Heterotic versus Type I

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I compare the (prototype) calculations of special $F^4$ and $R^4$ terms in the effective actions of the Spin$(32)/Z_2$ heterotic and type I string theories, compactified on $S^1$ and $T^2$. Besides checking duality, this elucidates the quantitative rules of D-brane calculus. I explain in particular (a) why D-branes do not run in loops, and (b) how their instanton contributions arise from orbifold fixed points of their moduli space.

1. Introduction

The conjectured duality \[\text{Type I} \leftrightarrow \text{Heterotic} \] between the type I and heterotic Spin$(32)/Z_2$ string theories is particularly intriguing. The massless spectrum of both theories, in 10 space-time dimensions, contains the (super)graviton and the (super)Yang Mills multiplets. Supersymmetry and anomaly cancellation fix completely the low-energy Lagrangian, and more precisely all two-derivative terms and the anomaly-cancelling, four-derivative Green-Schwarz couplings. One logical possibility, consistent with this unique low-energy behaviour, could have been that the two theories are self-dual at strong coupling. The conjecture that they are instead dual to each other implies that this unique infrared physics also has a unique consistent ultraviolet extrapolation.

One of the early arguments in favour of this duality was that the heterotic string appears as a singular solution of the type I theory. Strictly-speaking this is not an independent test of duality. The presence of a heterotic-string source will excite the massless heterotic backgrounds, and since the effective Lagrangian is unique we should not be surprised to find the same solution on the type-I side. The real issue is whether consistency of the theory forces us to include such excitations in the spectrum. This can for instance be argued in the case of type II string theory near a conifold singularity of the Calabi-Yau moduli space. Interestingly enough it can also be argued for the simplest field-theoretic analog, the Dirac monopole of an abelian theory in four dimensions. To render the ultraviolet theory consistent, one must embedd the photon field into a spontaneously-broken, asymptotically-free gauge model. The point-like monopoles are then promoted to smooth solitons, which can be pair-created and must thus be included.

I am not aware of such a direct argument in the case of the heterotic string solution. What is, however, known is that it has an exact conformal description as a D(irichlet) string of type I theory. In certain ways D-branes lie between fundamental quanta and smooth solitons so, even if we admit that they are intrinsic, we must still decide on the rules for including them in a semiclassical calculation. Do they contribute, for instance, to loops like fundamental quanta? And with what measure and degeneracy should we weight their Euclidean trajectories? In this talk I will review a prototype calculation in which these questions can be answered. Some related calculations in open string theory can be found in refs. The rules consistent with duality turn out to be natural and simple. D-strings, like smooth solitons, do not enter explicitly in loops, while their (wrapped) Euclidean trajectories contribute to the saddle-point sum, without topological degeneracy if one takes into account correctly the non-abelian structure of D-branes. The way in which the various pieces of the calculation fall in place is, I believe, further evi-

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2A (light) soliton loop can of course be a useful approximation to the exact instanton sum, as is the case near the strong-coupling singularities of the Seiberg-Witten solution.
dence for an underlying unique and elegant structure.

2. BPS states and the unfolding trick

The prototype calculation is that of $F^4$ and $R^4$ terms in the effective quantum action, after toroidal compactification to $d > 4$ dimensions. Except for a particular CP-even combination \[^{[8]}\], all these terms are special for the following reasons: (a) they are non-trivial, since supersymmetry and anomaly cancellation do not fix them completely below 10 dimensions, and (b) they are believed to be sensitive only to the BPS sector of the theory, meaning that they are only corrected by short multiplets at one loop, and by maximally-supersymmetric saddle points. In these respects they are analogous to $F^2$ and $R^2$ terms in vacua with 8 unbroken supercharges \[^{[8]}\].

