Anderson localization in bipartite lattices

Michele Fabrizio\textsuperscript{1,2} and Claudio Castellani\textsuperscript{3}

\textsuperscript{1} Istituto Nazionale di Fisica della Materia, and International School for Advanced Studies (SISSA), via Beirut 2-4, I-34014 Trieste, Italy,

\textsuperscript{2} International Center for Theoretical Physics (ICTP), Trieste, Italy

\textsuperscript{3} Istituto Nazionale di Fisica della Materia and Università degli Studi di Roma “La Sapienza”, Piazzale Aldo Moro 2, I-00185, Roma, Italy

Abstract

We study the localization properties of a disordered tight-binding Hamiltonian on a generic bipartite lattice close to the band center. By means of a fermionic replica trick method, we derive the effective non-linear \( \sigma \)-model describing the diffusive modes, which we analyse by using the Wilson–Polyakov renormalization group. In addition to the standard parameters which define the non-linear \( \sigma \)-model, namely the conductance and the external frequency, a new parameter enters, which may be related to the fluctuations of the staggered density of states. We find that, when both the regular hopping and the disorder only couple one sublattice to the other, the quantum corrections to the Kubo conductivity vanish at the band center, thus implying the existence of delocalized states. In two dimensions, the RG equations predict that the conductance flows to a finite value, while both the density of states and the staggered density of states fluctuations diverge. In three dimensions, we find that, sufficiently close to the band center, all states are extended, independently of the disorder strength. We also discuss the role of various symmetry breaking terms, as a regular hopping between same sublattices, or an on-site disorder.
I. INTRODUCTION

An interesting and still debated issue in the physics of the Anderson’s localization concerns the existence of delocalized states in dimensions $d \leq 2$, the conditions under which they appear, and their properties. This problem, which is, for instance, of relevance in the theory of the integer quantum Hall effect [1], got recently a renewed interest after evidences of a metal–insulator transition in two dimensions have been discovered [2].

One of the cases in which localization does not occur in any dimension is at the band center energy of a tight binding model on a bipartite lattice, when both the regular hopping and the disorder only couple one sublattice to the other, i.e. in the so called two sublattice model [4–6]. Although this is not a common physical situation, its consequences are surprising, and seem to escape any quasi-classical interpretation, which, on the contrary, provides simple physical explanations of other delocalization mechanisms [3]. Already in 1976, Theodorou and Cohen [4] realized that a one dimensional tight binding model with nearest neighbor random hopping has a single delocalized state at the band center (see also Ref. [5]). Afterwards, Wegner [6] and Oppermann and Wegner [7] showed that a delocalized state indeed exists under the above conditions in any dimension, within a large–$n$ expansion, being $n$ the number of orbitals per site. Later on, Wegner and Gade [8] argued that these models correspond to a particular class of non-linear $\sigma$-models for matrices in the zero replica limit. They were able to show that the quantum corrections to the $\beta$-function which controls the scaling behavior of the conductance vanish at the band center at all orders in the disorder strength, thus implying a metallic behavior at this value of the chemical potential. Moreover, they showed that, contrary to the standard case, the $\beta$-function of the density of states is finite. These results were based upon the non-linear $\sigma$-model derived by Gade [9] by means of a boson-replica trick method, in a particular two sublattice Hamiltonian with broken time reversal invariance.

More recently, the model without time reversal invariance has got a renewed interest for its implications in different physical contexts, for instance models with non-Hermitean
stochastic operators, or random flux models in two dimensions (see e.g. Refs. 10–12).

In this paper, we present an analysis of a generic disordered tight-binding Hamiltonian on a generic bipartite lattice. The starting model is therefore of quite general validity, also describing systems with time reversal invariance, and reduces in particular cases to the model discussed by Gade [9], or, in the honeycomb lattice, to models of Dirac fermions [12], or, finally, to random flux models [10,11]. By means of a fermionic-replica trick method, we derive the generic non-linear $\sigma$-model describing the diffusive modes, which we analyse by the Renormalization Group (RG). Needless to say, the effective model belongs to the same class of non-linear $\sigma$-models identified by Wegner and Gade, demonstrating once more the universality of this description in the theory of Anderson localization [13].

Since the work is quite technical, we prefer to give in the following section a short summary of the main results.

A. Summary of the main results

In this section we shortly present the main results, with particular emphasis to the connections and differences with the standard theory of the Anderson’s localization.

We consider a generic bipartite lattice and work with a unit cell which contains two sites from opposite sublattices. The Pauli matrices $\sigma$’s act on the two components of the wavefunction, corresponding to the two sites within each unit cell. In this lattice, we study a disordered tight-binding Hamiltonian which has the peculiar property of involving only both Pauli matrices $\sigma_1$ and $\sigma_2$. In other words, $H$ satisfies the conditions

$$\{H, \sigma_3\} = 0, \quad \{H, \sigma_1\} \neq 0, \quad \{H, \sigma_2\} \neq 0,$$

where $\{\ldots, \ldots\}$ indicates the anticommutator. By means of a path-integral approach within a fermionic replica trick method, we find that the low-energy diffusive modes at the band center, $E = 0$, can be represented by the non-linear $\sigma$-model

$$S[U] = \frac{2\pi \sigma_{xx}}{16} \int dR \text{Tr} \left[ \vec{\nabla} U(R)^{-2} \cdot \vec{\nabla} U(R)^2 \right]$$
where $U(R)$ is unitary and belongs to the coset space $U(4m)/\text{Sp}(2m)$, being $m$ the number of replicas. At finite energy $E \neq 0$, the symmetry of $U(R)$ gets reduced to $\text{Sp}(2m)/\text{Sp}(m) \times \text{Sp}(m)$, as in the standard case [16]. The enlarged symmetry is accompanied by new diffusive modes which appear in the retarded-retarded and advanced-advanced channels, which are instead massive in the standard case. In (2), $\sigma_{xx}$ is the Kubo conductivity (in units of $e^2/\hbar$) in the Drude approximation. We find that the new coupling constant, $\Pi$, is proportional to the fluctuations of the staggered density of states, i.e. to the following correlation function

$$
\frac{1}{V} \sum_{R R'} e^{-i q (R - R')} \langle \rho_s(E, R) \rangle \langle \rho_s(E, R') \rangle,
$$

where the bar indicates the impurity average, and $\rho_s(E, R)$ is the staggered density of states at energy $E$,

$$
\rho_s(E, R) = \sum_n \phi_n(R) \notag \sigma_3 \phi_n(R) \delta(E - \epsilon_n),
$$

being $\phi_n(R)$ the two-component eigenfunction of energy $\epsilon_n$.

The structure of the above action was derived by Gade and Wegner [8] for a particular Hamiltonian. Here, we derive it for a generic bipartite lattice and random hopping. Moreover, we provide a simple physical interpretation of $\Pi$.

Going back to (2), Gade and Wegner [8] gave a beautiful proof, based just on symmetry considerations, that the quantum corrections to the $\beta$-function of $\sigma_{xx}$ vanish in the zero replica limit. In Appendix E, we outline how their proof works in our case, which is slightly, but not qualitatively, different from the $U(N)/\text{SO}(N)$ case they have considered. Essentially, one can show that the action (2) possesses an invariant coupling $\sigma_{xx} + m \Pi$, which, in the $m \to 0$ limit, implies that $\sigma_{xx}$ is not renormalized, apart from its bare dimensions.

On the contrary, both the density of states and $\Pi$ have non vanishing $\beta$-functions. In $d = 1$, the system flows to strong coupling, hence we can not access the asymptotic infrared behavior. Nevertheless, the starting flow of the running variables indicates that the density
of states diverges. In $d = 2, 3$, the system flows to weak coupling, hence we can safely assume that the infrared behavior is captured by the RG equations. Indeed, in two dimensions, the density of states diverges at $E = 0$, while, in $d = 3$, it saturates to a finite value, although exponentially increased in $1/\sigma_{xx}$. Moreover, $\Pi$ has an anomalous behavior in $d = 2$, where it is predicted to diverge logarithmically. We explicitly estimate how these quantities behave as $E \to 0$, by means of a two-cutoff scaling approach, as discussed by Gade [9].

We have also analysed various symmetry breaking terms. The simplest ones are those which spoil the particular symmetry Eq.(1) of the model at $E = 0$, i.e. an on-site disorder or a same-sublattice regular hopping. These perturbations bring the symmetry of $U(R)$ down to $\text{Sp}(2m)/\text{Sp}(m) \times \text{Sp}(m)$, as in the standard localization problem. However, by evaluating the anomalous dimensions of these terms, we can estimate the cross-over lengths above which the symmetry reduction is effective. While in $d = 1, 2$ these terms always lead to a localized behavior also at the band center, in $d = 3$ the vicinity to the band center leads to an increase of the window in which delocalized states exist.

Finally, if the impurity potential breaks time-reversal symmetry, the matrix field $U(R)$ is shown to belong to the coset space $U(2m)$, which indeed agrees with the analysis of Gade [9].

The paper is organized as follows. In section II, we introduce the Hamiltonian. In section III, we derive the path-integral representation of the model, by using Grassmann variables within the replica trick method, and, in section IV, we study the symmetry properties of the action. In section V, we evaluate the saddle point of the action, while in sections VI, VII, VIII we derive the effective non-linear $\sigma$-model describing the long-wavelength fluctuations around the saddle point. The Renormalization Group analysis is presented in section IX, and the behavior in the presence of on-site disorder, of a same-sublattice regular hopping or with broken time reversal invariance is studied in sections X, XI, and XII, respectively. Finally, section XIV is devoted to a discussion of the results. We have also included several appendices containing more technical parts.
II. THE MODEL

We consider a tight binding Hamiltonian on a bipartite lattice, of the form

\[ H = \sum_{R \in A} \sum_{R' \in B} h_{RR'} \left( c_R^\dagger c_{R'} + c_{R'}^\dagger c_R \right), \quad (4) \]

where \( A \) and \( B \) label the two sublattices and the hopping matrix elements \( h_{RR'} \) are randomly distributed. We take a unit cell which includes two sites from different sublattices. In some cases, like the honeycomb lattice, this is indeed the primitive unit cell. In other cases, like the square lattice, it is not.

In this representation, the Hamiltonian can be written as

\[ H = \sum_{R,R'} h_{12}^{RR'} \left( c_1^\dagger c_{2R'} + H.c. \right) \quad (5) \]

where \( 1 \) and \( 2 \) label now the two sites in the unit cell, while \( R \) and \( R' \) refer to the unit cells, and \( h_{12}^{RR'} = h_{21}^{R'R} \). By introducing the two component operators

\[ c_R = \begin{pmatrix} c_1^R \\ c_2^R \end{pmatrix}, \]

we can also write the Hamiltonian as

\[ H = \sum_{R,R'} c_R^\dagger H_{RR'} c_{R'} \]

\[ = \sum_{R,R'} \frac{1}{2} \left( h_{12}^{RR'} + h_{21}^{R'R} \right) c_R^\dagger \sigma_1 c_{R'}^\dagger + \frac{i}{2} \left( h_{12}^{RR'} - h_{21}^{R'R} \right) c_R^\dagger \sigma_2 c_{R'}, \quad (6) \]

where the \( \sigma_i \)'s (\( i = 1,2,3 \)) are the Pauli matrices. We notice that, quite generally, the Hamiltonian involves both \( \sigma_1 \) and \( \sigma_2 \), but neither \( \sigma_3 \) nor \( \sigma_0 \), so that it satisfies the conditions in Eq. (1).

We can write \( h_{12}^{RR'} = t_{12}^{RR'} + \tau_{12}^{RR'} \), where \( t_{12}^{RR'} \) are the average values, which represent the regular (translationally invariant) hopping matrix elements, while \( \tau_{12}^{RR'} \) are random variables with zero average, which we assume to be gaussian distributed with width

\[ \langle (\tau_{12}^{RR'})^2 \rangle = u^2 \left( t_{12}^{RR'} \right)^2. \]
The dimensionless parameter \( u \) is a measure of the disorder strength in units of the regular hopping. In this way, the Hamiltonian is written as the sum of a regular part, \( H^{(0)} \), plus a disordered part, \( H_{\text{imp}} \).

For the regular hopping, we define

\[
\begin{align*}
    t_{RR'} &= \frac{1}{2} \left( t_{12}^{RR} + t_{21}^{12} \right), \\
    w_{RR'} &= \frac{1}{2} \left( t_{12}^{RR} - t_{21}^{12} \right),
\end{align*}
\]

so that the non disordered part, \( H^{(0)} \), of the Hamiltonian is

\[
H^{(0)} = \sum_{R,R'} c_R^\dagger H_{RR'}^{(0)} c_{R'} = \sum_{R,R'} c_R^\dagger (t_{RR'} \sigma_1 + iw_{RR'} \sigma_2) c_{R'},
\]  

(7)

Since, for any lattice vector \( R_0 \), \( t_{RR'} = t_{R+R_0 R'+R_0} \), as well as \( w_{RR'} = w_{R+R_0 R'+R_0} \), and, moreover, \( t_{RR'} = t_{R'R} \) while \( w_{RR'} = -w_{R'R} \), the Fourier transforms satisfy \( t_k = t_k^* \) (\( t_k \) real) and \( w_k = -w_k^* \) (\( w_k \) imaginary). In momentum space, the Hamiltonian matrix, \( H_k^{(0)} = t_k \sigma_1 + iw_k \sigma_2 \), is diagonalized by the unitary transformation \( c_k = U_k d_k \), with

\[
U_k = e^{-i \frac{\pi}{4} \sigma_2} e^{\frac{i \theta_k}{2} \sigma_1},
\]

(8)

where

\[
\tan \theta_k = \frac{i w_k}{t_k} = -\frac{\text{Im} t_k^{12}}{\text{Re} t_k^{12}}.
\]

(9)

Indeed, \( U_k^\dagger H_k^{(0)} U_k = \epsilon_k \sigma_3 \), where \( \epsilon_k^2 = t_k^2 - w_k^2 = |t_k|^2 + |w_k|^2 \).

\[\text{A. Current operator}\]

The commutator of the density \( c_R^\dagger c_R \) with the non disordered Hamiltonian (7) is

\[
\sum_{R'} c_R^\dagger (t_{RR'} \sigma_1 + iw_{RR'} \sigma_2) c_{R'} - c_R^\dagger (t_{R'R} \sigma_1 + iw_{R'R} \sigma_2) c_R,
\]

or, in Fourier space,

\[
\sum_k (t_{k+q} - t_k) c_{k+q}^\dagger \sigma_1 c_{k+q} + i (w_{k+q} - w_k) c_{k+q}^\dagger \sigma_2 c_{k+q}.
\]

Therefore, in the long wavelength limit, the current operator in the absence of disorder is
\[ \vec{J}_q^{(0)} \sim \sum_k \vec{\nabla}_k t_k c_k^\dagger \sigma_1 c_{k+q} + i \vec{\nabla}_k w_k c_k^\dagger \sigma_2 c_{k+q} \]

\[ = \sum_k \vec{\nabla}_k \epsilon_k c_k^\dagger \vec{B}_k \cdot \vec{\sigma} c_k + \epsilon_k \vec{\nabla} \theta_k c_k^\dagger \vec{B}_{\perp, k} \cdot \vec{\sigma} c_k \]

\[ = \sum_k \vec{\nabla}_k \epsilon_k d_k^\dagger \sigma_3 d_{k+q} + \epsilon_k \vec{\nabla} \theta_k d_k^\dagger \sigma_2 d_{k+q}, \]

the last being the expression in the basis which diagonalizes the Hamiltonian \( H_0 \). In (10), \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \), and the vectors

\[ \vec{B}_k = ( \cos \theta_k, \sin \theta_k, 0 ), \quad \vec{B}_{\perp, k} = ( -\sin \theta_k, \cos \theta_k, 0 ), \]

describe intra and inter-band contributions to the current vertex. Notice that the regular hopping Hamiltonian can be simply written as \( H^{(0)} = \sum_k \epsilon_k c_k^\dagger \vec{B}_k \cdot \vec{\sigma} c_k \).

Moreover, since also the impurity part of the whole Hamiltonian, (4), does not commute with the density operator, in the disordered model the current operator acquires an additional term proportional to the random hopping matrix elements, which we discuss later.

