TOWARD A FIELD THEORETICAL DESCRIPTION OF
TOPOLOGICALLY LINKED POLYMERS

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In this work a field theoretical model is constructed to describe the statistical mechanics of an arbitrary number of topologically linked polymers in the context of the analytical approach of Edwards. As an application, the effects of the topological interactions are studied in the one loop approximation. A natural way to include in the treatment also more sophisticated link invariants than the Gauss linking number is outlined in the Conclusions.

I. INTRODUCTION

Long flexible polymer molecules provide an ideal environment to study the effects of topology in statistical mechanics and condensed matter physics, but so far a comprehensive theory of topologically linked polymers does not exist. Until now, only the physics of unentangled open chains has been understood to a satisfactory extent \([1]\). In this letter the problem is tackled in the context of the so-called analytical approach, consisting in a number of different methods based on Edwards pioneering work \([2]\), in which the fluctuations of each polymers are treated as a random walk and described via a path integral in the Wiener sense. Analytical approaches of this kind have already been applied to several complex situations \([3] - [12]\) and seem very promising in explaining the physics of topological entangled polymers. Despite of that, they are affected by two main limitations. First of all, the Gauss linking number, a weak topological invariant, is used to distinguish the topological states of the system. Unfortunately, more sophisticated knot invariants like those described in \([13]\)
and \( \mathbb{L} \) cannot be applied in practice, since they have no immediate relation to the physical conformation of the polymer \( \mathbb{P} \). On the other side, one of the advantages of the analytical approach is to provide a field theoretical description of a system of polymers, establishing in this way a connection to the theory of critical phenomena in condensed matter and high energy physics. However, a mapping from the statistical mechanical problem of polymers to field theories has been realized only in a limited number of cases, mainly concerning the fluctuations of a single test polymer in a background of static polymers or fixed obstacles. Remarkable exceptions occur when two polymers are very close \( \mathbb{L} \), \( \mathbb{L} \), a situation which is particularly relevant in the study of DNA molecules \( \mathbb{L} \), or when the number of polymers becomes large \( \mathbb{L} \). Despite of these successes, the exact analytical treatment of a system of \( N \) polymers has been a formidable problem already mentioned in the pioneering work on topological entanglement of Brereton and Shah \( \mathbb{L} \) and remained unsolved until the present days.

To solve it, we construct a Chern-Simons (C-S) \( \mathbb{L} \) based model, which generalizes a previous work of the authors in which the two polymer case was discussed \( \mathbb{L} \). With respect to ref. \( \mathbb{L} \), the main difficulty is that one needs \( \frac{N(N-1)}{2} \) independent topological numbers to distinguish the possible topological states of \( N \) polymers according to the Gauss link invariant. As a consequence, one has to find a suitable C-S field theory which decouples the actions of the polymers as in \( \mathbb{L} \) and simultaneously is able to accommodate the huge amount of topological configurations allowed by the system.

The material contained in this paper will be divided as follows. In the next Section a topological Ginzburg-Landau model is constructed which describes the fluctuations of a system of \( N \) topologically linked polymers. The topological relations are specified using the Gauss linking number. Section III contains a perturbative study of the case \( N = 2 \). In particular, the one loop effective potential is computed proving that the topological interactions do not influence the critical behavior of the system in this approximation. Moreover, it is shown that the second topological moment can be exactly computed in terms of one loop Feynman diagrams. Finally, a detailed discussion of the results has been given in Section 2.
IV. In that Section a natural method to include higher order topological invariants is outlined. To this purpose, a new C-S field theory is defined, which is abelian but exhibits cubic interactions among the C-S fields. In this way such theory can generate radiative corrections containing knot invariants with the same mechanism acting in the case of nonabelian string theories [18, 19].

