EXPONENTIAL EQUATIONS IN ACYLINDRICALLY HYPERBOLIC GROUPS

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Abstract. Let $G$ be an acylindrically hyperbolic group and $E$ an exponential equation over $G$. We show that if $E$ is solvable in $G$, then there exists a solution whose components, corresponding to loxodromic elements, can be linearly estimated in terms of lengths of the coefficients of $E$. We give a more precise answer in the case where $G$ is a relatively hyperbolic group. Under some assumption of general character, the solvability and the search problems for exponential equations over $G$ can be reduced to its peripheral subgroups.

1. Introduction

In 2015, Myasnikov, Nikolaev and Ushakov initiated the study of exponential equations in groups [19] which has become a topic of intensive investigations on the edge of group theory and complexity theory [4, 9–18]. The results obtained in [19] for hyperbolic groups motivated us to investigate the decidability of exponential equations in the wider classes of relatively hyperbolic and acylindrically hyperbolic groups.

Definition 1.1. An exponential equation over a group $G$ is an equation of the form

$$a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1,$$

(1.1)

where $a_1, g_1, \ldots, a_n, g_n$ are elements from $G$ and $x_1, \ldots, x_n$ are variables (which take values in $\mathbb{Z}$). A tuple $(k_1, \ldots, k_n)$ of integers is called a solution of this equation if $a_1 g_1^{k_1} a_2 g_2^{k_2} \ldots a_n g_n^{k_n} = 1$ in $G$.

The first main theorem of this paper, Theorem A, is formulated and proved in Section 7. Here we give a simplified version of this theorem, Theorem A’. It says that if $G$ is an acylindrically hyperbolic group and the above equation is solvable, then there exists a solution $(k_1, \ldots, k_n)$ such that $|k_j|$ corresponding to loxodromic $g_j$ can be linearly bounded in terms of the lengths of the coefficients of this equation.

Theorem A’. (see Theorem A) Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Then there exists a constant $M > 1$ such that for any exponential equation

$$a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1$$

2010 Mathematics Subject Classification. Primary 20F65, 20F70; Secondary 20F67.

Key words and phrases. exponential equations, acylindrically hyperbolic groups, relatively hyperbolic groups, knapsack problem, decidability problems.
with constants $a_1, g_1, \ldots, a_n, g_n$ from $G$ and variables $x_1, \ldots, x_n$, if this equation is solvable over $\mathbb{Z}$, then there exists a solution $(k_1, \ldots, k_n)$ with

$$|k_j| \leq \left( n^2 + \sum_{i=1}^{n} |a_i|_X + \sum_{i=1}^{n} |g_i|_X \right) \cdot M$$

for all $j$ corresponding to loxodromic $g_j$.

If, additionally, $G$ is generated by a finite subset $Y$, then the above estimation remains valid if we replace there $X$ by $Y$ and $M$ by $M_{\sup \{ |y|_X \mid y \in Y \}}$.

**Remark 1.2.** The main result of the paper [19] of Myasnikov, Nikolaev and Ushakov says that if $G$ is a hyperbolic group with a finite generating set $X$, then there exists a polynomial $p_n(x)$ such that for any exponential equation of the form (1.1), if this equation is solvable then there exists a solution $(k_1, \ldots, k_n)$ with

$$|k_j| \leq p_n \left( \sum_{i=1}^{n} |a_i|_X + \sum_{i=1}^{n} |g_i|_X \right)$$

for $j = 1, \ldots, n$.

We consider the more general case where $G$ is an acylindrically hyperbolic group. This case is more difficult since $G$ is not necessarily finitely generated in general. Moreover, even if $G$ is finitely generated, it can happen that $G$ does not act acylindrically on a locally finite graph.

Theorem $A'$ restricted to the case where $G$ is hyperbolic and $X$ is finite implies that the above polynomial $p_n(x)$ can be taken to be linear. Indeed, all non-loxodromic elements of $G$ have finite orders in this case, and these orders can be bounded from above by a universal constant depending only on $|X|$ and $\delta$, where $\delta$ is the hyperbolicity constant of $G$ with respect to $X$ (see [3] or [6]). Theorem A in Section 7 gives further improvements.

**Remark 1.3.** The following example shows that the word *loxodromic* in the formulation of Theorem $A'$ cannot be omitted even in the case of finitely presented relatively hyperbolic groups.

**Example.** Let $H$ be a finitely presented group containing the group of rational numbers $\mathbb{Q}$. Such group can be constructed using Higman’s embedding theorem. Then the free product $G = H \ast F_2$ is finitely presented and relatively hyperbolic with respect to the subgroups $H$ and $F_2$, and the elements of $H$ and $F_2$ and their conjugates are elliptic with respect to the generating set $X = H \cup F_2$. We consider the rational numbers $a = -1$ and $b_i = \frac{1}{i}$ for $i \geq 1$ as elements of $G$. For each $i \in \mathbb{N}$ the exponential equation $ab_i^x = 1$ has a unique solution (namely $i$), and the sum of lengths of its coefficients is $|a|_X + |b_i|_X = 2$. Thus, there does not exist a function $f$ such that, for all $i$, the solution of $ab_i^x = 1$ is bounded from above by $f(|a|_X + |b_i|_X)$.

Theorem B (see Subsection 8.2) deals with certain exponential equations in groups with hyperbolically embedded subgroups; we use it to deduce Theorem C.
Theorem C, comparing with Theorem A, gives more information in the case where $G$ is a finitely generated relatively hyperbolic group. It says that for any exponential equation $E$ over $G$, there exists a finite disjunction $\Phi$ of finite systems of exponential equations over peripheral subgroups of $G$ such that $E$ is solvable if and only if $\Phi$ is solvable. If some additional data are known, one can find such $\Phi$ algorithmically. Moreover, having a solution of $\Phi$, one can find a solution of $E$.

**Theorem C.** Let $G$ be a group relatively hyperbolic with respect to a finite collection of subgroups $\{H_1, \ldots, H_m\}$. Suppose that $G$ is finitely generated, each subgroup $H_i$ is given by a recursive presentation and has solvable word problem, $G$ is given by a finite relative presentation $P = \langle X | R \rangle$ with respect to $\{H_1, \ldots, H_m\}$, where $X$ is a finite set generating $G$, and that the hyperbolicity constant $\delta$ of the Cayley graph $\Gamma(G, X \cup H)$ is known, $H = \bigsqcup_{i=1}^m H_i$.

Then there exists an algorithm which for any exponential equation $E$ over $G$ finds a finite disjunction $\Phi$ of finite systems of equations,

$$\Phi := \bigvee_{i=1}^k \bigwedge_{j=1}^{\ell_i} E_{ij},$$

such that

1. each $E_{ij}$ is an exponential equation over $H_\lambda$ for some $\lambda \in \{1, \ldots, m\}$ or a trivial equation of kind $g_{ij} = 1$, where $g_{ij}$ is an element of $G$,
2. for any $i = 1, \ldots, k$, the sets of variables of $E_{ij_1}$ and $E_{ij_2}$ are disjoint if $j_1 \neq j_2$,
3. $E$ is solvable if and only if $\Phi$ is solvable.

Moreover, any solution of $\Phi$ can be algorithmically extended to a solution of $E$.

In the proof of Theorem A, which is a stronger version of Theorem A', we use the following theorem about conjugator lengths in acylindrically hyperbolic groups. This theorem seems to be interesting for its own sake.

**Theorem 1.4.** Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Let $\delta$ be the hyperbolicity constant of the Cayley graph $\Gamma(G, X)$ and let $N$ be the function from Definition 2.7. Then there exists a universal constant $C$ such that for any two conjugate elements $h_1, h_2 \in G$ of (possibly infinite) order larger than $N(8\delta + 1)$, there exists $g \in G$ such that $h_2 = gh_1g^{-1}$ and $|g|_X \leq C(|h_1|_X + |h_2|_X)$.

**Remark 1.5.** In [19], the problem about decidability of equations (1.1) in integer numbers is called the Integer Knapsack Problem (IKP) for the group $G$. If we are looking for nonnegative integer solutions, the problem is called the Knapsack Problem (KP). Clearly, the decidability of (IKP) for $G$ implies the decidability of (KP) for $G$. To our best knowledge the answer to the following problem is unknown.

**Problem.** Does there exist a finitely presented group $G$ for which the Integer Knapsack Problem is decidable and the Knapsack Problem is undecidable?
2. Definitions and preliminary statements

We introduce general notation and recall some relevant definitions and statements from the papers [2, 8, 21]. In this paper, all actions of groups on metric spaces are assumed to be isometric.

2.1. General notation. All generating sets considered in this paper are assumed to be symmetric, i.e., closed under taking inverse elements. Let \( G \) be a group generated by a subset \( X \). For \( g \in G \) let \( |g|_X \) be the length of a shortest word in \( X \) representing \( g \). The corresponding metric on \( G \) is denoted by \( d_X \) (or by \( d \) if \( X \) is clear from the context); thus \( d_X(a, b) = |a^{-1}b|_X \). The right Cayley graph of \( G \) with respect to \( X \) is denoted by \( \Gamma(G, X) \). By a path \( p \) in the Cayley graph we mean a combinatorial path; the initial and the terminal vertices of \( p \) are denoted by \( p^- \) and \( p^+ \), respectively. The length of \( p \) is denoted by \( \ell(p) \). The label of \( p \) (which is a word in the alphabet \( X \)) is denoted by \( \text{Lab}(p) \).

Recall that a path \( p \) in \( \Gamma(G, X) \) is called \((\kappa, \varepsilon)\)-quasi-geodesic, where \( \kappa \geq 1 \), \( \varepsilon \geq 0 \), if \( d_X(p^-, p^+) \geq \kappa \ell(q) - \varepsilon \) for any subpath \( q \) of \( p \).

2.2. Hyperbolic spaces. A geodesic metric space \( X \) is called \( \delta \)-hyperbolic if each side of any geodesic triangle \( \Delta \) in \( X \) lies in the \( \delta \)-neighborhood of the union of the other two sides of \( \Delta \). We will use the following standard facts about hyperbolic spaces.

**Lemma 2.1.** Let \( X \) be a \( \delta \)-hyperbolic space. Suppose that \( R \) is a geodesic \( n \)-gon in \( X \). Then any side of \( R \) is at distance at most \( (n-2)\delta \) from the union of the other sides of \( R \).

**Lemma 2.2.** (see [6, Chapter III.H, Theorem 1.7]) For all \( \delta \geq 0 \), \( \kappa \geq 1 \), \( \varepsilon \geq 0 \), there exists a constant \( \mu = \mu(\delta, \kappa, \varepsilon) > 0 \) with the following property:

If \( X \) is a \( \delta \)-hyperbolic space, \( p \) is a \((\kappa, \varepsilon)\)-quasi-geodesic in \( X \), and \([x, y]\) is a geodesic segment joining the endpoints of \( p \), then the Hausdorff distance between \([x, y]\) and the image of \( p \) is at most \( \mu \).

The following corollary is a slight generalization of the previous one.

**Corollary 2.3.** Let \( X \) be a \( \delta \)-hyperbolic space, let \( p \) and \( q \) be \((\kappa, \varepsilon)\)-quasi-geodesics in \( X \) with \( \max\{d(p^-, q^-), d(p^+, q^+)\} \leq r \). Then the Hausdorff distance between the images of \( p \) and \( q \) is at most \( r + 2\delta + 2\mu \), where \( \mu = \mu(\delta, \kappa, \varepsilon) \) is the constant from Lemma 2.2.

