SOME INTEGRAL INEQUALITIES FOR $\alpha$-, $m$-, $(\alpha, m)$-LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new Hadamard type inequalities using elementary well known inequalities for functions whose inequalities absolute values are $\alpha$-, $m$-, $(\alpha, m)$-logarithmically convex.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f \left( \frac{a + b}{2} \right) \leq \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions (see [1]-[8]).

In this section, we will present definitions and some results used in this paper.

Definition 1. Let $I$ be an interval in $\mathbb{R}$. Then $f : I \rightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

In [1], the concepts of $\alpha$-, $m$- and $(\alpha, m)$-logarithmically convex functions were introduced as follows.

Definition 2. [1] A function $f : [0, b] \rightarrow (0, \infty)$ is said to be $m$-logarithmically convex if the inequality

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0,1]$.

Obviously, if putting $m = 1$ in Definition 2, then $f$ is just the ordinary logarithmically convex on $[0, b]$.

Definition 3. [8] A function $f : [0, b] \rightarrow (0, \infty)$ is said to be $\alpha$-logarithmically convex if

$$f(tx + (1-t)y) \leq [f(x)]^t \alpha \ [f(y)]^{(1-t)\alpha}$$

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holds for all \(x, y \in [0, b]\), \(\alpha \in (0, 1]\) and \(t \in [0, 1]\).

Clearly, when taking \(\alpha = 1\) in Definition 3, then \(f\) becomes the ordinary logarithmically convex on \([0, b]\).

**Definition 4.** A function \(f : [0, b] \to (0, \infty)\) is said to be \((\alpha, m)\)-logarithmically convex if

\[
f (t x + m (1 - t) y) \leq [f (x)]^{\alpha} [f (y)]^{m(1 - \alpha)}
\]

holds for all \(x, y \in [0, b]\), \((\alpha, m) \in (0, 1] \times (0, 1]\), and \(t \in [0, 1]\).

Clearly, when taking \(\alpha = 1\) in Definition 4, then \(f\) becomes the standard \(m\)-logarithmically convex function on \([0, b]\), and, when taking \(m = 1\) in Definition 4, then \(f\) becomes the \(\alpha\)-logarithmically convex function on \([0, b]\).

**2. NEW HADAMARD-TYPE INEQUALITIES**

In order to prove our main theorems, we need the following lemma [2].

**Lemma 1.** Let \(f : I \subset \mathbb{R} \to \mathbb{R}\) be a differentiable mapping on \(I^\circ\), \(a, b \in I^\circ\) with \(a < b\). If \(f' \in L [a, b]\) for \(0 \leq a < b < \infty\), then the following equality holds:

\[
\frac{f (a) + f (b)}{2} - \frac{1}{b - a} \int_a^b f (x) dx = \frac{b - a}{2} \int_0^1 \int_0^1 \left[ f' (ta + (1 - t) b) - f' (sa + (1 - s) b) \right] (s - t) dtds.
\]

A simple proof of this equality can be also done integrating by parts in the right hand side (see [2]).

The next theorems gives a new result of the upper Hermite-Hadamard inequality for \((\alpha, m)\)-logarithmically convex functions.

**Theorem 1.** Let \(I \supset [0, \infty)\) be an open interval and let \(f : I \to (0, \infty)\) be a differentiable function on \(I\) such that \(f' \in L (a, b)\) for \(0 \leq a < b < \infty\). If \(|f' (x)|\) is \((\alpha, m)\)-logarithmically convex on \([0, b]\) for \((\alpha, m) \in (0, 1]^2\), then

\[
\left| \frac{f (a) + f (b)}{2} - \frac{1}{b - a} \int_a^b f (x) dx \right| \leq \left\{ \begin{array}{ll}
\frac{f' \left( \frac{b}{m} \right)}{\alpha} + \frac{\alpha^2 \ln^2 \eta - 2 \alpha \ln \eta + 2 \eta^2 - 2}{\alpha \ln \eta}, & \eta = 1 \\
\frac{f' \left( \frac{b}{m} \right)}{\alpha} + \frac{\alpha^2 \ln^2 \eta - 2 \alpha \ln \eta + 2 \eta^2 - 2}{\alpha \ln \eta}, & \eta < 1
\end{array} \right.
\]

