Noisy-Interference Sum-Rate Capacity for Vector Gaussian Interference Channels

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Abstract

New sufficient conditions for a vector Gaussian interference channel to achieve the sum-rate capacity by treating interference as noise are derived, which generalize the existing results. More concise conditions for multiple-input-single-output, and single-input-multiple-output scenarios are obtained.

I. INTRODUCTION

The interference channel (IC) was first introduced by Shannon [1], and was later studied by Ahlswede [2] who gave a limiting expression for the capacity region. Determination of the single-letter expression of the capacity region of an IC has been a long standing open problem ever since.

The first capacity region of the IC was obtained by Carleial in [3] for the very strong interference case, in which the capacity is achieved by decoding and subtracting the interference before decoding the useful signals. The Gaussian IC model with power constraint was also introduced in [3]. The result of [3] was later extended to discrete memoryless ICs in [4]. In [5], Carleial showed that any Gaussian IC can be written in the standard form, i.e., both direct links have unit channel gain and the Gaussian noise has unit variance. An inner bound on the capacity region was obtained in [5] using superposition coding and sequential decoding. The best inner bound was obtained in [6] using superposition coding and joint decoding. This inner bound was later simplified in [7] and [8]. Early outer bounds on the capacity region of the IC can be found in [9], [10] and [11]. The capacity region of Gaussian IC with strong interference was obtained in [6] and [12], in which jointly decoding both the interference and the useful signal achieves the capacity. This result was extended to discrete memoryless ICs in [13].

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degraded memoryless IC was studied in [14] and later in [15]. The degraded Gaussian IC was studied in [12] and the sum-rate capacity was obtained. It was shown in [16] that the capacity region of a Gaussian Z interference channel (ZIC) is equivalent to that of a degraded Gaussian IC. Therefore, the sum-rate capacity of a Gaussian ZIC is automatically obtained. The corner points of the capacity region of a Gaussian IC were also studied in [16] and this still remains an open problem [17]. In [18], it has been shown that Gaussian inputs do not achieve the capacity region of the Gaussian IC in the limiting expression of [2].

In [19], two outer bounds on the capacity region were derived. The first bound is based on a genie-aided approach in which additional information is provided to the receivers. The second bound of [19] is obtained by allowing cooperation between transmitters. It was speculated in [19] that there might be other genies which give tighter outer bound than [19, Theorem 1]. In [20] another outer bound was derived using different genies. Using this bound, the Han and Kobayashi inner bound [6] is shown to be within 1 bit of the capacity region. Motivated by [20], new outer bounds were derived in [21]–[23] and it was shown that the sum-rate capacity is achieved by treating interference as noise if the IC satisfies a simple condition. This kind of Gaussian IC is said to have noisy interference. This noisy-interference sum-rate capacity is extended to multi-user Gaussian ICs in [23]–[25]. Meanwhile, the sum-rate capacity for Gaussian ICs with mix-interference was determined in [22] and [26] using [19, Theorem 1].

In this paper, we study the capacity of the two-user multiple-input multiple-output (MIMO) IC. As shown in Fig. 1, the received signals are defined as

\[
\begin{align*}
\mathbf{y}_1 &= \mathbf{H}_1 \mathbf{x}_1 + \mathbf{F}_2 \mathbf{x}_2 + \mathbf{z}_1 \\
\mathbf{y}_2 &= \mathbf{H}_2 \mathbf{x}_2 + \mathbf{F}_1 \mathbf{x}_1 + \mathbf{z}_2
\end{align*}
\]

(1)

where \( \mathbf{x}_i, i = 1, 2, \) is the transmitted (column) vector signal of user \( i \) which is subject to the average power constraint

\[
\sum_{j=1}^{n} \text{tr} \left( \mathbb{E} [\mathbf{x}_{ij} \mathbf{x}_{ij}^T] \right) \leq nP_i
\]

(2)

where \( \mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{in}, \) is the transmitted vector sequence of user \( i, \) and \( P_i \) is the power constraint. The noise \( \mathbf{z}_i \) is a Gaussian random vector with zero mean and identity covariance matrix; and \( \mathbf{H}_i \) and \( \mathbf{F}_i, \) \( i = 1, 2, \) are the channel matrices known at both the transmitters and receivers. Transmitter \( i \) has \( t_i \) antennas and receiver \( i \) has \( r_i \) antennas. Without loss of generality, we assume \( \mathbf{H}_i \neq \mathbf{0} \) and \( P_i > 0. \)

The capacity of a MIMO IC was first studied in [27] which derived an outer bound on the capacity region and determined the capacity region for the single-input-multiple-output (SIMO) IC with strong
interference. A lower bound for the sum-rate capacity based on Han and Kobayashi’s region were discussed in [28]. Telatar and Tse [29] showed that Han and Kobayashi’s region is within one bit per receive antenna of the capacity region. Recent work in [30] and [31] extended the existing capacity results from scalar ICs to MIMO ICs under average power constraints. Specifically, [30] and [31] derived the capacity region for aligned-strong interference, and the sum-rate capacity for aligned-strong $Z$ interference, aligned-weak $Z$ interference, noisy interference and mixed interference under average power constraints.

In [31], we say that a MIMO IC has

- aligned-strong interference if $H_i = F_i A_i$, $i = 1, 2$; or aligned strong $Z$ interference: if $F_1 = 0$ and $H_2 = F_2 A_2$;
- aligned-weak $Z$ interference: if $F_1 = 0$ and $F_2 = H_2 A_2$;
- noisy interference if [31, (36)-(39)] are satisfied for all $S_i \succeq 0$ with $\text{tr}(S_i) \leq P_i$; and
- mixed interference if $H_1 = F_1 A_1$ and $F_2 = H_2 A_2$;

where $A_i$ is a matrix satisfying $A_i A_i^T \preceq I$, and $I$ is an identity matrix. It can be shown that the capacity region of the SIMO IC with strong interference [27] is a special case of that of the aligned-strong interference. Moreover, the capacity results for aligned-strong interference, aligned-strong or aligned-weak $Z$ interference and mixed-interference apply to other power constraints, e.g., a covariance matrix constraint, a peak power constraint and a per-antenna power constraint.

The noisy-interference condition for MIMO ICs was later studied in [32] which requires only the optimal covariance matrices of $x_1$ and $x_2$ to satisfy the conditions [31, (36)-(39)], as long as these optimal covariance matrices are of full rank. An application of this result is the noisy-interference sum-rate capacity for symmetric SIMO ICs, i.e., $H_i$ and $F_i$ are column vectors with $H_1 = H_2$ and $F_1 = F_2$ and the power constraints are identical $P_1 = P_2$.

The results of [31] and [32] on the MIMO IC with noisy interference obtain different power regions.
Intuitively, [31] obtains the low power region of the noisy interference and [32] obtains the comparatively high power region of the noisy interference. The reason is that, [31] requires the power to be low enough such that any power allocation satisfies conditions [31, (36)-(39)]; while [32] requires the power to be high enough such that each eigen-mode is allocated non-zero power, and [31, (36)-(39)] are satisfied.

There exist MIMO ICs with noisy interference but which are not in the categories of [31] or [32]. These MIMO ICs include the parallel Gaussian IC [33] in which $H_i$ and $F_i$ are diagonal matrices, and the symmetric multiple-input-single-output (MISO) IC [32] in which $H_i$ and $F_i$ are row vectors with $H_1 = H_2$ and $F_1 = F_2$ and the power constraints are identical $P_1 = P_2$. For the noisy-interference conditions of both the parallel Gaussian IC and the symmetric MISO IC, there may exist some power allocations that violate [31, (36)-(39)]. Furthermore, the optimal input covariance matrices for the parallel Gaussian IC can be singular, and the optimal input covariance matrices for the symmetric MISO IC is always rank-1. Therefore, neither [31] nor [32] applies to these two special cases.

The major difficulty in the determination of the noisy-interference sum-rate capacity of a MIMO IC is that the characterization of the optimal input covariance matrices by treating interference as noise is needed in the derivation. However, these optimal input covariance matrices are unknown due to the non-convex nature of the optimization problem for maximizing the sum rate of single-user detection. In [31] all the possible input covariance matrices are required to satisfy some conditions. The results in [32] and [33], although not requiring all the input covariance matrices to satisfy the conditions, they do have some assumptions, or have some knowledge on the optimal input covariance matrices:

- Special MIMO ICs in [32]: the optimal input covariance matrices are assumed to be of full rank.
- Parallel Gaussian IC in [33]: the optimal input covariance matrices are diagonal. More importantly, the optimal power allocated at each antenna satisfies the parallel supporting hyperplane condition, or in another words, the sum-rate function for each sub-channel has the same subgradient at the optimal power allocation.
- Symmetric MISO IC in [32]: beamforming achieves the largest sum-rate for treating interference as noise. Thus the optimal input covariance matrices are both rank-1. The optimality of beamforming was proved in [34] and [35]. The same result was reproduced using different methods in [36] and [37]. By restricting to rank-1 matrices and using the assumption that the MISO IC is symmetric, the closed-form optimal input covariance matrices are obtained, which is crucial in deriving the noisy interference condition.

In this paper, we revisit the sum-rate capacity of the MIMO IC and derive a new noisy-interference
condition, i.e., treating interference as noise achieves the sum-rate capacity. This new condition requires only the optimal input covariance matrices to satisfy [31, (36)-(39)] and an additional condition, but does not require the optimal input covariance matrices to be of full rank (when they are of full rank, this additional condition is automatically satisfied). Thus, this new noisy-interference condition includes those in [31] and [32] as special cases. In addition, this noisy-interference condition includes those of the parallel Gaussian IC [33] and the symmetric MISO IC [32] as special cases. More concise condition for the general asymmetric MISO or SIMO ICs are also obtained.

The rest of the paper is organized as follows: the noisy-interference sum-rate capacity for the MIMO IC is obtained in Section II; the MISO and SIMO ICs are discussed in Sections III and IV, respectively; numerical examples are given in Section V; and we conclude in Section VI.

Before proceeding, we introduce some notation that will be used in the paper.

- Italic letters (e.g. $X$) denote scalars; and bold letters $x$ and $X$ denote column vectors and matrices, respectively.
- $I$ denotes the identity matrix and $0$ denotes the all-zero vector or matrix. The dimensions of $I$ and $0$ are determined by the context.
- $|X|$, $X^T$, $X^{-1}$ and rank($X$) denote respectively the determinant, transpose, inverse, and rank of the matrix $X$, and $||x||$ denotes the Euclidean vector norm of $x$, i.e., $||x||^2 = x^T x$.
- radius($X$) is the numerical radius [38, p. 321] of the square real matrix $X$, and is defined as
  \[
  \text{radius}(X) = \max_{\alpha^T\alpha \leq 1} \text{abs} (\alpha^T X \alpha),
  \]
  where $\alpha$ is a vector, and $\text{abs}(\cdot)$ denotes the absolute value.
- $x^n = [x_1^T, x_2^T, \ldots, x_n^T]^T$ is a long vector that consists of a sequence of vectors $x_i, i = 1, \ldots, n$.
- $\text{diag}[X_1, \ldots, X_n]$ is a diagonal matrix with diagonal entries $X_i$.
- $\text{Vec}(A)$ denote the vectorization operator, i.e., let $A = [a_1, a_2, \ldots, a_n]$, and $a_i, i = 1, \ldots, n$ be the column vectors, then $\text{Vec}(A) = [a_1^T, a_2^T, \ldots, a_n^T]^T$.
- $x \sim N(0, \Sigma)$ means that the random vector $x$ has Gaussian distribution with zero mean and covariance matrix $\Sigma$.
- $E[\cdot]$ denotes expectation; Cov(\cdot) denotes covariance matrix; $I(\cdot; \cdot)$ denotes mutual information; $h(\cdot)$ denotes differential entropy with the logarithm base $e$, and $\log(\cdot) = \log_e(\cdot)$.

II. MIMO ICs

We first derive a lower bound and an upper bound on the sum-rate capacity. The lower bound is simply the single-user detection sum rate. The upper bound is obtained by providing the receivers with
appropriate side information. Both the lower and upper bounds are formulated as optimization problems in which the lower bound is a non-convex problem and the upper bound is a convex problem. The sum-rate capacity is obtained by determining conditions under which these two optimization problems have the same solution.

A. Lower bound on the sum-rate capacity

By treating interference as noise, the maximum of the following optimization problem is a lower bound on the sum-rate capacity:

\[
\max \frac{1}{2} \log \left| I + H_1 S_1 H_1^T (I + F_2 S_2 F_2^T)^{-1} \right| + \frac{1}{2} \log \left| I + H_2 S_2 H_2^T (I + F_1 S_1 F_1^T)^{-1} \right| \\
\text{subject to } \text{tr}(S_1) \leq P_1, \quad \text{tr}(S_2) \leq P_2
\]

\[
S_1 \succeq 0, \quad S_2 \succeq 0.
\]

(3)

The following lemma gives the necessary Karush-Kuhn-Tucker (KKT) conditions for the optimal input covariance matrices \(S_i^*, i = 1, 2\).

