Quantum encoding is suitable for matched filtering

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Matched filtering is a powerful signal searching technique used in several employments from radar and communications applications to gravitational-wave detection. Here we devise a method for matched filtering with the use of quantum bits. Our method’s asymptotic time complexity does not depend on template length and, including encoding, is $O(L(\log_2 L)^2)$ for a data with length $L$ and a template with length $N$, which is classically $O(NL)$. Hence our method has superior time complexity over the classical computation for long templates. We demonstrate our method with real quantum hardware on 4 qubits and also with simulations.

I. INTRODUCTION

Matched filtering is a powerful technique for detecting a signal with a known waveform in the presence of noise. It maximizes the signal-to-noise ratio (SNR) of the signal which becomes the most powerful test statistic under Gaussian noise [1]. It had been initially developed for radar and communication applications [2] and has been used in various settings for which examples are gravitational-wave detection [3, 4] and quantum tomography via correlation of noisy measurements [5].

For a known time domain sought template signal $\tilde{x}(t)$ and the data stream $y(t)$ in the presence of noise, in time domain SNR can be calculated with a convolution as

$$\rho(t) \propto \int_{-\infty}^{\infty} y(\tau + t)x(\tau)d\tau$$  \hspace{1cm} (1)

where $x(\tau)$ is the inverse Fourier transform of $\frac{\tilde{S}(f)}{2\pi f}$; $S(f)$ being the noise power spectral density and $\tilde{X}(f)$ being the Fourier transform of $\tilde{x}(t)$ . Since SNR is used as a test statistic, any monotonic function of it is as powerful, hence ignoring constants in its calculation doesn’t change the performance.

The computational cost of matched filtering of a data series grows proportional to the length of the data and the length of the searched signal. For concreteness, considering a digitized signal template with $N$ points and a digitized data stream with $L$ points ($N \leq L$), making a search with matched filtering has the computational time cost $O(NL)$ since for every signal start time there are $N$ multiplications and one addition to be made. The SNR at the $j^{th}$ time can be written as

$$\rho(j) = \sum_{i=1}^{N} y_{i+j}x_{i}$$  \hspace{1cm} (2)

Eq. (1) and can be seen as the magnitude of the inner product of two real vectors $< y(\tau + t), x(\tau) >$. Such inner products naturally arise in quantum mechanics. This similarity makes computation of SNRs of a matched filter with the use of quantum bits (qubits) reasonable. Relevantly to the subject of this article, quantum computing was also suggested to be used for finding the best matching template in multi-template matched filtering applications [6].

In the following section we explain a hybrid algorithm which uses classical and quantum computation together and parallelizes the matched filtering computation for real valued signals. This algorithm has time complexity of $O(L(\log_2 L)^2)$, which is superior to the classical computation for long templates $N > (\log_2 L)^2$. The main idea is to use quantum measurements instead of doing multiplications. Next, in the Sec. III, we demonstrate our method with quantum hardware and ideal simulations for a simple configuration, and discuss our results. Finally, in the last section we conclude.

II. QUANTUM MEASUREMENTS FOR MATCHED FILTERING

In this section we describe how quantum measurements can be used for matched filtering. We will discuss only positive valued signal and data and then generalize for negative values. But first we give an introduction for encoding with qubits.

A. Encoding with qubits

Quantum computation is a developing field which aims to perform computations with the help of circuits exhibiting quantum phenomena. The idea is similar to constructing electronic circuits that perform certain operations such as an integrator circuit. The advantage of quantum computation comes from the fact that quan-
Quantum bits (qubits) can be in a superposition of their two states unlike the classical bits which are either 0 or 1. This property expands the computational space exponentially such that $\log_2 n$ qubits can store and manipulate information which could be handled by as large as $n$ classical bits. However, as humans, we need to interact with the qubits to observe their final state. This interaction, which is commonly referred as measurement, collapses the final quantum state to a classical state, according to a probability distribution dependent on its state, which makes impossible to gather the full information at once. Therefore quantum computers are classified as probabilistic Turing machines where as classical computers are classified as deterministic Turing machines.

