Third post-Newtonian dynamics of compact binaries: Noetherian conserved quantities and equivalence between the harmonic-coordinate and ADM-Hamiltonian formalisms

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Abstract

A Lagrangian from which derive the third post-Newtonian (3PN) equations of motion of compact binaries (neglecting the radiation reaction damping) is obtained. The 3PN equations of motion were computed previously by Blanchet and Faye in harmonic coordinates. The Lagrangian depends on the harmonic-coordinate positions, velocities and accelerations of the two bodies. At the 3PN order, the appearance of one undetermined physical parameter $\lambda$ reflects an incompleteness of the point-mass regularization used when deriving the equations of motion. In addition the Lagrangian involves two unphysical (gauge-dependent) constants $r'_1$ and $r'_2$ parametrizing some logarithmic terms. The expressions of the ten Noetherian conserved quantities, associated with the invariance of the Lagrangian under the Poincaré group, are computed. By performing an infinitesimal “contact” transformation of the motion, we prove that the 3PN harmonic-coordinate Lagrangian is physically equivalent to the 3PN Arnowitt-Deser-Misner Hamiltonian obtained recently by Damour, Jaranowski and Schäfer.
I. MOTIVATION AND RELATION TO OTHER WORKS

The long-standing problem of the gravitational dynamics of compact bodies has become very important in recent years because of the need to construct accurate templates for detecting the gravitational waves from inspiralling compact binaries in future experiments like LIGO and VIRGO [1-3]. Concerning the two-body problem the current state of the art is the 3PN approximation, corresponding to the inclusion of all the relativistic corrections up to the order $1/c^6$ (where $c$ is the velocity of light) with respect to the Newtonian acceleration. Up to the 2.5PN or $1/c^5$ approximation the equations of motion are well known, as they have been derived by many different methods with complete agreement on the result [4-17]. They have already been used for constructing the 2.5PN-accurate templates of inspiralling compact binaries [18-20].

To the 3PN order, the problem of equations of motion has been pursued by two groups working independently with different methods: on one hand, Jaranowski and Schäfer [21,22] and Damour, Jaranowski and Schäfer [23-25] employ the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation of general relativity; on the other hand, Blanchet and Faye [26-29] work iteratively with the Einstein field equations in harmonic coordinates. Both groups use a regularization based on Hadamard’s concept of “partie finie” to overcome the problem of the infinite self-field of point-like particles. However the details are actually different; notably the second group developed for this problem an extended version of the Hadamard regularization and a theory of generalized functions [27,28]. Both groups found that there remains one and only one physical constant, $\omega_{\text{static}}$ in the ADM-Hamiltonian formalism [21,25] and $\lambda$ in the harmonic-coordinate approach [26,29], that is left undetermined by the point-mass regularization. Furthermore, in the harmonic-coordinate approach, the equations of motion (obtained in Ref. [29]) depend on two additional constants $r'_1$ and $r'_2$ parametrizing some logarithmic terms, but these constants are not physical in the sense that they can be removed by a coordinate transformation. The aim of the present paper is three-fold: (i) to present the Lagrangian of the 3PN dynamics of the compact binary in harmonic coordinates, (ii) to obtain explicitly from it the ten Noetherian conserved integrals of the motion in harmonic coordinates, (iii) to exhibit a contact transformation of the harmonic-coordinate motion to some pseudo-ADM coordinates in order to compare our results [26,29] with the ones obtained by the other group [21,25].

Concerning (i), we find a generalized Lagrangian (i.e. depending on the positions, velocities and accelerations of the bodies) whose variation yields the conservative part of the 3PN equations of motion in harmonic coordinates as found in Ref. [29]. Our second point (ii) is to use the fact that the Lagrangian incorporates the ten symmetries of the Poincaré group (notably the boost symmetry) to compute the ten integrals corresponding to the energy, the linear and angular momenta, and the center-of-mass position. In particular, we find that the energy agrees with the previous result of Ref. [29]. As all these integrals will probably be needed in future work we choose to display them explicitly, despite the length of the expressions. We also give the balance equations they satisfy when the radiation reac-
tion effect is turned on. Finally, the result of point (iii) is that there exists a unique contact transformation of the harmonic-coordinate dynamical variables that changes the generalized Lagrangian into an ordinary Lagrangian (depending on positions and velocities) whose associated 3PN Hamiltonian matches exactly the one given by Damour, Jaranowski and Schäfer [24]. This proves the complete equivalence of the results obtained from the two (rather different) methods followed by the two groups, and constitutes a strong support of the validity of both methods. This equivalence has also been shown independently by the other group [25] (who presents also the formulas needed for computing the conserved quantities).

Notice that it holds if and only if the undetermined constant \( \lambda \) in the harmonic-coordinate formalism and the ambiguity constant \( \omega_{\text{static}} \) in the ADM Hamiltonian are related to each other by

\[
\omega_{\text{static}} = -\frac{11}{3} \lambda - \frac{1987}{840},
\]

a result already obtained in Ref. [26] on the basis of the comparison of the invariant energy of binaries moving on circular orbits. Likely the appearance of the unknown constant \( \lambda \) is not due to a real physical ambiguity, but is associated with an incompleteness of the point-mass regularization. It is probably related to the fact that, starting from the 3PN order, many separate integrals constituting the equations of motion of extended bodies would depend on the internal structure of the objects (e.g. their density profile), even in the limiting case where the radius of the objects tends to zero. Further work is needed to compute the precise value of \( \lambda \). On the other hand, the constants \( r_1' \) and \( r_2' \) occuring in the harmonic-coordinate Lagrangian disappear from the ADM-Hamiltonian (where there are no logarithms), in accordance with the fact that they are pure gauge.

The plan of this paper is as follows. In Section II, motivated by the striking equivalence between the (regularization-related) unknown constants \( \lambda \) and \( \omega_{\text{static}} \), we discuss our method of point-mass regularization and contrast it to the method advocated in [21–25]. Section III is devoted to the theoretical investigations. First we recall the theory of Noetherian conserved quantities in the case of a generalized Lagrangian, and next we show how to eliminate the accelerations in the harmonic-coordinate Lagrangian by a contact transformation at the 3PN order. The reader interested only in the results at the 3PN order can go directly to Section IV, where we present the closed-form expressions of the Lagrangian and the conserved energy, momenta and center of mass in harmonic coordinates, and give the result for the contact transformation as well as the final expressions for the Lagrangian and Hamiltonian in pseudo-ADM coordinates.

II. DISCUSSION ON THE POINT-MASS REGULARIZATION

The equivalence between the respective formalisms of [21–23] and [26–29] is interesting because the two groups have adopted some different approaches regarding the point-mass regularization (chosen in both cases to be based on the Hadamard concept of “partie finie” of
a singular function or a divergent integral \([30,31]\). Essentially the group \([21–25]\) introduced systematically some “ambiguity” parameters in the ADM Hamiltonian whenever the standard Hadamard regularization yielded inconsistent results, while the group \([26–29]\) looked for the most general solution allowed by some basic physical requirements and following from a new, mathematically consistent, Hadamard-type regularization.

More precisely, in our approach \([26–29]\), we adopted some specific variants of the Hadamard regularization which were devised specifically for this problem \([27,28]\). Let \(F\) be a function which is singular at two isolated points \(y_1\) and \(y_2\), and is smooth everywhere else; \(y_1\) and \(y_2\) are the positions of the particles in harmonic coordinates at some given instant \(t\). The Hadamard partie finie of \(F\) at the point \(y_1\), denoted \((F)_1\), is defined as the angular average over all directions of approach to \(y_1\) of the finite term (zeroth order) in the singular expansion of the function around this point. We found that this definition yields a natural extension of the notion of Dirac distribution at the location of a singular point, that we constructed by means of the Riesz delta-function \([32]\). As a result, the “partie finie delta-function” at the point 1, denoted \(\text{Pf}\delta_1\) where \(\delta_1 \equiv \delta(x - y_1)\), is the linear form defined on the set of singular functions of the type \(F\), that associates to any \(F\) the real number \((F)_1\) (see Eq. (6.9) in \([27]\)). Using an integral notation this means that \(\int d^3x \ F \text{Pf}\delta_1 = (F)_1\). (The partie finie delta-function \(\text{Pf}\delta_1\) constitutes a mathematically well-defined version of the so-called “good delta function” of Infeld \([33]\).) In our derivation of the equations of motion at 3PN order, this prescription is employed systematically to compute all the “compact-support” integrals, whose integrand is made of the product of a singular potential with some mass density localized on the two particle world-lines.

By applying the latter definition to the product \(FG\) we obtain \(\int d^3x \ FG \text{Pf}\delta_1 = (FG)_1\), which permits us to give a sense to the more complicated object \(F \text{Pf}\delta_1 \equiv \text{Pf}(F \delta_1)\), composed of the product of a delta-pseudo-function with a function which is singular on its support (such a product being ill-defined in the standard distribution theory). Namely, \(\text{Pf}(F \delta_1)\) is the linear form which associates to any function \(G\) the real number \((FG)_1\). It is important to realize that \(\text{Pf}(F \delta_1) \neq (F)_1 \text{Pf}\delta_1\) in general. This is an immediate consequence of the so-called “non-distributivity” of the Hadamard partie finie, namely the fact that \((FG)_1 \neq (F)_1(G)_1\) for two singular functions \(F\) and \(G\) in general. As an example taken from \([17]\), we have \((U^4)_1 = [(U)_1]^4 + 2[(U)_1]^2[(U)_2]^2\), where \(U = Gm_1/r_1 + Gm_2/r_2\) denotes the Newtonian potential of two particles (with \(r_1 = |x - y_1|\) and \(r_2 = |x - y_2|\)). In the post-Newtonian iteration one can check that the functions involved become singular enough so that the non-distributivity plays an actual role at the 3PN order: for instance, in the example above, \(U^4\) will appear in the metric coefficient \(g_{00}\) with a factor \(1/c^8\) in front, which indeed corresponds to the 3PN order. However, there is no problem linked with the non-distributivity in the equations of motion up to the 2.5PN approximation \([17]\). Therefore, from the 3PN order (but only from that order), it is a mathematically inconsistent regularization prescription to assume at once that \(\int d^3x \ F \text{Pf}\delta_1 = (F)_1\) and \(\text{Pf}(F \delta_1) = (F)_1 \text{Pf}\delta_1\). Faced with this problem, the authors \([21–25]\) have advocated that the breakdown of the distributivity of the Hadamard regularization at the 3PN order is a source of ambiguities. [Actually, in
their first paper, see the Appendix A in Ref. [21], these authors did performed their basic 
computation using the inconsistent rule \( \text{Pf}(F\delta_1) = (F)_1\text{Pf}\delta_1 \). Later in Ref. [23] (see the 
 Appendix A there), they argued that their result was “stable” against a possible violation 
of the latter rule.] By contrast, the authors [24–29] have accepted the special features of the 
partie finie, such as its non-distributivity, and constructed by its mean a mathematically 
consistent regularization, able to give a precise sense to all computations at the 3PN order.

The Hadamard partie finie \((F)_1\) of a singular function involves a spherical average that 
is defined within the spatial hypersurface \( t = \text{const} \) of a global coordinate system like the 
harmonic coordinates. Clearly, this definition is incompatible with the framework of a 
relativistic field theory, and we expect at some level a violation of the Lorentz invariance 
of the equations of motion due to this regularization. Remarkably, such a violation occurs only 
at the 3PN order; up to the 2.5PN order the equations of motion in harmonic coordinates, 
as computed using the regularization \((F)_1\), are Lorentz-invariant [17]. To overcome this 
problem at the 3PN order, it has been necessary to define a “Lorentzian” regularization [28], 
which consists merely of applying the Hadamard partie finie within the spatial hypersurface 
orthogonal to the (Minkowskian) four-velocity of a particle. It was shown in Ref. [29] 
that the Lorentzian regularization adds some new terms to the 3PN equations of motion 
[computed with the standard regularization \((F)_1\)] which are mandatory in order to maintain 
their Lorentz invariance (see for instance Eq. (5.35) in [28]). The Lorentzian partie finie 
of a singular function \( F \), denoted \([F]_1\), enables one to define a “Lorentzian” partie finie 
delta-function \( \text{Pf}\Delta_1 \), namely a linear form whose action on any \( F \) gives the real number 
\([F]_1\). It also permits the precise definition, given by Eq. (5.11) in [28], of a model for the 
stress-energy tensor of point-particles in (post-Newtonian expansions of) general relativity.

