RATIONALITY OF DARMON POINTS OVER GENUS FIELDS OF NON-MAXIMAL ORDERS

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Abstract. Stark–Heegner points, also known as Darmon points, were introduced by H. Darmon in [Dar01] as certain local points on rational elliptic curves, conjecturally defined over abelian extensions of real quadratic fields. The rationality conjecture for these points is only known in the unramified case, namely, when these points are specializations of global points defined over the strict Hilbert class field $H^+_F$ of the real quadratic field $F$ and twisted by (unramified) quadratic characters of $\text{Gal}(H^+_F / F)$. We extend these results to the situation of ramified quadratic characters; more precisely, we show that Darmon points of conductor $c \geq 1$ twisted by quadratic characters of $G^+_c = \text{Gal}(H^+_c / F)$, where $H^+_c$ is the strict ring class field of $F$ of conductor $c$, come from rational points on the elliptic curve defined over $H^+_c$.

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1. INTRODUCTION

The theory of Stark–Heegner points, also known as Darmon points, began with the foundational paper by H. Darmon [Dar01] in 2001. In this work, Darmon proposed a construction of local points on rational elliptic curves, the Stark–Heegner points, which, under appropriate arithmetic conditions, he conjectured to be global points defined over strict ring class fields of real quadratic fields, which are non-torsion when the central critical value of the first derivative of the complex $L$-function of the elliptic curve over the real quadratic field does not vanish. Note that the absence of a theory of complex multiplication in the real quadratic case, available in the imaginary quadratic case, makes the construction of global points on elliptic curves over real quadratic fields and their abelian extensions a rather challenging problem. The idea of Darmon was to define locally a family of candidates for their points, and conjecture that these come from global points. Following [Dar01], many authors proposed similar constructions in different situations, including the cases of modular and Shimura curves, and the higher weight analogue of Stark–Heegner, or Darmon, cycles; with no attempt to be complete, see for instance [Das05], [Gre09], [LRV12], [LRV13], [Tri06], [GS16], [RS12], [GMcS15], [GM15b], [GM15a], [GM14], and [GM13].

The arithmetic setting of the original construction in [Dar01] is the following. Fix a rational elliptic curve $E$ of conductor $N = Mp$, with $p \nmid M$ an odd prime number and $M \geq 1$ and integer. Fix also a real quadratic field $F$ satisfying the following Stark–Heegner assumption: all primes $\ell \mid M$ are split in $F$, while $p$ is inert in $F$. Under these assumptions, the central
critical value $L(E/F, 1)$ of the complex $L$-function of $E$ over $F$ vanishes. Darmon points are local points $P_c$ for $E$ defined of finite extension of $F_p$, the completion of $F$ at the unique prime above $p$; their definition and the main properties are recalled in Section 2 below. The definition of these points depends on the choice of an auxiliary integer $c \geq 1$, called the conductor of a Darmon point $P_c$. The rationality conjecture predicts that these points $P_c$ are localizations of global points $P_c$ which are defined over the strict ring class field $H^+_c$ of $F$ of conductor $c$.

Only partial results are known toward the rationality conjectures for Darmon points, or more generally cycles. The first result on the rationality of Darmon points is due to Bertolini and Darmon in the paper [BD09], where they show that a certain linear combination of these points with coefficients given by values of genus characters of the real quadratic field $F$ comes from a global point defined over the Hilbert class field of $F$. The main idea behind the proof of these results is to use a factorization formula for $p$-adic $L$-functions to compare the localization of Heegner points and Darmon points. More precisely, the first step of the proof consists in relating Darmon points to the $p$-adic $L$-function interpolating central critical values of the complex $L$-functions over $F$ attached to the arithmetic specializations of the Hida family passing through the modular form attached to $E$. The second step consists in expressing this $p$-adic $L$-function in terms of a product of two Mazur–Kitagawa $p$-adic $L$-functions, which are known to be related to Heegner points by the main result of [BD07]. A similar strategy has been adopted by [GSS16], [Sev12] [LV14], [LV16] obtaining similar results.

All known results in the direction of the conjectures in [Dar01] involve linear combination of Darmon points twisted by genus characters, which are quadratic unramified characters of $\text{Gal}(H^+_c/F)$, where $H^+_c$ is the (strict) Hilbert class field of $F$. The goal of this paper is to prove a similar rationality result for more general quadratic characters, namely, quadratic characters of ring class fields of $F$, so we allow for ramification. In the remaining part of the introduction we briefly state our main result and the main differences with the case of genus characters treated up to now.

Let $E/\mathbb{Q}$ be an elliptic curve, and denote by $N$ its conductor. Let $F/\mathbb{Q}$ be a real quadratic field $F = \mathbb{Q}(\sqrt{D})$ of discriminant $D = D_F > 0$, prime to $N$. We assume one has a factorization $N = Mp$ with $p \nmid M$, such that all primes $\ell | M$ are split in $F$ and $p$ is inert in $F$.

Fix an integer $c$ prime to $D \cdot N$ and a quadratic character

$$\chi : G^+_c = \text{Gal}(H^+_c/F) \rightarrow \{\pm 1\},$$

where, as above, $H^+_c$ denotes the strict class field of $F$ of conductor $c$. Let $O_c$ be the order in $F$ of conductor $c$. Recall that $G^+_c$ is isomorphic to the group of strict equivalence classes of projective $O^+_c$-modules, which we denote $\text{Pic}^+(O_c)$, where two such modules are strictly equivalent if they are the same up to an element of $F$ of positive norm. We assume that $\chi$ is primitive, meaning that it does not factor through any $G^+_f$ with $f$ a proper divisor of $c$.

Fix embeddings $F \hookrightarrow \bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ throughout. Let $P_c \in E(F_p)$ be a Darmon point of conductor $c$ (see Section 2 below for the precise definition of these points) where, as above, $F_p$ is the completion of $F$ at the unique prime of $F$ above $p$. It follows from their construction that Darmon points of conductor $c$ are in bijection with equivalence classes of quadratic forms of discriminant $Dc^2$, and this can be used to define a Galois action $P_c \mapsto P_c^\sigma$ on Darmon points, where $P_c$ is a fixed Darmon point of conductor $c$ and $\sigma \in G^+_c$. We may then form the point

$$(1) \quad P_\chi = \sum_{\sigma \in G^+_c} \chi^{-1}(\sigma)P_c^\sigma$$

which lives in $E(F_p)$. Finally, let $\log_E : E(\mathbb{C}_p) \rightarrow \mathbb{C}_p$ denote the formal group logarithm of $E$. Note that, since $p$ is inert in $F$, it splits completely in $H^+_c$, and therefore for any point $Q \in E(H^+_c)$ the localization of $Q$ at any of the primes in $H^+_c$ dividing $p$ lives in $E(F_p)$. Our main result is the following:
Theorem 1.1. Assume that \( c \) is odd and coprime to \( DN \). Let \( \chi \) be a primitive quadratic character of \( G^+_c \). Then there exists a point \( P_\chi \) in \( E(H^+_c) \) and a rational number \( n \in \mathbb{Q}^\times \) such that
\[
\log_E(P_\chi) = n \cdot \log_E(P_\chi).
\]
Moreover, the point \( P_\chi \) is of infinite order if and only if \( L'(E/F, \chi, 1) \neq 0 \).

If \( c = 1 \), this is essentially the main result of [BD09]. To be more precise, the work [BD09] needed to assume \( E \) had two primes of multiplicative reduction because of this assumption in [BD07]. However, this assumption has been removed by very recent work of Mok [Mok], which we also apply here.

The proof in the general case follows a similar line to that in [BD09]. However, some modifications are in order. The first difference is that the genus theory of non-maximal orders is more complicated than the usual genus theory, and the arguments need to be adapted accordingly. More importantly, one of the main ingredients in the proof of the rationality result in [BD09] is a formula of Popa [Pop06] for the central critical value of the \( L \)-function over \( F \) of the specializations at arithmetic points of the Hida family passing through the modular form associated with the elliptic curve \( E \). However, this formula does not allow to treat \( L \)-functions twisted by ramified characters. Instead, we recast an \( L \)-value formula from [MW09] which allows for ramification, expressed in terms of periods of Gross–Prasad test vectors, in a more classical framework to get our result.

Remark 1.2. When \((cD, N) \neq 1\), it may be that \( \pi \) and \( \chi \) have joint ramification. In this case, we can instead use [FMP17] in lieu of [MW09], at least in the case that \( f \) has squarefree level.

Remark 1.3. The main result of this paper assumes that all primes dividing \( M \) are split and \( p \) is inert in \( F \). More generally, the same results are expected to hold under the following relaxed modified Heegner assumption: \( p \) is inert in \( F \) and there is a factorization \( M = M^+ \cdot M^- \) of \( M \) into coprime integers such that a prime number \( \ell \mid M^+ \) if and only if \( \ell \) in split in \( F \), and \( M^- \) is a product of an even number of distinct primes. If the conductor \( N \) of the elliptic curve \( E \) can be factorized as \( N = M^+ \cdot M^- \cdot p \) with \( p \nmid M \) and the discriminant \( D \) of the real quadratic field \( F \), and \( \chi \) is a primitive quadratic character of \( G^+_c \) with \( c \) odd and coprime with \( ND \), then one can show that there exists a point \( P_\chi \) in \( E(H^+_c) \) and a rational number \( n \in \mathbb{Q}^\times \) such that \( \log_E(P_\chi) = n \cdot \log_E(P_\chi) \); moreover, the point \( P_\chi \) is of infinite order if and only if \( L'(E/F, \chi, 1) \neq 0 \). The proof of this result can be obtained with the methods of this paper by replacing modular curves of level \( M \) with Shimura curves of level \( M^+ \) attached to quaternion algebras of discriminant \( M^- \), following what is done in [LV14] in the case \( c = 1 \). However, since the notation in the quaternionic case is quite different from the notation in the case of modular curves, we prefer to only treat in detail the case when \( M^- = 1 \). Details in the case \( M^- > 1 \) are left to the reader.

Remark 1.4. The main result of this paper plays a role in the forthcoming works by Darmon-Rotger [DR] and Bertolini-Seveso-Venerucci [BSV] proving the rationality of Darmon points (actually, cohomology classes closely related to Darmon points) in situations which go far beyond the case of quadratic characters considered in this paper. This application was one of the main motivations for this work.

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2. Darmon points

Let the notation be as in the introduction: \( E/\mathbb{Q} \) is an elliptic curve of conductor \( N = Mp \) with \( p \nmid M \), and \( F/\mathbb{Q} \) is a real quadratic field of discriminant \( D = D_F \) such that all primes dividing \( M \) are split in \( F \) and \( p \) is inert in \( F \). Finally, \( c \) is a positive integer prime to \( ND \). The aim of this section is to review the definition of Darmon points and some results in [BD09] and [BD07].

