On scattering off the extreme Reissner-Nordström black hole in $N = 2$ supergravity

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The scattering amplitudes for the perturbed fields of the $N = 2$ supergravity about the extreme Reissner-Nordström black hole is examined. Owing to the fact that the extreme hole is a BPS state of the theory and preserves an unbroken global supersymmetry ($N = 1$), the scattering amplitudes of the component fields should be related to each other. In this paper, we derive the formula of the transformation of the scattering amplitudes.

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I. INTRODUCTION

Solitons that are non-perturbative configurations play an important role for studying non-perturbative aspects of quantum field theories. A soliton is a classical solution which is stationary, regular and classically and quantum mechanically stable configuration with finite localized energy. Solitons often have some conserved charges. From the stability of the configurations in classical and quantum theory, we may think a soliton as the least energy state whose energy is given by the charges. Further we may expect the inequality between the mass and charges of a soliton. In fact, we have the inequality in supersymmetric theories and call saturated states BPS states \[1\]. Although BPS states are massive and break the supersymmetry, they still have some unbroken supersymmetry.

In the Einstein-Maxwell theory, extreme Reissner-Nordström solutions behave as gravitational solitons \[2,3\]. The Einstein and the Einstein-Maxwell system can be embedded in supergravity theories. In the asymptotically flat spacetime, we can obtain the global charges that generate the rigid supersymmetry \[4\]. Therefore we can follow the argument in the rigid supersymmetry formally. For the $N = 1$ supergravity the positivity of energy is suggested \[4,6\]. Subsequently Witten established the positivity for the general relativity using the trick of “Witten spinor” \[6\]. Further, using the “Witten spinor” motivated by the transformation law of gravitini in the $N = 2$ supergravity, Gibbons and Hull \[7\] established the inequality between the mass and the electromagnetic charges. Further they showed that the saturated configurations are Majumdar-Papapetrou(MP) solutions, which are assemblages of the extreme Reissner-Nordström holes, and that the MP solutions have unbroken supersymmetries. More generalizations of their results are available in Refs. \[8,9\].

In addition, the non-renormalization theorem of on-shell effective action for the MP solutions was established \[8,10\]. Although supergravity has better ultraviolet behavior than the general relativity, it is known that supergravity is non-renormalizable at the perturbative level and is not regarded as the final theory. However we may expect that the final theory should include supergravity and may think the non-renormalization for the MP solutions as a guiding principle to the final theory.

Thus, it is very significant to investigate extreme black holes in classical and semiclassical framework through the general relativity and supergravity. To more understand extreme holes that are regarded as a kind of “vacuum” state, we need to study the fluctuation(excitation) about them.

Originally, the study of the perturbation about the extreme Reissner-Nordström hole was motivated by the interest in the no hair conjecture in supergravity \[11\] and by the interpretation problem of the paradoxical thermal properties of extreme-dilaton black holes \[12\].

Recently in the study on the method of calculating quasinormal frequencies of the extreme Reissner-Nordström hole, Onozawa, et.al. numerically \[13\] found that the quasinormal frequencies of gravitational waves and electromagnetic waves about it coincide by a suitable shift of the angular momentum indices. Due to the fact that quasinormal frequency is resonance pole of scattering wave, we may expect that gravitational and electromagnetic wave have the same reflection and transmission amplitude. Subsequently, they established coincidence between S-matrices of perturbations of gravitational, electromagnetic and spin-3/2 fields(gravitini) about the extreme Reissner-Nordström

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We adopt unit $M = 1$, where $M$ is the mass of background black hole.

On the background of the Reissner-Nordström solution, the bosonic perturbations are described by the Regge-Wheeler equation and the fermionic ones by the similar equation thorough the Newmann-Penrose formalism.

For the perturbations with the helicity-$(+1, +\frac{3}{2}, 2)$, $Y_{+s}(s = 1, \frac{3}{2}, 2)$, in phantom gauge \cite{18}, their equations of radial parts are

$$\Lambda^2 Y_{+s} + P_s \Lambda Y_{+s} - Q_s Y_{+s} = 0,$$

$$\frac{d}{dr_s} = \frac{\Delta}{r^2} \frac{d}{dr}, \quad \Delta = r^2 - 2r + Q^2,$$

$$\Lambda_\pm \equiv \frac{d}{dr_s} \pm i\Omega,$$

(2.1) (2.2) (2.3)

where $r_*$ is the tortoise coordinate and we omit the index of distinguishing two gravitini because they follows the same equation as expected from O(2) symmetry between them.

