BRANES IN THE MODULI SPACE OF FRAMED INSTANTONS

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ABSTRACT. In the physicist’s language, a brane in a hyperkähler manifold is a submanifold which is either complex or lagrangian with respect to three Kähler structures of the ambient manifold.

By considering the fixed loci of certain involutions, we describe branes in Nakajima quiver varieties of all possible types. We then focus on the moduli space of framed torsion free sheaves on the projective plane, showing how the involutions considered act on sheaves, and proving the existence of branes in some cases.

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1. Introduction

Given a Kähler manifold \( Y \) with complex structure \( \Gamma \), Riemannian metric \( \eta \) and symplectic form

\[
\omega(\cdot, \cdot) := \eta(\cdot, \Gamma(\cdot)),
\]

we say that subvariety \( Y' \subset Y \) is a \( A \)-brane if it is lagrangian with respect to \( \omega \), and that it is a \( B \)-brane if it is a complex subvariety.

Now let \( Y \) be a hyperkähler manifold with complex structures \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \) satisfying the usual quaternionic relations, and Riemannian metric \( \eta \); denote

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by $\omega_1$, $\omega_2$ and $\omega_3$ the associated symplectic forms, constructed as in (1.1). A subvariety $Y' \subset Y$ is said to be a brane if it is either an $A$-brane (i.e. lagrangian) or a $B$-brane (complex) with respect to each symplectic form or complex structure. One then specifies the behaviour of $Y'$ by saying that $Y'$ is either a $(B, B, B)$, $(B, A, A)$, $(A, B, A)$ or $(A, A, B)$-brane; note that those are all the possible branes. A $(B, B, B)$-brane is a hyperkähler submanifold of $Y$, since the hyperkähler structure of $Y$ restricts to the brane. On the other hand, a $(B, A, A)$-brane is a complex lagrangian submanifold with respect to the holomorphic symplectic form $\Omega_1 = \omega_2 + i\omega_3$. Similarly, $(A, B, A)$ and $(A, A, B)$-branes correspond with complex lagrangian submanifolds with respect to $\Omega_2$ and $\Omega_3$.

In the groundbreaking paper of Kapustin and Witten [KW], branes in the moduli space of Higgs bundles play an important role in connection with the Geometric Langlands program and Mirror Symmetry. This explains the great interest that the topic has attracted in the recent years, achieving the description of branes within moduli space of Higgs bundles as fixed point loci of involutions associated with reductions of structure group, and real structures [BS1, BS2, BGH, BG, GP1, GP2, H].

In this paper, we study branes in Nakajima quiver varieties, another important class of hyperkähler manifolds introduced in [Na1, Na2]. These varieties are constructed as hyperkähler quotients of the vector space of representations of a given quiver. Using linear algebra transformations (transposition, multiplication by scalars, addition and conjugation) we construct involutions on this vector space. Since these involutions are compatible with the hyperkähler structure (they preserve the Riemannian metric and either commute or anticommute with the three complex structures) and with the action of the group (they preserve the orbits and the preimages of the moment maps), one obtains involutions in the quotient that are compatible with the hyperkähler structure. If non-empty, the fixed point loci of these involutions provide examples of all possible kinds of branes on these hyperkähler manifolds.

We then focus on the study of the Nakajima quiver variety for the Jordan quiver. The ADHM correspondence relates this variety with the moduli space of framed torsion free sheaves over $\mathbb{P}^2$. We describe the involutions previously obtained in terms of transformations of the framed torsion free sheaves:

- In the first case, the involution is described by means of the dual framed sheaf, so one has to restrict to the locally free locus of the moduli space. The fixed points are autodual framed bundles, symplectic and orthogonal, which have been widely studied in the literature, see for example [AB, CO, JMW, O]. We show that the autodual locus is a $(B, B, B)$-brane inside the moduli space of framed torsion free sheaves over $\mathbb{P}^2$.

- The following involutions considered can be understood in terms of pull-backs by unitary holomorphic involutions in $\mathbb{P}^2$ that preserve the line at infinite. Choosing projective coordinates $[x_0 : x_1 : x_2]$ such that the line at infinity is $x_0 = 0$, a unitary involution is constructed with an involutive element $M$ of the unitary group $U(2)$ acting on $x_1$ and $x_2$. The fixed point set is given by those framed locally free sheaves which are isomorphic to the pull-back by the unitary involution. If $\det M = 1$, this gives a $(B, B, B)$-brane, while if $\det M = -1$, one has a $(B, A, A)$-brane.

- The last involutions considered are given by the composition of the complex conjugation of the framed sheaf and the pull-back by an antiholomorphic (unitary) involution of $\mathbb{P}^2$. Note that the composition of this two operations is a functor within the holomorphic category. In this case, if $\det M = 1$, the
fixed point set is a \((A, B, A)\)-brane, and if \(\det M = -1\), it is a \((A, A, B)\)-brane.

For particular values of the rank and the charge, we prove the existence of fixed points showing the non-emptiness of the branes considered.

2. Preliminaries

We begin by revising the construction of Nakajima quiver varieties and describing how to construct branes on hyperkähler quotients using involution on the covering hyperkähler manifold.

2.1. The hyperkähler structure of Nakajima quiver varieties. Let \(Q\) be a quiver with set of vertices \(\Delta\) and set of arrows \(\Sigma\). Define the framed quiver \(Q^\circ\) as the quiver whose vertex set is \(\Delta \sqcup \Delta'\), where \(\Delta'\) is another copy of the vertex set \(\Delta\), and whose arrow set is \(\Sigma^\circ = \Sigma \sqcup \Sigma_{\Delta}\), where \(\Sigma_{\Delta}\) denotes \(\{(i, i')\} \text{ for each } i \in \Delta\).

For each arrow \(a \in \Sigma\) with head \(\text{head}(a) \in \Delta\) and tail \(\text{tail}(a) \in \Delta\), construct another arrow \(\tilde{a}\) with head \(\text{tail}(a)\) and tail \(\text{head}(a)\). Consider the opposite quiver \(Q^\vee\) to be the quiver with the same set of vertices \(\Delta\) and whose set of arrows is \(\Sigma^\vee = \{\tilde{a} : a \in \Sigma\}\). We also define the double quiver \(\bar{Q}\) as the quiver with vertex set \(\Delta\) and arrow set \(\Sigma \sqcup \Sigma^\vee\).

Let \(V\) and \(W\) be two collections of complex vector spaces defined as \(V = \{V_i\}_{i \in \Delta}\) and \(W = \{W_i\}_{i \in \Delta}\). Let us define the following vector spaces

\[
\text{Hom}^\Sigma(V, W) = \bigoplus_{a \in \Sigma} \text{Hom}(V_{\text{tail}(a)}, W_{\text{head}(a)})
\]

and

\[
\text{Hom}^\Delta(V, W) = \bigoplus_{i \in \Delta} \text{Hom}(V_i, W_i).
\]

We call \(\text{dim}(V) = (\text{dim}(V_i))_{i \in \Delta} \in \mathbb{N}^{[\Delta]}\) the dimension vector of the collection \(V\).

A representation for the quiver \(Q\) associated to the collection \(V\) is an element of \(\text{Hom}^\Sigma(V, V)\). Given a representation \(A \in \text{Hom}^\Sigma(V, V)\), set

\[
A^* := (A^*_a)_{a \in \Sigma^\vee} \in \text{Hom}^\Sigma^\vee(V^*, V^*),
\]

\[
A^t := (A^t_a)_{a \in \Sigma^\vee} \in \text{Hom}^\Sigma^\vee(V^\vee, V^\vee)
\]

and

\[
\bar{A} := (\bar{A}_a)_{a \in \Sigma} \in \text{Hom}^\Sigma(V, V).
\]

Given \(A \in \text{Hom}^\Sigma(V, V), B \in \text{Hom}^\Sigma(V', V''), J \in \text{Hom}^\Delta(V, V')\) and \(I \in \text{Hom}^\Delta(V', V'')\), define

\[
BA = \left( \sum_{\text{tail}(a) = i} B_{\tilde{a}} A_a \right)_{i \in \Delta} \in \text{Hom}^\Delta(V, V'')
\]

and

\[
IJ = (I_i J_i)_{i \in \Delta} \in \text{Hom}^\Delta(V, V'').
\]

Note that the quiver \(\bar{Q}^\circ\) has as vertex set \(\Delta \sqcup \Delta'\) and arrow set \(\bar{\Sigma}^\circ = \Sigma \sqcup \Sigma' \sqcup \Sigma_{\Delta} \sqcup \Sigma_{\Delta'}\). Denote the space of representations of \(\bar{Q}^\circ\) for the collections of vector spaces \((V, W)\) by

\[
M_{V, W}^\circ := \text{Hom}^\Sigma(V, V) \oplus \text{Hom}^\Sigma(V, V) \oplus \text{Hom}^\Delta(W, V) \oplus \text{Hom}^\Delta(W, V).
\]

Thus a representation \(X\) of \(\bar{Q}^\circ\) for the collections of vector spaces \((V, W)\), is a quadruple \((A, B, I, J)\), where

\[
A \in \text{Hom}^\Sigma(V, V), B \in \text{Hom}^\Sigma(V, V),
\]
I \in \text{Hom}^\Delta(W, V) \text{ and } J \in \text{Hom}^\Delta(V, W).

