On the Boundary Regularity of Weak Solutions to the MHD System

V. Vyalov, T. Shilkin

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Abstract

We prove the partial regularity of the boundary suitable weak solutions to the MHD system near the plane part of the boundary.

1 Introduction

Assume $\Omega \subset \mathbb{R}^3$ is a $C^2$–smooth bounded domain and $Q_T = \Omega \times (0, T)$. In this paper we investigate the boundary regularity of solutions to the principal system of magnetohydrodynamics (the MHD equations):

$$\begin{align*}
\partial_t v + (v \cdot \nabla)v - \Delta v + \nabla p &= \text{rot} H \times H \\
\text{div} v &= 0 \quad \text{in } Q_T, \quad (1.1)
\end{align*}$$

$$\begin{align*}
\partial_t H + \text{rot rot} H &= \text{rot}(v \times H) \\
\text{div} H &= 0 \quad \text{in } Q_T. \quad (1.2)
\end{align*}$$

Here unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$, and the magnetic field $H : Q_T \to \mathbb{R}^3$. We impose on $v$ and $H$ the initial and boundary conditions:

$$\begin{align*}
v|_{\partial\Omega \times (0,T)} &= 0, \quad H_{\nu}|_{\partial\Omega \times (0,T)} = 0, \quad (\text{rot } H)|_{\partial\Omega \times (0,T)} = 0, \\
v|_{t=0} &= v_0, \quad H|_{t=0} = H_0. \quad (1.3, 1.4)
\end{align*}$$

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Here by \( \nu \) we denote the outer normal to \( \partial \Omega \) and \( H_\nu = H \cdot \nu \), \((\text{rot } H)_r = \text{rot } H - \nu (\text{rot } H \cdot \nu)\). We introduce the following definition:

**Definition:** Assume \( \Gamma \subset \partial \Omega \). The functions \((v, H, p)\) are called a *boundary suitable weak solution* to the system (1.1), (1.2) near \( \Gamma_T \equiv \Gamma \times (0,T) \) if

1) \( v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T) \cap W^{2,1}_9 (Q_T), \)

\[ H \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T), \]

2) \( p \in L_{\frac{3}{2}}(Q_T) \cap W^{1,0}_{\frac{9}{8},\frac{3}{2}} (Q_T), \)

3) \( \text{div } v = 0, \text{ div } H = 0 \) a.e. in \( Q_T, \)

4) \( v|_{\partial \Omega} = 0, H_\nu|_{\partial \Omega} = 0 \) in the sense of traces,

5) for any \( w \in L_2(\Omega) \) the functions

\[ t \rightarrow \int \Omega v(x,t) \cdot w(x) \, dx \quad \text{and} \quad t \rightarrow \int \Omega H(x,t) \cdot w(x) \, dx \]

are continuous,

6) \((v, H)\) satisfy the following integral identities: for any \( t \in [0, T] \)

\[ \int \Omega v(x,t) \cdot \eta(x,t) \, dx - \int \Omega v_0(x) \cdot \eta(x,0) \, dx + \int_0^t \int \Omega \left( - v \cdot \partial_t \eta + (\nabla v - v \otimes v + H \otimes H) : \nabla \eta - (p + \frac{1}{2} |H|^2) \text{div } \eta \right) \, dx \, dt = 0, \]

for all \( \eta \in W^{1,1}_\frac{2}{2}(Q_t) \) such that \( \eta|_{\partial \Omega \times (0,t)} = 0, \)

\[ \int \Omega H(x,t) \cdot \psi(x,t) \, dx - \int \Omega H_0(x) \cdot \psi(x,0) \, dx + \int_0^t \int \Omega \left( - H \cdot \partial_t \psi + \text{rot } H \cdot \psi - (v \times H) \cdot \text{rot } \psi \right) \, dx \, dt = 0, \]

for all \( \psi \in W^{1,1}_\frac{2}{2}(Q_t) \) such that \( \psi|_{\partial \Omega \times (0,t)} = 0. \)

7) For every \( z_0 = (x_0, t_0) \in \Gamma_T \) such that \( \Omega_R(x_0) \times (t_0 - R^2, t_0) \subset Q_T \) and for any \( \zeta \in C_0^\infty(B_R(x_0) \times (t_0 - R^2, t_0)) \) such that \( \frac{\partial \zeta}{\partial \nu}|_{\partial \Omega} = 0 \)

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the following “local energy inequality near $\Gamma_T$” holds:

$$
\sup_{t \in (t_0-R^2,t_0)} \int_{\Omega_{R(t_0)}} \zeta \left( |v|^2 + |H|^2 \right) \, dx +
+ 2 \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} \zeta \left( |\nabla v|^2 + |\text{rot} \ H|^2 \right) \, dx \, dt \leq
\int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} \left( |v|^2 + |H|^2 \right) \left( \partial_t \zeta + \Delta \zeta \right) \, dx \, dt +
+ \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} \left( |v|^2 + 2\bar{p} \right) v \cdot \nabla \zeta \, dx \, dt +
- 2 \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} (H \otimes H) : \nabla^2 \zeta \, dx \, dt +
+ 2 \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} (v \times H)(\nabla \zeta \times H) \, dx \, dt
$$

We remark also that the following identity holds

$$
\int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} (v \times H)(\nabla \zeta \times H) \, dx \, dt =
= \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} (v \cdot \nabla \zeta)|H|^2 \, dx \, dt - \int_{t_0-R^2}^{t_0} \int_{\Omega_{R(t_0)}} (v \cdot H)(H \cdot \nabla \zeta) \, dx \, dt
$$

Here $L_{s,l}(Q_T)$ is the anisotropic Lebesgue space equipped with the norm

$$
\|f\|_{L_{s,l}(Q_T)} := \left( \int_0^T \left( \int_{\Omega} |f(x,t)|^s \, dx \right)^{\frac{l}{s}} \, dt \right)^{\frac{1}{l}},
$$

and we use the following notation for the functional spaces:

- $W_{s,l}^{1,0}(Q_T) \equiv L_1(0,T; W_{s}^1(\Omega)) = \{ u \in L_{s,l}(Q_T) : \nabla u \in L_{s,l}(Q_T) \}$,
- $W_{s,l}^{2,1}(Q_T) = \{ u \in W_{s,l}^{1,0}(Q_T) : \nabla^2 u, \partial_t u \in L_{s,l}(Q_T) \}$,
- $W_{s,l}^{2}(\Omega) = \{ u \in W_{s}^1(\Omega) : u|_{\partial\Omega} = 0 \}$,

and the following notation for the norms:

$$
\|u\|_{W_{s,l}^{1,0}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)};
$$

$$
\|u\|_{W_{s,l}^{2,1}(Q_T)} = \|u\|_{W_{s,l}^{1,0}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.
$$
**Theorem 1.1** For any sufficiently smooth divergent-free \( v_0, H_0 \) satisfying \((1.3)\) there exists at least one boundary suitable weak solution near \( \partial \Omega \times (0,T) \) which satisfies the initial conditions:

\[
\|v(\cdot,t) - v_0(\cdot)\|_{L^2(\Omega)} \to 0, \quad \|H(\cdot,t) - H_0(\cdot)\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to +0,
\]

and additionally satisfies the global energy inequality

\[
\|v\|_{L^2_\infty(Q_T)} + \|H\|_{L^2_\infty(Q_T)} + \|\nabla v\|_{L^2(Q_T)} + \|\text{rot } H\|_{L^2(Q_T)} \leq \|v_0\|_{L^2(\Omega)} + \|H_0\|_{L^2(\Omega)}
\]

The global existence of weak solutions to the MHD equations \((1.1) – (1.3)\) was established originally in \([6]\). We sketch the proof of Theorem 1.1 in the Section 3.

**2 Main Results**

Denote \( B(x_0, r) \) the open ball in \( \mathbb{R}^3 \) of radius \( r \) centered at \( x_0 \) and denote by \( B^+(x_0, r) \) the half-ball \( \{ x \in B(x_0, r) \mid x_3 > 0 \} \). For \( z_0 = (x_0, t_0) \) denote \( Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0) \), \( Q^+(z_0, r) = B^+(x_0, r) \times (t_0 - r^2, t_0) \). In this paper we shall use the abbreviations: \( B(r) = B(0, r), B^+(r) = B^+(0, r) \) etc, \( B = B(1), B^+ = B^+(1) \) etc.

The main results of the present paper are the following theorems:

**Theorem 2.1** There exists an absolute constant \( \varepsilon_* > 0 \) with the following property. Assume \( (v, H, p) \) is a boundary suitable weak solution in \( Q_T \) and assume \( z_0 = (x_0, t_0) \in \partial \Omega \times (0, T) \) is such that \( x_0 \) belongs to the plane part of \( \partial \Omega \). If there exists \( r_0 > 0 \) such that \( Q^+(z_0, r_0) \subset Q_T \) and

\[
\frac{1}{r_0^3} \int_{Q^+(z_0, r_0)} \left( |v|^3 + |H|^3 + |p|^2 \right) dxdt < \varepsilon_*,
\]

then the functions \( v \) and \( H \) are Hölder continuous on the closure of \( Q^+(z_0, r_0) \).

**Theorem 2.2** For any \( K > 0 \) there exists \( \varepsilon_0(K) > 0 \) with the following property. Assume \( (v, H, p) \) is a boundary suitable weak solution in
and assume \( z_0 = (x_0, t_0) \in \partial \Omega \times (0, T) \) is such that \( x_0 \) belongs to the plane part of \( \partial \Omega \). If

\[
\limsup_{r \to 0} \left( \frac{1}{r} \int_{Q(z_0, r)} |\nabla H|^2 \, dx \, dt \right)^{1/2} < K
\]

and

\[
\limsup_{r \to 0} \left( \frac{1}{r} \int_{Q(z_0, r)} |\nabla v|^2 \, dx \, dt \right)^{1/2} < \varepsilon_0,
\]

then there exists \( \rho_0 > 0 \) such that the functions \( v \) and \( H \) are Hölder continuous on the closure of \( Q^+(z_0, \rho_0) \).

**Theorem 2.3** Assume \((v, H, p)\) is a boundary suitable weak solution in \( Q_T \) and denote by \( \Gamma \) the plane part of \( \partial \Omega \). Then there exists a closed set \( \Sigma \subset \Gamma \) such that for any \( z_0 \in (\Gamma \setminus \Sigma) \times (0, T] \) the functions \((v, H)\) are Hölder continuous in some neighborhood of \( z_0 \), and

\[
\mathcal{P}^1(\Sigma) = 0,
\]

where \( \mathcal{P}^1(\Sigma) \) is the one-dimensional parabolic Hausdorff measure of \( \Sigma \).

Our Theorem 2.3 presents for the MHD equations a result which is a boundary analogue of the famous Caffarelli–Kohn–Nirenberg (CKN) theorem for the Navier-Stokes system, see [2], see also [7]. The boundary regularity of solutions to the Navier-Stokes equations was originally investigated by G. Seregin in [8] and [9] in the case of a plane part of the boundary and by G. Seregin, T. Shilkin, and V. Solonnikov in [11] in the case of a curved boundary. The internal partial regularity of solutions to the MHD system was originally proved by C. He and Z. Xin in [1], see also [15], [16]. Note that thought using the methods of our paper one can prove various \( \varepsilon \)-regularity conditions involving various scale-invariant functionals (such as in [4]), in the present paper we concentrate on the condition (2.6), (2.7) as this condition provides the optimal estimate of the Hausdorff measure of the singular set \( \Sigma \) in Theorem 2.3.

