PROOF OF THE BMR CONJECTURE FOR $G_{20}$ AND $G_{21}$

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Abstract. We prove two new cases of the Broué-Malle-Rouquier freeness conjecture for the Hecke algebras associated to complex reflection groups. These two cases are the complex reflection groups of rank 2 called $G_{20}$ and $G_{21}$ in the Shephard and Todd classification. This reduces the number of remaining unproven cases to 3.

1. Introduction

Two decades ago, M. Broué, G. Malle and R. Rouquier conjectured in [3] that the generalized Hecke algebras that they attached to an arbitrary complex reflection group satisfy the crucial structural property of the ordinary (Iwahori-)Hecke algebras attached to a finite Coxeter group, namely that they are free modules of rank equal to the order of the group. This is known as the BMR freeness conjecture, and it can be easily reduced to the case where the complex reflection group $W$ is irreducible. We refer to [13] for a general exposition of this conjecture and standard results about it.

The Shephard-Todd classification of irreducible complex reflection groups defines an infinite family $G(de,e,n)$ of such groups, for which the conjecture was already known to hold by work of Ariki and Ariki-Koike (see [1, 2]), and a long list of exceptional groups. Subsequent works have proved it for most of the exceptional groups, notably all the ones of rank at least 3 (see [13, 11, 14]), and most of the ones of rank 2 (see [4, 5]). In rank 2, the 5 remaining ones are named, in Shephard-Todd notation, $G_{17}$, $G_{18}$, $G_{19}$, $G_{20}$ and $G_{21}$. In this work, we prove the cases of $G_{20}$ and $G_{21}$, by a method of a different nature than in the previous works. This reduces the list of remaining cases to the 3 groups $G_{17}$, $G_{18}$ and $G_{19}$, for which it appears difficult to apply readily the methods of this paper.

In section 2 we recall the main definitions, and prove a technical property that will allow us to work over rings of definitions which are polynomial rings, instead of the usual Laurent polynomial rings. In section 3 we explain the general method: how we find a potential basis for the Hecke algebras and how we find a list of rewriting rules. Then, sections 4 and 5 contain the rewriting rules we used in the cases of $G_{20}$ and $G_{21}$, respectively.

The GAP4 programs used for $G_{21}$ can be found on my webpage [http://www.lamfa.u-picardie.fr/marin].

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2. Definitions and preliminaries

Let $W$ be a finite complex (pseudo-)reflection group. We let $B$ denote the braid group of $W$, as defined in [3] §2 B, and recall that a (pseudo-)reflection $s$ is called distinguished if its only nontrivial eigenvalue is $\exp(2i\pi/o(s))$, where $i \in \mathbb{C}$ is the chosen square root of $-1$ and $o(s)$ denotes the order of $s \in W$. 

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We let \( R = \mathbb{Z}[a_{s,i}, a_{s,0}^{-1}] \) where \( s \) runs over the distinguished reflections in \( W \) and \( 0 \leq i \leq o(s) - 1 \), where \( o(s) \) is the order of \( s \) in \( W \), with the convention \( a_{s,i} = a_{s',i} \) if \( s, s' \) are conjugates in \( W \). For the standard notion of a braided reflection associated to \( s \) we refer to [3], where they are described as ‘generators-of-the-monodromy’ around the divisors of the orbit space. The definition of the Hecke algebra associated to \( W \) reads as follows.

**Definition 2.1.** The generic Hecke algebra is the quotient of the group algebra \( RB \) by the relations 
\[
x_{\sigma o(s)} - a_{s,0}^{-1}\sigma_{o(s)}^{-1} - \cdots - a_{s,0} = 0 \text{ for each braided reflection } \sigma \text{ associated to } s.
\]

Actually, it is enough to choose one such relation per conjugacy class of distinguished reflection, as all the corresponding braided reflections are conjugates in \( B \). Although we are not going to use this result in our proof, we mention that it was already known by work of Etingof and Rains (see [8]) that the Hecke algebras of the groups considered here are modules of finite type. Our main result can now be stated as follows.

**Theorem 2.2.** When \( W \) is a complex reflection group of Shephard-Todd type \( G_{20} \) or \( G_{21} \), then the generic Hecke algebra of \( W \) is a free \( R \)-module of rank \( |W| \).

Let \( R_0 = \mathbb{Z}[b_{s,i}, 1 \leq i \leq o(s)] \) where \( s \) runs over the distinguished reflections, with the convention \( b_{s,i} = b_{s',i} \) if \( s, s' \) are conjugates in \( W \), and define \( H_0 \) as the quotient of \( R_0 B \) by the relations
\[
\sigma_{o(s)} - b_{s,0}^{-1}\sigma_{o(s)}^{-1} - \cdots - b_{s,1} \sigma = 1 = 0
\]
for each braided reflection \( \sigma \) associated to \( s \). Again, it is enough to choose one such relation per conjugacy class of distinguished reflection. We let \( H \) denote the usual Hecke algebra, defined over \( R \).

