I. INTRODUCTION

Understanding the interplay between strong correlations and quenched disorder in low-dimensional superconductors remains one of the key challenges in condensed matter physics [1]. While pair-breaking due to elastic impurity scattering is detrimental to superconductivity, spatial inhomogeneity can locally enhance pairing. Near a superconductor-insulator transition, the latter can occur via the spatial accumulation of Cooper pairs, due to the *multifractal rarefaction* of single-particle eigenstates [2–5]. Multifractality arises in quantum-critical states due to wave interference induced by multiple impurity scattering. Two-dimensional (2D) multifractal superconductivity has very recently come to the fore in studies of transition metal dichalcogenides [7–9].

An enduring mystery is the role of spatial inhomogeneity in the high-$T_c$ cuprate superconductors. These materials are characterized by two different energy scales in the underdoped regime: the pseudogap energy $\Delta_1$ and the smaller pairing energy $\Delta_0$; the former (latter) increases (decreases) with increased underdoping [10]. Maps of $\Delta_1(r)$ obtained via STM demonstrate strong nanoscale spatial inhomogeneity that increases with underdoping [11–13]. A key observation in STM scans of the local density of states (LDoS) is that the lower energy scale $\Delta_0$ appears to divide fermionic excitations into two distinct classes. States with energies smaller than $\Delta_0$ behave like dispersive, “coherent” Bogoliubov-de Gennes quasiparticles, showing robust quasiparticle interference. States with energies above $\Delta_0$ are instead termed “incoherent,” as they exhibit strong spatial fluctuations that vary little with energy [13, 14]. Renewed urgency for understanding spatial inhomogeneity comes from Ref. [17], which reported an increase in $T_c$ with increasing disorder.

Much recent work on the strange physics of the cuprates invokes gravitational descriptions of quantum criticality [15]. In this paper, we uncover a very different role possibly played by quantum criticality in these materials, due to a different type of “gravitational” physics. We show that the dichotomy between plane-wave-like, low-energy quasiparticle states and strongly inhomogeneous, finite-energy states can be reconciled in a simple model of noninteracting nodal Dirac quasiparticles, subject to random velocity disorder. Formally this can be cast as the effect of static spacetime curvature [19–21] (“quenched gravitational disorder” (QGD)). For massless Dirac fermions, propagation is analogous to the lensing of starlight through a fixed, randomly gravitating space-time.
time \textsuperscript{22}. The gravitational language precisely defines the gauge-invariant content of the disorder, through the induced curvature.

QGD is realized whenever Dirac carriers arise from a correlation gap that is spatially inhomogeneous. In a d-wave superconductor, spatial gap fluctuations modulate the nodal quasiparticle velocity, see Fig. 1. In the context of conventional 2D Dirac materials such as graphene or the surface states of 3D topological insulators, QGD has so far received little attention \textsuperscript{21} \textsuperscript{24} \textsuperscript{26}. This is because disorder can usually couple in a more relevant fashion, through gauge and mass potentials \textsuperscript{21} \textsuperscript{27} \textsuperscript{29}. These perturbations typically induce metallic or Anderson insulating behavior on the largest scales \textsuperscript{29} \textsuperscript{32}.

In this paper, we use exact diagonalization to probe the effect of QGD on 2D Dirac carriers. While the low-energy states near the Dirac point are only weakly affected, we find that most of the energy spectrum converges into a “stack” of critical wave states. These quantum-critical wave functions apparently exhibit universal multifractal LDoS fluctuations. A representative state is shown in Fig. 1(b), while LDoS maps of states with different energies appear in Fig. 2.

Our results are very surprising, in that QGD is a strongly irrelevant perturbation at low energies. QGD was instead expected to preserve the quasi-ballistic nature of the clean system, even away from the Dirac point \textsuperscript{24}. Moreover, critical wave states typically arise only at isolated energies, such as a mobility edge or at the plateau transition of a quantum Hall effect \textsuperscript{28} \textsuperscript{34} \textsuperscript{35}. The population of critical states induced by QGD is compared to the total density of states in Fig. 4. Our calculations are performed in momentum space; the largest size studied is a 109 × 109 grid. Since Dirac cones typically span \(\sim 10\%\) of the 2D Brillouin zone, this corresponds roughly to a (100 nm)\(^2\) map. More of the spectrum becomes more critical with increasing disorder (Fig. 5) or system size (Fig. 8).

The coexistence of low-energy plane-wave and finite-energy multifractal states induced by QGD is reminiscent of STM spectra observed in the high-\(T\)\(_c\) cuprate superconductor BSCCO \textsuperscript{11} \textsuperscript{16}, where static spatial fluctuations are detected in LDoS maps at energies above the scale \(\Delta_0\). As we review in Sec. II, other types of disorder are predicted to induce different physics: (I) One class of (effectively topological \textsuperscript{22}) perturbations should also produce critical scaling throughout the quasiparticle energy spectrum \textsuperscript{30} \textsuperscript{38}. In this scenario, though, zero-energy states are also predicted to be multifractal, and the low-energy DoS \(\nu(\varepsilon)\) would be enhanced via a sublinear power law, \(\nu(\varepsilon \to 0) \sim |\varepsilon|^\delta\) with \(\delta < 1\) \textsuperscript{27} \textsuperscript{39} \textsuperscript{43}. (II) Disorder that induces generic internode scattering is instead predicted to Anderson localize the entire quasiparticle spectrum \textsuperscript{27} \textsuperscript{44}. Neither scenario (I,II) is consistent with STM data \textsuperscript{16}.

We restrict our attention in this paper to on-average isotropic QGD. We take both components of the Dirac velocity \(v_x(\mathbf{r})\) and \(v_y(\mathbf{r})\) to be quenched random vari-ables, with average value \(v_{x(\mathbf{r})} = v_{y(\mathbf{r})} \equiv v_0 = 1\), and spatial fluctuations that are smaller than the average (so as to avoid curvature horizons). A more realistic model for velocity modulation in the cuprates would restrict disorder to the component of the velocity along the Fermi surface, \(v_{a(x(\mathbf{r}))} \sim \Delta(\mathbf{r})/k_F\), see Fig. 1.

QGD should also arise at the surface of a topological superconductor (TSC). Due to “topological protection,” velocity modulation is the only allowed coupling

\[\varepsilon^2 \sim |\mathbf{p}| \approx \frac{\hbar}{m} |(k_x + i k_y)/\pi| \approx 1/\Lambda,\]

where \(\Lambda\) is a momentum cutoff, and the hopping parameter \(t\) is measured in units of \(\hbar/\Lambda\). Critical states are identified by their multifractal spectrum, see Fig. 4(b). The dimensionless disorder strength is \(\lambda \equiv \Lambda/\Delta_0\). Results here are obtained from exact diagonalization in momentum space over a \((2N+1) \times (2N+1)\) grid of momenta; here \(N = 54\). Critical states are identified by their multifractal spectrum, see Fig. 4(b). The dimensionless disorder strength is \(\lambda \equiv \Lambda/\Delta_0\). The criterion for “strong disorder” is \(\lambda \gtrsim 0.393\) (Sec. III B).

- \(\Delta\): superconducting gap
- \(t\): hopping parameter
of charged impurities to the 2D massless Majorana fluid expected to form at the surface of a class DIII TSC \cite{46-49} with winding number \(|\nu| = 1\) \cite{24, 51}, as we show in Appendix B.

The stack of critical finite-energy states found here is quite unusual, but not unprecedented. In Ref. \cite{58}, we observed stacks of critical, class C spin quantum Hall plateau transition (QHPT) states at finite energy in a surface model for a class CI TSC. The finite-energy surface states of a class AIII TSC should sit at the class A integer QHPT \cite{38, 37, 39, 45}. We expect that the finite-energy critical states identified here correspond to a version of the class D thermal QHPT \cite{51, 57}.

A. Outline

This paper is organized as follows. In Sec. II we review results on impurity scattering in \(d\)-wave superconductors. We derive the low-energy 4-node Dirac theory from a microscopic model. We review how restrictions to 2- and 1-node models are effectively topological, i.e. correspond to models for surface states of bulk TSCs. We summarize results for the thermal conductivity, density of states, and multifractal spectra of low- and finite-energy wave functions. We discuss the quantum critical “stacking” of multifractal finite-energy states and the connection to plateau transitions in quantum Hall effects that has recently been found to occur in these models \cite{38, 45}.

