Spectral flow in the supersymmetric $t$-$J$ model with a $1/r^2$ interaction

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The spectral flow in the supersymmetric $t$-$J$ model with $1/r^2$ interaction is studied by analyzing the exact spectrum with twisted boundary conditions. The spectral flows for the charge and spin sectors are shown to nicely fit in with the motif picture in the asymptotic Bethe ansatz. Although fractional exclusion statistics for the spin sector clearly shows up in the period of the spectral flow at half filling, such a property is generally hidden once any number of holes are doped, because the commensurability condition in the motif is not met in the metallic phase.

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I. INTRODUCTION

Since the seminal work of Sutherland and Shastry [1,2], the ground state properties in quantum systems with adiabatic change of twisted boundary conditions have attracted much current interest. Although the period of the full spectrum should be $2\pi$ as a function of the twist angle, each eigenstate has a period larger than $2\pi$ in general, if we trace the flow of the initial state adiabatically. This period provides us various information for quantum spin systems, correlated electron systems, etc. For example, it can be a sensitive probe to examine whether particles form a bound state like Cooper pair [3,4].

One of the remarkable applications of this method is to observe (fractional) exclusion statistics [1] directly in the spectral flow of the ground state for one-dimensional quantum systems [1]. For this purpose, it is crucial to sweep away all irrelevant interactions from the model, because the period is quite sensitive to irrelevant interactions, leading to the period $4\pi$ if they exist [1,2], and hiding the nature of exclusion statistics. Among others, it is known that quantum many-body systems with $1/r^2$ interaction are provided with an ideal situation to observe exclusion statistics, i.e., these systems are completely free from irrelevant interactions. Motivated by this, we have recently studied the spectral flow in the Haldane-Shastry spin model with $1/r^2$ exchange by generalizing the model to include twisted boundary conditions [1]. We have indeed found that the period of the spectral flow is controlled by the statistical interaction in exclusion statistics.

In order to investigate exclusion statistics for correlated electron systems, we wish to ask a basic question, i.e. what happens for the spectral flow when holes are doped into the Haldane-Shastry spin model. To address this question, we investigate in this paper the exact spectral flow in the supersymmetric $t$-$J$ model with $1/r^2$ hopping and exchange, which was exactly solved for the first time by Kuramoto and Yokoyama with periodic boundary conditions [10–12]. We extend the model to include the twisted boundary conditions, and then discuss spectral properties of the model as a function of the twist angle.

The paper is organized as follows. In the next section, we briefly mention how to generalize the solvable supersymmetric $t$-$J$ model of $1/r^2$ interaction to be compatible with twisted boundary conditions. In section III, we then derive the exact eigenstates of the model by exploiting a Jastrow-type ansatz. In section IV, it is shown that the spectrum thus obtained can be given alternatively by the asymptotic Bethe ansatz (ABA). It is then demonstrated that the spectral flow of the model is completely described in terms of the motif in the ABA. We find that once holes are doped into the Haldane-Shastry spin chain, the nature of fractional statistics can not be observed in the period of the spectral flow, though elementary excitations are still characterized by exclusion statistics. We claim that this is related to the breaking of the commensurability condition in the motif picture. Section V is devoted to summary.

II. MODEL HAMILTONIAN

We consider the supersymmetric $t$-$J$ model with $1/r^2$ interaction [10]. The model Hamiltonian is given by

$$H = \mathcal{P} \sum_{i \neq j} \sum_{l=-\infty}^{\infty} \frac{1}{(l-j-lN)^2} \times$$
\[
\left[ -\sum_{\sigma} c_{i\sigma}^\dagger c_{j+iN\sigma} + \left( S_i \cdot S_{j+iN} - \frac{1}{4} n_i n_{j+iN} \right) \right] P, \tag{1}
\]

where \( P \) is the projection operator to exclude the double occupancies at each site, \( i \) and \( j \) denote the indices for sites \((i, j) = 1, 2, \ldots, N\), and \( \sigma \) is a spin index labeled by \( \sigma = \uparrow \) and \( \downarrow \). Note that the summation with respect to \( l \) is introduced to treat the system in a ring geometry. Kuramoto and Yokoyama have shown that this model can be solved exactly with periodic boundary conditions, in which the summation over \( l \) results in the sine-inverse-square interaction. This model, which includes the Haldane-Shastry spin chain at half filling, provides us with a paradigm of ideal exclusion statistics.

