Instanton Flow and Circulation PDF in Turbulence

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The Turbulence in incompressible fluid is represented as a Field Theory in 3 dimensions. There is no time involved, so this is intended to describe stationary limit of the Hopf functional. The basic fields are Clebsch variables defined modulo gauge transformations (symplectomorphisms). Explicit formulas for gauge invariant Clebsch measure in space of Generalized Beltrami Flow compatible with steady energy flow are presented. We introduce a concept of Clebsch confinement related to unbroken gauge invariance and study Clebsch instantons: singular vorticity sheets with nontrivial helicity. This is realization of the "Instantons and intermittency" program we started back in the 90ties. These singular solutions are involved in enhancing infinitesimal random forces at remote boundary leading to critical phenomena. The resulting exponential distribution for PDF of velocity circulation fits the numerical simulations including pre-exponential factor $1/\sqrt{T}$. We revised and extended the investigation of the master equation for a flat loop, which led to the same predictions for PDF but with different intermediate solutions, correcting some errors of the previous papers.

I. INTRODUCTION: STATISTICS AS FIXED POINT OF DYNAMICS

Turbulence is well studied at a phenomenological level using numerical simulations of forced Navier-Stokes equations and fitting the data for distribution of various observables (such as moments of velocity and vorticity fields, as well as velocity circulation). The data suggest multifractal scaling laws implying some significant modifications of traditional Kolmogorov scaling by finite size vorticity structures with nontrivial distributions by shape, size and vorticity filling.

The microscopic theory, such as an effective Hamiltonian for the Gibbs distribution in ordinary critical phenomena, is missing. It is as though we already know the Newtonian dynamics but do not yet know the Gibbs distribution. We can simulate the Navier-Stokes equations and average over time, but we lack basic definitions of stationary statistics for vorticity or velocity fields.

This statistics would be a fixed point of the evolution of the Hopf functional. If we knew such an analog of the Gibbs law, we would be able to solve the theory analytically (at least in some extreme regime such as a large circulation limit for large loops). We would also have powerful Monte-Carlo methods with the Metropolis algorithm for fast simulation of this equilibrium statistics.

In this paper summarizing preprints we are trying to fill this gap. We construct the distribution of vorticity and velocity in three dimensions which is manifestly conserved in Euler dynamics, while describing a steady energy flow. It involves a two-component Clebsch field, as well as two auxiliary fields: one Bose field and one Majorana Grassmann field, both transforming as vectors in physical space $R_3$.

This is a candidate for the Turbulence fixed point of the Hopf evolution, but is it the right one? We can find out by investigating this distribution on theoretical level and comparing it with numerical simulations of the Navier-Stokes equation.

In the same way as with critical phenomena in ordinary statistical physics, we expect Turbulence to be universal, independent on peculiar mechanisms of energy pumping nor the boundary conditions as long as this energy pumping is provided.

In the WKB limit the tails of the PDF for velocity circulation $\Gamma$ over large fixed loops $C$ are controlled by a classical field $\phi_C^\infty(r)$ (instanton) concentrated around the minimal surface bounded by $C$.

The field is discontinuous across the minimal surface which leads to the delta function term for the tangent components of vorticity as a function of normal coordinate. The flux is still determined by the normal component of vorticity, which is smooth.

We study minimal surfaces in great detail in Appendix A of and we derive explicit formulas for the Clebsch instanton in Appendix B of this paper. It has nontrivial topology which we study in Appendices C,D, deserving further investigation by mathematicians.

As for the scaling area law $\Gamma^2 \sim A_C$ that we derived in from consistency of the loop equation, it now follows from simple power counting in the instanton equation.

The surprise here is an explicit form of the circulation PDF (involving two or three phenomenological parameters depending on the symmetry of the loop $C$). This PDF perfectly matches the DNS data at large circulations where this WKB solution applies.

II. ENERGY FLOW FROM THE BOUNDARY FORCES

As is well known, the energy is pumped into the turbulent flow from the largest scales (pipes, ships, etc.), and dissipated at the smallest scales due to viscosity effects. Let us see how that happens in some detail. Using

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Navier-Stokes equation
\[ v_\alpha = \nu \partial^2_\beta v_\alpha - v_\beta \partial_\beta v_\alpha - \partial_\alpha p; \partial_\alpha v_\alpha = 0 \]  
(1)

we have

\[ \partial_t \int d^3 r \frac{1}{2} v_\alpha^2 = \int d^3 r \nu \partial^2_\beta v_\alpha - v_\alpha (v_\beta \partial_\beta v_\alpha + \partial_\alpha p) \]  
(2)

Integrating by parts using the Stokes theorem we reduce this to

\[ \partial_t \int d^3 r \frac{1}{2} v_\alpha^2 = -\nu \int_V d^3 \omega_\alpha^2 + \int_{\partial V} d\sigma_\beta \left( v_\beta \left( p + \frac{1}{2} v_\alpha^2 \right) + \nu v_\alpha (\partial_\beta v_\alpha - \partial_\alpha v_\beta) \right) \]  
(3)

Velocity is related to vorticity by the Biot-Savart law:

\[ v_\alpha(r) = -e_{\alpha\beta\gamma} \partial_\beta \int d^3 r' \frac{\omega_\gamma(r')}{4\pi |r - r'|} \]  
(4)

In case there is no vorticity at the bounding sphere, the radial velocity would decrease as \(1/|r|^3\) at infinity:

\[ v_\alpha^T(r) \rightarrow Q_\beta(\vec{f}) \partial_\alpha \partial_\beta \frac{1}{4\pi |r|} \]  
(5)

with dipole moment \(\vec{Q}\) of vorticity of the thermostat

\[ Q_\alpha(\vec{f}) = \int d^3 r e_{\alpha\beta\gamma} r_\beta \omega_\gamma^T \]  
(6)

We shall assume that there is no global vorticity in our fluid (which is a matter of proper boundary conditions

\[ \int d^3 r' \omega_\gamma(r') = 0 \]  
(7)

Note that this asymptotic flow is laminar and purely potential, as vorticity is left far away from the boundary.\[7\]. For a finite resulting flow after cancellation of powers of \(|r|\), the pressure should behave as:

\[ p(r) \rightarrow |r|g(n) \]  
(8)

where \(n_\alpha\) is the normal vector to the surface. This would correspond to finite force \(\partial_\alpha p \sim 1\) which is distributed on a surface:

The resulting energy flux is:

\[ -\dot{E} \rightarrow Q_\alpha f_\alpha \]  
(9)

\[ f_\alpha = \lim_{|V| \rightarrow \infty} \int_{r \in \partial V} \frac{d\sigma(r)}{4\pi |r|^3} n_\alpha(r)p(r) \]  
(10)

This formula works for a generic bounding surface \(S = \partial V\), in a limit when it is blown up to infinity. For a sphere, it becomes an average over unit vectors \(n \in S_2\):

\[ f_\alpha = \int_{n \in S_2} \frac{d^2 n}{4\pi} n_\alpha g(n) = (n_\alpha g(n))_{n \in S_2} \]  
(11)

This random force \(f_\alpha\) would have some unknown PDF depending upon the specific microscopic mechanism of energy pumping:

\[ dP(\vec{f}) = P(\vec{f})d^3 f \]  
(12)
The natural assumption is that this PDF is Gaussian, in accordance with the Central Limit Theorem for an average of large number of uncorrelated forces on a surface of a remote sphere.

The asymptotic behavior of the normal component of velocity is parameterized by the above vector \(Q\) in [5].

Note that while the dissipation part has the space integral supported on high vorticity regions, the incoming energy flow is concentrated on the bounding surface.

The energy balance requires that these net energy flow is equal to zero. All the energy pumped from the boundary dissipates by viscosity at small scales inside vorticity cells. This provides the relation between vorticity distribution and the random force:

\[- \int_V d^3r \nu \omega_3(r) + f_\alpha \int_V d^3r e_{\alpha\beta\gamma} r_\beta \omega_\gamma(r) = 0 \quad (13)\]

### III. CANONICAL VS MICROCANONICAL ENSEMBLE

It is significant that this relation involves distribution of vorticity in the cell, where all dissipation is taking place. The second term comes from the flow through the boundary at infinity, but it involves the vorticity inside the cell. All the boundary conditions at infinity are represented by a (Gaussian) random vector force \(\mathbf{f}\).

Note that this is not the same as postulating the energy spectrum of the pumping forces. We have only one vector with Gaussian distribution with some unknown variance.

In the following section, we are going to add this constraint not as a delta function in microcanonical distribution, but rather as exponential factor, inserted in canonical distribution with corresponding Lagrange multiplier \(\lambda\). The motivation is the same as in statistical mechanics. We assume there is a "thermostat" interacting with a subsystem, with subsystem exchanging energy flow with "thermostat".

This is not the Gibbs distribution, of course, and the term "thermostat" does not mean that this chemical potential \(\lambda\) is related to the temperature.

Here is a physical picture we see as an origin of this thermodynamics. Consider a subsystem – single vorticity cell. The energy flowing through the infinite boundary is related to the net dipole moment. This involves contributions from the other cells over the whole space, which act as a "thermostat".

The net energy flow constraint (13) tells us that net flow from the bounding surface is dissipated in all the cells, the subsystem as well as the thermostat. If we single out the dissipation inside the subsystem, then there is a missing piece, both the contribution of other cells to the net dipole moment (14) and the dissipation terms \(\int_V d^3r \nu \omega_3^2(r)\) inside these other cells.

Therefore, the equation (13) adds up from the subsystem and from the thermostat. If we single out the subsystem \(E\), there will be an extra fluctuating term \(\mathcal{E}^T\). The exponential distribution with Lagrange multiplier for the subsystem energy flow accounts for that extra term. The Lagrange multiplier comes about as logarithmic derivative of the phase space of the thermostat with respect to energy of the subsystem (in this case the energy flow).

Technically, we have (with \(dT^T\) representing the phase space element for the thermostat)

\[ \int dT^T \delta (\mathcal{E}^T + \mathcal{E}) \propto \exp (S(-\mathcal{E})) \quad (14) \]

where \(S(\mathcal{E})\) is an entropy (logarithm of total phase space volume of the hyper surface of energy flow constraints) of the thermostat. The statistical mechanics then proceeds with expanding this entropy in the (relatively small) contribution to the energy flow from the subsystem [8].

\[ S(-\mathcal{E}) \to S(0) - \lambda \mathcal{E} \quad (15) \]

\[ \lambda = S'(0) \quad (16) \]

In case of microcanonical distribution, this entropy counted the volume of the energy hyper surface in phase space and we had \(\lambda = \beta\).

In our case (see below) we are going to integrate over space of so called Generalized Beltrami Flows, so this entropy will count the volume of the hyper surface of energy flow constraint in the space of these flows. But the general philosophy of interaction between the thermostat and the subsystem via exchange of thermodynamic variables, fixed by certain fugacities (Lagrange multipliers for microscopic constraints) is the same here.

### IV. ENERGY FLOW AS A BOUNDARY CONDITION

We are studying a vorticity cell, localized around minimal surface encircling the loop \(C\). As we said before, in addition there is some background vorticity distributed in space. We assume that this vorticity also has a finite support, but much larger than our singular vorticity cell.

One may think of this background as made from large number of similar cells of all sizes and shapes. We are going only those effects from these cells, such as the Lagrange multiplier \(\lambda\) in effective Hamiltonian for the subsystem we are studying.

The net dipole moment of the vorticity inside our fluid, would depend on external sources. In real flow in the pipe or around the ship we would need to solve equations with appropriate boundary conditions and then these forces will influence the solution inside, which would lead to some dependence of \(Q_\alpha\) of these random forces at the boundary.

The energy flow \(\mathcal{E}^T\) into the thermostat at the boundary
(infinite sphere) with external pressure

\[ p^{ext} = |r|g(r); \]
\[ g(r) = r_0 \partial_0 p^{ext}; \]
\[ \mathcal{E}^T = 2 \int_{S_2} d\sigma_0 v_0^T p^{ext} f_0 Q_0(f) \]
\[ \langle \mathcal{E}^T \rangle = 2\sigma \left( \frac{\partial Q_0(f)}{\partial f_0} \right) \]  \hspace{1cm} (17)

which relates this unknown vector function \( Q_0(f) \) to the energy flow

\[ \langle \frac{\partial Q_0}{\partial f_0} \rangle = \frac{\langle \mathcal{E}^T \rangle}{2\sigma} \]  \hspace{1cm} (18)

So, the random forces working on a remote sphere induced mass flow through the surface, netting to zero as it should in steady state, but they also induce an energy flow which adds up to a finite mean value.

Note that we did not assume anything about velocity or vorticity in the thermostat at finite distances, just its asymptotic behavior on the infinite boundary. We demanded that velocity has a pole at infinity compatible with incompressibility and matching the energy flow \( \mathcal{E}^T \) produced by random forces.

We are using here specific coordinate frame centered at the origin, and we use the sphere as a boundary. With some obvious generalizations the results should come out the same for arbitrary shape off the boundary surface as it was discussed in [3].

In the next sections, we assume two components of vorticity field, both producing velocity decreasing as \( 1/|r|^3 \): the localized vorticity cell (singular vorticity sheet in a limit of large circulation flow), and the background thermostat vorticity, spread over space.

The net velocity adds up from these two components, with the thermostat contribution dominating at infinity in thermodynamic limit.

The thermostat velocity is involved in receiving and passing the energy flow generated by the outside random forces on a remote sphere. The localized cell is receiving this energy flow and dissipating it on vorticity singularities (which singularities, as we shall see later, are smeared by viscosity to become Zeldovich pancakes of the viscous width).

