ON THE 2-DIVISIBILITY OF CERTAIN HEENGER POINTS

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Abstract. Let $E$ be an elliptic curve defined over the rationals and let $N$ be its conductor. Assume $N$ is prime. In this paper we give numerical evidence that suggests some conjectures on the 2-divisibility of certain sums of Heeneger points of discriminant $D$ dividing $4N$ on the elliptic curve $E$. One of these conjectures suggests a possible link between the parity of the eigenvalue $a_A(2)$ and the parity of the Šafarevič-Tate group $\Sha(A)$ of certain elliptic curves $A$ of square conductor.

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1. Preliminaries

Let $E$ is an elliptic curve over $\mathbb{Q}$. Using Tate’s algorithm [12] we may compute a global minimal Weierstraß equation

$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6$$

the conductor $N_E$ of $E$, the local Tamagawa numbers $c_p$ for $p|N$, and the minimal discriminant $\Delta_E$. We assume $N_E$ odd to simplify the discussion. The $L$-function of $E$ is given by

$$L(E, s) = \sum_{n=1}^{\infty} a_E(n)n^{-s} = \prod_{p \text{ prime}} (1 - a_E(p)p^{-s} + \delta_p p^{1-2s})^{-1},$$
where \( a_E(p) = p + \delta_p - \#(E_p) \), with \( \delta_p = 0 \) if \( E \) has bad reduction at \( p \) and 1 otherwise. By the work of Wiles [13] and Breuil-Conrad-Diamond-Taylor [3] we know that

\[
(1.2) \quad f(\tau) = \sum_{n=1}^{\infty} a_E(n)q^n \quad (q = e^{2\pi \tau})
\]

is a normalized newform of weight 2 and level \( N_E \). Let \( E_f \) be the elliptic curve over the field of complex number \( \mathbb{C} \) determined by the periods of

\[
(1.3) \quad \omega_f = 2\pi if(\tau)d\tau,
\]

regarding \( \omega_f \) as a (holomorphic) differential on the modular curve \( X_0(N) \). By the work of Mazur and Swinnerton-Dyer [11] we know that \( E_f \) may be provided with a \( \mathbb{Q} \)-structure such that \( L(E_f, s) = L(f, s) \) and such that the natural map from \( X_0(N) \) to \( E_f \) sending \( \infty \) to the identity element \( O \) of \( E_f \) is defined over \( \mathbb{Q} \). In particular, by Faltings’ isogeny theorem [5] the curves \( E \) and \( E_f \) are isogenous over \( \mathbb{Q} \). Thus there is a non-constant morphism

\[
(1.4) \quad \varphi : X_0(N) \longrightarrow E.
\]

defined over \( \mathbb{Q} \). Unless otherwise stated, let us assume the sign in the functional equation \( \Lambda(E, s) = \pm \Lambda(E, 2 - s) \) is “-”, where

\[
(1.5) \quad \Lambda(E, s) = N_E^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(E, s).
\]

In other words \( \varphi \) factors through the Atkin-Lehner quotient \( X_0^+(N) \) defined as \( X_0(N) \) modulo the involution \( w_N \). Now suppose we have a pair \((D, r)\) that satisfies the so-called Heegner condition\(^1\) i.e. \( D \) is the discriminant of an imaginary quadratic order \( \mathcal{O}_D \) of conductor \( f \) such that \( \gcd(N, f) = 1 \) and \( r \in \mathbb{Z} \) is such that

\[
(1.6) \quad D \equiv r^2 \pmod{4N}.
\]

So we have a proper \( \mathcal{O}_D \)-ideal \( \mathfrak{A} = \mathbb{Z}N + \mathbb{Z}\frac{-r + \sqrt{D}}{2} \subset \mathcal{O}_D \) and \( \mathcal{O}_D/\mathfrak{A} \cong \mathbb{Z}/N\mathbb{Z} \). Using Gross’s notation as in [6], for each proper \( \mathcal{O}_D \)-ideal \( \mathfrak{A} \) consider the Heegner point \( x = (\mathcal{O}_D, \mathfrak{A}, [\mathfrak{A}]) \), which lies in \( X_0(N)_H \), where \( H \) is the ring class field attached to \( \mathcal{O}_D \), and \( [\mathfrak{A}] \) denotes the class of \( \mathfrak{A} \) in \( \text{Pic}(\mathcal{O}_D) \). To simplify our exposition let us assume \( E_{\mathbb{Q}} \cong \mathbb{Z} \). Define the weighted

