Rainbow metric from quantum gravity

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In this letter, we describe a general mechanism for emergence of a rainbow metric from a quantum cosmological model. This idea is based on QFT on a quantum spacetime. Under general assumptions, we discover that the quantum spacetime on which the field propagates can be replaced by a classical spacetime, whose metric depends explicitly on the energy of the field: as shown by an analysis of dispersion relations, quanta of different energy propagate on different metrics, similar to photons in a refractive material (hence the name “rainbow” used in the literature). In deriving this result, we do not consider any specific theory of quantum gravity: the qualitative behavior of high-energy particles on quantum spacetime relies only on the assumption that the quantum spacetime is described by a wave-function $\Psi_o$ in a Hilbert space $\mathcal{H}_G$.

It has been argued [1-5] that classical gravity could be a collective phenomenon emerging from quantum degrees of freedom, not unlike fluid dynamics emerges from microscopic molecular interactions. It is often stated that such an effective spacetime should be described by a so-called “rainbow metric” [6-7], i.e., a metric that depends somehow on the energy of the particles propagating on it: it is not difficult to conceive that probing such an effective spacetime with high enough energies leads eventually to corrections due to the underlying fundamental quantum structure.† A fundamental origin for rainbow metrics has been identified (in the principle of relative locality [8]), their phenomenology has been studied [9, 10], and tests (based on the Lorentz-violating nature of such energy-dependent metrics) have been proposed [11, 12].

What was missing until today – as far as we know – is a general mechanism which produces an emergent rainbow metric from a quantum spacetime. Indeed, while various proposals for quantum gravity can all be argued to reproduce classical gravity in the low energy limit, it seems to us that a clear procedure to extract this limit is yet to be formulated.

In this letter, we put forward such a proposal. In Section 1, we describe an effective metric for massive scalar field in quantum gravity in complete generality (we only require gravitational degrees of freedom to be described in terms of a state $\Psi_o$ in a Hilbert space $\mathcal{H}_G$, and to be “heavy” compared to the matter degrees of freedom in the Born-Oppenheimer sense). The idea for this mechanism is based on QFT on quantum spacetime in the form [13]. With no $ad-hoc$ input, we find that the effective metric describing the emergent spacetime is indeed of the rainbow type, as it depends on the wave-vector $k$ of the mode of the matter field. In Section 2, we perform a low-energy expansion of this metric, and show that the first correction to the “classical metric” $\bar{g}_{\mu\nu}$ is of order $\beta p^2/m^2$, where $p$ is the physical momentum of the mode, $m$ is the mass of the field, and $\beta$ is a simple function of $\Psi_o$. It is rather surprising that the only information needed to reconstruct the effective metric from the quantum spacetime is just parameter $\beta$. Finally, in Section 3, we study the modified dispersion relation of this emergent metric, and find that both heavy and light particles behave in a different way than in classical gravity. In particular, the velocity of light remains an upper bound, but is now dependent on $\beta$ and on the momentum $p$ of the particle considered.

I. EFFECTIVE METRIC FOR MASSIVE SCALAR FIELD

For definiteness, consider a scalar field $\phi$ of mass $m$ minimally coupled to gravity. Following the analysis presented in [10, 20], we can separate the homogeneous and inhomogeneous degrees of freedom, and the classical dynamics for mode $k$ of $\phi$ is generated (up to second order) by the Hamiltonian

$$H_k = H_o - \frac{1}{2}H_o^{-1}\left[\pi_k^2 + \left(k^2a^4 + m^2a^6\right)\phi_k^2\right]$$  (1)

Here, $(\phi_k, \pi_k)$ are the phase space variables representing mode $k$, while $(a, \pi_a)$ are the conjugated variables representing the homogeneous degrees of freedom of gravity (that is, the scale factor and its momentum). $H_o$ is the part of the Hamiltonian that accounts for the evolution of these gravitational degrees of freedom at the same order.‡

