Are neutrinos spinorial tachyons?

Jakub Rembieliński
Katedra Fizyki Teoretycznej, Uniwersytet Łódzki
ul. Pomorska 149/153, 90–236 Łódź, Poland

Abstract
Quantum field theory of space-like particles is investigated in the framework of absolute causality scheme preserving Lorentz symmetry. It is shown that tachyons are associated with unitary orbits of Poincaré mappings induced from \( SO(2) \) little group instead of \( SO(2, 1) \) one. Therefore the corresponding elementary states are labelled by helicity. A particular case of the helicity \( \lambda = \pm \frac{1}{2} \) is investigated in detail and a corresponding consistent field theory is proposed. In particular, it is shown that the Dirac-like equation proposed by Chodos et al. [1], inconsistent in the standard formulation of QFT, can be consistently quantized in the presented framework. This allows us to treat more seriously possibility that neutrinos can be fermionic tachyons as it is suggested by the present experimental data about neutrino masses [2].

1 Introduction
Almost all recent experiments, measuring directly or indirectly the electron and muon neutrino masses, have yielded negative values for the mass square [2, 1]. It suggests that these particles might be fermionic tachyons. This intriguing possibility was written down some years ago by Chodos et al. [1] and Recami et al. [3]. In the light of the mentioned experimental data we observe a return of interest in tachyons [2, 3].

On the other hand, in the current opinion, there is no satisfactory theory of superluminal particles. This persuasion creates a psychological barrier to take such possibility seriously. Even if we consider eventuality that neutrinos are tachyons, the next problem arises; namely a modification of the theory of electro-weak interaction will be necessary in such a case. But, as we known, in the standard formulation of special relativity, the unitary representations of the Poincaré group, describing fermionic tachyons, are induced from infinite dimensional unitary representations of the non-compact \( SO(2, 1) \) little group. Consequently, in the conventional approach, the neutrino field should be infinite-component one so a construction of an acceptable local interaction is extremely difficult.

In this paper we suggest a solution to the above dilemma. To do this we use the formalism developed in the papers [6, 9] based on the earlier works [7, 8].

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*This work is supported under the Łódź University grant no. 457.
†E-mail address: jaremb@mvii.uni.lodz.pl
where it was proposed a consistent description of tachyons on both classical and quantum level. The basic idea is to extend the notion of causality without a serious change of special relativity. This can be done by means of a freedom in the determination of the notion of the one-way light velocity, known as the “conventionality thesis” \[9, 10\].

In the presented approach the relativity principle is formulated in the framework of a non-standard synchronization scheme (the Chang–Tangherlini (CT) scheme). This allows to introduce an absolute causality for all kinds of events (time-like, light-like, space-like). For “standard particles” our scheme is fully equivalent to the usual formulation of special relativity. On the other hand, for tachyons it is possible to formulate covariantly proper initial conditions and there exists a covariant lower bound of energy. Moreover, the paradox of “transcendental” tachyons does not appear in this scheme. On the quantum level tachyonic field can be consistently quantized using CT synchronization procedure and they distinguish a preferred frame via mechanism of the relativity principle breaking \[7, 5\], however with the preservation of the Lorentz covariance and symmetry.

The main properties of the presented formalism are in the agreement with local properties of the observed world; namely, we can in principle distinguish locally a preferred inertial frame by investigation of the isotropy of the Hubble constant. In fact, it coincides with the frame in which the Universe appears spherically\[1\]. Obviously, such a (local) preferred frame should correlate with the cosmic background radiation frame. Moreover, present cosmological models incorporate an absolute time (cosmological time). Therefore it is very natural to look for a local (flat space) formalism incorporating both Lorentz covariance and a distinguished inertial frame. Notice that two paradigms of the standard understanding of the (flat) space-time, namely the assumption of equivalence of inertial reference frames and a “democracy” between time and space coordinates, are in conflict with the above mentioned local properties of the observed world.

In this paper we classify all possible unitary Poincaré mappings for space-like momenta. The important and unexpected result is that unitary orbits for space-like momenta are induced from the \(SO(2)\) little group. This holds because we have a bundle of Hilbert spaces rather than a single Hilbert space of states. Therefore unitary operators representing Poincaré group act in irreducible orbits in this bundle. Each orbit is generated from subspace with \(SO(2)\) stability group. Consequently, elementary states are labelled by helicity, in an analogy with the light-like case. This fact is extremely important because we have no problem with infinite component fields.

Now, let us begin with a brief review of the theory proposed in \[5, 7, 8\].

