Interchange of filtered 2-colimits and finite 2-limits

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Abstract

In this paper we go into the study of 2-limit and 2-colimit in the 2-category $\mathcal{CAT}$ the category of small categories. In particular we show the commutation of filtered 2-colimits and finite 2-limits. It is a generalization of a classical result in category theory (see for example [1]).

1 Introduction

Recently the 2-category theory has developed rapidly. It gives a very useful language in many field of mathematics. One of the main notions is the 2-limits and 2-colimits. For example the stalk of a stack is a 2-colimit. In this paper we generalize a classical result in category theory.

In a first part we recall the definition of 2-limit and 2-colimit in $\mathcal{CAT}$, the 2-category of small categories. We give an explicit description of a filtered 2-colimit and of a 2-limit. But we do not express the morphisms of these categories in a classical way. We give them as elements of a colimit or a limit.

In a second part we prove the interchange of filtered 2-colimits and finite 2-limits. To prove this theorem we use the explicit expression of the categories $\lim_\rightarrow\lim_\leftarrow\lim_\rightarrow\lim_\leftarrow a(i, j)$.

We do not recall the definitions of a 2-category, a 2-functor (sometimes called pseudo functor in the literature), a 2-natural transformation of functors and a 2-modification. The reader can find them in the chapter 7 of [1], the appendix of the paper [3] and the paper [2].

Let us recall the definition of a filtered category :

**Definition 1.** A category $\mathcal{I}$ is filtered if it satisfies the conditions (i)-(iii) below.

(i) $\mathcal{I}$ is non empty,

(ii) for any $i$ and $j$ in $\mathcal{I}$, there exist $k \in \mathcal{I}$ and morphisms $i \to k$, $j \to k$,

(iii) for any parallel morphisms $s, s' : i \to j$ there exists a morphism $h : j \to k$ such that $h \circ s = h \circ s'$.

2 2-colimits and 2-limits

Let us first recall the definitions of a 2-limit and a 2-colimit. This part is inspired by the appendix of [3]. We cite [2] as a classical reference.
Definition 2. Let $\mathcal{I}$ be a small category, and $b : \mathcal{I} \to \mathbf{CAT}$ a 2-functor. The system $b$ admits a 2-colimit if and only if there exist:

- a category $\lim\limits_{i \in \mathcal{I}} b(i)$ and
- a 2-natural transformation $\sigma : b \to \lim\limits_{i \in \mathcal{I}} b(i)$

such that for all category $\mathcal{C}$ the functor:

$$(\sigma \circ) : \text{Hom}_\mathcal{C}(\lim\limits_{i \in \mathcal{I}} b(i), \mathcal{C}) \to \text{Hom}(b, \mathcal{C})$$

is an equivalence of categories.

We say that a 2-colimit has the strong factorisation property if $(\sigma \circ)$ is an isomorphism of categories.

More concretely, a 2-functor $b : \mathcal{I} \to \mathbf{CAT}$ admits a 2-colimit if and only if there exist:

- a category $\lim\limits_{i \in \mathcal{I}} b(i)$,
- functors $\sigma_i : b(i) \to \lim\limits_{i \in \mathcal{I}} b(i)$ for any $i \in \mathcal{I}$,
- and a natural equivalence $\Theta^\sigma_{ij} : \sigma_i \sim \sigma_j b(s)$ for any morphism $s : i \to j$ of $\mathcal{I}$ visualized by:

\[
\begin{array}{ccc}
\lim\limits_{i \in \mathcal{I}} b(i) & \xrightarrow{\sigma_i} & b(i) \\
\downarrow^{\sim} & & \downarrow^{\sim} \\
\lim\limits_{i \in \mathcal{I}} b(i) & \xrightarrow{\sigma_j} & b(j) \\
\downarrow^{\sim} & & \downarrow^{\sim} \\
b(i) & \xrightarrow{\sim} & b(j)
\end{array}
\]

such that $\sigma : b \to \lim\limits_{i \in \mathcal{I}} b(i)$ is a 2-natural transformation.

Moreover these data satisfy condition assuring that $(\sigma \circ)$ is an equivalence. The first one translate the fact that $(\sigma \circ)$ is essentially surjective and the second one that $(\sigma \circ)$ is fully faithfull.

