Musielak-Orlicz-Hardy Spaces Associated with Operators Satisfying Reinforced Off-Diagonal Estimates

Abstract
Let $X$ be a metric space with doubling measure and $L$ a one-to-one operator of type $\omega$ having a bounded $H_{\omega,\omega}$-functional calculus in $L^2(X)$ satisfying the reinforced $(p_l, q_l)$ off-diagonal estimates on balls, where $p_l \in [1, 2]$ and $q_l \in (2, \infty]$. Let $f : X \times [0, \infty) \to [0, \infty)$ be a function such that $f(x, \cdot)$ is an Orlicz function, $qf(x, t) \in A_{\omega}(X)$ (the class of uniformly Muckenhoupt weights), its uniformly critical upper type index $\theta_l(\omega) \in (0, 1]$ and $\omega(x, t)$ satisfies the uniformly reverse Hölder inequality of order $(q_l, l(\omega))$, where $(q_l, l(\omega))^* \omega$ denotes the conjugate exponent of $q_l, \omega(\cdot, t)$. In this paper, the authors introduce a Musielak-Orlicz-Hardy space $H_{\omega, l}(X)$, via the Lusin-area function associated with $L$, and establish its molecular characterization. In particular, when $L$ is nonnegative self-adjoint and satisfies the Davies-Gaffney estimates, the atomic characterization of $H_{\omega, l}(X)$ is also obtained. Furthermore, a sufficient condition for the equivalence between $H_{\omega, l}(\mathbb{R}^n)$ and the classical Musielak-Orlicz-Hardy space $H_{\omega, l}(\mathbb{R}^n)$ is given. Moreover, for the Musielak-Orlicz-Hardy space $H_{\omega, l}(\mathbb{R}^n)$ associated with the second order elliptic operator in divergence form on $\mathbb{R}^n$ or the Schrödinger operator $L := -\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, the authors further obtain its several equivalent characterizations in terms of various non-tangential and radial maximal functions; finally, the authors show that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H_{\omega, l}(\mathbb{R}^n)$ to the Musielak-Orlicz space $L^q(\mathbb{R}^n)$ when $q(\omega) \in (0, 1]$, from $H_{\omega, l}(\mathbb{R}^n)$ to $H_{\omega, l}(\mathbb{R}^n)$ when $q(\omega) \in (\frac{1}{1+\theta_l(\omega)}, 1]$, and from $H_{\omega, l}(\mathbb{R}^n)$ to the weak Musielak-Orlicz-Hardy space $W H_{\omega, l}(\mathbb{R}^n)$ when $q(\omega) = \frac{1}{1+\theta_l(\omega)}$ is attainable and $\omega(\cdot, t) \in A_{\omega}(X)$, where $q(\omega)$ denotes the uniformly critical lower type index of $\omega$.

Keywords
Musielak-Orlicz-Hardy space • molecule • atom • maximal function • Lusin area function • Schrödinger operator • elliptic operator • Riesz transform

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1. Introduction

The real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, introduced by Stein and Weiss [70] and systematically developed in the seminal paper of Fefferman and Stein [31], plays a central role in various fields of harmonic analysis and partial differential equations (see, for example, [21, 66] and the references therein). One of the main features of the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is the atomic decomposition characterizations (see [20] for $n = 1$ and [54] for $n > 1$). Later, the theory of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ with Muckenhoupt weights $w$ has been studied by García-Cuerva [33], and Strömberg and Torchinsky [69]. Furthermore, Strömberg [68] and Janson [44] introduced the Orlicz-Hardy space which plays an important role in studying the theory of nonlinear PDEs (see, for example, [13, 14, 16, 36, 43]). Recently, in [49], the last two authors of the present paper studied Hardy spaces of Musielak-Orlicz type which generalize the Orlicz-Hardy space in [44, 68] and the weighted Hardy spaces in [33, 69]. Furthermore, several real-variable characterizations of the Hardy spaces of Musielak-Orlicz type were established in [42, 56]. More recently, the Hardy space of Musielak-Orlicz type was studied in [73]. It is worth pointing out that Musielak-Orlicz functions are the natural generalization of Orlicz functions (see, for example, [28, 29, 49, 59]) and the motivation to study function spaces of Musielak-Orlicz type is attributed to their extensive applications in many fields of mathematics (see, for example, [13–16, 28, 29, 49, 50, 55] for more details). However, it is now understood that there are many settings in which the theory of the spaces of Hardy type can not be applicable; for example, the Riesz transform $\nabla L^{-1/2}$ may not be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $L := -\text{div}(A\nabla)$ is a second order divergence elliptic operator with complex bounded measurable coefficients (see, for example, [40]).

Recently, there has been a lot of studies which pay attention to the theory of function spaces associated with operators. In many applications, the very dependence on the function spaces associated with the operators provides many advantages in studying the boundedness of singular integrals which may not fall within the scope of the classical Calderón-Zygmund theory. Here, we would like to give a brief overview of this research direction. Let $L$ be an infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t>0}$ on $L^2(\mathbb{R}^n)$ whose kernels satisfy the Gaussian upper bound estimates. The theory on Hardy spaces associated with such operators $L$ was investigated in [5, 30]. Later, Hardy spaces associated with operators which satisfy the weaker conditions, so-called Davies-Gaffney estimate conditions, were treated in the works of Auscher et al. [8], Hofmann and Mayboroda [40] and Hofmann et al. [39, 41]. In [45–47, 57, 71–74], the authors studied operators which satisfy Davies-Gaffney estimates, where the nonnegative potential $V$ satisfies the reverse Hölder inequality of order $\alpha$ (see Section 2.2 below for these definitions). A typical example of such a $V$ is

$$V(x, t) := \left[\log(1 + t(x, x_0))^\beta + \log(1 + t)^\gamma\right]^\alpha$$

for all $x \in \mathcal{X}$ and $t \in [0, \infty)$, with some $\alpha \in (0, 1]$, $\beta \in [0, n]$ and $\gamma \in [0, 2\alpha(1 + \ln 2)]$ (see Section 2.2 for more details). Then, the last two authors of the present paper [74] introduced a Musielak-Orlicz-Hardy space $H_{p, \Phi}(\mathcal{X})$.
the Lusin-area function associated with \( L \), and obtained two equivalent characterizations of \( H_{\mu, l}(\mathcal{X}) \) in terms of the atom and the molecule. Hence, it is natural to raise the question when the condition \( \varphi(x, \cdot) \in \mathbb{R}_{H_{\mu, l}(\mathcal{X})} \) can be relaxed.

One of the main aims of this paper is to give an affirmative answer to this question. Moreover, motivated by [7, 12, 18, 74], in this paper, we consider more general operators by assuming that the considered operator satisfies Assumptions (A) and (B) in Subsection 2.3 of this paper. Indeed, Assumption (A) is weaker than "the nonnegative and self-adjoint" condition imposed on the operator \( L \) in [74]. Meanwhile, in Assumption (B), we first introduce the notion of the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls in spaces of homogeneous type (see Definition 2.7 below), which is quite wide so that it can provide a framework to treat almost the results in previous works (see, for example, [5, 12, 30, 39, 40, 45, 46, 74]). Under Assumptions (A) and (B), we first introduce the Musielak-Orlicz-Hardy spaces \( H_{\mu, l}(\mathcal{X}) \) (see Definition 4.1 below), via the Lusin-area function associated with \( L \), and then characterize the spaces \( H_{\mu, l}(\mathcal{X}) \) in terms of the molecule with \( \varphi(x, \cdot) \in \mathbb{R}_{H_{\mu, l}(\mathcal{X})} \) (see Theorem 4.8 below), where \( (q_1, l/(\varphi')) \) denotes the conjugate exponent of \( q_1, l/(\varphi) \). In particular, when \( L \) is nonnegative self-adjoint and satisfies the Davies-Gaffney estimates, the atomic characterization of \( H_{\mu, l}(\mathcal{X}) \) is also obtained (see Theorem 5.4 below). It is important to notice that \( \mathbb{R}_{H_{\mu, l}(\mathcal{X})} \subset \mathbb{R}_{L_{\mu, l}(\mathcal{X})} \), whenever \( q_1 > 2 \) and hence the results in this paper improve significantly those in [74], by enlarging the range of the weights. In the particular case that the heat kernels associated with \( \{e^{-itL}\}_{t \geq 0} \) satisfy the Gaussian upper bound estimate, then \( p_1 = 1 \) and \( q_1 = \infty \) and hence the class of \( \varphi \) can be extended to \( \varphi(x, \cdot) \in \mathbb{A}_{\mu, l}(\mathcal{X}) \). Moreover, we also give a sufficient conditions on \( L \) so that our Musielak-Orlicz-Hardy space \( H_{\mu, l}(\mathcal{X}) \) coincides with the Musielak-Orlicz-Hardy space \( H_{\mu, l}(\mathcal{X}) \) introduced by the third author of this paper in [49] when \( \mathcal{X} = \mathbb{R}^n \) (see Theorem 6.7 below).

As applications, we consider Musielak-Orlicz-Hardy spaces \( H_{\mu, l}(\mathcal{X}) \) in some particular cases, for example, \( L \) being the second order elliptic operator in divergence form or the Schrödinger operator. More precisely, for the Musielak-Orlicz-Hardy space \( H_{\mu, l}(\mathbb{R}^n) \) associated with the second order elliptic operator in divergence form on \( \mathbb{R}^n \) with bounded measurable complex coefficients or the Schrödinger operator \( L := -\Delta + V \), where \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \), we further obtain its several equivalent characterizations in terms of the non-tangential and the radial maximal functions (see Theorems 7.5 and 8.3 below); finally, we show that the Riesz transform \( \nabla L^{-1/2} \) is bounded from \( H_{\mu, l}(\mathbb{R}^n) \) to the Musielak-Orlicz space \( L^p(\mathbb{R}^n) \) when \( (p, q) \in (0, 1] \) from \( H_{\mu, l}(\mathbb{R}^n) \) to \( H_{\mu, l}(\mathbb{R}^n) \) when \( (p, q) \in (\frac{n}{n+1}, 1] \), and from \( H_{\mu, l}(\mathbb{R}^n) \) to the weak Musielak-Orlicz-Hardy space \( WH_{\mu, l}(\mathbb{R}^n) \) when \( (p, q) = \frac{n}{n-1} \) is attainable and \( \varphi(\cdot, t) \in \mathbb{A}_{I, l}(\mathcal{X}) \) (see Theorems 7.8, 7.11, 8.5 and 8.6 below), where \( \varphi(t) \) denotes the uniformly critical lower type index of \( \varphi \).

One of the new ingredients appeared in this paper is the introduction of the notion of the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls in spaces of homogeneous type with \( m \in \mathbb{N} := \{1, 2, \ldots \} \). We remark that, to study the weighted Hardy space \( H^p_{\mu, l}(\mathbb{R}^n) \) on the Euclidean space \( \mathbb{R}^n \) and to relax the range of the weight \( \omega \) as wide as possible, the authors introduced a notion of the reinforced \((p_1, q_1)\) off-diagonal estimates in [12], which is particular useful for studying the weighted Hardy space associated with various differential operators of second order in the setting of Euclidean spaces. However, if we consider the differential operators on some more general spaces (for example, the Laplace-Beltrami operator on the Riemannian manifold with a doubling measure), the reinforced \((p_1, q_1)\) off-diagonal estimates in [12] seem no longer suitable (see Remark 2.9(a)). To overcome this difficulty, we introduce the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls by combining the ideas of the reinforced \((p_1, q_1)\) off-diagonal estimates from [12] and the off-diagonal estimates on balls from [7]. Also, the order \( m \in \mathbb{N} \) makes many differential operators of higher order fall into our scope (see Remark 2.9(c)). We also point out that, in [39, 45], the authors introduced a Hardy space associated with operators \( L \) in the space of homogeneous type by assuming that \( L \) satisfies the so-called Davies-Gaffney estimates. However, due to the fact that Davies-Gaffney estimates are equivalent to \( L^2 - L^2 \) off-diagonal estimates on balls (see Remark 5.1(ii)), their Hardy spaces can be viewed as a special case of ours. Another interesting ingredient appeared in this paper is the discussion of the role of the \( L^2(\mathcal{X}) \) norm in the definitions of the Hardy space \( H_{\mu, l}(\mathcal{X}) \) and the atomic or the molecular Hardy space. This discussion has two aspects, the first one is from [12], where the authors asked the question that what happen if we replace \( L^2(\mathbb{R}^n) \) by \( L^2(\mathbb{R}^n) \) with \( q \neq 2 \) in the definition of the Hardy space. For this question, in the present setting, we prove that the space \( H_{\mu, l}(\mathcal{X}) \) is invariant when we do this replacement for all \( q \in (p_1, q_1) \) (see Theorem 4.9), which coincides with the result obtained in [12] when \( \varphi(x, t) := \hat{p} \varphi(x), \) with \( p \in (0, 1], \) for all \( x \in \mathbb{R}^n \) and \( t \in [0, \infty). \) The second aspect of this discussion can be reduced to the following question: "what happen if we replace the \( L^2(\mathbb{R}^n) \)-convergence of the atomic (resp. molecular) representation by the \( L^s(\mathbb{R}^n) \)-convergence with \( s \neq 2 \) in the definition of the atomic and the molecular Hardy space?"

This question arises naturally when we study the boundedness of the fractional integral between two different Hardy spaces. For this question, we prove that the atomic and the molecular Hardy spaces are invariant when we do this replacement for all \( s \in (p_1, q_1) \) (see Theorems 5.9 and 4.8).
The organization of this paper is as follows. In Section 2, we discuss the settings which are considered in this papers. This includes the assumptions for the function \( \varphi \) and the operator \( L \). Then, we establish the results on the \( L^p(\mathcal{X}) \)-boundedness of two square functions which is useful in what follows.

Section 3 is dedicated to studying the Musielak-Orlicz tent spaces. Like the classical result for the tent spaces, we also give out the atomic decomposition for the Musielak-Orlicz tent spaces.

In Section 4, we first introduce the Musielak-Orlicz-Hardy space \( H^p_{\Lambda}(\mathcal{X}) \) via the Lusin-area function and prove that the operator \( \tau_{L,M} \) (see (4.2) below for its definition) maps the Musielak-Orlicz tent space \( T_{L}(\mathcal{X}_{+}) \) continuously into the space \( H^p_{\Lambda}(\mathcal{X}) \) (see Proposition 4.5 below), here and in what follows, \( \mathcal{X}_{+} := \mathcal{X} \times (0, \infty) \). By this and the atomic decomposition of the space \( T_{L}(\mathcal{X}_{+}) \), we establish the molecular characterization of \( H^p_{\Lambda}(\mathcal{X}) \) (see Theorem 4.8 below). Moreover, similar to [12, Theorem 3.4], we show that \( H^p_{\Lambda}(\mathcal{X}) \) is invariant if we replace \( L^2(\mathcal{X}) \) by \( L^q(\mathcal{X}) \) with \( q \in (p, q_{L}) \) in the definition of \( H^p_{\Lambda}(\mathcal{X}) \) (see Theorem 4.9 below). As a consequence, we see that \( L^q(\mathcal{X}) \cap H^p_{\Lambda}(\mathcal{X}) \) is dense in \( H^p_{\Lambda}(\mathcal{X}) \) whenever \( s \in (p_{L}, q_{L}) \) (see Corollary 4.10 below).

If \( L \) is a nonnegative self-adjoint operator in \( L^2(\mathcal{X}) \) satisfying the reinforced \( (p_{L}, p'_{L}, 1) \) off-diagonal estimates on balls with \( p_{L} \in [1, 2] \), in Section 5, we establish the atomic characterization of the space \( H^p_{\Lambda}(\mathcal{X}) \) (see Theorem 5.4 below) by using the finite propagation speed for the wave equation and a similar method used in Section 4.

The aim of Section 6 is to give an affirmative answer to the question "when do the Musielak-Orlicz-Hardy spaces \( H^p_{\Lambda}(\mathcal{R}^n) \) and \( H^p_{\Lambda}(\mathcal{R}^n) \) coincide?". More precisely, if the distribution kernel of the heat semigroup \( \{ e^{-t \Delta} \}_{t \geq 0} \) satisfies the Gaussian upper bound estimate, some Hölder regularity and the conservation (see Assumption (C) below for details), then the spaces \( H^p_{\Lambda}(\mathcal{R}^n) \) and \( H^p_{\Lambda}(\mathcal{R}^n) \) coincide with equivalent quasi-norms (see Theorem 6.7 below).

In Section 7, as a special case, we further study the Musielak-Orlicz-Hardy space \( H^p_{\Lambda}(\mathcal{R}^n) \) associated with the second order elliptic operator in divergence on \( \mathcal{R}^n \) with complex bounded measurable coefficients. By making full use of the special structure of the divergence form elliptic operator and establishing a good-\( \lambda \) inequality concerning the non-tangential maximal function and the truncated Lusin-area function, we obtain the radial and the non-tangential maximal function characterizations of \( H^p_{\Lambda}(\mathcal{R}^n) \) (see Theorem 7.5 below). We remark that the proof of Theorem 7.5 is similar to that of [74, Theorem 7.4] (see also the proof of [72, Proposition 3.2]). Theorem 7.5 completely covers [46, Theorem 5.2 and Corollary 5.1] by taking \( \varphi \) as in (1.1) with \( w \equiv 1 \) and \( \Phi \) concave. Moreover, we prove that the Riesz transform \( \nabla L^{-1/2} \), associated with \( L \), is bounded from \( H^p_{\Lambda}(\mathcal{R}^n) \) to \( L^p(\mathcal{R}^n) \) when \( \varphi \in [0, 1] \), from \( H^p_{\Lambda}(\mathcal{R}^n) \) to \( H^p_{\Lambda}(\mathcal{R}^n) \) when \( \varphi \in (1, \infty) \), and from \( H^p_{\Lambda}(\mathcal{R}^n) \) to the weak Musielak-Orlicz-Hardy space \( W H^p_{\Lambda}(\mathcal{R}^n) \) when \( \varphi \in (0, 1) \) and is attainable (see Theorems 7.8 and 7.11 below). We point out that Theorem 7.8 completely covers [46, Theorems 7.1 and 7.4] by taking \( \varphi \) as in (1.1) with \( w \equiv 1 \) and \( \Phi \) concave. Theorem 7.11 completely covers [19, Theorem 1.2] by taking \( \varphi \) as in (1.1) with \( w \equiv 1 \) and \( \Phi \)|\( (t) := t|^{(\alpha+1)} \) for all \( t \in [0, \infty) \).

In Section 8, we consider the Musielak-Orlicz-Hardy spaces \( H^p_{\Lambda}(\mathcal{R}^n) \) associated with the Schrödinger operator \( L := -\Delta + V \), where \( 0 \leq V \in L^1_{\text{loc}}(\mathcal{R}^n) \). Similar to Section 7, we establish several equivalent characterizations of \( H^p_{\Lambda}(\mathcal{R}^n) \) in terms of the radial and the non-tangential maximal functions associated with the heat and the Poisson semigroups of \( L \) (see Theorem 8.3 below). Moreover, we also study the boundedness of \( \nabla L^{-1/2} \) on the space \( H^p_{\Lambda}(\mathcal{R}^n) \) (see Theorems 8.5 and 8.6 below). It is worth pointing out that Theorems 8.3 and 8.5, respectively, improve [74, Theorem 7.4] and [74, Theorems 7.11 and 7.15] by extending the range of weights (see Remarks 8.4 and 8.7 below for details).

Finally we make some conventions on notation. Throughout the whole paper, we denote by \( C \) a \textit{positive constant} which is independent of the main parameters, but it may vary from line to line. We also use \( C_{(x,y,...)} \) to denote a \textit{positive constant depending on the indicated parameters} \( y, \beta, ... \). The symbol \( \mathcal{A} \subseteq \mathcal{B} \) means that \( \mathcal{A} \subseteq \mathcal{B} \) and \( \mathcal{B} \subseteq \mathcal{A} \). Then, we write \( \mathcal{A} \sim \mathcal{B} \). The symbol \( |s| \) for \( s \in \mathbb{R} \) denotes the maximal integer not more than \( s \). For any given normed spaces \( \mathcal{A} \) and \( \mathcal{B} \) with the corresponding norms \( \| \cdot \|_\mathcal{A} \) and \( \| \cdot \|_\mathcal{B} \), the symbol \( \mathcal{A} \subseteq \mathcal{B} \) means that for all \( f \in \mathcal{A} \), then \( f \in \mathcal{B} \) and \( \| f \|_\mathcal{B} \leq \| f \|_\mathcal{A} \). Also given \( \lambda > 0 \), we write \( \lambda \mathcal{B} \) for the \( \lambda \)-dilated ball, which is the ball with the same center as \( \mathcal{B} \) and with radius \( r_{\lambda \mathcal{B}} = \lambda r_{\mathcal{B}} \). We also set \( \mathbb{N} := \{ 1, 2, ... \} \) and \( \mathbb{Z}_+ := \{ 0 \} \cup \mathbb{N} \). For each ball \( \mathcal{B} \subseteq \mathcal{X} \), we set

\[
S_0(\mathcal{B}) = \mathcal{B} \quad \text{and} \quad S_j(\mathcal{B}) = 2^j \mathcal{B} \setminus 2^{j-1} \mathcal{B}
\]

for \( j \in \mathbb{N} \). For any measurable subset \( \mathcal{E} \subseteq \mathcal{X} \), we denote by \( \mathcal{E}^c \) the set \( \mathcal{X} \setminus \mathcal{E} \) and by \( \mathcal{X}^c \) its characteristic function. For any \( \varphi := (\theta_1, \ldots, \theta_N) \in \mathbb{Z}_+^N \), let \( |\varphi| := \theta_1 + \cdots + \theta_N \). For any subsets \( \mathcal{E}, \mathcal{F} \subseteq \mathcal{X} \) and \( x \in \mathcal{X} \), let

\[
d(E, F) := \inf_{x \in \mathcal{E}, y \in \mathcal{F}} d(x, y) \quad \text{and} \quad d(z, E) := \inf_{x \in E} d(z, x).
\]
For $1 \leq q \leq \infty$, we denote by $q'$ the \textit{conjugate exponent} of $q$, namely, $1/q + 1/q' = 1$. Finally, we use the notation

$$\int_B h(x)d\mu(x) := \frac{1}{\mu(B)} \int_B h(x)d\mu(x).$$

2. Preliminaries

In Subsection 2.1, we first recall some notions on metric measure spaces and then, in Subsection 2.2, we state some notions and assumptions concerning growth functions considered in this paper and give some examples which satisfy these assumptions; finally, we recall some properties of growth functions established in [49]. In Subsection 2.3, we describe some basic assumptions on the operator $L$ studied in this paper and then study the $L^p(X)$-boundedness of two square functions associated with $L$.

2.1. Metric measure spaces

Throughout the whole paper, we let $X$ be a set, $d$ a metric on $X$ and $\mu$ a nonnegative Borel regular measure on $X$. For all $x \in X$ and $r \in (0, \infty)$, let

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

and $V(x, r) := \mu(B(x, r))$. Moreover, we assume that there exists a constant $C \in [1, \infty)$ such that, for all $x \in X$ and $r \in (0, \infty)$,

$$V(x, 2r) \leq CV(x, r) < \infty. \quad (2.1)$$

Observe that $(X, d, \mu)$ is a \textit{space of homogeneous type} in the sense of Coifman and Weiss [23]. Recall that in the definition of spaces of homogeneous type in [23, Chapter 3], $d$ is assumed to be a quasi-metric. However, for simplicity, we always assume that $d$ is a metric. Notice that the doubling property (2.1) implies the following strong homogeneity property that, for some positive constants $C$ and $n$,

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \quad (2.2)$$

uniformly for all $\lambda \in [1, \infty)$, $x \in X$ and $r \in (0, \infty)$. There also exist constants $C \in (0, \infty)$ and $N \in [0, n]$ such that, for all $x, y \in X$ and $r \in (0, \infty)$,

$$V(x, r) \leq C \left[1 + \frac{d(x, y)}{r}\right]^N V(y, r). \quad (2.3)$$

Indeed, the property (2.3) with $N = n$ is a simple corollary of the triangle inequality for the metric $d$ and the strong homogeneity property (2.2). In the cases of Euclidean spaces and Lie groups of polynomial growth, $N$ can be chosen to be $0$.

Furthermore, for $p \in (0, \infty)$, the \textit{space of $p$-integrable functions on $X$} is denoted by $L^p(X)$ and the \textit{(quasi-)norm} of $f \in L^p(X)$ by $\|f\|_{L^p(X)}$.

2.2. Growth functions

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an \textit{Orlicz function} if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for $t \in (0, \infty)$ and $\lim_{t \to \infty} \Phi(t) = \infty$ (see, for example, [59, 62, 63]). The function $\Phi$ is said to be of upper type $p$ (resp. lower type $p$) for some $p \in [0, \infty)$, if there exists a positive constant $C$ such that for all $s \in [1, \infty)$ (resp. $s \in [0, 1]$) and $t \in [0, \infty)$, $\Phi(st) \leq Cs^p\Phi(t)$.

For a given function $\varphi : X \times [0, \infty) \to [0, \infty)$ such that for any $x \in X$, $\varphi(x, \cdot)$ is an Orlicz function, $\varphi$ is said to be of uniformly upper type $p$ (resp. uniformly lower type $p$) for some $p \in (0, \infty)$ if there exists a positive constant $C$ such that for all $x \in X$, $t \in [0, \infty)$ and $s \in [1, \infty)$ (resp. $s \in [0, 1]$), $\varphi(x, st) \leq Cs^p\varphi(x, t)$. We say that $\varphi$ is of positive uniformly upper type (resp. uniformly lower type) if it is of uniformly upper type (resp. uniformly lower type) for some $p \in (0, \infty)$. Moreover, let

$$I(\varphi) := \inf\{p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p\} \quad (2.4)$$
and

\[ i(\varphi) := \sup \{ \rho \in (0, \infty) : \varphi \text{ is of uniformly lower type } \rho \} \]  \hspace{1cm} (2.5)

In what follows, \( l(\varphi) \) and \( i(\varphi) \) are, respectively, called the uniformly critical upper type index and the uniformly critical lower type index of \( \varphi \). Observe that \( l(\varphi) \) and \( i(\varphi) \) may not be attainable, namely, \( \varphi \) may not be of uniformly upper type \( l(\varphi) \) and uniformly lower type \( i(\varphi) \) (see below for some examples).

Let \( \varphi : \mathcal{X} \times [0, \infty) \to [0, \infty) \) satisfy that \( x \mapsto \varphi(x, t) \) is measurable for all \( t \in [0, \infty) \). Following [49], \( \varphi(\cdot, t) \) is called uniformly locally integrable if, for all bounded sets \( K \) in \( \mathcal{X} \),

\[
\int_K \sup_{t \in [0, \infty)} \left\{ \varphi(x, t) \left( \int_K \varphi(y, t) \, d\mu(y) \right)^{-1} \right\} \, d\mu(x) < \infty.
\]

**Definition 2.1.**

Let \( \varphi : \mathcal{X} \times [0, \infty) \to [0, \infty) \) be uniformly locally integrable. The function \( \varphi(\cdot, t) \) is said to satisfy the uniformly Muckenhoupt condition for some \( q \in [1, \infty) \), denoted by \( \varphi \in A_q(\mathcal{X}) \), if, when \( q \in (1, \infty) \),

\[
A_q(\varphi) := \sup_{t \in [0, \infty)} \left\{ \varphi(y, t) \left( \int B \varphi(x, t) \, d\mu(x) \right)^{-q/q'} \right\} < \infty,
\]

where \( 1/q + 1/q' = 1 \), or

\[
A_1(\varphi) := \sup_{t \in [0, \infty)} \left\{ \varphi(x, t) \left( \sup_{y \in B} \varphi(y, t) \right)^{-1} \right\} < \infty.
\]

Here the first supremums are taken over all \( t \in [0, \infty) \) and the second ones over all balls \( B \subset \mathcal{X} \).

The function \( \varphi(\cdot, t) \) is said to satisfy the uniformly reverse Hölder condition for some \( q \in [1, \infty) \), denoted by \( \varphi \in RH_q(\mathcal{X}) \), if, when \( q \in (1, \infty) \),

\[
RH_q(\varphi) := \sup_{t \in [0, \infty)} \left\{ \varphi(x, t) \left( \int B \varphi(x, t) \, d\mu(x) \right)^{-1/q} \right\} < \infty,
\]

or

\[
RH_{\infty}(\varphi) := \sup_{t \in [0, \infty)} \left\{ \sup_{y \in B} \varphi(y, t) \left( \int B \varphi(x, t) \, d\mu(x) \right)^{-1} \right\} < \infty.
\]

Here the first supremums are taken over all \( t \in [0, \infty) \) and the second ones over all balls \( B \subset \mathcal{X} \).

Let \( A_q(\mathcal{X}) := \bigcup_{q \in [1, \infty)} A_q(\mathcal{X}) \) and define the critical indices of \( \varphi \in A_q(\mathcal{X}) \) as follows:

\[
q(\varphi) := \inf \{ q \in [1, \infty) : \varphi \in A_q(\mathcal{X}) \} \quad (2.6)
\]

and

\[
r(\varphi) := \sup \{ q \in (1, \infty) : \varphi \in RH_q(\mathcal{X}) \}.
\]

Now we introduce the notion of growth functions.

**Definition 2.2.**

A function \( \varphi : \mathcal{X} \times [0, \infty) \to [0, \infty) \) is called a growth function if the following hold true:

(i) \( \varphi \) is a Musielak-Orlicz function, namely,

\[ \text{(i)} \] the function \( \varphi(x, \cdot) : [0, \infty) \to [0, \infty) \) is an Orlicz function for all \( x \in \mathcal{X} \);
(i) the function \( \varphi(\cdot, t) \) is a measurable function for all \( t \in [0, \infty) \).

(ii) \( \varphi \in A_{\infty}(\mathcal{X}) \).

(iii) The function \( \varphi \) is of positive uniformly upper type 1 and of uniformly lower type \( p_2 \) for some \( p_2 \in (0, 1] \).

**Remark 2.3.**

From the definitions of the uniformly upper type and the uniformly lower type, we deduce that, if the growth function \( \varphi \) is of positive uniformly upper type \( p_1 \) with \( p_1 \in (0, 1] \), and of positive uniformly lower type \( p_2 \) with \( p_2 \in (0, 1] \), then \( p_1 \geq p_2 \).

Clearly, \( \varphi(x, t) := \omega(x)\Phi(t) \) is a growth function if \( \omega \in A_{\infty}(\mathcal{X}) \) and \( \Phi \) is an Orlicz function of lower type \( p \) for some \( p \in (0, 1] \) and of upper type 1. It is known that, for \( p \in (0, 1] \), \( \Phi(t) := t^p \) for all \( t \in [0, \infty) \), then \( \Phi \) is an Orlicz function of lower type \( p \) and of upper type \( p \); for \( p \in (\frac{1}{2}, 1] \), if \( \Phi(t) := t^p / \ln(e + t) \) for all \( t \in [0, \infty) \), then \( \Phi \) is an Orlicz function of lower type \( q \) for \( q \in (0, p) \) and of upper type \( p \); for \( p \in (0, \frac{1}{2}] \), if \( \Phi(t) := t^p \ln(e + t) \) for all \( t \in [0, \infty) \), then \( \Phi \) is an Orlicz function of lower type \( p \) and of upper type \( p \) for \( q \in (p, 1] \). Recall that if an Orlicz function is of upper type \( p \in (0, 1] \), then it is also of upper type 1.

Another typical and useful example of the growth function \( \varphi \) is as in (1.2). It is easy to show that \( \varphi \in A_{\infty}(\mathcal{X}) \), \( \varphi \) is of uniformly upper type \( \alpha \), \( I(\varphi) = I(\varphi) = \alpha \), \( i(\varphi) \) is not attainable, but \( I(\varphi) \) is attainable. Moreover, it is worths to point out that such function \( \varphi \) naturally appears in the study of the pointwise multiplier characterization for the \( \text{BMO} \)-type space on the metric space with doubling measure (see [60, 61]); see also [50–53] for some other applications of such functions. Throughout the whole paper, we always assume that \( \varphi \) is a growth function as in Definition 2.2. Let us now introduce the Musielak-Orlicz space.

The **Musielak-Orlicz space** \( L^\varphi(\mathcal{X}) \) is defined to be the set of all measurable functions \( f \) such that \( \int_{\mathcal{X}} \varphi(x, |f(x)|) \, d\mu(x) < \infty \) with **Luxembourg norm**

\[
\|f\|_{L^\varphi(\mathcal{X})} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathcal{X}} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) \, d\mu(x) \leq 1 \right\}.
\]

In what follows, for any measurable subset \( E \) of \( \mathcal{X} \) and \( t \in [0, \infty) \), we let

\[
\varphi(E, t) := \int_E \varphi(x, t) \, d\mu(x).
\]

The following Lemma 2.4 on the properties of growth functions is just [49, Lemmas 4.1 and 4.2].

**Lemma 2.4.**

(i) Let \( \varphi \) be a growth function as in Definition 2.2. Then \( \varphi \) is uniformly \( \alpha \)-quasi-subadditive on \( \mathcal{X} \times [0, \infty) \), namely, there exists a positive constant \( C \) such that, for all \( (x, t_j) \in \mathcal{X} \times [0, \infty) \) with \( j \in \mathbb{N} \), \( \varphi(x, \sum_{j=1}^{\infty} t_j) \leq C \sum_{j=1}^{\infty} \varphi(x, t_j) \).

(ii) Let \( \varphi \) be a growth function as in Definition 2.2. For all \( (x, t) \in \mathcal{X} \times [0, \infty) \), assume that \( \varphi(x, t) := \int_0^t \frac{\varphi(x, s)}{s} \, ds \).

Then \( \varphi \) is a growth function, which is equivalent to \( \varphi \); moreover, \( \varphi(x, \cdot) \) is continuous and strictly increasing.

(iii) Let \( \varphi \) be a growth function as in Definition 2.2. Then \( \int_{\mathcal{X}} \varphi(x, \frac{|f(x)|}{\|f\|_{L^\varphi(\mathcal{X})}}) \, d\mu(x) = 1 \) for all \( f \in L^\varphi(\mathcal{X}) \setminus \{0\} \).

We have the following properties for \( A_{\infty}(\mathcal{X}) \), whose proofs are similar to those in [34, 35].

**Lemma 2.5.**

(i) \( A_1(\mathcal{X}) \subset A_p(\mathcal{X}) \subset A_q(\mathcal{X}) \) for \( 1 \leq p < q < \infty \).