In order to perform a calculation with data, that data needs to be encoded in the qubits. There are different encoding options [7] although the true power of quantum computation can be used with amplitude encoding where each point of $n$ point data is encoded as an amplitude to the $n$ different states spanned by $\log_2 n$ qubits. Despite the logarithmic gain in the number of qubits, the computational complexity of amplitude encoding scales with $O(n)$ which at first sight seems to negate any gain. However, it has been shown that with the use of extra qubits (typically called as ancilla qubits) during the encoding stage, which are thrown after the encoding is done, the time complexity of amplitude encoding can be reduced. Specifically Ref. [8] discusses a divide and conquer algorithm which uses $O(n)$ qubits to encode $\log_2 n$ qubits with a circuit depth $O((\log_2 n)^2)$. This improvement enables advantageous uses of qubits for computation without losing the advantage in the beginning. We assume this encoding algorithm for the rest of the article.

To illustrate the improvement in the computation capability with our method, first assume the values of the vectors we want to have their inner product are positive and consider the square roots of the values are encoded with amplitude encoding.

$$\text{Data} = |y\rangle = N_{y}^{-1} \sum_{i=1}^{L} \sqrt{y_i} |i\rangle_y,$$  \hspace{0.5cm} (3a)

$$\text{Signal} = |x\rangle = N_{x}^{-1} \sum_{i=1}^{N} \sqrt{x_i} |i\rangle_x$$ \hspace{0.5cm} (3b)

where $|i\rangle$ are the $2^n$ possible basis states obtainable with $n$ qubits, i.e. $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ for 2 qubits. $N_x = \sum_{i=1}^{N} x_i$ and $N_y = \sum_{i=1}^{L} y_i$ are the normalizations.

The overall state with the encoded data and the signal is

$$|\psi\rangle = N_y^{-1}N_x^{-1} \sum_{i=1}^{N} \sum_{j=1}^{L} \sqrt{y_j} \sqrt{x_i} |j\rangle_y |i\rangle_x$$ \hspace{0.5cm} (4)

Now the SNR for a signal starting at the data point $j$ can be computed by obtaining the result

$$\rho(j) = N_y N_x \sum_{i=1}^{N} |\langle i + j | y \rangle |^2 |\langle i | x \rangle |^2$$

$$= \sum_{i=1}^{N} |\sqrt{y_i} + \sqrt{x_i}|^2 = \sum_{i=1}^{N} y_i + x_i$$ \hspace{0.5cm} (5)

So adding up the probabilities of measuring the state $|i + j\rangle_x |i\rangle_y$ for the encoded qubits over all the values of $i$ and for a fixed $j$ value gives the SNR for a search at point $j$ (up to the normalization constants). Moreover, this addition is not needed to be done after probability of each combination is calculated, because when we make a measurement on the qubits any possible outcome may be measured giving information for one of the possible $j$ values. Therefore each repetition of this procedure contains information about all of the $j$ values since the probability of any of them appearing is dependent on the encoded information. Therefore, after the measurements, a parallelized logic circuit can be constructed whose inputs are the measured qubits and which gives binary 1 for the states which are wanted to be add up and 0 otherwise for each $j$ value. For example, for a 2 point signal and 2 point data, there is only one meaningful SNR to be calculated: $x_0y_0 + x_1y_1$. This corresponds to estimating the probability of finding the qubits in the states $|0\rangle_x |0\rangle_y$ or $|1\rangle_x |1\rangle_y$. So the expectation value of $\text{SNR}(q_y, q_z)$ can be directly estimated rather than first estimating $P(|0\rangle_x |0\rangle_y)$ and $P(|1\rangle_x |1\rangle_y)$ and than adding up. There is going to be total of $\log_2 NL$ qubits to be measured and $NL$ possible different outcomes. SNR of

**B. Positive valued signal and data**

Inner products come naturally in quantum mechanics. For example, the probability of finding a state $\psi$ in an eigenstate of an observable $\phi$ is given by $|\langle \phi | \psi \rangle|^2$. This can be exploited for a faster calculation for SNR via matched filtering. Before moving on to our method, we also mention that another possible way of doing this calculation with quantum computing than our method is via a swap test, which directly calculates the overlap of two states $|\langle \phi | \psi \rangle|^2$. However, we could not find a way of employing it with a time complexity improvement over the classical computation. The main bottleneck seemed to be the computation for the design of encoding circuits.