Define the trace of \( \text{tr} A = (A_i)_{i \in \Delta} \in \text{Hom}^\Delta(V, V) \) as
\[
\text{tr} A := \sum_{i \in \Delta} \text{tr} A_i.
\]

One can endow \( M_{V, W}^Q \) with a natural (flat) hyperkähler structure in the following way. First, we consider the hermitian metric \( \eta \) on \( M_{V, W}^Q \) given by
\[
\eta(X, X') := \frac{1}{2} \text{tr} (A(A')^* + A'A^*) + \frac{1}{2} \text{tr} (B(B')^* + B'B^*) + \frac{1}{2} \text{tr} (I(I')^* + I'I^*) + \frac{1}{2} \text{tr} (J(J')^* + J'J^*),
\]
where \( X = (A, B, I, J) \) and \( X' = (A', B', I', J') \). Next, consider the following complex structures on \( M_{V, W}^Q \)
\[
\Gamma_1(A, B, I, J) := (iA, -iB, iI, iJ),
\]
\[
\Gamma_2(A, B, I, J) := (B^*, A^*, -J^*, I^*),
\]
\[
\Gamma_3(A, B, I, J) := (-B^*, iA^*, -iJ^*, iI^*).
\]
Observe that the \( \Gamma_k \) satisfy the quaternionic relations. Also, note that our metric is compatible with these complex structures,
\[
\eta(\Gamma_k(X), \Gamma_k(X')) = \eta(X, X'),
\]
and therefore, using \( (\mathbb{I}_3) \), one can define the symplectic forms on \( \omega_1, \omega_2 \) and \( \omega_3 \), which completes the description of \( M_{V, W}^Q \) as a hyperkähler manifold.

Next, consider the groups, defined by the unitary and general linear groups,
\[
U(V) := \Pi_{i \in \Delta} U(V_i)
\]
and
\[
\text{GL}(V) := \Pi_{i \in \Delta} \text{GL}(V_i).
\]
Both groups act on \( M_{V, W}^Q \) in the following manner; given an element \( g = (g_i)_{i \in \Delta} \) of \( \text{GL}(V) \) or \( U(V) \), define:
\[
Ag := (A_i g_{\text{tail}(a)} )_{a \in \Sigma},
\]
\[
gB := (g_{\text{head}(a)} B_i )_{a \in \Sigma},
\]
\[
gI := (g_i I_i)_{i \in \Delta},
\]
and
\[
Jg := (J_i g_i)_{i \in \Delta}.
\]
One can take the following action of \( \text{GL}(V) \) or \( U(V) \) on \( M_{V, W}^Q \) by setting
\[
g \cdot (A, B, I, J) := (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}).
\]
One easily checks that for any \( X \in M_{V, W}^Q \) we have
\[
\Gamma_k(g \cdot X) = g \cdot \Gamma_k(X) \text{ and } \eta(g \cdot X, g \cdot X) = \eta(X, X'),
\]
so this action preserves the three complex structures and the hermitian metric. It follows that the action preserves the three symplectic forms,
\[
\omega_k(g \cdot X, g \cdot X) = \omega_k(X, X').
\]

Analogously, we have the action of \( h \in U(W) \) on \( M_{V, W}^Q \) given by
\[
h \cdot (A, B, I, J) := (A, B, Ih, h^{-1}J).
\]
As before, this action commutes with the three complex structures \( \Gamma_k \), preserves the metric \( \eta \), and therefore, the three symplectic forms \( \omega_k \). One can easily see
that both actions (2.2) and (2.3) commute, so it is possible to define an action of 
\( U(V) \times U(W) \) on \( M^{Q}_{V,W} \) by setting
\[
(g, h) \cdot X := (gAg^{-1}, gBg^{-1}, gIh, h^{-1}Jg^{-1}).
\]

(2.4)

The action of \( U(V) \) given by (2.2) on the symplectic manifolds \( (M^{Q}_{V,W}, \omega_{1}), \) 
\( (M^{Q}_{V,W}, \omega_{2}) \) and \( (M^{Q}_{V,W}, \omega_{3}) \) defines the following moment maps (see [Na2] for instance)
\[
\mu_{1}(X) = -\frac{1}{2} ([A, B] + [A^{*}, B^{*}] + IJ - J^{*}I^{*}),
\]
\[
\mu_{2}(X) = -\frac{1}{2i}([A, B] - [A^{*}, B^{*}] + IJ + J^{*}I^{*})
\]
and
\[
\mu_{3}(X) = [A, A^{*}] + [B, B^{*}] + II^{*} - J^{*}J.
\]

Note that \( \mu_{1} \) and \( \mu_{2} \) can be recombined into
\[
\mu_{C}(X) = -\mu_{1}(X) - i\mu_{2}(X) = [A, B] + IJ.
\]

One checks that the action of \( U(V) \times U(W) \) preserves \( \mu_{k}^{-1}(0) \) for \( k = 1, 2, 3; \) in fact
\[
\mu_{k}((g, h) \cdot X) = g\mu_{k}(X)g^{-1}.
\]

Then, \( GL(V) \) preserves \( \mu_{C}^{-1}(0) \) and so, the \textit{affine Nakajima quiver variety} for \( Q \) is defined as the affine GIT quotient
\[
(N_{0}^{Q} := \mu_{C}^{-1}(0) \sslash GL(V)).
\]

Take the character
\[
\chi \colon GL(V) \rightarrow \mathbb{C}^{*}
\]
\[
g = (g_{i})_{i \in \Delta} \mapsto \Pi_{i \in \Delta} \det(g_{i}).
\]

Associated to \( \chi \), the \textit{projective Nakajima quiver variety} for the quiver \( Q \) is defined as the projective GIT quotient
\[
(N_{1}^{Q} := \mu_{C}^{-1}(0) \sslash_{\chi} GL(V)),
\]
and using the inverse character,
\[
(N_{-1}^{Q} := \mu_{C}^{-1}(0) \sslash_{\chi^{-1}} GL(V)).
\]

Following [K], we say that \( X = (A, B, I, J) \in \mu_{C}^{-1}(0) \) is \textit{stable} if there is no proper collection of subspaces \( V' \subset V \) such that \( A(V') \subset V', B(V') \subset V' \) and \( \text{im} I \subset V' \). Similarly, \( X \) is \textit{costable} if there is no proper collection of subspaces \( V' \subset V \) such that \( A(V') \subset V', B(V') \subset V' \) and \( V' \subset \ker J \). We say that \( X \) is \textit{regular} if it is both stable and costable. Denote by \( \mu_{C}^{-1}(0)^{st} \), \( \mu_{C}^{-1}(0)^{ct} \) and \( \mu_{C}^{-1}(0)^{reg} \) the \( GL(V) \)-invariant sets of stable, costable and regular points.

Semistability, polystability and stability coincide in the GIT quotients (2.7) and (2.8). In this case, [K Proposition 3.1] implies the following.

**Proposition 2.1.** A point \( X \in \mu_{C}^{-1}(0) \) has a closed orbit \( GL(V) \cdot X \subset \mu_{C}^{-1}(0) \) in the projective quotient (2.7) if and only if it is stable. Furthermore, every \( X \in \mu_{C}^{-1}(0)^{st} \) has trivial stabilizer.

Similarly, \( X \in \mu_{C}^{-1}(0) \) has a closed orbit \( GL(V) \cdot X \subset \mu_{C}^{-1}(0) \) in the projective quotient (2.8) if and only if it is costable. Every \( X \in \mu_{C}^{-1}(0)^{ct} \) has trivial stabilizer.
One has a description of $\mathcal{N}^Q_1$ and $\mathcal{N}^Q_{-1}$ in terms of simple quotients
\begin{equation}
\mathcal{N}^Q_1 \cong \mu_{c_1}^{-1}(0)^{ct} / GL(V)
\end{equation}
and
\begin{equation}
\mathcal{N}^Q_{-1} \cong \mu^{-1}(0)^{ct} / GL(V).
\end{equation}

Recall that $\mu_{c_1}^{-1}(0)$ with the restriction of the symplectic form $\omega_3$ is a symplectic manifold where the action of $U(V)$ gives the moment map $\mu_3$. Using Kempf-Ness theorem [KN] one can express the GIT quotients as symplectic quotients, that is
\begin{equation}
\mathcal{N}_{0}^Q \cong \mu_{c_1}^{-1}(0) \cap \mu_{3}^{-1}(0) / U(V) = \mu_{1}^{-1}(0) \cap \mu_{2}^{-1}(0) \cap \mu_{3}^{-1}(0) / U(V);
\end{equation}
\begin{equation}
\mathcal{N}_{1}^Q \cong \mu_{c_1}^{-1}(0) \cap \mu_{3}^{-1}(-r) / U(V) = \mu_{1}^{-1}(0) \cap \mu_{2}^{-1}(0) \cap \mu_{3}^{-1}(-r) / U(V)
\end{equation}
and
\begin{equation}
\mathcal{N}_{-1}^Q \cong \mu_{c_1}^{-1}(0) \cap \mu_{3}^{-1}(r) / U(V) = \mu_{1}^{-1}(0) \cap \mu_{2}^{-1}(0) \cap \mu_{3}^{-1}(r) / U(V).
\end{equation}

Note that $\mathcal{N}_{0}^Q$, $\mathcal{N}_{1}^Q$ and $\mathcal{N}_{-1}^Q$ are hyperkähler manifolds, defined as hyperkähler quotients of $M_Y^Q(W)$.

Finally, we define $\mathcal{N}_{reg}^Q$ as the open subset of $\mathcal{N}_{1}^Q$ consisting of regular points. It is also a hyperkähler manifold and recalling (2.9) and (2.10), one has
\begin{equation}
\mathcal{N}_{reg}^Q \cong \mu_{c_1}^{-1}(0)^{reg} / GL(V).
\end{equation}

Note that (2.5), (2.8), and the fact that $\mu_{c_1}^{-1}(0)^{reg}$ is preserved under the action of $GL(V)$ imply that $\mathcal{N}_{reg}^Q$ can naturally be understood as an open subset of $\mathcal{N}_{1}^Q$ and $\mathcal{N}_{-1}^Q$.

2.2. Branes from involutions on a hyperkähler quotient. In this section, we describe how to construct branes on a hyperkähler quotient using analytic involutions on the covering hyperkähler.

