Abstract
We determine the topological complexity of configuration spaces of graphs that are not necessarily trees, which was a crucial assumption in previous results. We do this for two very different classes of graphs: fully articulated graphs and banana graphs. We also complete the computation in the case of trees to include configuration spaces with any number of points, extending a proof of Farber. At the end we show that an unordered configuration space on a graph does not always have the same topological complexity as the corresponding ordered configuration space (not even when they are both connected). Surprisingly, in our counterexamples the topological complexity of the unordered configuration space is in fact smaller than for the ordered one.

Keywords Configuration spaces · Graphs · Topological complexity

1 Introduction

It is a fundamental problem in industrial robotics to coordinate the movements of automated guided vehicles along a system of roads or rails, in such a way that no collisions occur. These situations can be modeled [11] by the configuration space $\text{Conf}_n(G)$ of $n$ particles on a graph $G$, which is given by

$$\text{Conf}_n(G) := \{(x_1, \ldots, x_n) | x_i \neq x_j \text{ for } i \neq j \} \subset G^n.$$
Every collision-free movement between two configurations of \( n \) points on the graph \( G \) corresponds to a path in the space \( \text{Conf}_n(G) \). The motion planning problem consists in finding a function which assigns to any pair of points a path between them.

Given a topological space \( X \) let \( p_X: X^I \to X \times X \) denote the free path fibration on \( X \), with projection \( p_X(\gamma) = (\gamma(0), \gamma(1)) \). A continuous motion planner on \( X \) is precisely a section of \( p_X \). Such a continuous motion planner only exists in very special cases (in fact, it exists if and only if \( X \) is contractible). Motivated by this, Farber introduced the topological complexity of a space [7]. It is a numerical homotopy invariant which measures the minimal discontinuity of every motion planner on this space.

**Definition 1.1** The topological complexity of \( X \), denoted \( \text{TC}(X) \), is defined to be the minimal \( k \) such that \( X \times X \) admits a cover by \( k + 1 \) open sets \( U_0, U_1, \ldots, U_k \), on each of which there exists a local section of \( p_X \) (that is, a continuous map \( s_i: U_i \to X^I \) such that \( p_X \circ s_i = \text{incl}_i: U_i \hookrightarrow X \times X \)).

Note that here we use the reduced version of \( \text{TC}(X) \), which is one less than the original definition by Farber.

Let \( T \) be a tree and let \( |V_{\geq 3}| \) denote the number of essential vertices of \( T \) (i.e., vertices with valence at least 3). Farber showed that \( \text{TC} (\text{Conf}_n(T)) = 2|V_{\geq 3}| \) whenever \( n \geq 2|V_{\geq 3}| \); see [9] and also Farber’s survey article [10]. In particular, \( \text{TC} (\text{Conf}_n(T)) \) doesn’t depend on \( n \) within that range. Later Scheirer computed the topological complexity for some \( n \) outside the aforementioned range [14].

We complete this picture, extending Farber’s argument to compute \( \text{TC} (\text{Conf}_n(T)) \) for all \( n \), see Theorem T. It should be mentioned that Scheirer’s methods also apply to unordered configuration spaces, whereas ours do not.

Apart from a few isolated examples, the topological complexity \( \text{TC} (\text{Conf}_n(G)) \) has only been computed in the case when \( G \) is a tree. Of course, the requirement that \( G \) be a tree is too restrictive from the point of view of robotics. Indeed, a road system with no loops in it is bound to be highly inefficient.

A vertex is an articulation if removing it makes the graph disconnected and a connected graph is fully articulated if every essential vertex is an articulation. In Theorem A we extend Farber’s result to \( \text{TC} (\text{Conf}_n(G)) = 2|V_{\geq 3}| \) for all fully articulated graphs \( G \) (of which trees are a special case) for \( n \geq 2|V_{\geq 3}| \).

In the proof of the results just mentioned we use the cohomology ring structure of \( \text{Conf}_n(G) \); more precisely we use the zero-divisor cup-length (see Sect. 2). Our proof essentially generalizes that of Farber [9]. The key technical ingredient which enables this generalization is that of configuration spaces with sinks, which were introduced by Chettih and the first author [4].

The other main result in this paper features a class of graphs which have no articulations at all, the banana graphs \( B_k \) (here \( B_k \) denotes the graph with two vertices and \( k \) edges connecting them). In this case we needed a completely different approach. The starting point is the fact that \( \text{Conf}_3 (B_4) \) has a particularly nice homotopy type, namely that of an orientable surface of genus 13 [4, Prop. 4.3, p. 19]. The topological complexity of surfaces is known to be equal to the zero-divisor cup-length. Using a Mayer–Vietoris spectral sequence argument involving sinks, similar to arguments made by the first author [13], we are able to gain information about the cohomology
ring of $\text{Conf}_3(B_k)$ for $k \geq 4$. This allows us to compute $\text{TC}(\text{Conf}_n(B_k))$ in all cases except for $k = 3$ with $n \geq 4$, see Theorem B.

In the last section we discuss a conjecture of Farber regarding ordered configuration spaces of graphs in relation to the results of this paper. In particular, we show that the conjecture is not true if we replace “ordered” by “unordered” and give examples where the topological complexity differs from the ordered to the unordered setting.

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2 Topological Complexity

Recall the definition from the introduction.

**Definition 1.1** The topological complexity of $X$, denoted $\text{TC}(X)$, is defined to be the minimal $k$ such that $X \times X$ admits a cover by $k + 1$ open sets $U_0, U_1, \ldots, U_k$, on each of which there exists a local section of $p_X$ (that is, a continuous map $s_i : U_i \to X$ such that $p_X \circ s_i = \text{incl}_i : U_i \hookrightarrow X \times X$).

In the remainder of this section we state several well-known results about topological complexity which will be useful later on.

Firstly, the topological complexity $\text{TC}(X)$ is a homotopy invariant of $X$.

**Proposition 2.2** ([7]) If $X$ is a homotopy retract of $Y$, then $\text{TC}(X) \leq \text{TC}(Y)$. Furthermore, if $X$ is homotopy equivalent to $Y$, then $\text{TC}(X) = \text{TC}(Y)$.

The dimension of $X$ gives us a general upper bound for $\text{TC}(X)$.

**Proposition 2.3** ([7]) Let $X$ be a path-connected paracompact space. Then the topological complexity of $X$ is bounded above by the covering dimension of the product:

$$\text{TC}(X) \leq \dim(X \times X).$$

In particular this upper bound holds for all connected CW-complexes and in that case $\dim(X \times X)$ is the CW-dimension.

Knowing the cohomology ring of a space $X$ can yield lower bounds for $\text{TC}(X)$ as shown in the following.

