Distribution-Free Runs-Based Control Charts

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Abstract

We propose distribution-free runs-based control charts for detecting location shifts. Using the fact that given the number of total successes, the outcomes of a sequence of Bernoulli trials are random permutations, we are able to control the conditional probability of a signal detected at current time given that there is not alarm before at a pre-determined level. This leads to a desired in-control average run length and data-dependent control limits. Two common runs statistics, the longest run statistic and the scan statistic, are studied in detail and their exact conditional distributions given the number of total successes are obtained using the finite Markov chain imbedding technique. Numerical results are given to evaluate the performance of the proposed control charts.

Keywords: Distribution-free, control charts, runs and patterns, longest run, scan statistic, finite Markov chain imbedding

1. Introduction

Statistical process control (SPC) has been demonstrated to be an effective technique to monitor quality characteristics of a process. The main task of a control chart is to detect the location shift of a process as quickly as possible. The typical setting is as follows: suppose there are \( m \) historical independent and identically distributed (i.i.d.) data \( X_{-m+1}, \ldots, X_0 \), the future data are collected according to the change-point model

\[
X_t = \begin{cases} 
F_0(x; \mu_0) & \text{for } t = -m + 1, \ldots, 0, 1, \ldots, \tau - 1, \\
F_1(x; \mu_1) & \text{for } t = \tau, \tau + 1, \ldots, 
\end{cases}
\]

where \( \tau \) is the unknown change point. In a typical control chart, the plotted data points are assumed to follow a parametric distribution \( F_0 \) such as normal distribution. The problem is clear and that is if the distributional assumption is not valid, then the promised characteristic of the control chart, such as in-control average run length (ARL), can no longer be reliable. In other words, such control charts are not robust to the underlying distributions. It has been shown in recent applications that the underlying distribution might not be normal. New charts need to be designed without distributional assumptions. Many nonparametric or distribution-free control charts have been proposed for this purpose to control the desired in-control ARL. And this is also the main objective of this manuscript to develop distribution-free control charts for detecting mean shifts.

To control the in-control ARL for every continuous probability distribution, nonparametric or distribution-free control charts have received considerable attention in the past.
decade. The main stream of nonparametric or distribution-free control chart consists of rank-based control charts and likelihood-based control charts. Those nonparametric control charts can be classified into three types: exponentially weighted moving average (EWMA), cumulative sum (CUSUM) and runs-type control charts. The classic Shewhart control chart can be considered as a special case of runs-type control charts. EWMA and CUSUM are popular because they are superior in detecting small mean shifts. Examples of nonparametric EWMA control charts can be found in the papers of Zou and Tsung (2010), Amin and Searcy (1991) and Yang and Cheng (2011). Nonparametric CUSUM control charts are studied by, for example, Chatterjee and Qiu (2009), Chowdhury et al. (2015) and Yang and Cheng (2011). A comprehensive review of nonparametric control charts is given by Chakraborti et al. (2001).

In practice, we prefer the Shewhart control chart since it is easy to understand and interpret. Runs rules have also been used to enhance the Shewhart control chart for small mean shifts. It has been shown that runs-based, for example scan statistic (Shmueli and Cohen (2003)), control charts can increase the power of the classic Shewhart control charts for detecting small mean shifts. However, the applicability is limited and the performance is not guaranteed if normality or some parametric models are assumed (see, e.g., Champ and Woodall (1987), Koutras et al. (2007)). To release this restriction, Chakraborti et al. (2009) proposed non-parametric control charts based on precedence statistics with runs rules. Their charts signal when two consecutive plotting points, for example median, fall outside the control limits. Another result based on Wilcoxon signed-rand statistic with some runs type rules is given by Chakraborti and Eryilmaz (2007). Compared to nonparametric EWMA and CUSUM control cahrts, this area seems to be underdeveloped. We believe that part of the reason might be a lack of methods for handling general runs without any distributional assumptions.

The finite Markov chain imbedding (FMCI) technique has been used to evaluate the exact characteristics, such as ARL and its standard deviation, of various control charts, including Shewhart, EWMA and CUSUM control charts where the normality is assumed (see, e.g., Fu et al. (2002, 2003)). In this manuscript, we propose distribution-free control charts with run rules and the ARLs are obtained exactly by the FMCI technique. The idea is to view Bernoulli trials given the total number of successes as random permutations. The idea coincides with other methods where they proposed statistics whose conditional distributions are independent of the unknown parameter and/or the underlying distribution (see, e.g., Chen et al. (2016)).

