THE SIX GROTHENDIECK OPERATIONS ON O-MINIMAL SHEAVES

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Abstract. In this paper we develop the formalism of the Grothendieck six operations on o-minimal sheaves. The Grothendieck formalism allows us to obtain o-minimal versions of: (i) derived projection formula; (ii) universal coefficient formula; (iii) derived base change formula; (iv) Künneth formula; (v) local and global Verdier duality.

1. Introduction

The study of o-minimal structures ([16]) is the analytic part of model theory which deals with theories of ordered, hence topological, structures satisfying certain tameness properties. It generalizes piecewise linear geometry ([16, Chapter 1, §7]), semi-algebraic geometry ([4]) and globally sub-analytic geometry ([36], also called finitely sub-analytic in [15]) and it is claimed to be the formalization of Grothendieck’s notion of tame topology (topologie modérée). See [16] and [18].

The most striking successes of this model-theoretic point of view of sub-analytic geometry include, on the one hand, an understanding of the behavior at infinity of certain important classes of sub-analytic sets as in Wilkie’s ([55]) as pointed out by Bierstone and Milman [3], and on the other hand, the recent, somehow surprising, first unconditional proof of the André-Oort conjecture for mixed Shimura varieties expressible as products of curves by Pila [45] following previous work also using o-minimality by Pila and Zanieri ([46]), Pila and Wilkie ([47]) and Peterzil and Starchenko ([43]).

The goal of this paper is to contribute further to the claim that o-minimality does indeed realize Grothendieck’s notion of topologie modérée by developing the formalism of the Grothendieck six operations on o-minimal sheaves, extending Delfs results for semi-algebraic sheaves ([12]) as well as Kashiwara and Schapira ([38]) and Prelli’s ([52]) results for sub-analytic sheaves restricted to globally sub-analytic spaces (as we deal, for now, only with definable spaces and not locally definable spaces - the later more general case will be dealt with in a sequel to this paper).

In the semi-algebraic case nearly all of our results are completely new. Indeed, in [12, Section 8], Delfs constructs the semi-algebraic proper direct image functor and only proves two basic results about this functor (base change and commutativity...
with small inductive limits) and then conjectures in [12, Remark 8.11 ii)] that: “It seems that the results of this section suffice to prove a semi-algebraic analogue of Verdier duality (by the same proof as in the theory of locally compact spaces c.f [54]).”

In the globally sub-analytic case we introduce a new globally sub-analytic proper direct image functor which, unlike Kashiwara and Schapira ([38]) and Prelli’s ([52]) sub-analytic proper direct image functor, generalizes to arbitrary o-minimal structures including: (i) arbitrary real closed fields; (ii) the non-standard models of sub-analytic geometry ([17]); (iii) the non-standard o-minimal structure which does not came from a standard one as in [40, 33]. See Remark 3.15 for further details on this. Moreover, unlike [38, 52], our definition is compatible with the restriction to open subsets (see Remark 3.16).

The Grothendieck formalism developed here allows us to obtain o-minimal versions of: (i) derived projection formula; (ii) universal coefficient formula; (iii) derived base change formula; (iv) K"unneth formula; (v) local and the global Verdier duality. It also sets up the framework for: (a) defining new o-minimal homology theories $H_*(X, \mathbb{Q}) := H^*(a_{X_1} \circ a_X^!, \mathbb{Q})$ extending o-minimal singular homology or the o-minimal Borel-Moore homology $H^\text{BM}_*(X, \mathbb{Q}) := H^*(a_{X*} \circ a_X^!, \mathbb{Q})$; (b) the full development of o-minimal geometry including the theory of characteristic classes, Hirzebruch-Riemann-Roch formula and Atiyah-Singer theorem in the non-standard o-minimal context.

We expect that the theory developed here will have interesting applications even in algebraic analysis. With our definition it is possible to treat the global sub-analytic sites in which are defined sheaves of functions with growth conditions up to infinity (e.g. tempered and Whitney $C^\infty$ or holomorphic functions). This kind of objects are very important for applications as in [9] and in [10] (in which globally sub-analytic sheaves are hidden under the notion of ind-sheaves on a bordered space).

Some quicker applications have already been obtained in the theory of definable groups in arbitrary o-minimal structures. See [22]. Using the notion of orientability defined here and the K"unneth formula, we show that if $G$ is a definably connected, definably compact definable group, then $G$ is orientable and the o-minimal sheaf cohomology $H^*(G; k_G)$ of $G$ with coefficients in a field $k$ is a connected, bounded, Hopf algebra over $k$ of finite type. Furthermore, using a consequence the Alexander duality proved here we develop degree theory which, when combined with a new o-minimal fundamental group in arbitrary o-minimal structure, gives the computation of the subgroup of $m$-torsion points of $G$ when $G$ is abelian - extending the main result of [23] which was proved in o-minimal expansions of ordered fields using the o-minimal singular (co)homology. This result is enough to settle Pillay’s conjecture for definably compact definable groups ([50] and [34]) in arbitrary o-minimal structures. Proofs of Pillay’s conjecture were recently also obtained in o-minimal expansions of ordered groups ([27] and [42]). However, in view of ([28]), it seems that those new proofs do not generalize to arbitrary o-minimal structures. Pillay’s conjecture is a non-standard analogue of Hilbert’s 5th problem for locally compact topological groups, roughly it says that after taking the quotient by a “small subgroup” (a smallest type-definable subgroup of bounded index) the quotient when equipped with the the so called logic topology is a compact real Lie group of the
same dimension.

The structure of the paper is the following. In Section 2 we introduce the setting where we will work on, listing all the preliminary results that will be needed throughout the rest of the paper. In Section 3 we define the proper direct image operation on o-minimal sheaves and prove its fundamental properties. In Section 4 we obtain the local and the global Verdier duality and its consequences.

Finally we note that, for the readers convenience, at the beginning of each section, after we introduce the set up, we include an extended summary in which we compare the ideas involved in the proofs of the section with those present in the references cited in the bibliography.

2. Preliminaries

In this paper we work in an arbitrary o-minimal structure \( M = (M, <, (c)_c \in \mathcal{C}, (f)_{f \in F}, (R)_{R \in \mathbb{R}}) \) with definable Skolem functions. We refer the reader to [16] for basic o-minimality.

We let Def be the category whose objects are definable spaces and whose morphisms are continuous definable maps between definable spaces - here and below “definable” always means definable in \( M \) with parameters. (See [16, Chapter 10, §1]). If \( M \) is a real closed field \( (\mathbb{R}, <, 0, 1, +, \cdot) \), then Def is the category whose objects are semi-algebraic spaces over \( R \) and whose morphisms are continuous semi-algebraic maps between such semi-algebraic space ([16, Chapter 2] and [12, Chapter I, Example 1.1]); if \( M \) is \( \mathbb{R} = (\mathbb{R}, <, 0, 1, +, \cdot, (f)_{f \in \text{an}}) \) - the field of real numbers expanded by restricted analytic functions, then Def is the category whose objects are globally sub-analytic spaces together with continuous maps with globally sub-analytic graphs between such spaces ([14]); if \( M \) is an ordered vector space \( (V, <, 0, +, (d)_{d \in D}) \) over an ordered division ring \( D \), then Def is the category whose objects are the piecewise linear spaces in this vector space together with continuous piecewise linear maps between such spaces ([16, Chapter 1, §7]).

Objects of Def are equipped with a topology determined by the order topology on \((M, <)\). However, if \((M, <)\) is non-archimedean then infinite definable spaces are totally disconnected and not locally compact, so one studies definable spaces equipped with the o-minimal site and replaces topological notions (connected, normal, compact, proper) by their definable analogues (definably connected, definably normal, definably compact, definably proper). The o-minimal site ([20]) generalizes both the semi-algebraic site ([12]) and the sub-analytic site ([38], [52]). Given an object \( X \) of Def the o-minimal site \( X_{\text{def}} \) on \( X \) is the category \( \text{Op}(X_{\text{def}}) \) whose objects are open (in the topology of \( X \) mentioned above) definable subsets of \( X \), the morphisms are the inclusions and the admissible covers \( \text{Cov}(U) \) of \( U \in \text{Op}(X_{\text{def}}) \) are covers by open definable subsets of \( X \) with finite sub-covers.

As shown in [20, Proposition 3.2], if \( A \) is a commutative ring, then the category \( \text{Mod}(A_{X_{\text{def}}} \) of sheaves of \( A \)-modules on \( X \) (relative to the o-minimal site) is isomorphic to the category \( \text{Mod}(A_X) \) of sheaves of \( A \)-modules on a certain spectral topological space \( \check{X} \), the o-minimal spectrum of \( X \), associated to \( X \). The o-minimal spectrum \( X \) of a definable space \( X \) is the set of ultra-filters of definable subsets of \( X \) (also know in model theory as types on \( X \)) equipped with the topology generated by the subsets \( \check{U} \) with \( U \in \text{Op}(X_{\text{def}}) \). If \( f : X \to Y \) is a morphism in Def, then
one has a corresponding continuous map $\tilde{f} : \tilde{X} \to \tilde{Y} : \alpha \mapsto \tilde{f}(\alpha)$ where $f(\alpha)$ is the ultrafilter in $\tilde{Y}$ determined by the collection $\{A : f^{-1}(A) \in \alpha\}$. (See [20, Definitions 2.2 and 2.18] or [6] and [48] where these notions were first introduced).

If $M$ is a real closed field $(R, <, 0, 1, +, \cdot)$, and $V \subseteq R^n$ is an affine real algebraic variety over $R$, then $\tilde{V}$ is homeomorphic to Sper $R[V]$, the real spectrum of the coordinate ring $R[V]$ of $V$ ([4, Chapter 7, Section 7.2] or [12, Chapter I, Example 1.1]); and the isomorphism $\text{Mod}(A_{V, \text{def}}) \simeq \text{Mod}(A_{\tilde{V}})$ from [20] corresponds in this case to [12, Chapter 1, Proposition 1.4].

Below we denote by $\text{Def}$ the corresponding category of o-minimal spectra of definable spaces and continuous definable maps and

$$\text{Def} \to \tilde{\text{Def}}$$

the functor just defined. Due to the isomorphism $\text{Mod}(A_{X, \text{def}}) \simeq \text{Mod}(A_{\tilde{X}})$ for every object $X$ of $\text{Def}$ we will work in this paper in $\tilde{\text{Def}}$.

2.1. Extended summary. In this section we will prove some preliminary results that will be crucial for the rest of the paper. These preliminary results are necessary since the category $\text{Def}$ in which we will be working is rather different from the category Top of topological spaces.

An object of $\tilde{\text{Def}}$ is a $T_0$, quasi-compact and a spectral topological space, i.e., it has a basis of quasi-compact open subsets, closed under taking finite intersections and each irreducible closed subset is the closure of a unique point. Such objects are not $T_1$ (unless they are finite), namely not every point is closed, so they are not Hausdorff (unless they are finite). In particular, the usual (topological) definition of compact topological space cannot be used in $\tilde{\text{Def}}$. Similarly the usual (topological) definition of proper maps is of no use in $\text{Def}$.

Since $\text{Def}$ has cartesian squares (which one should point out are not cartesian squares in the category of topological spaces $\text{Top}$), in this section we introduce a category theory definition of these notions in $\text{Def}$ just like in semi-algebraic geometry ([13, Section 9]) (and also in algebraic geometry [32, Chapter II, Section 5.4]) and point out all the properties needed later.

A fiber $\tilde{f}^{-1}(a)$ of a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ in $\tilde{\text{Def}}$ is not in general an object of $\tilde{\text{Def}}$, but following [1, Lemma 3.1] in the affine case, such fiber is homeomorphic to an object $(\tilde{f_S})^{-1}(a)$ of $\tilde{\text{Def}}(S)$ where $a$ is a realization of the type $\alpha$ and $S$ is the prime model of the first-order theory of $M$ over $\{a\} \cup M$. Here $\tilde{\text{Def}}(S)$ is the same as $\tilde{\text{Def}}$ but defined in the o-minimal structure $S$. This phenomena is the model theoretic analogue of what happens in real-algebraic geometry ([12, Proposition 2.4], [53, Chapter II, 3.2]) (and also in algebraic geometry [32, Chapter II, Section 3]). Indeed, if $f : X \to Y$ is a morphism of real schemes over a real closed field $R$ and $\alpha \in Y$, then $f^{-1}(\alpha)$ with the underlying topology is homeomorphic to the real scheme $X \times_Y \text{Sper} k(\alpha)$ over the residue ordered field $k(\alpha)$ of $\alpha$ (recall that $\alpha$ is a prime cone). By [48] the real closure of $k(\alpha)$ is isomorphic to the prime model over $R$ and a realization of $\alpha$. 

Due to the homeomorphism \( \tilde{f}^{-1}(\alpha) \cong (f^\circ)^{-1}(\alpha) \), when \( \alpha \in \tilde{Y} \) is a closed point, we will be able to use the theory from [24, Section 3] of normal and constructible families of supports on the object \( (f^\circ)^{-1}(\alpha) \) also on the fiber \( \tilde{f}^{-1}(\alpha) \) after we show that working in a full subcategory \( \tilde{A} \) of \( \text{Def} \), the family \( c \) of complete supports on \( (f^\circ)^{-1}(\alpha) \) is normal and constructible. Normal and constructible families of supports are the o-minimal analogue of the semi-algebraic and paracompactifying families of supports in semi-algebraic geometry ([12, Chapter II, Section 1]) and our main results on this, Lemma 2.21 and Corollary 2.22, are similar to [12, Chapter II, Proposition 8.2 and Corollary 8.3].

The full subcategory \( \tilde{A} \) of \( \tilde{\text{Def}} \) on which we must work is the image under \( \text{Def} \to \tilde{\text{Def}} \) of a subcategory \( A \) of \( \text{Def} \) such that:

(A0) cartesian products of objects of \( A \) are objects of \( \tilde{A} \) and locally closed subsets of objects of \( A \) are objects of \( \tilde{A} \);
(A1) in every object of \( A \) every open definable subset is a finite union of open and definably normal definable subsets;
(A2) every object of \( A \) has a definably normal definable completion in \( A \).

Examples of categories \( A \) as above include: (i) regular, locally definably compact definable spaces in o-minimal expansions of real closed fields; (ii) Hausdorff locally definably compact definable spaces in o-minimal expansions of ordered groups with definably normal completions; (iii) locally closed definable subspaces of cartesian products of a given definably compact definable group in an arbitrary o-minimal structure. For details on this see Example 3.1 below.

In the semi-algebraic case the semi-algebraic analogues of (A0), (A1) and (A2) are also extensively used and they hold, as in our example (i), for regular locally complete semi-algebraic spaces. In fact what is used there is a stronger version of (A1), namely that every open semi-algebraic subset of a regular semi-algebraic space is semi-algebraically normal. In topology we do not have the problem of fibers described above and what we need from (A0), (A1) and (A2) (Propositions 2.15 and 2.16) holds on Hausdorff locally compact topological spaces. Finally note that even in algebraic geometry similar constraints occur since when one has to define the proper direct image functor of a separated morphism of finite type of schemes one has to use Nagata’s theorem on the existence of proper extensions of such morphisms (i.e., existence of completions as in (A2)).

2.2. Morphisms proper in \( \tilde{\text{Def}} \). Here we will introduce a category theory definition of the notions proper and complete in \( \tilde{\text{Def}} \) just like in semi-algebraic geometry ([13, Section 9]) (and also in algebraic geometry [32, Chapter II, Section 4] or [31, Chapter II, Section 5.4]) and point out the properties needed later.

If \( X \) is an object of \( \tilde{\text{Def}} \), then a subset \( Z \subseteq X \) is called constructible if \( Z \) is also an object of \( \text{Def} \).

Let \( f : X \to Y \) be a morphism in \( \tilde{\text{Def}} \). We say that:
• \( f : X \to Y \) is closed in \( \widetilde{\text{Def}} \) if for every closed constructible subset \( A \) of \( X \), its image \( f(A) \) is a closed subset of \( Y \).

• \( f : X \to Y \) is a closed (resp. open) immersion if \( f : X \to f(X) \) is a homeomorphism and \( f(X) \) is a closed (resp. open) subset of \( Y \).

Since \( \text{Def} \to \widetilde{\text{Def}} \) is an isomorphism of categories and cartesian squares exist in \( \text{Def} \) we have:

**Fact 2.1.** In the category \( \widetilde{\text{Def}} \) the cartesian square of any two morphisms \( f : X \to Z \) and \( g : Y \to Z \) in \( \text{Def} \) exists and is given by a commutative diagram

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{p_Y} & Y \\
p_X & & \bigg\downarrow g \\
X & \xrightarrow{f} & Z
\end{array}
\]

where the morphisms \( p_X \) and \( p_Y \) are known as projections. The Cartesian square satisfies the following universal property: for any other object \( Q \) of \( \widetilde{\text{Def}} \) and morphisms \( q_X : Q \to X \) and \( q_Y : Q \to Y \) of \( \text{Def} \) for which the following diagram commutes,

\[
\begin{array}{ccc}
Q & \xrightarrow{u} & X \times_Z Y \\
q_X & & \bigg\downarrow p_Y \\
X \times_Z Y & \xrightarrow{p_Y} & Y \\
p_X & & \bigg\downarrow g \\
X & \xrightarrow{f} & Z
\end{array}
\]

there exist a unique natural morphism \( u : Q \to X \times_Z Y \) (called mediating morphism) making the whole diagram commute. As with all universal constructions, the cartesian square is unique up to a definable homeomorphism.

