Study on the modeling and application of arc structure’s profile tolerance

ZH Yan1,2, NY Qiu1, JC Hu1 and XG Li1

1School of Mechanical and Aerospace Engineering, Jilin University, Changchun, 130022, China

Email: yanzhena@jlu.edu.cn

Abstract. Profile tolerance can play a comprehensive control role, and then it can replace some other kinds of geometric tolerances, so it is widely used in practical engineering. However, there are few studies on profile tolerance modeling. There are only several papers in public, and the modeling methods in these papers all have certain deficiencies. In addition, the arc structure is much more used in practical engineering to deal with approximation of free-form surfaces, and the arc structure is often controlled by profile tolerance. Therefore, a modeling method specifically for the arc profile tolerance is given, and a tolerance analysis method including arc structure is given by combining the profile tolerance modeling method and Vector Loop and Monte Carlo method. Statistical distribution of each tolerance is included in the tolerance analysis method, and an example is given.

1. Introduction

There must be geometry errors in every mechanical part, such as dimensional error, shape error, direction error, and position error, etc. These geometry errors affect the function of the parts. For example, the geometry errors can lead to interchangeability problem, assimilability problem, dynamics characteristic, and the sealing problem, and so on. Therefore, the tolerance design job should be done when a product is developed. And the tolerances are marked on the drawings to control various types of geometry errors. Twelve geometric tolerances are given in the standard of ASME Y14.5-2018[1], and fourteen geometric tolerances are given in the standard of ISO 1101-2017[2]. And there are various modifiers in the two standards, which can be chosen according to the functional requirements by engineers. In these geometric tolerance items, the profile tolerance has comprehensive control role, and then it can replace many other geometry tolerances, so it is widely used in practical engineering.

On the one hand, the tolerance design affects the functionality of products; on the other hand, it affects the manufacturing costs. The two aspects are contradictory with each other. So a balance point between the two aspects should be found [3-5]. The tolerance analysis methods provide tools for tolerance design. The effects of tolerances on the function of the products can be predicted. In addition, the tolerance-cost model can be included to optimize tolerances further. There are various tolerance analysis methods available currently, including Vector Loop method [6, 7], Uniform Jacobian-Torsor method[8, 9], T-Map method[10, 11], and skin model[12, 14], etc. Among these methods, there are few researched on the modeling of profile tolerance. The vector Loop model only reduces the profile tolerance to a vector [6] ignoring many possible variations of the actual profile in the tolerance zone. Regarding the application of T-Map method in profile tolerance modeling, several papers have been published, and a systematic T-Map modeling method for line profile tolerance is given [15-18]. In paper
[18], using the original T-Map unit and the Boolean intersection between the units, a general modeling method for arbitrary line profile is proposed to obtain the four-dimensional complex T-Map graph of line profile. But the modeling method cannot include the statistical distribution of profile tolerances. The method of skin model can theoretically realize the modeling of profile tolerance, but it is still in the development stage.

In the practical engineering structures, because of the difficulty of the manufacturing process, the arc structure is widely used to deal with approximation of free-form surfaces[19, 20]. Therefore, it is of practical significance to specifically study the tolerance modeling of the profile tolerance of the arc structure. In this paper, a new modeling method for the profile tolerance of arc structure is presented. And a tolerance analysis method including the statistical distribution by using the Vector Loop and Monte Carlo methods is proposed.

2. Profile tolerance modeling of arc structure

In this paper, a tolerance analysis method for the mechanical system of a part containing arc structure based on the Vector Loop and Monte Carlo method is established. Firstly, the actual arcs within the profile tolerance zone are randomly generated. It is assumed that the actual arc has only errors in radius and coordinates of the center of the circle in the plane, and ignores other forms of error. Then the three parameters of the radius of the actual arc and the coordinates of the center of the circle (the actual arc has 4 degrees of freedom in plane, but the angle of center is given, so an actual arc has 3 degrees in this paper) are introduced into the vector ring model. The profile tolerance modeling of a circular arc essentially establishes the relationship between these three parameters, that is, the possible interval of the arc radius, and the area where the arc center coordinates of the arc may be when the radius is in a certain interval, and finally these areas are expressed through mathematical models.

Although the profile tolerance modeling of the arc structure is relatively simple and programmable, the actual operation process is tedious and error-prone. Moreover, the assembly tolerance analysis method with arc structure does not need the actual arc structure itself, but the center coordinates of the actual arc structure (see Chapter 3).

