The probability representation as a new formulation of quantum mechanics

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Abstract. We present a new formulation of conventional quantum mechanics, in which the notion of a quantum state is identified via a fair probability distribution of the position measured in a reference frame of the phase space with rotated axes. In this formulation, the quantum evolution equation as well as the equation for finding energy levels are expressed as linear equations for the probability distributions that determine the quantum states. We also give the integral transforms relating the probability distribution (called the tomographic-probability distribution or the state tomogram) to the density matrix and the Wigner function and discuss their connection with the Radon transform. Qudit states are considered and the invertible map of the state density operators onto the probability vectors is discussed. The tomographic entropies and entropic uncertainty relations are reviewed. We demonstrate the uncertainty relations for the position and momentum and the entropic uncertainty relations in the tomographic-probability representation, which is suitable for an experimental check of the uncertainty relations.

1. Introduction

Recently a new formulation of conventional quantum mechanics was suggested [1–3]. In this formulation, the notion of a quantum state is identified with the probability distribution called the tomographic-probability distribution or the state tomogram. This new formulation of quantum mechanics is called probability representation of quantum mechanics, and it is closely related to quasiprobability distributions of quantum states like the Wigner function [4], the Husimi–Kano Q-quasidistribution [5, 6], or the quasidistributions determining the diagonal representation of the density operator (P(α) function [7] or ϕ(z) function [8]). All these quasidistributions are different representations of the density operator, and the idea for their introduction was to find a quantum state representation that is similar to the classical probability density on the phase space determining the classical-system state. However, all quasiprobability distributions introduced have the same property: they are not fair probability densities on the phase space, which cannot exist in view of the uncertainty relations [9–12] that prohibit simultaneous measurements of two random variables like position and momentum.
It turned out to be possible to introduce a tomographic-probability distribution as a notion of quantum states, which does not contradict the uncertainty relations, since this distribution depends only on one random position; it does not contain a dependence on the conjugate momentum. All statistic characteristics of quantum observables like means, variances, covariances, . . . can be calculated in terms of measurable tomographic-probability distributions.

The aim of this paper is to present a review of the probability representation of quantum mechanics and to discuss possible experiments to check the uncertainty relations and the other quantum inequalities like the entropic inequalities [13–18], which are based on the concepts of Shannon entropy [19] and Rényi entropy [20].

In section 2, we give the definition of optical and symplectic tomograms for both, classical- and quantum-system states. In section 3, we present the basic classical and quantum equations in the probability representation. In section 4, we consider the uncertainty relations in tomographic form and review the properties of entropy in the tomographic-probability representation, including quantum inequalities. In section 5, we demonstrate the probability-representation scheme for spin (qudit) systems. We conclude in section 6 with prospectives.

2. Optical and symplectic tomograms

Let us consider a single-mode field with density operators \( \hat{\rho} \), which are Hermitian (i.e., \( \hat{\rho}^\dagger = \hat{\rho} \)), nonnegative (i.e., \( \hat{\rho} \geq 0 \)), and normalized (i.e., \( \text{Tr} \hat{\rho} = 1 \)). The quantum states of this field can be identified with the probability density \( \omega(X, \theta) \) of a random position \(-\infty < X < \infty\) depending on an extra angle parameter \( 0 < \theta \leq 2\pi \) via the relation

\[
\hat{\rho} = \frac{1}{2\pi^2} \int_0^\pi d\theta \int_{-\infty}^{+\infty} d\eta \, \omega(X, \theta) |\eta| \exp \left[ i\eta(X - \hat{q}\cos\theta - \hat{p}\sin\theta) \right].
\]

(1)

\( \hat{q} \) and \( \hat{p} \) are the position and momentum operators, respectively, i.e., \([\hat{q}, \hat{p}] = i1\), and we assume the reduced Planck constant \( \hbar = 1 \).

The probability density \( w(X, \theta) \) is called the optical tomogram; in quantum optics the random variable \( X \) is called homodyne photon quadrature and the angle \( \theta \) is called the local oscillator phase. In experiments on homodyne detection of quantum photon states [21,22], the local oscillator phase is a parameter under control and the quadrature \( X \) is measured. The output in an experiment with homodyne photon-state detection is the function \( w(X, \theta) \). It has the property [23]

\[
w(X, \theta + \pi) - w(-X, \theta) = 0,
\]

(2)

which can be obtained from the inverse relationship

\[
w(X, \theta) = \text{Tr} \hat{\rho} \delta \left( X - \hat{q}\cos\theta - \hat{p}\sin\theta \right).
\]

(3)
The optical tomogram is a nonnegative and normalized function: \( w(X, \theta) \geq 0 \) and \( \int w(X, \theta) \, dX = 1 \). It was used in experiments with a homodyne photon detector for reconstructing the Wigner function

\[
W(q, p) = \text{Tr} \hat{\rho} \hat{U}(q, p),
\]

where the operator \( \hat{U}(q, p) \) is connected to the parity operator \( \hat{P} \) by

\[
\hat{U}(q, p) = 2 \exp [2 i (p \hat{q} - q \hat{p})] \hat{P}.
\]

The density operator \( \hat{\rho} \) is given in terms of the Wigner function as follows:

\[
\hat{\rho} = \frac{1}{\pi} \int W(q, p) \exp [2 i (p \hat{q} - q \hat{p})] \hat{P} \, dq \, dp.
\]

