Fixed points and their stability in the functional renormalization group of random field models

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Abstract. We consider the zero-temperature fixed points controlling the critical behavior of the $d$-dimensional random-field Ising, and more generally $O(N)$, models. We clarify the nature of these fixed points and their stability in the region of the $(N, d)$ plane where one passes from a critical behavior satisfying the $d \to d - 2$ dimensional reduction to one where it breaks down due to the appearance of strong enough nonanalyticities in the functional dependence of the cumulants of the renormalized disorder. We unveil an intricate and unusual behavior.

Keywords: classical phase transitions (experiments), critical exponents and amplitudes (theory), renormalisation group, disordered systems (theory)

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1. Introduction

In a recent series of papers [1–5], we have shown how the critical behavior of the $d$-dimensional random-field Ising, and more generally $O(N)$, models can be fully described through the functional renormalization group (FRG). We have in particular stressed that the solution to many puzzles associated with this critical behavior lies in the existence of a transition between a region satisfying the $d \rightarrow d - 2$ dimensional reduction, i.e. where the critical behavior of the random-field system is identical to that of the corresponding pure model in two dimensions less [6–9], and one where dimensional reduction is broken.

The above transition takes place at a nontrivial line in the $(N, d)$ plane, in contrast for instance with the case of an interface in a random environment where dimensional reduction is always wrong below the upper critical dimension [10–13]. For the random-field $O(N)$ model [RF$O(N)$M], the zero-temperature fixed point associated with dimensional reduction disappears below a line $d_{\text{DR}}(N)$ that is close to 5.1 when $N = 1$ and decreases as $N$ increases, reaching $d = 4$, the lower critical dimension for ferromagnetism in the presence of a continuous $O(N)$ symmetry, when $N = 18$ [2, 4]. Below this line, the zero-temperature fixed point controlling the critical behavior is characterized by strong enough nonanalyticities in the functional dependence of the cumulants of the renormalized random field. Physically, this results from the presence of large-scale collective events known as ‘avalanches’ (whose fractal dimension is then equal to the fractal dimension of the total magnetization at criticality) [14]. Formally, this leads to a failure of the Ward–Takahashi identities associated with the underlying supersymmetry of the model [9] and to a breaking of the latter [4].

In this paper, we take a closer look at the transition from the regime controlled by the dimensional-reduction fixed point to that controlled by the ‘cuspy’ fixed point. Although this takes place in an unphysical region in systems with short-range interactions and disorder correlations (but could nonetheless be studied in $d = 3$ when one...
allows for long-range interactions and disorder correlations as we have recently pointed out [5]), the issue is important to underpin the whole FRG-based description.

A first question is whether one can find, within the FRG, operators that become relevant as dimensional reduction breaks down. To this end we have studied in more detail the stability of the ‘cuspless’ fixed point associated with dimensional reduction to a ‘cuspy’ perturbation, i.e. a perturbation displaying a linear cusp in the cumulants of the renormalized random field, when \( d \geq d_{\text{DR}} \). Quite surprisingly, we find two different mechanisms for the appearance and disappearance of the stable (critical) fixed point, depending on the value of \( N \) (or \( d \)).

For \( N \) sufficiently large, the cuspless fixed point becomes unstable with respect to a cuspy perturbation and this occurs at a nontrivial dimension \( d_{\text{cusp}}(N) \) that is close to, but different from, \( d_{\text{DR}}(N) \). As a result, there is a range of dimensions \( d_{\text{DR}}(N) < d < d_{\text{cusp}}(N) \) (or, alternatively, of number of components, \( N_{\text{DR}}(d) < N < N_{\text{cusp}}(d) \)) for which models described by a cuspless initial condition flow at criticality to the cuspless fixed point associated with dimensional reduction whereas models already described by a cuspy initial condition flow to a cuspy fixed point for which dimensional reduction fails. In an enlarged space of functions including those with a linear cusp, only the latter fixed point is fully stable (except for the usual relevant direction needed to tune the critical-point condition).

For a threshold value \( N_x \) and, correspondingly, a threshold dimension \( d_x = d_{\text{DR}}(N_x) = d_{\text{cusp}}(N_x) \), the two critical lines \( d_{\text{DR}}(N) \) and \( d_{\text{cusp}}(N) \) meet: see figure 1. For \( N < N_x \) and \( d > d_x \), the cuspless fixed point that governs the critical physics remains stable under cuspy perturbations down to \( d = d_{\text{DR}}(N) \), at which point it disappears. A cuspy fixed point then emerges continuously from the cuspless one, through a boundary-layer mechanism.

We derive the above results by a combination of approaches. We investigate the RF\( O(N)M \) near its lower critical dimension, in \( d = 4+\epsilon \), through the perturbative FRG in section 2. In section 3, we illustrate the mechanisms for the appearance of cuspy fixed points and the disappearance of cuspless ones in a toy model inspired from the beta function of the RF\( O(N)M \). We finally address the short- and long-range versions of the random-field Ising model (RFIM) through the nonperturbative FRG in section 4.

2. The RF\( O(N)M \) in \( d = 4+\epsilon \)

The long-distance physics of the RF\( O(N)M \) is described by the following Hamiltonian or bare action,

\[
S[\varphi, h] = \int_x \left\{ \frac{1}{2} |\varphi(x)|^2 + \frac{1}{2} \left| \varphi(x) \right|^2 + \frac{u}{4!} (|\varphi(x)|^2)^2 - h(x) \cdot \varphi(x) \right\},
\]

where \( \int_x \equiv \int d^d x \), \( \varphi(x) \) is an \( N \) component field and \( h(x) \) is a random source (a random magnetic field in the language of magnetic systems) with zero mean and a variance \( \overline{h^\mu(x) h^\nu(y)} = \delta_{\mu \nu} \Delta_B(x-y) \), where \( \mu, \nu = 1, \ldots, N \) and an overline denotes an average over the random field. For the usual short-range model, the function \( \Delta_B(x-y) \) can be
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Figure 1. Schematic phase diagram of the RFO(N)M in the (N, d) plane. The full line is $N_{DR}(d)$, where the cuspless fixed points present for $N > N_{DR}(d)$ disappear and the dashed line is $N_{cusp}(d)$, where the cuspless critical fixed point becomes unstable to a cuspy perturbation. The two lines meet at $N_x \approx 14$ and $d_x \approx 4.4$ (estimated from a 2–loop perturbative FRG in $d = 4 + \epsilon$). To the right of the threshold point, the cuspy critical fixed point disappears at $N_{DR}(d)$ when it is still stable with respect to a cuspy perturbation. A stable cuspy fixed point then appears continuously but through a boundary–layer mechanism.