(ii) \( RH_{\infty}(\mathcal{X}) \subset RH_p(\mathcal{X}) \subset RH_q(\mathcal{X}) \) for \( 1 < q \leq p \leq \infty \).

(iii) If \( \varphi \in A_p(\mathcal{X}) \) with \( p \in (1, \infty) \), then there exists \( q \in (1, p) \) such that \( \varphi \in A_q(\mathcal{X}) \).
(iv) If \( \varphi \in \mathcal{RH}_q(\mathcal{X}) \) with \( q \in (1, \infty) \), then there exists \( p \in (q, \infty) \) such that \( \varphi \in \mathcal{RH}_p(\mathcal{X}) \).

(v) \( \mathcal{A}_\infty(\mathcal{X}) = \cup_{p \in [1, \infty)} \mathcal{A}_p(\mathcal{X}) \subseteq \cup_{q \in (1, \infty]} \mathcal{RH}_q(\mathcal{X}) \).

(vi) If \( p \in (1, \infty) \) and \( \varphi \in \mathcal{A}_p(\mathcal{X}) \), then there exists a positive constant \( C \) such that, for all measurable functions \( f \) on \( \mathcal{X} \) and \( t \in [0, \infty) \),

\[
\int_\mathcal{X} |\mathcal{M}(f)(x)|^p \varphi(x, t) \, d\mu(x) \leq C \int_\mathcal{X} |f(x)|^p \varphi(x, t) \, d\mu(x),
\]

where \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function on \( \mathcal{X} \), defined by setting, for all \( x \in \mathcal{X} \),

\[
\mathcal{M}(f)(x) := \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),
\]

where the supremum is taken over all balls \( B \ni x \).

(vii) If \( \varphi \in \mathcal{A}_p(\mathcal{X}) \) with \( p \in [1, \infty) \), then there exists a positive constant \( C \) such that, for all balls \( B_1, B_2 \subset \mathcal{X} \) with \( B_1 \subset B_2 \) and \( t \in [0, \infty) \),

\[
\frac{\mu(B_2 \setminus B_1)}{\mu(B_1)} \leq C \frac{\mu(B_2)}{\mu(B_1)}^{1-1/p}.
\]

(viii) If \( \varphi \in \mathcal{RH}_q(\mathcal{X}) \) with \( q \in (1, \infty] \), then there exists a positive constant \( C \) such that, for all balls \( B_1, B_2 \subset \mathcal{X} \) with \( B_1 \subset B_2 \) and \( t \in [0, \infty) \),

\[
\frac{\mu(B_2 \setminus B_1)}{\mu(B_1)} \geq C \frac{\mu(B_2)}{\mu(B_1)}^{1-1/q}.
\]

**Remark 2.6.**

By Lemma 2.5(iii), we see that if \( q(\varphi) \in (1, \infty) \), then \( \varphi \notin \mathcal{A}_q(\mathcal{X}) \). Moreover, there exists \( \varphi \notin \mathcal{A}_1(\mathcal{X}) \) such that \( q(\varphi) = 1 \) (see, for example, [48]). Similarly, if \( r(\varphi) \in (1, \infty) \), then \( \varphi \notin \mathcal{RH}_{1/q}(\mathcal{X}) \), and there exists \( \varphi \notin \mathcal{RH}_\infty(\mathcal{X}) \) such that \( r(\varphi) = \infty \) (see, for example, [26]).

### 2.3. Two assumptions on the operator \( L \)

Before giving the assumptions on operators \( L \), we first recall some notions of bounded holomorphic functional calculus introduced by McIntosh [58].

For \( \theta \in [0, \pi) \), the open and closed sectors, \( S_0^\theta \) and \( S_\theta \), of angle \( \theta \) in the complex plane \( \mathbb{C} \) are defined, respectively, by setting \( S_0^\theta := \{ z \in \mathbb{C} : |\arg z| < \theta \} \) and \( S_\theta := \{ z \in \mathbb{C} : |\arg z| \leq \theta \} \). Let \( \omega \in [0, \pi) \). A closed operator \( T \) in \( L^2(\mathcal{X}) \) is said to be of type \( \omega \), if

(i) the spectrum of \( T, \sigma(T) \), is contained in \( S_\omega \);

(ii) for each \( \theta \in (\omega, \pi) \), there exists a nonnegative constant \( C \) such that, for all \( z \in \mathbb{C} \setminus S_\omega \),

\[
|||T - zI|||_{L^2(\mathcal{X})} \leq C|z|^{-1},
\]

where above and in what follows, for any normed linear space \( \mathcal{H} \), \( |||S|||_{\mathcal{L}(\mathcal{H})} \) denotes the **operator norm** of the linear operator \( S : \mathcal{H} \rightarrow \mathcal{H} \).

For \( \mu \in [0, \pi] \) and \( \sigma, \tau \in (0, \infty) \), let \( H(S_0^\mu) := \{ f : f \text{ is a holomorphic function on } S_0^\mu \} \),

\[
H_\infty(S_0^\mu) := \{ f \in H(S_0^\mu) : ||f||_{L^\infty(S_0^\mu)} < \infty \}
\]

and

\[
\Psi_{\sigma, \tau}(S_0^\mu) := \{ f \in H(S_0^\mu) : \text{there exists a positive constant } C \text{ such that for all } \xi \in S_\sigma^\mu, ||f(\xi)|| \leq C \inf\{|\xi|^\sigma, |\xi|^{-\tau}\} \}.
\]

It is known that every one-to-one operator \( T \) of type \( \omega \) in \( L^2(\mathcal{X}) \) has a unique holomorphic functional calculus (see, for example, [58]). More precisely, let \( T \) be a one-to-one operator of type \( \omega \), with \( \omega \in [0, \pi) \), \( \mu \in (\omega, \pi) \), \( \sigma, \tau \in (0, \infty) \),
and \( f \in \Psi_{s, \mathfrak{T}}(S^0_\mu) \). The function of the operator \( T \), \( f(T) \), can be defined by the \( H_\infty \)-functional calculus in the following way,

\[
f(T) := \frac{1}{2\pi i} \int_{\Gamma} (\xi I - T)^{-1} f(\xi) \, d\xi,
\]

(2.8)

where \( \Gamma := \{ re^{i\nu} : \infty > r > 0 \} \cup \{ re^{-i\nu} : 0 < r < \infty \} \), \( \nu \in (\omega, \mu) \), is a curve consisting of two rays parameterized anti-clockwise. It is known that \( f(T) \) in (2.8) is independent of the choice of \( \nu \in (\omega, \mu) \) and the integral in (2.8) is absolutely convergent in \( \| f \|_{L^2(S^0_\mu)} \) (see [38, 58]).

In what follows, we always assume \( \omega \in [0, \pi/2] \). Then, it follows, from [38, Proposition 7.1.1], that for every operator \( T \) of type \( \omega \) in \( L^2(\mathcal{X}) \), \( -T \) generates a holomorphic \( C_0 \)-semigroup \( \{ e^{-tT} \} \in S^0_{\pi/2-\omega} \) on the open sector \( S^0_{\pi/2-\omega} \), such that \( \| e^{-zT} \|_{L^1(S^0_\mu)} \leq 1 \) for all \( z \in S^0_{\pi/2-\omega} \) and, moreover, every nonnegative self-adjoint operator is of type 0.

Let \( \Psi(S^0_\mu) := \cup_{\sigma, \tau > 0} \Psi_{s, \mathfrak{T}}(S^0_\mu) \). It is well known that the above holomorphic functional calculus defined on \( \Psi(S^0_\mu) \) can be extended to \( H_\infty(S^0_\mu) \) via a limit process (see [58]). Recall that, for \( \mu \in (0, \pi) \), the operator \( T \) is said to have a bounded \( H_\infty(S^0_\mu) \) functional calculus in the Hilbert space \( \mathcal{H} \), if there exists a positive constant \( C \) such that, for all \( \psi \in H_\infty(S^0_\mu) \), \( \| \psi(T) \|_{L^1(\mathcal{H})} \leq C \| \psi \|_{L^1(S^0_\mu)} \) and \( T \) is said to have a bounded \( H_\omega \) functional calculus in the Hilbert space \( \mathcal{H} \), if there exists \( \mu \in (0, \pi) \) such that \( T \) has a bounded \( H_\infty(S^0_\mu) \) functional calculus.

For any given \( f \in L^1_{L^2}(\mathcal{X}) \), each ball \( B \subset \mathcal{X} \) and \( j \in \mathbb{Z}_+ \), let

\[
\int_{S_j(B)} |f(x)| d\mu(x) := \frac{1}{\mu(2B)} \int_{S_j(B)} |f(x)| d\mu(x).
\]

Now we recall the notion of \( L^p - L^q \) off-diagonal estimates on balls, which was first introduced in [7].

**Definition 2.7.**

Let \( k \in \mathbb{N} \), \( p, q \in [1, \infty) \) with \( p \leq q \), and \( \{ A_t \}_{t \geq 0} \) be a family of sublinear operators. The family \( \{ A_t \}_{t \geq 0} \) is said to satisfy \( L^p - L^q \) off-diagonal estimates on balls of order \( m \), denoted by \( A_t \in \mathcal{O}_m(L^p - L^q) \), if there exist constants \( \gamma, \theta_1, \theta_2 \in [0, \infty) \) and \( C, \epsilon \in (0, \infty) \) such that, for all \( t \in (0, \infty) \) and all balls \( B \subset \mathcal{X} \) and \( f \in L^1_{L^2}(\mathcal{X}) \),

\[
\left\{ \int_B |A_t(\chi_B f)(x)|^q d\mu(x) \right\}^{1/q} \leq C \left\{ \int_{S_j(B)} |f(x)|^p d\mu(x) \right\}^{1/p},
\]

(2.9)

and, for all \( j \in \mathbb{N} \) with \( j \geq 3 \),

\[
\left\{ \int_{S_j(B)} |A_t(\chi_B f)(x)|^q d\mu(x) \right\}^{1/q} \leq C 2^{\theta_1} \left\{ \int_{2^{j/2}B} |f(x)|^p d\mu(x) \right\}^{\theta_2} e^{-\frac{\gamma t}{(2^{j/2}r_{\mathcal{X}}(2^{j/2}r_{\mathcal{X}} - 1)}} \left\{ \int_{S_j(B)} |f(x)|^p d\mu(x) \right\}^{1/p},
\]

(2.10)

and

\[
\left\{ \int_B |A_t(\chi_{S_j(B)} f)(x)|^q d\mu(x) \right\}^{1/q} \leq C 2^{\theta_1} \left\{ \int_{2^{j/2}B} |f(x)|^p d\mu(x) \right\}^{\theta_2} e^{-\frac{\gamma t}{(2^{j/2}r_{\mathcal{X}}(2^{j/2}r_{\mathcal{X}} - 1))}} \left\{ \int_{S_j(B)} |f(x)|^p d\mu(x) \right\}^{1/p},
\]

(2.11)

where \( \gamma(s) := \max\{ s, \frac{1}{s} \} \) for all \( s \in (0, \infty) \).

Similar to the comments below [7, Definition 2.1], we have the following properties on \( \mathcal{O}_m(L^p - L^q) \).
Remark 2.8.

(i) It is easy to see that, for \( p \leq p_1 \leq q_1 \leq q \),
\[
\mathcal{O}_m(L^p - L^q) \subset \mathcal{O}_m(L^{p_1} - L^{q_1}).
\]

(ii) Similar to [7, Proposition 2.2], we see that \( A_t \in \mathcal{O}_m(L^1 - L^\infty) \) if and only if the associated kernel \( p_t \) of \( A_t \) satisfies the Gaussian upper bound, namely, there exist positive constants \( c \) and \( C \) such that, for all \( x, y \in \mathcal{X} \) and \( t \in (0, \infty) \),
\[
|p_t(x, y)| \leq \frac{C}{V(x, t^{1/m})} \exp \left(-c \frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} \right).
\]

(iii) \( A_t \in \mathcal{O}_m(L^p - L^q) \) if and only if its dual, \( A_t^* \), belongs to \( \mathcal{O}_m(L^{q'} - L^{p'}) \).

Now, we make the following two assumptions on operators \( L \), which are used through the whole paper.

**Assumption (A).** Assume that the operator \( L \) is a one-to-one operator of type \( \omega \) in \( L^2(\mathcal{X}) \) with \( \omega \in [0, \pi/2) \), has dense range in \( L^2(\mathcal{X}) \) and a bounded \( H_\infty \)-functional calculus in \( L^2(\mathcal{X}) \).

**Assumption (B).** Let \( m \in \mathbb{N} \). Assume that there exist \( p_1 \in [1, 2) \) and \( q_1 \in (2, \infty] \), depending on \( L \), such that the family \( \{(tL_t^k e^{-t\lambda})\}_{t>0} \), with \( k \in \mathbb{Z}_+ \), satisfies the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls, namely, for all \( t \in (0, \infty) \) and \( p, q \in (p_1, q_1) \) with \( p \leq q \), \( (tL_t^k e^{-t\lambda}) \in \mathcal{O}_m(L^p - L^q) \).

Remark 2.9.

(a) We first point out that in Assumptions (A) and (B), if \( L \) is non-negative self-adjoint, \( \mathcal{X} \) is the Euclidean space \( \mathbb{R}^n \) and \( m = 1 \), from [7, Proposition 3.2], it follows that the notion of the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls is the same as the reinforced \((p_1, q_1)\) off-diagonal estimates introduced in [12] (see [11, 27, 32] and their references for the history of the off-diagonal estimates). Here, we use the off-diagonal estimates on balls, because they coincide with the off-diagonal estimates when \( \mathcal{X} = \mathbb{R}^n \), and the off-diagonal estimates on balls seem more suitable in a general space of homogeneous type. For example, the heat semigroup \( e^{-t\lambda} \) on functions for general Riemannian manifolds with a doubling measure is not \( L^p - L^q \) bounded when \( p < q \) unless the measure of any ball is bounded below by a power of its radius. However, if we assume the \( L^p - L^q \) off-diagonal estimates, it then implies the \( L^p - L^q \) boundedness (see also the discussions above [7, Proposition 3.2]).

(b) Denote by \( L^* \) the adjoint operator of \( L \) in \( L^2(\mathcal{X}) \). Let \( p_1, q_1 \) be as in Assumption (B), \( m \in \mathbb{N} \) and \( k \in \mathbb{N} \). If \( (tL_t^k e^{-t\lambda}) \) satisfies the reinforced \((p_1, q_1, m)\) off-diagonal estimates on balls, then \( (tL_t^k e^{-t\lambda}) \) also satisfies the reinforced \((q_1, p_1, m)\) off-diagonal estimates on balls. Recall that, for any \( p \in [1, \infty] \), \( 1/p + 1/p' = 1 \).

(c) Examples of operators which satisfy Assumptions (A) and (B) include:

(i) the second order divergence form elliptic operators with complex bounded coefficients as in [40] (see also (7.1) below for its precise definition);

(ii) the \( 2m \)-order homogeneous divergence form elliptic operators

\[ (-1)^m \sum_{|\alpha| = m} \partial^{\alpha} (a_{\alpha, \beta} \partial^\beta) \]

interpreted in the usual weak sense via a sesquilinear form, with complex bounded measurable coefficients \( a_{\alpha, \beta} \) for all multi-indices \( \alpha \) and \( \beta \) (see, for example, [10, 18]);

(iii) the Schrödinger operator \(-\Delta + V\) on \( \mathbb{R}^n \) with the nonnegative potential \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) (see, for example, [39, 45] and related references).
Let \( L \) be a Schrödinger operator \(-\Delta + V\) on \( \mathbb{R}^n \) with the suitable real potential \( V \) as in [3];

(v) the nonnegative self-adjoint operators satisfying Gaussian upper bounds, namely, there exist positive constants \( C \) and \( c \) such that, for all \( x, y \in \mathcal{X} \) and \( t \in (0, \infty) \),

\[
|p_t(x, y)| \leq \frac{C}{V(x, t^{1/2})} \exp \left\{-c \left[ \frac{d(x, y)\langle 2m(2m-1) \rangle}{t^{1/2}(2m-1)} \right]^2 \right\},
\]

where \( p_t \) is the associated kernel of \( e^{-tL} \) and \( m \in \mathbb{N} \);

(d) We point out that the condition that \( L \) is one-to-one is necessary for the bounded \( H_{\infty} \) functional calculus on \( L^2(\mathcal{X}) \) (see [25, 58]). Moreover, from [25, Theorem 2.3], it follows that if \( T \) is a one-to-one operator of type \( \omega \) in \( L^2(\mathcal{X}) \), then \( T \) has dense domain and dense range;

(e) If \( L \) is nonnegative self-adjoint on \( L^2(\mathcal{X}) \) satisfying the reinforced \( (p_L, p_L', m) \) off-diagonal estimates on balls, then the condition that \( L \) is one-to-one can be removed and we can introduce another kind of functional calculus by using the spectral theorem. More precisely, in this case, for every bounded Borel function \( F : [0, \infty) \to \mathbb{C} \), we define the operator \( F(L) : L^2(\mathcal{X}) \to L^2(\mathcal{X}) \) by the formula

\[
F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda),
\]

where \( E_L(\lambda) \) is the spectral resolution of \( L \) (see [39] for more details). Observe also that a one-to-one nonnegative self-adjoint operator is of type 0.

Assume that the operator \( L \) satisfies Assumptions (A) and (B). For all \( k \in \mathbb{N} \), the vertical square function \( G_{L,k} \) is defined by setting, for all \( f \in L^2(\mathcal{X}) \) and \( x \in \mathcal{X} \),

\[
G_{L,k}(f)(x) := \left\{ \int_0^\infty \left| (t^{2m}L)^k e^{-t^{2m}L} f(x) \right|^2 \, dt \right\}^{1/2},
\]

which is bounded on \( L^2(\mathcal{X}) \) (see, for example, [58]). When \( k = 1 \), we write \( G_L \) instead of \( G_{L,1} \).

**Theorem 2.10.**

Let \( L \) satisfy Assumptions (A) and (B), \( k \in \mathbb{N} \) and, \( p_L \) and \( q_L \) be as in Assumption (B). Then \( G_{L,k} \) is bounded on \( L^p(\mathcal{X}) \) for all \( p \in (p_L, q_L) \).

To prove Theorem 2.10, we need the following two criteria, which are due to [4] (see also [6]).

**Lemma 2.11.**

Let \( p_0 \in [1, 2) \) and \( \{A_i\}_{i \geq 0} \) be a family of linear operators acting on \( L^2(\mathcal{X}) \). Suppose that \( T \) is a sublinear operator of strong type \((2, 2)\). Assume that there exists a sequence \( \{\alpha(j)\}_{j \geq 2} \) of positive numbers such that, for all balls \( B := B(x_0, r_0) \) and \( f \in L^{p_0}(\mathcal{X}) \) supported in \( B \),

\[
\left\{ \int_{S(x_0, r_0)} |T(f - A_i f)(x)|^2 \, d\mu(x) \right\}^{1/2} \leq \alpha(j) \left\{ \int_B |f(x)|^{p_0} \, d\mu(x) \right\}^{1/p_0}
\]

when \( j \geq 3 \), and

\[
\left\{ \int_{S(x_0, r_0)} |A_i f(x)|^2 \, d\mu(x) \right\}^{1/2} \leq \alpha(j) \left\{ \int_B |f(x)|^{p_0} \, d\mu(x) \right\}^{1/p_0}
\]

when \( j \geq 2 \). If \( \sum_{j \geq 2} \alpha(j) 2^{nj} < \infty \), then \( T \) is of weak type \((p_0, p_0)\).
Lemma 2.12.
Let \( p_0 \in (2, \infty) \) and \( \{A_i\}_{i=0} \) be a family of linear operators acting on \( L^2(\mathcal{X}) \). Suppose that \( T \) is a sublinear operator acting on \( L^2(\mathcal{X}) \). Assume that there exists a positive constant \( C \) such that, for all balls \( B := B(x_B, r_B), y \in B \) and \( f \in L^p(\mathcal{X}) \) supported in \( B \),
\[
\left\{ \int_B |T(I - A_y)f(x)|^p \, d\mu(x) \right\}^{1/p_0} \leq C \left[ M(|f|^p)(y) \right]^{1/2}
\]
and
\[
\left\{ \int_B |TA_yf(x)|^p \, d\mu(x) \right\}^{1/p_0} \leq C \left[ M(|Tf|^p)(y) \right]^{1/2},
\]
where \( M \) denotes the Hardy–Littlewood maximal function (see Lemma 2.5(v)). Then \( T \) is of strong type \((p_0, p_0)\).

Now we prove Theorem 2.10 by using Lemmas 2.11 and 2.12.

Proof of Theorem 2.10.
For the sake of simplicity, we only give the proof for \( k = 1 \). Since \( G_\lambda \) is bounded on \( L^2(\mathcal{X}) \), we can assume that \( p_1 < 2 < q_1 \). We now consider the following two cases.

Case 1. \( p \in (p_1, 2) \).
In this case, we apply Lemma 2.11 with \( A_B := I - (I - e^{-r_B^2}t)^M, M \in \mathbb{N} \) and \( M > (p + \theta_1)/2m \). From Assumption (B), we deduce that \( e^{-I_B} \in \mathcal{O}_M(L^p = L^2) \). Thus, (2.12) holds. It remains to show that for all \( j \in \mathbb{N} \) with \( j \geq 3 \), balls \( B \) and \( f \in L^p(\mathcal{X}) \) supported in \( B \), it holds that
\[
\left\{ \int_{S_j(0)} |G_\lambda(I - e^{-r_B^2}t)^Mf(x)|^2 \, d\mu(x) \right\}^{1/2} \leq 2^{-j(nM - \theta_1)} \left\{ \int_B |f(x)|^p \, d\mu(x) \right\}^{1/p}.
\]

First, we write
\[
\left\{ \int_{S_j(0)} |G_\lambda(I - e^{-r_B^2}t)^Mf(x)|^2 \, d\mu(x) \right\}^{1/2} = \left\{ \int_{S_j(0)} \int_0^{\infty} t^{2m} e^{-r_B^2t}(I - e^{-r_B^2}t)^Mf(x) \, dt \, d\mu(x) \right\}^{1/2} \leq \left\{ \int_{S_j(0)} \int_0^{2\pi/B} \int_0^{2\pi/B} \int_0^{2\pi/B} \cdots \int_0^{2\pi/B} \right\}^{1/2}.
\]

We first estimate 1. Write
\[
(I - e^{-r_B^2}t)^M = \int_0^{2\pi/B} \int_0^{2\pi/B} \cdots \int_0^{2\pi/B} L^M e^{-(s1 + \cdots + sM)t} \, ds_1 \cdots ds_M.
\]

Thus,
\[
1 \leq \left\{ \int_{S_j(0)} \int_0^{2\pi/B} \int_0^{2\pi/B} \int_0^{2\pi/B} \cdots \int_0^{2\pi/B} t^{2m} L^{M+1} e^{-(s1 + \cdots + sM)t} \, ds_1 \cdots ds_M \, dt \, d\mu(x) \right\}^{1/2}.
\]

By the fact that \( (t^{2m} + s1 + \cdots + sM)L^{M+1} e^{-(s1 + \cdots + sM)t} \in \mathcal{O}_M(L^p - L^2) \), we know that
\[
\left\{ \int_{S_j(0)} t^{2m} L^{M+1} e^{-(s1 + \cdots + sM)t} f(x) \, d\mu(x) \right\}^{1/2}
\]
Likewise, we also see that Case 2).

Thus, (2.19) holds and hence (2.18) holds true. By this and Lemma 2.12, we see that

\[ G \]

Let

\[ I \]

is equivalent to that, for all

\[ \ell \]

claim that, for all

\[ x \]

\[ B \]

\[ \mathcal{M} \]

\[ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \]

\[ \left\{ \frac{f_{B}}{|B|} \right\} \left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/p} \]

which, together with Minkowski’s integral inequality and (2.16), implies that

\[
1 \lesssim \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \lesssim 2^{-2|\ell|} \int_{0}^{2\pi} |f(x)|^{p} d\mu(x) .
\]

(2.17)

Likewise, we also see that

\[ \mathcal{M} \]

This, combined with (2.15) and (2.17), shows that (2.14) holds as long as \( M > (n + \theta)_{1}/2m \). Thus, as a direct consequence of Lemma 2.12 and interpolation, we conclude that \( G_{\ell} \) is bounded on \( L^{p}(X) \) for all \( p \in (p_{\ell}, 2) \).

Case 2). \( p \in (2, q_{\ell}) \).

In this case, we first prove (2.13) for \( T := G_{\ell} \) and \( A_{B} := I - (I - e^{-i\phi_{B}^{2m}})^{M} \) with \( M > (n + \theta)_{1}/2m \). To do this, we decompose \( I = \sum_{j=0}^{N} I_{j} \), where for each \( j \in \mathbb{Z}_{5}, I_{j} := I_{X_{5}^{2}} \). Then, by an argument similar to that used in the proof of Case 1, we conclude that (2.13) holds true for \( T := G_{\ell} \) and \( A_{B} := I - (I - e^{-i\phi_{B}^{2m}})^{M} \) with \( M > (n + \theta)_{1}/2m \). We now claim that, for all \( f \in L^{2}(X) \), balls \( B := B(x, r_{B}) \) and \( y \in B \),

\[
\left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/p} \lesssim \mathcal{M} \left( \| T f \|^{2} \right) \left( g \right)^{1/2} .
\]

(2.18)

Since \( I - (I - e^{-i\phi_{B}^{2m}})^{M} = \sum_{j=1}^{M} c_{j,M} e^{-i\phi_{B}^{2m}} \), where \( \{ c_{j,M} \}_{j=0}^{M} \) are constants depending on \( j \) and \( M \), it follows that (2.18) is equivalent to that, for all \( j \in \{ 1, \ldots, M \} \),

\[
\left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/p} \lesssim \mathcal{M} \left( \| T f \|^{2} \right) \left( g \right)^{1/2} .
\]

(2.19)

To see this, by Minkowski’s inequality, we conclude that

\[
\left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/p} \lesssim \left\{ \frac{f_{B}}{|B|} \right\} \left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/2} .
\]

Let \( g := t^{2m}Le^{-i\theta_{B}}f \) and \( g_{i} := g_{X_{5}(B)} \) with \( i \in \mathbb{Z}_{5} \). Then, from the fact that \( e^{-i\phi_{B}^{2m}} \in \mathcal{O}(L^{2} - L^{p}) \), we deduce that, for all \( y \in B \),

\[
\left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |f(x)|^{p} d\mu(x) \right\}^{1/p} \lesssim \left\{ \int_{0}^{2\pi} \sum_{k \in \mathbb{Z}_{5}} \left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} 2^{|a|} 2^{|b|} 2^{|c|} |g_{i}(x)|^{p} d\mu(x) \right\}^{1/2} dt \right\} \lesssim \mathcal{M} \left( \| T f \|^{2} \right) \left( g \right)^{1/2} .
\]

Thus, (2.19) holds and hence (2.18) holds true. By this and Lemma 2.12, we see that \( G_{\ell} \) is bounded on \( L^{p}(X) \) for all \( p \in (2, q_{\ell}) \), which completes the proof of Theorem 2.10.
For all $k \in \mathbb{N}$, the non-tangential square functions $S_{L,k}$ is defined by setting, for all $f \in L^2(\mathcal{X})$ and $x \in \mathcal{X}$,

$$S_{L,k}(f)(x) := \left\{ \int_0^{\infty} \int_{B(x,T)} \left| (t^{2L}e^{-t^2}f(y))^{2} \frac{d\mu(y)}{V(x,T)} \right|^2 \frac{dt}{T} \right\}^{1/2}. \quad (2.20)$$

In particular case $k = 1$, we omit the subscript $k$ to write $S_L$. It is easy to show that, for all $f \in L^2(\mathcal{X})$, $||S_{L,k}(f)||_{L^2(\mathcal{X})} \lesssim ||G_{L,k}(f)||_{L^2(\mathcal{X})}$ and hence $S_{L,k}$ is bounded on $L^2(\mathcal{X})$. Moreover, we have the following boundedness of $S_{L,k}$ on $L^p(\mathcal{X})$.

**Theorem 2.13.** Let $L$ satisfy Assumptions (A) and (B), $k \in \mathbb{N}$ and $p_L$ and $q_L$ be as in Assumption (B). Then $S_{L,k}$ is bounded on $L^p(\mathcal{X})$ for all $p \in (p_L, q_L)$.

**Proof.** Without the loss of generality, we may assume that $k = 1$. Similar to the proof of Theorem 2.10, we also consider the following two cases for $p$.

**Case 1.** $p \in (p_L, 2)$. In this case, we apply Theorem 2.11 to this situation for $T := S_L$ and $A_{q_L} := I - (I - e^{-i\theta_L})^M$ with $M > (2n + \theta_L)/2m$. Due to the fact that $(tL)^k e^{-t^2} \in C_0(\ell^p - L^2)$ for all $k \in \mathbb{Z}_+$, we only need to show (2.12), namely, for all $j \in \mathbb{N}$ with $j \geq 3$, balls $B := B(x_0, d_B)$ and $I \in L^p(\mathcal{X})$ supported in $B$, it holds that

$$\left\{ \int_{S_{j}(B)} | S_L(I - e^{-i\theta_L}I)^M f(x) |^2 \frac{d\mu(x)}{V(x,T)} \right\}^{1/2} \lesssim 2^{-j(2mM - \theta_L)} \left\{ \int_{\mathcal{X}} |f(x)|^p \frac{d\mu(x)}{V(x,T)} \right\}^{1/p}. \quad (2.21)$$

To show this, we first write

$$\int_{S_{j}(B)} | S_L(I - e^{-i\theta_L}I)^M f(x) |^2 \frac{d\mu(x)}{V(x,T)} = \int_{S_{j}(B)} \int_0^{d_B} \int_{B(x,T)} \left| (t^{2L}e^{-t^2}f(y))^{2} \frac{d\mu(y)}{V(x,T)} \right|^2 \frac{dt}{T} \frac{d\mu(x)}{V(x,T)} + \int_{S_{j}(B)} \int_0^{d_B} \cdots =: l_1 + l_2.$$

Let us first estimate $l_1$. Let

$$F_j(B) := \left\{ x \in \mathcal{X} : \text{ there exists } x \in S_j(B) \text{ such that } d(x,z) < \frac{d(x,x_0)}{4} \right\}.$$

Then $F_j(B) \subset S_{j-1}(B) \cup S_j(B) \cup S_{j+1}(B) =: U_j(B)$. This, together with the fact that $\int_{d(x_0) \leq r} \frac{d\mu(x)}{V(x,T)} \lesssim 1$, implies that

$$l_1 \leq \frac{1}{\mu(2B)} \int_{F_j(B)} \int_0^{d_B} \int_{B(x,T)} \left| (t^{2L}e^{-t^2}f(y))^{2} \frac{d\mu(y)}{V(x,T)} \right|^2 \frac{dt}{T} \frac{d\mu(x)}{V(x,T)} \lesssim \frac{1}{\mu(2B)} \int_{U_j(B)} \int_0^{d_B} \left| (t^{2L}e^{-t^2}f(y))^{2} \frac{d\mu(y)}{V(x,T)} \right|^2 \frac{dt}{T} \frac{d\mu(y)}{V(x,T)}.$$

At this stage, by an argument used in Case 1) of the proof of Theorem 2.10, we conclude that

$$l_1 \lesssim 2^{-j(2mM - \theta_L)} \left\{ \int_{\mathcal{X}} |f(x)|^p \frac{d\mu(x)}{V(x,T)} \right\}^{2/p}.$$

Likewise, for $l_2$, we write

$$l_2 \leq \int_{S_{j}(B)} \int_0^{d_B} \int_{B(x,T)} \left| (t^{2L}e^{-t^2}f(y))^{2} \frac{d\mu(y)}{V(x,T)} \right|^2 \frac{dt}{T} \frac{d\mu(x)}{V(x,T)}$$
\[
\begin{align*}
L & \leq \frac{1}{\mu(2/B)} \int_{2^{-1}B}^{\infty} \int_{X} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} dt \\
L & \leq \frac{1}{\mu(2/B)} \int_{2^{-1}B}^{\infty} \int_{\mathbb{R}^n} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} dt \\
& \quad + \sum_{j=2}^{\infty} \frac{1}{\mu(2/B)} \int_{2^{-1}B}^{\infty} \int_{S_j(B)} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} dt =: K + \sum_{j=2}^{\infty} H_j,
\end{align*}
\]

where \( B := B(x_B, t) \).

Notice that in this situation, \( B \subset B_t \), and hence \( f = f \chi_{B_t} \). By an argument similar to that used in the proof of Theorem 2.10 and the fact that \((tL)^k e^{-t} \in \mathcal{O}_m(L^p - L^2)\) for all \( k \in \mathbb{Z}_+ \), we see that

\[
\begin{align*}
\left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} & \\
& \leq \int_0^{\infty} \cdots \int_0^{\infty} \left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} d\mathbf{s} \\
& \leq \int_0^{\infty} \cdots \int_0^{\infty} \left[ \frac{t^{2n}}{(2\mathcal{R})^n} \right]^{1/2} \left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} d\mathbf{s} \\
& \leq \frac{2^{\mathbf{s}_0}}{t^{2n}} \left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} \\
& \quad \times \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{1/p} \lesssim 2^{-j(4m-n)} \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{1/p}.
\end{align*}
\]

Likewise, for all \( j \in \mathbb{N} \) with \( j \geq 2 \), we have

\[
\begin{align*}
\left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} & \\
& \leq \int_0^{2\mathcal{R}} \cdots \int_0^{2\mathcal{R}} \left\{ \int_{B_t} \left| t^{2n} L e^{-tMf(y)} \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} d\mathbf{s} \\
& \leq 2^{\mathbf{s}_0} \left( \frac{2\mathcal{R}}{t} \right)^{2n} \times 2^{-j(4m-n)} \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{1/p}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
H_j & \lesssim 2^{-j(4m-n)} \int_0^{2\mathcal{R}} \cdots \int_0^{2\mathcal{R}} \frac{\mu(B_t)}{\mu(B)} \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{2/p} \\
& \lesssim 2^{-j(4m-n)} \int_0^{2\mathcal{R}} \cdots \int_0^{2\mathcal{R}} \frac{2^j t^{-n}}{2\mathcal{R}} \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{2/p} \\
& \lesssim 2^{-j} 2^{-j(4m-n)} \left\{ \int_{B_t} \left| f(x) \right|^p d\mu(x) \right\}^{2/p},
\end{align*}
\]
where \( n \) is the dimension of \( \mathcal{X} \) appearing in (2.2).

From these estimates of \( K \) and \( H_j \), we deduce that (2.21) holds and hence \( S_1 \) is bounded on \( L^p(\mathcal{X}) \) for all \( p \in (p_L, 2) \).