The inertial range in our theory is a physical space rather than symbolic range of wavelengths: this is a space between the bounding sphere, where the energy flow originates, and the vorticity peaks where it is dissipated.

Weak background vorticity is spread over this space, and dissipation there is proportional to vanishing viscosity, whereas at peaks this small viscosity is compensated by large density of vorticity.

We describe this as a stationary distribution with Gaussian force as a vector parameter, but this can be regarded as a random process, describing time evolution of velocity field at large distances, where nonlinear effects disappear

\[ \partial_0 v_0^T(r,t) = -\partial_0 \mathcal{F}^T(r,t); \]
\[ p^T(r,t) = -\partial_0 Q_0 \left( \tilde{f}(t) \right) \partial_0 \frac{1}{|r|} \]  \hspace{1cm} (19)

This local internal pressure \( p^T(r,t) \) describes time evolution of this \( Q_0(t) \) regarded as a time series. At large time the forces \( \tilde{\mathcal{F}} \) obey Gauss distribution. So, instead of averaging over time we average over this distribution as usual in stochastic differential equations.

We are in fact imposing Dirichlet boundary condition at the bounding sphere on solution of Poisson equation \( \partial^2 v_0 = \epsilon_{\alpha\beta\gamma} \partial_\beta \omega_\gamma \). These boundary conditions relate the thermostat velocity field to the random forces acting on that sphere and supplying the energy flow.

As the sphere is infinitely far from the support of vorticity, the Coulomb kernel \( 1/|r| \) can be used inside this support. The asymptotic form \( \tilde{f}(r) \) of velocity with this kernel matches our boundary conditions, so the only thing left is to adjust is the constant vector \( \tilde{Q} \) relating it to random force \( \tilde{f} \).

There is some ambiguity here: any function \( \tilde{Q}(f) \) with the same expectation value \( \langle \frac{\partial Q_0}{\partial f_0} \rangle \) would produce the same energy flow, so it will be equivalent for our purpose.

We expect the Turbulence to be a universal fixed point of Hopf evolution equation (and/or the Loop Equations [6]), which is independent of initial conditions nor the boundary conditions at infinity as long as some parameters of the energy flow are fixed.

So, we constructed the ad hoc boundary conditions to provide an energy flow and we pray to Ken Wilson with his dogma of universality classes of fixed points in critical phenomena.

There is one important detail here, which may justify this procedure. In a limit when the external force goes to zero \( \sigma \to 0 \) the effective PDF of observables like circulation stays finite because of the susceptibility which grows as \( \nu \to 0 \).

We shall demonstrate that enhancement in subsequent sections. So, in a limit of zero viscosity there is a critical phenomenon of enhancing external forces like spontaneous magnetization of a ferromagnet.

This is different from a Kolmogorov scenario, but maybe it is time to move on after 80 years of praying to great Andrey Kolmogorov. In a feat of intuition he discovered the heart of the turbulence phenomena, but as it turns out, turbulence has some other body parts as well, and some of these parts are easier to study than the others.

The picture described here seems adequate to the high Reynolds DNS [9] as far as the circulation distribution is concerned. The critical phenomena taking place in turbulent flow at smaller spatial scales in absence of large circulation, are so far beyond the reach in our approach.

At the qualitative level we may view these multi-scale fluctuations as coming from singular vorticity structures (surfaces) of various spatial scales, uniformly distributed over space. This is similar to duality in four dimensional
field theory. There, too, fluctuating fields in a strong coupling phase are equivalent to weakly fluctuating strings which are two-dimensional surfaces in space-time.

So, we are not trying to deny the complex multi-fractal distributions of local vorticity and velocity differences. Obviously, these phenomena are real – but we simply found the conditions when these fluctuations are decoupling from the main singular flow. We found the way to bypass this complexity and get some exact relations for other observable quantities by using dual language of singular vorticity sheets.

By the way, nobody has proven that the multi-fractal scaling phenomena are even universal – the physics of the ensemble of the vorticity structures of varying sizes could depend of the specifics of the energy pumping on a bounding sphere, simply because the farther away the more influence comes from the velocity correlation growing as \(r^{3/2}\). In our case there are special reasons which we discuss, for the circulation PDF to come out universal up to scale factors.

The interaction between vorticity sheet and thermostat is described by a master equation which we derive and solve below.

Once again, our picture is anisotropic and our coordinate frame is fixed, simply because we are studying conditional probability for large velocity circulation around some large loop in coordinate space. Local velocity fluctuations play little role in this situation, it is all dominated by some steady singular flow, parametrized by global random force, implicitly describing stochastic process.

V. GENERALIZED BELTRAMI FLOW

Let us go deeper into the hydrodynamics. We parameterize the vorticity by two-component Clebsch field \(\phi = (\phi_1, \phi_2) \in \mathbb{R}^2\):

\[
\omega_\alpha = \frac{1}{2} e_{\alpha \beta \gamma} e_{ij} \partial_\beta \phi_i \partial_\gamma \phi_j
\]

(20)

The Euler equations are then equivalent to passive convection of the Clebsch field by the velocity field:

\[
\partial_t \phi_a = -v_a \partial_t \phi_a \\
v_a(r) = \frac{1}{2} e_{ij} \left( \phi_i \partial_\alpha \phi_j \right)^\perp
\]

(21)

(22)

Here \(V^\perp\) denotes projection to the transverse direction in Fourier space, or:

\[
V_a^\perp(r) = V_a(r) + \partial_a \partial_\beta \int d^3 r' \frac{V_\beta(r')}{4\pi |r - r'|}
\]

(23)

One may check that projection \(V^\perp\) is equivalent to the Biot-Savart law \(\{1\}\).

The conventional Euler equations for vorticity:

\[
\partial_t \omega_\alpha = \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha
\]

(24)

follow from these equations.

The Clebsch field maps \(R_3\) to \(R_2\) and the velocity circulation around the loop \(C \in R_3\):

\[
\Gamma(C) = \oint_C dr_\alpha v_\alpha = \oint_{\gamma_2} \phi_1 d\phi_2 = \text{Area}(\gamma_2)
\]

(25)

becomes the oriented area inside the planar loop \(\gamma_2 = \phi(C)\). We discuss this relation later when we build the Clebsch instanton.

The most important property of the Clebsch fields is that they represent a \(p,q\) pair in this generalized Hamiltonian dynamics. The phase-space volume element \(d\phi = \prod_x d\phi_1(x) d\phi_2(x)\) is invariant with respect to time evolution, as required by the Liouville theorem. We will use it as a base of our distribution. The generalized Beltrami flow (GBF) corresponding to stationary vorticity is described by \(G_\alpha(x) = 0\) where:

\[
G_\alpha \overset{\text{def}}{=} \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha
\]

(26)

These three conditions are in fact degenerate, as \(\partial_\alpha G_\alpha = 0\). So, there are only two independent conditions, the same number as the number of local Clebsch degrees of freedom. However, as we see below, relation between vorticity and Clebsch field is not invertible.

There is some gauge invariance (canonical transformation in terms of Hamiltonian system, or area preserving diffeomorphisms geometrically)\(\{10\}\).

\[
\phi_a(r) \rightarrow M_a(\phi(r))
\]

(27)

\[
\det \frac{\partial M_a}{\partial \phi_b} = \frac{\partial M_1}{\partial (M_1, M_2)} = 1.
\]

(28)

These transformations manifestly preserve vorticity and therefore velocity. \(\{11\}\)

In terms of field theory, this is an exact gauge invariance, rather than the symmetry of observables, much like color gauge symmetry in QCD. This is why back in the early 90-ties I referred to Clebsch fields as "quarks of turbulence". To be more precise, they are both quarks and gauge fields at the same time.

It may be confusing that there is another gauge invariance in fluid dynamics, namely the volume preserving diffeomorphisms of Lagrange dynamics. Due to incompressibility, the volume element of the fluid, while moved by the velocity field, preserved its volume. However, these diffeomorphisms are not the symmetry of the Euler dynamics, unlike the area preserving diffeomorphisms of the Euler dynamics in Clebsch variables.

One could introduce gauge fixing, for example the one mapping some surface bounded by a loop \(C\) inside a disk with the same area in Clebsch plane. We study the instanton in this gauge for the case of a planar loop in a later section of this paper. This gauge condition is linear and therefore it does not require any extra Faddeev-Popov ghosts.

The space where the Clebsch fields belong to is not specified by their definition. For our theory it is important
that this space is compact, which leads to discrete winding numbers. We accept the $S^2$ definition \[ \omega_{\alpha} = \frac{1}{2} Z e_{ijk} e_{\alpha \beta \gamma} S_i \partial_j S_j S_k; \quad S_i^2 = 1 \] (29)

It can be rewritten in terms of our Clebsch fields using polar coordinates $\theta \in (0, \pi), \varphi \in (0, 2\pi)$ for the unit vector $S = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$:

\[
\phi_1 = Z \cos \theta; \quad \phi_2 = \varphi \quad (\text{mod } 2\pi)
\]

The second variable $\phi_2$ is multi-valued, but vorticity is finite and continuous everywhere. The helicity $\int d^3r v_\alpha \omega_\alpha$ was ultimately related to winding number of that second Clebsch field \[14\].

The volume element on $S^2$

\[ d^2 \phi = d \cos \theta d\varphi \]

is equivalent to $d\phi_1 d\phi_2$ up to the scale factor $Z$.

The global parameter $Z$ is arbitrary: it cancels in our invariant measure (see below) and serves as a zero mode in our distribution.

We are going to set $Z = 1$ until the section where we restore it from dimensional counting in effective Hamiltonian and integrate it out to compute circulation PDF.

We are going to work in this gauge, where $\phi_2$ is an angular variable, as this will be the simplest one for topological properties.

Note also that our condition \[26\] comes from the Poisson bracket with Hamiltonian $H = \int d^3 r \frac{1}{\pi} \eta_{\alpha}^2$

\[ G_\alpha(r) = [\omega_\alpha, H] = \int d^3 r \frac{\delta \omega_\alpha(r)}{\delta \phi_i(r')} e_{ij} \frac{\delta H}{\delta \phi_j(r')} = - \int d^3 r \frac{\delta \omega_\alpha(r)}{\delta \phi_i(r')} v_\alpha(r') \partial_3 \phi_i(r') \] (33)

We only demand that this integral vanish. The stationary solution for Clebsch would mean that the integrand vanishes locally, which is too strong. We could not find any finite stationary solution for Clebsch field even in the limit of large circulation over large loop.

The GBF does not correspond to stationary Clebsch field: the more general equation

\[ \partial_t \omega_\alpha = \int d^3 r \frac{\delta \omega_\alpha(r)}{\delta \phi_i(r')} \partial_3 \phi_i(r) \]

(34)

\[ \partial_t \phi_i = - v_\alpha \partial_\alpha \phi_i + e_{ij} \frac{\partial h(\phi)}{\partial \phi_j} \]

(35)

with some unknown function $h(\phi)$ would still provide the GBF. The last term drops from here in virtue of infinitesimal gauge transformation $\delta \phi_\alpha = e_{\alpha \beta} \frac{\partial h(\phi)}{\partial \phi_\beta}$ which leave vorticity invariant.

This means that Clebsch field is being gauge transformed while convected by the flow. For the vorticity this means the same GBF.

VI. OUR MAIN CONJECTURE

We propose the following grand canonical ensemble:

\[ d Z = dP(\vec{f}) dZ D\phi \delta_{FP} [G[\phi]] \exp \left( - \lambda \left( \int_V d^3 r \nu \omega_\alpha^2 - f_\alpha \int_V d^3 r e_{\alpha \beta \gamma} r_\beta \omega_\gamma \right) \right) \]

(36)

\[ \delta_{FP}[G[\phi]] = \frac{\delta G_\alpha}{\delta \phi_\beta} \int DU \exp \left( i \int d^3 x U_\alpha(x) G_\alpha(x) \right) \]

(37)

\[ [G_\alpha(x), G_\beta(y)] = \int d^3 z \frac{\delta G_\alpha(x)}{\delta \phi_\beta(z)} e_{\alpha \beta} \frac{\delta G_\beta(y)}{\delta \phi_\alpha(z)} \]

(38)

It is invariant with respect to symplectomorphisms as one can readily check.

\[ e_{\alpha \beta} \frac{\partial M_\alpha}{\partial \phi_{\alpha'}} \frac{\partial M_\beta}{\partial \phi_{\beta'}} = e_{\alpha' \beta'} \det \frac{\partial M_\alpha}{\partial \phi_\beta} = e_{\alpha' \beta'} \]

(39)

From the point of view of matrix products in functional space this Poisson bracket is a product of three operators $\frac{\delta}{\delta \phi} \times \hat{E} \times \frac{\delta}{\delta \phi}^T$, where $\hat{E}_{\alpha, \beta}(x, y) = e_{\alpha \beta} \delta(x - y)$. This makes determinant of Poisson bracket equal to the square of our determinant times det $E = 1$.

Henceforth our determinant can be defined as a pfaffian\[14\]

\[ \det \frac{\delta G_\alpha}{\delta \phi_\beta} \equiv \sqrt{\det [G_\alpha(x), G_\beta(y)]} = \text{pf} ([G_\alpha(x), G_\beta(y)]) \]

(40)
This invariance of our measure with respect to the Euler time evolution is a central point of our construction. Let us dwell some more on this issue.

