The affine $A_{n-1}^{(1)}$ Toda fields with boundary reflection

Takeo Kojima

Department of Mathematics, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan

E-mail: kojima@math.cst.nihon-u.ac.jp

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Abstract

We study the affine $A_{n-1}^{(1)}$ Toda fields with boundary reflection. Our approach is based on the free field approach. We construct free field realizations of the boundary state and its dual. For an application of these realizations, we present integral representations for the form factors of the local operators. In a limiting case $\rho \to \infty$, our integral representations reproduce those of form factors for the $SU(n)$ invariant massive Thirring model with boundary reflection [1].
1 Introduction

For without-boundary field theory, integrability is ensured by the factorized scattering theory [2]. The factorized scattering theory for boundary field theory was developed by [3, 4, 5]. More precisely, I. Cherednik [3] proposed the boundary Yang-Baxter equation, and S. Ghoshal, A. Zamolodchikov [4] proposed the boundary crossing symmetry. B. Hou, K. Shi and W. Yang [5] proposed higher rank generalization of the boundary crossing symmetry. E. Sklyanin [6] began Bethe Ansatz treatment and introduced the commuting transfer matrix for boundary model. M. Jimbo et al. [7] developed these ideas for boundary model, [3, 4, 6] and established the free field approach [7, 8] for boundary integrable model. They diagonalized the Hamiltonian of the massive XXZ model with a boundary, and derived integral representations for the spin correlation functions. The $U_q(\widehat{sl}_n)$ generalization of reference [7] was achieved in [8].

In this paper we study the affine $A^{(1)}_{n-1}$ Toda fields with boundary reflection, by means of the free field approach. For $A^{(1)}_1$ symmetry case, Hou et al. [11] studied the affine $A^{(1)}_1$ Toda fields (SG model) with boundary reflection, in terms of the free field approach, and constructed the boundary state. They derived integral representations for form factors of local fields. In reference [11, 12], constructions of the boundary state started with Lukyanov’s ultra-violet cut-off realization of Zamolodchikov-Faddeev operators [10], and the form factors were derived after removing the cut-off parameter at the final stage. In this paper we prefer to work directly with operators with cut-off parameter removed, as the same manner as reference [13], in which the massless XXZ chain with a boundary was studied. Using free field realizations of the Zamolodchikov-Faddeev operators [14] and the local operators [15], we construct the free field realizations of the boundary state and its dual, and give integral representations for form factors of local fields of the affine $A^{(1)}_{n-1}$ Toda fields with boundary reflection. In this paper we consider the case where the deformation parameter of quantum group $U_q(\widehat{sl}_n)$ is

$$q = e^{-\frac{2\pi i}{\rho n}}, \rho > 0.$$  \hspace{1cm} (1.1)

In a limiting case $\rho \to \infty$, our integral representations reproduce those of form factors for the $SU(n)$ invariant massive Thirring model with boundary reflection [1]. For $n = 2$ case (SG model), it is possible to generalize the boundary condition of this paper (Appendix C).
Now a few words about the organization of this paper. In section 2 we set up problem. In section 3 we present the free field realizations of Zamolodchikov-Faddeev operators [14], and local operators [13]. In section 4 we construct the free field realizations of the boundary state and its dual (4.3), (4.24). In section 5 we derive the boundary qKZ equations (5.15), (5.16), which govern form factors, and present integral representations for form factors of local fields (5.17), (5.18), (5.39). In Appendix A we summarize the multiple gamma functions. In Appendix B we summarize the contraction relations of the basic operators. In Appendix C we summarize a generalization of the boundary condition for the affine $A_{1}^{(1)}$ Toda fields (SG model).

2 Model

The purpose of this section is to set up the problem, and present briefly necessary tools concerning the completely integral models of quantum field theory with massive spectra. The affine $A_{n-1}^{(1)}$ Toda fields with boundary reflection is described by the bulk $S$-matrix and the boundary $K$-matrix [3], where the amplitudes of the bulk $S$-matrix and the boundary $K$-matrix are related by the boundary crossing symmetry [4]. Let $V$ be $n$-dimensional vector space $V = \oplus_{j=0}^{n-1} \mathbb{C} v_{j}$. The bulk $S$-matrix $S(\beta) \in \text{End}(V \otimes V)$ of the present model is given by

\[
S(\beta) v_{j_1} \otimes v_{j_2} = \sum_{k_1, k_2 = 0}^{n-1} v_{k_1} \otimes v_{k_2} S(\beta)^{j_1, j_2}_{k_1, k_2}, \quad (2.1)
\]

where nonzero entries are given by

\[
S(\beta)_{jj}^{jj} = s(\beta), \quad (2.2)
\]

\[
S(\beta)_{jk}^{jk} = s(\beta) \times \frac{-\text{sh} \left( \frac{2\pi i}{\rho} \beta \right)}{\text{sh} \left( \frac{\pi}{\rho} (\beta - \frac{2\pi i}{n}) \right)}, \quad (j \neq k), \quad (2.3)
\]

\[
S(\beta)_{kj}^{jk} = s(\beta) \times \left\{ \begin{array}{ll}
-\text{e}^{\frac{2\pi i}{\rho} \beta} \frac{2\pi i}{mn} & \text{if } j > k, \\
\text{sh} \left( \frac{\pi}{\rho} (\beta - \frac{2\pi i}{n}) \right) & \text{if } j = k, \\
\text{sh} \left( \frac{\pi}{\rho} (\beta - \frac{2\pi i}{n}) \right) & \text{if } j < k,
\end{array} \right. \quad (2.4)
\]
Here we have set
\[ s(\beta) = \frac{S_2(-i\beta|\rho, 2\pi)S_2(i\beta + \frac{2(n-1)\pi}{n}|\rho, 2\pi)}{S_2(i\beta|\rho, 2\pi)S_2(-i\beta + \frac{2(n-1)\pi}{n}|\rho, 2\pi)}. \] (2.5)

The double trigonometric function \( S_2(x|\omega_1\omega_2) \) is summarized in Appendix A. The boundary \( K \)-matrix \( K(\beta) \in \text{End}(V) \) of the present model is given by
\[ K(\beta)v_j = \sum_{k=0}^{n-1} v_k K(\beta)^k_j, \] (2.6)
where
\[ K(\beta)^k_j = \varphi(\beta)^{-1} \times \delta_{j,k}, \quad \varphi(\beta) = \frac{S_2(-2i\beta|\rho, 4\pi)S_2(2i\beta + \frac{2(n-1)\pi}{n}|\rho, 4\pi)}{S_2(2i\beta|\rho, 4\pi)S_2(-2i\beta + \frac{2(n-1)\pi}{n}|\rho, 4\pi)}. \] (2.7)

The boundary \( K \)-matrix \( K(\beta) \) satisfies the boundary Yang-Baxter equation \[ \] and the unitary condition,
\[ K(\beta)K(-\beta) = \text{id}. \] (2.9)

For general \( n \geq 2 \) case, the \( K \)-matrix \( K(\beta) \) satisfies the boundary crossing symmetry proposed by \[ \], in a limiting case \( \rho \to \infty \). Precisely, set the matrix \( \tilde{S}(\beta) \in \text{End}(V \otimes V) \) and \( \tilde{K}(\beta) \in \text{End}(V) \) by
\[ \tilde{S}(\beta)v_{j_1} \otimes v_{j_2} = \lim_{\rho \to \infty} \sum_{k_1,k_2=0}^{n-1} v_{k_1} \otimes v_{k_2} S(\beta)^{j_1,j_2}_{k_1,k_2} (-1)^{\delta_{j_1,j_2} + \delta_{j_2,k_1}} \tilde{K}(\beta) = \lim_{\rho \to \infty} K(\beta). \] (2.10)

The matrix \( \tilde{S}(\beta) \) and the matrix \( \tilde{K}(\beta) \) satisfy Yang-Baxter equation and the Boundary Yang-Baxter equation, too. The pair \( \tilde{K}(\beta) \) and \( \tilde{S}(\beta) \) exist in the class which Hou et al. studied \[ \], because the matrix \( \tilde{S}(\beta) \) satisfies the projection property, \( \tilde{S}(\frac{2\pi i}{n}) \sim P_2^{(-)} \), where \( P_2^{(-)} \) is antisymmetric project operator in \( V \otimes V \). The matrix \( \tilde{S}(\beta) \) and the \( K \)-matrix \( \tilde{K}(\beta) \) satisfy boundary crossing symmetry. See notation in reference \[ \].

\[ \tilde{K}^*(\beta)^k_j = \sum_{l,m=0}^{n-1} \tilde{S}(2\beta - 2\pi i)^{l,j}_{k,m} \tilde{K}(-\beta + 2\pi i)^m_l. \] (2.11)

For example, \( n = 3 \) case of the relation (2.11) in a limiting case \( \rho \to \infty \), becomes
\[ \varphi \left( \beta - \frac{2\pi i}{3} \right) \varphi(\beta) \varphi \left( \beta + \frac{2\pi i}{3} \right) = \frac{S_1(-2i\beta + \frac{4\pi}{3}|\rho)}{S_1(2i\beta - \frac{4\pi}{3}|\rho)} \to -\beta + \frac{2\pi i}{3} + \frac{2\pi i}{3}. \] (2.12)
For the description of the space of the physical states, we use the Zamolodchikov-Faddeev operators. The Zamolodchikov-Faddeev operators $Z_j^*(\beta), Z_j(\beta), (j = 0, \cdots, n - 1)$ of the present model are characterized by the following three conditions.

\begin{align*}
Z_{j_1}^*(\beta_1)Z_{j_2}^*(\beta_2) &= \sum_{k_{1,k_2}=0}^{n-1} Z_{k_2}^*(\beta_2)Z_{k_1}^*(\beta_1)S(\beta_1 - \beta_2)^{k_{1,k_2}}, \\ Z_{j_1}(\beta_1)Z_{j_2}(\beta_2) &= \sum_{k_{1,k_2}=0}^{n-1} S(\beta_1 - \beta_2)^{j_{1,j_2}}Z_{k_2}(\beta_2)Z_{k_1}(\beta_1),
\end{align*}

(2.13) (2.14)

The Zamolodchikov-Faddeev operators $Z_j(\beta), Z_j^*(\beta)$ satisfy the inversion relation.

\begin{align*}
Z_{j_1}^*(\beta_1)Z_k(\beta_2 + \pi i) = \frac{\delta_{j,k}}{\beta_1 - \beta_2} + \cdots, \quad (\beta_1 \rightarrow \beta_2),
\end{align*}

(2.15)

where "\cdots" means regular term. Free field realizations of the Zamolodchikov-Faddeev operators were given in [14].

In terminology of quantum field theory, the operator $O$ which commutes with the Zamolodchikov-Faddeev operators up to scalar multiplicity, is called the local operator.

\begin{align*}
Z_j^*(\beta)O &= m(\beta)OZ_j^*(\beta).
\end{align*}

(2.16)

In this paper we restrict our attention to a class of local operators $Z_j'(\delta), (j = 0, \cdots, n - 1)$ which commutes with the Zamolodchikov-Faddeev operators as follows.

\begin{align*}
Z_j^*(\beta)Z_k'(\delta) = \mathcal{L}(\beta - \delta)Z_k'(\delta)Z_j^*(\beta), \quad \mathcal{L}(\delta) = \frac{\sh(-\delta + \frac{\pi i}{n})}{\sh(\delta + \frac{\pi i}{n})}.
\end{align*}

(2.17)

Free field realizations of a class of local operators $Z_k'(\delta), (k = 0, \cdots, n - 1)$ were given in the reference [13], in which the authors considered spin chain problem.