**Case 2.** \( p \in (2, q_1) \).

In this case, for any \( h \in L^{p/(2)}(\mathcal{X}) \), from Fubini’s theorem and Hölder’s inequality, we infer that

\[
\int_{\mathcal{X}} |S_1 f(x)|^2 h(x) \, d\mu(x) = \int_{\mathcal{X}} \int_0^\infty \int_{B(x,t)} \left| t^{n} e^{-t^{n+1} f(y)} \right|^2 h(x) \, \frac{d\mu(y)}{V(x,t)} \, \frac{dt}{t} \, d\mu(x) \\
\leq \int_{\mathcal{X}} \int_0^\infty \left| t^{n} e^{-t^{n+1} f(y)} \right|^2 \mathcal{M}(h)(y) \, \frac{dt}{t} \, d\mu(y) \\
\leq \int_{\mathcal{X}} |G_1 f(x)|^2 \mathcal{M}(h)(y) \, d\mu(y) \lesssim \|G_1 f\|_{L^p(\mathcal{X})}^2 \|\mathcal{M}(h)\|_{L^{p/(2)}(\mathcal{X})}.
\]

At this stage, using Theorem 2.10 and the fact that \( \mathcal{M} \) is bounded on \( L^p(\mathcal{X}) \) for all \( p \in (1, \infty) \), we conclude that

\[
\int_{\mathcal{X}} |S_1 f(x)|^2 h(x) \, d\mu(x) \lesssim \|f\|_{L^p(\mathcal{X})}^2 \|h\|_{L^{p/(2)}(\mathcal{X})},
\]

which implies that \( S_1 \) is bounded on \( L^p(\mathcal{X}) \) for all \( p \in (2, q_1) \) and hence completes the proof of Theorem 2.13.

\[\Box\]

### 3. Musielak-Orlicz tent spaces

In this section, we study the Musielak–Orlicz tent space associated with the growth function. We first recall some notions as follows.

For any \( v \in (0, \infty) \) and \( x \in \mathcal{X} \), let \( \Gamma_v(x) := \{(y, t) \in \mathcal{X} : d(x, y) < vt\} \) be the *cone of aperture \( v \) with vertex \( x \in \mathcal{X} \), here and in what follows, we always assume that \( \mathcal{X}_+ := \mathcal{X} \times (0, \infty) \). For any closed subset \( F \) of \( \mathcal{X} \), denote by \( R_v F \) the union of all cones with vertices in \( F \), namely, \( R_v F := \bigcup_{x \in F} \Gamma_v(x) \) and, for any open subset \( O \) of \( \mathcal{X} \), denote the tent over \( O \) by \( T_v(O) \), which is defined as \( T_v(O) := \{R_v(O^2)\} \). It is easy to see that \( T_v(O) = \{(x, t) \in \mathcal{X}_+ : d(x, O^2) \geq vt\} \).

In what follows, we denote \( \Gamma_1(x) \) and \( T_1(O) \) simply by \( \Gamma(x) \) and \( O \), respectively.

For all measurable functions \( g \) on \( \mathcal{X}_+ \) and \( x \in \mathcal{X} \), define

\[
A(g)(x) := \left\{ \int_{\Gamma_v(x)} \left| g(y, t) \right|^2 \, \frac{d\mu(y)}{V(x,t)} \, \frac{dt}{t} \right\}^{1/2}.
\]

Coifman, Meyer and Stein [22] introduced the tent space \( T^p_v(\mathcal{X}_+) \) for \( p \in (0, \infty) \), here and in what follows, \( \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty) \). The tent space \( T^p_v(\mathcal{X}_+) \) on spaces of homogenous type was introduced by Russ [65]. Recall that a measurable function \( g \) is said to belong to the tent space \( T^p_v(\mathcal{X}_+) \) with \( p \in (0, \infty) \), if \( \|g\|_{T^p_v(\mathcal{X}_+)} := \|A(g)\|_{L^p(\mathcal{X})} < \infty \). Moreover, Harboure, Salinas and Viviani [37], and Jiang and Yang [45], respectively, introduced the Orlicz tent spaces \( T^\varphi(\mathcal{X}_+) \) and \( T^\varphi(\mathcal{X}_+) \).

Let \( \varphi \) be as in Definition 2.2. In what follows, we denote by \( T^\varphi(\mathcal{X}_+) \) the space of all measurable functions \( g \) on \( \mathcal{X}_+ \) such that \( A(g) \in L^p(\mathcal{X}) \) and, for any \( g \in T^\varphi(\mathcal{X}_+) \), its quasi-norm is defined by

\[
\|g\|_{T^\varphi(\mathcal{X}_+)} := \|A(g)\|_{L^p(\mathcal{X})} = \inf \left\{ \lambda \in (0, \infty) : \int_{\mathcal{X}_+} \varphi \left( \frac{A(g)(x)}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
\]

Let \( p \in (1, \infty) \). A function \( A \) on \( \mathcal{X}_+ \) is called a \((T^\varphi, p)\)-atom if

1. there exists a ball \( B \subset \mathcal{X} \) such that \( \text{supp } a \subset \bar{B} \);
2. \( \|A\|_{L^p(\mathcal{X})} \leq (\mu(B))^{1/p} \|a\|_{L^1(\mathcal{X})} \).

\[\|\| \]
Furthermore, if \( A \) is a \((T_p, p)\)-atom for all \( p \in (1, \infty) \), we then call \( A \) a \((\varphi, \infty)\)-atom. For functions in \( T_{\varphi}(\mathcal{X}_+) \), we have the following atomic decomposition.

**Theorem 3.1.**

Let \( \varphi \) be as in Definition 2.2. Then for any \( f \in T_{\varphi}(\mathcal{X}_+) \), there exist \( \{ \lambda_j \}_j \subset \mathbb{C} \) and a sequence \( \{ A_j \}_j \) of \((T_p, \infty)\)-atoms associated with \( \{ B_j \}_j \) such that, for almost every \( (x, t) \in \mathcal{X}_+ \),

\[
f(x, t) = \sum_j \lambda_j A_j(x, t).
\]

Moreover, there exists a positive constant \( C \) such that, for all \( f \in T_{\varphi}(\mathcal{X}_+) \),

\[
A(\{ \lambda_j A_j \}_j) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \varphi(B_j, \frac{|\lambda_j|}{\lambda |\mathcal{X}(B_j)|^{\psi_p}}) \leq 1 \right\} \leq C \|f\|_{T_{\varphi}(\mathcal{X}_+)}. \tag{3.1}
\]

The proof of Theorem 3.1 is similar to that of [74, Theorem 3.1]. We omit the details here.

**Corollary 3.2.**

Let \( p \in (0, \infty) \) and \( \varphi \) be as in Definition 2.2. If \( f \in T_{\varphi}(\mathcal{X}_+) \cap T^2_{2, h}(\mathcal{X}_+) \), then the decomposition (3.1) also holds in both \( T_{\varphi}(\mathcal{X}_+) \) and \( T^2_{2, h}(\mathcal{X}_+) \).

The proof of Corollary 3.2 is similar to that of [74, Corollary 3.5] and hence we omit the details here.

In what follows, let \( T^{\varphi}_p(\mathcal{X}_+) \) and \( T^{\varphi, h}_2(\mathcal{X}_+) \) with \( p \in (0, \infty) \) denote, respectively, the set of all functions in \( T_{\varphi}(\mathcal{X}_+) \) and \( T^2_{2, h}(\mathcal{X}_+) \) with bounded support. Here and in what follows, a function \( f \) on \( \mathcal{X}_+ \) is said to have bounded support means that there exist a ball \( B \subset \mathcal{X} \) and \( 0 < c_1 < c_2 < \infty \) such that \( \text{supp} f \subset B \times [c_1, c_2) \).

**Proposition 3.3.**

Let \( \varphi \) be as in Definition 2.2. Then \( T^{\varphi}_p(\mathcal{X}_+) \subset T^{\varphi, h}_2(\mathcal{X}_+) \) as sets.

The proof of Proposition 3.3 is an application of the uniformly lower type \( p_2 \) property of \( \varphi \) for some \( p_2 \in (0, 1] \), which is similar to that of [42, Proposition 3.5]. We omit the details.

### 4. The Musielak-Orlicz-Hardy space \( H_{\varphi, L}(\mathcal{X}) \) and its molecular characterization

In this section, we first introduce the Musielak-Orlicz-Hardy space \( H_{\varphi, L}(\mathcal{X}) \) associated with the operator \( L \) via the Lusin-area function. Then we establish an equivalent characterization of \( H_{\varphi, L}(\mathcal{X}) \) in terms of the molecule. We begin with some notions and notation.

Let \( L \) satisfy Assumptions (A) and (B), and \( m \in \mathbb{N} \) be as in (2.9). For all \( f \in L^2(\mathcal{X}) \), the Lusin-area function \( S_L \) is defined as in (2.20).

By Theorem 2.13, we know that, for any \( p \in (p_L, q_L) \), where \( p_L \) and \( q_L \) are as in Assumption (B), there exists a positive constant \( C_{\|p\|} \), depending on \( p \), such that, for all \( f \in L^p(\mathcal{X}) \),

\[
\|S_L(f)\|_{L^2(\mathcal{X})} \leq C_{\|p\|} \|f\|_{L^p(\mathcal{X})}. \tag{4.1}
\]

Now we introduce the Musielak-Orlicz-Hardy \( H_{\varphi, L}(\mathcal{X}) \) via the Lusin-area function \( S_L \).
Definition 4.1.
Let \( \varphi \) be as in Definition 2.2 and \( L \) satisfy Assumptions (A) and (B). Assume that \( p \) and \( q \) are as in Assumption (B). A function \( f \in L^p(X) \) with \( p \in (p_L, q_L) \) is said to be in \( \tilde{H}_{p, L, \varphi}(X) \) if \( S_\lambda(f) \in L^q(X) \) and, moreover, define

\[
\|f\|_{\tilde{H}_{p, L, \varphi}(X)} := \|S_\lambda(f)\|_{L^q(X)} := \inf \left\{ \lambda \in (0, \infty) : \int_X \varphi \left( x, \frac{S_\lambda(f)(x)}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
\]

The Musielak-Orlicz-Hardy space \( \tilde{H}_{p, L, \varphi}(X) \) is defined to be the completion of \( \tilde{H}_{p, L, \varphi}(X) \) with respect to the quasi-norm \( \| \cdot \|_{\tilde{H}_{p, L, \varphi}(X)} \).

In what follows, for the simplicity of the notation, we write \( H_{p, L}(X) := H_{p, L, \varphi}(X) \).

Remark 4.2.
From the Aoki-Rolewicz theorem in [2, 64], it follows that, there exist a quasi-norm \( \| \cdot \| \) on \( \tilde{H}_{p, L, \varphi}(X) \) and \( \gamma \in (0, 1] \) such that, for all \( f \in \tilde{H}_{p, L, \varphi}(X) \), \( \|f\| \sim \|f\|_{\tilde{H}_{p, L, \varphi}(X)} \), and, for any sequence \( \{f_j\}_j \subset \tilde{H}_{p, L, \varphi}(X) \),

\[
\left\| \sum f_j \right\|^\gamma \leq \sum \|f_j\|^\gamma.
\]

By the theorem of completion of Yosida [75, p. 56], it follows that \( \tilde{H}_{p, L, \varphi}(X), \| \cdot \| \) has a completion space \( (H_{p, L, \varphi}(X), \| \cdot \|) \); namely, for any \( f \in H_{p, L, \varphi}(X) \), there exists a Cauchy sequence \( \{f_k\}_{k=1}^\infty \subset H_{p, L, \varphi}(X) \) such that \( \lim_{k \to \infty} \|f_k - f\| = 0 \). Moreover, if \( \{f_k\}_{k=1}^\infty \) is a Cauchy sequence in \( H_{p, L, \varphi}(X) \), then there exists a unique \( f \in H_{p, L, \varphi}(X) \) such that \( \lim_{k \to \infty} \|f_k - f\| = 0 \). Furthermore, by the fact that \( \|f\| \sim \|f\|_{\tilde{H}_{p, L, \varphi}(X)} \) for all \( f \in \tilde{H}_{p, L, \varphi}(X) \), we know that the spaces \( (H_{p, L, \varphi}(X), \| \cdot \|_{\tilde{H}_{p, L, \varphi}(X)}) \) and \( (H_{p, L, \varphi}(X), \| \cdot \|) \) coincide with equivalent quasi-norms.

To introduce the molecular Musielak-Orlicz-Hardy space, we first introduce the notion of the molecule associated with the growth function \( \varphi \).

Definition 4.3.
Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions (A) and (B), \( p \) and \( q \) be as in Assumption (B). Let \( q \in (p_L, q_L) \), \( M \in \mathbb{N} \) and \( \varepsilon \in (0, \infty) \). A function \( \alpha \in L^q(X) \) is called a \( (\varphi, q, M, \varepsilon) \)-molecule associated with the ball \( B \subset X \) if, for each \( k \in \{0, \ldots, M\} \) and \( j \in \mathbb{Z}_+ \), it holds that

\[
\left\| \left( L^{2^{-2k} L^{-1}} \right)^k \alpha \right\|_{L^q(S(B_j))} \leq 2^{-\varepsilon q} \| B \|^{1/q} \|X\|^{1/q} \|\alpha\|_{L^q(X)}.
\]

Moreover, if \( \alpha \) is a \( (\varphi, q, M, \varepsilon) \)-molecule for all \( q \in (p_L, q_L) \), then \( \alpha \) is called a \( (\varphi, M, \varepsilon) \)-molecule.

Definition 4.4.
Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions (A) and (B), \( p \) and \( q \) be as in Assumption (B). Assume that \( q \in (p_L, q_L) \), \( M \in \mathbb{N} \) and \( \varepsilon \in (0, \infty) \). The equality \( f = \sum \alpha_j \) is called a molecular \( (\varphi, r, q, M, \varepsilon) \)-representation of \( f \) for some \( r \in (p_L, q_L) \), if each \( \alpha_j \) is a \( (\varphi, q, M, \varepsilon) \)-molecule associated to the ball \( B_j \subset X \); the summation converges in \( L^r(X) \) and \( \{\alpha_j\}_j \) satisfies

\[
\sum \varphi(B_j, \|\alpha_j\|_{L^r(X)}^\gamma) < \infty.
\]

Let

\[
\tilde{H}_{p, L}^{M, q, \varepsilon}(X) := \{ f : f \ has \ a \ molecular \ (\varphi, r, q, M, \varepsilon) \ - \ representation \ for \ some \ r \ in (p_L, q_L) \}.
\]
with the quasi-norm $||f||_{\mathcal{M},e^{r\cdot}(\mathcal{X})} \text{ given by setting, for all } f \in \widetilde{H}^{M,q,e}_{\psi,L}(\mathcal{X})$,

$$||f||_{\mathcal{M},e^{r\cdot}(\mathcal{X})} := \inf \left\{ \Lambda\{\lambda_j\alpha_j\} : f = \sum_j \lambda_j\alpha_j \text{ is a molecular } (\varphi, r, q, M, e)\text{-representation} \right\},$$

where $\Lambda\{\lambda_j\alpha_j\}$ is as in (3.2).

The molecular Musielak-Orlicz-Hardy space $\mathcal{H}^{M,q,e}_{\psi,L}(\mathcal{X})$ is then defined as the completion of $\widetilde{H}^{M,q,e}_{\psi,L}(\mathcal{X})$ with respect to the quasi-norm $||f||_{\mathcal{M},e^{r\cdot}(\mathcal{X})}$.

In what follows, let $L^2_0(\mathcal{X})$ denote the set of all functions $f \in L^2(\mathcal{X})$ with bounded support, $M \in \mathbb{N}$ and $M > \frac{m}{2m} \frac{q}{q-\frac{q}{2\theta_2}} + \frac{\theta_2}{\theta_1} - \frac{2-\theta_2}{\theta_1} \frac{1}{M}$ where $k, q, \theta, \theta_1$ and $\theta_2$ are respectively as in Definition 2.7, (2.6), (2.5), (2.10) and Assumption (B). For all $f \in L^2_0(\mathcal{X})$ and $x \in \mathcal{X}$, define

$$\pi_{L,M}(f)(x) := C_{(m,M)} \int_0^\infty (t^m L)^{\alpha_j} e^{-t\alpha_j}(f(t))(x) \frac{dt}{t}, \quad (4.2)$$

where $C_{(m,M)}$ is a positive constant such that

$$C_{(m,M)} \int_0^\infty e^{2m(M+2)} e^{-2x_0} \frac{dt}{t} = 1. \quad (4.3)$$

Here $m$ is as in Definition 2.7.

For the operator $\pi_{L,M}$, we have the following boundedness.

**Proposition 4.5.**

Assume that $L$ satisfies Assumptions (A) and (B), and $\pi_{L,M}$ is as in (4.2). Let $\varphi$ be as in Definition 2.2 with $\varphi \in \mathcal{R}(q_i/l(|\varphi|))(\mathcal{X})$, where $q_i$ and $l(|\varphi|)$ are, respectively, as in Assumption (B) and (2.4). Then

(i) the operator $\pi_{L,M}$, initially defined on the space $T^b_\varphi(\mathcal{X})$ with $p \in (p_1, q_1)$, extends to a bounded linear operator from $T^b_\varphi(\mathcal{X})$ to $L^p(\mathcal{X})$;

(ii) the operator $\pi_{L,M}$, initially defined on the space $T^b_\varphi(\mathcal{X})$, extends to a bounded linear operator from $T^\varphi_\psi(\mathcal{X})$ to $H^{M,q,e}_{\psi,L}(\mathcal{X})$.

**Proof.** The proof of (i) is similar to that of [46, Proposition 4.1(i)]. We omit the details. Now we prove (ii). Let $f \in T^\varphi_\psi(\mathcal{X})$. Then by Proposition 3.3, Corollary 3.2 and (i), we know that

$$\pi_{L,M}f = \sum_l \lambda_l \pi_{L,M}A_l =: \sum_j \lambda_j\alpha_j$$

in $L^2(\mathcal{X})$, where $\{\lambda_j\}$ and $\{A_j\}$ satisfy (3.1) and (3.2). Recall that for each $j$, supp $A_j \subset \hat{B}_j$ and $B_j$ is a ball of $\mathcal{X}$. Moreover, from the fact that $S_1$ is bounded on $L^2(\mathcal{X})$, we deduce that for almost every $x \in \mathcal{X}$, $S_1(\pi_{L,M}(f))(x) \leq \sum_j |\lambda_j| S_1(\alpha_j)(x)$. This, combined with Lemma 2.4(i), yields

$$\int_\mathcal{X} \varphi \left( x, S_1(\pi_{L,M}(f))(x) \right) d\mu(x) \leq \sum_j \int_\mathcal{X} \varphi \left( x, |\lambda_j| S_1(\alpha_j)(x) \right) d\mu(x).$$

We now claim that for some $\epsilon \in (ng(\varphi)/l(\varphi), \infty)$, $\alpha_j = \pi_{L,M}(A_j)$ is a $(\varphi, M, \epsilon)$-molecule, up to a harmless constant, associated to the ball $B_j$ for each $j$. Indeed, assume that $A$ is a $(T, \epsilon)$-atom associated to the ball $B := B(x_0, r_0)$ and $q \in (p_1, q_1)$. Since for $q \in (p_1, 2)$, each $(\varphi, 2, M, \epsilon)$-molecule is also a $(\varphi, q, M, \epsilon)$-molecule, to prove the above
claim, it suffices to show that \( \sigma := \pi_{\ell,M}(A) \) is a \((\varphi, q, M, \epsilon)_\ell\)-molecule, up to a harmless constant, adapted to \( B \) with \( q \in [2, q_i] \).

Let \( q \in [2, q_i] \). When \( j \in \{0, \ldots, 4\} \), by (i), we know that

\[
\|\sigma\|_{L^1(S_j(B))} = \|\pi_{\ell,M}(A)\|_{L^1(S_j(B))} \lesssim \|A\|_{L^1(S_j(B))} \lesssim \|\mu(B)\|^{1/q}\|X_B\|_{L^1}^{1/q} \sim 2^{-j\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q}.
\]

(4.4)

When \( j \in \mathbb{N} \) with \( j \geq 5 \), take \( h \in L^q(\mathcal{X}) \) satisfying \( \|h\|_{L^q(\mathcal{X})} \leq 1 \) and \( \text{supp } h \subset S_j(B) \). Then from Hölder’s inequality and \( q' \in (q_i', 2) \), we infer that

\[
|\langle \pi_{\ell,M}(A), h \rangle| \leq \int_X \int_0^\infty |A(x, t)(t^{2n}L^{M+1}e^{-2nL^t}(h)(x))| \frac{dt}{t} \, dx
\]

\[
\leq \|A\|_{L^1(\mathcal{X})} \left\| X_\ell(t^{2n}L^{M+1}e^{-2nL^t}(h)) \right\|_{L^q(\mathcal{X})}
\]

\[
\lesssim \|A\|_{L^1(\mathcal{X})} \|\mu(B)\|^{1/q} \left\| \left( \int_B (t^{2n}L^{M+1}e^{-2nL^t}(h)(x, t))^2 \frac{dx}{t} \right)^{1/2} \right\|^{1/2}.
\]

(4.5)

Moreover, by Assumption (B), we see that

\[
\int_B \left| (t^{2n}L^{M+1}e^{-2nL^t}(h)(x, t))^2 \right| \frac{dx}{t} \leq \int_0^{t_0} \left\{ 2^{2\varphi_0} \left( \frac{2r_0}{t} \right) \right\}^{\varphi_0} \left[ \|\mu(B)\|^{2/q} \left| \mu(2B) \right|^{-1/q} \exp \left[ -\left( \frac{2r_0}{t} \right)^{2n/(2n-1)} \right] \right] \frac{dt}{t}
\]

\[
\leq 2^{2\varphi_0}\|\mu(B)\|^{2/q} \left( \frac{2r_0}{t} \right)^{2\varphi_0} \left( \int_0^{t_0} \frac{dt}{t} \right)^{2\varphi_0/\varphi_0} \lesssim 2^{-2\varphi}\|\mu(B)\|^{2/q} \left( \frac{2r_0}{t} \right)^{2\varphi_0}
\]

which, together with (4.5), implies that

\[
|\langle \pi_{\ell,M}(A), h \rangle| \lesssim 2^{-\varphi}\|\mu(B)\|^{1/q}\|X_B\|_{L^1}^{1/q} \lesssim 2^{-\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q}.
\]

From this and the choice of \( h \), we deduce that, for each \( j \in \mathbb{N} \) with \( j \geq 5 \),

\[
\|\sigma\|_{L^1(S_j(B))} = \|\pi_{\ell,M}(A)\|_{L^1(S_j(B))} \lesssim 2^{-\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q}.
\]

(4.6)

Moreover, let \( k \in \{1, \ldots, M\} \). When \( j \in \{1, \ldots, 4\} \), take \( h \in L^q(\mathcal{X}) \) satisfying \( \|h\|_{L^q(\mathcal{X})} \leq 1 \) and \( \text{supp } h \subset S_j(B) \).

Then it follows, from Hölder’s inequality and the \( L^q(\mathcal{X})\)-boundedness of \( S_{\ell', M+1-k} \), that

\[
\left| \langle (r_B^{2n}L^{-1})^k \pi_{\ell,M}(A), h \rangle \right| \lesssim \int_0^{t_0} \int_B \left| A(x, t) \right| \left| (t^{2n}L^{M+1-k}e^{-2nL^t}(h)(x)) \right| \frac{dx}{t} \, dt
\]

\[
\lesssim \|A\|_{L^1(\mathcal{X})} \|S_{\ell', M+1-k}(h)\|_{L^q(\mathcal{X})}
\]

\[
\lesssim \|\sigma\|_{L^1(\mathcal{X})} \|\mu(B)\|^{1/q}\|X_B\|_{L^1}^{1/q} \lesssim 2^{-j\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q},
\]

which implies that, for each \( k \in \{1, \ldots, M\} \) and \( j \in \{0, \ldots, 4\} \),

\[
\left| \langle (r_B^{2n}L^{-1})^k \sigma, h \rangle \right|_{L^1(S_j(B))} \lesssim 2^{-j\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q}.
\]

(4.7)

When \( j \in \mathbb{N} \) with \( j \geq 5 \), similar to the proof of (4.5), we know that, for each \( m \in \{1, \ldots, M\} \),

\[
\left| \langle (r_B^{2n}L^{-1})^k \sigma, h \rangle \right|_{L^1(S_j(B))} \lesssim 2^{-j\varphi}\|\mu(2B)\|^{1/q}\|X_B\|_{L^1}^{1/q}.
\]
which, together with (4.4), (4.6) and (4.7), implies that $\alpha$ is a $(\varphi, q, M, \epsilon)_{L}$-molecule. Let $\epsilon > nq(\varphi)/l(\varphi)$ and $M \notin \mathbb{N}$ with $M > \frac{\epsilon}{nq(\varphi)/l(\varphi) + \frac{\theta_1}{\theta_2} - \frac{2}{\theta_2}}$. By $\varphi \in \mathbb{R}^{\mathbb{N}(l(\varphi), l(\varphi))} \cap \mathbb{R}^{\mathbb{N}(l(\varphi), l(\varphi))}$, we find that there exist $p_{1} \in [l(\varphi), 1]$, $p_{2} \in (0, l(\varphi))$, $q_{0} \in (q(\varphi), \infty)$ and $q \in [2, q_{2})$ such that $\varphi$ is of uniformly upper type $p_{1}$ and lower type $p_{2}$. Now we claim that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, M, \epsilon)_{L}$-molecule $\alpha$ associated with the ball $B \subset X_{\epsilon}^{{\text{molecule}}}$.

$$\int_X \varphi(x, S_{i}(\lambda\alpha)(x)) \, d\mu(x) \lesssim \varphi \left( B, \frac{|\lambda|}{||\lambda||_{L}||\lambda||_{X_{\epsilon}}} \right). \tag{4.8}$$

If (4.8) holds, from this, the facts that, for all $\lambda \in (0, \infty)$,

$$S_{i}(\pi_{L,M}(f/\lambda)) = S_{i}(\pi_{L,M}(f))/\lambda \quad \text{and} \quad \pi_{L,M}(f/\lambda) = \sum_{j} \lambda_{j} \alpha_{j}/\lambda,$$

and $S_{i}(\pi_{L,M}(f)) \lesssim \sum_{j} |\lambda_{j}| S_{i}(\alpha_{j})$, it follows that, for all $\lambda \in (0, \infty)$,

$$\int_X \varphi \left( x, \frac{S_{i}(\pi_{L,M}(f))(x)}{\lambda} \right) \, d\mu(x) \lesssim \sum_{j} \varphi \left( B_{j}, \frac{|\lambda_{j}|}{\lambda ||\lambda||_{L}||\lambda||_{X_{\epsilon}}} \right),$$

which, together with (3.2), implies that

$$||\pi_{L,M}(f)||_{r_{p}(L^{\theta})} \lesssim A(\{\lambda\alpha_{j}\}) \lesssim ||f||_{r_{p}(L^{\theta})},$$

and hence completes the proof of (ii).

Now we prove (4.8). By the definition of $\alpha$, we see that

$$\int_X \varphi(x, S_{i}(\lambda\alpha)(x)) \, d\mu(x) \lesssim \sum_{j=0}^{\infty} \int_X \varphi \left( x, \left\{ \int_{0}^{r_{j}} \int_{[B_{j}(x), t]} \left| \int_{0}^{r_{j}} \int_{R_{[B_{j}(x), t]}} \left| \int_{0}^{r_{j}} \int_{R_{[B_{j}(x), t]}} \left| t^{2\alpha} Le^{-t^{2\alpha}} \left( \lambda \alpha \chi_{L^{\theta}} \right)(y) \right|^{2} \, d\mu(y) \, dt \, d\mu(x) \right\}^{1/2} \right\} \right) \, dx$$

$$+ \sum_{j=0}^{\infty} \int_X \varphi \left( x, \left\{ \int_{r_{j}}^{\infty} \int_{[B_{j}(x), t]} \left| \int_{r_{j}}^{\infty} \int_{R_{[B_{j}(x), t]}} \left| t^{2\alpha} Le^{-t^{2\alpha}} \left( \lambda \alpha \chi_{L^{\theta}} \right)(y) \right|^{2} \, d\mu(y) \, dt \, d\mu(x) \right\}^{1/2} \right\} \, dx =: \sum_{j=0}^{\infty} E_{j} + \sum_{j=0}^{\infty} F_{j}. \tag{4.9}$$

For any $j \in \mathbb{Z}_{+}$, let $B_{j} := 2^{j}B$. Then

$$E_{j} = \sum_{i=0}^{\infty} \int_{S_{i}(B_{j})} \varphi \left( x, |\lambda| \left\{ \int_{0}^{r_{j}} \int_{[B_{j}(x), t]} \left| \int_{0}^{r_{j}} \int_{R_{[B_{j}(x), t]}} \left| t^{2\alpha} Le^{-t^{2\alpha}} \left( \alpha \chi_{L^{\theta}} \right)(y) \right|^{2} \, d\mu(y) \, dt \, d\mu(x) \right\}^{1/2} \right\} \right) \, dx =: \sum_{i=0}^{\infty} E_{i,j}.$$
Now we estimate $G_{i,j}$. From Hölder’s inequality, Theorem 2.13, $\varphi \in \mathbb{R}^{\mathcal{H}(x)\gamma(\mathcal{X})}$ and Lemma 2.5(vi), we deduce that

$$
G_{i,j} \lesssim \|x\|_{L^2(\mathcal{X})}^{p_1} \left\{ \int_{B_i(\theta)} \left| \sum_{a=1}^{n} \left( \alpha x S_i(\theta) \right) (x) \right|^q \, d\mu(x) \right\}^{p_1/q} \left\{ \int_{S_i(\theta)} \left[ \varphi \left( x, \|x\|_{L_2^2(\mathcal{X})} \right) \right]^{q(\gamma_0)} \, d\mu(x) \right\}^{1/[q\gamma(\mathcal{X})]}.
$$

(4.11)

For $H_{i,j}$, similarly, we have

$$
H_{i,j} \lesssim 2^{-p_2(\varepsilon-\varepsilon_0)p_2} \varphi \left( B, \|x\|_{L_2^2(\mathcal{X})} \right),
$$

which, together with (4.10) and (4.11), implies that, for each $j \in \mathbb{Z}_+$ and $i \in \{0, 1, \ldots, 4\}$,

$$
E_{i,j} \lesssim 2^{-p_2(\varepsilon-\varepsilon_0)p_2} \varphi \left( B, \|x\|_{L_2^2(\mathcal{X})} \right).
$$

(4.12)

For all $j \in \mathbb{Z}_+$ and $x \in \mathcal{X}$, let

$$
H_j(x) := \left\{ \int_{0}^{r_B} \left\{ \int_{B(x,t)} \left| t^{\alpha} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y) \right|^2 \, \frac{d\mu(y)}{V(x,t)} \right\}^{1/2} \right\}^{1/2}.
$$

Now we estimate $\int_{S_i(\theta)} [H_j(x)] \, d\mu(x)$. For any $i, j \in \mathbb{Z}_+$, let

$$
S_j(B_i) := \{ y \in \mathcal{X} : 2^{i-1}r_B \leq d(y, r_i) \leq 2^{i+1}r_B \}.
$$

It is easy to see that when $i \geq 5$, $d(S_i(B), S_j(B)) \geq 2^{i+1}r_B$. By $M > \frac{\Theta_0}{2m} + \frac{\Theta_1}{2} + \frac{\Theta_2}{2} + 1 > \frac{\Theta_m}{2m} + \frac{\Theta_1}{2} + \Theta_2q - 1$, let $s \in \{ \frac{\Theta_m}{2m} + \frac{1}{2} + \Theta_1 + \Theta_2 \}q - 1, 2mMq + \Theta_2q + q/2 + 1$. Then by Hölder’s inequality, Fubini’s theorem and Assumption (B), we conclude that

$$
\int_{S_i(\theta)} [H_j(x)] \, d\mu(x)
\leq \int_{S_i(\theta)} \left\{ \int_{0}^{r_B} \left\{ \int_{B(x,t)} \left| t^{\alpha} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y) \right|^2 \, \frac{d\mu(y)}{V(x,t)} \right\}^{1/2} \right\} \left\{ \int_{B(x,t)} \frac{d\mu(y)}{V(x,t)} \right\}^{q(\gamma_0)/q} \, d\mu(x)
\lesssim r_B^{(q-2)/2} \int_{S_i(\theta)} \left\{ \int_{0}^{r_B} \left| t^{\alpha} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y) \right|^2 \, \frac{d\mu(y)}{V(x,t)} \right\} \left\{ \int_{B(x,t)} \frac{d\mu(y)}{V(x,t)} \right\}^{q(\gamma_0)/q} \, d\mu(x)
\lesssim r_B^{(q-2)/2} \int_{S_i(\theta)} \left\{ \int_{0}^{r_B} \left| t^{\alpha} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y) \right|^2 \, \frac{d\mu(y)}{V(x,t)} \right\} \left\{ \int_{B(x,t)} \frac{d\mu(y)}{V(x,t)} \right\}^{q(\gamma_0)/q} \, d\mu(x)
\lesssim r_B^{(q-2)/2} \int_{S_i(\theta)} \left\{ \int_{0}^{r_B} \left( \frac{t^{\alpha+1} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y)}{t} \right)^{1/2} \right\} \left\{ \int_{B(x,t)} \frac{d\mu(y)}{V(x,t)} \right\}^{q(\gamma_0)/q} \, d\mu(x)
\lesssim 2^{-q(\gamma_0)+q(\gamma_0)/2} \mu(2^{i+1}B) \|x\|_{L_2^2(\mathcal{X})}^{q(\gamma_0)} \int_{0}^{r_B} \left( \frac{t^{\alpha+1} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y)}{t} \right)^{1/2} \, dt.
$$

(4.13)

By using (4.13), similar to the proof of (4.12), we know that for any $j \in \mathbb{Z}_+$ and $i \in \mathbb{N}$ with $i \geq 5$,

$$
E_{i,j} \lesssim 2^{-p_2(\varepsilon-\varepsilon_0)p_2} \varphi \left( B, \|x\|_{L_2^2(\mathcal{X})} \right).
$$

(4.14)

