Global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical Hartree equation in $\mathbb{R}^d$

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Abstract

We prove the global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation with low regularity data in $\mathbb{R}^d$, $d \geq 3$. Precisely, we show that a unique and global solution exists for initial data in the Sobolev space $H^s(\mathbb{R}^d)$ with $s > 4(\gamma - 2)/(3\gamma - 4)$, which also scatters in both time directions. This improves the result in [5], where the global well-posedness was established for any $s > \max\left(\frac{1}{2}, \frac{4(\gamma - 2)}{3\gamma - 4}\right)$. The new ingredients in our proof are that we make use of an interaction Morawetz estimate for the smoothed out solution $Iu$, instead of an interaction Morawetz estimate for the solution $u$, and that we make careful analysis of the monotonicity property of the multiplier $m(\xi) \cdot \langle \xi \rangle^p$. As a byproduct of our proof, we obtain that the $H^s$ norm of the solution obeys the uniform-in-time bounds.

Key Words: Almost Interaction Morawetz estimate; Well-posedness; Hartree equation; I-method; Uniform bound.

AMS Classification: 35Q40, 35Q55, 47J35.

1 Introduction

In this paper, we study the global well-posedness of the following initial value problem (IVP) for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation.

\[
\begin{aligned}
    iu_t + \Delta u &= (|x|^{-\gamma} * |u|^2)u, & d \geq 3, \\
    u(0) &= u_0(x) \in H^s(\mathbb{R}^d),
\end{aligned}
\]

(1.1)

where $H^s$ denotes the usual inhomogeneous Sobolev space of order $s$.

We adopt the following standard notion of local well-posedness, that is, we say that the IVP (1.1) is locally well-posed in $H^s$ if for any $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_s)$ depending only on the norm of the initial data, such that a solution to the IVP exists on the time interval $[0, T]$, is unique in a certain Banach space of functional $X \subset C([0, T], H^s)$, and the solution map from $H^s$ to $C([0, T], H^s)$ depends continuously. If $T$ can be taken arbitrarily large, we say that the IVP (1.1) is globally well-posed.
Local well-posedness for the IVP (1.1) in $H^s$ for any $s > \frac{\gamma}{2} - 1$ was established in [18]. A local solution also exists for $H^{\frac{\gamma}{2} - 1}$ initial data, but the time of existence depends not only on the $H^{\frac{\gamma}{2} - 1}$ norm of $u_0$, but also on the profile of $u_0$. For more details on local well-posedness see [18].

$L^2$ solutions of (1.1) enjoy mass conservation

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^d)}.$$ 

Moreover, $H^1$ solutions enjoy energy conservation

$$E(u)(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} |u(t, x)|^2 |u(t, y)|^2 \, dx \, dy = E(u)(0),$$

which together with mass conservation and the local theory immediately yields global well-posedness for (1.1) with initial data in $H^1$. A large amount of works have been devoted to global well-posedness and scattering for the Hartree equation, see [7]-[11], [13], [15], [17]-[23].

Existence of global solutions in $\mathbb{R}^3$ to (1.1) corresponding to initial data below the energy threshold was recently obtained in [5] by using the method of “almost conservation laws” or “I-method” (for a detailed description of this method, see [25] or section 3 below) and the interaction Morawetz estimate for the solution $u$, where global well-posedness was obtained in $H^s(\mathbb{R}^3)$ with $s > \max(1/2, 4(\gamma - 2)/(3\gamma - 4))$. Since authors in [5] used the interaction Morawetz estimate, which involves $\dot{H}^{1/2}$ norm of the solution, the restriction condition $s \geq \frac{1}{2}$ is prerequisite. In order to resolve IVP (1.1) in $H^s$, $s < \frac{1}{2}$ by still using the interaction Morawetz estimate, we need return to the interaction Morawetz estimate for the smoothed out version $Iu$ of the solution, which is initially used in [2], whereafter in [5].

In this paper, we consider the case $d \geq 3$ and we prove the following result:

**Theorem 1.1.** Let $2 < \gamma < 3 \leq d$, the initial value problem (1.1) is globally well-posed in $H^s(\mathbb{R}^d)$ for any $s > \frac{4(\gamma - 2)}{3\gamma - 4}$. Moreover the solution satisfies

$$\sup_{t \in [0, \infty)} \|u(t)\|_{H^s(\mathbb{R}^d)} \leq C(\|u_0\|_{H^s}),$$
and there is scattering for these solutions, that is, the wave operators exist and there is asymptotic completeness on all of $H^s(\mathbb{R}^d)$.

Remark 1.1. As for the case $3 \leq \gamma < 4 \leq d$, local well-posedness for the IVP (1.1) in $H^s$ holds for any $s > \frac{\gamma}{2} - 1$. Note that in this case, we have

$$\frac{\gamma}{2} - 1 \geq \frac{1}{2},$$

which satisfies the need of the regularity of the interaction Morawetz estimate. Hence we only combine “I-method” with the interaction Morawetz estimate for the solution to obtain the low regularity of the IVP (1.1), just as in [3].

For the case $d = 3$, Theorem 1.1 improves the result $s > \max\left(\frac{1}{2}, \frac{4(\gamma - 2)}{(3\gamma - 4)}\right)$ in [5] (see Figure 1), where the authors used “I-method” and the interaction Morawetz estimate for the solution just as in [3]. In general, in order to prove the almost conservation law, one doesn’t need to use the monotonicity property of the multiplier $m(\xi) : \langle \xi \rangle^p$. In the present paper, we prove Theorem 1.1 by combining I-method with an interaction Morawetz estimate for the smoothed out version $Iu$ of the solution. Such a Morawetz estimate for an almost solution is the main novelty of this paper, which can lower the need on the regularity of the initial data.

Last, we organize this paper as following: In Section 2, we introduce some notation and state some important propositions that we will used throughout this paper. In Section 3, we review the I-method, prove the local well-posedness theory for $Iu$ and obtain an upper bound on the increment of the modified energy. In Section 4, we prove the “almost interaction Morawetz estimate” for the smoothed out version $Iu$ of the solution. Finally in Section 5, we give the details of the proof of the global well-posedness stated in Theorem 1.1.

2 Notation and preliminaries

2.1 Notation

In what follows, we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$. If $A \lesssim B$ and $B \lesssim A$, we say that $A \approx B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition, $\langle a \rangle := 1 + |a|$ and $a \pm := a \pm \epsilon$ with $0 < \epsilon \ll 1$. The reader also has to be alert that we sometimes do not explicitly write down constants that depend on the $L^2$ norm of the solution. This is justified by the conservation of the $L^2$ norm.

2.2 Definition of spaces

We use $L^r_x(\mathbb{R}^d)$ to denote the Lebesgue space of functions $f : \mathbb{R}^d \to \mathbb{C}$ whose norm

$$\|f\|_{L^r_x} := \left( \int_{\mathbb{R}^d} |f(x)|^r \, dx \right)^{\frac{1}{r}}$$
is finite, with the usual modification in the case \( r = \infty \). We also use the space-time Lebesgue spaces \( L^q_t L^r_x \) which are equipped with the norm
\[
\|u\|_{L^q_t L^r_x} := \left( \int_J \|u(t, x)\|_{L^r_x}^q \, dt \right)^{\frac{1}{q}}
\]
for any space-time slab \( J \times \mathbb{R} \), with the usual modification when either \( q \) or \( r \) are infinity.

When \( q = r \), we abbreviate \( L^q_t L^r_x \) by \( L^q_{t,x} \).

