$\hbar$ as parameter of Minkowski metric in effective theory

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Abstract

With the proper choice of the dimensionality of the metric components matter field variables, the action for all fields becomes dimensionless. Such quantities as the vacuum speed of light $c$, the Planck constant $\hbar$, the electric charge $e$, the particle mass $m$, the Newton constant $G$ never enter equations written in the covariant form, i.e., via the metric $g^{\mu\nu}$. The speed of light $c$ and the Planck constant $\hbar$ are parameters of a particular two-parametric family of solutions of general relativity equations describing the flat isotropic Minkowski vacuum in effective theory emerging at low energy: $g_{\text{Minkowski}}^{\mu\nu} = \text{diag}(-\hbar^2, (\hbar c)^2, (\hbar c)^2, (\hbar c)^2)$. They parametrize the equilibrium quantum vacuum state. The physical quantities which enter the covariant equations are dimensionless quantities and dimensionful quantities of dimension of rest energy $M$ or its power. Dimensionless quantities include the running coupling ‘constants’ $\alpha_i$; the geometric $\theta$-parameters which enter topological terms in action; and geometric charges coming from the group theory, such as angular momentum quantum number $j$, weak charge, electric charge $q$, hypercharge, baryonic and leptonic charges, number of atoms $N$, etc. Dimensionful parameters have dimensions of the rest energy and its powers. They include the rest energies of particles $M_n$ (or/and mass matrices); the gravitational coupling $K$ with dimension of $M^2$; string tension in QCD $K_{\text{QCD}}$ with dimension $M^2$; cosmological constant with dimension $M^4$; etc. In effective theory, the interval $s$ has the dimension of $1/M$; it characterizes the dynamics of particles in the quantum vacuum rather than the geometry of space-time. The action is dimensionless reflecting equivalence between an action and the phase of a wave function in quantum mechanics. We discuss the effective action, and the measured physical quantities resulting from the action, including parameters of metrology triangle which enter the Josephson effect, quantum Hall effect, and quantum pumping.

PACS numbers:
Keywords:
I. INTRODUCTION

The system of units is based on the theoretical understanding of physical laws. The traditional approach to the system of units is based on the two great physical theories of the twentieth century: special relativity and quantum mechanics, which suggest to fix the speed of light \( c \) to connect space and time units, and the Planck constant \( \hbar \) to connect mass and time units \([1]\). In theoretical physics, these quantities are often considered as fundamental constants and are used as units: in these units, \( c = \hbar = 1 \)[2].

However, another great theory – general relativity (GR) – undermines this approach.

A. Speed of light

In mechanics (Galilean, relativistic or any other) the energy of a freely moving body in the limit of small momentum is expanded in terms of momentum:

\[
E(p) = M + \frac{1}{2} K^{ik} p_i p_k + K^{ikm} p_i p_k p_m p_n + \ldots \tag{1.1}
\]

In the Galilean invariant mechanics, equation (1.1) contains only the first two terms, the internal or the rest energy \( M \) and kinetic energy with the isotropic mass tensor \( K^{ik} = m^{-1} \delta^{ik} \):

\[
E(p) = M + \frac{p^2}{2m}. \tag{1.2}
\]

It contains only two parameters of the body: the rest energy \( M \) and the inertial mass \( m \) in the kinetic energy. The Lorentz invariant theory – special relativity – connects two parameters of the body, inertial mass \( m \) and the rest energy \( M \), via the speed of light,

\[
m = \frac{M}{c^2}, \tag{1.3}
\]

and equation (1.1) is transformed to equation with the parameter \( M \) and ‘fundamental constant’ \( c \):

\[
E^2 - c^2 p^2 - M^2 = 0. \tag{1.4}
\]

GR effectively removes the ‘fundamental constant’ from the equation (1.4), it transforms this equation to

\[
g^{\mu\nu} p_\mu p_\nu + M^2 = 0 , \quad p_\mu = (-E, p_i). \tag{1.5}
\]

Equation (1.5) contains only the parameter of a body, the rest energy \( M \). The speed of light \( c \) does not enter explicitly: it becomes the part of the metric. Because of that, \( c \) never enters explicitly any equation, which is written in the covariant form, i.e. when the equation is expressed in terms of metric field (see e.g. [3, 4]). It may enter only the solutions of equations, in particular the solutions which have Minkowski space-time asymptotically.
In GR, the relation between space and time is not universal but depends on the metric field $g_{\mu\nu}(r, t)$. The speed of light $c$, by definition of speed as an infinitesimal proper length divided by an infinitesimal proper time, is determined in the limit of vanishing distance between the point objects. But zero distance limit is mathematical construction which does not reflect the real physical world. The physically measured speed of propagation of light between two distant objects is coordinate dependent and thus depends on the trajectory of propagation. This implies that the parameter $c$ may make sense only in the Minkowski space-time, when gravity is fully absent:

$$g_{\text{Minkowski}}^{\mu\nu} = \text{diag} \left( -1, \frac{1}{c_1^2}, \frac{1}{c_2^2}, \frac{1}{c_3^2} \right).$$

(1.6)

In fundamental GR the limiting speed $c$ is fundamental, which allows us to put $c = 1$. In effective theory, the latter is problematic. For example, if the underlying microscopic system is anisotropic, the limiting speed of the low-energy excitations depends on the direction of propagation, and effective Minkowski space-time contains 3 parameters:

$$g_{\text{Minkowski}}^{\mu\nu}(c_1, c_2, c_3) = \text{diag} \left( -1, \frac{1}{c_1^2}, \frac{1}{c_2^2}, \frac{1}{c_3^2} \right).$$

(1.7)

Such anisotropy of the physical speed of light will be revealed only at high energy. For the internal low-energy observers, the world obeys Lorentz invariance and equivalence principle of GR. The measured limiting speed is coordinate independent, isotropic and universal for all species (at least in the Fermi-point scenario of emergent gravity [3]), if observers do not use “xylophones, yachts and zebras to measure intervals along the $x, y$ and $z$ axes” [5], but use the same xylophone in all directions. Since xylophones (rods and clocks) are made of anisotropic quasiparticles, the rescaling of measured time and distances automatically occurs due to the physical Lorentz–Fitzgerald contraction of length of rods and the physical Lorentz slowing down of clocks. This leads to apparent isotropy and universality in the low-energy limit, while for external high-energy observer the speed of light depends on coordinates and energy and is different for different species.

