Eulerian polynomials and descent statistics

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May 15, 2017

Abstract

We prove several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials, extending results of Stembridge, Petersen, and Brändén. Additionally, we find $q$-exponential generating functions for $q$-analogues of these descent statistic polynomials that also keep track of the inversion number or inverse major index. We also present identities relating several of these descent statistic polynomials to refinements of type B Eulerian polynomials and flag descent polynomials by the number of negative letters of a signed permutation. Our methods include permutation enumeration techniques involving noncommutative symmetric functions, the modified Foata–Strehl action, and a group action of Petersen on signed permutations. Notably, the modified Foata–Strehl action yields an analogous relation between Narayana polynomials and the joint distribution of the peak number and descent number over 231-avoiding permutations, which we also interpret in terms of binary trees and Dyck paths.

Keywords: permutations, signed permutations, Eulerian polynomials, descent statistics, noncommutative symmetric functions, modified Foata–Strehl action

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*MS 050, Waltham, MA 02453
2010 Mathematics Subject Classification. Primary 05A05; Secondary 05A15, 05E05, 05E18.
1. Introduction

Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation in \( S_n \), the set of permutations of \( [n] = \{1, 2, \ldots, n\} \), which are called \( n \)-permutations. Also, let \( |\pi| \) be the length of \( \pi \)—so that \( |\pi| = n \) whenever \( \pi \in S_n \)—and let \( G := \bigcup_{n=0}^{\infty} S_n \). We say that \( i \in [n-1] \) is a descent of an \( n \)-permutation \( \pi \) if \( \pi_i > \pi_{i+1} \). Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences—or equivalently, maximal consecutive subsequences containing no descents—which we call increasing runs. For example, the descents of \( \pi = 85712643 \) are 1, 3, 6, and 7, and the increasing runs of \( \pi \) are 8, 57, 126, 4, and 3.

Let \( \text{des}(\pi) \) denote the number of descents of \( \pi \). Then it is clear that the number of increasing runs of \( \pi \) is \( \text{des}(\pi) + 1 \) when \( |\pi| \geq 1 \), i.e., when \( \pi \) is nonempty. The polynomial

\[
A_n(t) := \sum_{\pi \in S_n} t^{\text{des}(\pi) + 1}
\]

for \( n \geq 1 \) is called the \( n \)th Eulerian polynomial. We set \( A_0(t) = 1 \) by convention.\(^2\) The exponential generating function for Eulerian polynomials is well known:

\[
\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1 - t}{1 - te^{(1-t)x}}.
\]

The Eulerian polynomials have a rich history and appear in many contexts in combinatorics; see \cite{18} for a detailed exposition.

\(^1\)By convention, we take \( S_0 \) to consist of only the empty word.

\(^2\)In general, for all polynomials defined in this paper counting permutations by various statistics—such as \( P_n^{pk}(t) \), \( P_n^{(pk,des)}(y, t) \), etc.—we set the 0th polynomial to be 1 by convention.
The Eulerian polynomials are closely related to the distribution of other descent statistics: permutation statistics that depend only on the descent set and length of a permutation. Specifically, it is known that polynomials counting permutations by various descent statistics—the number of peaks, left peaks, and biruns—can be expressed in terms of Eulerian polynomials.

- We say that \( i \) (where \( 2 \leq i \leq n - 1 \)) is a peak of \( \pi = \pi_1 \pi_2 \cdots \pi_n \) if \( \pi_{i-1} < \pi_i > \pi_{i+1} \), and let \( \text{pk}(\pi) \) be the number of peaks of \( \pi \). For example, the peaks of \( \pi = 85712643 \) are 3 and 6, and so \( \text{pk}(\pi) = 2 \). The peak polynomials
  \[
  P_{\text{pk}}^n(t) := \sum_{\pi \in S_n} t^{\text{pk}(\pi) + 1}
  \]
are related to the Eulerian polynomials by the identity
  \[
  A_n(t) = \left(\frac{1 + t}{2}\right)^{n+1} P_{\text{pk}}^n\left(\frac{4t}{(1+t)^2}\right)
  \]
for \( n \geq 1 \), which was first stated explicitly by Stembridge \[25\] as a result of his theory of enriched \( P \)-partitions, but also follows from an earlier construction of Shapiro, Woan, and Getu \[19\] which was later rediscovered by Brändén as the “modified Foata–Strehl action” \[2\], a variant of a group action on permutations originally defined by Foata and Strehl \[6\]. Making a substitution yields the equivalent identity
  \[
  P_{\text{pk}}^n(t) = \left(\frac{2}{1+v}\right)^{n+1} A_n(v)
  \]
where \( v = \frac{2}{t}(1 - \sqrt{1-t}) - 1 \).

- We say that \( i \in [n-1] \) is a left peak of \( \pi \in S_n \) if either \( i \) is a peak, or if \( i = 1 \) and 1 is a descent. The number of left peaks of \( \pi \) is denoted \( \text{lpk}(\pi) \). For example, the left peaks of \( \pi = 85712643 \) are 1, 3 and 6, and so \( \text{lpk}(\pi) = 3 \). Let
  \[
  P_{\text{lpk}}^n(t) := \sum_{\pi \in S_n} t^{\text{lpk}(\pi)}
  \]
be the polynomial counting \( n \)-permutations by left peaks. Using a modification of enriched \( P \)-partitions called “left enriched \( P \)-partitions”, Petersen \[15\] Observation 3.1.2] proved the identity
  \[
  \sum_{k=0}^{n} \binom{n}{k} 2^k (1-t)^{n-k} A_k(t) = (1+t)^n P_{\text{lpk}}^n\left(\frac{4t}{(1+t)^2}\right)
  \]
for all \( n \). Equivalently,
  \[
  P_{\text{lpk}}^n(t) = \frac{1}{(1+v)^n} \sum_{k=0}^{n} \binom{n}{k} 2^k (1-v)^{n-k} A_k(v)
  \]
where again \( v = \frac{2}{t}(1 - \sqrt{1-t}) - 1 \).

\[\text{Equivalently, } i \text{ is a left peak of } \pi \text{ if it is a peak of the permutation } 0\pi \text{ obtained by prepending a letter } 0 \text{ to } \pi. \text{ We can also define a “right peak” in the analogous way, but it is easy to see from symmetry that the number of left peaks and the number of right peaks are equidistributed over } S_n.\]
A birun[^4] of a permutation is a maximal monotone consecutive subsequence—that is, an increasing run of length at least two or a decreasing run of length at least two—and the number of biruns of \( \pi \) is denoted \( \text{br}(\pi) \). For example, the biruns of \( \pi = 85712643 \) are \( 85 \), \( 57 \), \( 71 \), \( 126 \) and \( 643 \), and thus \( \text{br}(\pi) = 5 \). Define

\[
P_n^{\text{br}}(t) := \sum_{\pi \in S_n} t^{\text{br}(\pi)}.
\]

Using differential equations, David and Barton \[4\] proved the identity

\[
P_n^{\text{br}}(t) = \left( \frac{1 + t}{2} \right)^{n-1} (1 + v)^{n+1} A_n \left( \frac{1 - v}{1 + v} \right)
\]

for \( n \geq 1 \), where \( v = \sqrt{1 + t} \).

In this paper, we establish several new identities which similarly express polynomials counting permutations by certain descent statistics in terms of Eulerian polynomials, including refinements of the known results on \( p_k \) and \( l p_k \) proved by Stembridge and Petersen, respectively. Furthermore, we find expressions for \( q \)-exponential generating functions for \( q \)-analogues of these descent statistic polynomials that also keep track of the inversion number (or inverse major index; see Subsection 4.4), although there are no analogous expressions in terms of the \( q \)-Eulerian polynomials \( A_n(q, t) = \sum_{\pi \in S_n} q^{\text{inv}(\pi)} t^{\text{des}(\pi)+1} \). In particular, the descent statistics that we consider are the ordered pairs \((p_k, \text{des})\) and \((l p_k, \text{des})\), the number of “up-down runs” \( u d r \), and the triple \((l p_k, \text{val}, \text{des})\) where \( \text{val} \) is the number of “valleys”[^5].

The idea of descents has been extended to other finite Coxeter groups, the most important of which (from the perspective of permutation enumeration) are the hyperoctahedral groups \( B_n \), consisting of “signed permutations”. In particular, two notions of descent number for signed permutations are the “type B descent number” \( \text{des}_B \) and the “flag descent number” \( \text{fdes} \), which give rise to analogues of Eulerian polynomials: type B Eulerian polynomials and flag descent polynomials. We refine these polynomials to also keep track of the number of “negative letters” \( \text{neg} \) of a signed permutation, and uncover similar identities that relate the distribution of \((l p_k, \text{des})\) over \( S_n \) to that of \((\text{neg}, \text{des}_B)\) over \( B_n \)—which specializes to a connection between \( l p_k \) and \( \text{des}_B \) previously discovered by Petersen—and relate the distribution of \((l p_k, \text{val}, \text{des})\) over \( S_n \) to that of \((\text{neg}, \text{fdes})\) over \( B_n \), which specializes to a previously unknown connection between \( u d r \) and \( \text{fdes} \).

The methods that we employ are twofold. Our main results are obtained using general permutation enumeration techniques involving noncommutative symmetric functions developed by Gessel \[9\] and later extended by Gessel and the present author \[8, 28\]. Later, we use two group actions—the modified Foata–Strehl action and an action of Petersen on signed permutations—to prove generalizations of some of our basic results for \((p_k, \text{des}), (l p_k, \text{des}), \)

[^4]: Biruns are more commonly called *alternating runs*, but since the term “alternating run” is used for a different concept in the related papers [8, 28], we use the term “birun” which was suggested by Stanley [22, Section 4].

[^5]: We shall see that \((l p_k, \text{val}, \text{des})\) is equivalent to \((u d r, \text{des})\), but the former is easier to work with due to technical constraints surrounding the latter.
and \((lpk, \text{val}, \text{des})\). Notably, the generalized result for \((pk, \text{des})\) specializes to a result relating the Narayana polynomials and the \((pk, \text{des})\) polynomials for 231-avoiding permutations, which we also interpret in terms of binary trees and Dyck paths.

The organization of this paper is as follows. In Section 2, we review basic definitions from permutation enumeration and some classical results on counting permutations with a prescribed descent set, establish some preliminary facts about the aforementioned refinements of type B Eulerian polynomials and flag descent polynomials, and prove several identities relating the Eulerian polynomials, refined type B Eulerian polynomials, and refined flag descent polynomials. In Section 3, we introduce parts of the theory of noncommutative symmetric functions relevant to this work. In Section 4, we prove several identities involving noncommutative symmetric functions and use them to obtain our main results. Finally, in Section 5, we introduce the group actions of Brändén and Petersen and use them to generalize some of our results from Section 4.

A summary of every statistic appearing in this paper is given in the appendix.

2. Permutations and descents

2.1. Descent sets, compositions, and statistics

We begin by reviewing some basic material from permutation enumeration relating to descents. Recall that a descent of an \(n\)-permutation \(\pi\) is an index \(i \in [n-1]\) such that \(\pi_i > \pi_{i+1}\). The set of descents, or descent set, of a permutation \(\pi\) is denoted \(\text{Des}(\pi)\). We observed that the descents of a permutation separate it into increasing runs. In fact, the lengths of the increasing runs of a permutation determine its descents, and vice versa. Sometimes it is actually more convenient to represent a descent set of an \(n\)-permutation with a composition of \(n\) which encodes the lengths of its increasing runs.

Given a subset \(S \subseteq [n-1]\) with elements \(s_1 < s_2 < \cdots < s_j\), let \(\text{Comp}(S)\) be the composition \((s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)\) of \(n\), and given a composition \(L = (L_1, L_2, \ldots, L_k)\), let \(\text{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}\) be the corresponding subset of \([n-1]\). Then, \(\text{Comp}\) and \(\text{Des}\) are inverse bijections. If \(\pi\) is an \(n\)-permutation with descent set \(S \subseteq [n-1]\), then we call \(\text{Comp}(S)\) the descent composition of \(\pi\), which we also denote by \(\text{Comp}(\pi)\). Note that the descent composition of \(\pi\) gives the lengths of the increasing runs of \(\pi\). Conversely, if \(\pi\) has descent composition \(L\), then its descent set \(\text{Des}(\pi)\) is \(\text{Des}(L)\).

We partially order compositions of \(n\) by reverse refinement, that is, \(L = (L_1, \ldots, L_k)\) covers \(M\) if and only if \(M\) can be obtained from \(L\) by replacing two consecutive parts \(L_i\) and \(L_{i+1}\) with \(L_i + L_{i+1}\). For example, we have \((7, 6) < (1, 2, 4, 5, 1)\). Note that if \(L\) and \(M\) are descent compositions, then \(L \leq M\) under this ordering if and only if \(\text{Des}(L) \subseteq \text{Des}(M)\); in other words, \(\text{Comp}\) and \(\text{Des}\) are order-preserving bijections.

A permutation statistic \(\text{st}\) is called a descent statistic if it depends only on the descent composition, that is, if \(\text{Comp}(\pi) = \text{Comp}(\sigma)\) implies \(\text{st}(\pi) = \text{st}(\sigma)\) for any two \(\pi, \sigma \in \mathcal{S}\). Equivalently, \(\text{st}\) is a descent statistic if it depends only on the descent set and length of a permutation. Examples of descent statistics include all of the statistics discussed in the introduction: the descent number des, the peak number pk, the left peak number lpk, and the number of biruns br. Ordered tuples of descent statistics such as \((pk, \text{des})\) and \((lpk, \text{des})\) are also descent statistics.
Two further examples of descent statistics are the number of “valleys” and the number of “up-down runs”, defined as follows. We say that $i$ (where $2 \leq i \leq n - 1$) is a valley of $\pi$ if $\pi_{i-1} > \pi_i < \pi_{i+1}$, and the number of valleys of $\pi$ is denoted $\text{val}(\pi)$. The valleys of $\pi = 85712643$ are 2 and 4, so $\text{val}(\pi) = 2$.

An up-down run of a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is either a birun of $\pi$ or the letter $\pi_1$ if it is an increasing run of length 1, and the number of up-down runs of $\pi$ is denoted $\text{udr}(\pi)$. The up-down runs of $\pi = 85712643$ are 8, 85, 57, 71, 126, and 643, so $\text{udr}(\pi) = 6$.

The peak number $\text{pk}$ and valley number $\text{val}$—as well as the ordered pairs $(\text{pk}, \text{des})$ and $(\text{val}, \text{des})$—are equidistributed on $S_n$ due to symmetry, so we will not consider the valley number per se. However, the valley number and left peak number are closely related to the number of up-down runs, which we establish in the next lemma.

**Lemma 2.1.** Let $\pi \in S_n$ with $n \geq 1$. Then:

(a) $\text{udr}(\pi) = \text{lpk}(\pi) + \text{val}(\pi) + 1$

(b) $\text{lpk}(\pi) = \left\lfloor \frac{\text{udr}(\pi)}{2} \right\rfloor$

(c) $\text{val}(\pi) = \left\lfloor \frac{(\text{udr}(\pi) - 1)}{2} \right\rfloor$

(d) If $n - 1$ is a descent of $\pi$, then $\text{lpk}(\pi) = \text{val}(\pi) + 1$. Otherwise, $\text{lpk}(\pi) = \text{val}(\pi)$.

**Proof.** Every up-down run except the final one ends with either a left peak or a valley, and in fact these up-down runs alternate between ending with a left peak and ending with a valley, beginning with a left peak. For example, if $\text{udr}(\pi) = 5$, then the first up-down run ends with a left peak, the second ends with a valley, the third ends with a left peak, and the fourth ends with a valley. It is clear that this accounts for every left peak and every valley, which proves (a). Now, note that either $\text{lpk}(\pi) = \text{val}(\pi) + 1$ or $\text{lpk}(\pi) = \text{val}(\pi)$; this depends completely on whether the penultimate up-down run ends with a left peak or a valley, which is determined by whether the final up-down run is increasing or decreasing (i.e., whether the final run is long or short); this proves (d). Finally, (b) and (c) follow from (a) and (d). □

Lemma 2.1 shows that not only does $(\text{lpk}, \text{val})$ determines $\text{udr}$, but $\text{udr}$ also determines $(\text{lpk}, \text{val})$. In other words, $\text{udr}$ and $(\text{lpk}, \text{val})$ are equivalent permutation statistics.

An example of a permutation statistic that is not a descent statistic is the “inversion number”. An inversion in an $n$-permutation is a pair of indices $(i, j)$ with $1 \leq i < j \leq n$ such that $\pi_i > \pi_j$. Then the number of inversions of $\pi$ is denoted $\text{inv}(\pi)$. For example, the inversions of $\pi = 1432$ are $(4, 3)$, $(4, 2)$, and $(3, 2)$, so $\text{inv}(\pi) = 3$. It is well known that the polynomial counting $n$-permutations by inversion number is given by the $n$th $q$-factorial

$$[n]_q! := (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}),$$

i.e.,

$$\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]_q!.$$

We give two key remarks before continuing. First, the definitions and properties of descents, increasing runs, descent compositions, and descent statistics extend naturally to
words on any totally ordered alphabet such as \([n]\) or \(\mathbb{P}\) (the positive integers) if we replace the strict inequality \(<\) with the weak inequality \(\le\), which reflects the fact that increasing runs are allowed to be weakly increasing in this setting. For example, \(i\) is a peak of the word \(w = w_1w_2\cdots w_n\) if \(w_{i-1} \le w_i > w_{i+1}\).

Finally, recall that by definition, two permutations (or words) with the same descent composition must have the same value of \(st\) if \(st\) is a descent statistic. Hence, we shall use the notation \(st(L)\) to indicate the value of a descent statistic \(st\) on any permutation (or word) with descent composition \(L\).

### 2.2. Counting permutations with a prescribed descent set

If \(L = (L_1, \ldots, L_k)\) is a composition of \(n\), then we let \(l(L)\) denote the number of parts of \(L\) and we write \(\binom{n}{L}\) for the multinomial coefficient \(\binom{n}{L_1, \ldots, L_k}\). Similarly, we write \(\binom{n}{L}_q\) for the \(q\)-multinomial coefficient

\[
\binom{n}{L_1, \ldots, L_k}_q := \frac{[n]_q!}{[L_1]_q! [L_2]_q! \cdots [L_k]_q!}.
\]

**Lemma 2.2.** Let \(L\) be a composition of \(n\). Then:

(a) The number of \(n\)-permutations with descent composition \(K \leq L\) —or equivalently, with descent set contained in \(\text{Des}(L)\)—is the multinomial coefficient \(\binom{n}{L}\).

(b) The polynomial counting \(n\)-permutations with descent composition \(K \leq L\) —or equivalently, with descent set contained in \(\text{Des}(L)\)—by inversion number is the \(q\)-multinomial coefficient \(\binom{n}{L}_q\). That is,

\[
\sum_{\pi \in S_n \atop \text{Comp}(\pi) \leq L} q^{\text{inv}(\pi)} = \binom{n}{L}_q.
\]

See [23, Examples 2.2.4 and 2.2.5] for proofs. This result on counting \(n\)-permutations with a descent set contained in a prescribed set can then be used to count those with a prescribed descent set.