No Cusp

Cusp

$\{d_x, N_x\}$

$N_{cusp}(d)$

$N_{DR}(d)$

Near the lower critical dimension for ferromagnetism ($d = 4$), the critical behavior of the RFO(N)M is captured by a nonlinear-sigma model that in turn can be studied through a perturbative but functional RG. The resulting FRG flow equations have been obtained at one-loop [2, 3, 15, 16] and two-loop [3, 17] orders. The central quantity is the renormalized second cumulant of the random field $\Delta_k(z)$ (noted $R_k''(z)$ in previous work), where $k$ is the running infrared cutoff and $z$ is the cosine of the angle between fields in two different replicas of the system [2, 3]. A ‘linear cusp’ in this parametrization corresponds to a term in $\sqrt{1-z}$ as $z \to 1$. We then use the terminolgy ‘cuspy’ to describe a function with this behavior and ‘cuspless’ if the function and its first derivative in $z = 1$, $\Delta(1)$ and $\Delta'(1)$, are finite.

For completeness we recall the FRG equation for $\Delta_k(z)$ at one-loop, in $d = 4+\epsilon$:

$$\frac{1}{\epsilon} \partial_t \Delta_k(z) = \Delta_k(z) - \left\{ (N-3) \Delta_k(1) \Delta_k(z) + z \Delta_k(z) \right. $$
$$+ (N-3 + 4z^2) \Delta_k(z) \Delta_k'(z) - (N+1) z \Delta_k(1) \Delta_k'(z) $$
$$- z(1-z^2) \Delta_k(z) \Delta_k''(z) + (1-z^2) \Delta_k(1) \Delta_k'''(z) $$
$$- 3z(1-z^2) \Delta_k'(z)^2 + (1-z^2)^2 \Delta_k'(z) \Delta_k''(z) \right\} \quad (2)$$

where we have rescaled $\Delta_k$ by $8\pi^2 \epsilon$. The RG ‘time’ $t$ is defined such that the long-distance physics is recovered when $t \to -\infty$, i.e. $t = \log(k/\Lambda)$ with $\Lambda$ the microscopic or ultraviolet scale. The two anomalous dimensions $\eta$ and $\eta'$ characterizing the
spatial dependence of the correlation functions at criticality in random-field systems are expressed in terms of $\Delta(1)$ at the fixed point:

$$\eta = \epsilon \Delta_\ast(1)$$

$$\sigma = \epsilon \left[ -1 + (N-1)\Delta_\ast(1) \right].$$

For sufficiently large $N$, the critical behavior is controlled by a fixed point at which $\Delta_\ast(z)$ has only a ‘subcusp’, with a leading nonanalytic behavior in $(1-z)^{\alpha_\ast(N)}$ and $\alpha_\ast(N) \geq 1$, which implies that $\Delta_k(1)$ and $\Delta'_k(1)$ remain finite during the flow. A direct calculation shows that, under this hypothesis, the evolution of $\Delta_k(1)$ only depends on $\Delta_k(1)$ and the evolution of $\Delta'_k(1)$ only depends on $\Delta_k(1)$ and $\Delta'_k(1)$. At the corresponding cuspless fixed point, one has [3, 15]

$$\frac{\Delta_\ast(1)}{\epsilon} = \frac{1}{N-2}$$

$$\frac{\Delta'_\ast(1)}{\epsilon} = \frac{(N-8) - \sqrt{(N-2)(N-18)}}{2(N-2)(N+7)}.$$

The square root in the expression of $\Delta'_\ast(1)$ implies that this fixed point exists only for $N \geq N_{\text{DR}} = 18$.

The determination of $\alpha_\ast(N)$ is obtained as follows. Suppose that the function $\Delta_k(z)$ has a leading singularity with exponent $\alpha$ and $\alpha > 1$: $\Delta(z) = \Delta_k(1) - \Delta'_k(1) (1-z) + \cdots + a_k (1-z)^\alpha + \cdots$. One can easily show that the flow of $a_k$ is linear in $a_k$ and that it depends on $\Delta_k(1)$ and $\Delta'_k(1)$ only:

$$\frac{1}{\epsilon} \partial_\ast a_k = a_k \Lambda_{\alpha+1}(\Delta_k(1), \Delta'_k(1))$$

with [3]

$$\Lambda_{\alpha}(\Delta(1), \Delta'(1)) = 1 - \Delta(1)[N(2-p) + 2p^2 + p - 4] - \Delta'(1)p(6p + N - 5).$$

The only way to have a nonvanishing amplitude for a subcusp with exponent $\alpha_\ast$ is that

$$\Lambda_{\alpha+1}(\Delta_\ast(1), \Delta'_\ast(1)) = 0.$$

By using the fixed-point solution given in equation (5), we then obtain an explicit expression for $\alpha_\ast(N)$, which we do not reproduce here. It is found that $\alpha_\ast(N)$ decreases as $N$ decreases until it reaches $\alpha_\ast(N = 18) = 3/2$ [2, 3]. Below $N = 18 = N_{\text{DR}}$, the only nontrivial fixed points have a linear cusp, with $\alpha_\ast(N) = 1/2$.

The eigenvalues describing the stability of the cuspless fixed point under consideration are obtained by linearizing the FRG flow equations around this fixed point. In our previous work we found that for $N \geq 18$ the cuspless fixed point described by equation (5) is stable with respect to cuspless perturbations, except of course for the relevant direction (here, $\Delta(1)$) that must be fine-tuned to reach the critical point. Starting from a cuspless initial condition for $\Delta_k(z)$ at the ultraviolet scale, one ends up, after fine-tuning the relevant parameter $\Delta(1)$, at a cuspless fixed point, and the critical exponents are given by the dimensional-reduction predictions. However, at the time, we did not systematically investigate the stability of the cuspless fixed point with

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respect to a cuspy perturbation. This had been done by Sakamoto et al \[18\] in the large \(N\) limit in an expansion in \(1/N\), at one- and two-loop orders. The outcome was that the cuspless fixed point is then stable to all perturbations, with and without a cuspy functional behavior.

