A graph discretized approximation of semigroups for diffusion with drift and killing on a complete Riemannian manifold

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Abstract

In the present paper, we prove that the $C_0$-semigroup generated by a Schrödinger operator with drift on a complete Riemannian manifold is approximated by the discrete semigroups associated with a family of discrete time random walks with killing in a flow on a sequence of proximity graphs, which are constructed by partitions of the manifold. Furthermore, when the manifold is compact, we also obtain a quantitative error estimate of the convergence. Finally, we give examples of the partition of the manifold and the drift term on two typical manifolds: Euclidean spaces and model manifolds.

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1 Introduction

Let $M = (M, g)$ be a smooth $n$-dimensional Riemannian manifold and $m$ be the Riemannian volume measure on $M$. We assume that $M$ is geodesically complete and connected, but not necessarily compact. We denote by $C_c^\infty(M)$ and $C_0(M)$ the spaces of smooth functions on $M$ with compact support and continuous functions on $M$ vanishing at infinity, respectively. Let $b$ be a smooth vector field on $M$, and $V$ be a non-negative smooth function defined on $M$. We consider a drifted Schrödinger operator $A = A_V$ having the form

$$A f(x) = -\Delta f(x) - (bf)(x) + V(x)f(x), \quad x \in M, \ f \in C_c^\infty(M),$$

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where $\Delta$ is the negative Laplacian on $M$. Since $V$ is non-negative, the maximal principle of the Laplacian $\Delta$ implies that $(-\mathcal{A}, C_c^\infty(M))$ is dissipative, thus is closable in $C_0(M)$ (see [See84, Lemma 2.1] for details). Moreover by the Lumer-Phillips theorem, the closure of $(-\mathcal{A}, C_c^\infty(M))$ generates a contraction $C_0$-semigroup $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ in $C_0(M)$ under the the condition

(A): $(\lambda + \mathcal{A})(C_c^\infty(M))$ is dense in $C_0(M)$ for some $\lambda > 0$.

See e.g., [Paz85, Ebe99] for the definition of dissipativity and the Lumer-Phillips theorem.

Many problems in analysis on Riemannian manifolds naturally lead to the study of the drifted Schrödinger semigroup $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ and hence it is an important problem to find an efficient approximation scheme of $e^{-t\mathcal{A}}f$ for a given $f \in C_0(M)$. When $M = \mathbb{R}^n$ and $\mathcal{A} = -\Delta$, we have the explicit formula for the heat kernel of the Laplacian $\Delta$. Thus the integral representation formula

$$e^{t\Delta}f(x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy, \quad f \in C_0(\mathbb{R}^n)$$

enables us to study discrete approximations for $e^{t\Delta}f$ by applying numerical integration methods. However, such an exact formula of the kernel function of $e^{-t\mathcal{A}}$ for general $M$ and $\mathcal{A}$ has been proved only in limited situations. When $M$ is compact, it is known that the spectrum of the positive Laplacian $-\Delta$ consists of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

and the corresponding (orthonormal) eigenfunctions $\{\phi_i\}^\infty_{i=0}$ are smooth on $M$. Then the heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ can be written by

$$e^{t\Delta}f(x) = \sum_{i=0}^\infty e^{-t\lambda_i} \phi_i(x) \int_M f(y) \phi_i(y) dm(y).$$

By this spectral expansion formula, we see that the data $\{\lambda_i\}^\infty_{i=0}$ and $\{\phi_i\}^\infty_{i=0}$ have an important role to approximate $e^{t\Delta}f$. Needless to say, on compact manifolds, eigenvalues and eigenfunctions themselves are interesting objects. See, for instance, [Cha84, Ros97, Fuj95, Ots03]. It is worth mentioning that this kind of argument has been received a lot of attention in the study of manifold learning. See [BN03, BN08, BIK14, Tew17, SW17, Ain21] and references therein for recent results. As a matter of fact, these papers motivated us to write the present paper. In contrast to the above situation, such an expansion for the drifted Schrödinger semigroup $\{e^{-t\mathcal{A}}\}_{t \geq 0}$ on a non-compact manifold $M$ has been less-developed in general.

On the other hand, probability theory provides us a functional integral point of view to the studies of Schrödinger operators on Riemannian manifolds. We return to the case when the manifold $M$ is not necessarily compact and consider a diffusion process $X = (x_t, \mathbb{P}_x)$ starting from $x \in M$, which is generated by $\Delta + b$. Thanks to smoothness of $M$ and $b$, we can construct this diffusion process up to the explosion time $\zeta(x) := \inf\{t > 0; x_t \notin M\}$.
by solving a stochastic differential equation on $M$. Obviously, $\zeta(x) = \infty$ for all $x \in M$ when $M$ is compact. The diffusion process $X$ gives a probabilistic representation for the semigroup $\{e^{-tA}\}_{t \geq 0}$. More precisely, by the Feynman-Kac formula, we have

$$e^{-tA}f(x) = \mathbb{E}^x \left[ \exp \left( -\int_0^t V(x_s) \, ds \right) f(x_t) ; t < \zeta(x) \right] , \quad t \geq 0 , \quad x \in M$$

(1.1)

for all $f \in C_0(M)$. See e.g., [IW89, Gün17] for details. In view of Feynman’s original path integral approach to quantum physics, there are many interests of finite dimensional integral approximations of the Feynman-Kac type functional integral (1.1) over Riemannian manifolds in many branches of mathematics such as functional analysis, geometric analysis and probability theory. Actually, this topic has been studied intensively and extensively by many authors. See e.g., [ET81, IM85, Ino86, AD99, BP08, MMRS23] and references therein for further related results. We should mention that, in the case $V = 0$, the central limit theorem (CLT, in short) for geodesic random walks on Riemannian manifolds also gives a finite dimensional integral approximation of (1.1). For early work in this direction, see e.g., [Jør75, Pin76, Sun81, Blu84].

Under these circumstances of finite dimensional integral approximations of the functional integral (1.1) together with developments of numerical analysis and manifold learning theory, it is fundamental and important to study a discrete approximation scheme for the drifted Schrödinger semigroup $\{e^{-tA}\}_{t \geq 0}$. To tackle this problem, in the present paper, we introduce a family of discrete time random walks in the flow generated by the drift $b$ with killing on a sequence of proximity graphs, which are constructed by partitions cutting the Riemannian manifold $M$ into small pieces. Due to the effect of the drift $b$, these random walks are not necessarily symmetric in general. This makes our problem difficult and at the same time interesting. As a main result, under condition (A), we prove that $\{e^{-tA}\}_{t \geq 0}$ in $C_0(M)$ is approximated by the discrete semigroups generated by the family of random walks with a suitable scale change (see Theorem 2.1 for the precise statement). Furthermore, when $M$ is compact, we also obtain a quantitative error estimate of the convergence (see Theorem 2.4). As we shall state in Corollary 2.2, these results give us a finite dimensional summation approximation of the Feynman-Kac type functional integral (1.1) which would be a theoretical basis of a new numerical method.

We note that it is possible to study our problem in $L^p(\mathfrak{m})$-setting in parallel to $C_0(M)$-setting, where $L^p(\mathfrak{m})$, $1 < p < \infty$, denotes the usual real $L^p$-space on $M$ with respect to the volume measure $\mathfrak{m}$ equipped with the $L^p$-norm $\|f\|_{L^p(\mathfrak{m})} = \left( \int_M |f(x)|^p \, d\mathfrak{m}(x) \right)^{1/p}$. In fact, under two conditions

(A1)$_p$: $(-\lambda - \mathcal{A}, C_c^\infty(M))$ is dissipative in $L^p(\mathfrak{m})$ for some $\lambda \geq 0$;

(A2)$_p$: $(\lambda' + \mathcal{A})(C_c^\infty(M))$ is dense in $L^p(\mathfrak{m})$ for some $\lambda' > \lambda$,

the closure of $(-\mathcal{A}, C_c^\infty(M))$ generates a $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$ in $L^p(\mathfrak{m})$, and we see that most of the arguments in the proof of the above results apply $L^p(\mathfrak{m})$-setting (see

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also Theorem 2.1 and Corollary 2.2 for details). Note that the semigroup \( \{e^{-tA}\}_{t \geq 0} \) is not necessarily contractive in \( L^p(\mathfrak{m}) \).

We now mention related works. Burago-Ivanov-Kurylev [BIK14] studied a discrete approximation of the Laplacian on a compact Riemannian manifold based on partitions of the manifold. Although our framework is influenced by this paper, we need to extend their argument slightly to the case where the underlying manifold is non-compact. Chen-Kim-Kumagai [CKK13] also studied a similar discrete approximation for a large class of symmetric jump processes on metric measure spaces satisfying the volume doubling condition. (Note that the volume doubling condition does not necessarily hold for general non-compact complete Riemannian manifolds.) In a series of papers [IKK17, IKN20, IKN21], we studied CLTs for non-symmetric random walks on infinite graphs having a periodic structure such as crystal lattices and nilpotent covering graphs. Introducing a family of random walks which interpolates between the original non-symmetric random walk with the symmetrized one, as the limit, we captured the Brownian motion with a constant drift on a suitable space in which the graph is realized. However, this kind of interpolation is different from the family of random walks in a flow introduced in the present paper. It is also worth mentioning that Kotani [Kot02] studied a semigroup CLT for a generalized Harper operator on a crystal lattice in \( L^2 \)-setting, and obtained a (uniform) magnetic Schrödinger semigroup on an Euclidean space as the limit.

The rest of the present paper is organized as follows: In Sections 2.2 and 2.3, we introduce our framework of the graph discretization of the manifold \( M \) and a random walk in a flow with killing on the graph. In Section 2.4, we state main results (Theorems 2.1, 2.4 and Corollaries 2.2, 2.3). In Section 3, we devote ourselves to prove main results. In particular, in Theorem 3.2, we obtain a convergence rate of the generators of the family of random walks under the suitable scale change mentioned above. Combining this theorem with Trotter’s approximation theorem (cf. [Tro58, Kur69]) and a recent result on its convergence rate (cf. [Nam23]), we obtain Theorems 2.1 and 2.4, respectively. In Section 4, we discuss sufficient conditions for conditions \((A), (A1)_{p}\) and \((A2)_{p}\). Finally in Section 5, we give examples of the partition of the manifold and the drift on two typical manifolds: Euclidean spaces and model manifolds.

2 Framework and main results

2.1 Notations

We introduce some notations to describe our results in detail. Let \( C(M) \) be the space of all real-valued continuous function on the Riemannian manifold \( M \). Let \( C_0(M) \) be the subspace of \( C(M) \) vanishing at infinity, i.e., \( \lim_{d(x,o) \to \infty} f(x) = 0 \), where \( d(x,o) \) denotes the geodesic distance between \( x \in M \) and a base point \( o \in M \). This is a Banach space endowed with the uniform convergence topology \( \|f\|_{\infty} := \sup_{x \in M} |f(x)| \). Let \( C^\infty(M) \) be the space of all real-valued smooth functions on \( M \). Obviously, if \( M \) is compact, \( C_0(M) \) and \( C^\infty_c(M) \) coincide with \( C(M) \) and \( C^\infty(M) \), respectively.
Let $\partial$ be a cemetery point added to $M$ so that $M_\partial := M \cup \{\partial\}$ is the one-point compactification of $M$. As usual, we may regard $\partial$ as the point at infinity and $X = (x_t, \mathbb{P}_x)$ as a continuous Markov process on $M_\partial$. Note that we may rewrite the explosion time as $\zeta(x) = \inf\{t > 0; x_t = \partial\}$. Since $M$ is complete, $x \to \partial$ is equivalent to $d(x, o) \to \infty$, and thus $C_0(M)$ may be identified with the space of all continuous functions $f : M_\partial \to \mathbb{R}$ satisfying $f(\partial) = 0$. For simplicity of notation, we write $\partial = \{\partial\}$.

Throughout the present paper, we use $c, C$ to denote positive constants which may change from line to line. We also use the Landau symbols $O(\cdot)$ and $o(\cdot)$. If the dependence of $C$, $O(\cdot)$ and $o(\cdot)$ are significant, we denote them like $C(N)$, $O_N(\cdot)$ and $o_N(\cdot)$, respectively. For $a \in \mathbb{R}$, we denote by $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the greatest integer less than or equal to $a$ (resp. the least integer greater than or equal to $a$). Unless otherwise specified, we use the Einstein summation convention, which means that an index variable that appears twice in an expression is implicitly summed over all its possible values.

### 2.2 Graph discretization

In what follows, we present a framework of a graph discretization of the Riemannian manifold $M$. For $x \in M$ and $r > 0$, let $B_r(x)$ be the open geodesic ball of radius $r > 0$ centered at $x \in M$. For any subset $A$ of $M$, we denote by $U_r(A)$ the $r$-neighborhood of $A$, i.e.,
\[ U_r(A) = \cup_{x \in A} B_r(x). \]

For two subsets $A$ and $B$ of $M$, we define the Hausdorff distance $d_H(A, B)$ between $A$ and $B$ by
\[ d_H(A, B) := \inf\{r > 0; A \subset U_r(B), B \subset U_r(A)\}. \]

See [BBI01, Section 7.3] for details of the Hausdorff distance. Moreover, we set
\[ d_H(\{\partial\}, A) := \infty, \quad A \subset M. \]

A countable collection $\mathbb{X}$ of connected measurable subsets of $M$ with finite measures is called a partition of $M$ if
\[ M = \bigcup_{X \in \mathbb{X}} X, \quad X \cap Y = \emptyset \quad (X, Y \in \mathbb{X}, X \neq Y). \tag{2.1} \]

Throughout the present paper, we assume

(B): $\|\{X \in \mathbb{X}; X \subset B_r(x)\} < \infty$ for all $x \in M$ and $r > 0$;

(C): $|\mathbb{X}| := \sup_{X \in \mathbb{X}} \text{diam}(X) < \infty$.

