Dealing with State Estimation in Fractional-Order Systems under Artifacts

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Abstract— Fractional-order dynamical systems are used to describe processes that exhibit long-term memory with power-law dependence. Notable examples include complex neurophysiological signals such as electroencephalogram (EEG) and blood-oxygen-level dependent (BOLD) signals. When analyzing different neurophysiological signals and other signals with different origin (for example, biological systems), we often find the presence of artifacts, that is, recorded activity that is due to external causes and does not have its origins in the system of interest. In this paper, we consider the problem of estimating the states of a discrete-time fractional-order dynamical system when there are artifacts present in some of the sensor measurements. Specifically, we provide necessary and sufficient conditions that ensure we can retrieve the system states even in the presence of artifacts. We provide a state estimation algorithm that can estimate the states of the system in the presence of artifacts. Finally, we present illustrative examples of our main results using real EEG data.

I. INTRODUCTION

Fractional-order dynamical systems (FODS) have been successfully used to accurately model dynamics which undergo nonexponential power-law decay, and have long-term memory or fractal properties [1]–[6]. In particular, fractional-order models have been used in domains such as gas dynamics [7], viscoelasticity [8], chaotic systems [9], and biological swarming [10], just to mention a few. With the advent of cyber-physical systems (CPS), the usefulness of FODS becomes even more apparent, since we have to model the relationship between the spatial and temporal evolution of complex networks [11], [12].

Inspired by the recent spate of application of FODS to model the spatiotemporal properties of complex physiological signals such as electroencephalogram (EEG), electrocardiogram (ECG), electromyogram (EMG), and blood-oxygen-level dependent (BOLD) imaging [13], [14] in the context of neurophysiological applications, the estimation of the states of a fractional-order dynamical system plays a key role in assessing how the corresponding states are evolving, which unveil information about brain function. We are particularly interested in EEG signals, that are known to be prone to disturbances which are not cerebral in origin, which are known as artifacts in the neuroscience literature [15]. There are innumerable sources for these artifacts, some of the common ones being artifacts due to the blinking of the eye (called blink artifact), the potential difference between the EEG electrode and its lead (called electrode pop artifact), and muscular motions of chewing or swallowing (called glossokinetic artifact). Therefore, it becomes imperative to develop a set of tools which can robustly estimate the dynamics of the EEG signals even in the presence of these artifacts. Note that although the term artifacts is almost always used to describe behavior which is not of a cerebral nature in EEG signals, the notion of artifacts can be readily extended to characterize the disturbances that are not consistent with the stationary evolution of the system’s dynamics - in particular FODS.

In this paper, we develop a systematic framework to estimate the states of a fractional-order dynamical system when artifacts are associated with the measurement process. To do this, we are inspired by recent research on resilient state estimation of dynamical systems in the presence of attacks. For instance, [16]–[29] deal with the problem of identification and state estimation of cyber-physical systems in the presence of these attacks. Notice that artifacts are mainly ‘attacks by nature’, which we try to cope in the context of FODS, and are not assumed to follow any stochastic process or dynamics. Hence, inspired by the techniques outlined in the former, we develop a method to estimate the states of a fractional-order dynamical system in the presence of artifacts, which, to the best of our knowledge, has not been previously analyzed in the context of FODS.

A few works already exist in the domain of state estimation of fractional-order systems [30]–[34]. However, most of these assume the noise associated with the measurement process to have a certain stochastic characterization. The work that comes closest to ours is [34], but even that outlines a method for artifact correction using fractional calculus and a median filter.

The main theoretical contributions of our paper are as follows. First, we provide necessary and sufficient conditions that ensure the estimation of the states of a discrete-time fractional-order dynamical system in the presence of arbitrary artifacts. Next, we propose an algorithm that can estimate the states in the presence of artifacts in the sensor measurements. Finally, we borrow techniques of compressive sensing to improve the algorithm such that it becomes computationally feasible.

The remainder of the paper is organized as follows. Section II introduces the elements of the state estimation problem. Section III presents necessary and sufficient conditions that guarantee the estimation of states of FODS in the presence of artifacts. Sections IV and V consider the problems of constructing the state estimator using the...
techniques of compressive sensing. Finally, in Section VI we present illustrative examples showing the performance of our algorithm on a pedagogical example, as well as real data collected from a wearable EEG device.

