Two-logarithm matrix model with an external field

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Abstract

We investigate the two-logarithm matrix model with the potential $X\Lambda + \alpha \log(1 + X) + \beta \log(1 - X)$ related to an exactly solvable Kazakov–Migdal model. In the proper normalization, using Virasoro constraints, we prove the equivalence of this model and the Kontsevich–Penner matrix model and construct the $1/N$-expansion solution of this model.

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1 Introduction

Matrix models with the coupling to external matrices play an important role in the contemporary mathematical and theoretical physics. Historically, the first model of such type was the Brezin–Gross (BG) model of the unitary matrix $U$ linearly coupled to an external matrix field $\Lambda$. But the real breakthrough in this field was caused by Kontsevich’s papers where the generating functional for the 2D topological gravity was proved to be the integral over the Hermitian matrices $X$ with the potential $X^3$, which are linearly coupled to an external matrix $\Lambda$. Simultaneously, the Witten hypothesis that this generating functional is a $\tau$-function of the KdV hierarchy was proved. The generalized Kontsevich model (GKM)—the model with an arbitrary polynomial potential $V(X)$ and coupling with an external field—turned out to be a $\tau$-function of the Kadomtsev–Petviashvili hierarchy.

Then, the interest to matrix models with logarithmic potentials appeared. The first such model with the external field coupling was proposed in (the authors named it the Kontsevich–Penner (KP) model) and was pushed forward in where its equivalence to the Hermitian one-matrix model with an arbitrary nonsingular potential was proved. Underlying geometrical structure is the discretized moduli space (d.m.s.) construction. Later on, the exact relation was proved that connects this model in the d.m.s. times with two copies of the Kontsevich integral taken at different time sets.

Both the Kontsevich and the KP models, as well as the BG model, admit an explicit solutions in the $1/N$-expansion. Such solutions arise from the loop equation (or the Virasoro algebra constraints), which are at most quadratic in fields. One can formulate the problem to find all external field matrix models that manifest this property. Another model of this kind was the so-called NBI matrix model of IIB superstrings with the potential $X\Lambda + X^{-1} + (2\eta + 1) \log X$ appeared in the context of the (M)atrix string theory. This model includes the BG model as a particular case ($\eta = 0$) and away of this point, it can be reduced, after the time changing, to the Kontsevich model. (In particular, this enables one to produce the answer for the NBI model in the moment technique as soon as the answer for the Kontsevich model is known.) Note that the proof of equivalence of these two models relies upon the coincidence of the Virasoro algebras.

The last model, which completes the list of matrix models with the loop equations quadratic in fields and which can be therefore solved in the $1/N$-expansion framework is the two-logarithm (2-log) model with the potential $X\Lambda + \alpha \log(1 - X) + \beta \log(1 + X)$. This model turns out to be closely related to the exactly solvable Kazakov–Migdal models and it was thoroughly investigated in the case of the unit matrix $\Lambda$, i.e., where it is reduced to the one-matrix model. Even in this case, this model manifests a rich phase structure.

In the present paper, we do not investigate all possible phases of the 2-log model and rather confine our consideration to the Kontsevich phase only, in which the expansion over traces of negative powers of the matrix $\Lambda$ makes sense. First, we solve this model in the leading order of the $1/N$-expansion; then, we find the constraint equations (the Virasoro algebra) and prove that in the proper normalization, these equations are exactly equivalent to the constraint equations of the KP model. Possible applications of the 2-log model are discussed.
2 Matrix model with two logarithms

We start with the following matrix integral, which appear, for instance, in the logarithmic Kazakov–Migdal model [15, 16]:

\[ Z = \int dX e^{-N \text{tr} [X \Lambda + \alpha \log(1-X) + \beta \log(1+X)]} \]  

(2.1)

This integral is of the most general form, since, rescaling and shifting the fields \( X \) and \( \Lambda \), we may change the logarithmic branch points; however, we cannot change the constants \( \alpha \) and \( \beta \), which are actual charges in the model (2.1).

The matrix integral (2.1) belongs to a class of generalized Kontsevich models (GKM) [17]. Such models with negative powers of the matrix \( X \) have been previously discussed in the context of \( c = 1 \) bosonic string theory [18].

For the models of this type, the large \( N \) solution is known explicitly only in some special cases. The models with cubic potential for \( X \) [19] and the combination of the logarithmic and quadratic potentials [6, 20] were solved by a method based on the Schwinger–Dyson equations, developed first for the unitary matrix models with external field [1, 21]. The same technique, being applied to the integral (2.1), also allows one to find its large \( N \) asymptotic expansions in the closed form for arbitrary \( \alpha \) and \( \beta \).

