Finite-time stability and stabilization of linear discrete
time-varying stochastic systems

Tianliang Zhang 1, Feiqi Deng 1 *, and Weihai Zhang 2

1 School of Automation Science and Engineering,
South China University of Technology, Guangzhou 510640, P. R. China
2 College of Information and Electrical Engineering,
Shandong University of Science and Technology, Qingdao 266510, P. R. China

Abstract- This paper studies the finite-time stability and stabilization of linear discrete
time-varying stochastic systems with multiplicative noise. Firstly, necessary and sufficient con-
ditions for finite-time stability are presented via state transition matrix approach. Secondly, this
paper also develops the Lyapunov function method to study finite-time stability and stabiliza-
tion of discrete time-varying stochastic systems based on matrix inequalities and linear matrix
inequalities (LMIs), so as to Matlab LMI Toolbox can be used. Two numerical examples are
given to illustrate the effectiveness of the proposed results.

Keywords: Finite-time stability, stochastic systems, multiplicative noise, state transition
matrix, Lyapunov function.

1 Introduction

As it is well-known that stability is the first consideration in system analysis and synthesis.
Since A. M. Lyapunov published his classical work [14] on stability of ordinary differential
equations (ODEs) in 1892, Lyapunov’s stability theory has been one of the most important
issues in mathematics and modern control theory. In particular, Lyapunov’s second method has
been extended to continuous-time stochastic Itô-type differential equations, and we refer the
reader to [7,9,16]. Stability in Lyapunov sense describes the asymptotic behaviour of the state
trajectory as time approaches infinity. However, in practice, even a system is stable, it may
be totally useless, because it possesses unsatisfactory transient performance. So, one should be interested in not only classical Lyapunov stability, but also finite-time transient performance. To this end, finite-time stability was proposed in 1950s \[5,11\]. Finite-time stability is different from classical stability in two aspects: First, the concerned system operation is confined to a prescribed finite-time interval instead of an infinite-time horizon. Second, the state trajectory lies within a specific bound over the given finite interval of time. Recently, finite-time stability has become a popular research topic due to its practical sense. Many nice results have been obtained, and we refer the reader to \[1,12\] for linear deterministic systems, stochastic Itô systems \[10,23,24,27,28,32\], switching systems \[3,12,25,26\]. It is worth mentioning that short-time stability is also referred to finite-time stability as discussed in \[10,15,23,24\]. The reference \[32\] extended finite-time stability and stabilization of \[1\] to linear time-invariant Itô systems. A mode-dependent parameter approach was proposed to give a sufficient condition for finite-time stability and stabilization for Itô stochastic systems with Markovian switching \[25\], and the same method was also used to deal with finite-time guaranteed cost control of Itô stochastic Markovian jump systems \[26\]. It can be found that, up to now, there are few results on finite-time stability and stabilization of discrete time-varying stochastic systems with multiplicative noise.

As said by J.P. LaSalle \[13\], “Today there is more and more reason for studying difference equations systematically. They are in their own right important mathematical models”, “Moreover, their study provides a good introduction to the study theory of differential equations, difference-differential equations, and functional differential equations”. It is expected that, along the development of computer techniques, the study on discrete-time systems will become more and more important, and attract a lot of researchers’ attention. In \[17–20\], discrete-time mean-field linear-quadratic optimal control problems have been systematically researched. In \[4,16,31\], the $H_{\infty}$ control of linear discrete-time stochastic systems were studied, especially, necessary and sufficient conditions for mean square stability were presented. In \[2\], robust stability and stabilization for a class of linear discrete-time time-varying stochastic systems with Markovian jump were investigated based on a small-gain theorem. The reference \[21\] was about finite-time stability of discrete time-varying systems with randomly occurring nonlinearity and missing measurements. Finite-time stochastic stability and stabilisation with partly unknown transition probabilities for linear discrete-time Markovian jump systems was considered in \[30\]. By choosing Lyapunov-Krasovskii-like functionals, sufficient conditions were given in \[22,29\] for
finite-time stability of linear deterministic systems with time-varying delay based on LMIs. Most results on finite-time stability of stochastic systems are sufficient but not necessary conditions, which are derived by Lyapunov function or Lyapunov-Krasovski-like functional method.

This paper will study the finite-time stability of the following linear discrete time-varying stochastic system with multiplicative noise

\[ x_{k+1} = A_k x_k + C_k x_k w_k, \]

and the finite-time stabilization of the following control system

\[ x_{k+1} = A_k x_k + B_k u_k + (C_k x_k + D_k u_k) w_k. \]

Note that in order to study the detectability and observability of the following linear discrete time-varying system

\[ \begin{align*}
    x_{k+1} &= A_k x_k + C_k x_k w_k, \\
    y(k) &= H_k x(k),
\end{align*} \]

proved in this paper are important, which have potential applications to the study of piecewise finite-time stability and other control issues.

(i) We develop a state transition matrix approach to present some necessary and sufficient conditions for finite-time stability of linear discrete stochastic systems. Specifically, the following two identities

\[ \phi'_{l,k}(I_2 \otimes R_l) \phi_{l,k} = R_k^\frac{1}{2} \phi'_{l,k} \phi_{l,k} R_k^\frac{1}{2} \]

and

\[ \psi'_{l,k}(I_2 \otimes R_l) \psi_{l,k} = R_k^\frac{1}{2} \psi'_{l,k} \psi_{l,k} R_k^\frac{1}{2} \]

proved in this paper are important, which have potential applications to the study of piecewise finite-time stability and other control issues.

(ii) In order to further study finite-time stabilization and obtain easily testing criteria, we apply the Lyapunov function method to present some sufficient conditions for finite-time stability and stabilization based on matrix inequalities and LMIs.

The paper is organized as follows: In Section 2, we define finite-time stability and stabilization for linear discrete time-varying stochastic systems. Several useful lemmas are presented. In Section 3, finite-time stability is studied based on state transition matrix approach, where some necessary and sufficient criteria are obtained for finite-time stability. In Section 4, we make use of Lyapunov function method to investigate finite-time stability and stabilization, and
several sufficient conditions for finite-time stability and stabilization are given based on matrix inequalities and LMIs. In Section 5, two examples are constructed to show the effectiveness of our obtained results.

For convenience, the notations adopted in this paper are as follows.