II. PROBLEM STATEMENT

A. Notation

Throughout this paper, we use the following notations. Given a set $S$, we denote its cardinality by $|S|$, and its complement by $S^c = U \setminus S$ (the universal set $U$ will be clear from the context of the discussion). If $x \in \mathbb{R}^n$ is a vector, the support of $x$, denoted by supp$(x)$, is the set of nonzero components of $x$

$$\text{supp}(x) = \{i \in \{1, \ldots, n\} \mid x_i \neq 0\}.$$  

The number of nonzero elements in $x$ will be denoted by $\|x\|_0$, that is

$$\|x\|_0 = |\text{supp}(x)|.$$  

Further, if $K \subseteq \{1, \ldots, n\}$, we define $P_K$ to be the projection map onto the components of $K$. Also, given a matrix $M \in \mathbb{R}^{m \times n}$, the $i$-th row of $M$, $i \in \{1, \ldots, m\}$ is denoted by $M_i \in \mathbb{R}^n$. The row support of $M$ is defined to be the set of nonzero rows of $M$

$$\text{rowsupp}(M) = \{i \in \{1, \ldots, m\} \mid M_i \neq 0_n\},$$

where $0_n$ is the vector of zeros with dimension $n$. For matrices, like vectors, the number of nonzero rows of $M$ will be denoted by $\|M\|_{\ell_0}$, and given by

$$\|M\|_{\ell_0} = |\text{rowsupp}(M)|.$$

B. Problem Statement

1) System Model: Consider a linear discrete-time fractional-order dynamical model described as follows

$$\begin{align*}
\Delta^\alpha x[k+1] &= Ax[k], \\
y[k] &= Cx[k] + e[k],
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$ is the state for time step $k \in \mathbb{N}$ and $y \in \mathbb{R}^p$ is the output vector. $A$ is the system matrix and $C$ is the sensor measurement matrix. The vector $e \in \mathbb{R}^p$ represents the model of errors, i.e., the artifacts. Note that for the special case of EEG signals, these could be some of the EEG artifacts described in the Introduction. If there are no artifacts in channel $i \in \{1, \ldots, p\}$, then $e_i[k] = 0$ and the output $y_i[k]$ of channel $i$ is not corrupted. As a consequence, the sparsity pattern of $e[k]$ gives the set of sensors where there are artifacts present. Note that the system model is similar to a classic discrete-time linear time-invariant model except for the inclusion of the fractional derivative, whose expansion and discretization for the $i$-th state, $1 \leq i \leq n$, can be written as

$$\Delta^\alpha x_i[k] = \sum_{j=0}^{k} \psi(\alpha_i, j)x_i[k - j],$$

(2)

where $\alpha_i$ is the fractional order corresponding to state $i$ and

$$\psi(\alpha_i, j) = \frac{\Gamma(j - \alpha_i)}{\Gamma(-\alpha_i)\Gamma(j + 1)},$$

with $\Gamma(\cdot)$ being the gamma function defined by $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s} ds$ for all complex numbers $z$ with $\mathbb{R}(z) > 0$ [35].

2) Artifact Model: We assume in this paper that the measurement channels in which the artifacts are present do not change over time, but can be arbitrary. Note that this is a realistic assumption for our problem since neurophysiological studies of EEG artifacts indicate that they generally occur in specific channels without switching channels arbitrarily and have highly conserved morphological structure [36]. Therefore, we will talk about the number of channels which are affected by artifacts, which will allow us to make statements about unambiguously recovering the initial state $x[0]$ even in the presence of a given number of artifacts.

3) Central Estimator Model: We will also assume the presence of a central estimator, whose job is to receive the output $y_i[k], i \in \{1, \ldots, p\}$ of each sensor at every time step $k$ and, from this information, estimate the initial state $x[0]$. We also assume that the estimator has knowledge of the system matrix $A$, the fractional-order coefficients $\{\alpha_i\}_{i=1}^n$, and the sensor measurement matrix $C$, which can be retrieved using the methods outlined in [37]. From (1), we note that the problems of estimating $x[k]$ and $x[0]$ are exactly equivalent. Therefore, our focus will be on estimating $x[0]$.

Based on the above ingredients, the problem we consider in this paper is as follows.

Problem 1. Given the system matrix $A \in \mathbb{R}^{n \times n}$, the sensor measurement matrix $C \in \mathbb{R}^{p \times n}$, the fractional-order coefficients $\{\alpha_i\}_{i=1}^n$ corresponding to each state and the outputs $y_i[k] \in \mathbb{R}^p$ of sensor $i$, $i \in \{1, \ldots, p\}$, then for every time step $k \in \mathbb{N}$ in [1], we aim to estimate the initial state $x[0] \in \mathbb{R}^n$.

In what follows, we first provide necessary and sufficient conditions that guarantee the feasibility of the problem, followed by an efficient algorithm to estimate the states.

