Constructing the Leptonic Unitarity Triangle

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Abstract

Following analogy of the ‘$db$’ triangle in the quark mixing case, we have explored the construction of the ‘$\nu_2, \nu_3$’ leptonic unitarity triangle using the MNS matrix obtained by Bjorken et al. through generalization of the tribimaximal scenario. In particular, for the $U_{e3}$ range $0.05 - 0.15$, the existence of leptonic unitarity triangle indicates a fairly good possibility of having non zero CP violation.

In the last few years, apart from establishing the hypothesis of neutrino oscillations, impressive advances have been made in understanding the phenomenology of neutrino oscillations through solar neutrino experiments [1], atmospheric neutrino experiments [2], reactor based experiments [3] and accelerator based experiments [4] enabling the determination of the basic form of the Maki-Nakagawa-Sakata (MNS) leptonic mixing matrix [5]. At present, one of the key issues in the context of neutrino oscillation phenomenology is the existence of CP violation in the leptonic sector.

Taking clues from the existence of the unitarity triangle and consequently CP violation in the quark sector [6], several attempts [7]-[9] have been made to explore such a possibility in the leptonic sector. In particular, Farzan and Smirnov [8] have discussed the desirability of exploring the construction of leptonic unitarity triangle for finding possible clues to the existence of CP violation in the leptonic sector. Considering the ‘$e$-$\mu$’ triangle, for $U_{e3}$ values in the range $0.09 - 0.22$, they have examined the detailed implications of different values of CP violating phase $\delta$ on the possible accuracy required in the measurement of various oscillation probabilities. Very recently, Bjorken et al. [9], have constructed a generalization of the tribimaximal scenario and have not only presented a very useful form of the MNS matrix but have also proposed a unitarity triangle, referred to as ‘$\nu_2, \nu_3$’ which could be leptonic analogue of the much talked about ‘$db$’ triangle in the quark sector. This immediately suggests a need for deeper study of the ‘$\nu_2, \nu_3$’ triangle using clues from the ‘$db$’ triangle in the quark sector. In particular, it would be very much desirable, as a complimentary approach to the scenario investigated by Farzan and Smirnov [8], to
find the probable values of CP violating phase $\delta$ suggested by generalized tribimaximal scenario of Bjorken et al. [9].

To this end, taking clues from ‘$db$’ unitarity triangle in the quark sector, the purpose of the present paper is to explore the possibility of the construction of the leptonic unitarity triangle as well as the existence of CP violation in the leptonic sector. In particular, in the MNS matrix constructed by Bjorken et al. [9] we have considered different values of $U_{e3}$ suggested by various theoretical models [10]. Further, we have explored in detail the possibility of finding a non zero value of $J_l$, the Jarlskog’s rephasing invariant parameter in the leptonic sector as well as the related Dirac-like CP violating phase $\delta$.

For ready reference as well as to facilitate discussion of results, we begin with the neutrino mixing phenomenon expressed in terms of a $3 \times 3$ neutrino mixing MNS matrix [5] given by

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]

where $\nu_e, \nu_\mu, \nu_\tau$ are the flavor eigenstates and $\nu_1, \nu_2, \nu_3$ are the mass eigenstates. Following PDG representation, involving three angles and the Dirac-like CP violating phase $\delta$ as well as the two Majorana phases $\alpha_1, \alpha_2$, the MNS matrix $U$ can be written as

\[
U =
\begin{pmatrix}
\frac{c_{12}c_{13}}{\sqrt{2}} & \frac{s_{12}c_{13}}{\sqrt{2}} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & -s_{23}c_{13} \\
\frac{s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}}{c_{23}c_{13}} & \frac{-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}}{c_{23}c_{13}} & 0
\end{pmatrix}
\begin{pmatrix}
e^{i\alpha_1/2} & 0 & 0 \\
0 & e^{i\alpha_2/2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The Majorana phases $\alpha_1$ and $\alpha_2$ do not play any role in neutrino oscillations and henceforth would be dropped from the discussion.

Unitarity implies nine relations, three in terms of normalization conditions, the other six can be defined as

\[
\sum_{i=1,2,3} U_{\alpha i}U_{\beta i}^* = \delta_{\alpha \beta} \quad (\alpha \neq \beta),
\]

\[
\sum_{\alpha=e,\mu,\tau} U_{\alpha i}U_{\alpha j}^* = \delta_{ij} \quad (i \neq j),
\]

where Latin indices run over the mass eigenstates (1, 2, 3) and Greek ones run over the flavor eigenstates ($e, \mu, \tau$). These six non-diagonal relations can also be expressed through six independent unitarity triangles in the complex plane.

