**IIB matrix model: Emergent spacetime from the master field**

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**Abstract**

We argue that the large-$N$ master field of the Lorentzian IIB matrix model may give rise to the points and metric of a classical spacetime.

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The IIB matrix model has been suggested as a nonperturbative formulation of type-IIB superstring theory. First results on the partition function of the Euclidean IIB matrix model were reported in Refs. \cite{3, 4}. Later, numerical simulations \cite{3, 7} of the Lorentzian IIB matrix model suggested the appearance of a 3 + 6 split of the nine spatial dimensions (matching Euclidean results were presented in Ref. \cite{8}). Still, the physical interpretation of the emergence of classical spacetime in Refs. \cite{1, 2, 5–8} is not really satisfactory, as there is no manifest small parameter to motivate a saddle-point approximation \cite{9}.

Recently, we have revived an old idea (the large-$N$ master field of Witten \cite{10}) for a possible origin of classical spacetime in the context of matrix models; see App. B of Ref. \cite{9} with the general argument. But, in that last reference, we did not give any details about where precisely in the master field the classical spacetime is encoded. In the present short paper, we try to be more explicit.

Before we set out on our search of classical spacetime, we have four preliminary remarks. First, we take the Lorentzian signature, because it is not clear how to interpret the “space-time” from the Euclidean IIB matrix model. Second, our discussion of the Lorentzian path integrals will be strictly formal, evading all convergence issues. Third, we introduce a length scale “$\ell$” into the IIB matrix model, in order to give a dimension of length to the bosonic matrix variable. Fourth, the focus of the present paper is solely on the IIB matrix model, but it may be that some of our results carry over to other matrix models \cite{11–13}.

We will now start by recalling the IIB matrix model and the concept of the master field, and will then turn to the emergence of the spacetime points and the spacetime metric.

The action of the Lorentzian IIB matrix model is given by

$$S[A, \Psi] = -\text{Tr} \left( \frac{1}{4} [A^\mu, A^\nu] [A^\rho, A^\sigma] \eta_{\mu\rho} \eta_{\nu\sigma} + \frac{1}{2} \overline{\Psi}_\beta \Gamma^\mu_\beta [A_\mu, \Psi_\alpha] \right),$$

(2.1a)

$$\eta_{\mu\nu} = \left[ \text{diag} (-1, 1, \ldots, 1) \right]_{\mu\nu},$$

(2.1b)

with vector indices $\mu \in (0, 1, \ldots, 9)$ and spinor indices $\alpha \in (1, 2, \ldots, 32)$. The vector $A^\mu$ and the Majorana–Weyl spinor $\Psi_\alpha$ are both $N \times N$ traceless Hermitian matrices, they live in a ten-dimensional spacetime consisting of a single point (a special case of the Eguchi–Kawai reduction \cite{14} operative in the large-$N$ limit; see Ref. \cite{15} for a review).
The action (2.1) is invariant under the following global gauge transformation:

\[ A^\mu \rightarrow \Omega A^\mu \Omega^\dagger, \]  
\[ \Psi_\alpha \rightarrow \Omega \Psi_\alpha \Omega^\dagger, \]  
\[ \Omega \in SU(N). \]  

In addition, there is the \( SO(1, 9) \) Lorentz invariance and an \( N = 2 \) supersymmetry. 

The partition function \( Z \) follows from the following Lorentzian “path” integral:

\[ Z = \int dA d\Psi \exp \left( i S[A, \Psi] / \ell^4 \right). \]  

Here, we have introduced a length scale “\( \ell \)”, so that \( A^\mu \) from (2.1) must have the dimension of length and \( \Psi_\alpha \) the dimension of \((\text{length})^{3/2}\).

As the fermions appear quadratically in the action, they can be integrated out and the partition function becomes

\[ Z = \int dA \exp \left( i S_{\text{eff}}[A] / \ell^4 \right), \]  

in terms of the effective action \( S_{\text{eff}}[A] \). For completeness, we mention that the integration measure \( dA \) in (2.3) and (2.4) is standard, except for the restriction to traceless matrices.

### III. MASTER FIELD

A particular gauge-invariant bosonic observable is given by

\[ w^{\mu_1 \ldots \mu_m} = \text{Tr} \left( A^{\mu_1} \ldots A^{\mu_m} \right). \]  

Its expectation values are given by the following Lorentzian path integrals:

\[ \langle w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots \rangle = Z^{-1} \int dA \left( w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots \right) \exp \left[ i S_{\text{eff}} / \ell^4 \right], \]  

with normalization factor \( Z \) from (2.4).

