Topology-Controlled Nonreciprocal Photon Scattering in a Waveguide

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Waveguide quantum electrodynamics with multiple atoms provides an important way to study photon transport. In this work, we study the photon transport in a one-dimensional waveguide coupled to a topological array. The interaction between light and topology-dressed atoms yields a rich variety of photon scattering phenomena. We find that the nonreciprocal photon reflection originates from the quantum interference of photons scattered by the dissipative edge and bulk modes in the topological atom array. The largest nonreciprocity is found at the magic atomic spacing \( d = 3 \lambda_0 / 4 \), where \( \lambda_0 \) is the characteristic wavelength of the waveguide. For an odd number of atoms with an extremely small free space decay, the broken inversion and time-reversal symmetries induce a giant anomalous photon loss from the waveguide to free space. Our model can be implemented in superconducting quantum circuits. This work opens a new avenue to manipulate photons via the interaction between light and topological quantum matter.

Introduction.---One-dimensional (1D) waveguides are essential light-matter interfaces and have fundamental applications in quantum devices and quantum networks [1–3]. The photon transport in a waveguide can be controlled by coupling to a single atom [4–24] or an atom array [25–33]. In the subwavelength regime, the interference of photons emitted from atoms at different positions [34–37] gives rise to the collective enhancement of photon transport [38–42] and directional photon emission [43–45]. In waveguide quantum electrodynamics (QED) systems, e.g., atoms trapped around nanofibers [46–50], the direct atom-atom interaction is in general negligible. However, the direct interaction between atoms is essential in superconducting quantum circuits. By engineering this interaction, one can simulate many models in condensed matter physics and high energy physics, including spin models [51–53], topological matter [54–56], and lattice gauge theories [57, 58].

Of particular interest, symmetry-protected topological phases of matter open a field in materials science [59], and provide extensive opportunities in quantum computation and information technology [60–65]. In 2008, Refs. [66, 67] proposed to manipulate photon transport using topology, which paved the way for topological photonics [68–72]. In 2D or higher dimensional topological materials, photons can be guided via channels supported by edge modes and surface modes [73–77]. Such transport is immune to imperfections, randomness, and disorder due to the large bandgap separating chiral edge modes and bulk modes. The topological protection of photon transport has been realized in different incarnations of optical systems [78–83].

In this Letter, by virtue of the interaction between light and topological atom array via a waveguide, we show the intriguing photon scattering induced by the nontrivial topology. Arrays with an odd number of equally spaced atoms are described by the Su-Schrieffer-Heeger (SSH) model [84], as shown in Fig. 1(a). We find that for a magic atomic spacing \( d \), in contrast to the reciprocal transmission the reflection...
is nonreciprocal. This is attributed to the interplay between the inversion symmetry breaking induced by the non-trivial topology and the time-reversal symmetry breaking due to dissipation. More precisely, the quantum interference of electromagnetic waves reflected by the dissipative edge and bulk states gives rise to the large nonreciprocity. We propose the implementation of our model in superconducting quantum circuits.

**Single-photon scattering by a topological atom array.—**

As shown in the schematic Fig. 1(a), we study a topological atom array coupled to photonic modes in a 1D waveguide with linear dispersion. The Hamiltonian of the waveguide is \( \hat{H}_{\text{wg}} = -i c \int dx \left( \hat{a}^\dagger_l(x) \frac{\partial}{\partial x} \hat{a}_r(x) - \hat{a}^\dagger_r(x) \frac{\partial}{\partial x} \hat{a}_l(x) \right), \) where \( \hat{a}^\dagger_l (\hat{a}_l) \) and \( \hat{a}^\dagger_r (\hat{a}_r) \) are the creation and annihilation operators for the left (right) propagating photons, respectively. The topological atom array is described by the SSH model [84]

\[
H_{\text{ssh}} = \left( J_- \sum_{i=\text{odd}} \sigma_i^+ \sigma_{i+1}^- + J_+ \sum_{i=\text{even}} \sigma_i^+ \sigma_{i+1}^- \right) + \text{H.c.},
\]

where \( \sigma_i^+ = |e_i\rangle \langle g_i| \) depicts the transition from the ground state \( |g_i\rangle \) to the excited state \( |e_i\rangle \) of the \( i \)-th atom, and the nearest neighbor flip-flop interactions change alternatively along the chain as \( J_\pm = J_0 (1 \mp \cos \varphi) \). Different from the two edge modes in the SSH lattice with even number of sites [85], only a single edge state exists at either the left \( (0 \leq \varphi < \pi/2) \) or the right \( (\pi/2 < \varphi \leq \pi) \) boundary of the topological array with an odd number of sites. Without loss of generality, we focus on the atom array with a left-localized edge state, i.e., \( 0 \leq \varphi < \pi/2 \).