For getting viable clues to the construction of the leptonic unitarity triangle, we first consider the case of quarks wherein the CKM matrix [11] is fairly well established as well as the CP violating phase $\delta$ has also been measured recently [12]-[15] with a good deal of accuracy. To begin with, we consider the quark mixing matrix given by PDG 2006 [12] and attempt to reconstruct the CP violating phase $\delta$ using the Jarlskog’s rephasing invariant parameter $J_l$, equal to twice the area of any of the unitarity triangle. In this
context, we consider the usual ‘$db$’ triangle, expressed as
\[ V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \tag{5} \]
from which one can obtain a histogram of $J$ by considering Gaussian distribution for the CKM matrix elements as well as imposing the constraints $|a| + |c| > |b|$ and $|b| + |c| > |a|$ for the three sides of the triangle $a$, $b$, $c$. From the histogram of $J$, not shown here, one can find
\[ J = (3.0 \pm 0.4) \times 10^{-5}. \tag{6} \]
Using the relation between the parameter $J$ and phase $\delta$, e.g.,
\[ J = s_{12}s_{23}c_{12}c_{23}^2 \sin \delta, \tag{7} \]
one can obtain the corresponding histogram of $\delta$, shown in figure 1, yielding
\[ \delta = 55.4^\circ \pm 10.0^\circ. \tag{8} \]
For further details we refer the reader to [16]. Interestingly, we find that the above mentioned $J$ value has an excellent overlap with that found by PDG through their recent global analysis [12]. Also, this value of $\delta$ is fully compatible with the experimentally determined $\delta$ given by PDG 2006 as well as found by some of the most recent analyses [12]-[15].

The above discussion immediately provides a clue for exploring the possibility of existence of CP violation in the leptonic sector, even when leptonic mixing matrix is approximately known. In this context, we have considered the MNS matrix obtained by Bjorken et al. [9], e.g.,
\[ U = X Y \tag{9} \]
where
\[
X = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}, \\
Y = \begin{pmatrix}
C & 0 & \sqrt{\frac{3}{2}}U_{e3} \\
0 & 1 & 0 \\
-\sqrt{\frac{3}{2}}U_{e3}^* & 0 & C
\end{pmatrix} \tag{10}
\]
with
\[ C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2}. \tag{11} \]
The matrix $X$ corresponds to the tribimaximal mixing matrix whereas the matrix $Y$ breaks this exact tribimaximal form by small perturbations due to the effects of the element $U_{e3}$. It may be added that the matrix $U$ is unitary by construction.

It needs to be mentioned that the experimental data deviates somewhat from the tribimaximal form, therefore we have modified the matrix $X$ by introducing modest error of 5% to its ‘12’ and ‘23’ elements, governed by solar and atmospheric neutrino data. These elements have been considered independent in the present case, the other elements
along with the respective errors have been obtained by using the constraints of unitarity. For \( U_{e3} \), it is very well recognized that its value would have deep implications for the neutrino oscillation phenomenology \([17]-[24]\). However at present only its upper limit is known, therefore while constructing the matrix \( Y \) we have taken its values to be 0.05, 0.10 and 0.15 and attached 10% errors. These values have been considered primarily following a recent detailed analysis by Albright and Chen \([10]\) wherein they have studied the implications of \( U_{e3} \) values on various leptonic and grand unified models of neutrino masses and mixings. The errors in the elements of the matrices \( X \) and \( Y \) have been introduced following Farzan and Smirnov \([8]\) and keeping in mind the accuracy with which these would be measured by the planned neutrino experiments \([8]\), \([22]\), \([25]-[27]\). The matrices corresponding to \( U_{e3} \) values 0.05 ± 0.005, 0.10 ± 0.01 and 0.15 ± 0.015 are respectively as follows

\[
U = \begin{pmatrix}
0.8150 \pm 0.0204 & 0.5774 \pm 0.0289 & 0.05 \pm 0.005 \\
0.4508 \pm 0.0646 & 0.5774 \pm 0.0144 & 0.6808 \pm 0.0356 \\
0.3642 \pm 0.0646 & 0.5774 \pm 0.0144 & 0.7308 \pm 0.0356 \\
\end{pmatrix}, \quad (12)
\]

\[
U = \begin{pmatrix}
0.8104 \pm 0.0203 & 0.5774 \pm 0.0289 & 0.1 \pm 0.01 \\
0.4918 \pm 0.0648 & 0.5774 \pm 0.0144 & 0.6518 \pm 0.0363 \\
0.3186 \pm 0.0648 & 0.5774 \pm 0.0144 & 0.7518 \pm 0.0363 \\
\end{pmatrix}, \quad (13)
\]

\[
U = \begin{pmatrix}
0.8026 \pm 0.0202 & 0.5774 \pm 0.0289 & 0.15 \pm 0.015 \\
0.5312 \pm 0.0652 & 0.5774 \pm 0.0144 & 0.6201 \pm 0.0376 \\
0.2714 \pm 0.0652 & 0.5774 \pm 0.0144 & 0.7701 \pm 0.0376 \\
\end{pmatrix}, \quad (14)
\]

wherein we have given the magnitude of the elements, as is usual.