The Wigner function is real and normalized, \( (2 \pi)^{-1} \int W(q, p) \, dq \, dp = 1 \), and the relation of the Wigner function to the optical tomogram is

\[
w(X, \theta) = \frac{1}{2 \pi} \int W(q, p) \delta(X - q \cos \theta - p \sin \theta) \, dq \, dp,
\]

with the inverse relation

\[
W(q, p) = \frac{1}{2 \pi} \int_0^{\pi} d\theta \int_{-\infty}^{\infty} d\eta \, dX \, w(X, \theta) |\eta| \exp [i \eta (X - q \cos \theta - p \sin \theta)].
\]

Due to relations (7) and (8) found in [24,25], the optical tomogram can be used as a tool to reconstruct the Wigner function, which in turn was identified with the quantum state. A proposal to identify the tomogram itself with the quantum state was given in [1]. The physical meaning of the optical tomogram \( w(X, \theta) \) can be elucidated if one considers this tomogram as the concept of a state in classical statistical mechanics. In fact, such a state is traditionally identified with the probability density \( f(q, p) \) on the phase space. This function is nonnegative \( f(q, p) \geq 0 \) and normalized \( \int f(q, p) \, dq \, dp = 1 \).

Applying the Radon transform to this function, we obtain the optical tomogram of the classical state as

\[
w(X, \theta) = \int f(q, p) \delta(X - q \cos \theta - p \sin \theta) \, dq \, dp.
\]

Since the position \( X \) is given as \( X = q \cos \theta + p \sin \theta \), it has the physical meaning of the particle’s position measured in a reference frame in the phase space with initial axes rotated by \( \theta \). Thus, the optical tomogram is the probability density of the particle position measured in a rotated reference frame in the phase space. The information on the classical state coded by the probability density \( f(q, p) \) of two random variables \( q \) and \( p \) is equivalent to the information coded by the probability density of one random position \( X \) and the extra parameter \( \theta \) labeling the reference frame where this position is measured. This physical meaning of the optical tomogram is preserved in the quantum domain as well. The statistical characteristics of the random variables in both, the classical and quantum domains, can be understood in view of this clarified interpretation of the optical tomogram.
In fact, for $\theta = 0$ the random position $X$ has the physical meaning of the particle coordinate $q$, and
\[ \int f(q, p)q^n dq dp = \int w(X, 0)X^n dX. \] (10)
For $\theta = \pi/2$, the random position $X$ has the physical meaning of the particle momentum $p$, and
\[ \int f(q, p)p^n dq dp = \int w(X, \pi/2)X^n dX. \] (11)
Analogously, the same formulas exist in the quantum domain, namely,
\[ \text{Tr} \hat{\rho} q^n = \int w(X, 0)X^n dX, \quad \text{Tr} \hat{\rho} p^n = \int w(X, \pi/2)X^n dX. \] (12)
One can easily see that in the probability representation of quantum mechanics the formulas for highest moments of position and momentum are identical in both classical and quantum domains and have the standard form provided by conventional probability theory.

The mean value of energy in classical statistical mechanics for a system with the standard Hamiltonian $H = (p^2/2) + V(q)$ can be written in the tomographic form
\[ \int f(q, p) \left[ (p^2/2) + V(q) \right] dq dp = \int \left( X^2/2 \right) w(X, 0) dX + \int V(X)w(X, \pi/2) dX. \] (13)
An analogous formula
\[ \text{Tr} \hat{\rho} \left[ (\hat{p}^2/2) + V(\hat{q}) \right] = \int \left( X^2/2 \right)w(X, 0) dX + \int V(X)w(X, \pi/2) dX \] (14)
is valid for the quantum system.

Thus, the formulas for kinetic and potential energy means in classical and quantum mechanics are identical when expressed in terms of the tomographic-probability distributions.

One can introduce another tomographic-probability density called the symplectic tomogram $M(X, \mu, \nu)$, where a random variable $X$ is measured in a specific reference frame and real parameters $\mu$ and $\nu$ can be interpreted as parameters that provide a scaling transform of the reference-frame axes in the particle phase space $q \rightarrow \mu q$ and $p \rightarrow \nu p$. Another interpretation of these parameters consists in the fact that they have the form $\mu = s \cos \theta$ and $\nu = s^{-1} \sin \theta$, where $\theta$ is the rotation angle used in the definition of optical tomogram and the scaling parameter $s$ provides the scaling transform of the axes $q \rightarrow sq$ and $p \rightarrow s^{-1} p$. In such an interpretation, $X$ is the particle position measured in a reference frame in the phase space with axes that were first rescaled (parameter $s$) and then rotated (parameter $\theta$).