**Lemma 2.3.** Let \(L\) be a composition of \(n\). Then:

(a) The number \(\beta(L)\) of \(n\)-permutations with descent composition \(L\) —or equivalently, with descent set \(\text{Des}(L)\)—is given by the formula

\[
\beta(L) = \sum_{K \leq L} (-1)^{|l(L) - l(K)|} \binom{n}{K}.
\]  

(2.1)

(b) The polynomial

\[
\beta_q(L) := \sum_{\pi \in S_n \atop \text{Comp}(\pi) = L} q^{\text{inv}(\pi)}
\]
counting $n$-permutations with descent composition $L$—or equivalently, with descent set $\text{Des}(L)$—by inversion number is given by the formula

$$
\beta_q(L) = \sum_{K \leq L} (-1)^{l(L) - l(K)} \binom{n}{K} q^l(L).
$$

(2.2)

The proof of Lemma 2.3 is immediate from Lemma 2.2 and the inclusion-exclusion principle. Part (a) of Lemmas 2.2 and 2.3 were originally due to MacMahon [14], whereas part (b) of these lemmas were due to Stanley [20].

2.3. Type B permutations, descents, and Eulerian polynomials

Let $\mathcal{B}_n$ be the set of permutations $\pi = \pi_{-n} \cdots \pi_{-1} \pi_0 \pi_1 \cdots \pi_n$ of \{-$n$, \ldots, -$1$, 0, 1, \ldots, $n$\} satisfying $\pi_{-i} = -\pi_i$ for all $-$n $\leq i \leq$ n; we call these signed $n$-permutations (or type B $n$-permutations). Let $\mathcal{B} := \bigcup_{n=0}^{\infty} \mathcal{B}_n$. For any signed $n$-permutation $\pi$, we must have $\pi_0 = 0$ and $\pi$ is completely determined by $\{\pi_1, \ldots, \pi_n\}$, so we can write $\pi$ as $\pi = \pi_1 \cdots \pi_n$ with the understanding that $\pi_0 = 0$ and $\pi_{-i} = -\pi_i$ for all $i$. In this way, we can think of $\mathcal{B}_n$ as the subset of signed permutations in $\mathcal{B}_n$ with no negative letters among $\{\pi_1, \ldots, \pi_n\}$.

For cleaner notation, let us write $\bar{i}$ rather than $-i$ when writing out the letters of a signed permutation. For example, if $\pi = \pi_1 \pi_2 \pi_3$ with $\pi_1 = 3$, $\pi_2 = -2$, and $\pi_3 = -1$, then we write $\pi = 3\bar{2}\bar{1}$.

We say that $i \in \{0\} \cup [n-1]$ is a descent (or type B descent) of $\pi \in \mathcal{B}_n$ if $\pi_i > \pi_{i+1}$. Note that we allow 0 to be a descent, which happens precisely when $\pi_1$ is negative. There are two notions of descent number for signed permutations that we consider. The descent number (or type B descent number) $\text{des}_B(\pi)$ is simply the number of descents of $\pi \in \mathcal{B}_n$, whereas the flag descent number $\text{fdes}(\pi)$ is defined by

$$
\text{fdes}(\pi) := \begin{cases} 
2 \text{des}_B(\pi), & \text{if } \pi_1 > 0 \\
2 \text{des}_B(\pi) - 1, & \text{if } \pi_1 < 0;
\end{cases}
$$

that is, every descent except 0 is counted twice. For example, let $\pi = \bar{472635}$. Then the descents of $\pi$ are 0, 2, 3, and 6, so $\text{des}_B(\pi) = 4$ and $\text{fdes}(\pi) = 7$.

We define

$$
B_n(t) := \sum_{\pi \in \mathcal{B}_n} t^{\text{des}_B(\pi)}
$$

and

$$
F_n(t) := \sum_{\pi \in \mathcal{B}_n} t^{\text{fdes}(\pi)},
$$

which are type B analogues of Eulerian polynomials using the descent number and flag descent number, respectively. We call $B_n(t)$ the $n$th type B Eulerian polynomial and $F_n(t)$ the $n$th flag descent polynomial.

The exponential generating function

$$
\sum_{n=0}^{\infty} \frac{B_n(t)}{(1-t)^{n+1} n!} x^n = \frac{e^x}{1 - te^{2x}}
$$

(2.3)
was found by Steingrímsson [24], and the analogous formula for the flag descent polynomials
\[
\sum_{n=0}^{\infty} \frac{F_n(t)}{(1-t)(1-t^2)^n} \frac{x^n}{n!} = \frac{e^x}{1-te^x}
\] (2.4)
directly follows from a result of Adin, Brenti, and Roichman [1, Theorem 4.2].

We consider another statistic on \(B\): the number of negative letters
\[
\text{neg}(\pi) := \# \{\pi_i | \pi_i < 0 \text{ and } i \in [n]\}.
\]
So given \(\pi = \bar{4726351}\), we have \(\text{neg}(\pi) = 3\). We refine the polynomials \(B_n(t)\) and \(F_n(t)\) by this statistic, defining
\[
B_n(y, t) := \sum_{\pi \in B_n} y^{\text{neg}(\pi)} t^{\text{des}(\pi)}
\]
and
\[
F_n(y, t) := \sum_{\pi \in B_n} y^{\text{neg}(\pi)} t^{\text{fdes}(\pi)}.
\]
Later, we will relate the polynomials \(B_n(y, t)\) to the joint distribution of \(\text{lpk}\) and \(\text{des}\) over \(S_n\), and similarly \(F_n(y, t)\) with the joint distribution of \(\text{lpk}, \text{val},\) and \(\text{des}\). In doing so, we shall need the following exponential generating functions for these polynomials.

**Theorem 2.4.**
\[
\sum_{n=0}^{\infty} \frac{B_n(y, t)}{(1-t)^{n+1}} \frac{x^n}{n!} = \frac{e^x}{1-te^{(1+y)x}}
\]
Note that setting \(y = 1\) yields Steingrímsson’s formula (2.3).

**Proof.** We begin by proving the identity
\[
\frac{B_n(y, t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} (ky + (k + 1)) t^k,
\] (2.5)
which was first established by Petersen [16] using a different method.

Consider the left-hand side. Each term in \(B_n(y, t)\) corresponds to a signed \(n\)-permutation with a vertical bar inserted after each letter corresponding to a descent (and an initial bar if 0 is a descent). For example, if we have \(\pi = \bar{4726351}\), then we write this as
\[|47|2|635|1|\.
\]
The \(1/(1-t)^{n+1}\) factor corresponds to inserting any number of bars in any of the \(n+1\) positions between letters, before the first letter, or after the final letter. So for example, continuing from above, we may have
\[|47|2||63||5|1|\.
\]
Thus the left-hand side of (2.5) counts the number of signed \(n\)-permutations with any number of bars inserted in any of the \(n+1\) positions and at least one bar in every position...
corresponding to a descent, where $y$ is weighting the number of negative letters and $t$ is weighting the number of bars.

We claim that the right-hand side counts these same barred signed $n$-permutations. Fix $k \geq 0$; this is the number of bars. The bars create $k + 1$ “boxes” for inserting letters. For every $i \in [n - 1]$, we make two choices: whether or not to make it negative, and which box to put it into. The letters in each box are then placed in increasing order. Note that the first box cannot contain any negative letters; otherwise, 0 would be a descent, but there would not be a bar preceding the first letter. Thus, if a letter is made negative, then it contributes a weight of $y$ and we can place it in any of the $k$ boxes after the first one. If a letter remains positive, then it can be placed into any of the $k + 1$ boxes. Since there are $n$ letters and the choices are made independently, we have a total contribution of $(ky + (k + 1)) t^k$ in the case where there are $k$ bars in total. Summing over all $k$ yields the right-hand side of (2.5).

Now, observe that

$$\sum_{n=0}^{\infty} \frac{B_n(y, t)}{(1-t)^{n+1}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (ky + (k + 1)) t^k \frac{x^n}{n!}$$

$$= \sum_{k=0}^{\infty} e^{(ky + (k+1))x} t^k$$

$$= e^x \sum_{k=0}^{\infty} (e^{(1+y)x})^k t^k$$

$$= \frac{e^x}{1 - te^{(1+y)x}}$$

thus completing the proof.

\textbf{Theorem 2.5.}

$$\sum_{n=0}^{\infty} \frac{F_n(y, t)}{(1-t)(1-t^2)^n} \frac{x^n}{n!} = \frac{e^x + te^{(1+y)x}}{1 - t^2 e^{(1+y)x}}$$

Setting $y = 1$ yields the formula (2.3) of Adin, Brenti, and Roichman.

\textbf{Proof.} We first prove the identity

$$\frac{F_n(y, t)}{(1-t)(1-t^2)^n} = \sum_{k=0}^{\infty} (ky + (k + 1)) t^{2k} + \sum_{k=0}^{\infty} ((k + 1)(y + 1)) t^{2k+1}. \quad (2.6)$$

Each term in $F_n(y, t)$ corresponds to a signed $n$-permutation with an arrangement of bars, but now each nonzero descent contributes a weight of $t^2$, which corresponds to two bars. We write each $\pi \in \mathfrak{B}_n$ as $\pi = \pi_{-n} \cdots \pi_{-1} \pi_1 \cdots \pi_n$ (without $\pi_0 = 0$). For each descent $i \in [n - 1]$, we insert a bar immediately after $\pi_i$ and a bar immediately before $\pi_{i+1}$. If $i = 0$ is a descent, then we insert a single bar between $\pi_{-1}$ and $\pi_1$. For example, for $\pi = 4726351$, we have $1|536|274|47|2|635|1$.

The $1/(1 - t)$ factor corresponds to inserting any number of bars in the central position (between $\pi_{-1}$ and $\pi_1$), and the $1/(1 - t^2)^n$ factor corresponds to inserting any number of
bars in any of the $n$ positions to the right of the central position, and for each of these bars, a corresponding bar in the position symmetric about the center. For example, we may have

$$|1|5||36||2|74|47|2||63||5|1|.$$  

We claim that the right-hand side of (2.6) counts the same arrangements. We consider two cases: the number of bars is even or the number of bars is odd.

- Suppose that the number of bars is $2k$ for some $k \geq 0$. These bars create $2k + 1$ boxes; we will be inserting letters into the right-most $k + 1$ boxes. Again, for each letter, we decide whether or not to make it negative and decide which box to put it in. If a letter is made negative, then it contributes a weight of $y$ and it can only be inserted into the final $k$ boxes; if a letter is not made negative, then we can insert it into any of the right-most $k + 1$ boxes. Order the letters in each box in increasing order, and for each letter, insert its negative into the position symmetric about the center. These are precisely the arrangements that we want; thus we have the term $(ky + (k+1))n t^{2k}$, and summing over $k$ gives the total contribution from having an even number of bars.

- Suppose that the number of bars is $2k + 1$ for some $k \geq 0$. Then these bars create $2k + 2$ boxes. In this case, both positive and negative letters can be inserted into any of the right-most $k + 1$ boxes; if a negative letter is inserted in the $(k+2)$nd box, then the central bar acts as the bar corresponding to the 0 descent. Hence, this contributes $((k+1)(y+1))n t^{2k}$, and summing over $k$ gives the total contribution from having an odd number of bars.

Now, observe that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (ky + (k + 1)) t^{2k} \frac{x^n}{n!} = \sum_{k=0}^{\infty} e^{(k+1)y+1} x t^{2k}$$

$$= e^x \sum_{k=0}^{\infty} (t^2 e^{(1+y)x})^k$$

$$= \frac{e^x}{1 - t^2 e^{(1+y)x}}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} ((k+1)(y+1)) t^{2k+1} \frac{x^n}{n!} = \sum_{k=0}^{\infty} e^{(k+1)y+1} x t^{2k+1}$$

$$= te^{(1+y)x} \sum_{k=0}^{\infty} (t^2 e^{(1+y)x})^k$$

$$= \frac{te^{(1+y)x}}{1 - t^2 e^{(1+y)x}};$$

adding these expressions completes the proof. \qed
2.4. Several new Eulerian polynomial identities

Before proceeding, we prove several new identities relating the Eulerian polynomials $A_n(t)$, refined type B Eulerian polynomials $B_n(y,t)$, and refined flag descent polynomials $F_n(y,t)$. Unlike the main results of this paper which will be presented in Section 4, these results can be obtained simply using the exponential generating functions established in the previous subsection and will not require the more sophisticated techniques introduced in the next section.

**Theorem 2.6.** For $n \geq 0$, we have

$$B_n(y,t) = \sum_{k=0}^{n} \binom{n}{k} (1+y)^k (1-t)^{n-k} A_k(t).$$

*Proof.* Taking Theorem 2.4, multiplying both sides by $1-t$, and then replacing $x$ with $(1-t)x/(1+y)$ yields

$$\sum_{n=0}^{\infty} B_n(y,t) \frac{x^n}{(1+y)^n n!} = \frac{1-t}{1-te(1+y)x}$$

$$= \left( \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} \right) \left( \sum_{n=0}^{\infty} \left( \frac{1-t}{1+y} \right)^n \frac{x^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1-t}{1+y} \right)^{n-k} A_k(t) \frac{x^n}{n!}.$$

Equating the coefficients of $x^n/n!$ and multiplying both sides by $(1+y)^n$ yields the result. \( \square \)

By setting $y = 1$, we obtain the following corollary.

**Corollary 2.7.** For $n \geq 0$, we have

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} 2^k (1-t)^{n-k} A_k(t).$$

**Theorem 2.8.** For $n \geq 1$, we have

$$F_n(y,t) = \frac{1}{1+t} \left( \frac{1+y}{t} \right)^n A_n(t^2) + \sum_{k=0}^{n} \binom{n}{k} (1+y)^k (1-t^2)^{n-k} A_k(t^2).$$

*Proof.* It is readily checked that the statement of Theorem 2.5 is equivalent to

$$\frac{1}{1-t} \left( \sum_{n=1}^{\infty} F_n(y,t) \frac{x^n}{(1-t^2)^n n!} \right) = \frac{1+te^x}{1-t^2e^{(1+y)x}}.$$
Multiplying both sides by \(1 - t^2\) and replacing \(x\) with \((1 - t^2)x/(1 + y)\) yields

\[
(1 + t) \left(1 + t \sum_{n=1}^{\infty} \frac{F_n(y, t)}{(1 + y)^n} n! \right) = \frac{1 - t^2}{1 - t^2 e^{(1+t^2)x}} (1 + te^{1+t^2x})
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{A_n(t^2)}{n!} x^n \right) (1 + t \sum_{n=0}^{\infty} \left( \frac{1 - t^2}{1 + y} \right)^n x^n n!)
\]

\[
= \sum_{n=0}^{\infty} \left( A_n(t^2) + t \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1 - t^2}{1 + y} \right)^{n-k} A_k(t^2) \right) x^n n!.
\]

Equating the coefficients of \(x^n/n!\) and dividing both sides by \(t(1 + t)/(1 + y)^n\) yields the result.

We can set \(y = 1\) to obtain an identity relating \(F_n(t)\) and \(A_n(t)\), but the nicer identity

\[
F_n(t) = \frac{(1 + t)^n}{t} A_n(t) \tag{2.7}
\]

can be obtained by directly comparing the generating functions of \(F_n(t)\) and \(A_n(t)\), and can also be recovered as a specialization of a more general identity of Adin, Brenti, and Roichman [1, Theorem 4.4].

**Theorem 2.9.** For \(n \geq 1\), we have

\[
F_n(y, t) = \frac{1}{1 + t} \left( B_n(y, t^2) + \frac{1}{t} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 - t^2)^{n-k} B_k(y, t^2) \right).
\]

**Proof.** Due to Theorem 2.8, it suffices to show that

\[
B_n(y, t^2) = \sum_{k=0}^{n} \binom{n}{k} (1 + y)^k (1 - t^2)^{n-k} A_k(t^2)
\]

and that

\[
\frac{(1 + y)^n}{t} A_n(t^2) = \frac{1}{t} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 - t^2)^{n-k} B_k(y, t^2).
\]

Note that Theorem 2.6 directly implies the former equation, whereas a simple application of inclusion-exclusion to Theorem 2.6 implies the latter equation.

Rather than stating the result of setting \(y = 1\) in Theorem 2.9 we give a simpler identity relating the polynomials \(F_n(t)\) and \(B_n(t)\) using (2.7).

**Corollary 2.10.** For \(n \geq 1\), we have

\[
F_n(t) = \frac{1}{t} \left( \frac{1 + t}{2} \right)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 - t)^{n-k} B_k(t).
\]

**Proof.** By applying inclusion-exclusion to Corollary 2.7 we obtain

\[
A_n(t) = \frac{1}{2n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 - t)^{n-k} B_k(t). \tag{2.8}
\]

Combining (2.7) and (2.8) yields the result.
3. Noncommutative symmetric functions

3.1. Definitions

In this section, we introduce relevant aspects of the theory of noncommutative symmetric functions, which were formally introduced by Gelfand, et al. [7] in 1995 but were implicitly utilized in previous work, such as [9]. Throughout this section, fix a field \( F \) of characteristic zero. (We can take \( F \) to be \( \mathbb{C} \) in subsequent sections.) Then \( F \langle \langle X_1, X_2, \ldots \rangle \rangle \) is the \( F \)-algebra of formal power series in countably many noncommuting variables \( X_1, X_2, \ldots \). Consider the elements

\[
h_n := \sum_{i_1 \leq \cdots \leq i_n} X_{i_1} X_{i_2} \cdots X_{i_n}
\]

of \( F \langle \langle X_1, X_2, \ldots \rangle \rangle \), which are noncommutative versions of the complete symmetric functions \( h_n \). Note that \( h_n \) is the noncommutative generating function for weakly increasing words of length \( n \) on the alphabet \( \mathbb{P} \). For example, the weakly increasing word \( 13449 \) is encoded by \( X_1 X_3 X_2^2 X_9 \), which appears as a term in \( h_5 \). Given a composition \( L = (L_1, \ldots , L_k) \), we let

\[
h_L := h_{L_1} \cdots h_{L_k}.
\]

Equivalently,

\[
h_L = \sum_L X_{i_1} X_{i_2} \cdots X_{i_n}
\]

where the sum is over all \( (i_1, \ldots , i_n) \) satisfying

\[
\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1}, \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2}, \ldots , \underbrace{i_{L_1+\ldots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k},
\]

so \( h_L \) is the noncommutative generating function for words in \( \mathbb{P} \) whose descent composition \( K \) satisfies \( K \leq L \) in the reverse refinement ordering.

The \( F \)-algebra generated by the elements \( h_L \) is called the algebra \( \text{Sym} \) of noncommutative symmetric functions with coefficients in \( F \), which is a subalgebra of \( F \langle \langle X_1, X_2, \ldots \rangle \rangle \).

Next, let \( \text{Sym}_n \) be the vector space of noncommutative symmetric functions homogeneous of degree \( n \), so \( \text{Sym}_n \) is spanned by \( \{ h_L \}_{L \vdash n} \) where \( L \vdash n \) indicates that \( L \) is a composition of \( n \), and \( \text{Sym} \) is a graded \( F \)-algebra with

\[
\text{Sym} = \bigoplus_{n=0}^{\infty} \text{Sym}_n.
\]

For a composition \( L = (L_1, \ldots , L_k) \), we also define

\[
r_L := \sum_L X_{i_1} X_{i_2} \cdots X_{i_n}
\]

\[6\text{The algebra } \text{Sym} \text{ is defined differently in [7], but the algebras are the same.} \]
where the sum is over all \((i_1, \ldots, i_n)\) satisfying
\[
\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} \succ \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} \succ \cdots \succ \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}.
\]

Then, \(r_L\) is the noncommutative generating function for words on the alphabet \(P\) with descent composition \(L\).