Out of the six triangles defined by equations (3) and (4), Bjorken et al. \([9]\) have considered the ‘\( \nu_2, \nu_3 \)’ triangle which is the leptonic analogue of the ‘\( db \)’ triangle of the quark sector and is expressed as

\[
U_{e2}U_{e3}^* + U_{\mu 2}U_{\mu 3}^* + U_{\tau 2}U_{\tau 3}^* = 0.
\]

(15)

Using the matrices constructed above and following the same procedure as in the quark case, for the ‘\( \nu_2, \nu_3 \)’ triangle we obtain the corresponding respective values of \( J_l \) as

\[
J_l = 0.009 \pm 0.003, \quad (16)
\]

\[
J_l = 0.018 \pm 0.006, \quad (17)
\]

\[
J_l = 0.024 \pm 0.009. \quad (18)
\]

Using the relation between \( J_l \) and phase \( \delta \), equation (7), as well as considering the above values of \( J_l \) and mixing angles to have Gaussian distributions, one can find the corresponding histograms of \( \delta \). Using the histograms, shown in figure 2, the \( \delta \) values corresponding
to $U_{e3}$ values $0.05 \pm 0.005$, $0.10 \pm 0.01$ and $0.15 \pm 0.015$ are respectively as follows

\[ \delta \simeq 47^\circ \pm 15^\circ, \] (19)

\[ \delta \simeq 45^\circ \pm 15^\circ, \] (20)

\[ \delta \simeq 42^\circ \pm 16^\circ. \] (21)

It is interesting to note that the CP violating phase $\delta$ comes out to be around $45^\circ$ and is not much sensitive to $U_{e3}$ in the range $0.05 - 0.15$. The above calculated values of $\delta$, indicating an almost $2.5 \sigma$ deviation from $0^\circ$, are in line with the suggestion by several authors [25], [26], [28] about the expected CP violation in the leptonic sector. In particular, the present analysis is broadly in agreement with a similar analysis carried out by Farzan and Smirnov [8] and also with a recent phenomenological analysis carried out by K.R.S. Balaji et al. [29]. Further, it is interesting to note that the present analysis carried out purely on phenomenological inputs is very much in agreement with several analyses based on expected outputs from different experimental scenarios [8], [22], [25], [27]. In particular, the analysis of Marciano and Parsa [25] carried out for the BNL-Homestake (2540 km) proposal is in complete agreement with the present analysis in respect of expected error in $\delta$ and the insensitivity of $\delta$ for values of $U_{e3} \gtrsim 0.05$. Therefore, the BNL-Homestake experiment would not only shed light on the existence of CP violation in the leptonic sector but would also have implications for the scenario of Bjorken et al. [9].

To summarize, following analogy of the ‘db’ triangle in the quark sector, we have explored the possibility of the construction of the ‘$\nu_2, \nu_3$’ leptonic unitarity triangle using the MNS matrix obtained by Bjorken et al. [9]. In particular, modifying this matrix and considering values of $U_{e3}$ suggested by different theoretical models we have obtained the leptonic mixing matrices. Using these as well as the ‘$\nu_2, \nu_3$’ leptonic unitarity triangle, we have constructed histograms for the Dirac-like CP violating phase $\delta$ in the leptonic sector. Interestingly, from these histograms one can find that for the $U_{e3}$ range $0.05 - 0.15$, there is an almost $2.5 \sigma$ likelihood of finding a non zero Dirac-like CP violating phase $\delta$ with its central value to be around $45^\circ$. The present analysis is largely in agreement with the analysis of Marciano and Parsa [25] regarding the expected outcome of the BNL-Homestake (2540 km) proposal.

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Figure 1: Histogram of $\delta$ plotted by considering ‘$db$’ triangle in the case of quarks

Figure 2: Histogram of $\delta$ plotted by considering ‘$\nu_2, \nu_3$’ triangle in the case of neutrinos for (a) $U_{e3} = 0.05 \pm 0.005$ (b) $U_{e3} = 0.1 \pm 0.01$ (c) $U_{e3} = 0.15 \pm 0.015$