Q-deformed KP hierarchy: Its additional symmetries and infinitesimal Bäcklund transformations

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Abstract

We study the additional symmetries associated with the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy. After identifying the resolvent operator as the generator of the additional symmetries, the $q$-KP hierarchy can be consistently reduced to the so-called $q$-deformed constrained KP ($q$-cKP) hierarchy. We then show that the additional symmetries acting on the wave function can be viewed as infinitesimal Bäcklund transformations by acting the vertex operator on the tau-function of the $q$-KP hierarchy. This establishes the Adler-Shiota-van Moerbeke formula for the $q$-KP hierarchy.

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I. INTRODUCTION

Recently the so-called $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy has being a subject of intensive study in the literature [1–10]. The deformation is performed to the Lax formulation by introducing a parameter $q$ such that the deformed system recovers the ordinary KP hierarchy as $q$ goes to 1. Furthermore several integrable structures associated with the ordinary KP hierarchy [11,12] are also maintained in the $q$-KP hierarchy, such as infinite conservation laws [2], bi-Hamiltonian structure [3], Virasoro and $W$-algebras [3,4], etc. More recently, the formulation in the spirit of Sato and his school has been carried out for the $q$-KP hierarchy and the concepts of vertex operators and tau-functions can be established [7,9,10] as well.

In this letter, we would like to study another interesting property of the $q$-KP hierarchy, called additional symmetries which have been connected to the ordinary KP hierarchy from different points of view such as conformal algebras [13] and string equation of matrix models in the 2d quantum gravity (see, e.g. Refs. [14–16]), etc. For the ordinary KP hierarchy, Adler, Shiota and van Moerbeke have shown [15] that the additional symmetries acting on the wave function is equivalent to infinitesimal Bäcklund transformations (BTs) acting on the corresponding tau-function of the KP hierarchy. The proof was then simplified by Dickey using the resolvent operator [17]. Therefore it is quite interesting to see whether we can carry out this correspondence in the $q$-deformed framework.

Our main results contain two part. The first part (Sec. III.) is to construct the generator of the additional symmetries using the resolvent operator. As a by-product of this discussion, we obtain a reduction of the $q$-KP hierarchy, called $q$-deformed constrained KP ($q$-cKP) hierarchy. The second part (Sec. IV.) is devoted to the proof of the Adler-Shiota-van Moerbeke formula for the $q$-KP hierarchy.

Since our approach relies on the use of $q$-deformed pseudodifferential operators ($q$-PDOs), let us recall some basic facts about $q$-PDOs in the following.

The $q$-deformed derivative $\partial_q$ is defined by

$$\partial_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

which recovers the ordinary differentiation $\partial_x f(x)$ as $q$ goes to 1. Let us define the $q$-shift operator $\theta$ such that $\theta(f(x)) = f(qx)$ and then we have $\partial_q \theta^k(f) = q^k \theta^k(\partial_q f), (k \in \mathbb{Z})$. Further, from the definition (1.1), we have $(\partial_q fg) = (\partial_q f)g + \theta(f)(\partial_q g)$ which implies that $\partial_q f = (\partial_q f) + \theta(f)\partial_q$. Let $\partial_q^{-1}$ denote the formal inverse of $\partial_q$ then $\partial_q^{-1} f = \sum_{k \geq 0} (-1)^k q^{-k(k+1)/2} q^{-k-1} (\partial_q^k f) \partial_q^{-k-1}$.

In general the following $q$-deformed Leibniz rule holds.

$$\partial_q^n f = \sum_{k \geq 0} \left( \begin{array}{c} n \\ k \end{array} \right)_q \theta^{n-k}(\partial_q^k f) \partial_q^{n-k} \quad n \in \mathbb{Z}$$

(1.2)

where the $q$-number and the $q$-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1}$$

$$\left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{(n)_q (n-1)_q \cdots (n-k+1)_q}{(1)_q (2)_q \cdots (k)_q}$$

$$\left( \begin{array}{c} n \\ 0 \end{array} \right)_q = 1.$$
For a $q$-PDO of the form $P = \sum_{i=0}^{n} p_i \partial_q^{i}$ it is convenient to separate $P$ into the differential part $P_+ = \sum_{i \geq 0} p_i \partial_q^{i}$ and the integral part $P_- = \sum_{i \leq -1} p_i \partial_q^{i}$, respectively. The residue of $P$ is defined by $\text{res}_\partial(P) = p_{-1}$ and the conjugate operation $\ast$ for $P$ is defined by $P^* = \sum_i (\partial_q^i)^* p_i$ with $\partial_q^i = -\partial_q \theta^{-1}$. Then a straightforward calculation shows that $(PQ)^* = Q^*P^*$ where $P$ and $Q$ are any two $q$-PDOs.

