NONLINEAR INVISCID DAMPING FOR A CLASS OF MONOTONE SHEAR FLOWS IN FINITE CHANNEL

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Abstract. We prove the nonlinear inviscid damping for a class of monotone shear flows in $T \times [0,1]$ for initial perturbation in Gevrey-$\frac{s}{2}$ class with compact support. The main idea of the proof is to use the wave operator of a slightly modified Rayleigh operator in a well chosen coordinate system.

1. Introduction

We consider the 2D Euler system in the vorticity formulation with a background shear flow $(u(y), 0)$:

\[
\begin{cases}
\omega_t + u(y)\partial_x \omega - u''(y)\partial_x \psi + U \cdot \nabla \omega = 0, \\
U = \nabla^\perp \psi, \quad \Delta \psi = \omega, \quad \psi(t, x, 0) = \psi(t, x, 1) = 0, \\
\omega|_{t=0} = \omega_{in}(x, y).
\end{cases}
\]

Here, $(x, y) \in T \times [0,1]$, $\nabla^\perp = (-\partial_y, \partial_x)$ and $(U, \omega)$ are periodic in $x$ variable with period normalized to $2\pi$. The physical velocity is $(u(y), 0) + U$ where $U = (U^x, U^y)$ denotes the velocity perturbation and the total vorticity is $-u'(y) + \omega$.

In this work, we are interested in the long time behavior of (1.1) for small initial perturbations $\omega_{in}$. In particular, we show that all sufficiently small perturbations in a suitable regularity class undergo 'inviscid damping' and satisfy $(u(y), 0) + U(t, x, y) \to (u(y) + u_\infty(y), 0)$ as $t \to \infty$ for some $u_\infty(y)$ determined by the evolution.

The field of hydrodynamic stability started in the nineteenth century with Stokes, Helmholtz, Reynolds, Rayleigh, Kelvin, Orr, Sommerfeld and many others. In [18], Orr observed an important phenomenon that the velocity will tend to 0 as $t \to \infty$. This phenomenon is so-called inviscid damping, which is the analogue in hydrodynamics of Landau damping found by Landau [13], which predicted the rapid decay of the electric field of the linearized Vlasov equation around homogeneous equilibrium. Mouhot and Villani [17] made a breakthrough and proved nonlinear Landau damping for the perturbation in Gevrey class (see also [6]). In this case, the mechanism leading to the damping is the vorticity mixing driven by shear flow or Orr mechanism [18]. See [19, 20] for similar phenomena in various system.

Due to the presence of the nonlocal operator for general shear flows, the inviscid damping for general shear flows is a challenge problem even at linear level. For the linear inviscid damping we refer to [25, 21, 11, 12] for the results for general monotone flows. For non-monotone flows such as the Poiseuille flow and the Kolmogorov flow, another dynamic phenomena should be taken into consideration, which is so called the vorticity depletion phenomena, predicted by Bouchet and Morita [7] and later proved by Wei, Zhang and Zhao [22, 23]. See also [3].

Due to the possible nonlinear transient growth, it is a challenging task extending linear damping to nonlinear damping. Even for the Couette flow there are only few results. Moreover, nonlinear damping is sensitive to the topology of the perturbation. Indeed, Lin and Zeng [15] proved that nonlinear inviscid damping is not true for perturbations of the Couette flow in $H^s$.
for $s < \frac{3}{2}$. Bedrossian and Masmoudi [5] proved nonlinear inviscid damping around the Couette flow in Gevrey class $2_+$. Recently Deng and Masmoudi [8] proved some instability for initial perturbations in Gevrey class $2_+$. We refer to [9, 12] and references therein for other related interesting results.

Our main result is

**Theorem 1.1.** Suppose $u(y)$ is a smooth function defined on $[0, 1]$ which satisfies

1. (Monotone) There exists $c_0 > 0$ such that $\lambda' \geq c_0 > 0$. 
2. (Compact support) There exists $\theta_0 \in (0, \frac{1}{10}]$ such that $\text{supp } u'' \subset [4\theta_0, 1 - 4\theta_0]$. 
3. (Linear stability) The Rayleigh operator $u\text{Id} - u''\Delta^{-1}$ has no eigenvalue and no embedding eigenvalue. 
4. (Regularity) There exist $K > 1$ and $s_0 \in (0, 1)$ such that for all integers $m \geq 0$,

$$\sup_{y \in \mathbb{R}} \left| \frac{d^m(u''(y))}{dy^m} \right| \leq K^m(m!) \frac{1}{m+1}.$$

For all $1 > s_0 \geq s > 1/2$ and $\lambda_{in} > 0$, there exist $\lambda_{in} > \lambda_{\infty} = \lambda_{\infty}(\lambda_{in}, K, \theta_0, s) > 0$ and $0 < \lambda_{0} = \lambda_{0}(\lambda_{in}, \lambda_{\infty}, \theta_0, s) \leq \frac{1}{s}$ such that for all $e \leq \lambda_{0}$ if $\omega_{in}$ has compact support in $\mathbb{T} \times [3\theta_0, 1 - 3\theta_0]$ and satisfies

$$\|\omega_{in}\|_{G^\lambda_{in}}^2 = \sum_{k} \int_{\mathbb{T}} |\tilde{\omega}_{in}(k, \eta)| e^{2\lambda_{in}|k, \eta|^2} d\eta \leq \epsilon^2, \quad \int_{\mathbb{T} \times [0, 1]} \omega_{in}(x, y) dx dy = 0$$

then the smooth solution $\omega(t)$ to (1.1) satisfies:

1. (Compact support) For all $t \geq 0$, $\text{supp } \omega(t) \subset \mathbb{T} \times [2\theta_0, 1 - 2\theta_0]$. 
2. (Scattering) There exists $f_\infty \in G^\lambda_{\infty}$ with $\text{supp } f_\infty \subset \mathbb{T} \times [2\theta_0, 1 - 2\theta_0]$ such that for all $t \geq 0$,

$$\|\omega(t, x + tu(y) + \Phi(t, y), y) - f_\infty(x, y)\|_{G^\lambda_{\infty}} \lesssim \frac{\epsilon}{\langle t \rangle},$$

where $\Phi(t, y)$ is given explicitly by

$$\Phi(t, y) = \frac{1}{2\pi} \int_{0}^{t} \int_{\mathbb{T}} U^x(\tau, x, y) d\tau dy = u_\infty(y) t + O(\epsilon^2),$$

with $u_\infty = \partial_y < \Delta^{-1}f_\infty >$

3. (Inviscid damping) The velocity field $U$ satisfies

$$\left\| \frac{1}{2\pi} \int_{\mathbb{T}} U^x(t, x, \cdot) dx - u_\infty \right\|_{G^\lambda_{\infty}} \lesssim \frac{\epsilon^2}{\langle t \rangle^2},$$

$$\left\| U^x(t) - \frac{1}{2\pi} \int_{\mathbb{T}} U^x(t, x, \cdot) dx \right\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle},$$

$$\|U^y(t)\|_{L^2} \lesssim \frac{\epsilon}{\langle t \rangle^2}.$$

Let us give some remarks about Theorem 1.1:

- The compact support assumption of $u''$ is to prevent boundary effects.
- If the background flow satisfies $u''(y) \geq 0$, the Rayleigh operator $u\text{Id} - u''\Delta^{-1}$ has no eigenvalue and no embedding eigenvalue.
- The regularity assumption means that $u''$ is in some Gevrey-$\frac{1}{s_0}$ class.
- The solution will be less regular than the background flow $u(y)$. 

Let also mention a very recent paper by Ionescu and Jia [10], where a similar result to ours was proved with a different method. The two papers are independent.

Compared to (1.5) in [5], (1.2) loses an $\epsilon$, due to the linear nonlocal effect. We can recover the $\epsilon^2$ by using the wave operator $D_u$ defined in section 2:

**Corollary 1.2.** Let us define

$$D_u\omega(t,x,y) = \sum_{k=0}^{\infty} D_{u,k}(\hat{\omega}_k(t,y))e^{ikx}.$$  

Then there exists $f_\infty$ such that for all $t \geq 0$,

$$\|D_u\omega\|_{L^2} \lesssim \epsilon^2 \langle t \rangle.$$  

In (1.8), we can improve $L^2$ to Gevrey class by modulating the uniform estimate of the wave operator $D_{u,k}$ in Gevrey class.

1.1. Notation and conventions. For $f(x,y)$ in Schwartz space with compact support in $(0,1)$, we define the Fourier transform in the first direction $F_1 f(k,y)$, the Fourier transform in the second direction $F_2 f(x,\eta)$ and the Fourier transform in both directions $\hat{f}_k(\eta)$ where $(k,\eta) \in \mathbb{Z} \times \mathbb{R}$,

$$F_1 f(k,y) = \frac{1}{2\pi} \int_T f(x,y)e^{-ixk}dx,$$

$$F_2 f(x,\eta) = \int_\mathbb{R} f(x,y)e^{-i\eta y}dy,$$

$$\hat{f}_k(\eta) = \frac{1}{2\pi} \int_{T \times \mathbb{R}} f(x,y)e^{-ixk-i\eta y}dxdy.$$  

A convention we generally use is to denote the discrete $x$ (or $\tilde{z}$, $z$) frequencies as subscripts. By convention we always use Greek letters such as $\eta$ and $\xi$ to denote frequencies in the $y$, $u$ or $v$ direction and lowercase Latin characters commonly used as indices such as $k$ and $l$ to denote frequencies in the $x$, $\tilde{z}$ or $z$ direction (which are discrete). Another convention we use is to denote $K, M, N$ as dyadic integers $K, M, N \in D$ where

$$D = \left\{ \frac{1}{2}, 1, 2, 2^2, \ldots \right\}.$$  

When the sum is written with indices $K, M, M', N$ or $N'$ it will always be over a subset of $D$. This will be useful when defining Littlewood-Paley projections and paraproduct decompositions, see section A.1 in [5].

We use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \leq Cg$ (we analogously $f \gtrsim g$ define). Similarly, we use the notation $f \approx g$ when there exists $C > 0$ such that $C^{-1}g \leq f \leq Cg$. We sometimes use the notation $f \lesssim_\alpha g$ if we want to emphasize that the implicit constant depends on some parameter $\alpha$.

We will denote the $l^1$ vector norm $|k, \eta| = |k| + |\eta|$, by which convention is the norm taken in our work. Similarly, given a scalar or vector in $\mathbb{R}^d$ we denote

$$\langle \psi \rangle = \langle 1 + |\psi|^2 \rangle^{\frac{1}{2}}.$$  

We use a similar notation to denote the $x$ (or $z$) average of a function: $f_0 = \langle f \rangle = \frac{1}{2\pi} \int_T f(x,y)dx = P_0f$. We also frequently use the notation $P_x f = f - f_0$. We denote the standard $L^p$ norm by
\(\|f\|_{L^p}\). For any \(f\) with compact support in \((0, 1)\), we make common use of the Gevrey-\(\frac{1}{s}\) norm with Sobolev correction defined by
\[
\|f\|_{G^{\lambda,\sigma};s} = \sum_k \int \left| \hat{f}_k(\eta) \right|^2 e^{2\lambda|k,\eta|^s} \langle k, \eta \rangle^{2\sigma} d\eta.
\]
We refer to section A.2 in [5] for a discussion of the basic properties of this norm and some related useful inequalities. For \(\eta \geq 0\), we define \(E(\sqrt{|\eta|}) \in \mathbb{Z}\) to be the integer part. We define for \(\eta \in \mathbb{R}\) and \(1 \leq |k| \leq E(\sqrt{|\eta|})\) with \(\eta_k \geq 0\),
\[
t_k,\eta = \left| \eta_k \right| - |\eta| \left| k \right| (|k| + 1)
\]
and the critical intervals
\[
I_{k,\eta} = \begin{cases} [t_{|k|,\eta}, t_{|k|-1,\eta}] & \text{if } \eta k \geq 0 \text{ and } 1 \leq |k| \leq E(\sqrt{|\eta|}), \\ \emptyset & \text{otherwise.} \end{cases}
\]
For minor technical reasons, we define a slightly restricted subset as the resonant intervals
\[
I_{k,\eta} = \begin{cases} 1_{k,\eta} & \sqrt{|\eta|} \leq t_{k,\eta}, \\ \emptyset & \text{otherwise.} \end{cases}
\]
We use \(1_A\) be the characteristic function which means \(1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}\).
We also use the smooth cut-off functions \(\chi_1\) and \(\chi_2\) in Gevrey-\(\frac{2}{s_0+1}\) class which satisfies
\[
\text{supp } \chi_1 \subset [\theta_0, 1 - \theta_0], \quad \chi_1(y) \equiv 1 \text{ for } y \in [1 - 2\theta_0, 2\theta_0],
\]
\[
\text{supp } \chi_2 \subset \left[ \frac{2}{\theta_0}, 1 - \frac{\theta_0}{2} \right], \quad \chi_2(y) \equiv 1 \text{ for } y \in [\theta_0, 1 - \theta_0],
\]
and there exist \(K_1 > 1\) such that for all integers \(m \geq 0\)
\[
(1.9) \quad \sup_{y \in \mathbb{R}} \left| \frac{d^m \chi_i(y)}{dy^m} \right| \leq K_1^m (m!)^{\frac{2}{s_0+1}} (m+1)^{-2}, \quad i = 1, 2.
\]
One may refer [9, 12] for the construction of such cut-off functions.

1.2. Discussion. The key idea of this paper is to use the wave operator in a well chosen coordinate system. We are able to reduce the length of the paper, since we use [5] and [21] as black boxes. There are four propositions (see Proposition 2.12, 2.13, 2.15 and 2.19) which are identical with the propositions in [5]. The elliptic estimate (see Proposition 2.14) is slightly different, since the linear change of coordinates twists the Laplace operator \(\Delta_{x,y}\). In the energy estimate, we mainly focus on the new terms. One of them comes from the application of the wave operator (see Proposition 2.16), the other is coming from the nonlocal term (see Proposition 2.17 and 2.18).

2. Proof of Theorem 1.1

Next we give the proof of Theorem 1.1, starting the primary steps as propositions which are proved in the subsequent sections.

2.1. Time-dependent norm. We will use the same multiplier \(A(t, \nabla)\) introduced in [5].
\[
A_{\lambda}(t, \eta) = e^{\lambda(t)|k,\eta|^s} \langle k, \eta \rangle^{\sigma} J_k(t, \eta).
\]
The index \(\lambda(t)\) is the bulk Gevrey-\(\frac{1}{s}\) regularity and will be chosen to satisfy
\[
\lambda(t) = \frac{3}{4} \lambda_0 + \frac{1}{4} \lambda', \quad t \leq 1.
\]
we remark here that for
\[
\dot{\lambda}(t) = -\frac{\delta_\lambda}{(t)2^q}(1 + \lambda(t)), \quad t \in (1, \infty)
\]
where \(\lambda_0, \lambda'\) are parameters which depend on the regularity of the background flow \(u(y)\) and chosen by the proof, \(\delta_\lambda \approx \lambda_0 - \lambda'\) is a small parameter that ensures \(\lambda(t) \geq \frac{\lambda_0}{2} + \frac{\lambda'}{2}\) for \(t \geq 0\) and \(\frac{1}{2} < \tilde{q} \leq \frac{\tilde{q}}{3} + \frac{7}{16}\) is a parameter chosen by the proof.

Roughly speaking, we need the Gevrey bandwidth \(\lambda(t)\) in the multiplier \(A\) to be sufficiently small (if \(s = s_0\)) and determined by the background flow so that the wave operator is bounded in \(\{ f \in C^\infty : \|Af\|_2 < \infty\}\). See Proposition 2.5, Remark 2.7 and Remark 2.8 for more details.

The main multiplier for dealing with the Orr mechanism and the associated nonlinear growth is
\[
J_k(t, \eta) = \frac{e^{\mu(\eta)^\frac{1}{2}}}{w_k(t, \eta)} + e^{\mu|k|\frac{1}{2}},
\]
where
\[
w_k(t, \eta) = \begin{cases}
w_k(t_{E(\sqrt{\eta}), \eta}, \eta) & t < t_{E(\sqrt{\eta}), \eta} \\
w_{NR}(t, \eta) & t \in [t_{E(\sqrt{\eta}), \eta}, 2\eta] \setminus I_{k, \eta} \\
w_R(t, \eta) & t \in I_{k, \eta}
\end{cases}
\]
(2.2)
Here \((w_R(t, \eta), w_{NR}(t, \eta))\) is defined in the following way:

Let \(w_{NR}\) be a non-decreasing function of time with \(w_{NR}(t, \eta) = 1\) for \(t \geq 2\eta\). For definiteness, we remark here that for \(|\eta| \leq 1\), \(w_{NR}(t, \eta) = 1\), which will be a consequence of the definition. For \(k = 1, 2, 3, ..., E(\sqrt{\eta})\), we define
\[
w_{NR}(t, \eta) = \left(\frac{k^2}{\eta}(1 + b_{k, \eta}|t - \frac{\eta}{k}|)\right)^{C_k} w_{NR}(t_{k-1, \eta}, \eta),
\]
\(\forall t \in I_{k, \eta} = \left[\frac{\eta}{k}, t_{k-1, \eta}\right]\)
(2.3)
and
\[
w_{NR}(t, \eta) = \left(1 + a_{k, \eta}|t - \frac{\eta}{k}|\right)^{-1-C_k} w_{NR}\left(\frac{\eta}{k}, \eta\right),
\]
\(\forall t \in I_{1, \eta} = \left[t_{k, \eta}, \frac{\eta}{k}\right]\)
(2.4)
The constant \(b_{k, \eta}\) is chosen to ensure that \(\frac{k^2}{\eta}(1 + b_{k, \eta}|t_{k-1, \eta} - \frac{\eta}{k}|) = 1\), hence for \(k \geq 2\)
\[
b_{k, \eta} = \frac{2(k-1)}{k} \left(1 - \frac{k^2}{\eta}\right)
\]
and \(b_{1, \eta} = 1 - \frac{1}{\eta}\) and similarly,
\[
a_{k, \eta} = \frac{2(k+1)}{k} \left(1 - \frac{k^2}{\eta}\right)
\]
(2.5)
On each interval \(I_{k, \eta}\), we define \(w_{R}(t, \eta)\) by
\[
w_{R}(t, \eta) = \frac{k^2}{\eta} \left(1 + b_{k, \eta}|t - \frac{\eta}{k}|\right) w_{NR}(t, \eta), \quad \forall t \in I_{k, \eta} = \left[\frac{\eta}{k}, t_{k-1, \eta}\right],
\]
(2.7)
\[
w_{R}(t, \eta) = \frac{k^2}{\eta} \left(1 + a_{k, \eta}|t - \frac{\eta}{k}|\right) w_{NR}(t, \eta), \quad \forall t \in I_{1, \eta} = \left[t_{k, \eta}, \frac{\eta}{k}\right].
\]
(2.8)
We also define $J^R(t, \eta)$ and $A^R(t, \eta)$ to assign resonant regularity at every critical time:

$$
J^R(t, \eta) = \begin{cases} 
\epsilon^{i|\eta|} R^{-1} R^{\sqrt{\eta}}(t, \eta, \eta) & t < t_{E(\eta, \eta)} \\
\epsilon^{i|\eta|} R^{-1} R^{\sqrt{\eta}}(t, \eta) & t \in [t_{E(\eta, \eta)}, 2\eta] \\
\epsilon^{i|\eta|} R^{-1} R^{\sqrt{\eta}} & t \geq 2\eta,
\end{cases}
\tag{2.9}
$$

Moreover, there exist $A, B$ be two self-adjoint operators in the Hilbert space $H$, then the wave operator $D$ related to $A$ and $B$ satisfies

$$
A D = D B
$$

It can be defined by

$$
D = \lim_{t \to \infty} e^{-itA} e^{itB}.
$$

However the wave operator related to non-self-adjoint operators is usually not easy to construct and estimate. Recently, this was successfully used to solve important problems in fluid mechanics. In [14], the authors use the wave operator method to solve Gallay’s conjecture on pseudospectral and spectral bounds of the Oseen vortices operator. In [23], the wave operator method was used to solve Beck and Wayne’s conjecture.

In order to absorb the nonlocal term, we also construct a wave operator $D_{u,k}$ which is related to the Rayleigh operator $R_{u,k} = u(y) \text{Id} - u''(y) \Delta_{k}^{-1}$ where $\Delta_{k} = \partial_{y}^2 - k^2$. We have the following proposition.

**Proposition 2.1.** Suppose $k \neq 0, u \in C^4([0, 1])$ is a strictly monotone function and $R_{u,k}$ has no eigenvalue and no embedding eigenvalue. Then there exists $D_{u,k}$ and $D_{u,k}^1$ such that

$$
D_{u,k} R_{u,k} = u D_{u,k},
\tag{2.10}
$$

and for $f, g \in L^2(0, 1)$

$$
\int_{0}^{1} D_{u,k}(f(y)) D_{u,k}^1(g(y)) dy = \int_{0}^{1} f(y) g(y) dy.
\tag{2.11}
$$

Moreover, there exist $C > 1$ independent of $k$ such that

$$
C^{-1} \leq \|D_{u,k}\|_{L^2 \to L^2} \leq C, \quad C^{-1} \leq \|D_{u,k}^1\|_{L^2 \to L^2} \leq C.
\tag{2.12}
$$

Note that

$$
D_{u,k}^1 = (D_{u,k}^{-1})^*.
\tag{2.13}
$$

The proof can be found in Section 7.1. One can also refer to [21] for more details.

**Remark 2.2.** We have the following representation formula of the wave operator and its inverse:

$$
D_{u,k}(\omega)(k, y) = d_1(k, y) \omega(y) + u''(y) d_2(k, y) \int_{0}^{1} \frac{e(k, y', y)}{u(y') - u(y)} \omega(y') dy',
\tag{2.14}
$$

$$
D_{u,k}^1(\omega)(k, y) = d_1(k, y) \omega(y) + d_2(k, y) \int_{0}^{1} \frac{e(k, y', y) u''(y')}{u(y') - u(y)} \omega(y') dy',
\tag{2.15}
$$

$$
D_{u,k}^{-1}(\omega)(k, y) = d_1(k, y) \omega(y) + u''(y) \int_{0}^{1} \frac{e(k, y, y')}{u(y) - u(y')} d_2(k, y') \omega(y') dy'.
\tag{2.16}
$$
where $d_1(k, y)$, $d_2(k, y)$ and $e(k, y', y)$ are defined in (7.5), (7.6) and (7.3).

