HOMOLOGICAL FLIPS AND HOMOLOGICAL FLOPS

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Abstract. We introduce a notion of homological flips and homological flops. The former includes the class of all flips between Gorenstein normal varieties; while the latter includes the class of all flops between Cohen-Macaulay normal varieties whose contracted variety is quasi-Gorenstein. We prove that certain local cohomology complexes are dual to each other under homological flips/flops. We also develop the technique of weight truncation in the context of wall-crossings in birational cobordisms, parallel to that in [13, 7]. Combining these two techniques, we show that a homological flip (resp. homological flop) satisfying certain regularity conditions determines a fully faithful functor (resp. exact equivalence) between the derived categories of the stacky versions of the varieties in question. As an application, we construct a derived equivalence for any flop between Calabi-Yau varieties arising from a wall-crossing in smooth birational cobordism whose relatively unstable loci has codimension $\geq 2$.

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1. Introduction

A large amount of information about the geometry of a variety is encoded in its derived category. This rich interplay between the homological algebra of derived categories and the geometry of algebraic varieties is especially apparent in birational geometry. Indeed, in the seminal paper [4], Bondal and Orlov
put forth the following principle:

\[(1.1)\] The minimal model program amounts to minimizing the derived category in a given birational class.

Bondal and Orlov have provided some evidence for this principle in [4]. Since then, more and more results of the same spirit has been established by the work of many other mathematicians (see, e.g., the surveys [20, 21] and the references therein). A closely related phenomenon is also given by the “DK-conjecture” of Kawamata (see, e.g., [21 Conjecture 1.2]), which could be viewed as a variation of the same theme.

Recall that there are three basic birational operations in the minimal model program, namely divisorial contractions, flips, and flops. According to the principle \[(1.1)\], the derived category should shrink under divisorial contractions and flips, and it should remain equivalent under flops.

There is, however, an ambiguity of what it means by the “derived category”, and what it means for the derived category to “shrink”. For example, the derived category of a variety \(X\) may refer to either \(\mathcal{D}^b_{\text{coh}}(X)\) or \(\mathcal{D}^b_{\text{perf}}(X)\), which coincide if and only if \(X\) is regular; and shrinking may refer to a fully faithful functor, a localization, or a semi-orthogonal decomposition. In this introduction, we shall take the liberty to switch between these alternative viewpoints.

If we interpret the “derived category” to mean \(\mathcal{D}^b_{\text{perf}}(X)\), and “shrinking” to mean fully faithful embedding, then it is easy to see that the derived category often shrinks under divisorial contractions, for example if we work with normal varieties with, say, \(\mathbb{Q}\)-factorial terminal singularities. Indeed, these varieties have rational singularities, and hence any birational contraction \(f : X \to Y\) between them satisfies \(f_*(\mathcal{O}_X) = \mathcal{O}_Y\) and \(R^if_*(\mathcal{O}_X) = 0\) for all \(i > 0\). This condition is equivalent to the functor \(Lf^* : \mathcal{D}^b_{\text{qcoh}}(Y) \to \mathcal{D}^b_{\text{qcoh}}(X)\) being fully faithful, which then implies that its restriction \(Lf^* : \mathcal{D}^b_{\text{perf}}(Y) \to \mathcal{D}^b_{\text{perf}}(X)\) is fully faithful.

For flips and flops, such a direct comparison between the derived categories is absent because there is no direct morphism between the varieties under flips and flops. The main purpose of this paper is to study the change in derived categories under flips and flops.

In the rest of this introduction, we will illustrate the main ideas of our constructions and results. At some point, this discussion will become somewhat more technical, but since it is still simpler than the main text, we think that it will be useful for the reader.

Consider a log flip between normal varieties

\[(1.2)\] \[
\begin{array}{ccc}
X^- & \xrightarrow{\pi^-} & Y \\
\text{\ } & \searrow & \swarrow \\
\text{\ } & \text{\ } & X^+ \\
\end{array}
\]

By definition of a log flip, there is a Weil divisor \(D^-\) on \(X^-\) such that, if we denote by \(D^+\) its strict transform on \(X^+\), then we have

\[(1.3)\] (1) \(-D^-\) is \(\mathbb{Q}\)-Cartier and \(\pi^-\)-ample;
(2) \(D^+\) is \(\mathbb{Q}\)-Cartier and \(\pi^+\)-ample.

If we denote by \(D_Y\) their common strict transform to \(Y\), and let \(\mathcal{A}\) be the sheaf of \(\mathbb{Z}\)-graded algebras \(\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} O_Y(iD)\), then we have \(X^- = \text{Proj}(A_{\leq 0})\) and \(X^+ = \text{Proj}(A_{\geq 0})\).

Recall that our task is to relate the derived categories under flips and flops. Since the derived categories, or rather their DG enhancements, satisfy descent, it suffices to consider the case when \(Y\) is affine. In this case, the log flip \[(1.2)\] is therefore completely determined by a finitely generated \(\mathbb{Z}\)-graded algebra \(A\).

Notice that for each homogeneous element \(f \in A\) of \(\deg(f) > 0\), the canonical map \((A_{\geq 0})_f \to A_f\) of graded algebras is an isomorphism. Thus, the projective spaces is covered by the open subschemes \(\text{Spec}((A_f)_0)\). Moreover, for any graded \(A\)-module \(M \in \text{Gr}(A)\), the modules \((M_f)_0 \in \text{Mod}((A_f)_0)\) piece together to a quasi-coherent sheaf on \(X^+ = \text{Proj}^+(A) := \text{Proj}(A_{\geq 0})\). This gives a functor

\[(-)^{-} : \text{Gr}(A) \to \text{QCoh}(X^+) , \quad M \mapsto \widetilde{M}\]

Moreover, the (derived) pushforward functor also has an interpretation within \(\text{Gr}(A)\), as a Čech complex.
Indeed, let $f_1, \ldots, f_r$ be homogeneous elements in $A$. For any graded module $M \in \text{Gr}(A)$, we define the Čech complex (see Definition 2.23 for details)

$$
\tilde{\mathcal{C}}(f_1, \ldots, f_r)(M) := \left[ \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \longrightarrow \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0}f_{i_1}} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_r} \right]
$$

(1.4)

as well as the extended Čech complex (see Definition 2.17 for details)

$$
R\Gamma(f_1, \ldots, f_r)(M) := \left[ M \longrightarrow \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \longrightarrow \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0}f_{i_1}} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_r} \right]
$$

(1.5)

It turns out that the isomorphism type of $\tilde{\mathcal{C}}(f_1, \ldots, f_r)(M)$ and $R\Gamma(f_1, \ldots, f_r)(M)$ in the derived category $D(\text{Gr}(A))$ of graded modules depends only on the graded ideal $I$ generated by $(f_1, \ldots, f_r)$. We therefore denote them simply as $\tilde{\mathcal{C}}_I(M)$ and $R\Gamma_I(M)$ respectively. It is clear that there is an exact triangle in $D(\text{Gr}(A))$:

$$
\cdots \rightarrow R\Gamma_I(M) \xrightarrow{\delta} \tilde{\mathcal{C}}_I(M) \xrightarrow{\pi} R\Gamma_I(M)[1] \rightarrow \cdots
$$

(1.6)

If we take $I = I^+ := A_{>0} \cdot A$, then we clearly have

$$
R^j\pi_+^+(\tilde{\mathcal{C}}_I(M)) \cong H^j(\tilde{\mathcal{C}}_{I^+}(M))
$$

in $\text{QCoh}(Y) \simeq \text{Mod}(R)$, where $Y = \text{Spec} R$ for $R = A_0$. The same is true for $\pi^-$ and $I^- := A_{<0} \cdot A$.

The observations (1.6) and (1.7) are instances of the following result, which combines Serre equivalence with parts of Greenlees-May duality (see Sections 2 and 3):

**Theorem 1.8.** There is a semi-orthogonal decomposition

$$
D(\text{Gr}(A)) = \langle D_{I^+, \text{triv}}(\text{Gr}(A)), D_{\text{Tor}^+}(\text{Gr}(A)) \rangle
$$

such that the exact triangle associated to any $M \in D(\text{Gr}(A))$ is given by (1.6). Moreover, the triangulated category $D_{I^+, \text{triv}}(\text{Gr}(A))$ is generated by the objects $R\Gamma_I(A)(i)$, for $i \in \mathbb{Z}$, so that we have

$$
M \in D_{I^+, \text{triv}}(\text{Gr}(A)) \iff (\text{Hom}_{D(\text{Gr}(A))}(R\Gamma_I(A)(i)[j], M) = 0 \; \text{for all} \; i, j \in \mathbb{Z})
$$

(1.9)

Suppose that the divisor $D^+$ in $\text{Gr}(A)$ is chosen to be Cartier, then there is a derived equivalence

$$
D(\text{QCoh}(X^+)) \xrightarrow{\sim} D_{I^+, \text{triv}}(\text{Gr}(A))
$$

sending $\tilde{M}$ to $\tilde{\mathcal{C}}_{I^+}(M)$.

This theorem, as well as its version for $X^-$ and $I^-$, are useful because it tells us two things:

1. The derived categories $D(\text{QCoh}(X^-))$ and $D(\text{QCoh}(X^+))$ both sit as semi-orthogonal components of the same derived category $D(\text{Gr}(A))$.
2. They are the (right) orthogonals of $R\Gamma_{I^-}(A)$ and $R\Gamma_{I^+}(A)$ respectively.

Thus, it suggests that, to relate the derived categories under a log flip, one should look for relations between $R\Gamma_{I^-}(A)$ and $R\Gamma_{I^+}(A)$. One of the main results of this paper is that, for certain classes of flips and flops, there are indeed such a relation. For expository convenience, we only consider the case of flows in this introduction. More precisely, we impose the following assumptions:

1. The varieties $X^-$, $X^+$ and $Y$ are Gorenstein normal varieties with rational singularities;
2. The divisors $D^-$ and $D^+$ in $\text{Gr}(A)$ are Cartier and satisfies

$$
R^j\pi_+^+(\mathcal{O}(-iD^-)) = 0 \quad \text{and} \quad R^j\pi_+^+(\mathcal{O}(iD^+)) = 0 \quad \text{for all} \; i, j > 0
$$

(1.10)

Notice that (1) is satisfied whenever $X^-$, and hence $X^+$, is a Gorenstein Calabi-Yau variety with at most canonical singularities (see the arguments in Remark 5.79). Moreover, (2) is always satisfied if we replace $D^\pm$ by a suitable multiple $mD^\pm$.

Under the assumption (1.10)(1), since $Y$ is Gorenstein, the small contraction $\pi^+$ always satisfies the crepancy condition $(\pi^+)^*K_Y = K_X$. This can be rewritten as

$$
(\pi^+)^*(\mathcal{O}_Y) \cong \mathcal{O}_{X^+}
$$

(1.11)
If we apply the local adjunction isomorphism between \( R\pi^+_X \) and \((\pi^+)^I\), then we have an isomorphism
\[
R\pi^+_X \mathcal{O}_{X^+}(i) = R\pi^+_X \mathcal{H}\text{om}_{\mathcal{O}_{X^+}}(\mathcal{O}_{X^+}(-i), \mathcal{O}_{X^+}) \cong R\mathcal{H}\text{om}_{\mathcal{O}_Y}(R\pi^+_X \mathcal{O}_X(-i), \mathcal{O}_Y)
\]
(1.12)

Notice that, by (1.7), the left hand side of (1.12) is precisely the weight \(i\) component of \( \check{C}_{I^+}(A) \). On the other hand, if we consider the functor
\[
\mathcal{D}_Y : \mathcal{D}(\text{Gr}(A))^{\text{op}} \to \mathcal{D}(\text{Gr}(A))
\]
characterized by \( \mathcal{D}_Y(M)_i = R\text{Hom}_R(M_{-i}, R) \) (see 1.13 for more details), where we recall that \( Y = \text{Spec} R \) for \( R = A_0 \), then the right hand side of (1.12) can be rewritten as \( \mathcal{D}_Y(\check{C}_{I^+}(A)) \).

Thus, this argument suggests that, there should be an isomorphism
\[
(1.13) \quad \Phi^+ : \check{C}_{I^+}(A) \to \mathcal{D}_Y(\check{C}_{I^+}(A))
\]
in \( \mathcal{D}(\text{Gr}(A)) \). However, our argument establishes this only at the cohomology level. To show that this holds at the level of \( \mathcal{D}(\text{Gr}(A)) \), we need to keep track of the graded \( A \)-module structures at the chain level. We accomplish this by interpreting Grothendieck duality as Greenlees-May duality under Serre equivalence. See Sections 2 and 3 for details.

The same is true for \( X^- \), and we have an isomorphism
\[
(1.14) \quad \Phi^- : \check{C}_{I^-}(A) \to \mathcal{D}_Y(\check{C}_{I^-}(A))
\]
Recall that we are eventually interested in relating the derived categories under flips/flops. However, the maps (1.13) and (1.14) only involves one side of the flop. In order to relate two, a crucial observation is that they satisfy a certain compatibility condition, as expressed by the commutativity of the diagram (1.16) below, for \( a = 0 \). As we will see in Theorem 1.17 below, the setting (1.10) gives an example of the following main notion of this paper:

**Definition 1.15.** A homological flip (resp. homological flop) with affine base consists of a sextuple \((R, \omega_R^\bullet, A, \Phi^-, \Phi^+)\) where
1. \( R \) is a Noetherian ring with a dualizing complex \( \omega_R^\bullet \in \mathcal{D}^b_{\text{coh}}(R) \). Denote by \( Y := \text{Spec} R \).
2. \( A \) is a Noetherian \( \mathbb{Z} \)-graded ring such that \( A_0 \) is finite over \( R \).
3. \( a > 0 \) (resp. \( a = 0 \)) is an integer
4. \( \Phi^- \) and \( \Phi^+ \) are isomorphisms in \( \mathcal{D}(\text{Gr}(A)) \)
   \[
   \Phi^+ : \check{C}_{I^+}(A)(a) \to \mathcal{D}_Y(\check{C}_{I^+}(A))
   \]
   \[
   \Phi^- : \check{C}_{I^-}(A)(a) \to \mathcal{D}_Y(\check{C}_{I^-}(A))
   \]
where \( \mathcal{D}_Y : \mathcal{D}(\text{Gr}(A))^{\text{op}} \to \mathcal{D}(\text{Gr}(A)) \) is defined using the dualizing complex \( \omega_R^\bullet \), so that
\[
\mathcal{D}_Y(M)_i \cong R\text{Hom}_R(M_{-i}, \omega_R^\bullet).
\]
The maps \( \Phi^- \) and \( \Phi^+ \) are required to be compatible in the sense that the diagram
\[
(1.16)
\]
\[
\begin{array}{ccc}
\check{C}_{I^+}(A)(a) & \xrightarrow{\Phi^+} & \mathcal{D}_Y(\check{C}_{I^+}(A)) \\
A(a) & \xrightarrow{\eta^+} & \mathcal{D}_Y(\check{C}_{I^+}(A)) \\
\end{array}
\]
\[
\begin{array}{ccc}
\check{C}_{I^-}(A)(a) & \xrightarrow{\Phi^-} & \mathcal{D}_Y(\check{C}_{I^-}(A)) \\
A(a) & \xrightarrow{\eta^-} & \mathcal{D}_Y(\check{C}_{I^-}(A)) \\
\end{array}
\]
commutes in \( \mathcal{D}(\text{Gr}(A)) \), where \( \eta^+ : A \to \check{C}_{I^\pm}(A) \) are the maps in (1.9).

This notion generalizes directly to the case when \( Y \) is not necessarily affine (see Definition 5.69 for details). We will see in a moment that (1.10) gives an example of a homological flop. In fact, in the case of (1.10), one can say even more. Namely, the condition (1.10) (2) translates into a notion of stability (see Definition 5.3 for more details), while the condition of rational singularities in (1.10) (1) translates into a notion of pseudo-rationality (see Definition 5.71 for more details). We then have the following

**Theorem 1.17.** In the situation (1.10), the maps (1.13) and (1.14) determine a stable and pseudo-rational homological flop. The same is true for non-affine \( Y \).
An analogue of this Theorem holds in the case of flips as well. More precisely, any log flip \((1.2)\) whose chosen divisors \((1.3)\) satisfies the following two conditions determines a homological flip:

(1) \(O_X \pm (iD) = K_X\) is maximal Cohen-Macaulay for every \(i \in \mathbb{Z}\);
(2) \(aD = K_X\) for some \(a > 0\).

For example, if \((1.2)\) is a flip between Gorenstein normal varieties, then one can always choose \(a = 1\) and \(D = K_X\). Likewise, in the case of flops, the conditions \((1.10)\) can also be weakened. For a general version of results along these lines, see Theorem 5.72, which is a culmination of all the preparatory work in Sections 2, 3, 4, and 5. In particular, the proof of the compatibility condition \((1.16)\) brings together all these preparatory work.

A main advantage of being a homological flip/flop is the following result, which uses the compatibility condition \((1.16)\) in a crucial way (see Corollary 5.78):

**Theorem 1.18.** Let \((R, \omega_R, A, a, \Phi^-, \Phi^+)\) be a stable and pseudo-rational homological flip/flop, then there is an isomorphism \(\Psi : R\Gamma_I^-(A)(a)[1] \cong D_Y(R\Gamma_I^+(A))\). Moreover, the ring \(A\) is Gorenstein. The same is true for non-affine \(Y\).

As was suggested in the discussion following Theorem 1.8, this duality between the complexes \(R\Gamma_I^-(A)\) and \(R\Gamma_I^+(A)\) should allow one to relate the derived categories under the flip/flop in question. For example, in the situation \((1.10)\), one should be able to obtain a derived equivalence from this. However, the functor \(D_Y\) is not involutive on \(D(Gr(A))\). It is involutive only on certain subcategories of (locally) coherent complexes. We have not been able to find a suitable subcategory which contains both \(R\Gamma_I^-(A)\) and \(R\Gamma_I^+(A)\), exhibiting the derived categories of \(X^-\) and \(X^+\) as their orthogonals, and on which \(D_Y\) is involutive.

In order to make use of Theorem 1.18, we introduce the second major technique in this paper: weight truncation. The technique of weight truncation has a diverse origin. For example, though concerned with different problems, Kawamata [18] and Orlov [27] independently came up with techniques which could be viewed as an instance of a “grade restriction rule”. On the other hand, the paper [16] also considered a grade restriction rule in the study of B-branes in Landau-Ginzburg models, which was then interpreted mathematically by Segal [32]. These techniques were then developed to more sophistication in [13] and [7]. Parts of the techniques in [13] also have their origin in [35].

For the experts, we devote this paragraph to situate our contribution to this technique of weight truncation in relation to the prevailing literature. We focus on the abelian case, more precisely the case of wall-crossings in birational cobordisms. We believe that our techniques can be generalized directly to the non-abelian case. This will be investigated in the future. We do not impose any smoothness assumptions on the ambient stack that underlies the birational cobordism, and we construct a semi-orthogonal decomposition for the bounded above derived categories of coherent sheaves on this ambient stack, exhibiting that of the stacky GIT quotients as semi-orthogonal components. We obtain this by characterizing the semi-orthogonal components by certain local cohomology complexes on certain small preadditive category. The paper [14] also contains similar results, but obtained by rather different methods. It is not clear to us what are the relations between these two approaches. When the ambient stack is smooth, we show that our semi-orthogonal decomposition restricts to one on the bounded derived category of coherent sheaves, and coincides with that of [13] and [7]. Even in this case, our characterization of the semi-orthogonal decomposition in terms of local cohomology complexes is crucial in the method we use to relate the derived categories under flips and flops. Namely, this characterization is naturally expressed in terms that are close to Theorem 1.18 and hence allows us to combine this theorem with the technique of weight truncation.

We summarize our main results on weight truncation in Theorem 1.19 below. Notice however that the theorem contains some notions that are not defined in this introduction. Such notions require an extended discussion, and we refer the reader to Section 6 for details. A large part of Section 6 is independent of the rest of the paper.
Moreover, $X$ has a special kind of variation of GIT quotients, where we consider the group to be the derived category under these classes of flips/flops. This could be viewed as a rather strong result, because usually results of the form (1.20)(2) is quite difficult to prove, and often require a detailed case-by-case analysis. In any case, we show that, if Theorem 1.19 holds, then (1.20)(2) holds almost automatically. Although we have not been able to prove it, we expect that, if the conclusion of Theorem 1.18 holds, then (1.20)(2) holds whenever (1.20)(1) does.

Theorem 1.19. For any Noetherian $\mathbb{Z}$-graded ring $A$, there exists a semi-orthogonal decomposition (see [6.30] and Theorem [6.22])

$$D(\text{Gr}(A)) = \langle D_{<w}(\text{Gr}(A)), L_{\geq w}(D_{I^+\text{triv}}(\text{Gr}(A))), D_{\text{Tor}^+,\geq w}(\text{Gr}(A)) \rangle$$

where the middle component can be characterized as

$$L_{\geq w}(D_{I^+\text{triv}}(\text{Gr}(A))) = \{ M \in D_{\geq w}(\text{Gr}(A)) | R\Gamma_{I^+}(M) \in D_{<w}(\text{Gr}(A)) \}$$

and is equivalent to $D_{I^+\text{triv}}(\text{Gr}(A))$ via the quasi-inverse pair of exact functors (see Theorem [6.24])

Moreover, all these restricts to results on the full subcategory $D_{\text{coh}}^-(\text{Gr}(A))$. More precisely, we have (see Theorem [6.41] and [6.42])

$$D_{\text{coh}}^-(\text{Gr}(A)) = \langle D_{\text{coh},<w}(\text{Gr}(A)), L_{\geq w}(D_{\text{coh}(I^+\text{triv})}(\text{Gr}(A))), D_{\text{coh},\text{Tor}^+,\geq w}(\text{Gr}(A)) \rangle$$

$$L_{\geq w}(D_{\text{coh}(I^+\text{triv})}(\text{Gr}(A))) = \{ M \in D_{\text{coh},\geq w}(\text{Gr}(A)) | R\Gamma_{I^+}(M) \in D_{<w}(\text{Gr}(A)) \}$$

$$L_{\geq w} : D_{\text{coh}(I^+\text{triv})}(\text{Gr}(A)) \xrightarrow{\sim} L_{\geq w}(D_{\text{coh}(I^+\text{triv})}(\text{Gr}(A))) : \breve{\mathcal{C}}_{I^+}$$

We use this weight truncation to relate the derived categories under homological flips/flops. Namely, if we choose the divisor $D^-$ in (1.3) so that both $D^-$ and $D^+$ are Cartier, then we have equivalences $D_{\text{coh}}^-(X^\pm) \simeq D_{\text{coh}(I^+\text{triv})}^-(\text{Gr}(A))$. In fact, more generally, if we replace $X^\pm = \text{Proj}^\pm(A)$ by its stacky version $\mathcal{X}^\pm = \text{Proj}^\pm(A)$ (see Remark [3.30]), then there are equivalences $D_{\text{coh}}^-(\mathcal{X}^\pm) \simeq D_{\text{coh}(I^+\text{triv})}^-(\text{Gr}(A))$.

Moreover, $X^\pm$ coincides with $X^\pm$ if and only if $D^\pm$ is Cartier. This allows us to construct a functor

$$D_{\text{coh}}^-(\mathcal{X}^+) \simeq D_{\text{coh}(I^+\text{triv})}^-(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{\geq w}} L_{\geq w}(D_{\text{coh}(I^+\text{triv})}(\text{Gr}(A))) \xrightarrow{\mathcal{C}_{I^+}} D_{\text{coh}(I^+\text{triv})}^-(\text{Gr}(A)) \simeq D_{\text{coh}}^-(\mathcal{X}^-)$$

In order for this functor to be useful, we want two properties to hold:

(1) It restricts to a functor $D_{\text{coh}}^+(\mathcal{X}^+) \to D_{\text{coh}}^+(\mathcal{X}^-)$.

(2) This restriction is fully faithful in the case of flips; and an equivalence in the case of flops.

These properties are studied in detail in Section 8. In particular, we show that if the conclusion of Theorem 1.18 holds, then (1.20)(2) holds almost automatically. Although we have not been able to prove it, we expect that, if the conclusion of Theorem 1.18 holds, then (1.20)(2) holds whenever (1.20)(1) does. This could be viewed as a rather strong result, because usually results of the form (1.20)(2) is quite difficult to prove, and often require a detailed case-by-case analysis. In any case, we show that, if $A$ is smooth over a field $k$ of characteristic zero, then both conditions (1.20)(1)(2) hold, which then gives the desired relation between the derived categories under these classes of flips/flops.

Besides flips/flops, sheaves of $\mathbb{Z}$-graded rings (as in (1.1)) also arise in wall-crossing in birational cobordism. These are special kinds of variations of GIT quotients, where we consider the group to be $G = \mathbb{G}_m$, and we vary the GIT quotient of a projective-over-affine variety $X$ by twisting the linearization of the given (ample) line bundle $L$ by a (fractional) character $t \in \mathbb{Q}$. Then, there is a finite set $T \subset \mathbb{Q}$ such that the open subset of semi-stable points $X^{ss}(L(t)) \subset X$, for $t \in \mathbb{Q}$, remain the same when $t$ varies along each of the intervals between consecutive points in $T$, but undergoes a change (called wall-crossing) when $t$ pass through a point in $T$ (see Section 7 for a discussion). In fact, wall-crossing in birational cobordism is a rather general phenomenon. For example, by the “master space construction” of Thaddeus [33], every wall-crossing in GIT can be realized as a wall-crossing in birational cobordism.

In the case of a wall-crossing, i.e., if $[t^-, t^+] \cap T = \{t_0\}$, for some rational numbers $t^- < t_0 < t^+$, then we have

$$X^{ss}(L(t^-)) \subset X^{ss}(L(t_0)) \supset X^{ss}(L(t^+))$$
which then determines a log-flip-like diagram between the scheme-theoretic GIT quotients
\[ (1.22) \quad \xymatrix{ X^{ss}(L(t^-)) / \mathbb{G}_m \ar[r] & Y \ar[l] \quad X^{ss}(L(t^+)) / \mathbb{G}_m } \]
as well as between the stacky GIT quotients
\[ (1.23) \quad \xymatrix{ [X^{ss}(L(t^-)) / \mathbb{G}_m] \ar[r] & Y \ar[l] \quad [X^{ss}(L(t^+)) / \mathbb{G}_m] } \]

We say that the wall-crossing is small if the complements of the inclusion (1.21), which we call the relatively unstable loci, has codimension \( \geq 2 \). This is closely related to (1.22) being a log flip. Namely, if the variety \( X \) is normal, then the corresponding map (1.22) associated to any small wall-crossing is a log flip (see Proposition 7.7 and Remark 5.25). In this case, all of the techniques in this paper are applicable, and we have the following result (see Corollary 8.30):

**Theorem 1.24.** Let (1.22) be a small wall-crossing in a smooth birational cobordism (i.e., the space \( X \) is smooth). Suppose that the varieties \( X^{ss}(L(t^\pm)) / \mathbb{G}_m \) under the wall-crossing have trivial canonical divisors (also known as quasi-Calabi-Yau), then the corresponding stacky GIT quotients (1.23) are derived equivalent. i.e., there is an exact equivalence
\[ \mathcal{D}^b_{\text{coh}}([X^{ss}(L(t^+)) / \mathbb{G}_m]) \xrightarrow{\sim} \mathcal{D}^b_{\text{coh}}([X^{ss}(L(t^-)) / \mathbb{G}_m]) \]

We believe that this holds in certain non-smooth settings, as well as over more general GIT quotients. This will be investigated in the future.

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**Conventions.** A variety is always assumed to be reduced and irreducible. We will always work with cochain complexes. We denote the category of cochain complexes of an abelian category \( \mathcal{C} \) by \( \text{Ch}(\mathcal{C}) \).

2. Derived categories of graded modules

In this section, we extend some results on the derived category of a commutative ring, especially Greenlees-May duality, to the graded case. This extension is rather straightforward. We include some details partly to fix conventions and notations.

2.1. Graded rings and graded modules.

**Definition 2.1.** A \( \mathbb{Z} \)-graded ring is a commutative ring \( A \) with a \( \mathbb{Z} \)-grading \( A = \bigoplus_{n \in \mathbb{Z}} A_n \). Here, by commutative we mean \( xy = yx \), not \( xy = (-1)^{|x||y|}yx \).

A graded module over \( A \) will always mean a \( \mathbb{Z} \)-graded module \( M = \bigoplus_{n \in \mathbb{Z}} M_n \).

We first recall the following result (see, e.g., [3, Theorem 1.5.5]):

**Proposition 2.2.** Let \( A \) be a \( \mathbb{Z} \)-graded ring. The the followings are equivalent:

1. \( A \) is a Noetherian ring;
2. every graded ideal of \( A \) is finitely generated;
3. \( A_0 \) is Noetherian, and both \( A_{>0} \) and \( A_{\leq 0} \) are finitely generated over \( A_0 \);
4. \( A_0 \) is Noetherian, and \( A \) is finitely generated over \( A_0 \).

Following [1], we denote by \( \text{Gr}(A) \) the category of graded modules over \( A \), whose morphisms are maps of graded modules of degree 0; and by \( \text{gr}(A) \subset \text{Gr}(A) \) be the full subcategory consisting of finitely generated graded modules.

Given two graded modules \( M, N \in \text{Gr}(A) \), then the \( A \)-module \( M \otimes_A N \) has a natural grading where \( \deg(x \otimes y) = \deg(x) + \deg(y) \) for homogeneous \( x, y \in A \). Moreover, one can define a graded \( A \)-module \( \text{Hom}_A(M, N) \) whose degree \( i \) part is the set of \( A \)-linear homomorphism from \( M \) to \( N \) of homogeneous
degree $i$. Thus, in particular, we have $\text{Hom}_A(M, N) := \text{Hom}_{\text{Gr}(A)}(M, N) = \text{Hom}_A(M, N)_0$. These form the internal Hom objects with respect to the graded tensor product. More precisely, there is a canonical isomorphism of graded $A$-modules

$$\text{Hom}_A(M \otimes_A N, P) \cong \text{Hom}_A(M, \text{Hom}_A(N, P))$$

In fact, this is part of a more general tensor-Hom adjunction for a bigraded bimodule, in the sense of the following

**Definition 2.3.** Let $A, B$ be two $\mathbb{Z}$-graded rings. Then a bigraded $(A, B)$-bimodule is an $(A, B)$-bimodule $N$ with a $\mathbb{Z}^2$-grading $N = \bigoplus_{i,j} N_{i,j}$ such that $A_p \cdot N_{i,j} \cdot B_q \subset N_{i+p,j+q}$. The component $N_{i,j}$ is said to have $A$-grading $i$ and $B$-grading $j$.

Given a bigraded $(A, B)$-bimodule $N$. Then for any graded $A$-module $M$, the tensor product $M \otimes_A N$ is again a bigraded $(A, B)$-bimodule in the obvious way. Denote by $M \otimes_A^0 N$ the component of degree 0 in the $A$-grading, which is then a graded $B$-module. Similarly, for any graded $B$-module $P$, one can define a bigraded $(A, B)$-bimodule $\text{Hom}_B(N, P)$ by declaring that its component of degree $i$ in the $A$-grading is $\text{Hom}_B((N_{-i,*}), P_*)$. If we restrict to the component of degree 0 in the $B$-grading, then we have a graded $A$-module $\text{Hom}_B(N, P) \in \text{Gr}(A)$, defined by

$$\text{Hom}_B(N, P)_i := \text{Hom}_B((N_{-i,*}), P_*)$$

This gives rise to an adjunction

$$\text{Hom}_B(N, P)_i \cong \text{Hom}_B((N_{-i,*}), P_*)$$

Notice that, in general, neither of these functors commute with the twist functors.

Any map $f : A \to B$ of $\mathbb{Z}$-graded rings induces a bigraded $(A, B)$-bimodule $\tilde{B}$ defined by $\tilde{B}_{i,j} := B_{i+j}$, with the obvious $(A, B)$-bimodule structure. In this case, the adjunction (2.5) with respect to $\tilde{N}$ reduces to the simple form

$$- \otimes_A^0 B : \text{Gr}(A) \xrightarrow{\sim} \text{Gr}(B) : (-)_A$$

where $(-)_A$ is the functor of restriction of scalars along $f$.

The abelian category $\text{Gr}(A)$ is a Grothendieck category, with a set $\{A(-i)\}_{i \in \mathbb{Z}}$ of generators. The same set is also a set of compact generators in the derived category $\mathcal{D}(\text{Gr}(A))$. Since $\text{Gr}(A)$ is a Grothendieck category, the category of complexes has enough K-injectives (see, e.g., [30 Tag 079P]). Moreover, as in the ungraded case, it also has enough K-projectives (see, e.g., [30 Tag 06XX]). As a result, the bifunctors $- \otimes_A -$ and $\text{Hom}_A(-, -)$ admit derived functors

$$\text{RHom}_A(-, -) : \mathcal{D}(\text{Gr}(A)) \otimes \mathcal{D}(\text{Gr}(A)) \to \mathcal{D}(\text{Gr}(A))$$

which can in turn be used to define $\text{Ext}^*_A(M, N)$ and $\text{Tor}^*_A(M, N)$, the former of which does not coincide with the ungraded version in general.

Since $\text{Tor}^*_A(M, N)$ is the same whether computed in $\text{Gr}(A)$ or in $\text{Mod}(A)$, the notion of finite Tor dimension is unambiguous. On the other hand, we have the following lemma, which shows that in the Noetherian case, the notion of finite injective dimension is also unambiguous:

**Lemma 2.7.** If $A$ is a Noetherian $\mathbb{Z}$-graded ring, then we have

$$\text{inj} \dim_{\text{Gr}(A)}(M) \leq \text{inj} \dim_{\text{Mod}(A)}(M) \leq \text{inj} \dim_{\text{Gr}(A)}(M) + 1$$

**Proof:** A graded module (resp. module) $M$ is injective in $\text{Gr}(A)$ (resp. $\text{Mod}(A)$) if and only if for all graded ideal (resp. ideal) $a \subset A$, the induced map $M \to \text{Hom}_A(a, A)$ (resp. $A \to \text{Hom}_A(a, A)$) is surjective. Since $A$ is Noetherian, $a$ is finitely generated, and hence we have $\text{Hom}_A(a, A) = \text{Hom}_A(a, A)$ (see, e.g., [5 Exercise 1.5.19]). This proves the first inequality. For the second inequality, see, e.g., [5 Theorem 3.6.5].

$\square$
We now extend some standard results on the derived categories of modules to the graded case. We start with the following

**Definition 2.8.** An object $M \in \mathcal{D}(\text{Gr}(A))$ is said to be pseudo-coherent if it can be represented by a bounded above complex of finitely generated projective graded $A$-modules. Denote by $\mathcal{D}_{pc}(\text{Gr}(A)) \subset \mathcal{D}(\text{Gr}(A))$ the full subcategory consisting of pseudo-coherent objects.

**Definition 2.9.** Given a Noetherian $\mathbb{Z}$-graded ring $A$, then for $\mathfrak{m} \in \{+, -, b\}$, define $\mathcal{D}_\mathfrak{m}(\text{Gr}(A))$ the full subcategory of $\mathcal{D}(\text{Gr}(A))$ consisting of complexes $M \in \mathcal{D}(\text{Gr}(A))$ such that $H^p(M)$ is finitely generated for all $p \in \mathbb{Z}$.

Then an application of Lemma A.12 shows the following

**Proposition 2.10.** Given a Noetherian $\mathbb{Z}$-graded ring $A$, then for any $M \in \mathcal{D}(A)$, the followings are equivalent:

1. $M \in \mathcal{D}_{pc}(\text{Gr}(A))$;
2. $M \in \mathcal{D}_{coh}(\text{Gr}(A))$;
3. $M$ is quasi-isomorphic to a bounded above complex of free graded modules of finite rank.

Let $\mathcal{D}_{perf}(\text{Gr}(A))$ be the smallest split-closed triangulated subcategories containing the set $\{A(-i)\}_{i \in \mathbb{Z}}$ of objects. By [29, Theorem 4.22] (see also [26, Lemma 2.2]), $\mathcal{D}_{perf}(\text{Gr}(A))$ is precisely the full subcategory of compact objects in $\mathcal{D}(\text{Gr}(A))$. Moreover, as in the ungraded case, we have the following standard

**Theorem 2.11.** If the underlying ungraded ring of a Noetherian $\mathbb{Z}$-graded ring $A$ is regular, then we have $\mathcal{D}_{coh}(\text{Gr}(A)) = \mathcal{D}_{perf}(\text{Gr}(A))$.

We also mention the following graded analogues of [30, Tag 0A68], [30, Tag 0A69] and [30, Tag 0ATK], whose proofs are completely parallel to the ungraded case:

**Proposition 2.12.** For any $M \in \mathcal{D}_{pc}(\text{Gr}(A))$, $N \in \mathcal{D}^+(\text{Gr}(A))$ and $L \in \mathcal{D}(\text{Gr}(A))$ of finite injective dimension, the canonical map
\[ M \otimes_A R\text{Hom}_A(N, L) \rightarrow R\text{Hom}_A(R\text{Hom}_A(M, N), L) \]
is an isomorphism in $\mathcal{D}(\text{Gr}(A))$.

**Proposition 2.13.** Suppose $M \in \mathcal{D}(\text{Gr}(A))$ is such that the truncation $\tau_{\leq m}M$ is pseudo-coherent for each $m \in \mathbb{Z}$, and suppose that $N, L \in \mathcal{D}(\text{Gr}(A))$ have finite injective dimension, then the canonical map
\[ M \otimes_A R\text{Hom}_A(N, L) \rightarrow R\text{Hom}_A(R\text{Hom}_A(M, N), L) \]
is an isomorphism in $\mathcal{D}(\text{Gr}(A))$.

**Proposition 2.14.** For any $N \in \mathcal{D}_{pc}(\text{Gr}(A))$, $L \in \mathcal{D}^+(\text{Gr}(A))$, and $M \in \mathcal{D}(\text{Gr}(A))$ of finite Tor dimension, the canonical map
\[ M \otimes_A L \rightarrow \text{Hom}_A(N, M \otimes_A L) \]
is an isomorphism in $\mathcal{D}(\text{Gr}(A))$.

For later use, we also include the following simple

**Lemma 2.15.** Let $A$ be a Noetherian $\mathbb{Z}$-graded ring, and let $f \in A$ be a homogeneous element. If $L \in \mathcal{D}^b(\text{Gr}(A))$ has finite injective dimension, then so is $L_f := L \otimes_A A_f$.

**Proof.** Recall that an object $L \in \mathcal{D}(\text{Gr}(A))$ has finite injective dimension if and only if there exists integers $a \leq b$ such that $H^i(R\text{Hom}_A(N, L)) = 0$ for all $i \not\in [a, b]$, for all finitely generated module $N$ (see, e.g., [30, Tag 0A5T] for the ungraded version). Apply Proposition 2.14 to $M = A_f$ gives the desired result.

Alternatively, one can also prove this Lemma by writing $L_f$ as a directed colimit
\[ L_f = \text{colim} \left[ L \xrightarrow{f} L(m) \xrightarrow{f} L(2m) \xrightarrow{f} \ldots \right] \]
where \( m := \deg(f) \). By Lemma 3.36, it is therefore quasi-isomorphic to the corresponding homotopy colimit, which is a cone of a map between infinite sums of weight twistings of \( M \). Since \( A \) is Noetherian, infinite sums of injective modules remain injective, which then proves the claim. \( \square \)

2.2. Greenlees-May duality on graded rings.

**Definition 2.16.** Let \( I \) be a graded ideal in a \( \mathbb{Z} \)-graded ring \( A \). Given any graded module \( M \) over \( A \), an element \( x \in M \) is said to be \( I^\infty \)-torsion if there exists some \( n > 0 \) such that \( I^n x = 0 \). The graded module \( M \) is said to be \( I^\infty \)-torsion if every element in it is \( I^\infty \)-torsion. Denote by \( I^\infty \text{-Tor} \subset \text{Gr}(A) \) the full subcategory consisting of \( I^\infty \)-torsion modules.

It is clear that \( I^\infty \text{-Tor} \subset \text{Gr}(A) \) is a Serre subcategory. Thus the full subcategory \( \mathcal{D}_{I^\infty \text{-Tor}}(\text{Gr}(A)) \subset \mathcal{D}(\text{Gr}(A)) \) is a triangulated subcategory. One can show that this last inclusion always has a right adjoint. In fact, this right adjoint has a simple and useful description when \( I \) is finitely generated. To this end, we recall the following

**Definition 2.17.** Let \( f_1, \ldots, f_r \) be homogeneous elements in \( A \), of degrees \( d_1, \ldots, d_r \) respectively. For any graded module \( M \in \text{Gr}(A) \), we define the extended Čech complex of \( M \) with respect to the tuple \((f_1, \ldots, f_r)\) to be the cochain complex of graded modules

\[
\Gamma(f_1, \ldots, f_r)(M) := \left[ M \xrightarrow{d^0} \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \xrightarrow{d^1} \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0} f_{i_1}} \xrightarrow{d^2} \ldots \xrightarrow{d^{r-1}} M_{f_1 \ldots f_r} \right]
\]

whose differentials are defined by \( d^m := \sum_{j=0}^m (-1)^j d_{ij}^m \), where \( d_{ij}^m \) is the direct product of the canonical maps \( d_{ij}^m : A_{i_0 \ldots i_j \ldots i_m} \to A_{i_0 \ldots i_m} \). Here, the first term \( M \) is put in cohomological degree 0.

For a cochain complex \( M \in \text{Ch}(\text{Gr}(A)) \) of graded modules, we define \( R\Gamma(f_1, \ldots, f_r)(M) \) to be the total complex of the double complex \( C^{p,q} = \Gamma(f_1, \ldots, f_r)(M^p) \).

It is clear that we have

\[
\Gamma(f_1, \ldots, f_r)(M) \cong \Gamma(f_1, \ldots, f_r)(A) \otimes_A M \cong \Gamma(f_1)(A) \otimes_A \cdots \otimes_A \Gamma(f_r)(A) \otimes_A M
\]

The extended Čech complex may be written as a directed colimit of (cohomological) Koszul complexes

\[
\Gamma(f_1, \ldots, f_r)(M) \cong \text{colim}_{(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r} K^\bullet(M; f_1^{m_1}, \ldots, f_r^{m_r})
\]

which can be computed, as is often done in the literature (see, e.g., [30] Tag 0913), \([5] \text{Theorem 3.5.6}\) in the ungraded case, via the colimit on the cofinal system \((m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r\).

Whenever a homogeneous element \( g \in A \), of degree \( m \), lies in the ideal generated by \( f_1^{m_1}, \ldots, f_r^{m_r} \), the map in \( \text{Ch}(\text{Gr}(A)) \)

\[
g : K^\bullet(M; f_1^{m_1}, \ldots, f_r^{m_r}) \to K^\bullet(M; f_1^{m_1}, \ldots, f_r^{m_r})(m)
\]

is homotopic to zero, and hence induces the zero map in cohomology. Indeed, the graded analogue of, say, [30] Tag 0626] establishes this for \( M = A \), which then implies that it holds for all \( M \in \text{Ch}(\text{Gr}(A)) \), in view of \( (2.19) \). Since directed colimit commutes with taking cohomology, we see by \( (2.20) \) that the cohomology modules of the extended Čech complex are \( I^\infty \)-torsion, where \( I = (f_1, \ldots, f_r) \) is the ideal generated by the elements \( f_i \).

Since the complex \( R\Gamma(f_1, \ldots, f_r)(A) \) is flat over \( A \), the functor \( M \mapsto R\Gamma(f_1, \ldots, f_r)(M) \) on \( \text{Ch}(\text{Gr}(A)) \) is exact, and hence descends to a functor at the level of derived categories. Moreover, we have seen that this functor has image inside the full subcategory \( \mathcal{D}_{I^\infty \text{-Tor}}(\text{Gr}(A)) \). Thus, this gives a functor

\[
\Gamma_I := R\Gamma(f_1, \ldots, f_r) : \mathcal{D}(\text{Gr}(A)) \to \mathcal{D}_{I^\infty \text{-Tor}}(\text{Gr}(A))
\]

Moreover, the map \( \epsilon_M : R\Gamma(f_1, \ldots, f_r)(M) \to M \) defined by projecting to the first component of \( (2.18) \) gives rise to a natural transformation

\[
\epsilon : \iota \circ R\Gamma_I \Rightarrow \text{id}
\]

where \( \iota : \mathcal{D}_{I^\infty \text{-Tor}}(\text{Gr}(A)) \to \mathcal{D}(\text{Gr}(A)) \) is the inclusion functor. The cone of \( \epsilon_M \) is homotopic to the kernel of \( \epsilon_M \) shifted by 1, which is given by the following
Definition 2.23. The Čech complex of a graded module $M$ with respect to a tuple $(f_1, \ldots, f_r)$ of homogeneous elements is the cochain complex of graded modules

$$(2.24) \quad \check{C}(f_1, \ldots, f_r)(M) := \prod_{1 \leq i_0 \leq r} M_{f_{i_0}} \xrightarrow{-d_1} \prod_{1 \leq i_0 < i_1 \leq r} M_{f_{i_0} f_{i_1}} \xrightarrow{-d_2} \cdots \xrightarrow{-d_{r-1}} M_{f_1 \cdots f_r}$$

given as a subcomplex of $(2.13)$, shifted by one. As in Definition 2.17, this definition can be extended to cochain complexes $M \in \text{Ch} (\text{Gr}(A))$ by taking the total complex.

Clearly, the natural transformation $(2.22)$ is an isomorphism for $M \in D(\text{Gr}(A))$ if and only if $\check{C}(f_1, \ldots, f_r)(M)$ has zero cohomology. Now if $M \in D_{I^\infty \cdot \text{Tor}}(\text{Gr}(A))$, then each of the terms in $(2.24)$, thought of as a double complex, has zero cohomology. Thus, as an iterative cone of complexes with zero cohomology, the total complex $\check{C}(f_1, \ldots, f_r)(M)$ also has zero cohomology. This shows that

$$\text{if } M \in D_{I^\infty \cdot \text{Tor}}(\text{Gr}(A)), \text{ then the natural transformation } \epsilon_M \text{ is an isomorphism in } D(\text{Gr}(A)).$$

As a formal consequence of $(2.21)$, $(2.22)$ and $(2.24)$, we have the following

**Theorem 2.26.** The functor $(2.21)$ is a right adjoint to the inclusion $\iota: D_{I^\infty \cdot \text{Tor}}(\text{Gr}(A)) \to D(\text{Gr}(A))$, with counit given by $(2.22)$.

For each $M \in D(\text{Gr}(A))$, there is an exact triangle

$$(2.27) \quad \cdots \to R\Gamma_I(M) \xrightarrow{\eta_M} M \xrightarrow{\delta_M} R\Gamma_I(M)[1] \to \cdots$$

where $\eta_M = -d^0$, the negative of the first differential in $(2.13)$, and $\delta_M$ is the inclusion.

If $A$ is Noetherian, then the functors $R\Gamma_I$ and $\check{C}_I$ have alternative descriptions as right derived functor. Namely, let $\Gamma_I: \text{Gr}(A) \to I^\infty \cdot \text{Tor}$ be the right adjoint to the inclusion $I^\infty \cdot \text{Tor} \to \text{Gr}(A)$, and let $0\check{C}_I: \text{Gr}(A) \to \text{Gr}(A)$ be the functor $0\check{C}_I := H^0(\check{C}_I(M))$, then we have

**Proposition 2.28.** If $A$ is Noetherian, then the canonical functor $D(I^\infty \cdot \text{Tor}) \to D_{I^\infty \cdot \text{Tor}}(\text{Gr}(A))$ is an equivalence. Moreover, under this equivalence, the functor $(2.21)$ is identified with the right derived functor of $\Gamma_I : \text{Gr}(A) \to I^\infty \cdot \text{Tor}$.

Similarly, the functor $\check{C}_I : D(\text{Gr}(A)) \to D(\text{Gr}(A))$ is the right derived functor of $0\check{C}_I : \text{Gr}(A) \to \text{Gr}(A)$.

**Proof.** The first statement is the direct graded analogue of a well-known statement (see, e.g., [30, Tag 0955]). For the second statement, notice that for any $M \in \text{Gr}(A)$, there is an exact sequence

$$(2.29) \quad 0 \to \Gamma_I(M) \to M \to 0\check{C}_I(M) \to H^1(R\Gamma_I(M)) \to 0$$

Now if we apply this termwise to a K-injective complex of injective graded modules $M^\bullet$, then the first statement of this Proposition shows that $H^i(R\Gamma_I(M^\bullet)) = 0$ for all $i \in \mathbb{Z}$, and we therefore have a short exact sequence of cochain complexes

$$0 \to \Gamma_I(M^\bullet) \to M^\bullet \to 0\check{C}_I(M^\bullet) \to 0$$

By the first statement of this Proposition again, the map $\Gamma_I(M^\bullet) \to M^\bullet$ here is isomorphic in $D(\text{Gr}(A))$ to the map $\epsilon_M : R\Gamma_I(M^\bullet) \to M^\bullet$ in $(2.27)$. Thus, the cone of these two maps are also identified. In other words, we have $0\check{C}_I(M^\bullet) \cong \check{C}_I(M^\bullet)$ in $D(\text{Gr}(A))$. \qed

In view of this Proposition, we make the following

**Definition 2.30.** The complex $R\Gamma_I(M)$ is called the local cohomology complex of $M$ with respect to the graded ideal $I \subset A$.

Now we discuss a closely related notion of derived complete graded modules, following [30]. Let $M \in \text{Ch} (\text{Gr}(A))$ be a chain complex, and let $f \in A$ be a homogeneous element, say of degree $m$. Denote by $T(M, f)$ the cochain complex in $\text{Gr}(A)$ defined as the homotopy limit (see [B34])

$$(2.31) \quad T(M, f) := \text{holim} \left[ M \xrightarrow{\delta} M(-m) \xrightarrow{\delta} M(-2m) \xrightarrow{\delta} \cdots \right]$$
This cochain complex can be written as a Hom-complex. Indeed, if we denote by \( A_{(f)} \) the homotopy colimit (see (2.33))
\[
A_{(f)} := \text{hocollim } [A \xrightarrow{d} A(m) \xrightarrow{d} A(2m) \xrightarrow{d} \ldots]
\]
(2.32) then, since the corresponding ordinary colimit is simply \( A_f \), we have, by Lemma (3.36), a quasi-isomorphism
\[
A_{(f)} \xrightarrow{\sim} A_f,
\]
thus giving a free resolution \( A_{(f)} \) of \( A_f \). This allows us to rewrite (2.31) as
\[
T(M, f) \cong \text{Hom}_A(A_{(f)}, M) \cong R\text{Hom}_A(A_f, M)
\]
(2.33)

Since the association \( M \mapsto T(M, f) \) is exact, it descends to a functor \( T(\_ , f) : D(\text{Gr}(A)) \rightarrow D(\text{Gr}(A)) \).

We start with the following graded analogue of [30, Tag 091P]:

**Lemma 2.34.** Let \( f \in A \) be a homogeneous element of degree \( d \). Then for any complex \( M \in D(\text{Gr}(A)) \),
the followings are equivalent:
1. \( T(M, f) \) has zero cohomology;
2. \( R\text{Hom}_A(E, M) \) has zero cohomology for all \( E \in D(\text{Gr}(A_f)) \);
3. for every \( p \in \mathbb{Z} \), we have \( \text{Hom}_A(A_f, H^p(M)) = 0 \) and \( \text{Ext}_A^1(A_f, H^p(M)) = 0 \).
4. for every \( p \in \mathbb{Z} \), the complex \( T(H^p(M), f) \) has zero cohomology.

