The Five Columns Rule in Solving Definite Integration by Parts Through Transformation of Integral Limits

Mardeli Jandja$^1$ and Mohammad Lutfi$^2$

$^1$Department of Mathematics Education, Faculty of Education and Teacher Training, Tadulako University, Palu, 94118, Indonesia
$^2$Department of Petroleum Engineering, STT MIGAS, Balikpapan, 76127, Indonesia

lutfi_pllhd@yahoo.co.id

Abstract: This research introduces alternative method in solving definite integration by parts. The integral is solved using an algorithm of the tabular integration by parts through transformation of integral limits. The results revealed that the integration by parts formula changed to a new form after transformation. The implication of this method revealed that the final solution between this technique and standard techniques in calculus text books are exactly the same. The problem in this method is the procedures become longer than usual, therefore, the authors has derived the five columns rule to simplify the problem. The explanation of each columns are as follows: the first column writes the positive and negative signs alternately, the second column is $f(x)$, and then successive its derivative that lies below it repeatedly such that the derivative equal to zero, the third column is $g'(x)$, and then its integral lies below it, the fourth column writes the lower limit ($x = a$) and the upper limit ($x = b$), and the fifth column is the transformation of integral limits, i.e. the lower double limit $\{u(a)v(a)\}$ and upper double limit $\{u(b)v(b)\}$ on the integral respectively.

1. Introduction

The solution of integral can be obtained by analysing the integrand such that the techniques of integration can be applied and meet the requirements to integrable[1]. One of the techniques is integration by substitution. If the solution can not be obtained, it may be possible to use a double substitution, namely integration by parts. The indefinite integration by parts formula[2-6] is

$$\int u \, dv = uv - \int v \, du$$

(1)

The definite integrals can be calculated by imposing the limits on the integration from $a$ to $b$[7], so that eq. (1) becomes the definite integration by parts formula[8]

$$\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du$$

(2)

which obtained from the integration of the first derivative of a product of two functions [9] namely

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \bigg|_a^b - \int_a^b g(x)f'(x) \, dx$$

(3)
with using double substitution \( u = f(x) \) and \( dv = g'(x) \, dx \) without changing the integral limits. For example [10] Find \( \int_1^2 \ln x \, dx \). Solution by using eq. (2) as follows: Let \( u = \ln x \) and \( dv = dx \), then

\[
du = \left( \frac{1}{x} \right) dx \quad \text{and} \quad v = x,
\]

so that \( \int_1^2 \ln x \, dx = \left[ x \ln x \right]_1^2 - \int_1^2 \left( \frac{1}{x} \right) dx \approx 0.386 \). The same solution is obtained using eq. (3) by letting \( u = \ln x \) and \( dv = dx \).

There are several techniques to simplify solution of integration by parts. Most of them use the tabular method was developed from the integration by parts technique. In article by Dence [11], the two columns function method was introduced. Another method is the algorithm of the tabular integration by parts which is based on the algorithm given by Horowitz [12] was modified by Alcantara [13] by for ming a table consisting of three columns by means the LIATE rule. The word LIATE is stands for logarithmic, inverse trigonometric, algebraic, trigonometric, and exponential function. This method was first mentioned by Kasube [14]. For examples: Find the solution of

\[
\int x^n \sin ax \, dx, n, a \in \mathbb{R}, \quad \int x^2 \sin x \, dx, \quad \text{and} \quad \int_0^\pi x^2 \sin x \, dx \quad \text{respectively. Solution: Let } u = x^n \text{ and } dv = \sin ax \, dx.
\]

| Sign | Derivative | Integral |
|------|------------|----------|
| +    | \( x^n \)  | \( \sin ax \) |
| -    | \( n x^{n-1} \) | \( -\cos ax \) |

\[
\int x^n \sin ax \, dx = (x^n) \left( -\frac{\cos ax}{a} \right) - \int (n x^{n-1}) \left( -\frac{\cos ax}{a} \right) dx = -\frac{x^n \cos ax}{a} + \frac{n}{a} \int x^{n-1} \cos ax \, dx.
\]

Let \( n = 2 \) and \( a = 1 \), so that \( \int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx \). Solution of \( 2 \int x \cos x \, dx \) is obtained by letting \( u = 2x \) and \( dv = \cos x \, dx \)

| Sign | Derivative | Integral |
|------|------------|----------|
| +    | \( 2x \)   | \( \cos x \) |
| -    | 2          | \( \sin x \) |
| 0    |            | \( -\cos x \) |

\[
\int 2x \cos x \, dx = 2x \sin x - 2(-\cos x) + c = 2x \sin x + 2 \cos x + c.
\]

So that

\[
\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c. \quad \text{By imposing the explicit limits on the integration from 0 to } \pi, \quad \text{then } \pi \int_0^\pi x^2 \sin x \, dx = -x^2 \cos x \bigg|_0^\pi + 2x \sin x + 2 \cos x \bigg|_0^\pi = \pi^2 - 4.
\]