The expectation value (3.2) has the following factorization property:

\[ \langle w^{\mu_1 \ldots \mu_m} w^{\nu_1 \ldots \nu_n} \ldots w^{\omega_1 \ldots \omega_z} \rangle \overset{N}{=} \langle w^{\mu_1 \ldots \mu_m} \rangle \langle w^{\nu_1 \ldots \nu_n} \rangle \ldots \langle w^{\omega_1 \ldots \omega_z} \rangle, \]  

which holds to leading order in \( N \) (see Sec. III A of Ref. [15] for further discussion). From (3.3) follows the result that, to leading order in \( N \), the expectation value of the square of \( w \) equals the square of the expectation value of \( w \),

\[ \langle \left( w^{\mu_1 \ldots \mu_m} \right)^2 \rangle \overset{N}{=} \left( \langle w^{\mu_1 \ldots \mu_m} \rangle \right)^2. \]
which is a truly remarkable result for a statistical (quantum) theory.

According to Witten [10], the factorization results (3.3) and (3.4) imply that the path integrals (3.2) are saturated by a single configuration, the master field \( \hat{A}^\mu \). For the single observable \( w \) from (3.1), we then have

\[
\langle w^{\mu_1 \cdots \mu_m} \rangle_N = \text{Tr} \left( \hat{A}^{\mu_1} \cdots \hat{A}^{\mu_m} \right).
\]

(3.5)

In principle, it is possible that there is more than one master field, as long as these master fields give the same results for all possible observables of the type (3.1). For simplicity, we will talk, in the following, about a single master field.

The explicit expression for the IIB-matrix-model master field \( \hat{A}^\mu \) is not known. But it is possible to give an algebraic equation for it. Based on previous work by Greensite and Halpern [17], the IIB-matrix-model master field takes the following form [9]:

\[
\hat{A}_{ab}^\mu (\tau_{eq}) = e^{i(\hat{p}_a - \hat{p}_b)\tau_{eq}} \hat{a}_{ab}^\mu,
\]

(3.6a)

where \( \tau_{eq} \) must take a sufficiently large value (it traces back to the fictitious Langevin time \( \tau \) of stochastic quantization) and where the \( \tau \)-independent matrix \( \hat{a}^\mu \) on the right-hand side solves the following algebraic equation:

\[
i \left( \hat{p}_a - \hat{p}_b \right) \hat{a}_{ab}^\mu = -\frac{\delta S_{\text{eff}}}{\delta A_{\mu ba}}\bigg|_{A=a} + \hat{\eta}_{ab}^\mu,
\]

(3.6b)

in terms of the master momenta \( \hat{p}_a \) (uniform random numbers) and the master noise matrices \( \hat{\eta}_{ab}^\mu \) (Gaussian random numbers); see Ref. [17] for further details and Refs. [18, 19] for some interesting results.

IV. EMERGENT SPACETIME POINTS

As argued in App. B of Ref. [9], the only place where “classical spacetime” can reside is the master field \( \hat{A}^\mu \). But precisely where? In the following, we present a few rather naive ideas (hopefully, not too naive).

Following Refs. [5, 7], we begin by making a particular global gauge transformation [2.2a],

\[
\hat{A}^\mu = \Omega \hat{A}^\mu \Omega^+, \quad \Omega \in SU(N),
\]

(4.1a)
so that the transformed 0-component [singled out by the Minkowski “metric” (2.1b)] is diagonal and has ordered eigenvalues \( \alpha_i \in \mathbb{R} \),

\[
\hat{A}^0 = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N),
\]

\[
\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{N-1} \leq \alpha_N,
\]

\[
\sum_{i=1}^{N} \alpha_i = 0,
\]

where the last equality from tracelessness implies that some \( \alpha \)'s are negative and some positive.

Next, we introduce a continuous function \( \tilde{x}^0(\tilde{\sigma}) \equiv \tilde{c} \tilde{t}(\tilde{\sigma}) \) for \( \tilde{\sigma} \in (0, 1] \) by identifying (cf. Ref. [16])

\[
\tilde{x}^0(i/N) \equiv \tilde{c} \tilde{t}(i/N) \equiv \alpha_i,
\]

with \( i \in \{1, \ldots, N\} \) and a velocity \( \tilde{c} \) which is expected to be related to the vacuum velocity of light in the low-energy theory.

