GENERALIZED LYAPUNOV-RAZUMIKHIN METHOD FOR RETARDED DIFFERENTIAL INCLUSIONS: APPLICATIONS TO DISCONTINUOUS NEURAL NETWORKS

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Abstract. In this paper, a general class of nonlinear dynamical systems described by retarded differential equations with discontinuous right-hand sides is considered. Under the extended Filippov-framework, we investigate some basic stability problems for retarded differential inclusions (RDI) with given initial conditions by using the generalized Lyapunov-Razumikhin method. Comparing with the previous work, the main results given in this paper show that the Lyapunov-Razumikhin function is allowed to have a indefinite or positive definite derivative for almost everywhere along the solution trajectories of RDI. However, in most of the existing literature, the derivative (if it exists) of Lyapunov-Razumikhin function is required to be negative or semi-negative definite for almost everywhere. In addition, the Lyapunov-Razumikhin function in this paper is allowed to be non-smooth. To deal with the stability, we also drop the specific condition that the Lyapunov-Razumikhin function should have an infinitesimal upper limit. Finally, we apply the proposed Razumikhin techniques to handle the stability or stabilization control of discontinuous time-delayed neural networks. Meanwhile, we present two examples to demonstrate the effectiveness of the developed method.

1. Introduction. Due to the natural phenomena such as the finite processing time of signals and the energy propagating with a finite speed, the future states of a large number of practical dynamical systems are not only dependent of the present states but also are determined by the past states. Even, in some problems it is meaningless not to have dependence on the past \([2, 21, 33]\). That is, time-delays are inevitable. On the other hand, discontinuities are also typical phenomena which are often caused by control actions of many interesting engineering tasks, such as
the on-off controllers of thermostats for regulating room temperature, control synthesis of uncertain systems, variable structure control and sliding mode control, etc [5, 13, 15, 22, 24, 30, 32, 35]. Especially, in the field of neural networks, discontinuous dynamical neuron systems have been proved really useful as ideal models to solve various control problems such as programming problems and constrained optimization problems [10, 16, 25]. It should be pointed out that the additional difficulties will arise if both the time-delays and discontinuities are considered in a dynamical system. This class of dynamical system is usually described by the retarded differential equation (RDE) with discontinuous right-hand side. In this case, many results in the classical theory of RDE have been shown to be invalid for discontinuous dynamical systems with time-delays since the given vector field is no longer continuous. In order to analyze the dynamical behaviors of discontinuous system, Filippov developed the theory and framework of differential inclusion (DI) in 1964. Such a new framework named Filippov-DI-framework [14] has become a standard and very useful tool to deal with ordinary differential equations (ODE) possessing discontinuous property. In fact, by constructing the set-valued map in the sense of Filippov, the solution of ODE could be transformed into a solution of DI, which is also called as the Filippov-regularization. Therefore, the DI could be regarded as a generalization of ODE. After that, the retarded differential inclusion (RDI) or differential inclusions with memory was further investigated [2, 19, 20, 23, 31]. So an extension of Filippov-framework named Filippov-RDI-framework has been proposed to handle the RDE with discontinuous right-hand side. In 1981, a systematic introduction for the solution sets of functional differential inclusion (FDI) or RDI was given by Haddad [19]. In the monograph of Aubin and Cellina [2], the properties of the set of trajectories for FDI (RDI) have been presented. At the beginning of this century, Benchohra and Lupulescu gave the the existence results of the solutions for convex and nonconvex FDI (RDI) in [4] and [31], respectively. Also, Hong obtained the existence of a class of FDI with infinite delay in [23]. However, until now the theory of RDE with discontinuous right-hand side is still incomplete and we are also facing many challenges in studying the complex dynamical behaviors of discontinuous systems with time-delays owing to the lack of effective analysis tools and methods.

Because of the discontinuity of the vector field for ODE or RDE, numerous important questions should be treated. One of the fundamental questions we must face is the stability of solution. As well as we know, the main tool for investigating the stability problems is the powerful Lyapunov theory. In [14], Filippov developed the Lyapunov theory for ODE with discontinuous right-hand side via differential inclusion, but the Lyapunov functions are required to be smooth. Unfortunately, smooth Lyapunov functions do not suffice to analyze the stability of discontinuous systems. In the subsequent literature [3, 13, 15, 17, 24], the methods based on non-smooth Lyapunov functions are successfully applied in discussing the stability of DI or ODE with discontinuity. When the time-delay is introduced into the ODE with discontinuous right-hand side or DI, the Lyapunov-Krasovskii functional method [26] is an effective and important tool to investigate the stability problems. For example, in [36] and [34], the Lyapunov functionals are utilized to study the stability and asymptotical stability for RDI. Also in [27], the authors addressed stability problems of delay differential inclusions based on Lyapunov-Krasovskii functionals. Nevertheless, the suitable Lyapunov-Krasovskii functional is not easy to be constructed because its structure might be very complex. In addition, the
authors of [28] developed Razumikhin-type theorems for hybrid system with memory. In [43], the Razumikhin and Krasovskii stability theorems were generalized for time-varying time-delay systems. However, the time-delayed systems of [28] and [43] are required to be continuous. For these reasons, the classical Lyapunov-Razumikhin theorem should be generalized to deal with the stability problems of RDI or RDE possessing discontinuous property. This new method differs from those considered in the existing literature, where the Lyapunov-Razumikhin functions are easily constructed and are allowed to be non-smooth or local Lipschitz continuous. More importantly, the derivative (if it exists) of Lyapunov-Razumikhin function along the solution-trajectories of RDI is relaxed to be indefinite for almost everywhere. In a word, it is necessary and significant to develop generalized Lyapunov-Razumikhin method and further analyze the stability behaviors of RDI or RDE with discontinuous property in virtue of extended Filippov-framework.

The structure of this paper is arranged as follows. Section 2 collects some preliminary knowledge. Our main results and some remarks are contained in Section 3, where some sufficient conditions are given to guarantee the stability, uniform stability and asymptotic stability of the zero solution for retarded differential inclusion by employing generalized Lyapunov-Razumikhin method. Section 4 applies the generalized Lyapunov-Razumikhin method to deal with the stability and stabilization problems of discontinuous time-delayed neural networks and provides two examples which together show the feasibility of our method. Finally, Section 5 states some conclusions and gives the hints for future work.

Notations: Let \( \mathbb{R} \) be the set of real numbers, \( \mathbb{R}_+ \) denote the set of all nonnegative real numbers, \( \mathbb{N}_+ \) denote the set of all positive integers, and \( 2^{\mathbb{R}^n} \) denote the family of all nonempty subsets of \( \mathbb{R}^n \). Given \( a, b \in \mathbb{R} \), \( a \land b \) denotes the minimum of \( a \) and \( b \), while \( a \lor b \) denotes the maximum of \( a \) and \( b \). For the column vectors \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n \), \( \langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \) denotes the scalar product of \( x, y \), where the superscript \( T \) means the transpose operator. If \( x \in \mathbb{R}^n \), \( \| x \| \) denotes any vector norm of \( x \). Given a set \( E \subset \mathbb{R}^n \), by \( \text{meas}(E) \) we mean the Lebesgue measure in \( \mathbb{R}^n \) of \( E \) and by \( \overline{co}(E) \) we mean the closure of the convex hull of \( E \). Given the single-valued function \( V(x) : \mathbb{R}^n \to \mathbb{R} \), \( \nabla V(x) \) denotes the gradient of \( V(x) \). Given function \( \varphi, \varphi^{-1} \) represents the inverse function of \( \varphi \).

2. Preliminaries. In this section, we present some definitions and lemmas concerning retarded differential inclusions and non-smooth analysis, which will be used throughout this paper. First of all, let us give some basic concepts and known facts from set-valued analysis. For more details, the interested readers may refer to the books [1][2][11][12][14][24] and the papers [6][13][15].

Let \( \mathbb{R}^n (n \geq 1) \) denote an \( n \)-dimensional real Euclidean space with inner product \( \langle \cdot, \cdot \rangle \) and induced norm \( \| \cdot \| \). Given \( X \subseteq \mathbb{R}^n \), let us introduce following notation:

\[
P_{ke}(X) = \{ A \subset X : \text{convex nonempty and compact} \}.
\]

Suppose that to every point \( x \) of a set \( E \subset \mathbb{R}^n \) there corresponds a nonempty set \( F(x) \subset \mathbb{R}^n \), then \( x \mapsto F(x) \) is called a set-valued map from \( E \rightarrow \mathbb{R}^n \). Suppose \( E \subset \mathbb{R}^n \), then a set-valued map \( F \) with nonempty values is said to be upper semi-continuous (USC) at \( x_0 \in E \), if for any open set \( L \) containing \( F(x_0) \), there exists a neighborhood \( M \) of \( x_0 \) such that \( F(M) \subset L \). \( F(x) \) is said to have a closed (compact, convex) graph \( \{(x, y) \in E \times \mathbb{R}^n : y \in F(x)\} \) if for each \( x \in E \), \( F(x) \) is closed.
(compact, convex). If \( F \) has nonempty closed values, \( E \) is closed, and \( F \) is bounded in a neighborhood of each point \( x \in \mathbb{E} \), then \( F \) is upper semi-continuous on \( E \) if and only if its graph is closed.