### 2 Formalism

According to the papers \[3, 4\], transformation between two coordinate frames \(x^\alpha\) and \(x'^\alpha\) has the following form

\[
x' = D(\Lambda, u)(x + a),
\]

\[1\] It is well known such a situation is typical for Robertson–Walker space-times, see e.g. \[11\].
\[ u' = D(\Lambda, u)u. \]  

Here \( \Lambda \) belongs to the Lorentz group \( \mathcal{L} \), whilst \( u \) is a four-velocity of a privileged inertial frame \( x' \), as measured by an observer using \( x'^\mu \) coordinates. The \( a^\mu \) are translations. The transformations (1–2) have standard form for rotations i.e. \( D(R, u) = R \), whereas for boosts the matrix \( D \) takes the form

\[
D(\vec{V}, u) = \begin{pmatrix}
\gamma & 0 \\
-I + \frac{\vec{V} \otimes \vec{V}^T}{c^2} & \gamma + \sqrt{\gamma^2 + \frac{\vec{V}^2}{c^2}} \gamma_0^{-1} \\
\gamma_0^{-1} & -I + \frac{\vec{\sigma} \otimes \vec{\sigma}^T}{c^2} \gamma_0^{-4}
\end{pmatrix}
\]  

where we have used the following notation

\[
\gamma_0 = \left[ \frac{1}{2} \left( 1 + \sqrt{1 + \left( \frac{\vec{\sigma}}{c} \right)^2} \right) \right]^{1/2} = \frac{c}{u^0},  
\]

\[
\gamma(\vec{V}) = \left( \left( 1 + \frac{\vec{\sigma} \vec{V}}{c^2} \gamma_0^{-2} \right)^2 - \left( \frac{\vec{V}}{c} \right)^2 \right)^{1/2},  
\]

\[
\frac{\vec{\sigma}}{c} = \frac{\vec{u}}{u^0}.  
\]

Here \( \vec{V} \) is the relative velocity of \( x' \) frame with respect to \( x \) whilst \( \vec{\sigma} \) is the velocity of the preferred frame measured in the frame \( x \). The transformations remain unaffected the line element

\[
ds^2 = g_{\mu\nu}(u)dx^\mu dx^\nu
\]

with

\[
g(u) = \begin{pmatrix}
1 & \frac{u^0 u^T}{c^2} \\
\frac{u^0 u^T}{c^2} & -I + \frac{\vec{u} \otimes \vec{u}^T}{c^4} (u^0)^2
\end{pmatrix} = \begin{pmatrix}
1 & \frac{\vec{\sigma}^T}{c} \gamma_0^{-2} \\
\frac{\vec{\sigma}^T}{c} \gamma_0^{-2} & -I + \frac{\vec{\sigma} \otimes \vec{\sigma}^T}{c^2} \gamma_0^{-4}
\end{pmatrix},
\]

\[u^2 = g_{\mu\nu}(u)u^\mu u^\nu = c^2.\]

From (7) we can calculate the velocity of light propagating in a direction \( \vec{n} \)

\[
\vec{c} = \frac{\vec{c} \vec{n}}{1 - \frac{\vec{n} \vec{c}}{c} \gamma_0^{-2}}.
\]

It is easy to verify that the average value of \( \vec{c} \) over a closed path is always equal to \( c \).  

\(^2\)A necessity of a presence of a preferred frame for tachyons was stressed by many authors (see, for example, [12, 13]).
Now, according to our interpretation of the freedom in realization of the Lorentz group as freedom of the synchronization convention, there should exist a relationship between $x^h$ coordinates and the Einstein-Poincaré (EP) ones denoted by $x^E_h$. Indeed, we observe, that the coordinates

$$x_E = T^{-1}(u)x,$$

$$u_E = T^{-1}(u)u,$$

where the matrix $T$ is given by

$$T(u) = \begin{pmatrix} 1 & -\frac{u^0u^T}{c^2} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\sigma^T}{c}\gamma^{-2} \\ 0 & I \end{pmatrix}. \tag{12}$$

transform under the Lorentz group standardly i.e. (1–2) and (10–11) imply

$$x'_E = \Lambda x_E, \tag{13}$$

$$u'_E = \Lambda u_E. \tag{14}$$

It holds because $D(\Lambda, u) = T(u')\Lambda T^{-1}(u)$. Moreover, $ds^2 = ds^2_E$, $\tilde{c}_E = c\tilde{\sigma}$, $u^2_E = c^2$ and $g_E = \eta \equiv \text{diag}(+, -, -, -)$. Thus the CT synchronization scheme, defined by the transformations rules (1–2), is at first glance equivalent to the EP one. In fact, it lies in a different choice of the convention of the one-way light propagation (see (9)) under preserving of the Lorentz symmetry. Notwithstanding, the equivalence is true only if we exclude superluminal signals. Indeed, the causality principle, logically independent of the requirement of Lorentz covariance, is not invariant under change of the synchronization (10–11). It is evident from the form of the boost matrix (3); the coordinate time $x^0$ is rescaled by a positive factor $\gamma$ only. Therefore $\varepsilon(dx^0)$ is an invariant of (1–2) and this fact allows us to introduce an absolute notion of causality, generalizing the EP causality. Consequently, as was shown in [5], all inconsistencies of the standard formalism, related to the superluminal propagation, disappear in this synchronization scheme.

