Linear regression under model uncertainty

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Abstract

We reexamine the classical linear regression model when the model is subject to two types of uncertainty: (i) some of covariates are either missing or completely inaccessible, and (ii) the variance of the measurement error is undetermined and changing according to a mechanism unknown to the statistician. By following the recent theory of sublinear expectation, we propose to characterize such mean and variance uncertainty in the response variable by two specific nonlinear random variables, which encompass an infinite family of probability distributions for the response variable in the sense of (linear) classical probability theory. The approach enables a family of estimators under various loss functions for the regression parameter and the parameters related to model uncertainty. The consistency of the estimators is established under mild conditions on the data generation process. Three applications are introduced to assess the quality of the approach including a forecasting model for the S&P Index.

Keywords: Robust regression; G-normal distribution; distribution uncertainty; heteroscedastic error; S&P index

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1 Introduction

Robust regression has been actively developed during the years 1970-2000. A long catalogue of robust estimates for the regression coefficients has appeared in the literature that includes the $L_1$, $M$, $GM$, $RM$, $LMS$ and $LTS$, $S$, $MM$, $\tau$ and $SRC$ estimates among others.\footnote{Actually Huber complained that “the collection of estimates to choose from has become so extensive that it is worse than bewildering, namely counterproductive”. [8, page 195]} According to Huber, a robust procedure (or stability, see [8, page 5], is “in the sense that small deviations from the model assumptions should impair the performance only slightly, that is, the latter (described, say, in terms of the asymptotic variance of an estimate, or of the level and power of a test) should be close to the nominal value calculated at the model”. The robust regression estimates above have been designed to achieve such robustness while improving estimation efficiency and protecting against unexpected procedure breakdown.

Note that a central assumption in this robust statistics literature is that the majority of the data under analysis follows a distribution given by an assumed model. Although the assumed model can be very generic, it however must be unique as requested by statistical theory in order to enable inference about the model. When the data under analysis significantly deviates from the assumed model, inference runs out of the set-up of traditional robust statistics. Quoting again Huber, “the interpretation of results obtained by blind robust estimators becomes questionable when the fraction of contaminants is no longer small.” [8, page 198]

Originated from the field of mathematical finance, model uncertainty is a concept that can help statisticians deal with “no longer small” deviations of the data from an assumed model in some precise contexts. In an early work, [7] proposed to tackle model ambiguity aversion by the family of max-min expected utility functions, in a framework where data may follow an infinite family of models (or distributions). The concept of model uncertainty and its applications in mathematical finance are successively developed in the papers [2, 11, 3, 4, 5]. Particularly, coherent risk measures were introduced in [1] to study both market risks and non-market risks. Over the last decade, a fundamental concept of sublinear expectation was developed in [12, 13] which provides a general theory for quantifying uncertainty about probability distribution of random variables, and more generally, of stochastic processes.\footnote{In fact, the theory covers nonlinear expectations which are more general than the concept of sublinear expectations. However for the purpose of this paper, it is sufficient to consider sublinear expectations.} One important result of the theory is a central limit theorem (under sublinear expectation) that bridges the general theory and statistical data analysis under model or distribution uncertainty. Parallel to the role of a classical central
limit theorem to classical statistical inference, a *nonlinear normal distribution* is introduced to approximate asymptotic distributions of large sums of independent variables. This nonlinear normal distribution under sublinear expectation is the celebrated *G-normal distribution*. This theory is fully developed in the recent monograph [14]. (A review of relevant results in Appendix A).

This new theory of sublinear expectation leads to many questions to explore in data analysis in situations where distribution uncertainty is inherent to the data generation process under consideration. An example of such exploration is a recent work [15] where we constructed a new VaR predictor for financial indexes which shows a significant advantage over most of the existing benchmark VaR predictors. A fundamental idea underlying [15] is that, in parallel to classical data analysis where the normal distribution is a natural choice for measurement errors or data fluctuations, the G-normal distribution can serve as a primary tool for analyzing data fluctuations when distributions of data are subject to high uncertainty. Such high distribution uncertainty is indeed common in financial indexes such as the NASDAQ and S&P 500 indexes. The results obtained in [15] for VaR prediction provide a new confirmation of the existence of such distribution uncertainty. They also showcase the power and usefulness of the new theory of sublinear expectation for data analysis under model or distribution uncertainty.

In this paper we explore the implication of such model uncertainly in the context of regression analysis. Precisely, consider a $q$-dimensional deterministic covariate vector $X \in \mathbb{R}^q$ and a univariate dependent random variable $Y \in \mathbb{R}$ within a regression model of the form

$$Y = \beta^TX + \eta + \varepsilon,$$  

(1.1)

where $\beta \in \mathbb{R}^q$ is the vector of regression coefficients. The novelty here is the terms $\eta$ and $\varepsilon$ which account for mean uncertainty, and variance uncertainty, respectively. In layman’s language, we can say that $\beta^TX$ accounts for the contribution to the response mean from the given covariates $X$, while the unexplained or remaining part of the mean is non-accessible either because no more significant covariates are available, or it is varying through a somehow unknown mechanism. This uncertain part of the mean is modeled by the *nonlinear random variable* $\eta$. Furthermore, the fluctuation of $Y$ around its true mean, that is, the error $\varepsilon$, cannot be determined by a single classical probability distribution; rather it will follow the *nonlinear G-normal distribution* in order to capture the underlying uncertainty. The model (1.1) is referred as *distribution-uncertain regression model*.

Under both uncertainties about the mean and variance of the response variable, is it still possible to consistently estimate the regression parameter $\beta$ in (1.1)? To answer the question,
we consider a general loss function $\phi$ and introduce two population optimal parameters, under model uncertainty, namely,

$$
\bar{\beta}^*(\phi) = \arg \min_{\beta \in \mathbb{R}^q} \mathbb{E}[\phi(Y - \beta^T X - \eta)],
$$

(1.2)

and

$$
\underline{\beta}^*(\phi) = \arg \min_{\beta \in \mathbb{R}^q} -\mathbb{E}[-\phi(Y - \beta^T X - \eta)].
$$

(1.3)

The particular feature here is that $\mathbb{E}$ is the sublinear expectation operator. Possible choices for the loss function are $\phi(z) = z^2$ for the square loss, $\phi(z) = [\alpha - I(z < 0)]z$ for quantile loss at a given level $\alpha \in (0, 1)$, and $\phi(z) = I(z \leq 0)$ for the Value-at-Risk (VaR) loss. In general the optimal parameters $\bar{\beta}^*(\phi)$ and $\underline{\beta}^*(\phi)$ depend on the loss function $\phi(\cdot)$ under model uncertainty. On the other hand, if $Y$ had neither mean uncertainty nor variance uncertainty, that is, $\eta$ was a real constant and $\epsilon$ a classical centred noise variable, the model (1.1) would become a classical linear regression model, and we would have $\bar{\beta}^*(\phi) = \underline{\beta}^*(\phi) \equiv \beta$ for a large class of possible loss functions $\phi$.

As a main contribution of the paper, Theorem 2.1 in Section 2 characterizes the population optimal parameters $\bar{\beta}^*(\phi)$ and $\underline{\beta}^*(\phi)$ for a wide class of convex loss functions $\phi$. Next, in Section 3 we apply this characterization to the case of the square loss $\phi(z) = z^2$. Based on this characterization, we propose a class of estimators for both the regression parameter $\beta$ and those parameters that involve in the mean-uncertainty variable $\eta$ and the variance-uncertainty variable $\epsilon$. Under appropriate conditions on the data observation process, we establish large sample consistency of these estimators.

The related literature on regression analysis under model uncertainty is actually quite limited. When the error distribution in the regression model belongs to a finite family, [9] constructed a $k$-sample maximum expectation regression over the given finite family of distribution. Using the square loss, several estimators are proposed which are consistent and asymptotically normal. In a follow-up work, still under the assumption of finite-number uncertainty, [10] investigated a more general form of maximum expectation regression estimators and established their consistency and asymptotic normality under appropriate conditions.