When performing tolerance analysis based on vector loop and Monte Carlo method, firstly, the actual radius of the arc is randomly generated, and then it is determined which interval the radius is in, and then the possible region of the center of the circle is clarified, which is hereinafter referred to as circle center domain, and then the coordinates of the center of the circle are randomly generated in an area, and these three parameters are used in the vector loop model. The radius of the actual arc is divided into several sections, because the arc radius is in different sections, the shape of the possible coordinates of the center coordinates is different, and different mathematical models need to be expressed. The interval of the actual arc radius will be given below, and the shape of the center of the circle corresponding to each actual radius will be analyzed.

2.1. Actual arc radius interval division

Let the radius of the arc structure of the part be \( r \), the central angle be \( \alpha \), the central half angle be \( \theta \), and the profile tolerance value be \( t \) (as shown in Figure 1 (a)). Extract the arc abstraction, as shown in Figure 1 (b). In the coordinate system xoy with the center of the theoretical arc as the origin (the arc is symmetrically divided by the x-axis), the tolerance zone is the area between the two theoretical arcs with the distance \( t \) shown by the blue dotted line in the figure, then the actual arc needs to be in this area to qualify. The two blue dotted circles of the tolerance zone are centered on the center of the theoretical arc, and the radii are \((r - t/2)\) and \((r + t/2)\) respectively.
The arc profile tolerance modeling method given in this paper takes into account the translation and rotation of the actual arc profile in the tolerance zone, and change of the arc radius, so it has three degrees of freedom. Since the vector loop method is used in this paper, the parameters of the actual arc used are the center coordinates and the radius. Therefore, the modeling of the arc profile is to create an arc profile that does not exceed the tolerance zone within the tolerance zone of the arc, giving its center coordinates and radius. The modeling process of arc profile tolerance is as follows.

First, determine the three intervals of the actual arc radius. Considering that the radius of the actual arc structure is much larger than its tolerance value, the actual radius is within the tolerance zone and the radius is minimum when passing through the three points A, B and F; The radius is the largest when the actual arc is within the tolerance zone and passes through C, D, and E. In the coordinate system shown in Figure 1(b), the coordinates of these six points are as follows:

- Point A coordinates: \((x_A, y_A) = ((r - t/2)\cos \theta, (r - t/2)\sin \theta)\)
- Point B coordinates: \((x_B, y_B) = ((r - t/2)\cos \theta, -(r - t/2)\sin \theta)\)
- Point C coordinates: \((x_C, y_C) = ((r + t/2)\cos \theta, (r + t/2)\sin \theta)\)
- Point D coordinates: \((x_D, y_D) = ((r + t/2)\cos \theta, -(r + t/2)\sin \theta)\)
- Point E coordinates: \((x_E, y_E) = (r - t/2, 0)\)
- Point F coordinates: \((x_F, y_F) = (r + t/2, 0)\)

The coordinates of the three points (A, B, and F) that the actual minimum arc passes through are introduced into the standard equation (1) of the circle.

\[
(x - a)^2 + (y - b)^2 = R^2
\]  

Where a, b, and R are the center coordinates and the actual radius of the actual arc, and an algebraic equation with three unknowns can be obtained, and the center coordinates \((a_{\text{min}}, b_{\text{min}})\) and the radius \(r_{\text{min}}\) of the actual minimum arc can be obtained.

\[
(a_{\text{min}}, b_{\text{min}}) = \left(2rt/(-2r\cos \theta + 2r + t\cos \theta + t), 0\right)
\]

\[
r_{\text{min}} = \frac{\left[-16r^4\sin^2 \theta + 2\cos \theta - 2\right] - t^4\sin^2 \theta - 2\cos \theta - 2 + 8r^2 t^2 \sin^2 \theta} {2(-2r\cos \theta + 2r + t\cos \theta + t)}^{1/2}
\]

Similarly, the coordinates of the three points (C, D, E three points) that the actual maximum arc passes through are brought into the standard equation (1) of the circle. The center coordinates of the actual maximum arc are \((a_{\text{max}}, b_{\text{max}})\) and the radius is \(r_{\text{max}}\).

\[
(a_{\text{max}}, b_{\text{max}}) = \left(2rt/(2r\cos \theta - 2r + t\cos \theta + t), 0\right)
\]

\[
r_{\text{max}} = \frac{\left[-16r^4\sin^2 \theta + 2\cos \theta - 2\right] - t^4\sin^2 \theta - 2\cos \theta - 2 + 8r^2 t^2 \sin^2 \theta} {2(2r - 2r\cos \theta - t\cos \theta - t)}^{1/2}
\]
Assuming a random arc with an actual radius of $r_{\text{real}}$, if it is only translated in the x-axis direction within the profile tolerance zone, its center of the circle is a straight line segment that coincides with the x-axis (as shown in Figure 2(a)); If it only rotates within the profile tolerance zone, its center of the circle is an arc with a radius of $r_{\text{real}}$ (as shown in Figure 2(b)). If both translation and rotation are within the profile tolerance zone, the center of the circle is a closed polygon formed by a multi-segment arc (as shown in Figure 2(c)). When the random arc translates to a certain position and rotates by a certain angle (this angle is the maximum angle at which the random arc can rotate within the profile tolerance zone, hereinafter referred to as the maximum angle), the center of the random arc is located on the boundary contour of circle center domain. Therefore, the boundary contour can be determined by the rotation angle of a given random arc.

