FINITELY PRESENTED SUBGROUPS OF SYSTOLIC GROUPS ARE SYSTOLIC

GAŠPER ZADNIK

ABSTRACT. In this note we prove that every finitely presented subgroup of a systolic group is itself systolic.

1. Introduction

In early eighties, Gromov deduced several properties of Riemannian manifolds of non-positive sectional curvature without using Riemannian structure, but only the property of the induced distance function, which he called CAT(0) inequality [BG85]. Gromov proved that for a cube complex equipped with the piecewise Euclidean metric, one can locally check CAT(0) condition in terms of combinatorial structure of the complex, see [BH99, Theorem II.5.20].

In [Hag03, JS06] there was introduced the following simplicial analogue of CAT(0) spaces. It is called simplicial non-positive curvature.

Definition 1. A simplicial complex is flag if every finite set of vertices that are pairwise connected by edges spans a simplex. A loop of length $m$ in a simplicial complex $X$ is a simplicial embedding of an $m$-cycle into $X$. An edge connecting two non-consecutive vertices of a loop is called a diagonal. The property that every loop of length at least four and less than $m$ has a diagonal is called $m$-largeness. Let $m \geq 6$. A simply connected $m$-large flag simplicial complex is called systolic. We write only systolic instead of 6-systolic. A group acting properly and cocompactly by automorphisms on a systolic complex is called systolic.

Note that this definition of systolic complex differs from the original one, but is equivalent [JS06, Fact 1.2.(4) and Corollary 1.6].

The purpose of this note is to prove the following theorem.\footnote{The author was told that the result was also proven independently in [GHMP13].}

Theorem 2. Any finitely presented subgroup of a systolic group is systolic.

Theorem 2 was proven by Wise for torsion-free systolic groups, see [Wis03, §5]. Wise considers the quotient of the systolic complex under the group action, and his proof does not generalize to groups with torsion. Note that Theorem 2 is not true if we replace “systolic” with “CAT(0)” in the statement, see [BRS07, §2.3.3].

1.1. Notation and outline. All the paths in any simplicial complex are taken in its one-skeleton. We use the symbol $d_X$ to denote the shortest-path distance in one-skeleton of a simplicial complex $X$. Let $H \leq G$ have a finite presentation $(S|R)$ with $S$ symmetric. Let $C_S(H)$ be the oriented Cayley graph of $H$ with respect to the generating set $S$. This means that an edge connecting $h$ and $hs$ for $h \in H$ and a generator $s \in S$ comes equipped with two orientations, one for $s$ and one for $s^{-1}$, except when $s^2 = 1$, when there are two edges connecting $h$ and $hs$. Denote by $C_S^X(H)$ such subdivision of $C_S(H)$ that there exists a simplicial $H$-equivariant map $\phi : C_S^X(H) \to X$. Let $e_s$ denote the path between 1 and $s \in S$ in $C_S^X(H)$ that comes from subdivision of the edge connecting 1 and $s$ in $C_S(H)$. Let $x_0 = \phi(1)$ and $\gamma_s = \phi(e_s)$ for $s \in S$. Denote by $L$ the maximum of the lengths of $\gamma_s$. Denote also $\Gamma = \phi(C_S^X(H))$. We will frequently use $s_1, \ldots, s_m$ to denote generators from $S$. We write $\gamma_{s_1 \cdots s_m}$ for the path which is the concatenation $\gamma_{s_1} * (s_1 \gamma_{s_2}) * \cdots * (s_1 \cdots s_{m-1} \gamma_{s_m})$.

The outline of our proof is as follows. We find a neighborhood $N$ of $\Gamma$ in $X$ such that every loop in $\Gamma$ can be contracted in $N$. But new loops can appear in $N$. We thus have to consider $Y$, a disjoint union of $H$-translates of some large ball in $X$ modulo appropriate equivalence relation. Then $Y$ encodes an appropriate neighborhood of $\Gamma$ and moreover does not give rise to any new homotopically nontrivial loop. Finally, we extend
Y in appropriate category to a maximal $H$-cocompact simply connected flag simplicial complex and prove that it is $6$-large.