Lemma 1: Let \(S_i^*\) and \(S_i^*\) be optimal for problem (3), if \(P_1, P_2 > 0\), then there exist scalars \(\lambda_i\) and matrices \(W_i, i = 1, 2\), such that

\[
G_i + \lambda_i I - W_i = 0
\]

\[
\lambda_i \begin{cases} > 0 & \text{if } \text{tr}(S_i^*) = P_i \\ = 0 & \text{if } \text{tr}(S_i^*) < P_i \end{cases} \quad i = 1, 2
\]

\[
\text{tr}(S_i^* W_i) = 0, \quad W_i \succeq 0 \quad i = 1, 2
\]

(6)

where

\[
G_1 = -\frac{\partial R_{11}}{\partial S_1} \bigg|_{S_1 = S_1^*} - \frac{\partial R_{21}}{\partial S_1} \bigg|_{S_1 = S_1^*}
\]

\[
G_2 = -\frac{\partial R_{11}}{\partial S_2} \bigg|_{S_2 = S_2^*} - \frac{\partial R_{21}}{\partial S_2} \bigg|_{S_2 = S_2^*}
\]

(8)

(9)

\[
\frac{\partial R_{11}}{\partial S_1} \bigg|_{S_1 = S_1^*} = \frac{1}{2} H_1^T (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} H_1
\]

\[
\frac{\partial R_{11}}{\partial S_2} \bigg|_{S_2 = S_2^*} = -\frac{1}{2} F_2^T \left[ (I + F_2 S_2^* F_2^T)^{-1} - (I + H_2 S_2^* H_2^T + F_2 S_2^* F_2^T)^{-1} \right] F_2
\]

\[
\frac{\partial R_{21}}{\partial S_2} \bigg|_{S_2 = S_2^*} = -\frac{1}{2} F_2^T \left[ (I + F_2 S_2^* F_2^T)^{-1} - (I + H_2 S_2^* H_2^T + F_2 S_2^* F_2^T)^{-1} \right] F_2
\]

(10)

(11)

(12)
\[
\frac{\partial R_{2l}}{\partial \mathbf{S}_2} \bigg|_{\mathbf{S}_1 = \mathbf{S}_i} = \frac{1}{2} \mathbf{H}_2^T (\mathbf{I} + \mathbf{H}_2 \mathbf{S}_2 \mathbf{H}_2^T + \mathbf{F}_1 \mathbf{S}_1 \mathbf{F}_1^T)^{-1} \mathbf{H}_2
\]

and

\[
R_{1l}(\mathbf{S}_1, \mathbf{S}_2) = \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1 \mathbf{S}_1 \mathbf{H}_1^T (\mathbf{I} + \mathbf{F}_2 \mathbf{S}_2 \mathbf{F}_2^T)^{-1} \right|
\]

\[
R_{2l}(\mathbf{S}_1, \mathbf{S}_2) = \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_2 \mathbf{S}_2 \mathbf{H}_2^T (\mathbf{I} + \mathbf{F}_1 \mathbf{S}_1 \mathbf{F}_1^T)^{-1} \right|.
\]

**Proof:** Conditions (4)-(7) are the KKT conditions for problem (3). Here, we only need to prove that problem (3) satisfies some constraint qualifications denoted by CQ5 in [39, p. 306] such that \(\lambda_i\) and \(\mathbf{W}_i\) do exist. The rest of the proof is included in Appendix A. \(\blacksquare\)

**B. Upper bound on the sum-rate capacity**

The following is an upper bound on the sum-rate capacity of a MIMO IC.

**Theorem 1:** The sum-rate capacity of the MIMO IC is upper bounded by the maximum achieved in the following optimization problem:

\[
\max \quad \frac{1}{2} \log \left| \mathbf{I} + \left[ \begin{array}{c|c} \mathbf{H}_1 & \mathbf{F}_1 \\ \hline \mathbf{F}_1^T & \mathbf{E}_1 \end{array} \right] \mathbf{S}_1 \left[ \begin{array}{c|c} \mathbf{H}_1 & \mathbf{F}_1 \\ \hline \mathbf{F}_1^T & \mathbf{E}_1 \end{array} \right]^T \left( \mathbf{E}_1 + \left[ \begin{array}{c|c} \mathbf{F}_2 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^T & \mathbf{E}_2 \end{array} \right] \left[ \begin{array}{c|c} \mathbf{F}_2 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^T & \mathbf{E}_2 \end{array} \right]^T \right)^{-1} \right|
\]

\[+ \frac{1}{2} \log \left| \mathbf{I} + \left[ \begin{array}{c|c} \mathbf{H}_2 & \mathbf{F}_2 \\ \hline \mathbf{F}_2^T & \mathbf{E}_2 \end{array} \right] \mathbf{S}_2 \left[ \begin{array}{c|c} \mathbf{H}_2 & \mathbf{F}_2 \\ \hline \mathbf{F}_2^T & \mathbf{E}_2 \end{array} \right]^T \left( \mathbf{E}_2 + \left[ \begin{array}{c|c} \mathbf{F}_1 & \mathbf{S}_1 \\ \hline \mathbf{S}_1^T & \mathbf{E}_1 \end{array} \right] \left[ \begin{array}{c|c} \mathbf{F}_1 & \mathbf{S}_1 \\ \hline \mathbf{S}_1^T & \mathbf{E}_1 \end{array} \right]^T \right)^{-1} \right|
\]

subject to \(\text{tr} (\mathbf{S}_1) \leq P_1, \quad \text{tr} (\mathbf{S}_2) \leq P_2\)

\[
\mathbf{S}_1 \succeq 0, \quad \mathbf{S}_2 \succeq 0
\]

where \(\mathbf{E}_i, i = 1, 2\), can be any symmetric positive definite matrix satisfying

\[
\mathbf{E}_i = \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{A}_i \\ \hline \mathbf{A}^T_i & \mathbf{\Sigma}_i \end{array} \right] \succ 0
\]

\[
\mathbf{\Sigma}_1 \preceq \mathbf{I} - \mathbf{A}_2 \mathbf{\Sigma}_2^{-1} \mathbf{A}_2^T
\]

\[
\mathbf{\Sigma}_2 \preceq \mathbf{I} - \mathbf{A}_1 \mathbf{\Sigma}_1^{-1} \mathbf{A}_1^T.
\]

**Proof:** Let \(\mathbf{n}_i^n, i = 1, 2\), be a length-\(n\) sequence of independent and identically distributed (i.i.d.) Gaussian vectors, each having joint distribution with \(\mathbf{z}_i\) given by

\[
\begin{bmatrix} \mathbf{z}_i \\ \mathbf{n}_i \end{bmatrix} \sim \mathcal{N} (\mathbf{0}, \mathbf{E}_i) = \mathcal{N} \left( \mathbf{0}, \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{A}_i \\ \hline \mathbf{A}^T_i & \mathbf{\Sigma}_i \end{array} \right] \right).
\]

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Let \( x_i^n \) be the input sequence of user \( i \), and

\[
\sum_{j=1}^n \text{Cov} (x_{ij}) = nS_i \tag{21}
\]

\[
\text{tr} (S_i) \leq P_i \tag{22}
\]

Let \( \epsilon > 0 \) and \( \epsilon \to 0 \) when \( n \to \infty \). Then for any achievable rate \( R_1 \) and \( R_2 \), we have

\[
n(R_1 + R_2) - n\epsilon
\]

\[
\leq I (x_1^n; H_1 x_1^n + F_2 x_2^n + z_1^n) + I (x_2^n; H_2 x_2^n + F_1 x_1^n + z_2^n)
\]

\[
\leq I (x_1^n; H_1 x_1^n + F_2 x_2^n + z_1^n, F_1 x_1^n + n_1^n) + I (x_2^n; H_2 x_2^n + F_1 x_1^n + z_2^n, F_2 x_2^n + n_2^n)
\]

\[
= h (F_1 x_1^n + n_1^n) - h (n_1^n) + h (H_1 x_1^n + F_2 x_2^n + z_1^n | F_1 x_1^n + n_1^n) - h (F_2 x_2^n + z_2^n | n_2^n)
\]

\[
+ h (F_2 x_2^n + n_2^n) - h (n_2^n) + h (H_2 x_2^n + F_1 x_1^n + z_2^n | F_2 x_2^n + n_2^n) - h (F_1 x_1^n + z_1^n | n_1^n)
\]

\[
\leq h (F_1 x_1^n + n_1^n) - nh (n_1^n) + nh (H_1 x_1G + F_2 x_2G + z_1 | F_1 x_1G + n_1) - h (F_2 x_2^n + z_2^n | n_2^n)
\]

\[
+ h (F_2 x_2^n + n_2^n) - nh (n_2^n) + nh (H_2 x_2G + F_1 x_1G + z_2 | F_2 x_2G + n_2) - h (F_1 x_1^n + z_1^n | n_1^n)
\]

\[
\leq nh (F_1 x_1G + n_1) - nh (n_1^n) + nh (H_1 x_1G + F_2 x_2G + z_1 | F_1 x_1G + n_1) - nh (F_2 x_2G + z_1 | n_1)
\]

\[
+ nh (F_2 x_2G + n_2) - nh (n_2^n) + nh (H_2 x_2G + F_1 x_1G + z_2 | F_2 x_2G + n_2) - nh (F_1 x_1G + z_2 | n_2)
\]

\[
= nl (x_{1G}; [H_1 \quad F_2] x_{1G} + [F_1 \quad 0] x_{2G} + [z_1 \quad n_1]) + nl (x_{1G}; [H_2 \quad F_2] x_{2G} + [F_1 \quad 0] x_{1G} + [z_2 \quad n_2]) \tag{23}
\]

where in (a) we define \( x_{iG} \sim \mathcal{N} (0, S_i) \) and the inequality is by [31, Lemma 2], and (b) is by (18), (19) and [31, Lemma 3].

The following lemma establishes the convexity of the optimization problem (16) and the proof is included in Appendix B.

**Lemma 2:** The optimization problem (16) is a convex optimization problem.

Theorem 1 is derived using the same method that has been used in [31]. The maximum achieved in problem (16) for any choice of \( A_i \) and \( \Sigma_i \) that satisfy (17)-(19) is an upper bound on the sum-rate capacity of this MIMO IC regardless of whether it has noisy interference or not.

### C. Sum-rate capacity

When the MIMO IC has noisy interference, we can choose appropriate \( A_i \) and \( \Sigma_i \) such that the lower and upper bounds converge. Before proceeding, we first introduce the following matrix identity which will be used repeatedly in the proof of our main result.
Lemma 3: Assuming all the matrices have feasible dimension and the relevant matrices are invertible, we have
\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}^{-1} = \begin{bmatrix}
A_{11}^{-1} & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
A_{11}^{-1}A_{12} \\
-A_{11}^{-1}
\end{bmatrix}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \begin{bmatrix}
A_{21}^{-1} \\
-A
\end{bmatrix}.
\] (25)

Proof:
\[
\begin{aligned}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}^{-1} &= \begin{bmatrix}
(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12} (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\
-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
A_{11}^{-1} + A_{11}^{-1}A_{12} (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12} (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\
-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{bmatrix} \\
&= \begin{bmatrix}
A_{11}^{-1} & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
A_{11}^{-1}A_{12} \\
-A_{11}^{-1}
\end{bmatrix}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \begin{bmatrix}
A_{21}^{-1} \\
-A
\end{bmatrix}.
\]
\]
where (a) is by the block matrix inversion lemma [38, p. 18], and (b) is by the Woodbury matrix identity [38, p. 19]:
\[
(C + UBV)^{-1} = C^{-1} - C^{-1}U(B^{-1} + VC^{-1}U)^{-1}VC^{-1}.
\] (26)

The noisy-interference sum-rate capacity of a MIMO IC is obtained in the following theorem:

Theorem 2: For the MIMO IC defined in (1) and $P_i > 0, i = 1, 2$, if the optimal solution of problem (3) has $\text{tr}(S_i^r) > 0$, and there exist matrices $A_i$ and $\Sigma_i$ that satisfy (17)-(19) and
\[
S_i^rF_i^T = S_i^rH_i^T(I + F_2S_2^rF_2^T)^{-1}A_i
\] (27)
\[
S_2^rF_2^T = S_2^rH_2^T(I + F_1S_1^rF_1^T)^{-1}A_2
\] (28)
\[
W_1 \succeq O_1
\] (29)
\[
W_2 \succeq O_2
\] (30)
where
\[
W_1 = G_1 - \frac{\text{tr}(S_1^rG_1)}{P_1}I
\] (31)
\[
W_2 = G_2 - \frac{\text{tr}(S_2^rG_2)}{P_2}I
\] (32)
\[ O_1 = \frac{1}{2} \left[ A_1^T (I + F_2 S_2^2 F_2^T)^{-1} H_1 - F_1 \right]^T \left[ \Sigma_1 - A_1^T (I + F_2 S_2^2 F_2^T)^{-1} A_1 \right]^{-1} \]
\[ \cdot \left[ A_1^T (I + F_2 S_2^2 F_2^T)^{-1} H_1 - F_1 \right] \] (33)

\[ O_2 = \frac{1}{2} \left[ A_2^T (I + F_1 S_1^2 F_1^T)^{-1} H_2 - F_2 \right]^T \left[ \Sigma_2 - A_2^T (I + F_1 S_1^2 F_1^T)^{-1} A_2 \right]^{-1} \]
\[ \cdot \left[ A_2^T (I + F_1 S_1^2 F_1^T)^{-1} H_2 - F_2 \right] \] (34)

and \( G_1 \) and \( G_2 \) are defined in (8) and (9), respectively, then the sum-rate capacity is the maximum in problem (3) and is achieved by Gaussian input \( x_i^* \sim \mathcal{N}(0, S_i^*) \) and treating interference as noise.

**Proof:** It suffices to show that under conditions (17)-(19) and (27)-(30), the upper bound on the sum-rate capacity, i.e., the maximum in problem (16) for the given \( A_i \) and \( \Sigma_i \), is the same as the maximum in problem (3); and the maximum in (16) is also achieved by \( S_i^* \).

The proof has two stages. In stage one, we rewrite the objective function of problem (16) and show that this objective function, by choosing \( S_i = S_i^* \), equals the maximum achieved in problem (3). In stage two, we compare the KKT conditions of problems (3) and (16), and show that if the conditions in this theorem are all satisfied, then problem (16) is solved by the same \( S_i^* \) that maximizes (3).

Define

\[ R_{1u}(S_1, S_2) = \frac{1}{2} \log \left| \left[ H_1 \right]^T S_1 \left[ H_1 \right]^T \right| \left( E_1 + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right)^{-1} \left( E_1 + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right) \] (35)

\[ R_{2u}(S_1, S_2) = \frac{1}{2} \log \left| \left[ H_2 \right]^T S_2 \left[ H_2 \right]^T \right| \left( E_2 + \left[ F_1 \right] S_1 \left[ F_1 \right]^T \right)^{-1} \left( E_2 + \left[ F_1 \right] S_1 \left[ F_1 \right]^T \right) \] (36)

Before proceeding, we first show the following equality since it will be used repeatedly in the sequel:

\[ \left[ H_i \right]^T \left( E_i + \left[ F_j \right] S_j \left[ F_j \right]^T \right)^{-1} \left[ H_i \right] = \left[ H_i \right]^T \left[ I + F_j S_j F_j^T \right]^{-1} \left[ A_i \right]^{-1} \left[ H_i \right] \]

\[ \left[ F_i \right]^T \left[ I + F_j S_j F_j^T \right]^{-1} \left[ A_i \right]^{-1} \left[ F_i \right] \]

\[ \left( \Sigma_i - A_i^T (I + F_j S_j F_j^T)^{-1} A_i \right)^{-1} \left[ A_i^T (I + F_j S_j F_j^T)^{-1} - I \right]^{-1} \left[ H_i \right] \]

\[ \left[ F_i \right] \]
where (a) is by Lemma 3, \( i, j \in \{1, 2\} \), and \( i \neq j \), and we define \( \overline{\Omega}_i \) in the same way as in (33) and (34) by replacing \( S^*_i \) with \( S_i \).