If we exclude tachyons then, as was mentioned above, physics cannot depend of synchronization. Thus in this case any inertial frame can be chosen as the preferred frame, determining a concrete CT synchronization. This statement is in fact the relativity principle articulated in the CT synchronization language.

What happens, when tachyons do exist? In such a case the relativity principle is obviously broken: If tachyons exist then only one inertial frame is the true privileged frame. Therefore, in this case, the EP synchronization is inadequate to description of reality; we must choose the synchronization defined by (1–9). Moreover the relativity principle is evidently broken in this case as well as the conventionality thesis: The one-way velocity of light becomes (a priori) a really measured quantity.

To formalize the above analysis, in [5, 8] it was introduced notion of the synchronization group $L_S$. It connects different synchronizations of the CT-type and it is isomorphic to the Lorentz group:

$$x' = T(u')T^{-1}(u)x = D(A_S, u)T(u)A_S^{-1}T^{-1}(u)x, \tag{15}$$

$$u' = D(A_S, u)u, \tag{16}$$
with $\Lambda_S \in L_S$.

For clarity we write the composition of transformations of the Poincaré group $L \times T^4$ and the synchronization group $L_S$ in the EP coordinates

$$x'_E = \Lambda(x_E + a_E),$$

(17)

$$u'_E = \Lambda_S u_E.$$  

(18)

Therefore, in a natural way, we can select three subgroups:

$$L = \{(I, \Lambda)\}, \quad L_S = \{ (\Lambda_S, I) \}, \quad L_0 = \{ (\Lambda_0, \Lambda_0^{-1}) \}.$$

By means of (17–18) it is easy to check that $L_0$ and $L_S$ commute. Therefore the set $\{ (\Lambda_S, \Lambda) \}$ is simply the direct product of two Lorentz groups $L_0 \otimes L_S$. The intersystemic Lorentz symmetry group $L$ is the diagonal subgroup in this direct product. From the composition law (17–18) it follows that $L$ acts as an automorphism group of $L_S$.

Now, the synchronization group realizes in fact the relativity principle: If we exclude tachyons then transformations of $L_S$ are canonical ones. On the other hand, if we include tachyons then the synchronization group $L_S$ is broken to the $SO(3)_u$ subgroup of $L_S$; here $SO(3)_u$ is the stability group of $u^\mu$. In fact, transformations from the $L_S/SO(3)_u$ do not leave the absolute notion of causality invariant. On the quantum level $L_S$ is broken down to $SO(3)_u$ subgroup i.e. transformations from $L_S/SO(3)_u$ cannot be realized by unitary operators [5, 8].

3 Quantization

The following two facts, true only in CT synchronization, are extremely important for quantization of tachyons [5, 8]:

- Invariance of the sign of the time component of the space-like four-momentum i.e. $\varepsilon(k^0) = \text{inv}$,

- Existence of a covariant lower energy bound; in terms of the contravariant space-like four-momentum $k^\mu$, $k^2 < 0$, this lower bound is exactly zero, i.e. $k^0 \geq 0$ as in the lime-like and light-like case.

This is the reason why an invariant Fock construction can be done in our case [5, 8]. In the papers [5, 8] it was constructed a quantum free field theory for scalar tachyons. Here we classify unitary Poincaré mappings in the bundle of Hilbert spaces $H_u$ for a space-like four-momentum. Furthermore we find the corresponding canonical commutation relations. As result we obtain that tachyons correspond to unitary mappings which are induced from $SO(2)$ group rather than $SO(2,1)$ one. Of course, a classification of unitary orbits for time-like and light-like four-momentum is standard, i.e., it is the same as in EP synchronization; this holds because the relativity principle is working in these cases (synchronization group is unbroken).
3.1 Tachyonic representations

As usually, we assume that a basis in a Hilbert space $H_u$ (fibre) of one-particle states consists of the eigenvectors $|k, u; \ldots\rangle$ of the four-momentum operators $P^\mu$ namely

$$P^\mu |k, u; \ldots\rangle = k^\mu |k, u; \ldots\rangle$$

where

$$\langle k', u; \ldots |k, u; \ldots\rangle = 2k_0^0 \delta^3(k' - k)$$

i.e. we adopt a covariant normalization. The $k^0_0 = g^{0\mu}k^\mu_0$ is positive and the energy $k_0^+ = g_{0\nu}k^\nu_0$ is the corresponding solution of the dispersion relation

$$k^2 = g^\mu\nu k^\mu k^\nu = -\kappa^2.$$ 

Namely

$$k_{0+} = -\frac{u^0}{u^0 k} + \omega_k \left(\frac{c}{u^0}\right)^2$$

with

$$\omega_k = \frac{u^0}{c} \sqrt{\left(\frac{u^k}{c}\right)^2 + (|k|^2 - \kappa^2)}.$$ 

Notice that $k^0_0 = \omega_k$ and the range of the covariant momentum $k$ is determined by the following inequality

$$|k| \geq \kappa \left(1 + \left(\frac{c}{u^0}\right)^2 - 1\right) \left(\frac{u^k}{|u||k|}\right)^2$$

i.e. values of $k$ lie outside the oblate spheroid with half-axes $a = \kappa$ and $b = \kappa \frac{u^0}{c}$. The covariant normalization in (20) is possible because in CT synchronization the sign of $k^0$ is an invariant (see the form of the matrix $D$ in the eq. (3)). Thus we have no problem with an indefinite norm in $H_u$.