Besides the compact-support integrals computed before, the equations of motion contain 
many “non-compact” integrals, whose support extends up to infinity and which are divergent 
at the location of the particles. To them we assign systematically the value given by the 
Hadamard partie-finie of a divergent integral: \( \text{Pf}\int d^3x F \), see Eq. (3.1) in [27]. Furthermore, 
to any \( F \) in this class, we associate the pseudo-function \( \text{Pf}F \) which by definition is the linear 
form whose action on any other \( G \) gives the real number \( \text{Pf}\int d^3x FG \). Given then two 
pseudo functions their product is chosen to be the “ordinary” one \( \text{Pf}F \cdot \text{Pf}G = \text{Pf}(FG) \).

An important feature of the Hadamard partie-finie integral is that the integral of a 
gradient is not zero in general, \( \text{Pf}\int d^3x \partial_i F \neq 0 \), since it is equal to the sums of the 
parties finies of the surface integrals surrounding the singularities when the surface areas 
tend to zero; see Eq. (3.4) in [27]. This means that the ordinary derivative of singular 
functions shows a fundamental difference with the case of regular sources, since in this case 
the integral of a gradient is always zero (provided that the integrand decreases sufficiently 
fast at infinity). One can check that some non-vanishing integrals of a gradient start to 
appear precisely at the 3PN order. Confronted with this problem, the authors [21–25] 
have considered that this signals the presence of ambiguities at the 3PN order, notably 
because their ADM-Hamiltonian density is defined only modulo a total divergence, that 
one certainly does not want to contribute even in the case of singular sources. On the
other hand, the authors [26–29] have accepted this feature and introduced a new kind of (spatial or temporal) distributional derivative acting on the pseudo-functions of the type $PfF$ (for instance $\partial_i PfF$) in order to ensure that the integral of a gradient is always zero. It was found [27] that it is impossible to define a derivative which satisfies the Leibniz rule for the derivation of a product, i.e. $\partial_i (PfFG) \neq F\partial_i PfG + G\partial_i PfF$ in general, but that when one replaces the Leibniz rule by the weaker rule of “integration by parts”, an interesting mathematical structure exists. By rule of integration by parts, we refer to the relation $\int d^3x \left[ F\partial_i PfG + G\partial_i PfF \right] = 0$, for $F$ and $G$ arbitrary functions (see Eq. (7.2) in [27] where we use a more appropriate bracket notation for the spatial integral). While the rule of integration by parts is nothing but an integrated version of the “pointwise” Leibniz rule, the Leibniz rule itself is a stronger requirement, which is not satisfied in general as there are triplets of singular functions $F, G, H$ for which $\int d^3x H\partial_i (PfFG) \neq \int d^3x H\partial_i PfF$. The motivation for requiring the rule of integration by parts is that it is clearly valid in the case of regular fluid systems. Notably it implies that the integral of a gradient of any singular function of type $F$ is zero. However, because it violates the Leibniz rule, the distributional derivative cannot be completely satisfying on the physical point of view.

Actually two different distributional derivatives, and therefore two different regularizations, were introduced in Ref. [27]. A “particular” derivative, defined by Eq. (7.7) in [27], was first chosen for its simplicity. The two main properties of this derivative are: that (i) it reduces to the ordinary derivative, i.e. $\partial_i PfF = Pf(\partial_i F)$, whenever $F$ is bounded near the singularities (in addition of being smooth everywhere else), (ii) it obeys the rule of integration by parts. Though the particular derivative is especially convenient to use in practical computations, it does not follow from some “unicity” theorem. A more interesting derivative, on the mathematical point of view, is the so-called “correct” derivative (we follow the terminology of Ref. [29]) which does satisfy a unicity theorem. Namely, this derivative is obtained in Theorem 4 of [27] as the unique derivative satisfying the properties (i) and (ii) above, and, in addition, (iii) the rule of commutation of successive derivatives (Schwarz lemma). As it turned out, the “correct” derivative, given by Eq. (8.12) of [27], depends on one arbitrary numerical constant $K$. (Note that both the particular and correct derivatives reduce to the derivative of the standard distribution theory [31] when applied on smooth test functions with compact support.)

Summarizing, it is possible to construct a consistent regularization based on the Hadamard partie finie, thus one can give a precise meaning to any integral encountered in the computation, but there are several possible prescriptions associated with different distributional derivatives (and the Leibniz rule is not satisfied). Our strategy has been to perform two computations of the equations of motion, associated respectively with the “particular” and “correct” derivatives. Then the following was shown [28].

(I) The 3PN equations of motion, when computed by means of the Lorentzian regularization and the particular derivative, are in agreement with the known equations of motion up to the 2.5PN order, have the correct test-mass limit and most importantly are Lorentz invariant.
(II) Looking for the most general solution, allowed by the regularization, for the 3PN equations of motion to admit a conserved energy and a Lagrangian description, we find that they depend on two unphysical gauge-constants $r'_1$ and $r'_2$ (associated with the appearance of logarithms), and on one and only one physical constant $\lambda$ which cannot be determined within the method. The equations of motion possess all the physical properties that we expect, but the presence of the unknown constant $\lambda$ is somewhat baffling, as it probably reflects a physical incompleteness of the regularization.

(III) When the correct distributional derivative is used instead of the particular one, the equations of motion depend on $K$ in addition to $r'_1$, $r'_2$ and $\lambda$. In this case we find that they are no longer Lorentz invariant in general, but that there is a unique value of $K$ for which the Lorentz invariance is recovered: $K = \frac{41}{160}$. For this value the equations of motion have also all the physical properties we expect.

(IV) The different equations of motion as obtained by means of the “particular” and “correct” prescriptions (with $K = \frac{41}{160}$ in the second case) are physically equivalent in the sense that they differ from each other by an infinitesimal change of coordinates. This satisfying result indicates that the distributional derivatives introduced in Ref. [27] constitute merely some technical tools devoid of physical meaning.

In the scenario (III) one may wonder why after having used the Lorentzian regularization defined in Ref. [28] one still has to adjust the constant $K$ to a certain value in order to get finally the Lorentz invariance. The likely reason is that the distributional derivatives we use (the particular and correct ones) have not been defined in a Lorentz-invariant way, as their distributional terms are made of the delta-pseudo-function $\text{Pf}\delta_1$ instead of the “Lorentzian” delta-pseudo-function $\text{Pf}\Delta_1$ (see Eq. (3.36) in [28]). As a result, we find in the scenario (III) that although most of the terms satisfy the requirement of Lorentz invariance, notably the terms proportional to the combination of masses $m_1^2 m_2$ in the acceleration of particle one (these terms are shown to behave correctly thanks to the Lorentzian regularization), there still exists a limited class of terms, proportional to $m_2^3$, that do not obey the Lorentz invariance unless $K$ is adjusted to the value $\frac{41}{160}$. [In the scenario (I) where there is no constant to adjust the latter terms behave correctly.]

The problem of the Lorentz invariance of the equations of motion was solved in a quite different way by the other group [21, 25]. We recall that the harmonic-coordinate equations of motion are manifestly Lorentz-invariant because the harmonic gauge condition preserves the Poincaré symmetry. By contrast, the coordinate conditions associated with the ADM Hamiltonian formalism do not respect the Poincaré group, and therefore the authors [21, 25] had to prove that their Hamiltonian is compatible with the existence of generators in phase-space such that the usual Poincaré algebra is satisfied. More precisely, they constructed a generic “ambiguous” dynamics at the 3PN order, parametrized by some unknown ambiguity parameters associated notably with the non-distributivity of the Hadamard partie finie and to the fact that the integral of a gradient, in an ordinary sense, is not zero. They showed that
there were only two ambiguity parameters they denoted $\omega_{\text{kinetic}}$ and $\omega_{\text{static}}$. (Actually, in the first paper [21], they considered only the ambiguity constant $\omega_{\text{kinetic}}$ and obtained the value $\omega_{\text{static}} = \frac{1}{8}$. The static ambiguity was introduced in the second paper [22].) By imposing in an *ad hoc* manner the existence of the Poincaré generators for their ambiguous Hamiltonian, they showed [24] that the parameter $\omega_{\text{kinetic}}$ is fixed uniquely to the value $\frac{41}{24}$. This result was in fact obtained earlier [26] by comparing their expression of the energy of circular orbits [23] to the expression we got by means of the explicitly Lorentz-invariant formalism described in the scenario (I) above. Finally, having fixed $\omega_{\text{kinetic}}$, there still remained in the ADM-Hamiltonian formalism one and only one undetermined constant $\omega_{\text{static}}$, that we shall find to be equivalent, in the sense of Eq. (1.1), to the constant $\lambda$ appearing in harmonic coordinates. [Note that, despite the resemblance between the value $K = \frac{41}{160}$ in the scenario (III) and the result $\omega_{\text{kinetic}} = \frac{41}{24}$, the constant $K$ can be fixed to this unique value only if the sophisticated Lorentzian regularization is used before. Without such a regularization, several other terms not parametrized by $K$ would not behave correctly under Lorentz transformations, and therefore no value of $K$ could be chosen in order to restore the Lorentz invariance. In this sense the constant $K$ is more “specialized” than the constant $\omega_{\text{kinetic}}$.]

Finally, choosing one or the other of the two approaches advocated in Refs. [21–25] and [26–29] for the regularization is a matter of taste. In view of the equivalence of the final results, it is a good state of affairs that the two approaches are different conceptually and technically.

**III. THEORY**

**A. Noetherian conserved quantities for a generalized Lagrangian**

At the 1PN order, the equations of motion of two compact objects in General Relativity, as derived in Refs. [4,5], can be deduced from an ordinary Lagrangian, depending on the positions and velocities of the bodies, which was obtained by Fichtenholz [6]. At the next 2PN order, the equations of motion in harmonic coordinates, as obtained in [8,10,11], can only be deduced from a “generalized” Lagrangian, depending not only on the positions and velocities but also on the accelerations of the particles [8]. In particular, this confirmed a result of Martin and Sanz [34] that $N$-body systems cannot admit an ordinary Lagrangian description beyond the 1PN order, provided that the gauge conditions preserve the Lorentz invariance (as it is the case for the harmonic gauge). However, it has been shown by Damour and Schäfer [12] that there exists a special class of coordinates, which includes the ones associated with the ADM formalism, such that the Lagrangian at the 2PN order expressed by means of such coordinates becomes ordinary, i.e. does not depend on accelerations anymore. This means that we can eliminate the accelerations in the harmonic-coordinate Lagrangian at the 2PN order by going to the ADM coordinates [12]. In this paper, we shall find that the 3PN terms in the Lagrangian in harmonic coordinates depend also on accelerations, and that, like at the 2PN order, these accelerations can be eliminated by
a suitable coordinate transformation to some “pseudo-ADM” coordinates, following the
general method of redefinition of position variables [12, 35–37].

Strictly speaking, the dynamics of two compact bodies does not derive from a Lagrangian
at the 3PN approximation because of the radiation reaction damping effect at the previous
2.5PN order. When speaking of a 3PN Lagrangian or Hamiltonian, we always refer to the
conservative part of the dynamics, which corresponds to the “even” post-Newtonian orders
1PN, 2PN and 3PN. As we shall see, the radiation reaction effect manifests itself in the
non-conservation at the 2.5PN approximation of the conserved quantities associated with
the conservative 3PN dynamics [see Eqs. (4.7)].

