Conversion of Thermal Equilibrium States into Superpositions of Macroscopically Distinct States

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A simple procedure for obtaining superpositions of macroscopically distinct states is proposed and analyzed. We find that a thermal equilibrium state can be converted into such a state when a single global measurement of a macroscopic observable, such as the total magnetization, is made. This method is valid for systems with macroscopic degrees of freedom and finite (including zero) temperature. The superposition state is obtained with a high (low) probability when the measurement is made with a high (low) resolution. We find that this method is feasible in an experiment.

I. INTRODUCTION

Superpositions of macroscopically distinct states have been attracting much attention [1–28]. Such superposition is craved in quantum metrology [29–35], quantum computation [36–44], quantum teleportation [45–48], quantum repeaters [49–51], and fundamental tests of quantum mechanics [52–54]. Its experimental realizations have been reported in various systems [55–66]. However, most of them are limited to either extremely low temperature, such as in the superconducting quantum interference devices [57], or systems with small degrees of freedom, such as the single-mode photons [58]. These limitations were necessary for a long coherence time and good controllability.

For the coherence time, it seems possible to overcome the limitation; recent experiments showed that a long coherence time can be obtained even at room temperature, such as ∼2 ms [67–69] in a negatively charged nitrogen-vacancy (NV−) center in diamond [70–72]. A macroscopic system composed of a large number of such systems is expected to have both a long coherence time and large degrees of freedom, even at moderate temperature such as 300 K. If superpositions of macroscopically distinct states are realized in such a macroscopic system, it is interesting not only from a fundamental viewpoint that macroscopic quantum coherence is proved possible at such high temperature, but also from a viewpoint on potential applications operating at room temperature.

For the controllability, however, a challenge seems to remain; a macroscopic system at finite temperature T is expected to be in a thermal equilibrium state, i.e., in the Gibbs state ρeq. Since ρeq for T > 0 is a classical mixture of an exponentially large number of quantum states, it may seem very difficult to convert it into superpositions of macroscopically distinct states. A successful example is the interference experiment using a C60 molecule at 900 K [73, 74]. In this molecule, however, the center-of-mass motion, which exhibits the interference, is completely decoupled from the internal motion, which is responsible for the high temperature and high mixture. Such a perfect decoupling cannot be expected for general systems. Then, how can we convert equilibrium states of general macroscopic systems into such superpositions?

In this paper, we propose a simple method for obtaining superpositions of macroscopically distinct states from a thermal equilibrium state. For systems with N spins, for example, we find that such a state is obtained through just a single global measurement of the total magnetization of an equilibrium state in a magnetic field. Although the obtained state is a classical mixture of exp(Θ(N)) states and has non-vanishing temperature, it contains superpositions of macroscopically distinct states with a significant ratio, and therefore has potential applications like other cat states. This method is applicable to a wide class of systems including spin systems, atomic systems, quantum optical systems, and quantum dots. Through estimation using a state-of-the-art magnetometer and a system with long coherence time, we find our method feasible in an experiment.

II. RECIPE FOR CONVERSION

Our method is summarized as the following simple recipe. Although the recipe is applicable to various systems (see below), we explain it by taking, as an illustration, a quantum system composed of N spins with S = 1/2. First, attach a heat bath of inverse temperature β (= 1/T), apply a magnetic field h = (h, 0, 0) parallel to the z axis, and let the system equilibrate thermally. Then the state of this system becomes ρeq := e−βH / Zeq. Here, H is the Hamiltonian of the system in the presence of h, and Zeq := Tr[e−βH]. We will discuss details of H below. Second, measure the z component of the total magnetization Mz = ∑N i=1 Szi in such a way that the measurement operator Pz = |1/2⟩⟨1/2| is the projection operator onto the Mz = M subspace, where M denotes the outcome of the measurement. The case of a more general Pz will be discussed below. If preferred, one may remove the heat bath and h just before the measurement.
(quickly, so as not to change the state). Then, if the measurement is completed before the system decoheres, the post-measurement state $\hat{\rho}_{\text{post}}$ is given by

$$\hat{\rho}_{\text{post}} = \hat{P}_z e^{-\beta \hat{H}} \hat{P}_z / Z_{\text{post}},$$

where $Z_{\text{post}} := \text{Tr}[\hat{P}_z e^{-\beta \hat{H}} \hat{P}_z]$ is the ‘partition function,’ which is not the ordinary one because we will show that $\hat{\rho}_{\text{post}}$ is not a normal equilibrium state. Note that $[\hat{H}, \hat{M}_z] \neq 0$ because of the interaction $-\hbar \dot{M}_z$ with the magnetic field. This noncommutativity makes the properties of $\hat{\rho}_{\text{post}}$ highly nontrivial. We will show that this state is a generalized cat state, containing superpositions of macroscopically distinct state with a significant ratio (precise definition below).

III. GENERALIZED CAT STATE

Two states are macroscopically distinct if there is a macroscopic observable whose values are macroscopically distinct between them. Using this general observation, several measures have been proposed to quantify superposition of macroscopically distinct states $\cfrac{3}{3} \cfrac{8}{8} \cfrac{18}{18} \cfrac{78}{78}$, which are motivated, e.g., by stability $\cfrac{3}{3}$ and Fisher information $\cfrac{18}{18} \cfrac{23}{23}$. For pure states, all these measures are found to be equivalent to that of Ref. $\cfrac{8}{8}$. When a pure state is a superposition of macroscopically distinct states according to this established measure, we call it a pure cat state.

As we will show below, $\hat{\rho}_{\text{post}}$ is a mixture of $\exp(\Theta(N))$ states for $T > 0$. In order to detect superpositions of macroscopically distinct states in such a state, we use the index $q$ $\cfrac{8}{8} \cfrac{18}{18} \cfrac{22}{22} \cfrac{26}{26}$ (briefly reviewed in Appendix $\cfrac{C}{C}$) because it correctly detects pure cat states (if contained) in any mixed state. It is sufficient for the present purpose to introduce $q$ as a real number satisfying

$$\max_{\hat{A}, \hat{n}} \text{Tr}[\hat{\rho} \hat{C}_{\hat{A}, \hat{n}}] = \Theta(N^q)$$

for a general state $\hat{\rho}$. Here, $\hat{A}$ is an additive observable, $\hat{n}$ is a projection operator, and $\hat{C}_{\hat{A}, \hat{n}} := [\hat{A}, [\hat{A}, \hat{n}]]$. It is easily shown that $q \leq 2$. Supported by reasonable observations (see also Appendix $\cfrac{C}{C}$ for details), Ref. $\cfrac{8}{8}$ showed that if some $\hat{A}$ and $\hat{n}$ assure $q = 2$ for a quantum state, then the state contains pure cat states, i.e., superpositions of states whose values of $\hat{A}$ differ from each other by $\Theta(N)$, with a significant ratio. We call such a state a generalized cat state of $\hat{A}$. By “general,” we mean a state with $q \geq 2$ is not necessarily a pure state or a superposition of only two states like Schrödinger’s cat state. For example, the index $q$ correctly identifies the mixed state (which beats the standard quantum limit using quantum superposition) of $\text{Ref.} \cfrac{78}{78}$ as a generalized cat state. When necessary, one can also use the value of $\langle \hat{C}_{\hat{A}, \hat{n}} \rangle$ as a quantitative measure (Appendix $\cfrac{C}{C}$).

IV. EXAMPLE OF FREE SPINS

Though we will show below that our recipe is applicable to varieties of Hamiltonians, we start with the case of free spins to grasp the idea. The Hamiltonian, after applying the magnetic field, is

$$\hat{H}^0 := -\hbar \sum_{i=1}^N \hat{\sigma}_x = -\hbar \dot{M}_x,$$

where the superscript “0” denotes the absence of interactions between spins. The system is left to equilibrate in a heat bath of inverse temperature $\beta$. The state of the system becomes $\hat{\rho}_{\text{eq}}^0 := e^{-\beta \hat{H}^0} / Z_{\text{eq}}^0$, where

$$Z_{\text{eq}}^0 := \text{Tr}[e^{-\beta \hat{H}^0}] = 2^N \cosh^N(\beta \hbar).$$

For this $\hat{\rho}_{\text{eq}}^0$, $\dot{M}_z$ is measured, and the outcome $M$ is obtained. Then the post-measurement state $\hat{\rho}_{\text{post}}^0$ is given by Eq. $\cfrac{1}{1}$ with $\hat{H} = \hat{H}_0$, and $Z_{\text{post}}$ (denoted as $Z_{\text{post}}^0$) by

$$Z_{\text{post}}^0 := \text{Tr}[\hat{P}_z e^{-\beta \hat{H}^0} \hat{P}_z] = \left( \frac{N}{N + M} \right) \cosh^N(\beta \hbar).$$

While $\langle \dot{M}_z \rangle_{\text{eq}} = \Theta(N)$, we can easily show that $\langle M_{\text{post}} \rangle_{\text{post}} = \langle M_{\text{eq}} \rangle_{\text{post}} = 0$, where, throughout this paper, $\langle \cdot \rangle_{\text{eq}}$ and $\langle \cdot \rangle_{\text{post}}$ denote the expectation values in the pre-measurement equilibrium state and the post-measurement state, respectively. Therefore, the uncertainty relation $\delta M_{\text{eq}} \delta M_{\text{post}} > |\langle \dot{M}_z \rangle_{\text{eq}}|$ is used in discussions on spin squeezing $\cfrac{16}{16} \cfrac{27}{27} \cfrac{81}{81} \cfrac{82}{82}$, does not tell anything nontrivial for $\hat{\rho}_{\text{post}}^0$.

To show $q = 2$ for $\hat{\rho}_{\text{post}}^0$, it is sufficient by definition to show $\langle \hat{C}_{\hat{A}, \hat{n}} \rangle_{\text{post}} = \Theta(N^2)$ for some $\hat{A}$ and $\hat{n}$. Now, let us take $\hat{A} = \dot{M}_z$ and $\hat{n} = \hat{P}_z$. Then, we have

$$\langle \hat{C}_{\dot{M}_z, \hat{P}_z} \rangle_{\text{post}} = 2N + (N^2 - M^2) \tanh^2(\beta \hbar).$$

The factor $N^2 - M^2$ is $\Theta(N^2)$ except when $M = \pm N + o(N)$, which is the case very unlikely to happen for the following reason. From $\cfrac{1}{1}$ and $\cfrac{3}{3}$, the probability of obtaining $M$ as the outcome of the $\dot{M}_z$ measurement is calculated as $\text{Pr}[\dot{M}_z = M] = Z_{\text{post}}^0 / Z_{\text{eq}}^0 = (\langle N + M \rangle / 2)^N$. Therefore, $\text{Pr}[\dot{M}_z = \pm N + o(N)] = N o(N) / o(N)! 2N$, which is exponentially small in $N$. Hence, we hereafter exclude the case $M = \pm N + o(N)$ and assume

$$N^2 - M^2 = \Theta(N^2), \text{ i.e., } N - |M| = \Theta(N).$$

Then, since $\tanh^2(\beta \hbar) = \Theta(1)$ for any $|\beta \hbar| = \Theta(1)$, we have $\langle \hat{C}_{\dot{M}_z, \hat{P}_z} \rangle_{\text{post}} = \Theta(N^2)$, and thus $q = 2$. Therefore, an equilibrium state of a spin system with the Hamiltonian $\hat{H}^0 = -\hbar \dot{M}_z$ at any finite temperature can be converted into a generalized cat state of $\dot{M}_z$ by measuring $\dot{M}_z$ just once.
V. FEATURES OF THE POST-MEASUREMENT STATE

We note that the purity of \( \tilde{\rho}_{\text{post}}^0 \) is \( \text{Tr}[(\tilde{\rho}_{\text{post}}^0)^2] \leq 1/\exp(\Theta(N)) \) when \( |\beta h| = \Theta(1) \). This low purity is due to two facts: The purity of the pre-measurement state \( \tilde{\rho}_{\text{eq}}^0 \) is also \( 1/\exp(\Theta(N)) \), and the subspace onto which \( \hat{P}_z \) projects has a dimension of \( \exp(\Theta(N)) \), i.e., \( \text{Tr}[\hat{P}_z] = \binom{N}{(N+M)/2} \). Therefore, \( \tilde{\rho}_{\text{post}}^0 \) is a mixture of \( \exp(\Theta(N)) \) states when \( |\beta h| = \Theta(1) \) despite containing superpositions of macroscopically distinct states. This does not mean that the ratio of pure cat states (to non-cat states) is exponentially small, according to the definition of the index \( q \). In fact, the factor \( \tanh^2(\beta h) \) in Eq. (6) quantifies the contribution from pure cat states; it quantifies the ratio of pure cat states and how distinct are the values of \( M_z \) between the states that are superposed (Appendix C). At intermediate temperatures where \( \beta h = \Theta(1) \), there is \( \tanh^2(\beta h) = \Theta(1) \) of contribution from pure cat states.

We can show that \( \langle H^0 \rangle_{\text{post}} = 0 \) and \( \langle (\Delta H^0)^2 \rangle_{\text{post}} = \Theta(N^2) \) for the generalized cat state \( \tilde{\rho}_{\text{post}} \), where \( \Delta H^0 := H^0 - \langle H^0 \rangle_{\text{post}} \). Hence, if the energy of this state is measured, the outcome \( E \) will vary by \( \Theta(N) \) from run to run. Let us define ‘temperature’ of a nonequilibrium state (like the generalized cat state) as the temperature of the equilibrium state of the same energy. [This definition is reasonable from an operational viewpoint, as discussed in Appendix A] Then, the ‘temperature’ of \( \tilde{\rho}_{\text{post}} \) varies from run to run. Their values are non-zero except when \( E = \pm hN \), the highest and lowest energies. In this sense, the obtained generalized cat state has non-vanishing temperature.

