A new approach to the $\star$-genvalue equation

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Abstract

We show that the eigenvalues and eigenfunctions of the star-genvalue equation can be completely expressed in terms of the corresponding eigenvalue problem for the quantum Hamiltonian. Our method makes use of a Weyl-type representation of the star-product and of the properties of the cross-Wigner transform, which appears as an intertwining operator.

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1 Introduction and Motivation

One of the key equations in the deformation quantization theory of Bayen et al. [1, 2] is, no doubt, the star-genvalue (for short $\star$-genvalue) equation $H \star \Psi = E \Psi$ where $\star$ is the Moyal–Groenewold “star-product” [1, 2, 7]. In this Letter we show that the $\star$-genvalue equation can be completely solved

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in terms of the usual eigenvalue/eigenfunction problem \( \hat{H} \psi = E \psi \) where \( \hat{H} \) is the Weyl operator with symbol \( H \) (and vice versa). The underlying idea is simple: we first rewrite the equation \( H \star \Psi = E \Psi \) in the form

\[
H(x + \frac{1}{2}i \hbar \partial_p, p - \frac{1}{2}i \hbar \partial_x) \Psi(x, p) = E \Psi(x, p),
\]

where \( H(x + \frac{1}{2}i \hbar \partial_p, p - \frac{1}{2}i \hbar \partial_x) \) is the Weyl operator with symbol

\[
\mathbb{H}(z, \zeta) = H(x - \frac{1}{2} \zeta p, p + \frac{1}{2} \zeta x).
\]

We next show that the solutions of this equation and those of \( \hat{H} \psi = E \psi \) can be obtained from each other using a family of intertwining operators (which is countable when \( \hat{H} \) is essentially self-adjoint); these operators are up to a normalization factor, the cross-Wigner transforms \( \psi \mapsto \hat{W}(\psi, \phi) \) where \( \phi \) describes the set of eigenfunctions of \( \hat{H} \). Our approach is inspired by previous work [4] of one of us on the time-dependent Torres-Vega [8] Schrödinger equation in phase space.

**Notation**

We will write \( z = (x, p) \) where \( x \in \mathbb{R}^n \) and \( p \in (\mathbb{R}^n)^* \). Operators \( \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) are usually denoted by \( \hat{A}, \hat{B}, ... \) while operators \( \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n}) \) are denoted by \( \tilde{A}, \tilde{B}, ... \). The Greek letters \( \psi, \phi, ... \) stand for functions defined on \( \mathbb{R}^n \) while their capitalized counterparts \( \Psi, \Phi, ... \) denote functions defined on \( \mathbb{R}^{2n} \). We will make use of the symplectic Fourier transform defined for \( \Psi \in \mathcal{S}(\mathbb{R}^{2n}) \) by the formula

\[
\Psi^h_\sigma(z) = F^h_\sigma \Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \sigma(z, z')} \Psi(z')dz'.
\]

where \( \sigma(z, z') = p \cdot x' - p' \cdot x \) is the standard symplectic form on \( \mathbb{R}^n \times (\mathbb{R}^n)^* \equiv \mathbb{R}^{2n} \) (the dot \( \cdot \) stands for the duality bracket; in practice \( p \cdot x \) can be seen as the usual Euclidean scalar product under the identification \( (\mathbb{R}^n)^* \equiv \mathbb{R}^n \)). The symplectic Fourier transform is involutive: \( F^h_\sigma \circ F^h_\sigma = \text{identity on } \mathcal{S}'(\mathbb{R}^{2n}) \).