Let us consider a harmonic-coordinate generalized 3PN Lagrangian

\[ L_{\text{harmonic}} \equiv L[y_A(t), \dot{v}_A(t), a_A(t)], \quad (3.1) \]

depending on the instantaneous positions \( y_A(t) \equiv y^i_A(t) \) (with \( A = 1, 2 \) and \( i = 1, 2, 3 \)),
coordinate velocities \( v^i_A(t) \equiv \dot{v}_A(t) = dy_A/dt \), as well as coordinate accelerations \( a^i_A(t) \equiv \ddot{a}_A(t) = d\dot{v}_A/dt \). Our harmonic-coordinate 3PN Lagrangian is given by (4.1) below, but we
do not need to be so specific in the present Section, where most of the results hold in fact
for \( N \)-body systems (\( A = 1, \cdots, N \)). We assume that the dependence of the Lagrangian
(3.1) upon the accelerations is linear. As a matter of fact, it is always possible to eliminate
from a generalized post-Newtonian Lagrangian a contribution quadratic in the accelerations
by re-writing it in the form of a so-called “double-zero” term, which does not contribute to
the equations of motion, plus a term linear in the acceleration [12] (this argument can be
extended to any term polynomial in the accelerations).

The equations of motion of the \( A \)th body are deduced from the Lagrangian by taking
the functional derivative defined as

\[ \frac{\delta L}{\delta y^i_A} \equiv \frac{\partial L}{\partial y^i_A} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}^i_A} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial a^i_A} \right) = 0. \quad (3.2) \]

We consider first, very generally, an infinitesimal transformation of the path of the particle
\( A \) at some instant \( t \), i.e. \( \delta y_A(t) = y'_A(t) - y_A(t) \). The corresponding variations of its velocity
and acceleration are \( \delta \dot{v}_A(t) = d\delta y_A/dt \) and \( \delta a_A(t) = d\delta \dot{v}_A/dt \). Such a transformation of the
motion induces a variation of the Lagrangian, namely \( \delta L = L[y'_A, \dot{v}'_A, a'_A] - L[y_A, \dot{v}_A, a_A] \)
which is readily found to be expressible, at the linearized order in \( \delta y_A \), in the form

\[ \delta L = \frac{dQ}{dt} + \sum_A \frac{\delta L}{\delta y^i_A} \delta y^i_A + O (\delta y^2_A), \quad (3.3) \]

where the functional derivative \( \delta L/\delta y^i_A \) is given by (3.2) [it is zero “on shell”, i.e. when the
equations of motion are satisfied], and where we have introduced the total time-derivative
of a function \( Q \equiv Q[\delta y_A, \delta \dot{v}_A] \) defined by
\[ Q = \sum_A \left( p^i_A \delta y^i_A + q^i_A \delta v^i_A \right). \]  

(3.4)

Here, \( p^i_A \) and \( q^i_A \) denote the momenta that are conjugate to the positions \( y^i_A \) and velocities \( v^i_A \) of the particle \( A \) respectively, that is

\[
p^i_A = \frac{\delta L}{\delta v^i_A} = \frac{\partial L}{\partial v^i_A} - \frac{d}{dt} \left( \frac{\partial L}{\partial a^i_A} \right),
\]

(3.5a)

\[
q^i_A = \frac{\delta L}{\delta a^i_A} = \frac{\partial L}{\partial a^i_A}.
\]

(3.5b)

We now discuss the Noetherian conservation laws for generalized Lagrangians following Refs. \cite{9,11}. We know from Ref. \cite{29} that the 3PN equations of motion in harmonic coordinates are manifestly invariant (in a perturbative post-Newtonian sense) under the Lorentz and more generally the Poincaré group. Thus the dynamics associated with our 3PN generalized Lagrangian \cite{11} should stay the same after an infinitesimal Poincaré transformation of the dynamical variables \( y^\mu_A = (ct, y_A) \). In particular, this means that \( \delta L = 0 \) in the case of arbitrary infinitesimal constant spatial translations and rotations, \( \delta y^i_A = \epsilon^i \) and \( \delta y^i_A = \omega^i_j y^j_A \) with \( \omega_{ij} = -\omega_{ji} \). In this case Eq. (3.3) implies the conservation on-shell (all the \( \delta L/\delta y^i_A \)'s are zero) of the Noetherian linear and angular momenta given by

\[
P^i = \sum_A p^i_A,
\]

(3.6a)

\[
J^i = \varepsilon_{ijk} \sum_A \left( y^j_A p^k_A + v^j_A q^k_A \right).
\]

(3.6b)

Thus, \( dP^i/dt = 0 \) and \( dJ^i/dt = 0 \) on shell. On the other hand, we have \( \delta L = \tau dL/dt \) in the case of an infinitesimal constant time translation \( \delta t = \tau \), hence the conservation on-shell of the Noetherian energy from Eq. (3.3),

\[
E = \sum_A \left( v^i_A p^i_A + a^i_A q^i_A \right) - L.
\]

(3.7)

Thus, \( dE/dt = 0 \). We shall give the explicit expressions of these Noetherian energy and momenta at the 3PN order in harmonic coordinates in the next Section which is devoted to the results [see Eqs. (4.2)-(4.4)].

Finally, let us consider the symmetry of the Lagrangian that is associated with the invariance under Lorentz special transformations or boosts. Clearly, since the dynamics must stay the same after an infinitesimal constant Lorentz boost, the corresponding variation of the Lagrangian has to take essentially the form of a total time derivative. At the linearized order in the boost velocity \( W^i \), the transformation of the particle trajectories is given by \( \delta y^i_A = -W^i t + \frac{1}{c^2} W^j y^j_A v^i_A + \mathcal{O}(W^i W^i) \). There should exist a certain functional \( Z^i \) of the
positions, velocities and accelerations such that the 3PN Lagrangian variation reads $\delta L = W^i dZ^i/dt + \mathcal{O}(W^iW^i)$, plus some “double-zero” terms at the 3PN order (which are zero on-shell when applying the Noether theorem). By applying Eq. (3.3), we readily find the conservation on-shell of the Noetherian integral $K^i = G^i - P^i t$, where $P^i$ is the linear momentum (3.6a), and where $G^i$ represents the center-of-mass position:

$$G^i = -Z^i + \sum_A \left( -q^i_A + \frac{1}{c^2} \left[ y^i_A p^j_A v^j_A + y^i_A q^j_A a^j_A + v^i_A q^j_A v^j_A \right] \right).$$ (3.8)

Thus, $dK^i/dt = 0$, or equivalently $d^2G^i/dt^2 = 0$ (the center-of-mass vector $G^i$ is conserved in a frame where $P^i = 0$). The existence of the latter boost-symmetry of the Lagrangian is a confirmation of the Lorentz invariance of the 3PN equations of motion obtained in Ref. [29]. The Noetherian center-of-mass $G^i$ in harmonic coordinates at the 3PN order is given explicitly by Eq. (4.5) below.

The ten Noetherian quantities (3.6)-(3.8) have been found from our generalized Lagrangian as some functionals of the positions, velocities and accelerations of the particles. However, once they have been constructed, all the accelerations they involve can be order-reduced using the fact that they take on-shell some definite expressions depending on the positions and velocities as given by the equations of motion. Our final results presented in Section IV.A have all been order-reduced consistently with the 3PN approximation.

**B. Elimination of acceleration-dependent terms in a Lagrangian**

We start from the harmonic coordinate system $x^\mu = (ct, \mathbf{x})$ and perform an infinitesimal coordinate transformation to a new coordinate system $x'^\mu$, generally not obeying the harmonic gauge condition, of the type

$$x'^\mu = x^\mu + \varepsilon^\mu(x),$$ (3.9)

where $\varepsilon^\mu(x)$ is a function of the spatial coordinates $\mathbf{x}$ as well as a (local-in-time) functional of the trajectories $y_A(t)$ and velocities $v_A(t)$ parametrized by the coordinate time $t = x^0/c$. Namely,

$$\varepsilon^\mu(x, t) = \varepsilon^\mu[\mathbf{x}; y_A(t), v_A(t)].$$ (3.10)

Since the accelerations in the harmonic-coordinate Lagrangian appear only at the 2PN order, we suppose that the coordinate transformation starts at the same level. This means that $\varepsilon^i = \mathcal{O}\left(\frac{1}{c^4}\right)$ and $\varepsilon^0 = \mathcal{O}\left(\frac{1}{c^3}\right)$. In particular we can check that any term in the following which is at least quadratic in $\varepsilon^\mu$ is in fact of order $\mathcal{O}\left(\frac{1}{c^4}\right)$ and thus can be neglected in our study limited to the 3PN approximation. The trajectories and velocities in the new coordinates...
where \( x'^\mu = (ct', \mathbf{x}') \) are some functions \( y_A'(t') \) and \( v_A'(t') \) of the new coordinate time \( t' = x'^0 / c \). The “contact” transformation of the particle variables induced by the coordinate transformation \((3.9)-(3.10)\) is defined by

\[
\delta y_A^i(t) = y_A^i(t) - v_A^i(t) \tag{3.11}
\]

Neglecting all the terms of the order of the square of \( \varepsilon \) we obtain

\[
\delta y_A^i(t) = \varepsilon^i(y_A, t) - v_A^i c \varepsilon^0(y_A, t) + \mathcal{O} \left( \frac{1}{c^8} \right). \tag{3.11}
\]

In this paper we shall construct a contact transformation \( \delta y_A^i \), composed of 2PN and 3PN terms and neglecting \( \mathcal{O} \left( \frac{1}{c^8} \right) \), which is issued from some infinitesimal coordinate transformation \((3.9)-(3.10)\); however we shall not be so much interested in the coordinate transformation itself, in particular this means that we shall not investigate to which coordinate conditions it corresponds to (non-harmonic and/or ADM-type).

If the equations satisfied by the world-lines \( y_A(t) \) in some initial coordinate system derive from the Lagrangian \( L \), then the equations satisfied by the new world-lines \( y_A'(t') \) in a new coordinate system will derive from the new Lagrangian \( L' \) that is such that

\[
L'[y_A'(t), v_A'(t), a_A'(t), b_A'(t)] = L[y_A(t), v_A(t), a_A(t)] \tag{3.12}
\]

(see e.g. Eq. (5) of Damour and Schäfer [12]). Since we assumed that the contact transformation \( \delta y_A \) depends on the velocities, the new Lagrangian necessarily depends on positions, velocities, accelerations and also derivatives of accelerations: \( b_A(t) = da_A/dt \). Now the same computation as the one leading to Eq. (3.3) shows that, at the linearized order in \( \delta y_A \),

\[
L'[y_A, v_A, a_A, b_A] = L[y_A, v_A, a_A] + \frac{dQ}{dt} + \sum_A \frac{\delta L}{\delta y_A^i} \delta y_A^i + \mathcal{O} \left( \frac{1}{c^8} \right). \tag{3.13}
\]

Notice that both sides of this relation are expressed in terms of the same “dummy” variables, chosen to be the harmonic-coordinate ones, e.g. \( y_A \). At the end, when we obtain the new Lagrangian, we shall have to replace this dummy variable by the one corresponding to the new coordinate system, \( y_A' = y_A + \delta y_A \). The term with a total time-derivative is the same as the one found in Eq. (3.3), with \( Q \) given by (3.4). As one can see, the dependence of the Lagrangian \( L' \) upon derivatives of accelerations \( b_A \) comes only from this total time derivative. Therefore, by posing \( L'' = L' - \frac{dQ}{dt} \) we get a Lagrangian which is dynamically equivalent to the Lagrangian \( L' \) and depends like \( L \) on positions, velocities and accelerations only,

\[
L''[y_A, v_A, a_A] = L[y_A, v_A, a_A] + \sum_A \frac{\delta L}{\delta y_A^i} \delta y_A^i + \mathcal{O} \left( \frac{1}{c^8} \right). \tag{3.14}
\]

