Few-anyon systems in a parabolic dot

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(July 16, 2018)

The energy levels of two and three anyons in a two-dimensional parabolic quantum dot and a perpendicular magnetic field are computed as power series in $1/N$, where $J$ is the angular momentum. The particles interact repulsively through a coulombic $(1/r)$ potential. In the two-anyon problem, the reached accuracy is better than one part in $10^7$. For three anyons, we study the combined effects of anyon statistics and coulomb repulsion in the “linear” anyonic states.

PACS numbers: 03.65.Ge, 74.20.Kk

I. INTRODUCTION

Recent developments in semiconductor technology (e.g. MBE and electron lithography) opened the possibility to create totally confined electron systems, the so called artificial atoms or quantum dots. This is one of the examples of present-day-physics’ interest in confined finite systems, among which one can mention also the atomic and electronic traps, and the condensation of confined bosons.

Quantum dots exhibit very interesting properties like the possibility of varying their parameters (number of electrons, applied fields, dot’s geometry, temperature) over a wide range, the observation of conductance anomalies and coulomb repulsion in the “linear” anyonic states. Among these approaches one can mention the semiclassical quantisation, regularised perturbation theory, Padé-approximant techniques, and the $1/N$-expansions.

In the present paper, we continue the analytic-qualitative line of research and apply the $1/N$-expansion ($N$ is the absolute value of the angular momentum) to compute the energy levels of two and three anyons in a model parabolic dot. The particles interact through a coulombic $(1/r)$ repulsive potential. A magnetic field is applied perpendicularly to the plane of motion.

Numerical results for the two-anyon system were obtained by Myrheim et al. Exact analytic solutions at particular values of the coupling constants were found in. Bohr-Sommerfeld quantisation was applied to this system at low magnetic fields. We shall see that our method provides extremely accurate solutions for states with angular momentum $|J| \geq 2$. A picture for the “geometry” of the states (the spatial distribution of probability) is obtained also. In the three-anyon problem, however, to our knowledge there are no numerical or approximate calculations.

We start from the Hamiltonian of $N_a$ anyons moving in a two dimensional quantum dot in the presence of a perpendicular homogenous magnetic field. In the bosonic gauge, it is given by the expression

$$H = \frac{1}{2m} \sum_{i=1}^{N_a} \left[ \vec{p}_i^2 + \frac{e}{c} \vec{A}_i - \hbar \nu \vec{a}_i \right]^2 + \frac{m}{2} \omega_0^2 \sum_{i=1}^{N_a} r_i^2$$

in which the vector potential is taken in the symmetric gauge,

$$\vec{A}_i = \frac{1}{2} \vec{B} \times \vec{r}_i,$$

$\vec{a}_i$ is the statistical vector potential,

$$\vec{a}_i = \sum_{j \neq i} \vec{n} \times (\vec{r}_j - \vec{r}_i)/|\vec{r}_j - \vec{r}_i|^2.$$

$\vec{n}$ is the unit vector perpendicular to the plane of motion of the anyons, $e$ is the anyon’s charge, $\nu$ is the anyonic parameter, $\omega_0$ is the frequency of the parabolic potential needed to confine the anyons in the dot and $c$ is the dielectric constant of the medium. A dimensionless Hamiltonian is obtained by means of the change of variables $\vec{r}_i \rightarrow \sqrt{\hbar/(\Omega \beta)} \vec{r}_i$

$$\frac{H}{\hbar \Omega} = \frac{1}{2} \sum_{i=1}^{N_a} \vec{p}_i^2 + \frac{\omega_0}{2 \Omega} \vec{n} \cdot \sum_{i=1}^{N_a} \vec{r}_i \times \vec{p}_i$$

$$+ \nu \vec{n} \cdot \sum_{i>j} \frac{(\vec{r}_i - \vec{r}_j) \times (\vec{p}_i - \vec{p}_j)}{|\vec{r}_i - \vec{r}_j|^2}$$

$$+ \frac{1}{2} \sum_{i=1}^{N_a} r_i^2 + \frac{\omega_0}{4 \Omega} N_a (N_a - 1)$$

$$+ \frac{\nu^2}{2} \sum_{i\neq j,k} \frac{(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_k)}{|\vec{r}_i - \vec{r}_j|^2 |\vec{r}_i - \vec{r}_k|^2} + \beta^3 \sum_{i>j} \frac{1}{|\vec{r}_i - \vec{r}_j|},$$

where $\omega_0 = eB/(mc)$ is the cyclotronic frequency, $\Omega^2 = \omega_0^2/4 + \omega_0^2$, and $\beta^3 = \sqrt{me^4/(c^2\hbar^3)}$ is the square root.
of the ratio between the coulombic and oscillator characteristic energies. The problem has one exactly solvable limit: a low-density limit, which we call the Wigner limit, reached when \( \beta \to \infty \). In the \( \beta \to 0 \) (oscillator) limit, the two-anyon problem is exactly solvable[1] whereas the three-anyon system has an infinite family of exact linear states[2]. In real semiconductors, \( \beta \sim 1 \).

