When a quasi-particle Andreev reflects from a normal-superconducting (N-S) interface, the phase of the outgoing excitation is shifted by the phase of the superconducting order parameter [1]. Consequently if a phase coherent normal conductor is in contact with two superconductors with order parameter phases $\phi_1$, $\phi_2$, transport properties will be oscillatory functions of the phase difference $\phi = \phi_1 - \phi_2$. Spivak and Khmel’nitskii [2] argued that the ensemble average of the electrical conductance of a diffusive structure would have the form $\langle G(\phi) \rangle = A + B \cos(2\phi)$ where $A$ and $B$ are positive constants of magnitude less than the quantum of conductance $2e^2/h$. Thus the ensemble averaged conductance is predicted to have a maximum at zero phase and to have a fundamental periodicity of $\pi$. Later it was noted [3] that the conductance $G$ of a single sample should have a periodicity of $2\pi$, with an amplitude $< 2e^2/h$, although no comment on the possibility of a zero phase minimum was offered. The theories of references [2,3] were formulated before exact formulae [4,5] for the electrical conductance of N-S-N structures were available and therefore, to avoid significant supercurrents arising from Andreev scattering, were restricted to the regime where the distance between the superconductors is greater than the thermal coherence length $L_T = \sqrt{hD/k_B T}$.

The aim of this Letter is to address two crucial questions raised by recent experiments on such interferometers [6-10]. The first concerns the nature of the zero phase extremum in $G(\phi)$, which in the theory of [2] and all experiments using normal contacts is found to be a maximum, whereas in the low temperature limit $M < L_T$, theory [11] predicts that minima are allowed. A second concern is the amplitude of oscillation, which in the experiments [7,8,10] is found to be smaller than or of order $2e^2/h$, but in those of references [6,9] is several multiples of $2e^2/h$. If the latter are not an artifact of the experiments, then superconductivity enhanced quasi-particle interference effects (SEQUINs) of this kind should be present in the low temperature theory of references [4,5,11]. In what follows we demonstrate that large amplitude SEQUINs are indeed obtainable from exact solutions of the Bogoliubov - de Gennes equation and high-light conditions under which giant oscillations should be observable. Remarkably we predict that the amplitude vanishes for both very dirty and very clean samples and that to obtain a large effect, a degree of normal scattering must be introduced, in order that an approximate sum rule is broken. Further we predict that the experiments carried out to date are far from optimal and that amplitudes of oscillation many orders of magnitude greater than $e^2/h$ are attainable.

For simplicity we consider the zero temperature limit, where the electrical conductance between two normal reservoirs can be written [4,5] (in units of $2e^2/h$),

$$G = T_0 + T_a + \frac{2(R_a R'_a - T_a T'_a)}{R_a + R'_a + T_a + T'_a}$$

(1).

In this expression, $R_0, T_0$ (and $T_a$) are the coefficients for normal (Andreev) reflection and transmission for zero energy quasi-particles from reservoir 1, while $R'_0, T'_0$ (and $T'_a$) are corresponding coefficients for quasi-particles from reservoir 2. If each of the external leads connecting the reservoirs to the scatterer contains $N$ open channels, these satisfy $R_0 + T_0 + R_a + T_a = R'_0 + T'_0 + R'_a + T'_a = N$ and $T_0 + T_a = T'_0 + T'_a$. Furthermore, in the absence of a magnetic field, all reflection coefficients are even functions of $\phi$, while the transmission coefficients satisfy $T'_0(\phi) = T_0(-\phi)$, $T'_a(\phi) = T_a(-\phi)$. Consequently on quite general grounds, in the absence of a field, $G$ is predicted to be an even function of $\phi$ [12].

Figure 1 shows three examples of interferometers, for which results are presented below. Each has two superconducting regions with definite phases $\phi_1$ and $\phi_2$, in contact with a normal region (shown shaded). In each case the scattering region is connected to normal, external current carrying leads, with $N$ conducting channels. In figures 1a and 1b, a normal barrier (shown black) is placed at the N-S interface. In what follows we show numerical results obtained from a tight binding model, with diagonal elements $\epsilon_i$ and nearest neighbour hopping elements of magnitude unity. In the external leads $\epsilon_1 = 0$ and in the single line of sites forming the barrier $\epsilon_i = \epsilon_s$. In a disordered region of the sample, $\epsilon_s$ is chosen to be a random number uniformly distributed over the interval $\pm W$. As discussed in reference [5], the conductance is obtained by first evaluating the quantum mechanical scattering matrix and then evaluating the zero temperature conductance formula (1). The transfer matrix codes used [5] are extremely versatile and can be used to analyze arbitrary geometries, with multiple contacts.