In Sec. III we introduce the model studied here, a single 2D Dirac fermion with QGD. We then present the main numerical results of this paper; these consist of the density of states fluctuations. With the exception of the recent spectrum-wide criticality numerical results obtained in Refs. \cite{38, 45} and in this paper, all of the following is well known \cite{27, 58}.

Readers primarily interested in new results for the Dirac model with QGD can skip this section, and proceed directly to Sec. III.

II. DISORDER IN 2D \(d\)-WAVE SUPERCONDUCTORS, AND TOPOLOGICAL SUPERCONDUCTOR SURFACE FLUIDS

In this section we review a microscopic model for elastic impurity scattering in 2D \(d\)-wave superconductors. The generic model with elastic scattering between all four low-energy Dirac quasiparticle nodes Anderson localizes at all energies. We then show how restrictions on internode scattering produce independent, topologically protected sectors. The latter are equivalent to the surface states of TSCs in classes CI, AIII, and DIII. Finally, we review exact results for the thermal conductivity, density of states, and multifractal spectrum of local density of states fluctuations. With the exception of the recent spectrum-wide criticality numerical results obtained in Refs. \cite{38, 45} and in this paper, all of the following is well known \cite{27, 58}.

A. \(d\)-wave model

A 2D spin-singlet superconductor has the Bogoliubov-de Gennes Hamiltonian

\[
H = \int \Psi^\dagger(k) \hat{h}(k) \Psi(k), \quad \Psi^T(k) \equiv \left[ c^\dagger_\uparrow(k) c^\dagger_\downarrow(-k) \right],
\]

\[
\hat{h}(k) = \begin{bmatrix} \tilde{\varepsilon}(k) & \Delta(k) \\
\Delta^*(k) & -\tilde{\varepsilon}(-k) \end{bmatrix}.
\]

Here \(\Psi(k)\) is the Nambu spinor that annihilates spin-1/2 quasiparticles, \(T\) denotes the matrix transpose, \(\tilde{\varepsilon}(k) = \varepsilon(k) - \mu\) is the bare energy dispersion relative to the chemical potential \(\mu\), \(\Delta(k)\) is the mean-field BCS order parameter, and \(\int \equiv \int d^2k/(2\pi)^2\).

The two key symmetries we want to enforce are \(T^2 = -1\) time-reversal and spin SU(2) symmetry, the combination of which places the system in the Altland-Zimbauer

\[
\begin{align*}
\text{FIG. 3. The Dirac nodes of a 2D } d\text{-wave superconductor occur in two partnered pairs (1, 2) and (3, 4). The partners of a pair are related by time-reversal and spin rotation symmetries. Generic impurity scattering between all four nodes is believed to Anderson localize quasiparticle states at all energies.} \quad \text{On the contrary, if impurity scattering is restricted to occur only (i) between partners of a pair (1 } \leftrightarrow 2 \text{) or (3 } \leftrightarrow 4 \text{), or (ii) within each node separately, then the model decouples ("fractionalizes") into multiple topologically protected sectors.} \quad \text{These topologically protected sectors are equivalent to surface state theories of bulk (3D) TSCs, in classes CI, AIII, or DIII (see text).}
\end{align*}
\]
Imposing both effective low-energy, anisotropic Dirac theory, along the nodal directions, see Fig. 3. Linearizing the form of Ψ(ψn nodes i, t) project onto nodal pair (1, 2) (+) or (3, 4) (−), and where the Clifford algebra matrices are (\(\hat{\alpha}_1, \hat{\alpha}_2\) = (\(\hat{\sigma}^3, \hat{\sigma}^4\)). In Eq. (2.6), \(v_F\) is the bare Fermi velocity, while the perpendicular dispersion \(v_\Delta \approx \Delta_0/k_F\) arises from the pairing.

Non-magnetic impurities can couple to the low-energy theory via local, Hermitian fermion bilinear operators. The most relevant of these in the RG sense have no derivatives, and take the form of \(\bar{\Psi}\)ψn; \(\hat{c}\) project onto nodal pair (1, 2); \(\hat{\sigma}^3\). The remaining 8 allowed perturbations of the original model in Eq. (2.1) [27] cannot describe the isolated surface states of a strong TSC. This immediately implies that the generic 4-node Hamiltonian \(\hat{h}_2 = [\hat{k}_1 B_{\bar{a}, i}(\hat{r}) \hat{\lambda} + \hat{k}_2 C_{\bar{a}}(\hat{r})] \hat{\alpha}_{\bar{a}}\), (2.8) describing the isolated surface states of a strong TSC or insulator.

It is possible to derive the strengths and correlations of all 20 disorder potentials \(\{A_{a, i}^{(\pm)}, B_{a, i}, C_{a}\}\) in the effective low-energy field theory from microscopic perturbations of the original model in Eq. (2.1) [27], but we will not pursue this here. In fact, our main interest will be in these potential perturbations, but in the weaker "quenched gravitational disorder" (QGD, nodal velocity randomness), which we argue below should be present in the cuprate superconductors [18]. In Appendix B, we provide a derivation of QGD from the coupling of the electric potential to Majorana surface states in a microscopic model of a class DIII TSC.
B. Topological restrictions and TSC surface states: Localization, quantum criticality, and physical properties

1. Four coupled nodes: Spectrum-wide Anderson localization

Quantum wave interference induced by generic elastic scattering between all four nodes in the model described by Eqs. (2.6)–(2.9) is expected to Anderson localize all quasiparticle states, at both zero and finite single-particle energies.

The localization near zero energy is understood as follows. The Dirac model described by Eq. (2.9) can be averaged over all 20 disorder potentials using the replica or supersymmetry (SUSY) trick. For Gaussian white-noise distributions, the model flows to strong coupling under the perturbative renormalization group [24]. One then expects that the system can be described by a replicated or SUSY nonlinear sigma model in class CI. Although this sigma model admits a Wess-Zumino-Novikov-Witten (WZNW) term, it can be shown that no such term arises in the full four-node theory (e.g., it is forbidden by average parity invariance [24]; moreover, the WZNW model turns out to be topological [58], as reviewed below). Without the WZNW term, the 2D class CI sigma model also flows to strong coupling, interpreted as the tendency towards Anderson localization [44].

Anderson localization is anticipated at finite energy as well in the four-node model. The standard argument (which, however, needs to be revised for topological models as we emphasize below) goes as follows. At zero energy, 7 of the 10 Altland-Zirnbauer classes are characterized by a special particle-hole or chiral symmetry. This is true for all classes that describe spin-1/2, time-reversal-invariant superconductors in classes CI, AI, and DIII [58]. This symmetry also relates states at positive and negative energies, but does not tell us anything about the character of the states at some particular fixed energy \( \varepsilon \neq 0 \). On the other hand, for quasiparticle states in a superconductor, one can always consider \( |\varepsilon| \gg \Delta_0 \), where \( \Delta_0 \) is the pairing energy. Then the quasiparticle states should reside in the standard Wigner-Dyson class of the normal metal that hosts the superconductivity. Since \( \varepsilon = 0 \) is the only symmetry-distinguished energy, one concludes that all states with \( \varepsilon \neq 0 \) in the infinite-volume limit must reside in the orthogonal, unitary, or symplectic Wigner-Dyson class. For the four-node class CI model, the appropriate Wigner-Dyson class is the orthogonal metal class AI, which also preserves \( T \) and spin SU(2) symmetry. This class is believed to always localize in 2D [24].

Localization of all quasiparticle states is not borne out by experiments in the high-\( T_c \) cuprate superconductors, at or below optimal hole doping. First of all, localization is at variance with superconductivity itself, since \( T_c \) is not protected by Anderson’s theorem for non-s-wave pairing [27]. Second, STM data taken at energies less than a characteristic scale \( \Delta_0 \) (which is always below the pseudogap energy \( \Delta_1 \) on the underdoped side) show robust quasiparticle interference, a sign that there is just enough internode scattering to reveal the clean dispersion, but not enough to induce localization [15] [16].