where \(\eta = |f' (a)| / |f' \left( \frac{b}{m} \right)|\).
Proof. By Lemma 1 and since $|f'|$ is an $(\alpha, m)$-logarithmically convex on $[0, \frac{b}{m}]$, then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$

$$\leq \frac{b-a}{2} \int_0^1 \int_0^1 |f'(ta + (1-t)b) - f'(sa + (1-s)b)||s-t| \, dt \, ds$$

$$\leq \frac{b-a}{2} \left[ \int_0^1 \int_0^1 |s-t||f'(a)|^\alpha \left| f' \left( \frac{b}{m} \right) \right|^{m(1-\alpha)} \, dt \, ds \right]$$

$$+ \frac{b-a}{2} \left[ \int_0^1 \int_0^1 |s-t||f'(a)|^\alpha \left| f' \left( \frac{b}{m} \right) \right|^{m(1-\alpha)} \, dt \, ds \right]$$

If $0 < k \leq 1$, $0 < m, n \leq 1$

(2.3)  $k^{mn} \leq k^{mn}$.

When $\eta = 1$, by (2.3), we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$

$$\leq \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left[ \int_0^1 \int_0^1 |s-t| \, dt \, ds + \int_0^1 \int_0^1 |s-t| \, dt \, ds \right]$$

$$= \frac{b-a}{3} \left| f' \left( \frac{b}{m} \right) \right|^m$$

When $\eta < 1$, by (2.3), we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$

$$\leq \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left[ \int_0^1 \int_0^1 |s-t| \eta^{\alpha t} \, dt \, ds + \int_0^1 \int_0^1 |s-t| \eta^{\alpha s} \, dt \, ds \right]$$

$$= \frac{b-a}{3} \left| f' \left( \frac{b}{m} \right) \right|^m \left[ \frac{-\alpha^2 \ln^2 \eta - 2\alpha \ln \eta + 4\eta^\alpha + \alpha^2 \eta^\alpha \ln^2 \eta - 2\alpha \eta^\alpha \ln \eta - 4}{2\alpha^3 \ln^3 \eta} \right.$$

$$+ \frac{-\alpha \ln \eta + 2\eta^\alpha - \alpha \eta^\alpha \ln \eta - 2}{2\alpha^2 \ln^2 \eta}$$

which completes the proof. \qed

Corollary 1. Let $I \subseteq [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on $I$ such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is $m$-logarithmically convex on $[0, \frac{b}{m}]$ for $m \in (0, 1)$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \left\{ \begin{array}{ll} \frac{(b-a)}{2} \left| f' \left( \frac{b}{m} \right) \right|^m, & \eta = 1 \\ \frac{\ln^2 \eta - 2\ln \eta + 2\eta - 2}{m^2 \eta}, & \eta < 1 \end{array} \right.$$

where $\eta$ is same as Theorem 1.
Corollary 2. Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on $I$ such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is $\alpha$-logarithmically convex on $[0, b]$ for $\alpha \in (0, 1]$, then

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} \frac{b-a}{2} |f'(b)|, & \eta = 1 \\ \frac{b-a}{2} |f'(b)| \frac{\eta^\alpha - 4\alpha \ln \eta - 2\alpha^2 \ln^2 \eta - 1}{2\alpha^3 \ln^3 \eta}, & \eta < 1 \end{cases}
$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 2. Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on $I$ such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is an $(\alpha, m)$-logarithmically convex on $[0, b]$ for $(\alpha, m) \in (0, 1]^2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
(2.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} (b-a) |f'(b)|^m \left( \frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}}, & \eta = 1 \\ (b-a) |f'(b)|^m \left( \frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \times \left( \frac{\eta^{(q, \alpha)} - 4\alpha \ln \eta - 2\alpha^2 \ln^2 \eta - 1}{2\alpha^3 \ln^3 \eta} \right)^{\frac{1}{q}}, & \eta < 1 \end{cases}
$$

where $\eta(\alpha, \alpha)$ is same as Theorem 1.