**Lemma 2.4.** (see [7, Chapitre 3, Théorème 3.1]) For all \( \delta \geq 0 \), \( \kappa \geq 1 \), \( \varepsilon \geq 0 \), there exists a constant \( \mu = \mu(\delta, \kappa, \varepsilon) > 0 \) with the following property:

If \( X \) is a \( \delta \)-hyperbolic space and \( p \) and \( q \) are infinite \((\kappa, \varepsilon)\)-quasi-geodesics in \( X \) with the same limit points on the Gromov boundary \( \partial X \), then the Hausdorff distance between \( p \) and \( q \) is at most \( \mu(\delta, \kappa, \varepsilon) \).

The following lemma enables to estimate a displacement of a point on a segment of a hyperbolic space (under the action of an isometry) via displacements of the endpoints of this segment.
Lemma 2.5. Let $G$ be a group acting on a $\delta$-hyperbolic space $X$. Let $g \in G$ be an element and $[A, B]$ a geodesic in $X$. Suppose that $C$ is a point on $[A, B]$ such that $d(A, C) > d(A, gA) + 2\delta$ and $d(C, B) > d(B, gB) + 2\delta$. Then
\[
d(C, gC) \leq 4\delta + \min\{d(A, gA), d(B, gB)\}.
\]

Proof. By assumptions the distance from $C$ to $[A, gA] \cup [B, gB]$ is larger than $2\delta$. By Lemma 2.1, there exists a point $D \in [gA, gB]$ such that $d(C, D) \leq 2\delta$. Then
\[
d(C, gC) \leq d(C, D) + d(D, gC) = d(C, D) + |d(D, gA) - d(gC, gA)|
= d(C, D) + |d(D, gA) - d(C, A)|
\leq d(C, D) + d(C, D) + d(A, gA)
\leq 4\delta + d(A, gA).
\]
Analogously, we obtain $d(C, gC) \leq 4\delta + d(B, gB)$. \hfill $\Box$

Without loss of generality, we may assume that $\delta$ is integer.

The following lemma will be used in the proof of the elliptic case of Theorem 1.4.

Lemma 2.6. (see [2, Lemma 4.8]) For every $\delta \geq 0$, there exists $\epsilon_1 = \epsilon_1(\delta) \geq 0$ such that the following holds. Suppose that the Cayley graph of a group $G$ with respect to a generating set $X$ is $\delta$-hyperbolic for some integer $\delta \geq 0$. Let $a, b \in G$ be conjugate elements satisfying $|a|_X \geq |b|_X + 4\delta + 2$. Then there exist $x, y \in G$ with the following properties:
\begin{enumerate}
  \item $a = x^{-1}yx$;
  \item $|y|_X \in \{|b|_X + 4\delta + 1, |b|_X + 4\delta + 2\}$;
  \item any path $q_0q_1q_2$ in $\Gamma(G, X)$, where $q_0, q_1, q_2$ are geodesics with labels representing $x^{-1}y, x$, is a $(1, \epsilon_1)$-quasi-geodesic.
\end{enumerate}

2.3. Two equivalent definitions of acylindrically hyperbolic groups.

Definition 2.7. (see [5] and Introduction in [21]) An action of a group $G$ on a metric space $S$ is called acylindrical if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points $x, y$ with $d(x, y) \geq R$, there are at most $N$ elements $g \in G$ satisfying
\[
d(x, gx) \leq \epsilon \quad \text{and} \quad d(y, gy) \leq \epsilon.
\]

Given a generating set $X$ of a group $G$, we say that the Cayley graph $\Gamma(G, X)$ is acylindrical if the left action of $G$ on $\Gamma(G, X)$ is acylindrical. For Cayley graphs, the acylindricity condition can be rewritten as follows: for every $\epsilon > 0$ there exist $R, N > 0$ such that for any $g \in G$ of length $|g|_X \geq R$ we have
\[
|\{f \in G \mid f|_X \leq \epsilon, \ g^{-1}fg|_X \leq \epsilon\}| \leq N.
\]

Recall that an action of a group $G$ on a hyperbolic space $S$ is called elementary if the limit set of $G$ on the Gromov boundary $\partial S$ contains at most 2 points.
Definition 2.8. (see [21, Definition 1.3]) A group $G$ is called *acylindrically hyperbolic* if it satisfies one of the following equivalent conditions:

(AH$_1$) There exists a generating set $X$ of $G$ such that the corresponding Cayley graph $\Gamma(G, X)$ is hyperbolic, $|\partial \Gamma(G, X)| > 2$, and the natural action of $G$ on $\Gamma(G, X)$ is acylindrical.

(AH$_2$) $G$ admits a non-elementary acylindrical action on a hyperbolic space.

In the case (AH$_1$), we also write that $G$ is *acylindrically hyperbolic with respect to* $X$.

2.4. Elliptic and loxodromic elements in acylindrically hyperbolic groups.

The following definition is standard.

Definition 2.9. Given a group $G$ acting on a metric space $S$, an element $g \in G$ is called *elliptic* if some (equivalently, any) orbit of $g$ is bounded, and *loxodromic* if the map $\mathbb{Z} \to S$ defined by $n \mapsto g^n x$ is a quasi-isometric embedding for some (equivalently, any) $x \in S$. That is, for $x \in S$, there exist $\kappa \geq 1$ and $\varepsilon \geq 0$ such that for any $n,m \in \mathbb{Z}$ we have

$$d(g^n x, g^m x) \geq \frac{1}{\kappa} |n - m| - \varepsilon.$$

Let $X$ be a generating set of $G$. We say that $g \in G$ is *elliptic (respectively loxodromic) with respect to* $X$ if $g$ is elliptic (respectively loxodromic) for the canonical left action of $G$ on the Cayley graph $\Gamma(G, X)$. If $X$ is clear from a context, we omit the words “with respect to $X$”.

The set of all elliptic (respectively loxodromic) elements of $G$ with respect to $X$ is denoted by $\text{Ell}(G, X)$ (respectively by $\text{Lox}(G, X)$).

Note that for groups acting on geodesic hyperbolic spaces, there is only one additional isometry type of an element- parabolic (see e.g. [7, Chapitre 9, Théorème 2.1]).

Bowditch [5, Lemma 2.2] proved that every element of a group acting acylindrically on a hyperbolic space is either elliptic or loxodromic (see a more general statement in [21, Theorem 1.1]).

Recall that any loxodromic element $g$ in an acylindrically hyperbolic group $G$ is contained in a unique maximal virtually cyclic subgroup [8, Lemma 6.5]. This subgroup, denoted by $E_G(g)$, is called the *elementary subgroup associated with* $g$; it can be described as follows (see equivalent definitions in [8, Corollary 6.6]):

$$E_G(g) = \{ f \in G \mid \exists n \in \mathbb{N} : f^{-1} g^n f = g^{\pm n} \} = \{ f \in G \mid \exists k, m \in \mathbb{Z} \setminus \{0\} : f^{-1} g^k f = g^m \}.$$

Lemma 2.10. (see [21, Lemma 6.8]) Suppose that $G$ is a group acting acylindrically on a hyperbolic space $S$. Then there exists $L \in \mathbb{N}$ such that for every loxodromic element $g \in G$, $E_G(g)$ contains a normal infinite cyclic subgroup of index $L$.

Definition 2.11. Let $G$ be a group and $X$ be a generating set of $G$. For any two elements $u, v \in G$, we choose a geodesic path $[u, v]$ in $\Gamma(G, X)$ from $u$ to $v$ so that
\( w[u,v] = [wu,wv] \) for any \( w \in G \). With any element \( x \in G \) and any loxodromic element \( g \in G \), we associate the bi-infinite quasi-geodesic
\[
L(x,g) = \bigcup_{i=-\infty}^{\infty} x[g^i, g^{i+1}].
\]
We have \( L(x,g) = xL(1,g) \). The path \( L(1,g) \) is called the quasi-geodesic associated with \( g \).

**Corollary 2.12.** (\cite[Corollary 2.12]{[1]}) Let \( G \) be a group and \( X \) be a generating set of \( G \). Suppose that the Cayley graph \( \Gamma(G,X) \) is hyperbolic and acylindrical. Then there exist \( \kappa \geq 1 \) and \( \varepsilon_0 \geq 0 \) such that the following holds:

If an element \( g \in G \) is loxodromic and shortest in its conjugacy class, then the quasi-geodesic \( L(1,g) \) associated with \( g \) is a \((\kappa,\varepsilon)\)-quasi-geodesic.

We will use the following technical lemmas from \[2\].

**Lemma 2.13.** (see \cite[Lemma 4.7]{[2]}) Let \( G \) be a group and \( X \) be a generating set of \( G \). Suppose that the Cayley graph \( \Gamma(G,X) \) is hyperbolic and acylindrical. Then there exist real numbers \( \kappa \geq 1, \varepsilon_0 \geq 0 \) and a number \( n_0 \in \mathbb{N} \) with the following property.

Suppose that \( n \geq n_0 \) and \( c \in G \) is a loxodromic element. Let \( S(c) \) be the set of shortest elements in the conjugacy class of \( c \) and let \( g \in G \) be a shortest element for which there exists \( c_1 \in S(c) \) with \( c = g^{-1}c_1g \). Then any path \( p_0p_1 \ldots p_np_{n+1} \in \Gamma(G,X) \), where \( p_0, p_1, \ldots, p_n, p_{n+1} \) are geodesics with labels representing \( g^{-1}, c_1, \ldots, c_1, g \), is a \((\kappa,\varepsilon_0)\)-quasi-geodesic. In particular,
\[
|c^n|_X \geq \frac{1}{\kappa}(n|c_1|_X + 2|g|_X) - \varepsilon_0.
\]

2.5. **Stable norm.** Let \( G \) be a group and \( X \) is a generating set of \( G \). Recall that the stable norm of an element \( g \in G \) with respect to a generating set \( X \) is defined as
\[
||g||_X = \lim_{n \to \infty} \frac{|g^n|_X}{n},
\]
see \[7\]. It is easy to check that this number is well-defined, that it is a conjugacy invariant, and that \( ||g^k||_X = ||k|||g||_X \) for all \( k \in \mathbb{Z} \).

Bowditch \cite[Lemma 2.2]{[5]} proved that every element of a group acting acylindrically on a hyperbolic space is either elliptic or loxodromic (see a more general statement in \cite[Theorem 1.1]{[21]}). Moreover, he proved there that the infimum of the set of stable norms of all loxodromic elements for such an action is larger than zero (we assume that \( \inf \emptyset = +\infty \)).

**Lemma 2.14.** Let \( G \) be a group and \( X \) be a generating set of \( G \). Suppose that the Cayley graph \( \Gamma(G,X) \) is hyperbolic and acylindrical. For any loxodromic element \( a \in G \), which is shortest in its conjugacy class, we have
\[
||a||_X \geq \frac{|a|_X}{\kappa}, \tag{2.1}
\]
where \( \kappa \geq 1 \) is the universal constant from Corollary 2.12.
Proof. By Corollary 2.12 there exist universal constants $\kappa \geq 1$ and $\varepsilon \geq 0$ such that the path $L(1, a)$ is a $(\kappa, \varepsilon)$-quasi-geodesic. Then, for any natural $n$, we have

$$|a^n|_X \geq \frac{\ell(a^n) - \varepsilon}{\kappa} = \frac{n|a|_X - \varepsilon}{\kappa}.$$ 

Therefore

$$||a||_X = \lim_{n \to \infty} \frac{|a^n|_X}{n} \geq \frac{|a|_X}{\kappa}.$$

3. Proof of Theorem 1.4

Theorem 1.4 will be deduced from the following two lemmas, which say (simplified) that an element $h \in G$ can be conjugate to a shortest representative by an element $g$, whose length is bounded by a linear function of the length of $h$. The first lemma (about loxodromic $h$) follows directly from Lemma 2.13, while the second one (about elliptic $h$) seems to be not evident and needs an extended proof.