Now let us introduce the notion of Filippov solution by constructing the Filippov set-valued map (i.e., Filippov regularization, see [14]). Let \( \tau > 0 \) be a given real number. Suppose \( \phi \) is a continuous function on \( [-\tau, 0] \) and \( C = C([-\tau, 0], \mathbb{R}^n) \) denotes the Banach space of continuous functions \( \phi \) mapping the interval \( [-\tau, 0] \) into \( \mathbb{R}^n \) with the norm \( \| \phi \|_C = \sup_{-\tau \leq s \leq 0} \| \phi(s) \| \). If \( t_0 \in \mathbb{R}_+ \), and for \( b \in (0, +\infty] \), \( x(t) : [t_0 - \tau, b) \to \mathbb{R}^n \) is continuous, then \( x_t \in C \) is defined by \( x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0 \) for any \( t \in [t_0, b) \).

**Definition 2.1.** For each \( i = 1, 2, \ldots, n \), we say the real-valued function \( f_i : \mathbb{R} \times C \to \mathbb{R} \) is measurable, if for all \( s \in \mathbb{R} \), the point set \( \{(t, \phi) \in \mathbb{R} \times C : f_i(t, \phi) > s\} \) is a Lebesgue measurable set. A vector-valued function \( f : \mathbb{R} \times C \to \mathbb{R}^n \) is said to be measurable, if for each component \( f_i \) \( (i = 1, 2, \ldots, n) \) of \( f = (f_1, f_2, \ldots, f_n)^T \) is measurable.

Consider the following non-autonomous retarded differential equation of the vector form:

\[
\frac{dx}{dt} = f(t, x_t),
\]

where \( t \) denotes time; \( x_t(\cdot) \) represents the history of the state from time \( t - \tau \), up to the present time \( t \); \( dx/dt \) denotes the time derivative of \( x \) and \( f : \mathbb{R} \times C \to \mathbb{R}^n \) is measurable and essentially locally bounded. In this case, \( f(t, x_t) \) represents a vector field which is allowed to be discontinuous.

Let us construct the Filippov set-valued map \( F : \mathbb{R} \times C \to 2^{\mathbb{R}^n} \) given as

\[
F(t, x_t) = \bigcap_{\rho > 0 \text{ meas}(\mathbb{R}) = 0} \mathbb{R} \cup \{f(t, B(x_t, \rho) \setminus \emptyset) \}.
\]

Here \( \text{meas}(\mathbb{R}) \) denotes the Lebesgue measure of set \( \mathbb{R} \); intersection is taken over all sets \( \mathbb{R} \) of Lebesgue measure zero and over all \( \rho > 0 \); \( B(x_t, \rho) := \{x^*_t \in C \mid \|x^*_t - x_t\| < \rho\} \); \( \mathbb{C}(\mathbb{E}) \) is the closure of the convex hull of some set \( \mathbb{E} \).

**Definition 2.2.** A vector-valued function \( x(t) \) defined on a non-degenerate interval \( \mathbb{D} \subseteq \mathbb{R} \) is said to be a Filippov solution for retarded differential equation \( [1] \), if it is absolutely continuous on any compact subinterval \( [t_1, t_2] \) of \( \mathbb{D} \), and for almost everywhere \( t \in \mathbb{D} \), \( x(t) \) satisfies the following retarded differential inclusion

\[
\frac{dx}{dt} \in F(t, x_t).
\]

Since \( f(t, x_t) \) is essentially locally bounded, it is easy to check that the set-valued function \( F : \mathbb{R} \times C \to 2^{\mathbb{R}^n} \) is upper semi-continuous (USC) with nonempty, compact, convex values and locally bounded. Therefore, for every \( (t_0, x_{t_0}) \in \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}^n) \), there exists a local solution \( x(t_0, x_{t_0})(t) \) of the retarded differential inclusion \( [3] \) with the given initial state \( x_{t_0} \in C([-\tau, 0], \mathbb{R}^n) \) and initial time \( t = t_0 \geq 0 \). In general, the solutions in the sense of Filippov for \( [3] \) are not unique. Furthermore, under appropriate conditions, the maximal existing time interval of each solution \( x(t_0, x_{t_0})(t) \) with given condition can be extended to \( [t_0 - \tau, +\infty) \). Take \( [24] \) as an example, under the growth condition \( (\mathbf{g.c.}) \), the following lemma holds:
Lemma 2.3 (see [24]). If \( F \in \mathcal{S}(\mathbb{R} \times C, \mathbb{R}^n) = \{ F \mid F : \mathbb{R} \times C \to P_{\text{c}}(\mathbb{R}^n) \text{ is USC} \} \), and there exist Lebesgue integrable functions \( \mathcal{M}(t), \mathcal{N}(t) \) such that for almost everywhere \( t \), the following inequality holds
\[
|F(t, \phi)| = \sup_{\xi \in F(t, \phi)} |\xi| \leq \mathcal{M}(t) \|\phi\|_{C} + \mathcal{N}(t),
\]
then the retarded differential inclusion (3) has at least one solution on the interval \([t_0 - \tau, +\infty)\) with given initial condition.

Definition 2.4. If for any \( t \in \mathbb{R}, 0 \in F(t, 0) \), then \( x = 0 \) is said to be a zero solution of the retarded differential inclusion (3) or the retarded differential equation (1).

Definition 2.5. The zero solution of retarded differential equation (1) (or RDI (3)) is said to be
- stable if for any \( \varepsilon > 0, \forall t_0 \geq 0 \), there exists a \( \delta = \delta(\varepsilon, t_0) > 0 \) such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) = \{ x_{t_0} \in C : \| x_{t_0} \| < \delta \} \), each solution \( x(t_0, x_{t_0})(t) \) satisfies \( \| x(t_0, x_{t_0})(t) \| < \varepsilon \) for \( t \geq t_0 \); if \( \delta \) does not depend on \( t_0 \), then the zero solution of RDE (1) (or RDI (3)) is uniformly stable;
- attractive if \( \forall t_0 \geq 0 \), there exists a \( \delta = \delta(t_0) > 0 \) such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) \), each solution \( x(t_0, x_{t_0})(t) \) satisfies \( \lim_{t \to +\infty} \| x(t_0, x_{t_0})(t) \| = 0 \), that is to say, \( \forall \varepsilon > 0, \forall t_0 \in \mathbb{R}_+ \) there exists a \( \delta = \delta(t_0) > 0 \) and \( \exists T = T(\varepsilon, t_0, x_{t_0}) > 0 \) such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) \) and all \( t \geq t_0 + T \), each solution \( x(t_0, x_{t_0})(t) \) satisfies \( \| x(t_0, x_{t_0})(t) \| < \varepsilon \); if \( T \) does not depend on \( t_0 \) and \( x_{t_0} \), then the zero solution of RDE (1) (or RDI (3)) is uniformly attractive; if \( \delta \) can be arbitrarily large, then the zero solution of RDE (1) (or RDI (3)) is further said to be globally attractive or globally uniformly attractive;
- (uniformly) asymptotically stable if it is (uniformly) stable and (uniformly) attractive;
- exponential stable if for any positive number \( \varepsilon \in \mathbb{R}, \exists \lambda > 0, \exists \delta = \delta(\varepsilon, t_0) > 0 \), such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) \) and all \( t \geq t_0 \), each solution \( x(t_0, x_{t_0})(t) \) satisfies \( \| x(t_0, x_{t_0})(t) \| \leq e^{-\lambda(t-t_0)} \);
- globally exponentially stable if for any \( \delta > 0, \exists \lambda > 0, \forall t_0 \in \mathbb{R}_+, \exists M(\delta) > 0 \), such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) \) and all \( t \geq t_0 \), each solution \( x(t_0, x_{t_0})(t) \) satisfies \( \| x(t_0, x_{t_0})(t) \| \leq M(\delta)e^{-\lambda(t-t_0)} \).

Definition 2.6. A function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a \( \mathcal{K} \)-function which is denotes by \( \varphi \in \mathcal{K} \) if \( \varphi \) is continuous and strictly increasing with \( \varphi(0) = 0 \). A function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be a \( \mathcal{K}_\infty \)-function which is denoted by \( \varphi \in \mathcal{K}_\infty \) if it is a \( \mathcal{K} \)-function and also satisfies \( \lim_{\chi \to +\infty} \varphi(\chi) = +\infty \).