For the description of the space of the physical state we use the boundary state $|B\rangle$. The boundary state $|B\rangle$ and its dual $\langle B|$ are characterized by the following conditions.

\begin{align*}
K(\beta)^{j_1}_{j_2}Z_j^*(\beta)|B\rangle &= Z_j^*(-\beta)|B\rangle, \quad (j = 0, \cdots, n - 1), \\ K(\beta)^{j_1}_{j_2}\langle B|Z_j(-\beta + \pi i) &= \langle B|Z_j(\beta + \pi i), \quad (j = 0, \cdots, n - 1).
\end{align*}

(2.18) (2.19)

In this paper we shall construct the free field realizations of the boundary state and its dual. The space of state is spanned by the vectors.

\begin{align*}
Z_{j_1}(\beta_1) \cdots Z_{j_N}(\beta_N)|B\rangle.
\end{align*}

(2.20)
Let us set the local operator $Z$ by
\[ Z = Z_{k_1}'(\delta_1) \cdots Z_{k_Q}'(\delta_Q). \tag{2.21} \]
Consider the matrix element,
\[ \langle B | Z Z^*_j(\beta_1) \cdots Z^*_j(\beta_N) | B \rangle. \tag{2.22} \]
We call the above matrix elements "form factors". In this paper we give integral representations for form factors of local fields.

In this section we have introduced the basic tools of boundary field theory, i.e. the $S$-matrix $S(\beta)$, the boundary $K$-matrix $K(\beta)$, the Zamolodchikov-Faddeev operators $Z^*_j(\beta)$, local operator $Z$, the boundary state $|B\rangle$, its dual $\langle B |$, and form factors.

3 ZF operators, Local operators

The purpose of this section is to give the free field realizations of Zamolodchikov-Faddeev operators [14] and local operators [15] of the present model.

3.1 ZF operators

The purpose of this subsection is to give free field realizations of the Zamolodchikov-Faddeev operators $Z^*_j(\beta), Z_j(\beta), (j = 0, \cdots, n - 1)$. Let us introduce the free bosons, $a_j(t)(j = 1, \cdots, n - 1; t \in \mathbb{R})$, satisfying the commutation relations,
\[ [a_j(t), a_k(t')] = A_{jk}(t)\delta(t + t'), \quad A_{jk}(t) = -\frac{1}{t} \frac{\text{sh}(a_j|a_k)\pi t}{n} \frac{\text{sh} \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t}{\text{sh} \frac{\pi}{n} \text{sh} \frac{\rho}{2}}. \tag{3.1} \]
Here $((a_j|a_k))_{1 \leq j,k \leq n - 1}$ is the Cartan matrix of type $A_{n-1}$. Explicitly the Cartan matrix of type $A_{n-1}$, $((a_j|a_k))_{1 \leq j,k \leq n - 1}$, is written as
\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}
\tag{3.2}
\]
Let us introduce the Fock space $\mathcal{F}$ generated by the vacuum vector $|\text{vac}\rangle$ satisfying
\[
a_j(t)|\text{vac}\rangle = 0 \quad \text{for } t > 0. \tag{3.3}
\]

A normal ordering : $A$ : of an element $A$ is defined as usual ; annihilation operators $a_j(t)(t > 0)$ are replaced on the right of the creation operators $a_j(-t)(t > 0)$, for example,
\[
a_j(t_1)a_k(-t_2) := a_k(-t_2)a_j(t_1), (t_1, t_2 > 0). \tag{3.4}
\]

Let us introduce the basic operators $V_j(\alpha)(j = 0, 1, \cdots, n)$ by
\[
V_j(\alpha) = : \exp \left( \int_{-\infty}^{\infty} a_j(t)e^{i\alpha t} \, dt \right) :, \quad (j = 1, \cdots, n-1), \tag{3.5}
\]
\[
V_0(\alpha) = : \exp \left( \int_{-\infty}^{\infty} a_1^*(t)e^{i\alpha t} \, dt \right) :, \tag{3.6}
\]
\[
V_n(\alpha) = : \exp \left( \int_{-\infty}^{\infty} a_n^*(t)e^{i\alpha t} \, dt \right) :, \tag{3.7}
\]
where
\[
a_1^*(t) = -\sum_{j=1}^{n-1} a_j(t) \frac{\sinh(n-j)\pi t}{n \sinh \pi t}, \quad a_{n-1}^*(t) = -\sum_{j=1}^{n-1} a_j(t) \frac{\sinh \pi t}{n \sinh \pi t}. \tag{3.8}
\]

We have
\[
[a_1^*(t), a_j(t')] = \delta_{1j} \frac{1}{t} \frac{\sinh \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t}{\sinh \frac{\rho}{2}} \delta(t + t'), \quad [a_j(t), a_{n-1}^*(t')] = \delta_{j,n-1} \frac{1}{t} \frac{\sinh \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t}{\sinh \frac{\rho}{2}} \delta(t + t'). \tag{3.9}
\]

Free field realizations of the Zamolodchikov-Faddeev operators $Z_j(\beta)$, of the affine $A_{n-1}^{(1)}$ Toda fields are given by
\[
Z_j(\beta) = c_j \int_{C_{j+1}} \, d\alpha_{j+1} \cdots \int_{C_{n-1}} \, d\alpha_{n-1} :V_{j+1}(\alpha_{j+1}) \cdots V_{n-1}(\alpha_{n-1}) V_n(\beta) : \
\times e^{\pi (\alpha_{j+1}-\beta)} \prod_{k=j+1}^{n-1} \Gamma \left( \frac{i(\alpha_k+\pi)}{\rho} - \frac{\pi}{n\rho} \right) \Gamma \left( \frac{i(\alpha_k-\alpha_{k+1})}{\rho} - \frac{\pi}{n\rho} \right), \tag{3.10}
\]
where $\beta = \alpha_n$. Here the integral contour $C_k, (k = 1, \cdots, n-1)$ for $\alpha_k$ is chosen so that the poles at $\alpha_{k+1} - \frac{\pi i}{n} + \rho i \mathbb{Z}_{\geq 0}$ are above $C_k$, and that the poles at $\alpha_{k+1} + \frac{\pi i}{n} - \rho i \mathbb{Z}_{\geq 0}$ are below $C_k$. Here constant $c_j$ is chosen so that the inversion relation (2.15) holds.
Free fields realizations of the Zamolodchikov-Faddeev operators $Z^*_j(\beta)$ are given by

$$Z^*_j(\beta) = \int_{C^*_i} \cdots \int_{C^*_j} \alpha_1 \cdots \alpha_j : V_0(\beta)V_1(\alpha_1) \cdots V_j(\alpha_j) :$$

$$\times e^{\pi(\beta-\alpha_j)} \prod_{k=1}^j \Gamma \left( \frac{i(\alpha_k - 1 - \alpha_k)}{\rho} - \frac{\pi}{n\rho} \right) \Gamma \left( \frac{i(\alpha_k - 1 - \alpha_k)}{\rho} - \frac{\pi}{n\rho} \right), \quad (3.11)$$

where $\beta = \alpha_0$. Here the integral contour $C^*_k, (k = 1, \cdots, n-1)$ for $\alpha_k$ is chosen so that the poles at $\alpha_k - 1 - \frac{\pi}{n} + \rho i \mathbb{Z}_{\geq 0}$ are above $C^*_k$, and that the poles at $\alpha_k - 1 + \frac{\pi}{n} - \rho i \mathbb{Z}_{\geq 0}$ are below $C^*_k$.

Proof of the commutation relation (2.13) was given in [14]. The commutation relation (2.14) is derived as the same manner as (2.13). The inversion relation (2.15) is derived as the same manner as those of local operators in [15].

In what follows we use the following abbreviations, $(j = 1, \cdots, n-1)$.

$$V^n_j(\alpha) = \exp \left( \int_0^\infty a^n_j(t)e^{iat}dt \right), \quad V^c_j(\alpha) = \exp \left( \int_0^\infty a_j(-t)e^{-iat}dt \right), \quad (3.12)$$

$$V^a_0(\alpha) = \exp \left( \int_0^\infty a^*_1(t)e^{iat}dt \right), \quad V^c_0(\alpha) = \exp \left( \int_0^\infty a^*_1(-t)e^{-iat}dt \right), \quad (3.13)$$

$$V^a_n(\alpha) = \exp \left( \int_0^\infty a^*_{n-1}(t)e^{iat}dt \right), \quad V^c_n(\alpha) = \exp \left( \int_0^\infty a^*_{n-1}(-t)e^{-iat}dt \right). \quad (3.14)$$

**Note.** Free field realizations of the Zamolodchikov-Faddeev operators $Z^*_j(\beta)$ were given in [14]. In this paper we give free field realizations of the dual operators $Z_j(\beta)$.

### 3.2 Local operators

In terminology of quantum field theory, the local operator is the one which commutes with the Zamolodchikov-Faddeev operators, up to scalar function multiplicity. In this subsection we present free field realization of a class of local operators $Z'_j(\beta)$ constructed in [14]. The local operators $Z'_j(\beta), (j = 0, \cdots, n-1)$ commute with the Zamolodchikov-Faddeev operators, up to multiplicity function.

$$Z^*_j(\beta)Z'_k(\delta) = \mathcal{L}(\beta-\delta)Z'_k(\delta)Z^*_j(\beta), \quad (3.15)$$

where we have set

$$\mathcal{L}(\beta) = \frac{\text{sh}(-\beta + \frac{\pi}{n})}{\text{sh}(\beta + \frac{\pi}{n})}. \quad (3.16)$$
Let us set the auxiliary fields $b_j(t), (1 \leq j \leq n - 1; t \in \mathbb{R})$ by

$$b_j(t) = \frac{\text{sh} \frac{\rho t}{2}}{\text{sh} \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t} \times a_j(t).$$

(3.17)

The bose field $b_j(t)$ satisfies the following commutation relation.

$$[b_j(t), b_k(t')] = -\frac{1}{t} \frac{\text{sh} \frac{(a_j|a_k)\pi t}{n} \text{sh} \frac{\rho t}{2}}{\text{sh} \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t} \delta(t + t').$$

(3.18)

Let us set

$$b_1^*(t) = -\sum_{j=1}^{n-1} b_j(t) \frac{\text{sh} \frac{(n-j)\pi t}{n}}{\text{sh} \pi t}.$$  

(3.19)

We have

$$[b_1^*(t), b_j(t')] = \delta_{j,1} \frac{1}{t} \frac{\text{sh} \frac{\rho t}{2}}{\text{sh} \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t} \delta(t + t').$$

(3.20)

Let us set the basic operators.

$$U_j(\delta) = : \exp \left(-\int_{-\infty}^{\infty} b_j(t)e^{i\delta t} \right) :, (1 \leq j \leq n - 1),$$

(3.21)

$$U_0(\delta) = : \exp \left(-\int_{-\infty}^{\infty} b_1^*(t)e^{i\delta t} dt \right) :.$$  

(3.22)

Free field realizations of the local operators $Z_j^\prime(\delta), (j = 0, \ldots, n - 1)$ are given by

$$Z_j^\prime(\delta) = \int_{-\infty}^{\infty} d\gamma_1 \cdots \int_{-\infty}^{\infty} d\gamma_j : U_0(\delta)U_1(\gamma_1) \cdots U_j(\gamma_j) : 
\times e^{\pi \frac{2\pi}{n} (\gamma_j - \delta)} \prod_{k=1}^{j} \Gamma \left( \frac{i(\gamma_k - \gamma_{k-1} + \frac{\pi}{n})}{\rho + \frac{2\pi}{n}} \right) \Gamma \left( \frac{i(\gamma_{k-1} - \gamma_k) + \frac{\pi}{n}}{\rho + \frac{2\pi}{n}} \right).$$  

(3.23)

Here we set $\gamma_0 = \delta$.

