Perfect 2-colorings of Hamming graphs

E. A. Bespalov, D. S. Krotov, A. A. Matiushev, A. A. Taranenko, K. V. Vorob'ev

Abstract

We consider the problem of existence of perfect 2-colorings of Hamming graphs with given parameters. We start with conditions on parameters of graphs and colorings that are necessary for their existence. Next we observe constructions of perfect colorings, including some new constructions giving new parameters of colorings. At last, we deduce which parameters of colorings are covered by these constructions and give tables of admissible parameters of 2-colorings in Hamming graphs $H(n,q)$ for small $n$ and $q$.

1 Introduction

An equitable $k$-partition (equitable partition, regular partition, or partition design) of a graph $G = (V,E)$ (in general, a multigraph, i.e., loops and multiedges are allowed) is a partition of the vertex set $V$ into $k$ nonempty cells $V_0, \ldots, V_{k-1}$ such that every cell induces a regular subgraph and the bipartite graph formed by the edges between any two different cells is biregular. Another equivalent name is perfect $k$-coloring, which is formally referred to a $k$-valued functions on $V$ that defines an equitable partition: each cell of the partition is the preimage of some value. Many classes of combinatorial configurations can be defined as perfect colorings with specific parameters (the parameters of the perfect colorings are the degrees of the corresponding regular and biregular subgraphs). Examples of such configurations in distance-regular graphs are $1$-perfect codes, $t$-$(v,t+1,\lambda)$-designs, latin squares and latin hypercubes, MDS codes with distance 2 and 3, transversals in latin squares. Closely related to perfect colorings is the concept of completely regular code, which is defined to be a set $C$ of vertices such that the coloring of the vertices $v$ of the graph by the distance from $v$ to $C$ is a perfect coloring (the original definition of Delsarte [1] is different, but equivalent to the definition given here for distance-regular graphs, see [23]). In some cases, optimal, by mean of some bound, objects are proven to be in one-to-one correspondence with perfect colorings with special parameters. For example, unbalanced boolean function attending the correlation-immunity bound [4]; some binary 1-codes attending the linear-programming bound, which are not completely regular, however induce perfect 4-, 5-, or 6-colorings of the hypercube [19], [21]; orthogonal arrays attending the Bierbrauer–Friedman bound [3, 10] induce perfect 2-colorings of the Hamming graph [29], [30]; binary orthogonal arrays attending the Bierbrauer–Gopalakrishnan–Stinson bound [1] induce perfect 3-colorings of the hypercube [23]. A very nice result of Potapov [30] shows a one-to-one correspondence between the perfect 2-colorings of the Hamming graph $H(n,q)$ and the boolean-valued functions on $H(n,q)$ attending a bound that connects the correlation immunity of the function, the density of ones, and the average 0-1-contact number (the number of neighbors with function value 1 for a given vertex with value 0).

In the current paper, we study perfect 2-colorings of the $q$-ary Hamming graph $H(n,q)$, mainly focusing on the nonbinary case $q > 2$ and on constructions that give new parameters of perfect 2-colorings. Parameters of 2-colorings of the binary Hamming graph $H(n,2)$, the hypercube, were studied in [8, 17, 23]. In [8], general constructions of perfect 2-colorings of the hypercube were described. In [7], a bound on the correlation immunity of boolean functions was proved, giving a powerful non-feasibility test for parameters of perfect 2-colorings of the hypercube. In [9], two special parameter sets of perfect 2-colorings of $H(n,2)$ were considered; in one case, perfect colorings were constructed, in the other case, the nonexistence was proved. In [23], among other results, the non-feasibility of an infinite series of parameters of 2-colorings in $H(n,2)$ was proved. In this paper, we consider generalizations of the constructions from [7] to the non-binary case $q > 2$ and some new constructions for non-prime $q$.

The Hamming graphs belong to the class of distance-regular graphs, and we note that perfect colorings of distance-regular graphs attract many attention not only because of connections with many classes of optimal objects, but also because of their algebraic-combinatorial properties, which can be derived from the algebraic-combinatorial properties of the distance-regular graph. Some of these properties establish connections between different kinds of distance distributions of perfect colorings of distance-regular graphs, and especially Hamming graphs [20], [33], [17], [22], [34]. Besides the Hamming graphs, distance-regular graphs where perfect colorings have been studied include Johnson graphs, see e.g. [1, 11], Latin-square graphs [2], halved hypercubes [21], Grassmann graphs $J_q(n,2)$, see e.g. [27]. As an example of the study of perfect colorings of non-distance-regular graphs yielding interesting and deep results we refer [31].
We emphasize that all the results of our paper that are applicable to the case \( q = 2 \) were proved for this case by Fon-Der-Flaass [5]. Moreover, some binary results (bounds in [3] and [4], possibility to partition one color into lines in the main construction of [5], special cases in [9]) were not generalized or generalized partially. On the other hand, new results for composite \( q \) have no analogs in the binary case.

The perfect 2-colorings are equivalent to a special case of completely regular codes, namely, to the completely regular codes of covering radius 1. In its turn, the completely regular codes with arbitrary covering radius \( \rho \) are equivalent to a special case of perfect \( (\rho + 1) \)-colorings, namely, to the perfect colorings with tridiagonal quotient matrix. In a recent survey [5] on completely regular codes, Borges, Rifà, and Zinoviev mentioned perfect 2-colorings saying that very little is known for \( q > 2 \). Since there are more questions than answers in this area, our current work can be considered as a step forward from “very little” to “little”.

Let us describe the structure of the paper. In Section 2 we introduce the main concepts related to perfect colorings in Hamming graphs and provide some easy observations.

In Section 3 we observe conditions that are necessary for the existence of perfect colorings with given parameters. Since Hamming graphs are distance-regular, we can specialize for them general results on weight distributions of colors in distance-regular graphs. We also consider some other necessary conditions based on algebraic and arithmetic properties of colorings and distributions of colors in faces of the Hamming graph.

Section 4 is devoted to constructions of perfect colorings. Firstly, we give a series of constructions based on coverings of Hamming graphs. Next, we consider two direct constructions of perfect colorings based on MDS codes and 1-perfect codes. Then, in Subsection 4.3 we describe several more complex constructions of perfect colorings combining some of the previous ones and providing new admissible parameters of perfect colorings.

We summarize results on the admissibility of parameters of perfect colorings in Section 5. In particular, for cases when \( q \) is a prime power we find sufficient conditions for the existence of \((b, c)\)-coloring in \( H(n, q)\) for \( n \) greater or equal some \( n_0 \) and estimate the minimum value \( n_0 \) for which such colorings exist.

Finally, in Appendix we provide tables with admissibility statuses for parameters of colorings in \( q \)-ary \( n \)-dimensional Hamming graphs for small \( n \) and \( q = 2, 3, 4, \) and 6.

## 2 Notions, definitions, and easy observations

Given \( n \) and \( q \), the Hamming graph \( H(n, q) \) is the graph with the vertex set \( \mathbb{Z}_q^n = \{x = (x_1, \ldots, x_n) | x_i \in \mathbb{Z}_q \} \) such that vertices \( x \) and \( y \) are adjacent if and only if the Hamming distance (the number of different positions) between them is 1. Observe that \( H(n, q) \) is a connected \( (nq - 1) \)-regular graph. For any vertex \( x \), its (Hamming) weight \( \text{wt} \) is defined to be the Hamming distance from \( x \) to the all-zero vertex \( 0^n \). The binary Hamming graph \( H(n, 2) \) is also known as a hypercube, or the \( n \)-cube.

For \( k \in \{0, \ldots, n\} \), a \( k \)-dimensional face, or simply a \( k \)-face, is a set of vertices of \( H(n, q) \) inducing a graph isomorphic to \( H(k, q) \). Faces of dimension 1 are essentially the maximal cliques of the graph \( H(k, q) \) and often called lines.

The spectrum of the Hamming graph is the spectrum of its adjacency matrix. It is known that the spectrum of \( H(n, q) \) consists of the eigenvalues \( \lambda_i = n(q - 1) - qi \) with multiplicity \( \binom{n}{i}(q - 1)^i \), \( i = 0, \ldots, n \).

A \( k \)-coloring of a graph \( G \) is a surjective function \( f \) from the vertex set to the set of colors \( \{1, \ldots, k\} \) such that each vertex \( x \) of color \( i \) is adjacent to exactly \( s_{i,j} \) vertices of color \( j \). The matrix \( S = \{s_{i,j}\} \) of order \( k \) is called the \textit{quotient matrix} of the perfect coloring \( f \).

Let us have a closer look at parameters of perfect 2-colorings. The quotient matrix of a perfect 2-coloring \( f \) is usually written as

\[
S = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\]

The quotient matrix \( S \) has two different eigenvalues: the trivial eigenvalue \( n(q - 1) - (b + c) \) and the main eigenvalue \( \lambda = n(q - 1) - (a + b) \).

We note that for entries of the quotient matrix of a perfect 2-coloring \( f \) in \( H(n, q) \) it holds \( a + b = c + d = n(q - 1) \), so parameters \( a, b, c, \) and \( d \) uniquely define the matrix. Thus, we will say that \( f \) is a \((b, c)\)-coloring if it has the quotient matrix \( S \) as above. Rearranging colors, we may assume that \( b \geq c \).

It is easy to prove that the number of vertices of the first color in the perfect \((b, c)\)-coloring of \( H(n, q) \) is equal to \( \frac{n(q - 1)}{b + c} \cdot q^c \), and the number of vertices of the second color is \( \frac{n(q - 1)}{b + c} \cdot q^b \). Reducing factors, we find that the densities of colors in the perfect \((b, c)\)-coloring \( f \) are \( \frac{b'}{b + c} \) and \( \frac{c'}{b + c} \) respectively, or, equivalently, \( \frac{c'}{b' + c'} \) and \( \frac{b'}{b' + c'} \), where \( b' = \frac{b}{\gcd(b, c)} \) and \( c' = \frac{c}{\gcd(b, c)} \). Later we will see that the parameters \( b' \) and \( c' \) not only define a proportion of colors in a perfect \((b, c)\)-coloring but play an important role in the characterization of admissible parameters of colorings.

At the end of this section, we recall one straightforward application of König’s theorem, which will be used in one of constructions.