Our paper is organized as follows: in Section 3 we outline the proof of Theorem 1.1. In Section 4 we prove that the weak solutions of the linearized MHD equations are Hölder continuous up to the boundary. Section 5 contains the proof of the Decay Lemma and the sketch of the proof of Theorem 2.1. Section 6 is concerned with the estimate of
some Morrey functional for weak solutions to the heat equation near the boundary. These estimates together with the estimates of the scale invariant energy functionals obtained in Section 7 turn to be crucial for the key estimate (8.3) in the Section 8. Finally, in Section 8 we present the proofs of Theorems 2.2 and 2.3.

3 Local Energy Inequality and Existence of Boundary Suitable Weak Solutions

Denote by $\omega$ the usual smooth Sobolev kernel and $\omega_\delta(x) = \delta^{-3}\omega(\delta x)$. Denote by $\omega_\delta * H$ the convolution of $H$ with $\omega_\delta$. Assume $v_\delta^0$ and $H_\delta^0$ are smooth divergent–free functions satisfying the boundary conditions (1.3) and additionally satisfying the properties

$$v_\delta^0 \to v_0, \quad H_\delta^0 \to H_0 \quad \text{in} \ L_2(\Omega) \quad \text{as} \ \delta \to 0.$$ 

Consider the problem

$$\begin{align*}
\partial_t v + ((\omega_\delta * v) \cdot \nabla)v - \Delta v + \nabla p &= \text{rot } H \times (\omega_\delta * H) \quad \text{in} \ Q_T, \\
\text{div } v &= 0 \\
\partial_t H + \text{rot rot } H &= \text{rot}(v \times (\omega_\delta * H)) \quad \text{in} \ Q_T, \\
\text{div } H &= 0 \\
v|_{\partial \Omega \times (0,T)} &= 0, \quad H_v|_{\partial \Omega \times (0,T)} = 0, \quad (\text{rot } H)_v|_{\partial \Omega \times (0,T)} = 0, \\
v|_{t=0} &= v_0^\delta, \quad H|_{t=0} = H_0^\delta.
\end{align*}$$

(3.1) (3.2)

Using Galerkin approximations and energy estimates it is not difficult to prove existence of the unique smooth solution $(v^\delta, H^\delta, p^\delta)$ to this problem. Moreover, these functions satisfy the global energy inequality

$$\begin{align*}
\|v^\delta\|_{L_{2,\infty}(Q_T)} + \|H^\delta\|_{L_{2,\infty}(Q_T)} + \|v^\delta\|_{W^{1,0}_2(Q_T)} + \|H^\delta\|_{W^{1,0}_2(Q_T)} &\leq c \left( \|v_0\|_{L_2(\Omega)} + \|H_0\|_{L_2(\Omega)} \right).
\end{align*}$$

From this estimate using the interpolation inequality we obtain the estimate

$$\left\|((\omega_\delta * v^\delta) \cdot \nabla)v^\delta\right\|_{L^{3,2}_{\sigma/2}(Q_T)} + \|\text{rot } H^\delta \times (\omega_\delta * H^\delta)\|_{L^{3,2}_{\sigma/2}(Q_T)} \leq c.$$
Applying the coercive estimates for the linear Stokes problem (see, for example, [14]), we obtain
\[
\|v^\delta\|_{W^{2,1}_0(Q_T)} + \|\nabla p^\delta\|_{L^{3/2}_0(Q_T)} \leq c.
\]
So, the only thing we need is to verify the local energy inequality (1.5). Then the result of Theorem 1.1 follows if we pass to the limit as \(\delta \to 0\).

To simplify notations below we omit the index \(\delta\) and denote by \((v,H,p)\) the smooth functions \((v^\delta,H^\delta,p^\delta)\). Take \(z_0 = (x_0,t_0) \in \partial \Omega \times (0,T)\) and choose \(R\) so that \(Q_T(x_0) \times (t_0 - R^2,t_0) \subset Q_T\). Without loss of generality we can put \(x_0 = 0\), \(t_0 = 0\). Assume \(\zeta \in C_0^\infty(B_R \times (-R^2,0])\) satisfies \(\frac{\partial \zeta}{\partial n} \mid_{\partial \Omega} = 0\) and multiply the equation (3.1) by the test–function \(\eta = \zeta v\). Integrating the result over \(\Omega\), integrating by parts and taking into account the relation
\[
\int_\Omega (\omega_\delta \ast v_i)v_{j,i}\zeta v_j\,dx = -\frac{1}{2} \int_\Omega |v|^2(\omega_\delta \ast v) \cdot \nabla \zeta\,dx.
\]
we obtain
\[
\begin{align*}
\frac{\partial}{\partial t} \int_\Omega \zeta |v|^2 + \int_\Omega \zeta |\nabla v|^2\,dx &= \frac{1}{2} \int_\Omega |v|^2(\partial_t \zeta + \Delta \zeta)\,dx + \\
&+ \int_\Omega \bar{p} v \cdot \nabla \zeta\,dx + \frac{1}{2} \int_\Omega |v|^2(\omega_\delta \ast v) \cdot \nabla \zeta\,dx + \\
&+ \int_\Omega (\text{rot } H \times (\omega_\delta \ast H)) \cdot \zeta v\,dx = 0. (3.3)
\end{align*}
\]
Now multiply the equation for the magnetic field by \(\zeta H\). Integrating by parts and taking into account the boundary conditions (1.3) after routine calculations we arrive at
\[
\begin{align*}
\frac{\partial}{\partial t} \int_\Omega \zeta |H|^2\,dx + \int_\Omega \zeta |\text{rot } H|^2\,dx &= \\
= \frac{1}{2} \int_\Omega |H|^2(\partial_t \zeta + \Delta \zeta)\,dx - \int_\Omega H \otimes H : \nabla^2 \zeta\,dx + \\
&+ \int_\Omega \text{rot}(v \times (\omega_\delta \ast H)) \cdot \zeta H\,dx.
\end{align*}
\]
Integrating by parts, we obtain
\[
\begin{align*}
\int_\Omega \text{rot}(v \times (\omega_\delta \ast H)) \cdot \zeta H\,dx &= \int_\Omega \zeta(v \times (\omega_\delta \ast H)) \cdot \text{rot } H\,dx + \\
&+ \int_\Omega (v \times (\omega_\delta \ast H)) \cdot (\nabla \zeta \times H)\,dx
\end{align*}
\]
Note that
\[
\int_\Omega \zeta(v \times (\omega_\delta \ast H)) \cdot \text{rot } H\,dx = \int_\Omega \zeta((\omega_\delta \ast H) \times \text{rot } H) \cdot v\,dx
\]
Hence we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \zeta |H|^2 \, dx + \int_{\Omega} |\zeta| \text{rot } H |^2 \, dx = \frac{1}{2} \int_{\Omega} |H|^2 (\partial_t \zeta + \Delta \zeta) \, dx - \int_{\Omega} H \otimes H : \nabla^2 \zeta \, dx + \int_{\Omega} (\omega_\delta \ast H) \cdot \zeta v \, dx + \int_{\Omega} (v \times (\omega_\delta \ast H)) \cdot (\nabla \zeta \times H) \, dx.
\]

Adding identities (3.3) and (3.4) together, using the conciliation
\[
\int_{\Omega} (\text{rot } H \times (\omega_\delta \ast H)) \cdot \zeta H \, dx + \int_{\Omega} ((\omega_\delta \ast H) \times \text{rot } H) \cdot \zeta v \, dx = 0
\]
and passing to the limit as \( \delta \to 0 \) we arrive at (1.5). Theorem 1.1 is proved.

In conclusion of this section we introduce the following version of the local energy inequality near the plane part of the boundary.

**Theorem 3.1** Assume \((v, H, p)\) is a boundary suitable weak solution satisfied the MHD equations in a domain containing the set \( Q^+ = B^+ \times (-1, 0) \) such that the plane part of \( Q^+ \) belongs to the boundary \( \partial \Omega \times (-1, 0) \). Obviously then we have
\[
v|_{x_3=0} = 0, \quad H_3|_{x_3=0} = 0, \quad H_{1,3}|_{x_3=0} = H_{2,3}|_{x_3=0} = 0.
\]

Let \( \zeta \in C_0^\infty (B \times (-1, 0]) \) be a cut-off function such that \( \zeta \, 3|_{x_3=0} = 0 \). Assume \( b \in \mathbb{R}^3 \) is an arbitrary constant vector of the form \( b = (b_1, b_2, 0) \). Then the following inequality holds
\[
\sup_{t \in (-1, 0)} \int_{B^+} \zeta \left( |v|^2 + |\bar{H}|^2 \right) \, dx + 2 \int_{Q^+} \zeta \left( |\nabla v|^2 + |\text{rot } \bar{H}|^2 \right) \, dz \leq \int_{Q^+} \left( |v|^2 + |\bar{H}|^2 \right) (\partial_t \zeta + \Delta \zeta) \, dz + \int_{Q^+} \left( |v|^2 + 2\bar{p} \right) v \cdot \nabla \zeta \, dz + 2 \int_{Q^+} (\bar{H} \otimes \bar{H}) : \nabla^2 \zeta \, dz + 2 \int_{Q^+} (v \times H)(\nabla \zeta \times \bar{H}) \, dz
\]

where \( \bar{H} = H - b \).

**Proof:** The relation (3.5) is a combination of (1.5) with the relation obtained from (1.2) multiplied by the test function \( \psi = \zeta b \) where \( b = (b_1, b_2, 0) \) is a constant vector. The proof is routine and we omit it.
4 Linear Estimate

**Theorem 4.1** For any $M > 0$ there exists $c(M) > 0$ such that for any $a \in \mathbb{R}^3$ such that $a = (a_1, a_2, 0)^T$ and $|a| \leq M$, and for any $(u, h, q)$ satisfying the linear system

\[
\begin{align*}
\partial_t u - \Delta u + \nabla q &= \text{rot} h \times a \\
\text{div} u &= 0 \\
\partial_t h - \Delta h &= \text{rot}(u \times a) \\
\text{div} h &= 0 \\
u|_{x_3=0} &= 0, \\
h_3|_{x_3=0} &= 0, \\
\frac{\partial h}{\partial x_3}|_{x_3=0} &= 0, \\
\alpha = 1, 2,
\end{align*}
\]

the following estimate holds:

\[
\begin{align*}
\|u\|_{C^{1,6}(Q)} + \|h\|_{C^{1,6}(Q)} &= c(M) \left(\|u\|_{L^3(Q)} + \|h - b\|_{L^3(Q)} + \|q - c\|_{L^{3/2}(Q)}\right)
\end{align*}
\]

Here $b \in \mathbb{R}^3$ is an arbitrary vector of the form $b = (b_1, b_2, 0)^T$, and $c \in \mathbb{R}$ is an arbitrary constant.

