} = e F \cdot \dot{x} = e (e_0 \wedge E) \cdot \dot{x} + e F \cdot \dot{x} = e (E \cdot \dot{x}) e_0 - e \dot{x}_0 E + e F \cdot \dot{x} \]

with time and space components

\[ m \frac{d^2 x_0}{d \tau^2} = e (E \cdot \dot{x}) \quad \text{and} \quad m \frac{d^2 \mathbf{x}}{d \tau^2} = e \dot{x}_0 \mathbf{E} + e \mathbf{F} \cdot \dot{x}. \]
Expressions (77) distinguish the roles of \(E\) and \(F\), showing that the electric-like force \(e\hat{x}^0 E\) can perform work on a test particle and may be nonzero in a co-moving frame, while the magnetic-like force \(eF \cdot \hat{x}\) vanishes in the co-moving frame, and is seen from

\[
[F \cdot \hat{x}] \cdot \hat{x} = F \cdot [\hat{x} \wedge \hat{x}] = 0
\] (78)

to be orthogonal to the velocity. The analog of the three-vector Maxwell equations are found by decomposing the \(N\)-divergence into

\[
d = \partial^0 e_0 + \nabla \quad \nabla = \partial^i e_i
\] (79)

and applying (72) to (74) for the inhomogeneous equation

\[
d \cdot F = d \cdot (e_0 \wedge E + F) = \partial_0 E - \nabla \cdot E e_0 + \nabla \cdot F = - (\rho e_0 + J)
\] (80)

which separates into the time and space components

\[
\nabla \cdot E = \rho \quad - \nabla \cdot F - \partial_0 E = J.
\] (81)

We expand the unit pseudoscalar in the time + space reference frame as

\[
i = e_0 i_{(\text{space})} \quad i_{(\text{space})} = e_1 \ldots e_{N-1} \quad i^2_{(\text{space})} = (-1)^{(N-1)(N-2)}
\] (82)

and notice that the taking the dual of the field strength

\[
\tilde{F} = iF = ie_0 \wedge E + iF
\]

\[
= e_0 i_{(\text{space})} e_0 E + e_0 i_{(\text{space})} F
\]

\[
= e_0 \wedge [i_{(\text{space})} F] + (-1)^N [i_{(\text{space})} E]
\] (83)

exchanges the roles of the electric-like vector and the magnetic-like bivector. Writing

\[
\tilde{F} = -i_{(\text{space})} F
\] (84)

and using (57) and (58), the time + space theory, in analogy to the three-vector Maxwell equations, is expressed as

\[
\nabla \cdot E = \rho \quad \left[(-1)^{(N-1)(N-2)} + N\right] i_{(\text{space})} \nabla \wedge \tilde{F} - \partial_0 E = J
\] (85)

\[
\nabla \cdot \tilde{F} = 0 \quad -i_{(\text{space})} \nabla \wedge E + \partial_0 \tilde{F} = 0.
\] (86)

In this form, the divergencelessness of the field \(\tilde{F}\) is equivalent to the sourcelessness of the field \(\tilde{F}\).
3.3 Duality and the Dirac monopole in $N = 4$ dimensions

The standard treatment of the magnetic monopole relies on two closely related simplifications of Maxwell theory that obtain only in four dimensions. First, the $(N - 1)$-component electric-like vector $E$ and the $\frac{(N-1)(N-2)}{2}$-component magnetic-like bivector $F$ can only have an equal number of degrees of freedom in the special case that

$$ (N - 1) = \frac{(N - 1)(N - 2)}{2} \quad \Rightarrow \quad N = 4. \quad (87) $$

Then, the Maxwell field strength tensor

$$ F = \left[ F^{01} (e_0 \wedge e_1) + F^{02} (e_0 \wedge e_2) + F^{03} (e_0 \wedge e_3) + F^{12} (e_1 \wedge e_2) + F^{13} (e_1 \wedge e_3) + F^{23} (e_2 \wedge e_3) \right]. \quad (88) $$

decomposes into the non-manifestly covariant form

$$ F = e_0 \wedge E + F = \left[ E^1 (e_0 \wedge e_1) + E^2 (e_0 \wedge e_2) + E^3 (e_0 \wedge e_3) + H^3 (e_1 \wedge e_2) - H^2 (e_1 \wedge e_3) + H^1 (e_2 \wedge e_3) \right], \quad (89) \quad (90) $$