We are then interested in the eigenvalue \(\lambda(N)\) associated with a cuspy eigenfunction \(f_N(z)\). It coincides with \(\Lambda_{3/2}\) obtained from equations (8) and (9) and reads:

\[
\frac{\lambda(N)}{\epsilon} = \frac{1}{4(N+7)} \left[ 3(N+4) \sqrt{\frac{N-18}{N-2}} - N + 8 \right]
\]

where a positive value means an irrelevant direction. One checks that the result of \[18\] is recovered in the large \(N\) limit:

\[
\frac{\lambda(N)}{\epsilon} = \frac{1}{2} - \frac{9}{2N} - \frac{57}{2N^2} + O(1/N^3).
\]

We have plotted the eigenvalue \(\lambda(N)\) in figure 2. It decreases as \(N\) decreases, reaches zero when \(N = N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \approx 18.3923\) and then changes sign. Its value in \(N = 18\) is equal to \(-1/10\). The cuspless fixed point therefore becomes unstable with respect to a cuspy perturbation at a value of \(N\) which is slightly larger than the value \(N_{\text{DR}} = 18\) below which cuspless fixed points no longer exist.

We have repeated the analysis for the other cuspless fixed point that is somehow conjugate to the critical one described above but has one more cuspless relevant direction \[3, 15\]. It is characterized by \(\Delta_{\ast}(1) = \epsilon/(N - 2)\) and \(\Delta'_{\ast}(1) = \epsilon[ (N-8) + \sqrt{(N-2)(N-8)} ] / [2(N-2)(N+7)].\) The two cuspless fixed points merge and disappear when \(N = N_{\text{DR}} = 18\). The eigenvalue associated with a cuspy perturbation around this unstable cuspless fixed point is now given by

\[
\frac{\lambda(N)}{\epsilon} = \frac{1}{4(N+7)} \left[ -3(N+4) \sqrt{\frac{N-18}{N-2}} - N + 8 \right].
\]

A cuspy perturbation is therefore a relevant direction from \(N \to \infty\), where it behaves as \(-1 + 12/N - 24/N^2 + O(1/N^3)\), down to \(N_{\text{DR}}\). This is also displayed in figure 2.

The destabilization of the cuspless critical fixed point at \(N_{\text{cusp}}\) occurs in a standard way. We find by a numerical integration of the beta function that there exists a third fixed point, characterized by a cuspy functional form, which coincides with the cuspless critical fixed point for \(N = N_{\text{cusp}}\) and is stable for \(N < N_{\text{cusp}}\). The second smallest eigenvalue for this cuspy fixed point is also shown in figure 2. The general scenario for the exchange of stability of the fixed points is therefore quite common and appears in many other situations, such as for instance the destabilization of the \(O(N)\) Wilson-Fisher fixed point upon adding anisotropic interactions \[19\]. We give in figure 3 a schematic description of the RG flows to illustrate the evolution of the different fixed points.

The previous discussion makes it clear that there is a small domain \(N_{\text{DR}} < N < N_{\text{cusp}}\) where the critical behavior is described by the dimensional-reduction property if the initial condition of the flow is cuspless, but where it is governed by a cuspy fixed point and a breakdown of dimensional reduction otherwise. One should keep in mind that the FRG framework considered here starts with a coarse-grained Landau–Ginzburg description of the system at the microscopic (ultraviolet) scale: see equation (1). As long as there exists only one fixed point and that we limit our investigation to the critical physics, the detailed properties of the microscopic system are irrelevant. However,
in the small region between $N_{\text{DR}}$ and $N_{\text{cusp}}$, the situation is more intricate. A discussion of the $d = 0$ (1-site) problem \cite{4} shows that the presence of a cusp at the microscopic level, which is associated with avalanches, is most probably the rule rather than the exception at $T = 0$. This implies that physical systems at $T = 0$ are likely to always flow to the cuspy fixed point when $N < N_{\text{cusp}}$.\footnote{On the other hand, when studying the critical behavior at finite temperature, the initial condition is cuspleess as avalanches are rounded at all nonzero temperatures. In this case one should also include the flow of the renormalized temperature which is associated with the presence of a thermal boundary layer as one approaches the zero-temperature fixed point. Whether or not the system then flows to the cuspy fixed point when $d_{\text{cusp}} > d > d_{\text{DR}}(N)$ is unclear to us (but of limited physical consequence anyhow).}

Once the cuspy fixed-point solution is (numerically) obtained, we can derive the critical exponents, in particular the two anomalous dimensions $\eta$ and $\eta'$. We focus here on the vicinity of $N_{\text{cusp}}$, which was not considered previously. We display in figure 4 the two anomalous dimensions normalized by their dimensional-reduction expression.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{RFO$(N)$M at one-loop order in $d = 4+\epsilon$: Eigenvalue $\lambda(N)/\epsilon$ associated with a cuspy perturbation around different fixed points. The full curve corresponds to the cuspleess critical fixed point associated with dimensional reduction (equation (10)), the dashed line corresponds to the unstable conjugate cuspleess fixed point (equation 10) and the crosses correspond to the cuspy fixed point.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3}
\caption{Schematic flow for the RFO$(N)$ model in the plane $(\Delta'(1), a)$ where $a$ represents the amplitude of the cusp. The blue and red points represent the stable and unstable analytic fixed points and the black point represents the cuspy fixed point. Left: for $N > N_{\text{cusp}}$; middle: $N_{\text{DR}} < N < N_{\text{cusp}}$; right: $N = N_{\text{DR}}$. In the presence of a cusp ($a \neq 0$), $\Delta'(1)$ should be interpreted as (minus) the coefficient of the linear term in $(1 - z)$ when $z \to 1$.}
\end{figure}
ε = −N/(2)\partial \partial R \partial \partial R. The numerical determination of the fixed-point solution near N_{cusp} is difficult because of the presence of several fixed points which are close one to another and we were not able to determine the cuspy fixed point with sufficient accuracy near N_{cusp}. It is however clear numerically that for N slightly larger than N_{DR} there exists indeed two fixed points, with different anomalous dimensions.

We now discuss the behavior of the eigenfunction associated with the cuspy perturbation around the cuspless fixed point. From the work of Sakamoto et al [18], one knows that the physical eigenfunction f_N(z) with a cusp is a linear combination of two solutions of the eigenvalue equation, f_N^{(+)}(z) and f_N^{(-)}(z), the former having a cusp when z \to 1, i.e. f_N^{(+)}(z) \approx \sqrt{1-z} [1 + O(1-z)] and the latter having only a subcusp, i.e. f_N^{(-)}(z) \approx (1-z)^{\alpha_{-(N)}} [1 + O(1-z)]. Both functions, f_N^{(-)}(z) and f_N^{(+)}(z), individually diverge in z = -1 and are therefore not acceptable eigenfunctions. It is however possible to choose the coefficients of the linear combination so that the divergence in z = -1 of the two functions cancel. By continuity, we expect that this mechanism, which has been checked to order 1/N^2, still applies as one decreases N.

