**I. INTRODUCTION**

Inflation [1–8] provides a mechanism for explaining the observed flatness, homogeneity, and the lack of relic monopoles in the early Universe [9–13], as well as generating the inhomogeneities observed in the cosmic microwave background radiation [14–17]. Realistic microphysical models of inflation, in which the expansion of the universe is governed by the energy density of an inflaton field \( \phi \), require the inflaton potential \( U = U(\phi) \) to be approximately flat in order for successful inflation to occur [18]. The inflationary stage lasts until the inflaton field begins to oscillate around the minimum of its potential, decaying into lighter degrees of freedom [19–23].

Bulk viscosity, which phenomenologically acts as a negative pressure term, might arise in the fluid description of a particle ensemble through various mechanisms like inter-particle interaction [24, 25] or the decay of particles within the fluid [26–28]. Dissipative processes, described by the relativistic theory of viscosity [29–31], might have driven inflation, as discussed within the theory of the “standard” inflation theory [32–47], warm inflation [48–53], or the present observed accelerated expansion of the Universe [54–58]. Viable mechanisms for the generation of bulk viscosity from a field description have been suggested [50–52, 56, 59–62].

In the literature, there has been some debate whether bulk viscosity can drive inflation. The authors in Refs. [34–36] add a viscous term to a relativistic fluid to obtain an inflationary stage, but see Ref. [38]. Inflation cannot be supported in a cosmology where the expansion is dominated by a fluid with a perfect gas equation of state [30], but the outcome changes if a different equation of state is used [31, 37, 39–44], or in a two-fluid model [63, 64].

In this Letter we point out that a field whose energy density is mainly in the kinetic component can drive an inflationary period, if a bulk viscosity term \( \zeta \) is present. Splitting the viscosity into a constant \( \zeta_0 \) and a time-varying component \( \zeta_1 \), we find that the constant part

\[ \delta H^2 = \frac{8 \pi G}{3} \rho \quad \text{and} \quad \dot{H} = -4 \pi G (p + \rho), \]

where the expression for \( \dot{H} \) follows from using Eq. (1). Varying the density and pressure fields as

\[ \rho \rightarrow \rho + \delta \rho \quad \text{and} \quad p \rightarrow p + \delta p, \]

the expressions for the first-order perturbation terms are

\[ \delta \dot{H} + 3 H (\delta \rho + \delta p) - 3 (p + \rho) \dot{\Psi} + \frac{\nabla^2}{a^2} (p + \rho) \delta u = 0, \]

where \( \Psi \) is the perturbation in the gravitational field and \( \delta u \) is the total covariant velocity perturbation. Assuming a scale of inflation \( H_1 \ll M_{Pl} \) allows us to neglect the perturbations in the gravitational field in Eq. (4), or [13]

\[ \delta \dot{H} + 3 H (\delta \rho + \delta p) + \frac{\nabla^2}{a^2} (p + \rho) \delta u = 0. \]
III. STANDARD INFLATIONARY MODEL

In the standard theory of inflation [1–17], the action describing the evolution of a single real scalar field ϕ under the arbitrary potential $U = U(\varphi)$ and minimally coupled to the metric $g_{\mu\nu}$ is

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g_{\mu\nu} \hat{\partial} \varphi \hat{\partial} \varphi - U(\varphi) \right],$$

where $g \equiv \det(g_{\mu\nu})$. Here, we consider the mostly negative Friedmann-Robertson-Walker (FRW) metric. From the action $S$ follows the energy-momentum tensor

$$T_{\mu\nu} = \partial \varphi \partial \varphi - g_{\mu\nu} \left[ g^{\rho\sigma} \partial \varphi \partial \varphi - U \right],$$

with the energy density and pressure of the fluid associated to $T_{\mu\nu}$ given respectively by

$$\rho = \frac{1}{2} g_{\mu\nu} \partial \varphi \partial \varphi + U, \quad p = \frac{1}{2} g_{\mu\nu} \partial \varphi \partial \varphi - U.$$

