ASYMPTOTIC BEHAVIORS OF MEANS OF CENTRAL VALUES OF AUTOMORPHIC $L$-FUNCTIONS FOR GL(2)

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Abstract. Let $\mathcal{A}$ be the adele ring of a totally real algebraic number field $F$. We push forward an explicit computation of a relative trace formula for periods of automorphic forms along a split torus in $GL(2)$ from a square free level case done by Masao Tsuzuki [12], to an arbitrary level case. By using a relative trace formula, we study central values of automorphic $L$-functions for cuspidal automorphic representations of $GL(2, \mathcal{A})$ corresponding to Maass forms with arbitrary level.

Introduction

Let $k \geq 4$ be an even integer. For a prime $N$, let $S^\text{new}_k(N)$ be the space of all cuspidal new forms on the Poincaré upper half plane of weight $k$ for $\Gamma_0(N)$. The space $S^\text{new}_k(N)$ has an orthogonal basis $S^\text{new}_k(N)$ consisting of normalized Hecke eigen forms. For $\varphi \in S^\text{new}_k(N)$, we denote by $L(s, \varphi)$ the completed automorphic $L$-function for $\varphi$ satisfying a functional equation that relates the values at $s$ and $1 - s$. Let $\eta$ be a quadratic Dirichlet character of conductor $D$ with $\eta(-1) = -1$. The Dirichlet $L$-series associated to $\eta$ is denoted by $L_{\eta}(s, \eta)$. For a fixed prime $p \nmid D$, $J_{p, \eta}$ denotes the set of all primes $N$ satisfying both $\gcd(p, N) = \gcd(D, N) = 1$ and $\eta(N) = -1$. Let $a_p(\varphi)$ denote the $p$-th Fourier coefficient of $\varphi$ multiplied by $p^{-(k-1)/2}$. Ramakrishnan and Rogawski [8] studied a sum of central values of $L(s, \varphi)$ and proved the following theorem.

Theorem 1. [8, Theorem A]

For any interval $J \subset [-2, 2]$, we have

$$\lim_{N \to \infty} \sum_{N \in J_{p, \eta}, \varphi \in S^\text{new}_k(N), \quad a_p(\varphi) \in J} \frac{L(1/2, \varphi)L(1/2, \varphi \otimes \eta)}{||\varphi||^2} = 2^{k-1}\frac{\{(k/2 - 1)!\}^2}{\pi(k-2)!} L_{\text{fin}}(1, \eta) \mu_p^\eta(J),$$

where $||\varphi||$ denotes the Petersson norm of $\varphi$ and $\mu_p^\eta$ denotes the probability measure on $[-2, 2]$ defined by

$$\mu_p^\eta(x) = \begin{cases} p & (p^{1/2} + p^{-1/2} - x)^2 \mu_{ST}(x) \quad (\eta(p) = 1), \\ p+1 & (p^{1/2} + p^{-1/2} - x)^2 \mu_{ST}(x) \quad (\eta(p) = -1). \end{cases}$$

Here, $\mu_{ST}(x)$ is the Sato-Tate measure $(2\pi)^{-1}\sqrt{4 - x^2}dx$.

Feigon and Whitehouse [2] generalized this result to the case of Hilbert modular forms. Tsuzuki [12] gave a similar kind of asymptotic formula for Maass cusp forms with square free level in terms of automorphic representations of $GL(2)$ over a totally real algebraic number field. The purpose of this paper is to generalize Tsuzuki’s results of [12] to the case of arbitrary level.

To state our results in this paper, we prepare some notations. Let $F$ be a totally real algebraic number field, $\mathcal{O}_F$ its integer ring and $\mathcal{A}$ the adele ring of $F$. We denote by $\Sigma_F$ (resp. $\Sigma_\infty$ and $\Sigma_{\text{fin}}$) the set of all

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places (resp. all infinite places and all finite places) of $F$. For each $v \in \Sigma_{\text{fin}}$, we fix a uniformizer $\varpi_v$ of $F_v$ and denote by $q_v$ the cardinality of the residue field of $F_v$. For an ideal $\mathfrak{a}$ of $\mathfrak{o}_F$, let $S(\mathfrak{a})$ denote the set of all $v \in \Sigma_{\text{fin}}$ such that $\text{ord}_v(\mathfrak{a}) \geq 1$. The absolute norm of $\mathfrak{a}$ is denoted by $N(\mathfrak{a})$.

Fix a quadratic character $\eta = \prod_{v \in \Sigma_F} \eta_v$ of $F^\times \setminus \mathbb{A}^\times$ of conductor $f_\eta$ so that $\eta_v$ is trivial for any $v \in \Sigma_{\infty}$. Fix a finite subset $S$ of $\Sigma_F$ such that $\Sigma_{\infty} \subset S$ and $S \cap S(f_\eta) = \emptyset$. Let $J_{S,\eta}$ be the set of all ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ satisfying the following three conditions:

1. $S(\mathfrak{n}) \cap S(f_\eta) = \emptyset$ and $S(\mathfrak{n}) \cap S = \emptyset$,
2. $\eta_v(\varpi_v) = -1$ for any $v \in S(n)$,
3. $\tilde{\eta}(\mathfrak{n}) = \prod_{v \in \Sigma_{\text{fin}}} \eta_v(\varpi_v^{\text{ord}_v(n)}) = 1$.

Let $K_{\infty}$ be the standard maximal compact subgroup of $\text{GL}(2, F \otimes \mathbb{R})$. For an ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, $K_{\mathfrak{n}}(\mathfrak{n})$ denotes the Hecke congruence subgroup of $\text{GL}(2, \mathbb{A}_{\text{fin}})$ of level $\mathfrak{n}$. Let $\Pi_{\text{cus}}(\mathfrak{n})$ denote the set of all irreducible cuspidal automorphic representations of $\text{PGL}(2, \mathbb{A})$ having nonzero $K_{\mathfrak{n}}(\mathfrak{n})$-invariant vectors. Let $\Pi_{\text{fin}}(\mathfrak{n})$ be the set of all $\pi \in \Pi_{\text{cus}}(\mathfrak{n})$ with conductor $f_\pi = \mathfrak{n}$. The standard $L$-function of $\pi$ is denoted by $L(S, \pi)$. Let $S_\Lambda$ denote the set of all $v \in \Sigma_{\text{fin}}$ satisfying $\text{ord}_v(f_\pi) \geq 2$.

For $\mathfrak{n} \in J_{S,\eta}$ and $\pi = \otimes_v \pi_v \in \Pi_{\text{cus}}(\mathfrak{n})$, $\pi_v$ is isomorphic to a unitarizable spherical principal series representation $\pi(| \cdot |^{1/2} v^{1/2}, \cdot | v^{-1/2} )$ of $\text{GL}(2, F_v)$ for any $v \in S$. The spectral parameter $\nu_\pi, S$ at $S$ of $\pi$ is defined as $\nu_\pi, S = (\nu_v)_{v \in S}$. It is known that $\nu_v, S \in X_{\mathfrak{n}}^0 = \prod_{v \in S} X_v^0$, where $X_v^0 = \mathbb{R}_{\geq 0} \cup (0, 1)$ for $v \in \Sigma_{\infty}$ and $X_v^0 = i[0, 2\pi(\log q_v)^{-1}] \cup \{ x + iy \mid x \in (0, 1), y \in \{0, 2\pi i (\log q_v)^{-1}\} \}$ for $v \in \Sigma_{\text{fin}} = S \cap \Sigma_{\text{fin}}$, respectively.

We define a measure $\lambda^0_\eta$ on $X_{\mathfrak{n}}^0$ by $4\pi D_{\mathfrak{F}}^{1/2} L(1, \eta) \otimes_{v \in S} \lambda^0_\eta$, where $D_{\mathfrak{F}}$ denotes the absolute value of the discriminant of $F$, and for any $v \in S$, the measure $\lambda^0_\eta$ on $X_v^0$ with support in $i\mathbb{R}$ is given by

$$d\lambda^0_\eta(iy) = \frac{L(1/2, \pi(| \cdot |^{1/2} v^{1/2}, \cdot | v^{-1/2} \otimes \eta_v))}{L(1/2, \pi)} \left( \begin{array}{c} \frac{1}{4\pi} \Gamma(iy/2)^{-2}dy \\ \frac{\log q_v}{4\pi} |1 - q_v^{-iy}/2| dy \end{array} \right) \quad (v \in \Sigma_{\infty}),$$

$$\frac{\log q_v}{4\pi} |1 - q_v^{-iy}/2| dy \quad (v \in \Sigma_{\text{fin}}).$$

We remark that when $F = \mathbb{Q}$ and $v = p < \infty$, $\lambda^0_\eta(iy)$ is exactly equal to $\mu^0_\eta(x)$ by changing a variable $x = p^{iy/2} + p^{-iy/2}$. The main theorem of this paper is stated as follows.

**Theorem 2.** Let $\Lambda$ be an infinite subset of $J_{S,\eta}$. For any $f \in C_c(X_{\mathfrak{n}}^0)$, we have

$$\frac{1}{N(\mathfrak{n})} \sum_{\pi \in \Pi_{\text{fin}}(\mathfrak{n})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^S(1, \pi ; Ad)} f(\nu_\pi, S) \sim C(n) \langle \lambda^0_\eta, f \rangle$$

with

$$C(n) = \prod_{v \in S_2(n)} \{ 1 - \left( q_v^2 - q_v \right)^{-1} \} \prod_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} (1 - q_v^{-2})$$

as $N(\mathfrak{n}) \to \infty$ in $n \in \Lambda$. Here $S_1(\mathfrak{n})$ (resp. $S_2(\mathfrak{n})$) denotes the set of all $v \in S(\mathfrak{n})$ such that $\text{ord}_v(\mathfrak{n}) = 1$ (resp. $\text{ord}_v(\mathfrak{n}) = 2$).

This asymptotic formula gives the following counterpart of [8 Corollary B].

**Theorem 3.** Let $\Lambda$ be an infinite subset of $J_{S,\eta}$ and let $\{ J_v \}_{v \in S}$ be a family of intervals such that $J_v$ is contained in $[1/4, \infty)$ for any $v \in \Sigma_{\infty}$ and in $[-2, 2]$ for any $v \in \Sigma_{\text{fin}}$. Then, for any $M > 0$, there exists an irreducible cuspidal automorphic representation $\pi$ of $\text{GL}(2, \mathbb{A})$ with trivial central character satisfying the following conditions:

1. The conductor $f_\pi$ of $\pi$ belongs to $\Lambda$ and $N(f_\pi) > M$ holds.
2. Both $L(1/2, \pi) \neq 0$ and $L(1/2, \pi \otimes \eta) \neq 0$ hold.
3. The spectral parameter $\nu_\pi, S = (\nu_v)_{v \in S}$ at $S$ of $\pi$ satisfies $(1 - \nu_v^2)/4 \in J_v$ for any $v \in \Sigma_{\infty}$ and $\nu_v - v^{1/2} + q_v^{1/2} \in J_v$ for any $v \in \Sigma_{\text{fin}}$. 

We remark that \( L(1/2, \pi)L(1/2, \pi \otimes \eta) > 0 \) if \( L(1/2, \pi)L(1/2, \pi \otimes \eta) \neq 0 \) by Guo’s result [3]. Let \( \{v_j\}_{j \in \mathbb{N}} \) be the set of all places \( v \in \Sigma_{\text{fin}} - (S \cup S(f_n)) \) such that \( \eta_v(z_v) = -1 \) and let \( \{p_j\}_{j \in \mathbb{N}} \) be the set of all prime ideals of \( \mathfrak{o}_F \) corresponding to \( \{v_j\}_{j \in \mathbb{N}} \). Here are some examples of \( \Lambda \) in Theorems 2 and 5.

1. \( \Lambda = \{n = p_1 \cdots p_{2n} | n \in \mathbb{N}\}\),
2. \( \Lambda = \{n = p_1^{2n} | n \in \mathbb{N}\}\),
3. \( \Lambda = \{n = p_a^{2n} | n \in \mathbb{N}\} \) for a fixed \( a \in \mathbb{N}\),
4. \( \Lambda = \{n = p_a^{an}p_2^{bn} | n \in \mathbb{N}\} \) for fixed odd integers \( a, b > 0\).

The case (1) was treated by Tsuzuki [12, Theorem 1.1 and Corollary 1.2].

Motohashi [7] studied the growth of the square mean of central values of automorphic \( L \)-functions attached to Maass forms with full level via Kuznetsov’s trace formula. Tsuzuki [12, Theorem 1.3] considered a similar growth in the case where the level is square free and the base field is totally real. We give a generalization of [12, Theorem 1.3] to the case of arbitrary level.

**Theorem 4.** We set \( d_F = [F : \mathbb{Q}] \). Let \( \mathfrak{n} \) be an arbitrary ideal of \( \mathfrak{o}_F \) and \( \eta: F^\times \backslash A^\times \rightarrow \{\pm1\} \) a character of conductor relatively prime to \( \mathfrak{n} \). Assume that \( \eta_v(n) = 1 \) and \( \eta_v(-1) = 1 \) for any \( v \in \Sigma_{\text{fin}} \). Let \( J \) be a compact subset of \( \mathcal{X}_{\Sigma_{\text{fin}}}^+ \cap \prod_{v \in \Sigma_{\text{inf}}} i\mathbb{R} > 0 \) with smooth boundary. Then, for any \( \epsilon > 0 \), we have

\[
\sum_{\pi \in \Pi_{\text{ram}}(\mathfrak{n}), \nu_{\tau, \Sigma_{\text{inf}}} \in J} w_\eta^2(n) \frac{L(1/2, \pi)L(1/2, \pi \otimes \eta)}{N(j_\pi)L^{S_\pi}(1, \pi; \text{Ad})} = \frac{4D_F^{3/2}}{(4\pi)^{d_F}} \left( 1 + \delta_{n, \varepsilon_F} \right) \text{vol}(J)t^{d_F}(d_F \text{Res}_{s=1} L(s, \eta) \log t + C^0(n, F))
\]

\[
+ O(t^{d_F - 1}(\log t)^3) + O(t^{d_F(1 + 4\theta + \varepsilon)}), \quad t \to \infty,
\]

where \( w_\eta^2(n) \) is a constant explicitly defined in Lemma [7],

\[
C^0(n, F) = C T_{s=1} L(s, \eta) + \text{Res}_{s=1} L(s, \eta) \left\{ \frac{d_F}{2} \left( C_{\text{Euler}} + 2 \log 2 - \log \pi \right) + \log(D_F N(n)^{1/2}) \right\}
\]

and \( \theta \in \mathbb{R} \) is a constant such that

\[
|L(1/2 + it, \chi)| \ll q(\chi \cdot |\pi|_{\mathcal{A}}^{1/2 + \theta} \cdot t \in \mathbb{R}
\]

holds uniformly for any character \( \chi \) of \( F^\times \backslash A^\times \). Here \( q(\chi \cdot |\pi|_{\mathcal{A}}) \) is the analytic conductor of \( \chi \cdot |\pi|_{\mathcal{A}} \).

Moreover, we obtain the following results on subconvexity bounds depending on \( \theta < 0 \).

**Theorem 5.** Let \( \mathfrak{n} \) be an arbitrary ideal of \( \mathfrak{o}_F \). Let \( J \subset \mathcal{X}_{\Sigma_{\text{fin}}}^+ \cap \prod_{v \in \Sigma_{\text{inf}}} i\mathbb{R} > 0 \) be a closed cone. Then, for any \( \epsilon > 0 \), we have

\[
|L_{\text{fin}}(1/2, \pi)| \ll (1 + ||\nu_{\tau, \Sigma_{\text{inf}}}||)^{d_F + \sup(2d_F \theta, -1/2) + \varepsilon}
\]

for \( \pi \in \Pi_{\text{ram}}(\mathfrak{n}) \). \( J = \{ \pi \in \Pi_{\text{ram}}(\mathfrak{n}) | \nu_{\tau, \Sigma_{\text{inf}}} \in J \} \).

We remark that Theorem 5 was proved by Tsuzuki [12, Corollary 1.4] when \( \mathfrak{n} \) is square free. Michel and Venkatesh [5] gave sharp subconvexity bounds for automorphic \( L \)-functions for \( GL(1) \) and \( GL(2) \) in more general case.

Our method to prove from Theorem 2 to Theorem 5 is based on that of [12]. We introduce adelic Green functions on \( GL(2, \mathcal{A}) \) for ideals of \( \mathfrak{o}_F \) and then we give a relative trace formula by computing the regularized period of a Poincaré series of an adelic Green function in two different ways. Regularized periods used in this paper are toral period integrals regularized by Tsuzuki in order to define periods for nonrapidly decreasing functions on \( GL(2, \mathcal{A}) \). Since regularized periods of automorphic forms on \( GL(2, \mathcal{A}) \) with arbitrary level were studied by a previous paper [10], we can compute the spectral side of our relative trace formula.

We explain the structure of this paper. In [11] we introduce notations used throughout this paper. In [2] we review results of [10] and prepare several lemmas. In [3] we introduce adelic Green functions on \( GL(2, \mathcal{A}) \). In [4] we regularize a Poincaré series of an adelic Green function. The regularized Poincaré
series is called the regularized automorphic smoothed kernel. In [3] we compute the regularized period of the regularized automorphic smoothed kernel by using the spectral expansion. In [4] we decompose the regularized period of the regularized automorphic smoothed kernel into the sum of terms derived from some orbits. In [5] we prove from Theorem 2 to Theorem 5.

**Notation.** We write $\mathbb{N}$ for the set of natural numbers and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For sets $A$ and $B$, the set $\text{Map}(A,B)$ denotes the set of mappings from $A$ to $B$. For $f,g \in \text{Map}(A,\mathbb{R}_{\geq 0})$, let us denote by $f(x) \ll g(x)$, $x \in A$ an inequality $f(x) \leq C g(x)$ with some constant $C > 0$. For a given condition $P$, $\delta(P) \in \{0,1\}$ is defined by $\delta(P) = 1$ (resp. $\delta(P) = 0$) if $P$ is true (resp. false).

Let $F$ be an totally real field with its degree $d_F$ and $\mathfrak{o}_F$ its integer ring. Let $\mathfrak{A}$ and $\mathfrak{A}_{\text{fin}}$ be the adele ring and the finite adele ring of $F$, respectively. The symbols $\Sigma_{\infty}$ and $\Sigma_{\text{fin}}$ denote the set of all infinite places and the set of all finite places of $F$, respectively. For a place $v \in \Sigma_F = \Sigma_{\infty} \cup \Sigma_{\text{fin}}$, let $| \cdot |_v$ denote the normalized valuation of the completion $F_v$ of $F$ at $v$. For each $v \in \Sigma_{\text{fin}}$, let $\varpi_v$ be a uniformizer of $F_v$. Then, $p_v = \varpi_o v$ for $v \in \Sigma_{\text{fin}}$ is a maximal ideal of the integer ring $\mathfrak{o}_v$ of $F_v$ and we have $|\varpi_v|_v = q_v^{-1}$, where $q_v$ is the cardinality of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$. For an ideal $a$ of $\mathfrak{o}_F$, let $S(a)$ denote the set of all $v \in \Sigma_{\text{fin}}$ such that $v$ divides $a$. For any $k \in \mathbb{N}$, we write $S_k(a)$ for the set of all $v \in S(a)$ with $\text{ord}_v(a) = k$, where $\text{ord}_v(a)$ is the order of $a$ at $v$. Let $N(a)$ denote the absolute norm of $a$.

Let $G$ be the algebraic group $GL(2)$ with unit element $e$. For any $F$-algebraic subgroup $M$ of $G$, we set $M_F = M(F)$, $M_v = M(F_v)$ (for $v \in \Sigma_F$), $M_h = M(\mathfrak{A})$ and $M_{\text{fin}} = M(\mathfrak{A}_{\text{fin}})$, respectively. The diagonal maximal split torus of $G$ is denoted by $H$. Then, the Borel subgroup $B = HN$ of $G$ consists of all upper triangular matrices of $G$, where $N$ is the subgroup of $G$ consisting of all unipotent matrices of $G$. The center of $G$ is denoted by $Z$. We put $K_v = O(2,\mathbb{R})$ (resp. $K_v = GL(2,\mathfrak{o}_v)$) for $v \in \Sigma_{\infty}$ (resp. $v \in \Sigma_{\text{fin}}$). Then, $K = \prod_{v \in \Sigma_{\infty}} K_v$ is a maximal compact subgroup of $G$. Set $K_\infty = \prod_{v \in \Sigma_{\infty}} K_v$ and the regularized period of the regularized automorphic smoothed kernel into the sum of terms derived from some orbits. In [5] we prove from Theorem 2 to Theorem 5.

