Polynomial relations among principal minors of a $4 \times 4$-matrix

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Abstract

The image of the principal minor map for $n \times n$-matrices is shown to be closed. In the 19th century, Nansen and Muir studied the implicitization problem of finding all relations among principal minors when $n = 4$. We complete their partial results by constructing explicit polynomials of degree 12 that scheme-theoretically define this affine variety and also its projective closure in $\mathbb{P}^{15}$. The latter is the main component in the singular locus of the $2 \times 2 \times 2 \times 2$-hyperdeterminant.

1 Introduction

Principal minors of square matrices appear in numerous applications. A basic question is the Principal Minors Assignment Problem [4] which asks for necessary and sufficient conditions for a collection of $2^n$ numbers to arise as the principal minors of an $n \times n$-matrix. When the matrix is symmetric, this question was recently answered by Oeding [12] who extended work of Holtz and Sturmfels [5] to show that the principal minors of a symmetric matrix are characterized by certain hyperdeterminantal equations of degree four.

This question is harder for general matrices than it is for symmetric matrices. For example, consider our Theorem [1] which says the image of the principal minor map is closed. The same statement is trivially true for symmetric matrices, but the proof for non-symmetric matrices is quite subtle.

We denote the principal minors of a complex $n \times n$-matrix $A$ by $A_I$ where $I \subseteq [n] = \{1, 2, \ldots, n\}$. Here, $A_I$ is the minor of $A$ whose rows and columns are indexed by $I$, including the $0 \times 0$-minor $A_{\emptyset} = 1$. Together, they form a vector $A_*$ of length $2^n$. We are interested in an algebraic characterization of
all vectors in \( \mathbb{C}^{2^n} \) which can be written in this form. The map \( \phi_a: \mathbb{C}^{n^2} \to \mathbb{C}^{2^n}, A \mapsto A_a \) is called the affine principal minor map for \( n \times n \)-matrices.

**Theorem 1.** The image of the affine principal minor map is closed in \( \mathbb{C}^{2^n} \).

This result says that \( \text{Im} \, \phi_a \) is a complex algebraic variety. The dimension of this variety is \( n^2 - n + 1 \). This number is an upper bound because the principal minors remain unchanged under the transformation \( A \mapsto DAD^{-1} \) for diagonal matrices \( D \), and it is not hard to see that this upper bound is attained \cite{15}. What we are interested in here is the prime ideal of polynomials that vanish on the irreducible variety \( \text{Im} \, \phi_a \). We determine this prime ideal in the first non-trivial case \( n = 4 \). Here, we ignore the trivial relation \( A_\emptyset = 1 \).

**Theorem 2.** When \( n = 4 \), the prime ideal of the 13-dimensional variety \( \text{Im} \, \phi_a \) is minimally generated by 65 polynomials of degree 12 in the unknowns \( A_I \).

This theorem completes the line of research started by MacMahon, Muir and Nanson \cite{9,10,11} in the late 19th century. Our proof of Theorem 2 takes advantage of their classical results, and it will be presented in Section 3.

Algebraic geometers would consider it more natural to study the projective version of our problem. We define the projective principal minor map as

\[
\phi : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{2^n}, (A, B) \mapsto (\det(A_I B_{[n]\setminus I}))_{I \subseteq [n]}.
\]

Here we take two unknown \( n \times n \)-matrices \( A \) and \( B \) to form an \( n \times 2n \)-matrix \((A, B)\), and for each \( I \subseteq [n] \), we evaluate the \( n \times n \)-minor with column indices \( I \) on \( A \) and column indices \([n]\setminus I\) on \( B \). The image of \( \phi \) is a closed affine cone in \( \mathbb{C}^{2^n} \), to be regarded as a projective variety in \( \mathbb{P}^{2^n-1} = \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) \).

What makes the projective version more interesting than the affine version is that \( \text{Im} \, \phi \) is invariant under the natural action of the group \( G = \text{GL}_2(\mathbb{C})^n \). This was observed by JM Landsberg (cf. \cite{5,12}). In Section 4 we shall prove

**Theorem 3.** When \( n = 4 \), the projective variety \( \text{Im} \, \phi \) is cut out scheme-theoretically by 718 linearly independent homogeneous polynomials of degree 12. This space of polynomials is the direct sum of 14 irreducible \( G \)-modules.

The polynomials described in Theorems 2 and 3 are available online at

\[\text{http://math.berkeley.edu/~shaowei/minors.html} \].

A geometric interpretation of our projective variety is given in Section 5. We shall see that \( \text{Im} \, \phi \) is the main component in the singular locus of the
2\times2\times2\times2-hyperdeterminant. This is based on work of Weyman and Zelevinsky [16], and it relates our current study to the computations in [6]. We begin in Section 2 by rewriting principal minors in terms of cycle-sums. The resulting combinatorial structures are key to our proof of Theorem 1.

After posting the first version of this paper on the arXiv, Eric Rains informed us that some of our findings overlap with results in Section 4 of his article [1] with Alexei Borodin. The relationship to their work, which we had been unaware of, will be discussed at the end of this paper, in Remark 14.

2 Cycle-Sums and Closure

In this section, we define cycle-sums and determine their relationship to the principal minors. We then prove that a certain ring generated by monomials called cycles is integral over the ring generated by the principal minors. This integrality result will be our main tool for proving the closure theorem.

Let \( X = (x_{ij}) \) be an \( n \times n \)-matrix of indeterminates and \( \mathbb{C}[X] \) the polynomial ring generated by these indeterminates. Let \( P_I \in \mathbb{C}[X] \) denote the principal minor of \( X \) indexed by \( I \subseteq [n] \), including \( P_\emptyset = 1 \). Together, these minors form a vector \( P^\ast \) of length \( 2^n \). Thus \( A^\ast = P^\ast(A) \) if \( A \) is a complex \( n \times n \)-matrix. Now, given a permutation \( \pi \in \mathfrak{S}_n \) of \([n]\), define the monomial \( c_\pi = \prod_{i \neq \pi(i)} x_{i\pi(i)} \in \mathbb{C}[X] \) where the product is taken over the support of \( \pi \).