One should therefore find two eigenfunctions, one with a cusp and one with a sub-cusp in (1 - z)^{\alpha_{-(N)}}, to ensure that a linear combination of the two has a proper behavior in z = -1. The expression for \alpha_{+(N)} is obtained by imposing

\Lambda_{\alpha_{+,1}}(\Delta_{+(1)}, \Delta'_{+(1)}) = \lambda(N) / \epsilon, \quad (12)

where \Lambda_{\alpha_{+,1}} is obtained from equation (8). We thus get

\alpha_{+(N)} = \frac{1}{4} \left( N - 10 + \sqrt{(N-2)(N-18)} \right). \quad (13)

We show in figure 5 the behavior of \alpha_+ and \alpha_\ast as a function of N.

We observe that \alpha_+ is smaller than \alpha_\ast for large N (where \alpha_+ and \alpha_\ast behave as N/2 - 5 + O(1/N) and N/2 - 19/2 + O(1/N), respectively). For smaller values of N,
$\alpha_+$ becomes larger than $\alpha_*$ (in particular $\alpha_*(N = 18) = 3/2$ and $\alpha_+(N = 18) = 2$). The two curves cross exactly at $N_{\text{cusp}}$. This is not a surprise since for $N = N_{\text{cusp}}$, $\lambda = 0$, which implies that the conditions in equations (12) and (9) are degenerate.

The same analysis can be carried out at the two-loop order. To do so, we have used the FRG equations derived in [3]. We can then obtain the eigenvalue $\lambda(N)$ at order $\epsilon = d - 4$, which allows us to determine $N_{\text{cusp}}$ at first order in $\epsilon$:

$$N_{\text{cusp}}(d) = 2(4 + 3\sqrt{3}) - 3\left(\frac{2 + 3\sqrt{3}}{2}\right)\epsilon. \quad (14)$$

When compared to the result for $N_{\text{DR}}$, $N_{\text{DR}}(d) = 18 - \frac{49}{5}\epsilon$ [3], it can be seen that the absolute value of the slope (with $\epsilon$ or $d$) is larger for $N_{\text{cusp}}(d)$ than for $N_{\text{DR}}(d)$. By extrapolating the results, we therefore find that the two lines meet for $d = d_\times \approx 4.4$ and $N = N_\times \approx 14$. For the RFIM, where $N = 1$ and $d_{\text{DR}} \approx 5.1$, one should thus expect another scenario for the destabilization of the dimensional-reduction fixed point than the one found near $d = 4$. Indeed, for $d > 4.4$, the cuspless critical fixed point disappears at $N_{\text{DR}}(d)$ when it is still stable with respect to a cuspy perturbation.

We would like to emphasize again that the annihilation and disappearance of a pair of fixed points, with a square-root behavior of a coupling constant like in equation (5), is a rather common phenomenon in field theories, when several marginal operators are compatible with the symmetries of the problem. This situation is encountered in the Potts model [20, 21], in superconductors [22, 23], in Josephson junction arrays [24], in He₃ [25, 26], in smectic liquid crystals [27], in electroweak phase transitions [28, 29] and in frustrated magnets [30]. In all of these cases, the two fixed points meet and annihilate at some critical dimension. Beyond this dimension, the fixed-point characteristics acquire an imaginary part and are no longer of physical relevance. In the absence of a stable fixed point, the RG flow typically leads the system toward a region where the potential is unbounded from below because of operators of higher orders ($\phi^6$ terms for instance). This is in general interpreted as signaling the occurrence of a first-order transition.
This is however not what we find in the numerical analysis of the FRG flow equations for the RFIM. The typical situation is that there does exist a fixed point beyond the line where the two cuspless fixed points annihilate. In the next section, we present a toy model which we use to illustrate how a cuspy fixed point can emerge continuously from the annihilation of two cuspless fixed points. This unusual situation is made possible because we are renormalizing a full function, while in the situations previously mentioned, only a finite number of coupling constants are considered.

3. Toy model

We treat here a partial differential equation which is a generalization of the 1-loop flow equation of the RF\(O(N)\)M in equation (2). We consider a function \(\Delta(z)\) where \(z \in [-1, 1]\). The evolution under the RG flow is given by the following equation:

\[
\frac{\partial \Delta}{\partial t} = \Delta - \Delta \Delta' - \Delta - \Delta'' + \frac{1}{2}(1-z^2)\Delta'(z)\left[2z\Delta'(z) - (1-z^2)\Delta''(z)\right].
\]

The beta function depends on two parameters, \(A\) and \(B\), which replace the two parameters \(d\) and \(N\) of the RF\(O(N)\)M.

We start our study of the toy model by a determination of the region of parameters where cuspless fixed points exist. Assuming for now that the function \(\Delta_k\) is sufficiently regular, (i.e., that the first derivative is finite in \(z = 1\)), we get the following flow equations:

\[
\frac{\partial \Delta_k}{\partial t} = \Delta_k(1) - \Delta_k(1)\left[1 - \Delta_k(1)\right]
\]

\[
\frac{\partial \Delta_k'}{\partial t} = -B[\Delta_k(1) + \Delta_k'(1)]^2 + \Delta_k'(1)\left[1 - (1 + 2A)\Delta_k'(1)\right].
\]

Note that one has the property of the RF\(O(N)\)M that the flow of \(\Delta_k(1)\) depends on \(\Delta_k(1)\) only and that the flow of \(\Delta_k'(1)\) depends on \(\Delta_k(1)\) and \(\Delta_k'(1)\) only. The ‘critical’ fixed-point solution of interest is \(\Delta_\ast(1) = 1\), which is once unstable in the direction \(\Delta(1)\) (the associated eigenvalue is negative). The beta function for \(\Delta'(1)\), which is a polynomial in \(\Delta'(1)\), admits a real fixed-point solution for

\[
B \leq B_{DR}(A) = \frac{1}{8(1+A)}.
\]