**III. PATH INTEGRAL**

The starting point of our analysis is a path-integral representation of the generating functional, in terms of Grassmann variables, following the work by Efetov, Larkin and Khmelnitskii [16]. To this end, we introduce, for each unit cell \( R \), the Grassmann variables \( c_{R,a,p,\alpha} \) and their complex conjugates \( \bar{c}_{R,a,p,\alpha} \), where \( a = 1, 2 \) is the sublattice index, \( p = \pm \) is the index of the advanced (+) and retarded (-) components, and the index \( \alpha \) runs over \( m \) identical copies of the model, as in the usual replica trick method. In what follows, by convention, the Pauli matrices \( \sigma \)'s act in the two sublattice space, the \( \tau \)'s in the space of the Grassmann fields \( c \) and \( \bar{c} \), and the \( s \)'s in the \( \pm \) space.

In order to treat on equal footing both the particle-hole and the particle-particle diffusive modes (diffusons and cooperons, respectively), as implied by time-reversal invariance, it is convenient to introduce the Nambu spinors \( \Psi_R \) and \( \bar{\Psi}_R \) defined through

\[ \Psi_R = \frac{1}{\sqrt{2}} \begin{pmatrix} c_R \\ \bar{c}_R \end{pmatrix}, \]
where \( \tau_R \) and \( c_R \) are column vectors with components \( \tau_{R,a,p,\alpha} \) and \( c_{R,a,p,\alpha} \), respectively, and

\[
\Psi_R = [c\Psi]^t,
\]

where \( c = -i\tau_2 \) is the charge conjugation operator. The action in terms of the spinors is [see Eq.(8)]

\[
S = - \sum_{RR'} \Psi_R \left( E + i\frac{\omega}{2} s_3 - H_{RR'} \right) \Psi_{R'} \\
= - \sum_{RR'} \Psi_R \left( E + i\frac{\omega}{2} s_3 - H^{(0)}_{RR'} \right) \Psi_{R'} + \sum_{RR'} \Psi_R H_{imp,RR'} \Psi_{R'} \\
= S_0 + S_{imp}, \tag{13}
\]

where \( E \pm i\omega/2 \) are the complex energies of the advanced/retarded components.

### A. Disorder average

Before taking the disorder average, we notice that, in the spinor notation,

\[
c_{1R}^\dagger c_{2R'} + H.c. \rightarrow 2\overline{\Psi}_1 R \Psi_2 R' = 2\overline{\Psi}_2 R' \Psi_1 R.
\]

Therefore, the impurity part of the action can be written as

\[
S_{imp} = \sum_{R,R'} 2\tau_{12}^{RR'} \overline{\Psi}_1 R \Psi_2 R'.
\]

The generating functional, within the replica method, is

\[
\mathcal{Z} = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \mathcal{D}\tau P[\tau] e^{-S_0 - S_{imp}}, \tag{14}
\]

where \( P[\tau] \) is the gaussian probability distribution of the random bonds \( \tau_{12}^{RR'} \). The average over disorder changes the impurity action into

\[
S_{imp} = - \sum_{R,R'} 2u^2 \left( t_{12}^{RR'} \right)^2 \left( \overline{\Psi}_1 R \Psi_2 R' \right)^2 \\
= - \sum_{R,R'} 2u^2 \left( t_{12}^{RR'} \right)^2 \left( \overline{\Psi}_1 R \Psi_2 R' \right) \left( \overline{\Psi}_2 R' \Psi_1 R \right). \tag{15}
\]

We define
\[ W_{RR'} = 2u^2 \left( \frac{t_{RR'}}{2} \right)^2 \in \mathbb{R}, \]

so that \( W^*_q = W_{-q} \), and introduce

\[ X^\alpha_1 \psi = \frac{1}{\sqrt{1 + \gamma}} \psi, \]

where \( \alpha \) is a multilabel for Nambu, advanced/retarded and replica components, and analogously \( X^\alpha_2 \), as well as their Fourier transforms. By these definitions,

\[ S_{imp} = \frac{1}{V} \sum_q \sum_{\alpha, \beta} W_{-q} X^\alpha_1 \psi \Lambda^\alpha_2, \quad (16) \]

This form, as compared to (15), has the advantage to allow a simple Hubbard–Stratonovich transformation. Notice that the use of Nambu spinors has the great advantage to involve just a single Fourier component of \( W_{RR'} \). If we write

\[ W_q = \omega_q e^{i\phi_q}, \]

where \( \omega_q > 0 \) and \( \phi_q = -\phi_{-q} \), and define

\[ Y_{1q} = e^{-i\frac{\phi}{2}} X_{1q}, \quad Y_{2q} = e^{i\frac{\phi}{2}} X_{2q}, \]

\( S_{imp} \) takes the simple form

\[ S_{imp} = \frac{1}{V} \sum_q \omega_q Y^\alpha_1 \Lambda^\alpha_2 = \frac{1}{V} \sum_q \frac{\omega_q}{2} Tr \left[ Y^1_{1q} Y^2_{-q} + Y^2_{2q} Y^1_{-q} \right] \]

\[ = \frac{1}{V} \sum_q \frac{\omega_q}{4} Tr \left[ Y^1_{0q} Y^1_{-q} - Y^3_{3q} Y^3_{3q} \right], \quad (17) \]

where we have introduced \( Y_0 = Y_1 + Y_2 \) as well as \( Y_3 = Y_1 - Y_2 \). Moreover, our choice of the impurity potential, which does not break, on average, the spatial symmetries of the lattice, implies that \( \phi_q = -\theta_q \), see Eq.(13).

We notice that, if a term is written as

\[ \lambda \frac{A^2}{4} \sum_{\alpha, \beta} X^\alpha_1 \Lambda^\alpha_2 = \lambda \frac{A^2}{4} Tr \left( X^2 \right), \]

where \( X = X^\dagger \) and \( \lambda = \pm 1 \), one can always decouple it, by introducing an hermitean matrix \( Q \), by the following Hubbard–Stratonovich transformation
\[
\exp \left[ \frac{A^2}{4} Tr \left( X^2 \right) \right] = N \int \mathcal{D}Q \exp \left[ -A^{-2} Tr(Q^2) + \sqrt{\lambda} Tr \left( Q X^t \right) \right],
\]
where the normalization factor \(N^{-1} = \int \mathcal{D}Q \exp \left[ -A^{-2} Tr(Q^2) \right]\). In the specific example, (17) can be transformed into

\[
S_{\text{imp}} = \frac{1}{V} \sum_q \frac{1}{\omega_q} Tr \left[ Q_{0q} Q_{0-q} + Q_{3q} Q_{3-q} \right] - \frac{i}{V} \sum_q Tr \left[ Q_{0q} Y_{0-q}^t + i Q_{3q} Y_{3-q}^t \right] - \frac{i}{V} \sum_q i Tr \left[ Q_{0q} \left( \cos \frac{\phi_q}{2} X_{0-q}^t + i \sin \frac{\phi_q}{2} X_{3-q}^t \right) \right] - \frac{i}{V} \sum_q i Tr \left[ Q_{3q} \left( \cos \frac{\phi_q}{2} X_{3-q}^t + i \sin \frac{\phi_q}{2} X_{0-q}^t \right) \right].
\]

If we define \(Q_q = Q_{0q} \sigma_0 + i Q_{3q} \sigma_3\), we obtain

\[
S_{\text{imp}} = \frac{1}{V} \sum_q \frac{1}{2 \omega_q} Tr \left[ Q_q Q_q^t \right] - \frac{i}{V} \sum_{p,q} \Psi_p Q_{-q} e^{-\frac{i}{2} \phi_q \sigma_3} \Psi_{p+q} - \frac{i}{V} \sum_q i Tr \left[ Q_{0q} \left( \cos \frac{\phi_q}{2} X_{0-q}^t + i \sin \frac{\phi_q}{2} X_{3-q}^t \right) \right] - \frac{i}{V} \sum_q i Tr \left[ Q_{3q} \left( \cos \frac{\phi_q}{2} X_{3-q}^t + i \sin \frac{\phi_q}{2} X_{0-q}^t \right) \right].
\]

If we now transform the spinors in Eq. (20) to the diagonal basis, the coupling to the \(Q_q\) matrix transforms as \(U_{p+q}^\dagger Q_q e^{\frac{i}{2} \phi \sigma_3} U_p\). The simplest consequence is that, in the diagonal basis, \(Q_{k_1,k_2} = Q_{0,k_1,k_2} \sigma_0 + i Q_{1,k_1,k_2} \sigma_1\) depends on two wavevectors. However, in the case of cubic lattices, see Eq. (A1),
\[ U_{p+q}^\dagger Q_q e^{i\phi_q\sigma_3} U_{p+q} = e^{-\frac{i}{2}(\theta_q+\phi_q)\sigma_1} (Q_{0,q} - iQ_{3,q}\sigma_1) e^{i\phi_q\sigma_1} e^{-\frac{i}{2}\phi_q\sigma_1} \]
\[ = e^{-\frac{i}{2}(\theta_q+\phi_q)\sigma_1} (Q_{0,q} - iQ_{3,q}\sigma_1) = Q_{0,q} - iQ_{3,q}\sigma_1, \]
since \( \phi_q = -\theta_q \). Therefore, for cubic lattices, we can also write

\[ S_{imp} = \frac{1}{V} \sum_q \frac{1}{2\omega_q} Tr \left[ Q_q Q_q^\dagger \right] - i \sum_R \Phi_R Q(R)\Phi_R, \tag{23} \]

where now \( \Phi_R \) is the Grassmann field of the \( d_R \) operators, and the matrix \( Q(R) = Q_0(R)\sigma_0 + iQ_1(R)\sigma_1 \), with \( Q_1(R) = -Q_3(R) \) defined above.

Finally, we notice that, in the case of the honeycomb lattice, at the wavevector \( q_* \), connecting the two Dirac cones of the non-disordered dispersion band, \( \omega_{q_*} = 0 \). This observation will turn useful when discussing the long-wavelength behavior of the model.

**IV. SYMMETRIES**

The action (13), at \( E = \omega = 0 \), i.e. at the band center with zero complex frequency, is invariant under a transformation \( \Psi_R \rightarrow T \Psi_R \) if

\[ c T^t c^t H_{RR'} T = H_{RR'}. \]

Since the random matrix elements \( H_{RR'} \) involve both Pauli matrices \( \sigma_1 \) and \( \sigma_2 \), \( T \) has to satisfy at the same time \( c T^t c^t \sigma_1 T = \sigma_1 \) and \( c T^t c^t \sigma_2 T = \sigma_2 \). This implies that

\[ c T^t c^t = \sigma_1 T^{-1} \sigma_1 \]
\[ \sigma_1 T^{-1} \sigma_1 = \sigma_2 T^{-1} \sigma_2. \tag{25} \]

The condition (25) can be fulfilled only by a transformation \( T = T_0\sigma_0 + T_3\sigma_3 \)

Under such a transformation

\[ Q \rightarrow c T^t c^t QT = \sigma_1 T^{-1} \sigma_1 QT \equiv \sigma_2 T^{-1} \sigma_2 QT \]

Since \( \sigma_1 Q \sigma_1 = Q^t \), then
\[ T^{-1} \sigma_1 QT \sigma_1 = T^\dagger Q^\dagger \sigma_1 \left( T^{-1} \right)^\dagger \sigma_1 = T^\dagger \sigma_1 Q \left( T^{-1} \right)^\dagger \sigma_1. \]

Hence, the transformation is also unitary, \( T^\dagger = T^{-1} \). Moreover, such a transformation leaves the \( Q \)-manifold invariant, which implies that our Hubbard-Stratonovich decoupling scheme, which makes use of \( Q = Q_0 \sigma_0 + i Q_3 \sigma_3 \), is exhaustive.

The unitary transformation, \( T \), can be written as
\[
T = \exp \left[ \frac{W_0}{2} \sigma_0 + \frac{W_3}{2} \sigma_3 \right],
\]
where
\[
W_0^\dagger = -W_0, \quad W_3^\dagger = -W_3.
\]
In addition, we must impose the charge conjugacy invariance, which, through Eq.(24), implies that
\[
cW_0^tc^t = -W_0 = W_0^\dagger, \quad cW_3^tc^t = W_3 = W_3^\dagger.
\]
The number of independent parameters turns out to be \( 16m^2 \), which suggests that \( T \) is related to a unitary group, specifically \( U(4m) \), as argued by Gade and Wegner \cite{8}. In fact, we can alternatively write
\[
T = \left( \begin{array}{cc} e^{\frac{W_0 + W_3}{2}} & 0 \\ 0 & e^{\frac{W_0 - W_3}{2}} \end{array} \right) \equiv \left( \begin{array}{cc} U & 0 \\ 0 & cU^\dagger c^t \end{array} \right),
\]
where \( U \) is indeed a unitary transformation belonging to \( U(4m) \). The invariance of (13) at finite frequency, \( \omega \neq 0 \), implies the additional condition \( cT^d c^t s_3 T = s_3 \), which reduces the number of independent parameters to \( 8m^2 + 2m \), lowering the symmetry of \( U \) down to \( \text{Sp}(2m) \).

If \( E \neq 0 \), \( T \) has to satisfy also
\[
cT^d c^t T = \sigma_1 T^{-1} \sigma_1 T = 1.
\]
This implies that, at finite energy, i.e. away from the band center, \( T \) does not contain anymore a \( \sigma_3 \)-component. Indeed \( E \) lowers the symmetry of \( U \) down to \( \text{Sp}(2m) \), which is further reduced to \( \text{Sp}(m) \times \text{Sp}(m) \) by a finite frequency, as in the standard situation \cite{16}.
V. SADDLE POINT

The full action

\[ S = - \sum_{k,q} \Psi_k \left( E \delta_{q0} + i \frac{\omega}{2} s_3 \delta_{q0} - H_k^{(0)} \delta_{q0} + \frac{i}{V} Q_{-q} e^{-\frac{i}{2} \hat{Q} \sigma_3} \right) \Psi_{k+q} \]

\[ + \frac{1}{V} \sum_q \frac{1}{2 \omega_q} Tr \left[ Q_q Q_q^\dagger \right], \quad (28) \]

by integrating over the Nambu spinors, transforms into

\[ S[Q] = \frac{1}{V} \sum_q \frac{1}{2 \omega_q} Tr \left[ Q_q Q_q^\dagger \right] - \frac{1}{2} Tr \ln \left[ E + i \frac{\omega}{2} s_3 - H^{(0)} + i Q - i \hat{L} Q \right]. \quad (29) \]

The saddle point equation for homogeneous solutions at \( E = 0 \) and \( \omega = 0^+ \) is

\[ Q = i \frac{\omega_0}{4} \int \frac{d^2 k}{4 \pi^2} \left( i 0^+ s_3 - \epsilon_k + i Q \right)^{-1} + \left( i 0^+ s_3 + \epsilon_k + i Q \right)^{-1}, \]

where \( \omega_0 = \sum_{R \rightarrow R'} 2 u^2 (t_{R-R'}^{12})^2 \). The general solution is

\[ Q_{sp} = \frac{\pi}{4} \omega_0 \rho(0) s_3 \equiv \Sigma s_3, \quad (30) \]

with \( \rho(0) \) being the density of states at \( E = 0 \). In order to distinguish transverse from longitudinal modes, it is convenient to parametrize the \( Q \)-matrix in the following way

\[ Q(R_p) = \sigma_1 T(R)^{-1} \sigma_1 \left[ Q_{sp} + P(R) \right] T(R) \equiv Q(R) + \sigma_1 T(R)^{-1} \sigma_1 P(R) T(R). \quad (31) \]

Here \( P(R) \) describes the longitudinal modes, which we discuss more in detail in section [VII], and \( T \) the transverse modes. Namely, \( T \) has the form given in Eq.(27),

\[ T(R) = \exp \left( \frac{W(R)}{2} \right) = \exp \left( \frac{W_0(R)}{2} \sigma_0 + \frac{W_3(R)}{2} \sigma_3 \right), \quad (32) \]

with \( \exp[(W_0 + W_3)/2] \) belonging now to the coset space \( U(4m)/Sp(2m) \). This amounts to impose that

\[ \{ W_0, s \} = 0, \quad [ W_3, s ] = 0, \]

by which it derives that
\[ Q(R) = \sigma_1 T(R)^{-1} \sigma_1 Q_{sp} T(R) = Q_{sp} e^{W_0(R)\sigma_0 + W_3(R)\sigma_3}. \]  

(33)

In the $\pm$ space, we can write

\[
W_0 = \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} iA & 0 \\ 0 & iC \end{pmatrix},
\]