Now, $ku \equiv k_u u^\mu$ is an additional invariant. Indeed, because the transformations of $L_S$ are restricted to $SO(3)_{u}$ subgroup by causality requirement, and $SO(3)_{u}$ does not change $u$ nor $k$, our covariance group reduces to the Poincaré mappings (realized in the CT synchrony). Summarizing, irreducible family of unitary operators $U(\Lambda, a)$ in the bundle of Hilbert spaces $H_u$ acts on an orbit defined by the following covariant conditions

- $k^2 = -\kappa^2$;
- $\varepsilon(k^0) = \text{inv}$; for physical representations $k^0 > 0$ so $\varepsilon(k^0) = 1$ which guarantee a covariant lower bound of energy $\frac{\kappa^2}{c}$.
- $q \equiv \frac{uk}{c} = \text{inv}$; it is easy to see that $q$ is the energy of tachyon measured in the privileged frame.

As a consequence there exists an invariant, positive definite measure

$$d\mu(k, \kappa, q) = d^3k \theta(k^0) \delta(k^2 + \kappa^2) \delta(q - \frac{uk}{c})$$

Notice that we have contravariant as well as covariant four-momenta related by $g_{\mu\nu}$; the physical energy and momentum are covariant because they are generators of translations.
in a Hilbert space of wave packets.

Let us return to the problem of classification of irreducible unitary mappings
\( U(\Lambda, a) \):
\[
U(\Lambda, a) |k, u; \ldots \rangle = |k', u'; \ldots \rangle;
\]
here the pair \((k, u)\) is transported along trajectories belonging to an orbit fixed
by the above mentioned invariant conditions. To follow the familiar Wigner
procedure of induction, one should find a stability group of the double \((k, u)\). To
do this, let us transform \((k, u)\) to the preferred frame by the Lorentz boost \(L_u^{-1}\).
Next, in the privileged frame, we rotate the spatial part of the four-momentum
to the \(z\)-axis by an appropriate rotation \(R_{\vec{n}}^{-1}\). As a result, we obtain the pair
\((k, u)\) transformed to the pair \((k', u')\) with
\[
k' = \begin{pmatrix}
q \\
0 \\
0 \\
\sqrt{\kappa^2 + q^2}
\end{pmatrix}, \quad u = \begin{pmatrix}
c \\
0 \\
0 \\
0
\end{pmatrix}.
\] (26)

It is easy to see that the stability group of \((k', u')\) is the \(SO(2) = SO(2,1) \cap SO(3)\)
group. Thus tachyonic unitary representations should be induced from the
\(SO(2)\) instead of \(SO(2,1)\) group! Recall that unitary representations of the
\(SO(2,1)\) non-compact group are infinite dimensional (except of the trivial one).
As a consequence, local fields was necessarily infinite component ones (except
of the scalar one). On the other hand, in the CT synchronization case unitary
representations for space-like four-momenta in our bundle of Hilbert spaces are
induced from irreducible, one dimensional representations of \(SO(2)\) in a close
analogy with a light-like four-momentum case. They are labelled by helicity \(\lambda\),
by \(\kappa\) and by \(q\) \((\varepsilon(k^0) = \varepsilon(q)\) is determined by \(q\); of course a physical choice is
\(\varepsilon(q) = 1\)).

Now, by means of the familiar Wigner procedure we determine the Lorentz
group action on the base vectors; namely
\[
U(\Lambda) |k, u; \kappa, \lambda, q \rangle = e^{i\lambda\varphi(\Lambda, k, u)} |k', u'; \kappa, \lambda, q \rangle
\] (27)
where
\[
e^{i\lambda\varphi(\Lambda, k, u)} = U \left( R_{\Omega\varpi}^{-1} \Omega R_{\varpi} \right)
\] (28)
with
\[
\Omega = L_u^{-1} \Lambda L_u.
\] (29)

Here \(k\) and \(u\) transform according to the law \((\Lambda, a)\). The rotation \(R_{\varpi}\) connects
\(k\) with \(D(L_u^{-1}, u)k\), i.e.
\[
R_{\varpi}k = D(L_u^{-1}, u)k.
\] (30)
It is easy to check that \(R_{\Omega\varpi}^{-1} \Omega R_{\varpi}\) is a Wigner-like rotation belonging to the
stability group \(SO(2)\) of \((k, u)\) and determines the phase \(\varphi\). By means of standard
topological arguments \(\lambda\) can take integer or half-integer values only i.e.
\(\lambda = 0, \pm 1/2, \pm 1, \ldots\).

