Study of degenerate parabolic system modeling the hydrogen displacement in a nuclear waste repository.

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Abstract

Our goal is the mathematical analysis of a two phase (liquid and gas) two components (water and hydrogen) system modeling the hydrogen displacement in a storage site for radioactive waste. We suppose that the water is only in the liquid phase and is incompressible. The hydrogen in the gas phase is supposed compressible and could be dissolved into the water with the Henry’s law. The flow is described by the conservation of the mass of each components. The model is treated without simplified assumptions on the gas density. This model is degenerated due to vanishing terms. We establish an existence result for the nonlinear degenerate parabolic system based on new energy estimate on pressures.

Keywords: Degenerate system, nonlinear parabolic system, compressible flow, porous media

1. Introduction

An important quantity of hydrogen can be produced by corrosion of ferrous materials in a storage site for radioactive waste. A direct consequence of this production is the growth of hydrogen pressure around alveolus. This increasing gas pressure could break the surrounding host rock and fractures

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could appear in the confinement materials. This problem renews the mathematical interest in the equation describing multiphase/multicomponent flows through porous media. The cases of immiscible and incompressible flows have been treated with "global pressure" introduced by G. Chavent and J. Jaffre [6] by many authors, we refer for instance to [12, 7, 8, 12] where existence results are obtained under various assumptions on physical data. For two immiscible compressible flows without exchange between the phase, we refer to [14, 15] where the authors obtain the existence of solution when the densities depend on the global pressure and to [17, 18] for the general case where the density of each phase depends on its own pressure. This approach is also used in [2, 3] to treat a homogenization problem of immiscible compressible water-gas flow in porous media. For miscible and compressible flow, we refer to [9, 10] for more details.

In [4], the authors derive a compositional model of compressible multiphase flow in porous media. They focus their study on models where the fluid is a mixture of two components: water (mostly liquid) and hydrogen (mostly gas). An existence result has been shown in [20] for this model under the assumptions of non-degeneracy and of strictly positive saturation.

Recently in [5], the authors studied a new model of two compressible and partially miscible phase flow in porous media, applied to gas migration in an underground nuclear waste repository in the case where the velocity of the mass exchange between dissolved hydrogen and hydrogen in the gas phase is supposed finite.

Let us state the physical model used in this paper. We consider herein a porous medium saturated with a fluid composed of two phases (liquid and gas) and a mixture of two components (water and hydrogen) studied in [19]. As reported in [19], the author establish the existence of a weak solution, under non-degeneracy and slow oscillation assumptions on the diagonal coefficients and with small data for the hydrogen. Our aim is to show global solution for the degenerate system without restriction on data. The water is supposed only in the liquid phase (no vapor of water due to evaporation). In order to define the physical model, we write the mass conservation of each component

\[
\begin{align*}
    &P \begin{cases}
    \partial_t (\Phi s_l \rho^h_l + \Phi s_g \rho^h_g) + \text{div}(\rho^h_l V_l + \rho^h_g V_g) - \text{div}(\rho_l D^h_l \nabla X^h_l) = r_g, \\
    \partial_t (\Phi s_l \rho^w_l) + \text{div}(\rho^w_l V_l) = r_w.
    \end{cases}
\end{align*}
\]

(1.1)

Here the subscript \( l \) and \( g \) represent respectively the liquid phase and the gas phase. Quantities \( \Phi, \rho^h_l, \rho^h_g, \rho, s, X^h_l, X^h_g = \rho^h_l / \rho_l (X^h_l + X^h_g = 1) \)
and $D_i^h$ represent respectively the porosity of the medium, the density of dissolved hydrogen, the density of the hydrogen in the gas phase, the density of the $\alpha$ phase ($\alpha = l, g$), the saturation of the $\alpha$ phase ($s_l + s_g = 1$), the mass fraction of the hydrogen in the liquid phase, the diffusion-dispersion tensor of the hydrogen in the liquid phase. The velocity of each fluid $V_\alpha$ is given by the Darcy’s law

$$V_\alpha = -K \frac{k_{r\alpha}(s_{\alpha})}{\mu_{\alpha}} (\nabla p_{\alpha} - \rho_{\alpha}(p_{\alpha})g),$$  \hspace{1cm} (1.3)

where $K$ is the intrinsic permeability tensor of the porous medium, $k_{r\alpha}$ the relative permeability of the $\alpha$ phase, $\mu_{\alpha}$ the constant $\alpha$-phase’s viscosity, $p_{\alpha}$ the $\alpha$-phase’s pressure and $g$ the gravity. For detailed presentation of the model we refer to the presentation of the benchmark Couplex-Gaz [21] and [4, 20]. The capillary pressure law is defined as

$$p_c(s_l) = p_g - p_l,$$

is decreasing, ($\frac{dp_c}{ds_l}(s_l) < 0$, for all $s_l \in [0, 1]$) and $p_c(1) = 0$.

The system (1.1)–(1.2) is not complete, to closing the system, we use the ideal gas law and the Henry’s law

$$\rho_g^h = \frac{M^h}{RT}p_g, \quad \rho_l^h = M^h \frac{H^h}{RT}p_g,$$  \hspace{1cm} (1.4)

where the quantities $M^h$, $H^h$, $R$ and $T$ represent respectively the molar mass of hydrogen, the Henry’s constant for hydrogen, the universal constant of perfect gases and $T$ the temperature.

By these formulation, the system (1.1)–(1.2) is closed and we choose the liquid and gas pressures as unknowns. From (1.4), the henry’s law combined to the ideal gas law, to obtain that the density of hydrogen gas is proportional to the density of hydrogen dissolved

$$\rho_g^h = C_1 \rho_l^h$$

where $C_1 = \frac{1}{H^h RT} (= 52.51)$.  \hspace{1cm} (1.5)

Remark that the density of water $\rho_l^w$ in the liquid phase is constant and from the Henry’s law, we can write

$$\rho_l \nabla X_l^h = C_2 \rho_l^w \nabla p_g,$$

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where \( C_2 \) is a constant equal to \( H^h M^h \).