We call \( c_\pi \) a \( k \)-cycle if \( \pi \) is a cycle of length \( k \). For \( I \subseteq [n] \), \( |I| \geq 2 \), define the cycle-sum \( C_I = \sum_{\pi \in \mathcal{C}_I} c_\pi \) where \( \mathcal{C}_I \) is the set of all cycles with support \( I \).

Also, let \( C_\emptyset = 1 \) and \( C_i = x_{ii}, i \in [n] \). Together, they form a vector \( C^\ast \) of length \( 2^n \). The cycles, cycle-sums and principal minors generate subrings \( \mathbb{C}[c^\ast], \mathbb{C}[C^\ast] \) and \( \mathbb{C}[P^\ast] \) of \( \mathbb{C}[X] \). The next result shows that \( \mathbb{C}[C^\ast] = \mathbb{C}[P^\ast] \).

**Proposition 4.** The principal minors and cycle-sums satisfy the following relations: for any subset \( I \subseteq [n] \) of cardinality \( d \geq 1 \), we have

\[
P_I = \sum_{I = I_1 \sqcup \ldots \sqcup I_k} (-1)^{k+d}C_{I_1} \cdots C_{I_k} \tag{2}
\]

\[
C_I = \sum_{I = I_1 \sqcup \ldots \sqcup I_k} (-1)^{k+d(k-1)!}P_{I_1} \cdots P_{I_k} \tag{3}
\]

where the sums are taken over all set partitions \( I_1 \sqcup \ldots \sqcup I_k \) of \( I \).

**Proof.** The first equation is Leibniz’s formula for the determinant. The second equation is derived from the first formula by applying the Möbius inversion formula [13, §3.7] for the lattice of all set partitions of \([n]\).
Let \( \psi : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n} \) be the polynomial map given by (3). We say that \( u_* \in \mathbb{C}^{2^n} \) is realizable as principal minors if \( u_* = P_*(A) \) for some complex matrix \( A \). Similarly, \( u_* \) is realizable as cycle-sums if \( u_* = C_*(A) \) for some \( A \).

**Corollary 5.** A vector \( u_* \in \mathbb{C}^{2^n} \) is realizable as principal minors if and only if its image \( \psi(u_*) \) is realizable as cycle-sums.

A monomial in \( \mathbb{C}[X] \) can be represented by a directed multigraph on \( n \) vertices as follows: label the vertices 1,\( \ldots \), \( n \) and for each \( x^k_{ij} \) appearing in the monomial, draw \( k \) directed edges from vertex \( i \) to vertex \( j \). Cycles \( c_{i_1 \ldots i_k} \) correspond to cycle graphs \( i_1 \to \ldots \to i_k \to i_1 \) which we write as \( (i_1 \ldots i_k) \).

We are interested in studying when a product of cycles can be written as a product of smaller ones. This is equivalent to decomposing a union of cycles into smaller cycles. For instance, the relation \( c_{(123)}c_{(132)} = c_{(12)}c_{(23)}c_{(13)} \) says that the union of these two 3-cycles can be decomposed into three 2-cycles.

**Lemma 6.** Let \( \pi_1, \pi_2, \ldots, \pi_m \) be \( m \geq 2 \) distinct cycles of length \( k \geq 3 \) with the same support. Then, the product \( c_{\pi_1}c_{\pi_2} \cdots c_{\pi_m} \) can be expressed as a product of strictly smaller cycles.

*Proof.* We may assume that all the cycles have support \([k]\). Note that it suffices to prove our lemma for \( m = 2, 3 \). The following is our key claim: given an \( l \)-cycle \( c_{(1s(s+1) \ldots k)} \), \( l \leq k \), \( s \neq 2 \), not equal to \( c_{(1s(s+1) \ldots k)} \), the product \( c_{(1 \ldots k)}c_{(1s(s+1) \ldots k)} \) can be expressed as a product of cycles of length smaller than \( k \). Indeed, suppose that no such expression exists. Let the graphs of \( c_{(1 \ldots k)} \) and \( c_{(1s(s+1) \ldots k)} \) be \( G_1 \) and \( G_2 \) respectively. Color the edges of \( G_1 \) red and \( G_2 \) blue. Then \( G = G_1 \cup G_2 \) contains the cycle \( C_1 = (1s(s+1) \ldots k) \) where the first edge \( 1 \to s \) is blue while every other edge is red. Since \( s \neq 2 \), \( C_1 \) has fewer than \( k \) vertices. The following algorithm decomposes \( G \setminus C_1 \) into cycles:

1. Initialize \( i = 1 \) and \( v_1 = s \).
2. Begin with vertex \( v_i \) and take the directed blue path until a vertex \( v_{i+1} \) from the set \( \{1, 2, \ldots, v_i - 1\} \) is encountered.
3. Take the red path from \( v_{i+1} \) to \( v_i \). Call the resulting cycle \( C_{i+1} \).
4. If \( v_{i+1} = 1 \), we are done. Otherwise, increase \( i \) by 1 and go to step 2.

