Hermite–Laguerre–Gaussian beams in strongly nonlocal nonlinear media

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Abstract
We obtain an exact Hermite–Laguerre–Gaussian (HLG) beam solution of the Snyder–Mitchell mode, which is confirmed by the simulation of the nonlocal nonlinear Schrödinger equation. HLG breathers and solitons can be obtained by taking different input power. The HLG beams are a unity of Hermite–Gaussian beams (HGBs) and Laguerre–Gaussian beams (LGBs), which form the exact and continuous transition modes between HGBs and LGBs when an additional parameter changes continuously.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Optical vortices [1, 2] have become very important for tasks such as trapping and manipulation of small particles [3–5]. Optical spatial solitons have received much attention in nonlocal nonlinear media [6–35] due to their potential applications to photonic switching [6], all-optical switching and logic gating [7], and all-optical signal processing [8]. In recent years, the so-called bright vortex (or spinning) solitons have also drawn much interest [31–38]. Yakimenko et al [32] and Briedis et al [33] demonstrated vortex stabilization due to strongly nonlocal nonlinearity in the frameworks of different models in 2005. The first observation of stable vortex-ring solitons was performed in strongly nonlocal nonlinear media (SNNM) by Rotschild et al [34]. Yakimenko et al [35] found vortex solitons with a strong azimuthal instability which was eliminated only in the strongly nonlocal regime in 2006. Recent theoretical and experimental results demonstrate that nematic liquid crystal [11, 13] and lead glass are examples of strongly nonlocal nonlinear materials, which stimulate further theoretical studies of the optical beams in SNNM [26–28]. Hermite–Gaussian beams (HGBs) and Laguerre–Gaussian beams (LGBs) [26–28] are the exact solutions of the Snyder and Mitchell model (SMM) [6] in Cartesian and cylindrical coordinates, respectively. Ince–Gaussian solitons (IGSs) are exact and orthogonal soliton solutions of the SMM in elliptic coordinates [39–41]. The propagation of three-dimensional soliton clusters in strongly nonlocal nonlinear media has been investigated analytically and numerically [42]. However, to the best of our knowledge, the exact analytical Hermite–Laguerre–Gaussian (HLG) breathers of the SMM have not been studied.

Here, exact HLG beams are investigated analytically and numerically in SNNM. The HLG beams are a unity of HGBs and LGBs, which form the exact and continuous transition modes between HGBs and LGBs when an additional parameter changes continuously.

2. HLG solutions of the Snyder–Mitchell model
The complex amplitude \(\Phi(r, z)\) of a \((1 + 2)\)-dimensional paraxial light field in the nonlocal cubic nonlinear media satisfies the nonlocal nonlinear Schrödinger equation (NNLSE) [43–49],

\[
i \frac{\partial \Phi}{\partial z} + \mu \Delta_\perp \Phi + k \frac{\Delta n}{n_0} \Phi = 0,
\]

where \(\mu = 1/(2k)\), \(k = \omega n_0/c\) and \(n_0\) are the wave numbers and the linear refractive index of the media, \(\Delta_\perp = \delta^2/\delta x^2 + \delta^2/\delta y^2\), \(\Delta n = n_2 \int R(r - r')|\Phi(r', z)|^2 \, d^2r'\) and \(n_2\) are the nonlinear perturbation of refraction index and the nonlinear index coefficients, \(r\) and \(r'\) are the two-dimensional transverse coordinate vectors and \(R(\cdot)\) is the normalized symmetrical real spatial response function of the media. Assanto et al
applied the basic equation (1) to nonlinear guided waves in liquid crystals as this is a major area of application of this equation and Assanto et al. [48, 49] were the first group to derive the equation in this context. For liquid crystals, the appropriate kernel is related to the modified Bessel function $K_0$ [19, 49] or the exponential function [18]. To treat analytically the ensuing wave dynamics of equation (1), we choose a Gaussian response function.