Finally, the experimentally measured low-temperature longitudinal thermal conductivity [59] [60] is nonzero, and close to the universal (disorder-independent) theoretical result

\[
\frac{k}{T} = \frac{k_B^2}{3h} \left( \frac{v_F}{v_\Delta} + \frac{v_\Delta}{v_F} \right). \tag{2.10}
\]

This result was originally obtained via an approximate, self-consistent semiclassical calculation [14] [61] [62]. Eq. (2.10) is better understood as the exact result for the \( T \to 0 \) limit of the Landauer thermal conductivity in the clean four-node model, the analog (via Wiedemann-Franz) of the “ballistic” conductivity \( \sigma_{xx} = 4e^2/h \) for pristine graphene doped exactly to charge neutrality. The absence of impurity scattering combined with the vanishing density of states produces a finite, universal result due to evanescent wave propagation [63]. A very special feature of the topological models reviewed below is that Eq. (2.10) remains exact in the presence of disorder [39] [58] [44] [65], and is even predicted to be independent of virtual interaction (Altshuler-Aronov) corrections in these special models [66]. By contrast, the Anderson localized model in Eq. (2.9) would have \( k/T \to 0 \) as \( T \to 0 \).

2. Two coupled nodes: class CI TSC surface states and spectrum-wide spin quantum Hall criticality

If we restrict ourselves to intrapair scattering, such that the node pair (1,2) decouples from (3,4) (Fig. 3), then \( \hbar = \hbar_0 + \hbar_1 \) [Eqs. (2.6) and (2.7)]. Since the pairs are independent, we focus on one pair (1,2). After a rescaling of \( (x,y) \) and a basis change, the two-node Hamiltonian can be written as

\[
\hat{h}_{\text{CI2}} = \hat{\sigma} \cdot \left[ -i\nabla + A_i(r) \hat{\tau}^i \right], \tag{2.11}
\]

where \( i \in \{1, 2, 3\} \) and \( \hat{\sigma} \equiv \hat{\sigma}^1 \hat{\tau}^x + \hat{\sigma}^2 \hat{\tau}^y \). Here \( \hat{h}_{\text{CI2}} \) is a 4 \( \times \) 4 matrix differential operator acting in the composite (particle-hole) \( \otimes \) (valley) \( (\sigma \otimes \tau) \) space. Since nodes (1,2) are related by \( T \), the model still resides in class CI. The symmetry operations in Eq. (2.5) become

\[
T: \quad -\hat{\sigma}^3 \hat{h} \hat{\sigma}^3 = \hat{h}, \tag{2.12a}
\]

\[
P: \quad -\hat{\sigma}^1 \hat{\tau}^2 \hat{h}^\dagger \hat{\sigma}^1 \hat{\tau}^2 = \hat{h}. \tag{2.12b}
\]

Given the (transformed) form of the Clifford algebra in Eq. (2.11), the effective chiral/physical time-reversal symmetry condition in Eq. (2.12a) is anomalous, and cannot be realized without fine-tuning in two spatial dimensions [58] [67]. It is naturally realized on the surface of a
class CI TSC [15, 68], with minimal winding number $|\nu| = 2$. The sigma model takes the same form as the four-node model, except that it is now augmented with a WZNW term at level $k = 1$ [48, 49, 68].

The density of states $\nu(\varepsilon)$ exhibits critical scaling with energy $\varepsilon$ in the limit $|\varepsilon| \to 0$. In particular,

$$\lim_{\varepsilon \to 0} \nu(\varepsilon) \sim |\varepsilon|^{x_1/z}, \quad (2.13)$$

where $x_1/z = 1/7$ [60]. The zero-energy wave function $\psi_0(\mathbf{r})$ exhibits quantum critical fluctuations on all length scales; these are characterized by the multifractal spectrum $\tau(q)$ (for a review, see e.g. [29]). If the system has size $L \times L$, one divides this up into $N^2$ boxes of size $b$, with $N \equiv L/b$. Then one introduces the box probability

$$\mu_i \equiv \int d^2r |\psi_0(\mathbf{r})|^2,$$

where the integral is performed over $b_i$ the $i^{th}$ box. The multifractal spectrum is defined via the scaling behavior of moments of the box probability,

$$\sum_{i=1}^{N^2} (\mu_i)^q \sim \left(\frac{b}{L}\right)^{2(1-q)}, \quad (2.14)$$

Box probabilities can also be obtained by normalizing a spatial map of the local density of states in an STM experiment. In the limit $L \to \infty$, $\tau(q)$ is self-averaging. For the topological class CI model in Eq. (2.11), the spectrum is perfectly parabolic. The exact result is

$$\tau(q) = 2(q-1) + \Delta(q),$$

$$\Delta(q) = \theta q(1-q), \quad 0 \leq |q| \leq q_c, \quad q_c \equiv \sqrt{2}/\theta, \quad (2.15)$$

with $\theta = 1/4$ [41, 42].

Very recently, the authors considered the question of the finite-energy states of the model in Eq. (2.11). On one hand, the same argument presented in Sec. II.B leads to the conclusion that all finite-energy states should be Anderson localized in the orthogonal class AI; this was the “conventional wisdom” [29]. On the other hand, Eq. (2.11) also describes a surface quasiparticle fluid that forms at the boundary of a bulk TSC [58, 68], protected by the anomalous form of time-reversal symmetry in Eq. (2.12a). From this perspective, the idea that only the zero-energy single-particle wave function $\psi_0(\mathbf{r})$ escapes Anderson localization appears very strange. Indeed, it would correspond to a very weak form of “topological protection,” since in other topological phases such as quantum Hall liquids, 2D and 3D topological insulators, it is the entire band of edge or surface excitations that is protected from Anderson localization [40, 69].

The only alternative to localization was argued in Ref. [85] to be a “stacking” of identical, quantum critical wave functions at all nonzero energies. It was argued that each such wave function should sit at the class C, spin quantum Hall plateau transition (SQHPT) [70, 72]. The latter shares a few critical exponents with classical 2D percolation [34, 55, 71], a logarithmic conformal field theory [73, 74].

The numerical results of Ref. [38] are consistent with the “critical stacking” scenario. Near zero energy, the DoS and multifractal spectrum confirm the predictions of the WZNW theory [Eqs. (2.13) and (2.15), with $x_1/z = 1/7$ and $\theta = 1/4$]. At intermediate energies, however, a wide swath of states is found to exhibit weaker universal multifractality, given approximately by Eq. (2.15), but now with $\theta \simeq 1/8$. The latter is consistent with the SQHPT [34, 55]. More states exhibit SQHPT phenomenology upon increasing the disorder strength or system size [88].

Finally, we note that the results described above are clearly incompatible with experiments in the cuprates. This is not surprising, because in the 2D $d$-wave model (Fig. 5) it is difficult to microscopically suppress scattering between node pairs $(1, 2) \leftrightarrow (3, 4)$, while simultaneously retaining significant internode scattering between partners $(1 \leftrightarrow 2)$ and $(3 \leftrightarrow 4)$. The low-energy density of states vanishes as a strongly sublinear power-law $\nu(\varepsilon) \sim |\varepsilon|^{1/7}$. The two-node model exhibits the strongest multifractality at zero energy [Eq. (2.15) with $\theta = 1/4$] and weaker multifractality at finite energies [Eq. (2.15) with $\theta \simeq 1/8$ [38]]. These features are opposite the observations in STM on BSCCO, which show minimal spatial inhomogeneity in low-energy LDOS maps, with stronger inhomogeneity above the energy scale $\Delta_0$; in addition, the low-energy DoS retains the linear character of the clean system [16].

3. One node, vector potential disorder: class AIII TSC surface states and spectrum-wide quantum Hall criticality

We can further restrict to pure intranode scattering. The effective Hamiltonian is “half” of Eq. (2.11), i.e. a single 2-component Dirac fermion subject to $U(1)$ vector potential randomness,

$$\hat{h}_{AIII} = \hat{\sigma} \cdot [-i\nabla + A(\mathbf{r})]. \quad (2.16)$$

Formally this single-node Hamiltonian is the same as the surface fluid of a class AIII TSC with minimal winding number $|\nu| = 1$. The anomalous (topological) time-reversal symmetry is still encoded by Eq. (2.12a).