VI. EXTENTION OF PROJECTION OPERATOR ONTO A FINITE INTERVAL

So far, the projection operator \( \hat{P}_z \) which we used projects states onto the subspace of one exact eigenvalue \( M \) of \( M_z \). Although such a measurement seems feasible with the present-day technologies (as discussed below), there may be the cases where it is challenging to, for example, distinguish a state of \( M_z = 0 \) from a state of \( M_z = 2 \) because of a low resolution. To model such a general case, we specifically consider the case with the projection operator \( \hat{P}_z \) onto the subspace corresponding to a finite interval \( M_z < M_z < M_z \). We here show that even with a low-resolution, one can in principle obtain a generalized cat state. As an illustration, we study the system with \( N \) free spins. We assume \( |M_+| < M \) without loss of generality, and evaluate the index \( q \) of the post-measurement state \( \hat{P}_z e^{-\beta H^0} \hat{P}_z' \). Here, \( Z_{\text{post}}^0 := \text{Tr}[\hat{P}_z e^{-\beta H^0} \hat{P}_z' \hat{Z}_{\text{post}}^0] = \text{Tr}[\hat{P}_z e^{-\beta H^0} \hat{P}_z' \hat{Z}_{\text{post}}^0] \) is the ‘partition function,’ calculated as \( Z_{\text{post}}^0 = \sum_{k=0}^{(M_n-M_-)/2} \binom{N}{(N+M_n)/2+k} \cosh^N(\beta h) \). Then, using a function \( I(N, M_+, M_-) \) given in Appendix H that satisfies \( 0 \leq I \leq 1 \), we have

\[
\langle \hat{C}_{M_+M_-} \rangle_{\text{post}} = N^2 \tanh^2(\beta h) \frac{I(N, M_+, M_-)}{I(M_+, M_-)} + O(N). \tag{8}
\]

This becomes \( \Theta(N^2) \) when \( I(N, M_+, M_-) = \Theta(1) \). After some algebra, we find that, when \( M_+ = M_- = \Theta(1) \), the post-measurement state is a generalized cat state as long as \( N - |M_-| = \Theta(N) \). On the other hand, when \( M_+ = M_- > \Theta(1) \), the post-measurement state is a generalized cat state only when \( M_- = \Theta(N) \) and \( N - |M_-| = \Theta(N) \).

There is a trade-off between the resolution, \( M_+ - M_- \), and the success probability. If \( M_+ - M_- \) is as small as \( \Theta(1) \), one can obtain a generalized cat state through our recipe with the success probability of almost 100%. Easier to realize is a measurement with a lower resolution of \( M_+ - M_- > \Theta(1) \) such as \( \Theta(\sqrt{N}) \). In this case, as described above, \( M_- \) has to be \( \Theta(N) \) for obtaining a generalized cat state. That is, the measured \( M_z \) has to be \( \Theta(N) \). However, the probability of such a case is exponentially small because \( \Pr[\Theta(N) = M_- \leq M_z \leq M_+] = Z_{\text{post}}^0/\tilde{Z}_0^0 = e^{-\Theta(N)} \). Thus, when \( M_+ - M_- > \Theta(1) \), experiments should be run many times in order to obtain a generalized cat state. When designing experiments, these conditions should be taken into account according to one’s purpose.

VII. GENERALIZATION OF SYSTEMS AND INITIAL STATES

Up to this point, we have assumed spin-1/2 systems and the canonical Gibbs states as the pre-measurement states. Actually, any two-level system can be mapped to a spin-1/2 system. Thus our discussion is already applicable to other physical systems such as two-level atoms by mapping observables such as \( M_z \) and \( M_x \) appropriately. We here add even more general discussion by providing two conditions. At the same time, we show that the initial states need not be the Gibbs states.

Consider a macroscopic quantum system, which is not necessarily a spin system, in some state \( \hat{\rho}_{\text{pre}} \), which is taken as the pre-measurement state. Let \( \hat{A} \) and \( \hat{B} \) be additive operators of the system such that

\[
\hat{P}_b \hat{A} |b, \xi \rangle = 0 \tag{9}
\]

for all eigenstates \( |b, \xi \rangle \) of \( \hat{B} \), where \( \hat{B} |b, \xi \rangle = b |b, \xi \rangle \) and \( \xi \) labels degenerate eigenstates, and \( \hat{P}_b \) is the projection operator onto the \( \hat{B} = b \) subspace. When \( \hat{B} \) of \( \hat{\rho}_{\text{pre}} \) is measured, the post-measurement state \( \hat{\rho}_{\text{post}} = \hat{P}_b \hat{\rho}_{\text{pre}} \hat{P}_b/\text{Tr} \). We then obtain \( \langle \hat{A} \rangle_{\text{post}} = 0 \) and, by taking \( \hat{q} = \hat{P}_b \),

\[
\langle \hat{C}_{\hat{A}\hat{B}} \rangle_{\text{post}} = 2\text{Tr} \frac{\hat{P}_b \hat{\rho}_{\text{pre}} \hat{P}_b A^2}{\text{Tr} \hat{P}_b \hat{\rho}_{\text{pre}} \hat{P}_b}. \tag{10}
\]
then $\rho_{\text{post}}$ is a generalized cat state. [In the case of Eq. (6), for example, the set of $\{M_x, M_z, \rho_{\text{eq}}\}$ corresponds to $\{A, B, \hat{\rho}_{\text{pre}}\}$.] Since this result is applicable to any systems including quantum optical systems, atomic systems, and quantum dots, our recipe can be carried out in a wide class of physical systems.

The sufficient conditions [9] and [10] tell us that $\hat{\rho}_{\text{pre}}$ is not required to be the canonical Gibbs state $\rho_{\text{eq}}$. For example, in spin systems (or systems that can be mapped to spin systems), the pre-measurement state $\hat{\rho}_{\text{pre}}$ may be arbitrary if it has a macroscopic value of $M_x$, i.e., if $\langle M_x \rangle_{\text{pre}} = \Theta(N)$, because then the conditions are satisfied by the set of $\{M_k, \hat{M}_y, M_z, \hat{\rho}_{\text{pre}}\}$ with a non-vanishing probability (Appendix I2). This sufficient condition indicates that details of the system does not matter to our recipe. For example, our recipe is applicable even to systems with interactions whether the interactions are short-range or long-range. A detailed example of the XYZ model and a discussion using symmetries are in Appendix I5 and 16. Appendix I also suggests that the discussions of VII hold the same for interacting spins.

VIII. FEASIBILITY

To carry out our recipe, we need a spin system which does not decohere during the $M_z$ measurement. Long coherence times are realized in various systems such as those with ultracold atoms [84-87] and circuit QED systems [54, 57]. Among them, we here investigate the feasibility for the NV$^-$ centers [67-71], in which a spin has a long coherence time such as 470 $\mu$s (or 2 ms using a spin echo) even at room temperature [67-69]. (Since $S = 1$ is equivalent to two of $S = 1/2$ spins, we can apply our recipe to the NV$^-$ centers which have $S = 1$.) Following our recipe, we suppose that a system composed of $N$ NV$^-$ centers is left to equilibrate in the presence of a magnetic field. To obtain a generalized cat state with a coherence time of $\Theta(1)$ within $\tau_{\text{coh}}$, we need to measure $M_z$ with the resolution of $\Theta(1)$ within $\tau_{\text{coh}}$, the coherence time of the system. Since the coherence of an $N$-spin system is lost when just one spin decoheres, $\tau_{\text{coh}}$ is shorter than the coherence time $\tau$ of a single spin (for more details, see Appendix I). Here we assume a typical case $\tau_{\text{coh}} = \tau/N$ to discuss the feasibility. Fortunately, state-of-the-art magnetometers [58, 59], such as the one based on optically pumped potassium atoms operating in a spin-exchange relaxation free (SERF) regime [83, 84], are estimated to be sensitive enough: 160 aT/$\sqrt{Hz}$ [92], for example. Since one spin creates a magnetic field $\mu_B \mu_0/2\pi r^3$ at distance $r$, measurement of $\Theta(1)$ resolution can be performed within $\tau_{\text{coh}}/10$ even at room temperature when $N \lesssim 10^4$ and $r \approx 1 \mu$m. In particular, when $N = 10^2$ and $r \approx 1 \mu$m, a generalized cat state is obtained with a measurement time $\ll \tau_{\text{coh}}$ even at room temperature. Hence the obtained generalized cat state survives for most of $\tau_{\text{coh}} = 4.7 \mu$s after conversion.

More generally, temperature $T$ of a system is “high” if $k_B T$ is larger than any relevant energy scale of the system even if $T$ is much lower than room temperature. For free spins, for example, $T$ is high if $\beta \hbar \lesssim 1$. In this sense, a generalized cat state with a longer coherence time at sufficiently high temperature may be realized in other systems such as electron spins in donors in high-purity Si [93].

We can also show that within the system’s coherence time, the post-measurement state continue being a generalized cat state while evolving with time (Appendix I4).

IX. VERIFICATION OF THE CONVERSION

We also discuss how to verify the success of the creation of a generalized cat state in experiments. One way, which seems most practical, is to see the enhancement of performances of applications, such as the increase of the sensitivity in the application discussed above.

Another way is to investigate the state itself as follows. If $T \ll \hbar$, the pre-measurement state is the ground state, a pure state. In this case, the post-measurement state will also be a pure state. For a pure state $|\psi\rangle$, it was shown that if $|\langle\psi|\Delta A|\psi\rangle| = \Theta(N^2)$, where $\Delta A := A - \langle\psi|A|\psi\rangle$, then $|\psi\rangle$ is a pure cat state [3, 11, 15, 23-24, 42] (see Appendix I3 for a brief review). For the free spins, for example, the success can be verified by measuring $M_z$, and thereby confirming that the measured $\langle M_z \rangle_{\text{post}}$ agrees with the theoretical result, calculated as $N + (N^2 - M^2)/2$ for $T = 0$. If $T \gg \hbar$ and thus the post-measurement state is a mixed state, investigation of the state is very difficult in general. For example, state tomography requires an exponentially large number of procedures, addressing individual spins. However, fortunately, one can verify the success of conversion within the number of procedures that is polynomial in $N$ for our case. It is done by measuring $C_{M_z, P_z}$ and comparing the result with the theoretical one [10]. Since any observable of a spin system is a function of the Pauli operators, $C_{M_z, P_z}$ can be measured, by addressing individual spins (see Appendix I4 for details).

X. ADVANTAGES AND APPLICATIONS

Here we summarize advantages of our recipe. (i) The temperature of the system is arbitrary, i.e. ultra low temperature is not required. (ii) The procedure is simple: just one global measurement. (iii) Precise control of the pre-measurement state is unnecessary since Eqs. (9) and (10), or, more simply, $\langle M_z \rangle_{\text{pre}} = \Theta(N)$, is sufficient. (iv) It is applicable to many physical systems, as discussed above.

Our generalized cat state can be used for various applications like other cat states. For example, as shown in Refs. [73, 94], pure cat states and their mixture improve
the sensitivity of magnetometry, beating the standard quantum limit by a factor of $N^{1/4}$ even under the effect of decoherence. [Otherwise, enhancement is $N^{1/2}$, reaching the Heisenberg limit.] For the case of our generalized cat state, we must take account of the factor $\tanh^2(\beta h)$, which quantifies contribution from pure cat states, as discussed above. However, since $\tanh^2(\beta h)$ is independent of $N$, the overall sensitivity beats the standard quantum limit by a factor proportional to $N^{1/4}$.

The success probability is high (low) when $M_+ - M_- = \Theta(1)$ ($M_+ - M_- > \Theta(1)$). We showed two loose conditions that show the applicability of our recipe to various systems such as free spins, interacting spins, and more general systems including quantum optical systems and atomic systems. We also estimated that the method is feasible when the SERF magnetometer is used for measuring the magnetization of the NV$^-$ centers in diamond.

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Appendix A: Temperature of post-measurement state

Since the post-measurement state $\hat{\rho}_{\text{post}}$ (which is denoted as $\hat{\rho}_{\text{post}}^0$ for free spins) is not an ordinary equilibrium state, its temperature cannot be defined trivially. Therefore, it seems legitimate to define the ‘temperature’ operationally, i.e., to define it as the outcome that is obtained when the temperature of $\hat{\rho}_{\text{post}}$ is really measured.

There are two typical methods of measuring temperature. One is to use a thermometer which is much smaller than the system so that it does not alter the temperature of $\hat{\rho}_{\text{post}}$ of the system. In our case, however, it is not clear whether the composite system composed of the system and such a small thermometer reaches equilibrium, because $\hat{\rho}_{\text{post}}$ is far from equilibrium and we allow the system Hamiltonian to be integrable (such as free spins). Therefore, we employ the other typical method, which is to use heat baths that are much larger than the system.

Suppose that we have heat baths with various temperature, and many copies of the system in the same state $\hat{\rho}_{\text{post}}$. Then, suppose that one heat bath is attached to one copy of the system. After some amount of energy flows between the bath and the system, this composite system will reach equilibrium. If the net energy flow is zero, the temperature of the heat bath can be identified with the measured temperature of the system. If, on the other hand, the net energy flow is nonzero, retry this experiment using another heat bath of another temperature, and another copy of the system. This seems a reasonable and widely applicable method of measuring temperature.

When this method is used to measure the temperature $T_0$ of the post-measurement state $\hat{\rho}_0$ of free spins, the measured temperature varies from measurement to measurement because $\langle (\Delta H_0)^2 \rangle = \Theta(N^2)$ in this state. Since $\langle H_0 \rangle = 0$ is $\Theta(N)$ larger than the ground energy $-hN$, the average of the inverse temperature $1/T_0$, over many runs of measurements, is finite. In this sense, $\hat{\rho}_0$ has non-vanishing temperature. In a similar manner, we can show that the post-measurement state $\hat{\rho}_{\text{post}}$ of interacting spins also has non-vanishing temperature. That is, we can obtain by our recipe a generalized cat state of non-vanishing temperature.