The following is a very important remark that one should always have in mind:

**Remark 2.2.** The cartesian square in \( \widetilde{\text{Def}} \) of two morphisms \( f : X \to Z \) and \( g : Y \to Z \) in \( \text{Def} \) is not the same as the cartesian square in \( \text{Top} \) of the two morphisms \( f : X \to Z \) and \( g : Y \to Z \) in \( \text{Top} \). In particular, if \( \text{pt} \) denotes a fixed one point object of \( \text{Def} \), then the cartesian product

\[
\begin{array}{ccc}
X \times_{\text{pt}} Y & \xrightarrow{p_Y} & Y \\
p_X & & \bigg\downarrow a_Y \\
X & \xrightarrow{a_X} & \text{pt}
\end{array}
\]

in \( \widetilde{\text{Def}} \), also denoted by \( X \times Y \), of two objects \( X \) and \( Y \) in \( \widetilde{\text{Def}} \) is not the same as usual the cartesian product (in \( \text{Top} \)) of the objects \( X \) and \( Y \) in \( \text{Top} \).
Given a morphism \( f : X \to Y \) in \( \widetilde{\text{Def}} \), the corresponding diagonal morphism is the unique morphism \( \Delta : X \to X \times_Y X \) in \( \text{Def} \) given by the universal property of cartesian squares:

\[
\begin{array}{c}
\xymatrix{ X \\
X \ar[rr]^\Delta \\
X \ar[rr]^{\text{id}_X} \\
X \ar[rr]^{p_X} \\
X \ar[rr]_f & & Y }
\end{array}
\]

We say that:

- \( f : X \to Y \) is \textit{separated in} \( \widetilde{\text{Def}} \) if the corresponding diagonal morphism \( \Delta : X \to X \times_Y X \) is a closed immersion.

We say that an object \( Z \) in \( \widetilde{\text{Def}} \) is \textit{separated in} \( \widetilde{\text{Def}} \) if the morphism \( Z \to \{ \text{pt} \} \) to a point is separated.

Let \( f : X \to Y \) be a morphism in \( \widetilde{\text{Def}} \). We say that:

- \( f : X \to Y \) is \textit{universally closed in} \( \widetilde{\text{Def}} \) if for any morphism \( g : Y' \to Y \) in \( \text{Def} \) the morphism \( f' : X' \to Y' \) in \( \text{Def} \) obtained from the cartesian square

\[
\begin{array}{c}
\xymatrix{ X' \\
X \\
\ar[rr]_f & & Y \\
\ar[rr]^g & & Y' }
\end{array}
\]

in \( \text{Def} \) is closed in \( \widetilde{\text{Def}} \).

\textbf{Definition 2.3.} We say that a morphism \( f : X \to Y \) in \( \widetilde{\text{Def}} \) is \textit{proper in} \( \widetilde{\text{Def}} \) if \( f : X \to Y \) is separated and universally closed in \( \text{Def} \).

\textbf{Definition 2.4.} We say that an object \( Z \) of \( \widetilde{\text{Def}} \) is \textit{complete in} \( \widetilde{\text{Def}} \) if the morphism \( Z \to \text{pt} \) is proper in \( \text{Def} \).

Directly from the definitions (as in [31, Chapter II, Proposition 5.4.2 and Corollary 5.4.3], see also [13, Section 9]) one has the following. See [21] for a detailed proof in \( \text{Def} \) which transfers to \( \widetilde{\text{Def}} \):

\textbf{Proposition 2.5.} In the category \( \widetilde{\text{Def}} \) the following hold:

1. \textit{Closed immersions are proper in} \( \widetilde{\text{Def}} \).
2. \textit{A composition of two morphisms proper in} \( \widetilde{\text{Def}} \) \textit{is proper in} \( \widetilde{\text{Def}} \).
3. \textit{If} \( f : X \to Y \) \textit{is a morphism over} \( Z \) \textit{proper in} \( \text{Def} \) \textit{and} \( Z' \to Z \) \textit{is a base extension}, then the corresponding base extension morphism} \( f' : X \times_Z Z' \to Y \times_Z Z' \) \textit{is proper in} \( \widetilde{\text{Def}} \).
(4) If $f : X \to Y$ and $f' : X' \to Y'$ are morphisms over $Z$ proper in $\text{Def}$, then the product morphism $f \times f' : X \times_Z X' \to Y \times_Z Y'$ is proper in $\text{Def}$.

(5) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is proper in $\text{Def}$, then:
   (i) $f$ is proper in $\text{Def}$;
   (ii) if $g$ is separated in $\text{Def}$ and $f$ is surjective, then $g$ proper in $\text{Def}$.

(6) A morphism $f : X \to Y$ is proper in $\text{Def}$ if and only if $Y$ can be covered by finitely many open constructible subsets $V_i$ such that $f^{-1}(V_i) \to V_i$ is proper in $\text{Def}$.

From Proposition 2.5 we easily have:

**Corollary 2.6.** Let $f : X \to Y$ be a morphism in $\text{Def}$ and $Z \subseteq X$ a complete object of $\text{Def}$. Then the following hold:

1. $Z$ is a closed (constructible) subset of $X$.
2. $f|_Z : Z \to Y$ is proper in $\text{Def}$.
3. $f(Z) \subseteq Y$ is (constructible) complete in $\text{Def}$.
4. If $f : X \to Y$ is proper in $\text{Def}$ and $C \subseteq Y$ is a complete object of $\text{Def}$, then $f^{-1}(C) \subseteq X$ is (constructible) complete in $\text{Def}$.

Below, if $C$ is a subcategory of $\text{Def}$ we denote by $\tilde{C}$ its image under $\text{Def} \to \tilde{\text{Def}}$. The following is a standard consequence of Proposition 2.5. See [21] for a detailed proof in $\text{Def}$:

**Corollary 2.7.** Let $C$ be a full a subcategory of the category of definable spaces $\text{Def}$ whose set of objects is:

- closed under taking locally closed definable subspaces of objects of $C$,
- closed under taking cartesian products of objects of $C$.

Then the following are equivalent:

1. Every object $X$ of $\tilde{C}$ is completable in $\tilde{C}$ i.e., there exists an object $X'$ of $\tilde{C}$ which is complete in $\text{Def}$ together with an open immersion $i : X \hookrightarrow X'$ in $\tilde{C}$ with $i(X)$ dense in $X'$. Such $i : X \hookrightarrow X'$ is called a completion of $X$ in $\tilde{C}$.

2. Every morphism $f : X \to Y$ in $\tilde{C}$ is completable in $\tilde{C}$ i.e., there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{j} & Y'
\end{array}
$$

of morphisms in $\tilde{C}$ such that: (i) $i : X \to X'$ is a completion of $X$ in $\tilde{C}$; (ii) $j$ is a completion of $Y$ in $\tilde{C}$. 

Every morphism $f : X \to Y$ in $\tilde{\mathcal{C}}$ has a proper extension in $\tilde{\mathcal{C}}$ i.e., there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow f & & \downarrow \overline{f} \\
& Y & 
\end{array}
$$

of morphisms in $\tilde{\mathcal{C}}$ such that $i$ is a open immersion with $i(X)$ dense in $P$ and $\overline{f}$ a proper in $\text{Def}$.

Finally we observe the following which was proved in Def in the paper [21]:

**Proposition 2.8.** Let $\mathbb{S}$ be an elementary extension of $\mathbb{M}$. Let $f : X \to Y$ a morphism in $\text{Def}$. Then the following are equivalent:

1. $f$ is separated (resp. proper) in $\text{Def}$.
2. $f^\mathbb{S}$ is separated (resp. proper) in $\text{Def}(\mathbb{S})$.

We end the subsection by describing the closed subsets of objects of $\tilde{\text{Def}}$ and also showing that a morphism in $\tilde{\text{Def}}$ is closed if and only if it closed in $\text{Def}$.

**Remark 2.9.** Let $X$ be an object of $\tilde{\text{Def}}$ and $C = \bigcap_{i \in I} C_i \subseteq X$ with the $C_i$’s constructible subsets. The following hold:

- $C$ is quasi-compact subset (with the induced topology).
- A closed subset $B$ of $C$ is a quasi-compact subset. Indeed, $B = D \cap C$ with $D$ a closed subset of $X$. So $D$ is quasi-compact and $B$ is also quasi-compact.
- A quasi-compact subset $B$ of $C$ is a quasi-compact subset.

**Lemma 2.10.** Let $f : X \to Y$ be a morphism in $\tilde{\text{Def}}$. Then the following hold:

1. If $C = \bigcap_{i \in I} C_i \subseteq X$ with the $C_i$’s constructible subsets, then $f(C) = \bigcap_{i \in I} f(C_i)$.
2. If $D = \bigcap_{j \in J} D_j \subseteq Y$ with the $D_j$’s constructible subsets, then $f^{-1}(D) = \bigcap_{j \in J} f^{-1}(D_j)$.
3. $C \subseteq X$ is closed if and only if $C = \bigcap_{i \in I} C_i$ with each $C_i \subseteq X$ constructible and closed.

**Proof.** (1) Suppose that $C = \bigcap_{i \in I} C_i$ with the $C_i$’s constructible. Then $C$ is closed and hence compact in the Stone topology of $X$ (the topology generated by the constructible subsets). By [20, Remark 2.19], $f : X \to Y$ is continuous with respect to the Stone topologies on $X$ and $Y$, so $f(C)$ is compact in the Stone topology of $X$. Since $Y$ is Hausdorff in the Stone topology, $f(C)$ is closed (in the Stone topology). Therefore,

$$f(C) = \bigcap \{E : f(C) \subseteq E \text{ and } E \text{ is constructible}\}.$$ 

If $E$ is a constructible subset such that $f(C) \subseteq E$, then $C \subseteq f^{-1}(E)$ and by compactness, there is $C_i$ such that $C \subseteq C_i \subseteq f^{-1}(E)$ and hence $f(C) \subseteq f(C_i) \subseteq E$. Therefore, $f(C) = \bigcap_{i \in I} f(C_i)$. 


(2) Obvious.

(3) Suppose that \( C \) is closed. Then its complement is open and so a union of open constructible subsets. Then by [20, Remark 2.3], \( C \) is the intersection of closed constructible subsets. The converse is clear. \( \square \)

**Proposition 2.11.** Let \( f : X \rightarrow Y \) be a morphism in \( \widetilde{\text{Def}} \). Then the following are equivalent:

1. \( f \) is closed.
2. \( f \) is closed in \( \widetilde{\text{Def}} \).

**Proof.** Assuming (1) clearly (2) is immediate. Assume (2). If \( D \subseteq X \) be a closed constructible subset, then \( f(D) \) is closed (and constructible by [20, Remark 2.19]); if \( C \subseteq X \) is a general closed subset, we have \( C = \bigcap_{i \in I} C_i \) with the \( C_i \)'s closed and constructible (Lemma 2.10 (3)) and by Lemma 2.10 (1), \( f(C) = \bigcap_{i \in I} f(C_i) \) with each \( f(C_i) \) closed and constructible and hence, \( f(C) \) is closed as required. \( \square \)

### 2.3. Fibers in \( \widetilde{\text{Def}} \)

A fiber \( f^{-1}(\alpha) \) of a morphism \( f : X \rightarrow Y \) in \( \widetilde{\text{Def}} \) is not in general an object of \( \widetilde{\text{Def}} \), but a model theoretic trick such fibers can each be seen as objects of \( \text{Def}(S) \) for some elementary extension \( S \) of \( M \). This follows from our assumption that \( M \) has definable Skolem functions.

Before we proceed we recall the basic model theoretic consequences of our assumption on \( M \) that will be required later.

By [51, Theorem 5.1]:

**Fact 2.12.** Since \( M \) has definable Skolem functions, then for any parameter \( v \) in some model of the first-order theory of \( M \), there is a prime model \( S \) of the first-order theory of \( M \) over \( M \cup \{v\} \) such that, for every \( s \in S \), there is a definable map \( f : A \rightarrow M \) (definable in \( M \)), such that \( v \in A(S) \) and \( s = f^S(v) \).

By Fact 2.12 we have:

**Fact 2.13.** Since \( M \) has definable Skolem functions, if \( S \) is a prime model of the first-order theory of \( M \) over \( M \cup \{v\} \) where \( v \) is a parameter in some model of the first-order theory of \( M \), then if \( u \in S \) and \( K \) is a prime model of the theory of \( M \) over \( M \cup \{u\} \), then we have either \( K = M \) or \( K = S \).

Indeed, consider the model theoretic algebraic closure operator \( acl_M(\bullet) \) in \( S \) given by \( a \in acl_M(C) \) with \( C \subseteq S \) if and only if there is \( c \in C \) and a definable map \( f : A \rightarrow M \) (definable in \( M \)) such that \( c \in A(S) \) and \( f^S(c) = a \). By [51, Theorem 4.1], this model theoretic algebraic closure operator satisfies the exchange property: if \( a \in acl_M(Cb) \setminus acl_M(C) \), then \( b \in acl_M(Ca) \). Thus, by Fact 2.12, if \( u \in S \) and \( K \) is a prime model of the theory of \( M \) over \( M \cup \{u\} \), then we have either \( K = M \) or \( K = S \).

Let \( S \) be an elementary extension of \( M \). For each object \( \widetilde{X} \) of \( \widetilde{\text{Def}} \) we have a restriction map \( r : \widetilde{X}(S) \rightarrow \widetilde{X} \) such that given an ultrafilter \( \alpha \in \widetilde{X}(S) \), \( r(\alpha) \) is the ultrafilter in \( \widetilde{X} \) determined by the collection \( \{A : A(S) \in \alpha\} \). This is a continuous surjective map which is neither open nor closed.
The following result was proved in [1, Lemma 3.1] in the affine case - for continuous definable maps between definable sets, under the assumption that \( \mathcal{M} \) is an o-minimal expansion of an ordered group. However, the only thing needed from \( \mathcal{M} \) is that it has definable Skolem functions. In fact all that is required is Fact 2.12. A special case of this result, when the map is a projection \( X \times [a, b] \to X \), was proved before in [20, Claim 4.5].

**Lemma 2.14.** Let \( \tilde{f} : \tilde{X} \to \tilde{Y} \) be a morphism in \( \text{Def} \), \( \alpha \in \tilde{Y} \), \( \varphi \models \alpha \) a realization of \( \alpha \) and \( \mathcal{S} \) a prime model of the first-order theory of \( \mathcal{M} \) over \( \{a\} \cup \mathcal{M} \). Then there is a homeomorphism

\[
\rho_\varphi : (f^{\mathcal{S}})^{-1}(\alpha) \to \tilde{f}^{-1}(\alpha)
\]

induced by the restriction \( \rho : \tilde{X}(\mathcal{S}) \to \tilde{X} \).

**Proof.** Since \((f^{-1}(B))(\mathcal{S}) = (f^{\mathcal{S}})^{-1}(B(\mathcal{S}))\) for every definable subset \( B \subseteq Y \), we have a commutative diagram

\[
\begin{array}{ccc}
(f^{\mathcal{S}})^{-1}(\alpha) & \xrightarrow{\rho_\varphi} & \tilde{X}(\mathcal{S}) \\
| & | & | \\
\tilde{f}^{-1}(\alpha) & \xrightarrow{\tilde{f}} & \tilde{Y}(\mathcal{S}) \\
| & | & | \\
\tilde{f}^{-1}(\alpha) & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
\]

and so \( \rho_\varphi : (f^{\mathcal{S}})^{-1}(\alpha) \to \tilde{f}^{-1}(\alpha) \) is well defined.

We now have to show that \( \rho_\varphi : (f^{\mathcal{S}})^{-1}(\alpha) \to \tilde{f}^{-1}(\alpha) \) is a continuous and open bijection. Let \( Y_i \) be a definable chart of \( Y \) such that \( \alpha \in \tilde{Y}_i \) and let \( Z_i = f^{-1}(Y_i) \).

Since \((f_{|Z_i}^{\mathcal{S}})^{-1}(\alpha) = (f^{\mathcal{S}})^{-1}(\alpha) \) and \((f_{|Z_i}^{\mathcal{S}})^{-1}(\alpha) = \tilde{f}^{-1}(\alpha) \), the map we are interested on is the same as the restriction in the following commutative diagram

\[
\begin{array}{ccc}
((f_{|Z_i}^{\mathcal{S}})^{-1}(\alpha)) & \xrightarrow{\tilde{f}_{|Z_i}} & \tilde{Z}_i(\mathcal{S}) \\
| & | & | \\
\tilde{f}_{|Z_i}^{-1}(\alpha) & \xrightarrow{\tilde{f}} & \tilde{Y}_i
\end{array}
\]

Let \( X_j \) be a definable chart of \( X \). Since \((f_{|Z_i \cap X_j}^{\mathcal{S}})^{-1}(\alpha) = ((f_{|Z_i}^{\mathcal{S}})^{-1}(\alpha) \cap \tilde{X}_j(\mathcal{S})) \) and \((f_{|Z_i \cap X_j}^{\mathcal{S}})^{-1}(\alpha) = f_{|Z_i}^{-1}(\alpha) \cap \tilde{X}_j \), then the restriction in the following commutative diagram

\[
\begin{array}{ccc}
((f_{|Z_i \cap X_j}^{\mathcal{S}})^{-1}(\alpha)) & \xrightarrow{\tilde{f}_{|Z_i \cap X_j}} & (Z_i \cap \tilde{X}_j)(\mathcal{S}) \\
| & | & | \\
\tilde{f}_{|Z_i \cap X_j}^{-1}(\alpha) & \xrightarrow{\tilde{f}} & \tilde{Y}_i
\end{array}
\]

is a continuous and open bijection by [1, Lemma 3.1]. Therefore, since the \( \tilde{X}_j \)'s (resp. \( \tilde{X}_j(\mathcal{S}) \)) cover \( f_{|Z_i}^{-1}(\alpha) \) (resp. \((f_{|Z_i}^{\mathcal{S}})^{-1}(\alpha))\), it follows that \( \rho_\varphi : (f^{\mathcal{S}})^{-1}(\alpha) \to \tilde{f}^{-1}(\alpha) \) is a continuous and open surjection.
Let \( \beta_1, \beta_2 \in (\overline{f^3})^{-1}(a) = \overline{(f_{|Z_i})^3}^{-1}(a) \) be such that \( r_1(\beta_1) = r_1(\beta_2) \in \overline{f^{-1}}(a) = \overline{(f_{|Z_i})^{-1}}(a) \). Let \( X_j \) be a definable chart of \( X \) such that \( r_1(\beta_1) = r_1(\beta_2) \in \overline{X_j} \). Then \( r_1(\beta_1) = r_1(\beta_2) \in \overline{f_{|Z_i}}^{-1}(a) \cap \overline{X_j} = \overline{(f_{|Z_i \cap X_j})^{-1}}(a) \). On the other hand, we also have \( \beta_1, \beta_2 \in \overline{X_j(S)} \) and so \( \beta_1, \beta_2 \in ((f_{|Z_i})^3)^{-1}(a) \cap \overline{X_j(S)} = (((f_{|Z_i \cap X_j})^3)^{-1}(a) \)).