Definition 2.7 (Clarke’s Generalized Gradient [11]). Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a locally Lipschitz function, and let \( \Omega_V \subset \mathbb{R}^n \) denote the set of points where \( V \) fails to be differentiable, then for any \( x \in \mathbb{R}^n \) we can define the Clarke’s generalized gradient of \( V \) at point \( x \in \mathbb{R}^n \), as follows
\[
\partial V(x) = \bar{\partial}(\lim_{k \to \infty} \nabla V(x_k) : x_k \to x, x_k \not\in \mathbb{R} \cup \Omega_V),
\]
where \( \nabla \) denotes the nabla operator, \( \mathbb{R} \subset \mathbb{R}^n \) is a set with Lebesgue measure zero that can be arbitrarily chosen to simplify the computation.

Definition 2.8 (Regular [11]). Consider the function \( V : \mathbb{R}^n \to \mathbb{R} \) which is locally Lipschitz near \( x \in \mathbb{R}^n \). The usual one-sided directional derivative of \( V \) at \( x \) in the
direction $v \in \mathbb{R}^n$ is defined as
\[
D^+ V(x, v) = \lim_{h \to 0^+} \frac{V(x + hv) - V(x)}{h}
\]
when this limit exists. On the other hand, the generalized directional derivative of $V$ at $x$ in the direction $v \in \mathbb{R}^n$ is defined as
\[
D_C V(x, v) = \lim_{h \to 0^+, y \to x} \frac{V(y + hv) - V(y)}{h}.
\]
We say that a function $V : \mathbb{R}^n \to \mathbb{R}$ is regular at $x \in \mathbb{R}^n$, if for all $v \in \mathbb{R}^n$, the usual one-sided directional derivative of $V$ at $x$ in the direction $v \in \mathbb{R}^n$ exists, and $D^+ V(x, v) = D_C V(x, v)$. The function $V(x)$ is said to be regular in $\mathbb{R}^n$, if it is regular for each $x \in \mathbb{R}^n$.

**Definition 2.9 (C-regular [11, 15])**. A single-valued function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be C-regular, if and only if $V(x)$ is:

(i) regular in $\mathbb{R}^n$;
(ii) positive definite, that is, we have $V(x) > 0$ for $x \neq 0$, and $V(0) = 0$;
(iii) radially unbounded, that is, $V(x) \to +\infty$ as $\|x\| \to +\infty$.

It should be pointed out that a C-regular Lyapunov function $V$ is not necessarily differentiable. We call this type of function an Lyapunov-like function which is allowed to be non-smooth. Suppose that $x(t) : [t_0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact subinterval of $[t_0, +\infty)$. The next lemma gives a chain rule for computing the time derivative of a composed function $V(x(t)) : [t_0, +\infty) \to \mathbb{R}$.

**Lemma 2.10 (Chain Rule [11, 15])**. Suppose that the single-valued function $V : \mathbb{R}^n \to \mathbb{R}$ is C-regular, and $x(t) : [t_0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact subinterval of $[t_0, +\infty)$. Then, $x(t)$ and $V(x(t)) : [t_0, +\infty) \to \mathbb{R}$ are differential for almost everywhere $t \in [t_0, +\infty)$, and we have
\[
\frac{dV(x(t))}{dt} = \left\langle \zeta(t), \frac{dx(t)}{dt} \right\rangle, \forall \zeta(t) \in \partial V(x(t)).
\]

3. Main results. In this section, by using extended Filippov-framework and Rachunkinik techniques, we develop Lyapunov-based stability theory for the retarded differential inclusion of the form given by (3). We always assume that every solution $x(t_0, x_{t_0})(t)$ in the sense of Filippov for RDI (3) with given initial condition $(t_0, x_{t_0}) \in \mathbb{R}_+ \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, exists on the interval $[t_0 - \tau, +\infty)$. For the sake of convenience, we sometimes denote $x(t) = x(t_0, x_{t_0})(t)$. Before proceeding further, the following fundamental assumption is needed for the main results of this section.

- For any $t \in \mathbb{R}$, $0 \in F(t, 0)$.

Note that the stability of any solution for RDI (3) is equivalent to the stability of zero solution $x = 0$ for the corresponding retarded differential inclusion by a transformation.

**Theorem 3.1.** Suppose that there exists a C-regular and locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for any $t \in \mathbb{R}$, a function $\varphi_1 \in \mathcal{K}_\infty$, and a continuous function $G(t) : \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\varphi_1(\|x\|) \leq V(t, x)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$;
holds for almost everywhere.

Case 2. If \( \phi \) Recalling the condition (i), we have

\[
V(t + \theta, \phi(\theta)) = V(t, \phi(0))
\]

for any \( \phi \in C([-\tau, 0], \mathbb{R}^n) \) and all \( \theta \in [-\tau, 0] \);

(iii) \( \int_0^{+\infty} G^+(s)ds < +\infty \), where \( G^+(s) = G(s) \vee 0 \).

Then the zero solution of the retarded differential inclusion (3) is stable.

Proof. For \( t \in \mathbb{R} \) and \( \phi \in C \), we define

\[
W(t, \phi) = \sup_{-\tau \leq \theta \leq 0} V(t + \theta, \phi(\theta)).
\]

Since the function \( V \) is continuous, there exists a \( \theta_0 \in [-\tau, 0] \) such that \( W(t, \phi) = V(t + \theta_0, \phi(\theta_0)) \) and either \( \theta_0 = 0 \) or \( \theta_0 < 0 \).

Case 1. If \( \theta_0 < 0 \), then we have

\[
V(t + \theta, \phi(\theta)) < V(t + \theta_0, \phi(\theta_0)), \quad \text{for} \quad \theta \in (\theta_0, 0].
\]

So for sufficiently small \( h > 0 \), we can obtain

\[
W(t + h, x_{t+h}(t, \phi)) = W(t, \phi),
\]

which yields \( \frac{dW(t, \phi)}{dt} = 0 \).

Case 2. If \( \theta_0 = 0 \), we can derive from condition (ii) that the following inequality holds for almost everywhere.

\[
\frac{dW(t, \phi(0))}{dt} \leq G(t)W(t, \phi(0)).
\]

Based on the above two cases, we obtain that, for almost everywhere \( t \geq t_0 \), the inequality (6) is true. Now multiplying both sides of the inequality (6) by \( e^{-\int_{t_0}^t G(s)ds} \), an integration from \( t_0 \) to \( t \) leads to

\[
W(t, \phi(0)) \leq W(t_0, x_{t_0})e^{\int_{t_0}^t G(s)ds}, \quad \text{for all} \quad t \geq t_0.
\]

Since \( G(s) \leq G^+(s) \), it follows from above inequality that

\[
W(t, \phi(0)) \leq W(t_0, x_{t_0})e^{\int_{t_0}^t G^+(s)ds}
\]

\[
\leq W(t_0, x_{t_0})e^{\int_{t_0}^{+\infty} G^+(s)ds}, \quad \text{for all} \quad t \geq t_0.
\]

Notice that \( \int_{t_0}^{+\infty} G^+(s)ds < +\infty \) from condition (iii), we set \( G = e^{\int_{t_0}^{+\infty} G^+(s)ds} \) which is a positive constant. Due to the continuity of \( V(t + \theta, \phi) \) at \( \phi \) and \( V(t_0, 0) = 0 \), we can deduce that \( W(t_0, \phi) \) is continuous at \( \phi \) and \( W(t_0, 0) = 0 \). Therefore, for any \( \varepsilon > 0 \), \( \forall t_0 \in \mathbb{R}_+ \), there exists a \( \delta = \delta(\varepsilon, t_0) > 0 \) such that for any \( x_{t_0} \in \mathcal{B}(0, \delta) = \{x_{t_0} \in C : \|x_{t_0}\|_C < \delta\} \) implies that

\[
W(t_0, x_{t_0}) < \frac{\varphi_1(\varepsilon)}{G}.
\]

Recalling the condition (i), we have

\[
\varphi_1(\|\phi(\theta)\|) \leq V(t + \theta, \phi(\theta)) \text{ which yields that } \varphi_1(\|\phi(\theta)\|) \leq W(t, \phi) \text{ for any } (t, \phi) \in \mathbb{R} \times C. \text{ Consequently, we have } \varphi_1(\|\phi(0)\|) \leq W(t, \phi(0)).
\]
This, together with (8) and (9), leads to
\[
\|\phi(0)\| \leq \varphi_1^{-1}(W(t, \phi(0))) \\
\leq \varphi_1^{-1}(W(t_0, x_{t_0})G) \\
< \varphi_1^{-1}(\varphi_2(\varepsilon)G) = \varepsilon. \tag{11}
\]
Due to the arbitrariness of \(\phi \in C\), we can take
\[
\phi(\theta) = x_t(\theta), \text{ for } \theta \in [-\tau, 0].
\]
Then, we can obtain from (11) that
\[
\|x(t)\| = \|x_t(0)\| = \|\phi(0)\| < \varepsilon.
\]
This tells us that the zero solution of retarded differential inclusion (3) is stable. The proof is complete. \(\square\)

**Remark 1.** Comparing with existing Razumikhin-type stability results in [21], Theorem 3.1 does not require the Lyapunov-Razumikhin function \(V(t, \phi(0))\) has an infinitesimal upper limit. Moreover, the Lyapunov-Razumikhin function is not required to be differentiable with respect to time \(t\) for everywhere. In other words, we only request \(V(t, \phi(0))\) is continuous and differentiable for almost everywhere \(t \in \mathbb{R}\) (or absolutely continuous with respect to \(t\)).