allowing the magnetic-like bivector $F$ to be identified with a magnetic vector $H$ through

$$ F = H^1 (e_2 \wedge e_3) - H^2 (e_1 \wedge e_3) + H^3 (e_1 \wedge e_2), \quad (91) $$

$$ = e_1 e_2 e_3 \left[ H^1 e_1 + H^2 e_2 + H^3 e_3 \right], \quad (92) $$

$$ = \mathbf{i}_{(\text{space})} H. \quad (93) $$

Applying definition $\mathbf{84}$ to $\mathbf{93}$ we recover the identification

$$ \tilde{F} = -\mathbf{i}_{(\text{space})} \left[ \mathbf{i}_{(\text{space})} H \right] = H \quad (94) $$

and by recognizing that in three-space

$$ a \wedge b = a^i b^j (e_i \wedge e_j) = a_i b_j \left[ \epsilon^{ijk} \mathbf{i}_{(\text{space})} e_k \right] = \mathbf{i}_{(\text{space})} \left( \epsilon^{ijk} a_i b_j \right) e_k = \mathbf{i}_{(\text{space})} a \times b, \quad (95) $$

the Maxwell equations $\mathbf{85}$ and $\mathbf{86}$ assume the standard 3-vector form

$$ \nabla \cdot \mathbf{E} = \rho \quad \nabla \times \mathbf{H} - \frac{\partial}{\partial t} \mathbf{E} = \mathbf{J} \quad (96) $$

$$ \nabla \cdot \mathbf{H} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{H} = 0. \quad (97) $$
The second simplification follows from (56) which shows that the rank of \( \tilde{F} \) is \( N - 2 \), and therefore only in \( N = 4 \) is a bivector. In this case, the dual to the field strength assumes the form of an inequivalent field strength,

\[
\tilde{F} = iF = \frac{1}{4} \epsilon^{\alpha_1 \alpha_2 \beta \gamma} F_{\beta \gamma} e_{\alpha_1} \wedge e_{\alpha_2}
\]

\[
= [\epsilon^{0123} F_{01} e_2 \wedge e_3 + \epsilon^{0231} F_{02} e_3 \wedge e_1 + \epsilon^{0312} F_{03} e_1 \wedge e_2
\]

\[
+ \epsilon^{1230} F_{12} e_0 \wedge e_3 + \epsilon^{2310} F_{23} e_1 \wedge e_0 + \epsilon^{3120} F_{31} e_2 \wedge e_0]\]

\[= [E_1 e_2 \wedge e_3 - E_2 e_1 \wedge e_3 + E_3 e_1 \wedge e_2
\]

\[= - H_3 e_0 \wedge e_3 - H_1 e_0 \wedge e_1 - H_2 e_0 \wedge e_2]\]

\[= e_0 \wedge (-H) + i_{\text{space}} E,\]  

(98)

with electric and magnetic vectors exchanged on a one-to-one basis

\[
E \rightarrow H \quad H \rightarrow -E.
\]  

(99)

Although (83) shows that the duality operation exchanges the role of the electric-like vector and magnetic-like bivector in any dimension, it is only in \( N = 4 \) that the bivector can be associated with a vector in such a way that the dual system can be identified as a transformed electromagnetic system of the same rank.

The four dimensional Maxwell equations, in the divergence point of view, can be made symmetric under the duality operation

\[
F \leftrightarrow \tilde{F} \quad J^{(e)} \leftrightarrow J^{(m)}
\]  

(101)

by introducing the Dirac magnetic monopole current vector \( J^{(m)} \) as the source for the tensor \( \tilde{F} \) in the field equations

\[
d \cdot F = - J^{(e)}
\]  

(102)

\[
d \cdot \tilde{F} = - J^{(m)}.
\]  

(103)

Since equation (102) leads to (96) in the form

\[
\nabla \cdot E = \rho^{(e)} \quad \nabla \times H - \frac{\partial}{\partial t} E = J^{(e)}
\]  

(104)

and since \( F \rightarrow \tilde{F} \) is equivalent to the exchange (100), the second field equation (103) generalizes (97) in the form

\[
\nabla \cdot H = \rho^{(m)} \quad \nabla \times E + \frac{\partial}{\partial t} H = J^{(m)}.
\]  