We now show that there exists a contact transformation \( \delta y_A^i \) (actually, there exist infinitely many of them), together with a redefinition of the Lagrangian by the addition of...
a total time derivative, which eliminates all the accelerations in the Lagrangian up to the 3PN order. In other words, the 3PN Lagrangian that will follow is ordinary, i.e. depends on positions and velocities only. Damour and Schäfer \cite{12} have already shown how to eliminate the accelerations at the 2PN level. We shall see how to do this at the next 3PN order, but in fact the method is a particular application of a general algorithm to eliminate higher-derivative terms in a Lagrangian \cite{37}. Since the contact transformation (3.11) is assumed to start at the 2PN order, i.e. \( \delta y_i^A = O \left( \frac{1}{c^4} \right) \), we must control the functional derivative \( \frac{\delta L}{\delta y_i^A} \) appearing in the right side of Eq. (3.14) at the relative 1PN order. The standard Newtonian contribution is then followed by a certain 1PN correction, denoted \( m_A C_i^A \), hence

\[
\frac{\delta L}{\delta y_i^A} = m_A \left[ -a_i^A - \sum_{B \neq A} \frac{G m_B}{r_{AB}^2} n_{AB}^i + \frac{1}{c^2} C_i^A \right] + O \left( \frac{1}{c^4} \right). \tag{3.15}
\]

The 1PN term \( C_i^A \) can be straightforwardly computed from the Lagrangian (4.1). The point is that it does depend on accelerations, \( C_i^A \equiv C_i^A[y_B, v_B, a_B] \), with this dependence being linear. The presence of accelerations in \( C_i^A \) is the reason why the method used in Ref. \cite{12} to deal with the problem at the 2PN order cannot be extended immediately at the 3PN approximation. We shall see that the method necessitates the introduction in the contact transformation at the 3PN order of some “counter-term” \( X_i^A \) described below. Now, in view of the term \( -m_A a_i^A \) present in Eq. (3.15), it is clear that we will be able to remove all the accelerations at the 2PN order if we choose for the contact transformation the term \( \frac{1}{m_A} q_i^A \) (we recall that \( q_i^A \) is the conjugate momentum of the acceleration, \( q_i^A = \frac{\partial L}{\partial a_i^A} \)). Indeed the only possible accelerations at the 2PN order in the Lagrangian \( L'' \) would be contained in the combination \( L - \sum_A a_i^A q_i^A \), which clearly does not depend on accelerations because of the linearity of the original Lagrangian \( L \) upon \( a_i^A \). Furthermore, as discussed in \cite{12}, once we have eliminated the accelerations at the 2PN order, we are free to add to the contact transformation any term of the type \( \frac{1}{m_A} \frac{\partial F}{\partial v_i^A} \), where \( F \) is an arbitrary functional of the positions and velocities only, starting at the 2PN order. This follows immediately from the identity \( \frac{\partial F}{\partial t} = \sum_A \left( v_i^A \frac{\partial F}{\partial y_i^A} + a_i^A \frac{\partial F}{\partial a_i^A} \right) \), which shows that the further accelerations produced by this term are contained into the total time-derivative of \( F \), and so can be removed from the original Lagrangian without changing the dynamics. However, these procedures are no longer valid at the 3PN order because of the accelerations in the 1PN term \( C_i^A \) of (3.15), which will couple to the terms \( \frac{1}{m_A} \left[ q_i^A + \frac{\partial F}{\partial v_i^A} \right] \) as suggested before and produce some new accelerations. The solution of the problem is to add to the contact transformation some correction term that we shall find to be adjustable in a unique way so that it works.

As a result, we look for a contact transformation of the type

\[
\delta y_i^A = \frac{1}{m_A} \left[ q_i^A + \frac{\partial F}{\partial v_i^A} + \frac{1}{c^2} X_i^A \right] + O \left( \frac{1}{c^6} \right), \tag{3.16}
\]

where \( q_i^A \) is defined by Eq. (3.5b); \( F \) is a general functional of the positions and velocities, \( F \equiv F[y_A, v_A] \), and \( X_i^A \) denotes some “counter” term depending on positions and velocities.
only, $X_A^i \equiv X_A^i[y_B, v_B]$. We recall that $q_A^i$ is composed of 2PN and 3PN terms, which are easily computed from the Lagrangian (4.1). The function $F$ must start at the 2PN order; in addition we assume that it contains all possible generic terms at 3PN. Finally as explained above the counter term $X_A^i$ is purely of order 3PN. We now replace both Eqs. (3.15) and (3.16) into $L''$ given by (3.14) and investigate the occurrence of accelerations. Among the terms we recognize the combination $L - \sum_A a_A^i q_A^i$ which is free of any accelerations at the 3PN order. We also transfer several acceleration terms into the total time-derivative of $F$ as before. At last we find that the only remaining accelerations in $L''$ are contained into the particular combination of terms:

$$\sum_A \left( \frac{1}{c^2} \left[ q_A^i + \frac{\partial F}{\partial v'_A} \right] C_A^i - \frac{1}{c^6} a_A^i X_A^i \right) + O \left( \frac{1}{c^8} \right).$$

As all the terms in that combination are linear in the accelerations, we see that for any given function $F$ there is a unique choice of the term $X_A^i$ (for each particle) such that all the remaining accelerations are cancelled out, namely

$$\frac{1}{c^6} X_A^i = \sum_B \frac{1}{c^2} \left[ q_B^i + \frac{\partial F}{\partial v'_B} \right] \frac{\partial C_B^i}{\partial a_A^i} + O \left( \frac{1}{c^8} \right). \quad (3.17)$$

With the latter choice, the contact transformation (3.16), defined for any $F$, yields a Lagrangian $L''$ whose only accelerations come from (minus) the total time-derivative of $F$. Therefore, the 3PN Lagrangian $L''' = L'' + \frac{dF}{dt}$ is at once physically equivalent to $L''$, $L'$ and $L$, and free of accelerations. Our result reads then

$$L'''[y_A, v_A] = L + \sum_A \frac{\delta L}{\delta y_A^i} \delta y_A^i + \frac{dF}{dt} + O \left( \frac{1}{c^8} \right). \quad (3.18)$$

Remind the large freedom we still have on the definition of $L'''$, since we constructed it for any functional $F$ of the positions and velocities at the 2PN and 3PN orders.

In this paper we shall be able to determine uniquely the function $F$ by the requirement that the Lagrangian $L'''$ be exactly the ADM Lagrangian associated with the ADM (or ADM-type) Hamiltonian published by Damour, Jaranowski and Schäfer [24]. We shall not give the details of the computation since it consists merely of parametrizing the most general function $F$, constructed with the dynamical variables of the problem and having a compatible dimension, by means of some arbitrary constant parameters, and to show that all these constants are uniquely fixed by the condition of matching to the ADM Hamiltonian. We find indeed, in complete agreement with Ref. [23], that there is a unique set of constants for which this works. In particular the equivalence is possible if and only if the undetermined constant $\lambda$ appearing in the harmonic-coordinate formalism [29] is related to the constant $\omega_{\text{static}}$ of Jaranowski and Schäfer [22] by Eq. (1.1). Note that the latter matching shows also that the logarithms $\ln \left( \frac{r_1}{r_1'} \right)$ and $\ln \left( \frac{r_2}{r_2'} \right)$ present in the harmonic-coordinate Lagrangian...
where \( r'_1 \) and \( r'_2 \) denote some regularization constants, are eliminated by this contact transformation, in agreement with the fact proved in Ref. \[29\] that the logarithms, and the constants \( r'_1 \) and \( r'_2 \) therein, can be gauged away. See Eq. (4.4) below for the complete expression of the function \( F \).

At last, with \( F \) now fully specified by the equivalence with \[24\], we obtain the ordinary ADM-type Lagrangian

\[
L^{\text{ADM}} = L + \sum_A \frac{\delta L}{\delta y^i_A} \delta y^i_A + \frac{dF}{dt},
\]

(3.19)

given explicitly at the 3PN order by Eq. (4.11) below, in which, as mentioned above, we shall replace the "dummy" variables used in the computation, \( y^i_A \) and \( v^i_A \), by the real dynamical variables in pseudo-ADM coordinates, \( Y^i_A \) and \( V^i_A \). The ADM momentum conjugate to the velocity is

\[
P^i_A = \frac{\partial L^{\text{ADM}}}{\partial v^i_A} = p^i_A + \frac{\delta}{\delta v^i_A} \left( \sum_B \frac{\delta y^j_B}{\delta y^j_B} \frac{\delta L}{\delta y^j_B} \right) + \frac{\partial F}{\partial y^i_A},
\]

(3.20)

and the corresponding Hamiltonian follows from the ordinary Legendre transformation

\[
H^{\text{ADM}} = \sum_A P^i_A v^i_A - L^{\text{ADM}}.
\]

(3.21)

See Eq. (4.12) for the complete 3PN expression of this Hamiltonian (as a function of \( Y^i_A \) and \( P^i_A \)). [We have checked that the second equality in (3.20) is true at 3PN order.] Notice that, strictly speaking, \( H^{\text{ADM}} \) is not the ADM one, as it differs from it by a shift in phase-space coordinates at the 3PN order which is given in Ref. \[24\]. Indeed, the ADM Hamiltonian at the 3PN order is not ordinary, as it depends on the positions and momenta as well as on their derivatives \[21\]. But this is not a concern for our purpose, since we are interested in proving the equivalence between our approach \[20–24\] and the one of \[21–25\], that is in finding the existence of a unique transformation connecting both works, in whatever coordinate systems the two approaches found it convenient to be. We think that the equivalence found in this paper and in Ref. \[25\] convincingly confirms the correctness of the result. This equivalence is especially important in view of the different procedures adopted by the two groups to treat the point-mass divergencies (see Section II for a discussion).

IV. RESULTS

A. Conserved quantities in harmonic coordinates at the 3PN order

We first exhibit a generalized Lagrangian from which derive the 3PN equations of motion of two compact objects as they were obtained in harmonic coordinates; see Eqs. (7.16) in
The Lagrangian corresponds only to the conservative part of the equations, which excludes the radiation reaction term at the 2.5PN order. To compute it we proceed by guess-work, and find the occurrence of terms depending on accelerations at the 2PN and 3PN orders. The Lagrangian is chosen to be linear in the accelerations, and to agree at the 2PN approximation with the Lagrangian obtained in Ref. [11]. The result is

