VISCOELASTIC PLATE EQUATION WITH
BOUNDARY FEEDBACK

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Abstract. In this paper we consider a viscoelastic plate equation with a non-linear weakly dissipative feedback localized on a part of the boundary. Without imposing restrictive assumptions on the boundary frictional damping, we establish an explicit and general decay rate result that allows a wider class of relaxation functions and generalizes previous results existing in the literature.

1. Introduction. In this paper we are concerned with the following problem

\[
\begin{align*}
\begin{cases}
  u_{tt} + \Delta^2 u - \int_0^t g(t-s) \Delta^2 u(s) ds = 0, & \text{in } \Omega \times (0, \infty) \\
  u = \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_0 \times (0, \infty) \\
  \beta_1 u - \beta_1 \left( \int_0^t g(t-s) u(s) ds \right) = 0 & \text{on } \Gamma_1 \times (0, \infty) \\
  -\beta_2 u + \beta_2 \left( \int_0^t g(t-s) u(s) ds \right) + \theta(t) h(u) = 0 & \text{on } \Gamma_1 \times (0, \infty) \\
  u(x,y,0) = u_0(x,y), & \quad u_t(x,y,0) = u_1(x,y), & \text{in } \Omega,
\end{cases}
\end{align*}
\]

which is a viscoelastic plate equation with frictional damping at a part of the boundary and initial data in suitable function spaces. Here \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \) with a smooth boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), where \( \Gamma_0 \) and \( \Gamma_1 \) are closed and disjoint with \( \text{meas}(\Gamma_0) > 0 \), and \( n = (\nu_1, \nu_2) \) is the unit outward normal to \( \partial \Omega \), \( \eta = (-\nu_2, \nu_1) \) is the unit tangent positively oriented on \( \partial \Omega \), the integral term in (1.1) is the memory responsible for the viscoelastic damping where \( g \) is a positive function called the relaxation function, \( \theta \) is a time dependent coefficient of the frictional damping, and \( h \) is a specific function. We are denoting by \( \beta_1, \beta_2 \) the following differential operators:

\[
\beta_1 u = \Delta u + (1 - \mu) B_1 u, \quad \beta_2 u = \frac{\partial \Delta u}{\partial n} + (1 - \mu) \frac{\partial B_2 u}{\partial \eta}
\]

where

\[
B_1 u = 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \quad B_2 u = (\nu_1^2 - \nu_2^2) u_{xy} + \nu_1 \nu_2 (u_{yy} - u_{xx})
\]

and \( \mu \in (0, \frac{1}{2}) \) represents the Poisson coefficient. This system describes the transversal displacement \( u = u(x,y,t) \) of a thin vibrating plate subjected to internal viscoelastic damping and time-dependent boundary frictional damping.

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If the damping mechanism is only given by the boundary feedback, system (1.1), with \( \theta \equiv 1 \), has been widely studied in the literature. In Horn \[16\], Komornik \[21\], Lagnese \[23\], Lasiecka \[25\], and Ji and Lasiecka \[17\], it was proved that if \( h \) satisfies
\[
\min \left\{ |s|, |s|^q \right\} \leq |h(s)| \leq \max \left\{ |s|, |s|^{1/q} \right\}
\]
where \( c_1, c_2 \) are positive constants, then for \( q = 1 \) the energy decay rate is exponential while for \( q > 1 \) we obtain a polynomial decay rate. Similar results were also obtained for internal frictional damping (see Ammari and Tucsnak \[4\], Cavalcanti et al. \[9\], Guzman and Tucsnak \[13\], Komornik \[20\], Pazoto et al. \[44\], and Vascconcellos and Teixeira \[46\]). Decay results for arbitrary growth of the frictional damping term have been given by Amroun and Benaissa \[5\] motivated by the works done by Lasiecka \[24\], Lasiecka and Tataru \[27\], Liu and Zuazua \[30\], and Martinez \[31\] and \[32\] for damped wave equations. They established an explicit formula for the energy decay rates that need not to be of exponential or polynomial types. Similarly, Han and Wang \[14\] studied a coupled system of plate and wave equations and used internal frictional damping terms without imposing growth conditions near zero to achieve the stability and controllability of the system. In the presence of the time dependent coefficient \( \theta(t) \), Mustafa \[40\] and \[42\] established for the wave equation a general energy decay result depending on both \( h \) and \( \theta \).

On the other hand, when the unique damping mechanism in (1.1) is given by the memory term, we refer to Lagnese \[22\] and Rivera et al. \[37\], where it was proved that the energy decays exponentially if the relaxation function \( g \) decays exponentially and polynomially if \( g \) decays polynomially. The same results were obtained by Alabau-Boussouira et al. \[2\] for a more general abstract equation and by Santos and Junior \[45\] for boundary viscoelastic dampings. In \[38\] and \[39\] Rivera et al. investigated a class of abstract viscolastic systems of the form
\[
u_{tt}(t) + Au(t) - (g \ast A^\beta u)(t) = 0,
\]
where \( A \) is a strictly positive, self-adjoint operator with \( D(A) \) a subset of a Hilbert space and \( \ast \) denotes the convolution product in the variable \( t \). The authors showed that solutions for (1.2), when \( 0 < \beta < 1 \), decay polynomially even if the kernel \( g \) decays exponentially. While, in the case \( \beta = 1 \), the solution energy decays at the same decay rate as the relaxation function.