III. MAIN RESULTS

Prior to going into our main results, we review some essential theory for fractional-order systems, including closed-form expressions for the state dynamics. Using the expansion of the fractional-order derivative in [2], the evolution of the state vector can be written as follows

$$x[k+1] = Ax[k] - \sum_{j=1}^{k+1} D(\alpha, j)x[k + 1 - j],$$

(3)

where $D(\alpha, j) = \text{diag}(\psi(\alpha_1, j), \psi(\alpha_2, j), \ldots, \psi(\alpha_n, j))$. Alternatively, (3) can be written as

$$x[k+1] = \sum_{j=0}^{k} A_j x[k - j],$$

(4)
Proof. The proof of this Theorem readily follows from the distinguishability arguments in the context of observable states, where, given two distinct values \( x_a, x_b \) with \( x_a \neq x_b \) explaining the data collected with less than \( q \) affected channels, it is impossible to unambiguously recover \( x[0] \).

We then have a simple necessary and sufficient condition in terms of the matrices \( C \) and \( G_k \) to characterize the effect of correctly estimating the system states with artifacts being present in \( q \) channels.

**Theorem 2.** Let \( k \in \mathbb{N} \setminus \{0\} \). Then, the following statements are equivalent:

1. The central estimator \( C \) can recover \( x[0] \) in \( k \) time steps in the presence of artifacts across \( q \) channels.
2. For all \( z \in \mathbb{R}^n \setminus \{0\} \), \( \text{supp}(CG_0z) \cup \text{supp}(CG_1z) \cup \ldots \cup \text{supp}(CG_{k-2}z) > 2q \).

Comparisons with LTI systems

Similar results have been obtained by Fawzi et al. [28] for LTI systems. However, there are some crucial differences. Fawzi et al. consider the problem of dealing with errors injected by a malicious agent in the case of a discrete-time linear time-invariant system without inputs. Their version of the above theorem is a generalized criterion for the observability of a linear dynamical system with attacks, which implies that the initial state \( x[0] \) can only be recovered in \( k \) time steps if the observability matrix given by \( O_{LTI} = [C^T \ A^T \ C^T \ \ldots \ (A^T)^{k-1} \ C^T]^T \) has full rank. Furthermore, the maximum number of correctable errors cannot increase beyond \( k = n \) measurements, which follows immediately from the Cayley-Hamilton theorem, since, for any \( z, k \geq n \), we have \( \text{supp}(CA^kz) \subset \text{supp}(Cz) \cup \ldots \cup \text{supp}(CA^{k-1}z) \).

In general, for a linear discrete-time fractional-order system modeled by (1) and the matrices \( G_k \), the necessary and sufficient condition for observability is the existence of a finite time \( k' \) (which may be greater than \( n \)) such that \( \text{rank}(\Xi_{k'}) = n \), where the observability matrix \( \Xi_{k'} = [CG_0 \ CG_1 \ \ldots \ CG_{k'-1}]^T \) [38]. Therefore, a remarkable fact is that for a discrete-time fractional-order system, the maximum number of correctable artifacts can increase beyond \( n \) measurements. This is due to the fact that the terms in the observability matrix for fractional-order systems do not constitute a power series; hence we cannot apply the Cayley-Hamilton theorem as in the linear time-invariant case. Specifically, the terms \( G_{k'} \) in the observability matrix \( \Xi_{k'} \) are composed of the terms \( A_j, j = 0, \ldots, k - 1 \), which increase with the time step. Put simply, fractional-order dynamics aggregate the effects of all time for each new iteration of the state, and hence, this long-term memory becomes relevant in determining the minimum number of time steps in which the system becomes observable. Also, the property of being able to increase the number of correctable artifacts beyond \( n \) measurements is ideal from the perspective of our problem, since it assures us that taking more measurements will not go in vain.

Next, we present a result on the number of correctable artifacts as a function of the number of measurements \( \tau \).

**Theorem 3.** Let \( \tau \in \mathbb{N} \setminus \{0\} \) be such that \( \tau r \geq k' \), \( p \in \mathbb{N} \setminus \{0\} \) where \( k' \geq n \) is the minimum index such that \( \text{rank}(\Xi_{k'}) = n \). If the initial state \( x[0] \) can be estimated in \( \tau \) time steps in the presence of artifacts in \( q \) channels, then

\[
q < \frac{p - \left(\frac{(k' - 1)}{\tau}\right)}{2} \leq \frac{p - k'/\tau + 1}{2}.
\]
Remark 1. The problem of actually computing the number of channels affected by artifacts is nontrivial and involves checking that the nullspace of
\[
\begin{bmatrix}
P_K CG_0 \\
P_K CG_1 \\
\vdots \\
P_K CG_{r-1}
\end{bmatrix}
\]
is not trivial, which, in the worst case, requires computing the rank of \(2^p\) matrices, when \(K \subset \{1, \ldots, p\}\).

