Effective dynamics of stochastic wave equation with a random dynamical boundary condition

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Abstract: This work is devoted to the effective macroscopic dynamics of a weakly damped stochastic nonlinear wave equation with a random dynamical boundary condition. The white noises are taken into account not only in the model equation defined on a domain perforated with small holes, but also in the dynamical boundary condition on the boundaries of the small holes. An effective homogenized, macroscopic model is derived in the sense of probability distribution, which is a new stochastic wave equation on a unified domain, without small holes, with a usual static boundary condition.

Key words: Stochastic partial differential equations; random dynamical boundary condition; effective dynamics; stochastic homogenization; perforated domain.

AMS subject classifications (2010): 60H15, 37L55, 37D10, 37L25, 37H05.

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1 Introduction

Nonlinear wave equations, as a class of important mathematical models, describe the propagation of waves in certain systems or media, such as sonic booms, traffic flows, optic devices and quantum fields ([33, 40]). In the deterministic case, they have been studied extensively due to their wide applications in engineering and science (e.g., [20, 22, 31, 34]). On bounded domains or media, the effect of the boundary often needs to be considered. Dirichlet, Neumann and Robin boundary conditions are called static boundary conditions, as they are not involved with time derivatives of the system state variables. On the contrary, dynamic boundary conditions contain time derivatives of the system state variables and arise in many physical problems (see [17, 19, 32]).

In some physical problems, such as wave propagation through the atmosphere or the ocean, due to stochastic force, uncertain parameters, random sources and random boundary conditions, the realistic models take the random fluctuation into account ([6, 10, 13]). This leads to stochastic nonlinear wave equations, which have drawn quite attentions recently ([7, 10, 11, 15, 23, 24, 26, 43]).

In this paper, we are concerned with the effective, macroscopic dynamics of the following “microscopic” weakly damped stochastic nonlinear wave equation with a random dynamical boundary condition on a domain $D$ perforated with small holes

$$\begin{align*}
\varepsilon^2 \partial_{tt} u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon + u^\varepsilon - f(u^\varepsilon) &= \dot{W}_1 \quad \text{in } D^\varepsilon \times [0, \tau^*), \\
\varepsilon^2 \partial_{tt} \delta^\varepsilon + \partial_t \delta^\varepsilon + \varepsilon^2 \delta^\varepsilon &= -\varepsilon^2 \partial_t u^\varepsilon + \varepsilon^2 \dot{W}_2 \quad \text{on } \partial S^\varepsilon \times [0, \tau^*), \\
\delta^\varepsilon &= 0 \quad \text{on } \partial D \times [0, \tau^*), \\
\varepsilon^2 \partial_t u^\varepsilon &= 0 \quad \text{on } \partial S^\varepsilon \times [0, \tau^*). 
\end{align*}$$

(1.1)

Here $\varepsilon$ is a small positive parameter, and the domain $D^\varepsilon$ is a subset of an open bounded domain $D$ in $\mathbb{R}^3$, obtained by removing $S^\varepsilon$, the collection of small holes of size $\varepsilon$, periodically distributed in $D$. Also, $W_1$ and $W_2$ are two independent Wiener processes. This will be given in details in the next section. The symbol $\tau^*$ denotes a stopping time on $(0, +\infty)$, and $\frac{\partial}{\partial n}$ denotes the unit outer normal derivative on the boundary $\partial S^\varepsilon$. In particular, in this paper we will only concern with the case of the nonlinear term $f(u^\varepsilon) = \sin u^\varepsilon$ (the Sine-Gordon equation).

The system (1.1), when the white noises, $\dot{W}_1, \dot{W}_2$, and the parameter $\varepsilon$ are absent, arises in the modeling of gas dynamics in an open bounded domain $D$, with points on boundary acting like a spring reacting to the excess pressure of the gas (see [16, 25]). In this deterministic case, Beale [3, 4] and Mugnolo [27] established the well-posedness and analyzed some properties of the spectrum in some special cases. Cousin, Frota and Larkin [12] studied the global solvability and
asymptotic behavior. Frigeri [16] considered large time dynamical behavior. Furthermore, for the stochastic system (1.1), when the parameter $\varepsilon$ are absent, Chen and Zhang [7] investigated the long time behavior of the solutions.

Homogenization plays an important role in understanding multiscale phenomena in material science, climate dynamics, chemistry and biology [8, 36]. For the deterministic system defined on heterogeneous media, there have been some relevant works for heat conduction [28, 29, 35] and for wave propagation [9, 37]. Several authors also considered homogenization problems for the random partial differential equations (PDEs with random coefficients) [21, 30] and for the partial differential equations on randomly heterogeneous domains [5, 41, 42]. However, for the stochastic partial differential equations (PDEs with white noises), especially for the stochastic partial differential equations with random dynamical boundary conditions, due to the effect of both nonlinear dynamical boundary condition and the nonclassical fluctuation of driving white noises, the study of stochastic homogenization problem is still in its infancy (see [38, 39]).

Therefore, in this paper, we are especially interested in the stochastic homogenization problem of Equation (1.1). Our aim is to establish the effective macroscopic equation of Equation (1.1). For this purpose, the key step is to verify the compactness of the solutions in some function space for the deterministic systems. But it does not hold for stochastic Equation (1.1). Therefore, we will instead consider the tightness of the distributions of the solutions, so that the effective macroscopic equation is established in the sense of probability distribution. More precisely, we first analyze the microscopic model Equation (1.1) to establish the well-posedness. Since the energy relation of this stochastic system does not directly imply the a priori estimate of the solutions, we then introduce a pseudo energy argument to infer almost sure boundedness of the solutions. Furthermore, we use the a priori estimate to establish the tightness of distribution of the solutions. Finally, we derive the effective homogenized equation in the sense of probability distribution, which is a new stochastic wave equation on a unified domain without small holes but with a static boundary condition. The solutions of the original model Equation (1.1) converge to those of the effective homogenized equation in probability distribution, as the size of small holes $\varepsilon$ diminishes to zero.

This paper is organized as follows. In the next section, we will formulate the basic setup of the homogenization problem. In section 3, we will prove the well-posedness, almost sure boundedness and tightness of distribution of the solutions for the microscopic model Equation (1.1). In section 4, we will derive the effective homogenized equation in probability distribution.
2 Basic setup of the problem

Let the physical medium $D$ be an open bounded domain in $\mathbb{R}^3$ with piece-wise smooth boundary $\partial D$, and let $\varepsilon \in (0, 1)$ be a small real parameter. Denote by $Y = [0, l_1) \times [0, l_2) \times [0, l_3)$ a representative elementary cell in $\mathbb{R}^3$ and let $S$ be an open subset of $Y$ with smooth boundary $\partial S$ such that $S \subset Y$. The elementary cell $Y$ and the small cavity of hole $S$ inside it are used to model small scale obstacles or heterogeneities in a physical medium $D$. Define $\varepsilon S = \{\varepsilon y : y \in S\}$ and $S_{\varepsilon, k} = kl + \varepsilon S$ with $kl = (k_1l_1, k_2l_2, k_3l_3)$ and $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$. Let $S^\varepsilon$ be the set of all the holes contained in $D$, i.e.,

$$S^\varepsilon = \bigcup \{S_{\varepsilon, k} | \overline{S_{\varepsilon, k}} \subset D, \text{ and } k \in \mathbb{Z}^3\}.$$ 

Define $D^\varepsilon = D \setminus S^\varepsilon$. Then $D^\varepsilon$ is a periodically perforated domain with holes of the same size as period $\varepsilon$. Notice that the holes are assumed to have no intersection with the boundary $\partial D$, which implies that $\partial D^\varepsilon = \partial D \cup \partial S^\varepsilon$. See Fig. 1. This assumption is only needed to avoid technicalities and the results of our paper will remain valid without the assumption [2].

![Geometric setup in $\mathbb{R}^3$](image)

In the following, we introduce some other notations. Define $Y^* = Y \setminus S$ and $\nu = \frac{|Y^*|}{|Y|}$, with $|Y|$ and $|Y^*|$ the Lebesgue measure of $Y$ and $Y^*$ respectively. Denote the indicator function $\chi$ as follows

$$\chi(Y) = \begin{cases} 1, & \text{on } Y^*, \\ 0, & \text{on } S, \end{cases} \quad \text{and} \quad \chi(D) = \begin{cases} 1, & \text{on } D^\varepsilon, \\ 0, & \text{on } S^\varepsilon. \end{cases}$$

We also denote $\tilde{u}$ to be the zero extension to the whole domain $D$ for any function $u$ defined on the domain $D^\varepsilon$ as follows

$$\tilde{u} = \begin{cases} u, & \text{on } D^\varepsilon, \\ 0, & \text{on } S^\varepsilon. \end{cases}$$
In addition, let the Wiener processes $W_1(t)$ and $W_2(t)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$, be the two-sided in time with values in $L^2(D)$. Furthermore, assume that $W_1(t)$ and $W_2(t)$ are independent and that their covariance operators, $Q_1$ and $Q_2$, are symmetric nonnegative operators satisfying $TrQ_1 < +\infty$ and $TrQ_2 < +\infty$, respectively. Their expansions are given as follows

$$W_1(t) = \sum_{i=1}^{+\infty} \sqrt{\alpha_{1i}} \beta_{1i} e_i, \quad \text{with} \quad Q_1 e_i = \alpha_{1i} e_i,$$

$$W_2(t) = \sum_{i=1}^{+\infty} \sqrt{\alpha_{2i}} \beta_{2i} e_i, \quad \text{with} \quad Q_2 e_i = \alpha_{2i} e_i,$$

where $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal bases of $L^2(D)$, $\alpha_{1i}$ and $\alpha_{2i}$ are eigenvalues of $Q_1, Q_2$ respectively, and $\{\beta_{1i}\}_{i \in \mathbb{N}}$ and $\{\beta_{2i}\}_{i \in \mathbb{N}}$ are two sequences of mutually independent (two-sided in time) standard scalar Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

