PERTURBATIONS ON $K$-FUSION FRAMES

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Abstract. $K$-fusion frames are generalizations of fusion frames in frame theory. This article characterizes various kinds of property of $K$-fusion frames. Several perturbation results on $K$-fusion frames are formulated and analyzed.

1. Introduction

The concept of Hilbert space frames was first introduced by Duffin and Schaeffer [11] in 1952. Later, in 1986, frame theory was reintroduced and popularized by Daubechies, Grossman and Meyer [9]. Since then frame theory has been widely used by mathematicians and engineers in various fields of mathematics and engineering, namely, operator theory [16], harmonic analysis [13], signal processing [12], sensor network [7], data analysis [5], etc.

Frame theory literature became richer through several generalizations—fusion frame (frames of subspaces) [3, 6], $G$-frame (generalized frames) [22], $K$-frame (atomic systems) [14], $K$-fusion frame (atomic subspaces) [3], etc. and these generalizations have been proved to be useful in various applications.

This article focuses on study, characterize and explore several properties of $K$-fusion frame. It is organized as follows. Section 2 is devoted to the basic definitions and results related to frames and fusion frames. The characteristics of $K$-fusion frames are discussed in Section 3. Finally, results related to perturbation and erasure properties are established in Section 4.

Throughout the paper, $\mathcal{H}$ is a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and $\mathcal{L}(\mathcal{H})$ for $\mathcal{L}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, we denote $D(T), N(T)$ and $R(T)$ for domain, null space and range of $T$, respectively. For a collection of closed subspaces $\mathcal{W}_i$ of $\mathcal{H}$ and scalars $v_i$, $i \in I$, the weighted collection of closed subspaces $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ is denoted by $\mathcal{W}_v$. We consider the index set $I$ to be finite or countable.
2. Preliminaries

In this section we recall basic definitions and results needed in this paper. We refer the books of Ole Christensen [8] and Casazza et.al. [5] for an introduction to frame theory.

2.1. Frame. A collection \( \{f_i\}_{i \in I} \) in \( \mathcal{H} \) is called a frame if there exist constants \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2,
\]

for all \( f \in \mathcal{H} \). The numbers \( A, B \) are called frame bounds. The supremum over all \( A \)'s and infimum over all \( B \)'s satisfying above inequality are called the optimal frame bounds. If a collection satisfies only the right inequality in (1), it is called a Bessel sequence.

Given a frame \( \{f_i\}_{i \in I} \) of \( \mathcal{H} \). The pre-frame operator or synthesis operator is a bounded linear operator \( T : l^2(I) \to \mathcal{H} \) and is defined by \( T\{c_i\} = \sum_{i \in I} c_i f_i \). The adjoint of \( T \), \( T^* : \mathcal{H} \to l^2(I) \), given by \( T^* f = \{\langle f, f_i \rangle\} \), is called the analysis operator. The frame operator, \( S = TT^* : \mathcal{H} \to \mathcal{H} \), is defined by

\[
Sf = TT^* f = \sum_{i \in I} \langle f, f_i \rangle f_i.
\]

It is well-known that the frame operator is bounded, positive, self adjoint and invertible.

2.2. Fusion Frame. Consider a weighted collection of closed subspaces, \( \mathcal{W}_v \), of \( \mathcal{H} \). Then \( \mathcal{W}_v \) is said to be a fusion frame for \( \mathcal{H} \), if there exist constants \( 0 < A \leq B < \infty \) satisfying

\[
A \|f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2,
\]

where \( P_{W_i} \) is the orthogonal projection from \( \mathcal{H} \) onto \( W_i \). The constants \( A \) and \( B \) are called fusion frame bounds. A collection of closed subspaces, satisfying only the right inequality in (2), is called a fusion Bessel sequence.

Additionally, it is to be noted that for every fusion frame \( \mathcal{W}_v \), there are frames \( \{f_{ij}\}_{j \in J_i} \) for each \( W_i \) and these are called local frames for \( \mathcal{W}_v \). It is a well-known fact that \( \mathcal{W}_v \) is a fusion frame if and only if \( \{w_if_{ij}\}_{j \in J_i, i \in I} \) is a frame for \( \mathcal{H} \), for details readers are referred to Theorem 3.2 in [4].