Proof. Since $|f'|^q$ is an $(\alpha, m)$-logarithmically convex on $[0, \frac{b}{m}]$, from Lemma 4 and the well known Hölder inequality, we have

$$
(2.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \int_0^1 \int_0^1 |(f'(ta + (1-t)b)) - (f'(sa + (1-s)b))| |s-t| \, dt \, ds 
$$

$$
\leq \frac{b-a}{2} \int_0^1 \int_0^1 \left| f'(a) \right|^{\alpha} \left| f' \left( \frac{b}{m} \right) \right|^{m(1-\alpha)} \, dt \, ds 
$$

$$
+ \frac{b-a}{2} \int_0^1 \int_0^1 |s-t| \left| f'(a) \right|^{\alpha} \left| f' \left( \frac{b}{m} \right) \right|^{m(1-\alpha)} \, dt \, ds 
$$

$$
\leq \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left( \int_0^1 \int_0^1 |s-t|^p \, dt \, ds \right)^{\frac{1}{p}} 
$$

$$
\times \left[ \left( \int_0^1 \int_0^1 \eta^{\alpha} \, dt \, ds \right)^{\frac{1}{q}} + \left( \int_0^1 \int_0^1 \eta^{\alpha} \, dt \, ds \right)^{\frac{1}{q}} \right]
$$
If \( \eta = 1 \), by (2.3), we obtain
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq (b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \left( \int_0^1 \int_0^1 |s - t|^p \, dt \, ds \right)^{\frac{1}{p}}
\]
\[
= (b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \left( \frac{2}{(p + 1)(p + 2)} \right)^{\frac{1}{p}}
\]
If \( \eta < 1 \), by (2.3), we obtain
\[
(2.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left( \int_0^1 \int_0^1 |s - t|^p \, dt \, ds \right)^{\frac{1}{p}} \times \left[ \left( \int_0^1 \eta^{\alpha q} \, dt \, ds \right)^{\frac{1}{p}} + \left( \int_0^1 \eta^{\alpha q} \, dt \, ds \right)^{\frac{1}{q}} \right] \leq (b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \left( \frac{2}{(p + 1)(p + 2)} \right)^{\frac{1}{p}} \times \left( \eta(\alpha q, \alpha q) - 1 \right)^{\frac{1}{q}}
\]
which completes the proof.

\[\Box\]

**Corollary 3.** Let \( I \supseteq [0, \infty) \) be an open interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L(a, b) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is an \( m \)-logarithmically convex on \( [0, \frac{1}{m}] \) for \( m \in (0, 1) \) and \( p = q = 2 \), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} 
(b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \frac{1}{\sqrt{m}}, & \eta = 1 \\
(b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \frac{1}{\sqrt{m}} \left( \frac{\eta(2, 2) - 1}{\ln \eta(2, 2)} \right)^{\frac{1}{2}}, & \eta < 1 
\end{cases}
\]

**Corollary 4.** Let \( I \supseteq [0, \infty) \) be an open interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L(a, b) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is \( \alpha \)-logarithmically convex on \( [0, b] \) for \( \alpha \in (0, 1) \), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} 
(b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \left( \frac{2}{(p + 1)(p + 2)} \right)^{\frac{1}{p}}, & \eta = 1 \\
(b - a) \left| f' \left( \frac{b}{m} \right) \right|^m \left( \frac{2}{(p + 1)(p + 2)} \right)^{\frac{1}{p}} \left( \eta(\alpha q, \alpha q) - 1 \right)^{\frac{1}{q}}, & \eta < 1 
\end{cases}
\]
where \( \eta = |f'(a)| / |f'(b)|. \)

**Theorem 3.** Let \( I \supseteq [0, \infty) \) be an open interval and let \( f : I \to (0, \infty) \) be a differentiable function on \( I \) such that \( f' \in L(a, b) \) for \( 0 \leq a < b < \infty \). If \( |f'(x)|^q \) is
where \( \eta (\alpha, \alpha) \) is same as Theorem 1, and we take \( \eta (aq, aq) = \varphi \).

