Lower-Dimensional Regge-Teitelboim Gravity
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Abstract—We study modified gravity theory known as the Regge–Teitelboim approach, in which gravity is represented by the dynamics of a surface isometrically embedded in a flat bulk. We obtain some particular solutions of Regge–Teitelboim equations corresponding to a circularly symmetric vacuum 2+1-dimensional space-time. In contrast to GR, this vacuum space-time is not flat, so it is possible for the gravitational field to exist even without matter or a cosmological constant.

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1. INTRODUCTION

Although gravity was the first fundamental interaction that humankind discovered, it remains the last to be satisfactorily understood. Since the appearance of general relativity, there were numerous attempts to extend or modify it. Such modifications aimed at resolving the discrepancies of GR predictions and astrophysical (“dark matter”) and cosmological (“dark energy”) observations, as well as to provide a suitable framework for quantization. A satisfactory theory of gravity, as perceived by many, should meet the following criteria:

1. It should adequately describe all observable processes taking place in our universe.
2. One should be able to find its explicit solutions corresponding to physically relevant systems and processes.
3. It should have a sound physical and/or mathematical motivation.

Unfortunately, it is quite tricky to find a theory which could meet all these criteria at once, so we end up happy when at least two of them are met. Typical examples of phenomenological theories satisfying the first two criteria could be MOND or mimetic gravity \cite{1}, while various lower-dimensional models could definitely satisfy the last two ones, but fail to meet the first one because our universe is four-dimensional.

In this paper we consider the Regge–Teitelboim approach \cite{2} which is a string-inspired description of gravity as a dynamics of a surface in a flat higher-dimensional bulk. It possesses non-Einsteinian solutions as it belongs to the same class as mimetic gravity (both theories appearing after a differential field transformation in GR) \cite{3}, but has a clear geometric interpretation. Unfortunately, the field equations of the theory (the Regge–Teitelboim equations, RT) turn out to be much harder to analyze than the Einstein ones \cite{4}, and only a few particular solutions have been found (for recent developments, see \cite{5–7} and references therein). For that reason, in this paper we want to investigate another regime of this theory: we sacrifice the first criterion (applicability to our universe) in favor of the second one (ability to find a solution) and consider a lower-dimensional version of RT gravity. We also restrict ourselves to the circularly symmetric static vacuum case.

2. ISOMETRIC EMBEDDING OF A STATIC METRIC WITH CIRCULAR SYMMETRY

The metric of 2+1-dimensional space-time with $SO(2) \times \mathbb{R}^1$ symmetry can be chosen as follows:

\[
ds^2 = A(r)dt^2 - B(r)dr^2 - r^2d\phi^2.
\] (1)

Let us consider an isometric embedding of this metric into a 5-dimensional flat bulk. There are six types of $SO(2) \times \mathbb{R}^1$-symmetric surfaces which could have such symmetry \cite{8}. Let us restrict the consideration to four of them that can be put together in one expression \cite{9}:

\[
y^0 = kt + \frac{h(r)}{\alpha},
\]

\[
y^1 = \frac{f(r)}{\alpha}\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}(\alpha t + w(r))),
\]

\[
y^2 = r, \quad y^3 = x, \quad y^4 = y.
\]
\[ y^2 = \frac{f(r)}{\alpha} \cos(\varepsilon(at + w(r))), \]
\[ y^3 = r \cos \phi, \quad y^4 = r \sin \phi. \] (2)

Here \( k \) and \( \alpha \) are constants, and \( \varepsilon = \pm 1 \). The signature is \((\lambda, \mu, \varepsilon, \mu, -1, -1)\), where \( \lambda = \pm 1 \) and \( \mu = \pm 1 \). The functions \( f(r) \), \( w(r) \) and \( h(r) \) can be found using the induced metric conditions
\[ \partial_{\mu} y^{a} \partial_{\nu} y^{b} \eta_{ab} = g_{\mu \nu}, \] (3)

which give
\[ f(r) = \sqrt{\frac{A - k^{2} \lambda}{\mu \varepsilon}}, \quad w(r) = - \int \frac{k \lambda h' \mu}{\mu \varepsilon f^2} dr, \]
\[ h(r) = \int \frac{\alpha^2}{\lambda A}(1 - B)(A - \lambda k^2) - \frac{A^2}{4 \lambda \varepsilon A} dr. \] (4)

3. VACUUM REGGE-TEITELBOIM EQUATIONS

The main equations of Regge-Teitelboim approach can be obtained from the EH action in which the substitution (3) has been made. Variation w.r.t. \( y^a \) gives
\[ \partial_{\mu}(\sqrt{-g} G^{\mu \nu} \partial_{\nu} y^a) = 0. \] (5)

Although the index \( \alpha \) can take five values, it can be shown [10] that in the 2+1-dimensional case there are only two independent RT equations.