Appendix B: Index $p$ for pure states

In this section, we review index $p$, which for pure states detects superposition of macroscopically distinct states [3, 8, 11, 13, 17, 39, 42]. Although such states are called “anomalously fluctuating states” in Ref. [3] and “macroscopically entangled states” in Refs. [8, 11, 39], we here call them pure cat states (or, generalized cat states for mixed states) to be more comprehensible.

Although we have mainly used $q$ in the text, it seems necessary to understand $p$ for understanding $q$. We therefore review $p$ first in this section, and then $q$ in Appendix C.
We limit ourselves to pure states that are macroscopically uniform spatially. (The case of non-uniform states was formulated in Ref. [42].)

1. Motivation

It is not trivial to define superposition of macroscopically distinct states. For example, while the cat state

$$|\text{cat}+\rangle \equiv \frac{1}{\sqrt{2}} |000\cdots 0\rangle + \frac{1}{\sqrt{2}} |111\cdots 1\rangle$$  \hspace{1cm} (B1)$$

is obviously such a superposition, how about the following states?

$$|\psi_1\rangle \equiv \sqrt{1 - \frac{1}{N}} |000\cdots 0\rangle + \sqrt{\frac{1}{N}} |111\cdots 1\rangle,$$

$$|\psi_2\rangle \equiv \frac{|000\cdots 0\rangle + |100\cdots 0\rangle + |110\cdots 0\rangle + \cdots + |111\cdots 1\rangle}{\sqrt{N+1}}.$$  \hspace{1cm} (B2)

In order to identify superposition of macroscopically distinct states unambiguously, a reasonable index was proposed in Refs. [3, 11], as follows.

2. Macroscopically distinct states

We start with defining ‘macroscopically distinct states.’ Such states should be defined as those between which some macroscopic observable takes distinct values. But, what is a ‘macroscopic observable’? According to thermodynamics and statistical mechanics, macroscopic observables should be additive observables. Here, we say $\hat{A}$ is an additive observable if it is the sum of local observables $\hat{a}(r)$,

$$\hat{A} = \sum_r \hat{a}(r),$$  \hspace{1cm} (B4)$$

where the sum is taken over the whole system. We assume, for simplicity, that

$$\|\hat{a}(r)\| \equiv \text{Tr}|\hat{a}(r)| = \Theta(1).$$  \hspace{1cm} (B5)$$

Since $\hat{A} = O(N)$, two values of $\hat{A}$ are ‘distinct’ if they are different by $\Theta(N)$. According to this observation, the following definition is reasonable.

**Definition:** Macroscopically distinct states

Two (or more) states are macroscopically distinct if the values of some additive observable $\hat{A}$ are different by $\Theta(N)$ between them.

3. Superposition of macroscopically distinct states

Let $|A\nu\rangle$ be an eigenvector of $\hat{A}$ corresponding to eigenvalue $A$, where $\nu$ labels degenerate eigenvectors. If $|\psi\rangle$ does not contain a superposition of states with macroscopically distinct values of $A$, i.e., if it is just a superposition of $|A\nu\rangle$’s with macroscopically non-distinct values of $A$, then $|\langle A\nu|\psi\rangle|^2$ takes significant values only for $A$ such that $|A - \langle \psi|\hat{A}|\psi\rangle| = o(N)$. In this case, $\langle \psi|(\Delta \hat{A})^2|\psi\rangle = o(N^2)$. Hence, by contradiction, if $\langle \psi|(\Delta \hat{A})^2|\psi\rangle \neq o(N^2)$, i.e., if $\langle \psi|(\Delta \hat{A})^2|\psi\rangle = \Theta(N^2)$, then $|\psi\rangle$ contains a superposition of states with macroscopically distinct values of $A$. We are thus led to the following definition:

**Definition:** (for pure states) Superposition of macroscopically distinct states

A pure state $|\psi\rangle$ contains a superposition of macroscopically distinct states if there exists an additive observable $\hat{A}$ such that $\langle \psi|(\Delta \hat{A})^2|\psi\rangle = \Theta(N^2)$. 

4. Index p

It is then convenient to define the index $p$ as follows.

**Definition:** Index $p$ for pure states

Let $A$ be an additive observable. For a pure state $|\psi\rangle$, the index $p$ is defined as a real number such that

$$\max_A \langle \psi | (\Delta A)^2 | \psi \rangle = \Theta(N^p). \quad (B6)$$

It is easy to show that $1 \leq p \leq 2$. Using this index, the above definition can be rephrased as follows:

**Definition:** Pure cat state

For a pure state $|\psi\rangle$, if $p = 2$ then $|\psi\rangle$ contains a superposition of macroscopically distinct states, which we call a pure cat state.

Note that we have never assumed that only two macroscopically distinct states are superposed to form $|\psi\rangle$ with $p = 2$. Therefore, a pure cat state contains a superposition of two or more macroscopically distinct states.

For example, the cat state $|\text{cat}+\rangle$ has $p = 2$, as expected, because $\langle (\Delta M_z)^2 \rangle = O(N^2)$. By contrast, $|\psi_1\rangle$ has $p = 1$, and hence is not a pure cat state. On the other hand, $|\psi_2\rangle$ has $p = 2$ because $\langle (\Delta M_z)^2 \rangle = \Theta(N^2)$, and hence is a pure cat state. This may be understood because $|\psi_2\rangle$ is a superposition of states with $M_z = \Theta(N)$ and $M_z = -\Theta(N)$.

Note that a state with $p = 2 - \epsilon$ ($0 < \epsilon \ll 1$) is close to, but not completely, a pure cat state. In this paper, we are not interested in such an incomplete superposition of macroscopically distinct states.

It was shown that $p$ is directly related to physics. For example, fundamental stabilities of quantum many-body states are determined by $p$, as described in Appendices B.7 and B.9 and Refs. [3, 13]. Furthermore, $p = 2$ is necessary for quantum computational speedup [39, 42]. It is known that index $p$ agrees with other measures for superposition of macroscopically distinct states [3, 13, 78]. These facts also support that $p$ is a reasonable index. Furthermore, there is an efficient method of computing $p$ for a given pure state, as described in Appendix B.7 and Ref. [11]. The reader, if not interested in these facts, may jump to Appendix C in which the index $q$ is reviewed.

5. Decoherence rate of states with $p = 2$

Let us consider the decoherence rate $\Gamma$ of a pure state $|\psi\rangle$ by a classical noise (or a perturbation from environments), under a physical assumption that the interaction between the noise (or environments) and the system is the sum of local interactions. It was shown in Refs. [3, 32] that, with increasing the system size $N$, $\Gamma$ scales as

$$\Gamma \leq \Theta(N^p), \quad (B7)$$

where $p$ is the index $p$ of $|\psi\rangle$. This is a universal result, independent of any details of the system and noise. It implies, for example, that $\Gamma$ of a state with $p = 1$ grows at most as $\Theta(N)$.

It was also shown in Refs. [3, 32] that the equality in Eq. (B7) is achievable, i.e., a noise achieving the equality is in principle possible [36] that satisfies the above assumption on interaction between the noise and the system. In particular, if $|\psi\rangle$ has $p = 2$ then a noise that satisfies the assumption is in principle possible such that $|\psi\rangle$ decoheres as fast as $\Gamma = \Theta(N^2)$.

6. Stability against local measurements

Consider a quantum state $\hat{\rho}$, either pure or mixed, which is translationally invariant. Let $\hat{a}(x)$ and $\hat{b}(y)$ be local operators on (spatial regions around) the positions $x$ and $y$, respectively, which therefore commute with each other if $|x - y| >$ some constant. By $P(a \mid P(b))$ we denote the probability of getting the outcome $a$ when $\hat{a}(x) \mid \hat{b}(y)\rangle$ is measured. By $P(a, b)$, we denote the probability of getting the outcome $a$ and $b$ when $\hat{a}(x)$ and $\hat{b}(y)$ are measured simultaneously. We assume that $\hat{a}(x), \hat{b}(y)$ do not depend on $N$, and that

$$\langle |\hat{a}(x)| \rangle, \langle |\hat{b}(y)| \rangle \leq \text{some constant independent of } N \quad (B8)$$

for all $\hat{a}(x), \hat{b}(y)$, where $\langle \cdot \rangle$ denotes the expectation value in $\hat{\rho}$.

In terms of these quantities, we here define the stability against local measurements in a manner slightly different from that of Ref. [3], as follows [97]:
This implies that the outcome of measurement of ă(x) at x does not correlate with the outcome of measurement of ẵ(y) at y if |x − y| is large enough. When P(b) > 0, we can rewrite (B9) as

\[ |P(a; b) − P(a)| ≤ \epsilon P(a) \] for ∀x, y s.t. |x − y| > ℓ, and for ∀ā̃(x), ẵ(y), a, b. \hspace{1cm} (B10)

Here, \( P(a; b) := P(a, b)/P(b) \) is the conditional probability of getting a by measurement of ẵ(x) when b is obtained by measurement of ẵ(y). Relation (B10) implies that measurement of a local observable does not affect the outcome of measurement of a local observable at a distant point. We therefore call this property stability against local measurements.

While this stability is expected for any stable macroscopic states, it can be shown that a pure state with \( p = 2 \) does not have this stability.

More generally, it can be shown that (B10) is equivalent to the “cluster property” in the following sense:

\[
\left| \langle \Delta ă(x) \Delta ẵ(y) \rangle \right| \leq \epsilon \langle |ẵ(x)| |ẵ(y)| \rangle \] for ∀x, y s.t. |x − y| > ℓ, and for ∀ā̃(x), ẵ(y), a, b. \hspace{1cm} (B11)

Here, \( \Delta ă(x) := ă(x) − \langle ă(x) \rangle \) and \( \Delta ẵ(y) := ẵ(y) − \langle ẵ(y) \rangle \).

This is a generalization of the cluster property of infinite systems to finite systems. For any pure state with \( p = 2 \), we can show that it does not have the cluster property, and therefore it is not stable against local measurements.

7. Variance-covariance matrix and off-diagonal long-range order

There is an efficient method of calculating \( p \) [11]. For simplicity, we assume that each site of the lattice is a spin-1/2 system. For a given pure state |ψ\⟩, we define the variance-covariance matrix (VCM) by

\[
V_{\alpha i, \beta j} := \langle \psi | \Delta ă^i_{\alpha} \Delta ẵ^j_{\beta} | \psi \rangle = \langle \psi | \hat{\sigma}^i_{\alpha} | \psi \rangle \langle \psi | \hat{\sigma}^j_{\beta} | \psi \rangle - \langle \psi | \hat{\sigma}^i_{\alpha} \hat{\sigma}^j_{\beta} | \psi \rangle,
\]

where \( \alpha, \beta = x, y, z \), and \( i, j = 1, 2, \cdots, N \). The VCM is a \( 3N \times 3N \) Hermitian non-negative matrix. If \( \epsilon_{\text{max}} \) is the maximum eigenvalue of the VCM, it is shown that

\[
\epsilon_{\text{max}} = \Theta(N^{p-1}).
\]

One therefore has only to evaluate \( \epsilon_{\text{max}} \) to calculate \( p \). If \( \epsilon_{\text{max}} = \Theta(N) \), then \( p = 2 \) and \( |\psi\rangle \) contains superposition of macroscopically distinct values of the additive observable that is obtained from the eigenvector of the VCM corresponding to \( \epsilon_{\text{max}} \) [11, 17]. It is also shown that the number of eigenvalues that scales as \( \Theta(N) \) is at most \( \Theta(1) \). Equation (B12) also show that the off-diagonal long-range order does not necessarily imply \( p = 2 \) [8, 98].

Appendix C: Index q for mixed states

When \( \hat{\rho} \) is a mixed state, \( \text{Tr}[\hat{\rho}(\Delta \hat{A})^2] = \Theta(N^2) \) does not necessarily imply the existence of a superposition of macroscopically distinct states, because such an equality is satisfied also for a (classical) mixture of macroscopically distinct states. Hence, the index for mixed states cannot be a trivial generalization of \( p \). To correctly identify superposition of macroscopically distinct states for mixed states, the index \( q \) was proposed in Ref. [8], which we review in this section.

We assume that \( \hat{\rho} \) is macroscopically uniform spatially.
1. Motivation

Consider the mixture

$$\hat{\rho}_{\text{ex1}} \equiv \frac{1}{N} \sum_{i=1}^{N} |\psi_i\rangle \langle \psi_i|$$

(C1)

of $N$ different cat states,

$$|\psi_i\rangle = \frac{1}{\sqrt{2}} (|0_i\rangle + |1_i\rangle), \quad (i = 1, 2, \cdots, N),$$

(C2)

where

$$|0_i\rangle := |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |1_i\rangle, \quad |1_i\rangle := |1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |1\rangle.$$  

(C3, C4)

Every $|\psi_i\rangle$ is a cat state because $|0_i\rangle$ and $|1_i\rangle$ are eigenvectors of $\hat{M}_z$ with macroscopically distinct eigenvalues $\pm (N-2)$,

$$\hat{M}_z |0_i\rangle = -(N-2) |0_i\rangle, \quad \hat{M}_z |1_i\rangle = +(N-2) |1_i\rangle.$$  

(C5)

However, since the weight of each $|\psi_i\rangle$ in $\hat{\rho}_{\text{ex1}}$ is as small as $1/N$, which vanishes as $N \to \infty$, it may be nontrivial whether $\hat{\rho}_{\text{ex1}}$ contains a superposition of macroscopically distinct states.

To inspect whether the superposition is contained, let us introduce the following witness observable:

$$\hat{W} := \sum_{i=1}^{N} (|0_i\rangle \langle 1_i| + |1_i\rangle \langle 0_i|),$$

(C6)

whose eigenvalues are $0, \pm 1$. Obviously, it can detect quantum coherence between $|0_i\rangle$ and $|1_i\rangle$, states with macroscopically distinct values of $\hat{M}_z$. By noting $\langle \psi_i|\hat{W}|\psi_i\rangle = 1$, we find

$$\text{Tr}[\hat{\rho}_{\text{ex1}} \hat{W}] = 1,$$

(C7)

which shows that $\hat{\rho}_{\text{ex1}}$ does contain superposition of macroscopically distinct states. This may be understood by noting that $\hat{\rho}_{\text{ex1}}$ is a mixture of the ‘same sort’ of superpositions of macroscopically distinct states in the sense that all $|\psi_i\rangle$’s are superpositions of states with $M_z = \pm (N-2)$.