Now, the orthogonality relation \((\Lambda, a)\) reads
\[
\langle k', u; \kappa', \lambda', q | k, u; \kappa, \lambda, q \rangle = 2\omega_k\delta^{3}(k' - k)\delta_{\lambda', \lambda}.
\] (31)
3.2 Canonical quantization

Following the Fock procedure, we define canonical commutation relations

\[ [a_\lambda(k, u), a_\tau(p, u)]_\pm = [a_\lambda^\dagger(k, u), a_\tau^\dagger(p, u)]_\pm = 0, \quad (32) \]

\[ [a_\lambda(k, u), a_\lambda^\dagger(p, u)]_\pm = 2\omega_k\delta(k-p)\delta_{\lambda\tau}, \quad (33) \]

where \(-\) or \(+\) means the commutator or anticommutator and corresponds to the bosonic (\(\lambda\) integer) or fermionic (\(\lambda\) half-integer) case respectively. Furthermore, we introduce a Poincaré invariant vacuum \(|0\rangle\) defined by

\[ \langle 0|0\rangle = 1 \quad \text{and} \quad a_\lambda(k, u)|0\rangle = 0. \quad (34) \]

Therefore the one particle states

\[ a_\lambda^\dagger(k, u)|0\rangle \]

are the base vectors belonging to an orbit in our bundle of Hilbert spaces iff

\[ U(\Lambda)a_\lambda^\dagger(k, u)U(\Lambda^{-1}) = e^{i\lambda\varphi(\Lambda,k,u)}a_\lambda^\dagger(k', u'), \quad (36) \]

\[ U(\Lambda)a_\lambda(k, u)U(\Lambda^{-1}) = e^{-i\lambda\varphi(\Lambda,k,u)}a_\lambda(k', u'), \quad (37) \]

and

\[ [P_\mu, a_\lambda^\dagger(k, u)] = k^\mu_\lambda a_\lambda(k, u). \quad (38) \]

Notice that

\[ P_\mu = \int d^4k \theta(k^0) \delta(k^2 + \kappa^2) k_\mu \left( \sum_\lambda a_\lambda^\dagger(k, u)a_\lambda(k, u) \right) \quad (39) \]

is a solution of (38).

Let us determine the action of the discrete transformations, space and time inversions, \(P\) and \(T\) and the charge conjugation \(C\) on the states \(|k, u; \kappa, \lambda, q\rangle\).

\[ P|k, u; \kappa, \lambda, q\rangle = \eta_u|k^\pi, u^\pi; \kappa, -\lambda, q\rangle, \quad (40) \]

\[ T|k, u; \kappa, \lambda, q\rangle = \eta_t|k^\pi, u^\pi; \kappa, \lambda, q\rangle, \quad (41) \]

\[ C|k, u; \kappa, \lambda, q\rangle = \eta_c|k, u; \kappa, \lambda, q\rangle, \quad (42) \]

where \(|\eta_u| = |\eta_t| = |\eta_c| = 1, k^\pi = (k^0, -\vec{k}), u^\pi = (u^0, -\vec{u})\), the subscript \(c\) means the antiparticle state and \(P, C\) are unitary, while \(T\) is antunitary.

Consequently the actions of \(P, T\) and \(C\) in the ring of the field operators read

\[ Pa_\lambda^\dagger(k, u)P^{-1} = \eta_u a_{-\lambda}^\dagger(k^\pi, u^\pi), \quad (43) \]

\[ Ta_\lambda^\dagger(k, u)T^{-1} = \eta_t a_{\lambda}^\dagger(k^\pi, u^\pi), \quad (44) \]

\[ Ca_\lambda^\dagger(k, u)C^{-1} = \eta_c b_{\lambda}^\dagger(k^\pi, u^\pi), \quad (45) \]

where \(b_\lambda \equiv a_{\lambda}^\dagger\) — antiparticle operators.

Finally we can deduce also the form of the helicity operator:

\[ \lambda(u) = -\frac{W^\mu u_\mu}{c\sqrt{(Pu/c)^2 - P^2}} \quad (46) \]
where

\[ W^\mu = \frac{1}{2} \epsilon^{\mu \sigma \lambda \tau} J_{\sigma \lambda} P_\tau \]

is the Pauli-Lubanski four-vector.