\( M' \): the transpose of the matrix \( M \) or vector \( M \); \( M > 0 \) \((M < 0)\): the matrix \( M \) is a positive definite (negative definite) symmetric matrix; \( I_n \): \( n \times n \) identity matrix; \( \mathbb{R}^n \): the \( n \)-dimensional real Euclidean vector space; \( \mathbb{R}^{n \times m} \): the space of all \( n \times m \) matrices with entries in \( \mathbb{R} \); \( A \otimes B \): the Kronecker product of two matrices \( A \) and \( B \). \( \lambda_{\min}(A)(\lambda_{\max}(A)) \): the minimum (maximum) eigenvalue of a real symmetric matrix \( A \); \( N := \{0, 1, 2, \cdots\} \); \( N_+ := \{1, 2, \cdots\} \); \( N_T := \{0, 1, \cdots, T\} \) where \( T \in \mathbb{N}_+ \).

2 Preliminaries

Consider the following discrete-time time-varying stochastic difference system described by

\[
\begin{aligned}
    x_{k+1} &= A_k x_k + C_k x_k w_k, \\
    x_0 &\in \mathbb{R}^n, k \in N_{T-1}.
\end{aligned}
\]  

(2)

where \( x_k \in \mathbb{R}^n \) is the \( n \)-dimensional state vector. \( \{w_k\}_{k \in N_{T-1}} \) is a sequence of one-dimensional independent white noise processes defined on the complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k \in N_T}, \mathbb{P})\), where \( \mathcal{F}_k = \sigma(w_0, w_1, \cdots, w_{k-1}) \), \( \mathcal{F}_0 = \{\phi, \Omega\} \). Assume that \( E[w_k] = 0, E[w_k w_j] = \delta_{kj} \) where \( E \) stands for the mathematical expectation operator, and \( \delta_{kj} \) is a Kronecker function defined by \( \delta_{kj} = 0 \) for \( k \neq j \) while \( \delta_{kj} = 1 \) for \( k = j \). Without loss of generality, \( x_0 \) is assumed to be a deterministic vector. \( A_k \) and \( C_k \) are \( n \times n \) time-varying matrices with respect to \( k \).

Definition 1 Given a positive integer \( T \), two positive scalars \( 0 < c_1 \leq c_2 \), and a finite positive definite symmetric matrix sequence \( \{R_k > 0\}_{k \in N_T} \). The system (2) is said to be finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in N_T})\), if

\[
\|x_0\|^2_{R_0} \leq c_1 \Rightarrow E\|x_k\|^2_{R_k} < c_2, \quad \forall k \in N_T,
\]  

(3)

where \( \|x\|^2_R := x'Rx \).

This paper will also study the feedback stabilization of the following linear discrete time-varying control system

\[
\begin{aligned}
    x_{k+1} &= A_k x_k + B_k u_k + (C_k x_k + D_k u_k) w_k, \\
    x_0 &\in \mathbb{R}^n, k \in N_{T-1}.
\end{aligned}
\]  

(4)
where \( u_k \in \mathbb{R}^m \) is the \( m \)-dimensional control input.

**Definition 2** System (11) is said to be finite-time stabilizable with respect to \((c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})\), if there exists a linear state feedback control law \( u_k = K_k x_k, \ k \in \mathbb{N}_{T-1} \), such that the resulting closed-loop system

\[
\begin{align*}
    x_{k+1} &= (A_k + B_k K_k)x_k + (C_k + D_k K_k)x_k w_k, \\
    x_0 &\in \mathbb{R}^n, \ k \in \mathbb{N}_{T-1}
\end{align*}
\]

is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})\). In order to investigate the finite-time stability of system (2), we need to introduce some useful lemmas.

**Lemma 1** [33] For system (2), we have

(i) \( E\|x_l\|^2 = E\|\phi_{l,k} x_k\|^2 \) for \( l \geq k \), where \( \phi_{j,j} = I_n \) for \( j \in \mathcal{N} \), and

\[
\phi_{l,k} = \begin{bmatrix}
(I_{2^{l-k}-1} \otimes A_{l-1}) \phi_{l-1,k} \\
(I_{2^{l-k}-1} \otimes C_{l-1}) \phi_{l-1,k}
\end{bmatrix}, \quad l > k.
\]  

(ii) \( E\|x_l\|^2 = E\|\psi_{l,k} x_k\|^2 \) for \( l \geq k \), where \( \psi_{j,j} = I_n \) for \( j \in \mathcal{N} \), and

\[
\psi_{l,k} = \begin{bmatrix}
\psi_{l+1,k} A_k \\
\psi_{l+1,k} C_k
\end{bmatrix}, \quad l > k.
\]

(iii) \( x_k \in l^2_{T_k} \) if \( A_i \) and \( C_i \) are uniformly bounded for \( i \in \mathcal{N} \).

**Remark 1** The matrices \( \phi_{l,k} \) and \( \psi_{l,k} \) defined in Lemma 1 can be viewed as the state transition matrices in the mean square sense. In [33], \( \phi_{l,k} \) and \( \psi_{l,k} \) were introduced to present detectability conditions of linear discrete time-varying stochastic systems.

From Lemma 1, the state transition matrix is not unique, since that \( \psi_{l,k} = \phi_{l,k} \) is not necessarily true; see the following example.

**Example 1** It is easy to compute that

\[
\psi_{l,l-1} = \phi_{l,l-1} = \begin{bmatrix} A_{l-1} \\ C_{l-1} \end{bmatrix},
\]

\[
\psi_{l,l-2} = \begin{bmatrix} A_{l-1} A_{l-2} \\ C_{l-1} A_{l-2} \\ A_{l-1} C_{l-2} \\ C_{l-1} C_{l-2} \end{bmatrix}, \quad \phi_{l,l-2} = \begin{bmatrix} A_{l-1} A_{l-2} \\ A_{l-1} C_{l-2} \\ C_{l-1} A_{l-2} \\ C_{l-1} C_{l-2} \end{bmatrix}.
\]
Hence, $\phi_{l,l-2} \neq \psi_{l,l-2}$ for $A_{l-1} \neq C_{l-1}$, but we always have $\phi_{l,k}'\phi_{l,k} = \psi_{l,k}'\psi_{l,k}$ for $l \geq k$.

**Lemma 2** (Schur’s complement) For a real symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{T1} & S_{22} \end{bmatrix}$, the following three conditions are equivalent:

(i) $S < 0$;

(ii) $S_{11} < 0$, $S_{22} - S_{T1}S_{11}^{-1}S_{12} < 0$;

(iii) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{T1} < 0$.

**Lemma 3** For matrices $A$, $B$, $C$ and $D$ of suitable dimensions, we have

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

(8)

3 State Transition Matrix-based Approach for Finite-time Stability

In this section, we mainly use the state transition matrix approach developed in [31, 33] to study the finite-time stability and stabilization of the system (2).