If this condition is fulfilled, the solution reads

\[
\Delta_\ast'(1) = \frac{1 - 2B - \sqrt{1 - 8B(1 + A)}}{2(1 + 2A + B)}.
\]

There is also a conjugate fixed point, with a plus sign in front of the square root, which has at least two unstable directions and is therefore not associated with a critical point.
We now consider the vicinity of the cuspless critical fixed point and derive the eigenvalue $\Lambda_{p+1}$ associated with a perturbation whose functional form near $z = 1$ starts with $\sim (1 - z)^p$. A simple calculation leads to

$$\Lambda_{p+1} = 1 + \Delta_*(1)[p - 1 - B(p + 1)] - \Delta'_*(1)(p + 1)[1 + 2Ap + B].$$

(20)

The eigenvalue $\lambda = \Lambda_{3/2}$ is of particular interest because it is associated with the cuspy direction. In the region $B < B_{DR}$ where the cuspless critical fixed point exists we find that the cuspy direction is a relevant perturbation around the latter for $B_{cusp}(A) < B < B_{DR}(A)$ and $A < A_x = 3/2$, with

$$B_{cusp}(A) = \frac{-16 - 21A - 9A^2 + (4 + 3A)\sqrt{25 + 18A + 9A^2}}{36(1 + A)}.$$  

(21)

Observe that $B_{cusp}(A = A_x = 3/2) = B_{DR}(A = A_x = 3/2) = 1/20$ and that the eigenvalue $\lambda$ is then equal to zero, which means that, for this particular value of $A$, the cusp is marginal when the cusped fixed point vanishes. We summarize these findings in figure 6.

Another way of presenting the results is to evaluate $\lambda$ along the curve $B = B_{DR}(A)$. We then find

$$\lambda(A, B_{DR}(A)) = \frac{-3 + 2A}{4(3 + 4A)}.$$ 

(22)

For $A < A_x = 3/2$, the cuspy direction is already relevant when $B = B_{DR}(A)$. This is the typical situation encountered close to $d = 4$ in the RF$O(N)M$ at one-loop order. On the contrary, for $A > A_x = 3/2$, the cuspy direction is still irrelevant when the cusped fixed points annihilate for $B = B_{DR}(A)$, which is the typical situation for the RFIM close to $d = 5.1$.

We now study how a cuspy fixed point can appear when the cusped fixed points annihilate and disappear. We focus on the immediate vicinity of $B_{DR}$ and define, for a given $A$, $B = B_{DR}(A) + \epsilon'$. We anticipate that the cusp should appear in a boundary layer around $z = 1$ that shrinks to zero as $\epsilon'$ goes to zero. We therefore make the following ansatz:

$$\Delta_\epsilon(z) = 1 - \epsilon_k\left(\frac{1 - z}{\epsilon}\right),$$

(23)

where $\epsilon \to 0$ when $\epsilon' \to 0$. After inserting this expression in the flow equation, equation (15) and expanding at leading order in $\epsilon$, we get

$$\partial_\epsilon f_\epsilon(y) = -\frac{1}{16(1 + A)}\left\{\frac{9 + 8A}{y}f_{\epsilon}'(y)[f_{\epsilon}(y) - f_{\epsilon}(0)] + 2y^2 + 2f_\epsilon(y) + (7 + 8A)[2f_\epsilon(0) - yf_{\epsilon}'(y)] + 4A(1 + A)f_{\epsilon}'(y)[f_{\epsilon}'(y) + yf_{\epsilon}'(y)]\right\},$$

(24)

where $y = \sqrt{(1 - z)}/\epsilon$. Note that $\epsilon'$, which measures the distance to $B_{DR}$, does not appear in this equation. This means that, at least at this level, we are unable to relate...
the typical size of the boundary layer to the distance to $B_{DR}$. We simply assume here that both tend to zero simultaneously.

We are interested in the fixed-point solution of the above flow equation. For $y \gg 1$, i.e. outside the boundary layer, the fixed-point function behaves as

$$f_*(y) \approx y^2 / (3 + 4A).$$

When inserted in equation (23) this leads to $\Delta_*(z) = 1 - (1 - z)/(3 + 4A)$, which coincides with the expansion near $z = 1$ of the cuspless fixed point in $B = B_{DR}(A) = 1/[8(1+A)]$ (see equation (19)), i.e. for $\epsilon' = 0$.

Expanding now equation (24) for small $y$ (inside the boundary layer, where $y \ll 1$), we find that the flow equation of $f_*(y)$ depends only on the $p + 1$ first derivatives at the origin. We can therefore solve iteratively the fixed-point solution and express the derivatives of $f_*(y)$ at the origin as a function of one unknown, $f_*(0)$. We find in particular that

$$f'_*(0) = -16(1 + A) / (3 + 2A)^2,$$

$$f''_*(0) = 2(5 + 8A) / 3(9 + 16A + 8A^2).$$

This allows us to make predictions for the behavior of the original function. When $1 - z \to 0$ (inside the boundary layer), we expand $\Delta_*(z)$ as

$$\Delta_*(z) = \Delta_*(1) - a_* \sqrt{1 - z} + \Delta_{*,1}(1 - z) + \cdots$$

Figure 6. Toy model phase diagram in the $(A, B)$ plane. The curve $B_{DR}(A)$ (full red line) is the boundary at which the cuspless fixed points present below it merge and annihilate. The curve $B_{cusp}(A)$ (dashed blue line) is where the cuspless critical fixed point becomes unstable to a cuspy perturbation. The black point at $(A_0 = 3/2, B_0 = 1/20)$ is the intersection between the two curves. To the left of this point, when $B$ is increased, the cuspless fixed point first exchanges stationarity with a cuspy fixed point for $B = B_{cusp}$, before it disappears at $B = B_{DR}$. To the right of this point, the cuspless critical fixed point is stable up to $B = B_{DR}$. Between the two lines, the cuspless critical fixed point exists but is unstable with respect to a cuspy perturbation. Note the similarity with figure 1, except that the regions with or without cusp are not located in the same place.
Δ*,1 should not be interpreted here as the first derivative of Δ*(z) in z = 1 because of the singular dependence in √1−z. We then derive that

\[ \Delta_* = \frac{a_*^2}{1 - \Delta_*(1)^2} = \frac{f_*'(0)^2}{f_*'(0)} = -\frac{16(1 + A)}{(3 + 2A)^2} \]  

\[ \Delta_{*,1} = \frac{f_*''(0)}{2} = \frac{5 + 8A}{3(9 + 16A + 8A^2)}. \]