Then, a natural question was raised: how does the energy behave as the kernel function does not necessarily decay polynomially or exponentially? Han and Wang gave an answer to the above question when treating (1.2), for \( \beta = 1 \), in \[15\]. They considered relaxation functions satisfying
\[
g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0
\]
where \( \xi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nonincreasing differentiable function with
\[
\frac{\xi'(t)}{\xi(t)} \leq k, \quad \forall t \geq 0
\]
for some constant \( k \) and showed that the rate of the decay of the energy is exactly the rate of decay of \( g \) which is not necessarily of polynomial or exponential decay type. These conditions (1.3) and (1.4) on \( g \) where first used by Messaoudi \[33\] and \[34\] in studying a viscoleastic wave equation. After that, Messaoudi and Mustafa \[35\]...
and Mustafa and Messaoudi [41] eliminated condition (1.4) and used only (1.3) to establish more general stability results of viscoelastic Timoshenko beams. A similar condition to (1.3) was used by Kang [19] to treat plate equation with memory-type boundary conditions and by Ferreira and Messaoudi [11] to treat a nonlinear viscoelastic plate equation with a $-\partial^2_p(x, t)$-Laplacian operator. Another step forward is the work of Alabau-Boussouira and Cannarsa [1] who considered wave equation with memory whose relaxation function is satisfying

$$g' \leq -\chi(g(t))$$

where $\chi$ is a non-negative function, with $\chi(0) = \chi'(0) = 0$, and $\chi$ is strictly increasing and strictly convex on $(0, k_0]$, for some $k_0 > 0$. They also required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} x \frac{dx}{\chi(x)} < 1, \quad \lim_{s \to 0^+} \inf \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2}$$

and proved an energy decay result. In addition to these assumptions, if

$$\lim_{s \to 0^+} \sup \frac{\chi(s)/s}{\chi(s)} < 1 \quad \text{and} \quad g' = -\chi(g(t))$$

then, in this case, an explicit rate of decay is given. Here, a new theorem was announced which was applied to some new examples giving optimal decay rates. These assumptions imposed on $\chi$ do not appear intrinsic to the result claimed, but rather to the method based on weighted energy inequalities with the use of convexity. Later on, Mustafa and Messaoudi [43] similarly used (1.5) and provided another variant of that approach which was able to remove some of the constraints imposed in [1] and obtain an explicit and general decay rate formula.

The interaction between viscoelastic and frictional dampings was considered by several authors. Cavalcanti and Oquendo [10] looked into wave equation of the form

$$u_{tt} - \Delta u + \int_0^t \text{div}[a(x)g(t-s)\nabla u(s)]ds + b(x)h(u_t) + f(u) = 0, \quad x \in \Omega, t > 0 \quad (1.7)$$

and established exponential stability for $g$ decaying exponentially and $h$ linear and polynomial stability for $g$ decaying polynomially and $h$ having a polynomial growth near zero. For $g$ of general-type decay and $h$ having no restrictive growth assumption near the origin, Cavalcanti et al. [8] established decay rate estimates. Using (1.3), with time dependent coefficient and $a(x) = b(x) = 1$, Liu [29] proved a general decay result. Similarly, Guesmia and Messaoudi [12] studied Timoshenko systems with frictional versus viscoelastic damping and Messaoudi and Mustafa [36] studied viscoelastic wave equation with boundary feedback and obtained energy decay estimates. Once again, Kang [18] imposed the condition (1.3) on the relaxation function for an internal viscoelastic damping in a von Karman plate model which is also subject to a boundary frictional damping and they proved a general stability result.

Our aim in this work is to investigate (1.1) with both weak boundary frictional damping and internal viscoelastic damping. We obtain a general relation between the decay rate for the energy (when $t$ goes to infinity) and the functions $g$, $\theta$, and $h$ without imposing any growth assumption near the origin on $h$ and strongly weakening the usual assumptions on $g$. The result of this paper generalizes previous related results where it allows a larger class of functions $g$ and $h$, from which the energy decay rates are not necessarily of exponential or polynomial types and takes into account the effect of a time dependent coefficient $\theta(t)$. The proof is based on the
multiplier method and makes use of some properties of convex functions including the use of the general Young’s inequality and Jensen’s inequality. These convexity arguments were introduced by Lasiecka and Tataru [27] and used by Liu and Zuazua [30] and Alabau-Boussouira [3]. The paper is organized as follows. In section 2, we present some notation and material needed for our work. Some technical lemmas and the proof of our main result will be given in section 3.

2. Preliminaries. We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Throughout this paper, $c$ is used to denote a generic positive constant. We first consider the following hypotheses

(A1) $θ : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing $C^1$ function.

(A2) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function such that

\[ g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = t > 0, \quad (2.1) \]

and there exists a positive function $H \in C^1(\mathbb{R}_+)$ and $H$ is linear or strictly increasing and strictly convex $C^2$ function on $(0, r)$, $r < 1$, with $H(0) = H'(0) = 0$, such that

\[ g'(t) \leq -H(g(t)), \quad \forall t > 0. \]

(A3) $h : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing $C^0$ function and there exist constants $c_1, c_2 > 0$ such that

\[ c_1 |s| \leq |h(s)| \leq c_2 |s| \quad \text{if} \quad |s| \geq r, \]

\[ s^2 + h^2(s) \leq H^{-1}(sh(s)) \quad \text{if} \quad |s| \leq r. \]

In the sequel we assume that system (1.1) has a unique solution

\[ u \in L^\infty(\mathbb{R}_+; H^4(\Omega) \cap W^{1,\infty}(\mathbb{R}_+; W) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)) \]

where $W = \{ w \in H^2(\Omega) : w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_0 \}$. This result can be proved, for initial data in suitable function spaces, using standard arguments such as the Galerkin method (see [45]).

