One Modulo Three Mean Labeling Of Graphs

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Abstract

The concept of one modulo three mean labeling graph is, if there is an injective function \( \varphi \) from the vertex set of \( G \) to the set \( \{ a/0 \leq a \leq 3q - 2 \text{ and either } a \equiv 0(\text{mod}3) \text{ or } a \equiv 1(\text{mod}3) \} \) where \( q \) is the number of edges of \( G \) and \( \varphi \) induces a bijection \( \varphi^* \) from the edge set of \( G \) to \( \{a/1 \leq a \leq 3q - 2, a \equiv 1(\text{mod}3) \} \) given by \( \varphi^*(uv) = \left[ \frac{\varphi(u) + \varphi(v)}{2} \right] \) and the function \( \varphi \) is called one modulo three mean labeling of \( G \). In this paper, we obtain the results of one modulo three mean labeling of some several graphs.

Keywords: One modulo three mean labeling, One modulo three mean graphs, Triangular book Graph \((B_3^n)\), Duplication subdivision of the central edge of the Bistar \((B_{n,n})\), \(G_n^{+'}(n \geq 3)\) graph, Slanting ladder \(SL_{n'}(n \geq 3)\).

1. INTRODUCTION

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [1]. Given a graph \( G \), the symbols \( V(G) \) and \( E(G) \) denote the vertex set and edge set of the graph \( G \), respectively. Let \( G = (p, q) \) be a graph with \( p = |V(G)| \) vertices and \( q = |E(G)| \) edges. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. S.Somansundaram and R. Ponraj introduced mean labeling of graphs in [3] and further studied in [4]. Different kinds of mean labeling are further studied by Gayathri and Gopi [5-8]. V. Swaminathan and C. Sekar introduced the concept of one modulo three graceful labeling in [10]. Jeyanthi and Maheswari [11] introduced the concept of one modulo three mean labeling and proved that some standard graphs are one modulo three mean graphs. A graph \( G \) is said to be one modulo three mean graph if there is an injective function \( \varphi \) from the vertex set of \( G \) to the set \( \{a/0 \leq a \leq 3q - 2 \text{ and either } a \equiv 0(\text{mod}3) \text{ or } a \equiv 1(\text{mod}3) \} \) where \( q \) is the number of edges of \( G \) and \( \varphi \) induces a bijection \( \varphi^* \) from the edge set of \( G \) to
\(1 \leq \alpha \leq 3q - 2\), \(\alpha \equiv 1 \pmod{3}\)\) given by \(\varphi^*(uv) = \left\lfloor \frac{\varphi(u) + \varphi(v)}{2} \right\rfloor\) and the function \(\varphi\) is called one modulo three mean labeling of \(G\). Further some more results on one modulo three mean labeling of graphs in [13]. In [14], M. Kannan, R. Vikrama Prasad and R. Gopi are investigated Even vertex odd mean labeling of some graphs and also S. Meena and A. Ezhil are discussed about Total prime labeling of some graphs[15]. Motivated by the work of these authors, we prove that some several graphs are one modulo three mean labeling of graphs. We use the following definition in the subsequent sections.

### 1.1 Definition

One edge union of cycles of some length is called a book. The common edge is called base of the book. If we consider \(n\) copies of cycles of length \(t \geq 3\). The book is denoted by \(B_t^n\). If \(t=3\) the book is called Triangular Book Graph.

### 1.2 Definition

Duplication of a vertex \(v_k\) of a graph \(G\) produces a new graph \(G_1\) by adding a vertex \(v_{kl}\) with \(N(v_{kl}) = N(v_k)\). In other words a vertex \(v_{kl}\) is said to be a duplication of \(v_k\) if all the vertices which are adjacent to \(v_k\) are now adjacent to \(v_{kl}\) also.

### 1.3 Definition

Bistar is the graph obtained by joining the apex vertices of two copies of star \(k_{1,n}\)

### 1.4 Definition

A subdivision of a graph \(G\) is a graph resulting from the subdivision of edges in \(G\). The subdivision of some edge \(e\) with end points \(\{u,v\}\) yields a graph containing one new vertex \(w\), and with an edge set replacing \(e\) by two new edges \(\{u,w\}\) & \(\{w,v\}\).

For example

\[\text{u} \quad \text{e} \quad \text{v}\]

An be subdivided into two edges \(e_1\) & \(e_2\) connecting to a new vertex \(w\).

\[\text{u} \quad e_1 \quad w \quad e_2 \quad v\]

### 1.5 Definition

The Bistar \(B_{n,n}\) is the graph obtained from \(k_2\) by joining \(n\) pendent edges to one end of \(k_2\) and \(n\) pendent edges to the other end of \(k_2\), the edge of \(k_2\) is called the central edge \(B_{n,n}\).
1.6 Definition

Let G be a mean tree with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and let $G'$ be a copy of G with $\{v'_1, v'_2, \ldots, v'_n\}$. Then the graph $G_n^{(\pm)}$ is obtained by joining the vertex $v_i$ with $v'_i$ by an edge for all $1 \leq i \leq n$.