2. Proof of the theorem

We proceed in several steps as mentioned above. In the first step, we find a constant $R$ such that loops in $\Gamma$ are homotopically trivial in $N_X^R(\Gamma)$. Important properties of loops in $\Gamma$ deduced in the proof of Step 1 are collected in Fact 1, since we will need them later on.

Step 1. There exists a constant $R < \infty$ such that every loop in $\Gamma$ is homotopically trivial in $N_X^{R-L}(\Gamma)$.

Proof. After replacing a loop with its $H$-translate, it is enough to consider loops in $\Gamma$ containing a point at distance at most $L$ from $x_0$. We distinguish two main cases.

Case (1). Loop of the form $\gamma_{s_1 \cdots s_m}$ with $s_1 \cdots s_m x_0 = x_0$. There are three subcases.

(a) The word $s_1 \cdots s_m$ belongs to $R$. Because $R$ is finite, there is a number $R_1$ such that every such loop is homotopically trivial in $N_X^{R_1}(z)$ for every vertex $z \in \gamma_{s_1 \cdots s_m}$.

(b) The word $s_1 \cdots s_m = 1$ but it does not belong to $R$. Then $s_1 \cdots s_m$ is a concatenation of conjugates of relators from $R$, but each such conjugate is homotopically trivial in $N_X^{R_1}(\Gamma)$ hence the whole loop is homotopically trivial in $N_X^{R_1}(\Gamma)$.

(c) The point $x_0$ is fixed by $s_1 \cdots s_m$ but $s_1 \cdots s_m \neq 1$. Without loss of generality, we can assume that $s_1 \cdots s_m$ is the shortest representative of the corresponding group element since all the other representatives differ from the shortest by concatenation with words considered in Subcases (1.a, 1.b). By properness of $G$-action and hence of $H$-action, the number of elements $h \in H$ fixing $x_0$ is finite. Hence we can choose a constant $R_1$ such that $\gamma_{s_1 \cdots s_m}$ is homotopically trivial in $N_X^{R_1}(\Gamma)$ for every vertex $z \in \gamma_{s_1 \cdots s_m}$.

Case (2). Loop coming from a path $\gamma_{s_1 \cdots s_m}$ with self-intersection $x \notin Hx_0$. Without loss of generality we can assume that $x = \gamma_{s_1} \cap (s_1 \cdots s_m-1)\gamma_{s_m}$. Figure 1 shows such configuration. Observe that in this case, $d_X(x_0, s_1 \cdots s_m x_0) \leq 2L$. By properness of the $H$-action, there is an upper bound $N$ such that $s_1 \cdots s_m = p_1 \cdots p_k$, where the number $k$ of terms $p_i \in S$ is at most $N$. Since

$$s_1 \cdots s_m p_k^{-1} \cdots p_1^{-1} = 1,$$

the big loop $\gamma_{s_1 \cdots s_m p_k^{-1} \cdots p_1^{-1}}$ is a loop from Cases (1.a, 1.b), hence it can be contracted in $N_X^{R_1}(\Gamma)$. Thus to contract the original loop it suffices to contract

$$\gamma_{p_1 \cdots p_k} \ast \gamma,$$

where $\gamma$ is a path in $\Gamma$ from $s_1 \cdots s_m x_0$ to the intersection of $\gamma_{s_1}$ and $s_1 \cdots s_m-1\gamma_{s_m}$ concatenated with a path from $x$ to $x_0$. But the total length of the loop $(\hat{\gamma})$ is at most $(N + 2)L$. Let $R_2$ be a number such that any loop $\gamma_{p_1 \cdots p_k} \ast \gamma$ of type $(\hat{\gamma})$ can be contracted in $N_X^L(z)$ for any vertex $z \in \gamma_{s_1 \cdots s_m}$.}

Figure 1. Loop from Case (2); possibly $x = x_0$ or $x = s_1 \cdots s_m x_0$. If both equalities hold, this example is covered in Case (1).

