A note on Mellin transform, Eisenstein Series and distribution $d\epsilon_{it}$ on $\text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C})$

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Abstract

Let $\text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C})$ be the frame bundle of the Picard variety $\text{PSL}(2, \mathbb{Z}[i]) \backslash \mathbb{H}^3$, $f$ a smooth function with compact support defined on $\text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C})$ and $\mathcal{M}(f, s)$ the Mellin transform of $f$, in this note we prove in the proposition (2.3) that

$$\mathcal{M}(f, s) = \sum_{L \geq 0} \sum_{k,m \equiv -L/2}^{L/2} (-1)^{m-k} \cdot T_{kk}(I) \int f_{k,m}(z + \lambda j) \cdot E_{m,-m,-k}^{1/2}(z + \lambda j, it) \frac{dxdyd\lambda}{\lambda^3},$$

(1)

where $f_{k,m}(z + \lambda j)$ are the Fourier expansion coefficients of $f$ on the fibers, $E_{k,m}^L(g, s)$ are the Eisenstein series defined on $\text{SL}(2, \mathbb{C})$ using the representations of $\text{SU}(2)$ (see formula (26)) and $T_{kk}$ are a basis of eigenfunctions for Laplacian defined on $\text{SU}(2)$ (see formula (16)). The previous result is a generalization of the proposition 2.6 in Sarnak [19] for functions in the unit tangent bundles of a hyperbolic surfaces. Using (1) we can define the micro-local lift $d\epsilon_{it}$ (to $\text{SL}(2, \mathbb{C})$) of distribution $|E(z + \lambda j, it)|$, $dV$ as follows:

$$d\epsilon_{it} := E(z + \lambda j, it) \sum_{L \geq 0} \sum_{k,m \equiv -L/2}^{L/2} (-1)^{m-k} \cdot T_{kk}(I) \cdot E_{m,-m,-k}^{1/2}(z + \lambda j, -it) \cdot T_{km}.$$

(2)

The distribution $d\epsilon_{it}$ is analogous to the distribution $d\epsilon_{1/4 + it}$ which appears in Zelditch [26] for hyperbolic surfaces. We use ideas from Luo and Sarnak [13], Jakobson [7] and Koyama [9] to calculate $(f, d\epsilon_{it})$ for $f$ a cuspidal form (see formula 70) and for an incomplete Eisenstein series (see formula 92). We also establish asymptotic estimates when $t$ tends to $\infty$ (propositions (3.2) and (4.3)).

We conjecture that the new positive distribution $d\epsilon_{it}^F$, constructed with the Friedrichs’ symmetrization technique applied to $d\epsilon_{it}$, satisfies the same asymptotic estimates that $d\epsilon_{it}$ as in the propositions (3.2) and (4.3). This is what happens in the case of $\text{PSL}(2, \mathbb{Z}) \backslash \text{PSL}(2, \mathbb{R})$, see Jakobson [7]. This would imply that if $\Omega_1$ and $\Omega_2$ (vol $(\Omega_2) \neq 0)$ be compact Jordan measurable sets in $\text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C})$ then

$$\lim_{t \to \infty} \frac{\int_{\Omega_1} d\epsilon_{it}^F}{\int_{\Omega_2} d\epsilon_{it}^F} = \frac{\text{vol} (\Omega_1)}{\text{vol} (\Omega_2)}.$$ 

(3)

This last assertion is the quantum ergodicity for Eisenstein series on $\text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C})$.

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Introduction

The idea of integrating an invariant function with an Eisenstein series was introduced by Rankin and Selberg. Let \(\Sigma = \text{PSL}(2, \mathbb{Z}) \setminus \mathbb{H}^2\) the modular surface and \(f \in C_0^\infty(\Sigma)\), the Mellin transform of \(f\) in \(s\) can be obtained by integrating \(f\) with the Eisenstein series \(E(z, s)\) associated with \(\text{PSL}(2, \mathbb{Z})\). If \(f \in C_0^\infty(\text{PSL}(2, \mathbb{Z}) \Sigma)\), with \(\Sigma\) denote the unit tangent bundle to \(\Sigma\), Sarnak [19] gives a similar formula using the Eisenstein series \(E_f\) with the Eisenstein series \(E_f\) for any compact Jordan measurable sets \(\Omega\) and \(\Omega_1 \text{ and } \Omega_2 \text{ (area}(\Omega_2) \neq 0)\) in \(\Sigma\) is true that

\[
\lim_{k \to \infty} \int_{\Omega} |\varphi_{j_k}|^2 dA = \frac{\text{area}(\Omega)}{\text{area}(X)}.
\]

For the continuous spectrum of hyperbolic Laplacian given by the Eisenstein series, Luo and Sarnak [13] proved that for any compact Jordan measurable sets \(\Omega_1\) and \(\Omega_2\) \((\text{area}(\Omega_2) \neq 0)\) in \(\Sigma\) is true that

\[
\lim_{t \to \infty} \frac{\int_{\Omega_1} |E(z, \frac{1}{2} + it)|^2 \frac{dxdy}{y^2}}{\int_{\Omega_2} |E(z, \frac{1}{2} + it)|^2 \frac{dxdy}{y^2}} = \frac{\text{area}(\Omega_1)}{\text{area}(\Omega_2)},
\]

which follows from

\[
\int_{\Sigma} f(z) \cdot |E(z, \frac{1}{2} + it)|^2 \frac{dxdy}{y^2} \sim \frac{48}{\pi} \ln t \int_{\Sigma} f(z) \frac{dxdy}{y^2}.
\]

Jakobson [7] formulated an analog of (5) and then of (4) for \(\text{PSL}(2, \mathbb{Z}) \setminus \text{PSL}(2, \mathbb{R})\), for this, it was required a micro-local lift of the positive distribution \(|E(z, \frac{1}{2} + it)|^2 dA\) to \(\Sigma\). Zelditch [26] gave the following distribution:

\[
d\epsilon_{\frac{1}{2} + it} = E(\cdot, \frac{1}{2} + it) \sum_{k=0}^\infty E_{2k}(\cdot, \frac{1}{2} - it) e^{-2ik\theta}.
\]

Zelditch noted that the distribution \(\epsilon_{\frac{1}{2} + it}\) is not positive. A new positive distribution \(d\epsilon^\text{F}_{\frac{1}{2} + it}\) built using the Friedrichs’ symmetrization technique is required, see Zelditch [26].
Theorem 0.2 (Jakobson). Let \( \Omega_1 \) and \( \Omega_2 \) \((\text{vol}(\Omega_2) \neq 0) \) be arbitrary compact Jordan measurable sets in \( \text{PSL}(2, \mathbb{Z}) \backslash \text{PSL}(2, \mathbb{R}) \). Then

\[
\lim_{t \to \infty} \int_{\Omega_1} \frac{dE_{ it}}{d} = \frac{\text{vol}(\Omega_1)}{\text{vol}(\Omega_2)}
\]

(7)

Koyama [9] got the analog result at (4) for arithmetic 3-orbifolds \( M_D = \text{PSL}(2, \mathcal{O}_D) \backslash \mathbb{H}^3 \) with only one cusp, where \( \mathbb{Q}(\sqrt{-D}) \) be a imaginary quadratic field and \( \mathcal{O}_D \) its ring of integers, this includes the Picard variety that corresponds to the field \( \mathbb{Q}(\sqrt{-1}) \).

Theorem 0.3 (Koyama). Let \( \Omega_1 \) and \( \Omega_2 \) \((\text{vol}(\Omega_2) \neq 0) \) be arbitrary compact Jordan measurable sets in \( M_D \). Then

\[
\lim_{t \to \infty} \int_{\Omega_2} \frac{d\eta_{it}}{d} = \frac{\text{vol}(\Omega_1)}{\text{vol}(\Omega_2)}
\]

(8)

where \( d\eta_{it} = |E(z + \lambda j, \frac{1}{2} + it)|^2 dV \) with \( E(z + \lambda j, s) \) being the Eisenstein series for \( \text{PSL}(2, \mathcal{O}_D) \) and \( dV \) is the volume element of \( \mathbb{H}^3 \).

For \( \text{PSL}(2, \mathbb{Z}[i]) \backslash \text{PSL}(2, \mathbb{C}) \) the distribution in (2) is the analogue of the distribution in (6). The formula (7) of Jakobson’s theorem corresponds to the formula in (3).

In section 1 of preliminaries we recall some concepts such as: Wigner matrix, Eisenstein series and your Fourier expansion. In section 2 we review the Mellin transform and we prove the formula in (1). In section 3 we will calculate the interior product \((f, de_{it})\) for when \( f \) is a cusp form and we estimate when \( t \to \infty \). In section 4 we consider the case where \( f \) is an incomplete Eisenstein series. In the appendix we gather some auxiliary results.

1. Preliminaries

Let \( K := \mathbb{Q}(\sqrt{-1}) \) be an imaginary quadratic field and \( \mathcal{O} = \mathbb{Z}[i] \) its ring of integers. Let \( \tilde{\Gamma} := \text{PSL}(2, \mathbb{Z}[i]) \) be the corresponding Bianchi subgroup in \( \text{PSL}(2, \mathbb{C}) \) and \( \Gamma = \text{SL}(2, \mathbb{Z}[i]) \) the Bianchi subgroup in \( \text{SL}(2, \mathbb{C}) \). The quotient \( M = \tilde{\Gamma} \backslash \mathbb{H}^3 \) are a tridimensional Bianchi orbifold with only one cusp in \( \infty \). Since \( \text{PSL}(2, \mathbb{C}) \) can be identified with the orthonormal frame bundle \( F(\mathbb{H}^3) \) of hyperbolic 3-space then \( F(M) = \tilde{\Gamma} \backslash \text{PSL}(2, \mathbb{C}) \) can be identified with the orthonormal frame bundle of \( M \).

The upper half-space model for hyperbolic space will be used \( \mathbb{H}^3 \):

\[
\mathbb{H}^3 = \{(z, \lambda); z \in \mathbb{C}, \lambda \in \mathbb{R}^+\} = \{(x, y, \lambda); x, y \in \mathbb{R}, \lambda > 0\},
\]

provided with the hyperbolic Riemannian metric: \( ds^2 = \frac{dx^2 + dy^2 + dz^2}{\lambda^2} \). A point \( P \in \mathbb{H}^3 \) is denoted as follows:

\[
P = (z, \lambda) = z + \lambda j,
\]

where \( z = x + iy, \lambda > 0, \) and \( j \) is the well know quaternion. The groups \( \text{PSL}(2, \mathbb{C}) \) and \( \text{SL}(2, \mathbb{C}) \) act on \( \mathbb{H}^3 \) as follows:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + \lambda j) = \frac{(az + b)(\bar{c}z + d) + \bar{a}c\lambda^2}{|cz + d|^2 + |c|^2\lambda^2} + \frac{\lambda}{|cz + d|^2 + |c|^2\lambda^2} j.
\]

(9)

Remark 1.1. The action for \( \text{SL}(2, \mathbb{C}) \) is not effective since two matrices that differ by a sign determine the same Möbius transformation. The group \( \text{PSL}(2, \mathbb{C}) \) acts effectively on \( \mathbb{H}^3 \) by orientation-preserving isometries of the hyperbolic metric.