As usual, we define the Fourier transform of \( f(x) \in L^1_x \) by
\[
\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx.
\]

We define the fractional differentiation operator \( |\nabla_x|^\alpha \) for any real \( \alpha \) by
\[
\hat{|\nabla|^\alpha u}(\xi) := |\xi|^\alpha \hat{u}(\xi),
\]
and analogously
\[
\hat{\langle \nabla \rangle^\alpha u}(\xi) := \langle \xi \rangle^\alpha \hat{u}(\xi).
\]

The inhomogeneous Sobolev space \( H^s(\mathbb{R}^d) \) is given via
\[
\|u\|_{H^s} := \|\langle \nabla \rangle^s u\|_{L^2(\mathbb{R}^d)},
\]
while the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^d) \) is given via
\[
\|u\|_{\dot{H}^s} := \|\nabla^s u\|_{L^2(\mathbb{R}^d)}.
\]

Let \( S(t) \) denote the solution operator to the linear Schrödinger equation
\[
\imath u_t + \Delta u = 0, x \in \mathbb{R}^d.
\]

We denote by \( X^{s,b}(\mathbb{R} \times \mathbb{R}^d) \) the completion of \( \mathcal{S}(\mathbb{R} \times \mathbb{R}^d) \) with respect to the following norm
\[
\|u\|_{X^{s,b}} = \|S(-t)u\|_{H^s_x H^b_t} = \|\tau + |\xi|^2 h(\xi)^s \tilde{u}(\tau, \xi)\|_{L^2_x L^2_t(\mathbb{R} \times \mathbb{R}^d)},
\]
where \( \tilde{u} \) is the space-time Fourier transform
\[
\tilde{u}(\tau, \xi) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R} \times \mathbb{R}^d} e^{-i(x \cdot \xi + t\tau)} u(t, x) \, dt \, dxdx.
\]

Furthermore for a given time interval \( J \), we define
\[
\|u\|_{X^{s,b}(J)} = \inf \left\{ \|v\|_{X^{s,b}} ; \ v = u \text{ on } J \right\}.
\]
2.3 Some known estimates

Now we recall a few known estimates that we shall need. First we state the following Strichartz estimate [1], [14]. Let $d \geq 3$, we recall that a pair of exponents $(q, r)$ is called admissible if

$$\frac{2}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right), \quad 2 \leq q, r \leq \infty.$$

**Proposition 2.1.** Let $d \geq 3$, $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any two admissible pairs. Suppose that $u$ is a solution to

$$iu_t + \Delta u = F(t, x), \quad t \in J, x \in \mathbb{R}^d$$

$$u(0) = u_0(x).$$

Then we have the estimate

$$\|u\|_{L^q_t L^r_x(J \times \mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^\tilde{q}_t L^\tilde{r}_x(J \times \mathbb{R}^d)},$$

where the prime exponents denote Hölder dual exponents.

Let us say that a function $u$ has spatial frequency $N$ if its Fourier transform is supported on the annulus $\{ |\xi| \approx N \}$. From Strichartz estimate

$$\|u\|_{L^q_t L^r_x(J \times \mathbb{R}^d)} \lesssim \|u_0\|_{X^{0, \frac{1}{2}+}},$$

for admissible $(q, r)$ and Sobolev embedding theorem, we have

**Proposition 2.2.** For $r < \infty$, $0 \leq \frac{2}{q} \leq \min \{ \delta(r), 1 \}$, we have

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{\delta(r) - \frac{2}{q}, \frac{1}{2}+}}.$$ 

While for $2 \leq q \leq \infty$, $r = \infty$, we have

$$\|u\|_{L^q_t L^\infty_x} \lesssim \|u\|_{X^{\frac{2}{q} - \frac{1}{2}, \frac{1}{2}+}}.$$

3 the I-method and the modified local well-posedness

3.1 the I-operator and the hierarchy of energies

Let us define the operator $I$. For $s < 1$ and a parameter $N \gg 1$, let $m(\xi)$ be the following smooth monotone multiplier:

$$m(\xi) := \begin{cases} 
1, & \text{if } |\xi| < N, \\
\left( \frac{N}{|\xi|} \right)^{-s}, & \text{if } |\xi| > 2N.
\end{cases}$$

We define the multiplier operator $I : H^s \rightarrow H^1$ by

$$\hat{I}u(\xi) = m(\xi)\hat{u}(\xi).$$

The operator $I$ is smoothing of order $1 - s$ and we have that

$$\|u\|_{H^s} \lesssim \|Iu\|_{H^{s+1-s}} \lesssim N^{1-s}\|u\|_{H^{s+1}},$$

$$\|u\|_{X^{s, b_0}} \lesssim \|Iu\|_{X^{s+1-s, b_0}} \lesssim N^{1-s}\|u\|_{X^{s, b_0}}.$$
for any \( s_0, b_0 \in \mathbb{R} \).

We set
\[
\tilde{E}(u) = E(Iu),
\]
(3.1)
where
\[
E(u)(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4} \int \int \frac{1}{|x - y|^\gamma} |u(t, x)|^2 |u(t, y)|^2 \, dx \, dy.
\]

We call \( \tilde{E}(u) \) the modified energy. Since we will focus on the analysis of the modified energy, we collect some facts concerning the calculus of multilinear forms used to define the modified energy.

If \( k \geq 2 \) is an even integer, we define a spatial multiplier of order \( k \) to be the function \( M_k(\xi_1, \xi_2, \cdots, \xi_k) \) on
\[
\Gamma_k = \left\{ (\xi_1, \xi_2, \cdots, \xi_k) \in (\mathbb{R}^d)^k : \sum_{j=1}^k \xi_j = 0 \right\},
\]
which we endow with the standard measure \( \delta(\xi_1 + \xi_2 + \cdots + \xi_k) \). If \( M_k \) is a multiplier of order \( k \), \( 1 \leq j \leq k \) is an index and \( l \geq 1 \) is an even integer, the elongation \( X^j_k(M_k) \) of \( M_k \) is defined to be the multiplier of order \( k + l \) given by
\[
X^j_k(M_k)(\xi_1, \xi_2, \cdots, \xi_{k+l}) = M_k(\xi_1, \cdots, \xi_{j-1}, \xi_j + \cdots + \xi_{j+l}, \xi_{j+l+1}, \cdots, \xi_{k+l}).
\]

Also if \( M_k \) is a multiplier of order \( k \) and \( u_1, u_2, \cdots, u_k \) are functions on \( \mathbb{R}^d \), we define the \( k \)-linear functional
\[
\Lambda_k(M_k; u_1, u_2, \cdots, u_k) = \text{Re} \int_{\Gamma_k} M_k(\xi_1, \xi_2, \cdots, \xi_k) \prod_{j=1}^k u_j(\xi_j)
\]
and we adopt the notation \( \Lambda_k(M_k; u) = \Lambda_k(M_k; u, \overline{u}, \cdots, u, \overline{u}) \). We observe that the quantity \( \Lambda_k(M_k; u) \) is invariant

1. if one permutes the even arguments \( \xi_2, \xi_4, \cdots, \xi_k \) of \( M_k \);
2. if one permutes the odd arguments \( \xi_1, \xi_3, \cdots, \xi_{k-1} \) of \( M_k \);
3. if one makes the change of
\[
M_k(\xi_1, \xi_2, \cdots, \xi_{k-1}, \xi_k) \mapsto M_k(-\xi_2, -\xi_1, \cdots, -\xi_k, -\xi_{k-1}).
\]

If \( u \) is a solution of (1.1), the following differentiation law holds for the multiplier forms \( \Lambda_k(M_k; u) \)
\[
\partial_t \Lambda_k(M_k; u) = \Lambda_k(iM_k \sum_{j=1}^k (-1)^j |\xi_j|^2; u) + \Lambda_{k+2} \left( i \sum_{j=1}^k (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X^2_j(M_k; u) \right)
\]
(3.2)
where we use the notational convention \( \xi_{a,b} = \xi_a + \xi_b, \xi_{a,b,c} = \xi_a + \xi_b + \xi_c, \) etc.
Using the above notation, the modified energy \( \tilde{E}(u) \) can be written as follows:

\[
\tilde{E}(u) = \Lambda_2 \left( - \frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u \right) + \Lambda_4 \left( \frac{1}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u \right)
\]

where we abbreviate \( m(\xi_j) \) as \( m_j \).

Together with the the differentiation rules and the symmetry properties of k-linear functional \( \Lambda_k(M_k; u) \), we obtain

\[
\partial_t \Lambda_2 \left( - \frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u \right) = \Lambda_2 \left( - \frac{i}{2} \xi_1 m_1 \cdot \xi_2 m_2 \sum_{j=1}^2 (-1)^j |\xi_j|^2; u \right) + \Lambda_4 \left( - \frac{i}{2} \sum_{j=1}^2 (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X_j^2 (\xi_1 m_1 \cdot \xi_2 m_2; u) \right) = \Lambda_4 (i|\xi_{2,3}|^{-(d-\gamma)} m_1^2 |\xi_1|^2; u),
\]

and

\[
\partial_t \Lambda_4 (i|\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u)
\]

\[
= \Lambda_1 \left( \frac{i}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4 \sum_{j=1}^4 (-1)^j |\xi_j|^2; u \right) + \Lambda_6 \left( \frac{i}{4} \sum_{j=1}^3 (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X_j^2 (|\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u) \right)
\]

\[
= -\Lambda_4 (i|\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 m_2 m_3 m_4; u) - \Lambda_6 (i|\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_{1,2,3} m_4 m_5 m_6; u)
\]

The fundamental theorem of calculus together with these estimates implies the following proposition, which will be used to prove that \( \tilde{E} \) is almost conserved.

