Levinson Theorem for Differential Equations with Piecewise Constant Argument Generalized

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Abstract

In this work, it is presented an adaptation of an asymptotic theorem of N. Levinson of 1948, to differential equation with piecewise constant argument generalized, which were introduced by M. Akhmet in 2007. By simplicity and without loss of generality, the case where the argument is delayed is considered. The N. Levinson’s theorem which is adapted is that dealt by M. S. P. Eastham in his work which is present in this bibliography. The more relevant hypotheses of this theorem are highlighted and it is established a version of this theorem with these hypotheses for ordinary differential equations. Such a version is that which is adapted to differential equation with piecewise constant argument generalized. The adaptation is proved by mean the Banach fixed point where contractive operator is built from a suitable version of the constant variation formula.

2010 AMS Subject Class: 34A38, 34C27, 34D09, 34D20.

Key words: Differential Equations, Piecewise constant argument, Asymptotic Formula, Levinson Theorem.

1 Introduction

Let $n \in \mathbb{N}$. We will consider $\mathbb{K}^N := \mathcal{M}_{N \times 1}(\mathbb{K})$, i.e., as column vectors, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We will denote by $| \cdot |$ to the Euclidean norm for $\mathbb{K}^N$. $\mathcal{M}_N(\mathbb{C})$ will denote the $N \times N$ matrix with complex entries. On $\mathcal{M}_N(\mathbb{C})$, $\| \cdot \|$ will

*Supported by DIUBB 074108 1/R
denote the classic norm operator which is defined for $A \in \mathcal{M}_N(\mathbb{C})$, by $\|A\| = \sup_{v \in \mathbb{C}^N - \{0\}} \frac{|Av|}{|v|}$.

A Differential Equations with Piecewise Constant Argument Generalized (DEPCAG) is a differential equation of the form

$$\frac{dx}{dt} = f(t, x(t), x(\gamma(t))),$$

where $\gamma$ is of the form given above. A function $x = x(t)$ is understood as solution of the DEPCAG (1) if:

1. $x$ is continuous on $[t_0, +\infty[$;

2. the derivative $\frac{dx}{dt}$ of $x$ with the possible exception in $t = t_n$ for $n \in \mathbb{N}_0$, where is unilateral derivative exists;

3. $x$ is a solution of (1) with the possible exception in $t = t_n$ for all $n \in \mathbb{N}_0$.

Notice that (1) is an ordinary differential equation in each interval $[t_n, t_{n+1}]$ for all $n \in \mathbb{N}_0$, but the leaps be between those intervals creates a difference system of the form

$$x(\xi_n) = x(t_n) + \int_{t_n}^{\xi_n} f(\zeta, x(\zeta), x(t_n)),
$$

$$x(t_{n+1}) = x(\xi_n) + \int_{\xi_n}^{t_{n+1}} f(\zeta, x(\zeta), x(t_n)),$$

for all $n \in \mathbb{N}_0$.

The name generalized in those equations, is explained by the inclusion of the differential equations with piecewise argument whose study seems to be started by K. Cooke and J. Wiener in [6, 11, 12]. They consider equations of the form (1) with $\gamma(t) = [t]$ or $\gamma(t) = 2\left[\frac{t + 1}{2}\right]$, where $[\cdot]$ is the function assigns to each real number, the greater integer number less than that it. The first known generalization was made by M. Akhmet [1].

In this work we establish a version of an asymptotic Levinson’s theorem for the DEPCA

$$\frac{dy}{dt} = A(t)y(t) + B(t)y(\gamma(t)) + R(t)y(\gamma(t)), \ t \in [t_0, +\infty[, \quad (2)$$

where $A(t)$, $B(t)$ y $R(t)$ are matrices in $\mathcal{M}_N(\mathbb{C})$, whose coefficients are locally integrable functions of $t$. DEPCAG (2) will be seen as a perturbation of the DEPCAG

$$\frac{dz}{dt} = A(t)z(t) + B(t)z(\gamma(t)). \quad (3)$$

The original version of the Levinson’s Theorem to be considered, can be found in N. Levinson [8, (1948)] and it is the main result of the Eastham work [7]. This result is the following.
Theorem 1 Consider the system $C^N$,
\[
\frac{dy}{dt} = (\Lambda(t) + R(t))y, \; t \geq 0,
\]
where $\{\Lambda(t)\}_{t \geq 0}$ and $\{R(t)\}_{t \geq 0}$ are two families in $\mathcal{M}_N(\mathbb{C})$ such that
1. $\Lambda(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_N(t))$ such that there is $k \in \{1, \ldots, N\}$ satisfying
   \[
   (a) \lim_{t \to +\infty} \int_0^t \Re(\lambda_j(\xi) - \lambda_k(\xi)) d\xi = -\infty, \text{ for } j < k; \\
   (b) \text{There is a nonnegative real constant } K \text{ such that } \int_s^t \Re(\lambda_j(\xi) - \lambda_k(\xi)) d\xi \leq K, \text{ for } j,k \in \{1, \ldots, N\} \text{ and } s,t \geq 0;
   \]
2. $\|R(\cdot)\| \in L^1$.
Then, the system (4) has a solution $y = y_k(t)$ defined for a large enough $t$ and satisfying
\[
y_k(t) = \exp \left( \int_0^t \lambda_k(s) ds \right) (e_k + w(t)),
\]
where $e_k$ is the $k$-th vector of the canonical base of $\mathbb{C}^N$ and $w(t) \to 0$ as $t \to +\infty$.

In the book of Eastham [7], many applications of Theorem 1 can be obtained.