(105)
Then, in the rest frame of a point magnetic source we find the monopole solution

\[ \mathbf{H} = -\frac{g\mathbf{x}}{(x^2)^{3/2}} = -\nabla \left( -\frac{g}{|\mathbf{x}|} \right). \]  

(106)

Using (57) and (55) we can rewrite (103) in the Jacobi point of view, using

\[ d \cdot \mathbf{F} = d \cdot (i\mathbf{F}) = (-1)^{N-1} i d \wedge F = -J^{(m)} \]  

(107)

which becomes

\[ d \wedge F = -iJ^{(m)}. \]  

(108)

Combining (108) and (102) Maxwell’s equations are

\[ d\mathbf{F} = d \cdot \mathbf{F} + d \wedge \mathbf{F} = -J^{(e)} - iJ^{(m)} \]  

(109)

which is form invariant under the duality transformation induced by the unit pseudoscalar \( i \). Since \( i^2 = -1 \) in \( N = 4 \), we may construct the continuous duality transformation

\[ U(\theta) = e^{\theta i} = \cos \theta + i \sin \theta \]  

(110)

which acts as

\[ U(\theta) \, d\mathbf{F} = -U(\theta) \left[ J^{(e)} + iJ^{(m)} \right] \]  

\[ [d \cdot \mathbf{F} + d \wedge \mathbf{F}] \cos \theta + i [d \cdot \mathbf{F} + d \wedge \mathbf{F}] \sin \theta = -[\cos \theta + i \sin \theta] \left[ J^{(e)} + iJ^{(m)} \right] \]  

(111)

so that using (58) and (57) and separating terms of equal rank, we obtain

\[ d \cdot \mathbf{F}' = -J^{(e)\prime} \]  

(113)

\[ d \wedge \mathbf{F}' = -iJ^{(m)\prime} \]  

(114)

where

\[ \mathbf{F}' = \mathbf{F} \cos \theta - i\mathbf{F} \sin \theta \]  

(115)

\[ J^{(e)\prime} = J^{(e)} \cos \theta - J^{(m)} \sin \theta \]  

(116)

\[ J^{(m)\prime} = J^{(m)} \cos \theta + J^{(e)} \sin \theta. \]  

(117)

Since \( \mathbf{F} \) and \( i\mathbf{F} \) are both bivectors, related by the exchange of electric and magnetic fields, the transformed field can be identified as an electromagnetic field. Dirac argued [15] that the absence of the magnetic monopole can be viewed as a convention, according to which we choose the continuous duality transformation \( U(\theta) \) with angle \( \theta \) that takes \( J^{(m)\prime} \rightarrow 0. \)
3.4 Magnetic source in $N > 4$ dimensions

The Maxwell equations were made duality symmetric in $N = 4$ by introducing a source for the second divergence equation in (103). Such a source may be introduced in any dimension, but for $N > 4$ the rank of the dual $\tilde{F}$ is $N - 2 > 2$, so the field equations retain duality asymmetry. Labeling the rank of multivectors explicitly, the Maxwell equations become

$$d \cdot F_{(2)} = -J^{(e)}_{(1)}$$
$$d \cdot \tilde{F}_{(N-2)} = -\tilde{J}^{(m)}_{(N-3)}$$

where it is convenient to express the magnetic current as

$$\tilde{J}^{(m)}_{(N-3)} = (-1)^{N-1} iJ^{(m)}_{(3)}.$$  

(120)

The current trivector $J^{(m)}_{(3)}$ has components

$$J^{(m)}_{(3)} = \frac{1}{3!} J^{(m)\alpha\beta\gamma} e_\alpha \wedge e_\beta \wedge e_\gamma = e_0 \wedge J^{(m)}_{(2)} + \rho^{(m)}_{(3)}$$

where the current bivector $J^{(m)}_{(2)}$ and trivector $\rho^{(m)}_{(3)}$ have only space components

$$J^{(m)}_{(2)} = \frac{1}{2} J^{(m)ij} e_i \wedge e_j \quad \rho^{(m)}_{(3)} = \frac{1}{3!} J^{(m)ijk} e_i \wedge e_j \wedge e_k \quad i, j, k = 1, \ldots, N - 1,$$