We have the functional integral

$$\int D\phi \text{pf} ([G_\alpha(x), G_\beta(y)]) \delta [G_\alpha, \phi]$$

(41)

where $G_\alpha[x, \phi] = \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha$ is a functional of $\phi$ depending also on the point $x$. Time evolution steps to symplectic transformation of $\phi$ in this functional

$$\phi_\alpha(x) \Rightarrow \hat{\phi}_\alpha(x) = \phi_\alpha(x) + \delta \phi_\alpha(x)$$

(42)

$$\delta \phi_\alpha = \epsilon \epsilon_{\alpha \beta} \frac{\partial H}{\partial \phi_\beta}$$

(43)

| \[ G_{\alpha}[x, \phi] = \omega_{\beta} \partial_{\beta} v_{\alpha} - v_{\beta} \partial_{\beta} \omega_{\alpha} \] |
| \[ \text{is a functional of } \phi \text{ depending also on the point } x. \] |

The evolution corresponds to above symplectic transformations of Clebsch fields without changing observables related to time evolution.

The Jacobian of this transformation is 1, which provides symplectic invariance of the linear measure $D\phi = \prod_x d^2\phi(x)$.

We can view the GBF space as Hilbert space with scalar product

$$\langle A, B \rangle = \int d^3x d^3y A_\alpha(x) \hat{g}_{ab}(x, y) B_\beta(y) = \langle A \times \hat{g} \times B \rangle$$

(44)

$$\hat{g}_{ab}(x, y) = \int d^3x \frac{G_{\alpha}[x, \phi]}{\delta \phi_\alpha(x)} \frac{G_{\beta}[y, \phi]}{\delta \phi_\beta(y)} = \left( \frac{\delta G}{\delta \phi} \right)^T \times \frac{\delta G}{\delta \phi}$$

(45)

| \[ \langle A, B \rangle = \int d^3x d^3y A_\alpha(x) \hat{g}_{ab}(x, y) B_\beta(y) = \langle A \times \hat{g} \times B \rangle \] |
| \[ \hat{g}_{ab}(x, y) \text{ is the metric transforms covariantly.} \] |
| \[ \frac{\delta G}{\delta \phi} \text{ is the Jacobian of this transformation.} \] |

This is an induced metric in GBF space corresponding to the hyper-surface of $G_\alpha[x, \phi] = 0$, with Clebsch fields playing the role of internal coordinates parametrizing this surface.

The determinant of this metric is equal to the square of our Pfaffian, at least this would be so in case of equal number of components of the constraints $G_\alpha$ and the Clebsch fields $\phi_\alpha$.

However, the total number of 3 components of the constraints is bigger than the number 2 of components of the Clebsch field, though in fact there are only 2 independent components of $G_\alpha$, due to incompressibility relation between them $\partial_\alpha G_\alpha = 0$.

So, we can no longer use the interpretation of the Faddeev-Popov delta function because there is no such thing as a determinant of rectangular $N \times M$ matrix $X = \frac{\delta G}{\delta \phi}$.

There are actually two definitions in such case: $\sqrt{\det (X \times E \times X^T)}$ and $\sqrt{\det (X^T \times X)}$.

First one corresponds to Poisson brackets, and the second one- to the Hilbert space metric.

Using so called singular value decomposition one can prove (in finite $N \times M$ matrix case) that non-zero eigenvalues for these two matrices coincide. The bigger matrix of the two, corresponding to the largest of the $N, M$ of the dimensions of $X$, has all the eigenvalues of the smaller one, plus there are also $|M - N|$ zero eigenvalues in addition to this list.

In our case, with our prescription of keeping only positive eigenvalues of the bigger matrix (Poisson brackets), these two determinants coincide.

Therefore, our measure is the standard invariant measure in this space.

$$D\phi \text{pf} ([G_\alpha(x), G_\beta(y)]) \delta [G] = D\phi \sqrt{\det \hat{g}[\phi]} [G]$$

(46)

When the internal coordinates $\phi_\alpha(x)$ are transformed by means of time evolution, the metrics transforms covariantly

$$\hat{g}[\phi] = \frac{\delta \phi}{\delta \phi} \times \hat{g}[\phi] \times \left( \frac{\delta \phi}{\delta \phi} \right)^T$$

(47)

The evolution corresponds to above symplectic transformations of these internal coordinates: the functional analog of reparametrization of a surface.

Remember- we can perform time dependent gauge transformations of Clebsch fields without changing observables. So, in addition to actual Euler dynamics moving the parameters $\phi_\alpha(x, t)$ of our surface, there can be a time dependent symplectomorphisms, resulting in evolution of $\phi_\alpha(x, t)$.

General covariance of our metric together with invariance of the linear measure $D\phi$ in Clebsch space with respect to Hamiltonian evolution guarantees invariance of our measure with respect to time evolution as well as symplectomorphisms.

$$D\phi \sqrt{\det \hat{g}[\phi]} \delta [G] = D\phi \sqrt{\det \hat{g}[\phi]} [G]$$

(48)

This relation means that the Euler time evolution reparametrizes the internal coordinates on the GBF space without changing the volume element. The unobservable parameters $\phi_\alpha(x)$ transform, covering the manifold $G_\alpha = 0$ with uniform weight, while observables related to vorticity $\vec{\omega}$ stay invariant.
We discuss this issue in the next section for a simple Hamiltonian system with discrete degrees of freedom: particle in potential in $N$ dimensional space. The stationary points where $\partial_\phi \dot{\phi} = 0$ in phase space do not move with Hamiltonian dynamics, and our measure is equivalent to summing over them with unit weights.

In case of degenerate stationary manifold, which is our case in Euler-Clebsch dynamics, time evolution can be supplemented by gauge transformations covering this manifold. The measure stays invariant with respect to both transformations: Euler and gauge.

There are zero modes associated with conservation

$$\frac{\partial}{\partial x_\alpha} [G_\alpha(x), G_\beta(y)] = 0, \quad \frac{\partial}{\partial y_\beta} [G_\alpha(x), G_\beta(y)] = 0$$

(49)

So this determinant formally would be zero, unless we project out these zero modes. Otherwise it is well defined invariant kernel with well defined eigensystem.

The Lagrange multiplier $\lambda$ is conjugate to the energy flow constraint, so we have to use the thermodynamic relation

$$\mathcal{E} = -\frac{\partial \log Z}{\partial \lambda}$$

(50)

where $\mathcal{E}$ is the energy flow from the "thermostat" to the subsystem under consideration.

Note that our distribution does not fix the scale of the Clebsch fields.

Here is one important point we have to discuss. The effective Hamiltonian in our exponential

$$H_{eff} = \int_V d^3 \nu \omega_\alpha^2 - f_\alpha \int_V d^3 r e_{\alpha \beta \gamma} r_\beta \omega_\gamma$$

(51)

is not in general time-invariant in Euler dynamics. However, in virtue of GBF-invariant in Euler dynamics. However, in virtue of GBF condition we imposed on our measure, it is in fact invariant.

$$\hat{H}_{eff} = [\hat{H}_{eff}, H] = 2 \int_V d^3 \nu \omega_\alpha G_\alpha - f_\alpha \int_V d^3 r e_{\alpha \beta \gamma} r_\beta G_\gamma = 0$$

(52)

where $G_\alpha = \dot{\omega}_\alpha = [\omega_\alpha, H]$ is our GBF constraint. So, in virtue of our local constraint $G_\alpha(\vec{r}) = 0$, imposed on the FP measure multiplying this effective Gibbs distribution $e^{-\hat{H}_{eff}t}$, our canonical ensemble is time-invariant.

In Appendix A we study our distribution for a well known example of a particle moving in potential in $N$ dimensional space. Everything is clear and well defined in this example, so it is great way to understand the meaning of our measure.

However, a sophisticated reader who is satisfied with above general formulas does not need to look into that Appendix.

\section{VII. Lyapunov Stability and Theta Factor}

In general case, we have to fix the gauge\cite{15} and eliminate all the unstable GBF.

This Lyapunov stability of GBF is in fact determined by another kernel

$$L_{\alpha \beta}(x, y) = \frac{\delta G_\alpha(x)}{\delta \omega_\beta(y)}$$

(53)

which is not symmetric. For stability of our flow we need its eigenvalues (Lyapunov exponents) to all have negative or zero real part. There should not be any eigenvalues in the right semi-plane.

The Lyapunov stability of GBF is in fact determined by another kernel

$$\frac{\partial}{\partial x_\alpha} [G_\alpha(x), G_\beta(y)] = 0, \quad \frac{\partial}{\partial y_\beta} [G_\alpha(x), G_\beta(y)] = 0$$

where

$$\mathcal{E} = -\frac{\partial \log Z}{\partial \lambda}$$

(50)

Note that this formula does not rely on quantization of $N_+(\hat{L})$ which may not be valid for operators in Hilbert space. Even if there is a continuous distribution of eigenvalues, this $N_+(\hat{L})$ will remain positive in case there are some eigenvalues distributed in the right semi-plane. For any distribution in the left semi-plane including imaginary axis this $N_+(\hat{L})$ would remain zero. For infinite number of eigenvalues in right semi-plane $N_+(\hat{L}) \to +\infty$ so that theta function still works.

If we introduce the extended operator $\hat{L}(z) = \hat{L} + \imath z$ and use identity

$$\text{tr} \frac{1}{\hat{L}(z)} = -\imath \partial_z \log \det \hat{L}(z)$$

(56)

then we can count these eigenvalues as follows
\[
N_+ (\hat{L}) = - \lim_{{\epsilon \to 0^+}} \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} \exp (i \epsilon z) \partial_z \log \det \hat{L}(z) = \frac{1}{2\pi} \Delta_+ (z) \arg \det \hat{L}(z) \tag{57}
\]

Here \(\Delta_+ (z) \arg F(z)\) stands for the total phase acquired by \(F(z)\) when \(z\) goes around the anti-clockwise loop in upper semi-plane surrounding zeroes of \(F(z)\). In other words, it counts all eigenvalues of \(\hat{L}\) in the right semi-plane.

So, we have stability selection factor

\[
\Theta [\omega] = \theta \left( \pi - \Delta_+ (z) \arg \det \hat{L}(z) \right) = \int_{-\infty}^{\infty} \frac{dy}{2\pi (i y + 1)} \exp \left( i y \left( \pi - \Delta_+ (z) \arg \det \hat{L}(z) \right) \right) \tag{58}
\]

Coming back to our distribution with prescription (58) we see that the distribution is uniformly covering stable generalized Beltrami flows, and therefore is conserved in Euler dynamics. The gauge invariance remains unbroken at this stage. We do not know the general prescription of unambiguous gauge fixing, but in case of our instanton we can present a unique gauge condition (see below).

This is clearly not the Gibbs distribution (which would be undesirable). We are looking for an alternative fixed point of the PDF evolution which is capable of describing fixed energy flow instead of fixed energy.

As we shall see below, the GBF provides an adequately rich space of steady solutions that can incorporate energy flow.

The velocity circulation PDF is generated by the further constraint in (36):

\[
P \left( \Gamma | C \right) = \int dP(\hat{f}) \int D\phi \delta \left[ G_\alpha \right] \text{pf} \left( \left[ G_\alpha, G_\beta \right] \right) \Theta [\omega] \\
\delta \left( \Gamma - \oint_{C} d\vec{r} \vec{v} \right) \exp \left( -\lambda \left( \nu \int_{V} \omega_\alpha(r) d^3 r - f_\alpha \int_{V} \epsilon_{\alpha\beta\gamma} r_\beta \omega_\gamma(r) \right) \right) \tag{59}
\]

By construction, this \(P \left( \Gamma | C \right)\) satisfies the Euler Loop equations, as they are equivalent to

\[
\left\langle \exp \left( i \gamma \oint_{C} d\vec{r} \vec{v} \right) \oint_{C} d\vec{r} \vec{v} \times \vec{\omega} \right\rangle = 0 \tag{60}
\]

which reduces by the Stokes theorem to the flow of \(\nabla \times \vec{v} \times \vec{\omega}\) through the surface bounded by \(C\). This flow vanishes by virtue of steady equations of motion (24) for \(\vec{\omega}\).

Moreover, the cancellation of the functional determinant between the delta function and Pfaffian means that our PDF reduces to the average over space of all stable GBF. To be more precise, we have constructed an invariant measure in this space.

VIII. GHOST FIELDS

With our modified Faddeev-Popov delta functional we can still use their ghost fields but to get Pfaffian we need one Grassmann field, not two:

\[
\int D\phi \ \text{pf} \left( \left[ G_\alpha, G_\beta \right] \right) \delta [G_\alpha] = \int D\phi DU D\Psi \exp \left( i \left\langle U_\alpha | G_\alpha \right\rangle + \left\langle \Psi_\alpha \right\rangle \left[ G_\alpha, G_\beta \right] \Psi_\beta \right) \tag{61}
\]

with \(\Psi_\alpha\) being Grassmann field and \(\left\langle A | B \right\rangle, \left\langle A | X | B \right\rangle\) stands for vector and matrix products in functional space.

One may verify the simple re-scaling of Clebsch field leaves the measure invariant except for random force PDF.

The fields transform according to their dimensions:

\[
\phi_\alpha \Rightarrow \lambda \phi_\alpha \tag{62}
\]
\[
U_\alpha \Rightarrow \lambda^{-4} U_\alpha \tag{63}
\]
\[
\Psi_\alpha \Rightarrow \lambda^{-3} \Psi_\alpha \tag{64}
\]

The scale factors of \(\lambda\) emerging in the measure \(D\phi D\Psi DU\)
will all cancel (the Grassmann variable measures transforms with inverse Jacobian). So, the measure is scale invariant. As a consequence it does not depend upon the zero mode $Z$ in the definition \cite{25} of the Clebsch fields.