The objective of this manuscript is to construct distribution-free control charts controlling exactly the desired in-control ARL. We first control the conditional probability of a signal detected at current time given that there is not alarm before at a pre-determined level $\alpha$. This leads to a desired in-control ARL and data-dependent control limits. The rest of this manuscript is organized as follows. The exact conditional distributions of the scan statistic and the longest run is given in Section 2. The charting procedures for distribution-free runs-based and scan-based control charts are given in Section 3. Numerical results are given in Section 4. Discussion and summary are given in Section 5.
2. Conditional distributions

2.1. Conditional scan statistics

Let \( X_1, \ldots, X_n \) be a sequence of i.i.d. Bernoulli trials with \( p_i = P(X_i = 1) \). The scan statistic is defined as

\[
S_n(r) = \max_{1 \leq t \leq n} S_n(r, t),
\]

where \( S_n(r, t) = \sum_{i=t}^{t+r-1} X_i \) and \( r \) is the window size. Using the FMCI technique, the exact conditional distributions of scan statistics have been derived by Fu et al. (2012) in a sequence of Bernoulli trials. A brief introduction of the FMCI technique is given here. This prepares what is needed for the conditional distribution of the longest run in the next section. Details can be found in the paper of Fu et al. (2012).

The distributions of scan statistics is calculated based on the FMCI technique through the dual relationship with the waiting time distribution of a compound pattern. An example is given below.

**Example 1.** Consider \( r = 5 \) and \( s = 2 \), the compound pattern associated with the event \( \{ S_n(5) < 2 \} \) is \( \Lambda_{5,2} = \{11, 101, 1001, 10001\} \). Thus, the probability \( P(S_n(5) < 2) \) can be obtained by \( P(S_n(5) < 2) = P(W(\Lambda_{5,2}) > n) \). In the conditional case, given the total number \( m \) of successes the dual relationship is still true, i.e.

\[
P \left( S_n(5) < 2 \mid \sum_{i=1}^{n} X_i = m \right) = P \left( W(\Lambda_{5,2}) > n \mid \sum_{i=1}^{n} X_i = m \right).
\]

Given \( r \) and \( s \), we define a set of simple patterns of lengths no longer than \( r \), denoted by

\[
\Lambda_{r,s} = \{\Lambda_i, i = 1, \ldots, \ell\},
\]

where \( \Lambda_i \) is a simple pattern that begins and ends with 1 and contains total \( s \) 1’s, and \( \ell \) is the total number of simple patterns corresponding the scan statistic with parameters \( r \) and \( s \). It has been shown that for each scan statistic probability \( P(S_n(r) < s) \) the total number of simple patterns is

\[
\ell = \sum_{\nu=0}^{r-s} \binom{s - 2 + \nu}{\nu}.
\]

Let

\[
P = \left\{ \pi = (\pi_1, \ldots, \pi_n) : \pi_i = 0, 1 \text{ and } \sum_{i=1}^{n} \pi_i = m \right\}
\]

be the family of random permutations with \( m \) 1’s and \( n - m \) 0’s. Then, the conditional distributions of runs and patterns given the total number \( m \) of successes for a sequence of \( n \) Bernoulli trials are the same as the distributions of runs and patterns in an \([n-m, m]\)-specified random permutation \( \pi = (\pi_1, \ldots, \pi_n) \). It is worth to mention that in an \([n-N, N]\)-specified random permutation, the distributions of runs and patterns are independent of \( p \).
To construct a finite Markov chain \( \{Y_t\} \), we first define \( E_{r,s} \) as a set of all subpatterns of \( \Lambda_{r,s} \). Then, an imbedded Markov chain can be defined on the state space

\[
\Omega = \{(l, \omega) : l = 0, 1, \ldots, m \text{ and } \omega \in E_{r,s}\} \cup \{\emptyset, \alpha\},
\]