1. Preliminaries

Let $\mathfrak{A}_Q$ be the adele ring of $Q$ and $\psi_Q = \prod_{p} \psi_p$ the additive character of $Q \setminus \mathfrak{A}_Q$ with archimedean component $\psi_\infty(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$. Then, $\psi_F = \psi \circ \text{tr}_{F/Q} = \prod_{v \in \Sigma_F} \psi_v$ is a nontrivial additive character of $F \setminus \mathfrak{A}$. Let $\mathcal{D}_{F/Q}$ be the global differrent of $F/Q$ and set $\text{ord}_v \mathcal{D}_{F/Q} = d_v$ for any $v \in \Sigma_{\infty}$.

For $v \in \Sigma_F$, let $dx_v$ be the self-dual Haar measure of $F_v$ with respect to $\psi_v$. We set $d^x v = (1 - q_v^{-1})^{-1} \frac{dx_v}{|x|_v}$ for $v \in \Sigma_{\text{fin}}$ and $d^x v = d x_v/|x|_v$ for $v \in \Sigma_{\infty}$, respectively. Then, $d^x v$ is a Haar measure of $F_v^x$ and the product measure $d^x x = \prod_{v \in \Sigma_F} d^x v$ gives a Haar measure on $\mathbb{A}^x$. For each $v \in \Sigma_F$, we take a Haar measure $dk_v$ on $K_v$ such that total volume is one, and take a Haar measure $dg_v$ on $G_v$ in the following way. Let $dk_v$ (resp. $dk_v$) denotes the Haar measure on $H_v$ (resp. $N_v$) induced via the isomorphism $H_v \cong F_v^x \times F_v^x$ (resp. $N_v \cong F_v^x$). Then, $dg_v = dh_v 

\text{For } v \in \Sigma_{\text{fin}}$ and a quasi-character $\chi_v$ of $F_v^x$, $\psi_v^{(\chi_v)}$ denotes the conductor of $\chi_v$. We define the Gauss sum associated with $\chi_v$ by

$$
\mathcal{G}(\chi_v) = \int_{\mathbb{A}^x} \chi_v(u \varpi_v^{-d_v - f(\chi_v)}) \psi_v(u \varpi_v^{-d_v - f(\chi_v)}) \, d^x u.
$$
For any quasi-character $\chi = \prod_{v \in \Sigma_F} \chi_v$ of $F^\times \backslash \mathbb{A}^\times$, we define the conductor of $\chi$ by the ideal $f_\chi$ of $\mathfrak{o}_F$ such that $f_\chi \mathfrak{p}_v = \mathfrak{p}_v^{f_\chi(v)}$ for all $v \in \Sigma_{\mathfrak{f}_n}$. We write $\chi_{\mathfrak{f}_n}$ for $\prod_{v \in \Sigma_{\mathfrak{f}_n}} \chi_v$. The Gauss sum associated with $\chi$ is defined by the product of $\mathcal{G}(\chi_v)$ over all $v \in \Sigma_{\mathfrak{f}_n}$. We set $\chi(a) = \prod_{v \in \Sigma_{\mathfrak{f}_n}} \chi_v(\varpi_v^{\text{ord}_v(a)})$ for any ideal $a$ of $\mathfrak{o}_F$. For $v \in \Sigma_F$, we denote the trivial character of $F_v^\times$ by $1_v$, and the trivial character of $\mathbb{A}_v^\times$ by $\mathbf{1}_v$.

Throughout this paper, any quasi-character $\chi$ of $F^\times \backslash \mathbb{A}^\times$ is assumed to satisfy $\chi(y) = 1$ for all $y \in \mathbb{R} > 0$.

Such a quasi-character is a character. For any $v \in \Sigma_F$ and any character $\chi_v$ of $F_v^\times$, let $b_v(\chi_v)$ denote $b_v \in \mathbb{R}$ (resp. $b_v \in [0, 2\pi(\log q_v)^{-1}]$) such that the restriction of $\chi_v$ to $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_v^\times$) is of the form $| \cdot |_v^{b_v}$. For any character $\chi$ of $F^\times \backslash \mathbb{A}^\times$, the analytic conductor $q(\chi)$ of $\chi$ is defined to be

$$q(\chi) = \prod_{v \in \Sigma_{\infty}} (3 + |b(\chi_v)|) N(f_\chi).$$

Let $n$ be an ideal of $\mathfrak{o}_F$. For an ideal $c$ of $\mathfrak{o}_F$, let $\Xi_0(c)$ be the set of all characters $\chi$ of $F^\times \backslash \mathbb{A}^\times$ such that $f_\chi = c$ and $\chi_v(-1) = 1$ for all $v \in \Sigma_{\infty}$. We write $\Xi(n)$ for $\bigcup_{c|n} \Xi_0(c)$. Let $U_F^+$ be the set of all totally positive units of $\mathfrak{o}_F$ and set

$$\log U_F^+ = \{ (v)_{v \in \Sigma_{\infty}} \mid (u)_{v \in \Sigma_{\infty}} \in U_F^+ \}.$$

Then, $\log U_F^+$ is a lattice of $\mathbb{Z}$-rank $d_F - 1$ in $V$, where $V = \{(u)_{v \in \Sigma_{\infty}} \in \mathbb{R}^{d_F} \mid \sum_{v \in \Sigma_{\infty}} x_v = 0 \}$. Set

$$L_0 = \{ (b)_{v \in \Sigma_{\infty}} \in V \mid \sum_{v \in \Sigma_{\infty}} b_v \in \mathbb{Z} \text{ for all } (l)_{v \in \Sigma_{\infty}} \in \log U_F^+ \}.$$

Then, $L_0$ is also a $\mathbb{Z}$-lattice in $V$. Let $\chi$ be a character of $F^\times \backslash \mathbb{A}^\times$. Since $\chi(y) = 1$ for any $y \in \mathbb{R}_{>0}$, we have $\sum_{v \in \Sigma_{\infty}} b(\chi_v) = 0$. Thus, if we denote by $b(\chi)$ the element $(b(\chi_v))_{v \in \Sigma_{\infty}}$ of $\mathbb{R}^{d_F}$, then $b(\chi) \in L_0$ holds. Therefore the mapping $\chi \mapsto b(\chi)$ is a surjection from $\Xi(n)$ onto $L_0$ and the kernel $\Xi_{\text{ker}}(n)$ of this mapping is a finite abelian group.

**Lemma 6.** Let $X(n)$ be the order of $\Xi_{\text{ker}}(n)$. Then, for any $\epsilon > 0$, the estimate

$$X(n) \ll N(n)^{1/2 + \epsilon}$$

holds with the implied constant independent of $n$.

**Proof.** For any ideal $a$ of $\mathfrak{o}_F$, we set $I_F(a) = \prod_{v \in \Sigma_{\infty}} \mathbb{R}^\times \times \prod_{v \in \Sigma_{\mathfrak{f}_n}} (1 + aa_v)$. Then, the ray class group $C_F(a)$ modulo $a$ is defined by $C_F(a) = F^\times / I_F(a)$. For any fixed $c$ satisfying $c^2|n$, the group $\Xi_0(c) \cap \Xi_{\text{ker}}(n)$ is equal to the set of all characters of $F^\times \backslash \mathbb{A}^\times$ of finite order contained in $\Xi_0(c)$. Hence

$$\#(\Xi_0(c) \cap \Xi_{\text{ker}}(n)) \leq \#(F^\times \backslash \mathbb{A}^\times / I_F(c)) = h_F \#(C_F(\mathfrak{o}_F)/C_F(c)) \leq h_F N(c) \leq h_F N(n)^{1/2}$$

holds, where $h_F$ is the class number of $F$. Noting $\sum_{c|n} 1 \ll \log(1 + N(n)) \ll N(n)^{\epsilon}$ for any $\epsilon > 0$, we obtain the assertion. \[\square\]

## 2. Regularized periods of automorphic forms

In this section, we recall an explicit formula in [10] of the regularized periods of automorphic forms on $G_\mathbb{A}$.

### 2.1. Zeta integrals of cusp forms

Let $\pi$ be a $\mathbb{K}_\infty$-spherical irreducible cuspidal automorphic representation of $G_\mathbb{A}$ with trivial central character, where the representation space $V_\pi$ is realized in the space of cusp forms. For any quasi-character $\eta$ of $F^\times \backslash \mathbb{A}^\times$ and $\varphi \in V_\pi$, we define the global zeta integral by

$$Z(s, \eta, \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right) \eta(t)|t|_\mathbb{A}^{s-1/2} d^\times t, \quad s \in \mathbb{C}.$$

The defining integral converges absolutely for any $s \in \mathbb{C}$, and hence $Z(s, \eta, \varphi)$ is an entire function in $s$. 
We fix a family \( \{ \pi_v \}_{\nu \in \Sigma_{v}} \) consisting of irreducible admissible representations such that \( \pi \cong \bigotimes_{v \in \Sigma_{p}} \pi_v \). The conductor of \( \pi \) is denoted by \( f_\pi \), which is the ideal of \( \mathfrak{o}_F \) defined by \( f_\pi \mathfrak{o}_v = \mathfrak{o}_v^{f_\pi} \) for all \( v \in \Sigma_{\text{fin}} \), where \( \mathfrak{o}_v^{f_\pi} \) is the conductor of \( \pi_v \). Let \( n \) be an ideal of \( \mathfrak{o}_F \) which is divided by \( f_\pi \).

Let \( n \) be the maximal nonnegative integer \( m \) such that \( S_m(nf_v^{-1}) \neq \emptyset \). For \( \rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda^\flat_0(n) = \prod_{k=1}^n \text{Map} \left( S_k(nf_v^{-1}), \{0, \ldots, k \} \right) \), let \( \varphi_{\pi, \rho} \) denote the cusp form in \( V_{\pi, \rho}^{K_{\infty}} \mathbf{K}_0(n) \) corresponding to

\[
\prod_{v \in \Sigma_{\infty}} \phi_{v, \nu} \otimes \prod_{v \in S_1(nf_v^{-1})} \phi_{v, \nu} \otimes \cdots \otimes \prod_{v \in S_n(nf_v^{-1})} \phi_{v, \nu} \otimes \prod_{v \in \Sigma_{\text{fin}} - S(nf_v^{-1})} \phi_{v, \nu}
\]

by the isomorphism \( V_{\pi} \cong \bigotimes_{v \in \Sigma_{p}} V_{\pi_v} \). Here, \( V_{\pi_v} \) denotes the Whittaker model of \( \pi_v \) with respect to \( \psi_{F_v} \), \( \phi_{v, \nu} \) is the spherical vector in \( V_{\pi_v} \) for \( v \in \Sigma_{\infty} \) given in [10, 1.4], and the function \( \phi_{v, \nu} \) is the \( \mathbf{K}_0(n) \)-invariant vector for \( v \in \Sigma_{\text{fin}} \), which is constructed in [10, §2 and §3]. Then, the finite set \( \{ \varphi_{\pi, \rho} \}_{\rho \in \Lambda^\flat_0(n)} \) is an orthogonal basis of \( V_{\pi, \rho}^{K_{\infty}} \mathbf{K}_0(n) \). Here \( V_{\pi} \subset L^2(Z_{\mathbf{A}} F \backslash G_{\mathbf{A}}) \) is equipped with the \( L^2 \)-inner product (cf. [10, Proposition 17]).

We consider a character \( \eta \) of \( F^\times \backslash \mathbf{A}^\times \) satisfying

\[
\eta^2 = 1, \quad \eta \neq 1, \quad f_\eta \text{ is relatively prime to } n \text{ and } \eta(n) = 1.
\]

For such a character \( \eta \) and \( \varphi \in V_{\pi, \rho}^{K_{\infty}} \mathbf{K}_0(n) \), we define the modified global zeta integral by

\[
Z^\ast(s, \eta, \varphi) = \eta_{\text{fin}}(x_{\eta, \text{fin}}) Z \left( s, \eta, \pi \left( \begin{array}{cc} 1 & x_{\eta} \\ 0 & 1 \end{array} \right) \varphi \right), \quad s \in \mathbb{C}.
\]

Here \( x_{\eta} = (x_{\eta,v})_{v \in \Sigma_{p}} \in \mathbf{A} \) is the adele whose \( v \)-component satisfies \( x_{\eta,v} = 0 \) and \( x_{\eta,v} = w_v^{-f_\nu} \) for \( v \in \Sigma_{\infty} \) and \( v \in \Sigma_{\text{fin}} \), respectively, and \( x_{\eta, \text{fin}} \) denotes the projection of \( x_{\eta} \) to \( \mathbf{A}_{\text{fin}} \).

### 2.2. Regularized periods of cusp forms

We recall a definition of regularized periods of automorphic forms on \( G_{\mathbf{A}} \) defined in [12, §7]. For \( C > 0 \), let \( \mathcal{B}(C) \) be the space of all holomorphic even functions \( \beta \) on \( \{ z \in \mathbb{C} \mid \text{Re}(z) < C \} \) satisfying \( |\beta(\sigma + it)| < (1 + |t|)^{-\ell} \), \( \sigma \in [a, b], t \in \mathbb{R} \) hold for any \( [a, b] \subset (-C, C) \) and any \( \ell > 0 \). Let \( \mathcal{B} \) be the space of all entire functions \( \beta \) on \( \mathbb{C} \) such that the restriction of \( \beta \) to \( \{ z \in \mathbb{C} \mid |\text{Re}(z)| < C \} \) is contained in \( \mathcal{B}(C) \) for any \( C > 0 \).

For \( \beta \in \mathcal{B} \) and \( \lambda \in \mathbb{C} \), we define a function \( \hat{\beta}_\lambda \) on \( \mathbb{R}_{>0} \) by

\[
\hat{\beta}_\lambda(t) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z + \lambda} t^2 \, dz, \quad (\sigma > -\text{Re}(\lambda)),
\]

where \( L_\sigma = \{ z \in \mathbb{C} \mid \text{Re}(z) = \sigma \} \).

For \( \beta \in \mathcal{B} \), \( \lambda \in \mathbb{C} \), a character \( \eta \) of \( F^\times \backslash \mathbf{A}^\times \) satisfying \( (*) \) and a function \( \varphi : \mathfrak{A} G_{F} \backslash G_{\mathbf{A}} \to \mathbb{C} \), we consider

\[
P^n_{\beta, \lambda}(\varphi) = \int_{F^\times \backslash \mathbf{A}^\times} \{ \hat{\beta}(\lambda|t) + \hat{\beta}(\lambda|-t) \} \varphi \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right) \eta(t) \eta_{\text{fin}}(x_{\eta, \text{fin}}) \, dt.
\]

Now we assume that for any \( \beta \in \mathcal{B} \), there exists a constant \( C \in \mathbb{R} \) such that if \( \text{Re}(\lambda) > C \) the integral \( P^n_{\beta, \lambda}(\varphi) \) converges and the function \( \{ z \in \mathbb{C} \mid \text{Re}(z) > C \} \ni \lambda \mapsto P^n_{\beta, \lambda}(\varphi) \) is continued meromorphically to a neighborhood of \( \lambda = 0 \). Then a constant \( P^n_{\text{reg}}(\varphi) \) is called the regularized \( \eta \)-period of \( \varphi \) if \( \forall \lambda = 0, P^n_{\beta, \lambda}(\varphi) = P^n_{\text{reg}}(\varphi) \beta(0) \) for all \( \beta \in \mathcal{B} \). Then the following was proved in [10].
Proposition 7. [10] Main Theorem A] For any \( \rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_0^n(\mathfrak{n}) \) and \( \eta \) satisfying (\*) , the period \( P_{reg}^n(\varphi_{\pi, \rho}) \) can be defined and we have

\[
P_{reg}^n(\varphi_{\pi, \rho}) = Z^*(1/2, \eta, \varphi) = \mathcal{G}(\eta) \prod_{k=1}^{n} \prod_{v \in S_k} Q_{\rho, v}^n(\eta_v, 1) L(1/2, \pi \otimes \eta),
\]

where the constants \( Q_{\rho, v}^n(\eta_v, 1) \) are given as follows:

- If \( c(\pi_v) = 0 \) and \((\alpha_v, \alpha_v^{-1})\) is the Satake parameter of \( \pi_v \), then

\[
Q_{k, v}^n(\eta_v, 1) = \begin{cases} 
1 & \text{(if } k = 0), \\
\eta_v(\varphi_v) - \frac{\alpha_v + \alpha_v^{-1}}{q_v^{1/2} + q_v^{-1/2}} & \text{(if } k = 1), \\
q_v^{-1}\eta_v(\varphi_v)^{k-2}(\alpha_v q_v^{1/2}\eta_v(\varphi_v) - 1)(\alpha_v^{-1} q_v^{-1/2}\eta_v(\varphi_v) - 1) & \text{(if } k \geq 2). 
\end{cases}
\]

- If \( c(\pi_v) = 1 \), then \( \pi_v \) is isomorphic to a special representation \( \sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2}) \) for some unramified character \( \chi_v \) of \( F_v^x \) and

\[
Q_{k, v}^n(\eta_v, 1) = \begin{cases} 
1 & \text{(if } k = 0), \\
\eta_v(\varphi_v)^{k-1}(\eta_v(\varphi_v) - q_v^{-1}\chi_v(\varphi_v)^{-1}) & \text{(if } k \geq 1). 
\end{cases}
\]

- If \( c(\pi_v) \geq 2 \), then \( Q_{k, v}^n(\eta_v, 1) = \eta_v(\varphi_v)^k \) for any \( k \in \mathbb{N}_0 \).

2.3. Preliminaries for regularized periods of Eisenstein series. We fix a character \( \chi = \prod_{v \in \Sigma_F} \chi_v \) of \( F^x \backslash \mathbb{A}^x \). For \( \nu \in \mathbb{C} \), we denote by \( I(\chi \cdot |v|^{\nu/2}) \) the space of all smooth \( \mathbb{C} \)-valued right \( \mathbb{K} \)-finite functions \( f \) on \( G_\mathbb{K} \) with the \( B_\mathbb{K} \)-equivariance

\[
f( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ) = \chi(a/d) |a/d|_{\mathbb{A}}^{(\nu+1)/2} f(g)
\]

for all \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_\mathbb{K} \) and \( g \in G_\mathbb{K} \). If \( \nu \in i\mathbb{R} \), then the space \( I(\chi \cdot |v|^{\nu/2}) \) is unitarizable and a \( G_\mathbb{K} \)-invariant hermitian inner product is given by

\[
(f_1 | f_2) = \int_K f_1(k) \overline{f_2(k)} dk
\]

for any \( f_1, f_2 \in I(\chi \cdot |v|^{\nu/2}) \).

For \( \nu \in \mathbb{C} \) and \( f(\nu) \in I(\chi \cdot |v|^{\nu/2}) \). The family \( \{ f(\nu) \}_{\nu \in \mathbb{C}} \) is called a flat section if the restriction of \( f(\nu) \) to \( \mathbb{K} \) is independent of \( \nu \in \mathbb{C} \). We define the Eisenstein series for \( f(\nu) \in I(\chi \cdot |v|^{\nu/2}) \) by

\[
E(f(\nu), g) = \sum_{\gamma \in B_F \backslash G_F} f(\nu)(\gamma g), g \in G_\mathbb{K}.
\]

The defining series converges absolutely if \( \text{Re}(\nu) > 1 \). If \( \{ f(\nu) \}_{\nu \in \mathbb{C}} \) is a flat section, then \( E(f(\nu), g) \) is continued meromorphically to \( \mathbb{C} \) as a function in \( \nu \). We remark that the function \( E(f(\nu), g) \) is holomorphic on \( i\mathbb{R} \). On the half plane \( \text{Re}(\nu) > 0 \), \( E(f(\nu), g) \) is holomorphic except for \( \nu = 1 \), and \( \nu = 1 \) is a pole of \( E(f(\nu), g) \) if and only if \( \chi^2 = 1 \).