The next proposition is useful in order to reduce the number of parameters involved in the computations.

**Proposition 2.3.**

(i) \( H_0 \) is spanned by \( |W| \) elements as a \( R_0 \)-module iff it is a free \( R_0 \)-module of rank \( |W| \).

(ii) \( H \) is a free \( R \)-module of rank \( |W| \) iff \( H_0 \) is a free \( R_0 \)-module of rank \( |W| \).

**Proof.** The proof of (i) is the same as the one of [13], proposition 2.4. We prove (ii). We have a ring morphism \( \phi_1 : R \rightarrow R_0 \) defined by \( a_{s,i} \mapsto b_{s,i} \) if \( i \geq 1 \), \( a_{s,0} \mapsto 1 \), for which \( H_0 = H \otimes_{\phi_1} R_0 \).

Therefore, if \( H \) is a free \( R \)-module of rank \( |W| \), we get the \( H_0 \simeq R_0^{|W|} \otimes_{\phi_1} R_0 \simeq R_0^{|W|} \) is also free of rank \( |W| \). We prove the converse. Assume that \( H_0 \) is \( R_0 \)-free of rank \( |W| \). Let \( A = \mathbb{Z}[x_s, x_{s}^{-1}] \) where \( s \) runs among the distinguished reflections of \( W \) with \( x_s = x_{s'} \) if \( s, s' \) are conjugates in \( W \). We have an injective ring morphism \( R \rightarrow A \otimes_{\mathbb{Z}} R_0 \) defined by \( a_{s,0} \mapsto x_{s,0}^{o(s)} = x_{s}^{o(s)} \otimes 1 \), and \( a_{s,i} \mapsto b_{s,i} x_{s}^{o(s)} = x_{s}^{o(s)} \otimes b_{s,i} \) for \( i \geq 1 \). We first note that \( A \otimes R_0 \) is a free \( R \)-module of finite rank, since it is easily checked that
\[
A \otimes R_0 = \bigoplus_{s \in S} \bigoplus_{0 \leq i < o(s)} x_s^i R
\]
where \( S \) is a system of representatives of the conjugacy classes of distinguished reflections.

We denote \( H_0 \) the quotient of the group algebra \((A \otimes_{\mathbb{Z}} R_0)B\) of \( B \) over \( A \otimes_{\mathbb{Z}} R_0 \) by the relations 
\[
(x_s \sigma)^{o(s)} - b_{s,0}^{-1}(x_s \sigma)^{-o(s)} - \cdots - b_{s,1}x_s^{o(s)} - x_s^{o(s)} = 0 \text{ for each braided reflection } \sigma \text{ associated to } s.
\]
We consider the composite map
\[
AB \xrightarrow{\Delta} (AB) \otimes_A (AB) \xrightarrow{\text{Id} \otimes_A \text{Id}} (AB) \otimes_A (AB) \xrightarrow{\text{Id} \otimes (s \mapsto x_s)} (AB) \otimes_A A \xrightarrow{\simeq} AB
\]
where $\Delta$ is the usual coproduct of the Hopf algebra $AB$, $Ab : B \rightarrow B^{ab}$ the abelianization morphism and, by abuse of notations, the associated linear map $AB \rightarrow AB^{ab}$, and `$s \mapsto x_s$’ denotes the map $B^{ab} \rightarrow A$ defined as follows. It is known (see e.g. [3]) that $B^{ab}$ is a free $\mathbb{Z}$-module admitting a natural basis indexed by the conjugacy classes of distinguished reflections. The map is defined by mapping the basis element associated to (a conjugacy class of) distinguished reflection $s$ to the scalar $x_s \in A$.

The composite map is easily checked to be an $A$-algebra isomorphism. Its natural extension $(A \otimes R_0)B \rightarrow (A \otimes R_0)B$ induces an isomorphism $\hat{H}_0 = H \otimes_R (A \otimes \mathbb{Z} R_0)$.

Now, if $H_0$ is $R_0$-free of rank $|W|$, then $\hat{H}_0 = H_0 \otimes R_0 A$ is $A \otimes R_0$-free of rank $|W|$. Since $A \otimes R_0$ is a free $R$-module of finite rank, this implies that $\hat{H}_0$ is a free $R$-module of finite rank, and also that, since $\hat{H}_0 = H \otimes_R (A \otimes \mathbb{Z} R_0)$, that the $R$-module $H$ is a direct factor of $\hat{H}_0$. Therefore $H$ is projective as a $R$-module and this implies that $H$ is free of rank $|W|$ by [13], proposition 2.5.

