Extreme value theory for constrained physical systems

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We investigate extreme value theory (EVT) of physical systems with a global conservation law which describe renewal processes, mass transport models and long-range interacting spin models. A special feature is that the distribution of the extreme value exhibits a non-analytical point in the middle of the support. We reveal three exact relationships between constrained EVT and random walks, density of condensation, and rate of renewals, all valid in generality beyond the mid point. For example for renewal processes at time $T$ describing blinking quantum dots, photon arrivals, zero crossings of Brownian motion and many other systems the cumulative distribution of the maximum $m$ is $F(m) = 1 - \varphi(m)(N(T - m))$ where $\langle N \rangle$ is the average number of jumps in the process and $\varphi$ is the survival probability. Our theory provides a general tool to describe the constrained EVT close and far from thermodynamic limit, and is therefore a unified extension of classical EVT.

Extreme events are a large class of phenomena in natural and man-made systems which are uncommon compared to the usual dynamics. Despite their rare occurrence they still can have influential consequences, e.g. the fastest sperm in fertilization, the longest trapping time in transport and first passage problems in Markov processes. The original problem considers a set of $N \in \mathbb{N}$ independent and identically distributed (IID) random variables $(x_1, \ldots, x_N)$ and describes the statistics of its maximum $x_{\max} = \max(x_1, \ldots, x_N)$. Let $\psi(x)$ be the probability density function (PDF) of the random variables, and $\Psi(x)$ the cumulative distribution function (CDF). The starting point in the analysis is the observation that when the maximum $x_{\max}$ has the value $m$ then all other random variables are less than or equal to $m$. So the CDF of the maximum is $\text{Prob}(x_{\max} \leq m) = \Psi^N(m)$ and hence the PDF of the maximum is obviously

$$f(m) = N\psi(m)\Psi^{N-1}(m). \quad (1)$$

A central result of this classical extreme value theory (EVT) is that the limiting maximum PDF for large $N$ converges to one of three classes of distributions called Weibull, Gumbel or Fréchet depending on the large x behavior of $\psi(x)$ when $m$ is shifted and rescaled appropriately.

However, for most systems the assumption of IID random variables has to be abandoned. Recently EVT were studied for a wide range of different models whose common property is the global confinement of their dynamics, see [1] for a review. This global conservation induces correlations among the random variables, and this represents a truly vast number of models including renewal processes (RP) [10–14], mass transport models such as zero range processes (ZRP) [15–19], and long-range interacting spin models such as the truncated inverse distance squared Ising model (TIDSI) [20–24]. These describe numerous physical systems, from zero crossing of Brownian motion, arrival times at a detector, to interacting systems to name only a few. It was shown previously how the constrain may modify completely the EVT in the sense of strong deviations from Fréchet’s law [22–24]. This is important in the context of condensation, where one element of the system is dominating the statistics, see details below.

Our work provides a complete set of relations between constrained EVT and much simpler quantifiers of the underlying stochastic dynamics. This is found beyond the critical point $m > C/2$ with the global constrain $C > 0$, a point where the statistics goes through a dynamical phase transition [23–24]. Somewhat similar to the classical ensembles of statistical physics, e.g. microcanonical ensembles with fixed energy, volume and number of particles and canonical ensembles where the temperature of the bath is the constrain, the different constrains discussed below also give rich physical behaviors specific to the ensemble. This is the reason why we find below classes of constrained EVT behaviors. For example for renewal processes we find a remarkably exact relation between EVT and the mean number of renewals $\langle N \rangle$. Thus we are able to map the problem of EVT with a global conservation rule, to well-studied stochastic quantifiers, and in that sense we go beyond previous studies, and solve the problem completely. It should be noted that our results are generally valid close and far from the thermodynamic limit. We present these results for the three ensembles of RP, ZRP and TIDSI.

Renewal processes are widely used in physics [10–14], for example the random arrival times of radioactive debris to a Geiger counter. Mathematically these processes are described with a PDF $\psi(\tau)$ of inter-arrival times, sometimes called waiting times. The process starts at time $0$ considered as the first event. To construct the process draw $\tau_1$ from the PDF $\psi(\tau)$, and this describes the timing of the second event, we then renew this process namely draw $\tau_2$ from the same PDF and then the third event takes place at time $\tau_1 + \tau_2$, etc. The PDF of $\psi(\tau)$ can be either thin tailed or fat tailed and this has major consequences on the behavior
of the extreme events. For example exponential PDF \( \psi(\tau) \) describes arrival times of independent photons to a detector. An example of a fat tailed process is the zero crossing of Brownian motion, where \( \psi(\tau) \sim \tau^{-3/2} \) similarly for blinking quantum dots \[23, 24\] or times between jumps in the anomalous continuous time random walks \[27, 28\]. Here the constrain is the total measurement time \( T \). Further the number of renewals in \([0, T]\) is denoted \( N \) and it is random unlike the classical EVT where the number of random variables is fixed. Let \( \tau_{\text{max}} \) be the largest time interval, namely \( \tau_{\text{max}} = \max(\tau_1, \ldots, \tau_{N-1}, \tau_B) \). Here \( \tau_B \) is called the backward recurrence time, i.e. the time interval between the last renewal event and measurement time \( T \), see Fig. 1. The constrain means \( T = \sum_{i=1}^{N-1} \tau_i + \tau_B \) hence these waiting times are not independent. The goal of EVT is to find the PDF of \( \tau_{\text{max}} \) denoted \( f(m; T) \) which was extensively studied in \[23\]. Here \( m \) is the value of the random \( \tau_{\text{max}} \).

We now present a formula relating constrained EVT and the mean number of renewals \( \langle N(t) \rangle \), i.e. the average number of renewal events in the time interval \([0, t]\). For the range \( T/2 < m < T \), we find the following relation

\[
f(m; T) = \varphi(m) R(T - m) + \psi(m) \langle N(T - m) \rangle.
\]

We consider power law waiting time PDFs \( \psi(\tau) \sim b_\alpha \tau^{-1-\alpha} \) and \( 0 < \alpha < 1 \). It is well-known that the maximum exhibits a condensation effect in the sense that the largest waiting time is of the order of the measurement time \( T \) \[23\]. This

![FIG. 1. Schematic figure of the three models renewal process, zero range process and truncated inverse distance squared Ising model presented in the main text. The maximum in each model is colored orange.](image)

![FIG. 2. Histogram of the maximum PDF \( f(m; T) \) of \( RP \) from Monte Carlo simulations (blue circles) compared with the theory of Eq. (2) (black line) with a) exponential \( \psi(\tau) = \exp(-\tau) \) and \( T = 2 \) and b) Pareto \( \psi(\tau) = 1/2\tau^{-3/2}, \tau > 1, \) and \( T = 10 \). The height of the gap at \( m = T/2 \) is exactly \( 2\psi(T/2)\varphi(T/2) \), see supplemental material (SM), which is a) \( \approx 0.27 \) and b) 0.04. The simulations were performed with \( 10^7 \) realizations.](image)
is related to the fact that the mean waiting time $\langle \tau \rangle$ diverges. Now from our main result Eq. 2 clearly the PDF of the maximum, depends on the rate $R$ and the mean $\langle N \rangle$. In the limit when $m$ is large $\psi(m) \sim b_\alpha m^{-1-\alpha}$ and $\varphi(m) \sim b_\alpha m^{-\alpha}/\alpha$. Hence from Eq. 2 we get
\[ f(m; T) \sim \frac{1}{T^\alpha} \frac{b_\alpha}{\alpha} R(T - m). \] (4)

Here we consider the case when $T - m = O(1)$. This behavior is presented in Fig. 3 together with numerical simulations. The solution is clearly nontrivial, and it is clear both from the figure and from Eq. 4 that by measuring the PDF of $m$ we get the average rate of renewals $R$. Of course vice versa is also true, namely one can in principle measure $R$ and get the PDF of $m$.