**Definition 2.4** Let $X$ be a topological space and $A$ a coefficient ring. A class $z \in H^*(X \times X; A)$ is called a zero-divisor if the pull-back under the diagonal is trivial: $\Delta^*(z) = 0$.

The zero-divisor cup-length $\text{zcl}_A(X)$ is the length of the longest non-trivial product of zero-divisors in $H^*(X \times X; A)$.

**Proposition 2.5** ([7]) Let $X$ be a topological space and $A$ a coefficient ring. Then the topological complexity of $X$ is bounded below by the zero-divisor cup-length:

$$\text{TC}(X) \geq \text{zcl}_A(X).$$
Using the previous propositions Farber computed the topological complexity in the following cases.

**Proposition 2.6** ([7]) If $\Sigma_g$ is an orientable surface of genus $g \geq 2$, then

$$\text{TC}(\Sigma_g) = 4.$$  

**Proposition 2.7** ([8]) If $G$ is a connected graph with first Betti number $b_1(G)$, then

$$\text{TC}(G) = \begin{cases} 0 & \text{if } b_1(G) = 0, \\ 1 & \text{if } b_1(G) = 1, \\ 2 & \text{if } b_1(G) \geq 2. \end{cases}$$

### 3 Configuration Spaces of Graphs

For a topological space $X$ and a finite set $S$ we define the *configuration space of $X$ with particles labeled by $S$* as

$$\text{Conf}_S(X) := \{ f : S \to X \text{ injective} \} \subset \text{map}(S, X).$$

For $n \in \mathbb{N}$ we write $n := \{1, 2, \ldots, n\}$ and $\text{Conf}_n(X) := \text{Conf}_n(X)$. This is usually called the $n$th ordered configuration space of $X$. Let $G$ be a finite connected graph (i.e., a connected 1-dimensional CW complex with finitely many cells). We are interested in the topological complexity of configurations of $n$ ordered particles in $G$, that is, $\text{TC}(\text{Conf}_n(G))$.

Unless explicitly stated, we assume without loss of generality that none of the graphs have vertices of valence 2.

A main ingredient in our computations is a modified configuration space in which particles can collide in some parts of the graph. This construction was introduced in [4] and allows taking quotients of the underlying space of a configuration space in the following way.

For a number $n \in \mathbb{N}$, a graph $G$ and a subset $W$ of $G$’s vertices define the following configuration space with sinks:

$$\text{Conf}_n(G, W) = \{ (x_1, \ldots, x_n) \in G^n \mid \text{for } i \neq j \text{ either } x_i \neq x_j \text{ or } x_i = x_j \in W \}.$$  

Looking at a collapse map $G \to G/H$ for a subgraph $H \subset G$, there is now an induced map on configuration spaces if we turn the image of $H$ under $G \to G/H$ into a sink:

$$\text{Conf}_n(G) \to \text{Conf}_n(G/H, H/H).$$

### 3.1 A Combinatorial Model

For unordered and ordered configuration spaces of graphs there are combinatorial models due to Abrams [1], Ghrist [11], Świątkowski [15] and the first author [12]. In
[4], a combinatorial model for configuration spaces with sinks inspired by the latter two models was constructed. This model is a deformation retract of the configuration space with the structure of a cube complex.

**Definition 3.1** (Cube Complex, see [2, Defn. I.7.32]) A cube complex $K$ is the quotient of a disjoint union of cubes $X = \bigsqcup_{\lambda \in \Lambda} [0, 1]^k_{\lambda}$ by an equivalence relation $\sim$ such that the quotient map $p: X \to X/\sim = K$ maps each cube injectively into $K$ and we only identify faces of the same dimensions by an isometric homeomorphism.

**Remark 3.2** The definition above differs slightly from the original definition by Bridson and Häfliger, in that it allows two cubes to be identified along more than one face.

**Proposition 3.3** ([4, Prop. 2.3, p. 4]) Let $G$ be a finite graph, $W$ a subset of the vertices and $n \in \mathbb{N}$. Then $\text{Conf}_n(G, W)$ deformation retracts to a finite cube complex of dimension $\min\{n, |V_{\geq 2}| + |E_W|\}$, where $V_{\geq 2}$ is the set of non-sink vertices of $G$ of valence at least two and $E_W$ is the set of edges incident to two sinks.

The basic idea of the combinatorial model is to keep all particles on any single edge equidistant at all times. Moving one of the outmost particles from an edge to an empty essential non-sink vertex is then given by decreasing the distance of this particle from the vertex while simultaneously increasing the distance between the particles on this edge. Once the particle reaches the vertex, all remaining particles on the edge will be equidistant again.

More formally, the 0-cubes of the combinatorial model are all those configurations where all particles in the interior of each edge cut the edge into pieces of equal length and no particle is in the interior of any edge incident to one or two sink vertices. There are no particles on degree 1 vertices which are not sinks. A $k$-dimensional cube is given by choosing such a 0-cell, $k$ distinct particles sitting on distinct vertices and for each of those particles an edge incident to the corresponding vertex. The $i$th dimension of the cube $[0, 1]^k$ then corresponds to moving the $i$th of those $k$ particles from their position on the vertex onto the edge, where at time zero the particle is on the vertex and at time 1 it is on the edge. Remember that if there are already particles on the edge then they continuously squeeze together to make room for the new particle (it is also possible that two particles move onto the same edge from different sides). In the case where the particle moves onto an edge whose other terminal vertex is a sink vertex, the particle moves directly into the sink instead. Such a choice of $k$ movements determines a $k$-cube if and only if we can realize the movements independently, namely if no two particles move towards the same non-sink vertex and no two particles move along the same edge incident to two sink vertices. This describes the cube complex as a subspace of the configuration space.

Each non-sink (essential) vertex can only be involved in one of those combinatorial movements at the same time, so the dimension of this cube complex is bounded above by the number of essential vertices plus the number of edges between sink vertices.

**Example 3.4** The combinatorial model of $\text{Conf}_2(Y)$ for the graph $Y$ shaped like the letter $Y$ is given by a circle with six leaves attached to it. More precisely it consists
The combinatorial model of Conf$_2$($Y$). Each edge corresponds to the movement of a single particle from the essential vertex onto one of the three edges. Moving along the embedded circle the two particles move alternatingly onto the edge that is not occupied by the other particle.

![Fig. 1](image1)

The combinatorial model of Conf$_2$([0, 1], {0, 1}) consists of four edges of 6 univalent vertices, 6 vertices of valence 2, 6 vertices of valence 3 and 18 edges, see Fig. 1. In particular, we have $H_1$(Conf$_2$($Y$)) $\cong \mathbb{Z}$. Remember that particles only move towards vertices of valence at least 2 (or sink vertices), so in this example there is only one vertex towards which any particle can move.