This is so by design. In case of finite number of degrees of freedom the total volume of the GBF space with our measure is equivalent to adding contribution of each stationary point with weight 1.

As for the phase counter $\Theta[\omega]$ it is obviously invariant as the scaling of operator $\hat{L}$ does not affect its phase.

Usually, there is a divergent volume term in every field theory except supersymmetric one, where these factors cancel between Bosons and Fermions. Such cancellation happens here as well, which is a hint for a hidden supersymmetry.

The distribution for the random force will break this scale invariance. The same is true with respect to time reversal, which corresponds to the interchange of $\phi_1, \phi_2$. The distribution again does not change, but the random force will break this invariance, if its PDF is not even with respect to reflection $f \Rightarrow -f$.

This representation of invariant measure with ghost fields is suitable for the perturbative expansion in a background of a classical solution (instanton), which, as we shall see, dominates the distribution in the case of large circulation around a large loop.

\section{IX. Clebsch Confinement}

Let us look more closely at our functional integral. By naive counting of degrees of freedom it is just a number, as we have two degrees of freedom at each point in space and two independent local constraints \cite{24}, so that the whole integral reduces to a trivial sum over solutions of these constraints, just as it did in the case of a particle in a potential well.

Fortunately, this is not so simple: there is in fact a functional degeneracy of these constraints. First, one could shift vorticity by velocity times the arbitrary local scalar field $\vec{\omega} \Rightarrow \vec{\omega} + \phi(r)\vec{v}$ as long as $v_\alpha \partial_\alpha \phi = 0$ (meaning this field does not change along the flow). Also, from $\nabla \times \vec{v} \times \vec{\omega} = 0$ we can have $\vec{v} \times \vec{\omega} = \nabla F$ with arbitrary $F(x)$.

Naturally, we implied the ambiguity of the primary constraints as functionals of velocity and vorticity. As you start solving these constraints you will find that $F(x) = \nabla \left( p + \frac{1}{2} \vec{v}^2 \right)$. This does not change the fact that these constraints are degenerate, as they do not involve pressure $p(r)$ and are satisfied with arbitrary pressure.

As for the Clebsch field itself, it can be transformed by arbitrary local area-preserving diffeomorphism, as noted in the previous section.

There is, however, a limit where the functional integral reduces to a classical flow (instanton) up to the symplectomorphism. This is the limit of large circulation $F$ over a large loop $C$.

Let us first describe a qualitative physical picture of our instanton. It is similar in spirit to the magnetic monopole in 3-dimensional gauge theories. In these theories the ground state has condensate of monopoles there which leads to a dual Meissner effect of pushing electromagnetic field from the vacuum, leading to collapse of this field in thin flux tubes between charges.

This was the origin of confinement in 3D gauge theories, but of course, literally the same mechanism is absent here. There is no gauge invariance associated with velocity playing the role of vector potential. There is no $U(1)$ symmetry and no associated charges, and hence no monopoles either.

Our gauge symmetry involves the Clebsch fields and our analogues of monopoles are singular sheets in physical space where our gauge potential $\phi_\alpha$ become multi-valued.

And our analog of confinement is confinement of Clebsches, and our analog of gluon field shrinking to minimal surfaces bounded by quark loops is the vorticity shrinking to minimal surface in case large circulation over large loop is present.

We expect confinement phenomenon here, except instead of magnetic monopoles we have found different singular solutions leading to condensation of vorticity (our analog of magnetic field).

Here is this picture of vorticity condensation.

Comparing our two constraints (energy dissipation and fixed circulation) we observe that to minimize dissipation in effective Hamiltonian $\lambda \nu \int d^3 r |\omega|^2$ at fixed circulation we need the vorticity to be concentrated in a thin layer (of viscous thickness $h \sim \nu$) around the minimal surface $S_{\text{min}}(C)$ with area $A_C$ surrounded by $C$ and directed along the normal $n_\alpha$ to this surface to maximize the flux\cite{19}.

There are, of course, other vorticity cells randomly distributed all over space, with their own energy dissipations. We are considering the energy dissipation per cell $E_\text{cell}$, assuming this cell covers the minimal surface bounded by the loop $C$.

\section{X. Clebsch Instanton}

We found in \cite{3} multi-valued fields with nontrivial topology which are relevant to large circulation asymptotic behavior. Vorticity is parametrized by famous Clebsch coordinates

$$\omega_\alpha = e_{\alpha\beta\gamma} \partial_\beta \phi_1 \partial_\gamma \phi_2$$

and velocity is given by a Biot-Savart integral \cite{4}.

\subsection{A. Gauge Invariance and Clebsch Confinement}

There are some gauge transformations (canonical transformation in terms of Hamiltonian system, or area preserving diffeomorphisms geometrically) \cite{27} which leave vorticity invariant.
Infinitesimal version of these transformation is
\[ \delta \phi_a = \epsilon_{ab} \frac{\partial h}{\partial \phi_b} \] (66)
with arbitrary function \( h(\phi_1, \phi_2) \).

The conventional time evolution for Clebsch fields in Euler Hamiltonian dynamics is just a passive convection
\[ \partial_t \phi_a = -v_a \partial_a \phi_a \] (67)

We, however, generalize this evolution by adding time dependent gauge transformation which produce equivalent Clebsch fields
\[ \partial_t \phi_a = -v_a \partial_a \phi_a + e_{ab} \frac{\partial h}{\partial \phi_b} \] (68)

Independently of the gauge function \( h(\phi) \) the vorticity satisfies the same equations
\[ \partial_t \omega_{\alpha} = \omega_{\beta} \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_{\alpha} \] (69)

This is a direct consequence of gauge invariance of the Clebsch parametrization of vorticity.

The Turbulence phenomenon in fluid dynamics in Clebsch variables resembles the color confinement in QCD.

We have no Yang-Mills gauge field here, but instead we have nonlinear Clebsch field participating in gauge transformations. These transformations are global as opposed to local gauge transformations in QCD, but the common part is that this symmetry stays unbroken and leads to confinement of Clebsch field.

The description of Clebsch field as nonlinear waves \[ \phi_2 \] which was appropriate at large viscosity, or weak turbulence, quickly gets hopelessly complex when one tries to go beyond the K41 law into fully developed turbulence. The basic assumption of the Gaussian distribution of Clebsch field breaks down at small viscosity.

The small viscosity in Navier-Stokes equations is a nonperturbative limit, like the infra-red phenomena in QCD, when the waves combine into non-local and nonlinear structures best described as solitons or instantons.

Nobody managed to explain color confinement in gauge theories as a result of strong interaction of gluon waves. On the contrary, the topologically nontrivial field configurations such as monopoles in 3D gauge theory and instantons in 4D led to the understanding of the color confinement.

This is what we are doing here as well, except our singular solutions are not point like singularities but rather singular vorticity sheets.

Vorticity sheets (so called Zeldovich pancakes [21]) were extensively discussed in the literature in the context of the cosmic turbulence. Superficially they look similar to my instantons but at closer look there are some important distinctions. For one thing they are unrelated to the minimal surfaces, and for another one, they seem to have no topological numbers.

The general physics of the "frozen" vorticity in incompressible flow, collapsing in the normal direction and expanding along the surface, is essentially the same. What is different here is an explicit singular solution with its tangent and normal components at the minimal surface, the Clebsch field topology and its consequences for the circulation PDF.

The relevance of classical solutions in nonlinear stochastic equations to the intermittency phenomena (tails of the PDF for observables) was noticed back in the 90-ties [11] when it was used [22] to explain intermittency in Burgers equation. However, nobody succeeded in finding the instanton solution in 3D fluid dynamics until now.

B. Discontinuity at the Minimal Surface

Let us now describe the proposed stationary solutions of Euler equations in Clebsch variables.

Our Clebsch field \( \phi_2 \) has \( 2\pi n \) discontinuity across the minimal surface \( S_C \) bounded by \( C \). As it is argued in Appendix B the minimal surface is compatible with Clebsch parametrization of conserved vorticity directed at its normal in linear vicinity of the surface.

We parametrize the minimal surface [23] as a mapping to \( R^3 \) from the unit disk in polar coordinates \( \rho, \alpha \)
\[ S_C : \vec{r} = \vec{X}(\xi), \xi = (\rho, \alpha) \] (70)

In the linear vicinity of the surface
\[ \delta S_C : \vec{r} = \vec{X}(\xi) + z\vec{n}(\xi) \] (71)

the Clebsch field \( \phi_2 \) is discontinuous
\[ \phi_2 (\vec{r} \in \delta S_C) = m\alpha + 2\pi n \theta(z) + O(z^2); m, n \in \mathbb{Z} \] (72)

while the other component is continuous
\[ \phi_1 (\vec{r} \in \delta S_C) = \Phi(\xi) + O(z^2) \] (73)

The vorticity has the delta-function singularity at the surface:
If you study the vorticity conservation
\[ \partial_\alpha \omega_\alpha (r \in \delta S_C) = 0 \quad (75) \]
you will arrive at the self-consistency equation [3]
\[ \partial_\alpha n_\alpha = 0 \quad (76) \]

The Stokes theorem ensures that the flux through any other surface bounded by the loop \( C \) would be the same, but in that case the singular tangent component of vorticity would also contribute. The simplest computation
\[ \Gamma_C = \oint_C v_\alpha dr_\alpha = \int_S d\phi_1 \wedge d\phi_2 = m \int_0^{2\pi} (\Phi(1, \alpha) - \Phi(0, \alpha)) d\alpha \quad (77) \]

The simplest case of our instanton is that of a flat loop in 3D space, which we assume to be in \( x, y \) plane. The minimal surface is a part \( D_C \) of \( x, y \) plane bounded by this flat loop.

The generic formula (78) simplifies here (here \( i, j = 1, 2 \)):
\[ v_\beta^{\text{inst}}(r) = 2\pi n (\delta_\beta \gamma \partial_\gamma - \delta_\alpha \beta \partial_\alpha) \int_{S_{\text{min}}} d\sigma_\gamma(\xi) \partial_\alpha \Phi(\xi) \frac{1}{4\pi|X(\xi) - \vec{r}|} \quad (78) \]

We are assuming that the Clebsch field falls off outside the surface so that vorticity is present only in an infinitesimal layer surrounding this surface. In this case only the delta function term contributes to the Biot-Savart integral though only a regular term contributes to the circulation.

Let us now consider the steady flow Clebsch equations derived in [3], which we call the master equation:
\[ v_\alpha \partial_\alpha \phi_\alpha = e_{ab} \frac{\partial h(\phi)}{\partial \phi_b} \quad (79) \]

Here the gauge function \( h(\phi) \) is arbitrary, and must be determined from consistency of the equation.

The master equation is much simpler than the vorticity equations for GBF.

The leading term in these equations near the minimal surface is the normal flow restriction
\[ v_\alpha(r)n_\alpha(r) = 0; r \in S_C \quad (80) \]
which annihilates the \( \delta(z) \) term on the left side of (79). The next order terms will already involve the gauge function \( h(\phi) \).

corresponding to the mean curvature being zero at the minimal surface.

This delta term in vorticity is orthogonal to the normal vector to the surface and thus does not contribute to the flux through the minimal surface, so this flux is still determined by the second (regular) term and circulation is related to this \( \Phi(\xi) \)

The vanishing tangent velocity means that the regular part of equation (79) is satisfied identically with \( h = 0 \).

As for the singular part, proportional to \( \delta(z) \) it requires \( v_3(r) = 0 \).

In fact, there is always extra contribution \( \vec{v}^T(r_0) \) to the normal velocity from the 3D Biot-Savart integral of over vorticity in the thermostat cells (see [3]). So, correct equation reads
XI. SMEARED VORTICITY AND DISSIPATION IN NAVIER-STOKES EQUATIONS

The square of delta function entering the dissipation from the instanton has to be smeared at viscous scales. This smearing comes from the viscosity terms in the Navier-Stokes equation:

$$\partial_t \omega_\alpha = \nu \partial^2 \omega_\alpha + \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha \quad (83)$$

With this term present, we introduce viscosity correction to vorticity

$$\delta \omega_\alpha = \nu \partial \delta(h,z) w_\alpha(\xi) \quad (86)$$

and we find in the first order by matching the delta-terms:

$$w_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta w_\alpha + 2\pi n e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) = 0 \quad (87)$$

There is a solution with $w_\alpha$ belonging to the tangent plane

$$w_k \partial_k v_i - v_k \partial_k w_i + 2\pi n e_{ik} \partial_k \Phi = 0 \quad (88)$$

As we see in the end of this paper, this vanishing $\Lambda$ is necessary for the existence of critical phenomena in PDF distribution.

We neglected the contribution from the instanton to the energy flow from the boundary, but it dominates the dissipation because of square of delta function compensating small viscosity in front of $\int d^3 r \omega_\alpha^2$.

In the steady flow the right side must vanish. The $\delta'$-terms coming from the $v_\beta \partial_\beta \omega_\alpha$ term cancel by themselves in virtue of vanishing normal velocity at the surface.