What is remarkable is that we need $R$ only for very short times (since $T - m < 6$ in Fig. 3) and we get the EVT statistics for large times. In Fig. 3 we also present in a dashed line previously obtained theory 22 which used other methods. These pioneering works searched for the scaling form $f(m, T) \sim g(m/T)/T$. The function $g(m/T)$ is easy to derive also from our formalism and is presented in Table 1. Note that this scaled solution hides the essence of the connection of the EVT to the rate $R$ and $\langle N \rangle$ which is the focus of our work.

To appreciate the relation Eq. 2 even better and to understand its meaning we now present a sketch of its derivation, more details can be found in the supplemental material (SM). Let $f(m; T) = \sum_{N=1}^{\infty} f_N(m; T)$ with $f_N(m; T)$ being the maximum PDF with exactly $N$ renewal events. The longest interval can be either the last interval $\tau_B$, or any other time interval in the process. The former is denoted $\tau_B$ in the schematic Fig. 1. Based on this we may split contributions to $f_N(m; T)$ into two terms with each a $(N - 1)$-multiple integral
\[ f_N(m; T) = \varphi(m) m \int_0^m d\tau \prod_{i=1}^{N-1} \psi(\tau_i) \delta(T - m - \|\tau\|_1) \]
\[ + (N - 1)\varphi(m) \int_0^m d\tau \prod_{i=1}^{N-2} \psi(\tau_i) \varphi(\tau_{N-1}) \delta(T - m - \|\tau\|_1). \] (5)

with the $(N - 1)$-vector $\tau = (\tau_1, \ldots, \tau_{N-1})$ and the taxicab norm $\|\tau\|_1 = \sum_{i=1}^{N-1} \tau_i$, see 22. The $N - 1$ in the second term of Eq. 5 represents the fact that we have $N - 1$ choices of the time interval becoming the largest one, and this $N - 1$ will give the mean number of renewals in our main formula of Eq. 2. Clearly in Eq. 5 the delta function represents the global constrain in the sense that the maximum is equal $m$ and hence the remaining time intervals must sum up to $T - m$ where as mentioned $T$ is the total measurement time or more generally the constrain.

We already declared our intention to derive the extreme value theory for $T/2 < m < T$. The first step is the recognition that we may extend the upper limit of the integration in Eq. 5 till infinity. This allows us to decouple the solution and simplify the problem dramatically. To see why this is permissible note that by definition all the waiting times are positive, and the delta function constrain implies that any individual $\tau_i$, excluding $\tau_{\text{max}}$, cannot be larger than $T - m$. So if $m > T/2$ the constrain forces all $\tau_i \neq \tau_{\text{max}}$ to be shorter than $T/2$. Hence extending the integration till infinity does not alter the result. The second step is to replace the delta function with inverse Fourier representation $\delta(\tau') = \int_{-\infty}^{\infty} \exp(i k \tau') dk/(2\pi)$. At this stage one can use $\hat{\psi}(ik) = \int_{-\infty}^{\infty} \psi(\tau) \exp(-ik\tau) d\tau$ which is the Laplace transform of $\psi(\tau)$, and here the upper limit of integration is the mentioned extension of the integration domain. One then transforms variables according to $ik = s$ to obtain the following expression
\[ f_N(m; T) = \mathcal{L}^{-1} \left[ \varphi(m) \hat{\psi}^{N-1}(s) + \psi(m)(N - 1) \hat{\psi}^{N-2} \right] \] (6)

where $\mathcal{L}^{-1}$ is the inverse Laplace transform $s \rightarrow T - m$. The third step is to identify the first and second terms on the right hand side of Eq. 6 with well-known expressions from renewal theory and hence we can show that
\[ f_N(m; T) = \varphi(m) Q_{N-1}(T - m) + \psi(m)(N - 1) P_{N-1}(T - m). \] (7)

Here $Q_{N-1}(t) dt$ is the probability of the $N$-th renewal event in time interval $(t, t + dt)$, its Laplace transform is
Fréchet's law

TABLE I. Collecting of limiting laws of \( f(m; C) \) for \( RP \) with \( C = T \) and \( TIDSI \) with \( C = L \) in the critical phase, see SM for a detailed derivation. The first half distribution with \( m \in (0, C/2) \) were studied for \( RP \) in [24] and for \( TIDSI \) in [24]. In the second half, i.e. \( m \in (C/2, C) \), the scaling \( m = O(C) \) is applied with the rescaled variable \( \xi = m/C \). We find again the laws of a) [23] and b) [23]. The expression of c) has been derived in [23] using other methods, they express the law as a sum of two hypergeometric functions while the expression in the table is simpler, see SM. The limiting laws d)-h) are first presented in this article thus marked red. Note that we present the expressions in this table without prefactors.

Zero range processes in equilibrium describe a system with a fixed number \( K \) of interacting particles. These particles are located in well separated traps or containers where transition times between the containers are very fast, see Fig. \( 4 \). We have \( N \) such traps, and in each trap \( i \in [1, N] \) we have \( \kappa_i \geq 0 \) particles. Clearly the constrain is \( K = \sum_{i=1}^N \kappa_i \). Importantly, the particles in each container interact and their energy is \( E(\kappa) \). The interactions are all short-ranged, in the sense that the particles are not interacting once they are in different containers. Here \( \psi(\kappa) \) is the probability of finding \( \kappa_i \) particles in container \( i \). In thermal equilibrium \( \psi(\kappa) \) is the Boltzmann factor which depends on temperature and the energy \( E(\kappa) \), however in general non-equilibrium situations, we use \( \psi(\kappa) \) which was extensively studied and depends on microscopical description of the transitions [20, 21]. In this model the number of containers \( N \) is fixed unlike the random number of renewals in the previous model. The EVT deals with maximum number of particles denoted \( \kappa_{\text{max}} = \max(\kappa_1, \ldots, \kappa_N) \) and the corresponding PDF \( f(m; K) \) where the random maximum \( \kappa_{\text{max}} \) takes the value \( m \). A well-studied phenomenon in this model is condensation [15,16]. When the density of the system \( K/N \) crosses a critical value, a macroscopic number of particles may occupy one container. It is then natural to wonder what is the distribution of the maximum, since that describes the statistical properties of the condensation [5,16,22].

In SM we analyse the ZRP using the same technique as before. We find that the maximum PDF \( f(m; K) \) in the range \( K/2 < m < K \) is related to the well-studied marginal PDF \( p_{\kappa}(\kappa; K) \), namely

\[
f_N(m; K) = N p_{\kappa}(\kappa; K).
\]

This result was obtained in [16] as a limiting law in the condensation phase of the model. Our result shows that it is exactly valid close and far from thermodynamic limit, and no matter if condensation happens or not. It is independent of the structure of \( \psi(\kappa) \). Hence it is a general connection between EVT and the marginal PDF.

The result Eq. (8) is technically related to random walk theory

\[
f_N(m; K) = \frac{1}{Z_N(K)} N \psi(m) \Phi_{N-1}(K-m)
\]

with \( \Phi_N \) being the PDF of the sum of \( N \) IID random variables. The partition function is \( Z_N(K) = \Phi_N(K) \). We see here a useful modification of the classical EVT result. The CDF \( \Psi_{N-1}(m) \) of Eq. (11) is now replaced by \( \Phi_{N-1}(K-m) \) divided by the normalization \( \Phi_N(K) \). When one site has the maximum mass \( m \) then all other masses sum up to \( K - m \) due to the constrain. Eq. (8) gives a connection between EVT of the constrained process and one of the most well-studied problems in probability theory: i.e. the sum of IID random variables, in physics this is simply the problem of a \( N \) step random walk.