Since no decomposition into smaller cycles exists, one of the cycles \( C_i \) has \( k \) vertices. In particular, \( C_i \) contains the vertex 1, so by the above construction,
$v_i = 1$. Let $\mathcal{P}$ be the blue path in $\mathcal{C}_i$ from $v_{i-1}$ to $v_i = 1$. Since $s$ cannot lie in the interior of the red path $\mathcal{C}_i - \mathcal{P}$ from 1 to $v_{i-1}$, it must lie on $\mathcal{P}$. The blue edge into $s$ emanates from 1. However, 1 is the last vertex of $\mathcal{P}$, so $s$ must be the first vertex of $\mathcal{P}$, i.e. $v_{i-1} = s$. This shows that $\mathcal{P}$ is a path from $s$ to 1 with vertex set $\{s, s+1, \ldots, k, 1\}$. Its union with the blue edge $1 \rightarrow s$ gives a blue cycle contained in $\mathcal{G}_2$, so $\mathcal{G}_2$ equals this cycle. Since $\mathcal{G}_2$ is not the cycle $(1s(s+1)\ldots k)$, it contains an edge $\alpha \rightarrow \beta$ with $s \leq \alpha < \beta \leq k$ and $\beta \neq \alpha + 1$. The same argument with $\alpha$ and $\beta$ relabeled as $1$ and $s$ now shows that the vertex set of $\mathcal{G}_2$ in the old labeling is $\{\beta, \beta + 1, \ldots, k, 1, \ldots, s, \ldots, \alpha\}$. This is a contradiction, which proves the key claim.

We return to the lemma. For $m = 2$ we simply use the key claim. Suppose $m = 3$. Let $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ be the three cycles. The $m = 2$ case tells us that $\mathcal{G}_1 \cup \mathcal{G}_2$ can be decomposed into smaller cycles $\mathcal{C}_1, \ldots, \mathcal{C}_r$. The trick now is to take the union of some $\mathcal{C}_i$ with $\mathcal{G}_3$ and apply the key claim. If $\mathcal{C}_i$ has at most $|\mathcal{C}_i| - 2$ directed edges in common with $\mathcal{G}_3$, we are done. Indeed, we can label the vertices so that $\mathcal{G}_3 = (12\ldots k)$ and $\mathcal{C}_i = (1si_3\ldots i_l)$ with $s \neq 2$. Also, $\mathcal{C}_i \neq (1s(s+1)\ldots k)$, otherwise it has $|\mathcal{C}_i| - 1$ edges in common with $\mathcal{G}_3$. Hence, the key claim applies. We are left with the case where each $\mathcal{C}_i$ has $|\mathcal{C}_i| - 1$ edges in common with $\mathcal{G}_3$. Assume further that $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are those constructed by the algorithm in the key claim. It is then not difficult to deduce that either $\mathcal{G}_1 = \mathcal{G}_3$ or $\mathcal{G}_2 = \mathcal{G}_3$. This contradicts the assumption that the three graphs are distinct. \qed

**Proposition 7.** The algebra $\mathbb{C}[c_{\pi}]$ is integral over its subalgebra $\mathbb{C}[P_{\pi}]$.

**Proof.** Let $R_k = \mathbb{C}[P_{\pi}, \{c_{\pi}\}_{|\pi| \leq k}] \subset \mathbb{C}[X]$ be the subring generated by the principal minors and cycles of length at most $k$. Note that $R_n = \mathbb{C}[c_{\pi}]$ and $R_2 = \mathbb{C}[P_{\pi}]$ since $c_{(ij)} = P_i P_j - P_{ij}$ for all distinct $i, j \in [n]$. Thus, it suffices to show that $R_k$ is integral over $R_{k-1}$ for all $3 \leq k \leq n$. In particular, we need to show that each $k$-cycle $c_{\pi}$ is the root of a monic polynomial in $R_{k-1}[z]$ where $z$ is an indeterminate.

We claim that the monic polynomial $p(z) = \prod_{\pi \in \mathcal{E}_I} (z - c_{\pi})$ is in $R_{k-1}[z]$ for all $I \subseteq [n]$. Indeed, the coefficient of $z^{N-d}, 1 \leq d \leq N = |\mathcal{E}_I|$, in $p(z)$ is

$$\alpha_d = (-1)^d \sum_{\{\pi_1, \ldots, \pi_d\} \subseteq \mathcal{E}_I} c_{\pi_1} c_{\pi_2} \cdots c_{\pi_d}.$$  

Observe that $\alpha_1 = -C_{[k]}$, which lies in $\mathbb{C}[P_{\pi}] \subseteq R_{k-1}$ by Proposition \[. For $d > 1$, we apply Lemma \[ which implies that each monomial in $\alpha_d$ can be expressed as a product of smaller cycles. This shows that $\alpha_d \in R_{k-1}$.

\[}
Corollary 8. If \( \{ A_k \}_{k>0} \) is a sequence of complex \( n \times n \)-matrices whose principal minors are bounded, then the cycles \( c_\pi (A_k) \) are also bounded.

Proof. Proposition 7 implies that \( c_\pi \) satisfies a monic polynomial with coefficients in \( \mathbb{C}[P_*] \). Since the principal minors are bounded, these coefficients are also bounded, so the same is true for \( c_\pi \).

Proof of Theorem 1. Suppose \( \{ A_k \}_{k>0} \) is a sequence of complex \( n \times n \)-matrices whose principal minors tend to \( u_* \in \mathbb{C}^{2^n} \). Since the cycle values \( c_\pi (A_k) \) are bounded, we can pass to a subsequence and assume that the sequence of values for each cycle converges to a complex number \( v_\pi \). Lemma 9 below states that the image of the cycle map is closed. Hence there exists an \( n \times n \)-matrix \( A \) such that \( c_\pi (A) = v_\pi \) for all cycles. The limit minor \( u_I \) is expressed in terms of the \( v_\pi \) using the formula (2). We conclude that the principal minors of the matrix \( A \) satisfy \( P_I (A) = u_I \) for all \( I \).

The following lemma concludes the proof of Theorem 1 and this section.