Now we deal with $F_j$. Let

$$
F_j = \sum_{i=0}^{\infty} \int_{S_i(\theta)} \left\{ \int_{0}^{r_B} \left( \frac{t^{\alpha+1} \mathrm{e}^{-t^{2\alpha}L} \left( \alpha x S_i(\theta) \right) (y)}{t} \right)^{1/2} \, dt \right\} \, d\mu(x)
\lesssim \sum_{i=0}^{\infty} F_{i,j}.
$$
When \( i \in \{0, 1, \ldots, 4\} \), similar to the proof of (4.12), we conclude that
\[
F_{e,i} \lesssim 2^{-|\rho_2|\varphi(w_{aq}(\rho_2))} \varphi(B, |X_{\theta}|^{1/\epsilon_{e,i}}), \tag{4.15}
\]
For each \( j \in \mathbb{Z}_+ \) and all \( x \in \mathcal{X} \), let
\[
G_j(x) := \left\{ \int_0^\infty \int_{B(0,y)} \left| 2^{2n} Le^{-r_1^2} \left( 2^{2n} q \right)^M \right| \left( X_{\theta}(t) \left( r_B^{-2n} L^{-1} \right)^M \alpha \right) (y) \right|^2 \frac{d\mu(y)}{V(x, t)^{1/2}} \right\}^{1/2}.
\]
Now we estimate \( \int_{S_i} [G_j(x)]^q d\mu(x) \). We first see that, for all \( x \in \mathcal{X} \),
\[
G_j(x) \leq \left\{ \int_0^\infty \int_{B(0,y)} \left| 2^{2n} Le^{-r_1^2} \left( 2^{2n} q \right)^M \right| \left( X_{\theta}(t) \left( r_B^{-2n} L^{-1} \right)^M \alpha \right) (y) \right|^2 \frac{d\mu(y)}{V(x, t)^{1/2}} \right\}^{1/2} + \left\{ \int_{2^{2n+1} r_1} \cdots \right\}^{1/2} =: G_{j,1}(x) + G_{j,2}(x). \tag{4.16}
\]
For \( G_{j,1} \), similar to (4.13), we conclude that, when \( i \in \mathbb{N} \) with \( i \geq 5 \),
\[
\int_{S_i} [G_{j,1}(x)]^q d\mu(x) \lesssim 2^{-i(1+\theta_t+\theta_t(q-2)/(1+2q))} \mu(2^{i+2} B ||X_{\theta}|^{1/\epsilon_{e,i}}). \tag{4.17}
\]
For \( G_{j,2} \), by Theorem 2.13, we find that
\[
\int_{S_i} [G_{j,2}(x)]^q d\mu(x) \lesssim \frac{L_B^{2m \alpha q}}{(2^{2n} r_1)^{2m \alpha q}} \int_{S_i} \left[ \left( X_{\theta}(t) \left( r_B^{-2n} L^{-1} \right)^M \alpha \right) (x) \right]^q \frac{d\mu(x)}{V(x, t)^{1/2}} \lesssim 2^{-2m \alpha q/2} \mu(2^{i+2} B ||X_{\theta}|^{1/\epsilon_{e,i}}).
\]
which, together with (4.16) and (4.17), implies that
\[
\int_{S_i} [G_j(x)]^q d\mu(x) \lesssim 2^{-i(1+\theta_t+\theta_t(q-2)/(1+2q))} \mu(2^{i+2} B ||X_{\theta}|^{1/\epsilon_{e,i}}).
\]
By using this estimate, similar to the proof of (4.14), we see that, for all \( j \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \) with \( i \geq 5 \),
\[
F_{i,j} \lesssim 2^{2m \alpha q/(q+\theta_t+\theta_t(q-2)/(1+2q))} \varphi(B, |X_{\theta}|^{1/\epsilon_{e,i}}),
\]
which, together with (4.9) through (4.15) and \( s > \frac{|\rho_0|}{p_2^2} + \frac{1}{2} + \theta_t + \theta_t(q-1) \), implies that (4.8) holds true, and hence completes the proof of Proposition 4.5. \( \square \)

**Proposition 4.6.**

Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions (A) and (B), \( \epsilon \in (nq(\varphi)/l(\varphi), \infty) \) and \( M \in \mathbb{N} \) with \( M > \frac{nq(\varphi)}{l(\varphi) + \frac{1}{q} - \frac{1}{p}} \). Then, for all \( f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X}) \), there exist \( \{ \lambda_j \}_i \subseteq C \) and a sequence \( \{ \alpha_j \}_i \) of \( (\varphi, M, \epsilon) \)-molecules, respectively, associated with the balls \( \{ B_j \}_j \) such that \( f = \sum_i \lambda_j \alpha_j \) in both \( H_{\varphi,L}(\mathcal{X}) \) and \( L^2(\mathcal{X}) \). Moreover, there exists a positive constant \( C \) such that, for all \( f \in H_{\varphi,L}(\mathcal{X}) \cap L^2(\mathcal{X}) \),
\[
\Lambda(\{ \lambda_j \}_i) := \inf \left\{ \lambda \in (0, \infty) : \sum_i \varphi \left( B_j, \frac{\lambda_j}{\lambda} \right) |X_{\theta}|^{1/\epsilon_{e,i}} \leq 1 \right\} \leq C \| f \|_{H_{\varphi,L}(\mathcal{X})}.
\]
Proof. Let \( f \in H_{p, l}(\mathcal{X}) \cap L^2(\mathcal{X}) \). Then by the \( H_{\infty} \)-functional calculi for \( L \) and (4.1), we know that

\[
f = C_{p, M} \int_0^{\infty} (t^{2m} L)^{M+2} e^{-t^{2m} L} f \frac{dt}{t} = \pi_{l, M} \left(t^{2m} Le^{-t^{2m} L} f\right)
\]

in \( L^2(\mathcal{X}) \). Moreover, from Definition 4.1 and the \( L^2(\mathcal{X}) \)-boundedness of \( S_L \), we infer that \( t^{2m} Le^{-t^{2m} L} f \in T_\varphi(\mathcal{X}_+) \cap T_\psi(\mathcal{X}_+) \).

Applying Theorem 3.1, Corollary 3.2 and Proposition 4.5 to \( L \) in \( \phi. \), \( \theta. \), \( \alpha. \), \( \beta. \) and \( \epsilon. \), we conclude that

\[
f = \pi_{l, M}(t^{2m} Le^{-t^{2m} L} f) = \sum_j \lambda_j \pi_{l, M} A_j =: \sum_j \lambda_j a_j
\]

in \( L^2(\mathcal{X}) \cap H_{p, l}(\mathcal{X}) \), and \( \lambda(\{\lambda_j a_j\}) \lesssim \|t^{2m} Le^{-t^{2m} L} f\|_{\mathcal{T}_\varphi(\mathcal{X}_+)} \sim \|f\|_{H_{p, l}(\mathcal{X})} \). Furthermore, by the proof of Proposition 4.5, we know that, for each \( j \), \( a_j \) is a \( (\varphi, M, \epsilon) \)-molecule up to a harmless constant, which completes the proof of Proposition 4.6.

The proofs of Propositions 4.5 and 4.6 imply immediately the following corollary.

Corollary 4.7.
Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions (A) and (B), \( p_1 \) and \( q_1 \) be as in Assumption (B), \( q \in (p_1, q_1) \) and \( M \in \mathbb{N} \) satisfying \( M > \frac{q}{2m} \int_0^{1/2} \frac{dt}{t^{2m+M}} + \frac{q}{2m} - \frac{1}{2q} \), where \( q(\varphi), l(\varphi) \) and \( \theta_1 \), respectively, as in (2.6), (2.5) and (2.10). Suppose that \( T \) is a linear (resp. nonnegative sublinear) operator which maps \( L^2(\mathcal{X}) \) continuously into weak \( L^2(\mathcal{X}) \). If there exists a positive constant \( C \) such that, for all \( \lambda \in \mathbb{C} \) and \( (\varphi, \epsilon, M, \epsilon) \)-molecule \( \alpha \) associated with the ball \( B \),

\[
\int_X \varphi(x, T(\lambda \alpha)(x)) \, d\mu(x) \leq C \varphi \left( B, \frac{|\lambda|}{\|\chi_B\|_{L^2(\mathcal{X})}} \right),
\]

then \( T \) can extend to be a bounded linear (resp. sublinear) operator from \( H_{p, l}^{M, q, \epsilon}(\mathcal{X}) \) to \( L^q(\mathcal{X}) \).

Theorem 4.8.
Let \( \varphi \) be as in Definition 2.2 and \( L \) satisfy Assumptions (A) and (B). Assume that \( q \in [2, q_1) \cap (r(\varphi)\|I(\varphi), q_1), M \in \mathbb{N} \) with \( M > \frac{q}{2m} \int_0^{1/2} \frac{dt}{t^{2m+M}} + \frac{q}{2m} - \frac{1}{2q} \), \( \epsilon \in (\eta q(\varphi)|l(\varphi), \infty) \), where \( q_1, r(\varphi), l(\varphi), q(\varphi) \) and \( l(\varphi) \) are, respectively, as in Assumption (B), (2.7), (2.4), (2.6) and (2.5). Then \( H_{p, l}(\mathcal{X}) \) and \( H_{p, l}^{M, q, \epsilon}(\mathcal{X}) \) coincide with equivalent quasi-norms.

Proof. We first prove that

\[
\widetilde{H}_{p, l}^{M, q, \epsilon}(\mathcal{X}) \cap L^2(\mathcal{X}) \subset H_{p, l}(\mathcal{X})
\]

and the inclusion is continuous. Let \( f \in \widetilde{H}_{p, l}^{M, q, \epsilon}(\mathcal{X}) \cap L^2(\mathcal{X}) \). Then there exist \( \{\lambda_j\}_{\mathbb{N}} \subset \mathbb{C} \) and a sequence \( \{a_j\}_{\mathbb{N}} \) of \( (\varphi, q, M, \epsilon) \)-molecules such that \( f = \sum_{j=1}^{\infty} \lambda_j a_j \), where the summation converges in \( L^2(\mathcal{X}) \) for some \( r \in (p_1, q_1) \). By Theorem 2.13, we see that, for each \( i \in \mathbb{N} \), \( S_i(\sum_{j=1}^{\infty} \lambda_j a_j - f)(x) \leq \sum_{j=i+1}^{\infty} |\lambda_j| S_i(a_j)(x) \) for almost every \( x \in \mathcal{X} \), which, together with (4.8) and \( f \in \widetilde{H}_{p, l}^{M, q, \epsilon}(\mathcal{X}) \), implies that

\[
S_i \left( \sum_{j=1}^{i} \lambda_j a_j - f \right) \in L^q(\mathcal{X})
\]

and

\[
\left\| S_i \left( \sum_{j=1}^{i} \lambda_j a_j - f \right) \right\|_{L^q(\mathcal{X})} \to 0
\]
as $i \to \infty$. Moreover, by the definition of $(\varphi, q, M, e)_l$-molecules, H"older's inequality and $q \geq 2$, we conclude that for each $j \in \mathbb{N}$, $\alpha_j \in L^2(\mathcal{X})$, which, together with $f \in L^2(\mathcal{X})$, implies that, for any $i \in \mathbb{N}$, $f - \sum_{j=1}^{i} \alpha_j \in L^2(\mathcal{X})$. From this and (4.18), it follows that $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. Furthermore, by the fact that $S_t(f) \leq \sum_{i=1}^{\infty} |\lambda_i||S_t(\alpha_j)|$ and (4.8), we see that

$$||f||_{\tilde{H}_{\mu,q,e}(\mathcal{X})} \lesssim ||f||_{H_{\mu,q,e}(\mathcal{X})}. \quad (4.19)$$

Now we prove that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \subset \tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and the inclusion is continuous. Let $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. Then by Proposition 4.6, we know that there exist $\{\alpha_j\} \subset \mathbb{C}$ and a sequence $\{\sigma_j\}$ of $(\varphi, q, M, e)_l$-molecules such that $f = \sum_j \lambda_j \sigma_j$ in $H_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\Lambda(\{\lambda_j\}) \lesssim ||f||_{\tilde{H}_{\mu,q,e}(\mathcal{X})}$, which implies that $f \in \tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and

$$||f||_{\tilde{H}_{\mu,q,e}(\mathcal{X})} \lesssim ||f||_{H_{\mu,q,e}(\mathcal{X})}. \quad (4.19)$$

From this and (4.19), we infer that $\tilde{H}_{\mu,q,e}(\mathcal{X}) = \tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and, for all $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$,

$$||f||_{\tilde{H}_{\mu,q,e}(\mathcal{X})} \sim ||f||_{\tilde{H}_{\mu,q,e}(\mathcal{X})}. \quad (4.19)$$

To finish the proof of Theorem 4.8, it suffices to prove that $\tilde{H}_{\mu,q,e}(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ are dense in $H_{\mu,q,e}(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X})$, respectively. Indeed, if these hold true, by these and a standard density argument, we conclude that $H_{\mu,q,e}(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X})$ coincide with equivalent quasi-norms. Obviously, $\tilde{H}_{\mu,q,e}(\mathcal{X})$ is dense in $H_{\mu,q,e}(\mathcal{X})$. Now we prove that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\tilde{H}_{\mu,q,e}(\mathcal{X})$. Let $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. Then there exist a sequence $\{\lambda_j\} \subset \mathbb{C}$ and a sequence $\{\alpha_j\}$ of $(\varphi, q, M, e)_l$-molecules such that $f = \sum_j \lambda_j \alpha_j$ in $L^2(\mathcal{X})$ with some $r \in (p_l, q_l)$. For any $N \in \mathbb{N}$, let $f_N := \sum_{j=1}^{N} \lambda_j \alpha_j$. From the definition of $(\varphi, q, M, e)_l$-molecules, $q \geq 2$ and H"older's inequality, we deduce that, for all $j \in \mathbb{N}$, $\alpha_j \in L^2(\mathcal{X})$, which implies that, for any $N \in \mathbb{N}$, $f_N \in L^2(\mathcal{X})$. Thus, for any $N \in \mathbb{N}$, $f_N \in \tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$, and $||f - f_N||_{\tilde{H}_{\mu,q,e}(\mathcal{X})} \to 0$ as $N \to \infty$. Thus, we see that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\tilde{H}_{\mu,q,e}(\mathcal{X})$ and hence dense in $H_{\mu,q,e}(\mathcal{X})$. This finishes the proof of Theorem 4.8.

**Theorem 4.9.**

Let $\varphi$ be as in Definition 2.2 and $\mu$ satisfy Assumptions (A) and (B). Assume that $\varphi \in \mathbb{R}H_{[q_1, [\varphi]]}(\mathcal{X})$. Then the spaces $H_{\mu,q,e}(\mathcal{X})$ and $H_{\mu,q,e}(\mathcal{X})$, with $s \in (p_l, q_l)$, coincide with equivalent quasi-norms.

**Proof.** Let $s \in (p_l, q_l)$. By the definitions of the spaces $H_{\mu,q,e}(\mathcal{X})$ and $H_{\mu,q,e}(\mathcal{X})$, we see that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ coincide with equivalent quasi-norms. Similar to the proof of Theorem 4.8, we need to prove that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ are dense in $H_{\mu,q,e}(\mathcal{X})$ and $\tilde{H}_{\mu,q,e}(\mathcal{X})$, respectively.

We first prove that $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\tilde{H}_{\mu,q,e}(\mathcal{X})$. Let $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. Then by Proposition 4.5, we know that there exist $\{\lambda_j\} \subset \mathbb{C} \cap \mathbb{R}$ and a sequence $\{\sigma_j\}$ of $(\varphi, q, M, e)_l$-molecules, with $q \in (\max\{s, 2\}, q_l)$, such that $f = \sum_{j=1}^{\infty} \lambda_j \sigma_j$ in $H_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$. From $q \in (s, q_l) \cap [2, \infty)$ and H"older's inequality, we deduce that, for each $j \in \mathbb{N}$, $\alpha_j$ is a $(\varphi, 2, M, e)_l$-molecule and also a $(\varphi, s, M, e)_l$-molecule, which implies that for any $N \in \mathbb{N}$, $\sum_{j=1}^{N} \lambda_j \sigma_j \in L^2(\mathcal{X}) \cap L^2(\mathcal{X})$. Moreover, by (4.8), we see that $S_t(\sum_{j=1}^{\infty} \lambda_j \sigma_j) \subset T_q(\mathcal{X})$. Thus, for any $N \in \mathbb{N}$, $\sum_{j=1}^{N} \lambda_j \sigma_j \in \tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$. Furthermore, from $f = \sum_{j=1}^{\infty} \lambda_j \sigma_j$ in $H_{\mu,q,e}(\mathcal{X})$, we infer that $||f - \sum_{j=1}^{N} \lambda_j \sigma_j||_{\tilde{H}_{\mu,q,e}(\mathcal{X})} \to 0$ as $N \to \infty$. Thus, $\tilde{H}_{\mu,q,e}(\mathcal{X}) \cap L^2(\mathcal{X})$ is dense in $\tilde{H}_{\mu,q,e}(\mathcal{X})$.

Let $f \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. By the definition of $H_{\mu,q,e}(\mathcal{X})$, we see that \[ t^{1/n} \text{ or } \tilde{t}^{1/n} \in T_\varphi(\mathcal{X}_t). \] For any $N \in \mathbb{N}$, let $f_N := \pi_{1,N}(t^{1/n} \text{ or } \tilde{t}^{1/n} \chi_{\mathcal{X}_t})$, where

\[ O_N := \{ (y, t) \in \mathcal{X} \times (0, \infty) : d(y, x_0) < N, t \in (N^{-1}, N) \} \]

with some $x_0 \in \mathcal{X}$. Then from Proposition 3.3, we infer that $t^{1/n} \text{ or } \tilde{t}^{1/n} \chi_{\mathcal{X}_t} \in \tilde{T}_\varphi(\mathcal{X}_t) \cap T^2_\mathcal{X}(\mathcal{X}_t)$, which implies that $f_N \in \tilde{H}_{\mu,q,e}(\mathcal{X})$. Moreover, by $f \in L^2(\mathcal{X})$ and the $L^2(\mathcal{X})$-boundedness of $S_t$, we conclude that $f_N \in L^2(\mathcal{X})$, which
implies that \( t^{2n}e^{-t^{2n}}f \in T_2^2(X_2) \). From this and the definition of \( T_2^2(X_2) \), it follows that \( t^{2n}Le^{-t^{2n}}f\chi_{\mathcal{O}_n} \in T_2^2(X_2) \), which, together with Proposition 4.5(i), implies that \( f_N \in L^2(\mathcal{X}) \). Thus, \( f_N \in \tilde{H}_{p_{1},L,2}(\mathcal{X}) \cap L^2(\mathcal{X}) \). Moreover,

\[
\|S_t(f_N - f)\|_{\nu_{p_{1},L}^2(\mathcal{X})} \leq \left\| t^{2n}Le^{-t^{2n}}f\chi_{\mathcal{O}_n}\|^2 \right\|_{t_\mathcal{O}_n(\mathcal{X}_1)} \to 0,
\]
as \( N \to \infty \). Thus, \( \tilde{H}_{p_{1},L,2}(\mathcal{X}) \cap L^2(\mathcal{X}) \) is dense in \( \tilde{H}_{p_{1},L}(\mathcal{X}) \), which completes the proof of Theorem 4.9. \( \blacksquare \)

As a corollary of Theorem 4.9, we have the following conclusion. We omit the details.

**Corollary 4.10.**

Let \( L \) satisfy Assumptions (A) and (B), and \( \varphi \) be as in Definition 2.2 with \( \varphi \in \mathbb{R}\mathbb{H}(q_{L,1}(\mathcal{X})) \), where \( q_{L,1} \) and \( l(\varphi) \) are respectively as in Assumption (B) and (2.4). Then, for all \( s \in (p_L, q_{L}) \), the space \( L^s(\mathcal{X}) \cap H_{p_{1},L}(\mathcal{X}) \) is dense in \( H_{p_{1},L}(\mathcal{X}) \).

5. The atomic characterization of \( H_{p_{1},L}(\mathcal{X}) \)

In this section, we establish the atomic characterization of the Musielak-Orlicz-Hardy space \( H_{p_{1},L}(\mathcal{X}) \). To obtain the support condition of \( H_{p_{1},L}(\mathcal{X}) \) atoms by using the finite propagation speed for the wave equation, we have to restrict to a special case of operators satisfying Assumptions (A) and (B). More precisely, throughout this section, we assume that the considered operator \( L \) satisfies the following assumptions as in [12]:

**Assumption \( (H_1) \).** \( L \) is a non-negative and self-adjoint operator in \( L^2(\mathcal{X}) \).

**Assumption \( (H_2) \).** There exists a constant \( p_{1} \in [1, 2] \) such that the semigroup \( \{e^{-tL}\}_{t \geq 0} \), generated by \( L \), satisfies the reinforced \((p_{1}, p'_{1}, 1)\) off-diagonal estimates on balls as in Assumption (B).

**Remark 5.1.**

(i) It is easy to see that if an operator \( L \) satisfying Assumptions \((H_1)\) and \((H_2)\) is one-to-one, then it falls in the scope of operators satisfying Assumptions (A) and (B). For the more general case, by using the functional calculus via the spectral theorem, all the results obtained in the above sections still hold true in this situation. Here, the Hardy space \( H_{p_{1},L}(\mathcal{X}) \) is defined as in Definition 4.1. This is a little different from the version of Holmann et al. in [39], where the dense subspace \( H^2(\mathcal{X}) \) of the Hardy space is defined to be the completion of the range of \( L \) in \( L^2(\mathcal{X}) \), \( \mathcal{R}(L) \) (see [39] for more details). Recall that \( L^2(\mathcal{X}) = \mathcal{N}(L) \oplus \mathcal{R}(L) \), where \( \mathcal{N}(L) \) denotes the kernel of \( L \). We know that these Hardy spaces are different from a kernel space \( \mathcal{N}(L) \), which is not essential for our purpose. We make this change in the definition of the Hardy space, because it brings us some conveniences; for example, when \( p = 2 \), we obtain \( H^p_{p_{1}}(\mathcal{X}) = L^2(\mathcal{X}) \).

(ii) The following definition of the \( L^q \) off-diagonal estimates is from [4]. For all \( q \in (1, \infty) \), a family \( \{T_t\}_{t \geq 0} \) of operators is said to satisfy the \( L^q \) off-diagonal estimates, if there exist two positive constants \( C \) and \( c \) such that

\[
\|e^{-tL}f\|_{L^q(\mathcal{X})} \leq C e^{-\frac{ct}{2^q}t^2} \|f\|_{L^q(\mathcal{X})}.
\]

holds true for every closed sets \( E, F \subseteq \mathcal{X}, t \in (0, \infty) \) and \( f \in L^q(\mathcal{X}) \). From [7], we deduce that \( \{T_t\}_{t \geq 0} \in \mathcal{O}_1(L^q - L^q) \) if and only if \( \{T_t\}_{t \geq 0} \) satisfies the \( L^q \) off-diagonal estimates. Thus, Assumption \((H_2)\) implies that \( \{T_t\}_{t \geq 0} \) satisfies the \( L^q \) off-diagonal estimates.

To establish the atomic characterization of \( H_{p_{1},L}(\mathcal{X}) \), we first introduce the notion of the following atoms.

**Definition 5.2.**

Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions \((H_1)\) and \((H_2)\), and \( p_{1} \) be as in Assumption \((H_2)\). Assume that \( q \in (p_{1}, p'_{1}) \), \( M \in \mathbb{N} \) and \( B \subseteq \mathcal{X} \) is a ball. A function \( a \in L^q(\mathcal{X}) \) is called a \((\varphi, q, M)_{L_{1}}\)-atom associated with \( B \), if there exists a function \( b \in \mathcal{D}(L^q) \) such that

\[
(i) \ a = L^Mb;
\]
Let \( L \) satisfy Assumptions \((H_1)\) and \((H_2)\), and \( p_L \) be as in Assumption \((H_2)\). Assume that \( q \in (p_L, p'_L) \) and \( M \in \mathbb{N} \). For \( f \in L^2(\mathcal{X}) \), \( f = \sum_j \lambda_j a_j \) is called an atomic \((\varphi, q, M)\)\(-\)representation of \( f \), if, for all \( j \), \( a_j \) is a \((\varphi, q, M)\)\(-\)atom associated with the ball \( B_j \subset \mathcal{X} \), the summation converges in \( L^2(\mathcal{X}) \) and \( \{\lambda_j\}_j \subset \mathbb{C} \) satisfies that

\[
\sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|X_{B_j}\|_{\mathcal{E}(\mathcal{X})}} \right) < \infty.
\]

Let

\[
\widetilde{H}^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X}) := \{ f : \ f \text{ has an atomic } (\varphi, q, M)_L\text{-representation} \}
\]

with the quasi-norm given by

\[
\| f \|_{\widetilde{H}^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X})} := \inf \left\{ \Lambda(\{\lambda_j a_j\}_j) : f = \sum_j \lambda_j a_j \text{ is an atomic } (\varphi, q, M)_L\text{-representation} \right\},
\]

where the infimum is taken over all the atomic \((\varphi, q, M)_L\)\(-\)representations of \( f \) and

\[
\Lambda \left( \{\lambda_j a_j\}_j \right) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\lambda \|X_{B_j}\|_{\mathcal{E}(\mathcal{X})}} \right) \leq 1 \right\}.
\]

The atomic Musielak-Orlicz-Hardy space \( H^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X}) \) is then defined as the completion of \( \widetilde{H}^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X}) \) with respect to the quasi-norm \( \| \cdot \|_{\widetilde{H}^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X})} \).

We have the following atomic characterization of the Musielak-Orlicz-Hardy space \( H^{\varphi,L,\mathcal{X}}(\mathcal{X}) \).

**Theorem 5.4.**

Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions \((H_1)\) and \((H_2)\), \( p_L \) be as in Assumption \((H_2)\) and \( M \in \mathbb{N} \) satisfying \( M > \frac{2(q \|M\|_{\mathcal{E}(\mathcal{X})})}{r(q) \|\varphi\|_{\mathcal{E}(\mathcal{X})} - \frac{1}{p'_L}} \), where \( r(q) \) and \( \|\varphi\|_{\mathcal{E}(\mathcal{X})} \) are respectively as in (2.6) and (2.5). Assume further that \( q \in (r(q), r(q) \cap (p_L, p'_L)) \), where \( r(q) \) and \( \|\varphi\|_{\mathcal{E}(\mathcal{X})} \) are, respectively, as in (2.7) and (2.4). Then, \( H^{\varphi,L,\mathcal{X}}(\mathcal{X}) \) and \( \tilde{H}^{M,q}_{\varphi, q,L,\mathcal{X}}(\mathcal{X}) \) coincide with equivalent quasi-norms.

**Remark 5.5.**

When \( \mathcal{X} := \mathbb{R}^n \) and for all \( x \in \mathbb{R}^n \) and \( t \in [0, \infty) \), \( \varphi(x, t) := t^p w(x) \) with \( p \in (0, 1] \) and \( w \) a Muckenhoupt weight, Theorem 5.4 is just [12, Theorem 3.8].
To prove Theorem 5.4, we need to introduce some operator \( \pi_{\theta, L, k} \), which can be viewed as a retraction operator from the Musielak-Orlicz–tent space \( T_p(\mathcal{X}_s) \), introduced in Section 3, to \( H_{p, L}(\lambda) \). To this end, we first give some notation. In what follows, for any operator \( T \), we let \( K_T \) be its integral kernel. Let \( \cos(t \sqrt{L}) \) with \( t \in (0, \infty) \) be the cosine function operator generated by \( L \). By [24, Theorem 3.4] (see also [39, Proposition 3.4]), we know that there exists a positive constant \( C_0 \) such that

\[
\text{supp} K_{\cos(t \sqrt{L})} \subset \{ (x, y) \in \mathcal{X} \times \mathcal{X} : \ d(x, y) \leq C_0 t \}.
\]  

(5.1)

Moreover, let \( \psi \in C_c^\infty(\mathbb{R}) \) be even and suppose \( \psi \subset (-C_0^{-1}, C_0^{-1}) \), where \( C_0 \) is as in (5.1). Let \( \Phi \) denote the Fourier transform of \( \psi \). Then, for all \( k \in \mathbb{N} \) and \( t \in (0, \infty) \), the kernel of \( (t^2 L)^k \Phi(t \sqrt{L}) \) satisfies that

\[
\text{supp} K_{(t^2 L)^k \Phi(t \sqrt{L})} \subset \{ (x, y) \in \mathcal{X} \times \mathcal{X} : \ d(x, y) \leq t \}.
\]  

(5.2)

Now, let \( M \in \mathbb{N} \) with \( M > \frac{q(2(p_1' - p_0'))}{q(p_0' - p_1)} \), where \( q(\varphi) \) and \( l(\varphi) \) are respectively as in (2.6) and (2.5). Assume that \( \Phi \) is as in (2.2). Then, for all \( k \in \mathbb{N} \), \( f \in L^2_0(\mathcal{X}_s) \) and \( x \in \mathcal{X} \), the operator \( \pi_{\theta, L, k} \) is defined by

\[
\pi_{\theta, L, k}(f)(x) := C(\varphi, k) \int_0^\infty \left( t^2 L \right)^{k+1} \Phi(t \sqrt{L})(f(\cdot, t))(x) \frac{dt}{t},
\]

where \( C(\varphi, k) \) is a positive constant such that

\[
C(\varphi, k) \int_0^\infty t^{2(k+1)} \Phi(t)t^2 e^{-t^2} \frac{dt}{t} = 1.
\]  

(5.3)

Using Minkowski’s integral inequality and the quadratic estimates (see also [39, (3.14)]), we easily see that \( \pi_{\theta, L, k} \) can be continuously extended from \( T^2(\mathcal{X}_s) \) to \( L^2(\mathcal{X}) \). Moreover, we have the following boundedness of \( \pi_{\theta, L, M} \), which can be viewed as an extension of [74, Proposition 4.6].

**Proposition 5.6.**

Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions (H1) and (H2), \( p_1 \) be as in Assumption (H2), \( q \in (p_1, p_1') \) and \( M \in \mathbb{N} \) satisfying \( M > \frac{q(2(p_1' - p_0'))}{q(p_0' - p_1)} \), where \( q(\varphi) \) and \( l(\varphi) \) are respectively as in (2.6) and (2.5). Assume further that \( \varphi \in \mathbb{R} \mathcal{H}_{p_1', l(\varphi)}(\mathcal{X}) \), where \( l(\varphi) \) is as in (2.4). Then the operator \( \pi_{\theta, L, M} \) is bounded from \( T^2(\mathcal{X}_s) \) to \( L^2(\mathcal{X}) \), extends to a bounded linear operator from \( T^2(\mathcal{X}_s) \) to \( H_{p, L}(\lambda) \).

**Proof.** Without loss of generality, we may only prove Proposition 5.6 under the assumption that \( q \in [2, p_1') \). For the case when \( q \in (p_1, 2) \), the following proof is still valid, only need to make a few modifications when using Hölder’s inequality. Let \( f \in T^2(\mathcal{X}_s) \). From Proposition 3.3, Theorem 3.1, Corollary 3.2 and the fact that \( \pi_{\theta, L, M} \) is bounded from \( T^2(\mathcal{X}_s) \) to \( L^2(\mathcal{X}) \), we deduce that there exist a family \( \{ A_j \} \) of \( (T, \varphi) \)-atoms associated respectively to the balls \( \{ B_j \} \) and \( \{ \lambda_j \}_j \subset \mathbb{C} \) such that

\[
\pi_{\theta, L, M}(f) = \sum_j \lambda_j \pi_{\theta, L, M}(A_j) =: \sum_j \lambda_j a_j
\]

in \( L^2(\mathcal{X}) \) and

\[
\Lambda(\{ \lambda_j A_j \}) \lesssim ||f||_{T^2(\mathcal{X}_s)},
\]  

(5.4)

where \( \Lambda(\{ \lambda_j A_j \}) \) is as in (3.2). Moreover, since the square function \( S_L \) is nonnegative (which means that, for all \( f \in D(S_L) \) and \( x \in \mathcal{X} \), \( S_L(f)(x) \geq 0 \)), sublinear and \( S_L \) is bounded on \( L^2(\mathcal{X}) \), we know that, for almost every \( x \in \mathcal{X} \), \( S_L(\pi_{\theta, L, M}(f))(x) \leq \sum \lambda_j S_L(a_j)(x) \). This, combined with Lemma 2.4(i), implies that, for all \( \lambda \in (0, \infty) \),

\[
\int_\mathcal{X} \varphi \left( x, \frac{S_L(\pi_{\theta, L, M}(f))(x)}{\lambda} \right) d\mu(x) \lesssim \int_\mathcal{X} \varphi \left( x, \sum_j \lambda_j S_L(a_j)(x) \right) d\mu(x).
\]  

(5.5)
We first prove that, for each \( j, a_j \) is a \((\varphi, M)\)-atom associated with \( B_j \). Indeed, let

\[ b_j := C_{\varphi, M} \int_0^\infty t^{\frac{2}{(M+1)}} L \Phi(\sqrt{t}) |A_j(\cdot, t)| \frac{dt}{t}, \tag{5.6} \]

where \( C_{\varphi, M} \) is as in (5.3). From (5.2), we infer that, for all \( k \in \{0, \ldots, M\} \), \( \text{supp} L^k b_j \subset B_j \), which is the support condition of a \((\varphi, q, M)\)-atom as in Definition 5.2.