The interaction \( H_{\text{int}} = g \sum_{i=x,l,r} \hat{a}^\dagger_i(x_i) \sigma_i^+ e^{-i s_a k_0 x_i} + \text{H.c.} \) between atoms and the waveguide is determined by the coupling strength \( g \) and the wave vector \( k_0 = \omega_0/c \), where \( s_a = \pm \) for the right- and left-moving photons, and \( \omega_0 \) and \( x_i \) are the transition frequency and the position of the \( i \)-th atom. The photons in the waveguide mediate the long-range interaction of atoms and induce the collective dissipation [34, 86]. By integrating out the photonic modes, we obtain the non-Hermitian effective Hamiltonian

\[
H_{\text{eff}} = H_{\text{ssh}} + H_{\text{fs}} + H_{\text{w}}, \tag{3}
\]

under the Markovian approximation. Here, \( H_{\text{fs}} = -i \Gamma_0 \sum_i \sigma_i^+ \sigma_i^- \) with \( \Gamma_0 \) being the decay rate to free space, and \( H_{\text{w}} = -i \Gamma \sum_{ij} e^{ik_0 |x_i-x_j|} \sigma_i^+ \sigma_j^- \), where \( \Gamma = g^2/c \) denotes the spontaneous emission rate to the waveguide. The interplay of the coherent dynamics governed by the SSH Hamiltonian \( H_{\text{ssh}} \) and the incoherent interaction \( H_{\text{w}} \) determines the topological features. One can expect in the strong coherent coupling regime, i.e., \( J_0 \gg \Gamma \), that the localized edge state survives. In Fig. 1(b), we show the real part \( \Delta = \text{Re}(E) \) of the energy spectrum \( E \) of \( H_{\text{eff}} \) for \( J_0 \gg \Gamma \). The spectrum has the periodicity \( \lambda_0 = 2\pi/k_0 \) in \( d \). As \( d \) varies from 0 to \( \lambda_0 \), the spectrum of bulk states is significantly changed due to the long-range interaction mediated by waveguide photons [50]. However, since the edge state is topologically protected from the bulk modes by the bandgap, it is only slightly shifted. In particular, both the left and right halves of the spectrum have the rotational symmetry by \( \pi \) with respect to \( (d, \Delta) = (\lambda_0/4, 0) \) and \( (3\lambda_0/4, 0) \), respectively.

Compared with the optical responses [34, 35] of the atom array without direct interaction, i.e., \( J_0 = 0 \), the topological atom array has a profound influence on the photon transport. For a single photon with frequency \( \omega = ck \), the transmission and reflection amplitudes can be obtained as [87]

\[
t(\omega) = 1 - i \Gamma \sum_j \frac{V^\dagger | \psi^R_j \rangle \langle \psi^L_j | V}{\omega - \Delta_j + i \Gamma_j}, \tag{4}
\]

\[
r(\omega) = -i \Gamma \sum_j \frac{V^\dagger | \psi^L_j \rangle \langle \psi^R_j | V}{\omega - \Delta_j + i \Gamma_j}, \tag{5}
\]

where \( V = (e^{\pm ik_0 x_1}, e^{\pm ik_0 x_2}, \ldots)^T \) for the left- and right-incident photons, respectively, and the right and left eigenvectors \( | \psi^R_j \rangle \) and \( | \psi^L_j \rangle \) of \( H_{\text{eff}} \). Eq. (3) form the biorthogonal basis, i.e., \( | \psi^R_j \rangle \langle \psi^R_j | = \delta_{jj} \) [88]. The real and imaginary parts of \( E_j \), i.e., \( \Delta_j \) and \( \Gamma_j = -\text{Im}(E_j) \), denote the energy shift and the effective decay of the \( j \)-th mode in \( H_{\text{eff}} \), respectively. And \( \Gamma_j = \Gamma_0 + \tilde{\Gamma}_j \), where \( \tilde{\Gamma}_j \) denotes the collective decay induced by the dissipative interaction \( H_{\text{w}} \). The numerators in Eqs. (4) and (5) characterize the overlaps of photon modes and eigenmodes of the effective Hamiltonian in the transmission and reflection processes.

The topological feature of the array is imprinted in the spatial profile of the eigenvectors, \( | \psi^R_j \rangle \) and \( | \psi^L_j \rangle \), and the structure of the spectrum, which eventually determines the photon transport. In Fig. 1(c), we show the transmission spectra for \( d = \lambda_0/4, \lambda_0/2 \) and \( 3\lambda_0/4 \). The different linewidths of the transmission at \( \omega = 0 \) are determined by the decay rate of the edge mode. When the incident photon is resonant with the bulk states of frequency around \( \pm 2J_0 \), dips appear in the transmission spectra.