The main advantage in our method comes from the fact that it works only with real valued signals for which no phases beyond their signs exist. That allows us to use measurements more effectively where the phase information is lost. We were inspired by combining quantum and classical logic operations by Ref. [9], in which similar approaches were discovered for equivalent constructions of swap test with the purpose of simplifying the original quantum circuit of the swap test with classical additions.
each data point \( j \) is proportional to the number of measurements of mutually exclusive \( N \) of these possible different outcomes. Therefore depth of the logic circuits constructed with OR gates with 2 inputs are \( O(\log_2 N) \). The expansion of \( \log_2 NL \) qubits to \( NL \) different logic functions can be done with 2 input AND gates and inverters with depth \( O(\log_2 \log_2 NL) \) [10]. Total number of needed AND gates and inverters, without any logical simplifications, e.g. with a Karnaugh map; will be \( O(L \log_2 NL) \) and OR gates will be \( O(NL) \).

The computation time in this case becomes

\[
T[\text{Encoding time} + \text{Measurement time} + O(\log_2(N \log_2 NL)) \\
\times \text{propagation time for logic gates}]
\]

(6)

where \( T \) is the number of measurements (shots) for achieving a necessary precision. For a fixed precision, \( T \) scales with \( L \) since each measurement outcome is distributed to one of the \( L \) possible times. Measurements on qubits can be done simultaneously so, the measurement time per shot is constant. The divide and conquer algorithm mentioned [8] does the amplitude encoding for \( n \) point signal with a circuit with depth \( O((\log_2 n)^2) \) and with width (total necessary qubit count) \( O(n) \). Hence, the encoding time dominates the expression in the parentheses and the total time complexity becomes \( O(L(\log_2 L)^2) \). There are two other computation times in hindsight which are the design of the encoding circuit which is done once and has time complexity \( O(N + L) \) and the square rooting the original values whose time complexity is also \( O(N + L) \). These do not affect the asymptotic behaviour of \( O(L(\log_2 L)^2) \). Compared to the classical computation’s time complexity \( O(NL) \), this method improves the computation time for long templates \( N > (\log_2 L)^2 \).

C. Dealing with negative values

The description in the previous subsection works only for the positive values since we need to encode the square roots of the values. Solution to the negative numbers is to shift the data and signal by the magnitude of their minimum negative value so that all the numbers become positive.

\[
\text{Data} = |y\rangle = N_y^{-1} \sum_{i=1}^{L} \sqrt{y_i + \Delta y} |i\rangle_y,
\]

(7a)

\[
\text{Signal} = |x\rangle = N_x^{-1} \sum_{i=1}^{N} \sqrt{x_i + \Delta x} |i\rangle_x
\]

(7b)

where \( \Delta x = -\min(x_i) \) and \( \Delta y = -\min(y_i) \), with new normalizations \( N_x = \sum_{i=1}^{N} (x_i + \Delta x) \) and \( N_y = \sum_{i=1}^{L} (y_i + \Delta y) \).

Of course, this effect needs to be corrected later when calculating SNR since

\[
\mathcal{N}_y \mathcal{N}_x \sum_{i=1}^{N} |\langle i + j | y \rangle \langle i | x \rangle |^2
\]

\[
= \sum_{i=1}^{N} \sqrt{y_{i+j} + \Delta y} \sqrt{x_i + \Delta x}^2
\]

\[
= \sum_{i=1}^{N} y_{i+j} x_i + \Delta y x_i + \Delta x y_{i+j} + \Delta y \Delta x
\]

(8)

The necessary correction is subtracting the extra terms as

\[
\rho(j) = \mathcal{N}_y \mathcal{N}_x \sum_{i=1}^{N} |\langle i + j | y \rangle \langle i | x \rangle |^2
\]

\[
- \sum_{i=1}^{N} \Delta y x_i + \Delta x y_{i+j} + \Delta y \Delta x
\]

(9)

In this correction only the \( \sum_{i=1}^{N} \Delta x y_{i+j} \) term is not a constant term. Although it seems that \( N \) additions needs to be made for calculating it for every \( L \) values of \( j \), since the difference between the consecutive values of the correction is just 2 numbers (first number of the previous step and the last number of the new step), asymptotically \( O(N + L) \) operations are needed for calculating every \( L \) value of it. Hence, addition of this correction does not affect the previous scaling which was \( O(L(\log_2 L)^2) \).