Note that
\[
h_L = \sum_{K \leq L} r_K,
\]
and by inclusion-exclusion,
\[
r_L = \sum_{K \leq L} (-1)^{(L)-(K)} h_K.
\]
Hence, the \(r_L\) are noncommutative symmetric functions, and are in fact noncommutative versions of the ribbon Schur functions \(r_L\).

Since \(r_L\) and \(r_M\) have no terms in common for \(L \neq M\), it is clear that \(\{r_L\}_{L=\mathbb{R}}\) is linearly independent. From (3.1), we see that \(\{r_L\}_{L=\mathbb{R}}\) spans \(\text{Sym}_n\), so \(\{r_L\}_{L=\mathbb{R}}\) is a basis for \(\text{Sym}_n\). Because \(\{h_L\}_{L=\mathbb{R}}\) spans \(\text{Sym}_n\) and has the same cardinality as \(\{r_L\}_{L=\mathbb{R}}\), we conclude that \(\{h_L\}_{L=\mathbb{R}}\) is also a basis for \(\text{Sym}_n\).

Finally, let us consider the noncommutative generating function
\[
e_n := \sum_{i_1 > \cdots > i_n} X_{i_1} X_{i_2} \cdots X_{i_n}
\]
for decreasing words of length \(n\) on the alphabet \(P\). If we let
\[
h(x) := \sum_{n=0}^{\infty} h_n x^n \in \text{Sym}[[x]]
\]
be the generating function for the noncommutative complete symmetric functions \(h_n\) and let
\[
e(x) := \sum_{n=0}^{\infty} e_n x^n
\]
be the generating function for the \(e_n\), then it can be shown (see [9, p. 38] or [7, Section 7.3]) that
\[
e(x) = h(-x)^{-1}.
\]
Since \(h(-x)^{-1} \in \text{Sym}[[x]]\), it follows that the \(e_n\) are also noncommutative symmetric functions. The generating functions for the ordinary (commutative) elementary symmetric functions \(e_n\) and for the complete symmetric functions \(h_n\) satisfy the same relation as (3.3), so we see that the \(e_n\) are noncommutative versions of the elementary symmetric functions \(e_n\).

Although we won’t need to use this fact in this paper, it is worth noting that for a composition \(L = (L_1, \ldots, L_k)\) of \(n\), we can define
\[
e_L := e_{L_1} e_{L_2} \cdots e_{L_k}
\]
and \(\{e_L\}_{L=n}\) is a third basis for \(\text{Sym}_n\). This can be proven using a noncommutative analogue of the \(\omega\) involution for ordinary symmetric functions (see [21, Section 7.6]).
3.2. Homomorphisms

Our main results in the next section are obtained by applying certain homomorphisms to various identities involving noncommutative symmetric functions. Although noncommutative symmetric functions are generating functions for words, we shall see that these homomorphisms allow us to move from the realm of word enumeration to that of permutation enumeration. We define these homomorphisms \( \Phi : \text{Sym} \rightarrow F[[x]] \) by \( \Phi(h_n) = x^n/n! \), and \( \Phi_q : \text{Sym} \rightarrow F(q)[[x]] \) by \( \Phi_q(h_n) = x^n/[n]_q! \). Then if \( L \) is a composition of \( n \), we have

\[
\Phi(h_L) = \frac{x^{L_1}}{L_1!} \cdots \frac{x^{L_k}}{L_k!} = \binom{n}{L} \frac{x^n}{n!}
\]

and

\[
\Phi_q(h_L) = \frac{x^{L_1}}{[L_1]_q!} \cdots \frac{x^{L_k}}{[L_k]_q!} = \binom{n}{L}_q \frac{x^n}{[n]_q!}.
\]

For our proofs, we also need to determine the effect of \( \Phi \) and \( \Phi_q \) on \( r_L, h(1) = \sum_{n=0}^{\infty} h_n \), and \( e(1) = \sum_{n=0}^{\infty} e_n \). Recall that \( \beta(L) \) is the number of \( n \)-permutations with descent composition \( L \) and that \( \beta_q(L) \) is the polynomial counting \( n \)-permutations with descent composition \( L \) by inversion number.

**Lemma 3.1.**

(a) Let \( L \) be a composition of \( n \). Then \( \Phi(r_L) = \beta(L)x^n/n! \).

(b) \( \Phi(h(1)) = e^x \).

(c) \( \Phi(e(1)) = e^x \).

**Proof.** Part (a):

\[
\Phi(r_L) = \Phi \left( \sum_{K \leq L} (-1)^{i(L) - i(K)} h_K \right), \text{ by (3.2)}
\]

\[
= \sum_{K \leq L} (-1)^{i(L) - i(K)} \Phi(h_K)
\]

\[
= \sum_{K \leq L} (-1)^{i(L) - i(K)} \binom{n}{K} \frac{x^n}{n!}
\]

\[
= \beta(L) \frac{x^n}{n!}, \text{ by (2.1)}.
\]

Part (b):

\[
\Phi(h(1)) = \Phi \left( \sum_{n=0}^{\infty} h_n \right) = \sum_{n=0}^{\infty} \Phi(h_n) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.
\]

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Part (c):

\[ \Phi(e(1)) = \Phi(h(-1)^{-1}), \text{ by (5.3)} \]
\[ = \left( \sum_{n=0}^{\infty} (-1)^n \Phi(h_n) \right)^{-1} \]
\[ = \left( \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right)^{-1} \]
\[ = (e^{-x})^{-1} \]
\[ = e^x. \]

Consider the \( q \)-exponential function

\[ \exp_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \]

and its variant

\[ \text{Exp}_q(x) := \sum_{n=0}^{\infty} q^{(2)}_n \frac{x^n}{[n]_q!} \]

both \( q \)-analogues of the classical exponential function \( e^x \). It is well known that \( \text{Exp}_q(x) = (\exp_q(-x))^{-1} \). Using the same ideas as in the proof of Lemma 3.1 it is easy to verify the following lemma.

**Lemma 3.2.**

(a) Let \( L \) be a composition of \( n \). Then \( \Phi_q(r_L) = \beta_q(L)x^n/[n]_q! \).

(b) \( \Phi_q(h(1)) = \exp_q(x) \).

(c) \( \Phi_q(e(1)) = \text{Exp}_q(x) \).

4. Main results

4.1. Peaks and descents

Consider the polynomial

\[ P_n^{(pk, \text{des})}(y, t) := \sum_{\pi \in \mathfrak{S}_n} y^{pk(\pi) + 1} t^{\text{des}(\pi) + 1} \]

which refines the Eulerian polynomial \( A_n(t) \) and the peak polynomial \( P_n^{pk}(t) \). We prove in our first theorem an identity expressing \( P_n^{(pk, \text{des})}(y, t) \) in terms of \( A_n(t) \). To do so, we need to establish the following noncommutative symmetric function identity.
Lemma 4.1.

\[(1 - te(yx)h(x))^{-1} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \subseteq n} \frac{t^{pk(L)+1}(y + t)^{des(L) - pk(L)}(1 + yt)^{n - pk(L) - des(L) - 1}(1 + y)^{2pk(L)+1}}{(1 - t)^{n+1}} x^n r_L\]

Proof. Let \(\mathbb{P} = \{1, 2, 3, \ldots\}\) denote the set of positive integers decorated with underlines, endowed with the usual total ordering of \(\mathbb{P}\). Let us say that a word \(w\) on the alphabet \(\mathbb{P} \cup \mathbb{P} \cup \{\|\}\) (that is, the positive integers, underlined positive integers, and a vertical bar) is a peak word if \(w\) can be written as a sequence of subwords of the form \(w_1 w_2 \|\) where \(w_1\) is a (possibly empty) strictly decreasing word containing only letters from \(\mathbb{P}\) and \(w_2\) is a (possibly empty) weakly increasing word containing only letters from \(\mathbb{P}\). For example,

\[864211|457|931||12338|56||942788\]  

(4.1)

is a peak word. It is clear that the left-hand side of the given equation counts peak words where \(t\) is weighting the number of bars, \(y\) is weighting the number of underlined letters, and \(x\) is weighting the length of the underlying \(\mathbb{P}\)-word. We want to show that the right-hand side also counts peak words with the same weights.

Let us say that a peak word is minimal if it is impossible to remove bars from it to yield a peak word. Given a word in \(\mathbb{P}\), there is a unique minimal peak word corresponding to every possible choice of underlines. Indeed, if \(w\) is a word in \(\mathbb{P}\) with a given choice of underlines (that is, if \(w\) is a word in \(\mathbb{P} \cup \mathbb{P}\)), then a minimal peak word corresponding to \(w\) must have no bar at the beginning and a bar at the end, and whether or not there needs to be a bar between two letters \(a\) and \(b\) is completely determined by whether \(a > b\), whether \(a\) is underlined, and whether \(b\) is underlined. Moreover, adding bars to a peak word yields another peak word, so every peak word can be obtained from a unique minimal peak word by adding bars. For example, the minimal peak word corresponding to

\[8642114579311233856942788\]

is

\[864211457|931|12338|56|942788,\]

which is the unique minimal peak word from which we can obtain \(\|\) as they share the same underlying \(\mathbb{P}\)-word and choice of underlines.

We show that

\[t(t + yt)^{pk(L)}(1 + y)^{pk(L)+1}(y + t)^{des(L) - pk(L)}(1 + yt)^{n - des(L) - pk(L) - 1} x^n r_L\]

(4.2)

counts nonempty minimal peak words with descent composition \(L \subseteq n\). Every term in \(r_L\) corresponds to a \(\mathbb{P}\)-word with descent composition \(L\), and we give it a choice of underlines and insert necessary bars. As our working example, take the \(\mathbb{P}\)-word \(11375438876544579756673\).

1. There must be a bar at the end, hence the initial factor \(t\): 

\[11375438876544579756673\|\].

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2. For each letter corresponding to a peak, we choose whether or not to underline it. If we do underline it, then we insert a bar immediately before it; otherwise, we insert a bar immediately after it. This corresponds to the \((t + yt)^{pk(L)}\) factor. For example, we may have

\[113|754388|7654457|97566|73].\]

3. The above step divides our word into \(pk(L) + 1\) segments, separated by bars. Take the left-most smallest letter of each segment and choose whether or not to underline it; this gives the \((1 + y)^{pk(L)+1}\) factor. For example, we may have

\[113|754388|7654457|97566|73].\]

Note that this step determines whether the left-most smallest letter in each segment is to be part of the underlined decreasing subword or the non-underlined weakly increasing subword.

4. Take each letter corresponding to a descent that is not a peak and choose to either underline it or to add a bar after it; this gives \((y + t)^{des(L)−pk(L)}\). For example, we may have

\[113|754|388|7654457|97566|73].\]

This step eliminates instances of underlined letters separated by non-underlined letters in the same segment, and it is evident that this gives the minimal peak word corresponding to our current choice of underlines.

5. Finally, iterate through every letter that is (a) not the final letter of the word, (b) not corresponding to a descent, and (c) not followed immediately by a letter corresponding to a peak, and choose either to do nothing or to underline the next letter and add a bar in between the two letters; this gives \((1 + yt)^{n−des(L)−pk(L)−1}\). For example, we may have

\[113|754|388|7654457|97566|73].\]

Note that adding these underlines requires the corresponding bars to be placed, so the result is still a minimal peak word.

Through these steps, we have considered whether to underline each letter in the word, so in fact \((4.2)\) accounts for the unique minimal peak word corresponding to each choice of underlines, and thus counts all minimal peak words with descent composition \(L \models n\).

Observe that \((4.2)\) is equal to

\[t^{pk(L)+1}(y + t)^{des(L)−pk(L)}(1 + yt)^{n−pk(L)−des(L)−1}(1 + y)^{2pk(L)+1}x^{n-r_L},\]

which appears in the statement of this lemma. Dividing by \((1−t)^{n+1}\) corresponds to inserting any number of bars in the \(n + 1\) possible positions, which allows us to move from nonempty minimal peak words to all peak words except those that only consist of bars, which are accounted for by the \(1/(1−t)\) term at the beginning. Hence the lemma is proven. \qed
We remark that Lemma 4.1 (as well as the noncommutative symmetric function identities stated in the next several subsections) can also be proven using the present author’s “generalized run theorem” [28] and making appropriate substitutions, but we prefer the combinatorial proof given above.

The following is our main result relating the Eulerian polynomials and the \((pk, des)\) polynomials.

**Theorem 4.2.** For \(n \geq 1\), we have

\[
A_n(t) = \left(\frac{1 + yt}{1 + y}\right)^{n+1} P_n^{(pk, des)} \left(\frac{(1 + y)^2 t}{(y + t)(1 + yt)} - \frac{y + t}{1 + yt}\right).
\]

(4.3)

Equivalently,

\[
P_n^{(pk, des)}(y, t) = \left(\frac{1 + u}{1 + uv}\right)^{n+1} A_n(v)
\]

(4.4)

where \(u = \frac{1 + t^2 - 2yt - (1-t)\sqrt{(1+t)^2 - 4yt}}{2(1-y)t}\) and \(v = \frac{(1+t)^2 - 2yt - (1+t)\sqrt{(1+t)^2 - 4yt}}{2yt}\).

Note that evaluating (1.3) at \(y = 1\) recovers Stembridge’s identity

\[
A_n(t) = \left(\frac{1 + t^2}{1 + t}\right)^{n+1} P_n^{pk} \left(\frac{4t}{1 + t^2}\right)
\]

mentioned in the introduction of this paper.

**Proof.** Taking Lemma 4.1 evaluating at \(x = 1\), and applying the homomorphism \(\Phi\) yields

\[
\frac{1}{1 - te^{(1+y)x}} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} t^{pk(\pi) + 1}(y + t)^{\text{des}(\pi) - pk(\pi)}(1 + yt)^{n - pk(\pi) - \text{des}(\pi) - 1}(1 + y)^{2pk(\pi) + 1} \frac{x^n}{n!}
\]

by Lemma 3.1. Rearranging some terms yields

\[
\frac{1}{1 - te^{(1+y)x}} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} \frac{1}{1 + y} \left(\frac{1 + yt}{1 - t}\right)^{n+1} \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \frac{y + t}{1 + yt} \frac{\text{des}(\pi) + 1}{n!} x^n
\]

Multiplying both sides by \(1 - t\) and then replacing \(x\) by \((1 - t)x/(1 + y)\) yields

\[
\frac{1 - t}{1 - te^{(1-t)x}} = 1 + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} \left(\frac{1 + yt}{1 + y}\right)^{n+1} \left(\frac{(1 + y)^2 t}{(y + t)(1 + yt)}\right)^{pk(\pi) + 1} \frac{y + t}{1 + yt} \frac{\text{des}(\pi) + 1}{n!} x^n
\]

Note that the left-hand side is the exponential generating function for the Eulerian polynomials; thus equating the coefficients of \(x^n/n!\) gives (4.3).
Finally, \([4.4]\) can be obtained by setting \(u = \frac{(1+y)^2t}{(y+t)(1+yt)}\) and \(v = \frac{y+t}{1+yt}\), solving for \(y\) and \(t\) (which can be done using a computer algebra system such as Maple), and simplifying\(^7\). \(\square\)

We give a combinatorial proof of this result using the modified Foata–Strehl action in Subsection 5.1.

Next, we obtain a similar result for the \(q\)-analogue of the \((pk,\text{des})\) polynomial

\[
P_n^{(\text{inv},pk,\text{des})}(q, y, t) := \sum_{\pi \in S_n} q^{\text{inv}(\pi)} y^{pk(\pi)+1} t^{\text{des}(\pi)+1}
\]

also keeping track of the inversion number.

**Theorem 4.3.** We have

\[
\frac{1 - t}{1 - t \text{Exp}_q(yx) \exp_q(x)} = 1 + \sum_{n=1}^{\infty} \frac{(1 + yt)^{n+1}}{(1 + y)(1 - t)^n} P_n^{(\text{inv},pk,\text{des})} \left( q, \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right) x^n \left[ \frac{n!}{q} \right]. (4.5)
\]

Equivalently,

\[
\sum_{n=1}^{\infty} P_n^{(\text{inv},pk,\text{des})} (q, y, t) x^n \left[ \frac{n!}{q} \right] = \frac{v(1 + u) \text{Exp}_q \left( \frac{u(1-v)}{1+uv} x \right) \exp_q \left( \frac{1-v}{1+uv} x \right) - 1}{1 + uv} - \frac{1}{1 + uv} \text{Exp}_q \left( \frac{u(1-v)}{1+uv} x \right) \exp_q \left( \frac{1-v}{1+uv} x \right)
\]

where \(u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}\) and \(v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}\).

**Proof.** We follow the proof of Theorem 4.2 but apply the homomorphism \(\Phi_q\) instead of \(\Phi\), which by Lemma 3.2 yields

\[
\frac{1}{1 - t \text{Exp}_q(yx) \exp_q(x)} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} q^{\text{inv}(\pi)} \frac{t^{pk(\pi)+1} (y + t)^{\text{des}(\pi)-pk(\pi)} (1 + yt)^{n-pk(\pi)-\text{des}(\pi)-1} (1 + y)^{2pk(\pi)+1}}{(1 - t)^{n+1}} x^n \left[ \frac{n!}{q} \right].
\]

Multiplying both sides by \(1 - t\) and rearranging some terms yields \([4.5]\).

Next, we replace \(x\) by \((1 - t)x/(1 + yt)\) to get

\[
\frac{1 - t}{1 - t \text{Exp}_q \left( \frac{u(1-v)}{1+uv} x \right) \exp_q \left( \frac{1-t}{1+yt} \right)} = \frac{1}{1 - t} + \frac{1 + yt}{1 + y} \sum_{n=1}^{\infty} P_n^{(\text{inv},pk,\text{des})} \left( q, \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right) x^n \left[ \frac{n!}{q} \right].
\]

\(^7\)We exchanged \(u\) and \(v\) with \(y\) and \(t\), respectively, in the statement of \([4.4]\) in this theorem, so that the \((pk,\text{des})\) polynomial would have variables \(y\) and \(t\) as in its definition. The same is done for all subsequent results involving similar substitutions.
Making the same substitutions yields
\[ 1 + \frac{1 + u v}{1 + u} \sum_{n=1}^{\infty} P_n^{(\text{inv, pk, des})}(q, y, t) \frac{x^n}{[n]_q!} = \frac{1 - v}{1 - v \exp_q \left( \frac{u(1-v)t}{1+uv} x \right) \exp_q \left( \frac{1-v}{1+uv} x \right)} \]
where \( u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t} \) and \( v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt} \). Subtracting both sides by 1 and dividing by \((1 + uv)/(1 + u)\) completes the proof.

Unfortunately, we cannot express \( P_n^{(\text{inv, pk, des})}(q, y, t) \) in terms of the \( q \)-Eulerian polynomial
\[ A_n(q, t) := \sum_{\pi \in S_n} q^{\text{inv} (\pi)} t^{\text{des} (\pi)} + 1, \]
but we can recover the known \( q \)-exponential generating function for the \( q \)-Eulerian polynomials from the above result.

