ELECTRICALLY CHARGED MATTER IN PERMANENT ROTATION AROUND MAGNETIZED BLACK HOLES: A TOY MODEL FOR SELF-GRAVITATING FLUID TORI

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ABSTRACT

We present an analytical approach for the equilibrium of a self-gravitating charged fluid embedded in a spherical gravitational and dipole magnetic fields produced by a central mass. Our scheme is proposed, as a toy model, in the context of gaseous/dusty tori surrounding supermassive black holes in galactic nuclei. While the central black hole dominates the gravitational field and remains electrically neutral, the surrounding material has a non-negligible self-gravitational effect on the torus structure. By charging mechanisms it also acquires non-zero electric charge density, so the two influences need to be taken into account to achieve a self-consistent picture. Using our approach we discuss the impact of self-gravity, represented by the term \( d_i \) (ratio of the torus total mass to the mass of the central body), on the conditions for existence of the equilibrium and the morphology and typology of the tori. By comparison with a previous work without self-gravity, we show that the conditions can be different. Although the main aim of the present paper is to discuss a framework for the classification of electrically charged, magnetized, self-gravitating tori, we also mention potential astrophysical applications to vertically stratified fluid configurations.

Key words: gravitation – magnetic fields – methods: numerical

1. INTRODUCTION

Studies of equilibria of toroidal structures of perfect fluids are important for understanding the physics of accretion disks in active galactic nuclei (AGNs) (Abramowicz et al. 1978, 1984; Kozlowski et al. 1978) and the dense self-gravitating tori around stellar mass black holes, which can be the result of the merger of a black hole–neutron star binary or a remnant after the collapse of a massive star (Shibata 2007; Otani et al. 2009; Fujisawa et al. 2013). Nuclei of galaxies contain dusty tori and a central compact body that is frequently associated with a supermassive black hole (the mass typically \( M_b \approx 10^6–10^9 M_\odot \) (Krolik 2004; Eckart et al. 2005)). At a distance of \( 10^2–10^5 \) gravitational radii \( (R_g = GM/c^2 \approx 1.5 (M_b/M_\odot) \text{ km}) \), where \( G \) is the gravitational constant) these tori become self-gravitating (Collin & Hureé 2001; Karas et al. 2004). At the same time, this distance is large enough to reduce the effects of General Relativity (essential near the center) to a negligible level (Shlosman & Begelman 1987; Hureé 1998). Therefore, an accurate and adequate description of the torus can be made using the fluid equations with Newtonian gravity (Frank et al. 1985). This subject has also been studied in great detail (Stuchlík et al. 2000; Font & Daigne 2002; Kucáková et al. 2011; Slaný et al. 2013; Kovář et al. 2014) within the framework of General Relativity.

On the other hand, at smaller distances from the central body, the role of the self-gravity of the fluid becomes less important relative to other forces. Here we will discuss a transitional region; we will take self-gravity into account while maintaining the Newtonian description of the central field to which we add as a new ingredient the effect of large-scale magnetic dipole field attached to the central body. The role of self-gravity is commonly described by Toomre’s criterion (Toomre 1964; Goldreich & Lynden-Bell 1965). The self-gravity of toroids plays an important role because it affects the shape of the equilibrium configurations. Figures of rotating self-gravitating fluids have been described in great detail for decades under various assumptions. Ostriker & Mark (1968), Clement (1974), Eriguchi & Mueller (1985), and Hachisu (1986) studied self-gravitating systems in rotation without magnetic fields. Tomimura & Eriguchi (2005) and Yoshida & Eriguchi (2006) included the effects of poloidal and toroidal magnetic fields in rotating magnetized stars. Studies of stability have been performed too (Masuda et al. 1998; Lu et al. 2000). This work has been extended to magnetized torus-central compact object systems by Otani et al. (2009) and Fujisawa et al. (2013). The morphology of the solutions, their equilibria, and stability depend on many factors, such as the rotation law, or the polytropic index. Many sequences have been found, such as, for instance, the Maclaurin, the Jacobi, and the one-ring (Hachisu 1986; Ansorg et al. 2003; Tomimura & Eriguchi 2005). These configurations of rotating fluids have also been studied within the General Relativity framework, with both analytical and numerical approaches (Lanza 1992; Nishida & Eriguchi 1994; Lu et al. 2000; Shibata 2007; Rezzolla et al. 2010).

The torus acts as a source of material that gradually sinks from the outer regions down to the core, where it becomes heated and then accreted, typically on a viscous timescale. This mechanism drives accretion and it helps to transfer the angular momentum and releases energy in the form of outflowing winds and radiation. High-energy X-rays originate near the inner rim of the accretion flow. The temperature of the X-ray illuminated dust particles grows until they sublimate (Czerny & Hryniewicz 2011). During the photoionization the dust grains develop electric charges that are attached to their surface, and thus a complex dusty plasma is formed (Horanyi 1996; Vladimirov et al. 2005).

A central body is necessary to maintain the gravitational stability of an accretion torus. Black holes in the centers of galaxies are generally believed to be electrically neutral because of selective charge accretion of the ambient plasma (Kovář et al. 2008, 2011). A small non-zero charge is possible
when external magnetic fields interact with a rotating body (Wald 1974). However, the surrounding dust particles may indeed keep non-vanishing charges while the system as a whole remains in neutral equilibrium (Draine & Salpeter 1979; Horanyi 1996; Krishna Swamy 2005). Let us note that the combined influence of magnetic fields and self-gravity should be included not only for the sake of completeness and consistency of the model. Their impact may be essential for the structure of accretion flows and the final rate of mass accretion. It has been argued that magnetized accretion disks are less susceptible to gravitational fragmentation (Pariev et al. 2003; Salvesen et al. 2016). Recently, in a different context of geometrically thin accretion disks, Śadowski (2016) examined the role that strong large-scale magnetic fields that thread the fluid can play on the thermal stabilization of the flow.

In this paper, we apply the Newtonian hydrodynamical approach. The torus is modeled by a perfect fluid with some net electric charge spread through the fluid. This model represents a different limit to the well-known ideal magnetohydrodynamics (MHD) with zero resistivity and vanishing volume charge density. The approximation of ideal MHD is accurate in many astrophysically relevant situations involving fluids in motion (Melrose 1980). In contrast to this approach, we are working with zero-conductivity, meaning a non-vanishing electric charge density of the fluid distribution, such as ionized plasma (Wardle & Ng 1999; Inoue & Inutsuka 2008; Pandey & Wardle 2008). Despite the fact that the net charge is negligible in the majority of astrophysical realistic systems, where the accreted material is described in terms of ideal fluid with high conductivity that satisfies the conditions of a force-free limit, there has been an ongoing debate about the role of charge separation that may be caused by processes operating in complex (two-component) cosmic plasmas. These include the influence of irradiation by photons emanating from the central source, and the effect of electric forces acting parallel to the magnetic field lines in the local co-moving frame of the magnetized plasma (see Section 6.1 for more discussions).