The each $P_s$ and $Q_s$ are given by, for $s = 1, 2$,

$$P_s \equiv \frac{d}{dr_*} \ln \left( \frac{r^3}{D_s} \right), \quad D_s \equiv \Lambda^2 \left( 1 + \frac{2q_s}{\mu^2 r} \right),$$

$$Q_s \equiv \mu^2 \frac{\Delta}{r^2} \left( 1 + \frac{2q_s}{\mu^2 r} \right) \left( 1 + \frac{q_{s'}}{\mu^2 r} \right), \quad (s, s' = 1, 2; s \neq s'),$$

(2.4a) (2.4b)

where $q_{1,2}$ are defined by

$$q_1 = 3 + \sqrt{9 + 4Q^2 \mu^2}, \quad q_2 = 3 - \sqrt{9 + 4Q^2 \mu^2},$$

(2.5)

and for $s = \frac{3}{2}$,

$$P_{\frac{3}{2}} \equiv \frac{3}{r^2} \left( r^2 - 3r + 2Q^2 \right),$$

$$Q_{\frac{3}{2}} \equiv \frac{\Delta}{r^6} \left( \lambda r^2 + 2r - 2Q^2 \right).$$

(2.6a) (2.6b)

The radial perturbations $Y_{+s}$ are constructed in two ways. One is by the perturbed Weyl scalar $\Psi_0$ and the perturbed spin connection $\kappa$ as

$$Y_{+s}(r) = \frac{\Delta^2}{r^3} F_{+s}(r), \quad F_{+s} = R_{+2}(r) + \frac{q_2 k(r)}{\mu},$$

$$\Psi_0 = R_{+2}(r) S_{+2}(\theta)e^{i(\Omega t + m\phi)}, \quad \kappa = \sqrt{2r^2} k(r) S_{+1}(\theta)e^{i(\Omega t + m\phi)},$$

(2.7) (2.8)

where the constant $\mu$ is an eigenvalue of the spin-weighted spherical harmonics,
\[\mathcal{L}_1 \mathcal{L}_2 S_{+2} = -\mu^2 S_{+2}, \quad \mathcal{L}_2 \mathcal{L}_1 S_{+1} = -\mu^2 S_{+1}, \quad (2.9)\]
\[\mu = \sqrt{(J - 1)(J + 2)}. \quad (2.10)\]

The operators \(\mathcal{L}_n\) and \(\mathcal{L}_n^\dagger\) are defined by
\[\mathcal{L}_n \equiv \partial_\theta + \frac{m}{\sin \theta} + n \cot \theta, \quad (2.11a)\]
\[\mathcal{L}_n^\dagger \equiv \partial_\theta - \frac{m}{\sin \theta} + n \cot \theta. \quad (2.11b)\]

Besides, the functions \(S_{+1}\) and \(S_{+2}\) are related in the manner
\[\mathcal{L}_2 S_{+2} = \mu S_{+1}, \quad \mathcal{L}_1^\dagger S_{+1} = -\mu S_{+2}. \quad (2.12)\]

Another is by the Weyl scalar \(\Psi_1\) and the spin connection \(\sigma\) as
\[G_{+s}(r) = R_{+1}(r) + \frac{q_s}{\mu} s(r), \quad (2.13)\]
\[\Psi_1 = \frac{1}{r \sqrt{2}} R_{+1}(r) S_{+1}(\theta) e^{i(\Omega t + m \phi)}, \quad \sigma = r s(r) S_{+2}(\theta) e^{i(\Omega t + m \phi)}, \quad (2.14)\]
and \(G_{+s}\) are related to \(Y_{+s}\) through the relations,
\[\Delta \left( D_2^\dagger - \frac{3}{r} \right) F_{+s} = \mu \left(1 + 2 \frac{q_s}{\mu^2 r}\right) G_{+s'}, \quad (s, s' = 1, 2; s \neq s'), \quad (2.15)\]
where the operator \(D_n\) is
\[D_n \equiv \partial_r + \frac{ir^2 \Omega}{\Delta} + 2n \frac{r - 1}{\Delta}, \quad D_n^\dagger = (D_n)^*. \quad (2.16)\]

For the helicity-\((-3/2)\) perturbations, the supersymmetric gauge invariant quantities are constructed by the supercovariant curvature of the spin-\(3/2\) fields \(\psi^i_\mu\) as
\[H_0^i = \Psi^{i(ABC)}_\mu A^B \sigma^C, \quad (2.17)\]
\[\Psi^{i(ABC)} = \frac{1}{2} \left[ D_{(B)(A')} \psi^{jA}_{C'} + i \epsilon^{ij} F^{C} (A'B') |_{A'C} \right], \quad (2.18)\]
where \(D_\mu\) is covariant derivative with respect to the spin connection \(\omega_{\mu ab} = \omega_{\mu AB} \epsilon_{A'B'} + \bar{\omega}_{\mu A'B'} \epsilon_{AB}, \) for example,
\[D_\mu \eta_A = \partial_\mu \eta_A + \omega_{\mu AB} \eta_B, \quad (2.19)\]
and the bold-face letters indicate the background quantities, \(\sigma^A\) is a principal spinor of \(F_{AB}\), and \(\bar{\sigma}_{AB}\) is a 2-spinor representation of the self-dual part of electromagnetic field strength \(F_{\mu
u}\). Hence we obtain the modes of the helicity-\((-3/2)\) perturbations,
\[Y_{+\frac{3}{2}} = \frac{\Delta^{\frac{3}{2}}}{r^2} R_{+\frac{3}{2}}^0, \quad (2.20)\]
\[H_0^3 = R_{+\frac{3}{2}}^0(r) S_{+\frac{3}{2}}(\theta) e^{i(\Omega t + m \phi)}, \quad (2.21)\]
where \(m\) is +1/2 or -1/2 and the spin-weight +3/2 spherical harmonics \(S_{+\frac{3}{2}}\) satisfies
\[\mathcal{L}_{\frac{1}{2}} \mathcal{L}_{\frac{5}{2}} S_{+\frac{3}{2}} = -\lambda S_{+\frac{3}{2}}, \quad (2.22)\]
\[\lambda = (J_s - \frac{1}{2})(J_s + \frac{3}{2}) \quad (J_s = \frac{3}{2}, \frac{5}{2}, ...). \quad (2.23)\]