The Schwinger–Dyson equations for (2.1) follow from the identity

\[ \left( \frac{1}{N^3} \frac{\partial}{\partial \Lambda_{jk}} \frac{\partial}{\partial \Lambda_{li}} - \frac{1}{N} \delta_{jk} \delta_{li} \right) \int dX \frac{\partial}{\partial X_{ij}} e^{-N \text{tr} [X \Lambda + \alpha \log(1-X) + \beta \log(1+X)]} = 0. \]  

(2.2)

Written in terms of the eigenvalues, these \( N \) equations read

\[ \left[ -\frac{1}{N^2} \lambda_i \frac{\partial^2}{\partial \lambda_i^2} - \frac{1}{N^2} \sum_{j \neq i} \lambda_j \left( \frac{\partial}{\partial \lambda_j} - \frac{\partial}{\partial \lambda_i} \right) + \frac{\alpha + \beta - 2}{N} \frac{\partial}{\partial \lambda_i} + (\beta - \alpha) + \lambda_i \right] Z(\lambda) = 0. \]  

(2.3)

It is convenient to set

\[ W(\lambda_i) = \frac{1}{N} \frac{\partial}{\partial \lambda_i} \log Z. \]  

(2.4)

We also introduce the eigenvalue density of the matrix \( \Lambda \):

\[ \rho(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i). \]  

(2.5)

The density obeys the normalization condition

\[ \int dx \rho(x) = 1 \]  

(2.6)

and in the large \( N \) limit becomes a smooth function.

A simple power counting shows that the derivative of \( W(\lambda_i) \) in the first term on the left hand side of equation (2.3) is suppressed by the factor \( 1/N \) and can be omitted at \( N = \infty \). The remaining terms are rewritten as follows:

\[ -xW^2(x) - \int dy \rho(y) y \frac{W(y) - W(x)}{y - x} + (\alpha + \beta - 2)W(x) + (\beta - \alpha) + x = 0, \]  

(2.7)
where \( \lambda_i \) is replaced by \( x \). Equation (2.7) can be simplified by the substitution
\[
\tilde{W}(x) = xW(x) - \frac{\alpha + \beta - 1}{2}.
\] (2.8)

After some transformations, using the normalization condition (2.6), we obtain
\[
\tilde{W}^2(x) + x \int dy \rho(y) \frac{\tilde{W}(y) - \tilde{W}(x)}{y - x} = x^2 + (\beta - \alpha)x + \frac{(\alpha + \beta - 1)^2}{4}.
\] (2.9)

The nonlinear integral equation (2.9) can be solved with the help of the anzatz
\[
\tilde{W}(x) = f(x) + \frac{x}{2} \int dy \frac{\rho(y)}{f(y)} \frac{f(y) - f(x)}{y - x},
\] (2.10)

where \( f(x) \) is an unknown function to be determined by substituting (2.10) into Eq. (2.9). The asymptotic behaviors of \( \tilde{W}(x) \) and \( f(x) \) as \( x \to \infty \) follow from Eq. (2.9): \( \tilde{W}(x) \sim x + (\beta - \alpha - 1)/2 \), and the analytic solution with minimal set of singularities is merely
\[
f(x) = \sqrt{ax^2 + bx + c}.
\] (2.11)

Let us introduce the moments of the external field
\[
I_0 = \int \frac{\rho(x)}{f(x)} dx, \quad J_0 = \int \frac{\rho(x)}{f(x)} x dx.
\] (2.12)

The parameters \( a, b, \) and \( c \) are unambiguously determined from Eq. (2.9). We find that
\[
c = (\beta + \alpha - 1)^2/4,
\]

and \( a \) and \( b \) are implicitly defined by the following two constraints:
\[
1 + \frac{1}{2}I_0 = \frac{1}{\sqrt{a}},
\]
\[
\sqrt{a}J_0 = \beta - \alpha - \frac{b}{a},
\] (2.13)

or, in terms of the eigenvalues,
\[
1 + \frac{1}{2N} \sum_j \frac{1}{\sqrt{a\lambda_j^2 + b\lambda_j + c}} = \frac{1}{\sqrt{a}},
\]
\[
\sqrt{a} \frac{1}{N} \sum_j \frac{\lambda_j}{\sqrt{a\lambda_j^2 + b\lambda_j + c}} = \beta - \alpha - \frac{b}{a}.
\] (2.14)

So, we have
\[
W(x) = \frac{\sqrt{ax^2 + bx + c}}{x} + \frac{1}{2} \int dy \rho(y) \frac{f(y) - f(x)}{f(y)(y - x)} + \frac{\alpha + \beta - 1}{2x}.
\] (2.15)
Then, integrating (2.15) w.r.t. $x$ and checking that the stationary conditions w.r.t. the variables $a$ and $b$ take place, we find the answer for the integral in the large $N$ limit,

$$
\log Z = N^2 (\beta - \alpha)^2 \left[ \frac{1}{8} \log(b^2 - 4ac) - \frac{1}{4} \log a \right] + N^2 (\beta - \alpha) \left[ \frac{1}{4} \log a - \frac{1}{4} \log(b^2 - 4ac) 
+ \sqrt{c} \arctanh \left( \frac{2\sqrt{ca}}{b} - \frac{b}{2a} \right) \right] + N^2 \left[ \frac{b^2}{8a^2} - \frac{c}{2a} + \frac{2c}{\sqrt{a}} + \frac{c}{2} \log(b^2 - 4ac) \right]
+ N \sum_i \left[ \frac{\alpha + \beta - 1}{2} \log \lambda_i + \frac{f(\lambda_i)}{\sqrt{a}} + \frac{1}{2} (\beta - \alpha) \log \left( \sqrt{a} \lambda_i + \frac{b}{2\sqrt{a}} + f(\lambda_i) \right) \right]
- \sqrt{c} \arctanh \left( \frac{\sqrt{c} + \lambda_i b/(2\sqrt{c})}{f(\lambda_i)} \right) - \frac{1}{4} \sum_{ij} \log(\lambda_i - \lambda_j)
+ \arctanh \left( \frac{a\lambda_i \lambda_j + (\lambda_i + \lambda_j) b/2 + c}{f(\lambda_i) f(\lambda_j)} \right) \right].
$$