The singular $\delta(z)$ terms must balance the first term $\nu \partial^2 \omega_\alpha$ at $z \to 0$. For that purpose the $z$ dependence must match. The only function smearing $\delta(z)$ and proportional to itself after two derivatives is

$$\delta(h,z) = \frac{1}{2h^2} \exp\left(-\frac{|z|}{h}\right) \quad (84)$$

The perturbation term in the steady Navier-Stokes equations coming from viscosity is

$$2\pi n e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) \delta(h,z) = 2\pi n \frac{\nu}{h^2} e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) \delta(h,z) \quad (85)$$

Now we can take a limit at $h = \nu/\Lambda \to 0$

$$\nu \delta(h,z)^2 = \frac{\Lambda}{4h} \exp\left(-\frac{2|h|}{h}\right) \to \frac{\Lambda}{4} \delta(z) \quad (89)$$

From above viscosity correction we see that $\Lambda$ must go to zero with viscosity faster then $\sqrt{\nu}$

$$\Lambda \ll \sqrt{\nu} \quad (90)$$

This brings us to the effective Hamiltonian, which we now can compute as an anomaly at small viscosity

$$H_{\text{eff}} = H_{\text{eff}}^{(2)} - H_{\text{eff}}^{(1)} \quad (91a)$$

$$H_{\text{eff}}^{(2)} = \frac{\Lambda}{4} \int_{s_{\text{min}}} \delta\sigma(\xi) \tilde{F}(\xi)^2 \quad (91b)$$

$$H_{\text{eff}}^{(1)} = \tilde{f} Q(\tilde{f}) \quad (91c)$$

$$\tilde{F} = 2\pi n \tilde{\nabla} \Phi \times \tilde{n} \quad (91d)$$

As we see in the end of this paper, this vanishing $\Lambda$ is necessary for the existence of critical phenomena in PDF distribution.

XII. INSTANTON ON FLAT SURFACE

Here we re-derive and correct the preliminary results described in the preprint M20b. Some of the assumptions made in that paper turned out to be incorrect. The general predictions for PDF stay the same but formulas describing the dependence of the shape of the loop change significantly.
The cylindrical coordinate system $\rho, \theta, z$ we are using has a fictitious singularity at the origin, where $\sqrt{g} = \rho = 0$. To keep the normal vorticity $\omega_n \propto \frac{\partial \Phi}{\partial \rho}$ finite at the origin, the Clebsch field have to obey extra condition

$$\partial_\rho \Phi(\bar{r} = 0) = 0$$

In other terms, the linear term of Taylor expansion of $\Phi$ at the origin must vanish otherwise the normal vorticity will have $1/|\bar{r}|$ pole.

### A. Minimization Problem

There is a way to reduce our master equation to a minimization of a quadratic form.

$$\frac{1}{4\pi^2} \int_{D_C} d^2r' \partial_\alpha \frac{1}{|\bar{r}' - \bar{r}|} \int_{D_C} d^2r'' H(\bar{r}'') \partial'_\alpha \frac{1}{|\bar{r}'' - \bar{r}'|} = R(\bar{r})$$

Here

$$R(\bar{r}) = \frac{\nu^T(\bar{r}, z)}{\int_{D_C} d^2r' v^T(\bar{r}, z)}$$

is normalized to unit integral over the domain.

As we are interested in large size of domain $D_C$ compared to the size of vorticity support in the thermostat, this $R(\bar{r})$ is concentrated inside a finite region near the center of $D_C$. Later we study this equation approximating $R(\bar{r})$ by a delta function. Now we proceed for a general $R(\bar{r})$.

We observe that this problem is equivalent to minimization of positive quadratic form

$$Q[H] = - \int_{D_C} d^2r H(r) R(\bar{r}) + \frac{1}{2} \int_{D_C} d^2r F^2_\alpha[H, \bar{r}]$$

where $\bar{r}_0$ is the center of the disk $D$

$$F_\alpha[H, \bar{r}] = \frac{1}{2\pi} \int_{D_C} d^2r' H(\bar{r}') \partial'_\alpha \frac{1}{|\bar{r}' - \bar{r}|}$$

As we shall see later, the position of the origin drops from asymptotic formulas at large area.

This $F_\alpha[H, \bar{r}]$ is proportional to $\partial_\alpha \Phi(\bar{r})$. Thus, the quadratic part of our target functional is just a kinetic energy of a free scalar field, but it is the linear term which forces us to use $H(\bar{r})$ as an unknown.

In order for $\Phi(\bar{r})$ and its gradients to remain finite at the boundary $C$ the new field $H$ should satisfy Dirichlet boundary condition

$$H(C) = 0$$

In order for vorticity to remain finite at the origin we have to have

$$F_\alpha[H, \bar{0}] = 0$$

Let us make the integral transformation

$$\Phi(\bar{r}) = \int_{D_C} d^2r' \frac{T(\bar{r}, z)}{n} \int_{D_C} d^2r' H(\bar{r'}) \frac{1}{2\pi|\bar{r} - \bar{r}'|}$$

and we are arrive at universal equation

Coulomb poles disappeared from this problem, being replaced by weaker, logarithmic singularities (see the next section).

The net vorticity of instanton which we set to zero, provides two more constraints

$$\int_{D_C} d^2r F_\alpha[H, \bar{r}] = 0$$

The circulation integral

$$\Gamma[C] = m \int d\theta \left( \Phi \left( R\bar{f}(\theta) \right) - \Phi(\bar{0}) \right)$$

with $C : \bar{r} = L\bar{f}(\theta)$ being the equation for the contour $C$ in polar coordinates on the plane.

In Appendix E we describe finite element method to solve this variational problem.

### XIII. Circulation PDF

In this section we are going to finally derive predictions for the circulation PDF.

The vorticity and velocity fields as determined from the homogeneous GBF equations have arbitrary overall scale $Z$ of the Clebsch field $|20|$. We can normalize the vorticity by minimizing over the zero mode factor
\[ \omega \Rightarrow Z\omega \text{ the effective Hamiltonian} \]
\[
H^{\text{eff}} = \min_{\mathcal{Z}} \left( Z^2 H^{\text{eff}}_2 - Z H^{\text{eff}}_1 \right) \tag{102a}
\]
\[
H^{\text{eff}}_2 = \int_{V_{\text{c}}} d^3 r \omega_\alpha^2 (r) \tag{102b}
\]
\[
H^{\text{eff}}_1 = \bar{Q} (\bar{f}) \tag{102c}
\]
\[
Z = \frac{H^{\text{eff}}_1}{2H^{\text{eff}}_2} \tag{102d}
\]

In the linear approximation at small force
\[
H^{\text{eff}}_1 \rightarrow f_\alpha f_\beta Q_{\alpha\beta} \tag{103}
\]
where the trace of this matrix \( Q_{\alpha\alpha} \) is proportional to energy flow. We see that this matrix has to be positive definite, otherwise \( Z \) could vanish which would lead to singularity.

After this global renormalization, our solution for \( \Phi \), which was linearly related to the normal global velocity

\[
\Gamma[C] \propto \frac{m}{n} \frac{f_\alpha f_\beta Q_{\alpha\beta}}{\Lambda \left( \int_{D_C} d^2 r v_I^T (\vec{r}, z) \right)^2} \int_0^{2\pi} d\theta \int_{D_C} d^2 r \frac{H(\vec{r})}{H} \left( \frac{1}{|\vec{r} - Lf(\theta)|} - 1 \right), \tag{107}\]
\[
\bar{H} = \frac{\int_{D_C} d^2 r H(\vec{r}) R(\vec{r})}{\int_{D_C} d^2 r R(\vec{r})} \tag{108}
\]

We remind that the origin is placed at geometric center of the domain \( D_C \).

The integral \( \int_{D_C} d^2 r H(r) R(\vec{r}) \) in \( \bar{H} \) is concentrated on finite scales \( \vec{r} \sim 1 \) due to decrease of \( R(\vec{r}) \), so this \( \bar{H} \) scales as \( H(0) \), same as \( H(\vec{r}) \) in the integral in the numerator.

Collecting scales of the remaining factors we see that \( \Gamma[C] = LF[C/L] \) in agreement with the loop equation arguments [22].

Taylor expansion of \( \bar{Q}(\bar{f}) \) would be justified if, just like in a critical phenomena in statistical physics, the susceptibility \( Q_{\alpha\beta} \) would grow to infinity to compensate small value of external force.

This is what happens in a ferromagnet near the Curie point, when infinitesimal external magnetic field is enhanced by large susceptibility, resulting in a spontaneous magnetization.

In our theory this happens because \( \Lambda \ll \sqrt{\nu} \) becomes small. This enhances the leading term \( f_\alpha Q_{\alpha\beta} \bar{f} \rightarrow Q_{\alpha\beta} f_\beta \sim \sigma \) so that the higher terms \( O(\sigma^2) \) of expansion would be negligible.

The critical phenomenon, which is transformation of the Gaussian distribution to an exponential one, happens because of our singular instanton solution, which produces another factor of Gaussian force multiplying at the surface will also get the same factor \( Z \).

The quadratic part is dominated by the instanton and is given by integral square of gradient of \( \Phi \)
\[
H^{\text{eff}}_2 \propto \Lambda \int_{D_C} d^2 r (n \nabla \Phi)^2 \tag{104}
\]
this is the same integral which enters our quadratic form minimization, so that for a solution of this minimization problem
\[
H^{\text{eff}}_2 \propto \Lambda \left( \int_{D_C} d^2 r v_I^T (\vec{r}, z) \right)^2 \int_{D_C} d^2 r H(\vec{r}) R(\vec{r}) \tag{105}
\]
Therefore this normalization factor up to a normalization constant
\[
Z \propto \frac{f_\alpha f_\beta Q_{\alpha\beta}}{\Lambda \left( \int_{D_C} d^2 r v_I^T (\vec{r}, z) \right)^2 \int_{D_C} d^2 r H(\vec{r}) R(\vec{r})} \tag{106}
\]
Another source of dependence of the shape of the loop comes from the circulation integral. Putting that together this one through the interaction of an instanton with the thermostat background.

Resulting square of Gaussian variable in above formula for the circulation transforms the Gaussian distribution to the exponential one.

Also, we observe that the sign of \( \Gamma \) is proportional to the sign of the ratio of winding numbers \( \frac{m}{n} \).

Clearly, in addition to solution with winding numbers \( m, n \) there are always mirror solutions with \( \pm m, \pm n \).

The effective Hamiltonian at this solution in is exactly the same as for the positive \( m, n \), so the contributions from these flows must be added.

This provides the negative branch of circulation PDF. Summing up contribution from both signs we obtain an explicit formula for a Wilson loop
\[
\langle \exp (i \gamma \Gamma_C) \rangle = \frac{1}{2} \left( W \left( \frac{m}{n} \gamma \right) + W \left( - \frac{m}{n} \gamma \right) \right) \tag{109}
\]
\[
W(\gamma) = \frac{1}{\sqrt{\prod_{i=1}^{3} (1 - i \gamma \mu_i \Sigma[C])}} \tag{110}
\]
where \( \mu_i \) are three positive eigenvalues of the matrix (in decreasing order)
This corresponds to asymptotic law

\[
P(\Gamma) \propto \sqrt{\frac{n}{m \Sigma[C]|\Gamma|}} \exp \left( - \frac{n \Gamma}{m \mu_1 \Sigma[C]} \right) \quad (112)
\]

The functional \( \Sigma[C] \) is completely universal and calculable in terms of the our universal minimization problem, except for the unknown function \( R(\vec{r}) = v_z^T(\vec{r}, z) \). Remaining non-universal parameters of the thermostat and random forces are hidden in the matrix \( \tilde{\mu} \).

This function \( v_z^T(\vec{r}, z) \) is concentrated on the finite sizes near the middle of our domain and falls off as \( 1/|r|^3 \). Therefore, at large sizes of the loop and the area of the domain \( D_C \) this integral can be approximated as

\[
\int_{D_C} d^2r v^T(r, z) = \text{const} \quad (113)
\]

\[
\bar{H} \approx H(\bar{\theta}) \quad (114)
\]

The same approximation can be made in the target functional of our minimization problem. After that, the solution for \( H(\vec{r}) \) and \( \Sigma[C] \) will be universal.

It is also assumed that the circulation is large compared to the viscosity, and by definition of the WKB approximation we were considering the tails of distribution, at \( |\Gamma| \gg \mu_1 |\Sigma[C]| \).

In that region the (even) moments \( M_p = \langle \Gamma^p \rangle \) grow as \( \Gamma(p + \frac{1}{2}) \).

Another interesting prediction we have here is a non-trivial dependence of the circulation scale \( \Sigma[C] \) from the shape of the loop \( C \).

This function can be computed numerically using the variational method we outlined above. In particular, for the rectangle all singular integrals are calculable, so this problem is tractable.

\textbf{XIV. TOPOLOGY OF INSTANTON AND CIRCULATION PDF}

The quantization of the circulation in a classical problem deserves further attention.

One may wonder what are the physical values of the winding numbers \( m, n \). Maybe only the lowest levels are stable, and higher ones must be discarded?

If you consider effective Hamiltonian contribution from this instanton \( \Phi_1(10) \) you observe that it does not depend of winding numbers as the solution for \( \Phi \) does not depend of \( m \) and is inversely proportional to \( n \).

Therefore, the circulation only depends of ratio of winding numbers \( \frac{m}{n} \). In general case we have to sum over all \( m, n \) with yet unknown weights

\[
\exp\left( i \frac{\Gamma C}{\sqrt{A C}} \right) \propto \sum_{m, n \in \mathbb{Z}, m, n \neq 0} \frac{W\left( \frac{m}{n} \right)}{\sqrt{m/n}} \mathcal{A}_{n,m} \quad (115)
\]

These weights \( \mathcal{A}_{n,m} \) would come from the functional determinant arising from integration over fluctuations around the instanton in our effective field theory in \( \mathbb{R} \).