We present a convenient formalism and we give examples of a viable physical setup where both the self-gravity of the torus material and non-vanishing electric charge density interact to define the radial and vertical structure of an equilibrium configuration torus. The idea is to use the same method as Slaný et al. (2013). They work with four families of specific charged distributions and a general form for the orbital velocity. In addition to Slaný et al. (2013), we include effects of self-gravity, which has previously been neglected for simplicity. Here we decided to work, for convenience, with two of these four families, namely family II and IV. These two distributions have the advantage of producing equatorial and off-equatorial solutions, so they capture the two qualitatively different cases.

The paper is organized as follows. In Section 2, we present the basic equations, the assumptions made, the normalization, and the general conditions of equilibrium existence. Section 3 is dedicated to the study of equatorial tori. We show how the self-gravity influences the conditions of existence of the tori, their charges, and their morphologies. The same study is done for off-equatorial tori in Section 4. In Section 5 we employ the numerical method of self-consistent field to compare the precision of our analytical approximation with a corresponding solution, where some of the restricting assumptions can be relaxed. Conclusions are given in Section 6.
where \( M_c \) is the mass of the central object. The self-gravitational torus potential is given by Poisson’s equation:

\[
\Delta \Psi_{\text{tg}} = 4\pi G \rho_m.
\]  

(8)

In our work \( \Psi_{\text{tg}} \) is approximated by the gravitational potential of a loop in the equatorial plane, coordinates \((r_c, 0)\), and mass \( m \), centered on the axis. In cylindrical coordinates \((R, \phi, Z)\), it is given by (Durand 1953)

\[
\Psi_{\text{tg}} \sim -\frac{Gm}{r_c \pi} \sqrt{\frac{r_c}{R}} kK(k),
\]  

(9)

with

\[
k = \frac{2\sqrt{r_c R}}{\sqrt{(r_c + R)^2 + Z^2}}.
\]  

(10)

The complete elliptic integral of the first kind \( K \) (Gradshteyn & Ryzhik 1965) diverges when its modulus \( k = 1 \) (i.e., when the field point \((R, Z)\) coincides with the loop radius \((r_c, 0)\)). To avoid this singularity we add a parameter \( \lambda \) to the modulus \( k \)

\[
\lambda = \frac{2\sqrt{r_c R}}{\sqrt{(r_c + R)^2 + Z^2}} \rightarrow \frac{2\sqrt{r_c R}}{\sqrt{(r_c + R)^2 + Z^2} + \lambda^2}.
\]  

(11)

This technique, initially developed to handle numerical N-body simulations, is used to compute the gravitational potential of gaseous self-gravitating disks (Papaloizou & Lin 1989; Tremaine 2001; Li et al. 2009). The free-parameter, \( \lambda \), called “the smoothing length,” takes into account the vertical and the radial extension of the torus. Various prescriptions have been chosen for this parameter: (i) a function of the disk parameter, (ii) a function of space, and (iii) a constant value (Masset 2002; Huré & Pierens 2005; Müller et al. 2012; Huré & Trova 2015). However, no universal value has been adopted; see Huré & Pierens (2009) for an non-exhaustive list. In our work, the softening length is a function of the loop radius (i.e., the location of the maximum pressure), \( \lambda = 0.4r_c \). Even if this parameter influences the value of the gravitational potential, the morphology of the solutions remains preserved.

The last potential of the four unknown potential functions to be determined in Equation (6) is the “magnetic potential” \( M \). Since we know the Lorentz force is given by

\[
\mathbf{F} = j \wedge \mathbf{B},
\]  

(12)

where \( j = \rho_e \mathbf{v}_e \mathbf{e}_\phi \) and \( \mathbf{B} \) are respectively the fluid’s current density and the external magnetic field, the \( M \)-function has to satisfy

\[
\nabla M = -\frac{\mathbf{F}}{\rho_m}.
\]  

(13)

To solve this equation we impose proportionality

\[
\rho_e = \rho_m q(R, Z).
\]  

(14)

\( M \) depends on the rotation law and the specific charge distribution \( q(R, Z) \); see Appendix A for explicit equations. \( M \) is the solution of the Equation (38). To solve the equilibrium equation, we set various hypotheses on the system.

### 2.1. Assumptions

We assume that:

1. The fluid is axially symmetrical and also symmetric with respect to the mid-plane.
2. The fluid is incompressible, \( \rho_m = \text{Const} \), so the enthalpy is then written as

\[
H = \frac{P}{\rho_m}.
\]  

(15)

3. The integrability condition of the Equation (5) leads to two unknown functions: the orbital velocity \( v_\phi(R, Z) \), i.e., the way of rotation of the fluid, and the specific charge \( q(R, Z) \).
4. The fluid is embedded in an external dipolar magnetic field, which is given in cylindrical coordinates by

\[
\begin{align*}
B_R &= \frac{3\mu \lambda}{(R^2 + Z^2)^{3/2}}, \\
B_\phi &= 0, \\
B_Z &= \frac{\lambda (2Z^2 - R^2)}{(R^2 + Z^2)^{3/2}}.
\end{align*}
\]  

(16)

Relations (16) imply the following expression for the electromagnetic potential \( A_\phi \),

\[
A_\phi = \frac{\mu R}{(R^2 + Z^2)^{3/2}}.
\]  

(17)

According to Slaný et al. (2013), analysis of integrability condition shows that \( v_\phi(R) = K_1 R^{K_1} \) (where \( K_1 \) and \( K_2 \) are constants). So the centrifugal potential is

\[
\Phi = \frac{K_2}{2K_1} R^{2K_1}.
\]  

(18)

The choice of specific charge distribution reflects the choice of the rotation law, in order to have an integrable system. We decided to study two of the four specific charge distributions described in Slaný et al. (2013), the family II and IV. We have in cylindrical coordinates \((R, \theta, Z)\)

\[
\begin{align*}
q(R, Z) &= C \frac{(R^2 + Z^2)^{3/2}}{R^{3K_1}}, \\
M(R, Z) &= 2\mu K_2 C \frac{(R^2 + Z^2)^{3K_2/2 - 3/4}}{(2K_1 - 1)R^{2K_1 - 1}},
\end{align*}
\]  

(19)

for family II, and

\[
\begin{align*}
q(R, Z) &= C \frac{R^2}{(R^2 + Z^2)^{3(1 - K_1)/2}}, \\
M(R, Z) &= \mu K_2 C \frac{R^{4 - 2K_1}}{(K_1 - 2)(R^2 + Z^2)^{3 - 3K_1/2}}.
\end{align*}
\]  

(20)

for family IV. In the following examples, we set the model parameters in such a way that the imposed central dipole field dominates over the magnetic field produced by the current of the charged rotating tori. This assumption is checked in the Section 3.2.
2.2. Normalization