On the other hand, for any \( h \in L^q(B_j) \cap L^2(B_j) \), by (5.6), Assumption (H1), Fubini’s theorem, the fact that \( \text{supp} A_j \subset \bar{B}_j \) and Hölder’s inequality, we conclude that, for all \( k \in \{0, \ldots, M\} \),

\[
\left| \int_X \left( r_{B_j} L \right)^k b_j(x) h(x) \, d\mu(x) \right| \sim \int_0^\infty \int_X A_j(x, t) \left( r_{B_j} L \right)^k \Phi(\sqrt{t}) h(x) \frac{d\mu(x)}{t} \, dt \\
\leq r_{B_j}^2 \int_0^\infty \int_X A_j(x, t) \left( t^2 L \right)^{k+1} \Phi(\sqrt{t}) h(x) \frac{d\mu(x)}{t} \, dt \\
\leq r_{B_j}^2 \left\| A_j(A) \right\|_{L^q(\mu)} \left\| \int_X \left( t^2 L \right)^{k+1} \Phi(\sqrt{t}) h(x) \right\|_{L^q(\mu)}^{\frac{2}{q}}. 
\]

Following the same argument as that used in the proof of [12, Lemma 5.3], we easily see that, for all \( q' \in (p_1, p'_1) \), \( \pi_{\varphi, L, k} \) is bounded on \( L^{q'}(\chi) \). This, together with the arbitrariness of \( h \) and the fact that \( A_j \) is a \((T_{\varphi}, \infty)\)-atom associated with \( B_j \), implies that

\[
\left\| \left( r_{B_j} L \right)^k b_j \right\|_{L^q(\mu)} \lesssim r_{B_j}^{2M+1} \left\| \mu(B) \right\|^{\frac{1}{q}} \left\| \chi_{B_j} \right\|_{L^{q'}(\mu)}^{-1},
\]

which is the size condition of a \((\varphi, q, M)\)-atom as in Definition 5.2(iii). Thus, we conclude that, for each \( j, a_j \) is a \((\varphi, M)\)-atom associated with \( B_j \). We claim that, to finish the proof of Proposition 5.6, it suffices to show that, for any \( \lambda \in \mathbb{C} \) and \((\varphi, M)\)-atom \( a \) associated with the ball \( B \subset \chi \),

\[
\int_X \varphi(x, S_\lambda(a)(x)) \, d\mu(x) \leq \varphi \left( B, \frac{|\lambda|}{\|X_B\|_{L^q(\mu)}} \right). \tag{5.7}
\]

Indeed, if (5.7) holds, then by (5.5), we see immediately that, for all \( f \in T_{\varphi}^b(\chi_\lambda) \) and \( \lambda \in (0, \infty) \),

\[
\int_X \varphi(x, S_\lambda(\pi_{\varphi, \lambda, L}(f))(x)) \, d\mu(x) \lesssim \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|X_{B_j}\|_{L^q(\mu)}} \right),
\]

which, together with (5.4), implies that \( \|\pi_{\varphi, L, M}(f)\|_{q_0(\lambda, \chi)} \lesssim 1 \{ \lambda, A_j \} \lesssim \|f\|_{T_{\varphi}(\chi_\lambda)}. \) Thus, \( \pi_{\varphi, L, M} \) can be extended to a bounded operator from \( T_{\varphi}(\chi_\lambda) \) to \( H_{\varphi, L}(\chi) \). This proves the claim. By \( M > \frac{1}{2} \left( \frac{d(\varphi_1)}{p_2} - \frac{1}{p'_1} \right) \), \( \varphi \in \mathbb{H}_{\varphi_1}(\chi_\lambda) \) and Lemma 2.5(iv), we know that there exist \( q_0 \in (q(\varphi), \infty) \), \( p_2 \in (0, i(\varphi)) \), \( p_1 \in (i(\varphi), 1) \) and \( q \in (i(\varphi)[r(\varphi)], p'_1) \) such that \( \varphi \) is of uniformly upper type \( p_1 \) and lower type \( p_2 \), \( \varphi \in A_{q_0}(\chi) \), \( M > \frac{1}{2} \left( \frac{d(\varphi_1)}{p_2} - \frac{1}{p'_1} \right) \) and

\[
\varphi \in \mathbb{H}_{\varphi_1}(\chi_\lambda). \tag{5.8}
\]

Now, for \( j \in \{0, \ldots, 4\} \), similar to the proof of (4.12), we conclude that

\[
\int_{\text{supp}(\theta)} \varphi(x, S_\lambda(a_\lambda)(x)) \, d\mu(x) \leq \varphi \left( B, \frac{|\lambda|}{\|X_B\|_{L^q(\mu)}} \right).
\]
Now, we turn to the case when \( j \in \mathbb{N} \) and \( j \geq 5 \). From the fact that \( \varphi \) is of uniformly upper type \( \rho_1 \) and lower type \( \rho_2 \), we deduce that

\[
\int_{S_j(B)} \varphi(x, S_j(\lambda)(x)) \, d\mu(x) \lesssim \int_{S_j(B)} \|S_j(\lambda)(x)\| \varphi \left( x, \frac{|\lambda|}{\|X_0\|} \right) \, d\mu(x) + \int_{S_j(B)} \|S_j(\lambda)(x)\| \varphi \left( x, \frac{|\lambda|}{\|X_0\|} \right) \, d\mu(x) =: I_j + I_j.
\]

To estimate \( I_j \), let \( \widetilde{I}_j := \|S_j(\lambda)\|_{L^\infty}^2 \). By Hölder’s inequality and (5.8), we find that

\[
I_j \lesssim \|X_0\|^{\rho_1} \|S_j(\lambda)\|_{L^\infty}^{\rho_1} \left\{ \int_{S_j(B)} \left[ \varphi \left( x, \frac{|\lambda|}{\|X_0\|} \right) \right]^{\frac{1}{\rho_1'}} \, d\mu(x) \right\}^{\frac{1}{\rho_1'}} \lesssim \|X_0\|^{\rho_1} \|S_j(\lambda)\|_{L^\infty}^{\rho_1} \varphi \left( S_j(B), \frac{|\lambda|}{\|X_0\|} \right). \tag{5.9}
\]

To estimate \( \widetilde{I}_j \), we write \( \widetilde{I}_j \) into

\[
\widetilde{I}_j \lesssim \int_{S_j(B)} \left[ \int_0^{\delta_{\lambda,x}} \int_{B(x,t)} \left| t^2 L e^{-r_1 \mu}(a)(y) \right|^2 \frac{d\mu(y)}{t V(x,t)} \right]^{\frac{2}{\rho_1'}} \, d\mu(x) + \int_{S_j(B)} \left[ \int_0^{\delta_{\lambda,x}} \int_{B(x,t)} \cdots \frac{d\mu(y)}{t V(x,t)} \right]^{\frac{2}{\rho_1'}} \, d\mu(x) =: A_j + B_j. \tag{5.10}
\]

We first estimate \( A_j \). For \( j \geq 5 \), let

\[
G_j(B) := \left\{ y \in \mathcal{X} : \text{there exists } x \in S_j(B) \text{ such that } d(y, x) < \frac{1}{4} d(x, x_0) \right\},
\]

where \( x_0 \) denotes the center of \( B \). Moreover, by the triangle inequality, we easily see that, for all \( y \in G_j(B) \), \( d(y, x_0) \leq 2^{k-1} r_0 \) and \( d(y, x_0) \geq 2^{k-2} r_0 \). Thus, \( G_j(B) \subset \bigcup_{j_0=1}^{\infty} S_j(B) =: \tilde{S}_j(B) \). This, combining with Hölder’s inequality, Fubini’s theorem, the definition of the function \( b \) as in (5.6), Assumption (H2) and the fact that \( a \) is a \( (\varphi, M) \)-atom, implies that

\[
A_j \leq (2r_0)^{\frac{j-1}{2}} \int_{S_j(B)} \left[ \int_0^{\delta_{\lambda,x}} \int_{B(x,t)} \left| t^2 L e^{-r_1 \mu}(a)(y) \right|^q \frac{d\mu(y)}{t^2 V(x,t)} \right] \, d\mu(x)
\]

\[
\leq (2r_0)^{\frac{j-1}{2}} \int_0^{2^{-1} r_0} \int_{S_j(B)} \left( t^2 L \right)^{M+1} e^{-r_1 \mu}(b)(y)^q \frac{dt}{t^{(\frac{1}{2} + 2M)q}} \lesssim (2r_0)^{\frac{j-1}{2}} \|b\|_{L^q(B)}^{q} \int_0^{2^{-1} r_0} \exp \left\{ -C \frac{(2r_0)^2}{t^2} \right\} \frac{dt}{t^{(\frac{1}{2} + 2M)q}} \lesssim 2^{-j q M} \mu(B) \|X_0\|^{-q} \|X_0\| \tag{5.11}
\]

The estimate of \( B_j \) is similar to that of \( A_j \), and, via replacing Assumption (H2) by the \( L^q(\mathcal{X}) \)-boundedness of the family of operators \( \{ (t^2 L)^{M} e^{-t^{1/2} \lambda} \}_{t>0} \), we conclude that

\[
B_j \leq (2r_0)^{\frac{j-1}{2} - 2M} \int_{S_j(B)} \left[ \int_0^{\delta_{\lambda,x}} \int_{B(x,t)} \left| t^2 L e^{-r_1 \mu}(b)(y) \right|^q \frac{d\mu(y)}{t^{M+1} V(x,t)} \right] \, d\mu(x)
\]

\[
\lesssim (2r_0)^{\frac{j-1}{2} - 2M} \|b\|_{L^q(B)}^{q} \int_0^{2^{-1} r_0} \frac{dt}{t^{M+1}} \lesssim 2^{-j q M} \mu(B) \|X_0\|^{-q} \|X_0\|.
\]
which, together with (5.10) and (5.11), shows immediately that

\[ \tilde{I}_j \lesssim 2^{-2j}M \mu(B)\|X_0\|^{0} \phi^\nu \|

Thus, from this, (5.9) and Lemma 2.5(vii), we deduce that

\[ I_j \lesssim 2^{-2j}M \left[ \mu(2^j B) \right]^{\frac{1}{p_2}} \mu(2^j B) \varphi \left( S_j(B), \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right) \]

\[ \lesssim 2^{-2j}M \left[ \mu(2^j B) \right]^{\frac{1}{p_2}} \mu(2^j B) \varphi \left( \tilde{0}_{\epsilon}, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right) \sim 2^{-2j} \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right), \]

(5.12)

where \( \epsilon_0 := 2p_1M - n(q_0 - p_1/q) \). The estimate of \( I_j \) is similar to that of \( I_j \). We only need to point out that, from Lemma 2.5(ii) and the fact that \( \left( \frac{p_1}{p_2} \right)^{'} < \left( \frac{p_1}{p_2} \right)^{''} \), it follows that \( \varphi \in RH_{\frac{p_1}{p_2}}(\mathcal{X}) \). Thus, we conclude that

\[ J_j \lesssim 2^{-2j2p_2-2n[1]} \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right) \sim 2^{-2j0} \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right), \]

(5.13)

where \( \tilde{\epsilon}_0 := 2p_2M - n(q_0 - p_2/q) \). Let \( \tilde{\epsilon}_0 := \min \{ \epsilon_0, \tilde{\epsilon}_0 \} > 0 \). Combining (5.12) and (5.13), we immediately conclude that

\[ \int_{\mathcal{X}} \varphi (x, S_j(\lambda x)) \, d\mu(x) \lesssim \sum_{j \in \mathbb{N}} 2^{-2j0} \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right) \lesssim \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right), \]

(5.14)

which completes the proof of (5.7) and hence Proposition 5.6.

Before turning to the proof of Theorem 5.4, we introduce a sufficient condition which guarantees a given operator to be bounded on the atomic Musielak-Orlicz-Hardy space.

**Lemma 5.7.**

Let \( \varphi \) be as in Definition 2.2, \( L \) satisfy Assumptions \((H_1)\) and \((H_2)\), \( p_1 \) be as in Assumption \((H_2)\), \( q \in (p_1, p_1') \) and \( M \in \mathbb{N} \) satisfying \( M > \frac{q}{p_1} \left( \frac{\phi_{\nu}}{\phi_{\nu}} - \frac{1}{p_1} \right) \), where \( q(\varphi) \) and \( \ell(\varphi) \) are respectively as in (2.6) and (2.5). Suppose that \( T \) is a linear (resp. nonnegative sublinear) operator which maps \( L^2(\mathcal{X}) \) continuously into weak-\( L^2(\mathcal{X}) \). If there exists a positive constant \( C \) such that, for any \( \lambda \in \mathbb{C} \) and \( (\varphi, q, M) \)-atom \( a \) associated with the ball \( B \),

\[ \int_{\mathcal{X}} \varphi (x, T(\lambda a(x))) \, d\mu(x) \leq C \varphi \left( B, \frac{|\lambda|}{\|X_0\|^{\nu}(B)} \right), \]

(5.15)

then \( T \) can extend to be a bounded linear (resp. sublinear) operator from \( H_{\mu, L, \mathcal{X}}^{M, q} \) to \( L^q(\mathcal{X}) \).

The proof of Lemma 5.7 is similar to that of [74, Lemma 5.6]. See also the proof of [46, Lemma 5.1] and [12, Lemma 4.1]. We omit the details here.

Now we prove Theorem 5.4 by using Proposition 5.6 and Lemma 5.7.

**Proof of Theorem 5.4.** By Definition 5.3, we see that, to show Theorem 5.4, it suffices to prove that

\[ L^2(\mathcal{X}) \cap H_{\mu, L}(\mathcal{X}) = L^2(\mathcal{X}) \cap H_{\mu, L, \mathcal{X}}^{M, q}(\mathcal{X}) \]

(5.16)

with equivalent norms. We divide the proof of (5.16) into the following two steps.
Step 1. We first prove the inclusion $L^2(\mathcal{X}) \cap H_{\varphi,L}(\mathcal{X}) \subset L^2(\mathcal{X}) \cap H^{M,q}_{\varphi,L}(\mathcal{X})$. For any $f \in L^2(\mathcal{X}) \cap H_{\varphi,L}(\mathcal{X})$, by the bounded functional calculus in $L^2(\mathcal{X})$, we know that there exists a positive constant $C_{\varphi,M}$ such that

$$f = C_{\varphi,M} \int_0^\infty \left( t^2 L \right)^{M+1} \Phi(t \sqrt{t}) t^2 e^{-t^2} f \frac{dt}{t} = \pi_{\varphi,L} \left( t^2 L e^{-t^2} f \right)$$

in $L^2(\mathcal{X})$. Moreover, from the fact that $t^2 L e^{-t^2} f \in T_\varphi(\mathcal{X})$, we deduce that there exist $\{\lambda_j\} \subset \mathbb{C}$ and $\{A_j\}$ of $\{T_\varphi, \infty\}$-atoms, respectively, associated with $\{B_j\}$ such that

$$t^2 L e^{-t^2} f = \sum_j \lambda_j A_j$$

in $T_\varphi(\mathcal{X})$ and $T_\varphi^2(\mathcal{X})$. Let $(\{\lambda_j A_j\}) \subseteq \|t^2 L e^{-t^2} f\|_{T_\varphi(\mathcal{X})}$, which, together with Proposition 5.6, implies that

$$f = \pi_{\varphi,L,M} \left( t^2 L e^{-t^2} f \right) = \sum_j \lambda_j \pi_{\varphi,L,M}(A_j)$$

in $L^2(\mathcal{X})$. This, together with the fact that $\pi_{\varphi,L,M}(A_j)$ is an $(\varphi, q, M, L)$-atom associated with $B_j$, immediately shows that $f \in L^2(\mathcal{X}) \cap H^{M,q}_{\varphi,L}(\mathcal{X})$. Thus, $L^2(\mathcal{X}) \cap H_{\varphi,L}(\mathcal{X}) \subset L^2(\mathcal{X}) \cap H^{M,q}_{\varphi,L}(\mathcal{X})$.

Step 2. We now prove the inclusion $L^2(\mathcal{X}) \cap H^{M,q}_{\varphi,L}(\mathcal{X}) \subset L^2(\mathcal{X}) \cap H_{\varphi,L}(\mathcal{X})$. By (5.14), we know that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, M, L)$-atom $a_j$ associated with the ball $B_j$ in $\mathcal{X}$, which, together with Step 1, completes the proof of (5.16) and hence Theorem 5.4.

Now, we consider the question of replacing the role of $L^2(\mathcal{X})$ norm by the more general $L^s(\mathcal{X})$ norm for $s \in (p_L', p'_L)$, in the definition of the atomic Musielak-Orlicz-Hardy space $H^{M,q}_{\varphi,L}(\mathcal{X})$. We also introduce the following notion of the $L^s(\mathcal{X})$-adapted atomic Musielak-Orlicz-Hardy space $\widetilde{H}^{M,q}_{\varphi,L}(\mathcal{X})$.

**Definition 5.8.**

Let $\varphi$ be as in Definition 2.2, $L$ satisfy Assumptions (H1) and (H2), and $p_L$ be as in Assumption (H2). Assume that $q, s \in (p_L, p'_L)$ and $M \in \mathbb{N}$. For $f \in L^s(\mathcal{X})$, $f = \sum \lambda_j a_j$ is called an atomic $(\varphi, q, M, L)$-representation of $f$, if each $a_j$ is a $(\varphi, q, M, L)$-atom associated with the ball $B_j \subset \mathcal{X}$, the summation converges in $L^s(\mathcal{X})$ and $\{\lambda_j\} \subset \mathbb{C}$ satisfies that

$$\sum \varphi \left( B_j, \frac{|\lambda_j|}{\|X_{\varphi q,M}||x||_{\mathcal{X}}} \right) < \infty.$$  

Let

$$\widetilde{H}^{M,q}_{\varphi,L}(\mathcal{X}) := \{ f : f \text{ has an atomic } (\varphi, q, M, L) \text{-representation} \}$$

with the quasi-norm given by

$$\|f\|_{\widetilde{H}^{M,q}_{\varphi,L}(\mathcal{X})} := \inf \left\{ \Lambda(\{\lambda_j a_j\}) : f = \sum \lambda_j a_j \text{ is an atomic } (\varphi, q, M, L) \text{-representation} \right\},$$

where the infimum is taken over all the atomic $(\varphi, q, M, L)$-representations of $f$ and

$$\Lambda(\{\lambda_j a_j\}) := \inf \left\{ \lambda \in (0, \infty) : \sum \varphi \left( B_j, \frac{|\lambda_j|}{\lambda \||x||_{\mathcal{X}}^q} \right) \leq 1 \right\}.$$

The atomic Musielak-Orlicz-Hardy space $H^{M,q}_{\varphi,L}(\mathcal{X})$ is then defined as the completion of $\widetilde{H}^{M,q}_{\varphi,L}(\mathcal{X})$ with respect to the quasi-norm $\|\cdot\|_{\widetilde{H}^{M,q}_{\varphi,L}(\mathcal{X})}$. 


From its definition, we know that the space $H_{p, q, l}^M(\mathcal{X})$ as in Definition 5.3 can be viewed as the $L^2(\mathcal{X})$-adapted atomic Musielak-Orlicz-Hardy space $H_{p, q, l}^{M,q/2}(\mathcal{X})$. Moreover, we have the following equivalence between $H_{p, q, l}^{M,q/2}(\mathcal{X})$ and $H_{p, q, l}^{M,q/2}(\mathcal{X})$.

**Theorem 5.9.**

Let $\varphi$ be as in Definition 2.2, $L$ satisfy Assumptions (H1) and (H2), $p_1$ be as in Assumption (H2), and $M \in \mathbb{N}$ satisfying $M > \frac{2(q(p(p_1)))^2}{p_1}$, where $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.6) and (2.5). Then, for all $s \in (p_1, p'_1)$ and $q \in (l(\varphi)[r(\varphi)], p'_1)$, where $l(\varphi)$ and $r(\varphi)$ are respectively as in (2.4) and (2.7), $H_{p, q, l}^{M,q/2}(\mathcal{X})$ and $H_{p, q, l}^{M,q/2}(\mathcal{X})$ coincide with equivalent quasi-norms.

To prove Theorem 5.9, we need a few lemmas. The first one is a variant of Lemma 5.7, whose proof is similar. We omit the details.

**Lemma 5.10.**

Let $\varphi$ be as in Definition 2.2, $L$ satisfy Assumptions (H1) and (H2), $p_1$ be as in Assumption (H2), $q$, $s \in (p_1, p'_1)$ and $M \in \mathbb{N}$ satisfying $M > \frac{2(q(p(p_1)))^2}{p_1}$, where $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.6) and (2.5). Suppose that $T$ is a linear (resp. nonnegative sublinear) operator which maps $L^1(\mathcal{X})$ continuously into weak-$L^1(\mathcal{X})$. If there exists a positive constant $C$ such that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, M)_-$-atom $a$ associated with the ball $B$,

$$
\int_X \varphi(x, T(\lambda a)(x)) \, d\mu(x) \leq C \varphi \left( B, \frac{|\lambda|}{||\mathcal{X}||_{\pi(\mathcal{X})}} \right),
$$

then $T$ can extend to be a bounded linear (resp. sublinear) operator from $H_{p, q, l}^{M,q/2}(\mathcal{X})$ to $L^q(\mathcal{X})$.

**Lemma 5.11.**

Let $\varphi$ be as in Definition 2.2, $L$ satisfy Assumptions (H1) and (H2), $p_1$ be as in Assumption (H2), $q$, $s \in (p_1, p'_1)$ and $M \in \mathbb{N}$ satisfying $M > \frac{2(q(p(p_1)))^2}{p_1}$, where $q(\varphi)$ and $i(\varphi)$ are respectively as in (2.6) and (2.5). Assume further that $\varphi \in RH_{(p, l)(\varphi)}(\mathcal{X})$, where $l(\varphi)$ is as in (2.4). Then, for all $k \in \mathbb{N}$,

(i) the operator $\pi_{\varphi, l,k}$, initially defined on the space $T_{2, l}^\varphi(\mathcal{X}_k)$, extends to a bounded linear operator from $T_{2, l}^\varphi(\mathcal{X}_k)$ to $L^q(\mathcal{X})$;

(ii) for all $t \in (0, \infty)$, the operator $t^2 \text{Le}^{-t\varphi}$, initially defined on $L^q(\mathcal{X})$, extends to a bounded linear operator from $L^q(\mathcal{X})$ to $T_{2, l}^\varphi(\mathcal{X}_k)$.

**Proof.** We first prove (i). Let $f \in T_{2, l}^\varphi(\mathcal{X}_k) \cap T_{2, l}^\varphi(\mathcal{X}_k)$. For any $g \in L^q(\mathcal{X}) \cap L^2(\mathcal{X})$, by Fubini’s theorem, Assumption (H1), Hölder’s inequality and the $L^2(\mathcal{X})$-boundedness of the square function $S_{\varphi, l,k}$, we conclude that

$$
\left| \int_X \pi_{\varphi, l,k}(f)(x)g(x) \, d\mu(x) \right| = \left| \int_0^\infty \int_X f(x, t) \{t^2 \mathcal{L} \}^\varphi \varphi(\text{Le}^{-t\varphi}(g))(x) \, d\mu(x) \right| dt \leq ||A||_{L^q(\mathcal{X})} ||S_{\varphi, l,k}(g)||_{L^q(\mathcal{X})} \leq ||f||_{\mathcal{X}_k} ||g||_{L^2(\mathcal{X})},
$$

which, together with the dual representation of $L^q(\mathcal{X})$ norm and a density argument, implies that $\pi_{\varphi, l,k}$ extends to a bounded linear operator from $T_{2, l}^\varphi(\mathcal{X}_k)$ to $L^q(\mathcal{X})$. This shows that (i) is valid.

We now turn to the proof of (ii). By the definition of the Hardy space $H_{p, q, l}^p(\mathcal{X})$ (with $p \in (0, \infty)$ associated with operators satisfying Assumptions (H1) and (H2) in [39], together with an argument similar to that used in the proof of [41, Proposition 9.1(v)], we see that, for all $s \in (p_1, p'_1)$, $H_{p, q, l}^p(\mathcal{X}) = L^s(\mathcal{X})$. This, combined with the definition of $H_{p, q, l}^p(\mathcal{X})$, immediately implies that the operator $t^2 \text{Le}^{-t\varphi}$ extends to a bounded linear operator from $L^q(\mathcal{X})$ to $T_{2, l}^\varphi(\mathcal{X}_k)$. This shows (ii), which completes the proof of Lemma 5.11.

We now turn to the proof of Theorem 5.9.
Proof of Theorem 5.9. The inclusion that $H_{s,q}^{M,q}(X) \subset H_{s,q}^{M,q}(X)$ follows immediately from Theorem 5.4, Lemma 5.11 and (5.7). We now prove the inclusion $H_{s,q}^{M,q}(X) \subset H_{s,q}^{M,q}(X)$. To this end, we first recall the following Calderón reproducing formula, which is deduced from the bounded functional calculus in $L^2(X)$. More precisely, let $\Phi$ be as in (5.3). For any $f \in L^2(X) \cap L^2(X) \cap H_{s,q}^{M,q}(X)$, we see that there exists a positive constant $C_{(s,M)}$ such that

$$ f = C_{(s,M)} \int_0^\infty (t^2 L)^{M+1} \frac{\Phi \left( t \sqrt{L} \right) t^2 L e^{-t^2 L} f}{t} dt $$

in $L^2(X)$. Moreover, by Lemma 5.11(ii), we know that $t^2 L e^{-t^2 L} f \in T^s_2(X) \cap T^s(X)$. Thus, by a slight modification of the proof of [42, Corollary 3.4], we conclude that there exist $\{A_j\} \subset \mathbb{C}$ and a sequence of $(T^s_2, \infty)$-atoms $\{A_j\}$ associated with the balls $\{B_j\}$ such that

$$ t^2 L e^{-t^2 L} f = \sum_j \lambda_j A_j $$

in $T^s(X)$ and $T^s(X)$. Now, let $g \in L^2(X) \cap L^2(X)$. From (5.17), Fubini’s theorem and Assumption (H), we deduce that

$$ \int_X f(x) g(x) \, d\mu(x) = \int_0^\infty \int_X t^2 L e^{-t^2 L} f(x) \{t^2 L\}^M \Phi(t \sqrt{L}) g(x) \frac{d\mu(x)}{dt} dt $$

$$ = \sum_j \int_X \int_0^\infty \lambda_j A_j(y, t) \{t^2 L\}^M \Phi(t \sqrt{L}) A_j(x) \frac{dt}{t} g(x) \, d\mu(x) $$

$$ =: \sum_j \int_X \lambda_j a_j(x) g(x) \, d\mu(x). $$

By the proof of Proposition 5.6, we conclude that, for each $j \in \mathbb{N}$, $a_j$ is a $(\varphi, q, M)_L$-atom associated with $B_j$. This, together with (5.18), implies that $f$ has a $(\varphi, q, s, M)_L$-atomic representation $f = \sum_j \lambda_j a_j$ as in Definition 5.8. Thus, $f \in H_{s,q}^{M,q}(X)$, which, together with the fact that $L^2(X) \cap L^s(X) \cap H_{s,q}^{M,q}(X)$ is dense in $H_{s,q}^{M,q}(X)$ and a density argument, completes the proof of the inclusion $H_{s,q}^{M,q}(X) \subset H_{s,q}^{M,q}(X)$ and hence Theorem 5.9. \qed

6. A sufficient condition for the equivalence between the spaces $H_{s,L}(\mathbb{R}^n)$ and $H_{s,L}(\mathbb{R}^n)$

In this section, we give a sufficient condition on the operator $L$, satisfying Assumptions (A) and (B), such that $H_{s,L}(\mathbb{R}^n)$ and $H_{s,L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. We first recall some notions and properties of $H_{s,L}(\mathbb{R}^n)$.

In what follows, we denote by $S(\mathbb{R}^n)$ the space of all Schwartz functions and by $S'(\mathbb{R}^n)$ its dual space (namely, the space of all tempered distributions). For $m \in \mathbb{N}$, define

$$ S_m(\mathbb{R}^n) := \left\{ \phi \in S(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \sup_{y \in \mathbb{Z}^n, |y| \leq m+1} \left| 1 + |x| \right|^{m+2|m+|} |\partial^\alpha \phi(x)| \leq 1 \right\}. $$

Then for all $x \in \mathbb{R}^n$ and $f \in S'(\mathbb{R}^n)$, the non-tangential grand maximal function $f^*_m$ of $f$ is defined by setting

$$ f^*_m(x) := \sup_{\phi \in S_m(\mathbb{R}^n), |y| \leq m+1} |f * \phi(y)|, $$

where for all $t \in (0, \infty)$, $\phi_t(x) := t^{-\alpha} \phi(x)$. When $m(\varphi) := \left\lceil n[q(\varphi)/i(\varphi)] - 1 \right\rceil$, where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.6) and (2.5), we denote $f^*_{m(\varphi)}$ simply by $f^*$. Now we recall the definition of the Musielak-Orlicz-Hardy $H_{s,L}(\mathbb{R}^n)$ introduced by Ky [49] as follows.
**Definition 6.1.**
Let \( \varphi \) be as in Definition 2.2. The *Musielak-Orlicz-Hardy space* \( H_p^\varphi(\mathbb{R}^n) \) is defined to be the space of all \( f \in S'(\mathbb{R}^n) \) such that \( f^* \in L^p(\mathbb{R}^n) \) with the quasi-norm \( ||f||_{H_p^\varphi(\mathbb{R}^n)} := ||f^*||_{L^p(\mathbb{R}^n)} \).

To introduce the molecular Musielak-Orlicz-Hardy space, we first introduce the notion of molecules associated with the growth function \( \varphi \).

**Definition 6.2.**
Let \( \varphi \) be as in Definition 2.2, \( q \in (1, \infty) \), \( s \in \mathbb{Z}_+ \) and \( \varepsilon \in (0, \infty) \). A function \( \sigma \in L^q(\mathbb{R}^n) \) is called a \((\varphi, q, s, \varepsilon)\)-molecule associated with the ball \( B \) if

(i) for each \( j \in \mathbb{Z}_+ \), \( ||\sigma||_{L^q(B_j)} \leq 2^{j/q} |B|^{1/q} ||\chi_B||_{L^\varphi(\mathbb{R}^n)}^{-1} \);

(ii) \( \int_{\mathbb{R}^n} \sigma(x) x^\varphi \, dx = 0 \) for all \( \beta \in \mathbb{Z}_+^n \) with \( |\beta| \leq s \).

**Definition 6.3.**
Let \( \varphi \) be as in Definition 2.2, \( q \in (1, \infty) \), \( s \in \mathbb{Z}_+ \) and \( \varepsilon \in (0, \infty) \). The *molecular Musielak-Orlicz-Hardy space*, \( H_{p,\text{mol}}(\mathbb{R}^n) \), is defined to be the space of all \( f \in S'(\mathbb{R}^n) \) satisfying that \( f = \sum \lambda_j \sigma_j \) in \( S'(\mathbb{R}^n) \), where \( \{\lambda_j\}_j \subset \mathbb{C} \) and \( \{\sigma_j\}_j \) is a sequence of \((\varphi, q, s, \varepsilon)\)-molecules respectively associated to the balls \( \{B_j\}_j \), and

\[ \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{||\chi_{B_j}||_{L^\varphi(\mathbb{R}^n)}} \right) < \infty, \]

where, for each \( j \), the molecule \( \sigma_j \) is associated with the ball \( B_j \). Moreover, define

\[ ||f||_{H_{p,\text{mol}}(\mathbb{R}^n)} := \inf \left\{ \lambda \left( \{\lambda_j, \sigma_j\}_j \right) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\lambda ||\chi_{B_j}||_{L^\varphi(\mathbb{R}^n)}} \right) \leq 1 \right\}, \]

where the infimum is taken over all decompositions of \( f \) as above and

\[ \lambda \left( \{\lambda_j, \sigma_j\}_j \right) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\lambda ||\chi_{B_j}||_{L^\varphi(\mathbb{R}^n)}} \right) \leq 1 \right\}. \]

**Definition 6.4.**
Let \( \varphi \) be as in Definition 2.2.

(i) For each ball \( B \subset \mathbb{R}^n \), the *space* \( L^q_\varphi(B) \) with \( q \in [1, \infty] \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \), supported in \( B \), such that

\[ ||f||_{L^q_\varphi(B)} := \left\{ \begin{array}{ll} \sup_{t \in [0, \infty)} \left[ t^{1/q} \int_{\mathbb{R}^n} |f(x)|^q \varphi(x, t) \, dx \right]^{1/q} < \infty, & q \in [1, \infty), \\ ||f||_{C(B)} < \infty, & q = \infty. \end{array} \right. \]

(ii) A triplet \((\varphi, q, s)\) is said to be *admissible*, if \( q \in (q(\varphi), \infty) \) and \( s \in \mathbb{Z}_+ \) satisfying \( s \geq [q(q(\varphi)) - 1] \), where \( q(\varphi) \) and \( i(\varphi) \) are respectively as in (2.6) and (2.5). A measurable function \( \sigma \) on \( \mathbb{R}^n \) is called a \((\varphi, q, s)\)-atom, if there exists a ball \( B \subset \mathbb{R}^n \) such that

(i) \( \text{supp} \, \sigma \subset B \);

(ii) \( ||\sigma||_{L^q_\varphi(B)} \leq ||\chi_B||_{L^\varphi(\mathbb{R}^n)}^{-1} \);

(iii) \( \int_{\mathbb{R}^n} \sigma(x) x^\varphi \, dx = 0 \) for all \( \sigma \in \mathbb{Z}_+^n \) with \( |\sigma| \leq s \).
(iii) The atomic Musielak-Orlicz-Hardy space, $H^{\varphi,q,s}(\mathbb{R}^n)$, is defined to be the space of all $f \in S'(\mathbb{R}^n)$ satisfying that $f = \sum \lambda_j a_j$ in $S'(\mathbb{R}^n)$, where $\{\lambda_j\} \subset \mathbb{C}$ and $\{a_j\}$ is a sequence of $(\varphi, q, s)$-atoms associated with $\{B_j\}$, and

$$\sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|X_{B_j}\|_{L^n(\mathbb{R}^n)}} \right) < \infty.$$ 

Moreover, let

$$\Lambda(\{\lambda_j a_j\}) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\lambda \|X_{B_j}\|_{L^n(\mathbb{R}^n)}} \right) \leq 1 \right\}.$$

The quasi-norm of $f \in H^{\varphi,q,s}(\mathbb{R}^n)$ is defined by $||f||_{H^{\varphi,q,s}(\mathbb{R}^n)} := \inf\{\Lambda(\{\lambda_j a_j\})\}$, where the infimum is taken over all the decompositions of $f$ as above.

Then we have the following conclusion, which is just [42, Theorem 4.11].

**Lemma 6.5.**

Let $\varphi$ be as in Definition 2.2. Assume that $(\varphi, q, s)$ is admissible, $\epsilon \in (\max\{n + s, nq(i\varphi)/i(\varphi)\}, \infty)$ and $p \in (q(\varphi)[r(\varphi)]', \infty)$, where $q(\varphi)$, $i(\varphi)$ and $r(\varphi)$ are, respectively, as in (2.6), (2.5) and (2.7). Then $H_p(\mathbb{R}^n)$, $H^{\varphi, q, s}(\mathbb{R}^n)$ and $H^{\varphi, q, s}_E(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

Throughout this section, we always assume that the operator $L$ satisfies the following additional assumption.