The symplectic tomogram of the quantum state is defined as
\[ M(X, \mu, \nu) = \text{Tr} \hat{\rho} \delta (X - \mu \hat{q} - \nu \hat{p}). \] (15)
For the classical state, the symplectic tomogram is defined in view of the probability density $f(q, p)$ on the phase space
\[ M(X, \mu, \nu) = \int f(q, p) \delta (X - \mu q - \nu p) dq dp. \] (16)
Both formulas (15) and (16) have an inverse; in the quantum domain, one has
\[
\hat{\rho} = \frac{1}{2\pi} \int M(X, \mu, \nu) \exp \left[ i \left( X - \mu \hat{q} - \nu \hat{p} \right) \right] dX d\mu d\nu,
\]
and in the classical domain, one has
\[
f(q, p) = \frac{1}{4\pi^2} \int M(X, \mu, \nu) \exp \left[ i \left( X - \mu q - \nu p \right) \right] dX d\mu d\nu.
\]
Since the Dirac delta-function is a homogeneous function, i.e., \( \delta(\lambda y) = (|\lambda|)^{-1} \delta(y) \), both quantum and classical tomograms have the homogeneity property \( M(\lambda X, \lambda \mu, \lambda \nu) = (|\lambda|)^{-1} M(X, \mu, \nu) \). This means that the symplectic tomogram satisfies
\[
\left( X \frac{\partial}{\partial X} + \mu \frac{\partial}{\partial \mu} + \nu \frac{\partial}{\partial \nu} + 1 \right) M(X, \mu, \nu) = 0.
\]
The symplectic tomogram given for \( \mu = \cos \theta \) and \( \nu = \sin \theta \) yields the optical tomogram
\[
w(X, \theta) = M(X, \cos \theta, \sin \theta)
\]
and, due to the homogeneity property, the optical tomogram provides the symplectic tomogram
\[
M(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \mu^2}} w \left( \frac{X}{\sqrt{\mu^2 + \mu^2}}, \tan^{-1} \frac{\beta}{\mu} \right),
\]
where relation (2) must be taken into account. In view of relation (20), the measurement of the optical tomogram of the photon state also provides the symplectic tomogram, and both tomograms can be used for obtaining the statistics of photon quadratures.

The set of optical tomograms of classical states and the set of optical tomograms of quantum states are different sets. In fact, for the optical tomogram \( w(X, \theta) \) of a given quantum state, integral (17) must be a nonnegative operator, and the nonnegativity condition for the operator provides the constrains for the set of quantum optical tomograms. Also for the optical tomogram \( w(X, \theta) \) of a given classical state, the probability density \( f(q, p) \) on the particle’s phase space reads
\[
f(q, p) = \frac{1}{4\pi^2} \int_{\theta}^{\pi} \int_{-\infty}^{+\infty} d\theta dX w(X, \theta) |\eta| \exp \left[ i\eta(X - q \cos \theta - p \sin \theta) \right].
\]
The probability density must be a nonnegative function, the optical tomogram of the given classical state must have the above Radon integral as a nonnegative function, and this nonnegativity condition for the integral provides the constrains on the set of all optical tomograms of classical states. Both sets of optical tomograms (of classical and quantum states) have a common part. This intersection contains all normal probability densities which respect the quadrature uncertainty relations; this means that the normal distribution has the form
\[
w(X, \theta) = \frac{1}{\sqrt{2\pi \sigma_{XX}}} \exp \left( -\frac{(X - \bar{X})^2}{2\sigma_{XX}^2} \right),
\]
where
\[
\sigma_{XX} = \cos^2 \theta \sigma_{qq} + \sin^2 \theta \sigma_{pp} + \sin 2\theta \sigma_{qp}.
\]
The real variances $\sigma_{qq}$ and $\sigma_{pp}$ and covariance $\sigma_{qp}$ have to satisfy the Schrödinger–Robertson uncertainty relation [9–12]

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq 1/4. \tag{24}$$

The symplectic tomogram of a classical Gaussian state with the probability density

$$f_G(q,p) = (4\pi^2 \det \sigma)^{-1/2} \exp \left[-\frac{1}{2} (\hat{p}, \hat{q}) \sigma^{-1} \left( \begin{array}{c} \hat{p} \\ \hat{q} \end{array} \right) \right], \tag{25}$$

where $\hat{p} = p - \langle p \rangle$, $\hat{q} = q - \langle q \rangle$, and the symmetric dispersion matrix $\sigma = \left( \begin{array}{cc} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{array} \right)$, is given by a formula analogous to (22),

$$M_G(X, \mu, \nu) = \left[2\pi \sigma_{XX}(\mu, \nu)\right]^{-1/2} \exp \left[-\frac{1}{2} \frac{(X - \bar{X}(\mu, \nu))^2}{\sigma_{XX}(\mu, \nu)} \right]. \tag{26}$$

Here, the mean value of position $X$ is expressed in terms of the position and momentum means $\langle q \rangle$ and $\langle p \rangle$ as $\bar{X}(\mu, \nu) = \mu \langle q \rangle + \nu \langle p \rangle$ and the dispersion $\sigma_{XX}(\mu, \nu)$ is

$$\sigma_{XX}(\mu, \nu) = \mu^2 \sigma_{qq} + \nu^2 \sigma_{pp} + 2 \mu \nu \sigma_{qp}. \tag{27}$$

For an arbitrary Gaussian quantum state with the density operator

$$\hat{\rho} = Z^{-1} \exp \left[ A\hat{q}^2 + B\hat{p}^2 + C \left( \hat{q}\hat{p} + \hat{p}\hat{q} \right) + D\hat{q} + E\hat{p} \right], \tag{28}$$

in which the quadratic form under the exponent is a Hermitian operator and $Z = \text{Tr} \left\{ \exp \left[ A\hat{q}^2 + B\hat{p}^2 + C \left( \hat{q}\hat{p} + \hat{p}\hat{q} \right) + D\hat{q} + E\hat{p} \right] \right\}$, the optical tomogram has the Gaussian form (22), and the symplectic tomogram has the Gaussian form (26) with the parameters (23) and (27) providing the normal distributions.