(34)

where $A^\dagger = A$, $C^\dagger = C$, and additionally, since $cW^tc^t = \sigma_1 W^\dagger \sigma_1 = -\sigma_1 W \sigma_1$, then $cA^tc^t = A$, $cC^tc^t = C$ and $cB^tc^t = B^\dagger$. By writing

\[
A = A_0 \tau_0 + i (A_1 \tau_1 + A_2 \tau_2 + A_3 \tau_3),
\]

(35)

\[
B = B_0 \tau_0 + i (B_1 \tau_1 + B_2 \tau_2 + B_3 \tau_3),
\]

(36)

\[
C = C_0 \tau_0 + i (C_1 \tau_1 + C_2 \tau_2 + C_3 \tau_3),
\]

(37)

we find that the above conditions imply that, for $i = 0, \ldots, 3$,

\[
B_i, A_i, C_i \in \mathcal{Re},
\]

(38)

and

\[
A_0 = A_0^t, \quad C_0 = C_0^t,
\]

(39)

while, for $j = 1, 2, 3$,

\[
A_j = -A_j^t, \quad C_j = -C_j^t.
\]

(40)

VI. EFFECTIVE ACTION

In this section, we derive the effective field theory describing the long wavelength transverse fluctuations of $Q(R)$ around the saddle point. In the case of honeycomb lattices, we should worry about the momentum component of $Q$ which couples the two Dirac cones. However, one can see that the free action of $Q$ diverges at this wavevector, so that we are allowed to ignore the fluctuations around this momentum.
A. Integration over the Grassmann fields

As we said, by integrating (28) over the Grassmann variables, we obtain the following action of $Q$:

$$- S[Q] = -\frac{1}{V} \sum_q \frac{1}{2\omega_q} Tr \left[ Q_q Q_q^\dagger \right] + \frac{1}{2} Tr \ln \left[ E + i\frac{\omega}{2}s_3 - H^{(0)} + iQ - i\hat{L}Q \right].$$  \hspace{1cm} (41)

We start by neglecting the longitudinal fluctuations. Then, since $Q = \tilde{T}^\dagger Q_{sp} T$, where we define $\tilde{T} = \sigma_1 T \sigma_1 \equiv \sigma_2 T \sigma_2$, we can rewrite the second term of $S[Q]$ as

$$\frac{1}{2} Tr \ln \left( E \tilde{T} T^\dagger + i\frac{\omega}{2} \tilde{T} s_3 T^\dagger - \tilde{T} H^{(0)} T^\dagger + iQ_{sp} - V \right),$$  \hspace{1cm} (42)

where we define

$$V = i\tilde{T} \hat{L} Q T^\dagger.$$

Since $H^{(0)}_{RR'}$ involves either $\sigma_1$ and $\sigma_2$, while $T$ involves $\sigma_0$ and $\sigma_3$, then

$$H^{(0)}_{RR'} T(R')^\dagger = \tilde{T}(R')^\dagger H^{(0)}_{RR'} = \tilde{T}(R)^\dagger H^{(0)}_{RR'} + \left( \tilde{T}(R')^\dagger - \tilde{T}(R)^\dagger \right) H^{(0)}_{RR'}$$

$$= \tilde{T}(R)^\dagger H^{(0)}_{RR'} - \vec{\nabla} \tilde{T}(R)^\dagger \cdot (\vec{R} - \vec{R}') H^{(0)}_{RR'} + \frac{1}{2} \partial_{ij} \tilde{T}(R)^\dagger (R_i - R'_i) (R_j - R'_j) H^{(0)}_{RR'}.$$

Therefore the term $\tilde{T} H^{(0)} T^\dagger$ which appears in (42) can be written at long wavelengths as

$$\tilde{T}(R) H^{(0)}_{RR'} T(R')^\dagger = H^{(0)}_{RR'} - i\tilde{T}(R) \vec{\nabla} \tilde{T}^{-1}(R) \cdot \tilde{J}^{(0)}(R')_{RR'}$$

$$+ \frac{1}{2} \tilde{T}(R) \partial_{ij} \tilde{T}(R)^{-1} (R_i - R'_i) (R_j - R'_j) H^{(0)}_{RR'}$$

$$\equiv H^{(0)}_{RR'} + U_{RR'},$$  \hspace{1cm} (43)

where we used the fact that the long-wavelength part of the current operator in real space is, see (B3),

$$\tilde{J}(R') = -i \sum_R \left( \vec{R} - \vec{R}' \right) c_R^\dagger H_{RR'} c_{R'}.$$

Notice that in (43) only the current vertex which derives from the regular hopping appears, which is not the full current operator.
Moreover, a further current-like coupling will arise from the expansion in $V$ (see below). To this end, in the long-wavelength limit, the operator $\hat{L}$, see (22), can be approximately written as

$$\hat{L}Q(R) \simeq -\vec{\beta} \cdot \vec{\nabla} Q(R)\sigma_3 - \frac{1}{2} (\vec{\beta} \cdot \vec{\nabla})^2 Q(R),$$

(44)

where $\vec{\beta} = \vec{\nabla} \phi(q = 0)/2$.

Having defined $U$, (42) can be written as

$$\frac{1}{2} Tr \ln \left( E\tilde{T}T^\dagger + \frac{i\omega}{2} s_3 \tilde{T}T^\dagger - U - V - H(0) + iQ_{sp} \right)
= -\frac{1}{2} Tr \ln G + \frac{1}{2} Tr \ln \left( 1 + G E\tilde{T}T^\dagger + G \frac{i\omega}{2} s_3 \tilde{T}T^\dagger - G U - G V \right),$$

(45)

where $G = (-H(0) + iQ_{sp})^{-1}$ is the Green’s function in the absence of transverse fluctuations.

The effective field theory is then derived by expanding $S[Q]$ up to second order in $U$ and $V$, and first order in $E$ and $\omega$. In this way we get the following terms.

**B. Expansion in the $Q$ free action**

The free part of the action

$$S_0[Q] = \frac{1}{V} \sum_q \frac{1}{2\omega_q} Tr \left[ Q_q Q_q^\dagger \right],$$

can be expanded at small $q$. Since $\omega_q = \omega_{-q}$, then

$$\omega_q \simeq \omega_0(1 - \gamma q^2),$$

leading to

$$S_0[Q] \simeq \frac{1}{2\omega_0} \int dR Tr \left[ Q(R) Q(R)^\dagger \right] + \frac{\gamma}{2V\omega_0} \sum_q q^2 Tr \left[ Q_q Q_q^\dagger \right]
= \frac{1}{2\omega_0} \int dR Tr \left[ Q_{sp}^2 \right] + \frac{\gamma}{2\omega_0} \int dR Tr \left[ \vec{\nabla} Q(R) \cdot \vec{\nabla} Q(R)^\dagger \right].$$

(46)

The second term is a contribution to the current-current correlation function of the part of the current vertices proportional to the random hopping.
C. Expansion in $E$

Expansion of (45) in $E$ gives

$$\frac{E}{2} Tr \left( G \tilde{T} T^\dagger \right) = -\frac{i}{\omega_0} E Tr \left( Q_{sp} \tilde{T} T^\dagger \right) = -\frac{i}{\omega_0} E Tr Q.$$ (47)

D. Expansion in $\omega$

Expansion in $\omega$ gives

$$i \frac{\omega}{4} Tr \left( G \tilde{T} \tilde{s} T^\dagger \right) = \frac{\omega}{2 \omega_0} Tr (s_3 Q).$$ (48)

E. Expansion in $U$

The second order expansion in $U$ contains the following terms:

$$- \frac{1}{2} Tr (G U),$$ (49)

and

$$- \frac{1}{4} Tr (G U G U),$$ (50)

Taking in (49), the component of $U$ containing second derivatives, we get

$$- \frac{1}{4} Tr \left\{ \tilde{T}(R) \partial_i \tilde{T}(R)^{-1} (R_i - R'_i) \left( R_j - R'_j \right) H^{(0)}_{RR'} G(R', R) \right\}.$$  

By means of the Ward identity (B6), the above expression turns out to be

$$- \frac{\chi^{++}_{ij}}{8} Tr \left\{ \tilde{T}(R) \partial_i \tilde{T}(R)^{-1} \right\},$$ (51)

which, integrating by part, is also equal to

$$- \frac{\chi^{++}_{ij}}{8} Tr \left\{ \tilde{T}(R) \partial_i \tilde{T}(R)^{-1} \tilde{T}(R) \partial_j \tilde{T}(R)^{-1} \right\}$$

$$= -\frac{1}{16} \chi^{++}_{ij} Tr \left[ D_i D_j - D_i s_3 \sigma_1 D_j s_3 \sigma_1 \right]$$ (52)

$$- \frac{1}{16} \chi^{++}_{ij} Tr \left[ D_i D_j + D_i s_3 \sigma_1 D_j s_3 \sigma_1 \right].$$ (53)
Here we have introduced a matrix $\tilde{D}(R)$ with the $i$-th component

$$D_i(R) = D_0,i(R)\sigma_0 + D_3,i(R)\sigma_3 \equiv \tilde{T}(R)\partial_i\tilde{T}(R)^{-1}. \quad (54)$$

The part of (19) which contains first derivatives gives rise to a boundary term

$$\left[ \frac{1}{V} \sum_k \tilde{\nabla}\theta_k - \frac{\epsilon_k^2}{\epsilon_k^2 + \Sigma^2} \right] \int dR Tr \left[ \tilde{\nabla}W(R)\sigma_3 \right], \quad (55)$$

where $\Sigma$ has been defined by Eq.(30), which we discard by taking appropriate boundary conditions.

Let us now analyse the term (50), where we have to keep of $U$ only the part containing first derivatives. By making use of (43), this term is, in momentum space,

$$\frac{1}{4} \sum_{kq} Tr \left\{ \left[ \tilde{T}\tilde{\nabla}\tilde{T}^{-1} \right]_q \cdot \tilde{J}_k^{(0)}G(k + q) \left[ \tilde{T}\tilde{\nabla}\tilde{T}^{-1} \right]_{-q} \cdot \tilde{J}_k^{(0)}G(k) \right\} \approx \frac{1}{4} \sum_{kq} Tr \left\{ \left[ \tilde{T}\tilde{\nabla}\tilde{T}^{-1} \right]_q \cdot \tilde{J}_k^{(0)}G(k) \left[ \tilde{T}\tilde{\nabla}\tilde{T}^{-1} \right]_{-q} \cdot \tilde{J}_k^{(0)}G(k) \right\}$$

$$= \frac{1}{4} \sum_k \sum_R Tr \left\{ \tilde{D}(R) \cdot \tilde{J}_k^{(0)}G(k) \tilde{D}(R) \cdot \tilde{J}_k^{(0)}G(k) \right\}, \quad (56)$$

valid for small $q$. Here the matrix $\tilde{J}_k^{(0)} = \tilde{\nabla}_k t_k \sigma_1 + i\tilde{\nabla}_k w_k \sigma_2$. The non vanishing terms in (56) have both $\tilde{D}$’s either $\tilde{D}_0\sigma_0$ or $\tilde{D}_3\sigma_3$ [see (54)].

In the diagonal basis, upon defining, as we did in Eq.(30), $Q_{sp} = \Sigma s_3$, with $\Sigma = \pi\omega_0\rho(0)/4$, the Green’s function is

$$G(k) = \frac{1}{\epsilon_k\sigma_3 + i\frac{\pi}{4}\omega_0\rho(0)s_3} = \frac{-i}{\epsilon_k^2 + \Sigma^2} \sigma_0s_3 - \frac{\epsilon_k}{\epsilon_k^2 + \Sigma^2} \sigma_3s_0$$

$$\equiv G_0(k)\sigma_0s_3 + G_3(k)\sigma_3s_0. \quad (57)$$

Going back to the original basis,

$$G(k) = U_kG(k)U_k^\dagger = G_0(k)\sigma_0s_3 + G_3(k)\tilde{B}_k \cdot \tilde{\sigma}s_0. \quad (58)$$

where $\tilde{B}_k$ has been defined in (12). Therefore,
\[ \sigma_3 G(k) \sigma_3 = G_0(k) \sigma_0 s_3 - G_3(k) \vec{B}_k \cdot \vec{s}_0 = -s_1 G(k) s_1. \] (59)

By means of (59), we find that
\[ \sigma_3 \vec{J}^{(0)}_k G(k)^+ \sigma_3 = -\vec{J}^{(0)}_k \sigma_3 G(k)^+ \sigma_3 = \vec{J}^{(0)}_k G(k)^-, \] (60)

from which it derives that (56) can be written as the sum of two different terms
\[ \frac{1}{16} \chi_{ij}^{+-} Tr [D_i D_j - D_i s_3 \sigma_1 D_j s_3 \sigma_1] \] (61)
\[ + \frac{1}{16} \chi_{ij}^{++} Tr [D_i D_j + D_i s_3 \sigma_1 D_j s_3 \sigma_1]. \] (62)

By summing (61), (62), (52) and (53), we get
\[ \frac{1}{16} \left( \chi_{ij}^{+-} - \chi_{ij}^{++} \right) Tr [D_i D_j - D_i s_3 \sigma_1 D_j s_3 \sigma_1], \]
which is equal to
\[ - \frac{2\pi}{32 \Sigma^2} \sigma_{ij}^{(0)} \int dR Tr \left( \partial_i Q(R) \partial_j Q(R)^\dagger \right), \] (63)
where
\[ \sigma_{ij}^{(0)} = \frac{1}{2\pi} \left( \chi_{ij}^{+-} - \chi_{ij}^{++} \right) \]
is the Kubo conductivity with the current vertices which involve only the regular hopping [cf. Eq. (B4)].

F. Expansion in V

The expansion in \( V \) up to second order, gives two terms
\[ - \frac{1}{2} Tr (GV), \] (64)
and
\[ - \frac{1}{4} Tr (GVGV), \] (65)
In addition, we must also consider the mixed term

\[-\frac{1}{2}Tr(GUGV).\] (66)

The first order term (64) is

\[-\frac{1}{2}Tr(GV) = \frac{i}{\omega_0}Tr(Q_{sp}V)\]

\[= \frac{1}{\omega_0} \int dRTr \left( (Q(R)^\dagger \vec{\beta} \cdot \vec{\nabla}Q(R)\sigma_3 \right)\] (67)

\[-\frac{1}{2\omega_0} \int dRTr \left( \vec{\beta} \cdot \vec{\nabla}Q(R)^\dagger \vec{\beta} \cdot \vec{\nabla}Q(R) \right).\] (68)

The first term (67) is another boundary term, which we neglect. The second order term, Eq. (65), gives

\[-\frac{1}{4} \sum_k (G_0(k)^2 + G_3(k)^2) \int dRTr \left[ \vec{\beta} \cdot \vec{\nabla}Q(R)^\dagger \vec{\beta} \cdot \vec{\nabla}Q(R) \right].\] (69)

Notice that the saddle point equation implies that

\[\frac{1}{V} \sum_k G_3(k)^2 - G_0(k)^2 = \frac{2}{\omega_0}.\]

By using the above equation in (68), we find that (69) plus (68) give

\[-\frac{1}{2V} \sum_k \frac{\varepsilon_k^2}{\sum^2} \int dRTr \left[ \vec{\beta} \cdot \vec{\nabla}Q(R)^\dagger \vec{\beta} \cdot \vec{\nabla}Q(R) \right]\]

(70)

The contribution of the mixing term, (66), can be evaluated in a similar way, giving

\[\frac{1}{2V} \sum_k \varepsilon_k^2 \int dRTr \left[ \vec{\nabla}\theta_k \cdot \vec{\nabla}Q(R)^\dagger \vec{\beta} \cdot \vec{\nabla}Q(R) \right].\] (71)

We notice that, since \(\phi_q = -\theta_q\), then (70) and (71) can be included in (63) if the following redefinition of the current vertex is assumed:

\[\vec{J}_k^{(0)} = \vec{\nabla}\varepsilon_k \vec{B}_k \cdot \vec{\sigma} + \varepsilon_k \left( \vec{\nabla}\theta_k - \vec{\nabla}\theta_0 \right) \vec{B}_{\perp,k} \cdot \vec{\sigma}.\] (72)

Therefore the Kubo conductivity which appears in (63) has to be calculated with the above expression of the regular current vertex. This has notable consequences. First of all, for
cubic lattices, the interband contribution vanishes, hence the Kubo conductivity coincides with the true one, since the enlargement of the unit cell was artificial. This is compatible with (23), where we showed that the impurity action is local in the basis which diagonalizes the regular hopping, implying that the regular current vertex contains only the intraband operator.