Lemma 9. Let \( M \) be the number of cycles and consider the map \( \gamma : \mathbb{C}^{n^2} \to \mathbb{C}^M \) whose coordinates are the cycle monomials \( c_\pi \) in \( \mathbb{C}[X] \). Then the image of the monomial map \( \gamma \) is a closed subset of \( \mathbb{C}^M \). (So, it is a toric variety).

Proof. The general question of when the image of a given monomial map between affine spaces is closed was studied and answered independently in [2] and in [7]. We can apply the characterizations given in either of these papers to show that the image of our map \( \gamma \) is closed. The key observation is that the cycle monomials generate the ring of invariants of the \( (\mathbb{C}^*)^n \)-action on \( \mathbb{C}[X] \) given by \( X \mapsto D \cdot X \cdot D^{-1} \) where \( D \) is an invertible diagonal matrix. Equivalently, the exponent vectors of the monomials \( c_\pi \) are the minimal generators of a subsemigroup of \( \mathbb{N}^{n^2} \) that is the solution set of a system of linear equations on \( n^2 \) unknowns. The geometric meaning of this key observation is that the monomial map \( \gamma \) represents the quotient map \( \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}/(\mathbb{C}^*)^n \) in the sense of geometric invariant theory. Now, the results on images of monomial maps in [2] §3 and [7] ensure that \( \text{Im} \gamma \) is closed.

3 Sixty-Five Affine Relations

We seek to identify generators for the prime ideal \( \mathcal{I}_n \) of polynomials in \( \mathbb{C}[A_*] \) that vanish on the image \( \text{Im} \phi_n \) of the affine principal minor map. Here the \( 2^n \)
coordinates $A_t$ of the vector $A$, are regarded as indeterminates. For $n \leq 3$, every vector $u_+ \in \mathbb{C}^{2^n}$, $u_0 = 1$, is realizable as the principal minors of some $n \times n$-matrix, so $I_n = 0$. In this section, we determine $I_n$ for the case $n = 4$.

Finding relations among the principal minors of a $4 \times 4$-matrix is a classical problem posed by MacMahon in 1894 and partially solved by Nanson in 1897 [9, 10, 11]. The relations were discovered by means of “devertebrated minors” and trigonometry. Here, we write the Nanson relations in terms of the cycle-sums. They are the maximal $4 \times 4$-minors of the following $5 \times 4$-matrix:

$$
\begin{pmatrix}
C_{12}C_{14} & C_{12}C_{13} & C_{13}C_{12} & 2C_{23}C_{12}C_{13}C_{14} + C_{13}C_{12}C_{14}C_{13} \\
C_{12}C_{23} & C_{12}C_{24} & C_{23}C_{21} & 2C_{13}C_{21}C_{23}C_{24} + C_{23}C_{12}C_{13}C_{12} \\
C_{13}C_{32} & C_{23}C_{31} & C_{13}C_{34} & 2C_{12}C_{31}C_{32}C_{34} + C_{23}C_{13}C_{14}C_{13} \\
C_{23}C_{41} & C_{13}C_{42} & C_{12}C_{43} & 2C_{13}C_{41}C_{42}C_{43} + C_{23}C_{13}C_{14}C_{14} \\
1 & 1 & 1 & C_{1234}
\end{pmatrix}
$$

Each of the cycle-sums in this matrix can be rewritten as a polynomial in the principal minors $P_i$ using the relations (3). An explicit example is

$$
C_{123} = 2A_1A_2A_3 - A_{12}A_3 - A_{13}A_2 - A_{23}A_1 + A_{123}.
$$

The maximal minors of (4) give us five polynomials in the ideal $I_4$. Each can be expanded either in terms of cycle-sums or in terms of principal minors.

Muir [10] and Nanson [11] left open the question of whether additional polynomials are needed to generate the ideal $I_4$, even up to radical. We applied computer algebra methods to answer this question. In the course of our experimental investigations, we discovered the 65 affine relations that generate the ideal. They are the generators of the ideal quotient $(K : g)$ where $K$ is the ideal generated by the five $4 \times 4$-minors above, and $g$ is the principal $3 \times 3$-minor corresponding to the first three rows and columns of (1). Thus the main stepping stone in the proof of Theorem 2 is the identity

$$
I_4 = (K : g).
$$

Before proving this, we present a census of the 65 ideal generators, and we explain why all 65 polynomials are needed and to what extent they are uniquely characterized by the equality of ideals in (5). The polynomial ring $\mathbb{C}[A_+]$ has 15 indeterminates that are indexed by non-empty subsets of $\{1, 2, 3, 4\}$. It has the following natural multigrading by the group $\mathbb{Z}^4$:

$$
\begin{align*}
\deg(A_1) &= [1, 0, 0, 0] , \quad \deg(A_2) = [0, 1, 0, 0] , \ldots , \quad \deg(A_4) = [0, 0, 0, 1], \\
\deg(A_{12}) &= [1, 1, 0, 0] , \ldots , \quad \deg(A_{234}) = [0, 1, 1, 1] , \quad \deg(A_{1234}) = [1, 1, 1, 1].
\end{align*}
$$
This $\mathbb{Z}^4$-grading is a positive grading, i.e., each graded component is a finite-dimensional $\mathbb{C}$-vector space. Both the ideal $\mathcal{K}$ and the prime ideal $\mathcal{I}_4$ are homogeneous in this $\mathbb{Z}^4$-grading. This means that the minimal generators of both ideals can be chosen to be $\mathbb{Z}^4$-homogeneous, and their number is unique.

Our computation revealed that this number of generators is 65. Moreover, we found that the generators lie in 63 distinct $\mathbb{Z}^4$-graded components. The component in degree $[5, 5, 5]$ happens to be three-dimensional. We chose a $\mathbb{C}$-basis for this 3-dimensional space of polynomials. All 62 other components are one-dimensional, and these give rise to generators with integer coefficients and content 1 that are unique up to sign. The complete census of all 65 generators is presented in Table 1. For each generator we list its multidegree and its number of monomials (“size”) in the two expansions, namely, in terms of cycle-sums $C_I$ and in terms of principal minors $A_I$.