To be precise, let $(Y, \Gamma_1, \Gamma_2, \Gamma_3, \eta)$ be a (finite dimensional) hyperkähler manifold, let $a : Y \to Y$ be an analytic involution, and denote by $Y^a$ its subvariety of fixed points. By abuse of notation, denote by $a : T_xY \to T_xY$ the induced involution in the tangent space of a fixed point $x \in Y^a$, and recall that the tangent space of $Y^a$ at $x$ is the invariant subspace $(T_xY)^a$. Suppose that
\begin{equation}
\eta(a(\cdot), a(\cdot)) = \eta(\cdot, \cdot),
\end{equation}
that is, $a$ is an isometry, and that for each $k = 1, 2, 3$, and
\begin{equation}
\Gamma_k a(\cdot) = \delta_k a \Gamma_k (\cdot) \quad \text{with} \quad \delta_k = \pm 1.
\end{equation}
If $\delta_k = 1$, that is, if $a$ commutes with the complex structure $\Gamma_k$, the involution is holomorphic with respect to $\Gamma_k$. On the other hand, if $\delta_k = -1$, i.e. $a$ anticommutes with the complex structure $\Gamma_k$ and we say that $a$ is antiholomorphic with respect to $\Gamma_k$. In the last case, one has the following result (see [BS1] Lemma 9) for instance).

**Lemma 2.2.** Let $Y$ be a complex manifold of (complex) dimension $n$ and let $a : Y \to Y$ be an antiholomorphic involution. If the fixed point locus $Y^a$ is not empty, then it is a smooth analytic subvariety of real dimension $n$.

**Remark 2.3.** Let us further consider $(Y, \Gamma, \eta)$ to be a Kähler manifold. Take an antiholomorphic involution $a : Y \to Y$ satisfying (2.15). Then the fixed point locus
is isotropic, that is, for every two \( u, v \in T_z(Y^a) = (T_zY)^a \), one has \( \omega(u, v) = 0 \), since

\[
\omega(u, v) = \eta(u, \Gamma v) = \eta(au, a\Gamma v) \\
= \eta(au, -\Gamma av) = -\eta(u, \Gamma v) \\
= -\omega(u, v).
\]

If the involution \( a \) satisfies (2.15) and (2.16), one has that:

- if \( \delta_k = 1 \) then \( Y^a \) is a \( B \)-brane (complex subvariety) with respect to \( \Gamma_k \) since the involution \( a \) is holomorphic with respect to \( \Gamma_k \), and
- if \( \delta_k = -1 \) then \( Y^a \) is an \( A \)-brane (lagrangian subvariety) with respect to \( \Gamma_k \), since in that case \( Y^a \) is isotropic by Remark 2.3 and has maximal dimension by Lemma 2.2.

Now let \( G \) be a compact group acting on \( Y \) isometrically with respect to \( \eta \). One has moment maps \( \mu_1, \mu_2 \) and \( \mu_3 \), each associated to a symplectic form \( \omega_k \), with \( \omega_k : Y \rightarrow g^* \). For \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \) with \( \zeta_i \in j^* \) (where \( j \) denotes the centre of \( g \)), the associated hyperkähler quotient is defined as

\[
\overline{Y}_\zeta := \frac{\mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)}{G}.
\]

The importance of this construction relies in the fact that \( \overline{Y}_\zeta \) is also a hyperkähler manifold, with metric \( \overline{\eta} \), complex structures \( \overline{\Gamma_1}, \overline{\Gamma_2} \) and \( \overline{\Gamma_3} \), and symplectic forms \( \overline{\omega}_1, \overline{\omega}_2 \) and \( \overline{\omega}_3 \). Furthermore, for the natural inclusion and projection maps of the following diagram

\[
\begin{array}{ccc}
\mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3) & \xrightarrow{i} & Y \\
\downarrow \pi & & \downarrow i \\
\overline{Y}_\zeta & &
\end{array}
\]

we have that \( \pi^*\overline{\eta}, \pi^*\overline{\Gamma}_k \) and \( \pi^*\overline{\omega}_k \) are equal to \( i^*\eta, i^*\Gamma_k \) and \( i^*\omega_k \).

The involution \( a : Y \rightarrow Y \) descends to an involution \( \overline{\pi} : \overline{Y}_\zeta \rightarrow \overline{Y}_\zeta \) in the hyperkähler quotient, if and only if \( a \) restricts to \( \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3) \), i.e.

\[
a (\mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)) \subseteq \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3),
\]

and, for every point \( Y \in \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3) \), the image of the orbit of \( Y \) is the orbit of the image of \( Y \),

\[
a(G \cdot x) \subseteq G \cdot a(x).
\]

Note that if \( a \) satisfies (2.15) and (2.16) for \( \eta \) and the \( \Gamma_k \), then \( \overline{\pi} \) also satisfies (2.15) and (2.16) for \( \overline{\eta} \) and \( \overline{\Gamma}_1, \overline{\Gamma}_2 \) and \( \overline{\Gamma}_3 \). Therefore, we have established the main result of this section.

**Lemma 2.4.** Let \( G \) be a compact Lie group acting isometrically on the hyperkähler manifold \((Y, \Gamma_1, \Gamma_2, \Gamma_3, \eta)\), giving the hyperkähler quotient \((\overline{Y}, \overline{\Gamma}_1, \overline{\Gamma}_2, \overline{\Gamma}_3, \overline{\eta})\), as above. Let \( a : Y \rightarrow Y \) be an involution satisfying conditions (2.15), (2.16), (2.17) and (2.18). Then \( a \) induces an involution \( \overline{\pi} : \overline{Y}_\zeta \rightarrow \overline{Y}_\zeta \) in the quotient, and the subvariety of fixed points \((\overline{Y}_\zeta^a)^F \) is a brane inside \( \overline{Y}_\zeta \), whose type is given by the values of \( \delta_1, \delta_2 \) and \( \delta_3 \).

### 3. BRANES ON NAKAJIMA QUIVER VARIETIES

In this section we describe four different involutions on the Nakajima quiver varieties, and we show that the corresponding fixed point sets are branes of all possible types.
First, we set up some notation. Given an involution \( a : \mathcal{M}^Q_{V,W} \to \mathcal{M}^Q_{V,W}, \) and elements \( g \in U(V) \) and \( h \in U(W), \) we define the automorphism

\[
(3.1) \quad a_{(g,h)} : \mathcal{M}^Q_{V,W} \to \mathcal{M}^Q_{V,W} \quad X \mapsto (g,h) \cdot a(X).
\]

Note that \( a_{(g,h)} \) is not always involution, but only if \( g \in U(V) \) and \( h \in U(W) \) satisfy certain conditions which will be given precisely in each case.

Since the action of \( U(V) \times U(W) \) commutes with the three complex structures \( \Gamma_k, \) and preserves the metric \( \eta \) and the symplectic forms \( \omega_k, \) it is not difficult to check that if \( a \) satisfies the conditions of Lemma 2.4 then so does \( a_{(g,h)} \) when it is an involution.

### 3.1. The transposition involution

The first example is given by the involution \( b : \mathcal{M}^Q_{V,W} \to \mathcal{M}^Q_{V,W}, \)

\[
(A, B, I, J) \mapsto (A', B', J', -I').
\]

**Lemma 3.1.** The involution \( b \) is an isometry that commutes with \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3. \) Furthermore, \( b \) preserves \( \mu_\omega^{-1}(0) \cap \mu_3^{-1}(0) \) and for every \( X \in \mathcal{M}^Q_{V,W} \) one has that

\[
b(U(V) \cdot X) = U(V) \cdot b(X).
\]

In other words, \( b \) satisfies the conditions (2.15), (2.16) with \( \delta_k = 1 \) in (2.17) with \( \zeta_k = 0, \) and (2.18).

**Proof.** Let \( X = (A, B, I, J) \) and \( X' = (A', B', I', J'), \)

\[
\eta(b(X), b(X')) = \frac{1}{2} \tr (A'^T\overline{A} + (A')^T\overline{A} + B'B + (B')^T\overline{B})
\]

\[
+ \frac{1}{2} \tr (J'\overline{J} + (J')^T\overline{J}) + \frac{1}{2} \tr (I'I + (I')^T\overline{I}) +
\]

\[
= \frac{1}{2} \tr ((A')^*A + A^*A' + (B')^*B + B^*B')
\]

\[
+ \frac{1}{2} \tr ((J')^*J + J^*J') + \frac{1}{2} \tr ((I')^*I + I^*I')
\]

\[
= \eta(X, X').
\]

One trivially has that \( \Gamma_1 b = b \Gamma_1. \) Also, \( \Gamma_2 b = b \Gamma_2, \) since

\[
\Gamma_2 b(X) = (-\overline{B}, \overline{A}, \overline{I}, -\overline{J}) = b \Gamma_2(X)
\]

Commutativity with \( \Gamma_3 \) follows from the commutativity with \( \Gamma_1 \) and \( \Gamma_2. \)

Note that

\[
\mu_\omega(b(X)) = [A^T, B^T] - J^T I
\]

\[
= - [A, B] - (I J)^T
\]

\[
= - \mu_\omega(X)^T,
\]

and

\[
\mu_3(b(X)) = [A^T, (A^*)^T] + [B^T, (B^*)^T] + J^T (J^*)^T - (I^*)^T I
\]

\[
= - [A, A^*]^T - [B, B^*]^T + (J^*)^T - (I^*)^T
\]

\[
= - \mu_3(X)^T.
\]
Finally, for every $k \in U(V)$, one has that
\[
b(k \cdot X) = b(kAk^{-1}, kBk^{-1}, kI, Jk^{-1})
\]
\[
= ((k^{-1})^t A^t k^t, (k^{-1})^t B^t k^t, (k^{-1})^t J^t, -I^t k^t)
\]
\[
= (k^{-1})^t \cdot b(X).
\]

Next, consider the automorphism $b(g, h) : M_{V, W}^Q \to M_{V, W}^Q$, defined as in (3.1).

Note that $(b(g, h))$ is an involution if and only if
\[
g = (g_i)_{i \in \Delta} = (e^{i\omega_k} g_i^1)_{i \in \Delta} \quad \text{and} \quad h = (h_i)_{i \in \Delta} = (-e^{i\omega_k} h_i^1)_{i \in \Delta}
\]
with
\[
\omega_{\text{tail}(a)} = -\omega_{\text{head}(a)}
\]
for every $a \in \Sigma$. If $Q$ has a closed path with an odd number of steps, then, either
- $g^t = g$ and $h^t = -h$, or
- $g^t = -g$ and $h^t = h$.