Assumption (C). The distribution kernels $h_t$ of $e^{-tL}$ satisfy that there exist positive constants $\bar{C}, \epsilon, \epsilon \in (0, \infty)$ and $\nu \in (0, 1]$ such that, for all $t \in (0, \infty)$ and almost every $x, y, h \in \mathbb{R}^n$ with $2|h| \leq t^{1/2m} + |x - y|$,

$$|h_t(x, y)| + |h_t(y, x)| \leq \frac{C}{t^{m/2m}} \exp \left\{ -\frac{c|y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\},$$

(6.1)

and

$$|h_t(x + h, y) - h_t(x, y)| + |h_t(x, y + h) - h_t(x, y)| \leq \frac{C}{t^{m/2m}} \left( \frac{|h|}{t^{1/(2m-1)}} \right)^\nu \exp \left\{ -\frac{\bar{C}|x - y|^{2m/(2m-1)}}{t^{1/(2m-1)}} \right\}$$

(6.2)

and

$$\int_{\mathbb{R}^n} h_t(x, y) \, dx \equiv 1 \equiv \int_{\mathbb{R}^n} h_t(x, y) \, dy.$$  

(6.3)

**Remark 6.6.**

(i) If the operator $L$ satisfies Assumption (C), then $L$ satisfies Assumption (B) with $p_L = 1$ and $q_L = \infty$.

(ii) Let $L := -\text{div}(A\nabla)$ be the divergence form elliptic operator in $L^2(\mathbb{R}^n)$, where $A$ has real entries when $n \geq 3$ and complex entries when $n \in \{1, 2\}$. By [9, Chapter 1], we know that the operator $L$ satisfies Assumptions (A) and (C).

We now in the position to state the main result of this section.

**Theorem 6.7.**

Let $\varphi$ be as in Definition 2.2 and $L$ satisfy Assumptions (A) and (C). Assume that $q(\varphi) < \frac{n + \nu}{n + \nu}$ and $i(\varphi) \in (\frac{q(\varphi)}{n + \nu}, 1]$, where $\nu$, $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (6.2), (2.6) and (2.7). Then $H_p(\mathbb{R}^n)$ and $H^{\varphi, p, q, s}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.
Now we estimate \( I \).

By this, the uniformly upper type 1 property of \( \phi \) and \( C \) and a sequence \( \{ a_j \} \) of \( (\varphi, q, 0) \)-atoms such that

\[
I = \sum_{j} \lambda_j a_j
\]

in \( S'(\mathbb{R}^n) \) and

\[
||f||_{\mathcal{L}^p(\mathbb{R}^n)} \sim \Lambda\{\lambda_j a_j\}.
\]

Moreover, by \( f \in \mathcal{L}^2(\mathbb{R}^n) \) and the proof of [49, Theorem 3.4], we know that (6.4) also holds in \( \mathcal{L}^2(\mathbb{R}^n) \). Thus, to prove \( f \in \mathcal{H}_{p, L}(\mathbb{R}^n) \), it suffices to show that, for any \( \varphi \in \mathcal{A}_{q/(\varphi)}(\mathbb{R}^n) \), there exists a positive constant \( C_1 \) such that for all \( t \in (0, \infty) \) and almost every \( x, y, h \in \mathbb{R}^n \) with \( |h| \leq t + |x - y| \),

\[
|q_\varphi(x + h, y) - q_\varphi(x, y)| + |q_\varphi(x, y + h) - q_\varphi(x, y)| \lesssim \frac{1}{t^{\varphi}} \left( \frac{|h|}{t + |x - y|} \right)^\varphi \exp\left\{ - \frac{C_1|x - y|^{2m/(2m - 1)}}{t^{m/(2m - 1)}} \right\}
\]

and

\[
\int_{\mathbb{R}^n} q_\varphi(x, y) \, dx = 0 = \int_{\mathbb{R}^n} q_\varphi(x, y) \, dy.
\]

Write

\[
\int_{\mathbb{R}^n} \varphi(x, S_1(\lambda a)(x)) \, dx = \int_{\varphi B} \varphi(x, S_1(\lambda a)(x)) \, dx + \int_{\varphi B^c} \cdots = 1 + 2.
\]

Moreover, since \( q > q(\varphi)[r(\varphi)]' \), it follows that there exists \( p \in (1, \infty) \) such that \( p > [r(\varphi)]' \) and \( q/p > q(\varphi) \), which implies that \( \varphi \in \mathcal{A}_{q/(\varphi)}(\mathbb{R}^n) \). From this, Hölder’s inequality and Definition 2.1, we infer that

\[
||a||_{\mathcal{L}^p(\mathbb{R}^n)} \lesssim \left\{ \int_{\varphi B} \varphi(x, |a||Xa||_{A^1(\varphi)}^{-1}) \, dx \right\}^{1/(q/p')} \lesssim ||a||_{\mathcal{L}^p(\varphi)}^{1/q} |B|^{1/p'}.
\]

which implies that

\[
||a||_{\mathcal{L}^p(\varphi)} \lesssim ||a||_{\mathcal{L}^p(\varphi)}^{1/q} |B|^{1/p'}.
\]

By this, the uniformly upper type 1 property of \( \varphi \), Hölder’s inequality and the \( L^q(\mathbb{R}^n) \)-boundedness of \( S_1 \), we see that

\[
1 \lesssim \int_{\varphi B} \varphi(x, |a||Xa||_{A^1(\varphi)}^{-1}) \left( 1 + ||a||_{\mathcal{L}^p(\varphi)} S_1(a)(x) \right) \, dx \lesssim \varphi \left( B, |a||Xa||_{A^1(\varphi)}^{-1} \right)
\]

Now we estimate \( I_2 \). For any \( x \in (\varphi B)^c \), we first write

\[
[S_1(a)(x)]^2 = \int_{\varphi B} \int_{\mathbb{R}^n} q_\varphi(y, z) a(z) \, dz \, dy + \int_{\varphi B^c} \cdots = E_1(x) + E_2(x).
\]
Notice that, for any $x \in (8B)^C$, $y \in B(x, t)$ with $t \in (0, r_\theta)$ and $z \in B$, it holds that

\[
|y - z| \geq |x - z| - |x - y| \geq |x - x_0| - |x - y| - r_\theta \geq |x - x_0| - 2r_\theta \geq \frac{1}{2}|x - x_0|.
\] (6.12)

Moreover, similar to the proof of (6.9), we see that

\[
\|a\|_{L^1(B)} \leq \|a\|_{L^2(v)}|B|.
\] (6.13)

Let $s \in [n + \nu, \infty]$. Then from (6.12), (6.13) and (6.1), we deduce that, for all $x \in (8B)^C$,

\[
E_t(x) \lesssim \int_0^{r_\theta} \int_{B(x,t)} \left[ \int_B \left| \frac{1}{t^n} e^{-\frac{x_0 \cdot z}{|x_0|^2}} \right| |a(z)| \, dz \right]^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_0^{r_\theta} \int_{B(x,t)} \left( \frac{t}{|x - x_0|} \right)^{2s} \|a\|_{L^2(v)}^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \frac{r_\theta^{2s}}{|x - x_0|^{2s}} \|a\|_{L^2(v)}^2 \lesssim \frac{r_\theta^{2s}}{|x - x_0|^{2s}} \|a\|_{L^2(v)}^2.
\] (6.14)

Now we deal with $E_2(x)$. By $n + \nu > n\varphi(\rho)/i(\varphi)$, we know that there exist $q_0 \in (q(\varphi), \infty)$, $p_0 \in (0, i(\varphi))$ and $y \in (0, v)$ such that $n + \nu > nq_0/p_0$, which further implies that there exists $y_1 \in (0, y)$ satisfying that

\[
n + \nu - y_1 > nq_0/p_0.
\] (6.15)

Moreover, for any $x \in (8B)^C$, $y \in B(x, t)$ with $t \in (r_\theta, \infty)$, and $z \in B$, it holds that $t + |y - z| \geq |x - z| \geq \frac{1}{2}|x - x_0|$, which, together with $\int_B a(x) \, dx = 0$, (6.7) and (6.13), implies that

\[
E_2(x) \lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \int_B \left| \frac{1}{t^n} e^{-\frac{x_0 \cdot z}{|x_0|^2}} \right| |a(z)| \, dz \right]^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \int_B \left( \frac{t}{t + |y - z|} \right)^{2s} \|a(z)| \, dz \right]^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \int_B \left( \frac{t}{t + |y - z|} \right)^{2s} \|a(z)| \, dz \right]^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \int_B \left( \frac{t}{t + |y - z|} \right)^{2s} \|a(z)| \, dz \right]^2 \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \frac{r_\theta^{2s}}{|x - x_0|^{2s}} \|a\|_{L^2(v)}^2 \right] \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \int_{r_\theta}^{\infty} \int_{B(x,t)} \left[ \frac{r_\theta^{2s}}{|x - x_0|^{2s}} \|a\|_{L^2(v)}^2 \right] \, dy \, dt \quad \frac{t}{t + 1}
\]
\[
\lesssim \frac{r_\theta^{2s}}{|x - x_0|^{2s}} \|a\|_{L^2(v)}^2 \lesssim \frac{r_\theta^{2s}}{|x - x_0|^{2s}} ||Xa||_{L^2(v)}^2.
\]

From this, (6.11) and (6.14), it follows that, for all $x \in (8B)^C$,

\[
S_t(a)(x) \lesssim \frac{r_\theta^{2s}}{|x - x_0|^{2s}} ||Xa||_{L^2(v)}^2,
\]

which, together with the uniformly lower type $p_0$ property of $\varphi$, Lemma 2.5(vi) and (6.15), implies that

\[
I_2 \lesssim \int_{(8B)^C} \varphi \left( x, \frac{|Xa|}{|Xa|} ||Xa||_{L^2(v)}^2 \right) \, dx.
\]
In this section, we study the Musielak-Orlicz-Hardy space $H_{a}$. We begin this subsection by recalling some necessary notions and notation. Let $\Bbb{R}^n$ be an $n \times n$ matrix with entries $\{a_{ij}\}_{i,j=1}^n \subset L^{\infty}(\Bbb{R}^n, \Bbb{C})$ satisfying the ellipticity condition, namely, there exist constants $0 < \lambda_\alpha \leq \Lambda_\alpha < \infty$ such that, for all $\xi, \zeta \in \Bbb{C}$ and almost every $x \in \Bbb{R}^n$,

$$
\lambda_\alpha |\xi|^2 \leq \text{Re}(A(x)\xi, \zeta) \quad \text{and} \quad |(A(x)\xi, \zeta)| \leq \Lambda_\alpha |\xi| |\zeta|.
$$

By this, (6.8) and (6.10), we see that (6.6) holds true.

Now we prove that $H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n) \subset H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$. By Theorem 4.8 and Lemma 6.5, we only need to show that, for any given $\psi, \psi_1, M, \varepsilon$-molecule $\sigma$, it holds that $\alpha$ is a $(\psi, \psi_1, 0, \varepsilon)$-molecule as in Definition 6.2, where $\psi_1 \in (q(\psi), \infty)$, $M \in \Bbb{N}$ with $M > \frac{nq(\psi)}{2nq(\psi)}$, and $\varepsilon \in (nq(\psi)/l(\psi), \infty)$. Indeed, compared with Definitions 4.3 and 6.2, we know that, to show our conclusion, it suffices to prove that

$$
\int_{\Bbb{R}^n} \sigma(x) \, dx = 0. \quad (6.16)
$$

By the $H_{\infty}$-bounded functional calculus, we know that, for all $f \in L^p(\Bbb{R}^n)$ with $p \in (1, p_1]$,

$$(I + L)^{-1}f = \int_0^\infty e^{-t}e^{-tI}f \, dt,$$

which, together with Fubini’s theorem and (6.3), implies that, for all $f \in L^p(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$,

$$
\int_{\Bbb{R}^n} (I + L)^{-1}f(x) \, dx = \int_0^\infty e^{-t} \int_{\Bbb{R}^n} e^{-tI}f(x) \, dx \, dt = \int_{\Bbb{R}^n} f(x) \, dx. \quad (6.17)
$$

Moreover, by the definition of $\alpha$, we know that $\alpha \in L^1(\Bbb{R}^n) \cap L^p(\Bbb{R}^n)$ and there exists $b \in L^1(\Bbb{R}^n) \cap L^p(\Bbb{R}^n)$ such that $\alpha = L b$. From this and (6.17), we deduce that

$$
\int_{\Bbb{R}^n} \alpha(x) \, dx = \int_{\Bbb{R}^n} (I + L)^{-1}Lb(x) \, dx = \int_{\Bbb{R}^n} (I + L)^{-1}(I + L)b(x) \, dx - \int_{\Bbb{R}^n} (I + L)^{-1}b(x) \, dx = 0,
$$

which completes the proof of (6.16).

By the above proofs, we see that $H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$ and $H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$ coincide with equivalent quasi-norms, which, together with the fact that $H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$ and $H_{\psi L}(\Bbb{R}^n) \cap L^2(\Bbb{R}^n)$ are, respectively, dense in $H_{\psi L}(\Bbb{R}^n)$ and $H_{\phi L}(\Bbb{R}^n)$, and a density argument, implies that the spaces $H_{\psi L}(\Bbb{R}^n)$ and $H_{\psi L}(\Bbb{R}^n)$ coincide with equivalent quasi-norms. This finishes the proof of Theorem 6.7.

7. The Musielak-Orlicz-Hardy space associated with the second order elliptic operator in divergence form

In this section, we study the Musielak-Orlicz-Hardy space $H_{\psi L}(\Bbb{R}^n)$ associated with the second order elliptic operator in divergence form on $\Bbb{R}^n$ with complex bounded measurable coefficients. By making full use of the special structure of the divergence form elliptic operator, we establish the radial and non-tangential maximal function characterizations of $H_{\psi L}(\Bbb{R}^n)$ based respectively on the heat and Poisson semigroups of $L$. Moreover, we establish the boundedness of the associated Riesz transform on $H_{\psi L}(\Bbb{R}^n)$.

7.1. Maximal function characterizations of $H_{\psi L}(\Bbb{R}^n)$

We begin this subsection by recalling some necessary notions and notation. Let $A$ be an $n \times n$ matrix with entries $\{a_{ij}\}_{i,j=1}^n \subset L^{\infty}(\Bbb{R}^n, \Bbb{C})$ satisfying the ellipticity condition, namely, there exist constants $0 < \lambda_\alpha \leq \Lambda_\alpha < \infty$ such that, for all $\xi, \zeta \in \Bbb{C}$ and almost every $x \in \Bbb{R}^n$,

$$
\lambda_\alpha |\xi|^2 \leq \text{Re}(A(x)\xi, \zeta) \quad \text{and} \quad |(A(x)\xi, \zeta)| \leq \Lambda_\alpha |\xi| |\zeta|.
$$
where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}$ and $\Re \xi$ denotes the real part of the complex number $\xi$. Then the second order elliptic operator $L$ in divergence form is defined by

$$Lf := -\text{div}(A \nabla f),$$

(7.1)

interpreted in the weak sense via a sesquilinear form. It is well known that there exists a positive constant $\omega \in [0, \pi/2]$ such that the operator $L$ is of type $\omega$ on $L^2(\mathbb{R}^n)$ and $L$ has a bounded $H_\omega$-functional calculus on $L^2(\mathbb{R}^n)$ (see, for example, [1, 41]). Moreover, let $(p_-(L), p_+(L))$ be the interior of the maximal interval of exponents $p \in [1, \infty]$ for which the semigroup $\{e^{-tL}\}_{t>0}$ generated by $L$, is $L^p(\mathbb{R}^n)$ bounded. By [4, Proposition 3.2] (see also [41, Lemma 2.25]), we conclude that, for all $p_-(L) < p < p_+(L)$, $\{e^{-tL}\}_{t>0}$ satisfy the $L^p - L^q$ off-diagonal estimates. Thus, $L$ satisfies Assumptions (A) and (B) with $k = 2$. Therefore, a corresponding theory of the Musielak-Orlicz-Hardy space $H_{\varphi, \lambda}(\mathbb{R}^n)$, including its molecular characterization (see Theorem 4.8) is already known.

We also recall the definitions of some maximal functions associated with $L$ from [40]. Let $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the radial maximal functions, $R^\varphi_x$ and $R^{\varphi \lambda}_x$, respectively associated with the heat semigroup and Poisson semigroup generated by $L$ are defined by setting, for all $\alpha \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$R^\varphi_x(f)(x) := \sup_{t>0} \left\{ \frac{1}{(\alpha t)^n} \int_{B(x, \alpha t)} \left| e^{-t\varphi(L)}f(y) \right|^2 dy \right\}^{\frac{1}{2}},$$

(7.2)

and

$$R^{\varphi \lambda}_x(f)(x) := \sup_{t>0} \left\{ \frac{1}{(\alpha t)^n} \int_{B(x, \alpha t)} \left| e^{-t\varphi \lambda(L)}f(y) \right|^2 dy \right\}^{\frac{1}{2}}.\quad (7.3)$$

Similarly, the non-tangential maximal functions, $N^\varphi_x$ and $N^{\varphi \lambda}_x$, respectively associated with the heat semigroup and Poisson semigroup generated by $L$ are defined by setting, for all $\alpha \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$N^\varphi_x(f)(x) := \sup_{\overline{y,t} \in \Gamma_\alpha(x)} \left\{ \frac{1}{(\alpha t)^n} \int_{B(y, \alpha t)} \left| e^{-t\varphi(L)}f(z) \right|^2 dz \right\}^{\frac{1}{2}},$$

(7.4)

and

$$N^{\varphi \lambda}_x(f)(x) := \sup_{\overline{y,t} \in \Gamma_\alpha(x)} \left\{ \frac{1}{(\alpha t)^n} \int_{B(y, \alpha t)} \left| e^{-t\varphi \lambda(L)}f(z) \right|^2 dz \right\}^{\frac{1}{2}}.\quad (7.5)$$

where above and in what follows, $\Gamma_\alpha(x) := \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < \alpha t \}$. In what follows, when $\alpha = 1$, we remove the superscript $\alpha$ for simplicity. We also define the Lusin-area functions, $S^\varphi$ and $S^{\varphi \lambda}$, associated respectively to the heat semigroup and Poisson semigroup by setting, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$S^\varphi(f)(x) := \left\{ \int_{\Gamma(x)} \left| t \nabla e^{-t\varphi(L)}f(y) \right|^2 \frac{dy\,dt}{t} \right\}^{\frac{1}{2}},$$

(7.6)

and

$$S^{\varphi \lambda}(f)(x) := \left\{ \int_{\Gamma(x)} \left| t \nabla e^{-t\varphi \lambda(L)}f(y) \right|^2 \frac{dy\,dt}{t} \right\}^{\frac{1}{2}}.\quad (7.7)$$

We first introduce the Musielak-Orlicz-Hardy space, defined via the above maximal functions, as follows.
**Definition 7.1.**
Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1), and $S_\varphi$ as in (7.7). The $S_\varphi$-adapted Musielak-Orlicz-Hardy space $H_{\varphi, S_\varphi}(\mathbb{R}^n)$ is defined to be the completion of the set

$$\left\{ f \in L^2(\mathbb{R}^n) : ||f||_{H_{\varphi, S_\varphi}(\mathbb{R}^n)} := ||S_\varphi(f)||_{L^1(\mathbb{R}^n)} < \infty \right\}$$

with respect to the quasi-norm $|| \cdot ||_{H_{\varphi, S_\varphi}(\mathbb{R}^n)}$.

In a similar way, the $S_b$-adapted, $R_\varphi$-adapted, $R_\varphi$-adapted, $N_b$-adapted and $N_\varphi$-adapted Musielak-Orlicz-Hardy spaces,

$$H_{\varphi, S_b}(\mathbb{R}^n), H_{\varphi, R_\varphi}(\mathbb{R}^n), H_{\varphi, R_\varphi}(\mathbb{R}^n)$$

are also defined.

Following [4], let $(q_- (L), q_+ (L))$ be the interior of the maximal interval of exponents $p \in [1, \infty]$, for which the family of operators, $\{ \sqrt{\nabla e^{-tL}} \}_{t>0}$, is $L^p(\mathbb{R}^n)$ bounded. From [4, Proposition 3.7], it follows that $q_-(L) = p_-(L)$ and $q_+(L) > 2$.

Moreover, by [4, Corollary 3.8 and Proposition 3.9], we know that, for all $q_-(L) < p \leq q < q_+(L)$, the family of operators, $\{ \sqrt{\nabla e^{-tL}} \}_{t>0}$, satisfies the $L^p - L^q$ off-diagonal estimates.

For the operator $S_\varphi$, we have the following boundedness.

**Lemma 7.2.**
Let $S_b$ and $S_\varphi$ be respectively as in (7.6) and (7.7). Then, for all $p \in (q_-(L), q_+(L))$, both $S_b$ and $S_\varphi$ are bounded on $L^p(\mathbb{R}^n)$.

The proof of Lemma 7.2 follows from a similar method used for the vertical Lusin-area function associated with the heat semigroup (see [4, Theorem 6.1]). We omit the details here.

**Lemma 7.3.**
Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1),

$$q \in (q_-(L), \min\{ q_+, p_+(L) \})$$

and $M \in \mathbb{N}$. Then, there exists a positive constant $C$ such that, for all $t \in (0, \infty)$, closed sets $E, F \subset \mathbb{R}^n$ satisfying $d(E, F) > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp } f \subset E$,

$$\left\{ \int_0^\infty \left( \int_0^L t \left( \frac{d s}{s} \right)^M \right) \frac{d s}{s} \right\}^{1/2} \leq C \left( \frac{t}{d(E, F)} \right)^{2M} || f ||_{L^2(E)}$$

and

$$\left\{ \int_0^\infty \left( \int_0^L t \left( \frac{d s}{s} \right)^M \right) \frac{d s}{s} \right\}^{1/2} \leq C \left( \frac{t}{d(E, F)} \right)^{2M} || f ||_{L^2(E)}.$$

**Proof.** We first prove (7.8). To this end, by the change of variable, we write

$$\left\{ \int_0^\infty \left( \int_0^L t \left( \frac{d s}{s} \right)^M \right) \frac{d s}{s} \right\}^{1/2} \leq \left\{ \int_0^\infty \left( \int_0^L t \left( \frac{d s}{s} \right)^{M+1} \right) \frac{d s}{s} \right\}^{1/2} + \left\{ \int_0^\infty \cdots \frac{d s}{s} \right\}^{1/2} := H + K.$$
For $H$, we deduce, from the binomial theorem, Assumption (B) and the fact that, when $t \geq s$, $(kt \nabla e^{-(kt)^2}L)[s \nabla e^{-(kt)^2(M^2+L)}]$ satisfies the $L^q$ off-diagonal estimates in $(kt)^2$ (see, for example, [41, Lemma 2.22]), that

\[
H \lesssim \left\{ \int_0^t \left\| s \nabla e^{-(M^2+L)}f \right\|_{L^q(E)}^2 \frac{ds}{s} \right\}^{\frac{1}{2}} + \sup_{1 \leq s \leq t} \left\{ \int_0^t \left\| \left( k t \nabla e^{-(kt)^2}L \right) \left( e^{-(kt)^2(M^2+L)} \right) f \right\|_{L^q(E)}^2 \frac{s^2 ds}{t^2} \right\}^{\frac{1}{2}} \lesssim \left\{ \int_0^t \exp \left\{ -\frac{|d(E,F)|^2}{s^2} \right\} \frac{ds}{s} \right\}^{\frac{1}{2}} \left\| f \right\|_{L^q(E)} \lesssim \left[ \frac{t}{d(E,F)} \right]^{2M} \left\| f \right\|_{L^q(E)},
\]

which, together with (7.10) and (7.11), shows immediately that (7.8) holds. The proof of (7.9) is similar to that of (7.8). We omit the details here. \qed

Now, we are in the position to state our first main result in this section.

**Proposition 7.4.**
Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1), $S_0$ and $S_\varphi$ respectively as in (7.6) and (7.7). Assume further that $\varphi \in \mathbb{RH}_{\min[p_1, l), q), l])/l(q)^{\prime}((\mathcal{X})$, where $l(q)^{\prime}$ is as in (2.4). Then $H_\varphi S_0(\mathbb{R}^n)$, $H_\varphi S_\varphi(\mathbb{R}^n)$ and $H_\varphi L(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

**Proof.** We prove Proposition 7.4 by first showing that $L^2(\mathbb{R}^n) \cap H_\varphi S_\varphi(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \cap H_\varphi L(\mathbb{R}^n)$ coincide with equivalent quasi-norms, whose proof is divided into two different directions of inclusions. We first prove the inclusion $L^2(\mathbb{R}^n) \cap H_\varphi S_\varphi(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H_\varphi L(\mathbb{R}^n)$. Let $f \in L^2(\mathbb{R}^n) \cap H_\varphi S_\varphi(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$, let

\[
\bar{S}_\varphi(f)(x) := \left\{ \int_{(\mathbb{R}^n)} \left| t^2 Le^{-tvT}(f(y)) \right|^2 \frac{dy dt}{t^{n+1}} \right\}^{\frac{1}{2}}.
\]

From the proof of [40, Lemma 5.4], we deduce that, for all $x \in \mathbb{R}^n$, $\bar{S}_\varphi(f)(x) \lesssim S_\varphi(f)(x)$, which immediately implies that

\[
\left\| \bar{S}_\varphi(f) \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| S_\varphi(f) \right\|_{L^q(\mathbb{R}^n)}. \tag{7.12}
\]

Moreover, by the $L^2(\mathbb{R}^n)$-boundedness of $S_\varphi$ (see, for example, [40, (5.15)]), we know that

\[
\left\| \bar{S}_\varphi(f) \right\|_{L^2(\mathbb{R}^n)} \lesssim \left\| S_\varphi(f) \right\|_{L^2(\mathbb{R}^n)} \lesssim \left\| f \right\|_{L^2(\mathbb{R}^n)}.
\]

Thus, $t^2 Le^{-tvT}f \in T_\varphi(\mathbb{R}^{n+1}) \cap T_\varphi(\mathbb{R}^{n+1})$. Using the bounded $H_m$-functional calculus in $L^2(\mathbb{R}^n)$, we see that $f = \pi_{L_\varphi}(t^2 Le^{-tvT}f)$ in $L^2(\mathbb{R}^n)$, where $\pi_{L_\varphi}$ is as in (4.2), which, combining with Proposition 4.5 and (7.12), implies that

\[
\left\| f \right\|_{H_\varphi L(\mathbb{R}^n)} \lesssim \left\| t^2 Le^{-tvT}f \right\|_{T_\varphi(\mathbb{R}^{n+1})} \sim \left\| \bar{S}_\varphi(f) \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| f \right\|_{H_\varphi S_\varphi(\mathbb{R}^n)} < \infty.
\]
This shows \( f \in L^2(\mathbb{R}^n) \cap H_{p,q}(\mathbb{R}^n) \) and hence the inclusion \( L^2(\mathbb{R}^n) \cap H_{p,q}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_{p,q}(\mathbb{R}^n) \).

Now, we prove the inclusion \( L^2(\mathbb{R}^n) \cap H_{p,q}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_{p,q}(\mathbb{R}^n) \). To this end, it suffices to show that the operator \( S_P \) is bounded from \( H_{p,q}(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \). Moreover, by Corollary 4.7, we only need to show that, for any \( \lambda \in \mathbb{C} \) and \((\varphi, q, M, \epsilon)\)–molecule \( \alpha \) associated with \( B \),

\[
\int_{\mathbb{R}^n} \varphi(x, S_P(\lambda \alpha)(x)) \, dx \leq \varphi \left( B, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right),
\]

(7.13)

where \( q \in (p_-(L), \min\{p_+(L), q_+(L)\}) \), \( \epsilon \in (0, \infty) \) and \( M \in \mathbb{N} \) can be chosen sufficiently large.

To prove (7.13), we first write

\[
\int_{\mathbb{R}^n} \varphi(x, S_P(\lambda \alpha)(x)) \, dx \leq \int_{\mathbb{R}^n} \varphi \left( x, S_P \left( I - e^{-\Delta} \right)^M (\lambda \alpha)(x) \right) \, dx
\]

\[
+ \int_{\mathbb{R}^n} \varphi \left( x, S_P \left( I - \left( I - e^{-\Delta} \right)^M \right)(\lambda \alpha)(x) \right) \, dx =: I + J.
\]

(7.14)

For \( I \), let \( p_1 \in [l(\varphi), 1) \) and \( p_2 \in (0, l(\varphi)) \) such that \( \varphi \) is of uniformly upper type \( p_1 \) and lower type \( p_2 \). By Minkowski's inequality and Lemma 2.4(i), we conclude that

\[
1 \leq \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int_{S(2B)} \varphi \left( x, |\lambda| S_P \left( I - e^{-\Delta} \right)^M \chi_{S(\lambda \alpha)} (x) \right) \, dx
\]

\[
\leq \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left\{ \int_{S(2B)} \left[ S_P \left( I - e^{-\Delta} \right)^M \chi_{S(\lambda \alpha)} (x) \right] \|X\|_{L^p(\mathbb{R}^n)}^{-p_1} \, dx \right\}^{p_1} \varphi \left( x, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right) \, dx
\]

\[
+ \int_{S(2B)} \left[ S_P \left( I - e^{-\Delta} \right)^M \chi_{S(\lambda \alpha)} (x) \right] \|X\|_{L^p(\mathbb{R}^n)}^{-p_1} \varphi \left( x, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right) \, dx \right\} =: \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left\{ \tilde{t}_{i,j} + \tilde{t}_{i,j} \right\}.
\]

(7.15)

We first estimate \( \tilde{t}_{i,j} \) in the case when \( j \in \{0, \ldots, 4\} \). Let \( q \geq 2 \) and

\[
q \in \{l(\varphi), l(\varphi)^{-1}, \min\{p_+(L), q_+(L)\}) \cap (p_- (L), \min\{p_+(L), q_+(L)\}) \}
\]

(7.16)

such that (5.8) holds true. Let \( \tilde{\varphi} \in (q(\varphi), \infty) \). Then \( \varphi \in RH_{p_1,q_1}(\mathbb{R}^n) \cap H_{p_2,q_2}(\mathbb{R}^n) \). This, together with Hölder’s inequality, Lemma 7.3 and the \( L^q(\mathbb{R}^n) \)-boundedness of the semigroup \( \{e^{it\Delta}\}_{t>0} \) for \( q \in (p_-(L), p_+(L)) \), implies that

\[
\tilde{t}_{i,j} \leq \left\{ \int_{S(2B)} \left[ S_P \left( I - e^{-\Delta} \right)^M \chi_{S(\lambda \alpha)} (x) \right] \|X\|_{L^p(\mathbb{R}^n)}^{-p_1} \right\}^{p_1} \varphi \left( x, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right) \, dx
\]

\[
\times \left\{ \int_{S(2B)} \left[ \varphi \left( x, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right) \right] \|X\|_{L^p(\mathbb{R}^n)}^{p_2} \, dx \right\} \frac{1}{2^j} |2^{l+1}B|^{1/p_2} \frac{1}{2^j} \frac{1}{\|X\|_{L^p(\mathbb{R}^n)}} \left\{ \int_{S(2B)} \varphi \left( x, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right) \, dx \right\}.
\]

From this, Definition 4.3 and Lemma 2.5(vii), we deduce that

\[
\tilde{t}_{i,j} \leq 2^{-j|\lambda|^{p_1 + rac{q_2}{p_2}} - \frac{|\lambda|}{p_1}} \varphi \left( B, \frac{|\lambda|}{\|X\|_{L^p(\mathbb{R}^n)}} \right),
\]

when \( \epsilon > n \left( \frac{2}{p_1} - \frac{1}{q} \right) \).
We continue to estimate $\tilde{I}_{i,j}$ in the case when $j \geq 5$. Similar to the case when $j \leq 4$, we first have

$$
\tilde{I}_{i,j} \lesssim 2^{-i+j/2} \frac{1}{(i+j+2)} \left| x_B \right| B \left\| \mathcal{T} \right\|_{L^p(x_B \mathbb{R}^n)}^p \frac{d}{dx} \left( B, \frac{1}{\left|x_B\right| B(x)} \right) \left\{ \int_{S_j(2^n)} \left| S_\partial \left( I - e^{-i \hat{B} \hat{T}} \right) \right| \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left\| x_B \right\|_{L^p(x_B \mathbb{R}^n)}^p \right\}^{\frac{d}{dx}}
$$

$$
= 2^{-i+j/2} \frac{1}{(i+j+2)} \left| x_B \right| B \left\| \mathcal{T} \right\|_{L^p(x_B \mathbb{R}^n)}^p \frac{d}{dx} \left( B, \frac{1}{\left|x_B\right| B(x)} \right) \left( A_{i,j} \right)^p.
$$

(7.17)

For $A_{i,j}$, since $q \geq 2$, by the dual norm representation of the $L^2(x_B \mathbb{R}^n)$-norm, we know that there exists $g \in L^{2'}(x_B \mathbb{R}^n)$, with $\left\| g \right\|_{L^{2'}(x_B \mathbb{R}^n)} \leq 1$, such that

$$
A_{i,j} \sim \left\{ \int_{S_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| dy \right\|_{P_{\mathcal{B}}}^\frac{1}{q} \right\}^{\frac{1}{q}}
$$

$$
\sim \left\{ \int_{S_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| dy \right\|_{P_{\mathcal{B}}}^\frac{1}{q} \right\}^{\frac{1}{q}}
$$

$$
\lesssim \left\{ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| dy \right\|_{P_{\mathcal{B}}}^\frac{1}{q} \right\}^{\frac{1}{q}}
$$

$$
= \tilde{A}_{i,j} + \sum_{k=0}^{i-2} \tilde{A}_{i,k}.
$$

(7.18)

where $\mathcal{M}$ denotes the classical Hardy-Littlewood maximal function.