The truncated inverse distance squared Ising model describes an one-dimensional system of spin domains with each having spins +1 or −1, see Fig. 1. The total length is \( L \). There is an inverse squared long-range
interaction between spins within the same domain. Let \( N \) be the random number of domains \( i \in [1, N] \) with each the domain length \( \lambda_i \geq 1 \). The constrain is clearly \( L = \sum_{i=1}^{N} \lambda_i \). The domain \( i \) of length \( \lambda_i \) is associated with the weight \( \psi(\lambda) \propto \lambda^{-\gamma} \) where the domain length decays with the parameter \( \gamma \geq 1 \) which is the product of the inverse temperature \( 1/(k_B T) \) and the long-range interacting coupling constant \( \Delta \). The relevance of \( TIDS \) is that it exhibits a mixed order phase transition, i.e. it shows features of phase transitions of first and of second kind. At the transition the magnetization is discontinuous and the correlation length diverges. Depending on the temperature there is either a ferromagnetic phase with a large number of domains or a paramagnetic phase with one domain of order \( L \). Thus the analysis of the extreme domain size \( \lambda_{\max} = \max(\lambda_1, \ldots, \lambda_N) \) is important \([3, 2]\).

All we have to do is to sum the ZRP result over all \( N \) while replacing the constrain \( K \) by \( L \). This is also true for the partition function of \( TIDS \), i.e. \( Z(L) = \sum_{N=1}^{\infty} \Phi_N(L) \). We find the second half maximum PDF with \( 2/L < m < L \) as

\[
f(m; L) = \frac{1}{Z(L)} \psi(m) \sum_{N=1}^{\infty} N \Phi_{N-1}(L-m).
\]

The interpretation is as for ZRP, while the maximum spin domain length is \( m \) all other lengths add up to the remaining \( L - m \). But now the number \( N \) is random, hence we had to sum. We can also relate this formula to the mean number of domains

\[
f(m; L) = \psi(m) \frac{Z(L-m)}{Z(L)} (\langle N(L-m) \rangle + 1)
\]

where \( \langle N(L-m) \rangle = \sum_{N=1}^{\infty} N p_N(L-m) \) is mean number of domains and \( p_N(L-m) = \Phi_N(L-m)/Z(L-m) \) the probability of having \( N \) domains up to \( L \), see SM for more details. In SM and Table I we present how our theory is suitable to easily derive limiting laws if compared to \( m < K/2 \). There we present the large \( L \) critical behavior of \( f(m; L) \) found between ferromagnetic and paramagnetic phases.

We presented exact results for constrained EVT. This is obtained due to decoupling of the problem, once \( m \) is larger than half of the constrain our results map constrained EVT to well-studied stochastic quantifiers: namely to the mean number of events/domains Eq. 2 and 11, to the marginal PDF Eq. 8, and to the sum of IID random variables Eq. 10. Rich behaviors are found since we deal with a wide variety of constrained ensembles, which describe many physical systems. With the exact results, we may derive statistical behaviors far and close to the thermodynamic limit. The emerging picture is vastly different if compared with classical EVT of IID random variables as presented in Table I

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[1] J.-P. Bouchaud and M. Mézard, J. Phys. A - Math. Gen. 30, 7997 (1997).
[2] S. Albeverio, V. Jentsch, and H. Kantz, Extreme Events in Nature and Society (Springer Verlag, Berlin Heidelberg, 2006).
[3] P. Embrechts, C. Klüppelberg, and T. Mikosch, Modelling Extremal Events: for Insurance and Finance, vol. 33 (Springer Verlag, Berlin Heidelberg, 2013).
[4] J.-Y. Fortin and M. Clausel, J. Phys. A - Math. Theor. 48, 183001 (2015).
[5] S. N. Majumdar, A. Pal, and G. Schehr, Phys. Rep. 840, 1 (2020).
[6] B. Meerson and S. Redner, Phys. Rev. Lett. 114, 198101 (2015).
[7] Z. Schuss, K. Basnayake, and D. Holcman, Phys. Life Rev. 28, 52 (2019).
[8] W. Wang, A. Vezzani, R. Burioni, and E. Barkai, Phys. Rev. Res. 1, 033172 (2019).
[9] D. Hertich and A. Godec, J. Phys. A - Math. Theor. 52, 244001 (2019).
[10] C. Godrèche and J. Luck, J. Stat. Phys. 104, 489 (2001).
[11] M. Niemann, E. Barkai, and H. Kantz, Math. Model. Nat. Pheno. 11, 191 (2016).
[12] W. Wang, J. H. Schulz, W. Deng, and E. Barkai, Phys. Rev. E 98, 042139 (2018).
[13] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2 (Wiley, New York, 1971).
[14] S. B. Lowen and M. C. Teich, Phys. Rev. E 47, 992 (1993).
[15] M. Evans, S. N. Majumdar, and R. Zia, J. Stat. Phys. 123, 357 (2006).
[16] S. Majumdar, Exact Methods in Low-dimensional Statistical Physics and Quantum Computing: Lecture Notes of the Les Houches Summer School: Volume 89, July 2008 407 (2010).
[17] S. N. Majumdar, M. Evans, and R. Zia, Phys. Rev. Lett. 94, 180601 (2005).
[18] R. Zia, M. Evans, and S. N. Majumdar, J. Stat. Mech. Theory E. 2004, L10001 (2004).
[19] M. R. Evans and T. Hanney, J. Phys. A - Math. Gen. 38, R195 (2005).
[20] A. Bar and D. Mukamel, Phys. Rev. Lett. 112, 015701 (2014).
[21] A. Bar and D. Mukamel, J. Stat. Mech. Theory E. 2014, P11001 (2014).
[22] M. R. Evans and S. N. Majumdar, J. Stat. Mech. Theory E. 2008, P05004 (2008).
[23] C. Godrèche, S. N. Majumdar, and G. Schehr, J. Stat. Mech. Theory E. 2015, P03014 (2015).
[24] A. Bar, S. N. Majumdar, G. Schehr, and D. Mukamel, Phys. Rev. E 93, 052130 (2016).
[25] F. D. Stefani, J. P. Hoogenboom, and E. Barkai, Phys. Today 62, 34 (2009).
[26] G. Margolin and E. Barkai, Phys. Rev. Lett. 94, 080601 (2005).
[27] R. Metzler and J. Klafter, Phys. Rep. 339, 1 (2000).
[28] R. Kutner and J. Masoliver, Eur. Phys. J. B 90, 50 (2017).
[29] A. Vezzani, E. Barkai, and R. Burioni, Phys. Rev. E 100, 012108 (2019).
Extreme value theory for constrained physical systems: Supplemental material

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We derive the details of the Letter in full detail.

GENERAL EXTREME STATISTICS FOR CONSTRAINED PHYSICAL MODELS

In the main text we consider the three models of renewal process (RP), zero range process (ZRP) and truncated inverse distance squared Ising model (TIDSI). Some relevant details are summarized in Table II.

| Model  | Random variables $x_i$ | Values of $x_i$ | Constrain $C$ | $N$   |
|--------|------------------------|----------------|--------------|------|
| RP     | Waiting times $\tau_i$ | Continuous     | Measurement time $T = \sum_{i=1}^{N-1} \tau_i + \tau_B$ | Random |
| ZRP    | Number of particles $\kappa_i$ | Discrete | Total number $K = \sum_{i=1}^{N} \kappa_i$ | Fixed |
| TIDSI  | Domain lengths $\lambda_i$ | Discrete     | Total length $L = \sum_{i=1}^{N} \lambda_i$ | Random |

TABLE II. Overview of the three models with a constrain. We present their relevant random variables, the constrain and the randomness of $N$.