II. Q-DEFORMED KP HIERARCHY

The $q$-KP hierarchy is defined by the Lax equation

$$\partial_n L = [L^n, L], \quad \partial_n \equiv \frac{\partial}{\partial t_n} \tag{2.1}$$

with Lax operator of the form

$$L = \partial_q + \sum_{i=0}^{\infty} u_i \partial_q^{-i}. \tag{2.2}$$

According to the Sato theory, we can express the Lax operator as a dressed operator

$$L = S \partial_q S^{-1} \tag{2.3}$$

where $S = 1 + \sum_{i=1}^{\infty} \omega_i \partial_q^{-i}$ is called the Sato operator and $S^{-1}$ is its formal inverse. In terms of $S$ the Lax equation (2.1) is equivalent to the Sato equation

$$\partial_n S = -(L^n)_- S. \tag{2.4}$$

Let us introduce the $q$-deformed exponential function

$$e_q^x = \exp \left[ \sum_{k=1}^{\infty} \frac{(1-q^k)}{k(1-q)} x^k \right] \tag{2.5}$$

which implies that $(e_q^{-x})^{-1} = e_q^{-x}$, $\partial_q e_q^x = ze_q^x$ and $q\partial_q^i (e_q^{xz})^{-1} = z(e_q^{xz})^{-1}$.

Using $S$ we can define the wave function $\omega_q$ and adjoint function $\omega_q^*$, respectively as follows

$$\omega_q = Se_q^{xz} e^{\xi(t,z)} = (1 + \sum_{i=1}^{\infty} \omega_i z^{-i}) e_q^{xz} e^{\xi(t,z)} = \hat{\omega}_q e_q^{xz} e^{\xi(t,z)},$$

$$\omega_q^* = (S^*)^{-1}|_{x/q}(e_q^{xz})^{-1} e^{-\xi(t,z)} = (1 + \sum_{i=1}^{\infty} \omega_i^* z^{i}) (e_q^{xz})^{-1} e^{-\xi(t,z)} = \hat{\omega}_q (e_q^{xz})^{-1} e^{-\xi(t,z)}. \tag{2.6}$$

where $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$ and $P|_{x/q} = \sum_i p_i (x/q) q^i \partial_q^i$. It is easy to show that $\omega_q$ and $\omega_q^*$ satisfy the following linear systems

$$L \omega_q = z \omega_q, \quad \partial_n \omega = (L^n)_+ \omega
\quad (L^*)|_{x/q} \omega_q^* = z \omega_q^*, \quad \partial_n \omega_q^* = -(L^*)^n|_{x/q} \omega_q^*. \tag{2.7}$$

Lemma 2.1. Let $P$ and $Q$ be two $q$-PDOs. Then
\[ \text{res}_z(P e^{x^z} Q^z |_{x/q}(e^{x^z})^{-1}) = \text{res}_q(PQ) \]  \hspace{1cm} (2.8)

where we denote two types of residues as \( \text{res}_z(\sum_i a_i z^i) = a_{-1} \) and \( \text{res}_q(\sum_i a_i \partial_q^i) = a_{-1} \), respectively.

**Lemma 2.2.** (see, e.g. Sec. 7.7 of Ref. [12]) If \( f(z) = \sum_{-\infty}^\infty a_i z^{-i} \) then

\[ \text{res}_z[\zeta^{-1}(1 - z/\zeta) + z^{-1}(1 - \zeta/z)]f(z) = f(\zeta) \]  \hspace{1cm} (2.9)

where \((1 - z/\zeta)\) and \((1 - \zeta/z)\) should be understood as series in \(\zeta^{-1}\) and \(z^{-1}\), respectively.