**Nonreciprocal reflection.—** Single-photon scattering by the topological atom array depends on the atomic spacing \( d \) and the direction of the incident photon. In Fig. 2(a), we show the transmission and reflection spectra for the left- and right-incident photons, where the transmission is reciprocal. However, reflections for the left- and right-incident photons are different, when the incident photon is resonant with the edge state. We define the reflection nonreciprocity

\[
\Delta R = |R_l - R_r|, \tag{6}
\]
FIG. 2. (a) Transmission and reflection of a single photon through a topological atom array. Red-dashed (black-dot-dashed) and blue-dotted (green-solid) curves denote reflection (transmission) for left- and right-incident photons, respectively. Reflections from left and right are different at resonance. (b) Nonreciprocity at resonance changes with atomic spacing. The atom array with spacing $d = 3\lambda_0/4$ has the largest nonreciprocity for given parameters. (c) Nonreciprocity for topological arrays with different $\phi$. Red-dotted, blue-dashed and black-solid curves correspond to arrays with different sizes. Red-dashed, blue-dotted and black-solid curves correspond to arrays with 11, 15 and 21 atoms. We consider $\phi = 0.3\pi$ in (a,b,d); $d = 3\lambda_0/4$ in (a,c,d); atom number $N = 11$ in (a,b,c). Other parameters for all these figures: $J_0/\Gamma = 8$, $\Gamma_0/\Gamma = 0.05$.

FIG. 3. (a) Nonreciprocity changes with atomic spacing $d$ and coupling parameter $J_0$. When $J_0$ is zero, there is no nonreciprocity. When $J_0$ is large, the nonreciprocity has a maximum at $d = 3\lambda_0/4$. (b) Nonreciprocal reflection: red-dotted (red-solid) and blue-dotted (blue-dashed) curves are reflections for left- and right-incident photons with $N = 11$ ($N = 21$). (c) Reflections and transmissions for left- and right-incident photons change with $\Gamma_0$. The reflection for right-incident photon $R_0$ (blue-solid) is sensitive to $\Gamma_0$ and reduces to zero when $\Gamma_0 \approx 0.0246$. The transmissions for left- and right-incident photons, denoted respectively by green-solid and black-dotted curves, are the same. (d) Parameter $\Gamma_0m$, defined by $R_0(\Gamma = \Gamma_0m) = 0$, versus $\phi$. We consider $J_0/\Gamma = 8$, $d = 3\lambda_0/4$ in (b,c,d), $\phi = 0.3\pi$ in (a,b,c), $\Gamma_0/\Gamma = 0.05$ in (a,b). In addition, $N = 11$ and $N = 21$ are studied in (a,c), respectively.

where $R_l = |r_l|^2$ and $R_r = |r_r|^2$, with $r_l$ ($r_r$) being the reflection amplitude for the left- (right-) incident photon. In Fig. 2(b) we show the nonreciprocal reflection for various atomic spacings. For $0 \leq d/\lambda_0 \leq 0.5$, the nonreciprocity is symmetric around $d = \lambda_0/4$, while for $0.5 \leq d/\lambda_0 \leq 1$ the nonreciprocity is symmetric around $d = 3\lambda_0/4$. The reflection is reciprocal for $d = 0, \lambda_0/2, \lambda_0$; and the maximal reflection nonreciprocity is found at $d = 3\lambda_0/4$. To study how the edge state affects the nonreciprocity, Fig. 2(c) shows $\Delta R$ for different values of $\phi$. As $\phi$ increases, the nonreciprocity is enhanced. In Fig. 2(d), we consider the array with different numbers of atoms, showing that the longer the array the larger the nonreciprocity. Since $J_0$ controls the bandgap [85], we study the effect of the bandgap to the nonreciprocity by plotting $\Delta R$ versus the spacing $d$ and the coupling strength $J_0$ in Fig. 3(a). As expected, for vanishing $J_0$, the reflection is reciprocal. As $J_0$ increases, the position of the spacing $d$ at which the maximal nonreciprocity appears changes accordingly. For relatively large $J_0$, the maximal nonreciprocity appears at $d = 3\lambda_0/4$, which we refer as “magic spacing”.

In Fig. 3(b), the red dot-dashed (red-solid) and blue-dotted (blue-dashed) curves represent the reflections of the left- and right-incident photons for an array of size $N = 11$ ($N = 21$) and lattice spacing $d = 3\lambda_0/4$. Here, the reflection of the left-incident photon at resonance is almost unchanged as $N$ increases, while the reflection of the right-incident photon is drastically reduced. For the right-incident photon, the reflection reduction has a non-trivial relation with the free space decay $\Gamma_0$. As shown by the transmission and reflection versus $\Gamma_0$ in Fig. 3(c), the reciprocity of the reflection for $\Gamma_0 = 0$ results from time-reversal symmetry. When $\Gamma_0$ increases, in contrast to the almost unchanged reflection of the left-incident photon, the reflection of right-incident photon exhibits non-monotonic behavior, which reaches its minimum at $\Gamma_0m$. The minimum $\Gamma_0m$ versus $\phi$ is plotted in Fig. 3(d) for $N = 11$ and 21. As $N$ increases, $\Gamma_0m$ is reduced; namely, a tiny free space decay is able to yield huge nonreciprocity. In addition, the transmissions of left- and right-incident photons are reciprocal and slightly changed as $\Gamma_0$ varies.