Set $\bar{x}_k = R_{1/2}^k x_k$, then $E[x'_k R_k x_k] = E[\bar{x}'_k \bar{x}_k]$, and system (2) is equivalent to the following system:

$$
\left\{
\begin{array}{l}
\bar{x}_{k+1} = \bar{A}_k \bar{x}_k + \bar{C}_k \bar{x}_k w_k, \\
\bar{x}_0 = R_{1/2}^0 x_0 \in \mathbb{R}^n, k \in N_{T-1},
\end{array}
\right.
$$

where $\bar{A}_k = R_{k+1/2}^k A_k R_{k-1/2}^{-1}$, $\bar{C}_k = R_{k+1/2}^k C_k R_{k-1/2}^{-1}$. Lemma 1 yields directly the following lemma:

**Lemma 4** The state transition matrix $\bar{\phi}_{l,k}$ for the system (9) is in the form of

$$
\bar{\phi}_{l,k} = \begin{bmatrix}
(I_{2l-1} \otimes A_{l-1}) \bar{\phi}_{l-1,k} \\
(I_{2l-1} \otimes C_{l-1}) \bar{\phi}_{l-1,k}
\end{bmatrix}, \quad l > k, \quad \bar{\phi}_{k,k} = I.
$$

(10)

Moreover, $E\|\bar{x}_l\|^2 = E\|\bar{\phi}_{l,k} \bar{x}_k\|^2$. Note that $E\|x_k\|_{R_k}^2 = E\|\bar{x}_k\|^2$, we can obtain the following result directly.

**Proposition 1** The system (2) is finite-time stable with respect to $(c_1, c_2, T, \{R_k\}_{k \in N_T})$ if and only if (iff) the system (9) is finite-time stable with respect to $(c_1, c_2, T, \{R_k = I_n\}_{k \in N_T})$. The next lemma establishes the relationship between $\phi_{k,0}$ and $\bar{\phi}_{k,0}$.
Lemma 5 The state transition matrix $\phi_{k,0}$ of system (2) and the state transition matrix $\bar{\phi}_{k,0}$ of system (3) have the following relation:

$$\phi'_{k,0}(I_{2k} \otimes R_k)\phi_{k,0} = R_0^2 \phi'_{k,0} \bar{\phi}_{k,0} R_0^2. \quad (11)$$

**Proof.** Lemma 5 can be shown by induction. Because

$$\phi'_{k,0}(I_{2k} \otimes R_k)\phi_{k,0} = \phi'_{k,0}(I_{2k} \otimes R_k^{1/2})(I_{2k} \otimes R_k^{1/2})\phi_{k,0},$$

we only need to prove the following identity:

$$(I_{2k} \otimes R_k^{1/2})\phi_{k,0} = \bar{\phi}_{k,0} R_0^{1/2}. \quad (12)$$

For $k = 0$, in view of $\phi_{0,0} = \bar{\phi}_{0,0} = I_n$, we have

$$(I_1 \otimes R_0^{1/2})\phi_{0,0} = R_0^{1/2} = \bar{\phi}_{0,0} R_0^{1/2}.$$

Hence, (12) holds for $k = 0$. Assume that for $k = j - 1$, (12) holds, i.e., $(I_{2j-1} \otimes R_j^{1/2})\phi_{j-1,0} = \bar{\phi}_{j-1,0} R_0^{1/2}$, then we shall prove $(I_{2j} \otimes R_j^{1/2})\phi_{j,0} = \bar{\phi}_{j,0} R_0^{1/2}$. It can be seen that

$$(I_{2j} \otimes R_j^{1/2})\phi_{j,0} = \begin{bmatrix} I_{2j-1} \otimes R_j^{1/2} & 0 \\ 0 & I_{2j-1} \otimes R_j^{1/2} \end{bmatrix} \begin{bmatrix} (I_{2j-1} \otimes A_{j-1})\phi_{j-1,0} \\ (I_{2j-1} \otimes C_{j-1})\phi_{j-1,0} \end{bmatrix} = \begin{bmatrix} (I_{2j-1} \otimes (R_j^{1/2} A_{j-1}))\phi_{j-1,0} \\ (I_{2j-1} \otimes (R_j^{1/2} C_{j-1}))\phi_{j-1,0} \end{bmatrix}.$$

By Lemma 3 and (6), we have

$$(I_{2j} \otimes R_j^{1/2})\phi_{j,0} = \begin{bmatrix} (I_{2j-1} \otimes (R_j^{1/2} A_{j-1}))\phi_{j-1,0} \\ (I_{2j-1} \otimes (R_j^{1/2} C_{j-1}))\phi_{j-1,0} \end{bmatrix} = \bar{\phi}_{j,0} R_0^{1/2}. \quad (13)$$

The proof of this lemma is completed. □

Repeating the same procedure as in Lemma 5, the following more general relation still holds.
Lemma 6 For any $l \geq k$, $l, k \in \mathcal{N}$, we have the following identity:

$$\phi_{l,k}^l(I_{2l-k} \otimes R_l)\phi_{l,k} = R_k^{\frac{1}{2}} \phi_{l,k}^l \psi_{l,k} R_k^{\frac{1}{2}}.$$  \tag{14}$$

Corresponding to the second form of the state transition matrix $\psi_{l,k}$ defined in (7), we denote $\tilde{\psi}_{l,k}$ as another state transition matrix of system (2) from the state $x_k$ to the state $x_l$ with $l \geq k$, which is defined by

$$\tilde{\psi}_{l,k} = \begin{bmatrix} \tilde{\psi}_{l,k+1}A_k \\ \tilde{\psi}_{l,k+1}C_k \end{bmatrix}, \quad \tilde{\psi}_{l,l} = I.$$

Now, a more general relation corresponding to (14) can be given based on $\tilde{\psi}_{l,k}$.

Lemma 7 The state transition matrix $\psi_{l,k}$ of system (2) and the state transition matrix $\tilde{\psi}_{l,k}$ of system (9) have the following relation:

$$\psi_{l,k}^l(I_{2l-k} \otimes R_l)\psi_{l,k} = R_{l,k}^{\frac{1}{2}} \psi_{l,k}^l \tilde{\psi}_{l,k} R_{l,k}^{\frac{1}{2}}.$$  \tag{15}$$

Proof. We still prove this lemma by induction. For $k = l$,

$$\psi_{l,k}^l(I_{2l-k} \otimes R_l)\psi_{l,k} = \psi_{l,l}^l(I_{2l} \otimes R_l)\psi_{l,l} = R_l,$$

while

$$R_{l,k}^{\frac{1}{2}} \psi_{l,k}^l \tilde{\psi}_{l,k} R_{l,k}^{\frac{1}{2}} = R_{l,k}^{\frac{1}{2}} \psi_{l,l}^l \tilde{\psi}_{l,l} R_{l,l}^{\frac{1}{2}} = R_l.$$