Let us now introduce the linear change of coordinates, namely $(t, x, y) \rightarrow (t, z, v)$:

(2.17) $\quad v = u(y), \quad z = x - tv.$

Under this linear change of coordinates, $\Delta_{x,y}$ becomes $\Delta_u$, which is defined by

(2.18) $\quad \Delta_u \overset{\text{def}}{=} \partial_{zz} + (u')^2(\partial_v - t\partial_z)^2 + \tilde{u}''(\partial_v - t\partial_z)$

where $\tilde{u}' = u' \circ u^{-1}(v)$ and $\tilde{u}'' = u'' \circ u^{-1}(v)$. Here and below, for any $\phi(y)$, $\tilde{\phi}(v) = \phi \circ u^{-1}(v)$.

We also denote $\Delta_{u,k} = -k^2 + (\tilde{u}''(\partial_v - itk))^2 + \tilde{u}''(\partial_v - itk)$ to be the Fourier transform of $\Delta_u$ in the first direction.

Let us now write (2.14) and (2.16) in the $(z, v)$ coordinate.

\[ D_{u,k}(F_1 f(t, k, \cdot))(t, k, v) = D_1(k, v)F_1 f(t, k, v) \]

\[ + u''(u^{-1}(v))D_2(k, v) \int_{u(0)}^{u(1)} E(k, v_1, v) \frac{F_1 f(t, k, v_1)e^{-i(v_1-v)tk}}{v_1 - v} (u^{-1})'(v_1)dv_1, \]

\[ D_{u,k}^{-1}(F_1 f(t, k, \cdot))(t, k, v) = D_1(k, u)F_1 f(t, k, v) \]

\[ + D_2(k, v) \int_{u(0)}^{u(1)} E(k, v_1, v) \frac{F_1 f(t, k, v_1)e^{-i(v_1-v)tk}}{v_1 - v} (u^{-1})'(v_1)dv_1. \]

We change $D_{u,k}^{-1}$ in the $(z, v)$ coordinate slightly (see Remark 2.4) to make the dual of $D_{u,k}$ in $v$ variable:

\[ D_{u,k}^1(F_1 f(t, k, \cdot))(t, k, v) = D_1(k, v)F_1 f(t, k, v) \]

\[ + D_2(k, v) \int_{u(0)}^{u(1)} E(k, v_1, v) \frac{u''(u^{-1}(v_1))F_1 f(t, k, v_1)e^{-i(v_1-v)tk}}{v_1 - v} (u^{-1})'(v_1)dv_1. \]

Here $D_1(k, v)$, $D_2(k, v)$ and $E(k, v_1, v)$ satisfy

(2.19)

\[ D_1(k, u(y)) = d_1(k, y), \quad D_2(k, u(y)) = d_2(k, y), \]

\[ E(k, w(y_1), u(y_1)) = e(k, y_1, y). \]

One can also refer to (7.13), (7.14) and (7.15) for the explicit formula.

**Proposition 2.3.** For $k \neq 0$, let $R_{u,k} = vId - u''(u^{-1}(v))\Delta_{u,k}^{-1}$ be a modified Rayleigh operator. Then it holds for any $f(t, z, v)$ and $g(t, z, v)$ that

(2.20) $\quad D_{u,k}R_{u,k} = vD_{u,k},$

(2.21) $\quad D_{u,k}D_{u,k}^{-1} = D_{u,k}^{-1}D_{u,k} = Id, \quad D_{u,k} = (D_{u,k}^1)^*,$

(2.22) $\quad [\partial_t, D_{u,k}](F_1 f(t, k, \cdot))(t, k, v) = [ikv, D_{u,k}](F_1 f(t, k, \cdot))(t, k, v).$

(2.23) $\quad \int D_{u,k}(F_1 f(t, k, \cdot))(t, k, v)D_{u,k}^{-1}(F_1 g(t, k, \cdot))(t, k, v)dv = \int F_1 f(t, k, u)\overline{F_1 g(k, v)}dv,$

and going back to $(x, y)$, it holds that

\[ D_{u,k}(F_1 f(t, k, \cdot))(t, k, v)e^{ikz} \bigg|_{z=x-u(y)t, \ u=u(y)} = D_{u,k}(F_1 f(t, k, \cdot))(t, k, u(y))e^{ik(x-tu(y))}. \]
Recall Proposition 2.5. The proposition shows that the wave operator does not move frequencies a lot.

Remark 2.7. It holds that
\[ \text{supp} \theta \text{ and } k \]

for any \( f(t, x, y) = f(t, z, v) \big|_{z=u(y)t, \quad v=u(y)} \).

The proof is similar to the proof of (3.39a) in [5].
2.3. **Nonlinear coordinate transform.** Let us recall the nonlinear change of coordinate from [5], namely \((t, x, y) \to (t, z, v)\):

\[
v(t, y) = u(y) + \frac{1}{t} \int_0^t < -\partial_y \psi > (t', y) \chi_1(y) dt' \\
(2.24) = u(y) - \chi_1(y) \frac{t}{t} \int_0^t \int \partial_y \psi(t', x, y) dx dt',
\]

\[z(t, x, y) = x - tv(t, y).
\]

Thus \(\text{Ran } u = \text{Ran } v = [u(0), u(1)]\). This is the nonlinear version of (2.17).

Let us also define

\[
\begin{align*}
\partial_t v(t, v) &= \partial_t v(t, y), \\
\partial_y v(t, v) &= \partial_y v(t, y), \\
\partial_{yy} v(t, v) &= \partial_{yy} v(t, y),
\end{align*}
\]

\[w''(t, v) = u''(y), \quad w'(t, v) = u'(y).
\]

Here and below, for any \(\varphi(y), \varphi(v(t, y)) = \varphi(y)\).

Define \(\Omega(t, z, v) = \omega(t, x, y)\) and \(\Psi(t, z, v) = \psi(t, x, y)\), hence the original 2D Euler system (1.1) is expressed as

\[
\begin{align*}
\left\{
\begin{array}{l}
\partial_t \Omega - u'' \partial_z P_x \Psi + U \cdot \nabla_{z,v} \Omega = 0, \\
\Delta_t \Psi(t, z, v) \overset{\text{def}}{=} \partial_z \Psi + (\partial_y v)^2 (\partial_v - t\partial_z)^2 \Psi + \partial_{yy} v (\partial_v - t\partial_z) \Psi = \Omega(t, z, v),
\end{array}
\right.
\end{align*}
\]

where \(U = (0, \partial_y) + \partial_y \Upsilon \nabla_{z,v} P_x (\Psi \Upsilon)\). Here we use a smoother cut-off function \(\Upsilon(v) \in G_{ph,1}^{M(\theta_0), \frac{\rho_1}{2}}\) which satisfies \(\text{supp } \Upsilon(v) \subset [u(\theta_0), u(1 - \theta_0)]\) and \(\Upsilon(v) \equiv 1\), for \(v \in [u(\theta_0), u(1 - \theta_0)]\) to replace \(\chi_2\).

We introduce \(\Delta_t^{-1} \Omega = \Upsilon \Delta_t^{-1} (\Upsilon \Omega)\), then by the compact support of \(\Omega\), we have

\[\Psi \Upsilon = \Delta_t^{-1} \Omega.
\]

Now let us introduce the good unknown

\[
f(t, z, v) \overset{\text{def}}{=} < \Omega > (t, v) + P_{|k| \geq k_M} \Omega(t, z, v) \\
+ \sum_{0 < |k| < k_M} D_{u,k} (\mathcal{F}_1 \Omega(t, k, \cdot))(t, k, v) e^{iz},
\]

where

\[P_{|k| \geq k_M} \Omega(t, z, v) = \sum_{|k| \geq k_M} \mathcal{F}_1 \Omega(t, k, v) e^{iz}.
\]

Here \(k_M\) is a large constant depending only on the background shear flow \(u(y)\) and will be determined in the proof.

**Remark 2.8.** It holds that

\[
\Omega(t, z, v) =< f > (t, v) + P_{|k| \geq k_M} f(t, z, v) + \sum_{0 < |k| < k_M} D_{u,k}^{-1} (\mathcal{F}_1 f(t, k, \cdot))(t, k, v) e^{iz}.
\]

which gives us that there exist \(C = C(k_M) \geq 1\) independent of \(t\) such that for any \(\lambda < \lambda_D\),

\[
C^{-1} ||\Omega||_{G^{\lambda,\sigma}\lambda T \times [u(0), u(1)]} \leq ||f||_{G^{\lambda,\sigma}\lambda T \times [u(0), u(1)]} \leq C ||\Omega||_{G^{\lambda,\sigma}\lambda T \times [u(0), u(1)]},
\]

and that there exist \(C = C(k_M) \geq 1\) independent of \(t\) such that for any \(\lambda(t) < \lambda_D\),

\[
C^{-1} ||A\Omega||_2^2 \leq ||Af||_2^2 \leq C ||A\Omega||_2^2.
\]
The proof of the last inequality is similar to the proof of the product Lemma 3.8 and (3.40) in [5], which follows from Lemma 3.5 and 3.4 in [5], we omit the details.

An easy calculation gives that
\[
D_{u,k} \left( F_1 (u'' \partial_z (\Delta^{-1}_t \Omega)) \right) (t, k, v) = D_{u,k} \left( \tilde{F}_1 (u'' \partial_z (\Delta^{-1}_t \Omega) - \tilde{u}'' \partial_z (\Delta^{-1}_u \Omega)) \right) (t, k, v)
\]
\[
= D_{u,k} \left( F_1 \left( \tilde{F}_1 \left( (u'' - \tilde{u}'') \partial_z (\Delta^{-1}_t \Omega) - \tilde{u}'' \partial_z (\Delta^{-1}_u \Omega - \Delta^{-1}_t \Omega) \right) \right) \right) (t, k, v)
\]
we obtain by (2.22) that
\[
\partial_t f = -P_0 (U \cdot \nabla_{z,v} \Omega) + u'' \partial_z P_{|k| \geq k_M} (\Delta^{-1}_t \Omega)
\]
\[
- P_{|k| \geq k_M} (U \cdot \nabla_{z,v} \Omega)
\]
\[
(2.27)
\]
\[
+ \sum_{0 \leq |k| \leq k_M} D_{u,k} \left( F_1 \left( (u'' - \tilde{u}'') \partial_z (\Delta^{-1}_t \Omega) - \tilde{u}'' \partial_z (\Delta^{-1}_u \Omega - \Delta^{-1}_t \Omega) \right) \right) (t, k, v) e^{i z k}
\]
\[
- \sum_{0 \leq |k| \leq k_M} D_{u,k} \left( F_1 \left( U \cdot \nabla_{z,v} \Omega \right) \right) (t, k, v) e^{i z k}.
\]

We also have
\[
\partial_t v(t, y) = -\frac{\chi_1(y)}{t^2} \int_0^t < -\partial_y \psi > (t', y) dt' + \frac{\chi_1(y)}{t} < -\partial_y \psi > (t, y),
\]
and
\[
\partial_{tt} v(t, y) = \partial_t (\partial_t v(t, v(t, y))) = \partial_t (\partial_t v)(t, v(t, y)) + \partial_t v(t, y) \partial_t (\partial_t v)(t, v(t, y)) + \partial_t v(t, y) \partial_t (\partial_t v)(t, v(t, y))
\]
which gives that
\[
\partial_{tt} v(t, y) = 2\frac{\chi_1(y)}{t^3} \int_0^t < -\partial_y \psi > (t', y) dt' - 2\frac{\chi_1(y)}{t^2} < -\partial_y \psi > (t, y) + \frac{\chi_1(y)}{t} < \partial_t U^x > (t, y)
\]
\[
\frac{-2}{t} \partial_t v(t, y) + \frac{\chi_1(y)}{t} < \partial_t U^x > (t, y).
\]

By using the fact that
\[
\partial_t U^x + u(y) \partial_x U^x + u''(y) U^y + \partial_x p + U^x \partial_x U^x + U^y \partial_y U^x = 0,
\]
we obtain that
\[
\partial_t (\partial_t v) + \partial_t v \partial_x (\partial_t v) + \frac{2}{t} (\partial_t v) = -\frac{\chi_1}{t} \partial_y v < \nabla_{z, v}^\perp P_{\Psi} (\Psi \Phi) \cdot \nabla_{z, v} \tilde{U}^x >,
\]
where \( \tilde{U}^x(t, z(t, x), v(t, y)) = U^x(t, x, y) \).

We also have
\[
\partial_y v(t, y) = u'(y) - \frac{1}{t} \int_0^t < \omega > (t', y) dt',
\]
then it holds that
\[
\partial_t \left( t(\partial_y v(t, y) - u'(y)) \right) = -< \omega > (t, y).
\]

Let \( h(t, v(t, y)) = \partial_y v(t, y) - u'(y) \), then \( h(t, v) \) satisfies
\[
\partial_t h + \partial_t v \partial_v h = \frac{1}{t} (-P_0 \Omega - h) \overset{\text{def}}{=} \tilde{h} = \partial_y v \partial_t v.
\]

By (2.28) and (2.25), we have
\[
\partial_t \tilde{h} = -\frac{\tilde{h}}{t} - \frac{1}{t} (\partial_t P_0 \Omega + \partial h) = -\frac{2\tilde{h}}{t} - \partial_t v \partial_t \tilde{h} + \frac{\partial v}{t} < \nabla_{v, z}^\perp P_{\Psi} (\Psi \Phi) \cdot \nabla_{v, \Omega} >.
\]

**Remark 2.9.** Recall that \( \varphi(t, v(t, y)) = \varphi(y) \) and \( \bar{\varphi}(u(y)) = \varphi(y) \), then \( \varphi(t, v) \) satisfies the following transport equation
\[
\partial_t \varphi(t, v) + \partial_t v \partial_v \varphi(t, v) = 0,
\]
and \( \varphi_\delta = \varphi(t, v) - \bar{\varphi}(v) \) satisfies
\[
\partial_t \varphi_\delta + \partial_t v \partial_v \varphi_\delta = -\partial_t v \partial_v \bar{\varphi}(v).
\]

Thus we have
\[
(\partial_t + \partial_t v(t, v) \partial_v) (\chi_1 - \tilde{\chi}_1) = -\partial_t v(t, v) \partial_v \tilde{\chi}_1(v),
\]
\[
(\partial_t + \partial_t v(t, v) \partial_v) (\tilde{u}' - \tilde{u}') = -\partial_t v(t, v) \partial_v \tilde{u}'(v),
\]
\[
(\partial_t + \partial_t v(t, v) \partial_v) (\tilde{u}'' - \tilde{u}'') = -\partial_t v(t, v) \partial_v \tilde{u}''(v).
\]

Note that on the right hand side, the functions \( \partial_v \tilde{\chi}_1(v), \partial_v \tilde{u}'(v) \) and \( \partial_v \tilde{u}''(v) \) only depend on the background \( u \) and are smoother than \( \partial_t v(t, v) \).

### 2.4. Main energy estimate.
Let us now introduce the main energy for large time:
\[
E(t) = \frac{1}{2} \| A(t, \nabla) f \|_2^2 + E_v(t) + E_d(t)
\]
and for some constants \( K_v, K_D \) depending only on \( s, \lambda_M, \lambda_\infty \) fixed by the proof
\[
E_v(t) = (t)^{2+2s} \left\| \frac{A}{(\partial_t)^s} \tilde{h}(t) \right\|_2^2 + (t)^{4-K_D} \left\| \partial_t v(t) \right\|_{\tilde{G}_{\lambda(t), \sigma - \delta}}^2 + \frac{1}{K_v} \| A^R \tilde{h}(t) \|_2^2.
\]
and
\[
E_d(t) = \frac{1}{K_v} \left( \| A^R (\tilde{\chi}_1 - \tilde{\chi}_1) \|_2^2 + \| A^R (\tilde{u}' - \tilde{u}') \|_2^2 + \| A^R (\tilde{u}'' - \tilde{u}'') \|_2^2 \right).
\]
We refer to Lemma 2.1 in [5] for the local wellposedness for 2D Euler in Gevrey spaces also see [2]. By this way, we may safely ignore the time interval [0, 1] by further restricting the size of the initial data.

The goal is next to prove by a continuity argument that this energy \( E(t) \) (together with some related quantities) is uniformly bounded for all time if \( \epsilon \) is sufficiently small.

We define the following controls referred to in the sequel as the bootstrap hypotheses for \( t \geq 1 \) and some constant \( C(k_M) \),

(B1) \( E(t) \leq 10C(k_M)\epsilon^2 \);

(B2) Compact support of \( \Omega \): \( \text{supp} \Omega \subset [2.5\theta_0 - 10C(k_M)\epsilon, 1 - 2.5\theta_0 + 10C(k_M)\epsilon] \);

(B3) ‘CK’ integral estimates

\[
\int_1^t \left[ C_{K\lambda}(\tau) + C_{Kw}(\tau) + C_{Kg}(\tau) + C_{Kw^{v,2}}(\tau) + C_{K\lambda^{v,2}}(\tau) \right.
\]

\[
+ K_v^{-1} \left( C_{Kw^{v,1}}(\tau) + C_{K\lambda^{v,1}}(\tau) \right) + K_v^{-1} \sum_{i=1}^2 \left( C_{CCKw^i}(\tau) + C_{CCK\lambda^i}(\tau) \right) \bigg] d\tau \leq 20C(k_M)\epsilon^2.
\]

The CK terms above arise from the time derivatives of \( A(t) \) and \( A^R(t) \) and are naturally controlled by the energy estimates we are making. See (2.40), (2.41), (2.47) for the definitions.

**Proposition 2.10** (Bootstrap). Let \( k_M \) be fixed and sufficiently large. There exists an \( \epsilon_0 \in (0, \frac{1}{2}) \) depending only on \( k_M, \lambda_0, \lambda', s \) and \( \sigma \) such that if \( \epsilon < \epsilon_0 \) and on \( [1, T^*] \) the bootstrap hypotheses (B1)-(B3) hold, then for \( \forall t \in [1, T^*] \),

1. \( E(t) \leq 8C(k_M)\epsilon^2 \),

2. Compact support of \( \Omega \): \( \text{supp} \Omega \subset [2.5\theta_0 - 8C(k_M)\epsilon, 1 - 2.5\theta_0 + 8C(k_M)\epsilon] \),

3. and the CK controls satisfy:

\[
\int_1^t \left[ C_{K\lambda}(\tau) + C_{Kw}(\tau) + C_{Kg}(\tau) + C_{Kw^{v,2}}(\tau) + C_{K\lambda^{v,2}}(\tau) \right.
\]

\[
+ K_v^{-1} \left( C_{Kw^{v,1}}(\tau) + C_{K\lambda^{v,1}}(\tau) \right) + K_v^{-1} \sum_{i=1}^2 \left( C_{CCKw^i}(\tau) + C_{CCK\lambda^i}(\tau) \right) \bigg] d\tau \leq 18C(k_M)\epsilon^2,
\]

form which it follows that \( T^* = +\infty \).

The remainder of this section is devoted to the proof of Proposition 2.10, the primary step being to show that on \([1, T^*]\), we have

\[
E(t) + \frac{1}{2} \int_1^t \left[ C_{K\lambda}(\tau) + C_{Kw}(\tau) + C_{Kg}(\tau) + C_{Kw^{v,2}}(\tau) + C_{K\lambda^{v,2}}(\tau) \right.
\]

\[
+ K_v^{-1} \left( C_{Kw^{v,1}}(\tau) + C_{K\lambda^{v,1}}(\tau) \right) + K_v^{-1} \sum_{i=1}^2 \left( C_{CCKw^i}(\tau) + C_{CCK\lambda^i}(\tau) \right) \bigg] d\tau \leq E(1) + C(k_M)\epsilon^2 + K\epsilon^3.
\]

for some constant \( K \) which is independent of \( \epsilon \) and \( T^* \). If \( \epsilon \) is sufficiently small then (2.37) implies Proposition 2.10. Indeed, the control of the compact support of \( \Omega \) is based on a standard argument using the Lagrangian coordinates and the inviscid damping result. The proof can be found at the end of the section.

