INDIVIDUAL ERGODIC THEOREMS IN NONCOMMUTATIVE ORLICZ SPACES

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Abstract. For a noncommutative Orlicz space associated with a semifinite von Neumann algebra, a faithful normal semifinite trace and an Orlicz function satisfying $(\delta_2, \Delta_2)$-condition, an individual ergodic theorem is proved.

1. Introduction

Development of the theory of noncommutative integration with respect to a faithful normal semifinite trace $\tau$ defined on a semifinite von Neumann algebra $\mathcal{M}$, has given rise to a systematic study of various classes of noncommutative rearrangement invariant Banach spaces. The noncommutative $L^p$-spaces $L^p(\mathcal{M}, \tau)$ [20, 22, 19] and, more generally, noncommutative Orlicz spaces $L^\Phi(\mathcal{M}, \tau)$ [16, 17, 13] are important examples of such spaces.

Since every $L^\Phi(\mathcal{M}, \tau)$ is an exact interpolation space for the Banach couple $(L^1(\mathcal{M}, \tau), \mathcal{M})$, for any linear operator $T : L^1(\mathcal{M}, \tau) + \mathcal{M} \rightarrow L^1(\mathcal{M}, \tau) + \mathcal{M}$ such that

$$\|T(x)\|_\infty \leq \|x\|_\infty \quad \forall \ x \in \mathcal{M} \quad \text{and} \quad \|T(x)\|_1 \leq \|x\|_1 \quad \forall \ x \in L^1(\mathcal{M}, \tau)$$

(such operators are called Dunford-Schwartz operators), we have

$$T(L^\Phi) \subset L^\Phi \quad \text{and} \quad \|T\|_{L^\Phi \rightarrow L^\Phi} \leq 1.$$ 

Thus, it is natural to study noncommutative Dunford-Schwartz ergodic theorem in $L^\Phi(\mathcal{M}, \tau)$. The first result in this direction was obtained in [23] for the space $L^1(\mathcal{M}, \tau)$ (as it is noticed in [3, Proposition 1.1], the class of operators $\alpha$ that was employed in [23] coincides with the class of positive Dunford-Schwartz operators). In [10], the result of [23] was extended to the noncommutative $L^p$-spaces with $1 < p < \infty$. For general noncommutative fully symmetric spaces with non trivial Boyd indexes, an individual ergodic theorem was established in [3].

Note that the class of Orlicz spaces $L^\Phi(\mathcal{M}, \tau)$ is significantly wider than the class of spaces $L^p(\mathcal{M}, \tau)$. Besides, there are Orlicz spaces $L^\Phi(\mathcal{M}, \tau)$, with the Orlicz function satisfying the so-called $(\delta_2, \Delta_2)$-condition, which have trivial Boyd index $p_{L^\Phi} = 1$ (see Remark 2.3 below). Therefore an individual ergodic theorem for positive Dunford-Schwartz operators in Orlicz spaces does not follow from the results mentioned above.

The aim of this article is to establish an individual ergodic theorem for a positive Dunford-Schwartz operator in a noncommutative Orlicz space $L^\Phi(\mathcal{M}, \tau)$ associated
with an Orlicz function \( \Phi \) satisfying \((\delta_2, \Delta_2)\)–condition. Our argument is essentially based on the notion of uniform equicontinuity in measure at zero of a sequence of linear maps from a normed space into the space of measurable operators affiliated with \((\mathcal{M}, \tau)\). This notion was introduced in \cite{2} and then applied in \cite{15} to provide a simplified proof of noncommutative individual ergodic theorem for positive Dunford-Schwartz operators in \(L^p(\mathcal{M}, \tau), 1 < p < \infty\).