So in the case $k = l$, (15) holds. Assume for $k = j$, (15) holds, i.e.,

$$\psi_{l,j}^l(I_{2l-j} \otimes R_l)\psi_{l,j} = R_{l,j}^{\frac{1}{2}} \psi_{l,j}^l \tilde{\psi}_{l,j} R_{l,j}^{\frac{1}{2}}.$$  \tag{16}$$

Then we only need to show that (15) holds for $k = j - 1$, i.e.,

$$\psi_{l,j-1}^l(I_{2l-j+1} \otimes R_l)\psi_{l,j-1} = R_{l,j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j-1} \psi_{l,j-1} R_{l,j-1}^{\frac{1}{2}}.$$  \tag{17}$$

By induction assumption (16), the right hand side of (17) can be computed as

\[
R_{l,j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j-1} \psi_{l,j-1} R_{l,j-1}^{\frac{1}{2}} = R_{l,j-1}^{\frac{1}{2}} \left[ R_{j-1}^{-\frac{1}{2}} A_{j-1} R_{j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j}^t \right. \left. R_{j-1}^{-\frac{1}{2}} C_{j-1}^{t} R_{j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j}^t \right] \tilde{\psi}_{l,j} R_{j-1}^{\frac{1}{2}} C_{j-1} R_{j-1}^{\frac{1}{2}}
\]

\[
= \left[ A_{j-1}^{t} R_{j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j}^t \right. \left. C_{j-1}^{t} R_{j-1}^{\frac{1}{2}} \tilde{\psi}_{l,j}^t \right] \tilde{\psi}_{l,j} R_{j-1}^{\frac{1}{2}} C_{j-1}
\]

\[
= \psi_{l,j-1}^l(I_{2l-j} \otimes R_l)\psi_{l,j-1}.
\]
Hence, (17) holds, this lemma is proved. □

Lemmas 6-7 have potential important applications to piecewise finite-time stability and mean square stability.

**Theorem 1** The system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})\) iff the following inequalities are satisfied:

\[
\phi'_{k,0}(I_{2k} \otimes R_k)\phi_{k,0} < \frac{c_2}{c_1} R_0, \quad \forall k \in \mathbb{N}_T. \tag{18}
\]

**Proof.** We first prove the sufficiency of Theorem 1. If

\[
x_0' R_0 x_0 \leq c_1, \tag{19}
\]

then, by Lemma 5, we have

\[
E\|x_k\|^2_{R_k} = E\|\tilde{x}_k\|^2 = \tilde{x}_0' \phi'_{k,0} (I_{2k} \otimes R_k) \phi_{k,0} \tilde{x}_0 = x_0' R_0^{1/2} \phi'_{k,0} \phi_{k,0} R_0^{1/2} x_0
\]

\[
= x_0' \phi'_{k,0}(I_{2k} \otimes R_k)\phi_{k,0} x_0 = \|x_0\|^2_{\phi'_{k,0}(I_{2k} \otimes R_k)\phi_{k,0}}. \tag{20}
\]

If \(x_0 = 0\), then

\[
E\|x_k\|^2_{R_k} = 0 < c_2, \forall k \in \mathbb{N}_T. \tag{21}
\]

If \(x_0 \neq 0\), then, by (19) and (20), it follows that

\[
E\|x_k\|^2_{R_k} < \|x_0\|^2_{\frac{c_2}{c_1} R_0} \leq c_2, \quad k \in \mathbb{N}_T. \tag{22}
\]

In general, for any \(x_0 \in \mathbb{R}^n\), we can always conclude \(E\|x_k\|^2_{R_k} < c_2\) from \(x_0' R_0 x_0 \leq c_1\), i.e., the system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})\).

Below, we prove the necessity part by contradiction. If the system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})\), while there exists \(\bar{k} \in \mathbb{N}_T\) such that

\[
\phi'_{\bar{k},0}(I_{2\bar{k}} \otimes R_{\bar{k}})\phi_{\bar{k},0} \geq \frac{c_2}{c_1} R_0, \tag{23}
\]

then by taking an \(x_0\) satisfying \(x_0' R_0 x_0 = c_1\), it yields that

\[
E\|x_{\bar{k}}\|^2_{R_{\bar{k}}} = E\|x_0\|^2_{\phi'_{\bar{k},0}(I_{2\bar{k}} \otimes R_{\bar{k}})\phi_{\bar{k},0}} \geq x_0' \frac{c_2}{c_1} R_0 x_0 \geq c_2. \tag{24}
\]

The inequality (24) contradicts the finite-time stability, which requires

\[
E\|x_k\|^2_{R_k} < c_2.
\]

The proof is completed. □
Corollary 1 The system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k\in N_T})\) if the following holds for \(k \in N_T\):

\[
\phi_{k,0}' \phi_{k,0} < \frac{c_2 R_0}{c_1 \max_{k\in N_T} \lambda_{\text{max}}(R_k)},
\]

(25)

where \(\lambda_{\text{max}}(R_k)\) denotes the maximum eigenvalue of the matrix \(R_k\).

Proof. Note that

\[
\phi_{k,0}' (I_{2k} \otimes R_k) \phi_{k,0} \leq \max_{k \in N_T} \lambda_{\text{max}}(R_k) \phi_{k,0}' \phi_{k,0}.
\]

Hence, by Theorem 1, the condition (25) is a sufficient condition for the finite-time stability of the system (2). \(\square\)

Because \(R_k > 0\), \(k \in N_T\), there is a nonsingular matrix sequence \(\{L_k\}_{k\in N_T}\), such that \(R_k = L_k^T L_k\). By Lemma 3, we obtain \(I_{2k} \otimes R_k = (I_{2k} \otimes L_k^T)(I_{2k} \otimes L_k)\). By Theorem 1, the solvability of inequality (13) is equivalent to that the system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in N_T})\). In addition, for \(\forall k \in N_T\), we have

\[
\phi_{k,0}' (I_{2k} \otimes R_k) \phi_{k,0} < \frac{c_2}{c_1} R_0
\]

(26)

Using Lemma 2 twice, the inequality (26) can be equivalently written as

\[
\begin{bmatrix}
-\frac{c_2}{c_1} I_n & R_0^{-\frac{1}{2}} \phi_{k,0}' (I_{2k} \otimes L_k^T) \\
(I_{2k} \otimes L_k) \phi_{k,0} R_0^{-\frac{1}{2}} & -I_{2k} \circ n
\end{bmatrix} < 0
\]

(27)

Let \(P_k = \phi_{k,0} R_0^{-\frac{1}{2}} \phi_{k,0}'\), then (27) leads to

\[
(I_{2k} \otimes L_k) P_k (I_{2k} \otimes L_k^T) - \frac{c_2}{c_1} I_{2k} \circ n < 0.
\]