To conclude, the effective action so far derived is therefore

$$S[Q] = \frac{1}{2\omega_0} \int dRT \left[ Q(R)Q(R)^\dagger \right]$$

$$+ \frac{\gamma}{2V\omega_0} \sum_q q^2 Tr \left[ Q_q Q_q^\dagger \right]$$

$$+ \frac{2\pi}{32\Sigma^2} \delta_{ij}^{(0)} \int dRTr \left( \partial_i Q(R) \partial_j Q(R)^\dagger \right)$$

$$+ \int dR i \frac{E}{\omega_0} Tr \left( Q(R) - \frac{\omega}{2\omega_0} Tr \left( s_3 Q(R) \right) \right).$$

(73)

VII. LONGITUDINAL FLUCTUATIONS

The full expression of the $Q$-matrix we must indeed consider is the one given by (31),

$$Q(R)_P = \tilde{T}^{-1} [Q_{sp} + P(R)] T(R) \equiv Q(R) + \tilde{T}^{-1} P(R) T(R)$$

$$\equiv Q(R) + S(R),$$

(74)

where $T(R)$ involves transverse fluctuations and

$$P(R) = (P_{00}s_0 + P_{03}s_3) \sigma_0 + i (P_{31}s_1 + P_{32}s_2) \sigma_3,$$

(75)

being all $P$’s hermitean. Charge conjugation implies that $cP^t c^t = P$. The field $P(R)$ takes into account longitudinal fluctuations which are massive. Let us define $\Gamma(R-R')$ the Fourier transform of $\omega_q^{-1}$. Then, the free action of the $Q_P$ field is

$$S[Q_P] = \frac{1}{2} \int dR dR' \Gamma(R-R') Tr \left[ Q_P(R)Q_P(R')^\dagger \right]$$

$$= \frac{1}{2\omega_0} \int dRT \left[ Q_P(R)Q_P(R)^\dagger \right]$$

$$- \frac{1}{4} \int dR dR' \Gamma(R-R') Tr \left[ \left( Q_P(R) - Q_P(R') \right) \left( Q_P(R)^\dagger - Q_P(R')^\dagger \right) \right].$$

(76)
Since $QQ^\dagger = Q_{sp}^2$, (76) gives
\[ \frac{1}{2\omega_0} \int dRT r \left[ P(R)P(R)^\dagger + 2Q_{sp}P(R) + Q_{sp}^2 \right]. \]

The second term cancels with the first order expansion of $Tr \ln G_P$, since $Q_{sp}$ is the saddle point solution. What is left, i.e.
\[ \frac{1}{2\omega_0} \int dRT r \left[ P(R)P(R)^\dagger \right], \]
(78)
is actually the mass term of the longitudinal modes, since the second order expansion in $P$ of $Tr \ln G_P$ is zero. The other term, (77), can be analysed within a gradient expansion of $Q_P(R) - Q_P(R') = \vec{\nabla}Q_P(R) \cdot (\vec{R} - \vec{R}') + \ldots$. The details of the calculations are given in Appendix C.1, so that, in this section, we just present the final results.

The free action of the longitudinal fields is found to be
\[ S_0[P] = \frac{1}{V} \sum_q \frac{1}{2\omega_q} Tr(P_qP_q^\dagger). \]
(79)
Here, we neglect the contribution of the invariant measure, which, in the zero replica limit, gives rise to fluctuations smaller by a factor $u^2$ than Eq.(79). The integration over $P$ with the above action has several important consequences for the action of the transverse modes (see Appendix C.1).

First of all, all the terms of the Kubo conductivity with the random current vertices are recovered. In addition, we find a new operator
\[ -\frac{2\pi}{8\cdot 32\Sigma^3} \int dRT r \left[ Q^\dagger(R)\vec{\nabla}Q(R)\sigma_3 \right] \cdot Tr \left[ Q^\dagger(R)\vec{\nabla}Q(R)\sigma_3 \right], \]
(80)
which has contributions from two different terms. The first one is obtained by expanding each Green’s function in (50) at first order, $G_P = G_0 - iG_0PG_0$, and the second is derived by (77). They are analogous to the components of the Kubo conductivity with regular and with random current vertices, respectively.
VIII. EFFECTIVE NON–LINEAR $\sigma$-MODEL

In conclusion the final expression of the action of the transverse modes in the long-wavelength limit is

$$S[Q] = \frac{2\pi}{32\Sigma^2} \sigma_{xx} \int dR \text{Tr} \left( \nabla Q(R) \cdot \nabla Q(R)^\dagger \right)$$

$$+ \int dR \frac{E}{\omega_0} \text{Tr} \left( Q(R) \right) - \frac{\omega}{2\omega_0} \text{Tr} \left( s_3 Q(R) \right)$$

$$- \frac{2\pi}{8 \cdot 32\Sigma^4} \Pi \int dR \text{Tr} \left[ Q^\dagger(R) \nabla Q(R) \sigma_3 \right] \cdot \text{Tr} \left[ Q^\dagger(R) \nabla Q(R) \sigma_3 \right], \quad (81)$$

where we make use of the fact that, in the models we consider, $\sigma_{ij} = \delta_{ij} \sigma_{xx}$. Since $Q(R) = Q_{sp} T(R)^2$, see Eq.(33), expressing $T(R)$ by means of $U(R)$ as in Eq.(27), the action at $E = \omega = 0$ can also be written as

$$S[U] = \frac{2\pi \sigma_{xx}}{16} \int dR \text{Tr} \left[ \nabla U(R)^{-2} \cdot \nabla U(R)^2 \right]$$

$$- \frac{2\pi \Pi}{32 \cdot 2} \int dR \left\{ \text{Tr} \left[ U(R)^{-2} \nabla U(R)^2 \right] \right\}^2, \quad (82)$$

as anticipated in the section I A. As compared to the non-linear $\sigma$-model which is obtained in the absence of the particle-hole symmetry, the above action differs first of all because of the symmetry properties of the matrix field $U(R)$, which describes now the Goldstone modes within the coset space $\text{U}(4m)/\text{Sp}(2m)$. Moreover, it also differs for the last term of (81), which, in the general case, even if present, is not related to massless modes. An analogous term was originally obtained by Gade in a two sublattice model described by two on-site levels with a regular hopping of the form $H_{RR'} = t_{RR'} \sigma_1$, and a local time-reversal symmetry breaking random potential $H_{\text{imp},R} = w_1 R \sigma_1 + w_2 R \sigma_2$, which we discuss in section XII.

Although the action may be parametrized by a simple unitary field $U(R)$ as in (82), we prefer to work with the matrix $Q(R)$ which has the more transparent physical interpretation $Q(R) \sim \Psi(R) \overline{\Psi}(R)$.

Finally, it is important to notice that either $\sigma_{xx}$ and $\Pi$ have contributions from both the regular and the random current vertices. This implies that, even in the limit of strong
disorder, in which the average hopping is negligible with respect to its fluctuations, these constants are finite and become of order unity \[7,11\].

A. Gaussian Propagators

At second order in $W$, the dispersion term

$$
\frac{2\pi\sigma_{xx}}{32\Sigma^2} \int dR \text{Tr} \left( \bar{\nabla} Q \bar{\nabla} Q \right) \simeq -\frac{2\pi\sigma_{xx}}{32} \int dR \text{Tr} \left( \bar{\nabla} W \bar{\nabla} W \right)
$$

$$
= \frac{2\pi\sigma_{xx}}{32} \int dR T r \left( \bar{\nabla} B \bar{\nabla} B \right) + 2 T r \left( \bar{\nabla} A \bar{\nabla} A + \bar{\nabla} C \bar{\nabla} C \right),
$$

where $A$, $B$ and $C$ are defined through Eqs.(35), (36) and (37). For the $B$’s we find the quadratic action

$$
\frac{\pi\sigma_{xx}}{2} \sum_{i=0}^{4} \sum_{k} \sum_{ab} k^2 B_{i,ab}(k)B_{i,ab}(-k),
$$

so that

$$
\langle B_{i,ab}(k)B_{j,cd}(-k) \rangle = \delta_{ij}\delta_{ac}\delta_{bd}D(k), \quad (83)
$$

where

$$
D(k) = \frac{1}{\pi\sigma_{xx}} \frac{1}{k^2}. \quad (84)
$$

For the $A$’s we have to take into account also the disconnected term:

$$
-\frac{2\pi\Pi}{32 \cdot 8\Sigma^4} \int dR T r \left[ Q^\dagger(R)\bar{\nabla} Q(R)\sigma_3 \right] \cdot T r \left[ Q^\dagger(R)\bar{\nabla} Q(R)\sigma_3 \right]
$$

$$
\simeq -\frac{2\pi\Pi}{64} \int dR T r \left[ \bar{\nabla} W_3 \right] \cdot T r \left[ \bar{\nabla} W_3 \right]
$$

$$
= \frac{2\pi\Pi}{64} \int dR T r \left[ \bar{\nabla} A + \bar{\nabla} C \right] \cdot T r \left[ \bar{\nabla} A + \bar{\nabla} C \right]
$$

$$
= \frac{2\pi\Pi}{16} \int dR T r \left[ \bar{\nabla} A_0 + \bar{\nabla} C_0 \right] \cdot T r \left[ \bar{\nabla} A_0 + \bar{\nabla} C_0 \right].
$$

The non vanishing propagators are
\langle A_{0,ab}(k)A_{0,cd}(-k) \rangle = D(k) (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})
- D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \delta_{ab}\delta_{cd}, \quad (85)

\langle C_{0,ab}(k)C_{0,cd}(-k) \rangle = D(k) (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})
- D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \delta_{ab}\delta_{cd}, \quad (86)

\langle A_{0,ab}(k)C_{0,cd}(-k) \rangle = -D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \delta_{ab}\delta_{cd}, \quad (87)

where \( m \) is the number of replicas, while for \( i = 1, 2, 3 \)

\langle A_{i,ab}(k)A_{i,cd}(-k) \rangle = D(k) (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \quad (88)

\langle C_{i,ab}(k)C_{i,cd}(-k) \rangle = D(k) (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) . \quad (89)

Notice that the particular symmetry of the two-sublattice model leads to additional diffusive modes in the retarded-retarded and advanced-advanced channels, which are not massless in the standard case \[16\].

**B. Physical Meaning of \( \Pi \)**

Let us introduce an external source which couples to the staggered density of states, which is accomplished by adding to the action a term

\[ \int dR \Psi_{R} s_{3} \sigma_{3} \hat{\lambda}(R) \Psi_{R}, \]

where, in the replica space, the source \( \hat{\lambda}_{\alpha,\beta} = \lambda_{\alpha}\delta_{\alpha,\beta} \). The fluctuations of the staggered density of states is obtained by the derivative of the partition function with respect, for instance, to \( \lambda_{\alpha} \) and \( \lambda_{\beta} \), with \( \alpha \neq \beta \). Inserting the source term in the action, and integrating over the Grassmann fields after introducing the matrix \( Q \), leads to the following expression of the staggered density of states fluctuation in terms of \( Q \):

\[ F(R, R') = \frac{1}{\pi^{2} \omega_{0}^{2}} (Tr [Q_{aa}(R)s_{3}\sigma_{3}] Tr [Q_{\beta\beta}(R')s_{3}\sigma_{3}]), \quad (90) \]

The gaussian estimate of the above correlation function at momentum \( k \) is given by
\[ F(k) = -16 \sum_{\pi^2 \omega_0^2} \left( [A_{0,\alpha\alpha}(k) + C_{0,\alpha\alpha}(k)] [A_{0,\beta\beta}(-k) + C_{0,\beta\beta}(-k)] \right) = \frac{64 \Sigma^2}{\pi^2 \omega_0^2} D(k) \frac{\Pi}{\sigma_{xx}} + \Pi m. \]

(91)

Therefore, \( \Pi \) is directly related to the singular behavior of the staggered density of states fluctuations.

IX. RENORMALIZATION GROUP

In this section, we will apply the Wilson–Polyakov Renormalization Group (RG) procedure \[14,16\] to analyse the scaling behavior of the action. Indeed, some of the calculations which we present are redundant, given the proof by Gade and Wegner that the \( \beta \)-function is zero \[8\] (see Appendix \[3\]). Nevertheless, other results besides the conductance \( \beta \)-function are important, so that we describe the whole RG procedure.

A. RG equations

In the spirit of Wilson–Polyakov RG approach \[16,14\], we assume that

\[ T(R) = T_f(R)T_s(R), \]

where \( T_f \) involves fast modes with momentum \( q \in [\Lambda/s, \Lambda] \), while \( T_s \) involves slow modes with \( q \in [0, \Lambda/s] \), being \( \Lambda \) the higher momentum cut-off, and the rescaling factor \( s > 1 \). Within an \( \epsilon \)-expansion, where \( \epsilon = d - 2 \), we define

\[ \int_{\Lambda/s}^{\Lambda} \frac{d\vec{k}}{(2\pi)^d} D(k) \equiv L = \frac{1}{4\pi^2 \sigma_{xx}} \ln s + \mathcal{O}(\epsilon). \]

It is straightforward to show the following result

\[ Tr \left[ \vec{\nabla} Q^\dagger \vec{\nabla} Q \right] = Tr \left[ \vec{\nabla} Q_f^\dagger \cdot \vec{\nabla} Q_f \right] + 2Tr \left[ \vec{D}_s \sigma_1 Q_f \vec{D}_s Q_f^\dagger \sigma_1 \right] - 2\Sigma^2 Tr \left[ \vec{D}_s \vec{D}_s \right] + 4Tr \left[ \vec{D}_s Q_f^\dagger \vec{\nabla} Q_f \right], \]

(92)
where \( Q_f = \tilde{T}_f Q_{sp} T_f \) and \( \vec{D}_s = T_s \vec{\nabla} T_s^\dagger \). Moreover,

\[
\frac{1}{\Sigma} Tr \left[ Q^\dagger \vec{\nabla} Q \sigma_3 \right] \cdot Tr \left[ Q^\dagger \vec{\nabla} Q \sigma_3 \right] \\
= Tr \left[ \left( \vec{\nabla} W_s + \vec{\nabla} W_f \right) \sigma_3 \right] \cdot Tr \left[ \left( \vec{\nabla} W_s + \vec{\nabla} W_f \right) \sigma_3 \right].
\] (93)

Since the fast and slow modes live in disconnected regions of momentum space, only the stiffness term (92) generates corrections. By expanding the terms coupling slow and fast modes up to second order in \( W_f \), the stiffness generates an action term for the slow modes which, after averaging over the fast ones, is

\[
\frac{2\pi \sigma_{xx}}{32\Sigma^2} \int dR Tr \left[ \vec{\nabla} Q_s^\dagger \vec{\nabla} Q_s \right] + \langle F_1 \rangle_f - \frac{1}{2} \langle F_2 \rangle^2_f,
\] (94)

where

\[
F_1 = \frac{2\pi \sigma_{xx}}{32\Sigma^2} \int dR - 2 Tr \left[ \vec{D}\sigma_1 Q_{sp} W_f \vec{D} W_f Q_{sp} \sigma_1 \right] \\
+ 2 Tr \left[ \vec{D}\sigma_1 Q_{sp} \vec{D} Q_{sp} \vec{W}_f^2 \sigma_1 \right],
\] (95)

and

\[
F_2 = 4 \frac{2\pi \sigma_{xx}}{32} \int dR Tr \left[ \vec{D} W_f \vec{\nabla} W_f \right].
\] (96)

The explicit evaluation of these terms is outlined in Appendix D. Here we just give the final results. The Kubo conductivity is renormalized according to

\[
\sigma_{xx} \rightarrow (1 - 4Lm) \sigma_{xx},
\] (97)

while the \( \Pi \) factor

\[
\Pi \rightarrow \Pi + 4L \sigma_{xx}.
\] (98)

For what it regards the renormalization of \( E \) and \( \omega \), we notice that

\[
Q = Q_{sp} T_s T_f^2 T_s \\
\simeq Q_{sp} T_s^2 + Q_{sp} T_s W_f T_s + \frac{1}{2} Q_{sp} T_s W_f^2 T_s.
\]
Since the slow and fast degrees of freedom are defined in different regions of momentum space, only the second term is relevant. By means of Eq. (D13), we find that

\[ Q \rightarrow Q_s \left(1 + \frac{1}{2} \left(2 - 8m + \frac{\Pi}{\sigma_{xx} + m\Pi}\right) L\right) \]  

This leads to similar corrections to \( E \) and \( \omega \), which will have the same scaling behavior.

