ON THE CAUCHY PROBLEM FOR
GROSS-PITAEVSKII HIERARCHIES

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Abstract. The purpose of this paper is to investigate the Cauchy problem for the Gross-Pitaevskii infinite linear hierarchy of equations on $\mathbb{R}^n$, $n \geq 1$. We prove local existence and uniqueness of solutions in certain Sobolev type spaces $H^\alpha_\xi$ of sequences of marginal density operators with $\alpha > n/2$. In particular, we give a clear discussion of all cases $\alpha > n/2$, which covers the local well-posedness problem for Gross-Pitaevskii hierarchy in this situation.

1. Introduction

Motivated by recent experimental realizations of Bose-Einstein condensation the theory of dilute, inhomogeneous Bose systems is currently a subject of intensive studies in physics [5]. The ground state of bosonic atoms in a trap has been shown experimentally to display Bose-Einstein condensation (BEC). This fact is proved theoretically by Lieb et al [15, 16, 17] for bosons with two-body repulsive interaction potentials in the dilute limit, starting from the basic Schrödinger equation. On the other hand, it is well known that the dynamics of Bose-Einstein condensates are well described by the Gross-Pitaevskii equation [12, 13, 18]. A rigorous derivation of this equation from the basic many-body Schrödinger equation in an appropriate limit is not a simple matter, however, and has only achieved recently in three spatial dimensions by Elgart, Erdős, Schlein and Yau [6, 7, 8, 9, 10, 11], based on the notion of so-called Gross-Pitaevskii hierarchies. In their program an important step is to prove uniqueness to the Gross-Pitaevskii hierarchy via Feynman graph (see [8, 14]).

Recently, T.Chen and N.Pavlović [2] started to investigate the Cauchy problem for the Gross-Pitaevskii hierarchy, using a Picard-type fixed point argument. In the present paper, we continue this line of investigation. We will prove local existence and uniqueness of solutions in certain Sobolev type spaces $H^\alpha_\xi$ (for definition see Section 2 below) of sequences of marginal density operators with $\alpha > n/2$. Instead of using a fixed point principle as in [2], here we use the fully expanded iterated Duhamel series, and a
Cauchy convergence criterion, without additional conditions on any space-time norms. The assumption of $\alpha > n/2$ allows us to significantly simplify the approaches and provide an improvement for the previous work [2] in all cases $\alpha > n/2$. Our proof involves the simple property that the interaction operators $B^{(k)}$ are bounded maps from the $k+1$-particle Hilbert space $H^n_{k+1}$ to the $k$-particle Hilbert space $H^n_k$ in the cubic case. A case of this type has previously been presented by Chen-Pavlović [4] in their derivation of the quintic NLS for $n = 1$. In the much more difficult situation $\alpha \leq n/2$, as done recently in [3], it is necessary to invoke the Strichartz estimates of the type introduced in the pioneering work of Klainerman-Machedon [14].

The paper is organized as follows. In Section 2, some notations and the main result are presented. Section 3 is devoted to present elementary estimates which will be used later. In particular, we will prove the fact that the interaction operators $B^{(k)}$ are bounded maps from the $k+1$-particle Hilbert space $H^n_{k+1}$ to the $k$-particle Hilbert space $H^n_k$ in the cubic case for $\alpha > n/2$. The proof is completely analogous to that of the classical Sobolev inequality $\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^n(\mathbb{R}^n)}$. In section 4, the main result is proved. Finally, in Section 5, we discuss the so-called quintic Gross-Pitaevskii hierarchy and extend the result obtained in the previous sections to that case.

2. Preliminaries and statement of the main result

2.1. Gross-Pitaevskii hierarchies. As follows, we denote by $x$ a general variable in $\mathbb{R}^n$ and by $x = (x_1, \ldots, x_N)$ a point in $\mathbb{R}^{Nn}$. We will also use the notation $x_k = (x_1, \ldots, x_k) \in \mathbb{R}^{kn}$ and $x_{N-k} = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{(N-k)n}$. For a function $f$ on $\mathbb{R}^{kn}$ we let

$$(\Theta_{\sigma} f)(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

for any permutation $\sigma \in \Pi_k$ ($\Pi_k$ denotes the set of permutations on $k$ elements). Then, each $\Theta_{\sigma}$ is a unitary operator on $L^2(\mathbb{R}^{kn})$. A bounded operator $A$ on $L^2(\mathbb{R}^{kn})$ is called $k$-partite symmetric or simply symmetric if

$$\Theta_{\sigma} A \Theta_{\sigma^{-1}} = A$$

for every $\sigma \in \Pi_k$. Evidently, a density operator $\gamma^{(k)}$ on $L^2(\mathbb{R}^{kn})$ (i.e., $\gamma^{(k)} \geq 0$ and $\text{tr} \gamma^{(k)} = 1$) with the kernel function $\gamma^{(k)}(x_k; x'_k)$ is $k$-partite symmetric if and only if

$$\gamma^{(k)}(x_1, \ldots, x_k; x'_1, \ldots, x'_k) = \gamma^{(k)}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x'_{\sigma(1)}, \ldots, x'_{\sigma(k)})$$

for any $\sigma \in \Pi_k$.