**2 Stargenvalue Equation: Short Review**

In view of Schwartz’s kernel theorem every linear continuous operator \( \hat{A} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) can be represented, for \( \psi \in \mathcal{S}(\mathbb{R}^n) \), in the form \( \hat{A} \psi(x) = ... \)
\langle K_A(x, \cdot), \psi \rangle$ with $K_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. By definition the contravariant (Weyl) symbol of $A$ is the tempered distribution $A$ defined by the Fourier transform

$$a(x, p) = \left< e^{\frac{i}{\hbar}p(x)}, K_A(x + \frac{1}{2} \cdot, x - \frac{1}{2} \cdot) \right>.$$  \hfill (1)

Assume that $\hat{B} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$; then the product $\hat{C} = \hat{A} \circ \hat{B}$ exists and its Weyl symbol is given by the Moyal product

$$a \star b(z) = \left(\frac{1}{4\pi \hbar}\right)^{2n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{\hbar}\sigma(u, v)} a(z + \frac{1}{2}u)b(z - \frac{1}{2}v)dudv.$$  \hfill (2)

The main observation we will exploit in this paper is the following: if we write $a = H$ and $b = \Psi$ then we can write

$$H \star \Psi = \tilde{H}\Psi,$$  \hfill (3)

where

$$\tilde{H} = H(x + \frac{1}{2}i\hbar\partial_p, p - \frac{1}{2}i\hbar\partial_x)$$

is a certain pseudodifferential operator on $\mathcal{S}(\mathbb{R}^{2n})$ we are going to identify.

Let us view the linear operator $\tilde{H} : \Psi \mapsto H \star \Psi$ on $\mathcal{S}(\mathbb{R}^{2n})$ as a Weyl operator. Using formula (2), the kernel of $\tilde{H}$ is the distribution

$$K_{\tilde{H}}(z, y) = \left(\frac{1}{2\pi \hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(u, z-y)} H(z - \frac{1}{2}u)du,$$  \hfill (4)

hence using (1) and the Fourier inversion formula the contravariant symbol of $\tilde{H}$ is

$$\mathbb{H}(z, \zeta) = \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\zeta\cdot\eta} K_{\tilde{H}}(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta)d\eta.$$  

Using (1) and performing the change of variables $u = 2z + \eta - z'$ we get

$$K_{\tilde{H}}(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta) = \left(\frac{1}{2\pi \hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(\eta, z')} H(\frac{1}{2}z')d\zeta';$$

setting $H(\frac{1}{2}z') = H_{1/2}(z')$ the integral is $(2\pi \hbar)^n$ times the symplectic Fourier transform $\hat{F}_\sigma H_{1/2}(-\eta) = (H_{1/2})_\sigma(-\eta)$ so that

$$\mathbb{H}(\frac{1}{2}z, \zeta) = \left(\frac{1}{2\pi \hbar}\right)^{n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\zeta\cdot\eta} e^{\frac{i}{\hbar}\sigma(\eta, z)} (H_{1/2})_\sigma(-\eta)d\eta$$

$$= \left(\frac{1}{2\pi \hbar}\right)^{n} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z, J\zeta\cdot\eta)} (H_{1/2})_\sigma(\eta)d\eta$$

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where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard symplectic matrix. Since the second equality is the inverse symplectic Fourier transform of $(H_{1/2})_\sigma$ calculated at the point $z + J\zeta$. We finally get

$$\mathbb{H}(z, \zeta) = H(x - \frac{1}{2}\zeta_p, p + \frac{1}{2}\zeta_x)$$

where we are viewing $\zeta = (\zeta_x, \zeta_p)$ as the dual variable of $z = (x, p)$; this justifies formula (3) viewing $\tilde{H}$ as the quantized Hamiltonian obtained from $H$ by the quantum rule

$$ (x, p) \mapsto (x + \frac{1}{2}i\hbar\partial_p, p - \frac{1}{2}i\hbar\partial_x).$$

(6)

### 3 $\tilde{H}$-Calculus

There is another very fruitful way of interpreting the Weyl operators $\tilde{H} = H \star \Psi$. Let us return to the expression (2) with $a = H$ and $b = \Psi$; performing the changes of variable $u = 2(z' - z)$ and $v = z_0$ this formula can be rewritten as

$$\tilde{H}\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z_0, z')} H(z') dz' \left[ e^{-\frac{i}{\hbar}\sigma(z, z_0)} \Psi(z - \frac{1}{2}z_0) dz_0. \right]$$