Finally, to describe the cross-over behavior in the presence of symmetry breaking terms, we also need the scaling behavior of the operator \( Tr [Q(R)^2] \). We find that

\[ \langle TrQ^2 \rangle_f = \langle Tr [QspT_sT_f^2T_sQspT_sT_f^2T_s] \rangle_f = \left[1 + 2 \left(2 - 4m + \frac{\Pi}{\sigma_{xx} + m\Pi}\right) L\right] TrQ_s^2 - L (TrQ_s)^2. \]  

To implement the RG, we have to rescale the momenta in order to recover the original range \([0, \Lambda]\). This is accomplished by the transformation \( k \rightarrow k/s \), or, equivalently, \( R \rightarrow Rs \). Therefore, the stiffness as well as the fluctuation terms acquire a scaling factor \( s^\epsilon \simeq 1 + \epsilon \ln s \), while the \( E \) and \( \omega \) terms a factor \( s^d \). Hence, after defining \( t = 1/(4\pi^2\sigma_{xx}) \), \( c = 1/(4\pi^2\Pi) \), and \( \lambda \) the coupling constant of the operator \( TrQ^2 \), we get the following \( \beta \)-functions

\[ \beta_t = -\epsilon t + 4mt^2, \]  
\[ \beta_c = -\epsilon c - 4c^2, \]  
\[ \beta_E = dE + E\frac{t}{2} \left(2 + \frac{t}{c + mt}\right), \]  
\[ \beta_\lambda = d\lambda + 2\lambda t \left(2 + \frac{t}{c + mt}\right). \]  

At finite energy, \( E \neq 0 \), we may use a two-cutoff scaling approach [9]. Namely, we can follow the previous RG equations up to a cross-over scale, \( s_{cross} = s(E) \), at which the energy has flowed to a value \( E_0 \) of order \( \Sigma \), which plays the role of the high-energy cut-off in the theory. Above this scale, we must neglect all contributions coming from the \( W_3 \) modes, which acquire a mass. That is, we must abandon the RG equations (101)-(104), and let the coupling constants flow in accordance with the standard RG equations, which amount only to a renormalization of \( t \) according to
\[ \beta_t = -\epsilon t + t^2. \] (106)

If, by integrating (106), the inverse conductance \( t(s) \) flows to infinity, signalling an insulating behavior, then we can define a localization length \( \xi_{\text{loc}}(E) \) as the scale at which \( t \) has grown to a value of order unity.

**B. RG in \( d = 2 \)**

In \( d = 2 \), the solution of the RG equations for \( t \), \( c \), and \( E \) is

\[
\begin{align*}
  t(s) &= t(1) = t_0, \\
  \frac{1}{c(s)} &= \frac{1}{c(1)} + 4 \ln s = \frac{1}{c_0} + 4 \ln s, \\
  \ln \left( \frac{E(s)}{E} \right) &= \left[ 2 + \frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} \right) \right] \ln s + \frac{1}{2} t_0^2 \ln^2 s
\end{align*}
\]

At finite energy, the crossover length, \( s(E) \), to the standard, non particle-hole symmetric, model, which, as previously discussed, is defined through \( E[s(E)] = E_0 \sim \Sigma \), is given by

\[
s(E) = \exp \left\{ \frac{1}{2A} \left( \sqrt{B^2 + 4A \ln \frac{E_0}{E}} - B \right) \right\}, \tag{107}
\]

being

\[
A = \frac{1}{2} t_0^2, \quad B = 2 + \frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} \right).
\]

Above \( s(E) \), \( t \) flows according to Eq. (106) with \( \epsilon = 0 \), hence it grows to infinity, implying that the wavefunctions are localized for any \( E \neq 0 \). The localization length \( \xi_{\text{loc}}(E) \sim s(E) \), apart from a multiplicative factor which is \( \sim \exp[(1 - t_0)/t_0] \) if \( t_0 \ll 1 \). We see that, for

\[
E \gg E_0 \exp \left\{ -\frac{1}{2t_0^2} \left( 2 + \frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} \right) \right)^2 \right\},
\]

the localization length has a power law behavior, namely

\[
\xi_{\text{loc}}(E) \propto \left( \frac{E_0}{E} \right)^{\frac{1}{2}},
\]

otherwise, at very small energy, it diverges slower,
\[ \xi_{\text{loc}}(E) \propto \exp \left( \sqrt{\frac{\ln(E_0/E)}{A}} \right). \]

The density of states renormalizes like

\[
\ln \frac{\rho(s)}{\rho_0} = \frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} \right) \ln s + \frac{1}{2} t_0^2 \ln^2 s.
\]

At finite energy, the density of states flows until \( s < S(E) \), after which it stays constant. This implies that the renormalized value is obtained by

\[
\ln \frac{\rho(E)}{\rho_0} = \ln \frac{\rho(s(E))}{\rho_0} = \ln \frac{E_0}{E} - 2 \ln s(E),
\]

leading to

\[
\rho(E) = \rho_0 \left( \frac{E_0}{E} \right) \frac{1}{s(E)^2} \simeq \rho_0 \left( \frac{E_0}{E} \right) \exp \left( -\sqrt{\frac{4 \ln(E_0/E)}{A}} \right),
\]

the last equality valid at small energy.

**C. RG in \( d = 3 \)**

In \( d = 3 \),

\[
t(s) = t_0 s^{-1},
\]

\[
c(s) = \frac{c_0 s^{-1}}{1 + 4c_0 - 4c_0 s^{-1}},
\]

\[
\ln \frac{E(s)}{E} = 3 \ln s + \frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} + 4t_0 \right) \left( 1 - \frac{1}{s} \right) - t_0^2 \left( 1 - \frac{1}{s^2} \right),
\]

hence both \( t \) and \( c \) running variables vanish for \( s \to \infty \). The cross-over length diverges, as we approach \( E = 0 \), approximately like

\[
s(E) \simeq \left( \frac{E_0}{E} \right)^{\frac{1}{3}} \exp^{-\frac{t_0}{6} \left( 2 + \frac{t_0}{c_0} + t_0 \right)}.
\]

Hence, the density of states

\[
\rho(E) = \rho_0 \left( \frac{E_0}{E} \right) \frac{1}{s(E)^3} \simeq \rho_0 e^{\frac{t_0}{2} \left( 2 + \frac{t_0}{c_0} + t_0 \right)},
\]
saturates at \( E = 0 \) to a value exponentially increased in \( t_0 \) with respect to the bare \( \rho_0 \).

At \( E \neq 0 \), the inverse conductance above \( s_{\text{cross}} = s(E) \) flows according to (106) with \( \epsilon = 1 \) and boundary condition \( t(s_{\text{cross}}) = t_0/s_{\text{cross}} \). We find that, if

\[
t_0 < s_{\text{cross}},
\]

the system is metallic, otherwise it is insulating, with a localization length

\[
\xi_{\text{loc}}(E) \sim \frac{s(E)}{t_0 - s(E)}.
\]

This results implies that, for any amount of disorder, sufficiently close to \( E = 0 \), all eigenfunctions are delocalized, in agreement with recent numerical results [15]. However, if the disorder is gaussian, as we assumed, the random hopping model with zero regular hopping seems to be characterized by an inverse Drude conductance, \( t_0 \), which is an increasing function of \(|E|\), being smaller than the critical value \( \epsilon \) at \( E = 0 \) (see also Ref. [7]). In this case, the presence of a finite mobility edge in \( d = 3 \), even for zero regular hopping, would not depend crucially upon the intermediate RG flow in the vicinity to the band center. Nevertheless, we expect that \( t_0 \) at \( E = 0 \) varies for different kinds of disorder, and eventually it may become greater than unity. In this case, it is just the vicinity to the band center which makes it possible a finite mobility edge.

**X. ON SITE DISORDER**

In this and in the following section, we analyse various symmetry breaking terms, which, in the two sublattice representation, contain \( \sigma_0 \) and \( \sigma_3 \), hence spoiling Eq. (1).

We start by adding an onsite disorder

\[
\delta S_{\text{imp}} = \sum_R u_{1,R} c_{1,R}^\dagger c_{1,R} + u_{2,R} c_{2,R}^\dagger c_{2,R} = \sum_R c_R^\dagger \left( \frac{u_{1,R} + u_{2,R}}{2} \sigma_0 + \frac{u_{1,R} - u_{2,R}}{2} \sigma_3 \right) c_R,
\]

where \( \langle u_{i,R} \rangle = 0 \) and \( \langle u_{i,R} u_{j,R'} \rangle = \delta_{ij} \delta_{RR'} v^2 \). Within the path integral, this term becomes, once average over disorder is performed,
\[ \delta S_{\text{imp}} = \frac{v^2}{2V} \sum_q \left( Y_{1,q}^{\alpha \beta} Y_{1,-q}^{\beta \alpha} + Y_{2,q}^{\alpha \beta} Y_{2,-q}^{\beta \alpha} \right) \]
\[ = \frac{v^2}{4V} \sum_q \left( Y_{1,q}^{\alpha \beta} Y_{0,-q}^{\beta \alpha} + Y_{3,q}^{\alpha \beta} Y_{3,-q}^{\beta \alpha} \right). \]

By adding this term to (17), we get
\[ S_{\text{imp}} + \delta S_{\text{imp}} = \frac{1}{4V} \sum_q \left( \omega_q + v^2 \right) Tr (Y_{0,q} Y_{0,-q}) \]
\[ - (\omega_q - v^2) Tr (Y_{3,q} Y_{3,-q}). \]

(111)

If we assume that the onsite disorder is weak, i.e. \( \omega_q > v^2 \) at small \( q \), the consequence is that the \( Q \) free action becomes
\[ S_{\text{imp}}^0 = \frac{1}{V} \sum_q \frac{1}{\omega_q + v^2} Tr [Q_{0,q} Q_{0,-q}] + \frac{1}{\omega_q - v^2} Tr [Q_{3,q} Q_{3,-q}] \]
\[ = \frac{1}{V} \sum_q \frac{1}{2(\omega_q + v^2)} Tr [Q_q Q_q^\dagger] + \frac{2v^2}{\omega_q - v^4} Tr [Q_{3,q} Q_{3,-q}]. \]

(112)

Therefore, the on site disorder introduces a mass in the \( Q_3 \) propagators. Specifically, since \( 2iQ_3 \sigma_3 = Q - Q^\dagger \), the mass term can be written as
\[ - \frac{v^2}{4(\omega_0^2 - v^4)} Tr \left[ \left( Q_q - Q_q^\dagger \right) \left( Q_q - Q_q^\dagger \right) \right]. \]

Close to the saddle point, \( QQ^\dagger = Q_{sp}^2 \), and, for small \( q \), we get
\[ - \frac{v^2}{4(\omega_0^2 - v^4)} \int dR Tr \left[ Q(R) Q(R) + Q(R)^\dagger Q(R)^\dagger \right], \]

which, at second order in \( W \), reads
\[ - \frac{v^2 \Sigma_2^2}{2(\omega_0^2 - v^4)} \int dR Tr \left[ W(R) W(R) + W(R) s_3 W(R) s_3 \right] \]
\[ = - \frac{2v^2 \Sigma_2^2}{\omega_0^2 - v^4} \int dR Tr \left[ W_3(R) W_3(R) \right]. \]

(113)

In the presence of this term, we could proceed, as before, in the framework of two cutoff scaling theory. That is, we apply the previous RG equations until the above term becomes of the order \( \Sigma \rho_0 \), i.e. up to the scale which, by Eq.(104), is
\[ s_{\text{cross}} = \exp \left\{ \frac{1}{2A} \left( \sqrt{B^2 + 4A \ln \lambda - B} \right) \right\}, \]

(114)
in $d = 2$, where $A = 2t_0^2$ and $B = 2 + t_0(2 + t_0/c_0)$, while $s_{\text{cross}} \simeq \lambda^{1/d}$ in $d > 2$, where

$$\lambda \propto \omega_0 v^2,$$

in the limit of small $v$. Above this scale, the $W_3$ propagator gets fully massive, and the inverse conductivity flows with the RG equation (106). In $d = 2$, this implies that, ultimately, the system gets localized, although the density of states has increased in the first stage of the RG.

**XI. SAME–SUBLATTICE REGULAR HOPPING**

We can also introduce a particle–hole symmetry breaking term, by adding to the Hamiltonian a regular term connecting same sublattices, e.g.

$$\delta H_{RR'} = t_{RR'}^{(0)} \sigma_0 + i t_{RR'}^{(3)} \sigma_3 \rightarrow \delta H_k = t_k^{(0)} \sigma_0 + t_k^{(3)} \sigma_3.$$  

Expanding the action in $\delta H$, after integrating over the Nambu spinors, we get an additional term

$$\delta S[Q] = \frac{1}{2} Tr \left[ \tilde{T} \delta H T^\dagger G \right].$$  

(115)

We define

$$\frac{4}{\omega_0} \lambda_0 Q_{\text{sp}} \equiv i \frac{1}{V} \sum_k t_k^{(0)} G(k),$$  

$$\frac{4}{\omega_0} \lambda_3 Q_{\text{sp}} \equiv -i \frac{1}{V} \sum_k t_k^{(3)} G(k),$$  

so that (115) becomes

$$- \frac{i}{\omega_0} Tr \left[ Q (\lambda_0 \sigma_0 + \lambda_3 \sigma_3) \right].$$  

(116)

The $\lambda_0$-term acts like an energy term. This implies that, if we just shift the chemical potential, we do recover the same scenario as in the absence of this term and at $E = 0$. On the contrary, the $\lambda_3$-term is always a relevant perturbation, whose strength increases under
RG iteration as the energy $E$. We can define a crossover scale $s_{\text{cross}}$, which has the same expression as $s(E)$ in (107) and (110), for $d = 2$ and $d = 3$, respectively, provided $E \to \lambda_3$. Above this scale, the $W_3$ modes get fully massive and their contribution to the RG flow drops out.

