The construction of sets with strong quantum nonlocality using fewer states

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In this paper, we investigate the construction of orthogonal product states with strong nonlocality in multiparty quantum systems. Firstly, we focus on the tripartite system and propose a general set of orthogonal product states exhibiting strong nonlocality in \(d \otimes d \otimes d\) quantum system, which contains \(6(d - 1)^2\) states. Secondly, we find that the number of the sets constructed in this way could be further reduced. Then using \(4 \otimes 4 \otimes 4\) and \(5 \otimes 5 \otimes 5\) quantum systems as examples, it can be seen that when \(d\) increases, the reduced quantum state is considerable. Thirdly, by imitating the construction method of the tripartite system, two 3-divisible four-party quantum systems are proposed, \(3 \otimes 3 \otimes 3 \otimes 3\) and \(4 \otimes 4 \otimes 4 \otimes 4\), both of which contain fewer states than the existing ones. Our research gives a positive answer to an open question raised in [Halder et al., PRL, 122, 040403 (2019)], indicating that there do exist fewer quantum states that can exhibit strong quantum nonlocality without entanglement.

I. INTRODUCTION

The quantum state discrimination problem is one of the research hotspots in quantum information theory. For a group of quantum systems with local discriminability, separate observers can perform measurements on their own quantum systems and use the result to determine which state they are sharing by local operations and classical communication (LOCC). This problem is also the basis for studying many other quantum problems, such as quantum data hiding, quantum secret sharing, etc.[1-7].

It had always been believed that quantum entanglement was closely related to the nonlocality of orthogonal quantum states. In 1999, Bennett et al.[16] found nine orthogonal product states that cannot be perfectly distinguished in a \(3 \otimes 3\) quantum system if the two parties are restricted to LOCC. This result strongly proved that entanglement is not a necessary condition for nonlocality. Therefore, the phenomenon "Nonlocality without entanglement" was found and aroused intense discussion. In 2000, Walgate et al.[17] proved that any two orthogonal quantum states, no matter whether they were entangled or not, could always be perfectly distinguished by LOCC. This puts an emphasis that entanglement is not a sufficient condition for nonlocality. Subsequently, Walgate and Hardy[18] proposed that the measurements performed locally by each party need to keep the post-measurement states still orthogonal to each other, and based on this proposition, a simpler proof was given for the \(3 \otimes 3\) orthogonal product state constructed by Bennett.

In order to understand the relationship between entanglement and nonlocality better, the construction and judgment methods of orthogonal product states sets with nonlocality are widely discussed[19-36]. For instance, Li et al.[32] established a \(n \otimes 2l\) indistinguishable quantum system which contains 2 \((n + 2l)\) orthogonal product states and a \(n \otimes (2k + 1)\) system which includes 2 \((n + 2k + 1) - 7\) states; Zhang et al.[33] presented a general method to construct indistinguishable multipartite orthogonal product states in \(d_1 \otimes d_2 \otimes \cdots \otimes d_n, (d_{1,2,\ldots,n} \geq 3, n \geq 4)\). Besides, given a quantum system, judging whether it is nonlocal is also a hot topic. Bandyopadhyay et al.[34] proposed that if the unitary operators can be perfectly distinguished, the state corresponding to this unitary transformation can also be exactly discriminated by one-way LOCC. Yang et al.[35] and Wang et al.[36] put forward some special sets respectively. Based on these sets, they proposed a general method to judge whether a generalized Bell state group can be perfectly distinguished.

In addition, Halder et al.[37] recently proposed a very interesting statement that if a set of quantum states is locally irreducible, it cannot be distinguished by LOCC. He also discovered that in some nonlocal tripartite systems, when two parties measure together, there is a possibility that the shared state can be distinguished. Therefore, the concept of strong nonlocality was given, that is, if any two parties in the nonlocal tripartite system cannot determine the secret state after joint measurement, this system is considered to be strongly nonlocal. Based on Halder’s work, Zhang et al.[38] gave two general definitions of strong nonlocality, and constructed a group of quantum states with strong nonlocality in \(3 \otimes 3 \otimes 3\) and \(3 \otimes 3 \otimes 3 \otimes 3\) cases. But the number of states in the given examples is pretty large, and he did not further discuss the general construction methods in multiparty systems.

Therefore, motivated by the above discovers, in this
work, we extend the concept of strong nonlocality and present some definitions to help describe the nonlocality of multiparty systems. Then, a universal group of orthogonal product states in the tripartite system with strong nonlocality is established. An intuitive example $4 \otimes 4 \otimes 4$ is also given. In order to further explore which quantum state lead to the local indistinguishability of the overall system, the number of orthogonal product states should be as small as possible. Therefore, in tripartite system, we first discuss the minimum number of states in the system, the number of orthogonal product states should be $\frac{d^2 - 1}{2}$. In order to show this proposition more intuitively, two corresponding quantum systems $4 \otimes 4 \otimes 4$ and $5 \otimes 5 \otimes 5$ was presented, which consist of fewer states than the previous states sets. Comparing the number of states in the general system with the number in the minimum system, it is obvious that as the dimension becomes larger, the number of reduced states will also increase. Following this method, we next discuss reducing the number of states in the 3-divisible four-party system and propose two quantum systems, $3 \otimes 3 \otimes 3 \otimes 3$ and $4 \otimes 4 \otimes 4 \otimes 4$, which is significantly less than the number presented by Halder et al.[40] and Zhang et al.[38].

The rest of this paper is organized as follows. In Sec. II, a general definition of strong quantum nonlocality is presented. Then, in Sec. III, we focus on the construction of the strong nonlocal tripartite quantum system without entanglement and propose $d \otimes d \otimes d$. Also, the $4 \otimes 4 \otimes 4$ system with 48 states and the $5 \otimes 5 \otimes 5$ system with 72 states are proposed in this section. Next, in Sec. IV, the 3-divisible quadrilateral system is mainly discussed. Finally, we summarize in Sec. V.

II. THE GENERAL DEFINITION OF STRONG NONLOCALITY

In this section, we propose the concept of strong nonlocality and present some definitions to clarify this property in the multiparty system. In Ref.[38], a set of orthogonal quantum states is defined as strongly nonlocal if it is locally irreducible in every $(n - 1)$-partition, where $(n - 1)$-partition means the whole quantum system is divided into $n - 1$ parts. However, there exist some limitations for multiparty systems since it cannot ensure the system is nonlocal when it is divided into $n - 2$, $n - 3$ or other parts. And obviously, a multiparty system which can be simultaneously divided into $n - 1$ and $n - 2$ parts has stronger nonlocality than a system that can only be divided into $n - 1$ parts. Therefore, we extend the definition given by Zhang et al.[38] and obtain the Definition 1.

Definition 1: In a multiparty system $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, $(d_1, 2, \ldots, n \geq 3, n \geq 4)$, if a set of orthogonal product states is arbitrarily divided into $i$ parts, and the entire system is still locally irreducible in every new $i$ parts, then the system is called $i$-divisible, $i = 2, 3, \ldots, n - 1$.

Definition 2: In a multiparty system $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, $(d_1, 2, \ldots, n \geq 3, n \geq 4)$, if a set of orthogonal product states are $(n - 1)$-divisible, $(n - 2)$-divisible...and 2-divisible simultaneously, it is said that this system is strongly nonlocal.