Let \( \mathfrak{n} \) be an ideal of \( \mathfrak{o}_F \). Throughout §2, we assume that a character \( \chi \) of \( F^x \backslash \mathbb{A}^x \) is contained in \( \Xi(\mathfrak{n}) \).
2.4. Zeta integrals of Eisenstein series. We consider Eisenstein series for \( f \in I(\chi \mid \nu/2)^{K_\infty K_0(n)} \). Let \( n \) be the maximal nonnegative integer \( m \) such that \( S_m(nf_{\nu/2}^{-2}) \neq \emptyset \). For each \( v \in \Sigma_F \), the space \( I(\chi_v \mid \nu/2) \) is defined in the same way as the global case. For \( \rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\chi(n) = \prod_{k=1}^n \text{Map}(S_k(nf_{\nu/2}), \{0, \ldots, k\}) \), let \( f^{(\nu)}_{1, \rho} \) denote the vector in \( I(\chi \mid \nu/2)^{K_\infty K_0(n)} \) corresponding to

\[
\otimes_{v \in \Sigma_{\infty}} f^{(\nu)}_{0, \chi_v} \otimes \otimes_{v \in S_1(nf_{\nu/2})} f^{(\nu)}_{\rho_1(v), \chi_v} \otimes \cdots \otimes \otimes_{v \in S_n(nf_{\nu/2})} f^{(\nu)}_{\rho_n(v), \chi_v} \otimes \otimes_{v \in \Sigma_{\text{fin}} \setminus S(nf_{\nu/2})} f^{(\nu)}_{0, \chi_v}
\]

by the isomorphism \( I(\chi \mid \nu/2) \cong \otimes_{v \in \Sigma_F} I(\chi_v \mid \nu/2) \), where for \( v \in \Sigma_{\infty} \), \( f^{(\nu)}_{0, \chi_v} \) is the spherical vector in \( I(\chi_v \mid \nu/2) \) normalized so that \( f^{(\nu)}_{0, \chi_v}(e) \) equals one and for \( v \in \Sigma_{\text{fin}} \), \( f^{(\nu)}_{\rho_1(v), \chi_v} \) is the \( K_0(n\eta_v) \)-invariant vector for \( v \in \Sigma_{\text{fin}} \), which is constructed in \([10]\) \S 7 and \([8]\). Then, for any \( \rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\chi(n) \), the family \( \{f^{(\nu)}_{\rho, v} \mid v \in \Sigma \} \) is a flat section. Moreover, if \( \nu \in i\mathbb{R} \), the finite set \( \{f^{(\nu)}_{\rho, v} \mid \rho \in \Lambda_\chi(n) \} \) is an orthonormal basis of \( I(\chi \mid \nu/2)^{K_\infty K_0(n)} \) (cf. \([10]\) Proposition 33).

Let \( \rho \in \Lambda_\chi(n) \) and set \( E_{\chi, \rho}(\nu, g) = E(f^{(\nu)}_{\chi, \rho}, g) \). The constant term of \( E(f^{(\nu)}_{\chi, \rho}, g) \) is defined by

\[
E_{\chi, \rho}^\circ(\nu, g) = \int_{F \setminus \mathbb{A}} E_{\chi, \rho}(\nu, g) \frac{1}{x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g} \, dx.
\]

For \( k \in \{1, \ldots, n\} \), the sets \( U_k(\rho) \), \( R_k(\rho) \) and \( R_0(\rho) \) are defined as follows:

\[
U_k(\rho) = \bigcup_{m=k}^{n} \rho_m^{-1}(k) - S(\mathfrak{f}_\chi), \quad R_k(\rho) = \bigcup_{m=k}^{n} \rho_m^{-1}(k) \cap S(\mathfrak{f}_\chi),
\]

\[
R_0(\rho) = \left( \bigcup_{m=0}^{n} \rho_m^{-1}(0) \cap S(\mathfrak{f}_\chi) \right) \bigcup \left( S(\mathfrak{f}_\chi) - S(nf_{\nu/2}) \right).
\]

Furthermore, for any \( k \in \mathbb{N}_0 \), set

\[
S_k(\rho) = \begin{cases} R_0(\rho) & \text{(if } k = 0), \\ U_k(\rho) \cup R_k(\rho) & \text{(if } k \geq 1). \end{cases}
\]

\( R(\rho) = \bigcup_{k=0}^{n} R_k(\rho) \) and \( S(\rho) = \bigcup_{k=0}^{n} S_k(\rho) \). Then, by \([10]\) Proposition 34 we have

\[
E_{\chi, \rho}^\circ(\nu, g) = f^{(\nu)}_{\chi, \rho}(g) + D_F^{-1/2} A_{\chi, \rho}(\nu) \frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)} f^{(-\nu)}_{\chi, \rho}(g),
\]

where

\[
A_{\chi, \rho}(\nu) = N(\mathfrak{f}_\chi)^{-\nu} \prod_{k=0}^{n} \prod_{v \in S_k(\rho)} \left\{ \sqrt{q_{\nu, v}^{1/2}} \epsilon(1 - \nu, \chi_v^{-1}, \psi_{F_v}) \epsilon(1 + \nu/2, \chi_v, \psi_{F_v}) \frac{L(1 + \nu, \chi_v^2)}{L(1 - \nu, \chi_v^2)} \right\}.
\]

We fix a character \( \eta \) of \( F^\times \setminus \mathbb{A}^\times \) satisfying \((*)\) in \S 2.1. For any \( v \in \Sigma_{\text{fin}} \setminus S(\mathfrak{f}_\eta) \) and \( k \in \mathbb{N}_0 \), let \( Q^{(\nu)}_{k, \chi_v}(\eta_v, X) \) be the polynomial defined in \([10]\) \S 9. Then, we have the following.

**Proposition 8.** \([10]\) Proposition 35] We set \( E_{\chi, \rho}^\circ(\nu, g) = E_{\chi, \rho}(\nu, g) - E_{\chi, \rho}^\circ(\nu, g) \). Then \( E_{\chi, \rho}^\circ(\nu, -) \) is left \( B_F \)-invariant and we have

\[
Z^*(s, \eta, E_{\chi, \rho}^\circ(\nu, -)) = \mathcal{G}(\eta) D_F^{\nu/2} N(\mathfrak{f}_\chi)^{1/2 - \nu} B_{\chi, \rho}(s, \nu) \frac{L(s + \nu/2, \chi \eta) L(s - \nu/2, \chi^{-1} \eta)}{L(1 + \nu, \chi^2)},
\]

where

\[
B_{\chi, \rho}^\circ(\nu, s) = D_F^{-s/2} \left\{ \prod_{k=0}^{n} \prod_{v \in S_k(\rho)} Q^{(\nu)}_{k, \chi_v}(\eta_v, q_{\nu, v}^{1/2-s}) L(1 + \nu, \chi_v^2) \right\}
\]
2.5. Regularized periods of Eisenstein series. For any characters \( \chi_1 \) and \( \chi_2 \) of \( F^\times \backslash \mathbb{A}^\times \), we put \( \delta_{\chi_1,\chi_2} = \delta(\chi_1 = \chi_2) \). The regularized period \( P^\eta_{\text{reg}}(E_{\chi,\rho}(\nu,-)) \) was computed as follows in [10].

**Proposition 9.** [10] Main Theorem B] Assume \( \nu \in i\mathbb{R} \). Then the integral \( P^\eta_{\lambda}(E_{\chi,\rho}(\nu,-)) \) converges absolutely for any \((\beta, \lambda) \in \mathcal{B} \times \mathbb{C}\) such that \( \operatorname{Re}(\lambda) > 1 \). Moreover \( P^\eta_{\text{reg}}(E_{\chi,\rho}(\nu,-)) \) can be defined, and we have

\[
P^\eta_{\text{reg}}(E_{\chi,\rho}(\nu,-)) = \varnothing(\eta) D_F^{-\nu/2} N(f)_{1/2-\nu} B^\eta_{\chi,\rho}(1/2,\nu) \frac{L((1+\nu)/2,\chi\eta)L((1-\nu)/2,\chi^{-1}\eta)}{L(1+\nu,\chi^2)}.
\]

We define two functions \( \varepsilon_{\chi,\rho,-1} \) and \( \varepsilon_{\chi,\rho,0} \) on \( G_\mathbb{A} \) by the Laurent expansion

\[
E_{\chi,\rho}(\nu,g) = \frac{\varepsilon_{\chi,\rho,-1}(g)}{\nu - 1} + \varepsilon_{\chi,\rho,0}(g) + \varnothing(\nu - 1), \quad (\nu \to 1).
\]

We explain the regularized \( \eta \)-periods of \( \varepsilon_{\chi,\rho,-1} \) and that of \( \varepsilon_{\chi,\rho,0} \). Set \( R_{\mathcal{F}} = \operatorname{Res}_{s=1} \zeta_{\mathcal{F}}(s) = \operatorname{vol}(F^\times \backslash \mathbb{A}^1) \), where \( \zeta_{\mathcal{F}}(s) \) is the completed Dedekind zeta function of \( F \). The regularized period \( P^\eta_{\text{reg}}(\varepsilon_{\chi,\rho,-1}) \) was computed as follows in [10].

**Proposition 10.** [10] Lemma 38 and Theorem 39] We have

\[
\varepsilon_{\chi,\rho,-1}(g) = \delta(\chi^2 = 1, f_\chi = \sigma_F, S(\rho) = \emptyset) \frac{D_F^{-1/2} R_{\mathcal{F}}}{\zeta_{\mathcal{F}}(2)} \chi(\det g)
\]

for any \( g \in G_\mathbb{A} \). Moreover, for \( \lambda \in \mathbb{C} \) such that \( \operatorname{Re}(\lambda) > 0 \), we have

\[
P^\eta_{\beta,\lambda}(\varepsilon_{\chi,\rho,-1}) = \delta(\chi = \eta, f_\chi = \sigma_F, S(\rho) = \emptyset) \frac{2 D_F^{-1/2} R_{\mathcal{F}}}{\zeta_{\mathcal{F}}(2)} \beta(0)
\]

and \( P^\eta_{\text{reg}}(\varepsilon_{\chi,\rho,-1}) = 0 \).

For any character \( \xi \) of \( F^\times \backslash \mathbb{A}^\times \), we define \( R(\xi) \), \( C_0(\xi) \) and \( C_1(\xi) \) by the Laurent expansion

\[
L(s,\xi) = \frac{R(\xi)}{s - 1} + C_0(\xi) + C_1(\xi)(s - 1) + \varnothing((s - 1)^2), \quad (s \to 1).
\]

We note \( R(\xi) = \delta_{\xi,1} R_F \) for any character \( \xi \) of \( F^\times \backslash \mathbb{A}^\times \). The regularized period \( P^\eta_{\text{reg}}(\varepsilon_{\chi,\rho,0}) \) is defined under some conditions, and \( P^\eta_{\beta,\lambda}(\varepsilon_{\chi,\rho,0}) \) was computed as follows in [10].

**Proposition 11.** [10] Theorem 40 and Corollary 41] Let \( \eta \) be a character of \( F^\times \backslash \mathbb{A}^\times \) satisfying \((*)\) in §2.1. The integral \( P^\eta_{\beta,\lambda}(\varepsilon_{\chi,\rho,0}) \) converges absolutely for any \((\beta, \lambda) \in \mathcal{B} \times \mathbb{C}\) such that \( \operatorname{Re}(\lambda) > 1 \). There
exists an entire function \( f(\lambda) \) on \( \mathbb{C} \) such that

\[
P_{\beta, \lambda}^{\eta}(\varepsilon, \rho, 0) = \delta_{\lambda, \eta} R_{F} f_{\beta}^{[1]}(\varepsilon) \left\{ \frac{1}{\lambda - 1} + \frac{1}{\lambda + 1} \right\} \beta(1) + 2 \delta_{\lambda, \eta} R_{F} D_{F}^{-1/2} f_{\beta}^{[1]}(\varepsilon) \left\{ \frac{1}{\lambda + 1} + \frac{1}{\lambda - 1} \right\} \beta(1) + f(\lambda) - \mathcal{G}(\eta) D_{F}^{-1/2} R_{F} \delta_{\lambda, \eta} \left\{ - \frac{\tilde{B}_{\lambda, \eta}(0)}{\lambda + 1} + \frac{\tilde{B}_{\lambda, \eta}(0)}{\lambda - 1} \right\} \beta(1)
\]

where \( \tilde{B}_{\lambda, \eta}(z) = (z, \chi^{-1} \eta) B_{\lambda, \eta}(-z + 1/2, 1) \). Moreover we have

\[
CT_{\lambda = 0} P_{\beta, \lambda}^{\eta}(\varepsilon, \rho, 0) = \frac{\mathcal{G}(\eta) D_{F}^{-1/2} N(f_{\beta})^{-1/2}}{L(2, \chi^{2})} \left\{ - \frac{1}{2} \delta_{\lambda, \eta} \tilde{B}_{\lambda, \eta}(0) R_{F}^{2} \beta''(0) + A_{\chi, \eta}(0) \beta(0) \right\},
\]

where

\[
a_{\chi, \eta}(0) = - \frac{1}{2} \delta_{\lambda, \eta} (\tilde{B}_{\lambda, \eta})''(0) R_{F}^{2} - 2 \delta_{\lambda, \eta} \tilde{B}_{\lambda, \eta}(0) R_{F} C_{1}(1) + \tilde{B}_{\lambda, \eta}(0) C_{0}(\chi \eta)^{2}.
\]

2.6. An orthonormal basis of \( V_{\pi}^{K \infty K_{0}(n)} \). Let \( \pi \) be a cuspidal automorphic representation of \( G_{\mathbb{A}} \) such that \( \pi \in \Pi_{\text{cus}}(n) \). We put

\[
P^{\eta}(\pi; K_{0}(n)) = \sum_{\varphi \in B} \mathcal{Z}(1/2, 1, \varphi) Z(1/2, \eta, \varphi),
\]

where \( B \) is an orthonormal basis of \( V_{\pi}^{K \infty K_{0}(n)} \). In this subsection, we examine \( P^{\eta}(\pi; K_{0}(n)) \). Set \( \varphi_{\pi}^{\text{new}} = \varphi_{\pi, \rho_{\pi}} \), where \( \rho_{\pi} \) is a unique element of \( \Lambda_{\mathbb{A}}^{\pi}(f_{\beta}) \).

**Lemma 12.** The value \( P^{\eta}(\pi; K_{0}(n)) \) is independent of the choice of an orthonormal basis of \( V_{\pi}^{K \infty K_{0}(n)} \) and we have

\[
P^{\eta}(\pi; K_{0}(n)) = D_{F}^{-1/2} \mathcal{G}(\eta) w_{\eta}^{\pi}(\pi) L(1/2, \pi) L(1/2, \pi \otimes \eta) \| \varphi_{\pi}^{\text{new}} \|^{2}.
\]

Here \( w_{\eta}^{\pi}(\pi) \) is an explicit nonnegative constant defined as

\[
w_{\eta}^{\pi}(\pi) = \prod_{k=1}^{n} \prod_{v \in S(\pi)} r(\pi_{v}, \eta, k) = \prod_{v \in S(\pi)} r(\pi_{v}, \eta, \text{ord}_{v}(n_{\pi}^{-1}))
\]

where \( r(\pi_{v}, \eta, k) \) is defined as follows:

- If \( c(\pi_{v}) \geq 2 \), then \( r(\pi_{v}, \eta, k) = \begin{cases} k + 1 & \text{(if } \eta_{v}(\varpi_{v}) = 1), \\ 2^{-1}(1 + (-1)^{k}) & \text{(if } \eta_{v}(\varpi_{v}) = -1). \end{cases} \)

- If \( c(\pi_{v}) = 1 \), then \( \pi_{v} \) is isomorphic to \( \sigma(\chi_{v} \cdot \varpi_{v}^{1/2}, \chi_{v} \cdot \varpi_{v}^{-1/2}) \) for some unramified character \( \chi_{v} \) of \( F_{v}^{\times} \). Then

\[
r(\pi_{v}, \eta, k) = \begin{cases} 1 + \frac{1 - \chi_{v}(\varpi_{v})}{1 + \chi_{v}(\varpi_{v})} q_{v}^{-1} & \text{(if } \eta_{v}(\varpi_{v}) = 1), \\ 2^{-1}(1 + (-1)^{k}) & \text{(if } \eta_{v}(\varpi_{v}) = -1). \end{cases}
\]

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- If \( c(\pi_v) = 0 \) and \( (\alpha_v, \alpha_v^{-1}) \) is the Satake parameter of \( \pi_v \), then

\[
r(\pi_v, \eta_v, k) = \begin{cases} 
\frac{2}{1 + Q(\pi_v)} + \frac{1 - Q(\pi_v)}{1 + Q(\pi_v)} q_v + 1 (k - 1) & \text{if } \eta_v(\varpi_v) = 1, \\
\frac{q_v + 1 + (-1)^k}{q_v - 1} & \text{if } \eta_v(\varpi_v) = -1,
\end{cases}
\]

where \( Q(\pi_v) = (\alpha_v + \alpha_v^{-1})(q_v^{1/2} + q_v^{-1/2})^{-1} \).

Moreover, \( \mathcal{A}(\eta)^{-1} \mathcal{P}^0(\pi; \mathbb{K}_0(n)) \) is nonnegative.

**Proof.** The first assertion is obvious. Thus we may take \( \{||\varphi_{\pi, \varphi}||^{-1} \varphi_{\pi, \varphi}\}_{\rho \in \Lambda^0_v(n)} \) as \( B \). By virtue of Proposition 4 we have

\[
\mathcal{P}^0(\pi; \mathbb{K}_0(n)) = \sum_{\rho \in \Lambda^0_v(n)} \frac{1}{||\varphi_{\pi, \varphi}||^2} Z^*(1/2, 1, \varphi) Z^*(1/2, \eta, \varphi)
\]

where \( Z^*(1/2, \eta, \varphi) \) is defined by the local Rankin-Selberg integral.

Then, we obtain the second assertion by setting

\[
w_{n}^{\eta}(\pi) = \sum_{\rho \in \Lambda^0_v(n)} \prod_{k=1}^{n} \prod_{v \in S_k(\pi_v^{-1})} \left\{ \frac{Q_{\rho_k(v)}(\pi_v, 1)}{\tau_{\pi_v}(\rho_k(v))} \right\}.
\]

Here \( \tau_{\pi_v}(j, j) = ||\phi_{v, j}||^2 \) for \( k \in \mathbb{N} \), where \( || \cdot ||_v \) is the norm on \( V_{\pi_v} \) defined by the \( G_v \)-invariant inner product normalized so that \( ||\phi_{v, v}|| = 1 \). We remark that an explicit formula of \( \tau_{\pi_v}(j, j) \) was given in [10] Corollaries 12 and 16. By definition and a direct computation, we have

\[
w_{n}^{\eta}(\pi) = \prod_{k=1}^{n} \left\{ \sum_{(j_v) \in \{0, \ldots, k\}^n} \prod_{v \in S_k(\pi_v^{-1})} r_{v, j_v} \right\} = \prod_{k=1}^{n} \prod_{v \in S_k(\pi_v^{-1})} \sum_{j=0}^{k} r_{v, j}
\]

and \( \sum_{j=0}^{k} r_{v, j} = r(\pi_v, \eta_v, k) \), where \( r_{v, j} = \frac{Q_{\rho_k(v)}^{\tau_{\pi_v}}(\eta_v, 1)}{\tau_{\pi_v}(\rho_k(v))} \).

Then, one can easily obtain \( w_{n}^{\eta}(\pi) \in \mathbb{R}_{\geq 0} \) by noting \( |Q(\pi_v)| < 1 \) when \( c(\pi_v) = 0 \). Since the estimate \( L(1/2, \pi)L(1/2, \pi \otimes \eta) \geq 0 \) holds by [3], we can prove the lemma. \( \square \)

Since \( \eta^2 = 1 \), we have \( \tilde{\eta}(n) = \pm 1 \). We consider only the case of \( \tilde{\eta}(n) = 1 \) because of the following reason.