We introduce dimensionless physical quantities denoted by “tilde.” For the normalization, we use various quantities: \( X = R/r \), \( Y = Z/r \), \( \rho_m \), \( P_{max} \), \( r_c \), \( G \), \( C \), \( \mu \), and \( K_2 \). The dimensional variables are given by

\[
\Psi_{sg} = \tilde{\Psi}_{sg} Gm/r, \\
\Psi_e = \tilde{\Psi}_e GMc/r, \\
\phi = \tilde{\Phi}_2 r_c^{K_2}, \\
H = \tilde{H} \rho_{max} = a \tilde{H}, \\
M = \begin{cases} \tilde{M}_1 \mu C K_2 r_c^{K-1/2} & \text{(II)}, \\ \tilde{M}_4 \mu C K_2 r_c^{K-2} & \text{(IV)}. \end{cases}
\]

With these new variables, Equation (6) becomes

\[
a \tilde{H} + d_i \tilde{\Psi}_{sg} + \tilde{\Psi}_e + b \tilde{\Phi} + e \tilde{M} = c,
\]

with

\[
\begin{align*}
a &= \frac{P_{max} r_c}{\rho_m GM_c}, \\
b &= \frac{K_2^2 r_c^{K_2 + 1}}{GM_c}, \\
c &= \frac{Constr_c}{GM_c}, \\
d_i &= \frac{m}{M_c}.
\end{align*}
\]

The c-constant determines the surface of zero pressure (i.e., the boundary of the torus).

2.3. General Conditions for the Existence of Equilibrium

An equilibrium solution exists if there is a local pressure maximum, or a local enthalpy maximum. We suppose that the maximum is located in \( R = r_c, Z = z_c \), i.e., in \( X = 1, Y = y_c \). The necessary condition to have a local extremum in this point is

\[
\nabla \tilde{H} = 0.
\]

This extremum corresponds to a maximum if

\[
\begin{align*}
\frac{\partial^2 \tilde{H}}{\partial X^2} &< 0 \quad \text{and} \\
\frac{\partial^2 \tilde{H}}{\partial X^2} \times \frac{\partial^2 \tilde{H}}{\partial Y^2} - \left( \frac{\partial \tilde{H}}{\partial X \partial Y} \right)^2 &> 0.
\end{align*}
\]

The expressions of these derivatives are given in Appendix C.

In the following sections, we will show that there are equatorial toroidal configurations, where the maximum pressure takes place in \((X = 1, Y = 0)\), and also off-equatorial toroidal configurations, where the maximum pressure is placed in \((X = 1, Y = y_c)\).

3. EQUATORIAL TORI: INCOMPRESSIBLE FLUID

For equatorial tori, the maximum, located at \((X, Y) = (1, 0)\) must satisfy Equations (24) and (25). For both distributions of specific charge, the maxima of pressure exists if \( b, d_i, \) and \( e \) satisfy the following conditions.

1. Family II

If \( K_i + \frac{1}{2} \gtrless 0 \) then

\[
b \lessgtr -1 + \frac{d_i}{K_i + 1/2} \left[ \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial X^2} - \frac{2 K_i - 3}{2} \frac{\partial \tilde{\Psi}_{sg}}{\partial X} \right] \]

and

\[
b > \frac{2}{3} - \frac{d_i}{3} \left[ \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial Y^2} - \frac{3}{3} \frac{\partial \tilde{\Psi}_{sg}}{\partial Y} \right],
\]

\[
e = b - d_i \frac{\partial \tilde{\Psi}_{sg}}{\partial X} - 1.
\]

2. Family IV

If \( K_i + 2 \gtrless 0 \) then

\[
b \lessgtr \frac{1 - K_i}{K_i + 2} + \frac{d_i}{K_i + 2} \left[ \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial X^2} - (K_i - 3) \frac{\partial \tilde{\Psi}_{sg}}{\partial X} \right] \quad \text{and}
\]

\[
b > \frac{2}{3} - \frac{d_i}{3} \left[ \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial Y^2} - \frac{3}{3} \frac{\partial \tilde{\Psi}_{sg}}{\partial Y} \right],
\]

\[
e = b - d_i \frac{\partial \tilde{\Psi}_{sg}}{\partial X} - 1,
\]

where the values of \( \partial \tilde{\Psi}_{sg}/\partial X, \partial^2 \tilde{\Psi}_{sg}/\partial X^2, \) and \( \partial^2 \tilde{\Psi}_{sg}/\partial Y^2 \) are calculated in \((X, Y) = (1, 0)\). Their analytical expressions are given in Appendix B. The conditions described above depend on the mass ratio between the central mass and the torus mass, on the rotation law (i.e., the specific charge) and on the gradient and laplacian of the self-gravitational potential of the ring, located in \((1, 0)\) with a good accuracy. So the conditions given above are valid to the first order. They can give us information on the role of the self-gravity, on the conditions of equilibrium and on the configuration of tori. We can play with various parameters \( d_i, e, b, \) and \( r_c \).

3.1. Influence of Self-gravity on the Equilibrium Conditions

The conditions vary with the rotation law, so they are functions of \( K_i \). For definiteness, here we are going to study two different rotation laws: a rotation with constant angular momentum, \( K_i = -1 \), and the rigid rotation (i.e., constant angular velocity), \( K_i = 1 \).

3.1.1. Constant Angular Momentum: \( K_i = -1 \).

In the case where the self-gravity is neglected, \( d_i = 0 \), for the family II and IV, the conditions (26) and (27) are respectively graphically represented in Figure 1. Now, we add the self-gravity by giving a positive value to \( d_i \). The result is shown in the Figure 1 at the top for family II and at the bottom for family IV. We chose two different values for \( d_i \): \( d_i = 0.1 \) (in the middle); the torus mass represents 1% of the central mass and \( d_i = 0.5 \) (on the right). We can clearly see the influence of the self-gravity on the conditions. For both distributions, the range rises. If we impose \( K_2/\sqrt{GM} = 2.5 \), the range of possible value of \( r_c \) changes. It is shown in Table 1 for both families.
3.1.2. Rigid Rotation: $K_i = 1$.