**Proposition 3.1.** Let \( u \) be an \( H^1 \) solution to (1.1). Then for any \( T \in \mathbb{R} \) and \( \delta > 0 \), we have

\[
\tilde{E}(u)(T + \delta) - \tilde{E}(u)(T) = \int_T^{T+\delta} \Lambda_4 (M_4; u) \, dt + \int_T^{T+\delta} \Lambda_6 (M_6; u) \, dt
\]

with

\[
M_4 = i|\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 (m_1 - m_2 m_3 m_4); \\
M_6 = i|\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_{1,2,3} (m_{1,2,3} - m_4 m_5 m_6).
\]

Furthermore if \( |\xi_j| \ll N \) for all \( j \), then the multipliers \( M_4 \) and \( M_6 \) vanish on \( \Gamma_4 \) and \( \Gamma_6 \), respectively.
3.2 Modified local well-posedness

In this subsection, we shall prove a local well-posedness result for the modified solution $Iu$ and some a priori estimates for it.

Let $J = [t_0, t_1]$ be an interval of time. We denote by $Z_I(J)$ the following space:

$$Z_I(J) = S_I(J) \cap X^{1, \frac{1}{2} - \epsilon}_I(J)$$

where

$$S_I(J) = \left\{ u; \sup_{(q,r) \text{ admissible}} \| \langle \nabla \rangle Iu \|_{L^q_t L^r_x(J \times \mathbb{R}^d)} < \infty \right\},$$

$$X^{1, \frac{1}{2} - \epsilon}_I(J) = \left\{ u; \| Iu \|_{X^{1, \frac{1}{2} - \epsilon}(J \times \mathbb{R}^d)} < \infty \right\}.$$

Proposition 3.2. Let $2 < \gamma < 3 \leq d$ and $s > \frac{d}{2} - 1$, and consider the IVP

$$iIu_t + \Delta Iu = I(|x|^{-\gamma} * |u|^2 u), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}$$

$$Iu(t_0, x) = Iu_0(x) \in H^s(\mathbb{R}^d).$$

Then for any $u_0 \in H^s$, there exists a time interval $J = [t_0, t_0 + \delta]$, $\delta = \delta(\| Iu_0 \|_{H^1})$ and there exists a unique $u \in Z_I(J)$ solution to (3.3). Moreover there is continuity with respect to the initial data.

**Proof:** The proof of this proposition proceeds by the usual fixed point method on the space $Z_I(J)$. Since the estimates are very similar to the ones we provide in the proof of Proposition 3.3 below, in particular (3.9) and (3.10), we omit the details.

Proposition 3.3. Let $2 < \gamma < 3 \leq d$ and $s > \frac{d}{2} - 1$. If $u$ is a solution to the IVP (3.3) on the interval $J = [t_0, t_1]$, which satisfies the following a priori bound

$$\| Iu \|^{4}_{L^{d+3}_{t} L^{\frac{d+4}{4}}_{x}(J \times \mathbb{R}^d)} < \mu,$$

where $\mu$ is a small universal constant, then

$$\| u \|_{Z_I(J)} \lesssim \| Iu_0 \|_{H^s}.$$

**Proof:** We start by obtaining a control of the Strichartz norms. Applying $\langle \nabla \rangle$ to (3.3) and using the Strichartz estimate in Proposition 2.1. For any pair of admissible exponents $\langle q, r \rangle$, we obtain

$$\| \langle \nabla \rangle Iu \|_{L^q_t L^r_x} \lesssim \| Iu_0 \|_{H^1} + \| \langle \nabla \rangle I(|x|^{-\gamma} * |u|^2 u) \|_{L^q_t L^r_x}^{\frac{4d}{d+4}}. \quad (3.4)$$

Now we notice that the multiplier $\langle \nabla \rangle I$ has symbol which is increasing as a function of $|\xi|$ for any $s \geq \frac{d}{2} - 1$. Using this fact one can modify the proof of the Leibnitz rule for fractional derivatives and prove its validity for $\langle \nabla \rangle I$. See also Principle A.5 in the appendix of [25]. This remark combined with (3.4) implies that

$$\| \langle \nabla \rangle Iu \|_{L^q_t L^r_x} \lesssim \| Iu_0 \|_{H^1} + \langle \nabla \rangle I(|x|^{-\gamma} * |u|^2 u) \|_{L^q_t L^r_x}^{\frac{4d}{d+4}} + \| |x|^{-\gamma} * |u|^2 \|_{L^q_t L^r_x}^{\frac{4d}{d+4}} \| \langle \nabla \rangle Iu \|_{L^q_t L^r_x}^{\frac{4d}{d+4}}, \quad (3.5)$$

$$\lesssim \| Iu_0 \|_{H^1} + \langle \nabla \rangle Iu \|_{L^q_t L^r_x}^{\frac{4d}{d+4}} \| u \|_{L^q_t L^r_x}^2 \| \langle \nabla \rangle Iu \|_{L^q_t L^r_x}^{\frac{4d}{d+4}}.$$
where we used Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality.

In order to obtain an upper bound on \( \|u\|_{L_t^q L_x^{\frac{6d}{3d + 4 - s'}}} \), we perform a Littlewood-Paley decomposition along the following lines. We note that a similar approach was used in [3]. We write

\[
u = \nu_{N_0} + \sum_{j=1}^{\infty} \nu_{N_j}, \tag{3.6}
\]

where \( \nu_{N_0} \) has spatial frequency support for \( \langle \xi \rangle \leq N \), while \( \nu_{N_j} \) is such that its spatial Fourier support transform is supported for \( \langle \xi \rangle \approx N_j = 2^{h_j} \) with \( h_j \geq \log N \) and \( j = 1, 2, \ldots \).

By the triangle inequality and Hölder’s inequality, we have

\[
\|u\|_{L_t^q L_x^{\frac{6d}{3d + 4 - s'}}} \lesssim \|\nu_{N_0}\|_{L_t^q H_x^{\frac{d-3}{4}}} \|\nu_{N_0}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} + \sum_{j=1}^{\infty} \|\nu_{N_j}\|_{L_t^q H_x^{\frac{d-3}{4}}} \|\nu_{N_j}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} \tag{3.7}
\]

On the other hand, by using the definition of the operator \( I \), the definition of the \( \nu_{N_j} \)'s and the Marcinkiewicz multiplier theorem, we observe that for some \( 0 < \theta_i < 1, i = 1, \cdots, 4, \sum_{j=1}^{4} \theta_i = 1 \)

\[
\begin{align*}
\|\nu_{N_0}\|_{L_t^q L_x^{\frac{6d}{3d + 4 - s'}}} & \lesssim \|\nu_{N_0}\|_{L_t^q H_x^{\frac{d-3}{4}}} \|\nu_{N_0}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} \|\nu_{N_0}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} \|\nu_{N_0}\|_{L_t^q L_x^{\frac{3d}{3d + 4}}} \\
& \lesssim \|\nu_{N_0}\|_{L_t^q H_x^{\frac{d-3}{4}}} \|\nu_{N_0}\|_{Z_I(J)}^{1-\theta_1} \\
\|\langle \nabla \rangle I \nu_{N_j}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} & \approx N_j \left( \frac{N_{N_j}}{N} \right)^{1-s} \|\nu_{N_j}\|_{L_t^q L_x^{\frac{6d}{3d + 4}}} , \quad j = 1, 2, \cdots \quad (3.8) \\
\|\langle \nabla \rangle I \nu_{N_j}\|_{L_t^q H_x^{\frac{d-3}{4}}} & \approx \left( \frac{N_{N_j}}{N} \right)^{1-s} \|\nu_{N_j}\|_{L_t^q H_x^{\frac{d-3}{4}}} , \quad j = 1, 2, \cdots 
\end{align*}
\]