Observing that the integral between brackets is $(2\pi\hbar)^n$ times the symplectic Fourier transform of $H$ we can write this formula in the form

$$\tilde{H}\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} H^h_{\sigma}(z_0) \tilde{T}(z_0) \Psi(z) dz_0$$

(7)

where $\tilde{T}(z_0)$ is the operator defined by

$$\tilde{T}(z_0) \Psi(z) = e^{-\frac{i}{\hbar}\sigma(z, z_0)} \Psi(z - \frac{1}{2}z_0).$$

(8)

Formula (7) is strongly reminiscent of the representation

$$\hat{H}\psi = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} H^h_{\sigma}(z_0) \hat{T}(z_0) \psi dz_0$$

(9)

of a Weyl operator $\hat{H}$ in terms of its covariant symbol $H^h_{\sigma} = F^h_{\sigma} H$ and the Heisenberg–Weyl operator

$$\hat{T}(z_0) \psi(x) = e^{\frac{i}{\hbar}(p_0 \cdot x - \frac{1}{2}p_0 \cdot x_0)} \psi(x - x_0),$$
except that $\widetilde{T}(z_0)$ is allowed to act on functions of $z$ and not only of $x$. This feeling is amplified when one notes after a straightforward calculation that the operators $\widetilde{T}(z_0)$ obey the relations

$$\widetilde{T}(z_0 + z_1) = e^{-\frac{i}{\hbar} \sigma(z_0, z_1)} \widetilde{T}(z_0) \widetilde{T}(z_1)$$  \hspace{1cm} (10)

$$\widetilde{T}(z_1) \widetilde{T}(z_0) = e^{-\frac{i}{\hbar} \sigma(z_0, z_1)} \widetilde{T}(z_0) \widetilde{T}(z_1)$$  \hspace{1cm} (11)

which are similar to those satisfied by the Heisenberg–Weyl operators. These facts suggest that $\widetilde{T}(z_0, t) = e^{it\hbar} \widetilde{T}(z_0)$ defines a unitary representation of the Heisenberg group. Let us prove this is indeed the case. For this we will need the linear mapping $W_{\phi} : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^{2n})$ defined by

$$W_{\phi} \psi = (2\pi \hbar)^{n/2} W(\psi, \phi) \hspace{1cm} (12)$$

where $\phi$ denotes an arbitrary function in $S(\mathbb{R}^n)$ such that $||\phi||_{L^2} = 1$. $W(\psi, \phi)$ is the cross-Wigner distribution; we thus have explicitly

$$W_{\phi} \psi(z) = \int_{\mathbb{R}^n} e^{i\hbar p \cdot (x+y)} \overline{\psi(x+\frac{1}{2}y)} \phi(x-\frac{1}{2}y) dy. \hspace{1cm} (13)$$

In view of Moyal’s identity

$$(W(\psi, \phi)|W(\psi', \phi'))_{L^2(\mathbb{R}^{2n})} = \left(\frac{1}{2\pi \hbar}\right)^n (\psi|\psi')_{L^2(\mathbb{R}^n)} (\phi|\phi')_{L^2(\mathbb{R}^n)}$$

the operator $W_{\phi}$ extends into an isometry of $L^2(\mathbb{R}^n)$ onto a subspace $\mathcal{H}_{\phi}$ of $L^2(\mathbb{R}^{2n})$; we are going to see in a moment $\mathcal{H}_{\phi}$ is closed in $L^2(\mathbb{R}^{2n})$, but let us first give a formula for the adjoint $W^*_{\phi}$ of $W_{\phi}$. We have

$$W^*_{\phi} \Psi(z) = \int_{\mathbb{R}^n} e^{i\hbar p \cdot (x-y)} \phi(2y-x) \overline{\Psi(y,p)} dp dy \hspace{1cm} (14)$$

(it follows from a straightforward calculation using the identity $(W_{\phi} \psi|\Psi)_{L^2(\mathbb{R}^{2n})} = (\psi|W^*_{\phi} \Psi)_{L^2(\mathbb{R}^n)}$).