**Proof:**

1. Similar to Theorem 3.1 we obtain

\[
\begin{align*}
\|u\|_{L^{\infty}(Q^+)} + \|h\|_{L^{\infty}(Q^+)} &+ \|\nabla u\|_{L^2(Q^+)} + \|\nabla h\|_{L^2(Q^+)} \\
&\leq c(M) \left(\|u\|_{L^3(Q)} + \|h - b\|_{L^3(Q)} + \|q - c\|_{L^{3/2}(Q)}\right)
\end{align*}
\]

Here $\tilde{q} \equiv q - c$. Note that the right-hand side of the last inequality can be estimated by the right-hand side of (4.1) via H"older inequality.

2. For the function $u$ satisfying the system (4.2) we have the following estimate with arbitrary $s, l \in (1, +\infty)$ and $0 < \rho < r < \frac{9}{10}$ (see [10]):

\[
\|u\|_{W^{2,1}_{s,l}(Q^+)} \leq c \left(\|\text{rot} h\|_{L_{s,l}(Q^+(r))} + \|u\|_{L^3(Q^+)} + \|\tilde{q}\|_{L^{3/2}(Q^+)}\right)
\]

Here $C$ depends on $M, r, \rho, s,$ and $l.$
3. Denote by $h^*$ the following extension of the function $h$ from $Q^+$ onto $Q$: we take $h^*(x, t) = h(x, t)$ for $x_3 \geq 0$ and take
\[
\begin{aligned}
  h^*_x(x_1, x_2, x_3, t) &= h(x_1, x_2, -x_3, t) \\
  h^*_x(x_1, x_2, x_3, t) &= -h_3(x_1, x_2, -x_3, t)
\end{aligned}
\quad \text{for } x_3 < 0.
\]
Denote $g = \text{rot}(u \times H)$ and let $g^*$ be the following extension on $g$ from $Q^+$ onto $Q$: the components $g^*_x$, $\alpha = 1, 2$ are the even extensions of $g_\alpha$, and the component $g^*_3$ is the odd extension of $g_3$. Then the functions $h^*$, $g^*$ satisfy the equation
\[
\partial_t h^* - \Delta h^* = g^* \quad \text{in } Q. \tag{4.7}
\]

4. For the function $h^*$ satisfying the heat equation $\tag{4.7}$ the estimate similar to $\tag{4.6}$ holds.
\[
\|h^*\|_{W^{2, 1}_{s, l}(Q(\rho))} \leq C \left( \|g^*\|_{L_{s, l}(Q(r))} + \|h^* - b\|_{L_3(Q)} \right)
\]
Here $s, l \in (1, +\infty)$ and $0 < \rho < r \leq \frac{\rho}{10}$ are arbitrary and $C$ depends on $M$, $r$, $\rho$, $s$, and $l$. This estimate provides the inequality
\[
\|h\|_{W^{2, 1}_{s, l}(Q(\rho))} \leq C \left( \|\nabla u\|_{L_{s, l}(Q(r))} + \|h - b\|_{L_3(Q^+)} \right) \tag{4.8}
\]

5. First we apply $\tag{4.8}$ with $s = l = 2$ and $\rho = \frac{4}{5}$, $r = \frac{9}{10}$. Taking into account the energy estimate $\tag{4.5}$ we obtain the estimate of the norm $\|h\|_{W^{2, 1}_{2, 2}(Q^+(\frac{4}{5}))}$ by the right-hand side of $\tag{4.4}$. Then using the imbedding theorem $W^{2, 1}_{2, 2}(Q^+(\frac{4}{5})) \hookrightarrow W^{1, 0}_{\frac{10}{11}}(Q^+(\frac{4}{5}))$ we obtain the estimate of the norm $\|\nabla h\|_{L^\frac{10}{11}(Q^+(\frac{4}{5}))}$.

6. Now applying $\tag{4.6}$ with $s = \frac{10}{3}$, $l = \frac{2}{3}$ and $\rho = \frac{7}{11}$, $r = \frac{4}{5}$ we obtain the estimate
\[
\|u\|_{W^{2, 1}_{\frac{10}{3}, \frac{2}{3}}(Q^+(\frac{7}{11}))} \leq C \left( \|\text{rot } h\|_{L^{\frac{10}{3}, \frac{2}{3}}(Q^+(\frac{7}{11}))} + \|u\|_{L_3(Q^+)} + \|\bar{q}\|_{L_{3/2}(Q^+)} \right)
\]
By Hölder inequality we estimate $\|\text{rot } h\|_{L^{\frac{10}{3}, \frac{2}{3}}(Q^+(\frac{7}{11}))}$ by the norm $\|\nabla h\|_{L^\frac{10}{11}(Q^+(\frac{4}{5}))}$ which was already estimated on the previous step. On the other hand, by the imbedding theorem we obtain
\[
\|\nabla u\|_{L^2(B^+(\frac{7}{11}))} \leq C\|u\|_{W^{2, 1}_{\frac{10}{3}, \frac{2}{3}}(Q^+(\frac{7}{11}))} \tag{4.9}
\]
7. The estimate (4.9) implies in particular that \( \nabla u \in L_{9, \frac{3}{2}}(Q^+(\frac{7}{10})) \).

Applying (4.8) with \( s = 2, \ l = \frac{3}{2} \) we obtain the estimate

\[ \| h \|_{W^{2,1}_{9, \frac{3}{2}}(Q^+(\frac{7}{10}))} \leq C \left( \| \nabla u \|_{L_{9, \frac{3}{2}}(Q^+(\frac{7}{10}))} + \| h - b \|_{L_3(Q^+)} \right) \]

8. Finally, we apply (4.6) with \( s = 9, \ l = \frac{3}{2} \) and obtain

\[ \| u \|_{W^{2,1}_{9, \frac{3}{2}}(Q^+(\frac{3}{5}))} \leq C \left( \| \mathrm{rot} \, h \|_{L_{9, \frac{3}{2}}(Q^+(\frac{3}{5}))} + \| u \|_{L_3(Q^+)} + \| q \|_{L_{3/2}(Q^+)} \right) \]

9. Gathering all the estimates together we arrive at the estimate

\[ \| u \|_{W^{2,1}_{9, \frac{3}{2}}(Q^+(\frac{3}{5}))} + \| h \|_{W^{2,1}_{9, \frac{3}{2}}(Q^+(\frac{3}{5}))} \leq \]

\[ \leq C(M) \left( \| u \|_{L_3(Q^+)} + \| h - b \|_{L_3(Q^+)} + \| q - c \|_{L_{3/2}(Q^+)} \right) \]

The statement of the theorem follows now from the imbedding \( W^{s,l}_{q, \frac{3}{2}}(Q_T) \hookrightarrow C^{\alpha, \frac{\alpha}{2}}(Q_T) \) as \( \alpha = 2 - \frac{3}{s} - \frac{3}{l} > 0 \), where \( s = 9, \ l = \frac{3}{2}, \) and \( \alpha = \frac{1}{3} \). Theorem 4.1 is proved.

**Corollary 4.1** Let us introduce the functional

\[ Y_\tau(v, H, p) := \left( \int_{Q^+(\tau)} |v|^3 \, dx \, dt \right)^{1/3} + \]

\[ + \tau \left( \int_{Q^+(\tau)} |p - [p]_{B^+(\tau)}|^{3/2} \, dx \, dt \right)^{2/3} + \]

\[ + \left( \int_{Q^+(\tau)} |H - b_\tau(H)|^3 \, dx \, dt \right)^{1/3} \] (4.10)

where \( b_\tau(H) := ((H_1)_{Q^+(\tau)}, (H_2)_{Q^+(\tau)}, 0) \). Then for any \( M > 0 \) there is a constant \( C_l(M) > 0 \) such that for any solution \((u, h, q)\) of the linear system (4.1), (4.2), (4.3), the following estimate holds:

\[ Y_\tau(u, h, q) \leq C_l(M) \tau^{1/3} Y_1(u, h, q). \]
In this section we consider the MHD system
\[
\begin{aligned}
\partial_t v + (v \cdot \nabla)v - \Delta v + \nabla p &= \text{rot} \times H \quad \text{in } Q^+, \\
\text{div } v &= 0,
\end{aligned}
\]
\[
\begin{aligned}
\partial_t H - \Delta H &= \text{rot} (v \times H) \quad \text{in } Q^+,
\end{aligned}
\]
\[
\begin{aligned}
H |_{x_3=0} = 0, \quad v |_{x_3=0} = 0, \quad \frac{\partial H_1}{\partial x_3} |_{x_3=0} = \frac{\partial H_2}{\partial x_3} |_{x_3=0} = 0
\end{aligned}
\]
We denote by \( Y_\tau(v, H, p) \) the functional introduced in (4.10). We also denote by \( Y_\tau(v) \), \( \tilde{Y}_\tau(H) \), and \( \hat{Y}_\tau(p) \) the functionals
\[
Y_\tau(v) := \left( \int_{Q^+} |v|^3 \, dx \, dt \right)^{1/3},
\]
\[
\tilde{Y}_\tau(H) := \left( \int_{Q^+} |H - b_\tau(H)|^3 \, dx \, dt \right)^{1/3},
\]
\[
\hat{Y}_\tau(p) := \tau \left( \int_{Q^+} |p - [p]|_{B^+(\tau)}^{3/2} \, dx \, dt \right)^{2/3}
\]

**Theorem 5.1** There exists an absolute constant \( \varepsilon_0 > 0 \) such that for any \( M > 0 \) there exists \( C_* = C_*(M) \) with the following properties. For any boundary suitable weak solution \((v, H, p)\) of the MHD system (5.1), (5.2), (5.3), the following implication holds: if
\[
Y_1(v, H, p) < \varepsilon_0
\]
and
\[
|(H_1)_{Q^+}| + |(H_2)_{Q^+}| \leq M
\]
then
\[
Y_\tau(v, H, p) \leq C_* \tau^{1/3} Y_1(v, H, p)
\]
Proof:

1. Arguing by contradiction we assume there exists a sequence of numbers \( \varepsilon_m \to 0 \), and a sequence of boundary suitable weak solutions \((v^m, H^m, p^m)\) such that
   \[
   Y_1(v^m, H^m, p^m) = \varepsilon_m \to 0
   \]
   and
   \[
   Y_\tau(v^m, H^m, p^m) \geq C_\tau \varepsilon_m^{1/3}
   \]

2. Let us introduce functions
   \[
   u^m(x,t) = \frac{1}{\varepsilon_m} v^m(x,t),
   q^m(x,t) = \frac{1}{\varepsilon_m} \left( p^m(x,t) - [p^m]_{B^+}(t) \right),
   h^m(x,t) = \frac{1}{\varepsilon_m} \left( H^m(x,t) - a^m \right),
   a^m = b_1(H^m)
   \]
   Then
   \[
   Y_1(u^m, h^m, q^m) = 1, \quad Y_\tau(u^m, h^m, q^m) \geq C_\tau \varepsilon_m^{1/3} \tag{5.5}
   \]
   and \((u^m, h^m, q^m)\) satisfy the following equations in \( D'(Q^+) \)
   \[
   \begin{align*}
   \partial_t u^m + \varepsilon_m (u^m \cdot \nabla) u^m - \Delta u^m + \nabla q^m &= \text{rot } h^m \times (\varepsilon_m h^m + a^m) \\
   \text{div } u^m &= 0 \\
   \partial_t h^m - \Delta h^m &= \text{rot } (u^m \times (\varepsilon_m h^m + a^m)) \\
   \text{div } h^m &= 0 \\
   \left. u^m \right|_{x_3=0} &= 0, \quad \left. \frac{\partial h^m}{\partial x_3} \right|_{x_3=0} = 0 \tag{5.6}
   \end{align*}
   \]