**Corollary 4.4.**
\[ \sum_{n=0}^{\infty} A_n(q, t) \frac{x^n}{[n]_q!} = \frac{1 - t}{1 - t \exp_q((1-t)x)} \]

**Proof.** Take (4.5), set \( y = 0 \), and replace \( x \) with \((1-t)x\).

**Theorem 4.3** also specializes to a corresponding result for the \((\text{inv, pk})\) polynomial
\[ P_n^{(\text{inv, pk})}(q, t) := \sum_{\pi \in S_n} q^{\text{inv} (\pi)} t^{\text{pk} (\pi)} + 1. \]

**Corollary 4.5.** We have
\[ \frac{1 - t}{1 - t \exp_q(x) \exp_q(x)} = 1 + \sum_{n=1}^{\infty} \frac{(1 + t)^{n+1}}{2(1-t)^{n}} P_n^{(\text{inv, pk})} \left( q, \frac{4t}{(1+t)^2} \right) \frac{x^n}{[n]_q!}, \tag{4.6} \]
Equivalently,
\[ \sum_{n=1}^{\infty} P_n^{(\text{inv, pk})}(q, t) \frac{x^n}{[n]_q!} = \frac{2v \ \exp_q \left( \frac{1-v}{1+v} x \right) \exp_q \left( \frac{1-v}{1+v} x \right) - 1}{1 + v \ 1 - v \ \exp_q \left( \frac{1-v}{1+v} x \right) \exp_q \left( \frac{1-v}{1+v} x \right)} \tag{4.7} \]
where \( v = \frac{2}{t} (1 - \sqrt{1-t}) - 1 \).

**Proof.** Equation (4.6) is obtained by taking (4.5) and setting \( y = 1 \). Then (4.7) follows by replacing \( x \) with \( x(1-t)/(1+t) \), making an appropriate substitution, and rearranging some terms.
4.2. Left peaks and descents

In this subsection, we study the \( (lpk, \text{des}) \) polynomials

\[
P_n^{(lpk, \text{des})}(y, t) := \sum_{\pi \in \mathfrak{S}_n} y^{lpk(\pi)} t^{\text{des}(\pi)}
\]

and their \( q \)-analogues

\[
P_n^{(inv, lpk, \text{des})}(q, y, t) := \sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} y^{lpk(\pi)} t^{\text{des}(\pi)}.
\]

Using the same method as before, we obtain analogues of Theorems \( 4.2 \) and \( 4.3 \) for left peaks and descents, as well as a connection to the refined type B Eulerian polynomials introduced in Subsection 2.3.

We begin by proving a suitable noncommutative symmetric function identity.

**Lemma 4.6.**

\[
\frac{h(x)(1 - te(yx)h(x))^{-1} - 1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \vdash n} t^{lpk(L)} (y + t)^{\text{des}(L) - lpk(L)} (1 + yt)^{n - lpk(L) - \text{des}(L)} (1 + y)^{2lpk(L)} \frac{x^n r_L}{(1 - t)^{n+1}}
\]

**Proof.** Let us say that a word \( w \) on the alphabet \( \mathbb{P} \cup \mathbb{P} \cup \{\} \) is a \textit{left peak word} if \( w \) begins with a (possibly empty) weakly increasing subword containing only letters from \( \mathbb{P} \), followed by a sequence of subwords of the form \( |w_1 w_2 \) where \( w_1 \) is a (possibly empty) strictly decreasing word containing only letters from \( \mathbb{P} \) and \( w_2 \) is a (possibly empty) weakly increasing word containing only letters from \( \mathbb{P} \). The left-hand side of the given equation counts left peak words where \( t \) is weighting the number of bars, \( y \) is weighting the number of underlined letters, and \( x \) is weighting the length of the underlying \( \mathbb{P} \)-word. We want show that the right-hand side also counts left peak words with the same weights.

Call a left peak word \( w \) \textit{minimal} if it is impossible to remove bars from \( w \) to yield a left peak word. Similar to peak words in the proof of Lemma \( 4.1 \), every left peak word can be obtained from only one minimal left peak word, which is the only minimal left peak word on those letters with the same choice of underlines. We claim that

\[
(t + yt)^{lpk(L)} (1 + y)^{lpk(L)} (y + t)^{\text{des}(L) - lpk(L)} (1 + yt)^{n - \text{des}(L) - lpk(L)} x^n r_L
\]

counts nonempty minimal peak words with descent composition \( L \vdash n \). Every term in \( r_L \) corresponds to a \( \mathbb{P} \)-word with descent composition \( L \), and we give it a choice of underlines and insert bars in a similar way as in the proof of Lemma \( 4.1 \).

1. For each letter corresponding to a left peak, we choose whether or not to underline it. If we do underline it, then we insert a bar immediately before it; otherwise, we insert a bar immediately after it. This corresponds to the \((t + yt)^{lpk(L)}\) factor.
2. If the first letter corresponds to a left peak and was underlined, then the bars inserted in the above step divide our word into lpk($L$) segments. In this case, take the left-most smallest letter of each segment and choose whether or not to underline it. Otherwise, the bars divide our word into lpk($L$) + 1 segments, in which case we take the left-most smallest letter of each but the first segment and choose whether or not to underline it. This gives the $(1 + y)^{\text{lpk}(L)}$ factor.

3. Take each letter corresponding to a descent that is not a left peak and choose to either underline it or to add a bar after it; this gives $(y + t)^{\text{des}(L) - \text{lpk}(L)}$. As in the proof of Lemma 4.1 this step eliminates underlined letters separated by non-underlined letters appearing in the same segment, and gives a minimal left peak word corresponding to our current choice of underlines.

4. Finally, iterate through every letter that is (a) not the final letter of the word, (b) not corresponding a descent, and (c) not followed by a letter corresponding to a left peak, and choose either to do nothing or to underline the next letter and add a bar in between the two letters. In addition, if the first letter does not correspond to a left peak, then choose to either do nothing or to underline the first letter and prepend a bar. This gives $(1 + yt)^{n - \text{des}(L) - \text{lpk}(L) - 1}$, and the result is still a minimal left peak word as the new bars are necessary to accommodate the new underlines.

Through these steps, we have considered whether to underline each letter in the word, so (4.8) counts every minimal left peak word with descent composition $L \models n$. Dividing by $(1 - t)^{n+1}$ allows us to insert any number of bars in any of the $n + 1$ possible positions, thus creating left peak words from minimal left peak words, and the $1/(1 - t)$ term accounts for words containing only bars.

**Theorem 4.7.** For $n \geq 0$, we have

\[
\sum_{k=0}^{n} \binom{n}{k} (1 + y)^k (1 - t)^{n-k} A_k(t) = (1 + yt)^n P_n^{(\text{lpk,des})} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right). \tag{4.9}
\]

Equivalently,

\[
P_n^{(\text{lpk,des})}(y, t) = \frac{1}{(1 + uv)^n} \sum_{k=0}^{n} \binom{n}{k} (1 + u)^k (1 - v)^{n-k} A_k(v) \tag{4.10}
\]

where $u = \frac{1 + t^2 - 2yt - (1-t)\sqrt{(1+t)^2 - 4yt}}{2(1-g)t}$ and $v = \frac{(1+t)^2 - 2yt - (1+t)\sqrt{(1+t)^2 - 4yt}}{2yt}$.

As with Theorem 4.2, evaluating (4.9) at $y = 1$ recovers a known result, Petersen’s identity

\[
\sum_{k=0}^{n} \binom{n}{k} 2^k (1 - t)^{n-k} A_k(t) = (1 + t)^n P_n^{\text{lpk}} \left( \frac{4t}{(1 + t)^2} \right)
\]

in this case.
Proof. Taking Lemma 4.6, evaluating at $x = 1$, applying the homomorphism $\Phi$, and rearranging some terms yields

$$
\frac{e^x}{1 - te^{(1+y)x}} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{\pi \in \mathcal{S}_n} \frac{(1 + yt)^n}{(1 - t)^{n+1}} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right)^{\text{lpk}(\pi)} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(\pi)} \frac{x^n}{n!}.
$$

Multiplying both sides by $1 - t$ and then replacing $x$ by $(1 - t)x/(1 + y)$ yields

$$
\frac{1 - t}{1 - te^{(1-t)x}} e^{\frac{1 - t}{1 + ty}x} = \sum_{n=0}^{\infty} \sum_{\pi \in \mathcal{S}_n} \left( \frac{1 + yt}{1 + y} \right)^n \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right)^{\text{lpk}(\pi)} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(\pi)} \frac{x^n}{n!}.
$$

Moreover,

$$
\frac{1 - t}{1 - te^{(1-t)x}} e^{\frac{1 - t}{1 + ty}x} = \sum_{n=0}^{\infty} \sum_{\pi \in \mathcal{S}_n} \left( \frac{1}{n!} \right) \left( \frac{1 - t}{1 + y} \right)^n \frac{x^n}{n!}.
$$

so

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \text{lpm}(t) \left( \frac{1 - t}{1 + y} \right)^{n-k} \frac{x^n}{n!} =

\sum_{n=0}^{\infty} \sum_{\pi \in \mathcal{S}_n} \left( \frac{1 + yt}{1 + y} \right)^n \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right)^{\text{lpk}(\pi)} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(\pi)} \frac{x^n}{n!}.
$$

Equating the coefficients of $x^n/n!$ and rearranging some terms gives (4.9). Then (4.10) can be obtained by making the same substitutions as in the proof of Theorem 4.2. \qed

Now, for the $q$-analogue.

**Theorem 4.8.** We have

$$
\frac{(1 - t) \exp_q(x)}{1 - t \operatorname{Exp}_q(yx) \exp_q(x)} = \sum_{n=0}^{\infty} \left( \frac{1 + yt}{1 - t} \right)^n \left( \frac{1 + y)^2 t}{(y + t)(1 + yt)} \right)^{\text{lpk}(\pi)} \left( \frac{y + t}{1 + yt} \right)^{\text{des}(\pi)} \frac{x^n}{n!}.
$$

Equivalently,

$$
\sum_{n=0}^{\infty} P_n^{(\text{inv}, \text{lpk}, \text{des})}(q, y, t) \frac{x^n}{n!} = \frac{(1 - v) \exp_q \left( \frac{1-v}{1+uv} x \right)}{1 - v \operatorname{Exp}_q \left( \frac{2(1-v)}{1+uv} x \right) \exp_q \left( \frac{1-v}{1+uv} x \right)}
$$

(4.11)

where $u = \frac{(1+t)^2 - 2yt - (1-t)\sqrt{(1+t)^2 - 4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2 - 2yt - (1+t)\sqrt{(1+t)^2 - 4yt}}{2yt}$.
Proof. Apply the homomorphism $\Phi_q$ to Lemma 4.6 evaluated at $x = 1$; then multiplying both sides by $1 - t$ yields (4.11).

Next, replace $x$ by $(1 - t)x/(1 + yt)$ to get

$$\frac{(1 - t) \exp_q \left( \frac{1 - t}{1 + yt} x \right)}{1 - t \exp_q \left( \frac{y(1 - t)}{1 + yt} x \right) \exp_q \left( \frac{1 - t}{1 + yt} x \right)} = \sum_{n=0}^{\infty} P_n^{(\text{inv,lpk,des})} \left( q, \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right) \frac{x^n}{[n]_q!}.$$  

Making the same substitutions as before yields (4.12).

We note that taking (4.11) and evaluating at $y = 0$ gives

$$\frac{(1 - t) \exp_q ((1 - t)x)}{1 - t \exp_q ((1 - t)x) \exp_q (x)} = \sum_{n=0}^{\infty} q^{\text{inv}(\pi)} t^{\text{des}(\pi)} \frac{x^n}{[n]_q!},$$

which is equivalent to Corollary 4.4. Evaluating at $y = 1$, on the other hand, gives us a result for the $(\text{inv,lpk})$ polynomials

$$P_n^{(\text{inv,lpk})}(q, t) := \sum_{\pi \in S_n} q^{\text{inv}(\pi)} t^{\text{lpk}(\pi)}.$$

Corollary 4.9. We have

$$\frac{(1 - t) \exp_q (x)}{1 - t \exp_q (x) \exp_q (x)} = \sum_{n=0}^{\infty} \left( \frac{1 + t}{1 - t} \right)^n P_n^{(\text{lpk})} \left( q, \frac{4t}{(1 + t)^2} \right) \frac{x^n}{[n]_q!}.$$  

Equivalently,

$$\sum_{n=0}^{\infty} P_n^{(\text{lpk})} (q, t) \frac{x^n}{[n]_q!} = \frac{(1 - v) \exp_q \left( \frac{1-v}{1+v} x \right)}{1 - v \exp_q \left( \frac{1-v}{1+v} x \right) \exp_q \left( \frac{1-v}{1+v} x \right)}$$

where $v = \frac{2}{t} (1 - \sqrt{1 - t}) - 1$.

Lastly, we state an identity connecting the $(\text{lpk,des})$ polynomials with the refined type B Eulerian polynomials $B_n(y, t) = \sum_{\pi \in B_n} y^{\text{neg}(\pi)} t^{\text{des}(\pi)}$.

Theorem 4.10. For all $n \geq 0$, we have

$$B_n(y, t) = (1 + yt)^n P_n^{(\text{lpk,des})} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt} \right).$$  

Equivalently,

$$P_n^{(\text{lpk,des})}(y, t) = \frac{B_n(u, v)}{(1 + uv)^n}$$

where $u = \frac{1 + t^2 - 2yt - (1 - t) \sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t}$ and $v = \frac{(1 + t)^2 - 2yt - (1 - t) \sqrt{(1 + t)^2 - 4yt}}{2yt}$.

By setting $y = 1$ in (4.13), we recover the identity

$$B_n(t) = (1 + t)^n P_n^{\text{lpk}} \left( \frac{4t}{(1 + t)^2} \right)$$

where $B_n(t) = \sum_{\pi \in B_n} t^{\text{des}(\pi)}$, another result of Petersen [16].
Proof. Observe that (4.13) follows immediately from Theorems 2.6 and 4.7. Making the same substitutions as before yields (4.14).

We give a combinatorial proof of this result in Subsection 5.3.

4.3. Up-down runs and descents

Our remaining aim in this section is to prove analogous results for the number of up-down runs \( \text{udr} \) and the joint distribution of \( \text{udr} \) and \( \text{des} \). In particular, we express the polynomial

\[
P_n^{\text{udr}}(y, t) := \sum_{\pi \in S_n} t^{\text{udr}(\pi)}
\]

in terms of the \( n \)th Eulerian polynomial, give a \( q \)-exponential generating function for its \( q \)-analogue, and relate it to the distribution of the flag descent number over \( \mathcal{B}_n \). Due to technical constraints that will become apparent later, we cannot do the same with the polynomial

\[
P_n^{(\text{udr, des})}(y, t) := \sum_{\pi \in S_n} y^{\text{udr}(\pi)} t^{\text{des}(\pi)}.
\]

Instead, we will work with

\[
P_n^{(\text{lpk, val, des})}(y, z, t) := \sum_{\pi \in S_n} y^{\text{lpk}(\pi)} z^{\text{val}(\pi)} t^{\text{des}(\pi)},
\]

which is equivalent to \( P_n^{(\text{udr, des})}(y, t) \) in light of Lemma 2.1.

Lemma 4.11.

\[
(1 - t^2 h(x) e(yx))^{-1}(1 + th(x)) = \frac{1}{1-t} + \sum_{L \in \mathcal{B}_n} \frac{t^{\text{udr}(L)}(1 + y)^{\text{udr}(L)} - 1}{(1 - t)(1 - t^2)^n} x^n r_L
\]

Proof. Let us say that a word \( w \) on the alphabet \( \mathbb{P} \cup \mathbb{P} \cup \{\} \) is a \textit{up-down run word} if \( w \) is either:

- A sequence of subwords of the form \( w_1|w_2| \) where \( w_1 \) is a (possibly empty) weakly increasing word containing only letters from \( \mathbb{P} \) and \( w_2 \) is a (possibly empty) strictly decreasing word containing only letters from \( \mathbb{P} \);

- Or, a sequence of subwords of the form \( w_1|w_2| \) as described above, but ending with a subword of the form \( w_3| \), where \( w_3 \) is a (possibly empty) weakly increasing word containing only letters from \( \mathbb{P} \).

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For example,

\[12\|246678|984|321|5\|23\]  \hspace{1cm} (4.15)

is an up-down run word. The left-hand side of the given equation counts up-down run words where, as before, \(t\) is weighting the number of bars, \(y\) is weighting the number of underlined letters, and \(x\) is weighting the length of the underlying \(P\)-word. We want show that the right-hand side also counts up-down run words with the same weights.

Call an up-down run word \(w\) minimal if it is impossible to remove bars from \(w\) to yield an up-down run word. As before, every up-down run word can be obtained from only one minimal up-down run word, which is the only minimal up-down run word on those letters with the same choice of underlines. For example, the minimal up-down run word on

\[12246678|984321523\]

is

\[12246678|984|321|523\],

which is the unique minimal up-down run word that \((4.15)\) can be obtained from. We claim that

\[
t(t + yt)^{udr(L)-1}(1 + yt^2)^{n-1-des(L)-val(L)}(y + t^{des(L)-lpk(L)})
\times (1 + yt)^{1-lpk(L)+val(L)}(y + t)^{lpk(L)-val(L)}x^nr_L \hspace{1cm} (4.16)
\]

counts nonempty minimal up-down run words with descent composition \(L \models n\). Every term in \(r_L\) corresponds to a \(P\)-word with descent composition \(L\), and we give it a choice of underlines and insert the necessary bars. Let us take 8543211344488932334513456 as our working example.

1. Every up-down run word must end with a bar, so insert a bar at the end of our word:

\[8543212344488932334513456\].

This gives the initial factor of \(t\).

2. For each letter corresponding to a left peak or valley (i.e., each letter that is at the end of an up-down run other than the last one), we choose whether or not to underline it. For a left peak, if we do underline it, then we insert a bar immediately before it; otherwise, insert a bar immediately after it. For a valley, if we do underline it, then we insert a bar immediately after it; otherwise, insert a bar immediately before it. This gives the \((t + yt)^{udr(L)-1}\) factor. For example, we may have

\[|85432|12344488|932|33445|1|3456|\].

3. For each letter corresponding to a descent that is not a left peak (i.e., each letter corresponding to a descent and is not the final letter of an up-down run), choose whether or not to underline it. If we do not underline the letter, then prepend and
append a bar to it. This gives the \((y + t^2)^{\des(L) - \lpk(L)}\) factor. For example, we may have

\[
|854\ 3|2\ 12\ 3|4\ 4488\ 932|33|4|45|\ 1|3456|.
\]

This step eliminates instances of non-underlined letters appearing in the same segment as an underlined letter, and by adding the bars, we have a minimal up-down run word corresponding to our current choice of underlines.

4. For each letter corresponding to an ascent\(^8\) that is not a valley (i.e., each letter corresponding to an ascent and is not the final letter of an up-down run), choose whether or not to underline it. If we underline the letter, then also prepend and append a bar to it. This gives the \((1 + yt^2)^{n-1 - \des(L) - \val(L)}\) factor. For example, we may have

\[
|854\ 3|2\ 12\ 3|4\ 4488\ 932|33|4|45|\ 1|3456|.
\]

Note that adding the bars is necessary so that the result is a minimal up-down run word.