It is natural to compute the time evolution of \( E(t) \). Indeed we have

\[
\frac{1}{2} \frac{d}{dt} \|A(t, \nabla f)\|_2^2
\]
where the CK stands for ‘Cauchy-Kovalevskaya’ and expressed as

\[ (2.43) \]

\[ \eta w^{-\frac{1}{2}} \sum_{k} \int A_{k}(\eta) \tilde{f}_{k}(\eta) A_{k}(\eta) F_{2}(i k D_{u,k} \left( F_{1}\left( (\tilde{u}'-\tilde{u}'')(\Delta^{-1}_{t}\Omega) \right) \right) (t, k, \cdot) ) d\eta \]

\[ (2.39) \]

\[ = -C_{K}\lambda - C_{Kw} + \frac{1}{2\pi} \Pi^{tr}_{1} + \frac{1}{2\pi} \Pi^{u''} + \frac{1}{2\pi} \Pi^{tr}_{2} + \frac{1}{2\pi} \Pi^{u''e_{1}} + \frac{1}{2\pi} \Pi^{u''e_{2}} + \frac{1}{2\pi} \Pi, \]

where the CK stands for ‘Cauchy-Kovalevskaya’ and expressed as

\[ (2.40) \]

\[ C_{Kw} = -\lambda(t) \left\| \nabla^{s/2} A f \right\|_{2}^{2}, \]

\[ (2.41) \]

\[ C_{Kw} = \sum_{k} \int \frac{\partial t w_{k}(t, \eta)}{w_{k}(t, \eta)} e^{\lambda(t)\left\| k, \eta^{s} \right\| } \sigma \frac{e^{\mu\left\| k, \eta^{s} \right\| /2}}{w_{k}(t, \eta)} A_{k}(t, \eta) |\tilde{f}_{k}(t, \eta)|^{2} d\eta. \]

We define

\[ (2.42) \]

\[ \tilde{J}_{k}(t, \eta) = \frac{e^{\mu\left\| k, \eta^{s} \right\| /2}}{w_{k}(t, \eta)}, \]

\[ (2.43) \]

\[ \tilde{A}_{k}(t, \eta) = e^{\lambda(t)\left\| k, \eta^{s} \right\| } \sigma \tilde{J}_{k}(t, \eta). \]

**Remark 2.11.** Proposition 2.5 gives us that for \( \lambda_{0} \) and \( \lambda' \) sufficiently small,

\[ -\lambda(t) \left\| \nabla^{s/2} A \Omega \right\|_{2}^{2} \lesssim C_{Kw}, \quad \left\| \sqrt{\frac{\partial t w}{w}} A \Omega \right\|_{2}^{2} \lesssim C_{Kw} + C_{Kw}. \]

**Proof.** The first inequality is easy to prove. One can refer to the proof of (3.39a) in [5] for more details.

One can also follow the proof of (3.39b) in [5] to prove the second inequality. Here, we want to point out the difference. By applying Proposition 2.5, we get that

\[ \left| \sqrt{\frac{\partial t w_{k}(\eta)}{w_{k}(\eta)}} A_{k}(\eta) \tilde{f}_{k}(\eta) \right| \lesssim \sum_{M \geq 8} \int_{\mathbb{R}} \sqrt{\frac{\partial t w_{k}(\eta)}{w_{k}(\eta)}} A_{k}(\eta)(e^{-\lambda_{D}\left\| \eta - \xi^{(0)} \right\|_{M}})_{M} \left| f_{k}(\xi) \right|_{M} d\xi \]

\[ + \sum_{M \geq 8} \int_{\mathbb{R}} \sqrt{\frac{\partial t w_{k}(\eta)}{w_{k}(\eta)}} A_{k}(\eta)(e^{-\lambda_{D}\left\| \eta - \xi^{(0)} \right\|_{M}})_{M} \left| f_{k}(\xi) \right|_{M} d\xi \]

\[ + \sum_{M' \in \mathcal{D}} \sum_{M' \leq 8 M'} \int_{\mathbb{R}} \sqrt{\frac{\partial t w_{k}(\eta)}{w_{k}(\eta)}} A_{k}(\eta)(e^{-\lambda_{D}\left\| \eta - \xi^{(0)} \right\|_{M'}})_{M'} \left| f_{k}(\xi) \right|_{M} d\xi \]
\[ I_{HL} + I_{LH} + I_R. \]

One can follow the same argument as in the proof of (3.39b) in [5] and obtain the estimates of \( I_{LH} \) and \( I_R \). To treat \( I_{HL} \), we use the fact that
\[
\sqrt{\partial_t w_k(\eta)} w_k(\eta) \lesssim |\eta| t \quad \text{for } t \geq 1,
\]
which implies that
\[
\|I_{HL}\|_2^2 \lesssim \|f\|_{G_{\sigma, \lambda}}^2 \|\langle t \rangle\|_2^2 \lesssim CK_{\lambda}.
\]
Thus we complete the proof of the remark. □

Now we control (2.39). The most difficult part is to treat the \( \Pi \) term. The key idea is to extract the transport structure from \( \Pi \) which can be treated with \( \Pi^{tr_1} \) and \( \Pi^{tr_2} \). That is
\[
\Pi = \Pi^{tr_3} + \Pi^{com}, \quad \text{also see (4.1)}.
\]
with
\[
\Pi^{tr_3} = - \sum_{0 < |k| < k_M} \int A_k(\eta) \tilde{\Omega}_k(\eta) A_k(\eta) (U \cdot \nabla_{z,v}\Omega)_k(\eta) d\eta.
\]

The estimate of \( \Pi^{tr} = \Pi^{tr_1} + \Pi^{tr_2} + \Pi^{tr_3} \) is similar to the (2.23) in [5]. We start with integrating by parts,
\[
\frac{1}{2\pi} \Pi^{tr} = \int A (U \cdot \nabla_{z,v}\Omega) A\Omega dx = -\frac{1}{2} \int \nabla \cdot U |A\Omega|^2 dx + \int A \Omega [A (U \cdot \nabla_{z,v}\Omega) - U \cdot \nabla_{z,v}A\Omega] dx.
\]
By Sobolev embedding, \( \sigma > 5 \) and the bootstrap hypotheses, we have
\[
(2.44) \quad \left| \int \nabla \cdot U |A\Omega|^2 dx \right| \lesssim \|\nabla U\|_\infty \|A\Omega\|_2^2 \lesssim \frac{\epsilon}{(t)^{2-K_\sigma/2}} \|A\Omega\|_2^2 \lesssim \frac{\epsilon^3}{(t)^{2-K_\sigma/2}}.
\]

Similar to the proof in [5], we use a paraproduct decomposition to deal with the commutator.
\[
T_N = 2\pi \int A\Omega [A (U_{<N/8} \cdot \nabla_{z,v}\Omega_N) - U_{<N/8} \cdot \nabla_{z,v}A\Omega_N] dx
\]
\[
R_N = 2\pi \int A\Omega [A (U_{N} \cdot \nabla_{z,v}\Omega_{<N/8}) - U_{N} \cdot \nabla_{z,v}A\Omega_{<N/8}] dx
\]
\[
R = 2\pi \sum_{N \in \mathcal{D}} \sum_{N \leq N' \leq 8N} \int A\Omega [A (U_{N} \cdot \nabla_{z,v}\Omega_{N'}) - U_{N} \cdot \nabla_{z,v}A\Omega_{N'}] dx.
\]

The treatment of the transport term is similar to section 5 in [5].

**Proposition 2.12** (Transport). Under the bootstrap hypotheses,
\[
\sum_{N \geq 8} |T_N| \lesssim \epsilon CK_{\lambda} + \epsilon CK_{w} + \frac{\epsilon^3}{(t)^{2-K_\sigma/2}}.
\]

To obtain the estimate of the reaction term, one can follow step by step the proof in Section 6 of [5]. We need to mention that we should replace \( \partial_y v - 1 \) in [5] by \( \partial_y v - \tilde{w}' \) and we will use the fact that \( \partial_y v - \tilde{w}' = h + (\tilde{w}' - \tilde{w}) \). One may refer to [5] for more details.
Proposition 2.13 (Reaction). Under the bootstrap hypotheses,

\[ \sum_{N \geq 8} |R_N| \lesssim \epsilon C K \lambda + \epsilon C K_w + \frac{\epsilon^3}{(t)^{2-K_D/2}} + \epsilon C K^{v,1}_\lambda + \epsilon C K^{v,1}_w \]

\[ + \epsilon \left\| \left( \frac{\partial_v}{t} \right)^{-1} (\partial_z^2 + (\partial_v - t \partial_z)^2) \left( \frac{|\nabla|^2}{(t)^{s}} A + \sqrt{\frac{\partial_t w}{w}} \right) \right\|_2^2. \]

The elliptic estimate given below is slightly different from [5]. We will regard \( \Delta_t \) as a perturbation of \( \Delta_L \) instead of \( \Delta_L \) in [5]:

Proposition 2.14 (Precision elliptic control). Under the bootstrap hypotheses,

\[ \left\| \left( \frac{\partial_v}{t} \right)^{-1} (\partial_z^2 + (\partial_v - t \partial_z)^2) \left( \frac{|\nabla|^2}{(t)^{s}} A + \sqrt{\frac{\partial_t w}{w}} \right) P_{|k| \geq k_M} (\Psi \Upsilon) \right\|_2^2 \]

\[ \leq C \left[ C K \lambda + C K_w + \epsilon^2 \left( \sum_{i=1}^2 C C K^{i}_\lambda + C C K^{i}_w \right) \right], \]

where the ‘coefficient Cauchy-Kovalevskaya’ terms are given by

\[ C C K^{1}_\lambda = -\lambda(t) \left\| |\partial_v|^{s/2} A^R \left( (u' \circ u^{-1})^2 - (\partial_v v)^2 \right) \right\|_2^2, \]

\[ C C K^{1}_w = \left\| \sqrt{\frac{\partial_t w}{w}} A^R \left( (u' \circ u^{-1})^2 - (\partial_v v)^2 \right) \right\|_2^2, \]

\[ C C K^{2}_\lambda = -\lambda(t) \left\| |\partial_v|^{s/2} A^R \left( u'' \circ u^{-1} - \partial_{w} v \right) \right\|_2^2, \]

\[ C C K^{2}_w = \left\| \sqrt{\frac{\partial_t w}{w}} A^R \left( u'' \circ u^{-1} - \partial_{w} v \right) \right\|_2^2. \]

The constant \( C \) is independent of \( k_M \).

Generally, it holds that

\[ \left\| \left( \frac{\partial_v}{t} \right)^{-1} (\partial_z^2 + (\partial_v - t \partial_z)^2) \left( \frac{|\nabla|^2}{(t)^{s}} A + \sqrt{\frac{\partial_t w}{w}} \right) \right\|_2^2 \]

\[ \lesssim k_M C K \lambda + C K_w + \epsilon^2 \left( \sum_{i=1}^2 C C K^{i}_\lambda + C C K^{i}_w \right), \]

The proof is in Section 3.

The estimate of the remainder term is similar to the proof of Proposition 2.6 in [5]:

Proposition 2.15 (Remainders). Under the bootstrap hypotheses,

\[ R \lesssim \frac{\epsilon^3}{(t)^{2-K_D/2}}. \]

The next 3 terms are new. The \( \Pi^{com} \) comes from the application of the wave operator to the nonlinear terms.
Proposition 2.16. Under the bootstrap hypotheses,

\[ |\Pi^{\text{com}}| \lesssim \epsilon CK_\lambda + \epsilon CK_w + \frac{\epsilon^2}{\langle t \rangle^{2-K_D \epsilon/2}} + \epsilon CK_v^{1;1} + \epsilon CK_v^{v,1} \]

\[ + \epsilon \left\| \left( \frac{\partial_v}{t \partial z} \right)^{-1} (\partial_z^2 + (\partial_v - t \partial_z)^2) \left( \frac{\nabla^2}{\langle t \rangle^s} \Lambda + \sqrt{\frac{\partial w}{w}} \Lambda \right) P_\neq (\Psi \Upsilon) \right\|_2. \]

The term \( \Pi'''' \) was not present in [5]. The key observation for this term is as follows: 1. we can treat it as the reaction term; 2. we can get an extra smallness if \( |k| \geq k_M \) with \( k_M \) sufficiently large.

Proposition 2.17. Under the bootstrap hypotheses, for any \( \kappa_0 > 0 \) there exists \( k_M \) such that

\[ |\Pi''''| \leq \kappa_0 \left( \frac{\epsilon^2}{\langle t \rangle^{2}} + CK_\lambda + CK_w + \left\| \left( \frac{\partial_v}{t \partial z} \right)^{-1} \frac{\nabla^s}{\langle t \rangle^s} \Delta_L P_{|k| \geq k_M} A(\Psi \Upsilon) \right\|_2 \]

\[ + \left\| L \sqrt{\frac{\partial w}{w}} \Delta_L \bar{\Lambda} P_{|k| \geq k_M} (\Psi \Upsilon) \right\|_2. \]

The treatment of the nonlocal error terms \( \Pi''',e_1 \) and \( \Pi''',e_2 \) is also inspired from the reaction term in section 6 of [5].

Proposition 2.18. Under the bootstrap hypotheses,

\[ |\Pi''',e_1| + |\Pi''',e_2| \lesssim \epsilon CK_\lambda + \epsilon CK_w + \frac{\epsilon^3}{\langle t \rangle^{2-K_D \epsilon/2}} + \epsilon CK_v^{1;1} + \epsilon CK_v^{v,1} \]

\[ + \epsilon \left\| \left( \frac{\partial_v}{t \partial z} \right)^{-1} (\partial_z^2 + (\partial_v - t \partial_z)^2) \left( \frac{\nabla^2}{\langle t \rangle^s} \Lambda + \sqrt{\frac{\partial w}{w}} \Lambda \right) P_\neq (\Psi \Upsilon) \right\|_2. \]

The next step in the bootstrap is to provide good estimates on the coordinate system.

Proposition 2.19 (Coordinate system controls). Under the bootstrap hypotheses, for \( \epsilon \) sufficiently small and \( K_v \) sufficiently large there is a \( K > 0 \) such that

\[ \|
\| A^R h \|_2^2 + E_{tl}(t) + \frac{1}{2} \int_1^t \sum_{i=1}^2 CCK_i^i(\tau) d\tau + \frac{1}{2} \int_1^t \sum_{i=1}^2 CCK_i(\tau) d\tau \leq \frac{1}{2} K_v \epsilon^2, \]

\[ \| \langle t \rangle^{2+2s} \left\| \frac{A}{\langle \partial \nu \rangle^s} \right\|_2^2 + \frac{1}{2} \int_1^t CCK_v^2(\tau) + CK_v^{v,2}(\tau) d\tau \leq C(k_M) \epsilon^2 + K \epsilon^3, \]

\[ \langle t \rangle^{4-K_D^s} \| \| \left( \frac{\partial w}{w} \right)^2 \right\|_{L^1(\tau, \sigma)} \leq C(k_M) \epsilon^2 + K \epsilon^3, \]

\[ \int_1^t CCK_\lambda^{1;1}(\tau) + CK_v^{1;1}(\tau) d\tau \leq \frac{1}{2} K_v \epsilon^2, \]
where the $\mathcal{C}^{k,i}_\nu$ terms are given by

\[
\begin{align*}
\mathcal{C}^{\nu,2}_w(\tau) &= \langle \tau \rangle^{2+2s} \left\| \sqrt{\frac{\partial_z w}{w}} \left\langle \frac{\partial}{\partial \nu} \right\rangle^s \right\|_2^2, \\
\mathcal{C}^{\nu,2}_\lambda(\tau) &= \langle \tau \rangle^{2+2s} \left\| \left\langle \frac{\partial}{\partial \nu} \right\rangle^s \right\|_2^2, \\
\mathcal{C}^{\nu,1}_w(\tau) &= \langle \tau \rangle^{2+2s} \left\| \sqrt{\frac{\partial_z w}{w}} \frac{\partial \nu}{\partial \nu} \right\|_2^2, \\
\mathcal{C}^{\nu,1}_\lambda(\tau) &= \langle \tau \rangle^{2+2s} \left\| \left\langle \frac{\partial}{\partial \nu} \right\rangle^s \frac{\partial \nu}{\partial \nu} \right\|_2^2.
\end{align*}
\] (2.53)

Compared to Proposition 2.5 in [5], the main difference is the estimate of $E_d(t)$. Note that the equations of $\chi_1 - \tilde{\chi}_1$, $u' - \tilde{u}'$ and $u'' - \tilde{u}''$ (2.31)-(2.33) have the same transport structure as the equation of $\tilde{h}$:

\[
(\partial_t + \partial_v \partial_\nu) U = F,
\] (2.54)

where $F$ represents one of the forcing terms $\partial_v \partial_\nu \tilde{\chi}_1$, $\partial_v \partial_\nu \tilde{u}'$ or $\partial_v \partial_\nu \tilde{u}''$. Due to the fact that $\tilde{\chi}_1$, $\tilde{u}'$ and $\tilde{u}''$ are smoother given functions compared to the solutions, $F$ has the same behavior as $\partial_t v$ which is better than the force term $\bar{h} = \partial_y v \partial_\nu \partial_t v$ in the equation of $h$ (2.29). By following the estimate of $h$ in Section 8.2 of [5], one can obtain the estimate of $E_d(t)$ in (2.49) easily. We omit the proof of this proposition.

Remark 2.20. It holds that

\[
\begin{align*}
(\tilde{u}')^2 - (\partial_v u')^2 &= -2(\tilde{u}') Y(v) \left( h \left( u' - \tilde{u}' \right) \right) - \left( h \left( u' - \tilde{u}' \right) \right)^2, \\
\tilde{u}'' - \partial_y v &= \left( \tilde{u}'' - \bar{u}'' \right) - \partial_y v Y(v) \partial_\nu h.
\end{align*}
\]

2.5. Conclusion of proof. Now, we prove the compact support result in Proposition 2.10 and conclude the proof. Under the bootstrap assumption, we have

$\Delta \psi(t, x, y) = \omega(t, x, y) = \Omega(t, x - tv(t, y), v(t, y))$, $\psi(x, 0) = \psi(x, 1) = 0$.

Taking the Fourier transform in $x$, we get

\[
\mathcal{F}_1 \psi(t, k, y) = \int_0^1 G_k(y, z) \Omega(t, k, v(t, z)) e^{-iktv(t, z)} dz,
\] (2.55)

and

\[
\partial_y \mathcal{F}_1 \psi(t, k, y) = \int_0^1 \partial_y G_k(y, z) \Omega(t, k, v(t, z)) e^{-iktv(t, z)} dz.
\] (2.56)

Integrating by parts in $z$ in the identities (2.55)(twice) and (2.56) (once), we get that

\[
\left| U^y(t, x, y) \right| \lesssim \frac{\epsilon}{(t)^2}, \quad \left| U^x - U^x \right| \lesssim \frac{\epsilon}{(t)^2}.
\] (2.57)

Then let us define

\[
\left\{ \frac{d(X^1, X^2)}{dt}(t, x, y) = (u(y) + U^x(t, X^1, X^2), U^y(t, X^1, X^2)), \right. \\
(X^1, X^2)|_{t=0} = (x, y).
\]
Then \( \frac{d}{dt} \omega(t, X^1, X^2) = (u'' \partial_x \psi)(t, X_1, X_2) \) and \(|X^2(t, x, y) - y| \leq C \epsilon \), thus the for \( y \notin [4\theta_0 - C \epsilon, 1 - 4\theta_0 + C \epsilon] \), \( \frac{d}{dt} \omega(t, X^1, X^2) = 0 \). Thus \( \omega(t) \) will always be away from the boundary.

This ends the proof of the bootstrap Proposition 2.10.

Let us now prove (1.2), applying the same method as in Section 2.4 of [5], we get for \( \omega_1(t, z, v) = \Omega(t, z, v) \) and \( \psi_1(t, z, y) = \Psi(t, z, v) = \psi(t, x, y) \),

\[ \partial_t \omega_1 + \nabla_{z,y}^\perp P_{\neq} \psi_1 \cdot \nabla_{z,y} \omega_1 - u'' \partial_x \psi_1 = 0. \]

Then we can define

\[ f_{\infty} = \omega_1(1) - \int_1^\infty \nabla_{z,y} P_{\neq} \psi_1(s) \cdot \nabla_{z,y} \omega_1 ds + \int_1^\infty u'' \partial_x \psi_1(s) ds. \]

Therefore there exists \( f_{\infty} \) such that,

\[ \| \omega(t, x + tu(y) + \Phi(t, y) \chi_1(y), y) - f_{\infty}(x, y) \|_{g_{\lambda \infty}} \lesssim \frac{\epsilon}{(t)^2}. \]

Note that \( \omega(t, x, y) = 0 \) for \( y \notin [2\theta_0, 1 - 2\theta_0] \) so is \( f_{\infty} \), thus we can replace \( \Phi(t, y) \chi_1(y) \) by \( \Phi(t, y) \) which gives (1.2).

Now, we prove (1.4). We have

\[ -\partial_y < U_x > = < \omega >, \quad \partial_t < U_x > + < U_y \omega > = 0, \]

which implies that \( < U_x > (t, y) = C_U \) for any \( t \geq 1 \) and \( y \in [0, 2\theta_0] \cup [1 - 2\theta_0, 1] \). Let \( \tilde{U}(t, z, y) = U_x(t, x, y) \), then \( < U_x > (t, y) - C_U \) has the same compact support as \( \omega_1 \) and satisfies

\[ \partial_t < \tilde{U} > (t, y) - C_U \]

\[ = - < \nabla_{z,y} P_{\neq} \psi_1 \cdot \nabla_{z,y} \tilde{U} > \chi_1(y) \]

\[ = - \partial_y P_{\neq} (\Upsilon \psi_1) \partial_y (\partial_y - t \partial_y v \partial_z) (\Upsilon \psi_1) + \partial_z P_{\neq} (\Upsilon \psi_1) \partial_y \partial_z (\Upsilon \psi_1) > . \]

Then (1.4) follows from the fact that

\[ \| \Upsilon \psi_1 \|_{g_{\lambda \infty, x, y}} \lesssim \frac{\epsilon}{(t)^2}. \]

This ends the proof of Theorem 1.1.

2.6 Proof of Corollary 1.2. Now let us start the proof of Corollary 1.2. Let \( F(t, z, v) = (\mathbb{D}_u \omega)(t, x + tu(y) + \Phi(t, y), v(t, y)) = (\mathbb{D}_u \omega)(t, x + tv(t, y), y) \) with \( z = x - tv(t, y) \), then

\[ F(t, z, v) = < \omega_1 > + \sum_{k \neq 0} \mathbb{D}_{u,k}(F_1 \omega_k)(t, k, y) e^{ikx - ikv(t, y)} \]

and

\[ \partial_t F = -P_0 (U \cdot \nabla_{z,v} \Omega) \]

\[ + \sum_{|k| > 0} \mathbb{D}_{u,k} \left( F_1 \left( \left( u'' - \bar{u}'' \right) \partial_z (\Delta_t^{-1} \Omega) - \bar{u}'' \partial_z (\Delta_u^{-1} \Omega - \Delta_t^{-1} \Omega) \right) \right) (t, k, v) e^{ikz} \]

\[ - \sum_{|k| > 0} \mathbb{D}_{u,k} \left( F_1 \left( U \cdot \nabla_{z,v} \Omega \right) \right) (t, k, v) e^{ikz}. \]

By (2.58) and the uniform \( L^2 \) to \( L^2 \) boundedness of \( \mathbb{D}_{u,k} \), we have that

\[ \left\| \sum_{|k| > 0} \mathbb{D}_{u,k} \left( F_1 \left( \left( u'' - \bar{u}'' \right) \partial_z (\Delta_t^{-1} \Omega) - \bar{u}'' \partial_z (\Delta_u^{-1} \Omega - \Delta_t^{-1} \Omega) \right) \right) (t, k, v) e^{ikz} \right\|_2^2 \]
Let us define $f_{\infty}(t, z, v) = F_{\infty}(t, z, v)$ with

$$
F_{\infty}(z, v) = F(1, z, v) - \int_{1}^{\infty} P_{0}(U \cdot \nabla_{z,v}) \omega(\tau) d\tau
$$

$$
+ \int_{1}^{\infty} \sum_{|k| > 0} D_{u,k} \left( F_{1} \left( (u'' - \bar{u}'') \partial_{z} (\Delta_{t}^{-1} \Omega) - u'' \partial_{z} (\Delta_{u}^{-1} \Omega - \Delta_{t}^{-1} \Omega) \right) \right) (\tau, k, v) e^{izk} d\tau
$$

$$
- \int_{1}^{\infty} \sum_{|k| > 0} D_{u,k} \left( F_{1} \left( U \cdot \nabla_{z,v} \Omega \right) \right) (\tau, k, v) e^{izk} d\tau.
$$

Together with (1.4), it gives Corollary 1.2.

3. Elliptic estimate

The purpose of this section is to provide a thorough analysis of $\Delta_{u}^{-1}$ and $\Delta_{t}^{-1}$.