3. The conditions \tag{5.5} imply in particular the boundedness
   \[
   \sup_m \left( \|u^m\|_{L_2(Q^+)} + \|h^m\|_{L_2(Q^+)} + \|q^m\|_{L_2(Q^+)} \right) < +\infty \tag{5.9}
   \]
   From the local energy inequality near the boundary \tag{3.5} we obtain
   \[
   \|u^m\|_{L_{2,\infty}(Q^+\left(\frac{\alpha}{10}\right))} + \|h^m\|_{L_{2,\infty}(Q^+\left(\frac{\alpha}{10}\right))} + \\
   + \|u^m\|_{W_{2,0}^{1,0}(Q^+\left(\frac{\alpha}{10}\right))} + \|h^m\|_{W_{2,0}^{1,0}(Q^+\left(\frac{\alpha}{10}\right))} \leq C. \tag{5.10}
   \]
From the equations \((5.6), (5.7), (5.8)\) we also obtain the estimate

\[
\| \partial_t u^m \|_{L^5_0(-1,0; W^{-1}_2(B^+))} + \| \partial_t h^m \|_{L^5_0(-1,0; W^{-1}_2(B^+))} \leq C.
\]

4. Hence we can extract subsequences

\[
\begin{align*}
  u^m & \rightharpoonup u \text{ in } L^3(\mathbb{R}^3), \\
  h^m & \rightharpoonup h \text{ in } L^3(\mathbb{R}^3), \\
  q^m & \rightharpoonup q \text{ in } L^{3/2}(\mathbb{R}^3), \\
  u^m & \rightharpoonup u \text{ in } W^{1,0}(Q^+), \\
  h^m & \rightharpoonup h \text{ in } W^{1,0}(Q^+), \\
  a^m & \rightarrow a \text{ in } \mathbb{R}^3.
\end{align*}
\]

5. Passing to the limit in the equations \((5.6), (5.7), (5.8)\) we obtain

\[
\begin{align*}
  \partial_t u - \Delta u + \nabla q &= \text{rot } h \times a \quad \text{in } Q^+, \\
  \text{div } u &= 0 \quad \text{in } Q^+, \\
  u|_{x_3=0} &= 0, \\
  \partial_t h - \Delta h &= \text{rot}(u \times a) \quad \text{in } Q^+, \\
  \text{div } h &= 0 \quad \text{in } Q^+, \\
  h_3|_{x_3=0} &= 0, \\
  \frac{\partial h_1}{\partial x_3}|_{x_3=0} &= \frac{\partial h_2}{\partial x_3}|_{x_3=0} = 0.
\end{align*}
\]

6. From \((5.13)\) we conclude

\[
\lim_{m \rightarrow +\infty} Y_\tau(u^m) = Y_\tau(u), \quad \lim_{m \rightarrow +\infty} \tilde{Y}_\tau(h^m) = \tilde{Y}_\tau(h)
\]

and hence

\[
\limsup_{m \rightarrow \infty} Y_\tau(u^m, h^m, q^m) \leq Y_\tau(u) + \tilde{Y}_\tau(h) + \limsup_{m \rightarrow \infty} \tilde{Y}_\tau(q^m). \quad (5.16)
\]

As \((u, h, q)\) is a solution to the linear problem \((5.14) - (5.15)\), from Theorem 4.1 we obtain

\[
Y_\tau(u) + \tilde{Y}_\tau(h) \leq C(M) \tau^{1/3} Y_1(u, h, q) \quad (5.17)
\]

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7. Now we are going to estimate \( \limsup_{m \to \infty} \hat{Y}_r(q^m) \). For this purpose we decompose \((u^m, q^m)\) and \((u, q)\) as

\[
\begin{align*}
    u^m &= u^m_1 + u^m_2, \\
    q^m &= q^m_1 + q^m_2, \\
    u &= u_1 + u_2, \\
    q &= q_1 + q_2,
\end{align*}
\]

where \((u^m_1, q^m_1) \in W^{2,1}_{\frac{3}{2}, \frac{3}{2}}(\Pi^+) \times W^{1,0}_{\frac{3}{2}, \frac{3}{2}}(\Pi^+)\) are determined as a solutions of the following initial boundary-value problems in \( \Pi^+ = \mathbb{R}^3_+ \times (-1, 0) \):

\[
\begin{align*}
    \partial_t u^m_1 - \Delta u^m_1 + \nabla q^m_1 &= f^m, & \text{in} & \quad \Pi^+, \\
    \text{div} u^m_1 &= 0, & \text{in} & \quad \Pi^+, \\
    u^m_1|_{t=-1} &= 0, & \quad u^m_1|_{x_3=0} &= 0,
\end{align*}
\]

where \( f^m \) is defined by the expression \( \text{rot} h^m \times (\varepsilon^m h^m + a^m) - \varepsilon^m u^m \). \( \nabla u^m \) on the set \( Q^+(\frac{9}{10}) \) and extended by zero onto the whole \( \Pi^+ \). Similarly, \((u_1, q_1)\) are determined by the relations

\[
\begin{align*}
    \partial_t u_1 - \Delta u_1 + \nabla q_1 &= f, & \text{in} & \quad \Pi^+, \\
    \text{div} u_1 &= 0, & \text{in} & \quad \Pi^+, \\
    u_1|_{t=-1} &= 0, & \quad u_1|_{x_3=0} &= 0,
\end{align*}
\]

where \( f \) determined by the expression \( \text{rot} h \times a \) on the set \( Q^+(\frac{9}{10}) \) and extended by zero onto the whole \( \Pi^+ \).

8. As functions \( u^m_1 - u_1, q^m_1 - q_1 \) are the solution of the first initial boundary-value problem in \( \Pi^+ \) with the right-hand side \( f^m - f \) and zero initial and boundary conditions, we obtain the estimate (see, for example, [13], Proposition 2.1)

\[
\begin{align*}
    \|u^m_1\|_{W^{2,1}_{\frac{3}{2}, \frac{3}{2}}(\Pi^+)} + \|\nabla q^m_1\|_{L^{\frac{3}{2}, \frac{3}{2}}(\Pi^+)} &\leq C\|f^m\|_{L^{\frac{3}{2}, \frac{3}{2}}(Q^+(\frac{9}{10}))} \\
    \|u^m_1 - u_1\|_{W^{2,1}_{\frac{3}{2}, \frac{3}{2}}(\Pi^+)} + \|\nabla q^m_1 - \nabla q_1\|_{L^{\frac{3}{2}, \frac{3}{2}}(\Pi^+)} &\leq C\|f^m - f\|_{L^{\frac{3}{2}, \frac{3}{2}}(Q^+(\frac{9}{10}))}.
\end{align*}
\]

Note that

\[
\begin{align*}
    \|f^m\|_{L^{\frac{3}{2}, \frac{3}{2}}(Q^+(\frac{9}{10}))} &\leq C(M) \\
    \|f^m - f\|_{L^{\frac{3}{2}, \frac{3}{2}}(Q^+(\frac{9}{10}))} &\to 0, \quad \text{as} \quad m \to \infty.
\end{align*}
\]

So, taking into account the imbedding \( W^{1,0}_{\frac{3}{2}, \frac{3}{2}}(Q^+(\frac{9}{10})) \hookrightarrow L^{\frac{3}{2}}_{\frac{3}{2}}(Q^+(\frac{9}{10})) \), we can conclude that

\[
q^m_1 \to q_1 \quad \text{in} \quad L^{\frac{3}{2}}_{\frac{3}{2}}(Q^+(\frac{9}{10}))
\]
and hence for any $\tau \in (0, \frac{9}{10})$

$$\lim_{m \to \infty} Y_\tau(q^m) = Y_\tau(q_1).$$

On the other hand, $(u_1, q_1)$ is a solution of the linear Stokes problem in $Q^+$. Hence from Corollary 4.1 we conclude

$$Y_\tau(q_1) \leq C(M) \tau^{1/3} Y_{\frac{9}{10}}(q_1).$$

9. We need to estimate $Y_{\frac{9}{10}}(q_1)$. From imbedding theorem $L_{\frac{3}{2}}(B^+(\frac{9}{10})) \hookrightarrow W^{1, 2}_{\frac{3}{2}}(B^+(\frac{9}{10}))$ we conclude

$$Y_{\frac{9}{10}}(q_1) \leq C \|\nabla q_1\|_{W^{1, 2}_{\frac{3}{2}}(B^+(\frac{9}{10}))}.$$

For the solution $(u_1, q_1)$ of the initial-boundary value problem (5.18) we have the estimate

$$\|u_1\|_{W^{2,1}_{\frac{3}{2}}(Q^+(\frac{9}{10}))} + \|\nabla q_1\|_{L_{\frac{3}{2}}(Q^+(\frac{9}{10}))} \leq C(M) \|\nabla h\|_{W^{2,1}_{\frac{3}{2}}(Q^+(\frac{9}{10}))}.$$

Using Hölder inequality $\|\nabla h\|_{L_{\frac{3}{2}}(Q^+(\frac{9}{10}))} \leq C \|\nabla h\|_{L_2(Q^+(\frac{9}{10}))}$ and taking into account the weak convergence (5.12) from which we conclude

$$\|\nabla h\|_{L_2(Q^+(\frac{9}{10}))} \leq \liminf_{m \to \infty} \|\nabla h^m\|_{L_2(Q^+(\frac{9}{10}))},$$

and using (5.10) we obtain

$$Y_{\frac{9}{10}}(q_1) \leq C(M).$$

10. Now we consider functions $(u_2^m, q_2^m)$ determined by relations

$$u_2^m := u^m - u_1^m, \quad q_2^m := q^m - q_1^m. \quad (5.21)$$

These functions satisfy the homogeneous Stokes problems in $Q^+(\frac{9}{10})$:

$$\begin{align*}
\partial_t u_2^m - \Delta u_2^m + \nabla q_2^m &= 0 \quad \text{in} \quad Q^+(\frac{9}{10}), \\
\operatorname{div} u_2^m &= 0 \quad \text{in} \quad Q^+(\frac{9}{10}), \\
u_2^m |_{x_3=0} &= 0, \\
\partial_t u_2 - \Delta u_2 + \nabla q_2 &= 0 \quad \text{in} \quad Q^+(\frac{9}{10}), \\
\operatorname{div} u_2 &= 0 \quad \text{in} \quad Q^+(\frac{9}{10}), \\
u_2 |_{x_3=0} &= 0.
\end{align*}$$
Then
\[
\| u^m_2 \|_{W^{1,2}(\Omega + (\frac{\tau}{4}))} + \| \nabla q^m_2 \|_{L^{\frac{9}{2}}(\Omega + (\frac{\tau}{4}))} \leq C \left( \| u^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))} + \| q^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))} \right)
\]