If \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) then by the complex Iwasawa decomposition \( g \) can be written in a unique way by the following formula:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{\bar{a}d - \bar{b}c}{|c|^2 + |d|^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{|c|^2 + |d|^2}} & 0 \\ 0 & \sqrt{|c|^2 + |d|^2} \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{|c|^2 + |d|^2}} & \frac{-\bar{c}}{\sqrt{|c|^2 + |d|^2}} \\ \frac{\bar{d}}{\sqrt{|c|^2 + |d|^2}} & \frac{\bar{d}}{\sqrt{|c|^2 + |d|^2}} \end{pmatrix}.
\]
We call \((z, \lambda, A)\) the Iwasawa coordinates of \(g\) with
\[
 z = \frac{a\tilde{c} + b\tilde{d}}{|c|^2 + |d|^2}, \quad \lambda = |d|^2 + |d|^2, \quad A = \left( \begin{array}{cc} \frac{\tilde{d}}{\sqrt{\lambda}} & \frac{-\tilde{c}}{\sqrt{\lambda}} \\ \frac{-\tilde{d}}{\sqrt{\lambda}} & \frac{\tilde{c}}{\sqrt{\lambda}} \end{array} \right) \in \text{SU}(2).
\] (10)

Using Iwasawa coordinates we can write \(g\) as \(g = n[z]a[\frac{1}{\lambda}]A\), where

\[
n[z] := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad a[\lambda] := \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}.
\]

Denote \(G := \text{SL}(2, \mathbb{C})\) and \(dg, dk\) the corresponding Haar measures in \(G\) and \(K\) respectively that satisfy
\[
\int_{\text{SU}(2)} dk = 1, \quad dg = \frac{dx dy d\lambda dk}{\lambda^3}.
\]

Also will be denoted by \(dV\) the hyperbolic volume measure in \(\mathbb{H}^3\), which is
\[
dV = \frac{dx dy d\lambda}{\lambda^3}.
\]

Let \(l \in \frac{1}{2}\mathbb{Z}, l \geq 0\), two representations of \(\text{SU}(2)\) of dimension \(2l + 1\) will be used, for that purpose, we consider vectorial spaces on \(\mathbb{C}\) generated by:

\[
V_{2l} := \{ z^{l-q}; q \in \frac{1}{2}\mathbb{Z}, q \equiv l \mod 1, q \in [-l, l] \},
\]

\[V_{2l} := \{ f_m(x, y); m \in \frac{1}{2}\mathbb{Z}, m \equiv l \mod 1, m \in [-l, l] \}, \quad \text{where } f_m(x, y) := \frac{z^{l+m}y^{-m}}{\sqrt{(l+m)!(l-m)!}}.
\]

Guleska [5] considers the following representation, \(K \in \text{SU}(2) \rightarrow \sigma_l(K) : V_{2l} \rightarrow V_{2l}\) given by
\[
\sigma_l(K[\alpha, \beta]) \ z^{l-q} := (\alpha z - \beta)^{l-q} (\beta z + \bar{\alpha})^{l+q}, \quad \text{where } K = K[\alpha, \beta] := \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}
\]

with \(\alpha, \beta \in \mathbb{C}\) that satisfy \(|\alpha|^2 + |\beta|^2 = 1\). We have the next scalar product
\[
\langle z^{l-q}, z^{l-k} \rangle = \begin{cases} 0 & \text{if } k \neq q \\ \frac{1}{(l+q)!(l-q)!} & \text{if } k = q. \end{cases}
\]

Using the basis \(V_{2l}\) the functions \(\Phi_{km}^l : \text{SU}(2) \rightarrow \mathbb{C}\), that are the matrix coefficients of the representation \(\sigma_l\), are defined by
\[
\Phi_{km}^l(K) := \frac{1}{(l+k)!(l-k)!} \langle \sigma_l(K)z^{l-m}, z^{l-k} \rangle, \quad \forall K \in \text{SU}(2).
\]

The basis of functions \(\{\Phi_{km}^l\}\) in \(\text{SU}(2)\) is orthogonal, this is
\[
\int_{\text{SU}(2)} \Phi_{km}^l(K) \cdot \Phi_{km'}^{l'}(K) dk = \frac{1}{2l+1} \cdot \frac{(l+m)!(l-m)!}{(l+k)!(l-k)!} \delta_{l,l'} \delta_{k,k'} \delta_{m,m'}.
\] (11)

Wigner [24] uses the next representation, \(K \in \text{SU}(2) \rightarrow P_{km}^l : V_{2l} \rightarrow V_{2l}\) given by
\[
P_{km}^l f_m(x, y) := f_m \left( K^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right).
\]

The functions \(U_{km}^l : \text{SU}(2) \rightarrow \mathbb{C}\) are determined by the next formula
\[
P_{km}^l f_m(x, y) = \sum_{k=-l}^{l} U_{km}^l(K) \cdot f_k(x, y).
\]

The relationship between the matrix coefficients in the different basis is given by
Lemma 1.2. Let \( l, k, m \in \frac{1}{2} \mathbb{Z}, l \geq 0 \), such that \( l \equiv k \equiv m \mod 1, k, m \in [-l, l] \). Then

\[
\Phi_{km}^l(K) = \frac{\sqrt{(l+m)!(l-m)!}}{\sqrt{(l+k)!(l-k)!}} \cdot \Omega_{km}^l (IKI^{-1}), \quad \forall K \in SU(2),
\]

where \( I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \).

Proof. Direct calculations.

We remember the double covering epimorphism \( \Phi : SU(2) \rightarrow SO(3) \) which is denoted for \( \Phi(A) = \Phi_A \), then

\[
\Phi_{AB} = \Phi_A \circ \Phi_B, \quad \forall A, B \in SU(2).
\]

(13)

This is of course the spin cover of \( SO(3) \). Explicitly, if \( A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SU(2) \), then

\[
\Phi_A = \begin{pmatrix} \text{Re}(\alpha^2 - \beta^2) & -\text{Im}(\alpha^2 + \beta^2) & 2 \text{Re}(\alpha \beta) \\ \text{Im}(\alpha^2 - \beta^2) & \text{Re}(\alpha^2 + \beta^2) & 2 \text{Im}(\alpha \beta) \\ -2 \text{Re}(\alpha \beta) & \text{Im}(\alpha \beta) & |\alpha|^2 - |\beta|^2 \end{pmatrix}.
\]

(15)

A consequence of the Peter-Weyl theorem is that the functions \( D_{km}^l : SU(2) \rightarrow \mathbb{C} \) are a basis of \( L^2(SU(2), dk) \). As in Lachieze-Rey [15], we choose the basis of \( L^2(SU(2), dk) \) next:

\[
\{ T_{km}^l : l \in \mathbb{Z}, l \geq 0 ; k, m \in \frac{1}{2} \mathbb{Z}, k \equiv m \equiv \frac{l}{2} \mod 1, \text{ and } k, m \in [-\frac{l}{2}, \frac{l}{2}] \},
\]

where

\[
T_{km}^l(A) := \sqrt{\frac{l+1}{2\pi^2}} \cdot D_{mk}^{l/2} (\Phi(A)), \quad \forall A \in SU(2).
\]

(16)

Let \( R \in SO(3) \), we can write \( R \) in terms of its Euler angles:

\[
R = ROT(\theta, \chi, \phi) := \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \chi & 0 & -\sin \chi \\
0 & 1 & 0 \\
\sin \chi & 0 & \cos \chi
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with \( \theta \in [0, 2\pi), \chi \in [0, \pi], \phi \in [-2\pi, 2\pi) \). In addition we have,

\[
D_{km}^l(R) = D_{km}^l(ROT(\theta, \chi, \phi)) = e^{i\theta} d_{km}^l(\chi) e^{im\phi},
\]

(17)

where \( d_{km}^l(\chi) \) is Wigner small \( d \)-matrix. We refer to the book of Wigner [24] in order to see how the coefficients \( D_{km}^l(R) \) are deduced.

Later we use the following property of Wigner matrices. Let \( R, T \in SO(3), l, k, m \in \frac{1}{2} \mathbb{Z}, l \geq 0 \) and such that \( k, m \in [-l, l] \). Then

\[
D_{km}^l(R \circ T) = \sum_{a=-l}^{l} D_{ka}^l(R) \cdot D_{am}^l(T).
\]

(18)

This is a consequence of the fact that for fixed \( l \), the entries \( D_{km}^l \) are the coefficients of a representation.

In order to see how the basis changes \( \{ T_{km}^l \} \) under rotations, we remember the action of \( SU(2) \) in the smooth functions defined in \( SU(2) \)

\[
A \in SU(2) \longrightarrow R_A : C^\infty(SU(2)) \rightarrow C^\infty(SU(2))
\]
by
\[ R_A : f \mapsto R_A f \text{ with } R_A f(K) := f(A^{-1} K), \quad \forall f \in C^\infty(SU(2)), \quad \forall K \in SU(2). \]  

(19)

Then,
\[ R_{A^{-1}} \mathcal{T}_{km}^l(K) = \sqrt{\frac{l+1}{2\pi^2}} \cdot D_{mk}^{l/2}(\Phi(AK)), \quad \forall A, K \in SU(2). \]

(20)

Replacing (16) and (20) in (18) it is concluded that the basis \{\mathcal{T}_{km}^l\} changes under rotations according to the next formula:
\[ R_{A^{-1}} \mathcal{T}_{km}^l = \sum_{a=-l/2}^{l/2} D_{ma}^{l/2}(\Phi(A)) \cdot \mathcal{T}_{ka}^l, \forall A \in SU(2). \]

(21)

Now we remember other properties of Wigner matrix that we further use
\[ \mathcal{M}_{km}^l(K) = D_{km}^l(\Phi(K)), \quad \forall K \in SU(2). \]

(22)

\[ D_{km}^l(\Phi(IK^{-1})) = (-1)^{m-k} \cdot D_{km}^l(\Phi(K)), \quad \forall K \in SU(2). \]

(23)

\[ D_{km}^l(\Phi(K)^{-1}) = (-1)^{m-k} \cdot D_{m,-k}^l(\Phi(K)) = \overline{D_{mk}^l(\Phi(K))}, \quad \forall K \in SU(2). \]

(24)

Let \( l, k, m \in \frac{1}{2} \mathbb{Z}, \; l \geq 0, \; l \equiv k \equiv m \mod 1 \) and such that \( k, m \in [-l,l] \). We denote by \( \Gamma'_{\infty} \) the set of right cosets of \( \Gamma_{\infty}' \) in \( \Gamma \). We remember that \( \Gamma'_{\infty} \) is the maximal unipotent subgroup, i.e.,
\[ \Gamma'_{\infty} = \left\{ \left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) ; \; z \in \mathbb{Z}[i] \right\}. \]

If \( s \in \mathbb{C} \) is such that \( \text{Re}(s) > 1 \), \( f_{km}^l(\cdot, s) : SL(2, \mathbb{C}) \to \mathbb{C} \) is defined by the formula:
\[ f_{km}^l(g, s) := D_{km}^l(\Phi(g)_{T(g)^{-1}}) \text{ Im } g(j)^{1+s}. \]

(25)

The Eisenstein series \( E_{km}^l(\cdot, s) : SL(2, \mathbb{C}) \to \mathbb{C} \) associated to \( \Gamma \) at the cusp \( \infty \) are defined by the following formula:
\[ E_{km}^l(g, s) := \sum_{\sigma \in \Gamma'_{\infty} \setminus \Gamma} f_{km}^l(\sigma g, s). \]

(26)

**Remark 1.3.** In [18] the previous series are denoted as \( \hat{E}_{km}^l(g, s) \). Furthermore, we omit the factor \( \frac{1}{\Gamma'_{\infty} \setminus \Gamma} \) that appears in this work.

The series \( E_{km}^l(g, s) \) are well defined, they are convergent to \( \text{Re}(s) > 1 \) and are invariants under \( \Gamma \). So we define the series \( E_{km}^l(\cdot, s) : \mathbb{H}^3 \to \mathbb{C} \) associated to \( \Gamma \) at the cusp \( \infty \) by the formula:
\[ E_{km}^l(z + \lambda j, s) := E_{km}^l(n[z]|a[\lambda], s). \]

(27)

The identity (27) implies that the series \( E_{km}^l(z + \lambda j, s) \) are well defined and are convergent in the half-plane \( \text{Re}(s) > 1 \). Although \( E_{km}^l(z + \lambda j, s) \) do not define in general (except for \( l = k = m = 0 \)) Eisenstein series in \( \mathbb{H}^3 \) (since they are not invariant under \( \Gamma \)), they do admit a Fourier expansion.

In [12] Langlands developed the general theory of Eisenstein series, proving that they satisfy a functional equation and the analytic or meromorphic continuation of \( E_{km}^l(g, s) \) with respect \( s \) to all the complex plane. Also, since the Bianchi groups are special, the analytic or meromorphic continuation can be obtained without using the Langlands work, see [1], page 44, for a proof in our particular case \( \Gamma = PSL(2, \mathbb{Z}[i]) \).
In the following sections, the Fourier expansion of the series \( E_{k,m}^l(z + \lambda_j, s) \) will be used, before this, it is necessary to recall some important facts. Let \( n \in 4\mathbb{Z} \), we define a Hecke character \( \chi_n \) for \( Q(\sqrt{-1}) \) as follows:

\[
\chi_n(c) := \left( \frac{c}{|c|} \right)^n, \; \; 0 \neq c \in \mathbb{Z}[i],
\]

and where \( (c) \) denotes the ideal generated by \( c \). The Hecke L function \( L(s, \chi_n) \) are defined by:

\[
L(s, \chi_n) = \sum_{a} \frac{\chi_n(a)}{|N(a)|^s} = \prod_{p \text{ primo}} \frac{1}{1 - \chi_n((p)) \cdot |p|^{-2s}},
\]

for \( s \in \mathbb{C} \) such that \( \text{Re}(s) > 1 \) and the sum runs over all integral ideals \( a \) not zero of \( \mathbb{Z}[i] \). We denoted by \( \zeta_K(s) \) to the Dedekind zeta function associated to \( Q(\sqrt{-1}) \). In particular, \( L(s, \chi_0) = \zeta_K(s) \).