III. RESULTS

In this section we show our results produced using the quantum hardware and noise-free simulators provided by IBM Quantum through Qiskit [11]. The current state of hardware is mainly limited by the connections between the qubits, two qubit operation error and measurement error. To a lesser degree single qubit operations also have errors. The only available two qubit operation is controlled-not (CNOT) and all the other multi qubit operations are constructed via CNOT gates. Due to these limitations we demonstrate the hybrid computation of SNR with a simple construction which has \( N=2 \) and \( L=4 \). The main obstacle for not having more data points is the significantly greater estimated error in the encoding when data is encoded to more than 2 qubits. The signal length was chosen less than the data length in order to have more than one SNR value calculated with the described method. In this case there are 3 SNR values calculated per one data set. They correspond to the measurement probabilities \( P(|0\rangle_x |00\rangle_y) + P(|1\rangle_x |01\rangle_y) + P(|0\rangle_x |01\rangle_y) + P(|1\rangle_x |10\rangle_y) \) and \( P(|0\rangle_x |10\rangle_y) + P(|1\rangle_x |11\rangle_y) \).
The experiments we show here were ran on the backend ibmq lima. The estimated total CNOT error probability and the total measurement error probabilities were both about 7% which are the main estimated sources of error. We have chosen the 2 point signal template arbitrarily as \([2,-1]\). Each data set of 4 numbers were chosen randomly from a normal distribution of zero mean and unity variance. In order to have all of the possible 8 measurement outcomes, the signal and data were arbitrarily shifted 0.1 more than their minimum negative values, i.e. \(\Delta x = -\min(x_i) + 0.1, \Delta y = -\min(y_i) + 0.1\). Here we show results from 100 different data sets for each \(2 \times 10^4\) measurements (shots) were made. Fig. 1 shows an example circuit from our experiment for the encoding of 3 qubits and their measurements. Qubits q1, q2, and q3 are used for encoding the data into the qubits q1 and q2. The signal template is encoded to q4.

Fig. 2 shows two scatter plots, one for the calculated \(100 \times 3 = 300\) SNRs with quantum measurements vs. the true SNR values and one for the SNR errors between them vs. the true SNRs. The correlation coefficient for the points in the left figure was found to be 0.99, which should have been 1 ideally, and in the right figure as -0.57. The anti-correlation between the errors and the true SNR is a clear indication that the errors have arisen due to the noise in the circuit. This is due to the fact that in order to have an SNR with a high magnitude, either the amplitude of the data should be high or more relevantly the relative magnitudes of the consequent data points need to have specific values. Any noise in the system can corrupt such delicate data segments. These corruptions increase the marginal entropies of encoded signal and data incoherently resulting in decrease in their mutual information. Low SNR points on the contrary do not get affected by such corruptions as their already low mutual information cannot decrease more. Another observation that can be made is the affinity to having positive SNR errors. This can be explained by the asymmetry in the encoding due to shifting to the positive values. Without the correction in Eq. (9), only the positive SNR values can be obtained. Therefore there is a fundamental lower limit on the SNR errors during the encoding. The effect of this lower limit is seen as having mostly positive SNR errors, especially for data segments which have true negative SNR since their SNR values without the correction are the closest to zero. In Fig. 3 we show the result of a noise-free ideal quantum simulation with the same data and signal. The error in this case is only due to the Poisson errors due to finite number of measurements. The correlation coefficient between the SNRs is 1.0 and between the errors and the true SNRs is -0.1.

IV. CONCLUSION

Here we devised a hybrid parallel computation method which uses quantum encoding for matched filtering. This method’s time complexity does not grow with the growing template length. Consequently it is advantageous for long templates. For a template with \(N\) digitized points and a data with \(L\) digitized points, the time complexity of our method is \(O(L \log_2 L)^2\) which is classically \(O(NL)\). In order to achieve this performance, \(O(L + N)\) number of qubits and \(O(NL)\) number of classical logic gates are required. We successfully demonstrated our method on real quantum hardware as well as with simulations for a simple configuration, which is too simple for being practically useful. The hardware runs showed the effects of the noise coupled with the methodology. The limitations we encountered in the hardware for trying more complicated configurations are the CNOT error rates and the limited connectivity between the qubits which increases the required number of CNOT gates for encoding. With the improvements in the hardware, a useful application of this method can be executed in the future.

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FIG. 1. One of the circuits ran in the experiments for encoding.

FIG. 2. Results of the experiments on the quantum backend *ibmq_lima*. Left figure shows a scatter of the 300 SNRs computed with the use of qubits vs the true SNRs. The orange line is $x=y$ just for reference. Right figure shows the dependency on the errors in the computation on the true SNRs.

FIG. 3. Results from an ideal noise free quantum simulation. The orange line is again $x=y$ just for reference. The only source of error is the Poisson uncertainty due to finite number of shots $= 2 \times 10^4$. 