Notice that

\[ P^\lambda(u) P^{-1} = -\hat{\lambda}(u^\tau), \]

\[ T^\lambda(u) T^{-1} = \hat{\lambda}(u^\tau), \]

\[ C^\lambda(u) C^{-1} = \hat{\lambda}(u), \]

as well as

\[ [\hat{\lambda}(u), a_\lambda^{\dagger}(u, k)] = \lambda a_\lambda^{\dagger}(u, k). \]

### 3.3 Local fields

As usually we define local tachyonic fields as covariant Fourier transforms of the creation–annihilation operators. Namely

\[ \varphi_\alpha(x, u) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\infty dq \rho(q) \int d\mu(k, \kappa, q) \sum_\lambda \]

\[ \times \left[ w_{\alpha \lambda}(k, u) e^{ikx} b_\lambda^{\dagger}(k, u) + v_{\alpha \lambda}(k, u) e^{-ikx} a_\lambda(k, u) \right], \]

where the amplitudes \( w_{\alpha \lambda} \) and \( v_{\alpha \lambda} \) satisfy the set of corresponding consistency conditions (the Weinberg conditions). Here we sum over selected helicities and over the invariant \( q \) with a measure \( \rho(q) dq \). It can be shown that the density \( \rho(q) \) determines the form of translation generators \( P_\mu \) deduced from the corresponding Lagrangian. On the other hand \( P_\mu \) are given by eq. (39). Both definitions coincide only for \( \rho(q) = 1 \). The above statement can be easily verified for the scalar tachyon field discussed in [5] and for the fermionic tachyon field discussed below. Therefore in the following we choose simply \( \rho(q) = 1 \). Thus the integration in (51) reduces to the integration with the measure \( d^4k \theta(k^0) \delta(k^2 + \kappa^2) \).

### 4 Fermionic tachyons with helicity \( \lambda = \pm \frac{1}{2} \)

To construct tachyonic field theory describing field excitations with the helicity \( \pm \frac{1}{2} \), we assume that our field transforms under Poincaré group like bispinor (for discussion of transformation rules for local fields in the CT synchronization see [7]); namely

\[ \psi'(x', u') = S(\Lambda^{-1}) \psi(x, u), \]

where \( S(\Lambda) \) belongs to the representation \( D^{\frac{1}{2}(0)} \oplus D^{\frac{3}{2}(0)} \) of the Lorentz group. Because we are working in the CT synchronization, it is convenient to introduce an appropriate (CT-covariant) base in the algebra of Dirac matrices as

\[ \gamma^\mu = T(u)^\mu_\nu \gamma^\nu_E, \]

where \( \gamma^\mu_E \) are standard \( \gamma \)-matrices, while \( T(u) \) is given by the eq. (12). Therefore

\[ \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}(u)I. \]
However, notice that the Dirac conjugate bispinor \( \bar{\psi} = \psi^\dagger \gamma^0 \). Furthermore \( \gamma^5 = -\frac{i}{4!} \epsilon_{\mu\nu\sigma\lambda} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\lambda = \gamma_5^E \).

Now, we look for covariant field equations which are of degree one with respect to the derivatives \( \partial_{\mu} \) and imply the Klein–Gordon equation

\[
(g_{\mu\nu}(u)\partial_{\mu}\partial_{\nu} - \kappa^2) \psi = 0,
\]

(55)

related to the space-like dispersion relation \( k^2 = -\kappa^2 \). We also require the \( T \)-invariance of these equations.

As the result we obtain the following family of the Dirac-like equations

\[
\left\{ \left( \frac{u\gamma}{c} \sin \alpha - 1 \right) \left( \left( \frac{u\gamma}{c} \right) \cos \beta - \kappa \sin \beta \right) \right. \\
-\gamma^5 \left[ (-i\gamma \partial) + \frac{i}{2} \left[ \gamma \partial, \frac{u\gamma}{c} \right] \sin \alpha \right. \\
+ \frac{u\gamma}{c} \left( \left( \frac{u\gamma}{c} \right) (1 + \cos \alpha \sin \beta) + \kappa \cos \alpha \cos \beta \right) \right] \psi(x, u) = 0,
\]

(56)

derivable from an appropriate hermitian Lagrangian density. Here \( u\gamma = u_{\mu} \gamma^\mu \), \( u\partial = u_{\mu} \partial_{\mu} \), \( \gamma \partial = \gamma^\mu \partial_{\mu} \) and \( \alpha, \beta \) — real parameters, \( \alpha \neq (2n + 1)\frac{\pi}{2} \). To guarantee the irreducibility of the elementary system described by (56), the equation (56) must be accompanied by the covariant helicity condition

\[
\hat{\lambda}(u)\psi(u, k) = \lambda \psi(u, k)
\]

(57)

where \( \hat{\lambda} \) is given by (46) taken in the coordinate representation (see below) and \( \lambda \) is fixed (\( \lambda = \frac{1}{2} \) or \( -\frac{1}{2} \) in our case). This condition is quite analogous to the condition for the left (right) bispinor in the Weyl’s theory of the massless field. It implies that particles described by \( \psi \) have helicity \( -\lambda \), while antiparticles have helicity \( \lambda \). For the obvious reason in the following we will concentrate on the case \( \lambda = \frac{1}{2} \).

Notice that the pair of equations (56,57) is not invariant under the composition of the \( P \) or \( C \) inversions separately for every choice of \( \alpha \) and \( \beta \).