**Proof.** Since \( \{A_f(n)\}_{n \in \mathbb{Z}} \) is a set of compact generators of \( D(\text{Gr}(A_f)) \) (see the paragraph preceding
Theorem 2.11), we see that \( R\text{Hom}_A(E, M) \cong 0 \) for all \( E \in D(\text{Gr}(A_f)) \) if and only if it holds for \( E = A_f \).

In view of (2.33), this proves the equivalence (1) \( \iff \) (2). The equivalence (3) \( \iff \) (4) is also obvious, since
the graded modules appearing in (3) are simply the cohomology of the complex \( T(H^p(M), f) \) in (4). For the
equivalence (1) \( \iff \) (3), simply take a spectral sequence.

**Remark 2.35.** Since \( A_f \) is flat over \( A \), the standard tensor-forgetful adjunction between \( D(\text{Gr}(A_f)) \) and
\( D(\text{Gr}(A)) \) identifies the former as the full subcategory of \( D(\text{Gr}(A)) \) consisting of complexes such that the
multiplication map \( f : E \rightarrow E(n) \) is an isomorphism in \( D(\text{Gr}(A)) \). Thus condition (2) of Lemma 2.34
may be rewritten with respect to such complexes \( E \).

Given \( M \in D(\text{Gr}(A)) \), let \( I_M \) be the subset of \( A \) consisting of elements \( f = \sum f_i \in A \) such that,
for each homogeneous component \( f_m \in A_m \) of \( f \), the complex \( T(M, f_m) \) has zero cohomology. Then exactly
the same proof as in [30, Tag 091Q] shows the following

**Lemma 2.36.** The subset \( I_M \subset A \) is a radical graded ideal.

**Definition 2.37.** Let \( I \subset A \) be a graded ideal. An object \( M \in D(A) \) is said to be derived complete with respect to \( I \) if for every homogeneous \( f \in I \), the complex \( T(M, f) \) has zero cohomology. In other words,
if \( I \subset I_M \).

A graded module \( M \in \text{Gr}(A) \) is said to be derived complete with respect to \( I \) if the corresponding complex \( M[0] \in D(\text{Gr}(A)) \) concentrated in cohomological degree zero is so.

Denote by \( D_{I\text{-comp}}(\text{Gr}(A)) \subset D(\text{Gr}(A)) \) the full subcategory consisting of objects that are derived complete with respect to \( I \). Since \( T(M, f) \) can be written as a derived Hom complex (2.33), this full subcategory is triangulated and split-closed. Moreover, by Lemma 2.34(4), an object \( M \in D(\text{Gr}(A)) \) is in \( D_{I\text{-comp}}(\text{Gr}(A)) \) if and only if all of its cohomology modules are in the subcategory \( I\text{-comp} \subset \text{Gr}(A) \). This, in turn, shows that \( I\text{-comp} \) is a weak Serre subcategory.

Now let us focus on the case when \( I \) is finitely generated, say by \( f_1, \ldots, f_r \). In this case, we first construct
a free resolution of the extended \( \mathbb{C} \) complex \( R\Gamma_{(f_1, \ldots, f_r)}(A) \) by replacing each term \( A_{(f_1, \ldots, f_r)} \) in (2.18)
by its free resolution \( A_{(f_1, \ldots, f_r)} \) given in (2.32). Indeed, for any two homogeneous elements \( f \) and \( g \), say of
degrees \( m \) and \( n \) respectively, then the canonical map \( A_f \rightarrow A_g \) lifts canonically to a map \( A_{(f)} \rightarrow A_{(g)} \)
of cochain complexes, induced by a map of the corresponding directed systems defining the respective
homotopy colimit, where the component \( A(m) \) in (2.32) is sent to the component \( A((m+n)i) \) via the
map \( g^i \). Thus we define
\[
\begin{equation}
\widetilde{R}\Gamma_{(f_1, \ldots, f_r)}(A) := [A \xrightarrow{d_0} \prod_{1 \leq i_0 \leq r} A_{(f_{i_0})} \xrightarrow{d_1} \prod_{1 \leq i_1 \leq r} A_{(f_{i_0}f_{i_1})} \xrightarrow{d_2} \cdots \xrightarrow{d_{r-1}} A_{(f_1 \cdots f_r)}]
\end{equation}
\]
understood as a total complex of a double complex. This gives a free resolution

\[(2.39) \quad \mathcal{H} \Gamma (f_1, \ldots, f_r)(A) \xrightarrow{\sim} \Gamma (f_1, \ldots, f_r)(A)\]

In fact, under the the identifications (2.20) above and (2.40) here (for \(M = A\)), this quasi-isomorphism is precisely the quasi-isomorphism (13.35) from the homotopy colimit to the colimit of a directed system.

\[(2.40) \quad \mathcal{H} \Gamma (f_1, \ldots, f_r)(M) \cong \text{holim}_{(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r} K^* (M; f_1^{m_1}, \ldots, f_r^{m_r})\]

**Definition 2.41.** For any cochain complex \(M \in \text{Ch}(\text{Gr}(A))\), define the cochain complex \(L \Lambda (f_1, \ldots, f_r)(M) \in \text{Ch}(\text{Gr}(A))\) to be the Hom-complex

\[
L \Lambda (f_1, \ldots, f_r)(M) := \text{Hom}_A (\mathcal{H} \Gamma (f_1, \ldots, f_r)(A), M)
\]

\[
= [M \xleftarrow{d_0} \prod_{1 \leq i_0 \leq r} T(M, f_{i_0}) \xleftarrow{d_1} \prod_{1 \leq i_0 < i_1 \leq r} T(M, f_{i_0} f_{i_1}) \xleftarrow{d_2} \cdots \xleftarrow{d_{r-1}} T(M, f_1 \ldots f_r)]
\]

where the last line is thought of as the total complex of a double complex.

As an immediate consequence of (2.40), we may write this complex as a homotopy limit

\[
L \Lambda (f_1, \ldots, f_r)(M) \cong \text{holim}_{(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r} K^* (M; f_1^{m_1}, \ldots, f_r^{m_r})
\]

Moreover, in view of (2.39), it also gives an explicit model for the derived Hom complex

\[
L \Lambda (f_1, \ldots, f_r)(M) \cong R \text{Hom}_A (\Gamma r (A), M)
\]

where \(I = (f_1, \ldots, f_r)\) is the ideal generated by the elements \(f_i\). Moreover, recall that \(\Gamma r (A)\) has \(I^\infty\)-torsion cohomology, so that \(A_f \otimes_A \Gamma r (A) \cong 0\) for all homogeneous elements \(f \in I\). As a result, we have

\[
T (L \Lambda (f_1, \ldots, f_r)(M), f) \cong R \text{Hom}_A (A_f, R \text{Hom}_A (\Gamma r (A), M)) \cong R \text{Hom}_A (A_f \otimes_A \Gamma r (A), M) \cong 0
\]

for any homogeneous element \(f \in I\). In other words, the complex \(L \Lambda (f_1, \ldots, f_r)(M)\) is always derived \(I\)-complete. Thus, the exact functor \(L \Lambda (f_1, \ldots, f_r)\) descends to a functor

\[(2.42) \quad L \Lambda I := L \Lambda (f_1, \ldots, f_r) : D(\text{Gr}(A)) \rightarrow D_I(\text{comp}(\text{Gr}(A)))\]

Moreover, the canonical map \(\mathcal{H} \Gamma (f_1, \ldots, f_r)(A) \rightarrow A\) defined by projecting to the first component in (2.38) induced a map \(\epsilon^r_M : M \rightarrow L \Lambda (f_1, \ldots, f_r)(M)\) of cochain complexes. Thus, there is a natural transformation

\[(2.43) \quad \epsilon^r_M : \text{id} \Rightarrow \iota \circ L \Lambda I\]

where \(\iota : D_I(\text{comp}(\text{Gr}(A)) \rightarrow D(\text{Gr}(A))\) is the inclusion functor. The cone of \(\epsilon^r_M\) is homotopic to the cokernel of the (injective) map \(\epsilon^r_M\). If we shift this cokernel by \(-1\), we obtain the following

**Definition 2.44.** Given a cochain complex \(M \in \text{Gr}(A)\), define

\[(2.45) \quad \mathcal{E} (f_1, \ldots, f_r)(M) := [\prod_{1 \leq i_0 \leq r} T(M, f_{i_0}) \xleftarrow{d^t} \prod_{1 \leq i_0 < i_1 \leq r} T(M, f_{i_0} f_{i_1}) \xleftarrow{d^{t-1}} \cdots \xleftarrow{d^2} T(M, f_1 \ldots f_r)]\]

understood as the total complex of a double complex.

Clearly the natural transformation (2.43) is an isomorphism for \(M \in D(\text{Gr}(A))\) if and only if \(\mathcal{E} (f_1, \ldots, f_r)(M)\) has zero cohomology. Now if \(M \in D_I(\text{comp}(\text{Gr}(A))\) then each of the terms in (2.45), thought of as a column in a double complex, has zero cohomology. As an iterated cone on such columns, the total complex \(\mathcal{E} (f_1, \ldots, f_r)(M)\) also has zero cohomology. This shows that

\[(2.46) \quad \text{If } M \in D_I(\text{comp}(\text{Gr}(A)), \text{ then the natural transformation } \epsilon^r_M \text{ is an isomorphism in } D(\text{Gr}(A)).\]

As a formal consequence of (2.42), (2.43) and (2.46), we have the following

**Theorem 2.47.** The functor (2.42) is left adjoint to the inclusion \(\iota : D_I(\text{comp}(\text{Gr}(A)) \rightarrow D(\text{Gr}(A))\), with unit given by (2.43).
Definition 2.48. The complex \( L \Lambda_I(M) \) is either called the local homology complex, or the derived completion, of \( M \) with respect to the graded ideal \( I \subset A \).

Dual to (2.27), we also have the following exact triangle

\[
\cdots \to \mathcal{E}_I(M) \xrightarrow{\eta_I} M \xrightarrow{\delta_I} \Lambda \Lambda_I(M) \xrightarrow{-\delta_I} \mathcal{E}_I(M)[1] \to \cdots
\]

(2.49)

In fact, (2.27) and (2.49) are dual in a very precise sense. Namely if we substitute \( M = A \) in (2.27) to get the exact triangle

\[
\cdots \to R \Gamma_I(A) \xrightarrow{\epsilon_I} A \xrightarrow{\eta_I} \hat{\mathcal{C}}_I(A) \xrightarrow{\delta} R \Gamma_I(A)[1] \to \cdots
\]

then we have

\[
(2.27) \simeq (2.50) \otimes_A^L M \quad \text{and} \quad (2.49) \simeq R \text{Hom}_A((2.50), M)
\]

We record the following graded analogue of [30] Tag 0A6S, 0A6D:

Lemma 2.51. Given \( E \in \mathcal{D}(\text{Gr}(A)) \), suppose that there exists a homogeneous element \( f \in I \), say of degree \( n \), such that the multiplication map \( f : E \to E(n) \) is an isomorphism in \( \mathcal{D}(\text{Gr}(A)) \), then we have \( R \Gamma_I(E) \simeq 0 \) and \( L \Lambda_I(E) \simeq 0 \).

Proof. The property that \( f : E \to E(n) \) is an isomorphism in \( \mathcal{D}(\text{Gr}(A)) \) is preserved under \( R \Gamma_I(A) \otimes_A^L - \), so that \( R \Gamma_I(E) \) also has this property. However, cohomology modules of \( R \Gamma_I(E) \) are \( I^\infty \)-torsion, so that in particular any element \( x \in H^p(R \Gamma_I(E)) \) is annihilated by some high enough powers of \( f \). Hence we must have \( x = 0 \).

To show that \( L \Lambda_I(E) \simeq 0 \), notice that, by the criterion in Lemma 2.34(2) (see also Remark 2.35), we have \( R \text{Hom}_A(E, M) \simeq 0 \) for all \( M \in \mathcal{D}_{I\text{-comp}}(\text{Gr}(A)) \). Since the derived \( I \)-completion functor is left adjoint to inclusion (see Theorem 2.47), this shows that \( L \Lambda_I(E) \simeq 0 \).

For any homogeneous element \( f \in I \), the module \( A_f \) clearly satisfies the condition of this Lemma. The same is therefore true for \( M_f = A_f \otimes_A M \) and \( T(M, f) \simeq R \text{Hom}_A(A_f, M) \). This observation allows us to prove the following

Proposition 2.52. For any \( M \in \mathcal{D}(\text{Gr}(A)) \), we have \( L \Lambda_I(\hat{\mathcal{C}}_I(M)) \simeq 0 \) and \( R \Gamma_I(\mathcal{E}_I(M)) \simeq 0 \). Moreover, \( R \Gamma_I(M) \simeq 0 \iff L \Lambda_I(M) \simeq 0 \).

Proof. For the first statement, notice that in the double complexes (2.24) and (2.35) defining \( \hat{\mathcal{C}}_I(M) \) and \( \mathcal{E}_I(M) \), each column satisfies the condition of Lemma 2.51 (see the paragraph preceding the present Proposition), so that we have \( L \Lambda_I(\hat{\mathcal{C}}_I(M)) \simeq 0 \) and \( R \Gamma_I(\mathcal{E}_I(M)) \simeq 0 \).

For the second statement, if \( R \Gamma_I(M) \simeq 0 \), then we have \( M \simeq \hat{\mathcal{C}}_I(M) \) by (2.27), so that \( L \Lambda_I(M) \simeq 0 \) by the first statement. The converse is completely symmetric, using (2.49) this time.

Corollary 2.53. For any \( M \in \mathcal{D}(\text{Gr}(A)) \), we have the following isomorphisms in \( \mathcal{D}(\text{Gr}(A)) \):

\[
R \Gamma_I(M) \xrightarrow{R \Gamma_I(\eta_M)} R \Gamma_I(L \Lambda_I(M)) \quad \text{and} \quad L \Lambda_I(R \Gamma_I(M)) \xrightarrow{L \Lambda_I(\delta_M)} L \Lambda_I(M)
\]

Definition 2.54. A complex \( M \in \mathcal{D}(\text{Gr}(A)) \) is said to be \( I \)-trivial if we have \( R \Gamma_I(M) \simeq 0 \), or equivalently \( L \Lambda_I(M) \simeq 0 \) by Proposition 2.52.

Thus, we have the following recollement (for two functors pointing in opposite horizontal directions, the functor on top of the other is implicitly understood to be the left adjoint of the other):

\[
\begin{array}{ccc}
\mathcal{D}_{I\text{-triv}}(\text{Gr}(A)) & \xrightarrow{\hat{\mathcal{C}}_I} & \mathcal{D}(\text{Gr}(A)) \\
\xrightarrow{\mathcal{E}_I} & \xrightarrow{\delta} & \xrightarrow{\epsilon} \\
\mathcal{D}(\text{Gr}(A)) & \xrightarrow{\Gamma_I} & \mathcal{D}_{I\text{-comp}}(\text{Gr}(A))
\end{array}
\]

\[
\begin{array}{ccc}
R \Gamma_I & \xrightarrow{R \Gamma_I(\eta_M)} & R \Gamma_I(L \Lambda_I(M)) \\
\xrightarrow{L \Lambda_I} & \xrightarrow{L \Lambda_I(\delta_M)} & \xrightarrow{L \Lambda_I(\epsilon_M)} \\
L \Lambda_I(M) & \xrightarrow{\delta_M} & L \Lambda_I(M)
\end{array}
\]
In particular, this gives rise to two semi-orthogonal decompositions
\[
\mathcal{D}(\text{Gr}(A)) = \langle \mathcal{D}_{\text{triv}}(\text{Gr}(A)), \mathcal{D}_{I=\text{Tor}}(\text{Gr}(A)) \rangle
\]
(2.55)
\[
\mathcal{D}(\text{Gr}(A)) = \langle \mathcal{D}_{I=\text{comp}}(\text{Gr}(A)), \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \rangle
\]
whose decomposition triangles are (2.27) and (2.49) respectively.

The notion of derived completeness is closely related to the usual notion of completeness with respect to a graded ideal. We will only mention one aspect of this relation, in the form of Proposition 2.56 below. Let \( A \) be a \( \mathbb{Z} \)-graded ring, and \( I \subset A \) be a graded ideal. Then we may form the inverse system \((A/I^n)_{n \geq 1}\) of graded rings, and take its inverse limit \( \hat{A} := \lim A/I^n \) in the category of graded rings, which is computed as a limit in each graded component. Similarly, for any graded module \( M \) over \( A \), one can also take the inverse limit \( \hat{M} := \lim M/I^n M \) in the category of graded modules, which is again computed as a limit in each graded component. As usual, completions come with canonical maps \( A \to \hat{A} \) and \( M \to \hat{M} \). We say that a graded module \( M \in \text{Gr}(A) \) is \( I \)-separated if the canonical map \( M \to \hat{M} \) is injective; it is \( I \)-precomplete if \( M \to \hat{M} \) is surjective; it is \( I \)-adically complete if \( M \to \hat{M} \) is bijective. Then we have the following graded analogue of [30, Tag 091R]:

**Proposition 2.56.** Given a graded module \( M \in \text{Gr}(A) \), then we have:

1. if \( M \) is \( I \)-adically complete, then it is derived complete with respect to \( I \);
2. if \( I \) is finitely generated, and if \( M \) is derived complete with respect to \( I \), then \( M \) is \( I \)-precomplete.

For later use, we consider some duality operators on certain coherent subcategories of \( \mathcal{D}(\text{Gr}(A)) \) and \( \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \). We will assume that \( A \) is a Noetherian \( \mathbb{Z} \)-graded ring in the remainder of this subsection. We start with the following well-known application of Proposition 2.13.

**Lemma 2.57.** Suppose that \( A \) has finite injective dimension over itself, then every object \( M \in \mathcal{D}_{\text{coh}}(\text{Gr}(A)) \) is derived reflexive. Thus, the functor \( D_A := R\text{Hom}_A(-, A) : \mathcal{D}(\text{Gr}(A))^\text{op} \to \mathcal{D}(\text{Gr}(A)) \) restricts to an involutive anti-equivalence of \( \mathcal{D}_{\text{coh}}(\text{Gr}(A)) \), which interchanges \( \mathcal{D}_{\text{coh}}^-(\text{Gr}(A)) \) and \( \mathcal{D}_{\text{coh}}^+(\text{Gr}(A)) \).

Next, we consider a duality operator on a certain coherent subcategory of \( \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \), given by the restriction of the functor
\[
D_{\tilde{\mathcal{C}}_I(A)} := R\text{Hom}_A(-, \tilde{\mathcal{C}}_I(A)) : \mathcal{D}(\text{Gr}(A))^\text{op} \to \mathcal{D}_{I=\text{triv}}(\text{Gr}(A))
\]
(2.58)
Notice that, for any \( M \in \mathcal{D}(\text{Gr}(A)) \), we have
\[
R\text{Hom}_A(R\Gamma_I(A), R\text{Hom}_A(M, \tilde{\mathcal{C}}_I(A))) \simeq R\text{Hom}_A(R\Gamma_I(A) \otimes^L_A M, \tilde{\mathcal{C}}_I(A)) \simeq 0
\]
so that the functor (2.58) indeed lands in \( \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \).

Applying Proposition 2.13 we have

**Lemma 2.59.** Suppose that \( \tilde{\mathcal{C}}_I(A) \) has finite injective dimension, then for all \( M \in \mathcal{D}_{\text{coh}}(\text{Gr}(A)) \), we have
\[
\tilde{\mathcal{C}}_I(A) \otimes_A M \simeq D_{\tilde{\mathcal{C}}_I(A)}(D_{\tilde{\mathcal{C}}_I(A)}(M))
\]

The hypothesis of Lemma 2.59 is satisfied, for example, when \( A \) has finite injective dimension over itself:

**Lemma 2.60.** If \( M \in \mathcal{D}(\text{Gr}(A)) \) has finite injective dimension, then so does \( \tilde{\mathcal{C}}_I(M) \).

**Proof.** By construction (2.28), \( \tilde{\mathcal{C}}_I(M) \) is a finite complex of localizations of \( M \). Thus, the result follows from Lemma 2.15. 

Now we consider the restriction of (2.58) to \( \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \). Observe that, by the orthogonality \( \mathcal{D}_{I=\text{Tor}}(\text{Gr}(A)) \perp \mathcal{D}_{I=\text{triv}}(\text{Gr}(A)) \), we have, for all \( M, N \in \mathcal{D}(\text{Gr}(A)) \), the quasi-isomorphism
\[
R\text{Hom}_A(\tilde{\mathcal{C}}_I(M), \tilde{\mathcal{C}}_I(N)) \simeq R\text{Hom}_A(M, \tilde{\mathcal{C}}_I(N))
\]
so that in particular, we have
\[
D_{\tilde{\mathcal{C}}_I(A)}(\tilde{\mathcal{C}}_I(M)) \simeq D_{\tilde{\mathcal{C}}_I(A)}(M)
\]
(2.61)
Lemma 2.62. If $M \in \mathcal{D}_c\text{oh}(\text{Gr}(A))$, then we have

$$\mathcal{D}_c\text{oh}(\text{Gr}(A)) \cong \mathcal{D}_c\text{oh}(\text{Gr}(A)) \cong \hat{C}_f(\mathcal{D}_c\text{oh}(\text{Gr}(A)))$$

Proof. The first quasi-isomorphism is (2.61). The second quasi-isomorphism is an application of Proposition 2.64.

Combined with Lemma 2.60, this gives a duality on certain coherent subcategories of $\mathcal{D}_{I\text{-triv}}(\text{Gr}(A))$.

Definition 2.63. For each $\blacklozenge \in \{+, -, b\}$, let $\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A)) \subset \mathcal{D}_{I\text{-triv}}(\text{Gr}(A))$ be the essential image of $\mathcal{D}_{c\text{oh}}(\text{Gr}(A))$ under the functor $\hat{C}_f : \mathcal{D}^\bullet(\text{Gr}(A)) \to \mathcal{D}_{I\text{-triv}}(\text{Gr}(A))$.

Proposition 2.64. Suppose that $\hat{C}_f(\text{Gr}(A))$ has finite injective dimension, then the functor (2.58) restricts to an involutive anti-equivalence of $\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A))$, which interchanges $\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A))$ and $\mathcal{D}_{c\text{oh}(I\text{-triv})}(\hat{C}_f(\text{Gr}(A)))$.

Proof. The fact that (2.58) is involutive on $\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A))$ follows directly from Lemma 2.59 and Proposition 2.64. The fact that it interchanges $\mathcal{D}^-$ and $\mathcal{D}^+$ is a consequence of $\hat{C}_f(\text{Gr}(A))$ having finite injective dimension.

Combined with Lemma 2.60, we have the following.

Proposition 2.65. Suppose that $A$ has finite injective dimension over itself, then the following diagrams commute up to isomorphism of functors:

$$\begin{array}{ccc}
\mathcal{D}_{c\text{oh}}(\text{Gr}(A)) & \xrightarrow{\mathcal{D}_A} & \mathcal{D}_{c\text{oh}}(\hat{C}_f(\text{Gr}(A)))^\text{op} \\
\downarrow \hat{C}_f & & \downarrow \hat{C}_f \\
\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A)) & \xrightarrow{\mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A))} & \mathcal{D}_{c\text{oh}(I\text{-triv})}(\hat{C}_f(\text{Gr}(A)))^\text{op} \\
& \xrightarrow{\mathcal{D}_{c\text{oh}(I\text{-triv})}(\hat{C}_f(\text{Gr}(A)))} & \mathcal{D}_{c\text{oh}(I\text{-triv})}(\text{Gr}(A))
\end{array}$$

Proof. We have seen in Lemma 2.62 that the left square is always commutative. If $A$ has finite injective dimension, then by Lemma 2.60 so does $\hat{C}_f(A)$. Thus, by Lemma 2.59 and Proposition 2.64, each horizontal arrow is an equivalence of categories, and the composition for each row is isomorphic to the identity functor. This allows us to deduce the commutativity of the right square from that of the left.

3. Graded rings and projective spaces

In this section, we develop Serre’s equivalence for $\mathbb{Z}$-graded rings. Besides generalizing from $\mathbb{N}$-graded rings to $\mathbb{Z}$-graded rings (which is straightforward), we also sharpen the statement of Serre’s equivalence by emphasizing a certain Cartier condition (see Theorem 5.15), although this sharpened form is already implicit in the usual proof of Serre’s equivalence. Although somewhat well-known, we provide some details to the construction of our version of Serre’s equivalence, firstly because we need to keep track of a web of compatibility conditions in the course of the construction, and secondly because an interpretation of Grothendieck duality in terms of Greenlees-May duality naturally flows out of this construction. Both of these aspects will be crucial to our purposes later.

3.1. Serre’s equivalence. Let $A$ be a $\mathbb{Z}$-graded ring. Denote by $I^+$ and $I^-$ the graded ideals $I^+ = A_{>0} \cdot A$ and $I^- = A_{<0} \cdot A$.

Let $X^+ = \text{Proj}^+(A) := \text{Proj}(A_{\geq 0})$. Thus, $X^+$ is covered by the standard open sets $D(f) \cong \text{Spec}((A_{\geq 0})_f)$, for homogeneous $f$ of positive degree. Since the canonical map $(A_{\geq 0})_f \to A_f$ is an isomorphism of graded rings for $\deg(f) > 0$, the standard open sets $D(f)$ can also be described as $D(f) \cong \text{Spec}(A_f)$.

In order to guarantee that $\text{Proj}^+(A)$ is quasi-compact, we assume that $I^+$ is finitely generated. More precisely, we make following assumption:

(3.1) Assume that the ideal $I^+$ is generated by homogeneous elements $f_1, \ldots, f_p$ of positive degrees $d_i := \deg(f_i) > 0$, and let $d > 0$ be a positive integer that is divisible by each of $d_i$.

Then we record the following elementary result from the proof of [30, Tag 0EGH]:
Lemma 3.2. Under the assumption (3.1), for any $N > dp - \sum_{i=1}^{p} d_i$, we have $A_N = A_d \cdot A_{N-d}$.

Proof. Given $f_1^e \cdot \ldots \cdot f_p^e \cdot g_1^l \cdot \ldots \cdot g_q^l \in A_N$ with $\deg(g_i) \leq 0$, we must have $e_i \geq d_i/d$ for some $1 \leq i \leq p$. Then $f_1^e \cdot \ldots \cdot f_p^e \cdot g_1^l \cdot \ldots \cdot g_q^l = (f_1^{d/d}) \cdot (f_1^{e_i-d/d} \cdot \ldots \cdot f_p^{d/d}) \cdot g_1^l \cdot \ldots \cdot g_q^l$.

For later use, we apply this Lemma to give an alternative description of $(I^+)^\infty$-torsion elements in the sense of Definition 2.16.

Lemma 3.3. Under the assumption (3.1), suppose we are given a homogeneous element $x$ in a graded module $M$, then the following is equivalent:

1. $x$ is $(I^+)^\infty$-torsion;
2. $x \cdot (A_d)^n = 0$ for all sufficiently large $n$.
3. $x \cdot A^n_s = 0$ for some $s \in \mathbb{Z}$.

Proof. Since $A_d \subset I^+$, the implication (1) $\Rightarrow$ (2) is immediate. To show that (2) $\Rightarrow$ (3), notice that Lemma 3.2 implies that $A_N = (A_d)^n \cdot A_{N-dn}$ for $N \geq dp + dn$, so that we have $x \cdot A_{dp+dn} = 0$. The implication (3) $\Rightarrow$ (1) is obvious since the ideal $I^+$ is generated by elements of positive degrees.

Each graded module $M \in \text{Gr}(A)$ is in particular a graded module over $A_{\geq 0}$, and hence gives an associated quasi-coherent sheaf $\widetilde{M}$ on $\text{Proj}^+(A)$. The association $M \mapsto \widetilde{M}$ is lax monoidal, meaning that there are canonical maps $\widetilde{M} \otimes_{\mathcal{O}_X^+} \widetilde{N} \to M \otimes_A N$, satisfying the usual associativity and unitality conditions. Indeed, on $D(f)$, this map is induced by the obvious $A(f)$ bilinear map $M(f) \times N(f) \to (M \otimes_A N)(f)$. Moreover, there is a canonical map

\begin{equation}
M_0 \to H^0(X^+, \widetilde{M})
\end{equation}

doing $A_0$-modules, which sends $\xi \in M_0$ to $\xi \in M(f) = \Gamma(D(f), \widetilde{M})$ on each $D(f)$, for $\deg(f) > 0$.

Let $\widetilde{A}$ be the sheaf of $\mathbb{Z}$-graded algebras on $\text{Proj}^+(A)$ given by $\widetilde{A}_i := \widetilde{A}(i)$. The graded algebra structure is induced by the lax monoidal structure $\widetilde{A(i)} \otimes_{\mathcal{O}_X^+} \widetilde{A(j)} \to \widetilde{A(i+j)}$. In general, these maps may not be isomorphisms. However, we have the following

Lemma 3.5. Assume that (3.1) holds, then we have the followings:

1. The sheaf $\widetilde{A}$ is an ample invertible sheaf on $\text{Proj}^+(A)$.
2. For any $M \in \text{Gr}(A)$ and any $i \in \mathbb{Z}$, the map $\widetilde{M} \otimes_{\mathcal{O}_X^+} \widetilde{A(di)} \to \widetilde{M(di)}$ is an isomorphism. In particular, for each $i, j \in \mathbb{Z}$, the multiplication map $\widetilde{A_i} \otimes_{\mathcal{O}_X^+} \widetilde{A_j} \to \widetilde{A_{i+j}}$ is an isomorphism.

Proof. By Lemma 3.2 the standard open sets $D(f)$, for $\deg(f) = d$, covers $\text{Proj}^+(A)$. Thus, condition (2) can be checked on $D(f)$, which is obvious. This also shows that $\widetilde{A}_d$ is invertible. Finally, it is ample again because the standard open sets $D(f)$, for $\deg(f) = d$, covers $\text{Proj}^+(A)$.

Definition 3.6. A $\mathbb{Z}$-graded ring $A$ is said to be positively $\frac{1}{d}$-Cartier, for an integer $d > 0$, if the conclusion of Lemma 3.5 holds. Similarly, it is said to be negatively $\frac{1}{d}$-Cartier if the conclusion of Lemma 3.5 holds for $X = \text{Proj}^+(A)$ in place of $X^+$. It is said to be $\frac{1}{d}$-Cartier if it is both positively and negatively $\frac{1}{d}$-Cartier. When $d = 1$, we simply call these (positively/negatively) Cartier.

For each $m > 0$, let $A^{(m)}$ be the $\mathbb{Z}$-graded ring $(A^{(m)})_i = A_{mi}$. Then there is a canonical isomorphism $\text{Proj}^+(A^{(m)}) \cong \text{Proj}^+(A)$ given by $(A^{(m)})_{i f^m} = A_{i f}$.

For any graded module $M \in \text{Gr}(A)$, one also has an associated graded module $\hat{M}^{(m)} \in \text{Gr}(A^{(m)})$, defined by $(\hat{M}^{(m)})_i := M_{mi}$. This allows us to extend results on $\text{Proj}^+(A)$ from the usual case, where $A_{\geq 0}$ is generated by $A_1$ over $A_0$, to the general case (3.1). For example, there is a partial converse to Lemma 3.2 given by Proposition 3.9 below. To this end, we first recall the following standard result (see, e.g., [30 Tags 01Q1, 01QJ]):

\footnote{We use the definition of ampleness in [30 Tag 01PS].}
Proposition 3.7. Let \( \pi : X \to Y = \text{Spec} R \) be a proper morphism, where \( R \) is Noetherian, and let \( \mathcal{L} \) be an ample invertible sheaf on \( X \). Define the \( \mathbb{Z} \)-graded algebra \( S \) by \( S_i := H^0(X; \mathcal{L}^i) \), then \( S_{\geq 0} \) is Noetherian, and there is a canonical isomorphism \( \varphi : X \to \text{Proj}^+(S) \) over \( \text{Spec} R \).

Moreover, for any quasi-coherent sheaf \( \mathcal{F} \) on \( X \), let \( M \in \text{Gr}(S) \) be the graded \( S \)-module given by \( M_i := H^0(X, \mathcal{F} \otimes \mathcal{L}^i) \), then there is a canonical isomorphism \( \varphi^*(\tilde{M}) \to \mathcal{F} \), such that the composition
\[
M_0 \xrightarrow{(3.4)} H^0(\text{Proj}^+(S), \tilde{M}) = H^0(X, \varphi^*(\tilde{M})) \xrightarrow{\sim} H^0(X, \mathcal{F}) =: M_0
\]
is the identity.

In later applications of flips and flops, we will consider ample Weil divisors on \( X \) that are only assumed to be \( \mathbb{Q} \)-Cartier, but may not be Cartier. We formulate this by considering the following situation:

On a scheme \( X \) proper over another Noetherian scheme \( Y \), there is a quasi-coherent sheaf of \( \mathbb{Z} \)-graded algebras \( \bigoplus_{i \in \mathbb{Z}} \mathcal{O}(i) \) such that each \( \mathcal{O}(i) \) is coherent, and there exists some positive integer \( d > 0 \) such that
\[
\text{(1) the sheaf } \mathcal{O}(d) \text{ is an invertible sheaf on } X \text{ ample over } Y;
\]
\[
\text{(2) for each } i, j \in \mathbb{Z}, \text{ the multiplication map } \mathcal{O}(di) \otimes \mathcal{O}(j) \to \mathcal{O}(di + j) \text{ is an isomorphism.}
\]

For notational simplicity, we consider here the case when \( Y \) is affine, say \( Y = \text{Spec} R \). Let \( B \) be the graded algebra over \( R \) given by \( B_i := H^0(X, \mathcal{O}(i)) \). Then we have the following

Proposition 3.9. The \( \mathbb{N} \)-graded ring \( B_{\geq 0} \) is Noetherian. Moreover, there is a canonical isomorphism \( \varphi : X \to \text{Proj}^+(B) \) over \( \text{Spec} R \), together with a canonical isomorphism \( \bigoplus_{i \in \mathbb{Z}} \varphi^*(B(i)) \to \bigoplus_{i \in \mathbb{Z}} \mathcal{O}(i) \) of sheaves of \( \mathbb{Z} \)-graded algebras on \( X \), such that the composition
\[
B_i \xrightarrow{(3.4)} H^0(\text{Proj}^+(B), \tilde{B}(i)) = H^0(X, \varphi^*(\tilde{B}(i))) \xrightarrow{\sim} H^0(X, \mathcal{O}(i)) =: B_i
\]
is the identity.

Furthermore, for any quasi-coherent sheaf \( \mathcal{F} \) on \( X \), let \( M \in \text{Gr}(B) \) be the graded \( B \)-module given by \( M_i := H^0(X, \mathcal{F} \otimes \mathcal{O}(i)) \), then there is a canonical isomorphism \( \varphi^*(\tilde{M}) \to \mathcal{F} \), such that the composition
\[
M_0 \xrightarrow{(3.4)} H^0(\text{Proj}^+(B), \tilde{M}) = H^0(X, \varphi^*(\tilde{M})) \xrightarrow{\sim} H^0(X, \mathcal{F}) =: M_0
\]
is the identity.

Proof. All these statements follows by applying Proposition 3.7 to the ample invertible sheaf \( \mathcal{L} := \mathcal{O}(d) \), since we have \( \psi : \text{Proj}^+(B(d)) \to \text{Proj}^+(B) \), together with a canonical isomorphism \( \psi^*(\tilde{M}) \cong \tilde{M}(d) \) for each \( M \in \text{Gr}(B) \).

Corollary 3.10. In the situation \((3.8)\) with \( Y = \text{Spec} R \), the \( \mathbb{Z} \)-graded algebra \( B := \bigoplus_{i \in \mathbb{Z}} H^0(X, \mathcal{O}(i)) \) is positively \( \frac{1}{d} \)-Cartier.

Proof. To verify condition (2) of Lemma 3.5 we argue as in [30, Tag 01MU]. Namely, Proposition 3.9 together with condition (2) in \((3.8)\) implies in particular that the map \( B(di) \otimes_{\mathcal{O}_{\text{Proj}^+(B)}} B(dj) \to B(di + dj) \) is an isomorphism. One can use this to show that the standard open subsets \( D(fg) \), for \( \text{deg}(f) - \text{deg}(g) = d \), covers \( \text{Proj}^+(B) \), which then implies condition (2) of Lemma 3.5.

Now we get back to the situation of a \( \mathbb{Z} \)-graded ring \( A \) such that \( A_{\geq 0} \) is Noetherian, so that it is positively \( \frac{1}{d} \)-Cartier for some \( d > 0 \). The maps \((3.4)\), taken for various choices of \( M \), assemble to give many other maps. For example, if we let \( \tilde{M}(i) := \tilde{M} \otimes_{\mathcal{O}_X} A(i) \), then these maps give rise to canonical graded \( A \)-module structures on
\[
N' := \bigoplus_{i \in \mathbb{Z}} H^0(X^+, \tilde{M}(i)) \quad \text{and} \quad N := \bigoplus_{i \in \mathbb{Z}} H^0(X^+, \tilde{M}(i))
\]
and there are canonical maps \( M \to N' \leftarrow N \) of graded \( A \)-modules. These map satisfies the following properties (see, e.g., [30, 0B5R] for the case when \( M \) is finitely generated, which implies the general case by taking a directed colimit):

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Proposition 3.11. Assume that $A_{\geq 0}$ is Noetherian and $A$ is positively $\frac{1}{d}$-Cartier, then

(1) for any $i \in \mathbb{Z}$, the map $N_{d_i} \to N_{d_i}'$ is an isomorphism;
(2) both the kernel and cokernel of $M \to N'$ are $(I^+)^{\infty}$-torsion in the sense of Definition 2.10.
(3) if $M_{\geq c}$ is finitely generated over $A_{\geq 0}$ for some $c \in \mathbb{Z}$, then so are $N_{\geq c'}$ and $N'_{\geq c'}$, for any $c' \in \mathbb{Z}$.

Proof. (1) is obvious. (3) follows from [30] 0B5R, statements (1),(2)]. Thus it suffices to show (2). For this, one can either modify the proof of [30] 0B5R, statement (5)], or simply notice that exact sequence (2.29) gives

$$\ker(M \to N') = H^0(R\Gamma_{I^+}(M)) \quad \text{and} \quad \coker(M \to N') = H^1(R\Gamma_{I^+}(M))$$

both of which are $(I^+)^{\infty}$-torsion by definition (see Theorem 2.20).

Propositions 3.9 and 3.11 can be combined to describe the abelian category of quasi-coherent sheaves on Proj$^+(A)$ in terms of a Serre quotient (see Theorem 3.15 below). We start with the following

Definition 3.12. A homogeneous element $x \in M$ in a graded module $M \in \text{Gr}(A)$ is said to be positively torsion if it is $(I^+)^{\infty}$-torsion in the sense of Definition 2.10. The graded module $M$ is said to be positively torsion if every homogeneous element in it is positively torsion. Denote by Tor$^+(A) \subset \text{Gr}(A)$ the full subcategory of positively torsion modules.

A map of graded modules $M \to N$ over $A$ is said to be an $m$-uple equivalence if the associated map $M^{(m)} \to N^{(m)}$ over $A^{(m)}$ is an isomorphism. A graded module $M \in \text{Gr}(A)$ is said to be $m$-trivial if the map $0 \to M$ is an $m$-uple equivalence.

A map of graded modules $M \to N$ over $A$ is said to be an $m$-uple positive equivalence if both the kernel and cokernel of $M^{(m)} \to N^{(m)}$ are positively torsion over $A^{(m)}$. A graded module $M \in \text{Gr}(A)$ is said to be $m$-positively torsion, if $0 \to M$ is an $m$-uple positive equivalence. Denote by Tor$^+_m(A) \subset \text{Gr}(A)$ the full subcategory of $m$-positively torsion modules.

Denote by $I^+_m(A)$ the graded ideal $I^+_m := (A^{(m)})_{>0} \cdot A^{(m)}$, and similarly $I^-_m := (A^{(m)})_{<0} \cdot A^{(m)}$.

If $A_0$ is Noetherian, then if $I^+ \subset A$ is finitely generated, so is $I^+_m \subset A^{(m)}$. The characterization of positively torsion elements in Lemma 3.3 then shows that Tor$^+_m(A) \subset \text{Tor}^+_m(A)$, so that the Serre subcategory Tor$^+_m(A)$ descends to a Serre subcategory Tor$^+_m(A) \subset \text{Q}_A$ of the Serre quotient

$$\text{Q}_A := \text{Gr}(A)/\text{Tor}^+(A)$$

and we have a canonical equivalence of abelian categories

$$\text{Q}_A^{(m)} := \text{Gr}(A)/\text{Tor}^+_m(A) \simeq \text{Q}_A/\text{Tor}^+_m(A)$$

Consider the two functors

$$(3.13) \quad (-)^{\sim} : \text{Gr}(A) \to \text{Q}_A(X^+), \quad M \mapsto \widetilde{M}$$

$$(3.14) \quad 0\mathcal{L}^+ : \text{Q}_A(X^+) \to \text{Gr}(A), \quad 0\mathcal{L}^+(\mathcal{F}_I) := H^0(X^+, \mathcal{F} \otimes \widetilde{A}(I))$$

Suppose that $A_{\geq 0}$ is Noetherian and $\frac{1}{d}$-Cartier, then clearly the functor $(-)^{\sim}$ vanishes on the Serre subcategory Tor$^+_d(A)$, and hence descends to functors

then Propositions 3.9 and 3.11 can be combined to show the following result (see also the similar statement in [30] Tag 0BXF) for coherent, instead of quasi-coherent, sheaves on Proj$^+(A)$, in the case when $A$ is concentrated in non-negative weights.)

Theorem 3.15 (Serre’s equivalence). If $A_{\geq 0}$ is Noetherian and $\frac{1}{d}$-Cartier, then the functors in (3.14) are quasi-inverse equivalences.
Proof. By Proposition 3.9 we see that the composition \((-)^\sim \circ 0\mathcal{L}^+\) is isomorphic to the identity functor on \(\text{QCoh}(X^+)\).

The composition \(0\mathcal{L}^+ \circ (-)^\sim\) assigns to each \(M \in \text{Gr}(A)\) the graded module \(N\) in the statement of Proposition 3.11. By this Proposition, \(M\) and \(N\) are linked by two \(d\)-uple positive equivalences, and hence are isomorphic in \(\text{Gr}^+(A)\). \(\square\)

We have seen that the functors (3.13) descend to equivalences after quotienting \(\text{Gr}(A)\) by \(\text{Tor}^+_d(A)\). For later use, we will also investigate the situation when we quotient \(\text{Gr}(A)\) only by \(\text{Tor}^+(A)\). In this case, instead of having an equivalence, we have an adjunction

\[
\begin{align*}
0\mathcal{L}^+ : & \text{QCoh}(X^+) \leftrightarrow \text{QCoh}(X^+) : (-)^\sim \\
0\mathcal{R}^+ : & \text{QCoh}(X^+) \rightarrow \text{Gr}(A), \quad 0\mathcal{R}^+(\mathcal{F})_i := \text{Hom}_{\mathcal{O}_{X^+}}(\tilde{A}(-i), \mathcal{F})
\end{align*}
\]

Indeed, by Proposition 3.9 we again have an isomorphism \(0\mathcal{L}^+(\mathcal{F}) \cong \mathcal{F}\), which gives the adjunction units. Moreover, the composition \(0\mathcal{L}^+ \circ (-)^\sim\) assigns to each \(M \in \text{Gr}(A)\) the graded module \(N\) in the statement of Proposition 3.11. Since the map \(M \rightarrow N\) has positively torsion kernel and cokernel by that Proposition, the correspondence \(M \rightarrow N\) descends to a map \(0\mathcal{L}^+(\mathcal{F})(\tilde{M}) \rightarrow M\) in the quotient category \(\text{Gr}(A)/\text{Tor}^+(A)\), which gives the adjunction counits.

It turns out that the functor \((-)^\sim\) also has a right adjoint, which is obtained by descending the following functor to the quotient \(\text{Gr}(A)/\text{Tor}^+(A)\):

\[
\begin{align*}
0\mathcal{R}^+ & : \text{QCoh}(X^+) \rightarrow \text{Gr}(A), \quad 0\mathcal{R}^+(\mathcal{F})_i := \text{Hom}_{\mathcal{O}_{X^+}}(\tilde{A}(-i), \mathcal{F})
\end{align*}
\]

Clearly, there is a canonical map \(0\mathcal{L}^+(\mathcal{F}) \rightarrow 0\mathcal{R}^+(\mathcal{F})\) induced by the multiplication \(\tilde{A}(i) \otimes \tilde{A}(i) \rightarrow \tilde{A}(2i)\). Since each \(A(di)\) is an invertible sheaf, this map is a \(d\)-uple equivalence. Since \((-)^\sim\) of a graded module depends only on its \(m\)-positive equivalence class, there is again a canonical isomorphism \(\mathcal{F} \cong 0\mathcal{R}^+(\mathcal{F})\). On the other hand, for any graded module \(M \in \text{Gr}(A)\), there is a canonical map

\[
M_i \rightarrow H^0(X^+, \tilde{M}(i)) \rightarrow \text{Hom}_{\mathcal{O}_{X^+}}(\tilde{A}(-i), \tilde{M}) = 0\mathcal{R}^+(\tilde{M})_i
\]

where the second map is induced by the multiplication map \(\tilde{M}(i) \otimes \tilde{A}(i) \rightarrow \tilde{M}\). This gives a map of graded modules \(M \rightarrow 0\mathcal{R}^+(\tilde{M})\). Together with functorial isomorphism \(\mathcal{F} \cong 0\mathcal{R}^+(\mathcal{F})\), they give respectively the units and counits of the adjunction

\[
(-)^\sim : \text{Gr}(A) \leftrightarrow \text{QCoh}(X^+) : 0\mathcal{R}^+
\]

The compositions \(\text{Gr}(A) \xrightarrow{(-)^\sim} \text{QCoh}(X^+) \xrightarrow{0\mathcal{L}^+} \text{Gr}(A)\) and \(\text{Gr}(A) \xrightarrow{(-)^\sim} \text{QCoh}(X^+) \xrightarrow{0\mathcal{R}^+} \text{Gr}(A)\) are closely related to the functor \(0\mathcal{L}_I^+ : \text{Gr}(A) \rightarrow \text{Gr}(A)\) we considered in Proposition 2.28. Namely, it is easy to see that there is a canonical isomorphism \(H^0(X^+, \tilde{M}(i)) \cong 0\mathcal{L}_I^+(\tilde{M})_i\). Therefore, the maps \(M \rightarrow N\) we considered in Proposition 3.11 can be rewritten as

\[
M \rightarrow 0\mathcal{L}_I^+(M) \leftarrow 0\mathcal{L}^+(\tilde{M})
\]

Similarly, the map (3.13) can be rewritten as

\[
M \rightarrow 0\mathcal{L}_I^+(M) \rightarrow 0\mathcal{R}^+(\tilde{M})
\]

The adjunction (3.19) can be lifted to the level of derived categories. Indeed, since \(\text{QCoh}(X^+)\) is a Grothendieck category, it has enough injectives, so that the right derived functor \(\mathcal{R}^+\) exists as a functor \(\mathcal{R}^+ : \mathcal{D}^+(\text{QCoh}(X^+)) \rightarrow \mathcal{D}^+(\text{Gr}(A))\). On the weight-\(i\) component, it is given by

\[
\mathcal{R}^+(\mathcal{F})_i \simeq \mathcal{R}\text{Hom}_{X^+}(\tilde{A}(-i), \mathcal{F})
\]
Thus, it has finite cohomological dimension, and hence extends to a functor \( R^+ : D(QCoh(X^+)) \to D(\text{Gr}(A)) \). Moreover, as a right adjoint to an exact functor, the functor (3.17) sends injective objects to injective objects, and so there is still an adjunction at the level of derived categories:

\[
(3.22) \quad (-)^* : D(\text{Gr}(A)) \rightleftarrows D(QCoh(X^+)) : R^+
\]

If we take the derived versions of (3.20), then there are canonical maps

\[
(3.23) \quad \check{C}_{I^+}(M) \to R^+(\check{M})
\]

functorial in \( M \in D(\text{Gr}(A)) \). In order to describe \( \check{C}_{I^+}(M) \), we give the following graded analogue of [30 Tag 09T2]:

**Lemma 3.24.** If \( A \) is Noetherian, then for any injective object \( M \in \text{Gr}(A) \), the associated sheaf \( \check{M} \) is flasque.

**Proof.** For any closed subset \( Z \subset \text{Proj}^+(A) \), say defined by a graded ideal \( I \subset I^+ \), let \( U = \text{Proj}^+(A) \setminus Z \), then, as in [30 Tag 01YB], any section \( s \in \Gamma(U, \check{M}) \) is represented by an element in \( \text{Hom}_{\text{Gr}(A)}(I^n, M) \), for some \( n \geq 0 \). Since \( M \) is injective, it extends to an element in \( \text{Hom}_{\text{Gr}(A)}(A, M) = M_0 \). \( \square \)

As we have observed above, there is a canonical isomorphism \( H^0(\text{Gr}(X^+, \check{M}(i))) \cong \check{C}_{I^+}(M)_i \), for each \( i \in \mathbb{Z} \). Moreover, we have shown in Proposition (2.25) that the total right derived functor of \( \check{C}_{I^+} : \text{Gr}(A) \to \text{Gr}(A) \) is precisely the functor \( \check{C}_{I^+} : D(\text{Gr}(A)) \to D(\text{Gr}(A)) \). Combining these two facts with Lemma 3.24, we see that, for each \( i \in \mathbb{Z} \), there is a canonical isomorphism in \( D(R) \):

\[
(3.25) \quad \check{C}_{I^+}(M)_i \cong R\Gamma(\text{Gr}(X^+, \check{M}(i)))
\]

so that under (3.21) and (3.25), the map (3.23) is given in weight degree \( i \) by

\[
R\Gamma(\text{Gr}(X^+, \check{M}(i))) \to R\text{Hom}_{\text{Gr}(A)}(\check{M}(i), \check{M})
\]

The identification (3.25) also allows us to show the following

**Lemma 3.26.** For any \( M \in D^b_{\text{coh}}(\text{Gr}(A)) \), we have \( \check{C}_{I^+}(M)_i \in D^b_{\text{coh}}(A_0) \) and \( R\Gamma_{I^+}(M)_i \in D^b_{\text{coh}}(A_0) \) for each weight grading \( i \in \mathbb{Z} \).

**Proof.** In view of (3.25), the statement for \( \check{C}_{I^+}(M)_i \) holds because \( \text{Proj}^+(A) \to \text{Spec} A_0 \) is proper under our standing assumption that \( A_{\geq 0} \) is Noetherian. The same therefore holds for \( R\Gamma_{I^+}(M)_i \) because of the exact triangle (2.27). \( \square \)

Another aspect of the classical Serre’s equivalence is that the scheme \( \text{Proj}^+(A) \) depends only on the “tail” of \( A \). We recall from [1] that this is also true when \( A \) is not necessarily concentrated in non-negative degrees (it is true even when \( A \) is non-commutative). More precisely, consider a map \( f : A \to B \) of \( \mathbb{Z} \)-graded rings, which then induces an adjunction (2.6). In general, the right adjoint \( (-)_A \) preserves the subcategories of positively torsion modules, and hence induces a functor \( (-)_A : Q^+\text{Gr}(B) \to Q^+\text{Gr}(A) \). However, the functor \( - \otimes_A B \) may not preserve torsion modules. There is however a special case where it does:

**Proposition 3.27** ([1], Proposition 2.5). Suppose that the map \( f : A \to B \) of \( \mathbb{Z} \)-graded rings induces isomorphisms \( f : A_n \to B_n \) for sufficiently large \( n \), then the functor \( - \otimes_A B : \text{Gr}(A) \to \text{Gr}(B) \) sends \( \text{Tor}^+(A) \to \text{Tor}^+(B) \), and induces an equivalence \( \text{Q}^+\text{Gr}(A) \cong \text{Q}^+\text{Gr}(B) \).