Introducing Jacobi coordinates,

\[
\vec{\rho}_k = \sqrt{\frac{2k}{k+1}} \left\{ \frac{1}{k} \sum_{i=1}^{k} \vec{r}_i - \vec{r}_{k+1} \right\},
\]

\[1 \leq k \leq N_a - 1,
\]

\[
\bar{\rho}_{N_a} = \frac{1}{\sqrt{N_a}} \sum_{i=1}^{N_a} \vec{r}_i,
\]

the centre-of-mass and relative motions are separated

\[
\frac{H}{\hbar} = \frac{H_{CM}}{\hbar} + \frac{H_{rel}}{\hbar},
\]

where

\[
\frac{H_{CM}}{\hbar} = \frac{1}{2} \bar{p}_{N_a}^2 + \frac{\omega_c}{2\hbar} \bar{\rho}_{N_a} \times \bar{\rho}_{N_a} + \frac{1}{2} \rho_{N_a}^2,
\]

is the centre of mass Hamiltonian and

\[
\frac{H_{rel}}{\hbar} = \sum_{i=1}^{N_a-1} p_i^2 + \frac{\omega_c}{2\hbar} \sum_{i=1}^{N_a-1} \bar{p}_i \times \bar{p}_i
\]

\[+ \nu \bar{n} \cdot \sum_{i>j} \frac{\vec{r}_{ij} \times \vec{p}_{ij}}{r_{ij}} + \frac{1}{4} \sum_{i=1}^{N_a-1} p_i^2 + \frac{\omega_c}{2\hbar} \nu N_a (N_a - 1)
\]

\[+ \frac{\nu^2}{2} \sum_{i \neq j, k} \frac{\vec{r}_{ij} \times \vec{r}_{ik}}{r_{ij} r_{ik}} + \beta^3 \sum_{i>j} \frac{1}{r_{ij}}.
\]

is the Hamiltonian of the relative motion. We introduced the following notation: \( r_{ij} = \vec{r}_i - \vec{r}_j \), and \( \bar{p}_{ij} = \bar{p}_i - \bar{p}_j \).

We will obtain approximate expressions for the energy eigenvalues of \( \frac{H_{rel}}{\hbar} \) for \( N_a = 2 \) and \( N_a = 3 \) as a function of \( \beta \) by means of the \( 1/|J| \)-expansion.

\[\text{II. THE TWO-ANYON SYSTEM}\]

In the two-anyon problem, we have only one Jacobi coordinate, \( \bar{p}_1 \), and the Hamiltonian of the internal motion reads

\[
\frac{H_{rel}}{\hbar} = p_1^2 + \frac{\omega_c}{2\hbar} \bar{n} \cdot (\bar{p}_1 \times \bar{p}_1) + 2\nu \bar{n} \cdot \bar{p}_1 \times \bar{p}_1 + \frac{1}{4} \rho_1^2
\]

\[+ \frac{\nu^2}{\rho_1^2} + \frac{\beta^3}{\rho_1} + \frac{\omega_c}{2\hbar} \nu.
\]

Notice that \( \bar{n} \cdot (\bar{p}_1 \times \bar{p}_1) = J \). After the scaling transformation \( \rho_1^2 \to |J|/\rho_1^2 \), we get

\[
h = \frac{1}{|J|} \left[ \frac{H_{rel}}{\hbar} + \frac{\omega_c (J + \nu)}{2\hbar} \right] = \frac{1}{|J|} \left( (\nu + 1)^2 \rho_1^2 + \frac{1}{4} \rho_1^2 + \frac{\beta^3}{\rho_1} \right.
\]

\[- \frac{1}{\hbar^2} \left( \frac{\partial^2}{\partial \rho_1^2} + \frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \right),
\]

where \( \rho_1 = |ar{p}_1| \). We have “renormalised” the coupling constant, \( \beta^3 = \beta^3/|J|^2 \), and the statistical parameter, \( \nu = \nu/|J| \), in order to take account of the Coulomb repulsion and the statistical interaction in a nonperturbative way when taking the formal limit \( |J| \to \infty \). We shall look for symmetric eigenfunctions of \( h \), i.e. \( |J| \) shall be even.