**Proposition 1 ([8]).** Let \( f \) be a perfect coloring of \( H(n, 2) \), and let \( a \) be one of the colors. If \( f^{-1}(a) \) is not an independent set, then it can be partitioned into edges (lines).
3 Necessary conditions on parameters of perfect colorings

We start with one simple algebraic condition on parameters of \((b, c)\)-colorings. It is well known \([22]\) Ch. 5, Lemma 2.2] that the spectrum of the quotient matrix of a perfect coloring in a graph is contained in the graph spectrum. Since all eigenvalues of the Hamming graph \(H(n, q)\) have the form \(\theta_i = n(q - 1) - qi\), where \(i = 0, \ldots, n\), and the main eigenvalue of a \((b, c)\)-coloring of \(H(n, q)\) is \(\lambda = n(q - 1) - (b + c)\), we have the following.

**Proposition 2.** If there exists a \((b, c)\)-coloring of \(H(n, q)\), then \(b + c = qi\) for some \(i \in \{1, \ldots, n\}\).

### 3.1 Conditions based on colorings in faces

For 2-colorings, it is known that all large faces of the Hamming graph have the same densities of colors as in the whole graph.

**Proposition 3** (see e.g. [23] Proposition 1]. Let \(f\) be a perfect \((b, c)\)-coloring of the Hamming graph \(H(n, q)\). Then for each \(k\), \(n - \frac{b+c}{q} + 1 \leq k \leq n\) all \(k\)-dimensional faces of \(H(n, q)\) contain the same number of vertices of each color.

Based on this fact, we deduce the following necessary condition on the quotient matrices of 2-colorings. A special case of this condition was considered in [15] to show the non-existence of some 1-perfect codes over non-prime-power alphabets.

**Theorem 1.** Let \(f\) be a \((b, c)\)-coloring of \(H(n, q)\). Then \(b' + c'\) divides \(q^n - \frac{b+c}{q} + 1\), where \(b' = \frac{b}{\gcd(b, c)}\) and \(c' = \frac{c}{\gcd(b, c)}\).

**Proof.** By Proposition 3 if \(k = n - \frac{b+c}{q} + 1\) then each \(k\)-face has the same densities of colors equal to the densities in the whole \(H(n, q)\). Recall that the colors of \(f\) have densities \(\frac{b'}{\frac{b-c}{q}}\) and \(\frac{c'}{\frac{b-c}{q}}\). Since each \(k\)-face has an integer number of vertices of each color, we conclude that \(b' + c'\) divides \(q^k = q^{n - \frac{b+c}{q} + 1}\). \(\square\)

This theorem yields an additional necessary condition on the existence of \((b, 1)\)-colorings.

**Proposition 4.** Let \(f\) be a perfect \((b, 1)\)-coloring of \(H(n, q)\). Then \(b\) is divisible by \(q - 1\). Moreover, if \(q = p^t\) for some prime \(p\), then \(b + 1 = q^r\) for some \(r \in \mathbb{N}\).

**Proof.** For a \((b, 1)\)-coloring \(f\), the neighborhood of each vertex of the second color has exactly one vertex of the first color.

It follows that each line of \(H(n, q)\) contains 0, 1 or \(q\) vertices of the first color and \(q - 1\) or 0 vertices of the second color. In particular, every line containing a given vertex of the first color has either \(q - 1\) or \(0\) vertices of the second color. So \(b\) is divisible by \(q - 1\).

Let \(b = c(q - 1)\) for some \(c \in \mathbb{N}\). Suppose that \(q = p^t\) for some prime \(p\). Since \(b\) and \(c\) are relatively prime, Theorem 1 implies that \(b + 1\) divides \(q^r\) for some \(r \in \mathbb{N}\) and \(b + 1 = p^t\) for some \(t \in \mathbb{N}\). It is not hard to see that \(b = p^t\) is divisible by \(q - 1 = p^{t-1}\) if and only if \(t = rs\) for some \(r \in \mathbb{N}\). Thus \(b + 1 = q^r\). \(\square\)

In hypercubes \(H(n, 2)\), there is an additional necessary condition on parameters of \((b, c)\)-colorings.

**Lemma 1 [22].** If there exists a \((b, c)\)-coloring of \(H(n, 2)\) with \(b \neq c\), then \(n \geq \frac{4}{3}(b + c)\).

This bound is a special case of the bound on the correlation immunity of boolean functions proved in [7] for boolean functions. Generalizing this bound to \(q > 2\) is an open research problem.

**Conjecture 1.** If there exists a \((b, c)\)-coloring in the Hamming graph \(H(n, q)\) with \(b' + c' > q\), where \(b' = \frac{b}{\gcd(b, c)}\), \(c' = \frac{c}{\gcd(b, c)}\), then \(n \geq \frac{q+1}{q+1}(b + c)\).

More generally, the same bound can be conjectured for any two-valued function, not only for perfect 2-colorings (the conjecture starts with \(q = 3\) because the case \(q = 2\) exactly corresponds to the bound on the correlation immunity proved in [7]).

**Conjecture 2.** Let \(f : \{0, \ldots, q - 1\}^n \rightarrow \{0, 1\}, q \geq 3,\) be a function with \(B\) zeros and \(C\) ones. Assume that for some \(k\) all \(k\)-faces have the same number \(\frac{q^k}{2^t + C}\) of ones of \(f\). If \(\frac{B + C}{\gcd(B, C)}\) does not divide \(q\), then \(k \geq \frac{n}{q+1} + 1\).
3.2 Weight distribution theorems and generalizations

To describe the distance invariance of perfect colorings in one of the most general forms, we need the concept of completely regular sets.

A set \( C \) of vertices of a graph is called completely regular of covering radius \( \rho \) if the distance coloring of the graph vertices with respect to \( C \) is a perfect \((\rho + 1)\)-coloring. The weight distribution of a \( k \)-coloring \( f \) with respect to a completely regular set \( C \) of covering radius \( \rho \) (or simply the weight distribution of \( f \) if \( C = \{0^n\} \) in \( H(n, q) \)) is the collection of values \((W_i^f)_{i=0}^{k} \), where \( W_i^f \) is the number of vertices of color \( i \) at distance \( j \) from \( C \). A special case of the following theorem is well known as the Shapiro–Zlotnik–Lloyd theorem \([22, 26]\) about the invariance of the weight distribution of perfect codes (the proof is straightforwardly generalized to the general case).

**Lemma 2** (see e.g. \([20]\)). The weight distribution of a \( k \)-coloring \( f \) with respect to a completely regular set \( C \) is completely determined by the quotient matrices of \( f \) and \( C \) and the values \((W_i^f)_{i=0}^{k} \).

The possibility to calculate the weight distribution starting from the initial values provides a necessary condition for the existence of perfect coloring. We do not know examples when this condition rejects some putative quotient matrix of a perfect coloring of Hamming graphs. However, the following two refinements of the weight distribution theorem in special cases do work, at least for perfect colorings of the hypercubes \( H(n, 2) \).

**Lemma 3** (on local weight distributions \([33, 34]\)). Let \( f \) be a perfect \( k \)-coloring of \( H(n, q) \), let \( m \in \{0, \ldots, n\} \), and let \( W_i^{k,j} \) denote the number of vertices \( z = (x, y) \) such that \( wt(x) = i, wt(y) = j \), and \( f(z) = l \), where \( x \in \{0, \ldots, q - 1\}^m \) and \( y \in \{0, \ldots, q - 1\}^{n-m} \). All the values \((W_i^{k,j})_{i=0}^{m} \), \( i = 0, \ldots, m, \) \( j = 0, \ldots, n - m, l = 1, \ldots, k, \) are completely determined by the local weight distribution \((W_i^{k,0})_{i=0}^{m} \).

In \([8]\), Thms. 2, 3, Fon-Der-Flaass proved two bounds on the parameters of perfect 2-colorings of \( H(n, 2) \), that are based on the connections between the coefficients \( W_i^{1,j} \) (with \( i + j \leq 3 \)). Later, both bounds were majorated by the correlation-immunity bound \([7]\); however, this shows that nonexistence of perfect colorings of Hamming graphs can be proved based on Lemma 3 and it is worth to try to derive consequences of this lemma for non-binary cases or for the case of more than two colors. Finally, the following theorem is applicable for perfect colorings of \( H(n, 2) \), but not \( H(n, q) \) in general (see counterexamples in \([22]\)).

**Lemma 4** (strong distance invariance of perfect colorings of hypercubes \([33]\)). Let \( f \) be a perfect \( k \)-coloring of \( H(n, 2) \) such that \( f(0^n) = a \), and let \( W_{a,b,c}^{k,i,j} \) denote the number of pairs \( x, y \) of vertices \( z = (x, y) \) such that \( f(x) = b, f(y) = c, wt(x) = i, wt(y) = j, \) and the distance between \( i \) and \( j \) is \( k \). The coefficients \( W_{a,b,c}^{k,i,j} \) are completely determined by the quotient matrix and the color \( a \) of \( 0^n \).

The positiveness of the interweight-distribution coefficients \( W_{a,b,c}^{k,i,j} \) is a very strong condition. Empirically (computational results for small \( n \)), for perfect 2-colorings, it is as strong as the correlation-immunity bound \([7]\). If the number of colors is more than 2, then the known variant of the correlation-immunity bound \([16]\) is rather weak to reject real putative quotient matrices, up to our knowledge; at the same time, the interweight-distribution test does work as a strong necessary condition \([15]\). While the straightforward generalization of the invariance of the interweight distribution fails for perfect colorings of \( H(n, q), q > 2 \), in general, finding similar strong multi-parameter invariants in non-binary case is an open research problem.

4 Constructions of perfect colorings

4.1 Covering-based constructions

In this paragraph, we consider a series of constructions that allow to get perfect colorings of a Hamming graph on the base of perfect colorings of a smaller Hamming graph. All these constructions are based on graph coverings. In order to describe them, it is convenient to present Hamming graphs as Cayley graphs.

Let \( \Gamma \) be a finite group and \( A \subseteq \Gamma \) be a (multi)set of \( \Gamma \) such that \( A = A^{-1} \). The Cayley (multi)graph Cay(\( \Gamma, A \)) with the connecting set \( A \) is a (multi)graph with the vertex set \( \Gamma \) and the edge set \( \{(g, ga) | g \in \Gamma, a \in A\} \).

The Hamming graph \( H(n, q) \) is the Cayley graph Cay(\( \mathbb{Z}_q^n, I(n, q) \)) with the connecting set \( I(n, q) \) consisting of all vertices of weight 1.

A multigraph \( G = (V, E) \) is said to cover a multigraph \( H = (U, W) \) if there exists a surjective function \( \varphi : V \to U \), called a covering, such that for each \( u \in V \) the equality \( \{\varphi(u), (u, v) \in E\} = \{\varphi(u), \varphi(v) \in W\} \) holds as for multisets. Such a function \( \phi \) is called a covering. It is straightforward that a covering is a perfect coloring of \( G \) with the quotient matrix \( S \) equal the adjacency matrix of \( H \).