Besides, it is worth noting that there exist some systems, when it is divided into $k$ parts, only some certain partitioning methods can maintain the nonlocality of the system, while other partitioning methods cannot guarantee this property. Based on this situation, we give the Definition 3.

Definition 3: In a multiparty system $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, $(d_1, 2, \ldots, n \geq 3, n \geq 4)$, if a set of orthogonal product states is $k$-divisible, the nonlocality strength is represented by $S_k$. If a set of orthogonal states has only some certain $k$-division methods that can maintain nonlocality, the nonlocality strength of this system is represented by $S_k^-$, $S_k > S_k^-$, $S_k > S_{k+1}$.

Proposition 1: If a set of orthogonal states is 2-divisible, this system must have strong nonlocality.

Proof: A 2-divisible system means that any $1 \sim n - 1$ parties have possibility to combine together after the system dividing into two parts and the new combined party must start with a trivial measurement. That is to say, according the measurement result of each participant, the new party cannot be distinguished from other parties. Under this circumstance, when this set of orthogonal states is divided into three parts, any $1 \sim n - 2$ parties need to combine together. Due to the combination of $1 \sim n - 1$ parties has already included all situations that may occur for $1 \sim n - 2$ parties, the measurement of the new combined party will also be a trivial measurement. Therefore, this set of orthogonal product states can maintain nonlocality of the system after being divided into any three parts, which means, this system is 3-divisible. By analogy, this system will be 4-divisible, 5-divisible, ..., $(n - 1)$-divisible. In conclusion, if a quantum system is 2-divisible, it must be strongly nonlocal.

In this section, we mainly extend the general concept of strong nonlocality in the multiparty system. In the next section, a general set of orthogonal product state with strong nonlocality in tripartite system is constructed. Also, the minimum size of the strongly nonlocal quantum system is demonstrated.

III. TRIPARTITE SYSTEMS WITH STRONG NONLOCALITY

A. A set of orthogonal product states in $d \otimes d \otimes d$ system with strong nonlocality

In this section, we propose a general method for constructing a set of orthogonal product states in tripartite system with strong nonlocality and give an example of $4 \otimes 4 \otimes 4$ intuitively.

First, the following $6(d - 1)^2$ states in $d \otimes d \otimes d$ quantum
system is presented, $d \geq 3$.
\[
\begin{align*}
|\varphi_i\rangle &= |0 \pm i\rangle_A |0\rangle_B |i\rangle_C \\
|\varphi_{i+2d-2}\rangle &= |0\rangle_A |i\rangle_B |0 \pm i\rangle_C \\
|\varphi_{i+4d-4}\rangle &= |i\rangle_A |0 \pm i\rangle_B |0\rangle_C
\end{align*}
\]
\[i = 1, 2, \ldots, d - 1\]
\[|\varphi_{i+6d-6}\rangle &= |i \pm j\rangle_A |j\rangle_B |i\rangle_C \\
|\varphi_{i+3d-3d}\rangle &= |0 \pm i\rangle_A |i\rangle_B |0 \pm i\rangle_C \\
|\varphi_{i+5d-9d}\rangle &= |j\rangle_A |0 \pm i\rangle_B |i\rangle_C
\]
\[i = 1, 2, \ldots, d - 2, j = 2, 3, \ldots, d - 1, i < j\]
\[
\begin{align*}
|\varphi_{i+6d-12d}\rangle &= |0 \pm i\rangle_A |i\rangle_B |0 \pm i\rangle_C \\
|\varphi_{i+6d-12d+2}\rangle &= |i\rangle_A |j\rangle_B |0 \pm i\rangle_C \\
|\varphi_{i+6d-12d+4}\rangle &= |j\rangle_A |0 \pm i\rangle_B |i\rangle_C
\end{align*}
\]
\[i = d - 1, j = 1\]
\[|e \pm f\rangle = |e\rangle + |f\rangle, 0 \leq e < f \leq d\]

**Theorem 1:** In $d \otimes d \otimes d$, the set of orthogonal product states (1) is strongly nonlocal.

**Proof:** First, we need to prove that this set is nonlocal, and then prove that this set is still nonlocal after being divided into any two parts.

The first step is to prove that this tripartite system is nonlocal.

From the structure of states (1), it is obvious that these states have a cyclic property as the cyclic property of the trace. In other words, the set has the same properties in the different divisions of $A[B|C, B|A|C$, $C|A|B$. Therefore, it only needs to prove that a nontrivial measurement cannot be started when Alice goes first, and then the same result will be obtained in the case of Bob goes first or Charlie goes first.

Without loss of generality, suppose Alice starts with a nontrivial measurement, represented by a set of POVM elements $M_{m}^d M_{m}$ on $d \times d$ matrix. We could write the POVM measurement in $\{|0\rangle, |1\rangle, \ldots, |d - 1\rangle\}_A$ basis, which corresponds to the states (1):

\[
M_{m}^d M_{m} = \begin{bmatrix}
    a_{00} & a_{01} & \cdots & a_{0(d-1)} \\
    a_{10} & a_{11} & \cdots & a_{1(d-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{(d-1)0} & a_{(d-1)1} & \cdots & a_{(d-1)(d-1)}
\end{bmatrix}
\]

The post-measurement states could be expressed as $(I \otimes M_{m}) |\varphi_i\rangle$, which should be mutually orthogonal, due to a necessary condition for perfectly distinguishing post-measurement states. Then we can obtain $\langle \varphi_i | (I \otimes M_{m}) |\varphi_i\rangle = 0$. Considering the orthogonality of quantum states $|\varphi_{i+2d-2}\rangle$ and $|\varphi_{i+4d-4}\rangle$, we have $\langle 0 | (M_{m}^d M_{m}) | i\rangle = 0$, thus $a_{0i} = a_{i0} = 0$, $i = 1, 2, \ldots, d - 1$. Similarly, according to $|\varphi_{i+4d-4}\rangle$ and $|\varphi_{i+2d-2}\rangle$, we can get $\langle i | (M_{m}^d M_{m}) | i\rangle = 0$, thus $a_{ij} = a_{ji} = 0$, $i = 1, 2, \ldots, d - 2, j = 2, 3, \ldots, d - 1, i < j$.

For the states $|\varphi_i\rangle$, we can get $\langle 0 - i | (M_{m}^d M_{m}) | 0 + i\rangle = 0$, then $\langle 0 | (M_{m}^d M_{m}) | 0 - i \rangle (M_{m}^d M_{m}) | i\rangle = 0$, thus $a_{00} = a_{11} = \cdots = a_{ii}, i = 1, 2, \ldots, d - 1$.

Therefore, the original matrix can be reduced to:

\[
M_{m}^d M_{m} = \begin{bmatrix}
    a & 0 & \cdots & 0 \\
    0 & a & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a
\end{bmatrix}
\]

Furthemore, Alice’s measurement matrix is proportional to the identity matrix, which shows that Alice starts with a trivial measurement. If there are no nontrivial measurements existed to preserve orthogonality, the set of states is locally irreducible, which means the set is nonlocal. The same goes for Bob and Charlie’s measurements. Thus, the states (1) are locally irreducible in tripartition.

The second step is to prove that after three parties arbitrarily divided into two parts, the system is still nonlocal.