Note that due to (5.21), (5.9) and the first inequalities in (5.19), (5.20) we have the estimate
\[
\| u^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))} + \| q^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))} \leq C(\| u^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))} + \| q^m_2 \|_{L^3(\Omega + (\frac{\tau}{4}))})
\]

On the other hand, from the Hölder inequality we obtain for any \( \tau \in (0, \frac{1}{6}) \)
\[
\hat{Y}_\tau(q^m_2) = \tau \left( \int_{\Omega + (\frac{\tau}{4})} |q^m_2 - [q^m_2]_{B^+(\tau)}|^\frac{2}{3} \, dx \, dt \right)^{\frac{3}{2}} \leq C \tau \left( \int_{\Omega + (\frac{\tau}{4})} |\nabla q^m_2|^\frac{2}{3} \, dx \, dt \right)^{\frac{3}{2}} \leq C \tau \|\nabla q^m_2\|_{L^{\frac{9}{2}}(\Omega + (\frac{\tau}{4}))} \leq C(M) \tau \frac{2}{3}
\]

11. Summarizing all previous estimates we arrive at
\[
\limsup_{m \to \infty} \hat{Y}_\tau(q^m) \leq \lim_{m \to \infty} \hat{Y}_\tau(q^m_1) + \limsup_{m \to \infty} \hat{Y}_\tau(q^m_2) \leq C(M) \tau \frac{1}{3}
\]

Taking into account (5.16) and (5.17), we finally obtain
\[
\limsup_{m \to \infty} Y_\tau(u^m, h^m, q^m) \leq C(M) \tau \frac{1}{3}
\]

This estimate contradicts (5.5) whenever \( C_* > C(M) \). Theorem 5.1 is proved.

Theorem 2.2 follows from Theorem 5.1 in the standard way by iterations of the estimate (5.4), scaling arguments, and combination of boundary estimates with the internal estimates obtained in [15]. See details in [8], [9], [11], [12].
6 Estimates of Solutions to the Heat Equation

In this section we study solutions of the heat equations with the lower order terms:
\[
\partial_t H - \Delta H = \text{div}(v \otimes H - H \otimes v) \quad \text{in} \quad Q^+.
\]
\[
v|_{x_3=0} = 0,
\]
\[
H_3|_{x_3=0} = 0, \quad H_{\alpha,3}|_{x_3=0} = 0, \quad \alpha = 1, 2.
\]
Namely, we assume the functions \((v, H)\) possess the following properties:
\[
v, H \in W^{1,0}_2(Q^+),
\]
\[
v|_{x_3=0} = 0, \quad H_3|_{x_3=0} = 0 \quad \text{in the sense of traces},
\]
for any \(\eta \in C_0^\infty(Q; \mathbb{R}^3)\) such that \(\eta_3|_{x_3=0} = 0\) the following integral identity holds
\[
\int_{Q^+} \left( -H \cdot \partial_t \eta + \nabla H : \nabla \eta \right) dxdt = -\int_{Q^+} G : \nabla \eta \ dxdt, \quad (6.2)
\]
where \(G = v \otimes H - H \otimes v\), and
\[
\text{div} v = 0, \quad \text{div} H = 0 \quad \text{a.e. in} \quad Q^+. \quad (6.3)
\]

We start with the auxiliary results:

**Theorem 6.1** Assume conditions (6.1), (6.2) hold. Denote by \(v^*\) and \(H^*\) the extensions of the functions \(v\) and \(H\) from \(Q^+\) onto \(Q\) obtained in the following way: the components \(v^*_\alpha, H^*_\alpha, \alpha = 1, 2\) are the even extensions of the components \(v_\alpha, H_\alpha, \alpha = 1, 2\), and the components \(v^*_3, H^*_3\) are the odd extensions of \(v_3, H_3\). Define also the function
\[
G^* = v^* \otimes H^* - H^* \otimes v^*.
\]

Then the following relation holds:
\[
\partial_t H^* - \Delta H^* = \text{div} \ G^* \quad \text{in} \quad \mathcal{D}'(Q). \quad (6.4)
\]
Moreover, if (6.3) is satisfied, then
\[
\text{div} v^* = 0, \quad \text{div} H^* = 0, \quad \text{in} \quad \mathcal{D}'(Q).
\]
Proof of Theorem 6.1: The result is a direct consequence of the boundary conditions \( v|_{x_3=0} = 0 \) and \( H_3|_{x_3=0} = 0 \). Theorem 6.1 is proved.

Now we introduce the following functionals

\[
E(r) = \left( \frac{1}{r} \int_{Q^+(r)} |\nabla v|^2 \, dxdt \right)^{1/2},
\]

\[
E_*(r) = \left( \frac{1}{r} \int_{Q^+(r)} |\nabla H|^2 \, dxdt \right)^{1/2},
\]

\[
F_q(r) = \left( \frac{1}{r^{5-q}} \int_{Q^+(r)} |H|^q \, dxdt \right)^{1/q}.
\]

(6.5)

Theorem 6.2 Assume conditions (6.1) — (6.3) hold. Then for any \( r \in (0, \frac{1}{2}) \) the following estimate holds

\[
\|H\|_{L^1(Q^+(r)), \infty} \leq cr^2 \left( 1 + E(2r) \right) \left( F_2(2r) + E_*(2r) + r \right)
\]

(6.6)

Proof of Theorem 6.2: Consider the function

\[
\eta = \frac{\zeta H^*}{(1 + |H^*|^2)^{1/2}},
\]

where \( \zeta \in C_0^\infty(Q(2r)) \) is a standard cut-off function. The following relations are true:

\[
\int_{Q(2r)} \nabla H^* : \nabla \eta \, dxdt = \int_{Q(2r)} \frac{\nabla H^* : H^* \otimes \nabla \zeta}{(1 + |H^*|^2)^{1/2}} \, dxdt + \int_{Q(2r)} \zeta \left( \frac{|
abla H^*|^2}{(1 + |H^*|^2)^{1/2}} - \frac{|\nabla H^*|^2}{4(1 + |H^*|^2)^{1/2}} \right) \, dxdt,
\]

and

\[
(1 + |H^*|^2)\nabla H^*|^2 - \frac{1}{4} |\nabla H^*|^2 \geq |\nabla H^*|^2.
\]

Hence we obtain the estimate

\[
\int_{Q(2r)} \nabla H^* : \nabla \eta \, dxdt \geq \int_{Q(2r)} \zeta \frac{|
abla H^*|^2}{(1 + |H^*|^2)^{1/2}} \, dxdt - \int_{Q(2r)} |\nabla H^*||\nabla \zeta| \, dxdt.
\]
Testing the equation (6.4) by the function \( \eta \) we obtain

\[
\sup_{t \in (-r^2, 0)} \int_{B(2r)} \zeta (1 + |H^*|^2)^{\frac{1}{2}} \, dx + \int_{Q(2r)} \frac{\nabla H^*}{(1 + |H^*|^2)^{\frac{1}{2}}} \, dx \leq c \int_{Q(2r)} |\nabla H^*| |\nabla \zeta| \, dx dt +
\int_{Q(2r)} \text{div} G^* \cdot \eta \, dx dt.
\]

Note that

\[
\text{div} G^* = (H^* \cdot \nabla) v^* - (v^* \cdot \nabla) H^* \quad \text{a.e. in } Q.
\]

Integrating by parts we obtain

\[
\int_{Q(2r)} (v^* \cdot \nabla) H^* \cdot \eta \, dx dt = \int_{Q(2r)} \zeta v^* \cdot \nabla (1 + |H^*|^2)^{1/2} \, dx dt =
\int_{Q(2r)} v^* \cdot \nabla \zeta (1 + |H^*|^2)^{1/2} \, dx dt
\]

Hence

\[
\int_{Q(2r)} \text{div} G^* \cdot \eta \, dx dt \leq c \|\nabla v\|_{L^2(Q^+(2r))} \|H\|_{L^2(Q^+(2r))} +
\frac{c}{r} \|v\|_{L^2(Q^+(2r))} \|H\|_{L^2(Q^+(2r))} + \frac{c}{r} \|v\|_{L^1(Q^+(2r))}
\]

Taking into account the Poincare inequality for \( v \) and the Hölder inequality, we obtain the estimate

\[
\|H\|_{L^1, \infty(Q^+(r))} \leq c \left( r^{\frac{1}{2}} + \|\nabla v\|_{L^2(Q^+(2r))} \right) \|H\|_{L^2(Q^+(2r))} +
cr^{\frac{3}{2}} \|\nabla H\|_{L^2(Q^+(2r))} + c (r^3 + r^{\frac{3}{2}} \|\nabla v\|_{L^2(Q^+(2r))})
\]

which implies (6.6). Theorem 6.2 is proved.

The main result of this section is following:
Theorem 6.3 Assume conditions (6.1) — (6.3) hold. Then there exist absolute positive constants \( \varepsilon_1, \alpha \) and \( c \) such that for any \( \varepsilon \in (0, \varepsilon_1) \) and any \( K > 0 \)

\[
\sup_{r \in (0,1)} E(r) < \varepsilon \quad \text{and} \quad \sup_{r \in (0,1)} E_+(r) < K \quad (6.7)
\]

then for any \( 0 < r < \rho \leq 1 \)

\[
F_2(r) \leq c \left( \frac{r}{\rho} \right)^\alpha F_2(\rho) + c\varepsilon (K + 1) \quad (6.8)
\]

Proof of Theorem 6.3

1. Denote by \( v^* \) and \( H^* \) the extensions of functions \( v \) and \( H \) from \( Q^+ \) onto \( Q \) described in Theorem 6.1. Fix arbitrary \( r \in (0,1) \) and let \( \zeta \in C^\infty(\overline{Q}) \) be a cut off function such that \( \zeta \equiv 1 \) on \( Q(r) \) and \( \text{supp} \zeta \subset B \times (-1,0] \). Denote \( \Pi = \mathbb{R}^3 \times (-1,0) \) and denote by \( \hat{G} \) the function which coincides with \( G^* \) on \( Q(\frac{r}{2}) \) and additionally possesses the following properties: \( \hat{G} \in W^{1,0}_1(\Pi) \cap L^{1,6}_{1,5}(\Pi) \), \( \hat{G} \) is compactly supported in \( \Pi \), and

\[
\|\hat{G}\|_{L^{1,6}_{1,5}(\Pi)} \leq c \|G^*\|_{L^{1,6}_{1,5}(Q(\frac{r}{2}))} \quad (6.9)
\]

2. We decompose \( H^* \) as \( H^* = \tilde{H} + \hat{H} \), where \( \tilde{H} \) is a solution of the Cauchy problem for the heat equation

\[
\begin{align*}
\partial_t \tilde{H} - \Delta \tilde{H} &= \text{div} \hat{G} \quad \text{in} \quad \Pi, \\
\tilde{H}|_{t=-1} &= 0,
\end{align*}
\]

defined by the formula \( \tilde{H} = \Gamma \ast \text{div} \hat{G} = -\nabla \Gamma \ast \hat{G} \), where \( \Gamma \) is the fundamental solution of the heat operator. The function \( \tilde{H} \) satisfies the homogeneous heat equation

\[
\partial_t \tilde{H} - \Delta \tilde{H} = 0 \quad \text{in} \quad Q(\frac{r}{2}) \quad (6.11)
\]