There are natural coverings of Cayley graphs based on group homomorphisms. Recall that \( \varphi : \Gamma \to \Gamma' \) is a homomorphism between groups \((\Gamma, \ast)\) and \((\Gamma', \ast)\) if for all \( g, h \in \Gamma \) we have \( \varphi(g \ast h) = \varphi(g) \ast \varphi(h) \).
Theorem 2. Let $\varphi : \Gamma \rightarrow \Gamma'$ be a homomorphism between groups $\Gamma$ and $\Gamma'$, and let $A \subseteq \Gamma$ be a multisiset such that $A = A^{-1}$. Then $\varphi$ is a covering of $\text{Cay}(\Gamma', \varphi(A))$ by $\text{Cay}(\Gamma, A)$.

Proof. Firstly, we note that $A = A^{-1}$ implies that $\varphi(A) = (\varphi(A))^{-1}$, so the graph $\text{Cay}(\Gamma', \varphi(A))$ is well defined.

Let us check that $\varphi$ is a covering. Consider a vertex $g$ of $\text{Cay}(\Gamma, A)$. The multiset of vertices $\varphi(h)$, where $g$ and $h$ are adjacent in $\text{Cay}(\Gamma, A)$, is exactly the multiset $\varphi(g \cdot A)$. The multiset of vertices adjacent to $\varphi(g)$ in $\text{Cay}(\Gamma', \varphi(A))$ is $\varphi(g) \varphi(A)$. These multisets coincide because $\varphi$ is a group homomorphism.

All constructions of perfect colorings of $H(n, q)$ in this paragraph are based on the following known and straightforward fact.

Lemma 5. Suppose that $f$ is a perfect coloring of a multigraph $H$ with a quotient matrix $S$. If $\varphi$ is a covering of $H$ by a multigraph $G$, then $\varphi \circ f$ is a perfect coloring of $G$ with the quotient matrix $S$.

Given a multigraph $G$ and $t \in \mathbb{N}$, we denote by $G + tI$ the multigraph obtained from $G$ by adding $t$ loops to each vertex of $G$. (We make agreement that each loop contributes 1 to the degree of the corresponding vertex.) If $G$ is a Cayley multigraph $\text{Cay}(\Gamma, A)$, then the multigraph $G + tI$ can be presented as $\text{Cay}(\Gamma, A \cup \{t \times 0^n\})$, where $t \times 0^n$ is the identity element $0^n$ of $\Gamma$ with multiplicity $t$.

The definitions of a perfect coloring and $G + tI$ imply the following.

Lemma 6. If $f$ is a perfect coloring of a multigraph $G$ with the quotient matrix $S$, then $f$ is a perfect coloring of the multigraph $G + tI$ with the quotient matrix $S + tI$.

Let us consider now a series of graphs and multigraphs that can be covered by Hamming graphs.

Proposition 5. The Hamming graph $H(n+1, q)$ covers the multigraph $H(n, q) + t(q - 1)I$.

Proof. Let us consider the group homomorphism $\varphi : \mathbb{Z}_q^{n+1} \rightarrow \mathbb{Z}_q^n$ defined by the equation

$$\varphi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n).$$

It is straightforward to see the following: if $I(n+1, q)$ is the set of all elements of $\mathbb{Z}_q^{n+1}$ at distance 1 from the identity element, then the multiset $B := \varphi(I(n+1, q))$ is equal to $I(n, q) \cup \{t(q - 1) \times 0^n\}$. Since $\text{Cay}(\mathbb{Z}_q^n, B)$ is exactly the multigraph $H(n, q) + t(q - 1)I$, Theorem 2 implies that $\varphi$ is a covering of $H(n, q) + t(q - 1)I$ by $H(n+1, q)$.

Given a multigraph $G$, let $tG$ be the multigraph where every edge of $G$ has the multiplicity $t$ times larger than in $G$. If $G = \text{Cay}(\Gamma, A)$, then $tG$ can be presented as $\text{Cay}(\Gamma, tA)$, where $tA$ is the multiset obtained from $A$ by multiplying all multiplicities by $t$. The definitions of a perfect coloring and $tG$ imply the following.

Lemma 7. If $f$ is a perfect coloring of a multigraph $G$ with the quotient matrix $S$, then $f$ is a perfect coloring of $tG$ with the quotient matrix $tS$.

Proposition 6. The Hamming graph $H(tn, q)$ covers $tH(n, q)$.

Proof. Let us consider the group homomorphism $\varphi : \mathbb{Z}_q^{tn} \rightarrow \mathbb{Z}_q^n$ defined by

$$\varphi(x_1, \ldots, x_{tn}) = (x_1 \ast \ldots \ast x_{tn} \mod q, \ldots, x_{(n-1)t+1} \ast \ldots \ast x_{tn}),$$

where $\ast$ is an arbitrary loop operation on $\mathbb{Z}_q$ with identity 0 (i.e., $0 \ast 0 = 0$). It is straightforward to see that if $I(tn, q)$ is the set of all elements of $\mathbb{Z}_q^{tn}$ at distance 1 from the identity element, then $B := \varphi(I(tn, q))$ is equal to $tI(n, q)$. Since $\text{Cay}(\mathbb{Z}_q^n, B)$ is exactly the multigraph $tH(n, q)$, Theorem 2 implies that $\varphi$ is a covering of $tH(n, q)$ by $H(tn, q)$.

Proposition 7. The Hamming graph $H(n, pq)$ covers $pH(n, q) + n(p - 1)I$.

Proof. Let us consider the group homomorphism $\varphi : \mathbb{Z}_q^{pn} \rightarrow \mathbb{Z}_q^n$ defined as

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n) \mod q; \quad x_i \in Z_{pq}.$$

It is straightforward to see that if $I(n, pq)$ is the set of all elements of $\mathbb{Z}_q^{pn}$ at distance 1 from the identity element, then the multiset $B := \varphi(I(n, pq))$ is equal to $pI(n, q) \cup \{n(p - 1) \times 0^n\}$. Since $\text{Cay}(\mathbb{Z}_q^n, B)$ is exactly the multigraph $pH(n, q) + n(p - 1)I$, Theorem 2 implies that $\varphi$ is a covering of $pH(n, q) + n(p - 1)I$ by $H(n, pq)$.

Remark. There are many other ways to choose homomorphisms $\varphi$ in proofs of Propositions 6 and 7. For instance, it can be checked that if $g^1, \ldots, g^m$ are arbitrary $t$-ary loops of order $q$ such that $g^i(0, \ldots, 0) = 0$ for all $i$ then the operation

$$\varphi(x_1, \ldots, x_{tn}) = (g^1(x_1, \ldots, x_t), \ldots, g^m(x_t(x_{t(n-1)+1}, \ldots, x_{tn})))$$

is a group homomorphism with properties required in Propositions 6.

Using Propositions 6, 7, Lemmas 5, 6, and 7, we obtain the following constructions of perfect colorings in Hamming graphs.
Theorem 3 ([5] Prop. 33(ii)). For every perfect coloring of the Hamming graph $H(n, q)$ with a quotient matrix $S$ and every $t$ in $\mathbb{N}$, there exists a perfect coloring of $H(n + t, q)$ with the quotient matrix $S + t(q - 1)I$. In particular, if there is a perfect $(b, c)$-coloring of $H(n, q)$, then there is a perfect $(b, c)$-coloring of $H(n + t, q)$, $t = 1, 2, \ldots$.

Theorem 4 ([5] Prop. 33(ii)). For every perfect coloring of $H(n, q)$ with a quotient matrix $S$ and every positive integer $t$, there exists a perfect coloring of $H(tn, q)$ with the quotient matrix $tS$. In particular, if there is a perfect $(b, c)$-coloring of $H(n, q)$, then there is a perfect $(tb, tc)$-coloring of $H(tn, q)$, $t = 1, 2, \ldots$.

Theorem 5. For every perfect coloring of $H(n, q)$ with a quotient matrix $S$ and every positive integer $p$, there exists a perfect coloring of $H(n, pq)$ with the quotient matrix $pqS + n(p - 1)I$. In particular, if there is a perfect $(b, c)$-coloring of $H(n, q)$, then there is a perfect $(pb, pc)$-coloring of $H(n, pq)$, $p = 1, 2, \ldots$.

4.2 MDS codes, 1-perfect codes, and the invasion construction

Under a code in the Hamming graph $H(n, q)$, we mean an arbitrary nonempty subset of the vertex set of $H(n, q)$. To each code in $H(n, q)$ we assign a 2-coloring in which the set of vertices of the first color coincides with the code. Also we often identify codes with the corresponding 2-colorings.

An MDS code with distance $d$ is a set of $q^{n-d+1}$ vertices in $H(n, q)$ such that the Hamming distance between any two different code vertices is not less than $d$. Equivalently, a distance-$d$ MDS code has exactly one element in every $(d-1)$-face.

The definition implies that every distance-2 MDS code in $H(n, q)$ is exactly a $(n(q - 1), n)$-coloring. With the help of Theorem 4 distance-2 MDS codes in $H(n, q)$ can be constructed from a $(q - 1, 1)$-coloring of the complete graph $H(1, q)$.

A t-fold MDS code in $H(n, q)$ is a code that has exactly $t$ elements in every 1-face (line). Some t-fold MDS codes can be obtained as the union of $t$ disjoint copies of distance-2 MDS codes. Any t-fold MDS code corresponds to a $(n(q - t), nt)$-coloring of $H(n, q)$; by Theorem 4 colorings with such parameters can be constructed from a $(q - t, t)$-coloring of the complete graph $H(1, q)$.

Perfect colorings corresponding to distance-2 MDS codes and multifold MDS codes have the main eigenvalue $\lambda = -n$, which is the smallest eigenvalue of $H(n, q)$.

A 1-perfect code in $H(n, q)$ is a set of vertices such that each radius-1 ball $B(x) = \{y|\rho(x, y) \leq 1\}$ contains exactly one code vertex. The minimal Hamming distance between different vertices of a nontrivial (with more than 1 vertex) 1-perfect code is equal to 3.

It is well known that 1-perfect codes exist in $H(n, q)$ if $q = p^s$ for some prime $p$ and $n = \frac{q^d - 1}{q - 1}$ for some $r \in \mathbb{N}$ (e.g., $q$-ary Hamming codes). If $q$ is not a prime power, then the existence of 1-perfect codes in $H(n, q)$ is a long-standing open problem. The definition implies that every 1-perfect code in $H(n, q)$ is a $(n(q - 1), 1)$-coloring.

