A Zeno Story*

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Abstract
I describe the early (1974–75) work I did on what is now called the Zeno problem in quantum mechanics. Then I propose a new formulation which may obviate a vexing problem of operator limits and which also may be more measurement-compatible.

1 Introduction
I was involved in the formulation of the quantum Zeno effect some years ago, in 1974–75, working with B. Misra in Boulder. I wrote up some of that in [1] at that time, as a first draft of a paper. I also lectured on the problem at the Rocky Mt. Mathematics Consortium summer school in Bozeman in summer 1975 [2]. These contributions were never published, although I did summarize briefly some of the issues in [3]. Therefore I would like to return to this problem, describe the situation in 1974–75, and also propose a new formulation of the Zeno problem, using my recent results in [4].

The Zeno’s paradox “a watched pot never boils” has been presented in [5] and has since been discussed by many authors. As I recall, the problem was presented to Misra and me by Professor Josef Jauch when he visited us in Boulder in April 1974. In my work on the problem, I preferred to call it “The Counter Problem” [1]. I had nothing to do with the later and more popular formulation as “Zeno’s Paradox”, although I note now that in [2] I did state...
"When \( \limsup (PU_{t/n}/P)^n \|P\psi\| \overset{a.e.}={} \|P\psi\|^2 \), then \( P \) (affirmative answer) = \( P \) (that part was in \( C \) initially) \( \sim \) “particle cannot decay”.

Here \( C \) refers to a counter which can be visualized as absorbing the wave packet. In fact we were mostly following the formulation of Friedman [6], who deserves more citation. Like Friedman, I could not resolve the main mathematical issue, which was the existence and properties of an operator limit

\[
\liminf_{n \to \infty} (PU(t/n)P)^n
\]

In (1.1) \( U \) is a unitary evolution group and \( P \) is an orthogonal projection on a Hilbert space \( \mathcal{H} \). As [1] and [2] reveal, I spent considerable time on this question, because it seems to represent an important new domain in the theory of unbounded operator limits. Briefly, neither the “Trotter” theory of operator limits found in [7], nor the monotone forms approach found in [8], could be made to apply to the limit (1.1). Misra and I did publish the two related papers [9,10] but (frankly) I was surprised when [5] appeared, because I thought the problem was not solved.

As it turns out, the publication of [5] created considerable interest, so that was a good thing for physics. However [5] is based on never-proven mathematical assumptions. “Paradoxes” are often incompletely-cast models, although how to most-correctly cast a situation may not be easy.

2 The Continual Observation Problem

Friedman [6] treated what is now called the Zeno problem by reducing it to the question (1.1) above and to related operator-theoretic considerations. I quote ([6], Section 5, p. 1007)

My original motivation for studying semigroup product formulas was the following situation: \( P^t \) is a contraction semigroup on the Hilbert space \( \mathcal{H} \), \( E \) an orthogonal projection in \( \mathcal{H} \). If strong \( \lim_{n \to \infty} (EP^t/nE)^n \) exists, we call it the compression of \( P^t \) by \( E \). Formally one expects the limit to be \( \exp(tEAE) \) where \( A \) is the infinitesimal generator of \( P^t \). However, \( EAE \) is not in general the generator of a contraction semigroup, nor is its closure.

Friedman [6] goes on to give some partial results when the infinitesimal generator \( A \) is a negative selfadjoint operator, but he does not resolve the strong limit question (1.1). In ([6], Section 6) he then goes to the quantum mechanical continual observation problem. First he discusses Feynman’s ideal measurement formulation, that of determining whether or not a trajectory of a particle lies in a given space-time region. Then he presents a modified formulation of ideal measurements attributed to Ed Nelson, which becomes ([6], p. 1010)

This makes it reasonable to define

\[
\lim_{n \to \infty} \| (E \exp(-itH_0/n)E)^n \psi(x) \|_2
\]
(if it exists) as the probability that—an ideal measurement of ‘continual observation’ during the interval $(0, t)$ for the purpose of determining whether the particle stays in $E$ yields the result that the particle is indeed constantly in $E$ during $(0, t)$.

I leave it to the interested reader to pursue further details in [6]. I preferred to call the problem that of a (opaque) counter in [1,2,3]. The Zeno interpretation was put in in [5]. Since then there is a large literature. For such, I just defer to [11,12,13] and citations therein.

Friedman [6] concludes:

The question of whether strong limit$_{n\to\infty} (E \exp(itH_0/n)E)^n$ exists in $L^2$ and whether it is a unitary group remains open; to answer this would seem to require a much deeper knowledge of semigroup product formulas."

Very recently I have found [14] an alternative approach to this question which I now present in the next section.