### 3 Microscopic Model

Write Equation (1.1) in the Itô form as follows

$$\begin{cases}
    d\epsilon^u = v^\epsilon dt \\
    d\epsilon^v = (\triangle \epsilon^u - u^\epsilon - v^\epsilon + \sin u^\epsilon) dt + dW_1 \\
    d\delta^\epsilon = \theta^\epsilon dt \\
    d\theta^\epsilon = (-\frac{1}{\epsilon} \theta^\epsilon - \delta^\epsilon - v^\epsilon) dt + dW_2 \\
    \epsilon^u(0) = u_0, \epsilon^v(0) = v_0, \delta^\epsilon(0) = \delta_0, \theta^\epsilon(0) = \theta_0,
\end{cases} \quad (3.1)$$

We supplement Equation (3.1) with the initial data

$$\epsilon^u(0) = u_0, \epsilon^v(0) = v_0, \delta^\epsilon(0) = \delta_0, \theta^\epsilon(0) = \theta_0, \quad (3.2)$$

which are $\mathcal{F}_0$-measurable.

Now define

$$A^\epsilon = \begin{pmatrix}
    0 & I & 0 & 0 \\
    \triangle - I & -I & 0 & 0 \\
    0 & 0 & 0 & I \\
    0 & -I & -I & -\frac{1}{\epsilon^2} I
\end{pmatrix}, \quad F^\epsilon(U^\epsilon) = \begin{pmatrix}
    0 & \sin u^\epsilon \\
    0 & 0 \\
    0 & 0 \\
    0 & W_1 \\
    0 & W_2
\end{pmatrix}, \quad W = \begin{pmatrix}
    0 \\
    W_1 \\
    0 \\
    W_2
\end{pmatrix}.$$ 

Let $U^\epsilon := (u^\epsilon, v^\epsilon, \delta^\epsilon, \theta^\epsilon)^T$ be in the space

$$\mathcal{H}_\epsilon := \{U^\epsilon \in H^1_c(D^c) \times L^2(D^c) \times L^2(\partial S^c) \times L^2(\partial S^c) | \frac{\partial u^\epsilon}{\partial n} = \theta^\epsilon \text{ on } \partial S^c\},$$

with

$$\|U^\epsilon\|_{\mathcal{H}_\epsilon}^2 = \|u^\epsilon\|_{H^1_c(D^c)}^2 + \|v\|_{L^2(D^c)}^2 + \|\delta^\epsilon\|_{L^2(\partial S^c)}^2 + \|\theta\|_{L^2(\partial S^c)}^2,$$
where $H^1_\epsilon(D^\epsilon)$ and $L^2_\epsilon(D^\epsilon)$ denote the space $H^1(D^\epsilon)$ and $L^2(D^\epsilon)$ vanishing on $\partial D$, respectively. The superscript “$T$” denotes the transpose for the matrix.

Thus Equation (3.1)-(3.2) can be rewritten as

$$
\begin{cases}
    dU^\epsilon = A^\epsilon U^\epsilon dt + F^\epsilon(U^\epsilon)dt + dW(t), \\
    U^\epsilon(0) = U_0^\epsilon = (u_0, v_0, \delta_0, \theta_0)^T.
\end{cases}
$$

(3.3)

For the Cauchy problem (3.3), it follows from Frigeri [16] that the operator $A^\epsilon$ generates a strongly continuous semigroup $S^\epsilon(t) = \{e^{tA^\epsilon}\}_{t \geq 0}$ on $H^\epsilon$. Then the solution of Equation (3.3) can be written in the mild sense

$$
U^\epsilon(t) = S^\epsilon(t)U^\epsilon(0) + \int_0^t S^\epsilon(t-s)F^\epsilon(U^\epsilon(s))ds + \int_0^t \iota_s S^\epsilon(t-s)dW(s).
$$

(3.4)

Furthermore, the variational formulation is

$$
\begin{align*}
&\int_0^t \int_D u^\epsilon_t \varphi dxdt + \int_0^t \int_D u^\epsilon \varphi dxdt + \int_0^t \int_D \nabla u^\epsilon \nabla \varphi dxdt + \int_0^t \int_{\partial D^\epsilon} u^\epsilon \varphi d\sigma dt \\
&- \int_0^t \int_D \sin u^\epsilon \varphi dxdt + \varepsilon^2 \int_0^t \int_{\partial D^\epsilon_\delta} \varphi d\sigma dt + \varepsilon^2 \int_0^t \int_{\partial D^\epsilon_\delta} \varphi d\sigma dt \\
&= \int_0^t \int_D W_1 \varphi dxdt - \varepsilon^2 \int_0^t \int_{\partial D^\epsilon_\delta} \varphi d\sigma dt + \varepsilon^2 \int_0^t \int_{\partial D^\epsilon_\delta} \varphi d\sigma dt,
\end{align*}
$$

(3.5)

for any $\varphi \in C_0^\infty([0, \tau^*) \times D^\epsilon)$.

**Proposition 3.1 (Local well-posedness)** Let the initial datum $U_0^\epsilon$ be a $F_\epsilon$-measurable random variable with value in $H^\epsilon$. Then the Cauchy problem (3.3) has a unique local mild solution $U^\epsilon(t)$ in $C([0, \tau^*), H^\epsilon)$, where $\tau^*$ is a stopping time depending on $U_0^\epsilon$ and $\omega$. Moreover, the mild solution $U^\epsilon(t)$ is also a weak solution in the following sense

$$
\langle U^\epsilon(t), \phi \rangle_{H^\epsilon} = \langle U^\epsilon(0), \phi \rangle_{H^\epsilon} + \int_0^t \langle A^\epsilon U^\epsilon(s), \phi \rangle_{H^\epsilon} ds + \int_0^t \langle F^\epsilon(U^\epsilon(s)), \phi \rangle_{H^\epsilon} ds + \int_0^t \langle dW(s), \phi \rangle_{H^\epsilon},
$$

(3.6)

for any $t \in [0, \tau^*)$ and $\phi \in H^\epsilon$.

**Proof.** We first define a cut-off function as follows. For any positive parameter $R$, let $\eta_R(\cdot)$ be a positive real valued $C^\infty$-function on $[0, +\infty)$ such that

$$
\eta_R(s) = \begin{cases} 
1, & \text{for } 0 \leq s \leq \frac{R}{2}, \\
\in (0, 1), & \text{for } \frac{R}{2} < s \leq R, \\
0, & \text{for } R < s < +\infty.
\end{cases}
$$

Then the truncated system of the Cauchy problem (3.3) is defined as follows

$$
\begin{cases}
    dU^\epsilon = A^\epsilon U^\epsilon dt + F^\epsilon(U^\epsilon)dt + dW(t), \\
    U^\epsilon(0) = U_0,
\end{cases}
$$

(3.7)

where $F^\epsilon_R(U^\epsilon) = (0, \eta_R(||U^\epsilon||^2_{H^\epsilon}) \sin u^\epsilon, 0, 0)^T$. 

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In the meantime, we easily examine that $F^\varepsilon_R(U^\varepsilon)$ satisfies the sublinear growth and the Lipschitz continuity as in Chen and Zhang \[7\]. Therefore, according to Theorem 7.4 of Da Prato and Zabczyk \[13\], the truncated system (3.7) has a unique mild solution $U^\varepsilon_R(t)$ in $\mathcal{H}_\varepsilon$ for each fixed positive $R$.

Define a stopping time
\[
\tau_R := \inf\{t : \|U^\varepsilon\|_{\mathcal{H}_\varepsilon}^2 > \frac{R}{2}\}. \tag{3.8}
\]
We have $U^\varepsilon(t) = U^\varepsilon_R(t)$ as $t < \tau_R$. Also from Da Prato and Zabczyk \[13\], the path $t \to U^\varepsilon(t)$ is continuous. Let $\tau^* = \lim_{R \to +\infty} \tau_R$. Then $U^\varepsilon(t)$ is the unique local solution of the Cauchy problem (3.3) with lifespan $\tau^*$. Furthermore, applying the stochastic Fubini theorem, it can be verified that the local mild solution is also the weak solution. The proof is complete. \hspace{1cm} \blacksquare

Because the energy relation of this stochastic system does not directly imply the a priori estimate of the solutions, we will introduce a pseudo energy argument (see Chow \[11\] and Chen and Zhang \[7\]) to establish the a priori estimate of the solutions for the Cauchy problem (3.3). Furthermore, applying the a priori estimate, we could obtain the global existence and almost sure boundedness of solutions, which further implies the tightness of distribution of solutions.

