Superpotential method for $F(R)$ cosmological models

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Abstract

We construct the $F(R)$ gravity models with exact particular solutions using the conformal transformation and the superpotential method for the corresponding models in the Einstein frame. The functions $F(R)$ are obtained explicitly. We consider exact solutions for the obtained $R^2$ gravity model with the cosmological constant in detail.

1 Introduction

The $F(R)$ gravity is one of the most popular generalizations of the general relativity [1,2,3]. The $F(R)$ gravity models are actively used to describe different epochs of the Universe evolution. For example, the Starobinsky $R^2$ inflationary model [4] (see also [5]) leads to the predictions that do not contradict to the observation data [6]. Dark energy $F(R)$ models are actively investigated and analyzed, for example, in papers [7,8,9,10,11,12,13,14,15]. Moreover, there are $F(R)$ gravity models that can describe both inflation and the late time cosmic acceleration [16,17,18].

Exact solutions play an important role in cosmology and the search of integrable $F(R)$ models as well as models with exact particular solutions is an interesting problem [19,20]. A $F(R)$ gravity model can be transformed into a model with a minimally coupled scalar field with a canonical kinetic term by the metric and scalar field transformations [21]. There are a few methods to construct models with minimally coupled scalar fields with exact cosmological solutions. One of the popular methods is the superpotential one [22,23] (also known as the Hamilton–Jacobi method or the first-order formalism). In this paper, we generalize this method on $F(R)$ gravity models to get such models with exact solutions. We show a few examples of such models with an explicit dependence of $F(R)$.

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2 The corresponding Einstein frame models

Let us consider an $F(R)$ gravity model:

$$S_R = \int d^4 \tilde{x} \sqrt{-\tilde{g}} F(R),$$

where $F(R)$ is a double differentiable function of the Ricci scalar $R$. Introducing a new scalar field $\sigma$ without the kinetic term, we rewrite $S_R$ as follows [21, 24]:

$$\tilde{S}_J = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ \partial F(\sigma) \frac{\partial}{\partial \sigma} (R - \sigma) + F(\sigma) \right].$$

By the conformal transformation of the metric $g_{\mu\nu} = \frac{2}{f(\sigma)} \tilde{g}_{\mu\nu}$, where $f \equiv \frac{df(\sigma)}{d\sigma}$, one gets the following action in the Einstein frame [25]:

$$S_E = \int d^4 x \sqrt{-g} \left[ \frac{M^2_{Pl}}{2} R_E - \frac{h(\sigma)}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V_E \right],$$

where

$$h(\sigma) = \frac{3M^2_{Pl}}{2f^2} \left( \frac{df}{d\sigma} \right)^2, \quad V_E = \frac{M^4_{Pl}}{4f^2} f \sigma - F.$$ 

Introducing the scalar field

$$\psi = \sqrt{\frac{3}{2}} M_{Pl} \ln \left( \frac{2}{M^2_{Pl}} f(\sigma) \right),$$

we obtain the action $S_E$ as follows:

$$S_E = \int d^4 x \sqrt{-g} \left[ \frac{M^2_{Pl}}{2} R_E - \frac{1}{2} \partial_\mu \psi \partial_\mu \psi - V_E(\psi) \right].$$

So, we get the Einstein frame model with a standard scalar field. To obtain inverse transformation we present the potential and its derivative in the following form:

$$V_E(\psi) = \frac{M^2_{Pl}}{2} Re^{-\frac{3\sqrt{6}}{2M_{Pl}}} - Fe^{-\frac{2\psi}{M_{Pl}}}, \quad \frac{dV_E(\psi)}{d\psi} = -\frac{M_{Pl}}{\sqrt{6}} Re^{-\frac{3\sqrt{6}}{2M_{Pl}}} + \frac{4}{\sqrt{6}M_{Pl}} Fe^{-\frac{2\psi}{M_{Pl}}}.$$