The groups we have interested in are the ones denoted $G_{20}$ and $G_{21}$ in the Shephard-Todd notation. They admit presentations symbolized by the following diagrams

$$\begin{array}{ccc}
3 & \otimes & 5 & \otimes & 3 & \otimes & 2 & \otimes & 10 & \otimes & 3
\end{array}$$

that is $G_{20} = \langle s_1, s_2 | s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2, s_1^3 = s_2^3 = 1 \rangle$ and $G_{21} = \langle s_1, s_2 | (s_1 s_2)^5 = (s_2 s_1)^5, s_1^2 = s_2^2 = 1 \rangle$. In these presentations, $s_1, s_2$ are distinguished reflections, and every distinguished reflection is a conjugate of one of them. Moreover, $s_1$ and $s_2$ are conjugates in $G_{20}$, as is readily deduced from the presentation itself. The corresponding braid groups admit the same presentations, with the order relations removed.

We use the above proposition to define the Hecke algebras of $G_{20}$ and $G_{21}$ over $R_0$, where $R_0 = \mathbb{Z}[a, b]$ for $G_{20}$ and $R_0 = \mathbb{Z}[a, b, q]$ for $G_{21}$, with relations

$$
\begin{array}{c|c}
G_{20} : & s_1^3 = as_1^2 + bs_1 + 1 \\
& s_2^3 = as_2^2 + bs_2 + 1 \\
G_{21} : & s_1^5 = qs_1 + 1 \\
& s_2^5 = as_2^4 + bs_2 + 1
\end{array}
$$

In the subsequent section we prove these Hecke algebras are spanned by the ‘right’ number of elements, and this proves theorem [2.2] by proposition [2.3].

3. General method

In this section we describe the general method we used to prove the conjecture in these cases. It proceeds in several steps.

(i) Heuristics/Experimentation

(ii) Incremental determination of computational rules

(iii) Right multiplication table

3.1. Heuristics/Experimentation. The first crucial element is of heuristic nature, provided by a software able to compute non-commutative Gröbner basis for finitely presented associative $\mathbb{Q}$-algebras. We used the GAP4 package GBNP (see [7]) with the standard (‘deglex’) ordering for monomials, taking as input the presentations of [3], where we specialized the Hecke algebras at more or less random parameters. For $G_{20}$ and $G_{21}$ it finished in reasonable time for all the specializations we tried, while for $G_{18}$ and $G_{19}$ it was not able to complete the computation after several months of running time, except for the simple case of the group algebra specialization, that is the presentation of $W$ viewed as a presentation of
the Hecke algebra at very special parameters. For all the groups of the so-called icosahedral series of complex reflection groups of rank 2, GBNP nevertheless finds a Gröbner basis of the rational group algebra of $W$.

It turns out that most if not all the specializations we tried for $G_{20}$ and $G_{21}$ (including the group algebra specialization) provided the same number of elements for the Gröbner basis. As an indication of the complexity of this heuristic data, we provide the following table, where $\#W$ is the order of $W$ and $\#gb$ is the number of elements in the Gröbner basis. The groups whose name appears in bold fonts are the ones for which the BMR freeness conjecture is now proved, after work of Chavli for $G_{16}$ (see [4, 5]), of Marin-Pfeiffer for $G_{22}$ (see [14]), and by the present work for $G_{20}$ and $G_{21}$.

The output of GBNP we are interested in is the collection $\mathcal{G}$ of leading monomials of the Gröbner basis. In case we had computed the Gröbner basis for several specializations this collection turned out to be independent of the specialization. From this one computes easily the set $\mathcal{B}$ of all words avoiding the patterns which belong to $\mathcal{G}$. As expected, it has cardinality $|W|$ and provides for these specializations a basis of the Hecke algebra.

| $W$  | $\#W$ | $\#gb$ | $W$  | $\#W$ | $\#gb$ |
|------|-------|--------|------|-------|--------|
| $G_{16}$  | 600   | 44     | $G_{20}$ | 360   | 36     |
| $G_{17}$  | 1200  | 49     | $G_{21}$ | 720   | 30     |
| $G_{18}$  | 1800  | 138    | $G_{22}$ | 240   | 66     |
| $G_{19}$  | 3600  | 558    |       |       |        |

