Generalized Gauge Transformations and Regularized $\lambda \varphi^4$-type Abelian Vertices

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Abstract

Abelian Lagrangians containing $\lambda \varphi^4$-type vertices are regularized by means of a suitable point-splitting scheme combined with generalized gauge transformations. The calculation is developed in details for a general Lagrangian whose fields (gauge and matter ones) satisfy usual conditions. We illustrate our results by considering some special cases, such as the $|\psi\psi|^2$ and the Avdeev-Chizhov models. Possible application of our results to the Abelian Higgs model, whenever spontaneous symmetry breaking is considered, is also discussed. We also pay attention to a number of features of the point-split action such as the regularity and non-locality of its new “interacting terms”.

1 Introduction

In the quantum field-theoretical framework of modern Physics, products of fields at the same space and/or time point are not well-defined since these fields are taken as operator-valued distributions. As a consequence of such ill-defined products, we are led to divergent results when calculating relevant quantities, such as, physical masses and coupling constants. Physically speaking, such divergences arise because we describe elementary particles as if they were point-like entities and, consequently, carrying infinite density of mass, charge, and so forth.

Even though there are several regularization methods to deal with such problems, those based on point-splitting may offer some advantages respect to others when performed in a suitable way. Essentially, the procedure works by taking the field products initially at the same point, and later on at different points, by splitting them. The result is such that the new Lagrangian contains only regularized interaction terms. Already in 1934, Dirac employed such an idea in order to split products of quantities at the same points which appeared in density matrices of electronic and positronic physical distributions (see Ref.[1], for details). Some time later, a similar approach was employed by Johnson in order to calculate the actual Green functions of the Thirring model [2].

More recently, several results have been obtained by means of this method for both Abelian (QED) and non-Abelian (Standard Model) cases. For example, the values of some important physical parameters such as the top quark and Higgs scalar masses, have been got free from divergences and were
shown to be in good agreement with other schemes. This procedure was also shown to respect the
gauge invariance of the theories (for details see the papers listed in Refs. [3, 4]).

Nevertheless, these works did not pay enough attention to the explicit construction and form of the
new point-split gauge transformations. Such an issue was the subject of a more recent paper, Ref.[5],
where the Abelian infinitesimal form of these new transformations (the so-called generalized gauge
transformations, denoted by ggt’s) was proven to exist to all orders in the gauge coupling constant.
The explicit forms of such ggt’s as well as of a generalized QED-Lagrangian were presented up to
fourth order. This new Lagrangian, obtained from the original QED action, was shown to be regular-
ized, i.e., its interaction terms (including some new ones which appear from the splitting) presented
no product of fields at the same point. On the other hand, those new terms also displayed non-locality
property. As expected, as we set the point-splitting parameter to zero, we recover the original results.

restriction in applying them to other Abelian theories containing usual vector gauge fields coupled to
matter fields in a suitable way. Therefore, we intend here to apply the scheme discussed above to a
quite general Lagrangian which contains, among others, a $\lambda \varphi^4$-type vertex, in order to obtain its generalized, say, point-split version. This new Lagrangian will be explicitly constructed up to the second
order in the gauge coupling constant. In addition, the present work sets out not only to show the
good applicability of this alternative regularization procedure to a quite interesting class of Abelian
interaction vertex, but also to motivate further investigation towards its non-Abelian version, which
includes among others, the so-celebrated Higgs mechanism of the Salam-Weinberg electroweak theory.

Here, it is worthy noticing that the gauge transformation parameter may explicitly appear in point-
split actions, depending on the number of matter fields involved in the interaction vertex. Actually,
while for 3-vertices (two matter plus one gauge field) the gauge parameter is generally absent from
the point-split action, in matter 4-vertices its presence turns out to be, as far as we have understood,
a natural ingredient to preserve gauge invariance under generalized gauge transformations to a given
order in the gauge coupling constant. This comes from the fact that, in the framework of ggt’s the
gauge invariance, in a generalized sense, has to be constructed and checked order by order in a gauge
coupling constant expansion. Such an issue will become clearer throughout this work.

Our paper follows the outline below. In Section 2 the Lagrangian which will be worked out is presented
as well as a survey of the point-splitting scheme combined with generalized gauge transformations.
Then, we apply such a procedure to our Lagrangian and step-by-step we worked out its generalized
(split) expression up to the second order in the gauge coupling constant. Section 3 is devoted to
applications of the results obtained in the previous section to some specific cases, say, $(\bar{\psi}\psi)^2$ and a
modified version of the Avdeev-Chizhov models. We close our paper by pointing out some Concluding
Remarks. Among others interests, we pay attention to the applicability of our results to the Abelian
Higgs model whenever spontaneous symmetry breaking is concerned.