We first show

\[
R_{1u} (S^*_1, S^*_2) = R_{1l} (S^*_1, S^*_2): 
\]

\[
\frac{1}{2} \log \left| I + S_1 \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \left( E_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix} S_2 \begin{bmatrix} F_2 \\ 0 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \right| 
\]

\[
= \frac{1}{2} \log \left| I + S_1 H_1^T (I + F_2 S_2 F_2^T)^{-1} H_1 + 2S_1 \overline{\Omega}_1 \right| 
\]

where (a) is by the matrix identity

\[
|I + CD| = |I + DC| 
\]

and (b) is from (37). Similarly, we have

\[
R_{2u} (S_1, S_2) = \frac{1}{2} \log \left| I + S_2 H_2^T (I + F_1 S_1 F_1^T)^{-1} H_2 + 2S_2 \overline{\Omega}_2 \right|. 
\]

Since (27) and (28) imply

\[
S^*_i O_i = 0 
\]

then we immediately have

\[
R_{1u} (S^*_1, S^*_2) = \frac{1}{2} \log \left| I + S_1^* H_1^T (I + F_2 S_2^* F_2^T)^{-1} H_1 \right| 
\]

\[
= \frac{1}{2} \log \left| I + H_1 S_1^* H_1^T (I + F_2 S_2^* F_2^T)^{-1} \right| 
\]

\[
= R_{1l} (S^*_1, S^*_2) 
\]

where the second equality is by (39). Similarly, we have

\[
R_{2u} (S^*_1, S^*_2) = R_{2l} (S^*_1, S^*_2). 
\]

Next, we prove that the maximum in problem (16) is achieved when \( S_i = S^*_i \). Since by Lemma 2, problem (16) is a convex optimization problem, it suffices to prove that there exist Lagrangian multipliers \( \lambda_i \) and \( \Sigma_i \) such that the following KKT conditions are satisfied:

\[
- \frac{\partial R_{1u}}{\partial S_1} \bigg|_{S_1 = S^*_1} - \frac{\partial R_{2u}}{\partial S_1} \bigg|_{S_1 = S^*_1} + \lambda_i I - \Sigma_i = 0 
\]

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\[ \begin{aligned}
- \frac{\partial R_{1u}}{\partial S_1} \bigg|_{s_1=s_i^*} &= - \frac{\partial R_{2u}}{\partial S_2} \bigg|_{s_2=s_i^*} + \lambda_2 I - \mathbf{W}_2 = 0 \\
\lambda_i \begin{cases}
> 0 & \text{if } \text{tr}(S_i^*) = P_i \\
= 0 & \text{if } \text{tr}(S_i^*) < P_i
\end{cases} \\
\text{tr} (S_i^* \mathbf{W}_i) = 0, \quad \mathbf{W}_i \succeq 0.
\end{aligned} \] (45)

We first compute
\[ \begin{aligned}
- \frac{\partial R_{1u}}{\partial S_1} \bigg|_{s_1=s_i^*} &= - \frac{1}{2} \frac{\partial}{\partial S_1} \left( \log \left| I + S_1 \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \right| \begin{bmatrix} E_1 + S_2 \begin{bmatrix} F_2 \\ 0 \end{bmatrix} & \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \right) \bigg|_{S_1=s_i^*} \\
&= - \frac{1}{2} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \left( E_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix} S_2^T \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \\
& \cdot \left( I + S_i^* \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \right) \left( E_1 + \begin{bmatrix} F_2 \\ 0 \end{bmatrix} S_2^T \begin{bmatrix} F_2 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^{-1} \\
&= - \frac{1}{2} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \begin{bmatrix} I + F_2 S_2^T F_2^T \\ \Sigma_1 \end{bmatrix}^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \left( I + S_i^* \left( \begin{bmatrix} H_1^T \\ F_1^T \end{bmatrix} \begin{bmatrix} I + F_2 S_2^T F_2^T \\ \Sigma_1 \end{bmatrix}^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \right) \right) \\
&= - \frac{1}{2} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix}^T \begin{bmatrix} I + F_2 S_2^T F_2^T \\ \Sigma_1 \end{bmatrix}^{-1} \begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \left( I + S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right) \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I + S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right)^{-1} \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I + S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right)^{-1} \\
&= - \frac{1}{2} \left( H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 + 2O_1 \right) \left( I + S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right)^{-1} \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I + S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right)^{-1} \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I - S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right) \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I - S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right) \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \left( I - S_i^* H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 \right) \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 - O_1 \\
&= - \frac{1}{2} H_1^T \left( I + F_2 S_2^T F_2^T \right)^{-1} H_1 - O_1
\end{aligned} \] (46)
\[
\frac{\partial R_{1u}}{\partial S_2} \bigg|_{s = s^*} = \frac{1}{2} \left( \log E_1 + \left[ H_1 \right] S_1 \left[ H_1 \right]^T + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right) - \log \left( E_1 + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right)
\]

where (a) is by the matrix identity (39), (b) and (d) are both by (37), (c) and (f) are both by (41), (e) is by the Woodbury matrix identity (26), and (g) is by the matrix identity [40, p. 151]:

\[
C (I + DC)^{-1} = (I + CD)^{-1} C.
\]

Then we compute

\[
\frac{\partial R_{1u}}{\partial S_2} \bigg|_{s = s^*} = \frac{1}{2} \left[ F_2 \right]^T \left( \log E_1 + \left[ H_1 \right] S_1 \left[ H_1 \right]^T + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right) - \log \left( E_1 + \left[ F_2 \right] S_2 \left[ F_2 \right]^T \right)
\]

\[
= \frac{1}{2} \left[ F_2 \right]^T \left( \log E_1 + \left[ \begin{array}{c} H_1 \\ S_1 \end{array} \right] \left[ \begin{array}{c} H_1 \\ F_1 \end{array} \right]^T + \left[ \begin{array}{c} F_2 \\ S_2 \end{array} \right] \left[ \begin{array}{c} F_2 \\ 0 \end{array} \right]^T \right) - \log \left( E_1 + \left[ \begin{array}{c} F_2 \\ S_2 \end{array} \right] \left[ \begin{array}{c} F_2 \\ 0 \end{array} \right]^T \right)
\]

\[
= \frac{1}{2} \left[ F_2 \right]^T \left( \log E_1 + \left[ \begin{array}{c} H_1 \\ S_1 \end{array} \right] \left[ \begin{array}{c} H_1 \\ F_1 \end{array} \right]^T + \left[ \begin{array}{c} F_2 \\ S_2 \end{array} \right] \left[ \begin{array}{c} F_2 \\ 0 \end{array} \right]^T \right) - \log \left( E_1 + \left[ \begin{array}{c} F_2 \\ S_2 \end{array} \right] \left[ \begin{array}{c} F_2 \\ 0 \end{array} \right]^T \right)
\]
Similarly, we have

\[ \left[ \frac{1}{2} \begin{bmatrix} F_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right)^{-1} (H_1 S_1^* F_1^T + A_1) \\ -I \end{bmatrix} \right] \\
\left( F_1 S_1^* F_1^T + \Sigma_1 - (F_1 S_1^* H_1^T + A_1^T) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} (F_1 S_1^* H_1^T + A_1^T)^T \right)^{-1} \\
\left[ \begin{bmatrix} \left( F_1 S_1^* H_1^T + A_1^T \right) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} -I \end{bmatrix} \right] \\
\begin{bmatrix} H_1 \\ F_1 \end{bmatrix} \right] \\
\left( I + F_2 S_2^* F_2^T \right)^{-1} F_2 \\
= \frac{1}{2} F_2^T \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right)^{-1} H_1 S_1^* H_1^T \left( I + F_2 S_2^* F_2^T \right)^{-1} F_2 \\
+ \frac{1}{2} \begin{bmatrix} F_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right)^{-1} (H_1 S_1^* F_1^T + A_1) \\ -I \end{bmatrix} \\
\left( F_1 S_1^* F_1^T + \Sigma_1 - (F_1 S_1^* H_1^T + A_1^T) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} (F_1 S_1^* H_1^T + A_1^T)^T \right)^{-1} \\
\left( (F_1 S_1^* H_1^T + A_1^T) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} H_1 - F_1 \right) \left( I + F_2 S_2^* F_2^T \right)^{-1} F_1 \\
\left( I + F_2 S_2^* F_2^T \right)^{-1} F_2 \\
= \frac{1}{2} F_2^T \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right)^{-1} F_2 + \frac{1}{2} F_2^T \left( I + F_2 S_2^* F_2^T \right)^{-1} F_2 \\
= \frac{1}{2} \frac{\partial}{\partial S_2} \left[ \log \left( I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T \right) - \log \left( I + F_2 S_2^* F_2^T \right) \right] \bigg|_{S_1 = S_1^*} \\
= - \frac{\partial R_{11}}{\partial S_2} \bigg|_{S_1 = S_1^*} \\
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équality (b) and (d) are both from Lemma 3, (c) is directly from (27), and (e) is also from (27) which implies

\[ \left( (F_1 S_1^* H_1^T + A_1^T) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} H_1 - F_1 \right) S_1^* \\
= \left( A_1^T (I + F_2 S_2^* F_2^T)^{-1} H_1 S_1^* H_1^T + A_1^T \right) (I + H_1 S_1^* H_1^T + F_2 S_2^* F_2^T)^{-1} H_1 S_1^* - F_1 S_1^* \\
= A_1^T (I + F_2 S_2^* F_2^T)^{-1} H_1 S_1^* - F_1 S_1^* \\
= 0. \] 

Similarly, we have

\[ - \frac{\partial R_{2u}}{\partial S_1} \bigg|_{S_1 = S_1^*} = - \frac{\partial R_{2l}}{\partial S_1} \bigg|_{S_1 = S_1^*} \] 

(53)
\[- \frac{\partial R_{2u}}{\partial S_2} \bigg|_{S_i = S_i^*} = - \frac{\partial R_{2l}}{\partial S_2} \bigg|_{S_i = S_i^*} - \mathbf{O}_2. \quad (54)\]

By \((4)\) and \((7)\), we have

\[S_i^* \mathbf{G}_i + \lambda_i S_i^* = 0. \quad (55)\]

Thus, by \((6)\) we have

\[\lambda_i = - \frac{\text{tr}(S_i^* \mathbf{G}_i)}{P_i}, \quad (56)\]

and hence from \((4)\) and \((5)\) we have

\[\mathbf{W}_i = \mathbf{G}_i - \frac{\text{tr}(S_i^* \mathbf{G}_i)}{P_i} \mathbf{I} \quad (57)\]

i.e., the \(\mathbf{W}_i\)'s defined in \((31)\) and \((32)\) are the Lagrangian multipliers in \((4)\) and \((5)\).

Then, we choose

\[\overline{\lambda}_i = \lambda_i \quad (58)\]

\[\overline{\mathbf{W}}_i = \mathbf{W}_i - \mathbf{O}_i \quad (59)\]

such that

\[- \frac{\partial R_{1u}}{\partial S_1} \bigg|_{S_i = S_i^*} - \frac{\partial R_{2u}}{\partial S_1} \bigg|_{S_i = S_i^*} + \lambda_1 \mathbf{I} - \overline{\mathbf{W}}_1 \\
= - \frac{\partial R_{1l}}{\partial S_1} \bigg|_{S_i = S_i^*} - \mathbf{O}_1 - \frac{\partial R_{2l}}{\partial S_1} \bigg|_{S_i = S_i^*} + \lambda_1 \mathbf{I} - (\mathbf{W}_1 - \mathbf{O}_1) \\
= 0 \quad (60)\]

where the last equality is from \((4)\). Therefore, condition \((44)\) is satisfied. Similarly, condition \((45)\) is also satisfied. Condition \((46)\) is satisfied because of \((6)\), and condition \((47)\) is satisfied by the assumptions \((29)\) and \((30)\) and conditions \((27)\) and \((28)\) which imply

\[S_i^* \overline{\mathbf{W}}_i = S_i^* (\mathbf{W}_i - \mathbf{O}_i) = -S_i^* \mathbf{O}_i = 0 \quad (61)\]

where in the second equality, we use the fact that \(S_i^* \mathbf{W}_i = 0\) when \(\text{tr}(S_i^* \mathbf{W}_i) = 0\) and \(S_i^* \geq 0\) and \(\mathbf{W}_i \geq 0\). Therefore, there exist Lagrangian multipliers such that \(S_i^*\) satisfies the KKT conditions for problem \((16)\). Since problem \((16)\) is a convex optimization problem, \(S_i^*\) achieves the maximum in problem \((16)\). By \((42)\) and \((43)\), we conclude that the maximum in \((3)\) is the sum-rate capacity of the MIMO IC.

**Remark 1:** On comparing the upper bound function \(R_{ui}\) in \((38)\) and \((40)\) with the lower bound function in \((14)\) and \((15)\), respectively, we note that there is an extra term \(2S_i \mathbf{O}_i\) in the logarithm function. It
is obvious that $\mathbf{O}_i \succeq 0$ under conditions (18) and (19). Although $2\mathbf{S}_i \mathbf{O}_i$ may not necessarily be a semi-positive definite matrix, this extra term still increases the rate upon $R_{1l}$, e.g.,

$$R_{1u} = \frac{1}{2} \log \left| \mathbf{I} + \mathbf{S}_i \mathbf{H}_i^T \left( \mathbf{I} + \mathbf{F}_2 \mathbf{S}_2 \mathbf{F}_2^T \right)^{-1} \mathbf{H}_i + 2\mathbf{S}_i \mathbf{O}_i \right|$$

$$= \frac{1}{2} \log \left| \mathbf{I} + \mathbf{S}^\dagger_i \left( \mathbf{H}_i^T \left( \mathbf{I} + \mathbf{F}_2 \mathbf{S}_2 \mathbf{F}_2^T \right)^{-1} \mathbf{H}_i \right)^{1/2} \mathbf{S}^\dagger_i \right|$$

$$\geq \frac{1}{2} \log \left| \mathbf{I} + \mathbf{S}_i \mathbf{H}_i^T \left( \mathbf{I} + \mathbf{F}_2 \mathbf{S}_2 \mathbf{F}_2^T \right)^{-1} \mathbf{H}_i \mathbf{S}_i \right|$$

$$= \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1 \mathbf{S}_i \mathbf{H}_i^T \left( \mathbf{I} + \mathbf{F}_2 \mathbf{S}_2 \mathbf{F}_2^T \right)^{-1} \right|$$

$$= R_{1l}.$$  

Conditions (27) and (28) are sufficient conditions for (41) to hold, which makes the lower and upper bounds converge. This extra term $2\mathbf{S}_i \mathbf{O}_i$ is also considered in the scalar Gaussian IC [21, p. 696] and the parallel Gaussian IC [33, eq. (64)], in which we have $\mathbf{O}_i = 0$ for both cases. Furthermore, conditions (27) and (28) also mean that [31, Lemma 5]

$$\mathbf{x}_{iG}^* \rightarrow \mathbf{H}_i \mathbf{x}_{jG}^* + \mathbf{F}_j \mathbf{x}_{jG}^* + \mathbf{z}_i \rightarrow \mathbf{F}_j \mathbf{x}_{jG}^* + \mathbf{n}_i \quad i, j \in \{1, 2\}, i \neq j$$

form a Markov chain, where $\mathbf{x}_{iG}^* \sim \mathcal{N}(0, \mathbf{S}_i^*)$.