Also, we set

$$L^2_\sigma(\mathbb{R}^{kn}) = \{ f \in L^2(\mathbb{R}^{kn}) : \Theta_{\sigma} f = f, \forall \sigma \in \Pi_k \},$$

equipped with the inner product of $L^2(\mathbb{R}^{kn})$. Clearly, $L^2_\sigma(\mathbb{R}^{kn})$ is a Hilbert subspace of $L^2(\mathbb{R}^{kn})$. It is easy to check that any $k$-partite symmetric operator on $L^2(\mathbb{R}^{kn})$ preserves $L^2_\sigma(\mathbb{R}^{kn})$. 
Definition 2.1. Given $n \geq 1$, the $n$-dimensional Gross-Pitaevskii (GP) hierarchy refers to a sequence $\{\gamma_t^{(k)}\}_{k \geq 1}$ of $k$-partite symmetric density operators on $L^2(\mathbb{R}^k)$, where $t \geq 0$, which satisfy the Gross-Pitaevskii infinite linear hierarchy of equations,

\begin{equation}
(2.2) \quad i\partial_t \gamma_t^{(k)} = [\Delta^{(k)}(x), \gamma_t^{(k)}] + \mu B^{(k)} \gamma_t^{(k+1)}, \quad \Delta^{(k)} = \sum_{j=1}^k \Delta_{x_j}, \quad \mu = \pm 1,
\end{equation}

with initial conditions

$$
\gamma_{t=0}^{(k)} = \gamma_0^{(k)}, \quad k = 1, 2, \ldots .
$$

Here, $\Delta_{x_j}$ refers to the usual Laplace operator with respect to the variables $x_j \in \mathbb{R}^n$ and the operator $B^{(k)}$ is defined by

$$
B^{(k)} \gamma_t^{(k+1)} = \sum_{j=1}^k \text{tr}_{k+1} \left[ \delta(x_j - x_{k+1}), \gamma_t^{(k+1)} \right]
$$

where the notation $\text{tr}_{k+1}$ indicates that the trace is taken over the $(k+1)$-th variable.

As in [2], we refer to (2.2) as the cubic GP hierarchy. For $\mu = 1$ or $\mu = -1$ we refer to the corresponding GP hierarchies as being defocusing or focusing, respectively. We note that the cubic Gross–Pitaevskii hierarchy accounts for two-body interactions between the Bose particles (e.g., see [5, 10] and references therein for details).

Remark 2.1. In terms of the kernel functions $\gamma_t^{(k)}(x_k; x'_k)$, we can rewrite (2.2) as follows:

\begin{equation}
(2.3) \quad i\partial_t + \Delta^{(k)}_{\pm} \gamma_t^{(k)}(x_k; x'_k) = \mu \left[ B^{(k)} \gamma_t^{(k+1)} \right](x_k; x'_k),
\end{equation}

where $\Delta^{(k)}_{\pm} = \sum_{j=1}^k (\Delta_{x_j} - \Delta_{x'_j})$, with initial conditions

$$
\gamma_{t=0}^{(k)}(x_k; x'_k) = \gamma_0^{(k)}(x_k; x'_k), \quad k = 1, 2, \ldots .
$$

In particular, the action of $B^{(k)}$ on density operators with smooth kernel functions, $\gamma^{(k+1)}(x_{k+1}; x'_{k+1}) \in \mathcal{S}(\mathbb{R}^{k+1} \times \mathbb{R}^{k+1})$, is given by

\begin{equation}
(2.4) \quad \left[ B^{(k)} \gamma_t^{(k+1)} \right](x_k; x'_k) = \sum_{j=1}^k \int dx_{k+1} dx'_{k+1} \gamma_t^{(k+1)}(x_k, x_{k+1}; x'_k, x'_{k+1})
\end{equation}

$$
\times \delta(x'_j - x_{k+1}) \left[ \delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1}) \right].
$$

The action of $B^{(k)}$ can be extended to generic density operators. This will be made precise in Lemma 3.2.
Remark 2.2. Let $\varphi \in H^1(\mathbb{R}^n)$, then one can easily verify that a particular solution to (2.3) with initial conditions

$$\gamma_{t=0}^{(k)}(x_k; x'_k) = \prod_{j=1}^{k} \varphi(x_j)\varphi(x'_j), \quad k = 1, 2, \ldots,$$

is given by

$$\gamma_t^{(k)}(x_k; x'_k) = \prod_{j=1}^{k} \varphi_t(x_j)\varphi_t(x'_j) \quad k = 1, 2, \ldots,$$

where $\varphi_t$ satisfies the cubic non-linear Schrödinger equation

$$(2.5) \quad i\partial_t \varphi_t = -\Delta \varphi_t + \mu |\varphi_t|^2 \varphi_t, \quad \varphi_{t=0} = \varphi,$$

which is defocusing if $\mu = 1$, and focusing if $\mu = -1$.

The Gross-Pitaevskii hierarchy (2.2) can be written in the integral form

$$(2.6) \quad \gamma_t^{(k)} = U_0^{(k)}(t)\gamma_0^{(k)} + \int_0^t ds \, U_0^{(k)}(t-s)\tilde{B}^{(k)} \gamma_s^{(k+1)}, \quad k = 1, 2, \ldots,$$

where $\tilde{B}^{(k)} = -i\mu B^{(k)}$. Hereafter, the free evolution operator is defined by

$$U_0^{(k)}(t)A = \exp(i t\Delta^{(k)}) A \exp(-i t\Delta^{(k)}), \quad k = 1, 2, \ldots,$$

for every operator $A$ on $L^2(\mathbb{R}^{kn})$. The action of $U_0^{(k)}(t)$ on kernel functions $\gamma^{(k)} \in L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$ is given by

$$(2.7) \quad U_0^{(k)}(t)\gamma^{(k)}(x_k, x'_k) = e^{-it\Delta^{(k)}} \gamma^{(k)}(x_k, x'_k).$$