5. The only remaining letter of our word that still requires consideration is the final letter, so the last step is to choose whether or not to underline it. If the word ends with an increasing run of length 1 (which is equivalent to \(\lpk(L) - \val(L) = 1\) by Lemma \[2.1\])\(^9\) and we do not underline the final letter, then prepend a bar to it. If the word ends with an increasing run of length at least 2 (which is equivalent to \(\lpk(L) - \val(L) = 0\) by Lemma \[2.1\]) and we underline the final letter, then prepend a bar to it. This gives \((1 + yt^2)^{1 - \lpk(L) + \val(L)}(y + t)^{\lpk(L) - \val(L)}\). For example, we may have

\[
|854\ 3|2\ 12\ 3|4\ 4488\ 932|33|4|45|\ 1|345|6|.
\]

Again, we have a minimal up-down run word.

We have chosen whether or not to underline each letter in the word, so \((4.16)\) counts every minimal up-down run word with descent composition \(L \models n\). Dividing by \((1 - t^2)^n\) allows us to insert bars in multiples of two at the beginning of the word or between any two letters; adding them in multiples of two is necessary for the result to remain an up-down word. However, any number of bars can be added at the end, hence dividing by \(1 - t\) as well. This accounts for all up-down words other than those only consisting of bars, which are accounted for by the \(1/(1 - t)\) term.

Corollary 4.12.

\[
(1 - t^2h(x)e(x))^{-1}(1 + th(x)) = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \models n} \frac{2^{\udr(L) - 1}t^{\udr(L)}(1 + t^2)^{n - \udr(L)}}{(1 - t)^2(1 - t^2)^{n-1}}x^n r_L
\]

\(^8\)We say that \(i \in [n - 1]\) is an ascent of the word \(w = w_1 \cdots w_n\) if \(w_i \leq w_{i+1}\), i.e., if it is not a descent. Ascents are defined in the analogous way for permutations (with the weak inequality \(\leq\) replaced by the strict inequality < since letters cannot repeat).

\(^9\)Although Lemma \[2.1\] was stated for permutations, it also holds for words.
Proof. Taking Lemma 4.11 and setting \( y = 1 \), we obtain

\[
(1 - t^2h(x)e(x))^{-1}(1 + th(x)) = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \models n} 2^{\text{udr}(L) - \text{udr}(L)} \frac{(1 + t^2)^{n-1 - \text{val}(L) - \text{lpk}(L)}}{(1 - t)(1 - t^2)^n} x^n r_L.
\]

Because \( \text{udr}(L) = \text{lpk}(L) + \text{val}(L) + 1 \) (Lemma 2.1), it follows that

\[
(1 - t^2h(x)e(x))^{-1}(1 + th(x)) = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \models n} 2^{\text{udr}(L) - \text{udr}(L)} \frac{(1 + t^2)^{n-1}}{(1 - t)(1 - t^2)^n} x^n r_L.
\]

Since \( \text{udr} \) is the only statistic present in the noncommutative symmetric function identity given in the above corollary, we can use it to obtain results for the up-down run polynomials \( P_{\text{udr}}^n(t) \) and their \( q \)-analogues

\[
P_{\text{udr}}^n(q, t) := \sum_{\pi \in S_n} q^\text{inv}(\pi) P_{\text{udr}}^n(\pi).
\]

Theorem 4.13. For \( n \geq 1 \), we have

\[
A_n(t) = \frac{(1 + t^2)^n}{2(1 + t)^n} P_{\text{udr}}^n \left( \frac{2t}{1 + t^2} \right).
\]

Equivalently,

\[
P_{\text{udr}}^n(t) = \frac{2(1 + v)^{n-1}}{(1 + v^2)^n} A_n(v)
\]

where \( v = \frac{1 - \sqrt{1 - t^2}}{t} \).

Proof. Taking Corollary 4.12 evaluating at \( x = 1 \), applying the homomorphism \( \Phi \), and rearranging some terms yields

\[
\frac{1}{1 - te^x} = \frac{1 + te^x}{1 - t^2 e^{2x}} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} \frac{(1 + t^2)^n}{2(1 - t)^2 (1 - t^2)^{n-1}} \left( \frac{2t}{1 + t^2} \right)^\text{udr}(\pi) x^n.
\]

Multiplying both sides by \( 1 - t \) and then replacing \( x \) by \( (1 - t)x \) yields

\[
\frac{1 - t}{1 - te^{(1-t)x}} = 1 + \sum_{n=1}^{\infty} \sum_{\pi \in S_n} \frac{(1 + t^2)^n}{2(1 + t)^n} \left( \frac{2t}{1 + t^2} \right)^\text{udr}(\pi) x^n.
\]

The left-hand side is precisely the exponential generating function for the Eulerian polynomials, so equating the coefficients of \( x^n/n! \) gives (4.17). Then (4.18) can be obtained by making the substitution \( v = 2t/(1 + t)^2 \) and solving for \( t \). \( \square \)
Theorem 4.14. We have
\[
\frac{(1 - t)(1 + t \exp_q(x))}{1 - t^2 \exp_q(x) \exp_q(x)} = 1 + \frac{1 + t}{2} \sum_{n=1}^{\infty} \left( \frac{1 + t^2}{1 - t^2} \right)^n p_n^{(\text{inv,udr})}(q, \frac{2t}{1 + t^2}) \frac{x^n}{[n]_q}.
\]
(4.19)

Equivalently,
\[
\sum_{n=1}^{\infty} p_n^{(\text{inv,udr})}(q, t) \frac{x^n}{[n]_q!} = \frac{2}{1 + v} \left( \frac{(1 - v)(1 + v \exp_q(\frac{1 - v^2}{1 + v^2}x))}{1 - v^2 \exp_q(\frac{1 - v^2}{1 + v^2}x) \exp_q(\frac{1 - v^2}{1 + v^2}x)} - 1 \right)
\]
(4.20)

where \( v = \frac{1 - \sqrt{1 - t}}{t} \).

Proof. Apply the homomorphism \( \Phi_q \) to Corollary 4.12 evaluated at \( x = 1 \); then multiplying both sides by \( 1 - t \) yields (4.19).

Next, replace \( x \) by \( x(1 - t^2)/(1 + t^2) \) to get
\[
\frac{(1 - t)(1 + t \exp_q(\frac{1 - v^2}{1 + v^2}x))}{1 - t^2 \exp_q(\frac{1 - v^2}{1 + v^2}x) \exp_q(\frac{1 - v^2}{1 + v^2}x)} = 1 + \frac{1 + t}{2} \sum_{n=1}^{\infty} p_n^{(\text{inv,udr})}(q, \frac{2t}{1 + t^2}) \frac{x^n}{[n]_q!}.
\]

Then rearranging some terms and making the same substitution as in the proof of Theorem 4.14 yields (4.20). \( \square \)

Recall that when setting \( y = 1 \) in Lemma 4.11 all instances of the statistics \( \text{lpk} \) and \( \text{val} \) either cancel out or reduce to \( \text{udr} \). This is not possible in the general form of Lemma 4.11, so we cannot directly work with the polynomials \( P_n^{(\text{udr,des})}(y, t) = \sum_{\pi \in \mathcal{S}_n} y^{\text{udr}(\pi)} t^{\text{des}(\pi)} \). Since the statistics \( \text{udr,des} \) and \( \text{lpk, val, des} \) are equivalent, we will instead give results for the polynomials \( P_n^{(\text{lpk, val, des})}(y, z, t) = \sum_{\pi \in \mathcal{S}_n} y^{\text{lpk}(\pi)} z^{\text{val}(\pi)} t^{\text{des}(\pi)} \) and their \( q \)-analogues
\[
P_n^{(\text{inv,lpk, val, des})}(q, y, z, t) := \sum_{\pi \in \mathcal{S}_n} q^{\text{inv}(\pi)} y^{\text{lpk}(\pi)} z^{\text{val}(\pi)} t^{\text{des}(\pi)}.
\]

Theorem 4.15. For \( n \geq 1 \), we have
\[
\frac{(1 + y)^n}{t} A_n(t^2) + \sum_{k=0}^{n} \binom{n}{k} (1 + y)^k (1 - t^2)^{n-k} A_k(t^2) = (1 + yt)(1 + ty)(1 + yt^2)^{n-1} \times P_n^{(\text{lpk, val, des})}\left( \frac{t(1 + y)(y + t)}{(y + t^2)(1 + yt)}, \frac{t(1 + y)(1 + yt)}{(1 + yt^2)(y + t)}, \frac{y + t^2}{1 + yt^2} \right).
\]

Theorem 4.16.
\[
\frac{(1 - t)(1 + t \exp_q(x))}{1 - t^2 \exp_q(x) \exp_q(yx)} = 1 + t(1 + yt) \sum_{n=1}^{\infty} \frac{(1 + yt^2)^{n-1}}{(1 - t^2)^n} \times P_n^{(\text{inv,lpk, val, des})}\left( q, \frac{t(1 + y)(y + t)}{(y + t^2)(1 + yt)}, \frac{t(1 + y)(1 + yt)}{(1 + yt^2)(y + t)}, \frac{y + t^2}{1 + yt^2} \right) \frac{x^n}{[n]_q!}.
\]
We omit the proofs of the above two theorems as they follow in essentially the same way as the proofs of Theorems 4.13 and 4.14, except that we would use Lemma 4.11 rather than its specialization (Corollary 4.12). Unlike in these theorems, however, it is not possible to invert the identities to give an explicit expression for \( P_{n}(lpk, val, des) \) or for the \( q \)-exponential generating function for \( P_{n}^{(inv, lpk, val, des)}(q, y, z, t) \).

Finally, we relate the \( (lpk, val, des) \) polynomials to the refined flag descent polynomials \( F_{n}(y, t) = \sum_{\pi \in \mathcal{B}_{n}} y^{\text{neg}(\pi)}t^{\text{fdes}(\pi)} \), which specializes to a relation between the udr polynomials and the flag descent polynomials \( F_{n}(t) = \sum_{\pi \in \mathcal{B}_{n}} t^{\text{fdes}(\pi)} \).

**Theorem 4.17.** For \( n \geq 1 \), we have

\[
F_{n}(y, t) = (1 + yt)(1 + yt^{2})^{n-1}P_{n}^{(lpk, val, des)} \left( \frac{t(1 + y)(y + t)}{(y + t^{2})(1 + yt)} \cdot \frac{y + t^{2}}{1 + yt^{2}} \right).
\]

**Proof.** Follows immediately from Theorems 2.8 and 4.15.

We give a combinatorial proof of this result in Subsection 5.3.

**Corollary 4.18.** For \( n \geq 1 \), we have

\[
F_{n}(t) = \frac{(1 + t)(1 + t^{2})^{n}}{2t}P_{n}^{\text{udr}} \left( \frac{2t}{1 + t^{2}} \right).
\]

Equivalently,

\[
P_{n}^{\text{udr}}(t) = \frac{2v}{(1 + v)(1 + v^{2})^{n}}F_{n}(v)
\]

where \( v = \frac{1 - \sqrt{1 - t^{2}}}{t} \).

**Proof.** We obtain (4.21) by taking the preceding theorem, setting \( y = 1 \), and rearranging a few terms. Then (4.22) is obtained from (4.21) by the same substitution as before.

### 4.4. Two remarks: the inverse major index and alternating analogues

We end this section with two important remarks.

First, our formulas for \( q \)-analogues of descent statistic polynomials are also valid for the “inverse major index”. For a permutation \( \pi \in \mathcal{S} \), the major index \( \text{maj}(\pi) \) is defined to be the sum of its descents, and the inverse major index \( \text{imaj}(\pi) \) is the major index of its inverse when considered as an element of the symmetric group. For example, take \( \pi = 85712643 \), whose inverse is \( \pi^{-1} = 45872631 \). Since \( \text{Des}(\pi) = \{1, 3, 6, 7\} \) and \( \text{Des}(\pi^{-1}) = \{3, 4, 6, 7\} \), the major index of \( \pi \) is

\( \text{maj}(\pi) = 1 + 3 + 6 + 7 = 17 \)

whereas the inverse major index of \( \pi \) is

\( \text{imaj}(\pi) = \text{maj}(\pi^{-1}) = 3 + 4 + 6 + 7 = 20 \).
A remarkable result by Foata and Schützenberger [5] states that the inversion number \( \text{inv} \) and inverse major index \( \text{imaj} \) are equidistributed over descent classes. That is, for any \( S \subseteq [n-1] \),

\[
\sum_{\pi \in \mathfrak{S}_n, \text{Des}(\pi) = S} q^{\text{inv} (\pi)} = \sum_{\pi \in \mathfrak{S}_n, \text{Des}(\pi) = S} q^{\text{imaj} (\pi)},
\]

this is equivalent to saying that the polynomial \( \beta_q (L) \)—defined in Subsection 2.2—counting \( n \)-permutations with descent composition \( L \) by inversion number also counts these same permutations by inverse major index. It follows that Theorems 4.3, 4.8, 4.14, and 4.16 and Corollaries 4.5 and 4.9 can be restated for the inverse major index, that is, by replacing every instance of \( \text{inv} \) with \( \text{imaj} \).

Second, we mention that there exist “alternating analogues” for nearly every concept in this paper relating to descents, as well as for several results in this section. Given a permutation \( \pi \in \mathfrak{S}_n \), we say that \( i \in [n-1] \) is an alternating descent of \( \pi \) if \( i \) is an odd descent or an even ascent, and an alternating run of \( \pi \) is a maximal consecutive subsequence of \( \pi \) containing no alternating descents. Then the alternating descent set, alternating descent composition, and the alternating descent number \( \text{altdes} \) can all be defined in the obvious way. The distribution of \( \text{altdes} \) over \( \mathfrak{S}_n \) is given by the polynomial

\[
\hat{A}_n (t) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{altdes} (\pi) + 1},
\]

the alternating analogue of the \( n \)th Eulerian polynomial. The exponential generating function for the alternating Eulerian polynomials is

\[
\sum_{n=0}^{\infty} \hat{A}_n (t) \frac{x^n}{n!} = \frac{1 - t}{1 - t (\sec((1-t)x) + \tan((1-t)x))},
\]

which is the exponential generating function for the ordinary Eulerian polynomials with the exponential function \( e^x \) replaced by \( \sec x + \tan x \).

There is in fact an alternating analogue for every descent statistic. For example, since \( i \) is a peak of \( \pi \) precisely when \( i - 1 \) is an ascent and \( i \) is a descent, we could define \( i \) to be an alternating peak if \( i - 1 \) is an “alternating ascent” and \( i \) is an alternating descent, that is, if \( \pi_{i-1} > \pi_i > \pi_{i+1} \) and \( i \) is odd or if \( \pi_{i-1} < \pi_i < \pi_{i+1} \) and \( i \) is even.

Alternating descents were introduced by Chebikin [3] and later brought into our non-commutative symmetric function framework by Gessel and the present author [8]. In [8], the authors defined a third homomorphism \( \hat{\Phi} : \text{Sym} \rightarrow F[[x]] \) by \( \hat{\Phi}(h_n) = E_n x^n / n! \), where \( E_n \) is the \( n \)th Euler number defined by \( \sum_{n=0}^{\infty} E_n x^n / n! = \sec x + \tan x \), and showed that \( \hat{\Phi}(r_L) = \hat{\beta}(L) x^n / n! \), where \( \hat{\beta}(L) \) is the number of \( n \)-permutations with alternating descent composition \( L \). Moreover, it is not hard to show that \( \hat{\Phi}(h(1)) = \hat{\Phi}(e(1)) = \sec x + \tan x \), and one can use these facts to obtain alternating analogues of Theorems 4.2, 4.7, 4.13, and 4.15 that express alternating analogues of descent statistic polynomials in terms of the alternating Eulerian polynomials.
5. Proofs via group actions

5.1. The modified Foata–Strehl action and \((pk, des)\) polynomials

In Theorem 4.2, we established a connection between the \(n\)th Eulerian polynomial—the polynomial encoding the distribution of the descent number \(des\) over \(\mathcal{S}_n\)—and the polynomial encoding the joint distribution of the peak number \(pk\) and \(des\) over \(\mathcal{S}_n\). We can give a generalization of this result which establishes the same connection between polynomials encoding the distribution of these statistics over subsets of \(\mathcal{S}_n\) that are invariant under a certain group action. We begin by defining this group action and later use it to prove the aforementioned generalization of Theorem 4.2.

In what follows, it will be convenient to alter the definitions of descent, peak, and valley to refer to the letter and not the index; that is, given a permutation \(\pi = \pi_1 \pi_2 \cdots \pi_n\), we call \(\pi_i\) (rather than \(i\)) a descent of \(\pi\) if \(\pi_i > \pi_{i+1}\), a peak of \(\pi\) if \(\pi_{i-1} < \pi_i > \pi_{i+1}\), and a valley of \(\pi\) if \(\pi_{i-1} > \pi_i < \pi_{i+1}\) [10]. In addition, \(\pi_i\) is called a double ascent of \(\pi\) if \(\pi_{i-1} < \pi_i < \pi_{i+1}\) and is called a double descent of \(\pi\) if \(\pi_{i-1} > \pi_i > \pi_{i+1}\). Denote the number of double ascents of \(\pi\) by \(\text{dasc}(\pi)\) and the number of double descents of \(\pi\) by \(\text{ddes}(\pi)\). We will be working with peaks, valleys, double ascents, and double descents of the word \(\infty \pi \infty\), where \(\infty\) is a letter defined to be greater than any integer in the usual ordering. For example, if \(\pi\) is a permutation with \(\pi_1 < \pi_2\), then \(\pi_1\) is a valley of \(\infty \pi \infty\) even though it is not a valley of \(\pi\).

Let \(\pi\) be an \(n\)-permutation and let \(x \in [n]\). We may write \(\pi = w_1 w_2 x w_4 w_5\) where \(w_2\) is the maximal consecutive subword immediately to the left of \(x\) whose letters are all smaller than \(x\), and \(w_4\) is the maximal consecutive subword immediately to the right of \(x\) whose letters are all smaller than \(x\); this is called the \(x\)-factorization of \(\pi\). For example, if \(\pi = 467125839\) and \(x = 5\), then \(\pi\) is the concatenation of \(w_1 = 467\), \(w_2 = 12\), \(x = 5\), the empty word \(w_4\), and \(w_5 = 839\).

Define \(\varphi_x : \mathcal{S}_n \to \mathcal{S}_n\) by \(\varphi_x(\pi) = w_1 w_4 x w_2 w_5\). It is easy to see that \(\varphi_x\) is an involution, and that for all \(x, y \in [n]\), the involutions \(\varphi_x\) and \(\varphi_y\) commute with each other. These involutions were defined by Foata and Strehl [6] in the context of the original Foata–Strehl action, but this was modified by Brändén [2] in the following way. If \(x \in [n]\), then define \(\varphi'_x : \mathcal{S}_n \to \mathcal{S}_n\) by

\[
\varphi'_x(\pi) = \begin{cases} 
\varphi_x(\pi), & \text{if } x \text{ is a double ascent or double descent of } \infty \pi \infty, \\
\pi, & \text{if } x \text{ is a peak or valley of } \infty \pi \infty.
\end{cases}
\]

Equivalently, \(\varphi'_x(\pi) = \varphi_x(\pi)\) if exactly one of \(w_2\) and \(w_4\) is nonempty, and \(\varphi'_x(\pi) = \pi\) otherwise. Returning to the example \(\pi = 467125839\) and \(x = 5\), we have \(\varphi'_x(\pi) = 467512839\) because \(x = 5\) is a double ascent. However, if \(x = 8\), then \(\varphi'_x(\pi) = \pi\) since \(x = 8\) is a peak. Note that if \(x\) is a double ascent of \(\pi\), then it is a double descent of \(\varphi'_x(\pi)\), and vice versa.