Most of the calculus text books states the solution in solving problem similar to the technique above and many papers have been written concerning it. We have not come across any references dealing with the method to solve definite integration by parts through transformation of integral limits and its implication to the problem solving.
2. Transformation of Integral Limits

An approachment of the double substitution which followed by the transformation of integral limits has led to formed the integral double limits \( u(x)v(x) \). The application of this approachment can be defined as follows: If \( x = a \), then \( u(a) = f(a) \) and \( v(a) = g(a) \), so that the lower integral double limit is \( u(a)v(a) = f(a)g(a) \). If \( x = b \), then \( u(b) = f(b) \) and \( v(b) = g(b) \), so that the upper integral double limit is \( u(b)v(b) = f(b)g(b) \). According to the definition above, then eq. (2) becomes

\[
\int u \, dv = uv \bigg|_{a}^{b} - \int v \, du
\]

or

\[
\int u \, dv = u v \bigg|_{a}^{b} - \int v \, du
\]

The implication of eq. (5) is used in solving problem. But the procedure becomes longer than usual, therefore, the authors has derived an algorithm of the tabular integration by parts through transformation of integral limits called the five columns rule.

The lower integral double limit \( u(a)v(a) \) can be returned to the lower integral double limit \( x = a \). The upper integral double limit \( u(b)v(b) \) also can be returned to the upper integral double limit \( x = b \).

The explanation of each columns are as follows: the first column writes the positive and negative signs alternately, the second column is \( f(x) \), and then successive its derivative that lies below it repeatedly such that the derivative equal to zero, the third column is \( g'(x) \), and then its integral lies below it, the fourth column writes the lower limit \( (x = a) \) and the upper limit \( (x = b) \), and the fifth column is the transformation of integral limits, i.e. the lower double limit \( \{u(a)v(a)\} \) and upper double limit \( \{u(b)v(b)\} \) on the integral respectively. The arrows illustrates the process of using eq. (5) repeatedly. The five columns rule in solving definite integration by parts:

| Sign | Derivative | Integral | Limits | Double Limits |
|------|------------|----------|--------|---------------|
| +    | \( f(x) \) | \( g'(x) \) | \( x = a \) | \( u(a)v(a) \) |
| -    | \( f'(x) \) | \( g(x) \) | \( x = b \) | \( u(b)v(b) \) |
At the first stage of using eq. (5), the problem remain unsolved due to the solution still contain \( \int_{a}^{b} g(x)f'(x) \, dx \) that must be solved by using double substitution \( \tilde{u} = g(x) \) and \( d\tilde{v} = f'(x) \, dx \). The procedure in using eq. (5) is repeated such that the derivative in second column equal to zero. If the form on the right side is \( k \int_{a}^{b} f(x)g'(x) \, dx \), then the problem is solved algebraically.

4. The Implication to the Problems Solving

Example 1. \( \int_{0}^{\pi} x^2 \sin x \, dx \). Solution: Let \( u = x^2 \) and \( dv = \sin x \, dx \), then \( du = 2x \, dx \) and \( v = -\cos x \).

If \( x = 0 \), then \( u(0) = 0 \) and \( v(0) = -\cos(0) = -1 \), so that the lower double limit on the integral are \( u(0)v(0) = 0 \). If \( x = \pi \), then \( u(\pi) = \pi^2 \) and \( v(\pi) = -\cos(\pi) = 1 \), so that the upper double limit on the integral are \( u(\pi)v(\pi) = \pi^2 \). So that

\[
\int_{0}^{\pi} x^2 \sin x \, dx = \int_{u(0)v(0)}^{u(\pi)v(\pi)} u \, du = uv|_{0}^{\pi} - \int_{0}^{\pi} v \, du = \pi^2 - \int_{0}^{\pi} 2x \cos x \, dx.
\]

The solution of \( 2x \cos x \, dx \) is solved as the initial procedure using double substitution as follows: Let \( \tilde{u} = 2x \) and \( d\tilde{v} = \cos x \, dx \), then \( d\tilde{u} = 2 \, dx \) and \( \tilde{v} = \sin x \). If \( x = 0 \), then \( \tilde{u}(0) = 0 \) and \( \tilde{v}(0) = \sin(0) = 0 \), so that the lower double limit on the integral is \( \tilde{u}(0)\tilde{v}(0) = 0 \). If \( x = \pi \), then \( \tilde{u}(\pi) = 2\pi \) and \( \tilde{v}(\pi) = \sin \pi = 0 \), so that the upper double limit on the integral is \( \tilde{u}(\pi)\tilde{v}(\pi) = 0 \), therefore

