Charged Particle with Magnetic Moment in the Aharonov-Bohm Potential

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Abstract

We considered a charged quantum mechanical particle with spin $\frac{1}{2}$ and gyromagnetic ratio $g \neq 2$ in the field of a magnetic string. Whereas the interaction of the charge with the string is the well-known Aharonov-Bohm effect and the contribution of magnetic moment associated with the spin in the case $g = 2$ is known to yield an additional scattering and zero modes (one for each flux quantum), an anomaly of the magnetic moment (i.e. $g > 2$) leads to bound states. We considered two methods for treating the case $g > 2$.

The first is the method of self-adjoint extension of the corresponding Hamilton operator. It yields one bound state as well as additional scattering. In the second we consider three exactly solvable models for finite flux tubes and take the limit of shrinking its radius to zero. For finite radius, there are $N + 1$ bound states ($N$ is the number of flux quanta in the tube).

For $R \to 0$ the bound state energies tend to infinity so that this limit is not physical unless $g \to 2$ along with $R \to 0$. Thereby only for fluxes less than unity the results of the method of self-adjoint extension are reproduced whereas for larger fluxes $N$ bound states exist and we conclude that this method is not applicable.

We discuss the physically interesting case of small but finite radius whereby the natural scale is given by the anomaly of the magnetic moment of the electron $a_e = (g - 2)/2 \approx 10^{-3}$.

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1 Introduction

There is a continuous interest in the study of scattering and bound states in the potential of a magnetic string

$$\vec{A} = \frac{\Phi}{2\pi r} \vec{e}_\phi,$$  \hspace{1cm} (1)

with the flux $\Phi$. The interaction of a charged particle with this potential is described by the minimal coupling

$$\vec{p} \rightarrow \vec{p} - \frac{e}{c} \vec{A}.$$  \hspace{1cm} (2)

The corresponding magnetic field

$$\vec{H} = \Phi \delta(x) \delta(y) \vec{e}_3$$  \hspace{1cm} (3)

vanishes everywhere except on the flux line where it is infinite.

The famous Aharonov-Bohm effects [1] consists in a nontrivial scattering of a charged particle off the potential (1). It is due to the interference of phaseshifts of the wave function which are influenced by the potential (1). In an ideal situation the wave function vanishes where the magnetic field in nonzero demonstrating the role of the potential. In the last years the AB effect was studied in connection with fractional spin and statistics in [3], its contribution to cosmic strings in [4], [5]. There is a close relation to the calculation of propagators in chromomagnetic background fields [6], [7].

While the initial investigation concerns with a scalar particle, the inclusion of spin is natural. In the case of a particle with spin $\vec{s}$ there is in addition to (2) the interaction of its magnetic moment

$$\vec{\mu} = g\mu_B \vec{s}/\hbar$$  \hspace{1cm} (4)

($g$ being its gyromagnetic ratio) with the magnetic field (3) contributing

$$\Delta \hat{H} = \vec{\mu} \hat{H}$$  \hspace{1cm} (5)

to the Hamiltonian. As it stands, this is a point interaction and must be treated in an appropriate manner (see [9, 8] for example). From the mathematical point of view, one has to consider the corresponding Hamilton operator on a domain of functions vanishing on the flux line so that the term with the $\delta$-function disappears. On this domain the operator is not self-adjoint and its self-adjoint extensions (a one parameter family labeled by $\lambda$) define just all possible point interactions (3).

In the case of a neutral particle with magnetic moment (i.e. with the interaction (4)) this can be found in the book [9] within a general mathematical framework. For a spinor particle, using the Dirac equation, this analysis has been done in [4], [11, 14]. There it was shown that the self adjoint extensions can be defined by proper boundary conditions on the wave function on the flux line. Also, an analysis using a regularized $\delta$-function was done in [4] and [16]. Thereby the possibility of a bound state was discussed. Similar results exist for the spin 1 case [12].
In general, the Dirac equation leads to a magnetic moment which is characterized by a gyromagnetic ratio of \( g = 2 \). This case is exceptional from the point of view of its interaction with a magnetic flux line because the repulsive force of the AB effect is exactly compensated by the attractive force from the interaction of the magnetic moment with the flux (in case when they are antiparallel). This produces zero modes, i.e. bound states of zero binding energy. This situation was probably first mentioned in \(^2\) where it was shown that a spin-\( \frac{1}{2} \) particle (described by a Pauli (with \( g = 2 \)) as well as by Dirac equation) in a (in general, nonsingular) magnetic field of total flux \( \frac{\Phi}{(hc/e)} = N + \delta, \ 0 < \delta < 1 \) has \( N \) zero-energy normalizable eigenstates. It has the remarkable property, that its Hamilton operator factorizes and both equations have essentially the same form. This is a example for a supersymmetric quantum mechanical system.