- For any category $\mathcal{C}$, any morphism of functors $\rho : b \to \mathcal{C}$, there exist a functor:

$F : \lim\limits_{i \in \mathcal{I}} b(i) \to \mathcal{C}$

and an isomorphism $\varphi^F : \rho \to F \sigma$, which is a modification given, for all $i \in \mathcal{I}$, by a natural equivalence $\varphi^F_i : \rho_i \to F \circ \sigma_i$. This may be visualized by:
The compatibility condition is given by:

\[(F \bullet \theta_s^* \circ \varphi_i^F = (\varphi_j^F \bullet a(s)) \circ \theta_s^p)\]

The pair \((F, \varphi_i^F)\) is called a lax factorization of the system \(\rho\).

- Let \(\rho\) and \(\rho'\) be two 2-natural transformations and \(\lambda : \rho \to \rho'\) a modification. Then for any lax factorization \(F : 2 \lim_{i \in I} b(i) \to C\) of \(\rho\) and \(G : 2 \lim_{i \in I} b(i) \to C\) of \(\rho'\) there exists a unique natural transformation \(\Lambda : F \to G\) such that:

\[\varphi_i^G \circ \lambda_i = (\Lambda_i \bullet \sigma_i) \circ \varphi_i^F\]

If the 2-colimit has the strong factorization property then there exists a unique factorization such that \(\varphi_i^F\) is the identity.

A contravariant 2-functor \(\epsilon : I \to CAT\) is a 2-functor \(\epsilon : I^{op} \to CAT\). A 2-limit is construction dual of the 2-colimit. Hence, we have the following definition:

**Definition 3.** Let \(I\) be a small category and \(\epsilon : I \to CAT\) a contravariant 2-functor. The system \(\epsilon\) admits a 2-limit if and only if there exist:

- a category \(2 \lim_{i \in I} \epsilon(i)\) and
- a 2-natural transformation \(\sigma : 2 \lim_{i \in I} b(i) \to \epsilon\)

such that for all category \(C\) the functor:

\[(\circ \sigma) : \text{Hom}(\epsilon, C) \to \text{Hom}_r(2 \lim_{i \in I} b(i), C)\]

is an equivalence of categories.

We say that a 2-limit has the strong factorisation property if \((\circ \sigma)\) is an isomorphism of categories.

It is well known that the 2-category \(CAT\) is complete and co-complete. The proof of this result consist an explicit definition of the 2-limit and 2-colimit. We are going to recall these definitions, but morphisms of these categories won’t be given in a classical way. Usually they are given as classes of morphisms between
objects of b(i) which satisfy some conditions. Here we are going to express them in term of limit and colimit.

Let b : I → CAT be a 2-functor. Let us give some useful notations.

Let i, i' ∈ I and I_{ii'} be the category defined by:

- the objects of I_{ii'} are:
  \[ \{(i'', s, s') \mid i'' \in \text{Ob} I, s \in \text{Hom}_I(i, i''), s' \in \text{Hom}_I(i', i'')\} \]

visualized by:

- the morphisms of I_{ii'} from (i''_1, s_1, s'_1) to (i''_2, s_2, s'_2) are:
  \[ \text{Hom}_{I_{ii'}}((i''_1, s_1, s'_1), (i''_2, s_2, s'_2)) = \{ t \in \text{Hom}_I(i''_1, i''_2) \mid t \circ s_1 = s_2, t \circ s'_1 = s'_2 \} \]

visualized by:

Lemma 4. If the category I is filtered then the category I_{ii'} is filtered.

Proof. The proof is straightforward.

Proposition 5. The category B defined below is a 2-colimit of b satisfying the strong factorization property.

- Objects of B are pairs (i, X) where i ∈ I and X ∈ b(i),

- let (i, X) and (i', Y) be two objects of B, the morphisms from (i, X) to (i', Y) are the elements of the colimit:
  \[ \text{Hom}_B(X, Y) = \lim_{(i'', s, s') \in I_{ii'}} \text{Hom}_{b(i'')}((b(s)_1(X), b(s')_1(Y)) \]

where if t ∈ Hom_{I_{ii'}}((i''_1, s_1, s'_1), (i''_2, s_2, s'_2)), its image in the inductive system is the following morphism:

\[ \text{Hom}_{b(i'')}(b(s)_1(X), b(s')_1(Y)) \rightarrow \text{Hom}_{b(i'')}(b(s)_2(X), b(s')_2(Y)) \]

\[ h \rightarrow b_{i', s'_1} \circ b(t) h \circ b_{i, s_1} \]

where b_{i, s_1} is the isomorphism given by the 2-functor b :

\[ b_{i, s_1} : b(t \circ s_1) \sim b(t) \circ b(s_1) \]

Proof. See for example [3].
If $\mathcal{I}_{i''}$ is filtered, the morphisms from $(i, X)$ to $(i'', Y)$ are the elements of the quotient:

$$\prod_{i'' \in \mathcal{I}_{i''}} \text{Hom}_{b(i'')}\left(b(s)X, b(s')Y\right) / \sim$$