Other sections of the paper are as follows. Section 4 reports simulation experiments to assess the finite-sample properties of the proposed estimators under model uncertainty. In Section 5, we develop three applications of our method to robust regression, regression under heteroscedastic error, and to an analysis of daily returns of the S&P 500 Index. In Appendix A, we recall useful
results from the theory of sublinear expectation which are relevant to the work in this paper. All technical proofs are gathered in Appendix B.

2 Linear regression under distribution uncertainty

Consider the distribution-uncertain regression model (1.1) and the associate population optimal parameters $\tilde{\beta}^*(\phi)$ and $\beta^*(\phi)$ in (1.2) and (1.3) for a given convex loss function $\phi$. As mentioned in Introduction, standard choices for the loss function cover the least squares estimator, quantile regression estimator and a VaR estimator.

Technically, we first construct a specific infinite family of probabilities. Consider a canonical probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = C([0, 1])$ is the space of real-valued continuous functions on $[0,1]$. Let $\{B_t\}_{0 \leq t \leq 1}$ be a Brownian motion. The parameter space we consider is $\Theta = L^2(\Omega \times [0,1], [\sigma, \bar{\sigma}])$, space of square-integrable, progressively measurable random processes on $[0,1]$ with values in the interval $[\sigma, \bar{\sigma}]$. Here the two parameters $0 < \sigma < \bar{\sigma}$ are the lower and the upper limit for parameter processes $\theta = (\theta_s)_{0 \leq t \leq 1} \in \Theta$, respectively. The family of probability measures $\{P_\theta\}$ is: for $A \in \mathcal{F}$,

$$P_\theta(A) = P \circ \xi^{-1}_\theta(A) = P(\xi_\theta \in A), \quad \text{where} \quad \xi_\theta(\cdot) = \int_0^\cdot \theta_s dB_s.$$

The infinite family of probabilities $\{P_\theta\}_{\theta \in \Theta}$ will govern the regression model (1.1). Precisely, under $P_\theta$, the mean uncertainty variable $\eta$ takes a constant $\mu_\theta \in [\mu, \bar{\mu}]$, $\theta \in \Theta$, while the variance uncertainty variable $\varepsilon$ follows a nonlinear G-normal distribution $\mathcal{N}_G(0, [\sigma^2, \bar{\sigma}^2])$, with lower and upper variance parameters $(\sigma^2, \bar{\sigma}^2)$. Note that the distribution uncertainty of the error $\varepsilon$ includes an infinite family of distributions $\{F_\theta(\cdot)\}_{\theta \in \Theta} = \mathcal{N}_G(0, [\sigma^2, \bar{\sigma}^2])$, where $F_\theta(\cdot)$ is determined by $P_\theta$.

By the representation theorem of sublinear expectation, Theorem A.1, we can express the nonlinear expectation of any function of $\varepsilon$ as

$$\mathbb{E}[\phi(\varepsilon)] = \max_{\theta \in \Theta} E_\theta[\phi(\varepsilon)],$$

where $E_\theta[\cdot]$ is the classical linear expectation under $P_\theta$. Therefore the population optimal parameters in (1.2)-(1.3) have the form

$$\tilde{\beta}^*(\phi) = \arg \min_{\beta \in \mathbb{R}^q} \mathbb{E}[\phi(Y - \beta^\top X - \eta)] = \arg \min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E_\theta[\phi(Y - \beta^\top X - \mu_\theta)], \quad (2.1)$$

\footnote{The details of G-normal distribution are given in Appendix A.1.}
and
\[ \beta^*(\phi) = \arg \min_{\beta \in \mathbb{R}^q} \mathbb{E}[-\phi(Y - \beta^T X - \eta)] = \arg \min_{\beta \in \mathbb{R}^q} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)]. \tag{2.2} \]

In other words, \( \beta^*(\phi) \) and \( \beta^*(\phi) \) are optimal for the min-max loss and the min-min loss strategies, respectively, over the infinite family of probabilities \( \{P_\theta\}_{\theta \in \Theta} \).

Next, we have a technical lemma of exchange rule between the maximization or minimization steps in (2.1) and (2.2).

**Lemma 2.1.** We assume that the loss function \( \phi(\cdot) \in C_{1,Lip}(\mathbb{R}) \) is convex. We have the exchange formulas for (2.1) and (2.2):
\[ \min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)] = \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)], \tag{2.3} \]
and
\[ \min_{\beta \in \mathbb{R}^q} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)]. \tag{2.4} \]

As a consequence of Lemma 2.1, the two optimal parameters \( \beta^*(\phi) \) and \( \beta^*(\phi) \) can actually be determined under two classical normal distributions \( N(\mu_\theta, \sigma^2) \) and \( N(\mu_\theta, \sigma^2) \), with some specific mean parameters \( \mu_\theta \) and \( \mu_\theta \). This characterization of the parameters are instrumental for the construction of their estimators presented in Section 3.

**Theorem 2.1.** We assume that the loss function \( \phi(\cdot) \in C_{1,Lip}(\mathbb{R}) \) is convex. There exists an optimal distribution parameter \( \theta_\phi(s) = \sigma, \ 0 \leq s \leq 1, \) such that
\[ \beta^*(\phi) = \arg \min_{\mu_\theta} \mathbb{E}_{\theta_\phi}[\phi(Y - \beta^T X - \mu_\theta)]. \]

Similarly, there exists another optimal distribution parameter \( \theta_\phi(s) = \sigma, \ 0 \leq s \leq 1, \) such that
\[ \beta^*(\phi) = \arg \min_{\mu_\theta} \mathbb{E}_{\theta_\phi}[\phi(Y - \beta^T X - \mu_\theta)]. \]

The proofs of Lemma 2.1 and Theorem 2.1 are given in Appendix B.1 and B.2, respectively.

In order to calculate the two optimal parameters \( \beta^*(\phi) \) and \( \beta^*(\phi) \), we can use Theorem 2.1 with the following two-step procedure.

(1) Find the optimal linear expectations \( \mathbb{E}_{\theta_\phi}[\cdot] \) and \( \mathbb{E}_{\theta_\phi}[\cdot] \) based on the criterion function \( \phi(\cdot) \) such that
\[ \mathbb{E}_{\theta_\phi}[\phi(Y - \beta^T X - \mu_\theta)] = \max_{\theta \in \Theta} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)], \]
and
\[ \mathbb{E}_{\theta_\phi}[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\theta \in \Theta} \mathbb{E}_\theta[\phi(Y - \beta^T X - \mu_\theta)]. \]
(2) Once \( E_{\phi}(\cdot) \) and \( E_{\phi}(\cdot) \) are found, perform standard regression analysis under the two linear expectations to find the optimal parameters

\[
\bar{\beta}(\phi) = \arg \min_{\beta \in \mathbb{R}^q} E_{\phi}(\phi(Y - \beta^TX - \mu_{\phi})), \quad \underline{\beta}(\phi) = \arg \min_{\beta \in \mathbb{R}^q} E_{\phi}(\phi(Y - \beta^TX - \mu_{\phi})).
\]

This two-step procedure defines a new mechanism for determining the optimal parameters \( \bar{\beta}(\phi) \) and \( \underline{\beta}(\phi) \) under the considered distribution uncertainty. The procedure is valid for a general convex loss function \( \phi(\cdot) \in C_{Lip}(\mathbb{R}^d) \).

3 Least squares regression under distribution uncertainty

We now develop the least squares procedure for the estimation of the regression parameter \( \beta \) under the distribution-uncertain model (1.1). The loss function is thus \( \phi(\cdot) = (\cdot)^2 \), and the two population optimal parameters in (2.1) and (2.2) are:

\[
\bar{\beta} = \arg \min_{\beta \in \mathbb{R}^q} \mathbb{E}[(Y - \beta^TX - \eta)^2], \quad \underline{\beta} = \arg \min_{\beta \in \mathbb{R}^q} -\mathbb{E}[-(Y - \beta^TX - \eta)^2].
\]

We call \( \bar{\beta} \) the upper-least squares parameter (U-LSE), and \( \underline{\beta} \) the lower-least squares parameter (L-LSE). Applying the general Theorem 2.1 to the present case, we get the following characterization of these parameters, as well as that of the two variance parameters \( \sigma \) and \( \overline{\sigma} \).