In this particular case, it was easy to guess the witness observable $\hat{W}$. For general mixed states, however, it will be difficult to find an appropriate $\hat{W}$. The idea of the index $q$ is to do this automatically.

2. Index $q$

Let $\hat{A}$ be an additive observable, $\hat{\eta}$ be a projection operator, and

$$\hat{C}_{\hat{A},\hat{\eta}} \equiv [\hat{A}, [\hat{A}, \hat{\eta}]],$$

(C8)

which is a correlation of local observables (see the example of the text). The index $q$ is defined by

**Definition: Index $q$ for mixed states**

For a mixed state $\hat{\rho}$, the index $q$ is defined as a real number such that

$$\max_{\hat{A},\hat{\eta}} \{ \max \text{Tr}(\hat{\rho} \hat{C}_{\hat{A},\hat{\eta}}), N \} = \Theta(N^q).$$

(C9)

We can show that $1 \leq q \leq 2$. Since $\hat{C}_{\hat{A},\hat{\eta}}$ is a traceless Hermitian operator, its eigenvalues are real numbers whose sum is 0. Using this property, the above definition can be rewritten as

$$\max_{\hat{A}} \{ \max \text{Tr}(|\hat{A}, [\hat{A}, \hat{\rho}]|), N \} = \Theta(N^q).$$

(C10)
For pure states, suppose that, for some \((A, \nu, A, \nu')\) is satisfied also for another \((A, \nu, A, \nu')\). One can take another \(|\phi_i\rangle\) in such a way that
\[
\langle \phi_j | A \nu \rangle |A \nu'\rangle \langle A \nu' | \hat{\rho} | A \nu'\rangle \langle A \nu' | \phi_j \rangle > 0.
\]
When (C15) is satisfied also for another \((A, \nu, A', \nu')\), one can take another \(|\phi_i\rangle\) in such a way that (C16) is also satisfied for this combination of \((A, \nu, A', \nu', |\phi_i\rangle\)). Consequently, both terms give \(\langle \hat{C}_{A, \hat{\eta}} \rangle = \Theta(N^2)\) when \(\eta\) is appropriately taken. This is the basic idea of defining a generalized cat state by \(q\). (Actually, more general cases can be treated by \(q\), as exemplified in the next subsection.) That is, it is defined as follows:

**Definition: Generalized cat state**

For a mixed state \(\hat{\rho}\), if \(q = 2\) then \(\hat{\rho}\) contains superpositions of macroscopically distinct states with a significant magnitude. We call such a state, for which \(\langle \hat{C}_{A, \hat{\eta}} \rangle = \Theta(N^2)\), a generalized cat state of \(\hat{A}\).

A state with \(q = 2 - \epsilon\ (0 < \epsilon \ll 1)\) is close to, but not completely, a generalized cat state. In this paper, we are not interested in such an incomplete superposition of macroscopically distinct states.

If one is interested only in states with \(q > 1\) (such as the generalized cat states), the definition (C9) of \(q\) reduces to a simpler one,
\[
\max_{A, \hat{\eta}} \text{Tr}(\hat{\rho} \hat{C}_{A, \hat{\eta}}) = \Theta(N^q).
\]
This simplified form is used in the paper because only states with \(q = 2\) are analyzed. Note that when studying all states, including those with \(q = 1\), the original definition (C9) should be used because otherwise some of the reasonable properties in the next section would be lost.

### 3. Properties of index \(q\)

The index \(q\) has the following properties:

1. \(q = 1\) for any separable state (i.e., mixture of product states).
2. For pure states,
\[
q = 2 \quad \Leftrightarrow \quad p = 2 \quad \text{(hence,} \quad q < 2 \quad \Leftrightarrow \quad p < 2). \quad \text{(C18)}
\]
\[
q = 1 \quad \Rightarrow \quad p = 1, \quad p = 1 \quad \Rightarrow \quad q \leq 1.5. \quad \text{(C19)}
\]
3. Mixing can decrease $q$.
For example, consider two cat states,
\[ |\text{cat}± \rangle \equiv \frac{1}{\sqrt{2}} |000 \cdots 0\rangle \pm \frac{1}{\sqrt{2}} |111 \cdots 1\rangle, \tag{C20} \]
which have $q = p = 2$. Their mixture
\[ \hat{\rho}_{\text{ex}2} \equiv \frac{1}{2} |\text{cat}+ \rangle \langle \text{cat} + | + \frac{1}{2} |\text{cat} - \rangle \langle \text{cat} - | \]
\[ = \frac{1}{2} |111 \cdots 1\rangle \langle 111 \cdots 1| + \frac{1}{2} |000 \cdots 0\rangle \langle 000 \cdots 0| \tag{C21} \]
is a separable state, hence $q = 1$ according to property \( \blacksquare \).

4. Mixing does not increase $q$, i.e.,
\[ \hat{\rho} = \sum_i \lambda_i \hat{\rho}_i \Rightarrow q \leq \max_i \{q_i\}. \tag{C22} \]
This is evident from the trivial inequality;
\[ \max_A \text{Tr}(\hat{\rho} \hat{C}_{A,\bar{A}}) \leq \sum_i \lambda_i \max_{A_i,\bar{A}_i} \text{Tr}(\hat{\rho}_i \hat{C}_{A_i,\bar{A}_i}). \tag{C23} \]
This inequality also shows the following.

5. If $\hat{\rho}$ has $q = 2$ there exists a state(s) with $q = 2$ in every decomposition. That is, when
\[ \hat{\rho} = \sum_i \lambda_i \hat{\rho}_i = \sum_i \lambda'_i \hat{\rho}'_i = \cdots , \tag{C24} \]
where $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$ and similarly for $\lambda'_i$, then there exists a state with $q = 2$ in each of $\{\hat{\rho}_i\}_i, \{\hat{\rho}'_i\}_i, \cdots$.

6. In particular, if $\hat{\rho}$ has $q = 2$ there exists a pure state(s) with $p = 2$ [which means $q = 2$ according to (C18)] in every pure-states decomposition. That is, when
\[ \hat{\rho} = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i| = \sum_i \lambda'_i |\psi'_i\rangle \langle \psi'_i| = \cdots , \tag{C25} \]
where $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$ and similarly for $\lambda'_i$, then there exists a pure state with $p = 2$ in each of $\{|\psi_i\rangle\}_i, \{|\psi'_i\rangle\}_i, \cdots$. Here, the pure states in the decomposition are not necessarily orthogonal to each other; e.g., we do not assume $\langle \psi_i | \psi_j \rangle = 0$ for $i \neq j$.

7. In every pure-state decomposition, if $\hat{\rho}$ has $q = 2$ then pure states with $p = 2$ should be contained with a significant weight, i.e.,
\[ \sum_{i \in p = 2} \lambda_i = \Theta(1). \]
This is a necessary condition for $q = 2$.

8. A sufficient condition for $q = 2$ is as follows. For an additive operator $\hat{A}$, suppose that pure states $|\psi_1\rangle, |\psi_2\rangle, \cdots$ satisfy
\[ \langle \psi_i | \psi_j \rangle = \delta_{i,j} \text{ for } i, j = 1, 2, \cdots , \tag{C26} \]
\[ \langle \psi_i | \hat{A} | \psi_j \rangle = 0 \text{ for } i \neq j, \tag{C27} \]
\[ \langle \psi_i | (\Delta_i \hat{A})^2 | \psi_i \rangle = \Theta(N^2) \text{ for } i \leq \Lambda, \tag{C28} \]
\[ \langle \psi_i | (\Delta_i \hat{A})^2 | \psi_i \rangle < \Theta(N^2) \text{ for } i > \Lambda, \tag{C29} \]
where \( \Delta_i \hat{A} \equiv \hat{A} - \langle \psi_i | \hat{A} | \psi_i \rangle \) and \( \Lambda \) is a positive integer. Consider a classical mixture of these states, \( \hat{\rho} = \sum_i \lambda_i |\psi_i \rangle \langle \psi_i | \), where \( 0 \leq \lambda_i \leq 1 \) and \( \sum_i \lambda_i = 1 \). If
\[
\sum_{i \leq \Lambda} \lambda_i = \Theta(1),
\]
then any such a mixture has \( q = 2 \).

4. Examples

In the example of Appendix C of (C2) satisfies all conditions (C26–C28) for \( \hat{A} = \hat{M}_z = \sum_r \hat{\sigma}_z(r) \) and \( \Lambda = N \). Therefore, according to property S of Appendix C of (C3) any mixtures of these states, such as \( \hat{\rho}_{ex1} \) of (C1), have \( q = 2 \). This is consistent with the result on \( \hat{\rho}_{ex1} \) in Appendix C of (C3) where we used the witness observable \( \hat{W} \) that is explicitly given by (C6). By using \( q \), we have obtained the same conclusion without using an explicit form of a witness observable.

Another instructive example is the case where
\[
| \varphi_i \rangle \equiv (|i \rangle + |\bar{i} \rangle) / \sqrt{2},
\]
where \( |i \rangle \) (\( |\bar{i} \rangle \)) is an arbitrary state in which \( i \) spins are up (down) and \( N - i \) spins are down (up). If we limit the range of \( i \) over, say, \( 1 \leq i \leq N/3 \), then conditions (C26)–(C28) are all satisfied for \( \hat{A} = \hat{M}_z \) and \( \Lambda = N/3 \). Therefore, any mixtures of these states, such as
\[
\hat{\rho}_{ex3} \equiv \frac{3}{N} \sum_{i=1}^{N/3} |\varphi_i \rangle \langle \varphi_i |,
\]
have \( q = 2 \). Intuitively, such mixtures are mixtures of the same sort of superpositions of macroscopically distinct states in the sense that all \( |\varphi_i \rangle \)'s are superpositions of states with positive and negative \( \hat{M}_z \).

Furthermore, consider mixtures of \( \hat{\rho}_{ex1}, \hat{\rho}_{ex3} \) and \( \hat{\rho}_{ex2} \) of (C21):
\[
\hat{\rho}_{ex1} \equiv w \hat{\rho}_{ex1} + (1 - w) \hat{\rho}_{ex2},
\]
\[
\hat{\rho}_{ex3} \equiv w \hat{\rho}_{ex3} + (1 - w) \hat{\rho}_{ex2}.
\]
They also have \( q = 2 \) if \( w > 0 \) and independent of \( N \), because conditions (C26)–(C28) are all satisfied. This may be understood because they contain states with \( q = 2 \) with significant weights.

These properties and examples show that \( q \) is a reasonable index for a generalized cat state. One can identify such a state by measuring \( \hat{C}_{\hat{A}, \hat{\eta}} \) (correlation of local observables) for an appropriate pair of \( \hat{A}, \hat{\eta} \); hence, the state tomography is unnecessary.

5. Quantifying superpositions of macroscopically distinct states

The index \( q \) is defined as the power of \( N \) in \( \langle \hat{C}_{\hat{A}, \hat{\eta}} \rangle \) because for large \( N \) the power is, obviously, more important than the coefficient. But, if necessary, one can use the value of \( \langle \hat{C}_{\hat{A}, \hat{\eta}} \rangle \) to quantify superpositions of macroscopically distinct states, as follows[90].

Consider two quantum states that are generalized cat states of the same additive observable \( \hat{A} \), say \( \hat{M}_z \). Let \( |m \rangle \) satisfy \( \hat{M}_z |m \rangle = m |m \rangle \). Then, a pure state \( \hat{\rho}_N := \frac{1}{2}(|N \rangle + |-N \rangle)(\langle N | + \langle -N |) \) gives \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle = 2N^2 \) when \( \hat{\eta} = \hat{\rho}_N \) (which maximizes \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \)). And another pure state \( \hat{\rho}_{N/2} := \frac{1}{2}(|N/2 \rangle + |-N/2 \rangle)(\langle N/2 | + \langle -N/2 |) \) gives \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle = N^2/2 \) when \( \hat{\eta} = \hat{\rho}_{N/2} \) (which maximizes \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \)). From these observations, we see that \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \) quantifies how distinct the values of \( \hat{M}_z \) between the states that are superposed in pure cat states. As the difference of the eigenvalues (\( m \)'s) of superposed state becomes larger, \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \) also becomes larger.

On the other hand, a mixed state \( \frac{1}{2} \hat{\rho}_N + \frac{1}{2} \hat{\rho}_{\text{non-cat}} \) gives \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle = N^2 + O(N) \) when \( \hat{\eta} = \hat{\rho}_N \) (which maximizes the leading-order term, i.e., the \( O(N^2) \) term, of \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \)). Hence, we also see that \( \langle \hat{C}_{\hat{M}_z, \hat{\eta}} \rangle \) decreases with decreasing the ratio of pure cat states in the mixed state.
As seen from these examples, $\langle \hat{C}_{M, \theta} \rangle$ quantifies superpositions of macroscopically distinct states, reflecting the above two factors.

Applying this idea to our generalized cat state for free spins, we see that the factor $\tanh^2 \beta h$ plays the essential role, quantifying the contribution from pure cat states. The ratio of the generalized cat state at $T > 0$ and the generalized cat state at $T = 0$, which is a pure cat state, is $[2N + (N^2 - M^2) \tanh^2 \beta h]/[2N + N^2 - M^2]$ which behaves $\sim \tanh^2 \beta h$ for large $N$. Therefore, the factor $\tanh^2 \beta h$ reflects the decrease of the contribution of pure cat states when increasing the temperature.