3.2. Incremental determination of computational rules. It so happens that all defining relations are included in the Gröbner bases provided by GBNP. We view these as the first step in the construction of an ordered list $\mathcal{L}$ of rewriting rules of the form $w \rightsquigarrow c_w$, where $w \in \mathcal{G}$ and $c_w$ is a $R_0$-linear combination of elements of $\mathcal{B}$, with the property that the equality $w = c_w$ holds inside the Hecke algebra $H_0$. More precisely, the defining relations of the braid groups of the form $b_1 = b_2$ are included under the form $b_1 \rightsquigarrow b_2$ for $b_1 > b_2$. One checks that $b_1 \in \mathcal{G}$ and $b_2 \in \mathcal{B}$ in all cases. The order relations, of the form $\sigma^m = b_{s,m-1}\sigma^{m-1} + \cdots + b_{s,1}\sigma + 1$, are also included under the form $\sigma^m \rightsquigarrow b_{s,m-1}\sigma^{m-1} + \cdots + b_{s,1}\sigma + 1$. We denote $\mathcal{L}_0$ the ordered list of leading terms $w \in \mathcal{G}$ of the rules in $\mathcal{L}$.

The incremental process aims at enlarging $\mathcal{L}$ so that it contains at the end as many elements as $\mathcal{G}$, with the set of elements inside $\mathcal{L}_0$ being equal to $\mathcal{G}$.

The way we enlarge $\mathcal{L}$ is as follows. We use an algorithm for computing a given word as a $R_0$-linear combination of words as follows.

- **Input**: a word $w$ in the generators and their inverses.
- **If $w$ contains the inverse of a generator**, replace $w$ by a linear combination of positive words, by applying the rewriting rules $\sigma^{-1} \rightsquigarrow \sigma^{m-1} - b_{s,m-2}\sigma^{m-1} - \cdots - b_{s,1}$ as many times as needed, and apply the present algorithm to these words.
- **If $w \in \mathcal{B}$**, then return $w$.
- **If not**, then look for the first element in $\mathcal{L}_0$ which appear as a subword in $w$. If there is none, return **fail**. If there is one $v$, with $w = avb$, then replace it with the linear combination $acvb$, where $v \rightsquigarrow c_v$ belongs to $\mathcal{L}$, and apply the algorithm to each monomial of this linear combination.

It is clear that, if the present algorithm terminates for a given word $w$, producing a $R_0$-linear combination $b_w$, then the equality $w = b_w$ holds inside $H_0$. Adding more elements in
\( \mathcal{L} \) will not change the result if the input is one for which the algorithm already terminated, but instead potentially increases the number of words for which it does provide a result.

Our strategy is then to establish a number of equalities inside \( H_0 \) of the form \( w_i = b_{w_i} \), where \( w_i \in \mathcal{G} \) and \( b_{w_i} \) is a linear combination of words with possibly negative powers, such that \( \mathcal{L} \) originally contains the first \( w_1, \ldots, w_{n_0} \) originating from the defining relations, and so that we can build incrementally \( \mathcal{L} \) as follows.

- If \( \mathcal{L} = (w_1 \sim c_{w_1}, \ldots, w_n \sim c_{w_n}) \), then apply the algorithm with \( \mathcal{L} \) to \( b_{w_{n+1}} \). It produces a linear combination \( c_{w_{n+1}} \). Add to \( \mathcal{L} \) the rule \( w_{n+1} \sim c_{w_{n+1}} \).
- Start again with the new \( \mathcal{L} \).

For the first group \((G_{20})\) we are interested in, we managed to produce a convenient list of rewriting rules \( w_i \sim b_{w_i} \), completely by hand (see section 4). For the group \( G_{21} \) the making of this list had to be partly automatized, too (see section 5).

### 3.3 Right multiplication table.
Completing the (right)multiplication table is then merely a way to check that \( H_0 \) is indeed spanned by the elements of \( \mathcal{B} \). It is sufficient to calculate, using the algorithm described in the previous subsection, each word \( \bar{w} \) where \( w \in \mathcal{B} \) and \( s \) a generator, as a \( R_0 \)-linear combination of the words in \( \mathcal{B} \).