We first set up some standard notation. For any field \( L \), let \( P_{k-2}(L) \) be the space of homogeneous polynomials in 2 variables of degree \( k - 2 \), and let \( V_{k-2}(L) \) be its \( L \)-linear dual. We let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(L) \) act from the right on \( P(x,y) \in P_{k-2}(L) \) by the formula
\[
(P|\gamma)(x,y) = P(ax+by, cx+dy)
\]
and we equip \( V_{k-2}(L) \) with the dual action. If \( G \) is any abelian group, let \( MS(G) \) be the group of \( G \)-valued modular symbols, i.e. the \( \mathbb{Z} \)-module of functions \( I : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to G \) such that \( I(x,y) + I(y,z) = I(x,z) \) for all \( x,y,z \in \mathbb{P}^1(\mathbb{Q}) \). The value \( I(r,s) \) of \( I \in MS(G) \) on \( (x,y) \) will be usually denoted \( I(x \to y) \). The group \( \text{GL}_2(\mathbb{Q}) \) acts from the left by fractional linear transformations on \( \mathbb{P}^1(\mathbb{Q}) \), and if \( G \) is equipped with a left \( \mathbb{P}^1(\mathbb{Q}) \)-action, then \( MS(G) \) inherits a right \( \text{GL}_2(\mathbb{Q}) \)-action by the rule \( I(\gamma)(x,y) = \gamma \cdot I(\gamma^{-1}x, \gamma^{-1}y) \). If \( \Gamma_0 \) is a subgroup of \( \text{GL}_2(\mathbb{Q}) \), we denote \( MS_{\Gamma_0}(G) \) the subgroup of those elements in \( MS(G) \) which are invariant under the action of \( \gamma \) for all \( \gamma \in \Gamma_0 \). If \( f \) is a cuspform of level \( \Gamma_0(M) \) and weight \( k \), we may attach to \( f \) the standard modular symbol \( I_f \in MS_{\Gamma_0(M)}(V_{k-2}(\mathbb{C})) \); explicitly, for \( r,s \in \mathbb{P}^1(\mathbb{Q}) \) and \( P(x,y) \in P_{k-2}(\mathbb{C}) \) an homogenous polynomial of degree \( k - 2 \), put
\[
I_f(r \to s)(P(x,y)) = 2\pi i \int_r^s f(z)P(z,1) \, dz.
\]
The matrix \( \omega_{\infty} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) acts on the group of modular symbols \( MS_{\Gamma_0(M)}(V_{k-2}(\mathbb{C})) \), and we let \( I_f^\pm \) denote the projections to the \( \pm \)-eigenspaces for this action. Suppose that \( f \) is a normalized eigenform and let \( K_f \) be the field generated over \( \mathbb{Q} \) by the Fourier coefficients of \( f \). Then there are complex periods \( \Omega_f^\pm \) for each choice of sign \( \pm \) such that their product equals the Petersson inner product \( \langle f, f \rangle \), and \( I_f^\pm := I_f^\pm /\Omega_f^\pm \) satisfies the condition that if \( P(x,y) \in P_{k-2}(K_f) \) then \( I_f(r \to s)(P(x,y)) \) belongs to \( K_f \).

2.1. Measure-valued modular symbols and Darmon points. Let \( f \) be the newform of level \( N \) attached to \( E \) by modularity. Denote \( B = M_2(\mathbb{Q}) \) the split quaternion algebra over \( \mathbb{Q} \) and let \( R \) be the \( \mathbb{Z}[1/p] \)-order in \( B \) consisting of matrices in \( M_2(\mathbb{Z}[1/p]) \) which are upper triangular modulo \( M \). Define
\[
\Gamma = \{ \gamma \in R^\times \mid \det(\gamma) = 1 \}.
\]
Let \( \text{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z}) \) denote the \( \mathbb{Z} \)-module of \( \mathbb{Z} \)-valued measures on \( \mathbb{P}^1(\mathbb{Q}_p) \) with total measure equal to 0. By [BD09, Proposition 1.3], for each choice of sign \( \pm \), there exists a unique function, which we call the measure-valued modular symbol attached to \( f \),
\[
\mu_f^\pm : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to \text{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})
\]
denoted \( (r,s) \mapsto \mu_f^\pm(r \to s) \), satisfying the following conditions:
1. \( \mu_f^\pm(r \to s)(\mathbb{Z}_p) = I_f^\pm(r \to s) \)
2. For all \( \gamma \in \Gamma \) and all open compact subsets \( U \subseteq \mathbb{P}^1(\mathbb{Q}_p) \),
\[
\mu_f^\pm(\gamma(r) \to \gamma(s))(U) = \mu_f^\pm(r \to s)(U),
\]
where we let \( \text{GL}_2(\mathbb{Q}_p) \) act on \( \mathbb{P}^1(\mathbb{Q}_p) \) by fractional linear transformations.
Let $H_p = \mathbb{C}_p \setminus \mathbb{Q}_p$ denote the $p$-adic upper half plane. The system of measures $\mu_f$ can be used to define, for any $r, s \in \mathbb{P}^1(\mathbb{Q})$ and $\tau_1, \tau_2 \in H_p$, a double multiplicative integral

$$
\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{t - \tau_2}{t - \tau_1} d\mu_f\{r \to s\}(t).
$$

(On the right, the notation $\oint$ refers to the fact that the integration is relative to the multiplicative structure of $\mathbb{C}_p^\times$, and therefore is a limit of Riemann products instead of Riemann sums.) Let $q$ be the Tate period of $E$ at $p$, and let $\log_q$ be the branch of the $p$-adic logarithm satisfying $\log_q(q) = 0$. Define the additive version of the double multiplicative integral to be

$$
\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \log_q \left( \oint_{\tau_1}^{\tau_2} \int_r^s \omega_f \right).
$$

We finally introduce the notion of indefinite integral. By [BD09, Proposition 1.5], there exists a unique function from $H_p \times \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$ to $\mathbb{C}$, denoted $(\tau, r, s) \mapsto \int_{\tau}^{r} \int_{s}^{\tau} \omega_f$, satisfying the following conditions:

1. The integral is $\Gamma$-invariant, in the sense that for all $\gamma \in \Gamma$, we have

$$
\int_{\gamma(\tau)}^{\gamma(s)} \omega_f = \int_{\tau}^{s} \gamma(\omega_f)
$$

2. For any pair $\tau_1, \tau_2 \in H_p$, we have

$$
\int_{\tau_1}^{\tau_2} \int_r^s \omega_f - \int_{\tau_1}^{\tau_2} \int_r^{\tau_1} \omega_f = \int_{\tau_1}^{\tau_2} \int_r^s \omega_f
$$

3. For all $r, s, t$ in $\mathbb{P}^1(\mathbb{Q})$ we have

$$
\int_{r}^{\tau} \int_s^t \omega_f + \int_{s}^{\tau} \int_t^r \omega_f = \int_{r}^{t} \int_s^t \omega_f.
$$

We now define Darmon points using indefinite integrals above. Since $p$ is inert in $F$, the set $H_p \cap F$ is not empty and one may define the order $\mathcal{O}_\tau$ associated with $\tau \in H_p \cap F$ as

$$
\mathcal{O}_\tau = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in R \mid a\tau + b = c\tau^2 + d\tau \right\}.
$$

The map $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \to ct + d$ induces an embedding $\mathcal{O}_\tau \hookrightarrow F$, and thus $\mathcal{O}_\tau$ may be viewed as an order in $F$. For $\tau \in H_p \cap F$, let $\gamma_\tau = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ denote the unique generator of the stabilizer of $\tau$ in $\Gamma$ satisfying $ct + d > 1$ (with respect to the chosen embedding $F \subseteq \mathbb{Q}$). Let $\Phi_{\text{Tate}} : \mathbb{C}_p / q\overline{\mathbb{Z}} \to E(\mathbb{C}_p)$ denote the Tate uniformization of $E$ at $p$. Attached to $\tau$, there is an indefinite integral

$$
J^\times_\tau = \oint_{t}^{\tau} \int_{t}^{\tau} \gamma_\tau(r) \omega_f \in \mathbb{C}_p
$$

where $r \in \mathbb{P}^1(\mathbb{Q}_p)$ is any base point, and one can show that $\Phi_{\text{Tate}}(J^\times_\tau)$ is a well-defined point in $E(\mathbb{C}_p)$ independently of the choice of $r$, up to its torsion subgroup $E(\mathbb{C}_p)_{\text{tors}}$.

**Definition 2.1.** Let $\tau \in H_p \cap F$. The point $P_\tau = \Phi_{\text{Tate}}(J^\times_\tau) \in E(\mathbb{C}_p) \otimes \overline{\mathbb{Z}} \mathbb{Q}$, with $J^\times_\tau$ defined in (2), is the *Darmon point* attached to $\tau$. 

2.2. Shimura reciprocity law. Fix an integer $c$ prime to $D \cdot N$, and let $\mathcal{O}_c$ be the order of $F$ of conductor $c$. Denote $\mathcal{Q}_{D,c}$ the set of primitive binary quadratic forms of discriminant $Dc^2$. Let $SL_2(\mathbb{Z})$ act from the right on the set $\mathcal{Q}_{D,c}$ via the formula

$$
(Q|\gamma)(x,y) = Q(ax + by, cx + dy)
$$

for $Q \in \mathcal{Q}_{D,c}$ and $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$. The set of equivalence classes $\mathcal{Q}_{D,c}/SL_2(\mathbb{Z})$ is equipped with a group structure given by the Gaussian composition law. If $H_{c}^+ = \text{SL}_2(\mathbb{Z})/\mathcal{Q}_{D,c}$ then its Galois group $G_c^+ = \text{Gal}(H_{c}^+/F)$ is isomorphic to the group $\mathcal{Q}_{D,c}/SL_2(\mathbb{Z})$ via global class field theory (see [Coh78, Theorem 14.19]).

Fix $\delta \in \mathbb{Z}$ such that $\delta^2 \equiv D \pmod{4M}$. Let $\mathcal{F}_{D,c}$ denote the subset of $\mathcal{Q}_{D,c}$ consisting of forms $Q(x,y) = Ax^2 + Bxy + Cy^2$ such that $M \mid A$ and $B \equiv \delta \pmod{2M}$. The group $\Gamma_0(M)$ acts on $\mathcal{F}_{D,c}$ by the formula (3). Since $(M,D) = 1$, we also have $(\delta,M) = 1$, and therefore, by [GKZ87, Proposition, p. 505], the map $Q \mapsto Q$ sets up a bijection between $\mathcal{F}_{D,c}/\Gamma_0(M)$ and $\mathcal{Q}_{D,c}/SL_2(\mathbb{Z})$. In particular, the set $\mathcal{F}_{D,c}/\Gamma_0(M)$ is equipped with a structure of principal homogeneous space under $G_c^+$. If $Q \in \mathcal{F}_{D,c}/\Gamma_0(M)$ and $\sigma \in G_c^+$, we denote $Q^\sigma$ for the image of $Q$ by $\sigma$.

Define

$$
\mathcal{H}_p^{(D,c)} = \{ \tau \in \mathcal{H}_p \cap F \mid \mathcal{O}_c \to \mathcal{O}_c \}.
$$

Given $Q(x,y) = Ax^2 + Bxy + Cy^2$ a quadratic form in $\mathcal{F}_{D,c}$, let $\tau_Q = \frac{-B+\sqrt{D}}{2A}$ be a fixed root of the quadratic polynomial $Q(x,1)$. Then $\tau_Q$ belongs to $\mathcal{H}_p^{(D,c)}$ (via the fixed $p$-adic embedding of $F$ into $\bar{\mathbb{Q}}_p$) and its image in $\Gamma \backslash \mathcal{H}_p^{(D,c)}$ does not depend on the $\Gamma_0(M)$-equivalence class of $Q$. Given $\sigma \in G_c^+$, we will sometimes write $\tau_Q^\sigma$ for $\tau_Q^\sigma$.

Let $P$ be a point in $E(H_c^+)$. Since $p$ is inert in $K$, it splits completely in $H_c^+$, and therefore, after fixing a prime of $H_c^+$ above $p$, the point $P$ localizes to a point in $E(F_p)$, where $F_p$ is the completion of $F$ at unique prime above $p$.

**Conjecture 2.2.** The Darmon point $P_{\tau_Q}$ is the localization of a global point $P_c$, defined over $H_c^+$, and the Galois action on this point is described by the following Shimura reciprocity law: if $P_c \in E(H_c^+)$ localizes to $P_{\tau_Q} \in E(F_p)$ then $P_{\tau_Q}$ localizes to $P_{\tau_Q}$.