In particular, if we apply this to the inclusion map \( f : A_{\geq 0} \to A \), then we have the following

**Corollary 3.28.** The adjunction (2.6) for the map \( f : A_{\geq 0} \to A \) of graded rings descend to give an equivalence \( \text{Q}^+\text{Gr}(A) \cong \text{Q}^+\text{Gr}(A_{\geq 0}) \).
Corollary 3.29. Corollary 3.29 gives an alternative way to deduce Theorem 3.15 from the classical case when $A$ is concentrated in non-negative weights. However, in order to compare Grothendieck duality and Greenlees-May duality for $\mathbb{Z}$-graded rings, we find it more convenient to directly establish Serre’s equivalence in this $\mathbb{Z}$-graded setting.

Remark 3.30. When $A$ is not Cartier, the category $Q^+\text{Gr}(A)$ also admit a description in terms of quasi-coherent sheaves on a stacky projective space. Namely, one can show that the map of $\mathbb{G}_m$-equivariant schemes $\text{Spec}(A) \to \text{Spec}(A_{\geq 0})$ induces an isomorphism on the $\mathbb{G}_m$-invariant open subschemes

$$\text{Spec}(A) \setminus \text{Spec}(A/I^+)^+ \cong \text{Spec}(A_{\geq 0}) \setminus \text{Spec}(A_0).$$

(cf. Corollary 3.28). If we denote this $\mathbb{G}_m$-equivariant scheme by $W^{ss}(+)$, and let $\text{Proj}^+(A)$ be the quotient stack $[W^{ss}(+)/\mathbb{G}_m]$, then we have an equivalence $Q^+\text{Gr}(A) \simeq \text{QCoh}(\text{Proj}^+(A))$. See, e.g., [2 Proposition 2.3] or [13 Example 2.15].

Now we relate the derived category $D(Q^+\text{Gr}(A))$ to the subcategory $D_{\text{t-triv}}(\text{Gr}(A)) \subset D(\text{Gr}(A))$ considered in Section 2.2.

Proposition 3.31. For each $\bullet \in \{+, -, b\}$, the Serre subcategory $(I^+)^\infty\cdot\text{Tor} \subset \text{Gr}(A)$ is $\mathbb{D}^\bullet$-localizing. The functor $R\phi_* : D(Q^+\text{Gr}(A)) \to D(\text{Gr}(A))$ is fully faithful, and there is a semi-orthogonal decomposition

$$\mathbb{D}^\bullet(\text{Gr}(A)) = (R\phi_*(\mathbb{D}^\bullet(Q^+\text{Gr}(A))), \mathbb{D}^\bullet((I^+)\infty\cdot\text{Tor}(\text{Gr}(A))))$$

and the composition $R\phi_* \circ \phi^* : \mathbb{D}^\bullet(\text{Gr}(A)) \to \mathbb{D}^\bullet(\text{Gr}(A))$ is given by $\hat{C}_{I^+} : \mathbb{D}^\bullet(\text{Gr}(A)) \to \mathbb{D}^\bullet(\text{Gr}(A))$.

As a result, there is an exact equivalence

$$\phi^* : D_{\text{t-triv}}(\text{Gr}(A)) \cong D^\bullet(Q^+\text{Gr}(A)) : R\phi_*$$

Proof. The Serre subcategory $(I^+)\infty\cdot\text{Tor} \subset \text{Gr}(A)$ is clearly part of a torsion theory on $\text{Gr}(A)$. By Proposition 3.31, it is therefore localizing. By Proposition 3.30 it is moreover $D$-localizing, so that $R\phi_*$ is fully faithful. Thus, the essential image of $R\phi_* : D(Q^+\text{Gr}(A)) \to D(\text{Gr}(A))$ is equal to the right orthogonal of $D((I^+)\infty\cdot\text{Tor}(\text{Gr}(A)))$, which is therefore equal to $D_{\text{t-triv}}(\text{Gr}(A))$ by the semi-orthogonal decomposition in the first row of (2.29). This also shows that $R\phi_* \circ \phi^* \cong \hat{C}_{I^+}$ as endofunctors on $D(\text{Gr}(A))$. Since the functor $\hat{C}_{I^+}$ has finite cohomological dimension, it restricts to each of the subcategories $\mathbb{D}^\bullet(\text{Gr}(A))$. Equivalently, this means that the functor $R\phi_* : D(Q^+\text{Gr}(A)) \to D(\text{Gr}(A))$ also restrict to give functors on each of the subcategories $\mathbb{D}^\bullet(-)$.

Now we consider how the equivalence (3.32) restricts to give equivalences on subcategories with coherently cohomological. We start with the following

Definition 3.33. Let $\text{gr}(A) \subset \text{Gr}(A)$ be the full subcategory consisting of finitely generated graded $A$-modules, and let $q^+\text{gr}(A) \subset Q^+\text{Gr}(A)$ be the essentially image of $\text{gr}(A)$ under $\phi^* : \text{Gr}(A) \to Q^+\text{Gr}(A)$.

For each $\bullet \in \{+, -, b\}$, let $\mathbb{D}_{\text{coh}}^\bullet(Q^+\text{Gr}(A)) \subset D^\bullet(Q^+\text{Gr}(A))$ be the essential image of $\mathbb{D}_{\text{coh}}^\bullet(\text{Gr}(A))$ under the functor $\phi^* : D^\bullet(\text{Gr}(A)) \to D^\bullet(Q^+\text{Gr}(A))$.

Recall also from Definition 2.26 that $\mathbb{D}_{\text{coh}}((I^+)\text{triv})(\text{Gr}(A)) \subset D_{\text{t-triv}}^\bullet(\text{Gr}(A))$ is the essential image of $\mathbb{D}_{\text{coh}}^\bullet(\text{Gr}(A))$ under the functor $\hat{C}_{I^+} : D^\bullet(\text{Gr}(A)) \to D_{\text{t-triv}}(\text{Gr}(A))$.

Remark 3.34. (1) The reader is cautioned that in general, $\mathbb{D}_{\text{coh}}((I^+)\text{triv})(\text{Gr}(A)) \neq \mathbb{D}_{\text{coh}}^\bullet(\text{Gr}(A)) \cap D_{\text{t-triv}}^\bullet(\text{Gr}(A))$, because the functor $\hat{C}_{I^+}$ does not preserve $\mathbb{D}_{\text{coh}}^\bullet(\text{Gr}(A))$.

(2) One can show that, under the equivalence $Q^+\text{Gr}(A) \simeq \text{QCoh}(\text{Proj}^+(A))$ in Remark 3.30, the subcategory $\mathbb{D}_{\text{coh}}^\bullet(Q^+\text{Gr}(A)) \subset \mathbb{D}^\bullet(Q^+\text{Gr}(A))$ corresponds to the subcategory $\mathbb{D}_{\text{coh}}^\bullet(\text{Proj}^+(A)) \subset \mathbb{D}^\bullet(\text{Proj}^+(A))$, (at least) for $\bullet \in \{-, b\}$.

Corollary 3.35. The equivalence (3.32) restricts to give an exact equivalence

$$\phi^* : D_{\text{coh}}^\bullet((I^+)\text{triv})(\text{Gr}(A)) \cong D_{\text{coh}}^\bullet(Q^+\text{Gr}(A)) : R\phi_*$$
Proof. The functors \( \phi^* : D^b(\text{Gr}(A)) \to D^b(Q^+\text{Gr}(A)) \) and \( \check{\mathcal{E}}_{I^+} : D^b(\text{Gr}(A)) \to D^b_{I^+\text{-triv}}(\text{Gr}(A)) \) gets identified under (3.32), hence so are the essential images of \( D_{\text{coh}}^b(\text{Gr}(A)) \) under both functors.

3.2. Greenlees-May duality and Grothendieck duality. Now we relate Greenlees-May duality and Grothendieck duality for projective morphisms \( \pi : X \to Y \). For simplicity, we will again work with the setting when the base \( Y \) is a Noetherian affine scheme \( Y = \text{Spec} \, R \). We first consider the following composition of three-way adjunctions

\[
D_{I^+\text{-triv}}(\text{Gr}(A)) \xrightarrow{i} D(\text{Gr}(A)) \xrightarrow{-\otimes_R A} D(\text{Gr}(R)) \xrightarrow{i_0} D(\text{Mod}(R))
\]

We rearrange some of these functors as the following composition of adjunctions

(3.36) \[
D(\text{Gr}(A)) \xrightarrow{\check{\mathcal{E}}_{I^+}} D_{I^+\text{-triv}}(\text{Gr}(A)) \xrightarrow{i} D(\text{Gr}(A)) \xrightarrow{-\otimes_R A} D(\text{Gr}(R)) \xrightarrow{i_0} D(\text{Mod}(R))
\]

A crucial observation is that, by \( 3.25 \), the composition of all the right-pointing arrows is naturally isomorphism to the functor \( M \mapsto R\pi_+(\check{\mathcal{E}}_{I^+}(M)) \), where \( \pi : X^+ = \text{Proj}^+(A) \to Y \) is the projection to \( Y = \text{Spec} \, R \). This allows us to compare (3.36) with the following composition of adjunctions:

(3.37) \[
D(\text{Gr}(A)) \xrightarrow{-\otimes_R A} D(\text{Qcoh}(X^+)) \xrightarrow{R\pi_+} D(\text{Mod}(R))
\]

where the left-most adjunction is given by (3.32).

Comparing (3.36) and (3.37), we then have the following

**Theorem 3.38.** For each \( L \in D(\text{Mod}(R)) \), there is a canonical isomorphism

\[
R^+ (\pi^!(L)) \cong \check{\mathcal{E}}_{I^+} (R\text{Hom}_A(A, L))
\]

4. The case of non-affine base

In this section, we generalize the constructions and results of Sections 2 and 3 to the case of sheaves of \( \mathbb{Z} \)-graded rings, more precisely the setting (1.1). To establish the results on the derived categories, we find it most convenient to first construct the relevant functors by abstract categorical arguments, and then establish its properties locally on affine bases, by appealing to results in Sections 2 and 3.

Recall that \( G_m \) is the group scheme dual to the commutative Hopf algebra \( (\mathbb{Z}[t^\pm 1], \Delta) \), with \( \Delta(t) = t \otimes t \). The data of a \( \mathbb{Z} \)-graded ring \( A \) is equivalent to a \( G_m \)-action on the affine scheme \( \text{Spec} \, A \). Namely, the action map \( G_m \times \text{Spec} \, A \to \text{Spec} \, A \) corresponds dually to a ring map \( A \to \mathbb{Z}[t^\pm 1] \otimes A \), which may be written as \( a \mapsto \sum_{i \in \mathbb{Z}} t^i a_i \). The associativity of the action translates to the fact that \( a \mapsto a_i \) is an idempotent operator, while the unitality of the action translates to the fact that \( \sum_{i \in \mathbb{Z}} a_i = a \).

Given a group scheme \( G \) acting on a scheme \( X \), one way to define \( G \)-equivariance structure on a quasi-coherent sheaf \( \mathcal{F} \) is to require that the \( G \)-action on \( X \) extends to one on \( \text{Spec} \, (\text{Sym}_{\mathcal{O}_X}(\mathcal{F})) \) that preserves the extra symmetric grading. From this, it is easy to see that the category \( \text{Gr}(A) \) of graded modules over \( A \) is equivalent to the category of \( G_m \)-equivariant quasi-coherent sheaves on \( \text{Spec} \, A \).

In this section, we provide the formal arguments to extend our previous discussion to the case of non-affine base. More precisely, we work in the following setting:

(4.1) \[
Y \text{ is a Noetherian separated scheme, and } \mathcal{A} \text{ is a quasi-coherent sheaf of Noetherian } \mathbb{Z} \text{-graded rings on } Y, \text{ such that } \mathcal{A}_0 \text{ (and hence every } \mathcal{A}_i \text{) is coherent over } \mathcal{O}_Y.
\]

Denote by \( \text{Gr}(\mathcal{A}) \) the category of quasi-coherent graded \( \mathcal{A} \)-modules. Then \( \text{Gr}(\mathcal{A}) \) is equivalent to the category \( \text{Qcoh}_{G_m}(W) \) of \( G_m \)-equivariant quasi-coherent sheaves on the relative spectrum \( W := \text{Spec}_Y \mathcal{A} \).

We denote this equivalence by

(4.2) \[
(-)^b : \text{Gr}(\mathcal{A}) \xrightarrow{\approx} \text{Qcoh}_{G_m}(W) : (-)^b
\]
We fix our convention so that

\[(4.3) \text{ The shift functor } \mathcal{M} \mapsto \mathcal{M}(1) \text{ on } \text{Gr}(A) \text{ corresponds under } \[4.2\] \text{ to the twist of equivariance structure by the identity character of } G_m.\]

For any \(\mathcal{M}, \mathcal{N} \in \text{Gr}(A)\), one can define \(\mathcal{M} \otimes_A \mathcal{N} \in \text{Gr}(A)\), which coincides with the graded tensor product over each affine open subset \(U \subset Y\). Moreover, we have \((\mathcal{M} \otimes_A \mathcal{N})^* \cong \mathcal{M}^* \otimes_{m^*} \mathcal{N}^*\), where the right hand side inherits an equivariance structure in the usual way.

Now we define local cohomology complex. Given any quasi-coherent sheaf of graded ideals \(\mathcal{I} \subset A\), we say that a quasi-coherent sheaf \(M \in \text{Gr}(A)\) of graded module is \(\mathcal{I}^\infty\text{-torsion}\) if \(M^*\) restricts to zero on the complement of \(V(\mathcal{I}) := \text{Spec}_Y(A/\mathcal{I})\) in \(W\). Notice that when \(Y\) is affine, this coincides with Definition 2.16. Denote by \(U(\mathcal{I}) := W \setminus V(\mathcal{I})\), and \(j : U(\mathcal{I}) \hookrightarrow W\) the inclusion. Then the subcategory \(\mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A)) \subset \mathcal{D}(\text{Gr}(A))\) consisting of complexes with \(\mathcal{I}^\infty\text{-torsion}\) cohomology sheaves corresponds to the kernel of \(j^* : \mathcal{D}(\text{Coh}_{G_m}(W)) \to \mathcal{D}(\text{Coh}_{G_m}(U(\mathcal{I})))\) under the equivalence \(\[4.2\]\).

Since the functor \(j^*\) has a right adjoint \(Rj_*\), satisfying \(j^* \circ Rj_* \cong \text{id}\), it follows formally that there is a semi-orthogonal decomposition

\[(4.4) \mathcal{D}(\text{Gr}(A)) = \langle \mathcal{D}_{\mathcal{I}, \text{-triv}}(\text{Gr}(A)), \mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A)) \rangle \]

where \(\mathcal{D}_{\mathcal{I}, \text{-triv}}(\text{Gr}(A))\) is the essential image of the fully faithful composition

\[\mathcal{D}_{\mathcal{I}, \text{-triv}}(\text{Gr}(A)) := \text{Essential Image } [\mathcal{D}(\text{Coh}_{G_m}(U(\mathcal{I})))] \xrightarrow{\mathcal{R}j_*} \mathcal{D}(\text{Coh}_{G_m}(W)) \xrightarrow{(-)^*} \mathcal{D}(\text{Gr}(A))\]

More precisely, for any \(\mathcal{M} \in \mathcal{D}(\text{Gr}(A))\), define \(\mathcal{C}_\mathcal{I}(\mathcal{M}) := (\mathcal{R}j_* (\mathcal{I}^*(\mathcal{M}^*)))^* \in \mathcal{D}(\text{Gr}(A))\), which comes with a natural adjunction unit \(\eta_\mathcal{M} : \mathcal{M} \to \mathcal{C}_\mathcal{I}(\mathcal{M})\). Then \(\mathcal{R}G_\mathcal{I}(\mathcal{M}) \in \mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A))\) may be defined by the exact triangle

\[(4.5) \ldots \to \mathcal{R}G_\mathcal{I}(\mathcal{M}) \xrightarrow{\epsilon_\mathcal{M}} \mathcal{M} \xrightarrow{\eta_\mathcal{M}} \mathcal{C}_\mathcal{I}(\mathcal{M}) \xrightarrow{\delta_\mathcal{M}} \mathcal{R}G_\mathcal{I}(\mathcal{M})[1] \to \ldots\]

which is the decomposition triangle associated to the semi-orthogonal decomposition \(\[4.4\]\). Over affine open subsets of \(Y\), this coincides with the corresponding notions in Section 2.2.

**Lemma 4.6.** The exact triangle \(\[4.5\]\) restricts to \(\[2.27\]\) over each affine open subset \(U \subset Y\).

**Proof.** Clearly, the association of \(\[4.5\]\) to \(\mathcal{M} \in \text{Gr}(A)\) is local on the base \(Y\), so that it suffices to assume that \(Y = U\) is affine. In this case, the notion of \(\mathcal{I}^\infty\text{-torsion}\) modules clearly coincides with Definition 2.16. Since \(\mathcal{R}G_I : \mathcal{D}(\text{Gr}(A)) \to \mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A))\) was known to be right adjoint to the inclusion by Theorem 2.26, it must coincide with \(\mathcal{R}G_\mathcal{I}\). \(\square\)

One can use it to show that the isomorphisms \(\mathcal{R}G_I(M) \cong \mathcal{R}G_I(A) \otimes^{L}_A M\) and \(\mathcal{C}_I(M) \cong \mathcal{C}_I(A) \otimes^{L}_A M\) in \(\mathcal{D}(\text{Gr}(A))\) still hold in the present non-affine case.

**Corollary 4.7.** For any \(\mathcal{M} \in \mathcal{D}(\text{Gr}(A))\), there are canonical isomorphisms \(\mathcal{R}G_\mathcal{I}(\mathcal{M}) \cong \mathcal{R}G_\mathcal{I}(\mathcal{A}) \otimes^{L}_A \mathcal{M}\) and \(\mathcal{C}_\mathcal{I}(\mathcal{M}) \cong \mathcal{C}_\mathcal{I}(\mathcal{A}) \otimes^{L}_A \mathcal{M}\) in \(\mathcal{D}(\text{Gr}(A))\).

**Proof.** The canonical map \(\epsilon_A : \mathcal{R}G_\mathcal{I}(\mathcal{A}) \to \mathcal{A}\) in \(\[1.6\]\) induces a map \(\varphi_\mathcal{M} : \mathcal{R}G_\mathcal{I}(\mathcal{A}) \otimes^{L}_A \mathcal{M} \to \mathcal{M}\). By Lemma 4.6 we see that \(\mathcal{R}G_\mathcal{I}(\mathcal{A}) \otimes^{L}_A \mathcal{M} \in \mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A))\). Since \(\mathcal{R}G_\mathcal{I} : \mathcal{D}(\text{Gr}(A)) \to \mathcal{D}_{\mathcal{I}, \text{-Tor}}(\text{Gr}(A))\) is right adjoint to the inclusion, the map \(\varphi_\mathcal{M}\) induces a map \(\mathcal{R}G_\mathcal{I}(\mathcal{A}) \otimes^{L}_A \mathcal{M} \to \mathcal{R}G_\mathcal{I}(\mathcal{M})\). By Lemma 4.6 again, this map is an isomorphism on each affine open subscheme \(U \subset Y\). The second statement can be proved in a completely parallel manner. \(\square\)

We now turn to local homology complex in the non-affine setting. Already in the ungraded case, this notion has a technical subtlety arising from the fact that the internal Hom between quasi-coherent sheaves may not be quasi-coherent (see, e.g., [3], Remark (0.4)) for a discussion). The notion of quasi-coherator from [34] Appendix B) is therefore highly relevant in this discussion. We recall this notion now.

On a quasi-compact separated scheme \(X\), the inclusion \(\iota : \text{Coh}(X) \to \text{Mod}(O_X)\) has a right adjoint, called the quasi-coherator \(Q_X : \text{Mod}(O_X) \to \text{Coh}(X)\). The derived functor \(RQ_X : \text{D}(X) \to \text{D}(\text{Coh}(X))\) then restricts to an equivalence \(RQ_X : \text{D}_{qcoh}(X) \to \text{D}(\text{Coh}(X))\), which is quasi-inverse
to the inclusion (see, e.g., [30] Tag 08DB). Throughout this paper, we work with \(\mathcal{D}(\text{QCoh}(X))\) instead of \(\mathcal{D}_{\text{qcoh}}(X)\). More precisely, we work with the following two conventions:

**Convention 4.8.** On a quasi-compact separated scheme \(X\), let \(\mathcal{D}(X) = \mathcal{D}(\text{Mod}(\mathcal{O}_X))\) be the derived category of all sheaves of \(\mathcal{O}_X\)-modules.

For any \(\mathcal{F}, \mathcal{G} \in \text{Mod}(\mathcal{O}_X)\), denote by \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \text{Mod}(\mathcal{O}_X)\) the be the internal Hom object in \(\text{Mod}(\mathcal{O}_X)\), defined by \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \text{Hom}_{\mathcal{O}_U}([\mathcal{F}|_U, \mathcal{G}|_U])\).

For any \(\mathcal{F}, \mathcal{G} \in \text{QCoh}(X)\), denote by \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \text{QCoh}(X)\) the be the internal Hom object in \(\text{QCoh}(X)\), defined by \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = Q_X(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))\).

Both of these internal Hom functors have derived functors

\[
\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\cdot, \cdot) : \mathcal{D}(\mathcal{X})^{\text{op}} \times \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X})
\]

so that for any \(\mathcal{F}, \mathcal{G} \in \mathcal{D}(\text{QCoh}(X))\), we have

\[
\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{R}\text{Q}(\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))
\]

In particular, if \(X\) is Noetherian, \(\mathcal{F} \in \mathcal{D}_{\text{coh}}(\text{QCoh}(X))\) and \(\mathcal{G} \in \mathcal{D}^+(\text{QCoh}(X))\), then the canonical map \(\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\) is an isomorphism in \(\mathcal{D}(\mathcal{X})\) (see, e.g., [30] Tag 0A6H). As a result, these two will often be implicitly identified.

**Convention 4.9.** Let \(f : X \to Y\) be a morphism between quasi-compact separated schemes. The functors (\(L f^*, R f_* , f^!\)) are regarded as functors between \(\mathcal{D}(\text{QCoh}(X))\) and \(\mathcal{D}(\text{QCoh}(Y))\). Namely, \(L f^*\) and \(R f_*\) are the derived functors of the functors \(f^*\) and \(f_*\) between \(\text{QCoh}(X)\) and \(\text{QCoh}(Y)\). Under the equivalence \(\mathcal{R}\mathbb{Q} : \mathcal{D}_{\text{qcoh}}(\cdot) \to \mathcal{D}(\text{QCoh}(\cdot))\), the functor \(R f_*\) coincides with the usual one between \(\mathcal{D}_{\text{qcoh}}(X)\) and \(\mathcal{D}_{\text{qcoh}}(Y)\) (see, e.g., [30] Tag 0CRX). By adjunction, the same holds for \(L f^*\), as well as \(f^!\), whenever it is well-defined (see Section 5.2 for a summary of the functor \(f^!\)).

To develop local homology in the graded context, a careful discussion of sheafified graded Hom complexes and the graded ( quasi-)coherator is in order. For the sake of a formal argument, we temporarily introduce the category \(\text{GrMod}(\mathcal{A})\) of (not necessarily quasi-coherent) sheaves of graded \(\mathcal{A}\)-modules. For any \(\mathcal{M}, \mathcal{N} \in \text{GrMod}(\mathcal{A})\), define \(\text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \in \text{GrMod}(\mathcal{A})\) by

\[
\text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})(U)_i := \text{Hom}_{\text{GrMod}(\mathcal{A}|_U)}(\mathcal{M}|_U, \mathcal{N}|_U(U(i))
\]

This gives a bifunctor

\[
\text{Hom}_{\mathcal{A}}(-, -) : \text{GrMod}(\mathcal{A})^{\text{op}} \times \text{GrMod}(\mathcal{A}) \to \text{GrMod}(\mathcal{A})
\]

Since we are working over the category \(\text{GrMod}(\mathcal{A})\) of not necessarily quasi-coherent sheaves, the restriction functor \(j^* : \text{GrMod}(\mathcal{A}) \to \text{GrMod}(\mathcal{A}|_U)\), for any open subscheme \(j : U \to Y\), has an exact left adjoint \(j_!\) (see, e.g., [30] Tag 0797). Thus, the restriction of any \(K\)-injective complex is \(K\)-injective. As a result, one can use \(K\)-injective complexes to define the sheafified derived Hom complex:

\[
(4.10) \quad R\text{Hom}_{\mathcal{A}}(-, -) : \mathcal{D}(\text{GrMod}(\mathcal{A}))^{\text{op}} \times \mathcal{D}(\text{GrMod}(\mathcal{A})) \to \mathcal{D}(\text{GrMod}(\mathcal{A}))
\]

This is the internal Hom object in \(\mathcal{D}(\text{GrMod}(\mathcal{A}))\), with respect to the monoidal product \(- \otimes_{\mathcal{A}} -\). In other words, for any \(\mathcal{M}, \mathcal{N}, \mathcal{K} \in \mathcal{D}(\text{GrMod}(\mathcal{A}))\), there is a canonical isomorphism

\[
(4.11) \quad \text{Hom}_{\mathcal{D}(\text{GrMod}(\mathcal{A}))}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{K}) \cong \text{Hom}_{\mathcal{D}(\text{GrMod}(\mathcal{A}))}(\mathcal{M}, R\text{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{K}))
\]

Following the arguments of [34] Appendix B, one can show that the inclusion functor \(i : \text{Gr}(\mathcal{A}) \to \text{GrMod}(\mathcal{A})\) has a right adjoint \(Q_A : \text{GrMod}(\mathcal{A}) \to \text{Gr}(\mathcal{A})\), known as the ( quasi-)coherator. Moreover, this functor can be derived to obtain \(RQ_A : \mathcal{D}(\text{GrMod}(\mathcal{A})) \to \mathcal{D}(\text{Gr}(\mathcal{A}))\), which is still right adjoint to the \(i : \mathcal{D}(\text{Gr}(\mathcal{A})) \to \mathcal{D}(\text{GrMod}(\mathcal{A}))\). This allows us to define the bifunctor

\[
(4.12) \quad R\text{Hom}_{\mathcal{A}}(-, -) : \mathcal{D}(\text{Gr}(\mathcal{A}))^{\text{op}} \times \mathcal{D}(\text{Gr}(\mathcal{A})) \to \mathcal{D}(\text{Gr}(\mathcal{A}))
\]
by post-composing \((4.10)\) with the derived (quasi-)coherator:

\[
\mathcal{RHom}_\mathcal{A}(\mathcal{M}, \mathcal{N}) := \mathcal{R}Q_A \mathcal{RHom}_\mathcal{A}(\mathcal{M}, \mathcal{N})
\]

Since \(\mathcal{R}Q_A : \mathcal{D}(\text{GrMod}(\mathcal{A})) \to \mathcal{D}(\text{Gr}(\mathcal{A}))\) is right adjoint to the inclusion, it follows from \((4.11)\) that, for any \(\mathcal{M}, \mathcal{N}, \mathcal{K} \in \mathcal{D}(\text{Gr}(\mathcal{A}))\), there is a canonical isomorphism

\[
(4.13) \quad \text{Hom}_{\mathcal{D}(\text{Gr}(\mathcal{A}))}(\mathcal{M} \otimes^L \mathcal{N}, \mathcal{K}) \cong \text{Hom}_{\mathcal{D}(\text{Gr}(\mathcal{A}))}(\mathcal{M}, \mathcal{R} \mathcal{RHom}_\mathcal{A}(\mathcal{N}, \mathcal{K}))
\]

so that \(\mathcal{R} \mathcal{RHom}_\mathcal{A}(-, -)\) is the internal Hom bifunctor in \(\mathcal{D}(\text{Gr}(\mathcal{A}))\).

In particular, if we define the functors

\[
\mathcal{L} \mathcal{A}_\mathcal{S} : \mathcal{D}(\text{Gr}(\mathcal{A})) \to \mathcal{D}(\mathcal{A}) \quad \mathcal{L} \mathcal{A}_\mathcal{S}(\mathcal{M}) := \mathcal{R} \mathcal{RHom}_\mathcal{A}(\mathcal{R} \mathcal{I} \mathcal{S}(\mathcal{A}), \mathcal{M})
\]

\[
(4.14) \quad \mathcal{E} \mathcal{S} : \mathcal{D}(\text{Gr}(\mathcal{A})) \to \mathcal{D}(\mathcal{A}) \quad \mathcal{E} \mathcal{S}(\mathcal{M}) := \mathcal{R} \mathcal{RHom}_\mathcal{A}(\mathcal{C} \mathcal{E}(\mathcal{S}(\mathcal{A})), \mathcal{M})
\]

then by \((4.13)\) and Corollary \(4.7\), there are canonical adjunctions

\[
(4.15) \quad \mathcal{R} \mathcal{I} \mathcal{S} : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\text{Gr}(\mathcal{A})) \rightleftarrows \mathcal{E} \mathcal{S} : \mathcal{D}(\mathcal{A})
\]

Now we prove the following

**Proposition 4.16.** The canonical functor \(i : \mathcal{D}(\mathcal{A}) \to \mathcal{D}_{Gr}(\text{GrMod}(\mathcal{A}))\) is an equivalence, with quasi-inverse given by \(\mathcal{R}Q_A\).

**Proof.** The corresponding statement for \(i : \mathcal{D}(\text{Qcoh}(\mathcal{Y})) \to \mathcal{D}_{\text{Qcoh}}(\text{Mod}(\mathcal{O}_\mathcal{Y}))\) is well-known (see, e.g., [34 Appendix B] or [30 Tag 08DB]). One can either adapt this proof to our present case, or formally deduce our statement from that, as follows:

For each \(i \in \mathcal{Z}\), consider the diagram of functors

\[
\begin{array}{ccc}
\mathcal{D}(\text{Gr}(\mathcal{A})) & \xrightarrow{i} & \mathcal{D}(\text{GrMod}(\mathcal{A})) \\
\xrightarrow{- \otimes^L_\mathcal{O}_\mathcal{Y} \mathcal{A}(-i)} & & \xleftarrow{- \otimes^L_\mathcal{O}_\mathcal{Y} \mathcal{A}(-i)} \\
\mathcal{D}(\text{Qcoh}(\mathcal{Y})) & \xrightarrow{i} & \mathcal{D}(\text{Mod}(\mathcal{O}_\mathcal{Y}))
\end{array}
\]

Since the left diagram commutes up to isomorphism of functors, if we take the right adjoints of all the functors involved, we see that the right diagram also commutes up to isomorphism of functors. Thus, if we forget about the \(\mathcal{A}\)-module structure, then the derived (quasi-)coherator \(\mathcal{R}Q_A(\mathcal{M})\) is simply obtained by applying the derived (quasi-)coherator \(\mathcal{R}Q_Y\) to each weight component \(\mathcal{M}_i\). Therefore, the fact that the adjunction unit \(\text{id} \Rightarrow \mathcal{R}Q_A \circ i\) is an isomorphism on \(\mathcal{D}(\text{Gr}(\mathcal{A}))\); while the adjunction counit \(i \circ \mathcal{R}Q_A \Rightarrow \text{id}\) is an isomorphism on \(\mathcal{D}_{Gr}(\text{GrMod}(\mathcal{A}))\), follows from the corresponding statements for the adjunction \(i \dashv \mathcal{R}Q_Y\). \(\Box\)

A disadvantage of \((4.12)\), and hence of the functors \((4.13)\), is that it often fails to be local. Namely, if \(U \subseteq Y\) is an affine open subscheme, and if we let \(A := \mathcal{A}(U)\), then it may happen that \(\mathcal{R} \mathcal{Hom}_\mathcal{Y}(\mathcal{M}, \mathcal{N})(U) \not\cong \mathcal{R} \mathcal{Hom}_\mathcal{Y}(\mathcal{M}(U), \mathcal{N}(U))\). There is, however, an important case where equality holds.

For any \(\mathcal{M} \in \text{GrMod}(\mathcal{A})\) and \(\mathcal{F} \in \text{Mod}(\mathcal{O}_Y)\), define the graded sheafified Hom \(\mathcal{RHom}_\mathcal{Y}(\mathcal{M}, \mathcal{F}) \in \text{GrMod}(\mathcal{A})\) by

\[
\mathcal{RHom}_\mathcal{Y}(\mathcal{M}, \mathcal{F})(U)_i := \text{Hom}_{\text{Mod}}((\mathcal{M}|_U)_i, \mathcal{F}|_U)
\]

By taking \(\mathcal{K}\)-injective representatives of any \(\mathcal{F} \in \mathcal{D}(\text{Mod}(\mathcal{O}_Y))\), one can define its derived version\(^2\). This gives a bifunctor

\[
(4.17) \quad \mathcal{R} \mathcal{Hom}_\mathcal{Y}(-, -) : \mathcal{D}(\text{GrMod}(\mathcal{A}))^{\text{op}} \times \mathcal{D}(\text{Mod}(\mathcal{O}_Y)) \to \mathcal{D}(\text{GrMod}(\mathcal{A}))
\]

which then allows us to define

\[
(4.18) \quad \mathcal{R} \mathcal{Hom}_\mathcal{Y}(-, -) : \mathcal{D}(\mathcal{A})^{\text{op}} \times \mathcal{D}(\text{Qcoh}(\mathcal{Y})) \to \mathcal{D}(\mathcal{A})
\]

\(^2\)We are implicitly appealing to the argument in the paragraph preceding \((4.10)\).
by post-composing (4.17) with the derived (quasi-)coherator:

$$R\text{Hom}_{\mathcal{O}_Y}(M, F) := RQ_A R\text{Hom}_{\mathcal{O}_Y}(M, F)$$

By the argument in the proof of Proposition 4.10 we see that the derived (quasi-)coherator commutes with taking weight components. Thus, we have

$$R\text{Hom}_{\mathcal{O}_Y}(M, F)_i \simeq R\text{Hom}_{\mathcal{O}_Y}(M_{-i}, F)$$

For each $M, N \in D(\text{Gr}(\mathcal{A}))$ and $F \in D(\text{Qcoh}(Y))$, there is a canonical isomorphism in $D(\text{Gr}(\mathcal{A}))$:

$$R\text{Hom}_{\mathcal{O}_Y}(M \otimes_A^L N, F) \cong R\text{Hom}_{\mathcal{A}}(M, R\text{Hom}_{\mathcal{O}_Y}(N, F))$$

Indeed, one starts with the isomorphism

$$R\text{Hom}_{\mathcal{O}_Y}(M \otimes_A^L N, F) \cong R\text{Hom}_{\mathcal{A}}(M, R\text{Hom}_{\mathcal{O}_Y}(N, F))$$

in $D(\text{GrMod}(\mathcal{A}))$. Since $M \in D(\text{Gr}(\mathcal{A}))$, the right hand side can be replaced by $R\text{Hom}_{\mathcal{A}}(M, R\text{Hom}_{\mathcal{O}_Y}(N, F))$. Taking $RQ_A$ of both sides then gives (4.20).

In certain cases of interest, the bifunctor (4.18) is local on the base:

**Lemma 4.21.** Suppose $M \in D(\text{Gr}(\mathcal{A}))$ is such that $M_i \in D^\text{coh}_{\mathcal{A}}(Y)$ for each weight component $i \in \mathbb{Z}$, then for any $F \in D^+(\text{Qcoh}(Y))$, the complexes $R\text{Hom}_{\mathcal{O}_Y}(M, F)$ and $R\text{Hom}_{\mathcal{O}_Y}(M, F)$ represent the same object in $D(\text{GrMod}(\mathcal{A}))$. In particular, for any affine open subscheme $U = \text{Spec} R \subset Y$, we have

$$R\text{Hom}_{\mathcal{O}_Y}(M, F)(U) \cong R\text{Hom}_{\mathcal{R}}(M(U), F(U))$$

as objects in $D(\text{Gr}(\mathcal{A}(U)))$.

**Proof.** Given an affine open subscheme $U = \text{Spec} R \subset Y$, let $A := \mathcal{A}(U)$, $M = M(U) \in D(\text{Gr}(\mathcal{A}))$ and $F := F(U) \in D^+(R)$. The condition $M_i \in D^\text{coh}_{\mathcal{A}}(Y)$ then implies that $M_i \in D(R)$ is pseudo-coherent. As a result, we have $R\text{Hom}_{R}(M, F)_f \cong R\text{Hom}_{R}(M_i, F)$ for each $f \in R$, so that $R\text{Hom}_{\mathcal{A}(U)}(M_{|U}, F_{|U}) \in D(\text{GrMod}(\mathcal{A}(U)))$ have quasi-coherent cohomology. Since this holds for each affine open $U \subset Y$, we conclude that $R\text{Hom}_{\mathcal{O}_Y}(M, F)$ have quasi-coherent cohomology. By Proposition 4.11 the first statement follows. The second statement follows from the first, because $R\text{Hom}_{\mathcal{O}_Y}(M, F)$ is always local.

An important special case is when $F$ is a dualizing complex $\omega_Y^* \in D^b_{\mathcal{coh}}(\text{Qcoh}(Y))$. In this case, $R\text{Hom}_{\mathcal{O}_Y}(M, F)$ is local on the base when $M \in D^b_{\mathcal{A}}(\text{Gr}(\mathcal{A}))$, in the sense of the following

**Definition 4.22.** A quasi-coherent sheaf $M \in \text{Gr}(\mathcal{A})$ of graded $\mathcal{A}$-modules is said to be *locally coherent* if each $M_i$ is coherent as a sheaf over $Y$. For each $\bullet \in \{ +, -, b \}$, denote by $D^\bullet_{\mathcal{A}}(\text{Gr}(\mathcal{A}))$ the full subcategory of $D^\bullet(\text{Gr}(\mathcal{A}))$ consisting of objects with locally coherent cohomology sheaves.

Given a dualizing complex $\omega_Y^* \in D^b_{\mathcal{coh}}(\text{Qcoh}(Y))$, consider the functor

$$D_Y : D^b_{\mathcal{A}}(\text{Gr}(\mathcal{A}))^{\text{op}} \to D^b_{\mathcal{A}}(\text{Gr}(\mathcal{A})), \quad M \mapsto R\text{Hom}_{\mathcal{O}_Y}(M, \omega_Y^*)$$

Since $R\text{Hom}_{\mathcal{O}_Y}(M, \omega_Y^*)$ can be computed in each weight component (see (4.19)), we have the following

**Lemma 4.24.** The functor $D_Y$ is an involution, i.e., for any $M \in D^b_{\mathcal{A}}(\text{Gr}(\mathcal{A}))$, the canonical map $M \to D_Y(D_Y(M))$ is an isomorphism in $D^b_{\mathcal{A}}(\text{Gr}(\mathcal{A}))$. This involution interchanges $D^+_\mathcal{A}(\text{Gr}(\mathcal{A}))$ and $D^-_\mathcal{A}(\text{Gr}(\mathcal{A}))$. Moreover, $D_Y$ is local on the base $Y$ in the sense that $D_Y(M_{|U}) \cong D_Y(M_{|U})$ for any open subscheme $U \subset Y$, where $D_U$ is defined using the restriction of the dualizing complex $\omega_Y^*$ to $U$.

For each open affine subscheme $U = \text{Spec} R \subset Y$, take $\mathcal{F}^+(U) := \mathcal{A}(U)_{>0}$, $\mathcal{A}(U) \subset \mathcal{A}(U)$. It is clear that $\mathcal{F}^+(U)_f = \mathcal{F}^+(U)_f$ for each $f \in R$, and hence $\mathcal{F}^+ \subset \mathcal{A}$ forms a quasi-coherent sheaf of graded ideals of $\mathcal{A}$. Define $\mathcal{F}^-$ in a similar way. We claim that

$$R\Gamma_{\mathcal{F}^+}(A)_i \in D^b_{\mathcal{coh}}(\text{Qcoh}(Y)) \text{ and } \mathcal{F}^+(A)_i \in D^b_{\mathcal{coh}}(\text{Qcoh}(Y)).$$

Indeed, it suffices to prove this for affine $Y = \text{Spec} R$. In view of Lemma 4.16 the claim follows from Lemma 3.26 because $A_0$ is assumed to be coherent over $\mathcal{O}_Y$ in our standing assumption (4.1).

Our discussion establishes the following
Proposition 4.26. There is a pair of adjunctions

\[
\xymatrix{ 
\mathcal{D}(\text{Gr}(A)) & \mathcal{D}(\text{Gr}(A)) \ar[l]_{\mathcal{E}_{\mathcal{F}+}} & \mathcal{D}(\text{QCoh}(Y)) \ar[r]^{R\mathcal{H}_{\mathcal{O}_Y}} & \mathcal{D}(\text{QCoh}(Y)) 
}
\]

where the composition \(\mathcal{E}_{\mathcal{F}+} \circ R\mathcal{H}_{\mathcal{O}_Y}(A,-)\) is isomorphic to the functor \(R\mathcal{H}_{\mathcal{O}_Y}(\mathcal{C}_{\mathcal{F}+}(A),-)^{\prime}\).

Moreover, if \(\mathcal{F} \in \mathcal{D}^+(\text{QCoh}(Y))\), then this composition is local on the base \(Y\) in the sense that, for any affine open subscheme \(U = \text{Spec } R \subset Y\), if we let \(A := \mathcal{A}(U)\) and \(F := \mathcal{F}(U)\), then there is a natural isomorphism

\[
R\mathcal{H}_{\mathcal{O}_Y}(\mathcal{C}_{\mathcal{F}+}(A),\mathcal{F})(U) \cong \mathcal{H}_{\mathcal{O}_U}(\mathcal{C}_{\mathcal{F}+}(A), F) \quad \text{in } \mathcal{D}(\text{Gr}(A))
\]

Proof. The adjunction on the left has been shown in (4.15). The one on the right is a standard adjunction.

Proposition 4.31. For any \(\mathcal{F} \in \mathcal{D}(\text{Gr}(A))\), the adjunction \((\mathcal{E}_{\mathcal{F}+}, \mathcal{F})\) is an instance of (4.19), for \(\mathcal{M} = \mathcal{C}_{\mathcal{F}+}(A)\) and \(\mathcal{N} = \mathcal{A}\). For the final statement, apply Lemma 4.21 to \(\mathcal{M} = \mathcal{C}_{\mathcal{F}+}(A)\), which is valid in view of (4.25).

We now discuss Serre’s equivalence in the setting (4.1) of non-affine base. In fact, all the discussion carries through in this case almost without change. We will provide the formal arguments to construct the necessary functors and natural transformations. Properties about them can then be checked locally on affine \(U \subset Y\), i.e., by resorting to results in Section 4 so that some of these will be skipped.

Let \(X^+ := \text{Proj}_Y(A)\) and \(\pi^+ : X^+ \rightarrow Y\) the projection, then there are functors

\[
(\text{4.28}) \quad \mathcal{E}^+_+(\mathcal{F})(\mathcal{M}) := \mathcal{M} \
\]

For any \(\mathcal{M} \in \mathcal{D}(\text{Gr}(A))\), let \(\mathcal{E}^+_+(\mathcal{M}) := \mathcal{M}
\]

By considering its restrictions to affine open subschemes \(U \subset Y\), one can alternatively describe it as \(\mathcal{H}^0(\mathcal{C}_{\mathcal{F}+}(\mathcal{M}))_i \cong \pi^+_i(\mathcal{M}(i))\). As in the affine case, there are canonical maps in \(\text{Gr}(A)\):

\[
\text{4.29} \quad \mathcal{M} \rightarrow \mathcal{C}_{\mathcal{F}+}(\mathcal{M}) \leftarrow \mathcal{C}_{\mathcal{F}+}(\mathcal{M}) \rightarrow \mathcal{E}^+_+(\mathcal{M})
\]

In the non-affine setting, the notion of being \(\frac{1}{2}\)-Cartier still makes sense. One can either define it in a way parallel to Definition 3.10 or simply resort to the affine case:

Definition 4.30. The pair \((Y, \mathcal{A})\) in (4.1) is said to be \((\text{positively/negatively}) \frac{1}{2}\)-Cartier if for some, and hence any, open affine cover \(\{U_a\}\) of \(Y\), each of the \(\mathcal{A}\)-graded rings \(\mathcal{A}(U_a)\) is so, in the sense of Definition 3.10.

We have the following analogue of Propositions 3.9 and 3.11.

Proposition 4.31. For any \(\mathcal{F} \in \mathcal{D}(\text{Gr}(A))\), there is a canonical isomorphism \(\mathcal{E}^+_+(\mathcal{F}) \cong \mathcal{F}\).

Moreover, if we assume that \(\mathcal{A}\) is positively \(\frac{1}{2}\)-Cartier, then for any \(\mathcal{M} \in \mathcal{D}(\text{Gr}(A))\), we have

\[
(1) \quad \text{for any } i \in \mathbb{Z}, \text{ the map } (\mathcal{E}^+_+(\mathcal{M}))_i \rightarrow (\mathcal{C}_{\mathcal{F}+}(\mathcal{M}))_i \text{ is an isomorphism.}
\]

\[
(2) \quad \text{both the kernel and cokernel of } \mathcal{M} \rightarrow \mathcal{C}_{\mathcal{F}+}(\mathcal{M}) \text{ are } (\mathcal{F}^+)^{\infty}\text{-torsion.}
\]

\[
(3) \quad \text{if } \mathcal{M}_{\geq c} \text{ is coherent over } \mathcal{A}_{>0} \text{ for some } c \in \mathbb{Z}, \text{ then so are } (\mathcal{C}_{\mathcal{F}+}(\mathcal{M}))_{\geq c} \text{ and } (\mathcal{E}^+_+(\mathcal{M}))_{\geq c}, \text{ for any } c \in \mathbb{Z}.
\]

The notions in Definition 3.12 are local on the base \(Y\), and therefore extends directly to notions on \(\text{Gr}(A)\) in the setting (4.1). This allows us to define the Serre quotients

\[
\mathcal{Q}^+(\mathcal{A}) := \mathcal{A}(\mathcal{A})/\mathcal{Tor}^+(\mathcal{A}) \quad \text{and} \quad \mathcal{Q}^+_{(m)}(\mathcal{A}) := \mathcal{A}(\mathcal{A})/\mathcal{Tor}^+_{(m)}(\mathcal{A})
\]

Proposition 4.31 then implies the following
Theorem 4.32 (Serre’s equivalence). If the pair \((Y, \mathcal{A})\) is \(\frac{1}{2}\)-Cartier, then the functors \((-)^\sim\) and \(0\mathcal{L}^+\) in (4.28) descend to quasi-inverse equivalences

\[
0\mathcal{L}^+ : \text{QCoh}(X^+) \rightleftharpoons Q^+_{\text{Gr}}(\mathcal{A}) : (-)^\sim
\]

Now we consider the derived version of the second row of (4.29). Proposition 2.28 can be generalized to the non-affine setting to show that \(\check{\mathcal{C}}\mathcal{I}^+ : D(\text{Gr}(\mathcal{A})) \rightarrow D(\text{Gr}(\mathcal{A}))\) is the right derived functor of \(0\check{\mathcal{C}}\mathcal{I}^+\). Thus, if we take \(R^+ : D(\text{QCoh}(X^+)) \rightarrow D(\text{Gr}(\mathcal{A}))\) as the right derived of \(0R^+\), then there is still an adjunction at the level of derived categories:

\[
(-)^\sim : D(\text{Gr}(\mathcal{A})) \rightleftharpoons D(\text{QCoh}(X^+)) : (R^+)^\sim
\]

Moreover, the universal property of \(\check{\mathcal{C}}\mathcal{I}^+\) as a total right derived functor gives a map

\[
\check{\mathcal{C}}\mathcal{I}^+(\mathcal{M}) \rightarrow R^+(\check{\mathcal{M}})\]

In the present non-affine setting, the analogue of Lemma 3.24 still holds, and hence gives a description of (4.34) in each weight component in terms of the following commutative diagram in \(D(\text{QCoh}(Y))\):

\[
\begin{array}{ccc}
\check{\mathcal{C}}\mathcal{I}^+(\mathcal{M})_i & \rightarrow & R^+(\check{\mathcal{M}})_i \\
\downarrow & & \downarrow \\
R\pi^+_*(\check{\mathcal{M}}(i)) & \rightarrow & R\pi^+_* R\mathcal{H}om_{\mathcal{O}_{X^+}}(\mathcal{A}(i), \check{\mathcal{M}})
\end{array}
\]

This gives all the ingredients to relate Greenlees-May duality to Grothendieck duality. Namely, if we replace (3.36) and (3.37) by (4.27) and (4.33) respectively, then we have the following analogue of Theorem 3.38:

**Theorem 4.36.** For each \(\mathcal{F} \in D(\text{QCoh}(Y))\), there is a canonical isomorphism in \(D(\text{Gr}(\mathcal{A}))\):

\[
R^+((\pi^+)^!(\mathcal{F})) \cong E_{\mathcal{F}^+}(R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}, \mathcal{F}))
\]

where the complex on the right hand side is described in Proposition 4.26.

5. **Homological flips and homological flops**

This section contains the definition of homological flips and homological flops, the main notion of study in this paper (see Definition 5.69). Sections 5.1 and 5.2 serve as motivation for this notion. The constructions there will also go into the proof that certain classes of flips and flops are in fact homological flips and homological flops (see Theorem 5.72). We also prove the main result concerning homological flips/flops, which relates certain local cohomology complexes under homological flips/flops (see Theorem 5.74 and Corollary 5.78).

5.1. **Log flips and graded rings.** We continue to work in the setting of (4.1), which we will simply refer to as a pair \((Y, \mathcal{A})\). The constructions in Section 4 applies for both the positive and the negative directions. This associates two projective morphisms

\[
\begin{array}{ccc}
X^- & \rightarrow & \text{Proj}_Y(\mathcal{A}) \\
\pi^- & \rightarrow & \pi^+
\end{array}
\]

On the scheme \(X^- = \text{Proj}_Y(\mathcal{A})\), we will still write \(\mathcal{O}_{X^-}(i) := \mathcal{A}(i)\). Thus, \(\mathcal{O}_{X^-}(-1)\) is \(\pi^-\)-ample on \(X^-\), perhaps contrary to some conventions.

This viewpoint of assigning the spaces (5.1) to a pair \((Y, \mathcal{A})\) leads to a simple proof of the following

**Lemma 5.2.** For any \(\mathcal{M} \in \text{Gr}(\mathcal{A})\) locally finitely generated over \(\mathcal{A}\), there exists \(c^+, c^- \in \mathbb{Z}\) such that \(R\Gamma_{\mathcal{F}^+}(\mathcal{M})_i \simeq 0\) for all \(i > c^+\) and \(R\Gamma_{\mathcal{F}^-}(\mathcal{M})_i \simeq 0\) for all \(i < c^-\). The same is therefore true for any \(\mathcal{M}^\bullet \in D^b_{\text{coh}}(\text{Gr}(\mathcal{A}))\).
Proof. Since the local cohomology complexes $R\Gamma_{\mathcal{I}^\pm}(\mathcal{M})$ are local on the base $Y$ (see Lemma 4.30), it suffices to prove this on each open affine subschemes $U = \text{Spec } R \subset Y$ is affine. Let $A := \mathcal{A}(U)$ and $M := \mathcal{M}(U) \in \mathcal{O}(A)$.

By (2) and (3) of Proposition 3.11, we see that $M_i \to N'_i = H^0(\tilde{\mathcal{G}}_{I^+}(M)_i)$ is an isomorphism for $i \geq c_0^+$ for some $c_0^+ \in \mathbb{Z}$. Thus, for $i \geq c_0^+$, we have $R\Gamma_{I^+}(M)_i \cong 0$ if and only if $R^i\pi^+_\ast(\overline{M(i)}) = 0$ for all $j > 0$. Since $A$ is Noetherian, it is $\mathcal{O}$-Cartier for some $d > 0$. Then the sequence $\overline{M(i)}, M(i + d), M(i + 2d), \ldots$ of coherent sheaves on $X^+$ is a sequence of twist by an ample invertible sheaf, and hence must eventually have zero higher cohomology. Apply this for $i = 0, \ldots, d - 1$ in order to find $c^+$. The integer $c^-$ can also be found in a similar way, by considering the sequence $\overline{A(i)}, A(i - d), A(i - 2d), \ldots$ on $X^-$.