For the structure of figure 1a, figure 2 shows numerical results for giant oscillations in the electrical conductance $G(\phi)$. In the absence of a barrier ($\epsilon_b = 0$), the amplitude of oscillation (in units of $2e^2/h$) is negligible compared with unity, whereas for $\epsilon_b = 1, 2, 3$ a large amplitude oscillation is present. As the barrier strength $\epsilon_b$ increases, the amplitude of oscillation initially increases to a value of order $Ne^2/h$, before decreasing in proportion to the zero phase conductance. These results show that at intermediate barrier strengths, the relative amplitude as well as the absolute amplitude is optimised. Figure 3 shows results for the phase periodic conductance of structures 1b (solid line) and 1c (dashed line). For the structure 1b in figure 3 we have presented results for the most favourable barrier strength, in the absence of disorder. Introducing normal disorder or changing the barrier strength decreases the amplitude of oscillation. In the case of structure 1c in figure 3 there is no barrier.
but a disorder comparable to the experiments of reference [9] has been used. We have examined the structures in figures 1b and 1c for a variety of disorders and barrier strengths and in no case have found an amplitude which is more than a few percent of $N2e^2/h$.

The crucial role of normal scattering in optimising this effect can be understood through a multiple scattering description of a N-S interface, which captures the essential physics of interferometers. Consider first an idealisation of the structure of figure 1a, in which the distance $M$ between the superconductors vanishes and therefore for a long enough sample there is no quasi-particle transmission. In this limit the total resistance reduces to a sum of two measureable boundary resistances and in what follows, we therefore focus attention on the left-hand boundary conductance [13]

$$G_B(\phi) = 2R_a = 2\text{Tr} r_a r_a^\dagger = \sum_{i,j=1}^{N} (R_a)_{ij}$$  \hspace{1cm} (2),

where $(R_a)_{ij} = |(r_a)_{ij}|^2$ is the Andreev reflection probability from channel $j$ to channel $i$. As in equation (1), the Andreev reflection coefficient is of the form $R_a = R_{\text{diag}} + R_{\text{off-diag}}$ where $R_{\text{diag}} = \sum_{i=1}^{N} (R_a)_{ii}$ and $R_{\text{off-diag}}$ is the remaining contribution from inter-channel scattering.

In the absence of disorder, for $M = 0$ and $\phi = 0$, translation symmetry in the direction transverse to the current flow guarantees that $R_{\text{off-diag}} = 0$. For the system of figure 1a, with no barrier, no disorder and $M = 45$, figure 4b, shows the behaviour of the coefficients $(R_a)_{ij}$ for $i \neq j$ and demonstrates that even for finite $M$, off-diagonal scattering at $\phi = 0$ is negligible. This figure leads us to a second observation, namely that even for non-zero $\phi$, almost all of the off-diagonal coefficients are negligibly small and that a given channel $i$ couples strongly to at most one other channel $j$. Consequently in the absence of disorder, a multiple scattering description involving pairs of channels captures the essential physics.

Consider a normal barrier to the left of a N-S interface. Particles (holes) impinging on the normal scatterer are described by a scattering matrix $s_{pp}$, $(sh_h)$, and those arriving at the N-S interface by a reflection matrix $\rho$, where

$$s_{pp} = \begin{pmatrix} r_{pp} & r'_{pp} \\ r_{tp} & r'_{tp} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_{pp} & \rho_{ph} \\ \rho_{hp} & \rho_{hh} \end{pmatrix}. $$