Notice that system (4) is seen as a perturbation of the diagonal system
\[
\frac{dx}{dt} = \Lambda(t)x.
\]

An asymptotic version of the Levinson theorem which is known by authors, is given by M. Akhmet [2]. He considers the DEPCA,
\[
\frac{dy}{dt} = C_0y + f(t, x(t), \gamma(t)),
\]
as a perturbation of the autonomous ordinary differential equation
\[
\frac{dx}{dt} = C_0x,
\]
where $\gamma$ is defined as above and the following hypotheses are given
\[
\exists L > 0 : \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L(\|x_2 - x_1\| + \|y_2 - y_1\|) \text{ y } f(t, 0, 0) = 0; \\
\tag{9a}
\exists T : 0 < t_{n+1} - t_n \leq T; \\
\tag{9b}
\exists M, m > 0 : m \leq \|e^{C(t-s)}\| \leq M, \forall t, s \in [t_n, t_{n+1}]; MLTe^{MT} < 1; \\
\tag{9c}
2MLT < 1; M^2LT \left( \frac{MLTe^{MT} + 1}{1 - MLTe^{MT}} \right) < m; \\
\tag{9d}
\exists \eta : \mathbb{R}^+ \to [0, L] : \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \eta(t)(\|x_2 - x_1\| + \|y_2 - y_1\|).
\begin{equation}
\ell_0 := \int_0^{+\infty} e^{tm_\beta+m_\alpha-2t(t(\beta-\alpha))}d\eta(t)dt < +\infty, \tag{9e}
\end{equation}

where \(\lambda_1, \ldots, \lambda_p\) are the characteristic values of \(C_0\), \(\alpha = \min_{j=1,\ldots,p} \Re(\lambda_j)\), \(\beta = \max_{j=1,\ldots,p} \Re(\lambda_j)\), \(m_\alpha\) and \(m_\beta\) the maximum orders of the characteristic values of \(C\) with real part equal to \(\alpha\) and \(\beta\) respectively, for all \(t \in \mathbb{R}^+\).

Then, Akhmet [2] provides the following result.

**Theorem 2** Assume that (9a)-(9e) hold. Then, for every solution \(y = y(t)\) of the DEPCAG (7) has a representation

\begin{equation}
y(t) = e^{C_0t}[c + w(t)], \tag{10}
\end{equation}

where \(w(t) \to 0\) as \(t \to +\infty\) and \(c \in \mathbb{R}^N\).

That result, consider a DEPCAG as a perturbation of an autonomous ordinary differential equation (8). In Theorem 2 it is seen the perturbation of the whole fundamental matrix of (8). In our result we see the perturbation of only one dimension of the solution space of (8), although our non perturbed equation is already a DEPCAG.

This paper is organized as follow. A section of preliminaries is given where some definition and classical proof of result are given. We make emphasis in the most relevant part of those proof to be adapted in our main result. For example, it will be presented a version of the Theorem 1 for Ordinary Differential Equations by omitting that the system to be perturbed is diagonal since it is not relevant in our proof. Since a DEPCAG can be seen as a mixture between differential and difference equations, a version of Levinson’s theorem for difference equation is presented. Necessary results on DEPCAG are given too. The last section is devoted to the main result and its proof.

## 2 Preliminaries

### 2.1 Adapted version of Levinson Theorem for Ordinary Differential Equations

In the theorem 1 system (4) is seen as a perturbation of the diagonal system (8), where \(\Lambda(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_N(t))\) is such that \(k \in \{1, \ldots, N\}\) such that

1. \(\lim_{t \to +\infty} \int_0^t \Re(\lambda_j(\xi) - \lambda_k(\xi))d\xi = -\infty, \text{ for } j < k;\)

2. There is a real non negative constant \(K\) such that \(\int_s^t \Re(\lambda_j(\xi) - \lambda_k(\xi))d\xi \leq K, \text{ for } j, k \in \{1, \ldots, N\}\) and \(s, t \geq 0\).
If $X$ is a fundamental matrix of the diagonal system (6), its Cauchy matrix $X(t, s) = X(t)X(s)^{-1}$ will be given by

$$X(t, s) = \text{diag} \left( \exp \left( \int_s^t \lambda_1(s) ds \right), \ldots, \exp \left( \int_s^t \lambda_N(s) ds \right) \right). \quad (11)$$

This can be written as

$$X(t, s) = X_1(t, s) + X_2(t, s),$$

where $X_1(t, s) = \text{diag} \left( \exp \left( \int_s^t \lambda_1(s) ds \right), \ldots, \exp \left( \int_s^t \lambda_{k-1}(s) ds, 0, \ldots, 0 \right) \right)$ and $X_2(t, s) = \text{diag} \left( 0, \ldots, \exp \left( \int_s^t \lambda_k(s) ds \right), \ldots, \exp \left( \int_s^t \lambda_N(s) ds \right) \right)$.

Let $\Xi_1(t, s) = e^{-\int_s^t \lambda_k(\xi) d\xi} X_1(t, s)$, $\Xi_2(t, s) = e^{-\int_s^t \lambda_k(\xi) d\xi} X_2(t, s), H_1(t, s) = ||\Xi_1(t, s)||$ and $H_2(t, s) = ||\Xi_2(t, s)||$. Then, $H_1(t, s) \leq \max_{j=1, \ldots, k-1} \int_s^t \text{Re}(\lambda_j(\xi) - \lambda_k(\xi)) d\xi$. So,

$$H_1(t, s) \to 0 \text{ as } t \to +\infty$$

and there is a real constant $C \geq \max \{e^K, 1\}$ so large that $H_1(t, s) \leq C$, for all $t \geq s \geq 0$.

Moreover,

$$H_2(t, s) \leq C \text{ if } s \geq t \geq 0. \quad (13)$$

This allows to write every solution of (11), $y = y(t)$ defined for $t \geq t_0 \geq 0$ and $t_0$ large enough by mean the integral equation,

$$y(t) = X(t, t_0)q(y) + e^{\int_{t_0}^t \lambda_k(\xi) d\xi} \times \left[ \int_{t_0}^t H_1(t, s) R(s) e^{-\int_s^t \lambda_k(\xi) d\xi} y(s) ds - \int_{t_0}^{+\infty} H_2(t, s) R(s) e^{-\int_s^t \lambda_k(\xi) d\xi} y(s) ds \right], \quad (14)$$

where $q(y) = y(t_0) + \int_{t_0}^{+\infty} H_2(t, s) R(s) e^{-\int_s^t \lambda_k(\xi) d\xi} y(s) ds$, by assuming the existence of the integrals $\int^{+\infty} \cdots$. Later, we will prove the existence of those integrals. It will be seen that such that existence due to the assumption $R(t) \in L^1$ and due to $H_2$ is bounded, by taking $t_0$ large enough.