Preserving the integrability, we wish to treat the model with twisted boundary conditions. Note that this type of generalization is not straightforward for the present model because of the long-range nature of hopping and interaction. As was the case for the spin chain, it turns out that the summation over \( l \) in eq. (1) after imposing twisted boundary conditions plays a crucial role to preserve the integrability. The boundary conditions we impose in this paper are

\[
c_i = e^{-2i\phi_i} c_i, \quad c^\dagger_i = e^{-2i\phi_i} c^\dagger_i. \tag{2}
\]

In what follows, the angle \( \phi_i \) in unit of \( 2\pi \) defined above is referred to as the twist angle. Note that we can impose twisted boundary conditions independently on the spin and the charge degrees of freedom with two kinds of twist angles. To diagonalize the Hamiltonian, it is convenient to treat the system after a gauge transformation,

\[
c_{i\sigma} \rightarrow e^{-2i\phi_{i\sigma} / N} c_{i\sigma}, \quad c^\dagger_{i\sigma} \rightarrow e^{2i\phi_{i\sigma} / N} c^\dagger_{i\sigma}, \tag{3}
\]
as was done previously. Hence the transformed operators satisfy the periodic boundary condition.

In the remainder of the paper, we restrict ourselves to the case of rational twist angles,

\[
\phi_i = \frac{p_i}{q}, \quad \phi \equiv \frac{p}{q}, \tag{4}
\]

where \( p_i \) and \( q \) are integers. This restriction enables us to carry out the summation over \( l \) in eq. (1), and the resultant Hamiltonian is well defined,

\[
H = \left( \frac{\pi}{N} \right)^2 P \sum_{i \neq j} \left[ -\sum_{\sigma} J_{\phi_{i\sigma}} (i - j) c_{i\sigma}^\dagger c_{j\sigma} + J_{\phi_{i\sigma}} (i - j) S_i^z S_j^z + J_0 (i - j) \left( S_i^z S_j^z - \frac{1}{4} n_i n_j \right) \right] P, \tag{5}
\]

where \( \phi_{i\sigma} = \phi_i - \phi \) and the effective coupling constant is

\[
J_{\phi_{i\sigma}} = \frac{1}{q^2} \sum_{m=0}^{q-1} e^{2\pi i p (m + n_N) / q} \sin -2 \left[ \frac{\pi (m + n_N)}{q} \right]. \tag{6}
\]

for \( \phi = p/q \). The effect of twisted boundary conditions is now incorporated in the effective coupling \( J_{\phi_{i\sigma}} \), and we can solve \( H \) with periodic boundary conditions. Putting \( p = 0 \), we have \( J_0 (n) = \sin -\frac{2\pi n}{N} \), reproducing the periodic model.

In order to obtain the exact eigenstates of the Hamiltonian, we take the fully polarized state \( | P \rangle = | \uparrow \uparrow \cdots \uparrow \rangle \) as a reference state, and introduce the fermionic hole operator \( h_j \) by

\[
c_{i\uparrow} = h_i, \quad c_{i\downarrow}^\dagger = h_i, \tag{7}
\]

and the bosonic spin operator \( b_j \) by

\[
S_i^- = b_i^\dagger, \quad S_i^+ = b_i, \quad S_i^z = \frac{n_i}{2} - b_i^\dagger b_i, \tag{8}
\]

where \( n_i = 1 - h_i h_i^\dagger \). The Hamiltonian now reads

\[
H = \left( \frac{\pi}{N} \right)^2 \left( T_\uparrow (\phi) + T_\downarrow (\phi) + T_s (\phi) + H_{\text{int}} + e \right), \quad \tag{9}
\]

where

\[
T_\uparrow (\phi) = \sum_{i \neq j} J_{\phi_{i\sigma}} (i - j) h_i^\dagger h_j, \quad T_\downarrow (\phi) = \sum_{i \neq j} J_{\phi_{i\sigma}} (i - j) c_{i\sigma}^\dagger c_{j\sigma}, \quad T_s (\phi) = \sum_{i \neq j} J_0 (i - j) \left( n_i^{(b)} h_j^{(b)} + n_i^{(h)} h_j^{(h)} \right), \quad H_{\text{int}} = \sum_{i \neq j} J_0 (i - j) \left( n_i^{(b)} h_j^{(b)} + n_i^{(h)} h_j^{(h)} \right), \quad e = -\frac{1}{3} M (N^2 - 1). \tag{10}
\]

Here and in what follows, we omit \( P \) for simplicity, though we regard the Hamiltonian as always accompanying this operator.