For a family of closed subspaces, \( \{\mathcal{W}_i\}_{i \in I} \) of \( \mathcal{H} \), the associated \( l^2 \) space is defined by

\[
\left( \sum_{i \in I} \bigoplus \mathcal{W}_i \right)_{l^2} = \{\{f_i\}_{i \in I} : f_i \in W_i, \sum_{i \in I} \|f_i\|^2 < \infty \} \text{ with the inner product } \langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_{\mathcal{H}}.
\]

Let \( \mathcal{W}_v \) be a fusion frame. Then the associated synthesis operator \( T_{\mathcal{W}} : (\sum_{i \in I} \bigoplus \mathcal{W}_i)_{l^2} \to \mathcal{H} \) is defined as \( T_{\mathcal{W}}(f) = \sum_{i \in I} v_if_i \) for all \( \{f_i\}_{i \in I} \in (\sum_{i \in I} \bigoplus \mathcal{W}_i)_{l^2} \) and the analysis operator \( T_{\mathcal{W}}^* : \mathcal{H} \to \)
(\sum_{i \in I} \oplus \mathcal{W}_i)_{\ell^2} is defined as \( T^*_W(f) = \{ v_i P_{\mathcal{W}_i}(f) \}_{i \in I} \). It is well-known that (see [4]) the synthesis operator \( T_W \) of a fusion frame is bounded, linear and onto, whereas the corresponding analysis operator \( T^*_W \) is (possibly into) an isomorphism. Corresponding fusion frame operator is defined as \( S_W(f) = T_W T^*_W(f) = \sum_{i \in I} v^2_i P_{\mathcal{W}_i}(f) \). \( S_W \) is bounded, positive, self adjoint and invertible.

Any signal \( f \in \mathcal{H} \) can be expressed by its fusion frame measurements \( \{ v_i P_{\mathcal{W}_i} f \}_{i \in I} \) as

\[
    f = \sum_{i \in I} v_i S_W^{-1}(v_i P_{\mathcal{W}_i} f).
\]

2.3. \( K \)-fusion frame. In [3], authors, introduced a generalization of fusion frame, \( K \)-fusion frame, and scrutinized the equivalence between atomic subspaces and \( K \)-fusion frames. \( K \)-fusion frame is used to reconstruct signals from range of a bounded linear operator \( K \).

**Definition 2.1.** (\( K \)-fusion frame) Let \( K \in \mathcal{L}(\mathcal{H}) \), \( \mathcal{W}_v = \{ (W_i, v_i) \}_{i \in I} \) be a weighted collection of closed subspaces of \( \mathcal{H} \). Then \( \mathcal{W}_v \) is said to be a \( K \)-fusion frame for \( \mathcal{H} \) if there exist positive constants \( A, B \) such that for all \( f \in \mathcal{H} \) we have

\[
    A\|K^*f\|^2 \leq \sum_{i \in I} v^2_i \|P_{\mathcal{W}_i}f\|^2 \leq B\|f\|^2.
\]

In the rest of this Section, we recall some fundamental results in Hilbert space that are necessary to present some outcomes of this article.

**Theorem 2.2.** (Douglas' factorization theorem [10]) Let \( \mathcal{H}_1, \mathcal{H}_2, \) and \( \mathcal{H} \) be Hilbert spaces and \( S \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}), T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}). \) Then the following are equivalent:

1. \( R(S) \subseteq R(T) \).
2. \( SS^* \leq \alpha TT^* \) for some \( \alpha > 0 \).
3. \( S = TL \) for some \( L \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \).

**Lemma 2.3.** (Moore-Penrose pseudo-inverse [18, 21, 8, 1]) Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Hilbert spaces and \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be a closed range operator, then the followings hold:

1. \( TT^\dagger = P_{T(\mathcal{H})}, T^\dagger T = P_{T^*(\mathcal{K})} \)
2. \( \|f\|_{T^\dagger} \leq \|T^*f\| \) for all \( f \in T(\mathcal{H}) \).
3. \( TT^\dagger T = T, T^\dagger TT^\dagger = T^\dagger, (TT^\dagger)^* = TT^\dagger, (T^\dagger T)^* = T^\dagger T \).