Thus in both cases, fundamental and emergent, the limiting speed $c$ drops out of any equation written in the covariant form. For fundamental GR this is trivial since one may put $c = 1$. In effective GR even such notion as the fundamental speed is simply absent and its introduction is artificial. Let us now show that the same occurs with another fundamental constant, the $\hbar$.

**B. Planck constant $\hbar$**

Planck constant relates the frequency of emitted photon with the energy levels of atom:

$$M_m - M_n = \hbar \omega_{mn}.$$

(1.8)
For an extended body the relation between the invariant mass $M$ and the non-covariant frequency is also coordinate dependent due to gravitational red shift. This means that the parameter $\hbar$ can be measured only in the ideal limit of a point object. The string theory deals with the extended objects – fundamental strings, and in effective GR the size of an object is also limited at least by Planck length. This suggests that the quantity $\hbar$ is also the characteristic of Minkowski space-time and can be absorbed by metric as $c$.

Such absorption occurs, if the quantity $\hbar$, which traditionally is the prefactor in the quantum mechanical operator of momentum $p_\mu = -i\hbar\nabla_\mu$, is moved from $p_\mu$ to the metric $g_{\mu\nu}$ in Eq. (1.5). As a result, the isotropic Minkowski metric is characterized by two parameters, $\hbar$ and $c$, with $\hbar$ being the conformal factor of the Minkowski space-time:

$$g_{\text{Minkowski}}^{\mu\nu}(c, \hbar) = \hbar^{-2}\text{diag}(-1, c^{-2}, c^{-2}, c^{-2}) , \quad g^{\text{Minkowski}}_{\mu\nu}(c, \hbar) = \hbar^2\text{diag}(-1, c^2, c^2, c^2).$$

(1.9)

Now when $\hbar$ is absorbed by the metric, it also does not enter explicitly any covariant equation. In particular, equation (1.8) becomes

$$M_m - M_n = \omega_{mn} \sqrt{g_{00}},$$

(1.10)

i.e. in GR the quantum mechanic equation (1.8) is hidden in the equation (1.10) describing the red shift.

With the choice of the dimensionality of the metric components in Eq. (1.9), the action becomes dimensionless, $[S] = 1$. An example is the the classical action for a freely moving massive particle:

$$S_M = M \int ds \ , \quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$  

(1.11)

This action is dimensionless. A dimensionless action leads to a natural formulation of quantum mechanics in terms of Feynman path integral with the integrand $e^{iS}$. For a single particle, $S$ is the phase of a semiclassical wave function, while equation (1.5) transforms to

$$g^{\mu\nu}k_\mu k_\nu + M^2 = 0 \ , \quad k_\mu = (-\omega, k_i).$$

(1.12)

If gravity is fundamental and $\hbar$ is fundamental constant, the proposal to eliminate $\hbar$ from equations by hiding it into the metric (1.9) by scale transformation would be a formal mathematical trick only. Nothing prevents to use the scale factor larger or smaller than $\hbar$, if it is accompanied by the rescaling of mass $M$. If all the equations are covariant, one may safely put $c = \hbar = 1$.

However, if gravity is effective and general covariance emerges together with gravity only in the low-energy corner of the underlying microscopic physics of quantum vacuum, then all 10 components of the metric tensor become physical including the parameters $c$ and $\hbar$, 

Thus, in this case, the metric (1.9) becomes

$$g_{\text{Minkowski}}^{\mu\nu}(c, \hbar) = \hbar^{-2}\text{diag}(-1, c^{-2}, c^{-2}, c^{-2}), \quad g^{\text{Minkowski}}_{\mu\nu}(c, \hbar) = \hbar^2\text{diag}(-1, c^2, c^2, c^2).$$

(1.13)

This leads to a natural formulation of quantum mechanics in terms of Feynman path integral with the integrand $e^{iS}$. For a single particle, $S$ is the phase of a semiclassical wave function, while equation (1.5) transforms to

$$g^{\mu\nu}k_\mu k_\nu + M^2 = 0 \ , \quad k_\mu = (-\omega, k_i).$$

(1.14)
entering the Minkowski metric of the equilibrium vacuum state. They are not universal and together with the other parameters of the metric may explicitly enter the higher order terms in the action, for which the covariance as emergent phenomenon is not valid. As a results the ratio \((M_m - M_n)/\omega_{mn}\), which is universal in the low-energy range, will depend on frequency at high energy. This makes impossible setting \(\hbar = 1\), because this quantity depends on the vacuum state and in principle is time and coordinate dependent.

An example is provided by the \(q\)-theory \[6\] of quantum vacuum. According to the \(q\)-theory, our quantum vacuum is a self-sustained medium. It has an equilibrium state in which the parameter \(q\) is self-adjusted to its equilibrium value \(q_0\). It was recently demonstrated that such a self-adjusted equilibrium state may serve as an attractor, i.e. the non-equilibrium vacua relax towards the flat space-time with this value of \(q\) \[7\]. In principle it is possible that in emergent theory the Universe relaxes to the distinguished Minkowski space-time with fixed values of parameters \(c(q_0)\) and \(\hbar(q_0)\). Such an equilibrium vacuum could be the final state which the system approaches in our part of the Universe.

