SOME CALCULATIONS OF ORLICZ COHOMOLOGY AND POINCARÉ–SOBOLEV–ORLICZ INEQUALITIES

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ABSTRACT. We carry out calculations of Orlicz cohomology for some basic Riemannian manifolds (the real line, the hyperbolic plane, the ball). Relationship between Orlicz cohomology and Poincaré–Sobolev–Orlicz-type inequalities is discussed.

Key words and phrases: differential form, Orlicz cohomology, torsion, Poincaré–Sobolev–Orlicz inequality

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INTRODUCTION

The article continues the study of Orlicz cohomology of Riemannian manifolds initiated in [7, 8].

Orlicz cohomology is a natural generalization of $L_{qp}$-cohomology (for a detailed discussion of $L_{qp}$-cohomology, the reader is referred, for example, to [4]).

Like Orlicz function spaces, the Orlicz spaces $L^\Phi$ of differential forms are a natural nonlinear generalization of the spaces $L^p$. Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [6] and then by Agarwal, Ding, and Nolder in [1]. Orlicz forms on an arbitrary Riemannian manifold were apparently first examined by Kopylov and Panenko in [7].

In [4], Gol’dshtein and Troyanov demonstrated close relationship between $L_{qp}$-cohomology and Sobolev-type inequalities on Riemannian manifolds and, basing on this and some “almost duality” techniques, performed calculations of $L_{qp}$-cohomology for some basic manifolds. It turns out that, with some significant corrections and sometimes under additional constraints on the $N$-functions from which the Orlicz cohomology is constructed, these methods prove to be fruitful in computing Orlicz cohomology.

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we recall some basic information on abstract Banach complexes. Section 3 contains definitions concerning Orlicz spaces of differential forms on a Riemannian manifold, Orlicz cohomology, and its interpretation in terms of Poincaré–Sobolev–Orlicz inequalities (Theorems 3.3 and 3.4). Then we calculate the $L^\Phi_{1,2}$-cohomology of $\mathbb{R}$ (Section 4) the hyperbolic plane (Section 5) and the $L^\Phi$-cohomology of the ball (“$L^\Phi$-Poincaré inequality”, Section 6).

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1. N-Functions and Orlicz Function Spaces

**Definition 1.1.** A nonnegative function \( \Phi : \mathbb{R} \to \mathbb{R} \) is called an N-function if

- (i) \( \Phi \) is even and convex;
- (ii) \( \Phi(x) = 0 \iff x = 0 \);
- (iii) \( \lim_{x \to 0} \frac{\Phi(x)}{x} = 0 \); \( \lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty \).

An N-function \( \Phi \) has left and right derivatives (which can differ only on an at most countable set, see, for instance, [10, Theorem 1, p. 7]). The left derivative \( \phi \) of \( \Phi \) is left continuous, nondecreasing on \((0, \infty)\), and such that \( 0 < \phi(t) < \infty \) for \( t > 0 \), \( \phi(0) = 0 \), \( \lim_{t \to \infty} \phi(t) = \infty \). The function \( \psi(s) = \inf\{t > 0 : \phi(t) > s\} \), \( s > 0 \), is called the left inverse of \( \phi \).

The functions \( \Phi, \Psi \) given by
\[
\Phi(x) = \int_0^{|x|} \phi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt
\]
are called complementary N-functions.

The N-function \( \Psi \) complementary to an N-function \( \Phi \) can also be expressed as
\[
\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.
\]

Throughout the article, given an N-function \( \Phi : \mathbb{R} \to [0, \infty) \), we denote by \( \Phi^{-1} \) its “positive” inverse \( \Phi^{-1} : [0, \infty) \to [0, \infty) \).

N-functions are classified in accordance with their growth rates as follows:

**Definition 1.2.** An N-function \( \Phi \) is said to satisfy the \( \Delta_2 \)-condition (for all \( x \)), which is written as \( \Phi \in \Delta_2 \) if there exists a constant \( K > 2 \) such that \( \Phi(2x) \leq K\Phi(x) \) for all \( x \geq 0 \); \( \Phi \) is said to satisfy the \( \nabla_2 \)-condition (for all \( x \)), which is denoted symbolically as \( \Phi \in \nabla_2 \), if there is a constant \( c > 1 \) such that \( \Phi(x) \leq \frac{1}{2c}\Phi(cx) \) for all \( x \geq 0 \).

It is not hard to see that an N-function \( \Phi \) satisfies the the \( \nabla_2 \)-condition if and only if its dual N-function satisfies the \( \Delta_2 \)-condition.

Henceforth, let \( \Phi \) be an N-function and let \((\Omega, \Sigma, \mu)\) be a measure space.

**Definition 1.3.** Given a measurable function \( f : \Omega \to \mathbb{R} \), we put
\[
\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu.
\]

**Definition 1.4.** The linear space
\[
L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \text{ measurable} : \rho_\Phi(af) < \infty \text{ for some } a > 0\}
\]
is called an Orlicz space on \((\Omega, \Sigma, \mu)\).