where \( \emptyset \) is the initial state and \( \alpha \) is the absorbing state meaning that the compound pattern \( \Lambda_{s,r} \) occurs if the chain enters the absorbing state. From Fu et al. (2012), the transition probabilities from state \( u = (m_{t-1}, \omega_{t-1}) \) to state \( v = (m_t, \omega_t) \) are given by

\[
p_{uv}(t) = P(Y_t = (m_t, \omega_t) | Y_{t-1} = (m_{t-1}, \omega_{t-1})),
\]

\[
= \begin{cases} 
N_{t-1} - m_{t-1} & \text{if } \pi_t = 1, m_t = m_{t-1} + 1 \text{ and } \omega_t = < \omega_{t-1}, 1 >_{E_{r,s}}, \\
n - N_t + m_{t-1} + 1 & \text{if } \pi_t = 0, m_t = m_{t-1} \text{ and } \omega_t = < \omega_{t-1}, 0 >_{E_{r,s}}, \\
1 & \text{if } \omega_t = \omega_{t-1} = \alpha, \\
0 & \text{if otherwise},
\end{cases}
\]

where \( < \omega_{t-1}, \pi_t >_E \) denotes the longest subpattern after \( \pi_t \) is observed. Thus, the transition matrices are of the form:

\[
M_t^1(m) = \begin{bmatrix} \sum_{i=1}^{n} X_i = m \\ 0 \\ \sum_{i=1}^{n} X_i = m \end{bmatrix}.
\]

Based on the above construction, the waiting time random variables \( W(\Lambda) \) is finite Markov chain imbeddable (Fu et al. (2003)) and the exact distribution is given by the following theorem.

\textbf{Theorem 1} (Fu et al. (2012)). Let \( X_1, \cdots, X_n \) be a sequence of independent Bernoulli trials. Then

\[
P \left( S_n(r) < s \Big| \sum_{i=1}^{n} X_i = m \right) = P \left( W(\Lambda_{r,s}) > n \Big| \sum_{i=1}^{n} X_i = m \right) = \xi_0 \prod_{t=1}^{n} N_t^1(m)1^T,
\]

where \( N_t^1(m), t = 1, \ldots, n, \) are the essential matrices whose entries are given in (4).

One can see that the conditional distribution in (6) is independent of \( p \).

\subsection{2.2. Conditional longest run statistic}

Let \( L_n \) denote the length of the longest run of 1 in a sequence of Bernoulli trials. Consider the run

\[
\Lambda_d = 1 \cdots 1.
\]

The event \( \{L_n < d\} \) occurs if and only if the run \( \Lambda_d \) does not appear in the sequence of Bernoulli trials. Thus, we have

\[
P(L_n < d) = P(W(\Lambda_d) > n).
\]

The above equation also holds for conditional distribution given the total number of successes.
Following the construct of the imbedded Markov chain for the scan statistic in the previous section, we can similarly define a set of subpatterns of $\Lambda_d$ as

$$E_d = \{1, 11, \ldots, \underbrace{1\ldots 1}_{d-1}\}.$$ 

Then, we can define a finite nonhomogeneous Markov chain $\{Y_t\}$ on the state space

$$\Omega = \{(l, \omega) : l = 0, 1, \ldots, m \text{ and } \omega \in E_d \cup \{\emptyset, \alpha\}\}.$$  \hfill (8)

And the transition probabilities from state $u = (m_{t-1}, \omega_{t-1})$ to state $v = (m_t, \omega_t)$ are given by

$$p_{uv}(t) = P(Y_t = (m_t, \omega_t) \mid Y_{t-1} = (m_{t-1}, \omega_{t-1})),$$

$$= \begin{cases} 
\frac{N-m_{t-1}}{n} & \text{if } \pi_t = 1, m_t = m_{t-1} + 1 \text{ and } \omega_t = < \omega_{t-1}, 1 >_{E_d}, \\
\frac{n-t+1}{n} & \text{if } \pi_t = 0, m_t = m_{t-1} \text{ and } \omega_t = < \omega_{t-1}, 0 >_{E_d}, \\
1 & \text{if } \omega_t = \omega_{t-1} = \alpha, \\
0 & \text{otherwise.} 
\end{cases}$$ \hfill (9)

Again, the transition matrices are of the form:

$$M^2_t(m) = \begin{bmatrix} N^2_t(m) & C^2_t(m) \\ 0 & 1 \end{bmatrix}.$$ \hfill (10)