3 A new formulation

Some recent authors [13] have rediscovered Friedman’s work [6] or have otherwise been led to wonder [11] if one cannot stay in the reversible regime (e.g., lecture on 11-26-01 at the Solvay Conference by S. Pascazio). Although a bit oversimplified, this would mean that one could say under general conditions that the operator limit (1.1) would be $e^{itHP}$ where $H$ was the original self-adjoint infinitesimal generator of the evolution group $U_t = e^{itH}$. Therefore one naturally poses the question: when is $PHP$ selfadjoint, i.e., when is $(PHP)^* = PHP$? Of course one asked this question in the beginning [6,1,2,3]. The difficulty is that the usual operator-theoretic conditions for product adjoints do not apply well to such projected Hamiltonians. Briefly, the usual conditions (e.g., see [15]) for operators $A$ and $B$ to satisfy $(BA)^* = A^*B^*$ are either $A$ and $B$ bounded, or if not, $A$ and $B$ Fredholm (closed ranges and finite index). The tacit assumption is, of course, that generally $A$ and $B$ are regarded as noncommuting operators.

What I propose [14] is to reformulate (1.1) as the question

$$(AHA)^* = A^*H^*A^*$$

where $A$ replaces $P$. What I have in mind is that the spatial (Dirichlet) projection $P$ used by Friedman [6], see also [13], may not be the best way to formulate a “Zeno” (continual) measurement. Not only will (3.1) be provable mathematically (below), but it also opens up the use of new physical observables $A$ representing the act of measurement.

Here is an example of my approach. In [4] I recently obtained the following result.
Lemma 3.1 [4] Let $A$ and $B$ be arbitrary densely defined operators in a Hilbert space $\mathcal{H}$ and suppose the domain $\mathcal{D}(AB)$ is dense, the range $\mathcal{R}(B) \supset \mathcal{D}(A)$, the domain $\mathcal{D}(B^*) \supset \mathcal{R}(A^*)$, and that $B$ is 1–1. Then $(AB)^* = B^*A^*$. In particular, when $A$ and $B$ are selfadjoint, then the conditions are $\mathcal{D}(AB)$ dense, $\mathcal{R}(B) \supset \mathcal{D}(A)$, $\mathcal{D}(B^*) \supset \mathcal{R}(A^*)$.

We may now apply this Lemma to the “Zeno” problem. For simplicity, let me first restrict here to observables $A$ which are bounded selfadjoint operators on $\mathcal{H}$. This stipulation may be relaxed later. Then we have

\[(AH^A)^* = (HA^*)^* \supset (A^*H^*)^* = AHA\] (3.2)

The relations in (3.2) follow because $A = A^* \in \mathcal{B}(\mathcal{H})$, I have assumed the domain $\mathcal{D}(HA)$ to be dense so that the adjoint operator $(HA)^*$ exists, and one always has the relation $(AB)^* \supset B^*A^*$ when all adjoint operators spoken of therein are defined. Thus the only remaining issue is to make the second relation in (3.2) an equality. To that end one may use the result of [4]. Therefore I have shown the following.

Theorem 3.1 Let $A$ be a “continual measurement observable” such that $A = A^* \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(A) \supset \mathcal{D}(H)$, $\mathcal{D}(HA)$ dense. Then $(AH^A)^*$ is selfadjoint and the exponentiation $e^{iAHAt}$ is unitary.

Note that the condition $\mathcal{R}(A) \supset \mathcal{D}(H)$ in the Theorem seems reasonable for “continual” measuring observables: we should have $A$ able to cover all possible wave functions $\psi$ in the domain of the Hamiltonian. Another way to view this requirement is that $A$ as measuror is also compatible with $A$ as preparor of state. The condition $\mathcal{D}(HA)$ dense is a technical one which could be loosely viewed as assuring that the measurement observable $A$ is not “too” incompatible with the Hamiltonian $H$ whose action is to be measured. One could call these measuring observables those of ‘full measurement’.

4 Elaboration (after the conference)

Here I would like to add a few clarifications to the above. Further development of this approach will be pursued elsewhere [16].

4.1 Measurement and Domains

The issues surrounding measurement in quantum mechanics are numerous and longstanding. See for example [11, 17–20] for recent treatments. Here I will just make a few comments related to this paper.

Friedman [6], like many others, equates observation with measurement. The space–time region is $(0,t) \times \text{Counter}$. His main example, although stated as a free particle evolution $e^{-itH_0}\psi_0$ where $H_0 = -\sum_{i=1}^{3} \partial^2 / \partial x_i^2$ is the three dimensional Laplacian operator, is almost classical. What I call the Counter is in [6]
a closed bounded three-dimensional real domain $\mathcal{E}$ with smooth boundary $\partial \mathcal{E}$. Variations of this example are studied in [12,13].