## 2 The Lagrangian and the regularization procedure

We shall start this section by considering the following Lagrangian (which has the form of the massive
scalar Electrodynamics with self-interaction term, or the Abelian Higgs model -provided that $m^2 < 0$)

\[ \mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^\dagger (D^\mu \varphi) - \frac{m^2}{2} \varphi^\dagger \varphi - \frac{\lambda}{4} (\varphi^\dagger \varphi)^2, \]

\[ \text{(1)} \]

\[ ^1 \text{We shall use Minkowski metric } \text{diag}(\eta_{\mu\nu}) = (+, -, -, -) \text{ and greek letters running 0,1,2,3.} \]
with $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Clearly, the matter fields are considered to be complex and their products are taken at the same space-time point, say, $x$. This Lagrangian is invariant under the usual local gauge transformations:

$$\delta A_\mu(x) = -\partial_\mu \Lambda(x); \quad \delta \varphi(x) = +ie\Lambda(x)\varphi(x); \quad \delta \varphi^\dagger(x) = -ie\Lambda(x)\varphi^\dagger(x). \quad (2)$$

Now, in order to obtain a point-split version of the Lagrangian above, i.e., a form free from same point product of fields in interaction terms, we begin by writing the generalized version of the gauge transformations, ggt’s (denoted by $\delta_g$) up to $e^2$ (see Ref. [3] for further details):

$$\delta_g A_\mu(x) = -\partial_\mu \Lambda(x) = \delta A_\mu(x), \quad (3)$$
$$\delta_g \varphi(x) = +ie\Lambda(1)\varphi(2) + \frac{1}{2}(ie)^2[\Lambda(1) + \Lambda(3)](1, 3)\varphi(4) + O(e^3), \quad (4)$$
$$\delta_g \varphi^\dagger(x) = -ie\Lambda(-1)\varphi^\dagger(-2) - \frac{1}{2}(ie)^2[\Lambda(-1) + \Lambda(-3)](-1, -3)\varphi^\dagger(-4) + O(e^3), \quad (5)$$

where we have defined:

$$\Lambda(\pm n) = \Lambda(x \pm na); \quad \varphi(\pm n) = \varphi(x \pm na); \quad \varphi^\dagger(\pm n) = \varphi^\dagger(x \pm na); \quad (6)$$

$$\pm(m, \pm n) = \lim_{b \to 0^+} \int_{x=ma+b}^{x=ma-b} A_\mu(\eta)d\eta^\mu. \quad (7)$$

the point-splitting being implemented by the (constant) 4-vector $a_\mu = a$.

From the definition above, we realize the first price to be paid in order to avoid product of fields at the same point: the non-locality of the new model. We shall come back to this point later.

These ggt’s can be shown to satisfy the generalized Abelian condition up to $e^2$, i.e., the commutator of two distinct ggt’s (each of them with its respective parameter $a_1$ and $a_2$) vanishes up to this order:

$$[\delta_{g1}, \delta_{g2}]\varphi(x) = O(e^3); \quad [\delta_{g1}, \delta_{g2}]\varphi^\dagger(x) = O(e^3).$$

is worthy noticing that, as the parameter $a$ is set to zero, all the results above recover the usual ones (hereafter, by consistency, the same should happen to all point-split results). Furthermore, we should stress that the point-splitting acts only in transformations which present same point product, which is the case for $\delta\varphi$ and $\delta\varphi^\dagger$, but not for $\delta A_\mu$.

Now, we discuss the invariance of the ordinary Lagrangian, eq. (1), under the ggt’s above (more precisely, up to order $e^2$). The kinetic gauge term is clearly invariant since

$$\Lambda(\pm 1) = \Lambda(x \pm na); \quad \varphi(\pm 1) = \varphi(x \pm na); \quad \varphi^\dagger(\pm 1) = \varphi^\dagger(x \pm na); \quad (6)$$

$$\Lambda(\pm 2) = \Lambda(x \pm 2na); \quad \varphi(\pm 2) = \varphi(x \pm 2na); \quad \varphi^\dagger(\pm 2) = \varphi^\dagger(x \pm 2na); \quad (6)$$

$$\Lambda(\pm 3) = \Lambda(x \pm 3na); \quad \varphi(\pm 3) = \varphi(x \pm 3na); \quad \varphi^\dagger(\pm 3) = \varphi^\dagger(x \pm 3na); \quad (6)$$

$$\Lambda(\pm 4) = \Lambda(x \pm 4na); \quad \varphi(\pm 4) = \varphi(x \pm 4na); \quad \varphi^\dagger(\pm 4) = \varphi^\dagger(x \pm 4na); \quad (6)$$

is worthy noticing that, as the parameter $a$ is set to zero, all the results above recover the usual ones (hereafter, by consistency, the same should happen to all point-split results). Furthermore, we should stress that the point-splitting acts only in transformations which present same point product, which is the case for $\delta\varphi$ and $\delta\varphi^\dagger$, but not for $\delta A_\mu$.

Now, we discuss the invariance of the ordinary Lagrangian, eq. (1), under the ggt’s above (more precisely, up to order $e^2$). The kinetic gauge term is clearly invariant since $\delta_g A_\mu = \delta A_\mu$. The mass term for matter fields can be shown to be invariant in its action form, $\int m^2\varphi^\dagger\varphi dx$, with suitable change of variables within the integration (see Ref. [3] for more details). Contrary, the other terms are not invariant and must have their points split up. We choose to do the point-splitting (P.S) in the following way (like as in $\delta_g$, $A_\mu(\pm n)$ stands for $A_\mu(x \pm n)$):

$$\left(D_\mu \varphi(x)\right)\dagger(D^\mu \varphi(x)) \xrightarrow{\text{P.S.}} \left(D_\mu \varphi\right)\dagger(D^\mu \varphi) = \left[\partial_\mu \varphi^\dagger(x) - ieA_\mu(-1)\varphi^\dagger(-2)\right]\left[\partial^\mu \varphi(x) + ieA_\mu(1)\varphi(2)\right], \quad (8)$$

$$\left(\varphi^\dagger(x)\varphi(x)\right)^2 \xrightarrow{\text{P.S.}} \left(\varphi^\dagger\varphi\right)^2 \xrightarrow{\text{P.S.}} \varphi^\dagger(-1)\varphi(1)\varphi^\dagger(-2)\varphi(2). \quad (9)$$

2For further applications to fermionic fields, the Hermitian conjugation must be changed to Dirac conjugation. On the other hand, if the matter fields are rank-2 tensors, then additional attention must be paid to their indices. See Section III for details. In addition, in dealing with the actual Abelian Higgs model a important question that now takes place is whether the present scheme is more suitable applied before or after the spontaneous symmetry breaking be performed. Such a point will be discussed later (see Concluding Remarks, for more details).