For a real parameter $r$ in $(0, 1)$, we define
\[
v^\varepsilon_r = v^\varepsilon + ru^\varepsilon \quad \text{and} \quad \theta^\varepsilon_r = \theta^\varepsilon + r\delta^\varepsilon, \tag{3.9}
\]
with $(u^\varepsilon, v^\varepsilon, \delta^\varepsilon, \theta^\varepsilon)^T$ being the solution of the Cauchy problem (3.1)-(3.2). Then the solution $U^\varepsilon_r = (u^\varepsilon_r, v^\varepsilon_r, \delta^\varepsilon_r, \theta^\varepsilon_r)^T \in \mathcal{H}_\varepsilon$ satisfies the following equation
\[
\begin{cases}
    du^\varepsilon_r = (v^\varepsilon_r - ru^\varepsilon_r)dt & \text{in } D^\varepsilon \times [0, \tau^*), \\
    dv^\varepsilon_r = (\Delta u^\varepsilon_r - (1 - r + r^2)u^\varepsilon_r - (1 - r)v^\varepsilon_r + \sin u^\varepsilon_r)dt + dW_1 & \text{in } D^\varepsilon \times [0, \tau^*), \\
    d\delta^\varepsilon_r = (\theta^\varepsilon_r - r\delta^\varepsilon_r)dt & \text{on } \partial S^\varepsilon \times [0, \tau^*), \\
    d\theta^\varepsilon_r = (-\frac{1}{\varepsilon^2} - r)\theta^\varepsilon_r - (1 - \frac{1}{\varepsilon^2} + r^2)\delta^\varepsilon_r - v^\varepsilon_r + ru^\varepsilon_r)dt + dW_2 & \text{on } \partial S^\varepsilon \times [0, \tau^*), \\
    u^\varepsilon_r = v^\varepsilon_r = 0 & \text{on } \partial D \times [0, \tau^*), \\
    \delta^\varepsilon_r = \frac{\partial u^\varepsilon_r}{\partial n} & \text{on } \partial S^\varepsilon \times [0, \tau^*), \\
    u^\varepsilon_r(0) = u^\varepsilon(0), v^\varepsilon_r(0) = v^\varepsilon_r + ru_0 := v^\varepsilon_r(0) & \text{in } D^\varepsilon, \quad \text{on } \partial S^\varepsilon, \\
    \delta^\varepsilon_r(0) = \delta^\varepsilon(0), \theta^\varepsilon_r(0) = \theta^\varepsilon_r + r\delta_0 := \theta^\varepsilon_r(0) & \text{in } D^\varepsilon, \quad \text{on } \partial S^\varepsilon. \tag{3.10}
\end{cases}
\]

Define the pseudo energy functional $E^\varepsilon_r(t)$ of the Cauchy problem (3.3) as follows
\[
E^\varepsilon_r(t) := \|v^\varepsilon_r(t)\|_{L^2(D^\varepsilon)}^2 + \|\nabla u^\varepsilon_r(t)\|_{L^2(D^\varepsilon)}^2 + (1 - r + r^2)\|u^\varepsilon_r(t)\|_{L^2(D^\varepsilon)}^2 + \|\theta^\varepsilon_r(t)\|_{L^2(\partial S^\varepsilon)}^2 + (1 - \frac{1}{\varepsilon^2} + r^2)\|\delta^\varepsilon_r(t)\|_{L^2(\partial S^\varepsilon)}^2 + 4\|\cos \frac{u^\varepsilon_r(t)}{2}\|_{L^2(D^\varepsilon)}^2 + 2r\langle u^\varepsilon_r(t), \delta^\varepsilon_r(t) \rangle_{L^2(\partial S^\varepsilon)}.
\]

**Proposition 3.2** Let the initial data $U^\varepsilon_r(0)$ be a $\mathcal{F}_0$-measurable random variable in $L^2(\Omega, \mathcal{H}_\varepsilon)$. 

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Then for any time $t \in [0, \tau^*)$, we have

\[
\mathcal{E}_r^\varepsilon(t) = E_r^\varepsilon(0) - \int_0^t \left[ 2(1-r) \|v_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2r\|\nabla v_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2r(1-r^2)\|u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 \right] ds + 2\int_0^t \langle \theta_r^\varepsilon, \delta_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} ds + 2\int_0^t \langle \theta_r^\varepsilon, \theta_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} ds - 2\int_0^t \langle u_r^\varepsilon, \delta_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} ds + 2\int_0^t \langle u_r^\varepsilon, \theta_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} ds + \int_0^t \langle 2v_r^\varepsilon, dW_1(s) \rangle_{L^2(D^\varepsilon)} ds + \int_0^t \langle 2\theta_r^\varepsilon, dW_2(s) \rangle_{L^2(\partial S^\varepsilon)} ds + tTrQ_1 + tTrQ_2.
\]

Moreover,

\[
\mathcal{E}_r^\varepsilon(t) = \mathcal{E}_r^\varepsilon(0) \quad \text{satisfies (3.13)}
\]

for any $t \in [0, \tau^*)$. Put $M(u_r^\varepsilon) := \int_{D^\varepsilon} |u_r^\varepsilon|^2 dx$. Then from Itô formula, we deduce that

\[
M(u_r^\varepsilon(t)) = M(u_r^\varepsilon(0)) + \int_0^t \left[ M'(u_r^\varepsilon) \right]_{L^2(D^\varepsilon)} ds + \frac{1}{2} \int_0^t \left[ M''(u_r^\varepsilon) \right]_{L^2(\partial S^\varepsilon)} ds + \frac{1}{2} \int_0^t \left[ \frac{1}{2} Tr \left[ M''(u_r^\varepsilon) \right] \right]_{L^2(\partial S^\varepsilon)} ds,
\]

with $M'(u_r^\varepsilon) = 2u_r^\varepsilon$ and $M''(u_r^\varepsilon) = 2\varphi$ for any $\varphi \in D^\varepsilon$. After some calculations, we get that

\[
\langle M'(u_r^\varepsilon), (\Delta u_r^\varepsilon - (1-r^2)u_r^\varepsilon - (1-r)v_r^\varepsilon + \sin u_r^\varepsilon) \rangle_{L^2(D^\varepsilon)} = -\frac{d}{ds} \left[ \| \nabla u_r^\varepsilon \|_{L^2(D^\varepsilon)}^2 + (1-r^2)\|u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 4\|\cos \frac{u_r^\varepsilon}{2}\|_{L^2(D^\varepsilon)}^2 \right] -2\|\nabla u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2(1-r^2)\|u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2(1-r)\|v_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2\langle v_r^\varepsilon, \nabla u_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} + 2\langle u_r^\varepsilon, \sin u_r^\varepsilon \rangle_{L^2(D^\varepsilon)}.
\]

It immediately follows from (3.13) and (3.14) that

\[
\|v_r^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 + \|\nabla v_r^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 + (1-r^2)\|u_r^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 + 4\|\cos \frac{u_r^\varepsilon(t)}{2}\|_{L^2(D^\varepsilon)}^2
\]

\[
= \|v_r^\varepsilon(0)\|_{L^2(D^\varepsilon)}^2 + \|\nabla v_r^\varepsilon(0)\|_{L^2(D^\varepsilon)}^2 + (1-r^2)\|u_r^\varepsilon(0)\|_{L^2(D^\varepsilon)}^2 + 4\|\cos \frac{u_r^\varepsilon(0)}{2}\|_{L^2(D^\varepsilon)}^2 + 2\int_0^t \langle 2(1-r)\|v_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2\|\nabla u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 + 2(1-r^2)\|u_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 \rangle ds + 2\int_0^t \langle 2v_r^\varepsilon, \sin u_r^\varepsilon \rangle_{L^2(D^\varepsilon)} ds + \int_0^t \langle 2v_r^\varepsilon, dW_1(s) \rangle_{L^2(D^\varepsilon)} ds + tTrQ_1.
\]

Second, we examine the fourth equation of (3.10) and $M(\theta_r^\varepsilon) := \int_{\partial S^\varepsilon} |\theta_r^\varepsilon|^2 dx$. Note that

\[
M(\theta_r^\varepsilon(t)) = M(\theta_r^\varepsilon(0)) + \int_0^t \left[ M'(\theta_r^\varepsilon) \right]_{L^2(D^\varepsilon)} ds + \frac{1}{2} \int_0^t \left[ M''(\theta_r^\varepsilon) \right]_{L^2(\partial S^\varepsilon)} ds + \frac{1}{2} \int_0^t \left[ \frac{1}{2} Tr \left[ M''(\theta_r^\varepsilon) \right] \right]_{L^2(\partial S^\varepsilon)} ds,
\]

with $M'(\theta_r^\varepsilon) = 2\theta_r^\varepsilon$ and $M''(\theta_r^\varepsilon) = 2\varphi$ for any $\varphi \in L^2(\partial S^\varepsilon)$. After some calculations, we conclude that

\[
\langle M'(\theta_r^\varepsilon), (-\varphi - \frac{\partial u_r^\varepsilon}{\partial n}) \rangle_{L^2(\partial S^\varepsilon)} = -\left[ (1-r^2)\|\delta_r^\varepsilon - \varphi\|_{L^2(D^\varepsilon)}^2 + 2\|\delta_r^\varepsilon\|_{L^2(D^\varepsilon)}^2 - 2(1-r^2)\|\theta_r^\varepsilon\|_{L^2(\partial S^\varepsilon)}^2 \right] -2\langle \partial \theta_r^\varepsilon, \nabla v_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} - 2\langle \delta_r^\varepsilon, v_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)} + 2\langle \theta_r^\varepsilon, v_r^\varepsilon \rangle_{L^2(\partial S^\varepsilon)}.
\]
It follows from (3.16) and (3.17) that
\[ \frac{d}{dt} \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) = \frac{d}{dt} \mathbb{E}(\|u^\varepsilon(0)\|_{L^2(\partial\Omega)}^2) + \int_0^t \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) \, ds \]

Thus, from (3.15) and (3.18), we have
\[ \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) = \mathbb{E}(\|u^\varepsilon(0)\|_{L^2(\partial\Omega)}^2) + \int_0^t \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) \, ds \]

Meanwhile, we note that
\[ \frac{d}{dt} \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) = \int_0^t \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) \, ds \]

which implies that
\[ \frac{d}{dt} \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) = \int_0^t \mathbb{E}(\|u^\varepsilon(t)\|_{L^2(\partial\Omega)}^2) \, ds \]

Then it follows from (3.19) and (3.20) that (3.11) and (3.12) hold.