### 4. Rules for \( G_{20} \)
We first provide the list of rewriting rules, and subsequently justify it.

|   |   |   |
|---|---|---|
| (1) | 111 | \( \sim a.11 + b.1 + \emptyset \) |
| (2) | 222 | \( \sim a.22 + b.2 + \emptyset \) |
| (3) | 21212 | \( \sim 12121 \) |
| (4) | 211212 | \( \sim 1212112 \) |
| (5) | 2121122 | \( \sim 1122121 + (a).212112 + (-a).122121 + (-b).22121 + (b).21211 \) |
| (6) | 22122121 | \( \sim 12122122 + (a).2212212 + (-a).1212212 + (b).1221212 + (-b).121221 \) |
| (7) | 2211212 | \( \sim 1212211 + (a).211212 + (-a).121212 + (b).122112 + (b).1211212 \) |
| (8) | 21211211 | \( \sim 11211212 + (a).212112 + (-a).11211212 + (b).212112 + (-b).212112 \) |
| (9) | 21211222 | \( \sim 11221211 + (a).21211212 + (-a).12122112 + (-b).2122121 + (b).2112121 \) |
| (10) | 22121122 | \( \sim 12122112 + (a).2112112 + (-a).12122112 + (b).1222112 + (b).121212 \) |
| (11) | 21221122 | \( \sim a.2112122 + b.211122 + a.212122 + b.2122 + a.21212 + b.2121 + \) |
| (12) | 21122112 | \( + a.2112 + b.21 + \emptyset + 2121 \) |
| (13) | 211212212 | \( \sim a.221211212 + b.221121212 + 221121212 \) |
| (14) | 21221211 | \( \sim a.2122121 + b.212212 + a.212121 + b.2121 + 2121212 \) |
| (15) | 211221122 | \( \sim a.2112112 + b.2121212 + a.211212 + b.21212 + a.21212 + b.2121 + \) |
| (16) | 212211212 | \( + 21212 + a.21212 + 21212 + 21212 \) |
| (17) | 2112112211 | \( \sim (a).2112211 + (b).1221212 + (a).212112 + b.212 + (a).21212 + (b).212 + 2121 + \) |
| (18) | 211211212 | \( \sim (a).2112112 + (b).1212112 + (a).2112112 + (b).2112 + (a).2112 + (b).212 + 21212 + \) |
| (19) | 21221112122 | \( \sim (a).2122112 + (b).2122112 + (a).2122112 + (b).2122112 + (b).2212112 + (a).2122112 + \) |
| (20) | 21122112121 | \( \sim (a).2212112 + (b).2212112 + (a).2121212 + (b).21121 + 221212 \) |
We now justify each one of the above rules. Rule (3) is a direct consequence of the braid relation, and (4) follows from \(2112121 = 21212 = 1212112\).

We have

\[
212112 = \bar{1}1212112 \\
= \bar{1}21211212 \\
= a.12212212 + b.\bar{1}2212212 + \bar{1}221212 \\
= a.\bar{1}2212212 + b.12212212 + 1(111 - a.11 - b.1)22121 \\
= a.\bar{1}2212212 + b.12212212 + (11 - a.1 - b.0)22121 \\
= a.\bar{1}2212112 + b.\bar{1}2121212 + 122121 - a.122121 - b.22121 \\
= a.212112 + b.21211 + 122121 - a.122121 - b.22121
\]

whence (5).

We have \(22122121 = 221221212 = 22121212 = 2121121212\) hence

\[
22122121 = 1212(a.11 + b.1 + 0)212 \\
= a.12121212 + b.12121212 + 121221 \\
= a.12121212 + b.1221212 + 121221 \\
= a.21221212 + b.122121 + 121221(22 - a.2 - b\emptyset) \\
= a.2122121 + b.122121 + 1212212 - a.122121 - b.121221
\]

whence (6).
We have $2211212 = 2211221212 = 22211222$ hence
\begin{align*}
2211212 &= (a.22 + b.2 + 0)12212212 \\
&= a.2221212212 + b.2221212212 + 12212212 \\
&= a.2121212212 + b.2121212212 + 1221221212 - a.121221 - b.12122 \\
&= a.21121212 + b.1121212212 + 1221221212 - a.121221 - b.12122 \\
&= a.211212 + b.1121212212 - a.121221 - b.12122 \\
\end{align*}
whence (7).