3In the Abelian case, $\delta_g A_\mu = \delta A_\mu$ holds, but in the non-Abelian scenario, where the ordinary gauge transformation for $A_\mu^a$ involves products at the same point, the point-splitting will act on it, and its non-Abelian ggt’s will be different from the usual one. Indeed, such ggt’s were already worked out for $SU(2)$ [3], and more recently for $SU(N)$ [3].
And the split Lagrangian takes the form:

\[ \mathcal{L}_{P,S}^{(0)} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{m^2}{2} \varphi^\dagger(x) \varphi(x) + (D_\mu \varphi)^\dagger (D^\mu \varphi)_{P,S} - \frac{\Lambda}{4} (\varphi^\dagger \varphi)^2_{P,S}. \]  

(10)

Here, it is worthy noticing that, while the kinetic matter term, \( \partial_\mu \varphi^\dagger(x) \partial^\mu \varphi(x) \), involves a product at the same point, it does not need to be split because the action of the ggt’s on it will produce regularized terms. Now, taking \( \delta_g \) of such split terms up to order \( e \), we get:

\[ \delta_g \left( (D_\mu \varphi)^\dagger (D^\mu \varphi)_{P,S} \right) = (ie) \left[ \Lambda(1) \partial_\mu \varphi^\dagger(x) \partial^\mu \varphi(2) - \Lambda(-1) \partial_\mu \varphi^\dagger(2) \partial^\mu \varphi(x) \right] + \mathcal{O}(e^2). \]

which is, at first glance, non-vanishing. But, if we take its action form, we can perform a change of variables to show that the integrals exactly cancel each other. In other words, the r.h.s. of the previous expression gives rise to a vanishing term in the full split action.

Next, for the self-interaction term, we get:

\[ \delta_g \left( (\varphi^\dagger \varphi)^2_{P,S} \right) = ie \left[ \Lambda(2) \varphi^\dagger(-1) \varphi(3) - \Lambda(-2) \varphi^\dagger(-3) \varphi(1) \right] \varphi^\dagger(-2) \varphi(2) \]  

\[ + ie \varphi^\dagger(-1) \varphi(1) \left[ \Lambda(3) \varphi^\dagger(-2) \varphi(4) - \Lambda(-3) \varphi^\dagger(-4) \varphi(2) \right] + \mathcal{O}(e^2). \]  

(11)

Contrary to the previous one, the term above seems to be intrinsically non-vanishing; in fact, we did not see any way to set it to zero, neither by a suitable change of variables nor by partial integration. Therefore, we must search for a new term, \( \Omega_{P,S}^{(1)} \), such that \( (\varphi^\dagger \varphi)^2_{P,S} + \Omega_{P,S}^{(1)} \) be invariant under \( \delta_g \) at least up to order \( e \). This term exists and can be explicitly written as:

\[ \Omega_{P,S}^{(1)} = -ie \left\{ -2,2 \right\} \varphi^\dagger(-2) \varphi(2) + \left\{ -3,3 \right\} \varphi^\dagger(-1) \varphi(1) \]  

(12)

with the definition:

\[ \{ -n, +n \} = \lim_{b \to 0^+} \int_{x-na-b}^{x+na+b} dy^\mu \partial_\mu \left[ \varphi^\dagger \left( \frac{y}{n} + \frac{n-1}{n} x - na \right) \varphi \left( \frac{y}{n} + \frac{n-1}{n} x + na \right) \right] (-\infty, y), \]  

(13)

where \((-\infty, y)\) stands for \( \int_{-\infty}^{y} A^\nu(\eta) d\eta^\nu \).

Therefore, the split Lagrangian, whose action is invariant under \( \delta_g \) up to first order, \( \mathcal{L}_{P,S}^{(0)} \), is the sum of \( \mathcal{L}_{P,S}^{(0)} \) and \(-\frac{1}{4} \Omega_{P,S}^{(1)}\) (eqs. (10) and (12)).

It is precisely in this sense that gauge invariance has to be taken in the framework of generalized gauge transformations, ggt’s. Actually, since the ggt’s themselves take the form of an infinite series in the gauge coupling constant, then it is expected that the split (and regularized) action also presents a similar form, with its “generalized gauge invariance” being constructed and checked order by order.