To set the scattering problem, we need the normalized in(out)-going wave forms for \(Y_{+s}\) at the asymptotic regions \((r_s \rightarrow \pm \infty)\). At \(r_s \rightarrow \infty\), its asymptotic form of each normalized perturbation \(Y_{+s}(s = 1, \frac{3}{2}, 2)\) become
\[ Y_{+s}^{(+\infty, in)} \sim -4\Omega^2 e^{i\Omega r_s} \quad \text{and} \quad Y_{+s}^{(+\infty, out)} \sim -\frac{K_s}{4\Omega^2 r^2 s!} e^{-i\Omega r_s}, \] (2.24)

where \( s = 2 \) for \( s = 1, 2 \) and \( s = \frac{3}{2} \) for \( s = \frac{1}{2} \). \( K_s \) are defined by

\[ K_s \equiv \mu^2 (\mu^2 + 2) + 2i\beta_s, \quad \beta_s^2 \equiv q_{s'}^2, \quad (s, s' = 1, 2; s \neq s'), \] (2.25)

\[ K_{\frac{3}{2}} \equiv 2\Omega(\kappa_{\frac{3}{2}} + 2i\beta_{\frac{3}{2}}), \] (2.26)

\[ \kappa_{\frac{3}{2}} \equiv \left[ (J_s - \frac{1}{2}) (J_s + \frac{1}{2}) \right]^2, \quad \beta_{\frac{3}{2}}^2 \equiv 4. \] (2.27)

Similarly, at \( r_s \to -\infty \) for \( s = 1, 2 \)

\[ Y_{+s}^{(-\infty, out)} \sim 4i\Omega \left( i\Omega - \frac{r_s - Q^2}{r_+^2} \right) \exp (+i\Omega r_s) \]  
\[ Y_{+s}^{(-\infty, in)} \sim \frac{K_s (1 + \frac{2q_s}{\mu^2 r_+^2}) \Delta ||s||}{4r_+^2 \left( i\Omega - \frac{r_+ - 1}{r_+^2} \right) \left( i\Omega - \frac{r_+ - Q^2}{r_+^2} \right)} \exp (-i\Omega r_s), \] (2.28)

and for \( s = \frac{3}{2} \)

\[ Y_{+s}^{(-\infty, out)} \sim 4i\Omega \left( i\Omega - \frac{r_+ - 1}{2r_+^2} \right) \exp (+i\Omega r_s) \] 
\[ Y_{+\frac{3}{2}}^{(-\infty, in)} \sim \frac{K_{\frac{3}{2}} \Delta ||s||}{4r_+^2 \left( i\Omega - \frac{r_+ - 1}{2r_+^2} \right) \left( i\Omega - \frac{3r_+ - 1}{4} \right)} \exp (-i\Omega r_s). \] (2.29)

Using the above basis, the scattering problems of the perturbations are set as

\[ Y_{+s} \sim Y_{+s}^{(+\infty, in)} + R_s(\Omega)Y_{+s}^{(+\infty, out)} \quad (r_s \to \infty), \]
\[ \sim T_s(\Omega)Y_{+s}^{(-\infty, out)} \quad (r_s \to -\infty), \] (3.30)

where \( R_s \) and \( T_s \) are the reflection and transmission coefficients respectively.

III. THE TRANSFORMATION LAW OF THE CURVATURES

In the previous section, we summarize the perturbation equations governing the physical modes. On the extreme Reissner-Nordström background, the quasinormal frequencies of the perturbations with the different helicity coincide by the suitable shift of the total angular momentum \([13,14]\). This fact suggests that the reflection and transmission amplitudes are equivalent among the perturbations with the different helicity.

It is well known that the extreme Reissner-Nordström background has an unbroken global supersymmetry in \( N = 2 \) supergravity \([9]\). This implies that the perturbations with the different helicity are related to each other.

