FUNCTORIALITY OF BOSE-MESNER ALGEBRAS AND PROFINITE ASSOCIATION SCHEMES

MAKOTO MATSUMOTO, KENTO OGAWA, AND TAKAYUKI OKUDA

Abstract. We show that taking the set of primitive idempotents of commutative association schemes is a functor from the category of commutative association schemes with surjective morphisms to the category of finite sets with surjective partial functions. We then consider projective systems of commutative association schemes consisting of surjections (which we call profinite association schemes), for which Bose-Mesner algebra is defined, and describe a Delsarte theory on such schemes. This is another method for generalizing association schemes to those on infinite sets, related with the approach by Barg and Skriganov. Relation with $(t, m, s)$-nets and $(t, s)$-sequences is studied. We reprove some of the results of Martin-Stinson from this viewpoint.

1. Introduction

Association schemes are central objects in algebraic combinatorics, with many interactions with other areas of mathematics. It is natural to try to take projective limits of a system of association schemes, but there seems to be some obstruction to have a natural Bose-Mesner algebra. We show that a surjective morphism between commutative association schemes behaves well with respect to the two product structures of Bose-Mesner algebras and primitive idempotents, which gives rise to a notion of profinite association schemes and their Delsarte theory. They are closely related with profinite groups. We study kernel schemes and ordered Hamming schemes introduced by Martin-Stinson [16] as examples. We reprove their characterization of $(t, m, s)$-net as a design in an ordered Hamming scheme, and give a characterization of $(t, s)$-sequences in terms of profinite association schemes.

2. Surjections among association schemes and functoriality

Let $\mathbb{N}$ denote the set of natural numbers including 0, and $\mathbb{N}_{>0}$ the set of positive integers.

2.1. Category of association schemes. Let us recall the notion of association schemes briefly. See [1][6] for details. We summarize basic terminologies.

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Proposition 2.4. Functoriality of Bose-Mesner algebra.

Proof. The cardinality of the fiber is the summation of the valencies of $R_i$ for $i$ with $g(i) = i'_0$, hence independent of the choice of $x'$, which implies the result. This is
well-known [25], and it holds under a weaker condition (for unital regular relation partitions, see [12]). □

Theorem 2.5.

(1) Let \((f, g) : (X, R, I) \to (X', R', I')\) be a morphism. It induces an injection \(C(I') \to C(I)\) and hence \(Ψ : A_{X'} \to A_X\). Then \(Ψ\) is a morphism of unital rings with respect to the Hadamard product.

(2) Suppose that \(f\) is surjective. We define the convolution product \(\bullet_X\) on \(A_X\) by

\[
(A \bullet_X B)(x, z) = \frac{1}{\#X} \sum_{y \in X} A(x, y)B(y, z)
\]

(i.e., the matrix product normalized by the factor of \(\#X\)). Then, \(A_X\) has a unit \(E_X := \#XA_{i_0}\), and \(Ψ\) is a morphism of rings with respect to the convolution product (which may not map the unit of \(A_{X'}\) to the unit of \(A_X\)).

(3) The vector space \(C(X)\) is a left \(A_X\)-module by the following action:

\[
A_X \times C(X) \to C(X), (A, h) \mapsto (A \bullet h)
\]

given by

\[
(A \bullet h)(x) = \frac{1}{\#X} \sum_{y \in X} A(x, y)h(y),
\]

and the unit \(E_X\) acts trivially. This module structure is compatible with \(Ψ\) in the sense that the following diagram commutes:

\[
\begin{array}{c c c}
A_X \times C(X) & \to & C(X) \\
Ψ \times f^\dagger & \uparrow & f^\dagger \\
A_{X'} \times C(X') & \to & C(X').
\end{array}
\]  

(2.2)

If we consider \(C(X') \subset C(X)\), then the element \(Ψ(E_{X'}) \in A_X\) acts on \(C(X)\) as a projector \(C(X) \to C(X')\).

Proof. For a morphism of finite sets \(X \to Y\), the induced map \(C(Y) \to C(X)\) is a morphism of unital rings with respect to the Hadamard product. The first statement follows from this and the commutative diagram (2.1). For the second part, take \(A, B \in A'_{X'}\). (In fact, one may take \(A, B \in C(X' \times X')\); the following arguments depend only on the fact that the cardinality of the fiber of \(f\) is constant.)
Then
\[
\Psi(A \bullet_X B)(x, z) = (A \bullet_X B)(f(x), f(z))
\]
\[
= \frac{1}{X'} \sum_{y' \in X'} A(f(x), y')B(y', f(z))
\]
\[
= \frac{1}{X'} \sum_{y \in X} \Psi(A)(x, y)\Psi(B)(y, z)
\]
\[
= \frac{1}{X} \sum_{y \in X} \Psi(A)(x, y)\Psi(B)(y, z)
\]
\[
= (\Psi(A) \bullet_X \Psi(B))(x, z).
\]

For the third part, it is a routine calculation to check that \(A_X\) acts on \(C(X)\) as a unital ring. For \(A \in A_X\) and \(h \in C(X')\), the compatibility follows from:
\[
f^\dagger((A \bullet h))(x) = (A \bullet h)(f(x))
\]
\[
= \frac{1}{X'} \sum_{y' \in X'} A(f(x), y')h(y')
\]
\[
= \frac{1}{X'} \sum_{y \in Y} \Psi(A)(x, y)(f^\dagger h)(y)
\]
\[
= \frac{1}{X} \sum_{y \in X} \Psi(A)(x, y)(f^\dagger h)(y)
\]
\[
= (\Psi(A) \bullet (f^\dagger h))(x).
\]
The compatibility implies that \(\Psi(E_X)\) trivially acts on the image of \(C(X')\) in \(C(X)\). For \(A \in A_X\) and \(h \in C(X)\),
\[
(\Psi(A) \bullet h)(x) = \frac{1}{X} \sum_{y \in X} \Psi(A)(x, y)h(y)
\]
\[
= \frac{1}{X} \sum_{y \in X} A(f(x), f(y))h(y),
\]
which depends only on \(f(x)\), and hence \(\Psi(A) \bullet h\) lies in \(f^\dagger(C(X')) \subset C(X)\). Hence \(\Psi(E_X)\) is a projection \(C(X) \to C(X')\).

\[\square\]

**Definition 2.6.** Let \(\mathsf{AS}_{\text{surj}}\) be the category of association schemes with surjective morphisms.

**Corollary 2.7.** Let \(\mathsf{Alg}_{HC}\) (HC means Hadamard and convolution) be the category of finite dimensional \(\mathbb{C}\)-vector spaces \(A\) with:

1. one associative multiplication (Hadamard product) which gives a commutative unital semi-simple \(\mathbb{C}\)-algebra structure to \(A\),

Let \(\mathsf{Alg}_{HC}\) be the category of association schemes with surjective morphisms.

**Corollary 2.7.** Let \(\mathsf{Alg}_{HC}\) (HC means Hadamard and convolution) be the category of finite dimensional \(\mathbb{C}\)-vector spaces \(A\) with:

1. one associative multiplication (Hadamard product) which gives a commutative unital semi-simple \(\mathbb{C}\)-algebra structure to \(A\),
(2) one (possibly non-commutative) associative multiplication (convolution product) which gives a unital semi-simple \(C\)-algebra structure to \(A\).

(Note that in this case, semi-simplicity is equivalent to that \(A\) is a direct product of a finite number of matrix algebras over \(C\). Morphisms are injective \(C\)-linear maps preserving the both two products and the unit for the Hadamard product (we don’t require preservation of unit for the convolution product).

Then, the correspondence \(X \mapsto A_X\) gives a contravariant functor from \(\text{AS}_{\text{surj}}\) to \(\text{Alg}_{HC}\).

Proof. It is proved that \(A_X\) has two products. Semi-simplicity is well-known, c.f. [25]. (It is enough to show that there is no nilpotent ideal, but for any nonzero \(A \in A_X\), the product with its unitary conjugate \(A \bullet A^* = \frac{1}{\#X} AA^*\) is not nilpotent, hence \(A_X\) is semi-simple.) Functoriality is easy to check, using Theorem 2.5. \(\square\)

Corollary 2.8. Let us consider the category Mod of \(R\)-modules, whose object is a pair of a unital commutative ring \(R\) and an \(R\)-module \(M\), and a morphism from \((R,M)\) to \((R',M')\) is a pair of a ring morphism \(f : R \to R'\) and a \(Z\)-module morphism \(g : M \to M'\) which makes the following diagram commute:

\[
\begin{array}{ccc}
R \times M & \rightarrow & M \\
\downarrow f \times g & & \downarrow g \\
R' \times M' & \rightarrow & M' \\
\end{array}
\]

Then, the correspondence \(X \mapsto (A_X,C(X))\) is a contravariant functor from \(\text{AS}_{\text{surj}}\) to \(\text{Mod}\).

Proof. The correspondence is given in Theorem 2.5. It is a contravariant functor, since each of the correspondences \(X \mapsto A_X\), \(X \mapsto C(X)\) is a contravariant functor. \(\square\)

Remark 2.9. In [8], French constructed a sub-category of association schemes, and a covariant functor from it to the category of algebra. Our construction produces a contravariant functor, which seems to be of different nature.

2.3. Commutative association schemes and primitive idempotents. By semisimplicity and Artin-Wedderburn Theorem, any Bose-Mesner algebra \(A_X\) (with convolution product) is isomorphic to a product of matrix algebras over \(C\). We assume that \(X\) is commutative. Then, \(A_X\) is, as a unital ring, isomorphic to a direct product of copies of \(C\). Namely, \(A_X \cong C \times \cdots \times C =: C^n\) for some \(n \in \mathbb{N}\), and this decomposition is unique as a direct product of rings.

Proposition 2.10. Let \(A\) be a ring isomorphic to \(C^n\) (with componentwise multiplication). An element \(j \in A\) corresponding to an element of \(C^n\) with one coordinate 1 and the other coordinates 0 is called a primitive idempotent of \(A_X\). It is characterized by the idempotent property \(j^2 = j\), \(j \neq 0\) (this is equivalent to that the each coordinate is 0 or 1 and at least one 1 exists) and that there is no nonzero idempotent \(j' \neq j\) such that \(jj' = j'\) holds (this says only one coordinate is 1). Any idempotent is uniquely a sum of primitive idempotents.

Definition 2.11. Let FinSets be the category of finite sets. Let FinSets_{ps} be the category of finite sets and partial surjective functions.

Recall that for sets \(X\) and \(Y\), a partial function \(f\) from \(X\) to \(Y\) consists of a pair of a subset \(\text{dom}(f) \subset X\) (which may be empty) and a function \(f|_{\text{dom}(f)} : \text{dom}(f) \rightarrow \)
Two partial functions are equal if they have the same domain and the same function on the domain. A partial function is denoted by \( f : X \rightarrow Y \). For \( S \subseteq Y \), \( f^{-1}(S) := \{ x \in X \mid x \in \text{dom}(f), f(x) \in S \} \). Composition \( g \circ f \) of partial functions \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) has domain \( f^{-1}(\text{dom}(g)) \). A partial function \( f \) is surjective if \( \text{Im}(f) := \{ f(x) \mid x \in \text{dom}(f) \} \subseteq Y \) is \( Y \).

**Proposition 2.12.** Let \( \text{Alg}_{\text{ps}} \) be the category of rings isomorphic to \( \mathbb{C}^n \) for some \( n \), with injective \( \mathbb{C} \)-linear ring morphisms which may not preserve the unit. For \( A \in \text{Alg}_{\text{ps}} \), we define \( J(A) \) as the finite set of primitive idempotents of \( A \). For a finite set \( I \), we define \( C(I) \) the set of maps from \( I \) to \( \mathbb{C} \) as defined above. These are contravariant functors, and give contra-equivalence between \( \text{Alg}_{\text{ps}} \) and \( \text{FinSets}_{\text{ps}} \).

**Proof.** Note first that the notion of \( f : X \rightarrow Y \) is the notion of a family of disjoint subsets \( S_y \subset X \) (\( y \in Y \)), where \( S_y = f^{-1}(y) \), and the surjectivity of \( f \) is equivalent to that \( S_y \neq \emptyset \) for all \( y \in Y \). To show that \( J \) is a functor, let \( \varphi : A \rightarrow B \) be an injective \( \mathbb{C} \)-algebra homomorphism. For \( j \in J(A) \), \( \varphi(j) \) is a nonzero idempotent. Thus, it is a non-empty sum of elements of \( J(B) \), which gives a non-empty subset \( S_j \) of \( J(B) \) by Proposition 2.10. We construct a partial surjection \( J(B) \rightarrow J(A) \) by: for \( j \in J(A) \), elements of \( S_j \) is mapped to \( j \). Note that if \( j \neq j' \in J(A) \), then \( jj' = 0 \) and hence \( S_j \cap S_{j'} = \emptyset \). Since \( S_j \neq \emptyset \), the partial function is surjective. To check the functoriality, we consider \( A \xrightarrow{\varphi} B \xrightarrow{\varphi'} C \). The construction of partial surjection for \( \varphi \) is done by assigning to each \( j \in J(A) \) the set \( S_j \subset J(B) \) such that the summation of elements in \( S_j \) is \( \varphi(j) \). Similarly, for each \( j' \in S_j \), we have \( S_{j'} \subset J(C) \). Thus, \( J(\varphi) \circ J(\varphi') \) maps \( \prod_{j' \in S_j} S_{j'} \) to \( j \). This is the same with \( J(\varphi \circ \varphi') \).

The functoriality of \( I \rightarrow C(I) \) can be also checked in an elementary manner. It is easy to check that the two functors give contra-equivalence. We omit the detail. \( \square \)

**Definition 2.13.** Let \( \text{AS}_{\text{surj}} \) be a full subcategory of the commutative association schemes of \( \text{AS}_{\text{surj}} \), and \( \text{Alg}_{cHC} \) a full subcategory of \( \text{Alg}_{cHC} \) consisting of algebras with commutative convolution multiplication. By the remark at the beginning of Section 2.3, we have a contravariant functor \( \text{AS}_{\text{surj}} \rightarrow \text{Alg}_{cHC} \).

**Corollary 2.14.** The composite of the three functors \( \text{AS}_{\text{surj}} \rightarrow \text{Alg}_{cHC} \rightarrow \text{Alg}_{cHC} \rightarrow \text{Alg}_{cHC} \) forgetting the Hadamard product, and \( \text{Alg}_{cHC} \rightarrow \text{FinSets}_{\text{ps}} \) gives a covariant functor \( \text{AS}_{\text{surj}} \rightarrow \text{FinSets}_{\text{ps}} \). We denote this composition functor by \( J \). Thus, \( (X, R, I) \rightarrow J(X, R, I) \) is a functor, which corresponds \( X \) to the sets of primitive idempotents of \( A_X \) with respect to (the normalized) convolution products.

**Definition 2.15.** Let \( j_0 \) denote the element of \( J(X, R, I) \), corresponding to \( J_X \), where \( J_X \) denotes the matrix in \( C(X \times X) \) with components all 1. This element acts on \( C(X) \) as a projection to the space of constant functions. We use the symbol \( j_0 \) for different association schemes, similarly to \( i_0 \).

2.4. **Trivial examples and remarks.** In the theory of commutative association schemes, the set \( I \) and the set \( J \) are considered to be “dual.” Thus, it is natural to seek the relation between \( J \)'s for a morphism of association schemes (while a morphism between \( I \)'s is given by definition). The previous result says that there is a natural relation, if the morphism of association schemes is surjective. We couldn’t generalize the result for general (non-surjective) morphisms.

For a surjective morphism between association schemes, the corresponding partial surjection may be a properly partial function, and may be not an injection, as follows.
Definition 2.16. Let $G$ be a finite group. The triple $(G, R, I)$ where $I = G$ and $R(x, y) = x^{-1}y$ is a (possibly non-commutative) association scheme called the thin scheme of $G$.

Remark 2.17. Let $G$ be a finite abelian group, and $f : G \to H$ be a surjective homomorphism of abelian groups. Then we have the thin scheme of $G$ and the thin scheme of $H$, and $f$ is a surjective morphism of the association schemes. In this case, $J$ for $G$ is naturally isomorphic to the character group $G^\vee$, and one can check that the functor $J$ gives $G^\vee \to H^\vee$, which comes from the natural inclusion $\iota : H^\vee \to G^\vee$ (for $x \in G^\vee$ maps to $y \in H^\vee$ if $\iota(y) = x$, and $x$ is outside the domain if there is no such $y$). This gives an example that $J(X) \to J(X')$ is not a map but a partial function.

Remark 2.18. For any association scheme $(X, R, I)$, the mappings $f = \text{id}_X$ and $g : I \to \{0, 1\}$, $g(i_0) = 0$ and $g(i) = 1$ if $i \neq i_0$, gives a morphism of association schemes $(X, R, I) \to (X, g \circ R, \{0, 1\})$. In this case, $J(X, g \circ R, \{0, 1\})$ consists of $j_0$ and $E_X - j_0$. If $(X, R, I)$ is commutative, then the image of $j \in J(X, R, I)$ in $J(X, g \circ R, \{0, 1\})$ is $j_0$ if $j = j_0$, and $E_X - j_0$ if $j \neq j_0$. This gives an example that $J(X) \to J(X')$ is not injective.