3.1. Basic estimate of the Green’s function. In this section, we study the operator $\Delta_{u}^{-1}$ in the linear change of coordinates (2.17). Here,

$$
\psi(t, x, y) = \bar{\psi}(x - tu(y), u(y)), \quad \omega(t, x, y) = \bar{\omega}(x - tu(y), u(y)),
$$

hence, $\Delta \psi = \omega \chi_{2}$ is equivalent to $\Delta_{u} \bar{\psi} = \bar{\omega} \chi_{2}$.

Therefore we get that for $k \neq 0$,

$$
F_{1} \psi(k, y_{1}) = \int_{0}^{1} G_{k}(y_{1}, y_{2}) \chi_{2}(y_{2}) F_{1} \omega(k, y_{2}) dy_{2},
$$

and for $k = 0$,

$$
F_{1} \psi(0, y_{1}) = \int_{0}^{1} G_{0}(y_{1}, y_{2}) \chi_{2}(y_{2}) F_{1} \omega(0, y_{2}) dy_{2},
$$

where

$$
G_{k}(y_{1}, y_{2}) = \frac{1}{k \sinh k} \times \begin{cases} 
\sinh(k(1 - y_{2})) \sinh ky_{1}, & y_{1} \leq y_{2}, \\
\sinh(ky_{2}) \sinh(k(1 - y_{1})), & y_{1} \geq y_{2},
\end{cases}
$$

and

$$
G_{0}(y_{1}, y_{2}) = \begin{cases} 
(1 - y_{2})y_{1}, & y_{1} \leq y_{2}, \\
y_{2}(1 - y_{1}), & y_{1} \geq y_{2},
\end{cases}
$$

Let $\tilde{G}_{k}(u(1), u(2)) = G_{k}(y_{1}, y_{2})$, then we have

$$
F_{1} \Delta_{u}^{-1} \tilde{\omega} = F_{1}(\tilde{\chi}_{2} \Delta_{u}^{-1} (\tilde{\chi}_{2} \tilde{\omega}))
$$

$$
= \int_{u(0)}^{u(1)} \tilde{G}_{k}(u_{1}, u_{2})(u^{-1})(u_{2}) \tilde{\chi}_{2}(u_{1}) \tilde{\chi}_{2}(u_{2}) e^{-ikt(u_{2} - u_{1})} F_{1} \tilde{\omega}(k, u_{2}) du_{2}.
$$
Let us denote the Fourier transform of the $t$-independent part of the kernel by

$$G(k, \xi_1, \xi_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{G}_k(u_1, u_2)(u_1^{-1})'(u_2)\tilde{\chi}_2(u_1)\tilde{\chi}_2(u_2)e^{-iu_1\xi_1 -iu_2\xi_2}du_1du_2.$$ 

Thus we have

$$(3.1) \quad \Delta^{-1}_u\tilde{\omega}(t, k, \xi_1) = \frac{1}{2\pi} \int_{\mathbb{R}} G(k, \xi_1 - kt, kt - \xi_2)\tilde{\omega}(t, k, \xi_2)d\xi_2.$$ 

We have the following lemma.

Lemma 3.1 ([12]). Suppose $u$ and $\chi_2$ are defined as above with $s_0 \geq s$, then there exist $\lambda_\Delta = \lambda_\Delta(K, \theta_0, s_0) > 0$, $C > 0$ such that for all $k$,

$$|G(k, \xi_1, \xi_2)| \leq C \min \left\{ \frac{e^{-\lambda_\Delta(\xi_1 + \xi_2)^{s_0}}}{1 + k^2 + \xi_1^2}, \frac{e^{-\lambda_\Delta(\xi_1 + \xi_2)^{s_0}}}{1 + k^2 + \xi_2^2} \right\}.$$ 

Remark 3.2. To treat $\Delta^{-1}_u$, we also need another Green’s function $G_2(k, \xi_1, \xi_2)$ with a different cut-off function $\Upsilon$,

$$G_2(k, \xi_1, \xi_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{G}_k(u_1, u_2)(u_1^{-1})'(u_2)\Upsilon(u_1)\Upsilon(u_2)e^{-iu_1\xi_1 -iu_2\xi_2}du_1du_2.$$ 

Thus we have

$$(3.2) \quad \mathcal{F}_2\mathcal{F}_1\left(\Upsilon\Delta^{-1}_u(\Upsilon\tilde{\omega})\right)(t, k, \xi_1) = \frac{1}{2\pi} \int_{\mathbb{R}} G_2(k, \xi_1 - kt, kt - \xi_2)\tilde{\omega}(t, k, \xi_2)d\xi_2.$$ 

Moreover, there exist $\lambda_\Delta = \lambda_\Delta(K, \theta_0, s_0) > 0$, $C > 0$ such that for all $k$,

$$|G_2(k, \xi_1, \xi_2)| \leq C \min \left\{ \frac{e^{-\lambda_\Delta(\xi_1 + \xi_2)^{s_0}}}{1 + k^2 + \xi_1^2}, \frac{e^{-\lambda_\Delta(\xi_1 + \xi_2)^{s_0}}}{1 + k^2 + \xi_2^2} \right\}.$$ 

The key idea of the proof is to use the definition of Gevrey class in physical side (see [12]).

3.1.1. Lossy estimate. The following is the fundamental estimate on $\Delta^{-1}_u$ or $\Delta^{-1}_t$ which allows to trade the regularity of $\tilde{\omega}$ in a high norm for decay of the stream function in a slightly lower norm. By the estimate of the kernel $G$ and $G_2$, $\Delta_L\Delta^{-1}_u\tilde{\omega}$ has the same regularity as $\tilde{\omega}$, where

$$(3.3) \quad \Delta_L = \partial_{zz} + (\partial_v - t\partial_z)^2.$$ 

We have the following lemmas.

Lemma 3.3. Under the same assumption of Lemma 3.1, it holds that

$$(3.4) \quad \|P_{\neq} \partial_z^2\Delta^{-1}_u\tilde{\omega}\|_{G^{\lambda(t), \sigma-2,s}_0} \lesssim \|\tilde{\omega}\|_{G^{\lambda(t), \sigma,s}_0}/(1+t^2),$$

and

$$(3.5) \quad \|AP_{\neq} \partial_z^2\Delta^{-1}_u\tilde{\omega}\|_2 + \|AP_0(\partial_z)^2\Delta^{-1}_u\tilde{\omega}\|_{L^2} \lesssim \|A\tilde{\omega}\|_{L^2}.$$ 

As can be easily seen from examining $\Delta_u$, we cannot expect to gain $O(t^{-2})$ decay without paying two derivatives. Notice that since the coefficient $\partial_{uu}v$ effectively contains a derivative on $\Omega$, the estimate below loses three derivatives.

Lemma 3.4. Under the bootstrap hypotheses, for $\epsilon$ sufficiently small,

$$\|P_{\neq}(\Upsilon\Psi)\|_{G^{\lambda(t), \sigma-3,s}_0} = \|P_{\neq}\Delta^{-1}_t\Omega\|_{G^{\lambda(t), \sigma-1,s}_0} \lesssim \|\Omega\|_{G^{\lambda(t), \sigma-1,s}_0}/(1+t^2).$$

Let us now start to prove the lemmas.
Proof. Proof of Lemma 3.3. Omitting the time-dependence in \( \lambda, \tilde{\omega} \), for \( \lambda \) chosen sufficiently small than \( \lambda_\Delta \), by Lemma A.2 in \([5]\), we obtain that

\[
(3.6) \quad \|\Delta_u \Delta^{-1}_u \tilde{\omega} \|_{g_{\lambda,\sigma;/}} \leq \sum_k \left| (k^2 + (\xi - kt)^2) e^{\lambda |k| \xi/ \xi} \left| G(k, \xi - kt, \xi_2) \tilde{\omega}_k(\xi_2) d\xi_2 \right| \right|^2 L_\xi^2 \leq \sum_k \left| \int_R e^{-\lambda |\xi - \xi_2|/0} e^{(1+c)\lambda |\xi - \xi_2|} \left| G(k, \xi - kt, \xi_2) \tilde{\omega}_k(\xi_2) d\xi_2 \right| \right|^2 L_\xi^2 \lesssim \|\tilde{\omega}\|_{g_{\lambda,\sigma;/}},
\]

which also implies the first part of (3.5). Next by the same argument as (4.3) in \([5]\), we have that

\[
(3.7) \quad \|P_{\neq} \partial^2_\xi \Delta^{-1}_u \tilde{\omega} \|_{g_{\lambda,\sigma;/}} \lesssim \frac{1}{t^2 + 1} \|\Delta L P_{\neq} \Delta^{-1}_u \tilde{\omega} \|_{g_{\lambda,\sigma;/}},
\]

which gives (3.4).

Similarly, we have for any \( \sigma_1 \leq \sigma \)

\[
(3.8) \quad \|P_{\neq} \partial^2_\xi \Delta^{-1}_u (\tilde{\Psi} \tilde{\omega}) \|_{g_{\lambda,\sigma;/}} \lesssim \frac{1}{t^2 + 1} \|P_{\neq} \tilde{\omega} \|_{g_{\lambda,\sigma;/}}.
\]

To prove (3.5), one can follow the proof in (3.6) and obtain that

\[
(3.9) \quad \Delta_u \Psi = \Omega + h_1(t, v)(\partial_\theta - t \partial_z)^2 (\tilde{\Psi} \tilde{\omega}) + h_2(t, v)(\tilde{\Psi} \tilde{\omega}) (\partial_\theta - t \partial_z)(\tilde{\Psi} \tilde{\omega}),
\]

where

\[
(3.10) \quad h_1(t, v) = (\tilde{u}')(t, v)^2 - (\partial_y v)^2, \quad h_2(t, v) = \tilde{u}'' - \partial_{yy}v.
\]

Here we use the fact that \( v(t, y) \equiv u(y) \) for \( y \not\in \[\theta_0, 1 - \theta_0\] \), which implies that \( v^{-1}(t, c) \equiv u^{-1}(c) \) for \( c \not\in [u(\theta_0), u(1 - \theta_0)] \) and then

\[
(3.11) \quad h_j(t, v) = h_j(t, v) \tilde{\Psi}(v), \quad h_j(t, v) \partial_\theta \tilde{\Psi}(v) = 0, \quad h_j(t, v) \partial_v \tilde{\Psi}(v) = 0 \quad j = 1, 2.
\]

Therefore,

\[
\|\Delta_u \Psi\|_{g_{\lambda,\sigma;/}} \lesssim \|\Omega\|_{g_{\lambda,\sigma;/}} + \|h_1\|_{g_{\lambda,\sigma;/}} \|\Delta L \Psi\|_{g_{\lambda,\sigma;/}} + \|h_2\|_{g_{\lambda,\sigma;/}} \|\partial_\theta - t \partial_z(\tilde{\Psi} \tilde{\omega})\|_{g_{\lambda,\sigma;/}} \lesssim \|\Omega\|_{g_{\lambda,\sigma;/}} + \epsilon \|\Delta_u \Psi\|_{g_{\lambda,\sigma;/}}.
\]

Together with (3.7), this implies Lemma 3.4 provided that \( \epsilon \) is sufficiently small. \(\square\)
3.2. **Precision elliptic control.** Now we turn to the proof of Proposition 2.14. By using (3.9), we get that

\begin{equation}
\Psi(k, \xi_1) = \int_{\mathbb{R}} G_1(k, \xi_1 - kt, k t - \xi_2) \tilde{\Omega}(k, \xi_2) \, d\xi_2
\end{equation}

Define the multipliers

\[ M_1(t, k, \xi) = \left\langle \frac{\xi}{kt} \right\rangle^{-1} \frac{|k, \xi|^{s/2}}{(t)^s} \; \tilde{A} \sigma \; P_{\neq}, \]

\[ M_2(t, k, \xi) = \left\langle \frac{\xi}{kt} \right\rangle^{-1} \sqrt{\frac{\partial_w w(t, \xi_1)}{w(t, \xi_1)}} \; \tilde{A} \sigma \; P_{\neq}. \]

By Lemma 3.1, we have

\[ \sum_{i=1,2} \left| \mathcal{M}_i(k^2 + (\xi - kt)^2) \right| \int_{\mathbb{R}} G_1(k, \xi_1 - kt, k t - \xi_2) \tilde{\Omega}(k, \xi_2) \, d\xi_2 \right| \leq \left\| \mathcal{M}_1 \Omega \right\|_2 \leq \frac{1}{(t)^{2s}} \| \nabla |^{s/2} \Omega \|_2^2. \]

By Lemma 3.6 and Lemma A.2 in [5], we have

\[ \left| \mathcal{M}_1(k^2 + (\xi - kt)^2) \right| \int_{\mathbb{R}} G_1(k, \xi_1 - kt, k t - \xi_2) \tilde{\Omega}(k, \xi_2) \, d\xi_2 \right| \leq \left\| \mathcal{M}_1 \Omega \right\|_2 \leq \frac{1}{(t)^{2s}} \| \nabla |^{s/2} \Omega \|_2^2. \]

By Lemma 3.3, Remark 8, Lemma 3.4 and the proof of Lemma 3.6 in [5], we have

\[ \left| \mathcal{M}_1(k^2 + (\xi - kt)^2) \right| \int_{\mathbb{R}} G_1(k, \xi_1 - kt, k t - \xi_2) \tilde{\Omega}(k, \xi_2) \, d\xi_2 \right| \leq \left\| \mathcal{M}_1 \Omega \right\|_2 \leq \frac{1}{(t)^{2s}} \| \nabla |^{s/2} \Omega \|_2^2. \]

and

\[ \frac{\tilde{J}_k(\xi_1)}{J_k(\xi_2)} \leq e^{10|\xi_1 - \xi_2|^2}. \]

Thus we get that

\begin{equation}
\sum_{i=1,2} \left| \mathcal{M}_i(k^2 + (\xi - kt)^2) \right| \int_{\mathbb{R}} G_1(k, \xi_1 - kt, k t - \xi_2) \tilde{\Omega}(k, \xi_2) \, d\xi_2 \right| \leq \sum_{i=1,2} \left\| \mathcal{M}_i \Omega \right\|_2^2.
\end{equation}
Define,
\begin{equation}
T^1 = h_1(\partial_v - t\partial_z)^2(\Psi Y), \quad T^2 = h_2(\partial_v - t\partial_z)(\Psi Y),
\end{equation}
and divide each via a paraproduct decomposition in the $v$ variable only
\begin{equation}
T^1 = \sum_{M \geq 8} (h_1)_M(\partial_v - t\partial_z)^2(\Psi Y)_{<M/8} + \sum_{M \geq 8} (h_1)_{<M/8}(\partial_v - t\partial_z)^2(\Psi Y)_M
\end{equation}
\begin{equation}
+ \sum_{M \in \mathcal{D}} \sum_{M = M' \leq 8M} (h_1)_{M'}(\partial_v - t\partial_z)^2(\Psi Y)_M
\end{equation}
\begin{equation}
= T^1_{HL} + T^1_{LH} + T^1_R ,
\end{equation}
\begin{equation}
T^2 = \sum_{M \geq 8} (h_2)_M(\partial_v - t\partial_z)(\Psi Y)_{<M/8} + \sum_{M \geq 8} (h_2)_{<M/8}(\partial_v - t\partial_z)(\Psi Y)_M
\end{equation}
\begin{equation}
+ \sum_{M \in \mathcal{D}} \sum_{M = M' \leq 8M} (h_2)_{M'}(\partial_v - t\partial_z)(\Psi Y)_M
\end{equation}
\begin{equation}
= T^2_{HL} + T^2_{LH} + T^2_R .
\end{equation}
By following the argument of the proof of Proposition 2.4 in [5], we have
\begin{equation}
\|M_1 T^1_{HL}\|_2^2 + \|M_1 T^1_{LH}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L P_\lambda(\Psi Y)\|_2^2
\end{equation}
\begin{equation}
\|M_2 T^1_{HL}\|_2^2 + \|M_2 T^1_{LH}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L P_\lambda(\Psi Y)\|_2^2 + \epsilon^2 \|M_2 \Delta_L P_\lambda(\Psi Y)\|_2^2 ,
\end{equation}
and
\begin{equation}
\|M_1 T^2_{HL}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2 + \epsilon^2 \text{CCK}_\lambda^2 ,
\end{equation}
\begin{equation}
\|M_1 T^2_{LH}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2 + \epsilon^2 \text{CCK}_\lambda^2 ,
\end{equation}
\begin{equation}
\|M_2 T^2_{HL}\|_2^2 + \|M_2 T^2_{LH}\|_2^2 \lesssim \epsilon^2 \|M_2 \Delta_L(\Psi Y)\|_2^2 + \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2
\end{equation}
\begin{equation}
+ \epsilon^2 \text{CCK}_\lambda^1 + \epsilon^2 \text{CCK}_w^1 + \epsilon^2 \text{CCK}_w^2 ,
\end{equation}
and
\begin{equation}
\|M_1 T^1_{RH}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2 ,
\end{equation}
\begin{equation}
\|M_2 T^1_{RH}\|_2^2 + \|M_2 T^2_{RH}\|_2^2 \lesssim \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2 + \epsilon^2 \|M_2 \Delta_L(\Psi Y)\|_2^2 .
\end{equation}
Together we have
\begin{equation}
\|M_1 \Delta_L(\Psi Y)\|_2^2 + \|M_2 \Delta_L(\Psi Y)\|_2^2
\end{equation}
\begin{equation}
\lesssim \|M_1 \Omega\|_2^2 + \|M_2 \Omega\|_2^2 + \epsilon^2 \|M_1 \Delta_L(\Psi Y)\|_2^2 + \epsilon^2 \|M_2 \Delta_L(\Psi Y)\|_2^2
\end{equation}
\begin{equation}
+ \epsilon^2 \text{CCK}_\lambda^1 + \epsilon^2 \text{CCK}_w^1 + \epsilon^2 \text{CCK}_w^2 .
\end{equation}
This completes the proof of the Proposition 2.14.

4. The $\Pi$ Term

In this section, we study $\Pi$ defined in (2.39). For $0 < |k| < k_M$, let
\begin{equation}
\Pi_k = \int A_k(\eta) \overline{F_2 D_{u,k}(F_\Omega)}(\eta) A_k(\eta) F_2 \left(D_{u,k} \left(F_\Omega \left(U \cdot \nabla \omega \right) \right) \right)(t,k) \left(\eta\right) d\eta
\end{equation}
then
\begin{equation}
\Pi_k = \int A_k(\eta) \overline{F_2 D_{u,k}(F_\Omega)}(\eta) A_k(\eta) F_2 \left(D_{u,k} \left(F_\Omega \left(U \cdot \nabla \omega \right) \right) \right)(t,k) \left(\eta\right) d\eta
\end{equation}
\[
\begin{align*}
&= \int A_k(\eta)F_2D_{u,k}(\bar{F}_1\Omega)k(\eta)A_k(\eta)F_2\left(\check{\chi}_2D_{u,k}^1\left(\check{\chi}_2F_1\left(U \cdot \nabla_{z,v}\Omega\right)\right)(t,k,\cdot)\right)(\eta)d\eta \\
&+ \int A_k(\eta)F_2D_{u,k}(\bar{F}_1\Omega)k(\eta)A_k(\eta)F_2\left(\check{\chi}_2D_{u,k}^1\left(\check{\chi}_2F_1\left(U \cdot \nabla_{z,v}\Omega\right)\right)(t,k,\cdot)\right)(\eta)d\eta \\
= \int A_k(\eta)\bar{\Omega}_k(\eta)A_k(\eta)(U \cdot \nabla_{z,v}\Omega)k(\eta)d\eta \\
&+ \int (A_k(\eta) - A_k(\xi))D(t,k,\eta,\xi)\Omega_k(\xi)A_k(\eta)F_2\left(\check{\chi}_2D_{u,k}^1\left(\check{\chi}_2F_1\left(U \cdot \nabla_{z,v}\Omega\right)\right)(t,k,\cdot)\right)(\eta)d\xi d\eta \\
&+ \sum_l \int F_2D_{u,k}\left(F_2^{-1}\left(A_k(\eta)\bar{\Omega}_k\right)\right)(\eta) \\
&\quad \times (A_k(\eta) - A_k(\xi))D^t(t,k,\eta,\xi)\left(\hat{U}_{k-l}(\xi - \xi_1) \cdot (il, i\xi_1)\hat{\Omega}_t(\xi_1)\right)d\xi d\eta \\
&+ \int A_k(\eta)\tilde{f}_k(\eta)A_k(\eta)F_2\left(\check{\chi}_2D_{u,k}^1\left(\check{\chi}_2F_1\left(U \cdot \nabla_{z,v}\Omega\right)\right)(t,k,\cdot)\right)(\eta)d\eta \\
&= \Pi_k^{tr+1} + \Pi_k^{com2} + \Pi_k^{com3} = \Pi_k^{tr+3} + \Pi_k^{com},
\end{align*}
\]
and then
\begin{equation}
(4.1) \quad \Pi = - \sum_{0<k<k_M} \Pi_k = - \sum_{0<k<k_M} \Pi_k^{tr+3} - \sum_{0<k<k_M} \Pi_k^{com} \overset{\text{def}}{=} \Pi_k^{tr+3} + \Pi_k^{com}.
\end{equation}