The density of states scales as in Eq. (2.13), with exponent [39]

$$x_1/z = (1 - \lambda_A)/(1 + \lambda_A). \quad (2.17)$$

Here $\lambda_A$ denotes the variance of the assumed white-noise vector disorder potential,

$$A_\alpha(\mathbf{r}) A_\alpha'(\mathbf{r}') = \pi \lambda_A \delta_{\alpha,\alpha'} \delta^{(2)}(\mathbf{r} - \mathbf{r}').$$

At zero energy, the multifractal spectrum is given by
TABLE I. Key properties of the effective 2D dirty Dirac Bogoliubov-de Gennes Hamiltonian for quasiparticles in a d-wave superconductor. The number of nodes coupled refers to elastic impurity scattering between and/or within nodes. The generic model is expected to Anderson localize at all energies. The models restricted to disorder that couples only 2 or 1 node(s) are all effectively topological, i.e. describe the surface states of 3D bulk TSCs. All of the restricted models have the low-energy thermal conductivity given by Eq. (2.10), independent of disorder [39, 64, 65] and interactions [66]. The exponent $x_1/z$ governs the low-energy scaling of the density of states [Eq. (2.13)]. The restricted models all exhibit “multifractal stacking” of quantum critical wave functions throughout the energy spectrum. The parameter $\theta(\varepsilon)$ characterizes the multifractal spectrum of wave functions according to Eq. (2.15), near energy $\varepsilon$. The stacked critical states for classes CI and AIII apparently belong to the spin and integer quantum Hall plateau transitions (SQHPT and IQHPT, respectively). In the last entry, TQHPT refers to the thermal quantum Hall plateau transition; see Appendix C for a discussion.

| # of nodes coupled | Class | Effective winding # $|\nu|$ | $x_1/z$ | $\theta(\varepsilon = 0)$ | $\theta(\varepsilon \neq 0)$ | Dirt type(s) |
|-------------------|-------|-----------------|--------|-----------------|-----------------|----------------|
| 4                 | CI    | N/A             | 1/5    | N/A             | N/A             | Vector, potential, (localized) SU(2) vector |
| 2                 | CI    | 2               | 1/7    | 1/4             | $\simeq 1/8$ (SQHPT) | SU(2) vector |
| 1                 | AIII  | 1               | $1-\lambda A$ | $\lambda A$ | $\simeq 1/4$ (IQHPT) | U(1) vector |
| 1                 | DIII  | 1               | 1 (clean) | 0 (clean)   | $\simeq 1/13$ (TQHPT?) | Velocity/QGD |

Eq. (2.15), with $\theta = \lambda A$.  

For a review of this model in the context of TSCs and its higher-winding number generalizations, see e.g. Ref. 5.

At finite energy, the eigenstates of Eq. (2.16) form a stack of quantum-critical wave functions as in the CI case. These were expected to reside at the ordinary class A integer quantum Hall plateau transition (IQHPT) [39], and this result has been confirmed numerically in Refs. 37, 45. The IQHPT has an approximately parabolic multifractal spectrum as in Eq. (2.15), with $\theta \simeq 1/4$ [29, 53]. The spectrum-wide “stacked” IQHPT multifractality is quite strong. Unlike the $|\nu| = 2$ class CI model, the low-energy class AIII model predictions for the DoS and multifractal spectrum depend on the nonuniversal parameter $\lambda A$. When the latter is strong enough to render the finite-energy states critical over a length scale that is not too large, one would expect to see multifractality extending all the way down to zero energy [Eq. (2.18)], as well as a nonlinear enhancement of the DoS [Eqs. (2.13) and (2.17)]. Neither of these features are seen in STM data on BSCCO [16].

4. One node, gravitational disorder due to spatial gap inhomogeneity: class DIII TSC surface states and spectrum-wide criticality

The simplest possible model neglects all forms of potential scattering. However, even in this case it is possible for disorder to produce a nonzero effect. In particular, a slow spatial modulation of d-wave the gap amplitude $\Delta_0 = \Delta_0(r)$ should induce spatial modulation of $v_\Delta \sim \Delta_0/k_F$ in Eq. (2.6), referred to in the sequel as QGD. Because the disorder couples to an operator with a spatial derivative, it is formally irrelevant in an RG sense at zero energy, in contrast with the potential perturbations in Eqs. (2.7) and (2.8). The latter are marginal (at tree level); these conclusions assume short-range correlated disorder. As a result, for the QGD-only model the low-energy DoS $\nu(\varepsilon) \sim |\varepsilon|$ as in the clean limit, and the states near zero energy are not multifractal. Although velocity disorder modifies the definition of the (spin) current operator, the low-temperature thermal conductivity should be unchanged from the clean Landauer result in Eq. (2.10), due to the irrelevance of disorder near $\varepsilon = 0$.

The effects of such velocity disorder at finite $\varepsilon \neq 0$ are the main focus of this paper; the setup and results are discussed in the next section. Formally, a single node with QGD resides in class DIII, and could be realized as a dirty Majorana cone on the surface a bulk TSC (such as the candidate material Cu$_x$Bi$_2$Se$_3$ [26]). This is discussed in detail in Appendix B.

A summary of the results discussed in this section appears in Table I.
III. 2D DIRAC FERMIONS WITH QUENCHED GRAVITATIONAL DISORDER: RESULTS

A. Model and applications

A single 2D Dirac fermion with quenched velocity disorder can be represented by the action

\[ S = \int dt \, d^2r \left[ \bar{\psi} i \partial_\tau \psi + \frac{1}{2} \sum_{\alpha=1,2} v_\alpha(r) \left( \bar{\psi} i \slashed{\partial}^\alpha \partial_\alpha \psi \right) \right], \tag{3.1} \]

where \( \{ \hat{\sigma}^{1,2} \} \) denote the Pauli matrices in the usual basis, \( r = \{ x_1, x_2 \} \), \( \hat{A} \partial B = A \partial B - (\partial A)B \), and the field \( \psi = \psi_\sigma \) is a 2-component Dirac spinor \( (\sigma \in \{ \uparrow, \downarrow \}) \). The action can be interpreted in terms of fermions propagating through curved spacetime, with a static metric given by \( g_{\mu\nu}(r) = \text{diag} \left( -v_1(r), v_2(r), \frac{v_3(r)}{v_4(r)} \right) \). Eq. (3.1) obtains from \( g_{\mu\nu}(r) \) and from the covariant action (see Appendix A)

\[ S = \int |g|^{\frac{1}{2}} d^3x \bar{\psi} E_{\hat{A}}^\mu \bar{\sigma}^A (i \partial_\mu - \frac{1}{2} \omega_{\mu}^{BC} \hat{S}_{BC}) \psi, \tag{3.2} \]

where \( \mu \in \{ t, x_1, x_2 \} \) and \( A, B, C \in \{ 0, 1, 2 \} \); repeated indices are summed. Here \( \sqrt{|g|} d^3x \) is the volume measure, \( \{ \hat{\sigma}^A \} \) are the gamma matrices, \( E_{\hat{A}}^\mu \) is the “dreibein,” \( \omega_{\mu}^{BC} \) is the spin connection, and \( \hat{S}_{BC} \) generates local Lorentz transformations \[22\]. Since the velocity modulation enters through the effects of spacetime curvature in Eq. (3.2), we alternatively refer to this as “quenched gravitational disorder” (QGD). The gauge-invariant content of the disorder can be characterized via the induced curvature, as shown in Appendix A.

QGD can affect massless 2+1-D Dirac carriers whenever the latter arise from a correlation gap. Strong spatial inhomogeneity has been observed in gap maps of the superconducting and pseudogap regimes in the \( d \)-wave cuprate superconductor BSCCO \[11–16\]. This should in turn imply the modulation of the nodal quasiparticle states \( \{ \bar{\psi}_{\sigma} \} \) for a review of these models and scenarios. We only emphasize two points here. First, the linear low-energy density of states and the coherent, plane-wave-like nature of the lowest-energy quasiparticles observed in STM studies \[13\] appear inconsistent with strong internode scattering. Second, the experimentally-observed thermal conductivity \[59, 60\] is not consistent with localization; in fact, the “universal result” in Eq. (2.10) is predicted to hold for any topologically restricted model of scattering, including Eq. (3.1) (see Sec. II B).