Other helpful detail is the inequality,

$$H_1(t, s) \leq H_1(t, T)H_1(T, s) \text{ if } t \geq T \geq s. \quad (15)$$
The conditions (15) and (12), will allow us to establish that in the first term of the sum in (14) satisfies

\[
\lim_{t \to +\infty} \int_0^t H_1(t, s) R(s) e^{-\int_0^s \lambda(x) dx} y(s) ds = 0. \tag{16}
\]

In the following lemma, the recent claim is made more detailed.

**Lemma 1** Let \( h : [0, +\infty]^2 \to [0, +\infty] \) be a bounded function \( 0 \leq s \leq T \leq t \) such that

\[
h(t, s) \leq h(t, T) h(T, s)
\]

and \( \lim_{t \to +\infty} h(t, s) = 0 \). Let \( f : [0, +\infty] \to [0, +\infty] \) a \( L^1 \) function. Then,

\[
\lim_{t \to +\infty} \int_0^t h(t, s) f(s) ds = 0.
\]

**Proof:** It is not hard to see that

\[
\int_0^t h(t, s) f(s) ds \leq \int_0^T h(t, s) f(s) ds + \int_T^t h(t, s) f(s) ds.
\]

Hence,

\[
\int_0^t h(t, s) f(s) ds \leq h(t, T) \int_0^T h(T, s) f(s) ds + \int_T^t h(t, s) f(s) ds.
\]

Since \( \lim_{t \to +\infty} h(t, s) = 0 \),

\[
\lim_{t \to +\infty} \int_0^t h(t, s) f(s) ds \leq \lim_{t \to +\infty} \int_T^t h(t, s) f(s) ds.
\]

Since \( \{h(t, s)\}_{t, s \geq 0} \) is bounded, there is a constant \( C \geq 0 \) such that

\[
\lim_{t \to +\infty} \int_0^t h(t, s) f(s) ds \leq C \int_T^t f(s) ds.
\]

Since \( f \in L^1 \), it’s enough to take \( T \to +\infty \) for obtaining that

\[
\lim_{t \to +\infty} \int_0^t h(t, s) f(s) ds = 0,
\]

which proved this lemma.

\( \square \)

In the following Theorem, we present an adaptation of the Levinson Theorem where we shall disregard the diagonality of the unperturbed system.
Theorem 3 Consider the system in $\mathbb{C}^N$,

$$\frac{dy}{dt} = (A(t) + R(t))y, \ t \geq 0, \quad (17)$$

where $\{A(t)\}_{t \geq 0}$ and $\{R(t)\}_{t \geq 0}$ are two families of matrices in $\mathcal{M}_N(\mathbb{C})$ which are locally integrable in $t$. Assume that there is a unitary vector $\hat{e} \in \mathbb{C}^N$ and a locally integrable function $\hat{\lambda} : [0, +\infty] \rightarrow \mathbb{C}$ such that

$$X(t,s)\hat{e} = \exp \left( \int_s^t \hat{\lambda}(\xi)d\xi \right)\hat{e}, \quad (18a)$$

where $X(t,s)$ is the Cauchy matrix of

$$\frac{dx}{dt} = A(t)x. \quad (19)$$

Assume that there is a projection $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$ such that

$$\hat{e} \in (I - P)(\mathbb{C}^N). \quad (20a)$$

Assume that there is a constant $M > 0$ and a bounded function such that $h : [0, +\infty]^2 = [0, +\infty] \times [0, +\infty] \rightarrow [0, +\infty]$ and

$$\|X(t,s)P\| \leq \exp \left( \int_s^t \Re\hat{\lambda}(\xi)d\xi \right) Mh(t,s), \ for \ t \geq s; \quad (21a)$$

$$\|X(t,s)(I - P)\| \leq M \exp \left( \int_s^t \Re\hat{\lambda}(\xi)d\xi \right), \ for \ t \leq s; \quad (21b)$$

$$h(t,s) \rightarrow 0 \ for \ t \rightarrow +\infty; \quad (21c)$$

$$h(t,s) \leq h(t,T)h(T,s) \ if \ t \geq T \geq s. \quad (21d)$$

Assume that

$$\|R(\cdot)\| \in L^1. \quad (22a)$$

Then, system (17) has a solution $y = \hat{y}(t)$ defined $t$ large enough such that

$$\hat{y}(t) = \exp \left( \int_0^t \hat{\lambda}(s)ds \right) (\hat{e} + w(t)), \quad (23)$$

where $w(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Proof: Let

$$\Xi(t,s) = e^{\int_s^t \hat{\lambda}(\xi)d\xi}X(t,s).$$

Let $f = \|R(\cdot)\|$. Since $f : [0, +\infty] \rightarrow [0, +\infty]$ is a $L^1$ function, by Lemma 1 we obtain

$$\lim_{t \rightarrow +\infty} \int_0^t h(t,s)f(s)ds = 0. \quad (24)$$
Moreover,
\[
||\Xi(t, s)(I - P)|| \leq M, \text{ if } t \leq s. \tag{25}
\]

For \(t_0 \geq 0\), let \(\mathcal{B}_{t_0}\) be the set of all functions \(y \in C([t_0, +\infty[ , \mathbb{C}^N)\) such that
\[
||y||_{t_0} := \sup_{t \geq t_0} \left| e^{-\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} y(t) \right| < +\infty. \]
Notice that \((\mathcal{B}_{t_0}, || \cdot ||_{t_0})\) is a Banach space.

Let \(G : [t_0, +\infty[^2 \to \mathbb{C}^N\) be function defined by
\[
G(t, s) = e^{\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} \times \begin{cases} 
\Xi(t, s)P, & \text{if } t \geq s \\
-\Xi(t, s)(I - P), & \text{if } t < s. 
\end{cases}
\]

Let \(\mathcal{N} : C([t_0, +\infty[, \mathbb{C}^N) \to C([t_0, +\infty[, \mathbb{C}^N)\) the operator defined by
\[
\mathcal{N} y(t) = e^{\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} \hat{\xi} + \int_{t_0}^{+\infty} G(t, s) R(s) y(s) ds.
\]

Notice that \(y = \mathcal{N} y\) implies that \(y\) is solution of system (17), on \([t_0, +\infty[\).