**Lemma 2.4.** ([15, 20]) Suppose \( \mathcal{H} \) and \( \mathcal{K} \) be two Hilbert spaces and \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \). consider \( W \) be a closed subspace of \( \mathcal{H} \) and \( V \) be a closed subspace of \( \mathcal{K} \). Then we have the followings:

1. \( P_W T^* P_{TV} = P_W T^* \).
2. \( P_W T^* P_V = P_W T^* \) if and only if \( TW \subseteq V \).
Definition 2.5. (Drazin inverse \[2, 17\]) Let \(S, T \in \mathcal{L}(\mathcal{H})\), \(S\) is said to be the Drazin inverse of \(T\) if we have the following:

1. \(STS = S\).
2. \(ST = TS\).
3. \(TST^k = T^k\), for some positive integer \(k\).

It is to be noted that \(T \in \mathcal{L}(\mathcal{H})\) has the Drazin inverse in \(\mathcal{L}(\mathcal{H})\) if and only if \(\lambda = 0\) is a pole of the resolvent operator \((\lambda I - T)^{-1}\). Moreover, the order of the pole is equal to the index of \(T\). In particular 0 is not an accumulation point in the spectrum \(\sigma(T)\).

3. Characterization of \(K\)-fusion frames

In this section, we characterize various properties of \(K\)-fusion frame. The following theorem provides a sufficient condition on a bounded, linear operator \(K\) under which the image of \(K\)-fusion frame remains a \(K\)-fusion frame.

Theorem 3.1. Let \(K \in \mathcal{L}(\mathcal{H})\) be an idempotent, closed range operator and \(W_v\) be a \(K\)-fusion frame for \(\mathcal{H}\) with \(K^\dagger K(W_i) \subset W_i\). Then \(\{(KW_i, v_i)\}_{i \in I}\) constitutes a \(K\)-fusion frame for \(\mathcal{H}\).

Proof. First we prove for all \(i \in I\), \(K(W_i)\) is a closed subspace in \(\mathcal{H}\). Since \(K^\dagger K(W_i) \subset W_i\), \(KK^\dagger K(W_i) \subset K(W_i)\). Therefore using the Lemma 2.5.2 in \[8\] we have \(KK^* (KK^*)^{-1} K(W_i) \subset K(W_i)\) and hence \(K(W_i)\) is a closed subspace in \(\mathcal{H}\) for all \(i \in I\). Since \(\{(W_i, v_i)\}_{i \in I}\) is a \(K\)-fusion frame in \(\mathcal{H}\), there exist \(A, B > 0\) such that for all \(f \in \mathcal{H}\) we have,

\[
A\|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B\|f\|^2.
\]

Again as \(K\) is idempotent, using the Lemma 2.4 we obtain,

\[
A\|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} K^* f\|^2 \leq \|K\|^2 \sum_{i \in I} v_i^2 \|P_{KW_i} f\|^2.
\]

Therefore \(\frac{A}{\|K\|^2} \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{KW_i} f\|^2\).

Again from the Lemma 2.4 we have \(P_{KW_i} = P_{KW_i} K^\dagger K P_{W_i} K^*\). Therefore for all \(f \in \mathcal{H}\) we obtain,

\[
\sum_{i \in I} v_i^2 \|P_{KW_i} f\|^2 \leq \|K^\dagger \|^2 \sum_{i \in I} v_i^2 \|P_{W_i} K^* f\|^2 \leq B\|K^\dagger \|^2 \|K\|^2 \|f\|^2.
\]

Hence our assertion is tenable. \(\square\)

In the next result, we further characterize \(K\)-fusion frame by means of Drazin inverse.
Lemma 3.2. Let $K \in \mathcal{L}(H)$ has non-zero Drazin inverse, $S$. Also suppose that $W_v$ is a $K$-fusion frame for $H$. Then the following hold:

1. $W_v$ is a $SKS$-fusion frame for $H$.
2. $W_v$ is a $SK$-fusion frame in $H$.