The \(q\)-KP hierarchy admits a reduction defined by \((L^n)_{-} = 0\). This reduction is compatible with the hierarchy flows because

\[ \partial_k(L^n)_{-} = [(L^k)_{+}, L^n]_{-} = [(L^k)_{+}, (L^n)_{-}]_{-}. \]  \hspace{1cm} (2.10)

Hence \((L^n)_{-}\) is identical to zero if its initial value is. The hierarchy associated with this reduction is called \(q\)-deformed \(n\)-th KdV hierarchy whose Lax operator is thus defined by the \(n\)-th order \(q\)-deformed differential operator \((L^n)_{+}\).

### III. ADDITIONAL SYMMETRIES

Let us turn to the additional symmetries of the \(q\)-KP hierarchy. Our approach follows closely that of Dickey [17] by considering the Lax equation (2.1) as a dressing of the commutation relation \([\partial_k - \partial_q^k, \partial_q] = 0\). On the other hand, we find another operator

\[ \Gamma_q = \sum_{i=1}^{\infty} \left[ i t_i + \frac{(1-q)^i}{(1-q^i)} x^i \right] \partial_q^{i-1} \]  \hspace{1cm} (3.1)

which also commutes with \(\partial_k - \partial_q^k\), i.e., \([\partial_k - \partial_q^k, \Gamma_q] = 0\). Dressing it gives

\[ \partial_k M = [(L^k)_{+}, M], \quad M \equiv S \Gamma_q S^{-1} \]  \hspace{1cm} (3.2)

which are hierarchy flows for the operator \(M\). Eq.(2.1) together with (3.2) implies that

\[ \partial_k (M^m L^l) = [(L^k)_{+}, M^m L^l]. \]  \hspace{1cm} (3.3)

In contrast to the Sato equation (2.4), we define the additional flow for each pair \(m, l\) as follows

\[ \partial_{ml}^* S = -(M^m L^l)_{-} S, \quad \partial_{ml}^* \equiv \frac{\partial}{\partial t_{ml}} \]  \hspace{1cm} (3.4)

Then

\[ \partial_{ml}^* L = -[(M^m L^l)_{-}, L], \quad \partial_{ml}^* M = -[(M^m L^l)_{-}, M]. \]  \hspace{1cm} (3.5)

Hence \((M^m L^l)_{-}\) serves as the generator of the additional symmetries along the trajectory parametrized by \(t_{ml}^*\). Moreover it can be easily shown that the additional flows \(\{\partial_{ml}^*\}\)
commute with the hierarchy flows, i.e., $[\partial^*_m, \partial_k]L = 0$ but do not commute with each other due to the fact that the operator $M$ depends on $t_k$ explicitly.

Let us linearly combine the vector fields $\partial^*_m$ to form a new vector field $\partial^*_\mu \lambda = \sum_{m=0}^\infty \sum_{l=-\infty}^\infty (\mu - \lambda)^m \lambda^{-m-l-1} / m! \partial^*_m \partial^*_l$ which also belongs to the additional flows. Then the generator associated with this new vector field is given by

$$Y_q(\mu, \lambda) = \sum_{m=0}^\infty \sum_{l=-\infty}^\infty (\mu - \lambda)^m \lambda^{-m-l-1}(M^m L^{m+l})_-.$$  \hspace{1cm} (3.6)

**Theorem 3.1.** (Dickey) The operator $Y_q(\mu, \lambda)$ can be expressed as

$$Y_q(\mu, \lambda) = \omega_q(t, \mu) \partial_q^{-1} \theta(\omega_q^*(t, \lambda)). \hspace{1cm} (3.7)$$

Before proving the theorem, let us prepare the following identities.