**Remark 2:** When all the conditions in Theorem 2 are satisfied, the optimal input covariance matrix $\mathbf{S}_i^*$ and the corresponding auxiliary matrix $\mathbf{E}_i^*$ in (17) (obtained by replacing $\mathbf{\Sigma}_i$ and $\mathbf{A}_i$ with $\mathbf{\Sigma}_i^*$ and $\mathbf{A}_i^*$ associated with $\mathbf{S}_i^*$), form a saddle point of the upper bound function defined as

$$R_{su}(\mathbf{S}_i, \mathbf{E}_i) = R_{1u}(\mathbf{S}_i, \mathbf{E}_i) + R_{2u}(\mathbf{S}_i, \mathbf{E}_i)$$

where $R_{su}(\mathbf{S}_i, \mathbf{E}_i)$ is defined in (38) and (40). We use this expression in this remark to emphasize that $\mathbf{E}_i$ is also a parameter.

To show that this optimal solution is the saddle point, we first have

$$\min_{\mathbf{E}_i} \max_{\text{tr}(\mathbf{S}_i) \leq P_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) \leq \max_{\text{tr}(\mathbf{S}_i) \leq P_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i^*) = R_{su}(\mathbf{S}_i^*, \mathbf{E}_i^*)$$

where the second equality is by the existence of the Lagrangian multiplier satisfying the KKT conditions, and the convexity of $R_{su}(\mathbf{S}_i, \mathbf{E}_i^*)$ over $\mathbf{S}_i$, which imply that $R_{su}(\mathbf{S}_i, \mathbf{E}_i^*)$ is maximized by $\mathbf{S}_i^*$. On the other hand, we have

$$\max_{\text{tr}(\mathbf{S}_i) \leq P_i} \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) \geq \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i^*, \mathbf{E}_i) = R_{su}(\mathbf{S}_i^*, \mathbf{E}_i^*)$$

where the second inequality is by (41). Since the following is always true

$$\min_{\mathbf{E}_i} \max_{\text{tr}(\mathbf{S}_i) \leq P_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) \geq \max_{\text{tr}(\mathbf{S}_i) \leq P_i} \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i)$$

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we have
\[ \min_{\mathbf{E}_i} \max_{\mathbf{u}(\mathbf{S}_i) \leq \mathbf{P}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = \max_{\mathbf{u}(\mathbf{S}_i) \leq \mathbf{P}_i} \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = R_{su}(\mathbf{S}_i^*, \mathbf{E}_i^*). \]
By [39, Proposition 2.6.1 p. 132], \((\mathbf{S}_i^*, \mathbf{E}_i^*)\) is the saddle point of \(R_{su}(\mathbf{S}_i, \mathbf{E}_i)\).

**Remark 3:** Denote by \(\tilde{\mathbf{S}}_i\) the covariance matrix constraint in [31, Theorem 6] and denote by \(\tilde{\mathbf{E}}_i\) the corresponding auxiliary matrix consisting of \(\tilde{\mathbf{A}}_i\) and \(\tilde{\Sigma}_i\) for this \(\tilde{\mathbf{S}}_i\) that satisfy condition (17)-(19), (27) and (28). If all the conditions in [31, Theorem 6] are satisfied, i.e., for any \(0 \preceq \mathbf{S}_i \preceq \tilde{\mathbf{S}}_i\) there exist corresponding \(\mathbf{A}_i\) and \(\Sigma_i\) such that (17)-(19), (27) and (28) are satisfied, then \((\tilde{\mathbf{S}}_i, \tilde{\mathbf{E}}_i)\) is also a saddle point of the upper bound function according to the covariance matrix constraint. This can be shown in a similar way as the result in Remark 2. First, we have
\[ \max_{0 \preceq \mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = \max_{0 \preceq \mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i(\mathbf{S}_i)) = \max_{0 \preceq \mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} \max_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = \max_{\mathbf{E}_i} R_{su}(\tilde{\mathbf{S}}_i, \tilde{\mathbf{E}}_i). \]
where the first equality is by the assumption of existence of \(\mathbf{A}_i\) and \(\Sigma_i\) that satisfy condition (17)-(19), (27) and (28) for each feasible \(\mathbf{S}_i\), and we denote such auxiliary matrix \(\mathbf{E}_i\) as \(\mathbf{E}_i(\mathbf{S}_i)\). The second equality is by the fact that \(R_{su}\) is an increasing function of \(\mathbf{S}_i\). On the other hand, we have
\[ \min_{\mathbf{E}_i} \max_{\mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = \min_{\mathbf{E}_i} R_{su}(\tilde{\mathbf{S}}_i, \mathbf{E}_i(\tilde{\mathbf{S}}_i)) = \min_{\mathbf{E}_i} R_{su}(\tilde{\mathbf{S}}_i, \tilde{\mathbf{E}}_i) = \min_{\mathbf{E}_i} \max_{\mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i). \]
Therefore, we also have
\[ \max_{\mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} \min_{\mathbf{E}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i) = \min_{\mathbf{E}_i} \max_{\mathbf{S}_i \preceq \tilde{\mathbf{S}}_i} R_{su}(\mathbf{S}_i, \mathbf{E}_i). \]
By [39, Proposition 2.6.1 p. 132], \((\tilde{\mathbf{S}}_i, \tilde{\mathbf{E}}_i)\) is also a saddle point for \(R_{su}(\mathbf{S}_i, \mathbf{E}_i)\) according to the covariance matrix constraint. Therefore, [31, Theorem 6] parallels Theorem 2 in the covariance matrix constraint.

**Remark 4:** Theorem 2 includes [32, Theorem 1] as a special case. In [32, Theorem 1], it was shown that if the \(\mathbf{S}_i^*\) is full rank and there exist \(\mathbf{A}_i\) and \(\Sigma_i\) satisfying (17)-(18), (27) and (28), then this MIMO IC has noisy interference. In this case, (27) and (28) imply
\[ \mathbf{F}_i^T = \mathbf{H}_i^T (\mathbf{I} + \mathbf{F}_j \mathbf{S}_j^* \mathbf{F}_j^T)^{-1} \mathbf{A}_i, \quad i, j \in \{1, 2\}, i \neq j \]
and thus
\[ \mathbf{O}_i = 0. \]
Therefore, (29) and (30) are both satisfied since \(\mathbf{W}_i \succeq 0\) has been shown in Lemma 1.

**Remark 5:** Theorem 2 includes the noisy-interference sum-rate capacity result for the parallel IC in [33] as a special case. The parallel IC is a special MIMO IC with diagonal channel matrices \(\mathbf{H}_i = \)
diag \([h_{i1}, \cdots, h_{it}]\) and \(F_i = \text{diag} \,[f_{i1}, \cdots, f_{it}]\). We define the \(i\)th subchannel as that consisting of only the \(i\)th transmit and receive antennas. The lower bound in (3) for this channel, by choosing the diagonal input covariance matrix \(S_i\) can be written as

\[
\max \ R_{sl}(S_i) = \sum_{j=1}^{t} r_j(s_{1j}, s_{2j})
\]

subject to \(\sum_{j=1}^{t} s_{ij} \leq P_i, \quad s_{ij} \geq 0, \quad i = 1, 2.\) (62)

where

\[
r_j(s_{1j}, s_{2j}) = \frac{1}{2} \log \left( 1 + \frac{h_{1j}^2 s_{1j}}{1 + f_{2j}^2 s_{2j}} \right) + \frac{1}{2} \log \left( 1 + \frac{h_{2j}^2 s_{2j}}{1 + f_{1j}^2 s_{1j}} \right).\] (63)

However, in [33] the lower bound on the sum-rate capacity is not formulated as above, but as

\[
\max \ \sum_{j=1}^{t} C_j(s_{1j}, s_{2j})
\]

subject to \(\sum_{j=1}^{t} s_{ij} \leq P_i, \quad s_{ij} \geq 0, \quad i = 1, 2\) (64)

where \(s_{ij}\) denotes the power allocated to the \(j\)th subchannel for user \(i\), and \(C_j(s_{1j}, s_{2j})\) denotes the sum-rate capacity of the \(j\)th subchannel under power constraint \(s_{ij}\), i.e., power \(s_{ij}\) is allocated to the \(j\)th transmit antenna of user \(i\). The upper bound on the sum-rate capacity is also formulated via optimization problem (16). However, if we choose the auxiliary matrices \(A_i\) and \(\Sigma_i\) as in [33, eqs. (41) and (42)], then the upper bound can be written as

\[
\max \ R_{su}(S_i) = \sum_{j=1}^{t} f_j(s_{1j}, s_{2j})
\]

subject to \(\text{tr}(S_i) = \sum_{j=1}^{t} s_{ij} \leq P_i, \quad s_{ij} \geq 0, \quad i = 1, 2.\) (65)

where \(S_i = \text{diag} [s_{i1}, \cdots, s_{it}]\) and \(f_j(\cdot)\) is defined in [33, eq.(64)]. The auxiliary matrix \(E_i\) is the same in both upper bounds. Therefore, [33] uses exactly the same side information as that in Theorem 2. Moreover, [33] shows that the matrices \(A^*_i\) and \(\Sigma^*_i\) are both diagonal matrices (see \(E_i\) in [33, eqs. (41) and (42))]. Thus, the upper bound \(R_{su}(S_i)\) is the sum of the upper bound for each subchannel \(f_j\).

It has been shown in [33] that if the power constraint \(P_i\) is in the set [33, eq. (18)], then by [33, Theorem 3] this parallel IC has noisy interference and the optimal input covariance matrix \(S^*_i = \text{diag} [s^*_{i1}, \cdots, s^*_{it}]\) has the properties [33, eqs.(18), (74) and (75)]

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} \in \bigcap_{j=1}^{t} \partial C_j(s^*_{1j}, s^*_{2j}) \neq \text{empty} \quad (66)
\]
we have
\[ \frac{\partial f_j}{\partial s_{ij}} \mid_{s_{ij} = s_{i2j}^*, s_{2j} = s_{2j}} = \frac{\partial C_j}{\partial s_{ij}} \mid_{s_{ij} = s_{i2j}^*, s_{2j} = s_{2j}} = \frac{\partial r_j}{\partial s_{ij}} \mid_{s_{ij} = s_{i2j}^*, s_{2j} = s_{2j}} \quad \text{for all } i = 1, 2, \quad j = 1, \ldots, t \] (67)

where \( \partial C_j (s_{1j}, s_{2j}^*) \) is the subdifferential of \( C_j (s_{1j}, s_{2j}) \) at \( (s_{1j}, s_{2j}^*) \), and \([\lambda_1, \lambda_2] \) is the common subgradient shared by all the subdifferentials. From the expression of \( \partial C_j (s_{1j}, s_{2j}^*) \) in [33, eq. (100)], we have
\[ \begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} = \begin{bmatrix}
\frac{\partial r_j}{\partial s_{ij}} \mid_{s_{ij} = s_{i2j}^*} \\
\frac{\partial r_j}{\partial s_{2j}} \mid_{s_{ij} = s_{i2j}^*}
\end{bmatrix} \begin{bmatrix}
w_{1j} \\
w_{2j}
\end{bmatrix}, \quad i = 1, 2, \quad j = 1, \ldots, t \] (68)

where \( w_{1j} \) and \( w_{2j} \) are nonnegative constants. Hence, we have
\[ \frac{\partial R_{sl}}{\partial S_i} \mid_{s_i = S_i^*, s_j = S_j^*} = \lambda_i I + W_i \] (69)

where \( W_i = \text{diag}[w_{11}, \ldots, w_{1t}] \geq 0 \). By (67), we have
\[ \frac{\partial R_{su}}{\partial S_i} \mid_{s_i = S_i^*, s_j = S_j^*} = \frac{\partial R_{sl}}{\partial S_i} \mid_{s_i = S_i^*, s_j = S_j^*} = \lambda_i I + W_i \] (70)

which implies \( O_i = 0 \). Therefore, if a parallel IC satisfies the noisy-interference condition in [33], it also satisfies Theorem 2. The lower bound \( \max R_{sl} \) and the upper bound \( \max R_{su} \) are optimized at the same \( S_i^* \) with the same Lagrangian multipliers. The Lagrangian multipliers \( \lambda_i \) associated with the power constraint \( \text{tr}(S_i) \leq P_i \) form the common subgradient of all the individual subchannel capacities \( C_j \) (as well as the individual lower bounds \( r_j \)) and upper bounds \( f_j \), i.e., \( C_j \) (or \( r_j \)) and \( f_j \) have parallel supporting hyperplanes with the subgradient \([\lambda_1, \lambda_2] \) at the optimal power allocation point.

We note that to formulate the lower bound as in (64) is important for [33] since the problem is then a convex optimization problem. Furthermore, condition (67) directly guarantees the optimality of \( s_{ij}^* \) for (64), and only through which we show the optimality of \( s_{ij}^* \) for (62) [33].

**Remark 6:** Theorem 2 determines the noisy-interference sum-rate capacity for general MIMO ICs. When the MIMO IC reduces to a MISO or SIMO IC, the conditions in Theorem 2 can be simplified. We defer these results in Sections III and IV, respectively. In [32], noisy-interference sum-rate capacities of symmetric MISO and SIMO ICs are obtained, i.e., ICs with \( H_1 = H_2, \ F_1 = F_2, \ P_1 = P_2, \) and where all the \( H_i \) and \( F_i \) are column or row vectors. These two results are both included as special cases of Theorem 2. In Sections III and IV, the MISO and SIMO ICs can be symmetric and asymmetric.

**Remark 7:** Equations (27) and (28) are special cases of the Sylvester equation [41]. Once \( S_i^* \) is obtained, the matrix \( A_i \) can be obtained by solving the following linear equations:
\[ I \otimes \left( S_i^* H_i^T \left( I + F_2 S_2^* F_2^T \right)^{-1} \right) \text{Vec}(A_i) = \text{Vec} \left( S_i^* F_1^T \right) \]
\[
I \otimes \left( S^*_2 H^T_2 \left( I + F_1 S^*_1 F^T_1 \right)^{-1} \right) \text{Vec}(A_2) = \text{Vec} \left( S^*_2 F^T_2 \right)
\]
where \( \otimes \) denotes the Kronecker product of matrices. Therefore, the existence of \( A_i \) can be determined by the theory of linear equations.