3. Profinite association schemes

3.1. Projective system of association schemes. We recall the notion of projective system.

Definition 3.1. (Projective system)
We work in a fixed category $C$. Let $\Lambda$ be a directed ordered set, namely, $\Lambda$ is non-empty and for any $\alpha, \beta \in \Lambda$, there is a $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. Let $X_\lambda (\lambda \in \Lambda$ be a family of objects. For any $\lambda \leq \lambda'$, a morphism $p_{\lambda, \lambda'} : X_\lambda \to X_{\lambda'}$ is specified, and they satisfy the commutativity condition

$$p_{\lambda', \lambda''}p_{\lambda', \lambda} = p_{\lambda'', \lambda}$$

for any $\lambda \leq \lambda' \leq \lambda''$, and $p_{\lambda, \lambda = \text{id}_X}$. Then the system $(X_\lambda, p_{\lambda, \lambda'})$ is called a projective system in $C$. The morphisms $p_{\lambda, \lambda'}$ are called the structure morphisms of the projective system. The dual notion (i.e. the direction of morphism is inverted in the definition) is called an inductive system.

Let $X_\lambda$ be a projective system of association schemes. Then, each of $X_\lambda, I_\lambda$ is a projective system of finite sets. If all morphisms are surjective, then by Theorem 2.15 we have an inductive system of Bose-Mesner algebras $A_{X_\lambda}$, and if they are commutative, by Corollary 2.14 a projective system $J_{X_\lambda} := J(X_\lambda, R_\lambda, I_\lambda)$ (with structure morphisms being partially surjective maps).

Definition 3.2. Let $\Lambda$ be a directed ordered set. A projective system $(X_\lambda, R_\lambda, I_\lambda)$ ($\lambda \in \Lambda$) in $\text{AS}_{\text{surj}}$ is called a profinite association scheme. (This includes the notion of usual association scheme, as the case where $\# \Lambda = 1$.) We define profinite sets (see Lemma 3.3) $X^\wedge := \varprojlim X_\lambda$ and $P^\wedge := \varprojlim I_\lambda$ with projective limit topologies. A commutative profinite association scheme is a projective system of commutative association schemes. In this case, we define $J^\wedge$ similarly: an element $j_\lambda \in J^\wedge$ is the set of pair $\lambda \in \Lambda$ and compatible family of elements $j_\mu \in J_\mu$ for all $\mu \geq \lambda$, divided by the equivalence relation $(\lambda, j_\mu) \sim (\lambda', j'_\mu)$ defined by the existence of $\lambda'' \geq \lambda$, $\lambda'' \geq \lambda'$ such that for all $\mu \geq \lambda'' j_\mu = j'_\mu$ holds. An open basis of $J^\wedge$ is given by the inverse image of $j_\lambda \in J_\lambda$ through $J^\wedge \to J_\lambda$. 
Proposition 3.3. The natural projections $X^\wedge \to X_\lambda$ and $I^\wedge \to I_\lambda$ are surjective, the induced map $R^\wedge : X^\wedge \times X^\wedge \to I^\wedge$ is surjective, and $J^\wedge \to J_\lambda$ is a partial surjection.

The first two statements are well-known for projective systems of compact Hausdorff spaces with continuous morphisms, [22 Proposition 1.1.10, Lemma 1.1.5]. We need to prove the third statement, for partial surjections. It follows from the first statement, but for the self-containedness and for explaining the natural (projective limit) topology, we give a proof for the both.

Lemma 3.4.

(1) Let $X_\lambda$ be a projective system of finite sets and surjections. Define $X^\wedge$ as $\lim_{\leftarrow} X_\lambda$. Then, the natural map $X^\wedge \to X_\lambda$ is surjective. The set $X^\wedge$ is called a profinite set, and has a natural topology, called profinite topology, which is compact Hausdorff. For every $x_\lambda \in X_\lambda$, its inverse image in $X^\wedge$ is clopen, and these form an open basis.

(2) Let $J_\lambda$ be a projective system of finite sets and partial surjections. Define $J^\wedge$ as above. Then, the natural partial map $J^\wedge \to J_\lambda$ is a partial surjection. The set $J^\wedge$ has a natural topology, which is locally compact and Hausdorff. For every $j_\lambda \in J_\lambda$, its inverse image in $J^\wedge$ is clopen, and these form an open basis.

(3) Let $X_\lambda, Y_\lambda$ be projective systems of finite sets over the same directed ordered set $\Lambda$, and let $f_\lambda : X_\lambda \to Y_\lambda$ be a family of maps which commute with the structure morphisms. This induces a continuous map

$$f^\wedge : \lim_{\leftarrow} X_\lambda \to \lim_{\leftarrow} Y_\lambda.$$

If every $f_\lambda$ is surjective, then $f^\wedge$ is surjective.

Proof. (1): Recall the construction of the projective limit $\lim_{\leftarrow} X_\lambda$ in the category of set. It is a subset of the direct product $\prod_{\lambda \in \Lambda} X_\lambda$ defined as the intersection of

$$S_{\mu^\prime,\mu} := \{(s_\lambda)_{\lambda} \in \prod_{\lambda \in \Lambda} X_\lambda \mid s_\mu = p_{\mu^\prime,\mu}(s_{\mu^\prime})\}$$

for all $\mu^\prime, \mu \in \Lambda$ with $\mu^\prime \geq \mu$. (This means that $(s_\lambda)_{\lambda}$ belongs to $\lim_{\leftarrow} X_\lambda$ if and only if they satisfy $p_{\mu^\prime,\mu}(s_{\mu^\prime}) = s_\mu$ for every $\mu^\prime \geq \mu$.) Each finite set is equipped with the discrete topology, which is compact and Hausdorff. By Tychonoff’s theorem, the product is compact and Hausdorff. Because each component is Hausdorff, $S_{\mu^\prime,\mu}$ is a closed subset, and the intersection $\lim_{\leftarrow} X_\lambda$ is compact and Hausdorff. Take any $x_\delta \in X_\delta$. We shall show that this is in the image from $\lim_{\leftarrow} X_\lambda$. For each $\alpha \in \Lambda$, $\alpha \geq \delta$, we consider the closed set

$$T_\alpha := \{(s_\lambda)_{\lambda} \in \prod_{\lambda \in \Lambda} X_\lambda \mid s_\delta = x_\delta, s_\beta = p_{\alpha,\beta}(s_\alpha) \text{ for all } \beta \leq \alpha\}. $$

There is an element $y_\alpha \in X_\alpha$ such that $p_{\alpha,\delta}(y_\alpha) = x_\delta$, since $p$ is surjective. Then, there is a system $p_{\alpha,\beta}(y_\alpha)$ which is an element of $T_\alpha$, and thus $T_\alpha$ is nonempty. Now we take the intersection of $T_\alpha$ for all $\alpha \geq \delta$. For any finite number of $\alpha_i$, we may take $\gamma$ as an upper bound of all $\alpha_i$. Then the finite intersection contains $T_\gamma$, which is nonempty. By compactness, the intersection of $T_\alpha$ for $\alpha \geq \delta$ is nonempty. Take an element $(y_\lambda)_{\lambda}$ in the intersection. It lies in $\lim_{\leftarrow} X_\lambda$, and it projects to $x_\delta$ via $\lim_{\leftarrow} X_\lambda \to X_\delta$. From the definition of the direct product topology, the inverse
image of \(x_\lambda \in X_\lambda\) for various \(\lambda\) forms an open basis. They are clopen, since \(x \in X_\lambda\) is clopen.

2. Choose an element \(*\) which is contained in no \(J_\lambda\)’s. Let \(J_\lambda^*\) be \(J_\lambda \cup \{*\}\). For a partial surjection \(p : J_\lambda \to J_\mu\), we associate a surjection \(p^* : J_\lambda^* \to J_\mu^*\) as follows. For \(x \in \text{dom}(p)\), \(p^*(x) = p(x)\). For \(x \notin \text{dom}(p)\), \(p^*(x) = *\). In particular, \(p^*(*) = *\).

Let us take the projective limit as in the previous case to obtain \(J^* := \lim_{\leftarrow} J_\lambda^*\).

Then for any \(\lambda\), by \([1]\), we have a surjection

\[p_\lambda^* : J_\lambda^* \to J_\lambda^*\]

It is clear that there is a unique element \(\hat{*} \in J_\lambda^*\) that is mapped to \(*\) for every \(J_\lambda^*\). We define \(J_\lambda' := (\lim J_\lambda^*) \setminus \{\hat{*}\}\), and a partial surjection \(p_\lambda : J_\lambda' \to J_\lambda\) by: for \(x \in J_\lambda',\) \(x\) is outside of the domain of \(p_\lambda\) if \(p_\lambda^*(x) = *\), and otherwise \(p_\lambda(x) = p_\lambda^*(x)\).

It is easy to check that \(p_\lambda\) is a partial surjection (using the surjectivity of \(p_\lambda^*\)), and \(p_\lambda\) for various \(\lambda\) is compatible with the structure morphisms \(\mu_{\mu',\mu}\).

Since \(J^*\) is compact Hausdorff, its open subset \(J_\lambda'\) is locally compact Hausdorff, and has an open basis as in the case \([1]\). We need to prove that \(J^*\) in Definition 3.2 is canonically isomorphic to \(J_\lambda'\). Let \(J\) be the set of pairs \((\lambda, (j_\mu)_\mu)\) where \((j_\mu)_\mu\) is a compatible system for \(\mu \geq \lambda\). For such a pair, we assign an element of \(J^*\) as follows. For any \(\alpha \in \Lambda\), take any \(\beta\) with \(\beta \geq \alpha\) and \(\beta \geq \lambda\). Define \(j_\alpha \) as \(p_\beta^*(j_\mu)\).

This is well-defined by the commutativity of the structure morphisms, and gives a map \(f : J \to J_\lambda'\). This is surjective. Two pairs \((\lambda, (j_\mu)_\mu), (\lambda', (j'_\mu')_\mu')\) have the same image if and only if there is a \(\lambda''\) with \(\lambda'' \geq \lambda\), \(\lambda'' \geq \lambda'\) such that for any \(\mu \geq \lambda''\), \(j_\mu = j'_\mu\) holds. This gives a bijection \(J^* := J/\sim \to J_\lambda'\).

3. See \([22]\) Lemma 1.1.5].

Remark 3.5. The above construction gives a category equivalence between the category of sets with partial maps and the category of sets with one base point. Using this, it is not difficult to show that \(J^*\) is the projective limit in the category of sets with partial surjections.

Later it will be proved that, for any commutative profinite association scheme, \(J^*\) has the discrete topology (Proposition 3.10).

Definition 3.6. For a profinite association scheme \((X_\lambda, R_\lambda, I_\lambda)\), its Bose-Mesner algebra is defined by \(A_X := \lim A_{X_\lambda}\). For a commutative profinite association scheme, through the natural isomorphism \(C(J_\lambda) \to A_{X_\lambda}\), we have an isomorphism \(\lim C(J_\lambda) \to A_X\) which maps the point-wise product to the convolution product, and an isomorphism \(\lim C(I_\lambda) \to A_X\) which maps the point-wise product to the Hadamard product.

Let \(C(X^\wedge \times X^\wedge)\) denote the set of complex valued continuous functions on \(X^\wedge \times X^\wedge\) (and similarly \(C(I^\wedge)\) be the set of continuous functions on \(I^\wedge\)). Since we have inductive families of injections

\[A_{X_\lambda} \to C(X_\lambda \times X_\lambda) \to C(X^\wedge \times X^\wedge),\]

by universality, we have a canonical injective morphism \(A_X \to C(X^\wedge \times X^\wedge)\). Similarly, we have an injection \(\lim C(I_\lambda) \to C(I^\wedge)\).

Definition 3.7. For \(j_\lambda \in J_\lambda\), we denote by \(E_{j_\lambda} \in A_{X_\lambda}\) the primitive idempotent \(j_\lambda\) (the notation introduced to distinguish an element of \(J\) with an element of \(A_{X_\lambda}\)). It is an element of \(C(X_\lambda \times X_\lambda)\), and acts on \(C(X_\lambda)\) by the normalized convolution defined in Theorem 2.5. Let \(C(X_\lambda)_{j_\lambda}\) be the image of \(E_{j_\lambda}\) by this action.
Proposition 3.8. Take $j_\lambda \in J_\lambda$. For $\mu \geq \lambda$, we may regard $C(X_\lambda) \subset C(X_\mu)$. Then $\Psi(E_{j_\lambda})$ given in Theorem 2.5 acts on $g \in C(X_\mu)$ by $\Psi(E_{j_\lambda}) \cdot g$. The operator $\Psi(E_{j_\lambda})$ gives a splitting $C(X_\mu) \rightarrow C(X_\lambda)_{j_\lambda}$ to $C(X_\lambda)_{j_\lambda} \subset C(X_\mu)$.

Proof. By Theorem 2.5, $\Psi(E_{j_\lambda}) \in \text{End}(C(X_\mu))$ is a projection to $C(X_\lambda)$. Then

$$\Psi(E_{j_\lambda}) = \Psi(E_{j_\lambda} \cdot E_{X_\lambda}) = \Psi(E_{j_\lambda}) \cdot \Psi(E_{X_\lambda})$$

is a projection $C(X_\mu) \rightarrow C(X_\lambda) \rightarrow C(X_\lambda)_{j_\lambda}$. \hfill \Box

Definition 3.9. Let $j$ be an element of $J^\wedge$. Then $j$ is said to be isolated at $\lambda$ to $j_\lambda$, if there is a $j_\lambda \in J_\lambda$ such that the inverse image of $j_\lambda$ is the singleton $\{j\}$.

Proposition 3.10. The cardinality of the inverse image of $j_\lambda$ in $J_\mu$ for $\mu \geq \lambda$ does not exceed the dimension of $C(X_\lambda)_{j_\lambda}$, which is finite. Consequently, for each $j \in J^\wedge$, there exists a $\lambda$ at which $j$ is isolated. Thus, $J^\wedge$ has the discrete topology.

Proof. Suppose that the inverse image of $j_\lambda$ in $J_\mu$ is $\{j_1, j_2, \ldots, j_n\}$. Then the operator $\Psi(E_{j_\lambda}) = \sum_{i=1}^{n} \Psi(E_{j_i})$ on $C(X_\mu)$ is a projection to $C(X_\lambda)_{j_\lambda}$. Thus $C(X_\mu)_{j_i}$, $(i = 1, \ldots, n)$ are nonzero direct summands of $C(X_\lambda)_{j_\lambda}$, and hence $n$ does not exceed the dimension of $C(X_\lambda)_{j_\lambda}$. This means that for any $\mu \geq \lambda$, the cardinality of the inverse image of $j_\lambda$ is bounded, and hence there is a $\mu'$ such that the cardinality is constant for all $\mu \geq \mu'$. Let us fix $j \in J^\wedge$. Take any $\lambda_0$ such that $j$ is in the domain of the partial map to $J_{\lambda_0}$. We put $j_{\lambda_0}$ to the image of $j$. Then by the arguments above, we can find a $\lambda$ such that the cardinality of the inverse image of $j_{\lambda_0}$ in $J_{\lambda'}$ is constant for all $\lambda' \geq \lambda$. We put $j_\lambda$ to the image of $j$ in $J_\lambda$. Then $j_\lambda$ is in the inverse image of $j_{\lambda_0}$ and the cardinality of the inverse image of $j_\lambda$ in $J_{\lambda'}$ should be one for any $\lambda' \geq \lambda$. This implies that the inverse image of $j_\lambda$ in $J^\wedge$ is just $\{j\}$, namely, $j$ is isolated at $\lambda$. We have proved that $\{j\}$ is a clopen subset of $J^\wedge$ for any $j \in J^\wedge$, and hence $J^\wedge$ has the discrete topology. \hfill \Box

Corollary 3.11. We have $\lim \lambda C(J_\lambda) = \bigoplus_{j \in J^\wedge} C \cdot j$. We write $E_j$ for the image of $j \in J^\wedge$ in $A_X^\wedge$. Then $\bigoplus_{j \in J^\wedge} C \cdot E_j = A_X^\wedge$. We may use the notation $C_c(J^\wedge)$ for $\lim \lambda C(J_\lambda)$, since it is the set of functions (automatically continuous) on $J^\wedge$ with finite (= compact) support. Through the isomorphism $C_c(J^\wedge) \cong A_X^\wedge$, $E_j \in A_X^\wedge$ corresponds to the indicator function $\delta_j \in C_c(J^\wedge)$.

Remark 3.12. In the Pontryagin duality, the dual of a compact abelian group is a discrete group. Since $F^\wedge$ is profinite and compact, the above discreteness of $J^\wedge$ is an analogue to this fact.

We summarize the results of this subsection, to compare with the case of finite commutative association schemes.

