ON INJECTIVITY OF MAPS BETWEEN GROTHENDIECK GROUPS INDUCED BY COMPLETION

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Abstract. We give an example of a local normal domain $R$ such that the map of Grothendieck groups $G(R) \to G(\hat{R})$ is not injective. We also raise some questions about the kernel of that map.

1. Introduction

Let $(R, m, k)$ be a local ring and $\hat{R}$ the $m$-adic completion of $R$. Let $\mathcal{M}(R)$ be the category of finitely generated $R$-modules. The Grothendieck group of finitely generated modules over $R$ is defined as:

$$G(R) = \bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]$$

$$\left( \langle [M_2] - [M_1] - [M_3] \mid 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is exact} \rangle \right)$$

In [KK], Kamoi and Kurano studied injectivity of the map $G(R) \to G(\hat{R})$ induced by flat base-change. They showed that such map is the injective in the following cases: 1) $R$ is Henselian, 2) $R$ is the localization at the irrelevant ideal of a positively graded ring over a field, or, 3) $R$ has only isolated singularity. Their results raise the question: Is the map between Grothendieck group induced by completion always injective?

In [Ho1], Hochster announced a counterexample to the above question:

**Theorem 1.1.** Let $k$ be a field. Let $R = k[x_1, x_2, y_1, y_2]/(x_1x_2 - y_1x_1^2 - y_2x_2^2)$. Let $P = (x_1, x_2)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \to G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.

Hochster’s example comes from the “direct summand hypersurface” in dimension 2 and is not normal. He predicted that there is also an example which is normal. The main purpose of this note is to provide such an example. We have:

**Proposition 1.2.** Let $R = \mathbb{R}[x, y, z, w]/(x^2 + y^2 - (w+1)z^2)$. $R$ is a normal domain. Let $P = (x, y, z)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \to G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.

This will be proved in Section 2. We note that Kurano and Srinivas has recently constructed an example of a local ring $R$ such that the map $G(R)_\mathbb{Q} \to G(\hat{R})_\mathbb{Q}$ is not injective (see [KS]). The ring in their example is not normal, and we do not know if a normal example exists in that context (i.e. with rational coefficients).

In section 3 we will discuss some questions on the kernel of the map $G(R) \to G(\hat{R})$.

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unpublished note [Ho2], which provided the key ideas for our example. We also
thank the referee for many helpful comments.

2. OUR EXAMPLE

We shall prove Proposition [Fo]. First we need to recall some classical results:

Corollary 2.1. (Swan, [SW], Corollary 11.8) Let $k$ be a field of characteristic not
$2$, $n > 1$ an integer and $R = k[x_1, ..., x_n]/(f)$ where $f$ is a non-degenerate quadratic
form in $k[x_1, ..., x_n]$. Then $G(R) = Z \oplus Z/2Z$ if $C_0(f)$, the even part of the Clifford
algebra of $f$, is simple.

Proposition 2.2. (Samuel, see [PS], Proposition 11.5) Let $k$ be a field of character-
istic not 2 and $f$ be a non-degenerate quadratic form in $k[x_1, x_2, x_3]$. Let
$R = k[x_1, x_2, x_3]/(f)$. If $f = 0$ has no non-trivial solution in $k$ then $Cl(R) = 0$.

Proposition 2.3. (Kamoi-Kurano) Let $S = \oplus_{n \geq 0} S_n$ be a graded ring over a field
$S_0$ and $S_+ = \oplus_{n > 0} S_n$. Let $A = S_{S_+}$. Then the map $G(S) \to G(A)$ induced by
localization is an isomorphism.

Proof. See the proof of Theorem 1.5 (ii) in [KK].

Proposition [Fo] now follows from the following Propositions (clearly, $R$ is normal,
since the singular locus $V(x, y, z)$ has codimension 2):

Proposition 2.4. $[\hat{M}] = 0$ in $G(\hat{R})$.