In this paper we restrict ourselves with the covariant equations, i.e. we consider the lowest order terms in the effective theory and neglect the higher-order terms, which violate the emergent low-energy symmetry. We demonstrate that the choice \(1.9\) for Minkowski metric allows us to remove \(c\) and \(\hbar\) from all the covariant equations without setting \(c = \hbar = 1\), i.e. without assumption that these quantity are universal and coordinate independent. On the contrary, when \(c\) and \(\hbar\) are inside the metric, the symmetry arguments (gauge invariance, general covariance, etc.) do not prohibit the space-time dependence of \(c\) and \(\hbar\). This is possibly the necessary intermediate step towards the “quantum gravity”, the underlying microscopic theory in which \(c\) and \(\hbar\) may and should depend on space and time.

II. EFFECTIVE ACTION

Let us first discuss the effective action for the gauge fields and gravity as it appears, say, in the Fermi-point scenario \[3\], and rewrite it in such a way that \(\hbar\) is hidden in the metric field. The main lesson is that in the low-energy corner, i.e. in the region of applicability of effective theory, the parameter \(\hbar\) never enters explicitly.

A. Electromagnetic action

We choose the dimensions of the vector potential \(A_\mu\) and field strength \(F_{\mu\nu}\) as they naturally follow from the geometric origin of the gauge field. Since \(A_\mu\) arises as a result of localization of the dimensionless \(U(1)\) field, \(\nabla_\mu \phi \rightarrow A_\mu\), one has:

\[
[A_0] = [t]^{-1}, \quad [A_i] = [l]^{-1}, \quad [F_{ik}] = [l]^{-2}, \quad [F_{i0}] = [t]^{-1}[l]^{-1}.
\] (2.1)
The interaction of a classical particle with electromagnetic field is

\[ S_{\text{int}} = q \int dx^\mu A_\mu . \]  \hspace{1cm} (2.2)

Here \( q \) is the geometric charge of a particle corresponding to the \( U(1) \) gauge group, with \( q_e = -1 \) for electron; \( q_u = 2/3 \) for up quark; \( q_d = -1/3 \) for down quark, etc. In quantum mechanics, equation (2.2) corresponds to the covariant derivative \( D_\mu = \nabla_\mu - iqA_\mu . \) The motion equation of a massive particle with electric charge \( q \) in electromagnetic field follows from (1.11) and (2.2):

\[ \frac{du^\mu}{ds} + \Gamma^\mu_{\lambda\sigma} u^\lambda u^\sigma = \frac{q}{M} F^{\mu\nu} u_\nu , \quad u^\mu = \frac{dx^\mu}{ds}. \]  \hspace{1cm} (2.3)

The action for the electromagnetic field is

\[ S_{\text{em}} = \int d^3x dt \sqrt{-g} \frac{1}{16\pi\alpha} F^{\mu\nu} F_{\mu\nu} = \int d^3x dt \sqrt{-g} \frac{1}{16\pi\alpha} F_{\alpha\beta} F^{\alpha\mu} g^{\mu\nu} , \]  \hspace{1cm} (2.4)

where the dimensionless parameter \( \alpha \) is the logarithmically running coupling – the fine structure ‘constant’. In effective theories, \( 1/\alpha \) naturally emerges as logarithmically divergent factor, and in principle it is space- and time-dependent. For example, in quantum electrodynamics with massless fermions which emerges in superfluid \( ^3\text{He}-\Lambda \) \( [3] \) one has \( 1/\alpha \propto \ln[1/(F^{\mu\nu} F_{\mu\nu})] \).

With the choice (2.1), the actions (2.4) and (2.2) are dimensionless:

\[ [S_{\text{int}}] = [S_{\text{em}}] = 1. \]  \hspace{1cm} (2.5)

In effective theory, voltage (the difference of electric potentials) has the same dimension as frequency and electric current (the current of electrons): \([J_e] = [A_0] = [\omega] = [t]^{-1}.\)

The electric resistance and conductance are dimensionless: \([R] = 1.\) This suggests the possibility that the dimensionless relations between voltage and frequency, between voltage and current, and between current and frequency may have quantized values in some systems. The corresponding Josephson effect, quantum Hall effect, and quantum pumping, which form the so-called metrology triangle are discussed in Sec. [IV].

**B. Action for gravity**

The gravitational action in effective theories is

\[ S_{\text{grav}} = \int d^3x dt \sqrt{-g} \left( \Lambda + \frac{K}{16\pi} \mathcal{R} + \ldots \right) , \]  \hspace{1cm} (2.6)

where \( K \) and \( \Lambda \) are gravitational coupling and cosmological constant respectively, and dots denote the higher order terms which include in particular the \( \mathcal{R}^2 \) terms. Using dimensions of the metric component

\[ [g_{00}] = [\hbar]^{-2} , \quad [g^{00}] = [\hbar]^2 , \quad [g_{ik}] = [\hbar c]^{-2} , \quad [g^{ik}] = [\hbar c]^2 , \quad [\sqrt{-g}] = [\hbar]^{-4}[c]^{-3} . \]  \hspace{1cm} (2.7)
one obtains that the dimension of the curvature $R$ is:

$$[R] = [M]^2, \quad [d^3xdt\sqrt{-g}] = [M]^{-4}. \quad (2.8)$$

As follows from effective theories, dimensions of the gravitational coupling and cosmological constant are

$$[K] = [M]^2, \quad [\Lambda] = [M]^4, \quad (2.9)$$

while the $R^2$ terms have dimensionless prefactors. As a result, the gravitational action is dimensionless, $[S_{\text{grav}}] = 1$. The first two terms in the gravitational action (2.6) contain parameters with dimension $M^n$ and thus they emerge only if conformal invariance is violated and the fundamental length or energy scale enters the underlying microscopic theory.

The dimension of the metric suggests that metric is not the quantity, which describes the space-time, but the quantity, which determines the dynamics of effective fields in the background of a given quantum vacuum.