(28)

Pre-multiplying \((I_{2k} \otimes L_k)^{-1}\) and post-multiplying \([(I_{2k} \otimes L_k)^{-1}]'\) on both sides of (28), it follows that

\[
P_k < \frac{c_2}{c_1} (I_{2k} \otimes R_k^{-1}).
\]

(29)

Recall the following properties of \(\phi_{\ldots}\) in Lemma 1:

\[
\begin{align*}
\phi_{0,0} &= I, & \phi_{k+1,0} &= 
\begin{bmatrix}
(I_{2k} \otimes A_k) \phi_{k,0} \\
(I_{2k} \otimes C_k) \phi_{k,0}
\end{bmatrix},
\end{align*}
\]

(30)
if we substitute \( \phi_{k+1,0} \) into \( P_{k+1} = \phi_{k+1,0} R_0 \), then

\[
P_{k+1} = \begin{bmatrix}
(I_{2^k} \otimes A_k) \phi_{k,0} \\
(I_{2^k} \otimes C_k) \phi_{k,0}
\end{bmatrix} R_0^{-1} \begin{bmatrix}
\phi'_{k,0} (I_{2^k} \otimes A'_k) & \phi'_{k,0} (I_{2^k} \otimes C'_k)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I_{2^k} \otimes A_k) P_k (I_{2^k} \otimes A'_k) & (I_{2^k} \otimes A_k) P_k (I_{2^k} \otimes C'_k) \\
(I_{2^k} \otimes C_k) P_k (I_{2^k} \otimes A'_k) & (I_{2^k} \otimes C_k) P_k (I_{2^k} \otimes C'_k)
\end{bmatrix}.
\tag{31}
\]

Summarize the previous discussion, we are in a position to obtain the following theorem, which is equivalent to Theorem 1.

**Theorem 2** The system \( (2) \) is finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \) iff there exists a symmetric matrix sequence \( \{P_k\}_{k \in \{0, 1, \ldots, T\}} \) solving the following constrained difference equation:

\[
\begin{align*}
P_0 &= R_0^{-1}, \\
P_{k+1} &= \begin{bmatrix}
(I_{2^k} \otimes A_k) P_k (I_{2^k} \otimes A'_k) & (I_{2^k} \otimes A_k) P_k (I_{2^k} \otimes C'_k) \\
(I_{2^k} \otimes C_k) P_k (I_{2^k} \otimes A'_k) & (I_{2^k} \otimes C_k) P_k (I_{2^k} \otimes C'_k)
\end{bmatrix}, \\
P_k &< \frac{c_1}{\varepsilon} (I_{2^k} \otimes R_k^{-1}), \quad k \in N_T.
\end{align*}
\]

If system \( (2) \) is finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \), we shall show that the following perturbed system

\[
\begin{align*}
x_{k+1} &= (A_k + \varepsilon I) x_k + (C_k + \varepsilon I) x_k w_k, \\
x_0 &\in \mathbb{R}^n, \quad k \in N_T - 1,
\end{align*}
\tag{32}
\]

is also finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \) for sufficiently small real number \( \varepsilon \).

**Theorem 3** If system \( (2) \) is finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \), then so is the system \( \varepsilon \).

**Proof.** From Theorem 2, system \( \varepsilon \) is finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \) iff the constrained difference equation is admissible.

\[
P_0^\varepsilon = R_0^{-1}, \\
P_{k+1}^\varepsilon = \begin{bmatrix}
[I_{2^k} \otimes (A_k + \varepsilon I)] P_k^\varepsilon [I_{2^k} \otimes (A_k + \varepsilon I)'] & [I_{2^k} \otimes (A_k + \varepsilon I)] P_k^\varepsilon [I_{2^k} \otimes (C_k + \varepsilon I)'] \\
[I_{2^k} \otimes (C_k + \varepsilon I)] P_k^\varepsilon [I_{2^k} \otimes (C_k + \varepsilon I)'] & [I_{2^k} \otimes (C_k + \varepsilon I)] P_k^\varepsilon [I_{2^k} \otimes (C_k + \varepsilon I)']
\end{bmatrix},
\tag{33}
\]

\[
P_k^\varepsilon < \frac{c_1}{\varepsilon} (I_{2^k} \otimes R_k^{-1}), \quad k \in N_T
\]

From \( \varepsilon \), we can see that \( P_k^\varepsilon \to P_k \) as \( \varepsilon \to 0 \), where \( P_k \) is defined by \( \varepsilon \). If system \( \varepsilon \) is finite-time stable with respect to \( (c_1, c_2, T, \{R_k\}_{k \in N_T}) \), then, by Theorem 2, \( P_k < \frac{c_1}{\varepsilon} (I_{2^k} \otimes R_k^{-1}) \), \( k \in N_T \), which implies \( P_k^\varepsilon < \frac{c_1}{\varepsilon} (I_{2^k} \otimes R_k^{-1}) \) for sufficiently small \( \varepsilon > 0 \). This theorem is proved. \( \square \)
4 Lyapunov Function-based Approach for Finite-time Stability and Stabilization

The state transition matrix-based approach presents necessary and sufficient conditions for finite-time stability of system (2), which is elegant in theory. However, there is some difficulty in applying the state transition matrix-based approach to study finite-time stabilization of system (4). Below, we focus our attention on Lyapunov function-based approach to present some sufficient criteria for finite-time stability and stabilization that are easily tested via Matlab LMI Toolbox.

Theorem 4 System (2) is finite-time stable with respect to $(c_1, c_2, T, \{R_k\}_{k \in \mathbb{N}_T})$, if there exist a scalar $\alpha \geq 0$, and a symmetric positive definite matrix sequence $\{P_k > 0\}_{k \in \mathbb{N}_T}$ solving the following matrix inequalities:

$$
(\alpha + 1)^T c_1 \Lambda - c_2 \Delta < 0, \tag{34}
$$

$$
A_j^T R_{j+1}^\frac{1}{2} P_{j+1}^\frac{1}{2} A_j + C_j^T R_{j+1}^\frac{1}{2} P_{j+1}^\frac{1}{2} C_j - (\alpha + 1) R_j^\frac{1}{2} P_j R_j^\frac{1}{2} < 0, \quad j \in \mathbb{N}_{T-1}, \tag{35}
$$

where $\Delta = \min_{k \in \mathbb{N}_T} \lambda_{\min}(P_k)$, $\Lambda = \lambda_{\max}(P_0)$.

