UNIQUENESS OF THE KONTSEVICH-VISHIK TRACE

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Abstract. Let $M$ be a closed manifold. We show that the Kontsevich-Vishik trace, which is defined on the set of all classical pseudodifferential operators on $M$, whose (complex) order is not an integer greater than or equal to $-\dim M$, is the unique functional which (i) is linear on its domain, (ii) has the trace property and (iii) coincides with the $L^2$-operator trace on trace class operators.

Also the extension to even-even pseudodifferential operators of arbitrary integer order on odd-dimensional manifolds and to even-odd pseudodifferential operators of arbitrary integer order on even-dimensional manifolds is unique.

1. Introduction

We denote by $M$ a compact $n$-dimensional manifold without boundary. A classical pseudodifferential operator $(\psi do)$ $A$ acting on sections of a vector bundle over $M$ is said to have order $\mu \in \mathbb{C}$ if it belongs to the Hörmander class $S_{1,0}^{\Re \mu}(M)$ and the local symbols $a = a(x, \xi)$ of $A$ have asymptotic expansions

$$a \sim \sum_{j=0}^{\infty} a_{\mu-j},$$

where the $a_{\mu-j}$ are positively homogeneous of degree $\mu - j$ for large $\xi$. We shall write ord $A = \mu$ to express that the order of $A$ is $\mu$.

In two remarkable papers, Kontsevich and Vishik in 1994 and 1995 analyzed the properties of determinants of elliptic $\psi$do’s, [7], [8]. One important tool was the construction of a trace-like mapping $\text{TR}$ defined on the set of all classical $\psi$do’s whose order is not an element of $\mathbb{Z}_{\geq -n}$, the set of integers greater than or equal to $-n$; see also the Remarks, below.

We shall denote this domain by $D$. As the sum of two operators of orders $\mu$ and $\mu'$ in $D$ is an element of $D$ only if $\mu - \mu'$ is an integer, $D$ is not a vector space. Thus it does not make sense to speak about linear functionals on $D$. The map $\text{TR} : D \to \mathbb{C}$, however, is as linear as it can be expected to be:

$$\text{TR}(cA + dB) = c\text{TR}(A) + d\text{TR}(B) \quad \text{for } c, d \in \mathbb{C}, A, B, cA + dB \in D.$$
Moreover, TR behaves like a trace:

\[ \text{TR}(AB) = \text{TR}(BA), \quad \text{whenever } AB, BA \in \mathcal{D}. \]

Finally, the Kontsevich-Vishik trace (sometimes also canonical trace) TR coincides with the $L^2$-operator trace Tr on trace class $\psi$do's:

\[ \text{TR}(A) = \text{Tr}(A) \quad \text{if } \text{Re ord}(A) < -n. \]

It is clear that the Kontsevich-Vishik trace cannot be extended to a trace on the algebra of all $\psi$do's on $M$: The only trace there (up to multiples) is the Wodzicki residue [13], which is known to vanish on trace class operators. There is also a simple direct way to see this: We know – e.g. from the Atiyah-Singer index theorem – that there exists an elliptic pseudodifferential operator $P$ on $M$ with nonzero index. Using order reducing operators, we may assume the order of $P$ to be zero. Let $Q$ be a parametrix to $P$ modulo smoothing operators. Then

\[ \text{Index } P = \text{Tr}(1 - PQ) - \text{Tr}(1 - QP). \]

If we could extend TR to a trace on all pseudodifferential operators, the right hand side could be rewritten as the trace of the commutator $[P, Q]$ and therefore would have to be zero – a contradiction.

It has been observed, however, by Kontsevich-Vishik and Grubb [4] that TR extends to a slightly larger domain. Recall that the symbol $a$ of an integer order operator $A$ is said to be even-even if the homogeneous components satisfy

\[ a_{\mu-j}(x, -\xi) = (-1)^{\mu-j} a_{\mu-j}(x, \xi). \]

It is called even-odd, if

\[ a_{\mu-j}(x, -\xi) = (-1)^{\mu-j+1} a_{\mu-j}(x, \xi). \]

The Kontsevich-Vishik trace $\text{TR}(A)$ for a $\psi$do $A$ of order $\mu$ then can also be defined if $\mu \in \mathbb{Z}_{\geq -n}$, provided that

- (EE) $n$ is odd, and the symbol of $A$ is even-even, or
- (EO) $n$ is even, and the symbol of $A$ is even-odd.