Proof. $\hat{R} = \mathbb{R}[[x, y, z, w]]/(x^2 + y^2 - (w + 1)z^2)$. We want to show that $\hat{R}/P\hat{R} = 0$
in $G(\hat{R})$. Let $\alpha = \sqrt{w} + 1$ which is a unit in $\hat{R}$. Let $Q = (x, y - \alpha z)\hat{R}$. Then clearly
$Q$ is a height 1 prime in $\hat{R}$ and $P\hat{R} = Q + (y + \alpha z)\hat{R}$. The short exact sequence:

$$0 \to \hat{R}/Q \to \hat{R}/Q \to \hat{R}/P\hat{R} \to 0$$

where the second map is the multiplication by $y + \alpha z$ shows that $[\hat{R}/P\hat{R}] = 0$ in
$G(\hat{R})$.

Proposition 2.5. $[\hat{M}] \neq 0$ in $G(\hat{R})$.

Proof. It is enough to show that $[M_P] \neq 0$ in $G(R_P)$. Let $K = \mathbb{R}(w)$ then $R_P \cong
K[x, y, z]_{(x, y, z)}/(f)$ where $f = x^2 + y^2 - (w + 1)z^2$. Let $S = K[x, y, z]/(f)$. Clearly
$f$ is a non-degenerate quadratic form. Since the rank of $f$ is 3, an odd number,
$C_0(f)$ is a simple algebra over $K$ (see, for example, [La], Chapter 5, Theorem 2.4). By [2.1]
and [2.3] $G(R_P) = G(S) = Z \oplus Z/(2)$. We claim that $f$ has no non-
trivial solution in $K$. Suppose it has. Then by clearing denominators, there are polynomials $a(w), b(w), c(w) \in \mathbb{R}[w]$ such that

$$a(w)^2 + b(w)^2 = (w + 1)c(w)^2.$$ 

The degree of $a(w)^2 + b(w)^2$ is always even. The degree of $(w + 1)c(w)^2$ is odd
unless $c(w) = 0$. But then $a(w)^2 + b(w)^2 = 0$ which forces $a(w) = b(w) = 0$,
a contradiction. By the claim and [2.2] $Cl(R_P) = Cl(S) = 0$. Thus $[R_P]$ and
$[R_P/P_R P]$ generate $G(R_P) = Z \oplus Z/(2)$ (since the Grothendieck group is generated
by $[R_P/Q], Q \in \text{Spec} R_P$ and $\dim R_P = 2$). Since $Z \oplus Z/(2)$ can not be generated
by one element, $[R_P/P_R P]$ must be nonzero (it is easy to see that $[R_P/P_R P]$ is
2-torsion). 

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3. On the kernel of the map $G(R) \to G(\hat{R})$

In this section we raise some questions about the kernel of the map $G(R) \to G(\hat{R})$. First we fix some notations. Throughout this section we will assume, for simplicity, that $R$ is excellent, and is a homomorphic image of a regular local ring $T$. Let $d = \dim R$. Let $A_i(R)$ be the $i$-th Chow group of $R$, i.e.,

$$A_i(R) = \bigoplus_{P \in \text{Spec} R, \dim R/P = i} \mathbb{Z} \cdot [\text{Spec} R/P] \langle \text{div}(Q, x) \mid Q \in \text{Spec} R, \dim R/Q = i + 1, \ x \in R\backslash Q \rangle$$

where

$$\text{div}(Q, x) = \sum_{P \in \text{Min} R/((Q, x)R)} l_{RP} (R_P/(Q, x)R_P)[\text{Spec} R/P].$$

For an abelian group $A$, we let $A \otimes \mathbb{Q}$. The Chow group of $R$ is defined to be $A_*(R) = \bigoplus_{i=0}^{d} A_i(R)$. It is well known that there is a $\mathbb{Q}$-vector space isomorphism:

$$\tau_{R/T} : G(R)_{\mathbb{Q}} \to A_*(R)_{\mathbb{Q}}$$

(It is unknown whether this is independent of $T$). We also remark that the Grothendieck group $G(R)$ admits a filtration by $F_i G(R) = \{ [M] \in G(R) \mid \dim M \leq i \}$.