From (24) and (25) we have
\[
\left| \int_{t_0}^{+\infty} G(t, s) R(s) y(s) ds \right| \leq \Theta(t) ||y||_{t_0},
\]
for all \(y \in \mathcal{B}_{t_0}\), where \(\Theta(t) = \left[ \int_0^t \hat{h}(t, s) f(s) ds + M \int_t^{+\infty} f(s) ds \right]\) and that \(\Theta(t) \to 0\), as \(t \to +\infty\). Then, we chose \(t_0 \geq 0\) such that \(\theta = \sup_{t \geq t_0} \Theta(t) < 1\).

It’s not hard to see that \(\mathcal{N}(\mathcal{B}_{t_0}) \subseteq \mathcal{B}_{t_0}\). Then, we can consider the restriction \(\hat{\mathcal{N}}(\mathcal{B}_{t_0}) \subseteq \mathcal{B}_{t_0}\). Notice that \(||\mathcal{N} y_1 - \mathcal{N} y_2||_{t_0} \leq \theta ||y_1 - y_2||_{t_0}\), for all \(y_1, y_2 \in \mathcal{B}_{t_0}\), i.e., \(\hat{\mathcal{N}}\) is contractive. Then, by the Banach Fixed Point Theorem \(\mathcal{N}\) has an only one fixed point \(\hat{y} \in \mathcal{B}_{t_0}\).

Since \(\lim_{t \to +\infty} \Theta(t) = 0\),
\[
\lim_{t \to +\infty} e^{-\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} \hat{\xi} = \hat{\xi}.
\]

In particular, \(\lim_{t \to +\infty} e^{-\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} y_0(t) = \hat{\xi}\).

By considering \(\hat{y} = e^{\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} \hat{y}\) we have \(y(t) = e^{-\int_{t_0}^{t} \hat{\lambda}(\xi) d\xi} \hat{y}(t) - \hat{\xi}\), The asymptotic formula (23) is obtained.

\(\square\)

To prove Theorem 1 by using the Theorem 3, it’s enough to consider \(\hat{\lambda} = \lambda_k\), \(\hat{\xi} = \hat{e}_k\) \(P = \text{diag}(1_{k_1}, 1_{k_2}, \ldots, 1_{k_N})\), where
\[
1_{k_j} = \begin{cases} 
1, & \text{if } j < k \\
0, & \text{if } j \geq k
\end{cases}
\]
and
\[
\Xi(t, s) = \text{diag} \left( e^{\int_{t_0}^{t} (\lambda_1(\xi) - \lambda_k(\xi)) d\xi} , \ldots , e^{\int_{t_0}^{t} (\lambda_N(\xi) - \lambda_k(\xi)) d\xi} \right).
\]
Then, \( \| \Xi(t,s)P \| \leq h(t,s) \), where \( h(t,s) = H_1(t,s) = \max_{j=1,...,k-1} e^{ \int_s^t \Re (\lambda_j(\xi) - \lambda_k(\xi)) \, d\xi } \). So,
\[
h(t,s) \leq h(t,T)h(T,s),
\]
by (15), \( h(t,s) \to 0 \) as \( t \to +\infty \) by (16) and
\[
\| \Xi(t,s)(I-P) \| \leq M, \quad \text{if } t \leq s,
\]
as (13), where \( M = \max\{1,e^K\} \). So, we obtain that the system (4) has a solution \( y_0 = \hat{y} \) with the asymptotic formula (5).

2.2 Discrete version of Levinson Theorem

Since a DEPCAG is an hybrid system, i.e., it can be seen as a mixture of differential equations and difference equations the we present the following result which pretend to help the form of the asymptotic formula for a DEPCAG.

**Theorem 4** [4, Castillo-Pinto (2002)] Consider the difference system \( \mathbb{C}^N \),
\[
\Delta y(n) = (\Lambda(n) + R(n))y(n), \quad n \in \mathbb{N}_0 = \mathbb{Z} \cap [0, +\infty], \quad (27)
\]
where \( \Delta y(n) = y(n+1) - y(n) \), \( \{\Lambda(n)\}_{n \geq 0} \) and \( \{R(n)\}_{n \geq 0} \) are two families of matrices \( \mathcal{M}_N(\mathbb{C}) \) such that

1. \( I + \Lambda(n) \) is invertible for all \( n \in \mathbb{N}_0 \);
2. \( \Lambda(n) = \text{diag}(\lambda_1(n),\ldots,\lambda_N(n)) \) where there is \( k \in \{1,\ldots,N\} \) such that
   \[
   (a) \quad \lim_{n \to +\infty} \left| \prod_{\xi=0}^{n-1} \frac{1 + \lambda_j(\xi)}{1 + \lambda_k(\xi)} \right| = 0, \quad \text{for } j < k;
   \]
   \[
   (b) \quad \text{There is a non negative constant } K \text{ such that } \left| \prod_{\xi=m}^{n-1} \frac{1 + \lambda_j(\xi)}{1 + \lambda_k(\xi)} \right| \leq K, \quad \text{for } j,k \in \{1,\ldots,N\} \text{ and } m,n \in \mathbb{N}_0;
   \]
3. \( \frac{1}{|1 + \lambda_k|} \| R(\cdot) \| \in \ell^1. \)

Then, the system (27) has a solution \( y = y_k(n) \) defined for \( n \) large enough with the following asymptotic formula
\[
y_k(n) = \left[ \prod_{\xi=0}^{n-1} (1 + \lambda_k(\xi)) \right] (e_k + w(n)), \quad (28)
\]
where \( e_k \) is the \( k \)-th vector of the canonical base of \( \mathbb{C}^N \) and \( w(n) \to 0 \) as \( n \to +\infty. \)
2.3 DEPCAG

In this section sets the stage to present the main result.