A t-fold 1-perfect code in $H(n, q)$ is a set of vertices such that each radius-1 ball contains exactly $t$ code vertices. Some t-fold 1-perfect codes can be obtained as the union of $t$ disjoint copies of 1-perfect codes. t-Fold 1-perfect codes correspond to $(n(q - 1) - t + 1, t)$-colorings of $H(n, q)$; they exist if (but not necessarily “only if”) 1-perfect codes exist. The main eigenvalue of 1-perfect codes and multifold 1-perfect codes is $-1$.

For the future, we establish the following correspondence between 1-perfect codes in the Hamming graph $H(q + 1, q)$ and certain MDS codes in $H(q, q)$.

Proposition 8. If there exists a 1-perfect code in $H(q + 1, q)$, then there exists a partition of the vertex set of $H(q, q)$ into distance-2 MDS codes $M^0, \ldots, M^{q-1}$ such that each $M^i$ can be decomposed into the union of $q$ disjoint distance-3 MDS codes.

Proof. Let $C$ be a 1-perfect code (equivalently, a distance-3 MDS code) in $H(q + 1, q)$, and let $C_j$ be the subset of $C$ consisting of all codewords with last symbol $j$, $j = 0, \ldots, q - 1$. Denote by $M$, $L_0$, $\ldots$, $L_{q-1}$ the last-coordinate projections of $C$, $C_0$, $\ldots$, $C_{q-1}$, respectively. By the definition, $M$ is a distance-2 MDS code, and $L_j$, $j = 0, \ldots, q - 1$, are distance-3 MDS codes; moreover, $M = \cup_j L_j$. It follows that the codes $M^i = M + (i, 0, \ldots, 0)$, $i = 0, \ldots, q - 1$, form a required partition.

At last, we give one more construction of perfect 2-colorings in Hamming graphs on the base on a set of 2-colorings and a perfect coloring into an arbitrary number of colors. We will use this construction in our future results.

Let $f$ be a $k$-coloring of $H(n, q)$ and let $g_1, \ldots, g_k$ be 2-colorings of $H(m, q)$. Define the invasion $h = f \times (g_1, \ldots, g_k)$ of the coloring $f$ by colorings $g_1, \ldots, g_k$ to be a 2-coloring of $H(n + m, q)$ such that

$$h(x, y) = g_f(x)(y), \quad x \in \mathbb{Z}_q^n, \quad y \in \mathbb{Z}_q^m.$$ 

Roughly speaking, the invasion coloring $h$ of $H(n + m, q)$ is obtained by replacing each vertex of color $i$ in the coloring $f$ of $H(n, q)$ by the graph $H(m, q)$ colored into the coloring $g_i$.

\[^1\text{There is a misprint in the parameters of the resulting coloring in [5] Prop. 33(ii).}\]
Assume that $M^1, \ldots, M^9$ is a partition of the vertex set of $H(m, q)$ into $q$ pairwise disjoint distance-2 MDS codes. Given $t$, $0 \leq t \leq q$, define $G_t^i$ to be the coloring corresponding to the $t$-fold MDS code $\bigcup_{j=i}^{t+1-1} M^j$ (index $j$ goes cyclically modulo $q$) in $H(m, q)$ if $m \geq 1$. In the case $m = 0$, let $G_t^i$ be the coloring of a vertex into the first color if $1 \leq i \leq t$, and the coloring of a vertex into the second color if $t + 1 \leq i \leq q$.

**Proposition 9.** 1. Let $f$ be a perfect $(q + 1)$-coloring of $H(n, q)$ with the quotient matrix

$$
\begin{pmatrix}
\alpha' & \cdots & \alpha & \beta \\
\vdots & \ddots & \vdots & \vdots \\
\alpha & \cdots & \alpha' & \beta \\
\gamma & \cdots & \gamma & \delta
\end{pmatrix}
$$

If $m = \gamma - \alpha \geq 0$ then for each $t = 1, \ldots, q$ the invasions $h_t = f \times (G_t^1, \ldots, G_t^9)$, where $l = 1, 2$ and $l$ is the solid coloring of $H(m, q)$ into the color $l$, are $(h_t, c_t)$-colorings of $H(n + m, q)$ with $h_t = \gamma(q - t) + \beta(l - 1)$ and $c_t = \gamma t + \beta(2 - l)$.

2. Let $f$ be a perfect 2$q$-coloring of $H(n, q)$ with the quotient matrix

$$
\begin{pmatrix}
\alpha & \cdots & \alpha & \beta & \cdots & \beta \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha & \cdots & \alpha & \beta & \cdots & \beta \\
\gamma & \cdots & \gamma & \delta & \cdots & \delta \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\gamma & \cdots & \gamma & \delta & \cdots & \delta
\end{pmatrix}
$$

If $m = \gamma - \alpha = \beta - \delta \geq 0$, then for each $t_1, t_2 = 0, \ldots, q$ the invasion $h = f \times (G_{t_1}^1, \ldots, G_{t_1}^9, G_{t_2}^1, \ldots, G_{t_2}^9)$ is a $(b, c)$-coloring of $H(n + m, q)$ with $b = q(\gamma + \beta) - \gamma t_1 - \beta t_2$ and $c_t = \gamma t_1 + \beta t_2$.

**Proof.** 1. By the construction, the invasion $h_t = f \times (g_1, \ldots, g_k)$ is a coloring of $H(n + m, q)$ into colors 1 and 2. Each vertex $z \in H(n + m, q)$ is considered as an ordered pair of vertices $(x, y)$, where $x$ is a vertex of $H(n, q)$ and $y$ is a vertex of $H(m, q)$. Therefore, the number of vertices of color 1 (color 2) in the coloring $h_t$ adjacent to the vertex $z = (x, y)$ is equal to the number of vertices of color 1 (color 2) adjacent to $y$ in the coloring $G_t^i(x)$ or in the coloring $l$ plus the number of color-1 (color-2) vertices $h(x', y)$, where $x'$ runs over all the neighbors of $x$.

We recall that for each $i = 1, \ldots, q$, the coloring $G_t^i$ is a $(m(q - t), mt)$-coloring of $H(m, q)$.

Suppose that $h_t(z) = 1$, $z = (x, y)$. If $1 \leq f(x) \leq q$, then the number of vertices of $h_t^{-1}(2)$ adjacent to $z$ is $m(q - t) + \alpha(q - t) + \beta(l - 1)$. In the case $f(x) = q + 1$, the number of the vertices of $h_t^{-1}(2)$ adjacent to $z$ is equal to $\gamma(q - t) - \gamma t$. The condition $m = \gamma - \alpha$ implies that the equality $m(q - t) + \alpha(q - t) = \gamma(q - t)$ holds for the coloring $h_1$.

Suppose now that $h_t(z) = 2$, $z = (x, y)$. If $1 \leq f(x) \leq q$, then the number of the vertices of $h_t^{-1}(1)$ adjacent to $z$ is equal to $mt + \alpha t + \beta(2 - l)$. In the case $f(x) = q + 1$, the number of the vertices of $h_t^{-1}(1)$ adjacent to $z$ is equal to $\gamma t$. Again, the condition $m = \gamma - \alpha$ implies that the equality $mt + \alpha t = \gamma t$ holds for the coloring $h_2$.

Thus, the invasions $h_t$ are perfect $(b_1, c_1)$-colorings of $H(n + m, q)$ with $b_1 = \gamma(q - t) + \beta(l - 1)$ and $c_1 = \gamma t + \beta(2 - l)$.

2. As before, let $z = (x, y)$ be a vertex of $H(n + m, q)$.

Suppose $h(z) = 1$. By the definition of $h$, if $1 \leq f(x) \leq q$ then the number of neighbors of $z$ in $h^{-1}(2)$ is equal to $m(q - t_1) + \alpha(q - t_1) + \beta(q - t_2)$. In the case $q + 1 \leq f(x) \leq 2q$ the number of color-2 vertices adjacent to $z$ is equal to $m(q - t_2) + \gamma(q - t_1) + \delta(q - t_2)$. The condition $m = \gamma - \alpha = \beta - \delta$ implies that the equality $m(q - t_1) + \alpha(q - t_1) + \beta(q - t_2) = m(q - t_2) + \gamma(q - t_1) + \delta(q - t_2)$ holds for all $t_1$ and $t_2$.

Now suppose $h(z) = 2$. If $1 \leq f(x) \leq q$ then the number of neighbors of $z$ in $h^{-1}(1)$ is equal to $mt_1 + \alpha t_1 + \beta t_2$. In the case $q + 1 \leq f(x) \leq 2q + 1$, the number of color-1 vertices adjacent to $z$ is equal to $mt_2 + \gamma t_1 + \beta t_2$. Again, the condition $m = \gamma - \alpha = \beta - \delta$ implies that the equality $mt_1 + \alpha t_1 + \beta t_2 = mt_2 + \gamma t_1 + \beta t_2$ holds for all $t_1$ and $t_2$.

Thus, the invasion $h$ is a perfect $(b, c)$-coloring of $H(n + m, q)$ with $b = \gamma(q - t_1) + \beta(q - t_2)$ and $c = \gamma t_1 + \beta t_2$.

**4.3 Splitting construction I**

In this and next sections, we provide constructions that allow to obtain perfect 2-colorings with new proportions of colors. We call them splitting constructions. The idea of the first construction is to start with multiplying the parameters of a 2-coloring by a factor $q$, as in Theorem 3 after this, with the help of 1-perfect code in $H(q, q + 1)$, one of the colors (say, the second one) is split into $q$ “equivalent” colors; and finally, after some intermediate step,
t of these colors are unified with the first one, while the remaining $q-t$ colors form the new second color. So, the ratio of the colors changes as follows:

$$c : b = qc : qb \rightarrow qc : b : \ldots : b \rightarrow qc + tb : (q-t)b.$$ 

The construction generalizes the main construction in [5] from the case $q = 2$ to the case of an arbitrary $q$ such that there exists a 1-perfect code in $H(q+1,q)$.

We divide the proof into several lemmas. Firstly, for certain $q$ and for every perfect $(b,c)$-coloring of $H(n,q)$, we construct an appropriate perfect $2q$-coloring.

**Lemma 8.** Let $f$ be a perfect coloring in $H(n,q)$ with the quotient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If there exists a 1-perfect code in $H(q+1,q)$, then there exists a perfect $2q$-coloring $g$ of $H(qn,q)$ with the quotient matrix $T = \begin{pmatrix} a & \ldots & a & b & \ldots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a & \ldots & a & b & \ldots & b \\ c & \ldots & c & d & \ldots & d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c & \ldots & c & d & \ldots & d \end{pmatrix}$.