Now, calculating \( \delta_g \mathcal{L}_{P,S}^{(1)} \) at order \( e^2 \), we get (after suitable change of variables in the action form of the terms):

\[ \delta_g \left( (D_\mu \varphi)^\dagger (D^\mu \varphi)_{P,S} \right) |e^2 = (ie)^2 \left[ \Lambda(1) A_\mu(3) - \Lambda(3) A_\mu(1) \right] \varphi^\dagger(x) \varphi(4), \]  

(14)

\[ \text{with } U \uparrow \downarrow V = U \partial V - (\partial V) U. \]  

Again, we cannot set this term to zero. Instead, according to \( \Omega_{P,S}^{(1)} \), we must search for a new term, \( \Sigma_{P,S}^{(2)} \), such that \( (D_\mu \varphi)^\dagger (D^\mu \varphi)_{P,S} + \Sigma_{P,S}^{(2)} \) be invariant under \( \delta_g \) at least up to order \( e^2 \). Such term can be found and its simplest form is:

\[ \Sigma_{P,S}^{(2)} = -(ie)^2 \Sigma_\mu \left[ \varphi^\dagger(x) \varphi(4) \right]. \]
with $\Sigma_\mu$ being a function of $\Lambda$ and $A_\mu$. In fact, $\Sigma_\mu$ must be an object such that $\delta_y \Sigma_\mu = \Lambda(1) A_\mu(3) - \Lambda(3) A_\mu(1)$. It is easy to check that the following expression satisfies such a requirement:

$$\Sigma_{P,S}^{(2)} = -(ie)^2 \left\{ [A_\mu(1) + A_\mu(3)] (1,3) + ([1] + [3]) \int_{x+a}^{x+3a} F_{\mu\nu} (\xi) d\xi^\nu \right\} \varphi^\dagger (x) \partial^\mu \varphi (4),$$  \hspace{1cm} (15)

with $[\pm n] = \frac{1}{2} \left[ (\infty, \pm n) + (\infty, \pm n) \right]$. In addition, we may see that as $a \to 0$ then $\Sigma_{P,S}^{(2)}$ vanishes. [The expression inside $\{ \}$ was already obtained in Ref.\[5\] for the interacting vertex of QED; for details, see eq. (24) in that paper\[4\].]

Now, for the self-interaction sector, we get:

$$\delta_y \left( (\varphi^\dagger \varphi)^2_{P,S} + \Omega_{P,S}^{(1)} \right) |_{e^2} = \frac{1}{2} (ie)^2 \left\{ 2 \{-2,2\} \left[ \Lambda(-3) \varphi^\dagger(-4) \varphi(2) - \Lambda(3) \varphi^\dagger(-2) \varphi(4) \right] + \\
+ 2 \{-3,3\} \left[ \Lambda(-2) \varphi^\dagger(-3) \varphi(1) - \Lambda(2) \varphi^\dagger(-1) \varphi(3) \right] + \\
- \Lambda(-2) + \Lambda(-4) \right\} (2,4) \left[ \varphi^\dagger(-5) \varphi(1) \varphi^\dagger(-2) \varphi(2) + \varphi^\dagger(x) \varphi(2) \varphi^\dagger(-5) \varphi(3) \right] + \\
+ 2 \Lambda(-4) (-\infty,2) \left[ \varphi^\dagger(-1) \varphi(5) \varphi^\dagger(-2) \varphi(2) + \varphi^\dagger(-3) \varphi(5) \varphi^\dagger(-2) \varphi(x) \right] + \\
+ 2 \left[ \Lambda(3) (-\infty,3) + \Lambda(-3) (-\infty,3) \right] \varphi^\dagger(-4) \varphi(4) \varphi^\dagger(-1) \varphi(1) \right\}. \hspace{1cm} (16)$$

The non-vanishing of this term is evident. Searching for a new term, $\Omega_{P,S}^{(2)}$, such that $(\varphi^\dagger \varphi)^2_{P,S} + \Omega_{P,S}^{(1)} + \Omega_{P,S}^{(2)}$ be invariant under $\delta_y$ at least up to order $e^2$, is more difficult than for the former ones, $\Omega_{P,S}^{(1)}$ and $\Sigma_{P,S}^{(2)}$. Such difficulties arise from its rather complicated structure. Fortunately, an explicit expression may be indeed found. For that, we notice that the six last terms above have similar structure, say, $\Lambda(\pm n) (\pm m, \pm p) \varphi^\dagger \varphi \varphi^\dagger \varphi$-type factors. Actually, for such terms, the simplest $\Omega_{P,S}^{(2)}$-type ‘counter-term’ has the general form:

$$\frac{1}{2} (ie)^2 \left\{ \frac{\Lambda(\pm n)}{\Lambda(\pm m)} (\pm m, \pm p)^2 \right\}. $$

By remembering the definitions of the above quantities, it is easy to see that such a expression vanishes as $a \to 0$.

On the other hand, for the first two terms in eq. (16), those proportional to $\{-n, +n\}$, the task of finding $\Omega_{P,S}^{(2)}$-type counter-terms appear to be very easy if we take into account that:

$$\delta_y \{-n, +n\} |_{e^0} = \Lambda(-n) \varphi^\dagger(-n-1) \varphi(n-1) - \Lambda(n) \varphi^\dagger(-n+1) \varphi(n+1).$$

In fact, as it can be readily checked, those first two terms have the following $\Omega_{P,S}^{(2)}$-type counter-term:

$$\frac{1}{2} (ie)^2 \left\{ -2 \{-2, +2\} \{-3, +3\} \right\}. $$

\[4\]An alternative, but apparently non-equivalent, form for $\Sigma_{P,S}^{(2)}$ was obtained in Ref.\[16\] and reads:

$$-(ie)^2 \left\{ [1] A_\mu(3) - \frac{1}{2} \left[ \frac{[1]^2}{\Lambda(1)} \partial_\mu \Lambda(3) - \frac{[3]^2}{\Lambda(3)} \partial_\mu \Lambda(3) \right] \right\} \varphi^\dagger (x) \partial^\mu \varphi (4).$$

Despite the explicit presence of the gauge parameter in the expression above, it can be verified that it vanishes as $a \to 0$ and leads us to a split action invariant under ggt’s up to the 2nd order.