Now we use these estimates to obtain the following upper bound on (3.7)

\[
\begin{align*}
\|u\|_{L_t^q L_x^{\frac{6d}{3d + 4 - s'}}} & \lesssim \|\nu_{N_0}\|_{L_t^q H_x^{\frac{d-3}{4}}} \|\nu_{N_0}\|_{Z_I(J)}^{1-\theta_1} \\
& + \sum_{j=1}^{\infty} \left( \frac{N_{N_j}}{N} \right)^{1-s} \|u_{N_j}\|_{Z_I(J)} \left( \left( \frac{N_{N_j}}{N} \right)^{1-2} \|U_{N_j}\|_{Z_I(J)} \right)^{\frac{3}{2} - 1} \\
& \lesssim \mu \frac{\theta_1}{\frac{1}{Z_I(J)}} + \frac{\theta_1}{\frac{1}{Z_I(J)}} + N^{-\left( 2 - \frac{3}{2} \right)} \|u\|_{Z_I(J)} \\
& \lesssim \mu \frac{\theta_1}{\frac{1}{Z_I(J)}} + N^{-\left( 2 - \frac{3}{2} \right)} \|u\|_{Z_I(J)}, \\
\end{align*}
\]

which together with (3.5) implies that

\[
\|\langle \nabla \rangle I u\|_{L_t^q L_x^{\frac{6d}{3d + 4 - s'}}} \lesssim \|I u\|_{H^1} + \mu \frac{\theta_1}{\frac{1}{Z_I(J)}} + N^{-\left( 2 - \frac{3}{2} \right)} \|u\|_{Z_I(J)}^{3} \\
\tag{3.9}
\]

Now we shall obtain a control of the \( X^{s,b} \) norm. We use Duhamel’s formula and the
First we give the quadrilinear estimate.

\[ \| Iu \|_{X^{1, \frac{1}{2}}} \lesssim \| Iu_0 \|_{H^1} + \| \langle \nabla \rangle I(\langle |x|^{-\gamma} \ast |u|^2 \rangle u) \|_{X^{0, -\frac{1}{2}}} \]

\[ \lesssim \| Iu_0 \|_{H^1} + \| \langle \nabla \rangle I(\langle |x|^{-\gamma} \ast |u|^2 \rangle u) \|_{L_t^2 L_x^{\frac{6d}{9d-1}}} \]

\[ \lesssim \| Iu_0 \|_{H^1} + \| \langle \nabla \rangle Iu \|_{L_t^2 L_x^{\frac{6d}{9d-s}}} + \| u \|_{L_t^2 L_x^{\frac{6d}{9d-s}}} \| u \|_{L_t^6 L_x^{\frac{6d}{9d-s}}} \] \quad (3.10)

An upper bound on \( \| u \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \) is given by (3.8). In order to obtain an upper bound on \( \| u \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \), we proceed as follows. First we perform a dyadic decomposition and write \( u \) as (3.6). The triangle inequality applied on (3.6) gives for any \( 0 < \delta < \frac{2}{5} - 1 \)

\[ \| u \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \lesssim \| u_{N_0} \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} + \sum_{j=1}^{\infty} \| u_{N_j} \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \]

\[ = \| Iu_{N_0} \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} + \sum_{j=1}^{\infty} N_j^{d-s} N_j^{s-1} \| \langle \nabla \rangle^{1-\delta} Iu_{N_j} \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \]

\[ \lesssim \| Iu \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} + \| \langle \nabla \rangle^{1-\delta} Iu \|_{L_t^{6+} L_x^{\frac{6d}{9d-s}}} \lesssim \| Iu \|_{X^{1, \frac{1}{2}}} \] \quad (3.11)

where we use Proposition 2.2. By applying the inequalities (3.8) and (3.11) to bound the right hand side of (3.10), we obtain

\[ \| Iu \|_{X^{1, \frac{1}{2}}} \lesssim \| Iu_0 \|_{H^1} + \mu \| u \|^{3-\theta_1} \| u \|^{3-\theta_1} + N^{-\theta_1} \| u \|^{\frac{3}{\theta_1}} \] \quad (3.12)

The desired bound follows from (3.9) and (3.12) by choosing \( N \) sufficiently large.

### 3.3 An upper bound on the increment of \( \tilde{E}(u) \)

**Decomposition remark.** Our approach to prove a decay for the increment of the modified energy is based on obtaining certain multilinear estimates in appropriate functional spaces that are \( L^2 \)-based. Hence, whenever we perform a Littlewood-Paley decomposition of a function we shall assume that the Fourier transforms of the Littlewood-Paley pieces are positive. Moreover, we will ignore the presence of conjugates. At the end we will always keep a decay factor \( C(N_1, N_2, \cdots) \) in order to perform the summations.

Now we proceed to prove the almost conservation law of the modified energy. In Proposition 3.1, we prove that an increment of the modified energy can be expressed as

\[ \tilde{E}(u)(T + \delta) - \tilde{E}(u)(T) = \int_T^{T+\delta} \Lambda_4(M_4; u) \, dt + \int_T^{T+\delta} \Lambda_6(M_6; u) \, dt \]

with

\[ M_4 = i|\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 (m_1 - m_2 m_3 m_4) \]

\[ M_6 = i|\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_{1,2,3} (m_{1,2,3} - m_{4,m5,m6}) \]

Hence in order to control the increment of the modified energy, we shall find an upper bound on the \( \Lambda_4(M_4; u) \) and \( \Lambda_6(M_6; u) \) forms, which we do in the following propositions. First we give the quadrilinear estimate.
Proposition 3.4. For any Schwartz function \( u \), and any \( \delta \approx 1 \) just as in Proposition 3.2, we have that
\[
\left| \int_T^{T + \delta} \Lambda_4(M_4; u) \, dt \right| \lesssim N^{-1+4 \Delta \frac{4+1}{4}}, \tag{3.13}
\]
for \( s > \frac{2}{3} - 1 \).

Proof: By Plancherel theorem, we aim to prove that
\[
\left| \int_T^{T + \delta} \int_{\Gamma_4} |\xi_2,3|^{-(d-\gamma)}|\xi_1| m_1(m_1 - m_2m_3m_4) \hat{u}_1(t, \xi_1) \hat{\mu}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{\mu}_4(t, \xi_4) \right|
\lesssim N^{-1+4 \Delta} C(N_1, N_2, N_3, N_4) \prod_{j=1}^4 \| u_j \|_{X^{1, 4+}}, \tag{3.14}
\]
where \( C(N_1, N_2, N_3, N_4) \) is a decay just as the remark above, and it allows us to sum over all dyadic shells. The analysis which follows will not rely on the complex conjugate structure in \( \Lambda_4(M_4; u) \). Thus, by symmetry, we may assume that \( N_2 \geq N_3 \geq N_4 \).

Case 1: \( N \gg N_2 \). According to the definition of \( m(\xi) \), the multiplier
\[
|\xi_2,3|^{-(d-\gamma)}m_1(m_1 - m_2m_3m_4)
\]
is identically 0, the bound (3.13) holds trivially.

Case 2: \( N_2 \gg N \gg N_3 \geq N_4 \). Since \( \sum_{j=1}^4 \xi_j = 0 \), we have \( N_1 \approx N_2 \). We aim for (3.14) with a decay factor
\[
C(N_1, N_2, N_3, N_4) = N_2^{0-}.
\]

By the mean value theorem, we have the following pointwise bound
\[
|m_1(m_1 - m_2m_3m_4)| = |m_1(m_2,3,4 - m_2m_3m_4)| \lesssim m_1|\nabla m(\xi) \cdot (\xi_3 + \xi_4)| \text{ where } |\xi| \sim |\xi_2|
\lesssim m_1m_2 \frac{N_3}{N_2}.
\]

Hence by H"older’s inequality and Hardy-Littlewood-Sobolev’s inequality and Proposition 2.2, we obtain
\[
\text{LHS of (3.14)} \lesssim N_1^2 m_1m_2 \frac{N_3}{N_2} \int_T^{T + \delta} \int_{\Gamma_4} |\xi_2,3|^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{\mu}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{\mu}_4(t, \xi_4) \bigg| 
\lesssim N_1^2 m_1m_2 \frac{N_3}{N_2} \| u_1 \|_{L^\infty_t L^\infty_x} || u_2 \|_{L^\infty_t L^\infty_x} || u_3 \|_{L^\infty_t L^\infty_x} \| u_4 \|_{L^\infty_t L^\infty_x}
\lesssim N_1^2 m_1m_2 \frac{N_3}{N_2} N_4^{\gamma - 2} \prod_{j=1}^4 \| u_j \|_{X^{0, \frac{4+1}{4}}}.
\]

It suffices to show that
\[
N_1^2 m_1m_2 \frac{N_3}{N_2} N_4^{\gamma - 2} \lesssim N^{-1+4 \Delta} m_1m_2N_2(N_3)(N_4).
\]
We reduce to show that
\[ N^{1 - N^{0+}_2} \lesssim N_2 \langle N_3 \rangle N_3^{-1} \langle N_4 \rangle N_4^{2 - \gamma}. \]
This is true since
\[ N_2 \gtrsim N^{1 - N^{0+}_2}; \quad \langle N_3 \rangle N_3^{-1} \gtrsim 1; \quad \langle N_4 \rangle N_4^{2 - \gamma} \gtrsim 1. \]

**Case 3:** \( N_2 \geq N_3 \gtrsim N \). In this case, we use the trivial pointwise bound
\[ \left| m_1(m_1 - m_2 m_3 m_4) \right| \lesssim m_1^2. \]
The frequency interactions fall into two subcategories, depending on which frequency is comparable to \( N_2 \).