Note. Free field realizations of operators $Z_j^\prime(\delta)$ were given in the reference [13], in which authors considered correlation functions of the critical $A^{(1)}_{n-1}$ chain.

4 Boundary state

The purpose of this section is to give the free field realizations of the boundary state $|B\rangle$ and its dual $\langle B|$. We give free field realizations of the boundary state $|B\rangle$,

$$K(\beta)Z_j^\ast(\beta)|B\rangle = Z_j^\ast(-\beta)|B\rangle, \ (j = 0, \ldots, n - 1),$$  

(4.1)
and its dual state $\langle B |$,

$$K(\beta)^j_j(B)Z_j(-\beta + \lambda i) = \langle B | Z_j(\beta + \lambda i), \ (j = 0, \cdots, n-1; \lambda > 0). \quad (4.2)$$

Here we have used $K(\beta)^j_j$ given in (2.7). When we set $\lambda = \pi$, dual state $\langle B |$ becomes the dual boundary state of the affine $A^{(1)}_{n-1}$ Toda fields. When we set $\lambda = 0$, vacuum expectation value, $\langle B | Z_j^* (\beta_1) \cdots Z_j^* (\beta_N) | B \rangle$ produces an eigenvector of $A^{(1)}_{n-1}$ analogue of finite XXZ chain with double boundaries. See reference [16].

### 4.1 Boundary state

In this subsection we give the free field realization of the boundary state $| B \rangle$, and show the reflection realtion (4.1).

The boundary state $| B \rangle$ is realized as follows.

$$| B \rangle = e^B | \text{vac} \rangle. \quad (4.3)$$

Here $B$ is a quadratic part of free bosons.

$$B = \sum_{j,k=1}^{n-1} \int_0^\infty \alpha_{j,k}(t)a_j(-t)a_k(-t)dt + \sum_{j=1}^{n-1} \int_0^\infty \beta_j(t)a_j(-t)dt, \quad (4.4)$$

where scalars $\alpha_{j,k}(t)(j, k = 1, \cdots, n-1)$ and $\beta_j(t)$($j = 1, \cdots, n-1$) are given by

$$\alpha_{j,k}(t) = -\frac{t}{2} \times \frac{\text{sh} \left( \frac{\rho t}{2} \right)}{\text{sh} \left( \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t \right)} \times I_{j,k}(t), \quad (4.5)$$

$$\beta_j(t) = -\frac{\text{ch} \left( \frac{\rho t}{4} \right)}{\text{sh} \left( \left( \frac{\rho}{4} + \frac{\pi}{2n} \right) t \right)} \times \frac{\text{sh} \left( \frac{\pi t}{2} \right)}{\text{sh}(\pi t)} \times I_{j,j} \left( \frac{t}{2} \right) \times \sum_{k=1}^{n-1} I_{k,j}(t). \quad (4.6)$$

Here we have set the symmetric matrix $(I_{j,k}(t))_{1 \leq j,k \leq n-1}$ by

$$I_{j,k}(t) = \frac{\text{sh} \left( \frac{j \pi t}{n} \right) \text{sh} \left( \frac{(n-k) \pi t}{n} \right)}{\text{sh} \left( \frac{\pi t}{n} \right) \text{sh}(\pi t)} = I_{k,j}(t), \quad (1 \leq j \leq k \leq n-1). \quad (4.7)$$
We remark that the matrix $(I_{j,k}(t))_{1\leq j,k\leq n-1}$ is the inverse matrix of
\[
\left(\frac{\text{sh}(\alpha_j|\alpha_k)\pi t}{\text{sh} \frac{\alpha_j}{n}}\right)_{1\leq j,k\leq n-1}.
\]
Let us show that the boundary state $|B\rangle$ satisfies the following reflection relations.

\[
Z_j^*(\beta)|B\rangle = \varphi(\beta)Z_j^*(-\beta)|B\rangle.
\]

(4.8)
The presence of $e^B$ has an effect of a Bogoliubov transformation.

\[
e^{-B}a(t)e^B = a(t) + a(-t) + \sum_{j=1}^{n-1} A_{ij}(t)\beta_j(t), \quad (t \geq 0)
\]

\[
= a(t) + a(-t) + \frac{1}{t} \frac{\text{ch} \left( \frac{\rho t}{4} + \frac{\pi}{2n} t \right)}{\text{sh} \left( \frac{\rho t}{2n} \right)} \text{sh} \left( \frac{\pi t}{2n} \right),
\]

(4.9)

\[
e^{-B}a(-t)e^B = a(-t), \quad (t > 0).
\]

(4.10)

Therefore the basic operators $V_j(\alpha)$ act on the boundary state $|B\rangle$ as follows.

\[
V_j^n(\alpha)|B\rangle = G_j(\alpha)V_j^e(-\alpha)|B\rangle, \quad (j = 0, 1, \cdots, n).
\]

(4.11)

We have set

\[
G_j(\alpha) = -i 2^{1+j\frac{2\pi}{n}} e^\gamma \times \alpha, \quad (j = 1, \cdots, n-1),
\]

(4.12)

\[
G_0(\alpha) = 2^{-\frac{2\pi}{n}} e^{\frac{n\alpha}{2} (1+\frac{2\pi n}{n\rho})} \times \frac{\Gamma_2 \left( -2i\alpha + 2\pi - \frac{2\pi}{n} \middle| \rho, 4\pi \right) \Gamma_2 \left( -2i\alpha + \rho + 2\pi + \frac{2\pi}{n} \middle| \rho, 4\pi \right)}{\Gamma_2 \left( -2i\alpha \middle| \rho, 4\pi \right) \Gamma_2 \left( -2i\alpha + \rho + 4\pi \middle| \rho, 4\pi \right)},
\]

(4.13)

\[
G_n(\alpha) = G_0(\alpha).
\]

(4.14)

Using the above formulæ of the action of the basic operators, we get the actions of the Vertex opertaors $Z_j^*(\beta)$ as follows.

\[
Z_j^*(\beta)|B\rangle = (2\pi)^{-j} (\rho e^\gamma e^{-\frac{\pi}{\rho}})^{\frac{n}{2}} \prod_{k=0}^{j-1} \int_{C_i} d\alpha_1 \cdots \int_{C_j} d\alpha_j \prod_{k=0}^{j} V_k^e(\alpha_k)V_k^e(-\alpha_k)|B\rangle
\]

\[
\times \prod_{k=0}^{j} G_k(\alpha_k) \prod_{k=1}^{j} \Delta(\alpha_{k-1},\alpha_k) \prod_{k=1}^{j} S(\alpha_{k-1},\alpha_k).
\]

(4.15)

Here we have set the auxiliary functions $\Delta(\alpha_1,\alpha_2)$ and $S(\alpha_1,\alpha_2)$ by

\[
\Delta(\alpha_1,\alpha_2) = \prod_{\epsilon_1,\epsilon_2 = \pm} \Gamma \left( \frac{i(\epsilon_1\alpha_1 + \epsilon_2\alpha_2)}{\rho} - \frac{\pi}{n\rho} \right),
\]

(4.16)

\[
S(\alpha_1,\alpha_2) = e^{\frac{2\pi}{n}\alpha_1} - q e^{-\frac{2\pi}{n}\alpha_2}, \quad q = e^{-\frac{2\pi}{n\rho}}.
\]

(4.17)
The auxiliary function \( \Delta(\alpha_1, \alpha_2) \) is invariant under the change of variables \( \alpha_j \to -\alpha_j \),

\[
\Delta(\alpha_1, \alpha_2) = \Delta(-\alpha_1, \alpha_2) = \Delta(\alpha_1, -\alpha_2) = \Delta(-\alpha_1, -\alpha_2).
\] (4.18)

Now we arrive at

\[
Z^*_j(\beta)|B\rangle - \varphi(\beta)Z^*_j(-\beta)|B\rangle \\
= 2^{-j+1}\pi^{-j}(\rho e^\gamma e^{-\frac{\rho x}{2}})^{-1}(\pi^2)^{ij} \times \text{sh}\left(\frac{2\pi}{\rho}\beta\right) \int_{C^*_1} \frac{d\alpha_1}{\alpha_1} \cdots \int_{C^*_j} \frac{d\alpha_j}{\alpha_j} \\
\times \prod_{k=0}^{j} V^c_k(\alpha_k)V^c_k(-\alpha_k)|B\rangle \prod_{k=1}^{j} \Delta(\alpha_{k-1}, \alpha_k) \prod_{k=2}^{j} S(\alpha_{k-1}, \alpha_k) \prod_{k=1}^{j} G_k(\alpha_k). 
\] (4.19)

Here we have used the relation,

\[
\varphi(\beta) = \frac{G_0(\beta)}{G_0(-\beta)}. 
\] (4.20)

We change the integral variable \( \alpha_k \leftrightarrow -\alpha_k \), and the corresponding new contour \( \tilde{C}^*_k \). We find that the corresponding new contour \( \tilde{C}^*_k \) can be deformed to the same as \( C^*_k \). We find that the following part in the integrand :

\[
\prod_{k=1}^{j} \Delta(\alpha_{k-1}, \alpha_k) \prod_{k=0}^{j} V^c_k(\alpha_k)V^c_k(-\alpha_k)|B\rangle, 
\] (4.21)

is invariant under the change of variable \( \alpha_k \leftrightarrow -\alpha_k \). Summing up all changing of integral variables \( \pm \alpha_k, (k = 1, \cdots, j) \), we know that a sufficient condition of the relation,

\[
Z^*_j(\beta)|B\rangle - \varphi(\beta)Z^*_j(-\beta)|B\rangle = 0,
\] (4.22)

becomes the following polynomial identity,

\[
\sum_{\epsilon_1, \cdots, \epsilon_j = \pm} \prod_{k=2}^{j} S(\epsilon_{k-1}\alpha_{k-1}, \epsilon_k\alpha_k) \prod_{k=1}^{j} G_k(\epsilon_k\alpha_k) = 0.
\] (4.23)

Now we have derived the reflection relation (4.1).