Proof. Since $W_v$ is a $K$-fusion frame for $H$, there exist $A, B > 0$ such that for all $f \in H$ we have, $A \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2$. Again as $S$ is the non-zero Drazin inverse of $K$, for all $f \in H$ we have

$$\frac{A}{\|S\|^2} \|(SKS)^* f\|^2 \leq A \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2.$$

and also

$$\frac{A}{\|S\|^2} \|(SK)^* f\|^2 \leq A \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2.$$

Hence the conclusions follow. □

Remark 3.3. It is to be noted that $W_v$ is also a KS-fusion frame for $H$ but this result is obvious as for any $(0 \neq) T \in \mathcal{L}(H)$,

$$\frac{A}{\|T\|^2} \|(KT)^* f\|^2 \leq A \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2,$$

for all $f \in H$.

In the following theorem we scrutinize the robustness of $K$-fusion frames under erasure property.

Theorem 3.4. Let $K \in \mathcal{L}(H)$ be a closed range operator and $W_v$ be a $K$-fusion frame for $H$ with bounds $A$ and $B$. Suppose $J \subseteq I$ such that $\sum_{i \in J} v_i^2 = C < \infty$ with $(A - C \|K^\dagger\|^2) > 0$. Then $\{(W_i, v_i)\}_{i \in I \setminus J}$ forms a $K$-fusion frame for $R(K)$ with bounds $(A - C \|K^\dagger\|^2)$ and $B$.

Proof. Since $K$ has closed range, for all $f \in R(K)$ we have for all $i \in I$,

$$\|P_{W_i} f\| \leq \|K^\dagger\| \|K^* f\|$$

and hence $\sum_{i \in J} v_i^2 \|P_{W_i} f\|^2 \leq C \|K^\dagger\|^2 \|K^* f\|^2$ for all $f \in R(K)$. Consequently for all $f \in R(K)$,

$$(A - C \|K^\dagger\|^2) \|K^* f\|^2 = A \|K^* f\|^2 - C \|K^\dagger\|^2 \|K^* f\|^2 \leq \sum_{i \in I \setminus J} v_i^2 \|P_{W_i} f\|^2.$$

The upper bound follows directly from the assumption. □
4. Perturbation Properties

In this section we analyze stability conditions of $K$-fusion frames under perturbations.

**Lemma 4.1.** Let $K_1 \in \mathcal{L}(\mathcal{H})$, $\mathcal{W}_v$ be a $K_1$-fusion frame for $\mathcal{H}$. Suppose $K_2 \in \mathcal{L}(\mathcal{H})$ and $a, b \geq 0$ such that

$$\|(K_1^* - K_2^*)f\| \leq a\|K_1^*f\| + b\|K_2^*f\|, \text{ for all } f \in \mathcal{H}. $$

Then $\mathcal{W}_v$ is also a $K_2$-fusion frame for $\mathcal{H}$ if $b < 1$.

**Proof.** Since $\mathcal{W}_v$ is $K_1$-fusion frame for $\mathcal{H}$, there exist $A, B > 0$, for all $f \in \mathcal{H}$ we have

$$A\|K_1^*f\|^2 \leq \sum_{i \in I} v_i^2\|P_{W_i}f\|^2 \leq B\|f\|^2. $$

Now for all $f \in \mathcal{H}$ we obtain

$$\|K_2^*f\| \leq \|(K_1^* - K_2^*)f\| + \|K_1^*f\| \leq (1 + a)\|K_1^*f\| + b\|K_2^*f\|. $$

Therefore for all $f \in \mathcal{H}$ we have

$$A \left( \frac{1 - b}{1 + a} \right)^2 \|K_2^*f\|^2 \leq A\|K_1^*f\|^2 \leq \sum_{i \in I} v_i^2\|P_{W_i}f\|^2 \leq B\|f\|^2. $$

Hence our assertion is tenable. \qed

**Corollary 4.2.** Let $K_1, K_2 \in \mathcal{L}(\mathcal{H})$ and $0 \leq a, b < 1$ so that for all $f \in \mathcal{H}$,

$$\|(K_1^* - K_2^*)f\| \leq a\|K_1^*f\| + b\|K_2^*f\|. $$

Also suppose, $\mathcal{W}_v$ is a weighted collection of closed subspaces of $\mathcal{H}$. Then $\mathcal{W}_v$ is $K_1$-fusion frame for $\mathcal{H}$ if and only if it is $K_2$-fusion frame for $\mathcal{H}$.