For the case of a rigid rotation, $K_i = 1$, if we neglect the self-gravity, i.e., $d_i = 0$, the conditions (26) and (27) are not satisfied, as seen in Slaný et al. (2013). No equilibrium configuration of equatorial tori can be found for this specific rotation law. In the case where $d_i = 0$, the conditions (26b) and (27b) become

1. Family II

$$b < -1 + \frac{2d_i}{3} \left[ \frac{\partial^2 \tilde{\psi}_{SG}}{\partial X^2} + \frac{1}{2} \frac{\partial \tilde{\psi}_{SG}}{\partial X} \right],$$  
\hspace{3cm} (28a)$$

$$b > \frac{2}{3} - \frac{d_i}{3} \left[ \frac{\partial^2 \tilde{\psi}_{SG}}{\partial Y^2} - 3 \frac{\partial \tilde{\psi}_{SG}}{\partial Y} \right].$$   
\hspace{3cm} (28b)$$

2. Family IV

$$b < \frac{d_i}{3} \left[ \frac{\partial^2 \tilde{\psi}_{SG}}{\partial X^2} + 3 \frac{\partial \tilde{\psi}_{SG}}{\partial X} \right],$$  
\hspace{3cm} (29a)$$

$$b > \frac{2}{3} - \frac{d_i}{3} \left[ \frac{\partial^2 \tilde{\psi}_{SG}}{\partial Y^2} - 3 \frac{\partial \tilde{\psi}_{SG}}{\partial Y} \right].$$   
\hspace{3cm} (29b)$$

For family II, the conditions above are valid if

$$\frac{5}{3} < \frac{d_i}{3} \left( \frac{\partial^2 \tilde{\psi}_{SG}}{\partial Y^2} + 2 \frac{\partial \tilde{\psi}_{SG}}{\partial X} - 2 \frac{\partial \tilde{\psi}_{SG}}{\partial X} \right),$$  
\hspace{3cm} (30)$$

and for family IV, if

$$\frac{2}{3} < \frac{d_i}{3} \left( \frac{\partial^2 \tilde{\psi}_{SG}}{\partial Y^2} + \frac{\partial \tilde{\psi}_{SG}}{\partial X} - \frac{\partial \tilde{\psi}_{SG}}{\partial X} \right).$$   
\hspace{3cm} (31)$$

where the value of $\partial \tilde{\psi}_{SG}/\partial X$, $\partial^2 \tilde{\psi}_{SG}/\partial X^2$, and $\partial^2 \tilde{\psi}_{SG}/\partial Y^2$ are calculated in $(X, Y) = (1, 0)$. The right sides of inequality (30) (full line) and inequality (31) (dashed line) are plotted as a function of $d_i$ and compared to the left sides in Figure 2.

We see that conditions (30) and (31) are valid when the parameter $d_i$ is greater than 1.06 and 0.6, respectively. This means that solutions in rigid rotation can exist for these
families if the strength of the self-gravity is high enough. It is an interesting result because in the case of self-gravitating tori without a magnetic field and central mass, such solutions in rigid rotation exist too (Hachisu 1986). On the other hand, as also seen in the case of test fluid orbiting in central gravitational and dipolar magnetic fields, it is possible to find specific charge distributions (other than families II and IV studied there) allowing a stationary equilibrium torus in rigid rotation.

3.2. Influence of Self-gravity on the Torus Equilibrium of Both Distributions

To show the influence of self-gravity, we choose three configurations with the same rotation law, specific charge distribution, and location of maximum of pressure, and we vary the value of \( d_1 = [0, 0.1, 0.5] \). For family II, according to Section 3.1.1, if we choose the pair \( (r_c = 6, K_2/\sqrt{GM} = 2.5) \) then equilibrium is possible. We perform three tests, described in Table 2. They are graphically presented in Figure 3. For all of them we choose \( c = -0.02r_c \). This constant \( c \) determines the surface where the pressure is equal to zero. It defines the boundary and the shape of the torus. For family II, we can see that the morphology of the solution does not change. The pressure field has a toroidal shape for all three figures, while the specific charge has a cylindrical topology. The main change appears in the charge of the torus. For \( d_1 = 0 \) and \( d_1 = 0.1 \) the torus is positively charged but it is negatively charged for \( d_1 = 0.5 \). This information is given by the value of \( \mu C/\sqrt{GM} \) in Table 2. Another interesting effect is that the maximum of pressure raises with the value of \( d_1 \). The torus grows with the strength of the self-gravity. To check our assumption about the dominance of the dipolar field over the magnetic field produced by the current, we compare their magnitude close to the outer edge (closer to the center, the central dipole component grows stronger). For the torus configuration given by the test in Tables 2 and 3, via numerical integration we compute the total electric charge. We estimate the angular velocity at the pressure maximum and we calculate the magnetic field of a rotating narrow charged ring with the same charge and the same angular velocity. We obtain, for all the tests presented in the paper, \( B_{\text{torus}}/B_{\text{dipole}} = 10^{-1} \sim 10^{-2} \). The dipole field is stronger than the one produced by the torus.

For family IV, we perform the same test but with \( r_c = 4, K_2/\sqrt{GM} = 2.5 \), and \( c = -0.09r_c \); see Table 3 for the value of \( \mu C/\sqrt{GM} \) for each test. We can see that the topology has changed. There is the formation of a cusp that corresponds to the minimum of the pressure. The resulting configurations are represented in Figure 4. The effect of the self-gravity on the increase of central pressure and the extension of the torus seems to be the same as in the case of family II.

| Test | \( d_1 \) | \( \mu C/\sqrt{GM} \) |
|------|--------|-----------------|
| 1    | 0      | 0.041           |
| 2    | 0.1    | 0.010           |
| 3    | 0.5    | -0.113          |

4. OFF-EQUATORIAL TORI: INCOMPRESSIBLE CASE

We perform the same study as performed previously. We suppose now that the maximum is located in \((1, Y_c)\), but we keep the ring in \((1, 0)\), which simplifies the equations for the conditions of the torus existence. First, we search these conditions for both specific charge distributions (Equations (19) and (20)). The value of the constant \( b \) is not free, just as before, but it is now fixed by conditions (24) and (25) and is the same for both families:

\[
b = \frac{2}{3\sqrt{1 + Y_c^2}} + d_1 \left[ \frac{\partial \tilde{V}_{\text{Sur}}}{\partial X} - \frac{\partial \tilde{V}_{\text{Sur}}}{\partial Y} \frac{1 - 2Y_c^2}{3Y_c} \right]
\]

(32)

According to Equation (25), the conditions are given by the following inequalities.

\[
b F_1(1, Y_c) - d_1 F_2(1, Y_c) < 0,
\]

(33)

and

\[
b^2 G_1(1, Y_c) - d_1 G_2(1, Y_c) > 0.
\]

(34)

The form is the same for both families II and IV, \( F_1, F_2, G_1, \) and \( G_2 \) are complicated functions of \( Y_c \) and the first, second, and mixed derivatives of the self-gravitational potential \( \tilde{V}_{\text{Sur}} \). The expressions of these functions depend on the family, and are shown in Appendix D. The constant \( e \) is given, for families II and IV, respectively, by

\[
e = \frac{-\left[ b - d_1 \left( \frac{\partial \tilde{V}_{\text{Sur}}}{\partial X} - \frac{\partial \tilde{V}_{\text{Sur}}}{\partial Y} \right) \right]}{2(1 + Y_c^2)^{3K_2/2 - 3/4}},
\]

(35a)

\[
e = \frac{-\left[ b - d_1 \left( \frac{\partial \tilde{V}_{\text{Sur}}}{\partial X} - \frac{\partial \tilde{V}_{\text{Sur}}}{\partial Y} \right) \right]}{2(1 + Y_c^2)^{3K_2/2 - 3}},
\]

(35b)

and is always negative. Then for \( K_2 > 0 \) (positive rotation) and \( \mu > 0 \) (given the orientation of the magnetic field), there are only negatively charged off-equatorial toroidal configurations, as in Slaný et al. (2013). Choosing the loci of the torus center at \( Y_c = 1 \) for both families, we plot conditions (33) and (34) for \( K_1 = -1 \) and \( K_1 = 1 \) in Figure 5.