For given $x \in M$, we denote by $X(x)$ the unique element $X \in \mathbb{X}$ containing $x$. We also define $X(\partial) = \partial$. For each $X \in \mathbb{X}$, we take a reference point $x \in X$ and write $x(X)$. Set $\mathcal{X} = \{x(X)\}_{X \in \mathbb{X}}$. It follows from (2.1) that $x(X) \neq x(Y)$ for $X \neq Y$.

If the manifold $M$ is compact, condition (B) implies that such a partition $\mathbb{X}$ is a finite set. It is always possible to construct a partition $\mathbb{X}$ with (B) and (C) in the
following manner: For fixed a constant $\varepsilon > 0$, there exists a maximal $\varepsilon$-separated subset $\mathcal{S} = \{x_i\}_{i \in \mathbb{N}}$ of $M$. Note $B_{\varepsilon/2}(x_i) \cap B_{\varepsilon/2}(x_j) = \emptyset$, $i \neq j$, and $\bigcup_{i=1}^{\infty} B_{\varepsilon}(x_i) = M$ (see e.g., [Kan85, Kan86] for details). We set

$$X_i := B_{\varepsilon}(x_i) \setminus \bigcup_{j<i} B_{\varepsilon}(x_j), \quad i \in \mathbb{N},$$

and divide each $X_i$ into a finite number of connected components $X_i^1, \ldots, X_i^{k(i)}$. Obviously, $\mathbb{X}_\mathcal{S} = \{X_i^k; k = 1, \ldots, k(i), i \in \mathbb{N}\}$ is a partition of $M$ satisfying (C) because $|\mathbb{X}_\mathcal{S}| \leq 2\varepsilon$. Besides, since $\mathcal{S}$ is $\varepsilon$-separated and $M$ is complete and smooth, we easily see that $B_r(x) \setminus \mathcal{S}$ is a finite set for all $x \in M$ and $r > 0$. This means that $\mathbb{X}_\mathcal{S}$ satisfies (B). As another example of the partition of $M$, we may consider the Voronoi decomposition of $M$ with respect to $\mathcal{S}$, which is defined by

$$X_1 := \tilde{X}_1, \quad X_i := \tilde{X}_i \setminus \bigcup_{j=1}^{i-1} X_j, \quad i = 2, 3, \ldots,$$

where

$$\tilde{X}_i := \{y \in M; d(x_i, y) \leq d(x_j, y), \quad j \in \mathbb{N}, i \neq j\}, \quad i \in \mathbb{N}.$$

For given $\rho > 0$, let $\mathbb{X}$ be a partition of $M$ with (B) and (C) satisfying $|\mathbb{X}| < \rho$. We say that $X$ and $Y$ in $\mathbb{X}$ are adjacent if $d_H(X, Y) < \rho$ and write $X \sim_\rho Y$. Then we define an oriented graph $G(\mathbb{X}, \rho) = (\mathcal{V}, \mathcal{E})$ called a ($\rho$-) proximity graph of the manifold $M$ by $\mathcal{V} := \mathbb{X}$ and

$$\mathcal{E} := \{(X, Y) \in \mathbb{X} \times \mathbb{X}; X \sim_\rho Y\}.$$  

Since $M$ is connected, $G(\mathbb{X}, \rho)$ is also connected, and (B) implies that $G(\mathbb{X}, \rho)$ is locally finite, that is, for any $X \in \mathbb{X}$, the ($\rho$-)neighborhood of $X$ defined by

$$N_\rho(X) := \{Z \in \mathbb{X}; Z \sim_\rho X\}$$

is a finite set. We should mention that $G(\mathbb{X}, \rho)$ is uniform, i.e., $\sup_{X \in \mathbb{X}} \#N_\rho(X) < \infty$, provided that the Ricci curvature of $M$ is bounded from below. See [Kan85, Lemma 2.3] for details.

For later purpose, we introduce a graph $G_\partial(\mathbb{X}, \rho) = (\mathcal{V}_\partial, \mathcal{E}_\partial)$ constructed by the sum of the proximity graph $G(\mathbb{X}, \rho)$ and the point $\partial$ mentioned above. To be precise, $\mathcal{V}_\partial := \mathcal{V} \cup \partial$ and

$$\mathcal{E}_\partial := \mathcal{E} \cup \{(X, \partial), (\partial, X); X \in \mathbb{X}\} \cup \{\partial, \partial\}.$$  

We denote by $C_0(G_\partial(\mathbb{X}, \rho))$ the space of all bounded functions $F : \mathbb{X} \cup \{\partial\} \to \mathbb{R}$ with $F(\partial) = 0$ endowed with

$$\|F\|_{C_0(G_\partial(\mathbb{X}, \rho))} := \sup_{X \in \mathbb{X}} |F(X)|,$$

and by $L^p(G_\partial(\mathbb{X}, \rho)), 1 < p < \infty$, the set of all functions $F : \mathbb{X} \cup \{\partial\} \to \mathbb{R}$ satisfying $F(\partial) = 0$ and

$$\|F\|_{L^p(G_\partial(\mathbb{X}, \rho))} := \left(\sum_{X \in \mathbb{X}} |F(X)|^p \mathfrak{m}(X)\right)^{1/p} < \infty.$$
For a given partition $\mathcal{X}$ and a set of reference points $\mathcal{X} = \{x(X)\}_{X \in \mathcal{X}}$ associated with $\mathcal{X}$, we define two kinds of discretization maps $P_{\mathcal{X}} : L^p(\mathcal{M}) \to L^p(G_\partial(\mathcal{X}, \rho))$ and $[.]_{\mathcal{X}} : C_0(M) \to C_0(G_\partial(\mathcal{X}, \rho))$ by

$$P_{\mathcal{X}}f(X) = \frac{1}{m(X)} \int_X f \, dm \quad \text{and} \quad [f]_{\mathcal{X}}(X) = f(x(X)),$$

respectively.

### 2.3 Random walk in a flow with killing

Let us recall that $\theta$ is a smooth vector field on the complete Riemannian manifold $M$ and $V$ is a non-negative smooth potential function on $M$.

We first introduce the notion of the flow generated by $\theta$. For each $x \in M$, there exist an open interval $I = I_x = (-t(x), t(x))$ around $0 \in \mathbb{R}$ and a smooth curve $(\varphi_t(x))_{t \in I}$ on $M$ satisfying

$$\varphi_0(x) = x, \quad \frac{d}{dt} \varphi_t(x) = \theta(\varphi_t(x)), \quad t \in I.$$

Since the solution to the above differential equation depends smoothly on the initial point $x \in M$, it induces a local flow $\varphi = (\varphi_t(x)) : \{(t, x) ; t \in I_x, x \in M\} \to M$ generated by the vector field $\theta$. Note that $t(x)$ is continuous with respect to $x$. We set

$$s(x) := \inf\{t > 0 ; \varphi_t(x) \notin M\}, \quad x \in M.$$

Obviously, $s(x) \geq t(x)$ holds for all $x \in M$. Moreover $\varphi_s(x) = \partial$ provided $s \geq s(x)$.

For $|X| < \rho$, $X \in \mathcal{X}$ and $s \geq 0$, we set

$$N_{\rho, X}(s; X) := \{Y \in \mathcal{X} ; Y \sim_{\rho} X(\varphi_s(x(X)))\}$$

and

$$\mathcal{N}_{\rho, X}(s; X) := \bigcup_{Y \in N_{\rho, X}(s; X)} Y.$$

In the case $b = 0$, $N_{\rho, X}(s; X) = N_{\rho}(X)$. If $s \geq s(x(X))$, $N_{\rho, X}(s; X) = \partial$.

Inspired by an idea of [Mad89], we introduce a random walk in the flow $\varphi$ with killing on the proximity graph $(G_\partial(\mathcal{X}, \rho), \mathcal{X})$. Let $\alpha, s \geq 0$. We define the transition probability $p_{\alpha, s} = p_{\alpha, s, X}$ on $(G_\partial(\mathcal{X}, \rho), \mathcal{X})$ as follows: For each $X \in \mathcal{X}$, $Y \in \mathcal{V}_\partial$ and $0 \leq s \leq s(x(X))$,

$$p_{\alpha, s}(X, Y) := \begin{cases} \min\{\alpha V(x(X)), 1\} & \text{if } Y = \partial, \\ \max\{1 - \alpha V(x(X)), 0\} \frac{m(Y)}{m(N_{\rho, X}(s; X))} & \text{if } Y \in N_{\rho, X}(s; X), \\ 0 & \text{otherwise} \end{cases}$$

and for $s \geq s(x(X))$

$$p_{\alpha, s}(X, Y) := \begin{cases} 1 & \text{if } Y = \partial, \\ 0 & \text{otherwise} \end{cases}.$$
Besides, we set
\[ p_{\alpha,s}(\partial, \partial) := 1, \quad p_{\alpha,s}(\partial, X) := 0, \quad X \in \mathbb{X} \] for all \( s \geq 0 \).

In the usual manner, the transition probability \( p_{\alpha,s} \) induces a time homogeneous Markov chain on \( G_\partial(\mathbb{X}, \rho) \). We call it the random walk in the flow \( \varphi \) with a killing rate \( \alpha V \).

The corresponding transition operator \( L_{\alpha,s} = L_{\alpha,s,\varphi}: C_0(G_\partial(\mathbb{X}, \rho)) \to C_0(G_\partial(\mathbb{X}, \rho)) \) is given by
\[ L_{\alpha,s}F(\partial) := F(\partial) = 0 \quad \text{and} \quad L_{\alpha,s}F(X) := \sum_{Y \in \mathcal{V}_\partial} p_{\alpha,s}(X, Y)F(Y), \quad X \in \mathbb{X}. \] (2.2)

This means
\[ L_{\alpha,s}F(X) = \begin{cases} \max \{ 1 - \alpha V(x(X)), 0 \} \sum_{Y \in \mathcal{N}_\rho,\varphi(s;X)} \frac{m(Y)}{m(N_\rho,\varphi(s;X))}F(Y) & \text{if } s < s(x(X)), \\ 0 & \text{if } s \geq s(x(X)). \end{cases} \]

We abbreviate \( L_{\alpha,\alpha} \) to \( L_\alpha \) for brevity.

Here we give a remark. Let \( b = 0 \) and \( V = 0 \), and put \( m(X) := m(X)m(N_\rho,\varphi(s;X)) \), \( X \in \mathbb{X} \), and \( m(\partial) = m(\partial) := 0 \). Then we easily see
\[ p_{\alpha,s}(X, Y)m(X) = m(X)m(Y) = p_{\alpha,s}(Y, X)m(Y), \quad X, Y \in \mathcal{V}_\partial, \]
which means that the corresponding random walk is \( m \)-symmetric. On the other hand, the random walk in the flow \( \varphi \) is not necessarily symmetric in general. Indeed, by taking \( s > 0 \) large enough and \( \rho > 0 \) small enough, the distances among \( x \in \mathbb{X} \), \( \varphi_s(x) \) and \( \varphi_{2s}(x) \) can be very large. In this situation, if \( Y \) is close to \( \varphi_s(x) \), we observe \( p_{\alpha,s}(X, Y) > 0 \) and \( p_{\alpha,s}(Y, X) = 0 \). This means that the random walk cannot be symmetric.

### 2.4 Main results

Now we are in a position to state the main results in the present paper. We start with presenting the following theorem.

**Theorem 2.1.** Let \( M \) be an \( n \)-dimensional complete manifold, \( b \) be a smooth vector field on \( M \) and \( V \) be a non-negative smooth potential function on \( M \). Let \( \mathcal{A} = -\Delta - b + V \) be a drifted Schrödinger operator on \( M \) satisfying condition (A). Let \( \{X_k\}_{k \in \mathbb{N}} \) be a sequence of partitions of \( M \) satisfying conditions (B), (C) and \( |X_k| \to 0 \) as \( k \to \infty \). Let \( \{k(\rho)\}_{\rho > 0} \) be a subsequence of \( \mathbb{N} \) satisfying \( k(\rho) \nearrow \infty \) and \( |X_{k(\rho)}| = o(\rho^2) \) as \( \rho \searrow 0 \). Then for any \( f \in C_0(M) \) and \( t > 0 \),
\[ \lim_{\rho \searrow 0} \left\Vert L_{\rho^2}^{\frac{2(\alpha+2)}{2(n+2)}} \mathcal{A}_{k(\rho)} - \left[ e^{-t\mathcal{A}} \mathcal{A}_{k(\rho)} \right] \right\Vert_{C_0(G_\partial(X_{k(\rho)}, \rho))} = 0. \] (2.3)

Moreover, this convergence also holds in \( L^p \)-setting for all \( 1 < p < \infty \). Namely, under conditions (A1)\( p \), (A2)\( p \), (B) and (C), for any \( f \in L^p(\mathcal{M}) \) and \( t > 0 \),
\[ \lim_{\rho \searrow 0} \left\Vert L_{\rho^2}^{\frac{2(\alpha+2)}{2(n+2)}} \mathcal{P}_{X_{k(\rho)}} f - \mathcal{P}_{X_{k(\rho)}} (e^{-t\mathcal{A}} f) \right\Vert_{L^p(G_\partial(X_{k(\rho)}, \rho))} = 0. \] (2.4)

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This theorem implies the following corollary immediately.