(121)

so the dual is

$$iJ^{(m)}_{(3)} = e_0 i_{(space)} \left( e_0 \wedge J^{(m)}_{(2)} + \rho^{(m)}_{(3)} \right) = (-1)^{N} i_{(space)} J^{(m)}_{(2)} + e_0 i_{(space)} \rho^{(m)}_{(3)}.$$  

Combining (118), which has the non-manifestly covariant form (85) and (119), the Maxwell equations with electric and magnetic sources assume the form

$$\nabla \cdot E_{(1)} = \rho^{(e)}_{(0)} + \left[ (-1)^{\frac{(N-1)(N-2)}{2} + N} i_{(space)} \nabla \wedge \tilde{F}_{(N-3)} - \partial_0 E_{(1)} = J^{(e)}_{(1)}$$

(122)

$$\nabla \cdot \tilde{F}_{(N-3)} = (-1)^{N} i_{(space)} \rho^{(m)}_{(3)} - i_{(space)} \nabla \wedge E_{(1)} - \partial_0 \tilde{F}_{(N-3)} = i_{(space)} J^{(m)}_{(2)}.$$  

(123)

Since

$$\text{rank} \left[ i_{(space)} A^{(space)}_{(r)} \right] = (N - 1) - r = 3 - r$$

(124)
in four dimensions, we recover (104) and (105) by recalling (95) and associating

\[ \tilde{F}(1) = H \quad \rho^{(m)}(0) = i_{\text{(space)}} \rho^{(m)}(3) \quad J^{(m)}(1) = i_{\text{(space)}} J^{(m)}(2). \] (125)

Since the multivectors in (118) and (119) are of different rank when \( N \neq 4 \), these equations are not duality invariant, and taking advantage of the Clifford algebraic features of this representation, it was shown in [16] that no alternative duality symmetry exists for this system. It is clear from (122) and (123) that in the general case, exchange of \( E(1) \) and \( \tilde{F}(N-3) \) is not possible and there is no natural exchange of the electric and magnetic sectors that would permit consideration of exchange symmetry.

We may put (118) and (119) into the Jacobi point of view by applying (57) to combine

\[ d \cdot F(2) = -J^{(e)}(1) \quad \text{and} \quad d \wedge F(2) = -J^{(m)}(3). \] (126)

as

\[ dF(2) = - \left( J^{(e)}(1) + J^{(m)}(3) \right). \] (127)

This system can be made duality symmetric by generalizing the field and currents to Clifford numbers combining multivectors of appropriate rank, as

\[ F = F_2 + G_{(N-2)} \quad J = J^{(e)}(1) + J^{(m)}(3) + J^{(e)}(N-3) + J^{(m)}(N-1). \] (128)

The generalized Maxwell equations

\[ dF = -J \] (129)

separate by rank into

\[ d \cdot F(2) = -J^{(e)}(1) \quad d \wedge F(2) = -J^{(m)}(3) \] (130)
\[ d \cdot G_{(N-2)} = -J^{(e)}(N-3) \quad d \wedge G_{(N-2)} = -J^{(m)}(N-1) \] (131)

and these terms transform under duality according to

\[ i d \cdot F(2) = (-1)^{N-1} d \wedge i F(2) = -i J^{(e)}(1) \] (132)
\[ i d \wedge F(2) = (-1)^{N-1} d \cdot i F(2) = -i J^{(m)}(3) \] (133)
\[ i d \cdot G_{(N-2)} = (-1)^{N-1} d \wedge i G_{(N-2)} = -i J^{(e)}(N-3) \] (134)
\[ i d \wedge G_{(N-2)} = (-1)^{N-1} d \cdot i G_{(N-2)} = -i J^{(m)}(N-1). \] (135)
The transformed expressions may be written

\[ d \cdot F'_{(2)} = -J^{(e)\nu}_{(1)} \quad \quad d \wedge F'_{(2)} = -J^{(m)\nu}_{(3)} \]  

(136)

\[ d \cdot G'_{(N-2)} = -J^{(e)\nu}_{(N-3)} \quad \quad d \wedge G'_{(N-2)} = -J^{(m)\nu}_{(N-1)} \]  