\(^1\)This condition was introduced by Birch [2].
trace $t_{D,r,x}$ of $\varphi(x)$ on $E$ by the equation

\begin{equation}
  u_D t_{D,r,x} = \sum_{\sigma \in \text{Gal}(H/K)} \varphi(x^\sigma) \in E_Q,
\end{equation}

where

\[ u_D = \begin{cases} 
\frac{1}{2} \#(O_D^x), & \text{if } \#(O_D^x) > 2, \\
2, & \text{if } \#(O_D^x) = 2 \text{ and } N|D, \\
1, & \text{otherwise.}
\end{cases} \]

and $K = \mathbb{Q}(\sqrt{D})$. Now define the generalized trace $y_{D,r,x} \in E_Q$ as the sum

\begin{equation}
  y_{D,r,x} = \sum_{e|D} t_{D,e} x^e
\end{equation}

where we define $t_{D,e} x^e = O_E$ if $(D,e,x)$ does not satisfy the Heegner condition. Assume further that the derivative $L'(E,1) \neq 0$, so that $E$ has analytic 1. By the work of Kolyvagin [10] we know that the rank of $E$ is 1. To simplify the exposition assume $E_Q$ is torsion free. Fix a generator $g_E$ of the Mordell-Weil group $E_Q$ of $E$ over $Q$. Define $\beta_{D,r} \in \mathbb{Z}$ by the equation

\[ (y_{D,r}^+)_f = \beta_{D,r} g_E, \]

where $(y_{D,r}^+)_f$ denotes the $f$-eigencomponent of $y_{D,r}^+$, and $y_{D,r}^+$ denotes the image of $y_{D,r}$ in the Jacobian $J_0^+(N)$ as in Gross, Kohnen and Zagier [8]. Since $w_N$ acts as complex conjugation on $y_{D,r}$, we see $y_{D,r}^+$ is defined over $Q$, and the definition of $\beta_{D,r}$ makes sense. To ease notation we write $\beta_D$ instead of $\beta_{D,r}$ if there is no risk of confusion.

2. The conjectures

Suppose $N$ divides $D$. So taking the weighted trace associated to $D$ involves dividing by 2 a sum over a Galois orbit over $Q$ of points on $E$ obtained as images of the Heegner points of discriminant $D$. Now consider in particular the discriminant $D = -4N$ and assume from now on that $N$ is prime. Numerical evidence shown below strongly suggests that (at least) the $f$-eigencomponents $(y_{D,r}^+)_f$ of the generalised trace $y_D$ may be further divided by 2 in $E(Q)$, provided $N$ satisfies a certain congruence condition.\footnote{A consequence of Proposition 3.1 (p. 347) of Gross [7] is that the Heegner points involved in the generalised trace of discriminant $D = -4N$ are precisely the fixed points of the Fricke involution $w_N$. But we do not use this fact here.} But we shall find convenient to state a slightly stronger conjecture first.
Table 1. $\beta_{-N,E}$ such that $a_E(2) \in 2\mathbb{Z}$ and $N \equiv 3 \pmod{4}$ prime.