Proof. Choose a Lyapunov function for system (2) as

$$
V(x, k) = x' \hat{P}_k x,
$$

where $\hat{P}_k = R_k^\frac{1}{2} P_k R_k^\frac{1}{2}$. Let $\triangle V(x_j, j) = V(x_{j+1}, j + 1) - V(x_j, j)$, $j \in \mathbb{N}_{T-1}$. By (35),

$$
A_j^T \hat{P}_{j+1} A_j + C_j^T \hat{P}_{j+1} C_j - \hat{P}_j < \alpha \hat{P}_j,
$$

which implies

$$
E \triangle V(x_j, j) = E[V(x_{j+1}, j + 1) - V(x_j, j)]
= E[x_j'(A_j^T \hat{P}_{j+1} A_j + C_j^T \hat{P}_{j+1} C_j - \hat{P}_j)x_j] \leq \alpha EV(x_j, j). \tag{36}
$$

By (36), it yields that

$$
E[V(x_{j+1}, j + 1)] \leq (\alpha + 1)EV(x_j, j) < \cdots
\leq (\alpha + 1)^j V(x_0, 0)
\leq (\alpha + 1)^T x_0' \hat{P}_0 x_0. \tag{37}
$$

It follows that

$$
EV(x_k, k) = E[x_k' \hat{P}_k x_k] = E\{x_k' R_k^\frac{1}{2} P_k R_k^\frac{1}{2} x_k\} \geq \Delta E[x_k' R_k x_k] \tag{38}
$$
V(x_0, 0) = x_0^T P_0 x_0 = \{x_0^T R_0^T P_0 R_0 x_0\} \leq \bar{\lambda} x_0^T R_0 x_0. \quad (39)

So from (37)-(39), we have

\[ \Delta E[x_k^T R_k x_k] \leq (\alpha + 1)^k \bar{\lambda} x_0^T R_0 x_0 \]
\[ \leq (\alpha + 1)^k \bar{\lambda} c_1. \]
\[ \leq (\alpha + 1)^T \bar{\lambda} c_1. \] \quad (40)

By (34), \( E[x_k^T R_k x_k] < c_2 \) for \( k \in N_T \), which means that system (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in N_T})\). This theorem is shown. \( \square \)

The inequalities (34)-(35) are not LMIs due to the appearances of \((\alpha + 1)^T \bar{\lambda} \) and \((\alpha + 1)^T P_j\), \( j \in N_{T-1} \), which leads to a hard computation in using LMI Toolbox to solve (34)-(35). However, when \( \alpha = 0 \), the inequalities (34)-(35) becomes LMIs.

**Corollary 2** System (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in N_T})\), if there exists a symmetric positive definite matrix sequence \( \{P_k > 0\}_{k \in N_T} \) solving the following LMIs:

\[ c_1 \bar{\lambda} - c_2 \Delta < 0, \] \quad (41)
\[ A_j^T R_{j+1}^T P_{j+1} R_{j+1} A_j + C_j^T R_{j+1}^T P_{j+1} R_{j+1} C_j - R_{j+1}^T P_{j+1} R_{j+1} < 0, \quad j \in N_{T-1}, \] \quad (42)

where \( \Delta = \min_{k \in N_T} \lambda_{\min}(P_k) \), \( \bar{\lambda} = \lambda_{\max}(P_0) \).

Repeating the above proof and noting that when \(-1 < \alpha < 0\), (40) should be modified as

\[ \Delta E[x_k^T R_k x_k] \leq (\alpha + 1)^k \bar{\lambda} x_0^T R_0 x_0 \]
\[ \leq (\alpha + 1)^k \bar{\lambda} c_1. \]
\[ \leq (\alpha + 1)^T \bar{\lambda} c_1, \quad \forall k \in \{1, 2, \cdots, T\}. \] \quad (43)

We immediately obtain the following result: **Theorem** System (2) is finite-time stable with respect to \((c_1, c_2, T, \{R_k\}_{k \in N_T})\), if there exist a scalar \( \alpha \) with \(-1 < \alpha < 0\), and a symmetric positive definite matrix sequence \( \{P_k > 0\}_{k \in N_T} \) solving the following matrix inequalities:

\[ (\alpha + 1) c_1 \bar{\lambda} - c_2 \Delta < 0, \] \quad (44)
\[ A_j^T R_{j+1}^T P_{j+1} R_{j+1} A_j + C_j^T R_{j+1}^T P_{j+1} R_{j+1} C_j - (\alpha + 1)^T R_{j+1}^T P_{j+1} R_{j+1} < 0, \quad j \in N_{T-1}, \] \quad (45)

where \( \Delta = \min_{k \in N_T} \lambda_{\min}(P_k) \), \( \bar{\lambda} = \lambda_{\max}(P_0) \).
Remark 2 It is easy to see that when $\alpha \leq -1$, (45) does not admit solutions $\{P_k > 0\}_{k \in N_T}$.

Theorem 5 System (4) is finite-time stabilizable with respect to $(c_1, c_2, T, \{R_k\}_{N_T})$ via a linear state feedback $u_k = K_k x_k$, if for all $i \in N_T$ and $j \in N_{T-1}$, there exists a symmetric matrix sequence $X_j, Y_j$, scalars $\alpha \geq 0, \hat{\lambda}_1 > 0, \hat{\lambda}_2 > 0$, and $\hat{\lambda}_1 \geq \hat{\lambda}_2$, such that

\[
\hat{\lambda}_2 R_i^{-1} \leq X_i \leq \hat{\lambda}_1 R_i^{-1}, \quad (46)
\]

\[(\alpha + 1)^T c_1 \hat{\lambda}_1 - c_2 \hat{\lambda}_2 < 0, \quad (47)
\]

\[
\begin{bmatrix}
-(\alpha + 1)X_j & (A_j X_j + B_j Y_j)' & (C_j X_j + D_j Y_j)'

-X_{j+1} & 0 & (51)

* & -X_{j+1} & \end{bmatrix} < 0.
\]

In addition, if there is a feasible solution for conditions (46)-(48), the controller gain can be computed by

\[K_j = Y_j X_j^{-1}, j \in N_T.\]

Proof. By Definition 2 and Theorem 4, system (4) is finite-time stabilizable with respect to $(c_1, c_2, T, \{R_k\}_{N_T})$ via linear state feedback controllers $u_k = K_k x_k$ if the following matrix inequalities are admissible with respect to $\{\alpha \geq 0, \bar{\lambda} > 0, \underline{\lambda} > 0, P_k > 0, k \in N_T\}$.

\[
\begin{align*}
\Delta I < P_i < \bar{\lambda} I, \quad i \in N_T, \quad (49) \\
(\alpha + 1)^T c_1 \bar{\lambda} - c_2 \underline{\lambda} < 0, \quad (50) \\
(\alpha + 1) \hat{P}_j < 0, \quad j \in N_{T-1}. \quad (51)
\end{align*}
\]