\[
L = \frac{G m_1 m_2}{2r_{12}} + \frac{m_1 v_1^2}{2} + \frac{1}{c^2} \left\{ - \frac{G^2 m_1^2 m_2}{2r_{12}^2} + \frac{m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}} \left( - \frac{1}{4} (n_{12} v_1) (n_{12} v_2) + \frac{3}{2} v_1^2 - \frac{7}{4} (v_1 v_2) \right) \right\} \\
+ \frac{1}{c^4} \left\{ \frac{G^3 m_1^2 m_2}{2r_{12}^3} + \frac{19 G^2 m_1^2 m_2}{8r_{12}^3} + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( \frac{7}{2} (n_{12} v_1)^2 - \frac{7}{2} (n_{12} v_1) (n_{12} v_2) + \frac{1}{2} (n_{12} v_2)^2 \right) \\
+ \frac{1}{4} v_1^2 - \frac{7}{4} (v_1 v_2) + \frac{7}{4} v_2^2 \right\} + \frac{G m_1 m_2}{r_{12}} \left( \frac{3}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 - \frac{7}{8} (n_{12} v_2)^2 v_1^2 + \frac{7}{8} v_1^4 \right) \\
+ \frac{3}{4} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) - 2 v_1^2 (v_1 v_2) + \frac{1}{8} (v_1 v_2)^2 + \frac{15}{16} v_1^2 v_2^2 + \frac{m_1 v_1^6}{16} \right\} \\
+ G m_1 m_2 \left( - \frac{7}{4} (a_1 v_2) (n_{12} v_2) - \frac{1}{8} (n_{12} a_1) (n_{12} v_2)^2 + \frac{7}{8} (n_{12} a_1) v_2^2 \right) \\
+ \frac{1}{c^6} \left\{ \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( \frac{13}{18} (n_{12} v_1)^4 + \frac{83}{18} (n_{12} v_1)^3 (n_{12} v_2) - \frac{35}{6} (n_{12} v_1)^2 (n_{12} v_2)^2 - \frac{245}{24} (n_{12} v_1)^2 v_1^2 \right) \\
+ \frac{179}{12} (n_{12} v_1) (n_{12} v_2) v_1^2 - \frac{235}{24} (n_{12} v_2)^2 v_1^2 + \frac{373}{48} v_1^4 + \frac{529}{24} (n_{12} v_1)^2 (v_1 v_2) \\
+ \frac{97}{6} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) - \frac{719}{24} v_1^2 (v_1 v_2) + \frac{463}{24} (v_1 v_2)^2 - \frac{7}{24} (n_{12} v_1)^2 v_2 \\
- \frac{1}{2} (n_{12} v_1) (n_{12} v_2) v_2^2 + \frac{1}{4} (n_{12} v_2)^2 v_1^2 + \frac{463}{48} v_1^2 v_2^2 - \frac{19}{2} (v_1 v_2)^2 v_1^2 + \frac{45}{16} v_1^4 + \frac{5 m_1 v_1^8}{128} \right\} \\
+ G m_1 m_2 \left( \frac{3}{8} (a_1 v_2) (n_{12} v_1) (n_{12} v_2)^2 + \frac{5}{12} (a_1 v_2) (n_{12} v_2)^3 + \frac{1}{8} (n_{12} a_1) (n_{12} v_1) (n_{12} v_2)^3 \right) \\
+ \frac{1}{16} (n_{12} a_1) (n_{12} v_2)^4 + \frac{11}{4} (a_1 v_2) (n_{12} v_2)^2 v_1^2 - (a_1 v_2) (n_{12} v_2)^2 v_1^2 - 2 (a_1 v_2) (n_{12} v_2) (v_1 v_2) \\
+ \frac{1}{4} (a_1 v_2) (n_{12} v_2) (v_1 v_2) + \frac{3}{8} (n_{12} a_1) (n_{12} v_2)^2 (v_1 v_2) - \frac{5}{8} (n_{12} a_1) (n_{12} v_1)^2 v_2^2 \\
+ \frac{15}{8} (a_1 v_1) (n_{12} v_2) v_2^2 - \frac{15}{8} (a_1 v_1) (n_{12} v_2) v_1^2 - \frac{1}{2} (n_{12} a_1) (n_{12} v_1) (n_{12} v_2) v_2^2 \\
- \frac{5}{16} (n_{12} a_1) (n_{12} v_2)^2 v_2^2 + \frac{G^2 m_1^2 m_2}{r_{12}} \left( - \frac{235}{24} (a_2 v_1) (n_{12} v_1) - \frac{29}{24} (n_{12} a_2) (n_{12} v_1)^2 \\
- \frac{235}{24} (a_1 v_2) (n_{12} v_2) - \frac{17}{6} (n_{12} a_1) (n_{12} v_2)^2 + \frac{185}{16} (n_{12} a_1) v_2^2 - \frac{235}{48} (n_{12} a_2) v_1^2 \\
- \frac{185}{8} (n_{12} a_1) (v_1 v_2) + \frac{20}{3} (n_{12} a_1) v_2^2 \right) + \frac{G m_1 m_2}{r_{12}} \left( - \frac{5}{32} (n_{12} v_1)^3 (n_{12} v_2)^3 \right) \right\} \right\} \right\} \right\}
In our notation, \( r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \), \( \mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12} \), and the scalar products are written e.g. \( (n_{12}v_2) = \mathbf{n}_{12} \cdot \mathbf{v}_2 \). To the terms given explicitly above, we have to add the terms corresponding to the relabeling \( 1 \leftrightarrow 2 \), including those which are symmetric under the label exchange. Notice the presence of the constant \( \lambda \) which is the only unknown physical parameter in this Lagrangian, and of the two unknown gauge constants \( r'_1 \) and \( r'_2 \) (we follow exactly the notation of [29]). The Lagrangian presented here is not the only admissible one, as we can always add to it an arbitrary total time derivative (double-zero terms would make the Lagrangian non-linear in the accelerations). We have checked that our Lagrangian \((4.1)\) differs indeed from the one given by Eqs. (5.4)-(5.10) in Ref. [25] by a total time derivative.

Next we present the expressions of the conserved integrals of the 3PN harmonic-coordinate motion as constructed in Section III.A. These expressions involve only the relativistic 1PN, 2PN and 3PN terms corresponding to the conservative part of the dynamics at the 3PN order. The radiation reaction damping effect is added afterwards. All the quantities we present depend only on the positions and velocities, because all accelerations therein have been systematically order-reduced by means of the equations of motion. The energy \( E \) reads

\[
E = \frac{m_1 v_1^2}{2} - \frac{Gm_1m_2}{2r_{12}} + \frac{1}{c^2} \left\{ \frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{3m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}} \left( -\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2) \right) \right\}
\]

In our notation, \( r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \), \( \mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12} \), and the scalar products are written e.g. \( (n_{12}v_2) = \mathbf{n}_{12} \cdot \mathbf{v}_2 \). To the terms given explicitly above, we have to add the terms corresponding to the relabeling \( 1 \leftrightarrow 2 \), including those which are symmetric under the label exchange. Notice the presence of the constant \( \lambda \) which is the only unknown physical parameter in this Lagrangian, and of the two unknown gauge constants \( r'_1 \) and \( r'_2 \) (we follow exactly the notation of [29]). The Lagrangian presented here is not the only admissible one, as we can always add to it an arbitrary total time derivative (double-zero terms would make the Lagrangian non-linear in the accelerations). We have checked that our Lagrangian \((4.1)\) differs indeed from the one given by Eqs. (5.4)-(5.10) in Ref. [25] by a total time derivative.

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\[
E = \frac{m_1 v_1^2}{2} - \frac{Gm_1m_2}{2r_{12}} + \frac{1}{c^2} \left\{ \frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{3m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}} \left( -\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2) \right) \right\}
\]

In our notation, \( r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \), \( \mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12} \), and the scalar products are written e.g. \( (n_{12}v_2) = \mathbf{n}_{12} \cdot \mathbf{v}_2 \). To the terms given explicitly above, we have to add the terms corresponding to the relabeling \( 1 \leftrightarrow 2 \), including those which are symmetric under the label exchange. Notice the presence of the constant \( \lambda \) which is the only unknown physical parameter in this Lagrangian, and of the two unknown gauge constants \( r'_1 \) and \( r'_2 \) (we follow exactly the notation of [29]). The Lagrangian presented here is not the only admissible one, as we can always add to it an arbitrary total time derivative (double-zero terms would make the Lagrangian non-linear in the accelerations). We have checked that our Lagrangian \((4.1)\) differs indeed from the one given by Eqs. (5.4)-(5.10) in Ref. [25] by a total time derivative.

Next we present the expressions of the conserved integrals of the 3PN harmonic-coordinate motion as constructed in Section III.A. These expressions involve only the relativistic 1PN, 2PN and 3PN terms corresponding to the conservative part of the dynamics at the 3PN order. The radiation reaction damping effect is added afterwards. All the quantities we present depend only on the positions and velocities, because all accelerations therein have been systematically order-reduced by means of the equations of motion. The energy \( E \) reads

\[
E = \frac{m_1 v_1^2}{2} - \frac{Gm_1m_2}{2r_{12}} + \frac{1}{c^2} \left\{ \frac{G^2m_1^2m_2}{2r_{12}^2} + \frac{3m_1v_1^4}{8} + \frac{Gm_1m_2}{r_{12}} \left( -\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2) \right) \right\}
\]
\[ + \frac{1}{c^4} \left\{ - \frac{G^2 m_1^2 m_2}{2r_{12}^3} - \frac{19G^3 m_1^2 m_2^2}{8r_{12}^3} + \frac{5m_1 v_1^6}{16} + \frac{Gm_1 m_2}{r_{12}} \left( \frac{3}{8} (n_{12}v_1)^3(n_{12}v_2) \right) \right. \\
\frac{3}{16} (n_{12}v_1) (n_{12}v_2)^2 - \frac{9}{8} (n_{12}v_1)(n_{12}v_2)^2 v_1^2 - \frac{13}{8} (n_{12}v_2)^2 v_1^2 + \frac{21}{8} v_1^4 \\
\frac{13}{8} (n_{12}v_1) (n_{12}v_2)^2 + \frac{3}{4} (n_{12}v_1)(n_{12}v_2)(v_1 v_2) - \frac{55}{8} v_1^2(v_1 v_2) + \frac{17}{8} (v_1 v_2)^2 + \frac{31}{16} v_1^2 v_2^2 \\
\left. + \frac{G^2 m_1^2 m_2}{r_{12}^3} \left( \frac{29}{4} (n_{12}v_1)^2 - \frac{13}{4} (n_{12}v_1)(n_{12}v_2) + \frac{1}{2} (n_{12}v_2)^2 - \frac{3}{2} v_1^2 + \frac{7}{4} v_2^2 \right) \right\} \\
+ \frac{1}{c^6} \left\{ \frac{35m_1 v_1^8}{128} + \frac{Gm_1 m_2}{r_{12}} \left( - \frac{5}{16} (n_{12}v_1)^5(n_{12}v_2) - \frac{5}{16} (n_{12}v_1)^4(n_{12}v_2)^2 \right) \right. \\
\frac{5}{32} (n_{12}v_1)^3(n_{12}v_2)^3 + \frac{19}{16} (n_{12}v_1)^3(n_{12}v_2)^2 v_1^2 + \frac{15}{16} (n_{12}v_1)^2(n_{12}v_2)^2 v_1^2 \\
+ \frac{3}{4} (n_{12}v_1)(n_{12}v_2)^3 v_1^2 + \frac{19}{16} (n_{12}v_1)^4 v_1^2 - \frac{21}{16} (n_{12}v_1)(n_{12}v_2)v_1^4 - 2(n_{12}v_2)^2 v_1^4 + \frac{55}{8} v_1^6 \\
\left. + \frac{19}{16} (n_{12}v_1)^4(v_1 v_2) - (n_{12}v_1)^3(n_{12}v_2)(v_1 v_2) - \frac{15}{32} (n_{12}v_1)^2(n_{12}v_2)^2(v_1 v_2) \right) \\
\frac{45}{16} (n_{12}v_1)^2 v_1(v_1 v_2) + \frac{5}{4} (n_{12}v_1)(n_{12}v_2)v_1^2(v_1 v_2) + \frac{11}{4} (n_{12}v_2)^2 v_1^2(v_1 v_2) \\
\frac{13}{16} v_1^4(v_1 v_2) - \frac{3}{4} (n_{12}v_1)^2(v_1 v_2)^2 + \frac{5}{16} (n_{12}v_1)(n_{12}v_2)(v_1 v_2)^2 + \frac{41}{8} v_1^2(v_1 v_2)^2 \\
\left. + \frac{1}{16} (v_1 v_2)^3 - \frac{45}{16} (n_{12}v_1)^2 v_1^2 v_2^2 - \frac{23}{32} (n_{12}v_1)(n_{12}v_2)v_1^2 v_2^2 + \frac{79}{16} v_1^4 v_2^2 - \frac{161}{32} v_1^2(v_1 v_2)v_2^2 \right) \\
\frac{G^2 m_1^2 m_2}{r_{12}^6} \left( - \frac{49}{8} (n_{12}v_1)^4 + \frac{75}{8} (n_{12}v_1)^3(n_{12}v_2) - \frac{187}{8} (n_{12}v_1)^2(n_{12}v_2)^2 \right) \\
\frac{247}{24} (n_{12}v_1)(n_{12}v_2)^3 + \frac{49}{8} (n_{12}v_1)^2 v_1^2 + \frac{21}{8} (n_{12}v_1)(n_{12}v_2)^2 v_1^2 - \frac{11}{2} v_1^4 \\
\frac{15}{2} (n_{12}v_1)^2(v_1 v_2) - \frac{3}{2} (n_{12}v_1)(n_{12}v_2)(v_1 v_2) + \frac{21}{4} (n_{12}v_2)^2(v_1 v_2) - 27v_1^2(v_1 v_2) \\
\frac{55}{2} (v_1 v_2)^2 + \frac{49}{4} (n_{12}v_1)^2 v_2^2 - \frac{27}{2} (n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{3}{4} (n_{12}v_2)^2 v_2^2 + \frac{55}{4} v_1^2 v_2^2 \\
-28(v_1 v_2)^2 v_2^2 + \frac{135}{16} v_2^4 \right) + \frac{G^3 m_1^3 m_2}{r_{12}^6} \left( \frac{5809}{280} - \frac{11}{3} \lambda - \frac{22}{3} \ln \left( \frac{r_{12}}{r_1} \right) \right) \\
\frac{G^3 m_1^3 m_2}{r_{12}^6} \left( - \frac{547}{12} (n_{12}v_1)^2 - \frac{3115}{48} (n_{12}v_1)(n_{12}v_2) - \frac{123}{64} (n_{12}v_1)^2 \pi^2 \right) \\
\frac{123}{64} (n_{12}v_1)(n_{12}v_2) \pi^2 - \frac{575}{18} v_1^2 + \frac{41}{64} \pi^2 v_1^2 + \frac{4429}{144} (v_1 v_2) - \frac{41}{64} \pi^2(v_1 v_2) \\
\frac{G^3 m_1^3 m_2}{r_{12}^6} \left( - \frac{44627}{840} (n_{12}v_1)^2 + \frac{32027}{840} (n_{12}v_1)(n_{12}v_2) + \frac{3}{2} (n_{12}v_2)^2 + \frac{24187}{2520} v_1^2 \right) \\
\frac{27967}{2520} (v_1 v_2) + \frac{5}{4} v_2^2 + 22(n_{12}v_1)^2 \ln \left( \frac{r_{12}}{r_1} \right) - 22(n_{12}v_1)(n_{12}v_2) \ln \left( \frac{r_{12}}{r_1} \right) \right\} \\
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\[-\frac{22}{3} v_1^2 \ln \left( \frac{r_{12}}{r_1^2} \right) + \frac{22}{3} (v_1 v_2) \ln \left( \frac{r_{12}}{r_1^2} \right) \right) + 1 \leftrightarrow 2 + \mathcal{O} \left( \frac{1}{c^2} \right), \quad (4.2)\]