As an example of a Gaussian quantum state, we present the optical and symplectic tomograms of the coherent state $|\alpha\rangle$

$$w_\alpha(X, \theta) = \pi^{-1/2} \exp \left[- \left( X - \sqrt{2} \text{Re} \alpha \cos \theta - \sqrt{2} \text{Im} \alpha \sin \theta \right)^2 \right] \tag{29}$$

and

$$M_\alpha(X, \mu, \nu) = \frac{1}{\sqrt{\pi (\mu^2 + \nu^2)}} \exp \left[- \frac{\left( X - \sqrt{2} \text{Re} \alpha \mu - \sqrt{2} \text{Im} \alpha \nu \right)^2}{\mu^2 + \nu^2} \right]. \tag{30}$$

One can see that, in the tomographic-probability representation, both classical and quantum Gaussian states are given by the same normal distributions. The difference in these Gaussian states is described by the difference in the set of parameters $\sigma_{qq}$, $\sigma_{pp}$, and $\sigma_{qp}$, which in the quantum domain must satisfy the Schrödinger–Robertson inequality (24).

The symplectic tomogram of a quantum state can be presented in terms of the wave function $\psi(y)$ [3, 26]

$$M_\psi(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \psi(y) \exp \left[ i \left( \frac{\mu}{2\nu} y^2 - \frac{X}{\nu} y \right) \right] dy \right|^2, \tag{31}$$

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq 1/4.$$
and, analogously, the optical tomogram is

\[ W_\psi(X, \theta) = \frac{1}{2\pi|\sin \theta|} \left| \int \psi(y) \exp \left[ i \left( \frac{\cos \theta}{2 \sin \theta} y^2 - \frac{Xy}{\sin \theta} \right) \right] dy \right|^2. \] (32)

The tomogram of any quantum pure state (31) satisfies the condition

\[ \frac{1}{2\pi} \int M_\psi(X, \mu, \nu) M_\psi(-Y, \mu, \nu) e^{i(X-Y)DX} dY d\mu d\nu = 1, \] (33)

where the integral on the left-hand side coincides with the state purity parameter \( \tilde{\pi} = \text{Tr} \hat{\rho}^2 \). For other quantum states, this integral must be nonnegative and smaller than unity, since the purity parameter belongs to the domain \( 0 < \tilde{\pi} \leq 1 \).

Since the density operator \( \hat{\rho} \) is expressed in terms of symplectic tomogram by equation (17), for tomograms \( M_\psi(X, \mu, \nu) \) satisfying equality (33) one can introduce the wave function, both in classical and quantum mechanics, viy the relation

\[ \psi(x)\psi^*(x') = \frac{1}{2\pi} \int M_\psi(X, \mu, \nu) e^{iX(x | e^{-i\mu\hat{q} - i\nu\hat{p}} | x')} dX d\mu d\nu. \] (34)

Employing the equality

\[ \langle x | e^{-i\mu\hat{q} - i\nu\hat{p}} | x' \rangle = e^{-i(\mu^2/2) - i\mu x} \delta(x - x' - \nu), \] (35)

one obtains the wave function from the expression

\[ \psi(x)\psi^*(x') = \frac{1}{2\pi} \int M_\psi(X, \mu, x - x') \exp \left( iX - i\mu \frac{x + x'}{2} \right) dX d\mu. \] (36)

This equality (36) can take place only for the cases where the matrix on its right-hand side is nonnegative since the density matrix \( \psi(x)\psi^*(x') \) is the matrix of a nonegative operator \( | \psi \rangle \langle \psi | \), which is a rank-one projector.

3. Quantum and classical evolution equations in the tomographic probability representation

In the previous section, we considered classical and quantum single-mode field states in the probability representation. Since this representation is a simple change of variables given by an integral transform of the density matrix or the probability distribution function on the phase space, one can easily obtain all the classical and quantum equations, employing this change of variables.

First, we generalize the above considerations to multimode field states. In the conventional form, the multimode classical state is described by normalized probability density \( f(\vec{q}, \vec{p}) \), where \( \vec{q} = \{q_k\}, \vec{p} = \{p_k\}, k = 1, 2, \ldots, N \). The optical tomogram of this classical state is

\[ w(\vec{X}, \vec{\theta}) = \int f(\vec{q}, \vec{p}) \prod_{k=1}^{N} \delta \left( X_k - q_k \cos \theta_k - p_k \sin \theta_k \right) d\vec{q} d\vec{p}, \] (37)

where \( \vec{X} = \{X_k\} \) and \( \vec{\theta} = \{\theta_k\} \).
The symplectic tomogram of a classical state is defined as

$$M(\vec{X}, \vec{\mu}, \vec{\nu}) = \int f(\vec{q}, \vec{p}) \left[ \prod_{k=1}^{N} \delta (X_k - \mu_k q_k - \nu_k p_k) \right] d\vec{q} d\vec{p},$$  

(38)

where $\vec{\mu} = \{\mu_k\}$ and $\vec{\nu} = \{\nu_k\}$.