**Lemma 13.** Let \( \pi \) be a \( \mathbb{K}_\infty \)-spherical irreducible cuspidal automorphic representation of \( G_\mathbb{A} \) with trivial central character. Let \( \eta \) be a character of \( F^\times \backslash \mathbb{A}^\times \) such that \( \eta^2 = 1 \) and \( \mathfrak{f}_\eta \) is relatively prime to \( \mathfrak{f}_\pi \).

Suppose that \( \eta_v(-1) = 1 \) for any \( v \in \Sigma_\infty \). Then, \( L(1/2, \pi)L(1/2, \pi \otimes \eta) \leq 0 \) unless \( \tilde{\eta}(\mathfrak{f}_\pi) = 1 \).

**Proof.** By the argument in the proof of [12] Lemma 2.3, it is enough to show \( \epsilon(1/2, \pi_v, \psi_{v_F}) \epsilon(1/2, \pi_v \otimes \eta_v, \psi_{F_v}) = \eta_v(\varpi_v^{c(\pi_v)}) \) for any \( v \in \bigcup_{k \geq 2} S_k(\mathfrak{f}_\pi) \). It follows immediately from fundamental properties of \( \epsilon \)-factors (cf. [9] 1.1). We note that \( \eta_v \) is unramified if \( v \in S(\mathfrak{f}_\pi) \). \( \square \)

2.7: Adjoint \( L \)-functions. Let \( \pi \) be a cuspidal automorphic representation of \( G_\mathbb{A} \) contained in \( \Pi_{\text{cusp}}(n) \). We examine an explicit description of \( ||\varphi_{\pi}^{\text{new}}||^2 \) in terms of the adjoint \( L \)-function of \( \pi \) by computing the Rankin-Selberg integral. For any \( v \in \Sigma_F \), we denote by \( Z_v(s) \) the local Rankin-Selberg integral

\[
\int_{\mathbb{K}_v} \int_{F_v^c} \phi_{0, v} \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} k_v \right) \phi_{0, v} \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} k_v \right) |k_v|^{-1} d^* t_v d k_v.
\]
Let \( \rho_0 \) denote a unique element of \( \Lambda_1(\mathfrak{n}) \).

**Lemma 14.** Set \( S_\pi = \{ v \in \Sigma_\infty \mid \text{ord}_v(f_\pi) \geq 2 \} \). We have

\[
\int_{Z_v G_F \backslash G_\mathfrak{a}} \varphi_v^{\text{new}}(g) \varphi_\sigma^{\text{new}}(g) E_1, \rho_0(2s - 1, g) dg
= [K_{\mathfrak{fin}} : K_0(f_\pi)]^{-1} N(f_\pi)^s D_F^{s - 3/2} \zeta_F(2s - 1)^{-1} \zeta_F(s) L(s, \pi, Ad) \prod_{v \in S_\pi} \frac{q_v^{a_v(3/2 - s)} q_v^{e(\pi_v)(1 - s)} Z_v(s)}{1 + q_v^{-1}} \frac{L(s, \pi_v, Ad)}{1 + q_v^{-s}}
\]

for \( \text{Re}(s) \gg 0 \). Moreover, we have \( ||\varphi_v^{\text{new}}||^2 = 2N(f_\pi)[K_{\mathfrak{fin}} : K_0(f_\pi)]^{-1} L^{s^*}(1, \pi; Ad) \).

**Proof.** If \( v \in \Sigma_F - S_\pi \), then \( Z_v(s) \) is computed in [12, Lemma 2.14]. Hence, it suffices to examine \( Z_v(s) \) when \( v \in S_\pi \). By \( [K_v : K_0(p_{\pi_v}(\pi_v))] = q_v^{e(\pi_v)}(1 + q_v^{-1}) \), we obtain the first assertion.

We note \( Z_v(1) = q_v^{-d/v^2} \) for \( v \in S_\pi \). Then we obtain the second assertion by taking the residue at \( s = 1 \) since \( \text{Res}_{s=1} E_1, \rho_0(s, g) = D_F^{-1/2} R_F \zeta_F(2)^{-1} \) holds by Proposition 10.

3. Adelic Green functions

We define the adelic Green function on \( G_\mathfrak{a} \) associated to an ideal \( \mathfrak{n} \) of \( \mathfrak{f}_F \). This was introduced in [12] in the case where \( \mathfrak{n} \) is square free. We define the function in the case where \( \mathfrak{n} \) is an arbitrary ideal of \( \mathfrak{f}_F \).

For \( v \in \Sigma_\infty \), the Green function on \( GL(2, F_v) \), denoted by \( \Psi_v^{(2)}(s, -) \), is a function with the following property.

**Proposition 15.** [12 §4] Suppose \( v \in \Sigma_\infty \). We fix \( s \in \mathbb{C} \) and \( z \in \mathbb{C} \). Let \( f : G_v \to \mathbb{C} \) be a smooth function such that \( f \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} gk \right) = |t_1/t_2| c f(g) \) for any \( \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right) \in H_v, k \in K_v \) and \( g \in G_v \).

Suppose

\[
\sum_{m=0}^{2} \begin{vmatrix} d^m/dr^n f & \cosh r \\ \sinh r & \cosh r \end{vmatrix} \ll (\cosh 2r)^{|\text{Re}(z)|}, \quad r \in \mathbb{R}.
\]

If \( \text{Re}(s) > 2|\text{Re}(z)| + 1 \), then the equality

\[
\int_{H_v \backslash G_v} \Psi_v^{(2)}(s, g) [R(\Omega_v - (s^2 - 1)/2)f](g) dg = f(e)
\]

holds with the integral being convergent absolutely. Here \( \Omega_v \) denotes the Casimir operator of \( G_v \cong GL(2, \mathbb{R}) \).

For \( v \in \Sigma_\mathfrak{fin} \), let \( T_v \) denote the characteristic function of \( K_v \left( \begin{smallmatrix} \omega_v & 0 \\ 0 & 1 \end{smallmatrix} \right) K_v \) on \( G_v \) divided by \( \text{vol}(K_v, dg_v) \).

The function \( T_v \) is an element of the spherical Hecke algebra \( \mathcal{H}(G_v, K_v) \) and is called the \( v \)-th Hecke operator. The Green function on \( GL(2, F_v) \), also denoted by \( \Psi_v^{(2)}(s, -) \), is a function with the following property.

**Proposition 16.** [12 §5] Suppose \( v \in \Sigma_\mathfrak{fin} \). We fix \( s \in \mathbb{C} \) and \( z \in \mathbb{C} \) such that \( \text{Re}(s) > 2|\text{Re}(z)| \). Let \( f : G_v \to \mathbb{C} \) be a smooth function such that \( f \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} gk \right) = |t_1/t_2|^{c} f(g) \) for any \( \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right) \in H_v, k \in K_v \) and \( g \in G_v \). Then, the equality

\[
\int_{H_v \backslash G_v} \Psi_v^{(2)}(s, g) [R(T_v - (q_v^{(1-s)/2} + q_v^{(1+s)/2})1_{K_v})f](g) dg = \text{vol}(H_v \backslash H_v K_v)f(e)
\]

holds as long as the integral on the left hand side converges absolutely. Here \( 1_{K_v} \) is the characteristic function of \( K_v \) on \( G_v \) divided by \( \text{vol}(K_v, dg_v) \).
For any \( v \in S(n) \), we set
\[
\Phi_{n,v}(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}) = |\frac{t_1}{t_2}|^z \delta(x \in \mathcal{O}_v) \delta(k \in K_0(n)_{\mathbb{O}_v})
\]
for any \( t_1, t_2 \in F_v^* \), \( x \in F_v \) and \( k \in K_v \), and put \( \Phi_{0,v}(z) = \Phi_{n,v}(\frac{1}{n}) \) for \( v \in \Sigma_{\text{fin}} - S(n) \).

Set \( X_S = \prod_{v \in \Sigma_v} \mathbb{C} \times \prod_{v \in S_{\text{fin}}(n)} \mathbb{C}/4\pi i(\log q_v)^{-1}Z \), and \( q(s) = \inf_{v \in S}(\text{Re}(s_v) + 1)/4 \) for any \( s \in X_S \). Fix a finite subset \( S \) of \( \Sigma_F \) such that \( \Sigma_{\infty} \subset S \). For any \( s \in X_S \) and \( z \in \mathbb{C} \) such that \( q(s) > |\text{Re}(z)| + 1 \), the adelic Green function is defined by
\[
\Psi(z)(n|s, g) = \prod_{v \in S_{\infty}} \Psi_v(z)(s_v, g_v) \times \prod_{v \in S_{\text{fin}}(n)} \Phi_{n,v}(g_v) \prod_{v \notin S_{\infty} \cup S(n)} \Phi_{0,v}(g_v)
\]
for any \( g = (g_v)_{v \in \Sigma_F} \in G_{A} \). Note that the function \( \Psi(z)(n|s, -) \) on \( G_A \) is right \( K_{\infty}K_0(n) \)-invariant and continuous on \( G_A \). Moreover, we have \( \Psi(z)(n|s; \frac{t_1}{t_2}) = |t_1/t_2|^z \Psi(z)(n|s; g) \) for any \( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in H_A \) and \( g \in G_A \).

To state the property of adelic Green functions, for any \( \varphi \in C_c^\infty(\mathfrak{A}G_F \backslash G_A) \) we consider the integral
\[
\varphi_{H,z}(g) = \int_{\mathfrak{A}H_F \backslash H_A} \varphi(hg) \chi_z(h) dh,
\]
where \( \chi_z : H_F \backslash H_A \to \mathbb{C}^\times \) is defined by
\[
\chi_z \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right) = |t_1/t_2|^z
\]
for any \( t_1, t_2 \in \mathbb{A}^\times \). The integral \( \varphi_{H, z}(g) \) converges absolutely and \( \varphi_{H, z}(h) = \chi_z(h)^{-1}\varphi_{H, z}(g) \) holds for any \( h \in H_A \) (cf. [12 §6.2]).

Let \( \mathfrak{g}(\mathbb{A}_\infty) \) be the center of the universal enveloping algebra of the complexification of the Lie algebra of \( GL(2, F \otimes \mathbb{Q}_\mathbb{R}) \). For \( s \in X_S \), the element \( \Omega_S(s) \) of the algebra \( \mathfrak{g}(\mathbb{A}_\infty) \otimes \{ \otimes_{v \in S_{\text{fin}}} \mathcal{H}(G_v, K_v) \} \) is defined as
\[
\Omega_S(s) = \bigotimes_{v \in S_{\infty}} \left( \Omega_v - \frac{s_v^2 - 1}{2} \right) \bigotimes_{v \in S_{\text{fin}}} \left( T_v - (q_v^{1-s_v}/2 + q_v^{1+s_v}/2)1_{K_v} \right).
\]

The following proposition is proved in a similar way to [12 Lemma 6.3].

**Proposition 17.** Suppose \( q(s) > 2|\text{Re}(z)| + 1 \). Then, for any \( \varphi \in C_c^\infty(\mathfrak{A}G_F \backslash G_A)K_{\infty}K_0(n) \), the function \( g \mapsto \Psi(z)(n|s, g) \varphi_{H, z}(g) \) is integrable on \( H_{\text{fin}} \backslash H_A \) and we have
\[
\int_{H_{\text{fin}} \backslash H_A} \Psi(z)(n|s, g) |R(\Omega_S(s))\varphi_{H, z}(g)| dg = \text{vol}(H_{\text{fin}} \backslash H_A K_0(n)) \varphi_{H, z}(c).
\]

4. Spectral expansions of renormalized Green functions

The set \( X_S = \prod_{v \in \Sigma_F} \mathbb{C} \times \prod_{v \in S_{\text{fin}}(n)} \mathbb{C}/4\pi i(\log q_v)^{-1}Z \) is considered as a complex manifold with respect to a usual complex structure. Let \( \mathcal{S}_S \) be the space of holomorphic functions \( \alpha(s) \) on \( X_S \) such that for any \( v \in S \) and \( s' \in X_S - \{ v \} \), the function \( s_v \mapsto \alpha(s', s_v) \) is contained in \( \mathcal{S}_S \).

For \( c \in \mathbb{R}^S \), we put \( L_S(c) = \{ s \in X_S \mid \text{Re}(s) = c \} \). A multidimensional contour integral of a holomorphic function \( f(s) \) on \( X_S \) along \( L_S(c) \) is defined as in [12 §6.1]. Its contour integral is defined inductively as
\[
\int_{L_S(c)} f(s) d\mu_S(s) = \int_{L_{c}(v)} \left\{ \int_{L_{S - \{ v \}}(c')} f(s', s_v) d\mu_{S - \{ v \}}(s') \right\} d\mu_v(s_v)
\]
for \( c = (c', c_v) \in \mathbb{R}^S \), where

\[
d\mu_v(s) = \begin{cases} 
    sds & \text{(if } v \in \Sigma_{\infty} \text{)} \\
    \frac{1}{2}(\log q_v)\left(q_v^{(1+s)/2} - q_v^{(1-s)/2}\right)ds & \text{(if } v \in \Sigma_{\text{fin}} \text{)}
\end{cases}
\]

and \( L(c_v) \) stands for \( c_v + i\mathbb{R} \) and \( c_v + \mathbb{C}/4\pi(\log q_v)^{-1}\mathbb{Z} \) for \( v \in \Sigma_{\infty} \) and \( v \in \Sigma_{\text{fin}} \), respectively. Then, for \( c \in \mathbb{R}^S \) and \( z \in \mathbb{C} \) such that \( q(c) > |\text{Re}(z)| + 1 \), the integral

\[
\hat{\Psi}^{(z)}(n|\alpha; g) = \left( \frac{1}{2\pi i} \right)^\#S \int_{L_S(c)} \hat{\Psi}^{(z)}(n|s, g)\alpha(s)d\mu_S(s)
\]

is absolutely convergent and is independent of the choice of \( c \), and the function \( z \mapsto \hat{\Psi}^{(z)}(n|\alpha; g) \) is entire. Furthermore, for \( \beta \in \mathcal{B}, \lambda \in \mathbb{C} \) and \( g \in G_A \), we consider the integral

\[
\hat{\Psi}_{\beta,\lambda}(n|\alpha; g) = \frac{1}{2\pi i} \int_{L_S} \beta(z) \left( \hat{\Psi}^{(z)}(n|\alpha; g) + \hat{\Psi}^{(z^\dagger)}(n|\alpha; g) \right)dz
\]

for \( \sigma \in \mathbb{R} \) such that \(-\inf(g(s) - 1, \text{Re}(\lambda)) < \sigma < q(s) - 1\). The integral of the right hand side is absolutely convergent and is independent of the choice of \( \sigma \). Moreover, for \( \alpha \in \mathcal{A}_S, \beta \in \mathcal{B} \) and \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) > 0 \), the Poincaré series of \( \hat{\Psi}_{\beta,\lambda}(n|\alpha; g) \) is defined to be

\[
\hat{\Psi}_{\beta,\lambda}(n|\alpha; g) = \sum_{\gamma \in H_F \setminus G_F} \hat{\Psi}_{\beta,\lambda}(n|\alpha; \gamma g)
\]

for \( g \in G_A \). We have the following in the same way as [12 Proposition 9.1]:

(1) The series \( \hat{\Psi}_{\beta,\lambda}(n|\alpha; g) \) is absolutely convergent locally uniformly in \( \{\text{Re}(\lambda) > 0\} \times G_A \). Moreover, the function \( \lambda \mapsto \hat{\Psi}_{\beta,\lambda}(n|\alpha; g) \) on \( \text{Re}(\lambda) > 0 \) is holomorphic and the function \( g \mapsto \hat{\Psi}_{\beta,\lambda}(n|\alpha; g) \) on \( G_A \) is continuous, left \( G_F \)-invariant and right \( K_{\infty}K_0(n) \)-invariant.

(2) For \( \text{Re}(\lambda) > 0 \), we have \( \hat{\Psi}_{\beta,\lambda}(n|\alpha; -) \in L^1(\mathcal{A}_FG_F \setminus G_A) \) for any \( l > 0 \) such that \( l(1 - \text{Re}(\lambda)) < 1 \).

Let us compute the spectral expansion of \( \hat{\Psi}_{\beta,\lambda}(n|\alpha; -) \) explicitly. Recall spectral parameters at \( S \) of automorphic forms (cf. [12 9.1.3]). If an automorphic form \( \varphi \) on \( G_A \) satisfies that there exists \( \nu_{\varphi,S} = (\nu_{\varphi,v})_v \in \mathcal{X}_S \) such that

\[
R(\Omega_v)\varphi = \frac{\nu_{\varphi,v}^2 - 1}{2}\varphi
\]

and

\[
R(\mathbb{T}_v)\varphi = (q_v^{(1-\nu_{\varphi,v})/2} + q_v^{(1+\nu_{\varphi,v})/2})\varphi
\]

hold for any \( v \in \Sigma_{\infty} \) and any \( v \in \Sigma_{\text{fin}} \), respectively, then we call \( \nu_{\varphi,S} \) the spectral parameter at \( S \) of \( \varphi \). Set

\[
C(n, S) = (-1)^\#S \text{vol}(H_{\text{fin}} \setminus H_{\text{fin}}K_0(n)).
\]

We remark \( \text{vol}(H_{\text{fin}} \setminus H_{\text{fin}}K_0(n)) = D_F^{-1/2}[K_{\text{fin}} : K_0(n)]^{-1} \). By using Proposition 17 and a similar argument of [12 Lemma 9.4], we have the following.

**Lemma 18.** Assume \( \text{Re}(\lambda) > 1 \). Then, for any automorphic form \( \varphi \) on \( G_A \) with spectral parameter \( \nu_{\varphi,S} \), we have

\[
(\hat{\Psi}_{\beta,\lambda}(n|\alpha; -)|\varphi)_{L^2} = C(n, S)\alpha(\nu_{\varphi,S})P_{\beta,\lambda}^1(\mathbf{v}),
\]

where \( (\cdot|\cdot)_{L^2} \) is the \( L^2 \)-inner product on \( L^2(Z_A GF \setminus G_A) \).

For any character \( \chi \) of \( F^\times \setminus A^\times \) and \( \alpha \in \mathcal{A}_S \), we define the function \( \hat{\alpha}_\chi \) on \( \mathbb{C} \) by

\[
\hat{\alpha}_\chi(\nu) = \alpha(\nu + 2ib(\chi_v))_{v \in S}
\]

and write \( \hat{\alpha}(\nu) \) for \( \hat{\alpha}_1(\nu) \).
Fix an orthonormal basis $\mathcal{B}_{\text{cusp}}(n)$ of $\sum_{\varphi \in \mathcal{B}_{\text{cusp}}(n)} \varphi^\mathcal{K}_\infty K_0(n)$ and let $\mathcal{B}_{\text{res}}(n)$ be the orthonormal system consisting of all functions $\text{vol}(Z_\chi G_F\backslash G_\chi)^{-1/2}$ on $G_\chi$ for any $\chi \in \Xi_0(\varphi_F)$ such that $\chi^2 = 1$. We write $\Lambda(n)$ for $\Lambda_1(n)$. By Lemma 19, the following spectral expansion of $\Psi_{\beta,\lambda}(n;\alpha;g)$ is given in the same way as [12] Lemma 9.6.

Lemma 19. Assume $\text{Re}(\lambda) > 1$. Then we have the expression

$$\Psi_{\beta,\lambda}(n;\alpha;g) = C(n,S) \left\{ \sum_{\varphi \in \mathcal{B}_{\text{cusp}}(n)} \alpha(\nu_{\varphi,S}) P_{\beta,\lambda}(\varphi)(g) + \sum_{\varphi \in \mathcal{B}_{\text{res}}(n)} \alpha(\nu_{\varphi,S}) P_{\beta,\lambda}(\varphi)(g) \right\}$$

$$+ \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \frac{R_F}{8\pi i} \int_{\mathcal{B}} \tilde{\alpha}_\chi(\nu) P_{\beta,\lambda}(E_{\chi,\rho}(\nu, -)) E_{\chi,\rho}(\nu, g) d\nu.$$

The series and integrals in the right-hand side converge absolutely and locally uniformly on $Z_\chi G_F\backslash G_\chi$.

Lemma 20. For any $g \in G_\chi$, the function $\lambda \mapsto \Psi_{\beta,\lambda}(n;\alpha;g)$ on $\text{Re}(\lambda) > 1$ is continued to a meromorphic function on $\text{Re}(\lambda) > -1/2$.