Now, it is clear, that an anomalous magnetic moment destroys this property. Having in mind realistic particles like the electron with its anomaly factor

\[
a_e \equiv \frac{g - 2}{2} = 0.001159
\]

we consider in the present paper a particle with spin \( \vec{s} \) and gyromagnetic ratio \( g \) in 3 different, exactly solvable models of regularization of the \( \delta \)-function by a flux tube of radius \( R \) and establish their connection with the approach of self adjoint extensions. We consider to what extend this models correspond to different extensions. In each model there are \( N + 1 \) bound states in the case of the gyromagnetic ratio \( g \) being larger than two and the magnetic moment directed anti parallel to the magnetic flux and no bound state in the opposite case. This is by one bound state more than there are zero modes in the case \( g = 2 \). When shrinking the radius of the flux tube to zero, the gyromagnetic ratio must tend to 2 in order to have a finite bound state energy.

The models, we use, are (\( \vec{H} = H(r)\hat{e}_3 \))

1. \( H(r) = \frac{\Phi}{\pi R^2} \Theta(R - r) \) (homogeneous magnetic field inside) \( (7) \)
2. \( H(r) = \frac{\Phi}{2\pi R} \Theta(R - r) \) (magnetic field proportional to \( 1/r \) inside) \( (8) \)
3. \( H(r) = \frac{\Phi}{2\pi R} \delta(r - R) \) (a cylindrical shell with \( \delta \)-function) \( (9) \)

We consider for simplicity the nonrelativistic Hamilton operator

\[
\hat{H} = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 + \vec{\mu} \vec{H}
\]

with \( \vec{\mu} \) given by \( (4) \).

Due to spin conservation the magnetic interaction \( (3) \) can be replaced by

\[
\pm g\mu_B H(r)/2,
\]

\( \pm \) corresponding to the spin projection to the flux line. In the following we restrict ourselves to the minus sign, i.e. to the case when the magnetic moment leads to
a binding force. Further we choose \( \Phi > 0 \); for \( \Phi < 0 \) the spin direction must be reversed.

The paper is organized as follows. In the next section we consider the point interaction as self adjoint extension and obtain boundary conditions on the wave function at the origin. In the following section we consider magnetic flux tubes (7)-(9) with finite radius \( R \), write down the wave functions for bound states and scattering states. In the fourth section we consider the limit \( R \to 0 \) and establish its connection with the self adjoint extension as well as physical consequences. Conclusions are given in the last section.

2 Self Adjoint Extension

The Schrödinger equation for the problem under consideration can be written in the form

\[
\left( \frac{1}{2m} \left( p - \frac{e}{c} A \right)^2 - \frac{g \mu_B H(r)}{2} \right) \psi = E \psi
\]

(12)

where \( A \) and \( H \) are given by (1), (3) for an infinitely thin flux tube resp. by (7)-(9) for a finite flux tube. After separation of the angular dependence and the translational motion parallel to the flux tube by

\[
\psi(x) = \sum_{m=-\infty}^{\infty} \psi_m(r) e^{-im\varphi} e^{ip_3 x_3} \sqrt{\frac{2}{2\pi}}
\]

(13)

the equation reads (for simplicity we set \( p_3 = 0 \))

\[
\left( -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{(m - \delta a(r))^2}{r^2} - \frac{g \delta h(r)}{2} \right) \psi_m(r) = \epsilon \psi_m(r)
\]

(14)

with

\[
A = \frac{\phi}{2\pi r} a(r)
\]

and \( \delta = \frac{\Phi}{(hc/e)} \) is the flux measured in units of the flux quantum, \( h(r) = \frac{1}{r} \frac{\partial}{\partial r} a(r) \) is the radial distribution of the magnetic field (it is normalized according to \( \int_0^\infty dr \ r \ h(r) = 1 \)) and with the energy \( \epsilon = \frac{E}{(2m/k^2)} \).

Consider the case of an infinitely thin flux tube: \( a(r) = 1 \) and \( h(r) = 0 \) (\( r \neq 0 \)). In the mathematical analysis one has to start with the Hamilton operator

\[
\hat{H}_0 = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{(m - \delta)^2}{r^2}
\]

(15)

on a domain of functions

\[
\mathcal{D}(\hat{H}_0) = \{ \psi \in L^2([0, \infty)) \mid \psi(0) = 0 \}
\]

(16)

with the measure \( dr \ r \). The eigenfunctions of this operator are

\[
\psi_m(r) = J_{m-\delta} \left( \sqrt{\epsilon} r \right)
\]

(17)
\( m = 0, \pm 1, \ldots \). By means of the scalar product

\[
(\varphi, H_0 \psi) = \int_0^\infty dr \, r \, \varphi(r) \hat{H}_0 \psi(r),
\]

\( \hat{H}_0 \) is symmetric if

\[
\lim_{r \to 0} \left( \varphi^* r \frac{\partial}{\partial r} \psi - r \frac{\partial}{\partial r} \varphi^* \psi \right) = 0.
\]