The only (known) supersymmetric saddle points on the heterotic side are Euclidean trajectories of solitonic five-branes. Since for $d > 4$ non-compact dimensions these have no finite 6-cycle to wrap around, we expect the heterotic one-loop result to be exact. For zero Wilson lines, this one-loop amplitude reads \[^{[18,19]}\]

\[
\mathcal{I}_{\text{1-loop}}^\text{heter} = -\frac{V(d)}{210\pi^{d/2}} \int d^2\tau \left(\tau_1^2\right)^{5-d/2} \times \left.G\left(F,R,\tau\right)\right|_{\text{fundamental}} \times \Gamma_{10-d,10-d} \tilde{A}(F,R,\tau)
\]

where $V(d)$ is the volume of the non-compact space-time, $\Gamma_{10-d,10-d}$ is the usual sum over momenta and windings on the compactification torus \[^{[3]}\], and $\tilde{A}$ is an (almost) holomorphic modular form of weight zero, closely related to the elliptic genus \[^{[21]}\]. More precisely, the Lagrangian form of the lattice sum is

\[
\Gamma_{10-d,10-d} = \left(\frac{2}{\tau_2}\right)^{5-d/2} \sqrt{\det G} \times \sum_{n',m'} e^{-\frac{1}{24}(G+B)_{ij}(m'\tau-n')(m'\tau-n')}\]

with $G_{ij}$ the metric and $B_{ij}$ the (constant) antisymmetric-tensor background on the torus. I use the conventions $\alpha' = \frac{1}{2}$, and $G = L^2$ for a circle with radius $L$. Finally, the elliptic genus $A$ is a chiral partition function, with extra weights for gauge-charge, R-charge and spin operators, within the Cartan subalgebra of $SO(32) \times SO(10-d) \times SO(d-2)$. Expanding it out to fourth order in the charges and/or spins, and regularizing in a modular-invariant way, yields $A\tilde{A}\pi^{d/2}$.

This heterotic amplitude can be most easily derived in the (light-cone) Green-Schwarz formulation \[^{[16]}\]. Let me concentrate for definiteness on the gauge part of the effective action. The coupling of the heterotic string to a constant (transverse) field-strength background reads

\[
\delta I_\sigma \propto \int \left(\mathcal{L}_{\text{v}} \cdot \mathcal{A}\right) \left(\mathcal{L}_{\text{v}} \cdot \mathcal{A}\right) - \frac{1}{8} \left(S^i \gamma^{ij} S\right)
\]

with $S$ the Green-Schwarz fermions, and $J_\alpha$ the $SO(32)$ world-sheet currents that can be represented as fermion bilinears, $J_\alpha = T_{\alpha}\lambda_\beta \lambda_\gamma$. Consider now the $\sigma$-model functional integral on the torus. To absorb the eight fermionic zero modes we must bring down at least four powers of the fermionic piece of $\delta I_\sigma$. The result is proportional to the (covariantized) eight-index tensor

\[
t^{i_1 j_1 \ldots i_4 j_4} = \int \left[D\mathcal{S}_0\right] \left(S^1 \gamma_{i_1 j_1} \mathcal{S}_0\right) \ldots \left(S^1 \gamma_{i_4 j_4} \mathcal{S}_0\right),
\]

times the momentum and winding sum, times the partition function of left-moving (holomorphic) states with four insertions of the $SO(32)$ generators. Notice that the bosonic and fermionic determinants cancel out on the right-moving (supersymmetric) side. Putting all this together leads to expression (1) with

\[
\tilde{A}(F,\tau) = t_{(8)} \text{tr} F^4 + \frac{1}{29} \cdot \frac{E_4}{\eta^{24}} + \left[\frac{E_4^2}{\eta^{24}} - \frac{E_6}{\eta^{24}} \right] t_{(8)} \left(\text{tr} F^2\right)^2,
\]

where the traces are in the fundamental representation of $SO(32)$ and I have suppressed Lorentz indices. The Eisenstein series $E_{2k}(q = e^{2\pi i\tau})$ are modular forms of weight $2k$. They are holomorphic with the exception of $E_2$, which requires a non-holomorphic regularization.
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ered by the fact that there are no oscillator excitations in the right-moving sector.

The same type of argument can be applied to the open superstring \[1\]. Since the left- and right-

moving sectors are not independent in this case, the only BPS states are the (Kaluza-Klein de-

scendants) of the SO(32) (super)gauge bosons. After Poisson resummation of the Kaluza-Klein

momenta we get

\[
\mathcal{I}_{1\text{-loop}}^{\text{open}} = - \frac{V^{(10)}}{210 \pi^6} t_{(8)} \text{tr}_{\text{adj}} F^4 \times \\
\times \int_0^\infty \frac{dt}{t^2} \sum_{n \neq \{0\}} e^{-2\pi G_{ij} n^i n^j/t} \tag{4}
\]

with the trace in the adjoint representation of SO(32). The quadratically-divergent \( n^i = \{0\} \) term has been subtracted explicitly from the sum. A careful calculation \[22\] shows that it

indeed corresponds to a (one-particle-reducible) diagram, with a massless graviton exchanged in

the transverse channel. This is the only way in which \([\text{4}]\) differs from the naive (super)Yang Mills

expression.