**Proposition 3.3** Let the initial datum \( U^\varepsilon_0(0) \) be a \( \mathcal{F}_0 \)-measurable random variable in \( L^2(\Omega, \mathcal{H}_\varepsilon) \). Then for any time \( t \in [0, \tau^*] \), and a sufficient small \( r \) in \( (0, 1) \), there exists a positive constant \( C \) such that
\[ \mathbb{E}(\|U^\varepsilon(t)\|_{H_\varepsilon}^2) \leq C \mathbb{E}(\|U^\varepsilon_0(0)\|_{H_\varepsilon}^2) + C \int_0^t [\mathbb{E}(\|U^\varepsilon(s)\|_{H_\varepsilon}^2) \, ds + C[t TrQ_1 + t TrQ_2 + t]. \]

**Proof.** On the one hand, it follows from the Cauchy inequality and the trace inequality that there exists a positive constant \( C_{TI} > 0 \) (here and hereafter \( C_{TI} \) denotes the positive
constant in the trace inequality) such that

\[ 0 \leq r \mathbb{E} \|u^\varepsilon(t)\|_{L^2(\partial S^c)}^2 + 2r \mathbb{E} \langle u^\varepsilon(t), \delta^\varepsilon(t) \rangle_{L^2(\partial S^c)} + r \mathbb{E} \|\delta^\varepsilon(t)\|_{L^2(\partial S^c)}^2 \]

\[ \leq r C_{TT}^2 \mathbb{E} \|u^\varepsilon(t)\|_{H^1(D^c)}^2 + 2r \mathbb{E} \langle u^\varepsilon(t), \delta^\varepsilon(t) \rangle_{L^2(\partial S^c)} + r \mathbb{E} \|\delta^\varepsilon(t)\|_{L^2(\partial S^c)}^2, \]

which implies that

\[ \mathbb{E} \mathcal{E}^\varepsilon(t) \geq \mathbb{E} \|v^\varepsilon(t)\|_{L^2(D^c)}^2 + (1 - r C_{TI}^2) \mathbb{E} \|u^\varepsilon(t)\|_{L^2(D^c)}^2 + (1 - \frac{r}{\varepsilon^2} - r + r^2) \mathbb{E} \|\delta^\varepsilon(t)\|_{L^2(\partial S^c)}^2 \]

\[ \geq \mathbb{E} \|v^\varepsilon(t)\|_{L^2(D^c)}^2 + (1 - r C_{TI}^2) \mathbb{E} \|u^\varepsilon(t)\|_{L^2(D^c)}^2 + (1 - \frac{r}{\varepsilon^2} - r + r^2) \mathbb{E} \|\delta^\varepsilon(t)\|_{L^2(\partial S^c)}^2. \]  \hfill (3.22)

On the other hand, it follows from the Hölder inequality, the Young inequality and the trace inequality that

\[ \mathbb{E} \langle u^\varepsilon, \theta^\varepsilon_r \rangle_{L^2(\partial S^c)} \leq \mathbb{E} \|u^\varepsilon\|_{L^2(\partial S^c)} \cdot \mathbb{E} \|\theta^\varepsilon_r\|_{L^2(\partial S^c)} \]

\[ \leq r \mathbb{E} \|u^\varepsilon\|_{L^2(\partial S^c)}^2 + \frac{1}{4r} \mathbb{E} \|\theta^\varepsilon_r\|_{L^2(\partial S^c)}^2 \]

\[ \leq r C_{TI}^2 \mathbb{E} \|u^\varepsilon\|_{H^1(D^c)}^2 + \frac{1}{4r} \mathbb{E} \|\theta^\varepsilon_r\|_{L^2(\partial S^c)}^2, \]

which implies that

\[ 4r \mathbb{E} \langle u^\varepsilon, \theta^\varepsilon_r \rangle_{L^2(\partial S^c)} \leq 4r^2 C_{TI}^2 \mathbb{E} \|u^\varepsilon\|_{L^2(\partial S^c)}^2 + 4r^2 C_{TI}^2 \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2 + \mathbb{E} \|\theta^\varepsilon_r\|_{L^2(\partial S^c)}^2. \]  \hfill (3.23)

At the same time, it follows from the Cauchy inequality and the trace inequality that

\[ -4r^2 \mathbb{E} \langle u^\varepsilon, \delta^\varepsilon \rangle_{L^2(\partial S^c)} \leq 2r^2 \mathbb{E} \|u^\varepsilon\|_{L^2(\partial S^c)}^2 + 2r^2 \mathbb{E} \|\delta^\varepsilon\|_{L^2(\partial S^c)}^2 \]

\[ \leq 2r^2 C_{TI}^2 \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2 + 2r^2 C_{TI}^2 \mathbb{E} \|\delta^\varepsilon\|_{L^2(D^c)}^2 + 2r^2 \mathbb{E} \|\delta^\varepsilon\|_{L^2(\partial S^c)}^2. \]  \hfill (3.24)

Also it follows from the Cauchy inequality that

\[ 2r \mathbb{E} \langle u^\varepsilon, \sin u^\varepsilon \rangle_{L^2(D^c)} \leq r \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2 + r \mathbb{E} \|\sin u^\varepsilon\|_{L^2(D^c)}^2 \]

\[ \leq r \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2 + C. \]  \hfill (3.25)

Notice that \( \varepsilon \in (0, 1) \). Then it follows from Proposition 3.2 and (3.23)-(3.25) that

\[ \mathbb{E} \mathcal{E}^\varepsilon(t) \leq \mathbb{E} \mathcal{E}^\varepsilon(0) - \int_0^t 2(1 - r) \mathbb{E} \|v^\varepsilon\|_{L^2(D^c)}^2 + 2r(1 - 3r C_{TI}^2) \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2 \]

\[ + r [1 + 2r - 6r C_{TI}^2 + 2r^2 \mathbb{E} \|u^\varepsilon\|_{L^2(D^c)}^2] \]

\[ + (1 - 2r) \mathbb{E} \|\theta^\varepsilon_r\|_{L^2(\partial S^c)}^2 + 2r(1 - r - \frac{r}{\varepsilon^2} + r^2) \mathbb{E} \|\delta^\varepsilon\|_{L^2(\partial S^c)}^2] ds \]

\[ + t TrQ_1 + t TrQ_2 + Ct. \]  \hfill (3.26)

Let \( r \) be sufficient small in \((0, 1)\) such that

\[ \min\{1 - 2r, 1 - 3r C_{TI}^2, 1 - r - \frac{r}{\varepsilon^2} + r^2, 1 - 2r - 6r C_{TI}^2 + 2r^2, 1 - r - r C_{TI}^2 + r^2\} > 0. \]  \hfill (3.27)

Therefore, from (3.22), (3.26) and (3.27), there exists a positive constant \( C \) such that

\[ \mathbb{E} \|v^\varepsilon(t)\|_{L^2(D^c)}^2 \leq C \int_0^t \mathbb{E} \|v^\varepsilon(s)\|_{L^2(D^c)}^2 + \mathbb{E} \|\delta^\varepsilon(s)\|_{L^2(\partial S^c)}^2 \]

\[ + \mathbb{E} \|\theta^\varepsilon_r(s)\|_{L^2(\partial S^c)}^2 + \mathbb{E} \|\sin u^\varepsilon(s)\|_{L^2(D^c)}^2 + \mathbb{E} \|\theta^\varepsilon_r(s)\|_{L^2(\partial S^c)}^2 \]

\[ + \mathbb{E} \|\delta^\varepsilon(s)\|_{L^2(\partial S^c)}^2] ds + C[t TrQ_1 + t TrQ_2 + t], \]
which implies \(3.21\).

**Proposition 3.4** Let the initial datum \(U_0^\varepsilon\) be a \(\mathcal{F}_0\)-measurable random variable in \(L^2(\Omega, \mathcal{H}_\varepsilon)\). Then the solution \(U^\varepsilon(t)\) of the Cauchy problem \((3.3)\) globally exists in \(\mathcal{H}_\varepsilon\), i.e. \(\tau^* = +\infty\) almost surely.

**Proof.** For any given positive \(T_0\), consider the case that \(t < \tau^* \leq T_0\). For any stopping time \(\tau\) satisfying \(\tau < \tau^*\), it follows from Proposition 3.3 and the Gronwall inequality that for arbitrary \(t \leq \tau \land \tau_R\),

\[
\mathbb{E}\|U^\varepsilon_r(t)\|_{\mathcal{H}_\varepsilon}^2 \leq C(T_0, TrQ_1, TrQ_2, \mathbb{E}\|U^\varepsilon(0)\|_{\mathcal{H}_\varepsilon}),
\]

where \(\tau_R\) is defined as \((3.8)\).