We have $21211211 = 1212211211 = 1212211211 = 121212111$ hence
\begin{align*}
21211211 &= 12112112(a.11 + b.1 + 0) \\
&= a.12112112 + b.12112112 + (11 - a.1 - b)212112 \\
&= a.12121212 + b.12121212 + 12122121 - a.121212 + b.21212 \\
&= a.1121212212 + b.1121212212 + 12121221 - a.12121212 - b.21212 \\
&= a.211212 + b.1121212212 + 12121221 - a.12121212 - b.21212 \\
\end{align*}
and this proves (8). Similarly, we have $212112122 = 122112122 = 1212121122 = 1212212122$ hence
\begin{align*}
212112122 &= 12122121(a.22 + b.2 + 0) \\
&= a.2221212212 + b.2122121212 + 1221221212 \\
&= a.2121212212 + b.2121212212 + 1221221212 + (11 - a.1 - b)21212212 \\
&= a.2121212212 + b.1212121212 + 12122121 - a.12121212 - b.21212212 \\
&= a.21121212 + b.1121212212 + 12121221 - a.12121212 - b.21212212 \\
\end{align*}
and this proves (9). Finally we have $222112112 = 22211212122 = 22211212122 = 2221212212$ hence
\begin{align*}
222112112 &= (a.22 + b.2 + 0)1221221212 \\
&= a.2221221212 + b.2221221212 + 1222121212 \\
&= a.2221221212 + b.2221221212 + 1222121212 + (11 - a.1 - b)22212212 \\
&= a.2221221212 + b.1221221212 + 1222121212 - a.121212 + b.22212212 \\
&= a.2221221212 + b.1221221212 + 1222121212 - a.12121212 - b.22212212 \\
\end{align*}
and this proves (10).

We have $21221122 = a.2121122 + b.211122 + 2121122$ and $2121122 = a.21222 + b.21222 + 212122 = a.212122 + b.2122 + 2122$. Then, $212122 = a.21212 + b.21212 + 21212$. Since $21212 = 121212$ this proves (11).

We have $22121212 = a.2112212 + b.112212 + 2112212$, then $2112212 = a.21212 + b.221212 + 212212$ and $212212 = a.21212 + b.2122 + 2122$. Finally, $21212 = 121212$ and this proves (12).

We have $21121212122 = 21122122122 = 212212212122 = 212212212122 = a.221221212122 + 2221212122$ and this proves (13).

We have $221212112 = a.2221212212 + b.2221212212 + 2221212212$, and $221212122 = a.2122212 + b.212212 + 212212$. Now, $2221212 = 2212122 = 21221212$ and this proves (14).

We expand $2(11)(22)(11)(22)$ by using four times the relation $x^2 = a.x + b + x^{-1}$ for $x \in \{1, 2\}$ at the four places between parenthesis we get $212212122 = a.212212122 + b.212212122 + a.2112212 + b.21122 + a.2112212 + b.212 + 2122. Now 12121 = 21212 and this proves (15).

Rule #16 is similar to rule # 15 : we expand (22)(11)(22)(11)2 and use 121212 = 21212.
Rule #17 is similar to rules # 15 and # 16 : expand 2112(11)(22)(11) and use 211212 = 21121222 = 21121212 = 2121212.
By expanding (22)(11)21121 we get 22112112121 = (a).211211121 + (b).1121121 + (a).2121121 + (b).1121 + 2121121. Since 2121121 = 121212121 = 12121211 this proves (18).

By expanding 21211211(11)(22) we get 21211212122 = (a).2121112212 + (b).212111222 + (a).21211212 + (b).21212121 + 21221121 and 21211212 = 212112121 = 21211212121 which proves (19).

By expanding 2(11)(22)121121 we get 2112112121 = (a).212112121 + (b).22121212 + (a).212121121 + (b).2211212121 and 212112121 = 22121212 = 22121212 which proves (20).

By expanding 2112(11)(22)12 we get 21121121221 = (a).211212212 + (b).221122212 + (a).221121212 + (b).2211221212 and 211211212 = 211211211 = 21121121213 which proves (21).

We have 211211212122 = 2112112121221 = 211211212122121 = 211211212122121 and this proves (22). We have 211211212212 = 2112112121221 and this proves (23). We have 21121212121212121 = 211212121212121 and this proves (24). We expand 2112112122(12)11 and get 21121121212111 + (a).211212121212 + (b).21211121212 + 211211212121 and 21121212121 = 2112112121212 = 211211212121212 and this proves (25). We have 211212121212122 = 2112121212121212 and this proves (26).

By expanding 2(11)(22)(11)21212 we get 211211212122121 = (a).21121121121212 + (b).22121212212 + (a).21211212212 + (b).22112121212 and 211211212121212 = 21121212121212122 = 2112112121212112 = 21121121212121212 and this proves (27). By expanding 2(11)(22)(12)21212 we get 2112121212122 = (a).21211212122 + (b).221212112212 + (a).21121212122 + (b).22112121212 and 2112121212121 = 2112121212121 and this proves (28). We have 2112121212121212 = 211212121212121212 = 211212121212121212 and this proves (29). We have 21121212121212122 = 211212121212121212 and this proves (30). We have 2112121212121212121 = 211212121212121212121212121 and this proves (31). By expanding 21121212112(11)(22) we get 2112112111212 = a.21121212122 + b.21121121122 + a.21121112121 + b.21121121121, and 211211211212 = 2112112112111 = 2112112112121, which proves (32).