Equation (1) simplifies into the SMM for the case of the strong nonlocality [6],

$$i \frac{\partial \Phi}{\partial z} + \mu \Delta \Phi - \frac{n_2}{2\mu_0} k \gamma P_0 r^2 \Phi = 0,$$

(2)

where $\gamma = -\partial^2_{rr} R(r - r') |_{r=r=0}$ is the material parameter [14, 15] and $P_0$ is the input power at $z = 0$. We suppose that a solution of equation (2) is a multiplication of two functions $\Phi_F(r, z)$ and $\Phi_G(r, z)$,

$$\Phi = \Phi_F(r, z) \Phi_G(r, z),$$

(3)

where

$$\Phi_G = \sqrt{P_0} \exp[i\theta(z)] \exp \left[ -\frac{r^2}{2w(z)^2} + ic(z)r^2 \right],$$

(4)

where $w(z)$ is the beam width of the Gaussian beam, $c(z)$ is the phase-front curvature of the beam and $\theta(z)$ denotes the phase of the complex amplitude. They can be expressed by [6, 14, 15], respectively,

$$w(z)^2 = w_0^2 \left( \frac{P_r}{P_0} \sin^2 \beta_0 z + \cos^2 \beta_0 z \right),$$

(5)

$$c(z) = \frac{k_0 \beta_0}{2 \cos \theta_0 + P_0 \sin \theta_0} \tan \theta_0,$$

(6)

and $P_r = n_0 \beta_0^2 \gamma^2 w_0^2 \int \int \int \int \left[ 1 - (1 + 1)^{m} (1 + 1)^{m} \right] d\xi d\eta d\zeta d\zeta$. $w_0$ is the initial beam width of the Gaussian beam, $\beta_0 = \gamma n_2 P_0 / n_0 P_r$ and $P_r$ is the critical power of the soliton propagation, we have assumed that $\gamma n_2 > 0$ due to the definition of the critical power $P_r$. It is easy to find from equation (5) that $w(z)$ is the beam width of the Gaussian function oscillates periodically along the propagation $z$ when $P_0 \neq P_r$.

Substituting equations (3) and (4) into equation (2), we derive that

$$i \frac{\partial \Phi_F}{\partial z} + \mu \Delta \Phi_F + 2\mu \left( 2i(\mu - \frac{1}{w(z)^2}) \right) \Phi_F = 0.$$  

(7)

If we make the variable transform

$$\xi = \frac{x}{w(z)}, \quad \eta = \frac{y}{w(z)}, \quad \zeta = z,$$

(8)

then we can reduce equation (6) to

$$\left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + 2\mu \left( \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \right) \right) \Phi_F + 2i k w(z)^2 \frac{\partial}{\partial \xi} \Phi_F = 0.$$  

(9)

To obtain the Hermite–Laguerre–Gaussian beams, letting $\Phi_F = T_{nm}(\xi, \eta, \theta)(\xi, \eta)$, where $T_{nm} = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} a_{nm}^m \cos^{s-\mu} \theta \sin^{\beta_0} \theta P_{n-m}^{\beta_0+m-1}(-\cos 2\theta), P_{\beta_0+m}^{\beta_0+m-1}(\xi) = \frac{(1 - \gamma)^{\beta_0+m-1}}{\Gamma(1 - \gamma)} \Gamma \left[ \frac{1}{\gamma} (1 - \gamma)^{\frac{\beta_0+m-1}{\gamma} + \gamma \xi} (1 - \gamma)^{\beta_0+m-1} \right]$ are Jacobi polynomials [50], and the angle between the symmetry axis of the HGBs and the elements of the cylindrical lenses for transforming HGBs into LGBs in experiment, equation (8) is separated into the following three differential equations:

$$\frac{d^2 X}{d\xi^2} - 2\xi \frac{dX}{d\xi} + (2r + n + m - s)X = 0,$$

(10)

$$\frac{d^2 Y}{d\eta^2} - 2\eta \frac{dY}{d\eta} + 2sY = 0,$$

(11)