In this section, we obtain the supersymmetric transformation laws between the curvatures of the perturbed fields through \( N = 2 \) supergravity. The action of the \( N = 2 \) supergravity is represented by

\[ \mathcal{L} = -\frac{M_p^2}{2} \mathcal{R} - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \left( \psi^{jA}_{\mu} \hat{e}^{A'}_{\nu} D_{\rho} \hat{\psi}^{j}_{A'\sigma} - \hat{\psi}^{j}_{A'\mu} \hat{e}^{A'}_{\nu} D_{\rho} \psi^{j}_{A} \right) \]
\[ - \frac{i}{32M_p^2} \epsilon_{\mu\nu\rho\sigma} \left[ (\epsilon^{ij} \psi^{jA}_{\mu} \hat{\psi}^{j}_{A} \psi^{j}_{A'} \hat{\psi}^{j}_{A'} \sigma) + (\epsilon^{ij} \psi^{jA}_{\mu} \hat{\psi}^{j}_{A'} \psi^{j}_{A'} \hat{\psi}^{j}_{A} \sigma) \right] \]
\[ - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{i}{8M_p^2} \epsilon_{\mu\nu\rho\sigma} \left( \hat{\psi}^{j}_{A} \hat{\psi}^{j}_{A'} \psi^{j}_{A'} \psi^{j}_{A} \sigma \right), \]
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{2M_p} \epsilon^{ij} \left( \hat{\psi}^{j}_{A} \psi^{j}_{A'} - \hat{\psi}^{j}_{A'} \psi^{j}_{A} \right) \sigma \quad (i, j, k, l = 1, 2). \] (3.31)
where $M_p = (8\pi G)^{-1/2}$ is the Planck mass and $\epsilon_{\mu}^{AA'} = \epsilon_{\mu}^{AB} \sigma^{AB}$, and $D_{\mu}$ is covariant derivative with respect to $\omega_{\mu ab}$. The connection $\omega_{\mu ab}$ including the torsion is given by tetrad and gravitini through variating the action with respect to $\omega_{\mu ab}$.

\[
\omega_{\mu ab} = \omega^{(0)\mu ab} + K_{\mu ab},
\]

\[
\omega^{(0)\mu ab} = \epsilon^{\mu \nu \lambda \rho} \partial_{\nu} e^{\lambda} e^{\rho} - (a \leftrightarrow b),
\]

\[
K_{\mu ab} = \frac{i}{2M_p} \left( \epsilon^{\mu \nu \lambda \rho} \sigma^{AB} \psi_{[a] \lambda}^{i} \psi_{b \nu] \rho}^{i} - \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \psi_{[a] \lambda}^{i} \bar{\psi}_{b \nu] \rho}^{i} \right) - (a \leftrightarrow b),
\]

and the curvature is given by

\[
R_{\mu \nu ab} = 2 \partial_{\mu} \omega_{\nu ab} + 2 \omega_{[\mu a} \omega_{\nu b]},
\]

\[
R_{ab} = e^{a}_{\mu} e^{b}_{\nu} R_{\mu \nu bc}.
\]

The action is invariant under the supersymmetric transformations,

\[
\delta \epsilon_{\mu a} = - \frac{i}{2M_p} \left( \alpha^{i}_{A} \sigma^{AB} \bar{\psi}_{A' \mu}^{i} + \bar{\alpha}^{i}_{A'} \sigma^{AB} \psi_{A \mu}^{i} \right),
\]

\[
\delta A_{\mu} = \frac{1}{2} \epsilon^{ij} \left( \alpha^{i}_{A} \psi_{\mu}^{j} - \bar{\alpha}^{j}_{A'} \bar{\psi}_{\mu}^{i} \right),
\]

\[
\delta \bar{\psi}_{A \mu}^{i} = M_p \left( D_{\mu} \alpha^{i}_{A} - \epsilon^{ij} \bar{F}_{A'B} \alpha^{j}_{B} \psi_{\mu}^{i} \right),
\]

where $\alpha^{i}_{A}$ are Grassmann odd transformation parameters and $\bar{F}_{AB}$ is 2-spinor representation of the self-dual part of $\hat{F}_{\mu \nu}$.

We can check that $\omega_{\mu ab}$ and $\hat{F}_{\mu \nu}$ are supercovariant, i.e., their transformations have no derivative of transformation parameters. For the spin-3/2 fields, we introduce the supercovariant curvatures of $\psi^{i}_{\lambda A'B'} = \epsilon^{i}_{AB'} \psi^{i}_{A \mu}$ in 2-spinor representation,

\[
\Psi^{iA}_{BC} = \frac{1}{2} \left( D_{(B \lambda'} A') \psi^{iA \lambda'}_{C]} + \frac{i}{M_p} \epsilon^{ij} \hat{F}_{A'B} \psi^{iA \lambda'}_{(C] \lambda'} \right),
\]

\[
\Psi^{iA}_{B'C'} = \frac{1}{2} \left( D_{B \lambda} \psi^{iAB}_{C'} - \frac{i}{M_p} \epsilon^{ij} \hat{F}_{A'B} \psi^{iA \lambda'}_{(B] \lambda'} \right).
\]

They are transformed according to

\[
\delta \Psi^{iA}_{BC} = \frac{M_p}{2} R_{BC \lambda D} \alpha^{i}_{D} + \frac{i}{2} \epsilon^{ij} \left[ D_{(B \lambda'} \hat{F}_{C) \lambda'} \right] \bar{\alpha}^{i}_{A'} + O(\psi^2),
\]

\[
\delta \Psi^{iA}_{B'C'} = \frac{M_p}{2} R_{B'C \lambda D} \alpha^{i}_{D} - \frac{i}{2} \epsilon^{ij} \left[ D_{B \lambda} \hat{F}_{A'B} \right] \bar{\alpha}^{i}_{A'}
\]

\[
+ \frac{1}{2M_p} \hat{F}_{A'B} \hat{F}_{B'C'} \alpha^{i}_{D} + O(\psi^2).
\]

Because we will analyze the perturbations about a purely bosonic background, it is sufficient to obtain the transformation laws at linear order of $\psi^{iA}_{A \mu}$.