Let us define the bilinear form $a(., .)$ as follows

\[ a(u, v) = \int_\Omega \left\{ u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1 - \mu)u_{xy}v_{xy} \right\} dx dy \quad (2.2) \]

and, as $\text{meas} \Gamma_0 > 0$, we know that $\sqrt{a(u, u)}$ is an equivalent norm on $W$; that is, for some positive constants $\alpha$ and $\beta$,

\[ \alpha \| u \|^2_{H^2(\Omega)} \leq a(u, u) \leq \beta \| u \|^2_{H^2(\Omega)}. \quad (2.3) \]

Young’s inequality gives, for any $\varepsilon > 0$,

\[ a(u, v) \leq \varepsilon a(u, u) + \frac{1}{4\varepsilon} a(v, v) \quad (2.4) \]

Also, we mention the following useful identity, see [23],

\[ \int_\Omega (\Delta^2 u)v dx = a(u, v) + \int_\Gamma \{ (\beta_2 u)v - (\beta_1 u) \frac{\partial v}{\partial n} \} d\Gamma. \quad (2.5) \]

Now, we introduce the energy functional

\[ E(t) := \frac{1}{2} \int_\Omega |u_t|^2 dx + \frac{1}{2} \left( 1 - \int_0^t g(s)ds \right) a(u, u) + \frac{1}{2} (g \circ u)(t), \quad (2.6) \]
where
\[(g \circ u)(t) = \int_0^t g(t-s)u(t-u(s), u(t) - u(s))ds.\]

Our main stability result is the following.

**Theorem 2.1.** Assume that (A1)-(A3) hold. Then there exist positive constants \(k_1, k_2, k_3\) and \(\varepsilon_0\) such that the solution of (1.1) satisfies
\[
E(t) \leq k_3 H_1^{-1}(k_1 \int_0^t \theta(s)ds + k_2) \quad \forall t \geq 0, \tag{2.7}
\]
where
\[H_1(t) = \int_0^t \frac{1}{sH_0'((\varepsilon_0s))}ds \quad \text{and} \quad H_0(t) = H(D(t))\]
provided that \(D\) is a positive \(C^1\) function, with \(D(0) = 0\), for which \(H_0\) is strictly increasing and strictly convex \(C^2\) function on \((0, r]\) and
\[
\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))}ds < +\infty. \tag{2.8}
\]
Moreover, if \(\int_0^1 H_1(t)dt < +\infty\) for some choice of \(D\), then we have the improved estimate
\[
E(t) \leq k_3 G^{-1}(k_1 \int_0^t \theta(s)ds + k_2) \quad \text{where} \quad G(t) = \int_t^1 \frac{1}{sH'(((\varepsilon_0s))}ds. \tag{2.9}
\]
In particular, this last estimate is valid for the special case \(H(t) = ct^p\), for \(1 \leq p < \frac{3}{2}\).

**Remarks.**
1. Using the properties of \(H\), one can show that the function \(H_1\) is strictly decreasing and convex on \((0, 1]\), with \(\lim_{t \to 0} H_1(t) = +\infty\). Therefore, Theorem 2.1 ensures
\[
\lim_{t \to +\infty} E(t) = 0.
\]
2. Hypothesis (A3) implies that \(sh(s) > 0\), for all \(s \neq 0\).
3. The condition (A3), with \(r = 1\) and \(\theta \equiv 1\), was introduced and employed by Lasiecka and Tataru [27] in their study of the asymptotic behavior of solutions of nonlinear wave equations with nonlinear frictional boundary damping where they obtained decay estimates that depend on the solution of an explicit nonlinear ordinary differential equation. It was also shown there that the monotonicity and continuity of \(h\) guarantee the existence of the function \(H\) with the properties stated in (A3). In our present work, we study the plate equation with both boundary frictional damping, modulated by a time dependent coefficient \(\theta(t)\), and viscoelastic damping. We investigate the influence of these simultaneous damping mechanisms on the decay rate of the energy and establish an explicit and general energy decay formula, depending on \(g, h, \theta\).
4. The usual exponential and polynomial decay rate estimates, already proved for \(g\) satisfying (2.1) and \(H(t) = ct^p, 1 \leq p < 3/2\), are special cases of our result. We will provide a “simpler” proof for these special cases. We should refer here to the references [7], [26] and [28] where the optimal polynomial decay was pushed up to \(p < 2\).
5. Our result allows decay rates which are not necessarily of exponential or polynomial decay. For instance, if

\[ g(t) = a \exp(-t^q) \]

for \( 0 < q < 1 \) and \( a \) chosen so that \( g \) satisfies (2.1), then \( g'(t) = -H(g(t)) \) where, for \( t \in (0, r], r < a \),

\[ H(t) = \frac{qt}{\ln(a/t)^{1/\alpha}} \]

which satisfies hypothesis (A2). Also, by taking \( D(t) = t^\alpha \), (2.8) is satisfied for any \( \alpha > 1 \). Therefore, if \( h \) satisfies (A3) with this function \( H \), then we can use Theorem 2.1 and do some calculations (see [43]) to deduce that the energy decays at the rate

\[ E(t) \leq c \exp(-k \left[ \int_0^t \theta(s)ds \right]^q). \]

One can show that this example of \( g \) does not satisfy (1.6), and so no explicit rate of decay for this case is given in [1].