Alltogether, we find a constant $R' = \max\{R_1, R_1', R_2\}$ such that every loop in $\Gamma$ is homotopic in $N_X^{R'}(\Gamma)$ to a trivial loop. Hence $R = R' + L$ works.

We will call loops from Cases (1.a, 1.c) and loops $(\hat{\gamma})$ form Case (2) short loops. From the proof of Step 1 we deduce the following.

Fact 1. Every loop in $\Gamma$ is a concatenation of conjugates of short loops. Let $\gamma$ be a short loop in $\Gamma$. Then for every vertex $z$ on $\gamma$, the loop $\gamma$ is fully contained in the ball $N_X^{R-L}(z)$.

We are now ready to define $Y$. For every $h \in H$, denote $B^0_h = N_X^R(hx_0)^0$ and denote the copy of the vertex $v \in X$ in $B^0_h$ by $v^h$. Let $\sim$ be an equivalence relation on $\bigcup_{h \in H} B^0_h$ generated by $v^h \sim u^g$ if and only if $v = u$ in $X$ and $g^{-1} h \in S$. Let

$$Y^0 = \left( \bigcup_{h \in H} B^0_h \right) / \sim.$$
Note that $B_{h}^{i}$ injects into $Y^{0}$. For $y \in Y^{0}$ we write $\tilde{y}$ for the vertex of $X$ such that $y = \tilde{y}^{h}$ for some $h \in H$. Next, we define $Y^{1}$. We connect two vertices $y, z \in Y^{0}$ by an edge if there exist representatives $\tilde{y}^{h}, \tilde{z}^{h}$ for $y$ and $z$ with $h = g$ and $g, z$ adjacent. Let $Y$ be the flag completion of $Y^{1}$ and let $B_{h}$ be the simplicial span of $B_{h}^{i} \leq Y$. We define a natural action of $H$ on $Y$, which is induced from $H$-action on $X$. Note that $hB_{1} = B_{h}$, hence $H$-action on $Y$ is proper and cocompact.

Observe that by the construction of $Y$, there exists a proper $H$-equivariant map $f : Y \to X$. It is defined by $f(y) = \tilde{y}$ for $y \in Y^{0}$ and extends to higher-dimensional simplices simplicially. Let us define local sections

$$(\bigtriangledown) \quad i_{h} : N_{X}^{R}(h_{x_{0}}) \to B_{h}$$

by $i_{h}(u) = u^{h}$ for every $u \in N_{X}^{R}(h_{x_{0}})^{0}$ and every $h \in H$. By definition of $Y$, each map $i_{h}$ is bijective on zero-skeletal. Furthermore, vertices $u^{h}, v^{h} \in B_{h}^{i}$ are adjacent if and only if $u, v \in N_{X}^{R}(h_{x_{0}})^{0}$ are adjacent. Hence $i_{h}$ is well defined isomorphism between one-skeletal of $N_{X}^{R}(h_{x_{0}})$ and $B_{h}$. Since a flag complex is determined by its one-skeleton, $i_{h}$ is an isomorphism. Note that $B_{h}$ might not be a ball in $Y$.

Observe that for any $h \in H$ and $s \in S$ the two maps $i_{h}$ and $i_{gs}$ agree on $N_{X}^{R}(h_{x_{0}}) \cap N_{X}^{R}(hs_{x_{0}})$ because they agree on the zero-skeleton of that intersection. By that fact, there is a natural map $\varphi : C_{X}^{Y}(H) \to Y$, sending edge path $h_{x_{0}}$ to $i_{h_{0}}(h_{x_{0}})$. We can as well describe the map $\varphi$ in terms of the $H$-action on $Y$, but the previous discussion is more useful for us. Next step ensures that $\varphi$ has good properties.