**Proposition 1** The range $\mathcal{H}_{\phi}$ of $W_{\phi}$ is closed, and hence a Hilbert space.

**Proof.** Set $P_{\phi} = W_{\phi} W^*_{\phi}$ where $W^*_{\phi}$ is the adjoint of $W_{\phi}$; we have $P_{\phi} = P^*_{\phi}$ and $P_{\phi} P^*_{\phi} = P_{\phi}$ hence $P_{\phi}$ is an orthogonal projection. Since $W^*_{\phi} W_{\phi}$ is the identity on $L^2(\mathbb{R}^n)$ the range of $W^*_{\phi}$ is $L^2(\mathbb{R}^n)$ and that of $P_{\phi}$ is therefore precisely $\mathcal{H}_{\phi}$. Since the range of a projection is closed, so is $\mathcal{H}_{\phi}$. \[\blacksquare\]
This result, together with formula (11) shows that \( \tilde{T}(z_0) \) and \( \hat{T}(z_0) \) are unitarily equivalent representations of the Heisenberg group \( H_n \); the irreducibility of the representation \( \tilde{T}(z_0) : H_n \rightarrow \mathcal{H}_\phi \) follows from von Neumann’s uniqueness theorem for the projective representations of the CCR.

Let us return to the operator \( \tilde{H} = H^\star \). A straightforward calculation showing that \( W_\phi \) satisfies the intertwining relations

\[
x \star W_\phi \psi = (x + \frac{1}{2} i \hbar \partial_x) W_\phi \psi = W_\phi (x \psi)
p \star W_\phi \psi = (p - \frac{1}{2} i \hbar \partial_p) W_\phi \psi = W_\phi (-i \hbar \partial_p \psi)
\]

an educated guess is then that more generally:

**Proposition 2**

(i) The operator \( W_\phi \) intertwines the operators \( \tilde{T}(z_0) \) and \( \hat{T}(z_0) \):

\[
W_\phi (\hat{T}(z_0) \psi) = \tilde{T}(z_0) W_\phi \psi; \quad (15)
\]

(ii) We also have

\[
\tilde{H} W_\phi = W_\phi \tilde{H} \quad \text{and} \quad W_\phi^\star \tilde{H} = \tilde{H} W_\phi^\star. \quad (16)
\]

**Proof.** Making the change of variable \( y = y' + x_0 \) in the definition (13) of \( W_\phi \) we get

\[
W_\phi (\tilde{T}(z_0) \psi, \phi)(z) = e^{-\frac{i}{\hbar} \sigma(z, x_0) \partial_r} W_\phi (z - \frac{1}{2} z_0)
\]

which is precisely (13). Applying \( W_\phi \) to both sides of (12), we get

\[
W_\phi \tilde{H} \psi = \left( \frac{1}{2 \pi \hbar} \right)^n \int_{\mathbb{R}^{2n}} H^h_\sigma (z_0) W_\phi [\tilde{T}(z_0) \psi] dz_0.
\]

and hence

\[
W_\phi \tilde{H} \psi = \left( \frac{1}{2 \pi \hbar} \right)^n \int_{\mathbb{R}^{2n}} H^h_\sigma (z_0) [\tilde{T}(z_0) W_\phi \psi] dz_0,
\]

which is the first equality (16) in view of formula (7). To prove the second equality it suffices to apply the first to \( W_\phi^\star \tilde{H} = (\tilde{H}^\star W_\phi)^\star \).

4 Spectral Results

We will need the following result, which is quite interesting by itself:
Lemma 3 Let \((\phi_j)_j\) be an arbitrary orthonormal basis of \(L^2(\mathbb{R}^n)\). Then the vectors \(\Phi_{j,k} = W_{\phi_j}\phi_k\) form an orthonormal basis of \(L^2(\mathbb{R}^{2n})\).