### Appendix D: Purity of the post-measurement state

For free spins, the post-measurement state is

$$\hat{\rho}_0^{\text{post}} = \frac{\hat{P}_z \hat{\rho}_0^{\text{eq}} \hat{P}_z}{\text{Tr}[\hat{P}_z \hat{\rho}_0^{\text{eq}} \hat{P}_z]} = \frac{\hat{P}_z e^{-\beta \hat{H}_0} \hat{P}_z}{Z_0^{\text{post}}(\beta h)}. \quad (D1)$$

Purity of this state is calculated as

$$\text{Tr}[\hat{\rho}_0^{\text{post}}^2] = \frac{\text{Tr}[\hat{P}_z e^{-\beta \hat{H}_0} \hat{P}_z e^{-\beta \hat{H}_0} \hat{P}_z]}{Z_0^{\text{post}}(\beta h)} \quad (D2)$$

$$\leq \frac{\text{Tr}[\hat{P}_z (e^{-\beta \hat{H}_0})^2 \hat{P}_z]}{Z_0^{\text{post}}(\beta h)} \quad (D3)$$

$$= \frac{Z_0^{\text{post}}(2\beta h)}{Z_0^{\text{post}}(\beta h)^2}. \quad (D4)$$

Since

$$Z_0^{\text{post}}(\beta h) = \left( \frac{N}{(N + M)/2} \right)^{\cosh N(\beta h)}, \quad (D5)$$

the right-hand side of (D4) is calculated as

$$2^N \frac{N^N}{(N + M)^{N^2}} \frac{2^N}{2N} \frac{2^N}{2N} = 2^N \frac{1}{2N} \frac{1}{N^2} \lim_{N \to \infty} \left[ \frac{[1 + 2/(e^{2\beta h} + e^{-2\beta h})]^N}{1 + 1/\cosh(\beta h)} \right] \quad (D6)$$

Using the Stirling formula, we obtain

$$\frac{2^N}{(N + M)^{N^2}} \simeq \sqrt{\frac{\pi N}{2}} \exp(M^2/2N) \text{ when } |M| \lesssim \sqrt{N}. \text{ Hence when } |M| \lesssim \sqrt{N} \text{ and } \beta h = \Theta(1) \text{ (independent of } N)$$

$$\text{Tr}[\hat{\rho}_0^{\text{post}}^2] \leq \frac{1}{e^{\Theta(N)}}. \quad (D8)$$

Thus the post-measurement state is a mixture of $e^{\Theta(N)}$ states. This is due to the properties of the pre-measurement state and the projection operator $\hat{P}_z$. The pre-measurement state $\hat{\rho}_0^{\text{eq}}$ is a mixture of $e^{\Theta(N)}$ states because its entropy $-\text{Tr}[\hat{\rho}_0^{\text{eq}} \ln \hat{\rho}_0^{\text{eq}}]$ is $\Theta(N)$ when $T > 0$. $\hat{P}_z$ is a projection onto $e^{\Theta(N)}$ dimensional space because

$$\text{Tr}[\hat{P}_z] = \left( \frac{N}{(N + M)/2} \right)^N \sim 2^N \quad (D9)$$

when $M \sim 0$.

When $\beta h \to \infty$, on the other hand, the pre-measurement state is a ground state, i.e., a pure state. In this case, the post-measurement state is also a pure state.
Appendix E: Calculation of energy for free spins

Here we calculate \( \langle \hat{H}_0 \rangle_{\text{post}} \) and \( \langle (\Delta \hat{H}_0)^2 \rangle_{\text{post}} \) for the post-measurement state. Using \( \hat{\sigma}_x \mid \uparrow \rangle = \mid \downarrow \rangle \) and \( \hat{\sigma}_x \mid \downarrow \rangle = \mid \uparrow \rangle \), we can calculate \( \langle \hat{H}_0 \rangle_{\text{post}} = \text{Tr} \left[ \hat{\rho}_{\text{post}} \hat{H}_0 \right] \) easily as

\[
Z_{\text{post}}^0(\beta, M) \langle \hat{H}_0 \rangle_{\text{post}} = \text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}_0} \hat{P}_z \hat{H}_0 \right] = -\hbar \sum_\xi \langle M, \xi | e^{-\beta \hat{H}_0} \hat{P}_z \hat{M}_x | M, \xi \rangle = 0, \tag{E1}
\]

where \( \hat{M}_z | M, \xi \rangle = M | M, \xi \rangle \) and \( \xi \) labels degenerate eigenstates. The last line comes from the fact that \( \hat{M}_x | M, \xi \rangle = \sum_{i=1}^N \hat{\sigma}_i^x | M, \xi \rangle \) is a sum of the states that differ from \( | M, \xi \rangle \) by one spin being flipped. Such states are also eigenstates of \( \hat{M}_z \), but their eigenvalues are not \( M \). Therefore, \( \hat{P}_z \hat{M}_x | M, \xi \rangle = 0 \).

Similarly, using

\[
e^{-\beta \hat{H}_0} = \left( \begin{array}{cc} \cosh(\beta h) & \sinh(\beta h) \\ \sinh(\beta h) & \cosh(\beta h) \end{array} \right) ^\otimes N \tag{E4}
\]

and

\[
\langle \uparrow | \left( \begin{array}{cc} \cosh(\beta h) & \sinh(\beta h) \\ \sinh(\beta h) & \cosh(\beta h) \end{array} \right) | \downarrow \rangle = \langle \downarrow | \left( \begin{array}{cc} \cosh(\beta h) & \sinh(\beta h) \\ \sinh(\beta h) & \cosh(\beta h) \end{array} \right) | \uparrow \rangle = \sinh(\beta h), \tag{E5}
\]

\( Z_{\text{post}}^0(\beta, M) \langle \hat{H}_0^2 \rangle_{\text{post}} \) is calculated as

\[
Z_{\text{post}}^0(\beta, M) \langle \hat{H}_0^2 \rangle_{\text{post}} = \text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}_0} \hat{P}_z \hat{H}_0^2 \right] = \hbar^2 \sum_\xi \langle M, \xi | e^{-\beta \hat{H}_0} \hat{P}_z \hat{M}_z^2 | M, \xi \rangle = \hbar^2 Z_{\text{post}}^0(\beta, M) \left( N + \frac{N^2 - M^2}{2} \tanh^2(\beta h) \right). \tag{E8}
\]

Thus we have

\[
\langle (\Delta \hat{H}_0)^2 \rangle_{\text{post}} = \langle \hat{H}_0^2 \rangle_{\text{post}} - \langle \hat{H}_0 \rangle_{\text{post}}^2 = \hbar^2 \left( N + \frac{N^2 - M^2}{2} \tanh^2(\beta h) \right) = \Theta(N^2), \tag{E9}
\]

\( q = 2 \) when

\[
N^2 - M^2 = \Theta(N^2). \tag{E10}
\]

Appendix F: Calculation for the XYZ model

For \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = -\hbar \hat{M}_x - \sum_{i=1}^N (J_x \hat{\sigma}_i^x \hat{\sigma}_i^{x+1} + J_y \hat{\sigma}_i^y \hat{\sigma}_i^{y+1} + J_z \hat{\sigma}_i^z \hat{\sigma}_i^{z+1}) \), the ‘partition function’ and the density matrix of the post-measurement state are

\[
Z_{\text{post}} := \text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}} \hat{P}_z \right], \tag{F1}
\]

\[
\hat{\rho}_{\text{post}} := \frac{\hat{P}_z e^{-\beta \hat{H}} \hat{P}_z}{Z_{\text{post}}}. \tag{F2}
\]

According to the discussion on the general systems and states in Sec. VII, \( \langle \hat{C} \rangle_{\text{post}} = \Theta(N^2) \) for interacting spins. Let us take a look at the coefficients of the \( \Theta(N^2) \) term.
1. Calculation of $Z_{\text{post}}$

We are going to calculate $Z_{\text{post}}$ up to $\beta^2$ order. This will give a good result when

$$|\beta J| \ll 1$$  \hspace{1cm} (F3)

and

$$|\beta h| \ll 1,$$  \hspace{1cm} (F4)

where $J = (J_x, J_y, J_z)$. Using the notation

$$\hat{K}_\alpha := -J_\alpha \sum_{i=1}^{N} \hat{\sigma}_\alpha^i \hat{\sigma}_{\alpha}^{i+1}$$  \hspace{1cm} (F5)

for $\alpha = x, y, z$, we can expand $e^{-\beta \hat{H}}$ as

$$e^{-\beta \hat{H}} = 1 - \beta \hat{H} + \frac{\beta^2}{2} \hat{H}^2 + O(\beta^3).$$  \hspace{1cm} (F6)

Substituting this into the definition of $Z_{\text{post}}$ and dropping the terms that are obviously zero, we have

$$Z_{\text{post}} = \sum_\xi \langle M, \xi | e^{-\beta \hat{H}} | M, \xi \rangle$$

$$= \sum_\xi \langle M, \xi | 1 - \beta \hat{K}_z + \frac{\beta^2}{2} \left( \hat{K}_x^2 + \hat{K}_y^2 + \hat{H}_0^2 - 2 J_x J_y \sum_{i=1}^{N} \hat{\sigma}_z^i \hat{\sigma}_{z}^{i+1} \right) | M, \xi \rangle + O(\beta^3)$$  \hspace{1cm} (F7)

$$= \left( \frac{N}{N + M} \right) \left( 1 + \frac{\beta^2}{2} \left( N(J_x^2 + J_y^2 + h^2) - (\beta J_z + \beta^2 J_x J_y) \sum_{\xi} \langle M, \xi | \hat{\sigma}_z^i \hat{\sigma}_{z}^{i+1} | M, \xi \rangle \right) + \frac{\beta^2 J_z^2}{2} \sum_{\xi} \sum_{i,j} \langle M, \xi | \hat{\sigma}_z^i \hat{\sigma}_{z}^{i+1} \hat{\sigma}_z^j \hat{\sigma}_{z}^{j+1} | M, \xi \rangle + O(\beta^3) \right).$$  \hspace{1cm} (F8)

After some algebra, we have

$$Z_{\text{post}} = \left( \frac{N}{N - M} \right) \left( 1 + \frac{\beta^2}{2} \left( N(J_x^2 + J_y^2 + h^2) - (\beta J_z + \beta^2 J_x J_y) \frac{M^2 - N}{N - 1} \right) \right)$$

$$+ \frac{\beta^2 J_z^2}{2} \left( -2M^4(N - 4) + (N - 3)(N - 2)N(N(N + 4) + 16) + M^2(N(N(N - 15) + 90)) - 112 \right)$$

$$\frac{2(M - N - 2)(N - 2)(N - 1)(N + M + 2)}{(N - M)} \left( 1 - \beta J_z \frac{M^2 - N}{N - 1} + O(\beta^2) \right).$$  \hspace{1cm} (F9)

As we will see later, details of the $O(\beta^2)$ terms of $Z_{\text{post}}$ will not be necessary for the purpose of knowing $\text{Tr} \left[ \hat{\rho}_{\text{post}} \hat{C}_{M_z} \hat{P}_z \right]$ up to $\beta^2$ order.

2. Index $q$ of $\hat{\rho}_{\text{post}}$

To evaluate $\text{Tr} \left[ \hat{\rho}_{\text{post}} \hat{C}_{M_z} \hat{P}_z \right] = 2 \text{Tr} \left[ \hat{\rho}_{\text{post}} \hat{\mathcal{Z}}_{\text{post}} \right]$, we evaluate $\text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}} \hat{P}_z M_x^2 \right]$. Assuming (F3) and (F4), we obtain

$$\text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}} \hat{P}_z M_x^2 \right]$$

$$= NZ_{\text{post}} + 2 \left( \frac{N}{N - M} \right) \left( -\beta J_x \frac{N^2 - M^2}{2(N - 1)} - \beta J_y \frac{N^2 - M^2}{2(N - 1)} \right)$$

$$+ \beta^2 J_x^2 \frac{N^2 - M^2}{2(N - 1)} + \beta^2 J_y^2 \frac{N^2 - M^2}{2(N - 1)} + \beta^2 h^2 \frac{N^2 - M^2}{4}$$

$$+ \beta^2 J_z (J_x + J_y) \frac{(N^2 - M^2)(M^2 - 2N + 4)}{2(N - 1)(N - 2)}. \hspace{1cm} (F12)$$
With this and $Z_{\text{post}}$ from the previous subsection, $\langle \hat{C}_{M_z,\hat{P}_z} \rangle_{\text{post}}$ for the post-measurement state is obtained as

$$
\langle \hat{C}_{M_z,\hat{P}_z} \rangle_{\text{post}} = 2\text{Tr} \left[ \hat{P}_z e^{-\beta \hat{H}} \hat{P}_z \hat{M}_z^2 \right] / Z_{\text{post}}
$$

$$= 2N - 2\beta \frac{N^2 - M^2}{N - 1} (J_x + J_y)
$$

$$+ 2\beta^2 (N^2 - M^2) \left( \frac{k^2}{2} + \frac{J_z^2 + J_y^2}{N - 1} \right) + O(\beta^3)$$

$$= 2N + \beta O(N) + \beta^2 h^2 (N^2 - M^2) + \beta^2 O(N) + O(\beta^3).$$

The third term in the right-hand side indicates that up to the order of $\beta^2$, the coefficient of the $O(N^2)$ term does not depend on $J$.

When $J_y = J_z$ (=: $J_\perp$), we can improve (F15) by expanding only $\exp(-\beta \hat{H}_\text{int})$ since $e^{-\beta \hat{H}} = \exp(-\beta \hat{H}_0) \exp(-\beta \hat{H}_\text{int})$. Calculating in the same manner, we obtain

$$\langle \hat{C}_{M_z,\hat{P}_z} \rangle_{\text{post}} = 2N + (N^2 - M^2) \tanh^2(\beta h) - 2\beta (J_x + J_\perp) \frac{N^2 - M^2}{N - 1}
$$

$$+ 2\beta^2 (N^2 - M^2) \left( \frac{J_\perp^2}{N - 1} + \frac{J_z J_\perp (M^2 - N^2 + 4N - 4)}{(N - 1)^2(N - 2)} \right) + O(\beta^3).$$

The second term in the right-hand side indicates that up to the order of $\beta^2$, the coefficient of the $O(N^2)$ term is the same for the free spins. This result is useful when a strong magnetic field is applied, i.e., $|\beta h| \gg 1$, while the interactions between spins are weak, i.e., $|\beta J| \ll 1$.