Theorem 3.13. Let $X_\lambda$ be a commutative profinite association scheme. Notations are as in Definition 3.6 and Corollary 3.7. We have the following commutative diagram, where all the vertical arrows are injections:

$$
\begin{array}{c}
\text{C}(F^\wedge) \\
\uparrow \\
\text{C}_c(J^\wedge)
\end{array}
\quad
\begin{array}{c}
\text{C}(X^\wedge \times X^\wedge) \\
\uparrow \\
\text{C}(F^\wedge)
\end{array}
\quad
\begin{array}{c}
\text{A}_X^\wedge \\
\uparrow \\
\text{C}_c(I_\lambda)
\end{array}
\quad
\begin{array}{c}
\lim \lambda \text{C}(I_\lambda) \\
\uparrow \\
\text{C}_c(F^\wedge)
\end{array}
$$

where:
Proof. (1) For $x_\lambda \in X_\lambda$, we denote by $B_{x_\lambda}$ the inverse image of $x_\lambda$ in $X^\wedge$, which we call a $\lambda$-ball. This gives an open basis, and thus every $O_a$ is a union of $\lambda$-balls for various $\lambda$. Since $C$ is compact and covered by these balls, we may find a finite number of these balls which covers $C$ and each contained in some $O_a$. We take an upper bound $\mu$ for these finite number of $\lambda$. Let $S$ be the image of $C$ in $X_\mu$. Then, $B_{a_\mu}(s_\mu \in S)$ are finite, clopen, and mutually disjoint balls covering $C$, such that each ball is contained in some $O_a$.

(2) Let $f$ be a locally constant function. Then, for any $x \in X^\wedge$, there is an open neighborhood $O_x$ so that $f$ is constant on $O_x$. These constitute an open covering of $X^\wedge$. Applying (1) for $C = X^\wedge$ to find $\mu$ with $S = X_\mu$, where each $\mu$-ball is contained in some $O_x$, and hence $f$ is constant on each ball. This means that $f$ is an image of an element in $C(X_\mu)$. Conversely, any function in $C(X_\mu)$ clearly maps to a locally constant function on $X^\wedge$.

We shall show the density. For any $f \in C(X^\wedge)$ and $\epsilon > 0$, by continuity, for all $x \in X^\wedge$ we have an open neighborhood $O_x$ such that $y \in O_x$ implies $|f(x) - f(y)| < \epsilon$. By applying (1) for $C = X^\wedge$ to find a $\mu$ with $S = X_\mu$. We see that the $\mu$-balls cover $X^\wedge$, and for each $x_\mu \in X_\mu$, we define $g(x_\mu) = f(x')$ by choosing any $x'$ in $B_{a_\mu}$. Denote the projection by $\text{pr} : X^\wedge \to X_\mu$. Then for any $y \in X_\mu$, $g(\text{pr}(y)) = f(x')$ for some $x' \in B_{\text{pr}(y)}$. Since $B_{\text{pr}(y)} \subset O_x$ for some $x$, the values of $f$ in this ball differ at most by $2\epsilon$, namely,

$$|g(\text{pr}(y)) - f(y)| = |f(x') - f(y)| < 2\epsilon.$$ 

This shows that

$$||\text{pr}^\dagger(g) - f||_{\text{sup}} < 2\epsilon,$$

which is the desired density. \qed

3.2. Inner product and orthogonality.

**Proposition 3.15.** For a profinite association scheme, $X^\wedge$ has a natural probability regular Borel measure $\mu$, whose push-forward is the normalized (probability) counting measure on $X_\lambda$ for each $\lambda$. If $X^\wedge$ is a profinite group $G$ considered as a projective system of $G/N_\lambda$ for (finite index) open normal subgroups, then the above $\mu$ coincides with the probability Haar measure.
Proposition 3.16. For a finite set $X$, we define a (normalized) Hermitian inner product on $C(X)$ by

$$ (f, g) := \frac{1}{\#X} \sum_{x \in X} f(x)\overline{g(x)}. $$

For a profinite association scheme, it is defined on $C(X^\Lambda)$ by

$$ (f, g) := \int_{x \in X^\Lambda} f(x)\overline{g(x)}d\mu(x). $$

Then $C(X^\Lambda) \rightarrow C(X^\Lambda)$ preserves the inner products.

This is clear since $X^\Lambda \rightarrow X_\Lambda$ preserves the measures.

Definition 3.17. Let $V, W$ be $\mathbb{C}$-vector spaces with Hermitian inner products. For a linear morphism $A : V \rightarrow W$, its conjugate $A^* : W \rightarrow V$ (which may not exists) is characterized by the property

$$ (Af, g)_W = (f, A^*g)_V. $$
for any $f \in V$, $g \in W$. It is unique if exists. An endomorphism $A : W \to W$ is said to be Hermitian if $A = A^*$, as usual.

Let $W$ be again a $\mathbb{C}$-vector space with Hermitian inner product, and $V$ is a subspace with restricted inner product. Let $V^\perp$ be the space of vectors which is orthogonal to any vector in $V$. If $V \oplus V^\perp$ is equal to $W$, then $V$ is said to be an orthogonal component of $W$.

**Definition 3.18.** Let $W$ be a $\mathbb{C}$-linear space with Hermitian inner product. For a linear subspace $V \subset W$, let $p : W \to W$ be a projection to $V$, i.e., the image of $p$ is $V$ and $p|_V = \text{id}_V$. Then by linear algebra $W$ is decomposed as the direct sum, $W = V \oplus \text{Ker} p$. We say that $p$ is an orthogonal projector to $V$ if $V \oplus \text{Ker} p$ is an orthogonal direct sum. An orthogonal projector to $V$ exists uniquely, if $V$ is an orthogonal component.

**Proposition 3.19.** Let $W$ be a $\mathbb{C}$-linear space with Hermitian inner product, $V \subset W$ be an orthogonal component with orthogonal projector $p$, and $\iota : V \hookrightarrow W$ be the inclusion. We denote by $q : W \to V$ the restriction of the codomain of $p$. Then we have $\iota^* = q$ and $q^* = \iota$. For $A : V \to V$, if $A^*$ exists, then

$$(\iota \circ A \circ q)^* = \iota \circ A^* \circ q : V \to V.$$

(3.3)

We sometimes denote $\iota \circ A \circ q$ simply by $A$. We make a remark when this abuse of notation is used.

**Proof.** For $w \in W$ and $v \in V$, write $w = w_V + w'_V$, with $w_V \in V$ and $w'_V \in V^\perp$. Then

$$(qw, v)_V = (w_V, v)_V = (\iota w_V, v)_W = (w_V + w'_V, v)_W = (w, v)_W,$$

which shows that $\iota^* = q$ and $q^* = \iota$. Thus (3.3) follows. □

**Proposition 3.20.** Let $W$ be a $\mathbb{C}$-vector space with Hermitian inner product, and $p : W \to W$ an idempotent, namely, $p^2 = p$. By linear algebra, $W = \text{Im}(p) \oplus \text{Ker}(p)$ and $p$ is a projector to $V := \text{Im}(p)$. Then, $p$ is an orthogonal projector if and only if $p = p^*$. If $A \in \text{End}(V)$ is an orthogonal projector to $U \subset V$, then $\iota A q : W \to W$ is a projector to $U \subset W$.

**Proof.** If $p^2 = p$, then $W = V \oplus \text{Ker} p$ for $V = \text{Im}(p)$. Thus any element $w, w' \in W$ is written as $w + k, w' + k'$, respectively. If $V \oplus \text{Ker} p$ is orthogonal,

$$(p(v + k), v' + k') = (pv, v' + k') = (v, v') = (v + k, pv') = (v + k, p(v' + k')),$$

which shows $p = p^*$. For the converse, assume $p = p^*$. Then for $v \in V$, $k \in \text{Ker} p$, $(v, k) = (pv, k) = (v, pk) = (v, 0) = 0$, and hence $W$ and $\text{Ker} p$ are mutually orthogonal. For the last, (3.3) implies that $\iota Aq$ is an Hermitian operator, and by $q \circ \iota = \text{id}_V$ is an idempotent, whose image is $U$, hence is an orthogonal projector to $U$. □

**Definition 3.21.** Let $(X_\lambda)$ be a profinite association scheme, fix $\lambda \in \Lambda$ and $x_\lambda \in X_\lambda$. For the surjection $\text{pr} : X^\wedge \to X_\lambda$, the fiber $\text{pr}^{-1}(x_\lambda)$ at $x_\lambda$ is a clopen subset of $X^\wedge$. Thus, we have the restriction of the measure $\mu$ on $X^\wedge$ to $\text{pr}^{-1}(x_\lambda)$, denoted by the same symbol $(\text{pr}^{-1}(x_\lambda), \mu)$. Note that the volume of the fiber $\text{pr}^{-1}(x_\lambda)$ is $1/\#X_\lambda$.

Since $X^\wedge = \bigsqcup_{x_\lambda \in X_\lambda} \text{pr}^{-1}(x_\lambda)$, the following proposition holds.
Proposition 3.22. For any $\lambda$ and $f \in C(X^\lambda)$, we have
\[
\int_{x \in X^\lambda} f(x) d\mu(x) = \sum_{x_{\lambda} \in X_\lambda} \int_{y \in pr^{-1}(x_{\lambda})} f(y) d\mu(y).
\]

Proposition 3.23. Let $(X_\lambda)$ be a profinite association scheme. Then, by the surjection $pr : X^\lambda \to X_\lambda$, $C(X_\lambda)$ is identified with a subspace of $C(X^\lambda)$ with the inner product given by restriction. There is an orthogonal projection $q : C(X^\lambda) \to C(X_\lambda)$ given by averaging over each fiber: for $f \in C(X^\lambda)$ define
\[
q(f)(x_{\lambda}) = \frac{1}{\mu(pr^{-1}(x_{\lambda}))} \int_{y \in pr^{-1}(x_{\lambda})} f(y) d\mu(y),
\]
where $\mu(pr^{-1}(x_{\lambda})) = \frac{1}{\#X_\lambda}$. Thus, $C(X_\lambda)$ is an orthogonal component of $C(X^\lambda)$.

Proof. The preservation of the inner products is shown in Proposition 3.16. For any $g \in C(X_\lambda)$ its image is $g \circ pr \in C(X^\lambda)$ and we have
\[
q(g \circ pr)(x_{\lambda}) = \frac{1}{\mu(pr^{-1}(x_{\lambda}))} \int_{y \in pr^{-1}(x_{\lambda})} g(pr(y)) d\mu(y)
\]
\[
= g(x_{\lambda}) \frac{1}{\mu(pr^{-1}(x_{\lambda}))} \int_{y \in pr^{-1}(x_{\lambda})} 1 \cdot d\mu(y)
\]
\[
= g(x_{\lambda}).
\]

This implies that $q$ is a projection to $C(X_\lambda)$.

To show that $q$ is an orthogonal projection, we take $f \in \text{Ker} q$. This means that the average of $f$ on each fiber of $pr$ is zero. To show that $q$ is orthogonal, it suffices to show that $(f, g \circ pr)_{X^\lambda} = 0$ for $g \in C(X_\lambda)$ by Definition 3.18 but by applying Proposition 3.22 we have
\[
(f, g \circ pr)_{X^\lambda} = \int_{x \in X^\lambda} f(x) g(pr(x)) d\mu(x)
\]
\[
= \sum_{x' \in X_\lambda} \int_{x \in pr^{-1}(x')} f(x) g(pr(x')) d\mu(x)
\]
\[
= \sum_{x' \in X_\lambda} \int_{x \in pr^{-1}(x')} f(x) d\mu(x)
\]
\[
= \sum_{x' \in X_\lambda} g(x') \cdot 0 = 0.
\]

\[\square\]

Proposition 3.24.

(1) The (convolution) algebra $A_{X^\lambda}$ acts on $C(X^\lambda)$, which makes $C(X^\lambda)$ an $A_{X^\lambda}$-module. This is compatible with the action of $A_{X_{\lambda}}$ on $C(X_{\lambda})$, in the sense of Corollary 2.8.

(2) The convolution unit $E_{X_{\lambda}} \in A_{X_{\lambda}} \subset A_{X^\lambda}$ acts as an orthogonal projector to $C(X_{\lambda})$.

(3) Let $j \in J$. Take a $\lambda$ where $j$ is isolated to $j_{\lambda}$, and consider $E_{j_{\lambda}} \in A_{X_{\lambda}}$. By the abuse of notation in Proposition 3.19, $E_{j_{\lambda}}$ extends to the orthogonal projector
\[
E_j : C(X^\lambda) \to C(X^\lambda)
\]
Comparing with $q$ where $E$ is the orthogonal projection to $C$.

By Proposition 3.22, this is the same with

This is independent of the choice of the $\lambda$.

(4) For $j \neq j'$, $E_j \cdot E_j' = 0$.

(5) We have an orthogonal decomposition

for the domain $J'$ of the partial map from $J^\circ$ to $J^\circ$. Note that $J'$ is finite.

Proof. The action is given by: $A \in A_{X^\lambda}$ acts on $f \in C(X^\lambda)$ by

By Proposition 3.22 this is the same with

Comparing with $q$ in Proposition 3.23 we have

which shows that

where $\iota : C(X^\lambda) \rightarrow C(X^\lambda)$ is the injection $pr^1_\lambda$ and $A$ in the right hand side is an endomorphism of $C(X^\lambda)$. This and $q \circ \iota = id_{C(X^\lambda)}$ show that $C(X^\lambda)$ is an $A_{X^\lambda}$-module. The well-definedness (independence of the choice of $\lambda$) follows by taking a sufficiently large $\mu$.

By 3.23 and $E_{X^\lambda} = id_{C(X^\lambda)}$ on $C(X^\lambda)$ show that

which is the orthogonal projection to $C(X^\lambda)$.

For $j \in J$, take a $\lambda$ where $j$ is isolated to $j_\lambda \in J^\circ$. By (3.3) we have

where $E_{j_\lambda}$ in the right hand side is an endomorphism of $C(X^\lambda)$. Hence to prove that $E_{j_\lambda}$ is an orthogonal projector $C(X^\lambda) \rightarrow C(X^\lambda)_{j_\lambda}$, it suffices to show that $E_{j_\lambda} \in A_{X^\lambda}$ is an orthogonal projector $C(X^\lambda) \rightarrow C(X^\lambda)_{j_\lambda}$ by Proposition 3.20. The orthogonality of $E_{j_\lambda} \in A_{X^\lambda}$ is well known [11]. (Since $A_{X^\lambda}$ is closed under the antilinear isomorphism $*$ of ring, $E_{j_\lambda}^*$ must be a primitive idempotent, and since $E_{j_\lambda}E_{j_\lambda}^* \neq 0$, they must coincide, hence $E_{j_\lambda}$ is Hermitian and an orthogonal projection by Proposition 3.20. We define $E_j := E_{j_\lambda}$, where

$E_j : C(X^\lambda) \rightarrow C(X^\lambda)_{j_\lambda} := C(X^\lambda)_{j_\lambda}$. 

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For the well-definedness, we take \( \lambda, \lambda' \) where \( j \) is isolated. There exists a \( \mu \) with \( \mu \geq \lambda, \mu \geq \lambda' \). Then \( j \) is isolated to \( j_\mu \) at \( \mu \), and by (3.55) it is not difficult to see that the \( E_j \) defined using \( \mu \) is the same as those defined using \( \lambda, \lambda' \). For \( E_j \cdot E_{j'} = 0 \), take a \( \lambda \) where both \( j, j' \) are isolated to \( j_\lambda, j'_\lambda \), respectively. Then, \( j_\lambda \neq j'_\lambda, E_{j_\lambda}, E_{j'_\lambda} \in A_{X_\lambda} \), and \( E_{j_\lambda} \cdot E_{j'_\lambda} = 0 \). Thus, the relation holds in the inductive limit \( A_{X^\infty} \).

For fixed \( \lambda \), it is well-known [1] that we have an orthogonal decomposition

\[
C(X_\lambda) = \bigoplus_{j_\lambda \in J_\lambda} C(X_\lambda)_{j_\lambda}.
\]

(This follows from the orthogonality of \( E_{j_\lambda} \).) Since each \( C(X_\lambda)_{j_\lambda} \) is an orthogonal direct sum of \( C(X^\infty)_{j} \) over the finite number of \( j \in J^\infty \) whose image is \( j_\lambda \), the claim follows. \( \square \)

The following is an analogue to Peter-Weyl Theorem.

**Theorem 3.25.** We have an orthogonal direct decomposition

\[
\lim_{j \in J} C(X_\lambda) = \bigoplus_{j \in J} C(X^\infty)_{j}.
\]

Hence the right hand side is dense in \( C(X^\infty) \) with respect to the supremum norm.

**Proof.** Proposition 3.24 implies that \( C(X^\infty)_{j} \) are mutually orthogonal, and their orthogonal sum is a subspace of \( C(X^\infty) \). For any \( \lambda, C(X_\lambda) \) is an orthogonal sum of a finite number of \( C(X^\infty)_{j} \). Thus \( \lim_{j \in J} C(X_\lambda) \) is the direct sum of \( C(X^\infty)_{j} \) for \( j \in J^\infty \). The density follows from Lemma 3.14. \( \square \)

3.3. **Measures and Fourier analysis.** Since \( X^\infty \) has a natural probability measure, \( F^\infty \) has a pushforward measure via \( X^\infty \times X^\infty \to F \). It will be shown that \( F \) has a natural measure, and these measures are closely related to the classical notions of the multiplicity \( m_{i_\lambda} \) for \( J_\lambda \), and the valency \( k_{i_\lambda} \) for \( I_\lambda \) (see [1] [6]).