\[5\]In fact, if $\varphi$ and $\varphi^\dagger$ are fermionic fields, then $\varphi$ (or $\varphi^\dagger$) has anticommutative property, but the bilinear $\varphi^\dagger \varphi$ has commutative behavior. Therefore, even for fermionic fields, we can change the order of $\{-2, +2\}$ by $\{-3, +3\}$ and vice-versa, without any extra minus sign.
Therefore, the full $\Omega_{P,S}^{(2)}$-term takes over the form:

$$\Omega_{P,S}^{(2)} = \frac{1}{2}(\epsilon e)^2 \left( 2 \{ -2, +2 \} \{ -3, +3 \} + \right.$$  

$$+ \left( \frac{\Lambda(2) + \Lambda(4)}{\Lambda(2) - \Lambda(4)} \right) \frac{(2, 4)^2}{2} \left[ \varphi^\dagger (-1) \varphi(5) \varphi^\dagger (-2) \varphi(2) + \varphi^\dagger (-2) \varphi(x) \varphi^\dagger (-3) \varphi(5) \right] +$$  

$$- \left( \frac{\Lambda(-2) + \Lambda(-4)}{\Lambda(-2) - \Lambda(-4)} \right) \frac{(-2, -4)^2}{2} \left[ \varphi^\dagger (-5) \varphi(1) \varphi^\dagger (-2) \varphi(2) + \varphi^\dagger (-2) \varphi(2) \varphi^\dagger (-5) \varphi(3) \right] +$$  

$$+ \left( \frac{\Lambda(4)}{\Lambda(2)} \right) (-\infty, 2)^2 \left[ \varphi^\dagger (-1) \varphi(5) \varphi^\dagger (-2) \varphi(2) + \varphi^\dagger (-3) \varphi(5) \varphi^\dagger (-2) \varphi(x) \right] +$$  

$$+ \left( \frac{\Lambda(-4)}{\Lambda(-2)} \right) (-\infty, -2)^2 \left[ \varphi^\dagger (-5) \varphi(1) \varphi^\dagger (-2) \varphi(2) + \varphi^\dagger (-5) \varphi(3) \varphi^\dagger (x) \varphi(2) \right] +$$  

$$- \left( \frac{\Lambda(2)}{\Lambda(-2)} \right) (-\infty, -2)^2 + \left( \frac{\Lambda(-2)}{\Lambda(2)} \right) (-\infty, 2)^2 \varphi^\dagger (-3) \varphi(3) \varphi^\dagger (-2) \varphi(2) +$$  

$$- \left( \frac{\Lambda(3)}{\Lambda(-3)} \right) (-\infty, -3)^2 + \left( \frac{\Lambda(-3)}{\Lambda(3)} \right) (-\infty, 3)^2 \varphi^\dagger (-4) \varphi(4) \varphi^\dagger (-1) \varphi(1) \right].$$  

(17)

Finally, the $L_{P,S}^{(2)}$ Lagrangian, whose action is invariant under $\delta g$ up to order $e^2$, may be written as:

$$L_{P,S}^{(2)} = L_{P,S}^{(0)} + \Sigma_{P,S}^{(2)} - \frac{\lambda}{4}(\Omega_{P,S}^{(1)} + \Omega_{P,S}^{(2)})$$  

(18)

with the expressions for the above terms being given by $\Box$, $\Box$, $\Box$, and $\Box$.

The form of eq. (18) deserves further remarks. The explicit presence of the gauge parameter, $\Lambda$, in eq. (17), may seem to be spurious, since it is well-known that (usual) gauge invariance is explicitly broken by the presence of the gauge parameter in the action. Actually, such a symmetry is broken whenever it is taken in the usual sense, but by reassessing the meaning of gauge invariance in the context of ggt’s, then the scenario may be changed. As we have already pointed out, gauge invariance has in this framework to be constructed and checked by means of an order by order (in the gauge coupling constant) algorithm.

Therefore, when finding out a split action, this has to be also done order by order and its ‘generalized gauge invariance’ must be verified according such a “perturbative” procedure. Hence, if we wish to verify whether a given split action is invariant under ggt’s, say, up to 2nd order, for concreteness, then we must check: first, if as the splitting parameter vanishes, $a \to 0$, the split action restores the original one; and, if the split action is actually invariant under ggt’s up to 2nd order, then $\delta g S_{P,S}^{(2)} = O(e^3)$.

In our present case, eq. (18), both requirements are verified, even though the gauge parameter is explicitly present in its expression.

Whether other good split actions without explicit presence of such a parameter may be found for $\lambda \varphi^4$-type vertices is not so clear to us. Indeed, in the case of 3-leg vertices, like as $\varphi^\dagger A_\mu \gamma^\mu \varphi$, we have found a $\Lambda$-explicitly dependent action which was shown to satisfy both of the requirements above (see footnote that follows eq. (13)).

3 Applications to some self-interacting models

Here, in order to illustrate the applicability of our results, we shall deal with some $\lambda \varphi^4$-type models. Whenever necessary, we shall pay attention to specific points which were not still presented.
i) The \((\bar{\psi}\psi)^2\) model

The model which will be worked out is described by the following Lagrangian:

\[
\mathcal{L}_\psi(x) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} (i D_\mu \gamma^\mu - m_f) \psi - g(\bar{\psi}\psi)^2,
\]

with \(D_\mu\) and \(F_{\mu\nu}\) previously defined.