Sometimes $t_{RR'}^{(3)} = 0$, as, for instance, for next-nearest neighbor hopping in a square lattice. In this case, $\lambda_3 = 0$ and we need to evaluate the second order term

$$\delta S[Q] = \frac{1}{4} Tr \left[ \tilde{T} \delta H T^\dagger G \tilde{T} \delta H T^\dagger G \right].$$

If we define $F(R) = \tilde{T}(R) T(R)^\dagger$, which contains either $\sigma_0$ and $\sigma_3$, this term can be written, at long wavelengths, as

$$\delta S[Q] = \frac{1}{4} \sum_k \left( t_k^{(0)} \right)^2 Tr \left[ F_q G(k) F_{-q} G(k) \right].$$

By introducing,

$$\Sigma_{pq}^{i} = \frac{1}{V} \sum_k \left( t_k^{(0)} \right)^2 Tr \left[ \sigma_i G^p \sigma_i G^q \right],$$

where $p, q = \pm$, the following results hold

$$\Sigma_0^{pq} = \frac{1}{2} (1 - pq) \frac{4C}{\omega_0} \quad \Sigma_3^{pq} = -\frac{1}{2} (1 + pq) \frac{4C}{\omega_0},$$

where $C$ is a constant of dimension energy square, with order of magnitude given by the typical value of $\left( t_k^{(0)} \right)^2$ close to the surface corresponding to $E = 0$. Therefore we can write,

$$\delta S[Q] = \frac{C}{2\omega_0} \int dR Tr \left[ F(R) \tilde{F}(R) - F(R) s_3 \sigma_1 \tilde{F}(R) s_3 \sigma_1 \right]$$
$$= \text{const.} + \frac{C}{2\omega_0 \Sigma^2} \int dR Tr \left[ Q(R)^2 \right],$$

which is a mass term for the $W_3$ propagators, similar to (113). Therefore, a same-sublattice hopping introduces a cross-over length analogous to (114), with

$$\lambda \propto \frac{\Sigma^2}{C}.$$

Above this scale, the contribution of the $W_3$ modes to the RG flow has to be dropped out.
XII. TIME-REVERSAL SYMMETRY BREAKING

If the random hopping breaks time-reversal symmetry, i.e.

\[ H_{\text{imp}} = \sum_{RR'} \tau_{12}^{RR'} C_R^{\dagger} C_{R'} + H.c., \]

with both real and imaginary part of \( \tau_{12}^{RR'} \) gaussian distributed, after averaging, the impurity action can be written as

\[ S_{\text{imp}} = \frac{1}{V} \sum_q W_{-q} Tr \left[ X_{1,0,q} X_{2,0,-q} + X_{1,3,q} X_{2,3,-q} \right], \quad (118) \]

where

\[ X_{1,0,R}^{\alpha\beta} = \Psi_{1R}^{\alpha} \tau_0 \Psi_{1R}^{\beta}, \quad X_{1,3,R}^{\alpha\beta} = \Psi_{1R}^{\alpha} \tau_3 \Psi_{1R}^{\beta}, \]

with the indices \( \alpha \) and \( \beta \) running only over the replicas and the advanced/retarded components. This implies that the manifold in which \( Q \) varies contains in this case only \( \tau_0 \) and \( \tau_3 \) components. Indeed, as in the time reversal invariant case we are able to parametrize the \( 8m \times 8m \) matrix \( T \) in terms of a \( 4m \times 4m \) matrix \( U \in U(4m)/\text{Sp}(2m) \) [see Eq.(27)], similarly, without time-reversal symmetry, \( T \) can be parametrized by means of a \( 2m \times 2m \) matrix \( U \in U(2m) \), in agreement with Gade [9]. The effective non-linear \( \sigma \)-model is not modified, but the expressions (D1), (D2), (D3), (D4), and (D5) have to be substituted by

\[ \langle B_{ab} P_{bc} B_{cd} \rangle = 2D(k) \delta_{ad} Tr \left( P_0 \right), \quad (119) \]

\[ \langle B_{ab} P_{bc} B_{cd} \rangle = 0, \quad (120) \]

\[ \langle A_{ab} P_{bc} A_{cd} \rangle = 2D(K) \delta_{ad} Tr \left( P_0 \right) - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad}, \quad (121) \]

\[ \langle C_{ab} P_{bc} C_{cd} \rangle = 2D(K) \delta_{ad} Tr \left( P_0 \right) - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad}, \quad (122) \]

\[ \langle A_{ab} P_{bc} C_{cd} \rangle = -D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad}. \quad (123) \]

where \( P = P_0 + iP_3 \tau_3 \). Hence, the RG equations at \( m = 0 \) are, in this case,

\[ \beta_t = -\epsilon t, \quad (124) \]

\[ \beta_c = -\epsilon c - 2c^2, \quad (125) \]

\[ \beta_E = dE + \frac{Et^2}{2c}, \quad (126) \]
which coincide with those obtained by Gade [9].

XIII. DISCUSSION AND COMPARISON WITH THE STANDARD LOCALIZATION THEORY

In this section, we summarize the main differences between the model (2) and the standard non-linear \( \sigma \)-model which is derived in the theory of Anderson localization [13,16], placing particular emphasis on the properties of the \( Q \)-matrix. The specific form of the off-diagonal disorder we consider, which only couples one sublattice to the other, leads, via the Hubbard-Stratonovich decoupling, to the introduction of a space-varying \( 8m \times 8m \) complex \( Q \)-matrix, \( Q = Q_0\sigma_0 + iQ_3\sigma_3 \). Here, \( Q_0 \) and \( Q_3 \) are \( 4m \times 4m \) hermitean matrices, of which matricial structure refers to the retarded/advanced, spinor particle/hole and \( m \) replica components. Contrary to the standard case, \( Q \) is not hermitean.

The evaluation of the saddle point, \( Q_{sp} = \sigma_0\tau_0s_3 \) (section V), as well as the derivation of the effective action (section VI) are analogous to the standard case [13,16]. (We recall that \( \sigma_i, s_i \) and \( \tau_i \) indicate the Pauli matrices, including the unit matrix, acting on sublattice, advanced/retarded and spinor components, respectively.) The non-linear \( \sigma \)-model, Eq. (81), is obtained by integrating out the longitudinal massive \( Q \)-fluctuations and only keeping the transverse soft modes. The real novelty with respect to localization theory is not in the structure of the effective action. Indeed, the new term in (81), namely

\[
\Pi \int dR \left[ Tr \left( (Q(R)^\dagger \vec{\nabla}Q(R)\sigma_3) \right)^2 \right],
\]

even if present, would be irrelevant in the standard case. On the contrary, the essential difference, as expected, lies in the ensemble spanned by the soft modes at the particle-hole symmetry point \( E = 0 \). We get \( Q_{\text{soft}} = \tilde{T}^{-1}Q_{sp}T \), where the unitary matrix \( T \) only contains \( \sigma_0 \) and \( \sigma_3 \),

\[
T = \exp \left( \frac{W_0\sigma_0 + W_3\sigma_3}{2} \right),
\]

and
These expressions derive by the conditions (1), which fully specify the model, as shown in section IV. In that section, we also showed that the ensemble can be expressed in terms of unitary $4m \times 4m$ matrices

$$U = \exp \left[ \frac{W_0 + W_3}{2} \right],$$

as argued by Gade and Wegner [8]. Selecting the subset which leaves the saddle point invariant gives Eq.(2) with $U \in U(4m)/Sp(2m)$. In terms of $T$, the condition $\tilde{T}^{-1} Q_{sp} T \neq Q_{sp}$, leads to the requirements $[W_0, s_3] \neq 0$ and $\{W_3, s_3\} \neq 0$, which implies that $W_0$ is off-diagonal in the energy retarded/advanced space (as in the standard localization theory), while $W_3$ is diagonal. In other words, the homogeneous and staggered modes, $W_0$ and $W_3$, respectively, have different structure in the energy space. The energy diagonal $W_3$-modes betray the presence, at $E = 0$, of diffusive poles in the disorder averaged products of retarded and advanced Green’s functions, $\overline{G_R G_R}$ and $\overline{G_A G_A}$, with $G_{R,A} = (-H \pm i0^+)^{-1}$. In the localization theory [13,16], only the mixed products $\overline{G_R G_A}$ have a singular behavior. This explains why singular corrections to the density of states (which involves connected diagrams with same energy Green’s functions) are present in the two-sublattice model, while they are absent in the localization theory.

In a square lattice, the energy diagonal modes have the transparent meaning of density fluctuations with wave-vector $q$ nearby the nesting vector $G = (\pi, \pi, \ldots)$, see Appendix A. Indeed

$$Q_3(q) \sim \sum_{R \in A} e^{-iqR} \left( \Psi_{1R} \Psi_{1R} - \Psi_{2R} \Psi_{2R} \right)$$

$$= \sum_{R \in A, B} e^{-i(q+G)R} \Psi_{R} \Psi_{R} = Q(q + G),$$

where $A$ and $B$ label the two sublattice, and, for $R \in A$, we have taken by definition $\Psi_{1R} = \Psi_{R}$ and $\Psi_{2R} = \Psi_{R+\hat{x}}$, being $a$ the lattice spacing and $\hat{x}$ the unit vector in the $x$-direction. As soon as $E \neq 0$, nesting is not more important and indeed $Q_3$ becomes massive.
Finally, because of $G_R G_R$ and $G_A G_A$, also the conductance acquires other corrections with respect to standard localization theory. Indeed, these corrections add to give a vanishing $\beta$-function for $\sigma_{xx}$, as first indicated by Gade and Wegner [8]. We have explained in the Introduction section (see also Appendix E) that this is a consequence of a simple abelian gauge symmetry generated by Eq.(11) at the particle-hole symmetry point $E = 0$. Similarly to the results for the density of states, this behavior of the $\beta$-function is at odds with the standard theory.

XIV. CONCLUSIONS

In this work, we have derived the effective non-linear $\sigma$-model of a disordered electronic system on a generic bipartite lattice. This model, if the hopping matrix elements as well as the disorder only couple one sublattice with the other, shows an interesting behavior close to the band center, i.e. to the particle-hole symmetry point. Namely, the wave-functions are always delocalized at the band center, in any dimension. By a Renormalization Group (RG) analysis, in the framework of an $\epsilon$-expansion, $\epsilon = d - 2$, we have found that the quantum corrections to the conductivity vanish if the chemical potential is exactly at the band center, thus implying a metallic behavior. In two dimensions, in particular, the Kubo conductivity flows to a fixed value by iterating the RG. On the contrary, we have found that the staggered density of states fluctuations, which are controlled by a new parameter in the non-linear $\sigma$-model, are singular. This result is reminiscent of what it is found in equivalent one-dimensional models. In fact, models of disordered spinless fermions in one-dimension can be mapped, by a Jordan-Wigner transformation, onto disordered spin chains. In many cases, it is known that, in spite of the presence of disorder, these spin chain models display critical behavior, as shown in great detail by D. Fisher for random Heisenberg antiferromagnets and random transverse-field Ising chains [17]. Indeed, as pointed out by Fisher, the staggered spin fluctuations, in a random antiferromagnetic chain, also display critical behavior in the form of a power law decay, $(-1)^{R(\langle S(R)\rangle \langle S(0)\rangle)} \sim R^{-2}$, where the bar
indicates impurity average. Since the staggered spin-density corresponds to the staggered
density of the spinless fermions, this result is consistent with the outcome of our analysis,
which further suggests that a similar scenario generally holds in such models. Moreover,
as in one-dimension [17,5], we find that the density of states is strongly modified by the
disorder at the band center, and it actually diverges in \( d = 2 \). In reality, a random Heisenberg
chain, away from the \( XXZ \) limit, maps onto a spinless fermion random hopping model
in the presence of a random nearest-neighbor interaction. However, even in the presence
of this additional interaction, the Hamiltonian has still the abelian gauge-like symmetry
described in Appendix E, which is at the origin of the delocalization of the band center
state. This observation is also compatible with Fisher’s result that the physical behavior
does not qualitatively change upon moving away from the \( XXZ \) limit towards the isotropic
\( XXX \) Heisenberg point.

Many of the results which we have derived were already known. The existence of de-
localized states at the band center of a two-sublattice model was argued already in 1979
by Wegner [6,7]. The effective non-linear \( \sigma \)-model when the disorder breaks time-reversal
invariance, as well as the RG equations, have earlier been derived by Gade [9], although in
a particular two-sublattice model. The extension to disordered systems with time-reversal
symmetry was later on argued by Gade and Wegner [8]. Finally, random flux models and
disordered Dirac fermion models have recently been the subject of an intensive theoretical
study [10–12], for their implications to a variety of different physical problems.

In spite of that, our analysis has several novelties with respect to earlier studies. First
of all, the two-sublattice model which we study is quite general. Secondly, the physical
interpretation of the parameters which appear in the non-linear \( \sigma \)-model is quite transpar-ent. Thirdly, the explicit derivation of the RG equations with time-reversal invariance is
presented.
XV. ACKNOWLEDGMENTS

We are grateful to A. Nersesyan, V. Kratsov, E. Tosatti, and Yu Lu for helpful discussions and comments.

APPENDIX A: SPECIFIC EXAMPLES

As an example, we consider a tight binding model with only nearest neighbor hopping on a square and honeycomb lattice.

In the case of square lattice, the enlarged unit cell is the $\sqrt{2} \times \sqrt{2}$ one. The new reciprocal lattice vectors are $\vec{G}_1 = 2\pi(1,-1)/a$, $\vec{G}_2 = 2\pi(1,1)/a$, and the angle $\theta_k$ of Eq.(1) is

$$\theta_k = \frac{a}{2}(k_1 + k_2) = k_x a. \quad (A1)$$

In the case of the honeycomb lattice, the unit cell contains already two lattice sites. The energy is given by

$$\epsilon_k^2 = \left[ 1 + 2 \cos \left( \frac{3}{2} k_x a \right) \cos \left( \frac{\sqrt{3}}{2} k_y a \right) \right]^2 + \left[ 2 \sin \left( \frac{3}{2} k_x a \right) \cos \left( \frac{\sqrt{3}}{2} k_y a \right) \right]^2,$$

and

$$\theta_k = \tan^{-1} \left( \frac{2 \sin \left( \frac{3}{2} k_x a \right) \cos \left( \frac{\sqrt{3}}{2} k_y a \right)}{1 + 2 \cos \left( \frac{3}{2} k_x a \right) \cos \left( \frac{\sqrt{3}}{2} k_y a \right)} \right).$$

The Brillouin zone is still honeycomb, with the $y$-axis one of its axes, and side equal to $4\pi/(3\sqrt{3}a)$.

APPENDIX B: WARD IDENTITY

Let us consider a generic Hamiltonian in the two sublattice representation

$$H = \sum_{R_1, R_2} \hat{c}^\dagger_{R_1} H_{R_1, R_2} c_{R_2}, \quad (B1)$$
where $H_{R_1, R_2}$ is a $2 \times 2$ Hermitean matrix. The current operator

$$\vec{J}(R) = \sum_{R_1, R_2} c_{R_1}^\dagger \vec{J}_{R_1, R_2}(R) c_{R_2},$$

can be obtained by the continuity equation, leading to

$$\vec{\nabla} \cdot \vec{J}(R) = i \sum_{R_1} c_R^\dagger H_{R, R_1} c_{R_1} - c_{R_1}^\dagger H_{R_1, R} c_R,$$

(B2)

being $\vec{\nabla}$ the discrete version of the differential operator. The long-wavelength expression for $\vec{J}(R)$ can be obtained by Fourier transformation, namely, through

$$i \sum_R \vec{q} \cdot \vec{J}(R)e^{-i\vec{q} \cdot \vec{R}} = i \sum_{R, R_1} \left( c_R^\dagger H_{R, R_1} c_{R_1} - c_{R_1}^\dagger H_{R_1, R} c_R \right) e^{-i\vec{q} \cdot \vec{R}},$$

and expanding both sides in $q$, we get, for the linear term,

$$i \sum_R \vec{q} \cdot \vec{J}(R) = \sum_{R, R_1} \vec{q} \cdot \vec{R} \left( c_R^\dagger H_{R, R_1} c_{R_1} - c_{R_1}^\dagger H_{R_1, R} c_R \right)$$

$$= \sum_{R, R_1} \vec{q} \cdot \left( \vec{R}_1 - \vec{R} \right) c_{R_1}^\dagger H_{R_1, R} c_R,$$

hence

$$\vec{J}(R) = -i \sum_{R_1} \left( \vec{R}_1 - \vec{R} \right) c_{R_1}^\dagger H_{R_1, R} c_R. \quad \text{(B3)}$$

Let us define the correlation functions

$$\chi_{\mu, i}(R, R'; t, t_1, t_2) = \langle T \left[ c_{R_1}^\dagger(t) J_{R_1, R_2}^\mu(R) c_{R_2}(t) c_{R_3}^\dagger(t_1) J_{R_3, R_4}^i(R') c_{R_4}(t_2) \right] \rangle, \quad \text{(B4)}$$

where $\mu = 0, 1, 2, 3$, $J_{R_1, R_2}^0(R) = \delta_{R R_1} \delta_{R R_2}$ are the density matrix elements, and $J^i$, for $i = 1, 2, 3$, are the matrix element components of the current. By the continuity equation, we find that

$$\partial_t \chi_{0, i} + \partial_j \chi_{j, i} = i \sum_{R_1} -\delta(t - t_1) Tr \left[ G(R_1, R; t_2 - t) J_{R_1, R_1}^i(R') \right]$$

$$+ \delta(t - t_2) Tr \left[ G(R, R_1; t - t_1) J_{R_1, R}^i(R') \right].$$

If we integrate both sides by
\[
\int dt_1 dt_2, e^{i(E+\omega)(t-t_1)} e^{iE(t_2-t)},
\]

at \( \omega = 0 \) we find
\[
\partial_j \chi_{j,i}(R, R'; E) = i \sum_{R_1} Tr \left[ G(R, R_1; E) \tilde{J}_{R_1,R}(R') \right] - Tr \left[ G(R_1, R; E) \tilde{J}_{R,R_1}^\dagger(R') \right]. \quad (B5)
\]

Using once more the continuity equation (B2), we find
\[
\partial_j \partial'_i \chi_{j,i}(R, R'; E) = -Tr \left[ G(R, R'; E) H_{R',R} + G(R', R; E) H_{R,R'} \right] + \delta_{RR'} \sum_{R_1} Tr \left[ G(R, R_1; E) H_{R_1,R} \right] + Tr \left[ G(R_1, R; E) H_{R,R_1} \right].
\]