Equations (9) and (10) are the well-known Hermite differential equation [33]. From equations (9)–(11), we can derive

$$X = H_{n+m-s}(\xi) = H_{n+m-s} \left( \frac{x}{w(z)} \right),$$

(12)

$$Y = H_s(\eta) = H_s \left( \frac{y}{w(z)} \right),$$

(13)

$$\Theta = \exp \left[ -i(n + m)\vartheta(z) \right].$$

(14)

By substituting equations (4), (12)–(14) into equation (3), the exact solutions of equation (2) can be obtained as

$$\Phi_{n,m}(x, y, z, \vartheta) = \frac{C_{nm}}{w(z)} \exp \left[ -\frac{4r^2}{2w(z)^2} + ic(z)r^2 \right] \times \exp \left[ -i(n + m)\vartheta(z) \right] T_{nm} H_{n+m-s}(\xi) H_s \left( \frac{y}{w(z)} \right),$$

(15)

where $C_{nm} = \sqrt{P_0 / (\pi 2^{m+n} n! m!)}$ are normalization constants which can be determined by $\int \int \int \Phi_{n,m}^* (x, y, z, \vartheta)^2 dx \, dy = \delta_{P_0, w(z), c(z), \theta(z), P_r}$ are the same as those in equation (4), $H_r(\cdot)$ is the Hermite polynomial. HLG beams occur in the laboratory when a HGB or LGB undergoes an astigmatic transformation, for example, to a cylindrical lens or variable-phase mode converter, the additional parameter $\vartheta$ being the orientational angle of the lens. HLG beams can be mathematically understood by using the analogy between Gaussian beam and the quantum two-dimensional harmonic oscillator. HGB states correspond to linear, LGB states to circular and HLG beam states to elliptic orbits.

3. Discussion about the solutions

3.1. Relations among the HGBs, LGBs and HLG beams

When $\vartheta = 0$ and $\pi / 4$, equation (15) can be changed into HGBs and LGBs as follows:

$$\Phi_{n,m}(x, y, z, \vartheta = 0) = \frac{i^n}{n!} A(r, z) H_n \left( \frac{x}{w(z)} \right) H_m \left( \frac{y}{w(z)} \right),$$

(16)

$$\Phi_{n,m}(x, y, z, \vartheta = \pi / 4) = \frac{B(r, z) L_n \left( \frac{x^2}{w(z)^2} \right) \exp \left[ i[(n - m)\vartheta] \right]}{n \geq m,}$$

(17)

and

$$D(r, z) L_m \left( \frac{x^2}{w(z)^2} \right) \exp \left[ i[(m - n)\vartheta] \right], \quad m \geq n,$$

(18)

where $A(r, z) = \frac{(1)^m}{(n + m)w(z) \exp \left[ -r^2 / 2w(z)^2 \right] + \text{ic}(z)r^2} \exp \left[ -i(n + m)\vartheta(z) \right], B(r, z) = A(r, z)^2 m! \left( \frac{x}{w(z)} \right)^{n-m},$ $D(r, z) = A(r, z)^2 \frac{(1)^m}{(n + m)w(z)^2 \left( \frac{x}{w(z)} \right)^{n-m}},$ $L_n^{m} \left( \cdot \right)$ is the associated Laguerre polynomial and $\vartheta = \arctan(y/x)$ is the azimuthal angle. As $m = n = 0$, equation (15) can be reduced
Figure 1. Propagation dynamics of the Hermite–Laguerre–Gaussian soliton in the Gaussian-shaped response material. (a)–(d) Transverse normalized intensity distributions from the analytical solution; (e)–(h) distributions from the numerical simulation ($\alpha = 0.05$); (i)–(l) distributions from the numerical simulation ($\alpha = 0.2$); (m)–(p) phase distributions from the numerical simulation ($\alpha = 0.05$) and (q)–(t) phase distributions from the numerical simulation ($\alpha = 0.2$). The different columns represent the different propagation distances given on the top of the figure. The parameters are chosen as $n = 5, m = 3, P_0/P_c = 1, \vartheta = \pi/16$; $z_R = kw_0^2$ is the Rayleigh range.