Proof. Since \( |f'|^q \) is an \((\alpha, m)\)-logarithmically convex on \([0, \frac{b}{m}]\), for \( q \geq 1 \), from Lemma 1 and the well known power mean integral inequality, we get

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \left\{ \left[ \frac{b-a}{m} \left| f' \left( \frac{b}{m} \right) \right|^m \right]^{\eta} \right. \frac{1}{b-a} \int_a^b \left| f' \right|^m \left[ \left[ \frac{2\varphi-2}{\ln \varphi} \right] - \left[ \frac{\varphi+1}{2\ln \varphi} \right] \right] \left[ \left[ \frac{\varphi-1}{2\ln \varphi} \right] - \left[ \frac{\varphi+1}{2\ln \varphi} \right] \right] \left\} \right. \eta = 1 \\
\left\{ \left[ \frac{b-a}{m} \left| f' \left( \frac{b}{m} \right) \right|^m \right]^{\eta} \right. \frac{1}{b-a} \int_a^b \left| f' \right|^m \left[ \left[ \frac{2\varphi-2}{\ln \varphi} \right] - \left[ \frac{\varphi+1}{2\ln \varphi} \right] \right] \left[ \left[ \frac{\varphi-1}{2\ln \varphi} \right] - \left[ \frac{\varphi+1}{2\ln \varphi} \right] \right] \left\} \right. \eta < 1
\]

When \( \eta = 1 \), by (2.3), we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \left. \frac{b-a}{2} \left( \int_0^1 \int_0^1 |s-t| \, dt \, ds \right) \right|^{\frac{1}{2}} \left( \int_0^1 \int_0^1 |s-t| \, f'(ta + (1-t)b) \right) \right|^{\frac{1}{2}} \\
+ \left. \frac{b-a}{2} \left( \int_0^1 \int_0^1 |s-t| \, dt \, ds \right) \right|^{\frac{1}{2}} \left( \int_0^1 \int_0^1 |s-t| \, f'(sa + (1-s)b) \right) \right|^{\frac{1}{2}}
\]

\[
\leq \left. \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left( \int_0^1 \int_0^1 |s-t| \, dt \, ds \right) \right|^{\frac{1}{2}} \left( \int_0^1 \int_0^1 |s-t| \, \eta^{aq} \right) \right|^{\frac{1}{2}} \\
+ \left. \frac{b-a}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left( \int_0^1 \int_0^1 |s-t| \, dt \, ds \right) \right|^{\frac{1}{2}} \left( \int_0^1 \int_0^1 |s-t| \, \eta^{aq} \right) \right|^{\frac{1}{2}}
\]
When $\eta < 1$, by (23), we obtain

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{2} \left( \frac{1}{3} \right)^{1-\frac{1}{\eta}} \left| f'(\frac{b}{m}) \right|^m \left( \int_0^1 \int_0^1 |s-t| \eta^{\alpha q} \, dt \, ds \right)^{\frac{1}{2}}$$

$$+ \frac{b-a}{2} \left( \frac{1}{3} \right)^{1-\frac{1}{\eta}} \left| f'(\frac{b}{m}) \right|^m \left( \int_0^1 \int_0^1 |s-t| \eta^{\alpha q} \, dt \, ds \right)^{\frac{1}{2}}$$