Now, in the bispinor realization the helicity operator \( \hat{\lambda} \) has the following explicit form

\[
\hat{\lambda}(u) = \frac{\gamma^5 \left[ -i\gamma \partial, \frac{u\gamma}{c} \right]}{4 \sqrt{\left( -i \frac{u\partial}{c} \right)^2 + \Box}}
\]

(58)

where the integral operator \( \left( \left( -i \frac{u\partial}{c} \right)^2 + \Box \right)^{-\frac{1}{2}} \) in the coordinate representation is given by the well behaving distribution

\[
\frac{1}{\sqrt{\left( -i \frac{u\partial}{c} \right)^2 + \Box}} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^4p \varepsilon \left( \frac{up}{c} \right) e_{ipx}}{\sqrt{\left( \frac{up}{c} \right)^2 - p^2}}.
\]

(59)

Now, let us notice that the equation (56), supplemented by the helicity condition (57), are noninvariant under the composition of the \( P \) and \( C \) inversions (see eqs. (43–45) and the Appendix), except of the case \( \sin \alpha = \cos \beta = 0 \).

\[\text{\footnotesize 4}^{\text{In the Ref. \( \text{[6]} \) we found a class of the second order equations under condition of the \( P \)-invariance.}}\]
Because (56–57) are T-invariant, therefore only for \( \sin \alpha = \cos \beta = 0 \) they are CPT-invariant. Taking \( \sin \beta = \cos \alpha = 1 \) we obtain from (56)

\[
\left\{ \kappa + \gamma^5 \left[ i \gamma \partial - 2 \frac{\mu \gamma}{c} \left( i \frac{\mu}{c} \partial \right) \right] \right\} \psi = 0,
\]

supplemented by (57). On the other hand, for \( \cos \alpha = - \sin \beta = 1 \) we obtain

\[
\left( \kappa - \gamma^5 \left( i \gamma \partial \right) \right) \psi = 0.
\]

The last equation is exactly the Chodos et al. (1) Dirac-like equation for tachyonic fermion. However, contrary to the standard EP approach, it can be consistently quantized in our scheme (if it is supplemented by the helicity condition (57)). In the following we will analyze the eqs. (61) and (57) by means of the Fourier decomposition

\[
\psi(x, u) = \frac{1}{(2\pi)^{3/2}} \int d^4k \, \delta(k^2 + \kappa^2) \theta(k^0) \left[ w_{\frac{1}{2}}(k)e^{i\kappa x}b_{\frac{1}{2}}(k) + v_{-\frac{1}{2}}(k)e^{-i\kappa x}a_{-\frac{1}{2}}(k) \right]
\]

of the field \( \psi \). The creation and annihilation operators \( a \) and \( b \) satisfy the corresponding canonical anticommutation relations (32–33), i.e., the nonzero ones are

\[
[a_{-\frac{1}{2}}(k), a_{-\frac{1}{2}}(p)]_+ = 2\omega_k \delta(k - p)
\]

\[
[b_{\frac{1}{2}}(k), b_{\frac{1}{2}}(p)]_+ = 2\omega_k \delta(k - p)
\]

In (62) \( b_{-\frac{1}{2}} \) and \( a_{\frac{1}{2}} \) do not appear because we decided to fix \( \lambda = \frac{1}{2} \) in (57) (compare with (50)). As the consequence of (57) the corresponding amplitudes \( w_{-\frac{1}{2}} \) and \( v_{\frac{1}{2}} \) vanish. The nonvanishing amplitudes \( w_{\frac{1}{2}} \) and \( v_{-\frac{1}{2}} \) satisfy

\[
(k + \gamma^5 \kappa \gamma)w_{\frac{1}{2}}(k, u) = 0,
\]

\[
\left( 1 - \frac{\gamma^5 [k \gamma, \frac{\mu \gamma}{c}]}{2\sqrt{q^2 + \kappa^2}} \right) w_{\frac{1}{2}}(k, u) = 0,
\]

\[
(k - \gamma^5 \kappa \gamma)v_{-\frac{1}{2}}(k, u) = 0,
\]

\[
\left( 1 - \frac{\gamma^5 [k \gamma, \frac{\mu \gamma}{c}]}{2\sqrt{q^2 + \kappa^2}} \right) v_{-\frac{1}{2}}(k, u) = 0.
\]

Here \( k \equiv k_+ \), \( q = \frac{\mu k_+}{c} \). The solution of (65–68) reads

\[
w_{\frac{1}{2}}(k, u) = \left( \frac{k + \gamma^5 \kappa \gamma}{2\kappa} \right) \frac{1}{2} \left( 1 + \frac{\gamma^5 [k \gamma, \frac{\mu \gamma}{c}]}{2\sqrt{q^2 + \kappa^2}} \right) w_{\frac{1}{2}}(k, u),
\]

\[
v_{-\frac{1}{2}}(k, u) = \left( \frac{k + \gamma^5 \kappa \gamma}{2\kappa} \right) \frac{1}{2} \left( 1 + \frac{\gamma^5 [k \gamma, \frac{\mu \gamma}{c}]}{2\sqrt{q^2 + \kappa^2}} \right) v_{-\frac{1}{2}}(k, u),
\]

where the amplitudes are normalized by the covariant conditions

\[
\bar{w}_{\frac{1}{2}}(k, u) \left( \frac{\mu \gamma}{c} \right)^5 \gamma^5 w_{\frac{1}{2}}(k, u) = \bar{v}_{-\frac{1}{2}}(k, u) \left( \frac{\mu \gamma}{c} \right)^5 \gamma^5 v_{-\frac{1}{2}}(k, u) = 2q,
\]