For quantum multimode field states with the density operator $\hat{\rho}$, the optical tomogram is defined as

$$w(\vec{X}, \vec{\theta}) = \text{Tr} \left[ \hat{\rho} \prod_{k=1}^{N} \delta (X_k - \hat{q}_k \cos \theta_k - \hat{p}_k \sin \theta_k) \right],$$  

(39)

and the symplectic tomogram reads

$$M(\vec{X}, \vec{\mu}, \vec{\nu}) = \text{Tr} \left[ \hat{\rho} \prod_{k=1}^{N} \delta (X_k - \mu_k q_k - \nu_k \hat{p}_k) \right].$$  

(40)

The conventional classical probability density obeys the Liouville evolution equation

$$\left[ \frac{\partial}{\partial t} + \sum_{k=1}^{N} \left( \frac{\partial}{\partial q_k} - \frac{\partial V}{\partial q_k} \frac{\partial}{\partial p_k} \right) \right] f(\vec{q}, \vec{p}, t) = 0.$$  

(41)

In the position representation, the conventional von Neumann equation for the density matrix $\rho(\vec{x}, \vec{x}', t)$ reads

$$i \frac{\partial}{\partial t} \rho(\vec{x}, \vec{x}', t) = \left[ -\frac{1}{2} \sum_{j=1}^{N} \left( \frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial x_j'^2} \right) + \left[ V(\vec{x}) - V(\vec{x}') \right] \right] \rho(\vec{x}, \vec{x}', t).$$  

(42)

For both classical and quantum evolution equations, we assume that in the corresponding Hamiltonian the masses are one in the kinetic energy term and $V$ is the potential energy term.

Applying transforms (37) and (38) to the probability density $f(\vec{q}, \vec{p}, t)$, we arrive at the optical and symplectic tomographic forms of the Liouville evolution equation

$$\left\{ \frac{\partial}{\partial t} - \sum_{j=1}^{N} \left[ \cos^2 \theta_j \frac{\partial}{\partial \theta_j} - \frac{\sin \theta_j}{2} \left( 1 + X_j \frac{\partial}{\partial X_j} \right) \right] - \sum_{k=1}^{N} \left[ \frac{\partial V}{\partial q_k} \left( q_1 \to \sin \theta_1 \frac{\partial}{\partial \theta_1} \left( \frac{\partial}{\partial X_1} \right)^{-1} + X_1 \cos \theta_1 + i \frac{\sin \theta_1}{2} \frac{\partial}{\partial X_1} \right) \right. \right.$$  

$$+ X_2 \cos \theta_2 + i \frac{\sin \theta_2}{2} \frac{\partial}{\partial X_2}, q_2 \to \sin \theta_2 \frac{\partial}{\partial \theta_2} \left( \frac{\partial}{\partial X_2} \right)^{-1} + X_2 \cos \theta_2 + i \frac{\sin \theta_2}{2} \frac{\partial}{\partial X_2}, \ldots, $$  

$$q_j \to \sin \theta_j \frac{\partial}{\partial \theta_j} \left( \frac{\partial}{\partial X_j} \right)^{-1} + X_j \cos \theta_j + i \frac{\sin \theta_j}{2} \frac{\partial}{\partial X_j}, \ldots, q_N \to \sin \theta_N \frac{\partial}{\partial \theta_N} \left( \frac{\partial}{\partial X_N} \right)^{-1} + X_N \cos \theta_N + i \frac{\sin \theta_N}{2} \frac{\partial}{\partial X_N} \right\} w(\vec{X}, \vec{\theta}, t) = 0,$$  

(43)

and

$$\left\{ \frac{\partial}{\partial t} - \nu_j \frac{\partial}{\partial \mu_j} - \sum_{k=1}^{N} \frac{\partial V}{\partial q_k} \left[ q_1 \to - \frac{\partial}{\partial \mu_1} \left( \frac{\partial}{\partial X_1} \right)^{-1}, q_2 \to - \frac{\partial}{\partial \mu_2} \left( \frac{\partial}{\partial X_2} \right)^{-1}, \ldots, \right.$$

$$q_j \to - \frac{\partial}{\partial \mu_j} \left( \frac{\partial}{\partial X_j} \right)^{-1}, q_N \to - \frac{\partial}{\partial \mu_N} \left( \frac{\partial}{\partial X_N} \right)^{-1} \right\} w(\vec{X}, \vec{\mu}, \vec{\nu}, t) = 0,$$  

(44)
respectively.

Applying transforms (39) and (40) to the density operator in the position representation, we arrive at the tomographic form of equation (42); thus the evolution equation for the optical tomogram reads [27]:

\[
\frac{\partial}{\partial t} - \sum_{j=1}^{N} \left[ \cos^2 \theta_j \frac{\partial}{\partial \theta_j} - \frac{1}{2} \sin 2 \theta_j \left( 1 + X_j \frac{\partial}{\partial X_j} \right) \right] \left( q_{1} \rightarrow \sin \theta_1 \frac{\partial}{\partial \theta_1} \left( \frac{\partial}{\partial X_1} \right)^{-1} \right) + X_1 \cos \theta_1 + i \frac{\sin \theta_1}{2} \frac{\partial}{\partial X_1}, \quad q_2 \rightarrow \sin \theta_2 \frac{\partial}{\partial \theta_2} \left( \frac{\partial}{\partial X_2} \right)^{-1} + X_2 \cos \theta_2 + i \frac{\sin \theta_2}{2} \frac{\partial}{\partial X_2}, \ldots,
\]