Physically, this method of division means that the subsystems $A$(Alice) and $B$(Bob) are treated together as a sixteen-dimensional subsystem $AB$ on which joint measurements are now allowed. However, since the divided two parts are not symmetrical, it is necessary to consider the situation where each new party starts the measurement first. From the previous step, we know that the system is nonlocal when Charlie goes first. So next, we only need to consider the situation where the $AB$ system goes first. In order to describe the proof more clearly, we first rewrite the original state, let $|00\rangle \rightarrow |0\rangle, \ldots, |0(d - 1)\rangle \rightarrow |d - 1\rangle, |10\rangle \rightarrow |d\rangle, \ldots, |1(d - 1)\rangle \rightarrow |2d - 1\rangle, |20\rangle \rightarrow |2d\rangle, \ldots, (|d - 1\rangle(d - 1)\rangle \rightarrow |d^2 - 1\rangle$, then the following group (2) is obtained:

\[
\begin{align*}
|\varphi_i\rangle &= |00 \pm i\rangle_AB |0\rangle_C \\
|\varphi_{i+2d-2}\rangle &= |0i\rangle_AB |0 \pm i\rangle_C \\
|\varphi_{i+4d-4}\rangle &= |0 \pm ii\rangle_AB |0 \pm i\rangle_C \\
|\varphi_{i+2d-2}\rangle &= |ij\rangle_AB |i \pm j\rangle_C \\
|\varphi_{i+2d-2}\rangle &= |ij\rangle_AB |i \pm j\rangle_C \\
|\varphi_{i+5d-9d}\rangle &= |j0 \pm ji\rangle_AB |i\rangle_C
\end{align*}
\]
\[i = 1, 2, \cdots, d - 1, j = 2, 3, \cdots, d - 1, i < j\]
\[
\begin{align*}
|\varphi_{i+3d-3d}\rangle &= |0i \pm ii\rangle_AB |0\rangle_C \\
|\varphi_{i+4d-6d}\rangle &= |0i \pm ii\rangle_AB |0\rangle_C \\
|\varphi_{i+5d-9d}\rangle &= |j0 \pm ji\rangle_AB |i\rangle_C
\end{align*}
\]
\[i = 1, 2, \cdots, d - 2, j = 2, 3, \cdots, d - 1, i < j\]
\[
\begin{align*}
|\varphi_{i+6d-12d}\rangle &= |0i \pm ii\rangle_AB |0\rangle_C \\
|\varphi_{i+6d-12d+2}\rangle &= |ij\rangle_AB |0 \pm i\rangle_C \\
|\varphi_{i+6d-12d+4}\rangle &= |j0 \pm ji\rangle_AB |i\rangle_C
\end{align*}
\]
\[i = d - 1, j = 1\]
\[|e \pm f\rangle = |e\rangle + |f\rangle, 0 \leq e < f \leq d\]

Suppose Alice starts with the nontrivial and non-disturbing measurement, $M_{m}^d M_{m}$, the POVM ele-
ment can be written as a $d^2 \times d^2$ matrix in $\{0\}, |1\rangle, \ldots, |d^2 - 1\rangle \}_A$ basis, which corresponds to the states $(2)$:

$$M^\dagger_m M_m = \begin{bmatrix}
a_{00} & a_{01} & \cdots & a_{0(d^2-1)} \\
a_{10} & a_{11} & \cdots & a_{1(d^2-1)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{(d^2-1)0} & a_{(d^2-1)1} & \cdots & a_{(d^2-1)(d^2-1)}
\end{bmatrix}$$

The post-measurement state is $(I \otimes M_m)|\psi\rangle$, which is mutually orthogonal. So $(\phi_{ij}|(I \otimes M^\dagger_m M_m)|\psi\rangle = 0$ can be obtained. Considering the orthogonality of quantum states $|\phi_{ij}\rangle$ and $|\phi_{i+2d^2-2}\rangle$, we have $\langle 00| (M^\dagger_m M_m)|00\rangle = 0$, thus, $a_{00}(00) = a_{00}(00) = a_{00}(0) = a_{00}(0) = 0$. In order to make the proof clearer, in the following steps, we will only write the form of $a_{nm}, m < n$, since there is always $a_{nn} = a_{nn}$. According to $\langle |\phi_{i+2d^2-2}\rangle$ and $|\phi_{i+6d^2-6}\rangle$, we have $\langle 0i| (M^\dagger_m M_m)|0i\rangle = 0$, thus $a_{0i}(0i) = a_{0i}(ii) = 0$. For the states $|\phi_{i-2d^2-2}\rangle$ and $|\phi_{i+2d^2-2}\rangle$, we know $\langle 0i| (M^\dagger_m M_m)|0i\rangle = 0$, thus $a_{0i}(0i) = a_{0i}(ii) = 0$. Considering $|\phi_{i+2d^2-3d}\rangle$ and $|\phi_{i+2d^2-3d}\rangle$, $\langle 00| (M^\dagger_m M_m)|00\rangle = 0$ is obtained, thus, $a_{00}(00) = a_{00}(ii) = 0$. According to $|\phi_{i}\rangle$, $|\phi_{i+2d^2-6}\rangle$ and $|\phi_{i+2d^2-6}\rangle$, in the same way, we have $\langle 0i| (M^\dagger_m M_m)|0i\rangle = 0$ and $\langle 0i| (M^\dagger_m M_m)|0i\rangle = 0$, thus, $a_{0i}(ii) = a_{0i}(ij) = 0$. For $|\phi_{i+2d^2-3d}\rangle$, $|\phi_{i+2d^2-3d}\rangle$, $\langle i+ij| (M^\dagger_m M_m)|i+ij\rangle = 0$ and $a_{0ij}(0ij) = a_{0ij}(ij) = 0$. Got considering $|\phi_{i+2d^2-3d}\rangle$, $|\phi_{i+2d^2-3d}\rangle$, $|\phi_{i+2d^2-3d}\rangle$, $\langle i+j| (M^\dagger_m M_m)|i+j\rangle = 0$, thus, $a_{ij}(0) = a_{ij}(ii) = 0$. In the following, for the states $|\phi_{i+2d^2-2}\rangle$, $|\phi_{i+2d^2-6}\rangle$ and $|\phi_{i+2d^2-6}\rangle$, $\langle i+j| (M^\dagger_m M_m)|i+j\rangle = 0$ is obtained, thus $a_{ij}(ij) = a_{ij}(ij) = 0$. In addition, according to $|\phi_{i}\rangle$, we can obtain $\langle 0i| (M^\dagger_m M_m)|0i\rangle = 0$, thus $a_{0i}(0i) = a_{0i}(ii)$. Similarly, considering $|\phi_{i+2d^2-4}\rangle$, $|\phi_{i+2d^2-4}\rangle$, $|\phi_{i+2d^2-4}\rangle$ and $|\phi_{i+2d^2-4}\rangle$, $a_{ii}(ii) = a_{ii}(ii) = a_{ii}(ij) = a_{ii}(ij) = a_{ii}(ii)$ is also obtained.

Therefore, the original matrix can be reduced to:

$$M^\dagger M_m = \begin{bmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{bmatrix}$$

In order to show the above conclusion more intuitively, we take $d = 4$ as an example. The following state group (3) can be obtained. In Appendix A, a complete proof of Example 1 is given.