Assume that for equation (3) we have that

\[ I + \int_{\xi_n}^t X(\xi_n, u)B(u)du \] is invertible \hspace{1cm} (29a)

for all \( t \in [t_n, t_{n+1}] \) and \( n \in \mathbb{N}_0 \).

Let’s define \( Z : [t_0, +\infty[ \rightarrow \mathcal{M}_N(\mathbb{C}) \) as a matrix function such that \( Z(s, s) = I \) and for all \( z_0 \in \mathbb{C}^N \), \( z = Z(\cdot, s)z_0 \) is a solution of (3) in the sense given in the introduction for equation (1) such that \( z(s) = z_0 \). Then,

\[ Z(t, s) = X(t, \xi_n) \left[ I + \int_{\xi_n}^t X(\xi_n, u)B(u)du \right] \left[ I + \int_{\xi_n}^s X(\xi_n, u)B(u)du \right]^{-1} X(\xi_n, s), \]

for all \( t, s \in [t_n, t_{n+1}] \) such that \( t \geq s \), \( n \in \mathbb{N}_0 \) and \( X(t, s)X(s)^{-1} y X \) is a fundamental matrix for the system

\[ \frac{dx}{dt} = A(t)x. \] \hspace{1cm} (30)

For simplicity, we will consider \( \xi_n = t_n \), for all \( n \in \mathbb{N}_0 \). It will be seen that restriction does not imply an important loss of generality.

Assume that there are an unitary vector \( \hat{e} \) and locally integrable functions \( \hat{\lambda} : [t_0, +\infty[ \rightarrow \mathbb{C} \) and \( \hat{\lambda}_\gamma : [t_0, +\infty[ \rightarrow \mathbb{C} \) such that

\[ Z(t, s)\hat{e} = \hat{e}(t, s)\hat{e}, \] \hspace{1cm} (31a)

where

\[
\hat{e}(t, s) = e^{\int_s^t \hat{\lambda}(\xi)d\xi} \times \left[ 1 + \int_t^s e^{-\int_{t_j}^{t_{j+1}} \hat{\lambda}(\xi)d\xi} \hat{\lambda}_\gamma(u)du \right]^{-1} \times \left( \prod_{j=k_t}^{k_{t+1}} \left[ 1 + \int_{t_j}^{t_{j+1}} e^{-\int_{t_j}^{t_{j+1}} \hat{\lambda}(\xi)d\xi} \hat{\lambda}_\gamma(u)du \right] \right) \times \left[ 1 + \int_t^s e^{-\int_s^t \hat{\lambda}(\xi)d\xi} \hat{\lambda}_\gamma(u)du \right] \] \hspace{1cm} (31b)

and \( k_t \in \mathbb{N}_0 \) is defined for all \( t \in [t_0, +\infty[ \) such that \( t \in [t_{k_t}, t_{k_t+1}] \).

Assume that there is a projection \( P : \mathbb{C}^N \rightarrow \mathbb{C}^N \) such that

\[ \hat{e} \in (I - P)(\mathbb{C}^N); \] \hspace{1cm} (32a)

Assume that there are a bounded function \( h : [t_0, +\infty[ \times [t_0, +\infty[ \rightarrow [0, +\infty[ \) and a constant \( M > 0 \) such that

\[ \|\hat{Z}(t, s)P\| \leq h(t, s)|\hat{e}(t, s)|, \] \hspace{1cm} (33a)
for all \( t, s \in [t_0, +\infty[ \) such that \( t \geq s \):

\[
\left\| \hat{Z}(t, s) \right\| \leq M|\hat{c}(t, s)|;
\] (33b)

for all \( t, s \in [t_0, +\infty[ \) such that \( t < s \), where

\[
\hat{Z}(t, s) = \begin{cases} 
Z(t, t_{n+1})\Gamma(t_{n+1}, s) & \text{if } t_n \leq s \leq t_{n+1} \\
\Gamma(t, s) & \text{if } t_n \leq s \leq t_{n+1},
\end{cases}
\] (33c)

\[
\Gamma(t, s) = \begin{cases} 
Z(t, \gamma(t))X(t, s) & \text{if } t_k \leq s \leq \gamma(t) \\
X(t, s) & \text{if } \gamma(t) \leq t \leq t_{k+1},
\end{cases}
\] (33d)

\( s \in [t_n, t_{n+1}], t \geq t_n \) and \( n \in [n_0, +\infty]\cap \mathbb{Z} \).

Assume that

\[
h(t, s) \to 0 \text{ as } t \to +\infty;
\] (33e)

\[
h(t, s) \leq h(t, T)h(T, s) \text{ if } t \geq T \geq s.
\] (33f)

Moreover, assume that

\[
\int_{t_0}^{+\infty} \left\| \hat{c}(s, t_{k+1})^{-1}R(s) \right\| ds < +\infty.
\] (34a)

Condition (29a) is analogous to invertibility condition (regressivity) of theorem 27 and it implies the invertibility of \( Z(t, s) \) which allows to define \( Z(s, t) = Z(t, s)^{-1} \) for \( t > s \). Conditions (31a), (32a), (33a), (33b), (33c), (33d) and (34a), the conditions (18a), (20a), (21a), (21b), (21c), (21d) and (22a) of Theorem 11 respectively.

It will be used the variation of parameters formula for DEPCAG [3, 9] as follows.

Let

\[
\Phi(n) = H(n - 1)H(n - 2)\cdots H(0),
\]

where

\[
H(n) = X(t_{n+1}, \xi_n)D_n(t_{n+1})D_n(t_n)^{-1}X(\xi_n, t_n),
\] (35)

and

\[
D_n(t) = I + \int_{\xi_n}^{t} X(\xi_n, u)B(u)du,
\] (36)

for all \( n \in \mathbb{N}_0 \) and for all \( t \in [t_n, t_{n+1}] \).