### III. EXACT SOLUTION FOR TWISTED BOUNDARIES

In this section, we obtain the exact eigenstates of \( H \) by exploiting a Jastrow-type ansatz for the corresponding wave functions. We first divide the total set of sites \( \{1, 2, \ldots, N\} \) into three subsets \( \{x_a\}, \{y_b\} \) and \( \{u_c\} \), which denote, respectively, the position of down-spins (bosons), holes (fermions) and up-spins (\( b \) and \( c \)). We use Greek and Latin indices for bosons and fermions, respectively, the numbers of which are denoted by \( M \) and \( Q \). Then, we have \( S^z = \frac{N - Q}{2} - M \).
As was found in [10], electronic wave functions of Jastrow type are the exact eigenfunctions for the \( t-J \) model with periodic boundary conditions. We can also expect such wave functions to be still eigenstates even for twisted boundaries [2] if we treat the twisted systems after the gauge transformation [3], as was indeed demonstrated for the Haldane-Shastry spin model [6]. These observations motivate us to consider a Jastrow-ansatz state as an eigenstate for the twisted \( t-J \) model,

\[
\Psi = \sum_{x_1, \ldots, x_M, y_1, \ldots, y_Q} \psi(\{x_\alpha\}, \{y_\beta\}; J_s, J_h) \times \prod_\alpha \hat{b}_{x_\alpha}^\dagger \prod_\lambda \hat{h}_{y_\lambda}^\dagger |P\rangle,
\]

where the wave function \( \psi \) is defined by

\[
\psi(\{x_\alpha\}, \{y_\beta\}; J_s, J_h) = \prod_\alpha \epsilon^{J_s x_\alpha} \prod_\lambda \epsilon^{J_h y_\lambda} \times \prod_{\alpha<\beta} d(x_\alpha - x_\beta)^2 \prod_{l<m} d(y_\alpha - y_\beta) \prod_{\alpha, l} d(x_\alpha - y_\beta).
\]

Here, \( d(n) = \sin(n\pi/N) \), and we have introduced

\[
z = e^{2\pi i/N}.
\]

Spin and charge currents are defined by \( J_s = Q/2 \mod 1 \) and \( J_c = (Q+M-1)/2 \mod 1 \). Note that the wave function [12] is the gauge-transformed one whose basis is constructed by the periodic operators \( \hat{b}_{x_\alpha+n}^\dagger = \hat{b}_{x_\alpha}^\dagger \) and \( \hat{h}_{y_\lambda+n}^\dagger = \hat{h}_{y_\lambda}^\dagger \) (see (11)). The original wave function, based on the twisted operators, is obtained via the transformation [3],

\[
\psi_{org}(\{x_\alpha\}, \{y_\beta\}; J_s, J_h) = \prod_\alpha \epsilon^{\phi_{x_\alpha}} \prod_\lambda \epsilon^{\phi_{y_\lambda}} \psi(\{x_\alpha\}, \{y_\beta\}; J_s, J_h).
\]

We will show below that the wave function is indeed an eigenstate of the present Hamiltonian. For this purpose, we first calculate the actions of hopping terms on the wave function [12] in several steps, following similar calculations to Refs. [10] and [12] except that hopping terms are now given by a complicated form \( \phi_{\sigma} \) in a nontrivial way.

A. Action of \( T_I(\phi_I) \)

The action of \( T_I \) on the wave function is

\[
\frac{T_I(\phi_I)\psi}{\psi} = \sum_{n=1}^{N-1} J_{\phi_I}(n) z^{J_h n} \prod_l F_{lm}^{(n)} \prod_\alpha F_{la}^{(n)},
\]

where

\[
F_{lm}^{(n)} = \cos \frac{\pi n}{N} + \sin \frac{\pi n}{N} \cot \Theta_{lm},
\]

with

\[
\Theta_{lm} = \frac{\pi (y_l - y_m)}{N},
\]

Here we have introduced the same notations as those in [10].

B. Action of \( T_I(\phi_I) \)

Note that \( T_I \) exchanges pairs of \( \{x_\alpha\} \) and \( \{y_\beta\} \), and is difficult to treat directly [10][12]. However, if we introduce a unitary operator \( U \) generating the \( \pi \)-rotation around the \( x \)-axis; \( U = \prod_l e^{\pi i (S_{y,l}^+ + S_{y,l}^-)/2} \), \( T_I \) can be expressed by

\[
T_I(\phi_I) = \sum_{i \neq j} J_{\phi_I}(i - j) U^\dagger h_{l_I}^\dagger h_{l_I} U.
\]

Furthermore, from the identity proved by Wang-Liu-Coleman [12],

\[
\prod_{\alpha<\beta} d(x_\alpha - x_\beta) \prod_{\alpha, l} d(x_\alpha - y_l) = A(M, Q) \prod_l z^{\frac{\pi}{2} y_l} \prod_{a<b} d(u_a - u_b) \prod_{a,l} d(u_a - y_l),
\]

where \( A \) is a constant independent of the coordinate, we find a generalized Wang-Liu-Coleman’s theorem including the twist angle,

\[
\frac{T_I(\phi_I)\psi(\{x\}, \{y\}; J_s, J_h)}{\psi(\{x\}, \{y\}; J_s, J_h)} = \frac{T_I(\phi_I)\psi(\{u\}, \{y\}; N - J_s, J_h - J_s + \frac{N}{2})}{\psi(\{u\}, \{y\}; N - J_s, J_h - J_s + \frac{N}{2})}.
\]