We can see that for the equatorial case for these two families, solutions with a rigid rotation law do not exist because conditions (33) and (34) are not satisfied (see the right panels of Figure 5). The main difference with the equatorial case is that there is always a solution for \( K_1 = -1 \). The conditions seem to be satisfied everywhere.

Here, as done above, we produce three maps of enthalpy for three values of \( d_1 \). We set \( r_c = 18.75 \sqrt{2}/2, Y_c = 1, c = -0.015r_c, \) and \( K_1 = -1 \) for family II. The value of \( \mu C/\sqrt{GM} \) for each case is given in the left part of Table 4. The results are shown in Figure 6. Next, we produce the same figures for family IV with the same parameters, except \( c = -0.02r_c \). The value of \( \mu C/\sqrt{GM} \) is given in the right part of Table 4. The map linked to this family is shown in Figure 7. We can see for both families, that for \( d_1 = 0 \), there are toroidal off-equatorial structures located above and under the equatorial plane. Given a value of \( c \), by increasing \( d_1 \), the morphology of the solutions changes. The off-equatorial toroidal structures are linked to each other by the equatorial plane.

5. COMPARISON WITH THE SELF-CONSISTENT FIELD METHOD

Whereas the above-described approximation has enabled us to develop a systematic classification across the parameter...
space of the constraints for the existence of topologically
different toroidal con-
figurations, the adopted limit of an
infinitesimally narrow gravitating ring is an idealization. An
astrophysically realistic model will require taking the spatially
extended distribution of the torus material and the corresp-
onding pressure and density distribution. In order to relax the
mentioned restriction we can compare the analytical description
with a corresponding spatially extended con-
figuration con-
structed numerically. This will allow us to assess the accuracy
of the approximation, although the numerical solution does not
provide a methodology for the classification. To this end we
employ the self-consistent field method (SCF) that was
developed initially by Ostriker & Mark (1968).

The SCF approach is based on the integral form of Euler’s
Equation (6). Subsequently, variants of this method were
developed to describe the internal structure of rotating stars and
to explore different figures of equilibrium, namely, the structure
of rotating polytropes (Blinnikov 1975; Hachisu 1986; Tomi-
mura & Eriguchi 2005). Originally the method was applied in
the non-magnetized case. We have thus programmed a
modi-
fied version of the SCF method, where we introduce a
non-vanishing imposed magnetic term,
and we make a
comparison with the analytical toy model approximation.

The numerical setup is based on an iterative scheme for the
density. In our case this translates to solving Equation (22). The
iterative loop is initiated by setting the rotation law
e.g., the
rigid rotation, or the constant angular momentum density,
the radial profile of the specific charge density (e.g., a power-law
dependence), the polytropic index \( n = 0 \) for the

Table 3

| Test  | \( \dot{d}_t \) | \( \mu C/\sqrt{GM} \) |
|-------|---------------|---------------------|
| Test 1| 0             | \( \sim 3.6 \)       |
| Test 2| 0.1           | \( \sim 3.4 \)       |
| Test 3| 0.5           | \( \sim 2.6 \)       |

Figure 3. Maps of enthalpy distribution in positively charged tori (\( \tilde{H} \)) for \( \dot{d}_t = 0 \) (top left), \( \dot{d}_t = 0.1 \) (top right) and in negatively charged tori for \( \dot{d}_t = 0.5 \) (bottom left) for Family II. The parameters used to plot these graphs are given in Table 2. At the bottom right, the corresponding equatorial pressure profiles are shown a solid line
line for \( \dot{d}_t = 0 \), a dashed line for 0.1, and a dashed-dotted line for 0.5.

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incompressible case), the location of inner and outer edges, and the value of constants $d_t$ and $e$.

Additionally, in the magnetized case we introduce the external (dipole-type) magnetic field, and we also set the electric charge density profile within the fluid. Let us note that the enthalpy vanishes at the inner and outer edges of the volume occupied by the rotating fluid. Then, the sequence of steps is as follows.

1. Set an initial profile of density.
2. By employing Poisson’s equation, determine the corresponding gravitational potential.
3. By using the Equation (22) in the inner and the outer edge, obtain constants $b$ and $c$.
4. Calculate the enthalpy distribution.
5. Assuming the polytropic equation of state for the fluid, the pressure and the enthalpy are linked to the density. Using this relation, the density profile can be updated for the subsequent iteration step.
6. Compare the new density with the initial one. The equilibrium has been achieved if the two profiles coincide with each other within a pre-defined accuracy; otherwise the routine loops back to the first step until the variables reach convergence.

For further details on our implementation of the SCF method we refer the reader to a more detailed exposition given elsewhere (A. Trova et al. 2016, in preparation). Here, for the sake of definiteness we compare the outcome of the above-described procedure with test 2 of Table 2. We set $\mu C/\sqrt{GM} = 0.010$, $d_t = 0.1$, and we assume the same rotation law and specific charge that are imposed to start the computation. The criterion for achieving the convergence requires that the relative difference between the constant $b$ (or $a$ and $c$) values at subsequent steps reaches five times the machine epsilon precision (around $10^{-6}$ in single precision). We find that the equilibrium is reached, typically, after 10 to 15 iterations. We obtained $K_2/\sqrt{GM} = 2.45$ and $c/r_e = -0.0247$. These results are consistent with the values used in our analytical approach; see Section 3.2.

The equatorial enthalpy profile is plotted in Figure 8 for both approaches. We can thus conclude that the equatorial density
obtained with our approximation method is close to the corresponding profile reached by the SCF method.

We produced the same comparison in the case of the off-equatorial tori. We expect that the results are less consistent with the SCF due to the one-ring approximation in the equatorial plane. We performed another test using a double-ring approximation for the gravitational potential. The first ring is located in \((Y_1, c)\) and the second one is in \((-Y_1, c)\). The enthalpy profile in \(Y_c = Z_c/r_c\) (altitude of the maximum of pressure) as a function of \(X\) is plotted in Figure 9. As expected, the analytical profile from the one-ring approximation comes out quite inaccurate, however, the accuracy is much improved with the double-ring approach. The latter is clearly more precise, although the precision is still not as good as the result from the SCF method. One interesting point is that the morphology does not change. Instead, for both cases we find the two lobes under and above the equatorial plane.