In the \( |J| \to \infty \) limit, the only term surviving in the Hamiltonian is the effective (classical) potential energy

\[
U_{eff} = \frac{(\nu + 1)^2}{\rho_1^2} + \frac{1}{4} \rho_1^2 + \frac{\beta^3}{\rho_1}.
\]

Minimising \( U_{eff} \), we obtain the leading contribution to the energy, \( \epsilon_0 = U_{eff}(\rho_0) \), where the radius of the “Bohr orbit” is obtained from

\[
\frac{1}{2} \rho_0^4 - \beta^3 \rho_0 = 2(\nu + 1)^2.
\]

Substituting \( \rho_1 = \rho_0 + y_1/|J|^{1/2} \) in the r.h.s. of (11) and expanding, we get

\[
h = \sum_{i=0}^{\infty} \frac{h_i}{|J|^{i/2}},
\]

where the operator coefficients are given by

\[
h_0 = \frac{3}{4} \rho_0^2 - \frac{(\nu + 1)^2}{\rho_0^2},
\]

\[h_1 = 0,
\]

\[
h_2 = -\frac{\partial^2}{\partial y_1^2} + \frac{1}{4} \left( 3 + \frac{4(\nu + 1)^2}{\rho_0^2} \right) y_1^2
\]

\[
h_i = (-1)^i \left\{ \frac{1}{2} \rho_0^{4i} + \frac{(i - 1)(\nu + 1)^2}{\rho_0^{4i+2}} \right\} y_1^i
\]

\[+ \frac{1}{\rho_0^{4i+2}} y_1^{i-3} \frac{\partial}{\partial y_1}, \quad i \geq 3.
\]

Similar series are written for the wave function and the scaled energy, that is,

\[
\psi = \sum_{i=0}^{\infty} \frac{\psi_i}{|J|^{i/2}},
\]

\[
\epsilon = \frac{1}{|J|} \left[ \frac{E_{rel}}{\hbar} + \frac{\omega_c (J + \nu)}{2\hbar} \right] = \sum_{i=0}^{\infty} \frac{\epsilon_i}{|J|^{i/2}}.
\]
terms of small oscillations around the equilibrium orbit, i.e. the wave function is

$$\Psi_0 = e^{iJ\theta} \mid n \rangle,$$

(21)

where $\theta$ is the angle associated to the vector $\vec{\rho}_1$, the $\mid n \rangle$ are two-dimensional harmonic oscillator radial states with frequency $\omega_1 = \sqrt{3 + 4(\nu + 1)^2}/\rho_0^3$, and the first two coefficients for the energy are

$$
\begin{align*}
\epsilon_0 &= \frac{3}{4} \frac{n^2}{\rho_0^3} - \frac{(\nu + 1)^2}{\rho_0^3}, \\
\epsilon_2 &= \omega_1 \left(n + \frac{1}{2} \right).
\end{align*}
$$

(22)

Afterward, we may take account of anharmonicities. The results for the next two coefficients are the following

$$
\begin{align*}
\epsilon_4 &= -\frac{1}{4\rho_0^3} + \frac{3}{2\omega^4_1\rho_0^3} \left(1 + \frac{6(\nu + 1)^2}{\rho_0^3} \right) \\
&- \frac{(3n^2 + 3n + 11)}{4\omega^4_1\rho_0^3} \left(1 + \frac{4(\nu + 1)^2}{\rho_0^3} \right)^2, \\
\epsilon_6 &= \frac{3(2n + 1)}{4\omega^4_1\rho_0^3} + \frac{(2n + 1)}{\omega^4_1\rho_0^3} \left[5n^2 + 5n + 9 \right] \\
&+ \frac{(\nu + 1)^2}{\rho_0^3} \left[50n^2 + 50n + 81 \right] - \frac{2n + 1}{2\omega^4_1\rho_0^3} \left[87n^2 + 87n + 86 \right] \\
&+ 87n + 86 + \frac{12(\nu + 1)^2}{\rho_0^3} \left[87n^2 + 87n + 86 \right] \\
&+ \frac{4(\nu + 1)^4}{\rho_0^3} \left[713n^2 + 713n + 709 \right] - \frac{9(2n + 1)}{2\omega^4_1\rho_0^3} \left[25n^2 + 25n + 19 \right] \\
&\times \left(1 + \frac{6(\nu + 1)^2}{\rho_0^3} \right) \left[47n^2 + 47n + 31 \right] \\
&\times \left(1 + \frac{4(\nu + 1)^2}{\rho_0^3} \right)^2.
\end{align*}
$$

(24)

Notice that in both Wigner ($\beta \to \infty$) and oscillator ($\beta \to 0$) limits the corrections $\epsilon_4$ and $\epsilon_6$ go to zero. The expressions found in $\Box$ are reproduced if we take $\nu = 0$.

We show in Fig. 1 the relative weight of $\epsilon_6$ in $\epsilon$ for the first states with $J = 2$ and 6. The parameter $\nu$ was fixed to 1/2 (semions). The relative contribution of $\epsilon_6$ is never greater than $5 \times 10^{-5}$ or $3 \times 10^{-6}$ for $J = 2$ or 6 respectively. This shows that the $1/|J|$-series is extremely well behaved.