**Figure 2.** The movement of the center of the circle corresponding to the movement of the actual arc in the tolerance zone: a) move only; b) only rotate; c) move and rotate.

Assume that the random arc is located at the extreme position of the left end, shifting a small displacement to the right along the x-axis. And at the same time, the maximum angle of rotation around the intersection with the X-axis is tangent to the boundary arc of the profile tolerance zone or pass through one or more points point e.g. A, B, C, and D, until the random arc is located at the right end limit position, and the track swept by the center of the random arc is circle center domain. By observing and analyzing the law of circular motion, the actual radius of the arc can be divided into three intervals: $[r_{\text{min}}, r - t/2], (r - t/2, r + t/2]$ and $(r + t/2, r_{\text{max}}]$. In one of the intervals, the random arc through point A(B) with the actual radius $r_{\text{real}}$ will be tangent to the right boundary arc of the profile tolerance zone and the tangent point is D(C) (as shown figure. 3(a).), the actual radius in this case is

![Diagram](image-url)
defined as the minimum tangent actual arc radius \( r_{real1} \); or passing through point D(C), being tangent to the left boundary arc of the profile tolerance zone and the tangent point is A(B) (as shown in figure. 3(b)), the actual radius in this case is defined as the maximum tangent actual arc radius \( r_{real2} \).

Which interval the critical tangential condition is distributed depends on the arc theoretical radius \( r \), the angle \( \alpha \) and the profile tolerance value \( t \), and must have a relationship \( r_{real1} \leq r \leq r_{real2} \), which is proved as follows:

The coordinates of points A and D and the minimum tangent actual arc radius \( r_{real1} \) are brought into equation (1), and using the tangent relationship of point D, the following equations can be obtained:

\[
\begin{align*}
(x_A - a_{real1})^2 + (y_A - b_{real1})^2 &= r_{real1}^2 \\
(x_D - a_{real1})^2 + (y_D - b_{real1})^2 &= r_{real1}^2 \\
(a_{real1})^2 + (b_{real1})^2 &= (r + t/2 - r_{real1})^2
\end{align*}
\]

(6)

The minimum tangency actual arc radius \( r_{real1} \) is:

\[
r_{real1} = \frac{4r^2 \sin^2 \theta - t^2 \sin^2 \theta + t^2}{2(2r \sin^2 \theta - tsin^2 \theta + t)}
\]  

(7)

Similarly, the maximum tangent actual arc radius \( r_{real2} \) is:

\[
r_{real2} = \frac{4r^2 \sin^2 \theta - t^2 \sin^2 \theta + t^2}{2(2r \sin^2 \theta + tsin^2 \theta - t)}
\]  

(8)

Comparing the formulas of \( r_{real1} \) and \( r_{real2} \), it can be seen that the numerator is the same and the denominator is different. It is obvious that the denominator of \( r_{real1} \) is always greater than or equal to the denominator of \( r_{real2} \), that is, \( r_{real1} \leq r_{real2} \).

Let \( r_{real1} = r_{real2} \). Taking into account the actual situation, that is \( t \neq 0 \), get \( \sin^2 \theta = 1 \). Substituting it into the \( r_{real1} \) and \( r_{real2} \) equations, and get:

\[
r_{real1} = r_{real2} = r
\]

In summary:

\[
r_{real1} \leq r \leq r_{real2}
\]

The certificate is completed.

Since the theoretical arc radius \( r \) differs from the tolerance value \( t \) by more than two orders of magnitude, the actual arc center will be on the right side of the theoretical arc only when the central angle \( \alpha \) is small. Therefore, based on Figure 1b, the actual arc conforming to the profile should meet the following three points:

(a) The actual arc intersects the line segment AC;
(b) The actual arc intersects the line segment BD;
(c) The actual arc should be located in the area formed by the arcs AB and CD, unless the actual arc coincides with the arc AB or CD.

2.2. Interval analysis

when the actual radius of the random arc is in different intervals, the motion law in the tolerance band will be different, and the shape of the random arc center region will be different, then the mathematical model expressing the random arc center region will be different. Therefore, it is necessary to discuss the motion law of the actual arc in each interval between the partitions.