Lemma 3.1. Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Then for any $h \in \text{Lox}(G, X)$, there exists $g \in G$ such that $ghg^{-1}$ is a shortest element in the conjugacy class of $h$ and $|g|_X \leq K|h|_X$, where $K > 0$ is a universal constant depending on the acylindricity data of the pair $(G, X)$.

Proof. By Lemma 2.13 there exists universal constants $n_0$, $\kappa$ and $\varepsilon_0$ such that $|g|_X \leq \frac{1}{2}\kappa(|h|_X + \varepsilon_0)$. Then the statement holds for $K = \frac{1}{2}\kappa n_0 + \varepsilon_0$. \qed

Lemma 3.2. Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Then for any $h \in \text{Ell}(G, X)$, there exists $g \in G$ such that $g(h)g^{-1} \subseteq B_1(8\delta + 1)$ and $|g|_X \leq K|h|_X$, where $K > 0$ is a universal constant depending on the acylindricity data of the pair $(G, X)$.

Proof. It is known that there exists $g \in G$ such that $g(h)g^{-1} \subseteq B_1(4\delta + 1)$, see [21] Corollary 6.7] (the proof there utilizes the proof of [6] Part III $\Gamma$, Theorem 3.2]). We start with some $g$ satisfying this property and modify it to get a (possibly) other $g$ with the desired length. For any integer $i$ we denote $h_i = gh^ig^{-1}$. Clearly $h_i = h^i$ and

$$|h_i|_X \leq 4\delta + 1. \quad (3.1)$$

We choose a geodesic path $[A, B]$ in $\Gamma(G, X)$ from $A = 1$ to $B = g$. For any $i \in \mathbb{Z}$ we consider the geodesic path $[A_i, B_i] = h_i[A, B]$ and choose geodesic paths $[A, A_i]$
and \([B, B_i]\). Note that the paths \([A, B]\) and \([A_i, B_i]\) are both labeled by \(g\) and the paths \([A, A_i]\) and \([B, B_i]\) are labeled by \(h_i\) and \(h_i\), respectively, see Fig. 1.

Fig. 1. Illustration to the proof of the main theorem.

If \(|g|_X < R(8\delta + 3)\), we are done with \(K = R(8\delta + 3)\). Therefore we assume that
\[
d(A, B) = |g|_X \geq R(8\delta + 3).
\]
Let
\[
I = \{i \in \mathbb{Z} | |h^i|_X \leq 8\delta + 3\}. \quad (3.3)
\]

Claim 1. We have \(#\{h^i | i \in I\} \leq N(8\delta + 3)\).

Proof. For any \(i \in \mathbb{Z}\), we have
\[
d(A, h_i A) = |h_i|_X \overset{(3.1)}{\leq} 4\delta + 1
\]
and for any \(i \in I\) we have
\[
d(B, h_i B) = d(g, h_i g) = |g^{-1} h_i g|_X = |h^i|_X \overset{(3.3)}{\leq} 8\delta + 3.
\]
From this, (3.2) and the definition of the acylindrical action we obtain the statement. \(\square\)

Now consider \(i \in I^c\), where \(I^c = \mathbb{Z} \setminus I\). By Lemma 2.6 applied to \(a = h_i\) and \(b = h_i\), there exist \(x_i, y_i \in G\) such that
\[
h^i = x_i^{-1} y_i x_i,
\]
\[
|y_i|_X \leq 8\delta + 3, \quad (3.4)
\]
and any path \(\ell_i = p_i q_i r_i\) in \(\Gamma(G, X)\), where \(p_i, q_i, r_i\) are geodesics with labels \(x_i^{-1}, y_i, x_i\) is a \((1, \epsilon_1)\)-quasi-geodesic. Here \(\epsilon_1 = \epsilon_1(\delta)\) is a universal constant. In particular, we have
\[
2|x_i|_X + |y_i|_X \leq |h^i|_X + \epsilon_1, \quad (3.5)
\]
Since the labels of $\ell_i$ and $[B, B_i]$ are both equal to $h^i$, we can choose $\ell_i$ so that $(\ell_i)_- = B$ and $(\ell_i)_+ = B_i$, see Fig. 1. Observe that

$$h_i \cdot (q_i)_- = gh^i g^{-1} \cdot gx_i^{-1} = (q_i)_+. \quad (3.6)$$

Since the label of $q_i$ is $y_i$, we deduce from (3.4) that

$$d((q_i)_-, (q_i)_+) \leq 8\delta + 3. \quad (3.7)$$

**Claim 2.** There exists a constant $\epsilon_2 > 0$ depending only on $\delta$ such that the following holds. For any $i \in I$, there exists a point $u_i \in [A, B]$ such that

$$d((q_i)_-, u_i) \leq \epsilon_2. \quad (3.8)$$

and

$$d(u_i, h_i u_i) \leq 2\epsilon_2 + 8\delta + 3. \quad (3.9)$$

**Proof.** We set $\mu_1 = \mu(\delta, 1, \epsilon_1)$, where the function $\mu$ is defined in Lemma 2.2 and prove that the statement is valid for $\epsilon_2 = \mu_1 + 10\delta + 3$.

We prove the first statement. Recall that $\ell_i = p_i q_i x_i$ is a $(1, \epsilon_1)$-quasi-geodesic with endpoints $B, B_i$. Then, by Lemma 2.2 there exists a point $w_i \in [B, B_i]$ such that $d(w_i, (q_i)_-) \leq \mu_1$. By Lemma 2.1, $w_i$ is at distance at most $2\delta$ from the union of three sides $[B, A], [A, A_i], [A_i, B_i]$.

**Case 1.** Suppose that there exists $z_i \in [B, A]$ such that $d(w_i, z_i) \leq 2\delta$. Then

$$d((q_i)_-, z_i) \leq d((q_i)_-, w_i) + d(w_i, z_i) \leq \mu_1 + 2\delta < \epsilon_2,$$

and we are done with $u_i = z_i$.

**Case 2.** Suppose that there exists $z_i \in [A, A_i]$ such that $d(w_i, z_i) \leq 2\delta$. Then

$$d((q_i)_-, A) \leq d((q_i)_-, w_i) + d(w_i, z_i) + d(z_i, A) \leq \mu_1 + 2\delta + (4\delta + 1) < \epsilon_2,$$

and we are done with $u_i = A$.

**Case 3.** Suppose that there exists $z_i \in [A_i, B_i]$ such that $d(w_i, z_i) \leq 2\delta$. We set $u_i = h_i^{-1} z_i$. Then $u_i \in [A, B]$ and we have

$$d((q_i)_-, u_i) \overset{(3.6)}{=} d(h_i^{-1}(q_i)_+, h_i^{-1} z_i)$$

$$= d((q_i)_+, z_i)$$

$$\leq d((q_i)_+, (q_i)_- + d((q_i)_-, w_i) + d(w_i, z_i)$$

$$\overset{(3.7)}{\leq} (8\delta + 3) + \mu_1 + 2\delta = \epsilon_2.$$

This completes the proof of the first statement. Now we prove the second statement:

$$d(u_i, h_i u_i) \leq d(u_i, (q_i)_-) + d((q_i)_-, (q_i)_+) + d((q_i)_+, h_i u_i)$$

$$\overset{(3.5)}{=} d(u_i, (q_i)_-) + d((q_i)_-, (q_i)_+) + d(h_i (q_i)_-, h_i u_i)$$

$$\overset{(3.6)}{\leq} \epsilon_2 + (8\delta + 3) + \epsilon_2.$$
Now we define the set

\[ J = \{ i \in I^c | d(A, u_i) > R(8\delta + 1) + 2\epsilon_2 + 16\delta + 5 \}. \]

**Claim 3.** We have \( \#\{ h^i | i \in J \} \leq N(8\delta + 1) \).

**Proof.** We assume that \( J \neq \emptyset \). Then there exists a point \( C \in [A, B] \) such that

\[ d(A, C) = R(8\delta + 1) + 6\delta + 2, \quad (3.10) \]

and we have \( C \in [A, u_i] \) for any \( i \in J \). First we prove that

\[ d(C, h_i C) \leq 8\delta + 1. \quad (3.11) \]

For that we apply Lemma 2.5 to the geodesic paths \([A, u_i]\) and \([h_i A, h_i u_i]\) and the points \( C \) and \( h_i C \). The assumptions of this lemma are satisfied:

(a) \( d(C, A) > 6\delta + 2 \geq d(A, A_i) + 2\delta. \)

(b) \( d(C, u_i) = d(A, u_i) - d(A, C) > 2\epsilon_2 + 10\delta + 3 \geq d(u_i, h_i u_i) + 2\delta. \)

(c) \( d(A, C) = d(h_i A, h_i C). \)

By this lemma, \( d(C, h_i C) \leq 4\delta + d(A, h_i A) \leq 4\delta + (4\delta + 1) \) that proves (3.11). By (3.1) we have \( d(A, h_i A) \leq 4\delta + 1 \) and by (3.10) we have \( d(C, A) > R(8\delta + 1) \). From this, (3.11) and the definition of the acylindrical action we obtain the statement. □

Now we are ready to complete the proof of the statement. It follows from Claims 1 and 3 that

\[ \#\{ h^i | i \in I \cup J \} \leq n, \quad \text{where} \quad n = N(8\delta + 3) + N(8\delta + 1) \quad (3.12) \]

**Case 1.** Suppose \( \#(h) \leq n. \)

Let \( \mathcal{M} = \max\{|h^i|_X : 1 \leq i \leq n\} \). Note that \( \mathcal{M} \leq n|h|_X \). If \( |g|_X \leq \mathcal{M} + 8\delta + 2 \), we are done. Suppose that \( |g|_X > \mathcal{M} + 8\delta + 2 \). Let \( C \) be the point on the side \([A, B]\) such that

\[ d(C, B) = \mathcal{M} + 2\delta + 1. \quad (3.13) \]

Then \( d(C, A) > 6\delta + 1 \). It follows that the distance from \( C \) to \([A, A_i] \cup [B, B_i]\) is larger than \( 2\delta \). We set \( C_i = h_i C \). Then, by Lemma 2.5

\[ d(C, C_i) \leq 8\delta + 1. \quad (3.14) \]

Let \( g_1 \) be the label of the path \([C, B]\) (and hence of the path \([C_i, B_i]\)). The concatenation of the paths \([C, B], [B, B_i], [B_i, C_i]\) has the same endpoints as the geodesic path \([C, C_i]\). Therefore the label (in \( G \)) of the path \([C, C_i]\) is \( g_1 h^i g_1^{-1}. \)

Using (3.14), we obtain \( |g_1 h^i g_1^{-1}|_X \leq 8\delta + 1 \) for any \( i \). Using (3.12) and (3.13), we deduce

\[ |g_1|_X = d(C, B) = \mathcal{M} + 2\delta + 1 \]

\[ \leq n|h|_X + 2\delta + 1 \]

\[ \leq (N(8\delta + 3) + N(8\delta + 1))|h|_X + 2\delta + 1. \]

This completes the proof in this case.
Case 2. Suppose \( \#(h) > n \).