The first four rows of Table 1 refer to the four maximal minors of the matrix (4) which involve the last row. The expansion of any of these four minors in terms of cycle-sums has 32 monomials and is of total degree 8. However, the expansion of that polynomial in terms of principal minors $A_I$ is much larger: it has 5234 monomials.

Proof of Theorem 2. We compute the ideal $(\mathcal{K} : g)$ and find that it has the 65 minimal generators above. We check that each of the five generators of $\mathcal{K}$ vanishes on $\text{Im} \phi_n$ but $g$ does not vanish on $\text{Im} \phi_n$. This implies $(\mathcal{K} : g) \subseteq \mathcal{I}_4$.

To prove the reverse inclusion we argue as follows. Computation of a Gröbner basis in terms of cycle-sums reveals that $(\mathcal{K} : g)$ is an ideal of codimension 2, and we know that this is also the codimension of the prime ideal $\mathcal{I}_4$. Therefore $\mathcal{I}_4$ is a minimal associated prime of $(\mathcal{K} : g)$. To complete the proof, it therefore suffices to show that $(\mathcal{K} : g)$ is a prime ideal. We do this using the following lemma:

Lemma 10. Let $J \subset k[x_1, x_2, \ldots, x_n]$ be an ideal containing a polynomial $f = gx_1 + h$, with $g, h$ not involving $x_1$ and $g$ a non-zero divisor modulo $J$. Then, $J$ is prime if and only if the elimination ideal $J \cap k[x_2, \ldots, x_n]$ is prime.

Lemma 10 is due to M. Stillman and appears in [14, Prop. 4.4(b)]. We apply this lemma to our ideal $J = (\mathcal{K} : g)$ in the polynomial ring $\mathbb{C}[A_1]$, with $x_1 = A_{1234}$ as the special variable, and we take the special polynomial $f$ to be the $4 \times 4$-minor of the matrix (4) formed by deleting the fourth row.

We have $f = gA_{1234} + h$ where $g, h$ are polynomials that do not involve $A_{1234}$. A computation verifies that $(J : g) = J$, so $g$ is not a zero-divisor
Table 1: Multidegrees of the 65 minimal generators of $I_4$

| No. | Size | Deg. | Multidegree | Cycle-sums | Size | Deg. | Multidegree |
|-----|------|------|-------------|------------|------|------|-------------|
| 1   | 32   | 8    | [5, 4, 5, 5]| 34         | 163  | 10   | [4, 5, 5, 7]| Cycle-sums |
| 2   | 32   | 8    | [5, 4, 5, 5]| 35         | 163  | 10   | [4, 5, 7, 5]| Principal Minors |
| 3   | 32   | 8    | [5, 5, 4, 5]| 36         | 163  | 10   | [4, 7, 5, 5]| Cycle-sums |
| 4   | 32   | 8    | [5, 5, 5, 4]| 37         | 163  | 10   | [5, 4, 5, 7]| Principal Minors |
| 5   | 42   | 9    | [4, 4, 6, 6]| 38         | 163  | 10   | [5, 4, 7, 5]| Cycle-sums |
| 6   | 42   | 9    | [4, 6, 4, 6]| 39         | 163  | 10   | [5, 5, 4, 7]| Principal Minors |
| 7   | 42   | 9    | [4, 6, 6, 4]| 40         | 163  | 10   | [5, 5, 7, 4]| Cycle-sums |
| 8   | 42   | 9    | [6, 4, 4, 6]| 41         | 163  | 10   | [5, 7, 4, 5]| Principal Minors |
| 9   | 42   | 9    | [6, 4, 6, 4]| 42         | 163  | 10   | [5, 7, 5, 4]| Cycle-sums |
| 10  | 42   | 9    | [6, 6, 4, 4]| 43         | 163  | 10   | [7, 4, 5, 4]| Principal Minors |
| 11  | 80   | 9    | [4, 5, 5, 6]| 44         | 163  | 10   | [7, 5, 4, 5]| Cycle-sums |
| 12  | 80   | 9    | [4, 5, 6, 5]| 45         | 163  | 10   | [7, 5, 5, 4]| Principal Minors |
| 13  | 80   | 9    | [4, 6, 5, 5]| 46         | 254  | 10   | [4, 5, 6, 6]| Cycle-sums |
| 14  | 80   | 9    | [5, 4, 5, 6]| 47         | 254  | 10   | [4, 6, 5, 6]| Principal Minors |
| 15  | 80   | 9    | [5, 4, 6, 5]| 48         | 254  | 10   | [4, 6, 6, 5]| Cycle-sums |
| 16  | 80   | 9    | [5, 5, 4, 6]| 49         | 214  | 10   | [6, 4, 6, 5]| Principal Minors |
| 17  | 80   | 9    | [5, 5, 6, 4]| 50         | 214  | 10   | [5, 6, 4, 6]| Cycle-sums |
| 18  | 80   | 9    | [5, 6, 4, 5]| 51         | 214  | 10   | [5, 6, 6, 4]| Principal Minors |
| 19  | 80   | 9    | [5, 6, 5, 4]| 52         | 254  | 10   | [6, 4, 5, 6]| Cycle-sums |
| 20  | 80   | 9    | [6, 4, 5, 5]| 53         | 254  | 10   | [6, 4, 6, 5]| Principal Minors |
| 21  | 80   | 9    | [6, 5, 4, 5]| 54         | 254  | 10   | [6, 5, 4, 6]| Cycle-sums |
| 22  | 80   | 9    | [6, 5, 5, 4]| 55         | 254  | 10   | [6, 5, 6, 4]| Principal Minors |
| 23  | 116  | 9    | [5, 5, 5, 5]| 56         | 254  | 10   | [6, 6, 4, 5]| Cycle-sums |
| 24  | 116  | 9    | [5, 5, 5, 5]| 57         | 254  | 10   | [6, 6, 5, 4]| Principal Minors |
| 25  | 116  | 9    | [5, 5, 5, 5]| 58         | 354  | 10   | [5, 5, 6, 5]| Cycle-sums |
| 26  | 91   | 10   | [3, 6, 6, 6]| 59         | 354  | 10   | [5, 5, 6, 5]| Principal Minors |
| 27  | 91   | 10   | [6, 3, 6, 6]| 60         | 354  | 10   | [5, 6, 5, 5]| Cycle-sums |
| 28  | 91   | 10   | [6, 6, 3, 6]| 61         | 364  | 10   | [6, 5, 5, 5]| Principal Minors |
| 29  | 91   | 10   | [6, 6, 6, 3]| 62         | 685  | 11   | [4, 6, 6, 6]| Cycle-sums |
| 30  | 834  | 11   | [5, 5, 5, 7]| 63         | 685  | 11   | [6, 4, 6, 6]| Principal Minors |
| 31  | 834  | 11   | [5, 5, 7, 5]| 64         | 685  | 11   | [6, 6, 4, 6]| Cycle-sums |
| 32  | 834  | 11   | [5, 7, 5, 5]| 65         | 685  | 11   | [6, 6, 4, 6]| Principal Minors |
| 33  | 834  | 11   | [7, 5, 5, 5]| 66         | 685  | 11   | [6, 6, 4, 6]| Cycle-sums |
modulo \( J \). It remains to show that the elimination ideal \( J \cap \mathbb{C}[A_r \setminus A_{1234}] \) is prime. Now, since \( J \) has codimension two, this elimination ideal is principal. Indeed, its generator is the 4\( \times \)4-minor of (I) given by the first four rows. This polynomial has \( \mathbb{Z}^4 \)-degree \([6,6,6,6] \). We check using computer algebra that it is absolutely irreducible, and conclude that \( J \) is prime.

