Invariant-mass distribution of $c\bar{c}$ in $\Upsilon(1S) \rightarrow c\bar{c} + X$

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Abstract

We calculate the invariant-mass distribution for the $c\bar{c}$ pair produced in the inclusive $\Upsilon(1S)$ decay based on the color-singlet mechanism of the nonrelativistic quantum chromodynamics factorization approach at leading order in the bottom-quark velocity $v_b$ in the meson rest frame. As the short-distance processes, we consider $b\bar{b} \rightarrow g^* g g$ followed by $g^* \rightarrow c\bar{c}$ and $b\bar{b} \rightarrow \gamma^* \rightarrow c\bar{c}$ at leading order in the strong coupling. The invariant-mass distribution of the $b\bar{b} \rightarrow c\bar{c} g g$ contribution has a sharp peak just above the threshold and that of the $b\bar{b} \rightarrow \gamma^* \rightarrow c\bar{c}$ channel is concentrated at the maximally allowed kinematic end point. We predict that $\Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X]/\Gamma[\Upsilon(1S) \rightarrow \text{light hadrons}] = 0.065 \alpha_s$, which is smaller than a previous result by about 20%.

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Recent experimental analyses being carried out by the CLEO III experiment on the inclusive charm production in bottomonia decays [1] have activated a series of theoretical studies [2–4] based on the nonrelativistic quantum chromodynamics (NRQCD) factorization approach [5, 6]. Predictions for the branching fractions and the charmed-hadron momentum distributions in inclusive $\chi_{bJ}$ ($J = 0, 1, \text{and} 2$) decays were presented in Ref. [2]. These processes are nice probes to investigate the color-octet mechanism of the NRQCD factorization approach which is distinguished from the color-singlet-model calculation in Ref. [7]. In addition to the spin-triplet $P$-wave ($^3P_J$) bottomonium decay, the CLEO Collaboration also analyzes the charm production in the spin-triplet $S$-wave ($^3S_1$) bottomonium decay. Early theoretical studies on the inclusive charm production in the inclusive $\Upsilon(1S)$ decay began in the late 1970s. In 1978, Fritzsch and Streng calculated the invariant-mass distribution of the $c\bar{c}$ pair produced in the inclusive $\Upsilon(1S)$ decay [8] by considering the QCD process $b\bar{b} \rightarrow g^*gg \rightarrow c\bar{c}gg$ within the color-singlet model, where they predicted the branching fraction $\text{Br}[\Upsilon(1S) \rightarrow c\bar{c} + X]$ to be a few percents in the limit that the charm-quark momentum $p_c^*$ in the $c\bar{c}$ rest frame can be neglected. In 1996, Cheung, Keung, and Yuan calculated the $J/\psi$ production rate with the same short-distance process, where they considered the decay of the color-singlet $b\bar{b}$ pair producing a $c\bar{c}$ pair in the color-octet spin-triplet state that evolves into the $J/\psi$ [9]. One can find more studies on the bottomonium decay in Refs. [10–12].

In this paper, based on the color-singlet mechanism of the NRQCD factorization approach, we compute the invariant-mass distribution of the $c\bar{c}$ pair in $\Upsilon(1S) \rightarrow c\bar{c} + X$. This is an extension of a recent work on the total charm production rate and the momentum distribution of the charm hadrons produced in the inclusive $\Upsilon(nS)$ decay [3]. As is studied in Ref. [3], we consider the decay of the color-singlet spin-triplet $b\bar{b}$ pair [$\bar{b}b_1(^3S_1)$] into $g^*gg$ followed by $g^* \rightarrow c\bar{c}$, which we call the QCD contribution. We also consider the short-distance process $\bar{b}b_1(^3S_1) \rightarrow \gamma^* \rightarrow c\bar{c}$, which we call the QED contribution. The perturbative calculations of the short-distance processes are carried out at leading order in the strong coupling $\alpha_s$ and the QED coupling $\alpha$. For the long-distance part of the NRQCD factorization formula, we consider the leading contribution with respect to the bottom-quark velocity $v_b$ in the $\Upsilon(1S)$ rest frame. The relevant NRQCD matrix element for the channel is $\langle \Upsilon(1S)|O_1(^3S_1)|\Upsilon(1S)\rangle$.
defined in Ref. [6], where \( \mathcal{O}_1(3S_1) \) is the color-singlet spin-triplet four-quark operator for the annihilation decay of the \( \Upsilon(1S) \). At higher orders in \( v_b \), the color-octet short-distance processes \( b\bar{b}_8(3S_1) \rightarrow g^* \rightarrow c\bar{c} \), \( b\bar{b}_8(1S_0) \rightarrow g^*g \rightarrow c\bar{c} g \), and \( b\bar{b}_8(3P_J) \rightarrow g^*g \rightarrow c\bar{c} g \) can also contribute to the charm production in the inclusive \( \Upsilon(1S) \) decay. The color-octet contributions are estimated to be about 10% of the color-singlet contributions. Therefore, we neglect the color-octet processes in this work.

This paper is organized as follows: In Sec. II, we present the NRQCD factorization formula for the inclusive charm production in \( \Upsilon(1S) \) decay and compute the short-distance coefficients for the \( c\bar{c} \) invariant-mass distribution of the process. Numerical analysis for the invariant-mass distribution is given in Sec. III, which is followed by a summary in Sec. IV.