(2.16)

One can verify directly that

$$
\frac{\partial}{\partial a} \log Z = \frac{\partial}{\partial b} \log Z = 0
$$

(2.17)

and $\frac{1}{N} \frac{\partial}{\partial \lambda_i} \log Z = W(\lambda_i)$, as far as Eq. (2.14) hold.

### 3 A large $N$ limit comparison

Let us establish a relation between the constraint equations of the 2-log model and KP model [6]. It is convenient to introduce new charges (parameters) instead of $\alpha$ and $\beta$

$$
\gamma \equiv (\beta - \alpha)/2, \quad \varphi \equiv - (\alpha + \beta - 1)/2 = \sqrt{c},
$$

(3.1)

and new variables

$$
\tilde{b} \equiv b/a, \quad \tilde{c} \equiv c/a.
$$

(3.2)

Shifting all eigenvalues $\lambda_i$ by the same constant $\xi$, we can rewrite the 2-log constraint equations as follows:

$$
\frac{\varphi}{\sqrt{c}} \pm \frac{1}{2N} \sum_i \frac{1}{\sqrt{\lambda_i^2 + (2\xi + \tilde{b}) \lambda_i + (\xi^2 + \xi \tilde{b} + \tilde{c})}} = 1,
$$

$$
\pm \frac{1}{N} \sum_i \frac{\lambda_i + \xi}{\sqrt{\lambda_i^2 + (2\xi + \tilde{b}) \lambda_i + (\xi^2 + \xi \tilde{b} + \tilde{c})}} = 2\gamma - \tilde{b},
$$

(3.3)

where “±” depends on the branch of the square root. Then, we make the following time change:

$$
\text{tr} \frac{1}{\lambda^n} = \text{tr} \frac{1}{\eta^n} \pm \left( -2\varphi \frac{N}{(-\xi)^n} + 2N \delta_{n,1} - N \delta_{n,2} \right),
$$

(3.4)
where the role of “±” is the same. Making the presented time change and connecting the variables of the two models

\[
2\xi + \tilde{b} = 4b, \\
\xi^2 + \xi\tilde{b} + \tilde{c} = 4c,
\]

where \(b\) and \(c\) are already the KP variables, we have

\[
2b \pm \frac{1}{N} \sum_{i} \frac{1}{\sqrt{\eta_i^2 + 4b\eta_i + 4c}} = 0,
\]

\[
\pm \frac{1}{2N} \sum_{i} \frac{\eta_i}{\sqrt{\eta_i^2 + 4b\eta_i + 4c}} + c - 3b^2 = \gamma - \varphi,
\]

i.e., exactly the constraint equations of the KP model with \(\gamma - \varphi \equiv \tilde{\alpha}\). Here \(\xi\) is an arbitrary parameter. Using the original parameters \(\alpha, \beta,\) and \(\alpha_{\text{KP}}\) (\(\alpha_{\text{KP}}\) is the parameter \(\alpha\) of the KP model, and \(\alpha_{\text{KP}} + 1/2 = \tilde{\alpha}\) in the notation of [6]), we see that \(\alpha_{\text{KP}} = \beta - 1\).

Naively, the parameter \(\beta\) is more preferred than \(\alpha\) for some reason. Indeed, they play equal roles. The obvious symmetry of the 2-log matrix integral is encoded in the transformations \(\lambda_i \rightarrow -\lambda_i\) \((i = 1, N)\) and \(\alpha \leftrightarrow \beta\). Under such a symmetry, \(\gamma \rightarrow -\gamma, \varphi \rightarrow \varphi,\) and \(\alpha_{\text{KP}} = \gamma - \varphi \rightarrow \alpha_{\text{KP}} = -\gamma - \varphi = \alpha - 1\).