So, we get the function $F(R)$ in a parametric form [26, 27, 28]:

$$R = \left[ \frac{\sqrt{6}}{M_{Pl}} \frac{dV_E}{d\psi} + \frac{4V_E}{M^2_{Pl}} \right] e^{\frac{\psi}{3M_{Pl}}},$$

$$F = \frac{M^2_{Pl}}{2} \left[ \frac{\sqrt{6}}{M_{Pl}} \frac{dV_E}{d\psi} + \frac{2V_E}{M^2_{Pl}} \right] e^{\frac{\psi}{3M_{Pl}}}.$$

So, if model with one minimally coupled scalar field has exact solutions, then the corresponding $F(R)$ gravity model has them as well. The goal of this paper is to find such potentials $V_E$ that the Einstein frame model has exact solutions and the function $F(R)$ can be found in the analytic form.
3 Construction of $F(R)$ gravity models

If Eq. (6) has the solution

$$R = C_1 + C_k e^{k \psi_{M_P}} ,$$  

where $C_1$ and $C_k \neq 0$ are arbitrary constants, then the function $F(R)$ can be obtained in the analytic form. From Eqs. (6) and (8), we get the following linear first order differential equation for the potential $V_E$:

$$\left[ \sqrt{6} \frac{dV_E}{M_{Pl}} + 4 \frac{V_E}{M_{Pl}^2} \right] e^{\frac{k}{3} \psi_{M_P}} = C_1 + C_k e^{k \psi_{M_P}} .$$  

Equation (9) has the general solution:

$$V_E(\psi) = \frac{M_{Pl}^2}{2} \left( C_2 e^{-\frac{2}{3} \psi_{M_P}} + C_1 e^{-\frac{\psi_{M_P}}{3}} + C_\omega e^{\frac{\psi_{M_P}}{3}} \right) ,$$  

where $C_2$ is an integration constant, $\omega = \sqrt{6k}/2 - 1$, and $C_\omega = \frac{\sqrt{6k}}{\sqrt{6+3k}} = \frac{C_k}{\omega+2}$.

Substituting the potential (10) into (6) and (7), we obtain the following expressions:

$$R = C_\omega (\omega + 2) e^{(\omega + 1) \frac{\psi_{M_P}}{3}} + C_1 , \quad F = \frac{M_{Pl}^2}{2} \left( C_\omega (\omega + 1) e^{(\omega + 2) \frac{\psi_{M_P}}{3}} - C_2 \right) .$$  

Finally, we get

$$F(R) = \frac{M_{Pl}^2}{2} \left( C_\omega (\omega + 1) \left( \frac{R - C_1}{C_\omega (\omega + 2)} \right)^{\frac{\alpha}{\omega+2}} - C_2 \right) , \quad \text{where } \alpha = \frac{\omega + 2}{\omega + 1} .$$  

It is easy to see that $\alpha \neq 1$ for any $\omega$, also $\alpha = 2$ corresponds to $\omega = 0$.

4 The search of exact solutions

Let us consider the potential (10) in the case of $C_1 = C_2 = 0$. If $\omega \neq 0$, then we get an exponential potential and an integrable cosmological model [22, 29, 30, 31]. The general solutions of this model can be found explicitly in a parametric time [30]. The general solutions of the corresponding $R^\alpha$ models with an arbitrary $\alpha$, but $\alpha \neq 2$ and $\alpha \neq 1$, can be obtained from the general solution of the model with an exponential potential by the conformal transformation of the metric. In the case of $\omega = 0$, we get the model with the cosmological constant that is integrable (the general solution is presented in [32, 33]) and corresponds to a pure $R^2$ gravity model.

To prove the integrability of the cosmological model with an exponential potential the superpotential method has been used in the paper [22]. This method is actively used to get cosmological models with exact particular solutions both with one scalar field [22, 23, 34, 35, 36, 37, 38] and with a few scalar fields [22, 39, 40, 41], as well as to construct inflationary models [42, 43, 44, 45]. We use this method to get models with exact solutions in the cases of $C_1^2 + C_2^2 \neq 0$.

For the spatially flat Friedmann–Lemaître–Robertson–Walker metric with

$$ds^2 = -dt^2 + a_E^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right) ,$$  

(13)
the Einstein equations can be written in the following form:

\[ \dot{\psi} = -2M_{Pl}^2 W', \]

\[ V_E = 3M_{Pl}^2 W^2 - 2M_{Pl}^4 W'^2, \]

where the Hubble parameter \( H_E \equiv \dot{a}/a = W(\psi) \) and \( W' = \frac{dW}{d\psi} \).