Quantum interference between waves reflected by edge and bulk modes.—The nonreciprocity originates from the inversion symmetry breaking due to the edge mode and the time-reversal symmetry breaking induced by the free space decay. For the magic spacing $d = 3\lambda_0/4$ and $0 \leq \phi < \pi/2$, the edge mode appears on the left boundary. The left-incident photon barely couples to the edge mode; however, the right-incident photon strongly couples to the edge mode, which can be seen from the interaction
FIG. 4. (a) Relative strength for edge state reflecting photons coming from left and right directions. For given parameters, it has a minimum at $d = 3\lambda_0/4$. (b) and (c) represent $|\xi_j|$ ($j = 1, 2, \cdots, N$) for photons coming from left and right, respectively. The vertical axis labels eigenmodes of $H_{\text{eff}}$. (d) Absolute values of $\xi_e = \xi_{j_0}$ (blue-dashed) and $\xi_b = \sum_{j \neq j_0} \xi_j$ (red-solid), which correspond to the edge mode and bulk modes, for the right-incident photon. (e) The scaling behaviors between $\ln \tilde{\Gamma}_{j_0}$ and $N$ at different $\varphi$. (f) The $\beta$ factor of edge state (blue square), and photon loss $\eta$ (red star) for right-incident photon. We consider $J_0/\Gamma = 8$, $d = 3\lambda_0/4$ for (a), (d), $\varphi = 0.3\pi$ for (a,d), $N = 21$ for (a,b,c,d,f), $\Gamma_0/\Gamma = 0.05$ for (a,b,c,e,f).

spectra

$$\Xi_j = V^l |\psi_j^R\rangle \langle \psi_j^L|V,$$

$$\tilde{\Xi}_j = V^l |\psi_j^R\rangle \langle \psi_j^L|V,$$

i.e., the numerators in Eqs. (4) and (5), representing the amplitudes of reflection and transmission produced by the $j$th mode. It turns out that for vanishing direct interactions, i.e., $J_0 = 0$, $\Xi_j$ ($\tilde{\Xi}_j$) are the same for the left- and right-incident photons. In Fig. 4(a), we show $|\Xi_{j_0}/\Xi_{j_0}|$ as a function of $d$, where $j_0$ denotes the edge state and $l$ ($r$) represents the left- (right-) incident photon. It is clear that $|\Xi_{j_0}/\Xi_{j_0}|$ has a minimum at $d = 3\lambda_0/4$, which implies small overlap of the left-propagating photon and the edge mode at $d = 3\lambda_0/4$. (The parameters $\tilde{\Xi}_j$ in Eq. (S10) for the transmission process are found to be reciprocal with different atomic spacings [87]).

Even though the left-incident photon does not couple to the edge mode, it is totally reflected by the detuned bulk modes. For the right-incident photon, due to the finite coupling to the edge mode, the interference of waves reflected by the edge mode and bulk modes gives rise to the left outgoing wave. The contributions from the edge mode and bulk modes to the reflection can be characterized via

$$\xi_j = \frac{\Xi_j \Gamma}{-\Delta_j + i(\Gamma_0 + \Gamma_j)},$$

where the incident photon is assumed to resonate with the edge state. In Figs. 4(b,c), the absolute values of different components $\xi_j$ explicitly show the tiny and large contributions from the edge mode for the left- and right-incident photons, respectively. In Fig. 4(d), absolute values of the contributions $\xi_e = \xi_{j_0}$ (blue-dashed) for the edge mode and $\xi_b = \sum_{j \neq j_0} \xi_j$ (red-solid) for bulk modes are shown for the right-incident photon at $\varphi = 0.3\pi$. For a closed system without free-space decay, the contributions from the edge mode and bulk modes are $\xi_e = 2e^{i\phi_0}$ and $\xi_b = -e^{i\phi_0}$, respectively, with $\phi_0 = \pi/2$. When the free-space decay $\Gamma_0$ is turned on, the reflection $\xi_b \sim -e^{i\phi_0}$ from bulk states is hardly affected by the small $\Gamma_0$, since $|\Delta_{j \neq j_0}|$ or $\Gamma_{j \neq j_0}$ are much larger than $\Gamma_0$. However, because the edge mode has zero energy at the magic spacing and a tiny decay rate $\Gamma_{j_0}$, the small free-space decay drastically reduces the reflection $\xi_e = e^{i\phi_0}$ from the edge mode by half when $\Gamma_0 = \Gamma_{j_0}$, which induces a vanishing reflection $\xi_e + \xi_b \sim 0$ and the maximal nonreciprocity. Accordingly, components of the edge state and bulk states are equal and the largest nonreciprocity takes place around $\varphi = 0.26\pi$ and $0.425\pi$ for Fig. 4(c) [87]. The waveguide-induced collective decay rate for the edge state at magic spacing exhibits a scaling $\tilde{\Gamma}_{j_0} \sim e^{-\nu N}$ for some values of $\varphi$, as shown in Fig. 4(e). Due to this scaling behavior, $\Gamma_{0m} \sim \tilde{\Gamma}_{j_0}$ decreases as the size of the array increases, in good agreement with Fig. 3(d).