Proof. We give a proof in the same way as [12] Lemma 9.8. Let $\Psi_{\text{cusp}}(\lambda) = \Psi_{\text{cusp}}(\lambda, \alpha, g)$, $\Psi_{\text{res}}(\lambda) = \Psi_{\text{res}}(\lambda, \alpha, g)$ and $\Psi_{\text{ct}}(\lambda) = \Psi_{\text{ct}}(\lambda, \alpha, g)$ be the cuspidal part, the residual part and the Eisenstein part divided by $C(n,S)$ in the spectral expansion of $\Psi_{\beta,\lambda}(n;\alpha;g)$ given in Lemma 19, respectively.

First we examine $\Psi_{\text{res}}(\lambda)$. For $\text{Re}(\lambda) > 0$, by applying Proposition 10, the function $\Psi_{\text{res}}(\lambda)$ is written as

$$\Psi_{\text{res}}(\lambda) = \sum_{\chi \in \Xi(\varphi_F), \chi^2 = 1} \alpha(\nu_{\chi,\chi}) P_{\beta,\lambda}(\overline{\varphi}_{\chi})(g) = 2\tilde{\alpha}(1) \frac{R_F}{\text{vol}(Z_\chi G_F\backslash G_\chi)} \frac{\beta(0)}{\lambda}$$

and has a meromorphic continuation to $\mathbb{C}$. From this, $\text{CT}_{\lambda=0} \Psi_{\text{res}}(\lambda) = 0$ holds.

Next we examine $\Psi_{\text{cusp}}(\lambda)$. By the same computation as the proof of [12] Lemma 9.8, the series $\Psi_{\text{cusp}}(\lambda)$ converges absolutely and the estimate

$$|\Psi_{\text{cusp}}(\lambda, \alpha, g)| \ll y(g)^{-m}, \quad g \in \mathfrak{S} \cap G_\chi^1$$

holds, where $\mathfrak{S}$ denotes a Siegel set of $G_\chi$ such that $G_\chi = G_F \mathfrak{S}$ and $y$ denotes the height function on $G_\chi$. Moreover, $\Psi_{\text{cusp}}(\lambda)$ is analytically continued to an entire function and we have

$$\text{CT}_{\lambda=0} \Psi_{\text{cusp}}(\lambda) = \sum_{\varphi \in \mathcal{B}_{\text{cusp}}(n)} \alpha(\nu_{\varphi,S}) P_{\text{reg}}(\varphi)(g).$$

Therefore, it is enough to examine $\Psi_{\text{ct}}(\lambda)$. The argument is more complicated than that of [12] Lemma 9.8. Assume $\text{Re}(\lambda) > 1$ and $\nu \in i\mathbb{R}$. By the proof of [10] Theorem 37, the integral $P_{\lambda,\beta}(E_{\chi,\nu,\rho}(-\nu, -))$ can be expressed as

$$P_{\lambda,\beta}(E_{\chi,\nu,\rho}(-\nu, -)) = P_{\chi,1}(1, \lambda, -\nu) + D_F^{-1/2} A_{\chi,\nu,\rho}(-\nu) \frac{L(-\nu, \chi^{-2})}{L(1-\nu, \chi^{-2})} P_{\chi,1}(1, \lambda, \nu)$$

$$+ Q_{\chi,1,\nu,\rho}^{+}(1, \lambda, -\nu) + Q_{\chi,1,\nu,\rho}^{-}(1, \lambda, -\nu),$$

where

$$P_{\chi,\pm}(\eta, \lambda, \pm \nu) = \int_{\mathcal{B}} (\pm \nu) \delta_{\chi,\eta} R_F \left\{ \frac{\beta((\pm \nu - 1)/2)}{\lambda - (\pm \nu + 1)/2} + \frac{\beta((\pm \nu + 1)/2)}{\lambda + (\pm \nu + 1)/2} \right\} \right.$$

and

$$Q_{\chi,\pm,\nu,\rho}^{\pm}(\eta, \lambda, -\nu) = \frac{1}{2\pi i} \int_{L_{\pm}} Z^*(-\nu + 1/2, \eta, E_{\chi,\nu,\rho}(-\nu, -)) \frac{\beta(z)}{\lambda + z} dz.$$

We remark $E_{\chi,\nu,\rho}(-\nu, -) = E_{\chi,\nu,\rho}(-\nu, -)$. Furthermore, by the residue theorem, we have

$$P_{\lambda,\beta}(E_{\chi,\nu,\rho}(-\nu, -))$$
Thus we have to consider only $\Phi_1^{\eta}(1, \lambda, -\nu)$ for $\Re(\lambda) > -\sigma$. Thus we express $\Psi_{\alpha \eta}(\lambda)$ as the sum of the following four terms:

$$
\Phi_1(\lambda) = \frac{1}{8\pi i} \sum_{\lambda \in \Lambda(n)} f_1^{(\nu)}(\nu) \int_{\mathbb{R}} \tilde{\alpha}(\nu) \beta((\nu - 1)/2) \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} E_{1, \rho}(\nu; g) d\nu,
$$

$$
\Phi_2(\lambda) = \frac{1}{8\pi i} \sum_{\lambda \in \Lambda(n)} f_2^{(\nu)}(\nu) \int_{\mathbb{R}} \tilde{\alpha}(\nu) D_F^{-1/2} A_{1, \rho}(\nu) \frac{\zeta_F(\nu)}{\zeta_F(1 - \nu)} \beta((\nu + 1)/2)
$$

$$
\times \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} E_{1, \rho}(\nu; g) d\nu,
$$

$$
\Phi_3(\lambda) = \frac{1}{8\pi i} \sum_{\lambda \in \Lambda(n)} \sum_{\rho \in \Lambda_{\lambda}(n)} \int_{\mathbb{R}} \tilde{\alpha}(\nu) Q_{\lambda, \rho}^{(\eta)}(1, \lambda, -\nu) E_{1, \rho}(\nu; g) d\nu,
$$

$$
\Phi_4(\lambda) = -\sum_{\lambda \in \Lambda(n)} \sum_{\rho \in \Lambda_{\lambda}(n)} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \left\{ \frac{\beta((\nu + 1)/2)}{\lambda + (\nu + 1)/2} \right\} E_{1, \rho}(\nu; g) d\nu
$$

By the functional equation

$$
D_F^{-1/2} A_{1, \rho}(\nu) \frac{\zeta_F(\nu)}{\zeta_F(1 - \nu)} E_{1, \rho}(\nu; g) = E_{1, \rho}(\nu, g)
$$

of the Eisenstein series, the following equalities hold:

$$
\Phi_2(\lambda) = \frac{1}{8\pi i} \sum_{\lambda \in \Lambda(n)} f_2^{(\nu)}(\nu) \int_{\mathbb{R}} \tilde{\alpha}(\nu) D_F^{-1/2} A_{1, \rho}(\nu) \frac{\zeta_F(\nu)}{\zeta_F(1 - \nu)} E_{1, \rho}(\nu; g) \beta((\nu + 1)/2)
$$

$$
\times \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} d\nu
$$

$$
= \frac{1}{8\pi i} \sum_{\lambda \in \Lambda(n)} f_2^{(\nu)}(\nu) \int_{\mathbb{R}} \tilde{\alpha}(\nu) E_{1, \rho}(\nu; g) \beta((\nu + 1)/2) \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} d\nu
$$

$$
= \Phi_1(\lambda).
$$

Thus we have to consider only $\Phi_1(\lambda)$, $\Phi_3(\lambda)$ and $\Phi_4(\lambda)$. 

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We take \( c > 1 \). Then \( \Phi_1(\lambda) \) is expressed as

\[
\Phi_1(\lambda) = \frac{1}{8\pi i} \sum_{\nu \in \mathbb{R}} f_1^{(-\nu)}(\nu) \int_{i\mathbb{R}} \hat{\alpha}(\nu)\beta((\nu + 1)/2) \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} E_{1,\rho}(\nu; g) d\nu
\]

\[
= \frac{1}{8\pi i} \sum_{\nu \in \mathbb{R}} f_1^{(-\nu)}(\nu) \int_{i\mathbb{R}} \hat{\alpha}(\nu)\beta((\nu + 1)/2) \frac{1}{\lambda + (\nu + 1)/2} E_{1,\rho}(\nu; g) d\nu
\]

\[
+ \frac{1}{8\pi i} \sum_{\nu \in \mathbb{R}} f_1^{(-\nu)}(\nu) \left\{ \int_{L_c} \hat{\alpha}(\nu)\beta((\nu + 1)/2) \frac{1}{\lambda - (\nu + 1)/2} E_{1,\rho}(\nu; g) d\nu
\]

\[
- 2\pi i\hat{\alpha}(0)\beta(0) \frac{2}{\lambda} \epsilon_{1,\rho,-1}(g) \right\}.
\]

The first term is holomorphic on \( \text{Re}(\lambda) > -1/2 \), the second term is holomorphic on \( \text{Re}(\lambda) > (-c + 1)/2 \) and the third term is holomorphic on \( \mathbb{C} - \{0\} \). Hence \( \Phi_1(\lambda) = \Phi_2(\lambda) \) has a meromorphic continuation to \( \text{Re}(\lambda) > -1/2 \). Since \( \Phi_3(\lambda) \) is described as an absolutely convergent double integral, \( \Phi_3(\lambda) \) has an analytic continuation to \( \mathbb{C} \). We note that the integral \( Q^0_{\chi-1,\rho}(1,\lambda,-\nu) \) is absolutely convergent and is entire as a function in \( \lambda \). In order to examine \( \Phi_4(\lambda) \), we consider the following residues:

\[
\text{Res}_{z=(\nu+1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) = D_{F}^{-1/2+\nu/2} \mathcal{N}(\chi_{1})^{1/2+\nu} B_{\chi-1,\rho}^{1}(\nu/2,\nu) L(-\nu,\chi^{-1}) L(1-\nu,\chi^{-2}) D_{F}^{1/2} R_{F},
\]

\[
\text{Res}_{z=(-\nu+1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) = D_{F}^{-1/2+\nu/2} \mathcal{N}(\chi_{1})^{1/2+\nu} B_{\chi-1,\rho}^{1}(\nu/2,\nu) L(-\nu,\chi^{-1}) L(1-\nu,\chi^{-2}) D_{F}^{1/2} R_{F},
\]

\[
\text{Res}_{z=(\nu-1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) = D_{F}^{-1/2+\nu/2} \mathcal{N}(\chi_{1})^{1/2+\nu} B_{\chi-1,\rho}^{1}(1-\nu/2,\nu) L(1-\nu,\chi^{-1}) L(1-\nu,\chi^{-2}) D_{F}^{1/2} R_{F},
\]

The functions \( \text{Res}_{z=(\pm\nu+1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) \) are holomorphic on \( i\mathbb{R} \) as functions in \( \nu \) and vanish unless \( \chi = 1 \). Therefore, the integral

\[
\int_{i\mathbb{R}} \left\{ \frac{\beta((\nu + 1)/2)}{\lambda + (\nu + 1)/2} \text{Res}_{z=(\nu+1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu
\]

is holomorphic on \( \text{Re}(\lambda) > -1/2 \).

Consider the integral

\[
\int_{i\mathbb{R}} \left\{ \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} \text{Res}_{z=(\nu+1)/2} f_1^{1}(\chi_{-1,\rho},-z,-\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu
\]

Set \( F_{\rho}^{+}(\nu) = \text{Res}_{z=(\nu-1)/2} f_1^{1}(\rho,-z,-\nu) \). We note that \( F_{\rho}^{+}(\nu) \) is entire. By taking \( c > 1 \), we obtain

\[
\int_{i\mathbb{R}} \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} \text{Res}_{z=(\nu-1)/2} f_1^{1}(\rho,-z,-\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu
\]

\[
- \frac{\beta(0)}{\lambda} F_{\rho}^{+}(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu - 2\pi i \frac{\beta(0)}{\lambda} F_{\rho}^{+}(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu - 2\pi i \frac{\beta(0)}{\lambda} F_{\rho}^{+}(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu; g) d\nu.
\]

The first term of the right hand side is holomorphic on \( \text{Re}(\lambda) > (-c + 1)/2 \) and the second term is meromorphic on \( \mathbb{C} \). Set \( F_{\rho}^{-}(\nu) = \text{Res}_{z=(\nu+1)/2} f_1^{1}(\rho,-z,-\nu) \). By the relation \( B_{1,\rho}(1 - \nu/2,\nu) =
\]
Lemma 21. Let \( \Re(\lambda) > -(c+1)/2 \) and the second term is meromorphic on \( \mathbb{C} \). Hence we can prove that \( \Phi_1(\lambda) \) has a meromorphic continuation to \( \Re(\lambda) > -1/2 \). This gives us a meromorphic continuation of \( \Psi_{\chi_1}(\lambda) \) to \( \Re(\lambda) > -1/2 \).

\[ B_{1,\rho}^1(1-\nu/2,\nu)A_{1,\rho}(-\nu), \text{ we have} \]

\[ F_\rho^-(\nu)D_F^{1/2}A_{1,\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} = B_{1,\rho}^1(1-\nu/2,\nu)D_F^{1/2}(-R_F)D_F^{1/2}A_{1,\rho}(-\nu) = F_\rho^+(\nu), \]

and hence, we obtain

\[
\int_{\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda + (\nu -1)/2} F_\rho^-(\nu)\hat{\alpha}(\nu)E_{1,\rho}(\nu, g) d\nu = \int_{\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda + (\nu -1)/2} F_\rho^-(\nu)\hat{\alpha}(\nu)E_{1,\rho}(-\nu, g) d\nu
\]

\[
= \int_{\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda + (\nu -1)/2} F_\rho^-(\nu)D_F^{1/2}A_{1,\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} \hat{\alpha}(\nu)E_{1,\rho}(\nu, g) d\nu
\]

\[
= \int_{\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda + (\nu -1)/2} F_\rho^+(\nu)\hat{\alpha}(\nu)E_{1,\rho}(\nu, g) d\nu - 2\pi i \frac{\beta(0)}{\lambda} F_\rho^+(1)\hat{\alpha}(1)\alpha_{1,\rho,-1}(g).
\]

Then, in the last line of the equalities above, the first term is holomorphic on \( \Re(\lambda) > -(c+1)/2 \) and the second term is meromorphic on \( \mathbb{C} \). Hence we can prove that \( \Phi_1(\lambda) \) has a meromorphic continuation to \( \Re(\lambda) > -1/2 \). This gives us a meromorphic continuation of \( \Psi_{\chi_1}(\lambda) \) to \( \Re(\lambda) > -1/2 \).

Lemma 21. We have

\[ \text{CT}_{\lambda=0} \Psi_{\chi,\lambda}(n; \alpha; g) = C(n, S) \left\{ \sum_{\varphi \in \mathcal{B}_{\mathcal{A}_n}(n)} \alpha(n, \delta) \frac{P_{\text{reg}}(\varphi)}{P_{\text{reg}}(\varphi)} \varphi(g) \right. \]

\[ + \sum_{\nu \in \mathcal{A}_n} \sum_{\rho \in \mathcal{A}_n} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\nu(\nu)P_{\text{reg}}(E_{\chi^{-1},\nu}(-\nu, -)) E_{\chi,\rho}(\nu, g) d\nu \]

\[ + \sum_{\rho \in \mathcal{A}(n)} \{ f_{1,\rho}^0(e) + \delta(S(\rho) = 0) \} \left\{ \hat{\alpha}(1)\alpha_{1,\rho,-1}(g) + \hat{\alpha}(1)\alpha_{1,\rho,0}(g) \right\} \beta(0). \]

Proof. In the proof of Lemma 20 we gave the constant terms of the cuspidal and residual parts at \( \lambda = 0 \). Therefore, it is enough to evaluate the constant term of the Eisenstein part \( \Psi_{\chi_1}(\lambda) = 2\Phi_1(\lambda) + \Phi_3(\lambda) + \Phi_4(\lambda) \). By the residue theorem, we have

\[ \text{CT}_{\lambda=0} \Phi_1(\lambda) = \frac{1}{8\pi i} \sum_{\rho \in \mathcal{A}(n)} f_{1,\rho}^0(e) \int_{\mathbb{R}} \hat{\alpha}(\nu)\beta((\nu-1)/2) \frac{-1}{(\nu-1)/2} E_{1,\rho}(\nu, g) d\nu \]

\[ + \frac{1}{8\pi i} \sum_{\rho \in \mathcal{A}(n)} f_{1,\rho}^0(e) \int_{\mathbb{R}} \hat{\alpha}(\nu)\beta((\nu-1)/2) \frac{1}{(\nu-1)/2} E_{1,\rho}(\nu, g) d\nu \]

\[ = \frac{1}{8\pi i} \sum_{\rho \in \mathcal{A}(n)} f_{1,\rho}^0(e) \left( \{ \hat{\alpha}'(1)\alpha_{1,\rho,-1}(g) + \hat{\alpha}(1)\alpha_{1,\rho,0}(g) \} \beta(0) \right) \]

and the integral \( Q_{\chi^{-1},\rho}^0(1,0,-\nu) \) is written as

\[ Q_{\chi^{-1},\rho}^0(1,0,-\nu) = \frac{1}{2\pi i} \int_{L_\rho} \left\{ f_{\chi^{-1},\rho}^0(z, -\nu) + f_{\chi^{-1},\rho}^1(-z, -\nu) \right\} \frac{\beta(z)}{z} dz \]

\[ = f_{\chi^{-1},\rho}^1(0, -\nu)\beta(0) + \{ \text{Res}_{z=1+\nu}/2 + \text{Res}_{z=1-\nu}/2 + \text{Res}_{z=-1+\nu}/2 \} \]
Thus the constant term of $\Phi_3(\lambda)$ is evaluated as
\[
\text{CT}_{\lambda=0}\Phi_3(\lambda) = -\frac{R^{-1}}{8\pi i} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) \frac{\nu}{(\nu+1)^2} \frac{\beta((\nu+1)/2)}{(\nu+1)/2} \left( \text{Res}_{\nu=0} + \text{Res}_{\nu=1} \right) (\nu) E_{\chi,\rho}(\nu, g) d\nu
\]
\[
+ \frac{R^{-1}}{8\pi i} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) f^{1,-}_\chi(0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \beta(0) + \int_{\mathbb{R}} \text{Res}_{z=(-1+\nu)/2} \left\{ f^{1,-}_\chi(z, -\nu) \frac{\beta(z)}{z} \right\} \hat{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu.
\]
We examine the constant term of $\Phi_4(\lambda)$. By the expression of $\Phi_4(\lambda)$ given in the proof of Lemma 20, we have
\[
\text{CT}_{\lambda=0}\Phi_4(\lambda) = -\frac{R^{-1}}{8\pi i} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) \frac{\nu}{(\nu-1)^2} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \left( \text{Res}_{\nu=0} + \text{Res}_{\nu=1} \right) (\nu) E_{\chi,\rho}(\nu, g) d\nu
\]
\[
- 2 \frac{R^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{\mathbb{R}} \frac{\nu}{(\nu-1)^2} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F^{+}_\rho(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu.
\]
Therefore we obtain
\[
\text{CT}_{\lambda=0}(\Phi_3(\lambda) + \Phi_4(\lambda)) = \frac{R^{-1}}{8\pi i} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) f^{1,-}_\chi(0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \beta(0)
\]
\[
+ \int_{\mathbb{R}} \text{Res}_{\nu=0} \left\{ f^{1,-}_\chi(z, -\nu) \frac{\beta(z)}{z} \right\} \hat{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu
\]
\[
- 2 \frac{R^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{\mathbb{R}} \frac{\nu}{(\nu-1)^2} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F^{+}_\rho(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu.
\]
By noting the relation
\[
\int_{\mathbb{R}} F^+_\rho(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu = \int_{\mathbb{R}} F^-_\rho(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu,
\]
we have
\[
\frac{R^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{\mathbb{R}} \text{Res}_{\nu=0} \left\{ f^{1,-}_\rho(z, -\nu) \frac{\beta(z)}{z} \right\} \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu
\]
\[
- 2 \frac{R^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{\mathbb{R}} \frac{\nu}{(\nu-1)^2} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F^+_\rho(\nu) \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu
\]
\[
= \frac{R^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{\mathbb{R}} \left\{ F^+_\rho(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} + F^-_\rho(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \right\} \hat{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu.
\]
\[ -2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{L_c} \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} F^+_\rho(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \]

\[ = 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{i\mathbb{R}} F^+_\rho(\nu) \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \]

\[ - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{L_c} \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} F^+_\rho(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \]

\[ = \frac{R_F^{-1}}{4\pi i} \sum_{\rho \in \Lambda(n)} (-2\pi i) \text{Res}_{\nu=1} \left\{ \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} F^+_\rho(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) \right\}. \]