IV. CONSTRUCTING THE CENTRAL ESTIMATOR

In this section, we will focus on actually constructing the central estimator that can estimate the initial state \(x[0]\). Consider the central estimator \(C_0^k : (\mathbb{R}^p)^k \to \mathbb{R}^n\) defined in a way that \(C_0^k(y[0], \ldots, y[k-1])\) is the optimal \(x\) solution for the optimization problem
\[
\begin{aligned}
\text{minimize} & \quad \|K\| \\
\text{subject to} & \quad \text{supp}(y[k']) - CG_k x \subset K, \quad (8)
\end{aligned}
\]
for \(k' \in \{0, \ldots, k-1\}\).

We note that the optimization problem as stated above can have multiple solutions, since the central estimator looks for the smallest set \(K\) of channels disturbed by artifacts. In such a case, we consider \(C_0^k(y[0], \ldots, y[k-1])\) to be any such solution.

In the next result, we show that the central estimator \(C_0^k\) is, in some sense, optimal.

Theorem 4. Assume that \(x[0]\) can be estimated in \(k\) time steps in the presence of \(q\) artifacts, or \(|\text{supp}(CG_0z) \cup \cdots \cup \text{supp}(CG_{k-1}z)| > 2q\) for all \(z \in \mathbb{R}^n \setminus \{0\}\). Then, \(C_0^k\) correctly estimates \(x[0]\), that is, for any \(x^0 \in \mathbb{R}^n\) and \(e[0], \ldots, e[k-1] \in \mathbb{R}^p\) such that \(|\text{supp}(e[k])| \leq q\), we have \(C_0^k(y[0], \ldots, y[k-1]) = x[0]\), where \(y[k] = CG_k x[0] + e[k]\).

The implication of Theorem 4 is that if any estimator can estimate \(x[0]\) in \(k\) time steps in the presence of \(q\) artifacts, then \(C_0^k\) also can. However, the optimization problem \(8\) is NP-hard in general [39].

V. THE \(\ell_1\) CENTRAL ESTIMATOR

Given the time step \(k \in \mathbb{N} \setminus \{0\}\), consider the linear mapping \(\Phi_k\) defined by
\[
\Phi_k : \mathbb{R}^n \to \mathbb{R}^{p \times k}
\]
\[
x \mapsto \begin{bmatrix} CG_0 x \\ CG_1 x \\ \vdots \\ CG_{k-1} x \end{bmatrix}
\]
Further, define \(Y_k\) as the \(p \times k\) matrix obtained by stacking \(y[0], \ldots, y[k-1]\) as columns, that is,
\[
Y_k = \begin{bmatrix} y[0] & y[1] & \cdots & y[k-1] \end{bmatrix} \in \mathbb{R}^{p \times k}
\]
We have already seen that the central estimator \(C_0^k\) computes \(x[0]\) from the measurements \(y[0], \ldots, y[k-1]\) as the solution of the “\(\ell_0\)-norm” optimization problem
\[
C_0^k(y[0], \ldots, y[k-1]) = \arg \min_{x \in \mathbb{R}^n} \|Y_k - \Phi_k x\|_{\ell_0}. \quad (9)
\]
Remark 2. In [40], Candes and Tao show that under certain mild conditions, the “\(\ell_0\)-norm” can be replaced by the \(\ell_1\)-norm (that is, an \(\ell_1\)-norm relaxation), which leads to a linear program, for which the solutions are more computationally tractable.

From the argument in Remark 2, we have that a central estimator where the “\(\ell_0\)-norm” is replaced by the \(\ell_1\)-norm can perform the same job without any change in the optimal value of the optimization problem. More generally, following a similar approach to that proposed in [28], given \(r \geq 1\), if we measure the magnitude of a row of a matrix by its \(\ell_r\)-norm in \(\mathbb{R}^k\), then, the so-called “\(\ell_1/\ell_r\) central estimator” can be represented as follows
\[
C_r^k(y[0], \ldots, y[k-1]) = \arg \min_{x \in \mathbb{R}^n} \|Y_k - \Phi_k x\|_{\ell_1/\ell_r}, \quad (10)
\]
where,
\[
\|M\|_{\ell_1/\ell_r} = \sum_{i=1}^p \|M_i\|_{\ell_r}.
\]
Note that \(10\) is a convex optimization problem and can be tractably solved.

A. State estimation capability of the \(\ell_1/\ell_r\) central estimator

Next, we present a result that quantifies the state estimation capability of the \(\ell_1/\ell_r\) central estimator.