Then the system (1.1)–(1.2) can be written as

\[
\begin{align*}
\partial_t (\Phi m(s_l) \rho^h_l) + \text{div} (\rho^h_l \mathbf{V}_l + C_1 \rho^h_l \mathbf{V}_g) - \text{div} (C_2 \chi^w_1 D^h_1 \nabla p_g) &= r_g, \quad (1.6) \\
\partial_t (\Phi s_l) + \text{div} (\mathbf{V}_l) &= \frac{r_w}{\rho^w_t}. \quad (1.7)
\end{align*}
\]

where \( m(s_l) = s_l + C_1 s_g \).

Note that the mass exchange between dissolved hydrogen and hydrogen in the gas phase is static using the Henry’s law opposite then supposed in [5].

2. Assumptions and main result

The main point is to handle a priori estimates on the approximate solution. Due to the degeneracy for dissipative terms \( \text{div}(\rho^h_l M_\alpha \nabla p_\alpha) \), we can’t control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So, we are going to use the feature of global pressure to obtain uniform estimates on the gradient of the global pressure and the gradient of a capillary term to treat the degeneracy of the dissipative terms. Let summarize some useful notations in the sequel.

We recall the conception of the global pressure as describe in [6]

\[
M(s_l) \nabla p = M_l(s_l) \nabla p_l + M_g(s_g) \nabla p_g,
\]

with the \( \alpha \)-phase’s mobility \( M_\alpha \) and the total mobility are defined by

\[
M_\alpha(s_\alpha) = k_{\tau_\alpha}(s_\alpha)/\mu_\alpha, \quad M(s_l) = M_l(s_l) + M_g(s_g).
\]

This pressure \( p \) can be written as

\[
p = p_g + \bar{p}(s_l) = p_l + \bar{p}(s_l),
\]

with

\[
\frac{d\bar{p}}{ds_l} = -\frac{M_l(s_l)}{M(s_l)} \frac{dp_c}{ds_l} \quad \text{and} \quad \frac{d\bar{p}}{ds_l} = \frac{M_g(s_g)}{M(s_l)} \frac{dp_c}{ds_l}.
\]

We also define the contribution of capillary terms by

\[
\gamma(s_l) = -\frac{M_l(s_l) M_g(s_g)}{M(s_l)} \frac{dp_c}{ds_l} (s_l) \geq 0 \quad \text{and} \quad \mathcal{B}(s_l) = \int_0^{s_l} \gamma(z) dz.
\]
We complete the description of the model (1.6)-(1.7) by introducing boundary conditions and initial conditions. Let \( T > 0 \) be the final time fixed and let be \( \Omega \) a bounded open subset of \( \mathbb{R}^d \) (\( d \geq 1 \)). We set \( Q_T = (0,T) \times \Omega \), \( \Sigma_T = (0,T) \times \partial \Omega \) and we note \( \Gamma_l \) the part of the boundary of \( \Omega \) where the liquid saturation is imposed to one and \( \Gamma_n = \Gamma \setminus \Gamma_l \). The chosen mixed boundary conditions on the pressures are

\[
\begin{align*}
    p_g(t,x) &= p_l(t,x) = 0 \text{ on } (0,T) \times \Gamma_l, \\
    \mathbf{V}_l \cdot \mathbf{n} &= \mathbf{V}_g \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_n, \\
    X^w \mathbf{D}^h \nabla p_g \cdot \mathbf{n} &= 0 \text{ on } (0,T) \times \Gamma_n,
\end{align*}
\]

where \( \mathbf{n} \) is the outward normal to \( \Gamma_n \).

The initial conditions are defined on pressures

\[ p_\alpha(t = 0) = p_\alpha^0 \text{ in } \Omega, \text{ for } \alpha = l, g. \quad (2.3) \]

Next we introduce a classically physically relevant assumptions on the coefficients of the system.

(H1) The porosity \( \phi \in W^{1,\infty}(\Omega) \) and there is two positive constants \( \phi_0 \) and \( \phi_1 \) such that \( \phi_0 \leq \phi(x) \leq \phi_1 \) almost everywhere \( x \in \Omega \).

(H2) There exists two positive constants \( k_0 \) and \( k_\infty \) such that

\[ \|K\|_{(L^\infty(\Omega))^{d \times d}} \leq k_\infty \text{ and } \langle K(x)\xi,\xi \rangle \geq k_0|\xi|^2, \forall \xi \in \mathbb{R}^d. \]

(H3) The functions \( M_l \) and \( M_g \in C^0([0,1],\mathbb{R}^+) \), \( M_\alpha(s_\alpha = 0) = 0 \). and there is a positive constant \( m_0 > 0 \) such that for all \( s_l \in [0,1] \),

\[ M_l(s_l) + M_g(s_g) \geq m_0. \]

(H4) The densities \( \rho_\alpha \) (\( \alpha = l, g \)) are in \( C^1(\mathbb{R}) \), increasing and there exists two positive constants \( \rho_m > 0 \) and \( \rho_M > 0 \) such that

\[ 0 < \rho_m \leq \rho_\alpha(p_\alpha) \leq \rho_M, \ \alpha = l, g. \]

(H5) The capillary pressure fonction \( p_c \in C^1([0,1];\mathbb{R}^+) \) and there exists \( \overline{p_c} > 0 \) such that \( 0 < \overline{p_c} \leq |\frac{dp_c}{ds_l}|. \)

(H6) The functions \( r_\omega, r_g \in L^2(Q_T) \) and \( r_\omega, r_g \geq 0 \) a.e. for all \( (t,x) \in Q_T \).
(H7) The diffusion-dispersion tensor $D^h_t$ (function of $x$ and $s_t$) is a nonlinear continuous function of the liquid saturation $s_t$ and is bounded for $x \in \Omega$ and $s_t \in [0, 1]$. In addition, there exist a constant $d^* > 0$ such that $\forall v \in \mathbb{R}^d$, $\forall x \in \Omega$, $\forall s_t \in [0, 1]$, $\langle D^h_t(x, s_t)v, v \rangle \geq d^*||v||^2$.

(H8) The function $\gamma \in C^1([0, 1]; \mathbb{R}^+)$ satisfies $\gamma(s) > 0$ for $0 < s < 1$ and $\gamma(0) = \gamma(1) = 0$. We assume that $B^{-1}$ (the inverse of $B(s_t) = \int_0^{s_t} \gamma(z)dz$) is an Hölder function of order $\theta$, with $0 < \theta \leq 1$, on $[0, B(1)]$.