A comparison with the exact solutions found in $\Box$ is carried on in Fig. 2. Where the relative difference $|\epsilon - \epsilon_{\text{exact}}|/\epsilon_{\text{exact}}$ is plotted against $\nu$. The state with $J = 6$, $n = 0$ is shown. It may be easily verified that $\epsilon_{\text{exact}} = \rho_1^{J + \nu} |1 + \rho_1 / \sqrt{2}J + \nu| + 1 \epsilon e^{-\nu^2}$, $\epsilon_{\text{exact}} = |J + \nu| + 2$, are exact solutions of the two-anyon problem at $\beta^3 = \sqrt{2|J + \nu| + 1}$. The comparison shows that the relative error of our estimate is not greater than $10^{-8}$.

III. THE THREE-ANYON SYSTEM

The internal Hamiltonian of the system of three anyons in Jacobi coordinates $\rho_1$ and $\rho_2$ is written as

$$
\begin{align*}
H_{\text{rel}} &= \sum_{i=1}^{2} \rho_i^2 + \frac{\omega_\nu \rho_i^2}{2\Omega} \sum_{i=1}^{2} \rho_i \times \vec{p}_i + \frac{3\omega_\nu \rho_2}{2\Omega} \\
&+ \nu \vec{p}_1 \cdot \left[\frac{\hat{\rho}_1 \times \vec{p}_1}{\rho_1^2} + 2 \left(\hat{\rho}_1 + \sqrt{3}\rho_2\right) \times \left(\hat{\rho}_1 + \sqrt{3}\rho_2\right) \right] \\
&+ 2 \left(\hat{\rho}_1 - \sqrt{3}\rho_2\right) \times \left(\hat{\rho}_1 - \sqrt{3}\rho_2\right) \right] + \frac{1}{2} \sum_{i=1}^{2} \rho_i^2 \\
&+ 9\nu^2 \left(\rho_1^2 + \rho_2^2\right)^2 + 4 \left(\rho_1^2 \rho_2^2 - (\rho_1 \cdot \rho_2)^2 \right) \\
&+ \beta^3 \left(1 + \frac{1}{\rho_1} + \frac{2}{\rho_1 + \sqrt{3}\rho_2} + \frac{2}{\rho_1 - \sqrt{3}\rho_2} \right).
\end{align*}
$$

(26)

Doing the same scaling transformation $\rho_i^2 \to |J| \rho_i^2$, and making explicit the dependence on $|J|$, we get

$$
\begin{align*}
\frac{1}{|J|} \left[H_{\text{rel}} - \omega_\nu (J + 3\nu) \right] &= \frac{1}{4} \left(\rho_1^2 + \rho_2^2\right) + \frac{1}{4} (\rho_1^2 + \rho_2^2) \\
&+ \beta^3 \left[\frac{1}{\rho_1} + \frac{2}{\rho_1 + \sqrt{3}\rho_2} + \frac{2}{\rho_1 - \sqrt{3}\rho_2} \right] \\
&+ 4\nu \rho_1^2 (2 - 3\cos^2 \theta) + 3\rho_2^2 (2 - \cos^2 \theta) + \frac{\nu}{\rho_1} \\
&+ 9\nu^2 \left(\rho_1^2 + \rho_2^2\right)^2 + 4 \rho_1^2 \rho_2^2 \sin^2 \theta + \frac{1}{J} \left[\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right] \frac{\partial}{\partial \theta} \\
&+ \nu \left(12\rho_1 \rho_2 \sin^2 \theta \left|\rho_1 + \sqrt{3}\rho_2\right| \left|\rho_1 - \sqrt{3}\rho_2\right| \left(\rho_2 \frac{\partial}{\partial \rho_2} - \rho_1 \frac{\partial}{\partial \rho_1}\right) \right) \\
&+ \frac{2}{\rho_1} \frac{\partial}{\partial \rho_1} - 8 \rho_1^2 (1 - 3\cos^2 \theta) + 3\rho_2^2 (1 + \cos^2 \theta) \frac{\partial}{\partial \theta} \\
&+ \frac{1}{J} \left[- \frac{\partial^2}{\partial \rho_1^2} + \frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} + \frac{\partial^2}{\partial \rho_2^2} + \frac{1}{\rho_2} \frac{\partial}{\partial \rho_2} + \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right) \frac{\partial^2}{\partial \theta^2} \right],
\end{align*}
$$

(27)

where $\rho_1 = |\hat{\rho}_1|$, $\rho_2 = |\hat{\rho}_2|$, and $\cos \theta = \hat{\rho}_1 \cdot \hat{\rho}_2 / (\rho_1 \rho_2)$. The “renormalised” $\beta^3 = \beta^3 / |J|^3$ and $\nu = \nu / |J|$ were introduced.