One easily checks that, under the conditions posed above, $b(g, h)$ satisfy the conclusions of Lemma 3.3.

Summing all up (Lemma 2.4, Lemma 3.1 and the observations above), we obtain the following statement.

**Corollary 3.2.** Take $(g, h)$ as specified by (3.2), and let $(\mathcal{N}^Q_b)^g$ be the fixed point locus of the involution $b(g, h)$ on $\mathcal{N}^Q_b$. Then $(\mathcal{N}^Q_b)^g$ is a $(B, B, B)$-brane and vice-versa, so that $b(g, h)$ preserves regularity and descends to an involution on $\mathcal{N}^Q_{\text{reg}}$.

**Corollary 3.3.** The fixed point locus $(\mathcal{N}^Q_{\text{reg}})^g$ of the involution given by $b(g, h)$ on $\mathcal{N}^Q_{\text{reg}}$, is a $(B, B, B)$-brane within $\mathcal{N}^Q_{\text{reg}}$.

### 3.2. The sign involution

Consider a function
\[
\gamma : \Sigma \sqcup \Sigma_\Delta \to \{1, -1\}
\]
and use it to define the involution
\[
c^\gamma : M_{V, W}^Q \to M_{V, W}^Q
\]
\[
(A, B, I, J) \mapsto ((\gamma(a) A_a)_{a \in \Sigma}, (\gamma(a) B_a)_{a \in \Sigma}, (\gamma(i) I_i)_{i \in \Delta}, (\gamma(i) J_i)_{i \in \Delta}).
\]
We assume that $\gamma$ is not the constant function $\gamma = 1$, so that $c^\gamma \neq 1$.

**Lemma 3.4.** For any $\gamma$ as in (3.3), the involution $c^\gamma$ is an isometry that commutes with $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ and preserves $\mu_\gamma^{-1}(0) \cap \mu_\gamma^{-1}(r)$ for every $r \in i\mathbb{R}$. Also, for every $X \in M_{V, W}^Q$, one has that
\[
c^\gamma(U(V) \cdot X) = U(V) \cdot c^\gamma(X).
\]

In other words, $c^\gamma$ satisfies the conditions (2.15), (2.16) with $\delta_k = 1$, (2.17) with $\zeta_1 = \zeta_2 = 0$, $\xi_3 = ir1$, and (2.19).

**Proof.** Consider $F' \in \text{Hom}^\Sigma(V, V)$ (resp. $\text{Hom}^\Delta(V, W)$) and $E' \in \text{Hom}^{\Sigma'}(V, V)$ (resp. $\text{Hom}^{\Delta'}(V, W)$). Setting
\[
F' = (\gamma(a) F_a)_{a \in \Sigma} \quad \text{(resp. } (\gamma(i) F_i)_{i \in \Delta}),
\]
and
\[
E' = (\gamma(a) E_a)_{a \in \Sigma} \quad \text{(resp. } (\gamma(i) E_i)_{i \in \Delta}),
\]
one can easily check that $E^sF^s = EF$. Then, it follows that

- $\eta(c^\gamma(X), c^\gamma(X')) = \eta(X, X')$,
- $\mu_\gamma(c^\gamma(X)) = \mu_\gamma(X)$, and
- $\mu_\gamma(c^\gamma(X)) \neq \mu_\gamma(X)$.

It is trivial that $\Gamma_k e^\gamma = c^\gamma \Gamma_k$ for $k = 1, 2, 3$, and also that for every $k \in U(V)$, one has that

$$c^\gamma(kAk^{-1}, kBk^{-1}, kI, Jk^{-1}) = (kA'k^{-1}, kB'k^{-1}, kI', J'k^{-1})$$

so $c^\gamma(k \cdot X) = k \cdot c^\gamma(X)$.

Given $g \in U(V)$ and $h \in U(W)$, consider $c^\gamma_{(g,h)}$ as defined in (3.4). Note that $(c^\gamma_{(g,h)})^2 = 1$ if and only if

$$g^2 = (e^{i\omega_k})_{k \in \Delta} \quad \text{and} \quad h^2 = (e^{-i\omega_k})_{k \in \Delta}$$

with

$$\omega_{\text{tail}(a)} = \omega_{\text{head}(a)}$$

for every $a \in \Sigma$. If $Q$ has a closed path with an odd number of steps, then

$$g^2 = (e^{i\omega_k})_{k \in \Delta} \quad \text{and} \quad h^2 = (e^{-i\omega_k})_{k \in \Delta}.$$

As observed in the beginning of this section, Lemma 3.4 extends for $c^\gamma_{(g,h)}$. It is therefore clear that $c^\gamma_{(g,h)}$ descends to an involution on the hyperkähler quotients $N^Q_0$, $N^Q_1$ and $N^Q_2$. Moreover, it is automatic to check that $c^\gamma_{(g,h)}$ preserves stability and costability; therefore it restricts to an involution on $N^Q_{reg}$.

**Corollary 3.5.** Take $\gamma$ as in (3.3), and $(g, h)$ as in (3.4). For $s = 0, 1, -1$ and $\text{reg}$, let $(N^Q_{reg})^c$ be the fixed point locus given by $c^\gamma_{(g,h)}$. Then $(N^Q_{reg})^c$ is a $(B, B, B)$-brane within $N^Q_{reg}$.

### 3.3. The recombination involution

Recall that, except for 1 and -1, the idempotent elements of $U(2)$ lie in the orbit of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

under the conjugation action on $U(2)$, that is the set

$$T := \left\{ \begin{pmatrix} t & z \\ \bar{z} & -t \end{pmatrix}, \text{ satisfying } t^2 + z\bar{z} = 1, \text{ with } t \in \mathbb{R} \text{ and } z \in \mathbb{C} \right\}.$$ 

Write $\Sigma = \Lambda \cup \Lambda^\perp$, where $\Lambda$ is the set of loops of the quiver,

$$\Lambda := \{ a \in \Sigma \text{ such that } \text{tail}(a) = \text{head}(a) \},$$

and $\Lambda^\perp$ is its complement,

$$\Lambda^\perp := \{ a \in \Sigma \text{ such that } \text{tail}(a) \neq \text{head}(a) \}.$$

Next, consider a function

$$\delta : \Lambda \cup \Lambda^\perp \cup \Sigma_{\Delta} \rightarrow T,$$

such that

$$\delta(a) = \begin{pmatrix} t_a & 0 \\ 0 & -t_a \end{pmatrix}$$

for every $a \in \Lambda \cup \Sigma_{\Delta}$. Note that

$$\delta(a) \begin{pmatrix} A_a \\ B_a \end{pmatrix} = \begin{pmatrix} t_a \\ \bar{z}_a \end{pmatrix} \begin{pmatrix} A_a \\ B_a \end{pmatrix} = \begin{pmatrix} t_aA_a + z_aB_a \\ \bar{z}_aA_a - t_aB_a \end{pmatrix}$$

where $\bar{z}_a$ is the complex conjugate of $z_a$.
Proof. (resp. Hom$^\Delta$) Then, we automatically have
\[ \delta \] and
\[ \Gamma \] for every
\[ \delta_1(A, B) = (t_a A_a + z_a B_a)_{a \in \Sigma}, \quad \text{and} \quad \delta_2(A, B) = (\bar{z}_a A_a - t_a B_{\bar{a}})_{a \in \Sigma} \] and
\[ \delta(\mathbf{I}) = (t_i I_i)_{i \in \Delta}, \quad \text{and} \quad \delta(\mathbf{J}) = (-t_i I_i)_{i \in \Delta}. \]
Given $\delta$ as in (3.6), define the following involution on $M^Q_{\mathbf{V}, \mathbf{W}}$,
\[ d^\delta : M^Q_{\mathbf{V}, \mathbf{W}} \rightarrow M^Q_{\mathbf{V}, \mathbf{W}} \]
\[ (A, B, I, J) \mapsto ((\delta_1(A, B), \delta_2(A, B), \delta(\mathbf{I}), \delta(\mathbf{J})). \]

Lemma 3.6. The involution $d^\delta$ commutes with $\Gamma_1$ and anticommutes with $\Gamma_2$ and $\Gamma_3$, it is an isometry that preserves $\mu^{-1}_C(0) \cap \mu^{-1}_H(r)$ for every $r \in \mathbb{R}$ and, for every $X \in M^Q_{\mathbf{V}, \mathbf{W}}$, one has that
\[ d^\delta (U(V) \cdot X) = U(V) \cdot d^\delta(X). \]

In other words, $d^\delta$ satisfies the conditions (2.15), (2.16) with $\delta_1 = 1$ and $\delta_2 = \delta_3 = -1$, (2.17) with $\zeta_1 = \zeta_2 = 0$, $\zeta_3 = ir$, and (2.18).

Proof. Consider $F, F' \in \text{Hom}^\Sigma(V, V)$ (resp. $\text{Hom}^\Delta(V, W)$) and $E, E' \in \text{Hom}^\Sigma(V, V)$ (resp. $\text{Hom}^\Sigma(W, V)$). Thanks to the fact that $\delta(a) \in U(2)$, one can prove that
\[ \delta_1(E, F)^* \delta_1(E', F') + \delta_2(E', F') \delta_2(E, F)^* = \left( \sum_{\text{tail}(a) = i} \begin{pmatrix} E_a^* & F_a^* \\ \bar{F_a} & \bar{E_a}^* \end{pmatrix} \right) \delta(a)^* \delta(a) \left( \begin{pmatrix} E_a^* & F_a^* \\ \bar{F_a} & \bar{E_a}^* \end{pmatrix} \right) \right)_{i \in \Delta} \]
\[ = \left( \sum_{\text{tail}(a) = i} \begin{pmatrix} E_a^* & F_a^* \\ \bar{F_a} & \bar{E_a}^* \end{pmatrix} \right) \right)_{i \in \Delta} \]
\[ = E' E' + F' F'. \]

Then, we automatically have
- $\eta(d^\delta(X), d^\delta(X')) = \eta(X, X')$, and
- $\mu_3(d^\delta(X)) = \mu_3(X)$.