For the interval with two sinks we first consider the cases of few particles. In the case with only one particle there is no 1-dimensional class and with two particles there is precisely one 1-class: both particles sit on the first sink, particle 1 moves to the second sink, particle 2 follows, particle 1 returns to the first sink and finally also particle 2 moves back to the first sink, see Fig. 2.

**Example 3.5** In Fig. 2 the combinatorial model of Conf$_2$([0, 1], {0, 1}) is visualized. It is a circle constructed by gluing four edges together.
4 Fully Articulated Graphs

Definition 4.1 A vertex of a connected graph is called an articulation if removing it makes the graph disconnected. A connected graph is fully articulated if all essential vertices are articulations.

Every tree is fully articulated, but the class of fully articulated graphs is much larger than just trees. For instance, every graph can be turned into a fully articulated graph by adding a leaf or a loop at every essential vertex (Fig. 3).

The number of articulations of a graph yields a lower bound for the topological complexity of configuration spaces on that graph:

Theorem 4.2 Let $G$ be a graph with $|V_{\geq 3}|$ essential vertices, $m \geq 2$ of which are articulations, and let $n \geq 4$. Then

$$2 \min\{\lfloor n/2 \rfloor, m\} \leq \text{TC}(\text{Conf}_n(G)) \leq 2 \min\{n, |V_{\geq 3}|\}.$$ 

The following two theorems will follow from Theorem 4.2.

Theorem A Let $G$ be a fully articulated graph with at least one essential vertex. Further assume that $G$ is not homeomorphic to the letter $Y$. Then the topological complexity of $\text{Conf}_n(G)$ for $n \geq 2|V_{\geq 3}|$ is given by

$$\text{TC}(\text{Conf}_n(G)) = 2|V_{\geq 3}|.$$ 

If $G$ is homeomorphic to the letter $Y$, then

$$\text{TC}(\text{Conf}_n(G)) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3. \end{cases}$$
**Theorem T**  
Let $T$ be a tree which is not homeomorphic to an interval or to the letter $Y$. Then the topological complexity of $\text{Conf}_n(T)$ is given by

$$\text{TC}(\text{Conf}_n(T)) = 2 \min\{\lfloor n/2 \rfloor, |V_{\geq 3}|\}.$$  

The proof of Theorem 4.2 generalizes Farber’s argument from [9] using configuration spaces with sinks. Before we can begin the proof of the three main theorems above we need some auxiliary propositions.

**Proposition 4.3** ([3, Thm. 2.2.4, p. 17]) Let $G$ be a tree. Then $\text{Conf}_n(G)$ has homotopy dimension $\min\{\lfloor n/2 \rfloor, |V_{\geq 3}|\}$.

Notice that the above dimension estimate is not stated in this way in [3], but instead a description of the critical cells of a discrete Morse flow on $\text{Conf}_n(G)$ for $G$ a tree is given. A critical cell of dimension $k$ in this description needs at least $2k$ particles and $k$ essential vertices, proving the statement.

**Proposition 4.4** ([11], [12, Prop. 3.5, p. 36]) Let $Y^l_k$ be a graph which is constructed by attaching $k$ leaves and $l$ loops to a single vertex, such that $k + 2l \geq 3$. Then $\text{Conf}_n(Y^l_k)$ is homotopy equivalent to a graph with first Betti numbers

$$1 + \frac{(n + k + l - 2)!}{(k + l - 1)!} \frac{(n + 2l) - (k + l + 1)}{1}.$$  

**Proposition 4.5** Let $G$ be a connected graph with at least one articulation $v$ and let $G_v$ be a $Y$-graph embedded around that vertex such that $G_v - \{v\}$ meets at least two connected components of $G - \{v\}$. Then the induced map

$$H_1(\text{Conf}_2(G_v); \mathbb{Z}) \rightarrow H_1(\text{Conf}_2(G); \mathbb{Z})$$

is injective. More precisely, there is a quotient graph with sinks $(G, \emptyset) \rightarrow (\overline{G}_v, W_v)$ such that the induced composition

$$H_1(\text{Conf}_2(G_v); \mathbb{Z}) \rightarrow H_1(\text{Conf}_2(G); \mathbb{Z}) \rightarrow H_1(\text{Conf}_2(\overline{G}_v, W_v); \mathbb{Z})$$

is injective. Furthermore, the image has rank 1 because $\text{Conf}_2(G_v)$ is homotopy equivalent to the circle.

**Proof** Denote by $E_v$ the set of all edges not incident to $v$, then we define $\overline{G}_v$ to be the graph $G$ with all those edges collapsed to points. Additionally, we add an artificial 2-valent sink vertex in the middle of each edge forming a self-loop at $v$. Let $W_v$ be the set of all vertices of $\overline{G}_v$ except for $v$. By the total separation assumption the set $W_v$ contains at least two elements, and by the choice of embedding of the graph there are at least two edges of the image of $G_v$ in $\overline{G}_v$ pointing towards distinct sink vertices.

The graph $\overline{G}_v$ has exactly one non-sink vertex and does not have any edge incident to two sink vertices, so by Proposition 3.3 its combinatorial model is a graph. We will now see that a representative of the generator of $H_1(\text{Conf}_2(G_v))$ is mapped to
a non-trivial cycle in $\text{Conf}_2(\overline{G}_v, W_v)$, therefore representing a non-trivial homology class. The representative of the single class in $H_1(\text{Conf}_2(G_v))$ is shown in Fig. 4. We now map this cycle into the combinatorial model of $\text{Conf}_2(\overline{G}_v, W_v)$ and then by the total separability assumption there are two possibilities: either the three edges point to three distinct sink vertices of the quotient or they point to only two different sink vertices. In the first case it is easy to check that the 12 distinct edges of the cycle map injectively into the combinatorial model of the quotient, so it remains to show that the image is a non-trivial linear combination of edges also for the second case.

The image of the cycle in this case is represented as indicated in Fig. 5. Canceling all possible edges in the linear combination we get a sum of four circle classes: each of the two particles moves along an embedded circle in the two possible directions with the other particle sitting on each of the two sink vertices. All edges in these four summands of the cycle are distinct edges of the combinatorial model, showing that the linear combination is non-trivial. The lack of 2-cells shows that this cycle represents a non-trivial homology class. 

**Proof of Theorem 4.2** The upper bound follows from Proposition 3.3 (we may assume there are no vertices of valence 2) and Proposition 2.3. 