II. THE $N$–POLYMERS PROBLEM

Let $P_1, \ldots, P_N$ be a set of topologically linked polymers, describing in the space the trajectories $\Gamma_i, \; i = 1, \ldots, N$ defined as follows:

$$\Gamma_i = \{r_i(s_i) | 0 \leq s_i \leq L_i \; ; \; r_i(0) = r_i', r_i(L_i) = r_i\}$$

where $L_i$ denotes the contour length of $\Gamma_i$. The polymers can be either open or closed ($r_i' = r_i$). The topological states will be distinguished using the Gauss linking number:

$$\chi(\Gamma_i, \Gamma_j) \equiv \frac{1}{4\pi} \int_0^{L_i} \int_0^{L_j} dr_i(s_i) \times dr_j(s_j) \cdot \frac{(r_i(s_i) - r_j(s_j))}{|r_i(s_i) - r_j(s_j)|^3}$$

where $\chi(\Gamma_i, \Gamma_j)$ takes integer values $m_{ij}$ if $\Gamma_i, \Gamma_j, \; i \neq j = 1, \ldots, N$, are closed trajectories. To describe the statistical mechanics of the polymers we define the configuration probability $G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0)$. This function measures the probability that the trajectories $\Gamma_i$ fulfill the constraints:

$$\chi(\Gamma_i, \Gamma_j) = m_{ij} \quad \; \; \; \; i \neq j = 1, \ldots, N$$

and have extrema (in the open case) at the points $r'_i$ and $r_i, \; i = 1, \ldots, N$. In our notations $\{m\}$ denotes the $n \times n$ symmetric matrix of topological numbers with elements

$$m_{ij} = m_{ji} \quad \text{for} \quad i \neq j \quad ; \quad m_{ii} = 0$$

while $\{r\} = r_1, \ldots, r_N, \; \{L\} = L_1, \ldots, L_N$ etc. Starting from the path integral approach of Edwards [2], one obtains the following expression of $G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0)$:
\begin{align}
G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0) &= \int_{-\infty}^{+\infty} \left[ \prod_{i=1}^{N-1} \prod_{j=2}^{N} \frac{d\lambda_{ij}}{2\pi} e^{-i\lambda_{ij} m_{ij}} \right] G_{\{\lambda\}}(\{r\}, \{L\}; \{r'\}, 0) \\
\text{where} \\
G_{\{\lambda\}}(\{r\}, \{L\}; \{r'\}, 0) &= \int_{r_1}^{r} \mathcal{D}r_1(s_1) \ldots \int_{r_N}^{r} \mathcal{D}r_N(s_N) \exp \left\{ -(A_0 + A_{ev} + A_{top}) \right\} 
\end{align}

In the above equation the action

\begin{equation}
A_0 = \frac{3}{2a^2} \sum_{i=1}^{N} \int_{0}^{L_i} \rho_i^2(s_i)
\end{equation}

takes into account the excluded volume interactions.

For convenience, \( v_{ij}^0 \) has been defined as follows:

\begin{equation}
\begin{cases}
\tilde{v}_{ij}^0 & \text{for } i = j \\
\tilde{v}_{ij}^0/2 & \text{for } i \neq j
\end{cases}
\end{equation}

where the \( \tilde{v}_{ij}^0 \) are coupling constants with the dimension of a volume.

Finally, the topological constraints (3) are responsible of the topological term:

\begin{equation}
A_{\text{top}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \chi(\Gamma_i, \Gamma_j) \lambda_{ij}
\end{equation}

coming from the well known Fourier transform of the Dirac \( \delta \)-function:

\begin{equation}
\delta(m_{ij} - \chi(\Gamma_i, \Gamma_j)) = \int_{-\infty}^{+\infty} \frac{d\lambda_{ij}}{2\pi} \exp \left\{ -i [m_{ij} - \chi(\Gamma_i, \Gamma_j)] \lambda_{ij} \right\}
\end{equation}