(137)

where

\[ F'_{(2)} = (-1)^{N-1} i G_{(N-2)} \quad \quad G'_{(N-2)} = (-1)^{N-1} i F_{(2)} \]  

(138)

\[ J^{(e)\nu}_{(1)} = i J^{(m)\nu}_{(N-1)} \quad \quad J^{(m)\nu}_{(N-1)} = i J^{(e)\nu}_{(1)} \]  

(139)

\[ J^{(e)\nu}_{(N-3)} = i J^{(m)\nu}_{(N-3)} \quad \quad J^{(m)\nu}_{(N-3)} = i J^{(e)\nu}_{(N-3)} \]  

(140)

and duality symmetry is summarized as

\[ dF' = -J' \quad F' = F'_{(2)} + G'_{(N-2)} \quad \quad J' = J^{(e)\nu}_{(1)} + J^{(m)\nu}_{(N-1)} + J^{(e)\nu}_{(N-3)} + J^{(m)\nu}_{(N-1)} \]  

(141)

Generalizations of this model were discussed in [16].

Writing \( G_{(N-2)} \) and its dual in the non-manifestly covariant time + space reference frame

\[ G_{(N-2)} = e_0 \varepsilon_{(N-3)} + G_{(N-2)} \]  

(142)

\[ iG_{(N-2)} = e_0 \wedge [i_{(space)} G_{(N-2)}] + (-1)^N \left[ i_{(space)} \varepsilon_{(N-3)} \right] \]  

(143)

the duality transformation (138) is realized in the exchange of the noncovariant field sectors

\[ E_{(1)} \leftrightarrow E'_{(1)} = (-1)^N \tilde{G}_{(1)} \quad \quad \tilde{F}_{(N-3)} \leftrightarrow \tilde{F'}_{(N-3)} = \varepsilon_{(N-3)} \]  

(144)

\[ \varepsilon_{(N-3)} \leftrightarrow \varepsilon'_{(N-3)} = \tilde{F}_{(N-3)} \quad \quad \tilde{G}_{(1)} \leftrightarrow \tilde{G}'_{(1)} = (-1)^N E_{(1)} \]  

(145)

where

\[ \tilde{G}_{(1)} = -i_{(space)} G_{(N-2)} \quad \quad \tilde{F}_{(N-3)} = -i_{(space)} F_{(2)}. \]  

Equations (144) and (145) generalize the four dimensional exchange of the electric and magnetic sectors, but demonstrate explicitly that the duality transformation does not mix the \( E_{(1)} \) and \( F_{(2)} \) sectors of \( F_{(2)} \) or the \( \varepsilon_{(N-3)} \) and \( G_{(N-2)} \) sectors of \( G_{(N-2)} \). Writing the current \( J^{(m)\nu}_{(N-1)} \) as

\[ J^{(m)\nu}_{(N-1)} = e_0 \wedge J^{(m)\nu}_{(N-2)} + \rho^{(m)\nu}_{(N-1)} \]  

(147)
Along with the noncovariant field equations (122) and (123), the covariant expression (131) can be written

\[ \nabla \cdot \varepsilon_{(N-3)} = \rho_{(N-4)}^{(e)} \left[ \frac{(N-1)(N-2)}{2} \right] \mathbf{i}_{\text{space}} \nabla \wedge \mathbf{\tilde{G}}_{(1)} - \partial_0 \varepsilon_{(N-3)} = J_{(N-3)}^{(e)} \]  

(148)

\[ \nabla \cdot \mathbf{\tilde{G}}_{(1)} = (-1)^N \mathbf{i}_{\text{space}} \rho_{(N-1)}^{(m)} - \mathbf{i}_{\text{space}} \nabla \wedge \varepsilon_{(N-3)} - \partial_0 \mathbf{\tilde{G}}_{(1)} = \mathbf{i}_{\text{space}} J_{(N-2)}^{(m)}. \]

(149)

We observe from equations (130) and (131) that the system remains duality symmetric when

\[ J_{(3)}^{(m)} = J_{(N-3)}^{(e)} = 0 \]  