Note that $\Gamma_2 d^\delta = -d^\delta \Gamma_2$, given by
\[ \delta_1(-B^*, A^*) = (-t_a B_{\bar{a}}^* + z_a A_a^*)_{a \in \Sigma} = (-t_a B_{\bar{a}} + \bar{z}_a A_a)_{a \in \Sigma} = \delta_2(A, B)^* \]
and
\[ \delta_2(-B^*, A^*) = (\bar{z}_a A_a^* - t_a B_{\bar{a}})_{a \in \Sigma} = -(t_a A_a + z_a B_a)_{a \in \Sigma} = -\delta_1(A, B)^* \]

Since $\Gamma_1 d^\delta = d^\delta \Gamma_1$, then $d^\delta$ anticommutes with $\Gamma_3 = \Gamma_1 \Gamma_2$.

We can check that
\[ [t_a A_a + z_a B_{\bar{a}}, \bar{z}_a A_a - t_a B_{\bar{a}}] = t_a \bar{z}_a A_a^2 - t_a^2 A_a B_{\bar{a}} + |z_a|^2 B_a A_a - z_a t_a B_{\bar{a}}^2 - \bar{z}_a A_a^2 - t_a \bar{z}_a A_a B_{\bar{a}} + |z_a|^2 B_a A_a + t_a z_a B_{\bar{a}}^2 \]
\[ = (t_a^2 + |z_a|^2) A_a B_{\bar{a}} + (t_a^2 + |z_a|^2) B_a A_a \]
\[ = [A_a, B_{\bar{a}}], \]
and therefore we know that
\[
\mu_C(d^\delta(X)) = [\delta_1(A, B), \delta_2(A, B)] + \delta(I)\delta(J)
\]
\[
= \left( \sum_{\text{tail}(a)=i} [t_a A_a + z_a B_a, \tau_a A_a - t_a B_a] \right)_{i \in \Delta} + (-t^2 I, J)_{i \in \Delta}
\]
\[
= \left( \sum_{\text{tail}(a)=i} -[A_a, B_a] \right)_{i \in \Delta} - (I, J)_{i \in \Delta}
\]
\[
= -\mu_C(X).
\]

Finally, for every \( k \in U(V) \), one has that
\[
d^\delta(k \cdot X) = d^\delta(kAk^{-1}, kIk^{-1}, Jk^{-1})
\]
\[
= (\delta_1(kAk^{-1}, kIk^{-1}), \delta_2(kAk^{-1}, kIk^{-1}), \delta(I), \delta(Jk^{-1}))
\]
\[
= (k\delta_1(A, B)k^{-1}, k\delta_2(A, B)k^{-1}, k\delta(I), k\delta(J)k^{-1}),
\]
\[
= k \cdot d^\delta(X).
\]

Given \( g \in U(V) \) and \( h \in U(W) \), consider \( d^\delta_{(g, h)} \) as defined in (3.1). Note that
\[
(d^\delta_{(g, h)})^2 = 1
\]
if and only if
\[
g^2 = (e^{i\omega})_{i \in \Delta} \quad \text{and} \quad h^2 = (e^{-i\omega})_{i \in \Delta}
\]
with
\[
\omega_{\text{tail}(a)} = \omega_{\text{head}(a)}
\]
for every \( a \in \Sigma \). In addition, if \( Q \) has a closed path with an odd number of steps, then
\[
g^2 = (e^{i\omega})_{i \in \Delta} \quad \text{and} \quad h^2 = (e^{-i\omega})_{i \in \Delta}.
\]

As observed in the beginning of this section, Lemma 3.4 extends for \( d^\delta_{(g, h)} \). It is therefore clear that \( d^\delta_{(g, h)} \) descends to an involution on the hyperkähler quotients \( N_0^Q, N_1^Q \) and \( N_2^Q \). Moreover, it is straightforward to check that \( d^\delta_{(g, h)} \) preserves stability and costability; therefore it restricts to an involution on \( N^Q_{\text{reg}} \).

**Corollary 3.7.** Take \( \delta \) as in (3.6) and \( (g, h) \) as in equation (3.7). For \( * = 0, 1, -1 \) and \( \text{reg} \), let \( (N^Q_*)^d \) be the fixed point locus given by \( d^\delta_{(g, h)} \). Then \( (N^Q_*)^d \) is a \((B, A, A)\)-brane within \( N^Q_{\text{reg}} \).

**Remark 3.8.** Consider the \( \mathbb{C}^* \)-action on \( N_0^Q \), given by
\[
\lambda \cdot (A, B, I, J) = (\lambda A, \lambda B, I, \lambda J).
\]
It is shown in [C] Section 5.4, that the subvariety of points fixed under this action \( (N^Q_0)^{\mathbb{C}^*} \) is a complex lagrangian subvariety for \( \Gamma_1 \). Note that this agrees with our description, since \( (N^Q_0)^{\mathbb{C}^*} \subset (N^Q_0)^d \) when
\[
\delta(a) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
for all \( a \in \Lambda \sqcup \Lambda^\perp \sqcup \Sigma_\Delta \).
3.4. The conjugation involution. Finally, we consider the following involution in $\mathcal{M}_V^\varnothing W$

\[
e : \mathcal{M}_V^\varnothing W \rightarrow \mathcal{M}_V^\varnothing W, \quad (A, B, I, J) \mapsto (\bar{A}, \bar{B}, \bar{I}, \bar{J}).\]

As before, we establish the basic properties of $e$.

**Lemma 3.9.** The involution $e$ is an isometry that commutes with $\Gamma_2$ and anti-commutes with $\Gamma_1$ and $\Gamma_3$. Furthermore, $e$ preserves $\mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ and for every $X \in \mathcal{M}_V^\varnothing W$ one has that

\[
e(U(V) \cdot X) = U(V) \cdot e(X).
\]

In other words, $e$ satisfies the conditions (2.15), (2.10) with $\delta_2 = 1$ and $\delta_1 = \delta_3 = -1$, (2.17) with $\zeta_1 = \zeta_2 = \zeta_3 = 0$, and (2.13).

**Proof.** Note that $e\Gamma_1(X) = e(iX) = -ie(X) = -\Gamma_1(X)$. On the other hand one can easily check that $\Gamma_2 e = e \Gamma_2$. The involution $e$ preserves the metric $\eta$, since for every two $X = (A, B, I, J)$ and $X' = (A', B', I', J')$, one has

\[
\eta(e(X), e(X')) = \frac{1}{2} \text{tr} \left( \bar{A}(A')^t + \bar{A}A' + \bar{B}(B')^t + \bar{B}B' \right) + \frac{1}{2} \text{tr} \left( \bar{I}(I')^t + \bar{I}I' + \bar{J}(J')^t + \bar{J}J' \right) + \\
= \frac{1}{2} \text{tr} \left( (A'A^*)^t + (A')^*A^t + (B'B^*)^t + (B')^*B^t \right) + \frac{1}{2} \text{tr} \left( (I'I^*)^t + (I^*I')^t + (J'J^*)^t + (J^*J')^t \right) + \\
= \frac{1}{2} \text{tr} \left( (A'A^*) + (A')^*A + (B'B^*) + (B')^*B \right) + \frac{1}{2} \text{tr} \left( (I'I^*) + (I^*I') + (J'J^*) + (J^*J') \right) + \\
= \eta(X, X').
\]

It is also possible to show that the involution $e$ preserves $\mu_1^{-1}(0)$ and $\mu_3^{-1}(0)$. Observe that

\[
\mu_C(e(X)) = [\bar{A}, \bar{B}] - [\bar{I}] = \mu_C(X)
\]

and\n
\[
\mu_3(e(X)) = [\bar{A}, \bar{A}] + [\bar{B}, \bar{B}] + [\bar{I}] - [\bar{J}] = \mu_3(X).
\]

The last statement follows from the observation that $\bar{X} \in U(V)$ provided that $k \in U(V)$, since

\[
e(k \cdot X) = e(kAk^{-1}, kBk^{-1}, kI, Jk^{-1}) = \left( \bar{k}Ak^{-1}, \bar{k}Bk^{-1}, \bar{k}I, \bar{k}Jk^{-1} \right) = \bar{k} \cdot e(X).
\]

Given $g \in U(V)$ and $h \in U(W)$, consider $e_{(g, h)}$ as defined in (5.1). Note that $(e_{(g, h)})^2 = 1$ if and only if

\[
g = (g_i)_{i \in \Delta} = (e^{i\omega k} g_i^t)_{i \in \Delta} \quad \text{and} \quad h = (h_i)_{i \in \Delta} = (e^{i\omega k} h_i^t)_{i \in \Delta}
\]

with

\[
\omega_{\text{tail}(a)} = \omega_{\text{head}(a)}
\]

\[\square\]
for every $a \in \Sigma$. In addition, if $Q$ has a closed path with an odd number of steps, then
\[ g^2 = (e^{i\omega})_{i \in \Delta} \quad \text{and} \quad h^2 = (e^{-i\omega})_{i \in \Delta}. \]

As observed in the beginning of this section, Lemma 3.9 extends for $e_{(g,h)}$. It is therefore clear that $e_{(g,h)}$ descends to an involution on the hyperkähler quotient $\mathcal{N}^Q_0$. Moreover, it is straightforward to check that $e_{(g,h)}$ preserves stability and costability; therefore it restricts to an involution on $\mathcal{N}^Q_{\text{reg}}$.

**Corollary 3.10.** Take $(g,h)$ as in equation (3.8). For $* = 0$ and $\text{reg}$, let the fixed point locus $(\mathcal{N}^Q_0)^*$ of the involution given by $e_{(g,h)}$ on $\mathcal{N}^Q_0$. Then $(\mathcal{N}^Q_0)^*$ is a $(A,B,A)$-brane within $\mathcal{N}^Q_0$.