The exact conditional distribution of $L_n$ is then given by

$$P\left(L_n < d \mid \sum_{i=1}^n X_i = m\right) = \xi_0 \prod_{t=1}^n N^2_t(m)1^\top.$$ 

It is worthwhile to mention that the conditional longest run statistics has also been studied by Lou (1996). However, our procedure of creating the finite Markov chain is different. It can be clearly seen from our construction of the finite Markov chain that the formula of the conditional distribution is independent of $p$, but Lou’s formula involves $p$, although the conditional distribution does not depend on $p$. Lou’s formula has an advantage that it allows the Bernoulli trials to be Markov dependent.

**Example 2.** Suppose that $n = 5, m = 3$ and $d = 3$. The pattern corresponding to $\{L_5 < 3\}$ is $\Lambda_3 = 111$. The ending block is $E_3 = \{1, 11\}$ and $E_3 \cup S = \{0, 1, 11\}$. The state space can be constructed according to (8) and is given by $\Omega = \{(0,0),(1,0),(1,1),(2,0),(2,1),(2,11),(3,0),(3,1),(3,11)\} \cup \{\emptyset, \alpha\}$. Some redundant states are removed from the state space. For example, the state $(1, 11)$ would never occur. Using Theorem 1, the probability that the length of the longest 1’s run is less than 3 is 0.7. The same probability can be obtained by enumeration. Given 3 successes (1’s) in 5 Bernoulli trials, there are 10 possible outcomes $\{11100,11010,10110,01110,11001,10101,01101,10011,01011,00111\}$. Apparently, seven outcomes $\{11010,10110,11001,10101,01101,10011,01011\}$ contain the longest run of length less than three, and hence the probability is $7/10$. 

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3. Distribution-free run-based control charts

In this section, we propose new distribution-free runs-based control charts with any underlying process distribution \( F_0 \). Note that we describe our methodology for one-sided control charts for detecting upward shifts throughout this paper. But it can also be applied to control charts with lower and upper limits for detecting downward and upward shifts.

Let \( Y_1, \ldots, Y_n \) be the sequential observations from an unknown underlying distribution \( F_0 \). In an one-sided control charts, a label “1” is assigned to an observation if its value is greater than a certain threshold and “0”, otherwise. I.e., we define

\[
X_t = \begin{cases} 
1 & \text{if } Y_t \geq c, \\
0 & \text{if } Y_t < c.
\end{cases}
\]  

(11)

The scan statistic can then be defined as

\[
S_n(r) = \max_{1 \leq i \leq n} S_n(r, i),
\]  

(12)

where \( S_n(r, i) = \sum_{t=i}^{i+r-1} X_t \) and \( r \) is the window size. It has been widely studied, that runs and scans rules can enhance the ability of a Shewhart-type control chart for detecting small shifts. When \( r = 1 \), the control charts with scans rule are equivalent to the conventional Shewhart control charts which are good for detecting large shifts. It is intuitive that one can use a large \( r \) with a smaller threshold \( c \) for small shifts. This idea has been verified by our numerical study.

A general runs and patterns rules, denoted by \( R(k, r, Z) \) is proposed in Shmueli and Cohen (2003). We describe the rules in our notations below: the control chart signals an out-of-control alert, if \( k \) of the last \( r \) tested points fall in the region \( Z \). Here, we consider two types of rules for various values of \( k, r \) and \( c \):

1. R-1 = \( T(k, r, (c, \infty)) \): scans rule
2. R-2 = \( T(k, k, (c, \infty)) \): runs rule

Let \( N_n = \sum_{i=1}^{n} X_i \). Given the total number of 1’s, distribution-free control charts with data-dependent control limits can be constructed with runs rules. Let \( L_n \) denoted the length of the longest 1’s run in a sequence of Bernoulli trials. For a pre-specified \( \alpha \), the charting procedure for the two distribution-free control charts are given below.