It remains to show the lower bounds. Let $d = \min\{\lfloor n/2 \rfloor, m\} \geq 2$. Choose $d$ Y-graphs $G_i$ embedded as subgraphs in $G$ as in Proposition 4.5, each around a different articulation. Denote by $(\overline{G}_i, W_i)$ the corresponding quotient graphs. We may assume that the $G_i$ are disjoint from each other by sufficiently subdividing $G$.

Denote by $\Phi_{i,j} : \text{Conf}_n(G) \to \text{Conf}_2(\overline{G}_i, W_i)$ the map which sends the configuration $(x_1, \ldots, x_n)$ to the image of $(x_{2j-1}, x_{2j})$ in the quotient graph, for $1 \leq j \leq d$.

Since the $G_i$ are disjoint by construction, for every permutation $\sigma \in \Sigma_d$ there is an embedding $\psi_\sigma$ which makes the following diagrams commute for every $1 \leq i \leq d$.
\[ \prod_{j=1}^{d} \text{Conf}_2(G_j) \xrightarrow{\psi_\sigma} \text{Conf}_n(G) \]

\[ \xrightarrow{\pi_i} \text{Conf}_2(G_i) \xrightarrow{\Phi_{i,\sigma(i)}} \text{Conf}_2(G, W_i) \]

Notice that there are choices involved, but the rest of this proof is independent of those choices.

By Proposition 4.5, \( H_1(\text{Conf}_2(G_i); \mathbb{Z}/2) \) embeds as a copy of \( \mathbb{Z}/2 \) inside the module \( H_1(\text{Conf}_2(G_i, W_i); \mathbb{Z}/2) \). Let \( x_i \in H^1(\text{Conf}_2(G_i, W_i); \mathbb{Z}/2) \) denote the cohomology class dual to the generator of this copy of \( \mathbb{Z}/2 \). Consider the classes \( u_{i,j} = \Phi_{i,j}^*(x_i) \in H^1(\text{Conf}_n(G)) \).

Write \( u_{j,k} = u_{j,k} - 1 - 1 \times u_{j,k} \). All elements of this form are zero-divisors. By Proposition 2.5 the length of each non-zero product of zero-divisors is a lower bound for the topological complexity. In the following we will show that the product \( \prod_{i=1}^{d} \bar{u}_{i,i} \cdot \prod_{i=1}^{d} \bar{u}_{i,i+1} \) of length \( 2d \) is non-zero (the indices in \( u_{i,i+1} \) are considered modulo \( d \)).

Let \( z_{\sigma} \in H_d(\text{Conf}_n(G)) \) be the image of the generator of

\[ H_d \left( \prod_{j=1}^{d} \text{Conf}_2(G_j); \mathbb{Z}/2 \right) \cong \mathbb{Z}/2 \]

under \( (\psi_\sigma)_* \). Denote by \( z \) and \( z_{\text{sh}} \) the tori \( z_{\sigma} \) for \( \sigma \) the identity and the shift \( i \mapsto i + 1 \), respectively. We will now prove that the product of zero-divisors is non-trivial by showing that

\[ \left\langle \prod_{i=1}^{d} \bar{u}_{i,i} \cdot \prod_{i=1}^{d} \bar{u}_{i,i+1}, z \times z_{\text{sh}} \right\rangle \neq 0 \in \mathbb{Z}/2. \]

To compute this product, we have to evaluate products of the form \( \left\langle \prod_{i=1}^{d} u_{j_i, k_i}, z \right\rangle \) for sets of \emph{distinct} pairs \( (j_i, k_i) \). By definition, this can be computed by evaluating the product of the corresponding

\[ x_{j_1} \cdots x_{j_d} \in H^d \left( \prod_{i=1}^{d} \text{Conf}_2(\overline{G}_{j_i}, W_{j_i}) \right) \]

on the image of the non-trivial element of \( \mathbb{Z}/2 \) under the map

\[ \mathbb{Z}/2 \cong H_d \left( \prod_{i=1}^{d} \text{Conf}_2(G_i) \right) \xrightarrow{\psi_{id}} H_d(\text{Conf}_n(G)) \]

\[ \xrightarrow{\prod \Phi_{j_i, k_i}} H_d \left( \prod_{i=1}^{d} \text{Conf}_2(\overline{G}_{j_i}, W_{j_i}) \right). \]
This map is the tensor product of maps
\[ \eta_{ji, ki} : \mathbb{Z}/2 \cong H_d \left( \prod_{i=1}^{d} \text{Conf}_2(G_i) \right) \to H_1(\text{Conf}_2(\overline{G}_{ji}, W_{ji})) , \]
so we need to evaluate \( x_{ji} \) on the image of the map above for each \( i \) and multiply the results. The image of the non-trivial class under \( \eta_{ji, ki} \) has a representative that only meets the vertex of \( \overline{G}_{ji} \) that is the image of the essential vertex of \( G_{ki} \subset G \). Every class in \( H_1(\text{Conf}_2(\overline{G}_{ji}, W_{ji})) \) not meeting the non-sink vertex is trivial, so the map \( \eta_{ji, ki} \) can only be non-trivial if \( ki = ji \). Since by definition
\[ \langle u_1, 1 \cdots u_d, z \rangle = 1 , \]
this means that \( \langle \prod_{i=1}^{d} u_{ji, ki, i} , z \rangle \) is non-zero if and only if we have (up to permutation) that \( ji = ki = i \) for all \( i \).

Repeating the analogous reasoning for \( z_{sh} \) we see that
\[ \left( \prod_{i=1}^{d} \overline{u}_{i, i} \times \prod_{i=1}^{d} \overline{u}_{i, i+1} , z \times z_{sh} \right) \]
has exactly one non-trivial summand, namely
\[ \left( \prod_{i=1}^{d} u_{i, i} \times \prod_{i=1}^{d} u_{i, i+1} , z \times z_{sh} \right) , \]
showing that the product is non-trivial.

**Proof of Theorem A** The claim follows from Theorem 4.2 in all cases except when \( |V_{\geq 3}| = 1 \). In this case \( G \) is homeomorphic to a wedge of \( k \) intervals and \( l \) circles, such that \( k + 2l \geq 3 \). By Proposition 4.4, \( \text{Conf}_n(G) \) is homotopy equivalent to the circle if \( G \) is homeomorphic to the letter Y and \( n = 2 \), and homotopy equivalent to a graph with first Betti number at least 2 otherwise. By Proposition 2.7 this implies that \( \text{TC}(\text{Conf}_n(G)) = 1 \) if \( G \) is homeomorphic to the letter Y and \( n = 2 \), and \( \text{TC}(\text{Conf}_n(G)) = 2 \) otherwise.