**Corollary 2.2.** Under the setting of Theorem 2.1, we have the following:

1. For any \( x \in M, f \in C_0(M) \) and \( t > 0 \),

\[
    e^{-tA}f(x) = \lim_{\rho \to 0} L^{\left[\frac{2n+2}{\rho^2}\right]} [f]_{L_{k}(\rho)}(X_{k}(\rho)(x)),
\]

where \( X_{k}(\rho)(x) \) is the unique element of \( X_{k}(\rho) \) containing \( x \).

2. For any bounded open set \( U \) in \( M, f \in L^p(m) \) and \( t > 0 \),

\[
    \int_U e^{-tA}f(x)dm(x) = \lim_{\rho \to 0} \sum_{X \in X_{k}(\rho)} L^{\left[\frac{2n+2}{\rho^2}\right]} [f]_{L_{k}(\rho)}X_{k}(\rho)f(X)m(X). \quad (2.6)
\]

We also obtain the following corollary by repeating arguments in the proof of Theorem 2.1 with a slight modification.

**Corollary 2.3.** (1) Suppose that the potential function \( V \) is bounded from below by \(-v_0\) for some constant \( v_0 \geq 0 \). Then under the same setting as Theorem 2.1,

\[
    \lim_{\rho \to 0} \left\| \int e^{-\rho t} L^{\left[\frac{2n+2}{\rho^2}\right]} [f]_{L_{k}(\rho)}[e^{-tA}f]_{L_{k}(\rho)} \right\|_{C_0(\partial_0(X_{k}(\rho),\rho))} = 0,
\]

where \( L_{\alpha} = L_{\alpha,\alpha} \) is the transition operator in the flow \( \varphi_{\alpha} \) with the killing rate \( \alpha(V + v_0) \) defined in (2.2).

2. Let \((M, g, \mu)\) be a weighted manifold with \( d\mu(x) = e^{-U(x)}dm(x) \) for a smooth function \( U \) on \( M \). Define the weighted Laplacian \( \Delta_{\mu} \) by

\[
    \Delta_{\mu}f = \text{div}_{\mu}(\nabla f) = \Delta f - \langle \nabla U, \nabla f \rangle,
\]

where \( \text{div}_{\mu}b = e^{U} \text{div}(e^{-U}b) \) (see [Gri09, Section 3.6] and [Shi12]). Then Theorem 2.1 with respect to the weighted measure \( \mu \) instead of \( m \) (used in the definition of \( L_{\alpha,s} \)) and \( A_{\mu} = -\Delta_{\mu} - b + V \) holds. In this situation, positive constants \( K_1 \) and \( K_2 \) appearing in the error estimate (3.7) in Theorem 2.2 depend also on \( U \).

Now we further impose that the manifold \( M \) is compact. In this case, we have the following quantitative error estimate.

**Theorem 2.4.** Let \( M \) be an \( n \)-dimensional closed Riemannian manifold. Under the setting of Theorem 2.1, we take a subsequence \( \{k(\rho)\} \) of \( \mathbb{N} \) satisfying \( k(\rho) \not\to \infty \) and \( |X_{k(\rho)}| = O(\rho^{2+\alpha}) \) as \( \rho \to 0 \), where \( \alpha > 0 \). Then for any \( f \in C^\infty(M) \), \( t \geq 0 \) and \( 0 < \rho < 1 \) satisfying condition (3.6) in Theorem 2.2 below, the following error estimate holds:

\[
    \left\| L^{\left[\frac{2n+2}{\rho^2}\right]} [f]_{L_{k}(\rho)}[e^{-tA}f]_{L_{k}(\rho)} \right\|_{C_0(\partial_0(X_{k}(\rho),\rho))} \leq C\rho^{\alpha+1}, \quad (2.7)
\]

where \( C = C(t, n, f, \|b\|_\infty, \|\nabla b\|_\infty, \|V\|_\infty) \) is a positive constant.
3 Proof of main results

3.1 Convergence of generators

Lemma 3.1. Let \( \{\mathcal{X}_k\}_{k=1}^{\infty} \) be a sequence of partitions of \( M \) satisfying \( |\mathcal{X}_k| \leq 0 \) as \( k \to \infty \), and \( \mathcal{X}_k \) be a set of reference points associated with \( \mathcal{X}_k \). Then the sequence of spaces \( L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) and \( C_0(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) together with the maps \( \mathcal{P}_{\mathcal{X}_k} : L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) and \( [\cdot]_{\mathcal{X}_k} : C_0(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) approximate \( L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) and \( C_0(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) in the sense of Trotter [Tro58], respectively. Namely, for any \( f \in L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \)

\[
\|\mathcal{P}_{\mathcal{X}_k}f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} \leq \|f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))},
\]

\[
\lim_{k \to \infty} \|\mathcal{P}_{\mathcal{X}_k}f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} = \|f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))},
\]

and for any \( f \in C_0(\mathcal{G}_0(\mathcal{X}_k, \rho)) \)

\[
\|[f]_{\mathcal{X}_k}\|_{C_0(\mathcal{G}_0(\mathcal{X}_k, \rho))} \leq \|f\|_{C_0(\mathcal{G}_0(\mathcal{X}_k, \rho))},
\]

\[
\lim_{k \to \infty} \|[f]_{\mathcal{X}_k}\|_{C_0(\mathcal{G}_0(\mathcal{X}_k, \rho))} = \|f\|_{C_0(\mathcal{G}_0(\mathcal{X}_k, \rho))}.
\]

Proof. The estimate (3.1) follows directly from Hölder’s inequality. Noting that \( \{\mathcal{X}_k\}_{k=1}^{\infty} \) satisfies \( d_H(\mathcal{X}_k, \mathcal{G}_0(\mathcal{X}_k, \rho)) \) as \( k \to \infty \), we also obtain the estimate (3.3) and the convergence (3.4) immediately.

To prove (3.2), it suffices to show that for any \( \varepsilon > 0 \) and \( f \in L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) there exists \( k_0 \in \mathbb{N} \) such that for any \( k \geq k_0 \)

\[
\|f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} < \varepsilon.
\]

Moreover, since \( C^\infty_c(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) is dense in \( L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \), it suffices to show the estimate in (3.5) for any \( f_0 \in C^\infty_c(\mathcal{G}_0(\mathcal{X}_k, \rho)) \). Indeed, for any \( f \in L^p(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) take \( f_0 \in C^\infty_c(\mathcal{G}_0(\mathcal{X}_k, \rho)) \) such that \( \|f - f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} < \varepsilon \). Then we obtain

\[
\|f\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} = \|f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} + \|f - f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} \leq \|f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} + \|f - f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))}.
\]

To prove \( \|f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} < \varepsilon \), it suffices to prove that

\[
\|f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} < \varepsilon
\]

by the continuity of function \( x^{1/p} \) defined on \( [0, \infty) \). Then we have

\[
\|f_0\|_{L^p(\mathcal{G}_0(\mathcal{X}_k, \rho))} = \sum_{X \in \mathcal{X}_k} \left( \frac{1}{m(X)} \int_X |f_0|^p \, dm - \left| \frac{1}{m(X)} \int_X f_0 \, dm \right|^p \right) m(X).
\]
\[
\leq m(B_{2\rho}(\text{supp } f_0)) \max_{X \subset B_{2\rho}(\text{supp } f_0)} \left( \frac{1}{m(X)} \int_X |f_0|^p \, dm - \frac{1}{m(X)} \int_X f_0 \, dm \right)^p.
\]
Since \( X \in \mathbb{X}_k \) is connected, the integral-type mean value theorem implies that there exists \( z \in X \) such that
\[
\frac{1}{m(X)} \int_X f_0 \, dm = f_0(z).
\]
Then we obtain
\[
\frac{1}{m(X)} \int_X |f_0|^p \, dm - \frac{1}{m(X)} \int_X f_0 \, dm = \frac{1}{m(X)} \int_X (|f_0(y)|^p - |f_0(z)|^p) \, dm(y).
\]
Since \( f_0 \) is uniformly continuous, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \text{diam} X < \delta \) then
\[
|f_0(y)|^p \leq (|f_0(z)| + \varepsilon)^p.
\]
Then we obtain for \( y \in X \)
\[
|f_0(y)|^p - |f_0(z)|^p \leq p(|f_0(z)| + \varepsilon)^{p-1} \varepsilon,
\]
which concludes that
\[
\max_{X \subset B_{2\rho}(\text{supp } f_0)} \left( \frac{1}{m(X)} \int_X |f_0|^p \, dm - \frac{1}{m(X)} \int_X f_0 \, dm \right)^p \leq \varepsilon p \left( \sup_{x \in M} |f_0(x)| + \varepsilon \right)^{p-1}.
\]
Hence, the proof of the convergence in (3.2) is completed. \( \square \)

Let \( f \in C_c^\infty(M) \) and \( b \) be a smooth vector field on \( M \). We denote by \( \varphi \) the flow generated by \( b \). Since \( U_2(\text{supp } f) \) is compact and \( t = t(x) \) is a positive continuous function,
\[
s(f) := \min \{ t(x) \mid x \in U_2(\text{supp } f) \}
\]
is positive, and \( \varphi_s \) gives a diffeomorphism between \( \varphi_{-s} U_2(\text{supp } f) \) and \( U_2(\text{supp } f) \) for any \( 0 \leq s \leq s(f) \). We then define a generalized support of \( f \) in the flow \( \varphi \) by
\[
S(f) := \bigcup_{0 \leq s \leq \min\{s(f), 1\}} \varphi_{-s} U_2(\text{supp } f),
\]
and set
\[
\|b\|_{\infty,S(f)} = \max_{x \in S(f)} |b(x)|_{T_xM}.
\]
Moreover, for a non-negative smooth potential function \( V \) on \( M \), we also set
\[
\|V\|_{\infty,S(f)} = \max_{x \in S(f)} V(x).
\]
We emphasize that \( S(f) \) is compact, \( 0 \leq \|b\|_{\infty,S(f)} < \infty \) and \( 0 \leq \|V\|_{\infty,S(f)} < \infty \).

For \( f \in C_c^\infty(M) \), we set
\[
\|f\|_{C^k} := \max_{|\alpha| \leq k} \sup_{x \in M} \left| \frac{\partial^{|\alpha|} f}{\partial \alpha_1 u(1) \cdots \partial \alpha_n u(n)}(x) \right|, \quad k \in \mathbb{N},
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is the multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( (u(1), \ldots, u(n)) \) is the Riemannian normal coordinates at \( x \in M \).