2. Preliminaries

Assume that \(\mathcal{M}\) is a semifinite von Neumann algebra with a faithful normal semifinite trace \(\tau\), and let \(\mathcal{P}(\mathcal{M})\) be the complete lattice of projections in \(\mathcal{M}\). If \(1\) is the multiplicative identity of \(\mathcal{M}\) and \(e \in \mathcal{P}(\mathcal{M})\), we denote \(e^\perp = 1 - e\). Let \(L^0 = L^0(\mathcal{M}, \tau)\) be the \(\ast\)-algebra of \(\tau\)-measurable operators. Recall that \(L^0\) is a metrizable topological \(\ast\)-algebra with respect to the measure topology that can be equivalently (see \cite[Theorem 2.2]{1}) defined by either of the families

\[ V(\epsilon, \delta) = \{x \in L^0 : \|xe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon \} \]

or

\[ W(\epsilon, \delta) = \{x \in L^0 : \|e_xe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon \}, \]

\(\epsilon > 0, \delta > 0\), of neighborhoods of zero \cite{13}.

For a positive operator \(x = \int_0^\infty \lambda d\lambda \in L^0\) one can define

\[ \tau(x) = \sup_n \tau\left(\int_0^n \lambda d\lambda\right) = \int_0^\infty \lambda d\tau(e_\lambda). \]

If \(1 \leq p < \infty\), then the noncommutative \(L^p\)–space associated with \((\mathcal{M}, \tau)\) is defined as

\[ L^p = (L^p(\mathcal{M}, \tau), \| \cdot \|_p) = \{x \in L^0 : \| x \|_p = (\tau(\|x\|^p))^{1/p} < \infty\}, \]

where \(\|x\| = (x^*x)^{1/2}\) is the absolute value of \(x\); naturally, \(L^\infty = \mathcal{M}\).

For detailed accounts on noncommutative \(L^p\)-spaces, see \cite{19} \cite{22}.

Given \(x \in L^p\), let \(\{e_\lambda\}_{\lambda \geq 0}\) be the spectral family of projections of \(|x|\). If \(t > 0\), the \(t\)-th generalized singular number of \(x\) \cite{9} is defined as

\[ \mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^\perp) \leq t\}. \]

A Banach space \((E, \| \cdot \|_E) \subset L^0\) is called fully symmetric if the conditions

\[ x \in E, \ y \in L^0, \ s \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt \ \ \forall \ s > 0 \]

imply that \(y \in E\) and \(\|y\|_E \leq \|x\|_E\).

If \(L \subset L^0\), the set of all positive operators in \(L\) will be denoted by \(L_+\).

A fully symmetric space \((E, \| \cdot \|_E)\) is said to possess Fatou property if the conditions

\[ x_\alpha \in E_+, \ x_\alpha \leq x_\beta \text{ for } \alpha \leq \beta, \text{ and } \sup_\alpha \|x_\alpha\|_E < \infty \]

imply that there exists \(x = \sup_\alpha x_\alpha \in E\) and \(\|x\|_E = \sup_\alpha \|x_\alpha\|_E\).

Let \(m\) be Lebesgue measure on the interval \((0, \infty)\), and let \(L^0(0, \infty)\) be the linear space of all (equivalence classes of) almost everywhere finite complex-valued \(m\)–measurable functions on \((0, \infty)\). We identify \(L^\infty(0, \infty)\) with the commutative von Neumann algebra acting on the Hilbert space \(L^2(0, \infty)\) via multiplication by
the elements from \( L^\infty(0, \infty) \) with the trace given by the integration with respect to Lebesgue measure. A fully symmetric space \( E \subset L^p(M, \tau) \), where \( M = L^\infty(0, \infty) \) and \( \tau \) is given by the Lebesgue integral, is called \textit{fully symmetric function space} on \( (0, \infty) \).

Let \( E = (E(0, \infty), \| \cdot \|_E) \) be a fully symmetric function space. For each \( s > 0 \) let \( D_s : E(0, \infty) \to E(0, \infty) \) be the bounded linear operator given by

\[
D_s(f)(t) = f(t/s), \quad t > 0.
\]

The \textit{Boyd indices} \( p_E \) and \( q_E \) are defined as

\[
p_E = \lim_{s \to \infty} \frac{\log s}{\log \| D_s \|_E}, \quad q_E = \lim_{s \to +0} \frac{\log s}{\log \| D_s \|_E}.
\]

It is known that \( 1 \leq p_E \leq q_E \leq \infty \) \cite{14}, II, Ch.2, Proposition 2.b.2]. A fully symmetric function space is said to have \textit{non-trivial Boyd indices} if \( 1 < p_E \) and \( q_E < \infty \). For example, the spaces \( L^p(0, \infty) \), \( 1 < p < \infty \), have non-trivial Boyd indices:

\[
\| L_p(0, \infty) \| = q_{L_p(0, \infty)} = p
\]

\cite{14}, II, Ch.2, 2.b.1].