‡ All the remaining degrees of freedom of gravity do not affect $\phi_k$ at this order, and can thus be disregarded in light of the Born-Oppenheimer test field approximation (see later).
After formal quantization of matter and gravity, defines the following Schrödinger-like equation
\[ -i h \frac{d}{dt} \Psi = \left( \hat{H}_o - \frac{1}{2} \left( \hat{H}_o^{-1} \otimes \hat{\Omega}^2 + \hat{\Omega}(k, m) \otimes \delta \right) \right) \]
where
\[ \hat{\Omega}(k, m) := k^2 H^{-1} a^4 + m^2 H^{-1} a^6 \]
and \( \Psi \in \mathcal{H} = \mathcal{H}_G \otimes L_2(\mathbb{R}, d\phi_k) \), with \( \mathcal{H}_G \) being the Hilbert space of quantum gravity.\(^4\) At this point, we take the test field approximation: we assume that the scalar field does not back-react on the gravitational part. It is therefore allowed to retain only the 0th order in the Born-Oppenheimer expansion of \( \Psi \): during the whole evolution \( \Psi = \Psi_o \otimes \varphi \), where \( \varphi \in L_2(\mathbb{R}, d\phi_k) \) and \( \Psi_o \in \mathcal{H}_G \) evolves via Schrödinger-like equation \(-i d\Psi_o/dt = \hat{H}_o \Psi_o \). This being the case, we can trace away the gravitational part in \( \Psi \) and obtain an equation for the matter part only:
\[ i h \frac{d}{dt} \varphi = \hat{H}_k^{\text{fun}} \varphi \]
where
\[ \hat{H}_k^{\text{fun}} := \frac{1}{2} \left( \langle \Psi_o | \hat{H}_o^{-1} | \Psi_o \rangle \hat{\varphi}_k^2 + \langle \Psi_o | \hat{\Omega}(k, m) | \Psi_o \rangle \delta \hat{\varphi}_k^2 \right) \]
The point first observed in \(^{18}\) and further analysed in \(^{21,24}\), is that equation \( \Psi \) resembles the Schrödinger equation for a quantum field \( \varphi \) on a suitably defined classical spacetime. Let the spacetime be classically described by a metric \( g_{\mu\nu} \) of the Robertson-Walker type:
\[ g_{\mu\nu} dx^\mu dx^\nu = -\bar{N}^2 dt^2 +\bar{a}^2 (dx^2 + dy^2 + dz^2) \]
Construcing regular QFT on such a curved spacetime, one obtains for mode \( k \) of \( \phi \) the following effective Schrödinger equation:
\[ i h \frac{d}{dt} \varphi = \hat{H}_k^{\text{eff}} \varphi \]
where
\[ \hat{H}_k^{\text{eff}} := \frac{1}{2} \left[ \frac{\bar{N}}{\bar{a}} \hat{\varphi}_k^2 + \frac{\bar{N}}{\bar{a}} (k^2 \bar{a}^4 + m^2 \bar{a}^6) \delta \hat{\varphi}_k^2 \right] \]
In other words, we can replace the fundamental theory described by \(^\Phi\) with regular QFT on curved spacetime \(^6\) and \(^8\), provided that the terms in the two Hamiltonians \(^6\) and \(^8\) match. This last requirement gives rise to a system of 2 equations for 2 unknowns:
\[ \frac{\bar{N}}{\bar{a}} = \langle \hat{H}_o^{-1} \rangle, \quad \frac{\bar{N}}{\bar{a}} (k^2 \bar{a}^4 + m^2 \bar{a}^6) = \langle \hat{\Omega}(k, m) \rangle \]
The solution of the system is
\[ \bar{N} = a \langle \hat{H}_o^{-1} \rangle, \quad \bar{a} = \bar{a}(k^2/m^2) \]
where \( \bar{a}(k^2/m^2) \) is the solution to the algebraic equation
\[ \bar{a}^6 + \frac{k^2}{m^2} \bar{a}^4 - \frac{1}{\delta} = 0, \quad \text{with } \delta := \frac{\langle \hat{\Omega}(k, m) \rangle}{m^2 \langle \hat{H}_o^{-1} \rangle} \]
It is a non-trivial fact that this equation has a unique positive solution for every \( k \geq 0 \). It is given explicitly by
\[ \bar{a}^2(k^2/m^2) = \begin{cases} u_+ + u_- - \frac{k^2}{m^2} & \text{if } \frac{4k^6}{27m^6} \leq \delta \\ 2k^2 \frac{3m^2 \cos \theta - k^2}{m^2} & \text{if } \frac{4k^6}{27m^6} > \delta \end{cases} \]
where
\[ u_+ := \sqrt{\frac{\delta}{2} - \frac{k^6}{27m^6}} \pm \sqrt{\frac{\delta^2}{4} - \frac{k^6}{27m^6} \delta} \]
and
\[ \cos \theta := \frac{1}{3} \arccos \left( -1 + \frac{27m^6}{2k^6} \delta \right) \]
The two functions in \(^{12}\) match continuously at \( k = k_o \), where \( k_o \) is the unique positive solution to equation \( \delta = \frac{4k^6}{27m^6} \). For \( k < k_o \) we are in the first case, while for \( k > k_o \) in the second.