**Remark 3.11.** Note that $e_{(g,h)}$ does not preserve $\mu_3^{-1}(-r)$ (nor $\mu_3^{-1}(r)$) for $r \in i\mathbb{R}^{>0}$ and as a consequence, it does not descend to an involution in $\mathcal{N}^Q_0$ (nor $\mathcal{N}^Q_{\text{reg}}$). On the other hand, one can check (see the proof [Na3], Lemma 2.6) for instance) that $e_{(g,h)}$ preserves regularity and therefore descends to an involution on $\mathcal{N}^Q_{\text{reg}}$.

3.5. **Combining the involutions.** The involutions $b$, $c^\gamma$ and $e$ commute between them. The involution $d^\delta$ always commutes with $c^\gamma$ but only commutes with $b$ and $e$ for choices of $\delta$ that satisfy $z_\alpha \in \mathbb{R}$ for all $a \in \Lambda_0 \cup \Sigma_\Delta$. Whenever they commute, any possible combination of these involutions gives another involution that preserves regularity and automatically satisfies the conditions (2.13), (2.15), (2.17) and (2.18) associated to the hyperkähler quotient $\mathcal{N}^Q_0$ described in (2.11). Therefore, any composition of these involutions gives an involution on the Nakajima quiver varieties $\mathcal{N}^Q_0$ and $\mathcal{N}^Q_{\text{reg}}$ whose fixed point locus is a brane.

In particular, $(A,A,B)$-branes can be obtained by considering fixed point loci of involutions of the form $(ed^\delta)_{(g,h)}$ and $(ebd^\gamma)_{(g,h)}$, with the choice of $\delta$ that we specified above. More generally, the table below summarizes all the possibilities we have considered.

| Involution $e_{(g,h)}$ | Equation for $g$ and $h$ | Type of brane |
|------------------------|---------------------------|---------------|
| $b_{(g,h)}$            | $g = \theta g^t$ and $h = -\theta h^t$ | $(B,B,B)$     |
| $c^\gamma_{(g,h)}$    | $g^2 = (e^{i\omega})_{i \in \Delta}$ and $h^2 = (e^{-i\omega})_{i \in \Delta}$ | $(B,B,B)$     |
| $d^\delta_{(g,h)}$    | $g^2 = (e^{i\omega})_{i \in \Delta}$ and $h^2 = (e^{-i\omega})_{i \in \Delta}$ | $(B,A,A)$     |
| $e_{(g,h)}$            | $g = \theta g^t$ and $h = \theta h^t$ | $(A,B,A)$     |
| $(eb)_{(g,h)}$         | $g^2 = (e^{i\omega})_{i \in \Delta}$ and $h^2 = (e^{-i\omega})_{i \in \Delta}$ | $(A,B,A)$     |
| $(ec^\gamma)_{(g,h)}$  | $g = \theta g^t$ and $h = \theta h^t$ | $(A,B,A)$     |
| $(ebc)_{(g,h)}$        | $g^2 = (e^{i\omega})_{i \in \Delta}$ and $h^2 = (e^{-i\omega})_{i \in \Delta}$ | $(A,B,A)$     |
| $(ebd^\delta)_{(g,h)}$| $g^2 = (e^{i\omega})_{i \in \Delta}$ and $h^2 = (e^{-i\omega})_{i \in \Delta}$ | $(A,A,B)$     |

**Table 1.** Brane types for composed involutions

However, it is not clear whether the fixed point set of any of the involutions we described in this section is actually non-empty. To study existence of branes, we consider the simplest, but already very interesting, Nakajima quiver variety, the one provided by the Jordan quiver.
4. BRANES IN THE MODULI SPACE OF FRAMED INSTANTONS

We denote by $J$ the Jordan quiver, that is, a single vertex and a single edge-loop,

$$
\begin{array}{c}
\bullet \\
\end{array}
$$

and note that $\tilde{J}$ is the ADHM quiver,

$$
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
$$

Since $J$ has a single vertex, one has that $V = V$ and $W = W$ are just vector spaces of dimensions, say, $n$ and $r$, respectively, and

$$
M^J_{V,W} = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W).
$$

A point $X = (A, B, I, J)$ of $M^J_{V,W}$ is said to satisfy the ADHM equation if

$$(4.1) \quad [A, B] + IJ = 0.$$ 

The closed subvariety of points of $M^J_{V,W}$ that satisfy $(4.1)$ is called the ADHM variety and denoted by $A_{V,W}$. 

Note that the preimage $\mu^{-1}\{0\} \subset M_{V,W}$ is precisely the ADHM locus $A_{V,W}$. The definitions of stability, costability and regularity given in Section 2.1 apply directly in this case, and so, we have the stable and regular locus, denoted respectively by $A_{V,W}^{st}$ and $A_{V,W}^{reg}$. The action of $U(V)$ on $A_{V,W}$ can be trivially extended to an action of the whole general linear group $GL(V)$, preserving $A_{V,W}^{st}$ and $A_{V,W}^{reg}$.

On the other hand, fix a line $\ell \subset \mathbb{P}^2$; recall that a framed torsion free sheaf on $\mathbb{P}^2$ is a pair $(E, \Phi)$ consisting of a sheaf torsion free $E$ of rank $r$ and an isomorphism $\Phi : E|_{\ell} \cong O_{\ell} \oplus r$, called a framing at $\ell$.

Let $M_{\mathbb{P}^2}(r, n)$ denote the moduli space of framed torsion free sheaves on $\mathbb{P}^2$ of rank $r$ and second Chern class, or charge, $n$. Let also $M^\mu_{\mathbb{P}^2}(r, n)$ be the open subset consisting of locally free framed sheaves; recall that $M^\mu_{\mathbb{P}^2}(r, n)$ may also be regarded as the moduli space of framed $SU(r)$-instantons on $\mathbb{R}^3$ of charge $n$ [D].

The following is proved by Nakajima in [Na3, Section 2].

**Theorem 4.1.** There is a 1-1-correspondence between

1. $GL(V)$-orbits of $X \in A^{st}_{V,W}$, and
2. isomorphism classes of framed torsion free sheaves with rank $r = \dim W$ and charge $n = \dim V$.

Recalling (2.9), there is an isomorphism

$$
N^J_1 \congiso M_{\mathbb{P}^2}(r, n),
$$

that restricts to the regular locus

$$
N^J_{reg} \congiso M^\mu_{\mathbb{P}^2}(r, n).
$$

The goal of this section is to interpret the branes described in the previous section under the correspondence of Theorem 4.1 and provide some explicit examples showing that certain fixed point sets are nonempty.

However, we must first recall the construction of framed torsion free sheaves on $\mathbb{P}^2$ out of representation of the ADHM quiver.
4.1. The monad construction. Choose homogeneous coordinates \([x_0 : x_1 : x_2]\) on \(\mathbb{P}^2\) and set \(\ell\) to be the line \(\{x_0 = 0\}\). For every \(X = (A, B, I, J) \in \mathbb{A}_{V,W}\) define the sheaf complex

\[
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{\alpha_X} \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) \xrightarrow{\beta_X} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V,
\]

by setting

\[
\alpha_X = \begin{pmatrix} x_0 A - x_1 \\ x_0 B - x_2 \\ x_0 J \end{pmatrix} : \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{} \mathcal{O}_{\mathbb{P}^2} \otimes \begin{pmatrix} V \\ V \\ W \end{pmatrix}
\]

and

\[
\beta_X = \begin{pmatrix} -x_0 B + x_2, x_0 A - x_1, x_0 I \end{pmatrix} : \mathcal{O}_{\mathbb{P}^2} \otimes \begin{pmatrix} V \\ V \\ W \end{pmatrix} \xrightarrow{} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V;
\]

note that

\(\beta_X \circ \alpha_X = 0\) if and only if \([A, B] + JJ = 0\).

It is not difficult to check that (we refer to [Na3, Section 2] for details):

1. \(\alpha_X\) is injective for any nonzero \(X\),
2. \(\beta_X\) is surjective if and only if \(X\) is stable, in which case

\[
\mathcal{E}^X := \ker(\beta_X) / \text{im}(\alpha_X)
\]

is a torsion free sheaf, and
3. \(\mathcal{E}^X\) is locally free if and only if \(X\) is regular.

Note that \(\ker(\beta_X)|_\ell = (-x_1, -x_2, W)^t\) and \(\text{im}(\alpha_X)|_\ell = (-x_1, -x_2, 0)^t\), so the monad construction gives us also a framing

\[
\Phi^X : \mathcal{E}^X|_\ell = \ker(\beta_X)|_\ell / \text{im}(\alpha_X)|_\ell \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^2}|_\ell \otimes W.
\]

The action of \(M \in \text{GL}(V)\) gives isomorphic framed sheaves, i.e. \((\mathcal{E}^X, \Phi^X) \cong (\mathcal{E}^M X, \Phi^M X)\). All framed torsion free sheaves on \(\mathbb{P}^2\) can be obtained in this way.

4.2. Autodual framed sheaves. Take \(X \in \mathbb{A}_{V,W}^{\tau}\), and let \(Y = b_{(g,h)}(X)\). It is not difficult to see that the framed locally free sheaf \(\mathcal{E}^Y\) is the cohomology of the dual monad

\[
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V^\vee \xrightarrow{(\beta_X)^\vee} \mathcal{O}_{\mathbb{P}^2} \otimes (V^\vee \oplus V^\vee \oplus W^\vee) \xrightarrow{(\alpha_X)^\vee} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V^\vee,
\]

It follows that if \(X = b_{(g,h)}(X)\), then, for any \(g \in \text{GL}(V)\) and \(h \in \text{GL}(W)\), the associated framed locally free sheaf is autodual, i.e. there exists an isomorphism

\[
\varphi : (\mathcal{E}^X, \Phi^X) \rightarrow ((\mathcal{E}^X)^\vee, (\Phi^X)^\vee),
\]

cf. [JM] Proposition 2.4.

As a special case of Corollary [33], we obtain the following statement.