Let \( \Psi \) be the complementary N-function to \( \Phi \).
Below we as usual identify two functions equal outside a set of measure zero.
If \( f \in L^\Phi \) then the functional \( \| \cdot \|_\Phi \) (called the Orlicz norm) defined by
\[
\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup\left\{ \int_\Omega fg\,d\mu : \rho_\Psi(g) \leq 1 \right\}
\]
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is a seminorm. It becomes a norm if \( \mu \) satisfies the finite subset property (see [10] p. 59): if \( A \in \Sigma \) and \( \mu(A) > 0 \) then there exists \( B \in \Sigma, \ B \subset A, \) such that \( 0 < \mu(B) < \infty \).

The equivalent gauge (or Luxemburg) norm of a function \( f \in L^\Phi \) is defined by the formula
\[
\|f\|_{\Phi} = \|f\|_{L^\Phi(\Omega)} = \inf \left\{ K > 0 : \rho_\Phi \left( \frac{f}{K} \right) \leq 1 \right\}.
\]
This is a norm without any constraint on the measure \( \mu \) (see [10, p. 54, Theorem 3]).

2. Banach Complexes

Like in the case of \( L_{q,p} \)-cohomology, treated in [4], we apply some abstract facts about Banach complexes to the Orlicz cohomology of Riemannian manifolds.

In this section, we recall some definitions and assertions about abstract Banach complexes given in [4].

**Definition 2.1.** A Banach complex is a sequence \( F^* = \{ F^k, d_k \}_{k \in \mathbb{N}} \) where \( F^k \) is a Banach space and \( d = d_k : F^k \to F^{k+1} \) is a bounded operator with \( d_k + 1 \circ d_k = 0 \).

**Definition 2.2.** Given a Banach complex \( F^* \), introduce the vector spaces:
• \( Z^k := \ker( d : F^k \to F^{k+1} ) \) (a closed subspace of \( F^k \));
• \( B^k := \operatorname{Im}( d : F^{k-1} \to F^k ) \subset Z^k \);
• \( H^k(F^*) := Z^k / B^k \) is the cohomology of the complex \( F^* \);
• \( \overline{H}^k(F^*) := Z^k / \overline{B}^k \) is the reduced cohomology of the complex \( F^* \);
• \( T^k(F^*) := \overline{B}^k / B^k = H^k / \overline{H}^k \) is the torsion of the complex \( F^* \).

As was observed in [4], the following easy assertion holds:
(a) \( \overline{H}^k, Z^k \) and \( \overline{B}^k \) are Banach spaces;
(b) The natural (quotient) topology on \( T^k := \overline{B}^k / B^k \) is coarse (any closed set is either empty or \( T^k \));
(c) there is a natural exact sequence
\[
0 \to T^k \to H^k \to \overline{H}^k \to 0.
\]

**Lemma 2.3.** [4, Lemma 4.4] For any Banach complex \( F^* \), the following are equivalent:
(i) \( T^k = 0 \);
(ii) \( \dim T_k < \infty \);
(iii) \( H^k \) is a Banach space;
(iv) \( B^k \subset F^k \) is closed.

**Lemma 2.4.** [4, Proposition 4.5] The following are equivalent:
(i) \( H^k = 0 \);
(ii) The operator \( d_{k-1} : F^{k-1} / Z^{k-1} \to Z^k \) admits a bounded inverse \( d_{k-1}^{-1} \);
(iii) There exists a constant \( C_k \) such that for any \( \theta \in Z^k \) there is an element \( \eta \in F^{k-1} \) with \( d \eta = \theta \) and
\[
\| \eta \|_{F^{k-1}} \leq C_k \| \theta \|_{F^k}.
\]

**Lemma 2.5.** [4, Propositions 4.6 and 4.7] The following conditions (i) and (ii) are equivalent:
(i) $T^k = 0$.
(ii) The operator $d_{k-1} : F^{k-1}/Z^{k-1} \rightarrow B^k$ admits a bounded inverse $d_{k-1}^{-1}$.
Any of these conditions implies
(iii) There exists a constant $C'_k$ such that for any $\xi \in F^{k-1}$ there is an element $\zeta \in Z^{k-1}$ such that
$$\|\xi - \zeta\|_{F^{k-1}} \leq C'_k \|d\xi\|_{F^k}. \quad (2.1)$$

Moreover, if $F^{k-1}$ is a reflexive Banach space then conditions (i)-(iii) are equivalent.

3. ORLICZ SPACES OF DIFFERENTIAL FORMS AND ORLICZ COHOMOLOGY

Let $X$ be a Riemannian manifold of dimension $n$. Given $x \in X$, denote by $(\omega(x), \theta(x))$ the scalar product of exterior $k$-forms $\omega(x)$ and $\theta(x)$ on $T_xX$. This gives a function $x \mapsto (\omega(x), \theta(x))$ on $X$.