Thus, the action of \( T_I \) can be evaluated in a similar way to \( T_I \) as

\[
\frac{T_I(\phi_I)\psi}{\psi} = \sum_{n=1}^{N-1} J_{\phi_I}(n) z^{J_h n} \prod_{m(\neq l)} F_{lm}^{(n)} \prod_\alpha F_{la}^{(n)},
\]

where \( J_I = J_h - J_s + N/2 \) and \( F_{la}^{(n)} \) is defined as eq.(16), replacing \( y_m \) by \( u_a \) in eq.(17).

C. Action of \( T_s(\phi_s) \)

The action of \( T_s \) on the wave function is

\[
\frac{T_s(\phi_s)\psi}{\psi} = \sum_{n=1}^{N-1} J_{\phi_s}(n) z^{J_s n} \prod_{\beta \neq \alpha} F_{\alpha\beta}^{(n)} \prod_l F_{la}^{(n)},
\]

where
with

\[
B^{(n)}_{\alpha\beta} = 1 - g^{(n)}_{\alpha\beta}
\]

(23)

Here we denote \( z \equiv z^x \).

### D. Eigenvalues

There are various many-body terms in eqs.\((17), (21)\) and \((22)\). However, more than three-body terms turn out to vanish similarly to the case of the periodic model. Key identities to prove this are

\[
S^\phi_{st}(J) = \frac{1}{4} \sum_{n=1}^{N-1} J(n)z^n(1-z^n)^s(1-z^{-n})^t
\]

\[
= \begin{cases} 
(-)^s & \text{for } s + t = 2 \\
(-)^s(J + \phi + \frac{N}{2}) - \frac{1}{4} & \text{for } s + t = 1 \\
2\varepsilon(J + \phi) + \frac{1}{4}(N^2 - 1) & \text{for } s + t = 0 \\
0 & \text{for others,}
\end{cases}
\]

(25)

where \( J \) is an integer \( 0 < J < N \), \( s \) and \( t \) are non-negative integers in the region \( 0 \leq s + t \leq \min(J + \phi, N - J) \).

The function \( \varepsilon(k) \) for any rational \( k \) is defined by

\[
\varepsilon(k) = (2k) - \left\lfloor k \right\rfloor \left\lceil k \right\rceil,
\]

(26)

where \( \left\lfloor k \right\rfloor \) denotes the Gauss symbol, denoting the maximum integer which does not exceed \( a \). It is quite characteristic in twisted \( 1/r^2 \) models that such a function including the Gauss symbol enters the expression. We will see later that \( \varepsilon(k) \) is nothing but a single-particle energy as a function of the momentum \( k \).

Substituting the above results to eqs.\((15), (21)\) and \((22)\), we have

\[
\frac{(T_s + T_t + T_\perp)}{\psi} = W_0 + W_2 + W_3.
\]

(27)

First, \( W_0 \) is the constant term

\[
W_0 = \frac{1}{3} M(N^2 - 1) + \frac{2}{3} M(M^2 - 1) + \frac{1}{2} QM(2M - 1) + \frac{2}{3} Q(N^2 - 1) + \frac{1}{2} Q^2(N - Q) + 2M\varepsilon(J_s + \phi_s) + 2Q\varepsilon(J_h + \phi_t)
\]

\[+2Q \left( \varepsilon(J_\perp + \phi_\perp) + \frac{N^2}{4} \right),
\]

(28)

where \( \varepsilon(J + \phi) \) is defined by

\[
\varepsilon(J + \phi) = \begin{cases} 
\varepsilon(J + \phi), & \text{for integer } J, \\
\frac{1}{2}(\varepsilon(J + \phi - \frac{1}{2}) + \varepsilon(J + \phi + \frac{1}{2}) - \frac{1}{2}), & \text{for half-integer } J.
\end{cases}
\]

(29)

Next, \( W_2 \) is the two-body term, evaluated as

\[
W_2 = -2i \left( J_s + \phi_s - \frac{N}{2} \right) \sum_{\alpha,\ell} \cot \Theta_{\alpha\ell}
\]

\[
-2i \left( J_h + \phi_\perp - \frac{N}{2} \right) \sum_{\alpha,\ell} \cot \Theta_{\alpha\ell}
\]

\[
-2i (J_h - J_s + \phi_\perp) \sum_{\alpha,\ell} \cot \Theta_{\alpha\ell}
\]

\[
- \sum_{\alpha \neq \beta} J_0(x_\alpha - x_\beta)
\]

\[
= - \sum_{\alpha \neq \beta} J_0(x_\alpha - x_\beta),
\]