To express the configurational probability (6) in terms of fields, we introduce \( N \) gaussian scalar fields \( \phi_i(r) \), \( i = 1, \ldots, N \) and \( N(N-1) \) free C-S fields \( \mathbf{A}_{ij}^{(i)} \) and \( \mathbf{B}_{ij}^{(i)} \), with \( i = 1, \ldots, N - 1, j = 2, \ldots, N \) and \( j > i \). The action of the scalar fields is given by:

\begin{equation}
A_{\phi} = \frac{a^2}{2} \sum_{i,j=1}^{N} \int d^3r \phi_i \left( v^0 \right)^{-1} [\phi_j]
\end{equation}
while C-S one is given by:

$$S_{CS} = \frac{\kappa}{4\pi} \int d^3 r \sum_{i=1}^{N-1} \sum_{j>i}^{N} A_{(i)}^{(j)} \cdot (\nabla \times B_{(j)}^{(i)})$$

(13)

The C-S theory will be quantized in the Landau gauge, where the fields $A_{(i)}^{(j)}$ and $B_{(j)}^{(i)}$ are completely transverse. It turns out that the following relation is valid for the excluded volume term:

$$e^{-A_{ev}} = \int D\phi_1 \ldots D\phi_N \exp \left\{ -A_{(\phi)} - i \sum_{i=1}^{N} \int d^3 r J_i(r) \phi_i(r) \right\}$$

(14)

where

$$J_i(r) = \int_0^{L_i} ds_i \delta^{(3)}(r - r_i(s_i))$$

(15)

To treat the topological interaction it will be convenient to define the following linear combinations of the fields $A_{(i)}^{(j)}$ and $B_{(j)}^{(i)}$:

$$C^{(1)} = \sum_{j=2}^{N} \alpha_{(j)}^{(1)} A_{(j)}^{(1)}$$

(16)

$$C^{(i)} = \sum_{j=3}^{N} \alpha_{(j)}^{(i)} A_{(j)}^{(i)} + \sum_{j=1}^{N-2} \beta_{(i)}^{(j)} B_{(j)}^{(i)} \quad i = 2, \ldots, N - 1$$

(17)

and

$$C^{(N)} = \sum_{i=1}^{N-1} \beta_{(N)}^{(i)} B_{(N)}^{(i)}$$

(18)

The parameters $\alpha_{(j)}^{(i)}$ and $\beta_{(j)}^{(i)}$ depend on the C-S coupling constant $\kappa$ and on the matrix elements $\lambda_{ij}$ as follows:

$$\alpha_{(j)}^{(i)} = \frac{\kappa}{4\pi} \quad \beta_{(j)}^{(i)} = \lambda_{ij}$$

(19)

In terms of these fields $C$ it is now possible to state the Gaussian identity:

$$\int DADB \exp \left\{ -i S_{CS} - i \sum_{i=1}^{N} \int_0^{L_i} C^{(i)}(r(s_i)) dr(s_i) \right\} = \exp \left\{ -i \sum_{i=1}^{N-1} \sum_{j=2}^{N} \lambda_{ij} \chi(\Gamma_i, \Gamma_j) \right\}$$

(20)
Eq. (20), where we have used the compact notation:

$$
\int DADB \equiv \int \prod_{i<j=1}^N DA_{(i)}^DDB_{(j)}^{(i)}
$$

relates the topological term (10) to the amplitude of $N$ holonomies and will be central in the subsequent discussion. Substituting eqs. (20) and (14) in (6), the configurational probability $G_{(\lambda)}(\{r\}, \{L\}; \{r\}', 0)$ becomes:

$$
G_{(\lambda)}(\{r\}, \{L\}; \{r\}', 0) = \langle \prod_{i=1}^N G(r_i, L_i; r_i', 0|\phi_i, C^{(i)}) \rangle_{\{\phi\}, \{A\}, \{B\}}
$$

(22)

In the above formula the functions $G(r_i, L_i; r_i', 0|\phi_i, C^{(i)})$ are given by:

$$
G(r_i, L_i; r_i', 0|\phi_i, C^{(i)}) = \int_{r_i'}^{r_i} Dr_i(s_i) \exp \left\{ -\int_0^{L_i} ds_i L_{\phi_i}(r_i(s_i)) - i \int_0^{L_i} ds_i \dot{r}_i(s_i) \cdot C^{(i)}(r_i(s_i)) \right\}
$$