### III. PARAMETERS OF EFFECTIVE THEORY AND INVARIANT QUANTITIES

Let us consider physical quantities which in principle can be measured, and express them in terms of the invariant parameters, which enter the action. They can be distributed in the following groups: (i) quantities, which contain $\hbar$ and $c$ due to historical reasons, and do not contain these parameters when expressed in terms of the parameters emerging in effective theory; (ii) quantities, which still contain $\hbar$ even when expressed in terms of the effective theory parameters, but do not contain it after the rescaling of the metric in (2.7); (iii) dimensionless geometrical and topological charges.

#### A. quantities containing $\hbar$ due to historical reasons

Effective theory emerging in the low energy corner contains such parameters as fine structure constant $\alpha$, gravitational coupling $K$, angular momentum quantum number $j$, charge quantum number $q$, and rest energies $M$ of particles, etc. However, in the traditional description, which reflects the historical process of development of physical ideas, these quantities are splitted into the electric charge of a system $Q$, elementary electric charge $e$, speed of light $c$, Newton constant $G$, Planck constant $\hbar$, angular momentum $J$ and masses $m$:

$$K = \frac{\hbar c^5}{G}, \quad \alpha = \frac{e^2}{\hbar c}, \quad M = mc^2, \quad j = \frac{J}{\hbar}, \quad q = \frac{Q}{e}. \quad (3.1)$$

In effective theories such splitting is not justified, and moreover it is not necessary since the measured quantities do not contain the traditional parameters explicitly. There are some examples below.
1. Electron energy in a Coulomb field

The energy levels of electron in the Coulomb field of proton[8]:

\[ M_{n,j} = M_e \left( 1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left( \frac{n}{j + \frac{3}{4}} \right) + \ldots \right). \] (3.2)

It is expressed via the rest energy of a free electron \( M_e \); the fine structure constant \( \alpha \); quantum number \( n \) and angular momentum quantum number \( j \).

2. Energy levels in a Newton potential

Rest energy of system of two point bodies interacting via Newton gravitational potential:

\[ M_n = (M_1 + M_2) \left( 1 - \frac{1}{2n^2K^2} \frac{M_1^3M_2^3}{(M_1 + M_2)^2} + \ldots \right). \] (3.3)

It contains rest energies of the bodies \( M_1 \) and \( M_2 \); and gravitational coupling \( K \), which enters Einstein action in (2.6).

3. Black-hole temperature and entropy

Hawking temperature of a black hole with rest energy \( M_{BH} \) and its Bekenstein-Hawking entropy are:

\[ T_{BH} = \frac{K}{8\pi M_{BH}}, \quad S_{BH} = \frac{4\pi M_{BH}^2}{K}. \] (3.4)

4. Charged rotating black hole

Entropy of the rotating electrically charges black hole:

\[ S_{BH}(M, j, q) = \pi \left( 2M^2/K - q^2\alpha + 2\sqrt{M^4/K^2 - j(j + 1) - q^2\alpha} \right). \] (3.5)

Here \( j \) is the angular momentum quantum number of black hole; and \( q \) is its electric charge quantum number, i.e. the dimensional charge determined by the corresponding gauge group. The charge \( q \) is rational number with \( q_e = -1 \) for electron.

The above examples demonstrate that parameters \( Q, e, c, G, h \), angular momentum \( J \) and masses \( m \) are artificial. They reflect the long history of studies of the laws of physics, but they do not appear in effective theories where all the physical laws naturally and simultaneously emerge in the low-energy corner.
5. QCD vacuum energy

When weak or strong interaction is added, one also finds that strong and weak charges $g_S$ and $g_W$ do not enter explicitly. The Standard Model contains the corresponding running couplings and parameters of dimension $M^n$. Example is the vacuum energy density – cosmological constant – suggested in the QCD model [9]:

$$\Lambda \sim \frac{K_{QCD}^3}{K}.$$  \hfill (3.6)

It contains the gravitational coupling $K$ and string tension $K_{QCD}$ with $[K_{QCD}] = [\Lambda_{QCD}]^2 = [K] = [M]^2$, where $\Lambda_{QCD}$ is the energy scale of QCD.

B. quantities which do not contain $\hbar$ after rescaling of metric

1. Zeeman energy

The observable consequence of quantum electrodynamics is the Zeeman splitting of electron and muon energies in magnetic field $B$:

$$E_{Zeeman} = \frac{B}{M} \left(1 + \frac{\alpha}{2\pi} + \ldots\right),$$  \hfill (3.7)

where $M$ denotes either the electron $M_e$ or muon $M_\mu$ rest energy. Here the field strength $F_{ik}$ is given in geometric units in (2.1), $[F_{ik}] = [\ell]^{-2}$. The quantity $B^2 - E^2 = F_{\mu\nu}F^{\mu\nu}$ contains the metric elements, which enter $F_{\mu\nu}$. Since the dimensions of the metric elements are given by (2.7), the dimension of $B$ is $[B] = [M]^2$, and (3.7) contains only the parameters $M$ and $\alpha$. There is one more parameter, the charge, which is not shown explicitly because for electron and muon one has $q_e^2 = q_\mu^2 = 1$.

In traditional units one has (leaving only the first term):

$$E_{Zeeman} = \frac{e\hbar}{m} B.$$  \hfill (3.8)

2. Unruh effect

In effective theory, the Unruh temperature of the accelerated body is

$$T_U = \frac{a}{2\pi},$$  \hfill (3.9)

where $a$ is covariant acceleration:

$$a^2 = g_{\mu\nu} \frac{d^2x^\mu}{ds^2} \frac{d^2x^\nu}{ds^2}. \hfill (3.10)$$
Its dimension is \([a] = [M]\). In the traditional units, where the covariant acceleration has dimension \([a] = [l][t]^{-2}\), one has

\[
T_U = \frac{\hbar a}{2\pi c},
\]

while in (3.9) the Planck constant \(\hbar\) and \(c\) do not enter explicitly, because these parameters are absorbed by the metric \(g_{\mu\nu}\).