If \( E \) is a fully symmetric function space on \( (0, \infty) \), define

\[
E(M) = E(M, \tau) = \{ x \in L^0(M, \tau) : \mu(x) \in E \}
\]

and set

\[
\| x \|_{E(M)} = \| \mu(x) \|_E, \quad x \in E(M).
\]

It is shown in \cite{4} that \( (E(M), \| \cdot \|_{E(M)}) \) is a fully symmetric space.

If \( 1 \leq p < \infty \) and \( E = L^p(0, \infty) \), the space \( (E(M), \| \cdot \|_{E(M)}) \) coincides with the noncommutative \( L^p \)-space \( (L^p(M, \tau), \| \cdot \|_p) \) because

\[
\| x \|_p = \left( \int_0^\infty \mu^p_t(x) dt \right)^{1/p} = \| x \|_{L^p(M, \tau)}.
\]

\cite{22}, Proposition 2.4].

Since for a fully symmetric function space \( E \) on \( (0, \infty) \),

\[
L^1(0, \infty) \cap L^\infty(0, \infty) \subset E \subset L^1(0, \infty) + L^\infty(0, \infty)
\]

with continuous embeddings \cite{12} Ch.II, §4, Theorem 4.1], we also have

\[
L^1(M, \tau) \cap M \subset E(M, \tau) \subset L^1(M, \tau) + M,
\]

with continuous embeddings.

**Definition 2.1.** A convex continuous at 0 function \( \Phi : [0, \infty) \to [0, \infty) \) such that \( \Phi(0) = 0 \) and \( \Phi(u) > 0 \) if \( u \neq 0 \) is called an Orlicz function.

**Remark 2.1.** (1) Since an Orlicz function is convex and continuous at 0, it is necessarily continuous on \([0, \infty)\).

(2) If \( \Phi \) is an Orlicz function, then \( \Phi(\lambda u) \leq \lambda \Phi(u) \) for all \( \lambda \in [0, 1] \). Therefore \( \Phi \) is increasing, that is, \( \Phi(u_1) < \Phi(u_2) \) whenever \( 0 \leq u_1 < u_2 \).

We will need the following lemma.
Lemma 2.1. Let $\Phi$ be an Orlicz function. Then for any given $\delta > 0$ there exists $t > 0$ satisfying the condition
\[ t \cdot \Phi(u) \geq u \quad \text{whenever} \quad u \geq \delta. \]
In particular, $\lim_{u \to \infty} \Phi(u) = \infty$.

Proof. Since $\Phi(u) > 0$ as $u > 0$, it is possible to find $a > 0$ such that the equation $\Phi(u) = au$ has a solution $u = u_0 > 0$. Then, as $\Phi$ is convex, we have $\Phi(u) \geq au$ for all $u \geq u_0$.

Fix $\delta > 0$. If $\delta \geq u_0$, then we have
\[ \frac{1}{a} \cdot \Phi(u) \geq u \quad \forall \ u \geq \delta. \]
If $\delta < u_0$, then, since $\Phi(\delta) > 0$ and $\Phi$ is increasing on the interval $[\delta, u_0]$, there exists such $s > 1$ that $s \cdot \Phi(u) \geq au$, or
\[ \frac{s}{a} \cdot \Phi(u) \geq u, \quad \forall \ u \geq \delta. \]
\[ \Box \]

Remark 2.2. Since an Orlicz function $\Phi$ is continuous, increasing and such that $\lim_{u \to \infty} \Phi(u) = \infty$, there exists continuous increasing inverse function $\Phi^{-1}$ from $[0, \infty)$ onto $[0, \infty)$.