Theorem 5. The following statements are equivalent:
(1) The \(\ell_1/\ell_r\) central estimator \(C_r^k\) can correctly estimate \(x[0]\) in \(k\) time steps in the presence of \(q\) artifacts.
(2) For all \(K \subset \{1, \ldots, p\}\) with \(|K| = q\) and for all \(\Phi_k z \in z \in \mathbb{R}^n \setminus \{0\}\), we have
\[
\sum_{i \in K} \|\Pi_i\|_{\ell_r} < \sum_{i \in K^c} \|\Pi_i\|_{\ell_r}.
\]
Remark 3. Those who are familiar with the techniques of compressed sensing will immediately notice that the condition \(\sum_{i \in K} \|\Pi_i\|_{\ell_r} < \sum_{i \in K^c} \|\Pi_i\|_{\ell_r}\) is simply a restatement of the “nullspace property” that gives necessary and sufficient conditions for the recovery of sparse signals using \(\ell_1\)-norm relaxation [41].

B. An equivalent sufficient condition

As evident from the previous section, the \(\ell_1/\ell_r\) central estimator \(C_r^k\) can correctly estimate \(x[0]\) in \(k\) time steps in the presence of \(q\) artifacts, if and only if \(\sum_{i \in K} \|\Pi_i\|_{\ell_r} < \sum_{i \in K^c} \|\Pi_i\|_{\ell_r}\) for all \(K \subset \{1, \ldots, p\}\) with \(|K| = q\) and for all \(\Phi_k z \in z \in \mathbb{R}^n \setminus \{0\}\). However, it is not easy to check the above inequality for every \(z \in R^n \setminus \{0\}\). In this section, we will propose sufficient conditions that are easier to check and will let us conclude whether the central estimator \(C_r^k\) can estimate the system states in \(k\) time steps even in the presence of \(q\) artifacts. From Theorem 5 for a given \(K \subset \{1, \ldots, p\}\) and a given \(z \in \mathbb{R}^n \setminus \{0\}\), we
have
\[\sum_{i \in K} \| (\Phi_k z)_i \| \leq \sum_{i \in K^c} \| (\Phi_k z)_i \| \leq 1,\]
where \((\Phi_k z)_k \in \mathbb{R}^{[K] \times k}\) denotes the \([K] \times k\) matrix obtained by retaining only the rows in \(K\) from \(\Phi_k z\) (similarly for \((\Phi_k z)_{K^c}\)). Using this notation, we can say that if the \(\ell_1/\ell_r\) central estimator can recover the correct \(x[0]\) in the presence of \(q\) artifacts if and only if
\[
\sup_{K \subseteq \{1, \ldots, p\}} \sup_{z \in \mathbb{R}^n \setminus \{0\}} \| (\Phi_k z)_K \|_{\ell_1/\ell_r} < 1.
\]

We now propose a way which enables the efficient assessment of the above inequality. Specifically, we know that when the chosen norm is \(\ell_1 = \ell_2\) and \(L\) is a linear operator, we can use the minimum and maximum singular values of \(L\), \(\sigma_{\min}\) and \(\sigma_{\max}\) to write \(\|z\|_2 \leq \|Lz\|_2 \leq \sigma_{\max} \|z\|_2\).

We have, from the definition of the \(\ell_1/\ell_r\) norm, \(\| (\Phi_k z)_K \|_{\ell_1/\ell_r} = \sum_{i \in K} \| (\Phi_k z)_i \|_{\ell_1/\ell_r}\). Denote by \(\Phi_{k,i}\) the linear map from \(\mathbb{R}^n\) to \(\mathbb{R}^k\) such that \(\Phi_{k,i} z = (\Phi_k z)_i\) for all \(z \in \mathbb{R}^n\). This map can be represented by the matrix
\[
\begin{bmatrix}
P_{\{i\}} CG_0 \\
P_{\{i\}} CG_1 \\
\vdots \\
P_{\{i\}} CG_{K-1}
\end{bmatrix},
\]
where \(P_{\{i\}}\) is the projection map onto component \(i\). Now, assume \(K \subseteq \{1, \ldots, q\}\) is such that \(|K| = q\) is fixed. The numerator of the fraction \(\| (\Phi_k z)_K \|_{\ell_1/\ell_r} / \| (\Phi_k z)_{K^c} \|_{\ell_1/\ell_r}\) can be written as
\[
\| (\Phi_k z)_K \|_{\ell_1/\ell_r} = \sum_{i \in K} \| (\Phi_k z)_i \|_{\ell_1/\ell_r} \leq \sum_{i \in K} \| (\Phi_{k,i} z)_{\ell_r} \|_{\ell_r}.
\]