4.1. **Estimate of \( \Pi_k^{com1} \).** By Proposition 2.5, we have
\[
\begin{align*}
|\Pi_k^{com1}| &\lesssim \sum_l \int \left(\frac{A_k(\eta)}{A_k(\xi)} - 1\right)e^{-\lambda_D|\eta - \xi|^s}A_k(\xi)|\Omega_k(\xi)|A_k(\eta)e^{-\lambda_D|\eta - \xi_1|^s} \\
&\quad \times (\hat{U}_{k-l}(\xi_1 - \xi_2)||l,\xi_2||\hat{\Omega}_t(\xi_2))d\xi d\xi_2 d\xi_1 d\eta \\
&\lesssim \sum_l \int 1_D\left(\frac{A_k(\eta)}{A_k(\xi)} - 1\right)e^{-\lambda_D|\eta - \xi|^s}A_k(\xi)|\Omega_k(\xi)|A_k(\eta)e^{-\lambda_D|\eta - \xi_1|^s} \\
&\quad \times (\hat{U}_{k-l}(\xi_1 - \xi_2)||l,\xi_2||\hat{\Omega}_t(\xi_2))d\xi d\xi_2 d\xi_1 d\eta \\
&\quad + \sum_l \int 1_{D^c}\left(\frac{A_k(\eta)}{A_k(\xi)} - 1\right)e^{-\lambda_D|\eta - \xi|^s}A_k(\xi)|\Omega_k(\xi)|A_k(\eta)e^{-\lambda_D|\eta - \xi_1|^s} \\
&\quad \times (\hat{U}_{k-l}(\xi_1 - \xi_2)||l,\xi_2||\hat{\Omega}_t(\xi_2))d\xi d\xi_2 d\xi_1 d\eta \\
&= \Pi_{k,1} + \Pi_{k,2}^{com1}.
\end{align*}
\]
where \( D = \{\eta \geq 100k_M, |\eta - \xi| \leq \frac{1}{100}|\eta|, |\eta - \xi_1| \leq \frac{1}{100}|\eta|\} \) and \( 1_{D^c} = 1 - 1_D \). The second term is easy to deal with. We have that
\[
\frac{A_k(\eta)}{A_k(\xi)} \lesssim e^{C_\lambda|\eta - \xi|^s},
\]
then by the fact that \( \lambda \) is chosen much smaller than \( \lambda_D \), we have that for \( |k| < k_M \),
\begin{equation}
(4.2) \quad \Pi_{k,2}^{com1} \lesssim |A\Omega|_{L^2}||U||_{G^{2,3,3,s}}||\Omega||_{G^{2,3,3,s}} \lesssim \frac{e^3}{l^{2-K_D\epsilon/2}},
\end{equation}
with \( c \in (0,1) \).
By using the Bony decomposition, we write
\[
\Pi_{k,1}^{\text{com}} = \sum_{N \geq 8} \sum_{l} \int 1_D \left( \frac{A_k(\eta)}{A_k(\xi)} - 1 \right) e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| |A_k(\eta)| e^{-\lambda_D |\eta - \xi|^{\sigma_0}}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N \leq 8}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
\[
+ \sum_{N \geq 8} \sum_{l} \int 1_D \left( \frac{A_k(\eta)}{A_k(\xi)} - 1 \right) e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| |A_k(\eta)| e^{-\lambda_D |\eta - \xi|^{\sigma_0}}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N \leq 8}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
\[
+ \sum_{N' \in D} \sum_{N' \leq N \leq 8 N'} \sum_{l} \int 1_D \left( \frac{A_k(\eta)}{A_k(\xi)} - 1 \right) e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| |A_k(\eta)| e^{-\lambda_D |\eta - \xi|^{\sigma_0}}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N' \leq N}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
\[
= \Pi_{k,1,1,T}^{\text{com}} + \Pi_{k,1,1,R}^{\text{com}} + \Pi_{k,1,R}^{\text{com}}.\]

The \(\Pi_{k,1,1,R}^{\text{com}}\) term is better than the reaction term, since we gain derivatives from \(A_k(\eta) - A_k(\xi)\). We omit the proof and show the result. One can also refer to section 6 in [5] for more details.

\[
(\ref{eq:4.3}) \quad \Pi_{k,1,R}^{\text{com}} \lesssim \varepsilon \text{C}K + \varepsilon \text{C}w + \frac{\varepsilon^3}{(l/t)^{2-\kappa_0/2}} + \varepsilon \text{C}K^{\eta,1} + \varepsilon \text{C}K^{\xi,1}
\]
\[
+ \varepsilon \left\| \left\langle \frac{\partial_v}{\tilde{t} \tilde{S}_v} \right\rangle^{-1} (\partial_v^2 + (\partial_v - t \partial_s)^2) \left( \frac{\nabla |\tilde{S}|^2}{(l/t)^{\sigma}} A + \sqrt{\frac{\partial_v w}{w}} \tilde{A} \right) P_{\xi}(\Psi \Upsilon) \right\|_2^2.
\]

The remainder term is also easy to deal with, we have

\[
(\ref{eq:4.4}) \quad \Pi_{k,1,R}^{\text{com}} \lesssim \frac{\varepsilon^3}{(l/t)^{2-\kappa_0/2}}.
\]

Decompose the difference:

\[
(\ref{eq:4.5}) \quad A_k(\eta) - A_k(\xi) = A_k(\xi) \left[ e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} - 1 \right] + A_k(\xi) e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} \left[ J_k(\eta) [J_k(\xi)] - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma}
\]
\[
+ A_k(\xi) e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} \left[ \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma} - 1 \right].
\]

We write

\[
\Pi_{k,1,T}^{\text{com}} \lesssim \sum_{N \geq 8} \sum_{l} \int 1_D e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| \left[ e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} - 1 \right] A_k(\eta) e^{-\lambda_D |\eta - \xi|^{\sigma_0}}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N \leq 8}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
\[
+ \sum_{N \geq 8} \sum_{l} \int 1_D e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| \left[ e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma} A_k(\eta) e^{-\lambda_D |\eta - \xi|^{\sigma_0}}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N \leq 8}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
\[
+ \sum_{N \geq 8} \sum_{l} \int 1_D e^{-\lambda_D |\eta - \xi|^{\sigma_0}} A_k(\xi) |\Omega_k(\xi)| \left[ e^{\lambda \|k,\eta\|^\sigma - \lambda \|k,\xi\|^\sigma} - 1 \right] \frac{\langle k, \eta \rangle^\sigma}{\langle k, \xi \rangle^\sigma}
\times |\hat{U}_{k-l}(\xi_1 - \xi_2)_{l \leq N \leq 8}||l, \xi_2| \hat{\Omega}_l(\xi_2)| d\xi d\xi_2 d\xi_1 d\eta
\]
$$= \Pi_{\com,1,1,T} + \Pi_{\com,1,2,T} + \Pi_{\com,1,3,T}.$$ 

First we treat $\Pi_{\com,1,1,T}$. Since on the support of the integrand

$$|k, \xi| \approx |k, \xi_1|, \quad ||k, \xi_1| - |l, \xi_2|| \leq |k - l, \xi_1 - \xi_2| \leq \frac{6}{32} |l, \xi_2|,$$

it gives us that

$$26 \leq |l, \xi_2| \leq |k, \xi_1| \leq \frac{38}{32} |l, \xi_2|,$$

Similarly, we can treat $\Pi_{\com,1,3}$. Here we omit the proof.

(4.6) $$|\Pi_{\com,1,3,T}| \lesssim \frac{\epsilon^3}{(t)^2 - K_D \epsilon/2}.$$ 

We now treat $\Pi_{\com,1,2,T}$. We divide the integral as follows

$$\Pi_{\com,1,2,T} \lesssim \sum_{N \geq 8} \sum_{l} \int 1_{D} [1 + L]_1 e^{-\lambda_D |\eta - \xi|} \mathcal{A}_k(\xi) \Omega_k(\xi) e^{c|\eta|} \left[ \frac{J_k(\eta)}{J_k(\xi)} - 1 \right]$$

$$\times \mathcal{A}_k(\xi) e^{-\lambda_D |\eta - \xi|} \widehat{U}_{k-l}(\xi_1 - \xi_2) \langle N/|l, \xi_2| \rangle \mathcal{O}(\xi_2)_N \langle d\xi d\xi_2 d\xi_1 d\eta \rangle$$

$$= \Pi_{\com,1,2,T,S} + \Pi_{\com,1,1,2,T,L},$$

where $S = \{ t \leq \frac{1}{2} \min(\sqrt{\langle \xi \rangle}, \sqrt{\langle \eta \rangle}) \}$, $1_L = 1 - 1_S$. In $\Pi_{\com,1,2,T,S}$, we apply Lemma 3.7 in [5] to gain 1/2 derivatives. Indeed, we have

$$\Pi_{\com,1,2,T,S} \lesssim \frac{\epsilon}{(t)^2 - K_D \epsilon/2} \left\| \|\nabla^{s/2} \mathcal{A} \|_{1}\right\|_{2}^2 + \frac{\epsilon^3}{(t)^2 - K_D \epsilon/2}.$$ 

To treat $\Pi_{\com,1,2,T,L}$, we divide into two cases based on the relative size of $|l|$ and $|\xi_2|

$$\Pi_{\com,1,2,T,L} \lesssim \sum_{N \geq 8} \sum_{l} \int 1_{D} [1 + L]_1 e^{-\lambda_D |\eta - \xi|} \mathcal{A}_k(\xi) \Omega_k(\xi) e^{c|\eta|} \left[ \frac{J_k(\eta)}{J_k(\xi)} - 1 \right]$$

$$\times \mathcal{A}_k(\xi) e^{-\lambda_D |\eta - \xi|} \widehat{U}_{k-l}(\xi_1 - \xi_2) \langle N/|l, \xi_2| \rangle \mathcal{O}(\xi_2)_N \langle d\xi d\xi_2 d\xi_1 d\eta \rangle$$

$$= \Pi_{\com,1,2,T,L,z} + \Pi_{\com,1,2,T,L,v}.$$ 

In the first case $|k - l| \leq \frac{1}{8} |l|$ and $|k| \leq k_M$, we get that

$$\Pi_{\com,1,2,T,L,z} \lesssim \frac{\epsilon}{(t)^2 - K_D \epsilon/2}.$$ 

Turn now to $\Pi_{\com,1,2,T,L,v}$. Note that $|l, \xi_2| \lesssim |\xi_2| \approx |\xi| \approx t^2$. We have $|l, \xi_2| \lesssim |k, \xi|^{s/2}|l, \xi_2|^{s/2} t^{2-s}$

$$\Pi_{\com,1,2,T,L,v} \lesssim \frac{\epsilon}{(t)^{2s - K_D \epsilon/2}} \left\| \|\nabla^{s/2} \mathcal{A} \|_{1}\right\|_{2}^2.$$ 

Thus we conclude that

(4.7) $$\Pi_{\com,1,1,T} \lesssim \frac{\epsilon}{(t)^{2-K_D \epsilon/2}} \left\| \|\nabla^{s/2} \mathcal{A} \|_{1}\right\|_{2}^2 + \frac{\epsilon^3}{(t)^{2-K_D \epsilon/2}} + \frac{\epsilon}{(t)^{2s - K_D \epsilon/2}} \left\| \|\nabla^{s/2} \mathcal{A} \|_{1}\right\|_{2}^2.$$
4.2. Estimate of $\Pi_k^{com2}$. Now we treat $\Pi_k^{com2}$. We have

$$
\Pi_k^{com2} \lesssim \sum_l \int D(t, k, \eta, \xi_2)A_k(\xi_2)O_k(\xi_2)\left(A_k(\eta) - A_k(\xi)\right)
\times D^1(t, k, \eta, \xi)\left(\hat{U}_{k-l}(\xi - \xi_1) \cdot (il, i\xi)\hat{\Omega}(\xi_1)\right) d\xi_2 d\xi_1 d\eta
\lesssim \sum_l \int [1 \pm D^c]e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)O_k(\xi_2)\left[A_k(\eta)\right] d\xi_2 d\xi_1 d\eta
$$

$$
= \Pi^{com2}_{k,1} + \Pi^{com2}_{k,2},
$$

where $D = \{|\eta - \xi| \leq \frac{1}{100}|\xi|, |\xi| \geq 100k_M\}$ and $1D^c = 1 - 1_D$.

Then we have

$$
|\Pi^{com2}_{k,2}| \leq \frac{\varepsilon^3}{\langle l \rangle^{2-KDE/2}}.
$$

Similarly, we write

$$
\Pi^{com2}_{k,1} \lesssim \sum_{N \geq 8} \sum_l \int 1D e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)O_k(\xi_2)\left[A_k(\eta)\right] d\xi_2 d\xi_1 d\eta
\times e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)(\hat{U}_{k-l}(\xi - \xi_1)_{<N/8} l, \xi_1\hat{\Omega}(\xi_1)_{<N/8}) d\xi_2 d\xi_1 d\eta
$$

$$
+ \sum_{N \geq 8} \sum_l \int 1D e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)O_k(\xi_2)\left[A_k(\eta)\right] d\xi_2 d\xi_1 d\eta
\times e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)(\hat{U}_{k-l}(\xi - \xi_1)_{<N/8} l, \xi_1\hat{\Omega}(\xi_1)_{<N/8}) d\xi_2 d\xi_1 d\eta
$$

$$
+ \sum_{N \geq 8} \sum_l \int 1D e^{-\lambda \rho |\eta - \xi_2|^2} A_k(\xi_2)O_k(\xi_2)\left[A_k(\eta)\right] d\xi_2 d\xi_1 d\eta
$$

$$
= \Pi^{com2}_{k,1,T} + \Pi^{com2}_{k,1,R} + \Pi^{com2}_{k,1,R}.
$$

Similar to the $\Pi^{com1}_{k,1,R}$ and $\Pi^{com1}_{k,1,\hat{R}}$, we can use the same argument as the estimate of the reaction term and remainder term in section 6 and 7 of [5]. We omit the proof and show the result.

$$
|\Pi^{com2}_{k,1,R}| + |\Pi^{com2}_{k,1,\hat{R}}| \lesssim \varepsilon Ck + \varepsilon C\varepsilon + \frac{\varepsilon^3}{\langle l \rangle^{2-KDE/2}} + \varepsilon Ck + \varepsilon C\varepsilon
$$

$$
+ \varepsilon \left\| \left\langle \frac{\partial}{\partial z} \right\rangle^{-1} \left(\partial^2_z + (\partial - t e_z)^2\right) \left(\frac{\nabla^2}{\langle l \rangle^s} + \sqrt{\frac{\partial w}{w}} + \hat{A} \right) P e(\Psi Y) \right\|_2^2.
$$

We treat $\Pi^{com2}_{k,1,T}$. Since on the support of the integrand

$$
||k, \xi| - |l, \xi_1| | \leq |k - l, \xi - \xi_1| \leq \frac{6}{32}|l, \xi_1|,
$$

$$
\frac{26}{32}|l, \xi_1| \leq |k, \xi| \leq \frac{38}{32}|l, \xi_1|, 0 < |k| \leq k_M,
$$

$$
|k, \xi_2| + |\xi_2| \geq |k, \eta| \approx |\eta| \approx |k, \xi| \approx |\xi| \geq 100k_M.
$$
By using the composition in (4.5) and following the same argument as the treatments of $\Pi_{k,1,T}^{\text{com}}$, $\Pi_{k,1,2,T}^{\text{com}}$ and $\Pi_{k,1,3,T}^{\text{com}}$, we get that

$$
\Pi_{k,1,T}^{\text{com}} \lesssim \frac{\epsilon}{(t)^{2s-K_D^{-1/2}}} \left\| \nabla^{s/2} A \Omega \right\|^2_2 + \frac{\epsilon^3}{(t)^{2-K_D^{-1/2}}}.
$$

### 4.3. Estimate of $\Pi_k^{\text{com}}$

By Corollary 2.6, we get that

$$
\Pi_k^{\text{com}} = \sum_l A_k(\eta) f_k(\eta) A_k(\eta) D^{\text{com}}(t, k, \eta, \xi) \widehat{U}_{k-l}(\xi - \xi_1) \cdot (l, \xi_1) \Omega_{l}(\xi) d\xi d\eta
$$

$$
\lesssim \sum_l \int 1_{D^c} A_k(\eta) f_k(\eta) \frac{e^{-c\lambda_D \eta \cdot \xi}}{1 + |\eta - kt|^2 + |\xi - kt|^2} A_k(\xi) \widehat{U}_{k-l}(\xi - \xi_1) d\xi d\eta
$$

$$
+ \sum_{N \geq 8} \sum_l \int 1_{D^c} A_k(\eta) f_k(\eta) \frac{e^{-c\lambda_D \eta \cdot \xi}}{1 + |\eta - kt|^2 + |\xi - kt|^2}
$$

$$
	imes A_k(\xi) \widehat{U}_{k-l}(\xi - \xi_1) d\xi d\eta
$$

$$
+ \sum_{N \geq 8} \sum_l \int 1_{D^c} A_k(\eta) f_k(\eta) \frac{e^{-c\lambda_D \eta \cdot \xi}}{1 + |\eta - kt|^2 + |\xi - kt|^2}
$$

$$
	imes A_k(\xi) \widehat{U}_{k-l}(\xi - \xi_1) d\xi d\eta
$$

$$
= \Pi_{k,1,1}^{\text{com}} + \Pi_{k,1,2}^{\text{com}} + \Pi_{k,1,3}^{\text{com}} + \Pi_{k,1,4}^{\text{com}},
$$

where $D = \{ |\eta - \xi| \leq \frac{1}{100} |\xi|, \ |\xi| \geq 100k_M \}$ and $1_{D^c} = 1 - 1_D$.

The term $\Pi_{k,2}^{\text{com}}$ is easy to deal with, we have

$$
\Pi_{k,2}^{\text{com}} \lesssim \frac{\epsilon^3}{(t)^{2-K_D^{-1/2}}}.
$$

Similar to the above cases, $\Pi_{k,1,1}^{\text{com}}$ and $\Pi_{k,1,4}^{\text{com}}$ are easy to treat, by following the estimate in sections 6 and 7 of [5]. We have

$$
\Pi_{k,1,1}^{\text{com}} \lesssim \epsilon Ck^\lambda + \epsilon Ck^w + \frac{\epsilon^3}{(t)^{2-K_D^{-1/2}}} + \epsilon Ck^\lambda + \epsilon Ck^w
$$

$$
+ \epsilon \left\| \left( \frac{\partial_v}{t \partial_z} \right)^{-1} (\partial_x^2 + (\partial_v - t \partial_z)^2) \left( \frac{\nabla^s}{(t)^s} A + \sqrt{\frac{\partial w}{w}} A \right) P_\neq(\Psi \bar{Y}) \right\|^2_2.
$$

We treat $\Pi_{k,1,7}^{\text{com}}$. Since on the support of the integrand

$$
|k, \xi| - |l, \xi_1| \leq |k - l, \xi - \xi_1| \leq \frac{6}{32} |l, \xi_1|,
$$

$$
\frac{26}{32} |l, \xi_1| \leq |k, \xi| \leq \frac{38}{32} |l, \xi_1|, \ 0 < |k| \leq k_M,
$$

$$
|k, \eta| \approx |\eta| \approx |k, \xi| \approx |\xi| \geq 100k_M.
$$
We write
\[ \Pi_{k,1,T}^{\text{com}} = \sum_{N \geq 8} \sum_{l} \int 1_{D} [1_{R} + 1_{NR}] A_{k}(\eta) |f_{k}(\eta)| \frac{e^{-c_{\Delta} |\eta - \xi|^{4/3}}}{1 + |\eta - kt|^{2} + |\xi - kt|^{2}} \times A_{k}(\xi) |\widehat{U}_{k-l}(\xi - \xi_{1})|_{N/8} d\xi_{1} d\xi d\eta \]
\[ = \Pi_{k,1,T,1}^{\text{com}} + \Pi_{k,1,T,2}^{\text{com}}, \]
where \( R = \{ t \in I_{k,\eta} \cap I_{k,\xi} \} \) and \( 1_{NR} = 1 - 1_{R} \).

For \( \Pi_{k,1,T,2}^{\text{com}} \), by the fact that \( |\eta - kt|^{2} + |\xi - kt|^{2} \geq \frac{|\eta|^{2}}{k} \geq |\eta|^{2} \), we have
\[ |\Pi_{k,1,T,2}^{\text{com}}| \lesssim \frac{e^{3}}{(t)^{2-K_{D} \epsilon/2}}. \]

Now we treat \( \Pi_{k,1,T,1}^{\text{com}} \), we have
\[ \Pi_{k,1,T,1}^{\text{com}} \lesssim \sum_{N \geq 8} \sum_{l} \int 1_{D} 1_{R} A_{k}(\eta) |f_{k}(\eta)| \frac{e^{-c_{\Delta} |\eta - \xi|^{4/3} |k, \eta|}}{k^{2} + |\eta - kt|^{2}} \times A_{k}(\xi) |\widehat{U}_{k-l}(\xi - \xi_{1})|_{N/8} \Omega_{\xi}(\xi_{1}) d\xi_{1} d\xi d\eta. \]

It can be regarded as the action of the resonant modes \( |f_{k}(\eta)| \frac{|k, \eta|}{k^{2} + |\eta - kt|^{2}} \) on \( \Omega_{\xi}(\xi_{1}) \) which can be either non-resonant or resonant. Due to the fact that \( U \) decays as \( (t)^{-2 + K_{D} \epsilon/2} \), this term is better than \( R_{NR, R}^{N} \) and \( R_{NR}^{N} \) in section 6.1.3 and 6.1.4 of [5]. Thus we obtain that
\[ |\Pi_{k,1,T,1}^{\text{com}}| \lesssim e^{3} C_{\lambda} + e^{3} C_{w} + \frac{e^{3}}{(t)^{2-K_{D} \epsilon/2}}. \]

Together with (4.2), (4.3), (4.4), (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), (4.13) and (4.14), we prove Proposition 2.16.