Varying the inflaton field as $\varphi(x) = \phi(x) + \delta \phi(x)$ and using the definitions in Eq. (9), Eq. (1) gives

$$\delta \phi = \rho \hat{\partial} \varphi + U_{\varphi} = 0,$$

In scalar field theories, perturbations in the mode expansion $\delta \phi_k$ of the field operator,

$$\delta \phi(x) = \int \frac{d^3k}{(2\pi)^3} \hat{a}_k \delta \phi_k e^{ik \cdot r} + \text{h.c.},$$

satisfy the differential equation

$$\delta \ddot{\phi}_k + 3H \dot{\delta \phi}_k + \left( \frac{k^2}{a^2} + U_{\varphi \varphi} \right) \delta \phi_k = 0.$$  \hspace{1cm} (12)

During inflation, the Hubble rate in Eq. (2) is approximately constant due to the flatness of the potential $U$ and the fact that the kinetic term is negligible with respect to $U$. These conditions are equivalent to requiring that the two slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \text{and} \quad \eta \equiv \frac{U_{\varphi \varphi}^{\prime}}{3H^2}.$$ \hspace{1cm} (13)

satisfy $\epsilon \ll 1$ and $|\eta| \ll 1$. The latter condition is equivalent to imposing $\dot{\phi} \ll U_{\varphi \varphi}$. In the standard cosmological theory, the slow-roll parameters in Eq. (13) read [65]

$$\epsilon = \frac{1}{16\pi G} \left( \frac{U_{\varphi \varphi}^{\prime}}{U} \right)^2, \quad \text{and} \quad \eta = \frac{1}{8\pi G} \frac{U_{\varphi \varphi}^{\prime}}{\dot{H}}.$$ \hspace{1cm} (14)

Introducing $z = -1/aH$, setting $\delta \phi_k = \chi_k$, and neglecting $U_{\varphi \varphi} = 3H^2\eta \ll H^2$, Eq. (12) gives (\chi_k = d\chi_k/dz)

$$\chi_k^\prime \left( k^2 - 2 + 3(\epsilon - \eta) \right) \chi_k = 0,$$

whose solution (for $\epsilon = \eta = 0$) corresponding to an incoming wave and reducing to a plane-wave for $kz \gg 1$ is

$$\chi_k(z) = e^{-ikz} \left( \frac{1}{\sqrt{2k}} \left( 1 - i \frac{kz}{k} \right) \right).$$  \hspace{1cm} (16)

The inflaton field can thus be expanded in terms of lowering and raising operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ as

$$\hat{\chi}(z, r) = \int \frac{d^3k}{(2\pi)^3/2} \hat{a}_k \chi_k(z) e^{ik \cdot r} + \text{h.c.}. $$  \hspace{1cm} (17)

During inflation ($\dot{\phi} \ll H \dot{\phi}$), curvature perturbations generated on superhorizon scales are given by

$$\mathcal{R}_k = \frac{\mathcal{H} \delta \rho_k}{\rho} = \frac{U_{\varphi} \delta \phi_k}{3(p + \rho)} = \frac{H}{\phi} \delta \phi_k.$$ \hspace{1cm} (18)

The power spectrum of cosmological fluctuations is

$$\Delta_{\mathcal{R}}(k) = \langle |\mathcal{R}_k|^2 \rangle = \frac{4\pi H^2}{|\mathcal{H}|} \Delta^2(k),$$ \hspace{1cm} (19)

where the fluctuations per logarithmic $k$ range is

$$\Delta^2(k) = \frac{k^3}{2\pi^2} \frac{|\chi_k(z)|^2}{a^2}.$$ \hspace{1cm} (20)

In the limit where $k|z| \ll 1$, which corresponds to the region where the wavelength is larger than the Hubble radius, we finally obtain the scale-invariant spectrum

$$\Delta_{\mathcal{R}}(k) = \frac{G H^4}{\pi |\mathcal{H}|} \left( \frac{H^2}{2\pi\phi} \right)^2.$$ \hspace{1cm} (21)

At the scale $k_s = 0.05\text{Mpc}^{-1}$, the power spectrum is constrained at the 68% confidence level (CL) as [66]

$$\Delta_{\mathcal{R}}(k_s) = (2.215_{-0.037}^{+0.032}) \times 10^{-9}.$$ \hspace{1cm} (22)

The tensor-to-scalar ratio is constrained at 95% CL as [67]

$$r = \frac{\Delta_T(k_s)}{\Delta_{\mathcal{R}}(k_s)} = 16 \frac{G H^2}{\pi \Delta_{\mathcal{R}}(k_s)} < 0.12.$$ \hspace{1cm} (23)