**Proof.** The main idea of the construction is to split, in a right way, each color of a 2-coloring of $H(qn,q)$ obtained from Theorem 4 into $q$ colors.

Let $M^1, \ldots, M^n$ be a partition of the vertex set of $H(q,q)$ into $q$ distance-2 MDS codes such that each $M^i$ is partitioned into distance-3 MDS codes $L^1_i, \ldots, L^n_i$. The existence of such decomposition, provided a 1-perfect code in $H(q+1,q)$, is guaranteed by Proposition 5.

Define a $q$-ary quasigroup $h$ of order $q$ on the vertex set of $H(q,q)$ as $h(x_1, \ldots, x_q) = i$ if and only if the vertex $(x_1, \ldots, x_q)$ belongs to $M^i$.

Recall that every vertex $y = (y_1, \ldots, y_{qn})$ of $H(qn,q)$ can be considered as a tuple $y = (y^1, \ldots, y^n)$ of $n$ vertices $y^i = (y_{i-1}+(i-1)q+1, \ldots, y_{i}q)$ of $H(q,q)$. For shortness, let $X_y = (x_1, \ldots, x_q)$ be a vertex of $H(n,q)$ with $x_i = h(y^i)$ and $J_y = (j_1, \ldots, j_n)$ be a vertex of $H(n,q)$ such that $j_i$ is defined by the index of the distance-3 MDS code containing the vertex $y^i$: $y^i \in L^h_{j^i}(y)$. For an arbitrary $n$-ary quasigroup $R$ of order $q$ on the vertex set of $H(n,q)$, we define a coloring $g$ of $H(qn,q)$ as

$$g(y) = q(f(X_y) - 1) + R(J_y).$$

So $g$ is a $2q$-coloring of $H(qn,q)$ into colors $\{1, \ldots, q, q+1, \ldots, 2q\}$. Let us prove that $g$ is a perfect coloring with the quotient matrix $T$. Let $Z_{a}^i$ be a set of vertices $z$ adjacent to $y$ such that $z$ differs from $y$ in some component of $y^i$ from $H(q,q)$ with $\alpha = h(z^i) \neq h(y^i)$. Given a vertex $y$ with $f(X_y) = 1$, there are exactly $a$ sets $Z_{a}^i$ such that for all $z \in Z_{a}^i$ we have $f(X_z) = 1$ and $b$ sets $Z_{b}^i$ with $f(X_z) = 2$ for all $z \in Z_{b}^i$. Similarly, if $f(X_y) = 2$ then we have $c$ sets $Z_{c}^i$ such that for all $z \in Z_{c}^i$ it holds $f(X_z) = 1$ and $d$ sets $Z_{d}^i$ with $f(X_z) = 2$ for all $z \in Z_{d}^i$.

Note that for each $i$ and the cardinality of the set $Z_{a}^i$ is equal to $q$ and all these $q$ vertices $z$ are colored by $q$ in $q$ different colors. Indeed, their components $z^i$ belong to different distance-3 MDS codes $L^h_{j^i}$ (otherwise we have a contradiction with the minimal distance in $L^h_{j^i}$), and all other components $z^k$ of $z \in Z_{a}^i$ coincide and belong to the same codes $L^h_{j^i}$. Since the quasigroup $R$ takes all different values on the set of $q$ vectors $J_z$ different in one position, vertices $z \in Z_{a}^i$ are colored by $q$ in all $q$ possible colors.

Therefore, each vertex $y$ with $g(y) \in \{1, \ldots, q\}$ is adjacent to exactly $a$ vertices of each of the colors 1, \ldots, $q$ and is adjacent to $b$ vertices of each of the colors $q+1, \ldots, 2q$ in the coloring $g$. The same is true for vertices $y$ of colors $g(y) \in \{q+1, \ldots, 2q\}$.

In case when one of the colors of a perfect 2-coloring $f$ can be divided into $k$-dimensional faces, the similar method allows to construct the following perfect $(q+1)$-colorings.

**Lemma 9.** Let $f$ be a perfect coloring in $H(n,q)$ with the quotient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the set of vertices of the first color can be partitioned into $k$-faces. If there exists a 1-perfect code in $H(q+1,q)$, then there exists perfect colorings $g'$ and $g''$ in $q+1$ colors in $H(qn,q)$ with the quotient matrices

$$T' = \begin{pmatrix} a - k(q-1) & \ldots & a + k & qb \\ \vdots & \ddots & \vdots & \vdots \\ a + k & \ldots & a - k(q-1) & qb \\ c & \ldots & c & qd \end{pmatrix}, \quad T'' = \begin{pmatrix} a + k(q-1)^2 & \ldots & a - k(q-1) & qb \\ \vdots & \ddots & \vdots & \vdots \\ a - k(q-1) & \ldots & a + k(q-1)^2 & qb \\ c & \ldots & c & qd \end{pmatrix}.$$
Proof. In the proof of this lemma, we use the same notations as in Lemma [S]. In addition, if a vertex $x$ of $H(n, q)$ is colored with the first color in the coloring $f$, then let $I_x = \{i_1, \ldots, i_k\}$ be the set of $k$ free directions in the $k$-dimensional face $f$ containing the vertex $x$ in the demanded decomposition of this color into faces. We will say that $I_x$ is the set of special directions for the vertex $x$.

For an arbitrary $n$-ary quasigroup $R'$ of order $q$ on vertices of $H(n, q)$, define a coloring $g'$ of $H(qn, q)$ as

$$g'(y) = g'(y^1, \ldots, y^n) = \begin{cases} R'(j_1, \ldots, j_n) & \text{if } f(X_y) = 1, \quad \text{where } j_i = h(y^i) \text{ for } i \in I_x, \quad \text{and } y^i \in L_{j_i}^{h(y^i)} \text{ otherwise}; \\ q + 1 & \text{if } f(X_y) = 2, \end{cases}$$

and for a $(n - k)$-ary quasigroup $R''$ of order $q$ define a coloring $g''$ of $H(qn, q)$ as

$$g''(y) = g''(y^1, \ldots, y^n) = \begin{cases} R''(j_1, \ldots, j_{n-k}) & \text{if } f(X_y) = 1, \quad \text{where } i \notin I_x, \quad \text{and } y^i \in L_{j_i}^{h(y^i)}; \\ q + 1 & \text{if } f(X_y) = 2. \end{cases}$$

Let us prove that $g'$ and $g''$ are perfect $(q + 1)$-colorings with the quotient matrices $T'$ and $T''$ respectively.

As before, let $Z'_a$ be a set of $q$ vertices $z$ adjacent to $y$ such that $z$ differs from $y$ in some component of $y^i$ from $H(q, q)$ with $\alpha = h(z^i) \neq h(y^i)$.

1. We firstly prove that $g'$ is a perfect coloring.

Assume that a vertex $y$ is colored into one of the colors $\{1, \ldots, q\}$ by the coloring $g'$. Acting similar to the proof of Lemma [S] we see that all $qb$ vertices $z$ from the set $Z'_a$ with $i \notin I_x$ and $f(X_z) = 1$ are colored into all different colors. By definitions, all $qb$ vertices $z$ from sets $Z'_a$ with $f(X_z) = 2$ are different from $y$ in components $y^i$, where $i$ is not a special direction ($i \notin I_x$), so all these vertices are colored into color $q + 1$ in the coloring $g'$.

Consider the set of vertices $Z''_a$ adjacent to the vertex $y$ along a special direction $i \in I_x$. By definition of the coloring $g'$, all $q$ vertices $z$ from $Z''_a$ are colored into one color that is different from the color of the vertex $y$. Moreover, while $\alpha$ runs over all $q - 1$ values different from $h(y^i)$, the colors of the vertices in sets $Z''_a$ run over all $q - 1$ colors different from the color of $y$.

The above reasoning implies that in the coloring $g'$ every vertex $y$ of a color from the set $\{1, \ldots, q\}$ is adjacent to $a - k(q - 1)$ vertices of the same color, to $qb$ vertices of the color $q + 1$ and to $a + k$ vertices of each of the remaining colors.

If the vertex $y$ is colored into the color $q + 1$ by $g'$ then all vertices $z$ from sets $Z''_a$ are either colored into the color $q + 1$ (when $f(X_z) = 2$) or colored into all $q$ colors $\{1, \ldots, q\}$ (when $f(X_z) = 1$), that gives us the required parameters.

2. Let us prove that $g''$ is a perfect coloring.

The coloring $g''$ is different from $g'$ only in neighborhoods along special directions $i \in I_x$ of vertices $y$ for which $f(X_z) = 1$. In that case, all $q$ vertices $z$ from the sets $Z''_a$ are colored into the same color as the vertex $y$.

It implies that in the coloring $g''$ every vertex $y$ of a color from the set $\{1, \ldots, q\}$ is adjacent to $a + k(q - 1)^2$ vertices of the same color, to $qb$ vertices of the color $q + 1$ and to $a - k(q - 1)$ vertices of each of the remaining colors.

If the vertex $y$ is colored into the color $q + 1$ by the coloring $g''$, then we have the same coloring of its neighborhood as for the coloring $g'$.

Now we are ready to prove the main constructions. We start with a construction working for perfect colorings with the non-positive main eigenvalue.

**Theorem 6.** Assume that there is a $1$-perfect code in $H(q + 1, q)$. If $f$ is a $(b, c)$-coloring in $H(n, q)$ with the main eigenvalue $\lambda \leq 0$, then for all $t_1, t_2 = 1, \ldots, q$ there exist a $(q(b + c) - (ct_1 + bt_2), ct_1 + bt_2)$-coloring in $H(qn - \lambda, q)$ with the same main eigenvalue $\lambda$.

**Proof.** Since there is a $1$-perfect code in $H(q + 1, q)$, by Lemma [S] there is a perfect $(q + 1)$-coloring $g$ in the Hamming graph $H(qn, q)$ with the quotient matrix

$$T = \begin{pmatrix} a & \cdots & a & b & \cdots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a & \cdots & a & b & \cdots & b \\ c & \cdots & c & d & \cdots & d \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ c & \cdots & c & d & \cdots & d \end{pmatrix}.$$ 

Let $m = c - a = b - d = -\lambda$. By the hypothesis of the theorem, we have $m \geq 0$. With the help of Proposition [S], we obtain that for each $t_1, t_2 = 0, \ldots, q$ there exists a $(b, \tilde{c})$-coloring $F$ of $H(n + m, q)$ with $\bar{b} = q(b + c) - ct_1 - bt_2$ and $\tilde{c} = ct_1 + bt_2$. 