(23)

where

$$
L_{\phi_i}(r_i(s_i)) = \frac{3}{2a} \dot{r}_i^2(s_i) + i\phi_i(r_i)
$$

(24)

Moreover, everything inside the bracket $\langle \quad \rangle_{\{\phi\}, \{A\}, \{B\}}$ must be averaged with respect to the fields $\{\phi\}, \{A\}$ and $\{B\}$ by means of eqs. (14) and (20). As we see from eqs. (22)-(24) the polymer trajectories $\Gamma_1, \ldots, \Gamma_N$ are completely decoupled before averaging over the auxiliary fields. Furthermore, each of the factors $G(r_i, L_i; r_i', 0|\phi_i, C^{(i)})$ appearing inside the bracket $\langle \quad \rangle_{\{\phi\}, \{A\}, \{B\}}$ coincides formally with the evolution kernel of a particle subjected to random walk (in the Wiener sense of the path-integral formulation of statistical mechanics) and immersed in the electromagnetic field $(i\dot{\phi}_i, C^{(i)})$. Thus $G(r_i, L_i; r_i', 0|\phi_i, C^{(i)})$ satisfies the pseudo-Schrödinger equation [7]:

$$
\left[ \frac{\partial}{\partial L_i} - \frac{a}{6} D_i^2 + i\phi_i \right] G(r_i, L_i; r_i', 0|\phi_i, C^{(i)}) = \delta(L_i)\delta(r_i - r_i')
$$

(25)

The covariant derivatives $D_i$ are given by:

$$
D_i = \nabla + iC^{(i)} \quad i = 1, \ldots, N
$$

(26)
At this point it is convenient to consider the Laplace transform of the configurational probability $G_\lambda(\{r\}, \{L\}; \{r'\}, 0)$ with respect to the polymer lengths \[7\]:

$$G_\lambda(\{r\}, \{r\}', \{z\}) = \int_0^{+\infty} dL_1 \cdots \int_0^{+\infty} dL_N \exp \left\{ - \sum_{i=1}^N z_i L_i \right\} G_\lambda(\{r\}, \{L\}; \{r'\}, 0) \quad (27)$$

Each variable $z_i$ plays the role of a Boltzmann-like factor governing the distribution length of the $i$–th polymer. Applying the above Laplace transformations to the expression of $G_\lambda(\{r\}, \{L\}; \{r'\}, 0)$ given by eq. \(22\), one finds:

$$G_\lambda(\{r\}, \{r\}', \{z\}) = \langle N \prod_{i=1}^N G(r_i, r_i' ; z_i | \phi_i, C^{(i)}) \rangle_{\{\phi\}, \{A\}, \{B\}} \quad (28)$$

where the functions $G(r_i, r_i' ; z_i | \phi_i, C^{(i)})$ are the Laplace transforms of $G(r_i, L_i; r_i', 0 | \phi_i, C^{(i)})$. Therefore, they obey the stationary pseudo-Schroedinger equations:

$$\left[ z_i - a_6 D_i^2 + i \phi_i \right] G(r_i, r_i' ; z_i | \phi_i, C^{(i)}) = \delta(r_i - r_i') \quad (29)$$

The solution of eq. \(29\) can be given in terms of second quantized fields $\psi_i^*, \psi_i$, $i = 1, \ldots, N$:

$$G(r_i, r_i' ; z_i | \phi_i, C^{(i)}) = Z_i^{-1} \int D\psi_i D\psi_i^* \psi_i(r_i) \psi_i^*(r_i') e^{-F[\psi]} \quad (30)$$

where $F[\psi_i]$ represents the Ginzburg-Landau free energy of a superconductor in a fluctuating magnetic field:

$$F[\psi_i] = \int d^3r \left[ \frac{a}{6} |D_i \psi_i|^2 + (z_i + i \phi_i) |\psi_i|^2 \right] \quad (31)$$