3. Take arbitrary \( \theta \in (0, \frac{1}{2}) \). We estimate \( \|H\|_{L^2(Q^+((\theta r)))} \) in the following way

\[
\|H\|_{L^2(Q^+((\theta r)))} \leq \|H^*\|_{L^2(Q((\theta r)))} \leq \|\hat{H}\|_{L^2(Q((\theta r)))} + \|\tilde{H}\|_{L^2(Q((\theta r)))},
\]

(6.12)
For $\|\hat{H}\|_{L_2(Q(\theta r))}$ we have
\[
\|\hat{H}\|_{L_2(Q(\theta r))} \leq c \|\hat{H}\|_{L_2(Q(\frac{r}{2}))}. \tag{6.13}
\]
As $\tilde{H}$ satisfies (6.11) by local estimate of the maximum of $\tilde{H}$ via its $L_2$–norm we obtain
\[
\|\tilde{H}\|_{L_2(Q(\theta r))} \leq c \theta^{\frac{5}{2}} \|\tilde{H}\|_{L_2(Q(\frac{r}{2}))} \leq c \theta^{\frac{5}{2}} \left( \|H^\ast\|_{L_2(Q(r))} + \|\hat{H}\|_{L_2(Q(\frac{r}{2}))} \right) \tag{6.14}
\]

4. So, we need to estimate $\|\hat{H}\|_{L_2(Q(\frac{r}{2}))}$. As singular integrals are bounded on the anisotropic Lesbegue space $L_{s,l}$ (see, for example, [14]) for the convolution $\hat{h} = \Gamma \ast \hat{G}$ we obtain the estimate
\[
\|\hat{h}\|_{W^{2,1}_{\frac{s}{2},\frac{l}{2}}(Q(r))} \leq c \|\hat{G}\|_{L_{\frac{18}{11},\frac{6}{5}}(\Pi)}. \tag{6.15}
\]

On the other hand, from the 3D– parabolic imbedding theorem (see [1])
\[
W^{2,1}_{s,l}(Q) \hookrightarrow W^{1,0}_{p,q}(Q), \quad \text{as} \quad 1 - \left( \frac{3}{s} + \frac{2}{l} - \frac{3}{p} - \frac{2}{q} \right) \geq 0,
\]
for $p = q = 2$ and $s = \frac{18}{11}$, $l = \frac{6}{5}$ and for $\hat{H} = -\nabla \hat{h}$ we obtain
\[
\|\hat{H}\|_{L_2(Q(r))} \leq c \|\hat{G}\|_{L_{\frac{18}{11},\frac{6}{5}}(\Pi)}. \tag{6.19}
\]
(Note that the constant $c$ in this inequality does not depend on $r$). Taking into account (6.19) we arrive at
\[
\|\hat{H}\|_{L_2(Q(r))} \leq c \|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))}. \tag{6.15}
\]

5. From the definition of $G$ we obtain
\[
\|G\|_{L_{\frac{18}{11},\frac{6}{5}}(Q^+(\frac{r}{2}))} \leq c \left( \int_{-r^2/4}^{0} \|v \otimes H\|_{L_{\frac{6}{11}}(B^+(r/2))}^\frac{6}{5} dt \right)^\frac{5}{6}
\]
Applying the Hölder inequality and Sobolev imbedding $W^1_2(B^+(r)) \hookrightarrow L_6(B^+(r))$ for $v$ we obtain

$$
\|G\|_{L^{\frac{12}{11}}_2(Q^+)} \leq c \left( \int_{0}^{\frac{5}{2}} \|v\|_{L_6(B^+(r))}^{\frac{6}{5}} \|H\|_{L_2(B^+(r/2))}^{\frac{6}{5}} \, dt \right)^{\frac{5}{6}} 
\leq c \left( \int_{0}^{\frac{5}{2}} \|\nabla v\|_{L_2(B^+(r))}^{\frac{6}{5}} \|H\|_{L_2(B^+(r/2))}^{\frac{6}{5}} \, dt \right)^{\frac{5}{6}}.
$$

Interpolating $L^\frac{5}{4}$– norm between $L_1$ and $L_6$ and using the imbedding $W^1_2(B^+(r)) \hookrightarrow L_6(B^+(r))$ again we obtain

$$
\|H\|_{L_2(B^+(r/2))} \leq \|H\|_{L_1(B^+(r/2))}^{\frac{1}{3}} \|H\|_{L_6(B^+(r/2))}^{\frac{2}{3}} \leq \|H\|_{L_1(B^+(r/2))}^{\frac{1}{3}} \left( \|\nabla H\|_{L_2(B^+(r))}^{\frac{4}{3}} + r^{-\frac{4}{3}} \|H\|_{L_2(B^+(r))} \right)
$$

Hence we arrive at

$$
\|G\|_{L^{\frac{12}{11}}_2(Q^+)} \leq c \|H\|_{L_1,\infty(Q^+)}^{\frac{5}{3}} \times
\left( \int_{0}^{\frac{5}{2}} \|\nabla v\|_{L_2(B^+(r))}^{\frac{6}{5}} \left( \|\nabla H\|_{L_2(B^+(r))}^{\frac{4}{3}} + r^{-\frac{4}{3}} \|H\|_{L_2(B^+(r))} \right) \, dt \right)^{\frac{5}{6}}.
$$

Applying the Hölder inequality we obtain

$$
\|G\|_{L^{\frac{12}{11}}_2(Q^+)} \leq c \|H\|_{L_1,\infty(Q^+)}^{\frac{1}{3}} \|\nabla v\|_{L_2(Q^+)} \times
\left( \|\nabla H\|_{L_2(Q^+)}^{\frac{4}{3}} + r^{-\frac{4}{3}} \|H\|_{L_2(Q^+)} \right)
$$

6. Estimating $\|H\|_{L_1,\infty(Q^+)}$ using Theorem [6.2] we obtain

$$
\|G\|_{L^{\frac{12}{11}}_2(Q^+)} \leq c\left(1 + E(r)\right)^{\frac{1}{4}} \left( F_2(r) + E_2(r) + r \right)^{\frac{1}{4}} \times
\|\nabla v\|_{L_2(Q^+)} \left( \|\nabla H\|_{L_2(Q^+)}^{\frac{3}{4}} + r^{-\frac{3}{4}} \|H\|_{L_2(Q^+)} \right)
$$

(6.16)
7. Gathering estimates \(6.12 - 6.16\) together we arrive at
\[
\|H\|_{L^2(Q^+(\theta r))} \leq c \theta^{\frac{\alpha}{2}} \|H\|_{L^2(Q^+(r))} + c \, r^{\frac{\alpha}{2}} \|\nabla v\|_{L^2(Q^+(r))} \times \left(1 + E(r)\right)^{\frac{\alpha}{2}} \left(F_2(r) + E_1(r) + r\right) \left(\|\nabla H\|_{L^2(Q^+(r))} + r^{-\frac{\alpha}{2}} \|H\|_{L^2(Q^+(r))}\right)
\]
Dividing this inequality by \((\theta r)^{\frac{\alpha}{2}}\) we arrive at
\[
F_2'(\theta r) \leq c \, \theta F_2(r) + c(\theta) \, E(r) \left(1 + E(r)\right)^{\frac{\alpha}{2}} \left(F_2(r) + E_1(r) + r\right).
\]
Hence we obtain
\[
F_2(\theta r) \leq \left(c \theta + c(\theta) E(r) \left(1 + E(r)\right)^{\frac{\alpha}{2}}\right) F_2(r) + c(\theta) \, E(r) \left(1 + E(r)\right)^{\frac{\alpha}{2}} \left(E_1(r) + 1\right).
\]
Taking into account assumptions \(6.7\) for \(\varepsilon < 1\) we obtain
\[
F_2(\theta r) \leq \left(c \theta + c(\theta) \varepsilon\right) F_2(r) + c(\theta) \varepsilon(K + 1),
\]
valid for any \(r \in (0, 1)\) and any \(\theta \in (0, \frac{\pi}{2}]\).

8. Choosing \(\theta \in (0, \frac{\pi}{2}]\) so that
\[
c \theta = \frac{1}{4}
\]
and then choosing \(\varepsilon_1 \in (0, 1)\) so that
\[
\frac{1}{4} + c(\theta) \varepsilon_1 \leq \frac{1}{2}
\]
we obtain the estimate
\[
F_2(\theta r) \leq \frac{1}{2} F_2(r) + c \varepsilon(K + 1).
\]
Iterating this estimate we derive \(6.8\). Theorem 6.3 is proved.
7 Estimates of Energy Functionals

In the previous section we defined functionals $F_q(r)$, $E(r)$, and $E_*(r)$, see (6.5). Now we define few more functionals. Note that all these functionals are invariant with respect to the natural scaling of the MHD system. For $r \leq 1$, $q \in [1, \frac{10}{7}]$ and $s \in [1, \frac{9}{8}]$ we introduce the following quantities:

$$A(r) \equiv \left( \frac{1}{r} \sup_{t \in (-r^2,0)} \int_{B^+(r)} |v|^2 \, dy \right)^{1/2},$$

$$A_s(r) \equiv \left( \frac{1}{r} \sup_{t \in (-r^2,0)} \int_{B^+(r)} |H|^2 \, dy \right)^{1/2},$$

$$C_q(r) \equiv \left( \frac{1}{r^{1+q}} \int_{Q^+(r)} |v|^q \, dy \, dt \right)^{1/q},$$

$$D(r) \equiv \left( \frac{1}{r^2} \int_{Q^+(r)} |p - [p]|_{B^+(r)} |^{3/2} \, dy \, dt \right)^{2/3},$$

$$D_s(r) = R^{\frac{3}{2}-rac{1}{s}} \left( \int_{-r^2}^0 \left( \int_{B^+(r)} |\nabla p|^s \, dy \right)^{\frac{1}{s} - \frac{3}{4}} \, dt \right)^{2/3},$$

$$C(r) = C_3(r), \quad F(r) = F_3(r), \quad D_s(r) = D_{\frac{36}{35}}(r).$$

First we formulate the set of results following from the general theory of functions:

**Theorem 7.1** Assume $v$, $H \in W^{1,0}_{2}(Q^+)$ and $p \in W^{1,0}_{\frac{3}{2},\frac{9}{8}}(Q^+)$ are arbitrary functions. Assume $v|_{x_3 = 0} = 0$. Then the following inequalities hold:

$$C(r) \leq A^{\frac{1}{2}}(r)E^{\frac{1}{2}}(r), \quad F(r) \leq A^{\frac{1}{2}}(r)[E^{\frac{3}{2}}(r) + F^{\frac{3}{2}}(r)] \quad (7.1)$$

$$D(r) \leq cD_1(r), \quad D_1(r) \leq cD_s(r), \quad \forall s > 1. \quad (7.2)$$

**Proof of Theorem 7.1** The proof follows from interpolation inequalities and imbedding theorems. Proof of the similar inequalities for the Navier-Stokes system can be found in [5].