Consider now a circle compactification to 9 dimensions. Since there are no stable 2-cycles around which Euclidean D-string trajectories may wrap, we expect no instanton corrections on the type I side. Hence the action \([\text{4}]\) should be (almost) exact. But this raises an obvious puzzle: The heterotic spectrum contains an infinite tower of BPS states in arbitrary representations of \(\text{Spin}(32)/\mathbb{Z}_2\), all of which contribute to the one-loop integrand in \([\text{4}]\). On the type I side these correspond to states of a D-string, winding around the compactification circle. If treated as regular solitons, D-strings should not enter explicitly in loops. With so many more states “running” in the heterotic as compared to the type-I loop, how can the two expressions possibly match?

The answer to this puzzle makes use of an old trick familiar from the study of free-string ther-

modynamics \[23\]. The idea is to trade part of the lattice sum, so as to unfold the integration region from a fundamental domain into the strip,

\[
(2\pi^2 \tau_2) \hat{\mathcal{F}}_{1,1} = 2\pi L \left[ 1 + \sum_{n \neq 0} e^{-2\pi L^2 n^2/1m\mathcal{S}(\tau)} \right],
\]

where \( \mathcal{S} \) labels all modular transformations that leave \( \tau \) inside the strip \(-\frac{1}{2} \leq \tau_2 < \frac{1}{2} \). Using the modular invariance of \( \hat{\mathcal{A}} \), we find

\[
\mathcal{I}_{1\text{-loop}}^{\text{heter}} = - \frac{V^{(10)}}{210 \pi^6} \left[ \int_{\text{Fundom}} \frac{d^2 \tau}{\tau_2^2} + \int_{\text{Strip}} \frac{d^2 \tau}{\tau_2^2} \sum_{n \neq 0} e^{-2\pi L^2 n^2/\tau_2} \right] (\mathcal{A}(F, \tau)).
\]

The \( \tau_1 \) integration in the strip kills all but the \( q^0 \) piece of the elliptic genus,

\[
\left. \mathcal{A}\right|_{q^0} = t_{(8)} \text{tr}_{\text{adjoint}} F^4 - \\
- \frac{15}{16 \pi \tau_2} \frac{63}{64 \pi^2 \tau_2^2} \left( \text{tr} F^2 \right)^2.
\]

Plugging the first of these terms inside \([\text{6}]\) gives precisely the one-loop type I expression. The massive BPS states conspired with the stringy cutoff on the heterotic side, to reproduce the simple loop of SO(32) gauge bosons.

The heterotic one-loop contains in fact extra terms, besides the expression \([\text{4}]\). They can be organized as an expansion in inverse powers of the radius. Since the latter gets rescaled by duality, \( L^2 \rightarrow L^2/g_s \) with \( g_s \) the string coupling constant, these extra contributions must come from diagrams of genus \( \neq 1 \) on the type I side. The leading term in the decompactification limit is given by the integral of \( \hat{\mathcal{A}} \) over a fundamental domain. It is equal to the quartic piece of the Born-Infeld action, which arises from the type I disk diagram \([\text{10}]\). The subleading term is the one-loop contribution. The two remaining terms come from the non-holomorphic regularization of \( \hat{\mathcal{A}} \). They correspond to contact contributions in two- and three-loop open string diagrams. It is conceivable that matching all lower-dimensional operators in the effective heterotic and type I actions requires field redefinitions which absorb these terms \([\text{13}]\).
3. D-brane instantons

Let us now move one step further down and consider a $T^2$ compactification of the eighth and ninth spatial dimensions. The lattice sum on the heterotic side takes the form

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \sum_{M} e^{2\pi i T \det M - \frac{\pi T^2}{24}} |(1 \ U)^M(\tau)|^2$$

(7)

where $M$ runs over all $2 \times 2$ matrices with integer entries, and

$$U = (G_{SS} + i \sqrt{G})/G_{SS} \text{ and } T = \frac{1}{\alpha'} (B_{SS} + i \sqrt{G})$$

are the complex structure and Kähler moduli of $T^2$. The matrix $M$ describes the wrapping of the heterotic world sheet around the target-space torus. The two generators of the world-sheet torus are given, as vectors in the compactification lattice, by the columns of the matrix $M$. The exponent in equation (7) is the minimum Polyakov action for a given wrapping.