Observe that for any $\mathcal{M} \in \mathcal{D}(\mathcal{O}(A))$, there are isomorphisms

$$R\Gamma_{\mathcal{I}^+}(\mathcal{M})^{(m)} \cong R\Gamma_{\mathcal{I}^+}(\mathcal{M})^{(m)}$$

$$R\Gamma_{\mathcal{I}^-}(\mathcal{M})^{(m)} \cong R\Gamma_{\mathcal{I}^-}(\mathcal{M})^{(m)}$$

in $\mathcal{D}(\mathcal{O}(A^{(m)}))$. In the case of affine $Y$, this is because the isomorphism type of $R\Gamma_{I}(M)$ in $\mathcal{D}(\mathcal{O}(A))$ depends only on $\sqrt{I}$, so that $R\Gamma_{I^\pm}(M)$ may be computed by a set of elements $f_1, \ldots, f_r \in I^\pm(m)$ that generate $I^\pm(m)$ in $A^{(m)}$. In general, the affine case above implies that $R\Gamma_{\mathcal{I}^\pm}(\mathcal{M})^{(m)} \in \mathcal{D}(\mathcal{O}(Y)^{\infty})\mathcal{Tor}(\mathcal{O}(A^{(m)}))$, so that there is a induced map $R\Gamma_{\mathcal{I}^\pm}(\mathcal{M})^{(m)} \to R\Gamma_{\mathcal{I}^\pm}(\mathcal{M})^{(m)}$ in $\mathcal{D}(\mathcal{O}(A))$. The fact that this is an isomorphism can be checked locally on $U \subset Y$, which we have established above.

As a result, Lemma 3.2 applied to $\mathcal{M} = \mathcal{A}$ shows that, for any pair $(Y, A)$, its high enough uple components is stable in the sense of the following

**Definition 5.3.** The pair $(Y, A)$ is said to be pre-stable if $R\Gamma_{\mathcal{I}^+}(A)_i \cong 0$ for all $i > 0$ and $R\Gamma_{\mathcal{I}^-}(A)_i \cong 0$ for all $i < 0$. In other words, the integers $c^-, c^+ \in \mathbb{Z}$ in Lemma 3.2 for $\mathcal{M} = \mathcal{A}$ can be taken to be $c^- = c^+ = 0$.

The pair $(Y, A)$ is said to be stable if it is pre-stable and Cartier in the sense of Definition 4.30.

The main class of examples that we consider is the class of log flips between normal varieties. We first recall some standard terminology:

**Definition 5.4.** A morphism $f : X \to Y$ between varieties is said to be a contraction if it is projective, and if the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

A birational contraction $f$ is said to be small if the exceptional set $\text{Ex}(f) \subset X$ has codimension $\geq 2$.

If $X$ and $Y$ are normal varieties, then by Zariski’s main theorem, any birational projective morphism $f : X \to Y$ is a contraction. In this case, for any Weil divisor $D \in \text{WDiv}(X)$, there is a canonical inclusion $f_*\mathcal{O}(D) \hookrightarrow \mathcal{O}(f_*D)$ if we regard both as subsheaves of $\mathcal{K}_X = \mathcal{K}_Y$. One advantage of smallness is that

$$\text{If } f : X \to Y \text{ is a small birational contraction, then for any Weil divisor } D \in \text{WDiv}(X), \text{ the canonical inclusion } f_*\mathcal{O}(D) \hookrightarrow \mathcal{O}(f_*D) \text{ is an isomorphism.}$$

For any birational contraction $f : X \to Y$, the pushforward map $f_* : \text{WDiv}(X) \to \text{WDiv}(Y)$ on Weil divisors is surjective. It is bijective if and only if $f$ is small. As a result, for a composable pair of birational contractions $W \xrightarrow{g} X \xrightarrow{f} Y$, we have

The composition $f \circ g$ is small if and only if both $f$ and $g$ are small.

Recall that two birational small contractions

$$X^- \xrightarrow{\pi^-} Y \xleftarrow{\pi^+} X^+$$

of normal varieties over a field $k$ is said to be a log flip\(^3\) if there is a Weil divisor $D^-$ on $X^-$ such that, if we denote by $D^+$ its strict transform on $X^+$, then we have

\(^3\)The divisor $D^-$ is often written in the form $D^- = K_{X^-} + D'^-$
(1) $-D^-$ is $\mathbb{Q}$-Cartier and $\pi^-$-ample;
(2) $D^+$ is $\mathbb{Q}$-Cartier and $\pi^+$-ample.

In this case, if we denote by $D_Y$ their common strict transform to $Y$, then we have the following

**Proposition 5.7.** The quasi-coherent sheaf of $\mathbb{Z}$-graded rings $A := \bigoplus_{i \in \mathbb{Z}} O_Y(iD_Y)$ is Noetherian. Moreover, the maps (5.1) for the resulting pair $(Y, A)$ is canonically identified with (5.6). Under this identification, there are also canonical identifications of the sheaves of $\mathbb{Z}$-graded rings

$$\psi : \bigoplus_{i \in \mathbb{Z}} \widetilde{A}(i)_{X^\pm} \xrightarrow{\cong} \bigoplus_{i \in \mathbb{Z}} O_{X^\pm}(iD^\pm)$$

Thus, if $dD^-$ is Cartier, then $A$ is negatively $\frac{1}{d}$-Cartier; if $dD^+$ is Cartier, then $A$ is positively $\frac{1}{d}$-Cartier. The maps $\psi$ on $X^-$ and $X^+$ are moreover compatible in the sense that the following diagram commutes:

$$(5.8)$$

$$\begin{array}{ccc}
\pi^-(\widetilde{A}(i)_{X^-}) & \xrightarrow{\cong} & \pi^+(\widetilde{A}(i)_{X^+}) \\
\pi^-(\psi) \downarrow & & \downarrow \pi^+(\psi) \\
\pi^- O_{X^-}(iD^-) & \xrightarrow{\cong} & O_Y(iD_Y) \\
& & \pi^- O_{X^+}(iD^+) 
\end{array}$$

where the equalities in the second row is due to (5.5).

**Proof.** By (5.4), we have canonical isomorphisms $\pi^-(O(iD^-)) \cong A_i \cong \pi^+(O(iD^+))$, in fact equality as subsheaves of $\mathcal{O}_Y$. This shows that the sheaf $A$ is Noetherian (see Propositions 2.2 and 3.9). The claimed isomorphisms $\widetilde{A}(i)_{X^\pm} \cong O_{X^\pm}(iD^\pm)$, as well as the claimed commutativity of diagram, then follows from the obvious generalization of Proposition 3.9 to the case of non-affine $Y$, since the conditions in (3.8) are satisfied for the sheaves of $\mathbb{Z}$-graded algebras $\bigoplus_{i \in \mathbb{Z}} O_{X^\pm}(iD^\pm)$ on $X^\pm$. The statement about $\frac{1}{d}$-Cartier property then follows from Corollary 5.10.

Proposition 5.7 shows that for any log flip (5.6), the associated pair $(Y, A)$ determines a log flip in the sense of the following

**Definition 5.9.** Let $(Y, A)$ be a pair as in (4.4) satisfying $A_0 = O_Y$. We say that $(Y, A)$ *determines a log flip* if the associated diagram (5.1) consists of small birational contraction between normal varieties over a field $k$, and if there exists a Weil divisor $D^-$ on $X^-$ such that, if we denote by $D^+$ by its strict transform on $X^+$, then there are isomorphisms of sheaves of $\mathbb{Z}$-graded $k$-algebras

$$(5.10)$$

$$\psi : \bigoplus_{i \in \mathbb{Z}} \widetilde{A}(i)_{X^\pm} \xrightarrow{\cong} \bigoplus_{i \in \mathbb{Z}} O_{X^\pm}(iD^\pm)$$

which are compatible in the sense that the following diagram commutes:

$$(5.11)$$

$$\begin{array}{ccc}
\pi^-(\widetilde{A}(i)_{X^-}) & \xrightarrow{\cong} & \pi^+(\widetilde{A}(i)_{X^+}) \\
\pi^-(\psi) \downarrow & & \downarrow \pi^+(\psi) \\
\pi^- O_{X^-}(iD^-) & \xrightarrow{\cong} & O_Y(iD_Y) \\
& & \pi^- O_{X^+}(iD^+) 
\end{array}$$

Notice that the isomorphism (5.10) guarantees that $-D^-$ is $\mathbb{Q}$-Cartier and $\pi^-$-ample, while $D^+$ is $\mathbb{Q}$-Cartier and $\pi^+$-ample, so that the associated diagram (5.1) is indeed a log flip.

If, moreover, every sheaf $\widetilde{A}(i)_{X^\pm}$ is maximally Cohen-Macaulay, then we say that the pair $(Y, A)$ *determines a Cohen-Macaulay log flip*.

**Remark 5.12.** The diagram (5.11) induces a canonical map $A_i \to O_Y(iD_Y)$. In constrast to (5.8), we do not require this map to be an isomorphism. In fact, this map is an isomorphism if and only if the sheaf $A_i$ is reflexive. Indeed, the commutativity of the diagram (5.11) with the filled in map $A_i \to O_Y(iD_Y)$ shows that this map is an isomorphism on an open subset with complement of codimension $\geq 3$. If $A_i$ is

$4$ The equalities in the second row is due to (5.5).
reflexive, then this map is an isomorphism by Proposition 5.32 below, since a divisorial sheaf $O_Y(iD_Y)$ on a normal variety is always reflexive. The converse is obvious.

Now we study the converse problem: given a pair $(Y, \mathcal{A})$, deduce some properties for the diagram (5.1). We will be mostly interested in the case $O_Y = \mathcal{A}_0$. Otherwise, one may replace $Y$ by $Y := \text{Spec}_Y \mathcal{A}_0$. The first question we are interested in is when $\pi^+$ and $\pi^-$ are birational. Since this question is local on the base, we may assume that $Y$ is affine.

Thus, let $A$ be a Noetherian $\mathbb{Z}$-graded ring, and let $R := A_0$. Let $J := (I^- \cdot I^+)_0 \subset R$. It is clear that we have

$$(I^-)_0 = J = (I^+)_0$$

For any graded prime ideal $p \subset A$, denote by $p_0 \subset R$ its degree zero part. Suppose that $J \not\subset p_0$, then there exists some $f \in A_d$, $g \in A_{-d}$, for some $d > 0$, such that $fg \notin p$. Thus, if we denote by $A_{p_0}$ the localization of $A$ with respect to the multiplicative system $S := R \setminus p_0$, then we have $(A_{p_0})^{(d)} \cong R_p[k, t^{-1}]$. Thus, the morphisms (5.1) become isomorphisms when we localize at $p_0 \in \text{Spec} R \setminus V(J)$. We summarize this into the following

**Lemma 5.13.** For a pair $(Y, \mathcal{A})$ as in (4.1) satisfying $A_0 = O_Y$, let $\mathcal{I} := (\mathcal{I}^0 \cdot \mathcal{I}^+)_0 \subset O_Y$. Then for all $y \in Y \setminus V(\mathcal{I})$, there exists an open neighborhood $\eta \subset U \subset Y \setminus V(\mathcal{I})$ such that $A_{(d)}|_U \cong (A|_U)_0[t, t^{-1}]$ for some $d > 0$, hence the morphisms (5.1) are isomorphisms above $Y \setminus V(\mathcal{I})$.

Therefore, if $\mathcal{A}$ is furthermore assumed to be a sheaf of integral domains, and if $A_{<0} \neq 0$ and $A_{>0} \neq 0$, then the morphisms (5.1) are birational. Moreover, if we let $d := \min \{ |i| \in \mathbb{Z} \mid i \neq 0 \text{ such that } \mathcal{A}_i \neq 0 \}$, then there are embeddings of sheaves of $\mathbb{Z}$-graded rings $\mathcal{A} \hookrightarrow \mathcal{H}(t, t^{-1})$ and $\bigoplus_{i \in \mathbb{Z}} \mathcal{A}(i)_{X^\pm} \hookrightarrow \mathcal{H}(t, t^{-1})$, where $\deg(t) = d$, such that the following diagram commutes:

$$\begin{align*}
\bigoplus_{i \in \mathbb{Z}} \pi^+_i(\mathcal{A}(i)_{X^+}) & \longrightarrow A & \bigoplus_{i \in \mathbb{Z}} \pi^-_i(\mathcal{A}(i)_{X^-}) \\
\downarrow & & \downarrow \\
\pi^+_i \mathcal{H}(t, t^{-1}) & \longrightarrow \mathcal{H}(t, t^{-1}) & \pi^-_i \mathcal{H}(t, t^{-1})
\end{align*}$$

(5.14)

**Proof.** The preceding discussion establishes the first paragraph. For the second paragraph, notice that, since $\mathcal{A}$ is a sheaf of integral domains, and since $A_{<0} \neq 0$ and $A_{>0} \neq 0$, we have $\mathcal{A}_i \neq 0$ if and only if $i/d \in \mathbb{Z}$. It is then clear that any nonzero homogeneous element $f \in A(U)$ becomes invertible after passing the stalk at the generic point $\eta \in Y$. In particular, choosing such an element of degree $d$ gives an invertible element $f \in A_d = \mathcal{A} \otimes_{O_Y} \mathcal{H}$, and hence identifies $\mathcal{H}(t, t^{-1}) \cong \mathcal{A} \otimes_{O_Y} \mathcal{H}$ by sending $t$ to $f$. Since $\mathcal{A}$ is integral, we have $\mathcal{A} \hookrightarrow \mathcal{A} \otimes_{O_Y} \mathcal{H}$, which compose to the sought for embedding of $\mathcal{A}$.

Notice that $\bigoplus_{i \in \mathbb{Z}} \mathcal{A}(i)_{X^\pm}$ is a sheaf of $\mathbb{Z}$-graded integral domains on $X^\pm$. Moreover, recall that there is a canonical map $\mathcal{A}_i \rightarrow \pi^+_i(\mathcal{A}(i)_{X^\pm})$ for each $i \in \mathbb{Z}$, so that the section $f \in A(U)_d$ gives rise to a section $f \in \mathcal{A}(d)_{X^\pm}((\pi^\pm)^{-1}(U))$. The same argument then gives the embeddings for $\bigoplus_{i \in \mathbb{Z}} \mathcal{A}(i)_{X^\pm}$, making the claimed diagram commutes.

The preimage of the closed subspace $V(\mathcal{I}) \subset Y$ under $\pi^-$ and $\pi^+$ also has a simple description. We again focus on the affine case, keeping the above notation. Then we have

$$\sqrt{J \cdot A} = \sqrt{I^- \cdot I^+}$$

The inclusion “$\subset$” is clear. For the inclusion “$\supset$”, it suffices to show that $I^- \cdot I^+ \subset \sqrt{J \cdot A}$. Thus, let $\deg(f) = m > 0$ and $\deg(g) = -n < 0$, then for any $l \geq \max\{m, n\}$, we have $(fg)^l = (f^l g^m)(f^{l-m} g^{l-n})$, with $f^{l-m} g^{l-n} \in J$.

Recall that for any ring map $\varphi : A \rightarrow B$ and any ideal $I \subset A$, one has $\sqrt{I \cdot B} = \sqrt{I \cdot B}$. Thus, as a consequence of (5.15), we see that, for any $f \in I^+$, we have

$$\sqrt{J \cdot A_f} = \sqrt{I^- \cdot I^+ \cdot A_f} = \sqrt{I^- \cdot A_f}$$

Since $X^+ = \text{Proj}^+(A)$ is covered by the open subschemes Spec $(A_f)_0$, we have the following
Lemma 5.16. For a pair \((Y, A)\) as in (441) satisfying \(A_0 = \mathcal{O}_Y\), let \(\mathcal{J} := (\mathcal{J}^- \cdot \mathcal{J}^+)_0 \subset \mathcal{O}_Y\). Then the preimages of \(V(\mathcal{J})\) under the maps (5.1) are given by

\[
(\pi^-)^{-1}(V(\mathcal{J})) = \text{Proj}_Y(A/\sqrt{\mathcal{J}^+}) \quad \text{and} \quad (\pi^+)^{-1}(V(\mathcal{J})) = \text{Proj}_Y(A/\sqrt{\mathcal{J}^-})
\]

Now we investigate Cohen-Macaulay property (or more generally the \((S_r)\) condition) of \(X^-\) and \(X^+\). We start with the following

Lemma 5.17. Let \(\varphi : R \to S\) be an injective finite ring map between Noetherian rings. If \(S\) is catenary of finite Krull dimension then so is \(R\), and we have \(\dim(R_p) = \dim(S_q)\) for all prime ideal \(q \in \text{Spec} S\) above \(p \in \text{Spec} R\). Under this catenary condition, the following two conditions are equivalent:

1. \(S\) satisfies Serre’s condition \((S_r)\), i.e., \(\dim_{S_q}(S_q) \geq \min\{r, \dim(S_q)\}\) for all prime ideal \(q \in \text{Spec} S\).
2. \(S\) is “maximally \((S_r)\)” as a module over \(R\), i.e., it satisfies the condition \(\dim_{R_p}(S_q) \geq \min\{r, \dim(R_p)\}\) for all prime ideals \(p \in \text{Spec} R\), where we denote by \(S_p\) the localization of \(S\) with respect to the multiplicative subset \(R \setminus p\).

Proof. Recall that integral extensions preserve Krull dimensions (see, e.g., [30] Tags 00GQ, 00GT, 00GU]), so that we have \(\dim R = \dim S\). For any prime ideal \(q \in \text{Spec} S\) above \(p \in \text{Spec} R\), i.e., \(p = \varphi^{-1}(q)\), the map \(R/p \to S/q\) is still an integral extension, and hence has the same Krull dimension. In general, we have \(\dim R \geq \dim R_p + \dim R/p\). By the catenary condition (see, e.g., [30] Tag 02IG6) on \(S\), we have \(\dim R = \dim S\) and \(\dim R/p = \dim S/q\), we have \(\dim(R_p) \leq \dim(S_q)\). Therefore, we have equality, which in particular implies the catenary property of \(R\), proving the first statement. The claimed equivalence then follows from [30] Tag 0AUK] which, when applied to the finite ring map \(R_p \to S_p\), asserts that

\[
(5.18) \quad \dim_{R_p}(S_p) = \min\{ \dim_{S_q}(S_q) \mid q \in \text{Spec} S\text{ above } p \}
\]

\(\square\)

Corollary 5.19. Let \(B\) be a Noetherian \(\mathbb{Z}\)-graded ring such that \(B^{(d)} \cong B_0[t, t^{-1}]\) for some \(d > 0\). Suppose that \(B_0\) is universally catenary. Then the following two conditions are equivalent:

1. \(B\) satisfies Serre’s condition \((S_r)\).
2. Each \(B_i, i \in \mathbb{Z}\) is “maximally \((S_{r-1})\)” as a module over \(B_0\), i.e., it satisfies the condition \(\dim_{(B_0)_p}((B_i)_p) \geq \min\{r - 1, \dim((B_0)_p)\}\) for all prime ideals \(p \in \text{Spec} B_0\).

In particular, \(B\) is Cohen-Macaulay of dimension \(n + 1\) if and only if \(B_0\) is Cohen-Macaulay of dimension \(n\) and each \(B_i\) is maximal Cohen-Macaulay as a module over \(B_0\).

Proof. Apply Lemma 5.17 to the finite extension \(B^{(d)} \to B\), which is valid because \(B_0\) is assumed to be universally catenary. By assumption, we have \(B^{(d)} \cong B_0[t, t^{-1}]\). Moreover, by considering the \(\mathbb{Z}\)-grading mod \(d\), we see that, as a module over \(B^{(d)}\), \(B\) splits as a direct sum \(B = \bigoplus_{i=0}^{d-1} M(i)\), where \(M(i) := \bigoplus_{j \in \mathbb{Z}} B_{i+jd} \cong B_i[t, t^{-1}]\). Since the depth of a finite direct sum is equal to the minimal of the depths of the summands, we see that in this case, condition (2) in Lemma 5.17 says that each \(M(i)\) is maximally \((S_r)\) over \(B^{(d)}\). As we have observed above, we have \((B^{(d)}, M(i)) \cong (B_0[t, t^{-1}], B_i[t, t^{-1}])\), so that this last condition is furthermore equivalent to the condition that each \(B_i\) is maximally \((S_{r-1})\) over \(B_0\). Thus, the result follows from Lemma 5.17. \(\square\)

This can be used to prove the following

Proposition 5.20. If \(A\) is a Cohen-Macaulay \(\mathbb{Z}\)-graded Noetherian ring of (pure) dimension \(n + 1\), where \(A_0\) is universally catenary, then both \(X^-\) and \(X^+\) are Cohen-Macaulay of (pure) dimension \(n\). Moreover, for each \(i \in \mathbb{Z}\), the sheaf \(\mathcal{O}_{X^\pm}(i) := \tilde{A}(i)_{X^\pm}\) is maximally Cohen-Macaulay.

Proof. It suffices to prove this for \(X^+\). For any homogeneous element \(f \in A_d\) of degree \(d > 0\), the localization \(B := A_f\) is still Cohen-Macaulay of pure dimension \(n + 1\), and \(B_0\) is still universally catenary. Thus, we may apply Corollary 5.19 to obtain the desired result. \(\square\)
A similar proof shows the following

**Proposition 5.21.** Let $A$ be a $\mathbb{Z}$-graded normal domain finitely generated over a field $k$ of characteristic zero. If $\text{Spec } A$ has rational singularities, then so does $X^-$ and $X^+$. 

*Proof.* For any homogeneous element $f \in A_d$ of degree $d > 0$, the $k$-algebra $A_f$ still has rational singularities. Since $(A_f)^{(d)} \cong (A_f)_0[t, t^{-1}]$, there is a finite morphism $\text{Spec } A_f \to \text{Spec } (A_f)_0[t, t^{-1}]$. By (5.23) below, $(A_f)_0$, and hence $(A_f)[t, t^{-1}]$, is normal. Thus, an application of [17 Proposition 5.13] shows that $\text{Spec } (A_f)_0[t, t^{-1}]$ has rational singularities. Since $k[t, t^{-1}]$ is smooth over $k$, this shows that $\text{Spec } (A_f)_0$ has rational singularities as well. Since these covers the variety $X^+$, this shows that $X^+$ has rational singularities. By symmetry, it holds for $X^{-}$ as well. □

We also apply Corollary (5.19) to give sufficient conditions for a $\mathbb{Z}$-graded ring to give rise to a log flip. We start with the following

**Lemma 5.22.** Suppose that the $\mathbb{Z}$-graded ring $A$ is a normal domain, then the schemes $X^-$, $X^+$ and $Y := \text{Spec } A_0$ are normal integral schemes. If $A$ moreover satisfies Serre’s condition (S$_3$), and if $A_{< 0} \neq 0$ and $A_{> 0} \neq 0$, then for each $i \in \mathbb{Z}$, the sheaf $\mathcal{O}_{X^\pm}(i) := \mathcal{A}(i)_{X^\pm}$ is reflexive.

*Proof.* One of the definitions of a normal domain is that of an integrally closed domain. From this definition, one can directly verify that (5.23) If $A$ is a $\mathbb{Z}$-graded normal domain, then $A_0$ is also a normal domain.

Since $X^+$ is locally given by $\text{Spec } (A_f)_0$, and since localization preserves normality, the first statement for $X^+$ follows directly from (5.23). Similarly for $X^-$. The statement for $Y$ also follows from (5.23).

If $A$ satisfies Serre’s condition (S$_3$), then so does $A_f$, for $\deg(f) > 0$. Applying Corollary (5.19) to $B := A_f$, we see for each $i \in \mathbb{Z}$, the module $(A_f)_i$ over $(A_f)_0$ is maximally $(S_2)$. In other words, each of the sheaves $\mathcal{A}(i)_{X^\pm}$ is maximally $(S_2)$. Since $X^+$ is normal, this implies that $\mathcal{A}(i)_{X^+}$ is reflexive (see, e.g., [56 Proposition 1.4.1(b)]). □

**Proposition 5.24.** Let $(Y, A)$ be a pair as in (1.1). Suppose that

1. $Y$ is a variety over a field $k$, and $A_0 = \mathcal{O}_Y$.
2. $A$ is a sheaf of integral domains satisfying the condition $(S_3)$.
3. The closed subsets $V(\mathcal{F}^-)$ and $V(\mathcal{F}^+)$ of $\text{Spec } Y A$ both have codimension $\geq 2$.

then $(Y, A)$ determines a log flip in the sense of Definition 5.3. If the sheaf $A$ of algebras is furthermore Cohen-Macaulay, then it determines a Cohen-Macaulay log flip.

*Proof.* Normality of $X^-, X^+$ and $Y$ follows from Lemma 5.22. The assumption (3) implies that $A_{< 0} \neq 0$ and $A_{> 0} \neq 0$, so that Lemma 5.13 shows in particular that $\pi^-$ and $\pi^+$ are birational, and $\dim(Y) = \dim(A(U)) - 1$ for any affine open $U \subset Y$. Since $\dim(\text{Proj}(B)) \leq \dim(B) - 1$ for any Noetherian $\mathbb{N}$-graded ring $B$, we see from Lemmata 5.13 and 5.16 that the exceptional loci of $\pi^-$ and $\pi^+$ have codimension $\geq 2$, proving the first condition of Definition 5.9.

For the second condition, apply Lemma 5.13 again, which gives embeddings $\bigoplus_{i \in \mathbb{Z}} \mathcal{A}(i)_{X^\pm} \hookrightarrow \mathcal{X}_{X^\pm}[t, t^{-1}]$ of $\mathbb{Z}$-graded $k$-algebras. Since $\mathcal{A}(i)_{X^\pm}$ are reflexive sheaves (see Lemma 5.22), these embeddings identify $\mathcal{A}(i)_{X^\pm} = \mathcal{O}(D^\pm_i)$, for some Weil divisor $D^\pm_i$ on $X^\pm$. Since the embedding into $\mathcal{X}_{X^\pm}[t, t^{-1}]$ is multiplicative, we have $D^\pm_i + D^\pm_j \leq D^\pm_{i+j}$ for all $i, j \in \mathbb{Z}$. On the other hand, if we choose $d > 0$ such that the pair $(Y, A)$ is $\frac{1}{d}$-Cartier, then we have $D^\pm_i + D^\pm_d = D^\pm_{i+d}$. Thus, we must have $D^\pm_i + D^\pm_j = D^\pm_{i+j}$ for all $i, j \in \mathbb{Z}$. In particular, we have $D^\pm_i = iD^\pm$ for $D^\pm := D^\pm_1$. This gives the desired isomorphisms (5.11). The commutativity of (5.11) then follows from that of (5.14). Therefore, it suffices to show that $D^+$ is the strict transform of $D^-$. Indeed, the subsheaves $\mathcal{O}_{X^-}(D^-)$ and $\mathcal{O}_{X^+}(D^+)$ of $\mathcal{X}_{X^-}$ and $\mathcal{X}_{X^+}$ are defined as the images of the embedding $\mathcal{A}(i)_{X^\pm} \hookrightarrow \mathcal{X}_{X^\pm}$. By the commutativity of (5.11), they coincide on the canonically identified open subschemes $X^- \setminus \text{Proj}_Y(\mathcal{A}/\sqrt{\mathcal{F}^+}) \cong Y \setminus V(\mathcal{F}) \cong X^+ \setminus \text{Proj}_Y(\mathcal{A}/\sqrt{\mathcal{F}^-})$
(see Lemmas 5.13 and 5.16). Since these open subschemes have complement of codimension $\geq 2$, the Weil divisors $D^-$ and $D^+$ are strict transforms of each other.

For the last statement, simply apply Proposition 5.20.

Remark 5.25. Without the $(S)_1$ assumption on $\mathcal{A}$ in Proposition 5.24 (2), one can also show that the $d$-uple component $(Y, \mathcal{A}^{(d)})$ determines a log flip for any $d > 0$ such that $(Y, \mathcal{A})$ is $1/\Delta$-Cartier. This is, in effect, the proof of [33, Proposition 1.6]. Our Proposition 5.24 therefore gives a set of sufficient conditions for [34 Proposition 1.6] to hold.

5.2. Flips and flops. We first recall the standard notion of flips and flops. We also include a less standard condition of strong crepancy:

Definition 5.26. A flip is a log flip (5.6) in which the divisor $D^-$ may be taken to be $K_X^-$. A flop is a log flip (5.6) in which $K_X^-$ is $\mathbb{Q}$-Cartier and numerically $\pi^-$-trivial; and $K_X^+$ is $\mathbb{Q}$-Cartier and numerically $\pi^+$-trivial.

A log flip is said to be strongly crepant if the contracted variety $Y$ (and hence $X^-$ and $X^+$) is quasi-Gorenstein, i.e., if $K_Y$ is Cartier. This implies that it is a flop.

In this paper, we will only be concerned with flips, and strongly crepant flops, between Cohen-Macaulay normal varieties, because these classes of flips and flops behave well with respect to Grothendieck duality. We now recall some basic notions about the functor $\pi^*$, following [30, Tag 0DWE]. Recall that we work with Conventions [1.8 and 1.9] throughout.

Recall from [30 Tag 0F42] that, given a morphism $f : X \to Y$ of schemes, both separated and of finite type over a separated Noetherian base scheme $S$, then we may define the upper shriek functor $f^! : \mathcal{D}(\text{QCoh}(Y)) \to \mathcal{D}(\text{QCoh}(X))$ as follows: first factor $f$ as the composition $X \xrightarrow{j} \hat{X} \xrightarrow{\tilde{f}} Y$ for an open immersion $j$ and a proper morphism $\tilde{f}$, both over $S$, then define $f^!$ to be the composition $f^! = j^* \circ \tilde{f}^!$, where $\tilde{f}^!$ is right adjoint to $Rf_*$, and $j^*$ is the restriction. A different factoring $f = f' \circ f''$ gives rise to canonically isomorphic functors $j^* \circ f''^! \cong j^* \circ f^!$.

We also recall (see [30 Tag 0B6N]) that if $f : X \to Y$ is proper, then there is a canonical map

\[
\mathcal{F} \Rightarrow f^!(\mathcal{F}) \otimes_{\mathcal{O}_X} Lf^* \mathcal{G} \Rightarrow f^!(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})
\]

in $\mathcal{D}(\text{QCoh}(X))$ for each $\mathcal{F}, \mathcal{G} \in \mathcal{D}(\text{QCoh}(Y))$, which is adjoint to the map

\[
Rf_* f^!(\mathcal{F}) \otimes_{\mathcal{O}_X} Lf^* \mathcal{G} \Rightarrow Rf_* f^!(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) \cong \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}
\]

where $\epsilon$ is the adjunction counit. Moreover, (5.27) is an isomorphism if $\mathcal{G}$ is perfect.

A crucial feature of the upper shriek functor is that it sends dualizing complexes to dualizing complexes (see [30 Tags 0A7B, 0A87, 0AA3]). In particular, for a scheme $X$ separated and of finite type over a field $k$, if we denote by $\pi : X \to \text{Spec } k$ the projection map, then $\omega^*_X := \pi^*(k)$ is a dualizing complex, called the canonical dualizing complex of $X$. The following Lemma gives the correct shift in order to normalize the restriction $(\omega^*_X)_x$ to each stalk:

Lemma 5.28. For any point $x \in X$, if we let $\Delta_x \in \mathbb{Z}$ be the unique integer such that $(\omega^*_X)_x[-\Delta_x]$ is a normalized dualizing complex over $\mathcal{O}_{X,x}$ in the sense of [30 Tag 0A7M], then we have $\Delta_x = \dim(\{x\})$.

Proof. If $x$ is a closed point then this follows from the adjunctions

\[
R\text{Hom}_{\mathcal{O}_{X,x}}(k(x), (\omega^*_X)_x) \cong (R\text{Hom}_{\mathcal{O}_X}(i_x^*, k(x), \omega^*_X))_x \\
\cong ((i_x^*, R\text{Hom}_k(k(x), i^!_x \omega^*_X))_x \cong R\text{Hom}_k(k(x), k)
\]

where $i_x$ is the closed immersion $i_x : \text{Spec } k(x) \to X$, and the last quasi-isomorphism follows from the functoriality $\pi^! \cong i^! \pi^!$ for the composition $\pi_x : \text{Spec } k(x) \xrightarrow{i_x} X \xrightarrow{\pi} \text{Spec } k$. The general case then follows from the fact that $x \mapsto \Delta_x$ is a dimension function on $X$ (see [30 Tag 0A7Z]).

\footnote{One often also imposes a crepancy condition, meaning that the $\mathbb{Q}$-Cartier divisors $K_{X^-}$ and $K_{X^+}$ pull back to the same $\mathbb{Q}$-divisor on a common resolution.}

\footnote{See Convention [4.19]}
As a result, for any coherent sheaf \( \mathcal{F} \in \text{Coh}(X) \) of \( \dim(\text{supp}(\mathcal{F})) = d_x(\mathcal{F}) \) and \( \text{depth}(\mathcal{F}_x) = \delta_x(\mathcal{F}) \), we have by [30, Tag 0A7U]

1. If \( H^i(\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet))_x \neq 0 \) then \( -\Delta_x - d_x(\mathcal{F}) \leq i \leq -\Delta_x - \delta_x(\mathcal{F}) \);
2. \( H^{-\Delta_x-\delta_x(\mathcal{F})}(\mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet))_x \neq 0 \).

In particular, if \( X \) has pure dimension \( n \) (and is catenary since it is of finite type over a field), then these numbers may be rewritten as

\[
-\Delta_x - d_x(\mathcal{F}) = -n + (d_x(O_x) - d_x(\mathcal{F}))
\]

so that in particular \( \omega_X^\bullet \) itself has cohomology concentrated in degrees \( \geq -n \). Moreover, \( \mathcal{F} \) is maximal Cohen-Macaulay if and only if \( \mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet) \) has cohomology concentrated in degree \(-n\).

If \( X \) is moreover proper over Spec \( k \), then \( \pi^! \) is right adjoint to \( R\pi_\ast \), so that for each \( \mathcal{F} \in D^{\text{b}}_{\text{coh}}(X) \), there is a canonical isomorphism

\[
(\mathbb{H}^n(X, \mathcal{F}))^* \cong \text{Ext}^{-i}(\mathcal{F}, \omega_X^\bullet) = \mathbb{H}^{-i}(X, \mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\bullet))
\]

Thus, in particular, if we let \( \omega_X : = \mathcal{H}^{-n}(\omega_X^\bullet) \), then for any coherent sheaf \( \mathcal{F} \in \text{Coh}(X) \), the lowest nonzero degree for which \( \mathbb{H}^n(X, \mathcal{R}\text{Hom}(\mathcal{F}, \omega_X^\bullet)) \) could possibly be nonzero happens at \( \bullet = -n \), where it is given by \( H^n(X, \mathcal{H}(\mathcal{F}, \omega_X)) = \text{Hom}_X(\mathcal{F}, \omega_X) \). At this degree, (5.29) becomes

\[
H^n(X, \mathcal{F})^* \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)
\]

Thus, when \( X \) is proper over \( k \), \( \omega_X := \mathcal{H}^{-n}(\omega_X^\bullet) \) represents the functor \( \text{Coh}(X)^{\text{op}} \to \text{Mod}(k) \), \( \mathcal{F} \mapsto H^n(X, \mathcal{F})^* \), and is therefore unique up to canonical isomorphism, known as the dualizing sheaf on \( X \).

If \( X \) is not proper over \( k \), the coheret sheaf \( \omega_X := \mathcal{H}^{-n}(\omega_X^\bullet) \) is still well-defined, although it may not play the role of a dualizing sheaf. On any normal variety \( X \), it is in fact isomorphic to \( \mathcal{O}_X(K_X) \) (see Theorem 5.33 below). This follows by restricting to \( X \) the corresponding isomorphism [17, Proposition 5.75] on a compactification \( X \subset \overline{X} \). However, since we will need some details in the construction of such an isomorphism, we provide some details to the second proof in [17], written in a form more cogent to derived categories. We start by recalling some preparatory results:

**Proposition 5.30.** If \( X \) is a scheme of pure dimension \( n \), separated and of finite type over \( k \), then the stalks of the coherent sheaf \( \mathcal{H}^{-n}(\omega_X) \) satisfy \( \text{depth}_{\mathcal{O}_{X,x}}(\mathcal{H}^{-n}(\omega_X)_x) \geq \min\{2, \text{dim}(\mathcal{O}_{X,x})\} \).

**Proof.** We have seen above that the stalk \( \mathcal{H}^{-n}(\omega_X)_x \) is again the lowest degree cohomology of the normalized dualizing complex \( (\omega_X^\bullet)_x[-\Delta_x] \) over \( \mathcal{O}_{X,x} \). Thus the statement follows from (the proof of) [30, Tag 0AWE]. \( \square \)

**Corollary 5.31.** If \( X \) is a scheme of pure dimension \( n \), separated and of finite type over \( k \), and if \( \mathcal{O}_{X,x} \) is Gorenstein for all \( x \in X \) with \( \text{depth}(\mathcal{O}_{X,x}) \leq 1 \), then \( \mathcal{H}^{-n}(\omega_X) \) is reflexive. For example, this second condition is satisfied if \( X \) is normal.

**Proof.** This follows from Proposition 5.30 and the usual characterization of reflexive coherent sheaves by depth condition (see, e.g., [3, Proposition 1.4.1(b)]). \( \square \)

**Proposition 5.32 (13, Proposition 1.6).** If \( \mathcal{F} \) is a reflexive coherent sheaf on a normal integral scheme \( X \), then for any closed subset \( Z \subset X \) of codimension \( \geq 2 \), the canonical map \( \mathcal{F} \to j_*j^*\mathcal{F} \) is an isomorphism, where \( j : X \setminus Z \to X \) is the inclusion map.

Now, suppose we are given any proper morphism \( f : W \to X \) between schemes of pure dimension \( n \), separated and of finite type over \( k \), then the adjunction \( (\mathcal{R}f_* \rightleftharpoons f^! \mathcal{R}) \), together with the canonical identification \( \omega_W = f^!\omega_X^\bullet \), determines a counit morphism

\[
\text{Tr}_{W/X} : Rf_!(\omega_W^\bullet) \to \omega_X^\bullet
\]

so that if we take the lowest degree cohomology sheaves, we have a map of coherent sheaves

\[
0_{\text{Tr}_{W/X}} : f^!\mathcal{H}^{-n}(\omega_W^\bullet) \cong \mathcal{H}^{-n}(\mathcal{R}f_!(\omega_W^\bullet)) \xrightarrow{\mathcal{H}^{-n}(\text{Tr}_{W/X})} \mathcal{H}^{-n}(\omega_X^\bullet)
\]

This can be used to prove the following...
**Theorem 5.33.** Assume \( \text{char}(k) = 0 \), then on any normal variety \( X \) over \( k \) of dimension \( n \), there is an isomorphism \( \Phi_X : \mathcal{O}(K_X) \xrightarrow{\cong} \mathcal{H}^{-n}(\omega_X^*) \).

**Proof.** Choose a resolution of singularities \( f : W \to X \). i.e., \( W \) is smooth and \( f \) is a birational contraction. Since \( W \) is smooth, there is an isomorphism \( \Phi_W : \mathcal{O}(K_W) \xrightarrow{\cong} \mathcal{H}^{-n}(\omega_W^*) \). Then consider the diagram

\[
\begin{array}{ccc}
\mathcal{O}(K_W) & \xrightarrow{f_*} & \mathcal{H}^{-n}(\omega_W^*) \\
\downarrow & & \downarrow \Phi_{W/X} \\
\mathcal{O}(K_X) & \xrightarrow{\Phi_X} & \mathcal{H}^{-n}(\omega_X^*)
\end{array}
\]

(5.34)

Let \( Z = f(\text{Ex}(f)) \), which has codimension \( \geq 2 \) since \( f \) is a birational contraction. Since \( f \) is an isomorphism outside \( Z \), both of the vertical maps are isomorphisms over \( X \setminus Z \), so that the dashed arrow in (5.34) exists uniquely on \( X \setminus Z \). Since both \( \mathcal{O}(K_X) \) and \( \mathcal{H}^{-n}(\omega_X^*) \) are reflexive (see Corollary 5.31), we see by Proposition 5.32 that the dashed arrow \( \Phi_X \) in (5.34) exists uniquely over \( X \). \( \square \)

Assume from now on \( \text{char}(k) = 0 \). Then Theorem 5.33 allows us to formulate a crucial homological property of some classes of flips and flops. Suppose we are given a diagram of birational contractions between normal varieties, as in (5.4), which satisfies the condition

\[
(5.35)
\]

The canonical inclusion maps \( \pi^- \mathcal{O}(K_X-) \hookrightarrow \mathcal{O}(K_Y) \) and \( \pi^+ \mathcal{O}(K_X+) \hookrightarrow \mathcal{O}(K_Y) \) are isomorphisms.

This is automatic if both \( \pi^- \) and \( \pi^+ \) are small, in view of (5.5). It is also true if \( X^- \) and \( X^+ \) are quasi-Calabi-Yau, i.e., if \( K_{X^-} = 0 \) and \( K_{X^+} = 0 \).

Find a common resolution of singularities

\[
\begin{array}{ccc}
X^- & \xrightarrow{f^-} & W \\
\downarrow & & \downarrow \pi^- \\
Y & \xrightarrow{f^+} & X^+
\end{array}
\]

and use \( f^+ \) and \( f^- \) to define the maps \( \Phi_X, \Phi_Y \) and \( \Phi_Y \) as in the proof of Theorem 5.33. Since these are defined as the unique map making the diagram (5.34) commute, we see easily that the following diagram commutes:

\[
\begin{array}{ccc}
\pi_- \mathcal{O}(K_X-) & \xrightarrow{\Phi_X^-} & \mathcal{O}(K_Y) \\
\mathcal{H}^{-n}(\omega_X^-) & \xrightarrow{\Phi_Y^-} & \mathcal{H}^{-n}(\omega_Y^-)
\end{array}
\]

Now suppose that both \( X^- \) and \( X^+ \) are Cohen-Macaulay, then there is an isomorphism

\[
(5.37)
\]

\[
\mathcal{O}(K_Y) \xrightarrow{\Phi_Y^+} \mathcal{H}^{-n}(\omega_Y^*) \xrightarrow{\text{Tr}_{X^+}/Y} \mathcal{H}^{-n}(\omega_Y^*)
\]

and similarly for \( X^- \).

We may use the isomorphism (5.37) to express Grothendieck duality for the morphism \( \pi^+ \) in terms of the sheaf \( \mathcal{O}(K_X+) \) and the dualizing complex \( \omega_Y^*[n] \). Namely, for any \( \mathcal{F} \in \mathcal{D}_{\text{coh}}^b(\text{QCoh}(X^+)) \), there is a local adjunction isomorphism

\[
(5.38)
\]

\[
\mathcal{R}\pi_+^+ \mathcal{R}\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}(K_X+)) \xrightarrow{\cong} \mathcal{R}\mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{R}\pi_+^+(\mathcal{F}, \omega_Y^*[-n]))
\]

which sends an element \( \varphi \in \text{R} \Gamma(U; \mathcal{R}\pi_+^+ \mathcal{R}\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}(K_X+))) \) to \( \text{R}\mathcal{H}\text{om}_{(\pi^+)^{-1}U}(\mathcal{F}, \mathcal{O}(K_X+)) \) to the composition

\[
(5.39)
\]

\[
\mathcal{R}\pi_+^+ \mathcal{F} \xrightarrow{\mathcal{R}\pi_+^+(\varphi)} \mathcal{R}\pi_+^+(\mathcal{O}(K_X+)) \xrightarrow{\cong} \mathcal{R}\mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{R}\pi_+^+(\mathcal{O}(K_X+)), \omega_Y^*[-n]) \xrightarrow{\text{Tr}_{X^+}/Y} \omega_Y^*[-n]
\]

The commutativity of either square in (5.37) can be rewritten in terms of the local adjunction map (5.39). Namely, if we take \( \mathcal{F} = \mathcal{O}(K_X+) \) in (5.39), then the commutativity of the right square of (5.37)
guarantees that the canonical degree 0 global section of $R\pi_+^* R\mathcal{Hom}_X (\mathcal{O}(K_X), \mathcal{O}(K_X))$ is sent to the map $\Phi_Y$. Equivalently, we formulate this as the following Lemma, which will be useful later.

**Lemma 5.41.** The composition (where the last map is induced by $\mathcal{O}(K_Y) = \pi_+^* \mathcal{O}(K_X) \to R\pi_+^* \mathcal{O}(K_X)$)

\[
\mathcal{O}_Y \to R\pi_+^* \mathcal{O}_X \xrightarrow{R\pi_+^*(\Delta)} R\pi_+^* R\mathcal{Hom}_X (\mathcal{O}(K_X), \mathcal{O}(K_X)) \xrightarrow{\Phi_Y} \mathcal{O}(K_Y) \xrightarrow{\mathcal{H}^{-n}(\omega_X^*[-n])} R\mathcal{Hom}_Y (\mathcal{O}(K_Y), \omega_Y^*[-n])
\]

is the map $\mathcal{O}_Y \to R\mathcal{Hom}_Y (\mathcal{O}(K_Y), \omega_Y^*[-n])$ in $\mathcal{D}^b_{\text{coh}}(\text{QCoh}(Y))$ corresponding to the global map in $\mathcal{D}^b_{\text{coh}}(\text{QCoh}(Y))$ given as the composition

\[
\mathcal{O}(K_Y) \xrightarrow{\mathcal{H}^{-n}(\omega_Y^*[-n])} \omega_Y^*[-n]
\]

In other words, the composition (5.42) is equal to the composition

\[
\mathcal{O}_Y \xrightarrow{\Phi_Y} \mathcal{O}(K_Y) \xrightarrow{\mathcal{H}^{-n}(\omega_Y^*[-n])} \omega_Y^*[-n]
\]

Proof. By the description (5.40), this statement concerning (5.43) is equivalent to the commutativity of the outermost square of

\[
\begin{array}{ccc}
R\pi_+^* \mathcal{O}(K_X) & \xrightarrow{R\pi_+^*(\Phi_X^+)} & R\pi_+^* \mathcal{H}^{-n}(\omega_X^*[-n]) \\
\pi_+^* \mathcal{O}(K_X) & \xrightarrow{\pi_+^*(\Phi_X^+)} & \pi_+^* \mathcal{H}^{-n}(\omega_X^*[-n]) \\
\mathcal{O}(K_Y) & \xrightarrow{\Phi_Y} & \mathcal{H}^{-n}(\omega_Y^*[-n]) \\
\end{array}
\]

We verify that each of the square is commutative, where the crucial commutativity of the lower left square is given by (5.37). It is also clear that the statements concerning (5.43) and (5.44) are equivalent. \(\square\)

Notice that Lemma 5.41 asserts that, although the long composition there depends on $\pi^+ : X^+ \to Y$, the resulting map can be described in terms of $\Phi_Y$, without any reference to $X^+$. Thus, the corresponding compositions for $X^-$ and $X^+$ coincide. This compatibility between the local adjunction maps for $X^-$ and $X^+$ will be part of the condition for homological flips/flops to be introduced in the next subsection.

**Remark 5.45.** We discuss some of the assumptions we have imposed. Starting from the discussion (5.38), we have assumed that $X^-$ and $X^+$ are Cohen-Macaulay. Without this assumption, the maps (5.38), and hence (5.39), still exist, but may not be an isomorphism. Since all the maps are pointing to the “correct” directions, Lemma 5.41 also holds without the Cohen-Macaulay condition.

Another condition we have imposed is (5.35). Without this assumption, the map $\mathcal{O}(K_Y) \to R\pi_+^* \mathcal{O}(K_X)$ in the statement of Lemma 5.41 cannot be defined.

Now we work in the following setting:

The diagram (5.46) of birational contractions satisfies (5.35), both $X^-$ and $X^+$ are Cohen-Macaulay, and both $\pi^-$ and $\pi^+$ are strongly crepant in the sense that $K_Y$ is Cartier, and $(\pi^\pm)^* K_Y = K_{X^\pm}$.

This is satisfied for (1) strongly crepant flops between Cohen-Macaulay normal varieties; and (2) diagrams (5.46) of birational contractions where both $X^-$ and $X^+$ are Calabi-Yau (i.e., both are Cohen-Macaulay and have trivial canonical divisor $K_{X^\pm} \equiv 0$).

In this case, there are canonical identifications\(^7\)

\[
\mathcal{O}(K_{X^-}) = (\pi^-)^* \mathcal{O}(K_Y) \quad \text{and} \quad \mathcal{O}(K_{X^+}) = (\pi^+)^* \mathcal{O}(K_Y)
\]

\(^7\)We write equality signs because they are equal as subsheaves of the sheaves of rational functions $\mathcal{K}_{X^-}$ and $\mathcal{K}_{X^+}$. 
Since we assume that $K_Y$ is Cartier, the object $\omega^\bullet_Y := \omega^\bullet_Y[-n] \otimes O_Y \mathcal{O}(-K_Y)$ in $\mathcal{D}^b_{\text{coh}}(\text{QCoh}(Y))$ is a dualizing complex. Moreover, if we apply \([5.27]\) to $\mathcal{F} = \omega^\bullet_Y[-n]$ and $\mathcal{G} = \mathcal{O}(-K_Y)$, we see that there is an isomorphism

\[
\omega^\bullet_X[-n] \otimes O_X \mathcal{O}(-K_X) \xrightarrow{\cong} (\pi^+)^!(\omega^\bullet_Y)
\]

On the other hand, $\Phi_X^+$ gives a canonical isomorphism

\[
\mathcal{O}_X^+ = \mathcal{O}(K_X^+) \otimes O_X \mathcal{O}(-K_X^+) \xrightarrow{(\iota_{\Phi_X^+} \otimes \text{id}) \cong} \omega^\bullet_X[-n] \otimes O_X \mathcal{O}(-K_X^+)
\]

Composing with these two maps, we obtain a canonical isomorphism

\[
\Phi'_X : \mathcal{O}_X^+ \xrightarrow{\cong} \omega^\bullet_X[-n] \otimes O_X \mathcal{O}(-K_X^+) \xrightarrow{(\pi^+)^!} (\omega^\bullet_Y)
\]

We can again use this isomorphism to express Grothendieck duality for $\pi^+$ in terms of $\mathcal{O}_X^+$ and $\omega^\bullet_Y$. Namely, for any $\mathcal{F} \in \mathcal{D}(\text{QCoh}(X^+))$, there is a local adjunction isomorphism

\[
R\pi^+_* R\mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{F}, \mathcal{O}_X^+) \xrightarrow{\cong} R\mathcal{H}om_{O_Y}(R\pi^+_* \mathcal{F}, \omega^\bullet_Y)
\]

Unravelling the definitions, we see the Lemma\([5.11]\) can be rewritten in this case in the following form:

**Lemma 5.51.** The composition

\[
\mathcal{O}_Y \rightarrow R\pi^+_* \mathcal{O}_X^+ = R\pi^+_* R\mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{O}_X^+, \mathcal{O}_X^+)
\]

\[
\xrightarrow{\text{[5.50]}} R\mathcal{H}om_{O_Y}(R\pi^+_* \mathcal{O}_X^+, \omega^\bullet_Y) \rightarrow R\mathcal{H}om_{O_Y}(\mathcal{O}_Y, \omega^\bullet_Y) = \omega^\bullet_Y
\]

is equal to the composition

\[
\mathcal{O}_Y = \mathcal{O}(K_Y) \otimes O_Y \mathcal{O}(-K_Y) \xrightarrow{(\iota_{\Phi_Y^+} \otimes \text{id}) \cong} \omega^\bullet_Y[-n] \otimes O_Y \mathcal{O}(-K_Y) = \omega^\bullet_Y
\]

**Proof.** By definition, the map \([5.47]\) is adjoint to the map

\[
R\pi^+_*(\omega^\bullet_X[-n] \otimes O_X \mathcal{O}(-K_X^+)) \xrightarrow{\text{Tr}_{X^+/Y}} \omega^\bullet_Y[-n] \otimes O_Y \mathcal{O}(-K_Y) = \omega^\bullet_Y
\]

In particular, for $\mathcal{F} = \mathcal{O}_X^+$, the map \([5.50]\) sends the global section $1 \in \text{Hom}_{\mathcal{D}(\text{QCoh}(X^+))}(\mathcal{O}_X^+, \mathcal{O}_X^+)$ to the global section of $R\mathcal{H}om_{O_Y}(R\pi^+_* \mathcal{O}_X^+, \omega^\bullet_Y)$ given as the composition

\[
R\pi^+_* \mathcal{O}_X^+ \xrightarrow{\text{[5.48]}} R\pi^+_*(\omega^\bullet_X[-n] \otimes O_X \mathcal{O}(-K_X^+)) \xrightarrow{\text{[5.55]}} \omega^\bullet_Y
\]

Since a map from $\mathcal{O}_Y$ to any other object in $\mathcal{D}(\text{QCoh}(Y))$ is uniquely determined by where it sends the global section $1$ to, we see that \([5.52]\) can be rewritten as the left column of the following commutative
involution (4.23) defined using \( \omega \) of it. In fact, we don’t even require \( Y \) homological flips/flops. go into the proof of Theorem 5.72, which asserts that certain classes of actual flips and flops give rise to will mostly work over actual flips and flops, in order to motivate the definition. This discussion will also notion of homological flips/flops in Definition 5.69. In the discussion that leads up to this definition, we in (1) is the shift \( \omega \) from a pair \((X^-,X^+)\). Our discussion in the last subsection on flips and flops may be summarized as follows:

1. If (5.6) is a flip where \( X^- \) and \( X^+ \) are Cohen-Macaulay, then there are canonical isomorphisms \( \mathcal{O}(K_{X^+}) \to (\pi^+)^{\dagger}(\omega_Y^*[-n]) \) where \( \mathcal{O}(K_{X^+}) \) is \( \pi^+ \)-ample and \( \mathcal{O}(K_{X^-}) \) is \( \pi^- \)-anti-ample. Moreover, these two isomorphisms are compatible in the sense that Lemma 5.41 holds.
2. If (5.6) is a strongly crepant flop where \( X^- \) and \( X^+ \) are Cohen-Macaulay, then there are canonical isomorphisms \( \mathcal{O}_{X^+} \to (\pi^+)\langle\omega_Y^*\rangle \). Moreover, these two isomorphisms are compatible in the sense that Lemma 5.51 holds.