Let us define the following Sobolev space

$$H^1_Γ(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } Γ\}$$

this is an Hilbert space with the norm $||u||_{H^1_Γ(\Omega)} = ||\nabla u||_{(L^2(\Omega))^d}$.

Let us state the main result of this paper

**Theorem 1.** Let (H1)-(H8) hold and let the initial conditions $(p^0_0, p^0_l)$ belongs in $L^2(\Omega) \times L^2(\Omega)$ with $0 \leq s^0_0 \leq 1$. Then there exists a solution $(p_l, p_l)$ satisfying

$$p_l \in L^2(0, T; H^1_Γ(\Omega)) \text{ and } \sqrt{M_\alpha(s \alpha)}\nabla p_\alpha \in L^2(Q_T),$$

$s_t \geq 0$ a.e. in $Q_T$, $B(s_t) \in L^2(0, T; H^1(\Omega))$,

$$\Phi_\delta(p^h_l(p_l)m(s_t)) \in L^2(0, T; (H^1_Γ(\Omega))^\prime)$, $\Phi_\delta s_t \in L^2(0, T; (H^1_Γ(\Omega))^\prime)$, (2.6)

in the sense that for all $\varphi, \psi \in C^1(0, T; H^1_Γ(\Omega))$ with $\varphi(T, \cdot) = \psi(T, \cdot) = 0$, (2.7)

$$\begin{align*}
-\int_{Q_T} \Phi m(s_t)p^h_l(p_l)\partial_t \varphi dxdt &+ \int_{Q_T} K p^h_l(p_l)M_l(s_t)(\nabla p_l - \rho_l(p_l)g) \cdot \nabla \varphi dxdt \\
&+ C_1 \int_{Q_T} K p^h_l(p_l)M_l(s_t)(\nabla p_l - \rho_l(p_l)g) \cdot \nabla \varphi dxdt \\
&+ \int_{Q_T} C_2 X_l^w D^h_l \nabla p_l \cdot \nabla \varphi dxdt = \int_{Q_T} r_g \varphi dxdt \\
-\int_{Q_T} \Phi s_t \partial_t \psi dxdt &+ \int_{Q_T} \Phi s^0_t \psi(0, x) dx \\
&+ \int_{Q_T} K M_l(s_t)(\nabla p_l - \rho_l(p_l)g) \cdot \nabla \psi dxdt = \int_{Q_T} \frac{r_\omega}{\rho^0_l} \psi dxdt
\end{align*}$$

\[2\] This means that there exists a positive constant $b$ such that for all $a, b \in [0, B(1)]$, one has $|B^{-1}(a) - B^{-1}(b)| \leq b|a - b|^\theta$. 

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and the initial conditions are satisfied in the sense that for all \( \xi \in H^1_\Gamma(\Omega) \), the functions \( t \to \int_\Omega \Phi \rho^h_l(p_g)m(s_l)\xi dx \), and \( t \to \int_\Omega \Phi s_l\xi dx \), are in \( C^0([0,T]) \).

Furthermore, we have
\[
\left( \int_\Omega \Phi \rho^h_l(p_g)m(s_l)\xi dx \right)(0) = \int_\Omega \phi \rho^h_l(p_0^0)m(s_0^0)\xi dx,
\]
\[
\left( \int_\Omega \Phi s_l\xi dx \right)(0) = \int_\Omega \Phi s_0^0\xi dx.
\]

**Remark 1.** Remark that, the solutions obtained in the above theorem do not satisfy that \( s_l \leq 1 \), then the functions depend on the saturation are extended by continuity for \( s_l \geq 1 \).

### 3. Energy estimates

The notion of weak solutions is very natural provided that we explain the origin of the requirements (2.4)–(2.6). In this section, we give estimates on the gradient of the global pressure and on the gradient of the capillary term \( B \). In order to obtain these estimations, we define \( g_g \) and \( H_g \) by
\[
g_g(p_g) := \int_0^{p_g} \frac{1}{\rho^h_l(z)} dz \quad \text{and} \quad H_g(p_g) := \rho^h_l(p_g)g_g(p_g) - p_g.
\]

The function \( H_g \) verifies \( H'_g(p_g) = (\rho^h_l(p_g))' g_g(p_g) \), \( H_g(0) = 0 \), \( H_g(p_g) \geq 0 \) for all \( p_g \) and \( H_g \) is sublinear with respect to \( p_g \). This kind of function is introduced in [16, 17, 18].

By multiplying (1.6) by \( g_g(p_g) \) and (1.7) by \( C_1 p_l - p_g \), after integration and summation of equations, we deduce the equality
\[
\begin{align*}
\int_\Omega \Phi \left[ \partial_t \left( m(s_l)\rho^h_l(p_g) \right) g_g(p_g) + \partial_t s_l \left( C_1 p_l - p_g \right) \right] dx \\
+ \int_\Omega K M_l(s_l)(\nabla p_l - \rho_l(p_l)g) \cdot \nabla p_l dx \\
+ C_1 \int_\Omega K M_g(s_g)(\nabla p_g - \rho_g(p_g)g) \cdot \nabla p_g dx \\
+ \int_\Omega C_2 X^w_i D^h_l \nabla p_g \cdot \nabla p_g dx = \int_\Omega \left( r_g g_g(p_g) + \frac{r_w}{\rho^*_l(C_1 p_l - p_g)} \right) dx.
\end{align*}
\]
To treat the first term of the above equality, let
\[ M = \partial_t \left( m(s_l) \rho^h_i(p_g) \right) g_g(p_g) + \partial_t s_l \left( C_1 \ p_l - p_g \right) \]
\[ = \partial_t \left( m(s_l) \rho^h_i(p_g) g_g(p_g) \right) + \partial_t \left( s_l(C_1 p_l - p_g) \right) + C_1 s_l \partial_t(p_g - p_l) - C_1 \partial_t p_g \]
\[ = \partial_t \left( m(s_l) \rho^h_i(p_g) g_g(p_g) \right) + \partial_t \left( s_l(C_1 p_l - p_g) \right) + C_1 s_l \partial_t(p_c) - C_1 \partial_t p_g. \]

Consider \( N \) a primitive of \( s_i p'_c(s_i) \). We can write \( M = \partial_t \mathcal{E} \) where \( \mathcal{E} \) is defined by
\[ \mathcal{E} = m(s_l) \rho^h_i(p_g) g_g(p_g) + s_l(C_1 p_l - p_g) + C_1 N(s_l) - C_1 p_g \]
\[ = m(s_l) \left( \rho^h_i(p_g) g_g(p_g) - p_g \right) - C_1 s_i p_c(s_i) + C_1 N(s_l). \]