The minimum of the classical potential entering (27) is reached in the configuration of an equilateral triangle ($\rho_{01} = \rho_{02}$, $\theta = \pm \pi/2$). We choose, for example, $\rho_1 = \rho_{02}$, $\theta = \pi/2$. $\rho_{01}$ is obtained as the solution of the equation

$$
\rho_{01}^3 - 3\rho_{01}\beta^3 = (3\nu + 1)^2.
$$

(28)
Then, introducing \( \rho_1 = \rho_{01} + y_1 / \sqrt{J} \), \( \rho_2 = \rho_{01} + y_2 / \sqrt{J} \) and \( \theta = \pi / 2 + \zeta / \sqrt{J} \) in the r.h.s of (27), we obtain for the Hamiltonian \( \hbar \) a series like (14). The first operator coefficients are given by

\[
\begin{align*}
h_0 &= \frac{3}{2} \rho_{01}^2 - \frac{(3 \nu + 1)^2}{2 \rho_{01}^2}, \\
h_1 &= 0, \\
h_2 &= -\left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{2}{\rho_{01}^2} \frac{\partial^2}{\partial z^2} \right) \\
&\quad - \frac{2 \nu}{\rho_{01}^2} \text{sign}(J) (y_1 - y_2) \frac{\partial}{\partial z} + \frac{1}{4} \left( \frac{3}{\rho_{01}^2} + 1 \right) (y_1^2 + y_2^2) \\
&\quad + \frac{1}{16} \left( 1 - \frac{1}{\rho_{01}^2} \right) (5 y_1^2 + 6 y_1 y_2 + 5 y_2^2 + 3 \rho_{01}^2 z^2) \\
&\quad + \frac{9 \rho_{01}^2}{16 \rho_{01}^4} (y_1^2 + y_2^2 + 6 y_1 y_2 - \rho_{01}^2 z^2) \\
&\quad + \frac{3 \nu}{2 \rho_{01}^2} \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right) \\
&\quad - \frac{3 \nu}{\rho_{01}^2} \left( \text{sign}(J) (y_1 - y_2) \frac{\partial}{\partial z} \right), \\
h_3 &= -\frac{1}{\rho_{01}^4} \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right) + \frac{2}{\rho_{01}^2} (y_1 + y_2) \\
&\quad + \frac{3 \nu}{\rho_{01}^2} \text{sign}(J) \left( y_1^2 - y_2^2 \right) \frac{\partial}{\partial z} - \frac{1}{\rho_{01}^4} (y_1^3 + y_2^3) \\
&\quad - \frac{1}{64 \rho_{01}^4} \left( 1 - \frac{1}{\rho_{01}^4} \right) (19 y_1^3 + 3 y_1 y_2^2 + 33 y_1 y_2^2 + 9 y_2^2) \\
&\quad - 9 \rho_{01}^2 y_1 z^2 + 21 \rho_{01}^2 y_2 z^2 - \frac{9 \rho_{01}^2}{64 \rho_{01}^4} (y_1^3 - 5 y_1^3 - 21 y_1 y_2^2) \\
&\quad - 39 y_1 y_2^2 + 7 \rho_{01}^2 y_1 y_2 - 11 \rho_{01}^2 y_2 z^2) \\
&\quad + \frac{3 \nu}{\rho_{01}^4} \left( \text{sign}(J) \left( \frac{4 y_1^3 - 2 y_2^3 - 2 y_1 y_2^2 - \rho_{01}^2 z^2} {2 \rho_{01}^4} \right) \frac{\partial}{\partial z} \\
&\quad - \rho_{01}^2 z \left( y_1 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_1} \right) + 2 \rho_{01}^2 y_2 z \frac{\partial}{\partial y_1} \right) \\
&\quad - \frac{3}{32 \rho_{01}^4} (21 y_1^3 + 15 y_1^2 y_2 + 5 y_1 y_2^2 + 23 y_1 y_2^2) \\
&\quad + \rho_{01}^2 y_1 y_2 z^2 - 13 \rho_{01}^2 y_2^2 z^2 \right) \right], \\
h_4 &= \frac{1}{\rho_{01}^4} \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} - \frac{3}{\rho_{01}^2} (y_1^2 + y_2^2) \frac{\partial^2}{\partial z^2} \\
&\quad - \frac{4 \nu}{\rho_{01}^4} \text{sign}(J) (y_1^3 - y_2^3) \frac{\partial}{\partial z} - \frac{5}{4 \rho_{01}^4} (y_1^4 + y_2^4) \\
&\quad + \frac{1}{256 \rho_{01}^4} \left( 1 - \frac{1}{\rho_{01}^4} \right) \left( 329 y_1^4 - 25 \rho_{01}^2 y_1^2 \right) \\
&\quad + \frac{123}{2} y_1 y_2^2 + 135 y_1 y_2^2 + \frac{9}{4} y_2^4 - \frac{99}{2} \rho_{01}^2 y_1 y_2^2 \\
&\quad + 27 \rho_{01}^2 y_1 y_2 z^2 + \frac{141}{2} \rho_{01}^2 y_2^2 z^2 + \frac{41}{4} \rho_{01}^4 z^4 \right).
\end{align*}
\]