The problem, now, is how to extract the corresponding space coordinates \( \tilde{x}^m(\tilde{\sigma}) \) from the Hermitian \( \hat{A}_m \) matrices? The simplest idea (following Ref. [2]) is to calculate the eigenvalues of the nine matrices \( \hat{A}_m \), but, then, it is unclear how to order them with respect to the eigenvalues from (4.2). We will use a relatively simple procedure, which approximates the \( \hat{A}_m \) eigenvalues but still manages to order them along the diagonal.

We start from the following trivial observation: if \( M \) is an \( N \times N \) Hermitian matrix, then any \( n \times n \) block centered on the diagonal of \( M \) is also Hermitian, which holds for \( n \geq 1 \) and \( n \leq N \). With \( N \gg 1 \), we take \( n \) so that \( 1 \ll n \ll N \). Specifically, we proceed by the following six steps.

The first step is to let \( K \) be an odd divisor of \( N \), so that

\[
N = K n,
\]

\[
K = 2 L + 1,
\]

with both \( L \) and \( n \) positive integers (we have chosen an odd value of \( K \) for later convenience).

In the limit \( N \to \infty \), we also take \( K \to \infty \) but are not sure exactly how fast (with \( n \) staying finite or not).

The second step is to consider, in each of the ten matrices \( \hat{A}^\mu \) from (4.1) and (4.2), the \( K \) blocks of size \( n \times n \) centered on the diagonal.
The third step is to realize that we already know the diagonalized blocks of $\hat{A}_0^0$ from (4.2a), which allows us to define the following time coordinate $\hat{t}(\sigma)$ for $\sigma \in (0, 1]$:

$$\hat{x}^0(k/K) \equiv \hat{c} \hat{t}(k/K) \equiv \left( \frac{1}{n} \sum_{j=1}^{n} \hat{\alpha}_{k-1+j} \right) + \hat{c} \hat{t}_{\text{shift}},$$

(4.5)

with $k \in \{1, \ldots, K\}$, an arbitrary real constant $\hat{t}_{\text{shift}}$, and the velocity $\hat{c}$ mentioned below (4.3). The time coordinates from (4.5) are ordered,

$$\hat{t}(1/K) \leq \hat{t}(2/K) \leq \ldots \leq \hat{t}(1-1/K) \leq \hat{t}(1),$$

(4.6)

because the $\hat{\alpha}_i$ are, according to (4.2b). With an appropriate value of $\hat{t}_{\text{shift}}$ in (4.5), we can set $\hat{t} = 0$ for the half-way block at $k = L+1$. The blocks with $k < L+1$ will generically have negative time coordinates $\hat{t}$ and those with $k > L+1$ generically positive time coordinates $\hat{t}$.

The fourth step is to obtain the eigenvalues of the $n \times n$ blocks of the nine spatial matrices $\hat{A}_m^m$ and to denote these real eigenvalues $(\hat{\beta}_m^m)_i$, with $i \in \{1, \ldots, N\}$. How the $n$ eigenvalues are ordered in each block is irrelevant, as they will be averaged over in the next step.

The fifth step is to define, just as in step three, the following nine spatial coordinates $\hat{x}_m^m(\sigma)$ for $\sigma \in (0, 1]$:

$$\hat{x}_m^m(k/K) \equiv \frac{1}{n} \sum_{j=1}^{n} (\hat{\beta}_m^m)_{k-1+j},$$

(4.7)

with $k \in \{1, \ldots, K\}$.

The sixth and last step is, first, to observe that $\hat{t}(\sigma)$ from (4.5) and (4.6) is a nondecreasing function of $\sigma \equiv k/K$ and, then, to eliminate $\sigma$ between $\hat{t}(\sigma)$ from (4.5) and $\hat{x}_m^m(\sigma)$ from (4.7), in order to obtain

$$\hat{x}_m = \hat{x}_m(\hat{t}),$$

(4.8)

which corresponds to a particular foliation of what will become the classical spacetime.

If the master fields $\hat{A}_\mu$ are more or less block diagonal (as suggested by the numerical results from Refs. [5–7]) and if an appropriate value of $n$ can be chosen (for sufficiently large values of $N$), then the above steps may provide suitable spacetime points, which, in a somewhat different notation, are given by

$$\hat{x}_k = (\hat{x}_0^0, \hat{x}_k^m),$$

(4.9)

where $k$ runs over $\{1, \ldots, K\}$ and where each of the ten coordinates has the dimension of length, which traces back to the dimension of the bosonic matrix variable $A_\mu$ as discussed in Sec. [11]. These points (4.9) effectively build a spacetime manifold [with continuous (interpolating) coordinates $x^\mu$] if there is also an emerging metric $g_{\mu\nu}(x)$.
V. EMERGENT SPACETIME METRIC

In Sec. IV, we have obtained $\hat{x}_k^\mu$ as given by (4.9), which sample a ten-dimensional classical spacetime. (We have put a hat on our coordinates, in order to remind us of their master-field origin.) The idea is that low-energy fields propagate over these spacetime points. The low-energy fields include the matter fields (scalar, vector, spinor) and the metric field (tensor). In fact, Aoki et al. [2] have argued that the propagation of a matter field (for example, the propagation of a scalar field) determines the effective metric, which is found to depend on the density function of the spacetime points $\hat{x}_k^\mu$ and the correlations of these density functions.