Quantum scattering by edge and bulk modes yields anomalous photon transport. We introduce the beta factor [89, 90] $\beta = \tilde{\Gamma}_{j_0}/(\Gamma_{j_0} + \Gamma_0)$ to characterize the photon decay from the edge mode to the waveguide. In waveguide QED systems, atoms with high beta factor emit photons into the waveguide [91–93]. Our study shows that the photon in the waveguide totally emits to
free space even with $\beta \sim 1/2$ due to the interference with the reflective wave from bulk modes. In Fig. 4(f), we show the beta factor of the edge state and the photon loss $\eta = 1 - T_r - R_r$ from the waveguide to the free space for the right-incident photon, where $N = 21$. As presented in Fig. 4(f), when $\phi$ changes, the decay rate of the edge mode approaches the free-space decay and the $\beta$ factor reaches 1/2, giving rise to an enhanced photon loss close to unity.

Implementation.—The waveguide QED with a topological atom array can be implemented in different experimental platforms, e.g., superconducting quantum circuits [94–96]. Recently, waveguide QED with multiple superconducting artificial atoms has made enormous progress in experiments [93, 97–101]. A topological array with tunable interactions of atoms has been implemented with superconducting quantum circuits [54]. In our system, the nonreciprocity originates from the inversion symmetry breaking and the free-space decay of atoms, as shown in Fig. 3(c). To observe the nonreciprocity in experiments, one can tune the coupling parameter $\phi$, such that the position of $\Gamma_{\text{on}}$ shifts (see Fig. 3(d)) and the nonreciprocity changes accordingly. The nonreciprocal reflection also exists for the photon resonant with bulk modes [87]. However, due to the fact that the edge mode is topologically protected, for the incident photon resonant with the edge mode, the observation of the nonreciprocity in the reflection spectra is more feasible.

Discussions and Conclusions.—In this work, we study the photon scattering of a 1D waveguide coupled to a topological atom array. We find that the photon reflection by the topological atom array is nonreciprocal. It is attributed to destructive quantum interference between electromagnetic waves reflected by edge and bulk modes in the atom array. We show that, for the topological atom array with large bandgap, the nonreciprocity is maximal at a magic atomic spacing $d = 3\lambda_0/4$. The anomalous photon transport, i.e., the giant loss of the photon in the waveguide to free space, takes place, in spite of the relative decay $\Gamma/\Gamma_0 \gg 1$. Our work demonstrates the importance of topology in light-matter interacting phenomena, and sheds new light on topology-controlled quantum photonics.

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TOPOLOGY-CONTROLLED NONRECIPROCAL PHOTON SCATTERING IN A WAVEGUIDE:
SUPPLEMENTAL MATERIAL

I. Waveguide-mediated edge state

In the main text, we consider the effective Hamiltonian $H_{\text{eff}} = H_{\text{ssh}} + H'$. The Hamiltonian of the topological atom array is

$$H_{\text{ssh}} = \left( J_- \sum_{i=\text{odd}} \sigma_i^+ \sigma_{i+1}^- + J_+ \sum_{i=\text{even}} \sigma_i^+ \sigma_{i+1}^- \right) + \text{H.c.}, \quad (S1)$$

with $J_{\pm} = J_0 (1 \mp \cos \varphi)$. When the array has an odd number of atoms, only one edge state exists. The Hamiltonian can be diagonalized as $H_{\text{ssh}} = \sum_{j=1}^N \varepsilon_j |\alpha_j\rangle \langle \alpha_j|$, with $\varepsilon_{j-1} \leq \varepsilon_j \leq \varepsilon_{j+1}$. We assume that $j = j_0$ represents the edge state. In Fig. S1(a), we show the energy spectrum of $H_{\text{ssh}}$.

The edge state is localized to the left (right) boundary of the atom array for $0 \leq \varphi < 0.5\pi$ ($0.5\pi < \varphi \leq \pi$). The left edge state is $|\alpha_{j_0}\rangle = \frac{1}{N} \sum_{i=\text{odd}} (-J_-/J_+)^i |i\rangle$ with $|i\rangle = \sigma_i^+ |G\rangle$. And the right edge state is $|\alpha_{j_0}\rangle = \frac{1}{N} \sum_{i=\text{odd}} (-J_-/J_+)^{N-i} |i\rangle$. Here, $|G\rangle$ is the ground state of the atom array. In Fig. S1(b), we present wave functions of edge states at $\varphi = 0.3\pi$ and $\varphi = 0.7\pi$. They are symmetric to each other about the center of the atom array. Without loss of generality, we study the topological atom array with a left-localized edge state, i.e., $0 \leq \varphi < 0.5\pi$.