If $\Phi$ is an Orlicz function, $x \in L^p_\Phi$ and $x = \int_0^\infty \lambda d\xi$ its spectral decomposition, one can define $\Phi(x) = \int_0^\infty \Phi(\lambda) d\xi$. The noncommutative Orlicz space associated with $(\mathcal{M}, \tau)$ for an Orlicz function $\Phi$ is the set
\[ L^\Phi = L^\Phi(\mathcal{M}, \tau) = \left\{ x \in L^0(\mathcal{M}, \tau) : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) < \infty \quad \text{for some} \quad a > 0 \right\}. \]

The Luxemburg norm of an operator $x \in L^\Phi$ is defined as
\[ ||x||_\Phi = \inf \left\{ a > 0 : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) \leq 1 \right\}. \]

Theorem 2.1. \cite[Proposition 2.5]{13} $(L^\Phi, ||\cdot||_\Phi)$ is a Banach space.

Proposition 2.1. If $x \in L^\Phi$, then $\Phi(|x|) \in L^0$ and $\mu_t(\Phi(|x|)) = \Phi(\mu_t(x))$, $t > 0$. In addition, $\tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu_t(x))dt$.

Proof. As $x \in L^\Phi$, we have $\tau \left( \Phi \left( \frac{|x|}{a} \right) \right) < \infty$ for some $a > 0$. This implies that $\Phi \left( \frac{|x|}{a} \right) \in L^1$, so $\tau \left( \left\{ \Phi \left( \frac{|x|}{a} \right) > \lambda \right\} \right) < \infty$ for all $\lambda > 0$. Since
\[ \left\{ \Phi \left( \frac{|x|}{a} \right) > \lambda \right\} = \left\{ \Phi^{-1} \left( \Phi \left( \frac{|x|}{a} \right) \right) > \Phi^{-1}(\lambda) \right\} = \{ |x| > a\Phi^{-1}(\lambda) \}, \]
it follows that $\tau \left( \{ \Phi(|x|) > \mu \} \right) = \tau \left( \{ |x| > \Phi^{-1}(\mu) \} \right) < \infty$ for all $\mu > 0$, thus $\Phi(|x|) \in L^0$.

By \cite[Lemma 2.5, Corollary 2.8]{9}, given $x \in L^0$, we have $\mu_t(\varphi(|x|)) = \varphi(\mu_t(x))$, $t > 0$, for every continuous increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and, in addition, $\tau(\varphi(|x|)) = \int_0^\infty \varphi(\mu_t(x))dt$. Therefore $\mu_t(\Phi(|x|)) = \Phi(\mu_t(x))$ and $\tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu_t(x))dt$.

Next result follows immediately from Proposition 2.1.
Corollary 2.1. \( L^\Phi = \{ x \in L^0 : \mu(x) \in L^\Phi(0, \infty) \} \) and \( \| x \|_\Phi = \| \mu(x) \|_\Phi \) for all \( x \in L^\Phi \).

If \( (L^\Phi(0, \infty), \| \cdot \|_\Phi) \) is the Orlicz function space on \( (0, \infty) \) for an Orlicz function \( \Phi \), then, by \cite[Ch.2, Proposition 2.1.12]{S}, it is a rearrangement invariant function space. Since \( (L^\Phi(0, \infty), \| \cdot \|_\Phi) \) has the Fatou property \cite[Ch.2, Theorem 2.1.11]{S}, Corollary 2.1, \cite[Theorem 4.1]{M}, and \cite[Theorem 3.4]{L} yield the following.

Corollary 2.2. \( (L^\Phi, \| \cdot \|_\Phi) \) is a fully symmetric space with the Fatou property and an exact interpolation space for the Banach couple \( (L^1, \mathcal{M}) \).

We will also need the following property of the Luxemburg norm.

Proposition 2.2. If \( x \in L^\Phi \) and \( \| x \|_\Phi \leq 1 \), then \( \tau(\Phi(\| x \|)) \leq \| x \|_\Phi \).