We start our analysis of constrained extreme value theory with the cumulative distribution function (CDF) of these models under the condition of $N$ given random variables. This is for ZRP always fulfilled. The CDFs with given $N$ are

$$F_N^{(RP)}(m; T) = \int_0^m d\tau_1 \ldots \int_0^m d\tau_{N-1} \int_0^m d\tau_B \prod_{i=1}^{N-1} \psi(\tau_i) \varphi(\tau_B) \delta(T - \sum_{i=1}^{N-1} \tau_i - \tau_B),$$

$$F_N^{(ZRP)}(m; K) = \frac{1}{Z_N(K)} \sum_{\kappa_1 = 0}^{m} \ldots \sum_{\kappa_{N-1} = 0}^{m} \prod_{i=1}^{N} \psi(\kappa_i) \delta_{K - \sum_{i=1}^{N} \kappa_i},$$

$$F_N^{(TIDSI)}(m; L) = \frac{1}{Z(L)} \sum_{\lambda_1 = 0}^{m} \ldots \sum_{\lambda_{N-1} = 0}^{m} \prod_{i=1}^{N} \psi(\lambda_i) \delta_{L - \sum_{i=1}^{N} \lambda_i},$$

i.e. when the maximum has the value $m$ then all other values are less or equal $m$. These expressions of the CDFs can be found for RP in [1], for ZRP in [2] and for TIDSI in [3]. The partition functions for ZRP and TIDSI are given by convolutions $Z_N(K) = (\psi \ast \ldots \ast \psi)^{(N)}(K)$ and $Z(L) = \sum_{m=1}^{\infty} (\psi \ast \ldots \ast \psi)^{(N)}(L)$. The 2-fold convolution for discrete functions as in case of ZRP is defined as $(\psi_1 \ast \psi_2)^{(2)}(K) = \sum_{\kappa_1=0}^{K} \psi_1(\kappa_1) \psi_2(K - \kappa_1)$ and higher orders are defined successively. Similarly for TIDSI.

The maximum probability density function (PDF) with given $N$ for RP is $f_N(m; T) = d/dm F_N(m; T)$. It splits into two terms due to the differently distributed backward recurrence time $\tau_B$. The maximum probability mass function (PMF) with given $N$ for ZRP and TIDSI is the difference $f_N(m; C) = F_N(m; C) - F_N(m - 1; C)$ with $C = K$ and $L$. From the CDFs with given $N$ of Eq. (12) we get the maximum PDF/PMFs with given $N$ as

$$f_N^{(RP)}(m; T) = \varphi(m) \int_0^m d\tau_1 \ldots \int_0^m d\tau_{N-1} \prod_{i=1}^{N-1} \psi(\tau_i) \delta(T - m - \sum_{i=1}^{N-1} \tau_i)$$

$$+ (N-1)\psi(m) \int_0^m d\tau_1 \ldots \int_0^m d\tau_{N-2} \int_0^m d\tau_B \prod_{i=1}^{N-2} \psi(\tau_i) \varphi(\tau_B) \delta(T - m - \sum_{i=1}^{N-2} \tau_i - \tau_B),$$

$$f_N^{(ZRP)}(m; K) = \frac{1}{Z_N(K)} N \psi(m) \sum_{\kappa_1 = 0}^{m} \ldots \sum_{\kappa_{N-1} = 0}^{m} \prod_{i=1}^{N} \psi(\kappa_i) \delta_{K-m, \sum_{i=1}^{N-1} \kappa_i},$$

$$f_N^{(TIDSI)}(m; L) = \frac{1}{Z(L)} N \psi(m) \sum_{\lambda_1 = 0}^{m} \ldots \sum_{\lambda_{N-1} = 0}^{m} \prod_{i=1}^{N} \psi(\lambda_i) \delta_{L-m, \sum_{i=1}^{N-1} \lambda_i}.$$
An important observation is that the remaining \( N - 1 \) random variables are constrained with \( C - m \) with \( C = T, K, \) and \( L \) in the delta function and Kronecker delta. We will use this below. But before that we demonstrate now generally how the integrals/sums on the right hand side of Eq. (13) can be simplified for the range \( C/2 < m < C \).

### INTEGRALS WITH CONVOLUTION

We first consider \( RP \) and later summarize the results also for \( ZRP \) and \( TIDSI \). In Eq. (13) we have integrals of the form

\[
I_N(m, T') = \int_0^m d\tau_1 \ldots \int_0^m d\tau_N \prod_{i=1}^N g_i(\tau_i) \delta \left( T' - \sum_{j=1}^N \tau_j \right).
\]  

(14)

Note that in Eq. (13) there are \((N - 1)\)-multiple integrals but we consider now \( N \)-multiple integrals. The functions \( g_i(\tau_i) \) in Eq. (13) are the waiting time PDFs \( \psi(\tau_i) \) or the survival probability \( \varphi(\tau_i) \). Furthermore the parameter \( T' \) in Eq. (13) is the remaining time \( T - m \). Here we discuss general functions which must be positive \( g_i(\tau_i) \geq 0 \) with positive arguments \( \tau_i \geq 0 \). And we consider an arbitrary constrain \( T' > 0 \). The main result of this section is that the integral \( I_N(m, T') \) is identical to the convolution

\[
\int_0^m d\tau_1 \ldots \int_0^m d\tau_N \prod_{i=1}^N g_i(\tau_i) \delta \left( T' - \sum_{j=1}^N \tau_j \right) = (g_1 \ast \ldots \ast g_N)^{(N)}(T')
\]  

(15)

when the condition \( m > T' \) is fulfilled. This condition will lead to the range of the second half \( T/2 < m < T \) when \( T' = T - m \). The 2-fold convolution is \((g_1 \ast g_2)^{(2)}(T') = \int_0^{T'} d\tau_1 g_1(\tau_1) g_2(T' - \tau_1) \) and higher orders are defined successively.

We derive Eq. (15) with a proof by induction. Let us start with \( N = 2 \), i.e. we show now that

\[
I_2(m, T') = (g_1 \ast g_2)^{(2)}(T')
\]  

(16)

when \( m > T' \). Per definition we have

\[
I_2(m, T') = \int_0^m d\tau_1 \int_0^m d\tau_2 g_1(\tau_1) g_2(\tau_2) \delta (T' - \tau_1 - \tau_2).
\]  

(17)

For the inner integral we take both limits to infinity while putting two Heaviside functions into the integrand

\[
\int_0^m d\tau_2 g_2(\tau_2) \delta (T' - \tau_1 - \tau_2) = \int_{-\infty}^{+\infty} d\tau_2 g_2(\tau_2) \Theta(\tau_2) \Theta(m - \tau_2) \delta (T' - \tau_1 - \tau_2)
\]  

(18)

\[
= g_2(T' - \tau_1) \Theta(T' - \tau_1) \Theta(m - [T' - \tau_1]).
\]

Hence this inner integral is only nonzero under the condition

\[
T' - m < \tau_1 < T'.
\]  

(19)

The further analysis of the outer integral of Eq. (17) depends on this condition Eq. (19) and the relationship between \( T' \) and \( m \). We may consider the three regimes

(a) \( 0 < T' < m \),

(b) \( m < T' < 2m \),

(c) \( 2m < T' \).

(20)

Both conditions of Eq. (19) and Eq. (20) lead to

\[
I_2(m, T') = \begin{cases} 
  \int_0^{T'} d\tau_1 g_1(\tau_1) g_2(T' - \tau_1) & \text{for (a) } 0 < T' < m, \\
  \int_0^m d\tau_1 g_1(\tau_1) g_2(T' - \tau_1) & \text{for (b) } m < T' < 2m, \\
  \int_{T' - m}^0 d\tau_1 g_1(\tau_1) g_2(T' - \tau_1) & \text{for (c) } 2m < T'.
\end{cases}
\]  

(21)
See also Fig. 5 for three different areas of integration. We are only interested in the first regime when \(0 < T' < m\). Then the double integral is the convolution and hence Eq. (16) is shown for \(N = 2\).