We now proceed to abstract the situations (1) and (2) when the corresponding diagram (5.6) comes from a pair \((Y,\mathcal{A})\) as in (4.41). In this abstraction it does not matter that the dualizing complex appearing in (1) is the shift \( \omega_Y^*[-n] \) of the canonical dualizing complex; or the one appearing in (2) is a twist \( \omega_Y^* \) of it. In fact, we don’t even require \( Y \) to be defined over a field. Such an abstraction will lead to the notion of homological flips/flops in Definition 5.69. In the discussion that leads up to this definition, we will mostly work over actual flips and flops, in order to motivate the definition. This discussion will also go into the proof of Theorem 5.72 which asserts that certain classes of actual flips and flops give rise to homological flips/flops.

Let \((Y,\mathcal{A})\) be a pair as in (4.41). Fix a dualizing complex \( \omega_Y^{\dagger}\in\mathcal{D}_{\text{coh}}(\text{QCoh}(Y)) \), and consider the involution (4.23) defined using \( \omega_Y^{\dagger} \). By Theorem 4.36 and Proposition 4.20 the object \((\pi^+)\langle\omega_Y^{\dagger}\rangle \) has a

\[
\begin{align*}
\mathcal{O}_Y & \rightarrow \mathcal{O}(K_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}(-K_Y) \xrightarrow{(1:\Phi_Y)\otimes\text{id}} \omega_Y^*[-n] \otimes_{\mathcal{O}_Y} \mathcal{O}(-K_Y) \\
\Delta & \rightarrow \mathcal{R} \mathcal{H}_{\text{om}}_{\mathcal{O}_Y}(\mathcal{O}(K_Y),\mathcal{O}(K_Y)) \xrightarrow{(1:\Phi_Y)} \mathcal{R} \mathcal{H}_{\text{om}}_{\mathcal{O}_Y}(\mathcal{O}(K_Y),\omega_Y^*[-n])
\end{align*}
\]

Remark 5.56. (1) The same remarks as in Remark 5.45 about the Cohen-Macaulay condition on \( X^- \) and \( X^+ \) hold in the present context of (5.40).

(2) If \( Y \) is Cohen-Macaulay (hence Gorenstein), then the dualizing complex \( \omega_Y^* \) is isomorphic to \( \mathcal{O}_Y \) via the map \( \Phi_Y \).

5.3. Homological flips and homological flops. Our discussion in the last subsection on flips and flops may be summarized as follows:

1. If (5.6) is a flip where \( X^- \) and \( X^+ \) are Cohen-Macaulay, then there are canonical isomorphisms \( \mathcal{O}(K_{X^+}) \rightarrow (\pi^+)^{\dagger}(\omega_Y^*[-n]) \) where \( \mathcal{O}(K_{X^+}) \) is \( \pi^+ \)-ample and \( \mathcal{O}(K_{X^-}) \) is \( \pi^- \)-anti-ample. Moreover, these two isomorphisms are compatible in the sense that Lemma 5.41 holds.
2. If (5.6) is a strongly crepant flop where \( X^- \) and \( X^+ \) are Cohen-Macaulay, then there are canonical isomorphisms \( \mathcal{O}_{X^+} \rightarrow (\pi^+)\langle\omega_Y^*\rangle \). Moreover, these two isomorphisms are compatible in the sense that Lemma 5.51 holds.
description in terms of the functor $\mathbb{D}_Y$. Namely, there is a canonical isomorphism in $\mathcal{D}(\text{Gr}(\mathcal{A}))$:

\[(5.57)\quad \mathcal{R}^+((\pi^+)^!)(\omega_Y^\bullet)) \cong \mathbb{D}_Y(\hat{C}_{x^+}(\mathcal{A}))\]

If $\mathcal{O}$ is a flip between Cohen-Macaulay normal varieties, then we may take a divisor $D^-$ on $X^-$ satisfying the defining property of a log flip (see the discussion at (5.6)) such that $K_{X^-} = aD^-$ for some $a > 0$. For example, we can always take $D^- = K_{X^-}$ and $a = 1$. The corresponding pair $(Y, \mathcal{A})$ constructed in Proposition 5.7 then gives an example of the following situation:

\[(5.58)\quad \text{The pair } (Y, \mathcal{A}) \text{ determines a log flip in the sense of Definition 5.9, both } X^- \text{ and } X^+ \text{ are Cohen-Macaulay, and there exists } a > 0 \text{ such that } K_{X^\pm} = aD^\pm.\]

In the situation (5.58), there are isomorphisms

\[(5.59)\quad \mathcal{A}(a)_{X^\pm} \xrightarrow{\cong} \mathcal{O}_{X^\pm}(K_{X^\pm}) \xrightarrow{\cong} (\pi^\pm)^!(\omega_Y^\bullet[-n]) \quad \text{in } \mathcal{D}_{coh}^b(\text{QCoh}(X^\pm))\]

Similarly, if $\mathcal{O}$ is a strongly crepant flop between Cohen-Macaulay normal varieties, choose any Weil divisor $D^-$ on $X^-$ satisfying the defining condition for a log flip (see the discussion at (5.6)), then the corresponding pair $(Y, \mathcal{A})$ constructed in Proposition 5.7 gives an example of the following situation:

\[(5.60)\quad \text{The pair } (Y, \mathcal{A}) \text{ determines a log flip in the sense of Definition 5.9, both } X^- \text{ and } X^+ \text{ are Cohen-Macaulay, and } K_Y \text{ is Cartier.}\]

In the situation (5.60), there are isomorphisms

\[(5.61)\quad \mathcal{A}^\times_{X^\pm} = \mathcal{O}_{X^\pm} \xrightarrow{\cong} \pi^!(\omega_Y^\bullet) \quad \text{in } \mathcal{D}_{coh}^b(\text{QCoh}(X^\pm))\]

Both of the equations (5.59) and (5.61) can be written in the same way

\[(5.62)\quad \mathcal{A}(a)_{X^\pm} \xrightarrow{\cong} (\pi^\pm)^!(\omega_Y^\bullet)\]

where we take $\omega_Y^\bullet := \omega_Y^\bullet[-n]$ and $a > 0$ in the case (5.58); and $\omega_Y^\bullet := \omega_Y^\bullet$ and $a = 0$ in the case (5.60). Notice that $\omega_Y^\bullet$ is a dualizing complex in both cases.

For notational simplicity, we focus on the $X^+$-side in the following discussion. Applying (4.34) to $\mathcal{M} = \mathcal{A}(a)$, we see that there is a canonical map

\[(5.63)\quad \hat{C}_{x^+}(\mathcal{A}(a)) \rightarrow \mathcal{R}^+((\mathcal{A}(a)_{X^+}) \xrightarrow{\cong} \mathcal{R}^+((\pi^+)^!(\omega_Y^\bullet)))\]

where the weight $i$ component of the first map is given by

\[(5.64)\quad \mathcal{R}^i((\mathcal{A}(a + i)) \rightarrow \mathcal{R}^i_\mathcal{A}(\mathcal{A}(a + i))\]

by the description (4.35). Thus, if $\mathcal{A}$ is $\frac{1}{2}$-Cartier, then the map (5.63) is a quasi-isomorphism in weight components $i \in d\mathbb{Z}$. In fact, it is often a quasi-isomorphism for all weight components. Namely, if the pair $(Y, \mathcal{A})$ determines a Cohen-Macaulay log flip as in Definition 5.9 then the map (5.64) can be rewritten as the derived pushforward $\mathcal{R}^i_\mathcal{A}(\mathcal{A}(a + i))$ of the map (5.66) below, for $X := X^+$, $D := D^+$ and $D' := aD^+$, and hence is a quasi-isomorphism.

**Lemma 5.65.** Suppose that $X$ is a Cohen-Macaulay normal variety over $k$, and $D \in \text{WDiv}(X)$ is a Weil divisor on $X$ such that the sheaf $\mathcal{O}_X(-iD)$ is maximal Cohen-Macaulay. If $D' \in \text{WDiv}(X)$ is such that $\mathcal{O}_X(D')$ is a dualizing complex, then the canonical map

\[(5.66)\quad \mathcal{O}_X(D' + iD) \rightarrow \mathcal{R}_\mathcal{A}(\mathcal{O}_X(-iD), \mathcal{O}_X(D'))\]

is a quasi-isomorphism.

**Proof.** The 0-th cohomology sheaves of (5.66) is the map $\mathcal{O}_X(D' + iD) \rightarrow \mathcal{R}_\mathcal{A}(\mathcal{O}_X(-iD), \mathcal{O}_X(D'))$, which is always an isomorphism. Thus, it suffices to show that the complex $\mathcal{R}_\mathcal{A}(\mathcal{O}_X(-iD), \mathcal{O}_X(D'))$ has no higher cohomology sheaves. Since $\mathcal{O}_X(D')$ is assumed to be a dualizing complex, this is equivalent to the sheaf $\mathcal{O}_X(-iD)$ being maximal Cohen-Macaulay (see, e.g., [39, Tag 0A7U]). \qed
Combining (5.57) and (5.63), we obtain a map
\[(5.67)\]
\[\Phi^+ : \tilde{C}_{\mathcal{F}^+}(\mathcal{A})(a) \to \mathbb{D}Y(\tilde{C}_{\mathcal{F}^+}(\mathcal{A}))\]
in \(D(\text{Gr}(\mathcal{A}))\), which is an isomorphism in weight \(i\) if \(\mathcal{A}(\mathcal{I}_{X})\) is maximally Cohen-Macaulay.

Similarly, discussion holds for \(X^-\) and \(\mathcal{F}^-\) in place of \(X^+\) and \(\mathcal{F}^+\), and we likewise have a map
\[(5.68)\]
\[\Phi^- : \tilde{C}_{\mathcal{F}^-}(\mathcal{A})(a) \to \mathbb{D}Y(\tilde{C}_{\mathcal{F}^-}(\mathcal{A}))\]
in \(D(\text{Gr}(\mathcal{A}))\), which is an isomorphism in weight \(i\) if \(\mathcal{A}(\mathcal{I}_{X})\) is maximally Cohen-Macaulay. Moreover, as we will see in Theorem 5.72 below, the maps (5.67) and (5.68) thus obtained satisfy a certain compatibility condition, as formalized in the following

**Definition 5.69.** A weak homological flip (resp. weak homological flop) consists of a sextuple \((Y, \omega_Y^\bullet, \mathcal{A}, a, \Phi^-, \Phi^+)\) where

1. \(Y\) is a Noetherian scheme with a dualizing complex \(\omega_Y^\bullet \in \mathbb{D}^{bc}(\text{QCoh}(Y))\).
2. \(\mathcal{A}\) is a quasi-coherent sheaf of Noetherian \(\mathbb{Z}\)-graded rings on \(\mathcal{A}\).
3. \(a > 0\) (resp. \(a = 0\)) is an integer
4. \(\Phi^-\) and \(\Phi^+\) are maps in \(D(\text{Gr}(\mathcal{A}))\)
\[\Phi^+ : \tilde{C}_{\mathcal{F}^+}(\mathcal{A})(a) \to \mathbb{D}Y(\tilde{C}_{\mathcal{F}^+}(\mathcal{A}))\]
\[\Phi^- : \tilde{C}_{\mathcal{F}^-}(\mathcal{A})(a) \to \mathbb{D}Y(\tilde{C}_{\mathcal{F}^-}(\mathcal{A}))\]

that are \(d\)-uple quasi-isomorphisms if \(\mathcal{A}\) is \(\mathcal{I}\)-Cartier. Here \(\mathbb{D}Y\) is the functor (4.23) defined using \(\omega_Y^\bullet\).

The maps \(\Phi^-\) and \(\Phi^+\) are required to be compatible in the sense that the diagram
\[(5.70)\]
\[
\begin{array}{ccc}
\mathcal{A}(a) & \xrightarrow{\eta^+} & \tilde{C}_{\mathcal{F}^+}(\mathcal{A})(a) \\
\downarrow{\eta^-} & & \downarrow{\Phi^+} \\
\tilde{C}_{\mathcal{F}^-}(\mathcal{A})(a) & \xrightarrow{\Phi^-} & \mathbb{D}Y(\tilde{C}_{\mathcal{F}^-}(\mathcal{A})) \\
\end{array}
\]
commutes in \(D(\text{Gr}(\mathcal{A}))\), where \(\eta^+ : \mathcal{A} \to \tilde{C}_{\mathcal{F}^+}(\mathcal{A})\) are the adjunction units as in (4.5).

A weak homological flip/flop is said to be a homological flip/flop if the maps \(\Phi^-\) and \(\Phi^+\) are quasi-isomorphisms in all weight degrees.

We also introduce the following extra condition:

**Definition 5.71.** A (weak) homological flip/flop \((Y, \omega_Y^\bullet, \mathcal{A}, a, \Phi^-, \Phi^+)\) is said to be positively pseudo-rational if we have \(\mathcal{R}^! \tilde{C}_{\mathcal{F}^-}(\mathcal{A})_0 \simeq 0\) and \(\mathcal{R}^! \tilde{C}_{\mathcal{F}^+}(\mathcal{A})_a \simeq 0\); it is said to be negatively pseudo-rational if we have \(\mathcal{R}^! \tilde{C}_{\mathcal{F}^-}(\mathcal{A})_0 \simeq 0\) and \(\mathcal{R}^! \tilde{C}_{\mathcal{F}^-}(\mathcal{A})_a \simeq 0\); it is said to be pseudo-rational if it is both positively and negatively pseudo-rational.

Our discussion above leads to the following examples of (weak) homological flip/flop:

**Theorem 5.72.**
(1) In the situation (5.58), take \(\omega_Y^\bullet := \omega_Y^\bullet[-n]\), and let \(\Phi^-\) and \(\Phi^+\) be the maps (5.58) and (5.67), then the sextuple \((Y, \omega_Y^\bullet, \mathcal{A}, a, \Phi^-, \Phi^+)\) is a weak homological flip. If the pair \((Y, \mathcal{A})\) moreover determines a Cohen-Macaulay log flip, i.e., if each \(\mathcal{O}_{X}(i\mathcal{D})\) is maximal Cohen-Macaulay, then \((Y, \omega_Y^\bullet, \mathcal{A}, a, \Phi^-, \Phi^+)\) is a homological flip.

(2) In the situation (5.60), take \(\omega_Y^\bullet := \omega_Y^\bullet\) (see the discussion following (5.40)), and let \(\Phi^-\) and \(\Phi^+\) be the maps (5.68) and (5.67) for \(a = 0\), then the sextuple \((Y, \omega_Y^\bullet, \mathcal{A}, 0, \Phi^-, \Phi^+)\) is a weak homological flop. If the pair \((Y, \mathcal{A})\) moreover determines a Cohen-Macaulay log flop, i.e., if each \(\mathcal{O}_{X}(i\mathcal{D})\) is maximal Cohen-Macaulay, then \((Y, \omega_Y^\bullet, \mathcal{A}, 0, \Phi^-, \Phi^+)\) is a homological flop.

In either case, assume furthermore that \(\mathcal{A}_a\) is reflexive (this is automatic for \(a = 0\)). Under this assumption, if both \(X^+\) and \(Y\) have rational singularities, then the corresponding (weak) homological flip/flop is positively pseudo-rational; if both \(X^-\) and \(Y\) have rational singularities, then the corresponding (weak) homological flip/flop is negatively pseudo-rational.
Proof. In the discussion preceding Definition 5.69, we have already proved all the statements in both cases (1) and (2), except the commutativity of (5.70) in $D(\text{Gr}(A))$. For any $\mathcal{M} \in D(\text{Gr}(A))$, we have $\text{Hom}_{D(\text{Gr}(A))}(\mathcal{A}(a), \mathcal{M}) \cong \mathbb{H}^0(Y, \mathcal{M}_a)$, so that it suffices to show that the global section $1 \in \mathbb{H}^0(Y, \mathcal{A}(a)_a)$ is sent to the same element in $\mathbb{H}^0(Y, \mathcal{D}_Y(A)_a) = \text{Hom}_{D(\text{Qcoh}(Y))}((\mathcal{A}_a, \omega_Y^*)$ under the two routes in (5.70). Equivalently, it suffices to check that at weight $-a$, the two routes in (5.70) gives the same map in $D(\text{Qcoh}(Y))$. In fact, the upper route of (5.70) at weight $-a$ is given by

$$
\begin{aligned}
\mathcal{O}_Y & \to R^\pi_{a} \mathcal{O}_X^+ \\
R^\pi_{a} \mathcal{O}_X^+ & \to R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{A}(a)_X, \mathcal{A}(a)_X^+)
\end{aligned}
$$

(5.73)

For the case (2) of flips, with $a = 0$, we have $\mathcal{A}(a)_X = \mathcal{O}_X^+$, so that the composition (5.73) is precisely (5.52), and hence is equal to (5.53). Since the map (5.53) is defined purely in terms of structures on $Y$, it is the same for $X^+$ and $X^-$. For the case (1) of flips, the map (5.62) is given by (5.59), which factors through the isomorphism (5.10). Thus, the part " of (5.62) appearing in (5.73) may be rewritten as

$$
R^\pi_{a} \mathcal{O}_X^+ \to R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{A}(a)_X^+, \mathcal{O}(K_X^+))
$$

(5.63)

Replace it as such, and consider the commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
R^\pi_{a} \mathcal{O}_X^+ & \to & R^\pi_{a} \mathcal{O}_X^+ \\
\Delta & & \downarrow (5.10) \\
R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{O}(K_X^+), \mathcal{O}(K_X^+)) & \to & R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{A}(a)_X^+, \mathcal{O}(K_X^+)) \\
\downarrow (5.38) & & \downarrow (5.38) \\
R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{O}(K_X^+), (\pi^+)^!(\omega_Y^* [-n])) & \to & R^\pi_{a} \mathcal{H}om_{\mathcal{O}_X^+}(\mathcal{A}(a)_X^+, (\pi^+)^!(\omega_Y^* [-n])) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}(K_X^+), \omega_Y^* [-n]) & \to & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}(a)_X^+, \omega_Y^* [-n]) \\
\downarrow (5.10) & & \downarrow (5.10) \\
\mathcal{H}om_{\mathcal{O}_Y}(\pi^+((\mathcal{O}(K_X^+), \omega_Y^* [-n])) & \to & \mathcal{H}om_{\mathcal{O}_Y}(\pi^+((\mathcal{A}(a)_X^+, \omega_Y^* [-n])) \\
\Delta & & \downarrow (5.3) \\
\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}(K_X^+), \omega_Y^* [-n]) & \to & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{A}_a, \omega_Y^* [-n])
\end{array}
\end{array}
$$

where the horizontal arrow in the bottom row is the contravariant of the map $\mathcal{A}_a \to \mathcal{O}(aD_Y) = \mathcal{O}(K_Y)$ induced by the diagram (5.11) for $i = a$. By the commutativity of (5.11), this map is therefore the same for $X^+$ and $X^-$. Moreover, the left column, when precomposed with $\mathcal{O}_Y \to R^\pi_{a} \mathcal{O}_X^+$ is precisely the map (5.42). By Lemma (5.11) it is therefore the same for $X^+$ and $X^-$. This shows that (5.70) commutes in this case.

For the second statement, notice that the long exact sequence associated to (4.5) for $\mathcal{M} = \mathcal{A}$ gives at weight $i$ the exact sequence

$$
0 \to \mathcal{H}^i(\mathcal{R} \mathcal{S}_{\mathcal{X} \times \mathcal{A}(a)_i}) \to \mathcal{A}_i \xrightarrow{(4.3)} \pi^+_a(\mathcal{A}(a)_X) \xrightarrow{\delta} \mathcal{H}^{i+1}(\mathcal{R} \mathcal{S}_{\mathcal{X} \times \mathcal{A}(a)_i}) \to 0
$$

and the isomorphisms $R^i \pi^+_a(\mathcal{A}(a)_X)$ for $j \geq 1$. We have seen in Remark 5.12 that the map $\mathcal{A}_a \xrightarrow{(4.3)} \pi^+_a(\mathcal{A}(a)_X)$ is an isomorphism, so that we have $\mathcal{H}^{\leq 1}(\mathcal{R} \mathcal{S}_{\mathcal{X} \times \mathcal{A}(a)_a}) = 0$. Since the
same is true for \( i = 0 \), we have

\[
\text{For } i \in \{0, a\}, \quad R\Gamma_{\mathcal{F}_{\pm}}(A)_i \simeq 0 \quad \text{if and only if} \quad R^j\pi^\pm_*(\mathcal{A}(i)_{X_{\pm}}) \text{ for all } j > 0
\]

Since \( \mathcal{A}_{X_{\pm}} \cong \mathcal{O}_{X_{\pm}} \) and \( \mathcal{A}_{\mathcal{F}_{\pm}}(A)_{X_{\pm}} \cong \mathcal{O}(K_{X_{\pm}}) \), it is a standard fact about rational singularities\(^8\) that the higher direct images of these sheaves vanish. \( \Box \)

The following is the main result of this section:

**Theorem 5.74.** Suppose that \( (Y, \omega_y^{\bullet}, \mathcal{A}, a, \Phi^-, \Phi^+) \) is a \( \frac{1}{2} \)-Cartier weak homological flip/flop. Then there is a map \( \Psi : R\Gamma_{\mathcal{F}^+}(A)(a)[1] \to \mathcal{D}_y(\mathcal{D}_{\mathcal{F}^-}(A)) \) in \( \mathcal{D}(\text{Gr}(A)) \) such that, if \( c^-, c^+ \in \mathbb{Z} \) are integers as in Lemma 5.72 for \( \mathcal{M} = \mathcal{A} \), then \( \Psi \) is a quasi-isomorphism in weight \( i \) whenever \( i/d \in \mathbb{Z} \) and the following holds:

\[
(5.75) \quad i > \max\{-c^-, c^+ - a\} \quad \text{or} \quad i < \min\{-c^+, c^- - a\}
\]

If \( (Y, \omega_y^{\bullet}, \mathcal{A}, a, \Phi^-, \Phi^+) \) is a homological flip/flop, then the map \( \Psi \) can be chosen so that it is a quasi-isomorphism in all weight \( i \) whenever \((5.75)\) holds. If it is moreover pseudo-rational, then \( \Psi \) is also a quasi-isomorphism in weights 0 and \(-a\).

**Proof.** We first prove the case for homological flip/flop before tackling the weak case. Consider the diagram

\[
\begin{array}{c}
\mathcal{A}(a) \xymatrix{ \eta^+ \ar[r] & \mathcal{C}_{\mathcal{F}^+}(A)(a) \ar[r]^{\delta^+} & R\Gamma_{\mathcal{F}^+}(A)(a)[1] \ar[d]^-{\psi} \\
\mathcal{C}_{\mathcal{F}^-}(A)(a) \ar[r]_{\mathbb{D}_{\mathcal{A}}(\eta^-) \circ \Phi^+} & \mathbb{D}_{\mathcal{A}}(A) \ar[r]_{\mathbb{D}_{\mathcal{A}}(\mathcal{C}_{\mathcal{F}^-}(A)(a))} & \mathbb{D}_{\mathcal{A}}(R\Gamma_{\mathcal{F}^-}(A)) \ar[d]^-{\mathbb{D}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(\mathcal{C}_{\mathcal{F}^-}(A)(a)))} \\
R\Gamma_{\mathcal{F}^-}(A)(a)[1] \ar[r]_{\Psi} & \mathbb{D}_{\mathcal{A}}(R\Gamma_{\mathcal{F}^+}(A)) \ar[r] & Z
\end{array}
\]

(5.76)

Since \( \Phi^- \) and \( \Phi^+ \) are isomorphisms, the first two rows and the first two columns are parts of distinguished triangles (take \( 22.27 \) and their dual \( 22.27 \)). Moreover, the top left square is commutative by the compatibility condition \((5.71)\) of a homological flip/flop. Thus, by the 3 \times 3-lemma of a triangulated category (see, e.g., [6, Proposition 1.1.11] or [23, Lemma 2.6]), the object \( Z \) and the dotted arrows \( \Psi, \Psi' \) in diagram \((5.76)\) exists, making each row and column part of a distinguished triangle.

The cohomology of \( R\Gamma_{\mathcal{F}^+}(A) \) has weight \( \leq c^+ \), while the cohomology of \( R\Gamma_{\mathcal{F}^-}(A) \) has weight \( \geq c^- \). Thus, viewing \( Z \) as the cone of \( \Psi \), we see that the cohomology of \( Z \) could be nonzero only in weight \( i \leq \max\{-c^-, c^+ - a\} \). Similarly, viewing \( Z \) as the cone of \( \Psi' \), we see that the cohomology of \( Z \) could be nonzero only in weight \( i \geq \min\{-c^+, c^- - a\} \). This proves the first statement for homological flips/flops.

If the homological flip/flop in question is pseudo-rational, then both \( R\Gamma_{\mathcal{F}^+}(A)(a)[1] \) and \( \mathbb{D}_{\mathcal{A}}(R\Gamma_{\mathcal{F}^-}(A)) \) have zero cohomology in weight 0, and hence \( \Psi \) must be quasi-isomorphism in weight 0. Similarly, since both \( R\Gamma_{\mathcal{F}^+}(A)_0 \) and \( R\Gamma_{\mathcal{F}^-}(A)_0 \) have zero cohomology, \( \Psi \) must be quasi-isomorphism in weight \(-a\). This proves the second statement for homological flips/flops.

For weak homological flips/flops, the idea is that an analogous diagram \((5.76)\) still exists over \( \mathcal{D}(\text{Gr}(A)) \) if we start with the \( d \)-uple components \((d')^{(d)}\) of the top left square, and hence still satisfies the desired quasi-isomorphism in the said weight degrees. However, in order to define the map \( \Psi \) at the level of \( \mathcal{D}(\text{Gr}(A)) \), we need to slightly change our arguments. Thus, we rearrange the top left square of \((5.76)\)
into the top left square of the following diagram

\[
\begin{array}{c}
\mathcal{A}(a) \xrightarrow{\Phi^+ \circ \eta^+} \mathbb{D}_Y(\check{C}_{\varphi^+}(\mathcal{A})) \xrightarrow{\delta^+_i} W_1 \\
\downarrow \Phi^- \circ \eta^- \downarrow \mathbb{D}_Y(\eta^+) \downarrow \mathbb{D}_Y(\epsilon^{-}) \downarrow \Psi_1 \\
\mathbb{D}_Y(\check{C}_{\varphi^+}(\mathcal{A})) \xrightarrow{\delta^-_i} \mathbb{D}_Y(\mathcal{A}) \xrightarrow{\Theta} \mathbb{D}_Y(\mathcal{R}_\varphi^{-}(\mathcal{A})) \xrightarrow{\Psi_2} \mathbb{D}_Y(\mathcal{R}_\varphi'(\mathcal{A})) \xrightarrow{\delta^+_i} W_1' \\
\end{array}
\] (5.77)

The second row and the second column are parts of distinguished triangles, as they are the dual \(\mathbb{D}_Y(-)\) of the distinguished triangles \((2.27)\). We take \(W_1\) and \(W_1'\) to be the respectively cone of \(\Phi^+ \circ \eta^+\) and \(\Phi^- \circ \eta^-\), and then complete the diagram into \((5.77)\) by using the 3 x 3-lemma.

Then, we construct a map \(\Psi_2 : \mathcal{R}_\varphi'(\mathcal{A})[a][1] \to W_1\) by applying the axiom (TR3) to the diagram

\[
\begin{array}{c}
\mathcal{A}(a) \xrightarrow{\eta^+} \check{C}_{\varphi^+}(\mathcal{A})(a) \xrightarrow{\delta^+} \mathcal{R}_\varphi'(\mathcal{A})(a)[1] \xrightarrow{-e[1]} \mathcal{A}(a)[1] \\
\downarrow \Phi^+ \downarrow \mathbb{D}_Y(\check{C}_{\varphi^+}(\mathcal{A})) \downarrow \mathbb{D}_Y(\mathcal{R}_\varphi'(\mathcal{A})) \downarrow \mathbb{D}_Y(\mathcal{A}) \downarrow \Psi_2 \\
\mathcal{A}(a) \xrightarrow{\Phi^+} \check{C}_{\varphi^+}(\mathcal{A}) \xrightarrow{\delta^+_i} W_1 \xrightarrow{-e[1]} \mathcal{A}(a)[1] \\
\end{array}
\]

and similarly for \(\Psi'_2 : \mathcal{R}_\varphi'(\mathcal{A})(a'[1] \to W_1'\). Since \(\Phi^+\) and \(\Phi^-\) are quasi-isomorphism on any weight \(i \in \mathbb{Z}\), so are \(\Psi_2\) and \(\Psi'_2\). Thus, if we take \(\Psi = \Psi_1 \circ \Psi_2\) and \(\Psi' = \Psi'_1 \circ \Psi'_2\), then we obtain again the diagram \((5.76)\) where each square is commutative, and each row or column is part of a distinguished triangle on each weight components \(i \in \mathbb{Z}\). We can apply the same arguments as above on these weight components to conclude the first statement of this Theorem. □

As a Corollary, we prove the following

**Corollary 5.78.** Let \((Y, \omega_Y^*, \mathcal{A}, a, \Phi^-, \Phi^+\) be a pseudo-rational homological flip/flop such that the pair \((Y, \mathcal{A})\) is pre-stable in the sense of Definition \((5.3)\), then there is an isomorphism \(\Psi : \mathcal{R}_\varphi'(\mathcal{A})(a)[1] \xrightarrow{\simeq} \mathbb{D}_Y(\mathcal{R}_\varphi'(\mathcal{A}))\). As a result, for each open affine \(U \subset Y\), the ring \(\mathcal{A}(U)\) is Gorenstein.

**Proof.** The first statement follows directly from Theorem \((5.73)\).

In Section \(\mathbb{A}\) we have taken the care to ensure that all the functors that we use are local on the base (see in particular Lemma \((5.2)\)). Thus, any homological flip/flop restricts to one on any open subscheme \(U \subset Y\), so we may assume in the first place that \(Y = U = \text{Spec } R\) is affine, and show that \(A := \mathcal{A}(U)\) has finite injective dimension.

By the exact triangle \((2.33)\), it suffices to show that \(\mathcal{R}_\varphi'(\mathcal{A})\) and \(\check{C}_I'(\mathcal{A})\) have finite injective dimension. By definition of a homological flip/flop, there is an isomorphism \(\Phi^+ : \check{C}_I'(\mathcal{A})(a) \xrightarrow{\simeq} \mathbb{D}_Y(\check{C}_I'(\mathcal{A}))\) in \(\mathcal{D}(\text{Gr}(A))\). Also, we have just seen that there is an isomorphism \(\Psi : \mathcal{R}_\varphi'(\mathcal{A})(a)[1] \xrightarrow{\simeq} \mathbb{D}_Y(\mathcal{R}_\varphi'(\mathcal{A}))\). Since both \(\check{C}_I'(\mathcal{A})\) and \(\mathcal{R}_\varphi'(\mathcal{A})\) are represented by a finite complex of flat graded modules, the statement follows from the following simple fact, applied to \(N = \mathbb{D}_Y(\mathcal{A})\), and \(M\) being \(\mathcal{R}_\varphi'(\mathcal{A})\) or \(\check{C}_I'(\mathcal{A})\):

Suppose that \(M, N \in \mathcal{D}(\text{Gr}(A))\) are objects such that \(M\) has finite Tor-dimension, and \(N\) has finite injective dimension, then \(\mathcal{R}_\mathbb{H}_\mathcal{A}(M, N) \in \mathcal{D}(\text{Gr}(A))\) has finite injective dimension.

Indeed, this follows from the adjunction \(\mathcal{R}_\mathbb{H}_\mathcal{A}(L, \mathcal{R}_\mathbb{H}_\mathcal{A}(M, N)) \cong \mathcal{R}_\mathbb{H}_\mathcal{A}(L \otimes_A^\mathbb{L} M, N)\).

This result is particularly useful for homological flops. Namely, the notion of a (weak) homological flop is preserved under taking \(m\)-uple component. i.e., if \((Y, \omega_Y^*, \mathcal{A}, 0, \Phi^-, \Phi^+)\) is a (weak) homological flop, then so is \((Y, \omega_Y^*, \mathcal{A}(m), 0, (\Phi^-)^{(m)}, (\Phi^+)^{(m)})\), for any \(m > 0\). Thus, one can always take a high enough \(m\)-uple component to guarantee that it is stable in the sense of Definition \((5.3)\). Moreover, pseudo-rationality of a (weak) homological flop is also preserved by taking \(m\)-uple components. As a result, Corollary \((5.78)\) is applicable to a wide class of examples. See, for example, the following
Remark 5.79. If $X$ and $X'$ are two birational Gorenstein Calabi-Yau normal varieties with at most \( \mathbb{Q} \)-factorial terminal singularities, then a result \cite{19} of Kawamata shows that they are connected by a finite sequence of flops. If $X^− \rightarrow Y \leftarrow X^+$ is a log flip in which $X^−$ is Gorenstein Calabi-Yau with at most canonical singularities, then $K_Y \equiv 0$ and therefore the log flip is automatically strongly crepant. Moreover, if we choose a resolution $f : W \rightarrow X^−$, and apply the Grauert-Riemenschneider vanishing theorem \cite{11} (see also \cite{17} Corollary 2.68) to both $f$ and $\pi^− \circ f$, then we see that $R^i \pi^∗_\circ \mathcal{O}(K_{X^−}) = 0$ for all $j > 0$. Since $X^−$ is Calabi-Yau, we have $\mathcal{O}(K_{X^−}) \cong \mathcal{O}_{X^−}$, and this implies that $Y$ has rational singularities. As a result, any two birational Gorenstein Calabi-Yau normal varieties with at most \( \mathbb{Q} \)-factorial terminal singularities are connected by a sequence of pseudo-rational homological flops.

6. Weight truncation

In this section, we develop the technique of weight truncations. We mostly work with \( \mathbb{Z} \)-graded rings. The extension to the case of non-affine base is purely formal. In view of the discussion in Section 7, our technique is thus applicable to wall-crossings in birational cobordism. The relation between our constructions and those in the prevailing literature is also discussed in the introduction (see the discussion preceding Theorem 1.19).

6.1. Weight truncation. In this subsection, we use freely the language of modules over small preadditive categories. The reader is referred to Appendix A for details.

Given any \( \mathbb{Z} \)-graded ring $A$, let $\mathcal{F} = \mathcal{F}_A$ be the preadditive category with object set $\text{Ob}(\mathcal{F}) = \mathbb{Z}$, and Hom spaces $\mathcal{F}(i, j) := \Lambda_{i-j}$. Compositions in $\mathcal{F}$ are defined by multiplication in $A$ in the obvious way.

A right $\mathcal{F}$-module is nothing but a graded (right) $A$-module. More precisely, there is an equivalence of abelian categories

\[
(-)^\flat : \text{Gr}(A) \xrightarrow{\cong} \text{Mod}(\mathcal{F}), \quad (M^\flat)_i := M_i
\]

whose inverse will be denoted as $(-)^\flat : \text{Mod}(\mathcal{F}) \xrightarrow{\cong} \text{Gr}(A)$.

Since $A$ is assumed to be commutative, any graded module $M \in \text{Gr}(A)$ in fact induces an $\mathcal{F}$-bimodule. In other words, there is an additive functor

\[
\text{Gr}(A) \rightarrow \text{Mod}(\mathcal{F}^c), \quad M \mapsto \tilde{M}, \quad \text{where} \quad j(\tilde{M}_i) := M_i - j
\]

which recovers the functor (6.1) by restricting to $\text{gr}(\tilde{M})$.

The graded tensor products and graded Hom spaces between graded modules can be expressed naturally in terms of $\mathcal{F}$-bimodules. Namely, for any $M, N \in \text{Gr}(A)$, there are natural isomorphisms of $\mathcal{F}$-bimodules

\[
\tilde{M} \otimes_{\mathcal{F}} \tilde{N} \cong M \otimes_A N \quad \text{and} \quad \text{Hom}_\mathcal{F}(\tilde{M}, \tilde{N}) \cong \text{Hom}_A(M, N)
\]

In particular, if we only remember the right $\mathcal{F}$-module structure of $\tilde{M}$, then we have

\[
M^\flat \otimes_{\mathcal{F}} \tilde{N} \cong (M \otimes_A N)^\flat
\]

Since $A$ is assumed to be commutative, the preadditive category $\mathcal{F} = \mathcal{F}_A$ admits an involution, i.e., it comes equipped with an isomorphism $\mathcal{F} \cong \mathcal{F}^{\text{op}}$ of preadditive categories, defined by $i \mapsto -i$ on objects, and $\mathcal{F}(i, j) = \Lambda_{i-j} = \mathcal{F}^{\text{op}}(-i, -j)$ on Hom sets. In fact, the commutativity of $A$ is equivalent to the fact that this assignment $\mathcal{F} \rightarrow \mathcal{F}^{\text{op}}$ is a functor. This involution induces an isomorphism of categories

\[
(-)^\tau : \text{Mod}(\mathcal{F}) \xrightarrow{\cong} \text{Mod}(\mathcal{F}^{\text{op}}), \quad i(M^\tau) := M_{-i}
\]

whose inverse will also be denoted as $(-)^\tau$.

For any integer $a \in \mathbb{Z}$, let $\mathcal{F}_{[\geq a]}$ be the full subcategory of $\mathcal{F}$ on the subset $\text{Ob}(\mathcal{F}_{[\geq a]}) = \mathbb{Z} \geq a \subset \mathbb{Z} = \text{Ob}(\mathcal{F})$. Define $\mathcal{F}_{[\leq a]} \subset \mathcal{F}$ in the similar way. We will also write $\mathcal{F}_{[\leq a]} = \mathcal{F} = \mathcal{F}_{[a, \infty]}$. For any $-\infty \leq a' \leq a$, denote by $(-)^{[a,a']} : \text{Mod}(\mathcal{F}_{[a', a]}) \rightarrow \text{Mod}(\mathcal{F}_{[a]})$ the restriction functor.

Notice that the involution on $\mathcal{F}$ restricts to an isomorphism $(\mathcal{F}_{[\geq a]})^{\text{op}} \cong \mathcal{F}_{[\leq a]}$. As a result, there is again an isomorphism of categories

\[
(-)^\tau : \text{Mod}(\mathcal{F}_{[\geq a]}) \xrightarrow{\cong} \text{Mod}(\mathcal{F}_{[\leq a]})^{\text{op}}, \quad i(M^\tau) := M_{-i}
\]
whose inverse will also be denoted as \((-\tau)^{\circ}\). Similarly, the functor \((M^{\tau})_{i} := -_{i}M\) gives an isomorphism of categories \((-\tau)^{\circ} \colon \text{Mod}((\mathcal{F}_{[\geq a]})^{op}) \xrightarrow{\sim} \text{Mod}(\mathcal{F}_{[\leq -a]})\) whose inverse will also be denoted as \((-\tau)^{\circ}\).

One can also define transposition functors for bimodules. Namely, for any \(-\infty \leq a \leq b\), there is an isomorphism of categories

\[
(-\tau)^{\circ} : \mathcal{F}_{[\geq a]} \text{Mod}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{F}_{[\leq -a]} \text{Mod}_{\mathbb{Z}}, \quad i(M^{\tau})_{j} := -_{j}M_{-i}
\]

For any \(M \in \text{Gr}(A)\), the \(\mathcal{F}\)-bimodule \(\tilde{M}\) is induced by a symmetric \(A\)-bimodule, and is therefore self-transpose. In other words, there is a canonical isomorphism

\[
\tilde{\tau} \cong \tilde{M}, \quad \text{given by} \quad i(\tilde{\tau})_{j} = -_{j}\tilde{M}_{-i} = M_{j-i} = i\tilde{M}_{j}
\]

which can be checked to be \(\mathcal{F}\)-bilinear.

Transposition commutes with restriction functors. More precisely, for any \(-\infty \leq a' \leq a \leq b\), and for any \((\mathcal{F}_{[\geq a']}, \mathcal{F}_{[\geq b']})\)-bimodule \(M\), there is a canonical isomorphism

\[
(\geq a\mathcal{M}[\geq b])^{\tau} \cong \leq b\mathcal{M}(\leq a-\mathcal{F})
\]

Transpositions also commute with tensor products. For any \(a, b, c \geq -\infty\), for any any \(M \in \mathcal{F}_{[\geq a]} \text{Mod}_{\mathbb{Z}}\) and any \(\mathcal{N} \in \mathcal{F}_{[\geq b]} \text{Mod}_{\mathbb{Z}}\), there is a canonical isomorphism in \(\mathcal{F}_{[\leq c]} \text{Mod}_{\mathbb{Z}}\)

\[
(\mathcal{F} \otimes_{\mathcal{F}_{[\geq a]}} \mathcal{M})^{\tau} \cong \mathcal{F}^{\tau} \otimes_{\mathcal{F}_{[\leq -b]}} \mathcal{N}^{\tau}
\]

Similarly, for any \(M \in \mathcal{F}_{[\geq a]} \text{Mod}_{\mathbb{Z}}\) and any \(\mathcal{N} \in \mathcal{F}_{[\leq -c]} \text{Mod}_{\mathbb{Z}}\), there is a canonical isomorphism in \(\mathcal{F}_{[\leq -a]} \text{Mod}_{\mathbb{Z}}\)

\[
\left(\text{Hom}_{\mathcal{F}_{[\geq a]}}(\mathcal{M}, \mathcal{N})\right)^{\tau} \cong \text{Hom}_{\mathcal{F}_{[\leq -a]}}(\mathcal{M}^{\tau}, \mathcal{N}^{\tau})
\]

From now on, we fix an integer \(w \in \mathbb{Z}\). Notice that the inclusion functor \(\mathcal{F}_{[\geq w]} \hookrightarrow \mathcal{F}\) induces a three-way adjunction

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{F}_{[\geq w]}) & \xrightarrow{(-)_{[\geq w]}} & \text{Mod}(\mathcal{F}) \\
\text{Hom}_{\mathcal{F}}(\mathcal{F}_{[\geq w]}, \mathcal{F}) & \xrightarrow{(-)_{[\geq w]}} & \text{Mod}(\mathcal{F})
\end{array}
\]

\[\text{(6.4)}\]

In fact, under the equivalence \(\text{Mod}(\mathcal{F}) \simeq \text{Gr}(A)\), the right-pointing functor \(- \otimes_{\mathcal{F}_{[\geq w]}} \mathcal{F}\) on the top may be characterized as the unique cocontinuous functor satisfying

\[
\text{(6.5)} \quad \text{For each} \ i \geq w, \ \text{the functor} \ (- \otimes_{\mathcal{F}_{[\geq w]}} \mathcal{F})_{i} : \text{Mod}(\mathcal{F}_{[\geq w]}) \rightarrow \text{Gr}(A) \ \text{sends the free module} \ \mathcal{F}_{[\geq w]} \ \text{to the free graded module} \ A(-i)
\]

Since the inclusion functor \(\mathcal{F}_{[\geq w]} \rightarrow \mathcal{F}\) is fully faithful, we have

\[
\text{(6.6)} \quad (\mathcal{M} \otimes_{\mathcal{F}_{[\geq w]}} \mathcal{F}_{[\geq w]}) \cong M \otimes_{\mathcal{F}_{[\geq w]}} \mathcal{F}_{[\geq w]} \cong M
\]

\[
\text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}, \mathcal{M})_{[\geq w]} \cong \text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}_{[\geq w]}, \mathcal{M}) \cong M
\]

One can also relate \(\mathcal{F}_{[\geq w]}\)-modules to graded \(A_{\geq 0}\)-modules. Namely, given \(M \in \text{Mod}(\mathcal{F}_{[\geq w]})\), although the assignment \(M_{i} := M_{i}\) does not define a graded \(A\)-module, it nonetheless defines a graded \(A_{\geq 0}\)-module, which we denote as \(\mathcal{M}|_{A_{\geq 0}}\). This can be used to characterize finite generated \(\mathcal{F}_{[\geq w]}\)-modules:

\[\text{Lemma 6.7. Suppose that} \ A \ \text{is Noetherian, then for any} \ M \in \text{Mod}(\mathcal{F}_{[\geq w]}), \ \text{the followings are equivalent:}\]

\[\text{(1) \ } M \ \text{is a finitely generated} \ \mathcal{F}_{[\geq w]}\text{-module in the sense of Definition A.6}\]

\[\text{(2) there exists a finitely generated graded} \ A\text{-module} \ M \ \text{such that} \ M \cong M_{[\geq w]} := (M^{\sharp})_{[\geq w]};\]

\[\text{(3)} \mathcal{M}|_{A_{\geq 0}} \text{is a finitely generated graded} \ A_{\geq 0}\text{-module.}\]

\[\text{This is because for any} \ m > i - w \geq 0, \ \text{there is no natural action of} \ A_{-m} \ \text{on} \ M_{i} \ \text{inherited from the} \ \mathcal{F}_{[\geq w]}\text{-module structure of} \ M.\]

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Proof. For (1) \(\implies\) (2), simply notice that if there is an epimorphism \(\bigoplus_{j=1}^m i_j F_{[\geq w]} \to M\), then applying \(- \otimes_{F_{[\geq w]}} F\), we have by (6.3) an epimorphism \(\bigoplus_{j=1}^m (-i_j) \to (\mathcal{M} \otimes_{F_{[\geq w]}} F)^\flat\), which shows that \(M := (\mathcal{M} \otimes_{F_{[\geq w]}} F)^\flat\) is a finitely generated graded \(A\)-module. Moreover, it satisfies \(M \cong M^\sharp_{[\geq 0]}\) by (6.0).

For (2) \(\implies\) (3), we use the Noetherian condition. By Proposition 2.2, \(A_0\) is Noetherian, and each \(A_i\) is finitely generated as a module over \(A_0\). Moreover, as in Lemma 6.12, there exists integers \(d > 0\) and \(N_0 > 0\) such that whenever \(N \geq N_0\), we have \(A_N = A_d \cdot A_{N-d}\). Now if \(M\) is generated by \(\xi_1, \ldots, \xi_r\), over \(A\), then for any \(j \geq N_0 + \max_i \{\deg(\xi_i)\}\), we have \(M_j = A_d \cdot M_j-d\). Since each \(M_i\) is clearly finitely generated over \(A_0\), the same is true for \(\bigoplus_{w \leq j \leq N_0 + \max_i \{\deg(\xi_i)\}} M_i\), which then generates \(M^\sharp_{[\geq 0]}\) under the \(A_{\geq 0}\) action.

The implication (3) \(\implies\) (1) follows directly from the definitions. \(\square\)

**Corollary 6.8.** If the \(\mathbb{Z}\)-graded ring \(A\) is Noetherian, then the small preadditive category \(F_{[\geq w]}\) is Noetherian.

Proof. By the characterization of finitely generated right \(F_{[\geq w]}\)-modules in Lemma (6.7)(3), we see that \(F_{[\geq w]}\) is right Noetherian. The same argument also shows that \(F_{[\leq w]}\) is right Noetherian. By the isomorphism of categories \((-)^\tau : \text{Mod}((F_{[\geq w]})^{op}) \cong \text{Mod}(F_{[\leq w]}))\), we see that \(F_{[\geq w]}\) is left Noetherian as well. \(\square\)

Now we take the derived functors of the functors appearing in (6.4). Under the identification \(D(F) \cong D(\text{Gr}(A))\), we denote these derived functors as

\[
\mathcal{L}^{[\geq w]} : D(F_{[\geq w]}) \to D(\text{Gr}(A)), \quad \mathcal{L}^{[\geq w]}(M) := (\mathcal{M} \otimes_{F_{[\geq w]}} F)^\flat
\]

\[
\mathcal{R}^{[\geq w]} : D(F_{[\geq w]}) \to D(\text{Gr}(A)), \quad \mathcal{R}^{[\geq w]}(M) := (R\text{Hom}_{F_{[\geq w]}}(F, M))^\flat
\]

**Definition 6.9.** Let \(D_{\leq w}(\text{Gr}(A)) \subset D(\text{Gr}(A))\) be the full subcategory consisting of \(M \in D(\text{Gr}(A))\) such that \(M_i \cong 0\) for all \(i \geq w\).

Then we have a recollement

\[
\xymatrix{ D_{\leq w}(\text{Gr}(A)) \ar[rr]^i \ar[dr]_{\mathcal{L}_{[\geq w]}^{[\leq w]}} & & D(\text{Gr}(A)) \ar[rl]_{\mathcal{R}_{[\geq w]}^{[\leq w]}} \ar[dl]^{\mathcal{L}^{[\geq w]}} \ar[dr]_{\mathcal{R}^{[\geq w]}} \ar@/^/[rrr]^{(-)^{[\geq w]}} & & D(F_{[\geq w]}) \ar[ll]_{\mathcal{R}_{[\geq w]}^{[\leq w]}} \ar[dl]_{\mathcal{L}^{[\geq w]}} \ar@/_/[rrr]^{(-)^{[\geq w]}} & & D(F_{[\leq w]}) \ar[ll]_{\mathcal{R}^{[\geq w]}} }
\]

where we have written \(M_{[\leq w]}^{[\leq w]} := (M^\sharp_{[\geq w]}).