2.3. Real conjugation. Denote by $\tau_p \in \text{Gal}(H_c^+ / \mathbb{Q})$ the Frobenius element at $p$, well defined only up to conjugation. As recalled above, since $p$ is inert in $F$, it splits completely in $H_c^+$ and (after fixing as above a prime of $H_c^+$ above $p$), $\tau_p$ corresponds to an involution of $\text{Gal}(F_p/\mathbb{Q}_p)$. By [Dar01, Proposition 5.10], it is known that there exists an element $\sigma_\tau \in G_c^+$ such that

$$
\tau_p(J_\tau) = -w_M J_\tau \sigma_\tau
$$

and $\tau_p(P_\tau) = w_N P_{\tau \sigma_\tau}$ where $w_M$ and $w_N$ are the signs of the Atkin-Lehner involution $W_M$ and $W_N$, respectively, acting on $f$.

2.4. Families of measure-valued modular symbols and Darmon points. Let

$$
X = \text{Hom}^{\text{cont}}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)
$$

and embed $\mathbb{Z}$ into $X$ via $k \mapsto [x \mapsto x^{k-2}]$. There are rigid analytic functions $\kappa \mapsto a_n(\kappa)$ for integers $n \geq 1$, simultaneously defined on a suitable neighborhood $U$ of 2 (which we may assume containing only integers $k$ with $k \equiv 2 \pmod{p-1}$) such that the formal power series expansion

$$
f_\infty(\kappa) = \sum_{n \geq 1} a_n(\kappa) q^n
$$

when evaluated at $\kappa = k \in \mathbb{Z}$ is the $q$-expansion of a normalized eigenform on $\Gamma_0(N)$ of weight $k$, and such that $f_2 = f$, where recall that $f$ is the modular form attached to $E$ by modularity.
If \( k \neq 2 \), \( f_k \) is necessarily old, and we let \( f_k^\ast \) be the form of level \( \Gamma_0(M) \) and weight \( k \) whose \( p \)-stabilization coincides with \( f \); so \( f_k \) and \( f_k^\ast \) are related by the formula:

\[
f_k(z) = f_k^\ast(z) - p^{k-1}a_p(k)^{-1}f_k^\ast(pz).
\]

For \( k = 2 \) we simply put \( f_2^\ast = f_2 = f \).

Let \( \mathcal{W} = \mathbb{Q}_p^2 - \{(0, 0)\} \) and let \( \mathbb{D} \) denote the \( \mathbb{Q}_p \)-vector space of compactly supported \( \mathbb{Q}_p \)-valued measures on \( \mathcal{W} \). Let \( L_p = \mathbb{Z}_p^2 \). Say that \((x, y) \in L_p \) is primitive if \( p \) does not divide both \( x \) and \( y \) and let \( L'_p = (\mathbb{Z}_p^2)^\prime \) denote the subset of \( L_p \) consisting of of primitive vectors. Let \( \mathbb{D}_s \) be the subspace consisting of measures which are supported on the \( L'_p \). Let \( \Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \) be the Iwasawa algebra of \( \mathbb{Z}_p^\times \), identified with a subring of the ring of analytic functions on \( \mathcal{X} \). The \( \mathbb{Q}_p \)-vector space \( \mathbb{D} \) is equipped with a structure of \( \Lambda \)-algebra arising form the action of \( \mathbb{Z}_p^\times \) on \( \mathcal{W} \) and \( \mathbb{Z}_p^2 \) given by \((x, y) \mapsto (\lambda x, \lambda y) \) for \( \lambda \in \Lambda \). Also, \( \text{GL}_2(\mathbb{Q}_p) \) acts from the left on \( \mathbb{D}_s \) by translations, and \( \text{MS}_{\Gamma_0(M)}(\mathbb{D}_s) \) is naturally equipped with an action of Hecke operators.

In particular, we have a \( U_p \)-operator acting on \( \text{MS}_{\Gamma_0(M)}(\mathbb{D}_s) \) by the formula

\[
\int_{\mathcal{X}} \phi d(U_p \mu)(r \to s) = \sum_{a=0}^{p-1} \int_{\mathcal{X}} \phi d(\mu r^a) \int_{\mathcal{X}} \chi(x) d\mu(x),
\]

for any locally constant function \( \phi \) on \( \mathcal{W} \). Here \( \gamma_a = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \) and for any open compact subset \( X \subseteq \mathcal{W} \) and any locally constant function \( \phi \) on \( \mathcal{W} \), we put \( \int_X \phi d\mu = \int_{\mathcal{X}} \phi(x) d\mu(x) \), where \( \chi \) is the characteristic function of \( X \). In particular, we may define \( \text{MS}_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_s) \) to be the maximal submodule of \( \text{MS}_{\Gamma_0(M)}(\mathbb{D}_s) \) on which \( U_p \) acts invertibly. For each \( k \in U \) there is a specialization map

\[
\rho_k : \mathbb{D}_s^1 \longrightarrow \mathbb{V}_{k-2}(\mathbb{C}_p)
\]

deefined by

\[
\rho_k(\mu)(P(x, y)) = \int_{\mathbb{Z}_p^2} P(x, y) d\mu(x, y).
\]

Let \( \Lambda^1 \) denote the ring of \( \mathbb{C}_p \)-valued functions on \( \mathcal{X} \) which can be represented by a convergent power series expansion in some neighborhood of \( 0 \in \mathcal{X} \) and define \( \mathbb{D}_s^1 = \mathbb{D}_s \otimes_\Lambda \Lambda^1 \). For any \( \mu = \sum_{i} \lambda_i \mu_i \) with \( \lambda_i \in \Lambda^1 \) and \( \mu_i \in \mathbb{D}_s \), we call a neighborhood of regularity for \( \mu \) any neighborhood \( U_\mu \) of \( 0 \in U \) such that all \( \lambda_i \) converge in \( U_\mu \). The module \( \text{MS}_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_s) \) inherits a \( \Lambda \)-action from the \( \Lambda \)-module structure of \( \mathbb{D}_s \), and we may define \( \text{MS}_{\Gamma_0(M)}^{\text{ord},1}(\mathbb{D}_s) = \text{MS}_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_s) \otimes_\Lambda \Lambda^1 \). This \( \Lambda^1 \)-module is free of finite rank, and given \( \mu \in \text{MS}_{\Gamma_0(M)}^{\text{ord},1}(\mathbb{D}_s) \) it is possible to find a common neighborhood of regularity for all the measures \( \mu \{r \to s\} \), which we denote \( U_\mu \). The specialization map \( \rho_k \) induces a map, denoted by the same symbol,

\[
\rho_k : \text{MS}_{\Gamma_0(M)}^{\text{ord},1}(\mathbb{D}_s) \longrightarrow \text{MS}_{\Gamma_0(M)}^{\text{ord}}(\mathbb{V}_{k-2}(\mathbb{C}_p)),
\]

and [BD07, Theorem 1.5] shows that for each choice of sign \( \pm \) there exists a neighborhood \( U \) of \( 2 \in \mathcal{X} \) and \( \mu^\pm \in \text{MS}_{\Gamma_0(M)}^{\text{ord},1}(\mathbb{D}_s) \) such that \( \rho_k(\mu^\pm) = I^\pm_\mu \) and for all integers \( k \in U \), \( k \geq 2 \), there is \( \lambda^\pm(k) \in \mathbb{C}_p \) such that \( \rho_k(\mu^\pm) = \lambda^\pm(k) I^\pm_\mu \). Also, \( U \) can be chosen so that \( \lambda^\pm(k) \neq 0 \) for all \( k \in U \).

**Theorem 2.3.** If \( Q \in \mathcal{F}_{D,2} \), then

\[
\log_q(P_{\tau_Q}) = \int_{\mathbb{Z}_p^2} \log_q(x - \tau Q y) d\mu^\pm \{r \to s\}(x, y).
\]
Proof. This follows from [BD09, Theorem 2.5] as in [BD09, Corollary 2.6] noticing that the set
\[ \{ (x, y) \in \mathbb{Q}_p^2 \mid x - \tau_Q y \in \mathcal{O}_K \otimes \mathbb{Z}_p \} \]
coinsides with \( \mathbb{Z}_p^2 \). \qed

3. Complex \( L \)-functions of real quadratic fields

Here we recast the special value formula of the second author and Whitehouse [MW09], restricted to the setting of this paper, in a form convenient for our purposes.

Let \( f \in S_k(\Gamma_0(M)) \) be a even weight \( k \geq 2 \) newform for \( \Gamma_0(M) \). Let \( F/\mathbb{Q} \) be a real quadratic field of discriminant \( D > 0 \), prime to \( M \), and let \( \chi_D \) be the associated quadratic Dirichlet character; with a slight abuse of notation, we will denote by the same symbol \( \chi_D : \mathbb{A}_F^\times \to \mathbb{C}^\times \) the associated Hecke character, where \( \mathbb{A}_F \) is the adele ring of \( F \). We assume that all primes \( \ell \mid M \) are split in \( F \).

Let \( c \) be an integer prime to \( DM \) and let \( H_c^+ \) be the strict ring class field of \( F \) of conductor \( c \). Let \( G_c^+ = \text{Gal}(H_c^+/F) \). Let \( \chi : G_c^+ \to \mathbb{C}^\times \) be a primitive character, namely, a character which does not factor through \( G_c^+ \) for any proper divisor \( f \mid c \); with a slight abuse of notation, we will denote by the same symbol \( \chi : \mathbb{A}_F^\times \to \mathbb{C}^\times \) the associated Hecke character, where \( \mathbb{A}_F \) is the adele ring of \( F \).

3.1. Optimal embedding theory. We set up the theory of optimal embeddings, and its relation to the strict, or narrow, class group of \( \mathcal{O}_c \) and quadratic forms of discriminant \( Dc^2 \).

For more details, see [LV14, §4.3] or [LRV13, §4.1].

Let us denote by \( B = M_2(\mathbb{Q}) \) the split quaternion algebra over \( \mathbb{Q} \) and denote by \( R_0 \) the order in \( B \) consisting of matrices in \( M_2(\mathbb{Z}) \) which are upper triangular modulo \( M \). Let \( \mathcal{O}_c = \mathbb{Z} + c \mathcal{O}_F \) be the order of \( F \) of conductor \( c \), where \( \mathcal{O}_F \) is the ring of integers of \( F \). Let \( \text{Emb}(\mathcal{O}_c, R_0) \) be the set of optimal embeddings \( \psi : F \to \mathcal{O}_c \) into \( R_0 \) (so \( \psi(\mathcal{O}_c) = R_0 \cap \psi(F) \)). For every prime \( \ell \mid M \) fix orientations of \( R_0 \) and \( \mathcal{O}_c \) at \( \ell \), i.e., ring homomorphisms \( \Omega_\ell : R_0 \to \mathbb{F}_\ell \) and \( \sigma_\ell : \mathcal{O}_c \to \mathbb{F}_\ell \). Two embeddings \( \psi, \psi' \in \text{Emb}(\mathcal{O}_c, R_0) \) are said to have the same orientation at a prime \( \ell \mid M \) if \( \Omega_\ell \circ (\psi|_{\mathcal{O}_c}) = \Omega_\ell \circ (\psi'|_{\mathcal{O}_c}) \) and are said to have opposite orientations at \( \ell \) otherwise. An embedding \( \psi \in \text{Emb}(\mathcal{O}_c, R_0) \) is said to be oriented if \( \Omega_\ell \circ (\psi|_{\mathcal{O}_c}) = \sigma_\ell \) for all primes \( \ell \mid M \). We denote the set of oriented optimal embeddings of \( \mathcal{O}_c \) into \( R_0 \) by \( \mathcal{E}(\mathcal{O}_c, R_0) \). The action of \( \Gamma_0(M) \) on \( \text{Emb}(\mathcal{O}_c, R_0) \) from the right by conjugation restricts to an action on \( \mathcal{E}(\mathcal{O}_c, R_0) \). If \( \psi \in \mathcal{E}(\mathcal{O}_c, R_0) \) then \( \psi_\omega := \omega_\omega \psi \omega_\omega^{-1} \) belongs to \( \mathcal{E}(\mathcal{O}_c, R_0) \) as well, where recall that \( \omega_\omega = (\frac{1}{0 0 -1}) \), and \( \psi \) and \( \psi_\omega \) have opposite orientations at all \( \ell \mid M \). If \( \ell \) is a prime dividing \( M \) then \( \psi \) and \( \omega_\ell \psi \omega_\ell^{-1} \), where \( \omega_\ell = (\frac{0 1}{1 0}) \), have opposite orientations at \( \ell \) and the same orientation at all primes dividing \( M/\ell \).