Here the residue is expressed as

\[ \text{Res}_{\nu=1} \left\{ \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} F^+_\rho(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) \right\} \]

\[ = \text{Res}_{\nu=1} \left\{ \frac{\beta((\nu - 1)/2)}{(\nu - 1)^{1/2}} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) D_F^{-1/2 + \nu/2} B_{\chi, \rho}^{1/2} (1 - \nu/2, -\nu)(-R_F) \right\} \]

\[ = \{ 2 \tilde{\alpha}(1)e_{\rho, -1}(g) + 2 \tilde{\alpha}(1)e_{\rho, 0}(g) \} \delta(S(\rho) = \emptyset)(-R_F) \beta(0). \]

We note \( D^{(\nu-1)/2} B^\eta_{\rho}(1 - \nu/2, -\nu) = \tilde{\eta}(D_{\rho}/Q) \delta(S(\rho) = \emptyset) \) for any \( \eta \in \Xi_0(s_F) \) satisfying \( \eta^2 = 1 \). Therefore we obtain

\[ \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(n)} \int_{i\mathbb{R}} \{ \text{Res}_{\nu=1-\nu/2} + \text{Res}_{\nu=-1-\nu/2} \} \left\{ f_{1,\rho}^1(z, -\nu) \frac{\beta(z)}{z} \right\} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \]

\[ = \frac{R_F^{-1}}{4\pi i} \sum_{\rho \in \Lambda(n)} 2\pi i \{ 2 \tilde{\alpha}(1)e_{\rho, -1}(g) + 2 \tilde{\alpha}(1)e_{\rho, 0}(g) \} \delta(S(\rho) = \emptyset) R_F \beta(0) \]

\[ = \sum_{\rho \in \Lambda(n)} \delta(S(\rho) = \emptyset) \{ \tilde{\alpha}(1)e_{\rho, -1}(g) + \tilde{\alpha}(1)e_{\rho, 0}(g) \} \beta(0), \]

and hence

\[ \text{CT}_{\lambda=0} \Psi_{\lambda}(\lambda) = \frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) f_{\chi, \rho}^1(0, -\nu) E_{\chi, \rho}(\nu, g) d\nu \beta(0) \]

\[ + \sum_{\rho \in \Lambda(n)} \{ f_{1,\rho}^0(e) + \delta(S(\rho) = \emptyset) \} \{ \tilde{\alpha}(1)e_{\rho, -1}(g) + \tilde{\alpha}(1)e_{\rho, 0}(g) \} \beta(0). \]

This gives the expression of \( \text{CT}_{\lambda=0} \Psi_{\beta, \lambda}(\lambda|\alpha; g) \).

Then, we define the regularized smoothed kernel \( \tilde{\Psi}_{\text{reg}}(n|\alpha; g) \) by the relation

\[ \text{CT}_{\lambda=0} \tilde{\Psi}_{\beta, \lambda}(n|\alpha; g) = \tilde{\Psi}_{\text{reg}}(n|\alpha; g) \beta(0), \quad \beta \in \mathcal{B}. \]

Let \( \mathcal{S} \) be a Siegel set of \( G_\mathbb{A} \) such that \( G_\mathbb{A} = G_F \mathcal{S} \). We have the following estimate of \( \tilde{\Psi}_{\text{reg}}(n|\alpha; g) \).

**Lemma 22.** There exists \( N \in \mathbb{N} \) such that for any \( m \in \mathbb{N} \), the following estimates hold for any \( g \in G_\mathbb{A}^{\text{reg}} \cap \mathcal{S} \) uniformly.

1. \( \sum_{\varphi \in \mathcal{S}_{\text{cusp}}(n)} |\alpha(\nu, \varphi)| P_{\text{reg}}(\varphi)(\varphi(g)) \ll y(g)^{-m}, \)

2. \( \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} |\tilde{\alpha}_\chi(\nu) P_{\text{reg}}^1(E_{\chi, \rho}(-\nu, -)) E_{\chi, \rho}(\nu, g)| d\nu \ll y(g)^N, \)
Lemma 24. The function $\lambda \mapsto P_{\beta,\lambda}^\eta(\Psi_{\text{reg}}(n|\alpha; g))$ on $\text{Re}(\lambda) > N$ is analytically continued to a meromorphic function on $\mathbb{C}$. Its constant term at $\lambda = 0$ is given by

$$\text{CT}_{\lambda=0} P_{\beta,\lambda}^\eta(\Psi_{\text{reg}}(n|\alpha; g)) = \sum_{\rho \in \Lambda(n)} \left\{ f_{1,\rho}^{(0)}(e) + \delta(S(\rho) = \emptyset) \right\} \tilde{\alpha}_1(1) \frac{D_F^{-1/2}}{\zeta_F(2)}$$

$$\times \left\{ -\frac{1}{2} \delta_{n,1} \tilde{B}_1^1(0) R_F^{2} \beta''(0) + a_n^\eta(0) \beta(0) \right\}.$$ 

Proof. There exists a positive constant $C$ such that $L_{\text{fin}}(s, \chi)$ does not vanish for any non-quadratic character $\chi$ of $F^\times \setminus \mathbb{A}^\times$ if $\text{Re}(s) \geq 1 - C/\log(q(\chi)(3 + |\text{Im}(s)|))$ (cf. [5, Theorem 5.10]). Hence, by virtue of the proof of [11, Theorem 3.11], the estimate

$$\frac{1}{|L_{\text{fin}}(1, \chi)|} \ll \log q(\chi)$$

holds uniformly for non-quadratic characters $\chi$ of $F^\times \setminus \mathbb{A}^\times$. Next we give a generalized Siegel’s theorem for quadratic characters of $F^\times \setminus \mathbb{A}^\times$. By [6, Theorem 2.3.1], for any $\epsilon > 0$, the estimate

$$|L_{\text{fin}}(1, \chi)| \gg q(\chi)^{-\epsilon}$$

holds uniformly for quadratic characters $\chi$ of $F^\times \setminus \mathbb{A}^\times$. Indeed, [6, Theorem 2.3.1] works for general $L$-functions over $F$ in the sense of [5].

As a consequence, we have the estimate

$$\frac{1}{|L_{\text{fin}}(1 + \nu, \chi^2)|} \ll q(\chi^2) \cdot |\frac{\nu}{\lambda}|^\epsilon, \quad \nu \in i\mathbb{R}$$

with the implied constant independent of $\chi \in \Xi(n)$ and $n$. Combining this with the argument of the proof of [12, Lemma 9.9], we have the assertions. \qed

5. Periods of regularized automorphic smoothed kernels: spectral side

By (4) in Lemma [22], the integral $P_{\beta,\lambda}^\eta(\Psi_{\text{reg}}(n|\alpha; g))$ converges absolutely for $\text{Re}(\lambda) > N$ and is holomorphic on $\text{Re}(\lambda) > N$. We have the following expression in the same way as [12, Lemma 10.1].

Lemma 23. For $\text{Re}(\lambda) > N$, we have the expression

$$P_{\beta,\lambda}^\eta(\Psi_{\text{reg}}(n|\alpha; g)) = C(n, S) \{ P_{\text{cus}}^\eta(\beta, \lambda, \alpha) + P_{\text{fin}}^\eta(\beta, \lambda, \alpha) + P_{\text{res}}^\eta(\beta, \lambda, \alpha) \},$$

where

$$P_{\text{cus}}^\eta(\beta, \lambda, \alpha) = \sum_{\varphi \in \mathcal{M}(n)} \alpha(\varphi, S) P_{\text{reg}}^\eta(\varphi) P_{\beta,\lambda}^\eta(\varphi),$$

and

$$P_{\text{fin}}^\eta(\beta, \lambda, \alpha) = \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \tilde{\alpha}_\nu(\nu) P_{\text{reg}}^\eta(E_{\chi,\nu}^{-1,\rho}(-\nu, -)) P_{\beta,\lambda}^\eta(E_{\chi,\rho}(\nu, -)) d\nu$$

and

$$P_{\text{res}}^\eta(\beta, \lambda, \alpha) = \sum_{\rho \in \Lambda(n)} \{ f_{1,\rho}^{(0)}(e) + \delta(S(\rho) = \emptyset) \} (\tilde{\alpha}_1(1) P_{\beta,\lambda}^\eta(\epsilon_{1,\rho,-1}) + \tilde{\alpha}(1) P_{\beta,\lambda}^\eta(\epsilon_{1,\rho,0})).$$

Here the series converge absolutely and locally uniformly on $\text{Re}(\lambda) > N$.

The value $P_{\text{reg}}^\eta(\beta, \lambda, \alpha)$ is described by Propositions [10] and [11]. Then we have the following.

Lemma 24. The function $\lambda \mapsto P_{\text{reg}}^\eta(\beta, \lambda, \alpha)$ on $\text{Re}(\lambda) > N$ is analytically continued to a meromorphic function on $\mathbb{C}$. Its constant term at $\lambda = 0$ is given by

$$\text{CT}_{\lambda=0} P_{\text{reg}}^\eta(\beta, \lambda, \alpha) = \sum_{\rho \in \Lambda(n)} \left\{ f_{1,\rho}^{(0)}(e) + \delta(S(\rho) = \emptyset) \right\} \tilde{\alpha}_1(1) \frac{D_F^{-1/2}}{\zeta_F(2)}$$

$$\times \left\{ -\frac{1}{2} \delta_{n,1} \tilde{B}_1^1(0) R_F^{2} \beta''(0) + a_n^\eta(0) \beta(0) \right\}.$$
Here $\tilde{B}^n_{\chi,\rho}(z) = \epsilon(-z, \chi^{-1}\eta)B^n_{\chi,\rho}(-z + 1/2, 1)$ and

$$a^n_{1,\rho}(0) = -\frac{1}{2}(B^1_{1,\rho})'(0)\delta_{\eta,1}R^1_{\rho} - 2\tilde{B}^1_{1,\rho}(0)R_F C_1(1)\delta_{\eta,1} + \tilde{B}^1_{1,\rho}(0)C_0(\eta)^2.$$

**Lemma 25.** The function $\lambda \mapsto \mathbb{P}^n_{\text{ein}}(\beta, \lambda, \alpha)\mid \text{Re}(\lambda) > N$ is analytically continued to a meromorphic function on $\text{Re}(\lambda) > -1/2$.

**Proof.** By Proposition [S] we have

$$Z^*(s, \eta, E^\delta_{\chi,\rho}(\nu, -)) = \mathcal{G}(\eta)D^\nu/2\mathcal{N}(f_0^{\chi^{-1}})^{1/2-n}B^\eta_{\chi,\rho}(s, \nu)L(s + \nu/2, \chi\eta)L(s - \nu/2, \chi^{-1}\eta).$$

Set

$$\mathcal{L}^n_{\chi,\rho}(\nu) = D^\nu/2\mathcal{N}(f_0^{\chi^{-1}})^{1/2-n}B^{-1}_{\chi,\rho}(1/2, -\nu)L((1 + \nu)/2, \chi\eta)L((1 - \nu)/2, \chi^{-1}\eta)$$

and recall the expression

$$P^n_{\beta, \lambda}(E_{\chi,\rho}(\nu, -)) = P_\chi(\eta, \lambda, \nu) + D^{-1/2}_F A_{\chi,\rho}(\nu)\frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)}P_{\chi^{-1}}(\eta, \lambda, -\nu) + Q^n_{\chi,\rho}(\eta, \lambda, \nu) + Q^n_{\chi,\rho}(\eta, \lambda, \nu).$$

We remark

$$\mathbb{P}^n_{\text{ein}}(\beta, \lambda, \alpha) = \sum_{\chi \in \Xi(n)}\sum_{\rho \in \Lambda_\chi(n)}\frac{R^{1}_{F}}{8\pi i}\int_{i\mathbb{R}} \tilde{\alpha}(\nu)\mathcal{G}(\chi)(\nu)\left\{P_\chi(\eta, \lambda, \nu) + D^{-1/2}_F A_{\chi,\rho}(\nu)\frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)}P_{\chi^{-1}}(\eta, \lambda, -\nu) + Q^n_{\chi,\rho}(\eta, \lambda, \nu)ight\}d\nu.$$
and $B_{\eta, \rho}(1/2, \nu) = B_{\eta, \rho}(1/2, -\nu) A_{\eta, \rho}(\nu)$, we obtain $\Phi^+_1(\lambda) = \Phi^-_1(\lambda)$. The term $\Phi^+_1(\lambda)$ is expressed as

$$
\Phi^+_1(\lambda) = \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_{\chi, n}(\eta)} \{ \frac{R^{-1}_{\nu}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) f^{(0)}_{\chi, \rho}(e) \delta_{\chi, \rho} R_F \frac{1}{\lambda + (\nu + 1)/2} \beta((\nu + 1)/2) d\nu + \frac{R^{-1}_{\nu}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) f^{(0)}_{\chi, \rho}(e) \delta_{\chi, \rho} R_F \frac{1}{\lambda - (\nu + 1)/2} \beta((\nu + 1)/2) d\nu \}.
$$

Then the first term in the summation is holomorphic on $\text{Re}(\lambda) > -1/2$. For any fixed $\sigma > 1$, the second term in the summation is transformed into

$$
\int_{L_{-\sigma}} \frac{R^{-1}_{\nu}}{8\pi i} D^{-1/2}_{\nu} \left\{ \frac{\hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) f^{(0)}_{\chi, \rho}(e) \delta_{\chi, \rho} R_F}{\lambda - (\nu + 1)/2} \beta((\nu + 1)/2) d\nu + \delta_{\chi, \rho}(\eta) 2\pi i \text{Res}_{\nu = -1} \left( \frac{\hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) f^{(0)}_{\chi, \rho}(e) \delta_{\chi, \rho} R_F}{\lambda - (\nu + 1)/2} \beta((\nu + 1)/2) d\nu \right) \right\}.
$$

The first term in the expression above is meromorphic on $\text{Re}(\lambda) > -\sigma + 1/2$. In order to prove the meromorphicity of the second term in the expression above, we put

$$
D^{\nu/2} L((1 + \nu)/2, \eta) L((1 - \nu)/2, \eta) = \frac{D^\nu_{\nu-2}}{(\nu + 1)^2} + D^\nu_{\nu-1} + D^\nu_0 + O((\nu + 1)), \quad (\nu \to -1),
$$

$$
B_{\eta, \rho}(1/2, -\nu) = p_0(\rho) + p_1(\rho)(\nu + 1) + p_2(\rho)(\nu + 1)^2 + O((\nu + 1)^3), \quad (\nu \to -1)
$$

and

$$
\frac{\beta((\nu + 1)/2)}{\lambda - (\nu + 1)/2} \hat{\alpha}_\chi(\nu) = q_0(\lambda) + q_1(\lambda)(\nu + 1) + O((\nu + 1)^2), \quad (\nu \to -1).
$$

Then these give the following expressions:

$$
\text{Res}_{\nu = -1} \left\{ \frac{\beta((\nu + 1)/2)}{\lambda - (\nu + 1)/2} \hat{\alpha}_\chi(\nu) \right\} = p_0(\rho) q_0(\lambda) D^\nu_{\nu-2} + p_1(\rho) q_1(\lambda) D^\nu_{\nu-1} + p_2(\rho) q_2(\lambda) D^\nu_0,
$$

$$
q_0(\lambda) = \frac{\hat{\alpha}_\chi(0) \beta(0)}{\lambda}, \quad q_1(\lambda) = \left( \frac{\hat{\alpha}_\chi'(0)}{\lambda} + \frac{\hat{\alpha}_\chi(0)}{2\lambda^2} \right) \beta(0).
$$

Therefore $\Phi^+_1(\lambda) = \Phi^-_1(\lambda)$ has a meromorphic continuation to $\text{Re}(\lambda) > -1/2$. Since $\Phi_2(\lambda)$ is described as an absolutely convergent double integral, $\Phi_2(\lambda)$ is entire.

We examine $\Phi_3(\lambda)$. This is written as

$$
\Phi_3(\lambda) = - \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_{\chi, n}(\eta)} \frac{R^{-1}_{\nu}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) \sum_{a = (\pm \nu + 1)/2} \frac{\beta(a)}{\lambda + a} \text{Res}_{z = a} f^{\nu}_n(\lambda) d\nu
$$

$$
+ \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D^{-1/2}_{\chi, \rho}(\nu) \sum_{a = (\pm \nu - 1)/2} \frac{\beta(a)}{\lambda + a} \text{Res}_{z = a} f^{\nu}_n(\lambda) d\nu
.$$
for any fixed $\sigma > 1$. We note $\text{Res}_{\nu=-1} \Phi_{\eta,\rho}^1(\nu) = p_{\nu}^0(\rho)D_{-1}^\nu + p_{\nu}^1(\rho)D_{-2}^\nu$. Thus the part of $a = (\nu - 1)/2$ is meromorphic on $\text{Re}(\lambda) > -1/2$. Noting that the part of $a = (\nu - 1)/2$ equals that of $a = (\nu - 1)/2$, the function $\Phi_{\beta}(\lambda)$ has a meromorphic continuation to $\text{Re}(\lambda) > -1/2$. This completes the proof. □

**Lemma 26.** We have

$$\text{CT}_{\lambda=0} \Phi_{\eta,\rho}^1(\beta, \lambda, \alpha) + \text{CT}_{\lambda=0} \Phi_{\eta,\rho}^\eta(\beta, \lambda, \alpha)$$

$$= \left\{ \mathcal{G}(\eta)D_F^{-1/2}R_F^{-1} \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_{\chi}(n)} \frac{1}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) \mathfrak{g}_\eta^1(\nu) \mathfrak{g}_\eta^\eta(-\nu) d\nu \right.$$  

$$+ \delta(f_\eta) = \mathfrak{G}_F \left\{ Y_2^\eta(n) \hat{\alpha}_\chi^\eta(1) + Y_1^\eta(n) \hat{\alpha}_\chi^\eta(1) + Y_0^\eta(n) \hat{\alpha}_\chi^\eta(1) + Y_{-1}^\eta(n) \hat{\alpha}_\chi^\eta(1) \right\} \beta(0),$$

where we put

$$Y_2^\eta(n) = \sum_{\rho \in \Lambda(n)} D_F^{-1/2} \left\{ f_{\eta,\rho}^0(e) + \delta(S(\rho) = 0) \right\} \frac{1}{2} \hat{\alpha}_\chi^\eta(\nu) D_{-2}^\nu,$$

$$Y_1^\eta(n) = \sum_{\rho \in \Lambda(n)} D_F^{-1/2} \left\{ f_{\eta,\rho}^0(e) + \delta(S(\rho) = 0) \right\} \left\{ D_{-1}^\nu D_{-2}^\nu + D_{-1}^\nu D_{-2}^\nu \right\},$$

$$Y_0^\eta(n) = \sum_{\rho \in \Lambda(n)} D_F^{-1/2} \left\{ f_{\eta,\rho}^0(e) + \delta(S(\rho) = 0) \right\} \left\{ D_{-2}^\nu D_{-2}^\nu + D_{-1}^\nu D_{-2}^\nu + D_{-1}^\nu D_{-2}^\nu \right\}$$

and

$$Y_{-1}^\eta(n) = \sum_{\rho \in \Lambda(n)} \frac{\mathcal{G}(\eta)D_F^{-1/2} \left\{ f_{\eta,\rho}^0(e) + \delta(S(\rho) = 0) \right\}}{\mathfrak{G}_F(2)} a_{1,\rho}^\eta(0).$$