A direct comparison of equation (30), which is valid when \( B \rightarrow B_{DR}(A) \), with the result for \( B = B_{DR}(A) \), obtained from equation (19) with \( B = B_{DR}(A) = 1/[8(1+A)] \) or from the outer boundary-layer solution described above, shows that \( \Delta_{*,1} \) is in general discontinuous for \( B = B_{DR} \), except in \( A = 3/2 \) where \( \Delta_{*,1}|_{B_{DR}} = \Delta_{*,1}|_{B_{DR}} = 1/9 \). On the contrary, \( \Delta_*(1) \) is always continuous in \( B_{DR} \) and so is the amplitude of the cusp \( a_* \) that continuously goes to zero as \( B \rightarrow B_{DR} \). As a consequence, the critical exponents that depend only on \( \Delta_*(1) \) and on the amplitude of the cusp, which is the case of the exponent \( \nu \) of the correlation length and of the anomalous dimensions [3], are continuous. On the other hand, eigenvalues that depend on \( \Delta_{*,1} \) are not: this is the case for instance of the eigenvalue \( \lambda \) associated with a cuspy perturbation. Although \( \Delta_*(z) \) is a continuous function of \( B \) at fixed \( z \) when \( B \) increases from \( B_{DR} \), some properties of the fixed point may be discontinuous, which is a very unusual situation in the RG of critical phenomena.

We have checked by a direct numerical integration of the flow equation in equation (15) that the behaviors predicted above are indeed observed. This is illustrated for \( A = 8 > A_c \): figure 7 for \( \Delta_{*,1} \) and figure 8 for the eigenvalue \( \lambda \).

Finally, in the particular case \( A = 0 \), we have been able to solve analytically the fixed-point equation in equation (25). There is a unique family of solutions parametrized by \( f_0(0) \). It can be expressed in terms of the Lambert function \( W(x) \) (solution of \( W e^{W} = x \)).

Figure 7. \( \Delta_{*,1} \) as a function of \( B \) for \( A = 8 \). For \( B < B_{DR}(A = 8) = 1/72 \), the numerical solution approaches the predicted result in the absence of a cusp, 1/35, when finer meshes (larger number of points) are considered. For \( B \) approaching 1/72 from above, the numerical solution tend to 23/649, which is the solution extracted from the analysis of the boundary layer, see equation (30).
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\[
\begin{align*}
\{ & = + \quad - \\
\end{align*}
\]

\[
\begin{pmatrix}
+ & - \\
- & +
\end{pmatrix}
\]

\( f_y f_y W(0) \) \]

It is easily checked that the above solution satisfies the limiting behaviors described above when \( y \to \infty \) and \( y \to 0 \). Note that this case corresponds to the region where \( A < A_x \) and that the above cuspy fixed point which emerges from the merged cuspless ones is unstable. Another cuspy fixed is present and is stable for \( B > B_{\text{cusp}} \) (here, \( B_{\text{cusp}}(A = 0) = 1/9 \)). The generic situation is that there are two cuspy fixed points above \( B_{\text{DR}} \), one stable and one unstable, one that appears through a boundary-layer mechanism at \( B_{\text{DR}} \) and one that is already present below \( B_{\text{DR}} \).

4. The short- and long-range RFIM

We have next investigated the \( d \)-dimensional RFIM (\( N = 1 \)). In this case however, a nonperturbative FRG (NP-FRG) is required [1, 2, 4]. The central quantity is now the dimensionless cumulant of the renormalized random field \( \delta_k(\varphi_1, \varphi_2) \). It follows an FRG equation that is coupled to other functions describing the flow of the disorder-averaged effective action (the latter is described in a derivative expansion [1, 4]). The flow equations can be symbolically written as

\[
\begin{align*}
\partial_t u'_k(\varphi) &= \beta u'(\varphi), \\
\partial_t z_k(\varphi) &= \beta z(\varphi), \\
\partial_t \delta_k(\varphi_1, \varphi_2) &= \beta_k(\varphi_1, \varphi_2),
\end{align*}
\]

where as before \( t = \log(k/\Lambda) \); \( u_k(\varphi) \) is the dimensionless effective average potential (i.e., the local component of the disorder-averaged effective action) and \( z_k(\varphi) \) is the
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dimensionless function describing the renormalization of the field. The beta functions themselves depend on \( u'k, zk, \delta k \) and on their derivatives. Their expressions are given in [4] and are not reproduced here.

We consider first the usual short-range RFIM in which both the interactions and the random-field correlations are short-ranged (see equation (1) and below). Fixed points are studied by setting the left-hand sides of the equations in equation (32) to zero. The zero-temperature fixed point controlling the critical behavior has been determined in a previous investigation [4]: above a dimension \( d_{DR} \) close to 5.1, there exists a cuspless fixed point that can be reached when starting from a regular, cuspless, initial condition. The presence or absence of a cusp now refers to the dependence of \( \delta k(\varphi_1, \varphi_2) \) on the field difference \( \varphi_2 - \varphi_1 \). For its description, it turns out to be more convenient to change variable from \( \varphi_1 \) and \( \varphi_2 \) to \( \varphi = (\varphi_1 + \varphi_2)/2 \) and \( y = (\varphi_1 - \varphi_2)/2 \). The putative cusp is now in the variable \( y \). For \( d > d_{DR} \), the (critical) cuspless fixed point, which is characterized in the limit \( y \to 0 \) by

\[
\delta_*(\varphi, y) = \delta_{*,0}(\varphi) + \frac{1}{2} \delta_{*,2}(\varphi) y^2 + O(|y|^3),
\]

is stable with respect to cuspless perturbations, except of course for the relevant direction that corresponds to a fine-tuning to the critical point. As already stressed, such a fixed point corresponds to the \( d \to d - 2 \) dimensional reduction.

We have also investigated the stability of the cuspless, dimensional-reduction, fixed point with respect to a cuspy perturbation. We have followed the procedure described above for the ROO(N)M near \( d = 4 \). We search for a physical eigenfunction \( f(\varphi, y) \) with a linear cusp in \( y \) that is a linear combination of two solutions of the associated eigenvalue equation, \( f^-(\varphi, y) \) and \( f^+(\varphi, y) \), the former having a cusp when \( y \to 0 \), i.e. \( f^-(\varphi, y) \approx |y|f(\varphi) + O(y^2) \), and the latter having a subcusp only, i.e. \( f^+(\varphi, y) \approx |y|^{\alpha_+(d)} f(\varphi) + O(y^2) \), with \( \alpha_+(d) \) odd or noninteger. The linear combination should ensure that all divergences are cancelled and that the physical eigenfunction is defined for all values of \( y \).