Hence, by the above, \( \beta_1 = \beta_2 \). This proves that \( r_1 : (\overline{f^3})^{-1}(a) \to \overline{f^{-1}}(a) \) is also an injection as required. \( \square \)

2.4. The main assumptions. Here we introduce the main assumptions under which we develop the Grothendieck formalism of the six operations on \( \text{o-minimal sheaves} \). We also explain exactly which consequences of these assumptions are used later.

Below we will work in a full subcategory \( \tilde{\text{A}} \) of \( \tilde{\text{Def}} \) which is the image under \( \text{Def} \to \tilde{\text{Def}} \) of a subcategory \( \text{A} \) of \( \text{Def} \) such that:

(A0) cartesian products of objects of \( \text{A} \) are objects of \( \tilde{\text{A}} \) and locally closed subsets of objects of \( \text{A} \) are objects of \( \tilde{\text{A}} \);

(A1) in every object of \( \text{A} \) every open definable subset is a finite union of open and definably normal definable subsets;

(A2) every object of \( \text{A} \) has a definably normal definable completion in \( \text{A} \).

The only consequences of the assumptions (A0), (A1) and (A2) that will be crucial and will be used in the paper are the two proposition below. Later for some results we will have to add also an assumption (A3) which we will discuss in the appropriate place.

Condition (A1) alone gives us the following:

**Proposition 2.15.** Let \( X \) be an object of \( \tilde{\text{A}} \). Then the following hold:

1. In \( X \) every open subset is a finite union of open and normal constructible subsets.
2. (Shrinking lemma) If \( \{U_i : i = 1, \ldots, n\} \) is a covering of an open, normal constructible subset \( W \) of \( X \) by open subsets, then there are constructible open subsets \( V_i \) and constructible closed subsets \( C_i \) of \( W \) (1 \( i \) \( n \)) with \( V_i \subseteq C_i \subseteq U_i \) and \( W = \cup \{V_i : i = 1, \ldots, n\} \).
3. If \( \alpha \in X \), then there is an open, normal constructible subset \( U \) of \( X \) such that \( \alpha \in U \) and \( \alpha \) is closed in \( U \).

**Proof.** (1) Follows by (A1) and [20, Theorem 2.13].

(2) Is well known in normal spectral spaces, see for example [20, Proposition 2.17].

(3) The specializations of a point \( \alpha \) in an object \( Y \) of \( \tilde{\text{Def}} \) form finite chains by [20, Lemma 2.11] and if the object is normal, by [20, Proposition 2.12] or [7, Proposition 2], \( \alpha \) has a unique closed specialization \( \rho \) which must be the minimum of any chain of specialization of \( \alpha \). By (1) let \( U \) be an open, normal constructible open subset of \( X \) such that \( \alpha \in U \). Let \( \rho \) be the unique closed specialization of \( \alpha \) in \( U \). If \( \alpha \) is closed in \( U \) we are done, otherwise, \( U \setminus \{\rho\} \) is open (in \( U \) and so
also in \(X\), and so by (1) there is an open, normal constructible open subset \(V\) of \(U \setminus \{\rho\}\) (and so of \(X\)) such that \(\alpha \in V\). Since every specialization of \(\alpha\) in \(V\) is also a specialization of \(\alpha\) in \(U\) and the maximum length of chains of specializations of \(\alpha\) in \(V\) is smaller that the maximum length of chains of specializations of \(\alpha\) in \(U\), repeating the process finitely many times we get the result.

\[\square\]

Conditions (A0) and (A2) alone gives us the following:

**Proposition 2.16.** Let \(f : X \to Y\) be a morphism in \(\tilde{\mathbf{A}}\). Then the following hold:

1. \(f\) has a proper extension in \(\tilde{\mathbf{A}}\).
2. \(f\) is separated in \(\tilde{\text{Def}}\).
3. If \(Z \subseteq X\) is a closed constructible subset and \(f|_Z : Z \to Y\) is proper in \(\text{Def}\), then for every open constructible neighborhood \(W\) of \(Z\) in \(X\) there is a closed constructible neighborhood \(B\) of \(Z\) in \(W\) such that \(f|_B : B \to Y\) is proper in \(\tilde{\text{Def}}\).

**Proof.**

1. Follows by (A0), (A2) and Corollary 2.7.
2. Observe that the completions of \(X\) and \(Y\) in \(\tilde{\mathbf{A}}\) are normal if and only if they are o-minimal spectra of definably normal objects of \(\mathbf{A}\) ([20, Theorem 2.13]); a definably normal object of \(\text{Def}\) is Hausdorff; a morphism in \(\text{Def}\) between Hausdorff definable spaces has Hausdorff fibers and so is separated in \(\text{Def}\). Therefore by (A2), \(f\) is separated in \(\tilde{\text{Def}}\).
3. By (1) we have a commutative diagram

\[
\begin{array}{ccc}
X & \overset{\iota}{\longrightarrow} & P \\
\downarrow f & & \downarrow \overline{f} \\
Y & \end{array}
\]

of morphisms in \(\tilde{\text{Def}}\) such that \(\iota\) is an open immersion and \(\overline{f}\) is proper in \(\tilde{\text{Def}}\). Since \(f|_Z = \overline{f} \circ \iota|_Z\) is proper in \(\text{Def}\), by Proposition 2.5 (5) (i), \(\iota|_Z\) is proper in \(\tilde{\text{Def}}\) and hence it is closed in \(\tilde{\text{Def}}\). So \(\iota(Z)\) is a closed constructible subset of \(P\). On the other hand, \(\iota(Z)\) and \(P \setminus \iota(W)\) are closed disjoint constructible subsets of \(P\). Since \(P\) is normal, by the shrinking lemma, there is a closed constructible neighborhood \(C\) of \(\iota(Z)\) in \(P\) with \(C \subseteq \iota(W)\). Let \(B = \iota^{-1}(C)\). Then \(B\) is a closed constructible neighborhood of \(Z\) in \(X\) such that \(B \subseteq W\). Since \(f|_B = \overline{f} \circ \iota|_B\), \(\overline{f}\) is proper in \(\tilde{\text{Def}}\) and \(\iota|_B\) is a closed immersion, by Proposition 2.5 (1) and (2), \(f|_B : B \to Y\) is proper in \(\tilde{\text{Def}}\) as required.

\[\square\]

2.5. **On families of supports on fibers in \(\tilde{\text{Def}}\).** In the paper [24] normal and constructible families of supports on objects of \(\tilde{\text{Def}}\) played a fundamental role. Here we introduce the notion of a normal and constructible family of supports \(\Phi\) on a fiber \(\overline{f}^{-1}(\alpha)\) of a morphism \(\overline{f} : \tilde{X} \to \tilde{Y}\) in \(\tilde{\text{Def}}\). We show that working in \(\tilde{\mathbf{A}}\) will guaranty: (i) the normality and constructibility of the family \(\Phi^f\) of proper supports; (ii) the normality and constructibility of the family of complete supports on a fiber \(\overline{f}^{-1}(\alpha)\) of a closed point; (iii) the compatibility of the family of complete
supports on a fiber $\tilde{f}^{-1}(\alpha)$ of a closed point with the family of proper supports $\Phi_f$.

First recall the following definitions from [24, page 1267].

**Definition 2.17.** Let $X$ be an object in $\tilde{\text{Def}}$. A family $\Phi$ of closed subsets of $X$ is a family of supports on $X$ if:

- every closed subset of a member of $\Phi$ is in $\Phi$;
- $\Phi$ is closed under finite unions.

A family $\Phi$ of supports on $X$ is said to be constructible if:

- every member of $\Phi$ is contained in a member of $\Phi$ which is constructible.

A family $\Phi$ of supports on $X$ is said to be normal if:

- every member of $\Phi$ is normal;
- for each member $S$ of $\Phi$, if $U$ is an open neighborhood of $S$ in $X$, then there exists a closed constructible neighborhood of $S$ in $U$ which is a member of $\Phi$.

By Propositions 2.16 (3) and Proposition 2.16 (3) applied to the morphism $a_X : X \to \text{pt}$ to a point respectively we have:

**Example 2.18.** Let $f : X \to Y$ be a morphism in $\tilde{\text{A}}$ and $Z$ an object of $\tilde{\text{A}}$. Then:

1. The family $\Phi_f = \{ A : A \subseteq X \text{ is closed, } A \subseteq B \text{ for some closed constructible subset } B \text{ of } X \text{ such that } f|_B : B \to Y \text{ is proper in } \tilde{\text{Def}} \}$ of closed subsets of $X$ is a normal and constructible family of supports on $X$.
2. The family $c = \{ A : A \subseteq X \text{ is closed, } A \subseteq B \text{ for some closed constructible complete in } \tilde{\text{Def}} \text{ subset } B \text{ of } X \}$ of complete supports on $Z$ is a normal and constructible family of supports on $Z$.

Working with the previous definition in $\tilde{\text{Def}}(S)$ we can use the homeomorphism $r_\mid : (f^\natural)^{-1}(a) \to \tilde{f}^{-1}(\alpha)$ of Lemma 2.14 to define the notion of a normal and constructible family of supports on the fibers $\tilde{f}^{-1}(\alpha)$:

**Definition 2.19.** Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism in $\tilde{\text{Def}}$, $\alpha \in \tilde{Y}$, $a \models \alpha$ a realization of $\alpha$ and $S$ a prime model of the first-order theory of $M$ over $\{a\} \cup M$.

- A family of supports $\Psi$ on the quasi-compact space $\tilde{f}^{-1}(\alpha)$ is constructible if its inverse image $(r_\mid)^{-1}\Psi$ is a constructible family of supports on the object $(f^\natural)^{-1}(a)$ of $\tilde{\text{Def}}(S)$.
- A family of supports $\Psi$ on the quasi-compact space $\tilde{f}^{-1}(\alpha)$ is normal if its inverse image $(r_\mid)^{-1}\Psi$ is a normal family of supports on the object $(f^\natural)^{-1}(a)$ of $\tilde{\text{Def}}(S)$.

If there is no risk of confusion, and since $r_\mid$ is a homeomorphism, we also use $\Psi$ to denote the inverse image of $\Psi$ by $r_\mid$.

We also say that a subset $Z$ of $\tilde{f}^{-1}(\alpha)$ is constructible if its inverse image $(r_\mid)^{-1}(Z)$ is a constructible subset of the object $(f^\natural)^{-1}(a)$ of $\tilde{\text{Def}}(S)$. Of course,
a subset $Z$ of $\tilde{f}^{-1}(\alpha)$ is quasi-compact if its inverse image $(r_1)^{-1}(Z)$ is a quasi-compact subset of the object $(\tilde{f}^S)^{-1}(a)$ of $\text{Def}(S)$.

We can now introduce one of the main definitions of the paper:

**Definition 2.20.** Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism in $\tilde{\text{Def}}$, $\alpha \in \tilde{Y}$, $a \models \alpha$ a realization of $\alpha$ and $S$ a prime model of the first-order theory of $M$ over $\{a\} \cup M$.

- The family of complete supports on the object $(\tilde{f}^S)^{-1}(a)$ of $\tilde{\text{Def}}(S)$, denoted $c$, is the constructible family of supports on $(\tilde{f}^S)^{-1}(a)$ of all closed subsets $A$ of $(\tilde{f}^S)^{-1}(a)$ with $A \subseteq \tilde{Z}$ for some closed constructible complete in $\tilde{\text{Def}}(S)$ subset $\tilde{Z}$ of the object $(\tilde{f}^S)^{-1}(a)$ of $\tilde{\text{Def}}(S)$.

Throughout the paper we will also require that the family $c$ of complete supports on a fiber $\tilde{f}^{-1}(\alpha)$ of a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ in $\tilde{\text{Def}}$ with $\alpha \in \tilde{Y}$, is such that, if $\alpha$ is closed, then $c = \Phi_{\tilde{f}} \cap \tilde{f}^{-1}(\alpha)$ and $c$ is a normal and constructible family of supports.

First we need the following consequence of conditions (A0) and (A2):

**Lemma 2.21.** Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism in $\tilde{\text{A}}$. Let $\alpha \in \tilde{Y}$ be closed, $a \models \alpha$ a realization of $\alpha$ and $S$ a prime model of the first-order theory of $M$ over $\{a\} \cup M$. Then for every constructible complete in $\tilde{\text{Def}}(S)$ subset $K$ of $(\tilde{f}^S)^{-1}(a)$ there exists a closed constructible subset $B$ of $\tilde{X}$ such that $\tilde{f}_{|B} : B \to \tilde{Y}$ is proper in $\tilde{\text{Def}}$ and $K \subseteq B(S)$.

**Proof.** By Proposition 2.16 (1), we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\iota} & \tilde{P} \\
\tilde{f} \downarrow & & \tilde{h} \downarrow \\
\tilde{Y} & & \\
\end{array}
$$

of morphisms in $\tilde{\text{Def}}$ such that $\iota$ is an open immersion and $\tilde{h}$ is proper in $\tilde{\text{Def}}$. Since $K$ is a constructible complete in $\tilde{\text{Def}}(S)$ subset of $(\tilde{f}^S)^{-1}(a)$, by Corollary 2.6 (3) in $\tilde{\text{Def}}(S)$, we have that $\iota^S(K)$ is constructible complete in $\tilde{\text{Def}}(S)$ and hence closed in $(\tilde{h}^S)^{-1}(a) \subseteq \tilde{P}(S)$. The homeomorphism $r_1 : (\tilde{h}^S)^{-1}(a) \to \tilde{h}^{-1}(\alpha)$ (Lemma 2.14) implies that $r_1(\iota^S(K))$ is closed in $\tilde{h}^{-1}(\alpha)$ which is closed in $\tilde{P}$ ($\alpha$ is closed). Hence $r_1(\iota^S(K))$ is closed in $\tilde{P}$ and moreover $r_1(\iota^S(K)) \cap (\tilde{P} \setminus \iota(\tilde{X})) = \emptyset$. Since $\tilde{P}$ is normal, by the shrinking lemma, we can find disjoint constructible open neighborhoods $\tilde{U}$ and $\tilde{V}$ of $r_1(\iota^S(K))$ and $\tilde{P} \setminus \iota(\tilde{X})$ respectively in $\tilde{P}$. Set $\tilde{C} = \tilde{P} \setminus \tilde{V}$. Then $r_1(\iota^S(K)) \subset \tilde{C} \subset \iota(\tilde{X})$ with $\tilde{C}$ constructible and closed in both $\iota(\tilde{X})$ and $\tilde{P}$. 
From the commutative diagram,

\[
\begin{array}{ccc}
(f^\natural)^{-1}(a) & \xrightarrow{r}|_{f} & \tilde{f}^{-1}(a) \\
\downarrow{\iota} & \downarrow{\iota} & \downarrow{\iota}
\end{array}
\]

it follows that \(\iota(r_1(K)) \subset \tilde{C} \subset \iota(\tilde{X})\). Since \(\iota : \tilde{X} \to \iota(\tilde{X})\) is a homeomorphism, \(B = \iota^{-1}(\tilde{C})\) is a constructible and closed subset of \(\tilde{X}\) such that \(r_1(K) \subset B \subset \tilde{X}\). Furthermore, by Proposition 2.5 (1) and (2), \(\tilde{f}|_B = \tilde{h} \circ \iota|_B : B \to \tilde{Y}\) is proper in \(\text{Def}\) since \(\tilde{h}\) is proper in \(\text{Def}\) and \(\iota|_B : B \to Y\) is also proper \(\text{Def}\) being a closed immersion in \(\text{Def}\). On the other hand, since \(r(B(S)) = B\), we get \(K \subseteq B(S)\) as required. \(\square\)

By Lemma 2.21 we have the following consequence of conditions (A0) and (A2):

**Corollary 2.22.** Let \(\tilde{f} : \tilde{X} \to \tilde{Y}\) be a morphism in \(\tilde{A}\). Let \(\alpha \in \tilde{Y}\) be closed. Let \(c\) be the family of complete supports of \(\tilde{f}^{-1}(\alpha)\). Then

\[
c = \Phi_{\tilde{f}} \cap \tilde{f}^{-1}(\alpha) = \{ A \cap \tilde{f}^{-1}(\alpha) : A \in \Phi_{\tilde{f}} \}
\]

and \(c\) is a normal and constructible family of supports on \(\tilde{f}^{-1}(\alpha)\).

**Proof.** Let \(a \models \alpha\) be a realization of \(\alpha\) and \(S\) a prime model of the first-order theory of \(M\) over \(\{a\} \cup M\).

Let \(L \in c\). Then by definition of \(c\), there is a constructible complete in \(\text{Def}(S)\) subset \(K\) of \((f^\natural)^{-1}(a)\) such that \((r_1)^{-1}(L) \subseteq K\). By Lemma 2.21, there exists a closed constructible subset \(B\) of \(\tilde{X}\) such that \(r_1(K) \subset B \subset \tilde{X}\). Furthermore, by Proposition 2.8, \(\tilde{f}|_B = \tilde{h} \circ \iota|_B : B \to \tilde{Y}\) is proper in \(\text{Def}\) since \(\tilde{h}\) is proper in \(\text{Def}\) and \(\iota|_B : B \to Y\) is also proper \(\text{Def}\) being a closed immersion in \(\text{Def}\). On the other hand, since \(r(B(S)) = B\), we get \(K \subseteq B(S)\) as required. \(\square\)

By Example 2.18 (2), \(c\) is a normal and constructible family of supports \(\tilde{f}^{-1}(\alpha)\).

**3. Proper direct image**

In this section we will develop the theory of proper direct image in a full subcategory \(\tilde{A}\) of \(\text{Def}\) which is the image under \(\text{Def} \to \text{Def}\) of a certain full subcategory \(A\) of \(\text{Def}\).
Notation: For the rest of the paper we let $A$ be a noetherian ring (in fact we can assume more generally that $A$ is a commutative ring with finite weak global dimension, since this is all that we shall use about $A$ - see Remark 3.34).

If $X$ is a topological space we denote by $\text{Mod}(A_X)$ the category of sheaves of $A$-modules on $X$ and we call its objects $A$-sheaves on $X$.