**Proof of Theorem T** By Proposition 4.3 the homotopy dimension of \( \text{Conf}_n(T) \) is \( \min([n/2], |V_{\geq 3}|) \). Together with Proposition 2.3 this yields the upper bound. The lower bound for \( \min([n/2], |V_{\geq 3}|) \geq 2 \) follows immediately from Theorem 4.2 because trees are fully articulated.

Finally it remains to show the claim for \( \min([n/2], |V_{\geq 3}|) = 1 \). The case \( |V_{\geq 3}| = 1 \) is covered by Theorem A. The other possibility is that \( n \in \{2, 3\} \) and \( |V_{\geq 3}| \geq 2 \). By Proposition 4.3, the configuration space \( \text{Conf}_n(T) \) is in this case homotopy equivalent to a graph. By [4, Theorem A, p. 2], we can choose a basis of \( H_1(\text{Conf}_n(T); \mathbb{Z}/2) \) consisting of star classes and H-classes. Since there are at least two vertices there, is
at least one star class and one H-class, which then must be linearly independent. So, the graph has first Betti number at least 2 and we get \( \text{TC}(\text{Conf}_n(T)) = 2 \). \( \square \)

5 Banana Graphs

Definition 5.1 The banana graph \( B_k \) on \( k \geq 1 \) edges is the graph consisting of two vertices connected by \( k \) edges.

Theorem B The topological complexity of \( \text{Conf}_n(B_k) \) is given by

\[
\text{TC}(\text{Conf}_n(B_k)) = \begin{cases} 
4 & \text{if } k \geq 4 \text{ and } n \geq 3, \\
2 & \text{if } k \geq 3 \text{ and } n \leq 2 \text{ or } k = n = 3.
\end{cases}
\]

Remark 5.2 The case \( k \leq 2 \) is straightforward: by Proposition 3.3 the combinatorial model in these cases is 1-dimensional, so it reduces to computing the topological complexity of graphs using Proposition 2.7. For \( k \leq 2 \) and \( n > k \) the configuration space \( \text{Conf}_n(B_k) \) is disconnected, which means that it has infinite topological complexity. The remaining cases are

\[
\begin{align*}
\text{TC}(\text{Conf}_1(B_1)) &= \text{TC}(B_1) = 0, \\
\text{TC}(\text{Conf}_1(B_2)) &= \text{TC}(B_2) = \text{TC}(S^1) = 1, \\
\text{TC}(\text{Conf}_2(B_2)) &= \text{TC}(B_2) = \text{TC}(S^1) = 1.
\end{align*}
\]

The last equality holds because the projection \( \text{Conf}_2(B_2) \rightarrow \text{Conf}_1(B_2) = B_2 \) has a homotopy inverse given by putting the second particle antipodal to the first particle.

This means that Theorem B determines the topological complexity for all pairs \( (n, k) \) except for \( (n, 3) \) with \( n \geq 4 \).

The proof has four main ingredients, whose proofs will be the content of the rest of this section:

Proposition 5.3 ([4, Prop. 4.3, p. 19]) \( \text{Conf}_3(B_4) \) is homotopy equivalent to a closed surface of genus 13.

Proposition 5.4 For any \( k \geq 3 \) the space \( \text{Conf}_2(B_k) \) is homotopy equivalent to a connected graph of first Betti number at least \( k - 1 \).

Proposition 5.5 The map \( \text{Conf}_3(B_4) \rightarrow \text{Conf}_3(B_k) \) for \( k \geq 4 \) induces an injection

\[
H_1(\text{Conf}_3(B_4)) \rightarrow H_1(\text{Conf}_3(B_k)).
\]

Proposition 5.6 For each \( n \geq m \) and each graph \( G \) with at least one essential vertex there exists a map

\[
\text{Conf}_m(G) \rightarrow \text{Conf}_n(G)
\]
which composed with the forgetful map

\[ \text{Conf}_n(G) \to \text{Conf}_m(G) \]

is homotopic to the identity. In particular, we have that \( H_\ast(\text{Conf}_m(G)) \) is a direct summand of \( H_\ast(\text{Conf}_n(G)) \).

**Proof of Theorem B**  The second case follows from the fact that \( \text{Conf}_n(B_k) \) is in that case a connected graph with first Betti number at least two, and so has topological complexity 2 by Proposition 2.7. For \( n = 1 \) this is immediate and for \( n = 2 \) it follows from Proposition 5.4.

In remains to show that \( \text{Conf}_3(B_3) \) is homotopy equivalent to a 1-dimensional complex of first Betti number at least two. This can be seen by collapsing cells in the combinatorial model as follows. The combinatorial model is 2-dimensional because there are two essential vertices. If one of the moving particles in a 2-cell moves onto the edge where the bound particle (i.e., the particle that does not move freely) is, then the 1-cell in the boundary of this 2-cell where this moving particle is bound on the edge is not attached to any other 2-cell. Therefore, we can collapse the 2-cell onto the other three 1-cells in its boundary. After collapsing all such cells we can assume that in each 2-cell the bound particle is on an edge where none of the other two particles move along.

For a 2-cell where the two moving particles move on the same edge consider the 1-cell in its boundary where one of the particles is bound on the edge. Because we just collapsed all 2-cells where a particle is moving on the edge with a bound particle this 1-cell is not contained in any other 2-cell either, so we can collapse it and assume that in each 2-cell there is always exactly one particle on each edge.

Given such a 2-cell we now consider the 1-cell where one of the particles is on the vertex. There are two additional potential 2-cells that are incident to that 1-cell, corresponding to the bound particle leaving the vertex for one of the remaining two edges. But each of those 2-cells has two particles on a single edge, and since we collapsed all such 2-cells there is no other 2-cell attached and we can finish the collapse of the combinatorial model onto a 1-dimensional cube complex as claimed. The first Betti number of this graph is at least 2 because of Proposition 5.6.

We will now prove the first case. The dimension of the combinatorial model of \( \text{Conf}_n(B_k) \) for \( n \geq 3 \) and \( k \geq 4 \) is 2, giving an upper bound on the topological complexity of 4 via Proposition 2.3. We will now show that we can use Proposition 2.5 to provide a lower bound of 4 as well.

By Proposition 2.6 all closed surfaces of genus at least two have topological complexity 4, so the case \( \text{TC} (\text{Conf}_3(B_4)) = 4 \) follows from Proposition 5.3. By Propositions 5.6 and 2.2 it suffices to show \( \text{TC} (\text{Conf}_3(B_k)) = 4 \) for all \( k \geq 4 \).