Now we formulate a theorem concerning boundary suitable weak solutions to the MHD system.
**Theorem 7.2** Assume \((v, H, p)\) is a boundary suitable weak solution to the MHD equations in \(Q^+\). Then for any \(r \in (0, 1)\) and \(\theta \in (0, \frac{1}{2})\) the following inequalities hold

\[
A(r/2) + A_*(r/2) + E(r/2) + E_*(r/2) \leq c \left( C_2(r) + F_2(r) + C_{\frac{1}{2}}^2(r)D_{\frac{1}{2}}^2(r) + C_{\frac{1}{2}}^2(r) \right) +
\]

\[
+ c \left( C_{\frac{1}{2}}^2(r)A_{\frac{1}{2}}^2(r)E_{\frac{1}{2}}^2(r) + F_{\frac{1}{2}}^1(r)A_{\frac{1}{2}}^1(r)E_{\frac{1}{2}}^1(r) \right)
\]

\[
D_*(\theta r) \leq c \theta^\frac{4}{3} \left( D_*(r) + E(r) \right) +
\]

\[
+ c(\theta) \left( A_{\frac{4}{3}}^2(r)E_{\frac{4}{3}}^2(r) + A_{\frac{4}{3}}^2(r)F_{\frac{1}{2}}^1(r)E_*(r) \right)
\]

(7.3)

**Proof of Theorem 7.2** estimate (7.3):

1. Estimate (7.3) follows from (1.5) in a standard way. We just explain the specific estimates of the terms

\[
I_1 := \int_{Q^+(r)} |H|^2 (v \cdot \nabla \zeta) \, dxdt \quad \text{and} \quad I_2 := \int_{Q^+(r)} (v \cdot H)(H \cdot \nabla \zeta) \, dxdt.
\]

2. \(I_1\) we transform in the following way

\[
I_1 = \int_{Q^+(r)} \left( |H|^2 - [|H|^2]_{B^+(r)} \right) (v \cdot \nabla \zeta) \, dxdt
\]

Applying the Hölder inequality we obtain

\[
|I_1| \leq \frac{c}{r} \int_{-r^2}^0 \| |H|^2 - [|H|^2]_{B^+(r)} \|_{L_2^2(B^+(r))} \| v \|_{L_3^3(B^+(r))} \, dt
\]

Applying the inequality \(\| f - [f]_{B^+(r)} \|_{L_2^2(B^+(r))} \leq c\| \nabla f \|_{L_1^1(B^+(r))}\), we arrive at

\[
|I_1| \leq \frac{c}{r} \int_{-r^2}^0 \| \nabla |H|^2 \|_{L_1^1(B^+(r))} \| v \|_{L_3^3(B^+(r))} \, dt \leq
\]

\[
\leq \frac{c}{r} \int_{-r^2}^0 \| H \|_{L_2^2(B^+(r))} \| \nabla H \|_{L_2^2(B^+(r))} \| v \|_{L_3^3(B^+(r))} \, dt \leq
\]

\[
\leq \frac{c}{r^{5/3}} \| H \|_{L_2^2(Q^+(r))} \| \nabla H \|_{L_2^2(Q^+(r))} \| v \|_{L_3^3(Q^+(r))} \leq cr A_*(r)E_*(r)C(r)
\]

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3. For $I_2$ we obtain relations

$$I_2 = \int_{Q^+(r)} ((v \cdot H) - [v \cdot H]_{B^+(r)}) (H \cdot \nabla \zeta) \, dxdt$$

Hence

$$|I_2| \leq c r \int_{-r^2}^0 \|(v \cdot H) - [v \cdot H]_{B^+(r)}\|_{L^2(B^+(r))} \|H\|_{L^2(B^+(r))} \, dt \leq$$

$$\leq c r \|H\|_{L^2(Q^+(r))} \int_{-r^2}^0 \|\nabla(v \cdot H)\|_{L^2(B^+(r))} \, dt \leq c r \|H\|_{L^2(Q^+(r))} \times$$

$$\times \int_{-r^2}^0 \left( \|\nabla v\|_{L^2(B^+(r))} \|H\|_{L^2(B^+(r))} + \|\nabla H\|_{L^2(B^+(r))} \|v\|_{L^2(B^+(r))} \right) \, dt \leq$$

$$\leq \frac{c}{r^{2/3}} \|H\|_{L^2(Q^+(r))} \left( \|\nabla v\|_{L^2(Q^+(r))} \|H\|_{L^2(Q^+(r))} + \|\nabla H\|_{L^2(Q^+(r))} \|v\|_{L^2(Q^+(r))} \right)$$

So, we obtain

$$|I_2| \leq cr A_* (r) \left( E(r) F(r) + E_*(r) C(r) \right)$$

**Proof of Theorem 7.2 estimate (7.4):**

1. To obtain (7.3) we apply the method developed in [8], [10], see also [11]. Denote $\Pi_r = R^3_+ \times (-r^2, 0)$. We fix $r \in (0, 1]$ and $\theta \in (0, \frac{1}{2})$ and define a function $g : \Pi_r^+ \to R^3$ by the formula

$$g = \begin{cases} \text{rot} H \times H - (v \cdot \nabla)v, & \text{in } Q^+(r), \\ 0, & \text{in } \Pi_r^+ \setminus Q^+(r) \end{cases}$$

Then we decompose $v$ and $p$ as

$$v = \hat{v} + \tilde{v}, \quad p = \hat{p} + \tilde{p},$$

where $(\hat{v}, \hat{p})$ is a solution of the Stokes initial boundary value problem in a half-space

$$\begin{cases} \partial_t \hat{v} - \Delta \hat{v} + \nabla \hat{p} = g, & \text{in } \Pi_r^+, \\ \text{div} \hat{v} = 0, & \hat{v} \mid_{t=0} = 0, \quad \hat{v} \mid_{x_3=0} = 0, \end{cases}$$

$$\begin{cases} \partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{p} = 0, & \text{in } \Pi_r^+, \\ \text{div} \tilde{v} = 0, & \tilde{v} \mid_{t=0} = 0, \quad \tilde{v} \mid_{x_3=0} = 0, \end{cases}$$

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and \((\tilde{v}, \tilde{p})\) is a solution of the homogeneous Stokes system in \(Q^+(r)\):

\[
\begin{aligned}
\begin{cases}
\partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{p} = 0, \\
\text{div} \tilde{v} = 0
\end{cases}
\end{aligned}
\text{ in } Q^+(r),
\]

\[\tilde{v}|_{x_3=0} = 0.\]

2. For \(\nabla \hat{p}\) and \(\nabla \tilde{p}\) the following estimates hold (see [10], see also [13]):

\[
\|\nabla \hat{p}\|_{L^{3\frac{6}{5}}(Q^+(r))} + \frac{1}{r} \|\nabla \hat{\tilde{v}}\|_{L^{3\frac{6}{5}}(Q^+(r))} \leq c \left( \|H \times \text{rot} H\|_{L^{3\frac{6}{5}}(Q^+(r))} + \|(v \cdot \nabla)v\|_{L^{3\frac{6}{5}}(Q^+(r))} \right);
\]

\[
\|\nabla \tilde{p}\|_{L^{3\frac{6}{5}}(Q^+(\theta r))} \leq c \theta^{\frac{12}{11}} \left( \frac{1}{r} \|\nabla \tilde{v}\|_{L^{3\frac{6}{5}}(Q^+(r))} + \|\nabla \tilde{p}\|_{L^{3\frac{6}{5}}(Q^+(r))} \right).
\]

3. From the Hölder inequality we obtain

\[
\|H \times \text{rot} H\|_{L^{3\frac{6}{5}}(Q^+(r))} \leq c r^{\frac{2}{5}} \|H\|_{L^{2,\infty}(Q^+(r))} \|\nabla H\|_{L^2(Q^+(r))} \|H\|_{L^2(Q^+(r))}.
\]

\[
\|(v \cdot \nabla)v\|_{L^{3\frac{6}{5}}(Q^+(r))} \leq c r^{\frac{1}{2}} \|(v \cdot \nabla)v\|_{L^{3\frac{6}{5}}(Q^+(r))} \leq c r^{\frac{1}{2}} \|v\|_{L^{2,\infty}(Q^+(r))} \|\nabla v\|_{L^2(Q^+(r))}.
\]

Representing \(\tilde{\tilde{v}} = v - \tilde{\tilde{v}}, \tilde{\tilde{p}} = p - \tilde{\tilde{p}}\) and gathering all above estimates for \(\hat{p}\) and \(\tilde{\tilde{p}}\) we obtain

\[
D_\lambda(\theta r) \leq c \theta^{\frac{1}{4}} \left( D_\lambda(r) + E(r) + A_\lambda^\frac{1}{2}(r) E_r^\frac{1}{2}(r) + A_\lambda^\frac{5}{4}(r) E_r^F(r) \right) + c(\theta) \left( A_\lambda^\frac{1}{2}(r) E_r^\frac{1}{2}(r) + A_\lambda^\frac{5}{4}(r) E_r^F(r) \right)
\]

Theorem [7,2] is proved.
8 CKN condition and Partial Regularity of Solutions

In this section we present the proofs of Theorems 2.2 and 2.3.

**Theorem 8.1**

Denote by \( E(r) \) the following functional

\[
E(r) = A(r) + A_*(r) + D_*(r),
\]

and let \( \varepsilon_1 > 0 \) be the absolute constant defined in Theorem 6.3. For any \( K > 0 \) there exists a constant \( c(K) > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_1] \) and any boundary suitable weak solution \((v, H, p)\) of the MHD system in \( Q^+ \) if

\[
\sup_{r \in (0, 1)} E(r) \leq \varepsilon, \quad \sup_{r \in (0, 1)} E_*(r) \leq K, \tag{8.1}
\]

and

\[
F_2(1) \leq M, \tag{8.2}
\]

then for any \( 0 < r < \rho \leq 1 \)

\[
E(r) \leq c \left( \frac{r}{\rho} \right)^\beta E(\rho) + c(K)(1 + \rho^\alpha M). \tag{8.3}
\]

where \( \beta > 0 \) is some absolute constant.