(30)

which should be canceled out by one of the interaction terms in eq.\((10)\). Finally, the three-body term \( W_3 \) is

\[
W_3 = \frac{1}{2} \sum_{\alpha \neq \beta, l} \cot \Theta_{\alpha\beta} \cot \Theta_{\beta l} + \frac{1}{2} \sum_{l \neq m, a} \cot \Theta_{la} \cot \Theta_{ma}
\]

\[
- \frac{1}{2} \sum_{\alpha \neq \beta, l} \cot \Theta_{\alpha l} \cot \Theta_{\beta l}
\]

\[
+ \frac{1}{3} Q(M + 1) + \frac{1}{6} Q(N^2 - 1) + \frac{1}{3} Q^2(N - M) + \frac{1}{3} Q(Q^2 - 1).
\]

(31)

The three-body term thus reduces to the two-body term, and is also canceled out by the other interaction terms in eq.\((10)\). Collecting constant terms, we finally end up with the eigenvalue of the Hamiltonian

\[
E_t = \left( \frac{\pi}{N} \right)^2 (e + E)
\]

(32)

with

\[
e = \frac{1}{3} Q(N^2 - 1),
\]

\[
E = \frac{2}{3} M(M^2 - 1) + \frac{2}{3} Q(Q^2 - 1) + \frac{1}{2} Q(M + Q)(2M - Q) + 2M\varepsilon(J_s + \phi_\perp - \phi_\perp) + 2Q\varepsilon(J_h + \phi_\perp)
\]

\[+2Q \left( \varepsilon(J_\perp + \phi_\perp) + \frac{N^2}{4} \right).
\]

(33)

Here we have divided the energy into two parts \( e \) and \( E \) for later convenience, because \( E \) can be reproduced by the asymptotic Bethe ansatz (see eq.\((29)\)), whereas \( e \) simply corresponds to the chemical-potential term. The currents are subject to the constraints
\[
\frac{Q}{2} + M - 1 \leq J_s + \phi_\uparrow - \phi_\downarrow \leq N - \left(\frac{Q}{2} + M - 1\right),
\]
\[
\frac{1}{2}(Q + M - 1) \leq J_h + \phi_\uparrow \leq N - \frac{1}{2}(Q + M - 1),
\]
\[
-\frac{M + 1}{2} \leq J_h - J_s + \phi_\downarrow \leq \frac{M + 1}{2},
\]
(34)
in order for the formula (23) to be applicable. Note that when \(\phi_\uparrow = \phi_\downarrow = 0\), \(\tilde{\varepsilon}(J)\) reduces to the well-known form \(\tilde{\varepsilon}(J) = J(J - N)\), and the spectrum obtained here coincides with that found in [12].

The expression (33) gives the exact ground state energy with twisted boundary conditions. However, in order to trace the flow of the initial ground state correctly, we need more detailed information to specify the states, which may be supplied by the motif picture described below [13].

IV. SPECTRAL FLOW

A. Asymptotic Bethe Ansatz

Before discussing the spectral flow as a function of the twist angle, we first show that the above exact spectrum can be reproduced by the asymptotic Bethe ansatz (ABA), which naturally leads us to introduce the motif picture. To obtain the ABA equation for the supersymmetric t-J model, we recall that the present model is reduced to the Haldane-Shastry model at \(Q = 0\), and the spectral flow in this case [14] can be described by the ABA equation,

\[
\tilde{k}_\mu = I_\mu + \phi_\uparrow - \phi_\downarrow + \frac{1}{2} \sum_{\nu=1}^{M} \text{sgn}(\tilde{k}_\mu - \tilde{k}_\nu).
\]

(35)

We then have a simple relation \(\tilde{k}_\mu = k_\mu + \phi_s\), in terms of the rapidities \(k\)'s defined at \(\phi_s = 0\). Namely, the sole effect due to \(\phi_s\) is to shift \(k\)'s uniformly. To observe how the spectral flow occurs, it is convenient to introduce the description by the motif [13], which is briefly outlined here in the case of half filling. The essential point of the motif in this case is that one can describe the effect of two-body phase shifts in (8) by arranging 0 and 1 for a given configuration of rapidities. For example, the ground state of the Haldane-Shastry model is a singlet, denoted by the motif

\[
010 \cdots 1010101 \cdot \cdot \cdot 010,
\]
(36)

which means that the spacing of the occupied momenta, denoted by 1's, is enlarged twice as large as that for free fermions. From this motif we can identify the statistical interaction \(g = 2\) (\(g = 1\) corresponds to the free fermion case). The spectral flow of this state can be described by the flow of the motif, \(010 \cdots 010 \rightarrow 001 \cdots 101 \rightarrow 010 \cdots 010\), where the arrow means \(\delta \phi = 1\), or in other words the ring is threaded by a unit flux. Thus the period of the ground state turns out to be 2, which directly reflects the statistical interaction \(g = 2\) of exclusion statistics [5].