The elements of $s$ and $\rho$ are themselves matrices describing scattering between open channels of the external leads. For an ideal interface, where Andreev’s approximation is valid [1], $\rho_{pp}$ and $\rho_{hh}$ are negligible and in what follows will be set to zero. As a consequence, $\rho_{pp}$ and $\rho_{ph}$ are unitary and one obtains [14] $r_a = t_{pp}^\dagger \rho_{hp} M_{pp}^{-1} t_{pp}$, with $M_{pp} = 1 - r'_{pp} \rho_{ph} t_{hp}^\dagger \rho_{pp}$. In contrast with the analysis of [14], where $\rho_{hp}$ is proportional to the unit matrix, the interference effect of interest here is contained in the fact that $\rho_{hp}$ induces off-diagonal scattering. Substituting $r_a$ into equation (2) and taking advantage of particle-hole symmetry at $E = 0$, yields

$$G = 2\text{Tr} \left( T Q^{-1} T Q^\dagger \right)^{-1}$$  \hspace{1cm} (3)

where $Q = \rho_{pp} + (r')_{pp} \rho_{ph} (r')_{pp}^\dagger$, with $T = t_{pp}^\dagger t_{pp}$, the transmission matrix of the normal scattering region. This multiple scattering formula for the boundary conductance is valid in the presence of an arbitrary number of channels and in any dimension. Notice that if $T$ is equal to the unit matrix, $Q = \rho_{ph}$ and therefore $G = 2N$, irrespective of the phase periodic nature of $\rho_{ph}$. This demonstrates that at a clean interface, whatever the phase, the approximate unitarity of $\rho_{ph}$ yields the sum rule $R_{\text{diag}} + R_{\text{off-diag}} = N$ and therefore the conductance is independent of $\phi$. More generally, whenever normal reflection ($R_0$) and Andreev transmission ($T_a$) are negligibly small, unitarity imposes the sum rule $T_0 + R_a = N$ and since in this limit equation (1) reduces to $G = T_0 + R_a$, the amplitude of oscillation must vanish.

Equation (3) is very general and makes no assumption about the nature of matrices $\rho_{ph}$ and $s_{pp}$. We now introduce a two-channel model in which $\rho_{ph}$ is chosen to be an arbitrary two dimensional unitary matrix. In the absence of disorder, $t_{pp}$ and $r_{pp}$ are diagonal and therefore the only interchannel coupling is provided by $\rho_{ph}$. Substituting these matrices into equation (3), yields an expression for $r_a$ involving a single phase $\theta$, whose value is a linear combination of phase shifts due to normal reflection at the barrier, Andreev reflection at the N-S interface and the phase accumulated by an excitation travelling from the barrier to the interface. In what follows, we present results for the average value of $(R_a)_{ij}$, obtained by integrating over $\theta$. For a given value of $\phi$, once the normal barrier transmission coefficients $T_1$ and $T_2$ of the two channels are chosen, $(R_a)_{ij}$ is completely determined.

For the structure of figure 1a, figures 4a and 4b show numerical results for the diagonal $(R_a)_{ii}$ and off-diagonal coefficients $(R_a)_{ij}$ ($i \neq j$) respectively. Notice that at zero phase, most of the diagonal coefficients $(R_a)_{ii}$ are close to unity, although a small number of order $N|\Delta|/E_F$, where $|\Delta|$ is the order parameter magnitude and $E_F$ the Fermi energy, are suppressed, due to a breakdown of Andreev’s approximation for low angle scattering [15]. This slight breakdown of Andreev’s approximation yields a small amplitude oscillation even in the absence of normal potential scattering, but as emphasized by figure 2, the fractional amplitude is negligible. Figures 5a and 5b show corresponding results for the diagonal and off-diagonal coefficients in the presence of a barrier. At $\phi = 0$, there is no coupling between the channels and the scattering properties are those of $N$ independent channels, each with a barrier transmission coefficient $T_b$. The spectrum of the coefficients depends in detail on the shape of the barrier. The inset of figure 5a shows the boundary conductance $G(\phi)$ obtained by summing the curves in figures 5a nad 5b. Figures 5c and 5d show results from
the two channel evaluation of equation (3), obtained by choosing 10 pairs of transmission coefficients \( T_1, T_2 \), with \( T_2 = 0.2T_1 \). Clearly the qualitative features of the exact simulation are reproduced by this simple two channel analysis.