(1) When the actual arc radius \( r_{real} \) is in the interval \([r_{min}, r - t/2]\)

When the random arc radius \( r_{real} \) is in the interval \([r_{min}, r - t/2]\), the random arc is located at the left limit position and passes through two points A and B, as shown by the arc 1 in Figure 4(a). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will pass the A(B) point. When the arc is shifted to the right by a certain distance and reaches the maximum angle, the random arc will pass the point A(B) and be tangent to the right boundary arc of the tolerance zone, as shown by the arc 2 in Figure 4(a). In the process, the trajectory of the circle center is an arc with the point A (B) as the center and the random arc radius \( r_{real} \) as the radius. Since the radius is too large and the arc is too short, it appears to be a straight line in the figure. Continue to increase the moving distance and rotate the random arc to the maximum angle. The tangent point of the random arc and the right boundary arc of the tolerance zone will gradually approach the point F and finally coincides with the point F, as shown by arc 3 in Figure 4(a). In the process, the trajectory of the
circle center is an arc with the origin of the coordinate system as the center and \( r + t/2 - r_{real} \) as the radius. Combining the arcs of the above two processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 4(a).

\[ \text{Figure 4. The random arc center region when the actual arc radius } r_{real} \text{ is in the interval } [r_{min}, r - t/2]; \]

\( (a) \) \( r_{real1} \) is not in the interval \( [r_{min}, r - t/2] \), and \( r_{real} \) is in the interval \( [r_{min}, r - t/2] \); \( (b) \) \( r_{real1} \) is in the interval \( [r_{min}, r - t/2] \), and \( r_{real} \) is in the interval \( [r_{real1}, r - t/2] \).

case 1: \( r_{real1} \) is not in the interval \( [r_{min}, r - t/2] \)

case 2: \( r_{real1} \) is in the interval \( [r_{min}, r - t/2] \)

When the actual arc radius \( r_{real} \) is in the interval \( [r_{min}, r_{real1}] \), the shape of the random arc center region is the same as in case 1.

When the random arc radius \( r_{real} \) is in the interval \( [r_{real1}, r - t/2] \), the random arc is located at the left limit position and passes through two points A and B, as shown by the arc 1 in Figure 4(b). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will pass the A(B) point. When the arc is shifted to the right by a certain distance and reaches the maximum angle, the random arc will pass the point A(B) and point D(C), as shown by the arc 2 in Figure 4(b). In the process, the trajectory of the circle center is an arc with the point A (B) as the center and the random arc radius \( r_{real} \) as the radius. Since the radius is too large and the arc is too short, it appears to be a straight line in the figure. Continue to increase the moving distance and rotate the random arc to the maximum angle. And the random arc will pass the point C (D) and no longer pass the point A (B). When the arc is shifted to the right by a certain distance and reaches the maximum angle, the random arc will pass the point C(D) at a certain moment and be tangent to the right boundary arc of the tolerance zone, as shown by arc 3 in Figure 4(b). In the process, the trajectory of the circle center is an arc with the point D (C) as the center and the random arc radius \( r_{real} \) as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. And the random arc will be tangent to the right boundary arc of the tolerance zone. In the meantime, the arc no longer passes the point A (B) and the point C (D). Continue to increase the moving distance and rotate the random arc to the maximum angle. The tangent point of the random arc and the right boundary arc of the tolerance zone will gradually approach the point F and finally coincides with the point F, as shown by arc 4 in Figure 4(b). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and \( r + t/2 - r_{real} \) as the radius. Combining the arcs of the above three processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 4(b).

(2) When the actual arc radius \( r_{real} \) is in the interval \( [r - t/2, r + t/2] \)
Figure 5. The random arc center region when the actual arc radius $r_{real}$ is in the interval $[r - t/2, r + t/2]$ (a) $r_{real1}$ and $r_{real2}$ are not in the interval $[r - t/2, r + t/2]$, and $r_{real}$ is in the interval $[r - t/2, r + t/2]$ (b) only $r_{real1}$ is in the interval $[r - t/2, r + t/2]$, $r_{real}$ is in the interval $[r - t/2, r_{real1}]$ (c) only $r_{real2}$ is in the interval $[r - t/2, r + t/2]$, $r_{real}$ is in the interval $[r_{real2}, r + t/2]$.