The following theorem plays a crucial role in the proof of our main results.
Theorem 3.2. Let \( f \in C^\infty_c(M) \), and take \( \rho > 0 \) small enough such that
\[
\frac{\rho^2}{2(n+2)} \leq \min \left\{ s(f), \| V \|_{\infty,S(f)}, 1 \right\} \tag{3.6}
\]
and
\[
\rho < \text{inj}_M(S(f)),
\]
where \( \text{inj}_M(S(f)) \) is the infimum of the injectivity radius in \( S(f) \). Then for any partition \( \mathcal{X} \) satisfying \( |\mathcal{X}| < \rho/3 \), there exist positive constants \( K_1 \) and \( K_2 \) depending on \( \| f \|_{C^3}, \| b \|_{\infty,S(f)} \), \( \| \nabla b \|_{\infty,S(f)} \), \( \| V \|_{\infty,S(f)} \) and \( S(f) \) such that
\[
\left| \frac{2(n+2)}{\rho^2} (I - L_{\delta}) [f]_{\mathcal{X}}(X) - [Af]_{\mathcal{X}}(X) \right| \leq K_1 \frac{|\mathcal{X}|}{\rho^2} + K_2 \rho, \quad X \in \mathcal{X}. \tag{3.7}
\]

Proof. We set \( \delta = \frac{\rho^2}{2(n+2)} \) and fix \( X \in \mathcal{X} \). We denote the reference point \( x(X) \in \mathcal{X} \) by \( x \) for the simplicity of notation. Then we have
\[
\frac{1}{\delta} (I - L_{\delta}) [f]_{\mathcal{X}}(X) - [Af]_{\mathcal{X}}(X) = \frac{1}{\delta} (f(x) - f(\varphi_\delta(x)) + \delta bf(x)) \tag{3.8}
\]
\[
- V(x)(f(x) - f(\varphi_\delta(x)) + \delta bf(x)) + (\Delta f(x) - \Delta f(\varphi_\delta(x)) + \delta b\Delta f(x)) \tag{3.8}
\]
\[
+ \delta (V(x)bf(x) - b\Delta f(x)) \tag{3.8}
\]
\[
- \frac{1}{\delta} L_\delta [f]_{\mathcal{X}}(X) - V(x)f(\varphi_\delta(x)) + \Delta f(\varphi_\delta(x)) + \frac{1}{\delta} f(\varphi_\delta(x)). \tag{3.8}
\]
Applying the Taylor expansion formula, we have
\[
f(x) - f(\varphi_\delta(x)) = -\delta \int_0^1 bf(\varphi_{\delta \theta}(x))d\theta \tag{3.9}
\]
and
\[
f(x) - f(\varphi_\delta(x)) + \delta bf(x) = -\frac{\delta^2}{2} \int_0^1 \left\{ \text{Hess} f(b, b)(\varphi_{\delta \theta}(x)) + (\nabla b) f(\varphi_{\delta \theta}(x)) \right\} d\theta, \tag{3.10}
\]
where \( \nabla \) and \( \text{Hess} f \) stand for the Levi-Civita connection of the Riemannian metric \( g \) and the Hessian of \( f \), respectively. In a local coordinate \( \{ x^{(i)} \}_1^n \) of \( M \) with the natural basis \( \partial_i = \partial/\partial x^{(i)} \), \( i = 1, \ldots, n \), by using the expansion \( b = \sum_{i=1}^n b^i \partial_i \), we may write
\[
\text{Hess} f(b, b) = b^i b^j \frac{\partial^2 f}{\partial x^{(i)} \partial x^{(j)}} - b^i b^j \Gamma^k_{ij} \frac{\partial f}{\partial x^{(k)}}, \quad (\nabla b) f = b^i \left( \frac{\partial b^j}{\partial x^{(i)}} \partial_j + b^k \Gamma^k_{ij} \partial_k \right)f,
\]
where \( \{ \Gamma^k_{ij} \}_{i,j,k=1}^n \) are the Christoffel symbols of the Levi-Civita connection.

It follows from (3.9) and (3.10) that
\[
\left| \frac{1}{\delta} (f(x) - f(\varphi_\delta(x)) + \delta bf(x)) \right| \leq \frac{\delta}{2} (\| \text{Hess} f \|_\infty + \| (\nabla b) f \|_\infty).
\]
and
\[
\left| -V(x)(f(x) - f(\varphi_\delta(x)) + \delta bf(x)) + (\Delta f(x) - \Delta f(\varphi_\delta(x)) + \delta b\Delta f(x)) \right|
\leq 2\delta(\|V\|_{\infty,S(f)}\|bf\|_{\infty} + \|b\Delta f\|_{\infty}).
\]

Besides, we also have
\[
\left| \delta(V(x)b f(x) - b\Delta f(x)) \right| \leq \delta(\|V\|_{\infty,S(f)}\|bf\|_{\infty} + \|b(\Delta f)\|_{\infty}).
\]

Noting \(\max\{(1 - \delta V), 0\} = (1 - \delta V)\) in \(S(f)\), we expand the final line of the right-hand side of (3.8) as
\[
-\frac{1}{\delta}L_\delta[f_\mathcal{X}]X - V(x)f(\varphi_\delta(x)) + \Delta f(\varphi_\delta(x)) + \frac{1}{\delta}f(\varphi_\delta(x))
= \frac{\delta V(x) - 1}{\delta} \sum_{Y \in N_{\rho,\mathcal{X}}(\delta; X)} \frac{m(Y)}{m(N_{\rho,\mathcal{X}}(\delta; X))} f(x(Y))
+ \frac{1}{\delta}f(\varphi_\delta(x)) - V(x)f(\varphi_\delta(x)) + \Delta f(\varphi_\delta(x))
= \frac{1}{\delta m(N_{\rho,\mathcal{X}}(\delta; X))} \left\{ \sum_{Y \in N_{\rho,\mathcal{X}}(\delta; X)} m(Y)\left( f(\varphi_\delta(x)) - f(x(Y)) \right) - \int_{B(\varphi_\delta(x), \rho)} (f(\varphi_\delta(x)) - f(z)) \, dm(z) \right\}
+ (1 - \delta V(x)) \left\{ \frac{1}{\delta m(N_{\rho,\mathcal{X}}(\delta; X))} \int_{B(\varphi_\delta(x), \rho)} (f(\varphi_\delta(x)) - f(z)) \, dm(z) + \Delta f(\varphi_\delta(x)) \right\}
+ \delta V(x)\Delta f(\varphi_\delta(x)).
\]

Here we easily have
\[
|\delta V(x)\Delta f(\varphi_\delta(x))| \leq \delta\|V\|_{\infty,S(f)}\|\Delta f\|_{\infty} \leq \|\Delta f\|_{\infty}.
\]

To estimate the first term on the right-hand side of (3.11), we recall the definition of the Riemannian integral. Indeed, taking into account that
\[
B_{\rho - |\mathcal{X}|}(\varphi_\delta(x)) \subset N_{\rho,\mathcal{X}}(\delta; X) \subset B_{\rho + |\mathcal{X}|}(\varphi_\delta(x)) \quad (3.12)
\]
we have
\[
\sum_{Y \in N_{\rho,\mathcal{X}}(\delta; X)} m(Y)\left( f(\varphi_\delta(x)) - f(x(Y)) \right)
= \int_{N_{\rho,\mathcal{X}}(\delta; X)} (f(\varphi_\delta(x)) - f(z)) \, dm(z) + \sum_{Y \in N_{\rho,\mathcal{X}}(\delta; X)} \int_Y (f(z) - f(x(Y))) \, dm(z)
= \int_{B_{\rho}(\varphi_\delta(x))} (f(\varphi_\delta(x)) - f(z)) \, dm(z)
\]
\[
+ \int_{B_{r+\delta}(x) \setminus B_{r}(x)} (f(\varphi_{\delta}(x)) - f(z)) \, dm(z) \\
- \int_{B_{r+\delta}(x) \setminus N_{r}(\delta; X)} (f(\varphi_{\delta}(x)) - f(z)) \, dm(z) \\
+ \sum_{Y \in N_{r}(\delta; X)} \int_{Y} (f(z) - f(x(Y))) \, dm(z).
\]

Here we note that there exist positive constants \(c_1, c_2\) and \(c_3\) depending only on geometry on \(S(f)\) such that for any \(0 < h < r < 1\)

\[
m(B_{r+h}(\varphi_{\delta}(x))) - m(B_{r-h}(\varphi_{\delta}(x))) \leq c_1 h r^{n-1}
\]

and

\[
c_3 r^n \leq m(B_{r}(\varphi_{\delta}(x))) \leq c_2 r^n.
\]

Combining these estimates with the assumption of \(|X| < \rho/3\), we obtain

\[
\frac{1 - \delta V(x)}{\delta m(N_{r}(\delta; X))} \left\{ \sum_{Y \in N_{r}(\delta; X)} m(Y) (f(\varphi_{\delta}(x)) - f(x(Y))) \right. \\
- \int_{B_{r}(\varphi_{\delta}(x))} (f(\varphi_{\delta}(x)) - f(z)) \, dm(z) \\
+ 2 \int_{B_{r+\delta}(x) \setminus B_{r-\delta}(x)} \left| f(\varphi_{\delta}(x)) - f(z) \right| \, dm(z) \right\}
\leq \frac{1}{c_3} \left\{ \sum_{Y \in N_{r}(\delta; X)} \int_{Y} (f(z) - f(x(Y))) \, dm(z) \right. \\
+ 2 \int_{B_{r+\delta}(x) \setminus B_{r-\delta}(x)} \left| f(\varphi_{\delta}(x)) - f(z) \right| \, dm(z) \right\}
\leq \frac{2^n}{c_3 \rho^n} \left\{ \left\| f \right\|_{C^1} |X| m(B_{r+\delta}(\varphi_{\delta}(x))) \right. \\
+ 2 \left\| f \right\|_{C^1} (\rho + |X|) \left( m(B_{r+\delta}(\varphi_{\delta}(x))) - m(B_{r-\delta}(\varphi_{\delta}(x))) \right) \right\}
\leq \frac{2^n (4c_1 + 2^n c_2)(n + 2)}{c_3} \left\| f \right\|_{C^1} |X|/\rho^2.
\]

To estimate the second term on the right-hand side of (3.11), we now apply the spherical mean approximation of the Laplacian. Let us take a Riemannian normal coordinate system \((u^{(1)}, \ldots, u^{(n)})\) at \(\varphi_{\delta}(x) \in M\) via exponential map \(\exp_{\varphi_{\delta}(x)}: T_{\varphi_{\delta}(x)} M \to M\). More precisely, for an orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(T_{\varphi_{\delta}(x)} M\), the map

\[
(u^{(1)}, \ldots, u^{(n)}) \mapsto \sum_{j=1}^{n} u^{(j)} e_j \exp_{\varphi_{\delta}(x)} \mapsto \exp_{\varphi_{\delta}(x)} \left( \sum_{j=1}^{n} u^{(j)} e_j \right)
\]

gives a diffeomorphism between \(B_{\mathbb{R}^n}(r)\), the open ball of radius \(r\) in \(\mathbb{R}^n\) centered at origin and \(B_{r}(\varphi_{\delta}(x))\) for small \(r > 0\) and then the pair \((B_{r}(\varphi_{\delta}(x)), E^{-1} \circ \exp_{\varphi_{\delta}(x)}^{-1})\) can be regarded as a chart containing \(\varphi_{\delta}(x)\).
Now take \( z \in S(f) \) and \( r = \rho < \operatorname{inj}_M(S(f)) \). Applying the Taylor expansion formula to the function \( \tilde{f} = f \circ \exp_{\varphi(x)} \circ E \) on \( B^{\mathbb{R}^n}(r) \subset \mathbb{R}^n \) at the origin, we have for \( u = (u^{(1)}, \ldots, u^{(n)}) = E^{-1} \circ \varphi^{-1}(z) \)

\[
 f(z) = \tilde{f}(u^{(1)}, \ldots, u^{(n)}) = \tilde{f}(0) + \frac{\partial \tilde{f}}{\partial u^{(j)}}(0) u^{(j)} + \frac{1}{2} \frac{\partial^2 \tilde{f}}{\partial u^{(j)} \partial u^{(k)}}(0) u^{(j)} u^{(k)} + \mathcal{T}_{jkl}(\tilde{f})(u) u^{(j)} u^{(k)} u^{(l)},
\]

where

\[
 \mathcal{T}_{jkl}(\tilde{f})(u) = \int_0^1 d\theta_1 \int_0^{\theta_1} d\theta_2 \int_0^{\theta_2} d\theta_3 \frac{\partial^3 \tilde{f}}{\partial u^{(j)} \partial u^{(k)} \partial u^{(l)}}(\theta_3 u).
\]

Now we make use of the fact that the Riemannian volume element \( \sqrt{\det(\tilde{g}(u))} \) in the normal coordinates has the expansion

\[
 \sqrt{\det(\tilde{g}(u))} = 1 - \frac{1}{6} \text{Ric}_{jk}(0) u^{(j)} u^{(k)} + G_3(B_\rho(\varphi_\delta(x)); u),
\]

where \( \text{Ric}_{jk}(0) := \text{Ric}(\varphi_\delta(x))(e_j, e_k) \), the \((j, k)\)-component of the Ricci curvature tensor at \( \varphi_\delta(x) \), and \( G_3(B_\rho(\varphi_\delta(x)); u) \) is the reminder term. Here we should remark that there exists a continuous non-negative function \( G \) on \( M \) such that \( G_3(B_\rho(\varphi_\delta(x)); u) \) satisfies for all \( |u|_{\mathbb{R}^n} < \rho \)

\[
 \sup_{0 < \rho' < \rho} |G_3(B_{\rho'}(\varphi_\delta(x)); u)| \leq G(\varphi_\delta(x)) |u|_{\mathbb{R}^n}^3,
\]

(3.13)

where \( \delta' = \frac{(\delta^2)^2}{2(n+2)} \). See e.g., [Sak96, Lemma 3.5 in Chapter II] and [Sak71, Lemma 3.4] for details. We also mention here that

\[
 \int_{B^{\mathbb{R}^n}(\rho)} u^{(j)} du = 0, \quad \int_{B^{\mathbb{R}^n}(\rho)} u^{(j)} u^{(k)} du = \delta_{jk} \frac{\omega_n \rho^{n+2}}{n + 2},
\]

where \( \omega_n \) is the volume of the unit ball \( B^{\mathbb{R}^n}(1) \).