**Theorem 3.1.** Consider the distribution-uncertain regression model (1.1) under the square loss function \( \phi(z) = z^2 \).

(i). The U-LSE \( \bar{\beta} \) can be estimated by the observation samples from

\[
Y = \beta^TX + \mu_{\sigma} + \varepsilon',
\]

where \( \varepsilon' \) follows the classical normal distribution \( \mathcal{N}(0, \sigma^2) \).

(ii). The L-LSE \( \underline{\beta} \) can be estimated by the observation samples from

\[
Y = \beta^TX + \mu_{\overline{\sigma}} + \varepsilon'',
\]

where \( \varepsilon'' \) follows the classical normal distribution \( \mathcal{N}(0, \overline{\sigma}^2) \).

(iii). The variance parameters \( \sigma \) and \( \overline{\sigma} \) are characterized as follows:

\[
\sigma^2 = E_{\sigma}[(Y - \bar{\beta}X - \mu_{\sigma})^2], \quad \overline{\sigma}^2 = E_{\overline{\sigma}}[(Y - \underline{\beta}X - \mu_{\overline{\sigma}})^2],
\]

where \( (E_{\sigma}(\cdot), E_{\overline{\sigma}}(\cdot)) \) mean the expectations under \( \theta_{(\cdot)}(s) = \sigma, \theta_{(\cdot)}(s) = \overline{\sigma}, 0 \leq s \leq 1 \).
Results in Theorem 3.1 can be summarized as follows. When we adopt the min-max strategy,

$$\min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E_{\theta}[(Y - \beta X - \mu_\theta)^2],$$

the U-LSE $\bar{\beta}^*$ is the optimal parameter such that

$$\sigma^2 = E_{\sigma}[(Y - \bar{\beta}^* X - \mu_\sigma)^2] = \min_{\beta \in \mathbb{R}^q} \mathbb{E}[(Y - \beta^* X - \eta)^2]. \quad (3.5)$$

These characterizations will enable a sample counterpart of the U-LSE $\bar{\beta}^*$ which will be a consistent estimator for the parameter $\beta$, and subsequently, another consistent estimator for the upper variance $\sigma^2$.

Similarly, when we consider the min-min strategy, the L-LSE $\underline{\beta}^*$ is the optimal parameter such that

$$\underline{\sigma}^2 = E_{\underline{\sigma}}[(Y - \underline{\beta}^* X - \mu_{\underline{\sigma}})^2] = \min_{\beta \in \mathbb{R}^q} -\mathbb{E}[-(Y - \beta^T X - \eta)^2]. \quad (3.6)$$

Consistent estimators for both the parameter $\beta$ and the lower variance $\sigma$ can also be derived by using the sample counterparts of these parameters.

Consequently, we have the following results for U-LSE $\bar{\beta}^*$ and L-LSE $\underline{\beta}^*$.

**Corollary 3.1.** For the given square loss function $\phi(z) = z^2$, we have that

$$\bar{\beta}^* = \beta^* = \beta$$

for the distribution-uncertain regression model (1.1).

### 3.1 Consistent estimators for the regression parameter $\beta$ and distribution-uncertainty parameters $(\mu, \mu, \sigma^2, \sigma^2)$

In order to formulate a theory of consistent estimation, we need to define precisely the generation process of the data under consideration as follows.

**Data generation process:** The samples $\{(x_i, y_i)\}_{i=1}^T$ satisfy

$$y_i = \beta x_i + \eta_j + \varepsilon_i, \quad 1 + n_0(j - 1) \leq i \leq n_0 j, \quad 1 \leq j \leq K, \quad (3.7)$$

where $\eta_j \in [\mu, \mu]$, and $\varepsilon_i \sim \mathcal{N}(0, \sigma^2_j)$ with $\sigma^2_j \in [\sigma^2, \sigma^2]$. Thus, there are $K$ groups in the samples, and each group has $n_0$ elements with mean $\eta_j$ and variance $\sigma^2_j$. The total number of samples is $T = n_0 K$. 

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The main challenge here for estimating the diverse parameters in the distribution-uncertain model (1.1) is that the theoretical characterizations of the U-LSE and L-LSE parameters given in Theorem 3.1 cannot be used directly, because we do not have at our disposal samples from the two normal distributions $N(\mu, \sigma^2)$ and $N(\mu, \sigma^2)$ that appear in (3.2) and (3.3), respectively. The difficulty is also due to the fact that from one sample $(x_i, y_i)$ to next, the uncertain mean $\eta$ and uncertain error $\varepsilon$ can change significantly. We propose a method based on moving and overlapping blocks that lead to a family of intermediate residuals which are approximately distributed as $N(\mu, \sigma^2)$. These intermediate residuals are then used for consistent estimation of $(\beta, \sigma^2)$. Afterwards, we can build consistent estimations for $(\mu, \mu, \sigma^2)$.

Data generated under (3.7) can be seen as a practical instance of the general distribution-uncertain model (1.1). It defines a specification needed for the introduction of an estimation theory. It is possible to relax a few conditions of the process. For example, the group size $n_0$ may vary with the groups, and the uncertain mean $\eta$ and uncertain error $\varepsilon$ in the sample can have a controlled variation within each group. Particularly, only the samples $\{(x_i, y_i)\}_{i=1}^T$ are available to us, and we have no direct access to all other parameters and variables such as (i) the group partition and the group length $n_0$; (ii) the group means $(\eta_j)$ that account for the mean uncertainty; (iii) the error variances $(\sigma_j)$ that account for the error uncertainty. Therefore, the problem of parameter estimation here is not straightforward.

The main idea of our approach is to use moving blocks. The samples $\{(x_i, y_i)\}_{i=1}^T$ are scanned subsequently as $m = T - n + 1$ blocks of a given block length $n$ as in

$$\{1, \ldots, n\}, \{2, \ldots, n + 1\}, \ldots, \{T - n + 1, \ldots, T\}.$$  

Denote the data in the $l$th block by $B_l = \{(x_i, y_i)\}_{i=l+n-1}^T$, $1 \leq l \leq m$.

Estimators are constructed in several steps.

**Step 1. Estimators for the parameters $(\beta, \mu, \bar{\mu}, \sigma^2)$:**

(i) For each block $1 \leq l \leq m$ with data $B_l = \{(x_i, y_i)\}_{i=l+n-1}^T$, we run an ordinary LSE procedure using the standard regression model

$$y_i = \beta_l x_i + \mu + \varepsilon_{l,i}, \quad l \leq i \leq l + n - 1.$$  

Let $(\hat{\beta}_l, \hat{\mu}_l)$ be the obtained estimates for the regression parameter and mean parameter.

Denote by $z_i = y_i - \hat{\beta}_l x_i - \hat{\mu}_l$ the corresponding residuals. Define the mean squared error

$$\text{MSE} = \frac{1}{n} \sum_{l=1}^m \sum_{i=l+n-1}^T (y_i - \hat{\beta}_l x_i - \hat{\mu}_l)^2.$$  

(ii) For each $l$ and $k$ with $l \leq k$, we compute the residuals $z_{ik}$ for the regression parameter and mean parameter.
from the \(l\)th block by
\[
\hat{\sigma}_l^2 = \frac{1}{n-1} \sum_{i=l}^{l+n-1} \varepsilon_i^2.
\]

(ii) Find the block \(\hat{k}\) with minimum mean squared error, that is,
\[
\hat{k} = \arg \min_{1 \leq l \leq m} \hat{\sigma}_l^2.
\]

Let
\[
w_i = y_i - \hat{\beta}_k x_i, \quad 1 \leq i \leq T,
\]
\[
\tilde{\mu}_l = \frac{1}{n} \sum_{i=l}^{l+n-1} w_i, \quad 1 \leq l \leq m.
\]

We introduce the following estimators.

- The lower and upper means \(\{\mu, \tilde{\mu}\}\) are estimated, respectively, by
  \[
  \hat{\mu} = \min_{1 \leq l \leq m} \tilde{\mu}_l, \quad \tilde{\mu} = \max_{1 \leq l \leq m} \tilde{\mu}_l. \tag{3.8}
  \]

- The regression parameter \(\beta\) and the lower variance \(\sigma^2\) are estimated by
  \[
  \hat{\beta} = \hat{\beta}_k, \tag{3.9}
  \]
  \[
  \hat{\sigma}^2 = \hat{\sigma}_k^2, \tag{3.10}
  \]
  that is, the regression estimators from the minimum mean squared error block \(\hat{k}\).