Case 1: When $r_{real1}$ and $r_{real2}$ are not in the interval $[r - t/2, r + t/2]$

When the random arc radius $r_{real}$ is in the interval $[r - t/2, r + t/2]$, the random arc is located at the left limit position and is over and tangent to the E point, as shown by the arc 1 in Figure 5(a). Shift the random arc to the right by a small distance and rotate it to the maximum angle, and then the random arc will be tangent to the left boundary arc of the tolerance zone. When the arc is gradually shifted to the right by a small distance and reaches the maximum angle, the tangent point gradually moves away from point E and coincides with point A(B) at a certain moment, as shown by arc 2 in Figure 5(a). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r_{real} - (r - t/2)$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. The random arc will pass but no longer be tangent to the point A(B). Finally, the random arc will pass point A (B) and point D (C) at a certain moment, as shown by arc 3 in Figure 5(a). In the process, the trajectory of the circle center is an arc with the point A (B) as the center and $r_{real}$ as the radius. Repeat the above steps, the random arc will be tangent to the right boundary arc of the tolerance zone at point C (D), as shown by arc 4 in Figure 5(a). In the process, the trajectory of the circle center is an arc with the point C (D) as the center and $r_{real}$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. The random arc will be tangent to the right boundary arc of the tolerance zone. And the tangent point of the random arc and the right boundary arc of the tolerance zone will gradually approach the point F and finally coincides with the point F, as shown by arc 5 in Figure 5(a). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r + t/2 - r_{real}$ as the radius. Combining the arcs of the above processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 5(a).

Case 2: When only $r_{real1}$ is in the interval $[r - t/2, r + t/2]$

When the random arc radius $r_{real}$ is in the interval $[r - t/2, r_{real1}]$, the random arc is located at the left limit position and is over and tangent to the E point, as shown by the arc 1 in Figure 5(b). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will be tangent to the left boundary arc of the tolerance zone. When the arc is gradually shifted to the right by a small distance and reaches the maximum angle, the tangent point gradually moves away from point E and coincides with point A(B) at a certain moment, as shown by arc 2 in Figure 5(b). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r_{real} - (r - t/2)$ as the radius. When the arc is shifted to the right by a distance
and reaches the maximum angle, the random arc will pass the point A(B) and be tangent to the right boundary arc of the tolerance zone, as shown by the arc 3 in Figure 5(b). In the process, the trajectory of the circle center is an arc with the point A(B) as the center and the random arc radius $r_{real}$ as the radius. Since the radius is too large and the arc is too short, it appears to be a straight line in the figure. Continue to increase the moving distance and rotate the random arc to the maximum angle. The tangent point of the random arc and the right boundary arc of the tolerance zone will gradually approach the point F and finally coincides with the point F, as shown by arc 4 in Figure 5(b). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r + t/2 - r_{real}$ as the radius. Combining the arcs of the above two processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 5(b).

When the actual arc radius $r_{real}$ is in the interval $[r_{real1}, r + t/2]$, the shape of the random arc center region is the same as in case 1.

   case 3: When only $r_{real2}$ is in the interval $[r - t/2, r + t/2]$

   When the actual arc radius $r_{real}$ is in the interval $[r - t/2, r_{real2}]$, the shape of the random arc center region is the same as in case 1.

   When the random arc radius $r_{real}$ is in the interval $[r_{real1}, r + t/2]$, the random arc is located at the left limit position and is over and tangent to the E point, as shown by the arc 1 in Figure 5(c). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will be tangent to the left boundary arc of the tolerance zone. When the arc is gradually shifted to the right by a small distance and reaches the maximum angle, the tangent point gradually moves away from point E, and the random arc will pass the point C(D) at a certain moment and be tangent to the left boundary arc of the tolerance zone, as shown by the arc 2 in Figure 5(c). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r_{real} - (r - t/2)$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. And the random arc will pass the point C(D) at a certain moment and be tangent to the right boundary arc of the tolerance zone, as shown by arc 3 in Figure 5(c). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r + t/2 - r_{real}$ as the radius. Combining the arcs of the above processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 5(c).

   case 4: When both $r_{real1}$ and $r_{real2}$ are in the interval $[r - t/2, r + t/2]$

   When the actual arc radius $r_{real}$ is in the interval $[r - t/2, r_{real1}]$, the shape of the random arc center region is the same as the shape when the actual arc radius $r_{real}$ is in the interval $[r - t/2, r_{real1}]$ in case 2.

   When the actual arc radius $r_{real}$ is in the interval $[r_{real1}, r_{real2}]$, the shape of the random arc center region is the same as in case 1.

   When the actual arc radius $r_{real}$ is in the interval $[r_{real2}, r + t/2]$, the shape of the random arc center region is the same as the shape when the actual arc radius $r_{real}$ is in the interval $[r_{real2}, r + t/2]$ in case 3.