Since $\mathcal{D}$ is a subcategory of $\text{Top}$, if $X$ is an object of $\mathcal{D}$ then we have the classical operations $\mathcal{H}om_{A_X}(\bullet, \bullet)$, $\bullet \otimes_{A_X} \bullet$, $f_*$, $f^{-1}$, $(\bullet)_Z$, $\Gamma_Z(X; \bullet)$, $\Gamma(X; \bullet)$ on $A$-sheaves on $X$, where $Z \subseteq X$ is a locally closed subset. Below we may use freely these operations and refer the reader to [37, Chapter II, Sections 2.1 - 2.4] for the details on sheaves on topological spaces and on the properties of these basic operations.

Later in this section we may also use the derived versions of many of the properties relating the above operations and we refer to reader to [37, Chapter II, Section 2.6] for details. The reader can also see the classical references [5], [29] and [35] for similar details on sheaves on topological spaces.

3.1. Extended summary. Our goal below is to develop the theory of proper direct image in a full subcategory $\mathcal{A}$ of $\mathcal{D}$ which is the image under $\mathcal{D} \to \mathcal{D}$ of a subcategory $\mathcal{A}$ of $\mathcal{D}$ such that:

(A0) cartesian products of objects of $\mathcal{A}$ are objects of $\mathcal{A}$ and locally closed subsets of objects of $\mathcal{A}$ are objects of $\mathcal{A}$;

(A1) in every object of $\mathcal{A}$ every open definable subset is a finite union of open and definably normal definable subsets;

(A2) every object of $\mathcal{A}$ has a definably normal definable completion in $\mathcal{A}$.

For some of our results about the proper direct image (base change formula (Proposition 3.33), derived base change formula (Theorem 3.39), the Künneth formula (Theorem 3.40) and dual base change formula (Proposition 4.6)) we will also require that:

(A3) if $f : X \to Y$ is a morphism in $\mathcal{D}$ and if $u \in Y$, then for every elementary extension $S$ of $M$ and every $F \in \text{Mod}(A_X_{\text{def}})$ we have an isomorphism

$$H^*_c(f^{-1}(u); F_{|f^{-1}(u)}) \simeq H^*_c((f^S)^{-1}(u); F(S)_{|(f^S)^{-1}(u)})$$

where $\overline{F}(S) = r^{-1}F$, $r : \overline{X}(S) \to \overline{X}$ is the restriction and $H^*_c$ is the cohomology with definably compact supports ([24, Example 2-10 and Definition 2.12]).

We have:

Example 3.1. Categories $\mathcal{A}$ satisfying also (A3) include:

(i) Regular, locally definably compact definable spaces in o-minimal expansions of real closed fields. ((A1) is by [16, Chapter 10, Theorem (1.8)] and [16, Chapter 6, Lemma (3.5)]; (A2) and (A3) by [25, Fact 4.7 and Corollary 4.8].)

(ii) Hausdorff locally definably compact definable spaces in o-minimal expansions of ordered groups with definably normal completions. ((A1) by [16,
Chapter 6, Lemma (3.5)]; (A2) by assumption and (A3) by [25, Theorem 4.5].

(iii) Locally closed definable subspaces of cartesian products of a given definably compact definable group in an arbitrary o-minimal structure. (See [22].)

In these examples, (A3) is obtained in [25] after extending an invariance result for closed and bounded definable sets in o-minimal expansions of ordered groups from [2].

Convention: For the rest of the paper we will work in the category $\tilde{\text{Def}}$ (resp. $\tilde{A}$ or $\tilde{\text{Def}(S)}$) and omit the tilde on objects and morphisms. We will say “proper” (resp. “separated” or “complete”) instead of “proper in $\text{Def}$” (resp. “separated in $\text{Def}$” or “complete in $\text{Def}$”). We will assume that all morphism that we consider in $\text{Def}$ are separated (this is the case for morphisms in $A$ (Proposition 2.16 (2))). If $X$ is an object of $\text{Def}$ (resp. $\tilde{A}$), then $\text{Op}(X)$ denotes the category of open subsets of $X$ with inclusions and $\text{Op}^{\text{cons}}(X)$ is the full sub-category of constructible open subsets of $X$.

Since our definition of the o-minimal proper direct image functor $f_!$ associated to a morphism $f : X \to Y$ in $\text{Def}$ is similar to the definition of the topological proper direct image functor $f_!$, we only replace proper in Top by proper in $\text{Def}$, we follow closely the proofs in topology ([37, Chapter II, Sections 2.5 and 2.6]) but we have to deal with: (i) the fact that the fibers $f^{-1}(\alpha)$ are often not objects of $\text{Def}$; (ii) the fact that cartesian squares in $\text{Def}$ are not cartesian squares in Top; (iii) the fact that objects of $\text{Def}$ are not locally compact topological spaces.

The most technical result is the Fiber formula (Corollary 3.13) which is a consequence of the Relative fiber formula (Proposition 3.9) in whose proof we use heavily the consequences of our assumptions (A0), (A1) and (A2) discussed in the Section 2. Our methods are different from those in semi-algebraic case: the fiber formula there ([12, Chapter II, Corollary 8.8]) is obtained as a consequence of a Base change formula ([12, Chapter II, Theorem 8.7]) and no semi-algebraic analogue of the Relative fiber formula is proved by Delfs.

We then proceed with the theory of $f$-soft sheaves which gives us the $f_!$-injective objects (Proposition 3.26) and later the bound on the cohomological dimension of $f_!$ (Theorem 3.42). The $f$-soft sheaves are those whose restriction to fibers $f^{-1}(\alpha)$ are $c$-soft. We show our results about $f$-soft sheaves after we remark (Remark 3.19) that by our assumptions (A0), (A1) and (A2) we can always reduce to the case where $\alpha$ is a closed point in which case $c$ is a normal and constructible family of supports on $f^{-1}(\alpha)$ and the results follow, via $(f^S)^{-1}(a) \simeq f^{-1}(\alpha)$, from similar results already proved in [24, Section 3]. This theory has not been considered in the semi-algebraic case.

The Projection formula (Proposition 3.28) is obtained in a standard way but the Base change formula (Proposition 3.33) requires assumption (A3) and, after we prove a technical lemma (Lemma 3.32), we use (A3) more or less in a similar way the corresponding fact ([12, Chapter I, Theorem 6.10]) is used in the prove of the semi-algebraic Base change formula ([12, Chapter II, Theorem 8.7]). Using
the \( f \)-soft resolutions we obtain in a standard way the Derived projection formula (Theorem 3.35) and the Derived base change formula (Theorem 3.39) after we prove a technical lemma (Lemma 3.38) requiring assumption (A3). This part was also not considered in the semi-algebraic case.

The complication with the Base change formula and the Derived base change formula and the need to use (A3) arises from the fact that in a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y,
\end{array}
\]

in \( \text{Def} \), if \( \gamma \in Y' \), then we may have \((f'^{-1})^{-1}(a) \simeq f'^{-1}(\gamma) \) and \((f\circ g)^{-1}(b) \simeq f^{-1}(g(\gamma)) \) with \( K = \mathcal{M} \) and \( S \neq \mathcal{M} \). So after we identify \((f'^{-1})^{-1}(a) \) with \((f\circ g)^{-1}(b) \) via \( g'^{-1} \) we need to know that \( H^1_c((f'^{-1}(b)); F_{f'^{-1}(b)}) \simeq H^1_c((f\circ g)^{-1}(b); F_{(f\circ g)^{-1}(b)}) \).

### 3.2. Proper direct image

Here we define the proper direct image functor and prove some of its basic properties.

**Definition 3.2.** Let \( f : X \to Y \) be a morphism in \( \text{Def} \) and let \( F \in \text{Mod}(A_X) \). The **proper direct image** is the subsheaf of \( f_*F \) defined by setting for \( U \in \text{Op}^\text{cons}(Y) \)

\[
\Gamma(U; f_*F) = \lim_Z \Gamma_Z(f^{-1}(U); F),
\]

where \( Z \) ranges through the family of closed constructible subsets of \( f^{-1}(U) \) such that \( f|_Z : Z \to U \) is proper.

From the definition we have:

**Remark 3.3.** The functor \( f_* \) is clearly left exact: if \( f : X \to Y \) is proper, then \( f_! = f_* \); if \( i : Z \to Y \) is the inclusion of a locally closed subset, then \( i_! \) is the extension by zero functor (i.e. \( (i_!F)_\alpha \simeq F_\alpha \) or 0 according to \( \alpha \in Z \) or \( \alpha \not\in Z \)) and \( (\bullet)_Z = i_! \circ i^{-1}(\bullet) \). Compare with [37, Chapter II, Proposition 2.5.4].

**Remark 3.4.** If we consider the morphism \( a_X : X \to \{\text{pt}\} \) to a point in \( \text{Def} \), then we have

\[
(a_X)_*F \simeq \Gamma(\{\text{pt}\}; a_X)_*F = \Gamma_e(X; F).
\]

**Proposition 3.5.** Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms in \( \text{Def} \). Then \((g \circ f)_! \simeq g_! \circ f_! \).

**Proof.** Indeed, if \( V \in \text{Op}^\text{cons}(Z) \), then a section of \( \Gamma(V, g_! \circ f_!F) \) is represented by \( s \in \Gamma(f^{-1}(g^{-1}(V)); F) \) such that \( \text{supp}(s) \subset f^{-1}(S) \cap Z \), where \( Z, S \) are closed constructible in \( f^{-1}(g^{-1}(V)) \) and \( g^{-1}(V) \) respectively and such that \( f|_Z : Z \to g^{-1}(V), g|_S : S \to V \) are proper. The set \( Z \cap f^{-1}(S) \) is closed constructible and the restriction \((g \circ f)_! : Z \cap f^{-1}(S) \to V \) is proper (Proposition 2.5 (1) and (2)). Conversely if \( V \in \text{Op}^\text{cons}(Z) \), then a section of \( \Gamma(V, (g \circ f)_!F) \) is represented by \( s \in \Gamma_Z(f^{-1}(g^{-1}(V)); F) \) where \( Z \) is closed constructible in \( f^{-1}(g^{-1}(V)) \) such that \((g \circ f)|_Z : Z \to V \) is proper. But then \( f|_Z : Z \to g^{-1}(V) \) is proper and
$g_{|f(Z)} : f(Z) \to V$ is proper (Proposition 2.5 (5)).

In particular, we have:

**Remark 3.6.** For $f : X \to Y$ a morphism in $\text{Def}$, since $a_X = a_Y \circ f$, we have $\Gamma_c(Y; f_iF) \simeq \Gamma_c(X; F)$.

**Proposition 3.7.** Let $f : X \to Y$ be a morphism in $\text{Def}$. The functor $f_i$ commutes with filtrant inductive limits.

**Proof.** Let $(F_i)_{i \in I}$ be a filtrant inductive system in $\text{Mod}(A_X)$. Since open constructible sets are quasi-compact, then for any $V \in \text{Op}^{\text{cons}}(X)$ and any closed constructible set $S$ of $V$ the functors $\Gamma(V; \bullet)$ and $\Gamma(V \setminus S; \bullet)$ commute with filtrant limits ([24, Remark 2.7]). Hence $\Gamma_S(V; \bullet)$ commutes with filtrant limits. We have

$$\Gamma(U; \lim_{i} f_i F_i) \simeq \lim_{i} \Gamma(U; f_i F_i) \simeq \lim_{i, Z} \Gamma_Z(f^{-1}(U); F_i) \simeq \lim_{i, Z} \Gamma_Z(f^{-1}(U); \lim_{i} F_i) \simeq \Gamma(U; \lim_{i} F_i),$$

where $Z$ ranges through the family of closed constructible subsets of $f^{-1}(U)$ such that $f|_Z: Z \to U$ is proper.

The following lemma follows immediately from the definition of proper direct image and we leave the details to the reader.

**Lemma 3.8.** Let $f : X \to Y$ be a morphism in $\text{Def}$ and $W$ an open subset of $Y$. Consider the commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{f_i} & X \\
\downarrow & & \downarrow f \\
Y & \xrightarrow{f} & Y.
\end{array}$$

If $F \in \text{Mod}(A_X)$, then $(f_i F)|_W \simeq (f_i)|_W (F|_{f^{-1}(W)})$.

Remember that, given a morphism $f : X \to Y$ in $\text{Def}$, the family $\Phi_f$ is defined by $\Phi_f = \{A : A \subseteq X \text{ is closed}, A \subseteq B \text{ for some closed constructible subset } B \text{ of } X \text{ such that } f|_B : B \to Y \text{ is proper}\}$.

**Proposition 3.9** (Relative fiber formula). Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\textbf{A}$, $\beta \in Z$ and let $F$ be a sheaf in $\text{Mod}(A_X)$. Let $K$ be a subset of $g^{-1}(\beta)$. If $K \in \mathcal{E}$, then $\Gamma(K; f_i F) \simeq \Gamma_c(f^{-1}(K); F)$.

**Proof.** First we show that we may assume without loss of generality that $\beta$ is a closed point in $Z$ and $K$ is a closed subset of $Y$.

By Proposition 2.15 (3), there exists $Z' \in \text{Op}^{\text{cons}}(Z)$ with $\beta \in Z'$ such that $\beta$ is closed in $Z'$. Let $Y' = g^{-1}(Z')$ and $X' = f^{-1}(Y')$. Let also $f' = f|_{X'}$, and
Let $b \models \beta$ be a realization of $\beta$ and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{b\} \cup M$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
(f^\mathbb{S})^{-1}((g^\mathbb{S})^{-1}(b)) & \xrightarrow{r_1} & f^{-1}(g^{-1}(\beta)) \\
\downarrow (f')^\mathbb{S} & & \downarrow f_1 \\
(g^\mathbb{S})^{-1}(b) & \xrightarrow{r_1} & g^{-1}(\beta)
\end{array}
\]

given by Lemma 2.14 (where we have omitted the tildes and used the fact that $(g \circ f)^\mathbb{S} = (g^\mathbb{S}) \circ (f^\mathbb{S})$).

By Proposition 2.16 (1) and Corollary 2.7 (2), we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{i_Y} & Y' \\
\downarrow g & & \downarrow g' \\
Z & \xrightarrow{i_Z} & Z'
\end{array}
\]

of completions in $\tilde{A}$. So we can make a similar diagram

\[
\begin{array}{ccc}
(f^\mathbb{S})^{-1}((g^\mathbb{S})^{-1}(b)) & \xrightarrow{r_1} & f^{-1}(g^{-1}(\beta)) \\
\downarrow (f')^\mathbb{S} & & \downarrow f'_1 \\
(g^\mathbb{S})^{-1}(b) & \xrightarrow{r_1} & g^{-1}(\beta) \\
\downarrow (i_X)^\mathbb{S} & \downarrow (i_Y)^\mathbb{S} & \downarrow (i_Z)^\mathbb{S} \\
(f^\mathbb{S})^{-1}((g^\mathbb{S})^{-1}(b)) & \xrightarrow{r_1} & f^{-1}(g^{-1}(\beta))
\end{array}
\]

given by Lemma 2.14 applied also to $f'$ and $g'$.

First we observe the following:

**Claim 3.10.** $\Phi_f \cap f^{-1}(K) = c_{f^{-1}(K)}$.

Let $Z \in \Phi_f$. We have to show that $Z \cap f^{-1}(K) \in c$ in $f^{-1}(g^{-1}(\beta))$. We may assume that $Z$ is a closed constructible subset of $X$. Since $f_{|Z} : Z \rightarrow Y$ is proper in $\text{Def}$, $f^\mathbb{S}_{|Z(\mathbb{S})} : Z(\mathbb{S}) \rightarrow Y(\mathbb{S})$ is proper in $\bar{\text{Def}}(\mathbb{S})$ (Proposition 2.8). Since $K \in$
c in \( g^{-1}(\beta) \), by definition, there is a \( C \subseteq (g^S)^{-1}(b) \) constructible and complete in \( (g^S)^{-1}(b) \) such that \( K \subseteq r_j(C) \). By Corollary 2.6 (4), \( (f^S)^{-1}(C) \cap Z(S) \) is a constructible and complete in \( (f^S)^{-1}((g^S)^{-1}(b)) \). Therefore, \( r_j((f^S)^{-1}(C) \cap Z(S)) = Z \cap f^{-1}(r_j(C)) \) is a constructible and complete in \( f^{-1}(g^{-1}(\beta)) \). Since \( Z \cap f^{-1}(K) \) is a closed subset of \( Z \cap f^{-1}(r_j(C)) \) we have that \( Z \cap f^{-1}(K) \in c \) in \( f^{-1}(g^{-1}(\beta)) \).

Conversely, let \( Z \in c_{f^{-1}(K)} \), then there is \( C \) a constructible complete subset of \( f^{-1}(g^{-1}(\beta)) \) such that \( Z \subseteq C \subseteq f^{-1}(K) \). It follows from Corollary 2.22 that \( C \) is contained in a closed constructible subset \( B \) of \( X \) such that \( (g \circ f)_{|B} \) is proper, hence \( f_{|B} \) is proper (Proposition 2.5 (5)). So \( C \subseteq B \cap f^{-1}(K) \in \Phi_f \cap f^{-1}(K) \). \( \square \)

We proceed with the proof of the Proposition. We have

\[
\Gamma(K; f_{|Z} F) \simeq \lim_{U \to U; Z} \Gamma(U; f_{|U} F) \simeq \lim_{U \to U} \Gamma_Z(f^{-1}(U); F),
\]

where \( U \) ranges through the family of open constructible neighborhoods of \( K \) and \( Z \) is closed constructible in \( f^{-1}(U) \) and such that \( f_{|Z} : Z \to U \) is proper.

By Claim 3.10 we can conclude the proof after the following two steps.

**Claim 3.11.** We have an isomorphism

\[
\lim_{U \to U; Z} \Gamma_Z(f^{-1}(U); F) \simeq \lim_{U \to U} \Gamma_{\Phi_f \cap f^{-1}(U)}(f^{-1}(U); F),
\]

where \( U \) ranges through the family of open constructible neighborhoods of \( K \) and \( Z \) is closed constructible in \( f^{-1}(U) \) and such that \( f_{|Z} \) is proper.

Let \( s \in \Gamma(f^{-1}(U); F) \) whose support is contained in \( Z \) closed constructible in \( f^{-1}(U) \) such that \( f_{|Z} : Z \to U \) is proper. Since \( K \in c \), then \( r^{-1}(K) \in c \) in \( (g^S)^{-1}(b) \) and its image by \( r_Y \) belongs to \( c \) in \( (g^S)^{-1}(b) \). The homeomorphism \( r_1 : (g^S)^{-1}(b) \to g^{-1}(\beta) \) implies that \( r(r_Y(r^{-1}(K))) = i_Y(K) \) is closed in \( g^{-1}(\beta) \) and hence in \( Y' \), a completion of \( Y \) in \( \tilde{A} \). Since \( Y' \) is normal (\( (A2) \)) and [20, Theorem 2.12]], there exists an open constructible neighborhood \( V \) of \( K \) such that \( V \subseteq U \).