By (3.12), one of the elements 1, \( h, h^2, \ldots, h^n \) does not lie in the set \( \{ h^i \mid i \in I \cup J \} \). Then there exists \( 0 \leq i \leq n \) such that \( i \in I^c \setminus J \). In particular, \( d((q_i), u_i) \leq \epsilon_2 \) and \( d(A, u_i) \leq R(8\delta + 1) + 2\epsilon_2 + 16\delta + 5 \). Then
\[
|g|_X \leq d(B, (q_i)) + d((q_i), u_i) + d(u_i, A) \\
\leq |x_i|_X + \epsilon_2 + (R(8\delta + 1) + 2\epsilon_2 + 16\delta + 5).
\]

Finally we note that
\[
|x_i|_X \leq \frac{1}{2}(|h|_X + \epsilon_1) \leq \frac{1}{2} (n|h|_X + \epsilon_1).
\]

This completes the proof. \( \square \)

**Proof of Theorem 1.4.** (1) Suppose that \( h_1, h_2 \) are loxodromic elements. By Lemma 3.4, we may reduce the proof to the case that \( h_1, h_2 \) are shortest in their conjugacy class. Let \( g \in G \) be an arbitrary element such that \( h_1 = gh_2 g^{-1} \). We make two observations about the quasi-geodesics \( L(1, h_1) \) and \( L(g, h_2) \).

(a) Since \( h_1 = gh_2 g^{-1} \), the Hausdorff distance between \( L(1, h_1) \) and \( L(g, h_2) \) is at most \( |g|_X + \max\{|h_1|_X, |h_2|_X\} \). Therefore the limit points of these quasi-geodesics coincide.

(b) Since \( h_1 \) and \( h_2 \) are shortest in their conjugacy class, both \( L(1, h_1) \) and \( L(g, h_2) \) are \((\varkappa, \varepsilon)\)-quasi-geodesics, where \( \varkappa \) and \( \varepsilon \) are universal constants from Corollary 2.4.

It follows from (a) and (b) that the Hausdorff distance between \( L(1, h_1) \) and \( L(g, h_2) \) is at most \( k = \mu(\delta, \varkappa, \varepsilon) \), see Lemma 2.4. In particular, there exists a point \( z \in L(g, h_2) \) such that \( d(1, z) \leq k \). Let \( t = gh_2^i \) be the phase point on \( L(g, h_2) \) which is nearest to \( z \). In particular, \( d(t, z) \leq |h_2|_X \). Then
\[
|t|_X = d(1, t) \leq d(1, z) + d(z, t) \leq k + |h_2|_X.
\]

Moreover, \( h_1 = th_2 t^{-1} \), and we are done.

(2) Suppose that \( h_1, h_2 \) are elliptic elements. By Lemma 3.2, we may reduce the proof to the case that the subgroups \( \langle h_i \rangle, i = 1, 2 \), lie in the ball \( B_1(8\delta + 1) \). Let \( g \in G \) be an element such that \( h_1 = gh_2 g^{-1} \). Since the orders of \( h_i \) are larger than \( N(8\delta + 1) \) (by assumption), it follows from the definition of acylindricality that \( |g|_X \leq R(8\delta + 1) \). \( \square \)

4. **AN EXTENSION OF THE PERIODICITY THEOREM FROM [11]**

The main result of this section is Theorem 4.3 which is used in Section 6. It slightly extends the periodicity theorem from [11], see Theorem 4.2 below. Both theorems can be easily formulated in the case of free groups:

Let \( a, b \) be two cyclically reduced words in the free group \( F \) with basis \( X \). If the bi-infinite words \( L(a) = \ldots aaa \ldots \) and \( L(b) = \ldots bbb \ldots \) have a common subword of length \( |a| + |b| \), then some cyclic permutations of \( a \) and \( b \) are positive powers of some word \( c \).
For the case of acylindrically hyperbolic groups, we recall some notions. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and that $G$ acts acylindrically on $\Gamma(G, X)$. In [5, Lemma 2.2] Bowditch proved that the infimum of stable norms (see Section 2) of all loxodromic elements of $G$ with respect to $X$ is a positive number. We denote this number by $\text{inj}(G, X)$ and call it the injectivity radius of $G$ with respect to $X$.

**Definition 4.1.** Let $G$ be a group and $X$ a generating set of $G$. The right Cayley graph of $G$ with respect to $X$ is denoted by $\Gamma(G, X)$. For any two elements $u, v \in G$, we choose a geodesic path $[u, v]$ in $\Gamma(G, X)$ from $u$ to $v$ so that $w[u, v] = [wu, wv]$ for any $w \in G$. With any element $x \in G$ and any element $g \in G$ of infinite order, we associate the bi-infinite path $L(x, g) = \cdots p_{-1} op_0 \cdots$, where $p_n = [xg^n, xg^{n+1}]$, $n \in \mathbb{Z}$. The paths $p_n$ are called $g$-periods of $L(x, g)$. For a subpath $p \subset L(x, g)$ and a number $k \in \mathbb{N}$, we say that the path $p$ contains $k$ $g$-periods if there exists $n \in \mathbb{Z}$ such that $p_n p_{n+1} \cdots p_{n+k-1}$ is a subpath of $p$. The vertices $xg^n$, $n \in \mathbb{Z}$, are called the phase vertices of $L(x, g)$. Note that $L(x, g) = x L(1, g)$.

**Theorem 4.2.** (see [1, Theorem 1.4]) Let $G$ be a group and $X$ a generating set of $G$. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and that $G$ acts acylindrically on $\Gamma(G, X)$. Then there exists a constant $C > 0$ such that the following holds.

Let $a, b \in G$ be two loxodromic elements which are shortest in their conjugacy classes and such that $|a|_X \geq |b|_X$. Let $x, y \in G$ be arbitrary elements and $r$ an arbitrary non-negative real number. We set $f(r) = \frac{2r}{\text{inj}(G, X)} + C$.

Suppose that $p \subset L(x, a)$ and $q \subset L(y, b)$ are subpaths such that $d(p_-, q_-) \leq r$, $d(p_+, q_+) \leq r$, and $p$ contains at least $f(r)$ $a$-periods. Then there exist nonzero integers $s, t$ such that $(y^{-1} x)^s (x^{-1} y) = b^t$.

![Fig. 2. Illustration to Theorem 4.2](image-url)
The following theorem says that taking an appropriate linear function $F$ instead of $f$, we can guarantee that both numbers $s$ and $t$ are positive.

**Theorem 4.3.** Let $G$ be a group and $X$ a generating set of $G$. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and that $G$ acts acylindrically on $\Gamma(G, X)$. Then there exists a a linear function $F : \mathbb{R} \to \mathbb{R}$ with constants depending only on $(G, X)$ such that the following holds.

Let $a, b \in G$ be two loxodromic elements which are shortest in their conjugacy classes and such that $|a|_X \geq |b|_X$. Let $x, y \in G$ be arbitrary elements and $r$ an arbitrary non-negative real number. Suppose that $p \subseteq L(x, a)$ and $q \subseteq L(y, b)$ are subpaths such that $d(p_-, q_-) \leq r$, $d(p_+, q_+) \leq r$, and $p$ contains at least $F(r)$ $a$-periods. Then there exist positive integers $s, t$ such that

$$(y^{-1}x)a^s(x^{-1}y) = b^t.$$ 

**Proof.** We set $F(r) = f(r) + \kappa(4r + 4\delta + 5\mu) + \varepsilon + 1$, where $\delta$ is the hyperbolicity constant of the Cayley graph $\Gamma(G, X)$, $\kappa$ and $\varepsilon$ are from Corollary 2.12, and $\mu = \mu(\delta, \kappa, \varepsilon)$ is from Lemma 2.2. For brevity, we set $F = |F(r)|$.

First we show that, without loss of generality, we may assume that $p_-$ and $q_-$ are phase vertices of $L(x, a)$ and $L(y, b)$, respectively.

Let $A$ and $B$ be the leftmost phase vertices of $p$ and $q$, respectively. We set $a_1 = cac^{-1}$, where $c$ is the subpath of the $a$-period from $p_-$ to $A$ and we set $b_1 = dbd^{-1}$, where $d$ is the subpath of the $b$-period from $q_-$ to $B$, see Figure 3.

![Fig. 3. Reduction $L(x, a) = L(p_-, a_1)$.

Then $|a_1|_X = |a|_X$ and $L(x, a) = L(p_-, a_1)$ and also $|b_1|_X = |b|_X$ and $L(y, b) = L(q_-, b_1)$. Note that $p_-$ is a phase vertex of $L(p_-, a_1)$ and $q_-$ is a phase vertex of $L(q_-, b_1)$. Suppose we have proved that there exist positive integers $s, t$ such that

$$(q_-^{-1}p_-)^s(p_-^{-1}q_-) = b_1^t.$$ 

Substituting $a_1 = cac^{-1}$, $b_1 = dbd^{-1}$, $A = p_-c$ and $B = q_-, d$, we deduce

$$(B^{-1}A)a^s(A^{-1}B) = b^t.$$ 

Since $A$ is a phase vertex of $L(x, a)$, we have $A = xa^i$ for some $i \in \mathbb{Z}$. Analogously we have $B = yb^j$ for some $j \in \mathbb{Z}$. This implies $(y^{-1}x)a^s(x^{-1}y) = b^t$.

Thus, without loss of generality, we assume that $p_-$ and $q_-$ are phase vertices of $L(x, a)$ and $L(y, b)$, respectively. Then $L(x, a) = L(p_-, a)$ and $L(y, b) = L(q_-, b)$, and by Theorem 4.2 we have

$$(q_-^{-1}p_-)^s(p_-^{-1}q_-) = b^t.$$

(4.1)
for some nonzero integers \(s, t\). We may assume that \(s > 0\). Suppose that \(t < 0\). We set \(C = p_- a^s F\) and \(D = q_- b^t F\). Then \(C\) lies on \(L(p_-, a)\) to the right from \(p_+\) and \(D\) lies on \(L(q_-, b)\) to the left from \(q_-\), see Figure 4.

Let \(u\) be the subpath of \(L(p_-, a)\) from \(p_-\) to \(C\) and let \(v\) be the subpath of \(L(q_-, b^{-1})\) from \(q_-\) to \(D\). We have \(d(u_-, v_-) = d(p_-, q_-) \leq r\) by assumption in the theorem and we have \(d(u_+, v_+) \leq r\) since

\[
d(u_+, v_+) = d(p_- a^s F, q_- b^t F) = d(1, a^{-s} p_-^{-1} q_-^{-1} b^t F) \quad (4.1) = d(1, p_-^{-1} q_-) = d(p_-, q_-) \leq r.
\]

By Corollary 2.3 there exists a point \(E \in v\) such that

\[
d(p_+, E) \leq r + 2(\delta + \mu). \quad (4.2)
\]

We have

\[
d(E, q_-) \geq d(p_-, p_+) - d(p_-, q_-) - d(p_+, E) \geq d(p_-, p_+) - r - (r + 2(\delta + \mu)). \quad (4.3)
\]

The point \(q_-\) lies on the \((\varepsilon, \varepsilon)\)-quasi-geodesic \(L(y, b)\) between the points \(E\) and \(q_+\). Therefore, by Lemma 2.2 there exists a point \(q'_- \in [E, q_+]\) such that \(d(q_-, q'_-) \leq \mu\). Then

\[
d(E, q_-) \leq d(E, q'_-) + d(q'_-, q_-)
\leq d(E, q_+) + \mu
\leq d(E, p_+) + d(p_+, q_+) + \mu
\leq (r + 2(\delta + \mu)) + r + \mu. \quad (4.4)
\]

It follows from (4.3) and (4.4) that

\[
d(p_-, p_+) \leq 4r + 4\delta + 5\mu.
\]

On the other hand,

\[
\mathcal{F} \leq \mathcal{F}|a|_X \leq \ell(p) \leq \varepsilon d(p_-, p_+) + \varepsilon \leq \varepsilon(4r + 4\delta + 5\mu) + \varepsilon
\]

that contradicts the definition of \(\mathcal{F}\) at the beginning of the proof. Thus the assumption \(t < 0\) is not valid. \(\square\)
Lemma 5.3. Let \( k \) system \( G \) of two nonzero integers. In the case of \( \mathbb{Z} \) the index introduced in the following definition coincides with the index of the ideal \( (a, b) \) in the ideal \( (a) \).