In addition to the SSH interaction, the waveguide can induce indirect interactions between atoms. By taking into account the atomic dissipation to free space, we obtain

$$H' = -i\Gamma_0 \sum_i \sigma_i^+ \sigma_i^- - i\Gamma \sum_{ij} \cos(k_0|x_i - x_j|) \sigma_i^+ \sigma_j^- + \Gamma \sum_{ij} \sin(k_0|x_i - x_j|) \sigma_i^+ \sigma_j^-. \quad (S2)$$

At $J_0 \gg \Gamma$, the edge state is protected by the bandgap and is weakly perturbed by the waveguide-mediated interactions. The effective edge state can be approximately written as

$$|\psi^{R}_{j_0}\rangle \approx |\alpha_{j_0}\rangle + \sum_{j \neq j_0} \frac{\langle \alpha_j|H'|\alpha_{j_0}\rangle}{\varepsilon_{j_0} - \varepsilon_j} |\alpha_j\rangle, \quad (S3)$$

and

$$|\psi^{L}_{j_0}\rangle \approx |\alpha_{j_0}\rangle + \sum_{j \neq j_0} \frac{\langle \alpha_{j_0}|H'|\alpha_j\rangle}{\varepsilon_{j_0} - \varepsilon_j} |\alpha_j\rangle. \quad (S4)$$

Here, $|\psi^{R}_{j_0}\rangle$ and $|\psi^{L}_{j_0}\rangle$ are the right and left vectors of the edge mode in the effective Hamiltonian $H_{\text{eff}}$, respectively. Because of the non-Hermitian Hamiltonian $H'$, we know that $|\psi^{R}_{j_0}\rangle \neq |\psi^{L}_{j_0}\rangle$, but $\langle \psi^{R}_{j_0}|\psi^{R}_{j_0}\rangle = 1$. We can know from Eqs. (S3) and (S4) that the effective edge mode contains components of the bulk modes.

![Fig. S1. (a) Energy spectrum of $H_{\text{ssh}}$. (b) Wave function of edge state at $\varphi = 0.3\pi$ (red-dotted) and $\varphi = 0.7\pi$ (blue-starred). Here, we consider atom number $N = 11$.](image-url)
II. Single-photon scattering by a topological atom array

A. Photon scattering by a many-body quantum system

In this subsection, we present formulas for photon scattering by an atom array. The single-photon scattering by a many-body system in a waveguide can be studied using Green’s function [S1]. The amplitudes of reflection and transmission for a single photon are respectively

\[ r = -i \Gamma V^T G V, \]
\[ t = 1 - i \Gamma V^\dagger G V, \]

with Green’s function

\[ G = \frac{1}{\omega - H_{\text{eff}}}. \]

We consider the spectrum decomposition \( H_{\text{eff}} = \sum_j (\Delta_j - i \tilde{\Gamma}_j) |\psi_j^R \rangle \langle \psi_j^L| \), where \( |\psi_j^R \rangle \) and \( |\psi_j^L \rangle \) are the right and left vectors of the \( j \)th mode. For simplicity, we assume that \( j = 1, 2, \ldots, N \), label the eigenmodes of \( H_{\text{eff}} \) with ascending order of energy \( \Delta_j \); and \( j = j_0 \) represents the edge state. Note that \( |\psi_j^R \rangle \) and \( |\psi_j^L \rangle \) form a biorthogonal basis with \( \langle \psi_j^L | \psi_j^R \rangle = \delta_{jj'} \). Therefore, the reflection and transmission amplitudes can be written as

\[ r(\omega) = -i \sum_j \frac{V_j^T |\psi_j^R \rangle \langle \psi_j^L| V_j}{\omega - \Delta_j + i \tilde{\Gamma}_j}, \]
\[ t(\omega) = 1 - i \sum_j \frac{V_j^\dagger |\psi_j^R \rangle \langle \psi_j^L| V_j}{\omega - \Delta_j + i \tilde{\Gamma}_j}. \]

The above equations show how the eigenstates of the system (atom array + waveguide) scatter a single photon. The topological atom array coupled with the waveguide can be modelled as a superatom with multiple energy levels. We note that equations (S8) and (S9) can also be used to study photon scattering for a multilevel system without spatial extension, e.g., a three-level atom [S2], and giant atoms, which are nonlocally coupled to electromagnetic fields [S3].