Now I would like to make an observation about this setting. One of the difficulties of Friedman’s [6] model and analysis is that there are two different operators posed, both called $H_0$. The first $H_0$ is the free space Laplacian $\Delta$ in $L^2(\mathbb{R}^3)$. Its domain $\mathcal{D}(H_0)$, in order that it be selfadjoint, may be specified by appropriate Sobolev derivatives and appropriate decay rates at infinity. Then $L^2(\mathbb{R}^3)$ is projected to $L^2$ (Counter). Friedman’s projection $E$ is implemented by multiplication by the characteristic function $\chi(\mathcal{E})$. Within $L^2(\mathcal{E})$ we then find another selfadjoint operator $H_0$, the Laplacian $\Delta$ in $L^2(\mathcal{E})$ with specified Dirichlet boundary conditions. The domain $\mathcal{D}(H_0)$ of this operator is well known (see [21] and citations therein) but the trace nature of what means $\psi = 0$ on the boundary $\partial \mathcal{E}$ for $\phi$ in $\mathcal{D}(H_0)$ is as is well known a delicate matter. Remember that $\psi$ needed two Sobolev derivatives in the interior of $\mathcal{E}$. So to project the larger “quantum mechanical” evolution $e^{-itH_0}\psi_0$ simply by multiplying by the characteristic function $\chi(\mathcal{E})$ does nothing at all about attending to the matter of the regularity at the boundary needed by functions $\psi$ which are to be in the domain $\mathcal{D}(H_0)$ of the other operator $H_0$, namely, the Poisson–Dirichlet operator in $L^2(\mathcal{E})$. This, in my opinion, lies at the root of Friedman’s [6] inability to prove any rigorous result about continual measurement. Also, in my opinion, when Misra and Sudarshan must assume $\lim_{t_0 \to 0+} T(t) = E$, one is on unfirm ground of the same underlying nature. Stated another way and more generally, we once again find the quantum-classical interface between quantum probability wave and classical measuror, a delicate business.

On the other hand, the quantum mechanical evolutions $e^{-it\Delta}$ freespace and the classical evolution $e^{-it\Delta}$ Poisson–Dirichlet proceed easily on their way. The former is analyzed in [13] in one dimension and it is shown that, for initial state $\psi_0$ taken with support in $\mathcal{E} = [0,1]$, the evolution corresponds to the propagator of a particle in a square well with Dirichlet boundary conditions. In [13] “We prepare a particle in a state with support in $A$” (we may take $A \equiv \mathcal{E} \equiv [0,1] \equiv$ the counter) and this state $\psi_0$ is evolved by the free space evolution and frequently projected by $E = \chi(\mathcal{E})$ according to Friedman’s prescription. Much of the work in [13] then consists in using the explicit, Green’s function representation of the evolution $e^{-it\Delta}/2m$ to show Zeno nondecay. In this way they [13] are able to obviate a use of Trotter formula approach [12] which we have seen many times [1,2,6,7,12] does not quite go through.

Here I would like to make a second observation. Consider any Dirichlet domain $\mathcal{E}$ in $L^2(\mathbb{R}^n)$. By this I mean one for which the Poisson–Dirichlet problem

\[ \Delta \psi = f \text{ in } \mathcal{E}, \quad \psi = 0 \text{ on } \partial \mathcal{E} \]  \hspace{1cm} (4.1)

is well-posed. Let $H_0$ denote the corresponding selfadjoint operator in $L^2(\mathcal{E})$. Consider any initial $\psi_0$ in $\mathcal{D}(H_0)$. We notice that means that $\psi_0 = 0$ on $\partial \mathcal{E}$ and also that $\psi_0$ possesses the necessary regularity, especially up to the boundary, to be in $\mathcal{D}(H_0)$. Now we know rather generally (e.g., [8, Chapter IX]) that exponentiations $U_i$ commute with their infinitesimal generators, i.e., $U_i H_0 \subseteq$
He the free space evolution \(e^{-itH_0} \psi_0\) keeps vanishing on the surface of the counter. Moreover it is a unitary evolution in \(L^2(\mathcal{E})\). This means that had we started with the free space evolution \(e^{-it\Delta}\) and any initial state \(\psi_0\) in \(L^2(\mathbb{R}^n)\), then if we want to ‘count it’ in the way desired by Friedman [6] et al., we want a “projection” \(E\) which maps us not only to the counter \(\mathcal{E}\) but also to \(\mathcal{D}(H_0)\) where \(H_0\) is the Poisson–Dirichlet Hamiltonian. That means restricting \(L^2(\mathbb{R}^n)\) to \(L^2(\mathcal{E}) \cap \mathcal{D}(H_0)\). In other words, once you ‘project’ into the counter, also satisfying the Dirichlet trace boundary condition on the surface of the counter, then that portion of your original wave-packet free particle evolution continues as a unitary evolution within the counter. Once you are in the counter, you stay in the counter always. This, in my opinion, is the real meaning of the claim in [13] that “Zeno dynamics uniquely determines the boundary conditions, and they turn out to be of Dirichlet type.” In my opinion, they are of Dirichlet type only because one started with ‘Dirichlet type’ boundary conditions (neglecting the questions of needed regularity at the boundary) when one projected by means of multiplication by \(\chi(\mathcal{E})\). Stated another way: you can have any kind of selfadjoint boundary conditions on the surface of the counter, provided you ‘project’ to \(L^2(\mathcal{E}) \cap \mathcal{D}(H_0)\) where \(\mathcal{D}(H_0)\) now contains those boundary conditions. Then the above evolution-infinitesimal generator commutation property assures you of a Zeno dynamics in the counter thereafter.