**Definition 3.26.** Let \( \mu_F \) be the pushforward measure on \( F \) from \( X^\infty \times X^\infty \). This is characterized as follows. For \( i_\lambda \in I_\lambda \), let \( \Pr_{i_\lambda}^{-1}(i_\lambda) \subset F \) be the open set. Then

\[
\mu_F(\Pr_{i_\lambda}^{-1}(i_\lambda)) = \mu_{X^\infty \times X^\infty}(R_{i_\lambda} \circ \Pr_{X_\lambda \times X_\lambda}^{-1}(i_\lambda)).
\]

**Proposition 3.27.** In the above definition,

\[
\mu_F(\Pr_{i_\lambda}^{-1}(i_\lambda)) = \frac{k_{i_\lambda}}{\# X_\lambda}
\]

holds, where \( k_{i_\lambda} \) is the valency of the adjacency matrix \( A_{i_\lambda} \in A_{X_\lambda} \).

**Proof.** Recall that \( A_{i_\lambda} \) is the indicator function of \( R_{i_\lambda}^{-1}(i_\lambda) \), and the sum of the rows is an integer \( k_{i_\lambda} \) called the valency [1]. Thus, the volume \( \mu_F(\Pr_{i_\lambda}^{-1}(i_\lambda)) \) is the volume of \( \mu_{X^\infty}(\Pr_{X_\lambda \times X_\lambda}^{-1}R_{i_\lambda}^{-1}(i_\lambda)) \), which is

\[
\int_{x,y \in X^\infty \times X^\infty} A_{i_\lambda}(\Pr_{X_\lambda}(x), \Pr_{X_\lambda}(y))d\mu(x)d\mu(y) = \frac{1}{\# X_\lambda} \sum_{x',y' \in X_\lambda \times X_\lambda} A_{i_\lambda}(x', y') = \frac{k_{i_\lambda}}{\# X_\lambda}.
\]
Definition 3.28. The algebra $A_{X_{\lambda}}$ is closed under the adjoint $\ast$. We define the Hilbert-Schmidt inner product, which is Hermitian, as follows. For $A, B \in A_{X_{\lambda}}$, 

$$(A, B)_{\text{HS}} := \sum_{x \in X_{\lambda}} (A \ast B^*)(x, x).$$

For $A, B \in A_{X_{\lambda}}$, we define its Hilbert-Schmidt inner product by

$$(A, B)_{\text{HS}} := \int_{x \in X_{\lambda}} (A \ast B^*)(x, x)d\mu(x).$$

Then inclusion $A_{X_{\lambda}} \to A_{X_{\lambda}}$ preserves the inner products.

Proof. For $B \in A_{X_{\lambda}}$, $B^* \in A_{X_{\lambda}}$ follows because of the definition of association schemes, and the Hilbert-Schmidt inner product is defined. For $B \in A_{X_{\lambda}}$, we must show that $B^*$ exists in $A_{X_{\lambda}}$ and the integration converges. However, $B \in A_{X_{\lambda}}$ for some $\lambda$, and $B^* \in A_{X_{\lambda}}$. Since $C(X_{\lambda})$ is an orthogonal component of $C(X_{\lambda})$, $B^* = iB^\prime q$ is the adjoint as an operator on $C(X_{\lambda})$, by the abuse of notation in Proposition 3.19. The notation $(A \ast B^*)(x, x)$ comes from the middle vertical injection in (3.1). Since $A, B^* \in A_{X_{\lambda}} \subset C(X_{\lambda} \times X_{\lambda})$, 

$$\int_{x \in X_{\lambda}} (A \ast B^*)(x, x)d\mu(x) = \sum_{x \in X_{\lambda}} (A \ast B^*) (x', x'),$$

which is finite, satisfying the axioms of Hermitian inner product, and compatible with $A_{X_{\lambda}} \to A_{X_{\lambda}}$. □

Proposition 3.29. Through the natural isomorphism 

$$\lim_\downarrow C(I_{\lambda}) \to A_{X_{\lambda}},$$

we obtain Hermitian inner product on $\lim_\downarrow C(I_{\lambda})$, which is denoted by $(-, -)_{\text{HS}}$. Then for $f, g \in \lim_\downarrow C(I_{\lambda})$, we have 

$$(f, g)_{\text{HS}} = \int f(i)g(i)d\mu_F(i).$$

Proof. We may assume $f, g \in C(I_{\lambda})$ for some $\lambda$. Then by bilinearity it suffices to check the equality for $f = \delta_{i_1}, g = \delta_{i_2}$. Then, the right hand side is $\delta_{i_1, i_2} k_{i_1}/\#X_{\lambda}$ by Proposition 3.27. In $A_{X_{\lambda}}$, these functions correspond to $f \mapsto A_{i_1}, g \mapsto A_{i_2}$, and their inner product is 

$$\int_{x, y \in X_{\lambda} \times X_{\lambda}} A_{i_1}(pr_{X_{\lambda}}(x), pr_{X_{\lambda}}(y))A_{i_2}(pr_{X_{\lambda}}(y), pr_{X_{\lambda}}(x))^*d\mu(x)d\mu(y)$$

$$= \frac{1}{\#X_{\lambda}^2} \sum_{x, y' \in X_{\lambda} \times X_{\lambda}} A_{i_1}(x', y')A_{i_2}(x', y')$$

$$= \frac{k_{i_1}}{\#X_{\lambda}}.$$

This uses a well-known orthogonality of $A_{i_1}$: for any $x \in X_{\lambda}$ and $i_1, i_2 \in I_{\lambda}$, the $(x, x)$-component of the matrix product $A_{i_1} \ast A_{i_2}$ is the number of $y \in X_{\lambda}$ with $R(x, y) = i_1$ and $R(x, y) = i_2$, which is zero unless $i_1 = i_2$, and in the equal case $k_{i_1}$. □
Lemma 3.30. For a commutative association scheme \((X, R, I)\), we defined \(J\) as the set of primitive idempotents \(E_j\) with respect to the product \(\bullet\) in Theorem 2.20. We put a positive measure \(m_j\) for \(j \in J\), so that the inner product with respect to \(m_j\) in \(C(J)\) is isometric to that of \(A_X\) with respect to HS inner product. Then, \(m_j\) is the multiplicity of \(j\), defined as \(\dim(C(X)_j)\) [1].

Proof. Let \(E_j, E_{j'} \in A_X\) be two primitive idempotents. The computation
\[
(E_j, E_{j'})_{\text{HS}} = \sum_{x \in X} E_j \cdot E_{j'}(x, x) = \delta_{jj'} \text{tr}(E_j) = \delta_{jj'} m_j
\]
implies that if we define \(\mu_j(j) := m_j\), then the corresponding inner product is compatible with \(C(J) \to A_X\) with respect to the HS-inner product on \(A\).

Proposition 3.31. For a projective association scheme \((X, R, I)\), we define a measure \(\mu_{\lambda'}\) on the discrete set \(J^\lambda\) by \(\mu_{\lambda'}(j) = \dim(C(X^{\lambda'})_j)\). Then, the inner product on \(\lim C(J_\lambda)\) defined by
\[
(f, g)_{\text{HS}} = \sum_{j \in J^\lambda} f(j) \overline{g(j)} \mu_{\lambda'}(j)
\]
makes \(\lim C(J_\lambda) \cong A_{X^\lambda}\) isometric with respect to HS inner product.

Proof. Any \(f, g \in \lim C(J_\lambda)\) are contained in some \(C(J_\lambda)\), where any \(j\) in the support of \(f\) or \(g\) is isolated at \(\lambda\). The functions \(f, g\) are linear combinations of orthogonal basis \(E_j\) of \(C(J_\lambda)\), namely, \(f = \sum_j f(j) E_j\) and \(g = \sum_j g(j) E_j\) in \(A_{X_\lambda}\). The previous lemma says that \((E_j, E_{j'})_{\text{HS}} = \delta_{jj'} \dim(C(X^{\lambda'})_j)\), which proves the proposition.

In sum, we have the following form of Fourier transform.

Theorem 3.32. The isomorphism of vector spaces \(\lim C(J_\lambda) \cong A_{X^\lambda} \cong \lim C(I_\lambda)\) is an isometry between the space of compact support functions \(C_c(J^\lambda)\) with inner product defined in Proposition 3.31 and the space of locally constant functions \(C_c(F^\lambda)\) with inner product defined in Proposition 3.29 which maps the componentwise product in \(C_c(J^\lambda)\) to the convolution product in \(C_c(F^\lambda)\).

Remark 3.33. For a finite association scheme \((X, R, I)\), both \(E_j\) (\(j \in J\)) and \(A_i\) (\(i \in I\)) are orthogonal bases of \(A_X\) with respect to HS-inner product. The isometry
\[
C(I) \to C(J)
\]
is given by
\[
A_i \mapsto \sum_{j \in J} p_i(j) E_j.
\]
The isometry shows that
\[
(A_i, A_{i'})_{\text{HS}} = \delta_{ii'} \mu_{\lambda}(i) = \delta_{ii'} \frac{k_i}{\#X}
\]
is equal to
\[
\sum_{j \in J} p_i(j) \overline{p_{i'}(j)} m_j,
\]
which is one of the two orthogonalities of eigenmatrices as classically known [1]. (Here the factor \(\#X\) appears in a different manner because of the normalization in
Theorem 2.5. Our $E_j$ is $\#X$ times the classical primitive idempotent, which makes $p_i(j) = \frac{1}{\#X} P_i(j)$ where $P_i(j)$ are the components of the standard eigen matrices.)

The next is an analogue of the Fourier transform for $L^2$ spaces.

Proposition 3.34. The isometry $\lim C(J_\lambda) \to \lim C(I_\lambda)$ extends to a unique isometry $L^2(J^\wedge, \mu_J) \to L^2(I^\wedge, \mu_I)$.

Proof. We consider $L^2$-norm for all the spaces. The map $\lim C(J_\lambda) \to L^2(J^\wedge)$ is isometric, injective, and has a dense image, by considering the orthogonal basis $E_j$. Since $L^2(J^\wedge)$ is complete, this is a completion of $\lim C(J_\lambda)$. The map $\lim C(I_\lambda) \to L^2(I^\wedge)$ is also isometric, hence injective, and has a dense image (by Lemma 3.14 it is dense with respect to supremum norm in $C(I^\wedge)$, and since $I^\wedge$ is compact, dense with respect to the $L^2$-norm in $C(I^\wedge)$, and thus dense in $L^2(I^\wedge)$), so $L^2(I^\wedge)$ is a completion of $\lim C(I_\lambda)$. By the universality of the completion, they are isometric.

4. Some examples and relation with Barg-Skriganov theory

The results of this paper are closely related to the study by Barg and Skriganov [2]. They defined the notion of association schemes on a set $X$ with a $\sigma$-additive measure. See [2, Definition 1], where $I$ is denoted by $T$. Association scheme structure is given by a surjection $R : X \times X \to I$, with axioms similar to the finite case. The inverse image $R^{-1}(i) \subset X \times X$ of $i \in I$ is required to be measurable in $X \times X$, and intersection numbers $p^k_{ij}$ are defined as the measure of the set

$$\{z \in X \mid R(x, z) = i, R(z, y) = j\}$$

where $R(x, y) = k$, which is required to be independent of the choice of $x, y$. The set $I$ is required to be at most countably infinite. This countability assumption seems to be necessary, for example to avoid the case where $p^k_{ij}$ is all zero for any $i, j, k$.

Let $\chi_i(x, y)$ be the indicator function of $R^{-1}(i)$. They define adjacency algebras (Bose-Mesner algebras in our terminology) in [2, Section 3] as a set of finite linear combination of $\chi_i(x, y)$ with complex coefficient, with convolution product given by the integration

$$(a * b)(x, y) := \int_X a(x, z)b(z, y)d\mu(z).$$

However, it is not clear whether the adjacency algebra is closed under the convolution product. In fact, the authors remarked in the last of [2, Section 3.1] “our arguments in this part are of somewhat heuristic nature. We make them fully rigorous for the case of association schemes on zero-dimensional groups; see Section 8.” Later in Section 8, they define the adjacency algebra with an aid of locally compact zero-dimensional abelian groups and their duality. Some of the treated objects (compact cases) are examples of profinite association schemes in this paper.
4.1. **Profinite groups.** Let $G$ be a profinite group, that is, a projective limit of finite groups $G_\lambda$ with projective limit topology. Absolute Galois groups, arithmetic fundamental groups and $p$-adic integers are some examples appear in arithmetic geometry [22]. There are significant amounts of studies on analysis on abelian profinite groups, see for example [14].

4.1.1. **Schurian profinite association schemes.** We describe Schurian association schemes, and then profinite analogues. When we say an action of a group $G$, it means the left action, unless otherwise stated.

**Lemma 4.1.** Let $X$ be a topological space and $G$ a group acting on $X$ such that every element of $G$ induces a homeomorphism on $X$. Then the space of continuous functions $C(X)$ is a left $G$-module by $f(x) \mapsto f(g^{-1}x)$. Let $G\backslash X$ denote the quotient space with quotient topology. Let $C(G\backslash X) \subset C(X)$ denote the subspace of functions invariant by the action of $G$. There is a canonical injection

$$C(G\backslash X) \hookrightarrow C(X)$$

associated with the continuous surjection $X \to G\backslash X$. Then, there is a canonical isomorphism of $\mathbb{C}$-algebras

$$C(X)^G \to C(G\backslash X),$$

and hence we identify them.

**Proof.** An element $f$ of $C(X)^G$ is a continuous function which takes one same value for each $G$-orbit of $X$. Thus, it gives a mapping $G\backslash X \to \mathbb{C}$. By the universality of the quotient topology, this is continuous, hence lies in $C(G\backslash X)$. Conversely, a function in $C(G\backslash X)$ gives a continuous function in $C(X)$, which is invariant by $G$. □

**Proposition 4.2.** (Schurian association scheme) Let $G$ be a finite group, $H$ a subgroup, and let $X := G/H$ be the quotient. Then, we have a natural bijection

$$G\backslash (X \times X) \to H\backslash G/H,$$

where $G$ acts diagonally. We define

$I := G\backslash (X \times X)$. The quotient mapping

$$R : X \times X \to I$$

is an association scheme, which is called a Schurian scheme.

We give a proof, since descriptions used in the proof are necessary for the profinite case. The equality [14] is given by a surjection

$$(g_1H, g_2H) \mapsto Hg_1^{-1}g_2H,$$

which certainly factors through

$$G\backslash (g_1H, g_2H) \mapsto Hg_1^{-1}g_2H,$$

and the converse map is given by

$$HgH \mapsto (H, gH),$$

where the check of the well-definedness is left to the reader.
Lemma 4.3. Let $C(X)^\vee := \text{Hom}_{\mathbb{C}}(C(X), \mathbb{C})$ be the dual space, which is regarded as a left $G$-module, by letting $g \in G$ act on $\xi \in C(X)^\vee$ by $g(\xi)f = \xi(g^{-1}(f))$. Then we have an isomorphism of $G$-modules
\[ C(X) \rightarrow C(X)^\vee, \quad \delta_x \mapsto \text{ev}_x \]
for every $x \in X$, where $\delta_x$ is the indicator function of $x$ and $\text{ev}_x$ is the evaluation at $x$.

Proof. It is well-known that $\{\delta_x \mid x \in X\}$ is a linear basis of $C(X)$, and $\{\text{ev}_x \mid x \in X\}$ is a linear basis of $C(X)^\vee$. We compute
\[ g(\delta_x)(y) = \delta_x(g^{-1}y) = \delta_{gx}(y). \]
On the other hand,
\[ g(\text{ev}_x)(f) = \text{ev}_x(g^{-1}(f)) = (g^{-1}(f))(x) = f(gx) = \text{ev}_{gx}(f). \]
\[ \square \]

Proposition 4.4. The identification of a matrix and a linear map is obtained by an isomorphism of $G$-modules
\[ C(X \times X) = C(X) \otimes C(X) \cong C(X)^\vee \cong \text{Hom}_{\mathbb{C}}(C(X), C(X)). \quad (4.2) \]

Proof. An element $A \in C(X \times X)$ is mapped as
\[ A \mapsto \sum_{x,y \in X^2} A(x,y)\delta_x \otimes \delta_y \mapsto \sum_{x,y \in X^2} A(x,y)\delta_x \otimes \text{ev}_y. \]
The last element is regarded as an element of $\text{Hom}_{\mathbb{C}}(C(X), C(X))$ by mapping $f = \sum_{y \in X} f(y)\delta_y \in C(X)$ to
\[ (\sum_{x,y \in X^2} A(x,y)\delta_x \otimes \text{ev}_y)(f) = \sum_{x \in X} \sum_{y \in Y} A(x,y)f(y)\delta_x, \]
which is the multiplication of a matrix to a vector. \[ \square \]

Proposition 4.5. By taking the $G$-invariant part of (4.2), we have
\[ \text{Hom}_G(C(X), C(X)) \cong C(X \times X)^G = C(G \setminus (X \times X)) = C(H \setminus G/H), \quad (4.3) \]
where the left $\text{Hom}_G$ denotes the endomorphism as $G$-modules.