Definition 5.1. Let \( G \) be a group and let \( [a], [b] \) be two conjugacy classes of elements \( a, b \in G \) of infinite order. Suppose that \( a, b \) are commensurable. Then, by definition, there exist nonzero integers \( k, \ell \) such that the conjugacy classes of \( a^k \) and \( b^\ell \) coincide.

We take minimal \( k > 0 \) with this property and call the conjugacy class of \( a^k \) the least common multiple of the conjugacy classes of \( a \) and \( b \), and we denote it by \( [a] \lor [b] \). The number \( k \) is called the index of \( [a] \lor [b] \) with respect to \( [a] \) and is denoted by \( \text{Ind}_{[a]}([a] \lor [b]) \). Thus,

\[
\text{Ind}_{[a]}([a] \lor [b]) := \min \{ k > 0 \mid \exists s : a^k \sim b^s \}.
\]

Remark 5.2. The conjugacy class \( [a] \lor [b] \) does not depend on the choice of \( a \) and \( b \) in their conjugacy classes. The following lemma implies that if \( a \) and \( b \) are loxodromic elements of an acylindrically hyperbolic group \( G \), then \( [a] \lor [b] = \pm([b] \lor [a]) \). It also gives an estimation of \( \text{Ind}_{[a]}([a] \lor [b]) \) via the stable norm of \( b \).

In the following lemmas \( L \) is the constant from Lemma 2.10.

Lemma 5.3. Let \( G \) be an acylindrically hyperbolic group with respect to a generating system \( X \). Let \( a, b \) be two commensurable loxodromic elements of \( G \). Denoting \( k = \text{Ind}_{[a]}([a] \lor [b]) \) and \( \ell = \text{Ind}_{[b]}([b] \lor [a]) \), we have

\[
a^k \sim b^{\pm \ell},
\]

\[
k \cdot ||a||_X = \ell \cdot ||b||_X,
\]

\[
k \leq \frac{L^2}{\text{inj}(G, X)} \cdot ||b||_X.
\]

Proof. By definition we have \( a^k \sim b^s \) and \( b^\ell \sim a^t \) for some \( s, t \in \mathbb{Z} \). It follows \( k \cdot ||a||_X = |s| \cdot ||b||_X \) and \( \ell \cdot ||b||_X = |t| \cdot ||a||_X \). Hence \( k\ell = |s||t| \). By definition we have \( k \leq |t| \) and \( \ell \leq |s| \). This implies \( s = \pm \ell \) and hence (5.1) and (5.2).

We prove (5.3). By (5.1) we have \( a^k = z^{-1}b^{\pm \ell}z \) for some \( z \in G \). It follows that \( a \in E_G(z^{-1}b) \). Then, by Lemma 2.10 \( a^k \) and \( (z^{-1}b)^\ell \) belong to the same infinite cyclic group. Let \( c \) be a generator of this group. Then \( a^L = c^p \) and \( (z^{-1}b)^\ell = c^q \) for some nonzero integers \( p, q \). This implies

\[
a^{Lq} = z^{-1}b^{\ell p}z.
\]
From this and the definition of $k$, we have $k \leq L|q|$. It remains to estimate $|q|$. It follows from the definitions of stable norm and injectivity radius that

$$L||b||_X = ||b^L||_X = ||c^q||_X = |q| \cdot ||c||_X \geq |q| \cdot \text{inj}(G, X).$$

Hence

$$|q| \leq \frac{L||b||_X}{\text{inj}(G, X)}.$$

Substituting in the above established estimation $k \leq L|q|$, we complete the proof.

The following lemma estimates possible nonzero exponents $s, t$ in the equation $z^{-1}a^sz = b^t$ with given $a, b, z \in G$, where $G$ is acylindrically hyperbolic and $a, b$ are loxodromic.

**Lemma 5.4.** Let $G$ be an acylindrically hyperbolic group with respect to a generating system $X$. Let $a, b, z$ be elements of $G$, where $a$ and $b$ are loxodromic, such that $z^{-1}a^nz = b^m$ for some nonzero integers $n, m$. Then we have $z^{-1}a^sz = b^t$ with the same $z$, where

$$|s| = L \cdot \text{Ind}_{[a]}([a] \lor [b]) \quad \text{and} \quad |t| = L \cdot \text{Ind}_{[b]}([b] \lor [a]).$$

Moreover, if $n, m$ are positive, then $s, t$ can be also chosen to be positive.

**Proof.** We denote $k = \text{Ind}_{[a]}([a] \lor [b])$ and $\ell = \text{Ind}_{[b]}([b] \lor [a])$. By (5.1), there exists $z_1 \in G$ such that

$$z_1^{-1}a^kz_1 = b^{\pm \ell}.$$ 

From this and from the equation $z^{-1}a^nz = b^m$ we deduce

$$z_1^{-1}a^{mk}z_1 = b^{\pm m\ell} \quad \text{and} \quad z^{-1}a^{\ell t}z = b^{mt}.$$ 

We denote $e = zz_1^{-1}$. Then $ea^{mk}e^{-1} = a^{\pm \ell}$, hence $e \in E_G(a)$. By Lemma 2.10, $E_G(a)$ contains a normal infinite cyclic subgroup of index $L$. It follows that $e^{-1}a^L e = a^{\pm L}$. Then

$$z^{-1}a^{kL}z = z_1^{-1}e^{-1}a^{kL}ez_1 = z_1^{-1}a^{\pm kL}z_1 = b^{\pm \ell L}.$$ 

This shows that the first statement is valid for $s = kL$ and $t = \pm \ell L$.

Now we prove the second statement. Suppose that both $n, m$ are positive. From $z^{-1}a^nz = b^m$ and $z^{-1}a^sz = b^t$ follows $b^{ms} = b^{nt}$. Since $b$ has infinite order, we have $ms = nt$, hence $s$ and $t$ have the same sign. Changing the signs of $s$ and $t$ simultaneously, we may assume that both $s, t$ are positive.

6. AN AUXILIARY LEMMA

**Definition 6.1.** Let $G$ be a group and $g \in G$ be an element of infinite order. The set of elements of $G$ commensurable with $g$ is denoted by $\text{Com}(g)$. Thus,

$$\text{Com}(g) = \{ h \in G \mid g^t \text{ is conjugate to } h^s \text{ for some nonzero } s, t \}.$$
Lemma 6.2. Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Then there exists a constant $M > 1$ such that for any exponential equation
\[ a_1 g_1^{x_1} a_2 g_2^{x_2} \cdots a_n g_n^{x_n} = 1 \] (6.1)
with constants $a_1, g_1, \ldots, a_n, g_n$ from $G$ (where $g_1, \ldots, g_n$ are loxodromic and shortest in their conjugacy classes with respect to $X$) and variables $x_1, \ldots, x_n$, if this equation is solvable over $\mathbb{Z}$, then there exists a solution $(k_1, \ldots, k_n)$ with
\[ |k_j| \leq \left( n^2 + \sum_{i=1}^{n} \frac{|a_i|}{|g_j|} + \sum_{g_i \in \text{Com}(g_j)} \frac{|g_i|}{|g_j|} \right) \cdot M \] (6.2)
for all $j = 1, \ldots, n$.

Proof. Suppose that $(k_1, \ldots, k_n)$ is a solution of equation (6.1) with minimal sum $|k_1| + \cdots + |k_n|$. Because of symmetry, we estimate only $|k_1|$. In what follows we consider a polygon $P$ in the Cayley graph $\Gamma(G, X)$ corresponding to the equation (6.1). More precisely, let $P$ be a polygon in the Cayley graph $\Gamma(G, X)$ with consecutive sides $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$ such that the sides $p_i$ are geodesics with the labels $a_i$ and the sides $q_i$ are quasi-geodesics consisting of $k_i$ consecutive geodesic paths labelled by $g_i$. Note that each $q_i$ is a $(x, \varepsilon)$-quasi-geodesic path by Corollary 2.12.

By Lemmas 2.1 and 2.2, $q_1$ lies in the $\nu$-neighborhood of the union of the other sides of $P$, where
\[ \nu = (2n - 2)\delta + 2\mu. \]
For $i = 1, \ldots, n$, let $p_i'$ be the maximal phase subpath of $q_1$ such that the endpoints of $p_i'$ are at distance at most $\nu$ from $p_i$. Analogously, for $i = 2, \ldots, n$, let $q_i'$ be the maximal phase subpath of $q_1$ such that the endpoints of $q_i'$ are at distance at most $\nu$ from $q_i$.

![Fig. 5. The polygon $P$.](image)

Then the path $q_1$ is covered by the union of its subpaths $p_1', \ldots, p_n', q_2', \ldots, q_n'$ and at most $2n - 2$ additional $g_i$-periods. Therefore (and using the notation at the end of Section 4), we obtain
\[ N(q_1) \leq \sum_{i=1}^{n} N(p_i') + \sum_{i=2}^{n} N(q_i') + 2n - 2. \] (6.3)
We first estimate the numbers $N(p'_i)$:

$$N(p'_i) = \frac{\ell(p'_i)}{|g_1|_X} \leq \frac{\kappa d((p'_i)_-, (p'_i)_+) + \varepsilon}{|g_1|_X} \leq \frac{\kappa(\ell(p_i) + 2\nu) + \varepsilon}{|g_1|_X} \quad (6.4)$$

In Claims 2 and 3 below we estimate the numbers $N(q'_i)$. Since the endpoints of $q'_i$ are at distance at most $\nu$ from $q_i$, there exists a subpath $q''_i$ of $q_i$ or $\bar{q}_i$ such that $d((q'_i)_-, (q''_i)_-) \leq \nu$ and $d((q'_i)_+, (q''_i)_+) \leq \nu$. We need the following relation between $N(q''_i)$ and $N(q'_i)$.

**Claim 1.** We have

$$N(q''_i) \geq N(q'_i) \frac{|g_i|_X}{|g_i|_X} - 2\nu - 2. \quad (6.5)$$

**Proof.** The desired inequality follows from the following two estimations:

$$N(q''_i) \geq \frac{\ell(q''_i)}{|g_i|_X} - 2,$$

$$\ell(q''_i) \geq d((q''_i)_-, (q''_i)_+) \geq d((q'_i)_-, (q'_i)_+) - 2\nu = |g_i|_X N(q'_i) - 2\nu \geq N(q'_i) \frac{|g_i|_X}{|g_i|_X} - 2\nu.$$  \(\square\)

In the following part of the proof we will use the function $f$ from Theorem 4.2. We set

$$\alpha = \kappa(2\nu + 3 + f(\nu)). \quad (6.6)$$

**Claim 2.** If $g_1$ and $g_i$ are not commensurable, then

$$N(q'_i) \leq \alpha \frac{|g_i|_X}{|g_1|_X} + f(\nu). \quad (6.7)$$

**Proof.** First consider the case $|g_1|_X \geq |g_i|_X$. Suppose that (6.7) is not valid. Then $N(q'_i) > f(\nu)$. Then, by Theorem 4.2, $g_1$ and $g_i$ are commensurable that contradicts the assumption.