\[\square\]

### 4 A Pinch of Representation Theory

In this section we prove Theorem 3 and we explicitly determine the 14 polynomials of degree 12 that serve as highest weight vectors for the relevant irreducible \( G \)-modules. We begin by describing the general setting for \( n \geq 4 \).

Let \( \mathcal{V}_n \subset \mathbb{P}^{2^n-1} \) be the image of the projective principal minor map \( \phi \), and let \( \mathcal{J}_n \in \mathbb{C}[A_r] \) be the homogeneous prime ideal of polynomials in \( 2^n \) unknowns \( A_I \) that vanish on \( \mathcal{V}_n \). Clearly, \( \mathcal{J}_n \) is invariant under the action of \( S_n \) on \( \mathbb{C}[A_r] \) which comes from permuting the rows and columns of the \( n \times n \)-matrix. Let \( \text{GL}_2(\mathbb{C}) \) denote the group of invertible complex 2\( \times \)2-matrices, and consider the vector space \( V = V_1 \otimes V_2 \otimes \cdots \otimes V_n \) where each \( V_i \simeq \mathbb{C}^2 \) with basis \( e_i^0, e_i^1 \). We identify a basis vector \( e_{j_1}^i \otimes e_{j_2}^{i_2} \otimes \cdots \otimes e_{j_n}^{i_n} \) of the tensor product \( V \) with the unknown \( A_I \) where \( i \in I \) if and only if \( j_i = 1 \), for all \( i \).

The natural action of the \( n \)-fold product \( G = \text{GL}_2(\mathbb{C}) \times \cdots \times \text{GL}_2(\mathbb{C}) \) on \( V \simeq \mathbb{C}^{2^n} \) extends to its coordinate ring \( \mathbb{C}[A_r] \). This action commutes with the map \( \phi \) in (I). Here \( G \) acts on the parameter space of \( n \times 2n \)-matrices \((A,B)\) by having its \( i \)-th factor \( \text{GL}_2(\mathbb{C}) \) act on the \( n \times 2 \)-matrix formed by the \( i \)-th columns of \( A \) and \( B \). This argument, due to J. M. Landsberg, shows that the prime ideal \( \mathcal{J}_n \) is invariant under \( G \). See also [5, §6] and [12, Thm. 1.1].

**Corollary 11.** The set \( \mathcal{V}_n \) is closed in \( \mathbb{P}^{2^n-1} \), i.e. it is a projective variety.

**Proof.** For any index set \( I \subseteq [n] \), there exists a group element \( g \in G \) which takes \( A_\emptyset \) to \( A_I \). Thus, every affine piece \( U_I = \{ u_* \in \mathcal{V}_n : u_I \neq 0 \} \) of the constructible set \( \mathcal{V}_n = \text{Im} \phi \) is isomorphic to \( U_\emptyset = \text{Im} \phi_a \). The latter image is closed by Theorem 1. Therefore, \( \mathcal{V}_n \) is an irreducible projective variety. \( \square \)

The action of the Lie group \( G \) gives rise to an action of the Lie algebra \( \mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) \times \cdots \times \mathfrak{gl}_2(\mathbb{C}) \) on the polynomial ring \( \mathbb{C}[A_r] \) by differential operators. Here \( \mathfrak{gl}_2(\mathbb{C}) \) denotes the ring of complex 2\( \times \)2-matrices. Indeed, the vector

\[
\left( 0, \ldots, 0, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, 0, \ldots, 0 \right) \in \mathfrak{g},
\]

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whose entries are zero matrices of format $2 \times 2$ except in the $i$-th coordinate, acts on the polynomial ring $\mathbb{C}[A_n]$ as the linear differential operator

$$
\sum_{i \notin I} \left( w A_I \frac{\partial}{\partial A_I} + x A_{I \cup \{i\}} \frac{\partial}{\partial A_{I \cup \{i\}}} + y A_I \frac{\partial}{\partial A_{I \cup \{i\}}} + z A_{I \cup \{i\}} \frac{\partial}{\partial A_{I \cup \{i\}}} \right).
$$

Here the sum is over all $I \subseteq [n]$ not containing $i$. We extend this action to all of $\mathfrak{g}$ by linearity. If all the coordinate matrices of an element $h \in \mathfrak{g}$ are strictly upper triangular, we say that $h$ is a raising operator. Similarly, $h$ is a lowering operator if all the matrices are strictly lower triangular.