Proof. By \cite[Ch.2, Proposition 2.1.10]{S}, \( \int_0^\infty \Phi(|f|)dt \leq \| f \|_\Phi \) for \( f \in L^\Phi(0, \infty) \) with \( \| f \|_\Phi \leq 1 \). Thus the result follows from Proposition 2.1 and Corollary 2.1. \( \square \)

Definition 2.2. An Orlicz function \( \Phi \) is said to satisfy \( \Delta_2 \)–condition (\( \delta_2 \)–condition) if there exist \( k > 0 \) and \( u_0 > 0 \) such that

\[
\Phi(2u) \leq k\Phi(u) \quad \forall \; u \geq u_0 \quad \text{(respectively, } \Phi(2u) \leq k\Phi(u) \quad \forall \; u \in (0, u_0)) \).
\]

If an Orlicz function \( \Phi \) satisfies \( \Delta_2 \)–condition and \( \delta_2 \)–condition simultaneously, we will say that \( \Phi \) satisfies \( \delta_2, \Delta_2 \)–condition. In this case \( \Phi(2u) \leq c\Phi(u) \) for all \( u \geq 0 \) and some \( c > 0 \). Clearly, every space \( L^p, \; 1 \leq p < \infty \), is the Orlicz space for the function \( \Phi(u) = u^p \), \( u \geq 0 \), which satisfies \( \delta_2, \Delta_2 \)–condition.

Remark 2.3. (i) If an Orlicz function \( \Phi \) satisfies \( \Delta_2 \)–condition, then the Boyd index \( q_{L^\Phi(0, \infty)} < \infty \), that is, it is non-trivial (see \cite[II, Ch.2, Proposition 2.b.5]{M}).

(ii) The function \( \Phi_\alpha(u) = u \ln^\alpha(e + u), \; \alpha \geq 0 \), is an Orlicz function that satisfies \( \delta_2, \Delta_2 \)–condition for which the Boyd index \( p_{L^\Phi(0, \infty)} \) is trivial, that is, \( p_{L^\Phi(0, \infty)} = 1 \) \cite[§5]{M}.

A Banach space \( (E, \| \cdot \|_E) \subset L^0 \) is said to have order continuous norm if \( \| x_n \|_E \downarrow 0 \) for every net \( \{ x_n \} \subset E \) with \( x_n \downarrow 0 \).

Proposition 2.3. Let an Orlicz function \( \Phi \) satisfy \( \delta_2, \Delta_2 \)–condition. Then

(i) The fully symmetric space \( (L^\Phi, \| \cdot \|_\Phi) \) has order continuous norm.

(ii) The linear subspace \( L^1 \cap \mathcal{M} \) is dense in \( (L^\Phi, \| \cdot \|_\Phi) \).

Proof. (i) As shown in \cite[Ch.2, §2.1]{S}, the fully symmetric space \( (L^\Phi(0, \infty), \| \cdot \|_\Phi) \) has order continuous norm. Therefore, by \cite[Proposition 3.6]{S}, the noncommutative fully symmetric space \( (L^\Phi, \| \cdot \|_\Phi) \) also has order continuous norm.

(ii) Let \( x \in L^\Phi_+, \; n = 1, 2, \ldots, \) and \( e_n \) the spectral projection corresponding to the interval \( (n^{-1}, n) \). It is clear that \( \{ xe_n \} \subset \mathcal{M} \) and \( e_n^\perp \downarrow 0 \). Also, by (i) and \cite[Theorem 3.1]{W}, we have

\[
\| x - xe_n \|_\Phi = \| xe_n^\perp \|_\Phi \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, since \( \tau(\{ x > \epsilon \}) \) \( < \infty \) for all \( \epsilon > 0 \) (see proof of Proposition 2.1), it follows that \( \{ xe_n \} \subset L^1 \).

Since, for an arbitrary \( x \in L^\Phi \), we have \( x = x_1 - x_2 + i(x_3 - x_4) \), where \( x_i \in L^\Phi \), \( i = 1, \ldots, 4 \), the assertion follows. \( \square \)
3. Main Results

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful normal semifinite trace $\tau$, $L^0 = L^0(\mathcal{M}, \tau)$ the *-algebra of $\tau$-measurable operators affiliated with $\mathcal{M}$, $L^p = L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$, the noncommutative $L^p$-space associated with $(\mathcal{M}, \tau)$.