**Proof.** Let $\Phi_1^+, \Phi_2$ and $\Phi_3$ be the functions defined in the proof of Lemma 23. Then, we obtain $\text{CT}_{\lambda=0} \Phi_{\eta,\rho}^1(\beta, \lambda, \alpha) = \text{CT}_{\lambda=0} (2\Phi_1^+ + \Phi_2 + \Phi_3(\lambda))$. A direct computation gives us

$$\text{CT}_{\lambda=0} \Phi_1^+(\lambda) = \delta(f_\eta = \mathfrak{G}_F \sum_{\rho \in \Lambda(n)} \frac{1}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{g}_\eta^1(\nu) f_{\eta,\rho}^0(e) \frac{\beta(0)}{(\nu + 1)^2} d\nu$$

$$+ \delta(f_\eta = \mathfrak{G}_F \sum_{\rho \in \Lambda(n)} \frac{1}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{g}_\eta^1(\nu) f_{\eta,\rho}^0(e) \frac{\beta(0)}{(\nu + 1)^2} d\nu$$

and

$$\text{CT}_{\lambda=0} \Phi_2(\lambda) = \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_{\chi}(n)} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{g}_\eta^1(\nu) Q_{\chi,\rho}^\eta(\eta, 0, \nu) d\nu,$$

where

$$Q_{\chi,\rho}^\eta(\eta, 0, \nu) = f_{\chi,\rho}^\eta(0, \nu) \beta(0) + \sum_{a = (\pm \nu + 1)/2} \text{Res}_{z=a} \left\{ f_{\chi,\rho}^\eta(-z, \nu) \frac{\beta(z)}{z} \right\}.$$ 

The constant term of $\Phi_3(\lambda)$ at $\lambda = 0$ is evaluated as

$$\text{CT}_{\lambda=0} \Phi_3(\lambda)$$

$$= - \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_{\chi}(n)} \frac{R_F^{-1}}{8\pi i} \left\{ \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{g}_\eta^1(\nu) \sum_{a = (\pm \nu + 1)/2} \frac{\beta(a)}{a} \text{Res}_{z=a} f_{\chi,\rho}^\eta(-z, \nu) d\nu \right\}$$

$$- 2\delta(f_\eta = \mathfrak{G}_F \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{8\pi i} \left\{ \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{g}_\eta^1(\nu) \frac{\beta((-\nu - 1)/2)}{(-\nu - 1)^2} \text{Res}_{z=-(\nu - 1)/2} f_{\eta,\rho}^0(-z, \nu) d\nu \right\}.$$
Hence, we obtain
\[
\begin{align*}
\text{CT}_{\lambda=0} & = 2\delta(f_\eta = 0)(e) \sum_{\rho \in \Lambda(n)} \frac{1}{8\pi i} D_F^{-1/2} f_{\eta,\rho}^{(0)}(e) \left( \int_{\mathbb{R}} - \int_{L-\rho} \right) \frac{\alpha_\eta(\nu)\Omega_{\eta,\rho}(\nu) \beta((\nu + 1)/2)}{(\nu + 1)/2} d\nu \\
& + \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \Omega_{\chi,\rho}^1(\nu) f_{\chi,\rho}^\eta(0, \nu) \beta(0) d\nu \\
& + 2\delta(f_\eta = 0)(e) \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{8\pi i} \left( \int_{\mathbb{R}} - \int_{L-\rho} \right) \hat{\alpha}_\eta(\nu) D_F^{-1/2} \Omega_{\chi,\rho}^1(\nu) \frac{\beta((\nu - 1)/2)}{(-\nu - 1)/2} \\
& \times \text{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-\nu, \nu) \\
= & \delta(f_\eta = 0)(e) \sum_{\rho \in \Lambda(n)} \frac{1}{2} D_F^{-1/2} f_{\eta,\rho}^{(0)}(e) \text{Res}_{\nu=-1} \left\{ \hat{\alpha}_\eta(\nu)\Omega_{\eta,\rho}(\nu) \right\} \\
& + \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{8\pi i} \int_{\mathbb{R}} \hat{\alpha}_\chi(\nu) D_F^{-1/2} \Omega_{\chi,\rho}^1(\nu) \Omega_{\eta,\rho}(\nu) \beta((\nu - 1)/2) d\nu \\
& + \delta(f_\eta = 0)(e) \sum_{\rho \in \Lambda(n)} \frac{R_F^{-1}}{2} \text{Res}_{\nu=-1} \left\{ \hat{\alpha}_\eta(\nu) D_F^{-1/2} \Omega_{\eta,\rho}(\nu) \right\} \\
& \times \text{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-\nu, \nu) \\
& \text{We remark} \\
\text{Res}_{\nu=(-\nu-1)/2} f_{\eta,\rho}^\eta(-\nu, \nu) = \mathcal{G}(\eta)(-R_F) D_F^{-\nu/2} B_{\eta,\rho}(\nu/2 + 1, \nu) = -R_F \delta(S(\rho) = 0) \\
\text{and compute the residues as follows:} \\
\text{Res}_{\nu=-1} \left\{ \hat{\alpha}_\eta(\nu)\Omega_{\eta,\rho}(\nu) \right\} = -\text{Res}_{\nu=-1} \left\{ \hat{\alpha}_\eta(\nu)\Omega_{\eta,\rho}(\nu) \right\} \\
& = \hat{\alpha}_\eta(1)p_\eta(\rho)D_{\eta,\rho}^\nu(\rho) + 2\hat{\alpha}_\eta(1)p_\eta(\rho)p_{\eta,\rho}(\rho)D_{\eta,\rho}^\nu(\rho) + 2\hat{\alpha}_\eta(1)p_\eta(\rho)p_{\eta,\rho}(\rho)D_{\eta,\rho}^\nu(\rho) \beta(0) \\
& + \hat{\alpha}_\eta(1) \left\{ 2p_{\eta,\rho}(\rho)\beta(0) + D_{\eta,\rho}^\nu(\rho)\beta(0) + 2D_{\eta,\rho}^\nu(\rho)\beta(0) \right\}.
\end{align*}
\]
One can check that the sum of all terms containing $\beta''(0)$ in $\text{CT}_{\lambda=0}^{\eta, \text{eis}}(\beta, \lambda, \alpha)$ vanishes with the aid of Lemma 24. As a consequence, we obtain the assertion.

**Lemma 27.** For any $\epsilon > 0$, we have the following estimates
\[
|Y_j^n(n)| \ll N(n)^\epsilon, \quad j \in \{1, 0, 1, 2\},
\]
where the implied constant is independent of $n$.

**Proof.** The proof is given by describing $Y_j^n(n)$ for $j \in \{1, 0, 1, 2\}$ explicitly. Since $\eta$ is unramified, we have
\[
f_{\eta,\rho}^{(0)}(e) = \prod_{v \in S_1(\rho)} \eta_v(\varpi_v)q_v^{1/2} \prod_{k=2}^n \prod_{v \in S_k(\rho)} (1 - q_v^{-1}) \eta_v(\varpi_v)k \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{k/2}
\]
and
\[
p_\eta(\rho) = \tilde{\eta}(\mathfrak{D}_F/Q) \prod_{v \in S_1(\rho)} (1 - \eta_v(\varpi_v)) \left( \frac{q_v}{q_v - 1} \right)^{1/2} q_v^{-1/2}
\]
Thus, by noting \( S \), we obtain expressions

\[
p_1^q(\rho) = \hat{\eta}(\mathcal{D}_{F/Q}) \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) - \{w\}} Y_v^q(-1) \right\} (Y_w^q)'(-1)
\]

and

\[
p_2^q(\rho) = \frac{\hat{\eta}(\mathcal{D}_{F/Q})}{2} \sum_{w \in S(\rho)} \left[ \sum_{x \in S(\rho) - \{w\}} \left\{ \prod_{v \in S(\rho) - \{w, x\}} Y_v^q(-1) \right\} (Y_w^q)'(-1)(Y_x^q)'(-1) \right.
\]

\[
\left. + \left\{ \prod_{v \in S(\rho) - \{w\}} Y_v^q(-1) \right\} (Y_w^q)''(-1) \right],
\]

where we set

\[
C_v = \delta(v \in S_1(\rho)) + \delta \left( v \in \prod_{k=2}^n S_k(\rho) \right) \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2}
\]

and

\[
Y_v^q(\nu) = C_v \left( q_v + 1 + \eta_v(\varpi_v)(q_v^{1+\nu}/2 + q_v^{(1-\nu)/2}) \right) - \frac{q_v^{k\nu/2}}{q_v - q_v^\nu}.
\]

Further we have

\[
(Y_v^q)'(-1) = C_v \left( \log q_v \right) q_v^{-k/2 - \eta_v(\varpi_v)(q_v -1)^2/2 + k(1 + \eta_v(\varpi_v))q_v(q_v^2 - 1) + 2(1 + \eta_v(\varpi_v))q_v} \right)
\]

and

\[
(Y_v^q)''(-1) = C_v \left[ \eta_v(\varpi_v)(\log q_v) q_v^{-k/2}(1 + q_v)(q_v - q_v^{-1}) + (1 - q_v)\left( q_v - q_v^{-1} \right) + 2q_v^{-2} \right]
\]

\[
- \frac{4k^2(q_v - q_v^{-1})^2}{4k^2(1 + q_v)(1 - q_v^2)}
\]

\[
+ \frac{4k^2\eta_v(\varpi_v)}{4k^2(1 + q_v)(1 - q_v^2)}q_v^{-k/2}
\]

\[
+ \frac{(1 + \eta_v(\varpi_v))q_v}{k^2(q_v^2 - 1)^2(q_v - 1)} \left\{ \left( \frac{k^2}{4} (q_v^2 - 1) + 1 \right)(q_v^2 - 1)^2 + (k(q_v^2 - 1) + 2)(q_v^2 - 1)^2 \right\}.
\]

Thus, by noting \#S(\rho) \ll \log(1 + N(n))\), we obtain the estimates of \( Y_j^q(n) \) for \( j \in \{0, 1, 2\} \).

Next let us examine \( Y_1^q(n) \). We have the following expressions:

\[
\hat{B}_{1, \rho}(0) = \epsilon(0, \eta) B_{1, \rho}^q (1/2, 1),
\]

\[
B_{1, \rho}^q (1/2, 1) = \prod_{v \in S_1(\rho)} \frac{(\eta_v(\varpi_v) - 1)q_v^{-1/2}}{(1 - q_v^{-1})} \prod_{k=2}^n \prod_{v \in S_k(\rho)} \left\{ \frac{\eta_v(\varpi_v)k(\eta_v(\varpi_v) - 1)(\eta_v(\varpi_v) - q_v^{-1})}{1 - q_v^{-2}} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} \right\},
\]

\[
(\hat{B}_{1, \rho}^q)'(0) = \epsilon'(0, 1) B_{\rho}(0) + 2\epsilon'(0, 1) B_{\rho}'(0) + \epsilon(0, 1) B_{\rho}''(0).
\]
Here we set $B_v(z) = B_{1,v}(-z + 1/2, 1) = D_{F,F}^{-1} \prod_{v \in S(\rho)} B_v(z)$ and
\[
B_v(z) = \delta(v \in S_1(\rho))(q_v^{-1/2} \frac{q_v - 1}{q_v}) + \sum_{k=2}^{n} \delta(v \in S_k(\rho))(q_v^{-k} - q_v^{-k(1/2)^2} + q_v^{-k(2)z-1} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k/2} \frac{1}{1 - q_v^2}.
\]
A direct computation gives us
\[
B_v'(0) = (\log D_{F,F}) \prod_{v \in S(\rho)} B_v(0) + \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) \setminus \{w\}} B_v(0) \right\} B_v'(0),
\]
\[
B_v''(0) = (\log D_{F,F})^2 \prod_{v \in S(\rho)} B_v(0) + 2(\log D_{F,F}) \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) \setminus \{w\}} B_v(0) \right\} B_v'(0)
\]
\[
+ \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) \setminus \{w\}} B_v(0) \right\} B_w'(0) + \left\{ \prod_{v \in S(\rho) \setminus \{w\}} B_v(0) \right\} B_w''(0),
\]
\[
B_v'(0) = \delta(v \in S_1(\rho))(\log q_v) \frac{q_v^{-1/2}}{1 - q_v^{-1}} + \sum_{k=2}^{n} \delta(v \in S_k(\rho))(\log q_v) \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k/2} \frac{1}{1 + q_v^{-2}}
\]
and
\[
B_v''(0) = \delta(v \in S_1(\rho))(\log q_v)^2 \frac{q_v^{-1/2}}{1 - q_v^{-1}}
\]
\[
+ \sum_{k=2}^{n} \delta(v \in S_k(\rho))(\log q_v)^2 \frac{2k - 1 - 3q_v^{-1}}{k^2} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k/2} \frac{1}{1 - q_v^2}.
\]
This completes the proof of the estimate of $Y^\eta_{-1}(n)$.

By the aid of Lemmas 23 and 26, we obtain the expression of the spectral side of $P_{\text{reg}}^\eta(\Psi_{\text{reg}}(n|\alpha; -))$.

**Theorem 28.** The value $P_{\text{reg}}^\eta(\Psi_{\text{reg}}(n|\alpha; -))$ can be defined and we have
\[ P_{\text{reg}}^\eta(\Psi_{\text{reg}}(n|\alpha; -)) = C(n, S)[\mathbb{I}^\eta_{\text{cus}}(n|\alpha) + \mathbb{I}^\eta_{\text{els}}(n|\alpha) + \mathbb{D}^\eta(n|\alpha)]. \]
Here we put
\[ \mathbb{I}^\eta_{\text{cus}}(n|\alpha) = \sum_{\varphi \in \mathcal{M}_{\text{cus}}(n)} \alpha(\varphi, S) \frac{P_{\text{reg}}^1(\varphi)}{P_{\text{reg}}^\eta(\varphi)} P_{\text{reg}}^\eta(\varphi), \]
\[ \mathbb{I}^\eta_{\text{els}}(n|\alpha) = \sum_{\chi \in \Xi(n)} \sum_{\rho \in \Lambda_\chi(n)} \frac{R_{\rho}^{-1}}{8\pi i} \int_{\mathbb{R}} \alpha_\chi(\nu) P_{\text{reg}}^1(E_{\chi^{-1}, \rho}(-\nu, -)) P_{\text{reg}}^\eta(E_{\chi^{-1}, \rho}(\nu, -)) d\nu \]
and
\[ \mathbb{D}^\eta(n|\alpha) = \delta(\eta = \sigma_F)\{Y_2^\eta(n) \hat{\alpha}_\eta'(1) + Y_1^\eta(n) \hat{\alpha}_\eta(1) + Y_0^\eta(n) \hat{\alpha}_\eta(1) + Y_{-1}^\eta(n) \hat{\alpha}(-1)\}. \]
6. Periods of regularized automorphic smoothed kernels: geometric side

In this section, we describe the geometric expression of \( \hat{\Psi}_{\text{reg}}(n|\alpha; -) \) and its regularized \( \eta \)-period \( P_{\text{reg}}^{\eta}(\hat{\Psi}_{\text{reg}}(n|\alpha; -)) \). For \( \delta \in G_F \), we put \( \text{St}(\delta) = H_F \cap \delta^{-1} H_F \). By [12, Lemma 11.1], the following elements of \( G_F \) form a complete system of representatives of the double coset space \( H_F \backslash G_F / H_F \):

\[
e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

\[
u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad uw_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \nu w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},
\]

\[
\delta_b = \begin{pmatrix} 1 + b^{-1} & 0 \\ 1 & 1 \end{pmatrix}, \quad b \in F^\times - \{ -1 \}.
\]

Moreover, we have \( \text{St}(e) = \text{St}(w_0) = H_F \) and \( \text{St}(\delta) = Z_F \) for any \( \delta \in \{ u, \nu uw_0, \nu w_0 \} \cup \{ \delta_b | b \in F^\times - \{ -1 \} \} \). We note

\[
H_F \backslash G_F = \bigsqcup_{\delta \in H_F \backslash G_F / H_F} H_F / \delta H_F \cong \bigsqcup_{\delta \in H_F \backslash G_F / H_F} \text{St}(\delta) \backslash H_F.
\]

Thus we obtain the following expression for \( \text{Re}(\lambda) > 0 \):

\[
\Psi_{\beta,\lambda}(n|\alpha; \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}) = \sum_{\delta \in H_F \backslash G_F / H_F} \sum_{\gamma \in \text{St}(\delta) \backslash H_F} \Psi_{\beta,\lambda}(n|\alpha; \delta \gamma \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}).
\]

Set

\[
J_{\beta}(\lambda, \alpha; t) = \sum_{\gamma \in \text{St}(\delta) \backslash H_F} \Psi_{\beta,\lambda}(n|\alpha; \delta \gamma \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix})
\]

for any \( \delta \in H_F \backslash G_F / H_F \). We examine \( J_{\beta}(\lambda, \alpha; t) \). With a minor modification, we obtain Lemmas 29 and 30 by the same computation as [12 Lemma 11.2] and [12 Lemma 11.3], respectively.

**Lemma 29.** Both functions \( \lambda \mapsto J_{e}(\beta, \lambda, \alpha; t) \) and \( \lambda \mapsto J_{uw_0}(\beta, \lambda, \alpha; t) \) are analytically continued to entire functions. The values of these functions at \( \lambda = 0 \) are equal to \( J_{\alpha}(\alpha, t) \beta(0) \) and \( \delta(n = o_F)J_{\alpha}(\alpha, t) \beta(0) \), respectively, where

\[
J_{\alpha}(\alpha, t) = \delta(\eta = o_F) \left( \frac{1}{2\pi i} \right) \sum_{a \in \mathbb{A}\{x\}} \gamma_{3}(s) \alpha(s) d\mu_S(s)
\]

with

\[
\gamma_{3}(s) = \left\{ \prod_{v \in \Sigma_{\infty}} \frac{-1}{8} \frac{\Gamma((s_v + 1)/2)^2}{\Gamma((s_v + 3)/2)^2} \right\} \left\{ \prod_{v \in S_{\infty}} (1 - q_v^{-1}(s_v + 1)/2)^{-1}(1 - q_v^{-1}(s_v + 1)/2)^{-1} \right\}.
\]

We put

\[
J_{e}(\beta, \lambda, \alpha; t) = J_{e}(\beta, \lambda, \alpha; t) + J_{u}(\beta, \lambda, \alpha; t)
\]

and

\[
J_{uw_0}(\beta, \lambda, \alpha; t) = J_{uw_0}(\beta, \lambda, \alpha; t) + J_{u}(\beta, \lambda, \alpha; t).
\]

**Lemma 30.** For \( * \in \{ u, \bar{u} \} \), the function \( \lambda \mapsto J_{*}(\beta, \lambda, \alpha; t) \) is analytically continued to an entire function and the value at \( \lambda = 0 \) is equal to \( J_{*}(\alpha, t) \beta(0) \), where

\[
J_{e}(\alpha; t) = \left( \frac{1}{2\pi i} \right) \sum_{a \in \mathbb{A}\{x\}} \int_{\mathbb{A}\{x\}} \left\{ \Psi(0) \left( n | s; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) \right\} \alpha(s) d\mu_S(s)
\]

and

\[
J_{uw_0}(\beta, \lambda, \alpha; t) = J_{uw_0}(\beta, \lambda, \alpha; t) + J_{u}(\beta, \lambda, \alpha; t).
\]
Proposition 32. \( J_{\bar{\alpha}}(\alpha; t) = \left( \frac{1}{2\pi i}\right)^{\#S} \sum_{\alpha \in F^\times} \int_{U_{\alpha}(c)} \left\{ \hat{\Psi}(0) \left( n \mid s; \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) + \delta(n = o_F) \hat{\Psi}(0) \left( n \mid s; \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_n & 1 \end{pmatrix} w_0 \right) \alpha(s) d\mu_S(s) \right\} \) 

These series-integrals are absolutely convergent.

We put \[ J_{\text{hyp}}(\beta, \lambda, \alpha; t) = \sum_{b \in F^\times \setminus \{-1\}} J_{\bar{\alpha}}(\beta, \lambda, \alpha; t) = \sum_{b \in F^\times \setminus \{-1\}} \sum_{\alpha \in F^\times} \hat{\Psi}_{\beta,\lambda} \left( n \mid \alpha; \delta_{b} \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right). \]

We obtain Lemma 31 by the same proof as Lemma 11.21. In Lemma 11.21, it is assumed that \( n \) is square free. However, the argument in Lemma 11.21 works with a minor modification.