Writing $H^2 = \pi \Delta_{\mathcal{R}}(k_s) r/16G$ and $H^2 = (8\pi G/3)(U + \dot{\phi}^2/2)$, we derive

$$\dot{\phi} = \frac{\sqrt{\Delta_{\mathcal{R}}(k_s)}}{32G} r = 2.2 \times 10^{31} \text{GeV}^2 r_{0.1},$$ \hspace{1cm} (24)

$$U = \Delta_{\mathcal{R}}(k_s) r (48 - r) \approx \frac{3\Delta_{\mathcal{R}}(k_s) r}{128G^2} = 1.2 \times 10^{65} r_{0.1} \text{GeV}^4,$$ \hspace{1cm} (25)

where $r_{0.1} = r/0.1$. The scalar spectrum tilt is defined as

$$n_s - 1 = \frac{d\ln \Delta_{\mathcal{R}}(k_s)}{d\ln k} \bigg|_{k=k_s},$$ \hspace{1cm} (26)

or, using the result in Eq. (21),

$$n_s - 1 = \frac{d}{d\ln k} \left[ 4 \ln H - 2 \ln \dot{\phi} \right]_k = -6\epsilon_s + 2\eta_s,$$ \hspace{1cm} (27)

where $\epsilon_s$ and $\eta_s$ are the slow-roll parameters at $k = k_s$. 

IV. VISCOS INFLATION MODEL

A viscous term $\Pi = \Pi(t)$, possibly depending on time, may arise through either self-interaction [24, 25] or the decay [26-28, 34-37] of the inflation field, acting as a negative pressure $p \to p - \Pi$ [48-53]. In the FRW background, the equation of motion for the inflaton field reads

$$\dot{\varphi} \left( \varphi + \frac{k^2}{a^2} \varphi + U_\varphi \right) + 3H (\dot{\varphi}^2 - \Pi) = 0. \quad (27)$$

Writing $\varphi(x) = \phi(x) + \delta \phi(x)$ and introducing $\phi = u$, the expression for the background component is obtained as

$$\ddot{u} + 3H u + U_\phi = 3H \frac{\Pi}{u}, \quad (28)$$

which reduces to the usual result in Eq. (10) for $\Pi = 0$. Here instead, we show that Eq. (28) with a vanishing potential $U = 0$ also leads to an inflationary stage. We assume that viscosities are switched off whenever the momentum of the inflaton field is set to zero, or $\Pi = \zeta H = \zeta \lambda u$, where $\zeta$ is the viscosity coefficient, depending on $\phi$, and $H = \lambda u$ with $\lambda = \sqrt{4\pi G/3}$. We thus solve Eq. (28) with $U = 0$,

$$\ddot{u} + 3H u = \frac{3H^2 \zeta}{u}, \quad (29)$$

or, using $\ddot{u} = -H z u'$ with $z = -1/\lambda H$ and $u' = du/dz$,

$$z u' - 3(u - \lambda \zeta) = 0. \quad (30)$$

Assuming $\zeta = \zeta_0 + \zeta_1(z)$, where $\zeta_0$ is a constant and $\zeta_1(z)$ is a slowly-varying function, the solution to Eq. (30) reads

$$u = \lambda \zeta_0 - \lambda \zeta_1(z), \quad \text{with} \quad \zeta_1(z) = 3z^3 \int \frac{\zeta_1(z')}{(z')^4} dz'. \quad (31)$$

Using $\dot{H} = 4\pi G (\Pi - u^2) \approx 3\lambda^4 \zeta_0 \tilde{\zeta}_1$ and introducing

$$\epsilon = \frac{\dot{H}}{H^2} = \frac{3\tilde{\zeta}_1}{\zeta_0}, \quad (32)$$

we obtain $\epsilon \ll 1$ for $\tilde{\zeta}_1 \ll \zeta_0 / 3$. A second constrain is obtained by using the slow-roll condition $\phi \ll 3H \dot{\phi}$ as

$$\dot{u} \ll 3H u, \quad \text{or} \quad \beta \equiv \frac{z \tilde{\zeta}_1}{\zeta_0} = \frac{3 \tilde{\zeta}_1(z) + \zeta_1(z)}{\zeta_0} \ll 1, \quad (33)$$

where $\beta$ is a new slow-roll parameter and we used Eq. (31) to compute $\tilde{\zeta}_1$. Inflation ends when either one of these two conditions is no longer met, or $\epsilon(z_m), \beta(z_m) \sim 1$, where $z_m = \exp(N_\phi)$ with a sufficient number of e-folds $N_\phi$ for successful inflation.