**Appendix G: Symmetry consideration**

By revisiting the XYZ model from the symmetry point of view, we find that the discussion on the coefficient of $N^2$ is applicable to a broader class of Hamiltonian. To show it, we denote the rotation around $z$ axis by angle $\pi$, by $\hat{R}_z$. Since

$$\hat{R}_z \hat{H}_0 \hat{R}_z^\dagger = -\hat{H}_0,$$

$$\hat{R}_z \hat{H}_\text{int} \hat{R}_z^\dagger = \hat{H}_\text{int},$$

$$\hat{R}_z \hat{P}_z \hat{R}_z^\dagger = \hat{P}_z,$$

we find that $Z_{\text{post}}$ and $\langle \hat{C}_{M_z,\hat{P}_z} \rangle_{\text{post}}$ are even functions of $\beta h$, i.e.,

$$Z_{\text{post}}(\beta h, \beta \mathbf{J}, M) = \text{Tr} [\hat{P}_z e^{-\beta (\hat{H}_0 + \hat{H}_\text{int})} \hat{P}_z]$$

$$= \text{Tr} [\hat{R}_z \hat{P}_z \hat{R}_z^\dagger \hat{R}_z e^{-\beta (\hat{H}_0 + \hat{H}_\text{int})} \hat{P}_z \hat{R}_z \hat{R}_z^\dagger]$$

$$= Z_{\text{post}}(-\beta h, \beta \mathbf{J}, M),$$

$$C(\beta h, \beta \mathbf{J}, M)$$

$$:= \langle \hat{C}_{M_z,\hat{P}_z} \rangle_{\text{post}}$$

$$= 2\text{Tr} [\hat{P}_z e^{-\beta (\hat{H}_0 + \hat{H}_\text{int})} \hat{P}_z \hat{M}_z^2 / Z_{\text{post}}(\beta h, \beta \mathbf{J}, M)]$$

$$= 2\text{Tr} [\hat{R}_z \hat{P}_z \hat{R}_z^\dagger \hat{R}_z e^{-\beta (\hat{H}_0 + \hat{H}_\text{int})} \hat{R}_z \hat{P}_z \hat{R}_z \hat{R}_z^\dagger \hat{R}_z \hat{R}_z \hat{R}_z^\dagger] / Z_{\text{post}}(-\beta h, \beta \mathbf{J}, M)$$

$$= C(-\beta h, \beta \mathbf{J}, M).$$
This implies that, if we expand $C(\beta h, \beta \mathbf{J}, M)$ in a power series of $\beta h$ and $\beta \mathbf{J}$, then

$$C(\beta h, \beta \mathbf{J}, M)$$

$$= C^{(0)}(\beta h, \beta \mathbf{J}, M) + C^{(1)}(\beta h, \beta \mathbf{J}, M) + C^{(2)}(\beta h, \beta \mathbf{J}, M) + \cdots$$  \hspace{1cm} (G11)

$$= C^{(0)}(0, 0, M) + C^{(1)}(0, \beta \mathbf{J}, M) + C^{(2)}(\beta h, 0, M) + \cdots$$  \hspace{1cm} (G12)

According to the result for the free spins, $C(\beta h, 0, M) = (N^2 - M^2) \tanh^2(\beta h)$, and thus $C^{(2)}(\beta h, 0, M) = (N^2 - M^2)(\beta h)^2$, which indicates that up to the order of $\beta^2$, the coefficient of $N^2$ is the same as the free spins for any dimension and any lattice.

We immediately see that we can extend this discussion to a more general Hamiltonian. That is, if a system has the $z$-inv, then, regardless of the dimension nor the details of the lattice, the coefficient of $N^2$ is the same as the free spins for $O(\beta^2)$. Note that the above discussion does not restrict the range of interaction between spins. Hence it is applicable, e.g., even to the systems with long-range interactions.

**Appendix H: Projection onto a finite interval**

1. **Case for free spins**

We consider the free spins $\hat{H}_0$ and the projection operator $\hat{P}_z'$ onto the subspace corresponding to an interval $\alpha_+ N \leq \hat{M}_z \leq \alpha_- N$. We assume $|\alpha_-| < \alpha_+$, without loss of generality. In this case, the ‘partition function’ is

$$Z_{\text{post}}^0 := \text{Tr} \left[ \hat{P}_z' e^{-\beta \hat{H}_0} \hat{P}_z' \right]$$  \hspace{1cm} (H1)

$$= \sum_{|M, \xi \rangle \text{ s.t. } \alpha_+ N \leq M \leq \alpha_- N} \langle M, \xi | e^{-\beta \hat{H}_0} | M, \xi \rangle$$  \hspace{1cm} (H2)

$$= r(\alpha_-, \alpha_+) \cosh^N(\beta h),$$  \hspace{1cm} (H3)

where $|M, \xi \rangle$ is an eigenstate of $\hat{M}_z$, $\xi$ labels degenerate eigenstates, and

$$r(\alpha_-, \alpha_+) := \sum_{k=0}^{N + N_{\alpha_+ N}} \left( \begin{array}{c} N \\ \frac{N_{\alpha_+ N}}{2} \end{array} \right).$$  \hspace{1cm} (H5)

Then the post-measurement state is given by $\hat{\rho}_{\text{post}}^0 := \hat{P}_z' e^{-\beta \hat{H}_0} \hat{P}_z'/Z_{\text{post}}^0$. Taking $\hat{A} = \hat{M}_z$ and $\hat{\eta} = \hat{P}_z'$ for $\langle \hat{C}_{\hat{A}, \hat{\eta}} \rangle_{\text{post}}$ for $\hat{\rho}_{\text{post}}^0$, we have

$$\langle \hat{C}_{\hat{M}_z, \hat{P}_z'} \rangle_{\text{post}}$$

$$= \text{Tr} \left[ \hat{P}_z' e^{-\beta \hat{H}_0} \hat{P}_z' \left( \hat{M}_z^2 \hat{P}_z' - 2 \hat{M}_z' \hat{M}_z + \hat{P}_z' \hat{M}_z^2 \right) \right]/Z_{\text{post}}^0$$

$$= \text{Tr} \left[ \hat{P}_z' e^{-\beta \hat{H}_0} \hat{P}_z' \left( \hat{M}_z^2 \hat{P}_z' - 2 \hat{M}_z' \hat{M}_z + \hat{P}_z' \hat{M}_z^2 \right) \right]/Z_{\text{post}}^0$$

$$= \sum_{|M, \xi \rangle \text{ s.t. } \alpha_+ N \leq M \leq \alpha_- N} \langle M, \xi | e^{-\beta \hat{H}_0} \hat{P}_z' \hat{M}_z (1 - \hat{P}_z') \hat{M}_z | M, \xi \rangle.$$

$$Z_{\text{post}}^0$$
To calculate the right-hand side of this, we count the number of $|M, \xi \rangle$’s that go out of $[\alpha_-, \alpha_+]$ by operation of the first $\hat{M}_x$ [reading (H7) from right to left], and come back by the second $\hat{M}_x$. Then we find

$$
\sum_{|M, \xi \rangle \text{ s.t. } \alpha_+ N \leq M \leq \alpha_+ N} \frac{2 \langle M, \xi | e^{-\beta \hat{H}_0} \hat{P}_z \hat{M}_x \left( \hat{1} - \hat{P}_z \right) \hat{M}_x | M, \xi \rangle}{Z_{\text{post}}^{\alpha_0}}
$$

$$
= \frac{2}{Z_{\text{post}}^{\alpha_0}} \left( \cosh^N(\beta h) \left( \left( \frac{N}{N - \alpha_+ N} \right) \frac{N - \alpha_+ N}{2} + \left( \frac{N}{N + \alpha_+ N} \right) \frac{N + \alpha_+ N}{2} \right) + \sinh^2(\beta h) \cosh^N(\beta h) \left( \left( \frac{N}{N - \alpha_+ N} \right) \frac{N - \alpha_+ N}{2} \frac{N + \alpha_+ N}{2} \right) \right)
$$

$$
= \frac{N^2 \tanh^2(\beta h) I(N, \alpha_+ N, \alpha_- N) + O(N)}{N^2}.
$$

(H8)

where

$$
I(N, \alpha_+ N, \alpha_- N) := \frac{r(\alpha_+ N, \alpha_- N)(1 - \alpha_+^2)}{2r(\alpha_+ N, \alpha_- N)} + \frac{r(\alpha_- N, \alpha_- N)(1 - \alpha_-^2)}{2r(\alpha_- N, \alpha_- N)}.
$$

(H10)

The right-hand side of Eq. 8 becomes $\Theta(N^2)$ if $I(N, \alpha_+ N, \alpha_- N)$, which is by definition between 0 and 1, is $\Theta(1)$. Thus we investigate the conditions for $\alpha_+ N$ and $\alpha_- N$ to satisfy it. Since $r(\alpha_- N, \alpha_- N) > r(\alpha_+ N, \alpha_+ N)$, it is necessary and sufficient for $\langle \hat{C}_{\hat{M}_x} \hat{P}_z \rangle = \Theta(N^2)$ that $r(\alpha_- N, \alpha_- N)(1 - \alpha_+^2)/r(\alpha_- N, \alpha_+ N) = \Theta(1)$. We define $g(x)$ as follows:

$$
g(x) := r(x N, x N) = \left( \frac{N}{N + x} \right).
$$

(H11)

Using the Stirling formula, we have

$$
g(x) \sim \frac{1}{\sqrt{\pi(1 - x^2)} N/2} \left( \frac{2}{1 + x} \right)^{\frac{1 + x}{2} N} \left( \frac{2}{1 - x} \right)^{\frac{1 - x}{2} N}
$$

$$
= \frac{1}{\sqrt{\pi(1 - x^2)} N/2} \exp \left[ N \left( \frac{1 + x}{2} \ln \frac{2}{1 + x} + \frac{1 - x}{2} \ln \frac{2}{1 - x} \right) \right].
$$

(H12)

(H13)

When $x \ll 1$, in particular,

$$
g(x) \simeq \frac{2^N}{\sqrt{\pi N/2}} e^{-\frac{N + x}{2}}.
$$

(H14)

We find that $g(x)$ is convex up when $x < 1/\sqrt{N - 1} \sim 1/\sqrt{N}$, and convex down when $x > 1/\sqrt{N - 1} \sim 1/\sqrt{N}$. Using these, we can obtain the following conditions for obtaining a generalized cat state.

- When $\alpha_- > 1/\sqrt{N}$ and $\alpha_- \to 0$ ($N \to \infty$), $\alpha_+ - \alpha_- \leq \Theta(1/N)$ is the necessary and sufficient condition for $r(\alpha_- N, \alpha_- N)/r(\alpha_- N, \alpha_+ N) = \Theta(1)$.

- When $\alpha_+ < 1/\sqrt{N}$ and $\alpha_- > 0$, convergence of $N(\alpha_+ - \alpha_-)$ to a positive constant is the necessary and sufficient condition.

- When $0 < \alpha_- \leq 1/\sqrt{N} \leq \alpha_+$, $\alpha_+ - \alpha_- = \Theta(1/N)$ is the necessary and sufficient condition.

- Using the results we have obtained, we can show that $\alpha_+ \leq \Theta(1/N)$ is the necessary and sufficient condition when $\alpha_- \leq 0 \leq \alpha_+$.

- When $\alpha_- = \Theta(1)$ and $1 - \alpha_- = \Theta(1)$, generalized cat state can be obtained for any $\alpha_+$.
2. Case for interacting spins

It is easy to show that the discussion on \( \hat{P}'_z \) holds the same for the XYZ model. Using the symmetry
\[
\hat{R}_x \hat{P}'_z \hat{R}_x^\dagger = \hat{P}'_z.
\]
we find that
\[
\mathcal{Z}'_{\text{post}} := \text{Tr}[\hat{P}'_z e^{-\beta(\hat{H}_0 + \hat{H}_{\text{int}})} \hat{P}'_z]
\]
and
\[
\hat{P}'_{\text{post}} := \text{Tr}[\hat{P}'_z e^{-\beta(\hat{H}_0 + \hat{H}_{\text{int}})} \hat{P}'_z] / \mathcal{Z}'_{\text{post}}
\]
are even functions of \( \beta \hbar \), since \( \hat{R}_x \hat{H}_0 \hat{R}_x^\dagger = -\hat{H}_0 \) and \( \hat{R}_x \hat{H}_{\text{int}} \hat{R}_x^\dagger = \hat{H}_{\text{int}} \). Following the discussion in Appendix I (with \( \hat{P}_z \) replaced with \( \hat{P}'_z \)), we expect that if the conditions for \( q = 2 \) for \( \hat{H}_0 \) are satisfied, then \( q = 2 \) for \( \hat{H}_0 + \hat{H}_{\text{int}} \).

Appendix I: Feasibility in an experiment

A single NV\(^-\) center in diamond is known to have a long coherence time such as \( \tau = 470 \) \( \mu \)s at room temperature \(^{67,68}\). Let us consider a system composed of \( N \) NV\(^-\) centers.

Let \( N(t) \) be the number of NV\(^-\) centers that maintain coherence at time \( t \). When decoherence occurs independently in individual NV\(^-\) centers,
\[
N(t) = N \exp(-t/\tau).
\]
We are interested in how long all the spins maintain coherence, which defines the coherence time \( \tau_{\text{coh}} \) of this system. Thus we solve \( N(\tau_{\text{coh}}) = N - 1 \). When \( N \gg 1 \), we obtain
\[
\tau_{\text{coh}} = \tau/N.
\]
Therefore, \( \tau_{\text{coh}} = 4.7 \) \( \mu \)s and 47 ns for \( N = 100 \) and \( N = 10^4 \), respectively. Taking 3.5 \( \mu \)s (35 ns) as a duration of the measurement for \( N = 100 \) (\( N = 10^4 \)), we estimate how close the SERF magnetometer \(^{92}\) should be to detect a magnetic field by one spin.