Formally we can expand the solution $\gamma_t^{(k)}$ of (2.6) for any $m \geq 1$ as

$$(2.8) \quad \gamma_t^{(k)} = U_0^{(k)}(t)\gamma_0^{(k)} + \sum_{j=1}^{m-1} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j U_0^{(k)}(t-s_1)\tilde{B}^{(k)} \cdots \times U_0^{(k+j-1)}(s_{j-1} - s_j) \tilde{B}^{(k+j-1)} U_0^{(k+j)}(s_j) \gamma_{s_j}^{(k+j)}$$

$$+ \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m U_0^{(k)}(t-s_1)\tilde{B}^{(k)} \cdots \times U_0^{(k+m-1)}(s_{m-1} - s_m) \tilde{B}^{(k+m-1)} \gamma_{s_m}^{(k+m)},$$

with the convention $s_0 = t$. The terms in the summation contain only the initial data. The last error term involves the density operator at an intermediate time $s_m$. 
2.2. **Statement of the main result.** In order to state our main results, we require some more notation. We will use \( \gamma^{(k)}, \rho^{(k)} \) for denoting either (density) operators or kernel functions. For \( k \geq 1 \) and \( \alpha > 0 \), we denote by \( H^\alpha_k = \mathcal{H}^{\alpha}(\mathbb{R}^{kn} \times \mathbb{R}^{kn}) \) the space of measurable functions \( \gamma^{(k)} = \gamma^{(k)}(x_k, x'_k) \) in \( L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn}) \) such that
\[
\| \gamma^{(k)} \|_{H^\alpha_k} := \| S^{(k,\alpha)} \gamma^{(k)} \|_{L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})} < \infty,
\]
where
\[
S^{(k,\alpha)} := \prod_{j=1}^{k} [(1 - \Delta_{x_j}) \mathcal{E} (1 - \Delta_{x'_j}) \mathcal{F}].
\]
Evidently, \( H^\alpha_k \) is a Hilbert space with the inner product
\[
\langle \gamma^{(k)}, \rho^{(k)} \rangle = \langle S^{(k,\alpha)} \gamma^{(k)}, S^{(k,\alpha)} \rho^{(k)} \rangle_{L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})}.
\]
Moreover, the norm \( \| \cdot \|_{H^\alpha_k} \) is invariance under the action of \( U_0^{(k)}(t) \), that is,
\[
\| U_0^{(k)}(t) \gamma^{(k)} \|_{H^\alpha_k} = \| \gamma^{(k)} \|_{H^\alpha_k}
\]
because \( \exp \{ \pm i t \Delta^{(k)} \} \) commute with \( \Delta_{x_j} \) for any \( j \).

Let \( 0 < \xi < 1 \) and \( \alpha > 0 \), we define
\[
H^\alpha = \left\{ \Gamma = \{ \gamma^{(k)} \}_{k \geq 1} \in \bigotimes_{k=1}^{\infty} H^\alpha_k : \| \Gamma \|_{H^\alpha} := \sum_{k=1}^{\infty} \xi^k \| \gamma^{(k)} \|_{H^\alpha_k} < \infty \right\}.
\]
Evidently, \( H^\alpha \) is a Banach space equipped with the norm \( \| \cdot \|_{H^\alpha} \), which is introduced in [2]. We remark that similar spaces are used in the isospectral renormalization group analysis of spectral problems in quantum field theory (see [1]).

**Definition 2.2.** For \( T > 0 \), \( \Gamma_0 = \{ \gamma^{(k)}_0 \}_{k \geq 1} \in C([0, T], H^\alpha) \) is said to be a local (mild) solution to the Gross-Pitaevskii hierarchy (2.2) if for every \( k = 1, 2, \ldots \),
\[
\gamma^{(k)}_t = U_0^{(k)}(t) \gamma^{(k)}_0 + \int_0^t ds U_0^{(k)}(t - s) \tilde{B}^{(k)} \gamma^{(k+1)}_s \quad \forall t \in [0, T],
\]
holds in \( H^\alpha_k \).

Our main result in this paper is the following theorem.

**Theorem 2.1.** Assume that \( n \geq 1 \) and \( \alpha > n/2 \). Suppose \( \Gamma_0 = \{ \gamma^{(k)}_0 \}_{k \geq 1} \in H^\alpha \) for some \( 0 < \xi < 1 \). Then there exists a constant \( C = C_{\alpha,n} \) depending only on \( n \) and \( \alpha \) such that, for a fixed \( 0 < T < \xi/C \) with \( \eta = \xi - CT \), the following hold.

(i) There exists a solution \( \Gamma_t = \{ \gamma^{(k)}_t \}_{k \geq 1} \in C([0, T], H^\alpha) \) to the Gross-Pitaevskii hierarchy (2.2) with the initial data \( \Gamma_0 \) satisfying
\[
\| \Gamma_t \|_{C([0, T], H^\alpha)} \leq \frac{\eta}{C}\| \Gamma_0 \|_{H^\alpha}.
\]
(ii) For $T = \xi/(5C)$, if $\Gamma$ and $\Gamma'$ in $C([0, T], H^\alpha_{\eta})$ are two solutions to (2.2) with initial conditions $\Gamma_{t=0} = \Gamma_0$ and $\Gamma'_{t=0} = \Gamma'_0$ in $H^\alpha_{\xi}$ respectively, then

\begin{equation}
\|\Gamma_t - \Gamma'_t\|_{C([0,T], H^\alpha_{\eta})} \leq \frac{4}{5}\|\Gamma_0 - \Gamma'_0\|_{H^\alpha_{\xi}}.
\end{equation}

Consequently, the solution $\Gamma_t$ to the initial problem (2.2) with the initial data in $H^\alpha_{\xi}$ is unique in $C([0, T], H^\alpha_{\eta})$ for any $0 < T < \xi/C$.

**Remark 2.3.** We will prove this theorem by the method of infinitely iterating the Duhamel series, and proving Cauchy convergence without additional conditions on spacetime bounds, and however, our argument won’t work if $\alpha \leq n/2$. The assumption of $\alpha > n/2$ allows us to significantly simplify the approaches in the previous work [2] and improve the corresponding one. In the much more difficult situation $\alpha \leq \frac{n}{2}$, as done recently in [3], it is necessary to invoke the Strichartz estimates of the type introduced in the pioneering work of Klainerman-Machedon [14].