Choosing the superpotential

\[ W(\psi) = W_a e^{\frac{\sqrt{2}\psi}{2M_{Pl}} + W_b e^{\frac{\sqrt{2}\psi}{2M_{Pl}}}}, \]

where \( a, b, W_a \) and \( W_b \) are constants, we get the following potential:

\[ V_E = 6 \left( W_a^2 (1 - a^2) e^{\frac{\sqrt{2}\psi}{2M_{Pl}}} + 2W_a W_b (1 - ab) e^{\frac{\sqrt{2}\psi}{2M_{Pl}}} + W_b^2 (1 - b^2) e^{\frac{\sqrt{2}\psi}{2M_{Pl}}} \right). \]

For some values of parameters \( a \) and \( b \), we get potentials in the form \( \Box \) and the corresponding \( F(R) \) gravity models with exact solutions (see Table \( \Box \)). We can assume that \( a < b \) without loss of generality. In the cases \( a = -1/3, b = 1 \) and \( a = -1, b = 1/3 \) the resulting \( F(R) \) models are coincide. In the general case the particular solution \( \psi(t) \) can be obtained in quadratures by integrating Eq. \( \Box \).

| \( a, b \) | \( \omega \) | \( \alpha \) | \( C_2 \) | \( C_1 \) | \( C_\omega \) | \( 2F(R)/M_{Pl}^2 \) |
|--------|------|------|------|------|------|----------|
| \( a = -\frac{2}{3}, b = 0 \) | 0 | 2 | 10/3 \( W_a^2 \) | 12 \( W_a W_b \) | 6 \( W_b^2 \) | \( \frac{1}{24W_b^2} R^2 - \frac{W_a}{W_b} R + \frac{8W_a^2}{3} \) |
| \( a = -\frac{2}{3}, b = 1 \) | 1/2 | 5/3 | 10/3 \( W_a^2 \) | 0 | 20 \( W_a W_b \) | 30 \( W_a W_b \) | \( \sqrt{\frac{R}{50W_a W_b}} \) \( 5/3 \) - \( 10W_b^2/3 \) |
| \( a = -1, b = \frac{1}{3} \) | 1 | 3/2 | 0 | 16 \( W_a W_b \) | 6 \( W_a^2 \) | \( \frac{32}{3} W_b^2 \) | \( \frac{1}{16W_b^2} R - \frac{W_a}{W_b} \) \( 3/2 \) |
| \( a = -\frac{1}{3}, b = 1 \) | 1 | 3/2 | 0 | \( \frac{16}{3} W_a^2 \) | 16 \( W_a W_b \) | 32 \( W_a W_b \) | \( \frac{1}{8W_a W_b} R - \frac{W_a}{W_b} \) \( 3/2 \) |
| \( a = -3, b = -\frac{1}{3} \) | -9 | 7/8 | 0 | \( \frac{16}{3} W_b^2 \) | -48 \( W_a^2 \) | 384 \( W_a^2 \) | \( \frac{1}{336W_a} R - \frac{W_a}{W_b} \) \( 7/8 \) |
| \( a = -\frac{7}{3}, b = 1 \) | -7 | 5/6 | 40 \( W_a W_b \) | 0 | -\( \frac{80}{3} W_a^2 \) | 160 \( W_a^2 \) | \( \frac{3R}{400W_a^2} \) \( 5/6 \) - 40 \( W_a W_b \) |
| \( a = -\frac{5}{3}, b = 1 \) | -5 | 3/4 | 0 | 32 \( W_a W_b \) | -\( \frac{32}{3} W_a^2 \) | \( \frac{128}{3} W_a^2 \) | \( \frac{R}{32W_a^2} - \frac{W_a}{W_b} \) \( 3/4 \) |
| \( a = -\frac{3}{2}, b = -\frac{2}{3} \) | -9/2 | 5/7 | \( \frac{10}{3} W_b^2 \) | 0 | -\( \frac{15}{2} W_a^2 \) | \( \frac{105}{4} W_a^2 \) | \( \frac{4R}{75W_a^2} \) \( 5/4 \) - \( 10W_a^2/3 \) |
| \( a = -1, b = -\frac{2}{3} \) | -5/2 | 1/3 | \( \frac{10}{3} W_b^2 \) | 0 | 4 \( W_a W_b \) | 6 \( W_a W_b \) | \( \frac{1}{2W_a W_b} R \) \( 3/4 \) - \( 10W_a^2/3 \) |
| \( a = -\frac{2}{3}, b = -\frac{1}{3} \) | -3/2 | -1 | \( \frac{10}{3} W_a^2 \) | \( \frac{16}{3} W_b^2 \) | \( \frac{28}{3} W_a W_b \) | \( \frac{196W_a^2 W_b^2}{3(3R - 16W_b)} \) - \( 10W_b^2/3 \) |