9
The main eigenvalue of the perfect coloring $F$ is
\[ \tilde{\lambda} = (q - 1)(qn - \lambda) - \hat{b} - \hat{c} = (q - 1)(qn - \lambda) - q(b + c) = (q - 1)(qn - \lambda) - q((q - 1)n - \lambda) = \lambda. \]

The next theorem is a more general variant of the construction above, applicable when the first color of a perfect coloring $f$ can be divided into $k$-dimensional faces. If $k > 0$ ($k = 0$ corresponds to the previous theorem), starting from a perfect coloring with the negative main eigenvalue, it allows to get a perfect coloring with a smaller eigenvalue.

**Theorem 7.** Assume that there is a 1-perfect code in $H(q + 1, q)$. Let $f$ be a $(b, c)$-coloring of $H(n, q)$ with the main eigenvalue $\lambda$. If the vertex set of the first color of $f$ can be partitioned into $k$-faces, then the following holds:

1. If $\lambda + k \leq 0$, then for all $t = 1, \ldots, q$ there is a $(q(b + c) - tc, tc)$-coloring of $H(qn - \lambda - k, q)$ (with the main eigenvalue $\lambda - k(q - 1)$).

2. If $\lambda \leq k(q - 1)$, then for all $t = 1, \ldots, q$ there is a $(q(b + c) - tc, tc)$-coloring of $H(qn - \lambda + k(q - 1), q)$ with the main eigenvalue $\lambda + k(q - 1)^2$.

**Proof.**

1. Since there is a 1-perfect code in $H(q + 1, q)$, by Lemma 9 there is a perfect $(q + 1)$-coloring $g'$ in $H(qn, q)$ with the quotient matrix
\[ T' = \begin{pmatrix} a - k(q - 1) & \cdots & a + k & qb \\ \vdots & \ddots & \vdots & \vdots \\ a + k & \cdots & a - k(q - 1) & qb \\ c & \cdots & c & qd \end{pmatrix}. \]

Let $m$ be $c - a - k = -\lambda - k \geq 0$. With the help of Proposition 9(1), we obtain that for each $t = 0, \ldots, q$ there exists a $(\hat{b}, \hat{c})$-coloring $F$ of $H(n + m, q)$ with $\hat{b} = q(b + c) - ct$ and $\hat{c} = ct$.

The main eigenvalue of the perfect coloring $F$ is
\[ \hat{\lambda} = (q - 1)(qn - \lambda - k) - \hat{b} - \hat{c} = (q - 1)(qn - \lambda - k) - q(b + c) = \lambda - k(q - 1). \]

2. Since there is a 1-perfect code in $H(q + 1, q)$, by Lemma 9 there exists a perfect $(q + 1)$-coloring $g''$ of $H(qn, q)$ with the quotient matrix
\[ T'' = \begin{pmatrix} a + k(q - 1)^2 & \cdots & a - k(q - 1) & qb \\ \vdots & \ddots & \vdots & \vdots \\ a - k(q - 1) & \cdots & a + k(q - 1)^2 & qb \\ c & \cdots & c & qd \end{pmatrix}. \]

Let $m$ be $c - a + k(q - 1) = -\lambda + k(q - 1) \geq 0$. With the help of Proposition 9(1), we obtain that for each $t = 0, \ldots, q$ there exists a $(\hat{b}, \hat{c})$-coloring $F$ of $H(n + m, q)$ with $\hat{b} = q(b + c) - ct$ and $\hat{c} = ct$.

The main eigenvalue of $F$ is
\[ \hat{\lambda} = (q - 1)(qn - \lambda + k(q - 1)) - \hat{b} - \hat{c} = \lambda + k(q - 1)^2. \]

\[ \square \]

### 4.4 Splitting construction II

We conclude this section with one more splitting construction based on the covering of the graph $pH(n, q) + n(p - 1)f$ by $H(n, pq)$.

**Theorem 8.** Let $f$ be a $(q^2 - 1, 1)$-coloring of $H(q + 1, q)$ corresponding to a 1-perfect code such that the vertex set of the second (non-code) color can be partitioned into lines. Then for every $p \in \mathbb{N}$ and for each $t, 1 \leq t \leq p$ there exists a $((q^2 - 1)(p - t), (q^2 - 1)t + p)$-coloring of $H(q + 1, pq)$ (with the main eigenvalue $q(p - 1) - 1$).

**Proof.** For a vertex $x \in f^{-1}(2) \subset \mathbb{Z}_q^{q + 1}$, let $i_x$ denote the direction of the line containing this vertex in the demanded partition. We will say that $i_x$ is the special direction for the vertex $x$. For a vertex $y$ of $H(q + 1, pq)$, we denote by $X_y$ the vertex of $H(q + 1, q)$ defined as $X_y \equiv y \mod q$ entrywise, and by $X'_y$, the vertex of $H(q + 1, p)$ equal to $((\frac{b_1}{q^2}, \ldots, \frac{b_{q^2+1}}{q^2})).$
Let $g$ be a $(q(p - t), qt)$-perfect coloring in $H(q, p)$ corresponding to a $t$-fold MDS code in this graph. Define a coloring $h$ in $H(q + 1, pq)$ as follows

$$h(y) = h(y^1, \ldots , y^{r+1}) = \begin{cases} 1 & \text{if } f(X_y) = 1; \\ g(x_{i1}', \ldots , x_{ir}') & \text{if } f(X_y) = 2 \text{ and all } i \neq i_{X_y}, \end{cases}$$

where $(x_{i1}', \ldots , x_{ir}') = X_{y'}$. Let us show that the coloring $h$ is a perfect coloring of $H(q + 1, pq)$ with demanded parameters.

1. Let $y$ be a vertex of $H(q + 1, pq)$ colored by $h$ into the first color. Let us count the number of 2-colored vertices $z$ adjacent to $y$. Denote $Z^n_y$ to be the set of $p$ vertices adjacent to $y$ such that $z_i \equiv \alpha \neq y_i \mod q$.

Assume that $f(X_y) = 1$. Note that there are no adjacent vertices $z$ colored by $h$ into second color such that $f(X_z) = 1$. Note that for each vertex $y$ there are exactly $q^2 - 1$ sets $Z^n_y$ and, because $i$ is not a special direction for all $X_z$, $z \in Z^n_y$, each set $Z^n_y$ contains exactly $t$ vertices colored by $h$ into the first color and $p - t$ vertices colored into the second color. So each such vertex $y$ is adjacent to $(q^2 - 1)(p - t)$ vertices of the second color.

Suppose now that $f(X_y) = 2$. By the definition of the coloring $h$, there are $q(p - t)$ vertices $z$ adjacent to $y$ such that $y_i \equiv z_i \mod q$ for all $i$ and $h(z) = 2$. As before, for each vertex $y$ there are exactly $q^2 - 1$ sets $Z^n_y$. If $i = i_{X_y}$ is a special direction for the vertex $X_y$ in $H(q + 1, q)$, then for all $\alpha \neq y_i \mod q$ all vertices $z$ from the sets $Z^n_y$ are colored into the first color by $h$. On the other hand, if $i \neq i_{X_y}$ then each of $q^2 - q - 1$ sets $Z^n_y$ with $f(X_z) = 2$ contains exactly $t$ vertices $z$ such that $h(z) = 1$ and $p - t$ vertices $z$ for which $h(z) = 2$. The remaining set $Z^n_y$ with $z$ such that $f(X_z) = 1$ contains only vertices colored into the first color by $h$. Summing up, each such vertex $y$ with $h(y) = 1$ is adjacent to $(q^2 - 1)(p - t)$ vertices of the second color in the coloring $h$.

2. Let $y$ be a vertex of $H(q + 1, pq)$ colored by $h$ into the second color and let us count the number of vertices $z$ adjacent to $y$ and colored to the first color. Note that for all such vertices $y$ we have $f(X_y) = 2$.

By the definition of the coloring $h$, there are $qt$ vertices $z$ adjacent to $y$ such that $y_i \equiv z_i \mod q$ for all $i$ and $h(z) = 1$. Consider the sets $Z^n_y$ of vertices $z$ adjacent to $y$ such that $z_i \equiv \alpha \neq y_i \mod q$. If $i = i_{X_y}$ is the special direction for the vertex $X_y$ in $H(q + 1, q)$, then for all $\alpha \neq y_i \mod q$ all vertices $z$ from the sets $Z^n_y$ are colored into the second color by $h$. On the other hand, if $i \neq i_{X_y}$, then each of $q^2 - q - 1$ sets $Z^n_y$ with $f(X_z) = 2$ contains exactly $t$ vertices $z$ such that $h(z) = 1$ and $p - t$ vertices $z$ for which $h(z) = 2$. As before, the remaining set $Z^n_y$ with $z$ satisfying $f(X_z) = 1$ contains only vertices colored by $h$ into the first color. Summing up, each vertex $y$ with $h(y) = 2$ is adjacent to $(q^2 - 1)t + p$ vertices of the first color in the coloring $h$.

The main eigenvalue of $h$ is

$$\lambda = (q + 1)(pq - 1) - q^2p = q(p - 1) - 1.$$

Let us say few words on parameters for which the above constructions are applicable.

By Proposition 1 we can use the construction from Theorem 7 for all $(b, c)$-colorings in $H(n, 2)$ with the non-positive main eigenvalue. Theorem 7.2 can be used for constructing perfect colorings from colorings given after the application of Theorem 8 because this theorem gives colorings with large main eigenvalue and such that one of their colors can be split into faces.

As is stated in Proposition 1 the second color of a $(3, 1)$-coloring of $H(3, 2)$ can be partitioned into edges (1-faces, lines).

The second color of a $(8, 1)$-coloring (the complement of a 1-perfect code) in $H(4, 3)$ can be partitioned into lines. It is straightforward to verify that all such partitions are equivalent to the following

```
  .  .  .  |
  .  .  .  |
  .  .  .  |
  .  .  .  .
```

Here dots states for code vertices of a 1-perfect code and other symbols are used to show directions of 1-dimensional faces containing the vertex.

Therefore, Theorem 8 is applicable for $q = 2$ or $q = 3$ and gives us the following perfect colorings.

**Corollary 1.**

1. For each $p \in \mathbb{N}$ and $t = 0, \ldots , p$ there exists a perfect $(3p - 3t, p + 3t)$-coloring of $H(3, 2p)$ (with the main eigenvalue $\lambda = 2p - 3$).
2. For each $p \in \mathbb{N}$ and $t = 0, \ldots, p$ there exists a perfect $(8p - 8t, p + 8t)$-coloring of $H(4, 3p)$ (with the main eigenvalue $\lambda = 3p - 4$).

For cases $q \geq 4$ we need to answer the following question.