We now focus on the case $n = 4$. Let $\mathcal{T}^h$ be the ideal generated by the homogenizations of the 65 affine generators in Theorem 2 with respect to $A_8$. This is a subideal of the homogeneous prime ideal $\mathcal{J} = \mathcal{J}_4$ we are interested in. Our strategy is to identify a suitable intermediate ideal between $\mathcal{I}h \subset \mathcal{J}$.

Proof of Theorem 3. Let $\mathcal{K} = G \mathcal{T}^h$ be the ideal generated by the image of $\mathcal{T}^h$ under the group $G$. Then $\mathcal{K}$ is a $G$-invariant ideal of $\mathbb{C}[A_n]$ that is contained in the prime ideal $\mathcal{J}$. Since $G$ acts transitively on the affine charts $U_I$, and since $\mathcal{T}^h$ coincides with the unknown ideal $\mathcal{J}$ on the chart $U_8$, we conclude that the ideal $\mathcal{K}$ defines our projective variety $\mathcal{V}_4$ scheme-theoretically.

By definition, the ideal $\mathcal{K}$ is generated by its degree-12 component $\mathcal{K}_{12}$. To prove Theorem 3 we need to show that $\mathcal{K}_{12}$ is a $G$-module of $\mathbb{C}$-dimension 718, and that it decomposes into 14 irreducible $G$-modules. As a $G$-module, the graded component $\mathcal{K}_{12}$ is generated by the homogenizations of the 65 polynomials in Table 1. Representation theory as in [8, 12] tells us that the unique highest weight vectors of the irreducible $G$-modules contained in $\mathcal{K}_{12}$ can be found by applying raising operators $h \in \mathfrak{g}$ to these 65 generators.

Indeed, consider the 1st, 5th and 26th polynomials in Table 1. When written in terms of cycle-sums, these polynomials have 32, 42 and 91 terms respectively. Let $D$, $E$ and $F$ be their homogenizations with respect to $A_8$. By applying the raising and lowering operators in the Lie algebra $\mathfrak{g}$, one checks that all 65 generators lie in the $G \rtimes \mathfrak{S}_4$-orbit of $D$, $E$ and $F$. Thus,

$$
\mathcal{K}_{12} = M_D \oplus M_E \oplus M_F
$$

where $M_D$, $M_E$ and $M_F$ are the $G$-modules spanned by the $G \rtimes \mathfrak{S}_4$-orbits of the polynomials $D$, $E$ and $F$ respectively. Furthermore, as in [8], we write

$$
S_{ijkl} = S_{(12-i,j)}(\mathbb{C}^2) \otimes S_{(12-j,i)}(\mathbb{C}^2) \otimes S_{(12-k,k)}(\mathbb{C}^2) \otimes S_{(12-l,l)}(\mathbb{C}^2)
$$
for the tensor product of Schur powers of $\mathbb{C}^2$. The dimension of $S_{ijkl}$ equals $(13 - 2i)(13 - 2j)(13 - 2k)(13 - 2l)$. Our three $G$-modules decompose as

$$
M_D \simeq S_{4555} \oplus S_{5455} \oplus S_{5545} \oplus S_{5554}, \\
M_E \simeq S_{4466} \oplus S_{4646} \oplus S_{4664} \oplus S_{6464} \oplus S_{6446} \oplus S_{6644}, \\
M_F \simeq S_{3666} \oplus S_{6366} \oplus S_{6636} \oplus S_{6663}.
$$

The above three vector spaces have dimensions 540, 150 and 28 respectively. This shows that $\dim(K_{12}) = 718$, and the proof of Theorem 3 is complete.

Many questions about relations among principal minors remain open at this point, even for $n = 4$. The most basic question is whether the ideal $K$ is prime, that is, whether $K = J$ holds (cf. Remark 14). Next, it would be desirable to find a nice determinantal representation for the polynomials $E$ and $F$, in analogy to $D$ being the homogenization of the determinant of the last four rows in (4). We know little about the prime ideal $I_n$ for $n \geq 5$. It contains various natural images of the ideal $I_4$, but we do not know whether these generate. The most optimistic conjecture would state that $I_n$ is generated by the $\text{GL}_2(\mathbb{C})^n$-orbit of the polynomials $D$, $E$ and $F$. The work of Oeding [12] gives hope that at least the set-theoretic version might be within reach:

**Conjecture 12.** The variety $V_n \subset \mathbb{P}^{2n-1}$ is cut out by equations of degree $12$.

## 5 Singularities of the Hyperdeterminant

Our object of study is the projective variety $V_4$ which is parametrized by the principal minors of a generic $4 \times 4$-matrix. We have seen that $V_4$ is a variety of codimension two in the projective space $\mathbb{P}^{15}$. That ambient space is the projectivization of the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of $2 \times 2 \times 2 \times 2$-tables $a = (a_{ijkl})$. This section offers a geometric characterization of the variety $V_4$.