1.7 Definition

The slanting ladder $SL_n$ is a graph obtained from two paths $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ by joining each $u_i$ with $v_{i+1}$.

1.8 Theorem

The Triangular Book Graph $B_3^n$ is an one modulo three mean labeling graph.

Proof:

Let G be a triangular book graph, and let $\{v_i, 0 \leq i \leq n+2\}$ be the vertices and $\{e_i, 1 \leq i \leq 2n + 1\}$ be the edges which are denoted as in Fig 1.

Define $f : V(G) \to \{a / 1 \leq a \leq 6q - 5 \text{ and either } a \equiv 0 \text{ (mod 3)} \text{ or } a \equiv 1 \text{ (mod 3)}\}$

Vertex labeling are defined by

$$f(v_i) = \begin{cases} 
6l + 1 & ; i = 0, 1, 2 \\
12l - 11 & ; i = 3, 4, 5, 6 \ldots n + 2 
\end{cases}$$

Fig.1. Triangular Book Graph $B_3^n$

Edge labeling are defined by

$$f(e_i^+) : E(G) \to \{a / 1 \leq a \leq 3q + 1 \text{ and } a \equiv 1 \text{ (mod 3)}\}$$

$$f(e_i^-) = 3i + 1 ; i = 0, 1, 2, 3, \ldots,$$
From above the labeling of vertices and edges are distinct. Hence the graph is one modulo three mean labeling graph.

1.9 Example

One modulo three mean labeling of Triangular Book graph $B^6_3$ is shown in Fig 2.

Fig 2: Triangular Book graph $B^6_3$

1.10 Theorem

Duplication subdivision of the central edge of the Bistar SD($B_{n,n}$) is an one modulo three mean graph.

Proof:

Let \{u,v,w,u',v'\}, 1 \leq i \leq n and \{a, a_i (1 \leq i \leq n + 1), a_i' (1 \leq i \leq n)\} and \{b, b_i (1 \leq i \leq n + 1), b_i' (1 \leq i \leq n)\} be the vertices and edges which are denoted by the following Fig 3.

First we label the vertices as follows. Define

\[ f: V(G) \rightarrow \{a/0 \leq a \leq 3(p + q) - 8 \text{ and either } a \equiv 0 \text{ (mod3)} \text{ or } a \equiv 1 \text{ (mod3)}\}; \]

\[ f(u) = 1; \quad f(w) = 0; \quad f(v) = 12n + 13; \]

\[ f(u_i) = 6i + 1, \ 1 \leq i \leq n; \quad f(v_i) = 6i, \ 1 \leq i \leq n; \quad f(u_i') = 6n + 7; \]

\[ f(v_i') = 18n + 19. \]
Fig. 3. Duplication subdivision of the central edge of the Bistar SD(B_{n,n})

Edge labeling are defined by

\[ f(e_i^*): E(G) \rightarrow \{ a / 1 \leq a \leq 3q - 2 \text{ and } a \equiv 1 \ (\text{mod} \ 3) \} \]

Then the induced edge labels are,

- \( f'(a) = 1 \)
- \( f'(b) = 6n + 7 \)
- \( f'(a_i) = 3i + 1, 1 \leq i \leq n + 1 \)
- \( f'(b_i) = 3i + 1 + (6n + 7), 1 \leq i \leq n + 1 \)
- \( f'(a_i') = 3i + 3n + 4, 1 \leq i \leq n \)
- \( f'(b_i') = 3i + 9n + 10, 1 \leq i \leq n \)

The above defined function provides one modulo three mean labeling of duplication subdivision of the central edge of the Bistar SD(B_{n,n}). Hence the graph SD(B_{n,n}) is a one modulo three mean graph.

1.11 Example

One modulo three mean labeling is shown in Fig 4. (a) Duplication subdivision of the central edge of the Bistar SD(B_{4,4}), (b) Duplication subdivision of the central edge of the Bistar SD(B_{7,7})

Fig. 4. (a) Duplication subdivision of the central edge of the Bistar SD(B_{4,4})
1.12 Theorem

The graph $G_n^{(+) (n \geq 3)}$ is one modulo three mean graph for any $n$.

Proof:

Let $\{v_i, u_i, v'_i, u'_i\}$ be the vertices and $\{b_i, c_i, b'_i, c'_i, 1 \leq i \leq n \text{ and } a_i, a'_i, 1 \leq i \leq n - 1\}$ be the edges which are denoted as in the following Fig 5.

First we label the vertices as follows.