Indeed, if we denote by \(\mathcal{L}_{[\geq w]}^{[\leq w]}\) and \(\mathcal{R}_{[\geq w]}^{[\leq w]}\) the endofunctors on \(D(\text{Gr}(A))\) given by the compositions

\[
\mathcal{L}_{[\geq w]}(M) := \mathcal{L}_{[\geq w]}(M_{[\leq w]}^{[\leq w]}) \quad \text{and} \quad \mathcal{R}_{[\geq w]}(M) := \mathcal{R}_{[\geq w]}(M_{[\leq w]}^{[\leq w]})
\]

then the functors \(\mathcal{L}_{[\geq w]}^{[\leq w]}\) and \(\mathcal{R}_{[\geq w]}^{[\leq w]}\) are defined by the exact triangles

\[
\begin{align*}
\ldots & \to \mathcal{L}_{[\geq w]}(M) \to M \to \mathcal{L}_{[\leq w]}(M) \to \mathcal{L}_{[\geq w]}(M[1]) \to \ldots \\
\ldots & \to \mathcal{R}_{[\leq w]}(M) \to M \to \mathcal{R}_{[\leq w]}(M) \to \mathcal{L}_{[\leq w]}(M[1]) \to \ldots
\end{align*}
\]

We are mostly interested in the functors \(\mathcal{L}_{[\geq w]}^{[\leq w]}, \mathcal{L}_{[\geq w]}^{[\leq w]}\) and \(\mathcal{L}_{[\leq w]}^{[\leq w]}\) instead of their counterparts for \(\mathcal{R}\) and \(\mathcal{R}\). To this end, we make the following

**Definition 6.12.** Let \(D_{[\geq w]}(\text{Gr}(A)) \subset D(\text{Gr}(A))\) be the essential image of the fully faithful functor \(\mathcal{L}_{[\geq w]} : D(F_{[\geq w]}) \to D(\text{Gr}(A))\).

Alternatively, the subcategory \(D_{[\geq w]}(\text{Gr}(A)) \subset D(\text{Gr}(A))\) may be characterized as follows:

**Lemma 6.13.** \(D_{[\geq w]}(\text{Gr}(A))\) is the smallest strictly full triangulated subcategory containing the objects \(A(-i)\) for \(i \geq w\) and is closed under small coproducts. Therefore, we have

\[
D_{[\geq w]}(\text{Gr}(A)) \cap D(\text{Gr}(A)) = D_{[\geq w]}(\text{Gr}(A)) = \text{EssIm}(\mathcal{L}_{[\geq w]} : D_{\text{perf}}(F_{[\geq w]}) \to D(\text{Gr}(A)))
\]

where the subscript \((-)\) denotes the full subcategory of compact objects.
Proposition 6.19. The first statement follows from (6.5) and the fact that \( D_{[\geq w]} \) preserves small coproducts. For the second statement, the first equality is standard (see, e.g., [29] Lemma 2.2 or [29] Theorem 5.3). The second equality follows from the standard fact (see, e.g., (A.14)) that \( D_{\text{perf}}(F_{[\geq w]}) = D(F_{[\geq w]}) \).

It follows immediately from the recollection (6.11) that there is a semi-orthogonal decomposition

\[(6.14)\]

\[ D(\text{Gr}(A)) = \langle D_{< w}(\text{Gr}(A)), D_{[\geq w]}(\text{Gr}(A)) \rangle \]

Remark 6.15. The notation \( D_{[\geq w]} \) conveys the idea that these are objects “generated in weight \( \geq w \)”; while the notation \( D_{< w} \) means that these are objects “concentrated in weight \( < w \)”.

6.2. Local cohomology and weight truncation. Now we study local cohomology under weight truncation. For any finitely generated graded ideal \( I \subset A \), the objects \( R\Gamma_I(A) \) and \( \check{\Gamma}_I(A) \) in \( D(\text{Gr}(A)) \) gives rise to the objects \( [\geq w]I\Gamma_I(A)_{[\geq w]} \) and \( [\geq w]\check{\Gamma}_I(A)_{[\geq w]} \) in the derived category \( D((F_{[\geq w]})^r) \) of \( F_{[\geq w]} \)

\[ D \]

bimodules. Tensoring over these give rise to functors

\[(6.16)\]

\[ R\Gamma_{I,[\geq w]} : D(F_{[\geq w]}) \to D(F_{[\geq w]}), \quad M \mapsto M \otimes_{F_{[\geq w]}} R\Gamma_I(A)_{[\geq w]} \cong ((L_{[\geq w]}M) \otimes_A L_{[\geq w]} R\Gamma_I(A))_{[\geq w]} \]

\[ \check{\Gamma}_{I,[\geq w]} : D(F_{[\geq w]}) \to D(F_{[\geq w]}), \quad M \mapsto M \otimes_{F_{[\geq w]}} \check{\Gamma}_I(A)_{[\geq w]} \cong ((L_{[\geq w]}M) \otimes_A L_{[\geq w]} \check{\Gamma}_I(A))_{[\geq w]} \]

where the last isomorphisms on each line is obtained by applying (A.8) and (6.2).

We are most interested in the case \( I = I^+ \), where these functors behave very similarly to the corresponding ones on \( D(\text{Gr}(A)) \) (see Proposition 6.19 and Theorem 6.21 below). In fact, these properties are formal consequences of the following easy

Lemma 6.17. We have \( D_{< w}(\text{Gr}(A)) \subset D_{\text{Tor}^+}(\text{Gr}(A)) \)

which can be used to prove the following two results:

Lemma 6.18. If \( M \in D_{\text{Tor}^+}(\text{Gr}(A)) \) then \( L_{[\geq w]}(M) \in D_{\text{Tor}^+}(\text{Gr}(A)) \).

If \( R\Gamma_{I^+}(M) \in D_{< w}(\text{Gr}(A)) \) then \( L_{[\geq w]}(M) \otimes_A L_{[\geq w]} R\Gamma_{I^+}(A) \in D_{< w}(\text{Gr}(A)) \).

Proof. For any \( M \in D(\text{Gr}(A)) \), we have \( L_{[\geq w]}(M) \in D_{\text{Tor}^+}(\text{Gr}(A)) \) by Lemma 6.17 so that the first statement follows directly from the exact triangle in the first row of (6.11). For the second statement, apply \( R\Gamma_{I^+}(-) \) to the same exact triangle, and notice that \( L_{[\geq w]}(M) \otimes_A L_{[\geq w]} R\Gamma_{I^+}(A) \simeq L_{< w}(M) \in D_{< w}(\text{Gr}(A)) \) because \( L_{< w}(M) \in D_{\text{Tor}^+}(\text{Gr}(A)) \).

Proposition 6.19. The following two functors commute up to isomorphism of functors:

\[ D(\text{Gr}(A)) \xrightarrow{R\Gamma_{I^+}} D(\text{Gr}(A)) \xrightarrow{L_{[\geq w]}} D(F_{[\geq w]}) \]

\[ D(\text{Gr}(A)) \xrightarrow{\check{\Gamma}_{I^+}} D(\text{Gr}(A)) \xrightarrow{L_{[\geq w]}} D(F_{[\geq w]}) \]

Proof. Given any \( M \in D(\text{Gr}(A)) \), take the exact triangle in the first row of (6.11). By Lemma 6.17, we have \( L_{< w}(M) \in D_{\text{Tor}^+}(\text{Gr}(A)) \). Applying \( \check{\Gamma}_{I^+} \) to this exact triangle, we have \( \check{\Gamma}_{I^+}(L_{[\geq w]}M) \cong \check{\Gamma}_{I^+}(M) \) in \( D(\text{Gr}(A)) \). Applying \( (-)[\#_{[\geq w]}] \) to this isomorphism gives the commutativity of the second square. Similarly, applying \( R\Gamma_{I^+} \) to the same exact triangle, we have an exact triangle

\[ \ldots \to R\Gamma_{I^+}(L_{[\geq w]}M) \to R\Gamma_{I^+}(M) \to L_{< w}(M) \to R\Gamma_{I^+}(L_{[\geq w]}M)[1] \to \ldots \]

Applying \( (-)[\#_{[\geq w]}] \) therefore gives an isomorphism \( (R\Gamma_{I^+}(L_{[\geq w]}M))_{[\geq w]} \cong (R\Gamma_{I^+}(M))_{[\geq w]} \), proving the commutativity of the first square.

Definition 6.20. Let \( D_{I^{\text{triv}}}(F_{[\geq w]}) \) and \( D_{\text{Tor}^+}(F_{[\geq w]}) \) be the full subcategories of \( D(F_{[\geq w]}) \) defined by

\[ D_{I^{\text{triv}}}(F_{[\geq w]}) := \{ M \in D(F_{[\geq w]}), R\Gamma_{I^+(F_{[\geq w]})(M)} \simeq 0 \} \]

\[ D_{\text{Tor}^+}(F_{[\geq w]}) := \{ M \in D(F_{[\geq w]}), \check{\Gamma}_{I^+(F_{[\geq w]})(M)} \simeq 0 \} \]
Then we have the following

**Theorem 6.21.** For $I = I^+$, the functors (6.10) form a semi-orthogonal pair of idempotents, in the sense that the followings hold:

1. For any $\mathcal{M} \in \mathcal{D}(\mathcal{F}_{\geq w})$, we have $\mathcal{R} \mathcal{I}_{I^+, [\geq w]}(\mathcal{M}) \in \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})$ and $\mathcal{C}_{I^+, [\geq w]}(\mathcal{M}) \in \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w})$;
2. If $\mathcal{M} \in \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})$, then $\mathcal{R} \mathcal{I}_{I^+, [\geq w]}(\mathcal{M}) \cong \mathcal{M}$;
3. If $\mathcal{M} \in \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w})$, then $\mathcal{C}_{I^+, [\geq w]}(\mathcal{M}) \cong \mathcal{M}$;
4. There is a semi-orthogonal decomposition $\mathcal{D}(\mathcal{F}_{\geq w}) = \langle \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w}), \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w}) \rangle$.

**Proof.** Statement (1) follows immediately from Proposition 6.19. For statements (2) and (3), consider the exact triangle in the first row of (2.30). Take $(-)$ to obtain an exact triangle in $\mathcal{D}(\mathcal{F}^\circ)$. Restricting to $\mathcal{D}(\mathcal{F}_{\geq w})^\circ$ and applying $\mathcal{M} \otimes_{\mathcal{L} \mathcal{F}^\circ} L$ to it, we have an exact triangle

$$\cdots \rightarrow \mathcal{R} \mathcal{I}_{I^+, [\geq w]}(\mathcal{M}) \xrightarrow{\mathcal{C} \mathcal{M}} \mathcal{M} \xrightarrow{\mathcal{H} \mathcal{M}} \mathcal{C}_{I^+, [\geq w]}(\mathcal{M}) \xrightarrow{\mathcal{H} \mathcal{M}} \mathcal{R} \mathcal{I}_{I^+, [\geq w]}(\mathcal{M})[1] \rightarrow \cdots$$

which immediately shows (2) and (3). In fact, because of (1), this exact triangle also establishes the decomposition for (4), so that it suffices to show the semi-orthogonality $\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w}) \perp \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w})$.

Thus, let $\mathcal{M} \in \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})$ and $\mathcal{N} \in \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w})$, then by (2) and (3), we may write $\mathcal{M} \cong M_{\geq w}^I$ and $\mathcal{N} \cong N_{\geq w}^I$ for some $M \in \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$ and $N \in \mathcal{D}_{I^+ - \text{triv}}(\text{Gr}(A))$. By the adjunction (6.10), we have

$$\text{Hom}_{\mathcal{D}(\mathcal{F}_{\geq w})}(\mathcal{M}, \mathcal{N}) \cong \text{Hom}_{\mathcal{D}(\text{Gr}(A))}(\mathcal{L}_{\geq w}(M), N)$$

which is zero because of the first statement of Lemma 6.18.

Combined with (6.14), this gives the first statement of the following

**Theorem 6.22.** There is a semi-orthogonal decomposition

$$\mathcal{D}(\text{Gr}(A)) = \langle \mathcal{D}_{\leq w}(\text{Gr}(A)), \mathcal{L}_{\geq w}(\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})), \mathcal{L}_{\geq w}(\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})) \rangle$$

where the functor $\mathcal{L}_{\geq w} : \mathcal{D}(\mathcal{F}_{\geq w}) \rightarrow \mathcal{D}(\text{Gr}(A))$ is fully faithful. Moreover, the latter two semi-orthogonal components can be identified as

(6.23) $\mathcal{L}_{\geq w}(\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})) = \{ M \in \mathcal{D}_{\geq w}(\text{Gr}(A)) \mid \mathcal{R} \mathcal{I}^+(M) \in \mathcal{D}_{\leq w}(\text{Gr}(A)) \}$

$$\mathcal{L}_{\geq w}(\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})) = \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})(\text{Gr}(A)) : = \mathcal{D}_{\geq w}(\text{Gr}(A)) \cap \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$$

**Proof.** Only the identifications (6.23) of the semi-orthogonal components needs proof. It is clear that all the subcategories in (6.23) lie in $\mathcal{D}_{\geq w}(\text{Gr}(A))$. Moreover, given $M \in \mathcal{D}_{\geq w}(\text{Gr}(A))$, then by Proposition 6.19 we have $M_{\geq w}^I \in \mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w})$ if and only if $\mathcal{R} I^+(M)_{\geq w}^I = 0$. The latter condition is precisely $\mathcal{R} I^+(M) \in \mathcal{D}_{\leq w}(\text{Gr}(A))$, hence proving the first row of (6.23). For the second row, we similarly observe that, given any $M \in \mathcal{D}_{\geq w}(\text{Gr}(A))$, then by Proposition 6.19 we have $M_{\geq w}^I \in \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})$ if and only if $\mathcal{C}_{I^+}(M) \in \mathcal{D}_{\leq w}(\text{Gr}(A))$. But $\mathcal{C}_{I^+}(M)$ is always in $\mathcal{D}(I^+ - \text{triv})(\text{Gr}(A))$, so that the latter is true if and only if $\mathcal{C}_{I^+}(M) = 0$.

Unravelling the definitions, we see that this semi-orthogonal decomposition decomposes every $M \in \mathcal{D}(\text{Gr}(A))$ into the diagram

(6.24)

$$\begin{array}{ccc}
\mathcal{L}_{\geq w}^R \mathcal{I}^+(M) & \xrightarrow{\mathcal{L}_{\geq w}(\mathcal{C} \mathcal{M})} & \mathcal{L}_{\geq w}(M) \\
\downarrow \cong \quad \downarrow \cong \quad \downarrow \cong \\
\mathcal{L}_{\geq w} \mathcal{C}_{I^+}(M) & \xrightarrow{\mathcal{L}_{\geq w}(\mathcal{N} \mathcal{M})} & \mathcal{L}_{\leq w}(M) \\
\downarrow \cong \quad \downarrow \cong \\
\mathcal{L}_{\leq w}(M) & \xrightarrow{\text{unit}} & M
\end{array}$$

with the decomposition terms

1. $\mathcal{L}_{\leq w}(M) \in \mathcal{D}_{\leq w}(\text{Gr}(A))$,
2. $\mathcal{L}_{\geq w} \mathcal{C}_{I^+}(M) \in \mathcal{L}_{\geq w}(\mathcal{D}_{I^+ - \text{triv}}(\mathcal{F}_{\geq w}))$, and
3. $\mathcal{L}_{\geq w}^R \mathcal{I}^+(M) \in \mathcal{L}_{\geq w}(\mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})) = \mathcal{D}_{\text{Tor}^+}(\mathcal{F}_{\geq w})(\text{Gr}(A))$.
We also have the following two characterizations of semi-orthogonal components

**Lemma 6.25.** There is a semi-orthogonal decomposition

$$\mathcal{D}_{\text{Tor}^+}(\text{Gr}(A)) = (\mathcal{D}_{<w}(\text{Gr}(A)), \mathcal{D}_{\text{Tor}^+, [\geq w]}(\text{Gr}(A)))$$

**Proof.** For any $M \in \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$, take the exact sequence in the first row of (6.21). Since $L_{<w}M \in \mathcal{D}_{<w}(\text{Gr}(A)) \subset \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$, we have $\mathcal{L}_{[\geq w]}(M) \in \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$ as well. This proves the claimed decomposition. Orthogonality was already shown in Theorem 6.22. 

**Lemma 6.26.** The first two components in Theorem 6.22 can be identified as

$$\langle \mathcal{D}_{<w}(\text{Gr}(A)), \mathcal{L}_{[\geq w]}(\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})) \rangle = \{ M \in \mathcal{D}(\text{Gr}(A)) | R\Gamma_{I^+}(M) \in \mathcal{D}_{<w}(\text{Gr}(A)) \}$$

**Proof.** An object $M \in \mathcal{D}(\text{Gr}(A))$ lies in $\langle \mathcal{D}_{<w}(\text{Gr}(A)), \mathcal{L}_{[\geq w]}(\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})) \rangle$ if and only if the component $\mathcal{L}_{[\geq w]}R\Gamma_{I^+}(M)$ in (6.24) vanishes. This is true if and only if $R\Gamma_{I^+}(M) \in \mathcal{D}_{<w}(\text{Gr}(A))$. 

Our next goal is to show that the triangulated category $\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})$ is in fact equivalent to $\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$ (see Theorem 6.29 below). First, notice that, by the second statement of Lemma 6.18, the functor $(-)^{\geq w} : \mathcal{D}(\text{Gr}(A)) \rightarrow \mathcal{D}(\mathcal{F}_{[\geq w]})$ sends the subcategory $\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A)) \subset \mathcal{D}(\text{Gr}(A))$ to the subcategory $\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]}) \subset \mathcal{D}(\mathcal{F}_{[\geq w]})$, so that we have an exact functor

$$\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A)) \rightarrow \mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})$$

(6.27) $\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A)) \rightarrow \mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})$

In the other direction, there is the exact functor

$$\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]}) \rightarrow \mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$$

(6.28) $\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]}) \rightarrow \mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$

**Theorem 6.29.** The functors (6.27) and (6.28) are quasi-inverse equivalences.

**Proof.** First, notice that the functors (6.27) and (6.28) are restrictions of the composite adjunctions

$$\mathcal{D}(\mathcal{F}_{[\geq w]}) \xrightarrow{\mathcal{L}_{[\geq w]}} \mathcal{D}(\text{Gr}(A)) \xrightarrow{\mathcal{C}_{\text{I}^+}} \mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$$

and are therefore adjoints to each other.

The fact that the adjunction unit $\text{id} \Rightarrow (-)^{\geq w} \circ \mathcal{C}_{\text{I}^+}$ is an isomorphism on $\mathcal{D}_{I^+\text{-triv}}(\mathcal{F}_{[\geq w]})$ is precisely statement (3) of Theorem 6.21. This shows that, for any $M \in \mathcal{D}(\text{Gr}(A))$, the adjunction counit $\mathcal{C}_{\text{I}^+} \circ \mathcal{L}_{[\geq w]}(M^{\geq w}) \rightarrow M$ becomes an isomorphism after applying $(-)^{\geq w}$. In other words, its cone lies in $\mathcal{D}_{<w}(\text{Gr}(A)) \subset \mathcal{D}_{\text{Tor}^+}(\text{Gr}(A))$. If $M \in \mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$, then this cone also lies in $\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))$, which means it must be zero. 

The semi-orthogonal decomposition in Theorem 6.22 can be rewritten in the form

$$\mathcal{D}(\text{Gr}(A)) = (\mathcal{D}_{<w}(\text{Gr}(A)), \mathcal{L}_{[\geq w]}(\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A))), \mathcal{D}_{\text{Tor}^+, [\geq w]}(\text{Gr}(A)))$$

where $\mathcal{L}_{[\geq w]}(\mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A)))$ is the essential image of the functor $\mathcal{L}_{[\geq w]} : \mathcal{D}_{I^+\text{-triv}}(\text{Gr}(A)) \rightarrow \mathcal{D}(\text{Gr}(A))$, which is fully faithful by Theorem 6.29.

**Remark 6.31.** Our discussion so far about local cohomology on weight truncation have a completely parallel version for local homology (i.e., derived completion). Indeed, in place of Lemma 6.17, we have $\mathcal{D}_{<w}(\text{Gr}(A)) \subset \mathcal{D}_{\text{Tor}^+\text{-comp}}(\text{Gr}(A))$ by Proposition 2.50 which allows one to develop formal analogues of Proposition 6.19, Theorem 6.21, Theorem 6.22 and Theorem 6.29 where the functors $(R\Gamma_{\text{I}^+}, \mathcal{C}_{\text{I}^+}, \mathcal{L}, \mathcal{L})$ are replaced by $(\mathcal{L}_{\mathcal{A}_{\text{I}^+}}, \mathcal{E}_{\text{I}^+}, \mathcal{L}, \mathcal{L})$. However, this seems to be less useful for us because the results in the next subsection seems to have no analogue for this dual version.

For later use, we also show that the equivalence in Theorem 6.29 have finite cohomological dimension. This is clear for the functor (6.27) since it descends from an exact functor on the abelian categories. For the functor (6.28), we have the following
Lemma 6.32. The functor $\hat{C}_I \circ \mathcal{L}_{[\geq w]} : \mathcal{D}(\mathcal{F}_{[\geq w]}) \to \mathcal{D}_{I^+-\text{triv}}(\Gr(A))$ has finite cohomological dimension. Equivalently, the complex of bimodule $[\geq w] \hat{C}_I(A) \in \mathcal{D}(\mathcal{F}_{[\geq w]} \Mod_F)$ has finite Tor-dimension on the left.

Proof. Since the exact functor $(-)^{\sharp}_{[\geq w]} : \Gr(A) \to \Mod(\mathcal{F}_{[\geq w]})$ is essentially surjective, the statement in the Lemma is equivalent to the statement that the functor $\hat{C}_I \circ \mathcal{L}_{[\geq w]} : \mathcal{D}(\Gr(A)) \to \mathcal{D}_{I^+-\text{triv}}(\Gr(A))$ has finite cohomological dimension. To show this, simply apply $\hat{C}_I \circ -$ to the exact triangle in the first row of (6.11). By Lemma 6.17 we therefore have $\hat{C}_I(\mathcal{L}_{[\geq w]}(M)) \cong \hat{C}_I(M)$, which has cohomology in degrees $[p, q + r]$ if $M$ has cohomology in degrees $[p, q]$ (here $r$ is the number of generators of $I^+$).

6.3. Coherent subcategories. The goal of this subsection is to prove that the semi-orthogonal decomposition in Theorem 6.22 restricts to a semi-orthogonal decomposition on the subcategory $\mathcal{D}_{\text{coh}}(\Gr(A))$ (see Theorem 6.41 below). Then we show that the equivalence in Theorem 6.29 restricts to an equivalence between suitable bounded coherent subcategories (see Theorem 6.42).

From now on, assume that the $\mathbb{Z}$-graded ring $A$ is Noetherian. Then by Corollary 6.8, $\mathcal{F}_{[\geq w]}$ is also Noetherian, a fact that we will use without any more explicit mention. We start with the following

Proposition 6.33. If $M \in \mathcal{D}_{\text{coh}}(\Gr(A))$, then we have $\hat{C}_I(M)^{\sharp}_{[\geq w]} \in \mathcal{D}_{\text{coh}}^{b}(\mathcal{F}_{[\geq w]})$.

Proof. It suffices to assume that $M$ is a finitely generated graded $A$-module concentrated in cohomological degree 0. Then we have $H^p(\hat{C}_I(M)_i) \cong H^p(X^+, \hat{M}(i))$ where $(-)^{\sim} : \Gr(A) \to \Qcoh(X^+)$ denotes the associated sheaf on $X^+ := \Proj(A_{\geq 0})$. Thus, by Lemma 6.7 it suffices to show that $\bigoplus_{i \geq w} H^p(X^+, \hat{M}(i))$ is finitely generated over $A_{\geq 0}$.

Recall (see, e.g., [30, Tag 0B5T]) that if $\mathcal{F}$ is a coherent sheaf on $X^+$ and $\mathcal{L}$ is an ample invertible sheaf on $X^+$, then $\bigoplus_{i \geq w} H^p(\mathcal{F}, \mathcal{L}^\otimes i)$ is finitely generated over $B_{\geq 0} := \bigoplus_{i \geq 0} H^0(X^+, \mathcal{L}^\otimes i)$. In the present case, suppose that $A$ is positively $\frac{1}{d}$-Cartier, then our statement follows by applying this to $\mathcal{F} = \hat{M}, \hat{M}(1), \ldots, \hat{M}(d - 1)$ and $\mathcal{L} = \hat{A}(d)$, because the $\mathbb{N}$-graded algebra $B_{\geq 0} := \bigoplus_{i \geq 0} H^0(X^+, \hat{A}(di))$ is itself finite over $A^{(d)}$.

Remark 6.34. Proposition 6.33 (and its proof) is one of the major advantages of imposing the weight truncation. Namely, while $\hat{C}_I(M)$ has bounded cohomology, its cohomology groups $H^p(\hat{C}_I(M)) = \bigoplus_{i \in \mathbb{Z}} H^p(X^+, \hat{M}(i))$ are not finitely generated over $A$. This is because, while the sheaf $\bigoplus_{i \geq w} M(i)$ is finitely generated over the sheaf $\pi^\#(A_{\geq 0})$ of algebras, the sheaf $\bigoplus_{i \in \mathbb{Z}} \hat{M}(i)$ is not finitely generated over the sheaf $\pi^\#(A)$ of algebras (cf. [30, Tag 0897]).

Corollary 6.35. For each $\mathbf{ullet} \in \{+, -, b\}$, if $M \in \mathcal{D}_{\text{coh}}^{\mathbf{ullet}}(\Gr(A))$, then we have $\hat{C}_I(M)^{\mathbf{ullet}}_{[\geq w]} \in \mathcal{D}_{\text{coh}}^{\mathbf{ullet}}(\mathcal{F}_{[\geq w]})$.

Proof. Since $\hat{C}_I$ has bounded cohomological dimension, say $\hat{C}_I(\mathcal{D}^{p}(\Gr(A))) \subset \mathcal{D}^{p+m}(\Gr(A))$ and $\hat{C}_I(\mathcal{D}^{q}(\Gr(A))) \subset \mathcal{D}^{q+m}(\Gr(A))$, we have $H^p(\hat{C}_I(M)^{\#}_{[\geq w]}) \cong H^p(\hat{C}_I(\tau_{\geq p-m} M)^{\#}_{[\geq w]})$. Apply Proposition 6.33 to $\tau_{\geq p-m} M \subset \mathcal{D}_{\text{coh}}^{b}(\Gr(A))$ to conclude that $H^p(\hat{C}_I(M)^{\#}_{[\geq w]}) \in \Mod(\mathcal{F}_{[\geq w]})$ is finitely generated.

Definition 6.36. For each $\mathbf{ullet} \in \{+, -, b\}$, define the following subcategories of $\mathcal{D}(\mathcal{F}_{[\geq w]})$:

\[
\mathcal{D}_{\text{coh}}^{\mathbf{ullet}}(I^{+-\text{triv}}(\mathcal{F}_{[\geq w]}) := \mathcal{D}_{\text{coh}}^{\mathbf{ullet}}(\mathcal{F}_{[\geq w]}) \cap \mathcal{D}_{I^+-\text{triv}}(\mathcal{F}_{[\geq w]})
\]

Then Corollary 6.35 gives the following

Corollary 6.37. For each $\mathbf{ullet} \in \{+, -, b\}$, the semi-orthogonal decomposition in Theorem 6.24(4) restricts to a semi-orthogonal decomposition

\[
\mathcal{D}_{\text{coh}}^{\mathbf{ullet}}(\mathcal{F}_{[\geq w]}) = (\mathcal{D}_{\text{coh}}^{\mathbf{ullet}, I^{+-\text{triv}}}(\mathcal{F}_{[\geq w]}), \mathcal{D}_{\text{coh}}^{\mathbf{ullet}, \text{Tor}^+}(\mathcal{F}_{[\geq w]}))
\]
We wish to combine this with the semi-orthogonal decomposition \( \bullet \) to obtain a three-term semi-orthogonal decomposition on the full subcategory \( D_{\text{coh}}(\text{Gr}(A)) \). To this end, we have to show that the weight truncation functor \( L_{[w]} \), and hence \( L_{nw} \), preserve the full subcategory \( D_{\text{coh}}(\text{Gr}(A)) \subset D(\text{Gr}(A)) \). This follows from the following two lemmas:

**Lemma 6.38.** If \( M \in D_{\text{coh}}(\mathcal{F}_{[\geq w]}) \), then \( \mathcal{L}_{[\geq w]}(M) \in D_{\text{coh}}(\text{Gr}(A)) \).

**Proof.** Recall from Proposition 6.1 that \( D_{\text{coh}}(\mathcal{F}_{[\geq w]}) = D_{\text{pc}}(\mathcal{F}_{[\geq w]}) \), so that \( M \) may be represented by a bounded above complex \( P^* \) of free modules of finite rank. By (6.5), the complex \( (P^* \otimes \mathcal{F}_{[\geq w]})^\bullet \) is therefore in \( D_{\text{pc}}(\text{Gr}(A)) = D_{\text{coh}}(\text{Gr}(A)) \). \( \square \)

**Lemma 6.39.** For each \( \spadesuit \in \{ -, +, -h, b \} \), if \( M \in D_{\text{coh}}(\text{Gr}(A)) \), then \( M_{[\geq w]} \in D_{\text{coh}}(\mathcal{F}_{[\geq w]}) \).

**Proof.** By Lemma 6.40, the exact functor \( (-)^{\#}_{[\geq w]} : \text{Gr}(A) \to \text{Mod}(\mathcal{F}_{[\geq w]}) \) sends finitely generated graded modules to finitely generated modules. \( \square \)

As a result, if we let
\[
D_{\text{coh},[\geq w]}(\text{Gr}(A)) := D_{\text{coh}}(\text{Gr}(A)) \cap D_{[\geq w]}(\text{Gr}(A))
\]
\[
D_{\text{coh},<w}(\text{Gr}(A)) := D_{\text{coh}}(\text{Gr}(A)) \cap D_{<w}(\text{Gr}(A))
\]
then we have the following

**Proposition 6.40.** The full subcategory \( D_{\text{coh},[\geq w]}(\text{Gr}(A)) \subset D(\text{Gr}(A)) \) is the essential image of \( D_{\text{coh}}(\mathcal{F}_{[\geq w]}) \) under the fully faithful functor \( \mathcal{L}_{[\geq w]} : D_{\text{Gr}(A)}(\mathcal{F}_{[\geq w]}) \to D(\text{Gr}(A)) \). Moreover, the semi-orthogonal decomposition \( \bullet \) restricts to a semi-orthogonal decomposition
\[
D_{\text{coh}}(\text{Gr}(A)) = \langle D_{\text{coh},<w}(\text{Gr}(A)), D_{\text{coh},[\geq w]}(\text{Gr}(A)) \rangle
\]

Combining Corollary 6.37 and Proposition 6.40 we see that all the decomposition terms in \( \bullet \) lie in \( D_{\text{coh}}(\text{Gr}(A)) \). As a result, we have the following

**Theorem 6.41.** The semi-orthogonal decomposition in Theorem 6.22 restricts to a semi-orthogonal decomposition
\[
D_{\text{coh}}(\text{Gr}(A)) = \langle D_{\text{coh},<w}(\text{Gr}(A)), \mathcal{L}_{[\geq w]}(D_{\text{coh},<>_{w}}(\mathcal{F}_{[\geq w]})), \mathcal{L}_{[\geq w]}(D_{\text{coh},<w}(\mathcal{F}_{[\geq w]})) \rangle
\]
where the latter two semi-orthogonal components can be identified as
\[
\mathcal{L}_{[\geq w]}(D_{\text{coh},<>_{w}}(\mathcal{F}_{[\geq w]})) = \{ M \in D_{\text{coh},[\geq w]}(\text{Gr}(A)) \mid \text{R} \Gamma I^+(M) \in D_{<w}(\text{Gr}(A)) \}
\]
\[
\mathcal{L}_{[\geq w]}(D_{\text{coh},<w}(\mathcal{F}_{[\geq w]})) = D_{\text{coh},<w}(\text{Gr}(A)) \cap D_{\text{coh}}(\text{Gr}(A))
\]

Next we show that the equivalence in Theorem 6.29 restricts to an equivalence on coherent subcategories (see Theorem 6.42 below). Recall from Definition 6.33 that, for each \( \spadesuit \in \{ -, +, -h, b \} \), the full subcategory \( D_{\text{coh}}(\text{Gr}(A)) \subset D_{\text{coh}}(\text{Gr}(A)) \) is defined to be the essential image of \( D_{\text{coh}}(\text{Gr}(A)) \) under the functor \( \mathcal{C}_{I^+} : D_{\text{coh}}(\text{Gr}(A)) \to D_{\text{coh}}(\text{Gr}(A)) \). In view of Remark 6.34 and Corollary 6.35 this is the “correct” coherent subcategory to consider.

**Theorem 6.42.** For each \( \spadesuit \in \{ -, +, -h, b \} \), the equivalences in Theorem 6.24 restrict to equivalences
\[
\mathcal{C}_{I^+} \circ \mathcal{L}_{[\geq w]} : D_{\text{coh}}(\mathcal{F}_{[\geq w]}) \overset{\sim}{\longrightarrow} D_{\text{coh}}(\mathcal{F}_{[\geq w]}(\text{Gr}(A)) : (-)^{\#}_{[\geq w]}
\]

**Proof.** The fact that the functor \( \mathcal{L}_{[\geq w]} \) sends \( D_{\text{coh}}(\mathcal{F}_{[\geq w]})(\text{Gr}(A)) \) to \( D_{\text{coh}}(\mathcal{F}_{[\geq w]}(\text{Gr}(A)) \) is precisely the content of Corollary 6.35. For the other direction, notice that Lemma 6.38 establishes the statement for \( \spadesuit = - \). The general case then follows from Lemma 6.32 by using a standard truncation argument as in the proof of Corollary 6.35. \( \square \)

Now we investigate when the semi-orthogonal decomposition in Theorem 6.41 restricts to one in \( D_{\text{coh}}^b(\text{Gr}(A)) \). We make the following
Definition 6.43. The Noetherian $\mathbb{Z}$-graded ring $A$ is said to be positively regular if for all $M \in D^b_{coh}(\text{Gr}(A))$, we have $L_{[\geq w]}M \in D^b_{coh}(\text{Gr}(A))$ as well. It is said to be negatively regular if for all $M \in D^b_{coh}(\text{Gr}(A))$, we have $L_{[\leq w]}M \in D^b_{coh}(\text{Gr}(A))$ as well. It is said to be regular if it is both positively and negatively regular.

Since $(L_{[\geq w]}M)(-1) = L_{[\geq w+1]}(M(-1))$, all these notions are independent of the choice of $w \in \mathbb{Z}$.

The following is clear:

Proposition 6.44. If $A$ is positively regular, then the semi-orthogonal decomposition in Theorem 6.22 restricts to a semi-orthogonal decomposition

$$D^b_{coh}(\text{Gr}(A)) = (D^b_{coh, \leq w}(\text{Gr}(A)), L_{[\geq w]}(D^b_{coh, I^+\text{-triv}}(F_{[\geq w]})), L_{[\geq w]}(D^b_{coh, \text{Tor}^+}(F_{[\geq w]})))$$

where the latter two semi-orthogonal components can be identified as

$$L_{[\geq w]}(D^b_{coh, I^+\text{-triv}}(F_{[\geq w]})) = \{ M \in D^b_{coh, \leq w}(\text{Gr}(A)) | R^iI^+(M) \in D_{\leq w}(\text{Gr}(A)) \}$$

$$L_{[\geq w]}(D^b_{coh, \text{Tor}^+}(F_{[\geq w]})) = D^b_{coh, \text{Tor}^+}(F_{[\geq w]})(Gr(A)) := D^b_{coh, \leq w}(\text{Gr}(A)) \cap D_{\text{Tor}^+}(\text{Gr}(A))$$

6.4. A sufficient condition for regularity.

In this subsection, we provide an alternative characterization of the full subcategory $D^b_{coh, \leq w}(\text{Gr}(A))$ (see, e.g., Proposition 6.54 below). This allows us to show that our three-term semi-orthogonal decomposition in Theorem 6.41 coincides with that of [13, 7] when $A$ is smooth over a field $k$ of characteristic zero (see the discussion following Theorem 6.51 below).

We also follow the arguments of [13] to give a sufficient condition for a Noetherian $\mathbb{Z}$-graded ring $A$ to be (positively) regular in the sense of Definition 6.43 (see Theorem 6.58 below). By using Lemma 6.2 we are able to circumvent [13, Proposition 3.31] in the proof of an analogue of [13, Lemma 3.36]. This in turn allows us to weaken the assumption (A) in [13].

For any Noetherian $\mathbb{Z}$-graded ring $A$, let $\text{Gr}_{\leq w}(A) \subset \text{Gr}(A)$ be the Serre subcategory consisting of graded modules $M \in \text{Gr}(A)$ such that $M_i = 0$ for all $i \geq w$. Let $\text{gr}(A) \subset \text{Gr}(A)$ be the full subcategory of finitely generated graded modules, and let $\text{gr}_{\leq w}(A) = \text{gr}(A) \cap \text{Gr}_{\leq w}(A)$.

Throughout this subsection, we fix a graded ideal $I^+ \subset A$ such that

$$(6.45) \quad I^+ \subset I^+ \subset \sqrt{I^+}$$

Since the notion of $I^\infty$-torsion modules, $I$-trivial complexes, etc., depend only on $\sqrt{I}$, all our previous discussions remain valid if we replace $I^+$ by $I^+ \subset I'$ everywhere.

We start with the following simple

Lemma 6.46. The Serre subcategory $\text{gr}_{\leq w}(A) \subset \text{gr}(A)$ is the smallest Serre subcategory containing the essential image of $\text{gr}_{\leq w}(A/I^+)$ under the (fully faithful) functor $\text{gr}(A/I^+) \to \text{gr}(A)$.

Proof. Any module in $\text{gr}_{\leq w}(A)$ is positively torsion in the sense of Definition 8.12. Thus if $M \in \text{gr}_{\leq w}$ then let $M([I^+]^m) := \{ x \in M \mid (I^+)^m \cdot x = 0 \}$, we have an increasing filtration $0 \subset M[I^+]_1 \subset M[(I^+)^2] \subset \ldots$ whose union is $M$. Since $M$ is Noetherian, this must stabilize after finitely many terms. Since all the successive quotients in this filtration lies in $\text{gr}_{\leq w}(A/I^+)$, we have the desired result.

Corollary 6.47. The triangulated subcategory $D^b_{coh, \leq w}(\text{Gr}(A)) \subset D^b_{coh}(\text{Gr}(A))$ is the smallest triangulated subcategory of $D^b_{coh}(\text{Gr}(A))$ that contains the essential image of $D^b_{coh, \leq w}(\text{Gr}(A/I^+))$ under the functor $D^b_{coh}(\text{Gr}(A/I^+)) \to D^b_{coh}(\text{Gr}(A))$.

A completely parallel proof also shows the following

Lemma 6.48. The Serre subcategory $\text{gr}(A) \cap \text{Tor}^+(A) \subset \text{gr}(A)$ is the smallest Serre subcategory containing the essential image of $\text{gr}(A/I^+) \to \text{gr}(A)$. Therefore, the triangulated subcategory $D^b_{coh, \text{Tor}^+}(\text{Gr}(A)) \subset D^b_{coh}(\text{Gr}(A))$ is the smallest triangulated subcategory that contains the essential image of the functor $D^b_{coh}(\text{Gr}(A/I^+)) \to D^b_{coh}(\text{Gr}(A))$.

Proposition 6.49. For any $M \in D^b_{coh}(\text{Gr}(A))$, we have $M \in D^b_{coh, \leq w}(\text{Gr}(A))$ if and only if $M \otimes_A (A/I^+) \in D^b_{coh, \leq w}(\text{Gr}(A/I^+))$. 

Proof. The implication “⇒” is clear. For the converse, suppose that \( M \otimes^{L}_{B}(A/I'^{+}) \in \mathcal{D}_{[\geq w]}(\text{Gr}(A/I'^{+})) \). Since the functor \( \underline{-} \otimes^{L}_{A}(A/I'^{+}) \) is left adjoint to the restriction of scalar functor \((-)_{A} \), we have \( \mathcal{R}\text{Hom}_{A}(M, Q_{A}) \simeq 0 \) for each \( Q \in \mathcal{D}_{\text{coh}, \geq w}(\text{Gr}(A/I'^{+})) \). In view of Corollary 6.42, we have in particular \( \mathcal{R}\text{Hom}_{A}(M, K) \simeq 0 \) for each \( K \in \mathcal{D}_{\text{coh}, \geq w}(\text{Gr}(A)) \). If \( M \) is not in \( \mathcal{D}_{[\geq w], \text{coh}}(\text{Gr}(A)) \), then by Proposition 6.40, there exists a nonzero map to some \( N \in \mathcal{D}_{\text{coh}, \geq w}(\text{Gr}(A)) \), which must remain nonzero after passing to the truncation \( N \to \tau_{\geq m}(N) \) for some \( m \in \mathbb{Z} \). Since \( \tau_{\geq m}(N) \in \mathcal{D}_{\text{coh}, \geq w}(\text{Gr}(A)) \), this gives a contradiction. \( \square \)

Now suppose that \( B \) is a Noetherian \((-\mathbb{N})\)-graded ring, i.e., \( B \) is a \( \mathbb{Z} \)-graded ring such that \( B_{i} = 0 \) for all \( i > 0 \). Then we have \( I^{-}(B) = B_{\leq 0} \), and hence \( B/I^{-}(B) = B_{0} \). In this case, weight truncation can often be performed inductively:

**Lemma 6.50.** Suppose that \( M \in \mathcal{D}_{\leq w}(\text{Gr}(B)) \), then we have \( \mathcal{L}_{[\geq w-1]}(M) \cong M_{w-1}(w+1) \otimes^{L}_{B} B \).

**Proof.** Take the adjunction counit \( M_{w-1}(w+1) \otimes^{L}_{B} B \to M \), which is clearly a quasi-isomorphism in weight \( a-1 \), so that its cone lies in \( \mathcal{D}_{\leq w-1}(\text{Gr}(B)) \), and hence is equal to \( \mathcal{L}_{< w-1}(M) \), since we have \( M_{w-1}(w+1) \otimes^{L}_{B} B \in \mathcal{D}_{[\geq w]}(\text{Gr}(B)) \). \( \square \)

The functor \( \underline{-} \otimes^{L}_{B} B \) preserves both the subcategories \( \mathcal{D}_{[\geq w]} \) and \( \mathcal{D}_{\leq w} \):

**Lemma 6.51.** If \( M \in \mathcal{D}_{[\geq w]}(\text{Gr}(B)) \) then \( M \otimes^{L}_{B} B_{0} \in \mathcal{D}_{[\geq w]}(\text{Gr}(B_{0})) \).

If \( M \in \mathcal{D}_{\leq w}(\text{Gr}(B)) \) then \( M \otimes^{L}_{B} B_{0} \in \mathcal{D}_{\leq w}(\text{Gr}(B_{0})). \)

**Proof.** The first statement is obvious. For the second statement, we apply Lemma 6.50. It is clear that \( (M_{w-1}(w+1) \otimes^{L}_{B} B_{0}) \otimes^{L}_{B} B_{0} = \) concentrated in weight \( w-1 \). Thus, a repeated application of Lemma 6.50 gives a sequence of maps

\[
M = \mathcal{L}_{\leq w}(M) \to \mathcal{L}_{< w-1}(M) \to \mathcal{L}_{< w-2}(M) \to \ldots
\]

such that cone \( \mathcal{L}_{< i+1}(M) \to \mathcal{L}_{< i}(M) \otimes^{L}_{B} B_{0} \) is concentrated in weight \( i \). Thus, for any \( i < w \), the weight truncation \( \mathcal{L}_{[\geq i]}(M) = \text{cone}(M \to \mathcal{L}_{< i}(M))[-1] \) satisfies

\[
\mathcal{L}_{[\geq i]}(M) \otimes^{L}_{B} B_{0} \text{ is concentrated in weight } [i, w-1] \}

Since the sequence of maps

\[
\mathcal{L}_{[\geq w-1]}(M) \to \mathcal{L}_{[\geq w-2]}(M) \to \ldots \to M
\]

exhibits \( M \) as a homotopy colimit in \( \mathcal{D}(\text{Gr}(B)) \), and since homotopy colimit commutes with the functor \( \underline{-} \otimes^{L}_{B} B_{0} \), we have \( M \otimes^{L}_{B} B_{0} \in \mathcal{D}_{\leq w}(\text{Gr}(B_{0})). \) \( \square \)

We also have the following

**Lemma 6.52.** If \( M \in \mathcal{D}_{\text{coh}}^{\geq 0}(\text{Gr}(B)) \) is a nonzero object, then \( M \otimes^{L}_{B} B_{0} \in \mathcal{D}_{\text{coh}}^{\geq 0}(\text{Gr}(B_{0})) \) is also nonzero.

**Proof.** Take the highest nonvanishing cohomology degree \( H^{p}(M) \neq 0 \). Then by the Nakayama lemma for \( \mathbb{N} \)-graded rings, we have \( 0 \neq H^{p}(M) \otimes_{B} B_{0} = H^{p}(M \otimes^{L}_{B} B_{0}) \). \( \square \)

Combining these two lemmas, we have

**Proposition 6.53.** For any \( N \in \mathcal{D}_{\text{coh}}^{-}(\text{Gr}(B)) \), we have \( N \in \mathcal{D}_{\text{coh}, [\geq w]}(\text{Gr}(B)) \) if and only if \( N \otimes^{L}_{B} B_{0} \in \mathcal{D}_{\text{coh}, [\geq w]}(\text{Gr}(B_{0})). \)

**Proof.** The direction “⇒” is obvious. For the direction “⇐”, suppose that \( N \notin \mathcal{D}_{\text{coh}, [\geq w]}(\text{Gr}(B)) \) so that \( \mathcal{L}_{\leq w}(N) \neq 0 \). Then by Lemma 6.51, the exact triangle

\[
\ldots \to \mathcal{L}_{[\geq w]}(N) \otimes^{L}_{B} B_{0} \to N \otimes^{L}_{B} B_{0} \to \mathcal{L}_{[\leq w]}(N) \otimes^{L}_{B} B_{0} \to [1] \to \ldots
\]

is precisely the weight truncation sequence for \( N \otimes^{L}_{B} B_{0} \in \mathcal{D}(\text{Gr}(B_{0})). \) By Lemma 6.52, we have \( \mathcal{L}_{\leq w}(N) \otimes^{L}_{B} B_{0} \neq 0 \), which therefore shows that \( N \otimes^{L}_{B} B_{0} \notin \mathcal{D}_{\text{coh}, [\geq w]}(\text{Gr}(B_{0})). \) \( \square \)

**Proposition 6.54.** For any \( M \in \mathcal{D}_{\text{coh}}^{-}(\text{Gr}(A)) \), the followings are equivalent:
(1) $M \in \mathcal{D}_{\text{coh},[\geq w]}^{\oplus}(\text{Gr}(A))$
(2) $M \otimes^L_A (A/(I^- + I^+)) \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A/(I^- + I^+)))$.
(3) $M \otimes^L_A (A/\sqrt{I^- + I^+}) \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A/\sqrt{I^- + I^+}))$.

**Proof.** Take $B = A/I^+$ for $I^+ = I^+$ or $I^+ = \sqrt{I^+}$. In the former case, we have $B_0 = B/I^-(B) = A/(I^- + I^+)$. In the latter case, $B_0$ is a subring of the reduced ring $B$, and hence is reduced. In other words, $I^- + \sqrt{I^+} \subset A$ is equal to its radical, and must therefore be equal to $\sqrt{I^- + I^+}$. Thus, we have $B_0 = A/\sqrt{I^- + I^+}$, and it suffices to show that

$$M \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A)) \iff M \otimes^L_A B_0 \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(B_0))$$

for $B = A/I^+$, where $I^+$ is any graded ideal satisfying (6.55). Take $N := M \otimes^L_A B \in \mathcal{D}_{\text{coh}}^{-}(\text{Gr}(B))$. The result then follows from Propositions 6.49 and 6.53.

Now we give a sufficient condition for regularity in the sense of Definition 6.43 (see Theorem 6.58). The arguments for Lemma 6.56, Proposition 6.57, and Theorem 6.58 below are adapted from those in [13]. However, we weaken the assumption $(A)$ in loc. cit.

Take a graded ideal $I^+ \subset A$ satisfying (6.45). Consider the conditions

(a) The graded ring $B := A/I^+$ has finite Tor-dimension over the subring $B_0 \subset B$.
(b) As a quotient, the graded ring $B_0 = B/I^-(B)$ has finite Tor-dimension over $B$.
(c) If $M \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A/I^+))$ then $M \in \mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A))$.

Under the first two conditions, we have the following

**Lemma 6.56.** Suppose that (6.55) (a) holds, then $B$ is positively regular in the sense of Definition 6.43 i.e., for all $M \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$, we have $L_{<w}(M) \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$.

Suppose that (6.55) (b) holds, then for every $M \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$ there exists $i \in \mathbb{Z}$ such that $M \in \mathcal{D}_{\text{coh},[\geq i]}^{b}(\text{Gr}(B))$.

**Proof.** Since $B$ is $(-\mathbb{N})$-graded, there exists some $w' \in \mathbb{Z}$ such that $M \in \mathcal{D}_{<w'}(\text{Gr}(B))$. Apply Lemma 6.54, we see that $L_{[w'-1]}(M) \cong M_{w'-1}(-w' + 1) \otimes B_0$, which is in $\mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$ by the assumption (6.55) (a). Thus, $L_{<w'}(M) \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A))$. A repeated application of the argument then shows that $L_{<w}(M) \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A))$ for all $w \in \mathbb{Z}$.

For the second statement, the assumption (6.55) (b) guarantees that $M \otimes B_0 \in \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$. Since $B_0$ is concentrated in weight 0, any finitely generated graded module must be concentrated in finitely many weight components. Hence, $M \otimes B_0 \in \mathcal{D}_{[i]}(\text{Gr}(B))$ for some $i \in \mathbb{Z}$. By Proposition 6.53, this is precisely the sought for statement.

**Proposition 6.57.** Suppose that conditions (6.55) (a) and (6.55) (c) hold, then

(1) $\mathcal{D}_{\text{coh},[\geq w]}^{b}(\text{Gr}(A))$ is the smallest triangulated subcategory containing the essential image of the functor $\mathcal{D}_{\text{coh},[\geq w]}^{-}(\text{Gr}(A/I^+)) \rightarrow \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A))$.
(2) For any $M \in \mathcal{D}_{\text{coh},[\geq w]}^{b}(\text{Gr}(A))$, we have $L_{[w]}(M) \in \mathcal{D}_{\text{coh}}^{b}(M)$.

If (6.55) (b) also hold, then we also have

(3) for any $M \in \mathcal{D}_{\text{coh},[\geq w]}^{b}(\text{Gr}(A))$ there exists $i \in \mathbb{Z}$ such that $M \in \mathcal{D}_{\text{coh},[\geq i]}^{b}(\text{Gr}(A))$.