Here, due to the anti-commutative character of fermionic fields, we must pay special attention in changing the order of such fields. Moreover, the kinetic term is slightly different from that for scalar field and it must be taken apart. Fortunately, such a term was already studied in Ref.\[5\] and, if we perform the following splitting:

\[
\bar{\psi}(x) i D_\mu \gamma^\mu \psi(x) \overset{P.S.}{\rightarrow} (\bar{\psi} i D_\mu \gamma^\mu \psi)_{P.S.} = \bar{\psi}(x) i \partial_\mu \gamma^\mu \psi(x) - e \bar{\psi}(x-a) A_\mu(x) \gamma^\mu \psi(x+a),
\]

one can readily show that \(\int d^4x (\bar{\psi} i D_\mu \gamma^\mu \psi)_{P.S.}\) is invariant up to order \(e\). At second order, such a variation does not vanish, but it is exactly canceled by the following term (quite similar to eq.(15) above; see also eq. (24) in Ref.\[5\]):

\[
\Sigma^{(2)}_{\psi,P.S.} = -\frac{ie^2}{2} \bar{\psi}(-2) \gamma^\mu \psi(2) \left\{ [A_\mu(-1) + A_\mu(1)](-1, +1) + ([-1] + [1]) \int_{x-a}^{x+a} d\eta^\nu F_{\mu\nu}(\eta) \right\}.
\]

Now, the \((\bar{\psi}\psi)^2\)-term is split in the same way as \((\varphi^\dagger \varphi)^2\):

\[
(\bar{\psi}(x) \psi(x))^2 \overset{P.S.}{\rightarrow} (\bar{\psi}\psi)^2_{P.S.} = \bar{\psi}(-1) \psi(1) \bar{\psi}(-2) \psi(2).
\]

So, \(\mathcal{L}^{(0)}_{\psi,P.S.}\) for \(\psi\)-like fields reads:

\[
\mathcal{L}^{(0)}_{\psi,P.S.} = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - m_f \bar{\psi}(x) \psi(x) + i(\bar{\psi} D_\mu \gamma^\mu \psi)_{P.S.} - g(\bar{\psi}\psi)^2_{P.S.}.
\]

To get \(\mathcal{L}^{(2)}_{\psi,P.S.}\), we may use the \(\Omega^{(1)}_{\psi,P.S.}\) and \(\Omega^{(2)}_{\psi,P.S.}\) obtained in the previous section with suitable change of \(\varphi\) by \(\psi\) and \(\varphi^\dagger\) by \(\bar{\psi}\). Indeed, as we kept the original order of those matter fields in the previous results, we may write:

\[
\Omega^{(1)}_{\psi,P.S.} = \Omega^{(1)}_{\psi,P.S.}|_{\varphi\rightarrow\psi, \varphi^\dagger \rightarrow \bar{\psi}} \quad \text{and} \quad \Omega^{(2)}_{\psi,P.S.} = \Omega^{(2)}_{\psi,P.S.}|_{\varphi\rightarrow\psi, \varphi^\dagger \rightarrow \bar{\psi}}.
\]

Finally, we get:

\[
\mathcal{L}^{(2)}_{\psi,P.S.} = \mathcal{L}^{(0)}_{\psi,P.S.} + \Sigma^{(2)}_{\psi,P.S.} + g(\Omega^{(1)}_{\psi,P.S.} + \Omega^{(2)}_{\psi,P.S.}).
\]

ii) The ‘Avdeev-Chizhov’ model

Some years ago, Avdeev and Chizhov\[8\] proposed an Abelian model which includes antisymmetric rank-2 real tensors that describe matter, rather than gauge degrees of freedom. They are coupled to a usual vector gauge field as well as to fermions. The model was shown to reveal interesting properties: for instance, these new matter fields were shown to play an important role in connection with

\[\text{It is worthy noticing the (power-counting) non-renormalizability of this self-interaction vertex: } [g] = [\text{mass}]^{-1} \text{ in (3+1) dimensions. In addition, notice that this vertex is not of current-current-type, like as in the (1+1)D Thirring model. Thus, the regularization scheme of treating each current (two-leg) term separated (see Ref.\[3\] for details) does not work here.}\]
extended electroweak models in order to explain some observed decays like $\pi^- \rightarrow e^- + \bar{\nu} + \gamma$ and $K^+ \rightarrow \pi^0 + e^+ + \nu$, and a classical analysis of its dynamics has shown that some longitudinal excitations may carry “physical degrees of freedom” (see Ref. [10] for further details). In addition, some works have been devoted to the study of its supersymmetric generalization [11], as well as its connection with non-linear sigma models [12].

Starting off from these interesting features, it was shown that the coupling between tensorial and fermionic fields generates anomalies in the quantized version of the model and could also spoil its renormalizability [13]. The removal of the fermions has the additional usefulness of allowing us to write the new Lagrangian in a shorter form by means of complex field tensors, $\varphi_{\mu \nu}$ and $\tilde{\varphi}_{\mu \nu}$ [14]. Thus, the modified Avdeev-Chzhov model reads:

$$L_{AC}(x) = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (D_\mu \varphi^{\mu \nu})(D^{\alpha \varphi_{\alpha \nu}})^\dagger - \frac{\lambda}{4} \varphi^{\dagger}_{\mu \nu} \varphi^{\mu \nu} \varphi^{\dagger}_{\kappa \lambda} \varphi^{\kappa \lambda},$$

(24)

with $D_\mu$ and $F_{\mu \nu}$ already defined. Once $\varphi_{\mu \nu}$ is taken to satisfy a complex self-dual relation:

$$\varphi_{\mu \nu}(x) = +i \tilde{\varphi}_{\mu \nu}(x), \quad \tilde{\varphi}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \varphi^{\alpha \beta},$$

then it can be split into two (real tensors) parts:

$$\varphi_{\mu \nu}(x) = T_{\mu \nu}(x) + i \tilde{T}_{\mu \nu}(x) \quad \text{and} \quad \tilde{\varphi}_{\mu \nu}(x) = T_{\mu \nu}(x) - i \tilde{T}_{\mu \nu}(x),$$

where $T_{\mu \nu}$ and $\tilde{T}_{\mu \nu}$ are real and antisymmetric fields. Actually, regarded as matter fields, it is known that they describe spin-0 excitations (see, for example, Refs. [14, 17]).