Eq. (3.1) could be also realized on the surface of a class DII topological superconductor (TSC) \[46, 49\] with winding number \( |\nu| = 1 \) \[21, 31\]. In this case \( \psi \) is a real Majorana field with \( \psi_\sigma = i\psi_\sigma (\hat{\sigma}^1)_{\sigma,\bar{\sigma}} \), and one can show that (see Appendix B)

\[ v_1(r) = v_2(r) = v_0 \left( 1 + \theta \left[ e^{A_0}(r)/E_{\text{bulk}} \right] \right), \tag{3.3} \]

where \( v_0 \) is the bare surface Majorana velocity, \( A_0(r) \) is the screened electric potential due to (e.g.) static charged impurities, \( \theta \) is a constant, and \( E_{\text{bulk}} \) is the bulk gap energy of the TSC. The disorder can be characterized by a variance \( \lambda \propto n_{\text{imp}} \left[ e^2/(k_F r) \right]^2 \), where \( n_{\text{imp}} \) is the surface areal impurity density, and \( k_F (k_T) \) is the bulk Fermi (Thomas-Fermi screening) wave vector. In units such that \( v_0 = 1 \), \( \lambda \) is a squared-length; weak, short-range correlated QGD is therefore strongly irrelevant \[18\] at zero energy on the surface. Low-energy states are thus expected to be only weakly affected by QGD \[21\]. The fate of the finite-energy states is not obvious, however, since energy is itself a strongly relevant perturbation \[38, 39\].

B. Density of critical states (DoCS) and disorder-strength scaling

In order to work directly in the continuum, we diagonalize Eq. (3.1) exactly in momentum space.

\[ \hat{h}_{k, k'} = \begin{pmatrix} 0 & (k_x - i k_y) \nu \\ (k_x + i k_y) \nu & 0 \end{pmatrix} \delta_{k, k'} \]

\[ + \left( \frac{k_x + k'_x}{2} \right) \sigma^1 P_1 (k - k') \]

\[ + \left( \frac{k_y + k'_y}{2} \right) \sigma^2 P_2 (k - k'). \tag{3.4} \]

We set the parameter \( \nu = 1 \) in Eq. (3.4), appropriate to the relativistic clean system. A higher odd-integral value of \( \nu \in \{ 3, 5, \ldots \} \) can be used to represent a 2D class DII Majorana surface fluid that arises from a bulk topological superconductor with corresponding winding number \( \nu \) \[50\]. We used Eq. (3.4) to analyze the multifractal spectra of low-energy surface states with \( \nu \in \{ 3, 5, 7 \} \) in Ref. \[55\], where we tested predictions of the class DII SO(2n)\( _\nu \) \( (n \to 0) \) conformal field theory expected to describe the zero-energy surface states in these cases \[3\].

The velocity components in Eq. (3.4) are \( v_\alpha(r) = 1 + P_a(r) \), where the impurity potential is taken to be a composition of random phases in momentum space: \( P_a(k) = \left( \sqrt{\lambda}/L \right) \exp \left[ i \theta_a(k) - k^2 \xi^2 / 4 \right] \). Here \( \theta_a(-k) = -\theta_a(k) \), but these are otherwise independent, uniformly
distributed phase angles. We choose a short correlation length so as to approximate white noise disorder, appropriate (e.g.) to model the nanoscale gap inhomogeneity observed in the cuprates [10], or efficient screening of Coulomb impurities on the surface of a TSC: $\zeta \equiv (0.25)(2\pi/\Lambda)$, where $\Lambda$ is the momentum cutoff. Disorder becomes “strong” when the local variance of $P_n(r)$ in position space ($\equiv \Delta P$) becomes of order one. Disorder beyond this threshold regularly tips the velocity components through zero, which creates curvature horizons [20] [see Eq. (A7)]. Since $\Delta P = \sqrt{\lambda/(2\pi\zeta^2)}$, this corresponds to the condition $\lambda \equiv \tilde{\lambda}(\Lambda/2\pi)^2 = \pi/8 \simeq 0.393$. Here $\lambda$ denotes the dimensionless disorder strength.

Representative plots of the LDoS for states at different energies appear in Fig. 2 Wave function quantum criticality is characterized by the spectrum of exponents $\tau(q)$, reviewed in Sec. [11B2] [Eqs. (2.14) and (2.15)]. We calculate the multifractal spectrum $\tau(q)$ by the usual box-counting method; the reader is referred to Ref. [38] for technical details. In order to quantify the degree of criticality throughout the energy spectrum, we employ the following criterion. We compare the computed $\tau(q)$ spectrum for every state in regularly spaced energy bins to a quadratic ansatz [38], $\tau(q) = 2(q - 1)(1 - q^2_f)$ for $|q| \leq q_c$. We employ the “fitness” criteria, defined as follows [38]. For each eigenstate $\psi(r)$, we compute the error between the numerical spectrum $[\equiv \tau_N(q)]$ and the appropriate analytical prediction $[\equiv \tau_A(q)]$, error($q$) = $|\tau_N(q) - \tau_A(q)|/\tau_A(q)$. If the error is less than or equal to 4% for 85% of the evaluated $q$-points in the interval $0 \leq q \leq q_c$, we keep the state. We consider bins of 36 states each with consecutive eigenenergies.

![FIG. 4.](image)

FIG. 4. (a) The total density of states (DoS) for the dirty system described in Fig. 2, the density of critical states (DoCS), and the DoS histogram for the clean system. The DoCS counts the number of states with critical statistics (multifractal spectra) that match a universal ansatz with a certain fitness criterion. Also plotted is the second IPR $P_2$ (green dots), which only becomes appreciable in the high-energy Lifshitz tail. (b) The anomalous multifractal spectrum $\Delta(q) \equiv \tau(q) - 2(q - 1)$ for a narrow energy bin of states, selected from the DoCS with the highest percentage of critical states. The solid red curve denotes an average over 36 states with consecutive energy eigenvalues; the shaded red region indicates the standard deviation. The blue dashed curve is a parabolic ansatz for $\Delta(q)$. States contributing to the critical count in (a) match the ansatz within a certain threshold (see text) over the range $0 \leq q \leq q_c = 5.1$.

We empirically choose $q_c = 5.1$ for the parabolic ansatz; this corresponds to $\theta \simeq 1/13$ in Eq. (2.15). We exclude negative moments $q < 0$ from the fitness criterion, since evaluating these accurately requires significant coarse-graining; for this reason negative moments are typically not reported. We are unable to determine if the deviation seen between the ansatz and the data for $q < 0$ in Fig. 4(b) is intrinsic, or simply a finite-size...

![FIG. 5.](image)

FIG. 5. Total DoS and DoCS as in Fig. 4(a), but for four different dimensionless disorder strengths $\lambda$. Strong disorder corresponds to $\lambda \simeq 0.393$ (see text). The DoCS is computed by evaluating the fitness of every state residing in narrow energy bands of 36 states each, selected at regular intervals across the energy spectrum; it exhibits sample-to-sample variations. In (a), we plot the average DoCS (red dotted line) and the standard deviation (red shaded area) for 10 disorder realizations in each panel, computed for the smaller system size $N = 36$. Despite disorder-dependent fluctuations, the trend is clear: more of the spectrum becomes more critical with increasing disorder strength. In (b), we show results for the same disorder strengths, but now for typical disorder realizations and $N = 54$. The states near zero energy and at intermediate energies where the DoCS significantly deviates from the DoS, especially for weaker disorder strengths, are more weakly multifractal (sub-critical) than those included in the critical count. More strongly multifractal (super-critical) and/or localized states appear only in the high-energy Lifshitz tail, as indicated by the second IPR $P_2$ (green dots), which remains small outside the tail. This is consistent with the absence of Anderson localization in the main spectrum [20].
Anomalous multifractal spectra as in Fig. 4(b), but for the four different disorder strengths corresponding to the DoS/DoCS plots shown in Fig. 5. In each case, the spectrum is plotted for a narrow energy bin chosen so that the DoCS is maximized, in a given realization of the disorder. Results are shown for typical realizations (without disorder-averaging). The solid red line is the energy-average over spectra obtained from 36 consecutive eigenfunctions, while the shaded red region denotes the standard deviation over the energy bin.