(150)

in which case the noncovariant fields \( \mathbf{E}_{(1)} \) derives from the electric source \( J_{(1)}^{(e)} \) and \( \varepsilon_{(N-3)} \) derives from the magnetic source \( J_{(N-1)}^{(m)} \), while the noncovariant fields \( \mathbf{\tilde{F}}_{(N-3)} \) and \( \mathbf{\tilde{G}}_{(1)} \) remain sourceless. Thus, in \( N > 4 \) duality symmetry does not guarantee that each sector of the generalized field equations derives from a source. Nevertheless, the duality transformation exchanges, according to (144) and (145), an electric solution induced by a non-zero \( J_{(1)}^{(e)} \) with a magnetic solution induced by a non-zero \( J_{(N-1)}^{(m)} \), and may be regarded as a duality symmetric theory describing both electric and magnetic point sources. Alternatively, taking

\[ J_{(1)}^{(e)} = J_{(N-1)}^{(m)} = 0 \]  

(151)

we find a duality symmetric system in which case the noncovariant field \( \mathbf{\tilde{F}}_{(N-3)} \) derives from the magnetic source \( J_{(3)}^{(m)} \) and \( \mathbf{\tilde{G}}_{(1)} \) derives from the electric source \( J_{(N-3)}^{(e)} \), while the noncovariant fields \( \mathbf{E}_{(1)} \) and \( \varepsilon_{(N-3)} \) remain sourceless. In this case, the duality transformation exchanges an electric solution induced by a non-zero \( J_{(1)}^{(e)} \) with a magnetic solution induced by a non-zero \( J_{(N-1)}^{(m)} \), and may also be regarded as a duality symmetric theory describing both electric and magnetic point sources.

4 Monopole solution from Lorentz invariance

Bérard, Grandati, Lages and Mohrbach [5] studied the Lie algebra associated with the O(3) invariance of the nonrelativistic system described in (18) to (22). Calculating commutation relations with the angular momentum

\[ L_i = m \varepsilon_{ijk} \dot{x}^i \dot{x}^j \]  

(152)
the noncommutivity of the velocities leads to field-dependent terms,

\[
[x_i, L_j] = -i\hbar\epsilon_{ijk}x_k
\]

\[
[\dot{x}_i, L_j] = -i\hbar\epsilon_{ijk}\dot{x}^k + \frac{i\hbar}{m}\delta_{ij}(x \cdot B) - \frac{i\hbar}{m}x_iB_j
\]

\[
[L_i, L_j] = -i\hbar\epsilon_{ijk}L^k - i\hbar\epsilon_{ijk}x^k (x \cdot B)
\]

(153) \hspace{1cm} (154) \hspace{1cm} (155)

Generalizing the operator $\tilde{L}_i$ to

\[
\tilde{L}_i = L_i + Q_i
\]

(156)

it was shown that the standard angular momentum algebra

\[
[x_i, \tilde{L}_j] = -i\hbar\epsilon_{ijk}x_k
\]

\[
[\dot{x}_i, \tilde{L}_j] = -i\hbar\epsilon_{ijk}\dot{x}^k
\]

\[
[\tilde{L}_i, \tilde{L}_j] = -i\hbar\epsilon_{ijk}\tilde{L}^k.
\]

(157) \hspace{1cm} (158) \hspace{1cm} (159)

is restored by the choice

\[
Q_i = -x_i (x \cdot B),
\]

(160)

when the field $B$ satisfies the structural conditions

\[
x_jB_i + x_iB_j + x_jx_k\partial_iB^k = 0.
\]

(161)

Since \textbf{(161)} admits a solution of the form

\[
B_i = -\frac{g x_i}{x^3}
\]

(162)

the authors argue that the method has led to a magnetic monopole. Moreover, the total angular momentum $\tilde{L}_i$ is conserved under the classical motion.

In a previous paper [4] the Béardin, Grandati, Lages and Mohrbach construction was generalized to the full Lorentz group in $N$-dimensions. In the notation of the spacetime algebra formalism, the extended generators

\[
\tilde{M} = M + Q = m (x \wedge \dot{x}) + x \wedge (x \cdot W) - x^2W
\]

(163)

satisfy the canonical Lie algebra for the Lorentz group

\[
[D \cdot x, \tilde{M}] = -i\hbar x \wedge D \quad [D \cdot \dot{x}, \tilde{M}] = -i\hbar \dot{x} \wedge D
\]