## 3. Preliminary estimates

In the sequel, we will mostly work in Fourier (momentum) space. Following [8], we use the convention that variables $p, q, r, p', q', r'$ always refer to $n$ dimensional Fourier variables, while $x, x', y, y', z, z'$ denote the position space variables. With this convention, the usual hat indicating the Fourier transform will be omitted. For example, for $k \geq 1$ the kernel of a bounded operator $A$ on $L^2(\mathbb{R}^kn)$ in position space is $K(x_k; x'_k)$, then in the momentum space it is given by the Fourier transform

\begin{equation}
K(q_k; q'_k) = \langle K, e^{-i\langle \cdot, q_k \rangle} e^{i\langle \cdot, q'_k \rangle} \rangle = \int dx_k dx'_k e^{i\langle x_k; q_k \rangle} e^{-i\langle x_k; q'_k \rangle},
\end{equation}

with the slight abuse of notation of omitting the hat on left hand side. Here,

\begin{equation}
\langle x_k, q_k \rangle = \sum_{j=1}^{k} x_j \cdot q_j, \quad \forall x_k = (x_1, \ldots, x_k), q_k = (q_1, \ldots, q_k) \in \mathbb{R}^kn.
\end{equation}
Thus, on kernels in the momentum space $B^{(k)}$ in (2.4) acts according to

$$
[B^{(k)}]_{(k+1)}(p_k' ; p_k') = \sum_{j=1}^{k} \int dq_{k+1} dq_{k+1}' \times \left \{ \begin{array}{l}
\gamma^{(k+1)}(p_1, \ldots, p_j - q_{k+1} + q_{k+1}', \ldots, p_k, q_{k+1}; \vec{p}_k', q_k') \\
- \gamma^{(k+1)}(p_k, q_{k+1}; p'_1, \ldots, p'_j + q_{k+1} - q_{k+1}', \ldots, p'_k, q_{k+1}') \end{array} \right \}
$$

(3.1)

$$
= \sum_{j=1}^{k} \int dq_{k+1} dq_{k+1}' \left [ \prod_{l \neq j}^{k} \delta(p_l - q_l)\delta(p_l' - q_l') \right ] \\
\times \gamma^{(k+1)}(q_{k+1}; q_{k+1}') \left \{ \delta(p_j' - q_j')\delta(p_j - [q_j + q_{k+1} - q_{k+1}']) \\
- \delta(p_j - q_j)\delta(p_j' - [q_j' + q_{k+1}' - q_{k+1}']) \right \}.
$$

We begin with the following simple lemma.

**Lemma 3.1.** If $\beta > n$, then

$$
\sup_{p \in \mathbb{R}^n} \int_{\mathbb{R}^n} dq dq' \frac{(p)\beta}{(p+q-q')\beta(q)\beta(q')} < \infty.
$$

**Proof.** Since

$$
\frac{1}{(p+q-q')\beta(q)\beta(q')} \leq \frac{2^\beta}{(p+q')\beta(q')\beta} \left ( \frac{1}{(p+q-q')\beta} + \frac{1}{q} \right ) \\
\leq \frac{2^\beta}{(p)\beta} \left ( \frac{1}{(p+q')\beta} + \frac{1}{q'} \right ) \left ( \frac{1}{(p+q-q')\beta} + \frac{1}{q} \right )
$$

the inequality (3.2) is concluded from the assumption $\beta > n$. \qed

As in [14], we introduce for $\gamma^{(k+1)}(x_{k+1}, x_{k+1}') \in S(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n})$,

$$
[B^{(k)}_{j,k}]_{(k+1)}(x_k, x_k') = \int dx_{k+1} dx'_{k+1} \delta(x_{k+1} - x'_{k+1}) \delta(x_j - x_{k+1}) \gamma^{(k+1)}(x_{k+1}, x_{k+1}) \\
\text{and}
$$

$$
[B^{(k)}_{j,k}]_{(k+1)}^2(x_k, x_k') = \int dx_{k+1} dx'_{k+1} \delta(x_{k+1} - x'_{k+1}) \delta(x_j - x_{k+1}) \gamma^{(k+1)}(x_{k+1}, x_{k+1})
$$
where \( j = 1, \ldots, k \). Then, by (2.4) we have

\[
B^{(k)} = \sum_{j=1}^{k} (B_{j,k}^1 - B_{j,k}^2)
\]

acting on smooth kernel functions \( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n}) \).

The following estimate is crucial for the proof of Theorem 2.1. In fact, the proof of the lemma is completely analogous to that of \( \|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^\alpha(\mathbb{R}^n)} \) based on Fourier analysis.

**Lemma 3.2.** Suppose that \( \alpha > \frac{n}{2} \) and \( n \geq 1 \). Then, there exists a constant \( C_{\alpha,n} > 0 \) depending only on \( \alpha \) and \( n \) such that, for any \( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n}) \),

\[
\|B_{l,j,k}^l \gamma^{(k+1)}\|_{H_k^\alpha} \leq C_{\alpha,n} \|\gamma^{(k+1)}\|_{H_{k+1}^\alpha}, \quad l = 1, 2,
\]

for all \( k \geq 1 \), where \( j = 1, \ldots, k \). Consequently,

\[
(3.3) \quad \|B^{(k)} \gamma^{(k+1)}\|_{H_k^\alpha} \leq C_{\alpha,n} k \|\gamma^{(k+1)}\|_{H_{k+1}^\alpha}
\]

for any \( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n}) \) and all \( k \geq 1 \).

**Remark 3.1.** The estimate (3.3) indicates that the operator \( B^{(k)} \), originally defined on Schwarz functions, can be extended to a bounded operator from \( H_{k+1}^\alpha \) to \( H_k^\alpha \). In this case, we still denote it by \( B^{(k)} \).