We then obtain

\[
 \int_{B(\varphi_\delta(x), \rho)} (f(\varphi_\delta(x)) - f(z)) \, dm(z)
 = \int_{B^{\mathbb{R}^n}(\rho)} (\tilde{f}(0) - \tilde{f}(u)) \sqrt{\det(\tilde{g}(u))} du
 = \int_{B^{\mathbb{R}^n}(\rho)} \left( - \frac{\partial \tilde{f}}{\partial u^{(j)}}(0) u^{(j)} - \frac{1}{2} \frac{\partial^2 \tilde{f}}{\partial u^{(j)} \partial u^{(k)}}(0) u^{(j)} u^{(k)} + \mathcal{T}_{jkl}(\tilde{f})(u) u^{(j)} u^{(k)} u^{(l)} \right)
 \times \left( 1 - \frac{1}{6} \text{Ric}_{jk}(0) u^{(j)} u^{(k)} + G_3(B_\rho(\varphi_\delta(x)); u) \right) du
 = - \frac{\omega_n \rho^{n+2}}{2(n+2)} \Delta f(\varphi_\delta(x)) + \mathcal{R}_f(B_\rho(\varphi_\delta(x)))
 = - \frac{\omega_n \rho^{n+2}}{2(n+2)} \Delta f(\varphi_\delta(x)) + \mathcal{R}_f(B_\rho(\varphi_\delta(x))),
\]
where

\[ \mathcal{R}_f(B_\rho(\varphi_\delta(x))) := \frac{1}{6} \frac{\partial \tilde{f}}{\partial u^{(j)}}(0) \text{Ric}_{jk}(0) \int_{B^n(\rho)} u^{(j)} u^{(k)} u^{(l)} \, du \]

\[ + \int_{B^n(\rho)} T_{jkl}(\tilde{f})(u) u^{(j)} u^{(k)} u^{(l)} \, du \]

\[ - \frac{\partial \tilde{f}}{\partial u^{(j)}}(0) \int_{B^n(\rho)} u^{(j)} \mathcal{G}_3(B_\rho(\varphi_\delta(x)); u) \, du \]

\[ + \frac{1}{12} \frac{\partial^2 \tilde{f}}{\partial u^{(j)} u^{(k)} u^{(l)}}(0) \text{Ric}_{rs}(0) \int_{B^n(\rho)} u^{(j)} u^{(k)} u^{(l)} u^{(r)} u^{(s)} \, du \]

\[ - \frac{1}{2} \frac{\partial \tilde{f}}{\partial u^{(j)} u^{(k)}}(0) \int_{B^n(\rho)} u^{(j)} u^{(k)} \mathcal{G}_3(B_\rho(\varphi_\delta(x)); u) \, du \]

\[ - \frac{1}{6} \text{Ric}_{rs}(0) \int_{B^n(\rho)} T_{jkl}(\tilde{f})(u) u^{(j)} u^{(k)} u^{(l)} u^{(r)} u^{(s)} \, du \]

\[ + \int_{B^n(\rho)} T_{jkl}(\tilde{f})(u) u^{(j)} u^{(k)} u^{(l)} \mathcal{G}_3(B_\rho(\varphi_\delta(x)); u) \, du. \]

Repeating the same calculation as above, we also have

\[ m(B_\rho(\varphi_\delta(x))) = \omega_\rho \rho^n - \text{Scal}(0) \frac{\omega_\rho \rho^{n+1}}{n + 2} + \int_{B^n(\rho)} \mathcal{G}_3(B_\rho(\varphi_\delta(x)); u) \, du, \]

where Scal(0) stands for the scalar curvature at \( \varphi_\delta(x) \). Recalling (3.13), we observe that

\[ |\mathcal{R}_f(B_\rho(\varphi_\delta(x)))| + \int_{B^n(\rho)} \mathcal{G}_3(B_\rho(\varphi_\delta(x)); u) \, du \leq c_4 \omega_\rho \rho^{n+3}, \]

where the constant \( c_4 \) depends on \( \|f\|_{c^3} \) and the geometry on \( B_\rho(\varphi_\delta(x)) \). Using (3.12), we obtain

\[ \left| (1 - \delta V(x)) \left\{ \frac{1}{\delta m(N_\rho, x(\delta; X))} \right\} \int_{B_\rho(\varphi_\delta(x))} (f(\varphi_\delta(x)) - f(z)) \, dm(z) + \Delta f(\varphi_\delta(x)) \right| \]

\[ \leq \left| \frac{\omega_\rho \rho^n}{m(N_\rho, x(\delta; X))} - 1 \right| \cdot |\Delta f(\varphi_\delta(x))| + \frac{c_4 \omega_\rho \rho^{n+3}}{\delta^2 m(N_\rho, x(\delta; X))} \]

\[ \leq \frac{|\Delta f(\varphi_\delta(x))|}{m(B_\rho(\varphi_\delta(x)))} \left\{ |\omega_\rho \rho^n - m(B_\rho(\varphi_\delta(x)))| \right. \]

\[ \left. + m(B_\rho(\varphi_\delta(x)) \setminus B_{\rho - |X|}(\varphi_\delta(x))) + m(N_\rho, x(\delta; X) \setminus B_{\rho - |X|}(\varphi_\delta(x))) \right\} \]

\[ + \frac{2c_4 \omega_\rho(n + 2) \rho^{n+1}}{m(B_\rho(\varphi_\delta(x)))} \]

\[ \leq \frac{|\Delta f(\varphi_\delta(x))|}{c_3(\rho - |X|)^n} \left( |\text{Scal}(0)| \frac{\omega_\rho \rho^{n+2}}{n + 2} + c_4 \omega_\rho \rho^{n+3} + 2c_1 |X| \rho^{n-1} \right) + \frac{2c_4 \omega_\rho(n + 2) \rho^{n+1}}{c_3(\rho - |X|)^n} \]

\[ \leq \frac{2^n}{c_3} \left( |\text{Scal}(0)| \frac{\omega_\rho \rho^2}{n + 2} + c_4 \omega_\rho \rho^3 + 2c_1 \frac{|X|}{\rho} \right) \|f\|_\infty + \frac{2^{n+1} c_4 \omega_\rho(n + 2)}{c_3} \rho. \]

Putting it all together, we finally obtain our desired estimate (3.7). \( \square \)
Remark 3.3. Combining smoothness of $b$, $V$ and $f$, connectivity of each $X \in \mathcal{X}$, and compactness of $S(f)$, we can also show that the constants $K_1$ and $K_2$ in (3.7) do not depend on the choice of reference points $\mathcal{X}^r$.

Corollary 3.4. Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of partitions such that $|X_k| \searrow 0$ as $k \to \infty$. Then for any subsequence $\{k(\rho)\}_{\rho > 0}$ of $\mathbb{N}$ such that $k(\rho) \nearrow \infty$ and $|X_{k(\rho)}| = o(\rho^2)$ as $\rho \searrow 0$ and for any $f \in C_c^\infty(M)$,

$$\lim_{\rho \searrow 0} \left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) [f] X_{k(\rho)} - [Af] X_{k(\rho)} \right\|_{C_0(G_0(X_{k(\rho)}, \rho))} = 0, \quad (3.14)$$

and

$$\lim_{\rho \searrow 0} \left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \mathcal{P}_{X_{k(\rho)}} f - \mathcal{P}_{X_{k(\rho)}} (Af) \right\|_{L^p(G_0(X_{k(\rho)}, \rho))} = 0. \quad (3.15)$$

Proof. Applying Theorem 3.2, we easily have (3.14). Next, we prove (3.15). It is easy to see

$$\text{supp} \left( \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \mathcal{P}_{X_k} f - \mathcal{P}_{X_k} (Af) \right)$$

for any $0 < \rho \leq 1$ and $|X_k| \leq 1$. Then

$$m \left( \text{supp} \left( \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \mathcal{P}_{X_k} f - \mathcal{P}_{X_k} (Af) \right) \right)$$

is uniformly bounded by the finite constant $m(S(f))$ for all $0 < \rho \leq 1$ and $|X_k| \leq 1$. Hence we obtain

$$\left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \mathcal{P}_{X_{k(\rho)}} f - \mathcal{P}_{X_{k(\rho)}} (Af) \right\|_{L^p(G_0(X_{k(\rho)}, \rho))} \leq m(S(f))^{1/p} \left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \mathcal{P}_{X_{k(\rho)}} f - \mathcal{P}_{X_{k(\rho)}} (Af) \right\|_{C_0(G_0(X_{k(\rho)}, \rho))}$$

$$\leq m(S(f))^{1/p} \left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) \left( \mathcal{P}_{X_{k(\rho)}} f - [f] X_{k(\rho)} \right) \right\|_{C_0(G_0(X_{k(\rho)}, \rho))}$$

$$+ m(S(f))^{1/p} \left\| \frac{2(n+2)}{\rho^2} (I - L \frac{\rho^2}{2(n+2)}) [f] X_{k(\rho)} - [Af] X_{k(\rho)} \right\|_{C_0(G_0(X_{k(\rho)}, \rho))}$$

$$+ m(S(f))^{1/p} \left\| [Af] X_{k(\rho)} - \mathcal{P}_{X_{k(\rho)}} (Af) \right\|_{C_0(G_0(X_{k(\rho)}, \rho))}. \quad (3.16)$$

We observe that $\left\| I - L \frac{\rho^2}{2(n+2)} \right\|_{C_0(G_0(X_{k(\rho)}, \rho)) \to C_0(G_0(X_{k(\rho)}, \rho))} \leq 2$ and for any $X \in X_{k(\rho)}$

$$\mathcal{P}_{X_{k(\rho)}} (f)(X) - [f] X_{k(\rho)}(X) = \frac{1}{m(X)} \int_X (f(z) - f(x(X))) \, dm(z) \leq \sup_{z \in X} |f(z) - f(x(X))|$$
Theorem 1, we have $\rho > 0$ implies $e\gamma$ (e.g., [Tro58, Kur69]) to have (2.3) (resp. (2.4)).

In the proof of Theorem 3.5.

Proof of Theorems 2.1 and 2.4. deduced from (3.14).

Then the first and the third term in (3.16) can be estimated by

$$m(S(f))^{2(n+2)}(I - L_{\frac{\rho^2}{2(n+2)}})\left(P_{X_{\rho}} f - [f]_{X_{\rho}}\right)_{C_0(G_\rho(X_{\rho}, \rho))} \leq 2m(S(f))^{2(n+2)}\|f\|_{C^1(X_{\rho})},$$

and

$$m(S(f))^{2(n+2)}\|A f\|_{C^1(X_{\rho})} - \mathcal{P}_{X_{\rho}}(A f)_{C_0(G_\rho(X_{\rho}, \rho))} \leq m(S(f))^{1/p}\|A f\|_{C^1(X_{\rho})},$$

respectively. Since $|X_{\rho}| = o(\rho^2)$, these terms converges to 0. Consequently, (3.15) is deduced from (3.14).

3.2 Convergence of semigroups

Proof of Theorems 2.1 and 2.4. Combining Lemma 3.1, Corollary 3.4 with the condition (A) (resp. (A1)$_p$ and (A2)$_p$), we may apply Trotter’s approximation theorem (e.g., [Tro58, Kur69]) to have (2.3) (resp. (2.4)).

Now we prove Theorem 2.4. Noting that the hypoellipticity of the elliptic operator $A$ implies $e^{-tA}(C^\infty(M)) \subset C^\infty(M)$ for all $t \geq 0$, we may apply Namba [Nam23]. For given $\rho > 0$, we take $k(\rho) \in \mathbb{N}$ such that $|X_{k(\rho)}| < \rho^{2+\alpha}$. Combining Theorem 3.2 with [Nam23, Theorem 1], we have

$$\left\|L_{\frac{\rho^2}{2(n+2)}} [f]_{X_{k(\rho)}} - [e^{-tA}f]_{X_{k(\rho)}}\right\|_{C_0(G_\rho(X_{k(\rho)}))} \leq \sqrt{\frac{t}\rho^2}\left(K_1(0)\frac{|X_{k(\rho)}|}{\rho^2} + K_2(0)\rho + \|A f\|_{\infty}\right)$$

$$+ \frac{\rho^2}{2(n+2)}\left(K_1(0)\frac{|X_{k(\rho)}|}{\rho^2} + K_2(0)\rho + \|A f\|_{\infty}\right) + \int_0^t \left(K_1(s)\frac{|X_{k(\rho)}|}{\rho^2} + K_2(s)\rho\right)ds$$

$$\leq C\left\{t^{1/2} \rho^2 + t \max_{0 \leq s \leq t} (K_1(s))\rho^2 + t \max_{0 \leq s \leq t} (K_2(s))\rho\right\} = C\rho^{\alpha+1}, \quad f \in C^\infty(M), \quad t > 0,$$

where $K_i(s) = K_i(\|\cdot\|_{C^2}, \|b\|_{\infty}, \|\nabla b\|_{\infty}, \|V\|_{\infty}$, $i = 1, 2, s \geq 0$, are positive constants appearing in Theorem 3.2. This gives our desired estimate (2.7).

Remark 3.5. In the proof of Theorem 2.4, it is a key point to find a good core $\mathcal{D}$ such that $e^{-tA}(\mathcal{D}) \subset \mathcal{D}$. In the case where $M$ is compact, as mentioned above, it suffices to put $\mathcal{D} = C^\infty(M)$. On the other hand, in the case where $M$ is non-compact, this problem is not so trivial. We expect that

$$\mathcal{D} = \{f \in C_0(M) \cap C_0^3(M); Af \in C_0(M)\}$$
is a candidate of such a core. To check the stability of $\mathfrak{D}$ under the operation $e^{-tA}$, we need to show boundedness of the third order derivatives of $e^{-tA}f$ for $f \in \mathfrak{D}$. (Note that the first and the second order derivatives of $e^{-tA}f$ is obtained by $[\text{Tho}19, \text{Li}21]$ under several conditions on curvature and the derivative of the potential function $V$.) On the other hand, it is not so difficult to have (3.7) for $f \in \mathfrak{D}$ because the coefficients $K_1$ and $K_2$ depend on the derivatives up to the third order. Hence we conjecture that Theorem 2.4 still holds in the case where $M$ is non-compact by imposing additional conditions as mentioned above. We will discuss this problem in the future.