4.2 Dual Boundary State

The dual boundary state \( \langle B | \) is realized as follows.

\[
\langle B | = \langle \text{vac} | e^G.
\] (4.24)
Here $G$ is a quadratic form of free bosons.

\[ G = \sum_{j,k=1}^{n-1} \int_0^\infty \gamma_{j,k}(t) a_j(t) a_k(t) dt + \sum_{j=1}^{n-1} \int_0^\infty \delta_j(t) a_j(t) dt, \quad (4.25) \]

where scalar $\gamma_{j,k}(t)(j,k = 1, \ldots, n-1)$ and $\delta_j(t)(j = 1, \ldots, n-1)$ are given by

\[ \gamma_{j,k}(t) = e^{-2\lambda t} \alpha_{j,k}(t), \quad \delta_j(t) = e^{-\lambda t} \beta_j(t). \quad (4.26) \]

Here $\alpha_{j,k}(t)$ and $\beta_j(t)$ are introduced in (4.4) and (4.10).

Let us show the dual boundary state $\langle B|$ satisfies the following reflection relations.

\[ \langle B| Z_j(-\beta + \lambda i) = \varphi(\beta) \langle B| Z_j(\beta + \lambda i), \quad (j = 0, \ldots, n-1). \quad (4.27) \]

The presence of $e^G$ has an effect of a Bogoliubov transformation.

\[ e^G a_l(-t)e^{-G} = a_l(-t) + e^{-2\lambda t} a_l(t) + \sum_{j=1}^{n-1} A_{lj}(t) \delta_j(t), \quad (t \geq 0) \]

\[ = a_l(-t) + e^{-2\lambda t} a_l(t) + \frac{e^{-\lambda t} \text{ch} \left( \left( \frac{\rho}{4} + \frac{\pi}{2n} \right) t \right)}{\text{sh} \left( \frac{\pi t}{2n} \right)} \text{sh} \left( \frac{\rho t}{4} \right), \quad (4.28) \]

\[ e^G a_l(t)e^{-G} = a_l(t), \quad (t > 0). \quad (4.29) \]

Therefore the basic operators $V_j(\alpha)$ act on the dual boundary state $\langle B|$ as follows.

\[ \langle B| V_j^\ast(\alpha + \lambda i) = G_j^\ast(\alpha) \langle B| V_j^a(-\alpha + \lambda i), \quad (j = 0, \ldots, n). \quad (4.30) \]

We have set

\[ G_j^\ast(\alpha) = G_j(-\alpha), \quad (j = 0, \ldots, n). \quad (4.31) \]

Here $G_j(\alpha)$ is defined in (4.12), (4.13) and (4.14).

We get the actions of the Vertex operators $Z_j(\beta)$ as follows.

\[ \langle B| Z_j(\beta + \lambda i) = c_j(2\pi)^{-n-j-1}(\rho e^{-\frac{\pi}{2n}})^{-(1+\frac{j}{n})} \int_{C_{j+1}} \cdots \int_{C_{n-1}} \int_{C_n} da_{j+1} \cdots da_{n-1} \]

\[ \times \langle B| \prod_{k=j+1}^n V_k^a(\alpha_k + \lambda i) V_k^a(-\alpha_k + \lambda i) \]

\[ \times \prod_{k=j+1}^n \Delta(\alpha_k, \alpha_{k+1}) \prod_{k=j+1}^{n-1} S(\alpha_k, \alpha_{k+1}), \quad (4.32) \]
where $\Delta(\alpha_1, \alpha_2)$ and $S(\alpha_1, \alpha_2)$ are given in (4.16) and (4.17). Now we have

$$\langle B | Z_j(\beta + \lambda i) - \varphi(\beta)^{-1} | B | Z_j(-\beta + \lambda i) = 2^{-n+j+2} c_j e^{-\frac{2\pi^2}{\rho n}} \pi^{-n+j+1} (\rho e^{-\frac{\pi}{2}})^{-1} (2\rho e^{-\frac{\pi}{2}})^{(n-1-j)} \sin \left( \frac{2\pi}{\rho} \beta \right) G^*_n(\beta)$$

$$\times \int_{C_{j+1}} d\alpha_{j+1} \cdots \int_{C_{n-1}} d\alpha_{n-1} | B \rangle \prod_{k=j+1}^{n} V^a_k(\alpha_k + \lambda i) V^a_k(-\alpha_k + \lambda i)$$

$$\times \prod_{k=j+1}^{n-1} \Delta(\alpha_k, \alpha_{k+1}) \prod_{k=j+1}^{n-2} S(\alpha_k, \alpha_{k+1}) \prod_{k=j+1}^{n-1} G^*_k(\alpha_k). \quad (4.33)$$

As the same arguments as the case of the boundary state, we get a sufficient condition of the reflection relation (4.2),

$$\sum_{\epsilon_{j+1} \cdots \epsilon_{n-1} = \pm} \prod_{k=j+1}^{n-2} S(\epsilon_k \alpha_k, \epsilon_{k+1} \alpha_{k+1}) \prod_{k=j+1}^{n-1} G^*_k(\epsilon_k \alpha_k) = 0. \quad (4.34)$$

Now we have derived the reflection relation (4.2).

In this section, we have constructed free field realizations of the boundary state $| B \rangle$ and its dual $\langle B |$.

## 5 Form factors

The purpose of this section is to derive the difference equations which govern the form factors, and to give integral representations for form factors.

Let us introduce matrix element, called form factors.

$$f^Z(\beta_1, \cdots, \beta_N)_{j_1 \cdots j_N} = \frac{\langle B | Z Z^*_j(\beta_1) \cdots Z^*_j(\beta_N) | B \rangle}{\langle B | B \rangle}.$$  

Here $Z^*_j(\beta)$ is the Zamolodchikov-Faddeev operators, $| B \rangle$ is the boundary state, and $Z = Z^*_{k_1}(\delta_1) \cdots Z^*_{k_Q}(\delta_Q)$, ($k_1, \cdots, k_Q = 0, \cdots, n - 1$). Let us set

$$f^Z(\beta_1, \cdots, \beta_N) = \sum_{j_1 \cdots j_N = 0}^{n-1} v_{j_1} \otimes \cdots \otimes v_{j_N} f^Z(\beta_1, \cdots, \beta_N)_{j_1 \cdots j_N}. \quad (5.1)$$
5.1 Boundary qKZ equations

The purpose of this section is to derive the difference equations which govern form factors. Let us introduce notation $Z^*(\beta)$, $Z^*(1)(\beta_1)Z^*(2)(\beta_2)$ and $Z^*(2)(\beta_2)Z^*(1)(\beta_1)$ as follows.

\[ Z^*(\beta) = \sum_{j=0}^{n-1} Z^*_j(\beta) \otimes v_j, \tag{5.2} \]
\[ Z^*(1)(\beta_1)Z^*(2)(\beta_2) = \sum_{j_1,j_2=0}^{n-1} Z^*_j(\beta_1)Z^*_j(\beta_2) \otimes v_{j_1} \otimes v_{j_2}, \tag{5.3} \]
\[ Z^*(2)(\beta_2)Z^*(1)(\beta_1) = \sum_{j_1,j_2=0}^{n-1} Z^*_j(\beta_2)Z^*_j(\beta_1) \otimes v_{j_1} \otimes v_{j_2}. \tag{5.4} \]

In this notation the commutation relation of the Zamolodchikov-Faddeev operators is written by

\[ Z^*(1)(\beta_1)Z^*(2)(\beta_2) = S_{12}(\beta_1 - \beta_2)Z^*(2)(\beta_2)Z^*(1)(\beta_1), \tag{5.5} \]

where the $S$-matrix $S_{12}(\beta)$ acts on the space $V \otimes V$. As the same manner we write the reflection relations (4.1), (4.2) as follows.

\[ Z^*(\beta)|B\rangle = K(-\beta)Z^*(-\beta)|B\rangle, \tag{5.6} \]
\[ \langle B|Z(\beta + i\lambda) = K(\beta)\langle B|Z(-\beta + i\lambda). \tag{5.7} \]

As the same manner as (4.1), (4.2), we have the following reflection relations.

\[ \langle B|Z^*(\beta + i\lambda) = K(\beta)\langle B|Z^*(-\beta + i\lambda), \tag{5.8} \]
\[ Z(\beta)|B\rangle = K(-\beta)Z(-\beta)|B\rangle. \tag{5.9} \]

In this notation the function $f^Z(\beta_1, \ldots, \beta_N)$ is written as

\[ f^Z(\beta_1, \ldots, \beta_N) = \frac{\langle B|ZZ^*(1)(\beta_1) \cdots Z^*(N)(\beta_N)|B\rangle}{\langle B|B\rangle}. \tag{5.10} \]

Let us derive the difference equations, which govern the form factor $f^Z(\beta_1, \ldots, \beta_N)$. Consider the vacuum expectation value,

\[ \langle B|ZZ^*(1)(\beta_1) \cdots Z^*(r)(\beta_r) \cdots Z^*(N)(\beta_N)|B\rangle. \tag{5.11} \]
Moving ZF operator $Z^{(r)}(\beta_r)$ to the right of ZF operators $Z^{(r+1)}(\beta_{r+1}) \cdots Z^{(N)}(\beta_N)$, and acting ZF operator $Z^{(r)}(\beta_r)$ on state $|B\rangle$, we have

$$S_{r+1}(\beta_r - \beta_{r+1}) \cdots S_N(\beta_r - \beta_N)K_r(-\beta_r) \times \langle B|Z Z^{(1)}(\beta_1) \cdots Z^{(r-1)}(\beta_{r-1}) Z^{(r+1)}(\beta_{r+1}) \cdots Z^{(N)}(\beta_N)Z^{(r)}(-\beta_r)|B\rangle (5.12)$$

Moving ZF operator $Z^{(r)}(-\beta_r)$ to the left and acting the dual state $\langle B|$, we have

$$\prod_{k=1}^{Q} \mathcal{L}(\delta_k + \beta_r) \times S_{r+1}(\beta_r - \beta_{r+1}) \cdots S_N(\beta_r - \beta_N)K_r(-\beta_r) \quad (5.13)$$
$$\times \langle B|Z^{(r)}(\beta_r + 2i\lambda)Z Z^{(1)}(\beta_1) \cdots Z^{(r-1)}(\beta_{r-1}) Z^{(r+1)}(\beta_{r+1}) \cdots Z^{(N)}(\beta_N)|B\rangle.$$

Moving ZF operators $Z^{(r)}(\beta_r)$ to the right, we have

$$\prod_{k=1}^{Q} \mathcal{L}(\delta_k + \beta_r) \times S_{r+1}(\beta_r - \beta_{r+1}) \cdots S_N(\beta_r - \beta_N)K_r(-\beta_r) \quad (5.14)$$
$$\times \langle B|Z Z^{(1)}(\beta_1) \cdots Z^{(r-1)}(\beta_{r-1}) Z^{(r)}(\beta_r + 2i\lambda)Z^{(r+1)}(\beta_{r+1}) \cdots Z^{(N)}(\beta_N)|B\rangle.$$

Now we arrive at the following system of difference equations.