The crucial result in Ref. [2] is Eq. (4.16), which we rewrite as follows:

$$g^{\mu\nu}(x) \sim \int d^{10}y \langle\langle \rho(y) \rangle\rangle (x - y)^\mu (x - y)^\nu f(x - y) r(x, y), \quad (5.1)$$

where the average $\langle\langle \rho(y) \rangle\rangle$ corresponds, for the procedure used in Sec. IV, to averaging over different block sizes and block positions along the diagonal in the master field.

The quantities that enter the integral (5.1) are the density function

$$\rho(x) \equiv \sum_{k=1}^{K} \delta^{(10)}(x - \hat{x}_k), \quad (5.2)$$

the dimensionless density correlation function $r(x, y)$ defined by

$$\langle\langle \rho(x) \rho(y) \rangle\rangle \equiv \langle\langle \rho(x) \rangle\rangle \langle\langle \rho(y) \rangle\rangle r(x, y), \quad (5.3)$$

and a sufficiently localized function $f(x)$, which appears in the effective action of a low-energy scalar degree of freedom $\phi$ “propagating” over the discrete spacetime points $\hat{x}_k^\mu$.

$$S_{\text{eff}}[\phi] \propto \sum_{k,l} \frac{1}{2} f(\hat{x}_k - \hat{x}_l) (\phi_k - \phi_l)^2 + \sum_k \frac{1}{2} \mu^2 \ell^{-2} (\phi_k)^2, \quad (5.4)$$

where $f(x) = f(x^0, x^1, \ldots, x^9)$ has dimension $1/(\text{length})^2$, $\mu$ is dimensionless, and $\ell$ is the model length scale introduced in (2.3). Here, $\phi_k$ is the field value at the point $\hat{x}_k$ and the continuous field $\phi(x)$ has $\phi(\hat{x}_k) = \phi_k$. The continuous field $\phi(x)$ is then found to have a standard kinetic term $g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ in the action, with the inverse metric given by (5.1); see Sec. 4.2 of Ref. [2] for further details. Incidentally, the inverse metric (5.1) is seen to be dimensionless and the metric $g_{\mu\nu}(x)$ is obtained as the matrix inverse of (5.1).

The outstanding tasks are to obtain the master field $\hat{A}^\mu$, to identify $\phi$ from it (cf. Sec. 4.1 of Ref. [2]), and to recover the effective action (5.4). The explicit results for $\rho(x)$, $f(x)$, and
\( r(x, y) \) must also explain how the inverse metric from (5.1) acquires a Lorentzian signature. Working with \( \ell = 1 \), we have performed a toy-model calculation with the function \( f_{\text{test}}(x) = \gamma + x^0 x^1 \) inserted into the integral of (5.1), where we also set \( \rho(x) = r(x, y) = 1 \) and cut the integration ranges off symmetrically. The resulting inverse metric at \( x = 0 \) is found to change continuously from a Euclidean to a Lorentzian signature as the parameter \( \gamma \) changes continuously. The conclusion is that, in principle, it is possible to obtain a Lorentzian inverse metric from the expression (5.1).

For the record, we give a further result, based on Eq. (4.17) of Ref. [2], which concerns the background value of the dilaton field \( \Phi \),

\[
\sqrt{-g(x)} \exp \left[-\Phi(x)\right] \propto \langle \langle \rho(x) \rangle \rangle,
\]

with \( g \equiv \text{det} \, g_{\mu\nu} \) and the meaning of the average on the right-hand side explained by the lines below (5.1).

Returning to the expression (5.1) for the emergent inverse metric, we observe that it depends not only on the density distribution \( \rho \) of emerged spacetime points and their correlation function \( r \), but also on the localization function \( f \) from the scalar effective action (5.4). In this way, the metric only exists if matter is present, which reminds us of Dicke’s interpretation of spacetime (see, e.g., p. 50 of App. 4 and p. 60 of App. 5 in Ref. [20]).

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