Proof. This follows from (4.1), Lemma 4.1 and the definition of $G$-homomorphisms. \[ \square \]

Corollary 4.6. The construction in Proposition 4.2 gives an association scheme. The endomorphism ring $\text{Hom}_G(C(X), C(X))$ is isomorphic to the Bose-Mesner algebra $A_X$.

Proof. By the construction of $I = G \setminus (X \times X)$, the set of the adjacency matrices is closed under the transpose and contains the identity matrix. Their linear span is closed by the matrix product, since the adjacency matrices form a linear basis of (4.3) and the matrix product corresponds to the composition of the endomorphism ring $\text{Hom}_G(C(X), C(X))$. This gives an association scheme with Bose-Mesner algebra $\text{Hom}_G(C(X), C(X))$. \[ \square \]
Definition 4.7. Let $G$ be a finite group acting on a $\mathbb{C}$-linear space $V$. By the representation of finite groups, $V$ is decomposed into irreducible representations. For a fixed irreducible representation $W$, the multiplicity of $W$ in $V$ is the dimension of the vector space

$$\text{Hom}_G(W, V).$$

We say $V$ is multiplicity free, if the multiplicity is at most one for any irreducible representation $W$.

Corollary 4.8. The Schurian association scheme is commutative, if and only if $C(X)$ is multiplicity free. In this case, the set of the primitive idempotents is canonically identified with the set of the irreducible representations of $G$ in $C(X)$.

Proof. If the multiplicity is one for each irreducible representation, Schur’s lemma implies that $\text{Hom}_G(C(X), C(X))$ is a direct sum of copies of $\mathbb{C}$ (each copy corresponding to each irreducible representation in $C(X)$) as a ring, and hence is commutative. If there is an irreducible component with multiplicity $m \geq 2$, $\text{Hom}_G(C(X), C(X))$ contains a matrix algebra $M_m(\mathbb{C})$, and thus is not commutative. $\square$

See [22] for the following definition and properties (c.f. Section 3.1 above).

Definition 4.9. Let $G_\lambda (\lambda \in \Lambda)$ be a projective system of surjections of finite groups. A profinite group is a topological group which is isomorphic to $G^\wedge := \varprojlim G_\lambda$, where each $G_\lambda$ has the discrete topology. Because the projection $\text{pr}_\lambda : G^\wedge \to G_\lambda$ is surjective and continuous, we identify $G_\lambda$ with $G^\wedge / N_\lambda$, where $N_\lambda$ is the kernel of the $\text{pr}_\lambda$, which is a clopen normal subgroup of $G^\wedge$, since $G_\lambda$ is finite and discrete. By Tychonoff’s theorem, $G^\wedge$ is a compact Hausdorff group, with a basis of open neighborhood of the unit element $e$ consisting of $N_\lambda$. Being Hausdorff, the intersection of $N_\lambda$ is $\{e\}$.

We prepare a well-known lemma for describing profinite Schurian association schemes.

Lemma 4.10. Let $G^\wedge = \varprojlim G_\lambda = \varprojlim G_\lambda / N_\lambda$ be a profinite group. Let $H$ and $K$ be closed subgroups of $G^\wedge$. Give a quotient topology on $H \backslash G^\wedge / K$ by

$$q : G^\wedge \to H \backslash G^\wedge / K.$$

(1) The image of $K$ in $G_\lambda$ is $K N_\lambda$ (which is a union of cosets of $N_\lambda$, and hence a subset of $G_\lambda$). The same statement holds for $H$.

(2) We have the following commutative diagram of continuous surjections, where $\alpha$ is a bijection.

$$\begin{array}{ccc}
H \backslash G^\wedge / K & \to & H N_\lambda \backslash G^\wedge / K N_\lambda \\
\downarrow & & \downarrow \\
H N_\lambda \backslash G^\wedge / K N_\lambda & \sim & H N_\lambda \backslash G_\lambda / K N_\lambda.
\end{array}$$

(3) The $q : G^\wedge \to H \backslash G^\wedge / K$ is an open morphism, and $H \backslash G^\wedge / K$ is compact Hausdorff.

(4) The finite sets $H N_\lambda \backslash G_\lambda / K N_\lambda$ constitute a projective system, with a canonical homeomorphism

$$H \backslash G^\wedge / K \sim \varprojlim H N_\lambda \backslash G_\lambda / K N_\lambda.$$
The image of $K$ is the union of $kN_\lambda$ for $k \in K$, which is nothing but $KN_\lambda$.

Every set in the diagram is a quotient of $G^\alpha$ with various equivalence relation.

The image of $K_\alpha$ in the diagram is continuous, since the maps from $G^\alpha$ to the two sets at the bottom of the diagram factors through which is finite and discrete).

The vertical arrow is obviously well-defined. The mapping $G$ is bijective, since as a quotient of $G^\alpha$, they are the same. The slanting arrow is merely the composition of the two. We recall that $G^\alpha \to H\backslash G^\alpha/K$ is a quotient map. Thus, every map in the diagram is continuous, since the maps from $G^\alpha$ are continuous (where the mappings to the two sets at the bottom of the diagram factors through $G^\alpha/KN_\lambda$, which is finite and discrete).

The open sets $sN_\lambda$ for $s \in G^\alpha$ and various $\lambda$ constitute an open basis of $G^\alpha$. The image of $sN_\lambda$ in $H\backslash G^\alpha/K$ is $HsKN_\lambda$ (since $N_\lambda$ is normal), whose inverse image in $G^\alpha$ is again $HsKN_\lambda$, which is open. Thus, by the definition of quotient topology, $HsKN_\lambda$ is open in $H\backslash G^\alpha/K$, and hence $q$ is open.

Take two distinct elements $HgK$, $Hg'K$ in $H\backslash G^\alpha/K$. These are compact disjoint subsets of the Hausdorff space $G^\alpha$. It is well known that there are an open set $O_a \subset G^\alpha$ containing $HgK$ and an open set $O_b \supset Hg'K$ that separate $HgK$ and $Hg'K$. By (1) of Lemma 3.14 for the union of $HgK$ and $Hg'K$, we may take $\mu$ such that open sets of the form $sN_\mu \subset O_a$ with $sN_\mu \cap HgK \neq \emptyset$ cover $HgK$ and open sets $sN_\mu \subset O_b$ with $sN_\mu \cap Hg'K \neq \emptyset$ cover $Hg'K$, that is, $HgK N_\mu \subset O_a$ and $Hg'KN_\mu \subset O_b$ are disjoint. Since $q$ is open, $HgKN_\mu$ and $Hg'KN_\mu$ are open sets in $H\backslash G^\alpha/K$ which have no intersection and are neighborhoods of $HgK$, $Hg'K$ (which are considered as elements of $H\backslash G^\alpha/K$), respectively. Thus $H\backslash G^\alpha/K$ is Hausdorff.

If we see these sets as quotients of $G^\alpha$ as in (4.3), it is clear that these form a projective system. By the universality of projective limit, we have a continuous map

$$f : H\backslash G^\alpha/K \to \varprojlim HN_\lambda \backslash G_\lambda / KN_\lambda$$

by (2). The image of $f$ is dense, since for any element $\xi$ in the projective limit and its open neighborhood, there is a smaller open neighborhood which has the form $pr_\lambda^{-1}pr_\lambda(\xi)$ for some $\lambda$, since these form a basis of open neighborhoods of $\xi$. Because

$$H\backslash G^\alpha/K \to HN_\lambda \backslash G_\lambda / KN_\lambda$$

is surjective, the density of the image follows. Since $H\backslash G^\alpha/K$ is compact (being a quotient of a compact set $G^\alpha$), the image is compact and closed, hence $f$ is surjective. Since $H\backslash G^\alpha/K$ is Hausdorff, any two distinct points $HgK$, $Hg'K$ have disjoint open neighborhoods $HgKN_\lambda$, $Hg'KN_\lambda$, respectively, as proved above, hence their images in $H\backslash G^\alpha/K$ are distinct, which implies the injectivity of $f$. By the compactness of $H\backslash G^\alpha/K$, $f$ is closed, hence $f^{-1}$ is continuous, thus a homeomorphism.

**Proposition 4.11.** (profinite Schurian association scheme) Let $G^\alpha = \varprojlim G_\lambda = \varprojlim G^\alpha/KN_\lambda$ be a profinite group. We denote by $G_\lambda$ the quotient $G^\alpha/KN_\lambda$. For a closed subgroup $H < G$, its image in $G_\lambda$ is $H_\lambda := HN_\lambda$. Then we have a projective system of Schurian association schemes

$$X_\lambda := G_\lambda / H_\lambda$$

and

$$I_\lambda := H_\lambda \backslash G_\lambda / H_\lambda.$$
These systems give a profinite association scheme, which may be non-commutative. We call this type of profinite association scheme Schurian. We have canonical homeomorphisms
\[ G^\lor / H \cong \varprojlim X_\lambda = \varprojlim G^\lor / H N_\lambda \]
and
\[ H \setminus G^\lor / H \cong \varprojlim I_\lambda = \varprojlim H \setminus G^\lor / H_\lambda. \]

**Proof.** Everything is proved in Proposition 4.2 and Lemma 4.10, except that these mappings give surjective morphisms of association schemes. Take \( \lambda \geq \mu \). Since the surjectivity is proved in (2), it suffices to show the commutativity of
\[ X_\lambda \times X_\lambda \rightarrow I_\lambda \quad \downarrow \downarrow \]
\[ X_\mu \times X_\mu \rightarrow I_\mu, \]
which is easy to check: an element \((gH N_\lambda, g'H N_\lambda)\) at the left top is mapped to the right \( Hg^{-1}g'H N_\lambda\), then to the bottom \( Hg^{-1}g'H N_\mu\). This is the same via the left bottom corner. \(\square\)

**Proposition 4.12.** Let \( G^\lor / H \) be a Schurian profinite association scheme as in 4.11. Suppose that each finite association scheme \( X_\lambda \) is commutative. Then, \( J^\lor \) is the set of irreducible representations of \( G^\lor \) appearing in \( C(X_\lambda) \) for some \( \lambda \), where the multiplicity is at most one for any \( \lambda \).

**Proof.** We have an inductive system of injections \( C(X_\lambda \times X_\lambda) \), which gives via (4.2) an inductive system of injections
\[ \text{Hom}_C(C(X_\mu), C(X_\mu)) \rightarrow \text{Hom}_C(C(X_\lambda), C(X_\lambda)) \]
for \( \lambda \geq \mu \). To obtain \( A_{X_\lambda} \) and \( A_{X_\lambda} \), we take \( G^\lor \)-invariant part (4.3)
\[ \text{Hom}_{G^\lor}(C(X_\mu), C(X_\mu)) \rightarrow \text{Hom}_{G^\lor}(C(X_\lambda), C(X_\lambda)) \]
(note that there is a canonical surjection \( G^\lor \rightarrow G_\lambda \), hence a \( G_\lambda \)-module is a \( G^\lor \)-module.) By the surjectivity of \( G^\lor \rightarrow G_\lambda \), an irreducible representation in \( C(X_\mu) \) of \( G_\mu \) lifts to an irreducible representation in \( C(X_\lambda) \) of \( G_\lambda \).

We assume that every finite association scheme appeared is commutative. Then, the multiplicity of each irreducible representation \( V \subset C(X_\lambda) \) is one by Corollary 4.8 for any \( \lambda \). Thus, the partial surjection given in Corollary 2.14 is in this case the inverse to the injection \( J_\mu \rightarrow J_\lambda \), induced by \( G_\lambda \rightarrow G_\mu \), and
\[ J^\lor = \varprojlim J_\lambda \]
is identified with the set of irreducible representations of \( G^\lor \) appeared in some \( C(X_\lambda) \). \(\square\)

Indeed, the above irreducible representations are exactly the irreducible representations of \( G^\lor \) appearing in \( C(G^\lor / H) \). To show this, we recall a well known “no small subgroups” lemma.

**Lemma 4.13.** Let \( G \) be a Lie group over \( \mathbb{R} \) or \( \mathbb{C} \). Then, there is an open neighborhood of the unit \( e \), which contains no subgroup except \( \{e\} \).
Proof. We consider the Lie algebra \( g \) of \( G \). Give an arbitrary (Hermitian or positive definite) metric to \( g \). It is known that
\[
\exp : g \to G
\]
is homeomorphism when restricted to a small enough neighborhood of 0. We take the homeomorphic neighborhoods \( U \) of \( e \in G \) and \( V \) of \( 0 \in g \). We may assume that \( V \) is bounded with respect to the metric. Assume that \( U \) contains non-trivial subgroup of \( G \). Take an \( e \neq g \in G \) in the subgroup. Consequently, for any integer \( n, g^n \in U \). If \( h = \log(g) \in V \subset g \), \( nh = \log(g^n) \) corresponds to \( g^n \) and hence \( nh \in V \) for any \( n \). However, \( V \) is bounded, which contradicts the assumption. □

Corollary 4.14. Let \( G^\wedge \) be a profinite group, and \( f : G^\wedge \to G \) a group morphism to a real or complex Lie group \( G \). Then, \( \ker(f) \) is an open normal subgroup of \( G^\wedge \), and the image \( f(G^\wedge) \) in \( G \) is a finite subgroup of \( G \).

Proof. Take a small enough neighborhood of \( U \) in Lemma 4.13. Consider \( H = f^{-1}(U) \subset G^\wedge \), which is open in \( G^\wedge \). Thus, there exists \( \lambda \) such that the open normal subgroup \( N = N_\lambda \) is contained in \( H \). The image \( f(N) \subset G \) is a subgroup in \( U \), which must be \( \{ e \} \subset G \) by definition of \( U \). Thus, \( \ker(f) \) contains \( N \), and is a union of cosets of \( N \), hence an open normal subgroup of \( G^\wedge \). Since \( G^\wedge \) is compact and \( \ker(f) \) is open, \( f(G^\wedge) \cong G^\wedge / \ker(f) \) is a finite group. □

Proposition 4.15. Let \( G \) be a compact Hausdorff group, and \( H \) a closed subgroup. Consider the representation of \( G \) on \( C(G/H) \). Then, each irreducible component of \( C(G/H) \) is finite dimensional.

Proof. This follows from a variant of the Peter-Weyl theorem, see Takeuchi[24, Section 1]. □

Proposition 4.16. Let \( G^\wedge \) be a profinite group and \( H \) a closed subgroup. Consider the profinite association scheme arising from \( G^\wedge / H \). Let \( J(G^\wedge / H) \) be the set of the irreducible representations of \( G^\wedge \) on \( C(G^\wedge / H) \). Then, all finite association schemes \( X_\lambda \) in Proposition 4.11 are commutative, if and only if the representation \( C(G^\wedge / H) \) is multiplicity free. In this case, there is a canonical bijection
\[
f : J(G^\wedge / H) \to J^\wedge = \varinjlim J_\lambda.
\]

Proof. Put \( X^\wedge := G^\wedge / H \). Let \( V \subset C(X^\wedge) \) be an irreducible representation of \( G^\wedge \). By Proposition 4.15 \( V \) is finite dimensional. By Lemma 4.13 the representation factors through
\[
G \to G/H \to GL(V)
\]
for some \( \lambda \). By Lemma 4.1 this means that \( V \subset C(X^\wedge) \) lies in
\[
V \subset C(N_\lambda \setminus X^\wedge) = C(X_\lambda) \subset C(X^\wedge).
\]
Thus, \( C(X^\wedge) \) is multiplicity free, if and only if \( C(X_\lambda) \) is multiplicity free for every \( \lambda \in \Lambda \), in other words, if and only if every \( X_\lambda \) is a commutative association scheme, by Proposition 4.12. In the commutative case, the following observation shows that every irreducible component of \( C(X^\wedge) \) appears in \( C(X_\lambda) \) for some \( \lambda \). Because the multiplicity is one, for any \( V \) above, there is a unique representation isomorphic to \( V \) in \( \varinjlim C(X_\lambda) \). This gives a map \( f \) in (4.15). The injectivity follows from that \( C(X^\wedge) \) is multiplicity-free. The surjectivity follows from \( C(X_\lambda) \subset C(X^\wedge) \) is a \( G^\wedge \)-submodule, and any \( G_\lambda \)-irreducible component of \( C(X_\lambda) \) is a \( G^\wedge \)-irreducible component since \( G^\wedge \to G_\lambda \) is surjective. □
Remark 4.17. Kurihara-Okuda\textsuperscript{13} gives the notion of Bose-Mesner algebra for general homogeneous space $G/H$, where $G$ is compact Hausdorff and $H$ is a closed subgroup. In the case where $G$ is a profinite group and $C(G/H)$ is multiplicity free, their construction coincides with our definition, by Proposition 4.16.

4.1.2. Profinite abelian groups. For any finite abelian group $X$, we consider its thin scheme. Namely, $R : X \times X \to I = X$ with $(x, y) \mapsto y - x$. Then, the $i$-th adjacency matrix $R^{-1}(i)$ is the representation matrix of addition of $i$ on $C(X)$, and the Bose-Mesner algebra is isomorphic to the group ring $C(X)$, with usual Hadamard product and the convolution product

$$(f * g)(x) = \frac{1}{\#X} \sum_y f(x - y)g(y)$$

(the factor $\frac{1}{\#X}$ is not the standard, incorporated to be compatible with Theorem 2.5). Primitive idempotents are exactly the characters $\xi \in \hat{X}$, where $\hat{X} \subset C(X)$ is the group of characters of $X$.