6. The well-known Jensen’s inequality will be of essential use in establishing our main result. If \( F \) is a convex function on \([a, b]\), \( f : \Omega \to [a, b] \) and \( z \) are integrable functions on \( \Omega \), \( z(x) \geq 0 \), and \( \int_\Omega z(x)dx = k > 0 \), then Jensen’s inequality states that

\[ F \left[ \frac{1}{k} \int_\Omega f(x)z(x)dx \right] \leq \frac{1}{k} \int_\Omega F[f(x)]z(x)dx. \]

7. By (A2), we easily deduce that \( \lim_{t \to +\infty} g(t) = 0 \). Similarly, assuming the existence of the limit, we find that \( \lim_{t \to +\infty} (-g'(t)) = 0 \). Hence, there is \( t_1 > 0 \) large enough such that \( g(t_1) > 0 \) and

\[ \max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall \ t \geq t_1. \quad (2.10) \]

As \( g \) is nonincreasing, \( g(0) > 0 \) and \( g(t_1) > 0 \), then \( g(t) > 0 \) for any \( t \in [0, t_1] \) and

\[ 0 < g(t_1) \leq g(t) \leq g(0), \quad \forall \ t \in [0, t_1]. \]

Therefore, since \( H \) is a positive continuous function, then

\[ a \leq H(g(t)) \leq b, \quad \forall \ t \in [0, t_1] \]

for some positive constants \( a \) and \( b \). Consequently, for all \( t \in [0, t_1] \),

\[ g'(t) \leq -H(g(t)) \leq -a = -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t) \]

which gives, for some positive constant \( d \),

\[ g'(t) \leq -d g(t), \quad \forall \ t \in [0, t_1]. \quad (2.11) \]

8. If different functions \( H_1 \) and \( H_2 \) have the properties mentioned in (A2) and (A3) such that \( g'(t) \leq -H_1(g(t)) \) and \( s^2 + h^2(s) \leq H_2^{-1}(sh(s)) \), then there is \( r < \min\{r_1, r_2\} \) small enough so that, say, \( H_1(t) \leq H_2(t) \) on the interval \((0, r]\). Thus, the function \( H(t) = H_1(t) \) satisfies both (A2) and (A3), \( \forall \ t \geq t_1. \)
3. **Proof of the main result.** In this section we prove Theorem 2.1. For this purpose, we establish several lemmas.

**Lemma 3.1.** Under the assumptions (A1)-(A3), the energy functional satisfies, along the solution of (1.1), the estimate

\[
E'(t) = \frac{1}{2}(g' \circ u)(t) - \frac{1}{2}g(t) a(u, u) - \theta(t) \int_{\Gamma_1} u h(u_t) d\Gamma \leq 0. \tag{3.1}
\]

**Proof.** By multiplying equation (1.1) by \(u_t\) and integrating over \(\Omega\), using integration by parts, the boundary conditions, (2.5), hypotheses (A1)-(A3) and some manipulations, we obtain (3.1). \(\square\)

Now we are going to construct a Lyapunov functional \(L\) equivalent to \(E\), with which we can show the desired result.

**Lemma 3.2.** Under the assumptions (A1)-(A3), the functional

\[
I(t) := \int_{\Omega} uu_t dx
\]

satisfies, along the solution, the estimate

\[
I'(t) \leq -\frac{1}{2}a(u, u) + \int_{\Omega} u^2 dx + c g \circ u + c \int_{\Gamma_1} h^2(u_t) d\Gamma \tag{3.2}
\]

**Proof.** Direct computations, using (1.1), yield

\[
I'(t) = \int_{\Omega} u^2 dx - \int_{\Omega} u \Delta^2 u dx + \int_{\Omega} u \int_0^t g(t - s) \Delta^2 u(s) ds dx
\]

\[
= \int_{\Omega} u^2 dx - \int_{\Omega} u \Delta^2 u dx + \int_{\Omega} u \int_0^t g(t - s) \left( \int_0^s \Delta^2 u(s) - \Delta^2 u(t) \right) ds dx
\]

\[
+ \left( \int_0^t g(s) ds \right) \int_{\Omega} u \Delta^2 u(t) dx.
\]

Then, by (2.4)-(2.5), (1.1)\(_2\)-4, (A1)-(A3), and the Trace theorem, we get

\[
I'(t) = \int_{\Omega} u^2 dx - \left( 1 - \int_0^t g(s) ds \right) a(u, u) + \int_0^t g(t - s) a(u(s) - u(t), u(t)) ds
\]

\[
- \theta(t) \int_{\Gamma_1} uh(u_t) d\Gamma
\]

\[
\leq \int_{\Omega} u^2 dx - la(u, u) + \varepsilon \left( \int_0^t g(s) ds \right) a(u, u) + \frac{1}{4 \varepsilon} g \circ u + c \varepsilon \int_{\Gamma_1} u^2 d\Gamma
\]

\[
+ \frac{c}{4 \varepsilon} \int_{\Gamma_1} h^2(u_t) d\Gamma
\]

\[
\leq \int_{\Omega} u^2 dx - la(u, u) + \varepsilon a(u, u) + \frac{1}{4 \varepsilon} g \circ u + c \varepsilon \|u\|^2_{H^2(\Omega)}
\]

\[
+ \frac{c}{4 \varepsilon} \int_{\Gamma_1} h^2(u_t) d\Gamma.
\]

Using (2.3) and choosing \(\varepsilon\) small enough give (3.2). \(\square\)

**Lemma 3.3.** Under the assumptions (A1)-(A3), the functional \(K\) defined by

\[
K(t) := -\int_{\Omega} u_t \int_0^t g(t - s) (u(t) - u(s)) ds dx
\]
satisfies, along the solution, the estimate
\[
K'(t) \leq \varepsilon a(u, u) - \left( \int_0^t g(s)ds - \varepsilon \right) \int_\Omega u_t^2 dx + \varepsilon g \circ u - \varepsilon' g' \circ u + c \int_{\Gamma_1} h^2(u_t)d\Gamma
\] (3.3)

for any \(0 < \varepsilon < 1\).