Let \( a \in \mathcal{O}_c \) be an ideal representing a class \([a] \in \text{Pic}^+(\mathcal{O}_c)\) and let \( \psi \in \text{Emb}(\mathcal{O}_c, R_0) \). The left \( R_0 \)-ideal \( R_0 \psi(a) \) is principal; let \( a \in R_0 \) be a generator of this ideal with positive reduced norm, which is unique up to elements in \( \Gamma_0(M) \). The right action of \( \psi(\mathcal{O}_c) \) on \( R_0 \psi(a) \) shows that \( \psi(\mathcal{O}_c) \) is contained in the right order of \( R_0 \psi(a) \), which is equal to \( a^{-1} R_0 a \). This defines an action of \( \text{Pic}^+(\mathcal{O}_c) \) on conjugacy classes of embeddings given by \([a] \cdot \psi = [a \psi a^{-1}] \) in \( \text{Emb}(\mathcal{O}_c, R_0)/\Gamma_0(M) \). The principal ideal \( (\sqrt{D}) \) is a proper \( \mathcal{O}_c \)-ideal of \( F \); denote \( \mathcal{D} \) its class in \( \text{Pic}^+(\mathcal{O}_c) \) and define \( \sigma_F := \text{rec}(\mathcal{D}) \in G_c^+ \), where \( \text{rec} \) is the arithmetically normalized reciprocity map of class field theory. If \( a = (\sqrt{D}) \) then we can take \( a = \omega_\infty \psi(\psi(\sqrt{D}) \) in the above discussion, which shows that \( \mathcal{D} \cdot \psi = [\omega_\infty \psi \omega_\infty^{-1}] = [\psi^*] \). Using the reciprocity map of class field theory, for all \( \sigma \in G_c^+ \) and \( \psi \in \text{Emb}(\mathcal{O}_c, R_0)/\Gamma_0(M) \) define \( \sigma \cdot [\psi] := \text{rec}^{-1}(\sigma) \cdot [\psi] \). In particular, \( \sigma_F \cdot [\psi] = [\psi^*] \) for all \( \psi \in \text{Emb}(\mathcal{O}_c, R_0) \).

If \( \psi \) is an oriented optimal embedding then the Eichler order \( a^{-1} R_0 a \) inherits an orientation from the one of \( R_0 \) and it can be checked that we get an induced action of \( \text{Pic}^+(\mathcal{O}_c) \) (and \( G_c^+ \)) on the set \( \mathcal{E}(\mathcal{O}_c, R_0)/\Gamma_0(M) \), and this action is free and transitive. To describe a
(non-canonical) bijection between $E(\mathcal{O}_c, R_0)/\Gamma_0(M)$ and $G_c^+$, fix once and for all an auxiliary embedding $\psi_0 \in E(\mathcal{O}_c, R_0)$; then $\sigma \mapsto \sigma \cdot [\psi_0]$ defines a bijection $E: G_c^+ \to E(\mathcal{O}_c, R_0)/\Gamma_0(M)$ whose inverse, $G = E^{-1}: E(\mathcal{O}_c, R_0)/\Gamma_0(M) \to G_c^+$ satisfies the relation $G([\psi]) = \sigma_\cdot G([\psi])$ for all $\psi \in E(\mathcal{O}_c, R_0)$. Choose for every $\sigma \in G_c^+$ an embedding $\psi_\sigma \in E(\sigma)$, so that the family $\{\psi_\sigma\}_{\sigma \in G_c^+}$ is a set of representatives of the $\Gamma_0(M)$-conjugacy classes of oriented optimal embeddings of $\mathcal{O}_c$ into $R_0$. If $\gamma, \gamma' \in R_0$ write $\gamma \sim \gamma'$ to indicate that $\gamma$ and $\gamma'$ are in the same $\Gamma_0(M)$-conjugacy class, and adopt a similar notation for (oriented) optimal embeddings of $\mathcal{O}_c$ into $R_0$. For all $\sigma, \sigma' \in G_c^+$ one has $\sigma \cdot \psi_\sigma \sim \psi_{\sigma'}$ and $\psi_{\sigma} \sim \psi_{\sigma'\sigma}$ for all $\sigma \in G_c^+$.

Finally, note that the set $E(\mathcal{O}_c, R_0)/\Gamma_0(M)$ is in bijection with $\mathcal{F}_{Dc,2}/\Gamma_0(M)$, since both sets are in bijection with $G_c^+$; explicitly, to the class of the oriented optimal embedding $\psi$ corresponds the class of the quadratic form

$$Q_\psi(x, y) = Cx^2 - 2Axy - By^2$$

with $\psi(\sqrt{D}c) = (A_B/D_A)$.

### 3.2. Adelic ring class groups.

Below we will want to view the ring class group $G_c^+$ adelically. Since this is omitted from the literature on class field theory that we are aware of (adelic treatments usually explain ray class fields but not ring class fields, and expositions of ring class groups which treat real quadratic extensions, e.g., [Coh78], tend to not use adelic language), we explain briefly the passage from classical ring class groups to adelic ring class groups here. For a point of reference, we also describe the relation with ray class groups. As ring class groups which treat real quadratic extensions, e.g., [Coh78], tend to not use adelic

Let $c \in \mathbb{N}$ and $m = m_f \infty$ where $m_f = c \mathcal{O}_F$ and $m_\infty$ is a subset of the real places of $F$. For a real place $v$, let $\sigma_v$ be the associated embedding of $F$ into $\mathbb{R}$. Let $J_m$ be the group of fractional ideals of $\mathcal{O}_F$ which are prime to $m_f$. Let $F_m^\times$ be the subset of $F^\times$ consisting of $x \in F^\times$ such that $\sigma_v(x) > 0$ for each $v \in m_\infty$ and $v_p(x-1) > v_p(c)$ for $p | m_f$. Let $J_m^1$ denote the set of principal ideals generated by elements of $F_m^1$. Then the ray class group mod $m$ of $F$ is $\text{Cl}_m(F) = J_m/J_m^1$.

Let $F_m^\times$ be the set of $x \in F^\times$ such that $\sigma_v(x) > 0$ for each $v \in m_\infty$ and for each $p | m_f$ there exists $a \in \mathbb{Z}$ coprime to $c$ such that $v_p(x-a) \geq v_p(c)$. Let $P_m^\times$ be the set of principal ideals in $F$ generated by elements of $F_m^\times$. Then the ring class group mod $m$ of $F$ is $\text{Cl}_m(F) = J_m/P_m^\times$. Note we can write $F_m^\times = \bigcup_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} a F_m^\times$. Hence $\text{Cl}_m(F)/\text{Im}(\mathbb{Z}/c\mathbb{Z})^\times \simeq G_m(F)$, where $\text{Im}(\mathbb{Z}/c\mathbb{Z})^\times$ denotes the image of the natural map from $(\mathbb{Z}/c\mathbb{Z})^\times \to \text{Cl}_m(F)$, which is not in general injective.

Via the usual correspondence between ideals and ideles, $J_m$ is identified with $\hat{F_m}^\times/\hat{O}_m^\times$, where $\hat{F}_m$ consists of finite ideles $(\alpha_v)$ such that $\alpha_v \in O_{F,v}^\times$ for all $v | m_f$ and $\hat{O}_m^\times = \prod_v \mathbb{Q}_{F,v}^\times$. For $v < \infty$, we put $W_v = O_{F,v}^\times$ unless $v | m_f$, in which case $W_v = 1 + m_f O_{F,v}$. For $v | \infty$, we put $W_v = F_v^\times$ unless $v | m_\infty$, in which case $W_v = \mathbb{R}_{>0}$. Now define $W = \prod_v W_v$ and $A_{F,m}^1 = \prod_v A_{F,v}^1 \times \prod_v \mathbb{R}_{>0}$. Then we have $F_m^\times = F_m^\times \cap A_{F,m}^1$ and $J_m \simeq A_{F,m}^1/W$, so $\text{Cl}_m(F) = F_m^\times/A_{F,m}^1/W = F_m^\times/\hat{A}_m^\times/W$.

For the ring class group, again we can realize it as a quotient of the idele class group $F_m^\times/A_{F,m}^\times$, but now it will be a quotient by a subgroup $U = \prod U_{\ell} \times U_{\infty}$ which is a product over rational primes, rather than primes of $F$. As usual, for a rational prime $\ell < \infty$, write $O_{F,\ell}$ for $O_F \otimes \mathbb{Z}_\ell$, which is isomorphic to $\mathbb{Z}_\ell \otimes \mathbb{Z}_\ell$ if $\ell$ splits in $F$ and otherwise is $O_{F,e}$ if $\ell$ is the unique prime of $F$ above $\ell$. Now set $U_{\ell} = O_{F,\ell}^\times$ if $\ell \nmid c$ and $U_{\ell} = (\mathbb{Z}_\ell + c O_{F,\ell})^\times$ if $\ell | c$. We can uniformly write $U_{\ell} = O_{c,\ell}^\times$ for $\ell < \infty$, where $O_{c,\ell} = Z + c O_{F,\ell}$ and $O_{c,\ell} = O_{c,\ell} \otimes \mathbb{Z}_\ell$. For later use, we will write $\hat{O}_m^\times = \prod U_{\ell}$. Note that this is different from the product $\prod_{\ell < \infty} O_{c,\ell}^\times$ for $v$ running over primes of $F$ if $c$ is divisible by primes which split in $F$. Put $U_{\infty} = W_{\infty} = \prod_{v | \infty} W_{\infty}$ and
we write $U_{\infty} = F_{\infty}^+$. Thus we can write our strict ring class group as 
\[
G_{m}^+(F) = F_{m}^+ \backslash \mathcal{A}_{F,m}^\times U = F_{m}^+ \backslash \mathcal{A}_{F,m}^\times / U.
\]

In our case of interest, namely $F$ is real quadratic and $m_{\infty}$ contains both real places of $F$, 
we write $U_{\infty} = F_{\infty}^+$. Thus we can write our strict ring class group as 
\[
G_{c}^+ = F_{c}^+ \backslash \mathcal{A}_{F,c}^\times / \mathcal{O}_{c}^\times F_{\infty}^+.
\]

3.3. Special value formulas. We return to our case of interest where $F/\mathbb{Q}$ is real quadratic 
of discriminant $D$, $f$ is a weight $k$ newform for $\Gamma_0(M)$, $c$ is an integer coprime with $DM$, 
and $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_F$. Let $H_c$ be the corresponding ring class field and $h_c$ be the degree of $H_c/F$, 
which coincides with the cardinality of $\text{Pic}((\mathcal{O}_c)_c)$. Denote by $h_c^+$ the cardinality of $G_c^+$, 
so $h_c^+/h_c$ is equal to 1 or 2. Fix ideals $\mathfrak{a}_\sigma$ for all $\sigma \in G_c = \text{Gal}(H_c/F)$ in such a way 
that $\Sigma_c = \{\mathfrak{a}_\sigma \mid \sigma \in G_c\}$ is a complete system of representatives for $\text{Pic}((\mathcal{O}_c)_c)$. Clearly $\Sigma_c^+ = \Sigma_c$ is also a system of representatives for $\text{Pic}^+(\mathcal{O}_c)$ if $h_c^+ = h_c$, while if $h_c^+ \neq h_c$ the set $\Sigma_c^+$ of representatives of $\text{Pic}^+(\mathcal{O}_c)$ can be written as $\Sigma_c^+ \cup \Sigma_c^-$ with $\Sigma_c^+ = \{\mathfrak{a}_\sigma \mid \sigma \in G_c\}$ and $\mathfrak{d} = (\sqrt{D})$. Let $\epsilon_c > 1$ be the smallest totally positive power of a fundamental unit in $\mathcal{O}_c^\times$, 
and for all $\sigma \in G_c^+$ define $\gamma_\sigma = \psi_\sigma(\epsilon_c)$. Finally, define 
\[
\alpha = \prod_{\ell \in \mathbb{Z}, (\frac{D}{\ell}) = -1} \ell,
\]
where $\ell$ runs over all rational primes dividing $c$ which are inert in $F$.