In order to finish the proof of Eq. (16) we show it for \(N + 1\) while assuming that the statement is true for \(N\). We write again the definition of the integral

\[
I_{N+1}(m, T') = \int_0^m \ldots \int_0^m \prod_{i=1}^{N+1} g_i(\tau_i) \delta\left(T' - \sum_{j=1}^{N+1} \tau_j\right).
\]  

We rearrange the order of integration and separate \(-\tau_{N+1}\) in the delta function

\[
I_{N+1}(m, T') = \int_0^m \ldots \int_0^m \prod_{i=1}^{N} g_i(\tau_i) \delta\left(T' - \tau_{N+1} - \sum_{j=1}^{N} \tau_j\right) g_{N+1}(\tau_{N+1})
\]  

according to the assumption of the induction proof. The remaining integral over \(\tau_{N+1}\) is zero from \(T'\) to \(m\). The difference \(T' - \tau_{N+1} = \sum_{i=1}^{N} \tau_i\) is positive because all \(\tau_i\) are positive. So when \(\tau_{N+1} > T'\) we cannot fulfill the constrain. This property is controlled by the convolution in the integrand of Eq. (21) which is zero for negative arguments. So we get

\[
I_{N+1}(m, T') = \int_0^{T'} \prod_{i=1}^{N} g_i(\tau_i) \delta\left(T' - \tau_{N+1} - \sum_{j=1}^{N} \tau_j\right) g_{N+1}(\tau_{N+1})
\]  

and this is the convolution. Remember that we assumed \(T' < m\) in Eq. (24). Therefore we showed Eq. (16).

With the same arguments Eq. (16) can also be stated for discrete random variables with some arbitrary constrain \(C' > 0\). It is equivalently

\[
\sum_{y_1=0}^m \ldots \sum_{y_N=0}^m \prod_{i=1}^{N} g_i(y_i) \delta_{C', \sum_{j=1}^{N} y_j} = (g_1 \ast \ldots \ast g_N)(N)(C')
\]  

for \(m > C'\). Below we use Eq. (26) for ZRP with \(y_i = \kappa_i\) and \(C' = K - m\), and for TIDSI with \(y_i = \lambda_i\) and \(C' = L - m\). For ZRP and TIDSI the functions are \(g_i = \psi\) for all \(i\).

---

**FIG. 4.** Areas of integration of \(I_2(m; T')\) for three different regimes depending on the relationship between the maximum \(m\) to some parameter \(T'\), see Eq. (20). The most relevant integration is (a). Our claim is that in this case we may extent the integration in Eq. (14) till \(m = \infty\), since the constrain anyhow limits the relevant domain of the integration variables.
SECOND HALF MAXIMUM DISTRIBUTION

We see in Eq. (13) for the maximum PDF/PMFs that the argument of ρ depends on the remaining constrain T − m, K − m and L − m. We apply now the results of the last section: for RP we set T′ = T − m in Eq. (13), for ZRP we set C′ = K − m in Eq. (26), and for TIDSI we set C′ = L − m in Eq. (26). So we get for the ranges T/2 < m < T, K/2 < m < K and L/2 < m < L that the maximum PDF/PMFs with given N for all three models depend on convolutions

\[ f_N^{(RP)}(m; T) = \varphi(m)(\psi \ast \ldots \ast \psi)^{(N-1)}(T - m) + (N - 1)\psi(m)(\psi \ast \ldots \ast \psi \ast \varphi)^{(N-1)}(T - m), \]

\[ f_N^{ZRP}(m; K) = \frac{1}{Z_N(K)}N\psi(m)(\psi \ast \ldots \ast \psi)^{(N-1)}(K - m), \]

\[ f_N^{TIDSI}(m; L) = \frac{1}{Z(L)}N\psi(m)(\psi \ast \ldots \ast \psi)^{(N-1)}(L - m). \]

(27)

For RP and TIDSI we further sum over all N and get the second half maximum PDF/PMF

\[ f^{(RP)}(m; T) = \varphi(m) \sum_{N=1}^{\infty} (\psi \ast \ldots \ast \psi)^{(N-1)}(T - m) \]

\[ + \psi(m) \sum_{N=1}^{\infty} (N - 1)(\psi \ast \ldots \ast \psi \ast \varphi)^{(N-1)}(T - m), \]

\[ f^{(TIDSI)}(m; L) = \frac{1}{Z(L)}\varphi(m) \sum_{N=1}^{\infty} N(\psi \ast \ldots \ast \psi)^{(N-1)}(L - m) \]

(28)

For ZRP our main result is in Eq. (27), and for RP and TIDSI our main result is in Eq. (28).

**Relationship to other quantities**

*Sum of IID random variables*

For ZRP and TIDSI we can relate the second half maximum PMFs to random walks

\[ f_N^{ZRP}(m; K) = N\psi(m)\frac{\Phi_{N-1}(K - m)}{\Phi_N(K)}, \]

\[ f^{TIDSI}(m; L) = \psi(m)\sum_{N=1}^{\infty} N\Phi_{N-1}(K - m) \]

\[ \sum_{N=1}^{\infty} \Phi_N(K) \]

(29)

where Φ_N is the PMF of the sum of independent and identically distributed random variables.

**Marginal PDF/PMFs**

The second half maximum PDF/PMFs are related to the marginal PDF/PMFs. For RP it is

\[ f^{(RP)}(m; T) = \sum_{N=1}^{\infty} \rho_{N,B}(m) + \sum_{N=1}^{\infty} (N - 1)\rho_{N,NB}(m). \]

(30)

Here \( \rho_{N,B}(T - m) \) is the integration of the joint PDF, see Eq. (32) below, over the first \( N - 1 \) waiting times, the backward recurrence time takes the value \( m \). And \( \rho_{N,NB}(T - m) \) is the integration of the joint PDF over the first \( N - 2 \) waiting times and the backward recurrence time, one of the first \( N - 1 \) waiting times takes the value \( m \). The second half maximum PMFs for ZRP and TIDSI are related to the marginal PMFs via

\[ f_N^{ZRP}(m; T) = N\rho_N(m), \]

\[ f^{TIDSI}(m; L) = \sum_{N=1}^{\infty} N\rho_N(m) \]

(31)
where $\rho_N$ is integrated over $N - 1$ random variables.

These relationships to the marginal PDF/PMFs are based on the joint PDF/PMFs. For all three models these have been reported for $RP$ in [1], for $ZRP$ in [2] and for $TIDSI$ in [3]. They are

\[
p_N^{(RP)}(\tau_1, \ldots, \tau_{N-1}, \tau_B; T) = \prod_{i=1}^{N-1} \psi(\tau_i) \varphi(\tau_B) \delta \left( T - \sum_{i=1}^{N-1} \tau_i - \tau_B \right)
\]

\[
p_N^{(ZRP)}(\kappa_1, \ldots, \kappa_N; K) = \frac{1}{Z_N(K)} \prod_{i=1}^{N} \psi(\kappa_i) \delta K \sum_{i=1}^{N} \kappa_i^i,
\]

\[
p_N^{(TIDSI)}(\lambda_1, \ldots, \lambda_N; L) = \frac{1}{Z(L)} \prod_{i=1}^{N} \psi(\lambda_i) \delta \sum_{i=1}^{N} \lambda_i.
\]

The partition functions are the same as in Eq. (12).