**Example 1**: The following 54 states (3) is strongly nonlocal.

$$|\phi_{1,2}\rangle = |0 \pm 1\rangle_A |0\rangle_B |1\rangle_C$$

$$|\phi_{5,6}\rangle = |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C$$

$$|\phi_{9,10}\rangle = |0\rangle_A |0 \pm 2\rangle_B |0\rangle_C$$

$$|\phi_{13,14}\rangle = |0 \pm 3\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{17,18}\rangle = |0 \pm 3\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{21,22}\rangle = |0 \pm 2\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{25,26}\rangle = |0 \pm 2\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{29,30}\rangle = |0 \pm 3\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{33,34}\rangle = |0 \pm 3\rangle_A |0\rangle_B |0\rangle_C$$

$$|\phi_{37,38}\rangle = |0 \pm 1\rangle_A |1\rangle_B |0\rangle_C$$

$$|\phi_{41,42}\rangle = |0 \pm 1\rangle_A |1\rangle_B |0\rangle_C$$

$$|\phi_{45,46}\rangle = |0 \pm 1\rangle_A |1\rangle_B |0\rangle_C$$

$$|\phi_{49,50}\rangle = |0 \pm 3\rangle_A |1\rangle_B |0\rangle_C$$

$$|\phi_{53,54}\rangle = |0 \pm 3\rangle_A |1\rangle_B |0\rangle_C$$

(3)

**B. Construction of a strongly nonlocal system with fewer states**

In the previous section, a general tripartite quantum system with strong non-locality is established. However, the number of states in the group is considerable, especially when the dimension increases. It is not conducive for us to essentially explore which quantum states cause the local indistinguishability of the system. Therefore, in this section, we try to construct a quantum system that still has strong nonlocality, but contains fewer states.

**Proposition 2**: In the set of tripartite orthogonal product state with strong nonlocality, when all quantum states are composed of the product of $|i+j\rangle$ and $|\alpha\rangle$, the set includes at least $6 \left(\frac{d^2-1}{2}\right)$ states.

Proof: When judging whether a system is nonlocal, the following principle is often applied: if there do not exist nontrivial measurements to preserve orthogonality, this set of state is nonlocal[38]. In tripartite quantum system, when the system is divided into two parts, there will be a new party that combines two parties.So, the original system $d \otimes d \otimes d$ is transformed into a two-party system of $d^2 \otimes d$.

Since the original system is nonlocal, it is impossible to distinguish the state by starting measurement from the party with the dimension $d$. Hence, only the joint system with dimension $d^2$ needs to be considered. If the matrix of trivial measurement is always proportional to the identity matrix, there must be $a_{0\alpha} = a_{11} = \cdots = a_{ij}$ and at least $d^2-1$ pairs of orthogonal product basis $|i \pm j\rangle$ are required, $i, j = 0, 1, \cdots, d^2-1$.

In addition, in order to ensure the symmetry of the tripartite system, the positions of three parties are always circular. So the six states of the cycle can be regarded as
a small group. And obviously, this small group will contain at most two different \(|i \pm j\), \(i, j = 0, 1, \ldots, d^2 - 1\), which means that a system with the minimum number of states should contain at least \(\left\lceil \frac{d^2 - 1}{2} \right\rceil\) small groups. Therefore, the minimum number of states should be
\[
6 \left\lceil \frac{d^2 - 1}{2} \right\rceil.\]

According to the construction method proposed in this section, the following \(4 \otimes 4 \otimes 4\) quantum system is given as an example.

**Theorem 2:** In \(4 \otimes 4 \otimes 4\), the following 48 orthogonal product states \(\psi_i\) is strongly nonlocal.

\[
\begin{align*}
|\psi_{1,2}\rangle &= |0 \pm 1\rangle_A |0 \pm 1\rangle_B |1\rangle_C, \\
|\psi_{5,6}\rangle &= |1\rangle_A |0 \pm 1\rangle_B |0 \rangle_C, \\
|\psi_{9,10}\rangle &= |0 \rangle_A |2\rangle_B |0 \pm 2\rangle_C, \\
|\psi_{13,14}\rangle &= |1 \pm 3\rangle_A |0 \rangle_B |3\rangle_C, \\
|\psi_{17,18}\rangle &= |3\rangle_A |1 \pm 3\rangle_B |0 \rangle_C, \\
|\psi_{21,22}\rangle &= |1\rangle_A |2\rangle_B |0 \pm 1\rangle_C, \\
|\psi_{25,26}\rangle &= |0 \pm 3\rangle_A |2\rangle_B |1\rangle_C, \\
|\psi_{29,30}\rangle &= |3\rangle_A |1 \pm 2\rangle_B |0 \rangle_C, \\
|\psi_{33,34}\rangle &= |2\rangle_A |3\rangle_B |0 \pm 2\rangle_C, \\
|\psi_{37,38}\rangle &= |1 \pm 2\rangle_A |3\rangle_B |1\rangle_C, \\
|\psi_{41,42}\rangle &= |1\rangle_A |1 \pm 2\rangle_B |3\rangle_C, \\
|\psi_{45,46}\rangle &= |3\rangle_A |3\rangle_B |0 \pm 1\rangle_C.
\end{align*}
\]

(4)

Proof: Because the third parties \(A, B, C\) are symmetrical, it is only necessary to prove when the system is nonlocal while \(A\) \& \(B\) goes first, and the other two division methods will also obtain the same measurement results.

After rewriting the above state, we can get (5):

\[
\begin{align*}
|\psi_{1,2}\rangle &= |0 \pm 4\rangle_{AB} |1\rangle_C, \\
|\psi_{5,6}\rangle &= |4 \pm 5\rangle_{AB} |0 \rangle_C, \\
|\psi_{9,10}\rangle &= |2\rangle_{AB} |0 \pm 2\rangle_C, \\
|\psi_{13,14}\rangle &= |4 \pm 12\rangle_{AB} |3\rangle_C, \\
|\psi_{17,18}\rangle &= |13 \pm 15\rangle_{AB} |0 \rangle_C, \\
|\psi_{21,22}\rangle &= |6\rangle_{AB} |1\rangle_C, \\
|\psi_{25,26}\rangle &= |2 \pm 14\rangle_{AB} |1\rangle_C, \\
|\psi_{29,30}\rangle &= |4 \pm 7\rangle_{AB} |2\rangle_C, \\
|\psi_{33,34}\rangle &= |11\rangle_{AB} |0 \pm 2\rangle_C, \\
|\psi_{37,38}\rangle &= |7 \pm 11\rangle_{AB} |1\rangle_C, \\
|\psi_{41,42}\rangle &= |5 \pm 6\rangle_{AB} |3\rangle_C, \\
|\psi_{45,46}\rangle &= |15\rangle_{AB} |0 \pm 1\rangle_C.
\end{align*}
\]

(5)

Suppose Alice starts with the nontrivial and non-disturbing POVM element, \(M_m^\dagger M_m\), we could write the POVM element as a \(d^2 \times d^2\) matrix in \(\{|0\rangle, |1\rangle, \ldots, |15\rangle\}_A\) basis, which corresponds to the states (5):

\[
M_m^\dagger M_m = \begin{bmatrix}
av_0 & a_0\langle 14\rangle & a_0\langle 15\rangle & \cdots & a_{m0}\langle 14\rangle & a_{m0}\langle 15\rangle \\
av_{10} & a_{10}\langle 1\rangle & a_{10}\langle 14\rangle & \cdots & a_{m10}\langle 1\rangle & a_{m10}\langle 14\rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{1400} & a_{1400}\langle 1\rangle & a_{1400}\langle 14\rangle & \cdots & a_{m1400}\langle 1\rangle & a_{m1400}\langle 14\rangle \\
a_{1500} & a_{1500}\langle 1\rangle & a_{1500}\langle 14\rangle & \cdots & a_{m1500}\langle 1\rangle & a_{m1500}\langle 14\rangle \\
\end{bmatrix}
\]

The post-measurement states can be expressed as \((I \otimes M_m)|\psi_i\rangle\), which should be mutually orthogonal. So,

\[
\langle \psi_j | (I \otimes M_m^\dagger M_m) | \psi_i \rangle = 0.
\]

According to this principle, the original matrix could be transformed into:

\[
M_m^\dagger M_m = \begin{bmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{bmatrix}
\]

Table I shows the detailed derivation process.