and $Z_i$ is the partition function of the system:

$$Z_i = \int D\psi_i D\psi_i^* e^{-F[\psi]} \quad (32)$$

In order to eliminate the auxiliary fields $\phi_i$, we exploit the method of replica \[20\]. Let $\psi_i^{\omega_i}, \psi_i^{\nu_i}$ be a set of replica fields with $\omega_i = 1, \ldots, n_i$ and $i = 1, \ldots, N$, that form the $n_i$–ples $\Psi_i = (\psi_1^i, \ldots, \psi_{n_i}^i)$ and $\Psi_i^* = (\psi_1^{*i}, \ldots, \psi_{n_i}^{*i})$. In terms of these fields the Green functions $G(r_i, r_i' ; z_i | \phi_i, C^{(i)})$ become:
\[
G(r_i, r'_i; z_i | \phi_i, C^{(i)}) = \lim_{n_i \to 0} \int \mathcal{D} \Psi_i \mathcal{D} \Psi_i^* \psi_i^\omega(r_i) \psi_i^{\omega'}(r'_i) e^{-\sum_{i=1}^{n_i} F[\psi_i^\omega]} 
\]  
(33)

where the \( \omega \)'s are arbitrary integers chosen in the range \( 1 \leq \omega \leq n_i \). According to the replica method, we also assume that the limit for \( n_i \) going to zero commutes with the functional integrations over the C-S fields and the auxiliary fields \( \phi_i \). Substituting eq. (33) in (28) and performing the Gaussian integration over the \( \phi_i \), one obtains:

\[
G_{(\lambda)}(\{r\}, \{r'\}, \{z\}) = \lim_{n_1, \ldots, n_N \to 0} \int \mathcal{D} A \mathcal{D} B \prod_{i=1}^{N} [\mathcal{D} \Psi_i \mathcal{D} \Psi_i^*] \prod_{j=1}^{N} \left[ \psi_j^\omega(r_j) \psi_j^{\omega'}(r'_i) \right] \exp \left\{ -A(\{\Psi\}, \{A\}, \{B\}) \right\} 
\]  
(34)

where the polymer free energy \( A(\{\Psi\}, \{A\}, \{B\}) \) is given by:

\[
A(\{\Psi\}, \{A\}, \{B\}) = iS_{CS} + \sum_{i=1}^{N} \int d^3r \left[ \frac{a}{6} |D_i \Psi_i|^2 + z_i |\Psi_i|^2 \right] + \sum_{i,j=1}^{N} \frac{v_{ij}^0}{2a^2} \int d^3r |\Psi_i|^2 |\Psi_j|^2 
\]  
(35)

From the above equation we see that the fields \( \Psi_i, \Psi_i^* \) have the dimension of a mass, which is not the canonical dimensionality of the scalar fields in three dimensions. To remedy, we introduce a new mass parameter \( M \) \[21\]. It is nice to see that in this way it is possible to define an analog of the Planck constant:

\[
\hbar = \frac{Ma}{3} 
\]  
(36)

Performing the field rescaling

\[
\Psi'_i = \sqrt{\frac{M}{2}} \Psi_i \quad \Psi'^*_i = \sqrt{\frac{M}{2}} \Psi_i^* 
\]  
(37)

and working in units in which \( \hbar = 1 \), the action (35) becomes:

\[
A_N(\{\Psi'\}, \{A\}, \{B\}) = iS_{CS} + \sum_{i=1}^{N} \int d^3r \left[ \Psi'^*_i(-D_i^2 + m_i^2) \Psi'_i \right] + 
\]

\[
\sum_{i,j=1}^{N} \frac{2M^2v_{ij}^0}{a^2} \int d^3r |\Psi'_i|^2 |\Psi'_j|^2 
\]  
(38)
where the covariant derivatives are defined as before by eq. (26) and

\[ m_i^2 = 2 M z_i \quad (39) \]

From now on, we will always use the rescaled action (38) and the prime indices carried by the complex scalar fields will be dropped.