Then \( f_{|Z \cap f^{-1}(V)} : Z \cap f^{-1}(V) \to V \) is proper and composing with the inclusion \( V \hookrightarrow Y \), the morphism \( f_{|Z \cap f^{-1}(V)} : Z \cap f^{-1}(V) \to Y \) is proper (Proposition 2.5 (1)) and (2)). Hence \( Z \cap f^{-1}(V) \in \Phi_f \). The support of the restriction of \( s \) to \( f^{-1}(V) \) is contained in \( Z \cap f^{-1}(V) = Z \cap f^{-1}(V) \cap f^{-1}(V) \in \Phi_f \cap f^{-1}(V) \) and the result follows. \( \square \)

**Claim 3.12.** We have an isomorphism

\[
\lim_{U \to U} \Gamma_{\Phi_f \cap f^{-1}(U)}(f^{-1}(U); F) \simeq \Gamma_{\Phi_f \cap f^{-1}(K)}(f^{-1}(K); F),
\]

where \( U \) ranges through the family of open constructible neighborhoods of \( K \).

Let \( s \in \Gamma(f^{-1}(K); F) \) and let \( Z \in \Phi_f \) closed constructible in \( f^{-1}(U) \) containing its support. The section \( s \) is represented by a section \( t \in \Gamma(W; F) \) for some open constructible neighborhood \( W \) of \( f^{-1}(K) \). Since \( Z \in \Phi_f \) and \( K \in c \), we have \( Z \cap f^{-1}(K) \in c \) in \( f^{-1}(g^{-1}(\beta)) \) (Claim 3.10). Therefore, \( r^{-1}(Z \cap f^{-1}(K)) \in c \) in \( (f^S)^{-1}((g^S)^{-1}(b)) \) and its image by \( r_Y \) belongs to \( c \) in \( (f^S)^{-1}((g^S)^{-1}(b)) \). The homeomorphism \( r_1 : (f^S)^{-1}((g^S)^{-1}(b)) \to f'^{-1}(g'^{-1}(\beta)) \) implies that \( r(r_Y(r^{-1}(Z \cap
\( f^{-1}(K)) = i_X(Z \cap f^{-1}(K)) \) is closed in \( f^{-1}(g'^{-1}(\beta)) \) and hence in \( X' \), a completion of \( X \) in \( \widetilde{A} \). Since \( X' \) is normal ([A2] and [20, Theorem 2.12]) there exists an open constructible neighborhood \( V \) of \( Z \cap f^{-1}(K) \) with \( V' \subset W \). We have that \( f'_1|_{V'} \) is proper, so \( f'_1|_{V'} \) is proper (Proposition 2.5 (5) (i)) and \( V' \in \Phi_f \).

Since \( f^{-1}(K) = f^{-1}(K) \), we have \( f^{-1}(K) \cap ((\overline{V} \cap \text{supp}(t)) \setminus V) = 0 \) and so \( K \cap f'((\overline{V} \cap \text{supp}(t)) \setminus V) = 0 \). Since \( K \) and \( f'((\overline{V} \cap \text{supp}(t)) \setminus V) \) are closed subsets of \( Y' \) (recall that \( f'_1|_{V'} \) is closed) and \( Y' \) is normal, by shrinking lemma (Proposition 2.15 (2)), there exists an open constructible subset \( U \) of \( Y \) with \( K \subset U \) such that \( f^{-1}(U) \cap \overline{V} \cap \text{supp}(t) \subset V \). Let us define \( \overline{t} \in \Gamma(f^{-1}(U); F) \) by

\[
\overline{t}|_{f^{-1}(U) \cap \overline{V} \cap \text{supp}(t)} = 0, \\
\overline{t}|_{f^{-1}(U) \cap V} = t|_{f^{-1}(U) \cap V}.
\]

By construction \( \overline{t}|_{f^{-1}(K)} = s \). Note that \( \text{supp}(\overline{t}) \) is contained in \( f^{-1}(U) \cap \overline{V} \) and \( \overline{V} \in \Phi_f \).

Setting \( Y = Z \) and \( g = \text{id} \) we obtain

**Corollary 3.13** (Fiber formula). Let \( f : X \to Y \) be a morphism in \( \widetilde{A} \) and let \( F \) be a sheaf in \( \text{Mod}(A_X) \). Let \( \alpha \in Y \). Then \( (f_1|_F)_\alpha \simeq \Gamma_c(f^{-1}(\alpha); F) \).

We end the subsection comparing the o-minimal proper direct image functor \( f_1 \) with Verdier’s ([54]) topological proper direct image functor \( f_1 \) and Kashiwara and Schapira ([38]) and Prelli’s ([52]) sub-analytic proper direct image functor \( f_1 \).

To make the comparison more clear we will use the isomorphism \( \text{Mod}(A_X) \simeq \text{Mod}(A_{X_{\text{def}}}) \) and the fact that \( f : X \to Y \) is proper in \( \text{Def} \) if and only if \( \widetilde{f} : \widetilde{X} \to \widetilde{Y} \) is proper in \( \text{Def} \) to introduce \( f_1 : \text{Mod}(A_{X_{\text{def}}}) \to \text{Mod}(A_{Y_{\text{def}}}) \) as \( \widetilde{f}_1 : \text{Mod}(A_{\widetilde{X}}) \to \text{Mod}(A_{\widetilde{Y}}) \).

We now explain the relationship between the o-minimal proper direct image functor and Verdier’s ([54]) proper direct image functor.

**Remark 3.14.** Consider the category \( \text{Def} \) associated to an o-minimal expansion \( \mathbb{M} = (\mathbb{R}, <, (c)_{c \in C}, (f)_{f \in F}, (R)_{R \in \mathbb{R}}) \) of the ordered set of real numbers.

For a continuous map \( f : X \to Y \) between locally compact topological spaces Verdier’s proper direct image functor \( f_1 \) is given by: for \( G \in \text{Mod}(A_X) \) and \( U \in \text{Op}(Y) \) we have

\[
\Gamma(U; f_1G) = \lim_{Z \to U} \Gamma(Z; f^{-1}(U); G),
\]

where \( Z \) ranges through the family of closed subsets of \( f^{-1}(U) \) such that \( f_1|_Z : Z \to U \) is proper (in \( \text{Top} \)).

If \( X \) is an object of \( \text{Def} \) then \( X \) is also a topological space (with the usual topology generated by open definable subsets) and we have the natural morphism of sites \( \rho : X \to X_{\text{def}} \) induced by the inclusion \( \text{Op}(X_{\text{def}}) \subset \text{Op}(X) \).

Since if \( f_1|_Z : Z \to Y \) is proper in \( \text{Def} \) then \( f_1|_Z : Z \to Y \) is proper in \( \text{Top} \) ([21]), we have \( \Gamma(U; f_1|_Z \circ \rho_* F) \leq \Gamma(U; \rho_* \circ f_1 F) \) for \( U \in \text{Op}(X_{\text{def}}) \). However, in general, we have \( \rho_* \circ f_1 \neq f_1 \circ \rho_* \).
For example, suppose that $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate and $U = (0, +\infty)$. Let $\phi : (0, +\infty) \to (0, +\infty)$ be given by $\phi(t) = \frac{1}{t}$ (resp. $\phi(t) = \frac{1}{\ln(t)}$) if $M$ is semi-bounded ([18]) (resp. polynomially bounded ([18])). Set $S = \{(x, y) \in (0, +\infty) \times \mathbb{R} : y = \phi(x)\}$ and consider the sheaf $A_S$. Let $s \in \Gamma(f^{-1}(U); A_S)$. Then $f : \text{supp}(s) \to U$ is proper (it is a homeomorphism into its image), so $s \in \Gamma(U; f_!A_S) = \Gamma(U; \rho_!A_S)$, but $s \notin \Gamma(U; f_1 \circ \rho_!A_S)$ since there is no closed definable subset $Z$ of $f^{-1}(U)$ such that $f_!Z : Z \to U$ is proper in Def and $S \subset Z$.

Now we explain the relationship between the o-minimal proper direct image functor and Kashiwara and Schapira ([38]) and Prelli’s ([52]) sub-analytic proper direct image functor.

**Remark 3.15.** Consider the category Def associated to the o-minimal structure $\mathbb{R}_{\text{an}} = (\mathbb{R}, <, 0, 1, +, (f)_{f \in \text{an}})$ - the field of real numbers expanded by restricted globally analytic functions ([18]). As explained in [18], in this case, Def is the category of globally sub-analytic spaces with continuous maps with globally sub-analytic graphs.

If $X$ is a real analytic manifold, then we can equip $X$ with its sub-analytic site, denoted $X_{\text{sa}}$, where the objects are open sub-analytic subsets $\text{Op}(X_{\text{sa}})$ and $\mathcal{S} \subset \text{Op}(X_{\text{sa}}) \cap \text{Op}(U)$ is covering of $U \in \text{Op}(X_{\text{sa}})$ if for any compact subset $K$ of $X$ there is a finite subset $S_0 \subseteq S$ such that $K \cap \bigcup_{V \in S_0} V = K \cap U$, see [38]. In this context we have ([52]) a sub-analytic proper direct image functor $f_!$ given by: for $F \in \text{Mod}(A_{X_{\text{sa}}})$ and $U \in \text{Op}(Y_{\text{sa}})$ we have

$$
\Gamma(U; f_!F) = \lim_{K \to \infty} \Gamma_{K \cap f^{-1}(U)}(f^{-1}(U); F),
$$

where $K$ ranges through the family of compact sub-analytic subsets of $X$. We recall that this is a direct construction of Kashiwara and Schapira ([38]) sub-analytic proper direct image functor originally constructed as a special case of a more general construction within the theory of ind-sheaves.

If $X$ is also globally sub-analytic (i.e. a definable subset in the o-minimal structure $\mathbb{R}_{\text{an}}$) we have the natural morphism of sites $\nu : X_{\text{sa}} \to X_{\text{def}}$ induced by the inclusion $\text{Op}(X_{\text{def}}) \subseteq \text{Op}(X_{\text{sa}})$.

Since a compact sub-analytic subset is a compact globally sub-analytic subset (hence a definably compact definable subset in the o-minimal structure $\mathbb{R}_{\text{an}}$), we have $\Gamma(U; \nu_* \circ f_!F) \subseteq \Gamma(U; f_1 \circ \nu_*F)$ for $U \in \text{Op}(X_{\text{def}})$. However, in general, we have $\nu_* \circ f_1 \neq f_1 \circ \nu_*$.

For example, suppose that $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate and $U = (0, 1)$. Let $\phi : (0, 1) \to (0, +\infty)$ be given by $\phi(t) = \frac{1}{t}$ (resp., $\phi(t) = \frac{1}{\ln(t)}$) if $M$ is semi-bounded ([19]) (resp. polynomially bounded ([18])). Set $S = \{(x, y) \in (0, 1) \times \mathbb{R} : y = \phi(x)\}$ and consider the sheaf $A_S$. Let $s \in \Gamma(f^{-1}(U); A_S)$. Then $f_1 : \text{supp}(s) \to U$ is definably proper (and so proper in Def by [21]), $s \in \Gamma(U; f_1 \circ \nu_*A_S)$, but $s \notin \Gamma(U; \nu_!A_S)$ since there is no compact (globally) sub-analytic subset $K$ of $X$ such that $\text{supp}(s) \subset K$.

**Remark 3.16.** The o-minimal proper direct image functor $f_1$ in comparison with the sub-analytic proper direct image functor $f_!$ seems to be more natural since it commutes with restrictions in the sense of Lemma 3.8, as the classical Verdier
proper direct image functor $f_!$ does in topological spaces. The property of that lemma is not satisfied by the functor $f_!$ since there are less compact subsets on $f^{-1}(W)$ than there are intersections of $f^{-1}(W)$ with compact subsets of $X$.

3.3. $f$-soft sheaves. Here we introduce the $f$-soft sheaves where $f : X \to Y$ is a morphism in $\text{Def}$. We show that the full additive subcategory of $f$-soft sheaves is $f_!$-injective and is stable under $\varprojlim$ and whose flat objects are also stable under $\bullet \otimes$ for all $F \in \text{Mod}(A_X)$.

Consider a fiber $f^{-1}(\alpha)$ of a morphism $f : X \to Y$ in $\text{Def}$ and let $c$ be the family of complete supports on $f^{-1}(\alpha)$. Recall that a sheaf $F$ on $f^{-1}(\alpha)$ is $c$-soft if and only if the restriction $\Gamma(f^{-1}(\alpha); F) \to \Gamma(K; F)$ is surjective for every $K \in c$.

**Definition 3.17.** Let $f : X \to Y$ be a morphism in $\text{Def}$ and let $F$ be a sheaf in $\text{Mod}(A_X)$. We say that $F$ is $f$-soft if for any $\alpha \in Y$ the sheaf $F_{|f^{-1}(\alpha)}$ in $\text{Mod}(A_{f^{-1}(\alpha)})$ is $c$-soft.

By Remark 3.4 we have:

**Remark 3.18.** Let $a_X : X \to \{\text{pt}\}$ be the morphism to a point in $\text{Def}$ and let $F$ be a sheaf in $\text{Mod}(A_X)$. Then $F$ is $a_X$-soft if and only if it is $c$-soft.

**Remark 3.19.** Let $f : X \to Y$ be a morphism in $\text{Def}$. To prove a property of $f$-soft sheaves in $\text{Mod}(A_X)$ we have to take an arbitrary $\alpha \in Y$ and prove the corresponding property for $c$-soft sheaves in $\text{Mod}(A_{f^{-1}(\alpha)})$. We will use Proposition 2.15 (4) to be able to replace $f : X \to Y$ by a suitable morphism $f' : X' \to Y'$ in $A$ such that $\alpha \in Y'$ and is closed in $Y'$.

After that we take $a \models \alpha$ a realization of $\alpha$, $S$ a prime model of the first-order theory of $M$ over $\{a\} \cup M$ and $r : (f^S)^{-1}(a) \to f'^{-1}(\alpha)$ the homeomorphism of Lemma 2.14. It follows that $(f^S)^{-1}(a)$ is an object of $\text{Def}(S)$ and $c$ is a normal and constructible family of supports on $(f^S)^{-1}(a)$ because $f' : X' \to Y'$ is a morphism in $\text{Def}$ and by Corollary 2.22, $c$ is a normal and constructible family of supports on $f'^{-1}(\alpha)$. Therefore, we will be able to transfer results for $c$-soft sheaves on o-minimal spectral spaces ([24, Section 3]) to $c$-soft sheaves on the fiber $f^{-1}(\alpha)$ since all we need is that $c$ is a family of normal and constructible supports.

**Remark 3.20.** In the paper [24] we assumed that $A$ is a field, but that was only used to ensure that $\bullet \otimes G \simeq G \otimes \bullet$, for $G \in \text{Mod}(A_X)$, is exact. For this reason, our results here about $\bullet \otimes G \simeq G \otimes \bullet$ will come with the flatness assumption.

The first application of the above method is:

**Proposition 3.21.** Let $f : X \to Y$ be a morphism in $\text{Def}$. Then filtrant inductive limits of $f$-soft sheaves in $\text{Mod}(A_X)$ are $f$-soft.

**Proof.** Let $(F_i)_{i \in I}$ be a filtrant inductive family of $f$-soft sheaves in $\text{Mod}(A_X)$. We have to show that $\varinjlim F_i$ is an $f$-soft sheaf in $\text{Mod}(A_X)$, i.e., for every $\alpha \in Y$ the sheaf $(\varinjlim F_i)_{|f^{-1}(\alpha)} = \varinjlim (F_i_{|f^{-1}(\alpha)})$ in $\text{Mod}(A_{f^{-1}(\alpha)})$ is $c$-soft.
So fix $\alpha \in Y$ and by Proposition 2.15 (4), let $Y' \in \text{Op}^{\text{cons}}(Y)$ be such that $\alpha \in Y'$ and $\alpha$ is closed in $Y'$. Let $X' = f^{-1}(Y')$ and $f' = f|_{Y'}$. Then $f'^{-1}(\alpha) = f^{-1}(\alpha)$ and so $F_{i|f'^{-1}(\alpha)} = F_{i|f^{-1}(\alpha)}$. Hence the sheaf $\lim(F_{i|f'^{-1}(\alpha)})$ in $\text{Mod}(A_{f'^{-1}(\alpha)})$ is $c$-soft if and only if the sheaf $\lim(F_{i|f^{-1}(\alpha)})$ in $\text{Mod}(A_{f^{-1}(\alpha)})$ is $c$-soft.

Let $a \models \alpha$ a realization of $\alpha$, $S$ a prime model of the first-order theory of $\mathcal{M}$ over $\{a\} \cup M$ and $r_1 : (f^{S})^{-1}(a) \to f'^{-1}(\alpha)$ the homeomorphism of Lemma 2.14. Then the sheaf $\lim(F_{i|f'^{-1}(\alpha)})$ in $\text{Mod}(A_{f'^{-1}(\alpha)})$ is $c$-soft if and only if the sheaf $\lim(r^{-1}F_{i|(f^{S})^{-1}(a)})$ in $\text{Mod}(A_{(f^{S})^{-1}(a)})$ is $c$-soft.

But $(f^{S})^{-1}(a)$ is an object of $\text{Def}(S)$ and $c$ is a normal and constructible family of supports on $(f^{S})^{-1}(a)$ because $f' : X' \to Y'$ is a morphism in $\mathcal{A}$ and by Corollary 2.22, $c$ is a normal and constructible family of supports on $f'^{-1}(\alpha)$. Since the sheaves $F_{i|f'^{-1}(\alpha)}$ in $\text{Mod}(A_{f'^{-1}(\alpha)})$ are $c$-soft if and only if the sheaves $F_{i|f^{-1}(\alpha)}$ in $\text{Mod}(A_{f^{-1}(\alpha)})$ are $c$-soft if and only if the sheaves $r^{-1}F_{i|(f^{S})^{-1}(a)}$ in $\text{Mod}(A_{(f^{S})^{-1}(a)})$ are $c$-soft, by [24, Corollary 3.5] in $\text{Def}(S)$, we conclude that the sheaf $\lim(r^{-1}F_{i|(f^{S})^{-1}(a)})$ in $\text{Mod}(A_{(f^{S})^{-1}(a)})$ is $c$-soft as required. 