**Proof.** Consider the following full subcategories of $\mathcal{D}_{\text{coh}}^{b}(\text{Gr}(B))$:

$$\mathcal{E}_1 := \text{EssIm}(\mathcal{D}_{\text{coh},<w}^{b}(\text{Gr}(B)) \rightarrow \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A)))$$
$$\mathcal{E}_2 := \text{EssIm}(\mathcal{D}_{\text{coh},[\geq w]}^{b}(\text{Gr}(B)) \rightarrow \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A)))$$

For any full subcategory $\mathcal{E} \subset \mathcal{D}_{\text{coh}}^{b}(\text{Gr}(A))$, we denote by $\text{tri}(\mathcal{E})$ the smallest triangulated subcategory containing $\mathcal{E}$. In this notation, Corollary 6.47 asserts that $\text{tri}(\mathcal{E}_1) = \mathcal{D}_{\text{coh},<w}^{b}(\text{Gr}(A))$. Condition (6.55) (b) says that $\mathcal{E}_2 \subset \mathcal{D}_{\text{coh},[\geq w]}^{b}(\text{Gr}(A))$, so that $\mathcal{E}_1$ and $\mathcal{E}_2$ are strongly orthogonal in the sense of Corollary
By Lemma 6.50 we have $D^b_{coh}(\text{Gr}(B)) = \langle D^b_{\text{coh},<w}(\text{Gr}(B)), D^b_{\text{coh},[\geq w]}(\text{Gr}(B)) \rangle$ under condition (6.50a), so that

$$E := \text{EssIm}(D^b_{coh}(\text{Gr}(B)) \rightarrow D^b_{coh}(\text{Gr}(A))) \subset \langle E_1, E_2 \rangle$$

By Lemma 6.48 we have $\text{tri}(E) = D^b_{coh, Tor^+}(\text{Gr}(A))$. Combining these facts, we have

$$\text{tri}(E) \subset \text{tri}(E_1, E_2) \Rightarrow \text{tri}(E_1, \text{tri}(E_2))$$

$$\subset \langle D^b_{\text{coh},<w}(\text{Gr}(A)), D^b_{\text{coh}, Tor^+, [\geq w]}(\text{Gr}(A)) \rangle$$

$$\subset D^b_{\text{coh}, Tor^+}(\text{Gr}(A)) = \text{tri}(E)$$

Since the first and last term are the same, we must have equalities. Since semi-orthogonal components determine each other (see Lemma 6.9), the equality for the second inclusion implies that $\text{tri}(E_1) = D^b_{coh, Tor^+, [\geq w]}(\text{Gr}(A))$, which is the first sought for statement. The equality for the third inclusion is precisely the second sought for statement.

If (6.55b) holds, then applying the second statement of Lemma 6.50 together with (6.55c), we see that for every object $N \in E$ there is some $i \in \mathbb{Z}$ such that $N \in D^b_{coh, Tor^+, [\geq i]}(\text{Gr}(A))$. Since $D^b_{coh, Tor^+}(\text{Gr}(A)) = \text{tri}(E)$, every $M \in D^b_{coh, Tor^+}(\text{Gr}(A))$ also has this property. □

Theorem 6.58. Suppose that the conditions (6.55a), (b), (c) hold. Then

1. For every object $M \in D^b_{coh}(\text{Gr}(A))$, there exists some $i \in \mathbb{Z}$ such that $M \in D^b_{coh, [\geq i]}(\text{Gr}(A))$.
2. The $\mathbb{Z}$-graded ring $A$ is positively regular, i.e., for any $M \in D^b_{coh}(\text{Gr}(A))$, we have $L_{<w}(M) \subset D^b_{coh}(\text{Gr}(A))$.

Proof. Let $f_1, \ldots, f_r \in A$ be a set of elements of positive degrees $\deg(f_i) = d_i > 0$ that generate $I^+$, and let $K^\bullet(A, f_1, \ldots, f_r) = \bigwedge A(A \theta_1 \oplus \cdots \oplus A \theta_r)$ be the (cohomological) Koszul complex, which is a finite complex of free graded $A$-modules with a set $\{\land_{s \in S} \theta_s\}_{S \subset \{1, \ldots, r\}}$ of $2^r$ generators of weight $-\sum_{s \in S} d_s$ and cohomological degree $|S|$. Moreover, the differentials of the Koszul complex satisfies

$$(6.59) \quad d(K^\bullet(A, f_1, \ldots, f_r)) \subset I^+ \cdot K^\bullet(A, f_1, \ldots, f_r)$$

For any $M \in D^b_{coh}(\text{Gr}(A))$, let $K^\bullet(M, f_1, \ldots, f_r) := K^\bullet(A, f_1, \ldots, f_r) \otimes_A M$. Then (6.59) implies that

$$K^\bullet(M, f_1, \ldots, f_r) \otimes^L_A B \cong \bigoplus_{S \subset \{1, \ldots, r\}} (M \otimes^L_A B) (\sum_{s \in S} d_s)[-|S|]$$

where $B := A/I^+$. By Proposition 6.49, we therefore see that $K^\bullet(M, f_1, \ldots, f_r) \in D^b_{coh, [\geq i]}(\text{Gr}(A))$ if and only if $M \in D^b_{coh, [\geq i]}(\text{Gr}(A))$. Since we always have $K^\bullet(M, f_1, \ldots, f_r) \in D^b_{coh, Tor^+}(\text{Gr}(A))$ (see the discussion following (2.20)), the first statement of the present Theorem follows from Proposition 6.57(3).

Let $K^\bullet(A, f^1_1, \ldots, f^1_r)$ be the homological Koszul complex, i.e., it is the $A$-linear dual map of $K^\bullet(A, f^1_1, \ldots, f^1_r)\text{Tor}$. Thus, it is a finite complex of free graded $A$-modules with a set $\{\land_{s \in S} \theta^*_s\}_{S \subset \{1, \ldots, r\}}$ of $2^r$ generators of weight $-\sum_{s \in S} d_s$ and cohomological degree $-|S|$. Let $K^\bullet(M, f^1_1, \ldots, f^1_r) := M \otimes_A K^\bullet(A, f^1_1, \ldots, f^1_r)$, then by statement (1) we have just proved, we have

$$\text{cone}[M \rightarrow K^\bullet(M, f^1_1, \ldots, f^1_r)] \in D_{[\geq w]}(\text{Gr}(A)) \quad \text{for} \quad j \gg 0$$

As a result, we have $L_{<w}(M) \cong L_{<w}(K^\bullet(M, f^1_1, \ldots, f^1_r))$ for $j \gg 0$. Since $K^\bullet(M, f^1_1, \ldots, f^1_r) \in D^b_{coh, Tor^+}(\text{Gr}(A))$, the second statement follows from Proposition 6.57(2).

The following is the main class of examples of Noetherian $\mathbb{Z}$-graded rings that satisfies the conditions (6.55a), (b), (c):

Proposition 6.60. If $A$ is a $\mathbb{Z}$-graded ring finitely generated over a field $k$ of characteristic zero, and if the underlying ungraded algebra $A$ is smooth over $k$, then we have

1. The algebras $B^+ = A/\sqrt{T^+}$, $B^- = A/\sqrt{T^-}$ and $B_0 = A/\sqrt{T^+ - T^-}$ are smooth over $k$.  

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By Proposition 6.54, we may write the second component as

\[ (\rho^-)^{-1}(Z_i) = \dim(Z_i) + \dim(A). \]

Therefore, the conditions of (5.55) (a), (b), (c) are satisfied.

Proof. Notice that, in the beginning of the proof of Proposition 6.54, we have identified (6.55) and (6.62) with the semi-orthogonal decomposition \[ (a), (b), (c) \] are satisfied. First, notice that by regularity, the semi-orthogonal decomposition in Lemma 6.25 restricts to a statement follows from the first. to be smooth (hence regular), we have \[ D(A) \] an object \[ M \in D(A) \] of characteristic zero, and if the underlying ungraded algebra \[ A \] is smooth over \( k \), then

\[ (1) \quad A \text{ is regular in the sense of Definition 6.43} \]

\[ (2) \quad D^b_{\text{coh}}(\mathcal{F}_{[\geq w]}) = D_{\text{perf}}(\mathcal{F}_{[\geq w]}). \]

Proof. The first statement follows from Theorem 6.58 and Proposition 6.60. Notice that by Lemma 6.13 an object \( M \in D(\mathcal{F}_{[\geq w]}) \) is in \( D_{\text{perf}}(\mathcal{F}_{[\geq w]}) \) if and only if \( \mathcal{M}_{[\geq w]}(M) \in D_{\text{perf}}(\text{Gr}(A)) \). Since \( A \) is assumed to be smooth (hence regular), we have \( D_{\text{perf}}(\text{Gr}(A)) = D^b_{\text{coh}}(\text{Gr}(A)) \) by Theorem 2.11. Thus, the second statement follows from the first.

As a consequence, we have the following

Theorem 6.61. If \( A \) is a \( \mathbb{Z} \)-graded ring finitely generated over a field \( k \) of characteristic zero, and if the underlying ungraded algebra \( A \) is smooth over \( k \), then

\[ (1) \quad A \text{ is regular in the sense of Definition 6.43} \]

\[ (2) \quad D^b_{\text{coh}}(\mathcal{F}_{[\geq w]}) = D_{\text{perf}}(\mathcal{F}_{[\geq w]}). \]

Proof. The first statement follows from Theorem 6.58 and Proposition 6.60. Notice that by Lemma 6.13 an object \( M \in D(\mathcal{F}_{[\geq w]}) \) is in \( D_{\text{perf}}(\mathcal{F}_{[\geq w]}) \) if and only if \( \mathcal{M}_{[\geq w]}(M) \in D_{\text{perf}}(\text{Gr}(A)) \). Since \( A \) is assumed to be smooth (hence regular), we have \( D_{\text{perf}}(\text{Gr}(A)) = D^b_{\text{coh}}(\text{Gr}(A)) \) by Theorem 2.11. Thus, the second statement follows from the first.

Proposition 6.54 allows us to compare our construction of weight truncation with the ones in [13] and [7]. First, notice that by regularity, the semi-orthogonal decomposition in Lemma 6.25 restricts to a semi-orthogonal decomposition

\[ D^b_{\text{coh, Tor}^+}(\text{Gr}(A)) = \langle D^b_{\text{coh, Tor}^+}(\text{Gr}(A)), D^b_{\text{coh, Tor}^+, [\geq w]}(\text{Gr}(A)) \rangle \]

By Proposition 6.54 we may write the second component as

\[ D^b_{\text{coh, Tor}^+, [\geq w]}(\text{Gr}(A)) = \{ M \in D_{\text{Tor}^+}(\text{Gr}(A)) \mid M \otimes A(\sqrt{T^+} + T^+) \in D^b_{\text{coh, [\geq w]}(\text{Gr}(A/\sqrt{T^+} + T^+) \rangle \} \}

Comparing with [13] Definition 2.8 for \( \mathfrak{X} := [\text{Spec } A/G_m] \), we see that

\[ D^b_{\text{coh, Tor}^+, [\geq w]}(\text{Gr}(A)) = D^b_{\mathfrak{X}^w}(\mathfrak{X})_{\geq w} \]

Then, comparing (6.62) with [13] Theorem 2.10(5), we see that

\[ D^b_{\text{coh, <w}}(\text{Gr}(A)) = D^b_{\mathfrak{X}^w}(\mathfrak{X})_{< w} \]

Finally, if we compare the three term semi-orthogonal decomposition in Proposition 6.44 with the corresponding one in [13] Theorem 2.10(6), then we see that

\[ D^b_{\mathfrak{X}^w}(\mathfrak{X})_{\geq w} \]

This shows that the three-term semi-orthogonal decomposition of [13], and hence of [7], coincides with the one in Theorem 6.43 in the abelian case.

Moreover, [13] Lemma 3.37] also gives a semi-orthogonal decomposition

\[ D^b(\mathfrak{X}) = \langle D^b(\mathfrak{X})_{< w}, D^b(\mathfrak{X})_{\geq w} \rangle \]

Comparing this with Lemma 6.20 (or rather its restriction to \( D^b_{\text{coh}}(-) \)), we see that

\[ \{ M \in D^b_{\text{coh}}(\text{Gr}(A)) \mid \text{RG}_{\mathfrak{T}+}(M) \in D_{< w}(\text{Gr}(A)) \} = D^b_{\text{coh}}(\mathfrak{X})_{< w} \]

We have seen in Proposition 6.60 that \( \rho = \rho : \text{Spec } B^+ \to \text{Spec } B_0 \) is a locally trivial bundle of weighted affine spaces. i.e., we have \( B^+ = \text{Sym}_{B_0}(\mathcal{E}^+) \) for a projective graded module \( \mathcal{E}^+ = \oplus_{j>0} \mathcal{E}^+_j \) over \( B_0 \). Write \( \text{Spec } B_0 \) as a union of its connected components \( \text{Spec } B_0 = \bigcup Z_i \), i.e., \( B_0 = \prod B_0^{(i)} \). Let \( \eta_i^+ := \sum_j j \cdot \text{rank}(\mathcal{E}^+_j |_{Z_i}) \geq 0 \), so that \( \eta_i^+ > 0 \) if \( \rho^{-1}(Z_i) \neq Z_i \).
Proposition 6.64. Given $M \in D^b_{coh}(\text{Gr}(A))$, then $R\Gamma_{I^+}(M) \in D_{<w}(\text{Gr}(A))$ if and only if $M \otimes_{\mathbb{L}}^L B_0^{(i)} \in D_{<w}^+(\text{Gr}(B_0^{(i)}))$ for each $i$.

Proof. By (6.63), the first condition can be written as $M \in D^b(X)_{<w}$, so that the claimed equivalence follows from [13 Proposition 3.31] (see also the discussion preceding [13 Corollary 3.39]).

Corollary 6.65. In the situation of Proposition 6.64, let $\eta^+$ be defined as above. Then we have $\left. R\Gamma_{I^+}(A) \right|_j = 0$ for all $j > -\min\{\eta^+\}$.

6.5. The case of non-affine base. Now we consider weight truncation for pairs $(Y, A)$, i.e., in the setting (4.1). As in Section 4, it suffices to construct the relevant weight truncation functors that reduces to the ones above over any open affine subscheme Spec $R \subset Y$. Then the properties of such functors can be checked locally. We start with the following

Definition 6.66. Let $\mathcal{D}_{[\geq w]}(\text{Gr}(A)) \subset D(\text{Gr}(A))$ be the smallest strictly full triangulated subcategory closed under small coproducts, and containing the object of the form

\[(6.67) \quad F \otimes_{\mathbb{C}_Y} A(-i), \quad \text{where} \quad F \in D_{\text{perf}}(\text{QCoh}(Y)) \quad \text{and} \quad i \geq w\]

Clearly, each of the objects of the form (6.67) is compact in $D(\text{Gr}(A))$, and hence also in $D_{[\geq w]}(\text{Gr}(A))$. Thus, $D_{[\geq w]}(\text{Gr}(A))$ is compactly generated, and the inclusion functor $D_{[\geq w]}(\text{Gr}(A)) \rightarrow D(\text{Gr}(A))$ preserves small coproducts. By the Brown-Neeman representability theorem [25, Theorem 4.1], this inclusion therefore has a right adjoint, which will be denoted as

\[(6.68) \quad L_{[\geq w]} : D(\text{Gr}(A)) \rightarrow D_{[\geq w]}(\text{Gr}(A))\]

Since the full triangulated subcategory $D_{[\geq w]}(\text{Gr}(A)) \subset D(\text{Gr}(A))$ is right admissible, there exists a semi-orthogonal decomposition (see Proposition 5.12) of the form

\[(6.69) \quad D(\text{Gr}(A)) = (D_{<w}(\text{Gr}(A)), D_{[\geq w]}(\text{Gr}(A)))\]

where $D_{<w}(\text{Gr}(A)) := D_{[\geq w]}(\text{Gr}(A))^\perp$. Alternatively, it can be characterized as follows:

Lemma 6.70. An object $M \in D(\text{Gr}(A))$ is in $D_{<w}(\text{Gr}(A))$ if and only if its $i$-th weight component $M_i \in D(\text{QCoh}(Y))$ is zero for all $i \geq w$.

Proof. For any $M \in D(\text{Gr}(A))$, we have $M \in D_{[\geq w]}(\text{Gr}(A))^\perp$ if and only if

\[(6.71) \quad \text{Hom}_{D(\text{Gr}(A))}(F \otimes_{\mathbb{C}_Y} A(-i), M[j]) = 0 \quad \text{for all} \quad F \in D_{\text{perf}}(\text{Gr}(A)), i \geq w \quad \text{and} \quad j \in \mathbb{Z}\]

Indeed, by the simple fact (6.72) below, applied to $\mathcal{D} := D(\text{Gr}(A))$ and $X := M$, we see that the objects in (6.67) are in $^\perp(\Sigma M)$ if and only if $D_{[\geq w]}(\text{Gr}(A)) \subset ^\perp(\Sigma M)$.

Suppose $\mathcal{D}$ is a triangulated category that admits small coproducts. Then for any $X \in \mathcal{D}$, the full subcategory $^\perp(\Sigma X) := \{ Y \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(Y, X[i]) = 0 \quad \text{for all} \quad i \in \mathbb{Z} \}$ is a triangulated subcategory that is closed under small coproducts.

Notice that we have

\[\text{Hom}_{D(\text{Gr}(A))}(F \otimes_{\mathbb{C}_Y} A(-i), M[j]) \cong \text{Hom}_{D(\text{QCoh}(Y))}(F, M_i)\]

Recall that $D(\text{QCoh}(Y))$ is compactly generated by $D_{\text{perf}}(\text{QCoh}(Y))$ (see, e.g., [25 Corollary 2.3, Proposition 2.5] and [9, Theorem 3.1.1]). The result therefore follows from the characterization (6.71) of $D_{[\geq w]}(\text{Gr}(A))^\perp$. \hfill \square

Applying Proposition 5.12 again to the semi-orthogonal decomposition (6.69), we see that the inclusion $D_{<w}(\text{Gr}(A)) \rightarrow D(\text{Gr}(A))$ has a left adjoint, which we denote as

\[(6.73) \quad L_{<w} : D(\text{Gr}(A)) \rightarrow D_{<w}(\text{Gr}(A))\]

Apply (6.72) to the case $\mathcal{D} := D_{[\geq w]}(\text{Gr}(A))$. If each of the objects (6.67) is in $^\perp(\Sigma X)$, then we have $^\perp(\Sigma X) = D_{[\geq w]}(\text{Gr}(A))$, so that $X = 0$. This shows compact generation.
For any $\mathcal{M} \in \mathcal{D}(\text{Gr}(\mathcal{A}))$, the semi-orthogonal decomposition \[^{6.69}\] then gives us a decomposition sequence

\[
\ldots \rightarrow \mathcal{L}_{[\geq w]}(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow \mathcal{L}_{<w}(\mathcal{M}) \rightarrow \mathcal{L}_{[\geq w]}(\mathcal{M})[1] \rightarrow \ldots
\]

Given any open affine subscheme $U = \text{Spec} \mathcal{R} \subset Y$, let $\mathcal{A} := \mathcal{A}(U)$. If $\mathcal{M}$ is of the form \[^{6.67}\] then its restriction to $U$ has the form $\mathcal{K} \otimes_{\mathcal{F}} \mathcal{A}(-i)$, where $\mathcal{F} \in \mathcal{D}_{\text{perf}}(\mathcal{R})$ and $i \geq w$. Since $\mathcal{R}$ split generates $\mathcal{D}_{\text{perf}}(\mathcal{R})$, we see that these are all contained in $\mathcal{D}_{[\geq w]}(\text{Gr}(\mathcal{A}))$. Thus, we have

\[
\mathcal{D}_{[\geq w]}(\text{Gr}(\mathcal{A}))[U] \subset \mathcal{D}_{[\geq w]}(\text{Gr}(\mathcal{A}))
\]

By Lemma \[^{6.70}\] we also have

\[
\mathcal{D}_{<w}(\text{Gr}(\mathcal{A}))[U] \subset \mathcal{D}_{<w}(\text{Gr}(\mathcal{A}))
\]

Therefore the restriction of \[^{6.74}\] to $U$ becomes precisely the first row of \[^{6.11}\]. This allows us to verify properties of weight truncations locally.

Combining the weight truncation sequence \[^{6.74}\] with the local cohomology sequence \[^{1.5}\], this allows us to extend \[^{6.24}\] to the case of non-affine base:

\[
\begin{array}{cccc}
\mathcal{L}_{[\geq w]}R\Gamma_{\mathcal{F}+}(\mathcal{M}) & \xrightarrow{\mathcal{L}_{[\geq w]}(r,\mathcal{M})} & \mathcal{L}_{[\geq w]}\mathcal{M} & \xrightarrow{\text{unit}} \mathcal{M} \\
\mathcal{L}_{[\geq w]}\hat{\mathcal{C}}_{\mathcal{F}+}(\mathcal{M}) & \xleftarrow{[1]} & \mathcal{L}_{[\geq w]}(\eta_{\mathcal{M}}) & \xleftarrow{[1]} \mathcal{L}_{<w}(\mathcal{M})
\end{array}
\]

which gives the following generalization of \[^{6.30}\] to the non-affine case:

\[
\mathcal{D}(\text{Gr}(\mathcal{A})) = \langle \mathcal{D}_{<w}(\text{Gr}(\mathcal{A})), \mathcal{L}_{[\geq w]}(\mathcal{D}_{\mathcal{F}+,-\text{triv}}(\text{Gr}(\mathcal{A}))), \mathcal{D}_{\text{Tor}+,[\geq w]}(\text{Gr}(\mathcal{A})) \rangle
\]

where the component in the middle is the essential image of the functor $\mathcal{L}_{[\geq w]} : \mathcal{D}_{\mathcal{F}+,-\text{triv}}(\text{Gr}(\mathcal{A})) \rightarrow \mathcal{D}(\text{Gr}(\mathcal{A}))$, which is fully faithful with left quasi-inverse $\hat{\mathcal{C}}_{\mathcal{F}+}$. Alternatively, it may be described as

\[
\mathcal{L}_{[\geq w]}(\mathcal{D}_{\mathcal{F}+,-\text{triv}}(\text{Gr}(\mathcal{A}))) = \{ \mathcal{M} \in \mathcal{D}_{[\geq w]}(\text{Gr}(\mathcal{A})) | R\Gamma_{\mathcal{F}+}(\mathcal{M}) \in \mathcal{D}_{<w}(\text{Gr}(\mathcal{A})) \}
\]

7. Birational cobordisms

In this section, we remind the notion of a birational cobordism. This gives a natural context in which pairs $(Y, \mathcal{A})$ as in \[^{4.11}\] arises. This allows us to apply all our previous techniques to birational cobordisms.

Let $k$ be a field of characteristic zero, and $X$ a quasi-projective variety over $k$, acted on by a reductive algebraic group $G$. A $G$-linearized invertible sheaf $L \in \text{Pic}^G(X)$ is said to be $G$-effective if it is ample and if $X^{ss}(L) \neq \emptyset$. For any $G$-effective $L \in \text{Pic}^G(X)$, let $Y(L) := X^{ss}(L)/G$ be the categorical quotient (see \[^{21}\]). Thus $Y(L)$ is a quasi-projective variety equipped with a $G$-equivariant affine morphism $\pi : X^{ss}(L) \rightarrow Y(L)$, where $G$ acts trivially on $Y(L)$. As a result, there exists a sheaf $\mathcal{A}$ of $G$-algebra over $Y(L)$ such that $X^{ss}(L) = \text{Spec}Y(L)(\mathcal{A})$ as a $G$-variety. Moreover, there exists an ample invertible sheaf $M$ on $Y(L)$ such that $\pi^*(M) \cong L^\otimes n$ for some $n > 0$.

The open subset $X^{ss}(L) \subset X$ is preserved under certain changes in $L$. First of all, it is preserved under $G$-algebraic equivalence (see \[^{33}\] Proposition 2.1), so that it is well-defined for $L \in N^{G}(X)$. It is also preserved under the change $L \rightarrow L^\otimes_m$ for any $m > 0$, so that it is well-defined for $L \in N^{G}(X)_Q := N_{G}(X) \otimes \mathbb{Q}$, which is a finite dimensional vector space over $\mathbb{Q}$.

As $L$ varies in the finite dimensional vector space $N_{G}(X)_Q$, the open subset $X^{ss}(L) \subset X$, and hence the space $X^{ss}(L)/G$, changes. For many classes of $G$-varieties $X$ (see, e.g., \[^{10}\] \[^{33}\]), there is in fact a finite stratification of $N_{G}(X)_Q$ by connected subsets (called chambers and cells) such that if two points $L, L' \in N_{G}(X)_Q$ lie in the same strata, then we have $X^{ss}(L) = X^{ss}(L')$. In particular, there are only finitely many possible such GIT quotient.

We shall see this phenomenon in the simplest case when $G = \mathbb{G}_m$ and when $L$ varies by twisting by characters of $\mathbb{G}_m$. More precisely, given $L \in \text{Pic}^G_m(X)$, and any $t \in \mathbb{Z}$, let $L(t) \in \text{Pic}^G_m(X)$ be the $\mathbb{G}_m$-linearized invertible sheaf obtained by twisting the linearization by the character $z \mapsto z^t$ of $\mathbb{G}_m$. This operation $L \mapsto L(t)$ preserves $\mathbb{G}_m$-algebraic equivalence, and hence descends to an operation on $N_{G}^m(X)$. 
Moreover, if we descend further to $\text{NS}_m^G(X)_\mathbb{Q}$, then the twist $L(t)$ is well-defined for any $t \in \mathbb{Q}$. In this case, the variations of GIT quotients admit an elementary description.

Indeed, let $X$ be a projective-over-affine variety, and $L \in \text{Pic}^m(X)$ an ample $G_m$-linearized invertible sheaf on $X$. Let $R$ be the $\mathbb{N}$-graded ring defined by $R_m := H^0(X, L^\otimes m)$, then there is a canonical isomorphism $\varphi : X \xrightarrow{\cong} \text{Proj}(R)$ since $X$ is assumed to be projective-over-affine. Since $L$ is $G_m$-linearized, each of the vector spaces $H^0(X, L^\otimes m)$ admits an algebraic $G_m$-action, and is therefore $\mathbb{Z}$-graded. As a result, the algebra $R$ is in fact $\mathbb{N} \times \mathbb{Z}$-graded.

If we change $L$ to its twist $L(t)$ by a character $t \in \mathbb{Z}$, then $R$ is unchanged as an $\mathbb{N}$-graded algebra, but its $\mathbb{Z}$-grading changes. Indeed, by the convention \[\text{(4.3)},\] we see that the $\mathbb{Z}$-grading on $R_m$ is changed so that $H^0(X, L(t)^\otimes m) = H^0(X, L^\otimes m)(mt)$. In particular, if $f \in R$ is an element with bidegree $(m, n) \in \mathbb{N} \times \mathbb{Z}$ according to the $\mathbb{Z}$-grading given by $L$, then it has bidegree $(m, n - mt)$ according to the $\mathbb{Z}$-grading given by $L(t)$. Indeed, recall that the shift $M \mapsto M(mt)$ decreases degrees of elements by $mt$.

From now on, we always endow $R$ with the $\mathbb{Z}$-grading given by $L$, and write $R = \bigoplus_{(m,n) \in \mathbb{N} \times \mathbb{Z}} R_{m,n}$. Then the above discussion shows that

\[(7.1) \quad X^{ss}(L(t)) = \{ x \in X \mid \text{there exists } f \in R_{m,n}, m > 0, \text{ such that } n = mt \text{ and } f(x) \neq 0 \} \]

By replacing $L(t)$ by $L(t)^{\otimes d}$ if necessary, we see that \[\text{(7.1)}\] holds for $t \in \mathbb{Q}$ as well.

Choose elements $f_1, \ldots, f_r$ of $R$, homogeneous in both gradings, that generate $R$ over $R_{0,0}$. Suppose $f_i$ has bidegree $(m_i, n_i) \in \mathbb{N} \times \mathbb{Z}$. Let

\[\mathcal{S} = \{ S \subset \{1, \ldots, r\} \mid \text{there exists } i \in S \text{ such that } m_i > 0 \} \]

For each $t \in \mathbb{Q}$, let

\[\mathcal{S}_{\geq t} := \{ S \in \mathcal{S} \mid n_i > mt \text{ for all } i \in S \} \]

\[\mathcal{S}_{< t} := \{ S \in \mathcal{S} \mid n_i < mt \text{ for all } i \in S \} \]

\[\mathcal{S}_t := \mathcal{S} \setminus (\mathcal{S}_{\geq t} \cup \mathcal{S}_{< t}) \]

Clearly, the two subsets $\mathcal{S}_{\geq t}$ and $\mathcal{S}_{< t}$ of $\mathcal{S}$ are disjoint. We have the following elementary

**Lemma 7.3.** For any $S = \{i_1, \ldots, i_p\} \in \mathcal{S}$, we have $S \in \mathcal{S}_t$ if and only if there exists $e_{i_1}, \ldots, e_{i_p} \geq 0$ such that $\sum_{j=1}^p m_{i_j} e_{i_j} > 0$ and $\sum_{j=1}^p (n_{i_j} - tm_{i_j}) e_{i_j} = 0$.

**Proof.** The direction “$\Rightarrow$” is clear. For the direction “$\Leftarrow$”, suppose that $S \in \mathcal{S}_t$. By assumption, there exists some element, say $i_t \in S$, such that $m_{i_t} > 0$. There are three cases:

1. $n_{i_t} - tm_{i_t} = 0$. Then take $(e_{i_1}, \ldots, e_{i_p}) = (1, 0, \ldots, 0)$, and we arrive at the conclusion.

2. $n_{i_t} - tm_{i_t} > 0$. Then by the assumption $S \in \mathcal{S}_t$, there is some other elements, say $i_2 \in S$, such that $n_{i_2} - tm_{i_2} < 0$. Suppose that $t > 0$ is an integer such that $te \in \mathbb{Z}$. Then let $e_{i_2} = -e(n_{i_2} - tm_{i_2})$ and $e_{i_t} = e(n_{i_t} - tm_{i_t})$, and all other $e_{i_j}$ taken to be zero. The conclusion is then clearly satisfied.

3. $n_{i_t} - tm_{i_t} < 0$. This is completely symmetric to case (2).

\[\Box\]

For each $S \in \mathcal{S}_t$, let $X_S = \bigcap_{i \in S} X_{f_i} = X_{f_S}$, where $f_S = \prod_{i \in S} f_i$. Notice that $f_S$ has $\mathbb{N}$-grading $m_S = \sum_{i \in S} m_i$, which is positive since $S \in \mathcal{S}_t$.

**Proposition 7.4.** For any $t \in \mathbb{Q}$, we have $X^{ss}(L(t)) = \bigcup_{S \in \mathcal{S}_t} X_S$.

**Proof.** We use the characterization \[\text{(7.1)}\] of $X^{ss}(L(t))$. Any element $f \in R_{m,n}$ is an $R_{0,0}$-linear combination of monomials in $f_1, \ldots, f_r$. Thus the element $f$ in \[\text{(7.1)}\] may be taken to be such a monomial. In other words, we have

\[X^{ss}(L(t)) = \left\{ x \in X \mid \text{there exists } \{i_1, \ldots, i_p\} \subset \{1, \ldots, r\} \text{ with } e_{i_1}, \ldots, e_{i_p} \geq 0, \text{ such that } \sum_{j=1}^p e_{i_j} m_{i_j} > 0 \text{ and } \sum_{j=1}^p (n_{i_j} - tm_{i_j}) e_{i_j} = 0, \text{ and } f_i(x) \neq 0 \right\} \]

for each $j = 1, \ldots, p$.

By Lemma \[\text{(7.3)},\] the right hand side is precisely $\bigcup_{S \in \mathcal{S}_t} X_S$.

\[\Box\]
Let $T \subset \mathbb{Q}$ be the finite set of values of the form $n_i/m_i$, for some $m_i > 0$. Then the subsets of $\mathcal{S}$ changes only when $t$ passes through one of the values in $T$. By Proposition \ref{prop:7.4}, the open subset $X^{ss}(L(t))$ therefore remains unchanged on each of the intervals of $\mathbb{Q} \setminus T$.

**Definition 7.5.** For rational numbers $t_1 < t_2$, both not in $T$, the data $(X, L, t_1, t_2)$ is said to be a birational cobordism from $X_1 := X^{ss}(L(t_1))//\mathbb{G}_m$ to $X_2 := X^{ss}(L(t_2))//\mathbb{G}_m$.

If $[t^-, t^+] \cap T = \{t_0\}$, for some rational numbers $t^- < t_0 < t^+$, then the data $(X, L, t^-, t^+, t_0)$ is said to be a wall-crossing in birational cobordism.

The birational cobordism is said to be smooth if the variety $X$ is smooth.

We now investigate wall-crossings in birational cobordisms. Thus, let $[t^-, t^+] \subset T = \{t_0\}$, for some rational numbers $t^- < t_0 < t^+$. It is clear that

\[ \mathcal{S}_{t^-} \supset \mathcal{S}_{t^0} = \mathcal{S}_{t^+} \quad \text{and} \quad \mathcal{S}_{t^0} = \mathcal{S}_{t^0} \subset \mathcal{S}_{t^+} \]

so that $\mathcal{S}_{t^0} \subset \mathcal{S} \supset \mathcal{S}_{t^+}$. By Proposition \ref{prop:7.4}, we therefore have

\[ X^{ss}(L(t^-)) \subset X^{ss}(L(t_0)) \supset X^{ss}(L(t^+)) \]

Let $W := X^{ss}(L(t_0))$, and let $Y$ be the categorical quotient $Y := W//\mathbb{G}_m$. By construction (see \cite{24}), the morphism $\varphi : W \to Y$ is affine. Moreover, there exists an ample invertible sheaf $M$ on $Y$ such that $\varphi^*(M) \cong L(t_0)^{\otimes d}$ for some $d > 0$. Since $\varphi$ is affine, we have $W \cong \text{Spec}_Y(\varphi_*O_W)$. The $\mathbb{G}_m$-action on $W$ is such that $\varphi_*O_Y$ with the structure of a sheaf of $\mathbb{Z}$-graded algebra, so that we have a pair $(Y, A)$ as in \ref{def:7.1}. Moreover, this pair satisfies $A_0 = O_Y$.

By the isomorphism $\varphi^*(M) \cong L(t_0)^{\otimes d}$, we see that $W^{ss}(L(t)) = W^{ss}(\varphi^*(M)((t-t_0)d))$. Moreover, by \ref{eq:7.6}, we have $W^{ss}(L(t)) = X^{ss}(L(t))$ for $t = t^\pm$. As a result, we may identify the families

\[ \{ X^{ss}(L(t)) \}_{t^-, t^0 \leq t \leq t^+} = \{ W^{ss}(\varphi^*(M)(\epsilon)) \}_{(t-t_0)d \leq \epsilon \leq (t^+-t_0)d} \]

via the change of variables $\epsilon = (t-t_0)d$.

Since $W = \text{Spec}_Y A$, we see that

\[
\begin{align*}
W^{ss}(\varphi^*(M)(\epsilon)) &= (\text{Spec}_Y A) \setminus V(\mathcal{S}^+) \quad \text{for any } \epsilon > 0 \\
W^{ss}(\varphi^*(M)(\epsilon)) &= (\text{Spec}_Y A) \setminus V(\mathcal{S}^-) \quad \text{for any } \epsilon < 0
\end{align*}
\]

As a result, we have the following

**Proposition 7.7.** The varieties \ref{eq:7.6} can be identified with

\[ (\text{Spec}_Y A) \setminus V(\mathcal{S}^+) \subset (\text{Spec}_Y A) \supset (\text{Spec}_Y A) \setminus V(\mathcal{S}^-) \]

As a consequence, the stacky GIT quotients can be identified with the stacky projective spaces (see Remark \ref{rem:7.2})

\[ \left( \begin{array}{c} X^{ss}(L(t^-))//\mathbb{G}_m \\ Y \end{array} \right) \cong \left( \begin{array}{c} \text{Proj}^-_Y(A) \\ Y \end{array} \right) \]

\[ \left( \begin{array}{c} X^{ss}(L(t^+))//\mathbb{G}_m \\ Y \end{array} \right) \cong \left( \begin{array}{c} \text{Proj}^+_Y(A) \\ Y \end{array} \right) \]

while the scheme-theoretic GIT quotients can be identified with the projective spaces

\[ \left( \begin{array}{c} X^{ss}(L(t^-))//\mathbb{G}_m \\ Y \end{array} \right) \cong \left( \begin{array}{c} \text{Proj}_Y(A) \\ Y \end{array} \right) \]

\[ \left( \begin{array}{c} X^{ss}(L(t^+))//\mathbb{G}_m \\ Y \end{array} \right) \cong \left( \begin{array}{c} \text{Proj}^+_Y(A) \\ Y \end{array} \right) \]

Finally, we give a definition which imitates the condition that \ref{eq:7.5} be a log flip.

**Definition 7.10.** A wall-crossing in birational cobordism is said to be small if the relatively unstable loci have codimensions $\geq 2$. i.e., if both the closed subsets

\[ X^{ss}(L(t_0)) \setminus X^{ss}(L(t^\pm)) \subset X^{ss}(L(t_0)) \]

have codimension $\geq 2$. 

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8. Derived categories under flips and flops

In this section, we put together all our previous techniques in order to study the change in the derived category under homological flip/flop.

To illustrate the ideas, let’s first consider the case of affine base \( Y = \text{Spec} \, R \). We will focus on pre-stable pseudo-rational homological flips/flops, so that by Corollary 6.78 we have

(i) \( A \) is Gorenstein.

(ii) There is an isomorphism \( R\Gamma_I^+(A)(a)[1] \cong \mathcal{D}_Y(R\Gamma_I^-(A)) \) in \( \mathcal{D}(\text{Gr}(A)) \), where \( a \geq 0 \)

We will use this property to relate the derived categories

\[
\mathcal{D}^b_{\text{coh}(\mathcal{F}[\leq w])} \quad \text{and} \quad \mathcal{D}^b_{\text{coh}(\mathcal{F}[\geq w])}
\]

These are subcategories of \( \mathcal{D}^b_{\text{coh}(\mathcal{F}[\leq w])} \) and \( \mathcal{D}^b_{\text{coh}(\mathcal{F}[\geq w])} \) respectively. Notice that, by Proposition 6.42 (and its negative version), these are equivalent respectively to

\[
\mathcal{D}^b_{\text{coh}, I- \text{triv}(\mathcal{F}[1 \leq w])} \quad \text{and} \quad \mathcal{D}^b_{\text{coh}, I+ \text{triv}(\mathcal{F}[1 \geq w])}
\]

When applied to complexes of the form \( M^I_{[w]} \), for \( M \in \mathcal{D}(\text{Gr}(A)) \), this functor has an alternative expression

\[
(R\text{Hom}_{\mathcal{F}[\geq w]}(-, \mathcal{F}[\geq w]))^\sim \cong (R\text{Hom}_A(L_{[w]}(M), A))_{[1 \leq w]}
\]

Thus, we are led to consider the functor

\[
\mathcal{D}_A : \mathcal{D}(\text{Gr}(A))^\text{op} \rightarrow \mathcal{D}(\text{Gr}(A))
\]

Proposition 8.4. Suppose that \( \mathcal{S} \mathcal{L} \) (ii) holds. If \( M \in \mathcal{D}^b_{\text{coh}}(\text{Gr}(A)) \) satisfies \( R\Gamma_I^+(M) \in \mathcal{D}_{<w}(\text{Gr}(A)) \), then we have \( R\Gamma_I^-(\mathcal{D}_A(M)) \in \mathcal{D}_{>w+a}(\text{Gr}(A)) \).

Proof. Since \( \mathcal{D}_Y \) is involutive on complexes with locally coherent cohomology, condition \( \mathcal{S} \mathcal{L} \) (ii) can be rewritten as an isomorphism \( R\Gamma_I^-(A) \cong \mathcal{D}_Y(R\Gamma_I^+(A))(-a)[-1] \). Then we have

\[
R\text{Hom}_A(M, A) \otimes_L^A R\Gamma_I^-(A) \cong R\text{Hom}_A(M, R\Gamma_I^-(A))
\]

\[
\cong R\text{Hom}_A(M, \mathcal{D}_Y(R\Gamma_I^+(A))(-a)[-1])
\]

\[
\cong R\text{Hom}_A(M \otimes_L^A R\Gamma_I^+(A), \mathcal{D}_Y(A)(-a)[-1])
\]

\[
= \mathcal{D}_Y(R\Gamma_I^+(M))(-a)[-1]
\]

where the first quasi-isomorphism uses Proposition 2.14.

Corollary 8.5. Suppose that \( \mathcal{S} \mathcal{L} \) (ii) holds. Then the functor \( \mathcal{S} \mathcal{L} \) sends the subcategory \( \mathcal{D}^e_{\text{coh}, I+ \text{triv}(\mathcal{F}[\geq w])} \subset \mathcal{D}(\mathcal{F}[\leq w]) \) to the subcategory \( \mathcal{D}^e_{\text{coh}, I- \text{triv}(\mathcal{F}[\leq w])} \subset \mathcal{D}(\mathcal{F}[\geq w]) \).

Proof. Any \( M \in \mathcal{D}^e_{\text{coh}, I+ \text{triv}(\mathcal{F}[\geq w])} \) can be written as \( M = M^I_{[w]} \), where \( M := L_{[w]}(M) \) satisfies (see Theorem 6.41)

\[
M \in \mathcal{D}^e_{\text{coh}, [w]}(\text{Gr}(A)) \quad \text{and} \quad R\Gamma_I^+(M) \in \mathcal{D}_{<w}(\text{Gr}(A))
\]

Taking its dual \( \mathcal{D}_A(M) \), we see by Proposition 8.4 that

\[
\mathcal{D}_A(M) \in \mathcal{D}^e_{\text{coh}}(\text{Gr}(A)) \quad \text{and} \quad R\Gamma_I^-(\mathcal{D}_A(M)) \in \mathcal{D}_{>w+a}(\text{Gr}(A)) \subset \mathcal{D}_{>w}(\text{Gr}(A))
\]

Taking the restriction \( (-)^{[1]_{[w]} \leq w} \), we see by the negative versions of Lemma 6.39 and Proposition 6.19 that

\[
\mathcal{D}_A(M)^{[1]_{[w]} \leq w} \in \mathcal{D}^e_{\text{coh}}(\mathcal{F}[\leq w]) \quad \text{and} \quad R\Gamma_I^-[\leq w]((\mathcal{D}_A(M)^{[1]_{[w]} \leq w}) = 0
\]
which is precisely the statement that $\mathbb{D}_A(M)^{f \leq -w} \in \mathcal{D}_c^{+} \tau_{-\text{triv}}(\mathcal{F}_{\leq -w})$. Since $M^{f \leq -w} \cong M$ and $L_{[\geq w]} M = M$, we see by (8.3) that

$$(\mathcal{R} \text{Hom}_{\mathcal{F}_{[\geq w]}}(M, \mathcal{F}_{[\geq w]}))^{T} \cong \mathbb{D}_A(M)^{f \leq -w} \in \mathcal{D}_c^{+} \tau_{-\text{triv}}(\mathcal{F}_{\leq -w})$$

\[ \square \]

**Lemma 8.6.** Suppose that (8.1) (i) holds, and $A$ is positively regular in the sense of Definition 6.43, then the functor (8.2) sends the subcategory $\mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]}) \subset \mathcal{D}(\mathcal{F}_{[\geq w]})$ to the subcategory $\mathcal{D}_c^{b}(\mathcal{F}_{[\leq -w]}) \subset \mathcal{D}(\mathcal{F}_{[\leq -w]})$.

**Proof.** Given $M \in \mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]})$, let $M := L_{[\geq w]}(M)$. Then by positive regularity, we have $M \in \mathcal{D}_c^{b}(\text{Gr}(A))$. By the assumption (8.1)(i), we have $\mathbb{D}_A(M) \in \mathcal{D}_c^{b}(\text{Gr}(A))$, so that $\mathbb{D}_A(M)^{f \leq -w} \in \mathcal{D}_c^{b}(\mathcal{F}_{[\leq -w]})$. Since $M^{f \leq -w} \cong M = L_{[\geq w]}(M)$, the result follows from (8.3). \[ \square \]

For every $M \in \mathcal{D}(\mathcal{F}_{[\geq w]})$, there is a canonical map

$$M \rightarrow \mathcal{R} \text{Hom}(\mathcal{F}_{[\geq w]}^{\text{op}}(\mathcal{R} \text{Hom}(\mathcal{F}_{[\geq w]}(M, \mathcal{F}_{[\geq w]}))$$

in $\mathcal{D}(\mathcal{F}_{[\geq w]})$. We say that $M$ is derived reflexive if this map is an isomorphism. One can define the derived reflexivity of $M \in \mathcal{D}(\mathcal{F}_{[\geq w]}^{\text{op}})$ in a similar way.

**Proposition 8.7.** Suppose that the following holds:

1. Conditions (8.1) (i) (ii).
2. $A$ is positively regular in the sense of Definition 6.43.
3. Every object $M \in \mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]})$ is derived reflexive.

then the functor (8.2) restricts to a fully faithful functor

$$(\mathcal{R} \text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}_{[\geq w]}))^{T} : (\mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]}))^{\text{op}} \rightarrow \mathcal{D}_c^{b}(\mathcal{F}_{[\leq -w]})$$

**Proof.** The fact that (8.2) restricts to a functor (8.8) follows from Corollary 8.5 and Lemma 8.6. Fully faithfulness follows from Lemma A.14. \[ \square \]

We now focus on the case of homological flop, i.e., the case $a = 0$. In this case, Proposition 8.7 can be strengthened to a derived equivalence:

**Proposition 8.9.** Suppose that the following holds:

1. Conditions (8.1) (i), and (8.1) (ii) for $a = 0$.
2. $A$ is regular in the sense of Definition 6.43.
3. Every object $M \in \mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]})$ and $M' \in \mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]}^{\text{op}})$ are derived reflexive.

then the functor (8.2) restricts to an exact equivalence (8.8).

**Proof.** By Lemma 8.6 as well as its negative version, the functor (8.2), as well as its negative version, restricts to give functors

$$(\mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]}))^{\text{op}} \xrightarrow{\mathcal{R} \text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}_{[\geq w]})^{T}} \mathcal{D}_c^{b}(\mathcal{F}_{[\leq -w]})$$

These functors are quasi-inverse equivalences by assumption (3). By Corollary 8.5, as well as its negative version, both of these functors preserve the subcategories $\mathcal{D}_c^{b}(\mathcal{F}_{\geq w}^{+ \text{triv}})$, and hence restrict to equivalences for these subcategories. \[ \square \]

**Remark 8.10.** We expect that Condition (3) in Proposition 8.9 follow from condition (1) and (2).

For example, an analogue of [20] Tag 0A68 show that (3) holds if $\mathcal{F}_{[\geq w]}$ have finite left and right injective dimensions. Now, condition (1) and (2) guarantees that the functors $\mathcal{R} \text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}_{[\geq w]})^{\text{op}}$ and $\mathcal{R} \text{Hom}_{\mathcal{F}_{[\geq w]}}(\mathcal{F}_{[\geq w]})^{\text{op}}$ preserve the subcategories $\mathcal{D}_c^{b}(\mathcal{F}_{[\geq w]}^\text{op})$. This last condition seems very close to $\mathcal{F}_{[\geq w]}$ having finite left and right injective dimensions, but we have not been able to prove this implication.
A related point is that, by the assumption (8.11)(i), condition (3) would hold if $D_A$ sends $D^{b}_{\text{coh}, \geq w}(\text{Gr}(A))$ to $D_{\geq -w}(\text{Gr}(A))$, and $D^{b}_{\text{coh}, \leq -w}(\text{Gr}(A))$ to $D_{\geq w}(\text{Gr}(A))$ (see, e.g., [8.18] below). However, while $D_A$ sends the compact generators $\{A(-i)\}_{i \geq w}$ of $D_{\geq w}(\text{Gr}(A))$ to the compact generators $\{A(i)\}_{i \geq w}$ of $D(\text{Gr}(A))$ of $D_{\geq -w}(\text{Gr}(A))$, it may not send $D_{\geq w}(\text{Gr}(A))$ to $D_{\leq -w}(\text{Gr}(A))$ since the latter may not be closed under homotopy limits.

We now give classes of examples in which the assumptions of Proposition 8.7 and 8.9 are satisfied. We will work in the smooth case, although we believe that this assumption may often be relaxed.

Let $k$ be a field of characteristic zero, and let $A$ be a $\mathbb{Z}$-graded ring finitely generated over $k$. Assume the followings:

(8.11)  
(1) The underlying ungraded algebra $A$ is smooth over $k$.
(2) The closed subset $V(I^-)$ and $V(I^+)$ of Spec $A$ both have codimension $\geq 2$.

Then by Proposition 5.24, we see that, for $Y:=\text{Spec } A_0$, the pair $(Y, A)$ determines a Cohen-Macaulay log flip. In order to apply Theorem 5.72 to obtain examples of homological flips/flops, we impose the following extra conditions:

(8.12) There exists $a > 0$ such that $\widetilde{A(a)}_{X^\pm} \cong \mathcal{O}(K_{X^\pm})$, and $A_a$ is reflexive as a module over $A_0$.

(8.13) The variety $Y = \text{Spec } A_0$ is quasi-Gorenstein, i.e., $K_Y$ is Cartier.

By Theorem [6.72] a $\mathbb{Z}$-graded algebra satisfying (8.11) and the first condition of (8.12) gives rise to a homological flip; while one satisfying (8.11) and (8.13) gives rise to a homological flop. We now proceed to prove that these homological flips/flops are always pseudo-rational and pre-stable.

Recall from Proposition 6.60 that both of the maps

$$\text{Spec } B^- \xrightarrow{\varphi^-} \text{Spec } B_0 \xleftarrow{\varphi^+} \text{Spec } B^+$$

are locally trivial bundles of weighted affine spaces. As in the discussion preceding Proposition 6.60, write $B^- = \text{Sym}_{B_0}(\mathcal{E}^+)$ for a projective graded module $\mathcal{E}^+ = \oplus_{j > 0} \mathcal{E}^+_j$ over $B_0$, and similarly write $B^+ = \text{Spec}_{B_0}(\mathcal{E}^-)$, for a projective graded module $\mathcal{E}^- = \oplus_{j < 0} \mathcal{E}^-_j$. Write Spec $B_0$ as a union of its connected components Spec $B_0 = \bigcup Z_i$, i.e., $B_0 = \prod B_0^{(i)}$, and define

$$\eta_i^\pm := \sum_j j \cdot \text{rank}(\mathcal{E}^\pm_j |_{Z_i})$$

(8.14)

In terms of the vector bundles $\mathcal{E}^\pm$, Proposition 6.60(3) can be rewritten as

$$\text{rank}(\mathcal{E}^- |_{Z_i}) + \text{rank}(\mathcal{E}^+ |_{Z_i}) + \text{dim}(Z_i) = \text{dim}(A)$$

or equivalently

$$\text{rank}(\mathcal{E}^- |_{Z_i}) = \text{dim}(A) - \text{dim}((\rho^-)^{-1}(Z_i))$$
$$\text{rank}(\mathcal{E}^+ |_{Z_i}) = \text{dim}(A) - \text{dim}((\rho^+)^{-1}(Z_i))$$

By the assumption (8.11)(2), we therefore have $\text{rank}(\mathcal{E}^- |_{Z_i}) \geq 2$ and $\text{rank}(\mathcal{E}^+ |_{Z_i}) \geq 2$, so that they are in particular non-trivial, and hence

$$\eta_i^+ > 0 \quad \text{and} \quad \eta_i^- < 0$$

By Corollary 6.65 as well as its negative version, we therefore have

$$R\Gamma_{\varphi^+}(A)_j = 0 \quad \forall \ j \geq 0 \quad \text{and} \quad R\Gamma_{\varphi^-}(A)_j = 0 \quad \forall \ j \leq 0$$

(8.15)

which can be used to prove the following

**Theorem 8.16.**

(1) Any $\mathbb{Z}$-graded algebra $A$ over $k$ satisfying (8.11) and (8.12) gives rise to a pre-stable and pseudo-rational homological flip. As a result, the assumptions of Proposition 8.7 are satisfied.
(2) Any $\mathbb{Z}$-graded algebra $A$ over $k$ satisfying (8.11) and (8.13) gives rise to a pre-stable and pseudo-rational homological flop. As a result, the assumptions of Proposition 5.9 are satisfied.