The Fourier expansion of series \( E_{k,m}^l(z + \lambda_j, s) \) (see Guleska [5] or Watt [23], page 17) is given by:

\[
E_{k,m}^l(z + \lambda_j, s) = \delta_{k,m} \left[ \Gamma_{\infty} : \Gamma'_{\infty} \right] B_{k,m}^l \lambda^{1+s} + (-1)^{m+l} \frac{\Gamma(1+l-s) \Gamma(|m| + s)}{\Gamma(1+l+s) \Gamma(1+|m|-s)} \frac{L(s, \chi_{2m})}{L(s+1, \chi_{2m})} \delta_{-k,m} B_{k,m}^l \lambda^{1-s} \\
+ (-1)^{l+m} \left( 2\pi i \right)^{l-k-m} \sum_{0 \neq \omega \in \Lambda'} D_{k,m}(-\omega; s) \cdot |\omega|^{s-1} \cdot e^{-4\pi i \text{Re}(\omega z)} \left( \frac{\omega}{|\omega|} \right)^{-k-m} \\
\cdot \sum_{u=0} \xi_{l-k-m}(u) \frac{(2\pi |\omega| \lambda)^{1+l-u}}{\Gamma(1+l+s-u)} \cdot K_{s+l-|k+m|+u} \left( 4\pi |\omega| \lambda \right),
\]

with \( \Lambda := \mathbb{Z}[i] \) and

\[
\Lambda' := \{ z \in \mathbb{C} ; \text{Re}(z \alpha) \in \frac{1}{2} \mathbb{Z} \; \forall \alpha \in \Lambda \} = \frac{1}{2} \mathbb{Z}[i],
\]

\[
D_k(w; s) := \sum_{\begin{array}{c}
\alpha \\
\in \mathbb{R}
\end{array}} \frac{1}{|\alpha|^{2+2s}} \left( \frac{c}{|c|} \right)^{2k} \cdot e^{4\pi i \text{Re}(w \frac{c}{d})}, \; \forall k \in \mathbb{Z}
\]

where

\[
\mathbb{R} := \left\{ \left( \begin{array}{cc}
\ast & \ast \\
c & d
\end{array} \right) \in \Gamma ; \; c \neq 0, \; d \text{ mod } c \right\},
\]

and

\[
\xi_{l-k}(v, u) := \frac{u!(2l-u)!}{(l+k)!(l-k)!} \left( \begin{array}{c}
l - \frac{1}{2}(|v+k| + |v-k|) \\
u
\end{array} \right) \left( l - \frac{1}{2}(|v+k| - |v-k|) \right) \cdot B_{k,m}^l := \sqrt{(l+m)!(l-m)!} / \sqrt{(l+k)!(l-k)!}.
\]

We remember that for \( \nu \in \mathbb{C} \) a twisted divisor function is given by

\[
\sigma_{\nu}(w, p) := \frac{1}{4} \sum_{d|w} \chi_{4p}(d) \cdot |d|^{2\nu}, \; \forall w \in \mathbb{Z}[i], \; \forall p \in \mathbb{Z}.
\]

For any \( k \in 2\mathbb{Z} \)

\[
D_p(w; s) = \frac{16}{L(1+s, \chi_{2p})} \cdot \begin{cases} 
\sigma_{-s}(2w, \frac{p}{2}) & \text{if } w \neq 0, \; \text{Re}(s) > 0 \\
\frac{1}{4} L(s, \chi_{2p}) & \text{if } w = 0, \; \text{Re}(s) < 1.
\end{cases}
\]
2. Mellin transform

The aim in this section is to prove the formula in (1). Before we require some notation. Let \((q, \eta) \in \mathbb{H}^3\), \(F_{(q, \eta)} := \{ n[q] a[\eta] K; K \in \text{SU}(2) \}\), the “fiber” of \((q, \eta)\) in \(G\), and consider the natural diffeomorphism \(\Psi_{(q, \eta)} : F_{(q, \eta)} \to \text{SU}(2)\) defined by \(\Psi_{(q, \eta)}(n[q] a[\eta] K) = K\).

For \(\gamma \in G\), \(M_\gamma(q + \eta j) : \text{SU}(2) \to \text{SU}(2)\) is the function defined as follows:

\[
M_\gamma(q + \eta j) K := T(\gamma n[q] a[\eta] K) = T(\gamma n[q] a[\eta]) K, \ \forall K \in \text{SU}(2).
\]

Let \(f \in C^\infty(\Gamma \setminus G)\), equivalently, \(f : G \to \mathbb{C}\) (use the same letter) is a smooth function and \(\Gamma\) invariant. This is, \(f(g) = f(\gamma g), \ \forall \gamma \in \Gamma, \forall g \in G \iff f(n[z] a[\lambda] K) = f(n[z'] a[\lambda'] M_\gamma(z + \lambda j) K),\) where \(z' + \lambda' j = \gamma(z + \lambda j), \ \forall z + \lambda j \in \mathbb{H}^3, \ \forall \gamma \in \Gamma, \ \forall K \in \text{SU}(2)\). \hspace{1cm} (31)

Given that \(G \cong \mathbb{H}^3 \times \text{SU}(2)\) also use the notation \(f(z + \lambda j, K) = f((z + \lambda j, M_\gamma(z + \lambda j) K), \ \forall z + \lambda j \in \mathbb{H}^3, \ \forall \gamma \in \Gamma, \ \forall K \in \text{SU}(2)\).

On the other hand, restricting \(f\) to the fibers we have that \(f \circ \Psi_{(z, \lambda)}^{-1} : \text{SU}(2) \cong S^3 \to \mathbb{C}\) for every \((z, \lambda) \in \mathbb{H}^3\), for which, supposing that \(f \circ \Psi_{(z, \lambda)}^{-1}\) admits a Laplace series, we have the following expansion:

\[
f \circ \Psi_{(z, \lambda)}^{-1}(K) = \sum_{l \in \mathbb{Z}} \sum_{k,m=0}^{l/2} \tilde{f}_{k,m}^l(z + \lambda j) \cdot \hat{T}_{k,m}(K) (32)
\]

where

\[
\tilde{f}_{k,m}^l(z + \lambda j) = 2\pi^2 \int_{\text{SU}(2)} f \circ \Psi_{(z, \lambda)}^{-1}(K) \cdot \hat{T}_{k,m}(K) dk.
\]

Let \(\gamma \in \Gamma\), for \(33\)

\[
\tilde{f}_{k,m}^l(\gamma(z + \lambda j)) = 2\pi^2 \int_{\text{SU}(2)} f \circ \Psi_{\gamma(z, \lambda)}^{-1}(K) \cdot \hat{T}_{k,m}(K) dk, \hspace{1cm} (34)
\]

however for \(31\)

\[
f \circ \Psi_{\gamma(z, \lambda)}^{-1} = f \circ \Psi_{(z, \lambda)}^{-1} \circ M_\gamma(z + \lambda j)^{-1}. \hspace{1cm} (35)
\]

Replacing \(35\) in \(34\)

\[
\tilde{f}_{k,m}^l(\gamma(z + \lambda j)) = 2\pi^2 \int_{\text{SU}(2)} f \circ \Psi_{(z, \lambda)}^{-1} [M_\gamma(z + \lambda j)^{-1} K] \cdot \hat{T}_{k,m}(K) dk. \hspace{1cm} (36)
\]

We make the variable change in \(\text{SU}(2)\) given by

\[
M_\gamma(z + \lambda j)^{-1} K = K'.
\]

However if \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})\) is direct to verify that

\[
M_\gamma(z + \lambda j) K' = \frac{1}{\nu} \begin{pmatrix} cz + d & -\lambda \tau \\ \lambda c & cz + d \end{pmatrix} K', \ \forall K' \in \text{SU}(2),
\]

where \(\nu := \sqrt{\lambda^2 |c|^2 + |cz + d|^2}\). Therefore

\[
M_\gamma(z + \lambda j) K = T(\gamma g) K', \ \forall K' \in \text{SU}(2), \hspace{1cm} (38)
\]

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with \( g := n[z]e[\lambda] \).

We revisit the integral of \( \tilde{f}_{k,m}^l(\gamma(z + \lambda j)) \). Replacing (37) in the equation (36)

\[
\tilde{f}_{k,m}^l(\gamma(z + \lambda j)) = 2\pi^2 \int_{\mathbf{SU}(2)} f \circ \Psi^{-1}_{(z,\lambda)}(K') \cdot \overline{T}_{km}^l(M_\gamma(z + \lambda j)K') \, dk'.
\]  

(39)

Using the identity (38) and (19) it is known that

\[
\mathcal{T}_{km}^l(M_\gamma(z + \lambda j)K') = R(T_{(\nu)})^{-1} \mathcal{T}_{km}^l(K').
\]  

(40)

As see in (21), the basis of the \( \{ \mathcal{T}_{km}^l \} \) changes under rotations according to the following formula:

\[
R(T_{(\nu)})^{-1} \mathcal{T}_{km}^l = \sum_{u=-\frac{1}{2}}^{\frac{1}{2}} (\nu)^{\frac{1}{2}} (\Phi_{T_{(\nu)}}) \cdot \mathcal{T}_{ku}^l.
\]  

(41)

Summarizing, of (40) and (41),

\[
\mathcal{T}_{km}^l(M_\gamma(z + \lambda j)K') = \sum_{u=-\frac{1}{2}}^{\frac{1}{2}} (\nu)^{\frac{1}{2}} (\Phi_{T_{(\nu)}}) \cdot \mathcal{T}_{ku}^l(K').
\]  

(42)

Replacing (42), (33) and (24) in (39) it is obtained

**Lemma 2.1.** Let \( z + \lambda j \in \mathbb{H}^3, f \in C^\infty_0(\Gamma \setminus G), \gamma \in \Gamma \). For \( l \in \mathbb{Z}, l \geq 0, k, m \in \frac{1}{2} \mathbb{Z}, k, m \in [-\frac{1}{2}, \frac{1}{2}] \) and \( k \equiv m \equiv \frac{1}{2} \mod 1 \) it is true that

\[
\tilde{f}_{k,m}^l(\gamma(z + \lambda j)) = \sum_{u=-\frac{1}{2}}^{\frac{1}{2}} (\nu)^{u-m} (\nu)^{\frac{1}{2}} (\Phi_{T_{(\nu)}}) \cdot \tilde{f}_{k,u}(z + \lambda j).
\]  

Some known facts about the Mellin transform are recalled. For every \( \lambda \in \mathbb{R} \) with \( \lambda > 0 \) consider the surface:

\[ S_\lambda := \{(z + \lambda j, I) : z \in \mathbb{C} \} \subset G. \]

\( S_\lambda \) passes to the quotient \( \Gamma \setminus G \) as a closed orbifold denoted by \( \tilde{T}_\lambda \).

**Definition.** Let \( \lambda \in \mathbb{R} \) such that \( \lambda > 0 \), the probability measure \( \nu(\lambda) \) located at \( \tilde{T}_\lambda \subset \Gamma \setminus G \) are defined as follows:

\[
\nu(\lambda)(f) := \int_{\mathbb{R}^2/Z[i]} f(z + \lambda j, I) \, dx dy, \quad \forall f \in C^\infty_0(\Gamma \setminus G).
\]

**Definition 2.2.** If \( s \in \mathbb{C} \) such that \( \mathrm{Re}(s) > 1 \), the Mellin transform \( \mathfrak{M}(f, s) \) of \( f \in C^\infty_0(\Gamma \setminus G) \) is defined by the following equality:

\[
\mathfrak{M}(f, s) := \int_0^\infty \nu(\lambda)(f) \lambda^{s-2} \, d\lambda = \int_0^\infty \int_{\mathbb{R}^2/Z[i]} f(z + \lambda j, I) \lambda^{s+1} \frac{dx dy d\lambda}{\lambda^3}, \quad z = x + iy.
\]

Let \( S = \{(z, \lambda) \in \mathbb{H}^3: z \in [-\frac{1}{2}, \frac{1}{2}]^2 \} \), the next identity is true

\[
S = \bigcup_{\sigma \in \Gamma \setminus \Gamma} \sigma(F),
\]  

(43)

where \( F := \{(z, \lambda) \in \mathbb{H}^3: z \in [-\frac{1}{2}, \frac{1}{2}]^2, |z|^2 + \lambda^2 \geq 1 \} \).
Proposition 2.3. Let $f \in C_0^\infty(\Gamma \backslash G)$, $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$. Then

\[
\mathfrak{M}(f, s) = \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \cdot T_{kk}(I) \cdot \frac{dx dy d\lambda}{\lambda^3}.
\]

Proof. It is first observed that for (32)

\[
f(z + \lambda j, I) = \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \cdot T_{kk}(I).
\]

There is the following chain of equalities:

\[
\mathfrak{M}(f, s) = \int_{0}^{\infty} \int_{[-1/2, 1/2]^2} f(z + \lambda j, I) \text{Im} (z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3} \quad \text{definition (2.2)}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \text{Im} (z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3} \quad \text{by (44)}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \text{Im} (z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3}. \quad \text{by (43)}
\]

Now we take care of the integral obtained in the equation (45), for that we consider the next variable change:

\[
z + \lambda j = \sigma(z' + \lambda' j), \quad z' = x' + iy.
\]

As $\sigma : F \rightarrow \sigma(F)$ is an isometry, it is followed that:

\[
\int_{\sigma(F)} f_{l,k}(z + \lambda j) \text{Im} (z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3} = \int_{F} f_{l,k}(z + \lambda j) \text{Im} \sigma(z' + \lambda' j)^{1+s} \frac{dx dy d\lambda'}{\lambda'^{3}}.
\]

However for the lemma (2.1)

\[
\int_{\sigma(F)} f_{l,k}(z + \lambda j) \text{Im} (z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3} = \sum_{u \in [-1/2, 1/2]} (-1)^{u-1} \int_{F} \mathfrak{M}_{u,-k} (\Phi_{T(\sigma g)^{-1}}) \cdot f_{l,k}(z + \lambda j) \text{Im} \sigma(z + \lambda j)^{1+s} \frac{dx dy d\lambda}{\lambda^3}.
\]

Replacing the identity (46) in the equation (45) it is obtained the first identity of the following:

\[
\mathfrak{M}(f, s) = \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \left[ \sum_{u \in [-1/2, 1/2]} \mathfrak{M}_{u,-k} (\Phi_{T(\sigma g)^{-1}}) \text{Im} \sigma(z + \lambda j)^{1+s} \right] \frac{dx dy d\lambda}{\lambda^3}
\]

\[
= \sum_{l \in \mathbb{Z}} \sum_{l \geq 0} l^{1/2} \int_{F} \int_{\sigma \in \Gamma \backslash \mathfrak{M}} f_{l,k}(z + \lambda j) \cdot E_{u,-k}(z + \lambda j, s) \frac{dx dy d\lambda}{\lambda^3}.
\]
3. Cusp forms

We remember that the aim in this section is to encounter a formula for the inner product \( \left( F_{p,q}^{l}, d\epsilon_{it} \right) \), we also see that

\[
\lim_{t \to \infty} \left( F_{p,q}^{l}, d\epsilon_{it} \right) = 0.
\]

The following version of cusp forms is taken like Bruggeman and Motohashi [1], see formula 5.35 and lemma 5.1. It is also useful for the next definition the lemma 5.2.1 in Guleska [5].