Denote by $\pi_f$ and $\pi_\chi$ the automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ attached to $f$ and $\chi$, 
respectively.

**Theorem 3.1.** Let $c$ be an integer such that $(c, DM) = 1$. Let $\chi$ be a character of $G_c^+$ such 
that the absolute norm of the conductor of $\chi$ is $c(\chi) = c^2$. For any choice of the base point $\tau_0 \in H$, 
we have 
\[
L(\pi_f \otimes \pi_\chi, 1/2) = \frac{\sum_{\sigma \in \Sigma_c} \chi^{-1}(\sigma) \int_{\gamma_\sigma(\tau_0)} f(z) \psi_\sigma(z, 1)^{(k-2)/2} dz}{4 \alpha^2 \cdot (Dc^2)^{(k-1)/2}}.
\]

When $c = 1$, this is the positive weight case of [Pop06, Theorem 6.3.1], which also treats weight 0 Maass forms. 
If desired, one could similarly extend the above result to weight 0 Maass forms.

**Proof.** Write $\pi := \pi_f = \otimes'_v \pi_v$, where $v$ runs over all places of $\mathbb{Q}$, and let $n_\ell(\pi)$ be the conductor 
of $\pi_\ell$ for each prime number $\ell$. Define 
\[
U_f(M) = \prod_{\ell} U_\ell(M), \quad U_\ell(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_\ell) : c \equiv 0 \mod M \right\}.
\]

We associate to $f$ the automorphic form $\varphi_\pi = \varphi_f$ on $GL_2(\mathbb{A}_\mathbb{Q})$ given by 
\[
\varphi_\pi \colon Z(\mathbb{A}) GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) / U_f(M) \to \mathbb{C}
\]
\[
\varphi_\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 2 j(g; i)^k f \left( \frac{ai + b}{ci + d} \right),
\]

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, where we write $j(g; i) = \text{det}(g)^{1/2} (ci + d)^{-1}$ for the automorphy factor. Then $\varphi_\pi$ is $R_0$-$\ell$-invariant for each finite prime $\ell$. The scaling factor of 2 is present so 
that the archimedean zeta integral of $\varphi_\pi$ gives the archimedean $L$-factor.

For $\phi \in \pi$, let 
\[
(\phi, \phi) = \int_{Z(\mathbb{A}_\mathbb{Q}) \backslash GL_2(\mathbb{Q}) } \phi(t) \overline{\phi(t)} \, dt
\]
be the Petersson norm of $\phi$, where we take as measures on the groups $GL_2(\mathbb{A}_\mathbb{Q})$ and $Z(\mathbb{A}_\mathbb{Q})$ the products of the local Tamagawa measures. Here, as usual, we take the quotient measure on the quotient, giving $GL_2(\mathbb{Q}) \subset GL_2(\mathbb{A}_\mathbb{Q})$ the counting measure.

Denote by $\pi_F$ the base change of $\pi$ to $F$. Let $L(\pi_F \otimes \chi, s)$ be the $L$-function of $\pi_F$ twisted by $\chi$, which equals the Rankin–Selberg $L$-function $L(\pi_F \otimes \pi, s)$. Since $F$ splits at each prime $\ell$ where $\pi$ is ramified and at $\infty$, $c(\pi_F, \chi, 1/2) = +1$ for all places $v$ of $\mathbb{Q}$. Then, in our setting, the main result in [MW09, Theorem 4.2] states that

$$
|P_\chi(\phi)|^2 = L(\pi_F \otimes \chi, 1/2)(\varphi, \varphi) |\chi_{\ell}/\chi_{\ell}'|^{2} \prod_{v} \left(\frac{\ell}{\ell - \chi_D(\ell)}\right)^{2},
$$

where $\varphi \in \pi$ is a suitable test vector,

$$
P_\chi(\varphi) = \int_{F^\times \mathbb{A}_\mathbb{Q}\setminus \mathbb{A}_F^\times} \varphi(t)\chi^{-1}(t) dt,
$$

and $\varphi'$ (denoted $\varphi_\pi$ in loc. cit.) is a vector in $\pi$ differing from $\varphi_\pi$ only at $\infty$. We describe $\varphi$ and $\varphi'$ precisely below. Similar to before, we take the products of local Tamagawa measures on $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^\times$, and give $F^\times$ the counting measure.

First we describe the choice of the test vector $\varphi$, which we only need to specify up to scalars, as the left-hand side above is invariant under scalar multiplication. We will take $\varphi = \varphi_\pi\varphi_v$, where $v$ runs over all places of $\mathbb{Q}$.

For $\ell$ a finite prime $\mathbb{Q}$, let $c(\chi_\ell)$ denote the smallest $n$ such that $\chi_\ell$ is trivial on $(\mathbb{Z}_\ell + \ell^n\mathcal{O}_{F,\ell})^\times$. Since $\chi$ is a character of $G_\ell^+$, we have $c(\chi_\ell) \leq v_\ell(c)$ for all $\ell$. In particular, $\chi_\ell$ is trivial on $\mathbb{Z}_\ell^\times$, so $c(\chi_\ell)$ is the smallest $n$ such that $\chi_\ell$ is trivial on $(1 + \ell^n\mathcal{O}_{F,\ell})^\times$, and thus agrees with the usual definition of the conductor of $\chi_\ell$ when $\ell$ is inert in $F$. Similarly, if $\ell$ is ramified in $F$, say $\ell = p^2$, then $c(\chi_\ell)$ is twice the conductor of $\chi_\ell = \chi_0 : F^\times_p \to \mathbb{C}^\times$, though this case does not occur by our assumption $(c, D) = 1$. If $\ell = p_1p_2$ is split in $F$, then we can write $\chi_\ell = \chi_{p_1} \otimes \chi_{p_2}$ with $\chi_{p_1}, \chi_{p_2}$ characters of $\mathbb{Q}_\ell^\times$, which are inverses of each other on $\mathbb{Z}_\ell^\times$ as $\chi_\ell$ is trivial on $\mathbb{Z}_\ell^\times$. Hence $\chi_{p_1}$ and $\chi_{p_2}$ have the same conductor, which is $c(\chi_{p_1})$. Consequently, $c(\chi)$, the absolute norm of the conductor of $\chi_{p_1}$, is

$$
c(\chi) = \text{Norm} \left(\prod_{\ell} \ell^{c(\chi_{\ell})}\right) = \prod_{\ell} \ell^{2c(\chi_{\ell})}.
$$

Note that since $(c, M) = 1$, we have $c(\chi) = 0$ whenever $\pi_{\ell}$ is ramified, i.e., the conductor $c(\pi_{\ell}) > 0$. If $c(\chi_{\ell}) = 0$, let $R_{\chi, \ell}$ be an Eichler order of reduced discriminant $\ell^{c(\pi_{\ell})}$ in $M_2(\mathbb{Q}_\ell)$ containing $\mathcal{O}_{F,\ell}$. If $c(\chi_\ell) > 0$, so $\pi_{\ell}$ is unramified, let $R_{\chi, \ell}$ be a maximal order of $M_2(F_{\ell})$ which optimally contains $\mathcal{O}_{\ell} + \ell^{c(\chi_{\ell})}\mathcal{O}_{F,\ell}$. In either case, $R_{\chi, \ell}$ is unique up to conjugacy and pointwise fixes a 1-dimensional subspace of $\pi_{\ell}$. For $\ell < \infty$, take $\varphi_{\ell} \in \pi_{\ell}^{R_{\chi, \ell}}$ nonzero, normalized in such a way that $\otimes \varphi_{\ell}$ converges. For instance, we can take $\varphi_{\ell} = \varphi_{\pi, \ell}$ at almost all $\ell$. Each $\varphi_{\ell}$ is a local Gross–Prasad test vector, and our assumptions imply that the local Gross–Prasad test vectors $\varphi_{\ell}$ are (up to scalars) translates of the new vectors $\varphi_{\pi, \ell}$. (Gross–Prasad test vectors are not translates of new vectors in general.)

Embed $F$ into $M_2(\mathbb{Q})$ as follows. Consider a quadratic form

$$
Q(x, y) = -\frac{C}{2} x^2 + Axy + \frac{B}{2} y^2 \in \mathcal{F}_{Dc^2}.
$$

This means $Q$ is primitive of discriminant $Dc^2 = A^2 + BC, 2 \mid B$ and $2M \mid C$, which implies $A^2 \equiv Dc \mod 4$. Take the embedding of $F$ into $M_2(\mathbb{Q})$ induced by $\sqrt{Dc} \mapsto (\begin{smallmatrix} A & B \\ C & -A \end{smallmatrix})$. Then $\mathcal{O}_{c} = R_0 \cap F$, and

$$
F^\times = \left\{ g(x, y) := \begin{pmatrix} x + Ay & By \\ Cy & x - Ay \end{pmatrix} \in GL_2(\mathbb{R}) \right\}.
$$
For a prime $\ell \nmid c$, we have $O_{c,\ell} = O_{F,\ell} \subset R_{0,\ell}$. Thus we may take $R_{\chi,\ell} = R_{0,\ell}$ for $\ell \nmid c$ such that $\chi_\ell$ is unramified—in particular, for $\ell \nmid c D$. When $\chi_\ell$ is ramified, we may take $R_{\chi,\ell} = R_{0,\ell}$ if and only if $c(\chi_\ell) = v_\ell(c)$. By assumption, $c(\chi) = \prod_\ell \ell^{2c(\chi_\ell)} = c^2$, so we may take $R_{\chi,\ell} = R_{0,\ell}$ at each finite place $\ell$. Thus we may and will take $\varphi_\ell$ to be $\varphi_{\ell,\ell}$ at each $\ell$.

Now we describe $\varphi_\infty$. Note we can identify $F_\infty^\times / Q_\infty^\times$ with $F_1^1 / \{\pm 1\}$, where

$$F_\infty^1 = F_{\infty}^{1,+} \cup F_{\infty}^{1,-}, \quad F_\infty^1 = \{g(x,y) \in F_\infty^\times : \det g(x,y) = x^2 - Dc^2y^2 = \pm 1\}.$$  

Let

$$\gamma_\infty = \left(\begin{array}{cc} A + \sqrt{Dc} & A - \sqrt{Dc} \\ C & C \end{array}\right).$$

Then

$$\gamma_\infty^{-1} \left(\begin{array}{cc} A & B \\ C & -A \end{array}\right) \gamma_\infty = \left(\begin{array}{cc} \sqrt{Dc} & 0 \\ 0 & -\sqrt{Dc} \end{array}\right).$$

So

$$\gamma_\infty^{-1} F_\infty^1 \gamma_\infty = \left\{(x + y\sqrt{Dc}, 0, 0, x - y\sqrt{Dc}) : x^2 - Dc^2y^2 = \pm 1\right\}.$$  

The maximal compact subgroup of $F_\infty^1$ is

$$\Gamma_F = \gamma_\infty \left\{\begin{array}{cc} \pm 1 & 0 \\ 0 & \pm 1 \end{array}\right\} \gamma_\infty^{-1} = \{\pm I, \pm g(0, -(\sqrt{Dc})^{-1})\},$$

where one reads the $\pm$ signs independently. Let $U_\infty = \gamma_\infty O(2) \gamma_\infty^{-1}$, where $O(2)$ denotes the standard maximal compact subgroup of $GL_2(\mathbb{R})$. Then $U_\infty \supset \Gamma_F$, and the archimedean test vector in [MW09] is the unique up to scalars nonzero vector $\varphi_\infty$ lying in the minimal $U_\infty$-type such that $\Gamma_F$ acts by $\chi_\infty$ on $\varphi_\infty$. Specifically, we can take

$$\varphi_\infty = \pi_\infty(\gamma_\infty)(\varphi_{\infty,k} \pm \varphi_{\infty,-k}),$$

where $\varphi_{\infty,\pm k} = \frac{1}{2}\pi_\infty(\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}) \varphi_\pi$ is a vector of weight $\pm k$ in $\pi_\infty$, and the $\pm$ sign in (9) matches the sign of $\chi_\infty(\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix})$. This completely describes the test vector $\varphi$ chosen in [MW09].