We have 211211212212 = 211212121212121222 = 211212121212121222 which proves (33). We have 21121212121212 = 211212121212121222 = 211212121212121222 which proves (34). We have 211212121212212 = 211212121212121221 = 211212121212121221 which proves (35). We have 21121212121212 = 2112121212121212 which proves (36).

5. Rules for $G_{21}$

5.1. Semi-manual procedures. Let $Y$ be the alphabet $\{1, 2, \bar{1}, \bar{2}\}$, $M(Y)$ the free monoid over $Y$, and $F(Y) \subset M(Y)$ the subset of freely reduced words, that is the set of natural representatives of the free group on $\{1, 2\}$ viewed as a quotient of $M(Y)$. We denote $M^+(Y) = M(\{1, 2\}) \subset M(Y)$ the submonoid of positive words. We let red : $M(Y) \rightarrow F(Y)$ denote the usual reduction procedure, and red : $RM(Y) \rightarrow RF(Y)$ its natural linear extension, where we let $RM(Y)$ the monoid algebra over $R$ and $RF(Y)$ the (free) submodule spanned by $F(Y)$. We define pos : $RM(Y) \rightarrow RM(Y)$ and call positivation the (unique) algebra morphism mapping $1 \mapsto 1$, $2 \mapsto 2$, $\bar{1} \mapsto 22 - a.2 - b.\bar{0}$, $\bar{2} \mapsto 1 - a.\bar{0}$.

A more complicated procedure is what we call expansion. By convention we let $\bar{y} = y$ for all $y \in \{1, 2\}$. For $I \subset \mathbb{N}^*$, let us define the $I$-inversion map inv$_I : M(Y) \rightarrow M(Y)$ as follows.
If \( y = y_1 y_2 y_3 \ldots y_n \in M(Y) \) is a word in \( n \) letters, with \( y_k \in Y \), \( \text{inv}_I(y) = y' = y'_1 y'_2 y'_3 \ldots y'_{n-1} \in M(Y) \) is defined by \( y'_k = \bar{y}_k \) if \( k \in I \), \( y'_k = y_k \) if \( k \notin I \). We now define the partially defined \textit{expansion} map \( \exp_I : M(Y) \rightarrow M(Y) \) with respect to \( I \) by induction on the cardinality of \( I \). If \( I = \emptyset \), then \( \exp_\emptyset \) is the identity map. If not, let \( i_0 = \min(I) \), and let \( J \) such that \( I = J \cup \{i_0\} \). If \( y = y_1 y_2 y_3 \ldots y_n \in M(Y) \) is a word in \( n \) letters, with \( y_k \in Y \), then \( \exp_I(y) \) is defined if \( \exp_J(y) \) is defined, \( n \geq i_0 \) and if

- either \( y_{i_0} = 1 \), in which case \( \exp_I(y) = a.y' + z \) with \( y' = y'_1 y'_2 \ldots y'_{n-1} \) where \( y'_k = y_k \) for \( k < i_0 \), \( y'_k = y_{k+1} \) for \( k \geq i_0 \), and \( z = \exp_I(\text{inv}_{\{i_0\}}(y)) \)
- either \( y_{i_0} = y_{i_0} + 1 = 2 \), in which case \( \exp_I(y) = a.y' + b.y'' + z \) with
  - \( y'' = y''_1 y''_2 \ldots y''_{n-2} \) where \( y''_k = y_k \) for \( k < i_0 \), \( y''_k = y_{k+2} \) for \( k \geq i_0 \)
  - \( y' = y'_1 y'_2 \ldots y'_{n-1} \) where \( y'_k = y_k \) for \( k < i_0 \), \( y'_{i_0} = y_{i_0} = 2 \), \( y'_{k} = y_{k+1} \) for \( k \geq i_0 + 1 \).
  - \( z = \exp_J(\text{inv}_{\{i_0\}}(y'')) \).

It is easily checked that, when defined, \( \exp_I(y) = \text{tail}_I(y) + \text{head}_I(y) \) with \( \text{head}_I(y) \in M(Y) \) being characterized, with the above notations, by \( \text{head}_{\{i_0\} \cup J}(y) = \text{head}_J(z) \), and \( \text{head}_g(y) = y \).

5.2. Rules. We can now give the set of rules for \( G_{21} \), the justification that they correspond to genuine relations inside its Hecke algebra basically relying on the above sections.