Proof. By exploiting equations (1.1), integrating by parts, (2.5), and doing some manipulations, we have

\[
K'(t) = \int_0^t g(t-s)a(u(t) - u(s), u(t)) ds
- \int_0^t g(t-s)a(u(t) - u(s), \int_0^t g(t-\tau)u(\tau)d\tau) ds
+ \theta(t) \int_{\Gamma_1} h(u_t) \left( \int_0^t g(t-s)(u(t) - u(s)) ds \right) d\Gamma
- \int_\Omega u_t \int_0^t g'(t-s)(u(t) - u(s)) ds dx - \left( \int_0^t g(s) ds \right) \int_\Omega u_t^2 dx
\]

Then, we use (2.1) and (2.4) to get

\[
\int_0^t g(t-s)a(u(t) - u(s), u(t)) ds
\leq \frac{\varepsilon}{2} \left( \int_0^t g(s) ds \right) a(u, u) + \frac{1}{2\varepsilon} g \circ u \leq \frac{\varepsilon}{2} a(u, u) + \frac{1}{2\varepsilon} g \circ u
\]

and

\[
- \int_0^t g(t-s)a(u(t) - u(s), \int_0^t g(t-\tau)u(\tau)d\tau) ds
= \int_0^t g(t-s) \int_0^t g(t-\tau)a(u(t) - u(s), u(t) - u(\tau)) d\tau ds
- \left( \int_0^t g(s) ds \right) \int_0^t g(t-s)a(u(t) - u(s), u(t)) ds
\leq \left( 1 + \frac{1}{2\varepsilon} \right) g \circ u + \frac{\varepsilon}{2} a(u, u).
\]

Similarly, using (2.3), Hölder and Poincaré inequalities, and the Trace Theorem, yield

\[
\theta(t) \int_{\Gamma_1} h(u_t) \left( \int_0^t g(t-s)(u(t) - u(s)) ds \right) d\Gamma
\leq c \int_{\Gamma_1} h^2(u_t)d\Gamma + \int_{\Gamma_1} \left| \int_0^t g(t-s)(u(t) - u(s)) ds \right|^2 d\Gamma
\leq c \int_{\Gamma_1} h^2(u_t)d\Gamma + \int_{\Gamma_1} \left( \int_0^t g(t-s) \sqrt{g(t-s)} \right) (u(t) - u(s)) ds \left| g(t-s) \right| u(t) - u(s) \right|^2 d\Gamma
\leq c \int_{\Gamma_1} h^2(u_t)d\Gamma + \left( \int_0^t g(s) ds \right) \int_{\Gamma_1} \int_0^t g(t-s) |u(t) - u(s)|^2 ds d\Gamma
\]
On the other hand, we can choose which yields, for some constant $m > 0$,

$$N_1 \geq \varepsilon,$$

and

$$- \int_{\Gamma} u^2(t) + \int_{0}^{t} g'(t-s)(u(t) - u(s))dsdx$$

$$\leq \varepsilon \int_{\Omega} u^2dx + \frac{1}{4\varepsilon} \int_{\Omega} \left( \int_{0}^{t} g'(t-s)(u(t) - u(s))ds \right)^2 dx$$

$$\leq \varepsilon \int_{\Omega} u^2dx + \frac{1}{4\varepsilon} \left( \int_{0}^{t} g'(s)ds \right) \int_{\Omega} \left( \int_{0}^{t} g'(t-s) |u(t) - u(s)|^2 dsdx \right)$$

$$\leq \varepsilon \int_{\Omega} u^2dx - \frac{c}{\varepsilon} g' \circ u.$$

By combining all the above estimates, the assertion of Lemma 3.3 is proved. \hfill \Box

**Proof of Theorem 2.1.** For $N_1, N_2 > 1$, let

$$\mathcal{L}(t) := N_1 E(t) + N_2 K(t) + I(t)$$

and $g_1 = \int_{0}^{t} g(s)ds > 0$ where $1 > t_1 > 0$ was introduced in (2.10). By combining (3.1)-(3.3), and taking $\varepsilon = l/(4N_2)$, we obtain, for all $t \geq t_1$,

$$\mathcal{L}'(t) \leq -\frac{l}{4} a(u, u) - (N_2 g_1 - \frac{1}{4} - 1) \int_{\Omega} u^2dx$$

$$+ \left( \frac{4c}{\varepsilon} N_2^2 + c \right) (g \circ u)(t) + \left( \frac{1}{2} N_1 - \frac{4c}{\varepsilon} N_2^2 \right) (g' \circ u)(t) + (cN_2 + c) \int_{\Gamma} h^2(u_t)dt.$$ 

At this point, we choose $N_2$ large enough so that

$$\gamma := N_2 g_1 - \frac{1}{4} - 1 > 0,$$

then $N_1$ large enough so that

$$\frac{1}{2} N_1 - \frac{4c}{l} N_2^2 > 0.$$

So, we arrive at

$$\mathcal{L}'(t) \leq -\frac{l}{4} a(u, u) - \gamma \int_{\Omega} u^2dx + c(g \circ u)(t) + c \int_{\Gamma} h^2(u_t)dt,$$

which yields, for some constant $m > 0$,

$$\mathcal{L}'(t) \leq -m E(t) + c(g \circ u)(t) + c \int_{\Gamma} h^2(u_t)dt, \quad \forall t \geq t_1. \quad (3.4)$$