Proof. Since the \(W_{\phi_j}\) are isometries the vectors \(\Phi_{j,k}\) form an orthonormal system. It is sufficient to show that if \(\Psi \in L^2(\mathbb{R}^{2n})\) is orthogonal to the family \((\Phi_{j,k})_{j,k}\) (and hence to all the spaces \(\mathcal{H}_{\phi_j}\)) then \(\Psi = 0\). Assume that \((\Psi|\Phi_{j,k})_{L^2(\mathbb{R}^{2n})} = 0\) for all \(j,k\). Since we have

\[
(\Psi|\Phi_{j,k})_{L^2(\mathbb{R}^{2n})} = (\Psi|W_{\phi_j}\phi_k)_{L^2(\mathbb{R}^{2n})} = (W_{\phi_j}^*\Psi|\phi_k)_{L^2(\mathbb{R}^n)}
\]

it follows that \(W_{\phi_j}^*\Psi = 0\) for all \(j\) since \((\phi_j)_j\) is a basis; using the anti-linearity of \(W_{\phi}\) in \(\phi\) we have in fact \(W_{\phi}^*\Psi = 0\) for all \(\phi \in L^2(\mathbb{R}^n)\). Let us show that this implies that \(\Psi = 0\). In view of formula (14) for the adjoint of \(W_{\phi}\) the operator \(W_{\phi}^*\) has kernel

\[
\Phi_x(y, p) = \left(\frac{2}{\pi\hbar}\right)^{n/2} e^{\frac{ip}{\hbar}(x-y)}\phi(2y - x).
\]

Let us fix \(x\); the property \(W_{\phi}^*\Psi = 0\) for all \(\phi\) is then equivalent to \(\langle \Psi, \Phi_x \rangle = 0\) for all \(\Phi_x \in S(\mathbb{R}^{2n})\) (fixed \(x\)) and hence \(\Psi = 0\), which we set out to show. □

We now have everything we need to prove the main results of this Letter. We begin by stating the following general property:

Theorem 4 The following properties are true: (i) The eigenvalues of the operators \(\hat{H}\) and \(\tilde{H} = \hat{H}^{\dagger}\) are the same; (ii) Let \(\psi\) be an eigenfunction of \(\hat{H}\): \(\hat{H}\psi = \lambda\psi\). Then, for every \(\phi\), the function \(\Psi = W_{\phi}\psi\) is an eigenfunction of \(\tilde{H}\) corresponding to the same eigenvalue: \(\tilde{H}\Psi = \lambda\Psi\). (iii) Conversely, if \(\Psi\) is an eigenfunction of \(\tilde{H}\) then \(\psi = W_{\phi}^*\Psi\) is an eigenfunction of \(\hat{H}\) corresponding to the same eigenvalue.

Proof. That every eigenvalue of \(\hat{H}\) also is an eigenvalue of \(\tilde{H}\) is clear: if \(\hat{H}\psi = \lambda\psi\) for some \(\psi \neq 0\) then

\[
\tilde{H}(W_{\phi}\psi) = W_{\phi}\hat{H}\psi = \lambda(W_{\phi}\psi)
\]

and \(W_{\phi}\psi \neq 0\) because \(W_{\phi}\) is injective; this proves at the same time that \(W_{\phi}\psi\) is an eigenfunction of \(\tilde{H}\). Assume conversely that \(\tilde{H}\Psi = \lambda\Psi\) for \(\Psi \neq 0\) and \(\lambda \in \mathbb{R}\). For every \(\phi\) we have, using the second equality (16),

\[
\tilde{H}W_{\phi}^*\Psi = W_{\phi}^*\tilde{H}\Psi = \lambda W_{\phi}^*\Psi
\]
hence $\lambda$ is an eigenvalue of $\hat{H}$; $W^*_\phi \Psi$ is an eigenfunction if it is different from zero. Let us prove this is indeed the case. We have $W^*_\phi W^*_\phi \Psi = P^*_\phi \Psi$ where $P^*_\phi$ is the orthogonal projection on the range $\mathcal{H}_\phi$ of $W^*_\phi$. Assume that $W^*_\phi \Psi = 0$; then $P^*_\phi \Psi = 0$ for every $\phi \in S(\mathbb{R}^n)$, and hence $\Psi = 0$ in view of Lemma 3 above.