| $E$  | $a_E(2)$ | $\beta_{-N}$ | $E$  | $a_E(2)$ | $\beta_{-N}$ |
|------|----------|---------------|------|----------|---------------|
| 43A  | -2       | 2             | 8419A| 2        | 0             |
| 131A | 0        | 0             | 8747A| 0        | -4            |
| 163A | 0        | -2            | 8803A| 0        | -2            |
| 347A | -2       | -2            | 9539A| 2        | 0             |
| 443A | 0        | -2            | 9587A| 2        | -2            |
| 467A | 0        | 0             | 9811A| 0        | -4            |
| 811A | 0        | 2             | 10859A| 0       | 4             |
| 827A | 0        | 2             | 10859B| -2       | 4             |
| 1019A| 0       | 0             | 10987A| 0       | 0             |
| 1019B| -2       | -4            | 11867A| -2       | 6             |
| 1051A| 0       | 0             | 11923A| 0       | -4            |
| 1259A| 0       | -2            | 11939A| 0       | -2            |
| 1747A| 2       | 2             | 11939B| 2       | 0             |
| 1811A| 0       | -2            | 12163A| 2       | 4             |
| 1987A| 0       | -2            | 12619A| -2       | -6            |
| 2539A| -2       | 4             | 13043A| 0       | 0             |
| 2699A| 0       | 0             | 13523A| 2       | 0             |
| 3251A| 0       | 2             | 15083A| -2       | 6             |
| 3259A| -2       | -4            | 15091A| 2       | 2             |
| 3259B| -2       | -10           | 15131A| 0       | 4             |
| 3347A| 2       | 0             | 15227A| 0       | -2            |
| 3547A| 0       | -2            | 15971A| 2       | 2             |
| 3851A| -2       | 0             | 16883A| 0       | 2             |
| 3931A| 0       | -2            | 16963A| 2       | 10            |
| 3947A| 0       | 0             | 17387A| 0       | 0             |
| 4051A| 0       | 4             | 17387B| -2       | -8            |
| 4507A| -2       | -6            | 17483A| -2       | 6             |
| 4603A| -2       | -2            | 17747A| 2       | 2             |
| 5443A| 2       | 0             | 17827A| 0       | 4             |
| 5563A| 0       | 4             | 18059A| 0       | 0             |
| 6131A| 2       | 0             | 18251A| -2       | 8             |
| 6691A| 0       | -8            | 18859A| -2       | -2            |
| 7019A| -2       | -2            | 19387A| 0       | -4            |
| 7187A| 2       | 2             | 19387B| 0       | -6            |
| 7283A| 0       | 0             |       |          |               |
Conjecture 2.1. Assume that $E$ is such that $N_E \equiv 3 \pmod{4}$. If $a_E(2)$ is even then $\beta_{-N}$ is even. (See Table 1, below.)

By the work of Gross-Kohnen-Zagier [8] we know that for each prime $p$ such that $\gcd(p, N) = 1$ we have

$$\beta_{p^2D,pr,\varphi} + \left(\frac{D}{p}\right) \beta_{D,r,\varphi} = a_E(p)\beta_{D,r,\varphi},$$

provided $D$ is fundamental. So clearly Equation 2.1, with $p = 2$ and $D = -N$ together with Conjecture 2.1 imply the following.

Conjecture 2.2. Assume that $E$ is such that $N_E \equiv 3 \pmod{4}$. Then the integer $\beta_{-4N}$ is even.

Example 2.3. Suppose $E$ is elliptic curve 43A1 of Cremona’s Tables [4]. So $E$ is an elliptic curve with minimal Weierstrass equation

$$Y^2 + Y = X^3 + X^2,$$

$E_{\mathbb{Q}} = \mathbb{Z}g_E$, where $g_E = (0,0)$, and the eigenvalue $a_E(2) = -2$. Note that the pair $(-43, 129)$ satisfies the Heegner condition. We may find $\beta_{-43,129} = (-1, -1) \in E(\mathbb{Q})$. So $y_{-N} = 2g_E$, and thus $\beta_{-N} = 2$, which is even. So Conjecture 2.1 holds for curve 43A1. Now using Equation 2.1 we have $\beta_{-43} = (a_E(2) - \left(\frac{-43}{2}\right)) \beta_{-43}$. So $\beta_{-43} = (-2 + 1)\beta_{-43} = -\beta_{-43} = -2g_E$. In other words $\beta_{-43} = -2$ and Conjecture 2.2 holds for curve 43A1.

Using the following lemma we will actually prove part of Conjecture 2.2.

Lemma 2.4. Let $E$ be an elliptic curve of prime conductor $N$. If $N \equiv 7 \pmod{8}$, then $a(2) = \pm 1$. 
Proof. By brute force it may be found the list of the reduction modulo \( p = 2 \) of all possible Weierstraß models of \( E \) over \( \mathbb{Z} \) together with the corresponding eigenvalue \( a(p) \):

| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_6 \) | \( a(2) \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_6 \) | \( a(2) \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 1   | 0   | 0   | 0   | 1   | 0   | 0   | 0   | 1   | -1  |
| 0   | 0   | 1   | 0   | 1   | 0   | 1   | 0   | 0   | 1   | 0   | -1  |
| 0   | 0   | 1   | 1   | 0   | -2  | 1   | 0   | 1   | 0   | 1   | 1   |
| 0   | 0   | 1   | 1   | 1   | 2   | 1   | 0   | 1   | 1   | 1   | 1   |
| 0   | 1   | 1   | 0   | 0   | -2  | 1   | 1   | 0   | 0   | 1   | 1   |
| 0   | 1   | 1   | 0   | 1   | 2   | 1   | 1   | 0   | 1   | 0   | 1   |
| 0   | 1   | 1   | 1   | 0   | 0   | 1   | 1   | 1   | 0   | 0   | -1  |
| 0   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 0   | -1  |