**Proof of Theorem 8.1**

1. Without loss of generality we can assume \( K \geq 1 \). Then from (6.8) we obtain

\[
F_2(r) \leq cr^\alpha M + cK.
\]

From this inequality and (7.1) we obtain

\[
C(r) \leq c \mathcal{E}(r)^{\frac{1}{2}} \mathcal{E}(\rho)^{\frac{1}{2}}, \quad F(r) \leq c \mathcal{E}(r) \left( K^{\frac{1}{2}} + r^\alpha M^{\frac{1}{2}} \right) \tag{8.4}
\]

2. Assume \( r \in (0, 1) \) and \( \theta \in (0, \frac{1}{2}) \). From (7.3) with the help of (7.2) and the Young inequality we obtain

\[
\mathcal{E}(\theta r) \leq c \left( F_2(2\theta r) + D_*(2\theta r) \right) + \]

\[
+ c(\theta) \left( C_2(r) + C(r) + C_2^*(r) + C_2^*(r)A_2^*(r)E_2^*(r) + F_2^*(r)A_2^*(r)E_2^*(r) \right)
\]

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Taking into account (8.4) and (8.1) we obtain

\[
\mathcal{E}(\theta r) \leq c \left( F_2(2\theta r) + D_*(2\theta r) \right) + \\
+ c(\theta) \left( \varepsilon_1 + \mathcal{E}^\frac{1}{4}(r)\varepsilon_1^\frac{1}{4} + \mathcal{E}^\frac{3}{4}(r)\varepsilon_1^\frac{3}{4} + \varepsilon_1^\frac{1}{4} \mathcal{E}^\frac{3}{4}(r)K^\frac{1}{4} + (K^\frac{1}{4} + r^\alpha M^\frac{1}{4})\mathcal{E}^\frac{3}{4}(r)\varepsilon_1^\frac{3}{4} \right)
\]  

(8.5)

Applying the Young inequality \(ab \leq \varepsilon a^p + C_\varepsilon b^p\) we obtain

\[
\mathcal{E}(\theta r) \leq \frac{1}{4}\varepsilon \mathcal{E}(r) + c\left( F_2(2\theta r) + D_*(2\theta r) \right) + c(\theta)c(K) + c(\theta)r^\alpha M.
\]

3. From (6.8) and (7.4) we obtain

\[
F_2(2\theta r) + D_*(2\theta r) \leq c\theta^\alpha \left( F_2(r) + D_*(r) \right) + c\varepsilon_1(1 + K) + \\
+ c(\theta) \left( A_\varepsilon^\frac{3}{4}(r)E_\varepsilon^\frac{3}{4}(r) + A_\varepsilon^\frac{3}{4}(r)F_\varepsilon^\frac{3}{4}(r)E_\varepsilon(r) \right)
\]

Taking into account (8.4) and the obvious inequality \(F_2(r) \leq A_*(r)\) we arrive at

\[
F_2(2\theta r) + D_*(2\theta r) \leq c\theta^\alpha \mathcal{E}(r) + c(K) + \\
+ c(\theta) \left( \mathcal{E}^\frac{3}{4}(r)\varepsilon_1^\frac{3}{4} + \mathcal{E}^\frac{11}{12}(r)(K^\frac{11}{12} + r^\frac{\alpha}{12}M^\frac{11}{12})K \right)
\]

Applying the Young inequality we get

\[
F_2(2\theta r) + D_*(2\theta r) \leq \left( \frac{1}{4} + c\theta^\alpha \right)\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M)
\]

4. Gathering the estimates we obtain

\[
\mathcal{E}(\theta r) \leq \left( \frac{1}{4} + c\theta^\alpha \right)\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M).
\]

Fixing \(\theta \in (0, \frac{1}{2})\) so that

\[
\frac{1}{4} + c\theta^\alpha = \frac{1}{2}
\]

Hence

\[
\mathcal{E}(\theta r) \leq \frac{1}{2}\mathcal{E}(r) + c(\theta)c(K)(1 + r^\alpha M).
\]

Iterating this inequality we obtain (8.3). Theorem 8.1 is proved.
Theorem 8.2 Assume all conditions of Theorem 8.1 hold and fix \( \rho_0 \in (0,1) \) so that
\[
\rho_0^\alpha M \leq 1. \tag{8.6}
\]
Then for any \( 0 < r < \rho \leq \rho_0 \) the following estimates hold:
\[
A(r) + A_*(r) \leq c \left( \frac{r}{\rho} \right)^\gamma \left( A(\rho) + A_*(\rho) \right) + \varepsilon \frac{r}{\rho} D(\rho) + G(K,\varepsilon) \tag{8.7}
\]
\[
D(r) \leq c \left( \frac{r}{\rho} \right)^\gamma D(\rho) + c(K) \left( A_{12}^{11}(\rho) + A_{12}^{11}(\rho) \right) + G(K,\varepsilon) \tag{8.8}
\]
where \( \gamma > 0 \) is some absolute constant and \( G \) is a continuous function possessing the following property:
\[
\text{for any fixed } K > 0 \quad G(K,\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.9}
\]

Proof of Theorem 8.2

1. From (7.1) taking into account (8.6) we obtain
\[
C(r) \leq A_{12}^{12}(r) \varepsilon \frac{1}{\rho}, \quad F(r) \leq A_{12}^{12}(r)(K \varepsilon^{1/2} + 1) \tag{8.10}
\]

2. Take arbitrary \( r \in (0,\rho_0) \) and \( \theta \in (0,\frac{1}{2}) \). Denote by \( \mathcal{E}(r) \) the following functional
\[
\mathcal{E}_*(r) = A(r) + A_*(r),
\]
Then from (7.3) similar to (8.5) using (8.10) we derive
\[
\mathcal{E}_*(\theta r) \leq F_2(2\theta r) + C_{12}^{12}(2\theta r)D_{12}^{12}(2\theta r) + c(\theta) \left( \mathcal{E}_*(r) \varepsilon + \mathcal{E}_*(r) \varepsilon^{3/2} + \mathcal{E}_*(r)K \varepsilon^{1/2} + \mathcal{E}_*(r)(K^{1/2} + 1) \varepsilon^{1/2} \right)
\]
Applying the Young inequality and using (7.2) we obtain
\[
\mathcal{E}_*(\theta r) \leq \frac{1}{8} \mathcal{E}_*(r) + c(\theta)G(K,\varepsilon) + F_2(2\theta r) + C_{12}^{12}(2\theta r)D_{12}^{12}(2\theta r) \tag{8.11}
\]

3. From (6.8) we conclude
\[
F(2\theta r) \leq c\theta^\alpha \mathcal{E}_*(r) + G(K,\varepsilon). \tag{8.12}
\]
4. From (7.4) for \( r \leq \rho_0 \) with the help of (8.10) and the Young inequality we obtain

\[ D_∗(2θr) \leq c \theta^β D_∗(r) + c(θ)c(K)E_∗^{\frac{11}{12}}(r) + c(θ)G(K, ε) \]  (8.13)

Hence from (8.10) we obtain

\[ C^\frac{1}{2}(2θr)D^\frac{1}{2}(2θr) \leq c(θ)\epsilon^\frac{1}{4}D_∗^\frac{1}{2}(r) + c(θ)c(K)\epsilon^\frac{1}{4}E_∗^{\frac{17}{12}}(r) + c(θ)G(K, ε) \]

Applying the Young inequality we arrive at

\[ C^\frac{1}{2}(2θr)D^\frac{1}{2}(2θr) \leq \frac{1}{8}ε_∗(r) + \frac{1}{2}ε_∗D_∗(r) + c(θ)G(K, ε) \]  (8.14)

5. Gathering estimates (8.11) — (8.14) we obtain the inequality

\[ E_∗(θr) \leq \left( \frac{1}{4} + cθ^α \right) E_∗(r) + \frac{1}{2}ε_∗^\frac{1}{4}D_∗(r) + c(θ)G(K, ε) \]

Choosing \( θ \in (0, \frac{1}{2}) \) so that

\[ \frac{1}{4} + cθ^α = \frac{1}{2} \]

we obtain

\[ E_∗(θr) \leq \frac{1}{2} E_∗(r) + \frac{1}{2}ε_∗^\frac{1}{4}D_∗(r) + c(θ)G(K, ε) \]

Iterating this inequality we obtain (8.7).

6. Choosing in (8.13) \( θ \in (0, \frac{1}{2}) \) so that

\[ cθ^γ = \frac{1}{2} \]

and iteration the inequality we obtain we derive (8.8). Theorem 8.2 is proved.

**Theorem 8.3** For any \( K > 0 \) there exists a constant \( ε_0(K) > 0 \) such that if the condition (8.1) holds with \( ε \leq ε_0 \), then there exists \( ρ_∗ \in (0, 1) \) such that

\[ \left( C(ρ_∗) + F(ρ_∗) + D(ρ_∗) \right) < \frac{1}{ε_∗^\frac{1}{2}}, \]

where the constant \( ε_∗ > 0 \) is defined in Theorem 2.1.
Proof of Theorem 8.3

1. From (8.3) we obtain
\[ \limsup_{r \to 0} D_s(r) \leq c(K). \]

2. From (8.7) we derive
\[ \limsup_{r \to 0} \left( A(r) + A_s(r) \right) \leq \varepsilon \frac{1}{4} \limsup_{\rho \to 0} D(\rho) + G(K, \varepsilon) \leq \varepsilon \frac{1}{4} c(K) + G(K, \varepsilon). \]

3. From (8.8) we obtain
\[ \limsup_{r \to 0} D_s(r) \leq c(K) \limsup_{\rho \to 0} \left( A^{\frac{13}{12}}(\rho) + A_s^{\frac{11}{12}}(\rho) \right) + G(K, \varepsilon) \leq c(K) \left( \varepsilon \frac{1}{4} c(K) + G(K, \varepsilon) \right)^{\frac{11}{12}} + G(K, \varepsilon). \]

4. From (7.1) we conclude
\[ \limsup_{r \to 0} \left( C(r) + F(r) \right) \leq (\varepsilon \frac{1}{2} + K \frac{1}{2}) \limsup_{r \to 0} \left( A(r) + A_s(r) \right) \leq (\varepsilon \frac{1}{2} + K \frac{1}{2}) \left( \varepsilon \frac{1}{4} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}}. \]

5. Taking into account (8.9) for any \( K > 0 \) we can find \( \varepsilon_0(K) > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \)
\[ c(K) \left( \varepsilon \frac{1}{4} c(K) + G(K, \varepsilon) \right)^{\frac{11}{12}} + G(K, \varepsilon) < \frac{\varepsilon_s^{\frac{1}{4}}}{2} \]
and
\[ (\varepsilon \frac{1}{2} + K \frac{1}{2}) \left( \varepsilon \frac{1}{4} c(K) + G(K, \varepsilon) \right)^{\frac{1}{2}} < \frac{\varepsilon_s^{\frac{1}{4}}}{2}. \]
Hence for \( \varepsilon \in (0, \varepsilon_0) \)
\[ \limsup_{r \to 0} \left( C(r) + F(r) + D_s(r) \right) < \varepsilon \frac{1}{4}. \]

Theorem 8.3 is proved.
Proof of Theorem 2.2: Assume condition (2.6) holds and let $\varepsilon_0(K) > 0$ be the constant defined in Theorem 8.3. From (2.6), (2.7) we obtain there exists $R > 0$ such that

$$\sup_{r \in (0,R)} E(r) < \varepsilon_0 \quad \text{and} \quad \sup_{r \in (0,R)} E_*(r) < K.$$ 

Denote $(v^R, H^R, p^R)$ the functions

$$v^R(x,t) = Rv(x_0 + Rx, t_0 + R^2t), \quad H^R(x,t) = RH(x_0 + Rx, t_0 + R^2t),$$

$$p^R(x,t) = R^2p(x_0 + Rx, t_0 + R^2t).$$

Then functions $(v^R, H^R, p^R)$ satisfy all conditions of Theorem 8.3. The result follows from Theorem 8.3 and Theorem 2.1.

Proof of Theorem 2.3: The result is a direct consequence of Theorem 2.2 and measure theory, see [2], [7], [5], [8].

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