II. CHARM-QUARK PRODUCTION IN \( \Upsilon(1S) \) DECAY

In this section, we present the NRQCD factorization formula for the \( c\bar{c} \) invariant-mass distribution in \( \Upsilon(1S) \rightarrow c\bar{c} + X \) within the color-singlet mechanism at leading order in \( v_b \). As the short-distance contributions, we consider the QCD process \( b\bar{b}_1(3S_1) \rightarrow g^*gg \) followed by \( g^* \rightarrow c\bar{c} \) and the QED process \( b\bar{b}_1(3S_1) \rightarrow \gamma^* \rightarrow c\bar{c} \). We follow the formalism to calculate the inclusive charm production rate in the \( \Upsilon(nS) \) decay in Ref. [3].

A. NRQCD factorization formula

Within the color-singlet mechanism of the NRQCD factorization formalism the differential rate for the inclusive decay \( \Upsilon(1S) \rightarrow c\bar{c} + X \) at leading order in \( v_b \) is given by [3]

\[
d\Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X] = dC^{(c)}_1 \left( \frac{\langle \mathcal{O}_1(3S_1) \rangle_{\Upsilon(1S)}}{m_b^2} \right),
\]

where \( \langle \mathcal{O}_1(3S_1) \rangle_{\Upsilon(1S)} = \langle \Upsilon(1S) | \mathcal{O}_1(3S_1) | \Upsilon(1S) \rangle \) is the leading-order color-singlet NRQCD matrix element for the \( \Upsilon(1S) \) and \( m_b \) is the mass of the bottom quark. The short-distance coefficient \( dC^{(c)}_1 \) is insensitive to the long-distance nature of the \( \Upsilon(1S) \) and calculable as a perturbative series of the strong coupling \( \alpha_s \). At leading order in \( \alpha_s \) and \( \alpha \), the dimensionless short-distance coefficient \( dC^{(c)}_1 \) is the sum of the QCD and QED contributions:

\[
dC^{(c)}_1 = dC^{(QCD)}_1 + dC^{(QED)}_1.
\]
As is mentioned earlier, the QCD process denotes $b\bar{b}_1(^3S_1) \rightarrow c\bar{c}gg$ followed by $g^* \rightarrow c\bar{c}$ and the QED process stands for $b\bar{b}_1(^3S_1) \rightarrow \gamma^* \rightarrow c\bar{c}$. We use the identifiers (QCD) and (QED) to denote those contributions, respectively, in the remainder of this paper.

The short-distance coefficients can be determined by perturbative matching. If we replace the heavy-quarkonium state $|\Upsilon(1S)\rangle$ in Eq. (1) with the perturbative $b\bar{b}_1(^3S_1)$ state, $|b\bar{b}_1(^3S_1)\rangle$, then the long-distance matrix element $\langle O_1(^3S_1)\rangle_{\Upsilon(1S)}$ is replaced by the perturbative NRQCD matrix element $\langle O_1(^3S_1)\rangle_{b\bar{b}_1(^3S_1)}$, while the short-distance coefficient $dC_1^{(c)}$ remains the same:

$$d\Gamma[b\bar{b}_1(^3S_1) \rightarrow c\bar{c} + X] = dC_1^{(c)} \frac{\langle O_1(^3S_1)\rangle_{b\bar{b}_1(^3S_1)}}{m_b^2}. \quad (3)$$

Both the left side of Eq. (3) and the matrix element $\langle O_1(^3S_1)\rangle_{b\bar{b}_1(^3S_1)}$ are calculable perturbatively. By taking the ratio of the two quantities, we can determine the short-distance coefficient $dC_1^{(c)}$.

**B. Short-distance coefficients**

In this section, we describe the procedure to compute the short-distance coefficient $dC_1^{(QCD)}$ for the QCD process. We also provide the expression for $dC_1^{(QED)}$ by quoting a previous result in Ref. [3]. In both short-distance processes, the momenta of the $b$ and the $\bar{b}$ can be expressed in terms of the total momentum $P$ and the relative momentum $q$ of the $b\bar{b}$ pair:

$$p = \frac{1}{2}P + q, \quad (4a)$$
$$\bar{p} = \frac{1}{2}P - q, \quad (4b)$$

where $p$ and $\bar{p}$ satisfy the on-shell conditions $p^2 = \bar{p}^2 = m_b^2$ and $P \cdot q = 0$. In the rest frame of the $b\bar{b}$ pair, $P = (2E_b, 0)$ and $q = (0, \mathbf{q})$, where $E_b = \sqrt{m_b^2 + \mathbf{q}^2}$.