### Table 4

| \(d_t\) | Family II | Family IV |
|---|---|---|
| 0 | \(\mu C/\sqrt{GM} \sim -1.633\) | \(\mu C/\sqrt{GM} \sim -375\) |
| 0.1 | \(\mu C/\sqrt{GM} \sim -1.723\) | \(\mu C/\sqrt{GM} \sim -396\) |
| 0.5 | \(\mu C/\sqrt{GM} \sim -2.055\) | \(\mu C/\sqrt{GM} \sim -471\) |

6. DISCUSSION

From a conceptual point of view, the presented work introduces a pure topological survey of various globally charged and non-conductive perfect fluid toroidal configurations formed due to complex gravito-electromagnetic interactions. The combined actions of the fluid torus self-gravity, together with the electric charge distribution, were treated separately in previous papers. Here, considered together, they represent a basic theoretical view on the studied problem. There is, however, a promising astrophysical contextualization, as mentioned in the introduction. In the following, we comment on the circumstances essential for the astrophysical feasibility of the model: (i) a global non-zero charge distribution, (ii) ionization as a process for free charges generation, and (iii) an estimation of the particular physical...
characteristics of the considered circling matter. In the end, we also present a whole summary of this work.

6.1. Non-zero Charge and Ionization

The effect of the Lorentz force on the non-vanishing net charge carried by the torus material can help to support the vertical structure of the torus against its own self-gravity, thereby maintaining the geometrical thickness, which would otherwise tend to collapse into the equatorial plane. A non-vanishing net electric charge distribution can develop via various mechanisms, depending on the nature of the medium. Free charges are created by ionization of gas, the effects of intense irradiation of dust grains by the central source, and charge exchange within complex plasmas. Charge separation operates in organized magnetic fields (see Kovár et al. 2014 for further references), so large-scale regions of non-vanishing charge can develop even if the whole system is globally neutral. On the other hand, the electric charge of a central body (Wald’s charge) is supposed to be negligibly small because of selective accretion that helps to neutralize the body in the center.

AGNs are an important example of objects where the gas ionization ranges from small values (at large distance and low luminosity) up to a fully ionized medium in the energetic environment near the black hole (Krolik 1999). Typically, gas becomes partially ionized when it is the subject of irradiation by X-rays, and these become more intense toward the central source. The irradiation mechanism provides free electric charges and ions. The photoionization of the surface of the inner accretion disk (by a hot corona) can be characterized in terms of ionization parameter, \( \xi = 4 \pi F_X/n_H \), where \( F_X \) is the incident X-ray flux and \( n_H \) is the hydrogen number density (Ballantyne et al. 2011). A measurement of \( \xi \) can therefore provide information on the ionization profile and the density of the environment as functions of illuminating conditions, and thus the number of free charges. Typically, \( \xi \) spans from negligible values at the outer edge of the accretion disk up to \( \gtrsim 10^7 \) at a few gravitational radii, where the medium is almost fully ionized (Ballantyne et al. 2002; Róžańska et al. 2002).

However, with respect to electric charge content in realistic conditions, one needs to also take into account the emergence of dust (Laor & Draine 1993; Krishna Swamy 2005). Dust can develop when the grain temperature does not exceed above the sublimation temperature \( \sim 1500 \) K. Dust grains embedded within the gas will be charged to form complex plasma of a quasar. Charges bind dust to the surrounding partially ionized
The Eddington luminosity ratio is then reduced by a factor equal to dust cross-section (per proton, appropriately weighted for the spectral energy distribution profile) to the Thomson cross-section, which comes out of the order of $1: 10^3$ (Fabian 2012).

Figure 7. Same as Figure 6, but for Family IV.

Figure 8. The equatorial radial profile of enthalpy is shown for the two approaches. The result from the SCF method is plotted by a solid line, and the points from the analytical approach are indicated by crosses.

6.2. Physical Characteristics

The prime aim of this work is the discussion of a general framework for the combined gravito-electromagnetic actions...
studied through the enthalpy profiles of the formed tori mapping their geometries. An important challenge for further investigation is a detailed study of other physical characteristics (such as pressure, mass density, temperature, specific charge, etc.) describing more of the tori microphysics. This study requires precise adjustment of thermodynamical relations, representing a delicate problem. This means choosing reliable pressure–mass density and pressure–temperature relations. Moreover, the desired profiles of physical characteristics are sensitive to the central mass, the magnetic field strength, and the torus size, etc. A detailed quantitative discussion is beyond the scope of the present paper (a separate work in progress).

In order to give a rough view of the physical characteristics throughout the tori, we refer to our recent work (Kovář et al. 2016), where the same problem of electrically charged and non-conductive perfect fluid toroidal configurations in central gravitational and magnetic dipolar fields is discussed within a general relativity framework; there, however, the self-gravity of the tori is not apart of the discussion. In that paper, we concluded that the circling fluid with a constant specific angular momentum, under a polytropic equation of state, and with pressure being related to the temperature by the ideal gas relation form astrophysically relevant equatorial and off-equatorial tori with feasible pressure, mass density, and temperature profiles; in the presented sample cases reach their maxima in centers in intervals $p_{\text{max}} \simeq 10^{13} - 10^{16}$ Pa, $\rho_{\text{max}} \simeq 10^{-2} - 10^{3}$ kg m$^{-3}$, $T_{\text{max}} \simeq 10^{3} - 10^{4}$ K, and with average specific charges $10^{-11}$ of the proton one. We also showed that such tori must be relatively tiny in comparison with the central object. Being located close to the central object at radii $\sim 10GM/c^2$, their cross-sectional size is very small (a slender torus approximation). However, let us note that specific numerical values of the physical quantities can be tuned over a wide span in our general scheme because the initial assumptions contain a number of degrees of freedom for which we lack clear observational constraints.

7. CONCLUSION

In this paper, we discussed the impact of the self-gravity on the conditions of existence of charged fluid tori and their morphologies. The fluid, whose particles carry electrical charges, was assumed to be perfect and incompressible, the latter due to easier handling with equations. It was influenced by its own gravitational field, and by the gravitational potential and the dipolar magnetic field of the central mass. We base our study on the work of Slaný et al. (2013). We proceeded in the same way but included self-gravity. We analyzed the Euler’s equation to find stationary toroidal configurations for two families of specific charge.

The first interesting result is that the condition of existence of the tori changes with the strength of self-gravity, as characterized by the parameter $d_c$. The parameters allowing the existence of equatorial tori change with the value of $d_c$. For off-equatorial tori, as in the case without self-gravity, there is always a solution that is negatively charged. For both these families, for $K_2 > 0$ (positive direction of motion) and for $\mu > 0$ (orientation of the magnetic field), positively charged tori do not exist. The off-equatorial tori would have a positive charge only for $K_2 < 0$ or $\mu < 0$.