**Step 2.** The map $\varphi$ factors through $\phi : C_{X}^{Y}(H) \to \Gamma$.

**Proof.** We have to check that if two points of $C_{X}^{Y}(H)$ are identified under $\phi$, they are also identified under $\varphi$. To see this, observe that if two paths $\gamma_{1}$ and $\gamma_{2}$ in $Y$ cross in $\Gamma$ then, where $s, s' \in S$ and $h \in H$, there is a sequence of generators $p_{1}, \ldots, p_{k} \in S$ such that $p_{1} \cdots p_{k}$ is the shortest word representing $h$. In particular $p_{1} \cdots p_{k} \in S$ is a homotopy. It follows from Fact 1 that the ball $N_{X}^{R}(p_{1} \cdots p_{k}x_{0})$ contains the whole path $\gamma_{1} \cdots \gamma_{k}$ for all $l = 0, 1, \ldots, k$, where the empty word represents 1. Thus if we write $N_{X}^{R}$ for $\gamma_{1}$ with opposite orientation, we have that $N_{X}^{R}(p_{1} \cdots p_{k}x_{0})$ contains the path $\gamma = (h_{x_{0}}^{\ast} \gamma_{1} \cdots \gamma_{k})$ for all $l = 0, 1, \ldots, k$. Hence $B_{p_{1} \cdots p_{k}x_{0}}$ contains $i_{p_{1} \cdots p_{k}x_{0}}(\gamma)$ for all $l = 0, 1, \ldots, k$. Since two consecutive maps $i_{p_{1} \cdots p_{k}x_{0}}$ and $i_{p_{1} \cdots p_{k}x_{0}}$ agree on the intersection of their domains, the path $i_{l}(\gamma_{1})$ and $i_{l}(\gamma_{2})$ defines a homotopy on the intersections $\cap_{i=0}^{k} B_{p_{1} \cdots p_{k}x_{0}}$. Thus the point on $e_{s}$ is identified with the appropriate point on $h_{x_{0}}$ under $\varphi$. This finishes the proof.

By Step 2, there exists a lift $f_{1} : \Gamma \to Y$ of the map $f : Y \to X$. Obviously $f_{1}$ agrees with $i_{h}$ on $\Gamma \cap N_{X}^{R}(h_{x_{0}})$. From now on, we identify $\Gamma$ with its $f_{1}$-image.

**Step 3.** The complex $Y$ is simply connected.

As mentioned in the outline, we first prove that $Y$ encodes an appropriate neighborhood of $\Gamma$ such that loops in $\Gamma$ are homotopically trivial in $Y$. Then we exhibit a homotopy from any loop in $Y$ to a loop in $\Gamma$.

**Proof.** Take any loop $\gamma$ in $\Gamma \subseteq Y$. By Fact 1, it is a concatenation of short loops. In the same way as in the proof of Step 2 one can show that each short loop $\gamma'$ is fully contained in $B_{h}$ for each $h \in H$ such that $d_{Y}(h_{x_{0}}, \gamma') \leq L$. Pick such $h \in H$. We know that $B_{h}$ is isomorphic to $N_{X}^{R}(h_{x_{0}})$ via the map $i_{h}$ from $(\bigtriangledown)$. Invoking Fact 1 once again, we see that the loop $\gamma'$ is homotopically trivial in $B_{h}$, hence $\gamma$ is homotopically trivial in $Y$.

Finally we need to show that every loop in $Y$ is homotopic to a loop in $\Gamma$. Let $\beta : S^{1} \to Y^{1}$ be a loop in the one-skeleton of $Y$. We identify $S^{1}$ with $I/\partial I$, where $I = [0, 1]$ is an interval. Let $0 \leq t_{0} < t_{1} < \cdots < t_{k} < 1$ be cyclically ordered points on $S^{1}$ and $h_{0}, h_{1}, \ldots, h_{n}$ elements of $H$ such that $\beta(t_{i}) \in Y^{0}$ and $\beta(t_{i}, t_{i+1}) \in B_{h_{0}}$ for all $i = 0, 1, \ldots, n$, where indices are taken modulo $n + 1$. Since $B_{h_{0}}$ and $B_{h_{0}}$ both contain $\beta(t_{i})$, there is a sequence of generators $s_{1}, \ldots, s_{n}$ such that $\beta_{i} = h_{i-1} s_{1} \cdots s_{n}$ for $i = 0, 1, \ldots, n$. Recall that empty word stands for 1. This means that there exist geodesics