For our purposes, we would like to work with a different archimedean component than $\varphi_\infty$, corresponding to (a translate of) $\varphi_\pi$. Let $\varphi^-$ be the pure tensor in $\pi$ which agrees with $\varphi$ at all finite places, and is defined like $\varphi_\infty$ at infinity except using the opposite sign in the sum (9). Then necessarily any $\chi_\infty$-equivariant linear function on $\pi_\infty$ kills $\varphi_\infty^-$, so $P_\chi(\varphi^-) = 0$. Hence $P_\chi(\varphi) = P_\chi(\varphi')$ where $\varphi' = \varphi + \varphi^-$, and we can write $\varphi' = \otimes' \varphi_\pi'$, where $\varphi_\ell' = \varphi_\ell$ for finite primes $\ell$ and $\varphi_{\infty}' = \pi_\infty(\gamma_\infty) \varphi_{\pi}'$, i.e., $\varphi'(x) = \varphi_\pi(x \gamma_\infty)$.

Finally, we describe the vector $\varphi_\pi'$ appearing in (7). It is defined to be a factorizable function in $\pi$ whose associated local Whittaker functions are new vectors whose zeta functions are the local $L$-factors of $\pi$ at finite places, and at infinity is the vector in the minimal $O(2)$-type that transforms by $\chi_\infty$ under $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ such that the associated Whittaker function (restricted to first diagonal component) at infinity is $W_\infty(t) = 2\chi_\infty(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) |t|^{k/2} e^{-2\pi |t|}$. (This normalization gives $L(s, \pi) = \int_0^\infty W_\infty(t)|t|^{s-1/2} dt$.) Thus $\varphi_{\infty}'$ agrees with $\varphi_{\pi}'$ at all finite places and $\varphi_{\pi,\infty}' = 2(\varphi_{\infty,k} \pm \varphi_{\infty,-k})$, where the $\pm$ sign matches that in (9).

Hence $\varphi = \frac{1}{2} \pi(\gamma_\infty) \varphi_{\pi}'$, so by invariance of the inner product we have $(\varphi, \varphi) = \frac{1}{4}(\varphi, \varphi')$, and (7) becomes

$$|P_\chi(\varphi')|^2 = |P_\chi(\varphi)|^2 = L(\pi_F \otimes \chi, 1/2) \cdot \frac{1}{\sqrt{Dc}} \cdot \prod_{\ell \mid c} \left(\frac{\ell}{\ell - \chi_D(\ell)}\right)^2.$$  

Now we want to rewrite $P_\chi(\varphi')$. Recall that $\epsilon_c > 1$ is the smallest totally positive power of a fundamental unit in $\mathcal{O}_c^\times$. From (5), we obtain the isomorphism

$$A_Q^1 F_\infty^\times / A_Q^1 \mathcal{O}_c^\times \simeq G_\epsilon^+ \cdot (F_\infty^1 / \langle \epsilon_c \rangle Q_\infty^1) \simeq G_\epsilon^+ \cdot (F_\infty^1 / \langle \pm \epsilon_c \rangle).$$
We may identify

\[ F_{\infty}^{1,+}/(\pm \epsilon_c) = \left\{ \begin{pmatrix} x + Ay & By \\ Cy & x - Ay \end{pmatrix} \in SL_2(\mathbb{R}) : 1 \leq x + y\sqrt{D}c < \epsilon_c \right\}, \]

and the orbit of \( \gamma_{\infty}i \) in the upper half plane by this set is the geodesic segment connecting \( \gamma_{\infty}i \) to \( \epsilon_c\gamma_{\infty}i \), i.e., the image under \( \gamma_{\infty} \) of \( \{ iy : 1 \leq y \leq \epsilon_c^2 \} \subset H \). Let us call this arc \( \Upsilon \).

Since \( \mathbb{A}_Q^\times \subset F^\times \hat{\mathcal{O}}_c^\times F^\times_\infty \) and \( G_c^\times \simeq F^\times \backslash \mathbb{A}_F^\times /F^\times_\infty \) where \( F^\times_\infty = (\mathbb{R}_{>0})^2 \), we see that \( \chi \) is trivial on \( \mathbb{A}_Q^\times \hat{\mathcal{O}}_c^\times F^\times_\infty \). The Tamagawa measure gives \( \text{vol}(F^\times \mathbb{A}_Q^\times /\mathbb{A}_F^\times) = 2L(1, \eta) = 4\log \epsilon_F \text{vol}(\hat{O}^\times) \), where \( \eta \) is the quadratic character of \( \mathbb{A}_Q^\times /\mathbb{Q}^\times \) attached to \( F/\mathbb{Q} \) and \( \epsilon_F \) is the fundamental unit of \( F \). This implies \( \text{vol}(\mathbb{A}_Q^\times \hat{O}_c^\times /\hat{O}^\times) = 2\epsilon_c \text{len}(\Upsilon) \), where \( \text{len}(\Upsilon) = 2\log \epsilon_c \) is the length of \( \Upsilon \) with respect to the usual hyperbolic distance. Thus we compute

\[ P_\chi(\varphi') = 2\text{vol}(\hat{O}_c^\times) \sum_{t \in G_c^\times} \chi^{-1}(t) \int_{F_{\infty}^{1,+}/(\pm \epsilon_c)} \varphi(tg\gamma_{\infty}) \, dg \]

\[ = 4\text{vol}(\hat{O}_c^\times) \sum_{t \in G_c^\times} \chi^{-1}(t) \int_{1}^{\epsilon_c} j(t\gamma_{\infty}(\begin{smallmatrix} u & 0 \\ 0 & -1 \end{smallmatrix}) ; i) \, j(t\gamma_{\infty}(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) ; i) \, d^\times u \]

\[ = 2\text{vol}(\hat{O}_c^\times) \sum_{t \in G_c^\times} \chi^{-1}(t) \int_{1}^{\epsilon_c} j(t\gamma_{\infty}(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}) ; i) \, j(t\gamma_{\infty}(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) ; i) \, d^\times y, \]

where we use that \( f \) has trivial central character and the substitution \( y = u^2 \) at the last step.

For \( \ell \) a rational prime dividing \( c \), note that \( \mathcal{O}_{F,\ell}^\times /\mathcal{O}_{c,\ell}^\times \simeq \mathbb{Z}_\ell^\times /((1 + c\mathbb{Z}_\ell)) \) when \( \ell \) splits in \( F \) and \( \mathcal{O}_{F,\ell}^\times /\mathcal{O}_{c,\ell}^\times \simeq (\mathcal{O}_{F,\ell}/(1 + c\mathcal{O}_{F,\ell}))/((\mathbb{Z}_\ell^\times /((1 + c\mathbb{Z}_\ell))) \) when \( \ell \) is inert in \( F \). Hence, with our choice of measures,

\[ \text{vol}(\hat{O}_c^\times) = \text{vol}(\hat{O}_c^\times) \prod_{\ell \mid c} \text{vol}(\mathcal{O}_{F,\ell}^\times : \mathcal{O}_{c,\ell}^\times)^{-1} = \prod_{\ell \mid c} \frac{1}{\ell}, \]

where \( \ell \) runs over rational primes.

The next task is then to rewrite the integral appearing the in right hand side of the above formula. Let \( z = \gamma_{\infty}iy \). Then

\[ z = \frac{A}{C} + \sqrt{\frac{D}{C}} \left( 1 - \frac{2}{1 + iy} \right). \]

Since

\[ \frac{2iy}{(1 + iy)^2} = \frac{2}{1 + iy} - \frac{1}{2} \left( \frac{2}{1 + iy} \right)^2 = \frac{BC + 2ACz - C^2z^2}{2Dc^2}, \]

we have

\[ j(t\gamma_{\infty}(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}) ; i)^{-2} = \frac{C(1 + iy)^2}{2\sqrt{D}cy} = \frac{2i\sqrt{Dc}}{-Cz^2 + 2Az + B} = \frac{i\sqrt{Dc}}{Q(z, 1)}, \]

and

\[ dz = \frac{2i\sqrt{Dc}}{C(1 + iy)^2} \, d^\times y, \quad \text{i.e.} \quad d^\times y = \frac{2\sqrt{Dc}}{-Cz^2 + 2Az + B} \, dz = \frac{\sqrt{Dc}}{Q(z, 1)} \, dz. \]

Making the change of variable \( z = \gamma_{\infty}iy \), the above expression can be rewritten as

\[ P_\chi(\varphi') = \frac{2\text{vol}(\hat{O}_c^\times)}{i^{k/2} \cdot (\sqrt{D}c)^{(k-2)/2}} \sum_{t \in G_c^\times} \chi^{-1}(t) \int_{\Upsilon} f(tz) \cdot Q(z, 1)^{(k-2)/2} \, dz. \]
After another change of variable \( z' = tz \), the above integral becomes
\[
\int_{\mathcal{Y}} f(z') \cdot Q(t^{-1}z',1)^{(k-2)/2} dz' = \int_{\mathcal{Y}} f(z') \cdot (Q|t^{-1})(z',1)^{(k-2)/2} dz' = \int_{\mathcal{Y}} f(z') \cdot (Q|t^{-1})(z',1)^{(k-2)/2} dz',
\]
where \( \tau_t = t\gamma_{0i} \). Now, as long as \( t \) varies in \( G_+^\perp \), the quadratic forms \( Q|t^{-1} \) are representatives for the classes in \( \mathcal{F}_{D,t^2}/\mathcal{F}_0(M) \), as discussed in §3.1. Moreover, since \( \Upsilon \) is closed in \( \mathcal{H}/\mathcal{F}_0(N) \), this integral does not depend on the choice of base point. Plugging this into (10) gives the asserted formula. \( \square \)

3.4. Genus fields attached to orders. Assume from now on that \( c \) is odd. The genus field attached to the order \( \mathcal{O}_c \) of discriminant \( Dc^2 \) is the finite abelian extension of \( Q \), with Galois group isomorphic to copies of \( \mathbb{Z}/2\mathbb{Z} \), contained in the strict class field \( H^+_c \) of \( F \) of conductor \( c \) and generated by the quadratic extensions \( Q(\sqrt{D}) \) and \( Q(\sqrt{\ell^*}) \) where \( D = \prod_i D_i \) is any possible factorization of \( D \) into primary discriminants, \( \ell \mid c \) is a prime number and \( \ell^* = (-1)^{(\ell-1)/2}\ell \). See [Coh78, pp. 242-244] for details.

Fix a quadratic character \( \chi : G^+_c \to \{\pm 1\} \).

Definition 3.2. We say that \( \chi \) is primitive if it does not factor through \( H^+_f \) for a proper divisor \( f \mid c \).