Now, performing similar splittings in $L_{AC}(x)$, as we have done in former cases, we get, after some calculation, $L_{P.S}(\varphi^{\mu \nu})$:

$$L_{P.S}(\varphi^{\mu \nu}) = L_{P.S}^{(0)}(\varphi^{\mu \nu}) + \sum_{P.S}(\varphi^{\mu \nu}) - \frac{\lambda}{4} \left( \Omega_{P.S}^{(1)}(\varphi^{\mu \nu}) + \Omega_{P.S}^{(2)}(\varphi^{\mu \nu}) \right)$$

(25)

where the terms above have the following expressions:

$$L_{P.S}^{(0)}(\varphi^{\mu \nu}) = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (D_\mu \varphi^{\mu \nu})_{P.S}(D^{\alpha \varphi_{\alpha \nu}})^\dagger_{P.S} - \frac{\lambda}{4} \varphi^{\dagger}_{\mu \nu} \varphi^{\mu \nu} \varphi^{\dagger}_{\kappa \lambda} \varphi^{\kappa \lambda}_{P.S}$$

(26)

(with the splittings previously performed):

$$\sum_{P.S}(\varphi^{\mu \nu}) = -(i \epsilon)^2 \left[ \left[ [1] A^{\mu}(3) - [3] A^{\mu}(1) \right] - \frac{1}{2} \left( \frac{[1]^2}{\Lambda(1)} \partial^{\mu} \Lambda(3) - \frac{[3]^2}{\Lambda(3)} \partial^{\mu} \Lambda(1) \right) \right] \times \left[ \varphi^{\dagger}_{\mu \nu}(x) \partial_{\alpha} \varphi^{\alpha \nu}(4) - \varphi_{\mu \nu}(4) \partial_{\alpha} \varphi^{\dagger \alpha \nu}(x) \right]$$

(27)

(its slight difference with respect to $\sum_{P.S}$, eq. [13], is due to the tensor indices);

$$\Omega_{P.S}^{(1)}(\varphi^{\mu \nu}) = -(i \epsilon) \left( \{-2, +2\}^\mu_{\nu} \varphi^{\dagger}_{\mu \alpha} (-2) \varphi^{\alpha \nu}(2) + \{-3, +3\}^\mu_{\nu} \varphi^{\dagger}_{\mu \alpha} (-1) \varphi^{\alpha \nu}(1) \right),$$

(28)

and $\Omega_{P.S}^{(2)}(\varphi^{\mu \nu})$ which is easily obtained from $\Omega_{P.S}^{(1)}$ by making the interchanges:

$$\{-2, +2\} \rightarrow \{-3, +3\}, \quad \varphi^{\dagger}_{\mu \nu} \varphi^{\alpha \nu} \rightarrow \varphi^{\dagger}_{\mu \nu} \varphi^{\alpha \nu},$$

(29)

\footnote{We have already found a similar expression for this model in Ref. [14]. There, a slightly modified splitting was employed, as well as a lengthier form for $\sum_{P.S}(\varphi^{\mu \nu})$.}
where we have defined \([-n,+n]\) \(\mu_\nu\) in the same way as \([-n,+n]\), with \(\varphi^\dagger \varphi\) changed by \(\varphi^\dagger \varphi^{\alpha\nu}\) in its definition, eq. (13).

Usually, the 2-form field, \(\varphi_{\mu\nu}\), is treated as a gauge potential (the so-called Cremmer-Scherk-Kalb-Ramond field [17, 18]), and in (3+1) dimensions it describes a massless scalar excitation. However, when mixed to the Maxwell field, \(A_{\mu}\), by means of a topological mass term that links two Abelian factors, it yields a massive spin-1 excitation in the spectrum. On the other hand, we should stress that its coupling to charged matter may be realized only non-minimally, i.e., by means of its (3-form) field-strength [19].

Furthermore, from the point of view of the point-splitting procedure, since its usual gauge transformation, \(\delta \varphi_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x)\), does not involve products of quantities at the same space-time point, such a transformation does not undergo any change in going to the generalized case, say, \(\delta_g \varphi_{\mu\nu}(x) = \delta \varphi_{\mu\nu}(x)\). Therefore, by viewing \(\varphi_{\mu\nu}\) as a gauge potential, its ggt’s take easier expressions than when it is treated as a matter field (a general fact, at least in the Abelian framework, in dealing with such a procedure). This readily implies remarkable simplifications whenever working with \(\varphi_{\mu\nu}\) as a gauge potential.

4 Concluding Remarks

The point-splitting procedure combined with generalized gauge transformations has yielded regularized Lagrangians which contain \(\lambda \varphi^4\)-type interaction. The result is such that the generalized Lagrangians have their interacting terms defined at different space-time points. Nevertheless, this property introduces non-locality at the level of the regularized theory.