Therefore, \(A\) \& \(B\)’s measurement is proportional to the identity which means that \(AB\) joint system cannot start with a nontrivial measurement. In the same way, the other two division methods, \(BC|A\), \(AC|B\), will also obtain similar measurement results. So, this set of orthogonal product states is 2-divisible and has strong nonlocality. This completes the proof of the Theorem 2.

From the above results, we can see that the number of states in \(4 \otimes 4 \otimes 4\) system constructed in this way is 48, which is less than the set proposed by Yuan et al., and the set proposed by Halder et al.,[37]. When \(d\) is far away from \(4\), the reduced number of states may not be enough to attract attention, but when the dimension increases, the number of states will be greatly reduced compared to the previous method. For instance, in \(d \otimes d \otimes d\), when \(d\) is odd, the number of reduced states will reach \(3d^2 - 12d + 9\), and when \(d\) is even, the number will reach \(3d^2 - 12d + 6\).

FIG. 1 shows the comparison between the number of states in this paper and the previous ones. Therefore, it is significant to explore the minimum number of states in a system with strong nonlocality.

In addition, we give a set of orthogonal product state with strong nonlocality in \(5 \otimes 5 \otimes 5\), which contains 72 states. The proof of Theorem 3 is similar to the proof of Theorem 2.

**Theorem 3:** In \(5 \otimes 5 \otimes 5\), the following 72 orthogonal product states \(\psi_i\) is strongly nonlocal.

\[
\begin{align*}
|\psi_{1,2}\rangle &= |0 \pm 1\rangle_A |0 \pm 1\rangle_B |1\rangle_C, \\
|\psi_{5,6}\rangle &= |1\rangle_A |0 \pm 1\rangle_B |0 \rangle_C, \\
|\psi_{9,10}\rangle &= |0 \rangle_A |2\rangle_B |0 \pm 2\rangle_C, \\
|\psi_{13,14}\rangle &= |0 \pm 3\rangle_A |0 \rangle_B |3\rangle_C, \\
|\psi_{17,18}\rangle &= |3\rangle_A |0 \pm 3\rangle_B |0 \rangle_C, \\
|\psi_{21,22}\rangle &= |0\rangle_A |4\rangle_B |1 \pm 4\rangle_C, \\
|\psi_{25,26}\rangle &= |1 \pm 3\rangle_A |1\rangle_B |2\rangle_C, \\
|\psi_{29,30}\rangle &= |2\rangle_A |1 \pm 3\rangle_B |1\rangle_C, \\
|\psi_{33,34}\rangle &= |1\rangle_A |3\rangle_B |0 \pm 4\rangle_C, \\
|\psi_{37,38}\rangle &= |1 \pm 4\rangle_A |2\rangle_B |1\rangle_C, \\
|\psi_{41,42}\rangle &= |1\rangle_A |1 \pm 4\rangle_B |2\rangle_C, \\
|\psi_{45,46}\rangle &= |3\rangle_A |3\rangle_B |0 \pm 1\rangle_C, \\
|\psi_{53,54}\rangle &= |1\rangle_A |3 \pm 4\rangle_B |3\rangle_C, \\
|\psi_{57,58}\rangle &= |3\rangle_A |2\rangle_B |2 \pm 4\rangle_C, \\
|\psi_{61,62}\rangle &= |0 \pm 1\rangle_A |3\rangle_B |4\rangle_C, \\
|\psi_{65,66}\rangle &= |4\rangle_A |0 \pm 1\rangle_B |3\rangle_C, \\
|\psi_{69,70}\rangle &= |4\rangle_A |4\rangle_B |0 \pm 2\rangle_C.
\end{align*}
\]

(6)

In this section, we analyze the set of orthogonal product state with strong nonlocality in the tripartite system,
| POVM Element | Corresponding States |
|--------------|----------------------|
| $a_{0i} = a_{0i} = 0$ | $|\phi_{1,1}\rangle, |\phi_{1,2}\rangle, |\phi_{1,3}\rangle, |\phi_{1,4}\rangle, |\phi_{1,5}\rangle$ |
| $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ |
| $|\phi_{2,1}\rangle, |\phi_{2,2}\rangle, |\phi_{2,3}\rangle, |\phi_{2,4}\rangle, |\phi_{2,5}\rangle$ |
| $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
| $|\phi_{3,1}\rangle, |\phi_{3,2}\rangle, |\phi_{3,3}\rangle, |\phi_{3,4}\rangle, |\phi_{3,5}\rangle$ |
| $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $|\phi_{4,1}\rangle, |\phi_{4,2}\rangle, |\phi_{4,3}\rangle, |\phi_{4,4}\rangle, |\phi_{4,5}\rangle$ |
| $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ |
| $|\phi_{5,1}\rangle, |\phi_{5,2}\rangle, |\phi_{5,3}\rangle, |\phi_{5,4}\rangle, |\phi_{5,5}\rangle$ |
| $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ | $i = 11$ |
| $|\phi_{6,1}\rangle, |\phi_{6,2}\rangle, |\phi_{6,3}\rangle, |\phi_{6,4}\rangle, |\phi_{6,5}\rangle$ |
| $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ | $i = 16$ |
| $|\phi_{7,1}\rangle, |\phi_{7,2}\rangle, |\phi_{7,3}\rangle, |\phi_{7,4}\rangle, |\phi_{7,5}\rangle$ |
| $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ |
| $|\phi_{8,1}\rangle, |\phi_{8,2}\rangle, |\phi_{8,3}\rangle, |\phi_{8,4}\rangle, |\phi_{8,5}\rangle$ |
| $i = 8$ | $i = 9$ | $i = 10$ | $i = 11$ | $i = 12$ |
| $|\phi_{9,1}\rangle, |\phi_{9,2}\rangle, |\phi_{9,3}\rangle, |\phi_{9,4}\rangle, |\phi_{9,5}\rangle$ |
| $i = 13$ | $i = 14$ | $i = 15$ | $i = 16$ | $i = 17$ |
| $|\phi_{10,1}\rangle, |\phi_{10,2}\rangle, |\phi_{10,3}\rangle, |\phi_{10,4}\rangle, |\phi_{10,5}\rangle$ |
| $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ | $i = 8$ |
| $|\phi_{11,1}\rangle, |\phi_{11,2}\rangle, |\phi_{11,3}\rangle, |\phi_{11,4}\rangle, |\phi_{11,5}\rangle$ |
| $i = 9$ | $i = 10$ | $i = 11$ | $i = 12$ | $i = 13$ |
| $|\phi_{12,1}\rangle, |\phi_{12,2}\rangle, |\phi_{12,3}\rangle, |\phi_{12,4}\rangle, |\phi_{12,5}\rangle$ |
| $i = 14$ | $i = 15$ | $i = 16$ | $i = 17$ | $i = 18$ |
| $|\phi_{13,1}\rangle, |\phi_{13,2}\rangle, |\phi_{13,3}\rangle, |\phi_{13,4}\rangle, |\phi_{13,5}\rangle$ |
| $i = 5$ | $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ |
| $|\phi_{14,1}\rangle, |\phi_{14,2}\rangle, |\phi_{14,3}\rangle, |\phi_{14,4}\rangle, |\phi_{14,5}\rangle$ |
| $i = 10$ | $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ |
| $|\phi_{15,1}\rangle, |\phi_{15,2}\rangle, |\phi_{15,3}\rangle, |\phi_{15,4}\rangle, |\phi_{15,5}\rangle$ |
| $i = 15$ | $i = 16$ | $i = 17$ | $i = 18$ | $i = 19$ |
and calculate the minimum number of states that may exist in the tripartite system. To better reflect the conclusion, the corresponding systems $4 \otimes 4 \otimes 4$ and $5 \otimes 5 \otimes 5$ are given. In the next section, following the method in this section, the strongly nonlocal four-party systems with less states are discussed.