**Lemma 3.2.** Let $P$ be a $q$-PDO. Then we have

$$(P)_- = \sum_{i=1}^\infty \partial_q^{-i} \theta(\text{res}_{\partial_q}(\partial_q^i P)). \hspace{1cm} (3.8)$$

In particular,

$$f \partial_q^{-1} = \sum_{i=1}^\infty \partial_q^{-i} \theta(\partial_q^{-1} f). \hspace{1cm} (3.9)$$

**Proof of theorem 3.1.** Using Lemma 2.1 and Lemma 3.2 we have

$$(M^m L^{m+l})_- = (S \Gamma_q^m \partial_q^{m+l} S^{-1})_- = \sum_{i=1}^\infty \partial_q^{-i} \theta[\text{res}_{\partial_q}(\partial_q^i S \Gamma_q^m \partial_q^{m+l} S^{-1})]$$

$$= \sum_{i=1}^\infty \partial_q^{-i} \theta[\text{res}_z(\partial_q^i S \Gamma_q^m \partial_q^{m+l} e^{xz} e^{\xi(t,z)}(S^{-1})^*|_{x/q}(e^{xz})^{-1} e^{-\xi(t,z)}])$$

$$= \text{res}_z \left[ \sum_{i=1}^\infty \partial_q^{-i} \theta(z^{m+l}(z \omega_q^*) \partial_q^{-1} \theta(\omega_q^*)) \right] = \text{res}_z \left[ \sum_{i=1}^\infty \partial_q^{-i} \theta(z^{m+l}(z \omega_q) \partial_q^{-1} \theta(\omega_q^*)) \right]$$

where we have used $\Gamma_q^m \omega_q = \partial_z \omega_q$ to reach the last second equality. Next, using Lemma 2.2, we have

$$Y_q(\mu, \lambda) = \text{res}_z \left[ \sum_{m=0}^\infty (\mu - \lambda)^m \sum_{l=-\infty}^\infty \lambda^{-m-l-1}(z \omega_q \partial_q^{-1} \theta(\omega_q^*)) \right]$$

$$= \text{res}_z \left[ \frac{1}{z(1 - \lambda / z)} + \frac{1}{\lambda(1 - z / \lambda)} e^{(\mu - \lambda) \partial_q(\omega_q(t, z))} \partial_q^{-1} \theta(\omega_q^*(t, z)) \right]$$

$$= ((e^{(\mu - \lambda) \partial_q \omega_q(t, \lambda)}) \partial_q^{-1} \theta(\omega_q^*(t, \lambda)) = \omega_q(t, \mu) \partial_q^{-1} \theta(\omega_q^*(t, \lambda)). \hspace{1cm} \square$$

We further introduce a generator $Y(t)$ which is constructed from $Y_q(\mu, \lambda)$ by integrating the spectral parameters $\mu$ and $\lambda$ with some weightings. Let $\phi(t) = \int \rho(\mu) \omega_q(t, \mu) d\mu$ and $\psi(t) = \int \chi(\lambda) \theta(\omega_q^*(t, \lambda)) d\lambda$. Then we have $Y(t) = \phi(t) \partial_q^{-1} \psi(t)$ with
\[ \partial_k \phi = (L^k)^+ \phi, \quad \partial_k \psi = -(L^k)^+ \psi. \]  

(3.10)

Introducing new symmetries to a system always induces constraints to the system. In our present case, the additional symmetries generated by the operator \((M^mL^l)_-\) provides another reduction \((L^n)_- = (M^mL^l)_-\) to the q-KP hierarchy. From (2.11) and (3.3) it is easy to show that \(\partial_k ((L^n)_- - (M^mL^l)_-) = [(L^k)_+,[L^n]_-(M^mL^l)_-]\) and hence this reduction is compatible with the hierarchy flows as well. The situation still holds for the generator \(Y(t)\) since it is a special linear combinations of the operators \((M^mL^l)_-\). Therefore we obtain a reduction of the q-KP hierarchy defined by the Lax operator

\[ K = L^n = (L^n)_+ + \phi(t)\partial_q^{-1}\psi(t) \]  

(3.11)

which satisfies the Lax equation

\[ \partial_k K = [(K^{k/n})_+, K]. \]  

(3.12)

It is easy to show that (3.12) is compatible with Eq.(3.10). We call this hierarchy the q-deformed constrained KP (q-cKP) hierarchy which has rich integrable structures as well as q-KP hierarchy [15].