**Case 3a:** \( N_1 \approx N_2 \geq N_3 \gtrsim N \). In this case, we prove the decay factor
\[ C(N_1, N_2, N_3, N_4) = N_3^{0-}. \]
in (3.14). This allows us to directly sum in \( N_3 \) and \( N_4 \), and sum in \( N_1 \) and \( N_2 \) after applying Cauchy-Schwarz to those factors.

By Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality and Proposition 2.2, we obtain
\[
\text{LHS of (3.14)} \lesssim N_1^2 m_1^2 \int_{T^+} \int_{L_1^4} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4)|
\lesssim N_1^2 m_1^2 \| u_1 \|_{L_t^3 L_x^\infty} \| u_2 \|_{L_t^3 L_x^\infty} \| u_3 \|_{L_t^3 L_x^\infty} \| u_4 \|_{L_t^3 L_x^\infty}
\lesssim N_1^2 m_1^2 N_4^{\gamma-2} \prod_{j=1}^4 \| u_j \|_{X^{0, \frac{1}{2}+}}.
\]
It suffices to show that
\[ N_1^2 m_1^2 N_4^{\gamma-2} \lesssim N^{-1+} N_3^{0-} m_1 N_1 m_2 N_2 m_3 N_3 \langle N_4 \rangle. \]
We reduce to show that
\[ N^{1 - N^{0+}_3} \lesssim m_3 N_3 m_4 \langle N_4 \rangle N_4^{2 - \gamma}. \]
This is true since for \( s \geq \gamma - 2 \), we have
\[ m_3 N_3 m_4 \langle N_4 \rangle N_4^{2 - \gamma} \gtrsim m_3 N_3 m_4 \langle N_4 \rangle^{3 - \gamma}
\gtrsim m_3 N_3 \gtrsim N^{1 - N^{0+}_3}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-decreasing if \( s + p \geq 1 \). While for \( \frac{3}{2} - 1 < s < \gamma - 2 \), we have
\[ m_3 N_3 m_4 \langle N_4 \rangle N_4^{2 - \gamma} \gtrsim m_3 N_3 m_3 N_3^{3 - \gamma}
\gtrsim N^{4 - \gamma - N^{0+}_3} \gtrsim N^{1 - N^{0+}_3}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-increasing if \( s + p < 1 \).
Case 3b: $N_2 \approx N_3 \gtrsim N, N_2 \gtrsim N_1$. In this case, we prove the decay factor

$$C(N_1, N_2, N_3, N_4) = N_2^{0-}.$$ 

in (3.14). This will allow us to directly sum in all the $N_i$.

By Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality and Proposition 2.2 once again, we obtain

$$\text{LHS of (3.14)} \lesssim N_1^2 m_1^2 \int_T^{T+\delta} \int_{\Gamma_4} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4)|$$

$$\lesssim N_1^2 m_1^2 \|u_1\|_{L_t^4 L_x^{\frac{\delta}{d-4}}} \|u_2\|_{L_t^6 L_x^{\frac{\delta}{d-2}}} \|u_3\|_{L_t^6 L_x^{\frac{\delta}{d}}} \|u_4\|_{L_t^6 L_x^{\frac{\delta}{d-4}}}$$

$$\lesssim N_1^2 m_1^2 N_4^{\gamma-2} \prod_{j=1}^4 \|u_j\|_{X^{1, \frac{1}{2}+}}.$$

It suffices to show that

$$N_1^2 m_1^2 N_4^{\gamma-2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_3 N_3 \langle N_4 \rangle.$$ 

Note that

$$N_1^2 m_1^2 \lesssim m_1 N_1 m_2 N_2.$$ 

We reduce to show that

$$N_1^{1+} N_2^{0+} \lesssim m_3 N_3 \ m_4 \langle N_4 \rangle N_4^{2-\gamma}.$$ 

This is true since for $s \geq \gamma - 2$, we have

$$m_3 N_3 \ m_4 \langle N_4 \rangle N_4^{2-\gamma} \geq m_3 N_3 \ m_4 \langle N_4 \rangle^{3-\gamma} \geq m_3 N_3 \approx m_2 N_2 \geq N_1^{1-} N_2^{0+},$$

where we used the fact that $m(\xi) \langle \xi \rangle^p$ is monotone non-decreasing if $s + p \geq 1$. While for \(\frac{\gamma}{2} - 1 < s < \gamma - 2\), we have

$$m_3 N_3 \ m_4 \langle N_4 \rangle N_4^{2-\gamma} \geq m_3 N_3 \ m_3 N_3^{3-\gamma} \approx m_2 N_2^{4-\gamma} \geq N_1^{4-\gamma} N_3^{0+} \geq N_1^{1-} N_3^{0+},$$

where we used the fact that $m(\xi) \langle \xi \rangle^p$ is monotone non-increasing if $s + p < 1$. This completes the proof.

In order to make use of quadrilinear estimate (Proposition 3.4) to obtain sextilinear estimate, we first give a lemma

Lemma 3.1. Assume $u, \delta$ are as in Proposition 3.2 and $P_{N_{1,2,3}}$ the Littlewood-Paley projection onto the $N_{1,2,3}$ frequency shell. Then

$$\|P_{N_{1,2,3}}(I(u\nabla)^{-(d-\gamma)} |u|^2)\|_{L_t^4 L_x^{\frac{\delta}{d-4}}} \lesssim N_{1,2,3} \|u\|_{X^{1, \frac{1}{2}+}}^3.$$
\textbf{Proof:} We write $u = u_L + u_H$ where
\[ \text{supp } \tilde{u}_l(t, \xi) \subseteq \{ |\xi| < 2 \}, \]
\[ \text{supp } \tilde{u}_H(t, \xi) \subseteq \{ |\xi| > 1 \}. \]
Hence,
\[ \| P_{N,1,2,3} \left( I \left( u |\nabla|^{-d-\gamma} |u|^2 \right) \right) \|_{L^2_x L^\frac{6d}{d+4}} \]
\[ \lesssim \left\| P_{N,1,2,3} \left( I \left( u_L |\nabla|^{-d-\gamma} |u_L|^2 \right) \right) \right\|_{L^2_x L^\frac{6d}{d+4}} + \left\| P_{N,1,2,3} \left( I \left( u_H |\nabla|^{-d-\gamma} |u_H|^2 \right) \right) \right\|_{L^2_x L^\frac{6d}{d+4}}. \]

Consider the first term. By Hölder’s inequality, Hardy-Littlewood-Sobolev’s inequality and Proposition 2.2 we have
\[ \left\| P_{N,1,2,3} \left( I \left( u_L |\nabla|^{-d-\gamma} |u_L|^2 \right) \right) \right\|_{L^2_x L^\frac{6d}{d+4}} \lesssim \left\| u_L |\nabla|^{-d-\gamma} |u_L|^2 \right\|_{L^2_x L^\frac{6d}{d+4}} \lesssim \left\| u_L \right\|_{L^3_x L^{\frac{9d}{6d+4}}}^3 = \left\| I u_L \right\|_{L^3_x L^{\frac{18d}{9d+4}}}^3 \lesssim \left\| I u_L \right\|_{X^{1, \frac{3}{2}}} \lesssim N_{1,2,3} \left\| I u \right\|_{X^{1, \frac{3}{2}}} \]
since
\[ N_{1,2,3} \geq 1, \quad \text{and} \quad 0 \leq d \times \left( \frac{1}{2} - \frac{9d - 6\gamma - 4}{18d} \right) - \frac{2}{9} = \frac{\gamma}{3} \leq 1. \]