For the sake of brevity we shall denote this larger domain (depending on $n$) by $\mathcal{D}^+$. In both cases, the component $a_{-n}$ in the asymptotic expansion of the symbol of $A$ is odd in $\xi$ for large $|\xi|$, say for $|\xi| \geq 1$:

\[ a_{-n}(x, -\xi) = -a_{-n}(x, \xi). \]

Hence the density for the Wodzicki residue of the operator $A$ vanishes pointwise, i.e.

\[ \text{res}_x(A) = \int_{S^*_xM} \text{tr} a_{-n}(x, \xi) d\sigma(\xi) = 0 \quad \text{for each } x \in M. \]

Here, $d\sigma$ is the surface measure on the unit sphere $S^*_xM$ over $x$ in the cotangent bundle and tr is the fiber trace. The Wodzicki residue of $A$ is given by integration of res$_x$ $A$ over $M$ and therefore also vanishes.

The trace property (1.3) extends to the case where $A$ and $B$ have integer order and $AB$ and $BA$ belong to $\mathcal{D}^+$.

The Kontsevich-Vishik trace has received considerable attention and found interesting applications; see e.g. [5, 9, 10, 11, 12]. Moreover, it has been extended to boundary value problems in Boutet de Monvel’s calculus [3].
It seems, however, that it has never been noticed that the above properties make the Kontsevich-Vishik trace unique. This is what we show in this short note. The proof, which will be given in the next section, relies on ideas in [2].

**Theorem.** (a) Let $\tau : D \to \mathbb{C}$ be a map with properties (1.2), (1.3), and (1.4). Then $\tau = TR$.

(b) Also the extension of $\tau$ to $D^+$ is unique. In fact, $\tau$ is already unique on the space of all integer order $\psi$do's which satisfy (EE) or (EO) when $\mu \geq -n$.

## 2. Proof

In order to establish (a), choose a $\psi$do $A$ of order $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq -n}$ on $M$.

We find a covering of $M$ by open neighborhoods and a finite subordinate partition of unity $\{\varphi_j\}$ such that for every pair $(j, k)$, both $\varphi_j$ and $\varphi_k$ have support in one coordinate neighborhood. We write

$$A = \sum_{j, k} \varphi_j A \varphi_k.$$

Each operator $\varphi_j A \varphi_k$ may be considered a $\psi$do on $\mathbb{R}^n$. As the map $\tau$ has the linearity property (1.2), we may confine ourselves to the case where $A = \text{op} \ a$ with a symbol $a$ on $\mathbb{R}^n$ having an expansion (1.1). Moreover, we can assume that $A = \varphi A \psi$ whenever $\varphi, \psi \in C_\infty^\omega(\mathbb{R}^n)$ are equal to one on a sufficiently large set.

To simplify further, we write

$$A = \text{op} \ a_\mu + \text{op} \ a_{\mu-1} + \ldots + \text{op} \ a_{\mu-K} + \text{op} \ r,$$

where $a_{\mu-j}$ is a symbol on $\mathbb{R}^n$, homogeneous in $\xi$ of degree $\mu - j$ for $|\xi| \geq 1$, and $K$ is so large that $r \in S_{1,0}^{-n-1}$. For $\varphi, \psi \in C_\infty^\omega(\mathbb{R}^n)$ as above we then have

$$\tau(\text{op} \ a) = \tau(\varphi \text{op}(a) \psi) = \sum_{j=0}^{K} \tau(\varphi \text{op}(a_{\mu-j}) \psi) + \tau(\varphi \text{op}(r) \psi).$$

Since $\tau(\varphi \text{op}(r) \psi) = \text{tr}(\varphi \text{op}(r) \psi)$ by (1.4), we will know $\tau(\text{op} \ a)$ as soon as we know $\tau(\varphi \text{op}(a_{\mu-j}) \psi)$ for $j = 0, \ldots, K$.