**Proof.** We first consider \( B_{1,k}^1 \). For \( \gamma^{(k+1)} \in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n}) \), from the Plancherel’s theorem it is concluded that

\[
\|B_{1,k}^1 \gamma^{(k+1)}\|_{H_k^\alpha}^2 = \int \prod_{j=1}^{k} (p_j)^{2\alpha} (p_j')^{2\alpha} dp_k dp_k' \int dq dq' 
\gamma^{(k+1)}(p_1 + q' - q, p_2, \ldots, p_k, q; p_1', \ldots, p_k', q')^2.
\]
Then, we obtain
\[
\|B_{1,k}^1\gamma^{(k+1)}\|^2_{H^\alpha_k} \leq \int \prod_{j=1}^{k} \langle p_j \rangle^{2\alpha} \langle p_j' \rangle^{2\alpha} \, dp_k dp_k' \\
\times \left( \int dq dq' \frac{1}{\langle p_1 + q' - q \rangle^{2\alpha} \langle q' \rangle^{2\alpha}} \right) \\
\times \left[ \langle p_1 + q' - q, p_2, \cdots, p_k, q; p_1', \cdots, p_k', q' \rangle^2 \right] \\
\leq \sup_{p_1 \in \mathbb{R}^n} \int dq dq' \frac{\langle p_1 \rangle^{2\alpha}}{(p_1 + q' - q)^{2\alpha} \langle q' \rangle^{2\alpha}} \\
\times \left( \prod_{j=2}^{k} \langle p_j \rangle^{2\alpha} \langle p_j' \rangle^{2\alpha} \right) \, dp_k dp_k' dq dq' \\
\times \left\{ \langle p_1 + q' - q, p_1', \cdots, p_k, q; p_1', \cdots, p_k', q' \rangle^2 \right\} \\
= \sup_{p_1 \in \mathbb{R}^n} \int dq dq' \frac{\langle p_1 \rangle^{2\alpha}}{(p_1 + q' - q)^{2\alpha} \langle q' \rangle^{2\alpha}} \|\gamma^{(k+1)}\|^2_{H^\alpha_{k+1}} \\
\leq C_{\alpha,n} \|\gamma^{(k+1)}\|^2_{H^\alpha_{k+1}},
\]
where we have used Lemma 3.1 in the last inequality. For the operator $B_{1,k}^2$, we have the same estimate
\[
\|B_{1,k}^2\gamma^{(k+1)}\|^2_{H^\alpha_k} \leq C_{\alpha,n} \|\gamma^{(k+1)}\|^2_{H^\alpha_{k+1}}.
\]
Similarly, we can prove the same bound for $B_{j,k}^1$ and $B_{j,k}^2$ when $j = 2, \cdots, k$. Consequently, we conclude the estimate (3.3).

4. Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1. The proof is divided into two parts as follows.

Proof. (i) Let $\alpha > n/2$ and $0 < \xi < 1$. Given $\Gamma_0 = \{\gamma_0^{(k)}\}_{k \geq 1} \in H^\alpha_{\xi}$. For $m \geq 1$, set
\[
(4.1) \quad \gamma_{m,t}^{(k)} = U_0^{(k)}(t)\gamma_0^{(k)} + \int_0^t ds U_0^{(k)}(t-s)\tilde{B}^{(k)}\gamma_{m-1,s}^{(k+1)}, \quad t > 0, k \geq 1,
\]
\]
with the convention \( \gamma_{0,t}^{(k)} = \gamma_0^{(k)} \), where \( \tilde{B}^{(k)} = -i\mu B^{(k)} \) (e.g., (2.6)). By expansion, for every \( m \geq 1 \) one has

\[
\gamma_{m,t}^{(k)} = \mathcal{U}_0^{(k)}(t)\gamma_0^{(k)} + \sum_{j=0}^{m-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j \mathcal{U}_0^{(k)}(t-t_1)\tilde{B}^{(k)} \cdots \times \mathcal{U}_0^{(k)}(t-j-1)\tilde{B}^{(k+1)} \mathcal{U}_0^{(k)}(t_j)\gamma_0^{(k+j)} \\
+ \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \mathcal{U}_0^{(k)}(t-t_1)\tilde{B}^{(k)} \cdots \times \mathcal{U}_0^{(k+m-1)}(t_{m-1}-t_m)\tilde{B}^{(k+m-1)}\gamma_0^{(k+m)}
\]

\[
= \sum_{j=0}^m \Xi_{j,t}^{(k)},
\]

with the convention \( t_0 = t \). Then, for \( j = 1, \cdots, m - 1 \), by Lemma 3.2 we have

\[
\|\Xi_{j,t}^{(k)}\|_{H^0_k} \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j \|\mathcal{U}_0^{(k)}(t-t_1)\tilde{B}^{(k)} \cdots \times \mathcal{U}_0^{(k)}(t-j-1)\tilde{B}^{(k+1)} \mathcal{U}_0^{(k)}(t_j)\gamma_0^{(k+j)}\|_{H^0_k} \\
\leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j k \cdots (k+j-1)(C_{\alpha,n})^j \|\mathcal{U}_0^{(k+j)}(t_j)\gamma_0^{(k+j)}\|_{H^0_{k+j}} \\
= \left( \frac{t_j}{j!} \right) k \cdots (k+j-1)(C_{\alpha,n})^j \|\gamma_0^{(k+j)}\|_{H^0_{k+j}}
\]

\[
= \left( \frac{k+j-1}{j} \right)(C_{\alpha,n}t)^j \|\gamma_0^{(k+j)}\|_{H^0_{k+j}},
\]