The corresponding eigenvalue \( \lambda \) can then be determined by considering the vicinity of the fixed point with \( \delta_k(\varphi, y) \approx \delta_*(\varphi, y) + k^\lambda f(\varphi, y) \) and \( f(\varphi, y) \approx |y|f(\varphi) \) when \( y \to 0 \). By linearizing the flow equation for \( \delta_k \) around \( \delta_* \), fixing \( u'k(\varphi) \) and \( zk(\varphi) \) to their fixed-point values and expanding around \( y = 0 \), it is easy to derive that \( f_- (\varphi) \) satisfies the following eigenvalue equation:

\[
\lambda f_-(\varphi) = \frac{1}{2} (d-4+3\eta) f_-(\varphi) + \frac{1}{2} (d-4+\eta) \varphi f'_- (\varphi) \\
+ u_k \delta \int_0^\infty dx x^{d-1} \left\{ \frac{3}{2} f_-(\varphi) \left[ 4 z_*(\varphi) p_*(x, \varphi) p_*^{(0,1)}(x, \varphi) \\
+ 4 [z_*(\varphi) + s'(x)] p_*^{(0,1)}(x, \varphi)^2 + [z_*(\varphi) - \delta_{*,2}(\varphi)] p_*(x, \varphi)^2 \right] \right. \\
+ 3 f_-'(\varphi) p_*(x, \varphi) \left[ 2 [z_*(\varphi) + s'(x)] p_*^{(0,1)}(x, \varphi) + z_*'(\varphi) p_*(x, \varphi) \right] \\
+ f_-''(\varphi) [z_*(\varphi) + s'(x)] p_*(x, \varphi)^2 \left\}. \right.
\]

\[
(36)
\]
where $v_t^{-1} = 2^{d+1} \pi^{d/2} \Gamma(\frac{d}{2})$, partial derivatives are denoted by superscripts in parentheses and $x$ is the square of the dimensionless momentum; $p_\phi(x, \phi) = [x z_\phi(\phi) + s(x) + u'_\phi(\phi)]^{-1}$ is the (dimensionless) ‘propagator’, i.e. the so-called ‘connected’ 2-point correlation function, and $s(x)$ is a (dimensionless) cutoff function. (Choices of appropriate functional forms for $s(x)$ are discussed in [4].) Finally, $\tilde{\partial}$ is an operator acting only on the cutoff function $s(x)$ (appearing explicitly or through the dimensionless propagator) with $\tilde{\partial}s(x) \equiv (2 - \eta) s(x) - 2x s'(x)$. In deriving the above equation, we have used the fact that $\bar{\sigma} = \eta$ and $\delta_{*, 0}(\phi) = z_\phi(\phi)$, which are properties of the cusless, dimensionally-reduced, fixed point resulting from the underlying supersymmetry [4].

An equation for the fixed-point function $\delta_{*, 2}(\phi)$ that appears in equation (8) can be also derived by inserting the expansion in powers of $y$ of $\delta_{*, 0}(\varphi, y)$ (see equation (35)) in the corresponding beta function in equation (32). The algebra is straightforward but cumbersome and leads to:

$$0 = (d - 4 + 2 \eta_0) \delta_{*, 2}(\phi) + \frac{1}{2} (d - 4 + \eta_0) \phi \delta_{*, 2}'(\phi)$$

$$+ v_t \bar{\partial}_i \int_0^\infty dx \, x^{\frac{d}{2} - 1} \left\{ 4 \tilde{p}_\phi^{(0, 1)}(x, \phi) \tilde{p}_\phi^{(0, 2)}(x, \phi) z_\phi(\phi) \right\}$$

$$+ 5 \tilde{p}_\phi^{(0, 1)}(x, \phi)^2 (\phi)^2 + \tilde{p}_\phi(x, \phi)(2 \tilde{p}_\phi(x, \phi) z_\phi''(\phi) \right\}$$

$$+ 7 \tilde{p}_\phi^{(0, 2)}(x, \phi) z_\phi'(\phi)^2 + 4 z_\phi'(\phi)(2 z_\phi''(\phi) \tilde{p}_\phi^{(0, 1)}(x, \phi)$$

$$+ \tilde{p}_\phi(0, 3)(x, \phi) \right\} - 2 \tilde{p}_\phi(0, 2)(x, \phi)^2$$

$$- \tilde{p}_\phi(0, 1)(x, \phi) \tilde{p}_\phi(0, 3)(x, \phi) \right\} \left[ z_\phi(\phi) + s'(x) \right] + \frac{1}{2} \tilde{p}_\phi(x, \phi)^2 \right\}$$

$$+ 2 z_\phi^{(3)}(\phi) z_\phi'(\phi)) + 3 \delta_{*, 2}(\phi) \left(4 \tilde{p}_\phi^{(0, 1)}(x, \phi) \tilde{p}_\phi(x, \phi) z_\phi'(\phi)$$

$$+ 4 \tilde{p}_\phi^{(0, 1)}(x, \phi)^2 \right\}$$

$$+ \delta_{*, 2}(\phi) \left(4 \tilde{p}_\phi(x, \phi)(2 \tilde{p}_\phi^{(0, 1)}(x, \phi) \right\}$$

$$+ \delta_{*, 2}(\phi) \tilde{p}_\phi(x, \phi)^2 \left[ z_\phi(\phi) + s'(x) \right] - \frac{3}{2} \delta_{*, 2}(\phi)^2 \tilde{p}_\phi(x, \phi)^2 \right\}$$

where we have used that $\delta_{*, 0}(\phi) = z_\phi(\phi)$. From the knowledge of $u'_\phi(\varphi)$ and $z_\phi(\phi)$, which are obtained from two coupled equations (see [4]), we first solve the equation for $\delta_{*, 2}(\phi)$ and then use the input to solve equation (8). All partial differential equations are numerically integrated on a one-dimensional grid by discretizing the field $\varphi$. The resulting eigenvalue $\lambda(d)$ is plotted in figure 9. Note that $\lambda$ can be calculated exactly at the Gaussian fixed point that controls the critical behavior at and above the upper critical dimension $d = 6$ and one finds $\lambda = 1$ with an associated eigenfunction $f(\varphi, y) = |y|$. As seen in the figure, $\lambda$ is small but strictly positive when $d = d_{DR}$, in agreement with the phase diagram displayed in figure 1.