To estimate $\tilde{A}_{i,j}$, we need the following subordination formula,

$$
e^{-i \hat{T}} = C \int_0^\infty e^{-u} \frac{d}{du} e^{-i \hat{T}} du,
$$

(7.19)

where $C$ is a positive constant. By using Hölder’s inequality, (7.19), Minkowski’s integral inequality, the $L^{2'}(x_B \mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal function and Lemma 7.2, we conclude that

$$
\tilde{A}_{i,j} \lesssim \left\{ \int_0^\infty \left[ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| dy \right\|_{P_{\mathcal{B}}} \right] \left\| \frac{dt}{t} \right\|_{L^r} \right\}^{\frac{1}{q}}
$$

$$
\lesssim \int_0^\infty \left[ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| \frac{dt}{t} \right\|_{L^r} \right\}^{\frac{1}{q}}
$$

$$
\lesssim \int_0^\infty \left[ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left( y \right) \left\| \frac{dt}{t} \right\|_{L^r} \right\}^{\frac{1}{q}}
$$

(7.20)

We continue to estimate $\tilde{A}_{i,k}$. Similar to the estimates for $\tilde{A}_{i,j}$, we first conclude that

$$
\tilde{A}_{i,k} \lesssim \int_0^\infty \left[ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left\| \frac{dt}{t} \right\|_{L^r} \right\}^{\frac{1}{q}}
$$

$$
\lesssim \int_0^\infty \left[ \int_{\mathbb{S}_j(2^n)} \left[ \nabla e^{-i \hat{T}} \left( I - e^{-i \hat{B} \hat{T}} \right) \right]^M \left( \chi_{\mathbb{S}, \mathbb{B}}(x) \right) \left\| \frac{dt}{t} \right\|_{L^r} \right\}^{\frac{1}{q}}
$$

(7.21)
By the $L^q$ off-diagonal estimates (similar to the estimates used in (7.11)) and the change of variable (let $\bar{s} := \frac{2^{i+j}R^2}{1+w}$), we further find that

$$
\tilde{A}_{i,j,k} \lesssim ||a||_{L^q(S_i(\mathbb{R}))} \int_0^{\infty} e^{-u} \left\{ \int_0^{\infty} \frac{\exp \left\{ \frac{\alpha^2}{s(1+u)} \right\} \left( \frac{r_0^2}{s} \right)^{2M} ds}{s^2} \right\}^{\frac{1}{2}} du
$$

$$
\lesssim ||a||_{L^q(S_i(\mathbb{R}))} \int_0^{\infty} e^{-u} \left\{ \int_0^{\infty} \frac{e^{-s} \left( \frac{r_0^2 s(1+u)}{2^{i+j}R^2} \right)^{2M} ds}{s^2} \right\}^{\frac{1}{2}} du
$$

$$
\lesssim 2^{2i+j}M ||a||_{L^q(S_i(\mathbb{R}))} \int_0^{\infty} (1+u)^M e^{-u} \left\{ \int_0^1 s^{2M} e^{-s} ds \right\}^{\frac{1}{2}} du
$$

which, together with (7.17), (7.18), (7.20) and Definition 4.3, implies that, when $j \geq 5$,

$$
\tilde{T}_{i,j} \lesssim 2^{-(i+j)n(\frac{\alpha}{\beta})} ||Xa||_{L^p(\mathbb{R}^n)} ||B||^{-\alpha} \varphi \left( B, \frac{||\lambda||_{L^p(\mathbb{R}^n)}}{||Xa||_{L^p(\mathbb{R}^n)}} \right) \left( A_{i,j} \right)^{p^1} \lesssim 2^{-(i+j)p_1[2M+n(\frac{\alpha}{\beta})]} ||Xa||_{L^p(\mathbb{R}^n)} ||B||^{-\alpha} \varphi \left( B, \frac{||\lambda||_{L^p(\mathbb{R}^n)}}{||Xa||_{L^p(\mathbb{R}^n)}} \right) ||a||_{L^q(S_i(\mathbb{R}))}
$$

Similar to the estimates for $\tilde{T}_{i,j}$, we see that

$$
\tilde{I}_{i,j} \lesssim 2^{-(i+j)p_2[2M+n(\frac{\alpha}{\beta})]} 2^{-i\psi_{p_2}} \varphi \left( B, \frac{||\lambda||_{L^p(\mathbb{R}^n)}}{||Xa||_{L^p(\mathbb{R}^n)}} \right),
$$

which, combining with (7.15), implies that

$$
1 \lesssim \varphi \left( B, \frac{||\lambda||_{L^p(\mathbb{R}^n)}}{||Xa||_{L^p(\mathbb{R}^n)}} \right). \quad (7.21)
$$

Also, by following the same way as the estimates for $I$, we know that $J \lesssim \varphi \left( B, \frac{||\lambda||_{L^p(\mathbb{R}^n)}}{||Xa||_{L^p(\mathbb{R}^n)}} \right)$, which, together with (7.14) and (7.21), shows that (7.13) holds true.

The proof for the equivalence of $H_{\psi, S_i}(\mathbb{R}^n)$ and $H_{\psi, L}(\mathbb{R}^n)$ is similar. We omit the details here. This finishes the proof of Proposition 7.4.

Now, we state the maximal function characterizations of $H_{\psi, L}(\mathbb{R}^n)$ as follows.

**Theorem 7.5.**

Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1), $R_\beta$, $R_\rho$, $N_0$ and $N_\rho$ respectively as in (7.2), (7.3), (7.4) and (7.5). Assume further that $\varphi \in \mathcal{R}_{\min (\beta, \rho), (\alpha, \beta)}$ if $\varphi (\mathbb{R}^n)$, where $l(\varphi)$ is as in (2.4). Then $H_{\psi, S_i}(\mathbb{R}^n)$, $H_{\psi, L}(\mathbb{R}^n)$, $H_{\psi, N_0}(\mathbb{R}^n)$, $H_{\psi, N_\rho}(\mathbb{R}^n)$ and $H_{\psi, L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

**Remark 7.6.**

Theorem 7.5 completely covers [46, Theorem 5.2 and Corollary 5.1] by taking $\varphi$ as in (1.1) with $w \equiv 1$ and $\Phi$ concave.
To prove Theorem 7.5, we need a good-$\lambda$ inequality concerning the non-tangential maximal function and the truncated Lusin-area function associated with the heat semigroup. More precisely, let $\alpha \in (0, \infty)$ and $0 < \epsilon < R < \infty$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the truncated Lusin-area function $S_{h}^{\epsilon, R, \alpha}$, associated with the heat semigroup, is defined by setting,

$$S_{h}^{\epsilon, R, \alpha}(f)(x) := \left\{ \int_{\Gamma_{h}^{\epsilon, R}(x)} \left| \nabla e^{-\frac{1}{2}t} f(y) \right|^2 \frac{dy dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \quad (7.22)$$

where $\Gamma_{h}^{\epsilon, R}(x) := \{(y, t) \in \mathbb{R}^n \times (\epsilon, R) : |y - x| < \alpha t\}$. We have the following good-$\lambda$ inequality.

**Lemma 7.7.**

Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1). Assume that $\epsilon, R \in (0, \infty)$ with $\epsilon < R$. Then, there exist positive constants $\epsilon_0$ and $C$, independent of $\epsilon$ and $R$, such that, for all $y \in [0, 1]$, $\lambda$, $s \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ satisfying $||\mathcal{N}_h(f)||_{L^p(\mathbb{R}^n)} < \infty$,

$$\varphi \left\{ x \in \mathbb{R}^n : S_{h}^{\epsilon, R, \alpha}(f)(x) > 2\lambda, \mathcal{N}_h(f)(x) \leq \gamma \lambda \right\} , s \leq C \gamma^{\epsilon_0} \varphi \left\{ x \in \mathbb{R}^n : S_{h}^{\epsilon, R, \alpha}(f)(x) > \lambda \right\} , s.$$

**Proof.** Lemma 7.7 can be proved by using the same method as that used in the proof of [72, Lemma 3.3], where a good-$\lambda$ inequality was established in the setting of the strongly Lipschitz domain of $\mathbb{R}^n$. In the present situation, the proof is more simple, since we do not need to take care of the boundary condition and the diameter of the domain. Here, in order to avoid redundancy, we only give an outline for the proof of Lemma 7.7. Let

$$O := \{ x \in \mathbb{R}^n : S_{h}^{\epsilon, R, \alpha}(f)(x) < \infty \}.$$

From the $L^2(\mathbb{R}^n)$-boundedness of $S_{h}$, we deduce that $|O| < \infty$. By using Whitney’s covering lemma, we see that there exists a family $\{ Q_j \}$ of dyadic cubes, with the lengths $(l_j)_j$, satisfying that

(i) $O = \bigcup_j Q_j$ and $\{ Q_j \}$ are disjoint;

(ii) $2Q_j \subset O$ and $4Q_j \cap O^c \neq \emptyset$.

By this, to show the desired conclusion of Lemma 7.7, we only need to prove that, for all $j$,

$$\varphi \left\{ x \in Q_j : S_{h}^{\epsilon, R, \alpha}(f)(x) > 2\lambda, \mathcal{N}_h(f)(x) \leq \gamma \lambda \right\} , s \leq \gamma^{\epsilon_0} \varphi \left\{ Q_j, s \right\}. \quad (7.23)$$

Moreover, for all $x \in Q_j$ and $x_j \in 4Q_j \cap O^c$, using the fact that $f_{\max(10l_j, \epsilon)}(x) \subset f_{\max(10l_j, \epsilon)}(x_j)$ and the definition of $O$, we conclude that

$$S_{h}^{\max(10l_j, \epsilon), R, \alpha}(f)(x) \leq \lambda. \quad (7.24)$$

Thus, if $\epsilon > 10l_j$, we know that, for all $x \in Q_j$, $S_{h}^{\epsilon, R, \alpha}(f)(x) \leq \lambda$, which contracts with the condition $S_{h}^{\epsilon, R, \alpha}(f)(x) > 2\lambda$. This implies that $Q_j = \emptyset$. Hence, (7.23) holds in this case. If $\epsilon < 10l_j$, from the fact that, for all $x \in \mathbb{R}^n$, $S_{h}^{\epsilon, R, \alpha}(f)(x) \leq S_{h}^{\max(10l_j, \epsilon), R, \alpha}(f)(x) + S_{h}^{\max(10l_j, \epsilon), R, \alpha}(f)(x)$, (7.24) and Lemma 2.5(viii), we deduce that, to prove (7.23), it suffices to prove that, for all $j$,

$$\left\{ x \in Q_j \cap F : S_{h}^{\epsilon, R, \alpha}(f)(x) > \lambda \right\} \leq \gamma^2 |Q_j|, \quad (7.25)$$
where $F := \{x \in \mathbb{R}^n : \mathcal{N}_\delta(f)(x) \leq \eta \lambda \}$. We prove (7.25) by dividing two cases. If $\epsilon \geq 5l_j$, then similar to the proof of [72, Lemma 3.4] (replace the strong Lipschitz domain $\Omega$ and the non-tangential maximal function therein respectively by $\mathbb{R}^n$ and the present version of the non-tangential maximal function in (7.4)), we conclude that, for all $x \in \mathbb{R}^n$, $S^{e, R}_h \hat{\mathcal{M}}(f)(x) \leq \mathcal{N}_h(f)(x)$, which, combining with the definition of $F$, shows that

$$
\int_{\partial \mathcal{M}^c R_{\varepsilon, \delta, 10l_j}} S^{e, R}_h \hat{\mathcal{M}}(f)(x)^2 \, dx \leq \int_{\partial \mathcal{M}^c R_{\varepsilon, \delta, 10l_j}} \mathcal{N}_h(f)(x)^2 \, dx \leq (\eta \lambda)^2 |Q|.
$$

This, together with Chebyshev's inequality, implies the validity of (7.25). For the case when $\epsilon < 5l_j$, let $G := \{(y, t) \in \mathbb{R}^n \times (\epsilon, 10l_j) : d(y, Q_j \cap F) < \frac{t}{10}\}$. From (7.22) and Fubini's theorem, we infer that

$$
\int_{G} t |\nabla u(y, t)|^2 \, dy \, dt,
$$

where $u(y, t) := e^{-t^2}(f)(y)$. To estimate $\int_{G} t |\nabla u(y, t)|^2 \, dy \, dt$, we need to introduce some smooth cut-off function defined on a latter truncated cone. More precisely, let

$$
G_t := \left\{ (y, t) \in \mathbb{R}^n \times \left\{ \frac{\epsilon}{2}, 20l_j \right\} : d(y, Q_j \cap F) < \frac{t}{10} \right\}
$$

and $\eta \in C_0^{\infty}(G_t)$ satisfying $\eta \equiv 1$ on $G_t$, $0 \leq \eta \leq 1$ and $|\nabla \eta|_{L^\infty(G_t)} \lesssim \frac{1}{t}$ (see also the proof of [40, Lemma 5.4]). Then, by the ellipticity condition, we see that

$$
\int_{G_t} t |\nabla u(y, t)|^2 \, dy \, dt \leq \int_{G_t} t |\nabla u(y, t)|^2 \eta(y, t) \, dy \, dt \lesssim \Re \left( \int_{G_t} t A(y, \nabla u(y, t) \nabla u(y, t, \eta(y, t)) \, dy \, dt \right) =: \Re \text{el}.
$$

Using Leibniz's rule, the definition of $L$ and the fact that $\partial_t u(y, t) = -2tL(t, y)$, we know that

$$
L = \int_{G_t} t A(y, \nabla u(y, t) \nabla (\eta u)(y, t)) \, dy \, dt - \int_{G_t} t A(y, \nabla u(y, t) \eta u(y, t) \nabla \eta(y, t)) \, dy \, dt
$$

$$
= \int_{G_t} t L(y, t)[\eta \eta u(y, t)] \, dy \, dt - \int_{G_t} t A(y, \nabla u(y, t) u(y, t) \nabla \eta(y, t)) \, dy \, dt
$$

$$
= -\frac{1}{2} \int_{G_t} \partial_t u(y, t)[\eta \eta u(y, t)] \, dy \, dt - \int_{G_t} t A(y, \nabla u(y, t) u(y, t) \nabla \eta(y, t)) \, dy \, dt =: -\frac{1}{2} \rho_1 - \rho_2.
$$

Thus, from the fact that $\partial_t |u(y, t)|^2 = 2 \Re \partial_t u(y, t) \overline{u(y, t)}$, the integral by parts and $\eta \in C_0^{\infty}(G_t)$, we deduce that

$$
\Re \text{el} = -\Re \left[ \frac{1}{2} \rho_1 - \rho_2 \right]
$$

$$
= -\frac{1}{4} \int_{G_t} \partial_t |u(y, t)|^2 \eta(y, t) \, dy \, dt - \Re \left( \int_{G_t} t A(y, \nabla u(y, t) \nabla \eta(y, t)) \, dy \, dt \right)
$$

$$
= \frac{1}{4} \int_{G_t \setminus \bar{G}} |u(y, t)|^2 \partial_t \eta(y, t) \, dy \, dt - \Re \left( \int_{G_t \setminus \bar{G}} t A(y, \nabla u(y, t) \nabla \eta(y, t)) \, dy \, dt \right)
$$

$$
=: I_2 + I_3.
$$

The remaining estimates for $I_2$ and $I_3$ are obtained by using a decomposition of the set $G_t \setminus \bar{G}$, the properties of the cut-off function $\eta$, Besicovith's covering lemma and a Caccioppoli's inequality (see, for example, [72, (3.29)-(3.33)] for the detail calculations). We then conclude that $I_2 + I_3 \lesssim (\eta \lambda)^2 |Q|$. This, together with an application of Chebyshev's inequality, implies the validity of (7.25), which completes the proof of Lemma 7.7. 

\[\square\]
With the help of Lemma 7.7, we now prove Theorem 7.5.

**Proof of Theorem 7.5.** We first prove the following equivalence relationships

\[ H_{p,R_0}(\mathbb{R}^n) = H_{p,N_0}(\mathbb{R}^n) = H_{p,L}(\mathbb{R}^n). \]

The proof is divided into the following three steps.

**Step 1.** \( L^2(\mathbb{R}^n) \cap H_{p,N_0}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_{p,L}(\mathbb{R}^n) \). Let \( f \in L^2(\mathbb{R}^n) \cap H_{p,N_0}(\mathbb{R}^n) \). For any \( 0 < \epsilon < R < \infty \) and \( \gamma \in (0,1] \), by Lemma 2.4(ii), Fubini’s theorem and Lemma 7.7, we conclude that

\[
\int_{\mathbb{R}^n} \varphi \left( x, S^{c,R}_{\gamma,1}(f)(x) \right) \, dx \\
\sim \int_0^\infty \varphi \left( \left\{ x \in \mathbb{R}^n : S^{c,R}_{\gamma,1}(f)(x) > t \right\}, t \right) \frac{dt}{t} \\
\sim \int_0^\infty \varphi \left( \left\{ x \in \mathbb{R}^n : S^{c,R}_{\gamma,1}(f)(x) > t, N_0(f)(x) \leq \gamma t \right\}, t \right) \frac{dt}{t} + \int_0^\infty \varphi \left( \left\{ x \in \mathbb{R}^n : N_0(f)(x) > \gamma t \right\}, t \right) \frac{dt}{t} \\
\lesssim \gamma \int_0^\infty \varphi \left( \left\{ x \in \mathbb{R}^n : S^{c,R}_{\gamma,2}(f)(x) > \frac{t}{2} \right\}, t \right) \frac{dt}{t} + \frac{1}{\gamma} \int_0^\infty \varphi \left( \left\{ x \in \mathbb{R}^n : N_0(f)(x) > \gamma t \right\}, t \right) \frac{dt}{t} \\
\lesssim \gamma \int_{\mathbb{R}^n} \varphi \left( x, S^{c,R}_{\gamma,1}(f)(x) \right) \, dx + \frac{1}{\gamma} \int_{\mathbb{R}^n} \varphi \left( x, N_0(f)(x) \right) \, dx. \tag{7.26}
\]

By the change of variables and the fact that \( \varphi \) is of uniformly upper type 1, we further see that

\[
\int_{\mathbb{R}^n} \varphi \left( x, S^{c,R}_{\gamma,1}(f)(x) \right) \, dx \\
\lesssim \gamma \int_{\mathbb{R}^n} \varphi \left( x, S^{c,R}_{\gamma,2}(f)(x) \right) \, dx + \frac{1}{\gamma} \int_{\mathbb{R}^n} \varphi \left( x, N_0(f)(x) \right) \, dx.
\]

Moreover, using an argument similar to that used in the proof of [74, Lemma 7.7], we conclude that, for all \( 0 < \alpha < \beta < \infty \) and \( s \in (0, \infty) \),

\[
\int_{\mathbb{R}^n} \varphi \left( x, S^{c,R,s}_{\gamma,0}(f)(x) \right) \, dx \sim \int_{\mathbb{R}^n} \varphi \left( x, S^{c,R,\beta}_{\gamma,0}(f)(x) \right) \, dx.
\]

From this and (7.26) with \( \gamma \) sufficient small, it follows that

\[
\int_{\mathbb{R}^n} \varphi \left( x, S^{c,R}(f)(x) \right) \, dx \lesssim \int_{\mathbb{R}^n} \varphi \left( x, N_0(f)(x) \right) \, dx.
\]

Letting \( \epsilon \to 0 \) and \( R \to \infty \), we immediately know that

\[
\int_{\mathbb{R}^n} \varphi \left( x, S_0(f)(x) \right) \, dx \lesssim \int_{\mathbb{R}^n} \varphi \left( x, N_0(f)(x) \right) \, dx,
\]

which implies that \( \|f\|_{H_{p,N_0}(\mathbb{R}^n)} \lesssim \|f\|_{H_{p,L}(\mathbb{R}^n)} \). Thus, \( L^2(\mathbb{R}^n) \cap H_{p,N_0}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_{p,L}(\mathbb{R}^n) \).

**Step 2.** \( L^2(\mathbb{R}^n) \cap H_{p,R_0}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H_{p,R_0}(\mathbb{R}^n) \). Let \( f \in L^2(\mathbb{R}^n) \cap H_{p,R_0}(\mathbb{R}^n) \). By their definitions (see (7.2) and (7.3)), we see that \( N_0^\alpha(f) \leq R_0(f) \). Moreover, similar to [46, Lemma 5.3], we conclude that, for any \( 0 < \alpha < \beta < \infty \),

\[
\|N_0^\alpha(f)\|_{L^p(\mathbb{R}^n)} \sim \left\|N_0^\beta(f)\right\|_{L^p(\mathbb{R}^n)}.
\]
which immediately implies that

\[ \|\mathcal{N}_0(t)\|_{L^p} \sim \|\mathcal{N}_0^1(t)\|_{L^p} \leq \|\mathcal{R}_0(t)\|_{L^p}. \]

This establishes the inclusion \( L^2(\mathbb{R}^n) \cap H_{\varphi, R_0}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H_{\varphi, N_0}(\mathbb{R}^n) \).

**Step 3.** \( L^2(\mathbb{R}^n) \cap H_{\varphi, l}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap H_{\varphi, N_0}(\mathbb{R}^n) \). Let \( f \in L^2(\mathbb{R}^n) \cap H_{\varphi, l}(\mathbb{R}^n) \). By Theorem 4.9 and Lemma 5.7, it suffices to prove that, for any \( \lambda \in \mathbb{C} \) and \( (\varphi, q, M, \epsilon) \)-molecule \( \alpha \) associated with the ball \( B \),

\[ \int_{\mathbb{R}^n} \varphi(x, \mathcal{R}_0(\lambda \alpha)(x)) \, dx \leq \varphi \left( B, \frac{|\lambda|}{\|\mathcal{R}_0\|_{L^p}} \right), \quad (7.27) \]

where \( \epsilon \in (0, \infty) \) and \( M \in \mathbb{N} \) can be chosen sufficient large. The estimate (7.27) can be proved by using Assumption (B); see, for example, the proof of (4.8). We omit the details.

From Steps 1 through 3, we deduce that

\[ L^2(\mathbb{R}^n) \cap H_{\varphi, l}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_{\varphi, N_0}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap H_{\varphi, R_0}(\mathbb{R}^n) \]

with equivalent quasi-norms, which, together with the fact that

\[ L^2(\mathbb{R}^n) \cap H_{\varphi, l}(\mathbb{R}^n), \, L^2(\mathbb{R}^n) \cap H_{\varphi, N_0}(\mathbb{R}^n) \text{ and } L^2(\mathbb{R}^n) \cap H_{\varphi, R_0}(\mathbb{R}^n) \]

are, respectively, dense in \( H_{\varphi, l}(\mathbb{R}^n), \, H_{\varphi, N_0}(\mathbb{R}^n) \) and \( H_{\varphi, R_0}(\mathbb{R}^n) \), and a density argument, then implies that the spaces \( H_{\varphi, l}(\mathbb{R}^n), \, H_{\varphi, N_0}(\mathbb{R}^n) \) and \( H_{\varphi, R_0}(\mathbb{R}^n) \) coincide with equivalent quasi-norms. The proof for the equivalent relationships that \( H_{\varphi, l}(\mathbb{R}^n) = H_{\varphi, N_0}(\mathbb{R}^n) = H_{\varphi, R_0}(\mathbb{R}^n) \) is similar, we omit the details here. This finishes the proof of Theorem 7.5. \( \Box \)

### 7.2. Boundedness of the Riesz transform \( \nabla L^{-1/2} \)

In this subsection, we study the boundedness of the Riesz transform \( \nabla L^{-1/2} \) associated with \( L \) on \( H_{\varphi, l}(\mathbb{R}^n) \) for \( i(\varphi) \in \left( \frac{\alpha}{n+1}, 1 \right] \) and the associated weak boundedness at the endpoint \( i(\varphi) = \frac{\alpha}{n+1} \). Our main result is as follows.

**Theorem 7.8.**

Let \( \varphi \) and \( L \) be respectively as in Definition 2.2 and (7.1). Let \( \iota(\varphi), \, i(\varphi), \, q(\varphi) \) and \( r(\varphi) \) be, respectively, as in (2.4), (2.5), (2.6) and (2.7).

(i) If \( r(\varphi) \geq \frac{\min\{p(1), q(1)\} \gamma}{\varphi(1)} \), then \( \nabla L^{-1/2} \) is bounded from \( H_{\varphi, l}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \).

(ii) If \( i(\varphi) \in \left( \frac{\alpha}{n+1}, 1 \right], \, q(\varphi) \leq \frac{n+1}{\varphi(1)} \) and \( r(\varphi) \geq \frac{\min\{p(1), q(1)\} \gamma}{\varphi(1)} \), then \( \nabla L^{-1/2} \) is bounded from \( H_{\varphi, l}(\mathbb{R}^n) \) to \( H_{\varphi, l}(\mathbb{R}^n) \).

To prove Theorem 7.8, we need a new molecular characterization of the classical Musielak-Orlicz-Hardy space \( H_{\varphi}(\mathbb{R}^n) \). Similar to [42, Theorem 4.11], we have the following molecular characterization of \( H_{\varphi}(\mathbb{R}^n) \).

**Proposition 7.9.**

Let \( \varphi \) be as in Definition 2.2 and \( q \in (1, \infty) \). Assume that \( s \in \mathbb{N} \) satisfying \( s \geq [n\{\frac{q(1)}{\varphi(1)} - 1\}], \, \epsilon \in (\max\{n + s, \, n \{\frac{q(1)}{\varphi(1)}\} - \frac{s}{2}, \infty\} \), \( q(\varphi) \in [1, q] \) and \( r(\varphi) \geq \frac{q}{q - q(\varphi)} \), where \( i(\varphi), \, q(\varphi) \) and \( r(\varphi) \) are respectively as in (2.5), (2.6) and (2.7). Then, \( H_{\varphi, mol}(\mathbb{R}^n) \) and \( H_{\varphi}(\mathbb{R}^n) \) coincide with equivalent quasi-norms.

The proof of Proposition 7.9 is similar to that of [42, Theorem 4.11]. We omit the details here. Observe that, in [42, Theorem 4.11], the ranges of the exponents may be different from these of Proposition 7.9. More precisely, in [42, Theorem 4.11], the authors want to relax the range of the Musielak-Orlicz function \( \varphi \), by narrowing the range of the exponent \( q \). However, in the present case, we need more wider range of \( q \).
Proof of Theorem 7.8. The proof of (i) depends on Theorem 4.8 and Lemma 5.7. We only need to show that, for any $\lambda \in \mathbb{C}$ and $(\varphi, q, M, \epsilon)_L$-molecule $\alpha$ (with $M$ and $\epsilon$ large enough) associated with the ball $B$,

$$\int_{\mathbb{R}^n} \varphi(x, \nabla L^{-1/2}(\lambda \alpha))(x) \, dx \lesssim \varphi \left( B, \frac{|\lambda|}{|\alpha||_{\mathcal{L}(\mathbb{R}^n)}} \right).$$  \hspace{1cm} (7.28)

Using the $L^q(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$ for all $q \in (p_-(L), p_+(L))$ and the following off-diagonal estimates that

$$\left\| \nabla L^{-1/2} \left( I - e^{-tL} \right)^M f \right\|_{L^q(E)} \lesssim \left\{ \frac{t}{d(E, F)} \right\}^M \|f\|_{L^q(E)}$$

and

$$\left\| \nabla L^{-1/2} \left( tL e^{-tL} \right)^M f \right\|_{L^q(E)} \lesssim \left\{ \frac{t}{d(E, F)} \right\}^M \|f\|_{L^q(E)}$$

for closed sets $E, F \subset \mathbb{R}^n$ with $d(E, F) > 0$, we conclude (7.28) by using the same method as in (5.7). This shows (i). To prove (ii), let $q \in (p_-(L), p_+(L))$. For any $(\varphi, q, M, \epsilon)_L$-molecule $\alpha$ associated with $B$, similar to [74, (7.34)], we infer that there exists a large enough positive constant $c_0$ such that, for all $j \in \mathbb{Z}_+$,

$$\left\| \nabla L^{-1/2}(\alpha) \right\|_{L^q(B \setminus \mathbb{R}^n)} \lesssim 2^{-2q_0} |B| 2^{-1} \|\alpha\|_{L^q(\mathbb{R}^n)}.$$  \hspace{1cm} (7.29)

Moreover, since $1 \leq q(\varphi) < \frac{n+1}{n+q(\varphi)}$, we know $s := \left\lfloor \frac{n}{q(\varphi)} \right\rfloor = 0$, which, together with the fact that (see, for example, the proof of [46, Theorem 7.4] when $\varphi$ is an Orlicz function)

$$\int_{\mathbb{R}^n} \nabla L^{-1/2}(\alpha)(x) \, dx = 0,$$

immediately implies that, for each $(\varphi, q, M, \epsilon)_L$-molecule $m$ associated with the ball $B$, $\nabla L^{-1/2}(\alpha)$ is a $(\varphi, q, 0, \epsilon)_L$-molecule associated with the same ball $B$. This, together with the assumptions $q(\varphi) < \frac{n+1}{n+q(\varphi)}$, $r(\varphi) > \left( \frac{\min(p_-(L), q_+(L))}{q(\varphi)} \right)$ and Proposition 7.9, implies that $\nabla L^{-1/2}$ is bounded from $H_{q, L}(\mathbb{R}^n)$ to $H_{q, L}(\mathbb{R}^n)$, which completes the proof of (ii) and hence Theorem 7.8.

Now, we establish the weak boundedness of $\nabla L^{-1/2}$ at the endpoint $i(\varphi) = \frac{n}{n+1}$. Before stating our conclusions, we first recall some necessary definitions.

Definition 7.10. Let $\varphi$ be as in Definition 2.2, $\psi \in S(\mathbb{R}^n)$ satisfying $\text{supp} \psi \subset B(0, 1)$ and $\int_{\mathbb{R}^n} \psi(x) \, dx = 1$. The weak Musielak-Orlicz-Hardy space $WH_{\mathcal{H}}(\mathbb{R}^n)$ is defined to be the set $\{f \in S'(\mathbb{R}^n) : \|f\|_{WH_{\mathcal{H}}(\mathbb{R}^n)} < \infty\}$, where

$$\|f\|_{WH_{\mathcal{H}}(\mathbb{R}^n)} := \left\| \sup_{\rho \in (0, \infty)} \psi_\rho * f \right\|_{L^\infty(\mathbb{R}^n)}$$

$$:= \inf \left\{ \lambda \in (0, \infty) : \sup_{\rho \in (0, \infty)} \varphi \left( \left\{ x \in \mathbb{R}^n : \sup_{\rho \in (0, \infty)} \|\psi_\rho * f\|(x) > \eta \right\}, \frac{\eta}{\lambda} \right\} \leq 1 \right\}$$

and, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, $\psi_t(x) := t^{-n} \psi(t^{-1}x)$.

Theorem 7.11. Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (7.1). Assume that $i(\varphi), l(\varphi), q(\varphi)$ and $r(\varphi)$ are respectively as in (2.5), (2.4), (2.6) and (2.7). If $i(\varphi) = \frac{n}{n+1}$ is attainable, $\varphi \in A_1(\mathbb{R}^n)$, $l(\varphi) \in (0, 1)$ and $r(\varphi) > \left( \frac{\min(p_-(L), q_+(L))}{q(\varphi)} \right)$, then $\nabla L^{-1/2}$ is bounded from $H_{q, L}(\mathbb{R}^n)$ to $WH_{\mathcal{H}}(\mathbb{R}^n)$.
Remark 7.12.

Theorem 7.8 completely covers [46, Theorems 7.1 and 7.4] by taking \( \varphi \) as in (1.1) with \( \nu \equiv 1 \) and \( \Phi \) concave. Theorem 7.11 completely covers [19, Theorem 1.2] by taking \( \varphi(x,t) := t^{\alpha(n+1)} \) for all \( x \in \mathbb{R}^n \) and \( t \in [0, \infty) \).

To prove Theorem 7.11, we need the following superposition principle of weak type estimates.

**Lemma 7.13.**

Let \( \varphi \) be as in Definition 2.2 satisfying \( l(\varphi) \in (0, 1) \), where \( l(\varphi) \) is as in (2.4). Assume that \( \{a_j\}_j \) is a sequence of measurable functions and \( \{\lambda_j\}_j \subset \mathbb{C} \) such that there exists a sequence \( \{B_j\}_j \) of balls, it holds that

\[
\sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right) < \infty.
\]

Moreover, if there exists a positive constant \( C \) such that, for all \( \eta \in (0, \infty) \) and \( j \in \mathbb{N} \),

\[
\varphi \left( \{ x \in \mathbb{R}^n : |\lambda_j a_j(x)| > \eta \}, \eta \right) \leq C \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right).
\]

Then, there exists a positive constant \( \tilde{C} \) such that, for all \( \eta \in (0, \infty) \),

\[
\varphi \left( \left\{ x \in \mathbb{R}^n : \sum_j |\lambda_j a_j(x)| > \eta \right\}, \eta \right) \leq \tilde{C} \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right).
\]

**Proof.** For any given \( \eta \in (0, \infty) \), let \( E := \bigcup_j \{ x \in \mathbb{R}^n : |\lambda_j a_j(x)| > \eta \} \). From (7.30), we deduce that

\[
\varphi(E, \eta) \leq \sum_j \varphi \left( \{ x \in \mathbb{R}^n : |\lambda_j a_j(x)| > \eta \}, \eta \right) \leq \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right),
\]

which is desired. On the other hand, taking \( p_1 \in (l(\varphi), 1) \), then we know that \( \varphi \) is of uniformly upper type \( p_1 \). This, together with Chebyshev’s inequality and (7.30), implies that

\[
\varphi \left( \left\{ E^c : \sum_j |\lambda_j a_j(x)| > \eta \right\}, \eta \right) \leq \frac{1}{\eta} \sum_j \int_{\{ x \in \mathbb{R}^n : |\lambda_j a_j(x)| \leq \eta \}} |\lambda_j a_j(x)| \varphi(x, \eta) \, dx
\]

\[
\leq \frac{1}{\eta} \sum_j \int_0^{\eta} \varphi(\{ x \in \mathbb{R}^n : |\lambda_j a_j(x)| > \eta \}, \eta) \, dt
\]

\[
\leq \frac{1}{\eta} \int_0^{\eta} \frac{\eta^{p_1}}{t} \, dt \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right)
\]

\[
\leq \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{\|\lambda_j\|_{L^\infty(\mathbb{R}^n)}} \right),
\]

which, together with (7.31), implies the desired estimates and hence completes the proof of Lemma 7.13.

We now turn to the proof of Theorem 7.11.
Proof of Theorem 7.11. To prove this theorem, let \( q \in \{ p_+(L), \min\{ p_+(L), q_+(L)\} \} \). We first claim that it suffices to show that, for all \( \lambda \in \mathbb{C} \) and each \( (\varphi, q, \varepsilon, M)_L \)-molecules \( \sigma \) associated with the ball \( B \) (with \( \varepsilon \) and \( M \) large enough) and all \( \eta \in (0, \infty) \),

\[
\varphi \left\{ x \in \mathbb{R}^n : \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| > \eta \right\}, \eta \leq \varphi \left( B, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right). \tag{7.32}
\]

Indeed, if (7.32) holds true, then for all \( f \in L^2(\mathbb{R}^n) \cap H_{p, L}(\mathbb{R}^n) \), by Theorem 4.8, we know that, there exist a sequence \( \{ \lambda_j \} \subset \mathbb{C} \) and a sequence \( \{ \alpha_j \} \) of \( (\varphi, q, M, \varepsilon)_L \)-molecules associated with the balls \( \{ B_j \} \) such that \( f = \sum \lambda_j \alpha_j \) in \( L^2(\mathbb{R}^n) \) and \( ||f||_{H_{p, L}(\mathbb{R}^n)} \sim \Lambda(\{ \lambda_j \alpha_j \}) \). Moreover, by the assumption that \( I(\varphi) < 1 \), the change of variable and Lemma 7.13, we infer that

\[
||\nabla L^{-1/2}(f)||_{W_{H_{p, L}(\mathbb{R}^n)}} = \inf \left\{ \lambda \in (0, \infty) : \sup_{q \in (0, \infty)} \varphi \left\{ x \in \mathbb{R}^n : \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| \right\}, \eta \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda \in (0, \infty) : \sup_{q \in (0, \infty)} \varphi \left\{ x \in \mathbb{R}^n : \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| \right\}, \eta \leq 1 \right\}
\]

\[
\lesssim \inf \left\{ \lambda \in (0, \infty) : \sum_j \varphi \left( B_j, \frac{|\lambda_j|}{|X_{B_j}|_{L^p(\mathbb{R}^n)}} \right) \leq 1 \right\} \sim \Lambda(\{ \lambda_j \alpha_j \}) \sim ||f||_{H_{p, L}(\mathbb{R}^n)}.
\]

which, together with a density argument, implies that \( \nabla L^{-1/2} \) is bounded on \( H_{p, L}(\mathbb{R}^n) \) to \( W_{H_{p, L}(\mathbb{R}^n)} \). This shows the claim.