For a profinite abelian group $X^\wedge$, it is a projective limit of finite abelian groups $X_\lambda$. We have an inductive system of the character groups $J_\lambda := \hat{X_\lambda}$. For $\lambda \geq \mu$, the partial map $p_{\lambda, \mu} : J_\lambda \to J_\mu$ is given by the injection $i_{\mu, \lambda} : J_\mu \to J_\lambda$, where the domain of $p_{\lambda, \mu}$ is the image of $i_{\mu, \lambda}$, and in the domain $p_{\lambda, \mu}$ is the inverse of $i_{\mu, \lambda}$. Thus, $\hat{\Gamma}$ is isomorphic to $X^\wedge$, $\hat{\Gamma}$ is the inductive limit of the character groups $\hat{X}_\lambda$. The Bose-Mesner algebra is isomorphic to $C_c(\hat{\Gamma})$, the space of finite linear combinations of characters of $X^\wedge$, which is isomorphic to $C_c(\hat{\Gamma})$, the space of locally constant functions on $\hat{\Gamma} \simeq X^\wedge$. This example is outside of the scope of Barg-Skriganov theory, if $\hat{\Gamma} \simeq X^\wedge$ is uncountable.

Remark 4.18. Since $\hat{\Gamma}$ is discrete, $C_c(\hat{\Gamma})$ has a natural (and orthogonal) basis consisting of $E_j$ ($j \in \hat{\Gamma}$), but for the canonically isomorphic vector space $C_c(\hat{\Gamma})$, we do not have a natural basis, since $\hat{\Gamma}$ is compact with (possibly) uncountable cardinality. This makes a definition of eigenmatrices difficult.

4.2. Kernel schemes. Another example of profinite association schemes is a projective system of the kernel schemes. The kernel schemes are finite association schemes defined in Martin-Stinson\textsuperscript{16} to study $(t, m, s)$-nets. The $(t, m, s)$-nets are introduced by Niederreiter for quasi-Monte Carlo integrations, see\textsuperscript{18}.

We recall the kernel scheme, with a slight modification on the notation for $I$, to make a description as a projective system easier.

Definition 4.19. Let $n$ be a positive integer, and $V$ be a finite set of alphabet with cardinality $v \geq 2$. Let $X_n$ be $V^n$, and $I_n := \{1, 2, \ldots, n\} \cup \{\infty\}$. Define $R_n : X_n \times X_n \to I_n$ as follows. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be elements of $X_n$. Let $R(x, y)$ be the smallest index $i$ for which $x_i \neq y_i$. If $x = y$, then $R(x, y) = \infty$. This is a symmetric (and hence commutative) association scheme, with $R^{-1}(\infty)$ being the identity relation. This is called a kernel scheme, and denoted by $k(n, v)$.

A direct computation of the intersection numbers shows that this is a commutative association scheme, as shown in\textsuperscript{16}. We shall give an indirect proof in Corollary 4.27. (This is only for the self-containedness.)
Proposition 4.20. The kernel schemes \( k(n, v) \) forms a projective system, where \( X_{n+1} \to X_n \) is given by deleting the right most component, and \( I_{n+1} \to I_n \) is given by \( m \mapsto m \) for \( m \leq n \), and the both \( n+1, \infty \) are mapped to \( \infty \). Then \( X^\wedge = V^{\mathbb{N}_0} \) holds, and \( \text{pr}_n : X^\wedge \to X_n \) is obtained by taking the left (first) \( n \) components, while \( F^\wedge = \mathbb{N}_0 \cup \{ \infty \} \), with topology obtained by: each \( n \in \mathbb{N}_0 \) is clopen, and an open set containing \( \infty \) is a complement of a finite subset of \( \mathbb{N}_0 \), that is, \( F^\wedge \) is the one-point compactification of \( \mathbb{N}_0 \). We call this a pro-kernel scheme and denote by \( k(\infty, v) \).

Proof. It is easy to check that these make a projective system of association schemes. The topology of \( F^\wedge \) comes from the definition of projective limit topology. \( \Box \)

Remark 4.21. In this case \( F^\wedge \) is countable, and this is a special case of infinite association schemes examined in detail, named metric schemes, by Barg-Skriiganov [2, Section 8], for which the adjacency algebra is defined. In particular, in (8.33) they showed that it is an inductive limit of finite dimensional adjacency algebra, which indeed coincides with our definition.

Remark 4.22. This is merely an observation. For the set of \( p \)-adic integers \( \mathbb{Z}_p \), we have a valuation \( v : \mathbb{Z}_p \to \mathbb{N} \cup \{ \infty \} \) giving an ultrametric. The above pro-kernel scheme is isomorphic to this metric, up to the difference by one on the metric, i.e.,

\[
R : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{N} \cup \{ \infty \}, (x, y) \mapsto v(x - y).
\]

The same holds for a ring of formal power series \( \mathbb{F}_q[[t]] \), where \( \mathbb{F}_q \) denotes the finite field of \( q \) elements. The both yield the isomorphic pro-kernel schemes if \( p = q \).

To see \( J^\wedge \) of the pro-kernel scheme, we need to work with primitive idempotents. All necessary results are in Martin-Stinson[16], but because of the choice of the index \( I \) and normalization of the products necessary for making the projective system, the obtained constants slightly differ. To help the readers’ understandings, we recall the methods of Martin-Stinson. For the chosen integer \( v \geq 2 \), let \( V := \mathbb{Z}_v \) be the additive group \( \mathbb{Z}/v \). Then \( X = \mathbb{Z}_v^\wedge \) is an additive group, and the kernel scheme is a translation scheme [4 §2.10].

Definition 4.23. Let \( X \) be a finite abelian group. A translation scheme is an association scheme \( R : X \times X \to I \), which factors through \( X \times X \to X, (x, y) \mapsto y - x \), namely, there is an \( S : X \to I \) such that \( R(x, y) = S(y - x) \).

Definition 4.24. Return to the kernel scheme. We define \( \text{bot} : X_n \to I_n \) by

\[
\text{bot}(x_1, \ldots, x_n) := \min \{ i \mid x_i \neq 0 \},
\]

and \( \text{bot}(0, \ldots, 0) = \infty \). Then we have

\[
R_n : X_n \times X_n \to I_n, \quad (x, y) \mapsto \text{bot}(x - y).
\]

By \( S := \text{bot} \), the kernel scheme is a translation scheme.

Let \((X, R, I)\) be a translation scheme. For \( i \in I \), let \( S_i \subset X \) be \( S^{-1}(i) \) (\( S \) for the sphere). Let \( \hat{X} \) be the dual group of \( X \). Then, \( \xi \in \hat{X} \) is an orthonormal basis of \( C(X) \) under the normalized inner product in Definition 3.10. Let \( A_i \) denote the \( i \)-th adjacency matrix, and \( \chi_i : X \to \{0, 1\} \) be the indicator function for \( S_i \). Our
action in Theorem 2.5 shows that for $x \in X$

$$(A_i \cdot \xi)(x) = \frac{1}{\#X} \sum_{y \in X} A_i(x, y)\xi(y)$$

$$= \frac{1}{\#X} \sum_{y \in X} \chi_i(y-x)\xi(y)$$

$$= \frac{1}{\#X} \sum_{y \in x+S_i} \xi(y)$$

$$= \frac{1}{\#X} \sum_{a \in S_i} \xi(x+a)$$

$$= \frac{1}{\#X} (\sum_{a \in S_i} \xi(a))\xi(x).$$

Thus $\xi$ is an eigenvector with eigenvalue

$$\frac{1}{\#X} (\sum_{a \in S_i} \xi(a)). \quad (4.6)$$

The primitive idempotents are projectors to the common eigen spaces of $A_i$ for all $i \in I$. Let us assume $X = \mathbb{Z}_v^n$, and then $\hat{X} = \hat{Z}_v^n$, where the product of $\hat{Z}_v$ is written multiplicatively with unit 1. Thus,

$$(\xi_1, \ldots, \xi_n)(x_1, \ldots, x_n) = \prod_{i=1}^n \xi_i(x_i). \quad (4.7)$$

**Definition 4.25.** For $\xi = (\xi_1, \ldots, \xi_n) \in \hat{X}$, define $\text{top}(\xi)$ as the max $i$ such that $\xi_i \neq 1$. If $\xi = 1$, then $\text{top}(\xi) = 0$. Thus $\text{top} : \hat{X} \to \{0, 1, \ldots, n\}$. We shall denote $J_n := \{0, 1, \ldots, n\}$, because we shall soon show a natural correspondence between the set of primitive idempotents and $J_n$.

**Lemma 4.26.** For $\xi$ with $j = \text{top}(\xi)$, we denote by $p_i(j)$ the eigenvalue of $A_i$ for $\xi$. Then

$$p_i(j) = \begin{cases} v^{-n} & \text{if } i = \infty, \\ (v-1)v^{-i} & \text{if } j < i < \infty, \\ -v^{-i} & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}$$

Thus, for any $A_i$, the eigenvalue of $\xi \in \hat{X}$ depends only on $\text{top}(\xi)$, and if $\text{top}(\xi) \neq \text{top}(\xi')$, then there is an $i$ such that $A_i$ has different eigenvalues for $\xi$ and $\xi'$.

**Proof.** Recall that $I = \{1, 2, \ldots, n\} \cup \{\infty\}$. Put

$$B_i := \bigcup_{i \leq \ell \leq \infty} S_\ell.$$  

($B_i$ is the ball consisting of all the elements $(x_1, \ldots, x_n)$ with $x_\ell = 0$ for all $\ell < i$, with the cardinality $v^{n-i+1}$ for $i < \infty$ and 1 for $i = \infty$.) Fix $i$ and $\xi \in \hat{X}$ with
top(ξ) = j, then by (4.6) we have

\[ \tau(i, j) \xi := \sum_{i \leq \ell \leq \infty} A_\ell \cdot \xi \]
\[ = \frac{1}{\nu^n} \sum_{a \in B_i} \xi(a) \xi \]
\[ = \begin{cases} v^{-n} \xi & \text{if } i = \infty, \\ v^{-i+1} \xi & \text{if } j < i < \infty, \\ 0 & \text{if } j \geq i. \end{cases} \]

The last equality follows from that for \( i = \infty \), \( B_\infty = \{0\} \) and hence the summation is \( \sum_{a \in B_i} \xi(a) = 1 \), and if \( j < i < \infty \), then (4.7) is always 1 and the cardinality of \( B_i \) is \( v^{n-1} \), while if \( j \geq i \), \( \xi(a) \) depends on \( a \) and the summation over \( a \in B_i \) is zero, since \( \xi \) is a non-trivial character on the abelian group \( B_i \). Thus, the eigenvalue of \( A_i \) for \( \xi \) is the summation over \( S_i = B_i \setminus B_{i+1} \), that is,

\[
\begin{cases}
  v^{-n} & \text{if } i = \infty, \\
  (\tau(i, j) - \tau(i + 1, j)) = (v - 1)v^{-i} & \text{if } j < i < \infty, \\
  (\tau(i, j) - \tau(i + 1, j)) = -v^{-i} & \text{if } i = j, \\
  0 & \text{if } j > i,
\end{cases}
\]

as desired. \qed

**Corollary 4.27.** The kernel scheme \( \bar{\kappa(n, v)} \) is a commutative association scheme. A primitive idempotent is the projection to the subspace \( C(X)_j \) of \( C(X) \) spanned by \( \{ \xi \mid \text{top}(\xi) = j \} \) for each \( j \in J_n \), which identifies the set of primitive idempotents with \( J_n \).

**Proof.** Recall that we did not use the property of association schemes for the kernel schemes so far. Every \( A_i \) has \( C(X)_j \) as an eigenspace with an eigenvalue. Since \( C(X) \) is a direct sum of \( C(X)_j \) for \( j \in J_n \), this implies that the matrices \( A_i \) mutually commute. We consider the algebra generated by \( A_i \) by the matrix product. Each element of this algebra is determined by the \( n+1 \) eigenvalues corresponding to \( J_n \). Thus, the dimension of the algebra as a vector space does not exceed \( n+1 \), but \( A_i \)'s are \( n+1 \) linearly independent matrices, hence their linear combination is closed under the matrix multiplication. The rest of the axioms of association schemes are easy to verify. The description of primitive idempotents immediately follows. \qed

**Corollary 4.28.** The multiplicity \( m_j \) is 1 for \( j = 0 \) and \( m_j = (v - 1)v^{j-1} \) for \( j > 0 \). The valency \( k_i \) is 1 for \( i = \infty \) and \( k_i = (v - 1)v^{i-1} \) for \( i < \infty \).

**Proof.** The multiplicity is the number of \( \xi \) with \( \text{top}(\xi) = j \), which implies the first statement. The valency is the cardinality of the sphere \( S_i \), which implies the second. \qed

**Corollary 4.29.** For \( j \in J_n \), the projector \( E_j \) is given by

\[
E_j(x, y) = \sum_{\xi \in X, \text{top}(\xi) = j} \xi(x) \xi(y)
\]
\[ = \sum_{\xi \in X, \text{top}(\xi) = j} \xi(x - y).\]
We have

\[ A_i = \sum_{j \in J_n} p_i(j) E_j \]

where \( p_i(j) \) is defined in Lemma 4.26.

**Proof.** The first half describes the projection to the eigenspace corresponding to \( j \) by an orthonormal basis. The second is because \( A_i \) has the eigenvalue \( p_i(j) \) for that eigenspace. \( \square \)

For \( x - y \in S_i \), we define

\[ q_j(i) := E_j(x, y) = \sum_{\xi \in X, \text{top}(\xi) = j} \xi(x - y) \]

\[ = \begin{cases} 1 & \text{if } 0 = j \\ (v - 1)v^{j-1} & \text{if } 0 < j < i \leq \infty \\ -v^{j-1} & \text{if } j = i \\ 0 & \text{if } j > i. \end{cases} \]

This computation is dual to that for \( p_i(j) \), so omitted.

**Corollary 4.30.**

\[ E_j = \sum_{i \in I} q_j(i) A_i. \]

**Proof.** We evaluate the both sides at \( (x, y) \). The left hand side is \( q_j(i) \) for \( x - y \in S_i \), which is equal to the right hand side, since \( A_i(x, y) = \chi_i(x - y) \). \( \square \)

**Corollary 4.31.** In \( \overline{k(\infty, v)} \), the set \( J^c \) is the inductive limit of \( J_n \to J_{n+1} \), and hence naturally identified with \( \mathbb{N} \).

Now we pass to the pro-kernel scheme, i.e., take the inductive limit of \( A_{k(\infty, v)} \) to obtain \( A_{X^n} = \overline{A_{k(\infty, v)}} \).

**Proposition 4.32.** Let \( X_n \) denote the kernel scheme \( k(n, v) \) with \( I_n \) and primitive idempotents \( J_n \), and \( X^n \) the pro-kernel scheme \( k(\infty, v) \) with \( I^c \) and the primitive idempotents \( J^c \). For \( i \in I^c \setminus \{\infty\} \), we denote by \( A_i \) the image of \( \delta_i \in C_c(I^c) \) by the isomorphism \( C_c(I^c) \to A_{X^n} \). Note that \( \delta_\infty \notin C_c(I^c) \). Through \( A_{X_n} \to A_{X^n} \), \( E_j \) is mapped to \( E_j \), and \( A_i \) for \( i \in \{1, 2, \ldots, n\} \) is mapped to \( A_i \), while \( A_\infty \in A_{X_n} \) is mapped to an element corresponding to the indicator function \( \chi_{\{n+1, \ldots, \infty\}} \in C_c(I^c) \). It is convenient to denote the corresponding element in \( A_{X^n} \) by the symbol

\[ \sum_{i \in \{n+1, \ldots, \infty\}} A_i \]

since

\[ \chi_{\{n+1, \ldots, \infty\}} = \sum_{i \in \{n+1, \ldots, \infty\}} \delta_i \]

holds. (Note that \( A_\infty \) does not exist in \( A_{X^n} \).)

**Proof.** According to Proposition 4.20 we have \( I^c = \mathbb{N}_{>0} \cup \{\infty\} \) and \( J^c = \mathbb{N} \) by Corollary 4.31. Since \( J_n \to J^c \) is an injection, \( E_j \in A_{X_n} \) is mapped to \( E_j \in A_{X^n} \) (see Proposition 3.26). The projection \( I^c \to I_n \) is obtained by mapping \( i \mapsto i \) for
1 \leq i \leq n, and i \mapsto \infty for i > n. Since the indicator function \( \delta_i \in C(I_n) \) is mapped to \( \delta_X \in C_{lc}(I^0) \) for \( 1 \leq i \leq n \), \( A_1 \) is mapped to \( A_0 \). The description of the image of \( A_\infty \) is by the definition of \( A_X \rightarrow A_{X^0} \), which comes from \( I^0 \rightarrow I_n \).