The SERF magnetometer realized by a group at Princeton \(^{92}\) has magnetic field sensitivity \( \delta B = 160 \text{ aT}/\sqrt{\text{Hz}} \). This means, if the measurement lasts for \( 3\tau_{\text{coh}}/4 \approx 3.5 \) \( \mu \)s, the magnetometer can detect the magnetic field with
\[
(160 \times 10^{-18} \text{ T}/\sqrt{\text{Hz}}) \times \left( \frac{1}{2 \times 3.5 \times 10^{-6} \text{ s}} \right)^{1/2} = 60 \text{ fT}
\]
sensitivity. (600 fT if 35 ns.)

One spin has a magnetic moment of \( \mu_B = 9.3 \times 10^{-24} \text{ A m}^2 \). In a vacuum, the magnetic field \( B \) made by the spin is, when the distance from the spin is \( r \),
\[
B = \frac{\mu_B \mu_0}{2\pi r^3},
\]
where \( \mu_0 = 1.2 \times 10^{-6} \text{ T m/A}. \) If \( r = 3 \times 10^{-6} \text{ m} = 3 \mu \text{m} \), then
\[
B = 65 \text{ fT}.
\]
This is larger than the estimation of \(^{13}\). Thus, for \( N = 100 \), \( \dot{M}_z \) measurement of \( \Theta(1) \) resolution is possible if we assume that the outcome of the measurement is the time average of the measured magnetic field. After the measurement, a generalized cat state should be obtained, and it survives for the rest of the system’s coherence time, e.g., \( \tau_{\text{coh}}/4 \approx 1.2 \mu \)s.

For \( N = 10^4 \), smaller \( r \) is necessary since the coherent time \( \tau_{\text{coh}} \) of the system decreases. We find that \( r = 1 \mu \text{m} \) is sufficient. With \( r = 1 \mu \text{m} \), the magnetic field which one spin makes is \( B = 1.8 \text{ pT} \), which is larger than 1.7 pT, the minimum magnetic field the SERF can resolve within \( \tau_{\text{coh}}/10 \) for \( N = 10^4 \). Then the obtained cat state survives for \( 9\tau_{\text{coh}}/10 \). For \( N = 100 \) and \( r = 1 \mu \text{m} \), we find that \( \tau_{\text{coh}}/10^3 \) is sufficient for a measurement time, resulting in a long lifetime of the obtained cat state.

As discussed in \(^{15}\), decoherence rate could be, in principle, as large as \( \Theta(N^2) \). That is, coherence time of the obtained generalized cat state could be \( t = \tau/N^2 \). When there is such a noise, the above estimation for \( N = 100 \) is replaced with the case for \( N = 10^4 \). That is, for \( N = 100 \), distance of the magnetometer must be \( r = 1 \mu \text{m} \) and the duration of the measurement must be 4.7 ns.
Appendix J: Verification of the generalized cat state in experiments

1. Expression of $\hat{C}_{M_\alpha P_\alpha}$ by the Pauli operators

Here, as an illustration, we express $\hat{C}_{M_\alpha P_\alpha}$ of the free spins $\hat{H}_0$ of $N = 4$ spins with the outcome of the measurement $M_\alpha = 0$ by the Pauli operators, since what are easily measured in experiments would be $\sigma_\alpha (\alpha = x, y, z)$ of each spin. Then projection operator $\hat{P}_z$ is

$$\hat{P}_z = \left( \frac{\sum_{i=1}^{4} \sigma_z^i - 1}{4} \right) \left( \frac{\sum_{i=1}^{4} \sigma_z^i + 1}{4} \right) \left( \frac{\sum_{i=1}^{4} \sigma_z^i}{2} - 1 \right) \left( \frac{\sum_{i=1}^{4} \sigma_z^i}{2} + 1 \right) \tag{J1}$$

Then

$$\hat{C}_{M_\alpha P_\alpha} = [\hat{M}_x, [\hat{M}_x, \hat{P}_z]] \tag{J3}$$
$$= [\sum_{i=1}^{N} \sigma_z^i, [\sum_{i=1}^{N} \sigma_z^i, -\frac{1}{8} \sum_{l<m} \sigma_z^l \sigma_z^m + \frac{3}{8} \prod_l \sigma_z^l + 3 \frac{1}{8}]] \tag{J4}$$
$$= -\sum_{l<m} (\sigma_z^l \sigma_z^m - \sigma_z^l \sigma_z^m) + 3(2\sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x - \sigma_1^y \sigma_2^y \sigma_3^y \sigma_4^y - \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z - \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z - \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x - \sigma_1^y \sigma_2^y \sigma_3^y \sigma_4^y - \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z - \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z). \tag{J5}$$

The term $\sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$ can be measured, for example, by measuring $\sigma_z^1, \sigma_z^2, \sigma_z^3,$ and $\sigma_z^4$ simultaneously. This measurement also gives $\sigma_z^l \sigma_z^m$ for all pairs of $l, m$. On the other hand, $\sigma_1^y$ and $\sigma_1^z$ cannot be measured simultaneously (with vanishing errors) because of their noncommutativity. Hence, Eq. (J5) implies that to measure $\hat{C}_{M_\alpha P_\alpha}$ (and thereby to confirm the success of the conversion into a generalized cat state) for $N = 4$ and $M = 0$, it is sufficient to perform seven types of simultaneous measurement corresponding to the last seven terms in Eq. (J3).

2. How many types of measurements are necessary for general $N$?

We assume that $N$ is an even number. Then, the projection operator onto $\hat{M}_\alpha = M$ subspace is

$$\hat{P}_z := \prod_{L(\neq M)} \hat{M}_z - L \frac{M - L}{M - L} \tag{J6}.$$ 

Hence

$$\hat{C}_{M_\alpha P_\alpha} = [\hat{M}_x, [\hat{M}_x, \hat{P}_z]] = [\hat{M}_x, [\hat{M}_x, \sum_{i=1}^{N} a_i \hat{M}_z^i]] \tag{J7}$$

for a certain set of $\{a_i\} (l = 1, 2, ...N)$. To find how many types of measurements are necessary, we look at $l = N$ term. In particular, we investigate $[\hat{M}_x, [\hat{M}_x, \prod_{i=1}^{N} \sigma_z^i]]$. Using $[\sigma_x, \sigma_z] = -2i\sigma_y$ and $[\sigma_y, \sigma_z] = 2i\sigma_x$, we find that $[\hat{M}_x, [\hat{M}_x, \prod_{i=1}^{N} \sigma_z^i]]$ will give $\prod_{i=1}^{N} \sigma_z^i$ [one type], and products of $N - 2$ $\sigma_z$’s and two $\sigma_y$’s $[\binom{N}{2}$ types]. Thus, one must perform at most $(N^2 - N)/2 + 1$ types of simultaneous measurement.

Appendix K: Pre-measurement state may be non-equilibrium

1. When pre-measurement state is $\hat{P}_x$

We consider a quantum system with $N$ spins. Dimension of the Hilbert space is $D := 2^N$. To a state $\hat{\rho} := \frac{1}{D} \mathbf{1}$, \tag{K1}
we first operate a projection operator \( \hat{P}_x := \sum_\nu |M_x, \nu \rangle \langle M_x, \nu| \) onto the \( \hat{M}_x = M_x \) subspace, and then operate a projection operator \( \hat{P}_2 := \sum_\xi |M_z, \xi \rangle \langle M_z, \xi| \) onto the \( \hat{M}_z = M_z \) subspace, where \( \nu \) and \( \xi \) label degenerate eigenstates of \( \hat{M}_x \) and \( M_z \) respectively.

The state after operating \( \hat{P}_z \) is

\[
\hat{\rho}_{M_z} := \frac{\hat{P}_z \hat{\rho} \hat{P}_z}{\text{Tr} \left[ \hat{P}_z \hat{\rho} \hat{P}_z \right]}
\]

\[= \frac{\hat{P}_z \hat{1} \hat{P}_z}{\text{Tr} \left[ \hat{P}_z \right]}
\]

\[= \frac{\hat{P}_z}{(\hat{z}^N_{M_x})}.
\]

To this \( \rho_{M_z} \), \( \hat{P}_z \) is operated. Then the state becomes

\[
\hat{\rho}_{M_z} := \frac{\hat{P}_z \hat{\rho} \hat{P}_z}{\text{Tr} \left[ \hat{P}_z \hat{\rho} \hat{P}_z \right]}
\]

\[= \frac{\sum_\xi \xi |M_z, \xi \rangle \langle M_z, \xi| \left( \sum_\nu |M_x, \nu \rangle \langle M_x, \nu| \right) |M_z', \xi \rangle \langle M_z', \xi|}{\text{Tr} \left[ \hat{P}_z \hat{P}_z \hat{P}_z \right]}
\]

\[
\text{Does this } \hat{\rho}_{M_z} \text{ have } q = 2? \text{ Taking } \hat{A} = \hat{M}_z \text{ and } \eta = \hat{P}_z, \text{ we calculate } \langle \hat{C}_A \hat{\eta} \rangle \text{ as}
\]

\[
\text{Tr} \left[ \hat{\rho}_{M_z} \hat{C}_{M_z, \hat{P}_z} \right]
\]

\[= \frac{2 \text{Tr} \left[ \hat{P}_z \hat{P}_z \hat{P}_z \hat{M}_z^2 \right]}{\text{Tr} \left[ \hat{P}_z \hat{P}_z \hat{P}_z \right]}
\]

\[= 2N + \frac{4 \sum_\xi \sum_{|M_z, \xi}\langle M_z, \xi| \hat{P}_z^2 |M_z, \xi|'}{\text{Tr} \left[ \hat{P}_z \hat{P}_z \hat{P}_z \right]},
\]

where \( |M_z, \xi\rangle' \) is a state that differs from \( |M_z, \xi\rangle \) by one \( \uparrow \rangle \) and one \( \downarrow \rangle \) being flipped. After some algebra, we obtain

\[
\langle \hat{C} \rangle = 2N + (N^2 - M_z^2) \left( 1 - \frac{N^2 - M_z^2}{N(N-1)} \right)
\]

\[= 2N + (N^2 - M_z^2) \frac{M_z^2 - N}{N(N-1)}.
\]

Thus \( \hat{\rho}_{M_z} \) is a generalized cat state when \( (N^2 - M_z^2) = \Theta(N^2) \) and \( |M_z| = \Theta(N) \).

2. \( \langle \hat{M}_z \rangle_{\text{pre}} = \Theta(N) \) is a sufficient condition

More generally, the pre-measurement state \( \hat{\rho}_{\text{pre}} \) may be arbitrary if it has a macroscopic value of \( \hat{M}_x \), i.e. if \( \langle \hat{M}_z \rangle_{\text{pre}} = \Theta(N) \), because then the conditions are satisfied by the set of \( \{ \hat{M}_z, \hat{M}_y, \hat{M}_z, \hat{\rho}_{\text{pre}} \} \) for the following reason. [We thank M. Koashi for suggesting this point.]

The probability of getting the outcome \( \hat{M}_z = M \) is given by

\[
\text{Pr}(M) = \langle \hat{P}_{z,M} \rangle_{\text{pre}},
\]

(K12)
and the post-measurement state by
\[ \hat{P}_z \hat{\rho}_{\text{pre}} \hat{P}_z / \text{Pr}(M). \]
(K13)

Instead of investigating \( \langle \hat{M}_x^2 \rangle_{\text{post}} \) and \( \langle \hat{M}_y^2 \rangle_{\text{post}} \) separately, we study \( \langle \hat{M}_x^2 + \hat{M}_y^2 \rangle_{\text{post}} \). Furthermore, instead of investigating it for each value of \( M \), we consider its average over \( M \). Since \( \langle \hat{M}_x^2 + \hat{M}_y^2 \rangle_{\text{pre}} = 0 \), it is evaluated as
\[
\sum_M \text{Pr}(M) \langle \hat{M}_x^2 + \hat{M}_y^2 \rangle_{\text{post}} = \sum_M \text{Tr}[\hat{P}_z \hat{\rho}_{\text{pre}} (\hat{M}_x^2 + \hat{M}_y^2)] \\
= \langle \hat{M}_x^2 + \hat{M}_y^2 \rangle_{\text{pre}} \\
\geq \langle \hat{M}_x^2 \rangle_{\text{pre}} \\
\geq (\langle M_x \rangle_{\text{pre}})^2 \\
= \Theta(N^2). \quad \text{(K14)}
\]

This shows that a generalized cat is obtained with non-vanishing probability.

**Appendix L: Time evolution**

After the \( \hat{M}_z \) measurement, the system evolves autonomously with time. In some of the models we have investigated, such as in the free spins, \( \hat{M}_x \) is conserved because \([\hat{H}, \hat{M}_x] = 0 \). Since the post-measurement states are generalized cat states of \( \hat{M}_z \), the state will continue having \( \eta = 2 \) for these models. In fact, if \( \hat{M}_x \) is measured at \( t = 0 \) its post-measurement state \( \hat{\rho}_{\text{post}} \) evolves as \( \hat{U}_t \hat{\rho}_{\text{pre}} \hat{U}_t^\dagger \), where \( \hat{U}_t := e^{-i\hat{H}t} \). Hence, taking \( \hat{A} = \hat{M}_x \) and \( \hat{\eta} = \hat{U}_t \hat{\rho}_{\text{pre}} \hat{U}_t^\dagger =: \hat{P}(t) \) for \( \hat{C}_{\hat{A}_1} \), we have \( \text{Tr}[\hat{U}_t \hat{\rho}_{\text{post}} \hat{U}_t^\dagger \hat{C}_{\hat{M}_x, \hat{P}(t)}] = \text{Tr}[\hat{\rho}_{\text{post}} \hat{C}_{\hat{M}_x, \hat{P}_1}] = \Theta(N^2) \); thus \( \eta = 2 \). Note that taking \( \hat{\eta} = \hat{P}(t) \) is equivalent to taking \( \hat{C}_{\hat{A}_1} = \hat{U}_t \hat{C}_{\hat{M}_x, \hat{P}} \hat{U}_t^\dagger \) because \([\hat{U}_t, \hat{M}_x] = 0 \). Therefore, to verify the generalized cat state, one should measure the observable \([15] \) in which \( \hat{\sigma}_y \) and \( \hat{\sigma}_z \) are replaced with \( \hat{U}\hat{\sigma}_y\hat{U}_t^\dagger \) and \( \hat{U}\hat{\sigma}_z\hat{U}_t^\dagger \), respectively [which are \( \cos(2ht)\hat{\sigma}_y - \sin(2ht)\hat{\sigma}_z \) and \( \sin(2ht)\hat{\sigma}_y + \cos(2ht)\hat{\sigma}_z \) for free spins].