Given \( s, t \in [t_0, +\infty[ \), we have that

\[
Z(t, s) = X(t, \gamma(t))D_{k_s}(t)D_{k_s}(t_{k_s})^{-1}X(\gamma(t), t_{k_s})
\]

\[
\times \Phi(k_s)\Phi(k_s + 1)^{-1}
\]

\[
\times X(t_{k_s+1}, \gamma(s))D_{k_s}(t_{k_s+1})D_{k_s}(s)^{-1}X(\gamma(s), s).
\] (37)
Notice that \( X(t, s) \dot{e} = \exp \left( \int_s^t \hat{\lambda}(\xi) d\xi \right) \dot{e} \) and \( B(t) \dot{e} = \hat{\lambda}_d(t) \dot{e} \). Then,

\[
Z(t, s) \dot{e} = e^{\int_s^t \hat{\lambda}(\xi) d\xi} \times \left( 1 + \int_{\gamma(t)}^{t} e^{\int_{\gamma(s)}^u \hat{\lambda}(\xi) d\xi} \lambda_d(u) du \right)^{-1} \times \left( 1 + \int_{\gamma(s)}^{t} e^{\int_{\gamma(s)}^u \hat{\lambda}(\xi) d\xi} \lambda_d(u) du \right)^{-1} \times \dot{e}.
\]

Formula (31a) is naturally obtained under the assumption \( \xi_n = t_n \) for all \( n \in \mathbb{N}_0 \).

So, the solution of a DEPCAG

\[
\frac{d\psi}{dt} = A(t)\psi(t) + B(t)\psi(\gamma(t)) + f(t),
\]

(39)
can be written as,

\[
\psi(t) = Z(t, \gamma(t))\psi(\gamma(t)) + \int_{\gamma(t)}^{t} X(t, s)f(s)ds.
\]

and

\[
\psi(\gamma(t)) = Z(\gamma(t), t_{k_{t}})\psi(t_{k_{t}}) + \int_{t_{k_{t}}}^{\gamma(t)} X(\gamma(t), s)f(s)ds.
\]

Then,

\[
\psi(t) = Z(t, \gamma(t)) \left[ Z(\gamma(t), t_{k_{t}})\psi(t_{k_{t}}) + \int_{t_{k_{t}}}^{\gamma(t)} X(\gamma(t), s)f(s)ds \right] + \int_{\gamma(t)}^{t} X(t, s)f(s)ds,
\]

i.e.,

\[
\psi(t) = Z(t, t_{k_{t}})\psi(t_{k_{t}}) + \int_{t_{k_{t}}}^{t} \Gamma(t, s)f(s)ds.
\]

So,

\[
\psi(t_{n+1}) = H(n)x(t_{n}) + \int_{t_{n}}^{t_{n+1}} \Gamma(t_{n+1}, s)f(s)ds,
\]

for all \( n \in [n_0, +\infty] \cap \mathbb{Z} \). So,

\[
\psi(t) = Z(t, t_{k_{t}}) \left[ \Phi(t_{k_{t}})\Phi(n_0)^{-1}\psi(t_{n_0}) + \sum_{j=n_0}^{t_{k_{t}}-1} \Phi(t_{k_{t}})\Phi(j + 1)^{-1} \int_{t_j}^{t_{j+1}} \Gamma(t_{j+1}, s)f(s)ds \right] + \int_{t_{k_{t}}}^{t} \Gamma(t, s)f(s)ds.
\]
Hence,
\[
\psi(t) = Z(t, t_{n_0})\psi(t_{n_0}) + \sum_{j=n_0}^{t-1} \int_{t_j}^t Z(t, t_{j+1})\Gamma(t_{j+1}, s)f(s)ds + \int_{t_{n_0}}^t \Gamma(t, s)f(s)ds.
\]

(40)

Then, the integral equation (40) can be written as
\[
\psi(t) = Z(t, t_{n_0})\psi(t_{n_0}) + \int_{t_0}^t \dot{Z}(t, s)f(s)ds,
\]

(41)

for \( t \geq n_0 \).

Let \( F : [t_0, +\infty[ \times \mathbb{C}^N \rightarrow \mathbb{C}^N \) a locally integrable function in the first variable such that there is \( \eta : [t_0, +\infty[ \rightarrow \mathbb{R}_+^0 \) such that
\[
|F(t, \hat{a}) - F(t, \hat{b})| \leq \eta(t)|\hat{a} - \hat{b}|
\]

and
\[
\int_{t_0}^{+\infty} |E(t, \gamma(t))|^{-1}\eta(t)dt < +\infty,
\]

(42)

where \( E \) is given by the formula (38) which satisfies the relation \( Z(t, s)\dot{e} = E(t, s)e \).

Let’s extend the condition (33a) as
\[
|\dot{Z}(t, s)P| \leq M|E(t, s)|h(t, s), \text{ for } t \geq s,
\]

(44)

Let’s extend the condition (33b) as
\[
|\dot{Z}(t, s)(I - P)| \leq M|E(t, s)|, \text{ for } t \leq s,
\]

(45)

Let \( \Xi(t, s) = \dot{e}(t, s)^{-1}\dot{Z}(t, s) \), for \( t, s \geq 0 \). Sea \( G(t, s) : [0, +\infty[^2 \rightarrow \mathcal{M}_N(\mathbb{C}) \)

defined for
\[
G(t, s) = E(t, s) \times \begin{cases}
\Xi(t, s)P, & \text{if } t \geq s \\
-I\Xi(t, s)(I - P), & \text{if } t < s.
\end{cases}
\]

Then, \( \|G(t, s)\| \leq h(t, s)|\dot{e}(t, s)| \leq K|\dot{e}(t, s)| \), for all \( t, s \geq 0 \).

Let \( n_0 \in \mathbb{N}_0 \). Sea \( \mathcal{B}_{n_0} \) the set of functions \( y : [n_0, +\infty[ \rightarrow \mathbb{C}^N \) such that
\[
\dot{e}(\cdot, t_{n_0})^{-1}y \in L^\infty. \text{ For } y \in \mathcal{B}_{n_0}, \text{ sea } \|y\|_{n_0} = \sup_{t \geq t_{n_0}} \|\dot{e}(t, t_{n_0})^{-1}y(t)\|.
\]

Then, \( (\mathcal{B}_{n_0}, \| \cdot \|_{n_0}) \) is Banach space, which is isometrically isomorphic to the Banach space \( (L^\infty, \| \cdot \|_\infty) \).