**Definition 3.1.** Let \( l, p, q \in \mathbb{Z}_{\geq 0} \), \( l \geq 0 \), such that \( p, q \in [-l, l] \), \( p \equiv q \equiv l \mod 1 \). The cusp form \( F_{p,q}^{l} : \Gamma \backslash G \to \mathbb{C} \) associated to the value \( \lambda \) (with \( \frac{1}{l} + r^{2} = \lambda \)) is given for the following formula:

\[
F_{p,q}^{l}(z + \lambda j, K) = F_{p,q}^{l}(n[z]a[\lambda]K) = \sum_{0 \neq w \in \Lambda} \sum_{a=-l}^{l} c(w) j_{a}^{w}(ir; \lambda) e^{2\pi i \text{Re}(wz)} \Phi_{aq}^{l}(K),
\]

where

\[
j_{a}^{w}(s; \lambda) := 2 (-1)^{l-p} \pi^{s} |w|^{s-1} \left( \frac{iw}{|w|} \right)^{-p-a} \omega_{a}^{w}(s, p; \frac{|w|\lambda},
\]

moreover

\[
\omega_{a}^{w}(s, p; \lambda) := \sum_{v=0}^{l-\frac{1}{2}(|a+p|+|a-p|)} (-1)^{v} \epsilon_{p}(a, v) \frac{(2\pi \lambda)^{l+v-1}}{(1+l+s-v) \cdot K_{s+l-|a+p|}=0 (4\pi \lambda)}.
\]

The associated L-function to the cusp form \( F_{p,q}^{l} \) twisted by \( \chi_{m} \), an Hecke character for \( K_{D} \), is given by the formula:

\[
L(s, F_{p,q}^{l}, \chi_{m}) = \sum_{(w)} c(w) \cdot \chi_{m}(w) = \prod_{p \text{ primo}} \left( 1 - \frac{c(p) \cdot \chi_{m}(p)}{|p|^{2s}} + \frac{\chi_{m}(p)^{2}}{|p|^{4s}} \right)^{-1},
\]

the sum is over all the integral ideals \( (w) \) not zero of \( Z[i] \).

The cusp form in (47) for \( K = I \) is given by

\[
F_{p,q}^{l}(z + \lambda j, I) = \sum_{a=-l}^{l} c(w) j_{a}^{w}(ir; \lambda) e^{2\pi i \text{Re}(wz)} \Phi_{aq}^{l}(I) = \sum_{0 \neq w \in \Lambda} c(w) j_{a}^{w}(ir; \lambda) e^{2\pi i \text{Re}(wz)}.
\]

Later, of (50) and (47) it is seen that the following equation is valid for \( K \in \text{SU}(2) \) arbitrary

\[
F_{p,q}^{l}(z + \lambda j, K) = \sum_{a=-l}^{l} \Phi_{aq}^{l}(K) \cdot F_{p,a}^{l}(z + \lambda j, I).
\]

So for the identity in (2) we have that \( \left( F_{p,q}^{l}, d\epsilon_{it} \right) \)

\[
= \int_{\Gamma \backslash G} F_{p,q}^{l}(z + \lambda j, K) \cdot E(z + \lambda j, it) \sum_{a=-l}^{l} c(w) j_{a}^{w}(ir; \lambda) e^{2\pi i \text{Re}(wz)} \Phi_{aq}^{l}(K) \cdot \overline{T_{km}^{L}(K)} \cdot \overline{T_{km}^{L}(K)}
\]

\[
= \int_{\Gamma \backslash G} E(z + \lambda j, it) \sum_{a=-l}^{l} c(w) j_{a}^{w}(ir; \lambda) e^{2\pi i \text{Re}(wz)} \Phi_{aq}^{l}(K) \cdot \overline{T_{km}^{L}(K)} \cdot \overline{T_{km}^{L}(K)}
\]

\[
\int_{\text{SU}(2)} \Phi_{aq}^{l}(K) \cdot \overline{T_{km}^{L}(K)} \cdot dV.
\]

by (51)
Therefore

\[ \int_{\mathbf{SU}(2)} \Phi_{aq}(K) \cdot T_{km}^l(K) \, dk = (-1)^{m-k} \cdot T_{qq}^{2l}(I) \cdot \frac{\sqrt{(l+q)!(l-q)!}}{\sqrt{(l+a)!(l-a)!}} \cdot \frac{1}{2l+1} \cdot \delta_{l,\frac{1}{2}} \delta_{a,m} \delta_{q,k}. \] (53)

Replacing (53) in (52) we obtain that

\[ \left( F_{p,q}^l, d\epsilon_{it} \right) = \frac{1}{2\pi^2} \sum_{a=-l}^{l} \sum_{\sigma \in \Gamma' \setminus \Gamma} \int_{\sigma(\Gamma\backslash \mathbb{H}^3)} E(z + \lambda j, it) \cdot E_{-a,-q}(z + \lambda j, -it) \cdot F_{p,a}^l(z + \lambda j, I) \, dV. \] (54)

We make the following change of variable

\[ \sigma(z + \lambda j) = z' + \lambda' j \quad \Leftrightarrow \quad z + \lambda j = \sigma^{-1}(z' + \lambda' j). \]

Therefore

\[ n[z]a[\lambda] = \sigma^{-1} n[z']a[\lambda'] A^{-1}, \]

where \( A := T(\sigma^{-1} n[z']a[\lambda']) \). Then the inner product is given by

\[ \int_{\sigma(\Gamma\backslash \mathbb{H}^3)} E(\sigma^{-1}(z' + \lambda' j), it) \sum_{\sigma \in \Gamma' \setminus \Gamma} \frac{D_{-a,-q}(\Phi_{T(n[z']a[\lambda'] A^{-1})})}{\lambda^{1-it}} \cdot F_{p,a}^{l}(\sigma^{-1} n[z']a[\lambda'] A^{-1}) \frac{dx'dy'd\lambda'}{\lambda'^3}. \]

Now, for the invariance of the Eisenstein series and the functions \( F_{p,a}^{l} \) the inner product is

\[ \frac{1}{2\pi^2} \sum_{a=-l}^{l} \sum_{\sigma \in \Gamma' \setminus \Gamma} \int_{\sigma(\Gamma\backslash \mathbb{H}^3)} E(z' + \lambda' j, it) \sum_{\sigma \in \Gamma' \setminus \Gamma} \frac{D_{-a,-q}(\Phi_{T(\sigma^{-1} n[z']a[\lambda'] A^{-1})})}{\lambda^{1-it}} \cdot F_{p,a}^{l}(\sigma^{-1} n[z']a[\lambda'] A^{-1}) \frac{dx'dy'd\lambda'}{\lambda'^3}. \] (55)

For (51)

\[ F_{p,a}^{l}(n[z']a[\lambda'] A^{-1}) = \sum_{u=-l}^{l} \Phi_{u,a}(A^{-1}) \cdot F_{p,u}^{l}(z' + \lambda' j, I). \] (56)

Replacing (56) in (55), the inner product is given by

\[ \frac{1}{2\pi^2} \sum_{a=-l}^{l} \sum_{\sigma \in \Gamma' \setminus \Gamma} \int_{\sigma(\Gamma' \setminus \Gamma)} E(z' + \lambda' j, it) \cdot D^{l}_{-a,-q}(\Phi_{T}(A)) \cdot \lambda^{1-it} \cdot \Phi_{u,a}(A^{-1}) \cdot F_{p,u}^{l}(z' + \lambda' j, I) \frac{dx'dy'd\lambda'}{\lambda'^3} \]

\[ = \frac{1}{2\pi^2} \sum_{u=-l}^{l} \sum_{\sigma \in \Gamma' \setminus \Gamma} \int_{\sigma(\Gamma' \setminus \Gamma)} E(z' + \lambda' j, it) \cdot D^{l}_{-a,-q}(\Phi_{T}(A)) \cdot \lambda^{1-it} \cdot \Phi_{u,a}(A^{-1}) \cdot F_{p,u}^{l}(z' + \lambda' j, I) \frac{dx'dy'd\lambda'}{\lambda'^3} \]

\[ \left( F_{p,q}^l, d\epsilon_{it} \right) = \frac{1}{2\pi^2} \sum_{a=-l}^{l} \sum_{\sigma \in \Gamma' \setminus \Gamma} \int_{\sigma(\Gamma' \setminus \Gamma)} E(z' + \lambda' j, it) \cdot D^{l}_{-a,-q}(\Phi_{T}(A)) \cdot \lambda^{1-it} \cdot \Phi_{u,a}(A^{-1}) \cdot F_{p,u}^{l}(z' + \lambda' j, I) \frac{dx'dy'd\lambda'}{\lambda'^3} \]
\[
\int_{\sigma(G \setminus H^3)} E(z + \lambda_j, it) \cdot F_{p,u}^l (z + \lambda_j, I) \cdot \lambda^{1-it} \left[ \sum_{a \equiv -l \atop a \in \mathbb{Z}^+ \setminus \mathbb{Z}^*} \frac{\sqrt{(l+q)!}}{\sqrt{(l+a)!}} \cdot \Phi_{-a,-q}^l (\Phi(A)) \cdot \Phi_{ua}^l (A^{-1}) \right] dxdydl \lambda \wedge. \tag{57}
\]

Now we solve the summation between brackets, using (12), (24) and (18) we obtain
\[
\sum_{a \equiv -l \atop a \in \mathbb{Z}^+ \setminus \mathbb{Z}^*} \frac{\sqrt{(l+q)!}}{\sqrt{(l+a)!}} \cdot \Phi_{-a,-q}^l (\Phi(A)) \cdot \Phi_{ua}^l (A^{-1}) = \delta_{uq}. \tag{58}
\]

Then, replacing (58) in (57) we have
\[
\left( F_{p,q}^l, d\epsilon_{it} \right) = \frac{1}{2\pi^2} \sum_{\sigma \in G' \setminus G} \int_{\sigma(G \setminus H^3)} E(z + \lambda_j, it) \cdot F_{p,q}^l (z + \lambda_j, I) \cdot \lambda^{1-it} \frac{dxdydl \lambda}{\lambda^3} = \frac{1}{2\pi^2} \int_{G' \setminus \mathbb{H}^3} E(z + \lambda_j, it) \cdot F_{p,q}^l (z + \lambda_j, I) \cdot \lambda^{1-it} \frac{dxdydl \lambda}{\lambda^3}. \tag{59}
\]

For (50) and (48)
\[
F_{p,q}^l (z + \lambda_j, I) = \sum_{0 \neq w \in \Lambda} (-1)^{l-p} \pi^{ir} c(w) \cdot \frac{e^{2\pi i \text{Re}(wz)}}{\lambda^2} \cdot \Phi_{-1}^l (\Phi(p; \frac{|w|\lambda}{2})). \tag{60}
\]

Replacing (60) in (59)
\[
\left( F_{p,q}^l, d\epsilon_{it} \right) = \frac{1}{\pi^2} \sum_{0 \neq w \in \Lambda} (-1)^{l-p} \pi^{ir} c(w) \cdot \frac{e^{2\pi i \text{Re}(wz)}}{\lambda^2} \cdot \Phi_{-1}^l (\Phi(p; \frac{|w|\lambda}{2}) \cdot \int \int \int E(z + \lambda_j, it) \cdot \lambda^{1-it} \frac{dxdydl \lambda}{\lambda^3}. \tag{61}
\]