\[
\int_{0}^{\pi} 2x \cos x \, dx = \int_{\tilde{u}(0)\tilde{v}(0)}^{\tilde{u}(\pi)\tilde{v}(\pi)} \tilde{u} \, d\tilde{v} = \int_{0}^{\pi} \tilde{u} \, d\tilde{v} = \tilde{u}\tilde{v}|_{0}^{\pi} - \int_{0}^{\pi} \tilde{v} \, d\tilde{u} = 2 \cos x|_{0}^{\pi} = -4, \quad \text{so that}
\]

\[
\int_{0}^{\pi} x^2 \sin x \, dx = \pi^2 - 4. \quad \text{This procedure can be simplified by using the rule of five columns.}
\]

| Sign | Derivative | Integral | Limits | Double Limits |
|------|------------|----------|--------|--------------|
| +    | \( x^2 \)  | \( \sin x \) | \( 0 \) \( \pi \) | \( (0^2)(-\cos 0) = 0 \) |
| -    | \( 2x \)   | \(-\cos x \) | \( \pi \) | \( (\pi^2)(-\cos \pi) = \pi^2 \) |
| +    | \( 2 \)    | \(-\sin x \) | \( 0 \) \( \pi \) | \( (2.0)(-\sin 0) = 0 \) |
|      |            | \( \cos x \) | \( \pi \) | \( (2\pi)(-\sin \pi) = 0 \) |
|      |            | \( 0 \)    | \( 0 \) \( \pi \) | \( (2)(\cos 0) = 2 \) |
|      |            |            |        | \( (2)(\cos \pi) = -2 \) |

so that \( \int_{0}^{\pi} x^2 \sin x \, dx = \pi^2 - 4 \).
Example 2. Find \( \int_{0}^{\pi/b} e^{ax} \sin bx \, dx \)

Solution: Let \( u = e^{ax} \) and \( dv = \sin bx \, dx \), then \( du = ae^{ax} \, dx \) and \( v = -\frac{1}{b} \cos bx \). If \( x = 0 \), then \( u(0) = e^{0} = 1 \) and \( v(0) = -\frac{1}{b} \cos(0) = -\frac{1}{b} \), so that the lower double limit on the integral is \( u(0)v(0) = -\frac{1}{b} \). If \( x = \frac{\pi}{b} \), then \( u(\frac{\pi}{b}) = e^{\frac{a\pi}{b}} \) and \( v(\frac{\pi}{b}) = -\frac{1}{b} \cos \frac{\pi}{b} = -\frac{1}{b} (-1) = \frac{1}{b} \), so that the upper double limit on the integral is \( u(\frac{\pi}{b})v(\frac{\pi}{b}) = 1 \). Similarly to the previous procedure to solve \( \int_{0}^{\pi/b} e^{ax} \cos bx \, dx \), by letting \( \tilde{u} = e^{ax} \) and \( \tilde{v} = \cos bx \), then \( d\tilde{u} = ae^{ax} \, dx \) and \( \tilde{v} = \frac{1}{b} \sin bx \). If \( x = 0 \), then \( \tilde{u}(0) = e^{0} = 1 \) and \( \tilde{v}(0) = \frac{1}{b} \sin 0 = 0 \), so that the lower double limit on the integral is \( \tilde{u}(0)\tilde{v}(0) = 0 \). If \( x = \frac{\pi}{b} \), then \( \tilde{u}(\frac{\pi}{b}) = e^{\frac{a\pi}{b}} \) and \( \tilde{v}(\frac{\pi}{b}) = \frac{1}{b} \sin \frac{\pi}{b} = 0 \), so that the upper double limit on the integral is \( \tilde{u}(\frac{\pi}{b})\tilde{v}(\frac{\pi}{b}) = 0 \). Therefore, the solution becomes

\[
\int_{0}^{\pi/b} e^{ax} \sin bx \, dx = \frac{\pi}{b} \left( a \frac{e^{a\pi/b}}{b} + \frac{a}{b} \right) + \frac{a}{b} \left( -\frac{\pi}{b} \right) = \frac{b}{a^2 + b^2} \left( a e^{a\pi/b} + b \right). \]

Solution by using the five columns rule
therefore,

\[
\int_0^\pi a e^{ax} \sin bx \, dx = -\frac{\pi a\pi}{b^2} e^{bx} \sin bx 
\]

\[
\int_0^\pi a e^{ax} \sin bx \, dx = -\frac{\pi a\pi}{b^2} e^{bx} \sin bx 
\]

So that

\[
\int_0^\pi a e^{ax} \sin bx \, dx = -\frac{\pi a\pi}{b^2} e^{bx} \sin bx 
\]

The proof of this method have been conducted by comparing to many examples in this article references, where the results shows that the final solution are exactly the same.

5. Conclusions

The results revealed that the integrals could be solved using an algorithm of the tabular integration by parts through transformation of integral limits. The implication to the problem solving revealed that all the final solution of examples in many calculus text books and published articles are exactly the same compared to this method, which means that this method can be used as an alternative method in solving definite integration by parts. But the procedure becomes longer than usual, therefore, the authors has derived the five columns rule to simplify the problem.

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