For any subset \(S \subseteq [n]\), we define \(\varphi'_S : \mathcal{S}_n \to \mathcal{S}_n\) by \(\varphi'_S = \prod_{x \in S} \varphi'_x\). These \(\varphi'_S\) are also involutions, and for all \(S, T \subseteq [n]\), we still have that \(\varphi'_S\) commutes with \(\varphi'_T\). The group \(\mathbb{Z}_2^n\) acts on \(\mathcal{S}_n\) via the involutions \(\varphi'_S\) over all \(S \subseteq [n]\); this group action is called the modified Foata–Strehl action, abbreviated MFS-action. The MFS-action can also be characterized

\[\text{Note that the statistics des, pk, and val are the same regardless of which definition is used.}\]
in terms of “valley hopping” (see [17, p. 173] or [18, Section 4.2]), and can be defined for decreasing binary trees (see [2, p. 516]).

The involutions \( \varphi'_S \) were considered earlier by Shapiro, Woan, and Getu [19], but they did not seem to be aware of the connection to the work of Foata and Strehl. The MFS-action was used much more extensively by Brändén, whose results in [2] motivated much of the work in this section.

Given a set \( \Pi \) of \( n \)-permutations, let
\[
A(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{des}(\pi) + 1}
\]
be the \( n \)th descent polynomial restricted to permutations in \( \Pi \) and let
\[
P^{(pk, \text{des})}(\Pi; y, t) := \sum_{\pi \in \Pi} y^{pk(\pi) + 1} t^{\text{des}(\pi) + 1}
\]
be the \((pk, \text{des})\) polynomial for permutations in \( \Pi \).

**Theorem 5.1.** Suppose that \( \Pi \subseteq \mathfrak{S}_n \) for \( n \geq 1 \) is invariant under the MFS-action. Then
\[
A(\Pi; t) = \left( \frac{1 + yt}{1 + y} \right)^{n+1} P^{(pk, \text{des})}(\Pi; \frac{(1 + yt)^2}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}). \tag{5.1}
\]
Equivalently,
\[
P^{(pk, \text{des})}(\Pi; y, t) = \left( \frac{1 + u}{1 + uv} \right)^{n+1} A(\Pi; v) \tag{5.2}
\]
where
\[
u = \frac{1+2y+2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t} \quad \text{and} \quad v = \frac{(1+t)^2-2yt-(1-t)^2 \sqrt{(1+t)^2-4yt}}{2yt}.
\]

Note that taking \( \Pi = \mathfrak{S}_n \) yields Theorem 4.2.

**Proof.** Let \( \sigma \) be an \( n \)-permutation and let \( \text{Orb}(\sigma) = \{ \varphi'_S(\sigma) \mid S \subseteq [n] \} \) be the orbit of \( \sigma \) under the MFS-action. Let \( \tilde{\sigma} = \infty \pi \infty \) and \( \tilde{\sigma} = \infty \sigma \infty \), and for a subset \( S \subseteq [n] \), let \( \pi_S = \varphi'_S(\pi) \) and \( \tilde{\pi}_S = \infty \varphi'_S(\pi) \infty \). First we wish to prove the identity
\[
\left( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{des}(\pi)} \right) (1 + y)^{\text{dasc}(\tilde{\sigma}) + \text{ddes}(\tilde{\sigma})} = \sum_{\pi \in \text{Orb}(\sigma)} (1 + yt)^{\text{dasc}(\tilde{\sigma})} (y + t)^{\text{ddes}(\tilde{\sigma})} \cdot y^{pk(\tilde{\sigma})}, \tag{5.3}
\]
which we do by showing that the two sides of the equation encode the same objects.

We begin with the left-hand side. Each summand in the factor \( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{des}(\pi)} \) corresponds to a permutation in the orbit of \( \sigma \) weighted by its descent number. Each summand in the factor \( (1 + y)^{\text{dasc}(\tilde{\sigma}) + \text{ddes}(\tilde{\sigma})} \) corresponds to marking a subset of double ascents and double descents of \( \tilde{\sigma} \). Thus, the left-hand side counts permutations in \( \text{Orb}(\sigma) \) where \( t \) is weighting the descent number and \( y \) is weighting the number of marked letters.

Now, let us examine the right-hand side of (5.3). Each term on the right-hand side corresponds to taking a permutation \( \pi \in \text{Orb}(\sigma) \), choosing a subset \( S \) of double ascents and double descents of \( \tilde{\pi} \), applying \( \varphi'_S \) to \( \pi \), marking the letters of \( S \) in \( \pi_S \) — which are all double ascents or double descents of \( \tilde{\pi}_S \) — and weighting the marked letters by \( y \) and the descents...
of $\pi_S$ by $t$. The $(1 + yt)^{\text{dasc}(\pi)}$ factor corresponds to selecting the double ascents, and the $(y + t)^{\text{ddes}(\pi)}$ factor corresponds to selecting the double descents. The only remaining descents of $\pi_S$ are the peaks of $\pi_S$, which are precisely the peaks of $\pi$ since the MFS-action does not affect peaks; this contributes the factor of $\frac{\pi}{\pi}$. In summary, both sides of the equation count permutations in the orbit of $\sigma$ with a marked subset $S$ of letters by the same weights, but on the right-hand side, we are applying the involution $\varphi'_S$ to each $\pi \in \text{Orb}(\sigma)$ before doing the counting.

Next, observe the following:

- The number of peaks of $\pi$ is equal to $\text{pk}(\pi)$. More specifically, the peaks of $\pi$ are precisely the peaks of $\pi$, since neither $\pi_1$ nor $\pi_n$ can be peaks of $\pi = \infty \pi \infty$.
- The number of double descents of $\pi$ is equal to $\text{des}(\pi) - \text{pk}(\pi)$, since the set of descents of $\pi$ is equal to the set of double descents and peaks of $\pi$.
- The number of double ascents of $\pi$ is equal to $n - \text{pk}(\pi) - \text{des}(\pi) - 1$. To see why this is true, first observe that the number of valleys of $\pi$ is $\text{pk}(\pi) + 1$. Then, the double ascents of $\pi$ are the letters which are not peaks, valleys, or double descents of $\pi$. Thus,
  
  \[ \text{dasc}(\pi) = n - (\text{pk}(\pi)) + \text{val}(\pi) + \text{ddes}(\pi) \]
  
  \[ = n - (\text{pk}(\pi)) + \text{pk}(\pi) + 1 + \text{des}(\pi) - \text{pk}(\pi) \]
  
  \[ = n - \text{pk}(\pi) - \text{des}(\pi) - 1. \]
- The total number of double ascents and double descents of $\sigma$ is equal to $n - 2 \text{pk}(\sigma) - 1$, which follows from the previous two points.

Therefore, we have the equation

\[
\left( \sum_{\pi \in \text{Orb}(\sigma)} t^{\text{des}(\pi)} \right) (1 + y)^{n - 2 \text{pk}(\sigma) - 1} = \sum_{\pi \in \text{Orb}(\sigma)} (1 + yt)^{n - \text{pk}(\pi) - \text{des}(\pi) - 1} (y + t)^{\text{des}(\pi) - \text{pk}(\pi)} t^{\text{pk}(\pi)},
\]

and dividing both sides by $(1 + y)^{n - 2 \text{pk}(\sigma) - 1} = (1 + y)^{n - 2 \text{pk}(\sigma) - 1}$ gives us

\[
\sum_{\pi \in \text{Orb}(\sigma)} t^{\text{des}(\pi)} = \sum_{\pi \in \text{Orb}(\sigma)} \frac{(1 + yt)^{n - \text{pk}(\pi) - \text{des}(\pi) - 1} (y + t)^{\text{des}(\pi) - \text{pk}(\pi)} t^{\text{pk}(\pi)}}{(1 + y)^{n - 2 \text{pk}(\sigma) - 1}}.
\]

Then, summing over all orbits corresponding to permutations in $\Pi$ yields

\[
\sum_{\pi \in \Pi} t^{\text{des}(\pi)} = \sum_{\pi \in \Pi} \frac{(1 + yt)^{n - \text{pk}(\pi) - \text{des}(\pi) - 1} (y + t)^{\text{des}(\pi) - \text{pk}(\pi)} t^{\text{pk}(\pi)}}{(1 + y)^{n - 2 \text{pk}(\sigma) - 1}},
\]

and multiplying both sides by $t$ yields

\[
A(\Pi; t) = \sum_{\pi \in \Pi} \frac{(1 + yt)^{n - \text{pk}(\pi) - \text{des}(\pi) - 1} (y + t)^{\text{des}(\pi) - \text{pk}(\pi)} t^{\text{pk}(\pi) + 1}}{(1 + y)^{n - 2 \text{pk}(\sigma) - 1}}
\]

\[= \left( \frac{1 + yt}{1 + y} \right)^{n + 1} \left( \frac{(1 + y)^2 t}{(y + t)(1 + yt)} \right) \left( \frac{y + t}{1 + yt} \right), \]

which is (5.1). We obtain (5.2) by performing the same substitution as in the proof of Theorem 4.2. \qed
Setting $y = 1$ in (5.1) yields a main result of Brändén [2, Corollary 3.2], which we state below for completeness. For $\Pi \subseteq \mathfrak{S}_n$, let

$$P^{pk}(\Pi; t) := \sum_{\pi \in \Pi} t^{pk(\pi)+1}.$$  

**Corollary 5.2.** Suppose that $\Pi \subseteq \mathfrak{S}_n$ for $n \geq 1$ is invariant under the MFS-action. Then

$$A(\Pi; t) = \left(\frac{1+t}{2}\right)^{n+1} P^{pk}(\Pi; \frac{4t}{(1+t)^2}).$$

Equivalently,

$$P^{pk}(\Pi; t) = \left(\frac{2}{1+v}\right)^{n+1} A(\Pi; v)$$

where $v = \frac{2}{1} (1 - \sqrt{1 - t}) - 1$.

Besides $\Pi = \mathfrak{S}_n$, other subsets $\Pi$ of $\mathfrak{S}_n$ invariant under the MFS-action include the set of $r$-stack-sortable $n$-permutations $\mathfrak{S}_n^r$ (see [2 Corollary 4.2]) for any $r \geq 1$. Of particular interest is the set of 1-stack-sortable $n$-permutations $\mathfrak{S}_n^1$, which Knuth [12] showed to be precisely the set $\text{Av}_n(231)$ of 231-avoiding $n$-permutations: $n$-permutations such that there exist no $i < j < k$ for which $\pi_k < \pi_i < \pi_j$. The descent polynomial for 231-avoiding $n$-permutations is known to be the $n$th Narayana polynomial

$$N_n(t) := \sum_{\pi \in \text{Av}_n(231)} t^{\text{des}(\pi)+1} = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} t^k$$

for $n \geq 1$. Hence, we have the following corollary.

**Corollary 5.3.** For $n \geq 1$, we have

$$N_n(t) = \left(\frac{1+yt}{1+y}\right)^{n+1} P^{(pk,\text{des})}(\text{Av}_n(231); \frac{(1+y)^2t}{(y+t)(1+yt)}; \frac{y+t}{1+yt}).$$

(5.4)

Equivalently,

$$P^{(pk,\text{des})}(\text{Av}_n(231); y, t) = \left(\frac{1+u}{1+uv}\right)^{n+1} N_n(v)$$

where $u = \frac{(1+t)^2 - 2yt - (1-t) \sqrt{(1+t)^2 - 4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2 - 2yt - (1+t) \sqrt{(1+t)^2 - 4yt}}{2yt}$.

In the next subsection, we examine the polynomials $P^{(pk,\text{des})}(\text{Av}_n(231); y, t)$ in greater detail and interpret this corollary in terms of binary trees and Dyck paths. Setting $y = 1$ in (5.4) gives an identity relating the $n$th Narayana polynomial to the peak polynomial for 231-avoiding $n$-permutations, which is equivalent to Equation 4.7 of [18] p. 77.

We also state the result for 2-stack-sortable permutations, which can again be characterized in terms of pattern avoidance: the set of 2-stack-sortable $n$-permutations $\mathfrak{S}_n^2$ is equal to
the set $\text{Av}_n(2341, 35241)$ of $n$-permutations avoiding the pattern 2341 and the “barred pattern” 35241 (see [27, Theorem 4.2.18]). The descent polynomial for these $n$-permutations is given by the following formula, which was proven by Jacquard and Schaeffer [10] using a bijection with nonseparable planar maps:

$$A(\mathfrak{S}; t) = \sum_{k=1}^{n} \frac{(n + k - 1)!(2n - k)!}{k!(n - k + 1)!(2k - 1)!} t^k.$$ 

**Corollary 5.4.** For $n \geq 1$, we have

$$A(\mathfrak{S}; t) = \left(\frac{1 + yt}{1 + y}\right)^{n+1} P^{\text{des}}(\mathfrak{S}; (1 + y)^2 t, y + t) \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, 1 + yt)$$

Equivalently,

$$P^{\text{des}}(\mathfrak{S}; y, t) = \left(\frac{1 + u}{1 + uv}\right)^{n+1} A(\mathfrak{S}; v)$$

where $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)yt}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$.

Furthermore, there are various statistics $st$ that are constant on any orbit of the MFS-action. These include the number of occurrences of the generalized permutation pattern 23-1 and the number of occurrences of 13-2 (see [2, Theorem 5.1]) as well as the number of “admissible inversions” (see [13, Lemma 7]). Thus if we refine the polynomials $A(\Pi; t)$ and $P^{\text{des}}(\Pi; y, t)$ by defining

$$A^{\text{st}}(\Pi; t, w) := \sum_{\pi \in \Pi} t^{\text{des}(\pi)+1} w^{\text{st}(\pi)}$$

and

$$P^{\text{des}, \text{st}}(\Pi; y, t, w) := \sum_{\pi \in \Pi} y^{\text{pk}(\pi)+1} t^{\text{des}(\pi)+1} w^{\text{st}(\pi)},$$

then we have the following result.

**Corollary 5.5.** Suppose that $\Pi \subseteq \mathfrak{S}_n$ for $n \geq 1$ is invariant under the MFS-action, and let $st$ be a permutation statistic that is constant on any orbit of the MFS-action. Then

$$A^{\text{st}}(\Pi; t, w) = \left(\frac{1 + yt}{1 + y}\right)^{n+1} P^{\text{des}, \text{st}}(\Pi; (1 + y)^2 t, y + t) \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, 1 + yt, w),$$

Equivalently,

$$P^{\text{des}, \text{st}}(\Pi; y, t, w) = \left(\frac{1 + u}{1 + uv}\right)^{n+1} A^{\text{st}}(\Pi; v, w)$$

where $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)yt}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$.

---

11 A permutation is said to avoid the “barred pattern” 35241 if no 3241-pattern occurs without there being a larger letter between the letters corresponding to 3 and 2. Note that 35241 is not a signed permutation. For more general definitions on notions of pattern avoidance in permutations, see [11] for a comprehensive reference.
5.2. 231-avoiding permutations, binary trees, and Dyck paths

Corollary 5.3 shows that the polynomial

\[ P^{(pk, \text{des})}(\text{Av}_n(231); y, t) = \sum_{\pi \in \text{Av}_n(231)} y^{pk(\pi)+1} t^{\text{des}(\pi)+1} \]

can be expressed in terms of the \( n \)th Narayana polynomial \( N_n(t) \). However, it is worth noting that there is a simple formula for the polynomials \( P^{(pk, \text{des})}(\text{Av}_n(231); y, t) \) and their coefficients.

**Theorem 5.6.** For \( n \geq 1 \), we have

\[ P^{(pk, \text{des})}(\text{Av}_n(231); y, t) = \sum_{k=0}^{(n-1)/2} \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} y^{k+1} t^{k+1} (1+t)^{n-2k-1}, \] (5.5)

and the number of 231-avoiding \( n \)-permutations with exactly \( k \) peaks and \( j \) descents is

\[ \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} \binom{n-2k-1}{j-k}. \] (5.6)

By setting \( t = 1 \) in (5.5), we see that

\[ \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} 2^{n-2k-1} \]

is the number of 231-avoiding \( n \)-permutations with exactly \( k \) peaks; this is Corollary 4.5 of Brändén [2].

**Proof.** Define \( G = G(y, t, x) \) to be the ordinary generating function for the polynomials \( P^{(pk, \text{des})}(\text{Av}_n(231); y, t)/y \) for \( n \geq 1 \). Then \( G \) satisfies the functional equation

\[ G = x(yG^2 + tG + G + t); \]

this can be seen using the bijection with binary trees given later in this subsection. Then a straightforward application of Lagrange inversion yields (5.5), and (5.6) is easily obtained from (5.5) using the binomial theorem. \( \square \)

We recall the definitions of two families of objects closely related to 231-avoiding permutations. A binary tree\(^{12}\) is a rooted tree \( T \) satisfying the following two conditions:

1. Each node (i.e., vertex) of \( T \) has 0, 1, or 2 children.
2. Each child of each node is distinguished as a left child or a right child.

Let \( T_n \) be the set of binary trees with \( n \) nodes. For example, below is a binary tree with 6 nodes.

\[^{12}\text{These are sometimes also called binary plane trees or planar binary trees in the literature.}\]
We will work with two particular statistics defined on binary trees. If $T$ is a binary tree, then let $\text{nlc}(T)$ be the number of nodes of $T$ with no left children, and let $\text{tc}(T)$ be the number of nodes of $T$ with two children. For example, if $T$ is the binary tree given above, then $\text{nlc}(T) = 4$ and $\text{tc}(T) = 2$.

A Dyck path of semilength $n$ is a lattice path in the plane from $(0,0)$ to $(2n,0)$, consisting of up steps $(1,1)$ and down steps $(-1,1)$, and never going below the $x$-axis. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength $n$. A Dyck path can be encoded by a word on the alphabet $\{U,D\}$ in which $U$ corresponds to an up step and $D$ a down step; this is called the Dyck word corresponding to the Dyck path. For example, below is a Dyck path with semilength 6 corresponding to the Dyck word $UUUUUDDDDDUD$.

We define two statistics on Dyck paths: the number of “peaks” and the number of “hooks”. Define a peak of a Dyck path $\mu$ to be an occurrence of the subword $UD$ in the corresponding Dyck word of $\mu$, and a hook of $\mu$ to be an occurrence of the subword $DDU$. Let $\text{pk}(\mu)$ be the number of peaks of $\mu$ and $\text{hk}(\mu)$ the number of hooks of $\mu$. For example, if $\mu$ is the Dyck path given above, then $\text{pk}(\mu) = 3$ and $\text{hk}(\mu) = 1$.