**Theorem 3.2.** Assume that there exists a \(C\)-regular and locally Lipschitz continuous function \(V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+\) satisfying \(V(t, 0) = 0\) for any \(t \in \mathbb{R}\), functions \(\varphi_1, \varphi_2 \in K_\infty\), and a continuous function \(G(t) : \mathbb{R}_+ \to \mathbb{R}_+\) such that
\[
(i) \quad \varphi_1(\|x\|) \leq V(t, x) \leq \varphi_2(\|x\|), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n; \\
(ii) \quad \frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0)) \text{ for almost everywhere if } V(t + \theta, \phi(\theta)) \leq V(t, \phi(0)),
\]
for any \(\phi \in C([-\tau, 0], \mathbb{R}^n)\) and all \(\theta \in [-\tau, 0];
\]
(iii) \(\int_{\tau}^{+\infty} G^+(s)ds < +\infty\), where \(G^+(s) = G(s) \vee 0\).

Then the zero solution of the retarded differential inclusion (3) is uniformly stable.

**Proof.** Since the Lyapunov function \(V(t, x)\) has an infinitesimal upper limit, i.e., \(V(t, x) \leq \varphi_2(\|x\|)\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}^n\), then we can obtain
\[
V(t + \theta, \phi(\theta)) \leq \varphi_2(\|\phi(\theta)\|) \leq \varphi_2(\|\phi\|_{C}),
\]
which implies
\[
W(t, \phi) \leq \varphi_2(\|\phi\|_{C}), \text{ for } \forall (t, \phi) \in \mathbb{R} \times C.
\]
This leads to
\[
W(t_0, x_{t_0}) \leq \varphi_2(\|x_{t_0}\|_{C}). \tag{12}
\]
According to the proof of Theorem 3.1 we can derive from (8) and (12) that
\[
W(t, \phi(0)) \leq W(t_0, x_{t_0})e^{\int_{t_0}^{t} G^+(s)ds} \\
\leq \varphi_2(\|x_{t_0}\|_{C})G, \text{ for all } t \geq t_0. \tag{13}
\]
Recalling the condition (i), we can obtain from (10) and (13) that
\[
\|\phi(0)\| \leq \varphi_1^{-1}(W(t, \phi(0))) \\
\leq \varphi_1^{-1}(\varphi_2(\|x_{t_0}\|_{C})G). \tag{14}
\]
Due to the arbitrariness of $\phi \in C$, we can take
\[ \phi(\theta) = x_t(\theta), \text{ for } \theta \in [-\tau, 0]. \]
Then, we can obtain from (14) that
\[ \| x(t) \| = \| x_t(0) \| = \| \phi(0) \| \leq \varphi_1^{-1}(\varphi_2(\| x_{t_0} \|_C) G). \]

Hence, for any $\varepsilon > 0$, there exists a positive constant $\delta = \varphi_2^{-1}\left(\frac{\varepsilon}{\varphi_1}\right)$ which does not depend on $t_0$, such that, for any $x_{t_0} \in B(0, \delta) = \{ x_{t_0} \in C : \| x_{t_0} \|_C < \delta \}$, we have $\| x(t) \| < \varepsilon$. This means that the zero solution of the RDI (3) is uniformly stable. The proof is complete.

**Remark 2.** In Theorem 3.1 and Theorem 3.2, some conditions on the Lyapunov-Razumikhin function $V(t, \phi(0))$ have been relaxed. On the one hand, the Lyapunov-Razumikhin function is allowed to be non-smooth. Note that the non-smooth Lyapunov function is very effective to deal with the stability for RDE with discontinuous right-hand side or RDI. On the other hand, the derivative (if it exists) of the Lyapunov-Razumikhin function $V(t, \phi(0))$ along the trajectories of retarded differential inclusion system (3) is allowed to be positive definite for almost all $t \in [t_0, +\infty)$. This type of function is said to be almost indefinite Lyapunov-like function. However, the existing stability results involving Razumikhin techniques need the condition that the Lyapunov or Lyapunov-like function should have a negative or semi-negative definite derivative for almost everywhere. This makes the Lyapunov or Lyapunov-like function is difficult to be found. Therefore, our results are more general and they effectively improve the previously known results.

**Remark 3.** If the condition (iii) in Theorem 3.1 and Theorem 3.2 is replaced with the following condition (C), then the results of Theorem 3.1 and Theorem 3.2 are still correct by similar proof.

(C) $\int_{t_0}^{t_\infty} |G(s)| ds < +\infty$.

**Theorem 3.3.** Suppose that there exists a $C$-regular and locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for any $t \in \mathbb{R}$, functions $\varphi_1, \varphi_2 \in K_\infty$, a continuous nondecreasing function $p(\nu) > \nu$ for all $\nu > 0$, and a continuous function $G(t) : \mathbb{R}_+ \to \mathbb{R}$ such that
\begin{enumerate}
  \item [(i)] $\varphi_1(\| x \|) \leq V(t, x) \leq \varphi_2(\| x \|), \forall t, x \in \mathbb{R} \times \mathbb{R}^n$;
  \item [(ii)] $\frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0))$ for almost everywhere if
    \[ V(t + \theta, \phi(\theta)) < p(V(t, \phi(0))), \]
    for any $\phi \in C([\tau, 0], \mathbb{R}^n)$ and all $\theta \in [-\tau, 0]$;
  \item [(iii)] $\int_{t_0}^{t_\infty} |G(s)| ds < +\infty$, and there exists a constant $\lambda > 0$ such that for all $t \geq t_0$,
    \[ \int_{t_0}^{t} G^-(s) ds \geq \lambda(t - t_0), \quad (15) \]
    where $G^-(s) = [-G(s)] \lor 0$.
\end{enumerate}
Then the zero solution of the retarded differential inclusion (3) is uniformly asymptotically stable.
Proof. Obviously, Theorem 3.2 and Remark 3 imply the uniform stability of zero solution for RDI (3). Actually, if
\[ V(t, \phi(0)) \geq V(t + \theta, \phi(\theta)), \quad \theta \in [-\tau, 0]. \]  
(16)

Or, equivalently,
\[ V(t, x(t)) \geq V(t + \theta, x(t + \theta)), \quad \theta \in [-\tau, 0]. \]  
(17)

Recalling the properties of the function \( p(\nu) \), we can deduce from (16) or (17) that
\[ p(V(t, \phi(0))) > V(t + \theta, \phi(\theta)), \quad \theta \in [-\tau, 0]. \]  
(18)

According to the condition (ii), we can obtain from (18) that \( \frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0)) \) for almost everywhere. This shows that the conditions of Theorem 3.2 and Remark 3 are satisfied. Therefore, the zero solution of the RDI (3) is uniformly stable.

In order to complete the proof of this theorem, we need only to show the uniform attractiveness of the zero solution for RDI (3). For this purpose, we will use mathematical induction. Before proceeding further, we first do some necessary preparations. Due to the uniform stability of the zero solution, for given \( H > 0 \), there exists a constant \( \delta > 0 \) such that for any initial state \( x_{0} \in B(0, \delta) = \{x_{0} \in C : \|x_{0}\|_{C} < \delta\} \), each solution \( x(t) = x(t, x_{0})(t) \) satisfies \( \|x(t)\| < H \) for \( t \geq t_{0} \).

Again from the properties of the function \( p(\nu) \), for any \( \varepsilon > 0 \) \( (\varepsilon < H) \), we can choose a number \( a \) satisfying
\[ 0 < a < \inf_{\varepsilon \leq \varphi_{1}(\varepsilon)/2 \leq \varphi_{2}(H)} (p(\nu) - \nu), \quad a < \frac{\varphi_{1}(\varepsilon)}{2}. \]  
(19)

Let \( N \in \mathbb{N}_{+} \) (here \( \mathbb{N}_{+} \) denotes the set of all positive integers) such that
\[ \frac{\varphi_{1}(\varepsilon)}{2} + (N - 1)a < \varphi_{2}(H) \leq \frac{\varphi_{1}(\varepsilon)}{2} + Na. \]  
(20)

Since \( \int_{0}^{\infty} |G(s)|ds < +\infty \), then there exists a \( T = t_{0} + \mu \tau \), such that for \( t \geq T \), we have \( \int_{T}^{t} |G(s)|ds < \frac{a}{\varphi_{2}(H)} \), here \( \mu \geq 1 \) is a constant.