We introduce an expansion parameter $\lambda$ and replace fundamental fields, tetrad, connection, gravitini and electromagnetic potential about a background as, for example, $\psi^{i}_{A \mu} \rightarrow \psi^{i}_{A' \mu} + \lambda \psi^{i}_{A \mu}$, where we use the bold-face for the background quantities and standard letters for the perturbed quantities, respectively. Various equations and relations for perturbed fields are given by expanding with respect to $\lambda$.

The perturbed fields have gauge degree of freedoms originated from arbitrariness of correspondence between the perturbed world and the background world. Due to $\psi^{i}_{A \mu} = 0$, the bosonic quantities are invariant under the supersymmetric gauge transformations and the fermionic quantities transform, for example, into

\[
\delta_{g} \Psi^{i}_{ABC} = \frac{M_p}{2} R_{BC \lambda D} \beta^{i}_{D} + \frac{i}{2} \epsilon^{ij} \left[ D_{(B \lambda'} \hat{F}_{C) \lambda'} \right] \beta^{j}_{A'}
\]

where $\beta^{i}_{A}$ are any spinor parameters.
Let us consider supersymmetric transformation laws. Because of $\Psi_A^{\mu} = 0$, the bosonic background quantities are invariant under supersymmetric transformation. On the other hand, fermionic quantities generally change due to non-trivial bosonic background. For example, the background gravitini transform under the supersymmetric transformation into

$$\delta \Psi_A^{\mu} = M_p D_\mu \alpha_A - i e^{ij} F_A^B e_{\mu BA'} \bar{\alpha}^j A', \quad (3.9)$$

and the perturbed supercovariant curvatures of gravitini transform into

$$\delta \Psi_{ABC} = M_p \frac{1}{2} R_{BCA} D \alpha_D + \frac{i}{2} e^{ij} \left[ D_{(AB'} F_{C)A} \right] \bar{\alpha}_A^j + \frac{i}{2} e^{ij} \sigma_{a(B} A' \left[ \omega^a C)^D F_D A + \omega^a |A| D F_C )D \right] \bar{\alpha}_A^j. \quad (3.10)$$

Therefore, if there are some supercovariantly constant spinors (SCCS’s),

$$D_\mu \bar{\zeta}_A^i - \frac{i}{M_p} e^{ij} F_A^B e_{\mu BA'} \bar{\zeta}_A^j = 0, \quad (3.11)$$

the background configurations are invariant under the supersymmetric transformations that are induced by SCCS’s. And then, unbroken supersymmetry persists on the system consisting of the perturbed fields.

We introduce the quantities constructed by $\Psi_{ABC}^i$,

\begin{align*}
H_0^i & \equiv \Psi_{(AB)}^i o^A o_B o_C, \quad (3.12a) \\
H_1^i & \equiv \Psi_{(ABC)}^i o^A o_B l_C, \quad (3.12b) \\
H_2^i & \equiv \Psi_{(AB)}^i o^A l_B l_C, \quad (3.12c) \\
H_3^i & \equiv \Psi_{(ABC)}^i l^A l_B l_C. \quad (3.12d)
\end{align*}

where $o^A$ and $l^A$ are principal spinors of $F_{AB}$. Here we assume that the background spacetime is in the Petrov type D. Then the physical modes are described by $H_0^i$ or $H_3^i$ because they are diffeomorphic, local Lorentz gauge invariant due to the purely bosonic background and supersymmetric gauge invariant due to the type D character and Eq. (3.8).

Therefore we are interested in the transformation laws of $H_0^i$ generated by SCCS’s, $\zeta_A^i$,

$$\delta H_0^i = \frac{M_p}{2} \left[ \zeta_0^i (\Psi_1 - \zeta_1^i) \eta_0 \right] + \frac{i}{2} e^{ij} (D_\phi \phi_0) (\zeta_2^j m^a - \bar{\zeta}_2^j (\bar{s}^a_\mu \eta^a)) + i e^{ij} \phi_0 (\beta (\bar{s}^j_{(0)} - \bar{\epsilon} (\bar{s}^j_{(1)})) - i e^{ij} \phi_1 (\bar{\sigma} (\bar{s}^j_{(0)} - \kappa (\bar{s}^j_{(1)}))), \quad (3.13)$$

where $\zeta_0^i = o^A \zeta_A^i$ and $\zeta_1^i = l^A \zeta_A^i$ and they have the spin-weight $+ \frac{1}{2}$ and $- \frac{1}{2}$, respectively. And then, $\eta_0$, $\Psi_1$, $\phi_0$, $\sigma$ and $\kappa$ are perturbed Weyl scalars, Maxwell scalar and complex spin coefficients, respectively. Further $\epsilon$ and $\beta$ are background spin coefficients.