### 4.2 Domains and Zeno Measurors

In my alternate formulation (Section 3), we were able to give conditions under which the ‘projected’ infinitesimal generator \(EHE\) remained a generator. To do so I needed to let \(E\) be a ‘wider’ measuror \(A\), so that \(AHA\) remained selfadjoint in the original Hilbert space. It is desirable, I think, to be able to remain in the original Hilbert space, in contrast to what I have shown above to be a reduction of physical description to a unitary evolution trapped in a counter.

Therefore here I want to return to the approach of [4], viz. Lemma 3.1 above, and ask, rather than when \((AB)^* = B^*A^*\), the question: when is \((ABC)^* = C^*B^*A^*\)? This can then be specialized to: when is \(AHA\) selfadjoint? By the same analysis as [4] one can establish the following.

**Theorem 4.1** Let \(A, B, C\) be densely defined operators in a Hilbert space \(\mathcal{H}\). Suppose the domains \(\mathcal{D}(BC)\) and \(\mathcal{D}(ABC)\) are dense, ranges \(\mathcal{R}(BC) \supset \mathcal{D}(A)\), \(\mathcal{R}(C) \supset \mathcal{D}(B)\), domains \(\mathcal{D}((BC)^*) \supset \mathcal{R}(A^*)\), \(\mathcal{D}(C^*) \supset \mathcal{R}(B^*)\), and \(C\) and \(BC\) are 1–1. Then \((ABC)^* = C^*B^*A^*\).

**Corollary 4.1** Let \(A\) and \(H\) be selfadjoint operators in a Hilbert space \(\mathcal{H}\). Suppose \(\mathcal{D}(HA)\) and \(\mathcal{D}(AHA)\) are dense, \(\mathcal{R}(HA) \supset \mathcal{D}(A)\), \(\mathcal{R}(A) \supset \mathcal{D}(H)\), \(\mathcal{D}((HA)^*) \supset \mathcal{R}(A)\), \(\mathcal{D}(A) \supset \mathcal{R}(H)\). Then \((AHA)^* = AHA\).

The assumptions in Corollary 4.1, although appearing somewhat general, may be seen to reduce to rather precise statements about domains and ranges. I will state this as a further corollary.
Corollary 4.2 Let \( A \) and \( H \) be selfadjoint operators in a Hilbert space \( \mathcal{H} \). Then the sufficient conditions of Corollary 4.1 require necessarily that \( \mathcal{R}(H) = \mathcal{D}(A) \).

I remark that the quantum mechanical free space \( H_0 \) and Poisson–Dirichlet \( H_0 \) Hamiltonians we discussed above differ in this respect. Because 0 is in the continuous spectrum (e.g. see [22]) of the free space \( H_0 \), we know that \( \mathcal{R}(H_0) \) is properly dense in \( L^2(\mathbb{R}^n) \). So the measuring observable \( A \) should also be unbounded in that case. The Poisson–Dirichlet \( H_0 \) maps onto \( L^2(\mathcal{E}) \) and so the measuring observable \( A \) should be everywhere-defined, and hence bounded, in that case.

I will give a more complete accounting of these domain considerations elsewhere. For example, one may obtain another version of Theorem 4.1 by factoring \( ABC \) in the other way. There are versions where we may just obtain \( AHA \) symmetric. However I would like again to emphasize the point: these domain considerations, essential to selfadjointness and hence for application to obtain reversible Zeno evolutions, are delicate and (in my opinion) should be treated with care in measurement theory. In such a way they may provide useful insight into how measurers should measure. Thus one may regard my approach as a new viewpoint towards quantum measurement theory.

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