In addition, notice that there exists a constant \( \lambda > 0 \) such that \( \int_{t_{0}}^{t} G^{-}(s)ds \geq \lambda(t - t_{0}) \) for all \( t \geq t_{0} \). This means that there exists a \( T^{*} = \max\{1, \frac{1}{\lambda} \ln \frac{\Delta \varphi_{2}(H)}{Na}\} > 0 \) such that for all \( t \geq t_{0} + T^{*} \), the following estimation holds.
\[ e^{-\int_{t_{0}}^{t} G^{-}(s)ds} \leq e^{-\lambda(t - t_{0})} \leq \frac{Na}{\Delta \varphi_{2}(H)}, \]  
(21)

where \( \Delta = e^{\int_{0}^{\infty} |G(s)|ds} \) is a positive constant.

In the following, the discussion based on mathematical induction will be divided into three steps.

**Step 1.** By way of contradiction, we first prove that there exists a \( T_{1} > T \) such that
\[ V(T_{1}, x(T_{1})) < \frac{\varphi_{1}(\varepsilon)}{2} + (N - 1)a. \]  
(22)

If this is not true, then for \( \forall t > T \), we have
\[ V(t, x(t)) \geq \frac{\varphi_{1}(\varepsilon)}{2} + (N - 1)a \geq \frac{\varphi_{1}(\varepsilon)}{2}. \]  
(23)
It follows from (19), (20) and (23) that
\[ p(V(t, \phi(0))) = p(V(t, x(t))) \]
\[ > V(t, x(t)) + a \]
\[ \geq \frac{\phi_1(\varepsilon)}{2} + (N - 1)a + a \]
\[ = \frac{\phi_1(\varepsilon)}{2} + Na \]
\[ \geq \phi_2(H) \]
\[ \geq V(t + \theta, x(t + \theta)), \text{ for } \theta \in [-\tau, 0]. \] (24)

Therefore, we can obtain from the condition (ii) that
\[ \frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0)), \text{ for a.e. } t > T. \] (25)

Both sides of the above inequality (25) multiplied by \( e^{-\int_0^t G(s)ds} \), we can get
\[ e^{-\int_0^t G(s)ds} \frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0))e^{-\int_0^t G(s)ds}, \text{ for a.e. } t > T. \] (26)

An integration over the interval \([T, T + T^*]\) leads to
\[ V(T + T^*, x(T + T^*)) \leq V(T, x(T))e^{\int_T^{T + T^*} G(s)ds}. \] (27)

Notice that
\[ G(s) = G^+(s) - G^-(s), \] (28)
\[ G^+(s) + G^-(s) = |G(s)| \geq G^+(s). \] (29)

We can derive from (21) and (27)-(29) that
\[ V(T + T^*, x(T + T^*)) \leq V(T, x(T))e^{\int_T^{T + T^*} G^+(s)ds}e^{-\int_T^{T + T^*} G^-(s)ds} \]
\[ \leq \phi_2(||x(T)||)e^{\int_T^{T + T^*} |G(s)|ds}e^{-\int_T^{T + T^*} G^-(s)ds} \]
\[ \times e^{\int_T^{T + T^*} G^+(s)ds} \]
\[ = \phi_2(||x(T)||)e^{\int_0^{\mu T + T^*} |G(s)|ds}e^{-\int_0^{\mu T + T^*} G^-(s)ds} \]
\[ \leq \phi_2(H)e^{\int_0^{\infty} |G(s)|ds}e^{-\int_0^{\mu T + T^*} G^-(s)ds} \]
\[ \leq \phi_2(H) \frac{Na}{e^{\phi_2(H)}} = Na. \] (30)

On the other hand, it follows from (23) that
\[ V(T + T^*, x(T + T^*)) \geq \frac{\phi_1(\varepsilon)}{2} + (N - 1)a. \] (31)

Combining (30) and (31), we deduce that
\[ Na \geq \frac{\phi_1(\varepsilon)}{2} + (N - 1)a, \text{ i.e., } a \geq \frac{\phi_1(\varepsilon)}{2}. \] (32)
This contradicts the fact $a < \frac{\varphi_1(\varepsilon)}{2}$ given by (19). So (22) is true, here $T_1$ can be taken to be $T_1 = T + T^*$.

**Step 2.** Again by way of contradiction, we will prove that, for all $t \geq T_1$, the following inequality holds:

$$V(t, x(t)) < \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a + \frac{a}{2}$$

(33)

If we assume that (33) is not true, taking (22) into account, then there exists $t_2 > t_1 > T_1$ such that

$$V(t_1, x(t_1)) = \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a, \quad \text{(34)}$$

$$V(t_2, x(t_2)) = \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a + \frac{a}{2}. \quad \text{(35)}$$

Meanwhile, we can obtain

$$V(t_1, x(t_1)) \leq V(t, x(t)) \leq V(t_2, x(t_2)),$$

for $t_1 \leq t \leq t_2$. \quad \text{(36)}

From (19), (34) and (36), we can deduce that

$$p(V(t, \phi(0))) = p(V(t, x(t)))$$

$$> V(t, x(t)) + a$$

$$\geq V(t_1, x(t_1)) + a$$

$$= \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a + a$$

$$= \frac{\varphi_1(\varepsilon)}{2} + Na$$

$$\geq \varphi_2(H)$$

$$\geq V(t + \theta, x(t + \theta)),$$  for $\theta \in [-\tau, 0], \ t_1 \leq t \leq t_2$. \quad \text{(37)}

So, from the condition (ii), we have

$$\frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0)),$$  for a.e. $t_1 \leq t \leq t_2$. \quad \text{(38)}

Integrating both sides of (38) from $t_1$ to $t_2$, we obtain

$$V(t_2, x(t_2)) \leq V(t_1, x(t_1)) + \int_{t_1}^{t_2} G(s)V(s, \phi(0))ds$$

$$\leq V(t_1, x(t_1)) + \varphi_2(\|\phi(0)\|) \int_{t_1}^{t_2} |G(s)|ds$$

$$\leq V(t_1, x(t_1)) + \varphi_2(H) \int_{t_1}^{t_2} |G(s)|ds$$

$$< V(t_1, x(t_1)) + \varphi_2(H) \frac{a}{2\varphi_2(H)}$$

$$= \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a + \frac{a}{2}. \quad \text{(39)}$$

This contradicts (35). Hence, the inequality (33) holds for all $t \geq T_1$.

**Step 3.** By way of induction, we shall conclude that the zero solution of RDI (3) is uniformly attractive.
Now, we replace $T$ with $T_1$. Following the above discussion method given by Step 1 and Step 2, we can deduce that there exists $T_2 = T_1 + T^*$ such that for all $t \geq T_2$,

$$V(t, x(t)) < \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a. \quad (40)$$

Similarly, there also exists $T_3 = T_2 + T^*$ such that for all $t \geq T_3$, we have

$$V(t, x(t)) < \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a - \frac{a}{2}. \quad (41)$$

Continuing this process, there exists $T_{2N} = T_{2N-1} + T^*$, such that for all $t \geq T_{2N}$,

$$V(t, x(t)) < \frac{\varphi_1(\varepsilon)}{2} + (N - 1)a - (2N - 2)\frac{a}{2} < \frac{\varphi_1(\varepsilon)}{2} + a < \varphi_1(\varepsilon). \quad (42)$$

According to the condition (i), we can derive that

$$\|x(t)\| \leq \varepsilon, \text{ for all } t \geq T_{2N}. \quad (43)$$

Since $T_{2N} = T + 2NT^* = t_0 + \mu\tau + 2NT^*$ and $\mu\tau + 2NT^*$ is a positive constant which does not depend on $t_0$, then we can conclude that the zero solution of RDI (3) is uniformly asymptotically stable. The proof is complete. \qed

**Remark 4.** The criteria in Theorem 3.3 generalizes the results shown by Hale [21]. In the extended Filippov-framework, the Lyapunov-Razumikhin function $V(t, \phi(0))$ is allowed to have a indefinite or positive definite derivative for almost everywhere along the trajectories of RDI (3).

**Theorem 3.4.** Suppose that there exists a $C$-regular and locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for any $t \in \mathbb{R}$, functions $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$, and a continuous function $G(t) : \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\varphi_1(||x||) \leq V(t, x) \leq \varphi_2(||x||)$, $V(t, x) \in \mathbb{R} \times \mathbb{R}^n$;

(ii) $\frac{dV(t, \phi(0))}{dt} \leq G(t)V(t, \phi(0))$ for almost everywhere if $V(t) \leq V(t, \phi(0))$,

$$V(t + \theta, \phi(\theta)) \leq V(t, \phi(0)),$$

for any $\phi \in C([-\tau, 0], \mathbb{R}^n)$ and all $\theta \in [-\tau, 0]$;

(iii) $\int_0^\infty G^+(s)ds < +\infty$, and there exists a constant $\lambda > 0$ such that for all $t \geq t_0,$

$$\int_{t_0}^t G^-(s)ds \geq \lambda(t - t_0), \quad (44)$$

where $G^+(s) = G(s) \lor 0$, $G^-(s) = [-G(s)] \lor 0$.