The operator \( h_2 \) is diagonalised by changing variables

\[
y_1 = (y_1 + y_2) / \sqrt{2}, \quad y_2 = (y_1 - y_2) / \sqrt{2}, \quad z = \sqrt{2} z / \rho_{01},
\]

and making the “gauged” transformation \( h'_2 = e^{i \theta} h_2 e^{-i \theta} \) where \( f = \text{sign}(J) y_m z_m / (2 \rho_{01}) \), the results is

\[
h'_2 = h_s + h_m,
\]

where \( h_s \) describes the motion of a harmonic oscillator in the coordinate \( y_s \) (the symmetric mode),

\[
h_s = -\frac{\partial^2}{\partial y_s^2} + \frac{\omega_s^2}{4} y_s^2,
\]

with \( \omega_1 = \sqrt{3 + (3 \nu + 1)^2 / \rho_{01}^2} \), and \( h_m \) accounts for two dimensional motion in a “fictitious” magnetic field (the mixed mode)

\[
h_m = -\frac{\partial^2}{\partial y_m^2} - \frac{\partial^2}{\partial z_m^2} + \frac{\omega_m^2}{4} (y_m^2 + z_m^2)
\]

\[
- \frac{\nu \text{sign}(J)(3 \nu + 1)}{\rho_{01}^4} \left( y_m \frac{\partial}{\partial z_m} - z_m \frac{\partial}{\partial y_m} \right),
\]

where \( \omega_2 = \sqrt{3/2 - (3 \nu + 1)^2 / (2 \rho_{01}^2)} \).

The first two coefficients in the expansion for the energy are

\[
\epsilon_0 = \frac{3}{2} \rho_{01}^2 - \frac{(3 \nu + 1)^2}{2 \rho_{01}^2},
\]

\[
\epsilon_2 = \omega_1 \left(n_s + \frac{1}{2}\right) + \omega_2 (2n + |m| + 1) + \omega_3 \text{sign}(J)m,
\]

where \( \omega_3 = (3 \nu + 1) / \rho_{01}^2 \), and the quantum numbers \( n_s, n, \) and \( m \) may be used to approximately label the states.

Up to this order, the wave function is given by

\[
\Psi_0 = e^{iJz^2} | J, n_s, n, m \rangle,
\]

where \( \Xi \) accounts for overall rotations of the system, and \( | J, n_s, n, m \rangle \) are the eigenfunctions of \( h'_2 \). We notice that, when \( \beta \to 0 \) the energy becomes
$$E_0 = |J + 3\nu| + 2 + 2n_s + 2n + |m| + \text{sign}(J)m.$$  
(40)

These are the “linear” three-anyon states. We stress that they are obtained as harmonic excitations against the equilateral triangle configuration, and are not necessarily related to a cigar-like shape of the wave function \((\rho_1 >> \rho_2)\) as described in [2].

The set of numbers \(\{J, n_s, n, m\}\) compatible with the symmetry constraints (the wave function shall be symmetric) are obtained upon comparison with harmonic-oscillator wave functions at \(\nu = 0, \beta = 0\). Details may be found in [4]. An additional requirement is that the state \(\beta = 0\) should be a linear state. For example, the lowest linear states are the following: \(0, 0, 0, 0\) (the g.s., starting from \(E_0 = 2\) at the bosonic end), \(0, 1, 0, 0\), and \(2, 0, 0, 0 \) \(0, 1, 0, 1\) \(0, 1, 1, 0\) \(2, 1, 1, 0\). 

The lowest state with \(J < 0\) is \(-1, -1, 0, 0\), which starts at \(E_0 = 8\). Of course, states with small values of \(|J|\) cannot be described within our method.

In what follows, we restrict the analysis to levels with quantum numbers \(n_s = n = m = 0\). This leaves only the linear anyonic states with \(J = 3k\), where \(k\) is an integer.

The geometry of the state is an equilateral triangle. It can be seen from (29) that the side of the triangle increases with \(\nu\) when \(J > 0\), and decreases when \(J < 0\). Thus, the conoulomb repulsion is much more stronger for \(J < 0\) states, and the ordering of levels may dramatically change as \(\beta\) is increased. On the other hand, for \(\beta \to \infty\) the side grows like \(\rho_{01} \sim 3^{1/3}\beta\) and becomes independent of the anyonic parameter, as one expects. A strong coupling expansion [4] shows that the leading contribution to the energy (potential energy) is \(\sim \beta^2\), the next corrections (quantum fluctuations) are \(\sim 1\), the angular momentum and the statistical parameter enter the second order corrections, which are \(\sim 1/\beta^2\).