If \(\beta = \max_{i=1,\ldots,p} \| (\Phi_{k,i} z)_{\ell_r} \|_{\ell_r}\), then
\[
\| (\Phi_k z)_K \|_{\ell_1/\ell_r} \leq q\beta \|z\|_{\ell_r}, \tag{11}
\]

since \(|K| = q\). Next, consider the denominator of \(\| (\Phi_k z)_{K^c} \|_{\ell_1/\ell_r}\). Assume that \(\ell_r = \ell_2\). This allows us to write \(\| (\Phi_{k,i} z)_{\ell_2} \|_{\ell_2} \geq \sigma_{\min}(\Phi_{k,i}) \|z\|_{\ell_2}\), where \(\sigma_{\min}(\Phi_{k,i})\) is the smallest singular value of the linear map \(\Phi_{k,i}\). Next, denoting \(\alpha = \min_{i=1,\ldots,p} \sigma_{\min}(\Phi_{k,i})\), we get
\[
\| (\Phi_k z)_{K^c} \|_{\ell_1/\ell_r} = \sum_{i \in K^c} \| (\Phi_k z)_i \|_{\ell_1/\ell_r} \leq (p-q)\alpha \|z\|_{\ell_2}. \tag{12}
\]

Using (11) and (12), we get
\[
\sup_{z \in \mathbb{R}^n \setminus \{0\}} \| (\Phi_k z)_K \|_{\ell_1/\ell_r} / \| (\Phi_k z)_{K^c} \|_{\ell_1/\ell_r} < \frac{q\beta}{(p-q)\alpha}. \tag{13}
\]

Thus, the \(\ell_1/\ell_r\) central estimator \(C_{k,r}^1\) can correctly recover \(x[0]\) in \(k\) time steps in the presence of \(q\) artifacts if \(\frac{q\beta}{(p-q)\alpha} < 1\). Rearranging this inequality yields \(q < \frac{(p-q)\alpha}{\beta}\). So, artifacts in at least \([\frac{(p-q)\alpha}{\beta} + 1]\) channels can be corrected in \(k\) time steps. Simply speaking, we can do state estimation under setups where arbitrary artifacts affect at most \(q\) channels.

VI. ILLUSTRATIVE EXAMPLES

In this section, we show the performance of the proposed estimator first on a pedagogical toy example, and then on a more realistic system consisting of EEG signals under possible artifacts.

A. Pedagogical Example

First, we consider the performance of the \(\ell_1/\ell_2\) central estimator on a system of size \(n = p = 4\) with
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -0.0273 & -10.4940 & 0.8629 \\
0 & 0 & 0 & 1 \\
0.0053 & -0.0682 & -1.7351 & 2.1573
\end{bmatrix},
\]

\(C = I_{4 \times 4}\), and the fractional-order coefficients \(\alpha_1 = 0.10\), \(\alpha_2 = 0.15\), \(\alpha_3 = 0.60\), and \(\alpha_4 = 0.70\). The choice of \(C\) being the identity matrix if often suitable in physiological applications, where there are dedicated sensors to capture the evolution of each system state [42]. We assume that there are arbitrary artifacts associated with some measurements of the output \(y_1\), and that the artifacts are ten times the magnitude of the states. Figure 1 shows the comparison between the actual and estimated states which confirms the fact that the system states have been estimated correctly in the presence of artifacts. Note that we have also implicitly verified the statements in Theorems 2 and 3, since, with \(p = 4\), \(q = 1\), the number of measurements \(\tau = 5\), \(k' = 4\), and \(z = e_4\), where \(e_4\) is the 4-dimensional vector of ones, we have \(\|\text{supp}(CG_0 z)\| \cup \ldots \cup \|\text{supp}(CG_4 z)\| = 4 > 2q = 2\) and \(q \leq (p-k'/\tau + 1)/2\). The optimization problems were solved using CVX [43],[44].

B. EEG data

We also consider the performance of our algorithm on real neurophysiological signals. In particular, we use 150 measurements taken from 4 channels of a 64-channel EEG signal which records the brain activity of subjects. The distribution of the electrodes and the selected channels, along with the corresponding labels and numbers are shown in Figure 2. The subjects were asked to perform various motor
and imagery tasks, and the channels were selected because they are over the motor cortex of the brain, and enable us to predict motor actions such as movement of the hands and feet. The data was collected using the BCI2000 system with a sampling rate of 160Hz [45], [46]. We assume that there is an 'electrode pop' artifact in the first channel that lasts for 25 milliseconds, following which the electrode picks up noisy data only. The 150 measurements were partitioned into windows of size 6. This was done because the process under consideration is nonlinear, therefore, the error can become unbounded even under small perturbations, possibly due to the system identification process. The system was identified using the methods described in [37]. As before, we assume that there are dedicated sensors for each system state, that is, $C = I_{4x4}$.

Similar to the pedagogical example, we find that we can invoke the results of the paper, and, in particular, we can estimate the states with artifacts compromising one of the channels. Figure 2 shows the performance of the $\ell_1/\ell_2$ central estimator on the above data. We see that even in the presence of artifacts, we can estimate the system states fairly closely. Therefore, these simulations provide some evidence that the proposed approach might be used in the context of future neuro-wearable device applications.