**Definition 3.1.** Let $(X, \| \cdot \|)$ be a normed space, and let $Y \subset X$ be such that the neutral element of $X$ is an accumulation point of $Y$. A family of maps $A_\alpha : X \to L^0$, $\alpha \in I$, is called uniformly equicontinuous in measure (u.e.m.) (bilaterally uniformly equicontinuous in measure (b.u.e.m)) at zero on $Y$ if for every $\epsilon > 0$ and $\delta > 0$ there is $\gamma > 0$ such that, given $x \in Y$ with $\|x\| < \gamma$, there exists $e \in \mathcal{P}(\mathcal{M})$ such that
\[
\tau(e^{\perp}) \leq \epsilon \quad \text{and} \quad \sup_{\alpha \in I} \|A_\alpha(x)e\|_\infty \leq \delta \quad \text{(respectively,} \quad \sup_{\alpha \in I} \|eA_\alpha(x)e\|_\infty \leq \delta). \]

**Remark 3.1.** As explained in [15, Introduction], in the commutative case, the notion of uniform equicontinuity in measure at zero of a family $\{A_\alpha\}_{\alpha \in I}$ coincides with the continuity in measure at zero of the maximal operator associated with this family.

**Definition 3.2.** A sequence $\{x_n\} \subset L^0$ is said to converge to $x \in L^0$ almost uniformly (a.u.) (bilaterally almost uniformly (b.a.u.)) if for every $\epsilon > 0$ there exists such a projection $e \in \mathcal{P}(\mathcal{M})$ that $\tau(e^{\perp}) \leq \epsilon$ and $\|x - x_n\|_\infty \to 0$ (respectively, $\|e(x - x_n)e\|_\infty \to 0$).

A proof of the following fact can be found in [15, Theorem 2.1].

**Proposition 3.1.** Let $(X, \| \cdot \|)$ be a Banach space, $A_\alpha : X \to L^0$ a sequence of additive maps. If the family $\{A_\alpha\}$ is u.e.m. (b.u.e.m.) at zero on $X$, then the set
\[
\{x \in X : \{A_\alpha(x)\} \text{ converges a.u. (respectively, b.a.u.)}\}
\]
is closed in $X$.

**Definition 3.3.** A linear map $T : L^1 + L^\infty \to L^1 + L^\infty$ such that
\[
\|T(x)\|_\infty \leq \|x\|_\infty \quad \forall x \in \mathcal{M} \quad \text{and} \quad \|T(x)\|_1 \leq \|x\|_1 \quad \forall x \in L^1.
\]
is called a Dunford-Schwartz operator.

If $T$ is a Dunford-Schwartz operator (positive Dunford-Schwartz operator), we will write $T \in DS$ (respectively, $T \in DS^+$). If $T \in DS$, consider its ergodic averages
\[
A_n(x) = A_n(T, x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \quad x \in L^1 + L^\infty.
\]

(1)

Here is a noncommutative maximal ergodic inequality due to Yeadon [23] (for the assumption $T \in DS^+$, see a clarification given in [3, Proposition 1.1, Remark 1.2]):

**Theorem 3.1.** Let $T \in DS^+$ and $A_\alpha : L^1 \to L^1$, $n = 1, 2, \ldots$ be given by (1). Then for every $x \in L^1_+$ and $\nu > 0$ there exists a projection $e \in \mathcal{P}(\mathcal{M})$ such that
\[
\tau(e^{\perp}) \leq \frac{\|x\|_1}{\nu} \quad \text{and} \quad \sup_n \|eA_n(x)e\|_\infty \leq \nu.
\]
Now, let \( \Phi \) be an Orlicz function, \( L^\Phi = L^\Phi(\mathcal{M}, \tau) \) the corresponding noncommutative Orlicz space, \( \| \cdot \|_\Phi \) the Luxemburg norm in \( L^\Phi \).