For any \( \psi \in D \), there are - for the angular momentum \( m = N \), where \( N \) is the integer part of the flux

\[
\delta = N + \tilde{\delta} \quad (0 \leq \tilde{\delta} < 1),
\]

- functions \( \varphi \not\in D \), fulfilling this condition. They have the behaviour for \( r \to 0 \)

\[
\varphi \sim r^{-\tilde{\delta} + \lambda \tilde{\delta}}, \quad \lambda \text{ real}.
\]

So, the domain \( D \) of \( \hat{H}_0 \) can be enlarged by these singular modes (20). The parameter \( \lambda \) is arbitrary. Its dimension is \( r^{-2\tilde{\delta}} \).

Including this singular mode into the domain \( D \) of \( \hat{H}_0 \), it becomes self adjoint. Thereby different choices of \( \lambda \) lead to different self adjoint extensions.

The corresponding eigenfunctions in the continuous part of the spectrum \( (\epsilon = k^2) \) are

\[
\psi_S(r) = J_{\tilde{\delta}}(kr) + B_N(k) H_{\tilde{\delta}}^{(1)}(kr).
\]

In general, the scattering amplitude is defined by the asymptotics of the wave function for \( r \to \infty \)

\[
\psi(r) \approx e^{ikr \cos \varphi} + f(k, \varphi) \frac{e^{ikr}}{\sqrt{r}},
\]

where \( \varphi \) is the scattering angle. The usual Aharonov-Bohm scattering, i.e. without magnetic moment, corresponds to the first term in rhs of (21) and the scattering amplitude is well known. Expanding

\[
f(k, \varphi) = \sum_{m=-\infty}^{\infty} f_m(k) \frac{e^{-im\varphi}}{\sqrt{2\pi}},
\]

its contribution reads

\[
f_m^{AB}(k, \varphi) = \frac{1}{\sqrt{k}} \left( e^{i\pi(m-|m+\delta|)} - 1 \right) e^{-i\pi/4}.
\]

So, the presence of the contribution of the Hankel function in rhs of (21) which describes a outgoing cylindrical wave leads to an additional contribution to the scattering amplitude

\[
f_m(k) = f_m^{AB}(k) + \delta_{m,N} \frac{1}{\pi \sqrt{k}} B_N(k).
\]
There is one eigenfunction describing a bound state with binding energy $\kappa = -\epsilon$:

$$\psi_B(r) = K_\delta \left( \sqrt{\kappa r} \right).$$  \hfill (25)

Now, from the expansion of the solutions (25) and (21) for $r \to 0$ we obtain by means of (20) the connection of the bound state energy $\kappa$ with the parameter $\lambda$ of the self adjoint extension:

$$\lambda = -\frac{\Gamma(1 - \tilde{\delta})}{\Gamma(1 + \tilde{\delta})} \left( \sqrt{\kappa/2} \right)^{2\delta}. \hfill (26)$$

From this formula it follows that the bound state occurs in the case of negative parameter of the extension $\lambda$ only. For the scattering states we obtain from (22) and (20)

$$B_N(k) = i\sin \pi \tilde{\delta} + \frac{1}{\lambda(k/2)^2} \frac{\Gamma(1 + \tilde{\delta})}{\Gamma(1 - \tilde{\delta})} \left( \sqrt{\kappa/2} \right)^{2\delta}. \hfill (27)$$

So, for any parameter of the parameter $\lambda$ of the extension, there is an additional scattering and for $\lambda < 0$ there is a bound state. In that latter case the scattering amplitude can be expressed in terms of the bound state energy

$$B_N(k) = \frac{i\sin \pi \tilde{\delta}}{e^{-i\pi \tilde{\delta}} - \left( \frac{k}{\kappa} \right)^\delta}. \hfill (28)$$

### 3 Three Models

The regularization of the $\delta$-function interaction can be done by many different models for a finite flux tube. We consider here the simplest examples, which are exactly solvable. We write down the wavefunction inside the tube and stick them to the outside function.

The outside function ($r > R$) is a eigenfunction of the Hamilton operator (15). For $\epsilon < 0$ it is given by

$$\psi_m(r) = K_{m-\delta}(\sqrt{-\epsilon r}) \hfill (29)$$

and describes the bound state solution. Its logarithmic derivative reads

$$R_{ex} \equiv R \frac{\partial}{\partial r} \ln \psi_m(r)|_{r=R+0} = \sqrt{-\epsilon R} \frac{K'_{m-\delta} \left( \sqrt{-\epsilon R} \right)}{K_{m-\delta} \left( \sqrt{-\epsilon R} \right)}. \hfill (30)$$