On the other hand, we can choose $N_1$ even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t) \quad (3.5)$$

which means that, for some constants $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t).$$

Now, we consider the following partition of $\Gamma_1$

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t| \leq r\}, \quad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| > r\}.$$
and use (A1), (A3), (2.11) and (3.1) to conclude that, for any $t \geq t_1$,
\[
\theta(t) \int_0^{t_1} g(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds + \theta(t) \int_{\Gamma_{12}} h^2(u_\epsilon) d\Gamma \\
\leq -\frac{\delta}{2} \int_0^{t_1} \dot{g}(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds + c\theta(t) \int_{\Gamma_{12}} u_\epsilon h(u_\epsilon) d\Gamma \\
\leq -cE'(t).
\]
Next, we take $F(t) = \theta(t) L(t) + cE(t)$, which is clearly equivalent to $E(t)$ as $\theta$ is nonincreasing, and use (3.4) and (3.6), to get, for all $t \geq t_1$,
\[
F'(t) \leq -m\theta(t) E(t) + c\theta(t) \int_{t_1}^t g(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds \\
+ c\theta(t) \int_{\Gamma_{11}} h^2(u_\epsilon) d\Gamma, \quad \forall \ t \geq t_1.
\]
(1) $H(t) = ct^p$ and $1 \leq p < \frac{3}{2}$: This means, using Holder’s inequality, that
\[
\theta \int_{\Gamma_{11}} h^2(u_\epsilon) d\Gamma \leq c\theta \int_{\Gamma_{11}} [u_\epsilon h(u_\epsilon)]^{\frac{2}{p}} d\Gamma \leq c\theta \left[ \int_{\Gamma_{11}} u_\epsilon h(u_\epsilon) d\Gamma \right]^{\frac{1}{p}} \\
\leq c\theta^{\frac{p-1}{p}} [-E'(t)]^{\frac{1}{p}}.
\]
- **Case 1.** $p = 1$: Estimate (3.7) yields
\[
F'(t) \leq -m\theta(t) E(t) - c\theta(t) (g' \circ \alpha)(t) - cE'(t) \\
\leq -m\theta(t) E(t) - cE'(t), \quad \forall \ t \geq t_1
\]
which gives, as $J = F + cE \sim E$,
\[
J'(t) \leq -k\theta(t)J(t), \quad \forall \ t \geq t_1
\]
where $k$ is a positive constant. Thus,
\[
J(t) \leq J(t_1)e^{-k \int_{t_1}^t \theta(s) ds}, \quad \forall \ t \geq t_1.
\]
Hence, using the fact that $J \sim E$, we easily obtain
\[
E(t) \leq ce^{-k \int_{t_1}^t \theta(s) ds} = cG_1^{-1}(c' \int_0^t \theta(s) ds).
\]
- **Case 2.** $1 < p < \frac{3}{2}$: One can easily show that $\int_0^{+\infty} g^{1-\delta_0}(s)ds < +\infty$ for any $\delta_0 < 2 - p$. Using this fact and (3.1) and choosing $t_1$ even larger if needed, we deduce that, for all $t \geq t_1$,
\[
\eta(t) := \int_{t_1}^t g^{1-\delta_0}(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds \\
\leq c \int_{t_1}^t g^{1-\delta_0}(s)[\alpha(u(t), u(t)) + \alpha(u(t - s), u(t - s))]ds \leq cE(0) \int_{t_1}^t g^{1-\delta_0}(s)ds < 1.
\]
Then, Jensen’s inequality, (3.1), hypothesis (A2), and (3.8) lead to
\[
\int_{t_1}^t g(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds \\
= \int_{t_1}^t g^{\delta_0}(s)g^{1-\delta_0}(s)\alpha(u(t) - u(t - s), u(t) - u(t - s))ds
\]
Then, Young’s inequality gives

\[ \int_0^t g^{(p-1+\delta_0)}(s) g^{1-\delta_0}(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \leq \eta(t) \left[ \frac{1}{\eta(t)} \int_0^t g(s)^{p-1+\delta_0} g^{1-\delta_0}(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \]

\[ \leq \left[ \int_0^t g(s)^p a(u(t) - u(t-s), u(t) - u(t-s)) ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \]

\[ \leq c \left[ \int_0^t g'(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \leq c \left[ -E'(t) \right]^{\frac{\delta_0}{p-1+\delta_0}}. \]

Then, particularly for \( \delta_0 = \frac{1}{2} \), we find that (3.7) becomes

\[ F'(t) \leq -m \theta E(t) + c \theta \left[ -E'(t) \right]^{\frac{1}{p-1}} + c \theta^{\frac{p-1}{p}} \left[ -E'(t) \right]^\frac{1}{p} \]

Now, we multiply by \( E^{2p-2}(t) \) to get, using (3.1),

\[ (FE^{2p-2})' \leq F' E^{2p-2} \leq -m \theta E^{2p-1} + c \theta E^{2p-2} \left[ -E' \right]^{\frac{1}{p-1}} + c \theta^{\frac{p-1}{p}} E^{2p-2} \left[ -E' \right]^\frac{1}{p}. \]