**Remark 5** The result above is indeed quite general, because we do not make any assumption on the multiplicity of the (star)eigenvalues, nor do we assume that $\hat{H}$ is essentially self-adjoint. Notice that the proof actually works for arbitrary $\phi \in S'((\mathbb{R}^n))$. We present some examples at the end of this section.

**Corollary 6** Suppose that $\tilde{H}$ is an essentially self-adjoint operator on $L^2(\mathbb{R}^n)$ and that each of the eigenvalues $\lambda_0, \lambda_1, ..., \lambda_j, ...$ has multiplicity one. Let $\psi_0, \psi_1, ..., \psi_j, ...$ be a corresponding sequence of orthonormal eigenfunctions. Let $\Psi_j$ be an eigenfunction of $\tilde{H}$ corresponding to the eigenvalue $\lambda_j$. Then there exists a sequence $(\alpha_{j,k})_k$ of complex numbers such that

$$\Psi_j = \sum_k \alpha_{j,k} \psi_j$$

with $\Psi = W^*_\psi \psi \in \mathcal{H} \cap \mathcal{H}$. (17)

**Proof.** We know from Theorem 4 above that $\hat{H}$ and $\tilde{H}$ have same eigenvalues and that $\Psi_{j,k} = W^*_\psi \psi_j$ satisfies the eigenvalue equation $\hat{H} \Psi_{j,k} = \lambda_j \Psi_{j,k}$. Since $\tilde{H}$ is self-adjoint and its eigenvalues are distinct, its eigenfunctions $\psi_j$ form an orthonormal basis of $L^2(\mathbb{R}^n)$; it follows from Lemma 3 that the $\Psi_{j,k}$ form an orthonormal basis of $L^2(\mathbb{R}^2)$, hence there exist non-zero scalars $\alpha_{j,k}$ such that $\Psi_j = \sum_k \alpha_{j,k} \psi_j$. We have, by linearity and using the fact that $\tilde{H} \psi_j = \lambda_j \psi_j$,

$$\tilde{H} \Psi_j = \sum_k \alpha_{j,k} \tilde{H} \psi_j = \sum_k \alpha_{j,k} \psi_j \lambda_j$$

On the other hand we also have $\tilde{H} \Psi_j = \lambda_j \Psi_j$,

$$\tilde{H} \Psi_j = \lambda_j \Psi_j = \sum_k \alpha_{j,k} \lambda_j \Psi_k$$

which is only possible if $\alpha_{j,k} = 0$ for $k \neq j$; setting $\alpha_{j,\ell} = \alpha_{j,j,\ell}$ formula (17) follows. (That $\Psi_{j, \ell} \in \mathcal{H} \cap \mathcal{H}$ is clear using the definition of $\mathcal{H}$ and the sesquilinearity of the cross-Wigner transform.)

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We remark that the continuous spectrum can be dealt with in a similar fashion provided that one generalizes the transform $W_\phi$ by allowing the “parameter” to be a tempered distribution (in which case the normalization condition $\|\phi\|_{L^2} = 1$ does no longer make sense, of course); the same remark applies to the case where $\hat{H}$ is no longer essentially self-adjoint (cf. the remark following the proof of Theorem 4). To illustrate this, let us consider the two following typical examples (in dimension $n = 1$):