IV. INFINITESIMAL BÄCKLUND TRANSFORMATIONS

Through the bilinear identity it has been shown [14,15] that the (adjoint) wave functions of the q-KP hierarchy can be expressed in terms of a single function \(\tau_q(x; t)\) called tau-function such that

\[ \hat{\omega}_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; t)}, \quad \hat{\omega}_q^* = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} \]  

(4.1)

where \([z^{-1}] = (1/z, 1/2z^2, 1/z^3, \cdots)\). A peculiar property of \(\tau_q(x; t)\) is that it can be obtained from the tau-function \(\tau(t)\) of the ordinary KP hierarchy by shifting the time variables \(t\) to \(t+[x]_q\), i.e., \(\tau_q(x; t) = \tau(t+[x]_q)\) where \([x]_q = (x, (1-q)^2x^2/2(1-q^2), (1-q)^3x^3/3(1-q^3), \cdots)\). For convenience, from now on, the tau-function \(\tau_q(x; t)\) will be simply written by \(\tau_q(t)\) without any confusion.

Now let us define the vertex operators

\[ X_q(t, z) = e^{xz} \exp \left( \sum_{i=1}^{\infty} t_i z^i \right) \exp \left( -\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i \right) \]

\[ X_q^*(t, z) = (e^{xz})^{-1} \exp \left( -\sum_{i=1}^{\infty} t_i z^i \right) \exp \left( \sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i \right) \]  

(4.2)

then

\[ \omega_q = \frac{X_q(t, z)\tau_q(t)}{\tau_q(t)}, \quad \omega_q^* = \frac{X_q^*(t, z)\tau_q(t)}{\tau_q(t)} \]  

(4.3)

We further define a new vertex operator \(X_q(\mu, \lambda)\) which is constructed as follows:
two lemmas concerning the \( q \)-\( t \)s. \( q \) is a solution of the \( q \)-KP hierarchy then \( \tau_q + \epsilon X_q(\mu,\lambda)\tau_q \) a solution too. To connect such infinitesimal BTs to the additional symmetries described above, we prepare the following two lemmas concerning the \( q \)-analogue of the (differential) Fay identity.

**Lemma 4.1.** The tau-function \( \tau_q \) satisfies the following \( q \)-deformed Fay identity

\[
\sum_{(s_1,s_2,s_3)} (s_0 - s_1)(s_2 - s_3)\tau_q(t + [s_0] + [s_1])\tau_q(t + [s_2] + [s_3]) = 0 \tag{4.5}
\]

where \((s_1,s_2,s_3)\) denotes cyclic permutations.

The above identity is obtained from the Fay identity for tau-functions of the ordinary KP hierarchy by shifting the variables \( t \) to \( t + [x]_q \).

**Lemma 4.2.** The tau-function \( \tau_q \) satisfies the following \( q \)-deformed differential Fay identity

\[
\begin{align*}
\partial_q \tau_q(t - [s_3] + [s_1])\theta(\tau_q(t)) - \theta(\tau_q(t - [s_3] + [s_1]))(\partial_q \tau_q(t)) \\
= (s_3^{-1} - s_1^{-1})\theta(\tau_q(t + [s_1]))\tau_q(t - [s_3]) - s_3^{-1}\theta(\tau_q(t - [s_3] + [s_1]))\tau_q(t) \\
+ s_1^{-1}\tau_q(t - [s_3] + [s_1])\theta(\tau_q(t)) \\
\end{align*} \tag{4.6}
\]

**Proof of lemma 4.2.** From the Fay identity \((4.3)\), setting \( s_0 = 0 \) and divided by \( s_1s_2s_3 \) we have

\[
\sum_{(s_1,s_2,s_3)} (s_3^{-1} - s_2^{-1})\tau_q(t + [s_1])\tau_q(t + [s_2] + [s_3]) = 0. \tag{4.7}
\]

Letting \( s_2 = (1 - q)x \), shifted \( t \to t - [s_2] - [s_3] \) and using the relation \([qx]_q = [x]_q + [(1 - q)x]_q \) then \((4.6)\) follows. \( \square \)

**Theorem 4.3.** (Adler-Shiota-van Moerbeke and Dickey) The following formula

\[
X_q(\mu,\lambda)\omega_q(t,z) = (\lambda - \mu)Y_q(\mu,\lambda)\omega_q(t,z) \tag{4.8}
\]

holds, where it should be understood that the vertex operator \( X_q(\mu,\lambda) \) acting on \( \omega_q \) is generated by its action on \( \tau_q \) through the expression \((4.1)\).