Keeping the above results in mind, we now show how the ABA equation for twisted boundaries is generalized when holes are doped into the Haldane-Shastry model. To this end, we exploit the idea of the nested Bethe ansatz with graded symmetry (supersymmetry) [14], which leads to the Bethe equation for the supersymmetric t-J model with twisted boundary conditions,

\[
\tilde{k}_\mu = I_\mu^{(1)} + \phi_\uparrow - \phi_\downarrow + \frac{1}{2} \sum_{i=1}^{Q} \text{sgn}(\tilde{k}_\mu - \tilde{m}_i) + \frac{1}{2} \sum_{\nu=1}^{M+Q} \text{sgn}(\tilde{k}_\mu - \tilde{k}_\nu),
\]

(37)

\[
I_\mu^{(2)} + \phi_\downarrow = \frac{1}{2} \sum_{\nu=1}^{M+Q} \text{sgn}(\tilde{m}_i - \tilde{k}_\nu),
\]

(38)

where \(\mu = 1, 2, \ldots, M+Q\) and \(i = 1, 2, \ldots, Q\). The newly introduced rapidity \(\tilde{m}_i\) is concerned with the charge degrees of freedom. The energy \(E\) in eqs.(32) is given by

\[
E = 2 \sum_{\mu=1}^{M+Q} \varepsilon(\tilde{k}_\mu),
\]

(39)

where the single-particle energy \(\varepsilon(\tilde{k})\) is defined by eqs.(20) [15]. Note that the twist angles, \(\phi_\uparrow\) and \(\phi_\downarrow\), have been introduced in the above equations so as to be consistent with a nested procedure in the Bethe ansatz [6]. Both of the rapidities \(\tilde{k}\) and \(\tilde{m}\) are defined in the range \(0 \leq \tilde{k}, \tilde{m} < N\), which include the effect of the finite twist angle. The corresponding quantum numbers \(I_\mu^{(1)}\) and \(I_\mu^{(2)}\) should satisfy \(I_\mu^{(1)} = M/2 \mod 1\) and \(I_\mu^{(2)} = (M + Q)/2 \mod 1\).

Here we should mention the precise meaning of eqs.(38) which has been formally deduced by the nested Bethe ansatz, because it does not seem to hold for a fractional value of \(\phi_\downarrow\) at a first glance. This subtle problem comes from the phase shift with the step-wise sgn\((k)\) function. We briefly summarize how to treat this equation correctly. Consider, for example, the following configuration at \(\phi_\uparrow = \phi_\downarrow = 0\),

\[
\cdots < k_{\mu-1} < m_i < k_\mu < \cdots.
\]

(40)

If we put \(\phi_\downarrow = 1\), it is seen from eqs.(38) that \(I_\mu^{(2)}\) changes into \(I_\mu^{(2)} + 1\) and the above configuration should be changed to

\[
\cdots < k_{\mu-1} < k_\mu < m_i \cdots.
\]

(41)

Namely, \(m_i\) exchanges the position with \(k_\mu\) sitting on its right. Therefore, fractional \(\phi_\downarrow\) between 0 and 1 should
interpolate these two configurations smoothly. This is naturally realized if we introduce an infinitesimal width \( \eta \) in the step-wise phase shift and then take the limit of \( \eta \to 0 \). Based on this observation, we can separate the l.h.s. of eq.(38) as \( I^{(2)}_i + \phi_i \equiv (I^{(2)}_i + [\phi_i]) + (\phi_i - [\phi_i]) \), and the fractional part \( \phi_i - [\phi_i] \) can be absorbed into \( \text{sgn}(m_i - k_\mu) \equiv \text{sgn}(0) \) in the case of \( m_i = k_\mu \) for \( 0 < \phi_i < 1 \). Namely, the fractional portion of \( \phi_i \) can be incorporated into \( \text{sgn}(0) \) by taking the appropriate limit mentioned above \([19]\). As a consequence, we have from eq.(37),

\[
\frac{1}{2}\text{sgn}(m_i - k_\mu) = \phi_i - \frac{1}{2}
\]