For $\epsilon = k^2 > 0$ we obtain the outside scattering solution ($r > R$)

$$\psi_m(r) = J_{m-\delta}(kr) + B_m(k) H_{m-\delta}^{(1)}(kr) \hfill (31)$$

and its logarithmic derivative reads

$$R_{ex} \equiv r \frac{\partial}{\partial r} \ln \psi_m(r)|_{r=R+0} = \sqrt{\epsilon R} \frac{J'_{m-\delta} \left( \sqrt{\epsilon R} \right) + B_m(k) H_{m-\delta}^{(1)' \left( \sqrt{\epsilon R} \right)}}{J_{m-\delta} \left( \sqrt{\epsilon R} \right) + B_m(k) H_{m-\delta}^{(1)} \left( \sqrt{\epsilon R} \right)}. \hfill (32)$$
Below we are interested in the limit \( R \to 0 \). For \( \epsilon < 0 \) we note

\[
R_{ex} = \begin{cases} 
- |m - \delta| - 2|m - \delta| \Gamma(1 + \frac{|m - \delta|}{2}) \left( \frac{\sqrt{-\epsilon R}}{2} \right)^{2|m - \delta|} + \ldots & |m - \delta| < 1 \\
- |m - \delta| - 2 \frac{1}{|m - \delta| - 1} \left( \frac{\sqrt{-\epsilon R}}{2} \right)^{2} + \ldots & |m - \delta| > 1,
\end{cases}
\]

where two cases have to be distinguished.

### 3.1 Homogeneous Magnetic Field

In this model the magnetic field inside is homogeneous and zero outside. The functions \( h(r) \) and \( a(r) \) read:

\[
h(r) = \frac{2}{R^2} \Theta(R - r), \quad a(r) = \frac{r^2}{R^2} \Theta(R - r) + \Theta(r - R).
\]

The Schrödinger equation reads

\[
\left( - \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{(m - \delta)^2}{r^2} - \frac{g^2 \delta^2}{2 R^2} \right) \psi_m(r) = \epsilon \psi_m(r).
\]

The solution regular in \( r = 0 \) is given by

\[
\psi_m(r) = r^{|m|} \, _1F_1\left( \frac{2 - g}{4} + \frac{|m| - m}{2} - \frac{\epsilon R^2}{2\delta}, 1 + |m|, \frac{\delta r^2}{R^2} \right) e^{-\frac{2\delta r^2}{2 R^2}}.
\]

We need its logarithmic derivative at \( r = R \)

\[
R_1 \equiv R \frac{\partial}{\partial r} \psi_m(r)_{r=R=0}
\]

for \( m \geq 0 \) and use the notation \( x \equiv \sqrt{-\epsilon R} \):

\[
R_1 = |m| - \delta + \frac{2 - g}{4} + \frac{x^2}{4\delta} \, _1F_1\left( \frac{2 - g}{4} + 1 + \frac{x^2}{4\delta}, 2 + |m|; \delta \right) \, _1F_1\left( \frac{2 - g}{4} + \frac{x^2}{4\delta}, 1 + |m|; \delta \right).
\]

For \( x \to 0 \) we note

\[
R_1 = |m| - \delta + \frac{2 - g}{2} \delta m \alpha_1 + x^2 \beta_1 + \ldots
\]

with

\[
\alpha_1 = \frac{1}{2(1 + |m|)} \, _1F_1\left( \frac{2 - g}{4} + 1, 2 + |m|; \delta \right) \, _1F_1\left( \frac{2 - g}{4}, 1 + |m|; \delta \right), \quad \beta_1 = \frac{\partial}{\partial g} \frac{2 - g}{2} \alpha_1.
\]

The properties \( \alpha_1 > 0 \) and \( \beta_1 < 0 \) can be checked.
3.2 Magnetic Field Proportional to $1/r$

In this model we have

$$h(r) = \frac{1}{rR} \Theta(R - r), \quad a(r) = \frac{r}{R} \Theta(R - r) + \Theta(r - R).$$ (39)

The corresponding equation reads

$$\left( -\frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{(m - \delta \frac{r}{R})^2}{r^2} - \frac{g}{2} \frac{\delta}{rR} \right) \psi_m(r) = \epsilon \psi_m(r).$$ (40)

and it has a solution regular in $r = 0$

$$\psi_m(r) = r^{|m|} \frac{1}{|m|} \frac{\Gamma_{1F1}}{\Gamma_{1F1}} \left( \frac{1}{2} - |m| - \frac{\delta}{\delta} \right) - \frac{g}{4} \delta \frac{\delta}{\delta} \frac{1 + 2|m|; 2\delta \frac{r}{R}}{1 + 2\delta \frac{r}{R}} e^{-\delta \frac{r}{R}}$$ (41)

with the notation $\delta \equiv \sqrt{\delta^2 - \epsilon R^2}$. Its logarithmic derivative reads

$$R_2 \equiv r \frac{\partial}{\partial r} \psi_m(r)_{|r=R-0}$$

$$= |m| - \delta + 2 \frac{\left( \frac{1}{2} + |m| \right) \delta - (m + g/4) \delta}{1 + 2|m|} \frac{1}{\Gamma_{1F1}} \frac{\Gamma_{1F1}}{\Gamma_{1F1}} \left( \frac{1}{2} + |m| - (m + g/4) \delta \frac{2 + 2|m|; 2\delta}{1 + 2|m|; 2\delta} \right).$$