Mean number of random variables

For $ZRP$ the number of random variables $N$ is fixed. But for $RP$ and $TIDSI$ it is random. The probability of having $N$ random variables up to some arbitrary constrain $C' > 0$ with $C' = T'$ and $L'$ for these two models can be derived with the indicator function

\[
P_N^{(RP)}(T') = \left\{ I \left( \sum_{i=1}^{N-1} \tau_i < T' < \sum_{i=1}^{N} \tau_i \right) \right\} = (\psi \ast \ldots \ast \psi(\varphi)(N)(T'),
\]

\[
P_N^{(TIDSI)}(L') = \frac{1}{Z(L)} \left\{ I \left( \sum_{i=1}^{N} \lambda_i = L' \right) \right\} = \frac{1}{Z(L)} (\psi \ast \ldots \ast \psi)(N)(L').
\]

The average is executed over all possible trajectories $\langle \psi \rangle = \int_0^\infty d\tau_1 \ldots \int_0^\infty d\tau_N \prod_{i=1}^{N} \psi(\tau_i) \delta$ for $RP$ respectively $\langle \psi \rangle = \sum_{\lambda_1=0}^\infty \ldots \sum_{\lambda_N=1}^\infty \prod_{i=1}^{N} \psi(\lambda_i) \delta$ for $TIDSI$. The indicator function $I$ is 1 if its argument is valid and 0 if not. It can be expressed with two Heaviside functions $I \left( \sum_{i=1}^{N-1} \tau_i < T < \sum_{i=1}^{N} \tau_i \right) = \Theta(T - \sum_{i=1}^{N-1} \tau_i) \Theta(\sum_{i=1}^{N} \tau_i - T)$ in case of the $RP$ where the last waiting time $\tau_N$ is cut off to the backward recurrence time $\tau_B$. For $TIDSI$ the indicator function is the Kronecker delta $I \left( \sum_{i=1}^{N} \kappa_i = K \right) = \delta \sum_{i=1}^{N} \kappa_i = K$. Using this we find the convolutions on right hand side of Eq. (33).

Another important quantity for $RP$ is the probability $Q_N^{(RP)}(T') dT'$ of having the $(N + 1)$-th renewal event exactly in the time interval $(T', T' + dT')$. It can be calculated via

\[
Q_N^{(RP)}(T') = \left\{ \delta \left( T' - \sum_{i=1}^{N} \tau_i \right) \right\} = (\psi \ast \ldots \ast \psi)(N)(T').
\]

For $N = 0$ we have the initial condition $Q_0^{(RP)}(T') = \delta(T')$. With these quantities we obtain the mean number of random variables for $RP$ and $TIDSI$ by summing over all $N$, namely

\[
\langle N^{(RP)}(T') \rangle = \sum_{N=1}^{\infty} N P_N^{(RP)}(T'),
\]

\[
\langle N^{(TIDSI)}(L') \rangle = \sum_{N=1}^{\infty} N P_N^{(TIDSI)}(L')
\]

and the rate function only for $RP$

\[
R^{(RP)}(T') = \sum_{N=1}^{\infty} Q_N^{(RP)}(T').
\]
Thus with $T' = T - m$ and $L' = L - m$ we can write the second half PDF/PMF for $RP / TIDSI$ as

$$f(m; T) = \varphi(m) R^{(RP)}(T - m) + \psi(m) \langle N^{(RP)}(T - m) \rangle,$$

$$f(m; L) = \psi(m) \frac{Z(L - m)}{Z(L)} \left[ \langle N^{(TIDSI)}(L - m) \rangle + 1 \right]. \quad (37)$$

**GAP AT THE HALF TIME $T/2$**

For the continuous maximum PDF of $RP$ there is a clear gap at $T/2$ for small enough $T$. This gap is due to processes where next to the first event of the start only one more event occured before $T$. Hence we have two waiting times $(\tau_1, \tau_B)$ with $N = 2$. The maximum PDF $f_2(m; T)$ is zero for $m < T/2$ because one waiting time has to be larger than $T/2$. Otherwise the constrain $T = \tau_1 + \tau_B$ cannot be fulfilled. For $m > T/2$ the maximum PDF is

$$f_2(m; T) = \varphi(m) \psi(T - m) + \psi(m) \varphi(T - m). \quad (38)$$

So we get the height of the middle gap

$$\lim_{m \to (T/2)^-} f(m; T) - f(m; T) = f_2(T/2; T) = 2\psi(T/2)\varphi(T/2) \quad (39)$$

where $m \to (T/2)^\pm$ means we approach $T/2$ from left or right.

**LIMITING DISTRIBUTIONS**

We study limiting maximum PDFs/PMFs for $RP / TIDSI$ when the waiting time PDF/PMF follows a power law

$$\psi(x) \sim b_\alpha x^{-1-\alpha} \quad (40)$$

with $\alpha \in (0, 1)$ and $a \in (1, 2)$. The argument can be $x = \tau$ for $RP$ and $x = \lambda$ for $TIDSI$. For $TIDSI$ in particular, we study the critical maximum behavior between ferromagnetic and paramagnetic phases [3]. Because then the $z$-transform can be replaced by the Laplace transform [3] and we basically can repeat our derivation from $RP$, see below. So the behavior of $RP$ and $TIDSI$ in the critical phase are quite similar. When speaking of $TIDSI$ in the following, we always consider this critical phase. Then the weights are $\psi(\lambda) = \lambda^{-1-\alpha}/\zeta(1+\alpha)$ with the Riemann Zeta function $\zeta$.

Let us summarize some previous results of the maximum PDF $f(m; C)$ for $RP$ with $C = T$ [1] and for $TIDSI$ with $C = L$ [3]. For $\alpha \in (0, 1)$ the maximum is of the order of the constrain and the PDF scales

$$f(m; C) \sim \frac{1}{C} g(m/C). \quad (41)$$

This function has for both $RP$ and $TIDSI$ a different expression for $m \in (0, C/2)$ and $m \in (C/2, C)$. In the middle $C/2$ there is a kink, i.e. $g$ is non-differentiable. Below we calculate this function in the second half $m \in (C/2, C)$ where also the assumption $C - m = O(C)$ is needed. But when $m/C \to 1$ the function $g$ diverges. We cure this divergence by applying the scaling $C - m = O(1)$ which describes the very largest values of $m$.

For $\alpha \in (1, 2)$ the maximum behaves typically as Fréchet’s law

$$f(m; C) \sim \frac{1}{(C/\langle x \rangle)^{1/\alpha}} f \left( \frac{m}{(C/\langle x \rangle)^{1/\alpha}} \right) \quad (42)$$

with

$$f(\xi) = b_\alpha x^{-1-\alpha} \exp \left( -\frac{b_\alpha \xi^{-\alpha}}{\alpha} \right). \quad (43)$$

The mean is $\langle x \rangle = \langle \tau \rangle$ or $\langle x \rangle = \langle \tau \rangle = \zeta(\alpha)/\zeta(1 + \alpha)$. However, Fréchet’s law implies an unphysical infinite variance. For $RP$ this problem has been adressed in [4]. The found limiting law matches with Fréchet’s law, i.e. the gap at $T/2$ vanishes. We derive this result again rigorously by applying $m = O(C)$ and $C - m = O(C)$. And also for $TIDSI$ we find a similar correction to Fréchet’s law. In addition, the scaling $C - m = O(1)$ will also be applied.
Limiting distributions for RP

Let’s begin with $T - m = O(T)$. From Eq. (37) we see that we need the long-time behavior of $\langle N \rangle$ and $R$. It is suitable to study this problem in Laplace space because the convolution becomes a product. Eq. (37) reads

$$\hat{f}(m,s) = \psi(m) \frac{e^{-sm}}{s(1 - \psi(s))} + \varphi(m) \frac{e^{-sm}}{1 - \psi(s)}.$$ (44)