Since restriction is exact and commutes with $\otimes$, by the method of Remark 3.19 in combination with [24, Proposition 3.13] and Remark 3.20 we have:

**Proposition 3.22.** Let $f : X \to Y$ be a morphism in $\mathcal{A}$. If $G \in \text{Mod}(A_X)$ is $f$-soft and flat, then for every $F \in \text{Mod}(A_X)$ we have that $G \otimes F$ is $f$-soft.

By Relative fiber formula (Proposition 3.9) we have:

**Proposition 3.23.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{A}$ and let $F$ be a sheaf in $\text{Mod}(A_X)$. If $F$ is $(g \circ f)$-soft, then $f_* F$ is $g$-soft.

**Proof.** Let $\beta \in Z$, we shall prove that the restriction $(f_* F)_{|(g^{-1}(\beta))}$ is $c$-soft. Let $K \subset K'$ be elements of the family of complete supports $c$ on $g^{-1}(\beta)$. By Relative fiber formula (Proposition 3.9), there is a commutative diagram

$$
\begin{array}{ccc}
\Gamma(K'; f_* F) & \xrightarrow{\psi_{K'}} & \Gamma_c(f^{-1}(K'); F) \\
\downarrow & & \downarrow \\
\Gamma(K; f_* F) & \xrightarrow{\psi_{K}} & \Gamma_c(f^{-1}(K); F)
\end{array}
$$

induced by the restrictions, with $\psi_{K'}$ and $\psi_K$ isomorphisms. Since $F$ is $(g \circ f)$-soft, the arrow on the right is surjective by [24, Proposition 3.4 (3)]. Therefore, the arrow on the left is also surjective. This implies that, if $K'$ is a constructible element of the family of complete supports $c$ on $g^{-1}(\beta)$, then $(f_* F)_{|(K')}$ is soft, which implies $(f_* F)_{|(g^{-1}(\beta))}$ $c$-soft by [24, Proposition 3.4 (4)].

A special case which follows by Remark 3.6 and [24, Proposition 3.4] is the following. Compare also with [37, Chapter II, Proposition 2.5.7 (ii)].

**Remark 3.24.** Let $f : X \to Y$ be a morphism in $\mathcal{Def}$ and let $F$ be a sheaf in $\text{Mod}(A_X)$. Suppose that the family $c$ of supports on $X$ (resp. on $Y$) is such that every $C \in c$ has a neighborhood $D$ in $X$ (resp. in $Y$) such that $D \in c$. If $F$ is $c$-soft, then $f_* F$ is $c$-soft.
Lemma 3.25. Let \( f : X \to Y \) be a morphism in \( \Def \) with \( X \) an open subspace of a normal space in \( \Def \) and let \( F \) be a sheaf in \( \Mod(A_X) \). If \( F \) is injective and hence flabby, then \( F \) is \( f \)-soft. In particular, the full additive subcategory of \( \Mod(A_X) \) of \( f \)-soft sheaves is cogenerating.

Proof. Let \( \alpha \in Y \) and let \( K \) be an element of the family \( c \) of complete supports on \( f^{-1}(\alpha) \). Then \( K \) is quasi-compact in \( X \) (Remark 2.9). Since \( X \) is an open subspace of a normal space in \( \Def \), by [24, Lemma 3.2], the canonical morphism

\[
\lim_{K \subseteq U} \Gamma(U; F) \to \Gamma(K; (F|_{f^{-1}(\alpha)}))|_K
\]

where \( U \) ranges through the family of open constructible subsets of \( X \), is an isomorphism. Since \( F \) is flabby \( \Gamma(X; F) \to \Gamma(U; F) \) is surjective. The exactness of filtrant \( \lim \) implies that \( \Gamma(X; F) \to \Gamma(K; (F|_{f^{-1}(\alpha)}))|_K \) is surjective. This morphism factors through \( \Gamma(f^{-1}(\alpha); (F|_{f^{-1}(\alpha)}) \) and the result follows.

\( \Box \)

Proposition 3.26. Let \( f : X \to Y \) be a morphism in \( \tilde{A} \). Then the full additive subcategory of \( \Mod(A_X) \) of \( f \)-soft sheaves is \( f_! \)-injective, i.e.:

1. For every \( F \in \Mod(A_X) \) there exists an \( f \)-soft \( F' \in \Mod(A_X) \) and an exact sequence \( 0 \to F \to F' \).
2. If \( 0 \to F' \to F \to F'' \to 0 \) is an exact sequence in \( \Mod(A_X) \) and \( F' \) is \( f \)-soft, then \( 0 \to f_! F' \to f_! F \to f_! F'' \to 0 \) is an exact sequence.
3. If \( 0 \to F' \to F \to F'' \to 0 \) is an exact sequence in \( \Mod(A_X) \) and \( F \) and \( F' \) are \( f \)-soft, then \( F'' \) is \( f \)-soft.

Proof. (1) By Lemma 3.25 the full additive subcategory of \( \Mod(A_X) \) of \( f \)-soft sheaves is cogenerating.

(2) Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence in \( \Mod(A_X) \) with \( F' \) is \( f \)-soft. Then for every \( \alpha \in Y \), the sequence \( 0 \to F'_f|_{f^{-1}(\alpha)} \to F_{f^{-1}(\alpha)} \to F''_{f^{-1}(\alpha)} \to 0 \) is exact and \( F'_f|_{f^{-1}(\alpha)} \) is a \( c \)-soft sheaf in \( \Mod(A_{f^{-1}(\alpha)}) \). By the method of Remark 3.19 in combination with [24, Proposition 3.7 (2)], \( 0 \to \Gamma_c(f^{-1}(\alpha); F'_f|_{f^{-1}(\alpha)}) \to \Gamma_c(f^{-1}(\alpha); F''_f|_{f^{-1}(\alpha)}) \to 0 \) is an exact sequence for every \( \alpha \in Y \). By the Fiber formula (Corollary 3.13), the sequence \( 0 \to f_! F' \to f_! F \to f_! F'' \to 0 \) is exact.

(3) Let \( 0 \to F' \to F \to F'' \to 0 \) be an exact sequence in \( \Mod(A_X) \) with \( F' \) and \( F \) both \( f \)-soft. Then for every \( \alpha \in Y \), the sequence \( 0 \to F'|_{f^{-1}(\alpha)} \to F_{f^{-1}(\alpha)} \to F''|_{f^{-1}(\alpha)} \to 0 \) is exact and \( F'_f|_{f^{-1}(\alpha)} \) and \( F''_f|_{f^{-1}(\alpha)} \) are \( c \)-soft sheaves in \( \Mod(A_{f^{-1}(\alpha)}) \). By the method of Remark 3.19 in combination with [24, Proposition 3.7 (3)], \( F''|_{f^{-1}(\alpha)} \) is \( c \)-soft. Since \( \alpha \) was arbitrary, \( F'' \) is \( f \)-soft.

\( \Box \)

3.4. Projection and base change formulas. Here we prove the projection and base change formulas.

For our next proposition we need the following topological lemma which is an adaptation of [37, Lemma 2.5.12];
Lemma 3.27. Let $X$ be a quasi-compact topological space with a basis of open quasi-compact neighborhoods. Let $F \in \text{Mod}(A_X)$, $M$ a flat $A$-module and $\Phi$ a family of supports on $X$. Then there is a natural isomorphism.

$$\Gamma_{\Phi}(X; F) \otimes M \simeq \Gamma_{\Phi}(X; F \otimes M_X).$$

In particular, if $F$ is $\Phi$-soft, then $F \otimes M_X$ is $\Phi$-soft.

Proof. We may assume that $X \in \Phi$. If $X = \bigcup_j U_j$ is a finite cover of $X$ by open quasi-compact neighborhoods, then

$$0 \to \Gamma(X; F) \to \bigoplus_j \Gamma(U_j; F) \to \bigoplus_j \Gamma(U_j \cap U_i; F)$$

is exact. Applying the exact functor $\bullet \otimes M$, recall $M$ is flat, we obtain the commutative diagram with exact rows:

$$
\begin{array}{ccc}
0 & \to & \Gamma(X; F) \otimes M \\
\downarrow & & \downarrow \varphi \\
0 & \to & \Gamma_{\Phi}(X; F \otimes M_X)
\end{array}
\quad
\begin{array}{ccc}
\lambda & \to & \bigoplus_j \Gamma(U_j; F) \otimes M \\
\downarrow & & \downarrow \psi \\
\mu & \to & \bigoplus_j \Gamma(U_j \cap U_i; F) \otimes M
\end{array}
\quad
\begin{array}{ccc}
\lambda' & \to & \bigoplus_j \Gamma(U_j; F \otimes M_X) \\
\mu' & \to & \bigoplus_j \Gamma(U_j \cap U_i; F \otimes M_x)
\end{array}

Observe also that for every $x \in X$ we have

$$\varprojlim_U (\Gamma(U; F) \otimes M) \simeq \varprojlim_U \Gamma(U; F \otimes M_X),$$

where $U$ ranges through the family of open quasi-compact neighborhoods of $x$. In fact, both sides of this auxiliary isomorphism are isomorphic to $F_x \otimes M$. Thus, if $s \in \Gamma(X; F) \otimes M$ is such that $\varphi(s) = 0$, then we can find, by the auxiliary isomorphism and quasi-compactness of $X$, a finite covering $X = \bigcup_j U_j$ by open quasi-compact neighborhoods such that $\lambda(s) = 0$. Therefore, $s = 0$ and $\varphi$ is injective. If we apply the same argument to $U_j$ and $U_j \cap U_i$ instead of $X$ we see that $\psi$ and $\vartheta$ are also injective.

To show that $\varphi$ is surjective, take $t \in \Gamma(X; F \otimes M_X)$. By the auxiliary isomorphism above, there exists a finite covering $X = \bigcup_j U_j$ by open quasi-compact neighborhoods such that $\lambda'(t)$ is in the image of $\psi$. But by injectivity of $\vartheta$ it follows that $t$ is in the image of $\varphi$.

By the Fiber formula (Corollary 3.13) and Lemma 3.27 we have:

**Proposition 3.28 (Projection formula).** Let $f : X \to Y$ be a morphism in $\tilde{A}$, $F \in \text{Mod}(A_X)$ and $G \in \text{Mod}(A_Y)$. If $G$ is flat, then the natural morphism

$$f_* F \otimes G \to f_* (F \otimes f^{-1} G)$$

is an isomorphism.

Proof. The morphism $f^{-1} \circ f_* \to f^{-1} \circ f_* \to \text{id}$ induces the morphism $f^{-1}(f_* F \otimes G) \simeq f^{-1} \circ f_* F \otimes f^{-1} G \to F \otimes f^{-1} G$ and we obtain the morphism $f_* F \otimes G \to f_* (F \otimes f^{-1} G)$ by adjunction.
To prove that it is an isomorphism, let $\alpha \in Y$. Then
\[
(f_! (F \otimes f^{-1}G))_\alpha \cong \Gamma_c (f^{-1}(\alpha); (F \otimes f^{-1}G)|_{f^{-1}(\alpha)}) \\
\cong \Gamma_c (f^{-1}(\alpha); F_{|f^{-1}(\alpha)} \otimes (f^{-1}G)|_{f^{-1}(\alpha)}) \\
\cong \Gamma_c (f^{-1}(\alpha); F_{|f^{-1}(\alpha)} \otimes G_\alpha) \\
\cong \Gamma_c (f^{-1}(\alpha); F_{|f^{-1}(\alpha)} \otimes G) \\
\cong (f_! F)_\alpha \otimes G_\alpha \\
\cong (f_\times F)_\alpha \otimes G_\alpha,
\]
by the Fiber formula (Corollary 3.13), Lemma 3.27 and using also the fact that $(f^{-1}G)|_{f^{-1}(\alpha)} \cong G_\alpha$.

We now proceed to the proof of the base change formula. By Lemma 2.14, we have:

**Remark 3.29.** Let $f : X \to Y$ be a morphism in $\widetilde{\text{Def}}$ and $\alpha \in Y$. Let $a \models \alpha$ a realization of $\alpha$ and $S$ a prime model of the theory of $M$ over $\{a\} \cup M$. Then we have a commutative diagram
\[
\begin{array}{ccc}
(f_\times)^{-1}(a) & \to & X(S) \\
\downarrow r_\times & & \downarrow r \\
\downarrow f_\times & & \downarrow r \\
(f_\times)^{-1}(\alpha) & \to & X \\
\end{array}
\]
with the restriction $r_\times$ a homeomorphism (Lemma 2.14). If $F \in \text{Mod}(A_X)$ then
\[
F(S)|_{(f_\times)^{-1}(\alpha)} = (r_\times)^{-1}F|_{f_\times^{-1}(\alpha)}
\]
where here and below we use the notation $F(S) = r^{-1}F$.

**Lemma 3.30.** Consider a cartesian square in $\widetilde{\text{Def}}$
\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow g' & & \downarrow g \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Let $\gamma \in Y'$, $v \models \gamma$ a realization of $\gamma$ and $S$ a prime model of the first-order theory of $M$ over $\{v\} \cup M$. Let $u = g'v$ (so $u \models g(\gamma)$) is a realization of $g(\gamma)$). Then we have a commutative diagram
\[
\begin{array}{ccc}
(f_\times)^{-1}(v) & \to & X'(S) \\
\downarrow g_\times & & \downarrow g' \\
\downarrow f_\times & & \downarrow g \\
(f_\times)^{-1}(u) & \to & X(S) \\
\end{array}
\]
in $\widetilde{\text{Def}}(S)$ with $g_\times : (f_\times)^{-1}(v) \to (f_\times)^{-1}(u)$ an homeomorphism.
Proof. Indeed, after removing the tilde, we have a similar commutative diagram of continuous $S$-definable maps between $S$-definable spaces with the restriction to the $S$-definable fibers an $S$-definable homeomorphism. □

By Lemmas 2.14 and 3.30 we have:

Lemma 3.31. Consider a cartesian square in $\overline{\Def}$

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Let $\gamma \in Y'$, $v \models \gamma$ a realization of $\gamma$ and $S$ a prime model of the first-order theory of $M$ over $\{v\} \cup M$. Let $u = g^S(v)$ (so $u \models g(\gamma)$) and let $K$ is a prime model of the first-order theory of $M$ over $\{u\} \cup M$. If $K = S$, then we have a homeomorphism

\[g'_S : f'^{-1}(\gamma) \to f^{-1}(g(\gamma))\]

which induces an isomorphism

\[\Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}) \simeq \Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))})\]

for every $F \in \text{Mod}(A_X)$.

Proof. We have the following commutative diagram

\[
\begin{array}{ccc}
(f^S)^{-1}(v) & \xrightarrow{c} & X'(S) \\
\downarrow{g'^S} & & \downarrow{g^S} \\
(f^S)^{-1}(u) & \xrightarrow{r} & X(S) \\
\end{array}
\]

Since $g'^S : (f^S)^{-1}(v) \to (f^S)^{-1}(u)$ is a homeomorphism by Lemma 3.30 and the restrictions $(r_1)^{-1} : (f^S)^{-1}(v) \to f'^{-1}(\gamma)$ and $(r_1)^{-1} : (f^S)^{-1}(u) \to f^{-1}(g(\gamma))$ are also homeomorphisms (Lemma 2.14 and $K = S$) the result follows. □

From now on until the end of the subsection we need to assume also (A3).

Lemma 3.32. Consider a cartesian square in $\overline{\Def}$

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Suppose that $f : X \to Y$ satisfies (A3).

If $\gamma \in Y'$, then there exists an isomorphism

\[\Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}) \simeq \Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))})\]

for every $F \in \text{Mod}(A_X)$. 

Proof. Let $v \models \gamma$ be a realization of $\gamma$ and $S$ a prime model of the first-order theory of $\mathcal{M}$ over $\{v\} \cup M$. Set $u = g^3(v)$ and note that $u \models g(\gamma)$ is a realization of $g(\gamma)$. Let $K$ is a prime model of the first-order theory of $\mathcal{M}$ over $\{u\} \cup M$. Thus since $u \in S$, by Fact 2.13, we have either $K = \mathcal{M}$ and $\mathcal{M} \neq S$ or $K = S$. So we proceed with the proof by considering the two cases.

Case $K = \mathcal{M}$ and $\mathcal{M} \neq S$: We have $u = g(\gamma)$. Then we have
\[
\Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))}) = \Gamma_c(f^{-1}(u); F_{|f^{-1}(u)}) \\
\simeq \Gamma_c((f^0)^{-1}(u); F(\mathcal{S})_{|(f^0)^{-1}(u)}) \\
\simeq \Gamma_c((f^0)^{-1}(v); (g^0)^{-1}F(\mathcal{S})_{|(f^0)^{-1}(v)}) \\
\simeq \Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}),
\]
where the first isomorphism follows by (A3), the second follows from Lemma 3.30 and the third follows from Lemma 2.14 together with Remark 3.29.