For $m \geq 0$ and $x \equiv \sqrt{-\epsilon R} \rightarrow 0$ we note

$$R_2 = m - \delta + \frac{2 - g}{2} \delta \alpha_2 + \beta_2 x^2 + ...$$ (42)

with

$$\alpha_2 = \frac{1}{1 + 2m} \frac{1}{\Gamma_{1F1}} \frac{\Gamma_{1F1}}{\Gamma_{1F1}} \left( \frac{2 - g}{4} + 1, 2 + 2m; 2\delta \right)$$

and

$$\beta_2 = \frac{1}{2 \delta} \left( (1 + 2m) \alpha_2 - 1 + \frac{2 - g}{2} \left( \frac{\partial}{\partial \delta} - (g + 4m) \frac{\partial}{\partial g} \right) \alpha_2 \right).$$

Also in this case the properties $\alpha_2 > 0$ and $\beta_2 < 0$ can be checked.

3.3 Cylindrical Shell with $\delta$-function

Moving the $\delta$-function from $r = 0$ to $r = R$ one obtains a cylindrical shell on which the magnetic field is infinite

$$a(r) = \Theta(R - r), \quad h(r) = \frac{1}{R} \delta(r - R).$$

2This model is intensively used in [10].
The radial equation reads
\[
\left( -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{(m - \delta \Theta(R - r))^2}{r^2} - \frac{g}{2} \frac{1}{R} \delta(r - R) \right) \psi_m(r) = \epsilon \psi_m(r). \tag{43}
\]

In this case the \(\delta\)-function because moved away from the origin can be treated as usual in 1-dimensional case and substituted by the known boundary conditions
\[
\left. r \partial_r \psi_m(r) \right|_{R^+} = -\frac{g}{2} \delta \psi_m(r) \big|_{R^0}.
\tag{44}
\]

Then the solution of eq. (43) are Bessel functions
\[
\psi_m(r) = \begin{cases} 
\alpha J_{|m|}(\sqrt{\epsilon}r) & \text{for } r < R \\
J_{|m-\delta|}(kr) + B_m(k) H_{|m-\delta|}^{(1)}(kr) & \text{for } r > R
\end{cases}
\tag{45}
\]
with some coefficient \(\alpha\) and from condition (44) it follows
\[
R_3 \equiv -\frac{1}{2} g \delta + r \left. \frac{\partial \psi_m(r)}{\partial r} \right|_{r=R^0} = -\frac{1}{2} g \delta + \sqrt{\epsilon} R \left. \frac{J'_{|m|}(\sqrt{\epsilon}r)}{J_{|m|}(\sqrt{\epsilon}r)} \right|_{r=R^0} \quad \text{for } \epsilon > 0,
\]
\[
= -\frac{1}{2} g \delta + \sqrt{-\epsilon} R \left. \frac{I'_{|m|}(\sqrt{-\epsilon}r)}{I_{|m|}(\sqrt{-\epsilon}r)} \right|_{r=R^0} \quad \text{for } \epsilon < 0.
\]

For \(x \equiv \sqrt{-\epsilon} R \to 0\) we have
\[
R_3 = |m| - \delta + \frac{2 - g}{2} \delta \alpha_3 + \beta_3 x^2 + \ldots \tag{46}
\]
with
\[
\alpha_3 = 1, \quad \beta_3 = \frac{1}{2(1 + |m|)}.
\]

### 3.4 Bound State Energy and Scattering Amplitude

The solutions in all three models are determined by the condition
\[
R_{\text{ex}} = R_i \quad (i = 1, 2, 3). \tag{47}
\]

There are scattering solutions for all values of the parameters. They can be obtained by solving (17). The scattering amplitude reads
\[
B_m(k) = -\left( x \frac{\partial}{\partial x} - R_i \right) \left. J_{|m-\delta|}(x) \over (x \frac{\partial}{\partial x} - R_i) H_{|m-\delta|}^{(1)}(x) \right|_{x=kR}. \tag{48}
\]

Let us consider the bound state solutions. They does not exist for all values of the parameters. Consider the behaviour of \(R_i\) and \(R_{\text{ex}}\) as function of \(x \equiv \sqrt{-\epsilon} R\). It can be seen that \(R_{\text{ex}}\) decrease starting from \(R_{\text{ex}}(0) = -|m - \delta|\) (cf. (33)) while \(R_i(x)\)
increase starting from $R_i(x) = |m| - \delta + \frac{2-g}{2} \delta m \alpha_i$ (cf. (38), (42), (46)). So, solutions with binding energy $\kappa_m \equiv \sqrt{-\epsilon} = x/R$ of eq. (47) are possible for

$$g > 2, \quad 0 \leq m < \delta \left( 1 + \frac{g-2}{4} \alpha_i \right). \quad (49)$$

In the case $g = 2$ all solutions have vanishing energy, i.e. they correspond to zero modes. In general, the solution of (47) reads

$$x = f(\delta, g, m) \quad \text{with} \quad x = \kappa_m R, \quad (50)$$

where $f$ is some dimensionless function. Some lowest solutions of eq. (47) are shown in the figure for the model with the cylindrical $\delta$-shell.