The Fourier expansion of the classical Eisenstein series evaluated in \( s = it \) (with \( t \neq 0 \)) is given by:
\[
E(z + \lambda_j, it) = [\Gamma_\infty : \Gamma_\infty'] \lambda^{1+it} - \frac{i\pi}{\zeta_\infty (1+it)} \cdot \lambda^{1-it} + \frac{(2\pi)^{1+it} \lambda}{\Gamma(1+it)} \sum_{0 \neq \alpha \in \Lambda'} D_0 (-\alpha; it) \cdot |\alpha|^{it} \cdot K_{it} (4\pi |\alpha| \lambda) \cdot e^{-4\pi i \text{Re}(\alpha z)}. \tag{62}
\]

Then
\[
\int_{[0,1]^2} E(z + \lambda_j, it) \cdot e^{2\pi i \text{Re}(wz)} \frac{dxdy}{\lambda^3} = \frac{2\pi^{1+it} \lambda}{\Gamma(1+it)} D_0 (-\frac{w}{2}; it) \cdot |w|^{it} \cdot K_{it} (2\pi |w| \lambda). \tag{63}
\]

Replacing (62) in (61) is followed that
\[
\left( F_{p,q}^l, d\epsilon_{it} \right) = \frac{2\pi^{-1+it+ir}}{\Gamma(1+it)} (-1)^{l-p} \pi^{1+it} \left[ \sum_{0 \neq w \in \Lambda} c(w) \cdot \frac{e^{2\pi i \text{Re}(wz)}}{\lambda^2} \cdot \Phi_{-1}^l (\Phi(p; \frac{|w|\lambda}{2}) \cdot \frac{|w|^{it}}{\lambda^3} \cdot \frac{dxdydl \lambda}{\lambda^3} \right]. \tag{64}
\]
Nevertheless for (49)

\[ \mathcal{W}_q^l (ir, p; \frac{|w|\lambda}{2}) = \sum_{v=0}^{l-\frac{1}{2}(|q+p|+|q-p|)} (-1)^v \zeta_p^l (q, v) \frac{(\pi|w|\lambda)^{1+l-v}}{(1+i(l-v+ir))} \cdot K_{l-|q+p|-v+ir} (2\pi|w|\lambda). \]  

(64)

Replacing the equality (64) in (63) we obtain that

\[ \left( F^l_{p,q}, d\kappa_{it} \right) = \frac{2\pi^{l+it+ir}}{\Gamma(1+it)} (-1)^{l-p} \frac{p-q}{p-q} \sum_{v=0}^{l-\frac{1}{2}(|q+p|+|q-p|)} (-2)^v \zeta_p^l (q, v) \frac{(\pi|w|\lambda)^{1+l-v}}{(1+i(l-v+ir))} \]  

\[ \left[ \sum_{0 \neq w \in \Delta} c(w) |w|^{-v+i-ir+it} \mathcal{D}_0 (-\frac{w}{2}; it) \left( \frac{w}{|w|} \right)^{-p-q} \right] \int_0^\infty x^{l-v-it} \cdot K_{l-|q+p|-v+ir} (x) \cdot K_{it} (x) \, dx. \]  

(65)

The integral at the end of the last expression can be found in [4] page 692.

\[ \int_0^\infty x^{-z} \cdot K_{\mu}(x) \cdot K_{\nu}(x) \, dx = \frac{2^{2-z}}{\Gamma(1-z)} \frac{\Gamma(\frac{1-z}{2}+\mu+\nu)}{\Gamma(\frac{1-z}{2})} \frac{\Gamma(\frac{1-z}{2}+\mu-\nu)}{\Gamma(\frac{1-z}{2})} \frac{\Gamma(\frac{1-z}{2}-\mu+\nu)}{\Gamma(\frac{1-z}{2})} \cdot F\left(\frac{1-z}{2}+\mu+\nu, \frac{1-z}{2}-\mu+\nu; 1-z; 0\right), \]  

for Re(z) < 1 – |Re(\mu)| – |Re(\nu)|. Particularly,

\[ \int_0^\infty x^{l-v-it} \cdot K_{l-|q+p|-v+ir} (x) \cdot K_{it} (x) \, dx = \frac{2^{l-2-v-it}}{\Gamma(l+1-v-it)} \cdot \Gamma\left(\frac{1}{2}+l-v-\frac{1}{2}|q+p|+\frac{v}{2}\right) \cdot \Gamma\left(\frac{1}{2}+l-v-\frac{1}{2}|q+p|-\frac{v}{2}\right) \cdot \Gamma\left(\frac{1}{2}+l-v-\frac{1}{2}|q+p|-it-\frac{v}{2}\right) \cdot \Gamma\left(\frac{1}{2}+\frac{1}{2}|q+p|-it-\frac{v}{2}\right). \]  

(66)

On the other hand, for (30)

\[ S := \sum_{0 \neq w \in \Delta} c(w) |w|^{-1+i+2it} \mathcal{D}_0 (-\frac{w}{2}; it) \left( \frac{w}{|w|} \right)^{-p-q} = \frac{4^2}{\zeta_{K}(1+it)} \sum_{0 \neq w \in \Delta} c(w) |w|^{-1+i+2it} \left( \frac{w}{|w|} \right)^{-p-q} \sigma_{-it}(w, 0). \]  

(67)

However, for the lemma (5.5)

\[ \sum_{0 \neq w \in \mathbb{Z}[i]} c(w) \cdot |w|^\alpha \cdot \left( \frac{w}{|w|} \right)^\alpha \cdot \sigma_{\nu}(w, 0) = \frac{1}{4} \frac{L\left(\frac{1}{2} - \frac{s}{2}, \phi, \chi_\alpha\right) L\left(-\frac{s}{2}, \nu, \phi, \chi_\alpha\right)}{L(-s - \nu, \chi_{2\alpha})}. \]  

(68)

Then, replacing (68) in (67) it follows that

\[ S = \frac{4}{\zeta_{K}(1+it)} \cdot \frac{L\left(\frac{1}{2} - \frac{s}{2} - it, F^l_{p,q}, \chi_{-p-q}\right) \cdot L\left(\frac{1}{2} - \frac{it}{2}, F^l_{p,q}, \chi_{-p-q}\right)}{L(1-it, \chi_{-2p-2q})}. \]  

(69)
Finally, replacing (66) and (69) in (65) we obtain

\[
\left(F_{p,q}^l, d\epsilon_{it}\right) = (-1)^l \pi^{l-t} \Gamma(1+it) / \Gamma(1+it).
\]

\[
L \left( \frac{1}{2} - \frac{ir}{2} - it, F_{p,q}^l, \chi_{-p-q} \right) \cdot L \left( \frac{1}{2} - \frac{ir}{2}, F_{p,q}^l, \chi_{-p-q} \right) \cdot \Gamma \left( \frac{1}{2} + \frac{1}{2} |q + p| - \frac{ir}{2} \right) \cdot \Gamma \left( \frac{1}{2} + \frac{1}{2} |q + p| + \frac{ir}{2} - it \right)
\]

\[
\zeta_K(1+it) \cdot L \left( 1 - ir - it, \chi_{-2p-2q} \right)
\]

\[
\sum_{v=0}^{l-\frac{1}{2}(|q+p|+|q-p|)} (-1)^v \zeta_p(q, v) \cdot \Gamma \left( \frac{1}{2} + l - v - \frac{1}{2} |q + p| + \frac{ir}{2} \right) \cdot \Gamma \left( \frac{1}{2} + l - v - \frac{1}{2} |q + p| + \frac{ir}{2} - it \right) / \Gamma(1+it - v - it) / \Gamma(1+it - v - it).
\]

(70)

For the Stirling's formula $|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{-\frac{\pi}{2}|t|} |t|^\sigma - \frac{1}{2}$ as $t \to \infty$. Then, the absolute value of gamma factors in (70) where appears $t$ satisfy that

\[
|t|^{-\epsilon} \ll |\zeta_K(1+it)| \ll t^\epsilon, \quad \forall \epsilon > 0.
\]

(72)

The results of Michel and Venkatesh [16] and Garrett and Diaconu [3] proved that exists $\delta > 0$ such that for every $\epsilon > 0$

\[
L \left( \frac{1}{2} - it, F_{p,q}^l, \chi_{\alpha} \right) \ll |t|^{1-\delta + \epsilon} \quad \text{as } t \to \infty.
\]

(73)

The estimates (71), (72) and (73) in the formula (70) show the next:

**Proposition 3.2.** We have that

\[
\lim_{t \to \infty} \left(F_{p,q}^l, d\epsilon_{it}\right) = 0.
\]

4. **Incomplete Eisenstein Series**

Let $\psi(\lambda) \in C_0^\infty(0, \infty)$ be a rapidly decreasing function at 0 and $\infty$. Consider the Mellin transform of $\psi$

\[
H(s) = \int_0^\infty \psi(\lambda) \lambda^{-s} d\lambda.
\]

(74)

$H(s)$ is of Schwartz class in $t$ for each vertical line $\sigma + it$, we denote such line by $(\sigma)$.

The Mellin inversion formula affirms that

\[
\psi(\lambda) = \frac{1}{2\pi i} \int_{(\sigma)} H(s) \lambda^s ds
\]

(75)

for any $\sigma \in \mathbb{R}$.

**Definition 4.1.** Let $l, a, b \in \frac{1}{2}\mathbb{Z}$, $l \geq 0$, such that $l \equiv a \equiv b \mod 1$ with $a, b \in [-l, l]$, the incomplete Eisenstein series associated to $\psi$, denoted $F_{a,b}^l(\psi)$, are given by

\[
F_{a,b}^l(\psi)(g) := \sum_{\sigma \in \Gamma_\infty^l \setminus \Gamma} D_{ab}(\Phi_{F_l(\sigma) - 1}) \psi(\text{Im } \sigma g(j)) = \frac{1}{2\pi i} \int_{(3)} H(s) \cdot E_{a,b}^l(g, s - 1) ds.
\]

**Remark 4.2.** It will be used too the notation $F_{a,b}^l(\psi)(z + \lambda j, K)$. 

15
For (120)

\[ F_{a,b}^l(\psi)(z + \lambda j, K) = \frac{1}{2\pi i} \sum_{r=-l}^{l} \sum_{r \equiv l \text{mod 1}} D_{ar}^l(\Phi(K)^{-1}) \int H(s) \cdot E_{ra}^l(z + \lambda j, s - 1) \, ds. \]  

(76)

We have to \( \left( F_{a,b}^l(\psi), d\epsilon_{ll} \right) \)

\[ = \int_{\Gamma\backslash G} \left( \frac{1}{2\pi i} \sum_{r=-l}^{l} \sum_{r \equiv l \text{mod 1}} D_{ar}^l(\Phi(K)^{-1}) \int H(s) \cdot E_{rb}^l(z + \lambda j, s - 1) \, ds \right) \cdot E(z + \lambda j, it) \]

= \[ \frac{1}{2\pi i} \sum_{r=-l}^{l} \sum_{r \equiv l \text{mod 1}} D_{ar}^l(\Phi(K)^{-1}) \int H(s) \cdot E_{ra}^l(z + \lambda j, s - 1) \, ds \int_{\Gamma\backslash \mathbb{H}^3} E_{rb}^l(z + \lambda j, s - 1) \cdot E(z + \lambda j, it) \sum_{L \leq 0} \sum_{k,m} (-1)^{m-k} \cdot T_{kk}^L(I) \cdot E_{-m,-k}^l(z + \lambda j, -it) \cdot \overline{T_{km}^L(K)} \, dg \]

por (76)

\[ = \frac{1}{2\pi i} \sum_{r=-l}^{l} \sum_{r \equiv l \text{mod 1}} D_{ar}^l(\Phi(K)^{-1}) \int_{\Gamma\backslash \mathbb{H}^3} E_{ra}^l(z + \lambda j, s - 1) \cdot E(z + \lambda j, it) \sum_{L \geq 0} \sum_{k,m} (-1)^{m-k} \cdot T_{kk}^L(I) \cdot E_{-m,-k}^l(z + \lambda j, -it) \cdot \overline{T_{km}^L(K)} \, dg \]  

(77)

Working now with the integral in \( \mathbf{SU}(2) \) of the equation (77). Using the identities (24), (12), (16) y (11) it is seen that

\[ \int_{\mathbf{SU}(2)} D_{ar}^l(\Phi(K)^{-1}) \cdot \overline{T_{km}^L(K)} \, dk \overset{(78)}{=} \frac{1}{\sqrt{2\pi \sqrt{2l + 1}}} \delta_{l,l/2} \delta_{r,m} \delta_{a,k}. \]