We estimate the second term. By Sobolev’s inequality and using the Leibniz rule for the operator $|\nabla|^{2-\frac{\gamma}{2}} I$ and Proposition 2.2 we have
\[ \left\| \frac{1}{N_{1,2,3}} P_{N,1,2,3} \left( I \left( u_H |\nabla|^{-d-\gamma} |u_H|^2 \right) \right) \right\|_{L^2_x L^{\frac{6d}{d+4}}} \]
\[ \lesssim \left\| |\nabla|^{-d} P_{N,1,2,3} \left( I \left( u_H |\nabla|^{-d-\gamma} |u_H|^2 \right) \right) \right\|_{L^2_x L^{\frac{6d}{d+4}}} \]
\[ \lesssim \left\| |\nabla|^{2-\frac{\gamma}{2}} I u_H \right\|_{L^3_x L^{\frac{18d}{9d+14}}} \left\| u_H \right\|_{L^3_x L^{\frac{18d}{9d+14}}} \lesssim \left\| |\nabla|^{2-\frac{\gamma}{2}} I u_H \right\|_{X^{1, \frac{3}{2}}} \lesssim \left\| |\nabla| I u_H \right\|_{X^{1, \frac{3}{2}}}. \]

As for the third term. By Sobolev’s inequality and using the Leibniz rule for the operator
\[|\nabla|^2 \frac{2}{\gamma} I \text{ and Proposition 2.2 again, we have}
\]

\[
\| \frac{1}{N_{1,2,3}} P_{N_{1,2,3}} \left( I(u_L |\nabla|^{-(d-\gamma)} |u_H|^2) \right) \|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| |\nabla|-1 P_{N_{1,2,3}} \left( I(u_H |\nabla|^{-(d-\gamma)} |u_H|^2) \right) \|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| |\nabla|^2 \frac{2}{\gamma} I(u_H |\nabla|^{-(d-\gamma)} |u_H|^2) \|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| |\nabla|^2 \frac{2}{\gamma} I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-9\gamma+14}}} \| u_H \|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \| u_H \|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \\
\lesssim \| |\nabla| I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \| |\nabla| I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \| |\nabla|^{-2} I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \\
\lesssim \| |\nabla| I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \| |\nabla| I u_H \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \| |\nabla|^{-2} I u_L \|_{L_t^2 L_x^{\frac{18d}{9d-4}}} \\
\lesssim \| I u \|_{X^{1,\frac{1}{2}+}}^3.
\]

Now we estimate the fourth term. By Sobolev’s inequality and Hölder’s inequality, we obtain

\[
\| \frac{1}{N_{1,2,3}} P_{N_{1,2,3}} \left( I(u_H |\nabla|^{-(d-\gamma)} |u_L|^2) \right) \|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| |\nabla|-1 P_{N_{1,2,3}} \left( I(u_H |\nabla|^{-(d-\gamma)} |u_L|^2) \right) \|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| u_H |\nabla|^{-(d-\gamma)} |u_L|^2 \|_{L_t^2 L_x^{\frac{6d}{3d-4}}} \\
\lesssim \| u_H \|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+5}}} \| u_L \|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+5}}} \lesssim \| I u \|_{X^{1,\frac{1}{2}+}}^3.
\]

The remainder terms are similar to the third and fourth terms because we can ignore the complex conjugates. This completes the proof.

Now we proceed to prove the sextilinear estimate. Note that in the treatment of the quadrilinear form as in Proposition 3.4, we always took the \( \Delta u_1 \) factor in \( L_{s,t}^{\frac{6d}{3d-4}}(\mathbb{R}^d \times \mathbb{R}^d) \), estimating this by \( N_1 \| I u_1 \|_{X^{1,\frac{1}{2}}} \). Together with Proposition 3.4 and Lemma 3.1, we can obtain the following estimate.

**Proposition 3.5.** For any Schwartz function \( u \), and any \( \delta \approx 1 \) as in Proposition 3.2, we have that

\[
\left| \int_T^{T+\delta} \Lambda_0(M_0; u) \, dt \right| \lesssim N^{-1+} \| I u \|_{X^{1,\frac{1}{2}+}}^6
\]

for \( s > \frac{\gamma}{2} - 1 \).

### 4 Almost Interaction Morawetz estimate

In this section, we aim to prove the interaction Morawetz estimate for the smoothed out solution \( I u \), that is, “almost Morawetz estimate”. For this, we consider \( a(x_1, x_2) = |x_1 - x_2| : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \), a convex and locally integrable function of polynomial growth. In all of our
arguments, we will work with the Schwarz solutions. This will simplify the calculations and will enable us to justify the steps in the subsequent proofs. Then we approximate the $H^s$ solutions by the Schwarz solutions.

Theorem 4.1. Let $u$ be a Schwarz solution to

$$iu_t + \Delta u = \tilde{N}(u), \ (x, t) \in \mathbb{R}^d \times [0, T],$$

where $\tilde{N}(u) = (|x|^{-\gamma} * |u|^2)u$. Let $Iu$ be a solution to

$$iIu_t + \Delta Ig = I(\tilde{N}(u)), \ (x, t) \in \mathbb{R}^d \times [0, T].$$

Then

$$\|\nabla |\cdot|^{d-3/2} Iu\|_{L^4_t L^4_x} \lesssim \|Iu\|_{L^4_t H^1_x}\|Iu\|_{L^2_t L^2_x}^3 + \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{\tilde{N}_{bad}, Iu(t, x_1)Iu(t, x_2)\}_p \, dx_1 dx_2 dt. \quad (4.2)$$

with $\{\cdot, \cdot\}_p$ is the momentum bracket defined by

$$\{f, g\}_p = Re(f\nabla \bar{g} - g\nabla \bar{f}),$$

and

$$\tilde{N}_{bad} = \sum_{i=1}^2 (I\tilde{N}_i(u_i) - \tilde{N}_i(Iu_i)) \prod_{j=1, j \neq i}^2 Iu_j,$$

where $u_i$ is a solution to

$$iu_t + \Delta u = \tilde{N}(u), \ (x_i, t) \in \mathbb{R}^d \times \mathbb{R}, \ d \geq 3, \quad (4.3)$$

here $x_i \in \mathbb{R}^d$, not a coordinate. In particular, on a time interval $J_k$ where the local well-posedness Proposition 3.2 holds, we have that

$$\int_{J_k} \int_{\mathbb{R}^d} \nabla a \cdot \{\tilde{N}_{bad}, Iu(t, x_1)Iu(t, x_2)\}_p \, dx_1 dx_2 dt \lesssim \frac{1}{N^4} \|u\|_{Z_1(J_k)}^6.$$
**Proof of Theorem 4.1.** Now we rewrite the equation \( 4.1 \) as
\[
iIu_t + \Delta Iu = \tilde{N}(Iu) + (I(\tilde{N}(u)) - \tilde{N}(Iu))
\]
then by symmetry, the term \( \tilde{N}(Iu) \) will create a positive term that we can ignore, which is the same to the case in [20]. While the commutator \( I(\tilde{N}(u)) - \tilde{N}(Iu) \) will introduce an error term. Thus by Proposition 4.1, we have
\[
\int_0^T \int (- \Delta a) |Iu(t, x)|^2 dxdt \leq \sup_{t \in [0, T]} \left| \int \nabla a(x) \cdot \text{Im}(\overline{U}(x) \nabla Iu(x)) \, dx \right| + \left| \int_0^T \int \nabla a \cdot \{ I\tilde{N}(u) - \tilde{N}(Iu), Iu \}_p \, dxdt \right|.
\]
The second term on the right hand side of this inequality is what we call an error. We now turn to the details. The conjugation will play no crucial role in the forthcoming argument.

Now define the tensor product \( u := (u_1 \otimes u_2)(t, x) \) for \( x \) in
\[
\mathbb{R}^{d+d} = \{ x = (x_1, x_2) : x_1 \in \mathbb{R}^d, x_2 \in \mathbb{R}^d \}
\]
by the formula
\[
(u_1 \otimes u_2)(t, x) = u_1(t, x_1)u_2(t, x_2).
\]
Let us set
\[
IU(t, x) = \prod_{j=1}^2 Iu(t, x_j).
\]

If \( u \) solves \( 4.4 \) for \( d \) dimensions, then \( IU \) solves \( 4.4 \) for \( 2d \) dimensions, with right hand side \( \tilde{N}_I \) given by
\[
\tilde{N}_I = \sum_{i=1}^2 \left( I(\tilde{N}_i(u_i)) \prod_{j=1,j\neq i}^2 Iu_j \right).
\]