IV. FOUR-PARTY SYSTEMS WITH STRONG NONLOCALITY

Compared with the three-party system, the strong nonlocality of the four-party system requires that the quantum system is both $3$-divisible and $2$-divisible. In this section, we mainly consider the $3$-divisible system.

In the previous research, the $3$-divisible $3 \otimes 3 \otimes 3 \otimes 3$ quantum system proposed by Halder et al. [40] contains 81 states. Zhang et al. [38] proposed the $3 \otimes 3 \otimes 3 \otimes 3$ quantum system with the same properties includes 56 states. In this section, a $3 \otimes 3 \otimes 3 \otimes 3$ system which contains only 40 states is proposed. It can guarantee that this four-party system is $3$-divisible.

Theorem 4: In $3 \otimes 3 \otimes 3 \otimes 3$, the following 40 orthogonal product states (7) is $3$-divisible.

\[
\begin{align*}
\varphi_{1,2} &= |0 \pm 1\rangle_A|0\rangle_B|1\rangle_C|2\rangle_D \\
\varphi_{3,4} &= |0\rangle_A|1\rangle_B|2\rangle_C|0 \pm 1\rangle_D \\
\varphi_{5,6} &= |1\rangle_A|2\rangle_B|0 \pm 1\rangle_C|0\rangle_D \\
\varphi_{7,8} &= |2\rangle_A|0 \pm 1\rangle_B|0\rangle_C|1\rangle_D \\
\varphi_{9,10} &= |0 \pm 1\rangle_A|2\rangle_B|1\rangle_C|1\rangle_D \\
\varphi_{11,12} &= |2\rangle_A|1\rangle_B|1\rangle_C|0 \pm 1\rangle_D \\
\varphi_{13,14} &= |1\rangle_A|1\rangle_B|0 \pm 1\rangle_C|2\rangle_D \\
\varphi_{15,16} &= |1\rangle_A|0 \pm 1\rangle_B|2\rangle_C|1\rangle_D \\
\varphi_{17,18} &= |0 \pm 2\rangle_A|1\rangle_B|0\rangle_C|0\rangle_D \\
\varphi_{19,20} &= |1\rangle_A|0\rangle_B|0\rangle_C|0 \pm 2\rangle_D \\
\varphi_{21,22} &= |0\rangle_A|0\rangle_B|0 \pm 2\rangle_C|1\rangle_D \\
\varphi_{23,24} &= |0\rangle_A|0 \pm 2\rangle_B|1\rangle_C|0\rangle_D \\
\varphi_{25,26} &= |0 \pm 2\rangle_A|0\rangle_B|2\rangle_C|2\rangle_D \\
\varphi_{27,28} &= |0\rangle_A|2\rangle_B|2\rangle_C|0 \pm 2\rangle_D \\
\varphi_{29,30} &= |2\rangle_A|2\rangle_B|0 \pm 2\rangle_C|0\rangle_D \\
\varphi_{31,32} &= |2\rangle_A|0 \pm 2\rangle_B|0\rangle_C|2\rangle_D \\
\varphi_{33,34} &= |1 \pm 2\rangle_A|2\rangle_B|1\rangle_C|1\rangle_D \\
\varphi_{35,36} &= |2\rangle_A|1\rangle_B|1\rangle_C|1 \pm 2\rangle_D \\
\varphi_{37,38} &= |1\rangle_A|1\rangle_B|1 \pm 2\rangle_C|2\rangle_D \\
\varphi_{39,40} &= |1\rangle_A|1 \pm 2\rangle_B|2\rangle_C|1\rangle_D
\end{align*}
\]

In addition, we present another example in Appendix C, the $4 \otimes 4 \otimes 4 \otimes 4$ quantum system which contains 72 states. The proof of Theorem 5 is similar to the proof of Theorem 4.

In this section, we discuss the $3$-divisible states set in the four-party system, and presents the $3 \otimes 3 \otimes 3 \otimes 3$ and $4 \otimes 4 \otimes 4 \otimes 4$ quantum system as two example to reflect our proposition better.

V. CONCLUSION

In summary, we concentrated on the tripartite and four-party system and constructed some set of orthogonal product state which exhibits strong nonlocality. Firstly,
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Appendix A: Proof of Example 1

Example 1: The following 54 states (3) is strongly nonlocal.

Proof: Let $|00⟩ → |0⟩, |01⟩ → |1⟩, |02⟩ → |2⟩, |10⟩ → |3⟩, · · ·, |32⟩ → |14⟩, |33⟩ → |15⟩$. After rewriting the state (3), we can get the following states (A1):

\[
\varphi_{1,2} = |0 \pm 4⟩_A|0⟩_B|0⟩_C
\]

\[
\varphi_{3,4} = |1⟩_A|0 \pm 1⟩_B|0⟩_C
\]

\[
\varphi_{5,6} = |4 \pm 5⟩_A|0⟩_B|0⟩_C
\]

\[
\varphi_{7,8} = |0 \pm 8⟩_A|0⟩_B|0⟩_C
\]

\[
\varphi_{9,10} = |2⟩_A|0 \pm 2⟩_B|0⟩_C
\]

\[
\varphi_{11,12} = |8 \pm 10⟩_A|0⟩_B|0⟩_C
\]

\[
\varphi_{13,14} = |0 \pm 12⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{15,16} = |3⟩_A|0 \pm 3⟩_B|0⟩_C
\]

\[
\varphi_{17,18} = |12 \pm 15⟩_A|0⟩_B|0⟩_C
\]

\[
\varphi_{19,20} = |6 \pm 10⟩_A|0⟩_B|2⟩_C
\]

\[
\varphi_{21,22} = |10⟩_A|0 \pm 2⟩_B|0⟩_C
\]

\[
\varphi_{23,24} = |9 \pm 10⟩_A|0⟩_B|2⟩_C
\]

\[
\varphi_{25,26} = |7 \pm 15⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{27,28} = |15⟩_A|0 \pm 3⟩_B|0⟩_C
\]

\[
\varphi_{29,30} = |13 \pm 15⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{31,32} = |11 \pm 15⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{33,34} = |15⟩_A|0⟩_B|2⟩_C
\]

\[
\varphi_{35,36} = |14 \pm 15⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{37,38} = |1 \pm 5⟩_A|0⟩_B|2⟩_C
\]

\[
\varphi_{39,40} = |6⟩_A|0 \pm 1⟩_B|0⟩_C
\]

\[
\varphi_{41,42} = |8 \pm 9⟩_A|0⟩_B|1⟩_C
\]

\[
\varphi_{43,44} = |2 \pm 10⟩_A|0⟩_B|3⟩_C
\]

\[
\varphi_{45,46} = |11⟩_A|0 \pm 2⟩_B|0⟩_C
\]

\[
\varphi_{47,48} = |12 \pm 14⟩_A|0⟩_B|2⟩_C
\]

\[
\varphi_{49,50} = |3 \pm 15⟩_A|0⟩_B|1⟩_C
\]

\[
\varphi_{51,52} = |13⟩_A|0 \pm 3⟩_B|0⟩_C
\]

\[
\varphi_{53,54} = |4 \pm 7⟩_A|0⟩_B|3⟩_C
\]