**Proof.** The proof follows from above Lemma 4.1 by interchanging the roles of $K_1$ and $K_2$. \qed

Above results immediately provide the following result.

**Corollary 4.3.** Let $K \in \mathcal{L}(\mathcal{H})$, $\mathcal{W}_v$ be $K$-fusion frame for $\mathcal{H}$ with

$$\|K^*f - f\| \leq a\|K^*f\| + b\|f\|, \text{ for all } f \in \mathcal{H},$$

where $0 \leq a, b < 1$. Then $\mathcal{W}_v$ is fusion frame for $\mathcal{H}$.

Given a fusion Bessel sequence or $K$-fusion frame or fusion frame, can we construct a $K$-fusion frame? Following results address this using perturbations on projection operators.
Theorem 4.4. Let $a, b, c \geq 0$ and $\mathcal{W}_w, \mathcal{V}_v$ be two weighted collections of closed subspaces of $\mathcal{H}$ so that for all $f \in \mathcal{H}$,

$$
\left( \sum_{i \in I} \| (w_i P_{W_i} - v_i P_{V_i}) f \|^2 \right)^{\frac{1}{2}} \leq a \left( \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \right)^{\frac{1}{2}} + b \left( \sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \right)^{\frac{1}{2}} + c \Lambda(f),
$$

for some $\Lambda : \mathcal{H} \to \mathbb{R}_+$. Then the following results hold:

1. Let $\mathcal{W}_w$ be a fusion Bessel sequence in $\mathcal{H}$ and $b < 1$, $c = 0$. Then $\mathcal{V}_v$ is a $K$-fusion frame for $\mathcal{H}$ for any $K \in \mathcal{L}(\mathcal{H})$ satisfying $R(K) \subseteq R(T_{\mathcal{V}})$, where $T_{\mathcal{V}}$ is the associated synthesis operator of $\mathcal{V}_v$.

2. Let $K \in \mathcal{L}(\mathcal{H})$, $\mathcal{W}_w$ be a $K$-fusion frame for $\mathcal{H}$ with bounds $A, B > 0$ and $\Lambda(f) = \|K^* f\|$. Then if $a < 1$, $b < 1$, $0 \leq \frac{c-a}{1-a} < \sqrt{A}$, $\mathcal{V}_v$ is also a $K$-fusion frame for $\mathcal{H}$ with bounds $\left( \frac{\sqrt{A(1-a)-c}}{1+b} \right)^2$ and $\left( \frac{(1+a)\sqrt{B+c|K|}}{1-b} \right)^2$.

3. Let $\mathcal{W}_w$ be a fusion frame for $\mathcal{H}$ with bounds $A, B > 0$ and $\Lambda(f) = \|f\|$. Then $\mathcal{V}_v$ forms a fusion frame for $\mathcal{H}$ with bounds $\left( \frac{\sqrt{A-c-a\sqrt{B}}}{1+b} \right)^2$ and $\left( \frac{(1+a)\sqrt{B+c}}{1-b} \right)^2$, if $a\sqrt{B} + c < \sqrt{A}$, $b < 1$.

Proof. (1) Since $\mathcal{W}_w$ is a fusion Bessel sequence in $\mathcal{H}$, there exists $B > 0$ such that for all $f \in \mathcal{H}$ we have $\sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \leq B \| f \|^2$. Using Minkowski’s inequality, for all $f \in \mathcal{H}$, we obtain

$$
\left( \sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I} \| (w_i P_{W_i} - v_i P_{V_i}) f \|^2 \right)^{\frac{1}{2}} + \left( \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \right)^{\frac{1}{2}}
$$

$$
\leq (1 + a) \left( \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \right)^{\frac{1}{2}} + b \left( \sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \right)^{\frac{1}{2}},
$$

Hence

$$
\sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \leq \sqrt{B} \left( \frac{1 + a}{1 - b} \right)^2 \| f \|^2, \forall f \in \mathcal{H}.
$$

Therefore $T_{\mathcal{V}}^*$ and hence $T_{\mathcal{V}}$ is well-defined. The left hand inequality directly follows from Theorem 2.2.