Another interesting result is the impact of self-gravity on the charge of the equilibrium torus. As we saw in Section 3.2, the sign of the total charge can change. On the other hand, the morphologies of tori are similar to those for the non-self-gravitating case. We found the toroidal configuration, the closed iso-bars with cusps, and the toroidal off-equatorial configurations. The maximum of pressure, however, rises with the value of $d_c$, and the torus becomes thicker, which makes sense because higher gravity implies higher pressure to balance the gravitational and electric forces. Finally, the last interesting point is the possibility of these two families having solutions in rigid rotation, which exist for a self-gravitating torus without a spherical gravitational and a dipolar magnetic field too (Hachisu 1986), but not for non-self-gravitating tori with the specific charge distributions described in Slaný et al. (2013), from which we analyzed two out of four exemplary possibilities in the complete classification.

The approach described in this paper can serve as a useful test bed for comparisons with other methods. In particular, it provides us with better insight into conditions that define the form of the electrically charged configurations. The method allows us to produce a relatively precise approximation to their structure, taking self-gravity of the fluid into account. While the precision of the method can be verified numerically in the selected cases, e.g., by employing the SCF scheme, the closed analytical form provides a way to set constraints on the existence of different configurations.

While the combination of a large-scale organized (dipole-like) magnetic component and a non-vanishing net charge of the fluid are required to allow the emergence and stability of toroidal structures outside the equatorial plane, self-gravity acts against them. It was therefore interesting to verify, as we did in this paper, that the resulting lobes of matter above and below the equatorial plane can persist even when self-gravitational force is taken into account.

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APPENDIX A

DIFFERENTIAL EQUATIONS FOR THE MAGNETIC POTENTIAL

In cylindrical coordinates $(R, \phi, Z)$ we have,

$$\mathbf{B} = \text{rot} \mathbf{A} = \begin{bmatrix} \frac{\partial A_\phi}{\partial Z} \\ 0 \\ \frac{\partial}{\partial R} (RA_\phi) \bigg/ R \end{bmatrix}. $$ (36)
So the Lorentz force is given by
\[
\mathbf{F} = \frac{\rho v \mathbf{e}_\phi \times \mathbf{B}}{\rho_m} = q v \mathbf{e}_\phi \times \mathbf{B} = \frac{q v \mathbf{e}_\phi \times \mathbf{B}}{R} \frac{\partial (RA_\phi)}{\partial R}. \tag{37}
\]

We need \( \nabla M = -\mathbf{F}/\rho_m \). Then
\[
\begin{vmatrix}
\frac{q v \mathbf{e}_\phi \times \mathbf{B}}{R} & \frac{\partial (RA_\phi)}{\partial R} & 0 \\
\frac{q v \mathbf{e}_\phi \times \mathbf{B}}{R} & 0 & \frac{\partial (RA_\phi)}{\partial Z} \\
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial M}{\partial R} & 0 \\\n0 & \frac{\partial M}{\partial Z} \\
\end{vmatrix}. \tag{38}
\]

APPENDIX B
VALUE OF THE FIRST AND SECOND DERIVATIVES OF THE GRAVITATIONAL POTENTIAL OF THE RING
IN (1, \( Y_\text{c} \))

The normalized gravitational potential is given by
\[
\tilde{\psi}_{\text{Sg}} = -\sqrt{\frac{\pi}{X}} kK(k) \quad \text{with} \quad k = \frac{2\sqrt{X}}{\sqrt{(1 + X)^2 + Y^2 + \lambda^2}}. \tag{39}
\]

The first derivative of \( \tilde{\psi}_{\text{Sg}} \) with respect to the normalized radius \( X \) can be written as
\[
\frac{\partial \tilde{\psi}_{\text{Sg}}}{\partial X} = -\frac{1}{\pi} \sqrt{\frac{X}{k}} E(k) - 1 \frac{1}{2} \sqrt{X} kK(k) \tag{40}
\]
where \( \frac{\partial k}{\partial X} = \frac{k}{2X} - k^3 \frac{(1 + X)}{2X^2} \).

The second derivative with respect to \( X \) is given by
\[
\frac{\partial^2 \tilde{\psi}_{\text{Sg}}}{\partial X^2} = k(k) \left[ \frac{1}{X} \frac{\partial k}{\partial X} \right] \left[ \frac{1}{X} \frac{\partial k}{\partial X} \right] - 3k \frac{\sqrt{X}}{4X^3} \left[ \frac{\partial k}{\partial X} \right] \tag{41}
\]
with \( k^2 = 1 - k \). Then
\[
\frac{\partial^2 k}{\partial X^2} = \frac{1}{2X} \left( -1 - \frac{k}{2X} \right) - 3k \frac{\sqrt{X}}{4X^3} \left[ \frac{\partial k}{\partial X} \right] \tag{42}
\]
with \( k^2 = 1 - k^2 \). The first derivative with respect to \( Y \) is
\[
\frac{\partial \tilde{\psi}_{\text{Sg}}}{\partial Y} = \frac{Y}{\pi} \frac{k_l^2}{k^2} E(k) \tag{43}
\]
and
\[
\frac{\partial^2 k}{\partial Y^2} = -\frac{3k^2 \frac{\partial k}{\partial Y}}{4X} \tag{44}
\]
In (1, \( Y_\text{c} \)), Equation (45) becomes
\[
\frac{\partial^2 \tilde{\psi}_{\text{Sg}}}{\partial Y^2} = -\frac{Y^2 k_l^2}{16 k^2} \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] \tag{45}
\]
and in \( Y = 0 \)
\[
\frac{\partial^2 \tilde{\psi}_{\text{Sg}}}{\partial Y^2} = \frac{k_l^2}{4X} \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] \tag{46}
\]
To finish, we calculate the second-order mixed derivatives of \( \tilde{\psi}_{\text{Sg}} \). We differentiate Equation (45) with respect to \( X \).
\[
\frac{\partial \tilde{\psi}_{\text{Sg}}}{\partial X \partial Y} = \frac{1}{\pi} \frac{k_l^2}{k^2} \left[ \frac{1}{X} \frac{\partial k}{\partial X} \right] \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] - \frac{k_l^2}{4X} \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] \tag{47}
\]
In (1, \( Y_\text{c} \)), Equation (52) writes
\[
\frac{\partial^2 \tilde{\psi}_{\text{Sg}}}{\partial X \partial Y} = \frac{k_l^2}{8Y} \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] \left[ \frac{1}{X} \frac{\partial k}{\partial Y} \right] \tag{48}
\]
APPENDIX C

EXPRESSION OF THE DERIVATIVES OF THE ENTHALPY

1. Family II according to Equation (22), we have

\[
a \frac{\partial \tilde{H}}{\partial X} = \frac{X}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial \tilde{\Psi}_{sg}}{\partial X} + bX^{2K_t-1} \\
+ eX^{-2K_t}(X^2 + 2Y^2)^{3K_t/2-7/4}(2Y^2 - X^2),
\]