$$f^Z(\beta_1, \ldots, \beta_{r-1}, \beta_r - 2i\lambda, \beta_{r+1}, \ldots, \beta_N) = \prod_{k=1}^{Q} \mathcal{L}(\delta_k + \beta_r - 2i\lambda)\mathcal{L}(\beta_r - \delta_k)$$
$$\times S_{r+1}(\beta_r - \beta_{r+1} - 2i\lambda) \cdots S_N(\beta_r - \beta_N - 2i\lambda)K_r(-\beta_r + 2i\lambda)$$
$$\times S_{r+1}(\beta_N + \beta_r - 2i\lambda) \cdots S_{r+1}(\beta_{r+1} + \beta_r - 2i\lambda)$$
$$\times S_{r-1}(\beta_{r-1} + \beta_r - 2i\lambda) \cdots S_{r-1}(\beta_1 + \beta_r - 2i\lambda)K_r(-\beta_r + i\lambda)$$
$$\times S_{r-1}(\beta_r - \beta_1) \cdots S_{r-1}(\beta_r - \beta_{r-1})f^Z(\beta_1, \ldots, \beta_{r-1}, \beta_r, \beta_{r+1}, \ldots, \beta_N).$$

(5.15)
As the same argument as (5.15), we get a similar equation.

\[
\begin{align*}
 f^Z(\beta_1, \cdots, \beta_{r-1}, \beta_r + 2i\lambda, \beta_{r+1}, \cdots, \beta_N) &= \prod_{k=1}^Q L(\delta_k - \beta_r - 2i\lambda)L(-\beta_r - \delta_k) \\
 \times S_{r-1,r}(\beta_{r-1} - \beta_r - 2i\lambda) \cdots S_{1,r}(\beta_1 - \beta_r - 2i\lambda)K_r(\beta_r + i\lambda) \\
 \times S_{r,1}(\beta_r - \beta_1) \cdots S_{r,r-1}(\beta_r - \beta_{r-1}) \\
 \times S_{r+1,r}(\beta_r - \beta_{r+1}) \cdots S_{r,N}(\beta_r - \beta_N)K_r(\beta_r) \\
 \times S_{N,r}(\beta_N - \beta_r) \cdots S_{r+1,r}(\beta_{r+1} - \beta_r) f^Z(\beta_1, \cdots, \beta_{r-1}, \beta_r, \beta_{r+1}, \cdots, \beta_N).
\end{align*}
\]

(5.16)

We call this systems of difference equations (5.15), (5.16) “boundary quantum Knizhnik-Zamolodchikov equations” (boundary qKZ equations). When we set parameter \(\lambda = \pi\), boundary qKZ equations (5.15), (5.16) describe form factors for the affine \(A_{n-1}^{(1)}\) Toda fields with boundary reflection.

### 5.2 Integral Representations

The purpose of this section is to give integral representations of form factors \(f^Z(\beta_1, \cdots, \beta_N)\).

In order to evaluate the above vacuum expectation value, we invoke free field realization of various quantities, the Zamolodchikov- Faddeev operators \(Z'_\lambda(\beta)\), the local operators \(Z'_k(\delta)\), the boundary state \(|B\rangle\) and its dual \(\langle B|\).

At first we present results of simple cases.

\[
\begin{align*}
\frac{\langle B|Z'_\lambda(\delta)|B\rangle}{\langle B|B\rangle} &= \exp \left( \int_C \frac{dt}{2\pi i t} \log(-t) \frac{1}{1 - e^{-\lambda t}} \frac{\text{ch}(\frac{\delta t}{2})}{\text{sh}(\frac{\delta t}{2} + \frac{n \pi t}{2})} \frac{\text{sh} \frac{\pi t}{2} \text{sh} \frac{(n-1)\pi t}{2}}{\text{sh} \pi t} (e^{-\lambda t - i\delta t} + e^{i\delta t}) \right) \\
& \quad - \int_C \frac{dt}{2\pi i t} \log(-t) \frac{1}{1 - e^{-2\lambda t}} \frac{\text{sh}(\frac{\delta t}{2})}{\text{sh}(\frac{\delta t}{2} + \frac{n \pi t}{2})} \frac{\text{sh} \frac{(n-1)\pi t}{n}}{\text{sh} \pi t} \left( \frac{1}{2} e^{-2\lambda t - 2i\delta t} + e^{-2\lambda t} + \frac{1}{2} e^{2i\delta t} \right),
\end{align*}
\]

(5.17)
and

\[
f^{\lambda_0(\delta)}(\beta)_{0} = \frac{\langle B|Z_{0}^\prime(\delta)Z_{0}^*(\beta)|B \rangle}{\langle B|B \rangle}
= e^{-(\gamma+\log(2\pi)\frac{n+1}{2n})} \times \exp \left( - \int_C \frac{dt}{2\pi i t} \log(-t) \frac{1}{1-e^{-\lambda t}} \frac{\text{sh}(\frac{\pi t}{4}) \text{sh}(\frac{(n-1)\pi t}{2n})}{\text{sh}(\frac{\pi t}{2})} \right)
\times \left\{ \text{ch}\left( \frac{\rho}{2} + \frac{\pi}{2n} \right) t (e^{i\beta t} + e^{-\lambda t - i\beta t}) - \text{ch}\left( \frac{\rho}{2} + \frac{\pi}{2n} \right) t (e^{i\beta t} + e^{-\lambda t - i\beta t}) \right\}
+ \int_C \frac{dt}{2\pi i t} \log(-t) \frac{1}{1-e^{-\lambda t}} \frac{\text{sh}(\frac{(n-1)\pi t}{n})}{\text{sh}(\frac{\pi t}{2})} \left\{ (e^{-2\lambda t - i\beta t} - i\beta t) + e^{-2\lambda t + i\beta t} + e^{2\lambda t + i\beta t} \right\}.
\]

(5.18)

Here the integrand contour \( C \) is given in Appendix A. It is easily seen that the above formula is re-written by multiple Gamma functions \( \Gamma_r(x|\omega_1 \cdots \omega_r) \) summarized in Appendix A. When we set \( \lambda = \pi \), we get form factor of the affine \( A_{n-1} \) Toda fields with boundary reflection. In a limiting case \( \rho \to \infty \), our formula reproduce form factor for the \( SU(n) \) invariant massive Thirring model with boundary reflection \[1, 12\]. When we set \( \lambda \to 0 \), the quantity \( f^{\lambda}(\beta_1, \cdots, \beta_N) \) produces an eigenvector of \( A_{n-1}^{(1)} \) analogue of finite XXZ chain with double boundaries \[11\].

Next we present general formulae of the form factors \( f^{\lambda}(\beta_1, \cdots, \beta_N)_{j_1 \cdots j_N} \), and, at the same time, explain how to evaluate the vacuum expectation values. Let us fix the indexes \( \{j_1, \cdots, j_N\} \), where \( j_1, \cdots, j_N \in \{0, 1, \cdots, n-1\} \), and \( \{k_1, \cdots, k_Q\} \), where \( k_1, \cdots, k_Q \in \{0, 1, \cdots, n-1\} \). We associate the integration variables \( \alpha_{j,r}, (1 \leq r \leq N, 1 \leq j \leq j_r) \) to the basic operator \( V_j(\alpha_{j,r}) \) contained in the Zamolodchikov-Faddeev operator \( Z_{j,r}^*(\beta_r) \). We also use the notation \( \alpha_{0,r} = \beta_r \). We associate the integration variables \( \gamma_{k,s}, (1 \leq s \leq Q, 1 \leq k \leq k_s) \) to the basic operator \( U_k(\alpha_{k,s}) \) contained in the local operators \( Z_{k,s}^*(\delta_s) \). We also use the notation \( \gamma_{0,s} = \delta_s \). Let us set the index set \( A_j \) and \( G_k \) by

\[
A_j = \{ r | j_r \geq j \}, \quad G_k = \{ s | k_s \geq k \}.
\]

By normal-ordering the product of the Zamolodchikov-Faddeev operators and the local
operators, we have the following formulae.

\[
f^2(\beta_1, \cdots, \beta_N)_{j_1, \cdots, j_N} = E(\{\beta\}|\{\delta\}) \prod_{r=1}^N \prod_{j=1}^{j_r} d\alpha_{j,r} \prod_{s=1}^Q \prod_{k=1}^{k_s} d\gamma_{k,s} I(\{\alpha\}|\{\gamma\})^{k_1, \cdots, k_Q}_{j_1, \cdots, j_N}. \tag{5.20}
\]

Here the integral contour \(C_j^*\) was defined below (3.11).

Here we set the leading factor \(E(\{\beta\}|\{\delta\})\) by

\[
E(\{\beta\}|\{\delta\}) = e^{-\frac{\pi}{n} (\delta_1 + \cdots + \delta_n)} e^{\pi \sum (\beta_1 + \cdots + \beta_N)} \times \prod_{1 \leq k_1 < k_2 \leq Q} C_{0,0}^{UU}(\delta_{k_1} - \delta_{k_2}) \prod_{1 \leq j_1 < j_2 \leq N} C_{0,0}^{VV}(\beta_{j_1} - \beta_{j_2}) \prod_{k=1}^Q \prod_{j=1}^N C_{0,0}^{UV}(\delta_{k} - \beta_{j}). \tag{5.21}
\]

Here we used abbreviations \(C_{0,0}^{UU}(\delta), C_{0,0}^{VV}(\beta),\) and \(C_{0,0}^{UV}(\delta),\) which were introduced in Appendix B. Here we set the integrand function by

\[
I(\{\alpha\}|\{\gamma\})^{k_1, \cdots, k_Q}_{j_1, \cdots, j_N} = \prod_{s=1}^Q e^{\frac{\pi}{n} \gamma_{k,s}} \prod_{k=1}^{k_s} \Gamma \left( \frac{i(\gamma_{k,s} - \gamma_{k-1,s}) + \frac{\pi}{n}}{\rho + \frac{2\pi}{n}} \right) \Gamma \left( \frac{i(\gamma_{k-1,s} - \gamma_{k,s}) + \frac{\pi}{n}}{\rho + \frac{2\pi}{n}} \right) \times \prod_{r=1}^N \prod_{j=1}^{j_r} e^{\frac{\pi}{n} \alpha_{j,r}} \Gamma \left( \frac{i(\alpha_{j,r} - \alpha_{j-1,r}) - \frac{\pi}{n}}{\rho} \right) \Gamma \left( \frac{i(\alpha_{j-1,r} - \alpha_{j,r}) - \frac{\pi}{n}}{\rho} \right) \times \prod_{k=1}^Q \prod_{s_{1} \in \mathcal{G}_{k-1,s_{2}} \in \mathcal{G}_{k}} C_{k-1,k}^{UU}(\gamma_{k,s_{1}} - \gamma_{k,s_{2}}) \times \prod_{r=1}^N \prod_{j=1}^{j_r} C_{j,j}^{VV}(\alpha_{j,r_1} - \alpha_{j,r_2}) \times \prod_{r=1}^N \prod_{j=1}^{j_r} C_{j-1,j}^{VV}(\alpha_{j-1,r_1} - \alpha_{j,r_2}) \times \prod_{s_{1} \in \mathcal{G}_{k-1,s_{2}} \in \mathcal{G}_{k}} C_{k,k}^{UV}(\gamma_{k,s} - \alpha_{k,r}) \times \prod_{r=1}^N \prod_{j=1}^{j_r} C_{k,k}^{UV}(\gamma_{k-1,s} - \alpha_{k-1,r}) \times \prod_{s_{1} \in \mathcal{G}_{k-1,s_{2}} \in \mathcal{G}_{k}} C_{k-1,k}^{UV}(\gamma_{k-1,s} - \alpha_{k,r}) \times J(\{\alpha\}|\{\gamma\})^{k_1, \cdots, k_Q}_{j_1, \cdots, j_N}. \tag{5.22}
\]