and

\[
\|\Xi_{m,t}^{(k)}\|_{H^0_k} \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \|\mathcal{U}_0^{(k)}(t-t_1)\tilde{B}^{(k)} \cdots \times \mathcal{U}_0^{(k+m-1)}(t_{m-1}-t_m)\tilde{B}^{(k+m-1)}\gamma_0^{(k+m)}\|_{H^0_k} \\
\leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m k \cdots (k+m-1)(C_{\alpha,n})^m \|\gamma_0^{(k+m)}\|_{H^0_{k+m}} \\
\leq \frac{m}{m!} k \cdots (k+m-1)(C_{\alpha,n})^m \|\gamma_0^{(k+m)}\|_{H^0_{k+m}} \\
= \left( \frac{k+m-1}{m} \right)(C_{\alpha,n}t)^m \|\gamma_0^{(k+m)}\|_{H^0_{k+m}}.
\]
Then, for \( T > 0 \) (\( T \) will be fixed in the sequel) we obtain
\[
\| \gamma^{(k)}_{m,t} \|_{C([0,T],H^\alpha_k)} \leq \sum_{j=0}^{m} \| \Xi^{(k)}_j \|_{C([0,T],H^\alpha_k)}
\leq \sum_{j=0}^{m} \binom{k+j-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(k+j)} \|_{H^{\alpha}_{k+j}}.
\]
Hence, for \( 0 < \eta < 1 \) (which will be fixed later) one has
\[
\sum_{k=1}^{\infty} \eta^k \| \gamma^{(k)}_{m,t} \|_{C([0,T],H^\alpha_k)} \leq \sum_{j=0}^{m} \sum_{k=1}^{\infty} \eta^k \binom{k+j-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(k+j)} \|_{H^{\alpha}_{k+j}}
\leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \eta^k \binom{k+j-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(k+j)} \|_{H^{\alpha}_{k+j}}.
\]
By the direct computation, one has
\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \eta^k \binom{k+j-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(k+j)} \|_{H^{\alpha}_{k+j}}
= \sum_{j=0}^{\infty} \sum_{l=j+1}^{\infty} \eta^{l-j} \binom{l-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(l)} \|_{H^\alpha_l}
= \sum_{l=1}^{\infty} \sum_{j=0}^{l-1} \binom{l-1}{j} (C_{\alpha,n} T/\eta)^j (C_{\alpha,n} T)^l \| \gamma_0^{(l)} \|_{H^\alpha_l}
= \sum_{l=1}^{\infty} (1 + C_{\alpha,n} T/\eta)^{l-1} \eta^l \| \gamma_0^{(l)} \|_{H^\alpha_l}
= \frac{\eta}{\eta + C_{\alpha,n} T} \sum_{l=1}^{\infty} (\eta + C_{\alpha,n} T)^l \| \gamma_0^{(l)} \|_{H^\alpha_l}.
\]
Set \( \Gamma_{m,t} = \{ \gamma^{(k)}_{m,t} \} \). Let \( \eta = \xi - C_{\alpha,n} T \) with \( 0 < T < \xi/C_{\alpha,n} \). Then, it follows from (4.3) that
\[
\| \Gamma_{m,t} \|_{C([0,T],H^\alpha_k)} \leq \frac{\eta}{\xi} \sum_{l=1}^{\infty} \xi^l \| \gamma_0^{(l)} \|_{H^\alpha_l} = \frac{\eta}{\xi} \| \Gamma_0 \|_{H^\alpha_k}.
\]
Next, we prove that \( \{ \Gamma_{m,t} \}_{m \geq 1} \) converges to a solution. Indeed, by the above estimates for \( \Xi^{(k)}_{j,t} \) we have for any \( m, n \) with \( n > m \)
\[
\| \gamma^{(k)}_{m,t} - \gamma^{(k)}_{n,t} \|_{C([0,T],H^\alpha_k)} \leq 2 \sum_{j=m}^{n} \binom{k+j-1}{j} (C_{\alpha,n} T)^j \| \gamma_0^{(k+j)} \|_{H^{\alpha}_{k+j}}.
\]
Then,
\[ \| \Gamma_{m,t} - \Gamma_{n,t} \|_{C(\{0, T\}, \mathcal{H}_0^\alpha)} \leq 2 \sum_{j=m}^{n} \sum_{k=1}^{\infty} \eta^k \left( \binom{k + j - 1}{j} (C\alpha_n T)^{j} \| \gamma_0^{(k+j)} \|_{\mathcal{H}_k^0} \right). \]

An immediate computation as above yields that
\[ \| \Gamma_{m,t} - \Gamma_{n,t} \|_{C(\{0, T\}, \mathcal{H}_0^\alpha)} \leq \frac{2 \eta}{\xi} \sum_{l=m+1}^{\infty} \xi^l \| \gamma_0^{(l)} \|_{\mathcal{H}_l^0}. \]

Since \( \Gamma_0 = \{ \gamma_0^{(k)} \}_{k \geq 1} \in \mathcal{H}_0^\alpha \), it is concluded that \( \{ \Gamma_{m,t} \}_{m \geq 1} \) is a Cauchy sequence in \( C(\{0, T\}, \mathcal{H}_0^\alpha) \) and so converges to some \( \Gamma_t \in C(\{0, T\}, \mathcal{H}_0^\alpha) \). Taking the limit \( m \to \infty \) in (4.1) we prove that \( \Gamma_t \) is a solution to (2.2). Also, taking \( m \to \infty \) in (4.4) yields this solution satisfies (2.10).

(ii) Choose \( T = \xi/(5C) \) and suppose \( \Gamma_t, \Gamma'_t \in C(\{0, T\}, \mathcal{H}_0^\alpha) \) are two solutions to (2.2) with the initial datum \( \Gamma_0 \) and \( \Gamma'_0 \) in \( \mathcal{H}_0^\alpha \), respectively. Since (2.2) is linear, it suffices to consider \( \Gamma_t \) instead of \( \Gamma_t - \Gamma'_t \). By (2.8), for every \( m \geq 1 \) one has

\[ \gamma_t^{(k)} = \mathcal{U}_0^{(k)}(t) \gamma_0^{(k)} + \sum_{j=1}^{m-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{j-1}} dt_j \mathcal{U}_0^{(k)}(t - t_1) \mathcal{B}^{(k)} \cdots \times \mathcal{U}_0^{(k+j-1)}(t_{j-1} - t_j) \mathcal{B}^{(k+j-1)} \mathcal{U}_0^{(k+j)}(t_j) \gamma_0^{(k+j)} + \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \mathcal{U}_0^{(k)}(t - t_1) \mathcal{B}^{(k)} \cdots \times \mathcal{U}_0^{(k+m-1)}(t_{m-1} - t_m) \mathcal{B}^{(k+m-1)} \gamma_{t_m}^{(k+m)} \]