This is a hard mathematical task, though in principle doable, just as it was for instantons in gauge field theories.

The PDF tail from each term would be

\[
\frac{1}{\sqrt{\Gamma |\mu_1 \Sigma[C]|}} \exp \left( - \frac{|\Gamma n|}{|m \mu_1 \Sigma[C]|} \right) \mathcal{A}_{n,m} \sqrt{\frac{n}{m}} \quad (116)
\]

If we sum over all rational numbers \( \frac{m}{n} \) the exponential decay would become power-like contrary to numerical experiments \( [9] \) which strongly support a single exponential.

So, there is still something we do not understand about our measure on GBF : there are some topological superselection rules on top of the steadiness of the flow and minimization of effective Hamiltonian.

The conventional helicity integral for our solution is computed and discussed in \( [3] \) and also in Appendix D of this paper.

Another topological invariant which depends of these winding numbers was suggested in \( [3] \) where it was argued that it was distinguishing our solution from generic Clebsch field.

Consider the circulation \( \Gamma_{\delta C(\alpha)} \) around the infinitesimal loop \( \delta C(\alpha) \) which encircles our loop at some point with angular variable \( \alpha \) (Fig.1). It is straightforward to compute

\[
\Gamma_{\delta C(\alpha)} = \int_{\delta C(\alpha)} \phi_1 d\phi_2 = 2\pi n \phi_1 \quad (117)
\]

Clearly, this circulation stays finite in a limit of shrinking loop \( \delta C \) because of singular vorticity at the loop \( C \).

Now, integrating this over \( d\phi_2 = md\alpha \) we get our original circulation

\[
\int \Gamma_{\delta C(\alpha)} d\phi_2(\alpha) = 2\pi n \int \phi_1 d\phi_2 = 2\pi n \Gamma_C \quad (118)
\]

Geometrically, this is a volume of the solid torus in Clebsch space mapped from the tube made by sweeping the infinitesimal disk around our loop (see Fig.2).
FIG. 1. The infinitesimal loop $\delta C$ (red) encircling original loop $C$ (blue).

FIG. 2. The solid torus mapped into Clebsch space.

This volume stays finite in the limit of shrinking tube and equals $2\pi n$ times the velocity circulation $\Gamma_C$ in original space $R_3$.

This circulation by itself is an oriented area inside the loop in Clebsch space, which area is $m$ times the geometric area, as the area is covered $m$ times by the instanton field.

Let us look at the topology of the mapping from the physical space to the Clebsch space, assuming this space to be $S_2$ as suggested by [12, 13].

We cut out of $R_3$ the infinitesimal solid torus around our loop – this remaining space topologically also represents a solid torus. We cut this solid torus along the minimal surface $S_C$ bounded by $C$, and then glue it back with $2\pi n$ twist around the polar axis (path inside the solid torus).

The two sides of the minimal surface are mapped to the spheres $S_2$ which are rotated by $2\pi n$ around the polar axis. Apparently when we go through the minimal surface of the viscous thickness $h \sim \sqrt{\nu}$ we cover this $S_2$ precisely $n$ times.

This evolution of $\vec{S}(x, y, z)$ when $z$ goes from $-h$ to $+h$ describes this rapid rotation around the vertical axis. The tangential vorticity is related to the angular speed of this rotation, which goes to infinity as $1/h$. We discuss this evolution in some detail in Appendix C.

The corresponding vortex lines come from $z = -\infty$, enter the surface at $z \sim -h$ in the normal direction, then coil $n$ times, then exit at $z \sim h$ and go to $+\infty$ as shown at Fig.3.

This is the first cycle. The second one would correspond to the loop around the origin in polar coordinates we used. This contour does not pass through the surface, so it is topologically equivalent to a contractible loop drawn on a surface of this sphere $S_2$.

In general case such polar coordinates and such origin always exist on a minimal surface described by Enneper-Weierstrass parametrization [25].

However, this origin of polar coordinates is not a singularity of our space, this is just a singular system of coordinates.

As we discussed above, near the origin the $\phi_2$ field remains non-singular, with an extra condition $\partial_r \phi_2 = \partial_y \phi_2 = 0$ at the origin to avoid the $1/|\vec{r}|$ pole in normal component of vorticity near the surface.

This solid torus with cut surface is topologically equivalent to a 3D ball and our Clebsch field maps this ball onto $S_2$. The winding number $n$ counts the covering of the sphere by this map.

The second number $m$ would correspond to the periodicity in terms of the angle $\alpha$ in cylindrical coordinates.
There is no topological invariant which would protect such a periodic solution.

There is another way to arrive at the same conclusion. Topology of the Clebsch field was analysed in previous work [3] (see also Appendix C of this paper) and it was concluded that there is a helicity

\[ H = \int d^3r v_\alpha \omega_\alpha \]  

(119)

which is characterized by an integer. In Appendix D we compute helicity for our instanton in some general way and we found that it was proportional to the winding number \( n \).

\[ H = 2\pi n \int_C \tilde{\phi}_3 d\phi_1 \]  

(120)

Here \( \tilde{\phi}_3 \) is a third Clebsch field parametrizing velocity

\[ v_\alpha = -\tilde{\phi}_2 \partial_\alpha \phi_1 + \partial_\alpha \tilde{\phi}_3 \]  

(121)

This supports our argument that \( n \) has some topological meaning but \( m \) does not.

We therefore restrict ourselves with solutions with

\[ m = 1 \]  

(122)

which have quantized helicity but no fictitious axial singularities.

\[ \tilde{\phi}_3 = \frac{1}{2} \int_C (\partial_\alpha \phi_1 \partial_\alpha \phi_1 - \partial_\alpha \phi_1 \partial_\alpha \phi_1) \]  

(123)

\[ \mu_4 \]  

with \( \mu_4 \) being a phenomenological parameter and \( \Sigma[C] \) in [1119] being calculable in terms of the solution \( H(\vec{r}) \) of universal integral equation, corresponding to minimization of quadratic functional [96].

Contrary to some of my early conjectures, there is no universality in the area law, though there is a universal shape of decay of PDF, and the singular vorticity at the minimal surface is responsible for that decay.

The Wilson loop for each winding number is given by a simple algebraic expression

\[ \langle \exp \left( i \gamma C \right) \rangle_n = \frac{1}{\sqrt{\prod_{i=1}^{3} (1 - i \frac{\gamma \Sigma[C]}{n})}} \]  

(123)

\[ \mu_1 \]  

being calculable in terms of the solution \( H(\vec{r}) \) of universal integral equation, corresponding to minimization of quadratic functional [96].

For the observed rectangular shape these variation computations can be performed at a supercomputer, so we can compute this function with high accuracy and compare with existing DNS data.

The PDF is given by sum over positive integer winding numbers \( n \)

\[ P(\Gamma) = \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} e^{-i\gamma \Gamma} \left\langle \exp \left( i \gamma \int_C d^3r v_\alpha \right) \right\rangle \propto \frac{1}{\sqrt{\Gamma \mu_1 \Sigma[C]}} \sum_{n=1}^{\infty} \exp \left( -n \frac{|\Gamma|}{\mu_1 |\Sigma[C]|} \right) A_n \sqrt{n} \]  

(124)
Negative winding numbers are responsible for another branch of the PDF, so that resulting PDF is an even function of circulation at large $|\Gamma|$. Pre-exponential factors $A_n$ come from the next order in WKB approximation, the functional integral over the perturbations around our instanton. The leading terms produce determinants which are designed to cancel, so the corrections start in the next order of perturbation expansion.

The instanton expansion like this one was computed exactly in certain CFT in two and four dimensions. In general, it is a difficult task in gauge theories as the multi-instanton solution is quite complex. Our Abelian theory with multi-instanton simply corresponding to higher winding number in the same solution, is supposed to be much simpler than that of gauge theories, especially because of cancellation of determinants.

There is something remarkable with this exponential decay.

With circulation here being the sum of normal components of large number of local vorticities over the minimal surface, it is nontrivial for this circulation to have an exponential distribution, regardless of the local vorticity PDF as long as it has finite variance.

The Central Limit theorem tells us that unless these local vorticities are all strongly correlated, resulting flux (i.e. circulation) will have a Gaussian distribution.

The spectacular violation of this Gaussian distribution in the DNS [9] with seven decades of exponential tails, strongly suggest that there are large spatial structures with correlated vorticity, relevant for these tails.

In this paper, developing and correcting the previous one, we identified these spatial structures as coherent vorticity spread thin over minimal surface.

The steady state equations would be simply:

\[
\partial_t \mathbf{\phi} = (\mathbf{U}, \mathbf{p}) ;
\]

\[
G_\alpha = \frac{\partial \omega_\alpha}{\partial \phi_a} \partial_t \phi_a
\]

\[
\text{pf} \left[ G_\alpha, G_\beta \right] = \sqrt{\det \mathbf{g}}
\]

\[
\mathbf{g}_{ab} = \frac{\partial G_\alpha}{\partial \phi_a} \frac{\partial G_\beta}{\partial \phi_b}
\]

with $U_i = \partial_i U$, $U_{ij} = \partial_i \partial_j U$ etc. Note that the Jacobian $\det \mathbf{g}$ is not always positive in this Hamiltonian system, but our pfaffian is positive.

We assume now, that just as in case of continuous GBF equations, there are more constraints $\omega_\alpha, \alpha = 1, \ldots, M$ than dimension $2N$ of our phase space, but there are only $2N$ independent constraints because some of these $G_\alpha$ are linearly related.

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Sasha Polyakov read the draft of this paper and we had a productive discussion, helping me understand the meaning of my distribution.

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**Appendix A: Finite Dimensional Stationary Distribution**

Let us study our distribution for a simple example of $N$ dimensional particle moving in phase space $\mathbf{\phi}$ with Hamiltonian:

\[
\mathbf{\phi} = (p_1, q_1) \quad (A1)
\]

\[
H(\mathbf{\phi}) = \frac{p^2}{2} + U(\mathbf{q}) \quad (A2)
\]

Let us consider some vector functions $\mathbf{\omega}(\mathbf{\phi})$ in phase space which we would like to be stationary so we impose constraints

\[
\mathbf{\omega} = \partial_t \mathbf{\omega} = 0 \quad (A3)
\]

The steady state equations would be simply:

\[
\mathbf{\phi} = (\mathbf{U}, \mathbf{p}) ;
\]

\[
G_\alpha = \frac{\partial \omega_\alpha}{\partial \phi_a} \partial_t \phi_a
\]

\[
\text{pf} \left[ G_\alpha, G_\beta \right] = \sqrt{\det \mathbf{g}}
\]

\[
\mathbf{g}_{ab} = \frac{\partial G_\alpha}{\partial \phi_a} \frac{\partial G_\beta}{\partial \phi_b}
\]

So, do we have a theory of turbulence? Not yet IMHO, but we may be getting there.

Once again I am appealing to young string theorists: come and help me! This is no less beautiful than conformal field theories or matrix models. You would understand it and you can develop it into a Theory of Turbulence.
Let us consider linear vicinity of the stationary point \( \phi^* \) solving \( \partial_t \tilde{\phi}(\tilde{\phi}^*) = 0 \) and represent the \( M \) dimensional delta function as a Fourier integral

\[
\delta(\tilde{G}(\tilde{\phi})) = \int d^M u \exp \left(i \tilde{u} \tilde{G}(\tilde{\phi}) \right) \quad (A5)
\]

By definition \( G(\phi^*) = 0 \), so we can expand near this stationary point and we get (with \( \psi = \phi - \phi^* \))

\[
\int d^M u \exp \left(i \sum a \frac{\partial G}{\partial \phi_a} \psi_a \right) \quad (A6)
\]

Now we perform singular value decomposition \([17]\) of the rectangular matrix \( \frac{\partial G}{\partial \phi_a} \) (which is a pair of orthogonal transformations in left and right spaces preserving volume elements)

\[
\bar{u} = \sum_i \bar{u}^i \bar{U}^i; \quad (A7a)
\]

\[
\bar{\psi} = \sum_i \bar{\psi}^i \bar{V}^i; \quad (A7b)
\]

\[
\det \bar{U} = \det \bar{V} = 1, \quad (A7c)
\]

\[
\sum_i \bar{u}^i \lambda_i \bar{\psi}^i; \quad (A7d)
\]

and we are left with integrals over components \( \bar{u}^i \) with finite eigenvalues \( \lambda_i \) which lead to desired result

\[
\int' d^M u \exp \left(i \sum a \frac{\partial G}{\partial \phi_a} \psi_a \right) = \\
\int' d^M \bar{u} \exp \left(i \sum_i \bar{u}^i \lambda_i \bar{\psi}^i \right) \propto \\
\delta^{2N}(\bar{\psi}) \quad (A8)
\]

The integrals over the zero modes produce infinities and has to be eliminated by our prescription with the Pfaffian.

Following our prescription in this case would lead to the distribution:

\[
P(\tilde{\phi}) = \sqrt{\det \bar{g}(\bar{G})} \propto \sum_{\tilde{\phi}^*: \partial_t \tilde{\phi}^*(\tilde{\phi}^*) = 0} \delta(\tilde{\phi} - \tilde{\phi}^*) \quad (A9)
\]

which corresponds to the sum over all equilibrium states. Each such state \( \tilde{\phi}^* = (\tilde{0}, \tilde{r}) \) corresponds to a particle sitting at the local extremum \( \tilde{r} \) of the potential well with zero momentum, with net zero force acting at it.