As a consequence, replacing (78) in (77) the inner product is:

\[ \frac{1}{\sqrt{2\pi \sqrt{2l + 1}}} \int_{\Gamma\backslash \mathbb{H}^3} H(s) \int E(z + \lambda j, it) \sum_{m=-l}^{l} \sum_{m \equiv l \text{mod 1}} (-1)^{m-a} \cdot T_{aa}^{2l}(I) \cdot E_{mb}^l(z + \lambda j, s - 1) \cdot E_{-m,-a}^l(z + \lambda j, -it) \, dV \, ds \]

and developing \( E_{mb}^l(z + \lambda j, s - 1) \),

\[ = \frac{1}{2\pi^2} \int_{\Gamma\backslash \mathbb{H}^3} H(s) \int E(z + \lambda j, it) \sum_{m=-l}^{l} \sum_{m \equiv l \text{mod 1}} \sum_{\sigma \in \Gamma_\infty} D_{mb}^l(\Phi_{T(\sigma\in\mathbb{H}[a(]\lambda])^{-1}}) \]

\[ \cdot \text{Im} \sigma(z + \lambda j)^* \cdot E_{-m,-a}^l(z + \lambda j, -it) \, dV \, ds. \]  

(79)

As earlier, we make the variable change

\[ \sigma(z + \lambda j) = z' + \lambda' j \quad \iff \quad z + \lambda j = \sigma^{-1}(z' + \lambda' j). \]
So that
\[ n[z]a[λ] = σ^{-1} n[z']a[λ'] A^{-1}, \]
with \( A := T(σ^{-1} n[z']a[λ']) \). Using the invariance for the Eisenstein series we have that the integral in (79) is
\[ \int_{(3)} \frac{1}{2πi} H(s) \sum_{σ \in Γ' \setminus Γ} \int_{σ(Γ \setminus H^3)} E(z' + λj, it) \sum_{m=-l \atop \frac{m}{2} \in \mathbb{Z}} (1) \cdot D_{mb}(Φ(A)) \cdot \lambda^s \cdot E_{-m-a}(n[z']a[λ'] A^{-1}, -it) dV ds. \] (80)

For (120)
\[ E_{-m,-a}(n[z']a[λ'] A^{-1}, -it) = \sum_{u=\frac{1}{2} \mathbb{Z}} E_{u,-a}(z' + λj, -it). \] (81)

Replacing (81) in (80)
\[ I(t) := \frac{1}{2πi} \int_{(3)} \frac{1}{2πi} H(s) \sum_{σ \in Γ' \setminus Γ} \int_{σ(Γ \setminus H^3)} E(z' + λj, it) \sum_{m=-l \atop \frac{m}{2} \in \mathbb{Z}} (1) \cdot D_{mb}(Φ(A)) \cdot \lambda^s \]
\[ \cdot \sum_{u=\frac{1}{2} \mathbb{Z}} E_{u,-a}(z' + λj, -it) dV ds. \]
\[ = \frac{1}{2πi} \int_{(3)} \frac{1}{2πi} H(s) \sum_{σ \in Γ' \setminus Γ} \int_{σ(Γ \setminus H^3)} \lambda^s \cdot E(z' + λj, it) \sum_{u=\frac{1}{2} \mathbb{Z}} E_{u,-a}(z' + λj, -it) \]
\[ \left[ \sum_{m=-l \atop \frac{m}{2} \in \mathbb{Z}} (1) \cdot D_{mb}(Φ(A)) \cdot D_{-m,u}(Φ(A)) \right] dV ds. \] (82)

Now we take care of the sum in (82) between brackets, for (24) and (18)
\[ \sum_{m=-l \atop \frac{m}{2} \in \mathbb{Z}} (1) \cdot D_{mb}(Φ(A)) \cdot D_{-m,u}(Φ(A)) = (-1)^{a+b} \cdot δ_{u,-b}. \] (83)

Then, replacing (83) in (82)
\[ I(t) = (-1)^{a+b} \frac{1}{2πi} \int_{(3)} \frac{1}{2πi} H(s) \sum_{σ \in Γ' \setminus Γ} \int_{σ(Γ \setminus H^3)} \lambda^s \cdot E(z + λj, it) \cdot E_{-b,-a}(z + λj, -it) dV ds \]
\[ = (-1)^{a+b} \frac{1}{2πi} \int_{(3)} \frac{1}{2πi} H(s) \int_{[0,1]^2} \lambda^s \int_{[0,1]^2} E(z + λj, it) \cdot E_{-b,-a}(z + λj, -it) dx dy \frac{dλ}{λ^3} ds. \] (84)

The Fourier expansion of the classic Eisenstein series associated to \( Γ \) and evaluated in \( s = it \) (with \( t \neq 0 \)) is given by:
\[ E(z + λj, it) = [Γ : Γ_0] \lambda^{1+it} - \frac{i}{t} \frac{ζ_K(it)}{ζ_K(1+it)} \lambda^{1-it} \]
\[ + \frac{(2\pi)^{1+i\mu}}{\Gamma(1+i\mu)} \sum_{0 \neq \alpha \in \Lambda'} D_0(-\alpha; it) \cdot |\alpha|^{it} \cdot K_{it}(4\pi|\alpha|\lambda) \cdot e^{-4\pi i \text{Re}(\alpha x)}. \]  

The Fourier expansion of the Eisenstein series generalized and evaluated in \( s = -it \) (with \( t \neq 0 \)) is:

\[
E_{b,a}^t(z+\lambda j, -it) = [\Gamma_{\infty} : \Gamma'_{\infty}] \delta_{b,a} B_{b,a}^t \lambda^{1-it} + (-1)^{-a+|a|} \Gamma(1+l+it) \Gamma(|a|-it) \Gamma(1+|a|+it) \frac{L(-it, \chi_{-2a})}{\Gamma(1+l-it) \Gamma(1+|a|+it) L(1-it, \chi_{-2a})} \delta_{b,-a} B_{b,a}^{t} \lambda^{1+it} \\
+ (-1)^{l-a} \frac{(2\pi)^{it}}{\Gamma(1+it)} \sum_{0 \neq w \in \Lambda'} \sum_{u=0}^{l-\frac{i}{2}(|a+b|+|a-b|)} \sum_{u=0}^{l+\frac{i}{2}(|a+b|+|a-b|)} (-1)^u \xi_a^l(b, u) \frac{(2\pi|w|\lambda)^{1+it}}{\Gamma(1+l-u-it)} \cdot K_{t-|a+b|-u-it}(4\pi|w|\lambda). \]

From (84), (85) and (86) we have that

\[
\mathbf{I}(t) = (-1)^{a+b} \delta_{b,a} B_{b,a}^t [\Gamma_{\infty} : \Gamma'_{\infty}]^2 \frac{1}{2\pi^2} \int_{(3)} \frac{1}{2\pi i} H(s) \int_0^\infty \lambda^{-s-1} d\lambda ds + \left( \text{rapidly decreasing in } t \right) \\
+ (-1)^{l+b} \frac{(2\pi)^{it}}{\Gamma(1+it)} B_{b,a}^t \sum_{u=0}^{l-\frac{i}{2}(|a+b|+|a-b|)} \xi_a^l(b, u) \frac{(2\pi)^{-u}}{\Gamma(1+l-u-it)} \sum_{u=0}^{l-\frac{i}{2}(|a+b|+|a-b|)} \sum_{u=0}^{l+\frac{i}{2}(|a+b|+|a-b|)} (-1)^u \frac{(2\pi)^{-u}}{\Gamma(1+l-u-it)} \\
\left[ \int_{(3)} \frac{1}{2\pi i} H(s) \sum_{0 \neq w \in \Lambda'} D_{-a}(-w; -it) \cdot D_0(w; it) \cdot |w|^{-s} \cdot \left( \frac{w}{|w|} \right)^{a+b} \right] \\
\left[ \int_0^\infty \lambda^{1+s+l-u} \cdot K_{t-|a+b|-u-it}(4\pi|w|\lambda) \cdot K_{it}(4\pi|w|\lambda) \, d\lambda \right] ds. \]

And making the variable change \( x = 4\pi|w|\lambda \) in the last integral,

\[
\mathbf{I}(t) = (-1)^{a+b} \delta_{b,a} B_{b,a}^t [\Gamma_{\infty} : \Gamma'_{\infty}]^2 \frac{1}{2\pi^2} \int_{(3)} \frac{1}{2\pi i} H(s) \int_0^\infty \lambda^{-s-1} d\lambda ds + \left( \text{rapidly decreasing in } t \right) \\
+ (-1)^{l+b} \frac{(2\pi)^{it}}{\Gamma(1+it)} B_{b,a}^t \sum_{u=0}^{l-\frac{i}{2}(|a+b|+|a-b|)} \xi_a^l(b, u) \frac{(2\pi)^{-u}}{\Gamma(1+l-u-it)} \sum_{u=0}^{l-\frac{i}{2}(|a+b|+|a-b|)} \sum_{u=0}^{l+\frac{i}{2}(|a+b|+|a-b|)} (-1)^u \frac{(2\pi)^{-u}}{\Gamma(1+l-u-it)} \\
\left[ \int_{(3)} \frac{1}{2\pi i} H(s) \sum_{0 \neq w \in \Lambda'} D_{-a}(-w; -it) \cdot D_0(w; it) \cdot |w|^{-s} \cdot \left( \frac{w}{|w|} \right)^{a+b} \right] \\
\left[ \int_0^\infty x^{-1+s+l-u} \cdot K_{t-|a+b|-u-it}(x) \cdot K_{it}(x) \, dx \right] ds. \]  

The integral at the end of the expression (87) can be found in Gradshteyn and Ryzhik [4] page 692.

\[
\int_0^\infty x^{-z} \cdot K_{\mu}(x) \cdot K_{\nu}(x) \, dx = \frac{2^{-z-\nu}}{\Gamma(1-z)} \Gamma\left(\frac{1-z+\mu+\nu}{2}\right) \Gamma\left(\frac{1-z-\mu+\nu}{2}\right) \Gamma\left(\frac{1-z+\mu-\nu}{2}\right) \Gamma\left(\frac{1-z-\mu-\nu}{2}\right) \cdot F\left(\frac{1-z+\mu+\nu}{2},\frac{1-z-\mu+\nu}{2}; 1-\nu; 0\right),
\]

for \( \text{Re}(z) < 1 - |\text{Re}(\mu)| - |\text{Re}(\nu)| \). Particularly,

\[
\int_0^\infty x^{r-1+l-u} \cdot K_{t-|a+b|-u-it}(x) \cdot K_{it}(x) \, dx \\
= \frac{2^{s+l-3-u}}{\Gamma(s+l-u)} \Gamma\left(\frac{s}{2}+l-u-\frac{1}{2}|a+b|\right) \Gamma\left(\frac{s}{2}+\frac{1}{2}|a+b|+it\right) \Gamma\left(\frac{s}{2}+l-u-\frac{1}{2}|a+b|-it\right) \Gamma\left(\frac{s}{2}+\frac{1}{2}|a+b|\right). \]
for $-\lambda - \text{Re}(s) + u < -|\lambda| - |a| - |b| - u$.

As $\Lambda' = \frac{1}{2} \Lambda$ the summation of the expression (87) is the following:

$$S = 2^s \sum_{0 \neq w \in \mathbb{Z}[i]} \mathcal{D}_{-a}(\frac{w}{2}; -it) \cdot \mathcal{D}_{0}(\frac{w}{2}; it) \cdot |w|^{-a} \left( \frac{w}{|w|} \right)^{a+b}. \quad (89)$$

We will use the following identity Ramanujan type. For every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, $\mu, \nu \in \mathbb{C}$, in the convergence region the following formula is valid:

$$\sum_{0 \neq w \in \mathbb{Z}[i]} \left( \frac{w}{|w|} \right)^{4\alpha_1} \frac{1}{|w|^{4\alpha}} \cdot \sigma(\mu, w, \alpha_2) \cdot \sigma(\nu, w, \alpha_3) = \frac{1}{16} \frac{L(s, \chi_{4\alpha_1}) \cdot L(s - \mu, \chi_{4\alpha_1+4\alpha_2}) \cdot L(s - \nu, \chi_{4\alpha_1+4\alpha_2}) \cdot L(s - \mu - \nu, \chi_{8\alpha_1+4\alpha_2+4\alpha_3})}{L(2s - \mu - \nu, \chi_{8\alpha_1+4\alpha_2+4\alpha_3})}. \quad (90)$$

Replacing (30) in (89) and using the identity $\sigma_*(w, -p) = \sigma_*(w, p)$ for each $p \in \mathbb{Z}$ we have that

$$S = 2^s \frac{16}{L(1 + it, \chi_0)} \cdot \frac{16}{L(1 - it, \chi_{-2a})} \sum_{0 \neq w \in \mathbb{Z}[i]} \left( \frac{w}{|w|} \right)^{a+b} \sigma_*(w, 0) \cdot \sigma_*(w, -\frac{3}{2}). \quad (91)$$

Then, for (90)

$$S = 2^s \frac{4}{L(1 + it, \chi_0)} \cdot \frac{4}{L(1 - it, \chi_{-2a})} \frac{L(\frac{s}{2}, \chi_{a+b}) \cdot L(\frac{s}{2} + it, \chi_{a+b}) \cdot L(\frac{s}{2} - it, \chi_{a+b}) \cdot L(\frac{s}{2} + it, \chi_{a+b})}{L(s, \chi_{2b})}. \quad (93)$$