The first anharmonic corrections to the energy are given by

$$\epsilon_4 = \langle J, n_s, n, m | h_4' | J, n_s, n, m \rangle + \sum_{n_1', n_2', m'} \frac{\langle J, n_s, n, m | h_3' | J, n_1', n_1', m' \rangle}{\epsilon_2 n_1, n_2, m} \times \langle J, n_1', n_1', m' | h_4' | J, n_s, n, m \rangle,$$
(41)

where \(h_3'\) and \(h_4'\) are obtained from \(h_3\) and \(h_4\) by means of a gauge transformation, in the same way as explained above for \(h_2\).

For a state with quantum numbers \(|J, 0, 0, 0\rangle\), we get

$$\langle J, 0, 0, 0 | h_4' | J, 0, 0, 0 \rangle =$$

$$-\frac{3\nu}{8\rho_{01}^2} \left(9\omega_1^2 - \omega_1\omega_2 - 18\omega_2^2\right).$$

$$h_3' = A \frac{\partial}{\partial y_s} + By_s + Cy_s^3 + D,$$ 
(43)

where

$$A = \sqrt{2} \rho_{01} \left(-\frac{3\nu}{4\rho_{01}^2} \sin 2\alpha + \frac{\cos 2\alpha}{2} \right)$$

$$B = \sqrt{2} \rho_{01} \left(\frac{\sin^2 \alpha}{\rho_{01}^2} \frac{\partial^2}{\partial \xi^2} + \frac{\cos 2\alpha}{\rho_{01}^2} \frac{\partial^2}{\partial \alpha^2} \right)$$

$$+ \left(\frac{3\nu + 2}{\rho_{01}^2} \right) \frac{\partial}{\partial \alpha}$$

$$+ \left(\frac{3\nu + 2}{\rho_{01}^2} \right) \frac{\partial}{\partial \alpha} \right) \frac{\sin 2\alpha}{\rho_{01}^2} \frac{\partial^2}{\partial \alpha^2} + \frac{9\nu^2 - 1}{4\rho_{01}^2} \xi^2 \cos^2 \alpha$$

$$- \frac{3}{16} \frac{3\nu^2 - 2\nu - 1}{\rho_{01}^2} \xi^2 \right)$$

$$C = -\sqrt{2} \rho_{01} \left(1 + \frac{3\nu + 1}{\rho_{01}^2} \right)$$

$$D = \sqrt{2} \rho_{01} \left(\frac{5 + 2\nu^2 - 3\nu - 5}{\rho_{01}^2} \xi^2 \cos(4 \cos^2\alpha - 3)$$

$$- \frac{3\nu^2}{\rho_{01}^2} \xi^2 \sin(4 \cos^2\alpha - 1) \frac{\partial}{\partial \xi} \right)$$

$$+ \frac{3\nu^2}{\rho_{01}^2} \xi^2 \cos(4 \cos^2\alpha - 3) \frac{\partial}{\partial \alpha}.$$

Polar coordinates have been introduced according to \(\xi = y_s^2 + z_m^2\), \(\tan \alpha = z_m/y_m\). The only nonvanishing matrix elements entering the sum (41) are the following

$$\langle 0, 0 | A | 0, 0 \rangle = -\frac{\sqrt{2}}{\rho_{01}},$$
(48)

$$\langle 0, 0 | A | 0, \pm 2 \rangle = \frac{3 \text{sign}(J)\nu}{2\rho_{01}^3 \omega_2},$$
(49)

$$\langle 0, 0 | A | 0, 0 \rangle = \frac{3 \text{sign}(J)\nu}{2\rho_{01}^3 \omega_2},$$
(50)

$$\langle 0, 0 | B | 0, 0 \rangle = -\sqrt{2}\omega_2/2\rho_{01},$$
(51)

$$\langle 0, 0 | B | 0, 0 \rangle = \pm \text{sign}(J)\frac{3\nu + 2}{\rho_{01}^3 \omega_2} - \frac{1}{\rho_{01}^3} \frac{\omega_2}{4},$$

$$+ \frac{9\nu^2 - 1}{4\omega_2^2 \rho_{01}^2};$$

$$\langle 0, \pm 3 | D | 0, 0 \rangle = -\frac{\sqrt{6}}{16\rho_{01} \omega_2} (5\rho_{01}^4 - 5)$$

$$\pm 24\rho_{01}^2 \text{sign}(J)\omega_2 + 27\nu^2 - 6\nu).$$
(53)