Lemma 31. The function \( J_{\text{hyp}}(\beta, \lambda, \alpha; t) \) on \( \text{Re}(\lambda) > 1 \) is analytically continued to an entire function and the value at \( \lambda = 0 \) is \( J_{\text{hyp}}(\alpha, t)\beta(0) \), where \[ J_{\text{hyp}}(\alpha, t) = \sum_{b \in F^\times \setminus \{-1\}} \sum_{\alpha \in F^\times} \hat{\Psi}(0) \left( n \mid \alpha; \delta_{b} \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right). \]

The series converges absolutely and locally uniformly in \( t \in \mathbb{A}^\times \).

Therefore we obtain the geometric expression of \( \hat{\Psi}_{\text{reg}} \left( n \mid \alpha; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) \) by Lemmas 29, 30 and 31.

Proposition 32. Let \( n \) be an ideal of \( o_F \) and \( S \) a finite subset of \( \Sigma_F \) satisfying \( \Sigma_{\infty} \subset S \) and \( S \setminus S(n) = \emptyset \). Let \( \eta \) be a character satisfying \((\ast)\) in \( \S 2.1 \). Then, for any \( \alpha \in \mathcal{A}_S \), we have

\[ \hat{\Psi}_{\text{reg}} \left( n \mid \alpha; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix} \right) = (1 + \delta(n = o_F)) J_{\text{id}}(\alpha; t) + J_{\bar{\alpha}}(\alpha; t) + J_{\alpha}(\alpha; t) + J_{\text{hyp}}(\alpha; t), \quad t \in \mathbb{A}^\times. \]

Next let us compute \( P^n_{\beta,\lambda}(\hat{\Psi}_{\text{reg}}(n|\alpha; -)) \) explicitly. Define

\[ J^n_{\beta}(\beta, \lambda; \alpha) = \int_{F^\times \setminus \mathbb{A}^\times} J_{\lambda}(\alpha; t) \{ \hat{\beta}_{\lambda}(|t|_{\mathbb{A}}) + \hat{\delta}_{\lambda}(|t|_{\mathbb{A}}^{-1}) \} \eta(t)(\log x_{\mathbb{A}, \text{fin}})d^\times t \]

and

\[ \Upsilon^n_{S}(s) = \prod_{v \in \Sigma_{\infty}} \frac{-1}{8} \Gamma((s_v + 1)/4)^2 \left\{ \prod_{v \in \Sigma_{\text{fin}}} \left( 1 - q_v^{(s_v + 1)/2} \right)^{-1} \left( 1 - \eta_v \log q_v^{-1} \right) \right\}. \]

For any ideal \( \mathfrak{a} \) of \( o_F \), we set

\[ \mathcal{C}^n_{S, \mathfrak{a}}(s) = C_0(\eta) + R(\eta) \left\{ \log(D_F N(\mathfrak{a})) + \frac{d_F}{2} (C_{\text{Euler}} + 2 \log 2 - \log \pi) \right\} \]

\[ + \sum_{\mathfrak{p} \in \Sigma_{\text{fin}}} \frac{\log q_v}{1 - q_v^{(s_v + 1)/2}} + \frac{1}{2} \sum_{\mathfrak{p} \in \Sigma_{\infty}} \left( \psi \left( \frac{s_v + 1}{4} \right) + \psi \left( \frac{s_v + 3}{4} \right) \right). \]
where \( \psi(z) = \Gamma'(s)/\Gamma(s) \) is the digamma function and \( C_{\text{Euler}} \) is the Euler constant. We note that if \( \eta \neq 1 \), then \( c_{\alpha,n}(s) \) is independent of the choice of \( a \), and \( c_{\alpha,n}(s) = C_0(\eta) = L(1, \eta) \). Put

\[
\mathcal{R}_n(n|s) = \sum_{\mathcal{A}} \int_{\mathcal{A}} \Psi^{(0)} \left( \mathcal{C}_{\alpha,n}(s) \right) \eta(t) \eta_{\text{fin}}(x, \eta, \eta_n) \frac{d^x t}{t^s}.\]

The defining series-integral converges absolutely if we take \( c \in \mathbb{R} \) such that \( \text{Re}(s) = \zeta = (c+1)/4 > 1 \). By the expression of \( \tilde{\Psi}_{\text{reg}}(n|\alpha; -) \) in Proposition 32 and a similar computation as in the proof of [12] Theorem 12.1, we can express the geometric side of \( P_{\text{reg}}^\eta(\tilde{\Psi}_{\text{reg}}(n|\alpha; -)) \) as follows.

**Theorem 33.** For any \( \ast \in \{ \text{id}, \mu, \nu, \text{hyp} \} \), the integral \( \mathcal{J}_\ast^\eta(\beta, \lambda; \alpha) \) converges absolutely and locally uniformly in \( \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) > 1 \} \). The function \( \lambda \mapsto \mathcal{J}_\ast^\eta(\beta, \lambda; \alpha) \) is analytically continued to a meromorphic function on \( \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) > 1 \} \). Moreover, the constant term \( C \Gamma_{\lambda=0} \mathcal{J}_\ast^\eta(\beta, \lambda; \alpha) \) is equal to \( \mathcal{J}_\ast^\eta(n|\alpha, \beta)(0) \), where

\[
\mathcal{J}_\ast^\eta(n|\alpha, \beta)(0) = 0,
\]

\[
\mathcal{J}_\ast^\eta(n|\alpha)(0) = (1 + \delta(n = 0)) \frac{D_1^{1/2}/\mathcal{G}(\eta)}{L_\mathcal{C}(\alpha)} \int_{L_\mathcal{C}(\alpha)} \mathcal{Y}_\mathcal{S}^\eta(s) \mathcal{C}_{\alpha,n}(s) \alpha(s) d\mu_S(s),
\]

\[
\mathcal{J}_\ast^\eta(n|\alpha) = (1 + \delta(n = 0)) \frac{D_1^{1/2}/\mathcal{G}(\eta)}{L_\mathcal{C}(\alpha)} \int_{L_\mathcal{C}(\alpha)} \mathcal{Y}_\mathcal{S}^\eta(s) \mathcal{C}_{\alpha,n}(s) \alpha(s) d\mu_S(s)
\]

and

\[
\mathcal{J}_\ast^\eta(n|\alpha) = \left( \frac{1}{2\pi i} \right)^{\mathcal{S}} \int_{L_\mathcal{C}(\alpha)} \mathcal{R}_n(n|s) \alpha(s) d\mu_S(s).
\]

In particular, we have

\[
P_{\text{reg}}^\eta(\tilde{\Psi}_{\text{reg}}(n|\alpha; -)) = \mathcal{J}_\ast^\eta(n|\alpha) + \mathcal{J}_\ast^\eta(n|\alpha) + \mathcal{J}_\ast^\eta(n|\alpha).
\]

7. Proofs of main theorems

Fix a character \( \eta \) of \( F^\times \setminus \mathbb{A}^\times \) so that \( \eta^2 = 1 \) and \( \eta_{\mathbb{R}}(-1) = 1 \) for any \( \nu \in \Sigma_\infty \). Let \( S \) be a finite subset of \( \Sigma_F \) such that \( S \supset \Sigma_\infty \) and \( S_{\text{fin}} \cap S_{f_\infty} = \emptyset \). Let \( J'_S \) be the set of all ideals \( n \) of \( \mathcal{O}_F \) such that \( S(n) \cap (S \cup S(f_\infty)) = \emptyset \) and \( \tilde{\eta}(n) = 1 \). By Theorems 28 and 33, we obtain the relative trace formula

\[
C(n, S \mathcal{X}_{\text{cas}}(n|\alpha) + \mathcal{X}_{\text{cas}}(n|\alpha) + D^\eta(n|\alpha)) = \mathcal{J}_\ast^\eta(n|\alpha) + \mathcal{J}_\ast^\eta(n|\alpha) + \mathcal{J}_\ast^\eta(n|\alpha)
\]

for any \( \alpha \in \mathcal{S}_S \) and \( n \in J'_S \). The following estimate of \( \mathcal{J}_\text{hyp}(n|\alpha) \) is given by the same argument in the proof of [12] Lemma 12.9 with a minor modification.

**Lemma 34.** For any \( \alpha \in \mathcal{S}_S \) and \( q > 0 \), we have \( \mathcal{J}_\text{hyp}(n|\alpha) \ll N(n)^{-q} \) with the implied constant independent of \( n \in J'_S \).

**Lemma 35.** For any \( \epsilon > 0 \), we have

\[
|B_{\gamma, \rho}(1/2, \nu)| \ll N(f_\alpha)^{-1/2+\epsilon} N(n)^\epsilon, \quad \nu \in \mathbb{R}, \quad \rho \in \Lambda_{\alpha}(n), \quad \chi \in \Xi(n)
\]

with the implied constant independent of \( n \in J'_S \).

**Proof.** Assume \( \eta \in \mathbb{R} \). Then, the following estimate holds for any \( \epsilon > 0 \):

\[
|B_{\gamma, \rho}(1/2, \nu)| = \prod_{\kappa=0}^n \prod_{\nu \in \Sigma_{\kappa}(\rho)} \left| Q_{\kappa, \chi, \nu}(\eta, 1) \right| \left| L(1 + \nu, \chi_\nu^2) \right| \prod_{\nu \in \Sigma_{\kappa}(\rho)} (1 + q^{-\epsilon})
\]
Lemma 37. This completes the proof.

Proof. Letθ be a real number such that $|\chi| = |\chi|$, $\nu \in i\mathbb{R}$ holds with the implied constant independent of $\chi \in \Xi(n)$ and $n$. This was given in the proof of Lemma 22. Let $t \in \mathbb{R}$ uniformly for any $\chi \in \Xi(n)$ and $n$. We can take such $\theta$ so that $-1/4 < \theta < 0$ by 5. Thus, by the aid of Lemma 35 and Stirling’s formula, the explicit description of $P_{\text{reg}}(E_{\chi},\nu,-)$ in Proposition 12 gives us the estimate

\[
|P_{\text{reg}}(E_{\chi},\nu,-)| \ll N(f_{\chi})^{1/2} \sum_{\nu \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_{\nu})|)^{1/2 + 2\theta + \epsilon}
\]

for any $\epsilon > 0$, where the implied constant is independent of $\nu \in i\mathbb{R}, \chi \in \Xi(n)$ and $n \in J_{\mathbb{R}}(n)$. With the aid of Lemma 6, we have

\[
|C(n, S)|^{1/2} \ll N(n)^{-1} \sum_{\chi \in \Xi(n)} \sum_{\nu \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_{\nu})|)^{1/2 + 2\theta + \epsilon}
\]

Note $\sum_{a|n} 1 \ll N(n)^{\epsilon}$. Since we can take $\epsilon > 0$ so that $2\theta + 4\epsilon < 0$, we obtain the assertion. □

Lemma 36. For any $\alpha \in \mathcal{A}_{S}$, there exists $\delta > 0$ such that $|C(n, S)|^{1/2} \ll N(n)^{-\delta}$ with the implied constant independent of $n \in J_{S, \eta}$. Proof. We recall that for any $\epsilon > 0$, the estimate $|L_{\mathbb{R}}(1 + \nu, \chi^{2})|^{1/2} \ll q(\chi^{2}) · |\chi|^{\epsilon}$, $\nu \in i\mathbb{R}$ holds with the implied constant independent of $\chi \in \Xi(n)$ and $n$. This was given in the proof of Lemma 22. Let $t \in \mathbb{R}$ uniformly for any $\chi \in \Xi(n)$ and $n$. We can take such $\theta$ so that $-1/4 < \theta < 0$ by 5. Thus, by the aid of Lemma 35 and Stirling’s formula, the explicit description of $P_{\text{reg}}(E_{\chi},\nu,-)$ in Proposition 12 gives us the estimate

\[
|P_{\text{reg}}(E_{\chi},\nu,-)| \ll N(f_{\chi})^{1/2} \sum_{\nu \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_{\nu})|)^{1/2 + 2\theta + \epsilon}
\]

Note $\sum_{a|n} 1 \ll N(n)^{\epsilon}$. Since we can take $\epsilon > 0$ so that $2\theta + 4\epsilon < 0$, we obtain the assertion. □

Lemma 37. For any $\epsilon > 0$ and $\alpha \in \mathcal{A}_{S}$, we have $|C(n, S)|^{1/2} \ll N(n)^{-1+\epsilon}$ with the implied constant independent of $n \in J_{S, \eta}$.

Proof. This follows immediately from Lemma 22. □
For $n \in J'_{S,\eta}$, we set $(\lambda_{S}^{n}(n), \delta) = 2D_{\delta}^{1/2}g(\eta)^{-1}[K_{\text{fin}} : K_{0}(n)]^{-1} \sum_{\pi \in \Pi_{\text{cus}}(n)} \mathbb{P}^{n}(\pi, K_{0}(n)) f(\nu_{\pi}, S)$ for any $f \in C_{c}(X_{S}^{0+})$. The measure is extended to a measure on the Schwartz space $\mathcal{S}(X_{S}^{0+})$ (cf. [12, Lemma 13.16]). Combining Lemmas 12, 14, 34, 36 and 37 with the argument in [12, Lemma 13.18], we obtain the following theorem.

**Theorem 38.** For a fixed $\alpha \in \mathcal{A}_{S}$, there exists $\delta > 0$ such that for any infinite subset $\Lambda \subset J'_{S,\eta}$, we have

$$\langle \lambda_{S}^{n}(n), \alpha \rangle = \sum_{\pi \in \Pi_{\text{cus}}(n)} \frac{[K_{\text{fin}} : K_{0}(f_{\pi})]}{N(f_{\pi})} \mathbb{P}^{n}(\pi, \mathbb{P}^{n}(\pi, \nu_{\pi}, S)) = \langle \lambda_{S}^{n}, \alpha \rangle + O(N(n)^{-\delta})$$

as $N(n) \to \infty$ in $n \in \Lambda$.

We show the proof of Theorem 2. For $n \in J'_{S,\eta}$, let $n = \prod_{k=1}^{s} p_{k}^{a_{k}} \prod_{k=s+1}^{s+l} p_{k}$ with $a_{k} \geq 2$ be a prime ideal decomposition of $n$. For $\pi \in \Pi_{\text{cus}}(n)$ with $f_{\pi} = \prod_{k=1}^{s} p_{k}^{b_{k}} \prod_{k=s+1}^{s+l} p_{k}^{a_{k}}$, Lemma 12 gives us

$$\mathbb{P}^{n}(\pi) = \delta(\epsilon_{s+k}; k \in \{1\}^{s}, (a_{k} - b_{k})_{k} \in (2\mathbb{N})^{s}) \prod_{k=1}^{s} \left(\frac{\mathbb{N}(p_{k}) + 1}{\mathbb{N}(p_{k}) - 1}\right)$$

Hence, by setting $L(\pi) = \frac{L(1/2, \pi)L(1/2, \pi \otimes \eta)}{L(1, \pi, \text{Ad})}$ for a fixed $\alpha \in \mathcal{A}_{S}$, we obtain

$$\frac{1}{N(n)} \sum_{\pi \in \Pi_{\text{cus}}(n)} L(1/2, \pi) L(1/2, \pi \otimes \eta) \alpha(\nu_{\pi}, S)$$

$$\approx \left( \sum_{\pi \in \Pi_{\text{cus}}(n)} + \sum_{j=1}^{s+l} (-1)^{j} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq s+l} \sum_{\pi \in \Pi_{\text{cus}}(n)} \prod_{k=1}^{s} p_{k}^{-1} \right) \frac{[K_{\text{fin}} : K_{0}(f_{\pi})]}{N(f_{\pi})} \mathbb{P}^{n}(\pi, \mathbb{P}^{n}(\pi, \nu_{\pi}, S))$$

$$= \sum_{\pi \in \Pi_{\text{cus}}(n)} \frac{[K_{\text{fin}} : K_{0}(f_{\pi})]}{N(f_{\pi})} \mathbb{P}^{n}(\pi, \mathbb{P}^{n}(\pi, \nu_{\pi}, S)) + \sum_{j=1}^{s+l} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq s} (-1)^{j} \frac{[K_{\text{fin}} : K_{0}(n \prod_{k=1}^{s} p_{k}^{-1})]}{[K_{\text{fin}} : K_{0}(n)]}$$

$$\times \sum_{\pi \in \Pi_{\text{cus}}(n \prod_{k=1}^{s} p_{k}^{-1})} \frac{[K_{\text{fin}} : K_{0}(f_{\pi})]}{N(f_{\pi})} \mathbb{P}^{n}(\pi, \mathbb{P}^{n}(\pi, \nu_{\pi}, S)) \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( \frac{q_{v} + 1}{q_{v} - 1} \right) \mathbb{P}^{n}(\pi, \mathbb{P}^{n}(\pi, \nu_{\pi}, S)) \right)$$

$$= \langle \lambda_{S}^{n}, \alpha \rangle + O(N(n)^{-\delta}) + \sum_{j=1}^{s+l} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq s} \frac{N(n \prod_{k=1}^{s} p_{k}^{-1}) \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( 1 + q_{v}^{-1} \right)}{N(n) \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( 1 + q_{v}^{-1} \right)}$$

$$\times \left\{ \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( \frac{q_{v} + 1}{q_{v} - 1} \right) \right\}$$

$$= \left( 1 + \sum_{j=1}^{s+l} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq s} \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( 1 - q_{v}^{-1} \right)^{-1} \right) \langle \lambda_{S}^{n}, \alpha \rangle$$

$$+ O(N(n)^{-\delta}) \left( 1 + \sum_{j=1}^{s+l} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq s} \prod_{v \in S_{2}(n) \cap S(\prod_{k=1}^{s} p_{k})} \left( 1 - q_{v}^{-1} \right)^{-1} \right)$$
\[
\prod_{v \in S_2(n)} \left\{ 1 - (1 - q_v^{-2})^{-1} q_v^{-2} \right\} \prod_{v \in S(n) \setminus (S_1(n) \cup S_2(n))} (1 - q_v^{-2}) \langle \lambda_S^0, \alpha \rangle + \mathcal{O}(N(n)^{-\delta}).
\]

Here we note Theorem 38 and an explicit formula of \[w_{13.17}\]. By Theorem 2 and \(\text{vol}(\{\})\), we can generalize [12, Theorem 14.1] to the case of arbitrary level with a minor modification by the aid of the proof of [12, Theorem 13.17].

We show the proof of Theorem 3 which is same as in [12, Corollary 1.2]. Let \(J\) be the set of all \((\nu_v)_{v \in S} \in X^0_S\) such that \((1 - \nu_v^2)/4 \in J_v\) for any \(v \in \Sigma_{\infty}\) and \(q_v^{-\nu_v/2} + q_v^{\nu_v/2} \in J_v\) for any \(v \in S_{\text{fin}}\). Put
\[
I(n; J) = \frac{1}{N(n)} \sum_{\pi \in U_{\text{un}}(n)} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^S(1, \pi, \text{Ad})}.
\]

By Theorem 2 and \(\text{vol}(J, \lambda_S^0) > 0\), for any \(M > 0\), there exists \(n \in \Lambda\) with \(N(n) > M\) such that
\[
|I(n; J) - C(n) \text{vol}(J, \lambda_S^0)| < 2^{-1} \left\{ \prod_{v \in \Sigma_{\infty}} \left\{ 1 - (q_v^2 - q_v)^{-1} \right\} \right\} \zeta(2)^{-1} \text{vol}(J, \lambda_S^0).
\]

Therefore, \(I(n; J) > 2^{-1} \left\{ \prod_{v \in \Sigma_{\infty}} \left\{ 1 - (q_v^2 - q_v)^{-1} \right\} \right\} \zeta(2)^{-1} \text{vol}(J, \lambda_S^0) > 0\) holds by virtue of \(0 < \{\prod_{v \in \Sigma_{\infty}} \left\{ 1 - (q_v^2 - q_v)^{-1} \right\} \} \zeta(2)^{-1} < C(n)\). This completes the proof of Theorem 3.

We remark that Theorems 4 and 5 are proved in the same way as [12, Theorem 1.3, Corollary 1.4] since we can generalize [12, Theorem 14.1] to the case of arbitrary level with a minor modification by using the relative trace formula explained in [17].

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