(164)
when the field is given by
\[ W(x) = \mathbf{i} \frac{x \wedge U}{(x^2)^{3/2}} = \mathbf{i} G(x) \quad (165) \]
where \( D \) is an arbitrary constant vector, \( U \) is a fixed multivector of rank \( N - 3 \), and the dynamical evolution is restricted to the subspace
\[ x(\tau) \in x^U = \{ x \mid x \cdot U = 0 \}. \quad (166) \]
Similarly, the generators satisfy the nearly canonical relations
\[ \left[ (D^{(2)} \wedge D^{(1)}) \cdot \tilde{M}, \tilde{M} \right] = i \hbar \left[ D^{(2)} \wedge (D^{(1)} \cdot \tilde{M}) - D^{(1)} \wedge (D^{(2)} \cdot \tilde{M}) \right] + \Delta_2 \quad (167) \]
where
\[ \Delta_2 = i \hbar \frac{1}{(x^2)^{1/2}} \left[ (D^{(1)} \wedge x) \wedge (D^{(2)} \cdot U) - (D^{(2)} \wedge x) \wedge (D^{(1)} \cdot U) \right] \quad (168) \]
with arbitrary constant vectors \( D^{(1)}, D^{(2)} \). It was shown that the symmetry-breaking term \( \Delta_2 \) vanishes for the three generators of \( \text{O}(3) \) or \( \text{O}(2,1) \) that leave the subspace \( x^U \) invariant. Thus, the construction can be extended to any number of dimensions, but the canonical relations only obtain in a three-dimensional subspace of the full \( N \)-dimensional spacetime.

The solution \( (165) \) may be understood in the following way. In \( N = 4 \), a particle moving with uniform timelike four-velocity \( U \) produces a current
\[ J(x) = \int d\tau \ U \ \delta^4(x - U\tau) \quad (169) \]
inducing a field that may be found by solving the standard Maxwell equations
\[ d \cdot F = -J(x) \quad d \wedge F = 0 \]
via the wave equation
\[ d^2 F = -d \wedge J(x). \]
Using the standard Green’s function technique, the field is found to be the Coulomb-like Liénard-Wiechert field
\[ F(x) = -d \wedge \left\{ \frac{1}{2\pi} \int d^4x' \ J(x') \ \delta \left[ (x - x')^2 \right] \right\} = -d \wedge \frac{U}{(x^2)^{3/2}} = \frac{x \wedge U}{(x^2)^{3/2}} \quad (170) \]
where the four-velocity in \( N = 4 \) dimensions can be identified with the rank \( N - 3 \) multivector \( U \). The field \( W(x) \) that satisfies the requirements for the generator in \( (163) \) is then the dual
As shown in section 3.3, the dual $W(x) = iG(x)$ in this case is a field tensor with the electric and magnetic sectors exchanged. The solution $W(x)$ is seen to satisfy

$$
d \cdot W = 0 \quad d \wedge W = -J(x)$$

which we interpret to describe a magnetic field solution for a magnetic current, symmetric to the electric Coulomb solution for an electric current. Thus, in $N = 4$, $W(x)$ corresponds to the standard description of a magnetic monopole. In particular, by choosing $U = e_0$ along the time axis, so that the restriction (166) becomes

$$x(\tau) = (0, x)$$

the field $W(x)$ is

$$W(x) = E^i e_0 \wedge e_i + \frac{1}{2} \epsilon^{ijk} B_i e_j \wedge e_k = i \frac{e_0 \wedge x}{r^3} = -\frac{x^1}{r^3} e_2 \wedge e_3 + \frac{x^2}{r^3} e_1 \wedge e_3 - \frac{x^3}{r^3} e_1 \wedge e_2$$

where

$$r = \left[ (x^1)^2 + (x^2)^2 + (x^3)^2 \right]^{1/2}$$

providing the magnetic monopole solution

$$E = 0 \quad B = -\frac{1}{r^3} (x^1, x^2, x^3)$$

as in (162). In higher dimensions, the multivector $U$ is of rank $N - 3 > 1$ and although the field $G(x)$ retains the Liénard-Wiechert form, it has rank $N - 2 > 2$. Thus, the fields $G(x)$ and $W(x)$ can be identified with the fields $G_{(N-2)}$ and $F(2)$ given in (128). In this sense, $G_{(N-2)}$ is an electric-type field satisfying the equations