5. The case of the \( R^2 \) gravity

Let us consider in detail the case of the \( R^2 \) gravity model with

\[ F(R) = \frac{M_{Pl}^2}{2} \left( \frac{1}{24W_b^2} R^2 - \frac{W_a}{W_b} R + \frac{8W_a^2}{3} \right), \]
that corresponds to $W(\psi) = W_a \exp\left(-\frac{\sqrt{6} \psi}{3M_{Pl}}\right) + W_b$.

Equation (14) leads to

$$\frac{d\psi}{dt} = \frac{2\sqrt{6}}{3} M_{Pl} W_a e^{-\frac{\sqrt{6} \psi}{3M_{Pl}}}, \quad \Rightarrow \quad \psi = \frac{\sqrt{6}}{2} M_{Pl} \ln\left(\frac{4W_a}{3} (t - t_0)\right),$$

(19)

where $t_0$ is an integration constant.

Substituting $\psi(t)$ into $W(\psi)$, we get the Hubble parameter for the model with the scalar field:

$$H_E = \frac{3}{4(t - t_0)} + W_b.$$

(20)

In the initial $F(R)$ model, we get the Friedmann–Lemaître–Robertson–Walker metric

$$ds^2 = -\frac{M_{Pl}^2}{2f(R)} dt^2 + \tilde{a}^2 (dx_1^2 + dx_2^2 + dx_3^2), \quad \text{where} \quad \tilde{a}^2 = \frac{M_{Pl}^2}{2f(R)} a_E^2.$$

(21)

Using Eq. (11), we get

$$f(R) = \frac{M_{Pl}^2}{2} \exp\left(\frac{\sqrt{6}}{3} \frac{\psi}{M_{Pl}}\right) = \frac{2}{3} M_{Pl} W_a (t - t_0).$$

The cosmic time in this frame is

$$\tilde{t} = \sqrt{\frac{M_{Pl}^2}{2f(\sigma)}} dt = \sqrt{\frac{3(t - t_0)}{W_a}} + \tilde{t}_0.$$

(22)

The corresponding Hubble parameter can be presented in the form:

$$\tilde{H} = \tilde{a}^{-1} \frac{d\tilde{a}}{dt} = \sqrt{\frac{2f(R)}{M_{Pl}^2}} \left[ H_E - \frac{1}{2} \frac{d\ln(f)}{dt} \right] = \frac{1}{2(t - t_0)} + \frac{2}{3} W_b W_a (\tilde{t} - \tilde{t}_0).$$

(23)

The first term of this expression corresponds to the radiation dominated universe, whereas the second term is the Ruzmaikina–Rusmaikin solution [46].

6 Conclusions

In this paper, we have found a few $F(R)$ gravity models with exact solutions and shown that the superpotential method is a useful tool for this propose. The existence of a fundamental scalar field (the Higgs boson) gives good motivation to consider modified gravity models with an additional scalar field. The $F(R, \chi)$ gravity models with the scalar field $\chi$ [47] are very popular [24, 48, 49, 50, 51, 52] as models of inflation, in particular, the mixed Higgs–$R^2$ model [49, 50, 51]. We plan to generalize the investigation on the $F(R, \chi)$ models and to use the superpotential method developed for the search of exact solutions of the chiral cosmological models [41], or some other methods [53] to construct physically interesting $F(R, \chi)$ models with exact solutions.

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