Setting $X_j := \hat{P}_j^{-1}, Y_j := K_j X_j$, pre- and post-multiplying $X_j = X_j'$, the inequality (51) becomes

\[
(\alpha + 1)X_j + B_j Y_j)' X_j^{-1} (A_j X_j + B_j Y_j) + (C_j X_j + D_j Y_j) X_j^{-1} (C_j X_j + D_j Y_j) - (\alpha + 1)X_j < 0.
\]

which is equivalent to (48) according to Schur’s complement. Let $\hat{\lambda}_1 = 1/\underline{\lambda}, \hat{\lambda}_2 = 1/\bar{\lambda}$, and consider $\hat{P}_j = R^{1/2} P_j R^{1/2}$, then the inequalities (49) and (50) yield (46) and (47), respectively. This theorem is proved. \[\square\]

Unfortunately, the coupled inequalities (46)-(48) are not LMIs. However, when $\alpha = 0$, Theorem 5 leads to the following easily testing finite-time stabilization conditions. Corollary System (4) is finite-time stabilizable with respect to $(c_1, c_2, T, \{R_k\}_{N_T})$ via a linear state feedback
\( u_k = K_k x_k \), if for all \( i \in N_T \) and \( j \in N_{T-1} \), there exists a symmetric matrix sequence \( X_j, Y_j \), positive scalars \( \hat{\lambda}_1 > 0, \hat{\lambda}_2 > 0, \hat{\lambda}_1 \geq \hat{\lambda}_2 \) solving the following LMIs:

\[
\hat{\lambda}_2 R_i^{-1} \leq X_i \leq \hat{\lambda}_1 R_i^{-1},
\]

(53)

\[
c_1 \hat{\lambda}_1 - c_2 \hat{\lambda}_2 < 0,
\]

(54)

\[
\begin{bmatrix}
-X_j & (A_j X_j + B_j Y_j)' & (C_j X_j + D_j Y_j)'

-X_{j+1} & 0

* & -X_{j+1}
\end{bmatrix} < 0.
\]

(55)

In this case, the desired controller gains is given by

\[
K_j = Y_j X_j^{-1}, \quad j \in N_T.
\]

5 Numerical Examples

In this section, we give two examples to show the effectiveness of our proposed results.

**Example 1** Given the scalars \( c_1 = 0.25, c_2 = 8, T = 2, R_0 = 1, R_1 = 2, R_2 = 2 \). Consider system (2) with parameters as \( A_0 = C_0 = 1, A_1 = C_1 = 2 \) and \( x_0 = 0.5 \). By Lemma 1, the state transition matrices are computed as

\[
\phi_{1,0} = \begin{bmatrix}
A_0 \\
C_0
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix},
\]

\[
\phi_{2,0} = \begin{bmatrix}
A_1 & 0 & \phi_{1,0} \\
0 & A_1 \phi_{1,0} & \phi_{1,0} \\
C_1 & 0 & \phi_{1,0} \\
0 & C_1 \phi_{1,0} & \phi_{1,0}
\end{bmatrix} = \begin{bmatrix}
2 \\
2 \\
2 \\
2
\end{bmatrix}.
\]

By Theorem 1,

\[
\phi_{0,0}'(R_0) \phi_{0,0} = R_0 \leq \frac{c_2}{c_1} R_0,
\]

\[
\phi_{1,0}'(I_2 \otimes R_1) \phi_{1,0} = 4 \leq \frac{c_2}{c_1} R_0,
\]

\[
\phi_{2,0}'(I_4 \otimes R_2) \phi_{2,0} = 32 = \frac{c_2}{c_1} R_0.
\]

Therefore, the given system is finite-time stable with respect to \((0.25, 8, 2, \{1, 2, 2\})\).

**Example 2** Given the parameters \( c_1 = 2, c_2 = 10, T = 20, R_k = I_2, k \in N_{20} \) with the initial state \( x_0 = [1, 1]' \) in system (4). Assume the system (4) is a periodic system with coefficient
matrices as

\[
A_{2i+1} = \begin{bmatrix}
-1.023 & 0.195 \\
1.152 & 0.610 \\
\end{bmatrix}, \quad A_{2i} = \begin{bmatrix}
-0.204 & 1.255 \\
0.666 & 0.282 \\
\end{bmatrix},
\]

\[
C_{2i+1} = \begin{bmatrix}
-0.409 & 1.742 \\
-0.482 & -0.914 \\
\end{bmatrix}, \quad C_{2i} = \begin{bmatrix}
0.371 & 0.942 \\
0.326 & 1.748 \\
\end{bmatrix},
\]

\[
B_{2i} = B_{2i+1} = D_{2i} = D_{2i+1} = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix},
\]

where \( i \in \{0, 1, \ldots, \frac{T}{2} - 1 = 9\} \). We use Matlab to simulate the system state trajectories 1000 times to obtain the approximate value of \( E[x_k R_k x_k] \). Figs. 5 and 5 show the responses of \( x_k R_k x_k \) and \( E[x_k R_k x_k] \) of the uncontrolled system (2), respectively. From Figs. 5 and 5 it can be seen that the system state is divergent. It is worth pointing out that the curves of different colors in Fig. 5 represent different experiment results.

The response of \( R_k x_k^2 \) in system (2).

By Corollary 4, we can find a set of feasible solutions to (53)-(55) as follows:

\[
X_{2i+1} = \begin{bmatrix}
42.710 & -17.677 \\
-17.677 & 166.762 \\
\end{bmatrix}, \quad X_{2i} = \begin{bmatrix}
230.590 & 91.906 \\
91.906 & 170.080 \\
\end{bmatrix}, \quad Y_{2i+1} = \begin{bmatrix}
18.392 & -141.356 \\
\end{bmatrix},
\]

\[
Y_{2i} = \begin{bmatrix}
-141.855 & -206.872 \\
\end{bmatrix}, \quad \hat{\lambda}_2 = 0.0033, \quad \hat{\lambda}_1 = 0.0249.
\]

Hence, the periodic feedback gain matrices are given by

\[
K_k = \begin{cases}
\begin{bmatrix}
0.0835 & -0.8388 \\
\end{bmatrix}, & k = 2i + 1, \\
\begin{bmatrix}
-0.1855 & -1.0781 \\
\end{bmatrix}, & k = 2i.
\end{cases}
\]

(56)
Under the following state feedback controllers

\[ u_k = K_k x_k, \quad k \in N_T, \]

the closed-loop system of (4) is stabilizable with respect to \((2, 10, 20, \{R_k = I_2\}_{k \in N_20})\). From Fig. 5 we can see that \(E[x'_k R_k x_k] < c_2 = 10\) for any \(k \in N_20\).