Now, we turn to the proof of (7.32). Let \( p_1 \in (I(\varphi), 1) \) be a uniformly upper type of \( \varphi \). Then

\[
\varphi \left\{ x \in 16B : \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| > \eta \right\}, \eta \leq \varphi \left( B, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right).
\]

\[
\leq \int_{16B} \varphi \left( x, \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| \right) dx
\]

\[
\leq \int_{16B} \left( 1 + \left( \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| \right)^{p_1} \right)^{\frac{1}{p_1}} \varphi \left( x, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right) dx
\]

\[
\leq \int_{16B} \varphi \left( x, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right) dx + \int_{16B} \left( \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| \right)^{p_1} \varphi \left( x, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right) dx
\]

\[
= : A + B.
\]

Without loss of generality, we may only estimate \( B \), the estimate for \( A \) being similar and simpler. Let \( q \geq 2 \) and \( q \) be as in (7.16) such that (5.8) holds true. Let \( \tilde{q} \in (q(\varphi), \infty) \). Then \( \varphi \in \mathbb{R} \cup \{ q(\varphi) \} \cap A_q(\mathbb{R}^n) \). Moreover, from [4, Theorem 4.2], we know that \( \nabla L^{-1/2} \) is bounded on \( L^q(\mathbb{R}^n) \), which, together with Hölder’s inequality and the \( L^q(\mathbb{R}^n) \)-boundedness of the Hardy–Littlewood maximal function \( M \), implies that

\[
B \lesssim ||X_B||_{L^q(\mathbb{R}^n)} \left\{ \int_{16B} \left[ M \left( \nabla L^{-1/2}(\alpha) \right)(x) \right]^q dx \right\}^{\frac{1}{q}} \left\{ \int_{16B} \left[ \varphi \left( x, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right) \right]^{\frac{1}{p_1}} \right\}^{\frac{1}{p_1}} \left[ 16B \right]^{-\frac{1}{q}} \left[ B \right]^{-\frac{1}{q_1}}.
\]

\[
\lesssim ||X_B||_{L^q(\mathbb{R}^n)} \left\{ \int_{16B} |\alpha(x)|^q dx \right\}^{\frac{1}{q}} \varphi \left( B, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right) |B|^{-\frac{1}{q}}.
\]

Using Definition 4.3, we further see that \( B \lesssim \varphi(B, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}}) \). Thus, we have

\[
\varphi \left\{ x \in 16B : \sup_{t \geq 0} \left| \left( \varphi_t \ast (\nabla L^{-1/2}(\lambda \sigma)) \right)(x) \right| > \eta \right\}, \eta \leq \varphi \left( B, \frac{|\lambda|}{|X_B|_{L^p(\mathbb{R}^n)}} \right). \tag{7.33}
\]
Now, we turn to the case that \( x \in (16B)^c \). For all \( i \in \{5, 6, \ldots \} \), let

\[
I_i := \varphi \left( \left\{ x \in S_i(B) : \sup_{t \in (0, r_B)} \left| \psi_t \ast \left[ \nabla L^{-1/2}(\lambda \alpha) \right] (x) \right| > \frac{\eta}{2} \right\}, \eta \right)
\]

and

\[
J := \varphi \left( \left\{ x \in (16B)^c : \sup_{t \in (0, \infty)} \left| \psi_t \ast \left[ \nabla L^{-1/2}(\lambda \alpha) \right] (x) \right| > \frac{\eta}{2} \right\}, \eta \right).
\]

Assume that \( \tilde{S}_i(B) := 2^{i+1}B \setminus 2^{-2}B \) and \( \tilde{S}_i(B) := 2^{i+2}B \setminus 2^{i+3}B \). It is easy to see that, for all \( x \in S_i(B) \), \( t \in (0, r_B) \) and \( |y - x| < t \), it holds that \( y \in \tilde{S}_i(B) \). Thus, similar to the estimate for \( B \), we conclude that

\[
I_i \lesssim \varphi \left( S_i(B), \mathcal{M} \left( X_{S_i(B)} \nabla L^{-1/2}(\lambda \alpha) \right) (x) \right)
\]

\[
\lesssim \int_{S_i(B)} \left| \nabla L^{-1/2}(\alpha)(x) \right|^p \ dx \lesssim \| X_{\tilde{S}_i(B)} \|^p_{L^p_{\infty}} \varphi \left( 2^i B, \frac{|\lambda|}{\|X_{\tilde{S}_i(B)}\|_{L^p_{\infty}}} \right),
\]

which, together with (7.29), further implies that

\[
I_i \lesssim 2^{-i} \varphi \left( B, \frac{|\lambda|}{\|X_{\tilde{S}_i(B)}\|_{L^p_{\infty}}} \right),
\]

where \( \epsilon_0 \in (0, \infty) \) is a sufficiently large constant. Thus, we conclude that

\[
\sum_{i \in \mathbb{N}} I_i \lesssim \varphi \left( B, \frac{|\lambda|}{\|X_{\tilde{S}_i(B)}\|_{L^p_{\infty}}} \right).
\]

We now estimate \( J \). For all \( x \in \mathbb{R}^n \), let

\[
F_i(x) := \sup_{t \in (0, \infty)} \left| \frac{1}{\|X_{S_i(B)}\|_{L^p_{\infty}}} \int_{S_i(B)} \left[ \psi \left( \frac{x - y}{t} \right) - \psi \left( \frac{x - x_B}{t} \right) \right] \nabla L^{-1/2}(\lambda \alpha)(y) \ dy \right|.
\]

For all \( j \in \mathbb{N} \), \( y \in S_j(B) \), \( t \in [2^{-j}r_B, 2^{2-j}r_B] \), \( |x - y| < t \) or \( |x - x_B| < t \), we have \( |x - x_B| \leq |x - y| + |y - x_B| < (2^j + 2^j)r_B \). Thus, \( x \in (2^j + 2^j)B \). This, together with the mean value theorem, Hölder’s inequality and (7.29), implies that, there exists a sufficiently large constant \( \epsilon_0 \) such that

\[
F_i(x) \lesssim \sum_{j \in \mathbb{N}} \| X_{S_j(B)} \|_{L^p_{\infty}} \left( 2^j r_B \right)^{\frac{n}{p}} \| \nabla L^{-1/2}(\alpha) \|_{L^q_{\infty}} \lesssim \| |x - x_B| \|_{L^1_{\infty}} \left( 2^j r_B \right)^{\frac{n}{p}} \| \nabla L^{-1/2}(\alpha) \|_{L^q_{\infty}}
\]

\[
= C_0 \| X_{S_j(B)} \|_{L^p_{\infty}} \left( 2^j r_B \right)^{\frac{n}{p}} \| |x - x_B| \|_{L^1_{\infty}} \left( 2^j r_B \right)^{\frac{n}{p}}
\]

where \( C_0 \) is a positive constant independent of \( i \) and \( x \). Let

\[
\eta_0 := \max \left\{ j \in \mathbb{N} : C_0 \| X_{S_j(B)} \|_{L^p_{\infty}} > \frac{\eta}{2} \right\}.
\]
We know that, for all \( x \in \left( 2^i + 2^b \right) B \):
\[
F_i(x) \leq C_0 |\lambda| 2^{-i(\alpha + 1)} 2^{-\delta_0} \| \chi_B \|^{-1}_{L^\infty} \leq \frac{n}{2},
\]
which immediately implies that
\[
\left\{ x \in (16B)^C : |F_i(x)| > \frac{n}{2} \right\} \subset \left( 2^i + 2^b \right) B.
\]
Thus, from the assumption that \( i(\varphi) \) is attainable and \( \varphi \in A_1(\mathbb{R}^n) \), we infer that
\[
\varphi \left( \left\{ x \in (16B)^C : F_i(x) > \frac{n}{2} \right\}, \eta \right) \lesssim \varphi \left( \left( 2^i + 2^b \right) B, 2^{-i(\alpha+1)} 2^{-\delta_0} \frac{|\lambda|}{\| \chi_B \|_{L^\infty}} \right) \lesssim 2^{-i(\alpha + 1)\varphi(\varphi)} 2^{-\delta_0} \varphi \left( B, \frac{|\lambda|}{\| \chi_B \|_{L^\infty}} \right),
\]
which, together with \( i(\varphi) = \frac{n}{\pi^1} \), implies that
\[
\varphi \left( \left\{ x \in (16B)^C : F_i(x) > \frac{n}{2} \right\}, \eta \right) \lesssim 2^{-i(\varphi(\varphi)-n)} \varphi \left( B, \frac{|\lambda|}{\| \chi_B \|_{L^\infty}} \right).
\]
By this, together with Proposition 7.9 and the fact that \( \int_{\mathbb{R}^n} \nabla L^{1/2}(\lambda)(x) \, dx = 0 \), we find that
\[
I \lesssim \varphi \left( \left\{ x \in (16B)^C : \sum_{i=5}^{\infty} \sup_{t \in [y, \infty)} \left| \int_{S_2(t)} \frac{1}{t^2} \left( \psi \left( \frac{x-y}{t} \right) - \psi \left( \frac{x-x_B}{t} \right) \right) \nabla L^{-1/2}(\lambda \omega)(y) > \eta \right| \, dx \right), \eta \right)
\]
\[
\lesssim \sum_{i=5}^{\infty} 2^{-i(\varphi(\varphi)-n)} \varphi \left( B, \frac{|\lambda|}{\| \chi_B \|_{L^\infty}} \right) \sim \varphi \left( B, \frac{|\lambda|}{\| \chi_B \|_{L^\infty}} \right).
\]
This, combined with (7.33) and (7.34), implies that (7.32) holds true, which completes the proof of Theorem 7.11.

8. The Musielak-Orlicz-Hardy space associated with the Schrödinger operator

In this section, we establish several equivalent characterizations of the Musielak-Orlicz-Hardy space \( H_{p_1}(\mathbb{R}^n) \) associated with the Schrödinger operator \( L := -\Delta + V \), where \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \), in terms of the Lusin-area function associated with the Poisson semigroup of \( L \), the non-tangential and the radial maximal functions associated with the heat semigroup generated by \( L \) and the non-tangential and the radial maximal functions associated with the Poisson semigroup generated by \( L \). Moreover, we also consider the boundedness of the associated Riesz transform on \( H_{p_1}(\mathbb{R}^n) \).

Let
\[
L := -\Delta + V
\]
be a Schrödinger operator, where \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). Since \( V \) is a nonnegative function, from the Feynman-Kac formula, we deduce that the kernel of the semigroup \( e^{-tL} \), \( h_t \), satisfies that, for all \( t \in (0, \infty) \) and \( x, y \in \mathbb{R}^n \),
\[
0 \leq h_t(x, y) \leq (4\pi t)^{-n/2} \exp \left\{ -\frac{|x-y|^2}{4t} \right\}.
\]
Thus, \( L \) satisfies Assumptions (A) and (B) with \( k = 1 \), \( p_L = 1 \) and \( q_1 = \infty \). Moreover, \( L \) satisfies Assumptions \( (H_1) \) and \( (H_2) \) as in Section 5.

For all \( f \in L^1(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), define the Lusin-area function \( S_p \) associated with the Poisson semigroup of \( L \) by
\[
S_p(f)(x) := \left\{ \int t \left( t \int \left| t \nabla L e^{-t_1 T}(y) \right|^2 \frac{dy \, dt}{t^{(n+1)}} \right)^{1/2} \right\}^{1/2}.
\]
Similar to Definition 4.1, we introduce the space \( H_{p, S_p}(\mathbb{R}^n) \) as follows.
Definition 8.1. Let \( \varphi \) be as in Definition 2.2 and \( L \) as in (8.1). A function \( f \in L^2(\mathbb{R}^n) \) is said to be in \( \widetilde{H}_{\varphi, S_p}(\mathbb{R}^n) \) if \( S_p(f) \in L^q(\mathbb{R}^n) \); moreover, define
\[
||f||_{\widetilde{H}_{\varphi, S_p}(\mathbb{R}^n)} := ||S_p(f)||_{L^q(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi \left( x, \frac{S_p(f)(x)}{\lambda} \right) \, dx \leq 1 \right\}.
\]
The \( S_p \)-adapted Musielak-Orlicz-Hardy space \( H_{\varphi, S_p}(\mathbb{R}^n) \) is defined to be the completion of \( \widetilde{H}_{\varphi, S_p}(\mathbb{R}^n) \) with respect to the quasi-norm \( || \cdot ||_{H_{\varphi, S_p}(\mathbb{R}^n)} \).

Definition 8.2. Let \( L \) and \( \varphi \) be as in (8.1) and Definition 2.2, respectively. A function \( f \in L^2(\mathbb{R}^n) \) is said to be in \( \widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n) \) if \( N_\lambda(f) \in L^q(\mathbb{R}^n) \); moreover, let \( ||f||_{\widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n)} := ||N_\lambda(f)||_{L^q(\mathbb{R}^n)} \). The \( N_\lambda \)-adapted Musielak-Orlicz-Hardy space \( H_{\varphi, N_\lambda}(\mathbb{R}^n) \) is defined to be the completion of \( \widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n) \) with respect to the quasi-norm \( || \cdot ||_{H_{\varphi, N_\lambda}(\mathbb{R}^n)} \).

The spaces \( H_{\varphi, N_\lambda}(\mathbb{R}^n) \), \( H_{\varphi, N_\mu}(\mathbb{R}^n) \) and \( H_{\varphi, S_p}(\mathbb{R}^n) \) are respectively defined in a similar way.

Now, we give the following equivalent characterizations of \( H_{\varphi, L}(\mathbb{R}^n) \) in terms of maximal functions associated with \( L \).

Theorem 8.3. Assume that \( \varphi \) and \( L \) are as in Definition 8.2. Then
\[
H_{\varphi, L}(\mathbb{R}^n), H_{\varphi, N_\lambda}(\mathbb{R}^n), H_{\varphi, N_\mu}(\mathbb{R}^n), H_{\varphi, S_p}(\mathbb{R}^n), H_{\varphi, S_p}(\mathbb{R}^n) \text{ and } H_{\varphi, S_p}(\mathbb{R}^n)
\]
coincide with equivalent quasi-norms.

Remark 8.4. Theorem 8.3 generalizes [74, Theorem 7.4] by extending the range of the considered weights. More precisely, The radial and non-tangential maximal function characterizations of \( H_{\varphi, L}(\mathbb{R}^n) \) were obtained in [74, Theorem 7.4] under the assumption \( \varphi \in RH_{2,1/2-\|\varphi\|}(\mathbb{R}^n) \). However, in the above Theorem 8.3, the assumption \( \varphi \in RH_{2,1/2-\|\varphi\|}(\mathbb{R}^n) \) is not needed.

Proof of Theorem 8.3. The proof of Theorem 8.3 is divided into the following six steps.

Step 1. \( \widetilde{H}_{\varphi, L}(\mathbb{R}^n) \subset \widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n) \). Let \( M \) and \( q \) be as in Theorem 5.4. By the proof of Theorem 5.4, we know that \( \widetilde{H}_{\varphi, L}(\mathbb{R}^n) \) and \( \widetilde{H}_{\varphi, L, at}(\mathbb{R}^n) \) coincide with equivalent quasi-norms. Thus, we only need to prove \( \widetilde{H}_{\varphi, L, at}(\mathbb{R}^n) \subset \widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n) \). To this end, similar to the proof of (5.7), it suffices to show that, for any \( \lambda \in \mathbb{C} \) and \( \varphi, q, M \)-atom \( \alpha \) associated to the ball \( B \),
\[
\int_{\mathbb{R}^n} \varphi \left( x, N_\lambda(\alpha)(x) \right) \, dx \leq \varphi \left( B, ||\alpha||_{L^q(\mathbb{R}^n)}^{-1} \right).
\]

From the \( L^q(\mathbb{R}^n) \)-boundedness of \( N_\lambda \), similar to the proof of (5.7), it follows that the above estimate holds true. We omit the details here.

Step 2. \( \widetilde{H}_{\varphi, N_\lambda}(\mathbb{R}^n) \subset \widetilde{H}_{\varphi, S_p}(\mathbb{R}^n) \), which is deduced from the fact that, for all \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), \( \mathcal{R}_p(f)(x) \leq N_\lambda(f)(x) \).
Step 3. $\widetilde{H}_{p, r_0}(\mathbb{R}^n) \subset \widetilde{H}_{p, r}(\mathbb{R}^n)$, whose proof is similar to that of Step 3 in the proof of [74, Theorem 7.4]. We omit the details here.

Step 4. $\widetilde{H}_{p, r}(\mathbb{R}^n) \subset \widetilde{H}_{p, \alpha}(\mathbb{R}^n)$, whose proof is similar to that of Step 4 in the proof of [74, Theorem 7.4], and hence we omit the details here.

Step 5. $\widetilde{H}_{p, \alpha}(\mathbb{R}^n) \subset \widetilde{H}_{p, \beta}(\mathbb{R}^n)$, whose proof is similar to that of [74, Proposition 7.6]. We omit the details here.

Step 6. $\widetilde{H}_{p, \beta}(\mathbb{R}^n) \subset \widetilde{H}_{p, \gamma}(\mathbb{R}^n)$.

Let $f \in \widetilde{H}_{p, \gamma}(\mathbb{R}^n)$. Then $\sqrt{L}e^{-r \gamma T}f \in T_{p+1}(\mathbb{R}^n)$, which, together with Proposition 4.5(ii), implies that $\pi_{L, M}(t \sqrt{L}e^{-r \gamma T}f) \subset H_{p, \gamma, \alpha}(\mathbb{R}^n)$. Furthermore, from the $H_{p, \gamma, \alpha}$ functional calculus, we infer that

$$f = \frac{C_{(\alpha)}}{C_{(\gamma)}} \pi_{L, M}(t \sqrt{L}e^{-r \gamma T}f)$$

in $L^2(\mathbb{R}^n)$, where $C_{(\gamma)}$ is a positive constant such that $C_{(\alpha)} = \left(\frac{2}{a_{(p+1)}} \right)^{1/2} = 1$ and $C_{(\gamma)}$ is as in (4.3). This, combined with Proposition 5.6, implies that $f \in H_{p, \gamma}(\mathbb{R}^n)$. Therefore, we know that $\widetilde{H}_{p, \gamma}(\mathbb{R}^n) \subset \widetilde{H}_{p, \gamma}(\mathbb{R}^n)$.

From Steps 1 though 6, we deduce that

$$\widetilde{H}_{p, \gamma}(\mathbb{R}^n), \widetilde{H}_{p, \alpha}(\mathbb{R}^n), \widetilde{H}_{p, \beta}(\mathbb{R}^n), \widetilde{H}_{p, \gamma, \alpha}(\mathbb{R}^n), \widetilde{H}_{p, \gamma, \beta}(\mathbb{R}^n)$$

coincide with equivalent quasi-norms, which, together with the fact that

$$\widetilde{H}_{p, \gamma}(\mathbb{R}^n), \widetilde{H}_{p, \alpha}(\mathbb{R}^n), \widetilde{H}_{p, \beta}(\mathbb{R}^n), \widetilde{H}_{p, \gamma, \alpha}(\mathbb{R}^n), \widetilde{H}_{p, \gamma, \beta}(\mathbb{R}^n)$$

are, respectively, dense in $H_{p, \gamma}(\mathbb{R}^n)$, $H_{p, \alpha}(\mathbb{R}^n)$, $H_{p, \beta}(\mathbb{R}^n)$, $H_{p, \gamma, \alpha}(\mathbb{R}^n)$, and $H_{p, \gamma, \beta}(\mathbb{R}^n)$, and a density argument, then implies that the spaces $H_{p, \gamma}(\mathbb{R}^n)$, $H_{p, \alpha}(\mathbb{R}^n)$, $H_{p, \beta}(\mathbb{R}^n)$, $H_{p, \gamma, \alpha}(\mathbb{R}^n)$, and $H_{p, \gamma, \beta}(\mathbb{R}^n)$ coincide with equivalent quasi-norms, which completes the proof of Theorem 8.3. □

From now on, we study the boundedness of $\nabla L^{-1/2}$ on $H_{p, \gamma}(\mathbb{R}^n)$. Similar to Theorem 7.8, we have the following conclusions.

**Theorem 8.5.**

Let $\varphi$ and $L$ be as in Definition 2.2 and (8.1), respectively. Assume that $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $r \in (1, p_0)$ with some $p_0 \in (2, \infty)$. Let $i(\varphi), q(\varphi)$ and $r(\varphi)$ be, respectively, as in (2.5), (2.6) and (2.7).

(i) If $r(\varphi) > (p_0/|i(\varphi)|)'$, then $\nabla L^{-1/2}$ is bounded from $H_{p, \gamma}(\mathbb{R}^n)$ into $L^{q}(\mathbb{R}^n)$.

(ii) If $i(\varphi) \in (\frac{p}{p_1 - 1}, 1), \frac{q(\varphi)}{q_0(\varphi)} \in (1, \frac{p+1}{p})$ and $r(\varphi) > (p_0/|q(\varphi)|)'$, then $\nabla L^{-1/2}$ is bounded from $H_{p, \gamma}(\mathbb{R}^n)$ into $H_{p, \gamma}(\mathbb{R}^n)$.

**Proof.** The proof of Theorem 8.5(i) is similar to that of Theorem 7.8(i). We omit the details here. Now we prove (ii). Let $f \in H_{p, \gamma}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $M \in \mathbb{N}$ with $M > \frac{2}{a_{(p+1)}}$. Then there exist $p_2 \in (0, i(\varphi))$ and $q_0 \in (q(\varphi), \infty)$ such that $M > \frac{2}{a_{(p+1)}} \varphi \in \mathbb{A}_{(q_0)}(\mathbb{R}^n)$. Moreover, by Proposition 5.4, we know that there exist $\lambda_j \in \mathcal{C}$ and a sequence $\{a_j\}$ of $\varphi, q, M_\lambda$-atoms such that $f = \sum_j \lambda_j a_j$ in $L^2(\mathbb{R}^n)$ and $||f||_{H_{p, \gamma}(\mathbb{R}^n)} \sim ||f||_{H_{p, \gamma, \alpha}(\mathbb{R}^n)}$.

Moreover, we know that $\nabla L^{-1/2}(f) = \sum_j \lambda_j \nabla L^{-1/2}(a_j)$ in $L^2(\mathbb{R}^n)$.

Let $a$ be a $(\varphi, q, M_\lambda)$-atom associated with the ball $B$. For $i \in \mathbb{Z}_+$, let $\chi_i := \chi_{S_i(\mathbb{R}^n)}, \widetilde{\chi}_i := |S_i(B)|^{-1} \chi_i, m_i := \int_{S_i} \nabla L^{-1/2}(a)\chi_i \, dx$ and $M_i := \nabla L^{-1/2}(a)\chi_i - m_i \widetilde{\chi}_i$. Then we have

$$\nabla L^{-1/2}(a) = \sum_{i=0}^{\infty} M_i + \sum_{i=0}^{\infty} m_i \widetilde{\chi}_i.$$  \hspace{1cm} (8.2)

For $j \in \mathbb{Z}_+$, let $N_j := \sum_{i=j}^{\infty} m_i$. By an argument similar to that used in the proof of [45, Theorem 6.3], we know that $\int_{\mathbb{R}^n} \nabla L^{-1/2}(a)\chi_i \, dx = 0$, which, together with (8.2), yields

$$\nabla L^{-1/2}(a) = \sum_{i=0}^{\infty} M_i + \sum_{j=0}^{\infty} N_{j+1} (\overline{\chi}_{j+1} - \overline{\chi}_j).$$
Obviously, for all $i \in \mathbb{Z}_+$,
\[
\text{supp } M_i \subset 2^{i+1} B \quad \text{and} \quad \int_{\mathbb{R}^n} M_i(x) \, dx = 0. \tag{8.3}
\]
When $i \in \{0, \ldots, 4\}$, by Hölder’s inequality and the $L^q(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, we conclude that
\[
||M_i||_{L^q(\mathbb{R}^n)} \lesssim \left\{ \int_{S_{\rho}(B)} |\nabla L^{-1/2} a(x)|^q \, dx \right\}^{1/q} + \left\{ \int_{S_{\rho}(B)} |m_i \tilde{x}(x)|^q \, dx \right\}^{1/q} \lesssim ||a||_{L^q(\mathbb{R}^n)} \lesssim ||B||^{1/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)}.
\tag{8.4}
\]
Moreover, similar to (7.29), we know that, for all $i \in \mathbb{N}$ with $i \geq 5$,
\[
||M_i||_{L^q(\mathbb{R}^n)} \lesssim ||\nabla L^{-1/2} a||_{L^q(S_{\rho}(B))} \lesssim 2^{-2M_i} ||B||^{1/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)}.
\tag{8.5}
\]
Furthermore, by $\varphi \in \mathbb{R} \mathbb{H}_0(\rho, q')(\mathbb{R}^n)$, we see that there exist $q \in (2, \rho_0)$ and $\tilde{q} \in (q(\varphi), \infty)$ such that $\varphi \in A_{\tilde{q}}(\mathbb{R}^n) \cap \mathbb{R} \mathbb{H}_0(q')^{(1/\rho)}(\mathbb{R}^n)$. From this, Hölder’s inequality, (8.4) and (8.5), it follows that, for all $i \in \mathbb{Z}_+$ and $t \in (0, \infty)$,
\[
\left(\varphi(2^{i+1} B, t)^{-1} \int_{2^{i+1} B} |M_i(x)|^q \varphi(x, t) \, dx \right)^{1/q} \lesssim \left[ \int_{2^{i+1} B} |M_i(x)|^q \, dx \right] \left[ \int_{2^{i+1} B} \varphi(x, t)^q \, dx \right]^{1/q} \lesssim 2^{-2\tilde{q}M_i} ||B||^{\tilde{q}/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)} 2^{k_i+1} ||B||^{-\tilde{q}/q},
\]
which implies that
\[
||M_i||_{L^q(\mathbb{R}^n)} \lesssim 2^{-2(2M_i+\tilde{q})/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)}.
\tag{8.6}
\]
Then by (8.3) and (8.6), we conclude that, for each $i \in \mathbb{Z}_+$, $M_i$ is a constant multiple of a $(\varphi, \tilde{q}, 0)$-atom. Moreover, from (8.5), it follows that $\sum_{i=0}^{\infty} M_i$ converges in $L^q(\mathbb{R}^n)$.

Now we estimate $||N_i(\tilde{x}_{i+1} - \tilde{x})||_{L^q(\mathbb{R}^n)}$ with $i \in \mathbb{Z}_+$. By Hölder’s inequality and (8.4), we see that
\[
||N_i(\tilde{x}_{i+1} - \tilde{x})||_{L^q(\mathbb{R}^n)} \lesssim ||N_i||_{L^q(\mathbb{R}^n)} \lesssim \sum_{j=i+1}^{\infty} |m_{j+1}||2^j B|^{-\frac{q}{2}} \lesssim \sum_{j=i+1}^{\infty} |m_{j+1}||2^j B|^{-\frac{q}{2}} \lesssim \sum_{j=i+1}^{\infty} |2^j B|^{-\frac{q}{2}} |2^j B|^{\frac{q}{2}} ||\nabla L^{-1/2} a||_{L^q(S_{\rho}(B))} \lesssim 2^{-2M_i} ||B||^{\frac{1}{q}} ||X a||^{-1}_{L^q(\mathbb{R}^n)}.
\tag{8.7}
\]
From this and Hölder’s inequality, similar to the proof of (8.6), we deduce that, for all $i \in \mathbb{Z}_+$,
\[
||N_i(\tilde{x}_{i+1} - \tilde{x})||_{L^q(\mathbb{R}^n)} \lesssim 2^{-2(2M_i+\tilde{q})/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)}.
\tag{8.8}
\]
which, together with $\int_{\mathbb{R}^n}(\tilde{x}_{i+1}(x) - \tilde{x}(x)) \, dx = 0$ and $\text{supp } (\tilde{x}_{i+1} - \tilde{x}) \subset 2^{i+1} B$, implies that, for each $i \in \mathbb{Z}_+$, $N_i(\tilde{x}_{i+1} - \tilde{x})$ is a constant multiple of a $(\varphi, \tilde{q}, 0)$-atom. Moreover, by (8.7), we see that $\sum_{i=0}^{\infty} N_i(\tilde{x}_{i+1} - \tilde{x})$ converges in $L^q(\mathbb{R}^n)$. Thus, (8.2) is an atomic decomposition of $\nabla L^{-1/2} a$ and, furthermore, by (8.6), (8.8), the uniformly lower type $p_2$ property of $\varphi$ and $M > \frac{q}{2}(\frac{\rho_0}{2} - \frac{1}{2})$, we know that
\[
\sum_{i \in \mathbb{Z}_+} \varphi \left(2^{i+1} B, ||M_i||_{L^q(\mathbb{R}^n)} \right) + \sum_{i \in \mathbb{Z}_+} \varphi \left(2^{i+1} B, ||N_{i+1}(\tilde{x}_{i+1} - \tilde{x})||_{L^q(\mathbb{R}^n)} \right) \lesssim \sum_{i \in \mathbb{Z}_+} \varphi \left(2^{i+1} B, 2^{-2(2M_i+\tilde{q})/q} ||X a||^{-1}_{L^q(\mathbb{R}^n)} \right) \lesssim \sum_{i \in \mathbb{Z}_+} 2^{-2(2M_i+\tilde{q})/q} 2^{kn_{\rho_0}} \lesssim 1.
\tag{8.9}
\]
Replace $a$ by $a_j$ and, consequently, we then denote $M, N_i$ and $\tilde{\chi}_i$ in (8.2), respectively, by $M_{j,i}$, $N_{j,i}$ and $\tilde{\chi}_{j,i}$. Similar to (8.2), we know that

$$\nabla L^{-1/2}(f) = \sum_j \sum_{i=0}^\infty \lambda_j M_{j,i} + \sum_j \sum_{i=0}^\infty \lambda_j N_{j,i+1}(\tilde{\chi}_{j,i+1} - \tilde{\chi}_{j,i}),$$

where, for each $j$ and $i$, $M_{j,i}$ and $N_{j,i+1}(\tilde{\chi}_{j,i+1} - \tilde{\chi}_{j,i})$ are constant multiples of $(\varphi, \tilde{\varphi}, 0)$-atoms and the both summations hold true in $L^2(\mathbb{R}^n)$ and hence in $\mathcal{S}(\mathbb{R}^n)$. Moreover, from (8.9) with $B, M$ and $N_{j,i+1}(\tilde{\chi}_{j,i+1} - \tilde{\chi}_{j,i})$ replaced, respectively, by $B_j$, $M_{j,i}$ and $N_{j,i+1}(\tilde{\chi}_{j,i+1} - \tilde{\chi}_{j,i})$, we deduce that

$$\Lambda \{ (M_{j,i}j, i) \} + \Lambda \{ (N_{j,i+1}(\tilde{\chi}_{j,i+1} - \tilde{\chi}_{j,i}))j, i \} \lesssim \Lambda \{ (\lambda_j a_j) \} \lesssim \|f\|_{H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)}.$$

By this, we conclude that $\|\nabla L^{-1/2}(f)\|_{H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)}$, which, together with the fact that $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ and a density argument, implies that $\nabla L^{-1/2}$ is bounded from $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ to $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$. This finishes the proof of Theorem 8.6.

Moreover, similar to Theorem 7.11, for the Riesz transform $\nabla L^{-1/2}$ associated with the Schrödinger operator $L$, we also have the following endpoint boundedness.

**Theorem 8.6.**

Let $\varphi$ and $L$ be respectively as in Definition 2.2 and (8.1), and $i(\varphi)$, $l(\varphi)$, $q(\varphi)$ and $r(\varphi)$ be respectively as in (2.5), (2.4), (2.6) and (2.7). Assume that $\nabla L^{-1/2}$ is bounded on $L^r(\mathbb{R}^n)$ for all $r \in (1, p_0)$ with some $p_0 \in (2, \infty)$, and $\varphi \in A_1(\mathbb{R}^n) \cap \mathbb{R}^{H_{\varphi, q(\varphi)}}(\mathbb{R}^n)$. If $i(\varphi) = \frac{n}{n+1}$ is attainable and $l(\varphi) \in (0, 1)$, then $\nabla L^{-1/2}$ is bounded from $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ to $WH_\varphi(\mathbb{R}^n)$.

**Remark 8.7.**

(i) Theorem 8.5 improves [74, Theorems 7.11 and 7.15] by widening the range of weights. More precisely, it was proved in [74, Theorems 7.11 and 7.15], respectively, that $\nabla L^{-1/2}$ is bounded from $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ when $\varphi \in \mathbb{R}^{H_{\varphi, q(\varphi)}}(\mathbb{R}^n)$, and from $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ to $H_{\varphi, \tilde{\varphi}}(\mathbb{R}^n)$ when $\varphi \in \mathbb{R}^{H_{\varphi, q(\varphi)}}(\mathbb{R}^n)$. From the assumption $p_0 \in (2, \infty)$, it follows that $p_0/l(\varphi) < (2/l(\varphi)) = 2/[2 - l(\varphi)]$ and $(p_0/q(\varphi))' < (2/q(\varphi))' = 2/[2 - q(\varphi)]$, which, together with Lemma 2.5(iv), implies that

$$\mathbb{R}^{H_{\varphi, q(\varphi)}}(\mathbb{R}^n) \subset \mathbb{R}^{H_{\varphi, q(\varphi)}}(\mathbb{R}^n).$$

Thus, Theorem 8.5 essentially improves [74, Theorems 7.11 and 7.15].

(ii) Theorem 8.6 completely covers [19, Corollary 1.1] by taking $\varphi(x, t) := t^{|x|+1}$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$.

The proof of Theorem 8.6 is similar to that of Theorem 7.11. We omit the details here.

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