The probability in (13) can be calculated using Theorem 1. It follows from Theorem 1 and Bayes’ rule that

\[
P(S_n(r) < c_n(\alpha)|S_{n-1}(r) < c_{n-1}(\alpha), N_n = m) \\
\quad = \frac{P(S_n(r) < c_n(\alpha), S_{n-1}(r) < c_{n-1}(\alpha), N_n = m)}{P(S_{n-1}(r) < c_{n-1}(\alpha), N_n = m)} \\
\quad = \frac{P(S_n(r) < c_n(\alpha), S_{n-1}(r) < c_{n-1}(\alpha)|N_n = m)}{P(S_{n-1}(r) < c_{n-1}(\alpha)|N_n = m)}.
\]
We have that
\[
\begin{align*}
&\text{if } c_n(\alpha) > c_{n-1}(\alpha), \text{ then } \\
&P(S_n(r) < c_n(\alpha)|S_{n-1}(r) < c_{n-1}(\alpha), N_n = m) = 1, \text{ and } \\
&\text{if } c_n(\alpha) \leq c_{n-1}(\alpha), \text{ then } \\
&P(S_n(r) < c_n(\alpha), S_{n-1}(r) < c_{n-1}(\alpha)|N_n = m) = P(S_n(r) < c_n(\alpha)|N_n = m).
\end{align*}
\]

Hence,
\[
P(S_n(r) < c_n(\alpha)|S_{n-1}(r) < c_{n-1}(\alpha), N_n = m) =
\begin{cases} 
1 & \text{if } c_n(\alpha) > c_{n-1}(\alpha), \\
\frac{P(S_n(r)<c_n(\alpha)|N_n=m)}{P(S_{n-1}(r)<c_{n-1}(\alpha)|N_n=m)} & \text{if } c_n(\alpha) \leq c_{n-1}(\alpha).
\end{cases}
\]

We need to rewrite the probability in (15) as
\[
P(S_{n-1}(r) < c_{n-1}(\alpha)|N_n = m) =
\begin{align*}
&= \frac{P(S_{n-1}(r) < c_{n-1}(\alpha), N_n = m)}{P(N_n = m)} \\
&= \frac{P(N_{n-1} = m, X_n = 0)}{P(N_n = m)} P(S_{n-1}(r) < c_{n-1}(\alpha)|N_{n-1} = m) \\
&\quad + \frac{P(N_{n-1} = m-1, X_n = 1)}{P(N_n = m)} P(S_{n-1}(r) < c_{n-1}(\alpha)|N_{n-1} = m - 1) \\
&= P(N_{n-1} = m, X_n = 0|N_n = m) P(S_{n-1}(r) < c_{n-1}(\alpha)|N_{n-1} = m) \\
&\quad + P(N_{n-1} = m-1, X_n = 1|N_n = m) P(S_{n-1}(r) < c_{n-1}(\alpha)|N_{n-1} = m - 1),
\end{align*}
\]
where
\[
P(N_{n-1} = m, X_n = 0|N_n = m) = (n - m)/n \quad \text{and} \quad P(N_{n-1} = m - 1, X_n = 1|N_n = m) = m/n.
\]
Similarly, the charting procedure for the run rule R-2 control chart is given as follow.

**Algorithm 1: Scan rule control chart (R-1)**

1. Given a pre-specified $\alpha$, the control limits $\{c_n(\alpha)\}$ are determined by the following equations:
\[
P(S_n(r) \geq c_n(\alpha)|N_n) \leq \alpha, \text{ for } n = \nu, \\
P(S_n(r) \geq c_n(\alpha)|S_{n-1}(r) < c_{n-1}(\alpha), N_n) \leq \alpha, \text{ for } n > \nu.
\] (13)

2. Based on $\{c_n(\alpha)\}$ found in step 1., the run length is given by
\[
RL = \min\{n : S_n(\nu) \geq c_n(\alpha), n \geq \nu\}.
\] (14)
Algorithm 2: Longest run rule control chart (R-2)

1. Given a pre-specified $\alpha$ and a smaller integer $v$, the control limits $\{k_n(\alpha)\}$ are determined by the following equations:

$$
P(L_n \geq k_n(\alpha)|N_n) \leq \alpha, \text{ for } n = v, $$

$$
P(L_n \geq k_n(\alpha)|L_{n-1} < k_{n-1}(\alpha), N_n) \leq \alpha, \text{ for } n > v. $$

(16)

2. Based on $\{k_n(\alpha)\}$ found in step 1., the run length is given by

$$
RL = \min\{n : L_n \geq k_n(\alpha), n \geq 1\}. $$

(17)