Here we used abbreviations \(C_{j_1,j_2}^{VV}(\beta), C_{j_1,j_2}^{UU}(\beta)\) and \(C_{j_1,j_2}^{UV}(\beta),\) which were introduced in
Appendix B. Here we have set

$$J(\{\alpha\}|\{\gamma\})_{J_1,\ldots,J_N}^{k_1,\ldots,k_Q}$$

$$= \frac{1}{\langle B|B \rangle} \times \langle B | \exp \left( \int_0^\infty \sum_{j=1}^{n-1} X_j(t) a_j(-t) dt \right) \exp \left( \int_0^\infty \sum_{j=1}^{n-1} Y_j(t) a_j(t) dt \right) |B \rangle, \quad \text{(5.23)}$$

where

$$X_j(t) = \frac{\text{sh} \left( \frac{(n-j) \pi t}{n} \right)}{\text{sh} \pi t} \left( -\sum_{r=1}^{N} e^{-i\beta_r t} + \frac{\text{sh} \left( \frac{\pi t}{2} \right)}{\text{sh} \left( \frac{\pi n}{2} \right)} \sum_{s=1}^{Q} e^{-i\delta_s t} \right)$$

$$+ \sum_{r \in A_j} e^{-ia_{j,r} t} - \frac{\text{sh} \left( \frac{\pi t}{2} \right)}{\text{sh} \left( \frac{\pi n}{2} \right)} \sum_{s \in \Omega_j} e^{-i\gamma_{j,s} t}, \quad \text{(5.24)}$$

$$Y_j(t) = \frac{\text{sh} \left( \frac{(n-j) \pi t}{n} \right)}{\text{sh} \pi t} \left( \sum_{r=1}^{N} e^{i\beta_r t} + \frac{\text{sh} \left( \frac{\pi t}{2} \right)}{\text{sh} \left( \frac{\pi n}{2} \right)} \sum_{s=1}^{Q} e^{i\delta_s t} \right)$$

$$+ \sum_{r \in A_j} e^{ia_{j,r} t} - \frac{\text{sh} \left( \frac{\pi t}{2} \right)}{\text{sh} \left( \frac{\pi n}{2} \right)} \sum_{s \in \Omega_j} e^{i\gamma_{j,s} t}. \quad \text{(5.25)}$$

Next we evaluate the vacuum expectation value, $J(\{\alpha\}|\{\gamma\})_{J_1,\ldots,J_N}^{k_1,\ldots,k_Q}$, and get a formulae without free field operators. For our purpose we use the coherent state, $|\xi_1,\ldots,\xi_{n-1}\rangle$ and its dual state $|\bar{\xi}_1,\ldots,\bar{\xi}_{n-1}\rangle$, defined by

$$|\xi_1,\ldots,\xi_{n-1}\rangle = \exp \left( \sum_{k=1}^{n-1} \int_0^\infty \xi_k(s) a_k(-s) ds \right) |\text{vac}\rangle, \quad \text{(5.26)}$$

$$\langle \bar{\xi}_1,\ldots,\bar{\xi}_{n-1}| = \langle \text{vac} | \exp \left( \sum_{k=1}^{n-1} \int_0^\infty \bar{\xi}_k(s) a_k(s) ds \right). \quad \text{(5.27)}$$

The coherent state enjoy

$$a_j(t)|\xi_1,\ldots,\xi_{n-1}\rangle = \sum_{k=1}^{n-1} A_{j,k}(t) \xi_k(t)|\xi_1,\ldots,\xi_{n-1}\rangle, \quad (t > 0), \quad \text{(5.28)}$$

$$\langle \bar{\xi}_1,\ldots,\bar{\xi}_{n-1}| a_j(-t) = \sum_{k=1}^{n-1} A_{j,k}(t) \bar{\xi}_k(t) \langle \bar{\xi}_1,\ldots,\bar{\xi}_{n-1}|, \quad (t > 0). \quad \text{(5.29)}$$

The following completeness relation by means of Feynmann path integral is useful.

$$id = c_F \times \int \prod_{k=1}^{n-1} \prod_{s>0} d\xi_k(s) d\bar{\xi}_k(s)$$

$$\times \exp \left( - \sum_{k_1,k_2=1}^{n-1} \int_0^\infty A_{k_1,k_2}(s) \xi_{k_1}(s) \bar{\xi}_{k_2}(s) ds \right) |\xi_1,\ldots,\xi_{n-1}\rangle \langle \bar{\xi}_1,\ldots,\bar{\xi}_{n-1}|. \quad \text{(5.30)}$$
Here the integration $\int d\xi d\bar{\xi}$ is taken over the entire complex plane with the measure $d\xi d\bar{\xi} = -2idxdy$ for $\xi = x + iy$. Here $c_F$ is a constant.

In what follows we use the abbreviations, $\tilde{\beta}_j(t)$, $\tilde{\delta}_j(t)$, $\tilde{X}_j(t)$, and $\tilde{Y}_j(t)$ defined by

$$\tilde{\beta}_j(t) = sh \left( \frac{\pi t}{2n} \right) \times \frac{ch \left( \left( \frac{\rho}{4} + \frac{\pi}{2n} \right)t \right)}{sh \left( \frac{\rho t}{4} \right)} = t \times \sum_{k=1}^{n-1} A_{j,k}(t) \beta_k(t), \quad (5.31)$$

$$\tilde{\delta}_j(t) = e^{-\lambda t} \times \tilde{\beta}_j(t) = t \times \sum_{k=1}^{n-1} A_{j,k}(t) \delta_k(t), \quad (5.32)$$

$$\tilde{X}_j(t) = t \times \sum_{k=1}^{n-1} A_{j,k}(t) X_k(t) \quad (5.33)$$

$$\tilde{Y}_j(t) = t \times \sum_{k=1}^{n-1} A_{j,k}(t) Y_k(t) \quad (5.34)$$

and

$$J(\{\alpha\}|\{\gamma\})_{j_1\cdots j_N}^{k_1\cdots k_Q} = \frac{1}{\langle vac | e^G e^B | vac \rangle} \times \langle vac | e^G \exp \left( \int_0^\infty \sum_{j=1}^{n-1} Y_j(t) a_j(t) dt \right) \exp \left( \int_0^\infty \sum_{j=1}^{n-1} X_j(t) a_j(-t) dt \right) e^B | vac \rangle$$

$$\times \exp \left( \int_C \log(-t) \frac{sh \left( \frac{\rho t}{2} \right)}{2\pi it} \frac{sh \left( \left( \frac{\rho}{4} + \frac{\pi}{2n} \right)t \right)}{\left( c_1,t_2 \right) \sum_{l_1,l_2=1}^{n-1} \tilde{X}_{l_1}(t) I_{l_1,l_2}(t) \tilde{Y}_{l_2}(t) dt} \right). \quad (5.35)$$
When we insert the completeness relation of the coherent states, we have without-operator formula.

\[
\langle \text{vac} | e^{G} \exp \left( \int_{0}^{\infty} \sum_{j=1}^{n-1} Y_{j}(t) a_{j}(t) dt \right) \exp \left( \int_{0}^{\infty} \sum_{j=1}^{n-1} X_{j}(t) a_{j}(-t) dt \right) | \text{vac} \rangle = c_{F} \times \int \prod_{j=1}^{n-1} d\xi_{j}(t)d\bar{\xi}_{j}(t)
\]

\[
\times \exp \left( \int_{0}^{\infty} \frac{1}{t} \sum_{j=1}^{n-1} (\tilde{Y}_{j}(t) + \tilde{\delta}_{j}(t))\xi_{j}(t)dt + \int_{0}^{\infty} \frac{1}{t} \sum_{j=1}^{n-1} (\tilde{X}_{j}(t) + \tilde{\beta}_{j}(t))\bar{\xi}_{j}(t)dt 
\right.
\]

\[
+ \int_{0}^{\infty} \sum_{l_{1},l_{2}=1}^{n-1} A_{l_{1},l_{2}}(t) \left( \frac{e^{-2\lambda t}}{2} \xi_{l_{1}}(t)\xi_{l_{2}}(t) - \xi_{l_{1}}(t)\bar{\xi}_{l_{2}}(t) + \frac{1}{2} \xi_{l_{1}}(t)\bar{\xi}_{l_{2}}(t) \right) dt 
\]

\]

(5.36)

Performing the Gauss integral calculation, \( \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \), (5.36) becomes the following, up to constant multiplicity.

\[
\exp \left( - \int_{C} \frac{dt}{2\pi it} \log(-t) \frac{1}{1 - e^{-2\lambda t}} \frac{\sh \left( \frac{gt}{2} \right)}{\sh \left( \frac{g}{2} + \frac{\pi}{n} \right) t} \right)
\]

\[
\times \left( \sum_{l_{1},l_{2}=1}^{n-1} I_{l_{1},l_{2}}(t) \left( \frac{e^{-2\lambda t}}{2} (\tilde{X}_{l_{1}}(t) + \tilde{\beta}_{l_{1}}(t))(\tilde{X}_{l_{2}}(t) + \tilde{\beta}_{l_{2}}(t)) 
\right.
\]

\[
+ (\tilde{X}_{l_{1}}(t) + \tilde{\beta}_{l_{1}}(t))(\tilde{Y}_{l_{2}}(t) + \tilde{\delta}_{l_{2}}(t)) + \frac{1}{2}(\tilde{Y}_{l_{1}}(t) + \tilde{\delta}_{l_{1}}(t))(\tilde{Y}_{l_{2}}(t) + \tilde{\delta}_{l_{2}}(t)) \right) \right) 
\]

(5.37)
Now we arrive at the following formula.

\[
J(\{\alpha\}|\{\gamma\})_{j_1,\ldots,j_N}^{k_1,\ldots,k_Q} = \exp\left( - \int_C \frac{dt}{2\pi it} \log(-t) \right) \frac{1}{1 - e^{-2\lambda t}} \frac{\sh\left( \frac{\rho t}{2} \right)}{\sh\left( \left( \frac{\rho}{2} + \frac{\pi}{n} \right) t \right)} \frac{1}{\sh\left( \frac{\pi t}{n} \sh(\pi t) \right)}
\]

\[
\times \left( \sum_{l=1}^{n-1} \sh\left( \frac{\pi l t}{n} \right) \sh\left( \frac{(n - l) \pi t}{n} \right) \left( e^{\frac{2\lambda t}{2}X_l(t)^2} + e^{\frac{2\lambda t}{2}X_l(t)\bar{Y}_l(t)} + \frac{1}{2} \bar{Y}_l(t)^2 \right) \right)
\]

\[
+ \sum_{1 \leq l_1 < l_2 \leq n-1} \sh\left( \frac{\pi l_1 t}{n} \right) \sh\left( \frac{(n - l_2) \pi t}{n} \right) \left( e^{-2\lambda t} \bar{X}_{l_1}(t) \bar{Y}_{l_2}(t) + e^{-2\lambda t} \bar{X}_{l_1}(t) \bar{Y}_{l_2}(t) \right)
\]

\[
+ e^{-2\lambda t} \bar{X}_{l_1}(t) \bar{Y}_{l_1}(t) + \bar{Y}_{l_1}(t) \bar{Y}_{l_2}(t)) \right) - \int_C \frac{dt}{2\pi it} \log(-t) \frac{1}{1 - e^{-\lambda t}} \frac{\ch\left( \frac{\rho t}{4} \right)}{\sh\left( \left( \frac{\rho}{4} + \frac{\pi}{2n} \right) t \right)}
\]

\[
\times \left( \frac{\sh\left( \frac{\pi t}{2} \right)}{\sh\left( \frac{\pi t}{2n} \sh(\pi t) \right)} \sum_{l=1}^{n-1} \sh\left( \frac{l \pi t}{2n} \right) \sh\left( \frac{(n - l) \pi t}{2n} \right) \left( e^{-\lambda t} \bar{X}_l(t) + \bar{Y}_l(t) \right) \right).
\]