to the Gaussian beam. Hence, HLG beams are the more general beams containing Gaussian beam, HGBs, LGBs and intermediate HLG beams. HGBs and LGBs have zero lines excepting trivial cases. As $\vartheta > 0$, the intermediate HLG beams have real and imaginary components; all zeros in the intermediate HLG beams are separate points which are the zero points of intersection of real and imaginary parts of these functions. In these isolated zero points, the phase becomes singular. For small enough $\vartheta$, the negative vortices can appear on the row, which ultimately vanishes, i.e., the HLG beams become HGBs when $\vartheta = 0$. If $\vartheta = \pi/4$, some of the opposite sign zeros vanish, while the same sign zeros combine, and LGBs appear.

3.2. Comparison with numerical simulation of the NNLSE

Applying the split-step Fourier method [47], we will discuss the properties of the HLG mode in nonlocal nonlinear media. The exact analytical results of the SMM and the numerical simulations of equation (1) are shown in figures 1–3 for the HLG beams. To simulate the propagation of the HLG beams in the nonlocal nonlinear media, we use the different Hermite–Laguerre–Gaussian beams as the inputs at the $z = 0$, and assume the material response being the Gaussian function [9, 23, 14], i.e., $R(r) = 1/(2\pi w_m^2) \exp[-r^2/(2w_m^2)]$, where $w_m$ is the characteristic length of the material response function and $\alpha = w_0/w_m$ denotes the degree of nonlocality. It is easy to find from figures 1–3 that the analytical HLG solutions agree well with the numerical simulations for the case of strong nonlocality. When the degree of nonlocality becomes weaker, there are some differences between the analytical results of the SMM and the exact numerical ones of equation (1).

3.3. The HLG breathers and solitons

When $P_0 < P_c$, beam diffraction initially overcomes beam-induced refraction and the beam initially expands, and the beam initially contracts when $P_0 > P_c$. These are HLG breathers whose widths vibrate periodically during propagation. As $P_0 = P_c$, diffraction is exactly balanced by nonlinearity, and these are HLG solitons which preserve their widths along the propagation $z$-axis. In this case, equation (15) is reduced to HLG solitons,

$$\Phi_{n,m}(x, y, z|\vartheta) = \frac{C_{nm}}{w_0} \exp\left\{-\frac{r^2}{2w_0^2}\right\} \times \exp[-i(n + m)\vartheta z]T_{nm}H_{n+m-s}(\frac{x}{w_0})H_s(\frac{y}{w_0}).$$

(18)
Figures 1–3 show that the intensities and the phases of the HLG beams are stable under propagation in the case of strong nonlocality as $P_0 = P_c$. For the case of weaker nonlocality, the intensities and the phases of the HLG beams become unstable. $\Phi_{5,3}(x, y, z|0)$ is a (5, 3)-mode Hermite–Gaussian beam whose phase has only two values at a fixed $z$, and whose zeros are straight lines from the real-value structure in the case of strong nonlocality. Figures 1–3 present that all zeros in $\Phi_{5,3}(x, y, z|\pi/16)$, $\Phi_{5,3}(x, y, z|\pi/8)$ and $\Phi_{5,3}(x, y, z|3\pi/16)$ are separate points, and the vortex beam begins to rotate when the degree of nonlocality becomes weaker. $\Phi_{5,3}(x, y, z|\pi/4)$ is a (3, 2)-mode Laguerre–Gaussian vortex beam.

4. Conclusion

In conclusion, we have introduced exact HLG beams by analytical and numerical means in SNNM. The HLG beams are a unity of HGBs and LGBs, which form the exact and continuous transition modes between HGBs and LGBs when an additional parameter changes continuously. The beam widths of HLG breathers vibrate periodically during propagation when $P_0 \neq P_c$, and HLG solitons preserve their widths along the propagation $z$-axis as $P_0 = P_c$. The validity of the analytical results is confirmed by the exact numerical results of the NNSLE.

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