Let us recall the answer in the large \(N\) limit for the KP model [6]. Substituting \(\tilde{\alpha} = \gamma - \varphi,\) we have

\[
\log Z_{\text{KP}} = \frac{N^2}{2} \left(\gamma - \varphi - \frac{1}{2}\right) \log \left(b^2 - c\right) - \frac{5}{2} b^2 c - (\gamma - \varphi) c + \frac{c^2}{4} + 3 (\gamma - \varphi) b^2 + \frac{9}{4} b^4 \\
+ N \sum_i \left\{ \left(\frac{\eta_i}{2} - b\right) \sqrt{\frac{\eta_i^2}{4} + b\eta_i + c} + (\gamma - \varphi) \log \left(\eta_i + 2b + \sqrt{\frac{\eta_i^2}{4} + b\eta_i + c}\right) + \frac{\eta_i}{4} \right\} \\
- \frac{1}{4} \sum_{ij} \log \left(\frac{\eta_i \eta_j}{4} + \frac{b}{2} (\eta_i + \eta_j) + c + \sqrt{\frac{\eta_i^2}{4} + b\eta_i + c} \sqrt{\frac{\eta_j^2}{4} + b\eta_j + c}\right). \quad (3.7)
\]

Here we compare the large \(N\) limit answer for the 2-log model with (3.7). Further all equalities hold up to pure complex constant and irrelevant factors which can polynomially depend only on the parameters \(\alpha\) and \(\beta\) (the polynomial of no more than second degree) of the 2-log model. Obviously such additional terms cannot influence the critical behavior of the model. Making the eigenvalue shift by \(\xi\) and denoting \(d = b^2 - c,\) we obtain

\[
\log Z = \frac{N^2 \gamma^2}{2} \log d + N^2 \gamma \left[ - \left(\varphi + \frac{1}{2}\right) \log d + 2\varphi \log \left(\tilde{b} + 2\sqrt{\tilde{c}}\right) - \tilde{b} \right] + N^2 \left[ 2d + 2\varphi \sqrt{\tilde{c}} \right] \\
+ \frac{1}{2} \left(\varphi + \frac{1}{2}\right)^2 \log d - \varphi \left(\varphi + \frac{1}{2}\right) \log \tilde{c} \right] + N \sum_i \left\{ \gamma \log \left(1 + \frac{2b}{\lambda_i} + \sqrt{1 + \frac{4b}{\lambda_i} + \frac{4c}{\lambda_i^2}}\right) \\
- \varphi \log \left(\sqrt{1 + \frac{4b}{\lambda_i} + \frac{4c}{\lambda_i^2}} + \frac{b}{2\sqrt{c}} + \frac{\tilde{b} + \xi \tilde{b}/(2\sqrt{c})}{\lambda_i} \right) + \sqrt{\lambda_i^2 + 4b\lambda_i + 4c - \lambda_i} \right\} \\
+ \left(\gamma - \varphi - \frac{1}{2}\right) \log \lambda_i + \lambda_i \right\} - \frac{1}{4} \sum_{ij} f\left(\frac{1}{\lambda_i}, \frac{1}{\lambda_j}\right). \quad (3.8)
\]
where
\[ f(x, y) = \log \left( \frac{1}{2} + b(x + y) + 2cxy + \frac{1}{2} \sqrt{1 + 4bx + 4cx^2} \sqrt{1 + 4by + 4cy^2} \right). \] (3.9)

After some tedious algebra (similar to the one in [14]), we obtain
\[ \log Z = \log Z_{KP} + N \sum_i \left\{ \left( \gamma - \varphi - \frac{1}{2} \right) \log \left( \frac{\lambda_i}{\eta_i} \right) + \lambda_i - \frac{\eta_i^2}{2} \right\} + 2\varphi^2 \log \varphi. \] (3.10)

The difference between \( \log Z \) and \( \log Z_{KP} \) depends only on some normalization factors in the large \( N \) limit. As is worth noting, these factors differ from the original normalization factors of the two models, which can be obtained by the early developed scheme [14]. We show that the appeared normalization factors are indeed natural.

Let us investigate the Kontsevich regime of the two models (\( \Lambda \to \infty \) and \( \eta \to \infty \)). Then, for KP model we have (up to a constant)
\[ Z_{KP} = \mathcal{D} X e^{N \text{tr}[\eta X - \frac{X^2}{2} + \alpha \log X]} = e^{\frac{N}{2} \text{tr} \eta^2} (\det \eta)^{\beta} e^{N \text{tr} \alpha}. \]

There are two stationary points, \( X_0 = \pm 1 + \frac{Y}{\Lambda} \), in the Kontsevich regime for the 2-log model (\( Y \) is the new variable). Choosing \( X_0 = -1 + \frac{Y}{\Lambda} \) for definiteness (another stationary point gives the same answer after symmetry \( \Lambda \to -\Lambda \)), we obtain
\[ Z = (\det \Lambda)^{N(\beta - 1)} e^{N \text{tr} \alpha} \int DY e^{-N \text{tr}[Y + \alpha \log(2 - \frac{Y}{\Lambda}) + \beta \log Y]} \simeq (\det \Lambda)^{N(\beta - 1)} e^{N \text{tr} \alpha}. \]

This is nothing but our normalizing factors.