Then, Young’s inequality gives

\[ (FE^{2p-2})' \leq -m \theta E^{2p-1}(t) + \varepsilon \theta E^{2p-1}(t) + C_\varepsilon \theta(-E'(t)) + \delta \theta E^{2p} + C_\delta (-E'(t)). \]

Consequently, as \( E^{2p}(t) \leq E(0) E^{2p-1}(t) \), picking \( \varepsilon + \delta E(0) < m \), we obtain

\[ F_0'(t) \leq -m' \theta(t) E^{2p-1}(t) \]

where \( F_0 = FE^{2p-2} + CE \sim E \). Hence we have, for some \( a_0 > 0 \),

\[ F_0'(t) \leq -a_0 \theta(t) F_0^{2p-1}(t) \]

from which we easily deduce that

\[ E(t) \leq \frac{a}{\left( a' \int_0^t \theta(s) ds + a'' \right)^{\frac{1}{p-2}}} \]  \hspace{1cm} (3.9) \]

By recalling that \( p < 3/2 \) and using (3.9), we find that \( \int_0^{+\infty} \theta(t) E(t) dt < +\infty \).

Hence, by noting that

\[ \theta(t) \int_0^t a(u(t) - u(t-s), u(t) - u(t-s)) ds \leq c \int_0^t \theta(s) E(s) ds, \]

we get

\[ \theta(t) \int_0^t g(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \]

\[ = \theta(t) \left( g^p \frac{1}{p} \circ u \right)(t) \leq \theta \left( g^p \frac{1}{p} \circ u \right)(t) \]

\[ \leq c \theta^{\frac{p-1}{p}}(t) \left[ -E'(t) \right]^\frac{1}{p}. \]

So, estimate (3.7) becomes

\[ F'(t) \leq -m \theta(t) E(t) + c \theta^{\frac{p-1}{p}}(t) \left[ -E'(t) \right]^\frac{1}{p}. \]

Therefore, repeating the above steps, with multiplying by \( E^{p-1}(t) \), we arrive at

\[ E(t) \leq \frac{a}{\left( a' \int_0^t \theta(s) ds + a'' \right)^{\frac{1}{p-1}}} = c G^{-1}(t) \int_0^t \theta(s) ds + c'' \text{.} \]
(II) The general case: We define $I(t)$ by

$$I(t) := \int_{t_1}^t \frac{g(s)}{H_0^{-1}(-g'(s))} a(u(t) - u(t - s), u(t) - u(t - s)) \, ds$$

where $H_0$ is such that (2.8) is satisfied. As in (3.8), we find that $I(t)$ satisfies, for all $t \geq t_1$,

$$I(t) \leq 1. \quad (3.10)$$

We also assume, without loss of generality that $I(t) > 0$, for all $t \geq t_1$; otherwise (3.10) yields an exponential decay. In addition, we define $\lambda(t)$ by

$$\lambda(t) := -\int_{t_1}^t g'(s) \frac{g(s)}{H_0^{-1}(-g'(s))} a(u(t) - u(t - s), u(t) - u(t - s)) \, ds$$

and infer from (A2) and the properties of $H_0$ and $D$ that

$$\frac{g(s)}{H_0^{-1}(-g'(s))} \leq \frac{g(s)}{H_0^{-1}(H(g(s)))} \leq k_0$$

for some positive constant $k_0$. Then, using (3.1) and choosing $t_1$ even larger (if needed), one can easily see that $\lambda(t)$ satisfies, for all $t \geq t_1$,

$$\lambda(t) \leq -k_0 \int_{t_1}^t g'(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds \leq -cE(0) \int_{t_1}^t g'(s) \leq cg(t_1)E(0)$$

$$< \frac{1}{2} \min\{r, H(r), H_0(r)\}. \quad (3.11)$$

Since $H_0$ is strictly convex on $(0, r]$ and $H_0(0) = 0$, then $H_0(\mu x) \leq \mu H_0(x)$, provided $0 \leq \mu \leq 1$ and $x \in (0, r]$. The use of this fact, hypothesis (A2), (2.10), (3.10), (3.11), and Jensen’s inequality leads to

$$\lambda(t) = \frac{1}{I(t)} \int_{t_1}^t I(t)H_0[H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} a(u(t) - u(t - s), u(t) - u(t - s)) \, ds$$

$$\geq \frac{1}{I(t)} \int_{t_1}^t H_0[H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} a(u(t) - u(t - s), u(t) - u(t - s)) \, ds$$

$$\geq H_0 \left( \frac{1}{I(t)} \int_{t_1}^t I(t)H_0^{-1}(-g'(s)) \frac{g(s)}{H_0^{-1}(-g'(s))} a(u(t) - u(t - s), u(t) - u(t - s)) \, ds \right)$$

This implies that

$$\int_{t_1}^t g(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds \leq H_0^{-1}(\lambda(t)). \quad (3.12)$$

Now we estimate the last integral in (3.7). First, we can assume that $r$ is small enough such that

$$sh(s) \leq \frac{1}{2} \min\{r, H(r), H_0(r)\} \quad \text{for all } |s| \leq r. \quad (3.13)$$

Then, with $S(t)$ defined by

$$S(t) := \frac{1}{|\Gamma_{11}|} \int_{\Gamma_{11}} u_t h(u_t) \, d\Gamma,$$
(A3) and Jensen’s inequality give
\[ H^{-1}(S(t)) \geq c \int_{\Gamma_{11}} H^{-1}(u_t h(u_t))d\Gamma \geq c \int_{\Gamma_{11}} h^2(u_t)d\Gamma. \]  
(3.14)