Where, given $h_1 \in \text{Hom}_{b(i'')}\left(b(s_1)X, b(s'_1)Y\right)$ and $h_2 \in \text{Hom}_{b(i'')}\left(b(s_2)X, b(s'_2)Y\right)$, we write $h_1 \sim h_2$ if and only if there exist $(i_3, s_3, s'_3) \in \mathcal{I}_{i''}, t_{13} \in \text{Hom}\left((i_1, s_1, s'_1), (i_3, s_3, s'_3)\right)$ and $t_{23} \in \text{Hom}\left((i_2, s_2, s'_2), (i_3, s_3, s'_3)\right)$ such that:

$$b_{t_{13}, s_1}^{-1} \circ b(t_{13})h_1 \circ b_{t_{13}, s_1} = b_{t_{23}, s_2}^{-1} \circ b(t_{23})h_2 \circ b_{t_{23}, s_2}$$

Similarly we are going to give an explicit construction of a 2-limit. As before morphisms of this category will be given by a limit. Let $\mathcal{C}$ be a contravariant 2-functor:

$$\mathcal{C} : \mathcal{J}^{\text{op}} \longrightarrow \text{CAT}$$

**Proposition 6.** The category $\mathcal{C}$ define below, is a 2-limit of $\mathcal{C}$ satisfying the strong factorization property.

1. **The objects of $\mathcal{C}$ are pairs** $(X, \vartheta^X)$ **where**
   - $X = \{X_j\}_{j \in \mathcal{J}}$, for $X_j \in \text{Ob}(\mathcal{C}(j))$,
   - $\vartheta^X = \{\vartheta^X_t\}_{t \in \text{Hom}_{\mathcal{J}}(j, j')}$ **where**, for $t \in \text{Hom}_{\mathcal{J}}(j, j')$, $\vartheta^X_t$ **is an isomorphism**:
     $$\vartheta^X_t : X_j \xrightarrow{\sim} \mathcal{C}(t)X_{j'}$$
   **satisfying the following conditions**:

   A) **for any** $j \in \mathcal{J}$ **we have** $\text{Id}_j = c_j(X_j) \circ \vartheta^X_{\text{Id}_j}$, where $c_j$ **is the morphism** given by the 2-functor $\mathcal{C}$:
   $$c_j : \mathcal{C}(\text{Id}_j) \xrightarrow{\sim} \text{Id}_{\mathcal{C}(j)}$$

   B) **for any two composable morphisms** $t : j \rightarrow j'$ **and** $t' : j' \rightarrow j''$ **the following equation holds**:
   $$\mathcal{C}(t)(\vartheta^X_{t'}) \circ \vartheta^X_t = c_{t,t'}(X_{j''}) \circ \vartheta^X_{t't''}$$
   **where** $c_{t,t'}$ **is the isomorphism** given by the 2-functor $\mathcal{C}$:
   $$c_{t,t'} : \mathcal{C}(t \circ t') \xrightarrow{\sim} \mathcal{C}(t) \circ \mathcal{C}(t')$$

2. **Let** $(X, \vartheta^X)$ **and** $(Y, \vartheta^Y)$ **be two objects of** $\mathcal{C}$, **morphisms from** $(X, \vartheta^X)$ **to** $(Y, \vartheta^Y)$ **are elements of the limit**:
   $$\text{Hom}_C(X, Y) = \lim_{j \in \mathcal{J}} \text{Hom}_{\mathcal{C}(j)}(X_j, Y_j)$$
   **where if** $t \in \text{Hom}_{\mathcal{J}}(j, j')$, **its image in the projective system is the following morphism**:
   $$\text{Hom}_{\mathcal{C}(j)}(X_j, Y_j) \xrightarrow{h} \text{Hom}_{\mathcal{C}(j')}(X_{j'}, Y_{j'}) \xrightarrow{(\vartheta^Y_t)^{-1} \circ \mathcal{C}(t)(h) \circ \vartheta^X_t}$$

   This means that a morphism between two objects is the datum of $\{h_j\}_{j \in \mathcal{J}}$, **where** $h_j \in \text{Hom}_{\mathcal{C}(j)}(X_j, Y_j)$ **satisfies the equality**:
   $$h_j = (\vartheta^Y_t)^{-1} \circ \mathcal{C}(t)(h_{j'}) \circ \vartheta^X_t$$
   **for all** $t : j' \rightarrow j$ **morphisms of** $\mathcal{J}$.
3 Interchange of filtered 2-colimits and finite 2-limits

We are going to show that filtered 2-colimits commute with finite 2-limits. Let us give first a precise meaning to this sentence.