II. LOW-ENERGY LIMIT

Let us expand \(^{12}\) for \( k \ll m \). Up to order \( k^4/m^4 \), we have
\[ \bar{a}^2 \left( \frac{k^2}{m^2} \right) \approx \bar{a}_o^2 \left[ 1 + \frac{\beta}{3} \left( \frac{k}{\bar{a}_o} \right)^2 \right] \]
with
\[ \bar{a}_o^2 = \sqrt{\frac{\langle \hat{H}_o^{-1} \rangle}{\langle \hat{H}_o^{-1} \rangle} \langle \hat{H}_o^{-1} \rangle}, \quad \beta := \frac{\langle \hat{H}_o^{-1} \rangle}{\langle \hat{H}_o^{-1} \rangle} - 1 \]
From \( \bar{a} \) and \( \bar{a}_o \), we find \( \bar{N} \) and \( \bar{N}_o \) via the first equation in \(^{10}\). We can then identify two effective FLRW metrics: the low-energy one (experienced by a cosmological observer)
\[ \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\bar{N}_o^2 dt^2 + \bar{a}_o^2 (dx^2 + dy^2 + dz^2) \]
and the \( k \)-dependent one (seen by the particles)
\[ \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\bar{N}_o^2 dt^2 + \bar{a}^2 (dx^2 + dy^2 + dz^2) \]
Suppose that an observer with 4-velocity $u^\mu$ detects a particle with 4-momentum $k_\mu$. The energy and (norm of) momentum measured by the observer are

$$E = u^\mu k_\mu = \frac{k_\mu}{N_0}, \quad p^2 = (\tilde{g}^{\mu\nu} + u^\mu u^\nu)k_\mu k_\nu = \frac{k^2}{a^2}$$  \hspace{1cm} (19)$$

where we used the fact that $\tilde{g}^{\mu\nu} u^\mu u^\nu = -1$ to discover that $u^\mu = (1/N_0, 0, 0, 0)$. On the other hand, the particle satisfies the mass-shell relation in its metric [18]:

$$-m^2 = \tilde{g}^{\mu\nu}k_\mu k_\nu = -\frac{k_\mu^2}{N^2} + \frac{k^2}{a^2} = -f^2 E^2 + g^2 p^2$$  \hspace{1cm} (20)$$

having introduced the so-called “rainbow functions” [7]

$$f := \frac{N_0}{N}, \quad g := \frac{a}{\tilde{a}}$$  \hspace{1cm} (21)$$

From [15], it is immediate to compute $f$ and $g$, which explicitly depend on the physical momentum $p = k/\tilde{a}_0$:

$$f^2 = \left(1 + \frac{\beta \, p^2}{3 \, m^2}\right)^{-3}, \quad g^2 = \left(1 + \frac{\beta \, p^2}{3 \, m^2}\right)^{-1}$$  \hspace{1cm} (22)$$

We thus obtain a modified dispersion relation from [20],

$$E^2 = \frac{1}{f^2} \left(g^2 p^2 + m^2\right) \approx m^2 + (1 + \beta)p^2 + O(p^4)$$  \hspace{1cm} (23)$$

As expected, the standard dispersion relation $E = m$ is recovered in the limit $p \ll m$. The first correction in the case $p \approx m$ is precisely $\beta$, a quantity of exquisitely quantum gravitational origin. Note that – contrary to the general belief – no particular role is played by Planck energy, $E_{Pl} \approx 10^{28}$ eV. In fact, for a highly quantum spacetime we have $\beta \approx 1$, and hence the particles probe the quantum structure of spacetime already at $p \approx m$. For a proton, this would correspond to mild energies of order $10^9$ eV. On the other hand, it is clear that $\beta \approx 0$ for semiclassical states, and hence quantum gravity corrections are irrelevant for low-energy particles. We should mention that a similar result was recovered in the semiclassical limit in [25].