As \( L^\Phi \) is an exact interpolation space for the Banach couple \((L^1, \mathcal{M})\) (see Corollary 2.2),
\[
T(L^\Phi) \subset L^\Phi \quad \text{and} \quad \| T \|_{L^\Phi \to L^\Phi} \leq 1,
\]
hold for any \( T \in DS \), and we have the following.

**Proposition 3.2.** If \( T \in DS^+ \), then the family \( A_n : L^\Phi \to L^\Phi \), \( n = 1, 2, \ldots \), given by (1) is b.u.e.m. at zero on \((L^\Phi, \| \cdot \|_\Phi)\).

**Proof.** It is easy to verify (see [15, Lemma 4.1]) that it is sufficient to show that
\[
\text{Proposition 3.2.}
\]
by (1) is b.u.e.m. at zero on \( A \), \( \nu > 0 \) and \( 0 < \gamma \leq 1 \) be such that \( \nu \leq \frac{\lambda}{\gamma} \) and \( \frac{\lambda}{\gamma} \leq \epsilon \).

Take \( x \in L^\Phi_+ \) with \( \|x\|_\Phi \leq \gamma \), and let \( x = \int_0^\infty \lambda d\xi \) be its spectral decomposition. Then we can write
\[
x = \int_0^{\sqrt{\delta/2}} \lambda d\xi + \int_{\delta/2}^{\infty} \lambda d\xi \leq \delta + t \cdot \int_{\delta/2}^{\infty} \Phi(\lambda) d\xi \leq \delta + t \cdot \Phi(x),
\]
where \( x = \int_0^{\sqrt{\delta/2}} \lambda d\xi \) and \( \Phi(x) = \int_0^\infty \Phi(\lambda) d\xi \).

As \( \|x\|_\Phi \leq \frac{\delta}{2} \) and \( T \in DS^+ \), we have
\[
\sup_n \| A_n(x) \|_\Phi \leq \frac{\delta}{2}.
\]
Besides, by Proposition 2.2 \( \|x\|_\Phi \leq 1 \) implies that \( \| \Phi(x) \|_1 \leq \|x\|_M \leq \gamma \). Since \( \Phi(x) \in L^1_+ \), in view of Theorem 3.1, one can find a projection \( e \in \mathcal{P}(\mathcal{M}) \) such that
\[
\tau(e^+) \leq \frac{\| \Phi(x) \|_1}{\nu} \leq \frac{\gamma}{\nu} \leq \epsilon \quad \text{and} \quad \sup_n \| e A_n(\Phi(x)) e \|_\Phi \leq \nu \leq \frac{\delta}{2 \epsilon}.
\]
Consequently,
\[
\sup_n \| e A_n(x) e \|_\Phi \leq \sup_n \| e A_n(x) e \|_\Phi + t \cdot \sup_n \| e A_n(\Phi(x)) e \|_\Phi \leq \frac{\delta}{2} + t \cdot \frac{\delta}{2 \epsilon} = \delta,
\]
and the proof is complete.

Here is an individual ergodic theorem for noncommutative Orlicz spaces:

**Theorem 3.2.** Assume that an Orlicz function \( \Phi \) satisfy \((\delta_2, \Delta_2)\)–condition. Then, given \( T \in DS^+ \) and \( x \in L^\Phi \), the averages \( \{1\} \) converge b.a.u. to some \( \hat{x} \in L^\Phi \).

**Proof.** Since, by Proposition 2.2 the set \( L^1 \cap \mathcal{M} \subset L^2 \) is dense in \( L^\Phi \) and the averages \( \{1\} \) converge a.u., hence b.a.u., for every \( x \in L^2 \) (see, for example, [15, Theorem 4.1]), it follows from Propositions 3.2 and 3.1 that for any \( x \in L^\Phi \) the averages \( \{1\} \) converge b.a.u. to some \( \hat{x} \in L^0 \).