5. Nonlocal terms

First let us deal with the nonlocal term \( \Pi_{u''} \). We have
\[ \Pi_{u''} = \sum_{N \in \mathcal{D}, N \geq 8} \int A \left( \mu_{N}^{u''} 1_{|k| \geq k_{M}} P_{ \neq} \partial_{z} (\Psi \Upsilon)_{N} \right) A f d x \]
\[ + \sum_{N \in \mathcal{D}, N \geq 8} \int A \left( \mu_{N}^{u''} 1_{|k| \geq k_{M}} P_{ \neq} \partial_{z} (\Psi \Upsilon)_{N} \right) A f d x \]
\[ + \sum_{N \in \mathcal{D}, \frac{1}{8} N \leq N \leq 8N} \sum_{N \in \mathcal{D}, \frac{1}{8} N \leq N \leq 8N} \int A \left( \mu_{N}^{u''} 1_{|k| \geq k_{M}} P_{ \neq} \partial_{z} (\Psi \Upsilon)_{N} \right) A f d x \]
\[ = \Pi_{HL}^{u''} + \Pi_{LH}^{u''} + \Pi_{R}^{u''}. \]

In this section we use the fact that
\[ P_{|k| \geq k_{M}} \Psi \Upsilon = \Delta_{-1}^{-1} P_{|k| \geq k_{M}} f. \]

5.1. High-low interaction and high-high interaction. The estimate of \( \Pi_{HL}^{u''} \) and \( \Pi_{R}^{u''} \) are easy. We omit the proof and show the result that
\[ |\Pi_{HL}^{u''}| + |\Pi_{R}^{u''}| \lesssim \frac{e_{2}}{k_{M}(t)^{2}}. \]
5.2. Low-high interaction. We get that
\[
|\Pi_{\text{LH}}^{\mu'}| \lesssim \sum_{N \in \mathcal{D}, N \geq 8} \sum_{k \neq 0} \left| \int_{\xi, \eta} \overline{A f_k(\eta)} A_k(t, \eta) \left( \overline{u'}(\eta - \xi)_{< \frac{k}{8} N} 1_{|k| \geq k_M} |k| (\nabla \Psi)(\xi)_{N} \right) d\xi d\eta \right|
\]
\[
\lesssim \sum_{N \in \mathcal{D}, N \geq 8} \sum_{k \neq 0} \left| \int_{\xi, \eta} \left[ 1_{\text{NR}, \text{NR}} + 1_{\text{NR}, \text{R}} + 1_{\text{R}, \text{NR}} + 1_{\text{R}, \text{R}} \right] 1_{|k| \geq k_M} d\xi d\eta \right|
\]
\[
\times \overline{A f_k(\eta)} A_k(t, \eta) \left( \overline{u'}(\eta - \xi)_{< \frac{k}{8} N} 1_{|k| (\nabla \Psi)(\xi)_{N}} \right) d\xi d\eta \right|
\]
\[
= \Pi_{\text{LH}}^{\mu', \text{NR}, \text{NR}} + \Pi_{\text{LH}}^{\mu', \text{NR}, \text{R}} + \Pi_{\text{LH}}^{\mu', \text{R}, \text{NR}} + \Pi_{\text{LH}}^{\mu', \text{R}, \text{R}}
\]

**Treatment of \( \Pi_{\text{LH}}^{\mu', \text{NR}, \text{NR}} \).**

By (3.31), (6.1) and (A.11) in [5], we have
\[
|\Pi_{\text{LH}}^{\mu', \text{NR}, \text{NR}}| \lesssim \sum_{N \in \mathcal{D}, N \geq 8} \| u' \|_{L^\infty} \| \nabla |t|/2 A f_{\text{NR}} \|_{2} \left\| \frac{\partial_{z}}{(\nabla |t|/2)^{s/2}} 1_{|k| \geq k_M} P_{\neq} A(\nabla \Psi)_{N} \right\|_{2} .
\]

**Remark 5.1.** For any \( \kappa_0 > 0 \), there exist \( k_M \) such that for \( |k| > k_M \), it holds that
\[
\langle t \rangle^{s} \left\| \frac{\partial_{z}}{(\nabla |t|/2)^{s/2}} 1_{|k| \geq k_M} P_{\neq} A(\nabla \Psi) \right\|_{2} \leq \kappa_0 \left\| \frac{\partial_{z}}{(\nabla |t|/2)^{s/2}} 1_{|k| \geq k_M} P_{\neq} A(\nabla \Psi) \right\|_{2} .
\]

**Proof.** The remark follows from the following claim:

**Claim:** For any \( \kappa_0 > 0 \), there exists \( k_M \) such that for \( |k| > k_M \) and \( t \notin I_{k, \xi} \), it holds that
\[
\langle t \rangle^{s} \left| \frac{\xi}{t} \right|^{k} \leq \kappa_0 \left\langle \frac{\xi}{tk} \right\rangle^{-1} \left| \frac{\xi}{tk} \right|^{s} (k^{2} + (\xi - kt)^{2}) .
\]

Let consider the following three cases:

1. Case 1: \( \left| \frac{\xi}{tk} \right| \leq \frac{1}{2} \), then \( \left\langle \frac{\xi}{tk} \right\rangle^{-1} \approx 1 \) and \( (k^{2} + (\xi - kt)^{2}) \approx k^{2} t^{2} \), thus (5.2) is equivalent to prove that
\[
\frac{\langle t \rangle^{2s} |k|}{\langle k, \xi \rangle^{s}} \leq \kappa_0 k^{2} t^{2},
\]
which is obvious when \( |k| \geq k_M \).

2. Case 2: \( \left| \frac{\xi}{tk} \right| \geq \frac{3}{2} \), then \( \left\langle \frac{\xi}{tk} \right\rangle^{-1} \approx \frac{tk}{\xi} \), \( \langle k, \xi \rangle^{s} \approx |\xi|^{s} \) and \( (k^{2} + (\xi - kt)^{2}) \approx k^{2} + \xi^{2} \), thus (5.2) is equivalent to prove that
\[
\frac{|t|^{2s} |k| |\xi|}{|kt||\xi|^{s}(k^{2} + \xi^{2})} \leq \kappa_0 .
\]

Indeed we have for the left hand side that
\[
\frac{|t|^{2s} |k| |\xi|}{|kt||\xi|^{s}(k^{2} + \xi^{2})} \approx |t|^{2s-1} |\xi|^{-s-1} \leq |t|^{s-2} |k|^{-s-1} \leq \kappa_0 .
\]

3. Case 3: \( \frac{1}{2} \leq \left| \frac{\xi}{tk} \right| \leq \frac{3}{2} \), then \( |\xi| \approx |kt| \) and \( (k^{2} + (\xi - kt)^{2}) = k^{2}(1 + |t - \frac{\xi}{tk}|) \geq k^{2} + \frac{\xi^{2}}{t^{2}} \approx k^{2} + t^{2} \).

Thus
\[
\frac{\langle t \rangle^{2s} |k|}{\langle k, \xi \rangle^{s}(k^{2} + (\xi - kt)^{2})} \lesssim \frac{\langle t \rangle^{s} |k|^{1-s}}{(k^{2} + t^{2})} \leq \kappa_0 .
\]

Thus we proved the claim, which gives the remark. \( \square \)
Therefore we conclude that for \( k_M \) sufficiently large,
\[
|\Pi_{LH}^{\nu^*_{\text{NR}}}| \leq \frac{\kappa_0}{(t^*)^2} \left\| \nabla |v|^2 A f \right\|_2 + \kappa_0 \left\| \frac{\partial_v}{t \partial z} \right\|_{L^2}^{-1} \left\| \frac{\nabla |v|^2}{(t^*)^s} \Delta_L P_{|k| \geq k_M} A(\Psi \Upsilon) \right\|_2^2.
\]

**Treatment of \( \Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}}} \).**

By (6.1) and (A.7) in [5] and Remark 5.1, for some \( c \in (0, 1) \), and for any \( \kappa_0 > 0 \) there exists \( k_M \) such that
\[
|\Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}}}| \lesssim \sum_{N \in D, N \geq 8} \sum_{k \neq 0} \int_{\xi, \eta} |A f_k(\eta)| J_k(\eta) e^{\lambda|k, \xi|^2} e^{c \lambda|\eta - \xi|^2} (k, \xi)^\sigma |k|_{t \in I_{k, \xi}} \frac{1}{t^2} \frac{d\xi d\eta}{|k| \Lambda_1(\Psi \Upsilon)(k) N}.
\]

\[
|\Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}}}| \lesssim \sum_{N \in D, N \geq 8} \left\| \nabla |v|^2 A f_{\sim N} \right\|_2 \left\| \frac{\partial_v}{t^2} A f_{\sim N} \right\|_2 \left\| \frac{\nabla |v|^2}{(t^*)^s} \Delta_L P_{|k| \geq k_M} A(\Psi \Upsilon) \right\|_2.
\]

**Treatment of \( \Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}, R}} \).**

In this case \((k, \xi)\) is resonant and \((k, \eta)\) in non-resonant. It follows that \( 4|k|^2 \leq |\xi| \) and since \( N \geq 8 \), we have \( |\xi| \approx |\eta| \). By (3.16), (3.32), (A.7), (A.12), (A.11) and (6.6) in [5],
\[
|\Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}, R}}| \lesssim \sum_{N \in D, N \geq 8} \sum_{k \neq 0} \int_{\xi, \eta} J_k(\xi) \frac{w_R(\xi)}{w_{\text{NR}}(\xi)} \frac{w_k(t, \xi)}{w_k(t, \xi)} \left[ \frac{\partial_t w_k(t, \xi)}{w_k(t, \xi)} \right] |A f_k(\eta)_{\sim N}|.
\]

On the support of the integrand, we have \( A_k(\eta) \lesssim \hat{A}_k(\eta) \) and \( A_k(\xi) \lesssim \hat{A}_k(\xi) \). Moreover \( |\xi| \approx |k|t \).

Since \( t \in I_{k, \xi} \), thus by (3.5), (3.8) and (3.14) in [5], we have \( |\xi| \gtrsim k^2 \)
\[
\sqrt{\frac{w(t, \xi)}{\partial_t w(t, \xi)}} |k| \frac{w_R(t, \xi)}{w_{\text{NR}}(t, \xi)} \frac{1}{|k| \Lambda_1(\Psi \Upsilon)(k) N} \frac{1}{t^2} \frac{d\xi d\eta}{|k| \Lambda_1(\Psi \Upsilon)(k) N}.
\]

Thus for any \( \kappa_0 > 0 \), there is \( k_M \), such that for \( |k| \geq k_M \)
\[
(5.5)
|\Pi_{LH}^{\nu^*_{\text{R}}_{\text{NR}, R}}| \lesssim \sum_{N \in D, N \geq 8} \left\| \frac{\partial_t w}{|w|} \right\|_{L^2} \left( \left\| \frac{\partial_t w}{|w|} \right\|_{L^2} \left\| \frac{\nabla |v|^2}{(t^*)^s} A f_{\sim N} \right\|_2^2 + \left\| \frac{\nabla |v|^2}{(t^*)^s} A f_{\sim N} \right\|_2^2 \right).
\]
\[ \leq \kappa_0 \sum_{N \in \mathbb{D}, N \geq 8} \left( \| \sqrt{\frac{\partial w}{w}} \tilde{A} f_{-N} \|_2^2 + \| \frac{\nabla |s/2}{\langle t \rangle^{s/2}} A f_{-N} \|_2^2 + \| \frac{\partial w}{w} \Delta L \tilde{A} \mathcal{P}_{|k| \geq k_M} (\Psi \Upsilon) \|_2 \right) \]

\[ \leq \kappa_0 \left( \left\| \sqrt{\frac{\partial w}{w}} \tilde{A} f \right\|_2^2 + \kappa_0 \left( \left\| \frac{\nabla |s/2}{\langle t \rangle^{s/2}} A f \right\|_2^2 + \kappa_0 \left\| \frac{\partial w}{w} \Delta L \tilde{A} \mathcal{P}_{|k| \geq k_M} (\Psi \Upsilon) \right\|_2^2 \right) . \]

**Treatment of** \( \Pi_{LH}^{u'':R,R} \).

We have

\[ \left\| \Pi_{LH}^{u'':R,R} \right\| \lesssim \sum_{N \in \mathbb{D}, N \geq 8} \sum_{|k| \geq k_M} \left( \int_{\xi, \eta} 1_{R,R} |1_{|k| \geq k_M} \tilde{A} \tilde{f}_k (\eta) A_k (t, \eta) \left( \tilde{w}'' (\eta - \xi) \lesssim \frac{1}{\sqrt{N}} |k| |(\Psi \Upsilon)_k (\xi)_N \right) \right) \, d\xi d\eta . \]

By (3.31) and Lemma 3.4 in [5] and the fact that \( A_k (\eta) \lesssim \tilde{A}_k (\eta) \) and \( A_k (\xi) \lesssim \tilde{A}_k (\xi) \), we have

\[ \left\| \Pi_{LH}^{u'':R,R} \right\| \leq C \sum_{N \in \mathbb{D}, N \geq 8} \sum_{|k| \geq k_M} \left( \int_{\xi, \eta} \left| \tilde{A} \tilde{f}_k (\eta) \right| J_k (t, \xi) \frac{\partial w_k (t, \eta)}{w_k (t, \xi)} \sqrt{w_k (t, \xi)} |(\Psi \Upsilon)_k (\xi)_N| \right) \, d\xi d\eta \]

\[ \lesssim \sum_{N \in \mathbb{D}, N \geq 8} \sum_{|k| \geq k_M} \left( \int_{\xi, \eta} \left| \tilde{A} \tilde{f}_k (\eta) \right| e^{10 \mu |s| |k| \xi - \xi|} \frac{\partial w_k (t, \eta)}{w_k (t, \xi)} \sqrt{w_k (t, \xi)} |(\Psi \Upsilon)_k (\xi)_N| \right) \, d\xi d\eta \]

\[ \leq C \sum_{N \in \mathbb{D}, N \geq 8} \left\| \frac{\partial w}{w} \tilde{A} f_{-N} \right\|_2 \left\| \sqrt{\frac{\partial w}{w}} \tilde{A} f \right\|_2^2 + \left\| \frac{\partial w}{w} \Delta L \tilde{A} \mathcal{P}_{|k| \geq k_M} (\Psi \Upsilon) \right\|_2^2 \| \tilde{w}'' \|_{L^\lambda}. \]

The constant \( C \) is independent of \( k_M \). By (3.14) in [5],

\[ \sqrt{\frac{w_k (t, \xi)}{\partial w_k (t, \xi)}} |1_{|k| \geq k_M} \tilde{A} f_{-N} \|_2 \lesssim |k| \sqrt{1 + |t - \frac{\xi}{k}|} |1_{|k| \geq k_M} \tilde{A} f_{-N} |_2 \]

\[ \lesssim |k| \left( 1 + \frac{|t - \frac{\xi}{k}|}{k} \right) \sqrt{\frac{\partial w_k (t, \xi)}{w_k (t, \xi)}} |1_{|k| \geq k_M} \tilde{A} f_{-N} |_2 \]

which implies,

\[ \left\| \Pi_{LH}^{u'':R,R} \right\| \leq \frac{1}{k_M} \left\| \sqrt{\frac{\partial w}{w}} \tilde{A} f \right\|_2^2 + \frac{C}{k_M} \left\| \frac{\partial w}{w} \Delta L \tilde{A} \mathcal{P}_{|k| \geq k_M} \right\|_2^2 \]

for some constant \( C \) independent of \( k_M \).

Finally, summing together (5.1), (5.3), (5.5), (5.4) and (5.6) and by taking \( k_M \) sufficiently large, we proved Proposition 2.17.
5.3. **Nonlocal error terms.** In this subsection, let us treat $\Pi^{u'',\epsilon_1}$ and $\Pi^{u'',\epsilon_2}$. By using the $\epsilon$-smallness of $\tilde{u}'' - u''$ and the same argument as in section 6 of [5], it is easy to obtain that

$$|\Pi^{u'',\epsilon_1}| \lesssim c_{K_\lambda} + c_{K_w} + \frac{\epsilon^3}{\langle t \rangle^2 - k \delta \epsilon/2} + c_{K_{\lambda}^{v,1}} + c_{K_{w}^{v,1}}$$

$$+ \epsilon \left\| \frac{\partial_v}{t \partial_z} \right\|^{-1} \left( \partial_z^2 + (\partial_v - t \partial_z)^2 \right) \left( \frac{|\nabla|^2}{\langle t \rangle^s} A + \sqrt{\frac{\partial_t w}{w}} \right) P_{\chi}(\Psi \chi) \right\|_2^2. \tag{6.4}$$

To treat $\Pi^{u'',\epsilon_2}$, let us recall (3.9) and the definition of $T^1$ and $T^2$ (3.14). Thus we get that

$$\tilde{u}''(\Delta_1^{-1} \Omega - \Delta_{\chi}^{-1} \Omega) = -\tilde{u}''(\Delta_1^{-1})(T^1 + T^2).$$

Therefore, by using the same argument as in section 6 of [5] and the estimate in section 3.2, we obtain that

$$|\Pi^{u'',\epsilon_2}| \lesssim c_{K_\lambda} + c_{K_w} + \frac{\epsilon^3}{\langle t \rangle^2 - k \delta \epsilon/2} + c_{K_{\lambda}^{v,1}} + c_{K_{w}^{v,1}}$$

$$+ \epsilon \left\| \frac{\partial_v}{t \partial_z} \right\|^{-1} \left( \partial_z^2 + (\partial_v - t \partial_z)^2 \right) \left( \frac{|\nabla|^2}{\langle t \rangle^s} A + \sqrt{\frac{\partial_t w}{w}} \right) P_{\chi}(\Psi \chi) \right\|_2^2. \tag{6.5}$$

6. **Appendix: Gever spaces**

The physical space characterization of Gever functions is also used in this paper. We start with a characterization of the Gever spaces on physical side. See Lemma A.1 in [9] for the elementary proof.

**Lemma 6.1** ([9]). Suppose that $d = 1, 2, 0 < s < 1, K > 1$ and $g \in C^\infty(\mathbb{R}^d)$ with supp $g \subset [a, b]^d$ satisfies the bounds

(6.1) \[ |D^\alpha g(x)| \leq K^m(m + 1)^{m/s}, \quad x \in \mathbb{R}^d \]

for all integers $m \geq 0$ and multi-indeces $\alpha$ with $|\alpha| = m$. Then

(6.2) \[ |\hat{g}(\xi)| \lesssim_{K, s} L e^{-\mu|\xi|^s}, \]

for all $\xi \in \mathbb{R}^d$ and some $\mu = \mu(K, s) > 0$.

Conversely, assume that, for some $\mu > 0$ and $s \in (0, 1)$,

(6.3) \[ |\hat{g}(\xi)| \leq L e^{-\mu|\xi|^s}, \]

for all $\xi \in \mathbb{R}^d$. Then there is $K > 1$ depending on $s$ and $\mu$ such that

(6.4) \[ |D^\alpha g(x)| \lesssim_{K, s} K^m(m + 1)^{m/s}, \quad x \in \mathbb{R}^d \]

for all integers $m \geq 0$ and multi-indeces $\alpha$ with $|\alpha| = m$.

For $x \in [a, b]^d$ and parameters $s \in (0, 1)$ and $M \geq 1$, we define the spaces

(6.5) \[ \mathcal{G}_{ph}^{M,s}([a, b]^d) \overset{\text{def}}{=} \left\{ g : [a, b]^d \to \mathbb{C} : \|g\|_{\mathcal{G}_{ph}^{M,s}([a, b]^d)} < \infty \right\}. \]

where

(6.6) \[ \|g\|_{\mathcal{G}_{ph}^{M,s}([a, b]^d)} \overset{\text{def}}{=} \sup_{x \in [a, b]^d, m \geq 0, |\alpha| \leq m} \frac{|D^\alpha g(x)|}{(m + 1)^{m/s} M^m}. \]

Here ‘$ph$’ represents the physical side.
We define the spaces
\begin{equation}
\mathcal{G}^{M,s}_{ph,1}([a,b]^d) \defeq \left\{ g : [a,b]^d \to \mathbb{C} : \|g\|_{\mathcal{G}^{M,s}_{ph,1}([a,b]^d)} < \infty \right\},
\end{equation}
where
\begin{equation}
\|g\|_{\mathcal{G}^{M,s}_{ph,1}([a,b]^d)} \defeq \sup_{x \in [a,b]^d, m \geq 0, |\alpha| \leq m} \frac{|D^{\alpha}g(x)|}{\Gamma_s(m) M^m},
\end{equation}
with \( \Gamma_s(m) = (m!)^{\frac{1}{s}}(m+1)^{-\frac{1}{s}} \), also see [24] for more details.

By the Stirling's formula \( N! \sim \sqrt{2\pi N} (N/e)^N \), it is easy to check that there exist \( K_1 < K_2 \) such that
\[ K_1^m (m+1)^{m/s} \leq \Gamma_s(m) \leq K_2^m (m+1)^{m/s}, \]
which implies
\begin{equation}
\mathcal{G}^{K_1 M,s}_{ph,1}([a,b]^d) \subset \mathcal{G}^{M,s}_{ph,1}([a,b]^d) \subset \mathcal{G}^{K_2 M,s}_{ph,1}([a,b]^d).
\end{equation}
We also have
\begin{equation}
\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \Gamma_s(j) \Gamma_s(k-j) < \Gamma_s(k),
\end{equation}
which implies
\begin{equation}
\|fg\|_{\mathcal{G}^{M,s}_{ph,1}([a,b]^d)} \leq \|f\|_{\mathcal{G}^{M,s}_{ph,1}([a,b]^d)} \|g\|_{\mathcal{G}^{M,s}_{ph,1}([a,b]^d)}.
\end{equation}

**Remark 6.2.** Suppose \( g \in \mathcal{G}^{M,s}_{ph,1}(\mathbb{R}^d) \) with \( \supp g \subset [a,b]^d \), then for all integers \( m \geq 0 \) and multi-indeces \( \alpha \) with \( |\alpha| = m \). Then
\begin{equation}
|g(\xi)| \lesssim_{K,s} Le^{-\mu|\xi|^s},
\end{equation}
for all \( \xi \in \mathbb{R}^d \) and some \( \mu = \mu(K,s) > 0 \).

We also introduce the composition lemma in [24].

**Lemma 6.3** ([24]). Let \( I, J \) be real open intervals and let \( f : J \to \mathbb{R} \) be a \( C^\infty \)-function such that \( f' \in \mathcal{G}^{L,s}_{ph,1}(J) \) and \( g : I \to J \) be a \( C^\infty \)-function such that \( g' \in \mathcal{G}^{M,s}_{ph,1}(I) \). Let \( N \) be a real constant such that
\[ N \geq \max(M, L\|g'\|_{\mathcal{G}^{M,s}_{ph,1}(I)}). \]
Then the derivative \((f \circ g)'\) of the composite function \( f \circ g \) belongs to \( \mathcal{G}^{N,s}_{ph,1}(I) \) and satisfies
\begin{equation}
\| (f \circ g)' \|_{\mathcal{G}^{N,s}_{ph,1}(I)} \leq \frac{N}{L} \| f' \|_{\mathcal{G}^{L,s}_{ph,1}(J)}. \end{equation}

We also introduce the estimate of inverse function in Gevrey class.