[1] E. Schrödinger, Naturwissenschaften 23, 823 (1935).
[2] A. J. Leggett, Progress of Theoretical Physics Supplement 69, 80 (1980).
[3] A. Shimizu and T. Miyadera, Phys. Rev. Lett. 89, 270403 (2002).
[4] B. Yurke and D. Stöfer, Physical review letters 57, 13 (1986).
[5] N. D. Mermin, Physical Review Letters 65, 1838 (1990).
[6] S. Roy and V. Singh, Physical Review Letters 67, 2761 (1991).
[7] M. Kim and V. Bužek, Physical Review A 46, 4239 (1992).
[8] A. Shimizu and T. Morimae, Physical Review Letters 95, 090401 (2005).
[9] S. Mancini, V. Man’ko, and P. Tombesi, Physical Review A 55, 3042 (1997).
[10] S. Bose, K. Jacobs, and P. Knight, Physical Review A 56, 4175 (1997).
[11] T. Morimae, A. Sugita, and A. Shimizu, Physical Review A 71, 032317 (2005).
[12] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003).
[13] A. Sugita and A. Shimizu, Journal of the Physical Society of Japan 74, 1883 (2005).
[14] M. Paternostro, Physical Review Letters 106, 183601 (2011).
[15] F. Fröwis and W. Dür, New Journal of Physics 14, 093039 (2012).
[16] R. Inoue, S.-I.-R. Tanaka, R. Namiki, T. Sagawa, and Y. Takahashi, Phys. Rev. Lett. 110, 163602 (2013).
[17] T. Morimae and A. Shimizu, Phys. Rev. A 74, 052111 (2006).
[18] U. Akram, W. P. Bowen, and G. J. Milburn, New Journal of Physics 15, 093007 (2013).
[19] M. Vanner, M. Aspelmeyer, and M. Kim, Physical Review Letters 110, 010504 (2013).
[20] M. Arndt and K. Hornberger, Nature Physics 10, 271 (2014).
[21] R. Ghobadi, S. Kumar, B. Pepper, D. Bouwmeester, A. Lvovsky, and C. Simon, Physical Review Letters 112, 080503 (2014).
[22] T. Morimae, Physical Review A 81, 010101 (2010).
[23] G. Tóth and I. Apellaniz, Journal of Physics A: Mathematical and Theoretical 47, 095506 (2014).
[24] H. Jeong, M. Kang, and H. Kwon, Optics Communications 337, 2 (2015).
[25] F. Fröwis, N. Sanguoard, and N. Gisin, Optics communications 337, 2 (2015).
[26] T. Abad and V. Karimipour, Phys. Rev. B 93, 195127 (2016).

[27] N. Lambert, K. Debnath, A. F. Kockum, G. C. Knee, W. J. Munro, and F. Nori, Phys. Rev. A 94, 012105 (2016).

[28] U. B. Hoff, J. Kollath-Bönig, J. S. Neergaard-Nielsen, and U. L. Andersen, arXiv preprint arXiv:1601.01663 (2016).

[29] T. Ralph, Physical Review A 65, 042313 (2002).

[30] C. C. Gerry, A. Benmoussa, and R. Campos, Physical Review A 66, 013804 (2002).

[31] W. J. Munro, K. Nemoto, G. J. Milburn, and S. L. Braunstein, Physical Review A 66, 023819 (2002).

[32] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).

[33] V. Giovannetti, S. Lloyd, and L. Maccone, Nature photonics 5, 222 (2011).

[34] J. Joo, W. J. Munro, and T. P. Spiller, Physical review letters 107, 083601 (2011).

[35] A. Facio, E.-K. Dietsche, D. Grosso, S. Haroche, J.-M. Raimond, M. Brune, and S. Gleyzes, Nature 535, 262 (2016).

[36] P. T. Cochrane, G. J. Milburn, and W. J. Munro, Physical Review A 59, 2631 (1999).

[37] H. Jeong and M. S. Kim, Physical Review A 65, 042305 (2002).

[38] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Physical Review A 68, 042319 (2003).

[39] A. Ukena and A. Shimizu, Physical Review A 69, 022301 (2004).

[40] N. J. Cerf, G. Leuchs, and E. S. Polzik, Quantum information with continuous variables of atoms and light (Imperial College Press, 2007).

[41] A. Lund, T. Ralph, and H. Haselgrove, Physical review letters 100, 030503 (2008).

[42] A. Shimizu, Y. Matsuzaki, and A. Ukena, Journal of the Physical Society of Japan 82, 054801 (2013).

[43] R. W. Heeres, P. Reinhold, N. Ofek, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, arXiv preprint arXiv:1608.02430 (2016).

[44] W. Pfaff, C. J. Axline, L. D. Burkhart, U. Vool, P. Reinhold, L. Frunzio, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, arXiv preprint arXiv:1612.05238 (2016).

[45] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Physical Review Letters 80, 1121 (1998).

[46] S. Van Enk and O. Hirota, Physical Review A 64, 022313 (2001).

[47] H. Jeong, M. Kim, and J. Lee, Physical Review A 64, 052308 (2001).

[48] J. S. Neergaard-Nielsen, Y. Eto, C.-W. Lee, H. Jeong, and M. Sasaki, Nature Photonics 7, 439 (2013).

[49] N. Sangouard, C. Simon, N. Gisin, J. Laurat, R. Tualle-Brouri, and P. Grangier, JOSA B 27, A137 (2010).

[50] J. B. Brask, I. Rigas, E. S. Polzik, U. L. Andersen, and A. S. Sørensen, Physical review letters 105, 160501 (2010).

[51] J. Borregaard, J. B. Brask, and A. S. Sørensen, Physical Review A 86, 012330 (2012).

[52] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, American Journal of Physics 58, 1131 (1990).

[53] H. Jeong, W. Son, M. Kim, D. Ahn, and Č. Brukner, Physical Review A 67, 012106 (2003).

[54] M. Stobińska, H. Jeong, and T. C. Ralph, Physical Review A 75, 052105 (2007).

[55] C. Monroe, D. Meekhof, B. King, and D. J. Wineland, Science 272, 1131 (1996).

[56] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. Raimond, and S. Haroche, Physical Review Letters 77, 4887 (1996).

[57] J. R. Friedman, V. Patel, W. Chen, S. Tolpygo, and J. E. Lukens, Nature 406, 43 (2000).

[58] D. Leibfried, E. Knill, S. Seidelin, J. Britton, R. B. Blakestad, J. Chiaverini, D. B. Hume, W. M. Itano, J. D. Jost, C. Langer, et al., Nature 438, 639 (2005).

[59] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouri, and P. Grangier, Nature 448, 784 (2007).

[60] S. Deleglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J.-M. Raimond, and S. Haroche, Nature 455, 510 (2008).

[61] W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, and J.-W. Pan, Nature Physics 6, 331 (2010).

[62] D. J. Wineland, Reviews of Modern Physics 85, 1103 (2013).

[63] S. Haroche, Reviews of Modern Physics 85, 1083 (2013).

[64] B. Vlastakis, G. Kirchmair, Z. Leghtas, S. E. Nigg, L. Frunzio, S. M. Girvin, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Science 342, 607 (2013).

[65] G. Kirchmair, B. Vlastakis, Z. Leghtas, S. E. Nigg, H. Paik, E. Ginossar, M. Mirrahimi, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, Nature 495, 205 (2013).

[66] C. Wang, Y. Y. Gao, P. Reinhold, R. Heeres, N. Ofek, K. Chou, C. Axline, M. Reagor, J. Blumoff, K. Sliwa, et al., Science 352, 1087 (2016).

[67] G. Balasubramanian, P. Neumann, D. Twitchen, M. Markham, R. Kolesov, N. Mizuochi, J. Isoya, J. Achard, J. Beck, J. Tissler, et al., Nature materials 8, 383 (2009).

[68] T. Ishikawa, K.-M. C. Fu, C. Santori, V. M. Acosta, R. G. Beausoleil, H. Watanabe, S. Shikata, and K. M. Itoh, Nano letters 12, 2083 (2012).

[69] P. C. Maurer, G. Kucsko, C. Latta, L. Jiang, N. Y. Yao, S. D. Bennett, F. Pastawski, D. Hunger, N. Chisholm, M. Markham, et al., Science 336, 1283 (2012).

[70] F. Dolde, H. Fedder, M. W. Doherty, T. Nöbauer, F. Rempp, G. Balasubramanian, T. Wolf, F. Reinhard, L. C. Hollenberg, F. Jelezko, et al., Nature Physics 7, 459 (2011).

[71] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. Hollenberg, Physics Reports 528, 1 (2013).

[72] R. Schirhagl, K. Chang, M. Loretz, and C. L. Degen, Annual review of physical chemistry 65, 83 (2014).

[73] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. Van der Zouw, and A. Zeilinger, Nature 401, 680 (1999).

[74] O. Nairz, B. Brezger, M. Arndt, and A. Zeilinger, Physical Review Letters 87, 160401 (2001).

[75] In addition to the standard notation $O(N^k)$, we use $\Theta(N^k)$: For a function $f$ of $N$, we say $f = \Theta(N^k)$ if $f/N^k$ approaches a positive constant as $N \to \infty$. If $f = \Theta(N^k)$ then $f = O(N^k)$, but the inverse is not necessarily true.

[76] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge University Press, 2000).

[77] H. M. Wiseman and G. J. Milburn, Quantum Measure-
ment and Control (Cambridge University Press, 2010).

[78] C.-Y. Park, M. Kang, C.-W. Lee, J. Bang, S.-W. Lee, and H. Jeong, Physical Review A 94, 052105 (2016).

[79] J. A. Jones, S. D. Karlen, J. Fitzsimons, A. Ardavan, S. C. Benjamin, G. A. D. Briggs, and J. J. L. Morton, Science 324, 1166 (2009)

[80] M. Kitagawa and M. Ueda, Physical Review A 47, 5138 (1993).

[81] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature 413, 400 (2001).

[82] Z. Chen, J. G. Bohnet, S. R. Sankar, J. Dai, and J. K. Thompson, Physical review letters 106, 133601 (2011).

[83] By $f(N) > \Theta(N^k)$, we mean that for all $G > 0$, there exists $N_G$ such that $f(N)/N^k > G$ for $\forall N \geq N_G$.

[84] M. Greiner, O. Mandel, T. Esslinger, T. W. H"ansch, and I. Bloch, Nature 415, 39 (2002).

[85] I. Bloch, J. Dalibard, and W. Zwerger, Reviews of Modern Physics 80, 885 (2008).

[86] Y. Nakamura, Y. A. Pashkin, and J. Tsai, Nature 398, 786 (1999).

[87] I. Chiorescu, Y. Nakamura, C. M. Harmans, and J. Mooij, Science 299, 1869 (2003).

[88] H. Toida, Y. Matsuzaki, K. Kakuyanagi, X. Zhu, W. J. Munro, K. Nemoto, H. Yamaguchi, and S. Saito, Applied Physics Letters 108, 052601 (2016).

[89] W. Happer and H. Tang, Physical Review Letters 31, 273 (1973).

[90] J. Allred, R. Lyman, T. Kornack, and M. Romalis, Physical Review Letters 89, 130801 (2002).

[91] I. Savukov and M. Romalis, Physical Review A 71, 023405 (2005).

[92] H. Dang, A. Malloof, and M. Romalis, Applied Physics Letters 97, 151110 (2010).

[93] A. Tyryshkin, S. Tojo, J. Morton, H. Riemann, N. Abrosimov, P. Becker, H. Pohl, T. Schenkel, M. Thewalt, K. Itoh, and S. Lyon, Nature Materials 11, 143 (2012).

[94] Y. Matsuzaki, S. C. Benjamin, and J. Fitzsimons, Physical Review A 84, 012103 (2011).

[95] In addition to the standard notation $O(N^k)$, we use $\Theta(N^k)$: For a function $f$ of $N$, we say $f = \Theta(N^k)$ if $f/N^k$ approaches a positive constant as $N \rightarrow \infty$. If $f = \Theta(N^k)$ then $f = O(N^k)$, but the inverse is not necessarily true.

[96] We say “in principle possible” because we do not guarantee that such a noise really exists in a real physical system of interest.

[97] While Ref. [3] assumed natural states such as eigenstates of translationally invariant Hamiltonian with short-range interactions, the result of present section is applicable to wider classes of states, including artificial states which may appear in quantum information theory, if they are translationally invariant. The cluster property (B11) is defined also in a manner slightly different from that of Ref. [3].

[98] A. Shimizu and T. Miyadera, Phys. Rev. Lett. 85, 688 (2000).

[99] For pure states, one can use the value of $\langle (\Delta \hat{A})^2 \rangle$ to quantify superpositions of macroscopically distinct states, in a similar manner.

[100] To make quantitative comparison between generalized cat states of different additive observables, the normalization should be replaced with $||\hat{a}(r)|| = \text{Tr}||\hat{a}(r)|| = Q$, a constant common to all local observable $\hat{a}(r)$.