The articles [5, 12] show that the variety parametrized by the principal minors of a symmetric matrix is closely related to the $2 \times 2 \times 2$-hyperdeterminant. It is thus quite natural for us to ask whether such a relationship also exists in the non–symmetric case. We shall argue that this is indeed the case.

The *hyperdeterminant* of format $2 \times 2 \times 2 \times 2$ is a homogeneous polynomial of degree 24 in 16 unknowns having 2,894,276 terms [6]. Its expansion into
cycle-sums was found to have 13,819 terms. The hypersurface $\nabla$ of this hyperdeterminant consists of all tables $a \in \mathbb{P}^{15}$ such that the hypersurface

$$\mathcal{H}_a = \{(x, y, z, w) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 : \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} a_{ijkl}x_iy_jz_kw_l = 0\}$$

has a singular point. Weyman and Zelevinsky [16] showed that the hyperdeterminantal hypersurface $\nabla \subset \mathbb{P}^{15}$ is singular in codimension one. More precisely, by [16, Thm. 0.5 (5)], the singular locus $\nabla^{\text{sing}}$ of $\nabla$ is the union in $\mathbb{P}^{15}$ of eight irreducible projective varieties, each having dimension 13:

$$\nabla^{\text{sing}} = \nabla^{\text{node}}(\emptyset) \cup \bigcup_{1 \leq i < j \leq 4} \nabla^{\text{node}}(\{i, j\}) \cup \nabla^{\text{cusp}}. \tag{6}$$

Here, the node component $\nabla^{\text{node}}(\emptyset)$ is the closure of the set of tables $a$ such that $\mathcal{H}_a$ has two singular points $(x, y, z, w)$ and $(x', y', z', w')$ with $x \neq x'$, $y \neq y'$, $z \neq z'$ and $w \neq w'$. The extraneous component $\nabla^{\text{node}}(\{1, 2\})$ is the closure of the set of tables $a$ such that $\mathcal{H}_a$ has two singular points $(x, y, z, w)$ and $(x, y, z', w')$, and similarly for the other five extraneous components. Finally, the cusp component $\nabla^{\text{cusp}}$ parametrizes all tables $a$ for which $\mathcal{H}_a$ has a triple point. The connection to our study is given by the following result:

**Theorem 13.** The node component in the singular locus of the $2 \times 2 \times 2 \times 2$-hyperdeterminant coincides with the variety parametrized by the principal minors of a generic $4 \times 4$-matrix. In symbols, we have $\mathcal{V}_4 = \nabla^{\text{node}}(\emptyset)$.

**Proof.** We now dehomogenize by setting $x_0 = y_0 = z_0 = w_0 = 1$, $x_1 = x$, $y_1 = y$, $z_1 = z$ and $w_1 = w$. Let $(c_{ij})$ be a generic complex $4 \times 4$-matrix and consider the ideal in $\mathbb{C}[x, y, z, w]$ generated by the $3 \times 3$-minors of

$$\begin{pmatrix} c_{11} + x & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} + y & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} + z & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} + w \end{pmatrix}. \tag{7}$$

We claim that the variety of this ideal consists of two distinct points in $\mathbb{C}^4$. We prove this by a computation. Regarding the $c_{ij}$ as unknowns, we compute a Gröbner basis for the ideal of $3 \times 3$-minors with respect to the lexicographic term order $x > y > z > w$ over the base field $K = \mathbb{Q}(c_{11}, c_{12}, \ldots, c_{44})$. The output shows that the ideal is radical and the initial ideal equals $\langle x, y, z, w^2 \rangle$. 

13
The determinant of (7) is the affine multilinear form

$$F(x, y, z, w) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} a_{ijkl} \cdot x^i y^j z^k w^l$$

whose coefficients are the principal minors of the $4 \times 4$-matrix $(c_{ij})$. The claim established in the previous paragraph implies that the system of equations

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial w} = 0$$

has two distinct solutions over the algebraic closure of $K = \mathbb{Q}(c_{11}, \ldots, c_{44})$. These two solutions correspond to two distinct singular points of the hypersurface $H_4$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. From this we conclude that the table $a = (a_{ijkl})$ of principal minors of $(c_{ij})$ lies in the node component $\nabla_{\text{node}}(\emptyset)$.

Our argument establishes the inclusion $V_4 \subseteq \nabla_{\text{node}}(\emptyset)$. Both sides are irreducible subvarieties of $\mathbb{P}^{15}$, and in fact, they share the same dimension, namely 13. This means they must be equal.

Our results in Section 4 give an explicit description of the equations that define the first component in the decomposition (6). This raises the problem of identifying the equations of the other seven components. 

\textbf{Remark 14.} After posting the first version of this paper on the arXiv, we learned that some of the results in this paper have already been addressed in \cite{1} \S 4. Specifically, Landsberg’s observation that $V_n$ is $G$-invariant coincides with \cite{1} Theorem 4.2, and our Theorem 13 coincides with \cite{1} Theorem 4.6. Coincidentally, without reference to dimensions, we proved $V_4 \subseteq \nabla_{\text{node}}(\emptyset)$ directly while \cite{1} Theorem 4.6 gives the other inclusion. Moreover, at the very end of the paper \cite{1}, it is stated that “... the variety has degree 28, with ideal generated by a whopping 718 degree 12 polynomials”. This appears to prove our conjecture (at the end of Section 4) that the ideal $\mathcal{K}$ is actually prime. We verified that the ideal $\mathcal{K}$ has degree 28, but we did not yet succeed in verifying that $\mathcal{K}$ is saturated with respect to the irrelevant maximal ideal. We tried to do this computation by specializing the 16 unknowns to linear in fewer unknowns but this leads to an ideal which is not prime. It thus appears that the ideal $\mathcal{K}$ is not Cohen-Macaulay.

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