### III. THE TWO POLYMERS CASE

In the two polymers case the matrix \( m_{ij} \) of topological numbers is replaced by a single topological number \( m \) which describes the intersections of the trajectories \( \Gamma_1 \) and \( \Gamma_2 \). Its Fourier conjugate variable will be called \( \lambda \). It will also be convenient to put:

\[
\begin{align*}
A^{(1)}(1) &= A^{(1)}(2) = A^{(2)}(1) = A^{(2)}(2), \\
B^{(1)}(2) &= B^{(2)}(1) = B^{(2)}(2).
\end{align*}
\]

(40)

and

\[
\begin{align*}
\alpha^{(1)} &= \gamma_1, \quad \beta^{(1)} = \gamma_2, \\
\alpha^{(2)} &= \gamma_2, \quad \beta^{(2)} = \gamma_1.
\end{align*}
\]

(41)

In this new notation the action (38) becomes:

\[
\mathcal{A}_2(\{\Psi\}, \{A\}) = \frac{i K}{4 \pi} \int d^3 \mathbf{r} A^{(1)} \cdot (\nabla \times A^{(2)}) + \sum_{i=1}^{2} \int d^3 \mathbf{r} \left[ \Psi_i^* \left( -\nabla_i^2 + m_i^2 \right) \Psi_i \right] \\
+ \sum_{i,j=1}^{2} \frac{2 M^2 v_{ij}^0}{a^2} \int d^3 \mathbf{r} |\Psi_i|^2 |\Psi_j|^2
\]

(42)

where

\[ \nabla_i = \nabla + i \gamma_i A^{(i)} \quad (43) \]

Accordingly, the configurational probability \( G(\lambda)(\{\mathbf{r}\}, \{\mathbf{r}'\}, \{z\}) \) can be written as follows in the two polymers case:

\[ G(\lambda)(\{\mathbf{r}\}, \{\mathbf{r}'\}, \{z\}) = \]

9
\[
\lim_{n_1,n_2 \to 0} \int \prod_{i=1}^{2} [\mathcal{D}A^{(i)} \mathcal{D} \Psi_i \mathcal{D} \Psi^*_i] \prod_{j=1}^{2} \left[ \bar{\psi}^{\omega_j}(r_j) \psi^{\omega_j}(r'_j) \right] \exp \left\{ -A_2(\{ \Psi \}, \{ A \}) \right\}
\]  

(44)

Since we are interested in the effects of the topological interactions, we neglect the excluded volume interactions setting \( v_{ij}^0 = 0 \) in the remaining of this Section.

\[+ \ldots\]

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**FIG. 1.** Non-vanishing contribution to the one-loop effective potential due to the topological interactions. Solid lines represent the charged scalar bosons and dashed lines Chern-Simons vector fields.

The study the above topological Ginzburg-Landau model at one loop order we exploit the method of effective potential [22]. To this purpose one has to compute only the diagrams of fig. (1).

\[+ \ldots\]

**FIG. 2.** Diagrams containing at least one vertex of the coupling \( A^{\tau} \Psi^{\tau} \Psi^*_\tau \). These contributions to the effective potential are identically zero in the Landau gauge.

The other possible diagrams are in fact of the form given in fig. (2) and are suppressed in the Landau gauge [22]. First of all one needs to derive the Feynman rules starting from the action (42). Here we just state the result:

\[
\langle A^{(2)}(p) A^{(1)}(\mathbf{-p}) \rangle = \frac{4\pi}{\kappa} \epsilon_{\mu\nu\rho} \frac{p^\rho}{p^2} \quad \mu, \nu, \rho = 1, 2, 3
\]

(45)

\[
\langle \bar{\psi}^{\omega_i}(p) \psi^{\omega_j}(\mathbf{-p}) \rangle = \frac{\delta^{\omega_i \omega_j} \delta_{ij}}{p^2} \quad i, j = 1, 2
\]