\[ \Delta \sum_{j=0}^{m-1} \Xi^{(k)}_{m,j} = \Xi^{(k)}_{m,t}, \]

with the convention \( t_0 = t \). Note that,

\[ \| \Xi_{m,t}^{(k)} \|_{\mathcal{H}_k^0} \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \| \mathcal{U}_0^{(k)}(t - t_1) \mathcal{B}^{(k)} \cdots \times \mathcal{U}_0^{(k+m-1)}(t_{m-1} - t_m) \mathcal{B}^{(k+m-1)} \gamma_{t_m}^{(k+m)} \|_{\mathcal{H}_k^0} \]

\[ \leq \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m k \cdots \times (k + m - 1)(C\alpha_n)^m \| \gamma_{t_m}^{(k+m)} \|_{\mathcal{H}_k^0} \]

\[ \leq m(m + 1) \cdots (2m - 1)(C\alpha_n)^m \times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \| \gamma_{t_m}^{(k+m)} \|_{\mathcal{H}_k^0} dt_m. \]
Combining this estimate and (4.2) yields
\[
\|\gamma^{(k)}_t\|_{H^0_k} \leq \sum_{j=0}^{m-1} \binom{k+j-1}{j} (C_{\alpha,n} T)^j \|\gamma^{(k+j)}_0\|_{H^0_{k+j}} + m(m+1) \cdots (2m-1)(C_{\alpha,n})^m \\
\times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \|\gamma^{(k+m)}_{t_m}\|_{H^0_{k+m}} dt_m.
\]

Then for \(m \geq 1\) and \(0 < t < T\), we have
\[
\sum_{k=1}^{m} \eta^k \|\gamma^{(k)}_t\|_{H^0_k} \\
\leq \sum_{j=0}^{m-1} \sum_{k=1}^{m} \binom{k+j-1}{j} (C_{\alpha,n} T/\eta)^j \eta^{k+j} \|\gamma^{(k+j)}_0\|_{H^0_{k+j}} + m(m+1) \cdots (2m-1)(C_{\alpha,n}/\eta)^m \\
\times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} \sum_{k=1}^{m} \eta^{(k+m)} \|\gamma^{(k+m)}_{t_m}\|_{H^0_{k+m}} dt_m \\
\leq \frac{\eta}{\xi} \|\Gamma_0\|_{H^0_\xi} + \left( \frac{C_{\alpha,n} T}{\eta} \right)^m (2m-1) \int_0^T \|\Gamma_t\|_{C([0,T],H^0_\xi)} \|
\leq \frac{4}{5} \|\Gamma_0\|_{H^0_\xi} + \left( \frac{C_{\alpha,n} T}{\eta} \right)^m \int_0^T \|\Gamma_t\|_{C([0,T],H^0_\xi)}.
\]

By Stirling’s formula \(m! \approx m^{m+1/2} e^{-m}\) we have
\[
\binom{2m-1}{m} \approx \frac{4^m}{\sqrt{m}}.
\]

Then, taking \(m \to \infty\) in (4.5) yields that
\[
\|\Gamma_t\|_{C([0,T],H^0_\xi)} \leq \frac{4}{5} \|\Gamma_0\|_{H^0_\xi},
\]
because \(C_{\alpha,n} T/\eta = 1/4\). Thus, the space-time type bound (2.11) holds true.

The uniqueness of the solution in \(C([0,T],H^0_\eta)\) follows clearly from the estimate (2.11).

5. The quintic Gross–Pitaevskii hierarchy

In this section, we consider the so-called quintic Gross–Pitaevskii hierarchy. Recall that the quintic Gross–Pitaevskii hierarchy is given by
\[
(5.1) \quad i \partial_t \gamma^{(k)}_t = [-\Delta^{(k)}, \gamma^{(k)}_t] + \mu Q^{(k)} \gamma^{(k+2)}_t, \quad \Delta^{(k)} = \sum_{j=1}^{k} \Delta x_j, \quad \mu = \pm 1,
\]
in \( n \) dimensions, for \( k \in \mathbb{N} \), where the operator \( Q^{(k)} \) is defined by
\[
Q^{(k)}(k+2)_{t} = \sum_{j=1}^{k} \text{tr}_{k+1,k+2} \left[ \delta(x_j - x_{k+1})\delta(x_j - x_{k+2}), \gamma_{t}^{(k+2)} \right].
\]

It is defocusing if \( \mu = 1 \), and focusing if \( \mu = -1 \). We note that the quintic Gross–Pitaevskii hierarchy accounts for 3-body interactions between the Bose particles (see [4] and references therein for details).

**Remark 5.1.** In terms of kernel functions we can rewrite (5.1) as follows
\[
(i\partial_{t} + \triangle_{\pm}^{(k)}) \gamma_{t}^{(k)}(x_k; x'_k) = \mu(Q^{(k)}(k+2)_{t})(x_k; x'_k),
\]
where, the action of \( Q^{(k)}(k+2) \) on \( \gamma^{(k+2)}(x_{k+2}, x'_{k+2}) \in S(\mathbb{R}^{(k+2)n} \times \mathbb{R}^{(k+2)n}) \) is given by
\[
(Q^{(k)}(k+2))(x_k, x'_k) := \sum_{j=1}^{k} (Q_{j,k}^{(k+2)})(x_k, x'_k)
\]
\[
= \sum_{j=1}^{k} \int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2} \gamma^{(k+2)}(x_k, x_{k+1}, x_{k+2}; x'_k, x'_{k+1}, x'_{k+2})
\]
\[
\times \left[ \prod_{\ell=k+1}^{k+2} \delta(x_j - x_{\ell})\delta(x_j' - x'_{\ell}) - \prod_{\ell=k+1}^{k+2} \delta(x'_j - x_{\ell})\delta(x'_j' - x'_{\ell}) \right].
\]