Then the zero solution of the retarded differential inclusion (3) is uniformly asymptotically stable.

**Proof.** In Theorem 3.2 we have proven the uniform stability of the zero solution for RDI (3). Next, we need only to show the zero solution of the RDI (3) is uniformly attractive. Likewise, we can obtain the following equality by integrating both sides of (7) from $t_0$ to $t$.

$$W(t, \phi(0)) \leq W(t_0, x_{t_0})e^{\int_{t_0}^t G(s)ds}, \text{ for all } t \geq t_0. \quad (45)$$
Remark 5. In particular, if we take the asymptotic stability of the zero solution for RDI (3) by replacing the condition (44) with \( \phi \), then we can only deduce that the zero solution of the RDI (3) is asymptotically stable. It is obvious that the condition (44) yields the Lyapunov-Razumikhin function is not required to have an infinitesimal upper limit.

Due to the arbitrariness of \( \phi \in C \), we can take \( \phi(\theta) = x_1(\theta) \), for \( \theta \in [-\tau, 0] \).

Then, we can obtain from (48) that
\[
\|x(t)\| = \|x(t_0)\| = \|\phi(0)\| \leq \varphi_1^{-1} \left( \varphi_2(\|x_{t_0}\| C) G e^{-\lambda(t-t_0)} \right).
\]

The above inequality shows that the zero solution of the RDI (3) is uniformly asymptotically stable. The proof is complete. \( \square \)

Remark 5. In particular, if we take \( \varphi_1(\|x\|) = c \|x\|^r \), where \( c \) and \( r \) are positive constants, then we can further derive from (48) that the zero solution of the RDI (3) is globally exponential stable. It should be pointed out that, we can only prove the asymptotic stability of the zero solution for RDI (3) by replacing the condition (i) in Theorem 3.4 with \( \varphi_1(\|x\|) \leq V(t,x) \) for \( \forall(t,x) \in \mathbb{R} \times \mathbb{R}^n \), here the Lyapunov-Razumikhin function is not required to have an infinitesimal upper limit.

Remark 6. In Theorem 3.5, if we replace the condition (44) with \( \int_{t_0}^{+\infty} G^- (s) ds = +\infty \), then we can only deduce that the zero solution of the RDI (3) is asymptotically stable. It is obvious that the condition (44) yields \( \int_{t_0}^{+\infty} G^- (s) ds = +\infty \).

Our next result is also concerned with the uniform asymptotic stability, which can be implied by Theorem 3.4.

Theorem 3.5. Suppose that there exists a \( C \)-regular and locally Lipschitz continuous function \( V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+ \) satisfying \( V(t,0) = 0 \) for any \( t \in \mathbb{R} \), functions \( \varphi_1, \varphi_2 \in \mathcal{K}_\infty \), a continuous nondecreasing function \( p(\nu) > \nu \) for all \( \nu > 0 \), and a continuous function \( G(t) : \mathbb{R}_+ \to \mathbb{R} \) such that
\begin{align*}
(i) \quad & \varphi_1(\|x\|) \leq V(t,x), \forall(t,x) \in \mathbb{R} \times \mathbb{R}^n; \\
(ii) \quad & \frac{dV(t,\varphi(0))}{dt} \leq G(t)V(t,\varphi(0)) \text{ for almost everywhere if} \\
& V(t+\theta,\varphi(\theta)) < p(V(t,\varphi(0))), \text{ for any } \phi \in C([-\tau, 0], \mathbb{R}^n) \text{ and all } \theta \in [-\tau, 0].
\end{align*}
(iii) \( \int_0^{+\infty} G^+(s) ds < +\infty \), and there exists a constant \( \lambda > 0 \) such that for all \( t \geq t_0 \),

\[
\int_{t_0}^t G^-(s) ds \geq \lambda (t - t_0),
\]

where \( G^+(s) = G(s) \lor 0 \), \( G^-(s) = [-G(s)] \lor 0 \).

Then the zero solution of the retarded differential inclusion (3) is uniformly asymptotically stable.

**Remark 7.** To the best of our knowledge, there are few results on Razumikhin-type stability involving Filippov-framework and Lyapunov-like function. Moreover, the Lyapunov-Razumikhin function in the existing literature is required to be differentiable with respect to time \( t \) for everywhere, even its derivative is required to be negative or semi-negative definite. In contrast, Theorem 3.4 and Theorem 3.5 relax some condition that the derivative (if it exists) of the Lyapunov-Razumikhin function \( V(t, \phi(0)) \) along the trajectories of retarded differential inclusion system is allowed to be indefinite for almost all \( t \in [t_0, +\infty) \). In addition, the Lyapunov-Razumikhin function is relaxed to be non-smooth. It is noted that Theorem 3.4 and Theorem 3.5 are very useful to deal with the uniform asymptotic stability for retarded differential inclusions or retarded differential equations with discontinuous right-hand sides.

4. **Applications to discontinuous neural networks.** In practice, neural network models are usually described by the retarded differential equations possessing discontinuous property. These neural network models can be regarded as the special cases of retarded differential equation (1). Up to this point of time, various time-delayed neural networks have been established by means of discontinuous switching circuits and have further been extensively studied via retarded differential inclusions [7, 8, 18, 29, 37–42]. In this section, we apply the generalized Lyapunov-Razumikhin method to investigate the stability and stabilization problems of discontinuous time-delayed neural networks. Moreover, two numerical examples are provided to illustrate the applicability and effectiveness of our method.

We consider a class of neural networks described by the retarded differential equations with discontinuous right-hand sides:

\[
\frac{dx_i(t)}{dt} = d_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t - \tau(t))) + u_i(t),
\]

where \( i \in \mathcal{N} \triangleq \{1, 2, \ldots, n\} \), \( n \) corresponds to the number of units in neuron system (50); \( x_i(t) \) denotes the state variable of the potential of the \( i \)th neuron at time \( t \); \( d_i(t) \) denotes the self-inhibition of the \( i \)th neuron at time \( t \); \( b_{ij}(t) \) is the connection strength of \( j \)th unit on the \( i \)th unit related to the neurons with time-varying delay; \( f_j(\cdot) \) denotes the activation function of \( j \)th neuron; \( u_i(t) \) denotes external input; \( \tau(t) \) denotes the time-varying transmission delay at time \( t \) and is a continuous function satisfying

\[
0 \leq \tau(t) \leq \tau \text{ (here } \tau \text { is a positive constant).}
\]

Throughout this section, we always assume that \( d_i(t), b_{ij}(t) \) are continuous functions with respect to \( t \). The discontinuous neuron activations in (50) are assumed to satisfy the following properties:
(H1) For each $i \in \mathcal{N}$, $f_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_k^i\}$, where there exist finite right and left limits, $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$, respectively. Moreover, $f_i$ has at most a finite number of discontinuities on any compact interval of $\mathbb{R}$.

(H2) For every $i \in \mathcal{N}$, $0 \in \overline{\mathbb{C}[f_i(0)]}$ and there exist nonnegative constants $\alpha_i$ and $\beta_i$ such that

$$\sup_{\gamma_i \in \overline{\mathbb{C}[f_i(x_i)]}} |\gamma_i| \leq \alpha_i |x_i| + \beta_i, \forall x_i \in \mathbb{R},$$

where, for $\theta \in \mathbb{R}$,

$$\overline{\mathbb{C}[f_i(\theta)]} = \left[ \min\{f_i^-(\theta), f_i^+(\theta)\}, \max\{f_i^-(\theta), f_i^+(\theta)\} \right].$$

Let $x(t) = x(t_0, x_{t_0})(t)$ denote a solution of the retarded differential equation (50) with given initial condition $(t_0, x_{t_0}) \in \mathbb{R}_+ \times \mathbb{C}([-\tau, 0], \mathbb{R})$. By constructing the Filippov set-valued map (i.e. Filippov regularization), it is obvious that if $x(t)$ is a solution of the Filippov regularized differential equation (50), then it is a solution of the following retarded differential inclusion (51).

$$\frac{dx_i(t)}{dt} \in d_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)\overline{\mathbb{C}[f_j(x_j(t - \tau(t)))]} + u_i(t)$$

$$= F_i(t, x(t), x(t - \tau(t))), \text{ for a.e. } t \in [t_0, +\infty).$$

(51)

It is clear that the set-valued map $F = (F_1, F_2, \ldots, F_n)^T$ is USC with nonempty, compact, convex values and locally bounded. Hence, it is measurable. By the measurable selections Theorem 2, there exists a measurable function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T : [t_0 - \tau, +\infty) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{\mathbb{C}[f_j(x_j(t))]}$ for a.e. $t \in [t_0 - \tau, +\infty)$ and

$$\frac{dx_i(t)}{dt} = d_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)\gamma_j(t - \tau(t)) + u_i(t), \text{ for a.e. } t \in [t_0, +\infty).$$

(52)

Obviously, if $u_i(t) = 0$, then $x = (0, 0, \ldots, 0)^T$ is a zero solution in the sense of Filippov or equilibrium point. Our goal is to stabilize the discontinuous time-delayed neuron system (50) to the equilibrium point $x = (0, 0, \ldots, 0)^T$. To do so, let us design a novel switching controller $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ which is given by the following discontinuous function with respect to state.

$$u_i(t) = k_i x_i(t) + \ell_i \text{sign}(x_i(t)),$$

(53)

where $i \in \mathcal{N}$, the constants $k_i, \ell_i$ are gain coefficients to be determined.