We will need the following components of the Einstein tensor corresponding to (1):
\[ \sqrt{-g} G^tt = -\frac{1}{2 \sqrt{AB}} B', \]
\[ \sqrt{-g} G^{rr} = -\frac{1}{2 \sqrt{AB}} A', \]
\[ \sqrt{-g} G^{tr} = 0. \] (6)

The equation corresponding to \( a = 0 \) has only one term, so it can be immediately integrated:
\[ \sqrt{-g} G^{rr} \partial_{r} y^0 = C, \quad C = \text{const}, \] (7)

and solved w.r.t \( h(r) \):
\[ h(r) = \int \frac{2 \alpha C A B^2}{A' \sqrt{AB}} dr \] (8)

To obtain a compact form of another RT equation, it is convenient to take a linear combination of RT for \( a = 1 \) and \( a = 2 \) to get rid of the trigonometry:
\[ \partial_{y}^1 (\sqrt{g} G^{tt} \partial_{t} y^2 + \partial_{t} (\sqrt{g} G^{rr} \partial_{r} y^2)) \]
\[ - \partial_{y}^2 (\sqrt{g} G^{tt} \partial_{t} y^1 + \partial_{t} (\sqrt{g} G^{rr} \partial_{r} y^1)) = 0. \] (9)

Plugging (4) and (6) in (8) and (9), we have the following closed system:
\[ A'' A' A B - \frac{3 A^2} {2} - 3 A A' B' + \lambda \varepsilon \alpha^2 k^2 A'B(B-1) \]
\[ + \varepsilon \alpha^2 A^2 B'(\lambda k^2 - A) = 0, \] (10)
\[ A''/4 - \varepsilon \alpha^2 (B-1)(k^2 \lambda - A)A'' + 4 \lambda \varepsilon \alpha^2 A^2 B^2 C'' = 0. \] (11)

4. PARTICULAR SOLUTIONS

The resulting system of RT equations is quite cumbersome. Let us show a few ways to obtain its particular solutions. It can be done by imposing various additional constraints.

4.1. \( C = 0 \)

Suppose that the integration constant in the first RT equation (7) vanishes. It means that either \( G^{tt} = 0 \) or \( B' = 0 \). If \( G^{tt} = 0 \), then \( G^{tt} = 0 \) due to (9), so \( A = \text{const} \) and \( B = \text{const} \) due to (6), and our space-time is flat. Nontrivial solutions thus must correspond to \( h = \text{const} \). Solving (11) for \( B \), substituting it in (10) and integrating, we obtain
\[ A' = \pm \frac{\alpha \sqrt{2} A^{1/6} (A - \lambda k^2)^{5/6}}{\sqrt{A^{1/3} (A - \lambda k^2)^{2/3} - \alpha \varepsilon (A - \lambda k^2)}} \] (12)

Since there is no explicit solution of this ODE in terms of \( A(r) \), we are forced to make a physical assumption. Namely, let us consider the behavior of \( A \) at large \( r \) and suppose that \( A(r) \propto r^n \) with \( n > 0 \). Then from (12) it follows \( n - 1 = n/2 \), so
\[ A(r) \approx (r/R)^n. \] (13)

where \( R = \text{const} \). From (11) it follows that
\[ B(r) \approx \text{const} \] (14)

unless one performs a fine tuning of the constants. This result raises a question whether the system (10)–(11) has an exact solution corresponding to such a form of the metric. Let us try to find such a solution.

4.2. \( B(r) = \text{const} \)

In this section we show that the condition \( B(r) = \text{const} \) leads to a quadratic dependence of \( A \) on \( r \).

Let us assume that \( B(r) = b^2 = \text{const} \). Then (10) becomes
\[ A'' A - (A')^2/2 = (1 - b^2) \lambda \varepsilon \alpha^2 k^2. \] (15)

The general solution of (15),
\[ A(r) = a \cdot (1 + r/R)^2 + (1 - b^2) \lambda \varepsilon \alpha^2 k^2 r^2/2. \] (16)
where \( a \) is some constant, would satisfy (11) only if 
\( b = \pm 1 \) or \( k = 0 \) (the constant \( a \) then can be absorbed in the coordinate \( t \)). The case \( b = \pm 1 \) is physically undesirable since the quantity \( 1 - b^2 \) corresponds to the angular defect in BTZ geometry [11] and is related to the mass of a point source. To keep a nontrivial angular defect and thus a nonzero mass, one must conclude that \( k = 0 \). Therefore, let us discuss this case in more detail.