(1) 11 \( \rightsquigarrow \) (q).1 + \emptyset
(2) 222 \( \rightsquigarrow \) (a).22 + (b).2 + \emptyset
(3) 2121212121 \( \rightsquigarrow \) 1212121212
(4) 212121212121 \( \rightsquigarrow \) red(tail11(11 * w) + 121212121211)
(5) 221221212121 \( \rightsquigarrow \) red(tail1.3.4(w) + 1212121212)
(6) 21212121212121 \( \rightsquigarrow \) red(tail1.5.6.8(w * 22) + 22121212121212)

(11) 21212121212121 \( \rightsquigarrow \) red(tail1.13(1221 * w) + 12121212121212)

\[ w' = 12121212121212 \]

\[ w'' = 12121212121212 \]

(12) 21212121212121 \( \rightsquigarrow \) red(tail1.3.4.6(11 * w) + 1212121212)
(13) 21212121212121 \( \rightsquigarrow \) red(tail1.3.4.6.7.9.10.12(w) + 1212121212)
(14) 21212121212121 \( \rightsquigarrow \) red(tail4.1.6(w * 2112) + 212121212121212)
(15) 21212121212121 \( \rightsquigarrow \) red(tail8.9(w * 21212) + 2212121212121212)
(16) 21212121212121 \( \rightsquigarrow \) red(tail7.8(w * 22) + 2212121212121212)
(17) 21212121212121 \( \rightsquigarrow \) red(tail3.5.6.8.9(11 * w) + 12121212121212)
(18) 21212121212121 \( \rightsquigarrow \) red(tail3.5.6.9(11 * w) + 12121212121212)
(19) 21212121212121 \( \rightsquigarrow \) red(tail1.2.3.15.16.18(w * 2112) + 1212121212121212)
(20) 21212121212121 \( \rightsquigarrow \) red(tail8.9.11.12(w * 22) + 2212121212121212)
Table 1. Dominant terms of the Gröbner basis for $G_{20}$

| Num. | Word |
|------|------|
| 1    | 111  |
| 2    | 222  |
| 3    | 21212|
| 4    | 211212|
| 5    | 2121122|
| 6    | 22122121|
| 7    | 2211212|
| 8    | 21211211|
| 9    | 21211222|
| 10   | 221211212|
| 11   | 2211211212|
| 12   | 2212122|
| 13   | 2112122121|
| 14   | 21212212|
| 15   | 211221122|
| 16   | 21122112|
| 17   | 2112112|
| 18   | 22112112|
| 19   | 21221121|
| 20   | 21212121|
| 21   | 21221212|
| 22   | 211212|
| 23   | 211212212|
| 24   | 22121212|
| 25   | 2121212111|
| 26   | 212121222|
| 27   | 2112211221|
| 28   | 2122122121|
| 29   | 2121212212|
| 30   | 2121212121|
| 31   | 2121212121|
| 32   | 2112112122|
| 33   | 2121212121|
| 34   | 2121212112|
| 35   | 2121212121|
| 36   | 2121212121|

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| Φ | 112122 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
|---|--------|--------|----------|----------|----------|-------------|----------|----------|----------|----------|
| 1 | 112211 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 2 | 112212 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 11 | 112121 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 12 | 112122 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 21 | 112121 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 112 | 112121 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 122 | 112121 | 212212 | 2122121 | 21221212 | 21122122 | 211221122 | 212212122 | 212212122 | 211221122 | 211221122 |
| 21 | 211211 | 2112121 | 21121212 | 211212121 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 |
| 212 | 211211 | 2112121 | 21121212 | 211212121 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 |
| 22 | 211211 | 2112121 | 21121212 | 211212121 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 | 211212122 |

Table 2. Basis for $G_{20}$

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| Num. | Word       | Num. | Word       |
|------|------------|------|------------|
| 1    | 11         | 16   | 2212121221212121 |
| 2    | 222        | 17   | 2212121221212121 |
| 3    | 2121212121 | 18   | 2212121221212121 |
| 4    | 21212121221 | 19   | 212212121221212121 |
| 5    | 22121212121 | 20   | 2212121221212121 |
| 6    | 21212121221 | 21   | 2212121212212121 |
| 7    | 212212122121 | 22   | 2212121221212121 |
| 8    | 221212121212121 | 23   | 2212121221212121 |
| 9    | 212212121212121 | 24   | 2212121221212121 |
| 10   | 221212121212121 | 25   | 2212121221212121 |
| 11   | 212121212212121 | 26   | 2121212121212121 |
| 12   | 21221212212121 | 27   | 2212121221212121 |
| 13   | 221212121212121 | 28   | 2212212121212121 |
| 14   | 21212121212121 | 29   | 2212121221212121 |
| 15   | 221221221221221 | 30   | 2212121221212121 |

Table 3. Dominant terms of the Gröbner basis for $G_{21}$

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