**IV. THE RELATIONS OF THE REFLECTION AND TRANSMISSION COEFFICIENTS**

In the previous section, we obtained the transformation law between the perturbed curvatures of gravitini. Using it, we can relate the decoupled modes $Y_{+ \pm}$ on the extreme Reissner-Nordström black hole.

On the extreme hole, there exist the supercovariantly constant spinors,

\begin{align*}
\zeta_0^i &= \sqrt{2} \eta_{(0)}^i (\theta) \exp (im' \phi), \quad (4.1a) \\
\zeta_1^i &= \frac{\Delta^2}{r} \eta_{(1)}^i (\theta) \exp (im' \phi), \quad (4.1b)
\end{align*}

where $m'$ is $+ \frac{1}{2}$ or $- \frac{1}{2}$ and $\eta_A^i$ satisfy
\[ L^m_{-\frac{1}{2}} \eta^{(0)}_i = L^m_{-\frac{1}{2}} \eta^{(1)}_i = 0 , \quad (4.2a) \]
\[ L^m_{+\frac{1}{2}} \eta^{(0)}_i = \eta^{(1)}_i , \quad (4.2b) \]
\[ L^m_{+\frac{1}{2}} \eta^{(1)}_i = -\eta^{(0)}_i , \quad (4.2c) \]

where the operators \( L^m_n \) and \( L^m_{n+\frac{1}{2}} \) are the same operators as defined in Eqs. (2.11) in the previous section and we manifest azimuthal angular momentum dependence with index \( m \). The supercovariantly constant spinors satisfy the relation,

\[ \frac{i}{M_p} e^{i\phi} \Phi_i \zeta_A = -\psi \left( \frac{2r^2}{\Delta} A^A + \bar{\phi}^A \sigma^A \right) \zeta_A . \quad (4.3) \]

From Eq. (4.3), the transformation of \( H^j_0 \), Eq. (3.13) becomes

\[ \delta H^j_0 = \frac{M_p}{2} \left[ \frac{\Delta^2}{r} \left( \Psi_0 + 4\Psi_0 \frac{\Delta}{\Delta^2} \right) \eta^{(1)}_i - \sqrt{2} \left( \Psi_1 - 2\Psi_0 \right) \eta^{(0)}_i \right] e^{im_0} , \quad (4.4) \]

where we, of course adopt the phantom gauge \( q_0 = \phi_2 = 0 \).

According to Sec. II, we decompose \( \Psi_0, \Psi_1, \kappa \) and \( \sigma \) by the spin-weighted spherical harmonics, and we manifest angular momentum dependence. For example,

\[ \Psi_0 = R^J_{-\frac{1}{2}}(r) S^J_{-\frac{1}{2}}(0) e^{i(\Omega_{1/2} + m\phi)} , \quad (4.5) \]
\[ L^m_{-\frac{1}{2}} L^m_{-\frac{1}{2}} S^J_{-\frac{1}{2}} = -\mu_J S^J_{-\frac{1}{2}} , \quad (4.6) \]
\[ \mu_J = \sqrt{(J - 1)(J + 2)} . \quad (4.7) \]

And then

\[ \delta H^j_0 = \frac{M_p}{2} \left[ \frac{\Delta^2}{r} \left( R^J_{-\frac{1}{2}} + r \frac{d}{dr} \ln \left( \frac{\Delta}{r^2} \right) S^J_{-\frac{1}{2}} \right) \eta^{(1)}_i S^J_{-\frac{1}{2}} \right. \]
\[ - \frac{1}{r} \left( R^J_{+1} - r^3 \frac{d}{dr} \frac{\Delta}{r^2} S^J_{+1} \right) \eta^{(0)}_i S^J_{+1} \left] e^{i(\Omega_{1/2} + M\phi)} , \quad (4.8) \right. \]

where \( M = m + m' \).

The each quantity, \( \eta^{(1)}_i S^J_{+\frac{1}{2}} \) and \( \eta^{(0)}_i S^J_{+\frac{1}{2}} \), has spin-weight +3/2, but not an eigenstate of total angular momentum respectively. Hence we need decompose them into \( (S^J_{+1/2}) \) and \( (S^J_{-1/2}) \) that are eigenstates of the total angular momentum. It is easy to check the equations

\[ L^M_{-\frac{1}{2}} L^M_{-\frac{1}{2}} \eta^{(1)}_i S^J_{+\frac{1}{2}} = -\mu_J (\eta^{(1)}_i S^J_{+\frac{1}{2}}) - \mu_J^2 (\eta^{(1)}_i S^J_{+\frac{1}{2}}) , \quad (4.9a) \]
\[ L^M_{-\frac{1}{2}} L^M_{-\frac{1}{2}} \eta^{(0)}_i S^J_{+\frac{1}{2}} = -\mu_J (\eta^{(0)}_i S^J_{+\frac{1}{2}}) - (\mu_J^2 + 3)(\eta^{(0)}_i S^J_{+\frac{1}{2}}) . \quad (4.9b) \]