The contributions of degenerate matrices ($\det M = 0$) sum up to the perturbative type I result. This follows from an argument identical to the one used in the previous section. The novel feature are non-degenerate matrices, which correspond to heterotic world-sheet instanton corrections. Using a global world-sheet reparametrization, one can bring such a matrix $M$ to the canonical form

$$M = \pm \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \text{ with } 0 \leq j < k, \ p \neq 0.$$  

The sum over the $\text{PSL}(2,\mathbb{Z})$ orbits of these matrices can furthermore be traded against unfolding the fundamental domain integral onto (twice) the upper complex plane [24]. Performing explicitly the integral leads to the following expression for the one-loop heterotic action [11]

$$\mathcal{I}^\text{heter}_{1\text{-loop}} = \mathcal{I}_\text{degen} + \mathcal{I}_\text{inst}$$

where

$$\mathcal{I}_\text{inst} = \frac{\nu(10)}{2^{9/2} \pi^6} \sum_{\substack{0 \leq j < k \\ p > 0}} \frac{e^{2\pi i pkT}}{kpT} \times$$

$$\times \mathcal{O} \hat{A} \left( \frac{j + pu}{k} \right) + \text{c.c.}$$

(8)

Here $\mathcal{O} = 1 + ...$ is a differential operator, whose action is non-trivial only because of the non-holomorphicities of the elliptic genus. I will ignore this complication by assuming for instance that $\text{tr} F^2 = \text{tr} R^2$, in which case $\hat{A}$ is completely holomorphic.

Expression (8) is a sum over all distinct ways to wrap the target-space torus with the heterotic, or D-string (Euclidean) world sheet. The induced Kähler and complex moduli on this latter, for positive $p$'s, read

$$\tilde{T} = kpT \text{ and } \tilde{U} = \frac{j + pu}{k}.$$  

The three distinct ways to wrap the torus twice are, for instance, drawn in figure 1. They corre-

\[\text{Figure 1. The three distinct ways in which the (shaded) heterotic world sheet can wrap twice around the target-space torus.}\]
respond (from top to bottom) to
\[
M = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right), \quad \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right), \quad and \quad \left( \begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right),
\]

have complex structures
\[
\tilde{U} = 2U, \quad \frac{U}{2} \quad and \quad \frac{1 + U}{2},
\]

and (common) Kähler modulus \( \tilde{T} = 2T \). The weight of each surface in expression (8) can be recognized easily as the product of (a) the exponential of minus the Nambu-Goto action, \( I_{NG} = -2\pi iT \), (b) the elliptic genus at the induced complex structure \( \tilde{U} \), and (c) a factor of the inverse area. All this is in accordance with the naive expectation that the auxiliary world-sheet metric may be eliminated in favour of a heterotic Nambu-Goto action.

From the type-I point of view expression (8) is, however, still somewhat unnatural. The three configurations of figure 1 correspond to the same (singular) effective-field-theory solution, characterized by two units of the appropriate Ramond-Ramond charge. Why then should we count them as distinct saddle points? Furthermore, the fluctuations of the double D-brane are not described by the usual heterotic \( \sigma \)-model, but by its (non-abelian) \( 2 \times 2 \)-matrix generalization \( \mathbb{C} \), which is the low-energy limit of an open string theory. Why should then the result be proportional to the conventional elliptic genus?