4.3. \( k = 0 \)

Now we will show that the condition \( k = 0 \) alone leads to \( B(r) = \text{const} \). Let us assume that \( k = 0 \). Then Eq. (10) admits a first integral:

\[
\frac{A'^2}{2\varepsilon a^2 AB^{3/2}} + \frac{2}{\sqrt{B}} = K, \tag{17}
\]

so we can obtain an expression for \( A' \):

\[
A'^2 = 2\varepsilon a^2 AB^{3/2}(K - 2/\sqrt{B}). \tag{18}
\]

After the substitution of (18) into (11) we notice that \( A \) vanishes from the equation, making it algebraic with respect to \( B \). We thus conclude that \( B(r) = b^2 = \text{const} \), so the expressions we have obtained earlier represent the only solution of (10)–(11) in this case. Without loss of generality we can assume \( a = 1 \) (otherwise it can be absorbed in \( t \)) and \( K = 0 \) in (18) (as both \( K \) and \( k \) are constants, \( K \) can be absorbed in \( b \)). From \( K = 0 \) it follows that \( \varepsilon = -1 \), otherwise the signs of (18) do not match. We arrive at the conclusion that the only solution of (10)–(11) at \( k = 0 \) is

\[
A(r) = (1 + r/R)^2, \quad B(r) = b^2. \tag{19}
\]

5. SHAPE OF THE SURFACES

As we saw, the RT equations are satisfied by surfaces with the following metric:

\[
ds^2 = \left(1 + \frac{r}{R}\right)^2 dt^2 - b^2 dr^2 - r^2 d\phi^2. \tag{20}\]

Plugging the final form of the metric (19) together with the condition \( k = 0 \) into (4) and solving it w.r.t. \( f \) and \( h \), we obtain the following form of the embedding function:

\[
y^0 = \sqrt{-\lambda (b^2 - \beta^2 - 1)r},
\]

\[
y^1 = \beta(r + R) \sinh \left( \frac{t}{\beta R} \right),
\]

\[
y^2 = \beta(r + R) \cosh \left( \frac{t}{\beta R} \right),
\]

\[
y^3 = r \cos \phi,
\]

\[
y^4 = r \sin \phi. \tag{21}\]

For \( g_{tt} \) to be positive-definite, one should choose \( \mu = -1 \), and for \( g_{rr} \) to be able to reach \(-1\), one needs \( \lambda = 1 \), so the signature is \((+ - - - -)\).

To obtain a more familiar form of this surface, we can redefine the coordinates and constants following Staruszkiewicz [11]:

\[
r = \frac{\rho}{b}, \quad \phi = b\varphi, \quad \tilde{R} = R/b. \tag{22}\]

The constant \( \beta \) is still arbitrary, but it is convenient to identify it with \( b \), so the embedding function takes the form

\[
y^0 = \rho/b,
\]

\[
y^1 = (\rho + \tilde{R}) \sinh \left( \frac{t}{\tilde{R}} \right),
\]

\[
y^2 = (\rho + \tilde{R}) \cosh \left( \frac{t}{\tilde{R}} \right),
\]

\[
y^3 = \frac{\rho}{b} \cos(b\varphi),
\]

\[
y^4 = \frac{\rho}{b} \sin(b\varphi), \tag{23}\]

with the metric interval

\[
ds^2 = \left(1 + \frac{\rho}{\tilde{R}}\right)^2 dt^2 - \rho^2 dr^2 - \rho^2 d\varphi^2, \tag{24}\]

where \( \varphi \in [0, 2\pi/b] \). The projection of this surface onto \([y^0, y^3, y^4]\) represents a cone, so we have a singularity at \( r = 0 \) which can be associated with the presence of a point source [12]. The angular defect determined by \( b \) is related to the mass of this source [13].

6. DISCUSSION

Above we have shown that there exist non-Einsteinian solutions of vacuum RT equations for 5-dimensional symmetric embedding of 3-dimensional space-time. The simplest one of them corresponds to the metric

\[
ds^2 = \left(1 + \frac{r}{R}\right)^2 dt^2 - b^2 dr^2 - r^2 d\phi^2. \tag{25}\]

Such a metric gives rise to a nonzero Einstein tensor, so the corresponding space-time is nonflat and bears a gravitational field, whereas in 2+1-dimensional GR vacuum space-times must be flat. It is worth noting that, in addition to the parameter \( b \) (which, according to (22), can be interpreted as a bending parameter) related to the mass of a point source, there is also a constant \( R \) with dimension of length. Its appearance is connected with the form of the embedding function, whose components must have the dimension of length. In particular, when the metric has a shift symmetry w.r.t. a Cartesian coordinate (such as \( t \) in the present paper), a constant parameter appears.
in the components of the embedding function. As was shown in our previous papers [8, 9], there are several types of surfaces which are symmetric w.r.t. shifts of $t$. Some of them contain trigonometric or hyperbolic functions (there are the ones (2) we study here), while the others contain natural powers of $t$. All these types correspond to representations of the translation group whose most general form is $V(t) = \exp(\alpha W t)$, where $t$ is a translation parameter, $W$ is an arbitrary square matrix, and $\alpha$ is a multiplier with the dimension of inverse length. This multiplier will be present in all types of embedding function with such symmetry (except for a trivial one, in which $y^0 \propto t$ and no other component depends on $t$) and could appear in the metric. In the Schwarzschild case, when $[M] = L$, this parameter can be related to mass, whereas in a $2+1$-dimensional case, where $[M] = 1$, it must be related to other characteristics of the manifold. It would be especially interesting to study the interplay between this parameter and the value of the cosmological constant that could be added to the picture. However, the introduction of cosmological constant makes the analysis of equations much more difficult and requires an additional study.

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