**Theorem 4.1.** Suppose that the conditions (H1) and (H2) are satisfied, assume further that

(H3) For each $i \in \mathcal{N}$ and all $t \in \mathbb{R}$, the following inequalities hold

$$k_i + d_i(t) \leq \frac{1}{1 + t^2}, \ell_i + \sum_{j=1}^n |b_{ij}(t)| \beta_j \leq 0,$$

$$\max_{j \in \mathcal{N}} \left\{ \sum_{j=1}^n |b_{ij}(t)| \alpha_j \right\} \leq \frac{1}{1 + t^2}.$$
Proof. Let us choose a non-smooth Lyapunov-Razumikhin function for (51) as follows

\[ V(t, x) = \sum_{i=1}^{n} |x_i|. \]  

(54)

It is clear that \( V(t, x) \) is \( C \)-regular. Note that the function \( |x_i| \) is locally Lipschitz continuous in \( x_i \) on \( \mathbb{R} \). According to the definition of Clarke’s generalized gradient of function \( |x_i(t)| \) at \( x_i(t) \), we can obtain

\[ \partial(|x_i(t)|) = \partial_0[\text{sign}(x_i(t))] = \begin{cases} -1, & \text{if } x_i(t) < 0, \\ [-1, 1], & \text{if } x_i(t) = 0, \\ 1, & \text{if } x_i(t) > 0. \end{cases} \]

That is to say, for any \( \zeta_i(t) \in \partial(|x_i(t)|) \), we have \( \zeta_i(t) = \text{sign}(x_i(t)) \), if \( x_i(t) \neq 0 \); while \( \zeta_i(t) \) can be arbitrarily chosen in \([-1, 1]\), if \( x_i(t) = 0 \). Especially, we choose \( \zeta_i(t) = \text{sign}(x_i(t)) \). Obviously, it can be seen that \( |x_i(t)| = \zeta_i(t)x_i(t) \). By Lemma 2.10, we can calculate the time derivative of \( V(t, x) \) along the solution trajectory of (51) as follows:

\[
\frac{dV(t, x(t))}{dt} \bigg|_{(51)} = \sum_{i=1}^{n} \frac{dx_i(t)}{dt} \zeta_i(t) \\
= \sum_{i=1}^{n} \left[ d_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t - \tau(t)) + u_i(t) \right] \text{sign}(x_i(t)) \\
= \sum_{i=1}^{n} d_i(t)|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t - \tau(t))\text{sign}(x_i(t)) \\
+ \sum_{i=1}^{n} u_i(t)\text{sign}(x_i(t)) \\
\leq \sum_{i=1}^{n} d_i(t)|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}(t)|\gamma_j(t - \tau(t))\text{sign}(x_i(t)) \\
+ \sum_{i=1}^{n} u_i(t)\text{sign}(x_i(t)), \text{ for a.e. } t \geq t_0. \]

(55)

Substituting the controller (53) into the system (55), we have

\[
\frac{dV(t, x(t))}{dt} \leq \sum_{i=1}^{n} (d_i(t) + k_i)|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}(t)|\gamma_j(t - \tau(t))|\text{sign}(x_i(t))| \\
+ \sum_{i=1}^{n} \ell_i|\text{sign}(x_i(t))|, \text{ for a.e. } t \geq t_0. \]

(56)

Recalling the conditions (H1)-(H3), we derive from (56) that

\[
\frac{dV(t, x(t))}{dt} \leq \frac{1}{1 + t^2} \sum_{i=1}^{n} |x_i(t)| + \sum_{i=1}^{n} \ell_i|\text{sign}(x_i(t))| \\
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}(t)|(\alpha_j|x_j(t - \tau(t))| + \beta_j)|\text{sign}(x_i(t))|.
\]
\[
\begin{align*}
&\leq \frac{1}{1 + t^2} \sum_{i=1}^{n} |x_i(t)| + \max_{j \in \mathcal{N}} \left\{ \sum_{i=1}^{n} |b_{ij}(t)| \alpha_j \right\} \sum_{j=1}^{n} |x_j(t - \tau(t))| \\
&\quad + \sum_{i=1}^{n} \left( \ell_i + \sum_{j=1}^{n} |b_{ij}(t)| \beta_j \right) |\text{sign}(x_i(t))| \\
&\leq \frac{1}{1 + t^2} \sum_{i=1}^{n} (|x_i(t)| + |x_i(t - \tau(t))|), \text{ for a.e. } t \geq t_0. \tag{57}
\end{align*}
\]

If \(|x_i(t - \tau(t))| \leq |x_i(t)| (i \in \mathcal{N})\), i.e., \(V(t - \tau(t), x(t - \tau(t))) \leq V(t, x(t))\), it follows from (57) that
\[
\frac{dV(t, x(t))}{dt} \leq \frac{2}{1 + t^2} \sum_{i=1}^{n} |x_i(t)| = \frac{2}{1 + t^2} V(t, x(t)) \triangleq G(t)V(t, x(t)), \text{ for a.e. } t \geq t_0 \geq 0,
\]
where \(G(t) = \frac{2}{1 + t^2}\). Note that \(\int_{t_0}^{t} G(s)ds \leq \int_{t_0}^{\infty} \frac{2}{1 + s^2}ds \leq \pi < +\infty\) holds for all \(t_0 \in \mathbb{R}_+\). Now, we have checked that all of the conditions in Theorem 3.1 are satisfied. Therefore, we can conclude that the controlled discontinuous time-delayed neuron system (50) is stable under switching controller (53) with the given control gains \(k_i\) and \(\ell_i\). That is, the discontinuous time-delayed neuron system (50) can be stabilized.

If there is no external input in system (50) (i.e., \(u_i(t) = 0\)) and the condition (H2) is replaced with the following condition \((H4)\), then we can obtain the Theorem 4.2:

\((H4)\) For every \(i \in \mathcal{N}, 0 \in \mathcal{C}[f_i(0)]\) and there exist nonnegative constants \(\alpha_i\) such that
\[
\sup_{\gamma_i \in \mathcal{C}[f_i(x_i)]} |\gamma_i| \leq \alpha_i |x_i|, \quad \forall x_i \in \mathbb{R},
\]
where, for \(\theta \in \mathbb{R},\)
\[
\mathcal{C}[f_i(\theta)] = \left[ \min\{f_i^-(\theta), f_i^+(\theta)\}, \max\{f_i^-(\theta), f_i^+(\theta)\} \right].
\]

**Theorem 4.2.** Suppose that the conditions \((H1)\) and \((H4)\) are satisfied, assume further that

\((H5)\) For each \(i \in \mathcal{N}\) and all \(t \geq t_0 \geq 0\), the following inequalities hold
\[
d_i(t) \leq \frac{1}{1 + t^2} - t |\sin(t)|,
\]
\[
\max_{j \in \mathcal{N}} \left\{ \sum_{i=1}^{n} |b_{ij}(t)| \alpha_j \right\} \leq \frac{1}{2} t |\sin(t)|.
\]

Then the zero solution of the discontinuous time-delayed neural network system (50) without external input is asymptotically stable.