In general, non-local theories cannot be quantized with the usual methods and the interpretation of their results are not quite obvious. Moreover, we know that non-locality can lead to troubles as long as the causality of the theory is concerned. However, these problems arise only for the regularized theory, in much the same way as ghosts are present and unitarity is temporarily lost for regularized theories before the regularization parameter is removed.

Nevertheless, Osland and Wu [3] obtained some standard results in QED starting by a split Lagrangian (with regularity and non-locality properties). Their method works for the calculation of the quantities with a dependence in the splitting parameter, which is set to zero at the end of calculations in order to get the standard results.

What we may learn from these calculations is that, when point-splitting is combined with generalized gauge transformations in order to obtain regularized (Abelian) Lagrangians, the task becomes more difficult with the increasing of the number of matter fields at the same vertex; in general, the complications which arise from the presence of extra (Abelian) gauge fields are minor ones. So, the calculations involving \(\lambda \varphi^4\)-type vertices are harder to be performed than for ‘lower vertices’, \(\varphi A_\mu \varphi\), \(\varphi A_\mu A^\mu \varphi\), and so forth. In addition, higher-order terms in the coupling constant are, in general, more complicated to be handled than lower ones.

Our present study is a good example of such complications and the reason why they arise. For instance, in dealing with the 2nd order calculations of the split version of the self-interacting vertex

\(\lambda \varphi^4\)-type vertices, the generalized gauge covariance is not taken in its precise meaning.  

\[\text{It is worthy noticing that their Lagrangian (eq. (2.7) in Ref. [3]) is different from the ‘correct’ generalized QED-Lagrangian, up to fourth order (eq. (24) in Ref. [1]). Such a difference may be explained by noticing that, in Ref. [3], the generalized gauge covariance is not taken in its precise meaning.} \]
we have seen that a \textit{counter-term} for that term involved the explicit presence of the gauge parameter, scenario which is expected to become even more intricate at higher orders.

Another point that should be stressed is that this procedure is independent of the space-time dimension, and so, of the canonical dimension of the fields (matter or gauge ones)\footnote{In fact, the ggt’s depend on the splitting parameter, the constant vector $a_\mu$, and on the Abelian (or non-Abelian) character of the gauge fields.}. Hence, the expressions for our $\Sigma$ and $\Omega$ terms remain valid in other dimensions. Therefore, our present results could be equally well applied to four-matter (scalar, spinorial, and rank-2 tensorial) vertices in lower or higher space-time dimensions. However, special attention should be paid if dimensional reduction and/or compactification, spontaneous symmetry breaking, or other mechanisms are involved. As we shall discuss below for the case of the Higgs model, the present procedure is suitably applied only at the stage in which the true physical excitations are taken into account.

On the other hand, if we are dealing with a renormalizable theory (scalar, for simplicity) in $(2 + 1)$ dimensions, an extra $f\varphi^6$-term is allowed. In this case, our results could be applied to the model, including the $\lambda\varphi^4$-term, but the extra term should be worked out apart. As we have already said, the task of working out the split version of a matter vertex tends to become more difficult as the number of matter fields increases. Thus, we expect even more work in dealing with $\varphi^6$-like matter vertices than we had in the present case. Still concerning possible relevant applications in this space-time, we may think of applying the present procedure for studying some Abelian (and non-Abelian in a further stage, too) models connected with the Chern-Simons term. For instance, we may study some points concerning the radiation produced by accelerated point-like charges\cite{20}, an issue which still demands several answers (see Ref.\cite{21}, for more details).

Another relevant question that we may rise up here is the issue of the point-splitting in connection with the Abelian Higgs mechanism. The spontaneous symmetry breaking, as realized by a charged scalar, obliges a shift of the Higgs field around its vacuum expectation value and induces the appearance, among others, of trilinear matter couplings not present in the original action. Here, we may wonder whether the splitting should be performed before or after the breaking. Indeed, although our results are directly applied to the unbroken phase, it does not seem to be the best choice. Actually, we claim that the most suitable way to implement point-splitting is after the breaking takes place, for in the broken regime all possible vertices show up and we can really control the theory we are dealing with, since only at this stage we are quantizing the truly physical excitations. In this case, while our results are applicable to some terms of the Lagrangian written around the true ground state, such as the kinetic and quartic ones, the trilinear vertex, in turn, should be worked out apart (expected to be of easier manipulation than the present one).

We also hope that the present paper could help us whenever dealing with the non-Abelian case. In this scenario novel features will arise mainly because $\delta gA^a_\mu$ will take more complicated (and lengthier) forms, and they will imply in new ggt’s for the matter fields which, in turn, will also take lengthier expressions than those for the Abelian case.

Furthermore, in view of the special role that supersymmetry and supersymmetric gauge theories play in the programme of building up fundamental interaction models, it would be advisable to extend the point-splitting method to treat supersymmetric theories in superspace. Point-split super-actions both in the space-time and Grassmann coordinates may be an interesting issue since now gauge invariance and supersymmetry must be simultaneously checked and many features of the method must be revealed: the advantage of implementing the point- -splitting procedure in superspace is that supersymmetry is manifest and one needs not checking Ward identities (as it would be the case in a
component-field approach) to undertake that supersymmetry is kept upon point-splitting.

we claim that some questions concerning this issue should eventually become clearer. For example, how could Feynman rules for such a kind of Lagrangian be formulated? Or still, as we may see, there are some new ‘interaction terms’ in the generalized Lagrangian. Could these new terms have some physical interpretation and/or relevance?

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