Note that we count each such equilibrium state (stable or not!) with equal weight, which we normalize to 1.

In case there is some extra invariance of observables \( \tilde{\omega} \) with respect to transformation of original phase space coordinates \( \phi \), there will be some zero modes in the metric tensor \( \bar{g} \).

Integrating over these zero modes (gauge orbits) is not Gaussian, and has to be fixed by some gauge conditions with proper Faddeev-Popov Jacobian, which we do not consider here, as this is a well known procedure.

As for the time independence of the measure, this degeneracy does not affect it: each of these degenerate points does not move in Hamiltonian dynamics, regardless the fact that observables related to these points have the same values.

One could argue that prescription without absolute value of the Jacobian also has mathematical meaning, representing a topological invariant. In this case the meta-stable states with negative Jacobian will enter with negative sign.

For example, in one-dimensional case

\[
\int dx U'(x) \delta(U(x)) \quad (A10)
\]

one can start with an oscillator potential \( U(x) = \frac{1}{2}x^2 \) with only one minimum at the origin and add cubic and quartic terms, leading to the double-well potential with one maximum and two minima. Our pfaffian \( |U'(x)| \) would count \( 1 + 1 + 1 = 3 \) states in such a system, but the topological prescription would still have \( 1 - 1 - 1 = 1 \), same as for an initial oscillator.

The time-independence of this measure is obvious, as the stationary points by definition do not move with time

\[
\partial_t \bar{\phi}(\bar{\phi}^*) = 0 \quad (A11)
\]

Our canonical ensemble would be:

\[
\int d^{2N} \phi \exp \left(-\lambda \mathcal{H}_{\text{eff}} \left( \tilde{\omega}(\tilde{\phi}) \right) \right) P(\tilde{\phi}) \propto \sum_{\tilde{\phi}^*: \partial_t \tilde{\phi}^*(\tilde{\phi}^*) = 0} \exp \left(-\lambda \mathcal{H}_{\text{eff}} \left( \tilde{\omega}(\tilde{\phi}^*) \right) \right) \quad (A12)
\]

This is an example of so called "trivial" conservation laws, present in every Hamiltonian dynamics: place the system in its mechanical equilibrium, give it zero velocities and it will stay there.
Except in case there are many (or a continuous manifold) of these stationary states, our distribution gives equal weight to each of them. It is implied that the invisible forces from thermostat kick the system from one stationary state to another one, eventually leading to this uniform distribution over stationary states.

In the context of GBF this space of stationary points is not so trivial, in fact, as we shall see it is rich enough to describe the critical phenomena in turbulent flow.

Even in this elementary example we see a complication. Consider axial symmetric potential of sombrero hat.

There is a maximum at the origin and degenerate minimum. A sphere $q^2 = 1$. We get zero determinant at $N > 1$ at the minimum because of the zero modes corresponding to rotations of this minimal sphere.

This is clearly not what we need: to reject the maximum we need to demand that the tangent vorticity would be directed at the oriented normal to the surface and will be continuous, as only values of Clebsch field are jumping, but not the tangent plane derivatives. This applies only to the limits of vorticity from above and below the minimal surface (see the next section).

Such surface is shown at Fig. 5 for simplest Weierstrass-Enneper parametrization [25]:

![Fig. 5. The Enneper’s Minimal surface with $f = 1, g = z$](image)

\[
\bar{X}(\rho, \theta) = \bar{F}(\rho e^{i\theta}) \\
\bar{F}^z(z) = \left\{ \frac{1}{2}(1 - g^2)f, \frac{1}{2}(1 + g^2)f, gf \right\} \quad \text{(B3b)}
\]

with $g(z), f(z)$ being analytic functions inside the unit circle $|z| < 1$.

With $\phi_a(\xi)$ depending only on local coordinates $\xi = (\xi_1, \xi_2)$ on the minimal surface $r_\alpha = X_\alpha(\xi)$ we have:

\[
\Gamma = \int_{S_{\text{min}}^{(C)}} d\sigma_\alpha(r)\omega_\alpha(r), \quad \text{(B4a)}
\]

\[
\omega_\alpha(r) = n_\alpha(r)\Omega(r) \quad \text{(B4b)}
\]

\[
\Omega(r) = \frac{1}{\sqrt{G}} \frac{\partial(\phi_1, \phi_2)}{\partial(\xi_1, \xi_2)} \quad \text{(B4c)}
\]

where $G$ is determinant of the induced metric $G_{ij} = \partial_iX_\alpha \partial_jX_\alpha^*; i, j = 1, 2$. Geometrically, this $\Omega$ is the ratio of area element in Clebsch plane to that on a minimal surface.

It is important though that this $\Omega(r)$ factor can be extended in linear vicinity of the surface. Namely, in the linear vicinity in the normal direction it does not depend upon the normal coordinate $z$ as it follows from our condition [B3] on normal derivatives of Clebsch field (again, this excludes $z = 0$ where there are singular terms $\propto \delta(z)$).

\[
n_\alpha\partial_\alpha\Omega(r) = 0 \quad \text{(B5)}
\]
Let us verify it. In linear vicinity of local tangent plane to the surface its equation reads (with $K_1, K_2$ being principal curvatures at this point)

\[
\begin{align*}
&z - \frac{K_1}{2}x^2 - \frac{K_2}{2}y^2 = 0 \quad (B6a) \\
n_i = \frac{(-K_1x, -K_2y, 1)}{\sqrt{1 + K_1^2x^2 + K_2^2y^2}} \to (0, 0, 1) \quad (B6b) \\
&\Omega = n_\alpha \omega_\alpha \to \frac{1}{2} e_{ij} e_{\alpha \beta} \partial_i \phi_\alpha \partial_j \phi_\beta \quad (B6c) \\
n_\alpha \partial_\alpha \Omega(r) \to e_{ij} e_{\alpha \beta} \partial_j \phi_\beta \quad (B6d)
\end{align*}
\]

The mixed derivatives $\partial_i \partial_j \phi_\alpha$ vanish at $x = y = z = 0$ for our boundary conditions.

Self-consistency of this solution for Clebsch parameterization requires that this surface should be a minimal surface.

Indeed, let us assume that $\phi_\alpha$ has a discontinuity along some surface, with normal derivatives vanishing on both sides of the cut in $R_3$. In this case we would have vorticity proportional as the normal $n_\alpha$ to that surface with coefficient $\Omega(r)$ depending only on the local tangent coordinates, no $z$ dependence in linear vicinity.

The vorticity conservation $\partial_\alpha \omega_\alpha = 0$ would then lead to the equation

\[
0 = \partial_\alpha \omega_\alpha = \partial_\alpha (n_\alpha \Omega) = \Omega \partial_\alpha n_\alpha + n_\alpha \partial_\alpha \Omega \quad (B7)
\]

The term $\partial_\alpha n_\alpha$ here involves the surface derivatives as in $n_\alpha \partial_\beta n_\alpha = \frac{1}{2} \partial_\beta n^2 = 0$. Therefore

\[
\partial_\alpha n_\alpha = (\delta_{\alpha \beta} - n_\alpha n_\beta) \partial_\beta n_\alpha = -K_1 - K_2 \quad (B8)
\]

which is the divergence in the tangent plane, or trace of external curvature tensor (see Fig. 3 for detailed discussion).

We see, that for our boundary condition, with vanishing normal derivatives of Clebsch field and therefore vorticity, we arrive at the Plateau equation for the minimal surface $K_1 + K_2 = 0$.

This is quite remarkable: Clebsch field is allowed to have jumps across minimal surface as long as its normal derivatives vanish at each side of this surface!

**Appendix C: Non-singular gauge and Winding numbers**

Our singular gauge where $\phi_2$ is related to the angular variable in cylindrical coordinates and has $2\pi n$ discontinuity on a minimal surface raises obvious questions: maybe this is all an artefact of singular coordinates? What happens in a regular gauge where the Clebsch field is continuous?

Let us study the Clebsch field as a point on $S_2$, using the KM parametrization (29) (with $Z = 1$ for simplicity). The unit vector $\vec{S} \in S_2$ will have components

\[
\begin{align*}
&S_3 = \phi_1; \quad (C1a) \\
&S_1 + i S_2 = \sqrt{1 - S_3^2} e^{i \phi_2}; \quad (C1b) \\
&\omega_\alpha \propto e_{\alpha \beta \gamma} e_{ijk} S_i \partial_\beta S_j \partial_\gamma S_k \quad (C1c)
\end{align*}
\]

As the $2\pi n$ discontinuities of $\phi_2$ now "disappeared" in phase factor, how do we get our singular vorticity in this gauge?

Let us resolve this paradox in a physicist’s way. These discontinuities are, in fact, the approximation to the peaks of vorticity in Zeldovich pancakes. The Clebsch fields are not discontinuous with finite viscosity, they are rather changing in a thin layer of the thickness $\nu \sim \sqrt{\nu}$, imitating step function in a phase discontinuity.

\[
\begin{align*}
&\phi_2 \approx m \alpha + 2\pi n \Theta_h(z) + O(z^2); \quad (C2a) \\
&\Theta_h(z) = \frac{1}{2\nu} \int_{-\infty}^{z} \frac{\exp \left( -\frac{|u|}{\nu} \right)}{h} \, du; \quad (C2b) \\
&\frac{1}{2} (\text{sign}(u)(1 - \cosh(u)) + \sinh(u) + 1); \quad (C2c) \\
&u = \frac{z}{\nu} \quad (C2d)
\end{align*}
\]

The complex field $\Psi(x, y, z) = S_1 + i S_2$ now has some rapid changes in the region $|z| \sim \nu$ in normal direction to the minimal surface. Specifically, we have

\[
\frac{\partial \Psi}{\partial z} = 2\pi i n \Psi \Theta_h(z) + \text{reg terms} \quad (C3)
\]

The vorticity will have singular tangential components (with all factors $\sqrt{1 - S_3^2}$ cancel thanks to symplectomorphisms invariance of this representation)

\[
\omega_\alpha \propto 2\pi n e_{\alpha \beta \gamma} \partial_\beta S_3 \Theta_h' (z) \rightarrow \pi n e_{\alpha \beta \gamma} \partial_\beta \delta(z) \quad (C4)
\]

This smearing of a delta function exposed an interesting phenomenon. The two sides of the minimal surface are mapped to the spheres $S_2$ which are rotated by $2\pi n$ around the $z$ axis. Apparently when we go through the minimal surface we cover this $S_2$ precisely $n$ times.

This evolution of $\Psi(x, y, z)$ when $z$ goes from $-h$ to $+h$ describes this rapid rotation of $\vec{S}(x, y, z)$ around the vertical axis. The tangential vorticity is related to the angular speed of this rotation, which goes to infinity as $1/\nu$.

The corresponding vortex lines come from $z = -\infty$, enter the surface at $z \rightarrow -h$ in the normal direction, then coil $n$ times, then exit at $z \sim h$ and go to $+\infty$ as shown at Fig. 3.

There is still a potential singularity in this representation, namely at the axis of cylindrical coordinates, where the plane coordinates $x + i y \to 0$. Representing

\[
e^{i \alpha} = \frac{x + i y}{\sqrt{x^2 + y^2}} \quad (C5)
\]

and combining the square roots we have

\[
S_1 + i S_2 = \sqrt{1 - S_3^2} e^{i \phi_2} \rightarrow \sqrt{1 - (x^2 + y^2)}^n f^2(x, y, z) \quad (C6)
\]

This expression will have no singularities in coordinate space provided near this axis $x, y = 0$

\[
S_3 \to 1 - (x^2 + y^2)^n f^2(x, y, z) \quad (C7)
\]
In other words the axis of the cylindrical coordinates maps into one of the poles of the sphere \( S_2 \). In general case of the non-planar minimal surface this axial axis would be some path intersecting the surface in the normal direction and going to infinity.

So, we view our physical space as the solid torus \( (R_3 \) with infinitesimal tube around \( C \) cut out of it). This solid torus is cut across this minimal surface and glued back with \( 2\pi n \) twist around the angle \( \alpha \) around the axial origin (path in this solid torus crossing the minimal surface).

One could present a manifestly regular parametrization of the sphere, adequate to our instanton solution, in terms of the stereographic coordinates

\[
S_3 = \frac{1 - |u|^2|w|^2}{1 + |u|^2|w|^2}; \quad (C8a)
\]

\[
S_1 + i S_2 = \frac{2uw}{1 + |u|^2|w|^2}; \quad (C8b)
\]

\[
u = (x + ty)^m; \quad (C8c)
\]

\[
\arg w = \phi_2 - ma; \quad (C8d)
\]

The complex field \( w(\mathbf{x}, y, z) \), parametrizing the point \( \bar{S} \in S_2 \) is single-valued, and does not have any singularity in \( xyz \) space, except that its phase rapidly rotates \( n \) times around when the surface \( S \) is crossed.

This solid torus with the cut is now topologically equivalent to a ball (inside of \( S_2 \) sphere). This ball is mapped on a stereographic sphere \( S_2 \) with its pole corresponding to that axial path. The field does not have a singularity at this path.

The winding number \( n \) is counting covering of the sphere \( S_2 \) in this map from the ball and the number \( m \) would count periodicity or the Clebsch field with respect to the cylindrical axis rotation. Generic case would be \( m = 1 \), in which case no adjustment of parameters would be needed to cancel derivatives of \( S_1 + i S_2 \) at the cylindrical axis \( x = y = 0 \).