The probability in (16) is given by

$$
P(L_n(r) < k_n(\alpha)|L_{n-1}(r) < k_{n-1}(\alpha), N_n = m)
= \begin{cases} 
1 & \text{if } k_n(\alpha) > k_{n-1}(\alpha), \\
\frac{P(L_n(r) < k_n(\alpha)|N_n = m)}{P(L_{n-1}(r) < k_{n-1}(\alpha)|N_n = m)} & \text{if } k_n(\alpha) \leq k_{n-1}(\alpha).
\end{cases} $$

(18)

We denote RL$_0$, RL$_1$, ARL$_0$ and ARL$_1$ as the in-control run length, out-of-control run length, in-control average run length and out-of-control average run length. Note that in Algorithm 2, the probability $P(L_1 \geq k_1(\alpha))$ is either 0 or 1 depending on the value of $N_1$. Hence, it is suggested that the proposed longest run rule control chart should start monitoring process after a small number of observations (or samples) has been collected so that an appropriate $\alpha$ can be selected. By controlling the conditional probabilities in (13) and (16), the RL$_0$ of our proposed control charts follows the geometric distribution given in the following.

**Proposition 1.** Suppose the process is in-control, we have $P(RL_0 = n) = \alpha(1 - \alpha)^{n-1}$ for $n \geq 1$ and for any underlying distribution.

We omit the proof since it is similar as the proof of Theorem 1 in the paper of Chen et al. (2016). Since the distribution of RL$_0$ is the geometric distribution with parameter $\alpha$, the ARL$_0$ is equal to $1/\alpha$.

3.1. Randomized test and computational issues

The respective control limits $\{c_n\}$ and $\{k_n\}$ for scan rule and longest run rule control charts, needs to be determined for a given $\alpha$ or ARL. The scan and longest run statistics are discrete, and hence to maintain an exact pre-specified $\alpha$ or in-control ARL, randomized tests are needed. In what follows, we will show the detailed steps to determine $\{k_n\}$. Then, the control limits $\{c_n\}$ can be determined in a similar way.

For a given $\alpha$, define a randomized test at time $n$ as

$$
\phi_n = \begin{cases} 
1 & \text{if } L_n > k_n(\alpha), \\
\nu_n & \text{if } L_n = k_n(\alpha), \\
0 & \text{otherwise}.
\end{cases} $$

(19)
The probability in (16) becomes
\[
P(\text{reject } H_0 \text{ at time } n \mid \text{ accept } H_0 \text{ at time } n-1, N_n = m) = \frac{P(\text{reject } H_0 \text{ at time } n, \text{ accept } H_0 \text{ at time } n-1 \mid N_n = m)}{P(\text{ accept } H_0 \text{ at time } n-1 \mid N_n = m)}.
\]

By using (19), we have
\[
P(\text{reject } H_0 \text{ at time } n, \text{ accept } H_0 \text{ at time } n-1 \mid N_n = m) = E(\phi_n(1 - \phi_{n-1}) \mid N_n = m)
= E(\phi_n \mid N_n = m) - E(\phi_n \phi_{n-1} \mid N_n = m),
\]
where \(E(\phi_n \mid N_n = m) = P(L_n > k_n(\alpha) \mid N_n = m) + \nu_n P(L_n = k_n(\alpha) \mid N_n = m)\) and
\[
E(\phi_n \phi_{n-1} \mid N_n = m)
= P(L_n > k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N - n = m)
+ \nu_n P(L_n = k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N - n = m)
+ \nu_{n-1} P(L_n > k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N - n = m)
+ \nu_n \nu_{n-1} P(L_n = k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N - n = m).
\]

Using a conditional argument and Theorem 1, we can compute
\[
P(L_n = k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N_n = m)
= \frac{P(L_n = k_n(\alpha), L_{n-1} = k_{n-1}(\alpha), N_n = m)}{P(N_n = m)}
= \frac{P(L_n = k_n(\alpha) \mid L_{n-1} = k_{n-1}(\alpha), N_n = m) P(L_{n-1} = k_{n-1}(\alpha), N_n = m)}{P(N_n = m)}
= P(L_n = k_n(\alpha) \mid L_{n-1} = k_{n-1}(\alpha), N_n = m) P(L_{n-1} = k_{n-1}(\alpha) \mid N_n = m).
\]