$$d \cdot G_{(N-2)} = -J_{(N-3)}(x) \quad d \wedge G_{(N-2)} = 0$$

and $W(x)$ is the generalized monopole obtained through the duality transformation in $N$ dimensions, satisfying

$$d \cdot W_{(2)} = 0 \quad d \wedge W_{(2)} = -J_{(3)}(x).$$

We may construct a different kind of solution by replacing the timelike velocity $U = e_0$ with the spacelike vector $U = e_3$ along the z-axis, characteristic of the relative velocity of an interacting particle pair. Then, from

$$F(x) = \frac{x \wedge e_3}{\rho^3} = \frac{x^0}{\rho^3} e_0 \wedge e_3 + \frac{x^1}{\rho^3} e_1 \wedge e_3 + \frac{x^2}{\rho^3} e_2 \wedge e_3$$

(178)
we find the field

\[ W(x) = E^i e_0 \wedge e_i + \frac{1}{2} \epsilon^{ijk} B_j e_j \wedge e_k = \frac{i x \wedge e_3}{\rho^3} = \frac{x^0}{\rho^3} e_1 \wedge e_2 + \frac{x^1}{\rho^3} e_0 \wedge e_2 - \frac{x^2}{\rho^3} e_0 \wedge e_1, \quad (179) \]

where

\[ \rho = \left[ - (x^0)^2 + (x^1)^2 + (x^2)^2 \right]^{1/2} \quad (180) \]

generalizes the spatial separation in the action-at-a-distance problems of nonrelativistic mechanics in the subspace

\[ x = (x^0, x^1, x^2, 0) \in x^e_3 = \{ x \mid x \cdot e_3 = 0 \} \quad (181) \]

invariant under the O(2,1) subgroup of the full Lorentz group. The field strengths are

\[ E = \frac{1}{\rho^3} (-x^2, x^1, 0) \quad B = \frac{1}{\rho^3} (0, 0, x^0). \quad (182) \]

In this case, closed commutation relations hold among the O(2,1) generators \( \tilde{M}^{01}, \tilde{M}^{02}, \) and \( \tilde{M}^{12} = \tilde{L}_3, \) while the algebra of the generators is broken by field dependent terms for the boost \( \tilde{M}^{03} \) and the rotations \( \tilde{M}^{31} = \tilde{L}_2 \) and \( \tilde{M}^{23} = \tilde{L}_1. \) A description of the relativistic bound state problem for the scalar hydrogen atom was found \(^{[6]}\) in the context of the Horwitz-Piron \(^{[7]}\) formalism, using a potential of this form.

## 5 Conclusion

It has been shown \(^{[2]}\) that the assumption of commutation relations \(^{[1]}\) among quantum position and velocity operators and a second order force equation \(^{[3]}\) is sufficiently strong to establish a Lagrangian system of classical electrodynamics equipped with a canonical momentum related to velocity through minimum coupling to a vector gauge field \( A^\mu. \) The Lorentz invariance of this system guarantees conservation of the generator

\[ L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \]

Because the velocity operators do not commute in the presence of an electromagnetic field, commutation relations with the operator

\[ M^{\mu\nu} = m (x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu) \]
will contain terms depending on the gauge field, but only in combinations involving the
field strength, obtained as the exterior derivative of the gauge potential. It was shown in a
previous paper [4] that the operators $M^{\mu\nu}$ have a field-dependent extension that satisfies the
canonical commutation relations associated with $L^{\mu\nu}$, when the field $W^{\mu\nu}$ takes a particular
form. Despite the connection to an underlying $U (1)$ gauge theory, the field strength $W^{\mu\nu}$
was shown in three dimensions [5] and four dimensions [4] to represent a magnetic monopole.
In this paper, we examined the solution in $N > 4$ dimensions, and showed that it represents
a magnetic monopole in a generalized Maxwell theory, in which the Clifford-valued electromagnetic field strength is constructed to preserved duality symmetry in any dimension. This
notion of duality does not exchange the electric and magnetic sectors of the noncovariant
time + space decomposition of the field strength, but exchanges among the fields of different
rank in a covariant manner. Thus, the magnetic monopole solution is the dual of an electric
field solution of higher rank. The underlying gauge theory associated with this model will
be discussed in a subsequent paper.

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