We assume that \( \chi : G^+_c \to \{\pm 1\} \) is primitive. By (8), this means \( c(\chi) = c^2 \). Then \( \chi \) cuts out a quadratic extension \( H_\chi/F \) which, by genus theory for \( \mathcal{O}_c \), is biquadratic over \( Q \). Each quadratic extension of \( \mathcal{O}_c \) contained in the genus field of the order \( \mathcal{O}_c \) is of the form \( \mathcal{F}(\sqrt{D}) \) for some \( \Delta = D' \cdot \prod_{j=1}^{n} \ell^*_j \), with \( \ell_j \mid c \) and \( D' \) a fundamental discriminant dividing \( c \). Write \( H_\chi = \mathcal{F}(\sqrt{\Delta_1}, \sqrt{\Delta_2}) \), with \( \Delta_1 = D_1 \cdot \prod_{j=1}^{n} \ell^*_j \) for \( i = 1, 2 \) as above (so \( \ell_{i,j} \) are primes dividing \( c \)), and let \( K_1 = \mathcal{F}(\sqrt{\Delta_1}) \) and \( K_2 = \mathcal{F}(\sqrt{\Delta_2}) \). Since the third quadratic extension contained in \( H_\chi \) is the quadratic extension \( \mathcal{F}(\sqrt{D}) \), we have \( \Delta_1 \cdot \Delta_2 = D \cdot x^2 \) for some \( x \in \mathcal{O}_c^* \). We can write \( \Delta_1 = D_1 d \) and \( \Delta_2 = D_2 d \) for some \( d = \prod_{j=1}^{n} \ell^*_j \) with \( \ell_i \mid c \) primes and \( D = D_1 \cdot D_2 \) a factorization into fundamental discriminants, allowing \( D_1 = D_2 = D \) if \( d \neq \pm c \), then \( \chi \) factors through the extension \( H^+_d \neq H^+_f \) by the genus theory of the order of conductor \( Dd^2 \), and therefore \( \chi \) is not a primitive character of \( H^+_c \). So \( d = \pm c \). Thus we conclude that the quadratic fields \( K_1 = \mathcal{F}(\sqrt{D_1 d}) \) and \( K_2 = \mathcal{F}(\sqrt{D_2 d}) \) satisfy the following properties:

- \( D_1 \cdot D_2 = D \), where \( D_1 \) and \( D_2 \) are two coprime fundamental discriminants (possibly equal to 1).

- \( d = \pm c \) and \( d \) is a fundamental discriminant.

Let \( \chi_{D_1 d} \) and \( \chi_{D_2 d} \) be the quadratic characters attached to the extensions \( K_1 \) and \( K_2 \) respectively; thus \( \chi_{D_1 d}(x) = \left( \frac{D_1 d}{x} \right) \) and \( \chi_{D_2 d}(x) = \left( \frac{D_2 d}{x} \right) \). Similarly, let \( \chi_D \) be the quadratic character attached to the extension \( F/Q \), i.e., \( \chi_D(x) = \left( \frac{D}{x} \right) \). In particular, for all \( \ell \mid c \) we have
\[
(11) \quad \chi_D(\ell) = \chi_{D_1 d}(\ell) \cdot \chi_{D_2 d}(\ell).
\]

Say that \( \chi \) has sign \( +1 \) if \( H_\chi/F \) is totally real, and sign \( -1 \) otherwise. If \( \chi \) has sign \( w_\infty \in \{\pm 1\} \), put \( I_f = I^w_\infty \) and \( \Omega_f = \Omega^w_\infty \). Define
\[
\mathcal{L}(f, \chi) := \sum_{\sigma \in G^+_c} \chi^{-1}(\sigma) I_f(\tau_0 \to \gamma_{\psi_\sigma}(\tau_0))(Q_{\psi_\sigma}(x,y)^{(k-2)/2})
\]

Lemma 3.3. \( \mathcal{L}(f, \chi) = w_\infty \cdot \mathcal{L}(f, \chi) \).
Proof. This follows from the discussion in [Pop06, §6.1]. To simplify the notation, define

$$\Theta_\psi := I_f(\tau_0 \to \gamma_\psi(\tau_0))\left(Q_\psi(x, y)^{2(k-2)/2}\right),$$

which is independent of the choice of \(\tau_0\) and the \(\Gamma_0(M)\)-conjugacy class of \(\psi\). Let \(z \mapsto \bar{z}\) denote complex conjugation. A direct computation shows that \(\Theta_\psi = \Theta_{\psi^*}\), where recall that \(\psi^* = \omega^{-1}_f \psi \omega_\infty^{-1}\). From the discussion in §3.1 we have \(\sigma_F \cdot [\psi] = [\psi^*]\), and it follows that \(\Theta_{\psi} = \Theta_{\sigma_F \psi}\). Taking sums over a set of representatives of optimal embeddings shows that \(L(f, \chi) = \chi(\sigma_F) \cdot L(f, \chi)\). Let \(H_\chi\) be the field cut out by \(\chi\). The description of a system of representatives \(\Sigma_\chi\) and \(\Sigma_\chi^+\) of \(\text{Gal}(H_\chi/F)\) and \(\text{Gal}(H_\chi^+/F)\) in §3.3 shows that if \(\chi(\sigma_F) = 1\) then \(H_\chi\) is contained in \(H_c\), and therefore \(H_\chi\) is totally real. On the other hand, if \(\chi(\sigma_F) = -1\), then \(H_\chi\) cannot be contained in \(H_c\), and therefore it is not totally real, so it is the product of two imaginary extensions. By definition of the sign of \(\chi\), this means that \(L(f, \chi)\) is a real number when \(\chi\) is even, and is a purely imaginary complex number when \(\chi\) is odd, and the result follows. \(\square\)

Using the relation

$$L(\pi_g \times \pi_\chi, 1/2) = \frac{4}{(2\pi)^k} \left(\left(\frac{k-2}{2}\right)!\right)^2 L(f/F, \chi, k/2),$$

it follows from Lemma 3.3 that Theorem 3.1 can be rewritten in the following form:

$$L(f/F, \chi, k/2) = \frac{(2\pi)^{k-2} \cdot \Omega_2^2 \cdot w_\infty}{((\frac{k-2}{2})!)^2 \cdot \alpha^2 \cdot (Dc^2)^{(k-1)/2}} \cdot L(f, \chi)^2. \tag{12}$$

Remark 3.4. By the lemma, the sign \(w_\infty\) should also appear in equation (28) of [BD09], as the left hand side of that equation is not positive when \(\chi\) is odd. However, the main result in [BD09] still follows as this sign will cancel out with a sign arising from Gauss sums as in our argument below.

4. \(p\)-adic L-functions

Recall the notation introduced in §2.4: \(f_\infty\) is the Hida family passing through the weight two modular form \(f\) of level \(N = Mp\) associated to the elliptic curve \(E\) by modularity; \(U\) is a connected neighborhood of 2 in the weight space \(X\); \(\mu_\infty^\pm\) is a measure-valued modular symbol satisfying the property that for all integers \(k \in U, k \geq 2\), there is \(\lambda^\pm(k) \in \mathbb{C}_p^\times\) such that \(\rho_k(\mu_\infty^\pm) = \lambda^\pm(k) \Gamma_k^\pm\) and \(\lambda^\pm(2) = 1\).

4.1. \(p\)-adic L-function of real quadratic fields. For any \(Q \in \mathcal{F}_{Dc^2}\) and \(\kappa \in U\), define

$$Q(x, y)^{(\kappa-2)/2} = \exp_p\left(\frac{\kappa - 2}{2} \log_\kappa ((Q(x, y)))\right)$$

where \(\exp_p\) is the \(p\)-adic exponential and for \(x \in \mathbb{Q}_p\), we let \((x)\) denote the principal unit of \(x\), satisfying \(x = p^{\text{ord}_p(x)}\zeta(x)\) for a \((p-1)\)-th root of unity \(\zeta\). Recall the Hida family \(f_\infty\) introduced in §2.4.

**Definition 4.1.** Let \(Q \in \mathcal{F}_{Dc^2}\) and let \(\gamma_{\tau_Q}\) be the generator of the stabilizer of the root \(\tau_Q\) of \(Q(z, 1)\), chosen as in Definition 2.1.

1. Let \(r \in \mathbb{P}^1(\mathbb{Q})\). The partial square root \(p\)-adic L-function attached to \(f_\infty\), a choice of sign \(\pm\), and \(Q\) is the function of \(\kappa \in U\) defined by

$$L_{x}^{\pm}(f_\infty/F, Q, \kappa) = \int_{(\mathbb{Z}_p^2)^*} Q(x, y)^{(\kappa-2)/2} d\mu^\pm_{\kappa}\{r \to \gamma_{\tau_Q}(r)\}(x, y).$$
(2) Let \( \chi \) be a character of \( G_\mathbb{C}^+ \). The square root \( p \)-adic \( L \)-function attached to \( f_\infty \), a choice of sign \( \pm \), and \( \chi \) is the function of \( \kappa \) in \( U \) defined by

\[
\mathcal{L}^\pm_p(f_\infty/F, \chi, \kappa) = \sum_{\sigma \in G_\mathbb{C}^+} \chi^{-1}(\sigma)\mathcal{L}^\pm_p(f_\infty/F, Q^\sigma, \kappa).
\]

(3) The \( p \)-adic \( L \)-function attached to \( f_\infty \), the sign \( \pm \), and \( \chi \) is

\[
L^\pm_p(f_\infty/F, \chi, \kappa) = (\mathcal{L}^\pm_p(f_\infty/F, \chi, \kappa))^2.
\]

Let \( \chi : G_\mathbb{C}^+ \rightarrow \{\pm 1\} \) be a quadratic ring class character. Let \( \epsilon \) be the sign of \( \chi \) and set \( w_\infty = \epsilon \). Denote \( \mu_s = \mu_s^{w_\infty} \), \( \Omega_{f_k} = \Omega_{f_k}^{w_\infty} \), \( \lambda(k) = \lambda^{w_\infty}(k) \) and \( L_p(f_\infty/F, \chi, k) = L_{f_p}^{w_\infty}(f_\infty/F, \chi, k) \). Recall the newform \( f_k \) whose \( p \)-stabilization is the weight \( k \) specialization of the Hida family \( f_\infty \) introduced in §2.4. Define the algebraic part of the central value of the \( L \)-function of the newform \( f_k \) twisted by \( \chi \) to be

\[
L^{alg}(f_k^g/F, \chi, k/2) = \frac{((k/2)!)^2 \sqrt{Dc}}{(2\pi i)^{k-2} \cdot \Omega^2} \cdot L(f_k^g/K, \chi, k/2).
\]

**Theorem 4.2.** For all integers \( k \in U, k \geq 2 \), we have

\[
L_p(f_\infty/F, \chi, k) = \lambda(k)^2 \cdot \alpha^2 \cdot (1 - a_p(k)^{-2}p^{k-2})^2 \cdot (Dc^2)^{(k-2)/2} \cdot L^{alg}(f_k^g/F, \chi, k/2)
\]

where the rational number \( \alpha \) is defined in (6).