Case $K = S$: Then by Lemma 3.31 we have
\[
\Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))}) \simeq \Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}).
\]

We are now ready to prove the base change formula:

**Proposition 3.33 (Base change formula).** Consider a cartesian square in $\overline{\text{Def}}$

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]

and let $F \in \text{Mod}(A_X)$. Suppose that $f : X \to Y$ and $f' : X' \to Y'$ are in $\overline{A}$ and that $f : X \to Y$ satisfies (A3). Then
\[
g^{-1} \circ f'_* F \simeq f'_* \circ g'^{-1} F.
\]

**Proof.** Let us construct the morphism $g^{-1} \circ f'_* \to f'_* \circ g'^{-1}$. We shall construct first the morphism
\[
f'_* \circ g'_* \to g_* \circ f'_*.
\]

Let $U \in \text{Op}^{\text{cons}}(Y)$ and $G \in \text{Mod}(A_X)$. A section $t \in \Gamma(U; f_! \circ g'_* G)$ is defined by a section $s \in \Gamma((f \circ g')^{-1}(U); G)$ such that $\text{supp}(s) \subset g'^{-1}(Z)$ for a closed constructible subset $Z$ of $f^{-1}(U)$ such that $f_{1|Z} : Z \to U$ is proper. Since
\[
\begin{array}{ccc}
g'^{-1}(Z) & \xrightarrow{f'_1} & g^{-1}(U) \\
\downarrow{s'_1} & & \downarrow{g_1} \\
Z & \xrightarrow{f_1} & U
\end{array}
\]

is a cartesian square in $\overline{\text{Def}}$, $(f \circ g')^{-1}(U) = (g \circ f')^{-1}(U)$ and by Proposition 2.8 (5), the restriction $f'_{|g'^{-1}(Z)} : g'^{-1}(Z) \to g^{-1}(U)$ is proper. Therefore, $s \in \Gamma((g \circ
Note also that by Theorem 3.42, the functor \( Rf \) is proper. Then by the Fiber formula (Corollary 3.13) and

\[
\gamma = (f_1 F)_{g(g)} \quad \text{and} \quad (f'_1 \circ g'^{-1} F)_{g(g)} = \Gamma_c(f^{-1}(g(g)); F_{f^{-1}(g(g)))}
\]

Then by the Fiber formula (Corollary 3.13)

\[
(g^{-1} \circ f_1 F)_{g} = (f_1 F)_{g(g)} \quad \text{and} \quad \Gamma_c(f^{-1}(g(g)); F_{f^{-1}(g(g)))}
\]

To prove that \( g^{-1} \circ f_1 F \to f'_1 \circ g'^{-1} F \) is an isomorphism, let us take \( \gamma \in Y' \). But this is proved in Lemma 3.32.

3.5. Derived proper direct image. Here we derive the proper direct image and prove the derived projection and base change formulas. As corollaries we obtain the universal coefficients formula and the Künneth formula.

Remark 3.34. Since \( A \) is a commutative ring with finite weak global dimension, by the observations on page 110 in [37], if \( F \in D^b(A_X) \) (resp. \( F \in D^+(A_X) \)), then \( F \) is quasi-isomorphic to a bounded complex (resp. a complex bounded form below) of flat \( A \)-sheaves. Therefore, we may define the left derived functor

\[
\bullet \otimes \bullet : D^*(A_X) \times D^*(A_X) \to D^*(A_X)
\]

with \( * = -, +, b \).

Let \( f : X \to Y \) be a morphism in \( \tilde{A} \). We are going to consider the right derived functor of proper direct image

\[
Rf_1 : D^+(A_X) \to D^+(A_Y).
\]

If \( f \in D^+(A_X) \) and \( F' \) is a complex of \( f \)-soft sheaves quasi-isomorphic to \( F \) (which exists by Proposition 3.26), then

\[
Rf_1 F \simeq f_1 F'.
\]

Furthermore, if \( g : Y \to Z \) is another morphism in \( \tilde{A} \), then by Proposition 3.23,

\[
R(g \circ f)_1 \simeq Rg_1 \circ Rf_1.
\]

Note also that by Theorem 3.42, the functor \( Rf_1 \) induces a functor:

\[
Rf_1 : D^b(A_X) \to D^b(A_Y).
\]

Deriving the projection formula (Proposition 3.28) we have:
Theorem 3.35 (Derived projection formula). Let $f : X \to Y$ be a morphism in $\tilde{A}$. Let $F \in D^+(A_X)$ and $G \in D^+(A_Y)$. Then there is a natural isomorphism

$$Rf_! F \otimes G \cong Rf_!(F \otimes f^{-1}G).$$

Proof. First note that, if $G$ is flat, then by Lemma 3.27, $\bullet \otimes f^{-1}G$ sends $f$-soft sheaves to $f$-soft sheaves. (Indeed, for every $\alpha \in Y$, the restriction $(\bullet)|_{f^{-1}(\alpha)}$ commutes with $\otimes$ and $(f^{-1}G)|_{f^{-1}(\alpha)} = G_\alpha$ which is also flat.)

Now let $F \in D^+(A_X)$ and $G \in D^+(A_Y)$. Let $F'$ be a complex of $f$-soft sheaves quasi-isomorphic to $F$ (Proposition 3.26). By Remark 3.34, there exists a complex $G'$ of flat sheaves quasi-isomorphic to $G$. Then $F' \otimes f^{-1}G'$ is a complex of $f$-soft sheaves quasi-isomorphic to $F \otimes f^{-1}G$. Therefore, by Proposition 3.26,

$$Rf_! F \otimes G \cong f'_!(F' \otimes f^{-1}G') \cong Rf_!(F \otimes f^{-1}G),$$

where the second isomorphism follows from Proposition 3.28. \hfill \square

Corollary 3.36 (Universal coefficients formula). Let $X$ be an object of $\tilde{\text{Def}}$. Let $M$ be a $A$-module. Suppose that $c$ is a normal and constructible family of supports on $X$. Then there is an isomorphism

$$R\Gamma_c(X; M_X) \cong R\Gamma_c(X; A_X) \otimes M.$$

Proof. In the above proof if we use [24, Proposition 3.4] (resp. Lemma 3.27) instead of Proposition 3.26 (resp. Proposition 3.28), we obtain a natural isomorphism $R\Gamma_c(X; F) \otimes M \cong R\Gamma_c(F \otimes M_X)$. Since $M_X = a_X^{-1}M_{pt} \otimes A_X$, the result follows. \hfill \square

Corollary 3.37. Let $X$ be an object of $\tilde{\text{Def}}$. Let $M$ be a $A$-module. Suppose that $c$ is a normal and constructible family of supports on $X$. Then there is an exact sequence for each $p \in \mathbb{Z}$

$$0 \to H^p_c(X; A_X) \otimes M \to H^p_c(X; M_X) \to \text{Tor}_1(H^{p+1}_c(X; A_X), M) \to 0.$$

Recall that $F$ is acyclic with respect to a left exact functor $\varphi$ if $R^k\varphi F = 0$ if $k \neq 0$. In such a situation $F$ is also $\varphi$-injective.

In order to prove the derived base change formula, we need the following lemma in which we assume (A3):

Lemma 3.38. Consider a cartesian square

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}$$

in $\tilde{\text{Def}}$. Suppose that $f : X \to Y$ satisfies (A3). Then $g^{-1}(\bullet)$ sends $f$-soft sheaves to $f'_!$-acyclic sheaves.
Proof. Let $F \in \text{Mod}(A_X)$ be $f$-soft. Then $F_{|f^{-1}(\alpha)}$ is $\Gamma_c(f^{-1}(\alpha);\bullet)$-acyclic for every $\alpha \in Y$. We must show that the restriction $(g^{-1}F)_{|f^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma);\bullet)$-acyclic for every $\gamma \in Y'$.

So take $\gamma \in Y'$. Let $v \models \gamma$ be a realization of $\gamma$ and $S$ a prime model of the first-order theory of $\mathbb{M}$ over $\{v\} \cup M$. Set $u = g^S(v)$ and note that $u \models g(\gamma)$ is a realization of $g(\gamma)$. Let $K$ be a prime model of the first-order theory of $\mathbb{M}$ over $\{u\} \cup M$. Thus since $u \in S$, by Fact 2.13, we have either $K = \mathbb{M}$ and $\mathbb{M} \neq S$ or $K = S$. So we proceed with the proof by considering the two cases.

Case $K = \mathbb{M}$ and $\mathbb{M} \neq S$: We have $u = g(\gamma)$ and $F_{|f^{-1}(g(\gamma))} = F_{|f^{-1}(u)}$. Since $F_{|f^{-1}(u)}$ is $\Gamma_c(f^{-1}(u);\bullet)$-acyclic, by (A3), $F(S)_{|(f^S)^{-1}(u)}$ is $\Gamma_c((f^S)^{-1}(u);\bullet)$-acyclic. Since $(f^S)^{-1}(v) \rightarrow (f^S)^{-1}(u)$ is a homeomorphism (Lemma 3.30) and on the other hand, $((f^S)^{-1}F(S))_{|(f^S)^{-1}(u)} = (g^S)^{-1}((f^S)^{-1}(u))$, we conclude that $((f^S)^{-1}F(S))_{|(f^S)^{-1}(u)}$ is $\Gamma_c((f^S)^{-1}(v);\bullet)$-acyclic. As $(\gamma)^{-1} : f'^{-1}(\gamma) \rightarrow (f^S)^{-1}(v)$ is a homeomorphism (Lemma 2.14), by Remark 3.29, we have that $(g^{-1}F)_{|f'^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma);\bullet)$-acyclic.

Case $K = S$: Since $g^S : f'^{-1}(\gamma) \rightarrow f^{-1}(g(\gamma))$ is a homeomorphism (Lemma 3.31), $(g^{-1}F)_{|f'^{-1}(\gamma)} = (g^S)^{-1}(F_{|f^{-1}(g(\gamma))})$ and $F_{|f^{-1}(g(\gamma))}$ is $\Gamma_c(f^{-1}(g(\gamma));\bullet)$-acyclic, we have that $(g^{-1}F)_{|f'^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma);\bullet)$-acyclic. \hfill $\Box$

Let $f : X \rightarrow Y$ be a morphism in $\overline{A}$. The full additive subcategory of $\text{Mod}(A_X)$ of $f'$-acyclic sheaves is $f'$-injective. Therefore, if $F \in D^+(A_X)$ and $F'$ is a complex of $f'$-acyclic sheaves quasi-isomorphic to $F$, then

$$Rf_!F \simeq f'_!F'.$$

Deriving the base change formula (Proposition 3.33) we have:

**Theorem 3.39** (Derived base change formula). Consider a cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
$$

in $\overline{\text{Def}}$. Suppose that $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ are in $\overline{A}$ and that $f : X \rightarrow Y$ satisfies (A3). Then there is an isomorphism in $D^+(A_{Y'})$, functorial in $F \in D^+(A_X)$:

$$g^{-1} \circ Rf_!F \simeq Rf'_! \circ g'^{-1}F.'$$

Proof. Let $F \in D^+(A_X)$ and let $F'$ be a complex of $f'$-soft sheaves quasi-isomorphic to $F$. By Lemma 3.38, $g^{-1}$ sends $f_!$-soft sheaves to $f'_!$-acyclic sheaves and the $f'_!$-acyclic sheaves are $f'_!$-injective. So $g^{-1}F'$ is a complex of $f'_!$-acyclic sheaves quasi-isomorphic to $g^{-1}F$. Therefore, by Proposition 3.26 and Proposition
3.33 we have
\[ g^{-1} \circ Rf_1 F \simeq g^{-1} \circ f_1 F' \]
\[ \simeq f'_1 \circ g'^{-1} F' \]
\[ \simeq Rf'_1 \circ g'^{-1} F. \]

Combining the the derived projection and base change formulas we obtain:

**Theorem 3.40** (Küneth formula). Consider a cartesian square

\[
\begin{array}{ccc}
  X' & \xrightarrow{f'} & Y' \\
  \downarrow{\delta} & & \downarrow{g} \\
  X & \xrightarrow{f} & Y
\end{array}
\]

in $\tilde{\text{Def}}$ where $\delta = f \circ g' = g \circ f'$. Suppose that $f : X \to Y$ and $f' : X' \to Y'$ are in $\tilde{A}$ and that $f : X \to Y$ satisfies $(A3)$. There is a natural isomorphism

\[ R\delta_!(g'^{-1} F \otimes f'^{-1} G) \simeq Rf'_1 F \otimes Rg'_1 G \]

for $F \in D^+(A_X)$ and $G \in D^+(A_{Y'})$.

**Proof.** Using the derived projection formula and the derived base change formula we deduce

\[ Rf'_1 (g'^{-1} F \otimes f'^{-1} G) \simeq (Rf'_1 \circ g'^{-1} F) \otimes G \simeq (g^{-1} \circ Rf_1) \otimes G. \]

Using the derived projection formula once again we find

\[ Rg'_1 \circ Rf'_1 (g'^{-1} F \otimes f'^{-1} G) \simeq Rg'_1 ((g^{-1} \circ Rf_1 F) \otimes G) \simeq Rf'_1 F \otimes Rg'_1 G \]

and the result follows since $R\delta_! \simeq Rg'_1 \circ Rf'_1$. \qed

**Corollary 3.41.** Consider the cartesian square

\[
\begin{array}{ccc}
  X \times Y & \xrightarrow{p_Y} & Y \\
  \downarrow{p_X} & & \downarrow{a_Y} \\
  X & \xrightarrow{a_X} & \text{pt}
\end{array}
\]

in $\tilde{A}$. Suppose that $a_X : X \to \text{pt}$ satisfies $(A3)$. Then for any $k \in \mathbb{Z}$ there is a natural isomorphism

\[ H^c_c(X \times Y; A_{X \times Y}) \simeq \bigoplus_{p+q=k} (H^p_c(X; A_X) \otimes H^q_c(Y; A_Y)). \]

**Proof.** We have $A_{X \times Y} \simeq p_X^{-1} A_X \simeq p_Y^{-1} A_Y$. By Theorem 3.40 we obtain $Ra_{X \times Y} A_{X \times Y} \simeq Ra_X A_X \otimes Ray_A A_Y$. \qed
3.6. A bound for the cohomology of proper direct image. Here we finding a bound for the cohomology of the proper direct image.

The cohomological dimension of $f_!$ is the smallest $n$ such that $R^k f_! F = 0$ for all $k > n$ and all sheaves $F$ in $\text{Mod}(A_X)$.

**Theorem 3.42.** Let $f : X \to Y$ be a morphism in $\tilde{A}$. Then the cohomological dimension of $f_!$ is bounded by $\dim X$.

**Proof.** Let $F \in \text{Mod}(A_X)$ and let $\alpha \in Y$. Taking a $f$-soft resolution of $F$ (Proposition 3.26) one checks easily that $R^k f_! F_\alpha \simeq R\Gamma_c(f^{-1}(\alpha); F_{f^{-1}(\alpha)})$ (the Fiber formula - Corollary 3.13).

By the method of Remark 3.19 in combination with [24, Theorem 3.12], we see that $H^k(Rf_! F)_\alpha \simeq (R^k f_! F)_\alpha \simeq R^k \Gamma_c(f^{-1}(\alpha); F_{f^{-1}(\alpha)}) = 0$ if $k > \dim X$. Since $\alpha$ was arbitrary the result follows. \qed

4. **Poincaré-Verdier Duality**

In this section we prove the local and the global Verdier duality, we introduce the $A$-orientation sheaf and prove the Poincaré and the Alexander duality.

The proofs here follow in a standard way from the results already obtained in the previous section. Compare with the topological case in [37, Chapter III, Sections 3.1 and 3.3].

4.1. **Poincaré-Verdier duality.** Here we show that the derived proper direct image functor $Rf_!$ extends to $D(A_X)$ and both $Rf_!$ and its extension have a right adjoint $f^!$. We then deduce the basic properties of the right adjoint $f^!$ and the local and the global Verdier duality.

Let $f : X \to Y$ be a morphism in $\tilde{A}$. Let $\mathcal{J}$ be the full additive subcategory of $\text{Mod}(A_X)$ of $f$-soft sheaves. As a consequence of the results we proved for $f_!$ and for $f_!$-acyclic sheaves we have the following properties:

\[
\begin{aligned}
\mathcal{J} & \text{ is cogenerating;} \\
\mathcal{J} & \text{ is } f_! \text{-injective;} \\
\mathcal{J} & \text{ is stable under small } \oplus; \\
f_! & \text{ commutes with small } \oplus.
\end{aligned}
\]

Therefore, by [39, Proposition 14.3.4] the functor $Rf_! : D^+(A_X) \to D^+(A_Y)$ extends to a functor $Rf_! : D(A_X) \to D(A_Y)$ such that:

(i) for every $F \in D(A_X)$ we have

\[Rf_! F \simeq f_! F'.\]
where $F'$ is a complex of $f_*$-acyclic sheaves quasi-isomorphic to $F$;
(ii) $Rf_!$ commutes with small ∗.

**Theorem 4.1.** Let $f : X \to Y$ be a morphism in $\bar{A}$. Then the functor $Rf_! : D(A_X) \to D(A_Y)$ admits a right adjoint

$$f_! : D(A_Y) \to D(A_X).$$

The functor $f_!$ will thus satisfy an isomorphism

$$\text{Hom}_{D(A_Y)}(Rf_! F; G) \simeq \text{Hom}_{D(A_X)}(F; f_! G)$$

functorial in $F \in D(A_X)$ and $G \in D(A_Y)$. Moreover, the restriction

$$f_! : D^+(A_Y) \to D^+(A_X)$$

is well defined and it is the right adjoint to the restriction $Rf_! : D^+(A_X) \to D^+(A_Y)$.

**Proof.** The existence of right adjoint $f_! : D(A_Y) \to D(A_X)$ is a consequence of the Brown representability theorem (see [39, Corollary 14.3.7] for details).

For the second part we have to show that if $G \in D^+(A_Y)$, then $f_! G \in D^+(A_X)$. We may assume that $G \in D^{\geq 0}(A_Y)$. Let $N_0$ be the dimension of $X$. Then the cohomological dimension of $f_!$ is bounded by $N_0$ (Theorem 3.42). Set $a = -N_0 - 1$ for short. If $F \in D^{\leq a}(A_X)$, then $Rf_! F \in D^{\leq -1}(A_Y)$ and

$$0 = \text{Hom}_{D(A_Y)}(Rf_! F; G) \simeq \text{Hom}_{D(A_X)}(F, f_! G).$$

Hence for each $F \in D^{\leq a}(A_X)$ we have $\text{Hom}_{D(A_X)}(F, f_! G) = 0$. In particular, if $F = \tau^{\leq a} f_! G$, then

$$\text{Hom}_{D(A_X)}(\tau^{\leq a} f_! G, f_! G) \simeq \text{Hom}_{D^{\leq a}(A_X)}(\tau^{\leq a} f_! G, \tau^{\leq a} f_! G) = 0$$

and so $\tau^{\leq a} f_! G = 0$. This implies, by definition, that $f_! G \in D^+(A_X)$. \(\square\)

The bound of the $f$-soft dimension of $\text{Mod}(A_X)$ implies the following result:

**Proposition 4.2.** Let $f : X \to Y$ be a morphism in $\bar{A}$. Let $F \in \text{Mod}(A_X)$, $G \in \text{Mod}(A_Y)$. Let $N_0$ be the dimension of $X$. Then

$$\text{Hom}(F, H^{N_0} f_! G) \simeq \text{Hom}(R^{N_0} f_! F, G).$$

**Proof.** Let $I^\bullet$ be a complex of injective objects quasi-isomorphic to $f_! G$. As in the proof of Theorem 4.1 we have $f_! G \simeq \tau^{\geq -N_0} I^\bullet$ and hence we have the exact sequence

$$0 \to H^{N_0} f_! G \to I^{-N_0} \to I^{-N_0 + 1}$$

that implies the isomorphism

$$\text{Hom}(F, H^{N_0} f_! G) \simeq \text{Hom}_{D^{+}(A_X)}(F, f_! G[-N_0]).$$

On the other hand the complex $Rf_! F[N_0]$ is concentrated in negative degree and hence

$$\text{Hom}_{D^{+}(A_Y)}(Rf_! F[N_0], G) \simeq \text{Hom}(R^{N_0} f_! F[N_0], G) \simeq \text{Hom}(R^{N_0} f_! F, G).$$

Then the result follows from Theorem 4.1. \(\square\)
Hence, $\ id \simeq f^!$ is (isomorphic to) the extension by zero functor. What about $f^!$?