From these equations, we can decompose \( (\eta^{(1)}_i S^J_{+\frac{1}{2}}) \) and \( (\eta^{(0)}_i S^J_{+\frac{1}{2}}) \) as

\[ \eta^{(1)}_i S^J_{+\frac{1}{2}} = \xi^i q^J_{+\frac{1}{2}} - \xi^J_{-\frac{1}{2}} \eta^{(0)}_i S^J_{+\frac{1}{2}} , \quad (4.10a) \]
\[ \eta^{(0)}_i S^J_{+\frac{1}{2}} = \xi^i q^J_{+\frac{1}{2}} - \xi^J_{-\frac{1}{2}} (2\mu_J q_2 - q_1) , \quad (4.10b) \]

where \( \xi^i \) are arbitrary Grassmann odd constants and \( q_{1,2} \) are defined by

\[ q_1 \equiv 3 + \sqrt{9 + 4\mu_J^2} = 2(J + 2) , \]
\[ q_2 \equiv 3 - \sqrt{9 + 4\mu_J^2} = -2(J - 1) . \quad (4.11) \]

The expressions of \( q_j \) are the extreme limit of Eq. (2.3).
Because $\Delta = (r - 1)^2$ in the extreme case, Eq. (4.13) is rewritten as

$$
\delta H_0 = \frac{M_p \Delta^j \xi^i}{2r(q_2 - q_1)} \left[ S_{+\frac{J}{2}}^{j+\frac{J}{2}} \left( F_{+1}^{j} - \frac{q_1}{2\Delta^j \mu_j} G_{+2}^{j} \right) - S_{+\frac{J}{2}}^{j-\frac{J}{2}} \left( F_{+2}^{j} - \frac{q_2}{2\Delta^j \mu_j} G_{+1}^{j} \right) \right] e^{i(\Omega t + M\phi)},
$$

$$
= \frac{M_p \Delta^j \xi^i}{2r(q_2 - q_1)} \left[ S_{+\frac{J}{2}}^{j+\frac{J}{2}} \left\{ F_{+1}^{j} - \frac{q_1}{2 \mu_j^r} r^2 \left[ D_r^j - \frac{3}{r} \right] F_{+1}^{j} \right\} - S_{+\frac{J}{2}}^{j-\frac{J}{2}} \left\{ F_{+2}^{j} - \frac{q_2}{2 \mu_j^r} r^2 \left[ D_r^j - \frac{3}{r} \right] F_{+2}^{j} \right\} \right] e^{i(\Omega t + M\phi)},
$$

(4.12)

where we use the relation Eq. (2.15). Eq. (4.12) shows that the helicity-(+3/2) modes with $J + 1/2$ are generated by the unbroken supersymmetry from the helicity-(+1) mode $F_{+1}^{j}$ with total angular momentum $J$ or the helicity-(+2) mode $F_{+2}^{j}$ with $J + 1$.

From Eq. (4.12), radial parts $Y_{+\frac{J}{2}}^{j}$ of perturbed curvatures of gravitini are generated from $Y_{+s}^{j}$ as

$$
Y_{+\frac{J}{2}}^{j} = \xi^k \left[ Y_{+s}^{j} - C_{+s}^{j}(r) \Delta Y_{+s}^{j} \right],
$$

(4.13a)

or equivalently,

$$
\xi^k \left[ 2i\Omega + \frac{1}{C_s^j} - C_{+s}^{j}Q_s + \frac{1}{2} r P_s^j \right] Y_{s}^{j} = \left[ 2i\Omega + \frac{1}{C_s^j} + \frac{1}{2} r P_s^j \right] Y_{+s}^{j} + \Delta Y_{+s}^{j},
$$

(4.13b)

where

$$
C_{+s}^{j} = \frac{q_s r^2 \Delta^j}{2 \mu_j^s D_s}, \quad D_s = \Delta^2 \left( 1 + \frac{2q_s}{\mu_j^s} \right),
$$

(4.14)

and $J_s$ is $J + \frac{1}{2}$ for $s = 1$ and $J - \frac{1}{2}$ for $s = 2$. Eqs. (4.13) are our main result and in principle, we can also obtain the relations between potentials of perturbations with different helicities. Hereafter we omit the index $k$ which distinguishes two gravitini.

From Eqs. (4.13), we can obtain the relation between reflection and transmission amplitudes. From them, it follows that $Y_{+s}^{j}$ derived from $Y_{+s}^{j(+\infty, in)}$ and $Y_{+s}^{j(+\infty, out)}(s = 1, 2)$ have, respectively, the asymptotic behaviors at $r_s \to \infty$

$$
Y_{+\frac{J}{2}}^{j} \sim Y_{+\frac{J}{2}}^{j(+\infty, in)} \quad \text{and} \quad Y_{+\frac{J}{2}}^{j} \sim \frac{iq_s K_{+s}^{j}}{\mu_j^s} Y_{+\frac{J}{2}}^{j(+\infty, out)},
$$

(4.15)

Similarly, it follows that $Y_{+\frac{J}{2}}^{j}$ derived from $Y_{+s}^{j(-\infty, in)}$ and $Y_{+s}^{j(-\infty, out)}(s = 1, 2)$ have, respectively, the asymptotic behaviors at $r_s \to -\infty$

$$
Y_{+\frac{J}{2}}^{j} \sim Y_{+\frac{J}{2}}^{j(-\infty, in)} \quad \text{and} \quad Y_{+\frac{J}{2}}^{j} \sim \frac{iq_s K_{-s}^{j}}{\mu_j^s} Y_{+\frac{J}{2}}^{j(-\infty, out)},
$$