(A1)

Suppose $AB$ starts with the nontrivial and nondisturbing measurement, represented by a set of POVM elements $M^\dagger_m M_m$ on $d^2 × d^2$. the POVM measurement in $\{00,01,02,03,10,11,12,13,20,21,22,23,30,31,32\}$ basis can be written, which corresponds to the states (A1):

\[
M^\dagger_m M_m = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{0(14)} & a_{0(15)} \\
a_{10} & a_{11} & \cdots & a_{1(14)} & a_{1(15)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(14)0} & a_{(14)1} & \cdots & a_{(14)(14)} & a_{(14)(15)} \\
a_{(15)0} & a_{(15)1} & \cdots & a_{(15)(14)} & a_{(15)(15)} \\
\end{pmatrix}
\]

The post-measurement states can be expressed as $(I ⊗ M_m)|ϕ_i⟩$, which should be mutually orthogonal. Then $⟨ϕ_j|(I ⊗ M^\dagger_m M_m)|ϕ_i⟩ = 0$ is obtained. According to this principle, the original matrix could be transformed into:

\[
M^\dagger_m M_m = \begin{pmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a \\
\end{pmatrix}
\]

Table II shows the detailed derivation process.

Obviously, $AB$’s measurement matrix is proportional to the identity matrix, so it means that $AB$’s measurement starts with a trivial measurement, and cannot get any information about shared state distinction from the measurement result. As for the other two division methods, $BC|A, AC|B$, the proof method is similar to this. In summary, the multi-party divided into any two parts can keep the strong nonlocality of the quantum system.

Appendix B: Proof of Theorem 4

Theorem 4: In $3 ⊗ 3 ⊗ 3 ⊗ 3$, the following 40 orthogonal product states (7) is 3-divisible.

Proof: Let $|00⟩ → |0⟩, |01⟩ → |1⟩, |02⟩ → |2⟩, |10⟩ → |3⟩, |11⟩ → |4⟩, |12⟩ → |5⟩, |20⟩ → |6⟩, |21⟩ → |7⟩, |22⟩ → |8⟩$. After rewriting the above state, we can
| POVM Element | Corresponding States |
|--------------|---------------------|
| $a_{01} = a_{10} = 0$ | $|\psi_{1,2},\rangle,|\psi_{3},\rangle$ | $|\psi_{1,2},\rangle,|\psi_{3},\rangle$ | $|\psi_{1,2},\rangle,|\psi_{15},\rangle$ | $|\psi_{7,8},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{7,8},\rangle,|\psi_{5,6},\rangle$ |
| $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ |
| $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
| $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{11} = a_{11} = 0$ | $|\psi_{3},\rangle,|\psi_{9},\rangle$ | $|\psi_{3},\rangle,|\psi_{15},\rangle$ | $|\psi_{3},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{3},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{3},\rangle,|\psi_{5,6},\rangle$ |
| $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ |
| $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ | $i = 11$ |
| $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{21} = a_{12} = 0$ | $|\psi_{9},\rangle,|\psi_{115},\rangle$ | $|\psi_{9},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{9},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{9},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{9},\rangle,|\psi_{5,6},\rangle$ |
| $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ |
| $i = 8$ | $i = 9$ | $i = 10$ | $i = 11$ | $i = 12$ |
| $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{31} = a_{13} = 0$ | $|\psi_{15},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{15},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{15},\rangle,|\psi_{9},\rangle$ | $|\psi_{15},\rangle,|\psi_{9},\rangle$ |
| $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ | $i = 8$ |
| $i = 9$ | $i = 10$ | $i = 11$ | $i = 12$ | $i = 13$ |
| $i = 14$ | $i = 15$ |
| $a_{41} = a_{14} = 0$ | $|\psi_{1,2},\rangle,|\psi_{37,38},\rangle$ | $|\psi_{1,2},\rangle,|\psi_{39},\rangle$ | $|\psi_{1,2},\rangle,|\psi_{25,26},\rangle$ | $|\psi_{1,2},\rangle,|\psi_{25,26},\rangle$ |
| $i = 10$ | $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ |
| $i = 15$ |
| $a_{51} = a_{15} = 0$ | $|\psi_{37,38},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{37,38},\rangle,|\psi_{25,26},\rangle$ | $|\psi_{37,38},\rangle,|\psi_{41,42},\rangle$ | $|\psi_{37,38},\rangle,|\psi_{41,42},\rangle$ |
| $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
| $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{61} = a_{16} = 0$ | $|\psi_{39},\rangle,|\psi_{5,6},\rangle$ | $|\psi_{39},\rangle,|\psi_{41,42},\rangle$ | $|\psi_{39},\rangle,|\psi_{211},\rangle$ | $|\psi_{39},\rangle,|\psi_{211},\rangle$ |
| $i = 8$ | $i = 9$ | $i = 10$ | $i = 11$ | $i = 12$ |
| $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{71} = a_{17} = 0$ | $|\psi_{53,54},\rangle,|\psi_{41,42},\rangle$ | $|\psi_{53,54},\rangle,|\psi_{211},\rangle$ | $|\psi_{53,54},\rangle,|\psi_{41,42},\rangle$ | $|\psi_{53,54},\rangle,|\psi_{41,42},\rangle$ |
| $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{81} = a_{18} = 0$ | $|\psi_{7,8},\rangle,|\psi_{23,24},\rangle$ | $|\psi_{7,8},\rangle,|\psi_{21,\rangle}$ | $|\psi_{7,8},\rangle,|\psi_{45},\rangle$ | $|\psi_{7,8},\rangle,|\psi_{21},\rangle$ |
| $i = 10$ | $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ |
| $a_{91} = a_{19} = 0$ | $|\psi_{41,42},\rangle,|\psi_{211},\rangle$ | $|\psi_{41,42},\rangle,|\psi_{41,42},\rangle$ | $|\psi_{41,42},\rangle,|\psi_{211},\rangle$ | $|\psi_{41,42},\rangle,|\psi_{211},\rangle$ |
| $i = 15$ |
| $a_{101} = a_{110} = 0$ | $|\psi_{211},\rangle,|\psi_{45},\rangle$ | $|\psi_{211},\rangle,|\psi_{47,48},\rangle$ | $|\psi_{211},\rangle,|\psi_{29,30},\rangle$ | $|\psi_{211},\rangle,|\psi_{29,30},\rangle$ |
| $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{111} = a_{111} = 0$ | $|\psi_{45},\rangle,|\psi_{25,26},\rangle$ | $|\psi_{45},\rangle,|\psi_{25,26},\rangle$ | $|\psi_{45},\rangle,|\psi_{47,48},\rangle$ | $|\psi_{45},\rangle,|\psi_{29,30},\rangle$ |
| $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $a_{121} = a_{112} = 0$ | $|\psi_{13,14},\rangle,|\psi_{35,36},\rangle$ | $|\psi_{13,14},\rangle,|\psi_{35,36},\rangle$ | $|\psi_{13,14},\rangle,|\psi_{35,36},\rangle$ | $|\psi_{13,14},\rangle,|\psi_{35,36},\rangle$ |
| $i = 14$ | $i = 15$ |
| $a_{131} = a_{113} = 0$ | $|\psi_{35,36},\rangle,|\psi_{29,30},\rangle$ | $|\psi_{35,36},\rangle,|\psi_{29,30},\rangle$ |
| $i = 15$ |
| $a_{141} = a_{114} = 0$ | $|\psi_{35,36},\rangle,|\psi_{29,30},\rangle$ | $|\psi_{35,36},\rangle,|\psi_{29,30},\rangle$ |
| $i = 15$ |
| $a_{00} = a_{ii}$ | $|\psi_{37,38},\rangle$ | $|\psi_{39,45},\rangle$ | $|\psi_{23,24},\rangle$ | $|\psi_{41,42},\rangle$ |
| $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ |
| $i = 6$ | $i = 7$ | $i = 8$ | $i = 9$ | $i = 10$ |
| $i = 11$ | $i = 12$ | $i = 13$ | $i = 14$ | $i = 15$ |
| $|\psi_{31,32},\rangle$ | $|\psi_{13,14},\rangle$ | $|\psi_{29,30},\rangle$ | $|\psi_{35,36},\rangle$ | $|\psi_{17,18},\rangle$ |
get (B1):