As shown in figure 2, in the absence of zero phase interchannel scattering, the zero phase extremum is a minimum. From figure 4, it is clear that the nature of the extremum is the result of a competition between diagonal Andreev reflection coefficients, which exhibit a zero phase maximum and off-diagonal coefficients which possess a zero phase minimum. Analytically we find that, in the absence of disorder, the second derivative of two channel conductance satisfies \( d^2 G / d \phi^2 \geq 0 \), for all barrier strengths. In contrast, figure 6 shows numerical results for the structure of figure 1a, with \( M = 45 \), \( M' = 50 \), \( M'' = 15 \), but with the barrier replaced by a disordered normal square of width 30 sites. This shows that replacing the barrier by a disordered region, causes the extremum of the off-diagonal coefficients to switch from a minimum to a maximum. In addition, the channels no longer couple in pairs and a complete multi-channel scattering description is needed. It is noted \([3,5]\) that changing the microscopic impurity configuration shifts \( G(\phi) \) by an amount of order \( e^2/h \), so in the presence of giant oscillations, the nature of the zero phase extremum is insensitive to such changes.

In summary, we have demonstrated that giant SE-QUINs are obtainable from exact solutions of the Bogoliubov - de Gennes equation and can be observed by breaking a crucial sum rule. Remarkably, the structure of figure 1c used in the experiments of \([9]\) shows only a small percentage effect, which nevertheless, in view of the large number of channels in these experiments, yielded an amplitude greater than \( 2e^2/h \). We predict that a more optimal structure is obtained in the presence of a normal barrier at the interface and that in metallic samples, with very large \( N \), the amplitude of oscillation could be orders of magnitude larger than \( 2e^2/h \).

Since carrying out this work, we became aware of a related paper on diffusive conductors \([16]\), where it is predicted that \( (R_0)_{ii} \gg (R_0)_{ij} \) (\( i \neq j \)). This phenomenon of giant diagonal scattering (i.e. backscattering) was offered as the origin of large oscillations in \( G(\phi) \). In this Letter, we predict (see eg. fig.4) that giant oscillations can occur even when diagonal and off-diagonal scattering coefficients are comparable.

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FIG. 1. Three possible interferometers, each with two superconducting regions of width \( M'' \). In figures 1a and 1b, the superconductors are separated by a distance \( M \) and in 1c by a distance \( 3M \). In figure 1a, the scattering region is connected to normal, external current carrying leads, of width \( M + 2M'' \), in figures 1b and 1c of width \( M \). In figures 1a and 1b, a normal barrier (shown black) is placed at the N-S interface. The current flows from left to right between external reservoirs with potentials \( \mu_1 \) and \( \mu_2 \). In the tight binding model used in the numerical simulations, the barrier comprises a line of sites with diagonal elements \( \epsilon_i = \epsilon_b \).

FIG. 2. Numerical results for the conductance \( G \) of the structure of figure 1a, with \( M = 45 \), \( M' = 50 \), \( M'' = 15 \), and number of open channels \( N = 45 \). Results are shown for barrier potentials \( \epsilon_b = 0, 1, 2, 3 \). The number adjacent a given curve is the corresponding value of \( \epsilon_b \).

FIG. 3. Figure 3 shows results for the conductance \( G \) of the structure of figure 1b (solid line: left scale) with \( M = M' = M'' = 10, W = 0 \), and 1c (dashed line: right scale) with \( M = M' = M'' = 10, W = 0 \). \( N = 10 \) in both cases. For the structure 1b results are shown for a barrier potential \( \epsilon_b = 2 \).
FIG. 4. Figure 4a shows numerical results for the diagonal Andreev reflection coefficients \((R_a)_{ii}\) of the structure of figure 1a, with \(M = 45, M' = 50, M'' = 15, N = 45\), and no barrier present, \((\epsilon_b = 0)\). Figure 4b shows corresponding results for the off-diagonal coefficients \((R_a)_{ij}\) with \(i \neq j\).

FIG. 5. Figure 5a shows numerical results for the diagonal Andreev reflection coefficients \((R_a)_{ii}\) of the structure of figure 1a, with \(M = 45, M' = 50, M'' = 15, N = 45\), and barrier potential, \(\epsilon_b = 2\). Figure 5b shows corresponding results for the off-diagonal coefficients \((R_a)_{ij}\) with \(i \neq j\). Figures 5c and 5d show analytical results from a two channel calculation.

FIG. 6. Numerical results obtained from a tight binding model of the structure of figure 1a, but with the barrier replaced by a disordered region of length 30 sites. In these simulations, \(M = 45, M' = 15, M'' = 50, N = 45, \Delta_0 = 0.1\), and the disorder is \(W = 2.8\).