We have also checked that there is an additional solution $f^{(+)}(\varphi, y)$ that is associated with the same eigenvalue $\lambda$ and whose dependence on $y$ starts with a subcusp when $y \to 0$. In addition, we have repeated the analysis for the cusless unstable fixed point

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that is conjugate to the above critical one: it is characterized by the same
$u'(\varphi)$ and $z_*(\varphi) = \delta_{*,0} (\varphi)$ but corresponds to another solution $\delta_{*,2} (\varphi)$ of equation (37). The eigenvalue associated with a cuspy perturbation around this fixed point is plotted in the bottom panel of figure 9 and it merges with that for the other fixed point for $d = d_{\text{DR}}$. The cuspy eigenvalues for both cuspless fixed points have a square root behavior, as shown in the figure. We can fit these curves by a parabola, $d(\lambda) = 5.1503 - 0.0199\lambda + 0.8279\lambda^2$. We observe that $\lambda(d_{\text{DR}})$ is slightly positive, as already mentioned and that a cuspy perturbation around the unstable cusps point is marginal in a dimension slightly larger than $d_{\text{DR}}$. From the results of the preceding sections, this indicates that, in the case of the short-range RFIM, the breaking of dimensional reduction is associated with the appearance of a cuspy fixed point through a boundary layer. However, since $\lambda(d_{\text{DR}})$ is very small, we expect that the unusual features that signals the presence of a boundary-layer mechanism (in particular the discontinuity of the coefficient $\delta_{*,2}(\varphi)$ of the term in $y^2/2$ of the small $y$ expansion of $\delta_*(\varphi, y)$) to be almost unobservable.
Consequently, we have also investigated the RFIM in the presence of both long-range interactions, which decay in space as $|x - y|^{-(d + \sigma)}$, and long-range correlations of the random field that vary as $|x - y|^{-(d - \rho)}$. We have recently shown that for a specific choice of the exponents characterizing these long-range spatial dependences, namely $\rho = 2 - \sigma$, a supersymmetry can still be present in the associated superfield theory [5]. This supersymmetry leads to a $d \to d - 2$ dimensional-reduction property. The corresponding cusps fixed points exists below a critical value $\sigma_{DR}$, which in $d = 3$ is found between 0.71 and 0.72, depending on the precise choice of the dimensionless cutoff function $s(x)$, and disappears above. (In this case, the analogs of the lower and upper critical dimensions are a critical value $\sigma = 1$ above which there is no transition and a critical value $\sigma = 1/2$ below which the exponents are described by mean-field theory.)

In this long-range model, we have repeated the analysis of a cuspy perturbation around the stable and unstable cusps fixed points. For the stable (critical) fixed point the eigenvalue $\lambda$ decreases from 7/4 for $\sigma = 1/2$ to $\sim 0.7$ for $\sigma_{DR}$. The eigenvalue of the cuspy perturbation around the unstable fixed point increases from $-1/4$ for $\sigma = 1/2$ to 0.7 for $\sigma_{DR}$ and passes through zero for $\sigma \simeq 0.66$. This is displayed in figure 10. In this case, the eigenvalue associated with the cuspy perturbation is unambiguously strictly positive when the two cusps fixed points coalesce, as seen in figure 10 and the cuspy perturbation around the unstable fixed point becomes marginal for $\sigma_{cusp} \simeq 0.65$, which is significantly different from $\sigma_{DR}$. We therefore expect a cuspy fixed point to appear through a boundary layer for $\sigma > \sigma_{DR}$ with a sizable discontinuity in $\delta_{s,2}(\varphi)$ in $\sigma_{DR}$. (Note that $\delta_{s,2}(\varphi)$ is obtained as the second derivative of $\delta_{s}(\varphi, y)$ with respect to $y$ in $y = 0$ only in the absence of a cusp.)

To complement the above study of the stability of the cusps fixed points above $\sigma_{DR}$, we have integrated the flow equations (see equation (32)) without expanding in the $y$-direction. We focus on the long-range model. We find strong evidence for the occurrence of a boundary-layer mechanism for the appearance of a stable cusps fixed point above $\sigma_{DR}$. This is illustrated in figure 11 where we plot $\delta_{s,2}(\varphi)$ for two different values of $\varphi$ as a function of $\sigma$ around $\sigma_{DR}$. It can be seen that a discontinuity builds up as the mesh size is decreased, very much as in the toy model (see in particular figure 7).
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5. Conclusion

We have analyzed the mechanism by which the dimensional-reduction result breaks down in the $RFO(N)M$ by following the appearance, disappearance and change of stability of the (zero-temperature) fixed points. We have combined the perturbative FRG results near $d = 4$, at one and two loops, the nonperturbative FRG results, in particular for the short- and long-range RFIM and a toy model. Dimensional reduction for the critical behavior of the model is associated with a cuspless fixed point and breaking of dimensional reduction with a cuspy fixed point. (We recall that the cuspless or cuspy character refers to the functional dependence of the dimensionless second cumulant of the renormalized random field [1–5] and is physically associated with the subdominant or dominant role of the avalanches in the correlation functions [14].)

The outcome of our study is an intricate scenario which is illustrated in the $(N, d)$ phase diagram of figure 1. There are two different regimes separated by a threshold point whose estimated location is $(d_x \simeq 4.4, N_x \simeq 14)$. For smaller $d$ and larger $N$, the phenomenon of dimensional reduction takes place, while for larger $d$ and smaller $N$, the critical behavior is associated with a cuspless fixed point.
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the cuspless fixed point that leads to dimensional reduction is destabilized by a cuspy fixed point for some $N = N_{\text{cusp}}(d)$. This is a rather usual phenomenon, where two fixed points exchange their stability by crossing. For larger $d$ and smaller $N$, the critical cuspless fixed point annihilates with an unstable cuspless fixed point for some $N = N_{\text{DR}}(d)$. A new, cuspy, fixed point then emerges from these merged fixed points through a boundary-layer mechanism. This unusual phenomenon has some specific signatures in derivatives of the cumulants of the renormalized random field. These signatures appear too small to be detected in the standard short-range RFIM but can be numerically seen in the RFIM in the presence of long-ranged interactions and disorder correlations.

Acknowledgments

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