From the definition of the functions \( \mathcal{H}_g \) and \( N \), the expression of \( \mathcal{E} \) is equivalent to
\[ \mathcal{E} = m(s_l) \mathcal{H}_g(p_g) - C_1 \int_0^{s_i} p_c(z) dz. \]

Integrate (3.1) over \((0, T)\), we deduce by using the assumptions \((H1)-(H8)\), the positivity of \( \mathcal{H}_g \) and the sub-linearity of \( g_g(p_g) \), that
\[ \int_{Q_T} M_l |\nabla p_l|^2 dx dt + \int_{Q_T} M_g |\nabla p_g|^2 dx dt \]
\[ + \int_{Q_T} c_2 X_i^w D_t^h \nabla p_g \cdot \nabla p_g dx dt \leq C \left( 1 + \|p_l\|_{L^2(Q_T)} + \|p_g\|_{L^2(Q_T)} \right), \quad (3.2) \]
where \( C > 0 \) is constant. In term of global pressure, from the relation (2.1), we have the fundamental equality
\[ M |\nabla p|^2 + \frac{M_l M_g}{M} |\nabla p_c|^2 = M_l |\nabla p_l|^2 + M_g |\nabla p_g|^2, \quad (3.3) \]
The relation (2.2) between the global pressure and the pressure of each phase prove the following inequality
\[ \|p_l\|_{L^2(Q_T)} + \|p_g\|_{L^2(Q_T)} \leq \|p\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)} \]
\[ \leq C \|\nabla p\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)} + \|\tilde{p}\|_{L^2(Q_T)}. \]
The above inequality and the equality (3.3) combined to the estimate (3.2) ensures that \( p, p_g \in L^2(0, T; H^1_0(\Omega)) \) and \( B(s_l) \in L^2(0, T; H^1(\Omega)) \).
4. Construction of a regularized system

Before establishing Theorem 1, we introduce the existence of regularized solutions to system (1.1)–(1.2). First we are interested in a non-degenerate system by adding a dissipative term on saturation preserving the positivity of the liquid saturation. Precisely, we consider the non-degenerate system:

\[
\begin{aligned}
\mathcal{P}_\eta \left\{ 
\begin{array}{l}
\partial_t \left( \Phi m(s^n) \rho^h(p^n) - \text{div} \left( K \rho^h(p^n) M_l(s^n) (\nabla p^n - \rho_l(p^n) g) \right) \right ) \\
- C_1 \text{div} \left( K \rho^h(p^n) M_g(s^n) \left( \nabla p^n - \rho_g(p^n) g \right) \right ) - \text{div} \left( C_2 (X_l^w)(D_l^h)^{\eta} \nabla p_g \right ) \\
+ (C_1 - 1) \eta \text{div} (\rho^h(p^n) \nabla (p^n - p_l^n)) = r_g, \\
\partial_t \left( \Phi s^n \right ) - \text{div} \left( K M_l(s^n) (\nabla p^n - \rho_l(p^n) g) \right ) - \eta \text{div} (\nabla (p^n - p_l^n)) = \frac{r_w}{\rho_l^n}, \\
\end{array}
\right.
\end{aligned}
\]

completed with the initial conditions \((2.3)\), and the following mixed boundary conditions,

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
p^n(t, x) = p^0_n(t, x) = 0 \quad \text{on } (0, T) \times \Gamma, \\
\left( V^n_l + C_1 V^n_g + C_2 X_l^w D_l^h \nabla p_g + (C_1 - 1) \eta \nabla (p^n - p_l^n) \right ) \cdot n = 0 \quad \text{on } (0, T) \times \Gamma_n \\
\left( V^n_l - \eta \nabla (p^n - p_l^n) \right ) \cdot n = 0 \quad \text{on } (0, T) \times \Gamma_n
\end{array}
\right.
\end{aligned}
\tag{4.1}
\]

where \(n\) is the outward normal to the boundary \(\Gamma_n\) and \(V^n_\alpha = -K M_\alpha(s^n_\alpha)(\nabla p^n_\alpha - \rho_\alpha(p^n_\alpha) g)\).

We state the existence of solutions of the above system \((\mathcal{P}_\eta)\).

**Theorem 2.** Let \((H1)\)–\((H8)\) hold. Let \(p^0_g, p^0_l \in L^2(\Omega), 0 \leq s^0_l(x) \leq 1\). Then, for all \(\eta > 0\), there exists \((p^n_g, p^n_l)\) satisfying

\[
\begin{aligned}
p^n_\alpha &\in L^2(0, T; H^1_{\Gamma_l}(\Omega)), \quad \Phi \partial_t (\rho^h(p^n) m(s^n)) \in L^2(0, T; (H^1_{\Gamma_l}(\Omega))'), \\
s^n_l &\geq 0 \ a.e. \ in \ Q_T, \quad s^n_l \in L^2(0, T; H^1(\Omega)), \quad \Phi \partial_t s^n_l \in L^2(0, T; (H^1_{\Gamma_l}(\Omega))'), \\
\rho^h(p^n) m(s^n) &\in C^0 \left([0, T]; L^2(\Omega)\right), \quad s^n_l \in C^0 \left([0, T]; L^2(\Omega)\right).
\end{aligned}
\]
such that for all $\varphi, \psi \in L^2(0, T; H^1_1(\Omega))$

$$
\langle \Phi \partial_t (m(s^n_\eta) \rho_h^b(p^n_\eta)), \varphi \rangle + \int_{Q_T} K \rho_h^b(p^n_\eta) M_t(s^n_\eta) (\nabla p^n_\eta - \rho_l(p^n_\eta)g) \cdot \nabla \varphi dx dt
+ C_1 \int_{Q_T} K \rho_h^b(p^n_\eta) M_g(s^n_\eta) (\nabla p^n_\eta - \rho_g(p^n_\eta)g) \cdot \nabla \varphi dx dt
+ \int_{Q_T} C_2 (X^w_l(D^h_l)^\eta) \nabla p^n_\eta \cdot \nabla \varphi dx dt + (C_1 - 1) \eta \int_{Q_T} \rho_h^b(p^n_\eta) \nabla (p^n_\eta - p^n_l) \cdot \nabla \varphi dx dt
= \int_{Q_T} r_g \varphi dx dt,
$$

(4.2)

$$
\langle \Phi \partial_t s^n_\eta, \psi \rangle + \int_{Q_T} K M_t(s^n_\eta) (\nabla p^n_\eta - \rho_l(p^n_\eta)g) \cdot \nabla \psi dx dt
- \eta \int_{Q_T} \nabla (p^n_\eta - p^n_l) \cdot \nabla \psi dx dt = \int_{Q_T} \frac{r_\omega}{\rho_l^w} \psi dx dt,
$$

(4.3)

where the bracket $\langle ., . \rangle$ is the duality product between $L^2(0, T; (H^1_1(\Omega))')$ and $L^2(0, T; H^1_1(\Omega))$.