(4.16)

Therefore the asymptotic form of $Y_{+s}^{j}$ derived from the solution for $Y_{+s}^{j}(s = 1, 2)$ having the asymptotic behavior

$$
Y_{+s}^{j} \sim Y_{+s}^{j(+\infty, in)} + R_{+s}^{j} (\Omega) Y_{+s}^{j(+\infty, out)} \quad (r_s \to \infty),
$$

$$
\sim T_{s}^{j} (\Omega) Y_{+s}^{j(-\infty, out)} \quad (r_s \to -\infty),
$$

(4.17)

has the asymptotic behavior

$$
Y_{+\frac{J}{2}}^{j} \sim Y_{+\frac{J}{2}}^{j(+\infty, in)} + R_{+\frac{J}{2}}^{j} (\Omega) \frac{iq_s K_{+s}^{j}}{\mu_j^s} Y_{+\frac{J}{2}}^{j(+\infty, out)} \quad (r_s \to \infty),
$$

$$
\sim T_{\frac{J}{2}}^{j} (\Omega) Y_{+\frac{J}{2}}^{j(-\infty, out)} \quad (r_s \to -\infty).
$$

(4.18)
Accordingly, we obtain the relations of reflection and transmission coefficients,

\[ R^J_s(\Omega) = \gamma_s R^J_s(\Omega) \quad T^J_s(\Omega) = T^J_s(\Omega) \quad (s = 1, 2), \quad (4.19) \]

\[ \gamma_s = \frac{i\Omega q_s K_s}{\mu^2 J_s K_s}, \quad (4.20) \]

where \(|\gamma_s| = 1\). Thus, under the suitable shift of angular momentums, while the amplitudes of the transmitted waves are identically the same for three perturbed fields, the reflected amplitudes differ only in their phases.

V. SUMMARY

In the previous section, using the unbroken supersymmetry that remains on the extreme Reissner-Nordström black hole, we obtain the relation between the reflection and transmission coefficients of decoupled modes with (helicity, total angular momentum) = (1, J), \((\frac{1}{2}, J + \frac{1}{2})\), \((2, J + 1)\).

These relations are also expected for the perturbations about the superpartners of the extreme Reissner-Nordström black hole \([19]\) and for matter multiplets about them.

In the previous paper \([14]\), we observed that the Regge-Wheeler potential of gravitational perturbation coincides with one of electromagnetic perturbation by inversion of the tortoise coordinate, that is, exchange of the horizon for infinity, vice versa. It is interesting to understand the above correspondence by using the relations of the perturbations obtained in the previous section.

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[1] E. Witten and D. Olive, Phys. Lett. 78B, 97 (1978).
[2] P. Hajicek, Nucl.Phys. B185, 254 (1981).
[3] G.W. Gibbons, in Supersymmetry, Supergravity and Related Topics, proceedings of the XV th GIFT Seminar, edited by F. Augila, et.al. (World Scientific, 1985).
[4] C. Teitelboim, Phys. Lett. 69B, 240 (1977).
[5] D. Deser and C. Teitelboim, Phys. Rev. Lett. 39, 249 (1977);
    M. Grisaru, Phys. Lett. 37B, 249 (1978).
[6] E. Witten, Commun.Math.Phys.80, 381 (1981);
    J. M. Nester, Phys. Lett. 83A, 241 (1981).
[7] G. W. Gibbons and C. M. Hull, Phys. Lett. 109B, 190 (1982).
[8] R. Kallosh, A. Linde, T. Ortin and A. Peet, Phys. Rev. D 46, 5278 (1992).
[9] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend and J. Traschen, Nucl.Phys. B416, 850 (1994).
[10] R. Kallosh, Phys. Lett. 282B, 80 (1992).
[11] P. Cordero and C. Teitelboim, Phys. Lett. 78B, 80 (1978);
    R. Güven, ibid. 22, 2327 (1980);
    P.C. Aichelburg and R. Güven, ibid. 24, 2066 (1981);
    R. Güven, ibid. 25, 3117 (1982);
    P.C. Aichelburg and R. Güven, ibid. 27, 456 (1983).
[12] C.F.E. Holzhey and F. Wilczek, Nucl.Phys.B380, 447 (1992).
[13] H. Onozawa, T. Mishima, T. Okamura and H. Ishihara, Phys. Rev. D 53, 7033 (1996).
[14] H. Onozawa, T. Okamura, T. Mishima and H. Ishihara, Phys. Rev. D 55, (1997).
[15] S. Ferrara and P. van Nieuwenhuizen, Phys. Rev. Lett. 37, 1669 (1976).
[16] S. Chandrasekhar, The Mathematical Theory of Black Holes, (Clarendon, Oxford, 1983).
[17] G. F. Torres del Castillo and G. Silva-Ortigoza, Phys. Rev. D 46, 5395 (1992).
[18] See the [11], pp240.
[19] P. C. Aichelburg and F. Embacher, Phys. Rev. D 34, 3006 (1986).