At leading order in $\alpha_s$, the full QCD amplitude for the short-distance process $b(p)\bar{b}(\bar{p}) \rightarrow c(p')\bar{c}(\bar{p}')g(p_2)g(p_3)$ is given by

$$\mathcal{M}_{bb}^{(QCD)} = \frac{(4\pi\alpha_s)^2}{(p' + \bar{p})^2} \bar{u}(p')T^a\gamma_\lambda v(\bar{p}')\epsilon^b_\sigma(p_2)\epsilon^c_\tau(p_3)$$
$$\times \sum_{\text{perm}} \bar{\psi}(\bar{p}) \left[ \gamma^\lambda \frac{1}{\bar{p}' - \bar{p}_2 - \bar{p}_3 - m_b} \gamma^\sigma \frac{1}{\bar{p}' - \bar{p}_3 - m_b} \gamma^\tau \otimes T^a T^b T^c \right] u(p), \quad (5)$$
where $\sum_{\text{perm}}$ denotes the summation over the permutations of the three gluons attached to the bottom-quark line, $T^a$ is a generator of the SU(3)-color in the fundamental representation, $a$, $b$, and $c$ are color indices for the gluons, and $\epsilon_2$ and $\epsilon_3$ are polarization four-vectors of the external gluons with momenta $p_2$ and $p_3$, respectively.

In order to extract the $b\bar{b}_1(3S_1)$ contribution from the $b\bar{b}$ amplitude (5), we employ the covariant-projection method [13–15], which replaces the outer product $u(p)\bar{v}(\bar{p})$ of the $b\bar{b}$ spinors with the direct product of the color-singlet projector $\pi_1$ and the spin-triplet projector $\epsilon \cdot \Pi_3$, which are defined by

$$\pi_1 = \frac{1}{\sqrt{N_c}} 1,$$

(6a)

$$\epsilon \cdot \Pi_3 = -\frac{1}{4\sqrt{2}E_b(E_b + m_b)}(\not{p} + m_b)(\not{P} + 2E_b) \not{\epsilon} (\not{p} - m_b),$$

(6b)

where $N_c = 3$ is the number of colors, $1$ is the unit matrix of the SU(3)-color, and $\epsilon$ is the polarization four-vector of the $b\bar{b}_1(3S_1)$ state so that $P \cdot \epsilon = 0$. The projectors (6) are normalized as $\text{Tr}[\pi_1 \pi_1^\dagger] = 1$ and $\text{Tr}[(\epsilon \cdot \Pi_3)(\epsilon \cdot \Pi_3)^\dagger] = 4 p_0 \bar{p}_0$.

Because we are to compute the $c\bar{c}$ invariant-mass distribution, it is convenient to write the amplitude (5) as the product of the vector current $C^\mu$ for $g^* \rightarrow c\bar{c}$ and the amputated amplitude $B^\mu$ for $b\bar{b}_1(3S_1) \rightarrow ggg^*$, where

$$B^\mu = \sum_{\text{perm}} \text{Tr} \left[ \gamma^\mu \not{\epsilon} - \not{p}_2 - \not{p}_3 - m_b \not{\phi}_2 - \not{P} - \not{\phi}_3 - m_b \not{\phi}_3 \epsilon \cdot \Pi_3 \right],$$

(7a)

$$C^\mu = \frac{1}{(p' + \bar{p})^2} \bar{u}(p')\gamma^\mu \not{v}(p').$$

(7b)

Note that we do not include the color factor and the coupling in Eq. (7). At leading order in $v_b$, the amplitude for the QCD process $b\bar{b}_1(3S_1) \rightarrow c\bar{c}gg$ becomes

$$M^{(\text{QCD})}_{b\bar{b}_1(3S_1)} = \frac{(4\pi\alpha_s)^2}{4\sqrt{N_c}} d^{abc} T^q_{ij} B^\mu \bigg|_{q=0},$$

(8)

where $i$ and $j$ are color indices for the charm quark and the charm antiquark, respectively. In Eq. (8), we put $q = 0$ to take the $v_b$-leading contributions so that $E_b = m_b$, used $\text{Tr}[T^a T^b T^c] = (d^{abc} + i f^{abc})/4$, and chose only the symmetric component of the color factor. At $v_b = 0$, which we take at leading order in $v_b$, the amplitude (8) is infrared (IR) finite in the soft limits of any external gluons. However, in the limit that the charm quark becomes massless, the amplitude (8) may acquire collinear divergences. We will return to this point later in Sec. IID.
Squaring the amplitude (8) for $b\bar{b}_1(\bar{3}S_1) \rightarrow \bar{c}cgg$, averaging over the spin-triplet states, and summing over the spins of the final states, we obtain the differential annihilation rate for the QCD process. The contribution of the QCD process $b\bar{b}_1(\bar{3}S_1) \rightarrow \bar{c}cgg$ to the left side of Eq. (3) is

$$d\Gamma_{b\bar{b}_1(\bar{3}S_1)}^{(\text{QCD})} = \frac{(N_c^2 - 1)(N_c^2 - 4)}{N_c^2} \frac{8\pi^4\alpha_s^4}{3} B^{\mu\nu} C_{\mu\nu} \frac{d\Phi_4}{2!},$$  \tag{9}$$

where $d\Phi_4$ is the phase space element for the $\bar{c}cgg$ final state and the factors $1/3$ and $1/2!$ are for the average over the initial spins and for the two identical particles (gluons) in the final state, respectively. The tensors $B^{\mu\nu}$ and $C^{\mu\nu}$ in Eq. (9) are defined by

$$B^{\mu\nu} = \sum_{\text{spins}} B^{\mu} B^{\ast \nu}, \tag{10a}$$

$$C^{\mu\nu} = \sum_{\text{spins}} C^{\mu} C^{\ast \nu} = \frac{\text{Tr} \left[ (\not{p'} + m_c)\gamma^{\mu}(\not{p'} - m_c)\gamma^{\nu} \right]}{(p' + \not{p'})^4}. \tag{10b}$$