(46)
\[ \langle \psi_i \bar{\psi}_i (p_1) \psi_i \bar{\psi}_i (-p_2) A_{\mu}(p_3) \rangle = 2 \gamma_i \delta^{(3)}(p_1 - p_2 + p_3)(p_2)_\mu \] (47)

\[ \langle \psi_i \bar{\psi}_i (p_1) \psi_i \bar{\psi}_i (p_2) A_{\mu}(p_3) A_{\nu}(p_4) \rangle = \gamma_i^2 \delta_{\mu\nu} \delta^{(3)}(p_1 + p_2 + p_3 + p_4) \] (48)

After some calculations, one obtains the following expression of the effective potential at one loop order:

\[ V_{\text{eff}}(\{\Psi\}) = \sum_{i=1}^{2} m_i^2 |\Psi_i|^2 - \frac{2\Lambda^2}{\pi^2} |\Psi_1|^2 |\Psi_2|^2 + \frac{4\pi}{3} |\lambda|^3 (|\Psi_1||\Psi_2|)^3 \] (49)

In eq. (49) \( \Lambda \) is an ultraviolet cut-off with the dimension of a mass. Due to the fact that the microscopic scale of our model is given by the step length \( a \), phenomena whose spatial extensions are smaller than \( a \) have no meaning [7], so that we can put:

\[ \Lambda \sim a^{-1} \] (50)

From eq. (49) one sees that the total effect of the topological forces at one loop is to influence the strength of the repulsions among the polymers. In particular, topological forces do not affect the scaling behavior of the system. Indeed, only the last term appearing in the second member of eq. (49) may influence the critical behavior of the topological Ginzburg-Landau model (38), but it is of higher order in the numbers of replicas \( n_1 \) and \( n_2 \). Therefore, it does not contribute in the configurational probability (34), after the limit \( n_1, n_2 \rightarrow 0 \) is taken.

The simplest observable that can be computed within the above approach is the second topological moment

\[ \langle m^2 \rangle = \lim_{n_1, n_2 \rightarrow 0} \frac{Z_0^{-1} \int \prod_{i=1}^{2} [D\bar{A}_i D\bar{\psi}_i D\Psi_i] \prod_{j=1}^{2} [\bar{\psi}_j \bar{\psi}_j (r_j) \psi_j \bar{\psi}_j (r'_j)] \exp \{-A_0\}}{\int \prod_{i=1}^{2} [D\bar{A}_i D\bar{\psi}_i D\Psi_i] \prod_{j=1}^{2} [\bar{\psi}_j \bar{\psi}_j (r_j) \psi_j \bar{\psi}_j (r'_j)] \exp \{-A_0\}} \] (51)

Here \( G_m(\{r\}, \{r'\}, \{z\}) \) is easily evaluated by taking the inverse Fourier transformation with respect to \( \lambda \) of the configurational probability (14). This has been done in ref. [17]. In the case of the second topological moment one finds after some calculations:
where
\[ A_0 = i \frac{\kappa}{4\pi} \int d^3 r A^{(1)} \cdot (\nabla \times A^{(2)}) - \int d^3 r \Psi_1^2 (D_1^2 - m_1^2) \Psi_1 - \int d^3 r \Psi_2^2 (\Delta - m_2^2) \Psi_2 \]

and
\[ Z_0 = \int \prod_{i=1}^{2} [D A^{(i)}] \prod_{j=1}^{2} \left[ \psi_j^\dagger(r_j) \psi_i^\dagger(r_j) \right] \exp\left\{-A_0\right\} \]

Let us notice that in the action \( A_0 \) the field \( A^{(2)} \) plays the role of a Lagrange multiplier. As a consequence, after performing the integration in \( A^{(2)} \) in eq. (54), the field \( A^{(1)} \) becomes trivial, so that the path integral \( Z_0 \) actually describes the fluctuations of two free polymers.