(42)

for \( 0 < \phi_i < 1 \). Note that the initial order \( m_i < k_\mu \) at \( \phi_i = 0 \) indeed changes into \( k_\mu < m_i \) at \( \phi_i = 1 \). Substituting the above equation into eq.(37), we end up with the simple results,

\[
\tilde{k}_\mu = \begin{cases} 
  k_\mu + \phi_\uparrow, & \text{for } \cdots < m_i < k_\mu < \cdots, \\
  k_\mu + \phi_\downarrow - \phi_\downarrow, & \text{for } \cdots < k_\mu - 1 < k_\mu < \cdots,
\end{cases}
\]

(43)

for \( 0 \leq \phi_i < 1 \). This equation implies that \( \tilde{k} \) with (without) \( m \) in its left neighbor describes the charge (spin) degrees of freedom \([23]\). One can easily confirm that the above ABA equations indeed reproduce the exact energy \([23]\) obtained in sect.III by choosing the suitable quantum numbers \( I^{(1)}_i \) and \( I^{(2)}_i \).

### B. Spectral Flow in Metallic Phase

Having noticed that the exact solution obtained in sect.III can be reproduced by the ABA, we are now ready to discuss the spectral flow by comparing the numerical diagonalization results with the motif picture in the ABA. The numerical results for small systems can be complementary to the exact results in sect.III, because the analytical method there can supply only a special series of the eigenstates. In Fig.1, we have shown the exact spectral flow calculated numerically for two holes in the system with 6 sites (i.e. 4 electrons). In this system, the ground state is doubly degenerate concerning the spin degrees of freedom. For instance, two degenerate ground states in Fig.1 at \( \phi_\uparrow = \phi_\downarrow = 0 \) are classified by the motifs

\[
M1: \quad 0\text{i}1\text{e}1010, \\
M2: \quad 0\text{i}1\text{e}1010,
\]

(44)

and the first excited state by

\[
M3: \quad 0\text{i}1\text{e}1010.
\]

(45)

In order to express the motif in the doped case, we have introduced \( \circ \) in addition to 0 and 1, where \( \circ \) denotes the position of \( \tilde{m}_i \) in eqs.(37) and (38), namely, it specifies a doped hole in the motif. When boundaries are twisted, the pattern behavior in the spectral flow explicitly depends both on the configurations of \( k_\mu \)'s and \( m_i \)'s, because the motions of \( k_\mu \)'s are different from those of \( m_i \)'s as a function of the twist angle. These behaviors are correctly described by the flow of \( 1 \) and \( \circ \) in the motif. Since we can change two independent twist angles in the doped case, we have examined the spectral flow for two typical cases in Fig.1. One is the case where \( \phi_\uparrow = \phi_\downarrow = \phi \), which will be referred to as Type I. In this case, the twisted boundary condition is imposed only on the charge degree of freedom. The other is the case where \( \phi_\uparrow = \phi, \phi_\downarrow = 0 \), which will be referred to as Type II. It is seen from eq.(43) that the twisted boundary condition in Type II is equally imposed both on the charge and spin degrees of freedom. The motif \( M1 \), for example, behaves

\[
0\text{i}1\text{e}1010 \to 0\text{i}1\text{e}1010 \quad \text{for Type I}, \\
0\text{i}1\text{e}1010 \to 00\text{i}1\text{e}101 \quad \text{for Type II}
\]

(46)

when we increase the twist angle \( \delta \phi = 1 \). Therefore the corresponding spectral flow reads

\[
M1: \quad a1 \to a2 \to a3 \to a4 \to a4 \quad (= a1), \\
M2: \quad a2 \to a3 \to a1 \to a2, \\
M3: \quad b1 \to b2 \to b3 \to b1,
\]

(47)

for Type I shown in Fig.1(a), and

\[
M1: \quad c1 \to c2 \to c3 \to c4 \to c5 \to c6 \to c1, \\
M2: \quad c3 \to c4 \to c5 \to c6 \to c1 \to c2 \to c3,
\]

(48)

for Type II (\( M3 \) is the same) shown in Fig.1(b). Note that \( \to \) means \( \delta \phi = 1 \). In this way we can trace the natural spectral flow correctly following the flow of the motif. This in turn demonstrates that the spectral flow in the supersymmetric \( t-J \) model with \( 1/r^2 \) interaction completely fits in with the motif picture in the ABA even in the metallic case. We note that the spectral flows \( M1 \) and \( M2 \) for the ground states (Fig.1) agree with the exact results obtained in sect.III.