Remark 8: In Theorem 2 and its proof, we need to determine the existence of a positive definite \( \Sigma_i \). Sometimes the expression for \( \Sigma_i \) is not important (e.g., the parallel Gaussian IC discussed in Remark 5, and the symmetric SIMO IC discussed later in Remark 14). If we choose equality in both (18) and (19), we obtain two matrix equations which are special cases of a discrete algebraic Ricatti equation [42]. The existence of a positive definite solution is determined by [31, Lemma 9] using [42], which requires, for both \( i = 1 \) and 2:

\[
\text{radius} (\Phi_i) \leq \frac{1}{2} \tag{71}
\]

where

\[
\Phi_1 = \left( I - A^T_1 A_1 - A_2 A^T_2 \right)^{-\frac{1}{2}} A^T_1 A^T_2 \left( I - A^T_1 A_1 - A_2 A^T_2 \right)^{-\frac{1}{2}} \tag{72}
\]
\[
\Phi_2 = \left( I - A_1 A^T_1 - A^T_2 A_2 \right)^{-\frac{1}{2}} A^T_2 A^T_1 \left( I - A_1 A^T_1 - A^T_2 A_2 \right)^{-\frac{1}{2}}. \tag{73}
\]

Here we present a strengthened result of [31, Lemma 9] which requires (71) to be satisfied for only \( i = 1 \) or \( i = 2 \).

Lemma 4: For the following matrix equations for \( \Sigma_1 \) and \( \Sigma_2 \):

\[
\Sigma_1 = I - A_2 \Sigma^{-1}_2 A^T_2 \tag{74}
\]
\[
\Sigma_2 = I - A_1 \Sigma^{-1}_1 A^T_1 \tag{75}
\]

if \( \text{radius}(\Phi_1) \leq \frac{1}{2} \) or \( \text{radius}(\Phi_2) \leq \frac{1}{2} \) where \( \Phi_i \) is defined in (72) and (73), then there exist symmetric positive definite solutions for \( \Sigma_1 \) and \( \Sigma_2 \). Moreover, the solutions for both \( i = 1 \) and 2 satisfy

\[
\Sigma_i > A^T_i A_i \tag{76}
\]

or equivalently

\[
E_i = \begin{bmatrix} I & A_i \\ A^T_i & \Sigma_i \end{bmatrix} > 0. \tag{77}
\]

The proof is included in Appendix C.

For completeness, we give the noisy-interference condition of MIMO ZIC in the following proposition.
Proposition 1: For the MIMO IC defined in (1) with \( F_1 = 0 \) and \( P_i > 0, i = 1, 2, \) if the optimal solution of problem (3) has \( \text{tr}(S_i^*) > 0, \) and there exist matrices \( A_2 \) and \( \Sigma_2 \) that satisfy

\[
I \succeq A_2 A_2^T \\
S_2^* F_2^T = S_2^* H_2^T A_2 \\
W_2 \succeq O_2
\]

where

\[
W_2 = G_2 - \frac{\text{tr}(S_2^* G_2)}{P_2} I \\
O_2 = \frac{1}{2} (A_2^T H_2 - F_2)^T (I - A_2^T A_2)^{-1} (A_2^T H_2 - F_2)
\]

and \( G_2 \) are defined in (9), then the sum-rate capacity is the maximum in problem (3) and is achieved by Gaussian input \( x_i^* \sim \mathcal{N}(0, S_i^*) \) and treating interference as noise.

Proof: The proof is straightforward from Theorem 2 by choosing \( A_1 = 0, \Sigma_1 = I - A_2 A_2^T \) and \( \Sigma_2 = I. \) Condition (29) is automatically satisfied by \( W_1 \succeq 0 = O_1. \)

Remark 9: The aligned-weak interference condition in [31, Proposition 5] for the average power constraint is a special case of Proposition 1. The alignment weak interference means that if there exists a matrix \( A_2 \) with \( A_2 A_2^T \preceq I \) and \( F_2 = A_2^T H_2, \) then treating interference as noise achieves the sum-rate capacity. Obviously, in such a case, all the conditions in Proposition 1 are satisfied.

In Sections III and IV, we apply Theorem 2 to MISO and SIMO channels and simplify the noisy-interference conditions.

III. MISO ICs

In [32], it has been shown that the capacity of a two-user MISO IC is the same as that of a MISO IC with each transmitter having only two antennas. The main idea is to write the direct link channel vector as the sum of the interference channel vector and its orthogonal vector. The antenna reduction is also studied in [35] which shows that the single-user detection rate region of an \( m \)-user MISO IC with transmitter \( i, 1 \leq i \leq m, \) having \( t_i \) antennas, is the same as that of a MISO IC with transmitter \( i, \) having only \( \min\{t_i, m\} \) antennas. The antenna reduction is performed systematically using [35, eqs.(45)-(47)] which can also be used to show the equivalence of the capacity regions between the original \( m \)-user MISO IC and the new \( m \)-user MISO IC after antenna reduction. In the following, we apply the method in [35] to the two-user MISO IC to show the reduction process. On letting \( H_i = \hat{h}_i^T \) and \( F_i = \hat{f}_i^T, \)
In (1), the received signals of a MISO IC are

\[
\begin{align*}
Y_1 &= \hat{h}_1^T \hat{x}_1 + \hat{f}_2^T \hat{x}_2 + Z_1 \\
Y_2 &= \hat{h}_2^T \hat{x}_2 + \hat{f}_1^T \hat{x}_1 + Z_2
\end{align*}
\] (78)

where \( h_i \) and \( f_i \) are \( t_i \times 1 \) column vectors and we write the transmitted signal as \( \hat{x}_i \) with power constraint \( \hat{P}_i \). Define the singular value decomposition of \( f_i \) as

\[
\hat{f}_i = U_i \left[ \begin{array}{c} \|f_i\| \cos \theta_i \\
\|f_i\| \sin \theta_i \\
0 \end{array} \right]
\] (79)

where \( U_i^T U_i = I \) and the dimension of the zero vector is \( t_i - 1 \). Then we have

\[
U_i^T \hat{h}_i = \left[ \begin{array}{c} \|f_i\| \cos \theta_i \\
\|f_i\| \sin \theta_i \\
0 \end{array} \right]
\] (80)

where we define \( \theta_i \triangleq \angle (\hat{h}_i, \hat{f}_i) \), and \( g_i \) is a \( (t_i - 1) \times 1 \) vector. Equality (a) follows from the fact that the first row of \( U_i^T \) is \( \hat{f}_i / \|f_i\| \). Equality (b) is by the fact \( \|g_i\| = \|f_i\| \sin \theta_i \) and the singular value decomposition

\[
g_i = V_i \left[ \begin{array}{c} \|f_i\| \sin \theta_i \\
0 \end{array} \right]
\]

where \( V_i^T V_i = I \), and the dimension of the zero vector is \( t_i - 2 \).

Define

\[
\bar{x} \triangleq Q_i \hat{x}_i
\] (81)

where

\[
Q_i = \left[ \begin{array}{cc} 1 & 0 \\
0 & V_i \end{array} \right]^T U_i^T.
\] (82)

It is obvious that \( Q_i^T Q_i = I \). Then the received signals of the MISO IC can be written as

\[
Y_1 = \left[ \begin{array}{c} \|f_1\| \cos \theta_1 \\
\|f_1\| \sin \theta_1 \\
0 \end{array} \right]^T \bar{x}_1 + \left[ \begin{array}{c} \|f_2\| \\
0 \end{array} \right]^T \bar{x}_2 + z_1
\]
By removing irrelevant dimensions, we write the MISO IC in the following standard form:

\[
Y_1 = h_1^T x_1 + f_2^T x_2 + Z_1 \\
Y_2 = h_2^T x_2 + f_1^T x_1 + Z_2
\]  

(83)

where the dimension of all the vectors is 2, and the power constraint for user \( i \) is now \( P_i \), and

\[
P_i = \hat{P}_i \|h_i\|^2 \quad \text{(84)}
\]

\[
a_i = \frac{\|f_i\|^2}{\|h_i\|^2} \quad \text{(85)}
\]

\[
f_i = \begin{bmatrix} \sqrt{a_i} \\ 0 \end{bmatrix} \quad \text{(86)}
\]

\[
h_i = \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \end{bmatrix} \quad \text{(87)}
\]

Consequently, if \( S_i \) is the input covariance matrix of user \( i \) for equivalent channel (83), the corresponding input covariance for the original channel is

\[
S_i = \frac{1}{\|h_i\|^2} Q_i^T \begin{bmatrix} S_i & 0 \\ 0 & 0 \end{bmatrix} Q_i. 
\]

(88)

With the antenna reduction, we have the following result.

**Theorem 3:** For the MISO IC defined in (78) and its equivalent channel (83) with \( \cos \angle (h_i, f_i) \neq 0 \), \( f_i \neq 0, h_i \neq 0 \), \( i = 1, 2 \), denote \( S_i^* \) as the optimal solution of problem (3) for the equivalent channel (83), if \( S_i^* \neq 0 \) and

\[
\sigma_i^2 \geq \hat{\sigma}_i^2, \quad i = 1, 2
\]

(89)

\[
\text{abs}(A_1) + \text{abs}(A_2) \leq 1
\]

(90)

where

\[
\sigma_1^2 = \frac{1}{2} \left[ (1 + A_1^2 - A_2^2) + \sqrt{(1 + A_1^2 - A_2^2)^2 - 4A_1^2} \right] 
\]

(91)

\[
\sigma_2^2 = \frac{1}{2} \left[ (1 + A_2^2 - A_1^2) + \sqrt{(1 + A_2^2 - A_1^2)^2 - 4A_2^2} \right]
\]

(92)
\[
\hat{\sigma}_1^2 = -f_2^T S_2^* f_2 + \frac{\sqrt{a_2}}{\cos \theta_2} \left( 1 + h_2^T S_2^* h_2 + f_1^T S_1^* f_1 \right) \frac{f_2^T S_2^* h_1}{h_1^T S_2^* h_2} 
\]

(93)

\[
\hat{\sigma}_2^2 = -f_1^T S_1^* f_1 + \frac{\sqrt{a_1}}{\cos \theta_1} \left( 1 + h_1^T S_1^* h_1 + f_2^T S_2^* f_2 \right) \frac{f_1^T S_1^* h_1}{h_1^T S_1^* h_1} 
\]

(94)

\[
A_1 = \frac{f_1^T S_1^* h_1}{h_1^T S_1^* h_1} \left( 1 + f_2^T S_2^* f_2 \right) 
\]

(95)

\[
A_2 = \frac{f_2^T S_2^* h_2}{h_2^T S_2^* h_2} \left( 1 + f_1^T S_1^* f_1 \right) 
\]

(96)

then the sum-rate capacity is the maximum of problem (3) and is achieved by treating interference as noise.

Proof: We use Theorem 2 to prove the converse. We first consider the existence of \( A_i \) (i.e., \( A_i = A_i \) in the MISO case) in (27) and (28) which require

\[
S_1^* f_1 = S_1^* h_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} A_1 
\]

(97)

\[
S_2^* f_2 = S_2^* h_2 \left( 1 + f_1^T S_1^* f_1 \right)^{-1} A_2. 
\]

(98)

It has been shown in [35] that \( \text{rank}(S_i^*) \leq 1 \). With the assumption \( \text{tr}(S_i^*) > 0 \), we have

\[
\text{rank}(S_i^*) = 1. 
\]

(99)

Then we can write

\[
S_i^* = \gamma_i \gamma_i^T 
\]

(100)

where \( \gamma \) is a \( 2 \times 1 \) vector. We have

\[
\gamma_1 \gamma_1^T f_1 = \gamma_1 \gamma_1^T h_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} A_1. 
\]

(101)

Obviously, if \( \gamma^T h_1 = 0 \), then \( \gamma^T f_1 = 0 \) because otherwise transmitter 1 does not transmit anything to receiver 1 while still generating interference to receiver 2. In this case \( A_1 \) can choose any value. If \( \gamma^T h_1 \neq 0 \), we have

\[
A_1 = \frac{\gamma_1^T f_1}{\gamma_1^T h_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1}}. 
\]

(102)

Therefore, \( A_1 \) always exists. Similarly, we can show the existence of \( A_2 \). Another expression of \( A_i \) in (95) and (96) is obtained by left-multiply (97) and (98) with \( h_1^T \) and \( h_2^T \), respectively.

We then consider the existence of \( \Sigma_i \) (i.e., \( \Sigma_i = \sigma_i^2 \) in the MISO case) in (18) and (19). By choosing equality in both (18) and (19), we obtain \( \sigma_i^2 \) in (91) and (92). It can be shown that the existence of \( \sigma_i^2 \), or equivalently, that (91) and (92) are feasible, is guaranteed by (90) (details can be found in [21, p. 696]).
It remains to consider whether conditions (29) and (30) are satisfied. In the following, we do not verify these two conditions directly from (31) or (32). Instead, we use the equivalent conditions (4)-(7) since we have additional information (99) for \( S_i^* \).