**Corollary 4.2.** The subvariety of autodual framed locally free sheaves is a \((B, B, B)\)-brane inside \(\mathcal{M}_{\mathbb{P}^2}^{\tau}(r, n)\).
Two types of autodual bundles have been studied in the literature: symplectic (if the isomorphism in (4.6) satisfies $\phi^\vee = -\phi$) and orthogonal bundles (if the isomorphism in (4.6) satisfies $\phi^\vee = \phi$).

For the first case, considering $g = 1_V$ and $h$ some antisymmetric element of $GL(W)$, it follows from [JMW] Proposition 3.1] that the corresponding fixed point set $(\mathcal{N}_{reg})^b$ coincides with the moduli space $M_{reg}^{\text{sym}}(r, n)$ of framed symplectic locally free sheaves on $\mathbb{P}^2$.

Ottaviani has shown in [O Theorem 7.7] that the moduli space $\mathcal{S}_{22}(r, n)$ of (unframed) locally free symplectic sheaves on $\mathbb{P}^2$ of rank $r$ and charge $n$ is irreducible, nonsingular variety of dimension $(r + 2)n - \binom{r + 1}{2}$, nonempty when $r$ is even. Since $M_{reg}^{\text{sym}}(r, n)$ is a $Sp(W)$-principal bundle over $\mathcal{S}_{22}(r, n)$, denoting by $Sp$ the symplectic group, (cf. [JMW Proposition 3.2]), we conclude that $M_{reg}^{\text{sym}}(r, n)$ is a irreducible, nonsingular $(B, B, B)$-brane within $M_{reg}^{\text{lf}}(r, n)$.

Framed orthogonal bundles only exist in even charge, cf. [AB Proposition 3.4] or [ZNM] Lemma 4.3. So considering $g$ to be the standard symplectic form on $V$ and $h$ some symmetric element of $GL(W)$, it follows from [JMW Proposition 4.1] that the corresponding fixed point set $(\mathcal{N}_{reg})^b$ coincides with the moduli space $M_{reg}^{\text{ort}}(r, n)$ of framed orthogonal locally free sheaves on $\mathbb{P}^2$.

Regarding orthogonal bundles, Abuaf and Boralevi proved in [AB Main Theorem] that the moduli space $\mathcal{F}_{22}(r, n)$ of (unframed) locally free orthogonal sheaves on $\mathbb{P}^2$ trivial on a generic line and of rank $r$ and charge $n$ is an irreducible, nonsingular variety of dimension $(r - 2)n - \binom{r - 1}{2}$, nonempty when $r = n \geq 4$ and $r = n - 1 \geq 7$. Since $M_{reg}^{\text{ort}}(r, n)$ is an $O(W)$-principal bundle over $\mathcal{F}_{22}(r, n)$, denoting by $O$ the orthogonal group, (cf. [JMW Proposition 4.2]), we conclude that $M_{reg}^{\text{ort}}(r, n)$ is an irreducible, nonsingular $(B, B, B)$-brane within $M_{reg}^{\text{lf}}(r, n)$.

Using the correspondence between framed locally free sheaves on $\mathbb{P}^2$ and framed instantons on $\mathbb{R}^4$ we may rephrase the conclusions above in terms of instantons: framed $Sp(r)$- and $SO(r)$-instantons on $\mathbb{R}^4$ form irreducible, nonsingular $(B, B, B)$-branes in the moduli space of framed $SU(r)$-instantons on $\mathbb{R}^4$. Using standard notation, $SO$ and $SU$ denoted respectively the special orthogonal and special unitary groups.

4.3. Framed sheaves fixed by unitary involutions. Since $\text{Aut}(\mathbb{P}^2) = PSL(3, \mathbb{C})$, a holomorphic involution of $\mathbb{P}^2$ can be described by an idempotent element of $PSL(3, \mathbb{C})$. We say that an involution is unitary if it is contained in $PSU(3) \subset PSL(3, \mathbb{C})$.

Lemma 4.3. Every (non-trivial) unitary involution preserving the line $\ell = \{x_0 = 0\}$ is either

\begin{equation}
\sigma_1 : \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \\
\left[ x_0 : x_1 : x_2 \right] \mapsto \left[ -x_0 : x_1 : x_2 \right],
\end{equation}

or

\begin{equation}
\sigma_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \\
\left[ x_0 : x_1 : x_2 \right] \mapsto \left[ x_0 : tx_1 + zx_2 : \overline{x}_1 - tx_2 \right].
\end{equation}

Proof. If an involution preserves the line $\ell = \{x_0 = 0\}$, then it must be associated to a $\mathbb{C}^*$-class of matrices of the form

\[
\begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & * & 0 \\
0 & 0 & *
\end{pmatrix}.
\]
In this section we show that the fixed point sets $k$ and framed sheaves invariant under pullback by $\sigma$ are a $(B, B, B)$-brane for $k = 1$ and a $(B, A, A)$-brane for $k = 2$.

First, we analyze the relation between the points fixed by the involution

$$c : \mathcal{M}_{\mathbb{P}^2} \to \mathcal{M}_{\mathbb{P}^2} \quad (A, B, I, J) \mapsto (-A, -B, -I, -J).$$

and framed sheaves invariant under pullback by $\sigma_1$. Note that

$$\alpha^{c(X)} = \sigma_1^* \alpha^X = \begin{pmatrix} x_0(-A) - x_1 \\ x_0(-B) - x_2 \\ x_0(-J) \end{pmatrix},$$

and

$$\beta^{c(X)} = \sigma_1^* \beta^X = \begin{pmatrix} -x_0(-B) + x_2, x_0(-A) - x_1, x_0(-I) \end{pmatrix}.$$ 

It follows that $\mathcal{E}^{c(X)} = \sigma_1^* \mathcal{E}^X$.

**Lemma 4.4.** Let $X \in \mathcal{A}_{\mathbb{P}^2}$ with $X = c_{(g, h)}(X)$. Then there exist an isomorphism of framed sheaves

$$\varphi_1 : (\mathcal{E}^X, \Phi^X) \overset{\sim}{\longrightarrow} (\sigma_1^*(\mathcal{E}^X), \sigma_1^*(\Phi^X)).$$

Furthermore, the restriction to the line at infinity $\ell \subset \mathbb{P}^2$ is $\varphi_1|_{\ell} = h$.

**Proof.** Note that $\sigma_1^*(\mathcal{E}^X)$ is associated to the monad

$$\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{\sigma_1^* \alpha^X} \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) \xrightarrow{\sigma_1^* \beta^X} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V.$$

Given $g \in U(V)$ and $h \in U(W)$, one can set $f : V \oplus V \oplus W \to V \oplus V \oplus W$ as follows:

$$f = \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & h \end{pmatrix}.$$

If $X = c_{(g, h)}(X)$, the following isomorphism of monads

$$\begin{array}{c}
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{\alpha^X} \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) \xrightarrow{\beta^X} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V \\
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{\sigma_1^* \alpha^X} \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) \xrightarrow{\sigma_1^* \beta^X} \mathcal{O}_{\mathbb{P}^2}(1) \otimes V,
\end{array}$$

provides the desired isomorphism of framed sheaves.

Together with Corollary 3.3 we conclude:

**Corollary 4.5.** The fixed point locus $\mathcal{M}_{\mathbb{P}^2}(r, n)^{\sigma_1}$ is a $(B, B, B)$-brane in $\mathcal{M}_{\mathbb{P}^2}(r, n)$; in addition, the fixed point locus $\mathcal{M}_{\mathbb{P}^2}(r, n)^{\sigma_1}$ is a $(B, B, B)$-brane inside $\mathcal{M}_{\mathbb{P}^2}(r, n)$.

**Remark 4.6.** Fix $r = n = 2$, and consider the involution $c_\gamma : \mathcal{M}_{\mathbb{P}^2} \to \mathcal{M}_{\mathbb{P}^2}$ defined by

$$c_\gamma(A, B, I, J) = (-gAg^{-1}, -gBg^{-1}, gIh, h^{-1}Jg^{-1}),$$

where $g \in U(V)$ and $h \in U(W)$.
choosing $g \in GL(V)$ and $h \in GL(W)$ as

$$g = h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The fixed points of $c_{(g,h)}^*$ are defined by the following conditions

i): $A g + g A = 0$;

ii): $B g + g B = 0$;

iii): $I - g I h = 0$;

iv): $J - h^{-1} J g^{-1} = 0$.

Taking the matrices

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

we get a framed locally free sheaf; indeed, it is not difficult to check that the ADHM equation and the regularity condition are satisfied. In addition, it is fixed by the involution $c_{(g,h)}^*$ in (4.7).

Therefore, we obtain a nonempty $(B, B, B)$-brane inside $\mathcal{M}^\mathbb{T}_{\mathbb{P}^2}(2, 2)$. Moreover, note that $2k \times 2k$ matrices whose blocks form is as above, with each entry promoted to a $k \times k$ block being a multiple of the identity matrix, provides an explicit example of a point in $\mathcal{M}^\mathbb{T}_{\mathbb{P}^2}(2k, 2k)$ fixed by the involution (4.7), and hence a nonempty $(B, B, B)$-brane inside $\mathcal{M}^\mathbb{T}_{\mathbb{P}^2}(2k, 2k)$.

It would be interesting to determine precisely for which values of rank and charge such $(B, B, B)$-branes are nonempty, as well as checking irreducibility, smoothness, and computing its dimension.