We find that this energy is in agreement with the expression obtained in Ref. [29] by guesswork starting directly from the equations of motion. The logarithms \( \ln \left( \frac{r_{12}}{r_1^2} \right) \) and \( \ln \left( \frac{r_{12}}{r_1^2} \right) \) take the form of a gauge transformation of the energy (see Eq. (6.16) in [29]). Accordingly they will never enter a physical result such as the circular-orbit energy when expressed in terms of the orbital frequency of the circular motion (see Ref. [29]). Such is not the case of the constant \( \lambda \) which does enter the invariant energy. The total linear momentum \( P^i \) at the 3PN order is given by

\[
P^i = m_1 v_1^i \]

\[
+ \frac{1}{c^2} \left\{ - n_{12} \frac{G m_1 m_2}{2 r_{12}} (n_{12} v_1) + v_1^i \left( - \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} \right) \right\} 
\]

\[
+ \frac{1}{c^4} \left\{ n_{12} \left( \frac{G^2 m_1^2 m_2^2}{r_{12}^2} \left( \frac{29}{4} (n_{12} v_1) - \frac{9}{4} (n_{12} v_2) \right) + \frac{G m_1 m_2}{r_{12}} \left( \frac{3}{8} (n_{12} v_1)^3 + \frac{3}{8} (n_{12} v_1)^2 - \frac{7}{8} (n_{12} v_2)^2 + \frac{7}{4} (n_{12} v_1) (v_1 v_2) \right) \right) 
\]

\[
+ \frac{v_1^i}{r_{12}^2} \left( \frac{3 G^2 m_1^2 m_2^2}{2 r_{12}^2} + \frac{G^2 m_1^2 m_2^2}{8 r_{12}^2} + \frac{3 m_1 v_1^4}{8} \right) 
\]

\[
+ \frac{G m_1 m_2}{r_{12}} \left( \frac{13}{8} (n_{12} v_1)^2 - \frac{1}{4} (n_{12} v_1) (n_{12} v_2) - \frac{13}{8} (n_{12} v_2)^2 + \frac{5}{8} v_1^2 - \frac{7}{4} (v_1 v_2) + \frac{7}{8} v_2^2 \right) \right\} 
\]

\[
+ \frac{1}{c^6} \left\{ n_{12} \left( \frac{G^2 m_1^2 m_2^2}{r_{12}^2} \left( - \frac{45}{8} (n_{12} v_1)^3 + \frac{59}{8} (n_{12} v_1)^2 (n_{12} v_2) - \frac{179}{8} (n_{12} v_1) (n_{12} v_2)^2 \right) 
\]

\[
+ \frac{247}{24} (n_{12} v_2)^3 + \frac{135}{8} (n_{12} v_1) v_2^2 - \frac{87}{8} (n_{12} v_2) v_1^2 - \frac{53}{2} (n_{12} v_1) (v_1 v_2) + \frac{87}{4} (n_{12} v_2) (v_1 v_2) \right) 
\]

\[
+ \frac{53}{4} (n_{12} v_1) v_2^2 - 12 (n_{12} v_2) v_1^2 \right\} + \frac{G m_1 m_2}{r_{12}} \left( \frac{5}{16} (n_{12} v_1)^5 - \frac{5}{16} (n_{12} v_1)^4 (n_{12} v_2) 
\]

\[
- \frac{5}{16} (n_{12} v_1)^3 (n_{12} v_2)^2 + \frac{19}{16} (n_{12} v_1)^3 v_1^2 + \frac{15}{16} (n_{12} v_1)^2 (n_{12} v_2) v_1^2 + \frac{3}{4} (n_{12} v_1) (n_{12} v_2)^2 v_1^2 
\]

\[
+ \frac{5}{8} (n_{12} v_2)^3 v_1 - \frac{21}{16} (n_{12} v_1) v_1^4 - \frac{11}{16} (n_{12} v_2) v_1^4 - \frac{5}{4} (n_{12} v_1)^3 (v_1 v_2) \right) 
\]

\[
- \frac{9}{8} (n_{12} v_1)^2 (n_{12} v_2) (v_1 v_2) + \frac{15}{8} (n_{12} v_1) v_1^2 (v_1 v_2) + (n_{12} v_2) v_1^2 (v_1 v_2) - \frac{1}{8} (n_{12} v_1) (v_1 v_2)^2 
\]

\[
- \frac{15}{16} (n_{12} v_1) v_1^2 v_2^2 \right\} - \frac{175 G^3 m_1^2 m_2^2}{8 r_{12}^3} (n_{12} v_1) + \frac{G^3 m_1^3 m_2^2}{r_{12}^3} \left( \frac{46517}{840} (n_{12} v_1) 
\]

\[
+ \frac{34547}{840} (n_{12} v_2) + 22 (n_{12} v_1) \ln \left( \frac{r_{12}}{r_1^2} \right) - 22 (n_{12} v_2) \ln \left( \frac{r_{12}}{r_1^2} \right) \right) + v_1^i \left( \frac{5 m_1 v_1^6}{16} \right) 
\]
Next, the 3PN angular momentum $J^i$ is