The existing examples on the failure of injectivity for the map $G(R) \to G(\hat{R})$ and the affirmative results in [KK] motivate the following question:

**Question 3.1.** Suppose that $R$ satisfies $(R_n)$ (i.e. regular in codimension $n$). Then is $\ker(G(R) \to G(\hat{R}))$ contained in $F_{d-n-1} G(R)$?

Question 3.1 is closely related to the following:

**Question 3.2.** Suppose that $R$ satisfies $(R_n)$. Then is the map $A_i(R) \to A_i(\hat{R})$ injective for $i \geq d - n$?

In fact, if we allow rational coefficients, then the previous questions are equivalent. Let $G^i(R) = F_i G(R)/F_{i-1} G(R)$. Then clearly we have a decomposition:

$$G(R)_{\mathbb{Q}} = \bigoplus_{i=0}^{d} G^i(R)_{\mathbb{Q}}$$

Also, the Riemann-Roch map decomposes into isomorphisms $\tau^i : G^i(R)_{\mathbb{Q}} \to A_i(R)_{\mathbb{Q}}$, which make the following diagram:

$$\begin{align*}
G^i(R) & \xrightarrow{\tau^i_{R/T}} A_i(R) \\
\downarrow g_i & \quad \downarrow f_i \\
G^i(\hat{R}) & \xrightarrow{\tau^i_{\hat{R}/T}} A_i(\hat{R})
\end{align*}$$

commutative. It follows that

$$\ker(G(R)_{\mathbb{Q}} \to G(\hat{R})_{\mathbb{Q}}) \cong \bigoplus_{i=0}^{d} \ker(f_i) \cong \bigoplus_{i=0}^{d} \ker(g_i).$$

Thus we have:
Proposition 3.3. Let $R$ be an excellent local ring which is a homomorphic image of a regular local ring. Let $\dim R = d$ and let $0 < l \leq d$ be an integer. Then the maps $A_i(R)_{\mathbb{Q}} \to A_i(\hat{R})_{\mathbb{Q}}$ are injective for $i \geq l$ if and only if $\ker(G(R)_{\mathbb{Q}} \to G(\hat{R})_{\mathbb{Q}}) \subseteq F_{l-1} G(R)_{\mathbb{Q}}$.

We do not know if 3.2 is true even if $l = 1$. Note that if $R$ is normal, then both 3.1 and 3.2 are true for $l = 1$. In that situation $A_1(R) \cong Cl(R)$, and the map between class groups of $R$ and $\hat{R}$ is injective. Furthermore, it is well known that $G(R)/F_{d-2} G(R) \cong A_d(R) \oplus A_{d-1}(R)$ (see, for example [Ch], Corollary 1), so 3.1 is also true for $l = 1$.

Finally, one could formulate a stronger version of 3.1 as follows. Note that in both Hochster’s example and the example presented here, the support of the modules given actually equal to the singular locus of $R$. So one could ask:

**Question 3.4.** Let $R$ be an excellent local ring. Let $X = \text{Spec } R$, $Y = \text{Sing}(X)$, $\hat{X} = \text{Spec } \hat{R}$ and $\hat{Y} = \text{Sing}(\hat{X})$. One then has a commutative diagram:

$$
\begin{array}{ccc}
G(Y) & \longrightarrow & G(X) \\
\downarrow & & \downarrow \\
G(\hat{Y}) & \longrightarrow & G(\hat{X})
\end{array}
$$

(Here $G(X)$ denotes the Grothendieck group of coherent $O_X$-modules and the maps are naturally induced by closed immersions or flat morphisms). Is $\ker(g)$ contained in $\text{im}(f)$?

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