To specific, the probability \(P(L_{n-1} = k_{n-1}(\alpha) \mid N_n = m)\) can be computed using Theorem 1, and the probability \(P(L_n = k_n(\alpha) \mid L_{n-1} = k_{n-1}(\alpha), N_n = m)\) can be computed by finding the one-step transition probability with the initial probability \(P(L_{n-1} = k_{n-1}(\alpha) \mid N_n = m)\). Note that the value of \(k_n\) can only be either \(k_{n-1}\) or \(k_{n-1}+1\). Hence, if \(k_n(\alpha) = k_{n-1}(\alpha) + 1 = k + 1\), then
\[
P(L_n > k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N_n = m) = P(L_{n-1} > k \mid N_n = m)
\]
\[
P(L_n = k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N_n = m) = 0
\]
\[
P(L_n > k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N_n = m) = P(L_n > k, L_{n-1} > k - 1 \mid N_n = m) - P(L_{n-1} > k \mid N_n = m)
\]
\[
P(L_n = k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N_n = m) = P(L_n = k, L_{n-1} = k \mid N_n = m),
\]
and if \(k_n(\alpha) = k_{n-1}(\alpha) + 1 = k + 1\), then
\[
P(L_n > k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N_n = m) = P(L_n > k + 1, L_{n-1} > k \mid N_n = m)
\]
\[
P(L_n = k_n(\alpha), L_{n-1} > k_{n-1}(\alpha) \mid N_n = m) = P(L_{n-1} > k \mid N_n = m) - P(L_n > k + 1, L_{n-1} > k \mid N_n = m)
\]
\[
P(L_n > k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N_n = m) = 0
\]
\[
P(L_n = k_n(\alpha), L_{n-1} = k_{n-1}(\alpha) \mid N_n = m) = P(L_n = k + 1, L_{n-1} = k \mid N_n = m),
\]
3.2. Choice of k, r and c

In this section, we suggest the appropriate combinations of parameters k, r and c for detecting either small shift or large shift. In an unconditional scan-based control chart, the scan rule is used to enhance the control chart for small shift and when \( r = k = 1 \) the control chart reduces to a standard Shewhart control chart. In general, the combination of small \( r \) and large \( c \) is used to detect a large shift and the combination of large \( r \) and small \( c \) is used to detect a small shift.

Although the threshold \( c \) is not explicitly involved in the charting procedure, but it plays an important role in the performance of the control charts. If a small \( c \) is selected, then we tend to see a large number of 1’s. In this case, noise might mask the true signal. On the contrary, if a large \( c \) is selected, a small number of 1’s will be expected. In this case, the true signal might not be picked up due to the high threshold. For the ease of setting up control charts with an appropriate ARL\(_0\), we suggest to use small \( c \) and moderate \( k \) and \( r \). Such setting also increases the sensitivity for small shift.

4. Numerical results

To evaluate the performance of the two charting procedures introduced in Section 3, we set \( \alpha = 0.05 \) giving the in-control ARL = 200. For out-of-control ARLs, 1000 sequences are generated from normal distributions \( N(\mu, 1) \). Here, \( \mu \) is the mean difference between in-control and out-of-control processes. In the distribution-free control chart literature, it is usually assumed that there are \( m_0 \) historical (reference) data from the process that is in-control. We let \( m_0 = 20 \).

Table 1 gives the out-of-control ARLs of the R-2 control chart under three different mean shifts and three choices of \( c \). The outcome is apparent that the charting procedure won’t pick up the signal if the threshold \( c \) is much larger than \( \mu \). On the other hand, there will be too much noise if the threshold \( c \) is much smaller than \( \mu \). Table 1 indicates that for potential mean shift \( \mu \), a slightly smaller \( c \) performs better. Under the null hypothesis that there is no mean shift, the in-control ARL stays the same regardless the value of \( c \), but it largely drives the performance of a control chart. The values of \( c \) producing the smallest out-of-control ARLs for \( \mu = 1, 2 \) and 3 are 0, 1 and 2, respectively. A guideline for choosing \( c \) is to select a slightly smaller value than the anticipated mean shift \( \mu \). We suggest to use \( c = \mu - 1 \) for runs-based control charts. To inspect further, we consider a special case when \( \mu = 2 \) and compare the out-of-control ARLs for various values of \( c \). In Figure 1, the smallest out-of-control ARL occurs at about \( c = 1 \) which supports our suggestion.