\[
q_j \rightarrow \sin \theta_j \frac{\partial}{\partial \theta_j} \left( \frac{\partial}{\partial X_j} \right)^{-1} + X_j \cos \theta_j + i \frac{\sin \theta_j}{2} \frac{\partial}{\partial X_j}, \ldots, \quad q_N \rightarrow \sin \theta_N \frac{\partial}{\partial \theta_N} \left( \frac{\partial}{\partial X_N} \right)^{-1} + X_N \cos \theta_N + i \frac{\sin \theta_N}{2} \frac{\partial}{\partial X_N} - \text{c.c.}
\]

and the evolution equation for symplectic tomogram is [1–3]:

\[
\frac{\partial}{\partial t} - \sum_{j=1}^{N} \nu_j \frac{\partial}{\partial \mu_j} - \frac{1}{i} V \left( q_{1} \rightarrow \frac{\partial}{\partial \mu_1} \left( \frac{\partial}{\partial X_1} \right)^{-1} + i \frac{\nu_1}{2} \frac{\partial}{\partial X_1}, \quad q_2 \rightarrow \frac{\partial}{\partial \mu_2} \left( \frac{\partial}{\partial X_2} \right)^{-1} + i \frac{\nu_2}{2} \frac{\partial}{\partial X_2}, \ldots, q_j \rightarrow \frac{\partial}{\partial \mu_j} \left( \frac{\partial}{\partial X_j} \right)^{-1} + i \frac{\nu_j}{2} \frac{\partial}{\partial X_j}, \ldots, \quad q_N \rightarrow \frac{\partial}{\partial \mu_N} \left( \frac{\partial}{\partial X_N} \right)^{-1} + i \frac{\nu_N}{2} \frac{\partial}{\partial X_N} - \text{c.c.} \right) \}
\]

\[w(\vec{X}, \vec{\theta}, t) = 0. \]

By an analogous change of variables, one can obtain the tomographic form of the equation providing the energy spectrum of the quantum states [27]. In fact, one has the Schrödinger equation for the eigenvalues of the Hamiltonian

\[\left[ \frac{\hat{p}^2}{2} + V(\hat{q}) \right] | \psi_E \rangle = E | \psi_E \rangle, \]

and this equation can be rewritten in the form of an equation for the density operator \(\hat{\rho}_E = | \psi_E \rangle \langle \psi_E | \) as follows:

\[\frac{1}{2} \left\{ \frac{\hat{p}^2}{2} + V(\hat{q}), \hat{\rho}_T \right\} = E \hat{\rho}_E. \]

Applying transforms (39) and (40) to the density operator in the position representation, we arrive at the integro-differential equations for the energy spectrum of the Hamiltonian (47). These equations are equivalent to the standard Schrödinger equation for a fair probability density; they read

\[
E w_E(\vec{X}, \vec{\theta}) = \sum_{\sigma=1}^{N} \left\{ \cos^2 \theta_{\sigma} \left[ \frac{\partial^2}{\partial X_{\sigma}^2} + 1 \right] \frac{\partial}{\partial X_{\sigma}} - X_{\sigma} \left[ \frac{\partial}{\partial X_{\sigma}} \right]^{-1} \left( \cos^2 \theta_{\sigma} + \sin 2 \theta_{\sigma} \frac{\partial}{\partial \theta_{\sigma}} \right) + X_{\sigma}^2 \sin^2 \theta_{\sigma} - \frac{\cos^2 \theta_{\sigma}}{8} \frac{\partial^2}{\partial X_{\sigma}^2} \right\} w_E(\vec{X}, \vec{\theta})
\]

\[+ \left[ \text{Re} V \left\{ \sin \theta_{\sigma} \frac{\partial}{\partial \theta_{\sigma}} \left[ \frac{\partial}{\partial X_{\sigma}} \right]^{-1} + X_{\sigma} \cos \theta_{\sigma} + i \frac{\sin \theta_{\sigma}}{2} \frac{\partial}{\partial X_{\sigma}} \right\} \right] w_E(\vec{X}, \vec{\theta}). \]
for the optical tomogram and
\[
EM_E(\vec{X}, \vec{\mu}, \vec{\nu}) = \left[ \sum_{\sigma=1}^{N} \left\{ \frac{1}{2} \left[ \frac{\partial}{\partial X_{\sigma}} \right]^{-2} \left[ \frac{\partial}{\partial \nu_{\sigma}} \right]^{-2} - \frac{\mu_{\sigma}^2}{8} \left[ \frac{\partial}{\partial X_{\sigma}} \right]^{-2} \right\} \right] M_E(\vec{X}, \vec{\mu}, \vec{\nu})
\]
\[
+ \left[ \text{Re} V \left\{ - \left[ \frac{\partial}{\partial X_{\sigma}} \right]^{-1} \frac{\partial}{\partial \mu_{\sigma}} + \frac{i\nu_{\sigma}}{2} \frac{\partial}{\partial X_{\sigma}} \right\} \right] M_E(\vec{X}, \vec{\mu}, \vec{\nu}) \tag{50}
\]
for symplectic tomogram.