4 Constraint equations

Let us make the eigenvalue shift in the master equation of the 2-log model (\( \partial_i \equiv \frac{\partial}{\partial \lambda_i} \))
\[ \left[ -\frac{1}{N^2} (\lambda_i + \xi) \partial_i^2 - \frac{1}{N^2} \sum_{j \neq i} \frac{\lambda_j + \xi}{\lambda_j - \lambda_i} (\partial_j - \partial_i) + \frac{\alpha + \beta - 2}{N} \partial_i + \beta - \alpha + \lambda_i + \xi \right] Z = 0. \] (4.1)

Using our normalizing factor
\[ \prod_i \lambda_i^{N(\beta - 1)} e^{N \lambda_i} \] (4.2)
and pushing it through derivatives, we replace
\[ \partial_i \longrightarrow \partial_i + \frac{N(\beta - 1)}{\lambda_i} + N. \] (4.3)

Then, we obtain master equation for the normalized partition function
\[ \left[ -\frac{1}{N^2} (\lambda_i + \xi) \partial_i^2 - \frac{1}{N^2} \sum_{j \neq i} \frac{\lambda_j + \xi}{\lambda_j - \lambda_i} (\partial_j - \partial_i) - \frac{2\lambda_i}{N} \partial_i + \frac{\alpha - \beta - 2\xi}{N} \partial_i \right. \]
\[ - \frac{2\xi(\beta - 1)}{N \lambda_i} \partial_i - \frac{\xi(\beta - 1)^2}{\lambda_i^2} + \frac{\beta - 1}{\lambda_i} \left( \alpha - 2\xi + \frac{\xi}{N} \sum_j \frac{1}{\lambda_j} \right) \] (4.4)
Let us introduce the times of the 2-log model
\[ t_n = \frac{1}{n} \sum_i \frac{1}{\lambda_i^n}. \] (4.5)

Then, the constraint equations for \( Z(\{t_n\}) \) are obtained after some tedious algebra which we omit here. Collecting all coefficients to the term \( 1/(\lambda_k N^2) \), we obtain
\[ L_k Z(\{t_n\}) = 0, \quad k \geq -1, \] (4.6)
where
\[ L_k = V_{k+1} + \xi V_k + \xi N(\alpha + \beta - 1)\left(1 - \delta_{k,0} - \delta_{k,-1}\right) \frac{\partial}{\partial t_k} - N(\beta - 1)\delta_{k,0} + \xi \delta_{k,-1} N(\beta - 1)(t_1 - 2N), \] (4.7)
and
\[ V_k = -\sum_{m=1}^{\infty} m t_m \frac{\partial}{\partial t_{m+k}} - \sum_{m=1}^{k-1} \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_{k-m}} - N(\alpha - \beta + 1)(1 - \delta_{k,0} - \delta_{k,-1}) \frac{\partial}{\partial t_k} \\
+ 2N(1 - \delta_{k,-1}) \frac{\partial}{\partial t_{k+1}} + t_1 \delta_{k,-1} \frac{\partial}{\partial t_{k+1}} + N^2 \alpha(\beta - 1)\delta_{k,0}. \] (4.8)

Here, the derivatives over \( t_0 \) and \( t_{-1} \) are fictitious and are used only for compactifying the presentation.

For \( k, l \geq -1 \), \( L_k \) satisfy the algebra
\[ [L_k, L_l] = (l - k)(L_{k+l+1} + \xi L_{k+l}). \] (4.9)

Zero shift (\( \xi = 0 \)) results in the Virasoro algebra where the \( L_{-1} \) generator is absent,
\[ [V_k, V_l] = (l - k)V_{k+l}, \quad k, l \geq 0. \] (4.10)

We can also obtain the Virasoro algebra from the general algebra with nonzero shift by the replacement
\[ L_k = \sum_{s=0}^{\infty} \frac{(-1)^s}{\xi^{s+1}} L_{k+s}, \quad k \geq -1, \] (4.11)
which is singular at \( \xi = 0 \). Performing the replacement and using the relations \( \alpha_{KP} = \beta - 1 \) and \( \varphi = -(\alpha + \beta - 1)/2 \), we obtain
\[ L_k = -\sum_{m=1}^{\infty} m t_m \frac{\partial}{\partial t_{m+k}} - \sum_{m=1}^{k-1} \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_{k-m}} + 2N\alpha_{KP} \frac{\partial}{\partial t_k} + 2N^2 \frac{\partial}{\partial t_{k+1}} \\
- 2\varphi N \sum_{s=1}^{\infty} \frac{1}{(-\xi)^s} \frac{\partial}{\partial t_{k+s}} - 2N\alpha_{KP}(\delta_{k,0} + \delta_{k,-1}) \frac{\partial}{\partial t_k} - N^2 \alpha_{KP}^2 \delta_{k,0} \\
+ \delta_{k,-1} \left(t_1 - 2N + 2\varphi N / \xi\right) \left(N\alpha_{KP} + \frac{\partial}{\partial t_{k+1}}\right). \] (4.12)
After the time changing

\[ t_n = \tilde{t}_n - 2\varphi \frac{N}{(-\xi)^n} + 2N\delta_{n,1} - \frac{N}{2}\delta_{n,2}, \quad (4.13) \]

where

\[ \tilde{t}_n = \frac{1}{n} \sum_i \frac{1}{\eta_i^n} \quad (4.14) \]

are the times of the KP model, we obtain

\[ L_k = -\sum_{m=1}^{\infty} m \tilde{t}_{m+k} \frac{\partial}{\partial \tilde{t}_{m+k}} - \sum_{m=1}^{k-1} \frac{\partial}{\partial \tilde{t}_{k-m}} + 2N\alpha_{KP} \frac{\partial}{\partial \tilde{t}_k} + N \frac{\partial}{\partial \tilde{t}_{k+2}} \]

\[ - 2N\alpha_{KP} (\delta_{k,0} + \delta_{k,-1}) \frac{\partial}{\partial \tilde{t}_k} - N^2\alpha_{KP}^2 \delta_{k,0} + \tilde{t}_1 \delta_{k,-1} \frac{\partial}{\partial \tilde{t}_{k+1}} + N\alpha_{KP} \tilde{t}_1 \delta_{k,-1}. \quad (4.15) \]