Define $f : V \rightarrow \{a/ 0 \leq a \leq 3q, a \equiv 0 \text{ (mod 3)} \text{ and } a \equiv 1 \text{ (mod 3)}\}$

i.e., $f : V \rightarrow \{0, 1, 2, ..., 3q\}$
Then the induced edge labels are

For \( 1 \leq i \leq n - 1 \), \( f^*(a_i) = 6i - 2 \)

\[ f^*(a_i) = 12n + (6i - 5) \]

For \( 1 \leq i \leq n \), \( f^*(b_i) = 6i - 5 \)

\[ f^*(b_i') = 12n + (6i - 8) \]

\[ \begin{align*}
  f^*(c_i) & = (6n + (6i - 8)) \quad \text{if } i \text{ is odd} \\
  f^*(c_i') & = (6n + (6i - 5)) \quad \text{if } i \text{ is even}
\end{align*} \]

Therefore \( f^*(E) = \{1, 4, 7, ..., 3q - 2\} \)

Simply we can say \( f^*(E) = \{a/1 \leq a \leq 3q - 2, a \equiv 1 \mod 3\} \)

So \( f \) is one modulo three mean labeling. And hence the graph \( G_n^{(+)}(n \geq 3) \) is an one modulo three mean graph for any \( n \).

1.13 Example

One modulo three mean labeling is shown in Fig 6. (a) \( G_5^{(+)} \) graph, (b) \( G_4^{(+)} \) graph

![Diagram](https://via.placeholder.com/150)

Fig. 6. (a) \( G_5^{(+)}(n \geq 3) \) graph
1.14 Theorem

The graph $SL_n(n \geq 3)$ is one modulo three mean graph for any $n$.

Proof:

Case(i) : If $n$ is an even

Let $\{v_i, v'_i, 1 \leq i \leq n\}$ be the vertices and
$\{a_i, b_i, c_i, 1 \leq i \leq n-1\}$ be the edges which are denoted as in the following Fig 7.

First we label the vertices as follows.

Define $f: V \rightarrow \{a/ 0 \leq a \leq 3q - 2, a \equiv 0(mod 3)\ and\ a \equiv 1(mod 3)\}$
i.e., $f: V \rightarrow \{0, 1, 2, ..., 3q - 2\}$

For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} 
3(i - 1) &; i \text{ is an odd} \\
3i - 5 &; i \text{ is an even}
\end{cases}$$
Then the induced edge labels are,

For 1 \leq i \leq n - 1,

\[ f^*(a_i) = 3i - 2 \]
\[ f^*(b_i) = 3n + (3i - 5) \]
\[ f^*(c_i) = 6n + (3i - 8) \]

Case(ii) : If n is odd

Let \{u_i, u'_i; 1 \leq i \leq n\} be the vertices and
\{a'_i, b'_i, c'_i; 1 \leq i \leq n - 1\} be the edges which are denoted as in the following Fig 7.

First we label the vertices as follows.

Define \( f: V \to \{a/\ 0 \leq a \leq 3q, a \equiv 0 (mod 3) and a \equiv 1 (mod 3)\}\)

i.e., \( f: V \to \{0, 1, 2, ..., 3q\}\)

For 1 \leq i \leq n,

\[ f(u_i) = \begin{cases} 
3(i - 1) &; i \text{ is an odd} \\
3i - 5 &; i \text{ is an even} 
\end{cases} \]

\[ f(u'_i) = \begin{cases} 
6(n - 1) + 3(i - 1) &; i \text{ is an odd} \\
6(n - 1) + 3i - 5 &; i \text{ is an even} 
\end{cases} \]

Then the induced edge labels are,

For 1 \leq i \leq n - 1,

\[ f^*(a'_i) = 3i - 2 \]
\[ f^*(b'_i) = 3n + (3i - 5) \]
\[ f^*(c'_i) = 6n + (3i - 8) \]

\[ \text{Fig. 7. Slanting ladder } SL_n(\ n \geq 3) \text{ Graph} \]

Therefore \( f^*(E) = \{1, 4, 7, ..., 3q - 2\} \)

Simply we can say \( f^*(E) = \{a/\ 1 \leq a \leq 3q - 2, a \equiv 1 (mod 3)\}\)

So \( f \) is one modulo three mean labeling. And hence the graph \( SL_n(\ n \geq 3) \) is an one modulo three mean graph for any \( n \).
1.15 Example

One modulo three mean labeling is shown in Fig 8. (a) SL\(_4\) graph, (b) SL\(_5\) graph of case(i) case(ii) respectively

Case (i):

```
  18 19 19 22 24 25
  10 13 16
0 1 1 4 6 7 7
```

Fig. 8. (a) Slanting ladder SL\(_4\) graph

Case (ii):

```
  24 25 25 28 30 31 31 34 36
  13 16 19 22
0 1 1 4 6 7 7 10 12
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Fig. 8. (b) Slanting ladder SL\(_5\) graph

CONCLUSION

As all labelling graphs are One Modulo Three Mean Graphs. It is very interesting as well to investigate One Modulo Three Mean Labeling for Triangular Book graph, \(G^+_n\) (\(n \geq 3\)) graph and Slanting ladder \(SL_n\) (\(n \geq 3\)) graph.

Here we have constructed duplication subdivision of the Central edge of the bistar SD(B\(_{n,n}\)) is also one modulo three mean graph. Further investigate can be done to obtain the above labeling for some several graphs.
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