4. Uncertainty relations in the tomographic form

The Schrödinger–Robertson uncertainty relations (24) can be written for an arbitrary local oscillator phase \( \theta \); in this case, they take a form of the integral inequality for the measurable optical tomogram of the quantum state [3, 28, 29]:
\[
F(\theta) = \left( \int X^2 w(X, \theta) dX - \left[ \int X w(X, \theta) dX \right]^2 \right) \left( \int X^2 w \left( X, \theta + \frac{\pi}{2} \right) dX \right)
\]
\[
- \left[ \int X w \left( X, \theta + \frac{\pi}{2} \right) dX \right] \left[ \int X^2 w(X, \theta) dX \right] \left[ \int X^2 w \left( X, \theta + \frac{\pi}{2} \right) dX \right] - \left[ \int X w \left( X, \theta + \frac{\pi}{4} \right) dX \right] \left[ \int X^2 w \left( X, \theta + \frac{\pi}{4} \right) dX \right] \left[ \int X^2 w \left( X, \theta + \frac{\pi}{2} \right) dX \right] \left[ \int X^2 w \left( X, \theta + \frac{\pi}{4} \right) dX \right] \right)
\]
\[
- \left\{ \int \left[ \int X w \left( X, \theta + \frac{\pi}{4} \right) dX \right] \right\}^2 - \frac{1}{4} \geq 0. \tag{51}
\]

The positivity of the function \( F(\theta) \) for an arbitrary local oscillator phase cannot be violated in the quantum domain, but the optical tomograms of classical states can violate the positivity of \( F(\theta) \).

The uncertainty relations for position and momentum follow from the nonnegativity of the Hermitian density operator, which must have only nonnegative eigenvalues, but for systems like the oscillator the number of eigenvalues is infinite. In view of this, there exists an infinite hierarchy of the uncertainty relations, which include the inequalities not only for dispersions and means of the position and momentum, but also those for all highest moments of these quantum observables. The experimental check of the inequalities became possible only recently, when the homodyne photon detection was employed to obtain the optical tomogram of the photon quantum state [21].

The following inequality exists for quadratures that depend on two quantum states [30]
\[
\frac{1}{2} \left[ \int w_1(X, \theta) X^2 dX - \left( \int w_1(X, \theta) X dX \right)^2 \right] \left[ \int w_2(X, \theta + \pi/2) X^2 dX \right] - \left( \int w_2(X, \theta + \pi/2) X dX \right)^2 \right]
\]
\[
\times \left[ \int w_1(X, \theta + \pi/2) X^2 dX - \left( \int w_1(X, \theta + \pi/2) X dX \right)^2 \right] - \left\{ \int w_1(X, \theta + \pi/4) X^2 dX \right. - \left( \int w_1(X, \theta + \pi/4) X dX \right)^2 \right)
\]
\[
- \left( \int w_1(X, \theta + \pi/4) X dX \right)^2 - \frac{1}{2} \left[ \int w_1(X, \theta) X^2 dX - \left( \int w_1(X, \theta) X dX \right)^2 \right]
\]
\[-\frac{1}{2} \left[ \int w_1(X, \theta + \pi/2)X^2 dX - \left( \int w_1(X, \theta + \pi/2)X dX \right)^2 \right] \right] - \left( \int w_2(X, \theta + \pi/4)X dX \right) - \frac{1}{2} \left[ \int w_2(X, \theta + \pi/4)X^2 dX - \left( \int w_2(X, \theta)X dX \right)^2 \right] - \frac{1}{2} \left[ \int w_2(X, \theta + \pi/2)X^2 dX - \left( \int w_2(X, \theta + \pi/2)X dX \right)^2 \right] \geq \frac{1}{4}, \] (52)

which is a tomographic form of the state-extended uncertainty relations introduced by D. A. Trifonov [31, 32].

Other quantum inequalities exist, which must be fulfilled by measurable optical and symplectic tomograms.

For example, the Shannon entropy \( S_\theta = - \int w(X, \theta) \ln w(X, \theta) dX \) is associated with the optical tomogram \( w(X, \theta) = |\psi(X, \theta)|^2 \), where the complex function \( \psi(X, \theta) \) is given in the form of propagating initial wave function \( \psi(y) \) under the influence of the harmonic-oscillator potential [3]

\[ \psi(X, \theta) = \frac{1}{\sqrt{2\pi i \sin \theta}} \int \exp \left[ \frac{i}{2} \left( \cot \theta (y^2 + X^2) - \frac{2X}{\sin \theta} y \right) \right] \psi(y) dy, \] (53)

must satisfy the inequality

\[- \int w(X, \theta) \ln w(X, \theta) dX - \int w(X, \theta + \pi/2) \ln w(X, \theta + \pi/2) dX \geq \ln \pi e. \] (54)

The other inequality must be fulfilled as a property of Rényi entropy [3]; it reads

\[ (q - 1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[ w \left( X, \theta + \frac{\pi}{2} \right) \right]^{1/(1-q)} \right\} + (q + 1) \ln \left\{ \int_{-\infty}^{\infty} dX [w(X, \theta)]^{1/(1+q)} \right\} \geq \frac{1}{2} \{ (q - 1) \ln[\pi(1 - q)] + (q + 1) \ln[\pi(1 + q)] \}. \] (55)