We define the density of critical states (DoCS) as the number of states within an energy bin satisfying the above criterion. The ratio of the DoCS to the total density of states (DoS) is the effective energy-resolved distribution function for critical states. The DoS and the DoCS are shown in Fig. 4 along with representative multifractal spectra. In Fig. 5 we exhibit the DoS and DoCS for four different strengths of the disorder and two different system sizes. The main observation is that increasing the disorder makes more of the spectrum critical. States at low and intermediate energies in Fig. 5 that show the largest deviation between the DoCS and DoS are more weakly multifractal than the critical ansatz. The percentage of such sub-critical states in the middle of the spectrum decreases with increasing disorder strength. Super-critical and/or Anderson localized states are always confined to the high-energy Lifshitz tail, which extends well above the clean energy cutoff. The anomalous multifractal spectrum selected from an energy bin where the ratio of the DoCS to DoS is maximized is plotted for the same four disorder strengths in Fig. 6.

C. Plane-wave to critical-state crossover energy “Δ₀” versus disorder strength

We have chosen to study a model with on-average isotropic QGD. I.e., both components of the Dirac velocity \( v_{x,y}(\mathbf{r}) \) in Eq. (3.1) are taken to be random variables with the same average and variance. A more microscopically faithful model of QGD in the cuprates would take only one component of the velocity to be random, \( v_\Delta(\mathbf{r}) = \Delta(\mathbf{r})/k_F \), which controls the dispersion along the bare Fermi surface (see Figs. 4 and 5).

A key observation in STM studies of BSCCO in the optimal to underdoped regime is the qualitative separation between local density of states (LDoS) spectra obtained below and above a weakly doping-dependent scale \( \Delta_0 \). At energies below (above) \( \Delta_0 \), the LDoS maps exhibit robust energy-dispersing quasiparticle interference (strong energy-independent, nanometer-scale spatial inhomogeneity), interpreted as separating “coherent” low-energy quasiparticle states from “incoherent” intermediate-energy excitations [15, 16].

In our on-average isotropic QGD model, we can try to compute the scale \( \Delta_0 \) in terms of the energy cutoff \( \Lambda \), as a function of the dimensionless disorder strength \( \lambda \). We define \( \Delta_0 \) as the threshold where the DoCS becomes larger than a certain percentage of the total DoS. We choose the (arbitrary) criterion

\[
\text{DoCS} = (10\%) \, \text{DoS} \tag{3.5}
\]

in order to define \( \Delta_0 \), measured relative to the Dirac point. Results are shown in Fig. 5. We find a weak decrease of \( \Delta_0 \) with increasing disorder strength. This is qualitatively similar to cuprate data [15, 16] with increased underdoping, if the magnitude of the fluctuations in \( \Delta_1(\mathbf{r}) \) is taken as a proxy for the effective disorder strength (instead of the doping level).

We emphasize, however, that as with previously observed instances of critical wave function stacking at the surface of bulk topological superconductors [38, 45], we would generally expect that \( \Delta_0 \to 0 \) in the system size \( N \to \infty \) limit for any fixed, nonzero disorder strength \( \lambda \), assuming that the critical fluctuations of eigenstates are analyzed on sufficiently large scales. Our system sizes are too limited to extract a clear trend of \( \Delta_0 \) with \( N \) (see Fig. 5 below). This does not invalidate the concept of \( \Delta_0 \) at finite \( N \), or for analyzing STM LDoS maps over a finite sample area. A more detailed comparison of the critical crossover that can occur as a function of length scale versus energy, between low-energy and finite-
FIG. 8. Total DoS and DoCS as in Figs. 4 and 5 but for six different system sizes. The dimensionless disorder strength is not 0.240, as in Figs. 2 and 4. Strong disorder corresponds to λ ≃ 0.393. Results are presented for typical realizations of the disorder. The main effect of increasing the system size for fixed disorder strength is to convert more states near the middle of the spectrum from sub-critical to critical multifractality.

energy critical states with different statistics, is presented for surface states of a class AIII TSC in Ref. 15.

D. System-size scaling of the DoS and DoCS

As emphasized above, QGD is strongly irrelevant near the Dirac point. For an electrically charged Dirac field ψ, kinetic theory predicts a divergent dc conductivity for all nonzero temperatures 80. The finite-energy critical-state stack found here instead suggests a finite dc conductivity.

On the other hand, conventional symmetry arguments imply that the finite-energy states of a class DIII model should reside in the symplectic class AII 29, see Appendix C. This class can exhibit weak antilocalization, leading to a “supermetallic” phase in 2D 30, 31. In a finite-size system, it is therefore not surprising to find multifractal wave functions in 2D, since the scaling is only logarithmic in the system size. However, the curvature of the multifractal spectrum in that case should be related to the bare disorder strength. Moreover, more of the spectrum should become more weakly multifractal with increasing system size.

The leading order anomalous multifractal spectrum for a 2D symplectic metal is given by

$$\Delta(q) \equiv \tau(q) - 2(q - 1) = \frac{1}{8\pi^2G(L)}q(1-q), \quad (3.6)$$

where the dimensionless conductance $G(L)$ scales according to weak antilocalization 82

$$G(L) = G_0 + \frac{1}{2\pi^2} \log \left( \frac{L}{l_{el}} \right). \quad (3.7)$$

Here $L$ is the system size and $l_{el}$ denotes the elastic impurity scattering length. The bare conductance $G_0 = \nu(\varepsilon)D(\varepsilon)$, where $\nu(\varepsilon)$ is the density of states and $D(\varepsilon)$ is the diffusion constant, at the single-particle energy $\varepsilon$. Fixing the disorder strength fixes $G_0$ at each energy. Increasing the system size $L$ should then reduce multifractality according to Eq. (3.6), as the system flows slowly towards a “supermetallic” phase at infinite $L$.

We plot the evolution of the density of states (DoS) and density of critical states (DoCS) as a function of system size in Fig. 8. Results are presented for typical realizations of the disorder, with the same dimensionless disorder strength $\lambda = 0.240$ utilized to obtain Figs. 2 and 4. The main observation is that more of the spectrum at intermediate energies becomes critical (instead of plane-wave-like) with increasing system size. This is the opposite behavior expected from the symplectic metal class AII in Eqs. (3.6) and (3.7). Instead, the results shown in Fig. 8 indicate that more of the spectrum saturates to the universal, critical multifractal spectrum shown in Fig. 4(b). Finally, in Fig. 9 we show the finite-size scaling of two particular multifractal dimensions with system size. At each system size $N$, the dimensions are computed from an energy bin wherein the DoCS is maximized relative to the DoS.

E. Discussion and open questions

We have uncovered a second instance of quantum-critical wave function stacking 28, due here to QGD. In both Ref. 28 (class CI) and the present paper (class DIII), the stacking occurs in disordered Dirac models that can describe the 2D surface states of 3D topolog-
tical superconductors. A third instance (for topological superconductor surface states in class AIII) was touched upon in Ref. [37], and will be extensively explored in Ref. [15].

We argued that QGD naturally obtains whenever 2D Dirac quasiparticles arise from a spatially inhomogeneous gap. The critical-state stack found in this paper might provide a simple explanation for the division between plane-wave-like and spatially inhomogeneous LDoS fluctuations measured at low and finite energies, respectively, in the high-$T_c$ cuprates.

Avenues and questions for future work include the following. (1) One should incorporate velocity anisotropy, wherein only one component of the velocity $v_\Delta = \Delta/k_F$ is made random, see Figs. 1 and 3. (2) An immediate question regards transport versus temperature in the Dirac model with QGD. In particular, could the transition from “ballistic” low-energy states to critical, finite-energy ones explain the ubiquitous linear-in-$T$ resistivity observed in the strange metal phase above $T_c$? (3) What is the role of dephasing on transport? (4) Does the multifractal-stacking phenomenon enhance superconductivity in a self-consistent calculation of the gap?