**Proposition 4.3.** Let $f : X \to Y$ be a morphism in $\overline{A}$. If $f : X \to Y$ is a homeomorphism onto a locally closed subset of $Y$, then

$$f^! G \simeq f^{-1} \circ R\text{Hom}(A_f(X), G) \simeq f^{-1} \circ R\Gamma_{f(X)}(G)$$

for every $G \in D^+(A_Y)$. In particular:

- if $f : X \to Y$ is a closed immersion, then $\ id \simeq f^! \circ Rf_!$;
- if $f : X \to Y$ is an open immersion, then $f^! \simeq f^{-1}$.

**Proof.** Let $G \in D^+(A_Y)$ and $F \in D^+(A_X)$. Then

$$\text{Hom}_{D^+(A_Y)}(f^!_* F, G) \simeq \text{Hom}_{D^+(A_Y)}(f^!_* F, R\text{Hom}(A_f(X), G))$$

$$\simeq \text{Hom}_{D^+(A_X)}(f^{-1} \circ f^!_* F, f^{-1} \circ R\text{Hom}(A_f(X), G))$$

$$\simeq \text{Hom}_{D^+(A_X)}(F, f^{-1} \circ R\text{Hom}(A_f(X), G)).$$

Hence $f^! G \simeq f^{-1} \circ R\text{Hom}(A_f(X), G)$.

Suppose now that $f : X \to Y$ is a closed immersion. Then $f$ is proper, $Rf_* \simeq Rf_!$ and $f^{-1} \circ Rf_* \simeq \ id$. On the other hand, we have the isomorphisms

$$f^! \circ Rf_* F \simeq f^{-1} \circ R\text{Hom}(A_f(X), Rf_* F)$$

$$\simeq f^{-1} \circ Rf_* \text{Hom}(A_f(X), F)$$

$$\simeq f^{-1} \circ Rf_* F.$$

Hence, $\ id \simeq f^! \circ Rf_!$.

Suppose now that $f : X \to Y$ is an open immersion. Then $f^{-1} \circ Rf_* \simeq \ id$ and $R\Gamma_{f(X)} \simeq Rf_* \circ f^{-1}$. Hence, $f^! \simeq f^{-1}$. \(\square\)

We now prove several useful properties of the dual $f^!$ of the derived proper direct image functor $Rf_!$.

**Proposition 4.4.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\overline{A}$. Then $(g \circ f)^! \simeq f^! \circ g^!$.

**Proof.** This follows from $R(g \circ f)_! \simeq Rg_! \circ Rf_!$ and the adjunction in Theorem 4.1. \(\square\)

**Proposition 4.5 (Dual projection formula).** Let $f : X \to Y$ be a morphism in $\overline{A}$. Let $F \in D^b(A_Y)$ and $G \in D^+(A_Y)$. Then we have a natural isomorphism

$$f^! \circ R\text{Hom}(F, G) \simeq R\text{Hom}(f^{-1} F, f^! G).$$

**Proof.** This follows from the derived projection formula (Theorem 3.35), the adjunction in Theorem 4.1 and the adjunction

$$\text{Hom}_{D^+(A_Z)}(F \otimes H, G) \simeq \text{Hom}_{D^+(A_Z)}(H, R\text{Hom}(F, G)).$$

\(\square\)
Proposition 4.6 (Dual base change formula). Consider a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]

in $\tilde{\text{Def}}$. Suppose that $f : X \to Y$ and $f' : X' \to Y'$ are in $\tilde{A}$ and that $f : X \to Y$ satisfies (A3). Then there is an isomorphism in $\text{D}^+(A_X)$, functorial in $F \in \text{D}^+(A_Y)$:

\[
f^! \circ Rg_* F \simeq Rg'_* \circ f'^! F.
\]

**Proof.** This follows from the derived base change formula (Theorem 3.39), the adjunction in Theorem 4.1 and the adjunction

\[
\text{Hom}_{\text{D}^+(A_X)}(H, Rf_* G) \simeq \text{Hom}_{\text{D}^+(A_Y)}(h^{-1} H, G)
\]

for every morphism $h : X \to Z$ in $\tilde{A}$. □

Theorem 4.7 (Local and global Verdier duality). Let $f : X \to Y$ be a morphism in $\tilde{A}$. Then for $F \in \text{D}^0(A_X)$ and $G \in \text{D}^+(A_Y)$, we have the local Verdier duality

\[
Rf_* \circ R\text{Hom}(F, f^! G) \simeq R\text{Hom}(Rf_! F, G)
\]

and the global Verdier duality

\[
R\text{Hom}(F, f^! G) \simeq \text{Hom}(Rf_! F, G).
\]

**Proof.** We obtain the morphism

\[
Rf_* \circ R\text{Hom}(F, f^! G) \to R\text{Hom}(Rf_! F, G)
\]

by composing the canonical morphism

\[
Rf_* \circ R\text{Hom}(F, f^! G) \to R\text{Hom}(Rf_! F, Rf_! \circ f^! G)
\]

with the morphism $Rf_! \circ f^! G \to G$ obtained by adjunction (Theorem 4.1).

Let $V \in \text{Op}(Y)$. Then we have

\[
H^j(R\Gamma(V ; Rf_* \circ R\text{Hom}(F, f^! G))) \simeq \text{Hom}_{\text{D}^+(A_{f^{-1}(V)})}(F_{f^{-1}(V)}, f^! G[j]_{f^{-1}(V)})
\]

\[
\simeq \text{Hom}_{\text{D}^+(A_V)}(Rf_! F[V], G[j]_V)
\]

\[
\simeq H^j(R\Gamma(V ; R\text{Hom}(Rf_! F, G)))
\]

completing the proof of the first isomorphism. The second isomorphism is obtained from the first one by applying the functor $R\Gamma(Y ; \bullet)$. □

4.2. Orientation and duality. Here we introduce the $A$-orientation sheaf and prove the Poincaré and the Alexander duality theorems.

Remark 4.8. Let $X$ be an object of $\tilde{\text{Def}}$ such that $c$ is a normal and constructible family of supports on $X$. Then as the reader can easily verify all our previous results for the proper direct image $f_!$ of a morphism $f : X \to Y$ in $\tilde{A}$ that do not require (A3) hold also for the proper direct image $a_{X_!}$ of the morphism $a_X : X \to \text{pt}$. Indeed, (A0), (A1) and (A2) were used only to show:
(i) if \( \alpha \in Y \) is closed, then \( c = \Phi_f \cap f^{-1}(\alpha) \) and \( c \) is a normal and constructible family of supports on \( f^{-1}(\alpha) \); 
(ii) fiber formula; 
(iii) the theory of \( f \)-soft sheaves.

For the morphism \( a_X : X \to \text{pt} \) we have that (i) holds if we assume that \( c \) is a normal and constructible family of supports on \( X \), (ii) is Remark 3.4 and (iii) is the theory of \( c \)-soft sheaves developed already in [24, Section 3].

Let \( X \) be an object of \( \widetilde{\text{Def}} \) such that \( c \) is a normal and constructible family of supports on \( X \). Then the functor \( Ra_X : D^+(A_X) \to D^+(\text{Mod}(A)) \) admits a right adjoint

\[
a^!_X : D(\text{Mod}(A)) \to D(A_X)
\]

(Theorem 4.1) and we have the dual projection formula (Proposition 4.5) and local and the global Verdier duality (Theorem 4.7) for \( a^!_X \). In particular, we obtain the form of the global Verdier duality proved already in [24] (assuming \( A \) is a field), where \( a^!_XA_X \) is the dualizing complex:

**Theorem 4.9** (Absolute Poincaré duality). Let \( X \) be an object of \( \widetilde{\text{Def}} \) such that \( c \) is a normal and constructible family of supports on \( X \). Then we have a natural isomorphism

\[
R\text{Hom}(F, a^!_X A) \simeq R\text{Hom}(R\Gamma_c(X; F), A)
\]
as \( F \) varies through \( D^b(A_X) \).

Let \( X \) be an object of \( \widetilde{\text{Def}} \) such that \( c \) is a normal and constructible family of supports on \( X \). We say that \( X \) has an \( A \)-orientation sheaf if for every \( U \in \text{Op}^{\text{cons}}(X) \) there exists an admissible (finite) cover \( \{U_1, \ldots, U_\ell\} \) of \( U \) such that for each \( i \) we have

\[
H^p_{c}(U_i; A_X) = \begin{cases} 
A & \text{if } p = \dim X \\
0 & \text{if } p \neq \dim X.
\end{cases}
\]

(1)

If \( X \) has an \( A \)-orientation sheaf, we set \( \mathcal{O}_X := H^{-\dim X}a^!_X A \) and call it the \( A \)-orientation sheaf on \( X \). For \( A = \mathbb{Z} \) we simply say orientation sheaf on \( X \).

**Theorem 4.10.** Let \( X \) be an object of \( \widetilde{\text{Def}} \) of dimension \( n \) such that \( c \) is a normal and constructible family of supports on \( X \). Suppose that \( X \) has an \( A \)-orientation sheaf \( \mathcal{O}_X \). Then \( \mathcal{O}_X \) is the \( A \)-sheaf on \( X \) with sections

\[
\Gamma(U; \mathcal{O}_X) \simeq \text{Hom}(H^n_c(U; A_X), A).
\]

Moreover \( \mathcal{O}_X \) is locally constant and there is a quasi-isomorphism

\[
\mathcal{O}_X[n] \simeq a^!_X A.
\]
**Proof.** Setting $f = a_X, F = A_U, G = A$ in Proposition 4.2 we obtain
\[ \Gamma(U; \mathcal{O}_X) \simeq \text{Hom}(H^n_{\mathbb{Z}}(X; A_U), A) \simeq \text{Hom}(H^n_c(U; A_X), A), \]
where the second isomorphism follows from [24, Corollary 3.9].

On the other hand, (1), implies $R\Gamma(U; a_X^l A) \simeq R\text{Hom}(R\Gamma_c(U; A_X), A) \simeq A[-n]$, i.e. $a_X^l A$ is concentrated in degree $-n$ and the sheaf $H^{-n}a_X^l A$ is locally isomorphic to $A_X$.

\[ \square \]

In particular we recover [24, Theorem 4.11]:

**Remark 4.11** (Poincaré duality in cohomology). When $A$ is a field, setting $(\bullet)^\vee = \text{Hom}(\bullet, A)$ we obtain:
\[ H^p(X; \mathcal{O}_X) \simeq H_{n-p}^c(X; A_X)^\vee. \]

Using the pure homological algebra result [35, Proposition VI.4.6] we also have:

**Corollary 4.12.** Let $X$ be an object of $\text{Def}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ has a $\mathbb{Z}$-orientation sheaf $\mathcal{O}_X$. Then there is a short exact sequence of abelian groups:
\[ 0 \rightarrow \text{Ext}^1(H^{k+1}_c(X; \mathbb{Z}_X), \mathbb{Z}) \rightarrow H^{n-k}(X; \mathcal{O}_X) \rightarrow \text{Hom}(H^k_c(X, \mathbb{Z}_X), \mathbb{Z}) \rightarrow 0. \]
In particular $H^{n-k}(X; \mathcal{O}_X) \simeq \text{Hom}(H^k_c(X, \mathbb{Z}_X), \mathbb{Z})$ when $H^{k+1}_c(X, \mathbb{Z}_X)$ has no torsion.

**Proof.** By the pure homological algebra result [35, Proposition VI.4.6] we have
\[ 0 \rightarrow \text{Ext}^1(H^{k+1}C^\bullet, \mathbb{Z}) \rightarrow H^{-k}R\text{Hom}(C^\bullet, \mathbb{Z}) \rightarrow \text{Hom}(H^kC^\bullet, \mathbb{Z}) \rightarrow 0 \]
for any bounded complex $C^\bullet$ of abelian groups. Applying this to $C^\bullet = R\Gamma_c(X; \mathbb{Z}_X)$ and using $R\text{Hom}(R\Gamma_c(X; \mathbb{Z}_X), \mathbb{Z}) \simeq R\text{Hom}(\mathbb{Z}_X, a_X^l \mathbb{Z}) \simeq R\Gamma(X; \mathcal{O}_X[n])$ (by Theorems 4.9 and 4.10) the result follows.

\[ \square \]

**Corollary 4.13.** Let $X$ be an object of $\text{Def}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ has a $\mathbb{Z}$-orientation sheaf $\mathcal{O}_X$. Then there exists an isomorphism
\[ H^n(X; \mathcal{O}_X) \simeq \text{Hom}(H^0_c(X, \mathbb{Z}_X), \mathbb{Z}) \simeq \mathbb{Z}^l \]
where $l$ is the number of complete connected components of $X$.

**Proof.** By Corollary 4.12 (with $k = 0$) and since $H^0_c(X, \mathbb{Z}_X) = \mathbb{Z}^l$ where $l$ is the number of complete connected components of $X$, the result follows once we show that $H^1_c(X; \mathbb{Z}_X)$ is torsion free. But this is [5, Chapter I, Exercise 11 and Chapter II, Exercise 28].

\[ \square \]

By an $A$-orientation we understand an isomorphism $A_X \simeq \mathcal{O}_X$. We shall say that $X$ is $A$-orientable if an $A$-orientation exists and $A$-unorientable in the opposite case. For $A = \mathbb{Z}$ we simply say orientation, orientable or unorientable.

From [24, Theorem 3.12] (dim $X$ is a bound on the cohomological $c$-dimension of $X$) and Corollary 4.12, arguing as in [24, Proposition 4.13] we have:
Proposition 4.14. Let $X$ be an object of $\widetilde{\text{Def}}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ has an orientation sheaf $\mathcal{O}_r$. Then

1. $H^n_\mathcal{C}(X;\mathbb{Z}_X) \simeq \mathbb{Z}$ if $X$ is orientable.
2. $H^n_\mathcal{C}(X;\mathbb{Z}_X) \simeq 0$ if $X$ is unorientable.

If $Z$ a closed constructible subset of $X$, then setting $F = A_Z, G = A$ in Theorem 4.9 we obtain:

Theorem 4.15 (Alexander duality). Let $X$ be an object of $\widetilde{\text{Def}}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ is $A$-orientable. If $Z$ a closed constructible subset of $X$, then there exists a quasi-isomorphism

$$R\Gamma_Z(X;A_X) \simeq R\text{Hom}(R\Gamma_c(Z;A_X),A)[n].$$

In particular we recover [24, Theorem 4.14]:

Remark 4.16 (Alexander duality in cohomology). When $A$ is a field, setting $(\bullet)^\vee = \text{Hom}(\bullet,A)$ we obtain:

$$H^p_\mathcal{C}(X;A_X) \simeq H^{n-p}(Z;A_X)^\vee.$$

Using the pure homological algebra result [35, Proposition VI.4.6] we also have:

Corollary 4.17. Let $X$ be an object of $\widetilde{\text{Def}}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ is $\mathbb{Z}$-orientatable. Then there is a short exact sequence of abelian groups:

$$0 \to \text{Ext}^1(H^{k+1}_\mathcal{C}(Z;\mathbb{Z}_X),\mathbb{Z}) \to H^{n-k}_\mathcal{Z}(X;\mathbb{Z}_X) \to \text{Hom}(H^k_\mathcal{C}(Z;\mathbb{Z}_X),\mathbb{Z}) \to 0.$$

In particular $H^{n-k}_\mathcal{Z}(X;\mathbb{Z}_X) \simeq \text{Hom}(H^k_\mathcal{C}(Z;\mathbb{Z}_X),\mathbb{Z})$ when $H^{k+1}_\mathcal{C}(Z;\mathbb{Z}_X)$ has no torsion.

Proof. By the pure homological algebra result [35, Proposition VI.4.6] we have

$$0 \to \text{Ext}^1(H^{k+1}(C^\bullet,\mathbb{Z}) \to H^{n-k}R\text{Hom}(C^\bullet,\mathbb{Z}) \to \text{Hom}(H^kC^\bullet,\mathbb{Z}) \to 0$$

for any bounded complex $C^\bullet$ of abelian groups. Applying this to $C^\bullet = R\Gamma_c(Z;\mathbb{Z}_X)$ and using $R\text{Hom}(R\Gamma_c(Z;\mathbb{Z}_X),\mathbb{Z})[n] \simeq R\Gamma_Z(X;\mathbb{Z}_X)$ (by Theorem 4.15) the result follows.

Corollary 4.18. Let $X$ be an object of $\widetilde{\text{Def}}$ of dimension $n$ such that $c$ is a normal and constructible family of supports on $X$. Suppose that $X$ is $\mathbb{Z}$-orientatable. Then there exists an isomorphism

$$H^n_X(X;\mathbb{Z}_X) \simeq \text{Hom}(H^n_\mathcal{C}(Z;\mathbb{Z}_X),\mathbb{Z}) \simeq \mathbb{Z}^l$$

induced by the given orientation, where $l$ is the number of complete connected components of $Z$. 
Proof. By Corollary 4.17 (with $k = 0$) and since $H^0_c(Z; Z_X) = Z^l$ where $l$ is the number of complete connected components of $Z$, the result follows once we show that $H^1_c(Z; Z_X)$ is torsion free. But this is [5, Chapter I, Exercise 11 and Chapter II, Exercise 28]. □

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