$$= \frac{b-a}{2} \left( \frac{1}{3} \right)^{1-\frac{1}{\eta}} \left| f'(\frac{b}{m}) \right|^m \times \left\{ \frac{2\eta(\alpha, \alpha_q) - 2}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{1-\eta(\alpha, \alpha_q)}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

$$+ \left\{ \frac{\eta(\alpha, \alpha_q) - 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

which completes the proof. \qed

Corollary 5. Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on $I$ such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is $m$-logarithmically convex on $[0, \frac{b}{m}]$ for $m \in (0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)}{2} \left( \frac{1}{3} \right)^{1-\frac{1}{\eta}} \left| f'(\frac{b}{m}) \right|^m \times \left\{ \frac{2\eta(\alpha, \alpha_q) - 2}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{1-\eta(\alpha, \alpha_q)}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

$$+ \left\{ \frac{\eta(\alpha, \alpha_q) - 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

Corollary 6. Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (0, \infty)$ be a differentiable function on $I$ such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is $\alpha$-logarithmically convex on $[0, b]$ for $\alpha \in (0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)}{2} \left( \frac{1}{3} \right)^{1-\frac{1}{\eta}} \left| f'(\frac{b}{m}) \right|^m \times \left\{ \frac{2\eta(\alpha, \alpha_q) - 2}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{1-\eta(\alpha, \alpha_q)}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

$$+ \left\{ \frac{\eta(\alpha, \alpha_q) - 1}{[\ln(\eta(\alpha, \alpha_q))]^2} - \frac{\eta(\alpha, \alpha_q) + 1}{2\ln(\eta(\alpha, \alpha_q))} \right\}^{\frac{1}{2}}$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 4. Let $f : I \subset \mathbb{R} \to \mathbb{R}^+$ be differentiable on $I^c$, $a, b \in I$, with $a < b$ and $f' \in L([a, b])$. If $|f'|$ is an $(\alpha, m)$-logarithmically convex $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1]^2$.
and $\mu_1, \mu_2, \tau_1, \tau_2 > 0$ with $\mu_1 + \tau_1 = 1$ and $\mu_2 + \tau_2 = 1$, then

\begin{equation}
(2.8)
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \begin{cases}
\frac{(b-a)}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left\{ \frac{2\mu_1^2}{(2\mu_1 + 1)(\mu_1 + 1)} + \frac{2\mu_2^2}{(2\mu_2 + 1)(\mu_2 + 1)} + \tau_1 + \tau_2 \right\}, & \eta = 1 \\
\frac{(b-a)}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left\{ \frac{2\mu_1^2}{(2\mu_1 + 1)(\mu_1 + 1)} + \frac{2\mu_2^2}{(2\mu_2 + 1)(\mu_2 + 1)} + \tau_1 \frac{\eta(\frac{\tau_1}{\tau_2})^{-1}}{\ln \eta(\frac{\tau_1}{\tau_2})} + \tau_2 \frac{\eta(\frac{\tau_1}{\tau_2})^{-1}}{\ln \eta(\frac{\tau_1}{\tau_2})} \right\}, & \eta < 1
\end{cases}
\end{equation}

where $\eta(\alpha, \alpha)$ is same as Theorem [7].

Proof. Since $|f'|^q$ is an $(\alpha, m)$-logarithmically convex on $[0, \frac{b}{m}]$, from Lemma [1] we have

\begin{equation}
(2.9)
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \frac{(b-a)}{2} \int_0^1 \int_0^1 |(f' (ta + (1 - t) b)) - (f' (sa + (1 - s) b))| |s - t| \, dt \, ds
\leq \frac{(b-a)}{2} \left[ \int_0^1 \int_0^1 |s - t| |f'(a)|^\alpha \left| f' \left( \frac{b}{m} \right) \right|^m (1 - s^\alpha) \, dt \, ds \right]
+ \frac{(b-a)}{2} \left[ \int_0^1 \int_0^1 |s - t| |f'(a)|^\alpha \left| f' \left( \frac{b}{m} \right) \right|^m (1 - s^\alpha) \, dt \, ds \right]
= \frac{(b-a)}{2} \left| f' \left( \frac{b}{m} \right) \right|^m \left[ \int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds + \int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds \right]
\end{equation}