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**Fig.** The solutions $x = f(d, g, m)$ of eq. (50) for $m = 0, 1, \ldots, 4$ and $g = 2.2, 2.1, 2.05, 2.01$

Similar pictures can be drawn for the other two models. It can be seen that, in general, there is no simple rule for the energy levels $\kappa_m = x/R$ for general values of the parameters.

## 4 The Limit $R \to 0$

In the limit $R \to 0$, all other parameters fixed, the bound state energy increases unbounded as can be seen from (50). This indicates, that the limit $R \to 0$ in the models with finite flux tube is not physical at last in the nonrelativistic approximation choosen here.

One possibility to obtain a finite energy is to consider a small anomaly of the
magnetic moment, i.e. to consider the case

\[ a_e \equiv \frac{g - 2}{2} \to 0. \]

In that case all solutions \( x \) of eq. (50) tend to zero and the equation (47), which defines the bound states, can be solved. We obtain, using (33) and (38), (42), (46), for the highest angular momentum

\[ \frac{g - 2}{2} \delta \alpha_i = \left( \frac{\sqrt{\kappa_N R}}{2} \right)^{2\delta} \frac{2\Gamma(1 - \delta)}{\Gamma(1 + \delta)} \quad (m = N), \quad (51) \]

where \( N \) is the integer part of the flux, and

\[ \frac{g - 2}{2} \delta \alpha_i = \left( \sqrt{\kappa_m R} \right)^2 \left( \beta_i + \frac{1}{4(|\delta - m| - 1)} \right) \quad (m = 0, 1, \ldots, N - 1) \quad (52) \]

for the lower angular momenta.

For \( g \to 2, R \) fixed, \( \kappa_m \) \((m = 0, 1, \ldots, N - 1)\) tends to zero proportional to \( g - 2 \), whereas \( \kappa_N \) behaves like \((g - 2)^{1/\delta}\), i.e. tends more quickly to zero. Therefore, \( \kappa_m \) correspond to the zero modes (for \( g = 2 \)), whereas the state with \( \kappa_N \) has no correspondence in that case. It can be expected that its wave function vanishes.

For \( R \to 0 \), in the case \( N = 0 \), i.e. the flux being less than unity, a finite binding energy of the only bound state can be obtained by substituting

\[ \frac{g - 2}{2} \to \frac{2R^2}{g - 2} \quad (53) \]

in the initial equation (12) and after that performing the limit \( R \to 0 \). After the substitution (53), the bound state energy is determined by (instead of (51))

\[ \left( \frac{\sqrt{\kappa_0}}{2} \right)^{2\delta} = \frac{\Gamma(1 + \delta)}{\Gamma(1 - \delta)} \frac{g - 2}{2} \quad (54) \]

and we observe from eq. (26) that the parameter \( \lambda \) of the extension is just (up to the sign) the anomaly of the magnetic moment:

\[ \lambda = -\frac{g - 2}{2}. \quad (55) \]

This treatment of the \( \delta \)-function is equivalent to the general approach to 2-dimensional \( \delta \)-function in the Schrödinger equation by Berezin, Faddeev [8] and Albeverio [9], where the necessity of the renormalization of the coupling was pointed out.

In the case of fluxes larger than unity \((N \geq 1)\), there are bound states with energy \( \kappa_m \) larger than \( \kappa_N \) and the renormalization (53) is not sufficient to keep them finite. Instead, by means of (52), one must substitute

\[ \frac{g - 2}{2} \to \frac{g - 2}{2} R^2 \quad (56) \]
in the initial equation (12). Then the binding energies read (instead of (24))

\[ \kappa_m = \frac{g - 2}{2} \frac{\delta \alpha_i}{\beta_i + 1/(4(|\delta - m| - 1))} \]  (57)

\((m = 0, 1, \ldots N - 1)\). In this case we have \(\kappa_N = 0\).

A different way of understanding the limit \(R \to 0\) is to keep \(R\) small, but finite. In that case there is a natural scale given at the one hand side by the value of the anomaly of the magnetic moment of the electron

\[ a_e = \frac{g - 2}{2} = 0.001159 \]

and on the other hand side by the bound state energy to be nonrelativistic, i.e. smaller than the electron mass because we consider the nonrelativistic Schrödinger equation.