**Proof of Corollary 2.2.** First, we prove (1). For $x \in M$, let $y = x(X_{k(\rho)}(x)) \in X_{k(\rho)} \subset X_{k(\rho)}$. Then we obtain

$$
\left| e^{-tA}f(x) - L \frac{2(n+2)t}{\rho^2} \mathcal{F}_{k(\rho)}(X_{k(\rho)}(x)) \right|
$$

$$
\leq \left| e^{-tA}f(x) - e^{-tA}f(y) \right| + \left| e^{-tA}f \right| \mathcal{F}_{k(\rho)}(X_{k(\rho)}(x)) - L \frac{2(n+2)t}{\rho^2} \mathcal{F}_{k(\rho)}(X_{k(\rho)}(x))
$$

$$
\leq \left| e^{-tA}f(x) - e^{-tA}f(y) \right| + \left| e^{-tA}f \right| \mathcal{F}_{k(\rho)} - L \frac{2(n+2)t}{\rho^2} \mathcal{F}_{k(\rho)} \mathcal{G}_{0}(X_{k(\rho)}(\rho)) \tag{3.17}
$$

Because the function $e^{-tA}f$ is continuous and $d(x, y) \leq |X_{k(\rho)}| \to 0$ as $\rho \to 0$, the first term of the right-hand side in (3.17) converges to 0. By using (2.3), the second term in (3.17) converges to 0. Thus we obtain (2.5).

Next, we prove (2).

$$
\left| \int_{U} e^{-tA}f(x)dm(x) - \sum_{X \in X_{k(\rho)}, X \subset U} L \frac{2(n+2)t}{\rho^2} \mathcal{P}_{X_{k(\rho)}}f(X)m(X) \right|
$$

$$
\leq \left| \int_{U} e^{-tA}f(x)dm(x) - \sum_{X \in X_{k(\rho)}, X \subset U} \int_{X} e^{-tA}f(x)dm(x) \right|
$$

$$
+ \sum_{X \in X_{k(\rho)}, X \subset U} \int_{X} e^{-tA}f(x)dm(x) - \sum_{X \in X_{k(\rho)}, X \subset U} L \frac{2(n+2)t}{\rho^2} \mathcal{P}_{X_{k(\rho)}}f(X)m(X) \mathcal{G}_{0}(X_{k(\rho)}(\rho)) \tag{3.18}
$$

Because $U$ is a bounded open set and $|X_{k(\rho)}| \to 0$ as $\rho \to 0$,

$$
\left| \int_{U} e^{-tA}f(x)dm(x) - \sum_{X \in X_{k(\rho)}, X \subset U} \int_{X} e^{-tA}f(x)dm(x) \right| \to 0 \ (\rho \to 0).
$$

Then the first term of the right-hand side in (3.18) converges to 0.

By using Hölder’s inequality, we obtain

$$
\left| \sum_{X \in X_{k(\rho)}, X \subset U} \int_{X} e^{-tA}f(x)dm(x) - \sum_{X \in X_{k(\rho)}, X \subset U} L \frac{2(n+2)t}{\rho^2} \mathcal{P}_{X_{k(\rho)}}f(X)m(X) \right|
$$

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Hence (2.4) implies that the second term of the right-hand side in (3.1) converges to 0 as \( \rho \to 0 \). Then the proof of (2.6) is completed. \( \square \)

4 Sufficient conditions for (A), (A1)\(_p\) and (A2)\(_p\)

In this section, we discuss sufficient conditions for conditions (A), (A1)\(_p\) and (A2)\(_p\) on the drifted Schrödinger operator \( \mathcal{A}_V = -\Delta - b + V \) given in the Introduction. As mentioned in the proof of Theorem 1.3, these conditions hold in the case where \( M \) is compact. Hence, in this section, we focus on the case where \( M \) is non-compact. Although the non-negativity of the potential function \( V \) was always assumed in previous sections, we discuss our problems without assuming it in advance in this section.

First of all, we give a basic criterion for condition (A) essentially due to Seeley [See84, Theorem 2]. We give an outline of the proof for the reader’s convenience.

**Proposition 4.1.** Let \( M \) be a non-compact complete smooth Riemannian manifold, and \( V \) be a non-negative smooth function on \( M \). Suppose that there exists a sequence of smooth cut-off functions \( \{\chi_m\}_{m=1}^{\infty} \subset C_c^\infty(M) \) satisfying the following three conditions:

(a1): \( 0 \leq \chi_m(x) \leq 1 \) for all \( x \in M \) and \( m \in \mathbb{N} \).

(a2): For every compact set \( K \subset M \), there exists an integer \( m_0(K) \) such that \( \chi_m(x) = 1 \) for all \( x \in K \) and \( m \geq m_0(K) \).

(a3): There exists a constant \( C > 0 \) such that

\[
(\Delta + b)\chi_m(x) \leq C
\]

for all \( x \in M \) and \( m \in \mathbb{N} \).

Then condition (A) holds.

**Proof.** To prove that \( (1 + \mathcal{A}_V)(C_c^\infty(M)) \) is dense in \( C_0(M) \), it suffices to show that \( \nu \in C_0(M)^* \) equals to 0 provided

\[
C_0(M)^* \langle \nu, (1 + \mathcal{A}_V) \varphi \rangle_{C_0(M)} = 0 \quad \text{for all } \varphi \in C_c^\infty(M).
\]  

By the Riesz-Markov-Kakutani theorem, \( \nu \in C_0(M)^* \) is identified with a finite signed Borel measure \( d\nu \) on \( M \). Moreover combining (4.2) with the hypoellipticity of elliptic...
operator $A_V$, we have $d\nu(x) = u(x)dm(x)$, where $u \in C^\infty(M) \cap L^1(m)$ (see e.g., [IW89, Proposition 4.5]). Hence we may rewrite (4.2) as

$$
\int_M u(x) (1 + A_V) \varphi(x) \, dm(x) = 0 \quad \text{for all } \varphi \in C_c^\infty(M).
$$

(4.3)

Since $u \in L^1(m)$, for any $0 < \varepsilon < 1$, there exists a compact set $K = K_\varepsilon \subset M$ such that

$$
\int_{K^c} |u| \, dm \leq \varepsilon \int_M |u| \, dm,
$$

(4.4)

where $K^c$ is the complement of $K$. Namely,

$$
\int_K |u| \, dm \geq (1 - \varepsilon) \int_M |u| \, dm.
$$

(4.5)

By (a1), (a2) and (4.3), we obtain for all $m \geq m_0(K)$

$$
\int_K |u| \, dm \leq \int_M \chi_m(x)|u(x)| \, dm(x)
\leq -\int_M \chi_m(x)A_V^* u(x) \text{sgn}(u(x)) \, dm(x)
\leq -\int_M \chi_m(x)A_0^* u(x) \text{sgn}(u(x)) \, dm(x)
\leq -\int_M A_0 \chi_m(x)|u(x)| \, dm(x) = \int_M |u(x)|(\Delta + b) \chi_m(x) \, dm(x),
$$

(4.6)

where

$$
\text{sgn}(a) := \begin{cases} 
  a/|a| & \text{if } a \neq 0 \\
  0 & \text{if } a = 0
\end{cases}, \quad A_V^* u = -\Delta u + \text{div}(ub) + Vu
$$

and we used the non-negativity of $V$ for the third line. We also used Kato’s inequality (cf. [See84, Theorem 2])

$$
-A_0^* |u| \geq \text{sgn}(u)(-A_0^* u)
$$

for the fourth line.

We note here that (a2) implies

$$
(\Delta + b) \chi_m(x) = 0, \quad x \in K
$$

for all $m \geq m_0(K)$. Then by (a3), we have

$$
\int_M |u(x)|(\Delta + b) \chi_m(x) \, dm(x) \leq C \int_{K^c} |u| \, dm.
$$

(4.7)

Combining (4.4), (4.5), (4.6) with (4.7), we obtain

$$
(1 - \varepsilon) \int_M |u| \, dm \leq \int_K |u| \, dm
$$

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\[
\leq C \int_{K^c} |u| \, dm \leq C \varepsilon \int_M |u| \, dm,
\]
that is,
\[
0 \leq \{C \varepsilon - (1 - \varepsilon)\} \int_M |u| \, dm.
\]
We now take \(\varepsilon > 0\) small enough. Then \(u\) must satisfy
\[
\int_M |u| \, dm = 0,
\]
which implies \(u = 0\). Hence we complete the proof.

In \(L^p(m)\)-setting, the following criterion for conditions \((A1)_p\) and \((A2)_p\) is obtained by Shigekawa [Shi10, Shi12]. See Theorems 2.1, 4.1 and 4.2 in [Shi10] and Proposition 2 in [Shi12] for details of the proof.

**Proposition 4.2.** Suppose that there exists a constant \(\lambda \geq 0\) such that
\[
\frac{1}{p}(\text{div} b)(x) + V(x) \geq -\lambda, \quad x \in M, \quad (4.8)
\]
then condition \((A1)_p\) holds. Furthermore, suppose that there exist a base point \(o \in M\), a positive constant \(C\) and a positive non-increasing continuous function \(\kappa = \kappa(r) : [0, \infty) \to (0, 1]\) such that
\[
\int_0^\infty \kappa(r) \, dr = \infty \quad \text{and} \quad \kappa(r(x))br(x) \geq -C, \quad x \in M, \quad (4.9)
\]
then condition \((A2)_p\) holds, where \(r(x) := d(o,x)\) is the radial function from \(o \in M\). Hence, under (4.8) and (4.9), for any \(1 < p < \infty\), \((-A_V, C^\infty_c(M))\) is closable in \(L^p(m)\) and its closure generates a \(C^0\)-semigroup \(\{e^{-tA_V}\}_{t \geq 0}\) in \(L^p(m)\) and \(e^{-tA_V}f\) has the Feynman-Kac type functional integral representation (1.1) for all \(f \in C^\infty_c(M)\). Additionally, if we impose \(V \geq 0\), the semigroup \(\{e^{-tA_V}\}_{t \geq 0}\) is Markovian. Namely, for all \(f \in L^p(m)\), \(0 \leq f \leq 1\) implies \(0 \leq e^{-tA_V}f \leq 1\).

**Remark 4.3.** If we can take \(\lambda = 0\) in (4.8), the semigroup \(\{e^{-tA_V}\}_{t \geq 0}\) is contractive in \(L^p(m)\). (Note that it is also shown by [See84, Theorem 3].) If we assume \(V \geq 0\), the lower-boundedness of \(\text{div} b\) implies (4.8). However, needless to say, \(V \geq 0\) and (4.8) does not imply the lower-boundedness of \(\text{div} b\) in general.

**Example 4.4.** The functions
\[
\kappa(r) = 1, \quad \frac{1}{r}, \quad \frac{1}{r \log r}, \quad \frac{1}{r \log r \log r}, \ldots, \quad r \geq R
\]
are typical examples satisfying \(\int_R^\infty \kappa(r) \, dr = \infty\) with \(R = 0, 1, e, e^e, \ldots\), respectively.
For the later purpose, we fix a constant $\gamma > 1$ and take a smooth function $\phi = \phi_\gamma(t) : \mathbb{R} \to [0, 1]$ satisfying $\phi \equiv 1$ on $(-\infty, 1]$, $\phi \equiv 0$ on $[\gamma, \infty)$ and
\[
-\frac{2}{\gamma - 1} \leq \phi'(t) \leq 0, \quad |\phi''(t)| \leq \frac{8}{(\gamma - 1)^2} \quad \text{for all } 1 < t < \gamma. \tag{4.10}
\]
In the following two subsections, we consider the cases where
- $M$ has an empty cut-locus;
- $M$ has a variable lower Ricci curvature bound,
and give more concrete sufficient conditions for (A) in terms of the vector field $b$.

### 4.1 Manifolds with an empty cut-locus

**Proposition 4.5.** Let $M$ be a non-compact manifold with an empty cut-locus. Suppose that there exist some base point $o \in M$ and a positive non-increasing smooth function $\kappa = \kappa(r) : (0, \infty) \to (0, 1]$ such that
\[
\sup_{r > 0} |\kappa'(r)| < \infty, \quad \int_0^\infty \kappa(r) \, dr = \infty \tag{4.11}
\]
and the radial function $r(x) = d(o, x)$ satisfies
\[
\kappa(r(x)) \left( \Delta r(x) + br(x) \right) \geq -C \int_0^{r(x)} \kappa(r) \, dr, \quad x \in M \setminus \{o\} \tag{4.12}
\]
for some positive constant $C$. Then condition (A) holds.