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Appendix A: Random velocity modulation from Dirac fermions in curved spacetime

For the static metric given by $g_{\mu\nu}(r)$ below Eq. (3.1), the dreibein in Eq. (3.2) can be chosen as

$$E^\mu_A = \frac{\delta^\mu}{\phi_{\mu}(r)}, \quad \phi_0 = \sqrt{v_1(r) v_2(r)}, \quad \phi_1 = \phi_2^{-1} = \sqrt{\frac{v_2(r)}{v_1(r)}}. \quad (\text{A1})$$

We choose a basis for the gamma matrices,

$$\{\gamma^0, \gamma^1, \gamma^2\} = \{\sigma^3, i\sigma^2, -i\sigma^1\} \quad \Rightarrow \quad \hat{\gamma}^a \hat{\gamma}^a = -2 \eta_{\mu\nu} \hat{\gamma}^\mu \hat{\gamma}^\nu.$$

After absorbing the matrix $\gamma^0$ into $\bar{\psi}$ and using the fact that $(\gamma^0)^2 = 1$, the action in Eq. (3.2) reduces to

$$S = \int dt d^2r \left\{ \bar{\psi} i \partial_r \psi + \bar{\psi} \left[ \sum_{a=1,2} v_a(r) \hat{\sigma}^a \partial_a \right] \psi - \frac{1}{2} \bar{\psi} \left[ \omega^\mu_0(\mathbf{r}) + \sum_{a=1,2} v_a(r) \omega^\mu_a(\mathbf{r}) \hat{\sigma}^a \right] \bar{\gamma}^0 \right\}. \quad (\text{A2})$$

The spin connection merely plays the role of a counterterm to ensure Hermiticity of the single-particle Hamiltonian. Consider for simplicity the case with $v_1 = v_2 \equiv v(r)$, so that $\phi_1 = \phi_2 = 1$ and $\phi_0 = v$. Then the non-vanishing Christoffel symbols are

$$\Gamma^a_{00} = v(r) \partial_a v(r), \quad \Gamma^0_{0a} = \Gamma^0_{a0} = v^{-1}(r) \partial_a v(r). \quad (\text{A3})$$

The spin connection is

$$\omega^\mu_\nu A_B = E^A_\mu \Gamma^\lambda_{\mu A} E^\lambda_B - \left( \partial_\mu E^A_\lambda \right) E^\lambda_B = \Gamma^A_{\mu B} \frac{\phi_A}{\phi_B} - \delta^A_B \partial_\mu \frac{\phi_A}{\phi_B}, \quad (\text{A4})$$

the only nonzero components of which are

$$\omega^0_{0a} = -\omega^a_0 = \partial_a v(r). \quad (\text{A5})$$
Then Eq. (A2) reduces to
\[
S = \int dt d^2r \left\{ \bar{\psi} i \partial_t \psi + \bar{\psi} \sum_{a=1,2} [v(r) \hat{a}^\dagger \partial_a - \partial_a v(r) \hat{S}_{0a}] \psi \right\}.
\]

(A6)

Finally, using
\[
\hat{S}_{0a} = \frac{i}{4} [\hat{c}_0, \hat{z}_a] = -\frac{i}{2} \hat{a}^\dagger a,
\]
we can integrate by parts to get Eq. (3.1), specialized to \( v_1 = v_2 = v \).

In Eq. (3.1), the “Berry phase” term \( \bar{\psi} i \partial_t \psi \) has a homogeneous coefficient, corresponding to the physically flat spacetime hosting the Dirac material. Nevertheless, the Ricci tensor associated to \( g_{\mu\nu} \) is expected to form at the surface of a class DIII topological superconductor.

\[ R = -2v^{-1} \nabla^2 v. \]

(A7)

Appendix B: “Gravitational” coupling of electric potentials to the surface Majorana fluid of a class DIII topological superconductor

In this section we derive the form of the velocity modulation in Eq. (3.3), corresponding to the effect of an electric potential \( A^0(r) \) on the 2D Majorana fluid expected to form at the surface of a class DIII topological superconductor.

1. Bulk and surface states for solid-state \(^3\text{He}-B\)

As a simple model for a class DIII bulk topological superconductor with winding number \( |\varphi| = 1 \), we consider a solid-state analog of \(^3\text{He}-B\) [20, 47]. The bulk Bogoliubov-de Gennes Hamiltonian features isotropic \( \sigma \cdot k \) pairing, where \( \hat{\sigma} = \sigma^a \hat{e}_a \) is the vector of Pauli matrices acting on the physical spin-1/2 components \( \sigma \in \{\uparrow, \downarrow\} \), and \( a \) is summed over \( \{1, 2, 3\} \). The mean-field Hamiltonian is
\[ H = \frac{1}{2} \int k \chi^\dagger(k) \hat{h}(k) \chi(k), \]
\[ \hat{h}(k) = \left( \frac{k^2}{2m} - \mu \right) \hat{\tau}^3 + \Delta (\hat{\sigma} \cdot k) \hat{\tau}^2, \]
where \( \mu > 0 \) is the chemical potential and \( \Delta \) is the \( p \)-wave pairing amplitude. Here we have introduced the Balian-Werthammer (“Majorana”) spinor
\[ \chi(k) \equiv \begin{bmatrix} c(k) \\ \sigma^2 \hat{c}^\dagger(-k) \end{bmatrix}^T, \quad \chi^\dagger(k) = i \chi^T(-k) \hat{M}_p. \]

(B2)

The Pauli matrices \( \{\hat{\tau}^{1,2,3}\} \) act on particle-hole space. Particle-hole \( P \), time-reversal \( T \), and chiral \( S \) \((= T \times P)\) symmetries are defined via
\[ P : \quad -\hat{M}_p^{-1} \hat{h}^\dagger(-k) \hat{M}_p = \hat{h}(k), \]
\[ T : \quad \hat{M}_T^{-1} \hat{h}^\dagger(-k) \hat{M}_T = \hat{h}(k), \]
\[ S : \quad -\hat{M}_S \hat{h}(k) \hat{M}_S = \hat{h}(k), \]
(B3a)
\[ \hat{M}_p = \hat{\sigma}^2 \hat{\tau}^2 = \hat{M}_p^T, \quad (P^2 = +1), \]
\[ \hat{M}_T = i \hat{\sigma}^2 \hat{\tau}^3 = -\hat{M}_T^T, \quad (T^2 = -1), \]
\[ \hat{M}_S = \hat{\tau}^1, \]
(B3b)
consistent with class DIII [3][8].

As in [85], we implement hardwall boundary conditions at \( z = 0 \) in order to get surface states. Eq. (B1) separates into a \( k = 0 \) piece, and a nonzero \( k \) piece, where \( k \equiv \{k_x, k_y\} \) now accounts only for conserved transverse momenta.
\[ \hat{h} = \hat{h}_0 + \hat{h}_1, \]
\[ \hat{h}_0 = \left( -\frac{1}{2m} \partial_z^2 - \mu \right) \hat{\tau}^3 + \Delta (\hat{\sigma} \cdot k) \hat{\tau}^2, \]
\[ \hat{h}_1 = \left[ \frac{k^2}{2m} - eA^0(r) \right] \hat{\tau}^3 + \Delta (\hat{\sigma} \cdot k) \hat{\tau}^2. \]

(B4)
\[ \langle z|\varphi \rangle = \frac{1}{\sqrt{\mathcal{N}_0}} e^{-m\Delta z} \sin \left[ z\sqrt{2m\mu - m^2\Delta^2} \right], \quad \mathcal{N}_0 \]
(B6)

where \( \mathcal{N}_0 \) is a normalization factor.

2. \( k \cdot p \) perturbation theory

An effective Hamiltonian for the surface theory at \( k \neq 0 \) obtains by taking matrix elements of the following
operator between the $m_s = \pm 1$ zero modes in Eq. (B5),
\[ \hat{h}_s = \hat{h}_1 - \hat{h}_1 \hat{P}_1 \hat{h}_0^{-1} \hat{P}_1 \hat{h}_1 + \ldots \]
\[ = \hat{h}_1 \]
\[ - \sum_{m_s = \pm 1} \int_0^\infty \frac{dq}{2 \pi E_q} \hat{h}_1 \left[ |\psi_{q,m_s}\rangle \langle \psi_{q,m_s}| - \hat{\tau}^1 |\psi_{q,m_s}\rangle \langle \psi_{q,m_s}| \hat{\tau}^1 \right] \hat{h}_1 \]
\[ + \ldots, \] (B7)
where we have expanded the second term via the $k = 0$ resolution of the identity. The operator $\hat{P}_1$ on the first line is the projection out of the degenerate eigenspace of the zero modes, $\hat{P}_1 = 1 - \hat{P}_0$, where
\[ \hat{P}_0 = \sum_{m_s = \pm 1} |\psi_{0,m_s}\rangle \langle \psi_{0,m_s}|. \]
The state $|\psi_{q,m_s}\rangle$ is a positive-energy (gapped) bulk eigenstate of $\hat{h}_0$, parameterized by the standing wave momentum $q$, while $\hat{\tau}^1 |\psi_{q,m_s}\rangle$ is its negative-energy chiral conjugate [Eq. (B3)]. $E_q = \sqrt{(q^2/2m - \mu)^2 + q^2 \Delta^2}$ denotes the positive eigenenergy.