### Appendix D: Helicity

Let us now look at the helicity integral

\[
H = \int_{R_3 \setminus S_{\text{min}}} d^3 r \mathbf{\tilde{v}} \cdot \mathbf{\omega} \quad (D1)
\]

Note that in conventional form

\[
v_i = \phi_1 \partial_i \phi_2 + \partial_i \phi_3 \quad (D2)
\]

there will be singular terms in velocity \( \propto \delta(z) \). However, the Biot-Savart integral \( (22) \) demonstrates that these singular terms cancel between \( \phi_2 \) and \( \phi_3 \) leaving finite resulting velocity field.

To avoid these fictitious singularity, let us rewrite velocity in an equivalent form

\[
v_i = -\phi_2 \partial_i \phi_1 + \partial_i \tilde{\phi}_3 \quad (D3)
\]

\[
\tilde{\phi}_3 = \phi_1 \phi_2 + \phi_3 \quad (D4)
\]

This \( \tilde{\phi}_3 \) is single-valued, unlike the \( \phi_3 \). The discontinuity of the first term is compensated by that of the second one. In can be written as an integral over the whole space

\[
\tilde{\phi}_3(r) = -\partial_3 \int d^3 r' \phi_2(r') \partial_3 \phi_1(r') \frac{4\pi |r - r'|}{|r - r'|} \quad (D5)
\]

Now the singular component \( \phi_2 \) is not differentiated, so that there are no singularities. The helicity integral could now written as a map \( R_3 \mapsto (\phi_1, \phi_2, \tilde{\phi}_3) \)

Here is the most important point. There is a surgery performed in three dimensional Clebsch space: an incision is made along the surface \( \phi (S_{\text{min}}) \) and then it is glued back with \( 2\pi n \) twist around the axis of cylindrical coordinates.

Integrating over \( \phi_2 \) in \( (D6) \), using discontinuity

\[
\Delta \phi_2 (S_{\text{min}}) = 2\pi n \quad (D7)
\]

and then integrating

\[
\int_{S_{\text{min}}} d\tilde{\phi}_3 \wedge d\phi_1 \quad (D8)
\]

we find a simple formula

\[
H = 2\pi n \int_C \tilde{\phi}_3 d\phi_1 \quad (D9)
\]

One may wonder how can the pseudoscalar invariant like helicity be present in GBF: it is just the time reversal which is broken by energy flow, but not spacial parity.

The answer is that in virtue of the symmetry of the master equation there is always a GBF with an opposite helicity (negative \( n \)) and the same probability. We will take both solutions, instanton and anti-instanton into account when using the WKB methods to compute circulation PDF.

One may also wonder how do we get the nontrivial helicity if the velocity is orthogonal to vorticity at the surface where all action is happening. There are two answers.

Formally, helicity is created just by the discontinuity of the Clebsch field by the tangent component of vorticity in the infinitely thin boundary layer. This delta function
contributes to the helicity integral.

Another answer is that in the helicity integral over the remaining space $R_3 \setminus S_{\text{min}}$, the dot product $\vec{\omega} \vec{V}$ is not zero but rather reduces to a total derivative of the phase field $\phi_2$. After cancellations of all internal terms this integral is proportional to the total phase change from one side of the surface to another, which is $2\pi n$.

Regardless how we compute helicity we observe that resulting loop integral \( (D9) \) involves non-singular field $\hat{\phi}_3$ which depends upon the behavior of the basic Clebsch field $\phi_1, \phi_2$ in the whole remaining space, not just in linear vicinity of the minimal surface.

Our main physical assumption was that vorticity was inside the loop. Let us approximate the loop by the loop function on a surface. Then the following numerical distribution after all, nor it is relevant for distinguishing our going to zero.

As for the field $\hat{\phi}_3$ at the loop $C$ which is involved in helicity integral, it becomes arbitrarily small when effective thickness $h = \delta z$ goes to zero. Taking into account singularity of the Coulomb kernel we get an estimate $\hat{\phi}_3 \sim h \log h \to 0$.

We observe that in the limit when the effective thickness of vorticity layer goes to zero, we have helicity integral going to zero.

So, the helicity is not responsible for our vorticity distribution after all, nor it is relevant for distinguishing our instanton from some other Clebsch field.

**Appendix E: Finite Element Approximation**

Now, we assume that the function $H(\bar{x})$ is a smooth function on a surface. Then the following numerical approach would work.

Let us cover the domain $D_C$ by a square grid step 1 and assume that there are large number of these unit squares inside the loop. Let us approximate the loop by the loop drawn on this grid, passing through its cites.

Eventually we shall tend the area of $D_C$ to infinity, in which case this quantization will become irrelevant.

Now let us approximate $H(\bar{r})$ by its value at the center $\bar{c}$ inside each square $\square$

\[
H(\bar{r} \in \square) \approx h_{\square} = H(\bar{c}_{\square}) \tag{E1}
\]

The resulting integral over the square is calculable:

\[
I_\alpha(\square, \bar{r}) = \frac{1}{2\pi} \int_{\square} d^2 r' \partial'_\alpha \frac{1}{|\bar{r} - \bar{r}'|} = \sum_{i=0}^{3} (-1)^i A_\alpha \left( \bar{V}_i - \bar{r} \right) \tag{E2}
\]

Here $\bar{V}_i, i = 0, 1, 2, 3$ are the vertices of $\square$, counted anticlockwise starting with the left lower corner $\bar{V}_0$ and $A_\alpha(\bar{r}) = \frac{1}{2\pi} \arctan(\hat{r}_\alpha; \hat{r}) = \frac{\bar{r}}{|\bar{r}|}$

Thus we get an approximation

\[
F_\alpha[H, \bar{r}] \approx \sum_{\square \in D_C} h_{\square} I_\alpha(\square, \bar{r}) \tag{E4}
\]

After that the target functional $Q(H)$ becomes an ordinary quadratic form of a vector $h_{\square}, \square \in D_C$

The integral $\int_{D_C} d^2 r$ in $\square$ converges (there is logarithmic singularity in $A_\alpha(\bar{r})$ at $r_\alpha \to \pm |\bar{r}|$, but it is integrable). We have to compute symmetric matrix

\[
\langle \square_1 | M | \square_2 \rangle = \int_{D_C} d^2 r I_\alpha(\square_1, \bar{r}) I_\alpha(\square_2, \bar{r}) \tag{E5}
\]

and the linear term

\[
\int_{D_C} d^2 r R(\bar{r}) H(\bar{r}) \approx \sum_{\square} R(\bar{r}) h_{\square} \tag{E6}
\]

where $\square_0$ is the square at the origin (the center of the domain).

These integrals for the matrix elements as well as the linear term are calculable with 5 significant digits using adaptive cubature library [27], based on recursive subdivision of the multidimensional cube [28]. We wrote parallel code which works fast enough for millions of squares on a supercomputer.

For numerical stabilization we replaced the singular logarithm function in \( (E3) \) by cutoff function at $\epsilon = 10^{-6}$

\[
A_\alpha(\bar{r}) \approx \frac{\ln (1 + \hat{r}_\alpha, \epsilon) - \ln (1 - \hat{r}_\alpha, \epsilon)}{4\pi}; \tag{E7}
\]

\[
\ln(x, \epsilon) = \ln(\max \{|x|, \epsilon\}) \tag{E8}
\]

We also added to our target the stabilizer:

\[
Q[\bar{h}] = - \sum_{\square} h_{\square} R(\bar{r}) + \frac{1}{2} \sum_{\square_1, \square_2} h_{\square_1} \langle \square_1 | M | \square_2 \rangle h_{\square_2} + \frac{1}{2} \lambda(M) \sum_{\langle \square_1, \square_2 \rangle} (h_{\square_1} - h_{\square_2})^2 \tag{E9}
\]

Here $\square_0$ is the origin in our plane, $\langle \square_1, \square_2 \rangle$ denote squares sharing a side and

\[
\lambda(M) = \max |\delta M| \tag{E10}
\]
is maximal absolute error in computation of numerical integrals for matrix elements of \( M \), in our case \( \lambda \approx 10^{-6} \).

There are also three constraints (with \( \mathbf{0} \) representing the origin, which is a geometric center of the domain):

\[
C_1 : \sum_{\square} h_{\square} \int_{D_C} d^2r I_a(\square, r) = 0; \\
C_2 : h_{\square} = 0; \forall \square \in C; \\
C_3 : \sum_{\square} h_{\square} I_a(\square, \mathbf{0}) = 0
\]

(E11) (E12) (E13)

Once the matrix \( M \) is computed, the solution for the grid weights \( h_{\square} \) is given by the minimum of quadratic form \( Q \) with conditions \( C_1, C_2 \)

\[
h_{\square} = \arg \min \{ Q | C_1, C_2, C_3 \}
\]

(E14)

As for the symmetric positive definite matrix inversion, there are fast parallel libraries available in \( e^{++} \), so this looks achievable even for the grids with million squares.

We are planning to perform this computation for rectangles with various aspect ratios on a supercomputer and compare to available DNS data.

The circulation integral in terms of these coefficient \( h_{\square} \) reads

\[
\Gamma[C] = m \sum_{\square} h_{\square} \int_0^{2\pi} d\theta \int_0^\square d^2r \left( \frac{1}{|r|} - \frac{1}{|\mathbf{r}|} \right)
\]

(E15)

Note that in virtue of our boundary condition \( h_{\square} \in C = 0 \) the singular terms with the squares at the boundary \( C = \partial D \) are excluded from the sum.

The remaining terms contain integrals over the angle \( \theta \) of the double integrals \( \int_{D_C} d2r \) of Coulomb kernel.

These integrals are calculable. The basic integral reads

\[
B(x, y) = \int_0^x \int_0^y \frac{du dv}{\sqrt{u^2 + v^2}} = I(x, y) + I(y, x)
\]

\[
I(x, y) = 4x \arcsinh \left( \frac{xy}{\sqrt{x^2 + y^2}} \right)
\]

(E16a)

The integral over the square \( \square (\mathbf{P}, \mathbf{Q}) \) with corners at \( \mathbf{P} \) and \( \mathbf{Q} \) is given by sum of four terms

\[
G(\mathbf{P}, \mathbf{Q}) = \int_{\square (\mathbf{P}, \mathbf{Q})} \frac{d^2r}{|r|} = B(Q.x, Q.y) - B(P.x, Q.y) - B(Q.x, P.y) + B(P.x, P.y)
\]

(E17)

So, we represent the integral as (with \( \mathbf{C} = (\frac{1}{2}a, \frac{1}{2}b) \) corresponding to the middle of the rectangle)

\[
\int_0^{2\pi} d\theta \int_{\square (\mathbf{P}, \mathbf{Q})} d^2r \left( \frac{1}{|r - L\mathbf{f}(\theta)|} - \frac{1}{|r - \mathbf{C}|} \right) = J_x + J_y - 2\pi G (\mathbf{P} - \mathbf{C}, \mathbf{Q} - \mathbf{C});
\]

\[
J_x = \int_{-a/b}^{a/b} \frac{G(\mathbf{P}(t, 0), \mathbf{Q}) + G(\mathbf{P}(t, b), \mathbf{Q}(-b/a, b))}{1 + t^2} dt
\]

\[
J_y = \int_{-b/a}^{b/a} \frac{G(\mathbf{P}(0, t), \mathbf{Q}) + G(\mathbf{P}(-a/b, a), \mathbf{Q}(t, a))}{1 + t^2} dt
\]

(E18)

\[
\mathbf{P}(t, c) = \mathbf{P} - \left( \frac{a + bt}{2}, c \right)
\]

\[
\mathbf{Q}(t, c) = \mathbf{Q} - \left( \frac{b + at}{2}, c \right)
\]

(E19) (E20)

These \( \mathbf{P}(t), \mathbf{Q}(t) \) are equations of the sides of our polygon. Also note that in the limit of large size of the domain, when the number \( N \) of grid squares goes to infinity, the coefficients \( h_{\square} \) decrease as \( 1/N \).

In this limit, our sum over squares becomes the Riemann sum for an integral [97].

The reason for exactly computing the integrals over elementary squares with constant \( H(\mathbf{r}) \) inside each square was the Coulomb singularity. Resulting functions \( A_\alpha(\mathbf{r}), B(x, y) \) has only a logarithmic singularities, rather than the pole in Coulomb potential. So, the integrals involving these functions can be computed with high ac-
accuracy using cubature package [27] using regularization of logarithms with $\epsilon$ terms.

By exactly computing singular integrals we accelerated the convergence to a local limit $N \to \infty$. With Riemann sums for Coulomb kernel the errors would be $O\left(1/\sqrt{N}\right)$, but with replacing $H(\vec{r})$ by its values at the center the relative errors are related to second derivatives which is $O\left(1/N\right)$. So, with accessible $N \sim 10^6$ at modern supercomputers we expect to get 5 significant digits, which is beyond the statistical and systematic errors of the DNS at achievable lattices $24K^3$.

The hardest part of this computation is numerical integration needed for the kernel $\langle \Box_1 | M | \Box_2 \rangle$ for all the squares $\Box_1, \Box_2$. It has $O\left(N^3\right)$ complexity where $N$ is the number of squares inside $D_C$. Still, with $N \sim 100$ this (parallel) computation using adaptive cubature library [20] takes less than a minute on my server with 24 cores.

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In my work [25] the Clebsch variables were identified as major degrees of freedom in statistics of vortex cells and their potential relations to string theory was suggested. Finally, in recent work [34] I identified the surface degrees of freedom of the vortex cells as $U(1)$ compactified critical $c = 1$ string in two dimension, which was exactly solved by means of matrix models.
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