In order to answer these questions \( \mathbb{I} \) it is convenient to put the effective action (8) in the more elegant form
\[
\mathcal{I}^{\text{inst}} = -\frac{V^{(8)}}{2\pi^4} \sum_{N=1}^{\infty} e^{2\pi N T} \mathcal{H}_N \hat{A}(U) + \text{c.c.},
\]
with
\[
\mathcal{H}_N \hat{A}(U) \equiv \frac{1}{N} \sum_{\substack{\ell p = N \\ell \leq j < h}} \hat{A} \left( \frac{j + pU}{k} \right). \tag{10}
\]

In the mathematics literature \( \mathcal{H}_N \) is known as a Hecke operator \( \mathbb{K} \). We have just seen its geometric interpretation in terms of inequivalent \( N \)-fold wrappings of the torus by the (heterotic) world sheet. I will now describe an alternative interpretation, more appropriate on the type I side, in terms of the moduli space of instantons. The key will be to treat this moduli space as a symmetric orbifold \( \mathbb{O} \), an idea that is more familiar in the context of black-hole state counting \( \mathbb{B} \).

The low-energy fluctuations around a configuration of \( N \) instantonic D-branes are described by a heterotic matrix \( \sigma \)-model, with local SO(N) symmetry on the world sheet \( \mathbb{P} \). The coupling of a constant target-space background field reads
\[
\delta I_\sigma \propto \int F_{\alpha \beta}^{T \tau} \lambda_\tau^{T} \left( X^j \dot{D} X^j - \frac{1}{8} S^j \gamma^{ij} S \right) \lambda_\alpha.
\]

Under the SO(N) gauge symmetry the supercoordinates \( X^j \) and \( S^0 \) are symmetric matrices, the current-algebra fermions \( \lambda_\alpha \) are vectors, and \( \dot{D} \) is the antiholomorphic covariant derivative. We are interested in the functional integral of this \( \sigma \)-model, with four insertions of \( \delta I_\sigma \). Notice that contributions of massive string modes are expected to cancel out for this special amplitude, justifying the reduction of the calculation to the matrix model.

The moduli space of this multiinstanton has a Higgs branch along which the \( X^j \) have diagonal expectation values. In the type I language these label the positions of N D0-particles on the orientifold plane \( \mathbb{L} \). At a generic point in this moduli space there are \( 8N \) fermionic zero modes, corresponding to the diagonal components of the matrices \( S^k \). Since only eight of them can be absorbed by the insertions of the vertex \( \delta I_\sigma \), one would naively conclude that the sectors \( N > 1 \) do not contribute. This is wrong because of the residual gauge symmetry that permutes the positions of the D-branes. The moduli space is thus a symmetric orbifold and there are potential contributions from its fixed points.

Let me illustrate how this works in the case of two instantons. The massless fluctuations of the double D-brane are described by a conformal field

\footnote{This argument has been developed in collaboration with Pierre Vanhove \( \mathbb{L} \).}

\( \mathbb{L} \) There is also a Coulomb branch, corresponding to the motion of mirror pairs of D0-particles off the orientifold plane. Because of the SO(N)-gaugino zero modes, this part of the moduli space does not contribute.
theory with target space $\mathcal{M} \times \mathcal{M}/Z_2$, where $\mathcal{M}$ is the (transverse) target space of the heterotic string, and $Z_2$ is the exchange symmetry. There are four contributions to the amplitude, corresponding to the four boundary conditions on the torus. The untwisted sector has $2 \times 8$ fermionic zero modes and does not contribute. The contribution of the remaining three sectors is proportional to

$$\hat{A}(2U) + \hat{A}\left(\frac{U}{2}\right) + \hat{A}\left(\frac{U+1}{2}\right),$$

as can be shown using standard $Z_2$-orbifold techniques [12]. This is precisely the action of the Hecke operator $\mathcal{H}_2$, corresponding to the sum over the three surfaces of figure 1. The overall coefficient also checks, including the orbifold normalization of $\frac{1}{2}$, and the simple factor of the transverse volume characteristic of the twisted-sector contributions.

The generalization to any $N$ is straightforward. The target space is now the symmetric orbifold $\mathcal{M} \times \ldots \times \mathcal{M}$ N times $/ S_N$

The non-vanishing contributions to the amplitude come from those boundary conditions for which only the trace part of $S^a$ is (doubly) periodic on the torus. Up to a common overall normalization, the result is given by $\mathcal{H}_N \hat{A}(U)$, which is the matrix-model generalization of $\hat{A}$. The non-perturbative type I effective action is obtained by summing over all $N$, as in expression (1).

4. Outlook

It would be very interesting to extend these considerations to more involved settings, including in particular the type I D5 brane. This should lead to a simple combinatorial understanding, through instanton calculus, of the Seiberg-Witten solution.

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