From (7), the columns of \( \mathbf{W}_i \) are all in the eigenvector space of \( S_i^* \) associated with its zero eigenvalue. Since \( \text{rank}(S_i^*) = 1 \) and \( S_i^* \) is a \( 2 \times 2 \) matrix, the dimension of this eigenvector space is 1. By (97), the eigenvector is \( A_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} h_1 - f_1 \). Therefore, there exist a constant \( k \geq 0 \) such that

\[
\mathbf{W}_1 = k \left( A_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} h_1 - f_1 \right) \left( A_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} h_1 - f_1 \right)^T.
\] (103)

On the other hand, from (4) we have

\[
\mathbf{W}_1 = -\frac{h_1 h_1^T}{2 \left( 1 + h_1^T S_1^* h_1 + f_2^T S_2^* f_2 \right)} + \frac{h_2^T S_2^* h_2 \cdot f_1 f_1^T}{2 \left( 1 + f_2^T S_2^* f_2 \right)} + \lambda_1 \mathbf{I}. \quad (104)
\]

On comparing the element of \( \mathbf{W}_1 \) on the first row and the second column in expression (103) and (104), we have

\[
k = -\frac{\cos \theta_1}{2 \left( 1 + h_1^T S_1^* h_1 + f_2^T S_2^* f_2 \right)} \frac{f_1^T S_1^* h_1}{h_1^T S_1^* h_1} \frac{f_1^T S_1^* h_1}{h_1^T S_1^* h_1} \frac{\cos \theta_1 - \sqrt{a_1}}{\cos \theta_1 - \sqrt{a_1}}. \quad (105)
\]

From (33), we have

\[
O_1 = \left( A_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} h_1 - f_1 \right) \left( A_1 \left( 1 + f_2^T S_2^* f_2 \right)^{-1} h_1 - f_1 \right)^T \frac{\sigma_2^2 - A_2^2}{1 + f_2^T S_2^* f_2}. \quad (106)
\]

Therefore, condition (29) requires

\[
k \geq \frac{1}{\sigma_2^2 - A_2^2 \left( 1 + f_2^T S_2^* f_2 \right)} \quad (107)
\]

which is equivalent to (89). Similarly, (89) guarantees that (30) is satisfied. Therefore, under conditions (89) and (90), all the requirements of Theorem 2 are satisfied, and the MISO IC has noisy interference.

\[\square\]

Remark 10: Consider the computation of the noisy-interference sum-rate capacity of a MISO IC. Using [35, Theorem 1], the maximum of problem (3) is

\[
\max_{\phi_i \in [0, \text{abs}(\frac{\pi}{2} - \theta_i)]} \frac{1}{2} \log \left( 1 + \frac{P_1 \sin^2 (\theta_1 + \rho_1 \phi_1)}{1 + a_2 P_2 \sin^2 \phi_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2 \sin^2 (\theta_2 + \rho_2 \phi_2)}{1 + a_1 P_1 \sin^2 \phi_1} \right) \quad (108)
\]
where \( \rho_i = 1 \) if \( \theta_i \in [0, \pi/2] \) and \( \rho_i = -1 \) otherwise. If \( \phi_i^* \) is optimal, then the corresponding input covariance matrix is

\[
S_i^* = P_i \begin{bmatrix}
\sin^2 \phi_i^* & \rho_i \sin \phi_i^* \cos \phi_i^* \\
\rho_i \sin \phi_i^* \cos \phi_i^* & \cos^2 \phi_i^*
\end{bmatrix}.
\]

(109)

A closed-form expression for \( \phi_i^* \) is difficult to obtain for the general MISO ICs, or even MISO ZICs. However, if the MISO IC is symmetric with \( \theta_1 = \theta_2 = \theta \), \( a_1 = a_2 = a \) and \( P_1 = P_2 = P \), then we have:

\[
\tan \phi^* = \frac{1}{\left(1 + aP\right) \tan \theta}.
\]

(110)

**Remark 11:** If the MISO IC is symmetric as defined above, the noisy-interference condition is given in [32, Theorem 2], which can also be obtained from Theorem 3. In this case, the optimal \( S_i^* \) is given in (109) and (110). Conditions in Theorem 3 reduce to

\[
A = \frac{f^TS^*h}{h^TS^*h} \left(1 + f^TS^*f\right) \leq \frac{1}{2}
\]

(111)

\[
\sigma^2 = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4A^2} \leq \sigma^2 = -f^TS^*f + \frac{\sqrt{a}}{\cos \theta} \left(f^TS^*f + A\right).
\]

(112)

The above conditions are exactly [32, eq.(53)] which are satisfied under the conditions in [32, Theorem 2].

Theorem 3 applies to the case in which \( \cos \theta_i \neq 0 \) and \( \|f_i\| \neq 0 \). If any of these two conditions are satisfied, the MISO IC reduces to a MISO ZIC. The noisy-interference sum-rate capacity is obtain in the following proposition.

**Proposition 2:** For the MISO IC defined in (78) and its equivalent channel (83) with \( \cos \angle (h_1, f_1) = \pi/2 \), or \( f_1 = 0 \), denote by \( S_i^* \), \( i = 1, 2 \), the optimal solution of problem (3) for the equivalent channel (83). If \( S_1^* \neq 0 \) and

\[
f_1^T S_2^* f_2 \leq h_2^T S_2^* h_2 \]

(113)

\[
a_2 \left[ f_1^T S_2^* h_2 \left(1 + h_2^T S_2^* h_2\right)\right]^2 \leq \cos^2 \theta_2
\]

(114)

then the sum-rate capacity is the maximum in problem (3) and is achieved by treating interference as noise.

**Proof:** We first consider the case when \( f_1 = 0 \). From (91)-(96), we have

\[
\sigma_1^2 = 1 - A_2^2
\]

\[
\sigma_2^2 = 1
\]
\[
\sigma_1^2 = -f_2 S_2^* f_2 + \frac{\sqrt{a_2}}{\cos \theta_2} \left( 1 + h_2^T S_2^* h_2 \right) \frac{f_2^T S_2^* h_2}{h_2^T S_2^* h_2} \]
\[
\sigma_2^2 = 0
\]
\[
A_1 = 0
\]
\[
A_2 = \frac{f_2^T S_2^* h_2}{h_2^T S_2^* h_2}.
\]

Condition (113) guarantees that (90) is satisfied since
\[
A_2^2 = \left( \frac{f_2^T S_2^* h_2}{h_2^T S_2^* h_2} \right)^2 = \frac{f_2^T S_2^* f_2}{h_2^T S_2^* h_2}
\]
due to the fact that \( \text{rank} (S_2^*) = 1 \). Then it remains to consider (89) for \( i = 1 \), which is satisfied by (114) on the condition
\[
\frac{f_2^T S_2^* h_2}{\cos \theta_2} \geq 0
\]
which is true by (109):
\[
\frac{f_2^T S_2^* h_2}{\cos \theta_2} = \frac{\sin^2 \phi_2^* \cos \theta_2 + \rho_2 \sin \phi_2^* \cos \phi_2^* \sin \theta_2}{\cos \theta_2} \geq 0.
\]

In the case \( f_1 \neq 0 \) but \( \theta_1 = \frac{\pi}{2} \), the capacity region is outer bound by that of the same channel but with \( f_1 = 0 \). If (113) and (114) are satisfied, then the sum-rate capacity of the channel with \( f_1 = 0 \) is an outer bound on that of the channel with \( f_1 \neq 0 \) but \( \theta_1 = \frac{\pi}{2} \). The achievability is due to the fact that
\[
f_1^T S_1^* f_1 = 0
\]
since
\[
S_1^* = P_1 h_1 h_1^T.
\]

We note that Proposition 2 can also be proved by Proposition 1.

IV. SIMO ICs

On letting \( H_i = \hat{h}_i \) and \( F_i = \hat{f}_i \), \( i = 1, 2 \), in (1), the received signals of a MISO IC are
\[
\hat{y}_1 = \hat{h}_1 X_1 + \hat{f}_2 X_2 + \hat{z}_1
\]
\[
\hat{y}_2 = \hat{h}_2 X_2 + \hat{f}_1 X_1 + \hat{z}_2
\]
where \( h_i \) and \( f_i \) are \( t_i \times 1 \) vectors and we write the transmitted signal as \( \hat{x}_i \) with power constraint \( \hat{P}_i \).
We can follow the same process (79)-(81) in Section III to find the equivalent channel for (116) with reduced number of antennas. The difference is that we need to replace the $h_i$ in (80) with $h_j$ where $j \neq i$. Then we left-multiply $y_i$ with $Q_i$ and obtain the equivalent channel

$$y_1 = h_1 X_1 + f_2 X_2 + z_1$$

$$y_2 = h_2 X_2 + f_1 X_1 + z_2$$

(117)

where the dimension of all the vectors is 2, the power constraint for user $i$ is now $P_i$, and

$$P_i = \hat{P}_i$$

(118)

$$a_i = \frac{||f_i||^2}{||h_i||^2}$$

(119)

$$f_i = \begin{bmatrix} \sqrt{a_i} \\ 0 \end{bmatrix}$$

(120)

$$h_i = \begin{bmatrix} \cos \varphi_i \\ \sin \varphi_i \end{bmatrix}$$

(121)

$$\varphi_i = \angle (h_i, f_j) \quad i, j \in \{1, 2\}, j \neq i.$$  

(122)

We first present the noisy-interference sum-rate capacity of the SIMO ZIC as this is a special case of [31, Proposition 5].

**Proposition 3:** [31, Proposition 5] For the SIMO IC defined in (116) and its equivalent channel (117) with $\varphi_2 = \frac{\pi}{2}$ or $f_1 = 0$, if $||f_2|| \leq ||h_2||$, then the sum-rate capacity is

$$\frac{1}{2} \log \left| I + P_1 h_1 h_1^T (I + P_2 f_2 f_2^T)^{-1} \right| + \frac{1}{2} \log \left| I + P_2 h_2 h_2^T \right|. \quad (123)$$

**Proof:** We first consider the case when $f_1 = 0$. Then from [31, Proposition 5], if there exists a matrix $A_2$ such that

$$f_2 = A_2^T h_2$$

(124)

$$I \succeq A_2^T A_2$$

(125)

then the sum-rate capacity is

$$\max_{0 \leq S \leq P_i, i=1,2} \frac{1}{2} \log \left| I + S h_1 h_1^T (I + S f_2 f_2^T)^{-1} \right| + \frac{1}{2} \log \left| I + S h_2 h_2^T \right|. \quad (126)$$

Then we can choose

$$A_2^T = \frac{f_2 h_2^T}{||h_2||^2} = f_2 h_2^T$$

(127)
and (124) is satisfied. For (125), we observe
\[ A_T^2 A_2 = f_2 (h_2^T h_2) f_2^T = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \leq I \] (128)
where the last equality is by the assumption \( \|f_2\| \leq \|h_2\| \).

Then we need to show that \( S_i^* = P_i \) maximizes (126). On denoting the objective function of (126) by \( R_s \), we have
\[ \frac{\partial R_s}{\partial S} = \frac{1}{2} h_1^T \left( I + S_1 h_1 h_1^T + S_2 f_2 f_2^T \right)^{-1} h_1 \geq 0 \] (129)
and
\[ \frac{\partial R_s}{\partial S} = \frac{1}{2} f_2^T \left( I + S_1 h_1 h_1^T + S_2 f_2 f_2^T \right)^{-1} f_2 - \frac{1}{2} f_2^T \left( I + S_2 f_2 f_2^T \right)^{-1} f_2 + \frac{1}{2} h_2^T \left( I + S_2 h_2 h_2^T \right)^{-1} h_2 \]
\[ \geq -\frac{1}{2} f_2^T \left( I + S_2 f_2 f_2^T \right)^{-1} f_2 + \frac{1}{2} h_2^T \left( I + S_2 h_2 h_2^T \right)^{-1} h_2 \]
\[ (a) \quad -\frac{1}{2} \left( 1 + S_2 f_2^T f_2 \right)^{-1} f_2^T f_2 + \frac{1}{2} \left( 1 + S_2 h_2^T h_2 \right)^{-1} h_2^T h_2 \]
\[ = \frac{\|h_2\|^2 - \|f_2\|^2}{2 (1 + S_2 \|f_2\|^2) (I + S_2 \|h_2\|^2)} \]
\[ \geq 0 \] (130)
where (a) is by the matrix identity (49). Therefore \( R_s \) is maximized by \( S_i^* = P_i \).

In the case when \( f_1 \neq 0 \) and \( \phi_2 = \frac{\pi}{2} \), the converse can be proved by assuming \( f_1 = 0 \) to eliminate the interference, and the achievability is proved by left-multiplying \( y_2 \) with \( h_2 \) to null out the interference.

We note that Proposition 3 can also be proved by Proposition 1.

**Theorem 4:** For the SIMO IC defined in (116) and its equivalent channel (117), if for \( i = 1 \) or 2
\[ \text{radius } (\Phi_i) \leq \frac{1}{2} \] (131)
where
\[ \Phi_1 = \left( I - A_1^T A_1 - A_2 A_2^T \right)^{-\frac{1}{2}} A_1^T A_2 \left( I - A_1^T A_1 - A_2 A_2^T \right)^{-\frac{1}{2}} \] (132)
\[ \Phi_2 = \left( I - A_1 A_1^T - A_2^T A_2 \right)^{-\frac{1}{2}} A_2^T A_1 \left( I - A_1 A_1^T - A_2^T A_2 \right)^{-\frac{1}{2}} \] (133)
\[ A_1 \left( I + P_2 f_2 f_2^T \right) h_1 = f_1 \] (134)
\[ A_2 \left( I + P_1 f_1 f_1^T \right) h_2 = f_2 \] (135)
then the sum-rate capacity is
\[
\frac{1}{2} \log \left| I + P_1 h_1 h_1^T \left( I + P_2 f_2 f_2^T \right)^{-1} \right| + \frac{1}{2} \log \left| I + P_2 h_2 h_2^T \left( I + P_1 f_1 f_1^T \right)^{-1} \right|.
\] (136)

**Proof:** We prove Theorem 4 from Theorem 1 instead of Theorem 2 since the optimal solution is known for problem (16). If we choose \( A_i \) in (134) and (135), then by Lemma 3, given (131) there exists \( \Sigma_i \) such that
\[
A_1^T A_1 < \Sigma_1 = I - A_2 \Sigma_2^{-1} A_2^T \tag{137}
\]
\[
A_2^T A_2 < \Sigma_2 = I - A_1 \Sigma_1^{-1} A_1^T. \tag{138}
\]
Therefore, conditions (17)-(19) are satisfied. In the following, we show that the upper bound \( R_{1u}(S1, S2) + R_{2u}(S1, S2) \) is maximized at \( S^*_1 = P_1 \) and \( R_{1u}(P_1, P_2) + R_{2u}(P_1, P_2) = R_{1l}(P_1, P_2) + R_{2l}(P_1, P_2) \).

From (24) we have
\[
R_{1u} + R_{2u} = I \left( X_{1G}; \begin{bmatrix} h_1 \\ f_1 \end{bmatrix} X_{1G} + \begin{bmatrix} f_2 \\ 0 \end{bmatrix} X_{2G} + \begin{bmatrix} z_1 \\ n_1 \end{bmatrix} \right) + I \left( X_{1G}; \begin{bmatrix} h_2 \\ f_2 \end{bmatrix} X_{2G} + \begin{bmatrix} f_1 \\ 0 \end{bmatrix} X_{1G} + \begin{bmatrix} z_2 \\ n_2 \end{bmatrix} \right)
= h(f_1 X_{1G} + n_1) - h(n_1) + h(h_1 X_{1G} + f_2 X_{2G} + z_1 | f_1 X_{1G} + n_1) - h(f_2 X_{2G} + z_1 | n_1)
+h(f_2 X_{2G} + n_2) - h(n_2) + h(h_2 X_{2G} + f_1 X_{1G} + z_2 | f_2 X_{2G} + n_2) - h(f_1 X_{1G} + z_2 | n_2)
=-h(n_1) + h(h_1 X_{1G} + f_2 X_{2G} + z_1 | f_1 X_{1G} + n_1) - h(n_2) + h(h_2 X_{2G} + f_1 X_{1G} + z_2 | f_2 X_{2G} + n_2)
\] (139)

where the last equality is by (137) and (138) which mean
\[
\text{Cov}(n_i) = \text{Cov}(z_j | n_j), \quad i, j \in \{1, 2\}, i \neq j. \tag{140}
\]
Then it suffices to show that \( h(h_1 X_{1G} + f_2 X_{2G} + z_1 | f_1 X_{1G} + n_1) \) is an increasing function of \( \text{Cov}(X_{1G}) \).