For the sake of clarity, we omit the index $\eta$ in the problem $(\mathcal{P}_\eta)$. The sequel of this section is devoted to the proof of Theorem 2. The existence of solution of non-degenerated model $(\mathcal{P}_\eta)$ is splitted in three steps. The first one based on approached solutions solving a time discrete system with non-degenerate mobilities. For that, let be $T > 0$, $M \in \mathbb{N}^*$ a number of time step, $\delta t = T/M$ the time step and let initialize a sequence parametrized by $\delta t$ with the initial condition $p^n_\alpha$. Then, if we consider $(p^{n,\epsilon}_g, p^{n,\epsilon}_l) \in (L^2(\Omega))^2$ with $\rho_h^b(p^{n,\epsilon}_g)m(s^{n,\epsilon}_l) \geq 0$ and $s^{n,\epsilon}_l \geq 0$ at time $t_n = n\delta t$, we are searching for
a solution \((p_g^{n+1,\epsilon}, p_l^{n+1,\epsilon})\) of the following system

\[
\begin{align*}
\mathfrak{P}_{\eta, \delta t} \phi & \left( m(Z(s_t^{n+1,\epsilon})) \rho_t^h(p_g^{n+1,\epsilon}) - m(s_t^{n,\epsilon}) \rho_t^h(p_g^{n,\epsilon}) \right) \frac{\delta t}{\phi} \\
& - \text{div} \left( K \rho_t^h(p_g^{n+1,\epsilon}) (M_t'(s_t^{n+1,\epsilon}) \nabla p_t^{n+1,\epsilon} - M_t(s_t^{n+1,\epsilon}) \rho_t(p_t^{n+1,\epsilon}) g) \right) \\
& - C_1 \text{div} \left( K \rho_t^h(p_g^{n+1,\epsilon}) (M_t'(s_t^{n+1,\epsilon}) \nabla p_g^{n+1,\epsilon} - M_t(s_t^{n+1,\epsilon}) \rho_g(p_g^{n+1,\epsilon}) g) \right) \\
& + (C_1 - 1) \eta \text{div} \left( \rho_t^h(p_g^{n+1,\epsilon}) \nabla (p_g^{n+1,\epsilon} - p_t^{n+1,\epsilon}) \right) \\
& - \text{div} \left( C_2(X_t^w)^{n+1,\epsilon} D_h^\eta \nabla p_g^{n+1,\epsilon} \right) = \frac{r^{n+1}}{\rho_t^w},
\end{align*}
\]

where \(M_\alpha' = M_\alpha + \epsilon\), with \(\epsilon > 0\), with the boundary conditions (4.4). The regularization of the mobilities lead to the loss of the positivity on the liquid saturation. So, the functions \(M_\alpha\) and \(Z\) are extended on \(\mathbb{R}\) by continuity outside \([0, 1]\).

This technique of semi-discretization method in time has been used by Alt and Luckhaus [1] for degenerate parabolic system and has been employed in [15, 17, 5] for a porous medium. A Leray-Schauder’s fixed point theorem [22] allows to define a solution \((p_g^{n+1,\epsilon}, p_l^{n+1,\epsilon})\) for the system \(\mathfrak{P}_{\eta, \delta t}\).

The second step is devoted to pass to the limit when \(\epsilon\) goes to zero and to prove the positivity of the liquid pressure. A uniform estimate (with respect to \(\epsilon\)) based on the scalar product of (4.4) with \(g_g(p_g) := \int_0^{p_g} \frac{1}{\rho_t^g(z)} dz\) and (4.5) with \(\psi = C_1 p_t - p_g\) ensures by using (2.2) and (3.3) that

\[
(p^\epsilon, p_t^\epsilon) \quad \text{is uniformly bounded in } H^1_0(\Omega),
\]

\[
(B(s_t^\epsilon))_\epsilon \quad \text{is uniformly bounded in } H^1(\Omega),
\]

\[
(\nabla p^\epsilon(s_t^\epsilon))_\epsilon \quad \text{is uniformly bounded in } L^2(\Omega).
\]

Up to a subsequence, the sequences \((s_\alpha^\epsilon)_\epsilon, (p^\epsilon)_\epsilon, (p_t^\epsilon)_\epsilon\), verify the following
Then, pass to the limit as $\epsilon$ goes to zero in formulations (4.4)-(4.5) to obtain $(p_g, p_l) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ solution of

\[
\int_\Omega \frac{m(Z(s_l))\rho_l^h(p_g) - \rho^* m(s_l^*)}{\delta t} \varphi \, dx + \int_\Omega \mathbf{C}_2 \mathbf{X}_l^w 
abla p_g \cdot \nabla \varphi \, dx
\]
\[
+ \int_\Omega K \rho_l^h(p_g) M_l(s_l) (\nabla p_l - \rho_l(p_l) \mathbf{g}) \cdot \nabla \varphi \, dx
\]
\[
+ C_1 \int_\Omega K \rho_l^h(p_g) M_g(s_g) (\nabla p_g - \rho_g(p_g) \mathbf{g}) \cdot \nabla \varphi \, dx
\]
\[
+ (C_1 - 1) \eta \int_\Omega \rho_l(p_g) \nabla (p_g - p_l) \cdot \nabla \varphi \, dx = \int_\Omega r_g \varphi \, dx,
\]
\[
\int_\Omega \frac{Z(s_l) - s_l^*}{\delta t} \psi \, dx + \int_\Omega K M_l(s_l) (\nabla p_l - \rho_l(p_l) \mathbf{g}) \cdot \nabla \psi \, dx
\]
\[
- \eta \int_\Omega \nabla (p_g - p_l) \cdot \nabla \psi \, dx = \int_\Omega r^w \psi \, dx,
\]

to prove, that the liquid saturation is positive, we consider $\psi = (s_l)^-$ in (4.7) and according to the extension of the mobility of each phase $(M_l(s_l) s_l^- = 0)$ and the fact that $(Z(s_l) s_l^- = 0)$, we deduce that $s_l \geq 0$.