Now consider the case $|g_i|_X \geq |g_1|_X$. Suppose that (6.7) is not valid. Then

$$N(q'_i) > \alpha \frac{|g_i|_X}{|g_1|_X}. \quad (6.8)$$

Substituting (6.8) into (6.5), we deduce

$$N(q''_i) \geq \alpha \frac{|g_i|_X}{|g_1|_X} - 2\nu - 2 \geq \frac{\alpha}{\kappa} - 2\nu - 2 \geq f(\nu).$$

By Theorem 4.2 applied to $g_i$ and $g_1$, we obtain that these elements are commensurable. A contradiction.  \(\square\)
Now we set
\[
\beta = \kappa (2\nu + 3 + F(\nu))L,
\] (6.9)
where $F$ is the function from Theorem 4.3 and $L \geq 1$ is the constant from Lemma 2.10.

**Claim 3.** If $g_1$ and $g_i$ are commensurable, then
\[
N(q_i') \leq \beta \text{Ind}_{[g_1]}([g_1] \vee [g_i]).
\] (6.10)

**Proof.** Suppose the converse, i.e.
\[
N(q_i') > \beta \text{Ind}_{[g_1]}([g_1] \vee [g_i]).
\] (6.11)

Our nearest aim is to deduce the following two inequalities:
\[
N(q_i') > L \cdot \text{Ind}_{[g_1]}([g_1] \vee [g_i]) + F(\nu),
\] (6.12)
\[
N(q_i'') > L \cdot \text{Ind}_{[g_i]}([g_i] \vee [g_1]) + F(\nu).
\] (6.13)

The first inequality follows directly from the assumption (6.11) and the facts that $\beta \geq L + F(\nu)$ (since $\kappa \geq 1$ in (6.9)) and $\text{Ind}_{[g_1]}([g_1] \vee [g_i]) \geq 1$. We prove the second one.

\[
N(q_i'') \geq \frac{N(q_i')\|g_1\|_X}{\|g_i\|_X} - 2\nu - 2
\geq \frac{\beta \text{Ind}_{[g_1]}([g_1] \vee [g_i])\|g_1\|_X}{\|g_i\|_X} - 2\nu - 2
\geq \frac{\beta \text{Ind}_{[g_i]}([g_i] \vee [g_1])\|g_1\|_X}{\kappa \|g_i\|_X} - 2\nu - 2
\geq \frac{\beta \text{Ind}_{[g_i]}([g_i] \vee [g_1])}{\kappa} - 2\nu - 2
\geq L \cdot \text{Ind}_{[g_i]}([g_i] \vee [g_1]) + F(\nu).
\]

Thus, (6.12) and (6.13) are proved. By Theorem 4.3 and Lemma 5.4, there exist different phase vertices $x_1, x_2$ on $q_i'$ and different phase vertices $y_1, y_2$ on $q_i''$ such
that $x_1^{-1}y_1 = x_2^{-1}y_2$. Then we can cut out a piece from $\mathcal{P}$ and glue the remaining pieces as shown in Figure 6.

![Figure 6. Cutting out a piece from $\mathcal{P}$.

More precisely, let $\mathcal{P}_1$ be the subpath of the (cyclic) path $\mathcal{P}$ from $y_1$ to $x_1$ and $\mathcal{P}_3$ be the subpath of $\mathcal{P}$ from $x_2$ to $y_2$. We consider the polygon $\mathcal{P}'$ obtained by gluing the endpoints of $\mathcal{P}_1$ to the corresponding endpoints of the left translation $g\mathcal{P}_3$, where $g = x_1x_2^{-1} = y_1y_2^{-1}$. The new polygon $\mathcal{P}'$ corresponds to a solution of (6.1) with smaller value $|k_1| + \cdots + |k_n|$. A contradiction.

Thus, the summands in (6.3) are estimated in (6.4) and in Claims 2 and 3. This proves the inequality (6.2) for some universal constant $M$. \hfill $\square$

7. Theorem A and its proof

**Theorem A.** Let $G$ be an acylindrically hyperbolic group with respect to a generating set $X$. Then there exists a constant $M > 1$ such that for any exponential equation

$$a_1g_1^{x_1}a_2g_2^{x_2}\cdots a_ng_n^{x_n} = 1$$

(7.1)

with constants $a_1, g_1, \ldots, a_n, g_n$ from $G$ and variables $x_1, \ldots, x_n$, if this equation is solvable over $\mathbb{Z}$, then there exists a solution $(k_1, \ldots, k_n)$ with

$$|k_j| \leq \left( n^2 + \sum_{i=1}^{n} \frac{|a_i|}{g_j'}X + \sum_{g_i \in \text{Com}(g_j)} \frac{|g_i|}{g_j'}X + \sum_{g_i \in \text{Com}(g_j)} |g_i|X \right) \cdot M$$

(7.2)

for all $j$ corresponding to loxodromic $g_j$; here $g_j'$ is an element shortest in the conjugacy class of $g_j$ with respect to $X$.

This implies that if the equation (7.1) is solvable over $\mathbb{Z}$, then there exists a solution $(k_1, \ldots, k_n)$ with the universal estimation

$$|k_j| \leq \left( n^2 + \sum_{i=1}^{n} |a_i|X + \sum_{i=1}^{n} |g_i|X \right) \cdot M$$

(7.3)

for all $j$ corresponding to loxodromic $g_j$. 
From this case we have

\[ |h_i|_X \leq K |g_i|_X. \]  

(7.4)

For convenience we set \( g_0 = g_n \) and \( h_0 = h_n \). Now we rewrite (7.1) as

\[ a'_1(g'_1)^{x_1} a'_2(g'_2)^{x_2} \ldots a'_n(g'_n)^{x_n} = 1, \]

where \( a'_i = h_{i-1}^{-1} a_i h_i \) for \( i = 1, \ldots, n \). Since \( g'_i \) are loxodromic and shortest in their conjugacy classes, by Lemma \( \ref{lemma6.2} \) there exists a solution \((k_1, \ldots, k_n)\) of equation (7.1) with

\[ |k_j| \leq (n^2 + \sum_{i=1}^{n} |a'_i|_X + \sum_{g_i \in \text{Com}(g_j)} |g'_i|_X + \sum_{g_i \in \text{Com}(g_j)} \text{Ind}_{[g_j]}([g_j] \vee [g_i]) \cdot M_1 \]  

(7.5)

for all \( j = 1, \ldots, n \), where \( M_1 \) is a universal constant. We set

\[ M_2 = 2M_1 K \frac{L^2}{\text{inj}(G, X)}. \]

Then (7.2) with \( M = M_2 \) follows from (7.5) with the help of the following claim.

**Claim.** We have

1) \( |a'_i|_X \leq |a_i|_X + K(|g_{i-1}|_X + |g_i|_X) \).

2) \( |g'_i|_X \leq |g_i|_X \).

3)

\[ \text{Ind}_{[g_j]}([g_j] \vee [g_i]) \leq \frac{L^2}{\text{inj}(G, X)} \cdot |g_i|_X. \]

**Proof.** The first statement follows from the definition of \( a'_i \) and (7.4), the second from the definition of \( g'_i \), and the third from Lemma \( \ref{lemma5.3} \). \( \Box \)

Now we consider the general case. Let \( \mathcal{E} \) (resp. \( \mathcal{L} \)) be the set of the indexes \( i \in \{1, \ldots, n\} \) for which \( g_i \) is elliptic (resp. loxodromic). We have \( \mathcal{E} \cup \mathcal{L} = \{1, \ldots, n\} \). Note that by Lemma \( \ref{lemma3.2} \) if \( i \in \mathcal{E} \), then

\[ |g_i^{x_i}|_X \leq 2K |g_i|_X + (8\delta + 1) \leq 2K (8\delta + 2) |g_i|_X. \]  

(7.6)

for any choice of \( x_i \in \mathbb{Z} \). For any two consecutive numbers \( s, t \in \mathcal{L} \), let \( b_t \) be the product of the factors in (7.1) between \( g_t^{x_t} \) and \( g_t^{x_t} \), i.e.

\[ b_t = a_{s+1} g_{s+1}^{x_{s+1}} \ldots g_{t-1}^{x_{t-1}} a_t. \]  

(7.7)

Then we can reduce to the considered case (all \( g_i \) are loxodromic) by writing

\[ a_1 g_1^{x_1} \ldots a_n g_n^{x_n} = \prod_{t \in \mathcal{L}} b_t g_t^{x_t}. \]

From this case we have

\[ |k_j| \leq \left( n^2 + \frac{1}{|g'_j|_X} \left( \sum_{i \in \mathcal{L}} |b_i|_X + \sum_{i \in \mathcal{E}, g_i \in \text{Com}(g_j)} |g_i|_X \right) + \sum_{i \in \mathcal{E}, g_i \in \text{Com}(g_j)} |g_i|_X \right) \cdot M_2 \]  

(7.8)
for any \( j \in \mathcal{L} \) and some universal constant \( M_2 \). Now we estimate the sums in the internal brackets. First observe that for any choice of \( x_i \), we have

\[
\sum_{i \in \mathcal{L}} |b_i|X \leq \sum_{i \in \mathcal{E}} |a_i|X + \sum_{i \in \mathcal{E}} |g_i^n|X.
\]

\[
\leq \sum_{i = 1}^n |a_i|X + 2K(8\delta + 2) \sum_{i \in \mathcal{E}} |g_i|X
\]

\[
\leq \sum_{i = 1}^n |a_i|X + 2K(8\delta + 2) \sum_{i \in \mathcal{E}, g_i \notin \text{Com}(g_j)} |g_i|X.
\]

The last inequality is satisfied since the condition \( g_i \notin \text{Com}(g_j) \) is automatically satisfied for \( i \in \mathcal{E} \) (elliptic and loxodromic elements are not commensurable). Therefore the sum in the internal brackets in (7.8) does not exceed

\[
2K(8\delta + 2) \left( \sum_{i = 1}^n |a_i|X + \sum_{g_i \notin \text{Com}(g_j)} |g_i|X \right).
\]

Then (7.2) is satisfied for \( M = 2M_2K(8\delta + 2) \). The last statement of the main theorem follows from (7.2) and \( |g_j'|X \geq 1 \).

\[\square\]

8. Theorems B and C and their proofs

In subsection 8.1 we recall some definitions and statements about hyperbolically embedded subgroups and weakly hyperbolic groups. Theorem B is formulated and proved in subsection 8.2. Theorem C is deduced from Theorems A' and B in subsection 8.3.

8.1. Some definitions and statements from [8]. Let \( G \) be a group, \( \{H_\lambda\}_{\lambda \in \Lambda} \) a collection of subgroups of \( G \). A subset \( X \) of \( G \) is called a relative generating set of \( G \) with respect to \( \{H_\lambda\}_{\lambda \in \Lambda} \) if \( G \) is generated by \( X \) together with the union of all \( H_\lambda \). All relative generating sets are assumed to be symmetric. We define

\[
\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda.
\]

In this section, we always assume that \( X \) is a relative generating set of \( G \) with respect to \( \{H_\lambda\}_{\lambda \in \Lambda} \).

Definition 8.1. (see [8, Definition 4.1]) The group \( G \) is called weakly hyperbolic relative to \( X \) and \( \{H_\lambda\}_{\lambda \in \Lambda} \) if the Cayley graph \( \Gamma(G, X \sqcup \mathcal{H}) \) is hyperbolic.

We consider the Cayley graph \( \Gamma(H_\lambda, H_\lambda) \) as a complete subgraph of \( \Gamma(G, X \sqcup \mathcal{H}) \).