Denote by

\[ \phi_k : \text{Conf}_3(B_4) \leftrightarrow \text{Conf}_3(B_k) \]
the map induced by the inclusion $B_4 \hookrightarrow B_k$ for $k \geq 4$. By Proposition 5.5, the induced map

$$H^1(\phi_k): H^1(\text{Conf}_3(B_k); \mathbb{Q}) \rightarrow H^1(\text{Conf}_3(B_4); \mathbb{Q})$$

is surjective. In the proof of the lower bound of $\text{TC}(\text{Conf}_3(B_4)) = \text{TC}(\Sigma_1)$ in [7] Farber constructs four classes $u_1, u_2, u_3, u_4 \in H^1(\text{Conf}_3(B_4))$ such that the cup product of the associated zero divisors $u_1 \cdots u_4$ is non-trivial in the cohomology of $\text{Conf}_3(B_4)^{\times 2}$. Now choose preimages $v_1, v_2, v_3$ and $v_4$ of these four classes under $H^1(\phi_k)$ and look at the product of the corresponding zero-divisors $\bar{v}_1 \cdots \bar{v}_4$ in $H^4(\text{Conf}_4(B_k)^{\times 2})$. By construction, this element maps to $\bar{u}_1 \cdots \bar{u}_4 \neq 0$ under the ring map

$$H^*(\phi_k^{\times 2}): H^*(\text{Conf}_3(B_k)^{\times 2}; \mathbb{Q}) \rightarrow H^*(\text{Conf}_3(B_4)^{\times 2}; \mathbb{Q})$$

so it has to be non-trivial as well. It proves that $\text{TC}(\text{Conf}_3(B_k)) = 4$ for all $k \geq 4$. ☐

We will now prove the stated propositions.

**Proof of Proposition 5.4** Consider a 2-cube in the combinatorial model of $\text{Conf}_2(B_k)$ from Proposition 3.3. This cube has two coordinates, corresponding to the movement of the particles 1 and 2 towards the two vertices: increasing the horizontal coordinate moves particle 1 from the interior of some edge towards one of the two vertices, and increasing the vertical coordinate moves 2 in the same way towards the other vertex. Restricting to a face of the 2-cube corresponds to keeping the corresponding particle on the vertex or in the interior of the edge and moving the other particle towards a vertex. Exactly one of the four 1-cubes in the boundary of the 2-cube keeps particle 1 in the interior of some edge. This 1-cube is \emph{not} incident to any other 2-cube because particle 1 cannot move towards the same vertex as particle 2. Therefore, we can deform the 2-cell by collapsing this 1-cube onto the other three 1-cubes.

Repeating this process for all 2-cubes defines a homotopy equivalence to a graph. By Proposition 5.6, the first Betti number has to be at least $k - 1$. ☐

**Proof of Proposition 5.6** We will define a map between the combinatorial models with these properties, which by composition with the deformation retraction and inclusion determines a map of the ordinary configuration spaces. Choose an essential vertex $v$ and three edges $e_1, e_2, e_3$ incident to $v$.

For each $k$-cube in the combinatorial model of $\text{Conf}_m(G)$ where no particle moves from $e_1$ towards $v$ simply add the $n - m$ missing particles in ascending order onto $e_1$ between $v$ and all other particles on $e_1$.

Given a cube where one particle $p$ moves from $e_1$ towards $v$ we consider the following sequence of movements: move the $n - m$ new particles via $v$ onto $e_2$, move $p$ via $v$ onto $e_3$, move the $n - m$ particles back onto $e_1$ in the same way and finally move $p$ onto $v$. These movements are independent of the movements of the other particles in the chosen cube, so we can replace the movement of $p$ with this sequence. This defines a union of cells in the combinatorial model of $\text{Conf}_n(G)$, and we define our map to stretch the cube we started with onto this strip of cells, see Fig. 6. It is
Fig. 6 Replacing a 2-cell by a strip of 2-cells to construct a map $\text{Conf}_m(G) \rightarrow \text{Conf}_n(G)$ for $m < n$. The vertical direction in the cubes corresponds to the movement of particle 2, the seven small rectangles above are stretched to the seven cubes below.

It is straightforward to check that this gives a continuous map, i.e., that the restriction to the boundaries of a cell determines the correct map.

By construction, the composition with the map forgetting the $n - m$ new particles gives almost the identity, only the particles moving from $e_1$ towards $v$ briefly move onto $e_3$. Up to homotopy, however, the map is the identity.

5.1 Proof of Proposition 5.5

The proof of Proposition 5.5 is a combination of parts of proofs in [13]. For the convenience of the reader we reproduce the relevant parts here.

5.2 A Mayer–Vietoris Spectral Sequence for Configuration Spaces

In this section, we will use the basic classes in the first homology of configuration spaces of graphs to prove Proposition 5.5. These are given by star classes, H-classes and $S^1$-classes, which are elements in the first homology of configuration spaces of any star graph, the graph that looks like the letter H and the circle $S^1$, respectively. $S^1$-classes are given by all particles moving around the circle simultaneously, H-classes are given by “exchange movements” like in Example 3.5 and star classes are shufflings of particles sitting on the leaves via the central vertex. For more details on those classes, see for example [13].

We now recall the construction of the Mayer–Vietoris spectral sequence for configuration spaces discussed in [4].

**Definition 5.7** (Mayer–Vietoris spectral sequence) Let $J$ be a countable ordered index set and $\{V_j\}_{j \in J}$ an open cover of $X$, then we define the following countable open
cover \( \mathcal{U}(\{V_j\}) \) of \( \text{Conf}_n(X) \): for each \( \phi : n \to J \) we define \( U_\phi \) to be the set of all those configurations where each particle \( i \) is in \( V_{\phi(i)} \), i.e.,

\[
U_\phi := \bigcap_{i \in n} \pi_i^{-1}(V_{\phi(i)}).
\]

These sets are open and cover the whole space, so they define a spectral sequence

\[
E^1_{p,q} = \bigoplus_{\phi_0 \prec \cdots \prec \phi_p} H_q(U_{\phi_0} \cap \cdots \cap U_{\phi_p}; \mathbb{Q}) \Rightarrow H_*(\text{Conf}_n(X); \mathbb{Q}),
\]

where the indexing of the direct sum is over all ordered sets \( \phi_0 < \cdots < \phi_p \) of maps \( \phi_i \) as described above, converging to the homology of the whole space. Here, we chose an arbitrary ordering of the maps \( \phi_i \), for example lexicographic ordering. For an elementary proof of the convergence of this spectral sequence, see [13, Prop. 2.1, p. 25].