Let \( \varphi \in H^1(\mathbb{R}^n) \), then one can easily verify that a particular solution to (5.2) with initial conditions
\[
\gamma_{t=0}^{(k)}(x_k; x'_k) = \prod_{j=1}^{k} \varphi(x_j)\overline{\varphi(x'_j)}, \quad k = 1, 2, \ldots,
\]
is given by
\[
\gamma_{t}^{(k)}(x_k; x'_k) = \prod_{j=1}^{k} \varphi_{t}(x_j)\overline{\varphi_{t}(x'_j)} \quad k = 1, 2, \ldots,
\]
where \( \varphi_{t} \) satisfies the quintic non-linear Schrödinger equation
\[
i\partial_{t}\varphi_{t} = -\Delta\varphi_{t} + \mu|\varphi_{t}|^4\varphi_{t}, \quad \varphi_{t=0} = \varphi.
\]

**Remark 5.2.** The Gross–Pitaevskii hierarchy (5.1) can be written in the integral form
\[
\gamma_{t}^{(k)} = U_{0}^{(k)}(t)\gamma_{0}^{(k)} + \int_{0}^{t} ds \ U_{0}^{(k)}(t-s)\tilde{Q}^{(k)}(k+2), \quad k = 1, 2, \ldots,
\]
where $\tilde{Q}^{(k)} = -i\mu Q^{(k)}$. Formally we can expand the solution $\gamma_t^{(k)}$ of (5.4) for any $m \geq 1$ as

\begin{equation}
\gamma_t^{(k)} = U_0^{(k)}(t)\gamma_0^{(k)} + \sum_{j=1}^{m-1} \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{j-1}} ds_j U_0^{(k)}(t-s_1)\tilde{Q}^{(k)} \cdots \\
\times U_0^{(k+j-1)}(s_{j-1} - s_j)\tilde{Q}^{(k+j-1)} U_0^{(k+j)}(s_j)\gamma_0^{(k+j)} + \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m U_0^{(k)}(t-s_1)\tilde{Q}^{(k)} \cdots \\
\times U_0^{(k+m-1)}(s_{m-1} - s_m)\tilde{Q}^{(k+m-1)} \gamma_t^{(k+m)}.
\end{equation}

with the convention $s_0 = t$.

Let $q = (p_{k+1}, p_{k+2})$ and $q' = (p'_{k+1}, p'_{k+2})$ we have

\begin{align*}
(Q_j, k \gamma^{(k+2)})(p_k, p'_k) & = \int dq dq' \left[ \gamma^{(k+2)}(p_1, \ldots, p_j + p_{k+1} + p_{k+2} - p'_{k+1} - p'_{k+2}; \ldots, p_k, q; p'_{k+2}) \\
& - \gamma^{(k+2)}(p_{k+1}; p'_{k+1} + p'_{k+2} - p_{k+1} - p_{k+2}; \ldots, p'_k, q') \right]
\end{align*}

It is proved in [4] (Theorem 4.3 there) that for $\alpha > n/2$ there exists a constant $C = C_{n, \alpha} > 0$ depending only on $n$ and $\alpha$ such that

$$
\| Q_j, k \gamma^{(k+2)} \|_{\mathcal{H}_k^{\alpha}} \leq C \| \gamma^{(k+2)} \|_{\mathcal{H}_k^{\alpha}} \quad \forall j = 1, \ldots, k.
$$

Then, by slightly repeating the proof of Theorem 2.1, we can obtain the following theorem.

\textbf{Theorem 5.1.} Assume that $n \geq 1$ and $\alpha > n/2$. Suppose $\Gamma_0 = \{ \gamma_0^{(k)} \}_{k \geq 1} \in \mathcal{H}_\xi^{\alpha}$ for some $0 < \xi < 1$. Then there exists a constant $C = C_{\alpha, n}$ depending only on $n$ and $\alpha$ such that, for a fixed $0 < T < \xi/C$ with $\eta = \xi - CT$, the following hold.

(i) There exists a solution $\Gamma_t = \{ \gamma_t^{(k)} \}_{k \geq 1} \in C([0, T], \mathcal{H}_\eta^{\alpha})$ to the Gross-Pitaevskii hierarchy (5.1) with the initial data $\Gamma_0$ satisfying

\begin{equation}
\| \Gamma_t \|_{C([0, T], \mathcal{H}_\eta^{\alpha})} \leq \frac{1}{\eta \xi} \| \Gamma_0 \|_{\mathcal{H}_\xi^{\alpha}}.
\end{equation}

(ii) For $T = \xi/(5C)$, if $\Gamma_t$ and $\Gamma_t'$ in $C([0, T], \mathcal{H}_\eta^{\alpha})$ are two solutions to (5.1) with initial conditions $\Gamma_{t=0} = \Gamma_0$ and $\Gamma_{t=0}' = \Gamma_0'$ in $\mathcal{H}_\xi^{\alpha}$ respectively, then

\begin{equation}
\| \Gamma_t - \Gamma_t' \|_{C([0, T], \mathcal{H}_\eta^{\alpha})} \leq \frac{5}{4 \xi^2} \| \Gamma_0 - \Gamma_0' \|_{\mathcal{H}_\xi^{\alpha}}.
\end{equation}

Consequently, the solution $\Gamma_t$ to the initial problem (5.1) with the initial data in $\mathcal{H}_\xi^{\alpha}$ is unique in $C([0, T], \mathcal{H}_\eta^{\alpha})$ for any $0 < T < \xi/C$. 
We omit the details of the proof. This result also improves the corresponding one in [2] for the regime $\alpha > \frac{n}{2}$.

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