\[
J^i = \varepsilon_{ijk}m_1 y_j^i v_k^i \\
+ \frac{1}{c^5} \varepsilon_{ijk} \left\{ y_j^i v_1^k \left( \frac{3Gm_1m_2}{r_{12}} + \frac{m_1 v_1^2}{2} \right) - y_j^i v_2^k \frac{7Gm_1m_2}{2r_{12}} + y_j^i v_2^k \frac{Gm_2m_2}{2r_{12}^2} (n_{12}v_1) \right\} \\
+ \frac{1}{c^4} \varepsilon_{ijk} \left\{ -v_j^i v_2^k \frac{7Gm_1m_2}{4r_{12}^2} (n_{12}v_1) + y_j^i v_2^k \left( -\frac{5G^2m_1m_2}{4r_{12}^2} + \frac{7G^2m_2^2m_2}{2r_{12}^2} \right) \\
+ \frac{3m_1 v_1^4}{8} + \frac{Gm_1m_2}{r_{12}} \left( \frac{3}{2}(n_{12}v_2)^2 + \frac{7}{2}v_1^2 - 4(v_1v_2) + 2v_2^2 \right) \right\} \\
y_j^i v_2^k \left( \frac{7G^2m_1m_2}{4r_{12}^2} + \frac{Gm_1m_2}{r_{12}} \left( -\frac{1}{8}(n_{12}v_1)^2 - \frac{1}{4}(n_{12}v_1)(n_{12}v_2) \right) \\
+ \frac{13}{8}(n_{12}v_2)^2 - \frac{9}{8}v_1^2 + \frac{9}{4}(v_1v_2) - \frac{23}{8}v_2^2 \right) \right\} + y_j^i v_2^k \left( \frac{G^2m_1m_2}{r_{12}^3} \left( -\frac{29}{4} \right) (n_{12}v_1) \right) \\
+ \frac{9}{4}(n_{12}v_2) + \frac{Gm_1m_2}{r_{12}^2} \left( -\frac{3}{8}(n_{12}v_1)^3 - \frac{3}{8}(n_{12}v_1)^2(n_{12}v_2) + \frac{9}{8}(n_{12}v_1)v_1^2 \right) \\
+ \frac{7}{8}(n_{12}v_2)v_1^2 - \frac{7}{4}(n_{12}v_1)(v_1v_2) \right\} \right\} \\
+ \frac{1}{c^3} \varepsilon_{ijk} \left\{ v_j^i v_2^k \left( \frac{G^2m_1m_2}{r_{12}} \left( \frac{235}{24} (n_{12}v_1) - \frac{235}{24} (n_{12}v_2) \right) + Gm_1m_2 \left( \frac{5}{12} (n_{12}v_1)^3 \right) \\
+ \frac{3}{8}(n_{12}v_1)^2(n_{12}v_2) - \frac{15}{8}(n_{12}v_1)v_1^2 - (n_{12}v_2)v_1^2 + \frac{1}{4}(n_{12}v_1)(v_1v_2) \right) \right\} \\
y_j^i v_1^k \left( \frac{5m_1 v_1^6}{16} + \frac{Gm_1m_2}{r_{12}} \left( \frac{9}{8}(n_{12}v_2)^4 - \frac{7}{4}(n_{12}v_2)^2v_1^2 + \frac{33}{8}v_1^4 \right) \right)
\]
\[ G^i = m_1 y^i_i \\
+ \frac{1}{c^2} \left\{ y_1^i \left( - \frac{G m_1 m_2}{2r_{12}} + \frac{m_1 v_1^2}{2} \right) \right\} \\
+ \frac{1}{c^3} \left\{ v_1^i G m_1 m_2 \left( - \frac{7}{4} (n_{12} v_1) - \frac{7}{4} (n_{12} v_2) \right) + y_1^i \left( - \frac{5 G^2 m_1^2 m_2}{4r_{12}^2} + \frac{7 G^2 m_1 m_2^2}{4r_{12}^2} \right) \right. \\
+ \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}} \left( - \frac{1}{8} (n_{12} v_1)^2 - \frac{1}{4} (n_{12} v_1)(n_{12} v_2) + \frac{1}{8} (n_{12} v_2)^2 \right) \\
+ \left. \frac{19}{8} v_1^2 - \frac{7}{4} (v_1 v_2) - \frac{7}{8} v_2^2 \right) \right\} \\
+ \frac{1}{c^5} \left\{ v_1^i \left( \frac{235 G^2 m_1^2 m_2}{24 r_{12}} (n_{12} v_1) - (n_{12} v_2) \right) - \frac{235 G^2 m_1 m_2^2}{24 r_{12}} (n_{12} v_1) - (n_{12} v_2) \right\} \\
+ G m_1 m_2 \left( \frac{5}{12} (n_{12} v_1)^3 + \frac{3}{8} (n_{12} v_1)^2 (n_{12} v_2) + \frac{3}{8} (n_{12} v_1)(n_{12} v_2)^2 \right) \\
+ \frac{5}{12} (n_{12} v_2)^3 - \frac{15}{8} (n_{12} v_1) v_1^2 - (n_{12} v_2) v_1^2 + \frac{1}{4} (n_{12} v_1)(v_1 v_2) \\
+ \frac{1}{4} (n_{12} v_2)(v_1 v_2) - (n_{12} v_1) v_2^2 - \frac{15}{8} (n_{12} v_2)^2 v_2^2 \right) \right\} \\
+ y_1^i \left( \frac{5 m_1 v_1^6}{16} + \frac{G m_1 m_2}{r_{12}} \left( \frac{1}{16} (n_{12} v_1)^4 + \frac{1}{8} (n_{12} v_1)^3 (n_{12} v_2) + \frac{3}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 \right) \right. \\
+ \frac{1}{4} (n_{12} v_1)(n_{12} v_2)^3 - \frac{1}{16} (n_{12} v_2)^4 - \frac{5}{16} (n_{12} v_1)^2 v_1^2 - \frac{1}{2} (n_{12} v_1)(n_{12} v_2) v_1^2 \\
- \frac{11}{8} (n_{12} v_2)^2 v_1^2 + \frac{53}{16} v_1^4 + \frac{3}{8} (n_{12} v_1)^2 (v_1 v_2) + \frac{3}{4} (n_{12} v_1)(n_{12} v_2)(v_1 v_2) + \frac{5}{4} (n_{12} v_2)^2 (v_1 v_2) \\
- 5 v_1^2 (v_1 v_2) + \frac{17}{8} (v_1 v_2)^2 - \frac{1}{4} (n_{12} v_1)^2 v_2^2 - \frac{5}{8} (n_{12} v_1)(n_{12} v_2) v_2^2 + \frac{5}{16} (n_{12} v_2)^2 v_2^2 \\
+ \frac{31}{16} v_1^2 v_2^2 - \frac{15}{8} (v_1 v_2) v_2^2 - \frac{11}{16} v_2^4 \right) + \frac{G^2 m_1^2 m_2^2}{r_{12}^2} \left( \frac{79}{12} (n_{12} v_1)^2 - \frac{17}{3} (n_{12} v_1)(n_{12} v_2) \right) \\
+ \frac{17}{6} (n_{12} v_2)^2 - \frac{175}{24} v_2^2 + \frac{40}{3} (v_1 v_2) - \frac{20}{3} v_2^2 \right) + \frac{G^2 m_1 m_2^2}{r_{12}^2} \left( - \frac{7}{3} (n_{12} v_1)^2 \\
+ \frac{29}{12} (n_{12} v_1)(n_{12} v_2) + \frac{2}{3} (n_{12} v_2)^2 + \frac{101}{12} v_1^2 - \frac{40}{3} v_2^2 \right) + \frac{139}{24} v_2^2 \right) \\
- \frac{19 G^3 m_1^3 m_2^2}{8 r_{12}^3} + \frac{G^3 m_1^3 m_2^2}{r_{12}^3} \left( \frac{13721}{260} - \frac{22}{3} \ln \left( \frac{r_{12}}{r^1} \right) \right) \right\} 
\]
\[ + \frac{G^3 m_1 m_2}{r_{12}^3} \left( - \frac{14351}{1260} + \frac{22}{3} \ln \left( \frac{r_{12}}{r_2^2} \right) \right) \right \} + 1 \leftrightarrow 2 + \mathcal{O} \left( \frac{1}{c^7} \right). \]  

(4.5)

We checked that this expression of the harmonic-coordinate center of mass is changed under the contact transformation into the ADM-coordinate expression which is given by Eqs. (16)-(22) in Ref. [24]. Notice that the energy \( E \) is the only one among these integrals of the 3PN motion that depends on the unknown constant \( \lambda \). The other integrals \( P^i, J^i \) and \( G^i \) do not depend on \( \lambda \) and therefore are entirely determined.

The latter Noetherian quantities are no longer conserved when we take into account the radiation reaction effect at the 2PN order. In order to express the resulting balance equations in the best way, we modify all these quantities by certain terms of order 2.5PN and find that the right-hand-sides of the equations take the form appropriate to a radiative flux at infinity. We pose
\[
\tilde{E} = E + \frac{4G^2 m_2^2 m_1}{5c^5 r_{12}^3} (n_{12} v_1) \left[ (v_1 - v_2)^2 + \frac{2G(m_2 - m_1)}{r_{12}} \right] + 1 \leftrightarrow 2, \tag{4.6a}
\]
\[
\tilde{P}^i = P^i + \frac{4G^2 m_2^2 m_1}{5c^5 r_{12}^3} n_{12} \left( (v_1 - v_2)^2 - \frac{2G m_1}{r_{12}} \right) + 1 \leftrightarrow 2, \tag{4.6b}
\]
\[
\tilde{J}^i = J^i + \frac{4G m_1 m_2}{5c^5} \varepsilon_{ijk} \left[ v_1^2 v_1^j v_2^k + \frac{2G m_1}{r_{12}} v_1^j v_2^k - \frac{2G m_1}{r_{12}} (n_{12} v_1) (v_1^j v_2^k + v_1^k v_2^j) \right] - \frac{G m_1}{r_{12}} (v_1 - v_2)^2 y_1^j y_2^k + \frac{2G^2 m_1^2}{r_{12}^3} y_1^j y_2^k \right] + 1 \leftrightarrow 2, \tag{4.6c}
\]
\[
\tilde{G}^i = G^i + \frac{4G m_1 m_2}{5c^5} v_1^i \left( (v_1 - v_2)^2 - \frac{2G (m_1 + m_2)}{r_{12}} \right) + 1 \leftrightarrow 2. \tag{4.6d}
\]
as well as \( \tilde{K}^i = \tilde{G}^i - t \tilde{P}^i \). Then, the 3PN balance equations are given by
\[
\frac{d\tilde{E}}{dt} = - \frac{G}{5c^3} \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} + \mathcal{O} \left( \frac{1}{c^7} \right), \tag{4.7a}
\]
\[
\frac{d\tilde{P}^i}{dt} = \mathcal{O} \left( \frac{1}{c^7} \right), \tag{4.7b}
\]
\[
\frac{d\tilde{J}^i}{dt} = - \frac{2G}{5c^5} \varepsilon_{ijk} \frac{d^2 Q_{ij}}{dt^2} \frac{d^3 Q_{kl}}{dt^3} + \mathcal{O} \left( \frac{1}{c^7} \right), \tag{4.7c}
\]
\[
\frac{d\tilde{K}^i}{dt} = \mathcal{O} \left( \frac{1}{c^7} \right), \tag{4.7d}
\]
where the Newtonian trace-free quadrupole moment is \( Q_{ij} = m_1 (y_1^j y_2^i - \frac{1}{3} \delta^{ij} y_1^2) + 1 \leftrightarrow 2. \)

**B. Contact transformation and the ADM Hamiltonian at the 3PN order**

Our final result for the contact transformation (3.16) is as follows. The first term in (3.16) is composed of the conjugate momentum of the acceleration and is readily obtained by differentiating (4.1):
\[ q_i = \frac{1}{c^4} \left\{ n_{12}^i \left( -\frac{1}{8} G m_1 m_2 (n_{12} v_2)^2 + \frac{7}{8} G m_1 m_2 v_2^2 \right) - \frac{7}{4} G m_1 m_2 (n_{12} v_2) v_2^i \right\} \\
+ \frac{1}{c^6} \left\{ n_{12}^i \left( \frac{G^2 m_1^2 m_2}{r_{12}} \left( -\frac{17}{6} (n_{12} v_2)^2 + \frac{185}{16} v_1^2 - \frac{185}{8} (v_1 v_2) + \frac{20}{3} v_2^2 \right) \\
+ \frac{G^2 m_1 m_2^2}{r_{12}} \left( \frac{29}{24} (n_{12} v_2)^2 + \frac{235}{48} v_2^2 \right) + G m_1 m_2 \left( \frac{1}{8} (n_{12} v_1) (n_{12} v_2)^3 + \frac{1}{16} (n_{12} v_2)^4 \\
+ \frac{3}{8} (n_{12} v_2)^2 (v_1 v_2) - \frac{5}{8} (n_{12} v_1)^2 v_2^2 - \frac{1}{2} (n_{12} v_1) (n_{12} v_2) v_2^2 - \frac{5}{16} (n_{12} v_2)^2 v_2^2 \right) \right) \\
+ v_1^i \left( G m_1 m_2 \left( \frac{11}{4} (n_{12} v_2) v_1^2 - 2 (n_{12} v_2) (v_1 v_2) + \frac{15}{8} (n_{12} v_2) v_2^2 \right) \right) \\
+ v_2^i \left( -\frac{235 G^2 m_1^2 m_2}{24 r_{12}} (n_{12} v_2) + \frac{235 G^2 m_1 m_2^2}{24 r_{12}} (n_{12} v_2) \right) \\
+ G m_1 m_2 \left( \frac{3}{8} (n_{12} v_1) (n_{12} v_2)^2 + \frac{5}{12} (n_{12} v_2)^3 - (n_{12} v_2) v_1^2 \\
+ \frac{1}{4} (n_{12} v_2) (v_1 v_2) - \frac{15}{8} (n_{12} v_2) v_2^2 \right) \right\} + \mathcal{O} \left( \frac{1}{c^7} \right). \] (4.8)