**Proof.** We first put $\gamma = 4$, for instance, such that the function $\phi$ satisfies
\[
-1 \leq \phi'(u) \leq 0, \quad |\phi''(u)| \leq 1, \quad u \in \mathbb{R}. \tag{4.13}
\]
We define a sequence of smooth cut-off functions $\{\chi_m\}_{m=1}^\infty$ by
\[
\chi_m(x) := \phi\left(\frac{h(r(x))}{m}\right), \quad x \in M, \ m \in \mathbb{N},
\]
where
\[
h(r) = \int_0^r \kappa(s) \, ds, \quad r \geq 0.
\]
We should remark here that (4.11) implies $h(r) \not\to \infty$ as $r \to \infty$. Since we easily see
\[
\chi_m(x) = \begin{cases} 1 & \text{if } r(x) \leq h^{-1}(m), \\ 0 & \text{if } r(x) \geq h^{-1}(4m), \end{cases}
\]
it is sufficient to check (4.1) for all $h^{-1}(m) \leq r(x) \leq h^{-1}(4m)$. Combining a direct calculation with (4.13), $0 < \kappa \leq 1$, the basic Lipschitz estimate $|\nabla r(x)|_{T_xM} \leq 1$ and condition (4.12), we obtain
\[
\Delta \chi_m(x) + b\chi_m(x)
\]
\[
\kappa(r(x))^2 \leq \frac{1}{m^2} \phi''\left(\frac{h(r(x))}{m}\right) |\nabla r(x)|^2_{T_xM} + \frac{\kappa'(r(x))}{m} \phi'(\frac{h(r(x))}{m}) |\nabla r(x)|^2_{T_xM}
\]
\[
+ \frac{\kappa(r(x))}{m} \phi'(\frac{h(r(x))}{m}) \Delta r(x) + \frac{\kappa(r(x))}{m} \phi'(\frac{h(r(x))}{m}) b r(x)
\]
\[
\leq \frac{1}{m^2} \left( \sup_{r \geq R} |\kappa'(r)| \right) - \frac{1}{m} \left\{ \kappa(r(x)) \cdot (\Delta r(x) + b r(x)) \right\}
\]
\[
\leq C \left( 1 + \frac{h(r(x))}{m} \right) \leq 5C,
\]
where we used \( h(r(x)) \leq 4m \) for the final inequality. Thus we have shown (a3), and Proposition 4.1 leads us to the desired condition (A). This completes the proof. \( \square \)

As an example of manifolds without a cut-locus, we consider the case where \( M \) has a pole \( o \in M \), that is, the exponential map \( \exp : T_o M \to M \) is a diffeomorphism. In this case, we have the following criterion for condition (A).

**Corollary 4.6.** Assume that \( M \) has a pole \( o \in M \) and there exists a positive constant \( C \) such that
\[
\text{Ric}(x) \geq -C(1 + r(x)^2), \quad x \in M,
\]
where \( r(x) := d(o, x) \) is the radial function from the pole \( o \in M \). Furthermore assume that there exists a positive constant \( C \) such that
\[
\kappa(r(x)) b r(x) \geq -C \int_0^{r(x)} \kappa(r) dr, \quad x \in M \setminus \{o\},
\]
where the function \( \kappa \) is introduced in Proposition 4.5. Then condition (A) holds.

**Proof.** By [See84, Appendix], the reverse Laplacian comparison theorem
\[
\Delta r(x) \geq -C r(x), \quad x \in M \setminus \{o\}
\]
holds for some positive constant \( C \). Since the function \( \kappa : (0, \infty) \to (0, 1] \) is non-decreasing and smooth, it follows from (4.16) that
\[
\kappa(r(x)) \Delta r(x) \geq -C \kappa(r(x)) r(x) \geq -C \int_0^{r(x)} \kappa(r) dr.
\]
Combining this estimate with (4.15), we have (4.12). Thus we may apply Proposition 4.5 to complete the proof. \( \square \)

**Example 4.7.** If \( \kappa \equiv 1 \), (4.15) is equivalent to
\[
br(x) \geq -C r(x), \quad x \in M \setminus \{o\}.
\]
If there exists a sufficiently large \( R > 0 \) such that \( \kappa(r) = \frac{1}{r} \) and \( \kappa(r) = \frac{1}{r \log r} \) for all \( r > R \), we may read (4.15) as
\[
br(x) \geq -C r(x) \log r(x) \quad \text{and} \quad br(x) \geq -C r(x) \log r(x) \left( 1 + \log \log r(x) \right),
\]
respectively.
4.2 Manifolds with variable lower Ricci curvature bounds

We consider the case where the Ricci curvature is bounded from below by a (possibly unbounded) nonpositive function of the radial function \( r(x) := d(o, x) \) from some base point \( o \in M \). Throughout this subsection, we assume (4.14). Thanks to [BS18, Theorem 2.1], there exist a smooth exhaustion function \( r : M \to [0, \infty) \) and constants \( 0 < D_1 < D_2 \) and \( D_3 > 0 \) such that

\[
D_1 r(x)^2 \leq r(x) \leq D_2 \max\{1, r(x)^2\}, \quad x \in M; \tag{4.17}
\]

\[
|\nabla r(x)| \leq D_3 r(x), \quad x \in M \setminus B_1(o), \tag{4.18}
\]

\[
|\Delta r(x)| \leq D_3 r(x)^2, \quad x \in M \setminus B_1(o). \tag{4.19}
\]

**Corollary 4.8.** Assume (4.14) for some base point \( o \in M \). Furthermore assume that there exist two constants \( C > 0 \) and \( R \geq 1 \) such that

\[
b r(x) \geq -C r(x)^2, \quad x \in M \setminus B_R(o). \tag{4.20}
\]

Then condition (A) holds. In particular, if the vector field \( b \) satisfies

\[
|b(x)|_{T_x M} \leq C (1 + r(x)), \quad x \in M \tag{4.21}
\]

for some \( C > 0 \), (4.20) and hence condition (A) hold.

**Proof.** We put \( \gamma = D_2/D_1 \) and define a sequence of smooth cut-off functions \( \{\chi_m\}_{m=1}^\infty \) by

\[
\chi_m(x) := \phi\left(\frac{r(x)}{D_1 m^2}\right), \quad x \in M, \; m \in \mathbb{N}.
\]

Recalling (4.17) and noting \( \gamma > 1 \), we easily see

\[
\chi_m(x) = \begin{cases} 1 & \text{if } r(x) \leq \gamma^{-1/2} m, \\ 0 & \text{if } r(x) \geq \gamma^{1/2} m, \end{cases}
\]

and thus \( \Delta \chi_m(x) + b \chi_m(x) = 0 \) holds for \( r(x) < \gamma^{-1/2} m \) and \( r(x) > \gamma^{1/2} m \).

Hence it is sufficient to check condition (a3) for \( \gamma^{-1/2} m \leq r(x) \leq \gamma^{1/2} m \). Combining a direct calculation with (4.10), (4.18), (4.19) and (4.20), we have

\[
\Delta \chi_m(x) + b \chi_m(x) = \frac{1}{D_1^2 m^4} \phi''\left(\frac{r(x)}{D_1 m^2}\right) |\nabla r(x)|^2_{T_x M} + \frac{1}{D_1 m^2} \phi'\left(\frac{r(x)}{D_1 m^2}\right) \Delta r(x) + \frac{1}{D_1 m^2} \phi'\left(\frac{r(x)}{D_1 m^2}\right) b r(x)
\]

\[
\leq \frac{8D_3^2}{(D_2 - D_1)^2 m^2} r(x)^2 + \frac{2D_3}{(D_2 - D_1) m^2} r(x)^2 + \frac{2C}{(D_2 - D_1) m^2} r(x)^2. \tag{4.22}
\]
Since \( r(x)^2 \leq \gamma m^2 \), the right-hand side (4.22) is bounded from above by a positive constant \( C \) independent of \( m \). It means that we have shown (a3), and thus Proposition 4.1 leads us to the desired condition (A).

Besides, it follows from (4.18) and (4.21) that

\[
bn(x) \geq -b(x)|\nabla r(x)|_{T_xM}
\geq -C(1 + r(x))D_3r(x)
\geq -CD_3(r(x) + r(x))r(x) = -Cr(x)^2, \quad x \in M \setminus B_1(o).
\]

Thus we have shown (4.20) with \( R = 1 \). This completes the proof.

\[\square\]

5 Examples

In this section we study examples of a sequence of partitions and drift vector fields to satisfy conditions (A), (A1), (A2), (B) and (C) on two typical manifolds: Euclidean spaces and model manifolds.

5.1 Euclidean spaces

We consider the case \( M = \mathbb{R}^n \). Let \( x = (x^{(1)}, \ldots, x^{(n)}) \) be the standard Euclidean coordinates and we write

\[
r(x) = \left\{ \sum_{i=1}^{n} (x^{(i)})^2 \right\}^{1/2}, \quad b(x) = \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i}.
\]

We take a sequence of partitions \( X_k \) by

\[
X^k_{(i_1, \ldots, i_n)} = \left[ \frac{i_1}{k}, \frac{i_1 + 1}{k} \right) \times \cdots \times \left[ \frac{i_n}{k}, \frac{i_n + 1}{k} \right), \quad i = (i_1, \ldots, i_n).
\]

Since \( |X_k| = \frac{\sqrt{n}}{k} \), it is easy to check that each \( X_k \) satisfies conditions (B) and (C).

We first find a sufficient condition on the vector filed \( b \) for conditions (A1) and (A2) by applying Proposition 4.2 with \( \kappa \equiv 1 \). In this case, conditions (4.8) and (4.9) are rewritten by

\[
\sum_{i=1}^{n} \frac{\partial b^i}{\partial x^{(i)}}(x) \geq -\gamma, \quad x \in \mathbb{R}^n
\]

and

\[
\sum_{i=1}^{n} b^i(x)x^{(i)} \geq -Cr(x), \quad x \in \mathbb{R}^n \setminus \{0\},
\]

respectively. For example, we consider a vector field \( b \) given by

\[
b^i(x) = c_0^i(x) + c_1^i(x)x^{(i)} + c_2^i(x)(x^{(i)})^2 + \cdots + c_k^i(x)(x^{(i)})^{2k_i - 1}, \quad k_i \in \mathbb{N}, \ i = 1, \ldots, n, \quad (5.3)
\]
where $c^i_j = c^i_j(x)$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$, are smooth functions on $\mathbb{R}^n$. Note that we did not use the Einstein summation convention in (5.3). Here we further assume that for each $i = 1, \ldots, n$, the functions $c^i_j(x)$, $j = 1, \ldots, k_i$, are independent of $x^{(i)}$. Then (5.1) and (5.2) hold provided that

$$c^i_1(x), \ldots, c^i_{k_i}(x) \geq 0, \quad i = 1, \ldots, n, \quad x \in \mathbb{R}^n$$

and

$$ |(c^1_0(x), \ldots, c^n_0(x))|_{\mathbb{R}^n} \leq C', \quad x \in \mathbb{R}^n$$

with some positive constant $C'$, respectively.

Next, we find condition (A) by applying Proposition 4.5. In this case, the condition (4.12) with $\kappa \equiv 1$ is rewritten by

$$\sum_{i=1}^{n} b^i(x) x^{(i)} \geq -Cr(x)^2 - (n - 1), \quad x \in \mathbb{R}^n. \quad (5.4)$$

For example, we also consider the vector field $b$ given by (5.3). By a direct calculation, we easily see that (5.3) satisfies (5.4) provided that

$$ |(c^1_0(x), \ldots, c^n_0(x))|_{\mathbb{R}^n} \leq C' r(x), \quad x \in \mathbb{R}^n$$

and

$$c^i_1(x) \geq -C', \quad c^i_2(x), \ldots c^i_{k_i}(x) \geq 0, \quad i = 1, \ldots, n, \quad x \in \mathbb{R}^n$$

with some positive constant $C'$.

Since $\mathbb{R}^n$ is regarded as a model manifold, other examples of the vector field $b$ satisfying (A), (A1)$_p$ and (A2)$_p$ can be found in Section 5.2.4 below.

### 5.2 Model manifolds

#### 5.2.1 Basic facts for model manifolds

For $r_0 \in (0, +\infty]$ and a smooth positive function $\psi = \psi(r)$ on $(0, r_0)$, let $(M, g_\psi)$ be an $n$-dimensional model manifold with weight $\psi$, that is, there exists one chart on $M$ that covers all of $M$ and the image of this chart in $\mathbb{R}^n$ is $B_{\mathbb{R}^n}(r_0) = \{ x \in \mathbb{R}^n ; |x|_{\mathbb{R}^n} < r_0 \}$. The metric $g_\psi$ in the polar coordinates $(r, \theta) = (r, \theta_1, \ldots, \theta^{n-1})$ in the above chart has the form

$$g_\psi(r, \theta) = dr^2 + \psi(r)^2 g_{S^{n-1}}(\theta),$$

where $g_{S^{n-1}}$ is the standard Riemannian metric on the $(n - 1)$-dimensional unit sphere $S^{n-1}$ which has the form

$$g_{S^{n-1}}(\theta) = \gamma_{ij}(\theta) d\theta^i d\theta^j.$$

To avoid the singularity at the origin $o$, we always assume

$$\lim_{r \searrow 0} \psi(r) = 0, \quad \lim_{r \searrow 0} \psi'(r) = 1, \quad \lim_{r \searrow 0} \psi''(r) = 0.$$
Moreover, when \( r_0 < \infty \), we assume

\[
\lim_{r \to r_0} \psi(r) = 0, \quad \lim_{r \to r_0} \psi'(r) = -1, \quad \lim_{r \to r_0} \psi''(r) = 0
\]

to ensure the completeness and non-singularity of the manifold. Some of typical manifolds can be regarded as a model manifold. Indeed,

- \( \mathbb{R}^n \) with \( r_0 = \infty \) and \( \psi(r) = r \);
- \( S^n \) (without a pole) with \( r_0 = \pi \) and \( \psi(r) = \sin r, 0 < r < \pi \);
- \( H^n \) with \( r_0 = \infty \) and \( \psi(r) = \sinh r \)

are model manifolds. We refer to [Gri09, Section 3.10] for the precise definition and basic results of model manifolds.