This is exactly the Virasoro algebra that appears in the KP model with the normalizing factor

\[ \prod_i \eta_i^{\alpha_{KP} N} e^{\frac{N^2}{2} \eta_i^2}. \quad (4.16) \]

Indeed, we can perform the same operation for the KP model. First, we write the master equation for the normalized partition function

\[ \left[ -\frac{1}{N^2} \frac{\partial^2}{\partial \xi^2} - \frac{1}{N^2} \sum_{j \neq i} \frac{\partial}{\partial \eta_j} - \frac{\eta_i}{N} \frac{\partial}{\partial \xi} - \frac{2\alpha_{KP}}{N\eta_i} \frac{\partial}{\partial \eta_i} + \frac{\alpha_{KP}}{N\eta_i} \sum_j \frac{1}{\eta_j} - \frac{\alpha_{KP}^2}{\eta_i} \right] Z_{KP} = 0. \quad (4.17) \]

Then, using the KP model times \( \tilde{t}_n \) and collecting all coefficients to the term \( 1/(\eta_i^k N^2) \), we obtain

\[ L_k Z_{KP} = 0, \quad k \geq -1. \quad (4.18) \]

Therefore, we have proven the equivalence between the 2-log and KP models.

Now, we write the explicit relation between the normalized partition functions of the two models

\[ Z_{2\log} \left[ \left\{ \frac{1}{n} \operatorname{tr} \frac{1}{\lambda^n} \right\}; \alpha, \beta \right] = C(\alpha, \beta) \xi^{2\varphi(\beta-1)N^2} e^{N^2(2\beta-1)\xi} Z_{KP} \left[ \tilde{t}_n(\xi, \varphi), \alpha_{KP} \right], \quad (4.19) \]

where

\[ Z_{2\log} \left[ \left\{ \frac{1}{n} \operatorname{tr} \frac{1}{\lambda^n} \right\}; \alpha, \beta \right] = \frac{Z_{2\log} \left[ \lambda; \alpha, \beta \right]}{\prod_i \left\{ (\lambda_i - \xi)^{N(\beta-1)} e^{N(\lambda_i - \xi)} \right\}}, \]

\[ Z_{KP} \left[ \tilde{t}_n(\xi, \varphi), \alpha_{KP} \right] = \frac{Z_{KP} \left[ \eta(\xi, \varphi), \alpha_{KP} \right]}{\prod_i \left\{ (\eta_i)^{N\alpha_{KP}} e^{\frac{N}{2} \eta_i^2} \right\}}, \quad (4.20) \]

\( \alpha_{KP} = \beta - 1 \) and \( C(\alpha, \beta) \) is some constant depending on the parameters \( \alpha \) and \( \beta \).

Note that we use here unshifted initial field \( \lambda \) and explicitly show the dependence on the arbitrary parameter \( \xi \) by the following reason. For the unshifted \( \lambda \)-field, the Virasoro algebra
for the 2-log model does not possess the $L_{-1}$ generator. So, a question arises how we can obtain the $L_{-1}$ generator of the KP model when passing to the KP model. The reason is that after the time changing (4.13), the KP times $\tilde{t}_n$ become $\xi$-dependent. Differentiating (4.19) over $\xi$ and using the relation

$$\frac{d\tilde{t}_n}{d\xi} = (n + 1)\tilde{t}_{n+1} - N\delta_{n,1},$$

we obtain one more equation for $Z_{KP}$,

$$L_{-1} Z_{KP} = 0,$$

where $L_{-1}$ is just the generator of the KP Virasoro algebra.

5 Higher genus expressions

Let us recall the genus expansion for the KP model [8],

$$\log Z_{KP} = \sum_{g=0}^{\infty} N^{2-2g} F_g,$$  \hspace{1cm} (5.1)

where

$$F_g = \sum_{\alpha_j,\beta_j > 1} \langle \alpha_1 \cdots \alpha_s; \beta_1 \cdots \beta_l | \alpha \beta \gamma \rangle_g M_{\alpha_1} \cdots M_{\alpha_s} J_{\beta_1} \cdots J_{\beta_l} \frac{M_1^{\alpha} J_1^{\beta} d^4}{M_1^{\alpha} J_1^{\beta} d^4}, \quad g > 1, \hspace{1cm} (5.2)$$

and

$$F_1 = -\frac{1}{24} \log(M_1 J_1 d^4), \quad g = 1. \hspace{1cm} (5.3)$$

The moments were defined as follows ($k \geq 0$)