Later, we will show that under appropriate conditions, the estimators \((\hat{\beta}, \hat{\mu}, \tilde{\mu}, \tilde{\sigma}^2)\) converge to \((\beta, \mu, \tilde{\mu}, \tilde{\sigma}^2)\) as \(n \to \infty\) and \(K \to \infty\).

**Step 2. Estimator for the upper variance \(\tilde{\sigma}^2\):** To estimate the upper variance \(\tilde{\sigma}^2\), we need to remove the mean uncertainty which is present in the intermediate residuals \(w_i = y_i - \hat{\beta}_k x_i, \ 1 \leq i \leq T\). Let \(n_1 < n\) be a small window size and \(P = T/n_1\) (in practice, values like \(n_1 = 10, 20, 40\) are recommended). Let
\[
\tilde{w}_i = w_i - \frac{1}{n_1} \sum_{i=1+(j-1)n_1}^{jn_1} w_i, \quad 1 + (j-1)n_1 \leq i \leq jn_1, \quad 1 \leq j \leq P.
\]
This steps centralizes the data over a local window, and is expected to remove the fluctuation (uncertainty) about observation means. Define, for $1 \leq l \leq m$,

$$\hat{\sigma}_l^2 = \frac{1}{n - 1} \sum_{i=l}^{l+n-1} \hat{w}_i^2.$$  

Finally we estimate the upper variance by

$$\hat{\sigma}^2 = \max_{1 \leq j \leq m} \hat{\sigma}_j^2.$$  \hspace{1cm} (3.11)

The construction of the estimators above is motivated by the following observations.

(i) When two groups of samples, with respective sample means $(\mu_1, \mu_2)$ and sample variances $(\sigma_1^2, \sigma_2^2)$, are merged to one group, the mean of the resulting group takes value in the interval $[\mu_1 \land \mu_2, \mu_1 \lor \mu_2]$; its variance is larger than $\sigma_1^2 \land \sigma_2^2$.

(ii) If the two groups have a same sample mean $\mu$ and different sample variances $(\sigma_1^2, \sigma_2^2)$, the variance of the merged group belongs to the interval $[\sigma_1^2 \land \sigma_2^2, \sigma_1^2 \lor \sigma_2^2]$.

Furthermore, with reference to the data generation process (3.7), consider a data group of length $n_0$, $A_j = \{(x_i, y_i)\}_{i=1+(j-1)n_0}^{jn_0}$, where $1 \leq j \leq K$. When $n \leq n_0$, there exists a moving group $B_l = \{(x_i, y_i)\}_{i=l}^{l+n-1}$, where $1 \leq l \leq m$, such that $B_l \subset A_j$. We use the ordinary LSE to estimate the regression and mean parameters within each of the data blocks of $\{B_l\}_{l=1}^{m}$, and obtain $m$ estimates for the regression coefficient $\beta$ and the corresponding mean squared errors. Based on observation (i), we can use the minimum mean squared error from these $m$ data blocks as an estimator for the minimum variance of the data groups $\{A_j\}_{j=1}^{K}$. This is done with block $\hat{k}$ and the mean squared error $\hat{\sigma}_k^2$ from this block. Then, by Theorem 3.1 and Corollary 3.1, we can obtain the estimation $\hat{\beta}_k$ for $\beta$ based on the block $\hat{k}$.

The next question is to estimate $(\mu, \bar{\mu}, \bar{\sigma})$ via the $m$ sets of residuals $\{C_l\}_{l=1}^{m}$, where $C_l = \{w_i = y_i - \hat{\beta}_l x_i\}_{i=n-1}^{l+n-1}$. By observation (i), we can calculate means $\bar{\mu}_l$ of these $m$ sets of residuals, and their minimum and maximum values will be a good estimator for $\min_{1 \leq j \leq K} \eta_j$ and $\max_{1 \leq j \leq K} \eta_j$, respectively. As the latter values converge to the lower and upper mean, $\mu$ and $\bar{\mu}$, respectively, when $K \to \infty$, these estimators are consistent.

Finally for estimating the upper variance $\bar{\sigma}^2$ in Step 2, we first remove the mean uncertainty that is present in the intermediate residuals $\{C_l\}_{l=1}^{m}$ by using local averaging over smaller blocks of size $n_1 < n$. Then, by observation (ii), we can estimate $\bar{\sigma}$ with the maximum value of the mean squared errors from blocks $\{C_l\}_{l=1}^{m}$ after removing mean uncertainty.

The theoretical consistency of these estimators are established in the following theorem.
Theorem 3.2. Consider the data generation process (3.7), and assume that as $K \to \infty$,

$$
(\min_{1 \leq j \leq K} \eta_j, \max_{1 \leq j \leq K} \eta_j, \min_{1 \leq j \leq K} \sigma_j^2, \max_{1 \leq j \leq K} \sigma_j^2) \to (\mu, \bar{\mu}, \sigma^2, \bar{\sigma}^2).
$$

Assume also $n_0 \leq n$. Then as $K = n_0 \to \infty$,

(i) the lower variance estimator is strongly consistent, that is, $\hat{\sigma}^2 \to \sigma^2$, with probability 1;

(ii) the estimator $\hat{\beta}$ for the regression parameter is strongly consistent, that is, $\hat{\beta} \to \beta$, with probability 1;

(iii) the lower and upper mean estimators in (3.8) are strongly consistent, that is, $(\hat{\mu}, \bar{\mu})$ converge to $(\mu, \bar{\mu})$ with probability 1;

(iv) the upper variance estimator in (3.11) is strongly consistent, that is, $\hat{\sigma}^2 \to \sigma^2$ with probability 1.

The proof of the theorem is given in Appendix B.3.

4 Simulation experiments

Simulations are conducted to check the finite-sample performance of the Robust-LSE estimators proposed in Section 3.1. The design for the data generation process (3.7) is as follows: for $1 \leq j \leq K$, $1 + n_0(j - 1) \leq i \leq n_0j$,

- $\eta_j$ takes value in $[0, 5]$ uniformly, $\sigma_j$ takes value in $[0.1, 1]$ uniformly. Define

$$
(\eta_{\min}, \eta_{\max}) = (\min_{1 \leq j \leq K} \eta_j, \max_{1 \leq j \leq K} \eta_j), \quad (\sigma_{\min}, \sigma_{\max}) = (\min_{1 \leq j \leq K} \sigma_j, \max_{1 \leq j \leq K} \sigma_j).
$$

- $\varepsilon_i \in \mathcal{N}(0, \sigma_j^2)$;

- $y_i = x_i + \eta_j + \varepsilon_i$, \quad ($\beta = 1$).

Consider the estimators $(\hat{\beta}, \hat{\mu}, \bar{\mu}, \hat{\sigma}, \bar{\sigma})$ defined in Steps 1 and 2 in Section 3.1. We take $(n_0, n, n_1) = (200, 150, 20)$ and varying $T \in \{400, 800, 1600, 3200\}$ (or equivalently, $K = T/n_0 \in \{2, 4, 8, 16\}$).

For each combination of $(T, n_0, n, n_1)$, we generate 500 independent replications of the data set $\{\eta_j, \sigma_j\}_{j=1}^K$ and errors $\{\varepsilon_i\}_{i=1}^T$. The average values of the parameters $(\eta_{\min}, \eta_{\max}, \sigma_{\min}, \sigma_{\max})$ over the 500 replications are denoted as $(\bar{\eta}_{\min}, \bar{\eta}_{\max}, \bar{\sigma}_{\min}, \bar{\sigma}_{\max})$. 