From Eqs. (S8) and (S9) we know that photon reflection and transmission in the waveguide result from quantum interference between different scattering components, induced by eigenmodes of the effective Hamiltonian. Therefore, topological quantum systems, which have nontrivial spectrum structures and edge states, are of particular importance for scattering photons. In this work, we pinpoint the roles played by the bandgap and edge state in nonreciprocal photon reflection.

B. Nonreciprocal reflection controlled by bandgap and edge state

In the waveguide-coupled topological atom array studied in this work, photon transmission and reflection are respectively reciprocal and nonreciprocal when the edge state is resonantly driven. The difference of reciprocity for transmission and reflection comes from distinctive light-matter interactions in photon scattering processes. From Eqs. (S8) and (S9), scattering processes for reflection and transmission are characterized by

\[ \Xi_j = V_j^T |\psi_j^R \rangle \langle \psi_j^L| V_j, \]

and

\[ \tilde{\Xi}_j = V_j^\dagger |\psi_j^R \rangle \langle \psi_j^L| V_j. \]

Equations (S10) and (S11) represent light-matter interactions between propagating photons and eigenmodes in the topological atom array. From Eqs. (S3) and (S4), \( |\psi_j^R \rangle \langle \psi_j^L| \) for the edge state can be approximately expressed as

\[ |\psi_{j_0}^R \rangle \langle \psi_{j_0}^L| = |\alpha_{j_0} \rangle \langle \alpha_{j_0}| - \sum_{j \neq j_0} \frac{1}{\epsilon_j} (\langle \alpha_j | H' | \alpha_{j_0} \rangle \langle \alpha_j | \alpha_{j_0}| + \langle \alpha_{j_0} | H' | \alpha_j \rangle \langle \alpha_{j_0}| \alpha_j|) + \cdots \]

In the above equation, we assume that the bandgap of the topological atom array is large, i.e., \( J_0 \gg \Gamma \) [S4]. Therefore, the waveguide produces effective couplings between edge state and bulk states. To study the role of modes in
reflecting photons with different incident directions, we define relative strength of waves, which come from left and right directions, reflected by the $j$th eigenmode

$$\zeta_j = \frac{|\Xi_l^j|}{|\Xi_r^j|},$$  \hspace{1cm} (S13)

where $l$ and $r$ label the left- and right-incident photons. Similarly, $\tilde{\zeta}_j = |\tilde{\Xi}_l^j|/|\tilde{\Xi}_r^j|$ uncovers the directionality of transmitted waves through the $j$th eigenmode. In Fig. S2(a), we present $\zeta_{j0}$ and $\tilde{\zeta}_{j0}$ for the edge state for different atomic spacings $d$. We can see that the ratio is the same for the transmission, but it is spacing-dependent for reflection. In particular, there is a minimum of $\zeta_{j0}$ at $d = 3\lambda_0/4$. The difference between $\Xi_{j0}^l$ and $\Xi_{j0}^r$ makes the edge state critical in controlling photon transport in the waveguide. In Fig. S2(b), we present $|\zeta_{j0}|$ changed with $J_0$. When $J_0$ is large, $\zeta_{j0}$ is close to one. Therefore, the bandgap should not be too large such that the waveguide-mediated edge-bulk couplings, i.e., the second term in Eq. (S12), are not negligible.

Photon reflection depends on several parameters of the topological atom array. In Fig. S3(a), we present the reflectional nonreciprocity for atom arrays with different sizes. The size affects nonreciprocity enormously. For a small atom array, the nonreciprocity is small at $\varphi = 0$ and increases as $\varphi$ grows. However, when atom array is large, the nonreciprocity has a large value at $\varphi = 0$ and decreases with $\varphi$. When the atom array approaches the critical

$$\Delta R$$

FIG. S2. (a) Reflection and transmission factors for edge state. Red-solid and blue-dashed curves correspond to reflection and transmission, respectively. Here, we consider the incident photon is resonant with the edge state and $J_0/\Gamma = 8$. (b) $|\zeta_{j0}|$ for different values of $J_0$ with $d = 3\lambda_0/4$. As the increase of $J_0$, $|\zeta_{j0}|$ approaches one. Other parameters considered in (a) and (b) are: $\varphi = 0.3\pi$, $\Gamma_0/\Gamma = 0.05$ and atom number $N = 11$.

FIG. S3. (a) Reflectional nonreciprocity for different sizes of atom arrays. The red-solid, blue-dashed, green-dotted and black-dot-dashed curves correspond to arrays with atom numbers $N = 11, 21, 31$ and 41. Here, we consider $J_0/\Gamma = 8$. (b) Reflectional nonreciprocity for different values of $J_0$. The red-solid, blue-dashed, green-dotted and black-dot-dashed curves correspond to $J_0/\Gamma = 4, 6, 8$ and 10. The atom number is assumed to be $N = 11$. Other parameters in (a) and (b) are: $d = 3\lambda_0/4, \Gamma_0/\Gamma = 0.05$. 