It is well known that binary trees with $n$ nodes, Dyck paths with semilength $n$, and 231-avoiding $n$-permutations are all counted by the $n$th Catalan number $C_n$ given by

$$C_n := \frac{1}{n+1} \binom{2n}{n}.$$ 

A number of bijections exist between these three sets of objects. Below we will use two particular bijections which behave nicely with respect to the statistics defined above. The first bijection $\Theta : \text{Av}_n(231) \rightarrow \mathcal{T}_n$ satisfies $\text{des}(\pi) + 1 = \text{nlc}(\Theta(\pi))$ and $\text{pk}(\pi) = \text{tc}(\Theta(\pi))$. The second bijection $\Psi : \text{Av}_n(231) \rightarrow \mathcal{D}_n$ satisfies $\text{des}(\pi) + 1 = \text{pk}(\Psi(\pi))$ and $\text{pk}(\pi) = \text{hk}(\Psi(\pi))$. These bijections will allow us to give interpretations of Corollary 5.3 and Theorem 5.6 for binary trees and Dyck paths.

A decreasing binary tree $T$ with $n$ nodes is a binary tree with nodes labeled using distinct letters from $[n]$ such that if $i,j$ are nodes of $T$ with $j$ a child of $i$, then $i > j$\footnote{We are identifying each node with its label.} Let us briefly...
recall the classical recursive bijection \( \tilde{\Theta} \) between permutations\(^{14}\) and decreasing binary trees. If \( \pi \) is the empty permutation, then let \( \tilde{\Theta}(\pi) = \emptyset \). Otherwise, let \( n \) be the largest letter of \( \pi \), and factor \( \pi \) as \( \pi = \sigma n \tau \) where \( \sigma \) is the (possibly empty) subsequence of \( \pi \) consisting of all letters to the left of \( n \) and \( \tau \) is the (possibly empty) subsequence consisting of all letters to the right of \( n \). Then \( \tilde{\Theta}(\pi) \) is defined by letting \( n \) be the root of \( \tilde{\Theta}(\pi) \), attaching \( \tilde{\Theta}(\sigma) \) as a left subtree, and attaching \( \tilde{\Theta}(\tau) \) as a right subtree. It is clear that \( \tilde{\Theta} \) gives a bijection between \( S_n \) and \( T_n \) for all \( n \).

**Lemma 5.7.** Let \( \pi \in S_n \) with \( n \geq 1 \). Then \( \text{des}(\pi) + 1 = \text{nlc}(\tilde{\Theta}(\pi)) \) and \( \text{pk}(\pi) = \text{tc}(\tilde{\Theta}(\pi)) \).

*Proof.* Fix \( \pi \in S_n \) with \( n \geq 1 \), and write \( \pi = \sigma n \tau \). It is easy to verify that

\[
\text{des}(\pi) + 1 = \begin{cases}
\text{des}(\sigma) + 1 + \text{des}(\tau) + 1, & \text{if } |\sigma| \geq 1 \text{ and } |\tau| \geq 1, \\
\text{des}(\sigma) + 1, & \text{if } |\sigma| \geq 1 \text{ and } |\tau| = 0, \\
\text{des}(\tau) + 1 + 1, & \text{if } |\sigma| = 0 \text{ and } |\tau| \geq 1, \\
1, & \text{if } |\sigma| = |\tau| = 0,
\end{cases}
\]

and

\[
\text{nlc}(\tilde{\Theta}(\pi)) = \begin{cases}
\text{nlc}(\tilde{\Theta}(\sigma)) + \text{nlc}(\tilde{\Theta}(\tau)), & \text{if } |\sigma| \geq 1 \text{ and } |\tau| \geq 1, \\
\text{nlc}(\tilde{\Theta}(\sigma)), & \text{if } |\sigma| \geq 1 \text{ and } |\tau| = 0, \\
\text{nlc}(\tilde{\Theta}(\tau)) + 1, & \text{if } |\sigma| = 0 \text{ and } |\tau| \geq 1, \\
1, & \text{if } |\sigma| = |\tau| = 0,
\end{cases}
\]

so it follows from induction that \( \text{des}(\pi) + 1 = \text{nlc}(\tilde{\Theta}(\pi)) \) for all nonempty permutations \( \pi \). Similarly,

\[
\text{pk}(\pi) = \begin{cases}
\text{pk}(\sigma) + \text{pk}(\tau) + 1, & \text{if } |\sigma| \geq 1 \text{ and } |\tau| \geq 1, \\
\text{pk}(\sigma) + \text{pk}(\tau), & \text{otherwise},
\end{cases}
\]

and

\[
\text{tc}(\tilde{\Theta}(\pi)) = \begin{cases}
\text{tc}(\tilde{\Theta}(\sigma)) + \text{tc}(\tilde{\Theta}(\tau)) + 1, & \text{if } |\sigma| \geq 1 \text{ and } |\tau| \geq 1, \\
\text{tc}(\tilde{\Theta}(\sigma)) + \text{tc}(\tilde{\Theta}(\tau)), & \text{otherwise},
\end{cases}
\]

so we have \( \text{pk}(\pi) = \text{tc}(\tilde{\Theta}(\pi)) \) as well. \qed

For example, the permutation \( \pi = 132495876 \) is mapped by \( \tilde{\Theta} \) to the decreasing binary tree below.

\[\begin{array}{c}
9 \\
4 \quad 8 \\
3 \quad 5 \quad 7 \\
1 \quad 2 \quad 6
\end{array}\]

\(^{14}\)Here we are using the term “permutation” in a broader sense, referring to a linear ordering of any set of distinct integers.
Observe that $\text{des}(\pi) + 1 = \text{nlc}(\Theta(\pi)) = 5$ and $\text{pk}(\pi) = \text{tc}(\Theta(\pi)) = 3$. To define the map $\Theta : \text{Av}_n(231) \to \mathcal{T}_n$, first notice that the permutation $\pi = 132495876$ is $231$-avoiding, and that performing a post-order traversal of the corresponding tree $\Theta(\pi)$ yields the increasing permutation 123456789. Indeed, it is not hard to verify that a permutation $\pi \in S_n$ is $231$-avoiding if and only if a post-order traversal of $\Theta(\pi)$ yields $1 \cdots n$. Thus, if we define $\Theta(\pi)$ for $\pi \in \text{Av}_n(231)$ to be the binary tree obtained by taking $\Theta(\pi)$ and removing the labels, then we can recover $\Theta(\pi)$ from $\Theta(\pi)$ by labeling the nodes via post-order traversal; it follows that we have a well-defined bijection $\Theta : \text{Av}_n(231) \to \mathcal{T}_n$ satisfying the properties from Lemma 5.7. Therefore, the $n$th Narayana polynomial $N_n(t)$ gives the distribution of binary trees with $n$ nodes by the number of nodes without left children, and by defining

$$T_n^{(\text{tc, nlc})}(y, t) := \sum_{T \in \mathcal{T}_n} y^{\text{tc}(T)} t^{\text{nlc}(T)},$$

we have proven the following interpretation of Corollary 5.3 and Theorem 5.6.

**Corollary 5.8.** For any $n \geq 1$, we have

$$N_n(t) = \left(\frac{1 + yt}{1 + y}\right)^{n+1} T_n^{(\text{tc, nlc})} \left(\frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}\right).$$

Equivalently,

$$T_n^{(\text{tc, nlc})}(y, t) = \left(\frac{1 + u}{1 + uv}\right)^{n+1} N_n(v),$$

where $u = \frac{1 + t^2 - 2yt - (1-t)\sqrt{(1+t)^2 - 4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2 - 2yt - (1+t)\sqrt{(1+t)^2 - 4yt}}{2yt}$. Furthermore,

$$T_n^{(\text{tc, nlc})}(y, t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} y^{k+1} t^{k+1} (1 + t)^{n-2k-1}$$

and the number of binary trees with $n$ nodes, $k$ nodes with two children, and $j$ nodes with no left children is

$$\frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} \binom{n-2k-1}{j-k-1}.$$

Now we describe the bijection $\Psi : \text{Av}_n(231) \to \mathcal{D}_n$, which was defined by Stump in [26]. Let $\pi \in \text{Av}_n(231)$ with $\text{Comp}(\pi) = (L_1, L_2, \ldots, L_k) \equiv n$ and $\text{Comp}(\pi^{-1}) = (K_1, K_2, \ldots, K_k) \equiv n$. Then $\Psi(\pi)$ is defined to be the Dyck path with corresponding Dyck word $U^{K_1} D^{L_1} U^{K_2} D^{L_2} \cdots U^{K_k} D^{L_k}$, in which $U^m$ represents

$$\underbrace{UU \cdots U}_{m \text{ times}}$$

In defining this map, Stump proves that $\text{des}(\pi) = \text{des}(\pi^{-1})$ if $\pi$ avoids $\sigma$ for all $\sigma \in \{132, 231, 312, 213\}$. Thus, if $\pi \in \text{Av}_n(231)$, then the descent compositions of $\pi$ and $\pi^{-1}$ have the same number of parts.
and similarly for $D^{\mu}$. It is not obvious that this word actually corresponds to a Dyck path, or that $\Psi$ is a bijection even if it is well-defined; we refer the reader to Stump’s paper \cite{26} for proofs.

For example, take the permutation $\pi = 219438567$, which has inverse $\pi^{-1} = 215478963$. Then $\text{Comp}(\pi) = (1, 2, 1, 2, 3)$ and $\text{Comp}(\pi^{-1}) = (1, 2, 4, 1, 1)$. Then $\Psi(\pi)$ is the Dyck path corresponding to the Dyck word $UDUUDDUUUUDUDDUDDD$, pictured below.

\begin{center}
\includegraphics[width=0.5\textwidth]{dyck_path.png}
\end{center}

**Lemma 5.9.** Let $\pi \in S_n$ with $n \geq 1$. Then $\text{des}(\pi) + 1 = \text{pk}(\Psi(\pi))$ and $\text{pk}(\pi) = \text{hk}(\Psi(\pi))$.

**Proof.** It is obvious from the definition of $\Psi$ that $\text{des}(\pi) + 1$—the number of parts of $\text{Comp}(\pi)$—is equal to the number of peaks of $\Psi(\pi)$.

To show that $\text{pk}(\pi) = \text{hk}(\Psi(\pi))$, let $\pi \in \text{Av}_n(231)$ with descent composition $L = (L_1, L_2, \ldots, L_k)$. Since a peak of $\pi$ corresponds to an occurrence of an ascent followed by a descent, it follows that $\text{pk}(\pi)$ is equal to the number of non-final parts of $L$ of length at least 2 (that is, not counting $L_k$ if $L_k \geq 2$). Each part $L_i$ of $L$ contributes to $\Psi(\pi)$ a sequence of $L_i$ down steps, so if $L_i \geq 2$, then we have a sequence of at least 2 down steps. This sequence of down steps is followed by an up step if $i \neq k$ (i.e., if $L_i$ is not the final part of $L$), so each non-final part of $L$ of length at least 2 gives rise to a hook. Moreover, it is easy to see that every hook of $\Psi(\pi)$ arises in this way, so $\text{pk}(\pi) = \text{hk}(\Psi(\pi))$. \hfill \Box

For example, taking $\pi = 219438567$ as above, we have $\text{des}(\pi) + 1 = \text{pk}(\Psi(\pi)) = 5$ and $\text{pk}(\pi) = \text{hk}(\Psi(\pi)) = 2$. It follows from Lemma 5.9 that the Narayana polynomials $N_n(t)$ also give the distribution of Dyck paths with semilength $n$ by number of peaks, and if we define

$$D^{(\text{hk}, \text{pk})}_n(y, t) := \sum_{\mu \in D_n} y^{\text{hk}(\mu)+1} t^{\text{pk}(\mu)},$$

we obtain yet another interpretation of Corollary 5.3 and Theorem 5.6.

**Corollary 5.10.** For any $n \geq 1$, we have

$$N_n(t) = \left(\frac{1+yt}{1+y}\right)^{n+1} D^{(\text{hk}, \text{pk})}_n \left(\frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right).$$

Equivalently,

$$D^{(\text{hk}, \text{pk})}_n(y, t) = \left(\frac{1+u}{1+w}\right)^{n+1} N_n(v),$$

where $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$. Furthermore,

$$D^{(\text{hk}, \text{pk})}_n(y, t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} y^{k+1} t^{k+1} (1+t)^{n-2k-1}$$

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and the number of Dyck paths of semilength \( n \) with \( k \) hooks and \( j \) peaks is

\[
\frac{1}{k+1} \binom{2k}{k} \binom{n-1}{2k} \binom{n-2k-1}{j-k-1}.
\]

### 5.3. The Petersen action and \((\text{lpk, des})\) and \((\text{lpk, val, des})\) polynomials

Just as we showed that Theorem 4.2 can be generalized for subsets of \( S_n \) that are invariant under the modified Foata–Strehl action, we will generalize Theorems 4.10 and 4.17 for subsets of \( B_n \) that are invariant under a different \( \mathbb{Z}_2^n \)-action described by Petersen in [18, Section 13.2]. For this, we revert to our original definitions of descent, peak, and valley as referring to the index rather than the letter. Similarly, we redefine double ascent and double descent to refer to the index as well.

For a permutation \( \pi \in S_n \), let \( B(\pi) \) be the set of signed permutations that can be obtained by choosing any subset of the letters in \( \pi \) and replacing them with their negatives. For example, if \( \pi = 213 \), then \( B(\pi) = \{213, 2\overline{1}3, 21\overline{3}, 21\overline{3}, 2\overline{1}3, 2\overline{1}3, 21\overline{3}, 2\overline{1}3\} \). By thinking of \( \pi \) as a signed permutation, we see that \( \pi \) is the unique representative of \( B(\pi) \) without negative signs, and \( B(\pi) \) is the orbit of \( \pi \) under the \( \mathbb{Z}_2^n \)-action on \( B_n \) given by taking a subset of letters and reversing their signs.

Before proving our main results, we establish an important lemma that will allow us to determine the distribution of \((\text{neg, des}_B)\) or \((\text{neg, fdes})\) over the orbit of any given \( \pi \in S_n \) simply by knowing the number of peaks, valleys, double ascents, and double descents of the permutation \( \hat{\pi} = 0\pi \infty \) of \( \{0, \infty\} \cup [n] \).

**Lemma 5.11.** Let \( \pi \in S_n \), \( \hat{\pi} = 0\pi \infty \), and \( \sigma \in B(\pi) \).

- If \( i \) is a peak of \( \hat{\pi} \), then \( i-1 \) is a descent of \( \sigma \) if and only if \( \sigma_i < 0 \), and \( i \) is a descent of \( \sigma \) if and only if \( \sigma_i > 0 \).
- If \( i \) is a double ascent of \( \hat{\pi} \), then \( i-1 \) is a descent of \( \sigma \) if and only if \( \sigma_i < 0 \).
- If \( i \) is a double descent of \( \hat{\pi} \), then \( i \) is a descent of \( \sigma \) if and only if \( \sigma_i > 0 \).

Moreover, every descent of \( \sigma \) is accounted for exactly once in the above.

**Proof.** Suppose that \( i \) is a peak of \( \hat{\pi} \), so that \( \pi_{i-1} < \pi_i > \pi_{i+1} \). If \( \sigma_{i-1} > \sigma_i \), then \( \sigma_{i-1} < \pi_i = \pm \sigma_i \), which forces \( \sigma_i = -\pi_i \) and thus \( \sigma_i < 0 \). If \( \sigma_i > \sigma_{i+1} \), then \( \sigma_{i+1} < \pi_i = \pm \sigma_i \), which forces \( \sigma_i = \pi_i \) and thus \( \sigma_i > 0 \). On the other hand, if \( \sigma_i < 0 \), then \( \sigma_i = -\pi_i < -\pi_{i-1} \leq \sigma_{i-1} \) and \( \sigma_i = -\pi_i < -\pi_{i+1} \leq \sigma_{i+1} \), so \( i-1 \) is a descent and \( i \) is an ascent. The statements for double ascents and double descents follow using similar reasoning.

We now verify that every descent of \( \sigma \) is accounted for exactly once. Fix an \( i \) between 0 and \( n-1 \). If \( i \) is a peak or double descent of \( \hat{\pi} \), then \( i+1 \) is either a double descent or valley of \( \hat{\pi} \). Then the sign of \( \sigma_i \) determines whether or not \( i \) is a descent of \( \sigma \), whereas the sign of \( \sigma_{i+1} \) has no effect on whether or not \( i \) is a descent of \( \sigma \). If \( i \) is a valley or double descent, then \( i+1 \) is a peak or double ascent of \( \hat{\pi} \), and vice versa.

---

\[^{16}\text{This is assuming that } i \neq 1, \text{ but the case } i = 1 \text{ follows similarly.}\]
ascent of \( \hat{\pi} \) or if \( i = 0 \), then \( i + 1 \) is either a double ascent or peak of \( \hat{\pi} \). Then the sign of \( \sigma_{i+1} \) determines whether or not \( i \) is a descent of \( \sigma \), whereas the sign of \( \sigma_i \) has no effect on whether or not \( i \) is a descent of \( \sigma \). This shows that every descent is accounted for exactly once in the above.

Given a set \( \Pi \subseteq S_n \), let \( \mathfrak{B}(\Pi) := \bigcup_{\pi \in \Pi} \mathfrak{B}(\pi) \). Also, define the polynomials

\[
B(\Pi; y, t) := \sum_{\pi \in \mathfrak{B}(\Pi)} y^{\text{neg}(\pi)} t^{\text{des}(\sigma)}
\]

and

\[
P^{(\text{lpk, des})}(\Pi; y, t) := \sum_{\pi \in \Pi} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)}.
\]

**Theorem 5.12.** Let \( \Pi \subseteq S_n \) for \( n \geq 0 \). Then

\[
B(\Pi; y, t) = (1 + yt)^n P^{(\text{lpk, des})}(\Pi; \frac{(1 + y)^2 t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}).
\] (5.7)

Equivalently,

\[
P^{(\text{lpk, des})}(\Pi; y, t) = \frac{B(\Pi; u, v)}{(1 + uv)^n}
\] (5.8)

where \( u = \frac{1 + t^2 - 2yt - (1 - t)\sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t} \) and \( v = \frac{(1 + t)^2 - 2yt - (1 + t)\sqrt{(1 + t)^2 - 4yt}}{2yt} \).

By taking \( \Pi = S_n \), we recover Theorem 4.10.

**Proof.** For a fixed \( \pi \in \Pi \), let \( \hat{\pi} = 0 \pi \infty \). Then it follows from Lemma 5.11 that

\[
\sum_{\sigma \in \mathfrak{B}(\pi)} y^{\text{neg}(\sigma)} t^{\text{des}(\sigma)} = ((1 + y)t)^{\text{lpk}(\hat{\pi})} (1 + y)^{\text{val}(\hat{\pi})} (1 + yt)^{\text{dasc}(\hat{\pi})} (y + t)^{\text{ddes}(\hat{\pi})}.
\]

Now, observe the following:

- The number of peaks of \( \hat{\pi} \) is equal to \( \text{lpk}(\pi) \). Indeed, every peak of \( \pi \) is a peak of \( \hat{\pi} \), but if \( \pi \) begins with a descent, then that contributes an additional peak to \( \hat{\pi} \).

- The number of double descents of \( \hat{\pi} \) is equal to \( \text{des}(\pi) - \text{lpk}(\pi) \), since the set of descents of \( \pi \) is equal to the set of double descents and peaks of \( \hat{\pi} \).