**Question 1.** Given $q \geq 4$, does there exist a decomposition into $1$-dimensional faces of the complement of a $1$-perfect code in $H(q + 1, q)$?

## 5 Admissible parameters of perfect 2-colorings of $H(n, q)$

Firstly, we note that Theorem 3 implies the following alternative for the existence of perfect $(b, c)$-colorings in Hamming graphs.

**Proposition 10.** Given $q$, $b$ and $c$, either there are no $(b, c)$-colorings of $H(n, q)$ for all $n \in \mathbb{N}$, or there is $n_0 = n_0(b, c, q)$ such that $(b, c)$-colorings of $H(n, q)$ exist if and only if $n \geq n_0$.

We call parameters $q$, $b$ and $c$ satisfying the second alternative admissible, and we will say that $n_0 = n_0(b, c; q)$ is the threshold.

On the base of above theorems and some computation results, we put forward the following conjecture.

**Conjecture 3.** Parameters $q$, $b \neq 1$ and $c \neq 1$ are admissible if and only if $(b + c)/\gcd(b, c)$ divides $q^k$ for some $k \in \mathbb{N}$.

Necessity of this condition is established in Theorem 4 by Proposition 4, it is not sufficient if $c = 1$.

Combining together the above constructions of perfect 2-colorings, we confirm Conjecture 3 for some cases and provide some bounds on the threshold $n_0$. We start with some sufficient conditions on admissible parameters $(b, c)$ when $q$ is a prime power.

**Theorem 9.** Let $q = p^s$ be a power of prime $p$. Assume that $\frac{b + c}{\gcd(b, c)} = q^k$ for some $k \in \mathbb{N}$. Then the parameters $(b, c)$ are admissible. Moreover, the threshold $n_0 = n_0(b, c; q)$ satisfies the inequalities

$$\max \left\{ \frac{b}{q - 1}, \frac{b + c}{q} + k - 1 \right\} \leq n_0 \leq \frac{b + c - \gcd(b, c)}{q - 1}.$$  

Proof. Note that the parameters $(b, c)$ satisfy the necessary condition of Theorem 4. In addition, if $c = 1$ then $b$ and $c$ are relatively prime and coincide with $b'$ and $c'$. So, $b = q^k - 1$ and it is divisible by $q - 1$ as is claimed in Proposition 4.

To prove sufficiency, we need to construct a $(b, c)$-coloring $f$ in $H(n, q)$ for some $n$. Denote $m = \gcd(b, c)$, $b' = b/m$, and $c' = c/m$. We have $b' + c' = q^k$ and $b + c = mq^k$ for some $k$. As it was stated in Subsection 4.2 for $n' = \frac{q^k - 1}{q - 1}$ there is a $c'$-fold 1-perfect code in $H(n', q)$, corresponding to a $(b', c')$-coloring $f'$. With the help of Theorem 4 we multiply the perfect coloring $f'$ by $m$ and obtain a $(b, c)$-coloring in the graph $H(n, q)$ with

$$n = mn' = m\frac{q^k - 1}{q - 1} = \frac{b + c - \gcd(b, c)}{q - 1}.$$  

Let us prove the lower bound on the threshold $n_0$. Since for every $(b, c)$-coloring in $H(n, q)$ the parameter $b$ is not greater than the degree $n(q - 1)$ of the graph $H(n, q)$, we have $n_0 \geq \frac{b}{q - 1}$. By Theorem 4 if there exists a $(b, c)$-coloring in $H(n, q)$, then $q^k$ divides $q^{n - qd^{k - 1} + 1}$. Since $q$ is a prime power, we get the inequality $n_0 \geq mq^{k - 1} + k - 1$. 

As a direct corollary, we have that Conjecture 3 holds for the case of prime $q$.

**Theorem 10.** Assume that $q$ is a prime number.

1. The parameters $b$, $c$ and $q$ are admissible if and only if $b' + c' = q^k$ for some $k \in \mathbb{N}$.

2. If $b' + c' = q^k$ for some $k \in \mathbb{N}$ then the threshold parameter $n_0$ for existence of $(b, c)$-colorings satisfies the inequalities

$$\max \left\{ \frac{b}{q - 1}, \frac{b + c}{q} + k - 1 \right\} \leq n_0 \leq \frac{b + c - \gcd(b, c)}{q - 1}.$$  

Proof. It is only sufficient to note that in case of prime $q$ the sufficient condition from Theorem 4 ($b' + c'$ divides some power of $q$) is equivalent to that $b' + c'$ is a power of $q$. It only remains to use Theorem 9.

In certain cases, we can give the exact value of the threshold $n_0$:

**Corollary 2.** Let $q$ be a prime.

1. If $b$ and $c$ satisfy $\frac{b + c}{\gcd(b, c)} = q$, then $n_0 = \gcd(b, c)$.

2. If $b$ and $c$ are relatively prime such that $b + c = q^2$, then $n_0 = q + 1$. 

12
Acknowledgements

This work was funded by the Russian Science Foundation under grant 18-11-00136.
Appendix. Perfect colorings in Hamming graphs of small sizes

Lower bounds:

- Degree bound $a \geq 0$: $n = \lceil \frac{b}{q-1} \rceil$.
- Eigenvalue bound (Theorem 11): $n = \frac{b+c}{q} + k - 1$, where $k$ is the minimal integer such that $b' + c'$ divides $q^k$.

Constructions:

- Fon-Der-Flaass correlation immunity bound for $q = 2$ (Lemma 11): $n = \lceil \frac{2}{q} \rceil (b+c)$.

| $a + c$ | $b' + c'$ | ($b, c$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|--------|-----------|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2      | 2         | (1,1)    | + 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4      | 2         | (2,2)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4      | 4         | (3,1)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6      | 2         | (3,3)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8      | 8         | (4,4)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10     | 2         | (5,5)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12     | 2         | (6,6)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14     | 2         | (7,7)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16     | 2         | (8,8)    | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16     | 4         | (12,4)   | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16     | 10        | (16,10)  | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16     | 14        | (16,14)  | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16     | 16        | (16,16)  | - 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

5.1 Case $q = 2$

Relevant references are given after the tables.
5.2 Case $q = 3$ illustrating prime $q$

| $b + c$ | $d + e$ | $(b, c)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|--------|--------|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6      | 3      | (4, 2)   | - | + | + | + | + | + | + | + | + | + | + | + | + |
| 9      | 3      | (6, 3)   | - | - | - | a | + | p | ^{p,F} | + | + | + | + | + | + | + |
| 9      | 9      | (7, 2)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 9      | 9      | (8, 1)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 9      | 12     | (8, 4)   | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 15     | 3      | (10, 5)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 18     | 3      | (12, 6)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 18     | 9      | (10, 8)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 18     | 9      | (14, 4)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 18     | 9      | (16, 2)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 21     | 3      | (14, 7)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 24     | 3      | (16, 8)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 3      | (18, 9)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 9      | (15, 12) | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 9      | (21, 6)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 9      | (24, 3)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 12     | (14, 13) | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 12     | (16, 11) | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 22     | (17, 10) | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 19     | (19, 8)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 27     | (20, 7)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 27     | (22, 5)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 27     | (25, 4)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| 27     | 27     | (26, 1)  | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
### 5.3 Case $q = 4$ illustrating a prime power $q$

| $b+c$ | $b'+c'$ | $(b,c)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---------|---------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4     | 2       | (2,2)   | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| 4     | 4       | (3,1)   | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| 8     | 2       | (4,4)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 8     | 4       | (6,2)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 8     | 8       | (5,3)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 12    | 2       | (6,6)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 12    | 4       | (9,3)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 2       | (8,8)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 4       | (12,4)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 8       | (10,6)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 8       | (14,2)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 16      | (9,7)   | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 16      | (11,5)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 16      | (13,3)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 16    | 16      | (15,1)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 20    | 2       | (10,10)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 20    | 4       | (15,5)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 24    | 2       | (12,12)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 24    | 4       | (18,6)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 24    | 8       | (15,9)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 24    | 8       | (21,5)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 28    | 2       | (14,14)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 28    | 4       | (21,7)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 2       | (16,16)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 4       | (24,8)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 8       | (20,12)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 8       | (28,4)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 16      | (18,14)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 16      | (22,10)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 16      | (26,6)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 16      | (30,2)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (17,15)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (19,13)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (21,11)| - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (23,9)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (25,7)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (27,5)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
| 32    | 32      | (29,3)  | - | - | - | - | + | + | + | + | + | + | + | + | + | + |
5.4 Case $q = 6$ illustrating $q$ which is not a prime power