Moreover, we note that from Frigeri [16], for \(r \in (0, \frac{1}{2})\), \(\mathbb{E}\|U^\varepsilon_r\|_{\mathcal{H}_\varepsilon}^2 \geq \frac{1}{2} \mathbb{E}\|U^\varepsilon\|_{\mathcal{H}_\varepsilon}^2\). Then take \(r \in (0, \frac{1}{2})\) sufficiently small such that \((3.27)\) holds. Then for arbitrary \(t \leq \tau \land \tau_R\),

\[
\mathbb{E}\|U^\varepsilon_r(t)\|_{\mathcal{H}_\varepsilon}^2 \geq \frac{1}{2} \mathbb{E}\|U^\varepsilon(t)\|_{\mathcal{H}_\varepsilon}^2 \\
\geq C\mathbb{E}\|U^\varepsilon(t)\|_{\mathcal{H}_\varepsilon}^2 \cdot \chi(\{\tau_R \leq T_0\}) \\
\geq C \mathbb{E}\|U^\varepsilon(t)\|_{\mathcal{H}_\varepsilon}^2 \cdot \mathbb{P}\{\tau_R \leq T_0\},
\]

where \(\chi\) is the indicator function.

Therefore, from \((3.28)\) and \((3.29)\), we see that

\[
\mathbb{P}\{\tau_R \leq T_0\} \leq \frac{2C(T_0, TrQ_1, TrQ_2, \mathbb{E}\|U^\varepsilon(0)\|_{\mathcal{H}_\varepsilon})}{CR},
\]

which implies from the Borel-Cantelli lemma that

\[
\mathbb{P}\{\tau^* \leq T_0\} = 0,
\]

where \(\tau^* = \lim_{R \to +\infty} \tau_R\). In other words, we conclude that

\[
\mathbb{P}\{\tau^* = \infty\} = 1.
\]

Therefore the solution \(U^\varepsilon(t)\) of the Cauchy problem \((3.3)\) globally exists almost surely. This completes the proof.

**Proposition 3.5** Let the initial datum \(U_0^\varepsilon\) be a \(\mathcal{F}_0\)-measurable random variable in \(L^2(\Omega, \mathcal{H}_\varepsilon)\). Then the global solution \(U^\varepsilon(t)\) of the Cauchy problem \((3.3)\) is bounded in \(\mathcal{H}_\varepsilon\) almost surely.

**Proof.** From Proposition 3.4, we know that the solution \(U^\varepsilon(t)\) of the Cauchy problem \((3.3)\) globally exists on \([0, +\infty)\) almost surely. Therefore, it follows from Proposition 3.3 that for arbitrary \(t \in [0, +\infty)\),

\[
\frac{d}{dt} \mathbb{E}\|U^\varepsilon_r(t)\|_{\mathcal{H}_\varepsilon}^2 + C\mathbb{E}\|U^\varepsilon_r(t)\|_{\mathcal{H}_\varepsilon}^2 \leq C[TrQ_1 + TrQ_2 + 1],
\]

where \(C\) is a constant.
which immediately implies from the Gronwall inequality that

$$\mathbb{E}\|U_ρ^ε(t)\|_{\mathcal{H}_ε}^2 \leq \mathbb{E}\|U_ρ^ε(0)\|_{\mathcal{H}_ε}^2 e^{-Ct} + [TrQ_1 + TrQ_2 + 1](1 - e^{-Ct}).$$  \quad (3.33)

Note that for $r \in (0, \frac{1}{2})$, $\mathbb{E}\|U_ρ^ε\|_{\mathcal{H}_ε}^2 \geq \frac{1}{2}\mathbb{E}\|U_ρ^ε\|_{\mathcal{H}_ε}^2$. Thus we take $r \in (0, \frac{1}{2})$ sufficiently small such that (3.27) holds. It then follows from (3.33) that Proposition 3.5 holds.

Introduce a space

$$\Sigma_ε := \{U^ε \in H^2_ε(D^ε) \times H^1_ε(D^ε) \times H^1(\partial S^ε) \times H^1(\partial S^ε) : \frac{\partial u^ε}{\partial n} = \theta^ε \text{ on } \partial S^ε\},$$

where $H^2_ε(D^ε)$ and $H^1_ε(D^ε)$ denote the space $H^2(D^ε)$ and $H^1(D^ε)$ vanishing on $\partial D$, respectively.

**Proposition 3.6**  Let the initial datum $U_0^ε$ be a $\mathcal{F}_0$-measurable random variable in $L^2(\Omega, \Sigma_ε)$. Then the global solution $U^ε(t)$ of the Cauchy problem (3.5) is also bounded in $\Sigma_ε$ almost surely.

The proof of Proposition 3.6 is similar as Proposition 3.2, Proposition 3.3 and Proposition 3.5. It is omitted here.

In the following, for any $T > 0$, we consider the solution $(u^ε, v^ε)^T \in L^2(0, T; H^1_ε(D^ε) \times L^2_ε(D^ε))$ of Equation (3.1). Set

$$\mathcal{X} := H^1(D) \times L^2(D), \quad \mathcal{Y} := L^2(D) \times L^2(D), \quad \mathcal{Z} := H^{-1}(D) \times L^2(D).$$

We investigate the behavior of distribution of $(u^ε, v^ε)^T \in L^2(0, T; L^2_ε(D^ε) \times L^2_ε(D^ε))$ as $ε \to 0$, which needs the tightness of distribution (see [14]). Notice that the function space changes with $ε$, which is a difficulty for obtaining the tightness of distributions. Thus we will treat $\{\mathcal{L}((u^ε, v^ε)^T)\}_{ε>0}$ as a collection of distributions on $L^2((0, T), \mathcal{Y})$ by extending $(u^ε, v^ε)^T$ to the whole domain $D$, whose distribution is defined as $\mathcal{L}((\tilde{u}^ε, \tilde{v}^ε)^T)(A) = \mathbb{P}\{ω : (\tilde{u}^ε(\cdot, \cdot, ω), \tilde{v}^ε(\cdot, \cdot, ω))^T \in A\}$ for the Borel set $A \in L^2((0, T), \mathcal{Y})$.

**Proposition 3.7 (Tightness of distribution)**  Let the initial datum $U_0^ε$ be a $\mathcal{F}_0$-measurable random variable in $L^2(\Omega, \mathcal{H}_ε)$, which is independent of $W(t)$ with $\mathbb{E}\|U_0^ε\|_{\mathcal{H}_ε}^2 < \infty$. Then for any $T > 0$, $\mathcal{L}((u^ε, v^ε)^T)$, the distribution of $(u^ε, v^ε)^T$, is tight in $L^2((0, T), \mathcal{Y}) \cap C((0, T), \mathcal{Z})$.

**Proof.**  Firstly, we claim that $(u^ε, v^ε)^T$ is bounded almost surely in

$$G := L^2(0, T; \mathcal{X}) \cap (W^{1,2}(0, T; \mathcal{Z}) + W^{α,4}(0, T; \mathcal{Y})),
$$

where $W^{1,2}(0, T; \mathcal{Z})$ is a Banach space endowed with the norm

$$\|\varphi\|_{W^{1,2}(0, T; \mathcal{Z})}^2 = \|\varphi\|_{L^2(0, T; \mathcal{Z})}^2 + \|\frac{d\varphi}{dt}\|_{L^2(0, T; \mathcal{Z})}^2 < \infty, \quad \forall \ \varphi \in W^{1,2}(0, T; \mathcal{Z}),$$

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and $W^{\alpha,4}(0, T; \mathcal{Y})$ is another Banach space with $\alpha \in (\frac{1}{4}, \frac{1}{2})$ endowed with the norm

$$
\|\varphi\|_{W^{\alpha,4}(0, T; \mathcal{Y})}^4 = \|\varphi\|_{L^4(0, T; \mathcal{Y})}^4 + \int_0^T \int_0^T \frac{\|\varphi(t) - \varphi(s)\|^4_{\mathcal{Y}}}{|t - s|^{1+4\alpha}} \, ds dt < \infty, \quad \forall \ \varphi \in W^{\alpha,4}(0, T; \mathcal{Y}).
$$

By Proposition 3.5, we know that $(u^\varepsilon, v^\varepsilon)^T$ is bounded in $L^2(0, T; \mathcal{X})$ almost surely. Therefore, in the following, we only need to prove that $(u^\varepsilon, v^\varepsilon)^T$ is bounded in $W^{1,2}(0, T; \mathcal{Z}) + W^{\alpha,4}(0, T; \mathcal{Y})$ almost surely.