Let \( \mathcal{N} : \mathcal{B}_{n_0} \rightarrow \mathcal{B}_{n_0} \) be an operator defined by
\[
(\mathcal{N}y)(t) = E(t, t_{n_0})\dot{e} + \int_{t_{n_0}}^{+\infty} G(t, s)f(s, \gamma(s))ds,
\]

(46)
for all \( y \in \mathcal{B}_{n_0} \) and \( t \geq t_{n_0} \).

Notice that the operator \( \mathcal{N} \) is well defined. In fact, let \( y \in \mathcal{B}_{n_0} \). Then,

\[
|E(t, t_{n_0})^{-1}[(\mathcal{N}y)(t)]| \leq |\hat{e}| + M \int_{t_{n_0}}^{t} h(t, s) |E(s, \gamma(s))^{-1}F(s, \gamma(s))| |E(\gamma(s), t_{n_0})^{-1}y(\gamma(s))|ds
+ K \int_{t}^{\infty} |E(s, \gamma(s))^{-1}F(s, \gamma(s))| |E(\gamma(s), t_{n_0})^{-1}y(\gamma(s))|ds
\leq 1 + \Theta_{n_0}(t)\|y\|_{n_0},
\]

where \( \Theta_{n_0}(t) = M \int_{t_{n_0}}^{t} h(t, s) |E(s, \gamma(s))^{-1}|ds + M \int_{t}^{\infty} |E(s, \gamma(s))^{-1}1|ds \).

By (43), \( \Theta_{n_0}(t) \leq \sup_{\tau \geq t_{n_0}} \Theta_{n_0}(\tau) < +\infty \).

So, \( \mathcal{N}y \in \mathcal{B}_{n_0} \).

Let \( n_0 \) so large that \( \Theta_{n_0}(\infty) < 1 \). Since

\[ \|\mathcal{N}y_1 - \mathcal{N}y_2\|_{n_0} \leq \Theta_{n_0}(\infty)\|y_1 - y_2\|_{n_0}, \]

for all \( y_1, y_2 \in \mathcal{B}_{n_0} \), by the Banach Fixed Point Theorem, there is a only one \( y \in \mathcal{B}_{n_0} \) such that \( y = \mathcal{N}y \).

Moreover,

\[ |E(t, t_{n_0})^{-1}y(t) - \hat{e}| \leq \Theta_{n_0}(t)\|y\|_{n_0}. \tag{47} \]

Due to the conditions (32) y (33), we have that

\[ |E(\cdot, \gamma(\cdot))^{-1}1| \in L^1. \]

By lemma 4, we have that \( \lim_{t \to +\infty} \Theta_{n_0}(t) = 0 \).

By considering \( w(t) = E(t, t_{n_0})^{-1}y(t) - \hat{e} \), by the inequality (47), we have \( w(t) \to 0 \) as \( t \to +\infty \).

So, the constructed contractive operator allows us to state the following result.

**Theorem 5** Assume for that DEPCAG (3) the conditions (21a), (33a), (33b), (43), (47) are satisfied and the conditions (42) and (43) are satisfied for \( F \). Then, the operator \( \mathcal{N} \) defined by (40) satisfies:

1. \( \mathcal{N}(\mathcal{B}_{n_0}) \subseteq \mathcal{B}_{n_0} \);
2. \( E(t, t_{n_0})^{-1}[(\mathcal{N}y)(t) - \hat{e}] \to 0 \) as \( t \to +\infty \);
3. \( \mathcal{N} \) is contractive for \( n_0 \) large enough;
4. \( \mathcal{N} \) has a only one fixed point \( y_{n_0} \in \mathcal{B}_{n_0} \), i. e., \( y_{n_0} = \mathcal{N}(y_{n_0}) \);
5. The fixed point \( y_{n_0} \) satisfies the asymptotic formula

\[ y_{n_0}(t) = E(t, n_0)(\hat{e} + w(t)), \tag{48} \]

where \( w(t) \to 0 \) as \( t \to +\infty \).
3 Main Result

Now, we are in conditions to present our main result.

**Theorem 6** Assume that \( \xi_n = t_n \) for all \( n \in \mathbb{N}_0 \) an that conditions \( (29a), (31a), (33a), (33b), (33e), (33f) \) and \( (34a) \) are satisfied. Then, the DEPCAG (2) has a solution \( y = y(t) \) defined for \( t \geq t_{n_0} \) with \( n_0 \) large enough such that

\[
y(t) = \hat{e}(t, t_{n_0}) (\hat{e} + w(t)), \quad (49)
\]

where \( w(t) \to 0 \) as \( t \to +\infty \).

**Proof:** Since \( \xi_n = t_n \) for all \( n \in \mathbb{N}_0 \), \( \hat{e}(t, s) = E(t, s) \) and the conditions of Teorema 5 are satisfied, the operator \( \mathcal{N} \) defined by

\[
(\mathcal{N}y)(t) = \hat{e}(t, t_{n_0}) \hat{e} + \int_{t_{n_0}}^{+\infty} G(t, s) R(s)y(\gamma(s)) ds
\]

has a fixed point \( y = y(t) \) defined for all \( t \geq t_{n_0} \) with \( n_0 \) large enough with the asymptotic formula (48) which takes the form (49).