(53)

\[
a \frac{\partial \tilde{H}}{\partial Y} = \frac{Y}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial \tilde{\Psi}_{sg}}{\partial Y} \\
- 3eX^{-2K_t+1}Y(X^2 + 2Y^2)^{3K_t/2-7/4},
\]

(54)

\[
a \frac{\partial^2 \tilde{H}}{\partial X^2} = \frac{2X^2 - Y^2}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial X^2} + (2K_t - 1)bX^{2K_t-2} \\
- \frac{e}{2} \frac{(X^2 + Y^2)^{3K_t/2-11/4}}{X^{3K_t+1}}(-8K_tX^2Y^2) \\
+ 8K_tY^4 + 2K_tX^4 - 3X^2 + 18X^2Y^2),
\]

(55)

\[
a \frac{\partial^2 \tilde{H}}{\partial Y^2} = \frac{-X^2 - 2Y^2}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial Y^2} \\
- \frac{3e}{2} \frac{(X^2 + Y^2)^{3K_t/2-11/4}}{X^{3K_t+1}}(2X^2 - 5Y^2 + 6K_tY^2),
\]

(56)

\[
a \frac{\partial^2 \tilde{H}}{\partial X \partial Y} = \frac{3XY}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial X \partial Y} \\
- \frac{3e}{2} \frac{(X^2 + Y^2)^{3K_t/2-11/4}}{X^{3K_t+1}}Y \\
\times (2K_tX^2 - 4K_tY^2 - 5X^2 + 2Y^2).
\]

(57)

and

2. Family IV

\[
a \frac{\partial \tilde{H}}{\partial X} = \frac{X}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial \tilde{\Psi}_{sg}}{\partial X} + bX^{2K_t-1} \\
+ e \left( \frac{X}{\sqrt{X^2 + Y^2}} \right)^{3(1-K_t)} \frac{2Y^2 - X^2}{(X^2 + Y^2)^{\frac{3}{2}}} X^{K_t},
\]

(58)

\[
a \frac{\partial \tilde{H}}{\partial Y} = \frac{Y}{(X^2 + Y^2)^{\frac{3}{2}}} - d_t \frac{\partial \tilde{\Psi}_{sg}}{\partial Y} \\
- 3e \left( \frac{X}{\sqrt{X^2 + Y^2}} \right)^{3(1-K_t)} \frac{X^{K_t+1}Y}{(X^2 + Y^2)^{\frac{3}{2}}},
\]

APPENDIX D

DESCRIPTION OF THE F1, F2, G1, AND G2.

In this appendix, we give the explicit form of functions \( F_1, \ F_2, \ G_1, \) and \( G_2, \) which appear in Section 4 for both distributions. In each function, the first, second, and mixed derivatives of \( \Psi_{sg} \) are taken in \( X = 1 \) and \( Y = Y_c. \)

1. Family II

\[
F_1 = \frac{[(10K_t + 5) + Y_c^4(16K_t - 4) + Y_c^2(8K_t + 4)]}{4(1 + Y_c^2)^2},
\]

(63)

\[
F_2 = \frac{3}{2} \frac{(2 - Y_c^2)}{(1 + Y_c^2)^2} \left[ \frac{\partial \tilde{\Psi}_{sg}}{\partial X} - \frac{1}{3Y_c} \frac{\partial \tilde{\Psi}_{sg}}{\partial Y} \right] - \frac{\partial^2 \tilde{\Psi}_{sg}}{\partial X^2} \\
\times \frac{Y_c^2(18 - 8K_t) + 8K_tY_c^4 + (2K_t - 3)}{2(1 + Y_c^2)^2},
\]

(64)

\[
G_1 = \frac{3}{2} \frac{4K_t - 2Y_c^2}{(1 + Y_c^2)^2} \frac{(2K_t + 1)(3K_t - 1)}{2}.
\]

(65)

The function \( G_2 \) is a combination of various functions that depend, as \( F_2, \) on \( Y_c, \) the first, second, and mixed derivatives of the self-gravitational potential

\[
G_2 = b(F_1H_t + F_2H_t + 2H_3H_4) + d_i(F_2H_t - H_2),
\]

(66)
where

\[
H_1 = \frac{3(1 - 2Y_c^2)}{2(1 + Y_c^2)^2} \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{Y_c}{3Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right] - \frac{\partial^2 \bar{V}_{sg}}{\partial Y^2} - \frac{3(6K_1Y_c^2 - 5Y_c^2 + 2)}{2(1 + Y_c^2)^2} \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right].
\]

(67)

and

\[
H_2 = \frac{3Y_c^2}{4(1 + Y_c^2)^2}(6K_1 - 1),
\]

(68)

\[
H_3 = \frac{3Y_c}{4(1 + Y_c^2)^2}[(1 + 2K_1) + Y_c^2(2 - 4K_1)].
\]

(69)

2. Family IV

The functions are defined as follows.

\[
F_1 = \frac{(5K_1 + 1) + Y_c^4(8K_1 - 8) + Y_c^2(4K_1 + 8)}{(1 + Y_c^2)^2},
\]

(71)

\[
F_2 = -\frac{3}{2} \left( \frac{2 - Y_c^2}{1 + Y_c^2} \right)^2 \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{3Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right] - \frac{\partial^2 \bar{V}_{sg}}{\partial X^2} - \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right]
\]

\[
\times \frac{Y_c^2(15 - 4K_1) - Y_c^4(6 - 4K_1) - (3 - K_1)}{2(1 + Y_c^2)^2},
\]

(72)

\[
G_1 = \frac{3Y_c^2}{(1 + Y_c^2)^2}(-2 - 4K_1 + 3K_1^2).
\]

(73)

The function \(G_2\) is given by Equation (66) with the functions \(H_1, H_2, H_3,\) and \(H_4\) defined as follows.

\[
H_1 = \frac{3(1 - 2Y_c^2)}{2(1 + Y_c^2)^2} \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{Y_c}{3Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right] - \frac{\partial^2 \bar{V}_{sg}}{\partial Y^2} - \frac{3(1 - 7Y_c^2 + 3K_1Y_c^2)}{2(1 + Y_c^2)^2} \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right],
\]

(74)

\[
H_2 = \frac{3Y_c^2}{2(1 + Y_c^2)^2} (3K_1 - 5),
\]

(75)

\[
H_3 = \frac{3Y_c}{2(1 + Y_c^2)^2} [(K_1 - 1) + Y_c^2(4 - 2K_1)].
\]

(76)

\[
H_4 = -\frac{9Y_c}{2(1 + Y_c^2)^2} \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{3Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right] - \frac{\partial^2 \bar{V}_{sg}}{\partial X^2} - \frac{3Y_c(-2K_1Y_c^2 + K_1 + 4Y_c^2 - 4)}{2(1 + Y_c^2)^2} \times \left[ \frac{\partial \bar{V}_{sg}}{\partial X} - \frac{1}{Y_c} \frac{\partial \bar{V}_{sg}}{\partial Y} \right].
\]

(77)

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