(2) Since $\mathcal{W}_w$ is a $K$-fusion frame for $\mathcal{H}$ with bounds $A, B > 0$, for all $f \in \mathcal{H}$ we have $A \|K^* f\|^2 \leq \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \leq B \| f \|^2$. Now

$$
\left( \sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I} \| (w_i P_{W_i} - v_i P_{V_i}) f \|^2 \right)^{\frac{1}{2}} + \left( \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \right)^{\frac{1}{2}}
$$

$$
\leq (1 + a) \left( \sum_{i \in I} w_i^2 \| P_{W_i} f \|^2 \right)^{\frac{1}{2}} + b \left( \sum_{i \in I} v_i^2 \| P_{V_i} f \|^2 \right)^{\frac{1}{2}} + c \|K^* f\|,
$$
for all $f \in \mathcal{H}$. Hence
\[
\sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \leq \left( \frac{(1 + a)\sqrt{B} + c\|K\|}{1 - b} \right)^2 \|f\|^2, \ \forall \ f \in \mathcal{H}.
\]

Similarly, for the lower bound we have,
\[
\left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2} \leq a \left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2} + (1 + b) \left( \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \right)^\frac{1}{2} + c\|K^*f\|
\]
for all $f \in \mathcal{H}$. Hence we obtain
\[
\left( \frac{\sqrt{A}(1 - a) - c}{1 + b} \right)^2 \|K^*f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2, \ \forall \ f \in \mathcal{H}.
\]

(3) Since $\mathcal{W}_w$ is a fusion frame for $\mathcal{H}$ with bounds $A, B > 0$, for all $f \in \mathcal{H}$ we have
\[
A\|f\|^2 \leq \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \leq B\|f\|^2.
\]
Also for all $f \in \mathcal{H}$ we obtain
\[
\left( \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \right)^\frac{1}{2} \geq \left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2} - \left( \sum_{i \in I} \|(w_i P_{W_i} - v_i P_{V_i})f\|^2 \right)^\frac{1}{2}
\]
\[
\geq (\sqrt{A} - c)\|f\| - a \left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2} - b \left( \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \right)^\frac{1}{2},
\]
Therefore for all $f \in \mathcal{H}$ we obtain
\[
\left( \frac{\sqrt{A} - c - a\sqrt{B}}{1 + b} \right)^2 \|f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2.
\]

Moreover, for all $f \in \mathcal{H}$ we have
\[
\left( \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \right)^\frac{1}{2} \leq \left( \sum_{i \in I} \|(w_i P_{W_i} - v_i P_{V_i})f\|^2 \right)^\frac{1}{2} + \left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2}
\]
\[
\leq (1 + a) \left( \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \right)^\frac{1}{2} + b \left( \sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \right)^\frac{1}{2} + c\|f\|
\]
Hence for all $f \in \mathcal{H}$ we obtain
\[
\sum_{i \in I} v_i^2 \|P_{V_i}f\|^2 \leq \left( \frac{(1 + a)\sqrt{B} + c}{1 - b} \right)^2 \|f\|^2.
\]

\[\square\]

We acknowledge that recently Li and Leng [19] proved a similar result as stated in the second statement of above theorem. We present the result here as this work has been done almost simultaneously with the work of Li and Leng.
In the following proposition we discuss another perturbation condition on the projection operators to obtain a K-fusion frame.

**Proposition 4.5.** Let \( K \in \mathcal{L}(\mathcal{H}) \), \( \mathcal{W}_w \) be a K-fusion frame for \( \mathcal{H} \) with bounds \( A, B > 0 \). Also suppose \( \mathcal{V}_v \) is any weighted collection of closed subspaces of \( \mathcal{H} \) so that for all \( f \in \mathcal{H} \),

\[
\sum_{i \in I} |\langle f, (w_i^2 P_{V_i} - v_i^2 P_{V_i}) f \rangle| \leq R \|K^* f\|^2, \text{ where } 0 < R < A.
\]

Then \( \mathcal{V}_v \) forms a K-fusion frame for \( \mathcal{H} \) with bounds \((A - R)\) and \((B + R\|K\|)\).