In the case of flux less than unity the energy is nonrelativistic for \(\kappa_0 << 1/\lambda_c\), the inverse Compton wavelength of the electron, and the radius must by means of eq. (51) fulfill

\[ R >> \left( \frac{g - 2}{2} \frac{\Gamma(1 + \delta)}{\beta_i + 1/4(|\delta - m| - 1)} \right)^{1/(2\delta)} \lambda_c \]  (58)

Thereby it can be taken smaller than \(\lambda_c\) so that the flux tube can be considered as thin. Lets remark, that the considerations done here for small \(x = \sqrt{\kappa R}\) mean that the size of the orbit of the bound states is much larger than the radius \(R\).

Similar considerations apply to the case of the flux being larger than unity. Here, the radius must fulfill

\[ R >> \left( \frac{g - 2}{2} \frac{\delta \alpha_i}{\beta_i + 1/(4(|\delta - m| - 1))} \right)^{1/2} \lambda_c \]  (59)

This condition is stronger than (58). Nevertheless, \(R\) may be made smaller than \(\lambda_c\), so that the flux tube can be thin in this case too.

Let’s consider the limit \(R \to 0\) for scattering states, i.e. \(k^2 = \epsilon > 0\). Expanding \(B_m(k)\) (48), i.e. the additional scattering amplitude for a given angular momentum \(m\) and energy \(k^2\), for \(x \to 0\):

\[ B_m(k) = \frac{i \sin \pi \nu \left( \frac{x}{2} \right)^{2\nu}}{\Gamma(1+\nu)(2\nu+2\delta \alpha_i)} \left( 1 + \left( \beta_i - \frac{2\delta \alpha_i - 2}{8(1-\nu)} \right) x^2 \right) + \left( \frac{x}{2} \right)^{2\nu} e^{-i\nu} \]  (60)

with \(\nu = |m - \delta|\). From this formula it can be seen, that \(B_m(k)\) vanishes in the limit \(R \to 0\), all other parameters fixed. This is meaningful in the case \(g < 2\) where there are no boundstates.

For \(g > 2\), as shown above, in order to have finite bound state energies, the limit \(R \to 0\) must be performed together with \(g \to 2\).
For $0 \leq \delta < 1$, one must use the substitution (53) and obtains

$$B_0(k) \approx \frac{i \sin \pi \nu}{e^{-i\pi \nu} - \frac{g-2}{2} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \left(\frac{k}{2}\right)^{-2\delta}},$$

(61)

i.e. the same formula as in the method of self-adjoint extensions (27).

For flux larger than one, i.e. $\delta = N + \bar{\delta} > 1$, one has to use the substitution (56) and obtains

$$B_N(k) = \frac{1}{2} \left(e^{2i\pi \bar{\delta}} - 1\right) \quad (m = N)$$

(62)

and

$$B_m(k) \approx R^{2(\bar{\delta}-m-1)} \to 0 \quad (m = 0, 1, \ldots N-1)$$

(63)

So, the additional scattering takes place for the highest angular momentum only.

## 5 Conclusions

We considered a charged quantum mechanical particle with spin $\frac{1}{2}$ and gyromagnetic ratio $g \neq 2$ in the field of a magnetic string. Whereas the interaction of the charge with the string is the well known Aharonov-Bohm effect and the contribution of the magnetic moment associated with the spin in the case $g = 2$ is known to yield an additional scattering and zero modes (one for each flux quantum), an anomaly of the magnetic moment (i.e. $g > 2$) leads to bound states. We considered two methods for treating the case $g > 2$.

For an ideal string the interaction of the spin with the magnetic field (1) is pointlike and singular; i.e. the magnetic field contains the 2-dimensional $\delta$-function (3). A mathematical approach to treat this is the method of self-adjoint extension. It yields a family of operators labeled by a real parameter $\lambda$. For all values of this operator there is an additional scattering amplitude (24) resulting from the contribution of the magnetic moment and for $\lambda < 0$ there is one bound state (26). The main goal of the extension is to include a singular solution (21) resp. (25) into the domain of the Hamilton operator (15). It should be remarked that this method is - although mathematically correct (or, at last, may be made correct) - not satisfactory from the physical point of view because the parameter $\lambda$ of the extension is not correlated to physical parameters like the gyromagnetic ratio which does not enter this method at all.

A different method is to consider nonsingular flux tubes and shrinking its radius to zero. This is equivalent to regularize the $\delta$-function in the magnetic field by some less singular profile. We used 3 models for which the Schrödinger equation (12) (it is really a Pauli equation in this case) can be solved explicitly. The common result is that there is an additional scattering due to the magnetic moment and that for $g > 2$ there are bound states. This is not surprisingly since the existence of zero mode for $g = 2$ suggests the binding of a particle as soon as there is any additional attractive force. For $g$ slightly larger than 2 there are $N + 1$ bound states, where $N$
is the integer part of the flux. In general, the dependence of the energy levels \( \kappa_m \) on \( g \) and on the flux is complicated as can be seen from the figure.