IV. TOPOLOGICAL QUANTUM NUMBERS

Topological quantum numbers in action can be considered on the example of \(\theta\) term, quantum Hall effect (QHE) and Josepson effect.

A. \(\theta\)-term

The \(\theta\) term in action

\[
S_\theta = \frac{\theta}{16\pi^2} \int d^3x dt \ e^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu},
\]

(4.1)

It does not contain metric and is automatically dimensionless, \([S_\theta] = 1\), due to equations (2.1) which reflect the geometric nature of gauge fields.

The dimensional reduction of the \(\theta\)-term gives the topological term of Chern-Simons type in the 2 + 1 action:

\[
S_{CS} = \frac{k}{8\pi} \int d^2x dt e^{\alpha\beta} A_\alpha F_{\beta\gamma},
\]

(4.2)

where \(k\) is some dimensionless fundamental number. This action is dimensionless, \([S_{CS}] = 1\), and does not contain the metric field.

B. Quantum Hall effect

The quantum Hall effect (QHE) in condensed matter is characterized by a similar 2 + 1 action:

\[
S_{QHE} = \frac{q^2\nu}{8\pi} \int d^2x dt e^{\alpha\beta} A_\alpha F_{\beta\gamma}.
\]

(4.3)

Here \(q\) is electric charge of fermion, which is \(q = q_e = -1\) for electronic system, the systems of electrons. The dimensionless quantity \(\nu\) is some fundamental number, which characterizes the fermionic system; it is typically integer being related to the fermionic quantum topological number (Chern number) of the ground state of the system. The action (4.3) is also dimensionless, \([S_{QHE}] = 1\).
The corresponding current of electrons \( (q^2 = q_e^2 = 1) \):

\[
j^i = \frac{\delta S}{\delta A_i} = \frac{\nu}{2\pi} e^{0ij} F_{j0}, \tag{4.4}
\]

is transverse to electric field; this is the Hall current with quantized Hall conductivity:

\[
\sigma^q = \frac{\nu}{2\pi}. \tag{4.5}
\]

In effective theory, the Hall (and also the spin-Hall) conductance are expressed in terms of integer or rational number and \( \pi \).

In traditional units, where \( A_0 \) is substituted by \( e\tilde{A}_0/\hbar \), and action is multiplied by \( \hbar \), the Hall conductivity is given by

\[
\sigma_{xy} = 4\pi \alpha \sigma^d = 2\alpha \nu, \tag{4.6}
\]

where \( \alpha \) is the fine-structure constant. In this units, \( \sigma_{xy} \) is not expressed in terms of rational number and \( \pi \). The quantization is not exact, since the fine-structure constant is not a constant but is a running coupling. It depends on the infrared cut-off and thus is space- and time-dependent. The failure of the traditional description to obtain exact quantization comes from the unjustified splitting of the vector potential, \( A_\mu = (e/\hbar)\tilde{A}_\mu \), in the traditional description. As a result the field \( \tilde{F}_{\mu\nu} = \nabla_\mu \tilde{A}_\nu - \nabla_\nu \tilde{A}_\mu \) is not gauge invariant: under gauge transformation \( \tilde{A}_\mu \rightarrow \tilde{A}_\mu + (\hbar/e)\nabla_\mu \phi \), the field transforms as \( \tilde{F}_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} + \nabla_\mu (\hbar/e) \nabla_\nu \phi - \nabla_\nu (\hbar/e) \nabla_\mu \phi \). The field \( \tilde{F}_{\mu\nu} \) would be gauge invariant, only if the electric charge \( e \) and \( \hbar \) are the fundamental constants. But the charge \( e \) is certainly not a fundamental constant. This is the so-called ‘physical charge’, which is obtained by splitting of \( \alpha \) and thus is coordinate dependent. The gauge invariance and the true quantization of Hall conductivity are restored when the geometric vector potential \( A_\mu \) and the geometric \( U(1) \) charge of electron \( q_e = -1 \) are used.

C. Josephson effect

1. Josephson effect in superconductors

The ac Josephson effect in superconductors (charged superfluids) comes from the coupling of the phase \( \phi \) of the order parameter with electromagnetic field: due to gauge invariance the time derivative of \( \phi \) enters in action only in combination with the electric potential, \( \partial_t \phi - qA_0 \), where \( q = 2q_e = -2 \) is the electric charge of Cooper pairs. Taking into account the \( 2\pi \) periodicity of the phase \( \phi \) one obtains the following Josephson relation in superconductors:

\[
\omega = 2 \left( A_0^{(2)} - A_0^{(1)} \right). \tag{4.7}
\]

In effective theory, the Josephson relation contains only integer \( |q| = 2 \). This is because electromagnetic field is given in geometric presentation, in which \( A_0 \) has the same dimension
as frequency: \([A_0] = [\omega] = [t]^{-1}\). That is why the Josephson effect provides the standard of voltage in inverse unit of time.

In traditional units, the Josephson relation contains parameters \(e\) and \(\hbar\). The electric charge \(e\) is certainly non-fundamental, while \(\hbar\) can be non-fundamental. In both cases the quantization is lost, as in the case of quantum Hall effect in Sec. [IV.B]

2. Josephson effect in neutral systems

Let us now consider the ac Josephson effect in electrically neutral systems (superfluids). In superfluid liquids, such as superfluid \(^4\text{He}\) and \(^3\text{He}\), the role of voltage can be played by static gravitational field. That is why let us take into account gravity. In general relativity there is the Tolman law for temperature and for the chemical potential in thermodynamic equilibrium in the presence of a static gravitational field. It relates the local values \(T(r), \mu(r)\) with the global values \(T_T\) and \(\mu_T\) which are space-independent in equilibrium:

\[
T(r)\sqrt{-g_{00}(r)} = T_T, \quad \mu(r)\sqrt{-g_{00}(r)} = \mu_T,
\]