$\beta_{i} : (I, \partial I) \to \left( B_{h_{i-1}} \cdots s_{1}, \{ \beta(t_{i}), h_{i-1} \cdots s_{n}, s_{n} \} \right)$

for all $i = 0, 1, \ldots, n$ and $l = 0, 1, \ldots, s(i)$. Recall that $h_{i} \cdots s_{n}$ belongs also to $B_{h_{i-1}} \cdots s_{1}$. Since balls in systolic complexes are geodesically convex [J-S06, §7], the image of $\beta_{i}$ is contained in $B_{h_{i-1}} \cdots s_{1}$. Next, we can find some $\varepsilon > 0$ such that $t_{i} + \varepsilon < t_{i+1}$ for all $i$. After precomposing $\beta$ with a map $S^{1} \to S^{1}$, homotopic to the identity, we can assume that $\beta$ is constant on $[t_{i}, t_{i} + \varepsilon]$ for all $i = 0, 1, \ldots, n$. Hence

- for all $i = 0, 1, \ldots, n$ and $l = 1, 2, \ldots, n(\varepsilon)$ there is $H^{i} : [t_{j} + (l-1)\varepsilon, t_{j} + (l-1)\varepsilon] \times I \to B_{h_{i-1}} \cdots s_{1} \cap B_{h_{i-1}} \cdots s_{1}$ with $H^{i}_{t_{j} + (l-1)\varepsilon, t_{j} + (l-1)\varepsilon} = \beta_{j}^{i}$ and $H^{i}(t_{j} + \varepsilon, t_{j} + \varepsilon) = \beta_{j}^{i}(t)$, where $H^{i}_{t_{j} + \varepsilon, t_{j} + \varepsilon}$ is constant path $\beta(t)$ and $H^{i}(t, t_{j} + \varepsilon)$ is a path $h_{i-1} \cdots s_{1} \gamma_{i-1} \subseteq \Gamma$;

- for all $i = 0, 1, \ldots, n$ there is $H^{i} : [t_{i} + \varepsilon, t_{i} + \varepsilon] \times I \to B_{h_{i}}$ with $H^{i}(t_{i} + \varepsilon, t_{i} + \varepsilon) = \beta_{i}(t)$ and $H^{i}(t_{i} + \varepsilon, t_{i} + \varepsilon) = \beta_{i}(t)$, where $H(-, 0)$ is a constant path $h_{i}x_{0}$.

Since the homotopies from above agree on the intersections of their domains, they glue together to a homotopy $H : S^{1} \times I \to Y$ with $H(-, 0) = \beta$ and $H(-, 1)$ a loop in $\Gamma$. 

\[\square\]
In the following step, using $f$ we extend $Y$ to a systolic complex $\overline{Y}$, on which $H$ still acts properly and cocompactly and is thus a systolic group.

We say that a pair $(W, f_W)$ is an $f$-extension of $Y$ if the following holds. The complex $W$ is simply connected flag simplicial complexes containing $Y$ such that $Y^0 = W^0$ and that the $H$-action on $Y$ extends to an $H$-action on $W$. Furthermore, the map $f_W : W \to X$ is a simplicial $H$-equivariant map which extends $f$. Note that $f_W$ maps an edge of $W$ either to an edge or to a vertex of $X$.

Let $F$ be the family of all $f$-extensions of $Y$. Observe that $F$ is equipped with a natural partial order $\leq$, where $(W_1, f_{W_1}) \leq (W_2, f_{W_2})$ if there exists an $H$-equivariant embedding $i : W_1 \to W_2$ fixing $Y$ such that $f_{W_2} \circ i = f_{W_1}$. The family $F$ is nonempty since it contains $(Y, f)$ by Step 3. Let $(W_\lambda, f_{W_\lambda})_{\lambda \in \Lambda}$ be an increasing chain in $F$. Then the union $\bigcup_{\lambda \in \Lambda} W_\lambda \cup \bigcup_{\lambda \in \Lambda} f_{W_\lambda}$ is also in $F$, so it is an upper bound for the chain $(W_\lambda, f_{W_\lambda})_{\lambda \in \Lambda}$. By Kuratowski–Zorn Lemma, there exists a maximal element $(\overline{Y}, \overline{f}) \in F$.