(71)
\[ \tilde{w}_{\pm}(k^\mp, u) \frac{w_{\gamma}}{c} \gamma^5 v_{-\pm}(k, u) = 0. \]  \tag{72}

The amplitudes \( w_{\pm}(k, u) \) and \( v_{-\pm}(k, u) \), taken for the values \( \kappa \) and \( \gamma \) given in the eq. (26), have the following explicit form (for \( \gamma_5 \) matrix convention—see Appendix)

\[
w_{\pm}(k, u) = \begin{pmatrix} \sqrt{q + \sqrt{q^2 + \kappa^2}} \\ 0 \\ -\kappa \\ \sqrt{q + \sqrt{q^2 + \kappa^2}} \end{pmatrix}, \quad v_{-\pm}(k, u) = \begin{pmatrix} \sqrt{q + \sqrt{q^2 + \kappa^2}} \\ 0 \\ \kappa \\ \sqrt{q + \sqrt{q^2 + \kappa^2}} \end{pmatrix}. \tag{73}
\]

It is easy to see that in the massless limit \( \kappa \to 0 \) the eqs. (65–68) give the Weyl equations

\[ k \gamma w_{\pm} = k \gamma v_{-\pm} = 0, \quad \gamma^5 w_{\pm} = w_{\pm}, \quad \gamma^5 v_{-\pm} = v_{-\pm}, \]

as well as the amplitudes (69–70) have a smooth \( \kappa \to 0 \) limit (it is enough to verify (73)).

Now, the normalization conditions (71,72) generate the proper work of the canonical formalism. In particular, starting from the Lagrangian density \( \mathcal{L} = \bar{\psi} \left( \kappa - \gamma^5 (i \gamma \partial) \right) \psi \) we can derive the translation generators; with help of (62,63,64) and (71,72) we obtain

\[ P_{\mu} = \int \frac{d^3k}{2\omega_k} \bar{\psi} \left( a_{-\frac{1}{2}}^\dagger + b_{\frac{1}{2}} \right) \]

In agreement with (39). Thus we have constructed fully consistent free field theory for a fermionic tachyon with helicity \( \pm \frac{1}{2} \), quite analogous to the Weyl’s theory for a left spinor which is obtained as the \( \kappa \to 0 \) limit.

5 Conclusions

The main result of this work is that tachyons are classified according to the unitary representations of \( SO(2) \) rather than \( SO(2,1) \) group; so they are labelled by the eigenvectors of the helicity operator. In particular for the helicity \( \lambda = \pm \frac{1}{2} \) we have constructed family of \( T \)-invariant equations (56). Under condition of \( PCT \) invariance we selected two equations (60) and (61). The equation (61) coincide with the one proposed by Chodos et al. [1]. We show by explicit construction that, in our scheme, theory described by this equation, supplemented by the helicity condition (57) can be consistently quantized. This theory describe fermionic tachyon with helicity \( -\frac{1}{2} \). It has a smooth massless limit to the Weyl’s left-handed spinor theory. These results show that there are no theoretical obstructions to interpret the experimental data about square of mass of neutrinos [2] as a signal that they can be fermionic tachyons.

A Appendix

The discrete transformations \( P, T \) and \( C \), defined by the eqs. (10,12) are realised in the bispinor space standardly, i.e. \( P \) by \( \gamma_5 \), while \( T \) and \( C \) by \( T \) and \( C \).
satisfying the conditions

\[ T^\dagger T = I, \quad T^* T = -I, \quad T^{-1} \gamma^\mu T = \gamma^\mu, \]  

(75)

\[ C^\dagger C = I, \quad C^* C = -I, \quad C^T = -C, \quad C^{-1} \gamma^\mu C = -\gamma^\mu. \]  

(76)

Notice that the last condition in (75) and (76) can be formulated in terms of the standard \( \gamma_E^\mu \) exactly in the same form.

In explicit calculations of the amplitudes (73) we have used the following representations of the \( \gamma_E \) matrices: \( \tilde{\gamma}_E = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma_E^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \). In this representation the parity, charge conjugation and time inversion are given, up to a phase factor by

\[ \mathcal{P} = \gamma_E^0, \quad \mathcal{C} = i \gamma_E^0 \gamma_E^2, \quad \mathcal{T} = -i \gamma_E^0 \gamma_E^2 \gamma_E^5. \]

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