We considered the case \( g \to 2 \). Thereby all \( \kappa_m \) tend to zero. For \( m = 0, 1, \ldots, N-1 \), the solutions become the mentioned zero modes, for \( m = N \) the solution vanishes.

The limit \( R \to 0 \), all other parameters fixed, is not physical for \( g > 2 \). The reason is that the bound state energies \( \kappa_m \) enter the defining equation (50) multiplied by the flux tube radius \( R \), which follows from general dimensional considerations too. So, \( \kappa_m \) tend to infinity for \( R \to 0 \).

One possibility to treat this problem is to tend the gyromagnetic ratio \( g \) in the initial equation to 2 along with \( R \to 0 \). This can be viewed as some renormalization of \( g \). Thereby two cases have to be distinguished. Firstly, when the flux is less than unity, \( g - 2 \) tend to zero proportional to \( R^{2a} \), eq. (53). For \( g > 2 \), there is one bound state, its energy is given by eq. (54). This is essentially the same situation as in the method of self adjoint extension and the parameter \( \lambda \) of the extension can be related to the gyromagnetic ratio, eq. (55). Thereby the dependence on the parameters of the models used enters the renormalization (53) of the gyromagnetic ratio only. Such a renormalization is known in the mathematical approach [8], too. For the scattering amplitude there is a contribution in addition to the usual Aharonov-Bohm scattering (eq. (61)). It is given by the same formula as in the method of extension, eq. (28). So, for \( \delta < 1 \) both approaches are equivalent.

A different feature one observes for flux larger than unity. In order to keep all bound state energies finite, one is forced to tend \( g - 2 \) to zero proportional to \( R^2 \), eq. (56), i.e. much faster than in the previous case. Thereby the energy of the state \( \kappa_N \) (with the highest angular momentum \( m = N \)) tends to zero whereas for \( m = 0, 1, \ldots, N-1 \) the energies \( \kappa_m \) are finite. The additional scattering in this case takes place for \( m = N \) only, eq. (62): for \( m = 0, 1, \ldots, N-1, B_m(k) \) (eq. (53)) vanishes for fixed parameters. This is clear because the wave functions in this case are concentrated in the region of small \( R \). For sufficiently high momenta \( k \) scattering can be expected in this case too. So we conclude, that for flux larger than one both approaches yield different results. This must be not a contradiction by the following reason. In the method of self adjoint extension, the input information, which is contained in the Hamilton operator (15) makes no reference to the magnetic moment of the particle as well as not to the magnetic field and, moreover, no reference to the integer part \( N \) of the magnetic flux by means that it enters in the combination \( m - \delta \) only. So, as from the other method it is known, that the existence of bound states requires \( g > 2 \). In the same manner we conclude that the extension is applicable for flux less than one only, i.e. that a flux larger than one is too singular and cannot be described by the method of self adjoint extension.

The physically interesting case is to keep the flux tube radius small, but finite. The natural scale for the smallness of \( g - 2 \) is given by the anomaly \( a_e = (g - 2)/2 \) (3) of the magnetic moment of the electron and by the requirement the radius of the tube being not too small in order to have nonrelativistic bound state energies \( \kappa_m \ll m_e \) (\( m_e \) - the electron mass) because we use a nonrelativistic equation. Under these conditions there are approximately \( N + 1 \) bound states. The exact number is
given by eq. (49) and depends on the model used. However, this dependence is weak for small \( a_e \) as can be seen from eq. (49) where \( \alpha_i \) enters multiplied by \( g - 2 \). Furthermore, it should be emphasized that the physical restrictions to \( R \) and \( g - 2 \) allow the flux tube to be thin in that sense that \( \kappa_m R \ll 1 \) is possible, i.e. the orbit size of the bound states is much larger than the radius of the flux tube.

So, we conclude that for a gyromagnetic ratio larger than 2 the flux tube cannot be shrunk to a line for real physical parameters. A natural extension of these investigations would be the consideration of the Dirac equation with additional magnetic moment (i.e. including a term \( (g - 2) \sigma^{\mu\nu} F_{\mu\nu} \)). In that case, the limitation to the bound state energy being nonrelativistic is not necessary and smaller \( R \) can be considered. Furthermore, one can speculate that the anomaly of the magnetic moment, which is known to decrease in strong magnetic fields \([15]\), will eventually influence the limit \( R \rightarrow 0 \).

A further open question is, whether the interaction, which comes from the anomaly \( a_e \) of the magnetic moment, can be treated in perturbation theory with respect to \( a_e \) starting from the known solutions (especially from the zero mode of [2]) for an arbitrary profile of the magnetic field inside a finite flux tube.

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