   (3) When the actual arc radius $r_{real}$ is in the interval $[r + t/2, r_{max}]$
**Figure 6.** The random arc center region when the actual arc radius $r_{real}$ is in the interval $[r + t/2, r_{max}]$. (a) When $r_{real1}$ is not in the interval $[r + t/2, r_{max}]$, $r_{real}$ is in the interval $[r + t/2, r_{real1}]$. (b) When $r_{real2}$ is in the interval $[r + t/2, r_{max}]$, $r_{real}$ is in the interval $[r + t/2, r_{real2}]$. 

**case 1:** When $r_{real2}$ is not in the interval $[r - t/2, r + t/2]$ 

When the random arc radius $r_{real}$ is in the interval $[r + t/2, r_{max}]$, the random arc is located at the left limit position and is over and tangent to the E point, as shown by the arc 1 in Figure 6(a). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will be tangent to the left boundary arc of the tolerance zone. When the arc is gradually shifted to the right by a small distance and reaches the maximum angle, the tangent point gradually moves away from point E, and the random arc will pass the point C(D) at a certain moment and be tangent to the left boundary arc of the tolerance zone, as shown by arc 2 in Figure 6(a). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r_{real} - (r - t/2)$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. The random arc will pass point C and point D at a certain moment, as shown by arc 3 in Figure 6(a). In the process, the trajectory of the circle center is an arc with the point C (D) as the center and $r_{real}$ as the radius. Since the radius is too large and the arc is too short, it appears to be a straight line in the figure. Combining the arcs of the above processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 6(a).

**case 2:** When $r_{real2}$ is in the interval $[r + t/2, r_{max}]$ 

When the random arc radius $r_{real}$ is in the interval $[r + t/2, r_{real2}]$, the random arc is located at the left limit position and is over and tangent to the E point, as shown by the arc 1 in Figure 6(b). Shift the random arc to the right by a small distance and rotate the random arc to the maximum angle, and then the random arc will be tangent to the left boundary arc of the tolerance zone. When the arc is gradually shifted to the right by a small distance and reaches the maximum angle, the tangent point gradually moves away from point E and coincides with point A(B) at a certain moment, as shown by arc 2 in Figure 6(b). In the process, the trajectory of the circle center is an arc with the origin of the coordinate system as the center and $r_{real} - (r - t/2)$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. The random arc will pass but no longer be tangent to the point A (B). Finally, the random arc will pass point A (B) and point D (C) at a certain moment, as shown by arc 3 in Figure 6(b). In the process, the trajectory of the circle center is an arc with the point A (B) as the center and $r_{real}$ as the radius. Continue to increase the moving distance and rotate the random arc to the maximum angle. The random arc will pass point C and point D at a certain moment, as shown by arc 4 in Figure 6(b). In the process, the trajectory of the circle center is an arc with the point C (D) as the center and the random arc radius $r_{real}$ as the radius. Since the radius is too large and the arc is too
short, it appears to be a straight line in the figure. Combining the arcs of the above processes together can obtain the closed boundary contour of the random arc center region, as shown in Figure 6(b).

When the actual arc radius $r_{\text{real}}$ is in the interval $[r_{\text{real}2}, r_{\text{max}}]$, the shape of the random arc center region is the same as in case 1.

2.3. Establish a random arc center region model
Based on the above-mentioned motion law of the random arc center, the mathematical model of the arc center can be gained, that is, establish the mathematical model of the relationship between the random arc radius and its center region.

![Figure 7](image7.png)

Figure 7. The arc center region (2D) when the actual arc radius is small to large.

![Figure 8](image8.png)

Figure 8. The arc center region (3D) when the actual arc radius is small to large.
The arc center region pattern is given by a specific set of parameters as follows: \( r = 50 \text{mm}, \alpha = 60^\circ, t = 0.1 \text{mm} \). The 2D and 3D expressions of the random arc center region boundary given by the actual radius of the random arc are respectively shown in Figures 7 and 8. The boundary is composed of several curves or lines with different colors. The different colors are used to show they are obtained by different mathematical models. And it is convenient to understand the figures. Through the mathematical model of the random arc center region, Monte Carlo method can be used to randomly generate the actual center coordinates and the actual radius of the random arc.

Based on the above-mentioned the actual dimensions randomly generated that meet the tolerance requirements, a vector loop model of the assembly can be established for the tolerance analysis considering the contour tolerance and statistical distribution of the arc structure.

3. Assembly tolerance analysis method with arc structure

The platform structure for realizing low-frequency vibration isolation in the horizontal direction based on the principle of rolling vibration isolation is shown in Figure 9, which includes upper and lower platforms and rolling bodies. The parallelism between the upper and lower frame planes is required. In order to ensure the parallelism, it is necessary to carry out assembly tolerance analysis, that is, to establish a tolerance analysis model taking into account the tolerances of each part. Among them, the upper and lower frames contain circular arc structures, and it is controlled by the profile tolerance in the part drawing.