The dimensions of the local and global quantities are

\[
[T] = [\mu] = [M], \quad [T_T] = [\mu_T] = [\omega] = [t]^{-1}.
\]

In neutral superfluids the Josephson oscillations emerge due to difference of chemical potentials. Oscillations cannot emerge in equilibrium: they only occur if the Tolman potential have a jump across the Josephson junction. That is why the correct Josephson relation in neutral superfluids should be:

\[
\omega = \nu \left(\mu_T^{(2)} - \mu_T^{(1)}\right),
\]

where integer \(\nu = 1\) for superfluid \(^4\text{He}\) and \(\nu = 2\) for superfluid \(^3\text{He}\) with Cooper pairing mechanism of superfluidity. In traditional units, the parameter \(\hbar\) enters explicitly, and if \(\hbar\) is not fundamental, the quantization is not exact. This in principle will allow us to experimentally resolve between fundamental and non-fundamental \(\hbar\).

In terms of the local chemical potential \(\mu\), equation (4.10) becomes (we use \(\nu = 1\) assuming superfluid \(^4\text{He}\)):

\[
\omega = \sqrt{-g_{00}(2)\mu^{(2)} - g_{00}(1)\mu^{(1)}}.
\]

In case when gravity is the same across the junction, and only the values of the local chemical potential \(\mu\) are different on two sides of the contact, the Josephson relation (4.11) becomes

\[
\omega = \sqrt{-g_{00}(\mu^{(2)} - \mu^{(1)})}.
\]

In Minkowski vacuum this equation obtains the traditional form \(\omega = (\mu^{(2)} - \mu^{(1)})/\hbar\).
In the other case, when the chemical potential is the same on both sides of the contact, but the gravitational potential is different, the equation (4.11) becomes

\[ \omega = \mu \left( \sqrt{-g_{00}^{(2)}} - \sqrt{-g_{00}^{(1)}} \right). \] (4.13)

This situation takes place when the Josephson junction is represented by a channel connecting two vessels in which the level of the liquid has different height, \(h_2\) and \(h_1\). In the limit of a weak gravitational potential in the background of Minkowski space-time, one has

\[ g_{00} \approx -(1 - 2\Phi/c^2)/\hbar^2, \quad \sqrt{-g_{00}^{(2)}} - \sqrt{-g_{00}^{(1)}} \approx -(\Phi^{(2)} - \Phi^{(1)})/\hbar c^2, \quad \Phi^{(2)} - \Phi^{(1)} = g(h_2 - h_1), \]

and the equation (4.13) transforms to the familiar expression for the Josephson effect in superfluid \(^4\)He caused by gravitational field:

\[ \omega = \frac{mgh}{\hbar}, \quad h = |h_2 - h_1|. \] (4.14)

In superfluid \(^4\)He in the limit of zero temperature, the chemical potential equal the rest energy per unit atom of \(^4\)He: \(\mu = M\). This \(M\) slightly differs from the rest energy of an isolated \(^4\)He atom due to the energy added by interaction between the atoms and zero point motion of \(^4\)He atoms in superfluid, \(M \neq M_4\). That is why in effective theory, the Josephson relation (4.12) has the same form as both the gravitational red shift and the energy-frequency relation in quantum mechanics (1.10):

\[ \omega(\mathbf{r}) = \sqrt{-g_{00}(\mathbf{r})} \left( M^i - M^f \right), \] (4.15)

where \(M^i\) and \(M^f\) are the rest energies of an atom in initial state before radiation of photon and in the final state after the radiation correspondingly.

**D. Metrology triangle**

Another topological effect is quantum pumping, the transfer of fermions by periodic change of the parameters of the system: \(\dot{N} = \nu \omega/(2\pi)\), where \(\dot{N}\) is the number of fermions transferred per unit time between two subsystems, and \(\nu\) is topological quantum number. The quantum pumping in electronic systems (single-electron tunnelling [11, 12]) reflects the quantization of the number of electrons. Since electric current is the charge \(q\) transferred per unit time, \(J = q_e \dot{N}\), one obtains the relation between the current and frequency: \(J = q_e \nu \omega/(2\pi)\). The quantum pumping allows to calibrate frequency by measuring the current \(J\), or to calibrate current by tuning frequency. This completes the so-called metrology triangle: Josephson effect, QHE and quantum pumping relate respectively voltage and frequency, current and voltage, and frequency and current.

In effective theory, the current \(J\), the voltage \(\Delta A_0\) and frequency \(\omega\) are all expressed in inverse time units. The relations between these quantities are determined solely by the fundamental geometric or topological charges, integer or fractional, such as \(q\) and \(\nu\). These
three effects reflect different geometrical and topological properties of condensed matter system and provide two independent ways of calibration of voltage in terms of frequency: either directly by Josephson effect or by combination of QHE and quantum pumping.

Using the standard of voltage one may measure the potential produced by the geometric charge $q$:

$$A_0 = \frac{\alpha q}{r} \frac{1}{g^{rr}g^{00}\sqrt{-g}}.$$  \hspace{1cm} (4.16)

This allows to measure the combination of the metric elements after $\alpha$ is measured using (3.2). The other experiments measure other parameters and other combinations of the metric tensor. For example, $g_{00}$ can be found using equation (1.8).

V. SCHRÖDINGER EQUATION

A. Klein-Gordon equation for scalar field

In effective theory, the dimensionless action for a scalar field $\Phi(x)$ with dimension $[\Phi] = [M]$ has the form:

$$S_{\text{scalar}} = \frac{1}{2} \int d^3x dt \sqrt{-g} \left( g^{\mu\nu} \nabla_\mu \Phi^* \nabla_\nu \Phi - M^2 |\Phi|^2 \right).$$  \hspace{1cm} (5.1)

The kinetic term has the same dimension as the mass term without artificial introduction of the parameter $\hbar$.