**Proof.** Consider the same Lyapunov-Razumikhin function for (50) as (54). Similar to the proof of Theorem 4.1 under the conditions \((H1), (H4)\) and \((H5)\), we can
obtain
\[
\frac{dV(t, x(t))}{dt} \bigg|_{(51)} \leq \sum_{i=1}^{n} d_i(t)|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}(t)||\gamma_j(t - \tau(t))||\text{sign}(x_i(t))|
\]
\[
\leq \left( \frac{1}{1 + t^2} - t |\sin(t)| \right) \sum_{i=1}^{n} |x_i(t)|
\]
\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}(t)||\alpha_j|x_j(t - \tau(t))||\text{sign}(x_i(t))|
\]
\[
\leq \left( \frac{1}{1 + t^2} - t |\sin(t)| \right) \sum_{i=1}^{n} |x_i(t)|
\]
\[
+ \max_{j \in \mathcal{N}} \left\{ \sum_{i=1}^{n} |b_{ij}(t)||\alpha_j| \right\} \sum_{j=1}^{n} |x_j(t - \tau(t))|
\]
\[
\leq \left( \frac{1}{1 + t^2} - t |\sin(t)| \right) \sum_{i=1}^{n} |x_i(t)|
\]
\[
+ \frac{1}{2} t |\sin(t)| \sum_{i=1}^{n} |x_i(t - \tau(t))|, \quad \text{for a.e. } t \geq t_0.
\]
(58)

If \(|x_i(t - \tau(t))| \leq |x_i(t)| (i \in \mathcal{N})
that is to say, \(V(t - \tau(t), x(t - \tau(t))) \leq V(t, x(t))\)
we can derive from (58) that
\[
\frac{dV(t, x(t))}{dt} \leq \left( \frac{1}{1 + t^2} - \frac{1}{2} t |\sin(t)| \right) \sum_{i=1}^{n} |x_i(t)|
\]
\[
= \left( \frac{1}{1 + t^2} - \frac{1}{2} t |\sin(t)| \right) V(t, x)
\]
\[
\triangleq G(t)V(t, x), \quad \text{for a.e. } t \geq t_0 \geq 0,
\]
where \(G(t) = \frac{1}{1 + t^2} - \frac{1}{2} t |\sin(t)|\). It is obvious that
\[
\int_{t_0}^{+\infty} G^+(s)ds \leq \int_{t_0}^{+\infty} \frac{1}{1 + s^2}ds \leq \frac{\pi}{2} < +\infty,
\]
(59)
and
\[
\int_{t_0}^{+\infty} G(s)ds = \int_{t_0}^{+\infty} \frac{1}{1 + s^2}ds - \int_{t_0}^{+\infty} \frac{1}{2} s |\sin(s)|ds = -\infty.
\]
(60)

Noticing that \(-G^{-}(s) \leq G(s) \leq G^{+}(s)\), from (59) and (60), we can deduce
\[
\int_{t_0}^{+\infty} G^{-}(s)ds = +\infty.
\]
So, according to Theorem 3.4 and Remark 6, we can get that the zero solution of the discontinuous time-delayed neural network system (50) without external input is asymptotically stable.

Example 1. Let us consider the two-dimensional discontinuous retarded neural networks (50) with \(d_i(t) = b_{ij}(t) = \frac{1}{1 + t^2} (i, j \in \mathcal{N})\) and \(\tau(t) = 1\). The discontinuous neuron activation functions are given as
\[
f_i(\theta) = \begin{cases} 
\frac{1}{2} \theta + \frac{1}{10}, & \theta \geq 0, \\
\frac{1}{2} \theta - \frac{1}{10}, & \theta < 0,
\end{cases} \quad i = 1, 2.
\]
It is easy to check that the discontinuous neuron activation functions (see FIGURE 1) satisfy the conditions \((\mathcal{H}1)\) and \((\mathcal{H}2)\) with \(\alpha_i = \frac{1}{3}, \beta_i = \frac{1}{10} (i = 1, 2)\). Obviously, 0 is a discontinuous point of the activation function \(f_i(\cdot)\) and \(\text{co}[f_i(0)] = [-\frac{1}{10}, \frac{1}{10}]\).

Under the switching state-feedback controller (53), let us select the control gains \(k_i = \ell_i = -1\). It is not difficult to calculate that

\[
-1 + \frac{1}{1 + t^2} = k_i + d_i(t) \leq \frac{1}{1 + t^2},
\]

\[
\ell_i + \sum_{j=1}^{2} \left| b_{ij}(t) \right| \beta_j = -1 + \sum_{j=1}^{2} \frac{1}{1 + t^2} \cdot \frac{1}{10} \leq -1 + \frac{1}{5} \leq 0,
\]
\[
\frac{2}{3} \cdot \frac{1}{1 + t^2} = \max_{j \in \mathcal{N}} \left\{ \sum_{i=1}^{2} |b_{ij}(t)|\alpha_j \right\} \leq \frac{1}{1 + t^2}.
\]

Therefore, the conditions of Theorem 4.1 are satisfied. Then, we can stabilize the discontinuous time-delayed neural network system (50) by using switching controller (53) with the control gains \( k_i = \ell_i = -1 \). The simulation result is depicted in FIGURE 2 with initial values \( x_{t_0} = x_0 = (4, -4)^T \) for \( t \in [-1, 0] \). The above numerical simulation fit the theoretical results perfectly.

**Figure 3.** Discontinuous neuron activation functions of Example 2.

**Figure 4.** Time-domain behaviors of the state variables \( x_1(t) \) and \( x_2(t) \) for system (50) without external input in Example 2.
Example 2. Let us consider the two-dimensional discontinuous retarded neural networks (50) with 
\[ u_i(t) = 0, \quad d_i(t) = \frac{1}{t^2} + t|\sin(t)|, \quad b_{ij}(t) = \frac{1}{t^2} + \sin(t) \]
\((i,j \in \mathcal{N})\) and \(\tau(t) = 1\). The discontinuous neuron activation functions are given as
\[ f_i(\theta) = \begin{cases} \frac{1}{2}\theta, & 0 \leq \theta < 1, \\ \theta, & \theta < 0 \text{ or } \theta \geq 1, \end{cases} \]
i = 1, 2.

Obviously, the discontinuous neuron activation functions (see FIGURE 3) satisfy the conditions \((\mathcal{H}1)\) and \((\mathcal{H}4)\) with \(\alpha_i = 1\) \((i = 1, 2)\). Meanwhile, 1 is a discontinuous point of the activation function \(f_i(\cdot)\) and \([f_i^{-}(1), f_i^{+}(1)] = \left[\frac{1}{2}, 1\right]\). It is not difficult to check the condition \((\mathcal{H}5)\) holds. According to Theorem 4.2, we can conclude that the zero solution of the discontinuous time-delayed neural network system (50) without external input is asymptotically stable. Consider the initial conditions of system (50) without external input:
\[ x_{t_0} = x_0 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \]
for \(t \in [-1, 0]\).

Remark 8. In the monograph [21], Hale has developed the Lyapunov-Razumikhin method to deal with the stability of retarded differential equations and given some illustrative examples. However, by comparison, we can find that the right-hand sides of retarded differential equations are continuous in [21]. Thus, the method of [21] is invalid to investigate the stability and stabilization of the discontinuous time-delayed neural networks. Moreover, the Lyapunov-Razumikhin function \(V(t, \phi(0))\) of [21] is required to be differentiable with respect to time \(t\) for everywhere and its derivative is required to be negative or semi-negative definite. From Theorem 4.1 and Theorem 4.2, it can be seen that our Lyapunov-Razumikhin method can be used to handle the retarded differential equations with discontinuous right-hand sides via the framework of retarded differential inclusions. In addition, the Lyapunov-Razumikhin functions of Theorem 4.1 and Theorem 4.2 are allowed to have indefinite or positive definite derivative for almost everywhere. Therefore, our method and results are less restrictive than those of [21] in some sense.

5. Conclusions. In this paper, an extended Filippov-framework has been given to deal with the retarded differential equation with discontinuous right-hand side. That is to say, by constructing the Filippov set-valued map (i.e., Filippov regularization), we have introduced a new concept of Filippov solution and the initial value problem for retarded differential equation with discontinuous property. By doing so, the solution of retarded differential equation could be transformed into a solution of retarded differential inclusion. On this basis, we further investigate a large number of basic questions about the stability behaviors for retarded differential inclusions via the extended Filippov-framework. First of all, we have developed the generalized Lyapunov-Razumikhin method. Then, some new sufficient conditions have been established to guarantee the stability, uniform stability and asymptotic stability of the zero solution for retarded differential inclusion. These obtained results are novel since we have relaxed the requirement on the derivative (if it exists) of the Lyapunov-Razumikhin function and dropped some specific conditions. In other words, the derivative (if it exists) of Lyapunov-Razumikhin function is allowed to be almost indefinite instead of negative or semi-negative definite for almost everywhere in the earlier literature. Moreover, the Lyapunov-Razumikhin function has been relaxed to be non-smooth and the Lyapunov-Razumikhin function for handling the stability is not required to have an infinitesimal upper limit. In the end,
the generalized Lyapunov-Razumikhin method is applied to study the stability and stabilization issues of the discontinuous time-delayed neural networks. Meanwhile, two concrete examples are listed to illustrate our method and main results. An interesting open question is whether the results on Lyapunov-like stability for RDE with discontinuous property can be extended to a stochastic situation. This needs further development of the theory and method for stochastic RDE with discontinuous right-hand side and future work will be devoted to an investigation of this topic.

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