We may assume that $\mu$ is not an integer, since the operator trace determines $\tau$ on all operators of order $\mu < -n$. Now we let, similar to the proof of [2, Lemma 1.3(i)],

$$b_{\mu-j}(x, \xi) = \frac{1}{n + \mu - j} \sum_{k=1}^{n} \partial_{\xi_k} (\xi_k a_{\mu-j}(x, \xi)).$$

Euler’s relation for homogenous functions implies that, for $|\xi| \geq 1$,

$$b_{\mu-j} = \frac{1}{n + \mu - j} (na_{\mu-j} + (\mu - j)a_{\mu-j}) = a_{\mu-j}.$$

Hence we can write

$$\tau(\varphi \text{op}(a_{\mu-j}) \psi) = \tau(\varphi \text{op}(a_{\mu-j} - b_{\mu-j}) \psi) + \tau(\varphi \text{op}(b_{\mu-j}) \psi).$$

Since $a_{\mu-j} - b_{\mu-j}$ is regularizing, the first term on the right hand side is determined by property (1.4). Now we additionally choose $\chi \in C_\infty^\omega(\mathbb{R}^n)$ with $\chi \varphi = \varphi$ and
\chi \psi = \psi. The fact that \text{op}(\partial_{\xi_k} p) = -i \left[ x_k, \text{op} p \right] for an arbitrary symbol p implies that

\varphi \text{op}(b_{\mu-j}) \psi = -i \sum_{k=1}^{n} \left[ \chi x_k, \varphi \text{op}(\xi_k a_{\mu-j}) \psi \right].

Assuming that \tau has property (1.3), it vanishes on the last term in (2.9) which is a sum of commutators. Hence the proof of (a) is complete.

Next let us show (b). With the same considerations as before we may assume that \( A = \text{op} a \) is a pseudodifferential operator on \( \mathbb{R}^n \) with a representation as in (2.7), where now \( \mu \) is an integer \( \geq -n \) and the \( a_{\mu-j} \) have property (EE) or (EO).

We only have to show that \( \tau(\varphi \text{op}(a_{\mu-j}) \psi) \) is uniquely determined, \( j = 0, \ldots, \mu+n \). For \( \mu - j \neq -n \) the argument is as before, using the symbols in (2.8) and noting that \( a_{\mu-j}(x, \xi) \xi_k \) is even-even or even-odd whenever this is the case for \( a_{\mu-j} \).

Let us consider \( a_{-n} \). Now we apply the technique used in the proof of [2, Lemma 1.3(ii)]. The assumption that \( n \) is odd and \( a_{-n} \) even-even or \( n \) is even and \( a_{-n} \) even-odd implies that \( a_{-n} \) is odd in \( \xi \):

\[ a_{-n}(x, -\xi) = -a_{-n}(x, \xi) \quad \text{for } |\xi| \geq 1. \]

Hence, for each fixed \( x \), the integral over the unit sphere \( S = \{|\xi| = 1\} \) vanishes:

\[ \int_{S} a_{-n}(x, \xi) d\sigma(\xi) = 0. \tag{2.10} \]

The Laplace operator \( \Delta = \sum_{k=1}^{n} \partial^2 / \partial x_k^2 \) in polar coordinates takes the form

\[ \Delta = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_S, \]

where \( r = |\xi| \) is the radial variable and \( \Delta_S \) is the Laplace-Beltrami operator on \( S \).

Equation (2.10) implies that, for each \( x \), the function \( a_{-n}(x, \cdot)|_S \) is orthogonal to the constants which form the kernel of the symmetric operator \( \Delta_S \). Hence there is a unique function \( q(x, \cdot) \in C^\infty(S) \), orthogonal to the constants, such that \( \Delta_S q(x, \cdot) = a_{-n}(x, \cdot)|_S \). As \( \Delta_S \) commutes with the antipodal map \( \eta \mapsto -\eta \), we have \( \Delta_S q(x, -\cdot) = a_{-n}(x, -\cdot)|_S = -a_{-n}(x, \cdot)|_S \). Hence \( q(x, \cdot) + q(x, -\cdot) \) belongs to the kernel of \( \Delta_S \), and thus is constant. On the other hand, both \( q(x, \cdot) \) and \( q(x, -\cdot) \) are orthogonal to the constants. Therefore \( q(x, \cdot) + q(x, -\cdot) \) is zero, i.e., \( q(x, \cdot) \) is an odd function on \( S \).