B. Schrödinger equation

The nonrelativistic Schrödinger action can be obtained by expansion of Eq. (5.1). In case of space-time independent $g^{00}$ one introduces the Schrödinger wave function $\Psi$ with dimension $[\Psi] = [M]^{3/2}$

$$\Phi(r, t) = \frac{1}{\sqrt{M}} \exp \left( i M t / \sqrt{-g^{00}} \right) \Psi(r, t).$$  \hspace{1cm} (5.2)

After expansion over $1/M$ one obtains the Schrödinger-type action in the form

$$S_{\text{Sch}} = \int d^3x dt \sqrt{-g} L,$$

$$2L = i \sqrt{-g^{00}} (\Psi \partial_t \Psi^* - \Psi^* \partial_t \Psi) + \frac{ig^{0k}}{\sqrt{-g^{00}}} (\Psi \nabla_k \Psi^* - \Psi^* \nabla_k \Psi) + \frac{g^{ik}}{M} \nabla_i \Psi^* \nabla_k \Psi.$$  \hspace{1cm} (5.3)

For Minkowski space-time, (5.3) transforms after renormalization $\Psi \rightarrow \Psi(hc)^{3/2}$ to

$$S_{\text{Sch}}^{\text{Mink}} = \frac{1}{2} \int d^3x dt \left( i (\Psi \partial_t \Psi^* - \Psi^* \partial_t \Psi) + \frac{\hbar}{m} \nabla_i \Psi^* \nabla_i \Psi \right).$$  \hspace{1cm} (5.4)
The equation (5.4) contains a single parameter: the ratio of parameters $\hbar$ and $m$. In non-relativistic quantum mechanics, they always enter together. Examples are equation (4.14) for the Josephson effect in neutral superfluids and also the quantum of circulation of superfluid velocity: $\kappa = 2\pi \nu \hbar/m$. Here $m$ is the mass of an atom in a superfluid liquid; $\nu = 1$ for superfluid $^4$He; $\nu = 1/2$ for superfluid $^3$He-B; and $\nu = 1/4$ for the four-particle correlated state, which may occur in thin superfluid films, see e.g. Section 10 in Ref. [13].

The traditional expression for non-relativistic Schrödinger action is obtained after multiplication of the action (5.4) by $\hbar$. It contains the parameters of the equilibrium Minkowski vacuum, $c$ (via $m = M/c^2$) and $\hbar$. In the traditional form the spectrum of a nonrelativistic particle with mass $m$ in the 1D box of size $L_x$ with impenetrable walls is

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m L_x^2}. \quad (5.5)$$

However, in the covariant form it is

$$E_n = \frac{\pi^2 n^2}{2M(\Delta l)^2}, \quad (5.6)$$

where $\Delta l$ is the proper distance, $(\Delta l)^2 = L_x^2 g_{xx}$. Being written in the covariant form, the energy spectrum does not contain parameters $m$, $c$ and $\hbar$ explicitly. This demonstrates that even the non-relativistic quantum mechanics contains the rest energy $M$ instead of traditional mass $m$. This suggests that the Lorentz invariance and general covariance are not very important factors. The metric field $g_{\mu\nu}$ may serve the source of parameters $\hbar$ and $c$ even for the so called one half of general relativity [14], when the matter fields feel the metric field as effective geometry, but $g_{\mu\nu}$ does not necessarily obey the Einstein equations.

VI. DISCUSSION. EMERGENT VS FUNDAMENTAL PHYSICS

In general relativity, the parameters $\hbar$ and $c$ vanish from equations written in covariant form. This statement can be trivial or not, depending on whether gravity and other physical laws are fundamental or emergent.

In fundamental physics, the parameter $\hbar$ is a universal constant, that is why, in principle it can be hidden in the metric. It is important that, after it is included into the metric, $\hbar$ completely disappears from all physical equations if the action is represented in terms of the natural parameters. This could mean that $\hbar$ is the natural part of the metric, just in the same manner as the parameter $c$, which is artificially introduced for convention together with Boltzmann constant $k_B$.\footnote{Let us recall, that in fundamental general relativity there is no limiting velocity. For example, in expanding Universe the coordinate velocity of galaxies may exceed the speed of light, due to the ‘aether drift’. Only the relative velocity of two objects in the same point of space-time is limited by $c$. But GR does not} The Planck constant $\hbar$ becomes the second parameter of a
particular two-parametric family of solutions of general relativity equations describing the flat isotropic Minkowski vacuum in (1.9). After $\hbar$ becomes the part of the metric, it shares the same fate as parameter $c$. Since GR does not discriminate between metrics with different $\hbar$, the parameter $\hbar$ becomes the matter of convention too. Thus if GR is fundamental, the inclusion of $\hbar$ into the metric is equivalent to the choice of units $\hbar = c = 1$, which is trivial and does not lead to any new results.

In the emergent theories, there are no universal constants: the ‘fundamental’ constants may be given by vacuum expectation values of the $q$-field [6], of scalar fields [5], of a 4-index field strength [5, 6], etc. That is why $\hbar$ and $c$ could and should be space/time dependent in the same manner as $\alpha$. At first glance, the space-time dependence of the ‘fundamental constant’ $\hbar$ would violate many physical laws. This happens if the equations are written in the traditional form. The traditional form of the Feynman integral over the fields $\chi$ is $\int d\chi \exp (iS/\hbar)$. The quantum-mechanical phase factor is a compact U(1) quantity, which means that action acquires topological properties. If $\hbar$ is space-time dependent, it must be introduced within the integral: $S/\hbar \rightarrow \int d^3x dt \, \hbar^{-1} \sqrt{-g} L$. But this would violate the general covariance. This violation does not occur if $\hbar$ is a scalar field, but even in this case the gauge invariance of the topological terms in the action would be violated (see Sec. IV). That is why in the emergent gravity, it is simply necessary to include $\hbar$ into the metric in order to avoid such violations. As a result the action becomes dimensionless, and the Feynman integral becomes $\int d\chi \exp (iS)$. Just in the same way the gauge invariance requires the use of the geometric electric charge $q$ instead of the traditional electric charge $e$, which was introduced at the earlier stage of physics. The latter must be included into the geometric vector potential $A_\mu$ and into the fine structure ‘constant’ $\alpha$.