Finally, replacing (88) and (91) in (87) we conclude that

$$\left( F_{a,b}^l(\psi), d\epsilon_{it} \right) = F_1(t) + F_2(t) \quad (92)$$

where

$$F_1(t) = (-1)^{a+b} \delta_{b,a} B_{b,a}^l \left[ \Gamma_{\infty} : \Gamma'_{\infty} \right]^2 \frac{1}{2\pi i} \int \frac{1}{2\pi i} H(s) \int_0^\infty \lambda^{s-1} \lambda \, d\lambda \, d\lambda + \left( \text{rapidly decreasing in } t \right),$$

$$F_2(t) = (-1)^{l+b} b^{l+a} \frac{4}{\Gamma(1 + it)} \cdot \frac{1}{L(1 + it, \chi_0) \cdot L(1 - it, \chi_{-2a})} \sum_{u=0}^{l - \frac{1}{2}(|a+b|+|a-b|)} \left( -1 \right)^u \frac{\xi_u(b, u)}{\Gamma(1 + l - u - it)} \int \frac{1}{2\pi i} B(s) \, ds, \quad (93)$$

and

$$B(s) = \frac{H(s) \cdot L(\frac{s}{2}, \chi_{a+b}) \cdot L(\frac{s}{2} + it, \chi_{a+b}) \cdot L(\frac{s}{2} - it, \chi_{a+b}) \cdot L(\frac{s}{2} + it, \chi_{a+b})}{\Gamma(s + l - u) \cdot L(s, \chi_{2b}) \cdot \Gamma(\frac{s}{2} + l - u - \frac{1}{2}|a+b| + it) \cdot \Gamma(\frac{s}{2} + l - u - \frac{1}{2}|a+b| - it) \cdot \Gamma(\frac{s}{2} + \frac{1}{2}|a+b|)}.$$
\[ + 4(-1)^{l+b} b^a B_{b,a} \sum_{u=0}^{l-\frac{1}{2}(|a+b|+|a-b|)} (-1)^u \xi_a^l(b,u) \frac{\Gamma(1+it) \cdot \Gamma(1+l-u-it) \cdot L(1+it,\chi_0) \cdot L(1-it,\chi_{-2a})}{\Gamma(1+it) \cdot \Gamma(1+l-u-it) \cdot L(1+it,\chi_0) \cdot L(1-it,\chi_{-2a})} \text{Res}_{s=2} B(s). \] (94)

Now we take care of estimate the second adding in (94), for that we consider

\[ A(t) := \frac{1}{\Gamma(1+it) \cdot \Gamma(1+l-u-it) \cdot L(1+it,\chi_0) \cdot L(1-it,\chi_{-2a})} \int_{(1)} \frac{1}{2\pi i} B(s) \, ds. \] (95)

We make \( s = 1 + i\tau \) in the formula for \( B(s) \), then

\[
A(t) = \frac{1}{\Gamma(1+it) \cdot \Gamma(1+l-u-it) \cdot L(1+it,\chi_0) \cdot L(1-it,\chi_{-2a})} \int_{-\infty}^{\infty} H(1+i\tau) \cdot L\left(\frac{1}{2} + i\frac{\tau}{2},\chi_{a+b}\right) \cdot L\left(\frac{1}{2} + it + i\frac{\tau}{2},\chi_{a+b}\right) \cdot L\left(\frac{1}{2} - it + i\frac{\tau}{2},\chi_{-a+b}\right) \cdot L\left(\frac{1}{2} + i\frac{\tau}{2},\chi_{-a+b}\right) \cdot 2\pi^{2+i\tau} \Gamma(1+l-u+i\tau) \cdot L(1+i\tau,\chi_{2a}) \cdot \Gamma\left(l+\frac{1}{2}-u-\frac{1}{2}|a+b|+it+i\frac{\tau}{2}\right) \cdot \Gamma\left(l+\frac{1}{2}+\frac{1}{2}|a+b|+it+i\frac{\tau}{2}\right) \cdot \Gamma\left(l+\frac{1}{2}+\frac{1}{2}|a+b|+it+i\frac{\tau}{2}\right) \cdot \Gamma\left(l+\frac{1}{2}+\frac{1}{2}|a+b|+i\frac{\tau}{2}\right) d\tau. \] (96)

It is known that

\[ \ln^{-2} |t| \ll |\zeta_K(1+it)| \ll \ln^2 |t|. \] (97)

Using results in [11] and [21] the above can be generalized as follows:

\[ \ln^{-2} |t| \ll |L(1+it,\chi_p)| \ll \ln^2 |t|, \quad \forall p \in 4\mathbb{Z}. \] (98)

The bound for Heath-Brown [6] says that for every \( \epsilon > 0 \)

\[ \zeta_K\left(\frac{1}{2}+it\right) \ll_K t^{1/3+\epsilon}, \quad t \geq 1. \]

The version that generalizes the previous that we will use is due to Kaufman [8] and Sohne [20]

\[ L\left(\frac{1}{2}+it,\chi_p\right) \ll \left(1+|t|\right)^{1/3+\epsilon}, \quad \forall p \in 4\mathbb{Z}, \quad \forall \epsilon > 0. \] (99)

Stirling exponential asymptotics for \( B(1+i\tau) \) as a function of \( t \) gives

\[
\frac{e^{-\frac{\pi}{2} \left|\frac{\tau}{2}\right|} \cdot e^{-\frac{\pi}{2} \left|\frac{t+\tau}{2}\right|} \cdot e^{-\frac{\pi}{2} \left|\frac{t-\tau}{2}\right|} \cdot e^{-\frac{3}{4} \pi t}}{e^{-\frac{\pi}{2} \left|\tau\right|}} \leq e^{-\pi t},
\]

which cancels with the exponential growth of denominator in (95).

Using (97), (98), (99), Stirling’s formula and the rapid decay of \( H(1+i\tau) \) we are reduced to estimate in \( t \) the integral in (96), this is

\[
\ln^2 |t| \cdot \ln^2 |t| \int_{-\infty}^{\infty} H(1+i\tau) \cdot (1+|t+\frac{\tau}{2}|)^{1/3+\epsilon} \cdot (1+|t-\frac{\tau}{2}|)^{1/3+\epsilon} \, d\tau = O\left(t^{-1/3-(t-u)+\epsilon}\right).
\]

This concludes that

\[ A(t) = O\left(t^{-1/3-(t-u)+\epsilon}\right). \] (100)

Summarizing, from the formulas (94) and (100) we have that

\[
\left( F_{a,b}(\psi), \epsilon_i r \right) = O(1) + 4(-1)^{l+b} b^a B_{b,a}^l.
\]
\[
\sum_{u=0}^{l-\frac{1}{2}[(a+b)+|a-b|]} \frac{(-1)^u \zeta_{a,b}(b, u)}{\Gamma(1+it) \cdot \Gamma(1+l-u-it) \cdot L(1+it, \chi_0) \cdot L(1-it, \chi_{-2a})} \text{Res}_{s=2} B(s).
\] (101)

For the previous formula we require to find the residue of \(B(s)\) in \(s = 2\). This will be made for cases. First we consider \(a = b = 0\) and \(t \neq 0\). In this conditions we have that
\[
B(s) = \frac{H(s) \cdot \zeta_K(\frac{s}{2})^2 \cdot \zeta_K(\frac{s}{2} + it) \cdot \zeta_K(\frac{s}{2} - it) \cdot \Gamma(\frac{s}{2} + l - u) \cdot \Gamma(\frac{s}{2} + it) \cdot \Gamma(\frac{s}{2} + l - u - it) \cdot \Gamma(\frac{s}{2})}{\pi^s \Gamma(s + l - u) \cdot \zeta_K(s)}.
\]

Write \(B(s) = \zeta_K(\frac{s}{2})^2 \cdot G(s)\), with \(G(s)\) holomorphic in \(s = 2\). Put
\[
\zeta_K(\frac{s}{2}) = \frac{A_{-1}}{s-2} + A_0 + O(s-2) \quad \text{as} \quad s \to 2.
\]
Then
\[
B(s) = \left(\frac{A_{-1}}{s-2} + A_0 + O(s-2)\right)^2 \left(G(2) + G'(2)(s-2) + O(s-2)^2\right).
\]

The residue is
\[
\text{Res}_{s=2} B(s) = G(2) A_{-1} \left(2A_0 + A_{-1} \frac{G'(2)}{G(2)}\right).
\] (102)

As
\[
G(s) = \frac{H(s) \cdot \zeta_K(\frac{s}{2} + it) \cdot \zeta_K(\frac{s}{2} - it) \cdot \Gamma(\frac{s}{2} + l - u) \cdot \Gamma(\frac{s}{2} + it) \cdot \Gamma(\frac{s}{2} + l - u - it) \cdot \Gamma(\frac{s}{2})}{\pi^s \Gamma(s + l - u) \cdot \zeta_K(s)}
\]
it has that
\[
G(2) = \frac{H(2) \cdot \zeta_K(1 + it) \cdot \zeta_K(1 - it) \cdot \Gamma(1 + it) \cdot \Gamma(1 + l - u - it)}{\pi^s (1 + l - u) \cdot \zeta_K(2)}.
\] (103)

On the other hand,
\[
\frac{G'(2)}{G(2)} = \frac{H'(2)}{H(2)} + \frac{\zeta_K'(1 + it)}{2 \zeta_K(1 + it)} + \frac{\zeta_K'(1 - it)}{2 \zeta_K(1 - it)} + \frac{\Gamma'(1 + it)}{2 \Gamma(1 + it)} + \frac{\Gamma'(1 + l - u - it)}{2 \Gamma(1 + l - u - it)} + C,
\] (104)
where \(C\) is a constant that does not depend on \(t\).

The bound to Dirichlet L-functions by Landau says
\[
\frac{\zeta_K'(1 + it)}{\zeta_K(1 + it)} \ll_K \frac{\ln |t|}{\ln \ln |t|}
\] (105)
as \(t \to \infty\).

It is known that (see Laaksonen \[10\] formula (A.8) page 155)
\[
\frac{\Gamma'(1 + it)}{\Gamma(1 + it)} = \ln |t| + O(1).
\] (106)

Moreover
\[
A_{-1} = \text{Res}_{s=1} \zeta_K(s) = \frac{\pi}{4}.
\] (107)

For the lemma (5.3) if \(l - u \geq 1\)
\[
\frac{\Gamma'(1 + l - u - it)}{2 \Gamma(1 + l - u - it)} = \frac{1}{2} \sum_{k=0}^{l-u-1} \frac{1}{k+1-it} + \frac{\Gamma'(1 - it)}{2 \Gamma(1 - it)} = O(1) + \frac{1}{2} \ln t
\] (108)
as \( t \to \infty \).

Replacing (105), (106) and (108) in (104) we have
\[
\frac{G'}{G}(2) = O(1) + O\left(\frac{\ln t}{\ln \ln t}\right) + \ln t.
\] (109)

For (102) and (109),
\[
\text{Res}_{s=2} B(s) = G(2) \left( O(1) + A_{-1}^2 \ln t + O\left(\frac{\ln t}{\ln \ln t}\right) \right).
\] (110)

From (110) and (103) the formula (101) is transformed as follows
\[
\left( F^l_{0,0}(\psi), d\epsilon t \right) = O(1) + \sum_{u=0}^{l} \frac{(-1)^u \zeta_0(0, u)}{1 + l - u} \left[ O\left(\frac{\ln t}{\ln \ln t}\right) + (-1)^l \frac{H(2)}{4\zeta_K(2)} \ln t \right].
\] (111)

In particular,
\[
\left( F^0_{0,0}(\psi), d\epsilon t \right) = O(1) + O\left(\frac{\ln t}{\ln \ln t}\right) + \frac{H(2)}{4\zeta_K(2)} \ln t.
\] (112)

For \( l \geq 1 \) for the lemma (5.1) we know that the sum in \( u \) in the equation (111) is 0. Therefore
\[
\left( F^l_{0,0}(\psi), d\epsilon t \right) = O(1).
\] (113)

We consider now the case \( a = b \neq 0 \) and \( t \neq 0 \). Then,
\[
B(s) = \frac{H(s) \cdot L(\frac{s}{2}, \chi_{2a}) \cdot L(\frac{s}{2} + it, \chi_{2a}) \cdot \zeta_K(\frac{s}{2} - it) \cdot \zeta_K(\frac{s}{2})}{\pi^s \Gamma(s + l - u) \cdot L(s, \chi_{2a})}
\cdot \Gamma\left(\frac{s}{2} + l - u - |a|\right) \cdot \Gamma\left(\frac{s}{2} + |a| + it\right) \cdot \Gamma\left(\frac{s}{2} + l - u - |a| - it\right) \cdot \Gamma\left(\frac{s}{2} + |a|\right).
\]