Denote by $P$ the projection operator from $U^\varepsilon$ to $(u^\varepsilon, v^\varepsilon)^T$, i.e., $PU^\varepsilon = (u^\varepsilon, v^\varepsilon)^T$. Write Equation (3.34) as

$$
U^\varepsilon(t) = U^\varepsilon(0) + \int_0^t A^\varepsilon U^\varepsilon(\tau) d\tau + \int_0^t F^\varepsilon(U^\varepsilon(\tau)) d\tau + \int_0^t dW(\tau).
$$

Then

$$
PU^\varepsilon(t) = PU^\varepsilon(0) + \int_0^t [PA^\varepsilon U^\varepsilon(\tau) + PF^\varepsilon(U^\varepsilon(\tau))] d\tau + \int_0^t P dW(\tau). \quad (3.34)
$$

Denote

$$
I_1 := \int_0^t [PA^\varepsilon U^\varepsilon(\tau) + PF^\varepsilon(U^\varepsilon(\tau))] d\tau,
$$

and

$$
I_2 := \int_0^t P dW(\tau). \quad (3.36)
$$

For $I_1$, it follows from Proposition 3.5 and Proposition 3.6 that

$$
\mathbb{E}\|I_1\|_{L^2(0, T; \mathcal{Z})}^2 \\
= \mathbb{E}\int_0^T \|I_1(\tau)\|_2^2 d\tau \\
= \mathbb{E}\int_0^T \|PA^\varepsilon U^\varepsilon(\tau) + PF^\varepsilon(U^\varepsilon(\tau))\|_2^2 d\tau \\
\leq \mathbb{E}\int_0^T \left[ \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \right] d\tau \\
\leq C_T,
$$

and

$$
\mathbb{E}\|\frac{dI_1}{d\tau}\|_{L^2(0, T; \mathcal{Z})}^2 \\
= \mathbb{E}\int_0^T \|\frac{dI_1}{d\tau}\|_2^2 d\tau \\
= \mathbb{E}\int_0^T \|PA^\varepsilon U^\varepsilon(\tau) + PF^\varepsilon(U^\varepsilon(\tau))\|_2^2 d\tau \\
\leq \mathbb{E}\int_0^T \left[ \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \|\Delta u^\varepsilon(\tau) - u^\varepsilon(\tau) - v^\varepsilon(\tau) + \sin u^\varepsilon(\tau)\|_2^2 \right] d\tau \\
\leq C_T.
$$

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Here and hereafter, $C_T$ denotes various positive constants depending on the given $T > 0$. Then combining (3.37) and (3.38), we deduce that

$$\mathbb{E}\|I_1\|_{L^{2}(0,T;\mathbb{Z})}^{2} = \mathbb{E}\|I_1\|_{L^{2}(0,T;\mathbb{Z})}^{2} + \mathbb{E}\frac{dI_1}{dt}\|I_1\|_{L^{2}(0,T;\mathbb{Z})}^{2} \leq C_T. \quad (3.39)$$

Now we consider $I_2$. Put $M^\varepsilon(s,t) = \int_s^t dW(\tau)$. Then using the Burkholder-Davis-Gundy inequality and the Hölder inequality, we have

$$\mathbb{E}\|PM^\varepsilon(s,t)\|_{L^2(0,T;Y)}^4 = \mathbb{E}\left|\int_s^t PdW(\tau)\right|_{L^2(D)}^4 \leq C\mathbb{E}(\int_s^t TrQ_1 d\tau)^2 \leq C\mathbb{E}(\int_s^t 12^2 d\tau \cdot \int_s^t (TrQ_1)^2 d\tau) \leq C(t-s)^2. \quad (3.40)$$

Thus, it follows from (3.40) that

$$\mathbb{E}\|I_2\|_{L^4(0,T;Y)}^4 = \mathbb{E}\int_s^t |I_2(t)|_{L^4(0,T;Y)}^4 dt = \mathbb{E}\int_s^t \int_s^t PdW(\tau)\|_{L^2(D)}^4 dt = \mathbb{E}\int_s^t \|PM^\varepsilon(0,t)\|_{L^2(D)}^4 dt \leq \int_s^t C(t-s)^2 dt \leq C_T. \quad (3.41)$$

Also, for $\alpha \in (\frac{1}{3}, \frac{1}{2})$, by (3.40), we have

$$\mathbb{E}\int_s^t \int_0^T \frac{|I_2(t) - I_2(s)|_{L^4(0,T;Y)}^4}{|t-s|^{1+4\alpha}} ds dt = \mathbb{E}\int_s^t \int_0^T \frac{|PM^\varepsilon(0,t) - PM^\varepsilon(0,s)|_{L^2(0,T;Y)}^4}{|t-s|^{1+4\alpha}} ds dt \leq \int_s^t \int_0^T \frac{C(t-s)^2}{|t-s|^{1+4\alpha}} ds dt \leq C\int_s^t \int_0^T \frac{(t-s)^{1-4\alpha}}{ds dt} \leq C_T. \quad (3.42)$$

Thus, it follows from (3.41) and (3.42) that for arbitrary $\alpha \in (\frac{1}{3}, \frac{1}{2})$,

$$\mathbb{E}\|I_2\|_{W^{\alpha,4}(0,T;Y)}^4 = \mathbb{E}\|I_2\|_{L^4(0,T;Y)}^4 + \mathbb{E}\int_s^t \int_0^T \frac{|I_2(t) - I_2(s)|_{L^4(0,T;Y)}^4}{|t-s|^{1+4\alpha}} ds dt < C_T. \quad (3.43)$$

Immediately from (3.34)–(3.36), (3.39) and (3.43), we obtain that $(u^\varepsilon, v^\varepsilon)^T$ is bounded in $W^{1,2}(0,T;\mathbb{Z}) + W^{\alpha,4}(0,T;\mathbb{Y})$ almost surely, which completes the verification of the claim that $(u^\varepsilon, v^\varepsilon)^T$ is bounded almost surely in $G = L^2(0,T;\mathcal{X}) \cap (W^{1,2}(0,T;\mathbb{Z}) + W^{\alpha,4}(0,T;\mathbb{Y}))$.

By the Chebyshev inequality, we see that for any $\rho > 0$, there exists a bounded set $K_\rho \subset G$ such that $\mathbb{P}\{(u^\varepsilon, v^\varepsilon)^T \in K_\rho\} > 1 - \rho$. Moreover, notice that

$$L^2(0,T;\mathcal{X}) \cap W^{1,2}(0,T;\mathbb{Z}) \subset L^2(0,T;\mathbb{Y}) \cap C(0,T;\mathbb{Z}),$$
and for $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$,
\[
L^2(0, T; \mathcal{X}) \cap W^{\alpha,4}(0, T; \mathcal{Y}) \subset L^2(0, T; \mathcal{Y}) \cap C(0, T; \mathcal{Z}).
\]
We conclude that $K_\rho$ is compact in $L^2(0, T; \mathcal{Y}) \cap C(0, T; \mathcal{Z})$. Thus $\mathcal{L}((u^\varepsilon, v^\varepsilon)^T)$ is tight in $L^2(0, T; \mathcal{Y}) \cap C(0, T; \mathcal{Z})$.

4 Effective Model

In this section, we will use the two-scale method to derive the effective homogenized equation of Equation (1.1), in the sense of probability distribution. The solutions of the microscopic model Equation (1.1) converge to those of the effective homogenized equation in probability distribution, as the size of small holes $\varepsilon$ diminishes to zero. The main result is as follows.

**Theorem 4.1 (Homogenized model)** Let $(u^\varepsilon, \delta^\varepsilon)^T$ be the solution of Equation (1.1). Then for any $T > 0$, the distribution $\mathcal{L}(\tilde{u}^\varepsilon)$ converges weakly to $\mu$ in $L^2(0, T; L^2(D))$ as $\varepsilon \to 0$, with $\mu$ being the distribution of the solution $V$ of the following homogenized equation

\[
\begin{cases}
V_t(t, x) + V_i(t, x) - \nu^{-1}div_x A^*(\nabla_x V(t, x)) + V(t, x) - \sin(V(t, x)) = \nu \dot{W}_1, & \text{on } D \\
V(t, x) = 0, & \text{on } \partial D \\
V(0, x) = \frac{u_0}{\nu}, & V_i(0, x) = \frac{v_0}{\nu}, & \text{on } D,
\end{cases}
\]

(4.1)

where the effective matrix $A^* = (A^*_{ij})$ given by (4.21), $u_0$ and $v_0$ are the initial data supplemented in Equation (3.2), and the constant $\nu = \frac{|Y^*|}{|Y|}$. with $|Y|$ and $|Y^*|$ the Lebesgue measure of $Y$ and $Y^*$ respectively.

In the following, we will prove Theorem 4.1. We first provide some preliminaries. We will denote by $C^\infty_{\text{per}}(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^3$ that are periodic in $Y$. We also denote $L^2_{\text{per}}(Y)$ or $H^1_{\text{per}}(Y)$ the completion of $C^\infty_{\text{per}}(Y)$ in the usual norm of $L^2(Y)$ or $H^1(Y)$, respectively. In addition, we denote $D_T = [0, T] \times D$.

**Definition 4.1** A sequence of functions $u^\varepsilon(t, x)$ in $L^2(D_T)$ is called to be two-scale convergent to a limit $u(t, x, y) \in L^2(D_T \times Y)$, if for any function $\varphi(x, y) \in C^\infty_0(D_T; C^\infty_{\text{per}})$,

\[
\lim_{\varepsilon \to 0} \int_{D_T} u^\varepsilon(t, x) \varphi(t, x, \frac{x}{\varepsilon}) dx dt = \frac{1}{|Y|} \int_D \int_Y u(t, x, y) \varphi(t, x, y) dy dx dt,
\]

which is denoted by $u^\varepsilon \overset{2-s}{\rightharpoonup} u$.

**Lemma 4.1** Let $u^\varepsilon$ be a bounded sequence in $L^2(D_T)$. Then there exists a function $u \in L^2(D_T \times Y)$ and a subsequence $u_{\varepsilon_k} \to 0$ as $k \to \infty$ such that $u_{\varepsilon_k} \overset{2-s}{\rightharpoonup} u$. 