**Step 4.** For a maximal element $(\overline{Y}, \overline{f}) \in F$, the simplicial complex $\overline{Y}$ is a systolic complex, equipped with a proper and cocompact $H$-action.

**Proof.** We claim that the valence in $\overline{Y}$ is bounded from above. Recall that $\overline{f}$ and $f$ agree on $\overline{Y}^0 = Y^0$. Let $N_y \subseteq Y^0$ denote the set of all vertices adjacent to $y$ in $\overline{Y}$. For every $y' \in N_y$ either $f(y') = f(y)$ or $f(y')$ is adjacent to $f(y)$. This means that $f(N_y)$ is contained in $N_{\overline{f}(y)}(f(y))$. In other words, if we have $N_y \subseteq f^{-1}(N_{\overline{f}(y)}(f(y)))$. Because $X$ is proper, the ball $N_{\overline{f}(y)}(f(y))$ is compact. But $\overline{f}$ is a proper map, hence the set $f^{-1}(N_{\overline{f}(y)}(f(y)))$ is compact and the claim is proven. In particular, $\overline{Y}$ is a proper simplicial complex. Because the vertex set of $Y$ and $\overline{Y}$ coincide, the action of $H$ on $\overline{Y}$ is proper and cocompact. By definition, $\overline{Y}$ is flag and simply connected. It remains to prove 6-largeness.

Suppose for contradiction that there is some loop $\alpha$ of length four or five in $\overline{Y}$ without diagonals. If $\overline{f}$ maps $\alpha$ bijectively to $\alpha' = \overline{f}(\alpha) \subseteq X$, then there exist two non-consecutive vertices $u$ and $v'$ of $\alpha'$ connected by a diagonal because $X$ is systolic. Let $u$ and $v$ be the vertices of $\alpha$ mapped to $u'$ and $v'$ by $\overline{f}$. For every $h \in H$, we add an edge in $Y$ between $hu$ and $hv$ and extend $\overline{f}$ to the new edges naturally. Let us remind the reader that if for $n$ different $h_1, \ldots, h_n \in H$ all the sets $\{h_i u, h_i v\}$ coincide for $i = 1, 2, \ldots, n$, we only add one edge between $h_1 u$ and $h_1 v$ instead of $n$. This remark will be applied two more times without mentioning it. If $\overline{f}$ is not bijective on $\alpha$, there must be two vertices $u, v$ of $\alpha$, which are mapped by $\overline{f}$ to the same vertex. If they are non-consecutive in $\alpha$, we add edges between $hu$ and $hv$ for every $h \in H$ and extend $\overline{f}$ such that it maps any new edge to the common image of its endpoints. If $u$ and $v$ are consecutive, let $w \neq u$ be the other neighbor of $v$ in $\alpha$. Then we add edges between $hu$ and $hw$ for all $h \in H$. Note that since $\overline{f}(w) = \overline{f}(v)$, the point $\overline{f}(w)$ is either adjacent to $v$ or coincide with $\overline{f}(u)$ and hence we can extend $\overline{f}$ to the newly added edges.

In all cases, we added an $H$-orbit of an edge to $\overline{Y}$. After a flag completion, we obtain a flag simplicial complex $\hat{Y}$ on the set of vertices $Y^0$, properly containing $\overline{Y}$, together with a map $\overline{f}$, extending $\overline{f}$, and equipped with an $H$-action, extending $H$-action on $\overline{Y}$. The complex $\hat{Y}$ is also simply connected. Indeed, every edge $e$ in $\hat{Y}^1 - \overline{Y}^1$ is a diagonal of a loop $\alpha$ in $\overline{Y}^1$ of length less than six. This means that $e$ together with two consecutive edges of $\alpha$ form a triangle, which is filled after flag completion. Hence the path $e$ is homotopic relative to its endpoints to a path of length two in $\overline{Y}$. In other words, any loop in $\hat{Y}$ is homotopic to a loop in $\overline{Y}$ and the later is simply connected since it belongs to $F$. Hence $(\overline{Y}, \overline{f}) \leq (\hat{Y}, \hat{f}) \in F$, which contradicts the maximality of $(\overline{Y}, \overline{f})$.