6 Conclusions

This paper has studied the finite-time stability and stabilization of linear discrete time-varying stochastic systems. Necessary and sufficient conditions of finite-time stability and stabilization have been given based on the state transition matrix. Meanwhile, sufficient conditions that can be verified using Matlab LMI Toolbox have also been given. Two examples have been supplied
The response of $E[x_k' R_k x_k]$ in system (4) under $u_k = K_k x_k$.

to show the effectiveness of our main results.

References

[1] F. Amato, R. Ambrosino, M. Ariola, C. Cosentino, G. D. Tommasi, *Finite-Time Stability and Control*, London: Springer, 2014.

[2] S. Aberkane, S. Dragan, Robust stability and robust stabilization of a class of discrete-time time-varying linear stochastic systems, *SIAM J. Control Optim.* 53 (2015) 30-57.

[3] G. Chen, Y. Yang, New necessary and sufficient conditions for finite-time stability of impulsive switched linear time-varying systems, *IET Control Theory Appl.* 12 (2018) 140-148.

[4] V. Dragan, V. Morozan, A. M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, New York: Springer, 2010.

[5] P. Dorato, *Short time stability in linear time-varying systems*, *Proc. IRE Int. Convention Record Pt. 4* (1961) 83-87.

[6] A. El Bouhtouri, D. Hinrichsen, A. J. Pritchard, $H_{\infty}$-type control for discrete-time stochastic systems, *Int. J. Robust Nonlinear Control*, 9 (1999) 923-948.

[7] Z. Hu, F. Deng, Modeling and control of Itô stochastic networked control systems with random packet dropouts subject to time-varying sampling, *IEEE Trans. Autom. Control*, 62 (2017) 4194-4201.
[8] R. A. Horn, C. R. Johnson, *Matrix Analysis*, 2nd Edition, London: Cambridge University Press, 2012.

[9] R. Z. Has’minskii, *Stochastic Stability of Differential Equations*, Alphen: Sijtjoff and Noordhoff, 1980.

[10] S. Khoo, J. Yin, Z. Man, X. Yu, Finite-time stabilization of stochastic nonlinear systems in strict-feedback form, *Automatica*, 49 (2013) 1403-1410.

[11] G. Kamenkov, On stability of motion over a finite interval of time (in Russian), *J. Appl. Math. Mech.* 17 (1953) 529-540.

[12] X. Li, X. Lin, S. Li, Y. Zou, Finite-time stability of switched nonlinear systems with finite-time unstable subsystems, *J. Frankl. Inst.* 352 (2015) 1192-1214.

[13] J. P. LaSalle, *The Stability of Dynamical Systems*, SIAM: Philadelphia, 1976.

[14] A. M. Lyapunov (1892), The General Problem of the Stability of Motion (in Russian), A. T. Fuller, Ed., New York, NY, USA: Taylor & Francis, 1992.

[15] A. Lebedev, On stability of motion during a given interval of time (in Russian), *J. Appl. Math. Mech.* 18 (1954) 139-148.

[16] X. Mao, *Stochastic Differential Equations and their Applications*, 2nd Edition, Chichester:Horwood Publishing, 2007.

[17] R. Elliott, X. Li, Y. H. Ni, Discrete time mean-field stochastic linear-quadratic optimal control problems, *Automatica*, 49 (2013) 3222-3233.

[18] Y. H. Ni, R. Elliott, X. Li, Discrete-time mean-field Stochastic linear-quadratic optimal control problems, II: Infinite horizon case, *Automatica*, 57 (2015) 65-77.

[19] Y. H. Ni, X. Li, J. F. Zhang, Indefinite mean-field stochastic linear-quadratic optimal control: from finite horizon to infinite horizon, *IEEE Trans. Autom. Control*, 61 (2016) 3269-3284.

[20] Y. H. Ni, J. F. Zhang, X. Li, Indefinite mean-field stochastic linear-quadratic optimal control, *IEEE Trans. Autom. Control*, 60 (2016) 1786-1800.
[21] Y. Shi, Y. Tang, S. Li, Finite-time control for discrete time-varying systems with randomly occurring non-linearity and missing measurements, *IET Control Theory Appl.* 11 (2017) 838-845.

[22] S. B. Stojanovic, Robust finite-time stability of discrete time systems with interval time-varying delay and nonlinear perturbations, *J.Frankl. Inst.* 354 (2017) 4549-4572.

[23] H. Wang, Q. Zhu, Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form, *Automatica*, 54 (2015) 284-291.

[24] J. Yin, S. Khoo, Z. Man, X. Yu, Finite-time stability and instability of stochastic nonlinear systems, *Automatica*, 47 (2011) 1288-1292.

[25] Z. Yan, W. Zhang, G. Zhang, Finite-time stability and stabilization of Itô stochastic systems with Markovian switching: Mode-dependent parameter approach, *IEEE Trans. Autom. Control*, 60 (2015) 2428-2433.

[26] Z. Yan, J. H. Park, W. Zhang, Finite-time guaranteed cost control for Itô stochastic Markovian jump systems with incomplete transition rates, *Int. J. Robust Nonlinear Control*, 27 (2017) 66-83.

[27] Z. Yan, Y. Song, X. Liu, Finite-time stability and stabilization for Itô-type stochastic Markovian jump systems with generally uncertain transition rates, *Appl. Math. Comput.*, 321 (2018) 512-525.

[28] Z. Yan, G. Zhang, W. Zhang, Finite-Time Stability and Stabilization of Linear Itô Stochastic Systems with State and Control-Dependent Noise, *Asian J. Control*, 15 (2013) 270-281.

[29] Z. Zuo, H. Li, Y. Wang, New criterion for finite-time stability of linear discrete-time systems with time-varying delay, *J.Frankl. Inst.* 350 (2013) 2745-2756.

[30] Z. Zuo, Y. Liu, Y. Wang, H. Li, Finite-time stochastic stability and stabilisation of linear discrete-time Markovian jump systems with partly unknown transition probabilities, *IET Control Theory Appl.* 6 (2012) 1522-1526.

[31] W. Zhang, L. Xie, B. S. Chen, *Stochastic H2/H∞ Control: A Nash Game Approach*, Boca Raton, FL, USA: CRC Press, 2017.
[32] W. Zhang, X. An, Finite-time control of linear stochastic systems, *Int. J. of Innovative Computing Information and Control*, 4 (2008) 689-696.

[33] W. Zhang, W. Zheng, B. S. Chen, Detectability, observability and Lyapunov-type theorems of linear discrete time-varying stochastic systems with multiplicative noise, *Int. J. Control*, 90 (2017) 2490-2507.