Let \( I \) be a filtered category, \( J \) a finite category and \( a \) a 2-functor:

\[
a : I \times J^{op} \rightarrow \text{CAT}
\]

Let us denote \( 2\mathfrak{F}(C) \) the 2-category of 2-functors going from the category \( C \) to the 2-category \( \text{CAT} \). We have the following proposition, for a proof see for example [3]:

**Proposition 7.** The correspondence:

\[
2\mathfrak{F}(C) \rightarrow \text{CAT} \quad b \mapsto \lim_{i \in I} b(i)
\]

can be extended to a 2-functor between 2-categories. A similar statement holds for 2-limits.

Now let us consider, the natural 2-functor:

\[
\mathcal{J} \rightarrow 2\mathfrak{F}(\mathcal{I}) \quad j \mapsto a(\cdot, j).
\]

The composition of this 2-functor and the one defined in the proposition gives a functor from \( \mathcal{J} \) to \( \text{CAT} \). As \( \text{CAT} \) is complete we can consider its limit. Let us denote

\[
\lim_{j \in \mathcal{J}} \lim_{i \in I} a(i, j)
\]

this limit. We define in the same way the 2-colimit \( \lim_{i \in I} \lim_{j \in \mathcal{J}} a(i, j) \).

**Remark**

The composition, for all \( i \in \mathcal{I} \) and \( j \in \mathcal{J} \), of the functor define by the 2-colimit and the 2-limit:

\[
\lim_{j' \in \mathcal{J}} a(i, j') \rightarrow a(i, j) \rightarrow \lim_{i' \in \mathcal{I}} a(i', j)
\]

defines a functor:

\[
\Psi : \lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} a(i, j) \rightarrow \lim_{j \in \mathcal{J}} \lim_{i \in \mathcal{I}} a(i, j)
\]

**Theorem 8.** The natural functor

\[
\Psi : \lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} a(i, j) \rightarrow \lim_{j \in \mathcal{J}} \lim_{i \in \mathcal{I}} a(i, j)
\]

is an equivalence of categories.
Proof. The fact that \( \Psi \) is fully faithful comes directly from the expression of the morphisms of these two categories in terms of limits and colimits.

In detail, using the two propositions \([5]\) and \([6]\) we can give an explicit construction of the categories \( \lim_{i \in I} \lim_{j \in J} a(i, j) \) and \( \lim_{j \in J} \lim_{i \in I} a(i, j) \).

The category \( \lim_{i \in I} \lim_{j \in J} a(i, j) \) is defined as follows:

- its objects are the triples \((i, X, \vartheta^X)\) where
  - \(i\) is an object of \(I\),
  - \(X = \{X_j\}_{j \in J}\) for \(X_j\) is an object of \(a(i, j)\)
  - \(\vartheta^X = \{\vartheta^X_i\}_{t \in Hom_J}\), where, for \(t : j' \to j\) an isomorphism of \(J\), \(\vartheta^X_t\) is a morphism:
    \[\vartheta^X_t : X_j \to a(Id_i, t)X_{j'}\]

verifying the two following conditions:

- for any \(j \in \text{Obj}_J\) we have \(a_{ij}(X_{ij}) \circ \vartheta^X_{id_i} = Id_{X_i}\), where \(a_{ij}\) is the isomorphism \(a_{ij} : a(Id_i, Id_j) \sim Id_{a(i, j)}\) given by the 2-functor \(a\),
- for any two composable morphisms \(t : j \to j'\) and \(t' : j' \to j''\) the following equation holds:
  \[a(Id_i, t)(\vartheta^X_t) \circ \vartheta^X_{id_i} = a(Id_i, t) \circ a(Id_i, t')a(Id_i, t)\]

where \(a(Id_i, t'), \vartheta^X_{t'} : a(Id_i, t'ot) \sim a(Id_i, t)a(Id_i, t)\) is the morphism given by the 2-functor \(a\).