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Table 1 reports empirical statistics for the Robust-LSE estimators and for comparison purpose, the ordinary LSE estimators. For each case, we calculate the average and standard error for the two estimators of $\beta$. The method Robust-LSE indeed provides a better estimator $\hat{\beta}$ than the ordinary LSE, with smaller standard errors for $K \in \{2, 4\}$ and comparable standard errors for $K \in \{8, 16\}$. Note that, we have taken the parameters $\{\eta_j, \sigma_j\}_{j=1}^K$ uniformly from some intervals. The induced mean and variance uncertainties are less severe when the number of groups $K$ grows because in this case, an averaging effect appears to reduce such uncertainties, and thus the ordinary LSE method is able to provide an accurate estimate for the regression parameter. However, if the uncertain mean and variance values $\{\eta_j, \sigma_j\}$ do not obey any clearly defined distributions (as done here), the performance of the ordinary LSE is likely to worsen. Furthermore by construction, the Robust-LSE method provides consistent estimators $(\hat{\mu}, \hat{\sigma}^2, \hat{\sigma}^2)$ for the mean and variance uncertainty parameters in the samples.
Table 1: Empirical statistics of the Robust-LSE estimators and the ordinary LSE estimator from 500 replications. Average and standard errors are reported for $\beta$. Parameters are $\beta = 1$, $(n_0, n, n_1) = (200, 150, 20)$ and $T \in \{400, 800, 1600, 3200\}$.

| $T$  | $\hat{\beta}$ | $(\hat{\mu}, \hat{\rho})$ | $(\hat{\sigma}, \hat{\tau})$ |
|------|----------------|----------------------------|-----------------------------|
| 400  | 0.9729         | (1.7479, 3.2705)           | (0.3742, 0.7129)            |
|      | (0.5151)       |                            |                             |
| 800  | 0.9820         | (0.8607, 3.9908)           | (0.2703, 0.8399)            |
|      | (0.3583)       |                            |                             |
| 1600 | 0.9994         | (0.4028, 4.5665)           | (0.1861, 0.9164)            |
|      | (0.2387)       |                            |                             |
| 3200 | 0.9916         | (0.0730, 4.9132)           | (0.1427, 0.9776)            |
|      | (0.1153)       |                            |                             |

LSE

| $T$  | $\beta$  | $(\mu, \rho)$ | $(\sigma, \tau)$ |
|------|----------|---------------|------------------|
| 400  | 1.0299   | 2.4562        | 0.7352           |
|      | (1.4121) |               |                  |
| 800  | 0.9839   | 2.6162        | 1.1562           |
|      | (0.5803) |               |                  |
| 1600 | 1.0147   | 2.5216        | 1.3733           |
|      | (0.2105) |               |                  |
| 3200 | 0.9948   | 2.4905        | 1.4729           |
|      | (0.0812) |               |                  |
Figure 1: Samples of regression lines from the ordinary LSE and from the minimum mean squared error block \( \hat{k} \) LSE.

In Figure 1, we plot samples of regression lines for the ordinary LSE and from the minimum mean square error block \( \hat{k} \) LSE with parameters \( (\hat{\beta}, \hat{\mu}) \) given in the Robust-LSE method, respectively, and for sample size \( T \in \{400, 800, 1600, 3200\} \). It is clear that lines from minimum mean square error block \( \hat{k} \) LSE focus on the sub-samples with minimum mean squared errors, while lines from the ordinary LSE focus on the whole sample. This explains why in general the Robust-LSE method can provide a better estimator for the regression parameter \( \beta \).

Now, we fix the value of \( (T, n_0, n_1) = (1600, 200, 20) \), and verify properties of the Robust-LSE estimators when the block length \( n \) grows. We take \( n = 60, 80, 160, 200 \). Table 2 reports empirical statistics from 500 replications. From the estimators \( (\hat{\beta}, \hat{\mu}, \hat{\sigma}, \hat{\tau}) \), it is again observed that the method Robust-LSE performs better than the ordinary LSE. Furthermore, the conver-
gence of the Robust-LSE estimators \((\hat{\beta}, \hat{\mu}, \hat{\sigma}, \hat{\sigma})\) is verified when \(n\) grows. In Figure 2, we show as in Figure 1, sample regression lines from the ordinary LSE and the minimum mean squared error block \(\hat{k}\) from the Robust-LSE method. We observe that the latter can catch the groups with minimum variance in all the tested cases of block length \(n\).

Table 2: Empirical statistics of the Robust-LSE estimators and the ordinary LSE estimator from 500 replications. Average and standard errors are reported for \(\beta\). Parameters are \(\beta = 1\), \((T, n_0, n_1) = (1600, 200, 20)\) and \(n \in \{60, 80, 160, 200\}\).

| \(n\)   | \((\bar{\eta}_{\text{min}}, \bar{\eta}_{\text{max}})\) | \((\bar{\sigma}_{\text{min}}, \bar{\sigma}_{\text{max}})\) |
|--------|------------------------------------------------|----------------------------------|
| 60     | (0.5642, 4.4594) | (0.1965, 0.8901) |
| Robust-LSE | 0.9877 | (0.1597, 4.9009) | (0.1633, 0.9994) |
| LSE    | 1.0142 | 2.4645 | 1.3726 |
| 80     | (0.5683, 4.4583) | (0.2013, 0.9046) |
| Robust-LSE | 0.9950 | (0.3790, 4.8098) | (0.1755, 0.9810) |
| LSE    | 1.0054 | 2.4732 | 1.3782 |
| 160    | (0.5857, 4.4779) | (0.2034, 0.8995) |
| Robust-LSE | 1.0028 | (0.4650, 4.5696) | (0.1926, 0.9123) |
| LSE    | 0.9893 | 2.5715 | 1.3637 |
| 200    | (0.5610, 4.4350) | (0.2024, 0.8955) |
| Robust-LSE | 1.0016 | (0.5519, 4.5037) | (0.1959, 0.8887) |
| LSE    | 0.9980 | 2.5469 | 1.3712 |
Figure 2: Samples of regression lines from the ordinary LSE, and from the minimum mean squared error block $\hat{k}$ LSE. Parameters are $\beta = 1$, $(T, n_0, n_1) = (1600, 200, 20)$ and $n \in \{60, 80, 160, 200\}$.

5 Applications

5.1 Robust regression

We apply the Robust-LSE estimators to the traditional robust regression problem. Precisely, we compare our method with a benchmark robust regression estimator, namely the MM estimator. Actually, [17] has given an extensive review and comparison of the existing robust regression estimators under various scenarios of model contamination. Overall two estimators perform better than the other competitors, namely the MM estimator [16] and the REWLSE estimator [6].
Since these two best performers are close each other, we chose the MM estimator as a reference in this study.

Following a classical setting in the literature on robust regression, we consider a simple linear model with contamination of the form \( Y = X + \varepsilon \), with samples \( \{(x_i, y_i)\}_{i=1}^{T} \) where \( x_i = 1 + 0.01 \times i \), 1 \( \leq \) \( i \) \( \leq \) \( T \), and 6 scenarios for the errors \( \{\varepsilon_i\} \): for \( 1 \leq m \leq 6 \),

Scenario \( m \): \( \varepsilon_i \in N(0,1) \), \( 1 \leq i \leq a_m \times T \), \( \varepsilon_i \in N(0,100) \), \( a_m \times T < i \leq T \).

Here \( a_m \in \{0.95, 0.90, 0.80, 0.70, 0.60, 0.50\} \), and \( 1 - a_m \) is referred as the contamination rate of the base standard normal errors by a normal error with larger variance 100.

Under each scenario, we generate 500 replications of the data, and calculate the ordinary LSE, the MM estimator and the Robust-LSE estimator for the regression parameter \( \beta \). Table 3 reports the MSEs of the estimators from 500 replications; a companion plot for these MSEs is given at the bottom of the table. We can see that in general, the ordinary LSE has a large MSE. The Robust-LSE and the MM estimators have almost identical performances for scenarios 1 and 2 with light contamination. In contrast for scenarios 3, 4 and 5 with heavier contamination, the Robust-LSE clearly outperforms the MM estimator: especially in the last case with 50% contamination, the MM estimator shows a breakdown with a MSE almost the double of the one from the ordinary LSE (about 10 times of the one from the Robust-LSE estimator).