The second term in (3.16) involves the function \( F \) that constitutes the only possible freedom to adjust in order to match the harmonic-coordinate and ADM-Hamiltonian formalisms. This \( F \) was uniquely determined as

\[ F = \frac{1}{c^4} \left\{ G^2 m_1^2 m_2 \left( \frac{7}{4} (n_{12} v_1)^2 - \frac{1}{4} (n_{12} v_2)^2 \right) + \frac{G m_1 m_2}{4} (n_{12} v_2) v_1^2 \right\} \\
+ \frac{1}{c^6} \left\{ G^2 m_1^2 m_2 \left( -\frac{91}{144} (n_{12} v_1)^3 + \frac{21}{16} (n_{12} v_1)^2 (n_{12} v_2) - \frac{113}{24} (n_{12} v_1) v_1^2 \\
+ \frac{35}{8} (n_{12} v_2) v_1^2 + \frac{195}{16} (n_{12} v_1) (v_1 v_2) - \frac{3}{4} (n_{12} v_1) v_2^2 - \frac{1}{8} (n_{12} v_2) v_2^2 \right) \\
+ G m_1 m_2 \left( -\frac{1}{16} (n_{12} v_1) (n_{12} v_2) v_1^2 v_2^2 - \frac{5}{24} (n_{12} v_2)^3 v_1^2 + \frac{1}{8} (n_{12} v_2) v_1^2 v_2^2 + \frac{5}{16} (n_{12} v_1) v_1^2 v_2^2 \right) \\
+ \frac{G^3 m_1^2 m_2^2}{r_{12}^2} \left( \frac{245}{18} (n_{12} v_1) - \frac{21}{32} (n_{12} v_1) \pi^2 \right) \\
+ \frac{G^3 m_1^3 m_2}{r_{12}^2} \left( -\frac{25867}{2520} (n_{12} v_1) - \frac{3}{4} (n_{12} v_2) + \frac{22}{3} (n_{12} v_1) \ln \left( \frac{r_{12}}{r_1} \right) \right) \right\} + 1 \leftrightarrow 2 + \mathcal{O} \left( \frac{1}{c^7} \right). \] (4.9)
Notice the dependence of $F$ on the logarithms, \textit{viz}

$$\frac{22}{3} G^3 m_1^3 m_2 (n_{12} v_1) \ln \left( \frac{r_{12}}{r'_1} \right) - \frac{22}{3} G^3 m_1 m_2^3 (n_{12} v_2) \ln \left( \frac{r_{12}}{r'_2} \right),$$

which is necessary in order for the contact transformation to remove the logarithms of the harmonic-coordinate Lagrangian (3.19). This result can be checked to be in agreement with the contact transformation given by Eqs. (7.2) in Ref. [29]. The third term in (3.16) involves a correction term, purely of order 3PN, which is defined by (3.17). For this term we get

$$\frac{1}{c^6} X_1^i = \frac{1}{c^6} \left\{ \frac{n_{12}^i}{r_{12}^i} \left( - \frac{G^3 m_1^3 m_2}{4 r_{12}^i} - \frac{4 G^3 m_1^3 m_2^2}{4 r_{12}^i} - \frac{3 G^3 m_1 m_2^3}{4 r_{12}^i} \right) 
+ \frac{G^2 m_1^2 m_2}{r_{12}^i} \left( \frac{11}{8} (n_{12} v_1)^2 - \frac{1}{4} (n_{12} v_1)(n_{12} v_2) - \frac{27}{8} v_1^2 \right) 
+ \frac{G^2 m_1 m_2^2}{r_{12}^i} \left( \frac{3}{8} (n_{12} v_2)^2 - \frac{1}{4} v_1^2 - \frac{15}{8} v_2^2 \right) + Gm_1 m_2 \left( \frac{1}{16} (n_{12} v_2)^2 v_1^2 - \frac{5}{16} v_1^2 v_2^2 \right) \right) 
+ v_1 \left( \frac{35 G^2 m_1^2 m_2}{8 r_{12}^i} (n_{12} v_1) + \frac{G^2 m_1 m_2^2}{r_{12}^i} \left( - \frac{1}{4} (n_{12} v_1) - \frac{3}{2} (n_{12} v_2) \right) \right) 
+ v_2 \left( - \frac{7 G^2 m_1^2 m_2}{4 r_{12}^i} (n_{12} v_1) + \frac{21 G^2 m_1 m_2^2}{4 r_{12}^i} (n_{12} v_2) + Gm_1 m_2 \left( \frac{1}{8} (n_{12} v_2)^2 (n_{12} v_1) + \frac{7}{8} (n_{12} v_2)^2 \right) \right) 
+ \mathcal{O} \left( \frac{1}{c^7} \right).$$

The term $X_1^i$ is obtained by relabeling 1 ↔ 2. With those results we obtain the ADM Lagrangian (3.19) which is an ordinary Lagrangian, not containing any accelerations, and furthermore not containing any logarithms. Though the investigations in Section III.B were done with the harmonic-coordinate quantities taken as “dummy” variables, we must present here the ADM Lagrangian in terms of the variables corresponding to the motion in ADM coordinates. We denote them exactly like in harmonic coordinates but with upper-case letters, e.g. $R_{12} = |Y_1 - Y_2|$, $N_{12} = (Y_1 - Y_2)/R_{12}$, $(N_{12} V_2) = N_{12} V_2$.

$$L_{\text{ADM}} = \frac{Gm_1 m_2}{2 R_{12}} + \frac{1}{2} m_1 V_1^2 \left\{ - \frac{G^2 m_1^2 m_2}{2 R_{12}^3} + \frac{5 G^3 m_1^2 m_2^2}{8 R_{12}^3} + \frac{m_1 V_1^6}{16} \right\} + \frac{1}{c^4} \left\{ G^3 m_1 m_2 + \frac{5 G^3 m_1^2 m_2^2}{8 R_{12}^3} + \frac{G^2 m_1^2 m_2^2}{R_{12}^3} \left( \frac{15}{8} (N_{12} V_1)^2 + \frac{11}{8} V_1^2 - \frac{15}{4} (V_1 V_2) \right) \right\}.$$
Legendre transformation (3.21) as

The corresponding ADM (or, rather, ADM-type [24]) Hamiltonian is given by the ordinary Legendre transformation (3.21) as

\[
H^{\text{ADM}} = - \frac{G m_1 m_2}{2R_{12}} + \frac{P_0^2}{2m_1}
\]

(4.11)
\begin{align}
&+ \frac{1}{c^2} \left\{ - \frac{P_1^4}{8m_1^3} + \frac{G^2m_1^2m_2}{2R_{12}^3} + \frac{Gm_1m_2}{R_{12}} \left( \frac{1}{4} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{3}{2} \frac{P_1^2}{m_1^2} + \frac{7}{4} \frac{(P_1P_2)}{m_1m_2} \right) \right\} \\
&+ \frac{1}{c^6} \left\{ \frac{P_1^6}{16m_1^5} - \frac{G^3m_1^2m_2}{4R_{12}^3} - \frac{5G^2m_1^2m_2}{8R_{12}^3} + \frac{G^2m_2^2m_2}{R_{12}^3} \left( - \frac{3}{2} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} \right) \\
&+ \frac{19P_1^4}{4m_1^2} - \frac{27}{4} \frac{(P_1P_2)}{m_1m_2} + \frac{5P_2^2}{2m_2^2} \right\} + \frac{Gm_1m_2}{R_{12}} \left( - \frac{3}{16} \frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1m_2} \right) \\
&+ \frac{5}{8} \frac{(N_{12}P_1)^2P_2^2}{m_1^2m_2^2} + \frac{5P_1^4}{4} \frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{1}{16} \frac{(P_1P_2)^2}{m_1^2m_2^2} - \frac{11}{16} \frac{(P_1P_2)^2}{m_1^2m_2^2} \right\} \\
&+ \frac{1}{c^6} \left\{ - \frac{5P_1^8}{128m_1^7} + \left( \frac{G^4m_1^2m_2}{8R_{12}^4} + \frac{G^4m_1^2m_2}{R_{12}^4} \right) \left( \frac{993}{140} - \frac{11}{3} \frac{\lambda}{2} - \frac{21}{32} \pi^2 \right) \right\} \\
&+ \frac{G^3m_1^2m_2^2}{R_{12}^3} \left( - \frac{43}{16} \frac{(N_{12}P_1)^2}{m_1^2} + \frac{119}{16} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{3}{64} \frac{(N_{12}P_1)^2}{m_1^2} \right) \\
&+ \frac{3}{64} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{473}{48} \frac{P_1^2}{m_1^2} + \frac{1}{16} \frac{\pi^2 P_1^2}{m_1^2} + \frac{143}{16} \frac{(P_1P_2)}{m_1m_2} - \frac{1}{64} \frac{\lambda^2}{m_1m_2} \right\} \\
&+ \frac{G^3m_1^2m_2^2}{R_{12}^3} \left( \frac{5}{4} \frac{(N_{12}P_1)^2}{m_1^2} + \frac{21}{8} \frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{425}{48} \frac{P_1^2}{m_1^2} + \frac{77}{16} \frac{(P_1P_2)}{m_1m_2} - \frac{25P_2^2}{8m_2^2} \right) \\
&+ \frac{G^2m_2^2m_2}{R_{12}^3} \left( \frac{5}{12} \frac{(N_{12}P_1)^4}{m_1^4} - \frac{3}{2} \frac{(N_{12}P_1)^3(N_{12}P_2)}{m_1^3m_2} + \frac{10}{3} \frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2} \right) \\
&+ \frac{17}{16} \frac{(N_{12}P_1)^2P_2^2}{m_1^4} - \frac{15}{8} \frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{55}{12} \frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2} + \frac{P_1^4}{16} \frac{m_1m_2}{m_2} \\
&- \frac{11}{8} \frac{(N_{12}P_1)^2(P_1P_2)}{m_1^4} + \frac{125}{16} \frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{115}{16} \frac{(P_1P_2)^2}{m_1^4} \\
&+ \frac{25}{48} \frac{(P_1P_2)^2}{m_1^2m_2^2} - \frac{193}{48} \frac{(N_{12}P_1)^2P_2^2}{m_1^2m_2^2} + \frac{371}{48} \frac{P_2^2P_2^2}{m_1^2m_2^2} - \frac{27}{16} \frac{P_2^4}{m_1^2m_2^2} \right\} \\
&+ \frac{Gm_1m_2}{R_{12}} \left( \frac{5}{32} \frac{(N_{12}P_1)^3(N_{12}P_2)^3}{m_1^3m_2^3} + \frac{3}{16} \frac{(N_{12}P_1)^2(N_{12}P_2)^2P_1^2}{m_1^3m_2^3} \\
&- \frac{9}{16} \frac{(N_{12}P_1)(N_{12}P_2)^3P_2^2}{m_1^3m_2^3} + \frac{5}{16} \frac{(N_{12}P_1)^2P_2^4}{m_1^3m_2^3} - \frac{7}{16} \frac{P_1^6}{m_1^3m_2^3} \right) \\
&+ \frac{15}{32} \frac{(N_{12}P_1)^2(N_{12}P_2)(P_1P_2)}{m_1^3m_2^3} + \frac{3}{16} \frac{(N_{12}P_1)(N_{12}P_2)^2P_1^2}{m_1^3m_2^3} \\
&+ \frac{1}{16} \frac{(N_{12}P_2)^2P_2^4}{m_1^3m_2^3} - \frac{5}{16} \frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)^2}{m_1^3m_2^3} \\
&+ \frac{1}{8} \frac{P_2^2(P_1P_2)^2}{m_1^3m_2^3} - \frac{1}{16} \frac{(P_1P_2)^3}{m_1^3m_2^3} + \frac{5}{16} \frac{(N_{12}P_1)^2P_1^2P_2^2}{m_1^3m_2^3} \\
&+ \frac{7}{32} \frac{(N_{12}P_1)(N_{12}P_2)^2P_1^2P_2^4}{m_1^3m_2^3} + \frac{1}{4} \frac{P_1^4P_2^2}{m_1^3m_2^3} + \frac{1}{32} \frac{P_1^2(P_1P_2)^2}{m_1^3m_2^3} \right\} \\
&+ 1 \leftrightarrow 2 + \mathcal{O} \left( \frac{1}{c^4} \right). \quad (4.12)
\end{align}
This result is in perfect agreement with the expression obtained by Damour, Jaranowski and Schäfer [24]. (Note that in their published result, Eq. (12) in Ref. [24], the following terms are missing:

\[
\frac{G^2}{c^6 r_{12}^2} \left( - \frac{55}{12} m_1 - \frac{193}{48} m_2 \right) \frac{(N_{12} P_2)^2 P_1^2}{m_1 m_2} + 1 \leftrightarrow 2 .
\]

This is a misprint which has been corrected in an Erratum [24].) Finally, we recall that the agreement works if and only if our undetermined constant \( \lambda \) is related to their static-ambiguity constant \( \omega_{\text{static}} \) by Eq. (1.4), and their kinetic-ambiguity constant takes the value \( \omega_{\text{kinetic}} = \frac{41}{24} \). This completes the proof of the equivalence of the harmonic-coordinate and ADM-Hamiltonian approaches to the equations of motion of compact binaries at the 3PN order.

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