The polarizations for the external gluons and the $b\bar{b}_1(\bar{3}S_1)$ pair in $B^{\mu\nu}$ are summed as

$$\sum_{\lambda} \epsilon_1^\alpha(\lambda) \epsilon_1^{\ast \beta}(\lambda) = -g^{\alpha\beta} \text{ for } i = 2, 3, \tag{11a}$$

$$\sum_{\lambda} \epsilon_i^\alpha(\lambda) \epsilon_i^{\ast \beta}(\lambda) = -g^{\alpha\beta} + \frac{P^\alpha P^\beta}{P^2}. \tag{11b}$$

By substituting Eq. (9) to the left side of Eq. (3) and using the following value for the perturbative NRQCD matrix element

$$\langle O_1(\bar{3}S_1) \rangle_{b\bar{b}_1(\bar{3}S_1)} = 2N_c(2E_b)^2 = 8N_c m_b^2 + O(v_b^2), \tag{12}$$

we determine the short-distance coefficient $dC_1^{(\text{QCD})}$ as

$$dC_1^{(\text{QCD})} = \frac{(N_c^2 - 1)(N_c^2 - 4)}{N_c^3} \frac{8\pi^4\alpha_s^4}{3} B^{\mu\nu} C_{\mu\nu} \frac{d\Phi_4}{2!}. \tag{13}$$

For the QED process $b\bar{b}_1(\bar{3}S_1) \rightarrow \gamma^* \rightarrow c\bar{c}$, we quote the result given in Ref. [3], which was obtained by making use of the short-distance coefficient for the leptonic decay of $\Upsilon(1S)$:

$$dC_1^{(\text{QED})} = \frac{\pi}{3} e_Q^2 e_c^2 N_c \alpha_s^2 (2 + r) \sqrt{1 - r} \delta(1 - \xi) d\xi, \tag{14}$$

where $e_Q$ is the fractional electric charge of the heavy quark for $Q = c, b$ and the dimensionless variables $r$ and $\xi$ are defined by

$$r = \frac{m_c^2}{m_b^2}, \tag{15a}$$

$$\xi = \frac{m_{c\bar{c}}^2}{P^2}. \tag{15b}$$
where \( m_{c\bar{c}} \) is the invariant mass of the \( c\bar{c} \) pair and \( m_c \) is the charm-quark mass. The factor \( \sqrt{1 - r} \) in Eq. (14) is the ratio of the phase space for the \( c\bar{c} \) final state to the massless two-body phase space.

C. Phase-space integral for the QCD process

In order to compute the \( c\bar{c} \) invariant-mass distribution of the QCD process, it is convenient to factor out the two-body phase space of the \( c\bar{c} \) pair \( d\Phi_2(p_1 \rightarrow p' + \bar{p}') \) from the four-body phase space \( d\Phi_4 \) in Eq. (13), where \( p_1, p', \) and \( \bar{p}' \) are the momenta for the \( c\bar{c} \) pair, the \( c \), and the \( \bar{c} \), respectively. Then \( d\Phi_4 \) becomes the product of \( d\Phi_2(p_1 \rightarrow p' + \bar{p}') \) and the three-body phase space \( d\Phi_3(P \rightarrow p_1 + p_2 + p_3) \), where \( p_2 \) and \( p_3 \) are the momenta for the external gluons, convolved with the \( c\bar{c} \) invariant mass \( m_{c\bar{c}} \) as

\[
d\Phi_4(P \rightarrow p' + \bar{p}' + p_2 + p_3) = d\Phi_3(P \rightarrow p_1 + p_2 + p_3) \frac{d m_{c\bar{c}}^2}{2\pi} d\Phi_2(p_1 \rightarrow p' + \bar{p}').
\]  

(16)

The phase space (16) can further be simplified by using the dimensionless variable \( \xi \) in Eq. (15b) and the scaled energy fraction \( x_i \) in the \( \bar{b}b \) rest frame:

\[
x_i = \frac{2P \cdot p_i}{P^2}
\]

(17)

for \( i = 1, 2, \) and 3. Rewriting the phase space (16) in terms of \( \xi \) and \( x_i \), we get

\[
d\Phi_4 = \frac{dx_1 dx_2 dx_3}{128\pi^3} \delta(2 - x_1 - x_2 - x_3) \frac{P^4 d\xi}{2\pi} \frac{|p^*_c| d\Omega^*}{4m_{c\bar{c}}(2\pi)^2},
\]

(18)

where \( \Omega^* \) is the solid angle of the charm quark with the three momentum \( p^*_c \) in the \( c\bar{c} \) rest frame. The scaled energy fraction \( x_3 \) can be integrated out by using the energy delta function. Then the \( c\bar{c} \) invariant-mass distribution is obtained as a double integral of \( x_1 \) and \( x_2 \), where the physical ranges for \( \xi, x_1, \) and \( x_2 \) are given by

\[
\begin{align*}
  r &\leq \xi \leq 1, \\
  2\sqrt{\xi} &\leq x_1 \leq 1 + \xi, \\
  \frac{1}{2} \left(2 - x_1 - \sqrt{x_1^2 - 4\xi}\right) &\leq x_2 \leq \frac{1}{2} \left(2 - x_1 + \sqrt{x_1^2 - 4\xi}\right).
\end{align*}
\]