It is clear then that \( \infty \) is a simple pole of \( B(s) \) in \( s = 2 \). For (107)
\[
\text{Res}_{s=2} B(s) = \frac{H(2) \cdot L(1, \chi_{2a}) \cdot L(1 + it, \chi_{2a}) \cdot \zeta_K(1 - it)}{4\pi \Gamma(2 + l - u) \cdot L(2, \chi_{2a})}
\cdot \Gamma(1 + l - u - |a|) \cdot \Gamma(1 + |a| + it) \cdot \Gamma(1 + l - u - |a| - it) \cdot \Gamma(1 + |a|).
\] (114)

Replacing (114) in (101) we see that
\[
\left( F^l_{a,0}(\psi), d\epsilon t \right) = O(1) + (-1)^l \frac{H(2) \cdot L(1, \chi_{2a}) \cdot \Gamma(1 + |a|)}{\pi L(2, \chi_{2a})} \left[ \sum_{u=0}^{l-|n|} (-1)^u \zeta_0(a, u) \frac{\Gamma(1 + l - u - |a|)}{\Gamma(2 + l - u)} \right]
\cdot \frac{L(1 + it, \chi_{2a}) \cdot \zeta_K(1 - it)}{L(1 + it, \chi_{0})}
\cdot \frac{L(\frac{s}{2} - it, \chi_{-2a}) \cdot \zeta_K(it) \cdot \Gamma(\frac{s}{2} + l - u) \cdot \Gamma(\frac{s}{2} + l - u) \cdot \Gamma\left(\frac{s}{2} + l - u - it\right) \cdot \Gamma\left(\frac{s}{2} + |a|\right)}{\pi^s \Gamma(s + l - u) \cdot L(s, \chi_{-2a})}.
\] (115)

Applying (97), (98) and the Stirling’s formula in the identity (115) we conclude that
\[
\left( F^l_{a,0}(\psi), d\epsilon t \right) = O(1).
\] (116)

For the case \( a = -b \neq 0 \) and \( t \neq 0 \) we have that
\[
B(s) = \frac{H(s) \cdot \zeta_K(\frac{s}{2}) \cdot \zeta_K(\frac{s}{2} + it) \cdot L(\frac{s}{2} - it, \chi_{-2a}) \cdot L(\frac{s}{2}, \chi_{-2a}) \cdot \Gamma\left(\frac{s}{2} + l - u\right) \cdot \Gamma\left(\frac{s}{2} + it\right) \cdot \Gamma\left(\frac{s}{2} + l - u - it\right) \cdot \Gamma\left(\frac{s}{2}\right)}{\pi^s \Gamma(s + l - u) \cdot L(s, \chi_{-2a})}.
\]
As in the previous case $\infty$ is a simple pole of $B(s)$ in $s = 2$. For (107)
\[
\text{Res}_{s=2} B(s) = \frac{H(2) \cdot \zeta_K(1 + it) \cdot L(1 - it, \chi_{-2a}) \cdot L(1, \chi_{-2a}) \cdot \Gamma(1 + l - u) \cdot \Gamma(1 + it) \cdot \Gamma(1 + l - u - it)}{4\pi \Gamma(2 + l - u) \cdot L(2, \chi_{-2a})}.
\]

Replacing (117) in (101) and simplifying
\[
\left( F^l_{a, -a}(\psi), d\epsilon_{it} \right) = \mathcal{O}(1) + \left( \frac{-1}{l} \frac{H(2) \cdot L(1, \chi_{-2a})}{\pi L(2, \chi_{-2a})} \sum_{u=0}^{l-|a|} (-1)^u \frac{\zeta^l_u(\bar{a}, u)}{1 + l - u} \right) = \mathcal{O}(1).
\]

If $a \neq \pm b$ and $t \neq 0$ it has that $B(s)$ is analytic in $s = 2$ and then $\text{Res}_{s=2} B(s) = 0$. In this case, for (101)
\[
\left( F^l_{a, b}(\psi), d\epsilon_{it} \right) = \mathcal{O}(1).
\]

The results in (112), (113), lemma (5.4), (116), (118) and (119) imply the next:

**Proposition 4.3.** For $l = a = b = 0$, $t \neq 0$ we have to
\[
\left( F^0_{0,0}(\psi), d\epsilon_{it} \right) = \mathcal{O}(1) + \mathcal{O} \left( \frac{\text{ln} t}{\text{ln} \ln t} \right) + \frac{1}{4 \zeta_K(2)} \left( \int_{\Gamma \backslash G} F^0_{0,0}(\psi)(g) \, dg \right) \ln t,
\]

then
\[
\left( F^l_{0,0}(\psi), d\epsilon_{it} \right) \sim \frac{1}{4 \zeta_K(2)} \left( \int_{\Gamma \backslash G} F^l_{0,0}(\psi)(g) \, dg \right) \ln t.
\]

In other case and with $t \neq 0$
\[
\left( F^l_{a, b}(\psi), d\epsilon_{it} \right) = \mathcal{O}(1).
\]

5. **Appendix**

**Lemma 5.1.** Let $l \in \mathbb{Z}$, $l \geq 1$, we have
\[
\sum_{u=0}^{l} \frac{(-1)^u \zeta^l_u(0, u)}{1 + l - u} = 0.
\]

**Proof.** We denote by $S$ the previous sum, the following equations are valid

\[
S = \sum_{u=0}^{l} \frac{(-1)^u u!(2l - u)!}{1 + l - u} \frac{l!}{u!} \left( \frac{l}{u} \right) = \frac{1}{l!} \sum_{u=0}^{l} (-1)^u \frac{(2l - u)!}{(1 + l - u)!} \left( \frac{l}{u} \right).
\]

Making the change of variable $a = l - u$ and using property $\left( \frac{l}{k} \right) = \left( \frac{l}{l-k} \right)$ it follows that

\[
S = \frac{(-1)^l}{l!} \sum_{a=0}^{l} (-1)^a \left( \frac{l}{a} \right) \frac{\Gamma(a + l)}{\Gamma(a + 1)}.
\]

By formula 0.160 (2) in Gradshteyn and Ryzhik [4],
\[
S = \frac{(-1)^l}{l!} \frac{\Gamma(l)}{\Gamma(1 + l) \Gamma(1 - l)} = 0.
\]
Lemma 5.2. Let \( l, k, m \in \frac{1}{2}\mathbb{Z}, \ l \geq 0, \ l \equiv k \equiv m \mod 1 \) and such that \( k, m \in [-l, l] \). Moreover, \( B \in SU(2) \), \( g \in SL(2, \mathbb{C}) \). Then
\[
\begin{align*}
 f_{km}^l(gB, s) &= \sum_{a = -l}^{l} D_{ka}^l(\Phi_{B^{-1}}) \cdot f_{am}^l(g, s).
\end{align*}
\]
In particular, for \( g = n[z]a[\lambda]K \) we obtain
\[
\begin{align*}
 E_{km}^l(g, s) &= \sum_{a = -l}^{l} D_{ka}^l(\Phi_{K^{-1}}) \cdot E_{am}^l(z + \lambda j, s).
\end{align*}
\]

Proof. We have the following chain of identities
\[
\begin{align*}
 f_{km}^l(gB, s) &= D_{km}^l(\Phi_{T(gB^{-1})}) \cdot \Im gB(j)^{1+s} \\
&= D_{km}^l(\Phi_{B^{-1} \circ T(g^{-1})}) \cdot \Im g(j)^{1+s} \\
&= \sum_{a = -l}^{l} D_{am}^l(\Phi_{T(g^{-1})}) \cdot D_{ka}^l(\Phi_{B^{-1}}) \cdot \Im g(j)^{1+s} \quad \text{by (18)} \\
&= \sum_{a = -l}^{l} D_{ka}^l(\Phi_{B^{-1}}) \cdot f_{am}^l(g, s).
\end{align*}
\]

Lemma 5.3. Let \( s \in \mathbb{C}, \ m \in \mathbb{Z} \) such that \( m \geq 1 \) and \( s + k \neq 0 \) for \( k = \{0, 1, \ldots, m - 1\} \). Then we have
\[
\frac{\Gamma'(m + s)}{\Gamma(m + s)} = \sum_{k=0}^{m-1} \frac{1}{s + k} + \frac{\Gamma'(s)}{\Gamma(s)}.
\]

Proof. We know that \( \Gamma(1 + s) = s \Gamma(s) \), so
\[
\Gamma(m + s) = \prod_{k=0}^{m-1} (s + k) \cdot \Gamma(s).
\]

Derivating,
\[
\Gamma'(m + s) = \left[ \prod_{k=0}^{m-1} (s + k) \right]' \cdot \Gamma(s) + \left[ \prod_{k=0}^{m-1} (s + k) \right] \cdot \Gamma'(s).
\]

Dividing the expression in (122) by (121) we get the lemma.

Lemma 5.4. It is true that
\[
\int_{\Gamma \setminus G} F_{00}^0(\psi)(g) \, dg = H(2).
\]

Proof. We have the following equivalences
\[
\begin{align*}
\int_{\Gamma \setminus G} F_{00}^0(\psi)(g) \, dg &= \int_{\Gamma \setminus G} \sum_{\sigma \in \Gamma'_{\infty} \setminus \Gamma} D_{00}^0(\Phi_{T(\sigma g)^{-1}}) \psi(\Im \sigma (g(j))) \, dg \\
&= \int_{\Gamma'_{\infty} \setminus G} \psi(\Im g(j)) \, dg = \int_0^{\infty} \psi(\lambda) \int_{\mathbb{R}^2/\Gamma'_{\infty}} dx dy \int_{SU(2)} dk \frac{d\lambda}{\lambda^3} \quad \text{by definition (4.1)} \\
&= \int_0^{\infty} \psi(\lambda) \frac{d\lambda}{\lambda^3} = H(2), \quad \text{by (74)}
\end{align*}
\]
Lemma 5.5. The following identity is valid

$$\sum_{0 \neq w \in \mathbb{Z}[i]} c(w) \cdot |w|^s \cdot \left( \frac{w}{|w|} \right)^\alpha \cdot \sigma_\nu(w, 0) = \frac{1}{4} L\left(-\frac{s}{2}, F_{pq}; \chi_\alpha \right) L\left(-\frac{s}{2} - \nu, F_{pq}; \chi_\alpha \right) L(-s - \nu, \chi_{2\alpha}).$$

Proof. Let

$$R(s) := \sum_{0 \neq w \in \mathbb{Z}[i]} c(w) \cdot |w|^s \cdot \left( \frac{w}{|w|} \right)^\alpha \cdot \sigma_\nu(w, 0) = \prod_{p \mathbb{Z}[i]} R_p(s),$$

with

$$R_p(s) = \sum_{j=0}^\infty c(p^j) \cdot |p^j|^s \cdot \left( \frac{p^j}{|p^j|} \right)^\alpha \cdot \sigma_\nu(p^j, 0).$$

We observed that

$$\sigma_\nu(p^j, 0) = \frac{1}{4} \sum_{t=0}^j \left( |p^t|^2 \nu \right) = \frac{1}{4} \sum_{t=0}^j \left( |p|^2 \nu \right)^t = \frac{1}{4} \cdot \frac{1 - |p|^{2\nu(j+1)}}{1 - |p|^{2\nu}}.$$

Therefore,

$$R_p(s) = \frac{1}{4} \sum_{j=0}^\infty c(p^j) \cdot |p^j|^s \cdot \frac{p^{j\alpha}}{|p|^\alpha} \cdot \frac{1 - |p|^{2\nu(j+1)}}{1 - |p|^{2\nu}} = \frac{1}{4} \frac{1}{1 - |p|^{2\nu}} \sum_{j=0}^\infty \frac{c(p^j)}{|p|^\alpha} \cdot \frac{p^{j\alpha}}{|p|^\alpha} \cdot \left( 1 - |p|^{2\nu(j+1)} \right)$$

$$= \frac{1}{4} \frac{1}{1 - |p|^{2\nu}} \left[ \sum_{j=0}^\infty \left( p^{\alpha} \cdot |p|^{s - \alpha} \right)^j - |p|^{2\nu} \sum_{j=0}^\infty \left( p^{\alpha} \cdot |p|^{s - \alpha + 2\nu} \right)^j \right]$$

$$= \frac{1}{4} \frac{1}{1 - c(p) \cdot |p|^{s - \alpha} + p^{2\alpha} \cdot |p|^{2s - 2\alpha}} - \frac{|p|^{2\nu}}{1 - c(p) \cdot p^{\alpha} \cdot |p|^{s - \alpha + 2\nu} + p^{2\alpha} \cdot |p|^{2s - 2\alpha + 4\nu}}$$

$$= \frac{1}{4} \frac{1 - p^{2\alpha} \cdot |p|^{2s - 2\alpha + 2\nu}}{(1 - c(p) \cdot p^{\alpha} \cdot |p|^{s - \alpha + 2\nu} + p^{2\alpha} \cdot |p|^{2s - 2\alpha + 4\nu})}.$$

\[\square\]

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