Now let us decompose
\[
\tilde{N}_I = \tilde{N}_{\text{good}} + \tilde{N}_{\text{bad}}
\]
\[
\triangleq \sum_{i=1}^2 \left( \tilde{N}_i(Iu_i) \prod_{j=1,j\neq i}^2 Iu_j \right) + \sum_{i=1}^2 \left( (I(\tilde{N}_i(u_i)) - \tilde{N}_i(Iu_i)) \prod_{j=1,j\neq i}^2 Iu_j \right).
\]
The first term summand creates a positive term that we can ignore again. The term we call \( \tilde{N}_{\text{bad}} \) produces the error term. Now we pick \( a(x) = a(x_1, x_2) = |x_1 - x_2| \) where \( (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d \). Hence we have
\[
\|\nabla x \frac{d}{dx} IU\|_{L^4_t L^2_x}^4 \lesssim \|IU\|_{L^\infty_t H^1_x} \|IU\|_{L^2_t L^6_x}^3 \nu \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{\text{bad}}, Iu(t, x_1)Iu(t, x_2) \}_p \, dx_1dx_2dt.
\]
Note that the second term of the right hand side comes from the momentum bracket term in the proof of Proposition 4.1. Following with the same calculations in [2], we deduce that
\[
\mathcal{E} := \left| \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{\text{bad}}, Iu(t, x_1)Iu(t, x_2) \}_p \, dx_1dx_2dt \right| \lesssim \left( \|I(\tilde{N}(u)) - \tilde{N}(Iu)\|_{L^1_t L^2_x} + \|\nabla x (I(\tilde{N}(u)) - \tilde{N}(Iu))\|_{L^1_t L^2_x}\right) \|u\|_{Z_t^1(J)}^3.
\]
Now we proceed to estimate \( \| \nabla_x (I(\mathcal{N}(u)) - \mathcal{N}(Iu)) \|_{L^1_t L^2_x} \), which is the harder term. The term \( \| I(\mathcal{N}(u)) - \mathcal{N}(Iu) \|_{L^1_t L^2_x} \) can be estimated in the same way. Note that

\[
\mathcal{N}(u) = (|x|^{-\gamma} * |u|^2) u,
\]

we have

\[
\mathcal{F}_x \left( \nabla_x (I(\mathcal{N}(u)) - \mathcal{N}(Iu)) \right)(\xi) = \int \xi |\xi_{2,3}|^{-(d-\gamma)} (m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3)) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3.
\]

We decompose \( u \) into a sum of dyadic pieces \( u_j \) localized around \( N_j \), then

\[
\| \nabla_x (I(\mathcal{N}(u)) - \mathcal{N}(Iu)) \|_{L^1_t L^2_x} \leq \| \mathcal{F}_x \left( \nabla_x (I(\mathcal{N}(u)) - \mathcal{N}(Iu)) \right)(\xi) \|_{L^1_t L^2_x} \]

\[
\leq \sum_{N_1, N_2, N_3} \left\| \int_{|\xi_j| = N_j} |\xi| |\xi_{2,3}|^{-(d-\gamma)} |m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3)| \right. 
\times \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x}.
\]

Since the conjugation plays no crucial role here, without loss of generality, we assume that

\[ N_1 \geq N_2 \geq N_3. \]

Set

\[ \sigma(\xi_1, \xi_2, \xi_3) = |\xi_1 + \xi_2 + \xi_3| |m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)|, \]

then

\[ \sigma(\xi_1, \xi_2, \xi_3) = \sum_{j=1}^{4} \chi_j(\xi_1, \xi_2, \xi_3) \sigma(\xi_1, \xi_2, \xi_3) \]

\[ = \sum_{j=1}^{4} \sigma_j(\xi_1, \xi_2, \xi_3), \]

where \( \chi_j(\xi_1, \xi_2, \xi_3) \) is a smooth characteristic function of the set \( \Omega_j \) defined as follows:

- \( \Omega_1 = \{ |\xi_i| \approx N_i, i = 1, 2, 3; N \gg N_1 \}; \)
- \( \Omega_2 = \{ |\xi_i| \approx N_i, i = 1, 2, 3; N_1 \gg N \gg N_2 \}; \)
- \( \Omega_3 = \{ |\xi_i| \approx N_i, i = 1, 2, 3; N_1 \gg N_2 \gg N \gg N_3 \}; \)
- \( \Omega_4 = \{ |\xi_i| \approx N_i, i = 1, 2, 3; N_1 \gg N_2 \gg N_3 \gg N \}. \)
Hence, we have

\[ \| \nabla_x (I(\tilde{N}(u)) - \tilde{N}(Iu)) \|_{L^1 L^2} \]

\[ \lesssim \sum_{N_1, N_2, N_3} \sum_{j=1}^4 \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_j(\xi_1, \xi_2, \xi_3) \tilde{u}(\xi_1) \tilde{u}(\xi_2) \tilde{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1 L^2} \]

\[ := \sum_{N_1, N_2, N_3} \sum_{j=1}^4 L_j. \]

**Contribution of** \( L_1 \). Since \( \sigma_1 \) is identically zero when \( N \geq 4N_1 \), \( L_1 \) gives no contribution to the sum above.

**Contribution of** \( L_2 \). By the mean value theorem, we have the pointwise bound

\[ \sigma_2(\xi_1, \xi_2, \xi_3) \lesssim N_1 \cdot m_1 \frac{N_2}{N_1} = m_1 N_2. \]

Hence, by Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality, we obtain

\[ L_2 = \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_2(\xi_1, \xi_2, \xi_3) \tilde{u}(\xi_1) \tilde{u}(\xi_2) \tilde{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1 L^2} \]

\[ \lesssim m_1 N_2 \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \tilde{u}(\xi_1) \tilde{u}(\xi_2) \tilde{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1 L^2} \]

\[ \lesssim m_1 N_2 \left\| u_1 \right\|_{L^d L^{\frac{d}{d-4}}} \left\| u_2 \right\|_{L^d L^{\frac{d}{d-4}}} \left\| u_3 \right\|_{L^d L^{\frac{d}{d-6+\gamma}}} \]

\[ \lesssim m_1 N_2 N_3^{\gamma-2} \prod_{j=1}^3 \left\| u_j \right\|_{X^{0, \frac{d}{2}}} \]

It suffices to show that

\[ m_1 N_2 N_3^{\gamma-2} \lesssim N^{-1+ \gamma} N_1^{-1} m_1 N_1 \langle N_2 \rangle \langle N_3 \rangle. \]

We reduce to show that

\[ N_1^{-1} N_1^{0+} \lesssim N_1 \langle N_2 \rangle N_2^{-1} \langle N_3 \rangle N_3^{2-\gamma}. \]

This is true since

\[ N_1 \gtrsim N_1^{-1} N_1^{0+}; \]

\[ \langle N_2 \rangle N_2^{-1} \gtrsim 1; \]

\[ \langle N_3 \rangle N_3^{2-\gamma} \gtrsim 1. \]

**Contribution of** \( L_3 \). Note that

\[ \sigma_3(\xi_1, \xi_2, \xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1. \]
Hence, by Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality, we have

\[
L_3 = \left\| \int_{\|\xi\| = N_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_3(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_1^1 L_2^2} \\
\leq m_1 N_1 \left\| \int_{\|\xi\| = N_j} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_1^1 L_2^2} \\
\leq m_1 N_1 \left\| u_1 \right\|_{L_1^3 L_2^{6d-4}} \left\| u_2 \right\|_{L_1^3 L_2^{6d-4}} \left\| u_3 \right\|_{L_1^3 L_2^{6d-6+\gamma}} \\
\lesssim m_1 N_1 N_3^{\gamma-2} \prod_{j=1}^3 \left\| u_j \right\|_{X^{0, \frac{1}{4}+}}.
\]

It suffices to show that

\[m_1 N_1 N_3^{\gamma-2} \lesssim N^{-1+ N_2^0} - m_1 N_1 m_2 N_2 \langle N_3 \rangle.\]

We reduce to show that

\[N^{-1} - N_2^{0+} \lesssim m_2 N_2 \langle N_3 \rangle N_3^{2-\gamma}.\]

This is true since

\[m_2 N_2 \gtrsim N^{-1} - N_2^{0+}; \quad \langle N_3 \rangle N_3^{2-\gamma} \gtrsim 1.\]

**Contribution of** \(L_4\). **Note that**

\[\sigma_4(\xi_1, \xi_2, \xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1.\]

Hence, by Hölder’s inequality and Hardy-Littlewood-Sobolev’s inequality, we obtain

\[
L_4 = \left\| \int_{\|\xi\| = N_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_4(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_1^1 L_2^2} \\
\lesssim m_1 N_1 \left\| \int_{\|\xi\| = N_j} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_1^1 L_2^2} \\
\lesssim m_1 N_1 \left\| u_1 \right\|_{L_1^3 L_2^{6d-4}} \left\| u_2 \right\|_{L_1^3 L_2^{6d-4}} \left\| u_3 \right\|_{L_1^3 L_2^{6d-6+\gamma}} \\
\lesssim m_1 N_1 N_3^{\gamma-2} \prod_{j=1}^3 \left\| u_j \right\|_{X^{0, \frac{1}{4}+}}.
\]