• $H(x,p) = p$. In this case $\hat{H} = -i\hbar \partial / \partial x$ is a symmetric operator and the equation $\hat{H}\psi = E\psi$ has solutions for every real value of $E$; these solutions are the tempered distributions $\psi(x) = C \exp(ix/\hbar)$ ($C$ any complex constant). A straightforward calculation shows that

$$W_\phi\psi(x,p) = C' e^{\frac{2i}{\hbar}(E-p)x}F\phi(p)$$

where $C'$ is a new constant and $F\phi$ is the Fourier transform of $\phi$. If we let $\phi$ range over $S'(\mathbb{R}^n)$ and use the fact that the Fourier transform is an automorphism of $S'(\mathbb{R}^n)$ we see that $W_\phi\psi$ can be any distribution of the type

$$\Psi(x,p) = \Phi(p)e^{\frac{2i}{\hbar}(E-p)x}$$

with $\Phi \in S'(\mathbb{R}^n)$; these distributions are precisely the solutions of the stargenvalue equation

$$p \star \Psi = (p - \frac{1}{2i}\hbar \partial_p)\Psi = E\Psi$$

as a straightforward calculation shows.

• $H(x,p) = x$. Here $\hat{H}$ is the operator of multiplication by $x$; this a symmetric operator without any eigenvalues and eigenfunctions. It is however self-adjoint, and the solutions of $\hat{H}\psi = E\psi$ are the distributions $\psi = C\delta(x - E)$; one finds by an argument similar to that above that $W_\phi\psi$ can be any distribution of the type

$$\Psi(x,p) = \Phi(x)e^{\frac{2i}{\hbar}(E-x)p}$$

which is the general solution of the stargenvalue equation

$$x \star \Psi = (x + \frac{1}{2i}\hbar \partial_x)\Psi = E\Psi.$$
The previous treatment of the stargenvlue equation for operators with a continuous spectrum can be made rigorous in the setting of Gelfand triples. In our setting \((S(\mathbb{R}^n), L^2(\mathbb{R}^n), S'(\mathbb{R}^n))\) is the Gelfand triple of interest and the corresponding weak formulation of the eigenvalues and eigenvectors of an operator from \(S'(\mathbb{R}^n)\) to \(S(\mathbb{R}^n)\). For further information on Gelfand triples we refer the reader to the standard reference [3].

5 An Example and its Extension

As an illustration consider the harmonic oscillator Hamiltonian

\[
H = \frac{1}{2}(p^2 + x^2). \tag{18}
\]

In view of the results above the spectra of the operators \(\hat{H}\) and \(\tilde{H}\) are identical. Choosing for simplicity \(\hbar = 1\) the eigenvalues of \(\hat{H}\) are the numbers \(\lambda_N = N + \frac{1}{2}\) with \(N = 0, 1, 2, \ldots\). The normalized eigenfunctions are the rescaled Hermite functions

\[
\psi_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} \mathcal{H}_k(x). \tag{19}
\]

where

\[
\mathcal{H}_k(x) = (-1)^{km} e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}
\]

is the \(k\)-th Hermite polynomial. Using definition (12) of \(W_\phi\) together with known formulae for the cross-Wigner transform of Hermite functions (see for instance Wong [10], Chapter 24, Theorem 24.1) one finds that the eigenfunctions of \(\tilde{H}\) are linear superpositions of the functions

\[
\Psi_{j,k}(\zeta) = (-1)^j \left( \frac{j^k}{j+k} \right)^{\frac{1}{2}} 2^{\frac{k+1}{2}} \zeta^k \mathcal{L}_j^k(2|\zeta|^2) e^{-|\zeta|^2} \tag{20}
\]

where \(\zeta = x + ip\) and \(\Psi_{j,j+k} = \overline{\Psi_{j+k,k}}\) for \(k = 0, 1, 2, \ldots\); here

\[
\mathcal{L}_j^k(x) = \frac{1}{j!} x^{-k} e^x \left( \frac{d}{dx} \right)^j (e^{-x} x^{j+k}), \ x > 0
\]

is the Laguerre polynomial of degree \(j\) and order \(k\). (For similar results see Bayen et al. [2].)