(5.38)

Here \( \bar{X}_l(t) \), \( \bar{Y}_l(t) \) are defined in (5.33), (5.34). Here the integral contour \( C \) is as the same as those given in Appendix A. By this result, it is easily seen that \( J(\{\alpha\}|\{\gamma\})_{j_1,\ldots,j_N}^{k_1,\ldots,k_Q} \) is evaluated by multi-Gamma functions \( \Gamma_r(x|\omega_1 \cdots \omega_r) \), summarized in Appendix A. In a limiting case \( \rho \to \infty \), our integral formulae reproduce those of form factors for the \( SU(n) \) invariant massive Thirring model with boundary reflection \( \square \).

Let us summarize the result of this section. We present integral representations (5.20) for form factors of the local fields.

\[
f^Z(\beta_1,\ldots,\beta_N)_{j_1,\ldots,j_N}
\]

\[
= E(\{\beta\}|\{\delta\}) \prod_{r=1}^{N} \prod_{j_r=1}^{J_r} \int_{C^*_r} \frac{\alpha_{j_r,t}}{\gamma_{j_r,s}} \prod_{s=1}^{Q} \prod_{k_s=1}^{K_s} d\gamma_{k,s} I(\{\alpha\}|\{\gamma\})_{j_1,\ldots,j_N}^{k_1,\ldots,k_Q}.
\]

(5.39)

Here the factor \( E(\{\beta\}|\{\delta\}) \) is given by (5.24). The integrand \( I(\{\alpha\}|\{\gamma\})_{j_1,\ldots,j_N}^{k_1,\ldots,k_Q} \) is given in (5.22), where the factor \( J(\{\alpha\}|\{\gamma\})_{j_1,\ldots,j_N}^{k_1,\ldots,k_Q} \) is given in (5.38), and \( C^U_{j_1,j_2}(\alpha), C^V_{j_1,j_2}(\alpha) \) and \( C^T_{j_1,j_2}(\alpha) \) are given in Appendix B. The integral contour \( C^*_j \) is given below (3.14). For special cases we present more explicit formulae in (5.17), (5.18).
Fateev et al.\[18\] proposed an expression for vacuum expectation values of the special field, for boundary SG model, which is described by the following action

\[ \mathcal{A}_{FLZZ} = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \left( (\partial_x \varphi)^2 + (\partial_y \varphi)^2 \right) - \mu \int_{-\infty}^{\infty} dx \cos(\beta \varphi(x,0)). \] (5.40)

In this paper and Hou et al.\[11\], boundary SG model (affine \( A_1^{(1)} \) Toda fields) is the model which is described by the following action

\[ \mathcal{A}_{BSG} = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \left( (\partial_x \varphi)^2 + (\partial_y \varphi)^2 - \mu \cos(\varphi) \right). \] (5.41)

Our considering boundary SG model \( \mathcal{A}_{BSG} \) is different from Fateev et al.’s model \( \mathcal{A}_{FLZZ} \). V.Fateev, A.Zamolodchikov, Al.Zamolodchikov \[19\] studied Boundary Liouville conformal field theory. For a particular application, they present one point function of the special operator in the boundary SG model \( \mathcal{A}_{BSG} \). To reveal the connection between two formulae \[11\] and \[19\] is our future problem.

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## A Multiple Gamma function

Here we summarize the multiple gamma and the multiple sine functions, following [8]. Let us set the multiple Gamma function $\Gamma_r(x|\omega_1 \cdots \omega_r)$ by

$$
\log \Gamma_r(x|\omega_1 \cdots \omega_r) = \frac{(-1)^r}{r!} \gamma B_{r,r}(x|\omega_1 \cdots \omega_r) + \int_C \frac{e^{-xt} \log(-t)}{\prod_{j=1}^r (1-e^{-\omega_j t})} \frac{dt}{2\pi it},
$$

(A.1)

where the functions $B_{jj}(x)$ are the multiple Bernoulli polynomials defined by

$$
\frac{t^r e^{xt}}{\prod_{j=1}^r (e^{\omega_j t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r).
$$

(A.2)

Here $\gamma$ is Euler’s constant, $\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right)$. Here the contour of integral is given by

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0
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Contour $C$
Let us set the multiple sine function $S_r(x|\omega_1 \cdots \omega_r)$ by
\[
S_r(x|\omega_1 \cdots \omega_r) = \Gamma_r(x|\omega_1 \cdots \omega_r)^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x|\omega_1 \cdots \omega_r)^{(-1)^r}. \quad (A.3)
\]

The multiple Gamma function and the multiple sine function satisfy the recursion relations,
\[
\frac{\Gamma_r(x + \omega_1|\omega_1 \cdots \omega_r)}{\Gamma_r(x|\omega_1 \cdots \omega_r)} = \frac{1}{\Gamma_{r-1}(x|\omega_2 \cdots \omega_r)}, \quad (A.4)
\]
\[
\frac{S_r(x + \omega_1|\omega_1 \cdots \omega_r)}{S_r(x|\omega_1 \cdots \omega_r)} = \frac{1}{S_{r-1}(x|\omega_2 \cdots \omega_r)}. \quad (A.5)
\]

Here we understand $\Gamma_0 = x$. We have the following conditions.
\[
\Gamma_1(x|\omega) = \omega^{\frac{x}{\omega} - \frac{1}{2}} \frac{\Gamma\left(\frac{x}{\omega}\right)}{\sqrt{2\pi}}, \quad S_1(x|\omega) = 2 \sin\left(\frac{\pi x}{\omega}\right). \quad (A.6)
\]

We have
\[
\lim_{\rho \to \infty} S_1(x|\rho) = e^\gamma x, \quad \lim_{\rho \to \infty} S_2(x|\rho, \omega) = \frac{(2\pi)^{\frac{1}{2}}(\omega e^\gamma)^{\frac{1}{2} - \frac{x}{\omega}}}{\Gamma\left(\frac{x}{\omega}\right)}. \quad (A.7)
\]

Explicitly the multiple Bernoulli polynomials are written by
\[
B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2}, \quad (A.8)
\]
\[
B_{22}(x|\omega_1, \omega_2) = \frac{x^2}{\omega_1 \omega_2} - \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) x + \frac{1}{2} + \frac{1}{6} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right). \quad (A.9)
\]

**B Normal Ordering**

Here we list the formulas of the form
\[
X(\beta_1)Y(\beta_2) = C^{XY}(\beta_1 - \beta_2) : X(\beta_1)X(\beta_2) :, \quad (B.1)
\]

where $X, Y = U_j$, and $C^{XY}(\beta)$ is a meromorphic function on $\mathbb{C}$. These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral
\[
\int_0^\infty F(t)dt, \quad (B.2)
\]
which is divergent at \( t = 0 \). Here we adopt the following prescription for regularization: it should be understood as the contour integral,

\[
\int_C F(t) \frac{\log(-t)}{2\pi i} dt.
\]

(B.3)

Here we used the same contour \( C \) as the same as those in Appendix A.

The basic operators \( U_j(\beta), V_j(\beta) \) have the contraction relations, for \( 0 \leq j_1, j_2 \leq n \).

\[
U_{j_1}(\beta_1)U_{j_2}(\beta_2) = C_{j_1,j_2}^{UU}(\beta_1 - \beta_2) : U_{j_1}(\beta_1)U_{j_2}(\beta_2) :, \quad (B.4)
\]

\[
V_{j_1}(\beta_1)V_{j_2}(\beta_2) = C_{j_1,j_2}^{VV}(\beta_1 - \beta_2) : V_{j_1}(\beta_1)V_{j_2}(\beta_2) :, \quad (B.5)
\]

\[
U_{j_1}(\beta_1)V_{j_2}(\beta_2) = C_{j_1,j_2}^{UV}(\beta_1 - \beta_2) : U_{j_1}(\beta_1)V_{j_2}(\beta_2) :, \quad (B.6)
\]

\[
V_{j_1}(\beta_1)U_{j_2}(\beta_2) = C_{j_1,j_2}^{VU}(\beta_1 - \beta_2) : V_{j_1}(\beta_1)U_{j_2}(\beta_2) :, \quad (B.7)
\]

Here nonzero entries are given by

\[
C_{j,j}^{UU}(\alpha) = \frac{-i\alpha}{\rho + \frac{2\pi}{n}} e^{\frac{2\pi}{n}\alpha} (\gamma + \log(\rho + \frac{2\pi}{n})) \times \frac{\Gamma\left(\frac{-i\alpha + \rho}{\rho + \frac{2\pi}{n}}\right)}{\Gamma\left(\frac{-i\alpha + \frac{2\pi}{n}}{\rho + \frac{2\pi}{n}}\right)}, \quad (j = 1, \ldots, n - 1),
\]

(B.8)

\[
C_{j,j-1}^{UU}(\alpha) = C_{j-1,j}^{UU}(\alpha) = e^{-\frac{2\pi}{n}\alpha} (\gamma + \log(\rho + \frac{2\pi}{n})) \frac{\Gamma\left(\frac{-i\alpha + \rho}{\rho + \frac{2\pi}{n}}\right)}{\Gamma\left(\frac{-i\alpha + \rho + \frac{2\pi}{n}}{\rho + \frac{2\pi}{n}}\right)}, \quad (j = 1, \ldots, n),
\]

(B.9)

\[
C_{0,0}^{UU}(\alpha) = C_{n,n}^{UU}(\alpha) = e^{\frac{2\pi}{n}\alpha} \frac{\Gamma_2\left(-i\alpha + \rho + \frac{2\pi}{n}, 2\pi\right) \Gamma_2\left(-i\alpha + \rho + \frac{2\pi}{n}, 2\pi\right)}{\Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right) \Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right)}, \quad (B.10)
\]

\[
C_{0,n}^{UU}(\alpha) = C_{n,0}^{UU}(\alpha) = e^{\frac{2\pi}{n}\alpha} \frac{\Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right) \Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right)}{\Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right) \Gamma_2\left(-i\alpha + \frac{2\pi}{n}, 2\pi\right)}, \quad (B.11)
\]
\[ C_{j,j}^{VV}(\alpha) = -i\rho e^{2\gamma} e^{\frac{4\pi i}{\rho} (\gamma + \log \rho)} \alpha \times \frac{\Gamma\left(-\frac{i\alpha}{\rho} + 1 + \frac{2\pi}{n\rho}\right)}{\Gamma\left(-\frac{i\alpha}{\rho} - \frac{2\pi}{n\rho}\right)}, \quad (j = 1, \ldots, n-1), \]