(19a) (19b) (19c)

We observe that \( C^{\mu\nu} \) is the only factor that has the \( \Omega^* \) dependence in Eq. (13). The angular integral can easily be done if we express the momenta \( p' \) and \( \bar{p}' \) in terms of \( p_1 \) and
the relative momentum \( q' \) as

\[
p' = \frac{1}{2}p_1 + q', \quad (20a)
\]
\[
p' = \frac{1}{2}p_1 - q', \quad (20b)
\]

where \( p_1 = (m_{c\bar{c}}, 0) \) and \( q' = (0, p^*_c) \) in the \( c\bar{c} \) rest frame. After integrating out the solid angle \( \Omega^* \) analytically, we get

\[
\int d\Omega^* C^{\mu\nu} = \frac{8\pi}{m_{c\bar{c}}^2} \left( 1 - \frac{4|p^*_c|^2}{3m_{c\bar{c}}^2} \right) \left( -g^{\mu\nu} + \frac{p^{\mu}_1 p^{\nu}_1}{p^2_1} \right). \quad (21)
\]

Eq. (21) behaves like the massive spin-1 tensor with momentum \( p_1 \) with \( p^2_1 = m_{c\bar{c}}^2 \). Substituting Eqs. (16) and (21) into Eq. (13), setting \( P^2 = 4m_c^2 \) at leading order in \( v_b \), we simplify the differential short-distance coefficient for the QCD process:

\[
dC_1^{(\text{QCD})} = \frac{(N_c^2 - 1)(N_c^2 - 4)}{N_c^3} \frac{d\xi}{\xi} \sqrt{1 - \frac{r}{\xi}} \left( 1 + \frac{r}{2\xi} \right) \frac{\alpha_s^4}{36\pi} \int dx_1 dx_2 F(\xi, x_1, x_2). \quad (22)
\]

Here, the ranges of \( x_1 \) and \( x_2 \) integrals are given in Eq. (19) and the integrand \( F(\xi, x_1, x_2) \) is given by

\[
F(\xi, x_1, x_2) = \sum_{n=0}^{4} \frac{f_n(x_1, x_2) \xi^n}{(x_1 - 2\xi)^2x_2^2x_3^2}, \quad (23)
\]

where the coefficients \( f_n(x_1, x_2) \) are

\[
f_0(x_1, x_2) = x_1^2(x_1 - 1)^2 + x_2^2(x_2 - 1)^2 + x_3^2(x_3 - 1)^2, \quad (24a)
\]
\[
f_1(x_1, x_2) = -8 + 20(x_2 + x_3) - 20(x_2 + x_3)^2 + 12x_2x_3 + 8(x_2 + x_3)^3 - 14x_2x_3(x_2 + x_3) + x_2^2x_3, \quad (24b)
\]
\[
f_2(x_1, x_2) = 6 - 16(x_2 + x_3) + 12(x_2 + x_3)^2 + 2x_2x_3(x_2 + x_3 - 6), \quad (24c)
\]
\[
f_3(x_1, x_2) = -4 + 8(x_2 + x_3) + 2x_2x_3, \quad (24d)
\]
\[
f_4(x_1, x_2) = 2. \quad (24e)
\]

Note that we use \( x_3 = 2 - x_1 - x_2 \) in Eqs. (23) and (24). The QCD contribution to the \( c\bar{c} \) invariant-mass distribution in the inclusive \( \Upsilon(1S) \) decay is finally obtained by replacing \( dC_1^{(c)} \) in Eq. (1) with \( dC_1^{(\text{QCD})} \) in Eq. (22).

Our result can be compared with a previous result in Ref. [8]. Neglecting the term proportional to \(|p^*_c|^2/m_{c\bar{c}}^2\) in Eq. (21), we reproduce the function \( \rho(\xi) \) in Ref. [8] analytically, where \( \rho(\xi) \) is defined by

\[
\rho(\xi) = \frac{d\Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X]/d\xi}{\Gamma[\Upsilon(1S) \rightarrow \text{light hadrons}]} \quad (25)
\]
In Ref. [8] the authors used the order-$\alpha_s^3$ color-singlet contribution to $\Upsilon(1S) \to ggg$ for the $\Gamma[\Upsilon(1S) \to \text{light hadrons}]$. Imposing these approximations, we get

$$\rho(\xi) = \frac{dC_1^{(\text{QCD})}/d\xi}{F_1(3S_1)},$$

where $F_1(3S_1)$ is the short-distance coefficient of $\Upsilon(1S) \to ggg$ at leading order in $\alpha_s$ and $v_b$ [6, 10, 13]:

$$\Gamma[\Upsilon(1S) \to ggg] = F_1(3S_1) \frac{\langle O_1(3S_1) \rangle_\Upsilon(1S)}{m_b^2},$$

$$F_1(3S_1) = \frac{(N_c^2 - 1)(N_c^2 - 4)}{N_c^3} \frac{(\pi^2 - 9)}{18} \alpha_s^3.$$

Another check on the formula (22) can be done in comparison with a previous result for the color-octet spin-triplet $c\bar{c}$ contribution to the inclusive $J/\psi$ production in $\Upsilon(1S)$ decay: Eq. (22) is to be compared with Eq. (20) of Ref. [9]. The function $F(\xi, x_1, x_2)$ in Eq. (23) is equivalent to Eq. (21) of Ref. [9] up to an overall factor. After considering the differences in the phase space and the normalization for the states, we reproduce the results in Ref. [9] at the leading order in $\alpha_s$, $v_b$, and the charm-quark velocity $v_c$ in the $J/\psi$ rest frame ($p_c^* = 0$).