By inspection we may see that \( a(2) \in 2\mathbb{Z} \) implies \( a_1 \in 2\mathbb{Z} \). Now suppose \((a_1, a_2, a_3, a_4, a_6)\) is a global minimal Weierstraß equation of \( E \) over \( \mathbb{Q} \), normalised so that \(|a_3| \leq 1\). A straightforward computation shows that

\[
\Delta \equiv 7a_6a_1^6 + a_4a_3^5a_1 + 7a_3^2a_2a_4^4 + 4a_6a_2a_4^4 + a_7^2a_4^4 + a_3^3a_1^3 + 4a_6a_3^3a_1^3 + 2a_4a_3^2a_2^2 + 4a_3^2a_2a_1^2 + 5a_3^4 \quad (\text{mod} \ 8),
\]

where \( \Delta \) is the discriminant of the Weierstraß equation. So \( a_1 \in 2\mathbb{Z} \) implies \( \Delta \equiv 5a_3^4 \) (mod 8). But we assumed \( N \) prime. Therefore \( \Delta = \pm N \) and \( a_3 = \pm 1 \). Thus \( N \equiv \pm 5 \) (mod 8), i.e. \( N \not\equiv 7 \) (mod 8) and \( a(2) \) is odd. In other words \( a(2) = \pm 1 \) and the proof of the lemma is now complete. \( \square \)

**Theorem 2.5.** If \( E \) is such that \( N \equiv 7 \) (mod 8), then \( \beta_{-AN} \) is even.

**Proof.** Since \( N \) is prime and \( N \equiv 7 \) (mod 8), Lemma 2.4 implies \( a_E(2) \) is odd. So if we set \( p = 2 \) and \( D = -N \) in Equation 2.1 we see \( \beta_{-AN,E} \) is even. \( \square \)

**Remark 2.6.** The entries of Table 2 are again the numbers \( \beta_{-N} \), for prime \( N \leq 20000 \) such that \( N \equiv 3 \) (mod 4), now with \( a_E(2) \) odd. In sharp contrast with Table 1, in Table 2 we may see examples of both, even \( \beta_{-N} \) and odd \( \beta_{-N} \). There are 25 even \( \beta_{-N} \) and 33 odd \( \beta_{-N} \).

Let \( \text{III}(E) \) be the Šafarević-Tate group of \( E \) over \( \mathbb{Q} \) and let \( E^D \) be the twist of the elliptic curve \( E \) associated to a fundamental discriminant \( D \). Conjecture 2.1 suggests the following.
Table 2. \( \beta_{-N,E} \) such that \( a_E(2) = \pm 1 \) and \( N \equiv 3 \pmod{4} \) prime.

| \( E \) | \( a_E(2) \) | \( \beta_N \) | \( E \) | \( a_E(2) \) | \( \beta_N \) |
|---|---|---|---|---|---|
| 79A | -1 | 1 | 7723A | 1 | 5 |
| 83A | -1 | 1 | 8167A | 1 | -1 |
| 331A | -1 | 4 | 8623A | 1 | 3 |
| 359A | 1 | 2 | 9127A | -1 | -4 |
| 359B | -1 | 0 | 9491A | -1 | 4 |
| 431A | -1 | 1 | 9811B | -1 | -1 |
| 443B | -1 | 1 | 10163A | 1 | 0 |
| 503A | 1 | 1 | 10567A | 1 | -3 |
| 659A | 1 | 1 | 10799A | 1 | 5 |
| 1091A | -1 | -3 | 11119A | 1 | -1 |
| 1439A | 1 | 1 | 12007A | 1 | 13 |
| 1607A | -1 | -1 | 12227A | 1 | -5 |
| 3023A | -1 | 1 | 12547A | 1 | -2 |
| 3163A | 1 | -4 | 13451A | -1 | -7 |
| 3391A | -1 | 1 | 13619A | -1 | -2 |
| 3803A | 1 | 3 | 13723A | -1 | 10 |
| 4159A | 1 | 0 | 13723B | -1 | 4 |
| 4159B | 1 | -6 | 15551A | 1 | 1 |
| 4799A | -1 | 2 | 15859A | -1 | 1 |
| 4799B | -1 | 6 | 16411A | 1 | 4 |
| 5503A | -1 | -3 | 17299A | -1 | -6 |
| 5867A | 1 | 0 | 18059B | 1 | 2 |
| 5987A | 1 | 2 | 18127A | 1 | -8 |
| 6011A | 1 | 1 | 18523A | -1 | -7 |
| 6199A | -1 | -4 | 18899A | 1 | -5 |
| 6427A | 1 | 1 | 19211A | 1 | -6 |
| 6823A | -1 | -1 | 19583A | -1 | -1 |
| 6967A | 1 | 0 | 19927A | 1 | -3 |
| 7219A | -1 | -1 |
| 7699A | 1 | 2 |