This formula can serve as a numerical method to obtain $f(m;T)$ by applying numerical inverse Laplace transform techniques. The large $\tau$ behavior of the waiting time PDF $\psi(\tau) \sim b_\alpha \tau^{-1-\alpha}$ becomes the small $s$ behavior in Laplace space

$$\hat{\psi}(s) \sim \begin{cases} 1 - b_\alpha |\Gamma(-\alpha)| s^\alpha & \text{for } 0 < \alpha < 1, \\ 1 - \langle \tau \rangle s & \text{for } 1 < \alpha < 2 \end{cases}$$ (45)

with the mean waiting time $\langle \tau \rangle$. Hence for $0 < \alpha < 1$ we find

$$T f(m;T) \sim \frac{\sin(\pi \alpha)}{\pi} \xi^{-1-\alpha}(1 - \xi)^\alpha + \frac{\sin(\pi \alpha)}{\pi} \xi^{-\alpha}(1 - \xi)^{\alpha-1},$$ (46)

see also [1]. And for $1 < \alpha < 2$ we find

$$T^\alpha f(m;T) \sim \frac{b_\alpha}{\langle \tau \rangle} \xi^{-1-\alpha}(1 - \xi) + \frac{b_\alpha}{\alpha \langle \tau \rangle} \xi^{-\alpha},$$ (47)

see also [4]. The rescaled variable is $\xi = m/T$. On the other hand when $T - m = O(1)$ the second half maximum PDF scales for $0 < \alpha < 1$ and $1 < \alpha < 2$ as

$$T^\alpha f(m;T) \sim \frac{b_\alpha}{\alpha} R(T - m)$$ (48)

which is obtained because the large $m$ behavior of $\varphi$ dominates $\psi$ and both $R$ and $\langle N \rangle$ are constant.

Limiting distributions for TIDSI

The weights are generally

$$\psi(\lambda) = \frac{e^{-\beta \Delta}}{\Lambda^{1+\alpha}}$$ (49)

with the inverse temperature $\beta = 1/(k_B T)$, the chemical potential $\Delta$ and $\gamma = \beta J$ where $J$ is the strength of the inverse squared long-range interaction within a single spin domain, see [3]. Above derivation of limiting laws for RP can be repeated almost identically when the system is in the critical phase between ferromagnetic and paramagnetic phases. Here the marginal domain size decays algebraically. Then the weights are

$$\psi(\lambda) = \frac{1}{\zeta(1+\alpha) \lambda^{1+\alpha}}$$ (50)

with the Riemann Zeta function $\zeta(1+\alpha) = \sum_{N=1}^{\infty} N^{-1-\alpha}$, i.e. the fugacity becomes $e^{-\beta \Delta} = 1/\zeta(1+\alpha)$. It was shown in [3] that there are two regimes in the critical phase for $\alpha \in (0,1)$ and $\alpha > 1$. Here we restrict the second regime to $\alpha \in (1,2)$ in order to compare it to RP. As explained in [3] the analysis using $z$-transform can be replaced by Laplace transforms in the critical phase, we will use this now.

We start our analysis with the $z$-transform of the weights

$$\psi(\lambda) \rightarrow \hat{\psi}(z) = \frac{1}{\zeta(1+\alpha)} \sum_{L=1}^{\infty} L^{-1-\alpha} z^L$$ (51)
of the denominator of Eq. (28)

\[ \sum_{N=1}^{\infty} N(\psi \ldots \psi)^{(N-1)}(L - m) \xrightarrow{N \to \infty} \sum_{N=1}^{\infty} Nz^{m} \hat{\psi}^{N-1}(z) = \frac{z^m}{1 - \hat{\psi}(z)}^{2} \]  

(52)

and of the numerator of Eq. (28)

\[ \sum_{N=1}^{\infty} (\psi \ldots \psi)^{(N)}(L) \xrightarrow{N \to \infty} \sum_{N=1}^{\infty} \hat{\psi}^{N}(z) = \frac{\hat{\psi}(z)}{1 - \hat{\psi}(z)}. \]  

(53)

For the scaling \( m = O(L) \) and \( L - m = O(L) \) we need the large \( L \) limit of both the denominator and numerator. We set \( z = \exp(-s) \) and consider the small \( s \)-behavior of the weights

\[ \hat{\psi}(s) \sim \begin{cases} 
1 - \frac{\Gamma(-\alpha)}{\zeta(1+\alpha)} s^\alpha & \text{for } 0 < \alpha < 1, \\
1 - \langle \lambda \rangle s & \text{for } 1 < \alpha < 2.
\end{cases} \]  

(54)

This is equivalent to the asymptotic behavior of \( \hat{\psi}(z) \sim 1 - \frac{\Gamma(-\alpha)}{\zeta(1+\alpha)}(1-z)^{\alpha} = -\zeta(\alpha)(1-z) \) at the branch point \( z = 1 \), see [3]. For \( 0 < \alpha < 1 \) we get the inverse Laplace transform

\[ Lf(m; L) \sim \frac{\Gamma(\alpha)}{\Gamma(-\alpha)\Gamma(2\alpha)} \xi^{-1-\alpha}(1-\xi)^{2\alpha-1}. \]  

(55)

The same limiting law has been derived in [3] but with a different expression. Both results are identical, see discussion below following from Eq. (58). For \( 1 < \alpha < 2 \) we get

\[ L^{\alpha} f(m; T) \sim \frac{1}{\zeta(1+\alpha)\langle \lambda \rangle} \xi^{-1-\alpha}(1-\xi), \]  

(56)

see Fig. 5 where we compare this law with numerical simulation and Fréchet’s law [3]. The rescaled variable is \( \xi = m/L \). On the other hand when the remaining domain size scales as \( L - m = O(1) \) then the denominator stays constant. From Eq. (39) we get

\[ f(m; L) \sim \frac{\psi(L)}{Z(L)} Z(L-m) [(N(L-m)) + 1] \]

\[ = Z(L-m) [(N(L-m)) + 1] \times \begin{cases} 
\frac{\Gamma(-\alpha)\Gamma(\alpha)L^{-2\alpha}}{\zeta(1+\alpha)} & \text{for } 0 < \alpha < 1, \\
\frac{\zeta(1+\alpha)L^{\alpha} - 1}{\zeta(1+\alpha)\langle \lambda \rangle L^{-1-\alpha}} & \text{for } 1 < \alpha < 2.
\end{cases} \]  

(57)
Typical fluctuations for $0 < \alpha < 1$

In [3] the typical fluctuations of $f(m; L)$ in the second half $L/2 < m < L$ where calculated as

$$Lf(m; L) \sim \frac{1}{\xi^2} \frac{d}{du} H(u)|_{u=1/\xi}$$

with the function

$$H(u) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha + 1)\Gamma(-\alpha)} u^{1-\alpha}(u-1)^{2\alpha} F_1(1, 1 + \alpha, 1 + 2\alpha, 1 - u).$$

The hypergeometric function defined as

$$2 F_1(a, b, c, z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}$$

with the Pochhammer symbol $(a)_j = \Gamma(a + j)/\Gamma(a)$.