**Lemma 6.4** ([24]). Let \( I, J \) be real open intervals. Let \( f : I \to J \) be a \( C^\infty \)-surjection such that
\[ |f'(x)| \geq \frac{1}{A}, \quad x \in I \]
for some \( A > 0 \) and such that \( f'' \in \mathcal{G}^{L,s}_{ph,1}(I) \). Then \( f \) has a \( C^\infty \)-inverse \( f^{-1} : J \to I \) such that \((f^{-1})' \in \mathcal{G}^{M,s}_{ph,1}(J)\) for some \( M > 0 \).
7. Appendix: the wave operator

7.1. The existence of wave operator. In this subsection, we aim to construct the wave operator and prove Proposition 2.1. We will refer to some details from [21]. Let us first recall Definition 4.3, Lemma 4.4, Proposition 4.5 and Proposition 5.1 in [21] that, there exist \( \phi(k, y, y_c) \) and \( \phi_1(k, y, y_c) \) (here to make the notation good in this paper, we use \( \phi(k, y, y_c) \) and \( \phi_1(k, y, y_c) \) rather than \( \phi(k, y, c) \) and \( \phi_1(k, y, c) \) in [21]) such that \( \phi(k, y, y_c) = (u(y) - u(y_c))\phi_1(k, y, y_c) \) and

\[
\begin{align*}
\partial_{yy} \phi - k^2 \phi - \frac{u''(y)}{u(y) - u(y_c)} \phi &= 0, \quad \phi(k, y, y_c) = 0, \quad \partial_y \phi(k, y, y_c) = u'(y_c).
\end{align*}
\]

Moreover, it holds that

\[
(Id - k^2 T)(\phi_1)(k, y, y_c) = 1,
\]

and \((Id - k^2 T)\) is invertible in \( L^\infty_{y, y_c}([0, 1]^2) \), and

\[
\phi_1(k, y, y_c) \approx \frac{\sinh k(y - y_c)}{k(y - y_c)},
\]

where

\[
T\phi_1 = T_0 \circ T_2 \phi_1 = \int_{y_c}^y \frac{1}{(u(y') - u(y_c))^2} \int_{y_c}^{y'} (u(y'') - u(y_c))^2 \phi_1(y'', y_c) dy'' dy',
\]

with

\[
T_0 \phi_1 = \int_{y_c}^y \phi_1(y', y_c) dy',
\]

and

\[
T_2 \phi_1 = \frac{1}{(u(y) - u(y_c))^2} \int_{y_c}^y (u(y'') - u(y_c))^2 \phi_1(y'', y_c) dy''.
\]

Then by (7.1), (7.2) with \( t = 0 \) and (7.3) in [21], we obtain that for \( k \neq 0 \), it holds that

\[
\psi(k, y) = -\int_0^1 \frac{\rho(y_c) \Delta_k \psi(k, y, y_c) e(k, y, y_c)}{\sqrt{u'(y_c) \rho(y_c) p.v. \int_0^1 \frac{1}{\phi(k, y', y_c)^2} dy'}} dy_c,
\]

with

\[
e(k, y, y_c) = \begin{cases} 
\frac{1}{\phi(k, y', y_c)^2} & 0 \leq y < y_c, \\
\frac{1}{\phi(k, y, y_c)^2} & y_c < y \leq 1,
\end{cases}
\]

where

\[
\rho(y_c) = (u(y_c) - u(0))(u(1) - u(y_c)),
\]

\[
u'(y_c) \rho(y_c) p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \overset{\text{def}}{=} u(0) - u(1) - \rho(y_c) p.v. \int_0^1 \frac{u'(y) - u'(y_c)}{(u(y) - u(y_c))^2} dy + u'(y_c) \rho(y_c) \int_0^1 \frac{1}{(u(y) - u(y_c))^2} \left( \frac{1}{\phi_1(k, y, y_c)^2} - 1 \right) dy,
\]

\[
\quad + \phi_1(k, y, y_c) \int_0^1 \frac{1}{(u(y) - u(y_c))^2} \phi_1(k, y, y_c) dy.
\]
\[ \mathbb{D}_{u,k}(\omega)(k, y_c) = \frac{u'(y_c)\rho(y_c)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy}{\sqrt{\left( u'(y_c)\rho(y_c)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \right)^2 + \left( \pi \rho(y_c) \frac{u''(y_c)}{u'(y_c)^2} \right)^2}} + \frac{\rho(y_c)u''(y_c)p.v. \int_0^1 \frac{\int_0^y \omega(y')\phi_1(k, y', y_c)dy''}{\phi(k, y', y_c)^2} dy'}{\sqrt{\left( u'(y_c)\rho(y_c)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \right)^2 + \left( \pi \rho(y_c) \frac{u''(y_c)}{u'(y_c)^2} \right)^2}}. \]

(7.4)

Let us now define
\[ d_1(k, y) = \frac{u'(y)\rho(y)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy'}{\sqrt{\left( u'(y)\rho(y)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \right)^2 + \left( \pi \rho(y) \frac{u''(y)}{u'(y)^2} \right)^2}}, \]

(7.5)

\[ d_2(k, y) = -\frac{\rho(y)}{\sqrt{\left( u'(y)\rho(y)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \right)^2 + \left( \pi \rho(y) \frac{u''(y)}{u'(y)^2} \right)^2}}. \]

(7.6)

Thus we get that
\[ \mathbb{D}_{u,k}(f)(k, y) = d_1(k, y)f(y) + u''(y)d_2(k, y) \int_0^1 \frac{e(k, y', y)}{u(y') - u(y)} f(y') dy'. \]

(7.7)

Then we turn to the proof of Proposition 2.1.

Proof. By Lemma 6.1 and (8.6) in [21] and the no eigenvalue assumption of the Rayleigh operator, we have
\[ \left( u'(y_c)\rho(y_c)p.v. \int_0^1 \frac{1}{\phi(k, y, y_c)^2} dy \right)^2 + \left( \pi \rho(y_c) \frac{u''(y_c)}{u'(y_c)^2} \right)^2 \approx 1 + k^2 \rho(y_c)^2, \]

(7.8)

and then by (8.1) in [21], it holds that
\[ \|\mathbb{D}_{u,k}\omega\|_2 \leq C\|\omega\|_2, \]

(7.9)

here the constant \( C \) is independent of \( k \).

Let \( g \in H^2(0, 1) \cap H^1_0(0, 1) \) and \( \Delta_k \psi = (\partial_{yy} - k^2)\psi = \omega \), then by (7.2), it holds that
\[
\int_0^1 \omega(y)g(y)dy = \int_0^1 \psi(k, y)\left( \Delta_k g \right)(k, y)dy = \int_0^1 \frac{\int_0^y \Delta_k g(k, y'')\phi(k, y'', y_c)dy''}{\phi(k, y', y_c)^2} dy' \rho(y_c)\mathbb{D}_{u,k}(\omega)(k, y_c)dy_c.
\]

(7.10)

Integrating by parts, we obtain that
\[
\int_{y_c}^{y'} \Delta_k g(k, y'')\phi(k, y'', y_c)dy'' = \int_{y_c}^{y'} g(k, y'')\Delta_k \phi(k, y'', y_c)dy'' + \partial_y g(k, y')\phi(k, y', y_c) - g(k, y')\partial_y \phi(k, y', y_c) + u'(y_c)g(k, y_c),
\]
which gives us
\[
\int_0^1 \frac{\int_{y_c}^y \Delta_k g(k, y'') \phi(k, y'', y_c) dy''}{\phi(k, y', y_c)^2} dy' = \int_0^1 \frac{\int_{y_c}^y \frac{u''(y'') \phi(k, y'', y_c)}{u(y') - u(y_c)} dy''}{\phi(k, y', y_c)^2} dy' + \int_0^1 \frac{\int_{y_c}^y \frac{\partial_k g(k, y') \phi(k, y', y_c) - g(k, y') \partial_k \phi(k, y', y_c) + u'(y_c) g(k, y_c)}{\phi(k, y', y_c)^2} dy'}{\phi(k, y', y_c)^2} dy' + \int_0^1 \frac{\int_{y_c}^y \frac{\partial_y g(k, y') \phi(k, y', y_c)}{\phi(k, y', y_c)^2} dy'}{\phi(k, y', y_c)^2} dy' = \int_0^1 \frac{\int_{y_c}^y \frac{u''(y'') \phi(k, y'', y_c)}{\phi(k, y', y_c)^2} dy'' + g(k, y_c) u'(y_c) \rho(y_c) p.v. \int_0^1 \frac{1}{\phi(k, y', y_c)^2} dy}{\phi(k, y', y_c)^2},
\]
which implies that
\[
\int_0^1 \mathbb{D}_{u,k}(\omega)(k, y_c) \mathbb{D}_{u,k}^1(g)(k, y_c) dy_c = \int_0^1 \omega(k, y) g(k, y) dy,
\]
where
\[
\mathbb{D}_{u,k}^1(g)(k, y_c) \overset{\text{def}}{=} \frac{u'(y_c) \rho(y_c) p.v. \int_0^1 \frac{1}{\phi(k, y', y_c)^2} dy g(y_c)}{\sqrt{\left(\int_0^1 \frac{1}{\phi(k, y', y_c)^2} dy g(y_c)\right)^2 + \left(\pi \rho(y_c) \frac{u''(y_c)}{u'(y_c)^2}\right)^2 + \int_0^1 \frac{\int_{y_c}^y \frac{u''(y'') \phi_1(k, y'', y_c) dy''}{\phi(k, y', y_c)^2} dy'}{\phi(k, y', y_c)^2} dy' + \int_0^1 \frac{\int_{y_c}^y \frac{\partial_y g(k, y') \phi_1(k, y'', y_c) dy''}{\phi(k, y', y_c)^2} dy'}{\phi(k, y', y_c)^2} dy'}}.
\]
(7.10)

Thus by (8.1) in [21] and (7.8), we have
\[
\|\mathbb{D}_{u,k}^1 g\|_2 \leq C\|g\|_2,
\]
here the constant $C$ is independent of $k$.

Due to the fact that $H^2 \cap H_0^1$ is dense in $L^2$, for any $g \in L^2$, there is a sequence $\{g_n\}_{n \geq 1}$ such that $H^2 \cap H_0^1 \ni g_n \to g$ in $L^2$ and
\[
\int_0^1 \mathbb{D}_{u,k}(f)(y_c) \mathbb{D}_{u,k}^1(g_n)(y_c) dy_c = \int_0^1 f(y) g_n(y) dy.
\]

Then
\[
\left|\int_0^1 \mathbb{D}_{u,k}(f)(y_c) \mathbb{D}_{u,k}^1(g)(y_c) dy_c - \int_0^1 f(k, y) g(k, y) dy\right|
\]
\[
= \left|\int_0^1 \mathbb{D}_{u,k}(f)(y_c) \mathbb{D}_{u,k}^1(g_n)(y_c) dy_c - \int_0^1 f(k, y) (g(y) - g_n(y)) dy\right|
\]
\[
\leq C\|\mathbb{D}_{u,k}(f)\|_{L^2} \|\mathbb{D}_{u,k}^1(g - g_n)\|_{L^2} + \|f\|_{L^2} \|g - g_n\|_{L^2} \to 0.
\]

Thus we have proved (2.11) and (2.12).

In order to get (2.10), we integrate by parts and use $\phi(k, y_c, y_c) = 0, \partial_y \phi(k, y_c, y_c) = u'(y_c), \psi(k, 0) = \psi(k, 1) = 0$ to arrive at
\[
p.v. \int_0^1 \frac{\int_{y_c}^y u''(y') \psi(k, y') \phi_1(k, y', y_c) dy'}{\phi(k, y', y_c)^2} dy' = p.v. \int_0^1 \frac{\int_{y_c}^y \psi(k, y') (\partial_y^2 - k^2) \phi(k, y', y_c) dy'}{\phi(k, y, y_c)^2} dy
\]
where

$$
\mu \rightarrow \text{the nonlocal part into the following four types of integral operators:}
$$

$$
\text{Indeed, we will write}
$$

$$
(7.12)
$$

which gives (2.10).

By the (2.11), it is easy to obtain that the inverse of $D_{u,k}$ exists and has the following formula

$$
D_{u,k}^{-1}(f)(k,y) = d_1(k,y) f(y') + u''(y) \int_0^1 \frac{e(k,y,y')}{{u' - u' y'}} d_2(k,y') f(y') dy',
$$

which gives Remark 2.2.

7.2. **The Fourier transform of the integral operator.** In this section, we make preparations to study the Gevrey regularity of the nonlocal part in the wave operator. Indeed, we will write the nonlocal part into the following four types of integral operators:

$$
\Pi_m(F)(c) = \int_K(u,c) \mu_m(u - c) e^{-ikt(u - c)} F(u) du, \quad m = 1, 2, 3, 4,
$$

$$
\Pi_m^*(F)(u) = \int_K(u,c) \mu_m(u - c) e^{-ikt(u - c)} F(c) dc, \quad m = 1, 2, 3, 4,
$$

where $K(u_1, u)$ represent a smooth kernel with compact support which may vary from one line to the other and $\mu_1 \in C^\infty(R)$ with compact support such that

$$
\mu_1(u) = \begin{cases} 
1, |u| \leq u(1) - u(0), \\
0, |u| \geq 2(u(1) - u(0)),
\end{cases}
$$

$\mu_2(u) = \mu_1(u) 1_{R^+}(-v)$ and $\mu_3(u) = \mu_1(u) \ln |u|, \mu_4(u) = \text{p.v.} \frac{1}{u}$.

It is easy to check that $\Pi_m^* m = 1, 2, 3, 4$ are the dual operators of $\Pi_m, m = 1, 2, 3, 4$.

For a smooth kernel with compact support defined on $R^2$, let $\widehat{K}$ be the Fourier transform in both variables.

**Lemma 7.1.** Suppose $K(u, c) \in C^\infty(R^2)$ with compact support, then it holds for $m = 1, 2, 3, 4$ that

$$
\Pi_m(F)(\xi) = \frac{1}{2\pi} \int_{R^2} \hat{F}(\xi) \hat{K}(-\xi, \eta + \xi') \hat{\mu}_m(\xi' + kt) d\xi' d\xi,
$$

and

$$
\Pi_m^*(F)(\eta) = \frac{1}{2\pi} \int_{R^2} \hat{F}(\eta) \hat{K}(-\eta, \xi + \eta') \hat{\mu}_m(\xi' + kt) d\xi' d\eta.
$$

**Proof.** Let $K_L(u - c, c) = K(u, c)$, then we have that

$$
\Pi_m(F)(c) = \int_{R} K_L(u - c, c) \mu_m(u - c) e^{-ikt(u - c)} F(u) du
$$

$$
= \frac{1}{2\pi} \int_{R} K_L(u - c, c) \mu_m(u - c) e^{-ikt(u - c)} \int_{R} \hat{F}(\xi) e^{ikut} d\xi du
$$
\[
= \frac{1}{2\pi} \int_\mathbb{R} \hat{F}(\xi)e^{ix\xi} \int_\mathbb{R} K_L(u-c, c)\mu_m(u-c) e^{-ikt(u-c)} e^{ix(u-c)} du d\xi \\
= \frac{1}{2\pi} \int_\mathbb{R} \hat{F}(\xi)e^{ix\xi} F_1(K_L(\cdot, c)\mu_m)(-\xi + kt) d\xi \\
= \frac{1}{2\pi} \int_\mathbb{R} \hat{F}(\xi)e^{ix\xi} \hat{K}_L(-\xi - \xi', c)\mu_m(\xi' + kt) d\xi' d\xi.
\]

Thus we get that
\[
\hat{\Pi}_m(\hat{F})(\eta) = \frac{1}{2\pi} \int_\mathbb{R} \hat{F}(\xi)\hat{K}_L(-\xi - \xi', \eta - \xi)\mu_m(\xi' + kt) d\xi' d\xi.
\]

We also have
\[
\hat{\mu}_m(\xi, \eta) = \int_{\mathbb{R}^2} K_L(u, c) e^{-i\xi u - i\eta \eta} du dc \\
= \int_{\mathbb{R}^2} K_L(v-c, c) e^{-i(u-c)\xi - ic\eta} du dc \\
= \int_{\mathbb{R}^2} K_L(c, c) e^{-i(u-c)\xi - ic\eta} du dc = \hat{\mu}(\xi - \xi),
\]
which gives the lemma. \qed

**Remark 7.2.** Suppose \( K(u, c) \in \mathcal{O}^{M, s_0}_{ph, 1}(\mathbb{R}^2) \) with compact support in \([u(0), u(1)]^2\), then there exists \( \lambda = \lambda(M, s_0) \), such that for \( m = 1, 2, 3, 4 \),
\[
\left| \int \hat{K}(-\xi - \xi', \eta + \xi')\mu_m(\xi' + kt) d\xi' \right| \lesssim e^{-\lambda|\eta - \xi|^s_0}.
\]

This remark follows directly from Remark 6.2 and the fact that
\[
\int_\mathbb{R} e^{-\lambda'(|\xi + \xi'|^2 + |\eta + \xi'|^2)^{\frac{3p}{2}}} d\xi' \lesssim \int_\mathbb{R} e^{-\lambda'(|\xi - \eta|^s_0)} d\xi' \lesssim e^{-\lambda|\xi - \eta|^s_0},
\]
holds for \( 0 < \lambda < \lambda' \).

**7.3. Gevrey regularity.** In this subsection, we will study the Gevrey regularity of the coefficients \( d_1, d_2 \) and the kernel \( e \) of the wave operator and prove Proposition 2.5.

A direct calculation gives us that
\[
\phi(k, y, y_c) \int_j^y \frac{1}{\phi(k, y', y_c)^2} dy' \\
= \phi(k, y, y_c) \int_j^y \frac{1}{(u(y') - u(y_c))^2} \left( \frac{1}{\phi_1(k, y', y_c)^2} - 1 \right) dy' \\
+ \phi(k, y, y_c) \int_j^y \frac{1}{(u(y') - u(y_c))^2} dy' \\
= \phi(k, y, y_c) \int_j^y \frac{1}{(u(y') - u(y_c))^2} \left( \frac{1}{\phi_1(k, y', y_c)^2} - 1 \right) dy' \\
+ \phi(k, y, y_c) \int_{u(j)}^{u(y)} \frac{(u-1)'(u_1) - (u-1)'(u(y_c)) - (u-1)'(u(y_c))(u_1 - u(y_c))}{(u_1 - u(y_c))^2} du_1 \\
- \frac{1}{u'(y_c)} u'(y') - u(y_c) \bigg|_{y'=y} + \phi(k, y, y_c) \frac{w(u(y_c))}{u'(y_c)^2} \ln |u(y') - u(y_c)| \bigg|_{y'=j}.
\]
Thus we have that
\[
\phi(k, y, y_c) \int_j^y \frac{1}{\phi(k, y', y_c)} dy' = -\frac{1}{u'(y_c)} + \frac{u''(y_c)}{u'(y_c)^3} \phi(k, y, y_c) \ln |u(y) - u(y_c)| + \phi_j^{re}(k, y, y_c),
\]
where
\[
\phi_j^{re}(k, y, y_c) = \phi(k, y, y_c) \int_j^y \frac{1}{u(y') - u(y_c)} \left( \frac{1}{\phi_1(k, y', y_c)^2} - 1 \right) dy' + \phi(k, y, y_c) \int_{u(j)}^{u(y)} \frac{\phi_1(k, y, y_c) (u_1 - u(y_c))^2}{(u_1 - u(y_c))^2} du_1 + \frac{1}{u'(y_c)} (1 - \phi_1(k, y, y_c)) + \frac{1}{u'(y_c)} \frac{(u(y) - u(y_c)) \phi_1(k, y, y_c)}{u(j) - u(y_c)} - \phi(k, y, y_c) \frac{u''(y_c)}{u'(y_c)^3} \ln |u(j) - u(y_c)|.
\]
Therefore,
\[
e(k, y, y_c) = -\frac{1}{u'(y_c)} + \frac{u''(y_c)}{u'(y_c)^3} \phi(k, 1, y_c) \ln |u(1) - u(y_c)| + \phi_0^{re}(k, y, y_c)(1) + \phi_1^{re}(k, y, y_c)(1)\]
\[
\text{Let us define}
\]
\[
w_1(k, y_c) = \frac{-\frac{1}{u'(y_c)} + \frac{u''(y_c)}{u'(y_c)^3} \phi(k, 1, y_c) \ln |u(1) - u(y_c)| + \phi_0^{re}(k, 1, y_c)}{\phi_1(k, 1, y_c)} (u(y_c) - u(0)) u'(y_c),
\]
then, we get
\[
d_1(k, y_c) = \frac{w_1(k, y_c)}{\sqrt{w_1(k, y_c)^2 + \left( \pi \rho(y_c) \frac{u''(y_c)}{u'(y_c)^2} \right)^2}}.
\]
Making the change of coordinate \((y, y_c) \rightarrow (u, c)\) and using the definition (2.19), we also have

\[
(7.13) \quad D_1(k, c) = \frac{W_1(k, c)}{\sqrt{W_1(k, c)^2 + \left( \pi \tilde{\rho}(c) \frac{\tilde{u}''(c)}{\tilde{u}'(c)^2} \right)^2}},
\]
\[
(7.14) \quad D_2(k, c) = -\frac{\tilde{\rho}(c)}{\sqrt{W_1(k, c)^2 + \left( \pi \tilde{\rho}(c) \frac{\tilde{u}''(c)}{\tilde{u}'(c)^2} \right)^2}},
\]
\[
(7.15) \quad E(k, u, c) = -\frac{1}{u'(c)} + (u^{-1})''(c)(u - c) \Phi_1(k, u, c) \ln |u - c| + \Phi_0^{re}(k, u, c) (1) + \Phi_1^{re}(k, u, c) (1),
\]
where
\[
\tilde{\rho}(c) = (u(1) - c)(c - u(0)), \quad \tilde{u}'(c) = u' \circ u^{-1}(c) = \frac{1}{(u^{-1})'(c)}, \quad \tilde{u}''(c) = u'' \circ u^{-1}(c),
\]
\[
W_1(k, c) = \frac{-c + u(0)}{\Phi_1(k, u(1), c)} + \frac{(u^{-1})''(c) \tilde{\rho}(c) \ln |u(1) - c|}{(u^{-1})'(c)} + \frac{\Phi_0^{re}(k, u(1), c) c - u(0)}{\Phi_1(k, u(1), c) (u^{-1})'(c)},
\]
(7.16) \[ \Phi_{j}^{re}(k, u, c) = (u - c) \left[ \Phi_1(k, u, c) \int_{u(j)}^{u} \frac{1}{(u_1 - c)^2} \left( \frac{1}{\Phi_1(k, u_1, c)^2} - 1 \right) (u_1')'(u_1) du_1 \right. \]

\[ + \left. \Phi_1(k, u, c) \int_{u(j)}^{u} \frac{1}{u(c)} - (u_1')'(c) - (u_1'')'(c)(u_1 - c) \right] du_1 \]

\[ + \frac{1}{u'(c)} \frac{1 - \Phi_1(k, u, c)}{u - c} + \frac{1}{u'(c)} \frac{\Phi_1(k, u, c)}{u(j) - c} - \Phi_1(k, u, c)(u_1'')'(c) \ln |u(j) - c| \]

(7.17) \[ \Phi_1(k, u, c) \text{def} = (u - c) \Phi_{j,1}^{re}(k, u, c). \]

with \( \Phi_1(k, u, c) \) satisfying

\[ \Phi_1(k, u(y), u(y_c)) = \phi_1(k, y, y_c). \]

To study the Gevrey regularity of the wave operator, we only need to study the smooth kernels \( \Phi_1, \Phi_{0,1}^{re}, \Phi_{1,1}^{re} \) and the coefficients \( D_1, D_2 \). Indeed, all these functions are related to \( \Phi_1 \) closely. Once we obtain the estimate of \( \Phi_1 \), we can deduce the estimates of other functions.

Now let us study the regularity of \( \Phi_1 \).