Everything is well-defined, except \( i_0 = \infty \in I^0 \). In fact, the operators \( A_i, E_j \), the values \( p_i(j), q_j(i), \mu_j(j) = m_j, \mu_j(i) = k_i/\#X = (v-1)v^{-1} \) are all well-defined, except that \( A_\infty \) and \( p_\infty(j) \) are not well-defined, since there is no corresponding element to \( A_\infty \) in \( A_X^0 \). Indeed, \( A_\infty \) would correspond to the indicator function of \( \infty \in I^0 \), which is not continuous and hence is outside \( A_X^0 \). This forces to choose something which replaces \( A_\infty \). One possible way is to replace it with \( E_0 \) because

\[
E_0 = \sum_{i \in I^0} A_i
\]

holds as a function (constant value 1) on \( I^0 \). then

\[
\{ A_i \mid i \in \mathbb{N}_{>0} \} \cup \{ E_0 \}
\]
is a linear basis of \( C_{lc}(I^0) \), although this is not an orthogonal basis. This is the method chosen in [2, (8.3), (8.9)] to replace an infinite sum with a finite sum (\( \alpha_0 \) in (8.8) in their paper is nothing but \( E_0 \)), which gives a precise definition of the adjacency algebra. In our interpretation in Proposition 4.32, the expression in Corollary 4.29 becomes

\[
A_i = \sum_{j \in I^0} p_i(j)E_j
\]

which is a finite sum (see Lemma 4.26) and holds for \( i \neq \infty \) (since the equality holds in \( A_X^0 \)), except the problem that for \( i = \infty \), the value \( p_\infty(j) = v^{-n} \) converges to 0, while the summation is over infinitely many \( E_j \). In Corollary 4.30, the expression

\[
E_j = \sum_{i \in I^0} q_j(i)A_i
\]

seems an infinite sum, but \( q_j(i) = (v-1)v^{j-1} \) is constant for \( i > j \), and thus the right hand side lies in \( C_{lc}(I^0) \), or more precisely, the seemingly infinite sum is a finite sum by the usage of the symbol in Proposition 4.32, and the equality does hold (since it holds in \( A_X^0 \)). This may be translated into a finite summation formula for \( j \geq 1 \):

\[
E_j = \sum_{i \leq j} (q_j(i) - (v-1)v^{j-1})A_i + (v-1)v^{j-1}E_0
\]

\[
= \left( \sum_{i < j} - (v-1)v^{j-1}A_i \right) - v^jA_j + (v-1)v^{j-1}E_0.
\]

4.3. Ordered Hamming schemes. Martin and Stinson [16, Section 1.3] introduced ordered Hamming schemes as Delsarte’s extension of length \( s \) [2, Section 2.5] of the kernel schemes.

**Proposition 4.33.** (Extension of length \( s \))

Let \( R : X \times X \rightarrow I \) be a commutative association scheme. Let \( s \) be a positive integer, and \( S_s \) a symmetric group of degree \( s \), acting on the direct product \( I^s \). We obtain a quotient map \( I^s \rightarrow I^s/S_s \). Then, the composition

\[
X^s \times X^s \rightarrow I^s \rightarrow I^s/S_s
\]
is a commutative association scheme. For \( i \in I^s/S_s \), the corresponding Hadamard primitive idempotent is the sum of

\[
A_{i_1} \otimes \cdots \otimes A_{i_s}
\]

over \((i_1, \ldots, i_s) \in I^s \) which maps to \( i \). The set of primitive idempotents are naturally identified with \( J^s/S_s \), as follows. For \( j \in J^s/S_s \), the corresponding primitive idempotent is the sum of

\[
E_{j_1} \otimes \cdots \otimes E_{j_s}
\]

over \((j_1, \ldots, j_s) \in J^s \) which maps to \( j \). The corresponding eigenspace is the direct sum of the tensors of the eigenspaces

\[
C(X)_{j_1} \otimes \cdots \otimes C(X)_{j_s}
\]

over \((j_1, \ldots, j_s) \in J^s \) which maps to \( j \).

A proof is not given in the paper by Delsarte, but found in an unpublished (but reachable) paper by Godsil [9, 3.2 Corollary]. Because he did not mention the eigenspaces, which we need later, we shall recall a proof.

**Proof.** Let

\[
X^s \times X^s \to I^s
\]

be the direct product association scheme. Its Bose-Mesner algebra \( A_{X^s} = A_X^{\otimes s} \) is generated by the \( s \)-fold kronecker products of \( A_i \)'s for \( i \in I \), which satisfies the axiom of association scheme. The symmetric group \( S_s \) acts on \( A_{X^s} \) by permutation. We take the fixed part \( A_{X^s}^{S_s} \) of \( A_{X^s} \). Since \( S_s \) preserves the two products and the transpose, fix the \( i_0 \) and the \( j_0 \), \( A_{X^s}^{S_s} \) is a Bose-Mesner algebra. As a vector space, this is canonically isomorphic to \( C(I^s)^{S_s} \cong C(I^s/S_s) \), induced by \( I^s \to I^s/S_s \). For an \( i \in I^s/S_s \), its inverse image is the set of \((i_1, \ldots, i_s) \) mapped to \( i \), and the set of \( A_{i_1} \otimes \cdots \otimes A_{i_s} \) are mutually disjoint Hadamard idempotent, whose sum is an Hadamard idempotent \( A_i \). This must be primitive in \( A_{X^s}^{S_s} \), since the Hadamard product satisfies \( A_i \circ A_i = \delta_i \cdot I \), and their sum is the Hadamard unit \( J_{A_{X^s}^{S_s}} \). The same argument applies for the primitive idempotents \( E_j \). The description of the corresponding eigenspace follows. \( \square \)

This construction, when applied to the kernel schemes \( X_n, I_n = \{1, 2, \ldots, n\} \cup \{\infty\} \), and \( R_n : X_n \times X_n \to I_n \) given Definition [4.19], yields an ordered Hamming scheme.

**Definition 4.34.** The ordered Hamming scheme, denoted by \( \overrightarrow{H}(s, n, v) \), is defined as the extension of length \( s \) of the kernel scheme \( k(n, v) \).

The following is easy to check.

**Proposition 4.35.** The projections \( X_{n+1}^s \to X_n^s \) and \( I_{n+1}^s/S_s \to I_n^s/S_s \) coming from Proposition [4.20] give a projective system \( \overrightarrow{H}(s, n+1, v) \to \overrightarrow{H}(s, n, v) \) of commutative association schemes, namely, a profinite association scheme, which we call the pro-ordered Hamming scheme and denote by \( \overrightarrow{H}(s, \infty, v) \). The \( \overrightarrow{I} \) of this projective association schemes is \( (\mathbb{N}_{>0} \cup \{\infty\})^s/S_s \), and \( \overrightarrow{J} \) is \( \mathbb{N}^s/S_s \).

We shall mention on this profinite scheme later in the last of Section [5.3].
5. Delsarte theory for profinite association schemes

Here we extend Delsarte theory introduced in [6]. The method here follows Kurihara-Okuda [13], which generalizes Delsarte theory to compact homogeneous spaces. In this section, let \((X_\lambda, R_\lambda, I_\lambda)\) be a profinite association scheme, and \(X^\wedge, F^\wedge, J^\wedge, A_{X^\wedge}\) be those defined in Section 3.1.

5.1. Multiset and Averaging functional. We consider a finite multi-subset \(Y\) of \(X^\wedge\), which means that \(Y\) is a set in which finite multiplicity of elements is allowed and taken into account. To make the notion rigorous, we consider a map from a finite set \(Z\) to a set \(X\):

\[ g: Z \to X. \]

Then the image \(Y := g(Z)\) in \(X\) has a natural finite multiset structure, where the multiplicity of \(y \in Y\) is the cardinality of the fiber \(#g^{-1}(y)\). For \(S \subset X\), we use the notation

\[ #(Y \cap S) := #((g^{-1}(S)). \]

This merely means to count the number of elements in \(Y \cap S\) with taking the multiplicity into account. Thus, we call \(Y\) a “multi-subset” of \(X\), and use the notation

\[ Y \subset X \] by an abuse of language.

For any non-empty finite multiset \(Y \subset X^\wedge\), we would like to describe the notions of codes and designs. We begin with defining the averaging functional.

**Definition 5.1.** Let \(Y \subset X^\wedge\) be a finite multi-subset in the sense above. Define the averaging functional

\[ \text{avg}_Y : C(X^\wedge) \to \mathbb{C}, \quad f \mapsto \frac{1}{#Y} \sum_{x \in Y} f(x) := \frac{1}{#Z} \sum_{z \in Z} f(g(z)). \]

By (3.1), \(C_c(J^\wedge) \cong A_{X^\wedge} \cong C_{lc}(I^\wedge)\) holds, and by

\[ A_{X^\wedge} \to C(X^\wedge \times X^\wedge) \xrightarrow{\text{avg}_Y \otimes \text{avg}_Y} C, \]

\[ \text{avg}_Y^2 := \text{avg}_Y \otimes \text{avg}_Y \] defines a functional on \(A_{X^\wedge}\), on \(C_c(J^\wedge)\), and on \(C_{lc}(I^\wedge)\). For example, for \(f \in C_{lc}(F^\wedge)\), we have

\[ \text{avg}_Y^2(f) = \frac{1}{#Y^2} \sum_{x,y \in Y} f \circ R^\wedge(x, y) \]

\[ = \frac{1}{#Y^2} \sum_{x \in F} \sum_{y \in Y, R^\wedge(x, y) = i} f(i) \]

\[ = \frac{1}{#Y^2} \sum_{i \in F} #((Y \times Y) \cap R^\wedge^{-1}(i)) f(i). \]

Note that these coefficients are the inner-distribution of \(Y\) (Delsarte [6, Section 3.1]) multiplied by \(\frac{1}{#Y}\). Let us denote by \(C_{\oplus F^\wedge}\) the vector space whose basis is \(F^\wedge\), namely, the space of finite linear combinations of the elements of \(F^\wedge\). We have a mapping

\[ C_{\oplus F^\wedge} \to C_{lc}(F^\wedge)^\vee, \quad (5.1) \]

where \(\vee\) denotes the dual (i.e. \(\text{Hom}(C_{lc}(F^\wedge), \mathbb{C})\)), by the evaluation at \(i\):

\[ i \mapsto (f \mapsto f(i)), \]

which is injective since only a finite number of linear combinations appear in
\( C^{\oplus F^*} \), and their support can be separated by clopen subsets. The above computation shows that \( \text{avg}^2_Y \in C_{lc}(F^*)^\vee \) lies in \( C^{\oplus F^*} \), namely,

\[
\text{avg}^2_Y = \sum_{i \in I} \frac{1}{\#Y^2} \#((Y \times Y) \cap R^{n-1}(i)) \cdot i \in C^{\oplus F^*}. \tag{5.2}
\]

Note that every coefficient is non-negative. Next we compute \( \text{avg}^2_Y \) on \( C_{lc}(J^*) \). For \( j \in J^* \), we have the orthogonal projector \( E_j : C(X^*) \to C(X^*)_j \). For \( f \in C(X^*)_j \),

\[
f(x) = \int_{y \in X^*} E_j(x, y)f(y)d\mu(y) = (f, E_j(x, -))_{HS}.
\]

Thus, we define \( \text{avg}^j_Y \in C(X^*)_j \) by

\[
\text{avg}^j_Y(-) := \frac{1}{\#Y} \sum_{x \in Y} E_j(x, -), \tag{5.3}
\]

which represents the averaging functional in \( C(X^*)_j \):

\[
(f, \text{avg}^j_Y)_{HS} = \text{avg}^j_Y(f). \tag{5.4}
\]

Then

\[
\text{avg}^2_Y(E_j) = \frac{1}{\#Y^2} \sum_{x, y \in Y} E_j(x, y) = \frac{1}{\#Y} \sum_{y \in Y} \text{avg}^j_Y(y) = (\text{avg}^j_Y, \text{avg}^j_Y) = ||\text{avg}^j_Y||^2_{HS} \geq 0. \tag{5.5}
\]

These positivities make the LP method by Delsarte possible [6, Theorem 3.3].

**Definition 5.2.** Let us denote by \( Q_j \in C_{lc}(F^*) \) the image of \( E_j \in A_{X^*} \). This may be considered as a description of the canonical isomorphism

\[
Q : C_{lc}(J^*) \to C_{lc}(F^*), \quad \delta_j \mapsto Q_j,
\]

where \( \delta_j \) means the indicator function on \( J^* \) at \( j \in J^* \). (Note that \( E_j \in A_{X^*} \) corresponds to \( \delta_j \in C_{lc}(J^*) \) in Theorem 3.13; see Corollary 3.11). Then we have an injection

\[
C^{\oplus F^*} \to C_{lc}(F^*)^\vee \xrightarrow{Q^\vee} C_{lc}(J^*)^\vee,
\]

obtained by

\[
i \mapsto (f \mapsto f(i)) \mapsto (\delta_j \mapsto Q_j(i)).
\]

**Definition 5.3.** The above morphism \( Q^\vee|_{F^*} : C^{\oplus F^*} \to C_{lc}(J^*)^\vee \) is called the MacWilliams transform. In concrete, it maps

\[
Q^\vee|_{F^*} : \sum_{i \in I} a_i \cdot i \mapsto \sum_{j \in J^*} \sum_{i \in F^*} a_i Q_j(i) \text{ev}_j,
\]

where \( \text{ev}_j \) is the dual basis, i.e., \( \text{ev}_j(\delta_{j'}) = \delta_{jj'} \), or equivalently,

\[
\text{ev}_j : C_{lc}(J^*) \to \mathbb{C}, f \mapsto f(j).
\]

Note that the left is a finite sum, but the right may be an infinite sum (which causes no problem, since the dual of a direct sum is the direct product).
The following is a formal consequence:

$$Q'|_F : \text{avg}_Y^j \in \mathbb{C}^{\oplus F} \mapsto \text{avg}_Y^j \in C_c(J^\vee)^\vee, \quad (5.6)$$

5.2. Codes and designs. We define codes and designs. Let $I_C \subset F^\vee$ be a subset which does not contain $i_0$ ($C$ for code). Let $J_D \subset J^\vee$ be a subset which does not contain $j_0$ ($D$ for design). We define a convex cone $(I_C; J_D) := (I_C; J_D) \subset \mathbb{C}^{\oplus F}$ by

$$(I_C; J_D) := \left\{ \sum_{i \in I} a_i \cdot i \mid a_i = 0 \text{ for all but finite } i, \right.$$  

$$a_i \geq 0 \text{ for all } i \in F^\vee,$$

$$a_i = 0 \text{ for all } i \in I_C,$$

$$\sum_{i \in F^\vee} a_i q_j(i) \geq 0 \text{ for all } j \in J^\vee,$$

$$\sum_{i \in F^\vee} a_i q_j(i) = 0 \text{ for all } j \in J_D \right\}.$$

**Definition 5.4.** A non-empty finite multi-subset $Y \subset X^\wedge$ is called an $I_C$-free-code-$J_D$-design if $\text{avg}_Y^j$ lies in the cone $(I_C; J_D)$. Furthermore, an $I_C$-free-code-$\emptyset$-design [resp. an $\emptyset$-free-code-$J_D$-design] is simply called an $I_C$-free-code [resp. a $J_D$-design].

More explicitly, the $i$-component of $\text{avg}_Y^j$ in $\mathbb{C}^{\oplus F^\vee}$ is

$$a_i(Y) := \frac{1}{\# Y^2 \#((Y \times Y) \cap R^\wedge^{-1}(i))}, \quad (5.7)$$

which is non-negative [5.2] and required to be 0 for $i \in I_C$ (namely, there is no pair $(x, y) \in Y \times Y$ with relation $R^\wedge(x, y) \in I_C$), and its ev$_j$-component in $C_c(J^\vee)^\vee$ is

$$b_j(Y) := \text{avg}_Y^j(E_j) = \| \text{avg}_Y^j \|^2_{\text{HS}}, \quad (5.8)$$

which is non-negative [5.3] and required to be 0 for $j \in J_D$ (namely, the $j$-component of $\text{avg}_Y$ is zero for $j \in J_D$, or equivalently, for any $f \in C(X^\wedge)$, $\sum_{y \in Y} f(y) = 0$ for $j \in J_D$). Note that $i_0$ is removed from $I_C$, since

$$\frac{1}{\# Y^2 \#((Y \times Y) \cap R^\wedge^{-1}(i_0))} = \frac{1}{\# Y}$$

is positive, and $j_0$ is removed since $\| \text{avg}_Y^j \|_{\text{HS}} = 1$ (being the operator norm of the averaging for constant functions). Since

$$\# Y \text{avg}_Y^j(i_0) = 1 \text{ and } \sum_{i \in F^\vee} \# Y \text{avg}_Y^j(i) = \# Y,$$

we may consider an LP problem: under the constraint that $(a_i)_{i \in F^\vee}$ lies in the cone $(I_C; J_D)$ and $a_{i_0} = 1$, maximize/minimize $\sum_{i \in F^\vee} a_i$, which gives an upper/lower bound on the cardinality of $Y$ which is an $I_C$-free-code-$J_D$-design. The following is a formal consequence of [5.7], stating the relation with classic theory [8] Section 3).