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Lemma 4.2[1] If \( u^\varepsilon \xrightarrow{2-\varepsilon} u \), then \( u^\varepsilon \rightharpoonup \bar{u}(t,x) = \frac{1}{|\Omega|} \int_Y u(t,x,y)dy \).

Lemma 4.3[1] Let \( v^\varepsilon \) be a sequence in \( L^2(D_T) \) that two-scale converges to a limit \( v(x,y) \in L^2(D_T \times Y) \). Further assume that

\[
\lim_{\varepsilon \to 0} \int_{D_T} |v^\varepsilon(t,x)|^2 dxdt = \frac{1}{|Y|} \int_{D_T} \int_Y |v(t,x,y)|^2 dydxdt.
\]

Then for any sequence \( u^\varepsilon \in L^2(D_T) \), which two-scale converges to a limit \( u \in L^2(D_T \times Y) \), we have

\[
u^\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y u(\cdot,\cdot,y)v(\cdot,\cdot,y)dy, \text{ as } \varepsilon \to 0 \text{ in } L^2(D_T).
\]

Lemma 4.4[1] Let \( u^\varepsilon \) be a sequence of functions defined on \([0,T] \times D^\varepsilon\) which is bounded in \( L^2(0,T;H^1_c(D^\varepsilon)) \). There exists \( u(t,x) \in H^1_0(D_T), u_1(t,x,y) \in L^2(D_T;H^1_{per}(Y)) \) and a subsequence \( u^{\varepsilon_k} \) with \( \varepsilon_k \to 0 \) as \( k \to \infty \), such that

\[
\tilde{u}^{\varepsilon_k}(t,x) \xrightarrow{2-\varepsilon} \chi(Y)u(t,x), \quad k \to \infty,
\]

and

\[
\nabla_x u^{\varepsilon_k} \xrightarrow{2-\varepsilon} \chi(Y)[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)], \quad k \to \infty,
\]

where \( \chi(Y) \) is the indicator function as defined in Section 2.

For \( h \in H^{-1/2}(\partial S) \) and \( Y \)-periodic, define \( \lambda_h := \frac{1}{|\Omega|} \int_{\partial S} h(x)dx \). Also, for \( h \in L^2(\partial S) \) and \( Y \)-periodic, define \( \lambda^\varepsilon_h \in H^{-1}(D) \) as \( \langle \lambda^\varepsilon_h, \varphi \rangle_{H^{-1},H^1_0} = \varepsilon \int_{\partial S^e} \frac{h}{\varepsilon}(x)\varphi(x)dx \) with any \( \varphi \in H^1_0(D) \).

Lemma 4.5[13] Let \( \varphi^\varepsilon \) be a sequence in \( H^1_0(D) \) such that \( \varphi^\varepsilon \rightharpoonup \varphi \) in \( H^1_0(D) \) as \( \varepsilon \to 0 \). Then

\[
\langle \lambda^\varepsilon_h, \varphi^\varepsilon \rangle_{D^\varepsilon} \longrightarrow \lambda_h \int_D \varphi dx, \quad \varepsilon \to 0.
\]

Lemma 4.6 (Prohorov Theorem)[13] Suppose \( \mathcal{M} \) is a separable Banach space. The set of probability measures \( \{\mathcal{L}(X_n)\}_n \) on \( (\mathcal{M},\mathcal{B}(\mathcal{M})) \) is relatively compact if and only if \( \{X_n\} \) is tight.

Lemma 4.7 (Skorohod Theorem)[13] For an arbitrary sequence of Probability measures \( \{\mu_n\} \) on \( (\mathcal{M},\mathcal{B}(\mathcal{M})) \) weakly converges to probability measures \( \mu \), there exists a probability space \((\Omega,\mathcal{F},\mathbb{P})\) and random variables, \( X, X_1, X_2, \ldots, X_n, \ldots \) such that \( X_n \) distributes as \( \mu_n \) and \( X \) distributes as \( \mu \) and \( \lim_{n \to \infty} X_n = X, \mathbb{P}\)-a.s.
Proof of Theorem 4.1.

Let \((u^\varepsilon, v^\varepsilon)^T\) be the solution of Equation (1.1). On the one hand, as in [39], by the proof of Proposition 3.7, for any \(\rho > 0\), there is a bounded set \(K_\rho \subset G\) which is compact in \(L^2(0, T; \mathcal{Y})\) such that \(\mathbb{P}\{(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)^T \in K_\rho\} > 1 - \rho\). According to Lemma 4.6 and Lemma 4.7, we know that for any sequence \(\{\varepsilon_j\}_{j=1}^{\infty}\) with \(\varepsilon_j \to 0\) as \(j \to \infty\), there exists a subsequence \(\{\varepsilon_{j(k)}\}_{k=1}^{\infty}\), random variables \(\{(\tilde{u}_{*}^{\varepsilon_{j(k)}}, \tilde{v}_{*}^{\varepsilon_{j(k)}})^T\} \subset L^2(0, T; L^2_\varepsilon(D^\varepsilon) \times L^2_\varepsilon(D^\varepsilon))\) and \((u_*, v_*)^T \in L^2(0, T; \mathcal{Y})\) defined on a new probability space \((\Omega_*, \mathcal{F}_*, \mathbb{P}_*)\), such that for almost all \(\omega \in \Omega_*\),

\[
\mathcal{L}((\tilde{u}_{*}^{\varepsilon_{j(k)}}, \tilde{v}_{*}^{\varepsilon_{j(k)}})^T) = \mathcal{L}((\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)^T),
\]

and

\[
(\tilde{u}_{*}^{\varepsilon_{j(k)}}, \tilde{v}_{*}^{\varepsilon_{j(k)}})^T \rightarrow (u_*, v_*)^T \text{ in } L^2(0, T; \mathcal{Y}) \text{ as } k \to \infty. \quad (4.2)
\]

In the meantime, \((\tilde{u}_{*}^{\varepsilon_{j(k)}}, \tilde{v}_{*}^{\varepsilon_{j(k)}})^T\) solves

\[
\begin{cases}
    dP(U^\varepsilon) = PA^\varepsilon U^\varepsilon dt + PF^\varepsilon(U^\varepsilon) dt + PdW(t), \\
    PU^\varepsilon(0) = PU_0,
\end{cases}
\]

with \(W\) being replaced by a Wiener process \(W_*\), defined on the probability space \((\Omega_*, \mathcal{F}_*, \mathbb{P}_*)\) but with the same distributions as \(W\). Here \(P\) is the projection operator from \(U^\varepsilon\) to \((u^\varepsilon, v^\varepsilon)^T\) as defined in the proof of Proposition 3.7.

On the other hand, for \(u^\varepsilon\) in the set \(K_\rho\), it follows from Lemma 4.1 and Lemma 4.4 that there exist \(u(t, x) \in H^1_0(D_T)\) and \(u_1(t, x, y) \in L^2(D_T, H^1_{\text{per}}(Y))\) such that

\[
\tilde{u}^\varepsilon_k(t, x) \xrightarrow{2-s} \chi(Y) u(t, x), \quad k \to \infty, \quad (4.3)
\]

and

\[
\frac{\nabla_x u^\varepsilon_k}{\nabla_x u^\varepsilon_k} \xrightarrow{2-s} \chi(Y)[\nabla_x u(t, x) + \nabla_y u_1(t, x, y)], \quad k \to \infty. \quad (4.4)
\]

Furthermore, from Lemma 4.2, it follows that

\[
\tilde{u}^\varepsilon_k(t, x) \sim \frac{1}{|Y|} \int_Y \chi(Y) u(t, x) dy = \frac{1}{|Y|} \int_Y \chi(Y) dy \cdot u(t, x) = \nu u(t, x), \text{ in } L^2(D_T),
\]

which from the compactness of \(K_\rho\) immediately implies that

\[
\tilde{u}^\varepsilon_k(t, x) \rightarrow \nu u(t, x), \quad k \to \infty, \quad \text{in } L^2(D_T). \quad (4.5)
\]

Then combining the relationship of \(u^\varepsilon\) and \(v^\varepsilon\), (4.2) and (4.5), we have

\[
u_* = \nu u \quad \text{and} \quad v_* = \nu u_t. \quad (4.6)
\]
Now, in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we put \(\Omega_\rho = \{\omega \in \Omega : \bar{u}^\varepsilon(\omega) \in K_\rho\}\), \(\mathcal{F}_\rho = \{F \cap \Omega_\rho : F \in \mathcal{F}\}\), and \(\mathbb{P}_\rho(F) = \frac{\mathbb{P}(F \cap \Omega_\rho)}{\mathbb{P}(\Omega_\rho)}\), for \(F \in \mathcal{F}_\rho\). Then \((\Omega_\rho, \mathcal{F}_\rho, \mathbb{P}_\rho)\) forms a new probability space, whose expectation operator is denoted by \(\mathbb{E}_\rho\). In the following, we will work in the probability space \((\Omega_\rho, \mathcal{F}_\rho, \mathbb{P}_\rho)\) in stead of \((\Omega, \mathcal{F}, \mathbb{P})\).