Definition 8.2. (see [8, Definition 4.2]) For every \( \lambda \in \Lambda \), we introduce a relative metric \( \hat{d}_\lambda : H_\lambda \times H_\lambda \rightarrow [0, +\infty] \) as follows:

Let \( a, b \in H_\lambda \). A path in \( \Gamma(G, X \sqcup \mathcal{H}) \) from \( a \) to \( b \) is called \( H_\lambda \)-admissible if it has no edges in the subgraph \( \Gamma(H_\lambda, H_\lambda) \).

The distance \( \hat{d}_\lambda(a, b) \) is defined to be the length of a shortest \( H_\lambda \)-admissible path connecting \( a \) to \( b \) if such exists. If no such path exists, we set \( \hat{d}_\lambda(a, b) = \infty \).
Definition 8.3. (see [8, Definition 4.25]) Let $G$ be a group, $X$ a symmetric subset of $G$. A collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ of $G$ is called hyperbolically embedded in $G$ with respect to $X$ (we write $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$) if the following hold.

(a) The group $G$ is generated by $X$ together with the union of all $H_\lambda$ and the Cayley graph $\Gamma(G, X \cup H)$ is hyperbolic.

(b) For every $\lambda \in \Lambda$, the metric space $(H_\lambda, d_\lambda)$ is proper. That is, any ball of finite radius in $H_\lambda$ contains finitely many elements.

Definition 8.4. (see [8, Definition 4.5]) Let $q$ be a path in the Cayley graph $\Gamma(G, X \cup H)$. A non-trivial subpath $p$ of $q$ is called an $H_\lambda$-subpath, if the label of $p$ is a word in the alphabet $H_\lambda$. An $H_\lambda$-subpath $p$ of $q$ is called connected if there exists a path $\gamma$ in $\Gamma(G, X \cup H)$ that connects some vertex of $p_1$ to some vertex of $p_2$, and $\text{Lab}(\gamma)$ is a word consisting only of letters from $H_\lambda$.

Note that we can always assume that $\gamma$ has length at most 1 as every element of $H_\lambda$ is included in the set of generators. An $H_\lambda$-component $p$ of a path $q$ in $\Gamma(G, X \cup H)$ is isolated if it is not connected to any other component of $q$.

Given a path $p$ in $\Gamma(G, X \cup H)$, the canonical image of $\text{Lab}(p)$ in $G$ is denoted by $\text{Lab}_G(p)$.

Definition 8.5. (see [8, Definition 4.13]) Let $\kappa \geq 1$, $\varepsilon \geq 0$, and $m \geq 2$. Let $P = p_1 \ldots p_m$ be an $m$-gon in $\Gamma(G, X \cup H)$ and let $I$ be a subset of the set of its sides $\{p_1, \ldots, p_m\}$ such that:

1) Each side $p_i \in I$ is an isolated $H_{\lambda_i}$-component of $P$ for some $\lambda_i \in \Lambda$.

2) Each side $p_i \notin I$ is a $(\kappa, \varepsilon)$-quasi-geodesic.

We denote $s(P, I) = \sum_{p_i \in I} \hat{d}_{\lambda_i}(1, \text{Lab}_G(p_i))$.

Proposition 8.6. (see [8, Proposition 4.14]) Suppose that $G$ is weakly hyperbolic relative to $X$ and $\{H_\lambda\}_{\lambda \in \Lambda}$. Then for any $\kappa \geq 1$, $\varepsilon \geq 0$, there exists a constant $C(\kappa, \varepsilon) > 0$ such that for any $m$-gon $P$ in $\Gamma(G, X \cup H)$ and any subset $I$ of the set of its sides satisfying conditions of Definition 8.5 we have $s(P, I) \leq C(\kappa, \varepsilon)m$.

8.2. Elliptical exponential equations over a group with given hyperbolically embedded subgroups.

Theorem B. Let $G$ be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, and $X$ a symmetric relative generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$. Suppose that $\{H_\lambda\}_{\lambda \in \Lambda}$ is hyperbolically embedded in $G$ with respect to $X$. Then any exponential equation

$$a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1$$

(8.1)
with $a_1, \ldots, a_n \in G$ and $g_1, \ldots, g_n \in \mathcal{H} = \bigcup_{\lambda \in \Lambda} H_{\lambda}$ is equivalent to a finite disjunction of finite systems of equations,

$$\bigvee_{i=1}^{k} \bigwedge_{j=1}^{t_i} E_{ij},$$

such that

1. each $E_{ij}$ is an exponential equation over some $H_{\lambda}$, or a trivial equation of kind $g_{ij} = 1$, where $g_{ij}$ is an element of $G$,
2. for any $i = 1, \ldots, k$, the sets of variables of $E_{i,1}$ and $E_{i,2}$ are disjoint if $j_1 \neq j_2$.

Let $\Lambda_0 = \{\lambda_1, \ldots, \lambda_n\}$ be a subset of $\Lambda$ such that $g_{i} \in H_{\lambda_{i}}$, $i = 1, \ldots, n$, and let $L = n + \sum_{i=1}^{n} |a_i|_{\mathcal{X} \cup H}$. Then these systems of equations can be algorithmically written if for any $\lambda \in \Lambda_0$, there is an algorithm computing the following finite subsets of $H_{\lambda}$:

$$H_{\lambda,L} = \{h \in H_{\lambda} \mid \hat{d}_{\lambda}(1, h) \leq C(1,1) \cdot L\}, \quad (8.2)$$

where $C(1,1)$ is the constant from Proposition 8.6.

**Proof.** To describe the desired family of systems of equations formally, we first introduce definitions (a)-(b) below. Let $f : \{1, \ldots, n\} \to \Lambda_0$ be a map such that $g_{i} \in H_{f(i)}$ for $i = 1, \ldots, n$. For any $\lambda \in \Lambda_0$ we define the set

$$H^{*}_{\lambda} = H_{\lambda} \cup \{g^{x_i}_{i} \mid f(i) = \lambda\},$$

where $g^{x_i}_{i}$ is considered as a single letter. We also define $\mathcal{H}^{*} = \bigcup_{\lambda \in \Lambda_0} H^{*}_{\lambda}$. Thus, $\mathcal{H}^{*} = \mathcal{H} \cup \{g^{x_1}_{1}, \ldots, g^{x_n}_{n}\}$.

We represent each element $a_i$ by a word $A_i$ (not necessarily of minimal possible length) in the alphabet $\mathcal{X} \cup \mathcal{H}$. Then the expression on the left side of (8.1) can be represented by the word $\mathbf{W} = A_1 g^{x_1}_{1} A_2 g^{x_2}_{2} \ldots A_n g^{x_n}_{n}$ in the alphabet $\mathcal{X} \cup \mathcal{H}^{*}$. Let $L$ be the length of this word; we have $L = n + \sum_{i=1}^{n} |A_i|$.

We consider a closed disc $D$ such that its oriented boundary $\partial D$ is divided into $L$ consecutive paths $s_1, s_2, \ldots, s_L$ labelled by the elements of $\mathcal{X} \cup \mathcal{H}^{*}$ so that the label of $\partial D$ coincides with the cyclic word $\mathbf{W}$. Thus the equation (8.1) can be written in the form $\text{Lab}(\partial D) = 1$.

(a) Let $\lambda \in \Lambda_0$ and let $P$ be a nontrivial subpath of the cyclic combinatorial path $\partial D = s_1 s_2 \ldots s_L$. The subpath $P$ is called an $H^{*}_{\lambda}$-subpath of $\partial D$ if the label of $P$ is a word in the alphabet $\hat{H}^{*}_{\lambda}$. An $H^{*}_{\lambda}$-subpath $P$ of $\partial D$ is called an $H^{*}_{\lambda}$-component if $P$ is not contained in a longer subpath of $\partial D$ with this property. Sometimes we will skip the subscript $\lambda$ and call $P$ an $\mathcal{H}^{*}$-component of $\partial D$.

The cyclic combinatorial path $\partial D$ can be written as $\partial D = Q_1 P_1 \ldots Q_r P_r$, where $P_1, \ldots, P_r$ are all $\mathcal{H}^{*}$-components of $\partial D$. We say that an $\mathcal{H}^{*}$-component $P$ is special if the label of $P$ contains the letter $g^{x_j}_{j}$ for some $j \in \{1, \ldots, n\}$. 


(b) A region $R$ in $D$ homeomorphic to a closed disc is called an $H^*_\lambda$-region if its boundary has the form $U_1E_1\ldots U_sE_s$, where $U_1,\ldots,U_s$ are $H^*_\lambda$-components for the same $\lambda \in \Lambda_0$ and at least one of them is special, and $E_1,\ldots,E_s$ are simple paths in $D$ whose interiors lie in the interior of $D$, see Figure 7. We call these paths internal sides of $R$. We say that the internal sides of $R$ are boundedly labelled, if each $E_i$ is labelled by an element of $H^*_{\lambda,L}$, see (8.2).

![Fig. 7. An example of an $H^*_\lambda$-region, where $U_1$ and $U_2$ are special $\mathcal{H}^*$-components.](image)

Note that the set $H^*_{\lambda,L}$ is finite, since the metric space $(H^*_{\lambda}, \hat{d}_\lambda)$ is locally finite.

A collection of regions $\mathcal{R} = \{R_1,\ldots,R_t\}$, where each $R_i$ is an $H^*_{\lambda(i)}$-region for some $\lambda(i) \in \Lambda_0$ is called admissible if the intersection of $R_i$ and $R_j$ is either empty or consists of one or two points on $\partial D$. We do not distinguish two admissible collections of regions $\mathcal{R}$ and $\mathcal{R}'$ if there exists an isotopy of $D$ fixing $\partial D$ and carrying the elements of $\mathcal{R}$ to the elements of $\mathcal{R}'$.

A collection of regions $\mathcal{R} = \{R_1,\ldots,R_t\}$ is called complete if it is admissible and any special $\mathcal{H}^*$-component $P_i$ is contained in the boundary of some $R_j \in \mathcal{R}$.

(c) Let $\mathcal{R} = \{R_1,\ldots,R_s\}$ be any complete collection of regions with boundedly labelled internal sides. Let $R_{s+1},\ldots,R_{s+t}$ be the components of the closure of $D \setminus \bigcup \mathcal{R}$. Then $\mathcal{R}$ determines a system of exponential equations over $G$, namely

$$\text{Eq}(\mathcal{R}) : \begin{cases} \text{Lab}(\partial R_1) = 1, \\
\ldots \\
\text{Lab}(\partial R_{s+t}) = 1. \end{cases}$$
Indeed, since $p$ last edges of these properties. Therefore, $e$ is such that ($E$) for some $X$ word in the alphabet $X \sqcup H$ (i.e. it has no letters $g_i^{\pm}$).

**Claim.** The set $\mathcal{F}$ of all complete collections of regions with boundedly labelled internal sides is finite. Each solution of (8.1) satisfies the system $\text{Eq}(\mathcal{R})$ for some $\mathcal{R} \in \mathcal{F}$.

**Proof.** The finiteness of $\mathcal{F}$ follows from the finiteness of $H_{\lambda,L}$ for any $\lambda \in \Lambda_0$.

Suppose that $\overline{k} = (k_1, \ldots, k_n)$ is some solution of the equation (8.1). For brevity, we introduce the following two definitions.

**Definition 1.** Let $\Delta$ be a graph with edges labelled by elements of the alphabet $X \sqcup H^*$. A graph map $\psi : \Delta \to \Gamma(G,X \sqcup H)$ is called a $\overline{k}$-map, if $\psi$ maps edges labelled by elements of $X \sqcup H$ to edges labelled by the same elements, and edges labelled by $g_i^{\pm}$ to edges labelled by $g_i^{k_i}$.