The analysis of the relation between the points fixed by the involution

$$d^\delta : \mathcal{M}^\mathbb{T}_{V,W} \longrightarrow \mathcal{M}^\mathbb{T}_{V,W}$$

$(A, B, I, J) \mapsto (t A + z B, \overline{A} - t B, I, J)$,

and framed sheaves invariant under pullback by $\sigma_2$ is similar. Note that

$$\sigma_2^\alpha X = \begin{pmatrix} x_0 A - (tx_1 + zx_2) \\ x_0 B - (tx_1 + zx_2) \end{pmatrix}$$

and

$$\sigma_2^\beta X = \begin{pmatrix} -x_0 B + (\overline{A} - tx_2), x_0 A - (tx_1 + zx_2), x_0 I \end{pmatrix}.$$ 

Defining $f : V \oplus V \oplus W \to V \oplus V \oplus W$ in the following way

$$f = \begin{pmatrix} tg & zg & 0 \\ zg & -tg & 0 \\ 0 & 0 & h \end{pmatrix}$$

the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V & \xrightarrow{\sigma_2^\alpha X} & \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W) \\
1 \otimes (-g) & \downarrow & 1 \otimes f \\
\mathcal{O}_{\mathbb{P}^2}(-1) \otimes V & \xrightarrow{\sigma_2^\beta X} & \mathcal{O}_{\mathbb{P}^2} \otimes (V \oplus V \oplus W)
\end{array}$$

is an isomorphism of monads that induces the isomorphism $\sigma_2^E X \simeq \mathcal{O}_{\mathbb{P}^2}(X)$.

Together with Corollary 3.7, we conclude:
Corollary 4.7. The fixed point locus $\mathcal{M}_{2}(r,n)^{2}$ is a $(B, A, A)$-brane in $\mathcal{M}_{2}(r,n)$; in addition, $\mathcal{M}_{2}(r,n)^{2}$ is a $(B, A, A)$-brane in $\mathcal{M}_{2}(r,n)$.

Remark 4.8. Fix $r = n = 4$, and consider the involution $bd_{(g,h)}: \mathcal{M}_{V,W} \to \mathcal{M}_{V,W}$ defined by

$$bd_{(g,h)}(A, B, I, J) = (gA^{t}g^{-1}, -gB^{t}g^{-1}, -gJ^{t}h, -h^{-1}P^{t}g^{-1}),$$

choosing $g \in GL(V)$ and $h \in GL(W)$ as

$$g = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Observe that the fixed points of $bd_{(g,h)}$ are defined by the following conditions

- i): $Ag - gA^{t} = 0$;
- ii): $Bg + gB^{t} = 0$;
- iii): $I - gJ^{t}h = 0$;
- iv): $J - h^{-1}P^{t}g^{-1} = 0$.

Notice that conditions (iii) and (iv) are equivalent with the given choices of $g$ and $h$. Now consider the matrices

$$A = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & -1 & 0 \\ 0 & 1 & a & 0 \\ -1 & 0 & 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} b_{1} & b_{2} & b_{3} & 1 \\ 0 & -b_{1} & 1 & b_{4} \\ b_{4} & 1 & -b_{1} & 0 \\ 1 & b_{3} & -b_{2} & b_{1} \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Taking $J = I$, it is easy to check that $(A, B, I, J)$ are a fixed points of $bd_{(g,h)}$. Moreover, they satisfy the ADHM equation and, since $I$ and $J$ maximal rank, also fulfill the regularity condition. It follows that $(A, B, I, J)$ as above defines a framed locally free sheaf. Checking with Table 11 we obtain a nonempty $(B, A, A)$-brane inside $\mathcal{M}_{2}(4,4)$.

Furthermore, promoting each entry of the matrices above to a multiple of the $k \times k$ the identity matrix provides an explicit example of a point in $\mathcal{M}_{2}(4k,4k)$ fixed by the involution $[15]$, and hence a nonempty $(B, A, A)$-brane inside $\mathcal{M}_{2}(4k,4k)$.

Again, it would be interesting to determine precisely for which values of rank and charge such $(B, A, A)$-branes are nonempty, as well as checking irreducibility and smoothness.

4.4. Real framed sheaves. Finally, we consider the antiholomorphic involution on $\mathbb{P}^{2}$ given by conjugation:

$$\tau_{0} : \mathbb{P}^{2} \to \mathbb{P}^{2}, \quad [x_{0} : x_{1} : x_{2}] \mapsto [\overline{x}_{0} : \overline{x}_{1} : \overline{x}_{2}].$$

Recall that, given the sheaf $\mathcal{E}$, one can construct the conjugate sheaf $\overline{\mathcal{E}}$ by taking the set of conjugate local sections

$$\overline{\mathcal{E}(U)} := \overline{\mathcal{E}(U)}.$$

Lemma 4.9. Let $X \in \mathbb{A}_{V,W}^{2}$ with $X = e_{(g,h)}(X)$. Then the associated framed locally free sheaf satisfies

$$(\mathcal{E}^{X}, \Phi^{X}) \cong (\tau_{0}^{*}\overline{\mathcal{E}^{\overline{X}}}, \tau_{0}^{*}\overline{\Phi^{\overline{X}}}).$$
The fixed point locus \( \mathcal{M}_{\mathbb{P}^2}^{(r,n)}(r,n) \) is a \((A,B,A)\)-brane in \( \mathcal{M}_{\mathbb{P}^2}^{(r,n)} \).

4.5. Combining involutions. Note that \( \tau_0 \) commutes with \( \sigma_1, \sigma_2 \), and with taking duals. Thus we can consider the antiholomorphic involutions of \( \mathbb{P}^2 \)

\[
\tau_1 := \tau_0 \sigma_1 = \sigma_1 \tau_0
\]

and

\[
\tau_2 := \tau_0 \sigma_2 = \sigma_2 \tau_0.
\]

Let \( \theta \) be either 1 or \(-1\) and recall that \((\mathcal{E}^*, \Phi^*)\) denotes \((\mathcal{E}', \Phi')\). We obtain, thanks to Section 3.5, the description of a collection of branes inside \( \mathcal{M}_{\mathbb{P}^2}^{(r,n)} \), see Table 2.

### Table 2. Brane types for composed involutions

| \( b_{(g,h)} \) | \( c_{(g,h)} \) | \( d_{(g,h)} \) | \( e_{(g,h)} \) | \( (eb)_{(g,h)} \) | \( (ec)_{(g,h)} \) | \( (ebc)_{(g,h)} \) | \( (ed)_{(g,h)} \) | \( (ebd)_{(g,h)} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( g = \theta g' \) and \( h = -\theta h' \) | \( g^2 = e^{i\omega} \) and \( h^2 = e^{-i\omega} \) | \( g^2 = e^{i\omega} \) and \( h^2 = e^{-i\omega} \) | \( g = \theta g' \) and \( h = \theta h' \) | \( g^2 = e^{i\omega} \) and \( h^2 = e^{-i\omega} \) | \( g = \theta g' \) and \( h = \theta h' \) | \( g^2 = e^{i\omega} \) and \( h^2 = e^{-i\omega} \) | \( g = \theta g' \) and \( h = \theta h' \) | \( g^2 = e^{i\omega} \) and \( h^2 = e^{-i\omega} \) |
| \( (E, \Phi) \mapsto (E', \Phi') \) | \( (E, \Phi) \mapsto (\sigma_1^* E, \sigma_1^* \Phi) \) | \( (E, \Phi) \mapsto (\sigma_1^* E, \sigma_1^* \Phi) \) | \( (E, \Phi) \mapsto (\sigma_1^* E, \sigma_1^* \Phi) \) | \( (E, \Phi) \mapsto (\tau_0^* E, \tau_0^* \Phi) \) | \( (E, \Phi) \mapsto (\tau_0^* E, \tau_0^* \Phi) \) | \( (E, \Phi) \mapsto (\tau_0^* E, \tau_0^* \Phi) \) | \( (E, \Phi) \mapsto (\tau_0^* E, \tau_0^* \Phi) \) | \( (E, \Phi) \mapsto (\tau_0^* E, \tau_0^* \Phi) \) |
| \( (B, B, B) \) | \( (B, A, A) \) | \( (A, B, A) \) | \( (A, B, A) \) | \( (A, B, A) \) | \( (A, A, B) \) | \( (A, A, B) \) | \( (A, A, B) \) | \( (A, A, B) \) |

Proof. If \((\mathcal{E}^X, \Phi^X)\) is given by the monad (4.2) with maps (4.3) and (4.4), then, the framed locally free sheaf \((\mathcal{E}^X, \Phi^X)\) over \( \mathbb{P}^2 \) is \( \tau_0 (\mathbb{P}^2) \) is given by

\[
\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{E}^X \otimes (\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{V},
\]

where

\[
\alpha^X \times \left( \frac{x_0 A - x_1}{x_0 B - x_2} \right)
\]

and

\[
\beta^X \times \left( \frac{E - x_0 B + x_2, x_0 A - x_1, x_0 B} \right).
\]

Then, recalling that \( \tau_0^* \mathcal{F} = \mathcal{F} \), one has that \( \tau_0^* (\mathcal{E}^X) \) is given by the monad

\[
\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{V} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{V},
\]

with

\[
\tau_0^* \alpha^X \times \left( \frac{x_0 A - x_1}{x_0 B - x_2} \right)
\]

and

\[
\tau_0^* \beta^X \times \left( \frac{-x_0 B + x_2, x_0 A - x_1, x_0 B} \right).
\]

\( \Box \)

Together with Corollary 3.10, we conclude:

Corollary 4.10. The fixed point locus \( \mathcal{M}_{\mathbb{P}^2}^{(r,n)}(r,n) \) is a \((A,B,A)\)-brane in \( \mathcal{M}_{\mathbb{P}^2}^{(r,n)} \).
Remark 4.11. The matrices considered in examples of Remark [4.6] are real ones, and thus are fixed points of the composed involution $ee_{(g,h)}$, with $e_{(g,h)}$ defined as in equation (4.7).

Therefore, we obtain a nonempty $(A, B, A)$-brane inside $M_{P^2}(2k, 2k)$; it would be interesting to check the irreducibility and smoothness of this brane.

Remark 4.12. If the entries $a, b_1, b_2, b_3, b_4$ of the matrices in Remark [4.8] are chosen to be real, then they are fixed points of the composed involution $bd_{(g,h)}$, with $bd_{(g,h)}$ defined as in equation (4.8).

Therefore, we obtain a nonempty $(A, A, B)$-brane inside $M_{P^2}(4k, 4k)$; it would be interesting to check the irreducibility and smoothness of this brane.

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