In Equation \((3.5)\), we choose the test function \(\varphi\) as \(\varphi^\varepsilon(t, x) = \phi(t, x) + \varepsilon \Phi(t, x, \bar{y})\) with \(\phi(t, x) \in C^{\infty}_0(D_T)\) and \(\Phi(t, x, y) \in C^{\infty}_0(D_T; C^{\infty}_{per}(Y))\). Also, we notice that \((4.5)\) and \(\chi(D^\varepsilon) \to \nu\) in \(L^\infty(D)\). Then we have

\[
\int_0^T \int_{D_T} u_t^\varepsilon \varphi^\varepsilon dxdt = \int_0^T \int_{D_T} u_t^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= -\int_0^T \int_{D_T} u_t^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= -\int_0^T \int_{D_T} u_t^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= \int_0^T \int_{D_T} \Phi(u_t(t, x)) \phi(t, x) dxdt, \quad \varepsilon \to 0, \tag{4.7}
\]

and

\[
\int_0^T \int_{D_T} u_x^\varepsilon \varphi^\varepsilon dxdt = \int_0^T \int_{D_T} u_x^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= -\int_0^T \int_{D_T} u_x^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= -\int_0^T \int_{D_T} u_x^\varepsilon \phi(t, x) + \varepsilon \Phi(t, x, \bar{y}) dxdt \\
= \int_0^T \int_{D_T} \Phi(u_x(t, x)) \phi(t, x) dxdt, \quad \varepsilon \to 0, \tag{4.8}
\]

For \(\varphi^\varepsilon\), we have

\[
\nabla_x \varphi^\varepsilon = \nabla_x \phi(t, x) + \varepsilon \nabla_x \Phi(t, x, \bar{y}) \\
= \nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y) \\
\stackrel{2-s}{\rightarrow} \nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y), \quad \varepsilon \to 0, \tag{4.10}
\]

and

\[
\lim_{\varepsilon \to 0} \int_{D_T} |\nabla_x \varphi^\varepsilon|^2 dxdt = \lim_{\varepsilon \to 0} \int_{D_T} |\nabla_x \phi(t, x) + \varepsilon \nabla_x \Phi(t, x, \bar{y})|^2 dxdt \\
= \frac{1}{|Y|} \int_{D_T} \int_Y |\nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y)|^2 dy dx dt. \tag{4.11}
\]

Then it follows from \((4.4), (4.10), (4.11)\) and Lemma 4.3 that

\[
\int_0^T \int_{D_T} \nabla u^\varepsilon \cdot \nabla \varphi^\varepsilon dxdt \\
= \int_0^T \int_{D_T} \nabla u^\varepsilon \cdot \nabla (\phi(t, x) + \varepsilon \Phi(t, x, \bar{y})) dxdt \\
= \int_0^T \int_{D_T} \nabla u^\varepsilon \cdot [\nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y)] dxdt \\
= \int_0^T \int_{D_T} \nabla u^\varepsilon \cdot [\nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y)] dxdt, \quad \varepsilon \to 0. \tag{4.12}
\]
From (4.5) and note that \( \sin u \) is continuous and satisfies the global Lipshitz condition with respective to \( u \), we have

\[
\int_0^T \int_{D^c} \sin \kappa \varphi \, dx dt = \int_0^T \int_{D^c} \sin u [\phi(t, x) + \varepsilon \Phi(t, x, \frac{\tau}{\varepsilon})] \, dx dt \\
= \int_0^T \int_{D} [\sin \kappa \phi(t, x) + \varepsilon \sin \kappa \Phi(t, x, \frac{\tau}{\varepsilon})] \, dx dt \\
\to \int_0^T \int_{D} \sin (\nu u(t, x)) \phi(t, x) \, dx dt, \quad \text{as} \quad \varepsilon \to 0. \tag{4.13}
\]

Also realize that

\[
\int_0^T \int_{D^c} W_1 \varphi \, dx dt = \int_0^T \int_{D^c} \tilde{W}_1 [\phi(t, x) + \varepsilon \Phi(t, x, \frac{\tau}{\varepsilon})] \, dx dt \\
= \int_0^T \int_D [\chi(D^c) \phi(t, x) + \varepsilon \chi(D^c) \Phi(t, x, \frac{\tau}{\varepsilon})] \, dx d\tilde{W}_1(t) \\
\to \int_0^T \int_D \nu \varphi(t, x) \, dx dW_1(t), \quad \text{as} \quad \varepsilon \to 0. \tag{4.14}
\]

Moreover, from Proposition 3.6 and Lemma 4.4, we have

\[
\varepsilon^2 \int_0^T \int_{\partial S^c} \delta u \varphi \, dx dt = -\varepsilon^2 \int_0^T \int_{\partial S^c} \delta \kappa \varphi \, dx dt \\
= -\varepsilon \delta \int_{\partial S} \varepsilon \cdot \int_0^T \theta \varphi \, dx dt \\
= -\varepsilon \langle \lambda^\varepsilon, \int_0^T \theta \varphi \, dx dt \rangle_{D^c} \to 0, \quad \text{as} \quad \varepsilon \to 0, \tag{4.15}
\]

\[
\varepsilon^2 \int_0^T \int_{\partial S^c} \delta \kappa \varphi \, dx dt = \varepsilon \int_{\partial S^c} \varepsilon \cdot \int_0^T \delta \kappa \varphi \, dx dt \\
= \varepsilon \langle \lambda^\varepsilon, \int_0^T \delta \kappa \varphi \, dx dt \rangle_{D^c} \to 0, \quad \text{as} \quad \varepsilon \to 0, \tag{4.16}
\]

\[
\varepsilon^2 \int_0^T \int_{\partial S^c} u_t \varphi \, dx dt = -\varepsilon \int_{\partial S^c} \varepsilon \cdot \int_0^T u \varphi \, dx dt \\
= -\varepsilon \langle \lambda^\varepsilon, \int_0^T u \varphi \, dx dt \rangle_{D^c} \to 0, \quad \text{as} \quad \varepsilon \to 0, \tag{4.17}
\]

and

\[
\varepsilon^2 \int_0^T \int_{\partial S^c} W_2 \varphi \, dx dt = \varepsilon \int_{\partial S^c} \varepsilon \cdot \int_0^T \varphi \, dx dt \\
= \varepsilon \langle \lambda^\varepsilon, \int_0^T \varphi \, dx dt \rangle_{D^c} \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.18}
\]

Therefore, from (3.5), (4.7)-(4.9), (4.12)-(4.18), as \( \varepsilon \to 0 \), we have

\[
\int_0^T \int_D \nu u_t(t, x) \phi(t, x) \, dx dt + \int_0^T \int_D \nu u_t(t, x) \phi(t, x) \, dx dt \\
+ \frac{1}{|Y|} \int_0^T \int_D \int_{Y^*} [\nabla_x u(t, x) + \nabla_x u_1(t, x, y)][\nabla_x \phi(t, x) + \nabla_y \Phi(t, x, y)] \, dy \, dx dt \\
+ \int_0^T \int_D \nu u(t, x) \phi(t, x) \, dx dt - \int_0^T \int_D \sin (\nu u(t, x)) \phi(t, x) \, dx dt \\
= \int_0^T \int_D \nu \varphi(t, x) \, dx dW_1(t),
\]

which implies that

\[
\frac{\nu u_t(t, x) + \nu u_t(t, x)}{\partial (\nabla_x u(t, x) + \nabla_x u_1(t, x, y))} = 0, \quad \text{on} \quad \partial Y^* - \partial Y, \tag{4.19}
\]

where \( \mathbf{m} \) is the unit exterior norm vector on \( \partial Y^* - \partial Y \) and

\[
A(\nabla_x u(t, x)) = \frac{1}{|Y|} \int_{Y^*} [\nabla_x u(t, x) + \nabla_y u_1(t, x, y)] \, dy,
\]

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with $u_1$ satisfying for any $\Psi \in H^1_0(D_T; H^1_{per}(Y))$,
\[
\begin{aligned}
\int_{Y^*}[\nabla_x u(t,x) + \nabla_y u_1(t,x,y)] \nabla_y \Psi \, dy &= 0, \\
\text{$u_1$ is $Y$-periodic.}
\end{aligned}
\] (4.20)

Especially notice that Equation (4.20) has a unique solution for any given $u(t,x)$, which implies that $A(\nabla_x u(t,x))$ is well-defined. Please refer to [18] about the further properties of $A(\nabla_x u(t,x))$. Furthermore, from the classic theory of stochastic partial differential equation, the problem (4.19) is well-posed.

In addition, from the classical homogenization theory (see [8, 18]), we have
\[
u(t,x,y) = \sum_{i=1}^3 \frac{\partial u(t,x)}{\partial x_i}(w_i(y) - e_i(y))
\]
where $\{e_i\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$ and $w_i$ is the solution of the following elementary cell problem
\[
\begin{aligned}
\triangle_y w_i(y) &= 0, \quad \text{in } Y^*, \\
\partial w_i / \partial n &= 0 \text{ on } \partial S.
\end{aligned}
\]

Then $A \nabla u = A^* \nabla u$, with $A^* = (A^*_{ij})$ being the classical homogenized matrix defined as
\[
A^*_{ij} = \frac{1}{|Y|} \int_{Y^*} w_i(y) w_j(y) \, dy.
\] (4.21)

Then define $V(t,x) = \nu u(t,x)$. Combining (4.6) and (4.19), we know that Theorem 4.1 holds. The proof is thus complete. ■

Remark 4.1 By the classic stochastic partial differential equation theory, Equation (4.1) is well-posed. Here, we omit its proof.

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