The third step is devoted to pass to the limit as $\delta t$ goes to zero to prove the existence of a solution of the problem $(\mathfrak{P}_\eta)$. For this, we will show some uniform estimates with respect to $\delta t$ to obtain uniformly bounded on some quantities.

The next lemma gives us some uniform estimates with respect to $\delta t$.

**Lemma 1.** (Uniform estimates with respect to $\delta t$) The solution of (3.4) —
Let forget the exponent $n + 1$ in this proof and let denote with the exponent $\ast$ the physical quantities at time $t_n$. So, since $g_g$ is concave ($g''_g(p) \leq 0$), we have

$g_g(p_g) \leq g_g(p^\ast_g) + g'_g(p^\ast_g)(p_g - p^\ast_g),$

and from the definition of $H_g$, one gets

$$(\rho^h_t(p_g)m(s) - \rho^h_t(p^\ast_g)m(s^\ast))g_g(p_g) + (s - s^\ast) (C_1p_l - p_g)$$

$$
\geq H_g(p_g)m(s) - H_g(p^\ast_g)m(s^\ast) - C_1(s - s^\ast)p_c(s). \tag{4.9}
$$

Using the concavity of $P_c$ we have the inequality : $(s - s^\ast)p_c(s) \leq P_c(s) - P_c(s^\ast),$ and the above inequality (4.9), we obtain the following inequality

$$(\rho^h_t(p_g)m(s) - \rho^h_t(p^\ast_g)m(s^\ast))g_g(p_g) + (s - s^\ast) (C_1p_l - p_g)$$

$$
\geq H_g(p_g)m(s) - H_g(p^\ast_g)m(s^\ast) - C_1P_c(s) + C_1P_c(s^\ast). \tag{4.10}
$$

Now, to obtain the inequality (4.8), we just have to multiply (4.6) by $g_g(p_n^{n+1})$ and (4.7) by $(C_1p_n^{n+1} - p_n^{n+1})$, sum this two equations and use the inequality (4.10). \hfill \Box

The limit as $\delta t$ goes to zero is similar to the limit as $\epsilon$ goes to zero with additional difficulties on time derivative terms which are overcome in the same manner as in [15, 17, 18].
5. Existence of solutions of the degenerate system

We have shown in the previous section 4 the existence of a solution \((p^n_\eta, p^g_\eta)\) of the problem \(\mathcal{P}_\eta\). The aim of this section is to pass to the limit as \(\eta\) goes to the zero to prove the main result of this paper.

The first point to do this is to obtain uniform energy estimates with respect to \(\eta\). The second point is devoted to get uniform estimates on space and time translates which provide compactness results on solution by virtue of Kolmogorov’s theorem. Next, we will be able to pass to the limit as \(\eta\) goes to zero.

Now, we state the following two lemmas in order to establish uniform estimates with respect to \(\eta\).

**Lemma 2.** The sequences \((s^n_\eta)\) and \((p^n_\eta)\) satisfy

\[
\begin{align*}
&\text{\(s^n_\eta \geq 0\), almost everywhere in} \ Q_T, \quad (5.1) \\
&(\rho^n_\eta, (p^n_\eta)_\eta) \quad \text{is uniformly bounded in} \ L^2(0, T; H^1_{\Gamma_1}(\Omega)), \quad (5.2) \\
&(\sqrt{\eta} \nabla p^n_\eta(s^n_\eta))_\eta \quad \text{is uniformly bounded in} \ L^2(Q_T), \quad (5.3) \\
&(\Phi \partial_t (\rho^n_\eta(p^n_\eta)m(s^n_\eta)))_\eta \quad \text{is uniformly bounded in} \ L^2(0, T; (H^1_{\Gamma_1}(\Omega))'), \quad (5.4) \\
&(\Phi \partial_t (s^n_\eta))_\eta \quad \text{is uniformly bounded in} \ L^2(0, T; (H^1_{\Gamma_1}(\Omega))'). \quad (5.5)
\end{align*}
\]

**Proof.** The positivity of the saturation \((5.1)\) is conserved through the limit process. For the next four estimates, we just have to multiply \((4.2)\) by \(g_\eta(p^n_\eta) = \int_0^{p^n_\eta} \frac{1}{\rho^2(z)}dz\) and \((4.3)\) by \(C_1p^n_\eta - p^n_\eta\) and adding them. We follow the same calculation as in section 3 to provide the energy estimates \((5.2)-(5.5)\).

For all \(\varphi, \psi \in L^2(0, T; H^1_{\Gamma_1}(\Omega))\) and by using the formulation \((4.2)-(4.3)\) with the relation \((2.2)\) between the pressure of each phase and the global
pressure, one gets

\[
\left| \langle \Phi \partial_t (\rho_l^h (p_g^n) m(s^n_l)), \varphi \rangle \right| \leq (C_1 - 1) \eta \int_{Q_T} \rho_l^h (p_g^n) \nabla p_e (s^n_l) \cdot \nabla \varphi \, dx dt
\]

\[
+ \left| \int_{Q_T} K \rho_l^h (p_g^n) (M_l(s^n_l) \nabla p^n + \nabla \mathcal{B}(s^n_l)) \cdot \nabla \varphi \, dx dt \right|
\]

\[
+ \left| \int_{Q_T} X^w_1 D_1 (\rho_l^h (p_g^n)) \nabla p_g \cdot \nabla \varphi \, dx dt \right| + \left| \int_{Q_T} r_g \varphi \, dx dt \right|
\]

and

\[
\left| \langle \Phi \partial_t (s^n_l), \psi \rangle \right| \leq C_1 \eta \int_{Q_T} \nabla p_e (s^n_l) \cdot \nabla \psi \, dx dt
\]

\[
+ \left| \int_{Q_T} K (M_l(s^n_l) \nabla p^n + \nabla \mathcal{B}(s^n_l)) \cdot \nabla \psi \, dx dt \right|
\]

\[
+ \left| \int_{Q_T} \frac{r_w}{\rho_w} \psi \, dx dt \right|
\]

where the bracket \( \langle \cdot, \cdot \rangle \) represents the duality product between \( L^2(0, T; (H^1_{\Gamma_1}(\Omega))^\prime) \) and \( L^2(0, T; H^1_{\Gamma_1}(\Omega)) \).