Next we examine the large sample behaviour of the three estimators by gradually increasing the sample size from \( T = 200 \) to \( T = 1000 \). Among the 6 scenarios of contamination, we report the results for scenarios 1 and 4. The empirical MSEs are reported in Table 4, and displayed in a plot at its bottom. We can see that the Robust-LSE estimator performs better than the MM estimator in scenario 4 (medium contamination) while they are similar under scenario 1 (light contamination) while being both preferable than the ordinary LSE estimator. Besides, all the three estimators show consistency when the sample size increases.
Table 3: Empirical MSEs of the ordinary LSE, MM and Robust-LSE estimators for the regression coefficient $\beta$. Sample size $T = 200$ with 500 replications.

| Scenario | MM    | LSE    | Robust-LSE |
|----------|-------|--------|------------|
| 1        | 0.0205| 0.2075 | 0.0228     |
| 2        | 0.0264| 0.3459 | 0.0237     |
| 3        | 0.0603| 0.5733 | 0.0405     |
| 4        | 0.1823| 0.6875 | 0.0547     |
| 5        | 0.6079| 0.6970 | 0.1060     |
| 6        | 1.6230| 0.7033 | 0.1667     |
Table 4: Empirical MSEs of the ordinary LSE, MM and Robust-LSE estimators for the regression coefficient $\beta$. Sample size $T \in \{200, 400, 600, 800, 1000\}$ under scenarios 1 and 4 with 500 replications.

| Scenario | Method | $T = 200$ | $T = 400$ | $T = 600$ | $T = 800$ | $T = 1000$ |
|----------|--------|-----------|-----------|-----------|-----------|-----------|
| Scenario 1 | MM     | 0.0205    | 0.0024    | 0.0007    | 0.0003    | 0.0002    |
|          | LSE    | 0.2075    | 0.0282    | 0.0071    | 0.0031    | 0.0017    |
|          | R-LSE  | 0.0228    | 0.0025    | 0.0007    | 0.0003    | 0.0001    |
| Scenario 4 | MM     | 0.1823    | 0.0230    | 0.0064    | 0.0029    | 0.0013    |
|          | LSE    | 0.6875    | 0.0833    | 0.0251    | 0.0116    | 0.0052    |
|          | R-LSE  | 0.0547    | 0.0059    | 0.0020    | 0.0007    | 0.0004    |

5.2 Regression under heteroscedastic errors

In this section, we consider a special regression model under heteroscedastic errors:

$$Y_{ij} = \beta X_{ij} + \epsilon_{ij}, \ 1 \leq j \leq n_0, \ 1 \leq i \leq K, \ T = n_0K,$$

where $\epsilon_{ij} \sim N(0, \sigma_i^2)$, $1 \leq j \leq n_0$. We set $\beta = 1, K = 10, X_{ij} = 1 + 0.005(j + (i - 1)n_0)$ and

$$\sigma_i = \{0.6995, 0.5851, 0.3481, 0.1304, 0.7165, 0.3344, 0.4721, 0.5211, 0.1955, 0.4851\}$$

with $(\min_1^{10} \sigma_i, \max_1^{10} \sigma_i) = (0.1304, 0.7165)$. This list of variances is quite arbitrary; their exact values have no particular meaning in our discussion.
The particularity here is that the model has only variance uncertainty. We apply our Robust-LSE method, without prior knowledge about the heteroscedasticity of the data set, to obtain an estimation for the regression parameter $\beta$ and the underlying minimum and maximum volatility $(\bar{\sigma}_{\text{min}}, \bar{\sigma}_{\text{max}})$. Table 5 reports empirical averages of these estimates from 500 replications. The corresponding ordinary LSE estimates are also given for comparison. Figure 3 plots these empirical values. We find that the Robust-LSE can provide an estimator for $\beta$ which is as good as the ordinary LSE; it can also provide accurate estimations for the minimum and maximum volatilities while the ordinary LSE cannot.

Table 5: Heteroscedastic regression models with $(\beta, \bar{\sigma}_{\text{min}}, \bar{\sigma}_{\text{max}}) = (1, 0.1304, 0.7165)$. Averages of estimators from 500 replications and sample size $T \in \{500, 1000, 1500, 2000\}$.

| Parameters | $\beta$ | Min. volatility | Max. volatility |
|------------|---------|-----------------|-----------------|
| $T = 500$  | R-LSE   | 0.9770          | 0.1213          | 0.7844          |
|            | LSE     | 0.9981          | 0.4845          | 0.4845          |
| $T = 1000$ | R-LSE   | 1.0066          | 0.1246          | 0.7691          |
|            | LSE     | 1.0001          | 0.4864          | 0.4864          |
| $T = 1500$ | R-LSE   | 1.0050          | 0.1258          | 0.7544          |
|            | LSE     | 1.0002          | 0.4858          | 0.4858          |
| $T = 2000$ | R-LSE   | 1.0013          | 0.1267          | 0.7466          |
|            | LSE     | 1.0000          | 0.4854          | 0.4854          |
5.3 Real data analysis

We consider a simple linear model:

\[ Y = \beta_1 X_1 + \beta_2 X_2 + \varepsilon, \quad X_1, X_2 \in \mathbb{R}, \]  

(5.1)

where \( \varepsilon \) satisfies a normal distribution \( \mathcal{N}(0, \sigma^2) \). In real market, it is important to select the factors for the linear regression model. However, we may not observe the factor \( X_2 \) and ignore it. Thus, it is possible that we consider the following model:

\[ Y = \beta_1 X_1 + L + \varepsilon, \quad X_1 \in \mathbb{R}, \]  

(5.2)

where \( L \) is a constant. Note that, we can use the ordinal LSE to obtain the coefficient of model (5.2). Based on the distribution-uncertain regression model (1.1), we use a mean uncertain term
to represent the unknown factor $\beta_2 X_2$. The new model is

$$Y = \beta_1 X_1 + \eta + \varepsilon, \ X_1 \in \mathbb{R},$$

(5.3)

where $\eta$ takes value in a interval under sublinear expectation, and $\varepsilon$ has the $\mathcal{N}(0, \sigma^2)$ distribution.

We analyze the S&P500 Index to assess the performance of the models (5.2) and (5.3). The daily closing price data of the index covers the period from Jan. 3, 2000 to July 17, 2020. We consider a first order autoregression version of models (5.2) and (5.3):

$$X_{t+1} = \beta X_t + L + \varepsilon_{t+1}, \quad X_{t+1} = \beta X_t + \eta + \varepsilon_{t+1}.$$

Table 6: Regression results of LSE and Robust-LSE under criterion $F_{0.01}(2, 247) = 4.6921$

| Year       | Method | $\beta$ | $R^2$ | $F$-statistic |
|------------|--------|---------|-------|---------------|
| 201907–202007 | R-LSE | -0.4196 | 0.1802 | 27.1386       |
|            | LSE    | -0.3592 | 0.1290 | 18.2981       |
| 201807–201907 | R-LSE | 0.3250  | 0.0878 | 11.8815       |
|            | LSE    | 0.0117  | 0.0001 | 0.0168        |
| 201707–201807 | R-LSE | -0.2005 | 0.0380 | 4.8738        |
|            | LSE    | -0.0523 | 0.0027 | 0.3385        |
| 201607–201707 | R-LSE | -0.7813 | 0.3269 | 59.9758       |
|            | LSE    | -0.1764 | 0.0311 | 3.9661        |
| 201507–201607 | R-LSE | -0.5840 | 0.2115 | 33.1283       |
|            | LSE    | 0.0391  | 0.0015 | 0.1890        |

Table 6 shows that the model from the Robust-LSE performs better than the one from the ordinary LSE fit according to both the index $R^2$ coefficient and the $F$-statistic of goodness-of-fit. Furthermore beyond the 5 years reported in the table, we have also repeated the same comparison for all the past 20 years of the S&P500 Index: at 1% level, $F$-statistic is 18 times significant for the model fitted with the Robust-LSE, while it is the case for only one model fitted with the ordinary LSE.
6 Conclusion

In this study, a robust liner regression model under both mean and variance uncertainty in the response variable is investigated. We use a G-normal distribution to represent the variance uncertainty, and another nonlinear random variable for the mean uncertainty. These nonlinear random variables in fact encompass an infinite family of distributions for the response variable, instead of a single distribution in the classical regression model. For a given estimation loss criterion, two estimation strategies, namely the min-max and the min-min strategies are introduced for estimating the regression parameter. The theory of sublinear expectation allows us to characterize the optimal parameters for the two estimation strategies. By considering the square loss function, the method leads to the robust (upper and lower) least squares estimators that capture the maximum volatility and minimum volatility in the response variable. Under mild conditions on the data generation process, the consistency of the estimators for both the regression parameter and the parameters of mean and variance uncertainty is established. These theoretical results are confirmed by simulation experiments. The usefulness of the approach is assessed favorably in three applications in comparison to the existing regression methods including the ordinary LSE and a benchmark robust regression estimator.