Table 3 gives the out-of-control ARLs for the R-1 control chart. To ease the computational burden, we choose \( c = 2 \). The out-of-control ARLs suggest a larger window size and a smaller window size for small shifts and large shifts, respectively, for scan-based control charts. This coincides with the choice of \( r \) and \( c \) given in Section 3.2. The out-of-control ARLs for \( \mu = 1 \) are large because we did not pick the best \( c \) for each value of \( \mu \).

5. Conclusion and summary

Distribution-free runs-based control charts are proposed in this manuscript. The proposed control charts can be applied to both continuous and discrete data. The method is
Table 1: Out-of-control ARLs of R-2 control charts.

| $c$ = | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ |
|-------|-----------|-----------|-----------|
| 0     | 25.2273   | 10.4013   | 9.3220    |
| 1     | 52.8462   | 7.3930    | 4.8780    |
| 2     | 116.5683  | 11.2460   | 3.6970    |
| 3     | 193.4870  | 77.2250   | 6.7560    |

Table 2: Out-of-control ARLs of R-2 control charts with out of control t distribution.

| $r = 4$ | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ |
|---------|-----------|-----------|-----------|
|         | 126.0029  | 9.0210    | 4.1030    |

Table 3: Out-of-control ARLs of R-1 control charts and $c = 2$.

| $r$     | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ |
|---------|-----------|-----------|-----------|
| 6       | 123.7651  | 14.6070   | 3.8788    |
| 10      | 9.7356    | 5.0336    | 5.0336    |

Figure 1: Out-of-control ARLs when $\mu = 2$.

Table 4: Out-of-control ARLs of R-2 control charts.

| $m$     | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ |
|---------|-----------|-----------|-----------|
| 30      | 35.5940   | 28.5270   | 32.5760   |
| 50      | 7.3450    | 7.5150    | 7.9490    |
| 100     | 3.5390    | 3.7090    | 4.1000    |
illustrated using the longest run statistic and the scan statistic, but any other runs statistic can also be used. So readers can choose other runs statistics to suit their purpose. Data are first converted into \( \{0,1\} \) sequence based on (4). There are other transformations to convert data points into \( \{0,1\} \) sequence. For instance, the underlying continuous distribution \( F_0 \) can be used if it is known or can be accurately estimated. The transformed data then follow a uniform distribution and classic continuous scan statistics can be utilized to monitor characteristics of a process. The performance can be significantly improved with proper choice of the parameters \( c \) and \( k \). Some details are given in Section 4.

The advantages of our proposed control charts are three-fold: (i) they are truly distribution-free and the in-control ARL can be exactly controlled at a given level, (ii) historical (reference) data are not necessarily needed and (iii) they are easy to understand and operate. The only assumption is that samples are i.i.d. from some unknown underlying distribution \( F_0 \). In this manuscript, we describe the framework of distribution-free runs-based control charts. The implementation requires the FMCI technique to compute the necessary probabilities involving the charting procedures. The proposed method does not limit to detect only mean shifts, but it can be used for other characteristics, for example variance. One only needs to modify the way to convert the observed data to \( \{0,1\} \) sequence and chooses an appropriate runs statistic to detect shifts in the variance of a process.

However, there are also some drawbacks in our proposed control charts. The proposed control charts do not need to know the targeted values. The charts will signal if mean shifts occur. Thus, if the process is out of control in the beginning, then there is no mean shift and the proposed charts will not signal, unless there is a larger mean shift occurred later. Therefore, if the proposed control charts fail to detect the signal at the early stage, then the performance of the charts will deteriorate gradually. This is because that the charting procedure would adapt to observed sample sequence if there is no signal detected. If the proposed control chart monitors a process for a long time without an out-of-control signal, then it may mean two things: either the process is indeed in-control, or the process was out-of-control at an earlier time and the control chart adapted to it. Finally, we suggest our proposed control charts to be applied at an early stage of a process or to be used as supplementary charts in the beginning of a process.

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