**Proof.** Since \( \mathcal{W}_w \) is a K-fusion frame for \( \mathcal{H} \) with bounds \( A, B > 0 \), we have

\[
A \|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i} f\|^2 \leq B \|f\|^2, \text{ } \forall f \in \mathcal{H}.
\]

Now for all \( f \in \mathcal{H} \) we obtain

\[
\sum_{i \in I} w_i^2 \|P_{W_i} f\|^2 \leq \sum_{i \in I} |\langle f, (w_i^2 P_{V_i} - v_i^2 P_{V_i}) f \rangle| + \sum_{i \in I} v_i^2 \|P_{V_i} f\|^2 \leq R \|K^* f\|^2 + \sum_{i \in I} v_i^2 \|P_{V_i} f\|^2.
\]

Therefore for all \( f \in \mathcal{H} \) we have, \((A - R)\|K^* f\|^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i} f\|^2\).

Similarly, \( \sum_{i \in I} v_i^2 \|P_{V_i} f\|^2 \leq (B + R\|K\|) \|f\|^2 \), for all \( f \in \mathcal{H} \).

\[\square\]

In the following two results we analyze perturbation conditions under which a fusion Bessel sequence forms a K-fusion frame.

**Theorem 4.6.** Let \( \mathcal{W}_w \) be a fusion Bessel sequence in \( \mathcal{H} \) with bound \( B > 0 \) and \( J \subset I \) with \( T_{W} \) is the associated synthesis operator of \( \{(W_i, w_i)\}_{i \in I \setminus J} \). Let \( a, b \geq 0 \) and \( K \in \mathcal{L}(\mathcal{H}) \) satisfying \( \| (K^* - T_{W} T_{W}^*) f \| \leq a \|K^* f\| + b \|T_{W} f\| \) for all \( f \in \mathcal{H} \). Then \( \{(W_i, w_i)\}_{i \in I \setminus J} \) forms a K-fusion frame for \( \mathcal{H} \) with bounds \((\frac{1 - a}{b + \|T_{W}\|})^2\) and \( B \) if \( a < 1 \).

**Proof.** We have for all \( f \in \mathcal{H} \), \( \|K^* f\| \leq \|(K^* - T_{W} T_{W}^*) f\| + \|T_{W} T_{W}^* f\| \leq a \|K^* f\| + (b + \|T_{W}\|) \|T_{W} f\| \). Therefore

\[
\left(\frac{1 - a}{b + \|T_{W}\|}\right)^2 \|K^* f\|^2 \leq \sum_{i \in I \setminus J} w_i^2 \|P_{W_i} f\|^2 \leq \sum_{i \neq J} w_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2,
\]

for all \( f \in \mathcal{H} \).

\[\square\]
Theorem 4.7. Let $\mathcal{W}_w$ be a fusion Bessel sequence in $\mathcal{H}$ with bound $B > 0$ and let $J \subseteq I$ so that the associated synthesis operator of $\{(W_i, w_i)\}_{i \in I \setminus J}$ is $T_{W}$. Let $a, b, c \geq 0$ and $K \in \mathcal{L}(\mathcal{H})$ be a closed range operator such that $\|(K^* - T_{W}T_{W}^*)f\| \leq a \|K^*f\| + b \|T_{W}^*f\| + c \|f\|$ for all $f \in \mathcal{H}$. Then if $a + c\|K^\dagger\| < 1$, $\{(W_i, w_i)\}_{i \in I \setminus J}$ is a $K$-fusion frame for $R(K)$ with bounds
\[
\left(\frac{1-a-c\|K^\dagger\|}{b+\|T_{W}\|}\right) \|K^*f\| \leq (b + \|T_{W}\|) \|T_{W}^*f\|.
\]

Therefore for all $f \in R(K)$, we have the following:
\[
\left(\frac{1-a-c\|K^\dagger\|}{b+\|T_{W}\|}\right) \|K^*f\| \leq \|T_{W}^*f\| = \sum_{i \in I \setminus J} w_i^2 \|P_{W_i}f\|^2 \leq \sum_{i \in I} w_i^2 \|P_{W_i}f\|^2 \leq B \|f\|^2.
\]
Consequently, our declaration is sustainable. □

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