FIG. 3. This figure shows all the diagrams contributing to the second topological moment together with the weights they appear in the expression of \( \langle m^2 \rangle \).

The path integral appearing in the numerator of eq. (52) is instead more complicated. Nevertheless it is possible to show that, in the absence of excluded volume interactions, the only contributions to \( \langle m^2 \rangle \) come from the one loop diagrams of fig. (3). In this way the second topological moment can be exactly computed. Physically, one is interested to study the topological moments as a function of the length of the polymers. This requires a slight modification of the above approach to compute directly the configurational probability \( G(\lambda) \{\{r\}, \{L\}; \{r'\}, 0\} \). The details of this calculation will be presented elsewhere.
IV. CONCLUSIONS

In the previous Sections the statistical-mechanical problem of an arbitrary number of topologically entangled polymers has been mapped to a topological Ginzburg-Landau model, solving in this way a longstanding problem firstly pointed out in ref. [5]. In our model the C-S fields acquire a physical significance, since they propagate the forces acting on the polymers due to the presence of the topological constraints. Of course, the topological forces should not distinguish a polymer from the other, a principle that is apparently violated by our choice of coupling constants in eq. (19). However, explicit calculations show that this symmetry is not spoiled and actually puts severe constraints on the number of possible Feynman diagrams to be computed in the perturbative approach. For instance, one recognizes from eq. (19) and fig. (3) that the contributions to the one loop effective potential and to the second topological moment are completely symmetric in the fields $\Psi_1, \Psi_1^*$ and $\Psi_2, \Psi_2^*$. A related problem is the appearance of a spurious parameter in the polymer free energy (38), namely the C-S coupling constant $\kappa$. Also in this case it is easy to see that the difficulty is only apparent, since it is possible to remove the dependency on $\kappa$ with a simple rescaling of the fields $A^{(i)}_{(j)}$.

In our construction the topological forces acting on each couple of trajectories $\Gamma_i$ and $\Gamma_j$ are propagated only by the two fields $A^{(i)}_{(j)}$ and $B^{(i)}_{(j)}$. For that reason, the calculations performed in the case of two polymers can be extended also to an arbitrary number of polymers in a straightforward way. In this way it is possible to conclude from the analysis of Section III that the topological interaction does not change the critical behavior of the system at least in the one loop approximation.

As already mentioned in the Introduction, the Gauss linking number is a rather poor knot invariant. However, the field theoretical formulation of topologically linked polymers established here shows a natural way to include higher order topological interactions. This is achieved by adding to the action (38) a new contribution of the form:
\[ \mathcal{A}_N^{\text{int}} = \sum_{i,j,k=1}^{N-1} \sum_{l,m,n=2}^{N} \int d^3x \epsilon_{\mu \nu \rho} \left( \gamma_{lmn}^{ijk} A_{(l)\mu}^{(i)} A_{(m)\nu}^{(j)} A_{(n)\rho}^{(k)} + \delta_{lmn}^{ijk} A_{(l)\mu}^{(i)} A_{(m)\nu}^{(j)} B_{(n)\rho}^{(k)} + \ldots \right) \] (55)

Each term appearing in the above sum is topological and does not vanish apart from special combinations of the indices \( i, j, k, l, m, n \), in which two or three C-S fields coincide. Clearly, interactions as those given in eq. (55) generate corrections to the configurational probability \( G_{\{\lambda\}}(\{r\}, \{L\}; \{r'\}, 0) \) that contain higher order link invariants. In perturbation theory these corrections can be computed order by order with techniques similar to those already exploited in the case of nonabelian C-S field theories [18], [19]. To enforce the new topological constraints through Dirac \( \delta \)-functions as in eq. (3), one has to introduce additional Fourier variables besides the parameters \( \Lambda_{ij} \)'s. The possibility of defining the coupling constants \( \gamma_{lmn}^{ijk} \), \( \delta_{lmn}^{ijk} \) etc. in terms of these variables in a way that is suitable to impose relations on higher order link invariants is still an open problem and deserves more investigations.

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