Collecting everything, we arrive to
\[ \epsilon_4 = \langle J, 0, 0, 0 | h_4^+ | J, 0, 0, 0 \rangle - \left\{ \frac{0.3 | D | 0, 0)^2}{3\omega_2 + 3\omega_3 \text{ sign}(J)} \right\} + \frac{\langle 0, -3 | D | 0, 0 \rangle^2}{3\omega_2 - 3\omega_3 \text{ sign}(J)} - \frac{11}{\omega_1^2} \langle 0, 0 | C | 0, 0 \rangle^2 + \frac{\omega_1}{4} \left\{ \frac{(0, 0 | A | 0, 0)^2}{\omega_1} + \frac{(0, 0 | A | 0, 2)(0, 2 | A | 0, 0)}{\omega_1 + 2\omega_2 + 2\omega_3 \text{ sign}(J)} \right\} \]

\[ + \frac{\omega_1}{4} \left\{ \frac{(0, 0 | A | 0, -2)(0, -2 | A | 0, 0)}{\omega_1 + 2\omega_2 - 2\omega_3 \text{ sign}(J)} \right\} \]

\[ - \left\{ \frac{(0, 0 | A | 0, 2)(0, 2 | B | 0, 0)}{\omega_1 + 2\omega_2 + 2\omega_3 \text{ sign}(J)} \right\} + \frac{(0, 0 | A | 0, -2)(0, -2 | B | 0, 0)}{\omega_1 + 2\omega_2 - 2\omega_3 \text{ sign}(J)} \]

\[ - \frac{6}{\omega_1^2} \langle 0, 0 | B | 0, 0 \rangle \langle 0, 0 | C | 0, 0 \rangle - \frac{1}{\omega_1} \left\{ \frac{(0, 0 | B | 0, 0)^2}{\omega_1} + \frac{(0, 2 | B | 0, 0)^2}{\omega_1 + 2\omega_2 + 2\omega_3 \text{ sign}(J)} \right\} \]

\[ + \frac{(0, -2 | B | 0, 0)^2}{\omega_1 + 2\omega_2 - 2\omega_3 \text{ sign}(J)} \right\} \].

(54)

It may be checked that the corrections go to zero in both the Wigner ($\beta \to \infty$) and the oscillator ($\beta \to 0$) limits.

We show in Fig. 3 the relative weight of $\epsilon_4$ in $\epsilon$ for three semions in states with $J = 3$ and $J = 6$. The numbers are similar to those appearing in the two-anyon problem. Thus, we expect a similar accuracy to this order, i.e. one part in $10^3$ or better.

In Fig. 4, the levels with $J = \pm 6$ are drawn. $\beta$ is increased from 0.5 to 8. Notice that the coulomb effects are stronger for the state with negative $J$, and that the levels become flatter (as a function of $\nu$) as $\beta$ rises.

In conclusion, the energy levels of two and three anyons in a model parabolic dot were computed by means of the $1/|J|$-expansion. The qualitative picture emerging from the $1/|J|$-expansion is that of a rigid structure (an orbit in the two-anyon system, an equilateral triangle for three anyons) against which harmonic and anharmonic oscillations are developed. The coulomb repulsion is much stronger for negative-$J$ states. Comparison with exact particular solutions for two anyons shows excellent agreement.

Acknowledgements: A. G. acknowledges support from the Colombian Institute for Science and Technology (COLCIENCIAS).

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\[ \frac{\beta}{\nu} = \frac{1}{4} \left( \frac{1}{3} \right)^{1/3} \]

Fig. 1. Relative weight of $\epsilon_6$ in $\epsilon$. Two anyons in states with $n = 0$ and $\nu = 1/2$ are studied. a) $J = 2$, b) $J = 6$.

Fig. 2. Comparison between the $1/|J|$-estimate and the exact solution found in [16] for two anyons with $J = 6$, $n = 0$.

Fig. 3. Relative weight of $\epsilon_4$ for three anyons in states with $\nu = 1/2$. a) $J = 3$, b) $J = 6$.

Fig. 4. $|J|/\beta$ vs $\nu$ for three anyons in states with $J = \pm 6$. a) $\beta = 0.5$, b) $\beta = 8$. 

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\[ \frac{|\varepsilon - \varepsilon(\text{exact})|}{\varepsilon(\text{exact})} \]
Fig. 3a

\[
\frac{E_4}{J^2 \epsilon}
\]

Fig. 3b

\[
\frac{\beta}{\beta+1}
\]
Fig 4a

|J| \( \epsilon \)

\[ \begin{align*}
|J| &= 6 \\
J &= -6
\end{align*} \]

Fig. 4b

\[ \begin{align*}
|J| \epsilon &= 202.1 \\
&= 202.05 \\
&= 202 \\
&= 201.95 \\
&= 201.9 \\
&= 201.85 \\
&= 201.8
\end{align*} \]