We show now that Eq. (55) is identical to our result from Eq. (55). For that let us first take the derivative of the right hand side of Eq. (58) while $u = 1/\xi$:

$$u^2 \frac{d}{du} H(u) = u^2 \frac{\Gamma(\alpha)}{\Gamma(2\alpha + 1)\Gamma(-\alpha)} \left[ [(1-\alpha) u^{-\alpha}(u-1)^{2\alpha} + 2 \alpha u^{-\alpha}(u-1)^{2\alpha-1}] 2 F_1(1, 1 + \alpha, 2\alpha + 1, 1 - u) \right. \left. - \frac{1 + \alpha}{1 + 2\alpha} u^{-\alpha}(u-1)^{2\alpha} 2 F_1(2, 2 + \alpha, 2\alpha + 2, 1 - u) \right]$$

where we used $d/dz 2 F_1(a, b, c, z) = ab/cz F_1(1 + a, 1 + b, 1 + c, z)$. Now we take out the term $u^{-\alpha}(u-1)^{2\alpha-1}$ so that

$$u^2 \frac{d}{du} H(u) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha + 1)\Gamma(-\alpha)} u^{2-\alpha}(u-1)^{2\alpha-1} \left[ [(1-\alpha)(u-1) + 2 \alpha u] 2 F_1(1, 1 + \alpha, 2\alpha + 1, 1 - u) \right. \left. - \frac{1 + \alpha}{1 + 2\alpha} (u-1) 2 F_1(2, 2 + \alpha, 2\alpha + 2, 1 - u) \right].$$
To show the identity to Eq. (55) we have to show that the expression inside the big squared bracket of Eq. (62) is identical to $2\alpha$. Let us write this question shortly as

$$f(u)F(1, 1 - u) + g(u)F(2, 1 - u) = 2\alpha? \quad (63)$$

Here $f(u) = (1 - \alpha)(u - 1) + 2\alpha u$, $g(u) = (1 + \alpha)/(1 + 2\alpha)u(u - 1)$ and $F(i, 1 - u) = F_1(i + 1, i + 1 + \alpha, i + 1 + 2\alpha, 1 - u)$.

Since the hypergeometric function depends on $1 - u$ we consider the series expansion at $u = 1$ of the inner bracket. In principle any other point could be considered but the problem becomes simpler at $u = 1$. The Taylor series of Eq. (63) is

$$f(u)F(1, 1 - u) + g(u)F(2, 1 - u) = \sum_{j=0}^{\infty} \left( f(u)F(1, 1 - u) + g(u)F(2, 1 - u) \right) \frac{(u - 1)^j}{j!}. \quad (64)$$

We apply the general Leibniz rule of derivation

$$f(u)F(1, 1 - u) + g(u)F(2, 1 - u) = \sum_{j=0}^{\infty} \left( \sum_{k_1=0}^{j} \binom{j}{k_1} F^{(j-k_1)}(1, 1 - u) f^{(k_1)}(u) \right) \frac{(u - 1)^j}{j!} \quad (65)$$

The derivatives of $f$ and $g$ are

$$f^{(k_1)}(u) \bigg|_{u=1} = \begin{cases} 2\alpha & \text{for } k_1 = 0, \\ 1 + \alpha & \text{for } k_1 = 1, \\ 0 & \text{for } k_1 \geq 2, \end{cases} \quad (66)$$

$$g^{(k_2)}(u) \bigg|_{u=1} = \begin{cases} 0 & \text{for } k_2 = 0, \\ -\frac{1+\alpha}{1+2\alpha} & \text{for } k_2 = 1, \\ -\frac{2+\alpha}{1+2\alpha} & \text{for } k_2 = 2, \\ 0 & \text{for } k_2 \geq 3. \end{cases}$$

The two sums in Eq. (65) are only nonzero for $k_1 = 0, 1$ and $k_2 = 2, 3$. Thus we can write

$$f(u)F(1, 1 - u) + g(u)F(2, 1 - u) = \sum_{j=0}^{\infty} \left( \sum_{k_1=0}^{1} \binom{j}{k_1} F^{(j-k_1)}(1, 1 - u) f^{(k_1)}(u) \right) \frac{(u - 1)^j}{j!} \quad (67)$$

The binomial is zero when $k_1 > j$ and $k_2 > j$ so this expression is valid for all $j$. Now we express the hypergeometric function $F(2, 1 - u)$ by $F(1, 1 - u)$ via the relationship of their derivatives. The $j$-th derivative of the hypergeometric function at $u = 1$ is

$$F^{(j)}(1, 1 - u) \bigg|_{u=1} = (-1)^j \frac{(1)_j (1 + \alpha)_j}{(1 + 2\alpha)_j}, \quad (68)$$

thus

$$F^{(j)}(2, 1 - u) \bigg|_{u=1} = -\frac{1 + 2\alpha}{1 + \alpha} F^{(j+1)}(1, 1 - u). \quad (69)$$

So we can write

$$f(u)F(1, 1 - u) + g(u)F(2, 1 - u) = \sum_{j=0}^{\infty} \left( \sum_{k_1=0}^{1} \binom{j}{k_1} F^{(j-k_1)}(1, 1 - u) f^{(k_1)}(u) \right) \frac{(u - 1)^j}{j!}$$

$$- \frac{1 + 2\alpha}{1 + \alpha} \sum_{k_2=1}^{2} \binom{j}{k_2} F^{(j-k_2+1)}(1, 1 - u) g^{(k_2)}(u) \bigg|_{u=1} \frac{(u - 1)^j}{j!}. \quad (70)$$
We order according to the hypergeometric functions
\[ f(u)F(1, 1 - u) + g(u)F'(2, 1 - u) = \sum_{j=0}^{\infty} \left( F^{(j)}(1, 1 - u) \left[ {j \choose 0} f^{(0)}(u) - \frac{1 + 2\alpha}{1 + \alpha} {j \choose 1} g^{(1)}(u) \right] \bigg|_{u=1} 
+ F^{(j-1)}(1, 1 - u) \left[ {j \choose 1} f^{(1)}(u) - \frac{1 + 2\alpha}{1 + \alpha} {j \choose 2} g^{(2)}(u) \right] \bigg|_{u=1} \right) \frac{(u - 1)^j}{j!}. \] (71)

With Eq. (66) we get
\[ f(u)F(1, 1 - u) + g(u)F'(2, 1 - u) = \sum_{j=0}^{\infty} \left( 2\alpha + j \right) F^{(j)}(1, 1 - u) \bigg|_{u=1} + j(\alpha + j) F^{(j-1)}(1, 1 - u) \bigg|_{u=1} \frac{(u - 1)^j}{j!}. \] (72)

Now we split the summation over \( j \) for \( j = 0 \) and all other \( j \geq 1 \). For the latter we use the relationship between successive orders of the derivative for the hypergeometric function
\[ F^{(j)}(1, 1 - u) \bigg|_{u=1} = -j(\alpha + j) \frac{F^{(j-1)}(1, 1 - u) \bigg|_{u=1}}{2\alpha + j} \] (73)
valid for \( j \geq 1 \). This gives zero for all terms with \( j \geq 1 \) in Eq. (72) and only the term with \( j = 0 \) remains. With \( F^{(0)}(1, 1 - u) \big|_{u=1} = 1 \) we obtain
\[ f(u)F(1, 1 - u) + g(u)F'(2, 1 - u) = 2\alpha \] (74)

Thus we finally showed that indeed
\[ u^2 \frac{d}{du} H(u) = \frac{\Gamma(\alpha)}{\Gamma(2\alpha) \Gamma(-\alpha)} u^{2-\alpha} (u - 1)^{2\alpha-1}. \] (75)

Hence Eq. (58) is identical to our result from Eq. (54).

[1] C. Godrèche, S. N. Majumdar, and G. Schehr, J. Stat. Mech. Theory E. 2015, P03014 (2015).
[2] M. R. Evans and S. N. Majumdar, J. Stat. Mech. Theory E. 2008, P05004 (2008).
[3] A. Bar, S. N. Majumdar, G. Schehr, and D. Mukamel, Phys. Rev. E 93, 052130 (2016).
[4] A. Vezzani, E. Barkai, and R. Burioni, Phys. Rev. E 100, 012108 (2019).