The quantum entropic inequality for the symplectic tomogram can be written with respect to the photon distribution function as follows:

\[ H = - \sum_{n=0}^{\infty} \frac{1}{2\pi} \int \frac{e^{-X^2/(\mu^2 + \nu^2)}}{\sqrt{\pi(\mu^2 + \nu^2)}} \frac{1}{2^n n!} H_n^2 \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right) M(-Y, \mu, \nu)e^{i(X-Y)}dX dY d\mu d\nu \times \ln \left\{ \frac{1}{2\pi} \int \frac{e^{-X^2/(\mu^2 + \nu^2)}}{\sqrt{\pi(\mu^2 + \nu^2)}} \frac{1}{2^n n!} H_n^2 \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right) M(-Y, \mu, \nu)e^{i(X-Y)}dX dY d\mu d\nu \right\} \geq 0, \] (56)

which means that the quantum-state tomogram must satisfy this inequality since the photon distribution in any quantum state must be nonnegative normalized function of discrete variable which is the photon number. In the classical domain, the function expressed in terms of the classical-state tomogram can be negative or complex. In view of this fact, the entropy value calculated in terms of the measurable tomogram can provide an information on accuracy of the measurement, as well as an information on the classicality or quantumness of the state.
5. Qubit states in the probability representation

The spin states can also be described by probability distributions instead of spinors and density matrices [33,34]. The spin tomogram of a qudit state is determined by the formula

$$w(m, u) = \langle m | u^\dagger \rho u | m \rangle,$$  \hspace{1cm} (57)

where $\rho$ is the density matrix and $| m \rangle$ is the state corresponding to the $m$th spin projection, $m = -j, -j + 1, \ldots, j$. The matrix $u$ in (57) is the unitary matrix rotating the vectors in the Hilbert space of qudit states, and the normalization condition

$$\sum_{m=-j}^{j} w(m, u) = 1$$  \hspace{1cm} (58)

must hold.

The spin tomogram is a fair probability distribution of a random spin projection depending on extra parameters coded by unitary matrix $u$. As shown in [2, 33, 34], a reconstruction formula for the density matrix $\langle m | \hat{\rho} | m' \rangle$ exists, which can be expressed in terms of the spin tomogram $w(m, u)$. Thus all statistical properties of spin observables can be obtained from this tomogram. For example, the mean value of the spin projection on the $z$ axis is

$$\text{Tr} \left( \rho \hat{J}_z \right) = \sum_{m=-j}^{j} mw(m, u = 1).$$  \hspace{1cm} (59)

Formula (57) can be presented in another form: if the density matrix $\rho$ has nonnegative eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$, corresponding to the vector $\hat{\rho}$, and $N$ eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$, which can be considered as columns of the unitary matrix $u_0$, the probabilities $w(-j, u), w(-j + 1, u), \ldots, w(j, u)$ can be considered as components of a probability vectors $\vec{w}(u)$ which, in this case, reads

$$\vec{w}(u) = |uu_0|^2 \hat{\rho}.$$  \hspace{1cm} (60)

A matrix $|a|^2$, by definition, has the matrix elements expressed as $|a|^2_{kj} = |a_{kj}|^2$. Thus, the state density matrix $\rho$, determined by its eigenvalues $\rho_k$ and eigenvectors $\vec{w}(u)$, is mapped onto the probability vector $\vec{w}(u)$. This map is invertible and therefore the spin state can be identified with the probability distribution (or the probability vector) depending on extra parameters coded by the unitary matrix $u$, which rotates the reference frame in the Hilbert space of spin states.

One can introduce the tomographic entropy using Shannon formula

$$H_u = - \sum_{m=-j}^{j} w(m, u) \ln w(m, u).$$  \hspace{1cm} (61)

The von Neumann entropy of the spin state is the minimum of the tomographic entropy with respect to the parameters of the unitary group $u$

$$\min H_u = - \text{Tr} \rho \ln \rho.$$  \hspace{1cm} (62)
One can also introduce the tomographic Rényi entropy of the qudit state

\[ R_u = \frac{1}{1-q} \ln \left( \sum_{m=-j}^{j} (w((m,u))^q) \right) \]  

(63)

and show [35] that its minimum yields the conventional Rényi entropy of the quantum state

\[ \min R_u = \frac{1}{1-q} \ln \text{Tr} \rho^q. \]  

(64)

The relation of the unitary matrices with spin tomograms provides the following new inequality for the \( N \times N \) matrices:

\[ - \int \left( \sum_{j=1}^{N} |u_{jk}|^2 \ln |u_{jk}|^2 \right) du \geq \frac{1}{2} \ln N, \]  

(65)

where \( du \) is the Haar measure on the unitary group normalized as \( \int du = 1 \).

6. Conclusions

We showed the tomographic approach to describe the classical and quantum states by means of tomographic probability distributions, which are positive and normalized. These probability distributions were suggested for continuous and discrete variables like photon quadratures (position and momentum) and spin projections (qudits).

The basic classical and quantum evolution equations like the state evolution equation and the energy level equation were obtained in the form of equations for fair probability distributions. The unified classical–quantum state description appeared to be appropriate for introducing the joint quantum–classical dynamics, using the tomographic picture of states [36, 37].

The quantum inequalities like the basic uncertainty relations and entropic inequalities were expressed in terms of measurable optical tomograms; thus, all these inequalities can be checked in the experiments with homodyne photon detection [21, 29]. This check provides a new possibility to control the accuracy of homodyne detecting of the photon states.

The spin-tomographic probability distributions describe the qudit states and can be useful for studying the Bell inequalities and quantum correlations in multipartite spin systems.

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