We write \( X_{1G} = \bar{X}_{1G} + \hat{X}_{1G} \) where \( X_{1G} \) and \( \hat{X}_{1G} \) are independent Gaussian variables. Obviously, we have \( \text{Cov}(X_{1G}) \geq \text{Cov}(\hat{X}_{1G}) \) and
\[
\begin{align*}
\begin{split}
& h(h_1 X_{1G} + f_2 X_{2G} + z_1 | f_1 X_{1G} + n_1) \\
& \geq h(h_1 X_{1G} + f_2 X_{2G} + z_1 | f_1 X_{1G} + n_1, \hat{X}_{1G}, \hat{X}_{2G}) \\
& = h(h_1 \bar{X}_{1G} + f_2 \bar{X}_{2G} + z_1 | f_1 \bar{X}_{1G} + n_1).
\end{split}
\end{align*}
\] (141)
Therefore, the upper bound $R_{1u}(S_1, S_2) + R_{2u}(S_1, S_2)$ is maximized at $S_i^* = P_i$. From (38), (40), (134) and (135), we have

$$R_{iu}(P_1, P_2) = R_{il}(P_1, P_2).$$

Therefore, the upper bound is achievable and hence is the sum-rate capacity.

**Remark 12:** A simple way to choose matrix $A_i$ that satisfies (134) and (135) is to let

$$A_1 = \left( I + P_2 f_2 f_2^T \right) h_1 f_1^T$$

$$A_2 = \left( I + P_1 f_1 f_1^T \right) h_2 f_2^T.$$  

(143)

(144)

However, this may not always be the best choice for (131). An alternative way is to let [32, eq. (39)]

$$A_i = \frac{v_i f_i^T}{h_i^T \left( 1 + P_j f_j f_j^T \right) v_i}$$

(145)

where $v_i$ is a vector. Then, to satisfy (131), we need only

$$\min_{v_1, v_2} \text{radius}(\Phi_i) \leq \frac{1}{2}.$$  

(146)

**Remark 13:** Proposition 3 can also be obtained from Theorem 4. Let $f_1 = 0$, then we have $A_1 = 0$, $A_2 = h_2 f_2^T$ and $\Phi_i = 0$. Therefore, condition (131) is always satisfied. Notice that $(I - A_2 A_2^T)^{-\frac{1}{2}}$ and $(I - A_1 A_1^T)^{-\frac{1}{2}}$ must exist such that $\Phi_i$ exists. By [31, Lemma 7] this requires $A_2^T A_2 \preceq I$ which is (128).

**Remark 14:** If the SIMO IC is symmetric, i.e., $h_1 = h_2 = h$, $f_1 = f_2 = f$ and $P_1 = P_2 = P$, the noisy-interference condition is given in [32, Theorem 3]. We will show that the same result can be obtained from Theorem 4. Without loss of generality, we assume $\theta \in \left[0, \frac{\pi}{2}\right]$. The matrix $A$ that satisfies (134) and (135) can be chosen as

$$A = \frac{\sqrt{a}}{\cos \omega \cos \theta + \sin \omega \sin \theta} \begin{bmatrix} \cos \omega & \sin \omega \\ 0 & 0 \end{bmatrix}$$

(147)

where $\omega$ is a real number. Since $A_1 = A_2$, condition (131) reduces to radius($A$) $\leq \frac{1}{2}$, i.e.,

$$\frac{1}{2} \geq \min_{\omega} \max_{\phi} \abs{\begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}^T A \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} \begin{bmatrix} \cos \omega & \sin \omega \\ 0 & 0 \end{bmatrix}}$$

$$= \min_{\omega} \max_{\phi} \left| a \frac{\cos^2 \phi \cos \omega + \cos \phi \sin \phi \sin \omega}{\cos \omega \cos \theta + \sin \omega \sin \theta} \begin{bmatrix} \cos \omega \cos \theta + \sin \omega \sin \theta \\ \cos \omega \cos \theta - \sin \omega \sin \theta \end{bmatrix} \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} \right|$$
\[
\begin{align*}
= \min_\omega \sqrt{a} \frac{(1 + \operatorname{abs}(\cos \omega)) / 2}{\operatorname{abs} \frac{\cos \omega \cos \theta}{1 + aP} + \sin \omega \sin \theta} \\
= \min_{\omega \in [0, \pi]} \sqrt{a} \frac{(1 + \cos \omega) / 2}{\sqrt{r} \sin(\omega + \beta)}
\end{align*}
\]

where
\[
\begin{align*}
r &= \frac{\cos^2 \theta}{(1 + aP)^2} + \sin^2 \theta \\
\beta &= \arctan \frac{\cos \theta}{(1 + aP) \sin \theta} \in \left[0, \frac{\pi}{2}\right].
\end{align*}
\]

It can be shown that the optimal \(\omega\) for (148) is
\[
\begin{align*}
\omega &= \begin{cases} 
\frac{\pi}{2}, & \text{if } \beta \in \left[0, \frac{\pi}{4}\right] \\
\pi - 2\beta, & \text{if } \beta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].
\end{cases}
\end{align*}
\]

Then (148) becomes
\[
\begin{align*}
a \leq \sin^2 \theta & \quad \text{if } \frac{\cos \theta}{(1 + aP)} \leq \sin \theta \\
\frac{\cos^2 \theta}{(1 + aP)^2} - \frac{2\sqrt{a} \cos \theta}{1 + aP} + \sin^2 \theta \geq 0 & \quad \text{otherwise}
\end{align*}
\]

which are exactly the conditions in [32, Theorem 3].

V. NUMERICAL EXAMPLES

**Example 1:** Consider a MIMO IC with channel matrices:
\[
H_1 = \begin{bmatrix}
-1.4510 & -1.0078 \\
-1.8953 & 0.2184 \\
1.9125 & -1.6068
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
0.4255 & -0.1702 & 0.6865 \\
0.5133 & 0.1574 & 0.1805 \\
-0.4795 & -0.5019 & 0.4648
\end{bmatrix},
\]
\[
H_2 = \begin{bmatrix}
0.7739 & 1.4112 & -1.8231 \\
1.4817 & -0.4647 & 2.1620
\end{bmatrix} \quad \text{and} \quad F_1 = \begin{bmatrix}
-0.2636 & 0.2981 \\
-0.3483 & -0.1426
\end{bmatrix}
\]

and power constraints:
\[
P_1 = 1 \quad \text{and} \quad P_2 = 4.
\]

The optimal input covariance matrices for problem (3) are
\[
S_1^* = \begin{bmatrix}
0.9079 & -0.2892 \\
-0.2892 & 0.0921
\end{bmatrix} \quad \text{and} \quad S_2^* = \begin{bmatrix}
0.9458 & 0.1788 & 0.5314 \\
0.1788 & 0.6839 & -1.0601 \\
0.5314 & -1.0601 & 2.3703
\end{bmatrix}
\]
and both $S_1^*$ and $S_2^*$ are singular:

$$\text{rank}(S_1^*) = 1 \quad \text{and} \quad \text{rank}(S_2^*) = 2.$$ 

The $G_1$ and $G_2$ in (4) and (5) and the Lagrangian multipliers are

$$G_1 = \begin{bmatrix} -0.3624 & 0.0005 \\ 0.0005 & -0.3608 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -0.1368 & -0.0525 & -0.0294 \\ -0.0525 & -0.0591 & 0.0583 \\ -0.0294 & 0.0583 & -0.1305 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0.1740 & 0.5463 \\ 0.5463 & 1.7150 \end{bmatrix} \times 10^{-3}, \quad W_2 = \begin{bmatrix} 2.6419 & -5.2450 & -2.9381 \\ -5.2450 & 10.4117 & 5.8325 \\ -2.9381 & 5.8325 & 3.2674 \end{bmatrix} \times 10^{-2}$$

$$\lambda_1 = 0.3626 \quad \text{and} \quad \lambda_2 = 0.1632.$$ 

It is easy to verify that the KKT conditions in (4)-(7) are satisfied.

The $A_1$ and $A_2$ that satisfy (27) and (28) are

$$A_1 = \begin{bmatrix} -0.2821 & 0.4705 \\ 0.0254 & 0.2073 \\ -0.3814 & 0.1588 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0.0047 & 0.2392 & -0.4520 \\ 0.3215 & 0.2853 & -0.1663 \end{bmatrix}.$$ 

The $O_1$ and $O_2$ in (33) and (34) are

$$O_1 = 0 \quad \text{and} \quad O_2 = 0.$$ 

Therefore, (29) and (30) are satisfied. Hence the expressions for $\Sigma_1$ and $\Sigma_2$ are not relevant. As in Remark 8, we only need to show the existence of $\Sigma_1$ and $\Sigma_2$ that satisfy (17)-(19). We have that (71) is also satisfied:

$$\text{radius}(\Phi_1) = 0.4350 \quad \text{and} \quad \text{radius}(\Phi_2) = 0.3130.$$ 

Then, all the conditions in Theorem 2 are satisfied. Therefore, the sum-rate capacity is achieved by treating interference as noise and the optimal input covariances are $S_1^*$ and $S_2^*$.

**Example 2:** Consider a MISO IC in the form (78) with channel vectors:

$$\hat{h}_1 = \begin{bmatrix} -0.1481 \\ -1.7969 \\ 0.1331 \\ 0.6644 \end{bmatrix}, \quad \hat{f}_1 = \begin{bmatrix} 0.0201 \\ -0.0197 \\ -0.0729 \\ 0.7636 \end{bmatrix}, \quad \hat{h}_2 = \begin{bmatrix} 0.1050 \\ -0.0523 \\ 1.8070 \end{bmatrix}, \quad \hat{f}_2 = \begin{bmatrix} -0.4748 \\ -0.7711 \end{bmatrix}.$$
and power constraint
\[ P_1 = P_2 = 1. \]

The equivalent MISO IC in the form (83) has channel vectors
\[ h_1 = \begin{bmatrix} 0.3586 \\ 0.9335 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0.3985 \\ 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0.3818 \\ 0.9242 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0.5426 \\ 0 \end{bmatrix} \]
and power constraints
\[ P_1 = 3.7100 \quad \text{and} \quad P_2 = 3.2789. \]

The corresponding channel parameters are
\[ \theta_1 = 0.3833\pi, \quad \theta_2 = 0.3753\pi, \quad a_1 = 0.1588, \quad a_2 = 0.2944. \]

The optimal input covariance matrices for the equivalent channel are
\[ S_1^* = \begin{bmatrix} 0.2093 & 0.8561 \\ 0.8561 & 3.5007 \end{bmatrix} \quad \text{and} \quad S_2^* = \begin{bmatrix} 0.1345 & 0.6503 \\ 0.6503 & 3.1445 \end{bmatrix}. \]

The corresponding optimal covariance matrices for the original channel are
\[ \hat{S}_1^* = \begin{bmatrix} 0.0070 & 0.0808 & -0.0071 & -0.0187 \\ 0.0808 & 0.9356 & -0.0820 & -0.2168 \\ -0.0071 & -0.0820 & 0.0072 & 0.0190 \\ -0.0187 & -0.2168 & 0.0190 & 0.0502 \end{bmatrix} \quad \text{and} \quad \hat{S}_2^* = \begin{bmatrix} 0.0253 & 0.0204 & 0.1558 \\ 0.0204 & 0.0164 & 0.1253 \\ 0.1558 & 0.1253 & 0.9583 \end{bmatrix}. \]

The \( G_1, G_2 \) in (4) and (5) and the Lagrangian multipliers are
\[ G_1 = \begin{bmatrix} 0.0442 & -0.0357 \\ -0.0357 & -0.0929 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.0929 & -0.0420 \\ -0.0420 & -0.1017 \end{bmatrix} \]
\[ W_1 = \begin{bmatrix} 0.1459 & -0.0357 \\ -0.0357 & 0.0087 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.2033 & -0.0420 \\ -0.0420 & 0.0087 \end{bmatrix} \]
\[ \lambda_1 = 0.1016, \quad \lambda_2 = 0.1104. \]

It can be easily verified that the KKT conditions in (4)-(7) are satisfied.

The \( A_1 \) and \( A_2 \) that satisfy (97) and (98) (or (27) and (28)) are
\[ A_1 = 0.0992 \quad \text{and} \quad A_2 = 0.1156, \]
and the \( \sigma_i^2 \) and \( \bar{\sigma}_i^2 \) in (91)-(94) are
\[ \sigma_i^2 = 0.9874 > \bar{\sigma}_i^2 = 0.6277 \]
\[ \sigma^2_2 = 0.9891 > \sigma^2_2 = 0.4643. \]

Therefore, by Theorem 3, the sum-rate capacity of this MISO channel is achieved by treating interference as noise.

We can also verify condition (103) with

\[ k_1 = 1.0994 \text{ and } k_2 = 0.8133. \]

The \( O_1 \) and \( O_2 \) matrices in (33) and (34) are

\[
O_1 = \begin{bmatrix} 0.0679 & -0.0166 \\ -0.0166 & 0.0041 \end{bmatrix} \quad \text{and} \quad O_2 = \begin{bmatrix} 0.1280 & -0.0265 \\ -0.0265 & 0.0055 \end{bmatrix}.
\]

Since \( W_i \preceq O_i \), by Theorem 2, the sum-rate capacity of this MISO channel is achieved by treating interference as noise.