VII. CONCLUSION

In this paper, we investigated the problem of estimating the states of a discrete-time fractional-order dynamical system when there are artifacts present in some of the sensor measurements. In particular, we derive necessary and sufficient conditions that enable us to ensure that the estimation of the system states can be done even in the presence of artifacts. We further present some techniques which are inspired by tools used in compressive sensing to estimate the system states. Our results were illustrated on a toy fractional order system as well as more complex systems that represent data collected from a wearable EEG device.

Although FODS have found huge success in modeling the spatiotemporal properties of EEG, some of the properties accounted for by these models actually originate from unknown sources that are external to the system under consideration. Possible future work will be to model these external sources by unknown input stimuli, and then focus on state estimation of the resultant model with inputs. Also, real-time EEG activity can be monitored in order to self-regulate brain function. This is known in the literature as neurofeedback [47], and it would be interesting to study how the introduction of feedback to such a system changes our perspectives on this problem.

APPENDIX

Proof of Theorem 2: (1) $\implies$ (2): Assume for the sake of contradiction that there exists a $z \in \mathbb{R}^n \setminus \{0\}$ such that $|\text{supp}(CG_0 z) \cup \text{supp}(CG_1 z) \cup \ldots \cup \text{supp}(CG_{k-1} z)| \leq 2q$. Let $e_a[k]$ and $e_b[k]$ be such that $CG_k z = e_a[k] + e_b[k]$ with $\text{supp}(e_a[k]) \subset L_a$, $\text{supp}(e_b[k]) \subset L_b$ with $|L_a|, |L_b| \leq q$ where $L_a, L_b$ are any two subsets of $\{1, \ldots, p\}$ with cardinality less than or equal to $q$ satisfying $L_a \cup L_b = \text{supp}(CG_0 z) \cup \text{supp}(CG_1 z) \cup \ldots \cup \text{supp}(CG_{k-1} z)$. Now, suppose for $k' \in \{0, \ldots, k - 1\}$, $y[k'] = CG_{k'} z + e_b[k'] = CG_{k'} \cdot 0 + e_a[k']$. This means that if the central estimator can properly recover $x[0]$ in $k$ time steps even in the presence of $q$ artifacts, then $C(y[0], \ldots, y[k-1]) = z = 0$, which is impossible since $z \neq 0$.

(2) $\implies$ (1): The proof follows by contradiction. Suppose that the central estimator cannot recover $x[0]$ in $k$ time steps when there are $q$ artifacts. This immediately necessitates the existence of $x_a \neq x_b$ and error vectors $e_a[0], \ldots, e_a[k - 1]$ (supported on $L_a$, with $|L_a| \leq q$) and $e_b[0], \ldots, e_b[k - 1]$ (supported on $L_b$, with $|L_b| \leq q$) such that $CG_{k'} x_a + e_a[k'] = CG_{k'} x_b + e_b[k']$ for all $k' \in \{0, \ldots, k - 1\}$. Now, say $z = x_a - x_b \neq 0$. If we define $L = L_a \cup L_b$, then we must necessarily have $|L| \leq 2q$, and therefore, for all $k' \in \{0, \ldots, k - 1\}$, $\text{supp}(CG_{k'} z) \subset L$, which contradicts (2).

Proof of Theorem 3: Our goal will be to show that...
there exists a $z \neq 0$, such that $|\text{supp}(CG_{G\tau}z) \cup \ldots \cup \text{supp}(CG_{G_{\tau-1}z})| \leq p - \lfloor (k' - 1)/\tau \rfloor$. Assume $L$ to be any subset of $\{1, \ldots, p\}$ such that $|L| = \lfloor (k' - 1)/\tau \rfloor$. An obvious choice would be $L = \{1, \ldots, \lfloor (k' - 1)/\tau \rfloor\}$. Consider the linear operator $\Phi : z \in \mathbb{R}^n \mapsto (P_LCG_{G\tau}z, P_LCG_{G_1}z, \ldots, P_LCG_{G_{\tau-1}z}) \in \mathbb{R}^{K\tau}$, where $P_L$ is a projection onto the components of $L$. Now, since the codomain of $\Phi$ is $\mathbb{R}^{L'\tau}$, and since $|L| = \lfloor (k' - 1)/\tau \rfloor < k'/\tau$, the codomain of $\Phi$ has dimension strictly less than $k'$, which means that the nullspace of $\Phi$ has at least one vector other than the zero vector and is nontrivial. Hence, there exists a $z \neq 0$, such that $|\text{supp}(CG_{G\tau}z) \cup \ldots \cup \text{supp}(CG_{G_{\tau-1}z})| \leq |L'| = p - \lfloor (k' - 1)/\tau \rfloor$.