$\Delta R$
point $\varphi = 0.5\pi$, the nonreciprocity is reduced to zero. In Fig. 4(c) in the main text, $|\xi_{j0}|$ for the left-incident photon is reduced around $\varphi = 0.3\pi$ for $N = 21$. Accordingly, the nonreciprocity is reduced (see the blue-dashed curve in Fig. S3(a)). In particular, around $\varphi = 0.26\pi$ and $0.425\pi$, the nonreciprocity reaches maximum. In Fig. S3(b), we study the effect of $J_0$, which controls bandgap of the atom array. For relatively large values of $J_0/\Gamma$, the nonreciprocity $\Delta R$ is large at $\varphi = 0$. And the nonreciprocity can be tuned in a large regime by changing $\varphi$. However, as $J_0$ is increased, $\Delta R$ is reduced at $\varphi = 0$. The nonreciprocity is tunable in a small regime.

In the usual experiments with superconducting quantum circuits, the atom arrays have small sizes (around 10–20 qubits). One can obtain large reflectional nonreciprocity by choosing appropriate values of $J_0$ and $\varphi$.

### C. Nonreciprocal reflection for photons resonant with bulk states

In the main text, we study nonreciprocal reflection for photons which are resonant with the edge state. From the analysis in the previous subsection, we know that the waveguide-induced interaction modifies the edge state and gives rise to distinctive reflections for photons with different incident directions. Similar to the edge state, bulk states in the topological atom array are also modified by waveguide-induced interaction. Therefore, nonreciprocal reflection can be found for photons whose frequencies are resonant with bulk states.

In Fig. S4(a), we show $|\zeta_j|$ for bulk states with $j = N - 1$ (red-dotted) and $j = N$ (red-solid). The bulk states with $j = N - 1$ and $j = N$ have different reflectional coefficients $\Xi_j$ for left- and right-incident photons. However, the parameters $|\tilde{\zeta}_j|$ for $j = N - 1$ and $j = N$ are the same, as shown by the blue-dashed line. In Fig. S4(b), we show the transmission and reflection spectra. The transmission spectra are the same for left and right incident photons (green-solid and black-dot-dashed curves). But the reflection spectra are different. The arrows indicate bulk states with $j = N - 1$ and $j = N$. For the reason that frequencies between bulk states are small, it is challenging to observe reflectional nonreciprocity for photons with same frequencies as bulk states.

### D. $\Gamma_0$-dependent reflectional nonreciprocity

In the main text, we find that, with small $\Gamma_0/\Gamma$ and large $J_0/\Gamma$, the reflectional nonreciprocity is maximal at $d = 3\lambda_0/4$. The atomic decay to free space $\Gamma_0$ alters the nonreciprocity. In Fig. S5(a), we compare reflections at $d = \lambda_0/4$ and $d = 3\lambda_0/4$ for different values of $\Gamma_0/\Gamma$. At $d = 3\lambda_0/4$, the large nonreciprocity can be found at small value of $\Gamma_0/\Gamma$, as we studied in the main text. The increase of $\Gamma_0/\Gamma$ makes the nonreciprocity smaller. But for $d = \lambda_0/4$, a large nonreciprocity is found when $\Gamma_0$ is comparable with $\Gamma$. In Fig. S5(b), we show the nonreciprocity for $d = \lambda_0/4$ at different values of $J_0$. Different from $d = 3\lambda_0/4$ (see Fig. S3), $J_0$ does not have significantly changes the nonreciprocity. In the main text, we have shown that paired bulk states attribute to photon scattering at $d = 3\lambda_0/4$.  

![Diagram](image-url)
FIG. S5. (a) Photon reflections at different atomic spacings. Red-dot-dashed (red-dashed) and blue-solid (blue-dotted) curves denote reflections for photons coming from left and right at $d = \lambda_0/4$ ($d = 3\lambda_0/4$). Photon reflection for $d = \lambda_0/4$ has large nonreciprocity when $\Gamma_0/\Gamma$ is large. Here, we consider $\varphi = 0.3\pi$. (b) Reflectional nonreciprocity changes with $\varphi$. Red-solid, blue-dashed, green-dotted and black-dot-dashed curves correspond to $J_0/\Gamma = 4, 6, 8$ and $10$. (c) and (d) represent absolute values of $\xi_j$ for photons coming from left and right, respectively. We consider $\Gamma_0/\Gamma = 2$ in (b)-(d), and the number of atoms $N = 21$ in (a)-(d).

A similar situation is found for $d = \lambda_0/4$, as presented in Figs. S5(c) and S5(d). When the parameter $\varphi$ changes, different pairs of bulk states are coupled with propagating photons. For $d = 0, \lambda_0/2, \lambda_0$ or other values, pairs of bulk states do not contribute equally to photon scattering.

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