Notice that the example above can be generalized without difficulty to the case of arbitrary quadratic Hamiltonians of the type

\[
H = \frac{1}{2} M z \cdot z
\]
where $M$ is a positive-definite symmetric matrix. In fact, in view of Williamson's diagonalization theorem there exists a symplectic matrix $S$ such that

$$M = S^TDS, \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

where $\Lambda$ is the diagonal matrix whose entries are the moduli $\omega_j > 0$ of the eigenvalues $\pm i\omega_j$ of $JM$. We thus have

$$H \circ S = \sum_{j=1}^{n} \frac{\omega_j}{2} (x_j^2 + p_j^2)$$

and $\tilde{H} \circ S = \tilde{S}\tilde{H}\tilde{S}^{-1}$ where $\tilde{S}$ is anyone of the two metaplectic operators associated with $S$. The eigenvalues of $\tilde{H} \circ S$ and $\tilde{H}$ are the same; they are the numbers

$$\lambda_{N_1,...,N_n} = \sum_{j=1}^{n} (N_j + \frac{1}{2}) \omega_j$$

and then the eigenfunctions $\psi_S$ of $\tilde{H} \circ S$ and those, $\psi$, of $\tilde{H}$ by the formula $\psi_S = \tilde{S}\psi$. Now, the eigenfunctions of $\tilde{H}$ are tensor products of rescaled Hermite functions; using the fact that $\psi_S = \tilde{S}\psi$ together with the symplectic covariance formula

$$W(\tilde{S}\psi, \tilde{S}\phi)(z) = W(\psi, \phi)(S^{-1}z)$$

satisfied by the cross-Wigner distributions, the eigenvalues of $\tilde{H} = H^*$ are calculated in terms of tensor products of the functions of the functions \([20]\). We do not give the details of the calculations here since they are rather lengthy but straightforward.

\section{Concluding Remarks}

Due to limitation of length there are several aspects of our approach we have not discussed in this Letter. For instance, he methods developed here should apply with a few modifications (but in a rather straightforward way) to more general phase space (for instance co-adjoint orbits). A perhaps even more exciting problem is the following, which is closely related to our previous results \([5]\) on the relationship between the uncertainty principle
and the topological notion of symplectic capacity. A rather straightforward extension of the methods we used in [5] shows that if

\[
|W_{\phi}\psi(z)| \leq Ce^{-\frac{1}{\hbar}(a|x|^2 + b|y|^2)} \quad \text{for } z \in \mathbb{R}^{2n}
\]

(21)

then we must have \(ab \leq 1\). In particular we can have \(|W_{\phi}\psi(z)| \leq Ce^{-\frac{1}{\hbar}|z|^2}\) only if \(\varepsilon \geq \hbar\); it follows that the Hilbert spaces \(H_{\phi}\) do not contain any nontrivial function with compact support: assume in fact that \(\Psi \in H_{\phi}\) is such that \(\Psi(z) = 0\) for \(|z| \geq R > 0\). Then, given an arbitrary \(\varepsilon < \hbar\) one can find a constant \(C_\varepsilon\) such that \(|\Psi(z)| \leq C_\varepsilon e^{-\frac{1}{\hbar}|z|^2}\), which is impossible since \(\Psi = W_{\phi}\psi\) for some \(\psi \in L^2(\mathbb{R}^n)\). This suggests (taking Theorem 4 into account) that the solutions \(\Psi\) of the \(*\)-genvalue equation cannot be too concentrated around a point in phase-space. In fact we conjecture that if an estimate of the type \(|\Psi(z)| \leq Ce^{-\frac{1}{\hbar}Mz \cdot z}\) (\(M\) symmetric positive-definite) holds for an eigenfunction of the \(*\)-genvalue equation, then the symplectic capacity of the ellipsoid \(Mz \cdot z \leq \hbar\) must be at least \(\frac{1}{2}\hbar\). We will come back to this topic in a near future.

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