From (41) it can be easily proved that \( y \) is solution of (2). Therefore the DEPCAG (2) has a solution \( y = y(t) \) defined for \( t \geq t_{n_0} \) with \( n_0 \) large enough with the asymptotic formula (49).

\[ \square \]

**Remark:** If we had considered \( \xi_n \neq t_n \) as it was assumed, the asymptotic formula (49) is

\[
y(t) = E(t, t_{n_0}) (\hat{e} + w(t)), \quad (50)
\]

where \( E \) is given by

\[
E(t, s) = e^{\int_s^t \lambda(\xi) d\xi} \\
\times \left( 1 + e^{\int_s^t \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right) \left( 1 + e^{\int_s^t \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right)^{-1} \\
\times \left( \prod_{j=k_s+1}^{t} \left[ 1 + e^{\int_{\xi_j}^{\xi_{j+1}} \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right] \left[ 1 + e^{\int_{\xi_j}^{\xi_{j+1}} \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right]^{-1} \right) \\
\times \left( 1 + e^{\int_s^t \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right) \left( 1 + e^{\int_s^t \lambda(\xi) d\xi} \hat{\lambda}_d(u) du \right)^{-1}.
\]

From the proof the Theorem 6 it can be obtained that DEPCAG (2) has a solution with asymptotic formula (50). In similar way, the DEPCAG (2) can be changed by the DEPCAG:

\[
\frac{dy}{dt} = A(t)y(t) + B(t)\gamma(t) + F(t, \gamma(t)), \quad (51)
\]

where \( F \) satisfies (42) and (43). Then the DEPCAG (51) has a solution with asymptotic formula (49).
Example Consider the diagonal matrices in $\mathcal{M}_N(\mathbb{C})$,

$\Lambda_A(t) = \text{diag}(a_1(t), a_2(t), \ldots, a_N(t))$ and $\Lambda_B(t) = \text{diag}(b_1(t), b_2(t), \ldots, b_N(t))$.

Consider the system $\mathbb{C}^N$,

$$z'(t) = \Lambda_A(t)z(t) + \Lambda_B(t)z(\gamma(t)),$$

for $t \geq t_0$. Then, the solutions of (52) can be written as

$$y(t) = Z(t, s)y(s),$$

where $Z(t, s) = \text{diag}(e_1(t, s), e_2(t, s), \ldots, e_N(t, s))$,

$$e_i(t, s) = e^{\int_{t}^{s} a_i(\xi)d\xi} \left(1 + \int_{t}^{s} e^{-\int_{t}^{\omega} a_i(\xi)d\xi} b_i(\sigma)d\sigma\right)^{-1} \times \left[\prod_{k=1}^{\lambda-1} \left(1 + \int_{t}^{s} e^{-\int_{t}^{\omega} a_l(\xi)d\xi} b_l(\sigma)d\sigma\right)\right] \times \left(1 + \int_{t}^{s} e^{-\int_{t}^{\omega} a_l(\xi)d\xi} b_l(\sigma)d\sigma\right),$$

for $l \in \{1, \ldots, N\}$ and $t \geq s$.

We assume that

$$1 + \int_{t}^{s} e^{-\int_{t}^{\omega} a_l(\xi)d\xi} b_l(\sigma)d\sigma \neq 0,$$

for all $l \in \{1, \ldots, N\}$, $t \in [t_m, t_{m+1}]$ and $m \in \mathbb{N}_0$.

This is equivalent to (29a) and guarantees that $Z(t, s)$ is invertible for $t \geq s$ and the existence of $Z(t, s)$ for $t < s$. Moreover, $e_i(t, s) = \frac{1}{e_i(t, t)}$ for $t < \gamma$.

The following is a direct application of theorem (6).

**Corollary 1** Assume that there is $\kappa \in \{1, \ldots, N\}$ such that

(a) $\lim_{t \to +\infty} \frac{|e_i(t, s)|}{e_k(t, s)} = 0$, for $l < k$,

(b) There is $C > 0$ such that $\frac{|e_i(t, s)|}{e_k(t, s)} \leq C$, for $l \geq k$.

Let $\{R(t)\}_{t \geq 0}$ a family in $\mathcal{M}_N(\mathbb{C})$ such that $R(\cdot)$ is locally integrable and

$$\sum_{n=n_0}^{+\infty} \int_{t_n}^{t_{n+1}} |e_k(s, t_n)|^{-1} ||R(s)||ds$$

is convergent.

(54)

Then, the DEPCAG

$$y'(t) = \Lambda_A(t)y(t) + \Lambda_B(t)y(\gamma(t)) + R(t)y(\gamma(t)),$$

has a solution $y = y(t)$ defined for $t \geq t_0$ with $t_0$ a large enough such that

$$y(t) = e_k(t, t_0)(e_k + w(t)),$$

where $e_k$ is the $k$-th vector of the canonical base of $\mathbb{C}^N$ and $w(t) \to 0$ as $t \to +\infty$. 

16
References

[1] Akhmet, M. (2007) Integral manifolds of differential equations with piecewise constant argument of generalized type. Nonlinear Analysis TMA 66 no. 2, 367-383.

[2] Akhmet, M. (2008) Asymptotic behavior of solutions of differential equations with piecewise constant arguments. Applied Mathematics Letters 21 951-956.

[3] Akhmet, M. (2008) Stability of differential equations with piecewise constant arguments of generalized type 68 794-803

[4] Castillo, S., Pinto, M. (2002) Improvements in asymptotic formulae results for functional difference equations, New trends in difference equations, Taylor & Francis, London, 57-67.

[5] Coddington, E.A., Levinson, N (1955) Theory of ordinary differential equations. International series in pure and applied mathematics. McGraw-Hill.

[6] Cooke K., Wiener J. (1984) Retarded differential equations with piecewise constant delays. J. of Math. Anal. Appl. 99, 265-297

[7] Eastham, M.S.P. (1989) The Asymptotic Solution of Linear Differential Systems, Applications of the Levinson Theorem, Clarendon, Oxford,

[8] Levinson, N. (1948) The asymptotic nature of solutions of linear differential equations, Duke Math. J. 15, 111-126.

[9] Pinto, M. (2011) Pinto, Manuel. Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems. J. Difference Equ. Appl. 17, no. 2, 235–254

[10] Shah, S., Wiener, J. (1983) Advanced differential equations with piecewise constant argument deviations. Internat. J. Math. and Math. Sci. 6 no. 4, 71-703.

[11] Wiener, J. (1984) Differential equations with piecewise constant delays. Trends in theory and practice of nonlinear differential equations (Arlington, Tex., 1982), Lecture Notes in Pure and Appl. Math. 90 Dekker, New York, 547-552.

[12] Wiener, J. (1993) Generalized Solutions of Functional Differential Equations. World Scientific.