Based on these results for small systems, we now wish to generalize our statement to more generic cases, and deduce how the spectral flow behaves when any number of holes are doped into the Haldane-Shastry model with \( N \)-sites. We first note that the doped holes energetically prefer to occupy the positions in the center of the motif. For example, if we dope one or two holes, the corresponding motif should be

\[
010 \cdots 1010101 \cdots 010, \\
010 \cdots 1010101 \cdots 010.
\]

(49)

It is easily seen from these motifs that the period of the flow becomes macroscopic, i.e. \( N \) both for Types I and
II. Note that this is also the case for any doping rates away from half filling, namely, the period generally becomes \( N \) both for Type I and Type II. The period \( N \) for Type I (the charge sector) is naturally understood by the analogy to the free fermion case. On the other hand, the change of the period in Type II is remarkable, because it is given by 2 at half filling, reflecting fractional exclusion statistics with statistical interaction \( g = 2 \). Therefore once holes are doped, characteristic properties of ideal exclusion statistics for the spin sector are hidden in the spectral flow. This does not imply, however, that the nature of exclusion statistics is spoiled by doping, because various characteristic properties such as correlation functions can be still interpreted in terms of exclusion statistics \([7, 14]\). A reason why we cannot observe exclusion statistics in the spectral flow is related to breaking of the “commensurability” condition. For example, exclusion statistics with \( g = 2 \) is commensurate at half filling in the sense that the corresponding motif for the ground state fits in with the underlying lattice (see eq. (50)). In contrast, the motif for the metallic phase, for example, (49) does not meet the commensurability condition, hiding the \( g = 2 \) statistics in the flow. Therefore, we can say that the commensurability condition is essential to observe the ideal exclusion statistics in the spectral flow.

In this connection, it is instructive to think of a specific motif for an excited state in the case of \( N/3 \) hole-doping, which has a uniform configuration such that

\[
0101010\ldots01010.
\]

This motif satisfies the commensurability condition, and the resultant period of the state turns out to be 3. Although this kind of a special excitation may not be interesting for the present model, we wish to note that the same motif describes the “ground state” for the SU(3) Haldane-Shastry spin chain. Therefore, we can predict that the ground state of the latter model, or more generally, SU(\( \nu \)) spin model can have a period \( \nu \) in the spectral flow.

The above characteristic properties for the \( 1/r^2 \) model are to be compared with ordinary interacting electron systems, for which the ideal statistics may not be realized due to perturbations which deviate the system from the idealized situation. In such cases, it is difficult to observe exclusion statistics in the spectral flow even if the commensurability conditions are met, because irrelevant perturbations may dominate such properties.

V. SUMMARY

We have investigated the spectral flow in the supersymmetric \( t-J \) model with \( 1/r^2 \) interaction by analyzing the exact spectrum with twisted boundary conditions. At half filling, exclusion statistics with statistical interaction \( g = 2 \) can be observed in the period of the spectral flow. Away from half filling, two kinds of spectral flows for the charge and spin sectors appear. It has been found that the nature of the spectral flows nicely fits in with the motif picture even in the metallic phase. We have also addressed the question whether one can observe fractional exclusion statistics explicitly in the spectral flow in the metallic phase. By analyzing the exact spectrum in terms of the motif picture, we found that the period becomes \( N \) once any number of holes are doped, hiding characteristic properties of exclusion statistics with \( g = 2 \) for the spin sector. This drastic change has been shown to be related to breaking of the commensurability condition. Therefore, even in the idealized model for exclusion statistics, which is free from any irrelevant perturbations, it is crucial to satisfy the commensurability condition to observe the statistical interaction in the period of the spectral flow.

In this paper, we have been concerned with the cases of rational twist angles in unit of \( 2\pi \). We think that the present conclusion can be applied even for irrational cases, although we could not yet find the exact solution for such cases.

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In the case of the periodic model, the momentum \( k \) is given by an integer and thus the function \( (26) \) is reduced to the quadratic dispersion relation \( \varepsilon(k) = k(k - N) \) known for \( 1/r^2 \) models. A remarkable point is that the dispersion relation \( (26) \) for fractional cases is now essentially linear in \( k \). It was for the first time found in [6].

This kind of the interpretation is also necessary to reproduce the free electron spectrum with twisted boundary conditions by using the Bethe equation of the Hubbard model [2] in its zero interaction limit.

Here, we are concerned with the case \( \phi_\downarrow \geq 0 \). For the other case \( \phi_\downarrow \leq 0 \), \( m \) exchanges the position with \( k \) sitting on its left, and therefore, the role of “left” and “right” in the text exchanges.

Note added in proof

We have learned that the spectral flow for the \( XXZ \) model with short-range interaction was studied in F. C. Alcaraz, M. N. Barber and M. T. Batchelor, Ann. Phys. 182 (1988) 280.