D. Massless-charm-quark limit

As is discussed earlier in this section, the short-distance coefficient $dC_1^{(\text{QCD})}$ (22) for the QCD process is free of IR and collinear divergences as long as the charm quark is massive. However, in the limit where the charm quark becomes massless, $m_c \to 0$, $dC_1^{(\text{QCD})}$ may acquire collinear divergences. These singularities cancel only if we include the charm-quark contributions to the loop corrections to the gluon wave functions in the process $\Upsilon(1S) \to ggg$ [2]. In the remainder of this section, we check if $dC_1^{(\text{QCD})}$ (22) satisfies correct collinear behavior in the massless-charm-quark limit.

As the first step of the check, we can take the limit $\xi \to 0$ on the function $F(\xi, x_1, x_2)$ in Eq. (23). We find that $F(0, x_1, x_2) = f_0(x_1, x_2)$ and this value is proportional to the color-singlet short-distance coefficient $F_1(3S_1)$ in Eq. (27) for the decay $\Upsilon(1S) \to ggg$ at leading order in $\alpha_s$ and $v_b$ [6, 10, 13]:

$$F_1(3S_1) = \frac{(N_c^2 - 1)(N_c^2 - 4)}{N_c^3} \frac{\alpha_s^3}{18} \lim_{\xi \to 0} \int dx_1 dx_2 F(0, x_1, x_2).$$
Next, we investigate the asymptotic behavior of Eq. (13) in the limit \( m_{c\bar{c}} \to 2m_c \) and \( m_c \to 0 \). As \( m_{c\bar{c}} \to 0 \), \( |\mathbf{p}_c^*| \) approaches \( m_{c\bar{c}}/2 \) and gauge invariance requires \( p_1 \) to be orthogonal to \( \mathcal{B}_{\mu\nu} \). By making use of Eqs. (16), (21), and (13), we get

\[
\mathcal{B}_{\mu\nu} \int d\Phi_2(P \to p' + \bar{p'}) \mathcal{C}^{\mu\nu} \to \frac{\mathcal{B}_{\mu\nu}}{2\pi m_{c\bar{c}}^2} \frac{1}{3} (-g^{\mu\nu}). \tag{29}
\]

Note that, in this limit, \(-\mathcal{B}_{\mu\nu} g^{\mu\nu}\) becomes the squared amplitude for \( b\bar{b}_1(3S_1) \to ggg \). Substituting Eq. (16) into Eq. (13), and using the limiting value (29), we find that the short-distance coefficient \( C_1^{(QCD)} \) is divergent logarithmically in the limit \( m_{c\bar{c}} \to 2m_c \) and \( m_c \to 0 \).

\[
C_1^{(QCD)} \to \frac{\alpha_s}{2\pi} \left[ \frac{1}{8N_c} \left( \frac{\delta^{abc}}{4\sqrt{N_c}} \right)^2 \left( 4\pi \alpha_s \right)^3 \int \frac{d\Phi_3}{3!} \left( -\frac{1}{3} g^{\mu\nu} \mathcal{B}_{\mu\nu} \right) \right] \int (2m_b)^2 \frac{d\mathbf{m}_{c\bar{c}}^2}{m_{c\bar{c}}^2} \tag{30}
\]

where the factor \( 8N_c \) comes from the perturbative NRQCD matrix element in Eq. (12), the second and the third factors in the brackets are the color factor and the coupling for the process \( \Upsilon(1S) \to ggg \), respectively, \( 1/3! \) is the symmetry factor for the three gluons, and the factor \( 1/3 \) is for the average over the initial spin states. The quantity inside the square brackets in Eq. (30) is independent of \( m_{c\bar{c}} \) and is finite. Simplifying the leading divergent term in Eq. (30), we obtain the asymptotic form of \( C_1^{(QCD)} \). Because the collinear divergence is absent in the QED contribution, the collinear divergent contribution in the QCD process is the same as that in the short-distance coefficient \( C_1^{(c)} \):

\[
C_1^{(c)} \to \frac{\alpha_s}{\pi} F_1(3S_1) \log \frac{m_b}{m_c} \tag{31}
\]

It is explicit in Eq. (31) that the collinear divergent contribution, which is of order \( \alpha^4_s \), is proportional to the short-distance coefficient for \( \Upsilon(1S) \to ggg \) at order \( \alpha^3_s \).