Further investigation of the proposed method would include more extensive real data analysis. It is also worth researching on alternative data generation process for the general distribution-uncertain regression model (1.1).

A Preliminaries from the sublinear expectation theory

In the following, we introduce the sublinear expectation theory which is used to describe the infinite family of distributions. We suppose that there are an infinite family of probabilities \( \{P_\theta\}_{\theta \in \Theta} \) behind the error \( \epsilon \), and the related distribution is defined as \( F_\theta(z) = P_\theta(\epsilon \leq z), \ z \in \mathbb{R}, \ \theta \in \Theta \), where \( \Theta \) is a given set. Based on the given infinite family of probabilities \( \{P_\theta\}_{\theta \in \Theta} \), we introduce the representation results of a sublinear expectation \( \mathbb{E}[\cdot] \), which is defined on a linear space \( \mathcal{H} \) of real valued functions on \( \Omega \). A sublinear expectation \( \mathbb{E}[\cdot] : \mathcal{H} \to \mathbb{R} \) satisfies, for \( X,Y \in \mathcal{H} \):

(i). \( \mathbb{E}[X] \leq \mathbb{E}[Y], \ X \leq Y \);
(ii). \( \mathbb{E}[c] = c, \ c \in \mathbb{R} \);
(iii). \( \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \);
(iv). \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \; \lambda \geq 0. \)

The next result represents a sublinear expectation \( \mathbb{E}[\cdot] \) as a supremum over a family of classical linear expectations.

**Theorem A.1.** [14, Theorem 1.2.1] Let \( \mathbb{E}[\cdot] \) be a sublinear expectation on \( \mathcal{H} \). There exists an infinite family of linear expectation \( \{E_\theta, \; \theta \in \Theta\} \) such that

\[
\mathbb{E}[X] = \max_{\theta \in \Theta} E_\theta[X], \; X \in \mathcal{H}.
\] (A.1)

Define the space \( C_{l,Lip}(\mathbb{R}) \) of functions \( \phi(\cdot) \) which are locally Lipschitz: for some positive constants \( C \) and \( k \) depending on \( \phi \),

\[
|\phi(x) - \phi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \; x, y \in \mathbb{R}.
\]

We have the following nonlinear central limit theorem.

**Theorem A.2.** [14, Theorem 2.4.4] Let \( \{Z_i\}_{i=1}^\infty \) be a sequence of real-valued random variables on \( \mathcal{H} \). Further, let \( Z_{i+1} \) and \( Z_i \) be identically distributed and \( Z_{i+1} \) is independent from \( \{Z_1, Z_2, \cdots, Z_i\} \) for \( i \geq 1 \). In addition, we assume that

\[
\mathbb{E}[Z_1] = \mathbb{E}[-Z_1] = 0,
\]

and \( \mathbb{E}[|Z_1|^{2+\delta}] < \infty \) for some \( \delta > 0 \). Then, the sequence

\[
\left\{ \frac{Z_1 + Z_2 + \cdots + Z_n}{\sqrt{n}} \right\}_{n=1}^\infty
\]

converges to a G-normally distributed random variable \( Z \) under sublinear expectation \( \mathbb{E}[\cdot] \): that is, for \( \phi(\cdot) \in C_{l,Lip}(\mathbb{R}) \),

\[
\lim_{n \to \infty} \mathbb{E}\left[ \phi\left( \frac{Z_1 + Z_2 + \cdots + Z_n}{\sqrt{n}} \right) \right] = \mathbb{E}[\phi(Z)].
\]

The impact of this nonlinear central limit theorem on statistics is as follows. In parallel to the role of the normal distribution that appears in the limit of a classical central limit theorem, the nonlinear G-normal random variable \( Z \) that appears in this theorem can serve as a natural model for measurement errors in the nonlinear expectation framework, that is, when variables are subject not to a single distribution but to potentially infinite many and unknown distributions. In this paper, we apply this idea to the measurement error \( \varepsilon \) in a linear regression model as a way to catch up with its distribution uncertainty.

In the following, we develop more details on the G-normal distribution.
A.1 The G-normal distribution for variance uncertainty

In the following, we explicitly construct a random variable \( Z_1 \) which follows the G-normal distribution given in Theorem A.2. Recall the infinite family of probabilities \( \{ P_{\theta} \}_{\theta \in \Theta} \) introduced in Section 2. Let \( Z_1 \) satisfies

\[
\mathbb{E}[Z_1] = -\mathbb{E}[-Z_1] = x.
\]

Since \( \mathbb{E}[\cdot] = \max_{\theta \in \Theta} E_{\theta}[\cdot] \), this relationship means that

\[
\max_{\theta \in \Theta} E_{\theta}[Z_1] = \min_{\theta \in \Theta} E_{\theta}[Z_1] = x,
\]

that is, the maximum mean and the minimum mean of \( Z_1 \) over \( \theta \in \Theta \) are the same. In other words, \( Z_1 \) has no uncertainty on its mean. The expectations of \( Z_1 \) under \( \{ P_{\theta} \}_{\theta \in \Theta} \) are given by

\[
E_{\theta}[\phi(Z_1)] = \int_{\mathbb{R}} \phi(z) dF_{\theta}(z), \tag{A.2}
\]

where \( \phi \in C_{Lip}(\mathbb{R}) \) is some criterion (test) function.

In general, it is difficult to calculate the sublinear expectation \( \mathbb{E}[\phi(Z_1)] \). We construct a G-normal distribution using a partial differential equation. This is because the partial differential equation tool can help us to find the optimal parameter \( \theta_{\phi} \) such that

\[
\mathbb{E}[\phi(Z_1)] = E_{\theta_{\phi}}[\phi(Z_1)]
\]

and to calculate the expectation \( E_{\theta}[\phi(Z_1)] \) under linear expectation \( E_{\theta}[\cdot] \).

Assumption A.1. Let us assume \( \{ Z_t \}_{0 \leq t \leq 1} \) satisfies the following stochastic differential equation,

\[
dZ_t = \theta_t dB_t, \quad Z_0 = 0,
\]

under \( P_{\theta} \), \( \theta \in \Theta = L^2(\Omega \times [0, 1], [\sigma, \bar{\sigma}]) \), where \( \Theta \) is the set of all progressively measurable processes taking value on \( [\sigma, \bar{\sigma}] \).

The stochastic process \( \{ Z_t \}_{0 \leq t \leq 1} \) in Assumption A.1 admits a time-varying variance for the given probability measure \( P_{\theta} \). Therefore, there are infinite many distributions behind this process. We define the distribution of \( Z_1 \) as the G-normal distribution \( N_G(0, [\sigma^2, \sigma^2]) \).

Therefore, for a given criterion function \( \phi(\cdot) \in C_{Lip}(\mathbb{R}) \), we have

\[
\mathbb{E}[\phi(Z_t)] = \max_{\theta \in \Theta} E_{\theta}[\phi(Z_t)] = \max_{\theta \in \Theta} E_{\theta}[\phi(\int_0^t \theta_s dB_s)].
\]

Based on Assumption A.1, we use \( N_G(0, [\sigma^2, \sigma^2]) \) to represent the infinite family of distributions \( \{ F_{\theta} \}_{\theta \in \Theta} \) behind the random variable \( Z_1 \).
Proposition 2.2.10 of [14] showed that \( u(t, x) = \mathbb{E}[\phi(Z_t + x)] \) is the unique viscosity solution of the following partial differential equation:

\[
\partial_t u(t, x) - G(\partial_{xx}^2 u(t, x)) = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{A.3}
\]

with the initial condition \( u(0, x) = \phi(x), \quad x \in \mathbb{R} \), where the function \( G(\cdot) \) is defined as

\[
G(a) = \frac{1}{2} \left( \sigma^2 a^+ - \sigma^2 a^- \right), \quad a^+ = \max(a, 0), \quad a^- = \max(-a, 0). \tag{A.4}
\]

It should be noted that \( u(1, 0) = \mathbb{E}[\phi(Z_1)] \). Using the process \( \{Z_t\}_{0 \leq t \leq 1} \), we can calculate the characteristics of the \( G \)-normal random variable \( Z_1 \) for a given criterion function \( \phi(\cdot) \) under the infinite family of distributions \( \{F_\theta\}_{\theta \in \Theta} \).