On a model manifold \( M \), the Riemannian volume measure \( m \) is given in the polar coordinates by

\[
dm = \psi(r)^{n-1} dr \, d\theta,
\]

where \( d\theta \) stands for the Riemannian volume measure on \( S^{n-1} \). The Laplacian has the form

\[
\Delta = \frac{\partial^2}{\partial r^2} + (n-1) \frac{\psi'(r)}{\psi(r)} \frac{\partial}{\partial r} + \frac{1}{\psi(r)^2} \Delta_{S^{n-1}}.
\]

In particular,

\[
\Delta r = (n-1) \frac{\psi'(r)}{\psi(r)}.
\]

### 5.2.2 Ricci curvature on model manifolds

When \( n=2 \), we can compute the Ricci curvature on a model manifold by using the weight function \( \psi \) explicitly. Indeed, the Christoffel symbol \( \Gamma^k_{ij} \) can be given by

\[
\Gamma^r_{ij} = \begin{cases} 
-\psi(r) \psi'(r) & \text{if } (i,j) = (\theta, \theta) \\
0 & \text{otherwise}
\end{cases}, \quad \Gamma^\theta_{ij} = \begin{cases} 
\frac{\psi'(r)}{\psi(r)} & \text{if } (i,j) = (r, \theta) \text{ or } (\theta, r) \\
0 & \text{otherwise}
\end{cases}.
\]

Since the component of Ricci curvature tensor \( R_{ij} \) is given by

\[
R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij} \Gamma^m_{km} - \Gamma^k_{im} \Gamma^m_{jk},
\]

we obtain

\[
(R_{ij}) = \begin{pmatrix}
-\psi'(r) & 0 \\
0 & -\psi(r) \psi'(r)
\end{pmatrix}.
\]

Hence, we obtain

\[
\text{Ric} \geq -\frac{\psi''(r)}{\psi(r)}.
\]

When \( \psi \) is polynomial, that is, for some positive constant \( C_1 \) and for some \( \alpha \in \mathbb{R} \)

\[
\psi(r) = C_1 r^\alpha
\]
for all large $r > 0$, (5.5) implies that
\[
\text{Ric} \geq -\alpha(\alpha - 1)r^{-2}
\]
for all large $r > 0$.

When $\psi$ is exponential, that is, for some positive constant $C_1$ and some $\alpha, \beta \in \mathbb{R}$,
\[
\psi(r) = C_1 \exp(\alpha r^\beta)
\]
for all large $r > 0$, (5.5) implies that
\[
\text{Ric} \geq -\alpha \beta r^{\beta-2}((\beta - 1) + \alpha \beta r^\beta).
\]
In particular, if $\alpha > 0$ and $\beta > 2$, then
\[
\text{Ric} \geq -C'r^{2\beta-2}
\]
for some $C' > 0$ and all large $r > 0$, which fails the variable lower Ricci bound (4.14).

### 5.2.3 Partition of model manifolds

In this section, we give an example of the partition of the model manifold $(M, g_\psi)$ satisfying conditions (B) and (C). First of all, let us construct a partition of $\mathbb{S}^{n-1}$ by induction in dimension. We first consider the case $n = 2$, that is, $\mathbb{S}^1 \simeq [0, 2\pi)$. For any $K \in \mathbb{N}$, we take a partition consisting of $K$-pieces given by
\[
\left[0, \frac{2\pi}{K}\right), \left[\frac{2\pi}{K}, \frac{2\pi}{K} + \frac{2\pi}{K}\right), \cdots, \left[(K - 1)\frac{2\pi}{K}, \frac{2\pi}{K}\right) \right).
\]  

(5.6)

Next, as a partition of $\mathbb{S}^2$, take a spherical suspension (cf. [BBI01, Section 3.6.3]) of each piece in (5.6) as a subset of $\mathbb{S}^2$ and decompose it with same interval $\frac{\pi}{K}$ in the extended angle. This partition is nothing but a partition of $\mathbb{S}^2$ by $K$ longitude lines and $K$ latitude lines. Noting that for any piece of this partition $X \subset \mathbb{S}^2$,
\[
\text{diam}(X) \leq \frac{3\pi}{K}, \quad \text{vol}_{\mathbb{S}^2}(X) \leq \frac{2\pi^2}{K^2}.
\]

we repeat this procedure to construct a partition of $\mathbb{S}^{n-1}$. Then the given partition $\mathcal{Y}_K$ consisting of $K^{n-1}$ elements satisfies for any $Y \in \mathcal{Y}_K$
\[
\text{diam}(Y) \leq \frac{n\pi}{K}, \quad \text{vol}_{\mathbb{S}^{n-1}}(Y) \leq 2 \left(\frac{n\pi}{K}\right)^{n-1}.
\]

Now we construct a partition of a model manifold $M = (M, g_\psi)$ as follows: Let $\Pi : M \setminus \{o\} \to \mathbb{S}^{n-1}$ be the canonical projection, that is,
\[
\Pi(r, \theta) = \theta, \quad (r, \theta) \in M \setminus \{o\}.
\]
For \( l, m \in \mathbb{N} \), we set
\[
A_l(m) = B_{\frac{m}{l}}(o) \setminus B_{\frac{m-1}{l}}(o), \quad \psi_l(m) = \max_{\frac{m-1}{l} \leq r \leq \frac{m}{l}} \psi(r).
\]
(Note that \( A_l(m) = \emptyset \) if \( r_0 < \infty \) and \( \frac{m-1}{l} \geq r_0 \).)

Next we define a partition \( \{ X_{l,m}^k \}_{k=1,...,K^{n-1}} \) of \( A_l(m) \) by
\[
X_{l,m}^k = \Pi^{-1}(Y_k) \cap A_l(m), \quad k = 1, \ldots, K^{n-1}.
\]
We note that
\[
diam(X_{l,m}^k) \leq \frac{1}{l} + \frac{n\pi}{K} \psi_l(m)
\]
and
\[
m(X_{l,m}^k) \leq \frac{1}{l} \text{vol}_{S^{n-1}}(Y_k) \psi_l(m)^{n-1} \leq \frac{2}{l} \left( \frac{\pi}{K} \right)^{n-1} \psi_l(m)^{n-1}.
\]
Now choose \( K = K(l, m) := \lceil l \psi_l(m) n\pi \rceil \). Then for all \( k = 1, \ldots, K(l, m)^{n-1} \) and \( l, m \in \mathbb{N} \), we obtain
\[
diam(X_{l,m}^k) \leq \frac{2}{l}, \quad m(X_{l,m}^k) \leq \frac{2}{n^{n-1}l^n}.
\]
Consequently, for each \( l \in \mathbb{N} \), \( X_l := \{ X_{l,m}^k \}_{m \in \mathbb{N}, k=1,...,K(l,m)} \) is the partition of \( M \). It is easy to check conditions (B) and (C).

### 5.2.4 A sufficient condition for (A), (A1)\( p \) and (A2)\( p \)

In this section, we study a sufficient condition on a vector field \( b \) on the non-compact model manifold \((M, g_\psi)\) for conditions (4.8), (4.9) and (4.12) with \( \kappa \equiv 1 \). For a smooth vector field \( b = b(r, \theta) \) on \( M \), let \( b^r, b^\theta_1, \ldots, b^\theta_{n-1} \) be the coefficients of \( b \) in the polar coordinates \((r, \theta^{(1)}, \ldots, \theta^{(n-1)})\). Namely, they satisfy
\[
b(r, \theta) = b^r(r, \theta) \frac{\partial}{\partial r} + b^\theta_1(r, \theta) \frac{\partial}{\partial \theta^{(1)}} + \cdots + b^\theta_{n-1}(r, \theta) \frac{\partial}{\partial \theta^{(n-1)}}.
\]
Then conditions (4.8) and (4.9) are rewritten by
\[
(n-1) \frac{\psi'(r)}{\psi(r)} b^r(r, \theta) + \frac{\partial b^r}{\partial r}(r, \theta) + \text{div}_{S^{n-1}}(b^\theta(r, \theta)) \geq -\gamma
\]
and \( b^r(r, \theta) \geq -C \), respectively, where
\[
\text{div}_{S^{n-1}}(b^\theta(r, \theta)) = \frac{1}{\sqrt{\det(\gamma)}} \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta^{(i)}} \left( \sqrt{\det(\gamma)} b^\theta_i(r, \theta) \right).
\]
Also, condition (4.12) with \( \kappa \equiv 1 \) is rewritten by
\[
b^r(r, \theta) + (n-1) \frac{\psi'(r)}{\psi(r)} \geq -Cr.
\]
We now study a sufficient condition of $b$ in the cases where $\psi$ are polynomial and exponential separately.

(I) **Polynomial case:** For a positive constant $C_1$ and some $\alpha \in \mathbb{R}$, suppose

$$\psi(r) = C_1 r^\alpha$$

for all large $r$. In this case, for the vector field $b$ satisfying (4.8), assume that

$$\text{div}_{S^{n-1}}(b^\theta(r, \theta)) \geq -C'$$

(5.7)

for some constant $C'$. For example, this condition is true if $b_1^\theta, \ldots, b_{n-1}^\theta$ are constants for all large $r > 0$. Then the condition (4.8) is rewritten by

$$(n-1)\frac{\alpha}{r} b^r + \frac{\partial b^r}{\partial r} \geq C' - \gamma =: -\gamma'.$$

(5.8)

For example, we consider the case

$$b^r(r, \theta) = c_0(\theta) + c_1(\theta)r + \cdots + c_k(\theta)r^k,$$

(5.9)

where $c_0, \ldots, c_k$ are smooth functions on $S^{n-1}$. Then conditions (5.7) and

$$(n-1)\alpha + l) c_l(\theta) \geq 0, \quad \theta \in S^{n-1}, \quad l = 2, \ldots, k$$

imply (5.8) with some $\gamma'$. Hence condition (A1)$_p$ holds. Moreover, if $c_k$ in (5.9) is positive, then $b$ satisfies also (4.9) with some constant $C$, which implies condition (A2)$_p$.

On the other hand, condition (4.12) is rewritten by

$$b^r(r, \theta) \geq -Cr - (n-1)\frac{\alpha}{r}$$

For example, if

$$b^r(r, \theta) \geq -C' \frac{r}{r}$$

(5.10)

for some $C' > 0$, then $b$ satisfies condition (4.12), whence the condition (A) holds.

(II) **Exponential case:** For a positive constant $C_1$ and some $\alpha, \beta \in \mathbb{R}$, suppose

$$\psi(r) = C_1 \exp(\alpha r^\beta)$$

for all large $r$. To find a vector field $b$ satisfying (A1)$_p$ and (A2)$_p$, we assume (5.7). Then condition (4.8) is rewritten by

$$(n-1)\alpha \beta r^{\beta-1} b^r + \frac{\partial b^r}{\partial r} \geq -\gamma'.$$

(5.11)

For example, a vector filed $b$ with (5.7) and $b^r(r, \theta) = c(\theta)r$ with a smooth function $c(\theta)$ on $S^{n-1}$ satisfying

$$c(\theta)\alpha \beta \geq 0, \quad \theta \in S^{n-1}$$
satisfies (5.11), and thus condition (A1)$_p$ holds. Moreover, if $c(\theta)$ is positive, then $b$ satisfies also (4.9), which implies condition (A2)$_p$.

The condition (4.12) can be rewritten by

$$b'(r, \theta) \geq -Cr - (n - 1)\alpha\beta r^{\beta - 1}. \tag{5.12}$$

If $\beta \leq 2$, the leading term in the right-hand side of (5.12) is $-Cr$. Hence, for example, if

$$b'(r, \theta) \geq -C'r, \tag{5.13}$$

then $b$ satisfies the condition (4.12) for some positive constant $C'$ and then the condition (A) holds.

If $\beta > 2$, then the leading term in the right-hand side of (5.12) is $-(n - 1)\alpha\beta r^{\beta - 1}$. Hence, for example, $b$ satisfies the condition (4.12) if

$$b'(r, \theta) \geq -(n - 1)\alpha\beta r^{\beta - 1}. \tag{5.14}$$

In particular, the case where $n = 2$, $\alpha > 0$ and $\beta > 2$, the vector field $b$ with (5.7) and (5.14) gives a new example of $\mathcal{A}$ with (A) without the variable lower Ricci bound (4.14).

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