**Proof of theorem 4.3.** From \((4.1)\) we have

\[
X_q(\mu,\lambda)\frac{\tau_q(t - [z^{-1}])}{\tau_q(t)} = \frac{\tau_q(t)e^{-\sum_{i=1}^{\infty}z^{-i}/i\partial_{t_i}}X_q(\mu,\lambda)\tau_q(t) - \tau_q(t - [z^{-1}])X_q(\mu,\lambda)\tau_q(t)}{\tau_q^2(t)}
\]

\[
= e^{x\mu(x,\lambda)} e^x(t,\mu - \lambda) [\tau_q(t)\tau_q(t - [z^{-1}] - [\mu^{-1}] + [\lambda^{-1}])(z - \mu)(z - \lambda)^{-1} \\
- \tau_q(t - [z^{-1}])\tau_q(t - [\mu^{-1}] + [\lambda^{-1}])] / \tau_q^2(t)
\]

\[
= e^{x\mu(x,\lambda)} e^x(t,\mu - \lambda) (z - \lambda)^{-1} \left[ (z - \mu)\tau_q(t' + [z^{-1}] + [\mu^{-1}])\tau_q(t' + [\lambda^{-1}]) \\
- (z - \lambda)\tau_q(t' + [\mu^{-1}])\tau_q(t' + [z^{-1}] + [\lambda^{-1}]) \right] / \tau_q^2(t)
\]

\[
= e^{x\mu(x,\lambda)} e^x(t,\mu - \lambda) \left[ (z - \mu)\tau_q(t' + [z^{-1}])\tau_q(t' + [\lambda^{-1}]) \\
- (z - \lambda)\tau_q(t' + [\mu^{-1}])\tau_q(t' + [z^{-1}]) \right] / \tau_q^2(t)
\]
here $t' \equiv t - [z^{-1}] - [\mu^{-1}]$. Using the Fay identity (4.3) with $s_0 = 0, s_1 = \lambda^{-1}, s_2 = \mu^{-1}, s_3 = z^{-1}$ then the terms in the bracket can be simplified to $(\lambda - \mu)\tau_q(t' + [z^{-1}])\tau_q(t' + [\mu^{-1}] + [\lambda^{-1}])$.

Hence the following connection

$$X_q(\mu, \lambda)\omega_q(t, z) = e^{xz}e^{x\mu}(e^{x\lambda})^{-1}e^{t(z+\mu-\lambda)}(\lambda - \mu)\frac{\tau_q(t - [\mu^{-1}])\tau_q(t - [z^{-1}] + [\lambda^{-1}])}{(z - \lambda)}\tau_2(t)$$

$$= (\lambda - \mu)\omega_q(t, \mu)\partial_q^{-1}\theta(\omega_q(t, \lambda))\omega_q(t, z)$$

should be established for the proof. Multiplying both sides by $(\lambda - \mu)^{-1}\partial_q(\omega_q(t, \mu))^{-1}$ and using the expressions (4.1) we obtain

$$\partial_q\tau_q(t - [z^{-1}] + [\lambda^{-1}]) = (z - \lambda)\theta\left(\frac{\tau_q(t + [\lambda^{-1}])}{\tau_q(t)}\right)\frac{\tau_q(t - [z^{-1}] + [\lambda^{-1}])}{\tau_q(t)} - z\theta\left(\frac{\tau_q(t - [z^{-1}] + [\lambda^{-1}])}{\tau_q(t)}\right)$$

$$+ \lambda\frac{\tau_q(t - [z^{-1}] + [\lambda^{-1}])}{\tau_q(t)}$$

(4.9)

which is nothing but the $q$-deformed differential Fay identity (4.6) with $s_1 = \lambda^{-1}$ and $s_3 = z^{-1}$. This completes the proof of the theorem. □

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