From the estimations (5.2) and (5.5), we deduce

\[
\left| \langle \Phi \partial_t (\rho_l^h (p_g^n) m(s^n_l)), \varphi \rangle \right| \leq C \| \varphi \|_{L^2(0, T; H^1_{\Gamma_1}(\Omega))},
\]

and

\[
\left| \langle \Phi \partial_t (s^n_l), \varphi \rangle \right| \leq C \| \psi \|_{L^2(0, T; H^1_{\Gamma_1}(\Omega))},
\]

which establishes (5.6)–(5.7) and proves Lemma 2.

In the next lemma, we derive estimates on differences of space and time translates of the function \( U^n = \rho_l^h (p_g^n) m(s^n_l) \) which imply that the sequence \( (\rho_l^h (p_g^n) m(s^n_l))_n \) is relatively compact in \( L^1(Q_T) \).

**Lemma 3. (Space and time translate of U).** Under the assumptions (H1) – (H8), the following inequalities hold:

\[
\int_{\Omega \times (0, T)} |U^n(t, x + y) - U^n(t, x)| \, dx dt \leq \omega(|y|), \quad (5.8)
\]

\[
\int_{\Omega \times (0, T - \tau)} |U^n(t + \tau, x) - U^n(t, x)| \, dx dt \leq \tilde{\omega}(\tau), \quad (5.9)
\]

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for all \( y \in \mathbb{R}^3 \) and for all \( \tau \in (0, T) \); with \( \Omega' = \{ x \in \Omega, x + y \in \Omega \} \) and the function \( \omega \) and \( \tilde{\omega} \) are continuous, independent of \( \eta \) and satisfying \( \lim_{|y| \to 0} \omega(|y|) = 0 \) and \( \lim_{\tau \to 0} \tilde{\omega}(\tau) = 0 \).

**Proof.** For the space translates, we observe that

\[
\omega \lim
\]

\[
\text{Proof. For the space translates, we observe that}
\]

\[
\int_{(0,T) \times \Omega'} |U^{\eta}(t, x + y) - U^{\eta}(t, x)| \, dx \, dt
\]

\[
= \int_{(0,T) \times \Omega'} \left| \left( \rho_t^h(p_g^h)m(s_t^h) \right)(t, x + y) - \left( \rho_t^h(p_g^h)m(s_t^h) \right)(t, x) \right| \, dx \, dt
\]

\[
\leq \int_{(0,T) \times \Omega'} \left| m(s_t^h)(t, x + y) - \rho_t^h(p_g^h(t, x + y)) \rho_t^h(p_g^h(t, x)) \right| \, dx \, dt
\]

\[
+ \int_{(0,T) \times \Omega'} \left| \rho_t^h(p_g^h(t, x) - m(s_t^h)(t, x + y) - m(s_t^h)(t, x)) \right| \, dx \, dt
\]

\[
\leq E_1 + E_2,
\]

where \( E_1 \) and \( E_2 \) defined as follows

\[
E_1 = \rho_M \int_{(0,T) \times \Omega'} \left| s_t^h(t, x + y) - s_t^h(t, x) \right| \, dx \, dt, \tag{5.10}
\]

\[
E_2 = \int_{(0,T) \times \Omega'} \left| \rho_t^h(p_g^h(t, x + y)) - \rho_t^h(p_g^h(t, x)) \right| \, dx \, dt. \tag{5.11}
\]

To handle with the space translates on saturation, we use the fact that \( B^{-1} \) is an Hölder function, applying the Cauchy-Schwarz inequality and from (5.5), we deduce

\[
E_1 \leq C \left[ \int_{(0,T) \times \Omega'} \left| B(s_t^h(t, x + y)) - B(s_t^h(t, x)) \right| \, dx \, dt \right]^\theta
\]

\[
\leq C \left[ \int_0^T \int_{\Omega} \left( \int_0^1 \nabla B(s_t^h(t, x + ry)) \, dy \, dr \right) \, dx \, dt \right]^\theta \tag{5.12}
\]

\[
\leq C \left[ \int_0^T \int_{\Omega} \left( \int_0^1 \left| \nabla B(s_t^h(t, x + ry)) \right|^2 \, dy \, dr \right)^{\frac{1}{2}} \, dx \, dt \right]^\theta
\]

\[
\leq C |y|^\theta.
\]

To treat the space translates of \( E_2 \), we use the relationship between the gas pressure and the global pressure, namely : \( p_g = p - \tilde{p} \) defined in (2.2), then,
from the estimation on the global pressure (5.2) and the estimate (5.12) we have
\[ E_2 \leq C(|y| + |y|^2). \]

Define \( V^n = \Phi U^n \). From assumption (H1) on the porosity, we deduce the space translates on \( V^n \). The proof of the time translates of \( V^n \) can be found in [11] for more details.

From the previous two lemmas, we deduce the following convergences.

**Lemma 4.** (Strong and weak convergences). Up to a subsequence the sequence \((s^n_\eta)_\eta\), \((p^n := p^n_\eta + \mathcal{B}(s^n_\eta))_\eta\) and \((p^n_\alpha)_\eta\) verify the following convergences

\[
\begin{align*}
p^n_\eta &\longrightarrow p & \text{weakly in } L^2(0,T;H^1_\Gamma(\Omega)), \\
\mathcal{B}(s^n_\eta) &\longrightarrow \mathcal{B}(s_l) & \text{weakly in } L^2(0,T;H^1(\Omega)), \\
p^n_\eta &\longrightarrow p_g & \text{weakly in } L^2(0,T;H^1_\Gamma(\Omega)), \\
s^n_\eta &\longrightarrow s_l & \text{strongly in } L^2(Q_T), \text{ a.e. in } Q_T, \\
s_l &\geq 0 & \text{almost everywhere in } Q_T, \\
p^n_\alpha &\longrightarrow p_\alpha & \text{almost everywhere in } Q_T, \\
\Phi \partial_t(U^n) &\longrightarrow \Phi \partial_t(\rho^h(p_g)m(s_l)) & \text{weakly in } L^2(0,T;H^1_\Gamma(\Omega)), \\
\Phi \partial_t(s^n_\eta) &\longrightarrow \Phi \partial_t s_l & \text{weakly in } L^2(0,T;H^1_\Gamma(\Omega)),
\end{align*}
\]

where \( U^n = \rho^h(p^n_\alpha)m(s^n_\eta) \).