**Definition 2.** Let $\mathcal{R}$ be an admissible collection of regions in $D$. We denote by $D_\mathcal{R}$ the CW-complex obtained from $D$ by subdivision along all internal sides of all regions from $\mathcal{R}$. We use the notation $D_\mathcal{R}^{(1)}$ for the graph associated with the 1-skeleton of $D_\mathcal{R}$. Thus, the edges of $D_\mathcal{R}^{(1)}$ are the paths $s_1, \ldots, s_L$ and the internal sides of all regions from $\mathcal{R}$.

Observe that the above claim can be directly deduced from the following statement.

**Statement.** Let $\overline{k} = (k_1, \ldots, k_n)$ be an arbitrary solution of the equation (8.1) and let $\mathcal{P}$ be some closed path in $\Gamma(G,X \sqcup H)$ with the label $A_1g_1^{k_1} \cdots A_ng_n^{k_n}$. Then there exists a complete collection $\mathcal{R}$ of regions in $D$ with boundedly labelled internal sides such that the $\overline{k}$-map $\partial D \to \mathcal{P}$ extends to a $\overline{k}$-map $D_\mathcal{R}^{(1)} \to \Gamma(G,X \sqcup H)$.

It remains to prove this statement. Note that if $p$ is an arbitrary $H_\lambda$-component of $\mathcal{P}$, then there exists an edge $e$ in $\Gamma(G,X \sqcup H)$ such that $pe$ is a closed path in $\Gamma(G,X \sqcup H)$, and we have $\text{Lab}(e) \in H_\lambda$.

Let $p_{j_1}e_1p_{j_2}e_2 \cdots p_{j_m}e_m$ be a closed path in $\Gamma(G,X \sqcup H)$ such that $p_{j_1}, p_{j_2}, \ldots, p_{j_m}$ are $H_\lambda$-components of $\mathcal{P}$ for the same $\lambda \in \Lambda_0$, $j_1 < j_2 < \cdots < j_m$ (where we use the cyclic ordering on $\mathbb{Z}_r$), the corresponding component $P_{j_1}$ of $\partial D$ is special, $e_1, e_2, \ldots, e_m$ are edges in $\Gamma(G,X \sqcup H)$ with labels from $H_\lambda$, and $m$ is maximal with these properties.

Let $q_i$ be a subpath in $\mathcal{P}$ such that $(q_i)_- = (e_i)_-, (q_i)_+ = (e_i)_+$, $i = 1, \ldots, m$. Denote $\mathcal{P}_i = q_i e_i^{-1}$. We claim that $e_i^{-1}$ is an isolated $H_\lambda$-component in $\mathcal{P}_i$ for any $i$. Indeed, since $p_{j_i}$ and $p_{j_{i+1}}$ are $H_\lambda$-components of $\mathcal{P}$, the labels of the first and the last edges of $q_i$ do not lie in $H_\lambda$. Therefore $e_i^{-1}$ is an $H_\lambda$-component in $\mathcal{P}_i$. Since $m$ is maximal, this component is isolated in $\mathcal{P}_i$. By Proposition 8.6, we have $\text{Lab}(e_i) = b_i$ for some $b_i \in H_{\lambda,L}$.

Now we lift the edges $e_i$ to $D$, i.e., for any $e_i$ let $E_i$ be the directed chord in $D$ such that $(E_i)_- = (P_{j_i})_+, (E_i)_+ = (P_{j_{i+1}})_-$; we set $\text{Lab}(E_i) = b_i$, see Figure 8.
Let $R$ be the $H^*_\lambda$-region in $D$ with the boundary $\partial R = P_{j_1}E_1 \cdots P_{j_m}E_m$. Let $D_i$ be the closure of the component of $D \setminus R$, which contains $E_i$ in its boundary, $i = 1, \ldots, m$. By induction, there exists a complete collection $\mathcal{R}_i$ of regions in $D_i$ with boundedly labelled internal sides such that the $k$-map $\partial D_i \to \mathcal{P}_i$ extends to a $k$-map $(D_i)_{\mathcal{R}_i}^{(1)} \to \Gamma(G, X \sqcup \mathcal{H})$. Then the collection $\mathcal{R} = \{R\} \bigcup_{i=1}^m \mathcal{R}_i$ satisfies the above statement.

Fig. 8. Illustration to the proof of the statement.

8.3. Proof of Theorem C. We first prove two auxiliary lemmas about relatively hyperbolic groups, which have algorithmic character. We rely on the manuscript of Osin [20].

Remark 8.7. Let $G$ be a group relatively hyperbolic with respect to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, and let $X$ be a finite relative generating set of $G$. It is well known that any element of $G$ has exactly one of the following three types: (1) parabolic, (2) non-parabolic of finite order, (3) loxodromic with respect to $X \cup \mathcal{H}$.

Lemma 8.8. Let $G$ be a group which is relatively hyperbolic with respect to a finite collection of its subgroups $\mathbb{H} = \{H_1, \ldots, H_m\}$. Suppose that

(a) $G$ is finitely generated,

(b) each subgroup $H_i$ is given by a recursive presentation and has solvable word problem,
(c) $G$ is given by a finite relative presentation $\mathcal{P} = \langle X \mid R \rangle$ with respect to $\mathbb{H}$, where $X$ is a finite set generating $G$.

(d) the hyperbolicity constant $\delta$ of the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is known.

Then the question about the type of an element $g \in G$ (given as a word in the alphabet $X \cup \mathcal{H}$) is algorithmically decidable.

Proof. By [20, Theorem 5.6], we determine whether $g$ is parabolic or not. Suppose that $g$ is nonparabolic. We show how to determine whether the order of $g$ is finite or not.

By Lemma 4.5 from [20] (together with the last line of its proof) combined with Corollary 4.4 from [20], any element of finite order in $G$ is conjugate to an element of the set

$$S = \{ a \in G \mid |a|_X \leq B \cdot (8\delta + 1)^2 \},$$

where $B = 2C \max_{R \in \mathcal{R}} |R|_{X \cup \mathcal{H}}$, and $C$ is the constant in the relative Dehn function $D^\text{rel}_G$. Since $X$ is finite, we can find the set $S$ efficiently. Let $I = \{0, 1, \ldots, |S|\}$, for $i \in I$ we check whether $g^i$ is conjugate to an element of $S$, see Theorem 5.13 from [20]. If for some $i \in I$ the element $g^i$ is not conjugate to an element of $S$, then $g^i$ (and hence $g$) is loxodromic. If every element $g^i$, $i \in I$, is conjugate to an element of $S$, then there exist two different numbers $i, j \in I$ such that $g^i$ is conjugate to $g^j$. In this case $g$ cannot be loxodromic, hence $g$ has a finite order. $\square$

Lemma 8.9. Let $G$ be a finitely generated group which is relatively hyperbolic with respect to a finite collection of its subgroups $\{H_1, \ldots, H_m\}$. Suppose that $G$ is given by a finite relative presentation $\mathcal{P} = \langle X \mid R \rangle$ with respect to $\{H_1, \ldots, H_m\}$, where $X$ is a finite set generating $G$. Suppose we know the hyperbolicity constant $\delta$ of the Cayley graph $\Gamma(G, X \cup \mathcal{H})$. Then the constant $M$ from Theorem A can be algorithmically computed.

Proof. We may assume that all subgroups $H_i$ are proper. Then, by Proposition 5.2 from [21], $G$ is acylindrically hyperbolic with respect to $X \cup \mathcal{H}$. We claim that the following functions and constants can be computed in terms of $|X|$, $\delta$, and $\max_{r \in \mathcal{R}} |r|_{X \cup \mathcal{H}}$:

- the functions $R$ and $N$ from Definition 2.7,
- the constant $L$ from Lemma 2.10,
- the injectivity radius $\text{inj}(G, X \cup \mathcal{H})$, see the paragraph before Definition 4.1.

Indeed, by the proof of Proposition 5.2 from [21], one can take $R(\varepsilon) = 6\varepsilon + 2$, $N(\varepsilon) = (6\varepsilon + 2)|B_X(2\varepsilon)|$. By the proof of Lemma 6.8 from [21], one can compute $L$ in terms of $\delta$ with the help of the functions $R$ and $N$. Finally, one can compute $\text{inj}(G, X \cup \mathcal{H})$ in terms of $|X|$, $\delta$, and $\max_{r \in \mathcal{R}} |r|_{X \cup \mathcal{H}}$, see the proof of Theorem 4.25 from [20].

Following the proof of Theorem A', where these functions and constants were used, one can compute $M$. $\square$
Theorem C. Let $G$ be a group relatively hyperbolic with respect to a finite collection of subgroups $\{H_1, \ldots, H_m\}$. Suppose that $G$ is finitely generated, each subgroup $H_i$ is given by a recursive presentation and has solvable word problem, $G$ is given by a finite relative presentation $\mathcal{P} = \langle X \mid R \rangle$ with respect to $\{H_1, \ldots, H_m\}$, where $X$ is a finite set generating $G$, and that the hyperbolicity constant $\delta$ of the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is known, $\mathcal{H} = \bigsqcup_{i=1}^m H_i$.

Then there exists an algorithm which for any exponential equation $E$ over $G$ finds a finite disjunction $\Phi$ of finite systems of equations,

$$\Phi := \bigvee_{i=1}^k \bigwedge_{j=1}^{\ell_i} E_{ij},$$

such that

1. each $E_{ij}$ is an exponential equation over $H_\lambda$ for some $\lambda \in \{1, \ldots, m\}$ or a trivial equation of kind $g_{ij} = 1$, where $g_{ij}$ is an element of $G$,
2. for any $i = 1, \ldots, k$, the sets of variables of $E_{ij_1}$ and $E_{ij_2}$ are disjoint if $j_1 \neq j_2$,
3. $E$ is solvable if and only if $\Phi$ is solvable.

Moreover, any solution of $\Phi$ can be algorithmically extended to a solution of $E$.

Proof. Consider the exponential equation $E$, which is

$$a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1 \quad (8.3)$$

with $a_1, \ldots, a_n, g_1, \ldots, g_n \in G$. Let $A_{\text{par}}, A_{\text{fin}}, A_{\text{lox}}$, be the subsets of $\{g_1, \ldots, g_n\}$ consisting of parabolic elements, non-parabolic elements of finite order, and loxodromic elements, respectively. We have

$$\{g_1, \ldots, g_n\} = A_{\text{par}} \cup A_{\text{fin}} \cup A_{\text{lox}}.$$

If the equation $E$ has a solution then, by Theorem A, there exists a solution $(k_1, \ldots, k_n)$ with

$$|k_j| \leq \left( n^2 + \sum_{i=1}^n |a_i|_{X \cup \mathcal{H}} + \sum_{i=1}^n |g_i|_{X \cup \mathcal{H}} \right) : M$$

for all $g_j \in A_{\text{lox}}$. Hence, the solvability of $E$ is equivalent to the solvability of a finite disjunction of equations of type (8.3) with $A_{\text{lox}} = \emptyset$. Therefore, we assume that $A_{\text{lox}} = \emptyset$. For elements $g_j \in A_{\text{fin}}$, it is sufficient to look for solutions with $k_j \in \{0, 1, \ldots, m_j - 1\}$, where $m_j$ is the order of $g_j$. Therefore we may additionally assume that $A_{\text{fin}} = \emptyset$. Thus, we have reduced to the case where all elements $g_i$ are parabolic. For any parabolic $g_i$, there exists $h_i \in G$ such that $h_i^{-1} g_i h_i \in H_{\lambda(i)}$ for some $\lambda(i) \in \{1, \ldots, m\}$. This reduces the problem to Theorem B, which gives the desired $\Phi$. \qed
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