for all $t \in [0, 1]$. Using the well known inequality $rt \leq \mu r^\beta + \tau t^\tau$, on the right side of (2.9), we get

\begin{equation}
(2.10)
\int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds + \int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds
\leq \mu_1 \int_0^1 \int_0^1 |s - t| \eta_1^\alpha \, dt \, ds + \tau_1 \int_0^1 \int_0^1 \eta_1^\alpha \, dt \, ds
+ \mu_2 \int_0^1 \int_0^1 |s - t| \eta_2^\alpha \, dt \, ds + \tau_2 \int_0^1 \int_0^1 \eta_2^\alpha \, dt \, ds
\end{equation}

When $\eta = 1$, by (2.3), we get

\begin{equation}
(2.11)
\int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds + \int_0^1 \int_0^1 |s - t| \eta^\alpha \, dt \, ds
\leq \frac{2\mu_1^3}{(2\mu_1 + 1)(\mu_1 + 1)} + \frac{2\mu_2^3}{(2\mu_2 + 1)(\mu_2 + 1)} + \tau_1 + \tau_2
\end{equation}
When $\eta < 1$, by (2.3), we get
\begin{equation}
(2.12)
\int_{0}^{1} \int_{0}^{1} |s-t|^{\tau} f(t) dt ds = \int_{0}^{1} \int_{0}^{1} |s-t|^{\eta} f(t) dt ds + \int_{0}^{1} \int_{0}^{1} |s-t|^{\mu} f(t) dt ds
\end{equation}
\begin{align*}
\leq \mu_1 & \int_{0}^{1} \int_{0}^{1} |s-t|^{\tau} f(t) dt ds + \mu_2 \int_{0}^{1} \int_{0}^{1} |s-t|^{\mu} f(t) dt ds + \tau_1 \int_{0}^{1} \int_{0}^{1} |s-t|^{\frac{\eta}{\tau}} f(t) dt ds \\
\leq \mu_1 & \int_{0}^{1} \int_{0}^{1} |s-t|^{\tau} f(t) dt ds + \mu_2 \int_{0}^{1} \int_{0}^{1} |s-t|^{\mu} f(t) dt ds + \tau_1 \int_{0}^{1} \int_{0}^{1} |s-t|^{\frac{\eta}{\tau}} f(t) dt ds \\
= & \frac{2\mu_3}{(2\mu_1 + 1)(\mu_1 + 1)} + \frac{2\mu_3}{(2\mu_2 + 1)(\mu_2 + 1)} + \eta \frac{\eta}{\eta - 1} \ln \frac{\eta}{\eta - 1} + \tau_2 \ln \frac{\eta}{\eta - 1} + \eta \ln \frac{\eta}{\eta - 1}
\end{align*}
from (2.14)- (2.15), which completes the proof. \qed

**Corollary 7.** Under the assumptions of Theorem 4 and \( \mu = \mu_1 = \mu_2 > 0, \tau = \tau_1 = \tau_2 > 0 \) with \( \mu + \tau = 1 \), then we have
\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) dx \right| \leq \left\{ \begin{array}{ll}
\frac{(b-a)}{2} \left| \frac{f'(h)}{m} \right| \left[ \frac{4\mu_3}{(2\mu_1 + 1)(\mu_1 + 1)} + 2\tau \right], & \eta = 1 \\
\frac{(b-a)}{2} \left| \frac{f'(h)}{m} \right| \left[ \frac{4\mu_3}{(2\mu_2 + 1)(\mu_2 + 1)} + 2\tau \frac{(\eta + 1)^{\gamma} - 1}{\gamma} \right], & \eta < 1
\end{array} \right.
\end{equation}

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