By Fourier transform,
\[
\sum_{RR'} \partial_j \partial'_i \chi_{j,i}(R, R'; E) e^{-i\vec{q} \cdot (\vec{R} - \vec{R}')} = \sum_{RR'} q_i q_j \chi_{j,i}(R, R'; E) e^{-i\vec{q} \cdot (\vec{R} - \vec{R}')} \sum_{RR'} \left( 1 - e^{-i\vec{q} \cdot (\vec{R} - \vec{R}')} \right) Tr \left[ G(R, R'; E) H_{R',R} + G(R', R; E) H_{R,R'} \right].
\]

At small \( q \), the above expression is
\[
\sum_{RR'} q_i q_j \chi_{j,i}(R, R'; E) = \frac{1}{2} \sum_{RR'} q_i q_j \left( R_i - R'_i \right) \left( R_j - R'_j \right) Tr \left[ G(R, R'; E) H_{R',R} + G(R', R; E) H_{R,R'} \right],
\]

leading to
\[
\sum_{RR'} \chi_{j,i}(R, R'; E) = \sum_{RR'} \left( R_i - R'_i \right) \left( R_j - R'_j \right) Tr \left[ G(R, R'; E) H_{R',R} \right]. \quad (B6)
\]

**APPENDIX C: LONGITUDINAL MODES**

As discussed in Section VII, the expression of the \( Q \)-matrix which includes also the longitudinal modes is \( Q_P(R) = Q(R) + S(R) \), where \( Q(R) \) and \( S(R) \) have been defined through (74). The free action for these fields contains a local term, Eq.(76), and a non local one, Eq.(77). The latter is
\[
-\frac{1}{4} \int dR dR' \Gamma(R - R') Tr \left[ \Delta_R Q(R') \Delta_R Q(R')^\dagger \right] + \Delta_R S(R') \Delta_R S(R')^\dagger + 2\Delta_R Q(R') \Delta_R S(R')^\dagger \], \quad (C1)
\]
where we have defined the operator
\[
\Delta_R f(R') = f(R) - f(R') = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \vec{R} - \vec{R}' \right)^n \cdot \vec{\nabla}^n f(R).
\]
Let us apply this operator to \(Q(R)\) and \(S(R)\), keeping all terms which contains at most two derivatives which act to the transverse matrices \(T\). We obtain
\[
\Delta_R Q(R') \simeq \vec{\nabla} Q(R') \cdot \left( \vec{R} - \vec{R}' \right), \quad (C2)
\]
\[
\Delta_R S(R') \simeq \tilde{T}(R')\dagger \Delta_R P(R') T(R')
+ \left[ \left( \vec{\nabla} \tilde{T}(R')\dagger \right) P(R) T(R') + \tilde{T}(R')\dagger P(R) \left( \vec{\nabla} \tilde{T}(R') \right) \right] \cdot \left( \vec{R} - \vec{R}' \right) \quad (C3)
\]
\[
+ \frac{1}{2} \left[ \left( \vec{\nabla}^2 \tilde{T}(R')\dagger \right) P(R) T(R') + 2 \left( \vec{\nabla} \tilde{T}(R')\dagger \right) P(R) \left( \vec{\nabla} \tilde{T}(R') \right) \right]
+ \tilde{T}(R')\dagger P(R) \left( \vec{\nabla}^2 \tilde{T}(R') \right) \cdot \left( \vec{R} - \vec{R}' \right)^2. \quad (C4)
\]
The term which is obtained by \((C3)\) times its hermitean conjugate, together with the local piece \((C6)\) give the free action of the longitudinal modes
\[
S_0[P] = \frac{1}{V} \sum_q \frac{1}{2\omega_q} Tr \left[ P_q P_q\dagger \right]. \quad (C6)
\]
The mixed terms, after defining \(\tilde{D} = T\vec{\nabla}T\dagger\), give rise to the coupling between transverse and longitudinal modes
\[
S[Q, P] = -\frac{1}{2} \int dR dR' \Gamma \left( R - R' \right) \left( \vec{R} - \vec{R}' \right) Tr \left[ \tilde{D}(R') \left( P^\dagger(R') P(R) - P(R)\dagger P(R') \right) \right] \quad (C7)
- \frac{1}{4} \int dR dR' \Gamma \left( R - R' \right) \left( \vec{R} - \vec{R}' \right)^2 \cdot Tr \left[ \sigma_1 \tilde{D}(R')\sigma_1 P(R) \tilde{D}(R') P^\dagger(R') + \sigma_1 \tilde{D}(R')\sigma_1 P(R') \tilde{D}(R') P^\dagger(R) \right]
- \left( \tilde{D}(R')\tilde{D}(R') \left( P^\dagger(R) P(R') + P^\dagger(R') P(R) \right) \right)
- \frac{1}{4} \int dR dR' \Gamma \left( R - R' \right) \left( \vec{R} - \vec{R}' \right) \cdot Tr \left[ \vec{\nabla} Q(R') \Delta_R P(R')\dagger + H.c. \right]. \quad (C10)
\]
The last term, \((C10)\), gives rise to higher gradient contributions, hence can be neglected.

1. Longitudinal propagators

Before averaging over the longitudinal modes, we have to evaluate the longitudinal propagators. The matrix
\[ P = (P_{0,0}s_0 + P_{0,3}s_3)\sigma_0 + i (P_{3,1}s_1 + P_{3,2}s_2)\sigma_3, \]

has to satisfy \( cP^t c' = P \), and, in addition, all \( P_\alpha = P_\alpha^\dagger \). For \( \alpha = (0,0), (0,3), (3,1) \), by writing
\[
P_\alpha = P_\alpha^{(0)}\tau_0 + i\vec{P}_\alpha \cdot \vec{\tau},
\]
we find that
\[
P_\alpha^{(0)}, \vec{P}_\alpha \in \mathbb{R}e, \quad P_\alpha^{(0)} = \left(P_\alpha^{(0)}\right)^t, \quad \vec{P}_\alpha = -\left(\vec{P}_\alpha\right)^t.
\]

For \( \alpha = (3,2) \), by writing
\[
P_{3,2} = iP_{3,2}^{(0)}\tau_0 + \vec{P}_{3,2} \cdot \vec{\tau},
\]
we must impose
\[
P_{3,2}^{(0)}, \vec{P}_{3,2} \in \mathbb{R}e, \quad P_{3,2}^{(0)} = -\left(P_{3,2}^{(0)}\right)^t, \quad \vec{P}_{3,2} = \left(\vec{P}_{3,2}\right)^t.
\]

If \( P_\alpha^{(i)} \) is a symmetric real matrix, its propagator is
\[
\langle P_\alpha^{(i)} P_\alpha^{(i)} \rangle = \frac{G}{2} (\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}), \quad (C11)
\]
while if it is antisymmetric
\[
\langle P_\alpha^{(i)} P_\alpha^{(i)} \rangle = -\frac{G}{2} (\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}), \quad (C12)
\]
where \( G_k = \omega_k/8V \), and \((a, b, c, d)\) are replica indices. By means of these propagators, we readily find that, if \( M = M^{(0)} + i\vec{M}_i \cdot \vec{\tau} \), where \( M^{(i)} \) are matrices in the replica space, then, for \( \alpha = (0,0), (0,3), (3,1) \), the following results hold
\[
\langle P_\alpha MP_\alpha \rangle = -G_c M^t c' + 2GTr \left(M^{(0)}\right) = -G_c M^t c' + GTr (M), \quad (C13)
\]
while, for \( \alpha = (3,2) \),
\[
\langle P_{3,2} MP_{3,2} \rangle = G_c M^t c' + 2GTr \left(M^{(0)}\right) = G_c M^t c' + GTr (M). \quad (C14)
\]
More generally,

\[
\langle PMs_i\sigma_jP^\dagger \rangle = -GcM_i c^d (s_i\sigma_j + s_3s_is_i\sigma_j s_3 + \sigma_3s_is_i\sigma_j s_1\sigma_3 - \sigma_3s_2s_3s_i\sigma_j s_2\sigma_3) \\
+ GTr (M) (s_i\sigma_j + s_3s_is_i\sigma_j s_3 + \sigma_3s_is_i\sigma_j s_1\sigma_3 + \sigma_3s_2s_i\sigma_j s_2\sigma_3),
\]  

(C15)

\[
\langle PMs_i\sigma_jP \rangle = -GcM_i c^d (s_i\sigma_j + s_3s_is_i\sigma_j s_3 - \sigma_3s_is_i\sigma_j s_1\sigma_3 + \sigma_3s_2s_i\sigma_j s_2\sigma_3) \\
+ GTr (M) (s_i\sigma_j + s_3s_is_i\sigma_j s_3 - \sigma_3s_is_i\sigma_j s_1\sigma_3 - \sigma_3s_2s_is_i\sigma_j s_2\sigma_3).
\]  

(C16)

For \( j = 0, 3 \) the above expression simplifies to

\[
\langle PMs_i\sigma_jP^\dagger \rangle = -2Gc (Ms_i\sigma_j)^t c^d + 2G\sigma_j Tr (Ms_is_0),
\]  

(C17)

\[
\langle PMs_i\sigma_jP \rangle = -2Gs_3c (Ms_i\sigma_j)^t c^d s_3 + 2G\sigma_j Tr (Ms_is_3),
\]  

(C18)

while for \( j = 1, 2 \)

\[
\langle PMs_i\sigma_jP^\dagger \rangle = -2Gs_3c (Ms_i\sigma_j)^t c^d s_3 + 2G\sigma_j Tr (Ms_is_3),
\]  

(C19)

\[
\langle PMs_i\sigma_jP \rangle = -2Gc (Ms_i\sigma_j)^t c^d + 2G\sigma_j Tr (Ms_is_0).
\]  

(C20)

2. Averaging \( S[Q,P] \)

We have now all what it is needed to proceed in the averaging over \( P \). Here we just sketch the calculation, which is quite involved and requires the matrix properties of \( \vec{D} \) which are determined in Appendix D. We just remark that (C8) and (C9) do not reproduce the correct stiffness term. Indeed, it is (C7), which contributes at second order, which cancels the additional terms and allows to express everything in terms of the matrix \( Q \).

By means of the previously calculated propagators of the longitudinal modes, we find that

\[
\langle S[Q,P] \rangle_P = \frac{Y}{4d\Sigma^2} \int dR Tr \left[ \vec{\nabla}Q(R)^\dagger \vec{\nabla}Q(R) \right] + \frac{1}{8\Sigma^2} \left[ Tr \left( Q(R)^\dagger \vec{\nabla}Q(R)\sigma_3 \right) \right]^2,
\]  

(C21)

where

\[
Y = \int dR \Gamma^2(R) \Gamma(R)^{-1} R^2.
\]  

(C22)
We notice that the first term is a contribution to the Kubo conductivity and the second to the staggered density of states fluctuations of the diagrams where the current vertices are those proportional to the random hopping.

3. Additional terms

The last class of corrections which generate new operators is obtained by expanding each Green’s function in (50) at first order, \( G_P = G_0 - iG_0PG_0 \), leading to the term

\[
\frac{1}{4} \langle \text{Tr} (GPGUGPGU) \rangle_P. \tag{C23}
\]

For the sake of clarity, we will analyse this term only in the case of a cubic lattice, where the derivation is more straightforward. We will postpone a discussion about the general case at the end of the section.

In the cubic lattice, according to Eq.(23), the electron-\( Q \) coupling can be brought to a local one also in the diagonal basis. In this basis, \( Q = Q_0 + iQ_1 \sigma_1 \),

\[
P(R) = (P_{00}s_0 + P_{03}s_3) \sigma_0 + i(P_{11}s_1 + P_{12}s_2) \sigma_1, \tag{C24}
\]

the unitary matrix

\[
T(R) = \exp \left[ \frac{W_0(R)}{2} \sigma_0 + \frac{W_1(R)}{2} \sigma_1 \right], \tag{C25}
\]

where \( W_1 \) has the same form as \( W_3 \) defined by Eq.(34), and \( \tilde{T} = \sigma_3 T \sigma_3 \). Moreover, since \( H_{RR'} = \epsilon_{R-R'} \sigma_3 \) in the diagonal basis, being \( \epsilon_{R-R'} \) the Fourier transform of \( \epsilon_k \), the current operator which appears in the definition of \( U_{RR'} \), Eq.(43), has only matrix elements \( \vec{J}_k = \nabla \epsilon_k \sigma_3 \).

Once averaged over \( P \), (C23) gives, among other terms which correct the Kubo conductivity, a new term [see Eq.(Greenfunction)]

\[
\frac{1}{2} \sum_{a,b=1,2} \sum_{h=\pm} \sum_{pq} \omega_{p-k} Tr \left( U^{h,h;0,0}_{q} G^{a,0}_{p+q} G^{b,h}_{p} \right) Tr \left( U^{a,h;b,h}_{-q} g^{b,h}_{k} g^{a,0}_{k+q} \right)
\] 

\[
- Tr \left( U^{h,h;0,0}_{q} G^{a,0}_{p+q} G^{b,h}_{p} \right) Tr \left( U^{a,h;b,h}_{-q} g^{b,h}_{k} g^{a,0}_{k+q} \right),
\]
where the first piece derives from $P_0$ and the second from $P_1$. The structure in the energy/sublattice indices can be shortly represented by

$$\sum_{i=1,\ldots,4} \Lambda_i \otimes \Lambda_i = \frac{1}{4} (\sigma_2 \otimes \sigma_2 + s_3 \sigma_1 \otimes s_3 \sigma_1 + \sigma_3 \otimes \sigma_3 + s_3 \otimes s_3),$$

so that the above term can be written, at small $q$, as

$$\frac{1}{2} \sum_{i=1,\ldots,4} \sum_{pq} \omega_{p-k} Tr (U_q G_p \Lambda_i G_p) Tr (U_{-q} G_k \Lambda_i G_k)$$

We remind that

$$G_p U_q G_p = -i G_p \bar{D}_q \cdot J_p G_p,$$

where $\bar{D}_q$ is the Fourier transform of $\bar{T}(R) \nabla \bar{T}(R)^{-1}$. We notice that only the diagonal component in energy of $\bar{D}$ enters. Moreover, for the diagonal matrices $s_j = s_0, s_3$, since $\bar{D} = \bar{D}_0 \sigma_0 + \bar{D}_1 \sigma_1$, it derives that, for $i = 0, 1$, the following equality holds $Tr \bar{D} s_j \sigma_i = -Tr (\nabla \bar{W} s_j \sigma_i / 2)$. Therefore, just $W_1 \sigma_1$ contributes. By means of (54), for any $\Lambda$'s we have

$$-i Tr (\bar{D}_{1,q} s_1 J_p G(p) \Lambda_i G(p)) = + \nabla \bar{\epsilon}_p \cdot Tr (\bar{D}_{1,q} s_1 G(p) s_1 \sigma_2 \Lambda_i G(p))$$

$$= \nabla \bar{\epsilon}_p \cdot Tr \left[ \bar{D}_{1,q} \left( G_3^2 \sigma_3 \sigma_2 \Lambda_i \sigma_3 - G_0^2 \sigma_3 \sigma_2 \Lambda_i s_3 + G_3 G_0 s_3 \sigma_3, \sigma_2 \Lambda_i \right) \right].$$

Only for $\Lambda_1 = \sigma_2/4$ the trace over the $\sigma$'s is finite, leading to

$$2 \left( G_3^2 - G_0^2 \right) \nabla \bar{\epsilon}_p \cdot Tr (\bar{D}_{1,q}).$$

In conclusion, going back to the original sublattice representation, and defining

$$\Pi^{(0)} = \frac{1}{4\pi V^2 d} \sum_{pq} \nabla \bar{\epsilon}_k \cdot \nabla \bar{\epsilon}_p \omega_{p-k} Tr (\sigma_3 G_p^+ \sigma_3 G_p) Tr (\sigma_3 G_k^+ \sigma_3 G_k),$$

we obtain the following explicit expression of (C23) [notice that $Q = Q_0 \sigma_0 + i Q_1 \sigma_1 \rightarrow Q_0 \sigma_0 + i Q_3 \sigma_3$ in the sublattice basis]

$$\frac{2\pi}{8 \cdot 32 \Sigma^4} \Pi^{(0)} \int dR Tr \left[ Q^+(R) \nabla Q(R) \sigma_3 \right] \cdot Tr \left[ Q^+(R) \nabla Q(R) \sigma_3 \right].$$

This term, which derives from the regular current vertices, has the same form as the second term in (C21), which, on the contrary, is due to the random current vertices. Therefore, the coupling constant $\Pi$ which appears in the final expression is the sum of both terms.
The symmetry of the operator \( (C30) \), which, due to the trace, involves the Nambu, energy and sublattice components \( \tau_0, s_0 \) and \( \sigma_3 \), respectively, suggests that the prefactor \( \Pi \) represents the fluctuations of the staggered density of states, as discussed in section VIB. In the case of a generic bipartite lattice, we do expect an analogous term to appear, because the staggered-density of states fluctuations still acquire singular contributions, and the operator is not forbidden by the symmetry properties of the \( Q \)-matrix. The reason why we decided to show only the case of cubic lattices is that the distinction between the longitudinal from the transverse modes is a bit ambiguous at large momenta, where the both are in a sense massive. This is not a problem for cubic lattices, where one can show that the large momentum components of the transverse modes do not contribute, hence \( (C30) \) exhausts the whole contribution. On the contrary, in other cases, we do have to keep into account the contribution of the small-wavelength transverse modes to recover the full expression. This makes the calculations more involved than in the case of cubic lattices.