Proof. Let $Y := \text{Spec} \ A_0$. As we have seen above, the pair $(Y, A)$ determines a Cohen-Macaulay log flip by Proposition 5.23. Thus, by Theorem 5.72, a $\mathbb{Z}$-graded algebra satisfying (8.11) and the first condition of (8.12) gives rise to a homological flip; while one satisfying (8.11) and (8.13) gives rise to a homological flop. The condition (8.15) for $j \neq 0$ says that $A$ is pre-stable. Moreover, the same condition at weight $j = 0$ can be rewritten as

$$\pi^+_*(O_{X^+}) = O_Y \quad \text{and} \quad R^j\pi^+_* (O_{X^+}) = 0 \text{ for all } i > 0 \tag{8.17}$$

By Proposition 5.21 both $X^-$ and $X^+$ have rational singularities. Thus, (8.17) implies that $Y$ also has rational singularities. Thus, the second paragraph of Theorem 5.72 implies that these homological flips/flops are pseudo-rational.

In both cases, condition (1) in Proposition 8.24 (resp. 8.3) then follows from an application of Corollary 5.78; while the conditions (2),(3) follow from Theorem 6.61.

We now reformulate Propositions 8.9 and 8.7 so that it does not involve any duality functor, and does not involve the small preadditive category $\mathcal{C}$. This reformulation will allow a generalization of these results to the case of non-affine base. We first rewrite the functor (8.2) in terms of the full subcategories $\mathcal{D}_{[≥w]}(\text{Gr}(A))$ and $\mathcal{D}_{(≥w]}(\text{Gr}(A))$. Namely, by the equivalences

$$\mathcal{D}(\mathcal{C}_{≥w}) \xrightarrow{\mathcal{L}_{≥w}|_{≥w}} \mathcal{D}([≥w]) \xrightarrow{\mathcal{L}_{≥w}|_{≤w}} \mathcal{D}([≥w])$$

and by (8.3), the following diagram of functors are commutative up to isomorphism of functors:

$$\mathcal{D}_{[≥w]}(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}_{[≥w]}(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}_{[≥w]}(\text{Gr}(A))$$

and by (8.3), the following diagram of functors commutes up to isomorphism of functors:

$$\mathcal{D}_{[≥w]}(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w]) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w])$$

so that we may focus on the first row of this diagram, and neglect the small preadditive category $\mathcal{C}_{[≥w]}$.

Consider the subcategory (see (6.23) and (6.30), or rather their restrictions to $\mathcal{D}_{[≥w]}$), with the conditions (2),(3) follow from Theorem 6.61.

When the functor $\mathcal{D}_A$ is applied to objects $M$ in this subcategory, Proposition 8.4 shows that, under the assumption (8.1) (ii), we have $\mathcal{L}_{[≥w]}(\mathcal{D}_A(M)) = 0$, or equivalently

$$\mathcal{L}_{[≥w]}(\mathcal{D}_A(M)) = \mathcal{L}_{[≥w]}(\mathcal{D}_A(M))$$

Thus, the following diagram of functors commutes up to isomorphism of functors:

$$\mathcal{D}_{[≥w]}(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w]) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w])$$

which gives an alternative way to express Corollary 8.5.

Combining (8.18) and (8.19), we have the diagram

$$\mathcal{D}_{[≥w]}(\text{Gr}(A)) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w]) \xrightarrow{\mathcal{L}_{[≥w]|[≥w]}|_{[≥w]}} \mathcal{D}([≥w])$$

(8.20)
where the vertical equivalences are due to Theorem 6.42. Since the second row is the candidate functor that relate the derived categories in Propositions 8.9 and 8.7, we may focus our attention on the first row.

Now, we consider the composition $\tilde{\mathcal{C}}_f \circ \mathbb{D}_A$ appearing in the first row of (8.20). Recall that the functor $\tilde{\mathcal{C}}_f \circ \mathbb{D}_A$, for any graded ideal $I \subset A$, was studied at the end of Section 2.2. In particular, we have shown in Lemma 2.62 that the composition

$$D_{\text{coh}}(\text{Gr}(A)) \to D_{\text{coh}(I-\text{triv})}(\text{Gr}(A))$$

can be rewritten as the composition

$$(8.21) \quad D_{\text{coh}}(\text{Gr}(A)) \xrightarrow{\tilde{\mathcal{C}}_f} D_{\text{coh}(I-\text{triv})}(\text{Gr}(A)) \xrightarrow{\mathbb{D}_{\text{coh}(I-\text{triv})}} D_{\text{coh}(I-\text{triv})}(\text{Gr}(A))$$

Now we assume that $\tilde{\mathcal{C}}_f(A)$ has finite injective dimension. Recall from Lemma 2.60 that this follows from the assumption (8.1)(i). By Proposition 2.64, the duality functor $\mathbb{D}_{\tilde{\mathcal{C}}_f(A)}$ appearing in (8.21) is then an equivalence of categories, which moreover restricts to an anti-autoequivalence on the subcategory $D_{\text{coh}(I-\text{triv})}(\text{Gr}(A))$. We summarize this discussion into the following

**Proposition 8.22.** The following diagram commutes up to isomorphism of functors:

$$(8.23) \quad D_{\text{coh}}(\text{Gr}(A)) \xrightarrow{\tilde{\mathcal{C}}_f} D_{\text{coh}(I-\text{triv})}(\text{Gr}(A)) \xrightarrow{\mathbb{D}_{\text{coh}(I-\text{triv})}} D_{\text{coh}(I-\text{triv})}(\text{Gr}(A))$$

where the vertical arrows are equivalences of categories.

Moreover, if $\tilde{\mathcal{C}}_f(A)$ has finite injective dimension (which holds under the assumption (8.1)(i)), then the functor $\mathbb{D}_{\tilde{\mathcal{C}}_f(A)}$ appearing in the first row is also an equivalence of categories. As a result, we have in this case

1. The functor (8.21) restricts to a functor (8.8) if and only if the functor $\tilde{\mathcal{C}}_f \circ \mathcal{L}_{[\geq w]}$ restricts to a functor

   $$(8.24) \quad \tilde{\mathcal{C}}_f \circ \mathcal{L}_{[\geq w]} : D_{\text{coh}(I-\text{triv})}(\text{Gr}(A)) \to D_{\text{coh}(I-\text{triv})}(\text{Gr}(A))$$

2. If (1) holds, then the restriction (8.8) is fully faithful (resp. an equivalence) if and only if the restriction (8.24) is.

In the above discussion, we have seen that the functor in the second row of (8.23) often restrict to fully faithful functor or exact equivalences on the subcategories of bounded cohomologies. By Proposition 8.22, so does the functor $\tilde{\mathcal{C}}_f \circ \mathcal{L}_{[\geq w]}$, which is well-defined in the case when the base $Y$ is not necessarily affine. This supports the following ansatz that relates the derived categories under homological flips/flops:

**Ansatz 8.25.** Let $(Y, \omega_Y^\bullet, A, a, \Phi^-, \Phi^+)$ be a pseudo-rational homological flop (resp. flip) such that the pair $(Y, A)$ is pre-stable in the sense of Definition 5.3. Then the functor

$$(8.26) \quad D_{\text{coh}([\mathcal{F}+\text{triv}])(\text{Gr}(A))} \xrightarrow{\mathcal{L}_{[\geq w]\circ \mathcal{L}_{[\geq w]}(\mathcal{D}_{\text{coh}([\mathcal{F}+\text{triv}])(\text{Gr}(A)))}} \mathcal{D}_{\text{coh}([\mathcal{F}+\text{triv}])(\text{Gr}(A))}$$

tends to restrict to a functor

$$(8.27) \quad \tilde{\mathcal{C}}_f \circ \mathcal{L}_{[\geq w]} : D_{\text{coh}([\mathcal{F}+\text{triv}])(\text{Gr}(A))} \to D_{\text{coh}([\mathcal{F}+\text{triv}])(\text{Gr}(A))}$$

and this restriction tends to be an equivalence (resp. fully faithful functor).

**Remark 8.28.** Ansatz (8.25) is closely related to [13, Ansatz 4.11]. In fact, it generalizes it. i.e., if [13, Ansatz 4.11] holds, then so does Ansatz (8.25).

Our previous discussion then gives the following instance in which Ansatz (8.25) is satisfied:
Theorem 8.29. Let \((Y, \mathcal{A})\) be a pair satisfying \(\mathcal{A}_0 = \mathcal{O}_Y\). Suppose that Spec\(_Y\) \(\mathcal{A}\) is a smooth variety over \(k\), such that the closed subsets \(V(\mathcal{I}^-)\) and \(V(\mathcal{I}^+)\) of Spec\(_Y\) \(\mathcal{A}\) both have codimension \(\geq 2\). Then

(1) Suppose that there exists a \(a > 0\) such that \(\tilde{\mathcal{A}}(a)_{X^\pm} \cong \mathcal{O}_{X^\pm}\), and \(A_a\) is a reflexive sheaf on \(Y\), then the functor \((8.20)\) restricts to a fully faithful functor \((8.27)\).

(2) Suppose that the normal variety \(Y\) is quasi-Gorenstein, i.e., \(K_Y\) is Cartier, then the functor \((8.20)\) restricts to an equivalence \((8.27)\).

Notice that condition (2) in Theorem 8.29 is automatically satisfied if either \(X^-\) or \(X^+\) is Calabi-Yau. Thus, we have the following

Corollary 8.30. Let \((X, L, t^-, t^+)\) be a wall-crossing in a smooth birational cobordism (see Definition 7.3) that is small in the sense of Definition 7.10. Suppose that the scheme-theoretic GIT quotients \(X^{ss}(L(t^\pm)) / \mathbb{G}_m\) are quasi-Calabi-Yau (i.e., have trivial canonical divisors), then the corresponding stacky GIT quotients \((7.8)\) are derived equivalent. i.e., there is an exact equivalence

\[
D^b_{coh}([X^{ss}(L(t^+)) / \mathbb{G}_m]) \cong D^b_{coh}([X^{ss}(L(t^-)) / \mathbb{G}_m])
\]

Remark 8.31. (1) We expect that the smoothness condition in Theorem 8.29 can be relaxed. Such a relaxation will be important in relating the derived categories under flips/flops. This will be investigated in the future.

(2) We expect that Corollary 8.30 can be generalized to other types of variations of GIT quotients. This will be investigated in the future.

Appendix A. Modules over preadditive categories

A preadditive category is a category \(\mathcal{A}\) enriched over the monoidal category \((\text{Ab}, \otimes)\) of abelian groups. It is said to be small if the objects of \(\mathcal{A}\) form a set \(\text{Ob}(\mathcal{A})\). It is helpful to think of a small preadditive category as an “associative ring with many objects”, as in [22]. This allows us to define the notions of left/right modules, tensor products, Hom spaces, etc, which we recall now.

Given a small preadditive category \(\mathcal{A}\), a left \(\mathcal{A}\)-module is an additive functor \(\mathcal{A} \to \text{Ab}\), while a right \(\mathcal{A}\)-module is an additive functor \(\mathcal{A}^{op} \to \text{Ab}\). Maps between left or right modules are simply natural transformations. We will mostly work with right modules, and we denote the category of right \(\mathcal{A}\)-modules by \(\text{Mod}(\mathcal{A})\). In more concrete terms, a right \(\mathcal{A}\)-module associates an abelian group \(M_a\) to each \(a \in \text{Ob}(\mathcal{A})\), together with maps \(M_a \otimes \mathcal{A}(a', a) \to M_{a'}\), satisfying the obvious associativity and unitality conditions.

Given small preadditive categories \(\mathcal{A}\) and \(\mathcal{B}\), an \((\mathcal{A}, \mathcal{B})\)-bimodule consists of a collection \(M(b, a) = aM_b\) of abelian groups, one for each pair \(a \in \text{Ob}(\mathcal{A})\) and \(b \in \text{Ob}(\mathcal{B})\), together with maps \(\mathcal{A}(a, a') \otimes aM_b \otimes \mathcal{B}(b', b) \to a'M_{b'}\), satisfying the obvious associativity and unitality conditions. For example, \(\mathcal{A}\) can be canonically a bimodule over itself. Denote by \(_{\mathcal{A}}\text{Mod}_{\mathcal{B}}\) the category of \((\mathcal{A}, \mathcal{B})\)-bimodules.

If \(M \in \_{\mathcal{A}}\text{Mod}_{\mathcal{B}}\) and \(N \in \_{\mathcal{B}}\text{Mod}_{\mathcal{C}}\), then define \(M \otimes_{\mathcal{B}} N \in \_{\mathcal{A}}\text{Mod}_{\mathcal{C}}\) by

\[
_{\mathcal{A}}M \otimes_{\mathcal{B}} N := \bigoplus_{b \in \text{Ob}(\mathcal{B})} aM_b \otimes bN_c / (\xi f \otimes \eta - \xi \otimes f \eta)
\]

where we mod out the abelian subgroup generated by the displayed relations, for \(\xi \in aM_b\), \(f \in \mathcal{B}(b, b')\), and \(\eta \in bN_c\). In particular, if \(\mathcal{A} = \mathcal{C} = \ast\) is the preadditive category with one object, with endomorphism algebra \(\mathbb{Z}\), then this gives the notion of a tensor product \(M \otimes_{\mathcal{B}} N \in \text{Ab}\) between a right \(\mathcal{B}\)-module \(M\) and a left \(\mathcal{B}\)-module \(N\).

Similarly, if \(M \in \_{\mathcal{A}}\text{Mod}_{\mathcal{B}}\) and \(N \in \_{\mathcal{C}}\text{Mod}_{\mathcal{B}}\), then we define \(_{\mathcal{A}}\text{Hom}_{\mathcal{B}}(M, N) \in \_{\mathcal{C}}\text{Mod}_{\mathcal{A}}\) by

\[
_{\mathcal{A}}\text{Hom}_{\mathcal{B}}(M, N)_a := \{ (\varphi_b)_{b \in \text{Ob}(\mathcal{B})} \in \text{Hom}_{\text{Ab}}(aM_b, cN_d) \mid \varphi_b(\xi f) = \varphi_{b'}(\xi) f \ \forall \xi \in aM_b, f \in \mathcal{B}(b, b') \}
\]

In other words, \(_{\mathcal{A}}\text{Hom}_{\mathcal{B}}(M, N)_a\) is the Hom-space \(_{\mathcal{A}}\text{Hom}_{\mathcal{B}}(aM, cN)\) in the (big) additive category \(\text{Mod}(\mathcal{B})\).
As for usual associative algebras, there are canonical isomorphisms

\[(A.3) \quad M \otimes_B B \cong M \cong A \otimes_A M \quad \text{and} \quad \text{Hom}_B(B, M) \cong M\]

For any \( M \in \mathcal{A}\text{Mod}_B \), \( N \in \mathcal{B}\text{Mod}_C \), and \( L \in \mathcal{C}\text{Mod}_C \), there is a usual Hom-tensor adjunction, given by the canonical isomorphism of \((\mathcal{E}, \mathcal{A})\)-bimodules

\[(A.4) \quad \text{Hom}_C(M \otimes_B N, L) \cong \text{Hom}_B(M, \text{Hom}_C(N, L))\]

For each \( a \in \text{Ob}(\mathcal{A}) \), denote by \( \mathcal{A}_a := \mathcal{A}(a, a) \). A right module \( M \) is said to be free if there is an indexed set of objects \( \varphi : S \to \text{Ob}(\mathcal{A}) \), together with an isomorphism \( M \cong \bigoplus_{s \in S} (\varphi(s)) \mathcal{A} \). In more concrete terms, this means that there is a set \( S \) of elements \( \xi_s \in M_{\varphi(s)} \) such that, for any \( a \in \text{Ob}(\mathcal{A}) \), any element \( \xi \in M_a \) can be written uniquely as a finite sum \( \xi = \sum \xi_s f_s \), for \( f_s \in \mathcal{A}(a, \varphi(s)) \). The cardinality of \( S \) is said to be the \textit{rank} of the free module \( M \).

Clearly the category \( \text{Mod}(\mathcal{A}) \) of right modules is an abelian category, where limits and colimits are determined objectwise. Thus, it also satisfies the usual \text{Ab5} and \text{Ab3*} axioms of an abelian category. Moreover, the set \( \{ \mathcal{A}_a \}_{a \in \text{Ob}(\mathcal{A})} \) of right modules forms a set of generators for \( \text{Mod}(\mathcal{A}) \), so that \( \text{Mod}(\mathcal{A}) \) is a Grothendieck category (see, e.g., \([30\text{, Tag 079B}]\)). Projective objects in \( \text{Mod}(\mathcal{A}) \) are precisely retracts of free modules.

A projective right module is said to be of \textit{finite rank} if it is a retract of a free module of finite rank.

**Definition A.5.** We say that a right module \( M \in \text{Mod}(\mathcal{A}) \) is \textit{finitely generated} if there is an epimorphism \( \bigoplus_{s \in S} (\varphi(s)) \mathcal{A} \to M \) for a finite indexed set of objects \( \varphi : S \to \text{Ob}(\mathcal{A}) \). In more concrete terms, this means that there is a finite set \( S \) of elements \( \xi_s \in M_{\varphi(s)} \) such that, for any \( a \in \text{Ob}(\mathcal{A}) \), any element \( \xi \in M_a \) can be written as a finite sum \( \xi = \sum \xi_s f_s \), for \( f_s \in \mathcal{A}(a, \varphi(s)) \).

**Definition A.6.** A small preadditive category \( \mathcal{A} \) is said to be \textit{right Noetherian} (resp. \textit{left Noetherian}) if every submodule of a finitely generated right (resp. left) \( \mathcal{A} \)-module is finitely generated. It is said to be \textit{Noetherian} if it is both left and right Noetherian.

Since \( \text{Mod}(\mathcal{A}) \) is a Grothendieck category, it has enough injectives (see, e.g., \([30\text{, Tag 079H}]\)). Moreover, complexes in \( \text{Mod}(\mathcal{A}) \) admit K-injective resolutions (see, e.g., \([30\text{, Tag 079P}]\)). The category \( \text{Mod}(\mathcal{A}) \) clearly has enough projectives. Thus, by \([31\text{, Theorem 3.4}]\) (see also \([30\text{, Tag 06XX}]\)), complexes in \( \text{Mod}(\mathcal{A}) \) admit K-projective resolutions. This allows us to take derived functors of the above Hom functors and tensor functors. However, a subtlety arises when one wants to take the derived tensor product or the derived Hom bimodule between bimodules. For example, even if \( M \in \mathcal{A}\text{Mod}_B \) is projective in the category \( \mathcal{A}\text{Mod}_B \), it may not be true that each \( \mathcal{A}_a M \in \text{Mod}(\mathcal{B}) \) is projective, or even flat, so that it might be problematic if one wants to derive \((\mathcal{A}, 1)\) and \((\mathcal{A}, 2)\) naively. However, for our purposes, we will only need to consider the derived tensor product (or derived Hom) between a module and a bimodule. For these, there are no problems, and we may define

\[(A.7) \quad - \otimes^L_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}\text{Mod}_B) \to \mathcal{D}(\mathcal{B})\]

\[\text{RHom}_{\mathcal{A}}(\mathcal{A}, \mathcal{B}) : \mathcal{D}(\mathcal{A})^{\text{op}} \times \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}^{\text{op}})\]

In particular, given an additive functor \( F : \mathcal{A} \to \mathcal{B} \), then \( \mathcal{B} \) may be viewed as an \((\mathcal{A}, \mathcal{B})\)-bimodule in the obvious way, so that the extension functor

\[(A.8) \quad - \otimes^L_{\mathcal{A}} \mathcal{B} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})\]

is well-defined. Moreover, it satisfies the usual extension-restriction adjunction

\[
\text{RHom}_B(M \otimes^L_{\mathcal{A}} \mathcal{B}, N) \cong \text{RHom}_A(M, N)
\]

for any \( M \in \mathcal{D}(\mathcal{A}) \) and \( N \in \mathcal{D}(\mathcal{B}) \).

**Definition A.9.** An object \( M \in \mathcal{D}(\mathcal{A}) \) is said to be \textit{pseudo-coherent} if it is quasi-isomorphic to a bounded above complex \( F^\bullet \) of projective modules of finite rank. Denote by \( \mathcal{D}_{\text{pc}}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}) \) the full subcategory consisting of pseudo-coherent objects.
**Definition A.10.** Suppose \( \mathcal{A} \) is right Noetherian, then denote by \( \mathcal{D}^-_{\text{coh}}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}) \) the full subcategory consisting of objects \( M \in \mathcal{D}(\mathcal{A}) \) such that each \( H^p(M) \) is finitely generated, and \( H^p(M) = 0 \) for \( p \gg 0 \).

**Proposition A.11.** Suppose \( \mathcal{A} \) is right Noetherian, then for any \( M \in \mathcal{D}(\mathcal{A}) \), the followings are equivalent:

1. \( M \in \mathcal{D}_{\text{perf}}(\mathcal{A}) \);
2. \( M \in \mathcal{D}^-_{\text{coh}}(\mathcal{A}) \);
3. \( M \) is quasi-isomorphic to a bounded above complex of free modules of finite rank.

**Proof.** Clearly only the implication (2) \( \Rightarrow \) (3) needs proof. It follows from the well-known Lemma A.12 below.

**Lemma A.12.** Let \( \mathcal{C} \) be an abelian category, and let \( \mathcal{P} \subset \text{Ob}(\mathcal{C}) \) be a collection of projective objects closed under finite direct sum. Denote by \( Q(\mathcal{P}) \subset \text{Ob}(\mathcal{C}) \) the collection of objects \( M \) such that there exists an epimorphism \( P \to M \) from some \( P \in \mathcal{P} \). Suppose \( Q(\mathcal{P}) \) is closed under taking subobjects, then for any bounded above complex \( M^\bullet \) in \( \mathcal{C} \) whose cohomology objects lie in \( Q(\mathcal{P}) \), there exists a bounded above complex \( P^\bullet \) of objects in \( \mathcal{P} \), together with a quasi-isomorphism \( \varphi : P^\bullet \to M^\bullet \).

**Proof.** Assume that \( M^i = 0 \) for \( i > b \). Choose an epimorphism \( P^b \to M^b/d(M^b-1) \), and lift it to a map \( P^b \to M^b \). Suppose there is a complex \( P^\bullet \) of objects of \( \mathcal{P} \) concentrated in degrees \([a,b]\), together with a map \( \varphi : P^\bullet \to M^\bullet \), such that \( H^i(P^\bullet) \to H^i(M^\bullet) \) is an isomorphism for \( i > a \) and is surjective for \( i = a \). The last condition guarantees that the map

\[(A.13) \quad (\varphi,-d) : Z^a(P^\bullet) \oplus (M^{a-1}/d(M^{a-2})) \to Z^a(M^\bullet)\]

is surjective. Notice that \( Q(\mathcal{P}) \) is a Serre subcategory, so that \( \mathcal{D}_{Q(\mathcal{P})}(\mathcal{C}) \) is a triangulated subcategory. Since the two term complex \([M^{a-1}/d(M^{a-2}) \to Z^a(M^\bullet)] \) has cohomology objects \( H^{a-1}(M^\bullet) \) and \( H^a(M^\bullet) \), and since the map \( A.13 \), thought of as a two term complex, is an extension of it by \( Z^a(P^\bullet) \in Q(\mathcal{P}) \), we see that the kernel \( K \) of \( A.13 \) is in \( Q(\mathcal{P}) \) as well. Choose an epimorphism \( P' \to K \) with \( P' \in \mathcal{P} \). Then we have a commutative diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{d'} & Z^a(P^\bullet) \\
\psi \downarrow & & \varphi \downarrow \\
M^{a-1}/d(M^{a-2}) & \xrightarrow{d} & Z^a(M^\bullet)
\end{array}
\]

such that the induced map \( \varphi : Z^a(P^\bullet)/d'(P') \to Z^a(M^\bullet)/d(M^{a-1}) = H^a(M^\bullet) \) is an isomorphism. Moreover, choose an epimorphism \( \psi'' : P'' \to H^{a-1}(M^\bullet) \subset M^{a-1}/d(M^{a-2}) \) from some \( P'' \in \mathcal{P} \), and consider the commutative diagram

\[
\begin{array}{ccc}
P' \oplus P'' & \xrightarrow{(d',0)} & Z^a(P^\bullet) \\
(\psi',\psi'') \downarrow & & \varphi \downarrow \\
M^{a-1}/d(M^{a-2}) & \xrightarrow{d} & Z^a(M^\bullet)
\end{array}
\]

Let \( P^{a-1} := P' \oplus P'' \), and lift the map \((\psi',\psi'')\) to a map \( \varphi : P^{a-1} \to M^{a-1} \). This completes the desired inductive step. \( \square \).

Clearly, the set \( \{a\mathcal{A}\}_{a \in \text{Ob}(\mathcal{A})} \) forms a set of compact generators of \( \mathcal{D}(\mathcal{A}) \). Denote by \( \mathcal{D}_{\text{perf}}(\mathcal{A}) \) the smallest split-closed triangulated subcategory of \( \mathcal{D}(\mathcal{A}) \) containing the set \( \{a\mathcal{A}\}_{a \in \text{Ob}(\mathcal{A})} \) of objects, then it is a standard fact (see, e.g., [29] Theorem 4.22 or [26] Lemma 2.2) that

\[(A.14) \quad \mathcal{D}(\mathcal{A})_c = \mathcal{D}_{\text{perf}}(\mathcal{A})\]

where the subscript \((-)_c\) denotes the subcategory of compact objects.
Now we investigate the reflexivity property of modules. For any object \( M \in D(\mathcal{A}) \), denote by 
\[
\mathbb{D}_\mathcal{A}(M) := \mathcal{R}\text{Hom}_\mathcal{A}(M, \mathcal{A}) \in D(\mathcal{A}^{\text{op}}).
\]
This gives a functor \( \mathbb{D}_\mathcal{A}: D(\mathcal{A})^{\text{op}} \to D(\mathcal{A}^{\text{op}}) \). Moreover, then there is a canonical map
\begin{equation}
A.15 \quad M \to \mathbb{D}_\mathcal{A}(\mathbb{D}_\mathcal{A}(M))
\end{equation}
in the derived category \( D(\mathcal{A}) \), which is moreover natural in \( M \in D(\mathcal{A}) \). Indeed, one suffices to take a K-projective resolution \( P^\bullet \xrightarrow{\sim} M \) in \( \text{Ch}(\text{Mod}(\mathcal{A})) \), and a K-projective resolution \( \tilde{Q}^\bullet \xrightarrow{\sim} (P^\bullet)^\vee \) in \( \text{Ch}(\text{Mod}(\mathcal{A}^{\text{op}})) \). The map \( A.15 \) is then the induced map \( P^\bullet \to (\tilde{Q}^\bullet)^\vee \), which can be shown to be independent of choices, and is natural in \( M \).

**Definition A.16.** An object \( M \in D(\mathcal{A}) \) is said to be derived reflexive if the map \( A.15 \) is an isomorphism in \( D(\mathcal{A}) \).

Then we have the following

**Lemma A.17.** Suppose that \( M \in D(\mathcal{A}) \) is derived reflexive, then for any \( N \in D(\mathcal{A}) \), the functoriality map
\[
\mathcal{R}\text{Hom}_\mathcal{A}(N, M) \to \mathcal{R}\text{Hom}_\mathcal{A}(\mathbb{D}_\mathcal{A}(M), \mathbb{D}_\mathcal{A}(N))
\]
is a quasi-isomorphism.

**Proof.** We have the following series of adjunctions:
\[
\mathcal{R}\text{Hom}_{\mathcal{A}^{\text{op}}}(\mathbb{D}_\mathcal{A}(M), \mathbb{D}_\mathcal{A}(N)) \simeq \mathcal{R}\text{Hom}_{\mathcal{A}^{\text{op}}}(\mathbb{D}_\mathcal{A}(M) \otimes N, \mathcal{A}) \\
\simeq \mathcal{R}\text{Hom}_\mathcal{A}(N, \mathcal{R}\text{Hom}_{\mathcal{A}^{\text{op}}}(\mathbb{D}_\mathcal{A}(M), \mathcal{A})) \\
\simeq \mathcal{R}\text{Hom}_\mathcal{A}(N, M)
\]
where we use the derived reflexivity of \( M \) in the last step. A direct inspection shows that this quasi-isomorphism is precisely the functoriality map. \( \square \)

**Appendix B. Reminders on derived categories**

**Definition B.1.** Given full subcategories \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) of a triangulated category \( D \), we say that \( (\mathcal{E}_1, \ldots, \mathcal{E}_n) \) is directed if for all \( 1 \leq i < j \leq n \), we have \( \text{Hom}_D(E_j, E_i) = 0 \) for all \( E_i \in \mathcal{E}_i \) and \( E_j \in \mathcal{E}_j \).

Given a directed sequence \( (\mathcal{E}_1, \ldots, \mathcal{E}_n) \), we denote by \( \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle \) the full subcategory of \( D \) consisting of objects \( X \in D \) with the following property:
\begin{equation}
B.2 \quad \text{There exists a sequence of maps } X_{n+1} \to X_n \to \ldots \to X_1 \text{ in } D \text{ such that } X_{n+1} = 0, X_1 = X \text{ and } E_i := \text{cone}(X_{i+1} \to X_i) \in \mathcal{E}_i \text{ for each } 1 \leq i \leq n.
\end{equation}

In the special case \( n = 2 \), the notation \( \mathcal{E} = \langle \mathcal{E}_1, \mathcal{E}_2 \rangle \) implicitly means that
\begin{enumerate}
\item \( \text{Hom}_D(E_2, E_1) = 0 \) for all \( E_1 \in \mathcal{E}_1 \) and \( E_2 \in \mathcal{E}_2 \);
\item The subcategory \( \mathcal{E} \subset D \) consists of objects \( X \in D \) such that there is an exact triangle
\begin{equation}
B.3 \quad \ldots \to E_2 \to X \to E_1 \to E_2[1] \to \ldots
\end{equation}
where \( E_1 \in \mathcal{E}_1 \) and \( E_2 \in \mathcal{E}_2 \).
\end{enumerate}

The bracket notation \( \langle - , \ldots , - \rangle \) is useful because of the following well-known associativity property, which can be proved by a repeated application of the octahedral axiom of a triangulated category:

**Proposition B.4.** For any \( n \geq 1 \), and any \( 1 \leq p \leq n - 1 \), we have
\[
\langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle = \langle \langle \mathcal{E}_1, \ldots, \mathcal{E}_p \rangle, \langle \mathcal{E}_{p+1}, \ldots, \mathcal{E}_n \rangle \rangle.
\]

Proposition B.4 ensures that the extension \( \langle - , \ldots , - \rangle \) can be performed iteratively, so that one can understand it by focusing on the case \( n = 2 \). In this case, we have the following straightforward
Lemma B.5. Let $E = \langle E_1, E_2 \rangle$, then we have

$$E_1 = E_2^\perp := \{ E \in E | \text{Hom}_D(E_2, E) = 0 \text{ for all } E_2 \in E_2 \}$$

$$E_2 = E_1^\perp := \{ E \in E | \text{Hom}_D(E, E_1) = 0 \text{ for all } E_1 \in E_1 \}$$

More precisely, the collections on the right hand side are precisely those that are isomorphic to objects in $E_1$ and $E_2$ respectively.

Lemma B.6. Let $E = \langle E_1, E_2 \rangle$. If both $E_1$ and $E_2$ are triangulated, then so is $E$.

Proof. Closure of $E$ under the shift functor $[1]$ is obvious. For the closure of taking cone, suppose we are given a map $f : X \to X'$ in $E$, then consider the diagram

$$\cdots \to E_2 \xrightarrow{i} X \xrightarrow{j} E_1 \to \cdots$$

$$\cdots \to E_2' \xrightarrow{i'} X' \xrightarrow{j'} E_1' \to \cdots$$

where $E_1, E_1' \in E_1$ and $E_2, E_2' \in E_2$. Since $j' \circ f \circ i = 0$, the dashed arrows exist, which makes the left square commute. An application of the $3 \times 3$-lemma in a triangulated category (see, e.g., [6] Proposition 1.1.11 or [23] Lemma 2.6) then shows that cone($f$) $\in E$. □

Corollary B.7. Suppose $E_1$ and $E_2$ are strongly orthogonal, meaning that $\text{Hom}_D(E_2, E_1[i]) = 0$ for all $E_1 \in E_1$, $E_2 \in E_2$ and all $i \in \mathbb{Z}$. If we denote by $\text{tri}(E)$ the smallest triangulated subcategory of $D$ containing $E \subset D$, then we have $\text{tri}(E_2) \perp \text{tri}(E_1)$, and $\text{tri}(\langle E_1, E_2 \rangle) = \langle \text{tri}(E_1), \text{tri}(E_2) \rangle$.

Now we focus further to the case when $\langle E_1, \ldots, E_n \rangle$ is the entire triangulated category $D$:

Definition B.8. A directed sequence $(E_1, \ldots, E_n)$ of full subcategories of $D$ is said to be a generalized semi-orthogonal decomposition of $D$ if we have $D = \langle E_1, \ldots, E_n \rangle$. If each $E_i$ is a triangulated subcategory, then the directed sequence $(E_1, \ldots, E_n)$ is said to be a semi-orthogonal decomposition of $D$.

Remark B.9. While we will focus mainly on the case of semi-orthogonal decompositions, the more general case when the subcategories $E_i$ are not necessarily triangulated is also important for other purposes. For example, a t-structure is precisely a generalized semi-orthogonal decomposition for $n = 2$, such that $E_2[1] \subset E_2$, or equivalently, $E_1[-1] \subset E_1$. In particular, formulating Proposition B.4 in the generality of Definition B.1 allows one to give a simple proof of the result [6] on gluing of t-structures.

Lemma B.10. Given a generalized semi-orthogonal decomposition $D = \langle E_1, E_2 \rangle$ such that $E_2[1] \subset E_2$, or equivalently $E_1[-1] \subset E_1$, then the assignment $X \mapsto E_1$ and $X \mapsto E_2$ in $\text{B.3}$ are functorial, and gives a right (resp. left) adjoint to the inclusion $E_2 \hookrightarrow D$ (resp. $E_1 \hookrightarrow D$).

Proof. Recall that, for any $Z \in D$, the functor $\text{Hom}_D(-, Z) : D \to \text{Ab}$ sends exact triangles to long exact sequences. Applying this to $\text{B.3}$ for $Z \in E_1$ shows that there is a canonical bijection $\text{Hom}_D(E_1, Z) \xrightarrow{\sim} \text{Hom}_D(X, Z)$. The shows adjunction, and hence functoriality. □

Definition B.11. A full triangulated subcategory $E \subset D$ is said to be right admissible (resp. left admissible) if the inclusion functor $E \hookrightarrow D$ has a right (resp. right) adjoint. It is said to be admissible if it is both right and left admissible.

Proposition B.12. A full triangulated subcategory $E_2 \subset D$ is right admissible if and only if there exists a full triangulated subcategory $E_1 \subset D$ such that $D = \langle E_1, E_2 \rangle$. Dually, a full triangulated subcategory $E_1 \subset D$ is left admissible if and only if there exists a full triangulated subcategory $E_2 \subset D$ such that $D = \langle E_1, E_2 \rangle$.

Proof. We have already seen the implications “$\Leftarrow$” in Lemma B.10. Conversely, suppose that the inclusion functor $i : E_2 \hookrightarrow D$ has a right adjoint $r : D \to E_2$, then the adjunction counit $\text{id} \Rightarrow ri$ is an isomorphism since $ri$ is fully faithful (see, e.g., [30] Tag 07RB). Notice that $r : D \to E_2$ is also an exact functor (see, e.g., [30] Tag 0ASD). From this, one shows that, for all $X \in D$, we have cone($ir(X) \to X)[-1] \in E_2^\perp$, and hence there is a semi-orthogonal decomposition $D = \langle E_2^\perp, E_2 \rangle$. □
We now investigate semi-orthogonal decompositions arising from Serre subcategories. Our main results are Corollary B.28, Proposition B.30 and Proposition B.31. Since these are mostly rearrangement of arguments in [28], we skip most of the proofs. We first recall the following

**Definition B.13.** A full subcategory \( S \) of an abelian category \( C \) is called a Serre subcategory \(^1\) if for any short exact sequence \( 0 \to X' \to X \to X'' \to 0 \) in \( C \), \( X \) is in \( S \) if and only if both \( X' \) and \( X'' \) are in \( S \).

Given any Serre subcategory \( S \subset C \), there is an abelian category \( C/S \), together with an exact functor \( \phi^* : C \to C/S \), which is universal with respect to this property (see, e.g., [28 Section 4.3]).

**Definition B.14.** A Serre subcategory \( S \subset C \) is said to be a localizing subcategory if \( \phi^* : C \to C/S \) has a right adjoint \( \phi_* : C/S \to C \). Dually, it is said to be a colocalizing subcategory if the functor \( \phi^* : C \to C/S \) has a left adjoint \( \phi_! : C/S \to C \).

**Lemma B.15.** If \( S \subset C \) is a localizing subcategory, then the adjunction counit \( \phi^* \phi_* \to \text{id} \) is an isomorphism of functors on \( C/S \).

As we will see in Proposition B.19 below, for a Serre subcategory to be (co)localizing, it is necessary and sufficient for objects in \( C \) to be “approximated” by \( S \)-(co)closed objects in the sense of the following

**Definition B.16.** Given a Serre subcategory \( S \subset C \), an object \( M \in C \) is said to be \( S \)-closed if \( \text{Hom}_C(S, M) = \text{Ext}^1_C(S, M) = 0 \) for all \( S \in S \). Dually, it is said to be \( S \)-coclosed if \( \text{Hom}_C(M, S) = \text{Ext}^1_C(M, S) = 0 \) for all \( S \in S \).

Notice that we define \( \text{Ext}^1_C(X,Y) := \text{Hom}_{\mathbb{D}(C)}(X,Y[i]) \), so that it is well-defined, and gives rise to the standard long exact sequences, even when \( C \) does not have enough injectives or projectives. We now turn to the following characterization of \( S \)-closed objects:

**Lemma B.17.** Given any object \( M \in C \), the following conditions are equivalent:

1. \( M \) is \( S \)-closed.
2. Given \( f : X \to Y \) such that both \( \ker(f) \) and \( \text{coker}(f) \) are in \( S \), the induced map \( \text{Hom}_C(Y, M) \to \text{Hom}_C(X, M) \) is a bijection.
3. Given an injection \( f : X \to Y \) such that \( \text{coker}(f) \) is in \( S \), the induced map \( \text{Hom}_C(Y, M) \to \text{Hom}_C(X, M) \) is a bijection.
4. For any \( X \in C \), the induced map \( \text{Hom}_C(X, M) \to \text{Hom}_{C/S}(\phi^*(X), \phi^*(M)) \) is a bijection.

Now if \( S \subset C \) is a localizing subcategory, then by verifying condition (2) of Lemma B.17, one can show that an object in the essential image of \( \phi_* : C/S \to C \) is \( S \)-closed. Moreover, for any \( X \in C \), the adjunction unit \( \epsilon_X : X \to \phi_* \phi^*(X) \) constitutes an example of an \( S \)-closure, in the sense of the following

**Definition B.18.** Given a Serre subcategory \( S \subset C \), an \( S \)-closure of an object \( X \in C \) is a map \( \epsilon_X : X \to \bar{X} \) from \( X \) to an \( S \)-closed object \( \bar{X} \), such that \( \ker(\epsilon_X) \) and \( \text{coker}(\epsilon_X) \) are both in \( S \).

Notice that, by condition (2) of Lemma B.17, if an \( S \)-closure exists, then it is unique up to canonical isomorphism. This notion is useful because of the following

**Proposition B.19.** Given a Serre subcategory \( S \subset C \), the following statements are equivalent:

1. The Serre subcategory \( S \subset C \) is a localizing subcategory.
2. Every object in \( C \) has an \( S \)-closure.
3. Every object \( X \in C \) has a largest subobject \( X^S \) in \( S \), and \( X/X^S \) embeds into an \( S \)-closed object.

We record the following simple Lemma for later use:

**Lemma B.20.** Let \( S \subset C \) be a localizing subcategory, then short exact sequences in \( C/S \) can be functorially lifted to short exact sequences in \( C \).

**Proof.** Given \( 0 \to X' \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \), take the short exact sequence \( 0 \to \phi_* X' \xrightarrow{\phi_*(f)} \phi_* Y \to \text{coker}(\phi_*(f)) \to 0 \). The functor \( \phi^* \) then sends it to the one we start with because it is exact. \( \square \)

\(^1\)Often also called a dense subcategory, for example in [28]. See also the discussion of the terminology in [33 Tag 02MO].
We now give a class of Serre subcategories that are easily recognized to be localizing. We start with the following

**Definition B.21.** A torsion pair for an abelian category $\mathcal{C}$ consists of a pair of full subcategories $\mathcal{F}$ and $\mathcal{T}$ satisfying the following conditions:

1. $\mathcal{F} \cap \mathcal{T} = \emptyset$;
2. if $X$ is an object of $\mathcal{T}$, then any quotient object of $X$ is also in $\mathcal{T}$;
3. if $X$ is an object of $\mathcal{F}$, then any subobject of $X$ is also in $\mathcal{F}$;
4. for each $X \in \mathcal{C}$, there is an exact sequence $0 \to X^T \to X \to X_F \to 0$ with $X^T \in \mathcal{T}$ and $X_F \in \mathcal{F}$.

We write $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ for a torsion pair.

**Lemma B.22.** Let $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ be a torsion pair, then we have

$$\mathcal{F} = \{ X \in \mathcal{C} | \text{Hom}_\mathcal{C}(T, X) = 0 \ \text{for all} \ T \in \mathcal{T} \}$$

$$\mathcal{T} = \{ X \in \mathcal{C} | \text{Hom}_\mathcal{C}(X, F) = 0 \ \text{for all} \ F \in \mathcal{F} \}$$

More precisely, the collections on the right hand side are precisely those that are isomorphic to objects in $\mathcal{F}$ and $\mathcal{T}$ respectively.

**Definition B.23.** A torsion pair $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ is said to be injectively cogenerated if the following two conditions hold:

1. if $X$ is an object of $\mathcal{T}$, then any subobject of $X$ is also in $\mathcal{T}$;
2. for any $F \in \mathcal{F}$, there is a monomorphism $F \hookrightarrow I$ where $I$ is an injective object of $\mathcal{C}$ that lies in $\mathcal{F}$.

In view of Lemma B.22 and Proposition B.19, we have (the first statement of) the following

**Proposition B.24.** If $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ is an injectively cogenerated torsion pair, then

1. $\mathcal{T} \subset \mathcal{C}$ is a localizing subcategory;
2. if $I$ is an injective object of $\mathcal{C}$ that lies in $\mathcal{F}$, then $\phi^*(I)$ is injective in $\mathcal{C}/\mathcal{T}$;
3. the category $\mathcal{C}/\mathcal{T}$ has enough injectives.

**Lemma B.25.** If $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ is a torsion pair such that $\mathcal{T}$ is closed under subobjects (i.e., if it satisfies condition (1) of Definition B.23) then any essential extension of an object $F \in \mathcal{F}$ is still in $\mathcal{F}$.

**Corollary B.26.** If $\mathcal{C}$ admits injective envelope, then every torsion pair in $\mathcal{C}$ that satisfies condition (1) of Definition B.23 also satisfies condition (2).

This is useful because of the following well-known result (see, e.g., [28] Theorem 3.10.10):

**Theorem B.27.** Every Grothendieck category admits injective envelopes.

Combining Corollary B.26, Theorem B.27 and B.24, we have the following

**Corollary B.28.** Suppose $\mathcal{C}$ be a Grothendieck category. If $\mathcal{C} = (\mathcal{F}, \mathcal{T})$ is a torsion pair such that $\mathcal{T}$ is closed under subobjects (i.e., if condition (1) of Definition B.23 is satisfied), then the Serre subcategory $\mathcal{T}$ is localizing.

Let $\mathcal{C}$ be an abelian category. Denote by $D(\mathcal{C})$ its derived category. For any Serre subcategory $\mathcal{S} \subset \mathcal{C}$, denote by $D_S(\mathcal{C}) \subset D(\mathcal{C})$ the full subcategory consisting of complexes whose cohomology lie in $\mathcal{S}$. Then $D_S(\mathcal{C})$ is a split-closed triangulated subcategory of $D(\mathcal{C})$. In fact, since the canonical functor $\phi^* : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is exact, it descends to an exact functor $\phi^* : D(\mathcal{C}) \to D(\mathcal{C}/\mathcal{S})$, whose kernel is precisely $D_S(\mathcal{C})$. Similar statements hold when $D$ is replaced by $D^+$, $D^-$ or $D^b$.

**Definition B.29.** A Serre subcategory $\mathcal{S} \subset \mathcal{C}$ is said to be $\mathcal{D}^\bullet$-localizing (where $\bullet \in \{ +, -, b \}$) if the functor $\phi^* : D^\bullet(\mathcal{C}) \to D^\bullet(\mathcal{C}/\mathcal{S})$ has a right adjoint $R\phi_* : D^\bullet(\mathcal{C}/\mathcal{S}) \to D^\bullet(\mathcal{C})$ such that the adjunction counit $\epsilon : \phi^* \circ R\phi_* \Rightarrow \text{id}$ is an isomorphism.

Dually, $\mathcal{S}$ is said to be $\mathcal{D}^\bullet$-colocalizing if the functor $\phi^* : D^\bullet(\mathcal{C}) \to D^\bullet(\mathcal{C}/\mathcal{S})$ has a left adjoint $L\phi^* : D^\bullet(\mathcal{C}/\mathcal{S}) \to D^\bullet(\mathcal{C})$ such that the adjunction unit $\eta : \text{id} \Rightarrow \phi^* \circ L\phi^*$ is an isomorphism.

12This holds more generally for weak Serre subcategory in the sense of [30] Tag 02MO (see [30] Tag 06UQ)
Although we do not assume in the above Definition that $R\phi_*$ and $L\phi_!$ are derived functors of some functors at the abelian level, this will often be the case in applications. The following result gives the main class of example of $\mathcal{D}$-localizing subcategory:

**Proposition B.30.** Let $\mathcal{C}$ be a Grothendieck abelian category and $S \subset \mathcal{C}$ any localizing subcategory. Then we have

1. The Serre quotient $\mathcal{C}/S$ is a Grothendieck category.
2. For $\bullet \in \{+, -\}$, the functor $\phi_\bullet : \mathcal{C}/S \to \mathcal{C}$ has a total right derived functor $R\phi_\bullet : \mathcal{D}^\bullet(\mathcal{C}/S) \to \mathcal{D}^\bullet(\mathcal{C})$, which is right adjoint to $\phi^\bullet : \mathcal{D}^\bullet(\mathcal{C}) \to \mathcal{D}^\bullet(\mathcal{C}/S)$, and makes $S \subset \mathcal{C}$ a $\mathcal{D}^\bullet$-localizing subcategory.

**Proof.** The quotient functor $\phi^\bullet : \mathcal{C} \to \mathcal{C}/S$ preserves arbitrary colimits since it has a right adjoint. Thus $\mathcal{C}/S$ admits small colimits. Since short exact sequences in $\mathcal{C}/S$ can be functorially lifted to short exact sequences in $\mathcal{C}$ (see Lemma [B.20]), directed colimit is exact in $\mathcal{C}$. Moreover, the functor $\phi^\bullet$ sends any generating set of $\mathcal{C}$ to a generating set of $\mathcal{C}/S$. Thus, $\mathcal{C}/S$ is a Grothendieck category, proving (1).

Since any Grothendieck category has enough injectives (see, e.g., [30, Tag 079H]) and K-injectives (see, e.g., [30, Tag 079P]), the functor $\phi^\bullet : \mathcal{C}/S \to \mathcal{C}$ can be derived to $R\phi^\bullet : \mathcal{D}^\bullet(\mathcal{C}/S) \to \mathcal{D}^\bullet(\mathcal{C})$ for $\bullet \in \{+, -\}$. Moreover, as a right adjoint to an exact functor, the functor $\phi^\bullet : \mathcal{C}/S \to \mathcal{C}$ preserves injectives and K-injectives. Thus, the derived functor $R\phi^\bullet : \mathcal{D}^\bullet(\mathcal{C}/S) \to \mathcal{D}^\bullet(\mathcal{C})$ is right adjoint to $\phi^\bullet : \mathcal{D}^\bullet(\mathcal{C}) \to \mathcal{D}^\bullet(\mathcal{C}/S)$.

The fact that the adjunction counit $\epsilon : \phi^\bullet \circ R\phi_* \Rightarrow \text{id}$ is an isomorphism is also clear by applying it on any (K-)injective representative.

The usefulness of Definition [B.29] lies in the following obvious

**Proposition B.31.** Suppose that $S \subset \mathcal{C}$ is a $\mathcal{D}^\bullet$-localizing subcategory, then $R\phi_*$ is fully faithful, and gives rise to a semi-orthogonal decomposition

$$\mathcal{D}^\bullet(\mathcal{C}) = \langle R\phi_*(\mathcal{D}^\bullet(\mathcal{C}/S)), \mathcal{D}_S^\bullet(\mathcal{C}) \rangle$$

As a result, there is an equivalence of triangulated categories

$$\phi^\bullet : \mathcal{D}^\bullet(\mathcal{C})/\mathcal{D}_S^\bullet(\mathcal{C}) \xrightarrow{\sim} \mathcal{D}^\bullet(\mathcal{C}/S) : R\phi_*$$

**Remark B.32.** An analogue of Proposition [B.31] is claimed in [30, Tag 06XM]. However, the proof seems to be incomplete.

Let $\mathcal{C}$ be an abelian category with exact small coproducts (also known as an Ab 4-category), then for any direct system $X_\bullet = [X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots]$ of objects in $\text{Ch}(\mathcal{C})$, define its *homotopy colimit* to be the cochain complex defined by the cone

$$\text{holim}_{n \in \mathbb{N}} X_n := \text{cone} \left( \prod_{n \in \mathbb{N}} X_n \xrightarrow{\alpha} \prod_{n \in \mathbb{N}} X_n \right)$$

where the map $\alpha$ is the coproduct of the maps $X_n \xrightarrow{\text{id}_{X_n} - f_n} X_n \amalg X_{n+1}$.

Dually, if $\mathcal{C}$ be an abelian category with exact small products (also known as an Ab 4*-category), then for any inverse system $X^\bullet = [X^0 \xleftarrow{f^0} X^1 \xleftarrow{f^1} \ldots]$ of objects in $\text{Ch}(\mathcal{C})$, define its *homotopy limit* to be the cochain complex defined by the cocone

$$\text{holim}_{n \in \mathbb{N}} X^n := \text{cocone} \left( \prod_{n \in \mathbb{N}} X^n \xleftarrow{\beta} \prod_{n \in \mathbb{N}} X^n \right)$$

where the map $\beta$ is the product of the maps $X^n \oplus X^{n+1} \xrightarrow{\text{id}_{X^n} - f^n} X^n$.

In general, the ordinary (termwise) directed colimit of a directed system $X_\bullet$ in $\text{Ch}(\mathcal{C})$ is the cokernel of the map $\alpha$ in (B.33); while the ordinary (termwise) inverse limit of an inverse system $X^\bullet$ in $\text{Ch}(\mathcal{C})$ is the kernel of the map $\beta$ in (B.34). Thus, we have canonical maps

$$\text{holim}_{n \in \mathbb{N}} X_n \to \lim X_n \quad \text{for direct system } X_\bullet$$

$$\lim X^n \to \text{holim}_{n \in \mathbb{N}} X^n \quad \text{for inverse system } X^\bullet$$
The notion of homotopy colimits often coincides with the ordinary colimits, in view of the following Lemma (see, e.g., [30] Tag 0949):

**Lemma B.36.** If $C$ is an Ab 5 category (i.e., directed colimits exist and are exact), then for any directed system $X_n$, the canonical map $hocolim_{n\in\mathbb{N}}X_n \to \varinjlim X_n$ is a quasi-isomorphism.

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