The sum-rate capacity is

\[ R_1 + R_2 = 0.7533 + 0.7009 = 1.4543. \]

Example 3: Consider a SIMO IC with channel vectors:

\[
\tilde{h}_1 = \begin{bmatrix} -1.8356 \\ 0.0668 \\ 0.0355 \end{bmatrix}, \quad \tilde{f}_1 = \begin{bmatrix} 1.1136 \\ -0.0346 \\ -0.2537 \\ 0.1179 \end{bmatrix}, \quad \tilde{h}_2 = \begin{bmatrix} 0.2458 \\ 0.0700 \\ -0.6086 \\ -1.2226 \end{bmatrix}, \quad \tilde{f}_2 = \begin{bmatrix} 0.1583 \\ -0.6714 \\ -0.5161 \end{bmatrix}
\]

and power constraint

\[ P_1 = P_2 = 1. \]

The equivalent SIMO IC is

\[
h_1 = \begin{bmatrix} -0.2234 \\ 0.9747 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0.6252 \\ 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0.1764 \\ 0.9843 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0.6201 \\ 0 \end{bmatrix}
\]

with power constraint

\[ P_1 = 3.3753 \text{ and } P_2 = 1.9304. \]

The corresponding channel parameters are

\[ \varphi_1 = 0.5717\pi, \quad \varphi_2 = 0.4436\pi, \quad a_1 = 0.3909, \quad a_2 = 0.3845. \]
We simply choose matrices $A_1$ and $A_2$ as in (143) and (144):

$$A_1 = \begin{bmatrix} -0.2434 & 0 \\ 0.6094 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0.2537 & 0 \\ 0.6103 & 0 \end{bmatrix}.$$  

We have $I - A_1^T A_1 - A_2 A_2^T \succeq 0$, $I - A_1 A_1^T - A_2^T A_2 \succeq 0$ and

$$\text{radius} (\Phi_1) = 0.2784 \quad \text{and} \quad \text{radius} (\Phi_1) = 0.2815.$$  

Therefore, by Theorem 4 treating interference as noise achieves the sum-rate capacity and

$$R_1 + R_2 = 0.7297 + 0.5317 = 1.2614.$$  

We can also use Theorem 2 to verify the result. The $A_1$ and $A_2$ satisfy (27) and (28). The numerical radius condition guarantees the existence of $\Sigma_1$ and $\Sigma_2$ to satisfy (17)-(19). Furthermore, we have $W_1 = W_2 = O_1 = O_2 = 0$. Therefore, all the conditions in Theorem 2 are satisfied.

**Example 4:** In this example, we consider the maximum value of $a_i$ for MISO and SIMO ICs to have noisy interference with various choices of $P_i$ and $\theta_i$ or $\varphi_i$. For the symmetric MISO or SIMO IC, one can use Theorem 3 and 4 to generate the same result as [32, Fig. 2]. For the SIMO ZICs, the maximum $a_2$ is 1 regardless of $P_i$ and $\varphi_2$ by Proposition 3. For the MISO ZIC, the maximum $a_2$ is shown in Fig. 2 by Proposition 2.

**Example 5:** In this example, we show that a MISO ZIC in which the noisy-interference conditions in Proposition 2 are violated and treating interference as noise does not achieve the sum-rate capacity.

Consider a MISO ZIC with $P_1 = 1$, $P_2 = 10$, $a_1 = 0$, $a_2 = 0.4$, $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{4}$. As is shown in Fig. 2, this MISO IC does not satisfy the noisy-interference condition. The maximum sum-rate by treating interference as noisy is

$$R_1 + R_2 = 1.3725$$  

and is achieved by (108) and (109):

$$S_1^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S_2^* = \begin{bmatrix} 1.7566 & 3.8053 \\ 3.8053 & 8.2434 \end{bmatrix}.$$  

However, we consider a Han and Kobayashi achievable rate region [6], [7] for the MISO ZIC:

$$R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{1 + f_2^T S_p f_2} \right)$$  

$$R_2 \leq \frac{1}{2} \log \left( 1 + h_2^T (S_p + S_c) h_2 \right)$$
\[ R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + h_2^T S_p h_2 \right) + \frac{1}{2} \log \left( 1 + \frac{P_1 + f_2^T S_c f_2}{1 + f_2^T S_p f_2} \right) \]

where \( S_p \) and \( S_c \) are respectively the covariance matrices for the input vectors that carry the private and common messages. Then we can achieve a sum-rate of

\[ R_1 + R_2 = 1.4093 \]

by the same \( S_1^* \) and a different \( S_2^* = S_p^* + S_c^* \) with

\[ S_p^* = \begin{bmatrix} 1.1542 & 2.2652 \\ 2.2652 & 4.4458 \end{bmatrix} \quad \text{and} \quad S_c^* = \begin{bmatrix} 4.1906 & 0.9367 \\ 0.9367 & 0.2094 \end{bmatrix}. \]

VI. Conclusion

We have studied the noisy-interference sum-rate capacity of MIMO ICs. Sufficient conditions for a MIMO IC to achieve the sum-rate capacity by treating interference as noise have been obtained. For the special cases of MISO and SIMO ICs, simplified conditions have been derived. These conditions largely extend all the existing sufficient conditions.

Appendix

A. Proof of Lemma 1

If we write the optimization problem in the standard form:

\[
\begin{align*}
\text{min} & \quad f(\mathbf{x}) \\
\text{subject to} & \quad g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\
& \quad \mathbf{x} \in \mathcal{X}
\end{align*}
\]

then CQ5 in [39, p. 306] requires that there exist a vector \( \mathbf{y} \in N_{\mathcal{X}}(\mathbf{x}^*) \) such that

\[
\nabla g_j(\mathbf{x}^*)^T \mathbf{y} < 0 \quad \forall j \in A(\mathbf{x}^*)
\]

where \( \mathbf{x}^* \) is optimal for problem (153), \( \nabla g_j(\mathbf{x}^*) \) is the gradient of \( g_j(\mathbf{x}) \) at \( \mathbf{x}^* \), \( N_{\mathcal{X}}(\mathbf{x}^*) \) is the normal cone of \( \mathcal{X} \) at \( \mathbf{x}^* \), \( N_{\mathcal{X}}(\mathbf{x}^*)^* \) is the polar cone of \( N_{\mathcal{X}}(\mathbf{x}^*) \), and \( A(\mathbf{x}^*) \) is index set of all the active inequality constraints. Applying this theorem to our case, we need to find matrices \( K_i, i = 1, 2 \), such that

\[
\begin{align*}
K_i & \in N_{\mathcal{S}_i}(\mathbf{S}_i^*)^* = T_{\mathcal{S}_i}(\mathbf{S}_i^*) \\
\text{tr}(K_i) & < 0 \quad \text{if} \quad \text{tr}(\mathbf{S}_i^*) = P_i
\end{align*}
\]
Fig. 2. The maximum value of $a$ for a MISO ZIC with $P_1 = 1$ to have noisy interference.

where $\mathcal{S}_i$ is the set of symmetric positive semi-definite matrices with the same dimension as that of $\mathcal{S}_i^*$, and $T_{\mathcal{S}_i}(\mathcal{S}_i^*)$ is the tangent cone of $\mathcal{S}_i$ at $\mathcal{S}_i^*$. The equality of (155) is due to the convexity of $\mathcal{S}_i$ and [39, Proposition 4.6.3, p. 254].

Define a sequence of matrices $\{Y_k\}$:

$$Y_k = \mathcal{S}_i^* - \frac{1}{k} U \cdot \text{diag}[\eta_1, 0, \cdots, 0] \cdot U^T, \quad k = 1, 2, \cdots$$

(157)

where $U$ is a unitary matrix associated with the eigenvalue decomposition of $\mathcal{S}_i^*$, and $\eta_1$ is the largest eigenvalue of $\mathcal{S}_i^*$:

$$\mathcal{S}_i^* = U \cdot \text{diag}[^\eta_1, \eta_2, \cdots, \eta_t] \cdot U^T.$$  

(158)

Obviously, we have

$$\{Y_k\} \subseteq \mathcal{S}_1, \quad Y_k \neq \mathcal{S}_1 *$$

(159)
\[
\lim_{k \to \infty} Y_k = S_1^*
\]  \hspace{1cm} (160)
\[
\lim_{k \to \infty} \frac{Y_k - S_1^*}{\|\text{Vec}(Y_k - S_1^*)\|} = -U \cdot \text{diag}[\eta_1, 0, \ldots, 0] \cdot U^T \quad \text{if } \eta_1 < 0
\]  \hspace{1cm} (161)

Therefore, by [39, Definition 4.6.2, p. 248]
\[
K_1 \triangleq -U \cdot \text{diag}[\eta_1, 0, \ldots, 0] \cdot U^T \in T_{S_1^*}(S_1^*).
\]  \hspace{1cm} (162)

Since \(\eta_1\) is the largest eigenvalue of \(S_1^*\), we have
\[
\text{tr} (K_1) = -\eta_1 < 0 \quad \text{if } \text{tr}(S_1^*) = P_1 > 0.
\]  \hspace{1cm} (163)

We can similarly find \(K_2\) satisfying (155) and (156) for \(S_2^*\). Therefore, the constraint qualifications are satisfied and there exist Lagrangian multipliers \(\lambda_i\) and \(W_i\) satisfying (4)-(7).

\section*{B. Proof of Lemma 2}

To prove that the objective function of problem (16) is concave over \(S_1^*\) and \(S_2^*\), it is equivalent to prove that (23) is concave. By [31, Lemma 1], both the conditional entropies \(h(H_1|x_{1G} + F_2x_{2G} + z_2|F_1x_{1G} + n_1)\) and \(h(H_2|x_{2G} + F_1x_{1G} + z_1|F_2x_{2G} + n_2)\) are concave. Therefore, by symmetry, it suffices to prove that
\[
h(F_1x_{1G} + n_1) - h(F_1x_{1G} + z_2|n_2)\]

is concave over \(S_1\) and \(S_2\).

From (20) we have \(\text{Cov}(z_2|n_2) = I - A_2\Sigma_2^{-1}A_2^T\). From (18), there exists a Gaussian vector \(v \sim \mathcal{N}(0, \hat{\Sigma})\) where
\[
\hat{\Sigma} = (I - A_2\Sigma_2^{-1}A_2^T) - \Sigma_1.
\]

We further let \(\tilde{z}\) be independent of all other random vectors of interest, and then we have
\[
h(F_1x_{1G} + n_1) - h(F_1x_{1G} + z_2|n_2) = h(F_1x_{1G} + n_1) - h(F_1x_{1G} + n_1 + v)
\]
\[
= -I(v; F_1x_{1G} + n_1 + v).
\]  \hspace{1cm} (164)

Define a binary random variable \(Q\) with probability mass function \(Pr(Q = 0) = q\) and \(Pr(Q = 1) = 1 - q\) where \(0 \leq q \leq 1\). Let \(\tilde{x}_1\) have mixed Gaussian distribution with conditional distribution
\[
p(\tilde{x}_1|Q) = \begin{cases} p(\tilde{x}_1|Q = 0) = p(\tilde{x}_1^{(1)}) & \sim \mathcal{N}(0, S_1^{(1)}) \\ p(\tilde{x}_1|Q = 1) = p(\tilde{x}_1^{(2)}) & \sim \mathcal{N}(0, S_1^{(2)}) \end{cases}
\]  \hspace{1cm} (165)

where
\[
S_1 = qS_1^{(1)} + (1 - q)S_1^{(2)}.
\]  \hspace{1cm} (166)
Then we have
\[-qI(v; F_1x_1^{(1)} + n_1 + v) - (1 - q)I(v; F_1x_2^{(2)} + n_1 + v)\]
\[= -I(v; F_1x_1 + n_1 + v|Q)\]
\[= -h(v|Q) + h(v|F_1x_1 + n_1 + v, Q)\]
\[\leq -I(v; F_1x_1 + n_1 + v)\]
\[\leq -I(v; F_1x_1G + n_1 + v)\]  \hspace{1cm} (167)
where (a) is by the assumption that \(Q\) is independent of \(v\) and the fact that conditioning does not increase entropy. In (b), we let \(x_1G \sim \mathcal{N}(0, S_1)\). The inequality is by (166) and the fact that Gaussian noise is the worst additive noise [43]. Therefore, \(h(F_1x_1G + n_1) - h(F_1x_1G + z_2|n_2)\) is concave over \(S_1\) and \(S_2\). Similarly, we can prove that \(h(F_2x_2G + n_2) - h(F_2x_2G + z_1|n_1)\) is also a concave function of \(S_1\) and \(S_2\).

C. Proof of Lemma 3

In the proof of [31, Lemma 9], if radius(\(\Phi_1\)) \(\leq\frac{1}{2}\), then there exist \(\Sigma_1\) that satisfy
\[
\Sigma_1 = I - A_1(I - A_1\Sigma_1^{-1}A_1^T)^{-1}A_2
\]  \hspace{1cm} (168)
and \(\Sigma_1 - A_1^T A_1\) is positive definite. Then it suffice to prove that \(I - A_1\Sigma_1^{-1}A_1^T\) is positive definite since we can substitute \(\Sigma_1\) defined in (168) into (75) and obtain a positive definite \(\Sigma_2\).

Let \(\Sigma_1 = A_1^T A_1 + X\) where \(X \succ 0\); then we have
\[
\Sigma_2 = I - A_1\Sigma_1^{-1}A_1^T
\]
\[= I - A_1(X + A_1^T A_1)^{-1}A_1^T\]
\[\overset{(a)}{=} I - A_1(X + T\Lambda T^T)^{-1}A_1^T\]
\[\overset{(b)}{\geq} I - A_1(\eta I + \Lambda)^{-1}T^T A_1^T\]
\[= I - A_1 T(\eta I + \Lambda)^{-1}T^T A_1^T\]  \hspace{1cm} (169)
where in (a) we let \(A_1^T A_1 = T\Lambda T^T\) be the eigenvalue decomposition of \(A_1^T A_1\) and \(TT^T = I\) and \(\Lambda\) is a diagonal matrix with non-negative diagonal elements. In (b), we let \(\eta\) be the smallest eigenvalue of \(X\). Since \(X\) is symmetric positive definite, we have \(\eta > 0\). The inequality of (b) is by the fact \(X \succeq \eta I\).
Since $I - B^TB$ is positive definite if and only if $I - BB^T$ is positive definite, we only need to prove that $I - (\eta I + \Lambda)^{-\frac{1}{2}} T^T A_1^T A_1 T (\eta I + \Lambda)^{-\frac{1}{2}}$ is positive definite, which is obviously true since $T^T A_1^T A_1 T = \Lambda$ and $\eta > 0$.

We have proved that if radius $(\Phi_1) \leq \frac{1}{2}$, then there exist $\Sigma_1 \succ A_1^T A_1$ and $\Sigma_2 \succ 0$ that satisfy (74) and (75). Now we need to prove that $\Sigma_2 \succ A_2^T A_2$, which is true by the fact $I - A_2 \Sigma_2^{-1} A_2^T = \Sigma_1 \succ 0$ and [31, Lemma 6].

By symmetry, if radius$(\Phi_2) \leq \frac{1}{2}$, we also have positive definite solutions. The equivalence between (76) and (77) is by [31, Lemma 6].

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