\[
\begin{align*}
|\varphi_{1.2}\rangle &= |0 \pm 3\rangle_{AB}|1\rangle_C|2\rangle_D, \\
|\varphi_{2.4}\rangle &= |1\rangle_{AB}|2\rangle_C|0 \pm 1\rangle_D, \\
|\varphi_{5.6}\rangle &= |5\rangle_{AB}|0 \pm 1\rangle_C|0\rangle_D, \\
|\varphi_{7.8}\rangle &= |6 \pm 7\rangle_{AB}|0\rangle_C|1\rangle_D, \\
|\varphi_{9.10}\rangle &= |2 \pm 5\rangle_{AB}|1\rangle_C|1\rangle_D, \\
|\varphi_{11.12}\rangle &= |7\rangle_{AB}|1\rangle_C|0 \pm 1\rangle_D, \\
|\varphi_{13.14}\rangle &= |4\rangle_{AB}|0\rangle_C|2\rangle_D, \\
|\varphi_{15.16}\rangle &= |3 \pm 4\rangle_B|2\rangle_C|1\rangle_D,
\end{align*}
\]

(B1)

Suppose AB starts with the nontrivial and non-disturbing measurement, represented by a set of POVM elements \(M^\dagger M_m\) on \(d^2 \times d^2\). We could write the POVM measurement in \(\{0, 1, \cdots, |S\rangle\}_A\) basis, which corresponds to the states (B1):

\[
M^\dagger M_m = \begin{bmatrix}
    a_{00} & a_{01} & 0 & 0 & a_{05} & 0 & 0 & 0 \\
    a_{10} & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{20} & a_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{30} & a_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{40} & a_{41} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{50} & a_{51} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{60} & a_{61} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{70} & a_{71} & 0 & 0 & 0 & 0 & 0 & 0 \\
    a_{80} & a_{81} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The post-measurement states can be expressed as \((I \otimes M_m)|\varphi_i\rangle\), which should be mutually orthogonal. So, \((\langle \varphi_j | (I \otimes M^\dagger M_m) | \varphi_i \rangle = 0\) is obtained. According to this principle, the original matrix could be transformed into:

\[
M^\dagger M_m = \begin{bmatrix}
    a & 0 & \cdots & 0 \\
    0 & a & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a
\end{bmatrix}
\]

Table III shows the detailed derivation process.

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| POVM Element | Corresponding States |
|--------------|----------------------|
| $a_{0i} = a_{i0} = 0$ | $i = 1$ $|\psi_{21}\rangle, |\psi_{23}\rangle$ $i = 2$ $|\psi_{21}\rangle, |\psi_{27}\rangle$ $i = 3$ $|\psi_{21}\rangle, |\psi_{19}\rangle$ $i = 4$ $|\psi_{21}\rangle, |\psi_{13}\rangle$ |
| $a_{1i} = a_{i1} = 0$ | $i = 1$ $|\psi_{3}\rangle, |\varphi_{9,10}\rangle$ $i = 2$ $|\psi_{3}\rangle, |\varphi_{215,16}\rangle$ $i = 3$ $|\psi_{3}\rangle, |\varphi_{223,34}\rangle$ $i = 4$ $|\psi_{3}\rangle, |\varphi_{9,10}\rangle$ |
| $a_{2i} = a_{i2} = 0$ | $i = 1$ $|\varphi_{27}\rangle, |\varphi_{15,16}\rangle$ $i = 2$ $|\varphi_{27}\rangle, |\varphi_{15,16}\rangle$ $i = 3$ $|\varphi_{27}\rangle, |\varphi_{23,34}\rangle$ $i = 4$ $|\varphi_{27}\rangle, |\varphi_{7,8}\rangle$ |
| $a_{3i} = a_{i3} = 0$ | $i = 1$ $|\varphi_{19}, |\varphi_{13}\rangle$ $i = 2$ $|\varphi_{19}\rangle, |\varphi_{33,34}\rangle$ $i = 3$ $|\varphi_{19}\rangle, |\varphi_{7,8}\rangle$ $i = 4$ $|\varphi_{19}\rangle, |\varphi_{7,8}\rangle$ |
| $a_{4i} = a_{i4} = 0$ | $i = 1$ $|\varphi_{13}\rangle, |\varphi_{33,34}\rangle$ $i = 2$ $|\varphi_{13}\rangle, |\varphi_{7,8}\rangle$ $i = 3$ $|\varphi_{13}\rangle, |\varphi_{7,8}\rangle$ $i = 4$ $|\varphi_{13}\rangle, |\varphi_{33,34}\rangle$ |
| $a_{5i} = a_{i5} = 0$ | $i = 1$ $|\varphi_{5}\rangle, |\varphi_{7,8}\rangle$ $i = 2$ $|\varphi_{5}\rangle, |\varphi_{7,8}\rangle$ $i = 3$ $|\varphi_{5}\rangle, |\varphi_{7,8}\rangle$ $i = 4$ $|\varphi_{5}\rangle, |\varphi_{7,8}\rangle$ |
| $a_{6i} = a_{i6} = 0$ | $i = 1$ $|\varphi_{29}, |\varphi_{11}\rangle$ $i = 2$ $|\varphi_{29}, |\varphi_{29}\rangle$ $i = 3$ $|\varphi_{29}, |\varphi_{29}\rangle$ $i = 4$ $|\varphi_{29}, |\varphi_{29}\rangle$ |
| $a_{7i} = a_{i7} = 0$ | $i = 1$ $|\varphi_{11}\rangle, |\varphi_{29}\rangle$ $i = 2$ $|\varphi_{11}\rangle, |\varphi_{29}\rangle$ $i = 3$ $|\varphi_{11}\rangle, |\varphi_{29}\rangle$ $i = 4$ $|\varphi_{11}\rangle, |\varphi_{29}\rangle$ |

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