Now we choose a smooth function \( \omega \) on \( \mathbb{R} \) which vanishes for small \( r \) and is equal to 1 for \( r \geq 1/2 \). We let

\[ b_{-n} = \omega(r)r^{2-n}q = \omega(|\xi|)|\xi|^{2-n}q(x, \xi/|\xi|). \]

This is a smooth function on \( \mathbb{R}^n \) which is homogeneous of degree \( 2-n \) in \( \xi \) for \( |\xi| \geq 1 \). As \( a_{-n}(x, \xi) \) vanishes for \( x \) outside a compact set, so does \( b_{-n}(x, \xi) \). In particular, \( b_{-n} \) is an element in \( S^2_{0,0} \mathbb{R}^n \times \mathbb{R}^n \). Moreover, we have for \( |\xi| \geq 1 \)

\[ \Delta b_{-n} = \Delta (r^{2-n}q(x, \cdot)) = r^{-n}a_{-n}(x, \cdot)|_S = a_{-n}. \]

We write \( a_{-n} = (a_{-n} - \Delta b_{-n}) + \Delta b_{-n} \). The symbol \( a_{-n} - \Delta b_{-n} \) is regularizing and thus \( \tau(\varphi \text{op}(a_{-n} - \Delta b_{-n}) \psi) \) is determined by (1.4). The operator associated with \( \text{op}(\varphi \Delta b_{-n} \psi) \) on the other hand is a sum of commutators:

\[ \varphi \text{op}(\Delta b_{-n}) \psi = -i \sum_{k=1}^{n} \left[ \chi x_k, \varphi \text{op}(\partial_{\xi_k} b_{-n}) \psi \right]. \tag{2.11} \]
where χ is chosen as in the proof of (a). Hence τ vanishes on \( \varphi \text{op}(\Delta b_{-n})\psi \). This concludes the argument.

Remarks. (a) One way of defining TR is as follows [7], [4]: Choose an invertible positive \( \psi \text{do} \) \( P \) of order \( m > 0 \) with scalar principal symbol and define the complex powers \( P^z \), \( z \in \mathbb{C} \). The generalized ‘zeta function’ \( \zeta(A, P, z) = \text{Tr}(AP^{-z}) \) is holomorphic on \( \{ \text{Re} \, z > (n + \mu)/m \} \), \( \mu = \text{ord} \, A \), and extends meromorphically to \( \mathbb{C} \) with at most simple poles in the points \( z_j = (n + \mu - j)/m \). If \( \mu \notin \mathbb{Z}_{\leq -n} \) or if \( A \) satisfies (EE) or (EO), then there is no pole in \( z = 0 \), and \( \text{TR}(A) := \zeta(A; P, 0) \) is independent of \( P \).

One can also define TR for \( \mu \notin \mathbb{Z}_{\leq -n} \) by regularizing the integral \( \int k(x, x) \, dx \) over the local distributional kernel of \( A \), thus generalizing Lidskij’s formula for trace class operators. In this spirit (and in a more general framework) Connes and Moscovici prove another uniqueness result, [1, Lemma I.5]: TR is the unique holomorphic extension of the classical Lidskij formula through holomorphic families of \( \psi \text{do} \)’s of noninteger order.

(b) For noninteger \( \mu \), the uniqueness of the Kontsevich-Vishik trace can also be derived from a result by Lesch, [9, Proposition 4.7], which implies that a \( \psi \text{do} \) \( A \) of order \( \mu \neq \mathbb{Z} \) can be written in the form \( A = \sum [P_j, Q_j] + R \) with finitely many \( \psi \text{do} \)’s \( P_j, Q_j \), and \( R \) of orders 1, \( \mu \), and \( -\infty \), respectively. His proof relies on a construction by Guillemin [6, Theorem 6.2] which makes the argument less elementary than the one given here.

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