Conjecture 2.7. Let \( E \) be as above and assume \( N \equiv 3 \pmod{4} \), with \( N \) prime such that \( \beta_{-N} \neq 0 \). If \( a_E(2) \) is even then \(|\text{III}(E^{-N})|\) is even.
Recall that for each fundamental discriminant $D$ and each prime $p$
\begin{equation}
a_A(p) = \chi(p)a_E(p),
\end{equation}
where $\chi$ is the quadratic character attached to $\mathbb{Q}(\sqrt{D})$ and $A = E^D$. In particular, by putting $p = 2$ and $D = -N$ in Equation 2.2 (and noting $\chi(2) \neq 0$) we may see $a_A(2)$ and $a_E(2)$ have the same parity. Also note $N_A = N_E^2$, where $N_A$ denotes the conductor of $A$. So Conjecture 2.7 suggests a possible link between the parity of $a_A(2)$ and the parity of $|\text{III}(A)|$, for certain elliptic curves $A$ such that $N_A$ is square.

**Example 2.8.** Suppose $E$ is curve 43A1. (See Example 2.3, above.) With the help of Tate’s algorithm [12] it is easy to find a global minimal model for $A = E^D$ of conductor $N_A = 43$. Then using Cremona’s Tables [4] it may be found that $A$ is in fact curve 1849D1 of conductor $N_A = 43^2$. A short calculation shows that the eigenvalue $a_A(2) = 2$. This is consistent with the (refined version of the) Birch and Swinnerton-Dyer conjecture, which predicts that $|\text{III}(A)| = 4$.

We have a further conjecture. Table 3 suggests the following.

**Conjecture 2.9.** If $N \equiv 1 \pmod{4}$ then the integers $\beta_{-4N}$ and $\beta_{-4}$ have the same parity. (See Table 3, below.)

### 3. Further remarks

A well-known result of Gross, Kohnen, and Zagier [8] implies that the integers $\beta_D$ for $D = -3, -4, \ldots$ are the coefficients of the newform $g$ of weight $\frac{3}{2}$, that corresponds (modulo multiplication by a scalar) to the newform $f$ of weight 2 via the Shimura correspondence.\(^3\) Perhaps a divisibility theory of (suitably normalised) half-integral weight modular forms analogous to the divisibility theory of integral weight modular forms might lead to a proof of the conjectures stated above. However, not much is known about divisibility properties of modular forms of half-integral weight, apart from the work of Koblitz [9], and the work of Balog, Darmon, and Ono [1], which is about modular forms of half-integral weight and level $N = 4$. For example, by the work of Koblitz [9] we know that Ramanujan’s famous congruence
\[
\tau(n) \equiv \sigma_{11}(n) \pmod{691},
\]
\(^3\)Recall we assumed $N$ prime; for $N$ composite we would need to consider instead of newforms $g$ of weight $\frac{3}{2}$, Jacobi newforms of weight 2 and index $N$.\)
Table 3. $\beta_{-4N,E}$ and $\beta_{-4}$ with $N \equiv 1 \pmod{4}$ prime.