Recall that \( \Phi(k, u(y), u(y_c)) = \phi(k, y, y_c), \Phi_1(k, u(y), u(y_c)) = \phi_1(k, y, y_c), \Phi(k, u, c) = (u - c) \Phi_1(k, u, c) \) and

(7.19) \[ \Phi_1(k, u, c) = 1 + k^2 \int_{c}^{u} \frac{u(u_1')'(u_1)}{(u_1 - c)^2} \int_{c}^{u_1} (u_2 - c)^2 (u_1')'(u_2) \Phi_1(k, u_2, c) du_2 du_1 \]

\[ = 1 + k^2 T_0 \circ T_{2,2} \Phi_1, \]

with \( T_0 \Phi_1(k, u, c) = \int_{c}^{u} \Phi_1(k, u_1, c) du_1 \) and for \( j_1 \leq j_2 + 1, j_2 = 0, 1, 2, \ldots \),

\[ T_{j_1,j_2} \Phi_1(k, u_1, c) = \frac{(u_1')'(u_1)}{(u_1 - c)^{j_1}} \int_{c}^{u_1} (u_2 - c)^{j_2} (u_1')'(u_2) \Phi_1(k, u_2, c) du_2. \]
For any $k \neq 0$, let $M \geq 1$ be a large constant, then we define the following weighted Gevrey spaces:

\begin{equation}
G_{\phi h, \cosh}^{M|k|,s}([a, b]^2) = \{ f \in C^\infty([a, b]^2) : \| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)} < \infty \},
\end{equation}

with

\begin{equation}
\| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)} = \sup_{(u, c) \in [u(0), u(1)]^2, m \geq m_1 \geq 0} \frac{|\partial^{m-m_1}_u (\partial_c + \partial_u)^m f(u, c)|}{\Gamma_s(m) M^m |k|^m \cosh M|k| (u - c)}.
\end{equation}

Remark 7.3. It holds that

\[
\sup_{(u, c) \in [u(0), u(1)]^2, m \geq m_1 \geq 0} \frac{|\partial^{m-m_1}_u \partial_c^m f(u, c)|}{\Gamma_s(m) M^m |k|^m \cosh M|k| (u - c)} \leq \| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)}.
\]

Proof. We have

\[
\sup_{(u, c) \in [u(0), u(1)]^2, m \geq m_1 \geq 0} \frac{|\partial^{m-m_1}_u \partial_c^m f(u, c)|}{\Gamma_s(m) M^m |k|^m \cosh M|k| (u - c)} \\
\leq \sup_{m_1 \geq 0} 2^{-m_1} \sum_{m_2 = 0}^{m_1} \frac{m_1!}{m_2!(m_1 - m_2)!} \Gamma_s(m) M^m |k|^m \cosh M|k| (u - c)
\]

\[
\leq \sup_{m_1 \geq 0} 2^{-m_1} \sum_{m_2 = 0}^{m_1} \frac{m_1!}{m_2!(m_1 - m_2)!} \| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)} \leq \| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)}.
\]

Thus we proved the remark.

Remark 7.4. It holds that

\[
\| fg \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)} \leq C \| f \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)} \| g \|_{G_{\phi h, \cosh}^{M|k|,s}([a, b]^2)}.
\]

Proposition 7.5. Suppose $(u^{-1})^n \in G_{\phi h, \cosh}^{K_u,s_0}([u(0), u(1)])$, then there exists $M_0 = M_0(K_u) \geq 1$, such that for any $M \geq M_0$, it holds that

\[
\| T_0 \circ T_{2,2} \|_{G_{\phi h, \cosh}^{M|k|,s_0}([u(0), u(1)]^2)} \leq \| G_{\phi h, \cosh}^{M|k|,s_0}([a, b]^2) \}
\]

\[
\frac{C}{M^2 k^2}.
\]

Proof. It is easy to check that

\[
(\partial_u + \partial_c)^m T_0 \circ T_{2,2} f
\]

\[
= \sum_{k_1 + k_2 \leq m} \frac{m!}{(m - k_1 - k_2)!k_1!k_2!} \int_c^u \frac{(u^{-1})^{(k_1+1)}(u)}{(u_1 - c)^2} \int_c^{u_1} (u_2 - c)^2 (u^{-1})^{(k_2+1)}(u_2) (\partial_u + \partial_c)^{m_k} f(k, u_2, c) du_2 du_1,
\]

which gives us that

\[
\frac{\| (\partial_u + \partial_c)^m T_0 \circ T_{2,2} f \|}{\Gamma_s(m) M^m |k|^m \cosh M|k| (u - c)}
\]

\[
\leq \sum_{k_1 + k_2 \leq m} \frac{m!}{(m - k_1 - k_2)!k_1!k_2!} \Gamma_s(m - k_1 - k_2) \Gamma_s(k_1) \Gamma_s(k_2) \int_c^u \frac{1}{(u_1 - c)^2} \int_c^{u_1} (u_2 - c)^2 \cosh M|k| (u_2 - c) du_2 du_1.
\]
which implies

\[ (7.22) \]

\[
\sup_{m \geq 0} \left| \frac{(\partial_u + \partial_c)^m T_0 \circ T_{2,2} f}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right| \leq \frac{C}{M^2k^2} \sup_{m \geq 0} \left| \frac{(\partial_u + \partial_c)^m f}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|.
\]

We also have

\[
\partial_u T_0 \circ T_{2,2} f = T_{2,2} f
\]

\[
\partial_u^2 T_0 \circ T_{2,2} f = (u-c) \frac{(u-1)''(u)}{(u-1)'(u)} T_{3,2} f - 2T_{3,2} f + (u-1)'(u) f(u, c)
\]

**Claim:** It holds for any \( s \in (0, 1) \) and \( j \geq 0 \) that

\[ (7.23) \]

\[
\sup_{m_1, m_2 \geq 0} \left| \frac{\partial_u^{m_1}(\partial_u + \partial_c)^{m-m_1} \int_c^u (u-c)^j f(u', c) du'}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)} \leq C \sup_{m_1, m_2 \geq 0} \left| \frac{\partial_u^{m_2} f}{\cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}.
\]

Let admit the claim and finish the proof of the proposition first. Indeed, it is easy to check that

\[
\sum_{m=0}^{2} \sum_{m_1=0}^{m} \left| \frac{\partial_u^{m_1} \partial_u^{m-m_1} T_0 \circ T_{2,2} f}{(M|k|)^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)} \leq C \sum_{m=0}^{2} \left| \frac{\partial_u^{m_2} f}{\cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}.
\]

For more details of the above estimate one can refer to [21].

For \( m_1 \geq 2 \), by using the claim, we have that

\[
\left| \frac{(\partial_u + \partial_c)^{m-m_1} \partial_u^{m_1} T_0 \circ T_{2,2} f}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)} \leq \left| \frac{(\partial_u + \partial_c)^{m-m_1} \partial_u^{m_1-2} \int_c^u (u-c)^j f(u', c) du'}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}
\]

\[
+ 2 \left| \frac{(\partial_u + \partial_c)^{m-m_1} \partial_u^{m_1-2} T_{3,2} f}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}
\]

\[
+ \left| \frac{(\partial_u + \partial_c)^{m-m_1} \partial_u^{m_1-2} ((u-1)'(u) f(u, c))}{\Gamma_s(m)M^m|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}
\]

\[
\leq C \sup_{m_2 \leq m_1 - 2, m_3 \leq m - m_1} \left| \frac{(\partial_u + \partial_c)^{m-m_3} \partial_u^{m_2} f(u, c)}{\Gamma_s(m_2 + m_3)M^{m_2+m_3}|k|^m \cosh M|k|(u-c)} \right|_{L^\infty([u(0), u(1)]^2)}.
\]

Finally, let us prove the claim. Indeed we have

\[
\partial_u^{m_1} (\partial_u + \partial_c)^{m-m_1} \left( \frac{1}{(u-c)^{j+1}} \int_c^u (u' - c)^j f(u', c) du' \right)
\]

\[
= \partial_u^{m_1} (\partial_u + \partial_c)^{m-m_1} \left( \int_0^1 t^j f(c + (u-c)t, c) dt \right)
\]
Corollary 7.7.

Proof. Suppose \( \Phi \) is as in Proposition 7.5 and \( \Phi_1 \) satisfies (7.19), then there exist \( M_0 \) and \( C(k) > 0 \) such that for any \( |k| \neq 0 \), for all integers \( m \geq m_1 \geq 0 \) and \( M \geq M_0 \),

\[
|\partial_u^m \partial_c^{m-m_1} \Phi_1(k, u, c)| \leq C(k) \Gamma_{s_0}(m)(2M)^m |k|^m.
\]

Moreover there exists \( C \geq 1 \) such that

\[
|\partial_u^m \partial_c^{m-m_1} (\Phi_1(k, u, c) - 1) - \frac{1}{(u-c)^2}| \leq C(k) \Gamma_{s_0}(m)(CM)^m |k|^m.
\]

Thus we proved the proposition. \( \square \)

By (7.19), Proposition 7.5 and Remark 7.3, we have the following corollary:

**Corollary 7.6.** Suppose \( u \) is the same as in Proposition 7.5 and \( \Phi_1 \) satisfies (7.19), then there exist \( M_0 \) and \( C(k) > 0 \) such that for any \( |k| \neq 0 \), for all integers \( m \geq m_1 \geq 0 \) and \( M \geq M_0 \),

\[
|\partial_u^m \partial_c^{m-m_1} \Phi_1(k, u, c)| \leq C(k) \Gamma_{s_0}(m)(2M)^m |k|^m.
\]

Moreover there exists \( C \geq 1 \) such that

\[
|\partial_u^m \partial_c^{m-m_1} (\Phi_1(k, u, c) - 1) - \frac{1}{(u-c)^2}| \leq C(k) \Gamma_{s_0}(m)(CM)^m |k|^m.
\]

Proof. The second inequality follows directly from the first inequality and the fact that \( \Phi_1(k, u, c)|_{u=c} = 1 \) and \( \partial_c \Phi_1(k, u, c)|_{u=c} = \partial_u \Phi_1(k, u, c) = 0 \). So we only need to prove the first inequality.

By taking \( M_0 \) large enough, it follows from (7.19) and Proposition 7.5 that

\[
\|\Phi_1\|_{G^{M, s_0}_{\text{ph, cosh}}([a, b]^2)} \leq \|k^2 T_0 \circ T_{2, 2} \Phi_1\|_{G^{M, s_0}_{\text{ph, cosh}}([a, b]^2)} + C
\]

\[
\leq \frac{C}{M^2} \|\Phi_1\|_{G^{M, s_0}_{\text{ph, cosh}}([a, b]^2)} + C,
\]

which gives us that

\[
\|\Phi_1\|_{G^{M, s_0}_{\text{ph, cosh}}([a, b]^2)} \leq C.
\]

Then the corollary follows from Remark 7.3. \( \square \)

**Corollary 7.7.** For \( j = 0, 1 \), let \( \Phi_{j,1}^r \) and \( \Phi_{j,1}^c \) be defined as in (7.16) and (7.17), then there exists \( C = C(\theta_0, M, k) \) such that for any \( |k| \neq 0 \), for all integers \( m \geq m_1 \geq 0 \),

\[
\sup_{(c, u) \in [u(0), u(1)]^2} |\partial_u^m \partial_c^{m-m_1} \Phi_{j,1}^r(k, u, c)| \leq C \Gamma_{s_0}(m)(CM)^m |k|^m,
\]

and

\[
\sup_{(c, u) \in [u(0), u(1)]^2} |\partial_u^m \partial_c^{m-m_1} \Phi_{j,1}^c(k, u, c)| \leq C \Gamma_{s_0}(m)(CM)^m |k|^m.
\]

Proof. By the second inequality in Corollary 7.6, we obtain that for all integers \( m \geq m_1 \geq 0 \), there exist \( C \) such that for \( j = 0, 1 \)

\[
\sup_{(c, u) \in [u(0), u(1)]^2} \left| \partial_u^m \partial_c^{m-m_1} \int_{u(j)}^u \frac{1}{(u-1)^2} \left( \frac{1}{\Phi_1(k, u_1, c)^2} - 1 \right)(u-1)'(u_1)du_1 \right|
\]
By the fact that
\[
\frac{(u^{-1})'(u) - (u^{-1})'(c) - (u^{-1})''(c)(u - c)}{(u - c)^2}
\]
\[
= \frac{1}{(u - c)^2} \int_c^u \int_c^u (u^{-1})''(u_2)du_2du_1 = \int_0^1 \int_0^1 t(u^{-1})''(c + st(u - c))dsdt,
\]
we obtain that there is \( C \geq 1 \) such that
\[
\left\| \frac{(u^{-1})'(u) - (u^{-1})'(c) - (u^{-1})''(c)(u - c)}{(u - c)^2} \right\|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)]^2)} \leq \| (u^{-1})'' \|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)])},
\]
which gives us that for \( j = 0, 1 \),
\[
\left\| \int_{u(j)}^u \frac{(u^{-1})'(u) - (u^{-1})'(c) - (u^{-1})''(c)(u - c)}{(u - c)^2}du \right\|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)]^2)} \leq \| (u^{-1})'' \|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)])}.
\]
We also have that for some \( C(\theta_0) \),
\[
\sup_{c \in [u(\Phi_s),u(1-\Phi_s)]} \left( \left| \frac{1}{u(j) - c} \right| + \left| \log |u(j) - c| \right| \right) \leq Cm!.
\]
Thus together with Corollary 7.6, Lemma 6.4, we obtain the Corollary. \( \square \)

Let us now turn to the proof of Proposition 2.5.

**Proof.** By the definition of the Gevrey class \( \mathcal{G}_{ph,1}^{M,s_0}([u(0),u(1)]^d) \) it is easy to check that for any \( f \in \mathcal{G}_{ph,1}^{M,s_0}([u(0),u(1)]^d) \), its trace will be in the Gevrey class \( \mathcal{G}_{ph,1}^{M,s_0}([u(0),u(1)]^{d-1}) \) of the same regularity.

By (7.18) and Lemma 7.1, we get that there exists \( D(t,k,\xi_1,\xi_2) \) such that
\[
\mathcal{F}_2(D_{u,k}(\tilde{\mathcal{F}}_2 f(t,k,\cdot))) (t,k,\xi_1) = \int D(t,k,\xi_1,\xi_2) \tilde{f}_k(t,\xi_2)d\xi_2.
\]
The behavior of \( D(t,k,\xi_1,\xi_2) \) depends only on the regularity of \( D_1, D_2 \) and \( E \). Indeed, by Corollary 7.6, Corollary 7.7, Lemma 6.4 and (7.8), we obtain that there exist \( M_u \) such that for all integers \( m \geq m_1 \geq 0 \) it holds that
\[
\left\| \left( W_1(k,c)^2 + \left( \pi \tilde{\rho}(c) \frac{u''(c)}{u'(c)^2} \right)^2 \right)^{-\frac{4}{3}} \right\|_{\mathcal{G}_{ph,1}^{M_u,s_0}([u(\Phi_s),u(1-\Phi_s)])} \leq \frac{C}{k, \theta_0, \| (u^{-1})'' \|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)])}},
\]
which gives us that
\[
\| D_1(k,\cdot) \tilde{\mathcal{F}}_2 \|_{\mathcal{G}_{ph,1}^{M_u,s_0}(R)} + \| D_2(k,\cdot) \tilde{\mathcal{F}}_2 \|_{\mathcal{G}_{ph,1}^{M_u,s_0}(R)} + \| \Phi_{\xi_1}^{\xi_2}(k,u,c) \tilde{\mathcal{F}}_2(u) \|_{\mathcal{G}_{ph,1}^{M_u,s_0}(R^2)} \leq \frac{C}{k, \theta_0, \| (u^{-1})'' \|_{\mathcal{G}_{ph,1}^{K_u,s_0}([u(0),u(1)])}}.
\]
Therefore by Lemma 7.1, Remark 6.2 and Remark 7.2, we obtain Proposition 2.5. \( \square \)
7.4. Commutator. In this section, let us study the difference between $D^1_{u,k}$ and $D_{u,k}$. We have

\[
\tilde{\chi}_2 D_{u,k}(F_1 \tilde{f}) - \tilde{\chi}_2 D^1_{u,k}(F_1 \tilde{f})
= \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} \frac{F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}}{u_1 - u} \left[(u^{-1})'(u_1)\tilde{u}''(u) - \tilde{u}''(u_1)\right]du_1
+ \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} (u^{-1})''(u)\Phi_1(k, u_1, u) \ln|u_1 - u| F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}
\times [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1
\]

\[
- \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}\Phi_{0,1}^r(k, u_1, u)(1_R - (u_1 - u))
\times [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1
\]

\[
+ \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}\Phi_{1,1}^r(k, u_1, u)
\times [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1.
\]

Let $D_{com}(t, k, \xi, \xi_1)$ be the Fourier kernel of the operator $\tilde{\chi}_2 D_{u,k}(\tilde{\chi}_2 F_1 \tilde{f}) - \tilde{\chi}_2 D^1_{u,k}(\tilde{\chi}_2 F_1 \tilde{f})$, which means that

\[
F_2\left(\tilde{\chi}_2 D_{u,k}(\tilde{\chi}_2 F_1 \tilde{f}) - \tilde{\chi}_2 D^1_{u,k}(\tilde{\chi}_2 F_1 \tilde{f})\right)(\xi) = \int D_{com}(t, k, \xi, \xi_1) f_k(t, \xi_1)d\xi.
\]

Then by the same argument as in the proof of Proposition 2.5, we get that there exists $\lambda_D$ such that

\[
|D_{com}(t, k, \xi, \xi_1)| \leq e^{-\lambda_D|\xi - \xi_1|^\alpha}.
\]

Let us also study the derivate $(\partial_u - itk)$ acting on $\tilde{\chi}_2 D_{u,k}(F_1 \tilde{f}) - \tilde{\chi}_2 D^1_{u,k}(F_1 \tilde{f})$:

\[
(\partial_u - itk)\left(\tilde{\chi}_2 D_{u,k}(F_1 \tilde{f}) - \tilde{\chi}_2 D^1_{u,k}(F_1 \tilde{f})\right)
= \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} (u^{-1})''(u)\Phi_1(k, u_1, u) \ln|u_1 - u| F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}
\times [\partial_u [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1
\]

\[
+ \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}\Phi_{0,1}^r(k, u_1, u)(1_R - (u_1 - u))
\times [\partial_u [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1
\]

\[
- \tilde{\chi}_2 D_2(k, u) \int_{u(0)}^{u(1)} F_1 \tilde{f}(t, k, u_1)e^{-i(u_1-u)tk}\Phi_{1,1}^r(k, u_1, u)(1_R - (u_1 - u))
\times [\partial_u [(u^{-1})'(u_1)\tilde{u}''(u) - (u^{-1})'(u)\tilde{u}''(u_1)]du_1.
\]
which together with the fact that
\[ (u^{-1})'(u_1)\tilde{\omega}''(u) - (u^{-1})'(u)\tilde{\omega}''(u) \in G_{\phi,1}(\|u(0), u(1)\|^2), \]
for some \( K_u > 0 \).

Therefore, there exists \( D^{com,1}(t, k, \xi, \xi_1) \) such that
\[ (i\xi -ikt)\mathcal{F}_2\left( \tilde{\chi}_2D_{u,k}(\tilde{\chi}_2\mathcal{F}_1\tilde{f}) - \tilde{\chi}_2\mathcal{D}_u^1(\tilde{\chi}_2\mathcal{F}_1\tilde{f}) \right)(\xi) = \int D^{com,1}(t, k, \xi, \xi_1)\Phi_k(t, \xi_1)d\xi_1. \]

Moreover, Remark 7.2 gives us that there exists \( \lambda_D^r \) such that
\[ |D^{com,1}(t, k, \xi, \xi_1)| \lesssim e^{-\lambda_D^r|k-\xi_1|^\alpha}, \]
which together with the fact that
\[ i(\xi - kt)D^{com}(t, k, \xi, \xi_1) = D^{com,1}(t, k, \xi, \xi_1), \]
gives us that
\[ |D^{com}(t, k, \xi, \xi_1)| \lesssim \frac{e^{-\lambda_D|k-\xi_1|^\alpha}}{1 + |\xi - kt|}. \]

We can repeat the above argument once more and get that
\begin{align*}
(i\xi -ikt)^2\mathcal{F}_2\left( \tilde{\chi}_2D_{u,k}(\tilde{\chi}_2\mathcal{F}_1\tilde{f}) - \tilde{\chi}_2\mathcal{D}_u^1(\tilde{\chi}_2\mathcal{F}_1\tilde{f}) \right)(\xi) &= \int D^{com,2}(t, k, \xi, \xi_1)\Phi_k(t, \xi_1)d\xi_1.
\end{align*}

Moreover there exists \( \lambda_D^r \) such that
\[ |D^{com,2}(t, k, \xi, \xi_1)| \lesssim e^{-\lambda_D|k-\xi_1|^\alpha}, \]
which together with the fact that
\[ -(\xi - kt)^2D^{com}(t, k, \xi, \xi_1) = D^{com,2}(t, k, \xi, \xi_1), \]
gives us that
\[ |D^{com}(t, k, \xi, \xi_1)| \lesssim \frac{e^{-\lambda_D|k-\xi_1|^\alpha}}{1 + |\xi - kt|^2}. \]
By the fact that

\[
\frac{1 + |\xi_1 - kt|^2}{1 + |\xi - kt|^2} \lesssim \frac{1 + |\xi - kt|^2 + |\xi - \xi_1|^2}{1 + |\xi - kt|^2} \lesssim (\xi - \xi_1)^2;
\]

we obtain Corollary 2.6 for some \( \lambda_0 < \lambda'_D \).

Finally, let us make a remark about the optimality of the 2 derivatives gain in the commutator. The commutator of a smooth function \( u \) and the Hilbert transform \( [u, H]f(x) = \frac{1}{2\pi} \int_a^b \frac{u(x) - u(y)}{x-y} f(y) dy \) is smooth as long as \( f \in L^1 \). However if there is a lower singularity in the integral for example \( T(f) = \int_a^b \ln |x - y| f(y) dy \), then the commutator \( [u, T]f = \int_a^b (u(x) - u(y)) \ln |x - y| f(y) dy \) is not as good. Indeed, after taking derivatives twice, we have

\[
\partial^2_x [u, T]f(x) = 2 \int_a^b \frac{u'(x)}{x-y} f(y) dy + \int_a^b u''(x) \ln |x - y| f(y) dy + \int_a^b \partial_x \left( \frac{u(x) - u(y)}{x-y} \right) f(y) dy.
\]

Thus the regularity of \( \partial^2_x [u, T]f \) will depend on the regularity of \( f \) because of the singular integral.

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