**Proof.** The weak convergences (5.13)–(5.15) follows from the uniform estimates (5.2) and (5.5) of lemma 2.

By the Riesz-Frechet-Kolmogorov compactness criterion, the relative compactness of \( V^n \) in \( L^1(Q_T) \) is a consequence of Lemma 3 and then ensures the following strong convergences

\[
\Phi \rho^h(p^n_\alpha)m(s^n_\eta) \longrightarrow l \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T,
\]

and consequently

\[
\rho^h(p^n_\alpha)m(s^n_\eta) \longrightarrow U = l/\Phi \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T.
\]

In order to prove the convergence (5.16), we reproduce the previous Lemma 3 for \( V^n = \Phi s^n_\eta \) and as an application of the Riesz-Frechet-Kolmogorov compactness criterion we establish (5.16). And from (5.16), we deduce (5.17).
The convergence (5.21) combined with (5.16), to prove that
\[ p^\eta \rightarrow p_g \text{ a.e. in } Q_T, \]
then the convergence (5.18) for \( \alpha = g \) is then established. And again as consequence of (5.16) with the capillary pressure law, we deduce (5.18) for \( \alpha = l \). At last, the weak convergence (5.19) and (5.20) is a consequence of the estimate (5.6) and (5.7).

\[ \square \]

In order to achieve the proof of Theorem 1, it remains to pass to the limit as \( \eta \) goes to zero in the formulations (4.2)–(4.3), for all smooth test functions \( \varphi \) and \( \psi \) in \( C^1([0,T];H^1_\Gamma(O)) \) such that \( \varphi(T) = \psi(T) = 0 \),

\[
- \int_{Q_T} \Phi \rho^h(p^\eta_g) m(s^\eta_l) \partial_t \varphi dx dt + \int_{Q_T} C_2 X^w_i D^h_i \nabla p_g \cdot \nabla \varphi dx dt \\
+ \int_{Q_T} K \rho^h(p^\eta_g) M_i(s^\eta_l) (\nabla p^\eta_g - \rho_l(p_l) g) \cdot \nabla \varphi dx dt \\
+ C_1 \int_{Q_T} K \rho^h(p^\eta_g) M_g(s^\eta_l) (\nabla p^\eta_g - \rho_g(p_g) g) \cdot \nabla \varphi dx dt \\
+ (C_1 - 1) \eta \int_{Q_T} \rho^h(p^\eta_g) \nabla (p^\eta_g - p^\eta_l) \cdot \nabla \varphi dx dt \\
= \int_{Q_T} r_g \varphi dx dt + \int_{Q_T} \Phi \rho^h(p^\eta_g) m(s^\eta_l) \varphi(0,x) dx dt,
\]

(5.22)

\[
- \int_{Q_T} \Phi s^\eta_l \partial_t \psi dx dt + \int_{Q_T} K M_i(s^\eta_l) (\nabla p^\eta_g - \rho_l(p_l) g) \cdot \nabla \psi dx dt \\
- \eta \int_{Q_T} \nabla (p^\eta_g - p^\eta_l) \cdot \nabla \psi dx dt = \int_{Q_T} r\psi dx dt + \int_{Q_T} \Phi s^\eta_l \psi(0,x) dx dt,
\]

(5.23)

The first term in (5.22) and (5.23) converge due to the strong convergence of \( \rho^h(p^\eta_g)m(s^\eta_l) \) to \( \rho^h(p_g)m(s_l) \) in \( L^2(Q_T) \) and the strong convergence of \( s^\eta_l \) to \( s_l \) in \( L^2(Q_T) \).
The third and fourth term in (5.22) can be written as,

$$\int_{Q_T} K M_\alpha(s_\alpha^n) \rho_\alpha(p_\alpha^n) \nabla p_\alpha^n \cdot \nabla \phi dx dt = \int_{Q_T} K M_\alpha(s_\alpha^n) \rho_\alpha(p_\alpha^n) \nabla p_\alpha^n \cdot \nabla \phi dx dt$$

$$+ \int_{Q_T} K \rho_\alpha(p_\alpha^n) \nabla B(s_\alpha^n) \cdot \nabla \phi dx dt. \quad (5.24)$$

The two terms on the right hand side of the equation (5.24) converge arguing in two steps. Firstly, the Lebesgue theorem and the convergences (5.16) and (5.18), establish

$$\rho_\alpha(p_\alpha^n) M_\alpha(s_\alpha^n) \nabla \phi \longrightarrow \rho_\alpha(p_\alpha) M_\alpha(s_\alpha) \nabla \phi \quad \text{strongly in } (L^2(Q_T))^d,$$

$$\rho_\alpha(p_\alpha^n) \nabla \phi \longrightarrow \rho_\alpha(p_\alpha) \nabla \phi \quad \text{strongly in } (L^2(Q_T))^d.$$  

Secondly, the weak convergence on global pressure (5.13) and the weak convergence (5.14) combined to the above strong convergences allow the convergence for the terms of the right hand side of (5.24). In the same way for the second term of the equation (5.23). The fifth term of equations (5.22) can be written as

$$\eta \int_{Q_T} \rho_b^i(p_g^n) \nabla (p_g^n - p_l^n) \nabla \phi dx dt = \sqrt{\eta} \int_{Q_T} \rho_b^i(p_g^n) (\sqrt{\eta} \nabla p_c(s_l^n)) \nabla \phi dx dt,$$

the Cauchy-Schwarz inequality and the uniform estimate (5.3) ensure the convergence of this term to zero as $\eta$ goes to zero. In the same way for the third term of equation (5.23). The other terms converge using (5.16)–(5.18) and the Lebesgue dominated convergence theorem.

The weak formulations (2.7) and (2.8) are then established. And The main theorem \ref{thm:main} is then established.

6. References

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