**APPENDIX D: EXPLICIT DERIVATION OF THE RG EQUATIONS**

In this Appendix we outline the derivation of the RG equations. Before that, it is convenient to list some useful results.

1. Averages

Besides the propagators, we will also need the explicit expression of particular averages which enter in the derivation of the RG equations. The following results hold:

\[
\langle B_{ab} P_{bc} B^{\dagger}_{cd} \rangle = 4D(k)\delta_{ad}Tr(P_0), \quad (D1)
\]

\[
\langle B_{ab} P_{bc} B_{cd} \rangle = -2D(k)P^{\dagger}_{ad}, \quad (D2)
\]

where \( P = P_0 + i\vec{P} \cdot \vec{\tau} \) is a quaternion real matrix.

In addition,
\( \langle A_{ab} P_{bc} A_{cd} \rangle = -2D(k)P_{ad}^\dagger + 4D(K)\delta_{ad} Tr(P_0) - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad} \), \hspace{1cm} \text{(D3)}

\( \langle C_{ab} P_{bc} C_{cd} \rangle = -2D(k)P_{ad}^\dagger + 4D(K)\delta_{ad} Tr(P_0) - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad} \), \hspace{1cm} \text{(D4)}

\( \langle A_{ab} P_{bc} C_{cd} \rangle = -D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} P_{ad} \). \hspace{1cm} \text{(D5)}

For instance, if \( P = \hat{I} \), then

\( \langle B B^\dagger \rangle = 4D(k)m\hat{I} \), \hspace{1cm} \text{(D6)}

\( \langle B B \rangle = -2D(k)\hat{I} \), \hspace{1cm} \text{(D7)}

\( \langle A A \rangle = \left[ -2D(k) + 4D(k)m - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \right] \hat{I} \), \hspace{1cm} \text{(D8)}

\( \langle C C \rangle = \left[ -2D(k) + 4D(k)m - D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \right] \hat{I} \), \hspace{1cm} \text{(D9)}

\( \langle A C \rangle = -D(k) \frac{\Pi}{\sigma_{xx} + \Pi m} \hat{I} \). \hspace{1cm} \text{(D10)}

2. RG equations

First of all, we need to know the quaternion structure of the matrix \( D = T\nabla T^\dagger \). Since

\[ D = -D^\dagger, \quad cD^\dagger c^\dagger = -\sigma_1 D\sigma_1, \]

we can write \( D = D_0\sigma_0 + D_3\sigma_3 \), where in the \( \pm \)-space

\[ D_0 = \begin{pmatrix} A_0 & B_0 \\ -B_0^\dagger & C_0 \end{pmatrix}, \]

\hspace{1cm} \text{(D11)}

and

\[ D_3 = i \begin{pmatrix} A_3 & B_3 \\ B_3^\dagger & C_3 \end{pmatrix}. \]

\hspace{1cm} \text{(D12)}

We can write each of the above matrices in quaternion form, \( P = P^{(0)}\tau_0 + i\vec{P} \cdot \vec{\tau} \), where \( P^{(0)}, \vec{P} \in \mathcal{R}e \), but, in addition, we must impose that
\[
(A_0^{(0)})^t = -A_0^{(0)} \quad \vec{A}_0^t = \vec{A}_0 \quad (C_0^{(0)})^t = -C_0^{(0)} \quad \vec{C}_0^t = \vec{C}_0
\]
\[
(A_3^{(0)})^t = A_3^{(0)} \quad \vec{A}_3 = -\vec{A}_3 \quad (C_3^{(0)})^t = C_3^{(0)} \quad \vec{C}_3 = -\vec{C}_3
\]

By making use of the above properties, and by means of the Eqs.\((\text{D}2)\), \((\text{D}1)\), \((\text{D}3)\), \((\text{D}4)\) and \((\text{D}3)\), after defining
\[
L = \int_{\Lambda/s} d^2 k (2\pi)^2 D(k),
\]
we get the following results
\[
\langle W_f D_0 W_f \rangle_f = L_A \Gamma D_0
\]
\[
+ 2L_B \begin{pmatrix} 0 & B_0 \\ -B_0^\dagger & 0 \end{pmatrix} - 2L_A \begin{pmatrix} A_0 & 0 \\ 0 & C_0 \end{pmatrix}, \quad \text{(D13)}
\]
where \(\Gamma = \Pi/(\sigma_{xx} + m\Pi)\), and
\[
\langle W_f D_3 W_f \rangle_f = L_A \Gamma D_3
\]
\[
+ 2L_A i \begin{pmatrix} A_3 & 0 \\ 0 & C_3 \end{pmatrix} - 2L_B i \begin{pmatrix} 0 & B_3 \\ B_3^\dagger & 0 \end{pmatrix}
\]
\[
- 4L \hat{T} \text{r} \left( iA_3^{(0)} + iC_3^{(0)} \right). \quad \text{(D14)}
\]

For further convenience, when useful, we have labelled \(L_B\) the propagator of \(B_f\), and \(L_A\) the ones of \(A_f\) and \(C_f\). The next useful result is
\[
\langle W_f W_f \rangle_f = (-4L_B m - 4L_A m + 2L_A + L_A \Gamma) \hat{I}. \quad \text{(D15)}
\]

Through \((\text{D13})\) and \((\text{D14})\) we therefore get
\[
\langle F_1 \rangle_f = \frac{2\pi \sigma_{xx}}{32\Sigma^2} \int dR - 2\langle \text{Tr} \left[ \tilde{D} \sigma_1 Q_{sp} W_f \tilde{D} W_f Q_{sp} \sigma_1 \right] \rangle_f
\]
\[
+ 2\langle \text{Tr} \left[ \tilde{D} \sigma_1 Q_{sp} \tilde{D} Q_{sp} \tilde{W}_f^2 \sigma_1 \right] \rangle_f
\]
\[
= \frac{2\pi \sigma_{xx}}{32} \int dR \left( L_B - L_A + 2L_A m + 2L_B m \right) \text{Tr} \begin{pmatrix} 0 & B_0 \\ -B_0^\dagger & 0 \end{pmatrix}^2 \quad \text{(D16)}
\]
\[
+ 4 \left( 2L_A m + 2L_B m \right) \begin{pmatrix} iA_3 & 0 \\ 0 & iC_3 \end{pmatrix}^2 \quad \text{(D17)}
\]
\[ -2L [Tr (D\sigma_3)]^2 \]  
\[ -4 (2L_A m + 2L_B m + 2L_A) \begin{pmatrix} A_0 & 0 \\ 0 & C_0 \end{pmatrix}^2 \]  
\[ -4 (2L_A m + 2L_B m + L_A + L_B) \begin{pmatrix} 0 & iB_3 \\ iB_3^\dagger & 0 \end{pmatrix}^2. \]

The calculation of \( \langle F_2^2 \rangle_f \) is more involved, since one needs the average of four \( W_1 \)'s. For sake of lengthy, we just quote the final result that such a term cancels (D13) and (D20). We next notice that

\[
\frac{1}{\Sigma^2} Tr \left[ \nabla Q^\dagger \nabla Q \right] = 2Tr \left[ Ds_3\sigma_1 Ds_3\sigma_1 - DD \right] \\
= -4Tr \left( \begin{array}{cc} 0 & B_0 \\ -B_0^\dagger & 0 \end{array} \right)^2 - 4Tr \left( \begin{array}{cc} iA_3 & 0 \\ 0 & iC_3 \end{array} \right)^2,
\]

and that

\[
Tr \left[ Q^\dagger \nabla Q \sigma_3 \right] = 2\Sigma^2 Tr (D\sigma_3).
\]

Therefore, for \( L_A = L_B \),

\[
\langle F_1 - \frac{1}{2}F_2^2 \rangle_f \\
= -4Lm \frac{2\pi \sigma_{xx}}{32\Sigma^2} \int dR Tr \left[ \nabla Q^\dagger \nabla Q \right] \\
- \frac{1}{2} L \frac{2\pi \sigma_{xx}}{32\Sigma^4} \left[ Tr \left( Q^\dagger \nabla Q \sigma_3 \right) \right]^2.
\]

In the cases in which the \( A \) and \( C \) modes are gaped (\( L_A = 0 \), and no \( D_3 \)), we obtain the standard result

\[
\langle F_1 - \frac{1}{2}F_2^2 \rangle_f = -(2Lm + L) \frac{2\pi \sigma_{xx}}{32\Sigma^2} \int dR Tr \left[ \nabla Q^\dagger \nabla Q \right].
\]

**APPENDIX E: GADE AND WEGNER’S PROOF OF THE VANISHING \( \beta \)-FUNCTION**

The equations from (34) to (40) imply that \( W_3 \) is not a traceless matrix. Indeed, we can alternatively write \( W \) as
\[ W = W' + \frac{1}{4m} \text{Tr} (W_3) \sigma_3 \equiv W' + i\phi \sigma_3, \]  
(E1)

with \( W' = W_0 \sigma_0 + W_3' \sigma_3 \), now being \( W_3' \) a traceless matrix. Since \( \sigma_3 \) commutes with \( W' \), this means that

\[ T(R)^2 = e^{W(R)} = e^{i\phi(R) \sigma_3} e^{W'(R)} \equiv e^{i\phi(R) \sigma_3} V(R), \]  
(E2)

which also defines the matrix field \( V(R) \). By means of this parametrization, the non-linear \( \sigma \)-model (2) can also be written as

\[
S[T] = S[V, \phi] = \frac{2\pi \sigma_{xx}}{32} \int dR \text{Tr} \left[ \tilde{\nabla} V(R)^{-1} \cdot \tilde{\nabla} V(R) \right] + \frac{m\pi}{2} (\sigma_{xx} + m\Pi) \int dR \tilde{\nabla} \phi(R) \cdot \tilde{\nabla} \phi(R).
\]  
(E3)

Therefore the action of \( V \) is distinct from that of \( \phi \), and the latter, being a phase, is gaussian. This implies that the combination \( \sigma_{xx} + m\Pi \) is not renormalized and scales with its bare dimension \( \epsilon \), for any number of replicas. In turns, it means that, in the zero replica limit, it is \( \sigma_{xx} \) which is not renormalized! This is completely equivalent to the nice proof given by Gade and Wegner [8] that the quantum corrections to the \( \beta \)-function of the conductance of a \( \text{U}(N)/\text{SO}(N) \) model vanish at all orders in the \( N \to 0 \) limit.

The other important result concerns the renormalization of an operator

\[ T^{2q} = e^{i\phi \sigma_3} V^q. \]

Within RG,

\[ e^{i\phi \sigma_3} \to \frac{t}{2} q^2 \ln s \left( \frac{t}{c + mt} - \frac{1}{m} \right) e^{i\phi \sigma_3}. \]  
(E4)

The second term, which is singular in the \( m \to 0 \) limit, has to be canceled by the one-loop renormalization of \( V^q \). Gade and Wegner showed that this cancellation holds for any \( q \). Furthermore, they argued that, apart from the one-loop correction, the renormalization of \( V^q \) does not contain any other singular term in the \( m \to 0 \) limit. This, as they pointed out, has very important consequences. In \( 2d \), \( t \) does not scale, while \( c \) goes to zero. Therefore
the term which dominates the renormalization of $T^{2q}$ for $m = 0$ is just the first term in the right hand side of (E4). This argument implies that the one-loop correction, which we have derived for the density of states ($q = 1$ case), is sufficient to identify the correct asymptotic behavior.

To conclude, let us discuss more in detail the origin of this gaussian field $\phi$. In the Grassmann variable path-integral representation, the action for the particle-hole symmetric model at $E = \omega = 0$

$$S = \sum_{RR'} \overline{\Psi}_R H_{RR'} \Psi_{R'},$$

posseses a simple abelian gauge-like symmetry

$$\Psi \rightarrow e^{i\phi \sigma_3} \Psi,$$

because $\{\sigma_3, H_{RR'}\} = 0$. It is just this symmetry which causes the appearance of the gaussian part of the non-linear $\sigma$-model. Notice that this symmetry implies a particle-hole symmetric Hamiltonian, which is invariant under the transformation $c_{1,R} \rightarrow c_{1,R}^\dagger$ but $c_{2,R} \rightarrow -c_{2,R}^\dagger$. In fact, $\{\sigma_3, H_{RR'}\} = 0$ also means that $\{\tau_1 \sigma_3, H_{RR'}\} = 0$, being $H_{RR'} \propto \tau_0$. The interesting fact is that (E5) is not a symmetry of the fermion operators. Indeed, under this transformation, $c_R \rightarrow e^{i\phi \sigma_3} c_R$, but $\overline{c}_R \rightarrow e^{i\phi \sigma_3} \overline{c}_R$, and not $\overline{c}_R \rightarrow e^{-i\phi \sigma_3} \overline{c}_R$ as we would expect if $c_R$ and $\overline{c}_R$ had to be identified with the operators $c_R$ and $c_R^\dagger$. Finally, we notice that if, besides $\sigma_3$, the Hamiltonian commutes with another Pauli matrix (as it can be the case for specifically built particle-hole symmetric models), the above gauge symmetry would be non abelian, hence spoiling all peculiar properties which we have shown to occur. Indeed, in the last case, the system can be mapped into a standard localization model with an additional sublattice index.
REFERENCES

[1] See e.g. *The Quantum Hall Effect*, ed. by R.E. Prange, S.M. Girvin, Springer-Verlag
       New York (1990).

[2] Kravchenko *et al.*, Phys. Rev. B *50*, 8039 (1994); *ibid.* *51*, 7038 (1995).

[3] G. Bergmann, Phys. Rev. B *28*, 2914 (1983).

[4] G. Theodorou and M.H. Cohen, Phys. Rev. B *13*, 4597 (1976).

[5] T.P. Eggarter and R. Riedinger, Phys. Rev. B *18*, 569 (1978).

[6] F. Wegner, Phys. Rev. B *19*, 783 (1979).

[7] R. Oppermann and F. Wegner, Z. Phys. B *34*, 327 (1979).

[8] R. Gade and F. Wegner, Nucl. Phys. B *360*, 213 (1991).

[9] R. Gade, Nucl. Phys. B *398*, 499 (1993).

[10] T. Fukui, Nucl. Phys. B *562*, 477 (1999).

[11] A. Altland and B.D. Simons, cond-mat/9909152.

[12] S. Guruswamy, A. LeClair, and A.W.W. Ludwig, cond-mat/9909143.

[13] F. Wegner, Z. Phys. B *35*, 207 (1979); S. Hikami, Phys. Rev. B *24*, 2671, (1981); A.M.M. Pruisken and L. Schäfer, Nucl. Phys. B *200*, 20, (1982).

[14] G. Wilson and J. Kogut, Phys. Rep. *12*, 77 (1974); A.M. Polyakov, Phys. Lett. B*59*, 79 (1975).

[15] P. Cain, R.A. Römer, and M. Schreiber, Ann. Phys. (Leipzig) *8*, 507 (1999).

[16] K.B. Efetov, A.I. Larkin, and D.E. Khmel’nitsky, Zh. Eksp. Teor. Fiz. *79*, 1120 (1979) [Sov. Phys. JETP *52*, 568 (1980)].

[17] D. Fisher, Phys. Rev. B *50*, 3799 (1994); *ibid.* *51*, 6411 (1995).