For brevity, we will also write

\[
U_{\phi_0 \cdots \phi_p} := U_{\phi_0} \cap \cdots \cap U_{\phi_p}.
\]

The boundary map \( d_1 \) is given by the alternating sum of the face maps induced by

\[
U_{\phi_0} \cap \cdots \cap U_{\phi_p} \leftrightarrow U_{\phi_0} \cap \cdots \cap \widehat{U_{\phi_i}} \cap \cdots \cap U_{\phi_p}
\]

forgetting the \( i \)th open set from the intersection. Of course, this construction generalizes to configuration spaces with sinks. From now on we will suppress the rational coefficients in our notation.

**Proof of Proposition 5.5** Endow \( B_k \) with a path metric such that each edge has length 1. Consider the open cover of \( B_k \) given by the open balls \( V_1 \) and \( V_2 \) of radius \( 2/3 \) around \( v_1 \) and \( v_2 \), respectively. The intersection \( V_1 \cap V_2 \) is given by a disjoint union of intervals of length \( 1/3 \). Pulling the particles out of this intersection if possible one can see that each intersection \( U_{\phi_0 \cdots \phi_p} \) is homotopy equivalent to a disjoint union of spaces of the form

\[
\text{Conf}_{S_1}(\text{Star}_{v_1}) \times \text{Conf}_{S_1}(\bigsqcup_k I) \times \text{Conf}_{S_2}(\text{Star}_{v_2}),
\]

where \( \text{Star}_v \) is a small contractible neighborhood of \( v \), \( I \) is an interval and \( S_1 \sqcup S_\gamma \sqcup S_2 = \{1, 2, 3\} \).

Let us now compute the bottom row of the \( E^2 \)-page of the spectral sequence \( E^*_{*,*}[k] \) associated to the open cover of \( \text{Conf}_3(B_k) \). The bottom row of the \( E^1 \)-page is given at position \( (p, 0) \) by the direct sum of all terms of the form \( H_0(U_{\phi_0 \cdots \phi_p}) \). If we now write the same spectral sequence for \( \text{Conf}_3(B_k, V(B_k)) \), i.e., with both vertices turned into sinks, then we see by the identification (1) that the bottom rows of both \( E^1 \)-pages agree (including differentials). Since in the sink case all \( U_{\phi_0 \cdots \phi_p} \) have contractible path
components, all higher rows of the $E^1$-page are trivial. Therefore, the bottom row of the 
$E^2$-pages of both spectral sequences at position $p$ is given by $H_p(\text{Conf}_3(B_k, V(B_k)))$. 
In particular, this gives $E^\infty_{1,0}[k] \cong H_1(\text{Conf}_3(B_k, V(B_k)))$ for both spectral sequences.

By [13, Thm. D, p. 5] the first homology of $\text{Conf}_3(B_k, V(B_k))$ is generated 
by $S^1$-classes and H-classes. It is straightforward to check that all H-classes 
in $\text{Conf}_3(B_k, V(B_k))$ are trivial for $k \geq 3$, so we can choose a basis of 
$H_1(\text{Conf}_3(B_k, V(B_k)))$ consisting only of individual particles moving along an 
embedded circle in $B_k$ with both other particles sitting on one of the sinks. This 
means that the inclusions

$$\text{Conf}_{\{1\}}(B_k) \sqcup \text{Conf}_{\{2\}}(B_k) \sqcup \text{Conf}_{\{3\}}(B_k) \to \text{Conf}_3(B_k, V(B_k))$$

given by putting the remaining particles onto one of the sinks induce a surjection in first 
homology. Composition with the forgetful maps shows that the map is also injective in 
homology, so it in fact induces an $H_1$-isomorphism. Since $H_1(\text{Conf}_1(B_k)) = H_1(B_k)$
has $H_1(\text{Conf}_1(B_4)) = H_1(B_4)$ as a direct summand we get that also $E^\infty_{1,0}[4] = 
H_1(\text{Conf}_3(B_4, V(B_4)))$ is a direct summand of $E^\infty_{1,0}[k] = H_1(\text{Conf}_3(B_k, V(B_k)))$.

It remains to see that the same splitting is possible for $E^\infty_{0,1}[k]$. The module $E^1_{0,1}[k]$ 
is by the identifications (1) given by a direct sum of modules of the form

$$H_1(\text{Conf}_{S_\{1\}}(\text{Star}_{v_1}[k]) \times \text{Conf}_{S_\{2\}}(\sqcup_{k} I) \times \text{Conf}_{S_\{2\}}(\text{Star}_{v_2}[k]),$$

where $\text{Star}_{v_1}[k]$ and $\text{Star}_{v_2}[k]$ are the stars around the vertices $v_1$ and $v_2$. It is straightforward to check that the map

$$E^1_{0,1}[k] \supset H_1(\text{Conf}_3(\text{Star}_{v_1}[k])) \oplus H_1(\text{Conf}_3(\text{Star}_{v_2}[k])) \to E^2_{0,1}[k]$$

is a surjection and that the images of the two direct summands intersect trivially in $E^2_{0,1}[k]$. Define $Q^v_{11}[k]$ as the image of the map

$$\bigoplus_{e \in E(\text{Star}_{v_1}[k])} H_1(\text{Conf}_S(\text{Star}_{v_1}[k])) \to H_1(\text{Conf}_3(\text{Star}_{v_1}[k])),$$

where $S \subseteq \{1, 2, 3\}$ and the map is induced by adding the third particle onto the end of edge $e$ (away from $v_1$). This submodule is everything generated by classes with 
only two moving particles. We now write the module $H_1(\text{Conf}_3(\text{Star}_{v_1}[k]); \mathbb{Q})$ as $Q^v_{21}[k] \oplus Q^v_{31}[k]$ for some (arbitrary) choice of $Q^v_{31}[k]$. Because in $E^1_{1,1}[k]$ there is 
no term with all three particles in one of the two stars, this means that $Q^v_{21}[k]$ does 
not intersect the image of the boundary map $d_0$ and therefore is a direct summand of $E^1_{0,1}[k]$. The module $Q^v_{21}[k]$ by definition has a basis where each basis element has 
exactly one bound particle (i.e., a particle that does not move freely). By adding $d_0$ 
boundaries to the image of such a basis element we can arrange that this bound particle 
is always on a fixed edge $e_0$. One can now check that the image of $Q^v_{21}[k]$ under the 
collapse map to $E^2_{0,1}[k]$ is the same as the image of $Q^v_{21}[k]$, defined as the image of
the induced map
\[ \bigoplus_{|S|=2} H_1 \left( \text{Conf}_S(\text{Star}_{v_1}[k]) \right) \to H_1 \left( \text{Conf}_3(\text{Star}_{v_1}[k]) \right), \]

where the third particle is always put onto the edge \( e_0 \). There are no relations imposed onto the image of \( Q_2^{v_1}[k] \) for \( v \in \{v_1, v_2\} \) under the quotient map, so we get (by repeating the same argument for \( v_2 \))

\[ E^2_{0,1}[k] \cong Q_2^{v_1}[k] \oplus Q_3^{v_1}[k] \oplus Q_2^{v_2}[k] \oplus Q_3^{v_2}[k]. \]