- the set of morphisms from \((i, X, \vartheta^X)\) to \((s', Y, \vartheta'^Y)\) is given by the limit:
  \[\text{Hom}(X, Y) := \lim_{i' \in I, j' \in J} \lim_{i \in I} \text{Hom}_{a(i', j')}(a(s, Id_j)X_j, a(s', Id_j)Y_{j'})\]

The category \( \lim_{j \in J} \lim_{i \in I} a(i, j) \) is defined as follows:

- the objects are pairs \((X, \vartheta^X)\), where
  - \(X = \{(i_j, X_{ij})\}_{j \in J}\) with \(i_j \in I\) and \(X_{ij} \in a(i, j)\),
  - \(\vartheta^X = \{[\vartheta^X_i]\}_{t \in Hom_J}\) where, for \(t \in Hom_J(j, j')\), \([\vartheta^X_t]\) belongs to the quotient:
    \[\prod_{i' \in I, j', j''} \text{Hom}_{a(i', j')}\left(a(s, Id_j)X_{ij}, a(s', t)X_{ij}\right) / \sim\]

and where \([\vartheta^X_t]\) satisfies the following equalities:

\[a_{ij}(X_{ij}) \circ \vartheta^X_{id_i} = [Id_{X_i}]\] (1)

\[a(Id_i, t)(\vartheta^X_t) \circ \vartheta^X_{id_i} = [a(Id_i, t')X_{ij'} \circ \vartheta^X_{t'ot}]\] (2)

and for any two composable morphisms \(t : j \to j'\) and \(t' : j' \to j''\),
the set of morphisms from \((X, \theta^X)\) to \((Y, \theta^Y)\) is given by the limit:

\[
\lim_{j \in J} \lim_{i' \in I_j} \text{Hom}_{a(i', j)}(a(s, Id_j)X_j, a(s', Id_j)Y_j)
\]

The natural functor between the 2-limits is:

\[
\varepsilon : \lim_{i \in I} \lim_{J \in J} a(i, j) \rightarrow \lim_{i \in I} \lim_{J \in J} a(i, j)
\]

Moreover, if \(X = (i, X, \theta^X)\) and \(Y = (i', Y, \theta^Y)\) are two objects of \(\lim_{i \in I} \lim_{J \in J} a(i, j)\),

the morphism from \(\text{Hom}_{a} \lim_{i \in I} \lim_{J \in J} (X, a)\) to \(\text{Hom}_{a} \lim_{i \in I} \lim_{J \in J} (X, a)\)

induced by \(\varepsilon\) is the natural morphism:

\[
\lim_{i \in I} \lim_{J \in J} \text{Hom}(a(s, Id_j)X_j, a(s', Id_j)Y_j) \rightarrow \lim_{i \in I} \lim_{J \in J} \text{Hom}(a(s, Id_j)X_j, a(s', Id_j)Y_j)
\]

As filtered colimits commute with finite limits, the morphism above is an isomorphism and \(\varepsilon\) is fully faithful.

The proof that \(\Psi\) is essentially surjective is similar to the proof of the commutation of filtered limits and finite colimits.

Let \(\left\{i_j, X_j, \{[\theta^X_i]\}\right\}\) an object of \(\lim_{i \in I} \lim_{J \in J} a(i, j)\).

Using the property \((ii)\) of a filtered category inductively, one proves that there exist an object \(k' \in I\) and, for any \(j \in J\), a morphism \(s'_j : i_j \rightarrow k'\) in \(I\).

Thus, for all \(t\) morphism of \(J\), \([\theta^X_t]\) can be viewed as the class of an object \(\theta^X_t\) of \(\text{Hom}(a(s_j, Id_j)X_j, a(s'_j, t)X'_j)\). Remark that even if the class \([\theta^X_t]\) satisfies the equalities (1) and (2), the objects \(\theta^X_t\) may not satisfy then. Using the property \((iii)\) of a filtered category inductively, one proves that there exist an object \(k\) of \(I\) and a morphism \(s_k : k' \rightarrow k\) such that all the equalities hold also for the objects \(\theta^X_t\). Hence the triple:

\[
\left(k, \{a(s_k \circ s_j, Id_j)X_j\}, \{a_{s_k,s'_j}^{-1} \circ \theta^X_t \circ a_{s_k,s'_j}\}\right)
\]

is an object of \(\lim_{i \in I} \lim_{J \in J} a(i, j)\). For \(j \in J\), the objects \(X_{i_j}\) and \(a(s_k \circ s_j, Id_j)X_{i_j}\)

are isomorphic and, for all morphism \(t\) of \(J\), we have

\([\theta^X_t] = [\theta^X_t]\)

Hence we have shown that \(\Psi\) is an equivalence of categories.

\[\square\]

References

[1] S. Mac Lane. *Categories for the working mathematician. Second edition.* Graduate Texts in Mathematics, 5. Springer-Verlag, 1998.

[2] R. Street. Categorical structures. *Handbook of algebra*, 1996.

[3] I. Waschkies. The stack of microlocal perverse sheaves. *Bull. Soc. Math. France* 132, 2004.