In the emergent theories, the quantities $\hbar$ and $c$ do not enter the equations of general relativity, but they are two parameters which characterize the special solution of the equations. This solution describes the effective Minkowski metric emerging in the background of the equilibrium vacuum state of our Universe. In case if the equilibrium vacuum state corresponds to the effective de Sitter space-time, one more parameter will characterize our vacuum, the Hubble constant $H$ which enters the effective interval in the de Sitter Universe:

$$ds^2 = \frac{1}{\hbar^2} \left( -dt^2 + \frac{1}{c^2} (dr - H r dt)^2 \right).$$ (6.1)

The parameter $H$ also does not enter the GR equations, and in this respect it is similar to $c$ and $\hbar$. However, as distinct from Minkowski vacuum, de Sitter vacuum can be dynamically unstable [7] and thus its Hubble parameter cannot serve as fundamental constant.

In conclusion, the fundamental gravity is equivalent to the universal $\hbar$, while in emergent gravity $\hbar$ is non-universal. In the former case $\hbar$ can be included into metric, while in the
discriminate between Minkowski metrics with different $c$. On the contrary, in effective GR the existence of distinguished Minkowski metric with fixed parameters is typical.
latter case \( h \) must be included into metric. Inclusion of \( h \) into metric suggests that \( h \) becomes related to gravity (maybe to dilaton field in effective gravity) and this reflects the peaceful coexistence and interplay of gravity and quantum mechanics.

Finally, let us mention an example, where the difference between the fundamental and effective theories is most pronounced. If Minkowski vacuum emerges as a result of spontaneous symmetry breaking, it can be degenerate with different signs of emergent parameters \( h \) and \( c \). The bosonic fields cannot distinguish between the vacua with different signs of \( c \) and \( h \), because these quantities enter quadratically into the metric. For fermions these 4 vacua are physically different, because their dynamics is determined by the vierbein, rather than by metric, and vierbein depends linearly on \( h \) and \( c \). The physical laws, however, are the same in all these vacua, if the underlying symmetry which connects these vacua is exact, and in all the equations discussed here \( h \) must be substituted by \( \sqrt{h^2} \). The interesting consequence of such symmetry breaking would be the possibility of domain walls separating the vacua with different signs of \( h \) and \( c \): the \( h \)-wall where \( h \) changes sign; the corresponding \( c \)-walls; and combined walls. Within the \( h \)-wall, the parameter \( h \) crosses either the value \( h = 0 \) or the value \( h = \infty \). In condensed matter the analogs of such walls in the vierbein field can be found in Ref. [15]; these include the walls in which one, two or all the three speeds of light in Eq. (1.7) change sign (see also Ref. [16] and references therein).

It is a pleasure to thank Frans Klinkhamer, Gordey Lesovik and Yuriy Makhlin for discussions and criticism, and Michael Duff for e-mail correspondence. This work is supported in part by the Russian Foundation for Basic Research (grant 06-02-16002-a) and the Khalatnikov–Starobinsky leading scientific school (grant 4899.2008.2).

[1] C.J. Borde, Base units of the SI, fundamental constants and modern quantum physics, Phil. Trans. R. Soc. A\textbf{363}, 2177–2201 (2005).
[2] J.-P. Uzan, The fundamental constants and their variation: observational and theoretical status, Rev. Mod. Phys. \textbf{75}, 403–455 (2003).
[3] G.E. Volovik, \textit{The Universe in a Helium Droplet}, Clarendon Press, Oxford (2003).
[4] G.E. Volovik, Fundamental constants in effective theory, JETP Lett. \textbf{76}, 77–79 (2002), arXiv:physics/0203075.
[5] M.J. Duff, A party political broadcast on behalf of the Zero Constants Party, in: M.J. Duff, L.B. Okun and G. Veneziano, Triologue on the number of fundamental constants, JHEP 0203 (2002) 023, arXiv:physics/0110060.
[6] F.R. Klinkhamer and G.E. Volovik, Dynamic vacuum variable and equilibrium approach in cosmology, Phys. Rev. D \textbf{78}, 063528 (2008), arXiv:0806.2805.
[7] F.R. Klinkhamer and G.E. Volovik, Possible solution of the cosmological constant problem, arXiv:0907.3887.
[8] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press (1995).

[9] F.R. Klinkhammer and G.E. Volovik, Gluonic vacuum, $q$–theory, and the cosmological constant, Phys. Rev. D 79, 063527 (2009); arXiv:0811.4347.

[10] R.C. Tolman, *Relativity, Thermodynamics and Cosmology*, Clarendon Press, Oxford (1934).

[11] J.L. Flowers and B.W. Petley, Planck, units, and modern metrology, Ann. Phys. (Berlin) 17, 101–114 (2008).

[12] J.P. Pekola, J.J. Vartiainen, M. Möttönen, O.-P. Saira, M. Meschke, D.V. Averin, Hybrid single-electron transistor as a source of quantized electric current, Nature Physics, 4, 120–124 (2008); arXiv:0708.1995.

[13] G.E. Volovik, *Exotic properties of superfluid $^3$He*, World Scientific, Singapore, 1992.

[14] M. Visser, Heuristic approach to the Schwarzschild geometry, Int. J. Mod. Phys. D 14, 2051–2068 (2005); gr-qc/0309072.

[15] G.E. Volovik, Superfluid $^3$He-B and gravity, Physica B 162, 222–230 (1990).

[16] G.E. Volovik, Topological invariant for superfluid $^3$He-B and quantum phase transitions, Pis’ma ZhETF 90, 639–643 (2009); arXiv:0909.3084.