Notice that by symmetry we have in fact

\[ E^2_{0,1}[k] \cong Q_2^{v_1}[k] \oplus Q_3^{v_1}[k] \oplus 2. \quad (2) \]

Each \( H_1(\text{Conf}_S(\text{Star}_v[k])) \) for any finite set \( S \) has \( H_1(\text{Conf}_S(\text{Star}_v[4])) \) as a direct summand: choose a spanning tree in the graph \( \text{Conf}_S(\text{Star}_v[4]) \), then this defines a tree in \( \text{Conf}_S(\text{Star}_v[k]) \) via the inclusion. Extending this tree to a spanning tree shows that the first homology of \( \text{Conf}_S(\text{Star}_v[4]) \) is a direct summand of \( H_1(\text{Conf}_S(\text{Star}_v[k])) \). It is straightforward to check that this direct sum decomposition respects the splitting into \( Q_2^{v_1}[k] \oplus Q_3^{v_1}[k] \) in the sense that \( Q_2^{v}[4] \) is a direct summand of \( Q_2^{v_1}[k] \) and \( Q_2^{v_1}[4] \) and \( Q_3^{v_1}[k] \) can be chosen so that the former is a direct summand of the latter. This shows that \( E^2_{0,1}[4] \) is a direct summand of \( E^2_{0,1}[k] \). Denote the complement by \( E^2_{0,1}(k, 4) \).

It remains to show that the boundary map

\[ d_2: H_2(\text{Conf}_3(B_k, V(B_k))) \to E^2_{0,1}[k] \cong E^2_{0,1}[4] \oplus E^2_{0,1}(k, 4) \]

preserves this splitting for \( k > 4 \).

Consider the analogous spectral sequence \( \widetilde{E}^*_2[k] \) for the space \( \text{Conf}_3(B_k, \{v_2\}) \) instead of \( \text{Conf}_3(B_k) \). In this case we get that the term \( \widetilde{E}^2_{0,1}[k] \) is given by \( Q_2^{v_1}[k] \oplus Q_3^{v_1}[k] \). The combinatorial model of \( \text{Conf}_3(B_k, \{v_2\}) \) is one dimensional and all 1-cubes have the following form: one particle moves from the sink to the other vertex while the other particles stay in the sink. Therefore, its first homology is generated by individual particles moving along embedded circles with the remaining particles fixed on the sink. Each such class can be represented as an element of \( \widetilde{E}_1^\infty[k] \) by looking at the intersection of the two open sets where a particle \( p \) is in the neighborhood of \( v_1 \) or \( v_2 \), respectively, and all other particles are in the neighborhood of \( v_2 \). The circle class is then represented by the difference of \( p \) being in different connected components of the disjoint union of intervals given by the intersection of the two open subsets of \( B_k \).

This shows that \( \widetilde{E}_0^\infty[k] = \widetilde{E}_0^2[k] = 0 \), and therefore that

\[ \tilde{d}_2: H_2(\text{Conf}_3(B_k, V(B_k))) \to Q_2^{v_1}[k] \oplus Q_3^{v_1}[k] \]

is surjective in this case. Since the second homology of \( \text{Conf}_3(B_k, \{v_2\}) \) is trivial, this map in fact is an isomorphism. This shows with the identification (2) that for the
spectral sequence for $\text{Conf}_3(B_k)$ we get

$$E_{0,1}^\infty = E_{0,1}^3 \cong \overline{Q_2^{v_1}}[k] \oplus Q_3^{v_1}[k]$$

because the map $d_2$ is the product of the two isomorphisms $\tilde{d}_2$ from the cases where $v_1$ or $v_2$ is a sink, respectively (we can compare the spectral sequences using the induced maps, by naturality). Since we already saw that $\overline{Q_2^{v_1}}[4]$ and $Q_3^{v_1}[4]$ are direct summands of $\overline{Q_2^{v_1}}[k]$ and $Q_3^{v_1}[k]$, respectively, this concludes the proof. \qed

### 6 On a Conjecture of Farber and on Unordered Configuration Spaces

In [9] Farber formulated the following

**Conjecture 6.1** (Farber) Let $G$ be a connected graph with $|V_{\geq 3}| \geq 2$ and let $n \geq 2|V_{\geq 3}|$. Then

$$\text{TC}(\text{Conf}_n(G)) = 2|V_{\geq 3}|.$$  

In the same paper Farber proved that the conjecture holds for trees. The results in this paper provide further evidence for this conjecture by showing that it holds for the more general fully articulated graphs and for most banana graphs.

In Theorem A we also show that, for $T$ a tree,

$$\text{TC}(\text{Conf}_n(T)) = 2\lceil n/2 \rceil$$

grows steadily in $n$ while $n < 2|V_{\geq 3}|$ until it stabilizes at

$$\text{TC}(\text{Conf}_n(T)) = 2|V_{\geq 3}|$$

for $n \geq 2|V_{\geq 3}|$. This does not generalize to banana graphs: by Theorem B we have that

$$\text{TC}(\text{Conf}_3(B_k)) = 4,$$

but $3 < 2|V_{\geq 3}|$. This raises the problem of understanding the behavior of $\text{TC}(\text{Conf}_n(G))$ for small $n$, for a general graph $G$.

Another open question is the relationship between $\text{TC}(\text{Conf}_n(G))$ for ordered configuration spaces and $\text{TC}(\text{UConf}_n(G))$ for unordered configuration spaces. Scheirer [14] showed that they coincide in many cases and so one might be tempted to conjecture that they are always equal, provided that $\text{UConf}_n(G)$ is connected. However, this is in fact not the case.

A counterexample is given by $G = H$ and $n = 4$. We know from Theorem T that $\text{TC}(\text{Conf}_4(H)) = 4$. However by the work of Connolly and Doig [6, Propositions 8 and 11] we see that $\text{UConf}_4(H)$ is the classifying space for $F_{10} \ast (\mathbb{Z} \times \mathbb{Z})$ and so $\text{TC}(\text{UConf}_4(H)) = 3$ by [5], where the topological complexity of classifying spaces
of right-angled Artin groups is computed. This shows that Farber’s conjecture does not hold true in the unordered setting, not even for trees.

One can similarly construct infinitely many such counterexamples by gluing $m$ Y graphs together so that all essential vertices lie on an interval and taking $n = 2m$. It is worth noting that in all these cases the topological complexity in the unordered setting is in fact smaller than in the ordered one.

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