It is clear that a b.a.u. convergent sequence in \( L^0 \) converges in measure, hence \( A_n(x) \to \hat{x}, \quad x \in L^\Phi, \) in measure. Since, by Corollary 2.2 \( L^\Phi \) has the Fatou property, its unit ball is closed in the measure topology [6, Theorem 4.1], and 2, hence \( \sup_n \| A_n(x) \|_{L^\Phi \to L^\Phi} \leq \|x\|_\Phi \), implies that \( \hat{x} \in L^\Phi \).
Remark 3.2. In was shown in [3, Theorem 5.2] that if $E(0, \infty)$ is a fully symmetric function space with Fatou property and non-trivial Boyd indices and $T \in DS^+$, then for any $x \in E(M, \tau)$ the averages $A_n(x)$ converge b.a.u. to some $\hat{x} \in E(M, \tau)$.

According to Remark 2.3 (ii), there exists an Orlicz function $\Phi$ that satisfies $(\delta_2, \Delta_2)$-condition for which the Boyd index $\mu_{E^*(0, \infty)}$ is trivial. Thus, Theorem 3.2 does not follow from Theorem 3.3.

Now we shall turn to a class of Orlicz spaces for which the averages (1) converge a.u. The following fundamental result is crucial.

Theorem 3.3 (Kadison’s inequality [11]). If $S : M \to M$ is a positive linear operator such that $S(1) \leq 1$, then $S(x)^2 \leq S(x^2)$ for every $x^* = x \in M$.

Definition 3.4. We call a convex function $\Phi$ on $[0, \infty)$ 2-convex if the function $\tilde{\Phi}(u) = \Phi(\sqrt{u})$ is also convex.

For example, $\Phi(u) = \frac{u^p}{p}$, $u \geq 0$, is 2-convex that satisfies $(\delta_2, \Delta_2)$-condition whenever $p \geq 2$.

It is clear that if $\Phi$ is a 2-convex Orlicz function, then $\tilde{\Phi}$ is also an Orlicz function, and it is easy to verify the following.

Proposition 3.3. If $\Phi$ be a 2-convex Orlicz function, then $x^2 \in L^+_\Phi$ and $\|x^2\|_\Phi = \|x\|_\Phi^2$ for every $x \in L^+_\Phi$.

Proposition 3.4. Let $\Phi$ be a 2-convex Orlicz function. Then the family $\{A_n\}$ given by (1) is u.e.m. at zero on $(L^\Phi, \| \cdot \|_M)$.

Proof. As it was noticed earlier, it is sufficient to show that $\{A_n\}$ is u.e.m. at zero on $(L^\Phi, \| \cdot \|_M)$.

Fix $\epsilon > 0$, $\delta > 0$. By Proposition 3.2 $\{A_n\}$ is b.u.e.m. at zero on $(L^\Phi, \| \cdot \|_\Phi)$. Therefore there exists $\gamma > 0$ such that, given $y \in L^\Phi$ with $\|y\|_\Phi < \gamma$,

$$\sup_n \|eA_n(y)e\|_\infty \leq \delta^2$$

for some $e \in \mathcal{P}(M)$ with $\tau(e^\perp) \leq \epsilon$.

Now, let $x \in L^\Phi_+$ be such that $\|x\|_\Phi < \gamma^{1/2}$. Then, due to Proposition 3.3 $x^2 \in L^\Phi_+$ and $\|x^2\|_\Phi = \|x\|_\Phi^2 \leq \gamma$, implying that there is a projection $e \in \mathcal{P}(M)$ such that

$$\sup_n \|eA_n(x^2)e\|_\infty \leq \delta^2$$

and $\tau(e^\perp) \leq \epsilon$.

Then, by Kadison’s inequality,

$$\left[ \sup_n \|A_n(x)e\|_\infty \right]^2 = \sup_n \|A_n(x)e\|_\infty^2 = \sup_n \|eA_n(x^2)e\|_\infty \leq \delta^2,$$

which completes the proof. \(\square\)

Now, as in Theorem 3.2 we obtain the following.

Theorem 3.4. If an Orlicz function $\Phi$ satisfies $(\delta_2, \Delta_2)$-condition and is 2-convex, then, given $T \in DS^+$ and $x \in L^\Phi$, the averages (1) converge a.u. to some $\hat{x} \in L^\Phi$. 
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