- The number of valleys of \( \hat{\pi} \) is equal to \( \text{lpk}(\pi) \), but this is harder to see. Suppose that \( n - 1 \) is a descent of \( \pi \). Then \( \text{val}(\hat{\pi}) = \text{val}(\pi) + 1 \). We know from Lemma 2.1 that, in this case, \( \text{lpk}(\pi) = \text{val}(\pi) + 1 \), so indeed \( \text{val}(\hat{\pi}) = \text{lpk}(\pi) \). If \( n - 1 \) is not a descent of \( \pi \), then \( \text{val}(\hat{\pi}) = \text{val}(\pi) \), and since we know that \( \text{lpk}(\pi) = \text{val}(\pi) \) from Lemma 2.1, the claim follows.

- The number of double ascents of \( \hat{\pi} \) is equal to \( n - \text{lpk}(\pi) - \text{des}(\pi) \), since

\[
\text{dasc}(\hat{\pi}) = n - (\text{pk}(\hat{\pi}) + \text{val}(\hat{\pi}) + \text{ddes}(\hat{\pi}))
\]

\[
= n - (\text{lpk}(\pi) + \text{lpk}(\pi) + \text{des}(\pi) - \text{lpk}(\pi))
\]

\[
= n - \text{lpk}(\pi) - \text{des}(\pi).
\]
Hence, we have
\[
\sum_{\sigma \in B(\pi)} y^{\text{neg}(\pi)} t^{\text{des}_B(\pi)} = \left(1 + y^2 t\right)^{\text{lpk}(\pi)} (y + t)^{\text{des}(\pi)} - \text{lpk}(\pi)(1 + yt)^{n - \text{lpk}(\pi) - \text{des}(\pi)}
\]
\[
= (1 + yt)^n \left(\frac{(1 + y)^2 t}{(y + t)(1 + yt)}\right)^{\text{lpk}(\pi)} \left(\frac{y + t}{1 + yt}\right)^{\text{des}(\pi)}.
\]
Summing over all \( \pi \in \Pi \) yields (5.7), and (5.8) is obtained via the same substitutions as before.

By setting \( y = 1 \) in (5.7), we obtain the following specialization. For \( \Pi \subseteq S_n \), define
\[
B(\Pi; t) := \sum_{\pi \in B(\Pi)} t^{\text{des}_B(\pi)}
\]
and
\[
P^{\text{lpk}}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{lpk}(\pi)}.
\]

**Corollary 5.13.** Let \( \Pi \subseteq S_n \) for \( n \geq 0 \). Then
\[
B(\Pi; t) = (1 + t)^n P^{\text{lpk}} \left(\Pi; \frac{4t}{(1 + t)^2}\right).
\]
Equivalently,
\[
P^{\text{lpk}}(\Pi; t) = \frac{B(\Pi; v)}{(1 + v)^n}
\]
where \( v = \frac{2}{t}(1 - \sqrt{1 - t}) - 1 \).

Similarly to Corollary 5.5, we can refine Theorem 5.12 by statistics that are constant on any orbit of the described action. These are the statistics \( \text{st} \) on \( B \) such that when \( \sigma, \tau \in B(\pi) \)—that is, if \( \sigma \) and \( \tau \) reduce to the same \( \pi \in S \) when their negative signs are removed—we have \( \text{st}(\sigma) = \text{st}(\tau) \), or in other words, statistics \( \text{st} \) that depend only on the underlying unsigned permutation. Clearly, these are the permutation statistics on \( S \).\(^{17}\)

Thus, for a statistic \( \text{st} \) on \( S \) and a signed permutation \( \sigma \in B \), we let \( \text{st}(\sigma) \) mean the value of \( \text{st} \) on the unsigned permutation obtained by removing all negative signs from \( \sigma \).

For a permutation statistic \( \text{st} \) on \( S \), define
\[
B^{\text{st}}(\Pi; y, t, w) := \sum_{\pi \in B(\Pi)} y^{\text{neg}(\pi)} t^{\text{des}_B(\pi)} w^{\text{st}(\pi)}
\]
and
\[
P^{(\text{lpk, des, st})}(\Pi; y, t, w) := \sum_{\pi \in \Pi} y^{\text{lpk}(\pi)} t^{\text{des}(\pi)} w^{\text{st}(\pi)}.
\]

\(^{17}\)More precisely, each such statistic induces a permutation statistic on \( S \), and conversely, each permutation statistic on \( S \) induces a statistic on \( B \) that is constant on every orbit.
Corollary 5.14. Let \( st \) be a permutation statistic on \( \mathcal{S} \) and let \( \Pi \subseteq \mathcal{S}_n \) for \( n \geq 0 \). Then

\[
B^{st}(\Pi; y, t, w) = (1 + yt)^nP^{(lpk, des, st)}(\Pi; \frac{(1 + y)^2t}{(y + t)(1 + yt)}, \frac{y + t}{1 + yt}, w).
\]

Equivalently,

\[
P^{(lpk, des, st)}(\Pi; y, t, w) = \frac{B^{st}(\Pi; u, v, w)}{(1 + uv)^n}
\]

where \( u = \frac{1 + t^2 - 2yt - (1-t)\sqrt{(1+t)^2 - 4yt}}{2(1-y)t} \) and \( v = \frac{(1+t)^2 - 2yt - (1+t)\sqrt{(1+t)^2 - 4yt}}{2yt} \).

To conclude this section, we state the corresponding results for the statistics \((lpk, val, des)\) and \((neg, fdes)\). For \( \Pi \subseteq \mathcal{S}_n \), define

\[
F(\Pi; y, t) := \sum_{\pi \in B(\Pi)} y^\text{neg}(\pi) f^\text{des}(\pi)
\]

and

\[
P^{(lpk, val, des)}(\Pi; y, z, t) := \sum_{\pi \in \Pi} y^\text{lpk}(\pi) z^\text{val}(\pi) f^\text{des}(\pi).
\]

Recall that the reverse complement \( \pi^{rc} \) of a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n \) is given by

\[
\pi^{rc} := (n + 1 - \pi_n)(n + 1 - \pi_{n-1}) \cdots (n + 1 - \pi_1).
\]

That is, we first reverse the order of the letters in \( \pi \), and then replace the \( i \)th smallest letter with the \( i \)th largest letter for all \( 1 \leq i \leq n \). For example, if \( \pi = 1723465 \), then the reverse complement of \( \pi \) is given by \( \pi^{rc} = 3245617 \). Also, let the reverse complement \( \Pi^{rc} \) of a set of permutations \( \Pi \subseteq \mathcal{S}_n \) be the set of their reverse complements: \( \Pi^{rc} := \{ \pi^{rc} | \pi \in \Pi \} \).

In the proof of the next theorem, we will be working with the reverse complement of the permutation \( \hat{\pi} = 0\pi\infty \) of \( \{0, \infty\} \cup [n] \) where \( \pi \in \mathcal{S}_n \), which we define to be \( \hat{\pi}^{rc} := 0\pi^{rc}\infty \).

Theorem 5.15. Let \( \Pi \subseteq \mathcal{S}_n \) for \( n \geq 1 \). Then

\[
F(\Pi; y, t) = (1 + yt)(1 + yt^2)^{n-1}
\]

\[
\times P^{(lpk, val, des)}(\Pi^{rc}; \frac{t(1 + y)(y + t)}{(y + t^2)(1 + yt)}, \frac{t(1 + y)(1 + yt)}{(1 + yt^2)(y + t)}, \frac{y + t^2}{1 + yt^2}).
\]

By taking \( \Pi = \mathcal{S}_n \), we recover Theorem 4.17.

Proof. For a fixed \( \pi \in \Pi \) and \( \sigma \in \mathcal{B}(\pi) \), let \( \hat{\pi} = 0\pi\infty \). Then it follows from Lemma 5.11 that

\[
\sum_{\sigma \in \mathcal{B}(\pi)} y^\text{neg}(\sigma) f^\text{des}(\sigma) = c_1(\pi) \cdots c_n(\pi)
\]

where

\[
c_1(\pi) = \begin{cases} 1 + yt, & \text{if } \pi_1 < \pi_2, \\ t(y + t), & \text{if } \pi_1 > \pi_2, \end{cases}
\]
and for \( i > 1 \),

\[
c_i(\pi) = \begin{cases} 
    t^2(1 + y), & \text{if } i \text{ is a peak of } \pi, \\
    1 + y, & \text{if } i \text{ is a valley of } \pi, \\
    1 + yt^2, & \text{if } i \text{ is a double ascent of } \pi, \\
    y + t^2, & \text{if } i \text{ is a double descent of } \pi.
\end{cases}
\]

By taking the reverse complement of \( \pi \), we have that

\[
\sum_{\sigma \in \mathcal{B}(\pi)} y^{\neg(\sigma)} f^{\des(\sigma)} = d_1(\pi) \cdots d_n(\pi)
\]

where

\[
d_n(\pi) = \begin{cases} 
    1 + yt, & \text{if } \pi_{n-1}^{rc} < \pi_n^{rc}, \\
    t(y + t), & \text{if } \pi_{n-1}^{rc} > \pi_n^{rc},
\end{cases}
\]

and for \( i < n \),

\[
d_i(\pi) = \begin{cases} 
    t^2(1 + y), & \text{if } i \text{ is a valley of } \pi^{rc}, \\
    1 + y, & \text{if } i \text{ is a peak of } \pi^{rc}, \\
    1 + yt^2, & \text{if } i \text{ is a double ascent of } \pi^{rc}, \\
    y + t^2, & \text{if } i \text{ is a double descent of } \pi^{rc}.
\end{cases}
\]

It follows from Lemma 2.1 that \( d_n(\pi) = (t(y + t))^{\lpk(\pi^{rc}) - \val(\pi^{rc})} (1 + yt)^{1 + \val(\pi^{rc}) - \lpk(\pi^{rc})} \); the \( 1 + yt \) factor disappears if \( n \) is a valley of \( \pi^{rc} \), and the \( t(y + t) \) factor disappears if instead \( n \) is a double ascent of \( \pi^{rc} \), which are the only two possibilities.

Next, we determine the contribution of the \( d_i(\pi) \) for \( i < n \) where \( i \) is a peak or valley of \( \pi^{rc} \). The remaining number of valleys of \( \pi^{rc} \)—that is, not including \( n \)—is equal to \( \val(\pi^{rc}) \), and the number of peaks of \( \pi^{rc} \) is equal to \( \lpk(\pi^{rc}) \). When \( n \) is a valley of \( \pi^{rc} \) and thus \( \lpk(\pi^{rc}) = \val(\pi^{rc}) + 1 \), the total contribution from the peaks, valleys, and the final letter is

\[
t^{\lpk(\pi^{rc}) + \val(\pi^{rc}) - 1} (1 + y)^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (t(y + t))^{\lpk(\pi^{rc}) - \val(\pi^{rc})} (1 + yt)^{1 + \val(\pi^{rc}) - \lpk(\pi^{rc})} =
\]

\[
t^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (1 + y)^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (y + t)^{\lpk(\pi^{rc}) - \val(\pi^{rc})} (1 + yt)^{1 + \val(\pi^{rc}) - \lpk(\pi^{rc})}.
\]

On the other hand, when \( n \) is a double ascent of \( \pi^{rc} \) and thus \( \lpk(\pi^{rc}) = \val(\pi^{rc}) \), the contribution is

\[
t^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (1 + y)^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (t(y + t))^{\lpk(\pi^{rc}) - \val(\pi^{rc})} (1 + yt)^{1 + \val(\pi^{rc}) - \lpk(\pi^{rc})} =
\]

\[
t^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (1 + y)^{\lpk(\pi^{rc}) + \val(\pi^{rc})} (y + t)^{\lpk(\pi^{rc}) - \val(\pi^{rc})} (1 + yt)^{1 + \val(\pi^{rc}) - \lpk(\pi^{rc})},
\]

so in fact the contribution from the two cases are the same.

The remaining contribution comes from the double descents and (remaining) double ascents of \( \pi^{rc} \). Note that the number of double descents of \( \pi^{rc} \) is equal to \( \des(\pi^{rc}) - \lpk(\pi^{rc}) \). It is a bit trickier to see that the number of remaining double ascents of \( \pi^{rc} \) is equal to \( n - 1 - \val(\pi^{rc}) - \des(\pi^{rc}) \). We know that the total number of double ascents of \( \pi^{rc} \) is equal
to $n - \text{lpk}(\pi^{rc}) - \text{des}(\pi^{rc})$. If $n$ is a valley of $\pi^{rc}$, then $\text{lpk}(\pi^{rc}) = \text{val}(\pi^{rc}) + 1$, so the number of remaining double ascents of $\pi^{rc}$ is equal to

$$n - \text{lpk}(\pi^{rc}) - \text{des}(\pi^{rc}) = n - (\text{val}(\pi^{rc}) + 1) - \text{des}(\pi^{rc}) = n - 1 - \text{val}(\pi^{rc}) - \text{des}(\pi^{rc}).$$

Otherwise, if $n$ is a double ascent of $\pi^{rc}$, then $\text{lpk}(\pi^{rc}) = \text{val}(\pi^{rc})$, so the remaining number of double ascents of $\pi^{rc}$ is again equal to

$$(n - \text{lpk}(\pi^{rc}) - \text{des}(\pi^{rc})) - 1 = n - 1 - \text{val}(\pi^{rc}) - \text{des}(\pi^{rc}).$$

Thus, the contribution from the double descents and (remaining) double ascents is

$$(y + t^2)^{\text{des}(\pi^{rc})-\text{lpk}(\pi^{rc})}(1 + yt^2)^{n-1-\text{val}(\pi^{rc})-\text{des}(\pi^{rc})}.$$

Therefore,

$$\sum_{\sigma \in B(\pi)} y^{\text{neg}(\sigma)}t^{\text{fdes}(\sigma)} = d_1(\pi) \cdots d_n(\pi)$$

$$= (t(1+y))^{\text{lpk}(\pi^{rc})+\text{val}(\pi^{rc})}(y + t)^{\text{lpk}(\pi^{rc})-\text{val}(\pi^{rc})}(1 + yt)^{1+\text{val}(\pi^{rc})-\text{lpk}(\pi^{rc})}$$

$$\times (y + t^2)^{\text{des}(\pi^{rc})-\text{lpk}(\pi^{rc})}(1 + yt^2)^{n-1-\text{val}(\pi^{rc})-\text{des}(\pi^{rc})}$$

$$= (1 + yt)(1 + yt^2)^{n-1} \left( \frac{t(1+y)(y + t)}{(y + t^2)(1 + yt)} \right)^{\text{lpk}(\pi^{rc})}$$

$$\times \left( \frac{t(1+y)(1 + yt)}{(1 + yt^2)(1 + yt)} \right)^{\text{val}(\pi^{rc})} \left( \frac{y + t^2}{1 + yt^2} \right)^{\text{des}(\pi^{rc})},$$

and summing over all $\pi \in \Pi$ yields the result. \hfill \Box

By setting $y = 1$, we obtain the following specialization relating the polynomials

$$F(\Pi; t) := \sum_{\pi \in B(\Pi)} t^{\text{fdes}(\pi)}$$

and

$$P^{\text{udr}}(\Pi; t) := \sum_{\pi \in \Pi} t^{\text{udr}(\pi)},$$

thus generalizing Corollary 4.18.

**Corollary 5.16.** Let $\Pi \subseteq S_n$ for $n \geq 1$. Then

$$F(\Pi; t) = \frac{(1 + t)(1 + t^2)^n}{2t} P^{\text{udr}} \left( \Pi^{rc}; \frac{2t}{1 + t^2} \right).$$

Equivalently,

$$P^{\text{udr}}(\Pi; t) = \frac{2v}{(1 + v)(1 + v^2)^n} F(\Pi^{rc}; v)$$

where $v = \frac{1-\sqrt{1-t^2}}{t}$.  

49
Finally, we give a refinement of Theorem 5.15 that keeps track of a statistic which is constant on any orbit of the action, i.e., a statistic on $S$. For such a statistic $st$, let

$$F^{st}(\Pi; y, t, w) := \sum_{\pi \in B(\Pi)} y^{\text{neg}(\pi)} t^{\text{des}(\pi)} w^{st(\pi)}$$

and

$$P^{(\text{lpk, val, des, st})}(\Pi; y, z, t, w) := \sum_{\pi \in \Pi} y^{\text{lpk}(\pi)} z^{\text{val}(\pi)} t^{\text{des}(\pi)} w^{st(\pi)}.$$

**Corollary 5.17.** Let $st$ be a permutation statistic on $S$. Then

$$F^{st}(\Pi; y, t, w) = (1 + yt)(1 + yt^2)^{n-1} \times P^{(\text{lpk, val, des, st})}(\Pi^{re}, \frac{t(1+y)(y+t)}{(y+t^2)(1+yt)}, \frac{t(1+y)(1+yt)}{(1+yt^2)(y+t)}, 1 + yt^2, w).$$

**A. Tables of statistics**

Tables 1, 2, 3, 4 summarize the statistics that appear in this paper. For each statistic, we list the symbol used for the statistic, the name of the statistic, the (sub)section in this paper where the statistic is defined, and references to new results in this paper which involve the statistic.

| Statistic | Name of Statistic | Definition | Results |
|-----------|-------------------|------------|---------|
| des       | descent number    | §1         | Theorems 2.6, 2.8, 4.2, 4.7, 4.13, 4.15, 5.1; Corollaries 2.7, 5.3, 5.4 |
| pk        | peak number       | §1         | Corollary 5.13 |
| lpk       | left peak number  | §1         |         |
| br        | number of biruns  | §1         |         |
| Des       | descent set       | §2.1       |         |
| val       | valley number     | §2.1       |         |
| udr       | number of up-down runs | §2.1 | Theorem 4.13, Corollaries 4.18, 5.16 |
| inv       | inversion number  | §2.1       |         |

18 We note that every result in this paper for a joint statistic involving the inversion number inv gives rise to an analogous result for the joint statistic obtained by replacing inv with the inverse major index imaj, even though we do not explicitly state these statistics below. See Subsection 4.4 for more details.
Table 2: Type B permutation statistics

| Statistic | Name of Statistic       | Definition | Results |
|-----------|-------------------------|------------|---------|
| des_B     | (type B) descent number | §2.3       | Corollaries 2.7, 2.10, 5.13 |
| fdes      | flag descent number     | §2.3       | Corollaries 2.10, 4.18, 5.16 |
| neg       | number of negative letters | §2.3     |         |
| (neg, des_B) |                        |            | Theorems 2.3, 2.6, 2.9, 4.10, 5.12 |
| (neg, fdes) |                        |            | Theorems 2.3, 2.8, 2.9, 4.17, 5.15 |

Table 3: Binary tree statistics

| Statistic | Name of Statistic       | Definition | Results |
|-----------|-------------------------|------------|---------|
| nlc       | number of nodes with no left children | §5.2       | Corollary 5.8 |
| tc        | number of nodes with two children | §5.2       |         |
| (tc, nlc) |                        |            | Corollary 5.8 |
Table 4: Dyck path statistics

| Statistic | Name of Statistic | Definition | Results |
|-----------|-------------------|------------|---------|
| pk        | number of peaks (occurrences of $UD$) | §5.2       | Corollary 5.10 |
| hk        | number of hooks (occurrences of $DDU$) | §5.2       |          |
| (hk, pk)  |                   |            |          |

Acknowledgements. The author thanks Ira Gessel for suggesting this project, reading earlier versions of the manuscript, and offering many helpful suggestions which greatly improved the quality of this work; Kyle Petersen for providing feedback on several identities presented in this paper; and an anonymous referee for their constructive comments and suggestions.

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