The first term in Eq. (B7) gives the relativistic dispersion for the Majorana surface fluid,
\[ \hat{h}_s^{(1)} = \Delta \hat{\sigma} \wedge k, \] (B8)
where $A \wedge B = A_x B_y - A_y B_x$. This is consistent with the surface projection of the symmetry conditions in Eq. (B3),
\[ P : - (\hat{M}_s^{(5)})^{-1} \hat{h}_s^{(1)} (-k) \hat{M}_s^{(5)} = \hat{h}_s(k), \] (B9a)
\[ T : (\hat{M}_T^{(5)})^{-1} \hat{h}_s^{(1)} (-k) \hat{M}_T^{(5)} = \hat{h}_s(k), \] (B9b)
\[ S : - \hat{M}_s^{(5)} \hat{h}_s(k) = \hat{h}_s(k), \]
where
\[ \hat{M}_s^{(5)} = \hat{\sigma}^1 = (\hat{M}_s^{(5)})^T, \quad (P^2 = +1), \]
\[ \hat{M}_T^{(5)} = i \hat{\sigma}^2 = - (\hat{M}_T^{(5)})^T, \quad (T^2 = -1), \]
\[ \hat{M}_S^{(5)} = \hat{\sigma}^3. \]

Coupling to the vector potential $A^0$ obtains from the second term in Eq. (B7). Working to linear order in the potential, the relevant $+ -$ matrix elements take the form
\[ - \sum_{m_s = \pm 1} \int_0^\infty \frac{dq}{2 \pi E_q} \langle \tau^1 = +1 | +1 \langle \varphi | [-eA^0 \hat{\tau}^3] \left[ |\psi_{q,m_s}\rangle \langle \psi_{q,m_s}| - \hat{\tau}^1 |\psi_{q,m_s}\rangle \langle \psi_{q,m_s}| \hat{\tau}^1 \right] \left[ \Delta \hat{\sigma} \cdot k \hat{\tau}^2 \right] |\tau^1 = -1 \rangle | -1 \rangle | \varphi \rangle \]
\[ + \frac{eA^0}{2} \Delta \hat{k} \int_0^\infty \frac{dq}{\pi E_q} \langle \varphi | \langle \tau^1 = +1 | +1 \langle \psi_{q,+1} | e^{-A^0 \hat{\tau}^3} | \tau^1 = -1 \rangle | -1 \rangle | \varphi \rangle, \] (B10)
and
\[ - \sum_{m_s = \pm 1} \int_0^\infty \frac{dq}{2 \pi E_q} \langle \varphi | \langle \tau^1 = +1 | +1 \langle \psi_{q,-1} | e^{A^0 \hat{\tau}^3} | \tau^1 = -1 \rangle | -1 \rangle | \varphi \rangle \]
\[ = \Delta \hat{k} eA^0 \int_0^\infty \frac{dq}{\pi E_q} \langle \varphi | \langle \tau^1 = +1 | +1 \langle \psi_{q,-1} | \tau^1 = +1 \rangle | \varphi \rangle, \] (B11)

where $k \equiv k_x - ik_y$. Evaluating these leads to perturbation of the form
\[ \hat{h}_s^{(2)} = \frac{\partial}{2} \left[ \frac{eA^0(r)}{E_{\text{bulk}}} \hat{\sigma} \wedge k + \hat{\sigma} \wedge k \frac{eA^0(r)}{E_{\text{bulk}}} \right], \] (B12)
where $E_{\text{bulk}} \simeq k_F \Delta$ is the bulk excitation gap, and $\hat{\sigma}$ is a pure order-one number. Since the bare Majorana fluid velocity is $\Delta$, we recover Eq. (3.3).

**Appendix C: Symmetry class for finite-energy states; Connection to the class D thermal quantum Hall plateau transition (?)**

The 2D velocity-randomized Dirac Hamiltonian given by the sum $\hat{h}_s^{(1)} + \hat{h}_s^{(2)}$ in Eqs. (B8) and (B12) resides in class DIII, due to the $T^2 = -1$ and $P^2 = +1$ time-reversal and particle-hole symmetries encoded in Eq. (B9). These symmetries hold irrespective of whether the fermion field $\psi$ in Eq. (3.1) is a complex-valued Dirac or real-valued Majorana spinor.

Typically, we can associate an effective field theory, the nonlinear sigma model, to describe the wave functions of any single-particle Hamiltonian at some particular fixed energy. The nonlinear sigma model employs local operators to encode the probability statistics of the extended,
critical, or localized states. Using fermionic replicas to perform disorder-averaging, class DIII is associated to a sigma model with the target manifold $O(2n)$, where $n$ is proportional to the number of replicas. In two spatial dimensions, this can be seen via the nonabelian bosonization of the clean, zero-energy Majorana fermion field theory.

For the DIII theory, nonzero energy is a relevant perturbation that couples to the principal chiral field in the nonlinear sigma model. In this case, the $O(2n)$ symmetry of the zero-energy theory is broken down to the diagonal subgroup $O(2n)$. The energy perturbation will induce an RG flow to a new fixed point, which should be associated to a different sigma model with target manifold $O(2n)/H$. There are only two possibilities:

1. Class AII, the “symplectic” class associated to disordered metals with time-reversal symmetry and strong spin-orbit coupling. This class typically exhibits weak antilocalization in two dimensions. Class AII also describes the 2D surface states of 3D topological insulators. The target manifold is

$$G/H = \frac{O(2n)}{O(n) \times O(n)}.$$  

2. Class D, typically associated to superconductors with broken time-reversal symmetry and strong spin-orbit coupling. Class D should be realized in dirty $p + ip$ superconductors. The target manifold is

$$G/H = \frac{O(2n)}{U(n)}.$$ 

We argued in Sec. [I] that class AII appears incompatible with the numerical results obtained here for finite-energy states of the velocity-modulated Dirac Hamiltonian in Eq. (3.1). We therefore expect that our finding of critical states throughout the energy spectrum with universal multifractal spectra should instead be associated to class D.

There are three classes of time-reversal invariant topological superconductors (TSCs) in 3D, differing by the amount of spin rotational symmetry: CI [SU(2)], AIII [U(1)], and DIII (no spin symmetry, strong spin-orbit coupling). There are also three classes of time-reversal broken quantum Hall topological insulators or TSCs in 2D: class C [spin quantum Hall (SQH) effect], class A [ordinary integer quantum Hall (IQH) effect], and class D [thermal quantum Hall (TQH) effect]. All of these classes are characterized by integer (AIII, DIII, A, D) or twice-integer (CI, C) bulk topological winding numbers.

In our previous work, we performed a similar population analysis as presented in this paper for the finite-energy 2D surface states of a 3D class CI topological superconductor. Based on the numerical results obtained there, we concluded that the finite-energy states of the 2D class CI model take the form of a “stack” of critical wave functions with universal multifractal spectra. The spectra are energy independent, and consistent with the plateau transition of the spin quantum Hall effect in class C. A connection between classes CI and C based on symmetry considerations similar to Eq. (C1) was also presented in Ref. [38].

Finite-energy states in class AIII must reside in class A. For the 2D surface states of a 3D class AIII topological superconductor, whether these are critically delocalized or Anderson localized depends upon the presence or absence of a topological theta term (at $\theta = \pi$) in the effective nonlinear sigma model. In a separate work, we will present strong evidence that 2D finite-energy surface states of 3D class AIII TSCs form a “stack” of critical wave functions [14] [see also Ref. [37]]. The spectra are energy independent, and consistent with the plateau transition of the ordinary class A integer quantum Hall effect. For classes CI and AIII, the only alternative to the plateau-transition-stacking scenario is Anderson localization, but this is not observed in our numerical studies.

We therefore expect that the “stack” of critical states found in the present paper can be associated to the thermal quantum Hall plateau transition in class D. In comparison to classes C and A, relatively little is known about the thermal quantum Hall plateau transition. The global phase diagram for a 2D system in class D is complicated by the advent of a thermal metal phase in addition to Anderson localized thermal Hall plateaux. Further studies of the possible multi-critical point in the phase diagram of class D could shed light on the nature of the finite-energy states found here.

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