**Theorem 5.5.** The Mac-Williams transformation $Q'|_F$ maps $\{5.7\}$ to $\{5.8\}$:

$$\sum_{i \in F^\vee} a_i(Y) \cdot i \mapsto \sum_{j \in J^\vee} b_j(Y)_{\text{ev}}_j.$$
In concrete,
\[ b_j(Y) = \sum_{i \in I} a_i(Y)Q_j(i). \]

**Remark 5.6.** The meaning of the values \( b_j(Y) \) is clear, and called the inner distribution \[6, Section 3.1\] (up to a factor of \( \#Y \)), but there, the meaning of the values \( b_j(Y) = \sum_{i \in I} a_i(Y)Q_j(i) \) is not clear, just the semi-positivity is proved. Now the value \( b_j(Y) = \| \text{avg}_Y^j \|_{\text{HS}}^2 \) has a clear interpretation. On the space \( C(X^\infty)_j \), the operator norm of \( \text{avg}_Y^j \) is \( \| \text{avg}_Y^j \|_{\text{HS}} \), since

\[
\sup_{f \in C(X^\infty)_j \setminus \{0\}} \frac{|\text{avg}_Y^j(f)|}{\|f\|} = \sup_{f \in C(X^\infty)_j \setminus \{0\}} \frac{(f, \text{avg}_Y^j)}{\|f\|} \leq \frac{\|f\| \cdot \|\text{avg}_Y^j\|}{\|f\|} = \|\text{avg}_Y^j\|,
\]

and the equality holds for \( f = \text{avg}_Y^j \). This may be interpreted as the worst-case error in approximating the integration for \( C(X^\infty)_j \) by \( \text{avg}_Y^j \) (where the true integration value is zero for \( j \neq j_0 \), because of the orthogonality to the constant functions \( C(X^\infty)_{j_0} \)), see a comprehensive book by Dick-Pillichshammer \[7, Definition 2.10\] on quasi-Monte Carlo integration. Another remark: the injectivity \(5.1\) implies that \( Y \) and \( Y' \) have the same inner distribution

\[
\#((Y \times Y) \cap R^{i-1}(i)) = \#((Y' \times Y') \cap R^{i-1}(i)) \text{ for all } i \in I^\infty
\]

if and only if

\[
\| \text{avg}_Y^j \|_{\text{HS}} = \| \text{avg}_{Y'}^j \|_{\text{HS}} \text{ for all } j \in J^\infty.
\]

5.3. \((t, m, s)\)-nets and \((t, s)\)-sequences. The \((t, m, s)\)-nets are point sets for quasi-Monte Carlo integration, well-studied, see \[18\]. For \((t, m, s)\)-nets, the LP bound by ordered Hamming schemes introduced by Martin-Stinson \[16\] yields strong lower bounds on the cardinality of the point set \( P \), see \[16\] and Bierbrauer \[3\] for example. We do not have such an application, but profinite association schemes may be used to formalize the notion of \((t, s)\)-sequence \[18\], as mentioned at the last of this section.

The purpose of this section is to reprove some results of Marin-Stinson, and see a relation to pro-ordered Hamming schemes. The main result Corollary \[5.19\] of this section is proved in Martin-Stinson \[16, Theorem 3.4\] by using the notion of dual schemes and weight enumerator polynomials. Here we shall give a self-contained proof avoiding the use of these tools. Our proof matches to the formalization in Section \[5.2\] relying on the average functional \( \text{avg}_Y^j \) introduced in Definition \[5.1\].

Recall the kernel scheme in Definition \[4.19\] where the set of alphabets is \( Z_v := Z/\nu \), and \( X_n = Z_v^n, I_n = \{1, 2, \ldots, n\} \cup \{\infty\} \),

\[
\text{bot} : X_n \to I_n
\]

in Definition \[4.24\] give the kernel scheme as a translation scheme. We shall identify a point in \( X_n \) with a point in the real interval \([0, 1]\), by

\[
(x_1, \ldots, x_n) \mapsto 0.x_1x_2 \cdots x_n \in [0, 1),
\]

where the right expression means the \( v \)-adic decimal expansion. Then, we have a map

\[
X_n^s \to [0, 1]^s.
\]
We shall define \((t, m, s)\)-nets following Niederreiter [17, Definition 4.1]. An elementary interval of type \((d_1, \ldots, d_s)\) is a subset of \([0, 1]^s\) of the form
\[
\prod_{i=1}^{s}[a_i v^{-d_i}, (a_i + 1)v^{-d_i}),
\]
where \(a_i\) and \(d_i\) are non-negative integers such that \(a_i < v^{d_i}\). Its volume is
\[
\prod_{i=1}^{s} v^{-d_i} = v^{-\sum_{i=1}^{s} d_i}.
\]

**Definition 5.7.** Let \(0 \leq t \leq m\) be integers. A \((t, m, s)\)-net in base \(v\) is a multi-subset \(P \subset [0, 1]^s\) consisting of \(v^m\) points, such that every elementary interval of type \((d_1, \ldots, d_s)\) with \(d_1 + \cdots + d_s = m - t\) (hence volume \(v^{t-m}\)) contains exactly \(v^t\) points of \(P\).

Our purpose is to state these conditions in terms of designs \(Y \subset X^s\) for a finite multi-subset \(Y\).

**Definition 5.8.** A finite multi-subset \(Y\) in \(X\) is said to be uniform on \(X\) if \(#(Y \cap \{x\})\) is constant for \(x \in X\), that is, \(#g^{-1}(x)\) is constant for \(x \in X\) if \(Y\) is defined by a map \(g : Z \to X\).

**Definition 5.9.** For an \(s\)-tuple of non-negative integers \(d = (d_1, \ldots, d_s)\) with each component less than or equal to \(n\), define
\[
X_d := \prod_{i=1}^{s} Z_{v_{d_i}},
\]
and the projection
\[
pr_d : X^s_n \to X_d
\]
by taking the left most \(d_i\) components of the \(i\)-th \(X_n\) in \(X^s_n\), for \(i = 1, \ldots, s\).

**Definition 5.10.** A non-empty multi-subset \(Y \subset X^s_n\) is \(d\)-balanced, if the image of \(Y\) in \(X_d\) by the map \(X^s_n \to X_d\) is uniform on \(X_d\).

**Proposition 5.11.** A multi-subset \(Y \subset X^s_n\) of cardinality \(v^m\) gives a \((t, m, s)\)-net if and only if it is \(d\)-balanced for any non-negative tuples \(d = (d_1, \ldots, d_s)\) with \(d_1 + \cdots + d_s = m - t\).

The proof is immediate when we consider the mapping \(X^s_n \to X_d\), since then there is a natural one-to-one correspondence between \(X_d\) and the set of elementary intervals of type \((d_1, \ldots, d_s)\). We may state the conditions in terms of the dual group.

**Lemma 5.12.** Let \(X\) be a finite abelian group, and \(Y\) a finite multi-subset in \(X\). For \(f \in C(X)\), define
\[
I(f) := \text{avg}_X(f) = \frac{1}{\#X} \sum_{x \in X} f(x).
\]
(This is the true integral of \(f\) over the finite set \(X\)). The following are equivalent.

1. \(I(f) = \text{avg}_Y(f)\) holds for any \(f \in C(X)\).
2. The finite multi-subset \(Y\) is uniform on \(X\).
3. \(\text{avg}_Y(\xi) = 0\) holds for any non-trivial \(\xi \in \hat{X}\).
Proof. Suppose that the first condition holds. Let \( \chi_x \) be the indicator function of \( x \in X \). The condition \( I(\chi_x) = \text{avg}_Y(\chi_x) \) implies \( \frac{1}{\#X} = \frac{\#(Y \cap \{x\})}{\#Y} \), hence \( Y \) is uniform on \( X \). The converse is obvious.

Suppose again the first condition. For any non-trivial \( \xi \), \( I(\xi) = 0 \). This implies that \( \text{avg}_Y(\xi) = 0 \). For the converse, we assume the third condition. Note that \( \hat{X} \) is a base of \( C(X) \), so it suffices to show that \( I(\xi') = \text{avg}_Y(\xi') \) holds for any \( \xi' \in \hat{X} \).

This holds for non-trivial \( \xi \), and for the trivial \( \xi = 1 \), \( I(1) = 1 = \text{avg}_Y(1) \). \( \square \)

**Lemma 5.13.** The dual \( \hat{X}_d \) of \( X_d \) can be identified with the set of characters

\[ \{ \xi := (\xi_1, \ldots, \xi_s) \in \hat{X}_n^s \mid \text{top}(\xi_i) \leq d_i \text{ for } i = 1, \ldots, s \} \]

(see Definition 4.29 for the notation \( \text{top} \)).

**Corollary 5.14.** For a multi-subset \( Y \subset X_n^s \), the composition \( Y \to X_n^s \to X_d \) is uniform if and only if

\[ \text{avg}_Y(\xi) = 0 \]

holds for any non-trivial \( \xi \in \hat{X}_d \).

**Corollary 5.15.** A multi-subset \( Y \subset X_n^s \) of cardinality \( v^m \) is a \( (t, m, s) \)-net if and only if

\[ \text{avg}_Y(\xi) = 0 \]

holds for any \( \xi \in \hat{X}_d \setminus \{1\} \) and any non-negative \( \mathbf{d} = (d_1, \ldots, d_s) \) satisfying

\[ \sum_{i=1}^s d_i = m - t. \]

Niederreiter[17] and Rosenbloom-Tsfasman[23] defined the notion of NRT-weight, see also Niederreiter and Pirsic[24].

**Definition 5.16.** Define the NRT-weight \( \text{wt} \) by the composition

\[ \text{wt} : \hat{X}_n^s \xrightarrow{\text{top}} J_n^s \xrightarrow{\text{sum}} \mathbb{N}, \]

where \( \text{sum} \) is a function taking the sum of the \( s \) coordinates and \( \text{top} \) is defined in Definition 4.25.

**Theorem 5.17.** Let \( Y \subset X_n^s \) be a multi-subset of cardinality \( v^m \). Then it is a \( (t, m, s) \)-net if and only if for any \( \xi \in \hat{X}_n \) with \( \text{wt}(\xi) \leq m - t \) and \( \xi \neq 1 \), \( \text{avg}_Y(\xi) = 0 \).

This follows from Corollary 5.15. Following Martin-Stinson[16], let us define

\[ \text{shape} : J_n^s \to J_n^s / S. \]

In Proposition 4.33 and Lemma 4.26, we observed that for \( j \in J_n^s / S \), the corresponding eigenspace \( C(X_n)_j \) is spanned by the characters in the inverse image of \( \overline{j} \) by

\[ X_n^s \xrightarrow{\text{top}} J_n^s \to J_n^s / S. \]

It is easy to see that \( \text{wt} \) factors as

\[ \hat{X}_n^s \xrightarrow{\text{top}} J_n^s \xrightarrow{\text{shape}} J_n^s / S \xrightarrow{\text{sum}} \mathbb{N}, \]

where the definition of \( \text{sum} \) is clear from the context. (We remark that \( \text{sum} \) is denoted by “height” in Martin-Stinson[16 P.337], but we changed the notation to avoid the collision with a standard use of “height” in the context of the Young diagrams.) Thus,

\[ \{ \xi \in \hat{X}_n^s \mid \text{wt}(\xi) \leq m - t \} = \{ \xi \in \hat{X}_n^s \mid \text{sum}(\text{shape}(\text{top}(\xi))) \leq m - t \}. \]
Let $\text{Theorem 5.18.}$ property (5.4)) for these $j$ is the representation of $\text{avg}_Y$ in the right hand side, namely, $\text{avg}_Y = 0$ for each of $\xi \in C(X_n^s)_j$ for $j \in J_n^3/S_s \setminus \{j_0\}$, sumup$(j) \leq m - t$. This is equivalent to $\text{avg}_Y = 0$ (the left hand side is the representation of $\text{avg}_Y$ in the subspace $C(X_n^s)_j$ defined in (5.3) with property (5.4)) for these $j$.

Now we proved [16, Theorem 3.4]:

**Theorem 5.18.** Let $Y \subset X_n^s$ be a multi-subset of cardinality $v^m$. Then, $Y$ is a $(t, m, s)$-net if and only if $\text{avg}_{Y} = 0$ for any $j \neq j_0$, sumup$(j) \leq m - t$.

In terms of the designs (see Section 5.2), we may state

**Corollary 5.19.** Define

$$J_D := \{j \in J_n^3/S_s \mid j \neq j_0, \text{ sumup}(j) \leq m - t\}.$$  

Then, $Y \subset X_n^s$ is a $J_D$-design if and only if it is a $(t, m, s)$-net in $X_n^s$.

By this, Delsarte’s LP method works, see [16]. The next is an example where we may use a profinite association scheme to make a concept of $(t, s)$-sequence inside the design theory, (see [18, Definition 4.2] and [19, Remark 2.2]).

Let us consider the pro-ordered Hamming scheme $H(s, \infty, v) = (X^s, R^s)$ defined in Proposition 4.35. Note that $X^s = (\lim X_n^s)^s = \lim (X_n^s) = (Z_{v^{N^s}})^s$. For each $n$, we denote by $\pi_n$ the surjection from $X_n^s$ onto $X_n^s$. That is,

$$\pi_n(x^1, \ldots, x^s) := ([x^1]_n, \ldots, [x^s]_n) \quad (5.10)$$

for each $(x^1, \ldots, x^s) \in X^s = (Z_{v^{N^s}})^s$, where we put $[(x_1, \ldots)]_n := (x_1, \ldots, x_n)$ for each $(x_1, \ldots) \in Z_{v^{N^s}}$.

**Definition 5.20.** A sequence of points $p_0, p_1, \ldots \in X^s$ is a $(t, s)$-sequence in base $v$, if for any integers $k$ and $m > t$, the point set $Y_{k, m} \subset X_n^s \subset [0, 1]^s$ consisting of the $\pi_n(p_j)$ with $kv^m \leq j < (k + 1)v^m$ (considered as a multi-set) is a $(t, m, s)$-net in base $v$.

For a comparison with a standard Definition 5.22, see Remark 5.23. We may state the condition of $(t, s)$-sequences in terms of the designs in the pro-ordered Hamming scheme $H(s, \infty, v)$. Recall that $J^s = N^s/S_s$, and we may define sumup so that the composition

$$N^s \to J^s = (N^s/S_s) \xrightarrow{\text{sumup}} N$$

maps $(j_1, \ldots, j_s) \mapsto \sum_{i=1}^s j_i$.

**Theorem 5.21.** In the pro-ordered Hamming scheme, we define

$$J_{D_k} := \{h \in (N^s/S_s) = J^s \mid \text{sumup}(h) \leq \ell, h \neq j_0\}.$$  

A sequence of points $p_0, p_1, \ldots \in X^s$ is a $(t, s)$-sequence in base $v$, if and only if for any integers $k$ and $m > t$, the point set $Y_{k, m}$ consisting of the $\pi_n(p_j)$ with $kv^m \leq j < (k + 1)v^m$ is a $J_{D_{m-1}}$-design.

In the original definition, the sequence is taken in an infinite set $[0, 1]^s$, where infinite precision is necessary. Because of this infiniteness, a fixed finite scheme would not be able to describe the notion of $(t, s)$-sequence. Our profinite association scheme is a tool to deal with the infiniteness.
We close our paper with a discussion on the definitions of \((t, s)\)-sequences. The following is a definition by Niederreiter-Özbudak\cite{19} Definition 2.2 [to be precise, its special case], which is a slight variant of the original Niederreiter’s one: \cite{18} Definition 4.2).

**Definition 5.22.** Let \(p_0, p_1, \ldots \in [0, 1]^s\) be a sequence of points, with prescribed \(v\)-adic expansions. (In other words, each \(p_j\) is considered as an element in \(X^\wedge\).) It is a \((t, s)\)-sequence in base \(v\), if for any integers \(k\) and \(m > t\), the point set \(P_{k,m}\) consisting of the \(\pi_m(p_j)\) with \(kv^m \leq j < (k + 1)v^m\) is a \((t, m, s)\)-net in base \(v\).

**Remark 5.23.** It is easy to see that Definitions 5.20 and 5.22 are the same. In Definition 5.22, we use \(\pi_m\) in (5.10) for \(p_j\), where \(p_j\) is an element of \(X^\wedge\). In fact, \(\pi_m\) is not well-defined for \([0, 1]^s\), namely, for base \(v = 2\), for example, \(0.0111\cdots = 0.1000\cdots\). One way to avoid this subtle problem may be to require the \(v\)-adic expansion of a real number to be of the latter type, namely, to avoid expansions consisting of all \((v - 1)\)'s after some digit. However, in important constructions such as in Niederreiter-Xing\cite{21}, points with the former type of expansions may appear. Thus, to define the notion of \((t, s)\)-sequences as a sequence in \([0, 1]^s\) has a subtle problem. Definition 5.20 would be a natural definition of a \((t, s)\)-sequence. In other words, the notion would be better defined in terms of \(X^\wedge\) rather than \([0, 1]^s\). This is essentially stated in \cite{19} Remark 2.2. Of course, a large part of researchers would prefer to work in \([0, 1]^s\).

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