The only order-\( \alpha^4_s \) contributions to \( \Upsilon(1S) \to \) light hadrons that depend on \( m_c \) except for the \( q\bar{q}gg \) final state are the virtual charm-quark loop corrections to the gluon wave functions in \( \Upsilon(1S) \to ggg \). The leading divergent term of the virtual correction is

\[
C_1^{(c,\text{virtual})} \to -3i\Pi(0)F_1(3S_1)
\]

\[
= \frac{\alpha_s}{\pi} F_1(3S_1) \log \frac{m_c}{\mu}, \tag{32}
\]

where \( \Pi(0) \) is the virtual charm-quark loop contribution to the vacuum polarization for an on-shell gluon and \( \mu \) is the renormalization scale. We find that the collinear divergence cancels in the sum of the right sides of Eqs. (31) and (32). Therefore, \( C_1^{(c)} + C_1^{(c,\text{virtual})} \), which is the complete \( m_c \) dependent contributions to the hadronic decay of the \( \Upsilon(1S) \) at order \( \alpha^4_s \), is free of collinear divergence.
III. NUMERICAL ANALYSIS

In this section, we provide a phenomenological prediction for the $c\bar{c}$ invariant-mass distribution in inclusive $\Upsilon(1S)$ decay by making use of the NRQCD factorization formula obtained in Sec. II. As shown in Eq. (2), the short-distance coefficient $dC_1^{(c)}$ is the sum of the QCD and the QED contributions in Eqs. (22) and (14), respectively. Substituting the $dC_1^{(c)}$ into the NRQCD factorization formula (1), we obtain the differential rate depending on the scaled invariant mass $\xi$ defined in Eq. (15b). The resultant $c\bar{c}$ invariant-mass distribution is

$$
\frac{d}{dm_{c\bar{c}}^2} \Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X] = \frac{1}{m_{\Upsilon(1S)}^2} \frac{dC_1^{(c)}(\langle O_1(3S_1) \rangle_{\Upsilon(1S)})}{d\xi} / m_b^2,
$$

where we use $P^2 = m_{\Upsilon(1S)}^2$ and $m_{\Upsilon(1S)}$ is the mass of the $\Upsilon(1S)$. In our numerical analysis, we use the same input parameters as those used in Ref. [3], where the momentum distribution of the charm quark produced in the inclusive $\Upsilon(1S)$ decay is studied.

The short-distance coefficient $dC_1^{(c)}$ depends on the strong coupling $\alpha_s$ and the ratio $r$ defined in Eq. (15a). For the strong coupling, we take the running coupling $\alpha_s(m_{\Upsilon(1S)}/2) = 0.215$. As shown in Eq. (19a), the threshold of the phase space is determined by the ratio $r = m_c^2/m_b^2$. In order to make the end points of the phase space fit to the physical ones, we use $m_c = m_D$ and $m_b = m_{\Upsilon(1S)}/2$ in evaluating the ratio $r$, where $m_D = 1.87$ GeV is the average mass of the $D^0$ and $D^+$ and $m_{\Upsilon(1S)} = 9.46$ GeV [20]. Then the numerical value for the ratio becomes $r = 4m_D^2/m_{\Upsilon(1S)}^2 \approx 0.1563$. This choice of $m_c$ and $m_b$ for the ratio $r$ seems reasonable for the open-charm production in the $\Upsilon(1S)$ decay. For the bottom-quark mass $m_b$ that appears in the NRQCD factorization formulas (1) and (33), we use the one-loop pole mass $m_b = 4.6 \pm 0.1$ GeV. The numerical value for the long-distance NRQCD matrix element in Eqs. (1) and (33) is quoted from Ref. [3]:

$$
\langle O_1(3S_1) \rangle_{\Upsilon(1S)} = 3.07^{+0.21}_{-0.19} \text{ GeV}^3.
$$

For more details of the determination of the NRQCD matrix element in Eq. (34), we refer the readers to Refs. [16–19].

The $c\bar{c}$ invariant-mass distribution from the QCD process is obtained by substituting Eqs. (22) and (34) into Eqs. (1) and (2), and integrating out $x_1$ and $x_2$ over the ranges in Eq. (19). We plot the QCD contribution to the invariant-mass distribution $d\Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X]/d\xi$ in Fig. 1 as a function of the dimensionless variable $\xi$. Note that $d\Gamma[\Upsilon(1S) \rightarrow$
FIG. 1: Invariant-mass distribution of the $c\bar{c}$ pair produced in the inclusive $\Upsilon(1S)$ decay as a function of $\xi$, which is defined in Eq. (15b). Only the contribution from the QCD process $b\bar{b}(3S_1) \rightarrow c\bar{c}gg$ is shown. The contribution from the QED process $b\bar{b}(3S_1) \rightarrow \gamma^* \rightarrow c\bar{c}$, which is proportional to $\delta(1 - \xi)$, is not shown.

\[ c\bar{c} + X] / d\Gamma_{\Upsilon(1S)} = m_{\Upsilon(1S)} \frac{d\Gamma[\Upsilon(1S) \rightarrow c\bar{c} + X]}{d\xi}. \]

The distribution in Fig. 1 has a sharp peak of height 5.48 keV at $\xi = 0.20$, which is just above the threshold at $\xi = r$. The invariant-mass distribution of the $c\bar{c}$ from the QED process is obtained in a similar way by using Eq. (14). As shown in Eq. (14), the QED contribution is proportional to the delta function $\delta(1 - \xi)$ so that the contribution is concentrated at the kinematic end point $\xi = 1$. Therefore, the contribution of the QED process is well distinguished from the QCD contribution.