## B Proofs

### B.1 Proof of Lemma 2.1

In the first step, we prove that,

\[
\min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E_\theta[\phi(Y - \beta^T X - \mu_\theta)] = \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E_\theta[\phi(Y - \beta^T X - \mu_\theta)].
\]

For any given \( \beta \in \mathbb{R}^q \), since \( X \) is a deterministic vector variable, from (1.1), \( \varepsilon = Y - \beta^T X - \eta \) satisfies a \( G \)-normal distribution \( \mathcal{N}_G(a, [\sigma^2, \sigma^2]) \), where \( a \) is a constant, which depends on \( \beta \). Note that by assumption, \( \phi(\cdot) \) is convex. Let

\[
u(t, x) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \int_{-\infty}^{\infty} \phi(y + x) e^{-\frac{(y-x)^2}{2\sigma^2}} \, dy.
\]

Because the equation (A.3) admits a unique classical solution, we can verify that \( u(t, x) \) is this solution, with initial condition \( \lim_{t \to 0} u(t, x) = \phi(x) \). Thus, we can take \( \theta(s) = \sigma \), \( 0 \leq s \leq 1 \) such that

\[
u(1, x) = E_\sigma[\phi(Y - \beta^T X - \mu_\sigma)] = \mathbb{E}[\phi(Y - \beta^T X - \eta)].
\]

By Theorem A.1, we have

\[
\mathbb{E}[\phi(Y - \beta^T X - \eta)] = \max_{\theta \in \Theta} E_\theta[\phi(Y - \beta^T X - \mu_\theta)],
\]

and thus

\[
E_\sigma[\phi(Y - \beta^T X - \mu_\sigma)] = \max_{\theta \in \Theta} E_\theta[\phi(Y - \beta^T X - \mu_\theta)].
\]
Thus, we have
\[
\min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)].
\]
Obviously,
\[
\min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X) - \mu_\theta] \leq \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)],
\]
which implies that
\[
\min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)] \leq \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)].
\]
On the other hand, it is easy to verify that
\[
\min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)] \geq \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)].
\]
Thus, we have
\[
\min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)] = \max_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)], \tag{B.1}
\]
and
\[
\min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\theta \in \Theta} \max_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)]. \tag{B.2}
\]
Similarly, we can obtain the “min-min=min-min” exchange rule:
\[
\min_{\beta \in \mathbb{R}^q} \min_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\theta \in \Theta} \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)], \tag{B.3}
\]
and
\[
\min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\beta \in \mathbb{R}^q} \min_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)]. \tag{B.4}
\]
This completes the proof. \(\Box\)

### B.2 Proof of Theorem 2.1

Note that \(\phi(\cdot)\) is convex. By the representation results (B.2) and (B.4) of Lemma 2.1, we have
\[
E[\phi(Y - \beta^T X - \mu_\theta)] = \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)],
\]
and
\[
E[\phi(Y - \beta^T X - \mu_\theta)] = \min_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)].
\]
This implies that
\[
\bar{\beta}(\phi) = \arg \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)] = \arg \min_{\beta \in \mathbb{R}^q} \max_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)],
\]
and
\[
\underline{\beta}(\phi) = \arg \min_{\beta \in \mathbb{R}^q} E[\phi(Y - \beta^T X - \mu_\theta)] = \arg \min_{\beta \in \mathbb{R}^q} \min_{\theta \in \Theta} E[\phi(Y - \beta^T X - \mu_\theta)].
\]
This completes the proof. \(\Box\)
B.3 Proof of Theorem 3.2

For notation simplicity, we set $A_j = \{(x_i, y_i)\}_{i=1+n_0(j-1)}^{n_0j}$, $1 \leq j \leq K$, with $\eta_j \in [\mu, \bar{\mu}]$, and $\varepsilon_j \in N(0, \sigma_j^2)$, $\sigma_j^2 \in [\sigma^2, \bar{\sigma}^2]$, the total number of samples is $T = n_0K$. For each group $A_j$, when $n \leq n_0$, there exists integer $k_j$ such that the samples $\{(x_i, y_i)\}_{i=k_j}^{k_j+n-1} \subset A_j$. Thus, we can find a block $B_l = \{(x_i, y_i)\}_{i=1}^{l+n-1}$ belongs to the group of $\{A_j\}_{j=1}^K$ with the smallest variance $\min_{1 \leq j \leq K} \sigma_j^2$, as by assumption, the latter is assumed to converge to $\sigma_j^2$.

(i). Recall that within the $l$th block with data $B_l = \{(x_i, y_i)\}_{i=1}^{l+n-1}$, using ordinary LSE as defined in Step 1-(i) of the procedure, we obtain the ordinary LSE for the regression parameter and block mean, namely $\hat{\hat{\beta}}_l, \hat{\mu}_l$. The mean squared error $\hat{\sigma}_l^2$ in the block is also easily obtained. Recall the observation (ii) given below (3.11): if one $B_l$ overlaps with two $A_j$ groups, say $A_{j_1}$ and $A_{j_2}$, the mean squared error $\hat{\sigma}_l^2$ will be larger than if $B_l$ is contained in a single $A_j$ group. Therefore, the minimum of these mean squared errors will be achieved by one block $B_l$ which is included in a single $A_j$. Thus, by Theorem 3.1 and Corollary 3.1, when $n \leq n_0$, $\hat{\beta}_k$ and $\hat{\sigma}_k^2 = \min_{1 \leq l \leq m} \hat{\sigma}_l^2$ are consistent estimators for $\beta$ and $\min_{1 \leq j \leq K} \sigma_j^2$. As the latter is assumed to converge to $\sigma^2$ as $K \rightarrow \infty$, we have $\hat{\sigma}_k^2 \rightarrow \sigma^2$ with probability 1 as $K \land n \rightarrow \infty$. In a similar manner, we obtain $\hat{\beta}_k \rightarrow \beta$ with probability 1 as $K \land n \rightarrow \infty$.

(ii). In (i) above, we have obtained the consistency of $(\hat{\beta}_k, \hat{\sigma}_k^2)$ for the parameters $(\beta, \min_{1 \leq j \leq K} \sigma_j^2)$. Recall the observation (i) given below (3.11), the estimators $\hat{\mu} = \min_{1 \leq l \leq m} \hat{\mu}_l$ and $\hat{\bar{\mu}} = \max_{1 \leq l \leq m} \hat{\bar{\mu}}_l$ for minimum mean and maximum mean from the groups $(A_j)_{1 \leq j \leq K}$ are consistent, that is, $\mu = \min_{1 \leq l \leq m} \hat{\mu}_l$ and $\bar{\mu} = \max_{1 \leq l \leq m} \hat{\bar{\mu}}_l$ converge almost surely to $\min_{1 \leq j \leq K} \eta_j$ and $\max_{1 \leq j \leq K} \eta_j$. As by assumption, the latter values converge to $(\mu, \bar{\mu})$ as $K \rightarrow \infty$, the strong consistency of $(\hat{\mu}, \hat{\bar{\mu}})$ for $(\mu, \bar{\mu})$ is obtained.

(iii). Similar to the arguments given in (i), the upper variance estimator $\hat{\sigma}_2^2$ given in (3.11) converge to $\max_{1 \leq j \leq m} \sigma_j^2$. As by assumption, the latter is assumed to converge to $\bar{\sigma}_2^2$ as $K \rightarrow \infty$, we have $\hat{\sigma}_2^2 \rightarrow \bar{\sigma}_2^2$ with probability 1.

The proof is complete. □

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