Using Eqs. (21) and (22), the scalar power spectrum and the tensor-to-scalar ratio in the viscous inflation are

$$\Delta R(k_s) = \frac{\lambda^6 \zeta_0^3}{4\pi^2 \tilde{\zeta}_1^3}, \quad r = \frac{12 \lambda^6 \zeta_0^2}{\pi^2 \Delta R(k_s)}, \quad (34)$$

from which we obtain

$$\zeta_0 = \frac{\pi}{2\lambda^3} \sqrt{\frac{\Delta R(k_s)}{3}} r = 3.2 \times 10^{50} r_{0.1} \text{GeV}^3, \quad (35)$$

$$\tilde{\zeta}_1 = \frac{\pi}{96\lambda^3} \sqrt{\frac{\Delta R(k_s)}{3}} r^3 = 6.6 \times 10^{47} r_{0.1} \text{GeV}^3. \quad (36)$$

Notice that the tensor-to-scalar ratio and the first slow-roll parameter are related by $r = 16\epsilon$, as in the standard inflation model. The Hubble rate during inflation is

$$H_I = \lambda^2 \zeta_0 = 3.7 \times 10^{13} \text{GeV} r_{0.1}^{1/2}. \quad (37)$$

Using Eq. (6), we find that the Fourier transform of the scalar field perturbation $\delta \phi_k$ for the mode $k$, defined in Eq. (11), satisfies the equation

$$\ddot{\delta \phi_k} + 3H \dot{\delta \phi_k} + \left( \frac{k^2}{a^2} - \frac{3H^2 \Pi}{u} \right) \delta \phi_k = 0. \quad (38)$$

Since $\Pi = \zeta \lambda u = \Pi_u$ and

$$\Pi_u \equiv \frac{d\Pi}{d\phi} = -\frac{H z \Pi}{u} = \frac{H \lambda^2 \zeta_0^2}{u} \beta, \quad (39)$$

Eq. (38) reads

$$\ddot{\delta \phi_k} + 3H \dot{\delta \phi_k} + \left( \frac{k^2}{a^2} - 3H^2 \beta \right) \delta \phi_k = 0, \quad (40)$$

which is the same expression as Eq. (12), with $\Pi_u$ playing the role of the potential curvature $U_{\phi\phi}$. Switching to the variable $z$ and writing $\delta \phi_k = z \chi_k$, we have

$$\chi''_k + \left( \frac{k^2}{a^2} - \frac{2 + 3(\epsilon - \beta)}{z^2} \right) \chi_k = 0, \quad (41)$$

with solution $\chi_k = C_1 \sqrt{z H^{(1)}(\nu, z)}$, where $H^{(1)}(\nu, z)$ is the Hankel function of the first kind and

$$\nu = \frac{3}{2} \left( 1 + \frac{4}{3}(\epsilon - \beta) \right)^{1/2} \approx \frac{3}{2} + \epsilon - \beta. \quad (42)$$

We obtain that the power spectrum predicted by the viscous inflation model is almost scale-invariant since, using Eq. (25), the spectral tilt results in

$$n_s - 1 = -\frac{d \ln \Delta R(k_s)}{d \ln z} \bigg|_{z = z_*} = -4\epsilon_* + 2\beta_*: \quad (43)$$

As long as the conditions for an accelerated expansion $\epsilon, \beta \ll 1$ are met, Eq. (43) predicts a mild running of the power spectrum, while the viscous term $\zeta$ is approximately constant, see Eq. (35) with small corrections as in Eq. (36). The exact value of $\zeta$ depends on the tensor-to-scalar ratio $r$, and it is calculable in models where the Hubble rate during inflation is predicted.

ACKNOWLEDGMENTS

We acknowledge support by Katherine Freese through a grant from the Swedish Research Council (Contract No. 638-2013-8993).