| $E$   | $\beta_{-4}$ | $\beta_{-4N}$ | $E$   | $\beta_{-4}$ | $b_{-4N}$ | $E$   | $\beta_{-4}$ | $b_{-4N}$ |
|-------|--------------|---------------|-------|--------------|-----------|-------|--------------|-----------|
| 37A   | -1           | 3             | 3853A | 1            | 3         | 11353A| 1            | -5        |
| 53A   | -1           | 1             | 3877A | 1            | 5         | 11789A| 0            | 10        |
| 61A   | 1            | 1             | 4021A | 1            | -1        | 12097A| -1           | 1         |
| 89A   | 1            | -1            | 4481A | -2           | -10       | 12277A| -1           | -1        |
| 101A  | -1           | 1             | 4481B | 0            | 4         | 12289A| -1           | -7        |
| 197A  | -1           | -5            | 4493A | -2           | 4         | 12413A| 1            | 3         |
| 229A  | 0            | -2            | 5237A | -1           | 5         | 13093A| 0            | -6        |
| 269A  | -1           | 3             | 5309A | -1           | 7         | 13537A| 2            | 14        |
| 277A  | -1           | 3             | 5417A | 1            | 3         | 13789A| 2            | 14        |
| 373A  | 1            | 1             | 5417B | 4            | -8        | 14173A| -1           | 1         |
| 557A  | 1            | 1             | 5417C | 0            | -4        | 14461A| 0            | -8        |
| 593A  | 1            | -5            | 5653A | 0            | 2         | 15013A| -1           | 9         |
| 677A  | -1           | -1            | 5717A | 0            | -2        | 15101A| -1           | 15        |
| 797A  | 0            | 0             | 6373A | -2           | 10        | 15349A| 0            | -2        |
| 829A  | -1           | -9            | 6689A | 1            | -5        | 15641A| 1            | -5        |
| 997A  | 0            | 8             | 7109A | 1            | -1        | 15661A| 1            | -31       |
| 1549A | 1            | 5             | 7109B | 1            | 3         | 15661B| 1            | -5        |
| 1949A | 1            | 3             | 7213A | 1            | 15        | 15773A| 0            | -6        |
| 1973A | 1            | 1             | 7757A | 1            | 3         | 15889A| -1           | 5         |
| 2017A | 1            | 5             | 8069A | 0            | -4        | 16061A| -1           | 1         |
| 2089A | 0            | 8             | 8101A | -3           | -19       | 16189A| -1           | -11       |
| 2141A | -1           | -3            | 8597A | 0            | -4        | 16369A| 0            | -8        |
| 2161A | 1            | 1             | 8929A | 1            | 7         | 16649A| 0            | 4         |
| 2221A | 1            | 1             | 9109A | 1            | 3         | 16649B| 0            | -4        |
| 2269A | 1            | -3            | 9413A | 0            | -10       | 16889A| -1           | 5         |
| 2341A | -1           | 1             | 9829A | -4           | -4        | 16937A| 1            | 3         |
| 2357A | 1            | -1            | 9941A | -1           | -1        | 17093A| 1            | -5        |
| 2557A | 0            | 2             | 10061A| -1           | -3        | 17573A| 0            | 12        |
| 2609A | 1            | 1             | 10333A| 1            | -5        | 17837A| -1           | 9         |
| 2749A | 1            | -5            | 10333B| 1            | 7         | 18097A| 0            | 12        |
| 3109A | 1            | 7             | 10733A| 0            | -2        | 18269A| 0            | -2        |
| 3229A | 0            | 6             | 10789A| -1           | -1        | 18397A| 0            | 12        |
| 3313A | -1           | 3             | 10949A| 2            | -4        | 19469A| 2            | -4        |
| 3469A | -1           | -7            | 11321A| 1            | -1        |
| 3797A | -2           | -10           | 11321B| 0            | -4        |
where
\[ \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n \]
and \( \sigma_k(n) = \sum_{d|n} d^k \) descends via the Shimura correspondence to the congruence
\[ \alpha(n) \equiv -252c(n) \pmod{691}, \]
where \( \delta = \sum_{n=1}^{\infty} \alpha(n)q^n \) is the cusp form of weight \( \frac{13}{2} \) for \( \Gamma_0(4) \) that corresponds to \( \Delta \) under the Shimura lift, normalised so that \( \alpha(1) = 1 \) and \( c(n) \) is the coefficient of \( q^n \) in H. Cohen’s generalised class number Eisenstein series \( H_{13/2} = \sum_{n=1}^{\infty} c(n)q^n \).

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