![Figure 9. Vector loop model of horizontal vibration isolation platform.](image)

![Figure 10. Parallelism error assessment.](image)

Establish the coordinate system shown in Figure 9. When the horizontal displacement of the upper platform B relative to the lower platform A is \( \Delta s \), the vector loop is established as shown in Figure 9, and the vector \( \vec{v}_1 \) starts from the origin \( O_1 \) of the \( X_1O_1Y_1 \) coordinate system and points to the theoretical center \( A_1 \) of the left arc of the lower platform; Vector \( \vec{v}_2 \) starts from the theoretical center \( A_1 \) of the left arc and points to the center \( A_2 \) of the actual arc; Vector \( \vec{v}_3 \) starts from the actual center \( A_2 \) of the left arc of the lower platform and points to the actual center \( B_2 \) of the arc on the left side of the upper platform; Vector \( \vec{v}_4 \) starts from the theoretical center \( B_1 \) of the arc on the left side of the upper platform and points to its actual center \( B_2 \); Vector \( \vec{v}_5 \) starts from the origin \( O_2 \) of the \( X_2O_2Y_2 \) coordinate system and points to the theoretical center circle \( B_1 \) of the left side of the upper platform; Vector \( \vec{v}_6 \) starts from the origin \( O_1 \) of the \( X_1O_1Y_1 \) coordinate system and points to the origin \( O_2 \) of the \( X_2O_2Y_2 \) coordinate system. Similarly, a vector loop is also created at the right arc of the upper and lower platforms, where the vector \( \vec{v}_7 \) is specially arranged along the contour of the plane on the upper platform. Based on the vector ring graph in Figure 9, the following vector equation can be obtained:

\[
\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{v}_6 + \vec{v}_5 + \vec{v}_4 \\
\vec{v}_10 + \vec{v}_{11} + \vec{v}_{12} = \vec{v}_6 + \vec{v}_7 + \vec{v}_8 + \vec{v}_9 \\
v_6 \cos \beta_6 = \Delta s
\]
Where $\vec{v}_2, \vec{v}_4, \vec{v}_9, \vec{v}_{11}$ are the vectors of the center of each arc theoretical center pointing to the actual center, $|\vec{v}_2|$, and $|\vec{v}_{12}|$ are the difference between the actual radius of the arc and the actual size of the steel ball in contact with it. The actual center coordinates of each arc, the actual radius of the arc and the actual size of the steel ball are generated by the Monte Carlo method (Section 3). The actual arc center coordinates $(a_i, b_i)$ of the generated arc need to be converted into the global coordinate system of the assembly to participate in the calculation of $\vec{v}_2, \vec{v}_4, \vec{v}_9, \vec{v}_{11}$, which can be obtained:

$$\vec{v}_i = \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix} = \begin{bmatrix} \cos\gamma_i & \sin\gamma_i \\ -\sin\gamma_i & \cos\gamma_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad i = 2, 4, 9, 11$$

Where $\gamma_i$ is the angle between the positive x-axis of the global coordinate system of the part and the positive x-axis of the arc local coordinate system, and the counterclockwise is positive. Decomposing equation (10) along the x-axis and y-axis yields, which can be obtained:

$$l_{v1x} + v_{2x} + v_3\cos\beta_3 = v_6\cos\beta_6 + l_{v5x} + v_{4x}$$
$$l_{v1y} + v_{2y} + v_3\sin\beta_3 = v_6\sin\beta_6 - l_{v5y} + v_{4y}$$

$$l_{v10x} + v_{11x} + v_{12}\cos\beta_{12} = v_6\cos\beta_6 + v_7\cos\beta_7 - l_{v10x} + v_{9x}$$
$$l_{v10y} + v_{11y} + v_{12}\sin\beta_{12} = v_6\sin\beta_6 + v_7\sin\beta_7 - l_{v10y} + v_{9y}$$

$$v_6\cos\beta_6 = \Delta$$

Where $\beta_i$ is the angle between the positive direction of each vector and the positive direction of the x-axis. Considering the trigonometric function and the inverse trigonometric angle solution to solve the domain restriction problem, it is transformed into the following equation:

$$l_{v1x} + v_{2x} + v_3\cos\beta_3 = v_6\cos\beta_6 + l_{v5x} + v_{4x}$$
$$l_{v1y} + v_{2y} - v_3(1 - \cos\beta_3^2)^{1/2} = v_6(1 - \cos\beta_6^2)^{1/2} - l_{v5y} + v_{4y}$$

$$l_{v10x} + v_{11x} + v_{12}\cos\beta_{12} = v_6\cos\beta_6 + v_7(1 - \sin\beta_7^2)^{1/2} - l_{v10x} + v_{9x}$$
$$l_{v10y} + v_{11y} - v_{12}(1 - \cos\beta_{12}^2)^{1/2} = v_6(1 - \cos\beta_6^2)^{1/2} + v_7\sin\beta_7 - l_{v10y} + v_{9y}$$

$$v_6\cos\beta_6 = \Delta$$

Solving the above mathematical model can obtain $\sin\beta_7$, and further obtain the parallelism error $f = |v_7\sin\beta_7|$ from figure 7.