**Remark 3.** A first counterexample to Theorem 2 assuming only finitely generated instead of finitely presented subgroup is due to Stallings. Denote a free group of rank two generated by $x$ and $y$ by $(x, y)$. In [Sta63] (see also [BRS07, §2.4.2]) Stallings proved that the kernel $K$ of the homomorphism

$$
\tau : (a, b) \times (x, y) \to \mathbb{Z}, \quad \tau(a) = \tau(b) = \tau(x) = \tau(y) = 1,
$$

is finitely presented but not finitely presentable. Hence $K$ cannot be systolic. On the other hand, the direct product of two free groups of rank two is systolic, see [EP11].

Even in the case where $G$ is hyperbolic, one cannot hope for a generalization of the above. By the Rips Construction [Rip82], for any finitely presented group $Q$ and arbitrary $\lambda > 0$, there exists a finitely presented $C^\prime(\lambda)$ small cancellation group $G$ and a short exact sequence $\{1\} \to N \to G \to Q \to \{1\}$, where $N$ is finitely generated normal subgroup of $G$. Due to [Bio81], the group $N$ is finitely presentable if and only if $Q$ is finite. Hence, if we choose $Q = \mathbb{Z}$ and $\lambda = \frac{1}{6}$, then the Rips Construction gives a finitely presented $C^\prime(1/6)$ small cancellation group $G$ which is hyperbolic [Gro87] and $C(7)$ [LS77, Chapter V, §2]. By [Wis03], $C(7)$ group $G$ is 7-systolic. But it has a finitely generated not finitely presentable subgroup $N$, hence a finitely generated non-systolic subgroup. In particular, systolic and even 7-systolic groups are not coherent in general.

**Acknowledgement**

I would like to express my gratitude to the Institute of Mathematics of the Polish Academy of Sciences for hospitality during my work on this project, and especially to Piotr Przytycki for suggesting me this problem.
and for his indispensable advices and comments on previous drafts. I would also like to thank Tomasz Elsner for useful discussion on the problem.

References

[BGS85] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of non-positive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985.

[Bie81] Robert Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematical Notes, Queen Mary College, Department of Pure Mathematics, London, 1981.

[BRS07] Noel Brady, Tim Riley, and Hamish Short, *The geometry of the word problem for finitely generated groups*, Birkhäuser Verlag, Basel, 2007.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[EP11] Tomasz Elsner and Piotr Przytycki, *Square complexes and simplicial nonpositive curvature*, 2011, to appear in Proc. Amer. Math. Soc. Preprint: http://arxiv.org/abs/1107.4182.

[GHMP13] Richard Gaelan Hanlon and Eduardo Martínez-Pedroza, *Lifting group actions, equivariant towers and subgroups of non-positively curved groups*, 2013. Preprint: http://arxiv.org/abs/1307.2660.

[Gro87] Mikhael Gromov, *Hyperbolic groups*, Essays in group theory, Vol. 8, Math. Sci. Res. Inst. Publ., Springer, New York, 1987, pp. 75–263.

[Hag03] Frédéric Haglund, *Complexes simpliciaux hyperboliques de grande dimension*, 2003, Prépublication d’Orsay 2003-71.

[JŚ06] Tadeusz Januszkiewicz and Jacek Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. Inst. Hautes Études Sci. 104 (2006), 1–85.

[LS77] Roger C. Lyndon and Paul E. Shupp, *Combinatorial group theory*, Classics in Mathematics, vol. 89, Springer-Verlag, Berlin, 1977.

[Rip82] Eliyahu Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. 14 (1982), no. 1, 45–47.

[Sta63] John R. Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Amer. J. Math. 85 (1963), 541–543.

[Wis03] Daniel T. Wise, *Systolic complexes and their fundamental groups*, 2003, in preparation.