**Proof.** By definition,

\[
\mathcal{L}_p(f_\infty/F, Q, k) = \int_{\mathbb{Z}_p^g} Q(x, y)(k-2)/2 \, d\mu_\ast(r \rightarrow \gamma(r)}(x, y)
\]

\[
= \lambda(k)(1 - a_p(k)^{-2}p^{k-2})L^{\ast}(f_k^g(r \rightarrow \gamma(r)}(Q^{(k-2)/2}),
\]

where the last equality follows from [BD09, Proposition 2.4] and therefore we get, in the notation of Section 3.4,

\[
L_p(f_\infty/F, \chi, k) = \lambda(k)^2(1 - a_p(k)^{-2}p^{k-2})^2 \cdot L(f_k^g, \chi)^2.
\]

Using (12) gives the result. \[\square\]

4.2. Mazur–Kitagawa \( p \)-adic \( L \)-functions. Let \( \chi : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \{\pm 1\} \) be a primitive quadratic Dirichlet character of conductor \( m \). Suppose that \( \chi(-1) = (-1)^{(k-2)/2}w_\infty \) and put \( \Omega_{f_k} = \Omega_{f_k}^{w_\infty} \), \( \lambda(k) = \lambda^{w_\infty}(k) \) and \( \mu_\ast = \mu_s^{w_\infty} \). For \( k \in U \) a positive integer define

\[
L^{alg}(f_k^g, \chi, k/2) = \frac{\tau(\chi)((k-2)/2)!}{(-2\pi i)^{(k-2)/2} \Omega^2 L(f_k^g, \chi, k/2)}
\]

as the algebraic part of the central special value of \( L(f_k^g, \chi, s) \), where \( \tau(\chi) = \sum_{a=1}^m \chi(a)e^{2\pi ia/m} \) denotes the Gauss sum of the character \( \chi \). The Mazur–Kitagawa \( p \)-adic \( L \)-function is defined as

\[
L_p(f_\infty, \chi, k, s) = \sum_{a=1}^m \chi(pa) \int_{\mathbb{Z}_p^g \times \mathbb{Z}_p^g} \left( x - \frac{pa}{m} \right)^{s-1} y^{k-s-1} \, d\mu_\ast(\infty \rightarrow pa/m)
\]

and satisfies the following interpolation formula: for all integers \( k \in U \) with \( k \geq 2 \) we have

\[
L_p(f_\infty, \chi, k, k/2) = \lambda(k)(1 - \chi(p)a_p(k))^{-1}p^{(k-2)/2}L^{alg}(f_k^g, \chi, k/2).
\]
4.3. A factorization formula for genus characters. Let $\chi : G^+_c \to \{\pm 1\}$ be a primitive character, and let $\chi_{D_d} : \mathbb{Q}(\sqrt{D_d}) \to \{\pm 1\}$ and $\chi_{D_d} : \mathbb{Q}(\sqrt{D_d}) \to \{\pm 1\}$ be the associated quadratic Dirichlet characters.

**Theorem 4.3.** The following equality

$\quad L_p(f_\infty/F, \chi, \kappa) = \alpha^2 \cdot (Dc^2)^{(k-2)/2} \cdot L_p(f_\infty, \chi_{D_d}, \kappa, \kappa/2) \cdot L_p(f_\infty, \chi_{D_d}, \kappa, \kappa/2)$

holds for all $\kappa \in U$, where the rational number $\alpha$ is defined in (6).

**Proof.** Let $\chi_{D_d}$ denote the quadratic characters associated with the extension $\mathbb{Q}(\sqrt{D_d})$. Since $p$ is inert in $F$, we have $\chi_D(p) = -1$, and therefore from (11) we get

$\chi_{D_d}(p) = -\chi_{D_d}(p)$.

It follows that the Euler factor $(1 - a_p(k)^{-2}p^{-k/2})^2$ appearing in Theorem 4.2 is equal to the product of the two Euler factors

$\quad (1 - \chi_{D_d}(p)a_p(k)^{-1}p^{-k/2})^2 \quad \text{and} \quad (1 - \chi_{D_d}(p)a_p(k)^{-1}p^{-k/2})^2$

appearing in (13). By comparison of Euler factors, we see that for all even integers $k \geq 4$ in $U$ we have

$\quad L(f_k^2/F, \chi, s) = L(f_k^2, \chi_{D_d}, s) \cdot L(f_k^2, \chi_{D_d}, s)$.

Therefore, from Theorem 4.2 and the factorization formula (14) it follows that for all even integers $k \geq 4$ in $U$ the following factorization formula holds:

$\quad L_p(f_\infty/F, \chi, k) = \left(\frac{\alpha^2 \cdot \sqrt{D_c} \cdot (Dc^2)^{(k-2)/2} \cdot w_\infty}{\tau(\chi_{D_d}) \cdot \tau(\chi_{D_d})}\right) \cdot L_p(f_\infty, \chi_{D_d}, k, k/2) \cdot L_p(f_\infty, \chi_{D_d}, k, k/2)$.

Since $D_d$ are fundamental discriminants, $\tau(\chi_{D_d}) = \sqrt{D_d}$ (interpreting $\sqrt{x}$ as $i\sqrt{|x|}$ for $x < 0$), so $\frac{\sqrt{D_c}}{\tau(\chi_{D_d}) \cdot \tau(\chi_{D_d})} = w_\infty$, and the formula in the statement holds for all even integers $k \geq 4$ in $U$. Since $\mathbb{Z} \cap U$ is a dense subset of $U$, and the two sides of equation (15) are continuous functions in $U$, they coincide on $U$. \qed

5. The main result

Let the notation be as in the introduction: $E/\mathbb{Q}$ is an elliptic curve of conductor $N = Mp$ with $p \nmid M$, $p \neq 2$, and $F/\mathbb{Q}$ a real quadratic field of discriminant $D = D_F$ such that all primes dividing $M$ are split in $F$ and $p$ is inert in $F$. Finally, $c \in \mathbb{Z}$ is a positive integer prime to $ND$ and $\chi : G^+_c \to \{\pm 1\}$ is a primitive quadratic character of the strict ring class field of conductor $c$ of $F$. Let $w_\infty$ be the sign of $\chi$, and as above put $L_p(f_\infty/F, Q, \kappa) = L^{w_\infty}_p(f_\infty/F, Q, \kappa)$, $L_p(f_\infty/F, \chi, \kappa) = L^{w_\infty}_p(f_\infty/F, \chi, \kappa)$, and $L_p(f_\infty/F, \chi, \kappa) = L^{w_\infty}_p(f_\infty/F, \chi, \kappa)$.

We begin by observing that $L_p(f_\infty/F, Q, 2) = 0$, since its value is $\mu_f \{r \to \gamma_q(r)\} \mathbb{P}^1(\mathbb{Q}_p)$, and the total measure of $\mu_f$ is zero. For the next result, let $w_M$ be the sign of the Atkin–Lehner involution acting on $f$. Also, let $\log_E : E(C_p) \to \mathbb{C}$ denote the logarithmic map on $E(C_p)$ induced from the Tate uniformization and the choice of the branch $\log_q$ of the logarithm fixed above.

**Theorem 5.1.** For all quadratic characters $\chi : G^+_c \to \{\pm 1\}$ we have

$$\frac{d}{d\kappa} L_p(f_\infty/F, \chi, \kappa)_{\kappa=2} = \frac{1}{2} (1 - \chi_{D_d}(-M)w_M) \log_E(P_\chi),$$

where $P_\chi$ is defined as in (1).
Proof. We have
\[
\frac{d}{d\kappa} L_Q(f_\infty/F, \chi, \kappa)_{\kappa=2} = \frac{1}{2} \int_{(\mathbb{Z}/p^2)^*} \left( \log_q(x - \tau_Q y) + \log_q(x - \tilde{\tau}_Q y) \right) d\mu_\ast \{ r \rightarrow \gamma_{\tau_Q}(r) \}
\]
\[
= \frac{1}{2} (\log_E(P_{\tau_Q}) + \log_E(\tilde{\tau}_Q P_{\tau_Q})).
\]
By (4), \( \tau_p(J_{\tau_Q}) = -w_M J_{\ast \tau_Q} \) and by [BD09, Proposition 1.8] (whose extension to the present situation presents no difficulties) we know that \( \chi(\sigma) = \chi_{D_1d}(-M) \), so the result follows summing over all \( \mathcal{Q} \).

**Theorem 5.2.** Let \( \chi \) be a primitive quadratic character of \( \mathbb{Z}^+ \) with associated Dirichlet characters \( \chi_{D_1d} \) and \( \chi_{D_2d} \). Suppose that \( \chi_{D_1d}(-M) = -w_M \). Then:

1. There is a point \( \mathbf{P}_\chi \in E(H_1)^\chi \) and \( n \in \mathbb{Q}^\times \) such that \( \log_E(P_{\chi}) = n \cdot \log_E(P_{\chi}) \).
2. The point \( \mathbf{P}_\chi \) is of infinite order if and only if \( L'(E/F, \chi, 1) \neq 0 \).

Proof. By Theorem 5.1 we have
\[
\frac{1}{2} \frac{d^2}{d\kappa^2} L_p(f_\infty/F, \chi, \kappa)_{\kappa=2} = \log^2_E(P_{\chi}).
\]
On the other hand, by the factorization of Theorem 4.3 we have
\[
L_p(f_\infty/F, \chi, \kappa) = \alpha^2 \cdot (Dc^2)_{(k-2)/2} \cdot L_p(f_\infty, \chi_{D_1d}, \kappa, \kappa/2) \cdot L_p(f_\infty, \chi_{D_2d}, \kappa, \kappa/2),
\]
where the integer \( \alpha \) is defined in (6). Let sign(\( E, \chi_{D_2d} \)) = \( -w_N \chi_{D_1d}(-N) \), where \( w_N \) is the sign of the Atkin–Lehner involution at \( N \). This is the sign of the functional equation of the complex \( L \)-series \( L(E, \chi_{D_1d}, s) \). Since
\[
\chi_{D_1d}(-N) \cdot \chi_{D_2d}(-N) = \chi_D(-N) = -1,
\]
we may order the characters \( \chi_{D_1d} \) and \( \chi_{D_2d} \) in such a way that sign(\( E, \chi_{D_1d} \)) = \( -1 \) and sign(\( E, \chi_{D_2d} \)) = \( +1 \). So \( \chi_{D_1d}(-N) = w_N \) and since \( \chi_{D_1d}(-M) = -w_M \) it follows that \( \chi_{D_1d}(p) = -w_p = a_p \). So the Mazur–Kitagawa \( p \)-adic \( L \)-function \( L_p(f, \chi_{D_1d}, \kappa, s) \) has an exceptional zero at \( (\kappa, s) = (2, 1) \), and its order of vanishing is at least 2. We may apply [BD07, Theorem 5.4], [Mok11, Sec. 6] and [Mok, Theorem 3.1], which show that there is a global point \( \mathbf{P}_{\chi_{D_1d}} \in E(Q(\sqrt{D_1t})) \) and a rational number \( \ell_1 \in Q^\times \) such that
\[
\frac{d^2}{d\kappa^2} L_p(f_\infty, \chi_{D_1d}, \kappa, \kappa/2)_{\kappa=2} = \ell_1 \log^2_E(P_{\chi_{D_1d}}),
\]
and this point is of infinite order if and only if \( L'(E, \chi_{D_1d}, 1) \neq 0 \). Moreover, \( \ell_1 \equiv L^\text{alg}(f, \psi, 1) \mod (Q^\times)^2 \) for any primitive Dirichlet character \( \psi \) such that \( L(f, \psi, 1) \neq 0, \psi(p) = -\chi_{D_1d}(p) \), and \( \psi(\ell) = \chi_{D_1d}(\ell) \) for all \( \ell \mid M \). Now
\[
\ell_2 = \frac{1}{2} L_p(f_\infty, \chi_{D_2d}, 2, 1) = L^\text{alg}(E, \chi_{D_2d}, 1)
\]
is a rational number which is non-zero if and only if \( L(E, \chi_{D_2d}, 1) \neq 0 \). In this case, \( \ell_1 \ell_2 \) is a square: choose \( t \in Q^\times \) such that \( t^2 = \ell_1 \ell_2 \) if \( \ell_2 \neq 0 \) and \( t = 1 \) otherwise, and let \( \mathbf{P}_\chi = \mathbf{P}_{\chi_{D_1d}} \) in the first case and 0 otherwise. Now the first part of the theorem follows setting \( n = \alpha \cdot t \).
Finally, for the second part note that \( L(E, \chi_{D_2d}, 1) \neq 0 \) if and only if \( L'(E/F, \chi, 1) \neq 0 \) thanks to the factorization (14). \( \square \)
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