$$M_k = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\eta_i - x_+)^{k+1/2} (\eta_i - x_-)^{1/2}} - \delta_{k,1}, \hspace{1cm} (5.4)$$

$$J_k = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\eta_i - x_+)^{k+1/2} (\eta_i - x_-)^{1/2}} - \delta_{k,1},$$

where $x_\pm$ are the endpoints of the cut for the one-cut solution and $d = x_+ - x_-$. In our notation,

$$x_\pm = -2b \pm \sqrt{4b^2 - 2c}. \hspace{1cm} (5.5)$$

Let us introduce the moments for the 2-log model ($k \geq 0$)

$$N_k = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\lambda_i - y_+)^{k+1/2} (\lambda_i - y_-)^{1/2}}, \hspace{1cm} (5.6)$$

$$K_k = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\lambda_i - y_+)^{k+1/2} (\lambda_i - y_-)^{k+1/2}}.$$
where
\[ y_{\pm} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} . \]  
(5.7)

We are interested in the relation between the moments of the two models for \( k \geq 0 \) (for \( k = 0 \), the relation is given by constraint equations (3.3) and (3.4)). After making the eigenvalue shift \((y_{\pm} = x_{\pm} + \xi)\) and performing the time changing, we obtain \((k \geq 1)\)

\[ M_k = N_k + 2 \varphi \frac{(-1)^{k+1}}{y_+^{k+1/2} y_-^{1/2}}, \]  
\[ J_k = K_k + 2 \varphi \frac{(-1)^{k+1}}{y_+^{1/2} y_-^{k+1/2}}. \]  
(5.8)

So, for the 2-log model, we have
\[
\log Z = \sum_{g=0}^{\infty} N^{2-2g} F_{g}^{2 \log} ,
\]  
(5.9)

where
\[
F_{g}^{2 \log} = \sum_{\alpha_j>1, \beta_j>1} \langle \alpha_1 \ldots \alpha_s; \beta_1 \ldots \beta_l | \alpha \beta \gamma \rangle_g \prod_{i=1}^{s} \left( N_{\alpha_i} + 2 \varphi \frac{(-1)^{\alpha_i+1}}{y_+^{\alpha_i+1/2} y_-^{1/2}} \right) \times \prod_{i=1}^{l} \left( K_{\beta_i} + 2 \varphi \frac{(-1)^{\beta_i+1}}{y_+^{\beta_i+1/2} y_-^{1/2}} \right) \left( N_{1} + 2 \varphi \frac{1}{y_+^{1/2} y_-^{3/2}} \right)^{\alpha} \times \left( N_{1} + 2 \varphi \frac{1}{y_+^{1/2} y_-^{3/2}} \right)^{\beta} (y_+ - y_-)^{\gamma - 1}, \quad g > 1, \]  
(5.10)

and
\[
F_1 = -\frac{1}{24} \log \left\{ \left( N_{1} + 2 \varphi \frac{1}{y_+^{1/2} y_-^{3/2}} \right) \left( K_{1} + 2 \varphi \frac{1}{y_+^{1/2} y_-^{3/2}} \right) (y_+ - y_-)^{4} \right\}. \]  
(5.11)

Therefore, expression (2.16) for genus zero, taking into account the normalizing factor (4.2), and expressions (5.10), (5.11), completely determine the partition function of the model (2.1) at all genera.

### 6 Determinant formulas

The exact determinant formulas in our model can be easily found using the Itzykson–Zuber–Mehta technique for the integration over angular variables in multi-matrix models. The partition function can be expressed as follows

\[
Z = (2\pi)^{N^2/2} \frac{\Theta_2}{\Theta_1} \prod_{i} \left\{ dx_i (1 - x_i) - \alpha x_i (1 + x_i)^{-\alpha} e^{-N\alpha x_i} \right\} \frac{\Delta(x)}{\Delta(\lambda)},
\]  
(6.1)

where \( \Delta(x) = \prod_{i>j} (x_i - x_j) \) is the Van der Monde determinant and \( \theta_1, \theta_2 \) are some integration limits. We know that in the large \( N \) limit, the difference between partition functions
calculated in various integration limits is exponentially small and does not affect the $1/N$ perturbative expansion. So, we investigate several cases.

(i). For $\theta_1 = -1$ and $\theta_2 = 1$, we use the following integral representation $(a, b > 0)$

$$
\int_{-1}^{1} dx (1-x)^{a-1}(1+x)^{b-1} e^{-cx} = 2^{a+b-1} e^{-c} B(a, b) \Phi(a, a+b; 2c), \quad (6.2)
$$

where $\Phi(a, c; z) \equiv \, _1F_1(a, c; z)$ is the confluent hypergeometric function and $B(a, b)$ is the beta-function.