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
k + c & b + c & (a, c) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
6 & 2 & (3, 3) & + & + & + & + & + & + & + & + & + & + \\
6 & 3 & (4, 2) & + & + & + & + & + & + & + & + & + & + \\
6 & 6 & (5, 1) & + & + & + & + & + & + & + & + & + & + \\
12 & 2 & (6, 6) & + & + & + & + & + & + & + & + & + & + \\
12 & 3 & (8, 4) & + & + & + & + & + & + & + & + & + & + \\
12 & 4 & (9, 3) & + & + & + & + & + & + & + & + & + & + \\
12 & 6 & (10, 2) & + & + & + & + & + & + & + & + & + & + \\
12 & 12 & (7, 5) & + & + & + & + & + & + & + & + & + & + \\
18 & 2 & (9, 9) & + & + & + & + & + & + & + & + & + & + \\
18 & 3 & (12, 6) & + & + & + & + & + & + & + & + & + & + \\
18 & 6 & (15, 3) & + & + & + & + & + & + & + & + & + & + \\
18 & 9 & (10, 10) & + & + & + & + & + & + & + & + & + & + \\
18 & 9 & (14, 4) & + & + & + & + & + & + & + & + & + & + \\
18 & 9 & (16, 2) & + & + & + & + & + & + & + & + & + & + \\
18 & 18 & (11, 11) & + & + & + & + & + & + & + & + & + & + \\
18 & 18 & (13, 5) & + & + & + & + & + & + & + & + & + & + \\
24 & 2 & (12, 12) & + & + & + & + & + & + & + & + & + & + \\
24 & 3 & (16, 8) & + & + & + & + & + & + & + & + & + & + \\
24 & 4 & (18, 6) & + & + & + & + & + & + & + & + & + & + \\
24 & 6 & (20, 4) & + & + & + & + & + & + & + & + & + & + \\
24 & 8 & (15, 9) & + & + & + & + & + & + & + & + & + & + \\
24 & 8 & (21, 3) & + & + & + & + & + & + & + & + & + & + \\
24 & 12 & (14, 10) & + & + & + & + & + & + & + & + & + & + \\
24 & 12 & (22, 2) & + & + & + & + & + & + & + & + & + & + \\
24 & 24 & (17, 7) & + & + & + & + & + & + & + & + & + & + \\
24 & 24 & (19, 5) & + & + & + & + & + & + & + & + & + & + \\
30 & 2 & (15, 15) & + & + & + & + & + & + & + & + & + & + \\
30 & 3 & (20, 10) & + & + & + & + & + & + & + & + & + & + \\
30 & 6 & (25, 5) & + & + & + & + & + & + & + & + & + & + \\
36 & 2 & (18, 18) & + & + & + & + & + & + & + & + & + & + \\
36 & 3 & (24, 12) & + & + & + & + & + & + & + & + & + & + \\
36 & 4 & (27, 9) & + & + & + & + & + & + & + & + & + & + \\
36 & 6 & (30, 6) & + & + & + & + & + & + & + & + & + & + \\
36 & 9 & (20, 16) & + & + & + & + & + & + & + & + & + & + \\
36 & 9 & (28, 8) & + & + & + & + & + & + & + & + & + & + \\
36 & 9 & (32, 4) & + & + & + & + & + & + & + & + & + & + \\
36 & 12 & (21, 15) & + & + & + & + & + & + & + & + & + & + \\
36 & 12 & (33, 3) & + & + & + & + & + & + & + & + & + & + \\
36 & 18 & (22, 14) & + & + & + & + & + & + & + & + & + & + \\
36 & 18 & (26, 10) & + & + & + & + & + & + & + & + & + & + \\
36 & 18 & (34, 2) & + & + & + & + & + & + & + & + & + & + \\
36 & 36 & (19, 17) & + & + & + & + & + & + & + & + & + & + \\
36 & 36 & (25, 13) & + & + & + & + & + & + & + & + & + & + \\
36 & 36 & (29, 7) & + & + & + & + & + & + & + & + & + & + \\
36 & 36 & (31, 5) & + & + & + & + & + & + & + & + & + & + \\
36 & 36 & (35, 1) & + & + & + & + & + & + & + & + & + & + \\
\hline
\end{array}
\]

Special cases of admissibility of $(b, c)$-colorings in $H(n, q)$

$q = 2$

(9, 7)-coloring in $H(12, 2)$ + : existence is proved in [9].

(11, 5)-coloring in $H(12, 2)$ − : nonexistence is proved in [9].

$q = 3$

(14, 4)-coloring in $H(7, 3)$ − : if such a coloring exists then each 5-dimensional face of $H(7, 3)$ contains an orthogonal array $A$ with certain parameters. Nonexistence of such array $A$ follows from [13].

$q = 6$

(35, 1)-coloring in $H(7, 6)$ − : there are no 1-perfect codes in $H(7, 6)$ [13].
References

[1] S. V. Avgustinovich and I. Yu. Mogilnykh. Perfect 2-colorings of Johnson graphs $J(6,3)$ and $J(7,3)$. In Á. Barbero, editor, Coding Theory and Applications (Second International Castle Meeting, ICM-CTA 2008, Castillo de la Mota, Medina del Campo, Spain, September 15-19, 2008. Proceedings), volume 5228 of Lect. Notes Comput. Sci. pages 11–19. Springer-Verlag, Berlin Heidelberg, 2008. DOI: 10.1007/978-3-540-87448-5_2.

[2] R. A. Bailey, P. J. Cameron, A. L. Gavrilyuk, and S. V. Goryainov. Equitable partitions of Latin-square graphs. J. Comb. Des., 27(3):142–160, Dec. 2018. DOI: 10.1002/jcd.21634.

[3] J. Bierbrauer. Bounds on orthogonal arrays and resilient functions. J. Comb. Des., 3(3):179–183, 1995. DOI: 10.1002/jcd.3180030304.

[4] J. Bierbrauer, K. Gopalakrishnan, and D. R. Stinson. Orthogonal arrays, resilient functions, error-correcting codes, and linear programming bounds. SIAM J. Discrete Math., 9(3):424–452, Aug. 1996. DOI: 10.1137/S0895480194270950.

[5] J. Borges, J. Rif`a, and V. A. Zinoviev. On completely regular codes. Probl. Inf. Transm., 55(1):1–45, Jan. 2019. DOI: 10.1134/S0032946019010010.

[6] P. Delsarte. An Algebraic Approach to Association Schemes of Coding Theory, volume 10 of Philips Res. Rep., Supplement. N.V. Philips’ Gloeilampenfabrieken, Eindhoven, 1973.

[7] D. G. Fon-Der-Flaass. A bound on correlation immunity. Sib. Elektron. Mat. Izv., 4:133–135, 2007. Online: http://mi.mathnet.ru/eng/semr149.

[8] D. G. Fon-Der-Flaass. Perfect 2-colorings of a hypercube. Sib. Math. J., 48(4):740–745, 2007. DOI: 10.1007/s11202-007-0075-4 translated from Sib. Mat. Zh. 48(4) (2007), 923-930.

[9] D. G. Fon-Der-Flaass. Perfect colorings of the 12-cube that attain the bound on correlation immunity. Sib. Elektron. Mat. Izv., 4:292–295, 2007. In Russian. English translation: https://arxiv.org/abs/1403.8091.

[10] J. Friedman. On the bit extraction problem. In Foundations of Computer Science, IEEE Annual Symposium on, pages 314–319, Los Alamitos, CA, USA, 1992. IEEE Computer Society. DOI: 10.1109/SFCS.1992.267760.

[11] A. L. Gavrilyuk and S. V. Goryainov. On perfect 2-colorings of Johnson graphs $J(v,3)$. J. Comb. Des., 21(6):232–252, July 2013. DOI: 10.1002/jcd.21327.

[12] C. D. Godsil. Algebraic Combinatorics. Chapman and Hall, New York, 1993.

[13] S. W. Golomb and E. C. Posner. Rook domains, latin squares, and error-distributing codes. IEEE Trans. Inf. Theory, 10(3):196–208, 1964. DOI: 10.1109/TIT.1964.1053680.

[14] A. Hedayat, E. Seiden, and J. Stufken. On the maximal number of factors and the enumeration of 3-symbol orthogonal arrays of strength 3 and index 2. J. Stat. Plann. Inference, 58(1):43–63, March 1997. DOI: 10.1016/S0378-3758(96)00059-6.

[15] O. Heden and C. Roos. The non-existence of some perfect codes over non-prime power alphabets. Discrete Math., 311(14):1344–1348, 2011. DOI: 10.1016/j.disc.2011.03.024.

[16] J. Y. Hyun. A bound on equitable partitions of the Hamming space. IEEE Trans. Inf. Theory, 56(5):2109–2111, May 2010. DOI: 10.1109/TIT.2010.2043773.

[17] J. Y. Hyun. Local duality for equitable partitions of a Hamming space. J. Comb. Theory, Ser. A, 119(2):476–482, Feb. 2012. DOI: 10.1016/j.jcta.2011.10.006.

[18] D. Krotov, J. Koolen, and W. Martin. Completely regular codes: Tables. Online: https://sites.google.com/site/completelyregularcodes/.

[19] D. S. Krotov. On the binary codes with parameters of doubly-shortened 1-perfect codes. Des. Codes Cryptography, 57(2):181–194, 2010. DOI: 10.1007/s10623-009-9360-5.

[20] D. S. Krotov. On weight distributions of perfect colorings and completely regular codes. Des. Codes Cryptography, 61(3):315–329, 2011. DOI: 10.1007/s10623-010-9479-4.

[21] D. S. Krotov. On the binary codes with parameters of triply-shortened 1-perfect codes. Des. Codes Cryptography, 64(3):275–283, 2012. DOI: 10.1007/s10623-011-9574-1.
[22] D. S. Krotov. On calculation of the interweight distribution of an equitable partition. *J. Algebr. Comb.*, 40(2):373–386, 2014. DOI: 10.1007/s10801-013-0492-3

[23] D. S. Krotov. On the OA(1536,13,2,7) and related orthogonal arrays. *Discrete Math.*, 343:paper 111659, 2020. DOI: 10.1016/j.disc.2019.111659

[24] D. S. Krotov, I. Yu. Mogilnykh, and A. Yu. Vasil’eva. On completely regular codes of covering radius 1 in the halved hypercubes. E-print 1812.03159, arXiv.org, 2018.

[25] D. S. Krotov and V. K. Vorob’ev. On unbalanced Boolean functions attaining the bound $2n/3 - 1$ on the correlation immunity. E-print 1812.02166v2, arXiv.org, 2018.

[26] S. P. Lloyd. Binary block coding. *Bell Syst. Tech. J.*, 36(2):517–535, 1957. DOI: 10.1002/j.1538-7305.1957.tb02410.x

[27] I. A. Matkin. Cameron-Liebler line classes in $PG(n,5)$. *Trudy Instituta Matematiki i Mekhaniki URO RAN*, 24(2):158–172, 2018. DOI: 10.21538/0134-4889-2018-24-2-158-172

[28] A. Neumaier. Completely regular codes. *Discrete Math.*, 106-107:353–360, Sept. 1992. DOI: 10.1016/0012-365X(92)90565-W

[29] V. N. Potapov. On perfect colorings of Boolean $n$-cube and correlation immune functions with small density. *Sib. Ehlektron. Mat. Izv.*, 7:372–382, 2010. In Russian, English abstract.

[30] V. N. Potapov. On perfect 2-colorings of the $q$-ary $n$-cube. *Discrete Math.*, 312(6):1269–1272, 2012. DOI: 10.1016/j.disc.2011.12.004

[31] S. A. Puzynina. On periodicity of perfect colorings of the infinite hexagonal and triangular grids. *Sib. Math. J.*, 52(1):91–104, 2011. DOI: 10.1134/S0037446609010101 translated from *Sib. Mat. Zh.*, 52(1):115–132, 2011.

[32] H. S. Shapiro and D. L. Slotnick. On the mathematical theory of error correcting codes. *IBM J. Res. Dev.*, 3(1):25–34, 1959.

[33] A. Yu. Vasil’eva. Local and interweight spectra of completely regular codes and of perfect colorings. *Probl. Inf. Transm.*, 45(2):151–157, 2009. DOI: 10.1134/S0033946509020069 translated from *Probl. Peredachi Inf.*, 45(2):84-90, 2009.

[34] A. Yu. Vasil’eva. Local distributions for eigenfunctions and perfect colorings of $q$-ary hamming graph. *Des. Codes Cryptography*, 87(2-3):509–516, Mar. 2019. DOI: 10.1007/s10623-018-0559-1