It suffices to show that

\[m_1 N_1 N_3^{\gamma-2} \lesssim N^{-1+ N_2^0} - m_1 N_1 m_2 N_2 m_3 N_3.\]

We reduce to show that

\[N^{-1} - N_2^{0+} \lesssim m_2 N_2 m_3 N_3 N_3^{2-\gamma}.\]

This is true since for \(s \geq \gamma - 2\), we have

\[m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim m_2 N_2 \gtrsim N^{-1} - N_2^{0+}\]
where we used the fact that $m(\xi)\langle \xi \rangle^p$ is monotone non-decreasing if $s + p \geq 1$. While for $\frac{d}{2} - 1 < s < \gamma - 2$, we have

$$m_2 N_2 m_3 N_3^{-\gamma} \gtrsim m_2 N_2 N^{-\gamma} \gtrsim N^{4-\gamma} N_2^{0+} \gtrsim N^{1-N_2^{0+}}$$

where we used the fact that $m(\xi)\langle \xi \rangle^p$ is monotone non-increasing if $s + p < 1$.

5 Proof of Theorem 1.1

We first scale the solution. Suppose that $u(t, x)$ is a global in time solution to (1.1) with initial data $u_0 \in C_0^\infty(\mathbb{R}^d)$. Setting

$$u^\lambda(t, x) = \lambda^{\frac{1-s}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}),$$

we choose a parameter $\lambda$ so that

$$\|Iu^\lambda_0\|_{H^1} = O(1),$$

that is

$$\lambda \approx N^{\frac{1-s}{2}}.$$ (5.1)

Next, let us define

$$S := \{ 0 \leq t < \infty : \|Iu^\lambda\|_{L^4_t H^{-\frac{d-3}{4}}(0, t) \times \mathbb{R}^d)} \leq K\lambda^{\frac{3}{2}(\frac{d}{2}-1)} \},$$

with $K$ a constant to be chosen later. We claim that $S$ is the whole interval $[0, \infty)$. Indeed, assume by contradiction that it is not so, then since

$$\|Iu^\lambda\|_{L^4_t H^{-\frac{d-3}{4}}(0, t) \times \mathbb{R}^d)}$$

is a continuous function of time, there exists a time $T \in [0, \infty)$ such that

$$\|Iu^\lambda\|_{L^4_t H^{-\frac{d-3}{4}}(0, T) \times \mathbb{R}^d)} > K\lambda^{\frac{3}{2}(\frac{d}{2}-1)},$$ (5.2)

$$\|Iu^\lambda\|_{L^4_t H^{-\frac{d-3}{4}}(0, T) \times \mathbb{R}^d)} \leq 2K\lambda^{\frac{3}{2}(\frac{d}{2}-1)}.$$ (5.3)

We now split the interval $[0, T]$ into subintervals $J_k, k = 1, \cdots, L$ in such a way that

$$\|Iu^\lambda\|_{L^4_t H^{-\frac{d-3}{4}}(J_k \times \mathbb{R}^d)} \leq \mu,$$

with $\mu$ as in Proposition 3.3. This is possible because of (5.3). Then, the number $L$ of possible subintervals must satisfy

$$L \approx \frac{(2K\lambda^{\frac{3}{2}(\frac{d}{2}-1)})^4}{\mu} \approx \frac{(2K)^4\lambda^{3(\frac{d}{2}-1)}}{\mu}.$$ (5.4)

From Proposition 3.2 and Proposition 3.3 we know that

$$\sup_{t \in [0, T]} E(Iu^\lambda(t)) \lesssim E(Iu^\lambda_0) + \frac{L}{N^{1-}}$$
and by our choice \((5.1)\) of \(\lambda\), \(E(Iu^\lambda_0) \lesssim 1\). Hence, in order to guarantee that
\[
E(Iu^\lambda(t)) \lesssim 1
\]
holds for all \(t \in [0, T]\), we need to require that
\[
L \lesssim N^{1-}.
\]
According to \((5.1)\), this is fulfilled as long as
\[
\frac{(2K)^4 \lambda^{3(\frac{3}{2}-1)}}{\mu} \lesssim N^{1-}.
\] \((5.5)\)
From our choice of \(\lambda\), the expression \((5.5)\) implies that
\[
\frac{(2K)^4}{\mu} \lesssim N^{1-} \frac{1}{\lambda^{\frac{1}{2}+1} \lambda^{3(\frac{3}{2}-1)}}.
\]
Thus this is possible for \(s > \frac{4(\gamma-2)}{3\gamma-4}\) and a large number \(N\).

Now recall the a priori estimate \((4.2)\)
\[
\left\| \nabla \right\|^{-\frac{d-3}{4}} \left\| Iu^\lambda \right\|_{L^4_t L^4_x} \lesssim \left\| Iu^\lambda \right\|_{L^\infty_t H^1_x} \left\| Iu^\lambda \right\|_{L^3_t L^2_x}^3 + \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \left\{ \mathcal{N}_{bad}, Iu^\lambda(t, x) Iu^\lambda(t, x_2) \right\} dx_1 dx_2 dt.
\]
Set
\[
\text{Error}(t) := \int_{\mathbb{R}^d} \nabla a \cdot \left\{ \mathcal{N}_{bad}, Iu^\lambda(t, x_1) Iu^\lambda(t, x_2) \right\} dx_1 dx_2.
\]
By Theorem \((4.1)\) and Proposition \((3.3)\) on each interval \(J_k\), we have that
\[
\left| \int_{J_k} \text{Error}(t) dt \right| \lesssim \frac{1}{N^{1-}} \left\| u^\lambda \right\|_{L^3_t(J_k)}^6 \lesssim \frac{1}{N^{1-}}.
\]
Summing all the \(J_k\)’s, we have that
\[
\left| \int_0^T \text{Error}(t) dt \right| \lesssim \frac{L}{N^{1-}} \lesssim N^{0+}.
\]
Therefore, by our choice \((5.1)\) of \(\lambda\), we obtain
\[
\left\| \nabla \right\|^{-\frac{d-3}{4}} \left\| Iu^\lambda \right\|_{L^4_t L^4_x} \lesssim \left\| Iu^\lambda \right\|_{L^\infty_t H^1_x} \left\| Iu^\lambda \right\|_{L^3_t L^2_x}^3 + N^{0+} \lesssim C \lambda^{3(\frac{3}{2}-1)}.
\]
This estimate contradicts \((5.2)\) for an appropriate choice of \(K\). Hence \(S = [0, \infty)\). In addition, let \(T_0\) be chosen arbitrarily, we have also proved that for \(s > \frac{4(\gamma-2)}{3\gamma-4}\),
\[
\left\| Iu^\lambda(\lambda^2 T_0) \right\|_{H^1_x} = O(1).
\]
Then
\[
\left\| u(T_0) \right\|_{H^s} = \left\| u(T_0) \right\|_{L^2} + \left\| u(T_0) \right\|_{H^s} = \left\| u_0 \right\|_{L^2} + \lambda^{s-\frac{3}{2}+1} \left\| u^\lambda(\lambda^2 T_0) \right\|_{H^s} \lesssim \lambda^{s-\frac{3}{2}+1} \left\| Iu^\lambda(\lambda^2 T_0) \right\|_{H^1_x} \lesssim \lambda^{s-\frac{3}{2}+1} \approx N^{1-s}.
\]
Since $T_0$ is arbitrarily large, the a priori bound on the $H^s$ norm concludes the global well-posedness of the Cauchy problem (1.1).

Note that we have obtained that
$$\|Iu\|_{L^4 H^{-\frac{d-3}{2}}([0,+,\infty)\times\mathbb{R}^d)} \leq C(\|u_0\|_{H^s}),$$
this together with Proposition 2.2, Proposition 3.3 and the property of the operator $I$ implies that
$$\sup_{(q,r) \text{ admissible}} \|\langle \nabla \rangle^s u\|_{L^q L^r([0,+,\infty)\times\mathbb{R}^d)} \leq C(\|u_0\|_{Z_1([0,+,\infty))}) \lesssim C(\|Iu_0\|_{H^1}) \lesssim C(\|u_0\|_{H^s}),$$
then we can prove scattering by using the well-known standard argument [1], [3] etc.. This completes the proof.

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