4. Case

Based on the assembly tolerance analysis method of mechanical system including arc parts, an application example is given here. Taking the assembly model in figure 9 as an example, the specific parameters of the assembly are shown in figure 1(a).

The solution steps are as follows:

1. Determine the actual minimum radius of the arc $r_{min}$, actual maximum radius of the arc $r_{max}$, minimum tangent radius of the arc $r_{real1}$, maximum tangent radius of the arc $r_{real2}$. The specific parameter of the arc include radius $r = 50\text{mm}$, angle $\alpha = 60^\circ$, profile $t = 0.1\text{mm}$. From equations (3), (5), (7) and (8), the values of $r_{min}$, $r_{max}$, $r_{real1}$, and $r_{real2}$ can be obtained.

2. Determine the edge of its center region. That find: $r_{real1} \leq r - t/2 \leq r + t/2 \leq r_{real2}$.

3. Applying Monet Carlo method by means of uniform distribution mode to determine the steel balls’ actual size $d_i$, actual radius of each arc $r_i$ and the actual center coordinates $(a_i, b_i)$ of each arc. The distribution pattern is shown in Figure 11.

4. Determine the vector loop equation and substitute the size parameters into equation (12).

5. Determine the unknowns of the equation above and solve the parallelism error $f$.

$$v_3 = r_1 + r_2 - d_1$$
$$v_{12} = r_2 + r_4 - d_2$$
$$v_{2x} = \begin{bmatrix} \cos(-\pi/2) & -\sin(-\pi/2) \\ \sin(-\pi/2) & \cos(-\pi/2) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} -a_1 \\ b_1 \end{bmatrix}$$
$$v_{2y} = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_2 \\ a_2 \end{bmatrix}$$

12
\[
\begin{bmatrix}
\nu_{1\text{x}} \\
\nu_{1\text{y}}
\end{bmatrix} =
\begin{bmatrix}
\cos(-\pi/2) & -\sin(-\pi/2) \\
\sin(-\pi/2) & \cos(-\pi/2)
\end{bmatrix}
\begin{bmatrix}
\alpha_3 \\
\beta_3
\end{bmatrix} =
\begin{bmatrix}
\beta_3 \\
-\alpha_3
\end{bmatrix}
\]
\[
\begin{bmatrix}
\nu_{3\text{x}} \\
\nu_{3\text{y}}
\end{bmatrix} =
\begin{bmatrix}
\cos(\pi/2) & -\sin(\pi/2) \\
\sin(\pi/2) & \cos(\pi/2)
\end{bmatrix}
\begin{bmatrix}
\alpha_4 \\
\beta_4
\end{bmatrix} =
\begin{bmatrix}
-\beta_4 \\
\alpha_4
\end{bmatrix}
\]

(6) Change the \(\Delta s\) and use Monte Carlo method to calculate 2000 times randomly, and then observe the distribution range and law of parallelism. It is shown in Figure 12.

**Figure 11.** random distribution of the actual center of the circle (the colors of points are just to distinguish each other).

**Figure 12.** The distribution of parallelism error (a) \(\Delta s = -0.2\) (b) \(\Delta s = -0.1\) (c) \(\Delta s = 0.1\) (d) \(\Delta s = 0.2\).
5. Conclusion
A large number of arc structures are used in engineering. Such structures often use profile tolerance to control the geometry errors. There are few researches on modeling methods of the profile tolerance. Existing tolerance analysis methods have certain deficiencies. Therefore, a new modeling method of profile tolerance of arc structure is studied, and a tolerance analysis method including the statistic distribution based on the Vector Loop and Monte Carlo method. An example is given. The research results show that:

(1) The profile tolerance model established in this paper can reflect the position and radius error of the arc profile in the tolerance zone;

(2) There is a corresponding relationship between the possible area of the center of the actual arc and the radius of the arc. The shape of the area where the center of the circle lies is related to the radius of the arc;

(3) The tolerance analysis method for arc structure based on vector loop and Monte Carlo method can include the statistic distribution of each tolerance.

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