**Proof of Theorem 4:** Assumed that the ordered pair $(x[0], K)$ is not optimal in (8). This necessitates the existence of some other initial state $x_a \neq x[0]$ and error vectors $e_{a0}, \ldots, e_{a(k-1)}$ with $\text{supp}(e_{a}[k]) \subset K_a$ that generate the same sequence of sensor readings $y[0], \ldots, y[k-1]$. In addition, $|K_a| \leq |K| \leq q$. Therefore, we have two different initial conditions $x_a \neq x[0]$ and two different sets of error vectors $e[0], \ldots, e[k-1]$ and $e_a[0], \ldots, e_{a}[k-1]$ that explain the sequence of observed sensor readings while corresponding to less than $q$ correctable artifacts. This contradicts our assumption that $(x[0], K)$ is not optimal in (8), and we are done.

**Proof of Theorem 5:** (1) $\implies$ (2): We resort to contradiction and assume that (2) does not hold. Then, there exists $K \subset \{1, \ldots, p\}$ with $|K| = q$, and $\Pi = \Phi_kz \in \mathbb{R}^{p \times k}$ with $z \neq 0$ such that $\sum_{i \in K} \|\Pi_i\|_{\ell_1} \geq \sum_{i \in K^c} \|\Pi_i\|_{\ell_1}$.

Now, from the output equation $y[k'] = CG_{k'}x[0] + e[k'] = e[k']$, and $Y_k$ be the matrix obtained by stacking the sensor readings $y[0], \ldots, y[k-1]$ in columns. Of course, since $K$ denotes the set of channels in which artifacts are present, $\text{rowsupp}(Y_k) = K$ and $(Y_k)_i = (\Phi_kz)_i$ for all $i \in K$. Subsequently, we have

$$
\|Y_k - \Phi_kz\|_{\ell_1/\ell_1} = \sum_{i=1}^n \| (Y_k - \Phi_kz)_i \|_{\ell_1} = \sum_{i \in K} \|\Pi_i\|_{\ell_1} \leq \sum_{i \in K} \|\Pi_i\|_{\ell_1} = \sum_{i=1}^n \| (Y_k - \Phi_kz)_i \|_{\ell_1} = \|Y_k - \Phi_kz[0]\|_{\ell_1/\ell_1}.
$$

This means that there exists a $z \neq 0$ such that the value of the objective function in (11) is smaller at $z$ than at $x[0] = 0$. Therefore, $C_{1,\tau}^{r}$ fails to reconstruct the initial state from the sensor outputs, which implies that (2) must be true.

(2) $\implies$ (1): We suppose that (1) is not true, that is, there exists $x[0]$ with error vectors $e[0], \ldots, e[k-1]$ with $\text{supp}(e[k']) \subset K$ with $|K| = q$ such that $C_{1,\tau}^{r}(y[0], \ldots, y[k-1]) \neq x[0]$ with $y[k'] = CG_{k'}x[0] + e[k']$ for all $k' \in \{0, \ldots, k-1\}$. Since the central estimator fails to reconstruct $x[0]$ from the sensor outputs, this means that in the optimization problem (10), there exists an $x \neq x[0]$ that achieves a lower objective than $x[0]$. Next, define $\tilde{z} = \tilde{x} - x[0] \neq 0$, $\Pi = \Phi_kz = U - V$ with $U = Y_k - \Phi_kx[0]$ and $V = Y_k - \Phi_k\tilde{x}$. Then,

$$
\sum_{i \in K} \|\Pi_i\|_{\ell_1} = \sum_{i \in K} \|U_i - V_i\|_{\ell_1} \geq \sum_{i \in K} \|U_i\|_{\ell_1} - \|V_i\|_{\ell_1},
$$

where the last step follows from the reverse triangle inequality for the matrix $\ell_1$-norm. Now, since $\text{rowsupp}(U) \subset K$ and since $\tilde{x}$ achieves a lower objective in (10) than $x[0]$, $\sum_{i \in K} \|U_i\|_{\ell_1} \geq \sum_{i=1}^n \|V_i\|_{\ell_1} \geq \sum_{i=1}^n \|V_i\|_{\ell_1}$. So,

$$
\sum_{i \in K} \|\Pi_i\|_{\ell_1} = \sum_{i \in K} \|U_i\|_{\ell_1} - \sum_{i \in K} \|V_i\|_{\ell_1} = \sum_{i \in K} \|V_i\|_{\ell_1} = \sum_{i \in K} \|\Pi_i\|_{\ell_1},
$$

where the last equality follows from the fact that $\text{rowsupp}(U) \subset K$. Hence, (2) does not hold.

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