Integrating the invariant-mass distribution over the variable $\xi$, we obtain the total pro-
duction rate of the $c\bar{c}$ pair in the inclusive $\Upsilon(1S)$ decay:

$$\Gamma[\Upsilon(1S) \to c\bar{c} + X] = \Gamma^{(\text{QCD})} + \Gamma^{(\text{QED})},$$  \hspace{1.0cm} (35a)

$$\Gamma^{(\text{QCD})} = 1.44 \pm 0.36 \text{ keV},$$  \hspace{1.0cm} (35b)

$$\Gamma^{(\text{QED})} = 2.60 \pm 0.65 \text{ keV},$$  \hspace{1.0cm} (35c)

where the theoretical uncertainties in Eq. (35) come from the uncertainties of $m_b$, the NRQCD matrix element in Eq. (34), and uncalculated next-to-leading-order relativistic and QCD corrections, which we set to be 10\% ($v_b^2 \sim 0.1$) and 21.5\% ($\alpha_s = 0.215$) of the central value, respectively. The QED contribution $\Gamma^{(\text{QED})}$ to $\Upsilon(1S) \to c\bar{c} + X$ agrees with that in Ref. [3]. The QCD contribution $\Gamma^{(\text{QCD})}$ differs from that in Ref. [3] by about 2\% because we omit the contribution of the process $b\bar{b}(^3S_1) \to g^* g \gamma$ followed by $g^* \to c\bar{c}$ while this tiny contribution is included in Ref. [3]. As shown in Eq. (35), the QED contribution $\Gamma^{(\text{QED})}$ occupies about 60\% of the total charm production rate of the inclusive $\Upsilon(1S)$ decay. As is discussed in Ref. [3], we expect that the QCD next-to-leading-order corrections to the QED process may modify the shape of the invariant-mass distribution.

Our result for $\Gamma^{(\text{QCD})}$ can be compared with another previous result [8]. In Ref. [8], the ratio of the total charm production rate in the inclusive $\Upsilon(1S)$ decay to $\Gamma[\Upsilon(1S) \to \text{light hadrons}]$ is predicted to be $\Gamma[\Upsilon(1S) \to c\bar{c} + X]/\Gamma[\Upsilon(1S) \to \text{light hadrons}] \approx 0.081 \alpha_s$. This is greater by about 20\% than our result,

$$\frac{\Gamma[\Upsilon(1S) \to c\bar{c} + X]}{\Gamma[\Upsilon(1S) \to \text{light hadrons}]} = 0.065 \alpha_s.$$  \hspace{1.0cm} (36)

As we discussed in Sec. II, the authors of Ref. [8] made the approximation of neglecting $|p_c^*|^2/m_{c\bar{c}}^2$ in computing the function $\rho(\xi)$ in comparison with Eq. (21) of this paper. The approximation leads to the overestimation of the height of the peak in the invariant-mass distribution and this is the reason for the discrepancy between the two results. To check this point explicitly, we carry out the same calculation with the approximations that were used in Ref. [8]. The calculation shows that the height of the peak increases by about 8\% from 5.48 keV to 5.93 keV at the same horizontal position at $\xi = 0.20$. The difference increases as $\xi$ increases ranging up to 30\%. As a result, the total rate without approximation is smaller by about 20\%.
IV. SUMMARY

We have calculated the invariant-mass distribution of the $c\bar{c}$ pair produced in the inclusive decay of the $\Upsilon(1S)$ based on the color-singlet mechanism of the NRQCD factorization formalism at leading order in the bottom-quark velocity $v_b$ in the meson rest frame. As the short-distance processes, we considered the QCD process $b\bar{b}_1(^3S_1) \rightarrow g^* gg$ followed by $g^* \rightarrow c\bar{c}$ at leading order in $\alpha_s$ and the QED process $b\bar{b}_1(^3S_1) \rightarrow \gamma^* \rightarrow c\bar{c}$ at leading order in $\alpha$ and $\alpha_s$.

The QCD contribution to the $c\bar{c}$ invariant-mass distribution has a sharp peak just above the threshold, and that of the QED process is concentrated at the maximally allowed kinematic end point. In comparison with a previous analysis on the QCD process, our prediction for the peak of the QCD contribution is lower than that in Ref. [8] and the total production rate of the $c\bar{c}$ pair in the inclusive $\Upsilon(1S)$ is smaller than that in Ref. [8] by about 20%. The main reason for the discrepancy is that, in Ref. [8], the authors made an approximation of neglecting $|p^*_c|^2/m_{c\bar{c}}^2$ while we keep the full expression in Eq. (21) of this paper.

We also investigate the collinear divergences of the decay rate in the massless charm-quark limit $m_c \rightarrow 0$. Although the decay rate of leading order in $v_b$ is free of both IR and collinear divergences if the charm quark is massive, the rate acquires collinear divergences in the limit $m_c \rightarrow 0$. We have confirmed that our analytic expression for the differential decay rate reproduces the correct collinear behavior in this limit so that the divergence exactly cancels that of the charm-quark loop corrections to the gluon wave function for the $\Upsilon(1S) \rightarrow ggg$ process. The sum of the two contributions are the $m_c$ dependent contribution to the inclusive $\Upsilon(1S)$ decay into light hadrons at order $\alpha_s^4$.

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