### III. ANALYSIS AND DISCUSSION

Having the dispersion relation, it is possible to compute the velocity of the mode:

$$v = \frac{dE}{dp} = \frac{1 + \beta}{\sqrt{m^2 + (1 + \beta)p^2} \, p}$$  \hspace{1cm} (24)$$

For $m \to 0$ we do not recover 1, but rather $\sqrt{1 + \beta}$. At this order of approximation, the speed of light does not depend on $p$. However it does depend on the quantum state $\Psi_o$ of the spacetime, and hence on time. Moreover, repeating the computations of the previous section for the exact solution [17], one finds an explicit dependence of $E$ and $v$ on $p/m$. We represent these two functions in figures 1 and 2, for the choices

$$\frac{\langle H^{-1} \hat{a}^6 \rangle}{\langle \hat{H}^{-1} \rangle} = 0.9(\mathring{a})^3, \quad \frac{\langle H^{-1} \hat{a}^4 \rangle}{\langle \hat{H}^{-1} \rangle} = 1.1(\mathring{a})^{4/3}$$  \hspace{1cm} (25)$$

which give roughly $\beta \approx 0.2$ (a highly non-classical situation). Notice that the speed of massive particles approaches but never exceeds the (deformed) speed of light. For $p \ll m$, the classical speed of light ($v = 1$) is recovered.

In conclusion, our analysis shows that the quantum nature of spacetime unavoidably affects the propagation of test particles, producing (apparent) Lorentz-violating effects. This has been shown by constructing an effective metric from very general assumptions (in particular, we did not need to restrict to a specific quantum theory of gravity). Intuitively, the generality of the result can be understood by observing that – independently of the chosen quantum gravity theory – the quantum state $\Psi_o$ of the gravitational field is not an eigenstate of the “metric operator” in general. Hence, it is to be expected that

![FIG. 1: Dispersion relation $E = E(p)$ for a scalar field of mass $m$. Red = semiclassical spacetime ($\beta \approx 0$); Blue = quantum spacetime ($\beta \approx 0.2$).](image1)

![FIG. 2: Velocity $v = v(p)$ of different modes of the massive field. Red = semiclassical spacetime ($\beta \approx 0$); Blue = quantum spacetime ($\beta \approx 0.2$). The dashed lines represent the speed of light in the semiclassical spacetime (black) and in the quantum spacetime (green) respectively.](image2)
matter particles of different momenta \( p \) will couple differently to the quantum geometry, probing different aspects of it. In the present work, we have used a massive scalar field \( \phi \) – one of the simplest forms of matter – but there is no reason not to expect the same qualitative behavior for other species.

As for the intensity of the effect, we have shown that the only parameter governing the corrections is \( \beta \), a single function of \( \Psi_\alpha \). This is rather striking, considering that infinitely many quantum states can be found that give the same value \( \beta \). On the other hand, we should not be too surprised, since the same happens for photons propagating in refractive media: the scattering of light through a crystal is perfectly well described in terms of the refractive index \( n \), a single parameter in spite of the infinitely many possible microscopic configurations of the atoms.

While states \( \Psi_\alpha \) can be found for which \( \beta \) is not small, they are highly non-classical: indeed, for any reasonable coherent state, the expectation values of any two operators \( \hat{A} \) and \( \hat{B} \) satisfy \( \langle \hat{A}\hat{B} \rangle \approx \langle \hat{A} \rangle \langle \hat{B} \rangle \) up to small corrections. We thus conclude that the correction to classical gravity due to the underlying quantum structure of spacetime is completely irrelevant in experiments carried out today. However, this statement must be checked explicitly for each quantum gravity theory and each state \( \Psi_\alpha \). In fact, the requirement \( \beta \ll 1 \) might well be used as a test for classicality of states. On the other hand, many quantum gravity and quantum cosmology theories maintain that in the early stages of its life, the Universe should be described by a non-classical state. In this case \( \beta \approx 1 \), and the modified dispersion relations have to be taken into account when studying the behavior of primordial matter. We can conceive that such effects might have left a trace in the CMB, or even speculate that they might have contributed to the inflation and subsequent formation of structures.

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