Then, in the domain $\alpha, \beta < 1/N$, we obtain

$$
Z = (2\pi)^{N^2-2} 2^{(a+b)N^2+N(N+1)/2} \prod_i \left\{ B(-\alpha N+1, -\beta N+i) \right\}
\times e^{-N\text{tr} \Lambda} \frac{\det (\Delta(\lambda))}{\Delta(\lambda)} \prod_{1 \leq i, j \leq N} \|\Phi(-\alpha N+1, -(\alpha + \beta) N + j + 1; 2N\lambda_i)\|. \quad (6.3)
$$

(ii). For $\theta_1 = 1$ and $\theta_2 = \infty$, we use the relation $(a, c > 0)$

$$
\int_{1}^{\infty} dx (1-x)^{a-1}(1+x)^{b-1} e^{-cx} = (-1)^{a-1} 2^{a+b-1} e^{-c} \Gamma(a) \Psi(a, a+b; 2c), \quad (6.4)
$$

where

$$
\Psi(a, c; z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} \Phi(a-c+1, 2-c; z) \quad (6.5)
$$

is the other confluent hypergeometric function and $\Gamma(a)$ is the gamma-function.

Then, in the domain where $\alpha < 1/N$, $\beta$ is unrestricted, and $\lambda_i > 0$, we have

$$
Z = (2\pi)^{N^2-2} (-1)^{-aN^2} 2^{-(a+b)N^2+N(N+1)/2} \Gamma(-\alpha N+1)
\times e^{-N\text{tr} \Lambda} \frac{\det (\Delta(\lambda))}{\Delta(\lambda)} \prod_{1 \leq i, j \leq N} \|\Psi(-\alpha N+1, -(\alpha + \beta) N + j + 1; 2N\lambda_i)\|. \quad (6.6)
$$

This answer covers more general domain of the parameters $\alpha$ and $\beta$ than (6.3).

(iii). If $\alpha = 0$ we get the simplest answer setting $\theta_1 = -1$ and $\theta_2 = \infty$. In the domain $\beta < 1/N$ and $\lambda_i > 0$, we obtain

$$
Z \bigg|_{\alpha=0} = (2\pi)^{N^2-2} \prod_i \{ \Gamma(-\beta N+i) \} (\det \Lambda)^{(\beta-1)N} e^{N\text{tr} \Lambda}, \quad (6.7)
$$

which is the unshifted normalizing factor up to a constant.

7 String susceptibilities

Let us calculate the string susceptibility w.r.t. $\gamma$ and $\varphi$ for (2.16). By virtue of Eq. (2.17),

$$
\frac{d}{d\gamma} \log Z = \frac{\partial}{\partial\gamma} \log Z \quad \text{and the same holds true for } \varphi. \quad \text{Furthermore, an amazing fact is that}
$$
the expressions obtained are themselves stationary w.r.t. differentiation over \(a\) and \(b\). This means that the total second derivatives in \(\gamma\) and \(\varphi\) coincide with the corresponding partial derivatives, so we have

\[
\chi_1 = \frac{1}{N^2} \frac{d^2}{d\gamma^2} \log Z = \log(b^2 - 4ac) - 2 \log a ,
\]

\[
\chi_2 = \frac{1}{N^2} \frac{d^2}{d\gamma d\varphi} \log Z = -2 \operatorname{arctanh} \frac{2\sqrt{ac}}{b} = -\log \frac{b + 2\sqrt{ac}}{b - 2\sqrt{ac}} ,
\]

\[
\chi_3 = \frac{1}{N^2} \frac{d^2}{d\varphi^2} \log Z = \log(b^2 - 4ac) + 6 .
\]

(7.1)

Recalling the string susceptibility of the KP model in the KP variables \(b\) and \(c\) [6]

\[
\chi_{\text{KP}} = \log(b^2 - c)
\]

(7.2)

and using the relations (3.5), we obtain

\[
\chi_{\text{KP}} = \chi_1 .
\]

(7.3)

8 Conclusion

This paper concludes the series of papers [6, 10, 12, 14] devoted to studying the external field matrix problems with logarithmic potentials. We see that, at least in the \(1/N\)-expansion in terms of the corresponding moments, all these models can be reduced either to the Kontsevich model or to the Hermitian one-matrix model with an arbitrary potential. Here, the question arises whether this can be derived directly within the \(\tau\)-function framework [5]. The related question is which reductions of the Kadomtsev–Petviashvili hierarchy correspond to the NBI and 2-log models.

One can always say the origin of the logarithmic terms is due to additional degrees of freedom that were integrated out. Matrix integral (2.1) can be represented as the \(O(\alpha, \beta)\)-type [24] matrix integral

\[
Z = \int dX \prod_{i=1}^\alpha d\Psi_i d\Psi_i \prod_{j=1}^\beta d\Phi_j d\Phi_j e^{-N \text{tr} \left[ \overline{\Psi}_i \Psi_i + \overline{\Phi}_j \Phi_j + X (\Lambda - \Psi_i \overline{\Psi}_i + \Phi_j \overline{\Phi}_j) \right]} ,
\]

(8.1)

where the sum over repeated indices is implied and we assume the matrix fields \(\Phi\) and \(\Psi\) are Grassmann even. Action (8.1) is of a nonlinear sigma model type with free matrix fields \(\Phi\) and \(\Psi\) dwelling on the manifold \(\Lambda - \Psi_i \overline{\Psi}_i + \Phi_j \overline{\Phi}_j = 0\).

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