(B.12)

\[ C_{j,j-1}^{VV}(\alpha) = C_{j-1,j}^{VV}(\alpha) = e^{-\frac{2\pi i + n\pi}{n\rho}(\gamma + \log \rho)} \frac{\Gamma\left(-\frac{i\alpha}{\rho} - \frac{\pi}{n\rho}\right)}{\Gamma\left(-\frac{i\alpha}{\rho} + 1 + \frac{\pi}{n\rho}\right)}, \quad (j = 1, \ldots, n), \]

(B.13)

\[ C_{0,0}^{VV}(\alpha) = C_{n,n}^{VV}(\alpha) = e^{\frac{\pi i}{n}(1 + \frac{2\pi}{n\rho})} \frac{\Gamma_2(-i\alpha + \rho + \frac{2\pi}{\rho} | \rho, 2\pi) \Gamma_2(-i\alpha + 2\pi - \frac{2\pi}{\rho} | \rho, 2\pi)}{\Gamma_2(-i\alpha | \rho, 2\pi) \Gamma_2(-i\alpha + 2\pi + \rho | \rho, 2\pi)}, \]

(B.14)

\[ C_{0,n}^{VV}(\alpha) = C_{n,0}^{VV}(\alpha) = e^{\frac{\pi i}{n}(1 + \frac{2\pi}{n\rho})} \frac{\Gamma_2(-i\alpha + \pi | \rho, 2\pi) \Gamma_2(-i\alpha + \pi + \rho | \rho, 2\pi)}{\Gamma_2(-i\alpha + \pi - \frac{2\pi}{\rho} | \rho, 2\pi) \Gamma_2(-i\alpha + \pi + \frac{2\pi}{\rho} | \rho, 2\pi)}. \]

(B.15)

\[ C_{j,j}^{UV}(\alpha) = -e^{-2\gamma} \left(\frac{\alpha + \pi i}{n}\right)^{-1} \left(\frac{\alpha - \pi i}{n}\right)^{-1}, \quad (j = 1, \ldots, n-1), \]

(B.16)

\[ C_{j,j-1}^{UV}(\alpha) = C_{j-1,j}^{UV}(\alpha) = -i\alpha e^{\gamma}, \quad (j = 1, \ldots, n), \]

(B.17)

\[ C_{0,0}^{UV}(\alpha) = C_{n,n}^{UV}(\alpha) = e^{-(\gamma + \log 2\pi) \frac{\pi i}{n}} \frac{\Gamma\left(\frac{\alpha}{2\pi} + \frac{1}{2n}\right)}{\Gamma\left(\frac{\alpha}{2\pi} + 1 - \frac{1}{2n}\right)}, \]

(B.18)

\[ C_{0,n}^{UV}(\alpha) = C_{n,0}^{UV}(\alpha) = e^{-\frac{\pi}{n}(\gamma + \log 2\pi)} \frac{\Gamma\left(\frac{\alpha}{2\pi} + \frac{1}{2} - \frac{1}{2n}\right)}{\Gamma\left(\frac{\alpha}{2\pi} + \frac{1}{2} + \frac{1}{2n}\right)}, \]

(B.19)

and

\[ C_{j_1,j_2}^{UU}(\beta) = C_{j_1,j_2}^{UV}(\beta), \quad (0 \leq j_1, j_2 \leq n). \]

(B.20)

Here we have set the supplemental basic operators \( U_n(\alpha) \) by

\[ U_n(\alpha) =: \exp \left( -\int_0^\infty b_{n-1}^* t e^{i\beta t} dt \right); \quad b_{n-1}^* (t) = -\sum_{j=1}^{n-1} b_j(t) \frac{\sin \frac{j\pi t}{n}}{n}, \]

(B.21)
In this Appendix we give an additional result for $U_q(\hat{sl}_2)$ case, and give some comments for $U_q(\hat{sl}_n)$ case. Let us introduce the operators $\hat{\Psi}^*_j(\beta), (j = 0, 1)$ by

$$
\hat{\Psi}^*_0(\beta) = V_0(\beta),
\hat{\Psi}^*_1(\beta) = \int_{C_1^*} d\alpha : V_0(\beta)V_1(\alpha) : e^{\pi \beta} \Gamma \left( \frac{i(\beta - \alpha)}{\rho} - \frac{\pi}{2\rho} \right) \Gamma \left( \frac{i(\alpha - \beta)}{\rho} - \frac{\pi}{2\rho} \right),
$$

where the integral contour $C_1^*$ is given in section 3. The operators $\hat{\Psi}^*_j(\beta)$ is slightly different from realizations of $Z_j^*(\beta)$ (3.11). Factor $e^{-\alpha \rho}$ drops in $\hat{\Psi}^*_j(\beta)$. However the operators $\hat{\Psi}^*_j(\beta)$ satisfy the same commutation relation (2.13), too. Let us introduce the boundary $K$-matrix $\hat{K}(\beta) \in \text{End}(\mathbb{C}^2)$ by

$$
\hat{K}(\beta) = \frac{\hat{G}_0(-\beta)}{\hat{G}_0(\beta)} \begin{pmatrix} 1 & 0 \\ 0 & \frac{-\frac{2\beta}{\rho} e^{\frac{\pi}{2\rho}}}{\frac{2\beta}{\rho} + \frac{\pi}{2\rho}} \end{pmatrix},
$$

where $\hat{G}_0(\beta)$ is given by (C.10). $\hat{K}(\beta)$ is the general diagonal boundary $K$-matrix associated with $S$-matrix $S(\beta)$ (2.1). (See [9]). Let us set the state $|\hat{B}\rangle$ by

$$
|\hat{B}\rangle = e^{\hat{B}}|\text{vac}\rangle.
$$

Here we have set

$$
\hat{B} = \int_0^\infty \hat{\alpha}_{1,1}(t)a_1(-t)a_1(-t)dt + \int_0^\infty \hat{\beta}_1(t)a_1(-t)dt,
$$

where

$$
\hat{\alpha}_{1,1}(t) = -\frac{t}{2} \times \frac{\text{sh} \frac{\pi t}{2}}{\text{sh} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) t} \times \frac{\text{sh} \frac{\pi t}{2}}{\text{sh} \pi t},
\hat{\beta}_1(t) = -\frac{\text{sh} \frac{\pi t}{2}}{\text{sh} \pi t} \times \left( \frac{\text{sh}(i\mu - \frac{\pi}{2} - \frac{\pi}{2})t}{\text{sh} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) t} + \text{sh} \frac{\pi t}{4} \frac{\text{ch} \frac{\pi t}{2}}{\text{sh} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) t} \right).
$$

We have

$$
V_0(\beta)|\hat{B}\rangle = \hat{G}_0(\beta)V_0(-\beta)|\hat{B}\rangle,
V_1(\alpha)|\hat{B}\rangle = \hat{G}_1(\alpha)V_1(-\alpha)|\hat{B}\rangle,
$$

(C.8)
(C.9)
where

\[
\hat{G}_0(\beta) = 2^{-\frac{\pi}{\rho}} e^{\frac{\pi}{\rho}(-1+\frac{\beta}{2})} \times \frac{\Gamma_2(-2i\alpha + \rho, 4\pi)\Gamma_2(-2i\alpha + \rho + 3\pi|\rho, 4\pi)}{\Gamma_2(-2i\alpha|\rho, 4\pi)\Gamma_2(-2i\alpha + \rho + 4\pi|\rho, 4\pi)} \\
\times \frac{\Gamma_2(-i\alpha + i\mu|\rho, 2\pi)\Gamma_2(-i\alpha - i\mu + 2\pi|\rho, 2\pi)}{\Gamma_2(-i\alpha - i\mu + \pi|\rho, 2\pi)},
\]

(C.10)

\[
\hat{G}_1(\alpha) = -ie^{\gamma}2^{1+\frac{\beta}{\rho}}(\rho e^{\gamma})^{1+\frac{\beta}{2}} \times \alpha \times \frac{\Gamma(1 + \frac{\pi}{2\rho} + \frac{i(\mu - \alpha)}{\rho})}{\Gamma(-\frac{\pi}{2\rho} + \frac{i(\mu - \alpha)}{\rho})}.
\]

(C.11)

As the same arguments as this paper we get

\[
\hat{K}(\beta) \hat{\Psi}^*_j(\beta) |\vec{B}\rangle = \hat{\Psi}^*_j(-\beta) |\vec{B}\rangle, \quad (j = 0, 1).
\]

(C.12)

It seems that this result is “homogeneous” version of paper [11].

At last we give some comments on \(U_q(\hat{sl}_n)\) case. Let us introduce the operators \(\hat{\Psi}^*_j(\beta), \quad (j = 0, \cdots, n - 1)\) by

\[
\hat{\Psi}^*_j(\beta) = \int_{C_1^j} d\alpha_1 \cdots \int_{C_1^j} d\alpha_j : V_0(\beta)V_1(\alpha_1) \cdots V_j(\alpha_j) : \\
\times e^{\frac{\pi}{\rho} \sum k=1^j} \prod \Gamma \left( \frac{i(\alpha_{k-1} - \alpha_k)}{\rho} - \frac{\pi}{n\rho} \right) \Gamma \left( \frac{i(\alpha_k - \alpha_{k-1})}{\rho} - \frac{\pi}{n\rho} \right).
\]

(C.13)

The operators \(\hat{\Psi}^*_j(\beta)\) are slightly different from realizations of \(Z_j^*(\beta)\) [3, 11]. Factor \(e^{-\frac{\alpha_j}{\rho}}\) drops in \(\hat{\Psi}^*_j(\beta)\). It is possible to construct “boundary state” \(|\vec{B}\rangle\) for general diagonal boundary \(K\)-matrix \(\hat{K}(\beta)\), associated with the \(S\)-matrix \(S(\beta)\) [2, 1], as the same manner as \(n = 2\) case. Relating to general diagonal boundary \(K\)-matrix, see Appendix of the paper [4],

\[
\hat{K}(\beta) \hat{\Psi}^*_j(\beta) |\vec{B}\rangle = \hat{\Psi}^*_j(-\beta) |\vec{B}\rangle, \quad (j = 0, \cdots, n - 1).
\]

(C.14)

However we cannot derive the commutation relations [2, 13] for \(n > 2\), under the scheme of the papers [14, 15]. Therefore we select the realizations of \(Z_j^*(\beta)\) in section 3. When we consider a limiting case \(\rho \to \infty\), two kind of operators \(Z_j^*(\beta)\) and \(\hat{\Psi}^*_j(\beta)\) become free field realizations of the Zamolodchikov-Faddeev operators \(Z_j^*(\beta)\) given in [4]. Therefore, in this limiting case, we have constructed the boundary state \(|B\rangle\) for general diagonal boundary \(K\)-matrix. See the papers [4, 2].

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