Inserting the estimates (3.12) and (3.14) into (3.7), we obtain
\[ F'(t) \leq -m\theta(t) E(t) + c\theta(t) H_0^{-1}(\lambda(t)) + c\theta(t) H^{-1}(S(t)), \quad \forall \ t \geq t_1. \]

One can easily make use of the properties of \( H, D, H_0 \) and the fact that \( H_0^{-1}(S(t)) = D^{-1}(H^{-1}(S(t))) \), \( D^{-1}(0) = 0 \), and \( H^{-1}(S(t)) \leq r \) to deduce, for some positive constant \( c' \), that \( H^{-1}(S(t)) \leq cH_0^{-1}(S(t)) \). Therefore
\[ F'(t) \leq -m\theta(t) E(t) + c\theta(t) H_0^{-1}(\lambda(t)) + c\theta(t) H_0^{-1}(S(t)) \leq -m\theta(t) E(t) + c\theta(t) H_0^{-1}(\lambda(t) + S(t)). \]  
(3.15)

Now, for \( \varepsilon_0 < r \) and \( c_0 > 0 \), using (3.15) and the fact that \( E' \leq 0, H_0', H_0'' > 0 \) on \( (0, r] \), we find that the functional \( R_1 \), defined by
\[ R_1(t) := H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) F(t) + c_0 E(t), \]
satisfies, for some \( a_1, a_2 > 0 \),
\[ a_1 R_1(t) \leq E(t) \leq a_2 R_1(t) \]  
(3.16)

and, for all \( t \geq t_1 \),
\[ R_1'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) F(t) + H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) F'(t) + c_0 E'(t) \leq -m\theta(t) E(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\theta(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) H_0^{-1}(\lambda(t) + S(t)) + c_0 E'(t). \]  
(3.17)

Let \( H_0^* \) be the convex conjugate of \( H_0 \) in the sense of Young (see [6] p. 61-64), then
\[ H_0^*(s) = s(H_0'^{-1}(s) - H_0'[H_0'^{-1}(s)], \quad \text{if} \ s \in (0, H_0'(r)] \]  
(3.18)

and \( H_0^* \) satisfies the following Young’s inequality
\[ AB \leq H_0^*(A) + H_0(B), \quad \text{if} \ A \in (0, H_0'(r)], B \in (0, r]. \]  
(3.19)

With \( A = H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) \) and \( B = H_0^{-1}(\lambda(t) + S(t)) \), using (3.1), (3.11), (3.13) and (3.17)-(3.19), we arrive at
\[ R_1'(t) \leq -m\theta(t) E(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\theta(t) H_0^*(\varepsilon_0 \frac{E(t)}{E(0)}) \leq -m\theta(t) E(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\theta(t) E(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) E'(t) + c_0 E'(t). \]

Consequently, with a suitable choice of \( \varepsilon_0 \) and \( c_0 \), we obtain
\[ R_1'(t) \leq -k\theta(t) \left( \frac{E(t)}{E(0)} \right) H_0^*(\varepsilon_0 \frac{E(t)}{E(0)}) = -k\theta(t) H_2(\frac{E(t)}{E(0)}), \quad \forall t \geq t_1, \]  
(3.20)
where \( H_2(t) = tH'_0(\varepsilon_0 t) \).

Since

\[
H'_2(t) = H'_0(\varepsilon_0 t) + \varepsilon_0 tH''_0(\varepsilon_0 t)
\]

then, using the strict convexity of \( H_0 \) on \((0, r]\), we find that \( H'_2(t), H_2(t) > 0 \) on \((0, 1]\). Thus, with \( R(t) = \frac{\alpha_1 R_1(t)}{E(0)} \) and using (3.16) and (3.20), we have

\[
R(t) \sim E(t)
\]

and, for some \( k_1 > 0 \),

\[
R'(t) \leq -k_1 \theta(t) H_2(R(t)), \quad \forall t \geq t_1.
\]

Considering \( H_1(t) = \int_t^1 \frac{1}{\Pi_2(s)} ds \), we deduce that \((H_1(R))'(t) > 0, \forall t \geq t_1\), which implies that \( H_1(R(t)), t \geq t_1 \), is increasing. Thus,

\[
k_1 \int_{t_1}^t \theta(s) ds \leq \int_{t_1}^t (H_1(R))'(s) ds \leq H_1(R(t)) - H_1(R(t_1)),
\]

and so, for some \( k_2 > 0 \),

\[
R(t) \leq H_1^{-1}(k_1 \int_{t_1}^t \theta(s)ds + k_2), \quad \forall t \geq t_1.
\]

Here, we used, based on the properties of \( H_2 \), the fact that \( H_1 \) is strictly decreasing on \((0, 1]\). Using (3.21)-(3.22) and by virtue of continuity and boundedness of \( E \) and \( \theta \), we obtain (2.7).

Moreover, if \( \int_0^1 H_1(t) dt < +\infty \), then \( \int_0^{+\infty} H_1^{-1}(t)dt < +\infty \), and so, by (2.7), \( \int_0^{+\infty} E(t) dt < +\infty \). Then,

\[
\int_{t_1}^t a(u(t) - u(t - s), u(t) - u(t - s)) ds \leq c \int_0^t E(s) ds < +\infty.
\]

Therefore, we can repeat the same procedures with

\[
I(t) := \int_{t_1}^t a(u(t) - u(t - s), u(t) - u(t - s)) ds,
\]

and

\[
\lambda(t) := -\int_{t_1}^t g'(s)a(u(t) - u(t - s), u(t) - u(t - s)) ds,
\]

to establish (2.9). □

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