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be two complementary $N$-functions. Given a measurable $k$-form $\omega$, we put
$$\rho_\Phi(\omega) := \int_X \Phi(|\omega(x)|)d\mu_X.$$ 
Here $d\mu_X$ stands for the volume element of the Riemannian manifold $X$. We will identify $k$-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold $X$, introduce the space $L^\Phi(X, \Lambda^k)$ as the class of all measurable $k$-forms $\omega$ satisfying the condition $\rho_\Phi(\alpha \omega) < \infty$ for some $\alpha > 0$.

As in the case of Orlicz function spaces, the space $L^\Phi(X, \Lambda^k)$ is endowed with two equivalent norms: the gauge norm
$$\|\omega\|_{(\Phi)} = \inf \left\{ K > 0 : \rho_\Phi \left( \frac{\omega}{K} \right) \leq 1 \right\},$$
and the Orlicz norm ($\Psi$ is the complementary $N$-function to $\Phi$):
$$\|\omega\|_\Phi = \sup \left\{ \left| \int_X (\omega(x), \theta(x))d\mu_X \right| : \rho_\Psi(\theta) \leq 1 \right\}.$$ 
As in the case of function spaces, it can be proved that $L^\Phi(X, \Lambda^k)$ endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a $k$-form $\omega$ is nothing but the gauge norm of its modulus function $|\omega|$. The same holds for the Orlicz norm ($[\mathbb{L}]$ Lemma 2.1).

Unless otherwise specified, we endow the $L^\Phi$ spaces with the gauge norms; the quotient (semi)norm on each of the cohomology spaces to be defined below depends on the choice of the norms on $L^\Phi_1$ and $L^\Phi_2$, but the resulting topology does not.

**Definition 3.1.** A form $\theta \in L^{j+1}_{1, \text{loc}}(X)$ is called the (weak) differential $d\omega$ of $\omega \in L^j_{1, \text{loc}}(X)$ if
$$\int_U \omega \wedge du = (-1)^{j+1} \int_U \theta \wedge u.$$
for every orientable domain \( U \subset \text{Int} \, X \) and every form \( \alpha \in D^{n-j-1}(X) \) having support in \( U \).

Let \( \Phi_I \) and \( \Phi_{II} \) be \( N \)-functions. For \( 0 \leq k \leq n \), put

\[
\Omega^k_{\Phi_I, \Phi_{II}}(X) = \{ \omega \in L^{\Phi_I}(X, \Lambda^k) : d\omega \in L^{\Phi_{II}}(X, \Lambda^{k+1}) \}.
\]

This is a Banach space with the norm

\[
\|\omega\|_{(\Phi_I), (\Phi_{II})} = \|\omega\|_{(\Phi_I)} + \|d\omega\|_{(\Phi_{II})}.
\]

Consider also the spaces

\[
Z^k_{\Phi_I, \Phi_{II}}(X) = \{ \omega \in L^{\Phi_I}(X, \Lambda^k) : d\omega = 0 \};
\]

\[
B^k_{\Phi_I, \Phi_{II}}(X) = \{ \omega \in L^{\Phi_I}(X, \Lambda^k) : \omega = d\beta \text{ for some } \beta \in L^{\Phi_{II}}(X, \Lambda^{k-1}) \}.
\]

Denote by \( \overline{B}^k_{\Phi_I, \Phi_{II}}(X) \) the closure of \( B^k_{\Phi_I, \Phi_{II}}(X) \) in \( L^{\Phi_I}(X, \Lambda^k) \).

**Definition 3.2.** The quotient spaces

\[
H^k_{\Phi_I, \Phi_{II}}(X) := Z^k_{\Phi_I, \Phi_{II}}(X)/B^k_{\Phi_I, \Phi_{II}}(X)
\]

and

\[
\overline{T}^k_{\Phi_I, \Phi_{II}}(X) := \overline{Z}^k_{\Phi_I, \Phi_{II}}(X)/\overline{B}^k_{\Phi_I, \Phi_{II}}(X)
\]

are called the \( k \)-th \( L^{\Phi_I, \Phi_{II}} \)-cohomology and the \( k \)-th reduced \( L^{\Phi_I, \Phi_{II}} \)-cohomology of the Riemannian manifold \( X \), the latter cohomology being a Banach space. Define the \( \Lambda \)-cohomology torsion as

\[
T^k_{\Phi_I, \Phi_{II}}(X) := \overline{T}^k_{\Phi_I, \Phi_{II}}(X)/B^k_{\Phi_I, \Phi_{II}}(X).
\]

The torsion \( T^k_{\Phi_I, \Phi_{II}}(X) \) can be either \( \{0\} \) or infinite-dimensional. In fact, if \( \dim T^k_{\Phi_I, \Phi_{II}}(X) < \infty \) then \( B^k_{\Phi_I, \Phi_{II}}(X) \) is closed, hence \( T^k_{\Phi_I, \Phi_{II}}(X) = \{0\} \). In particular, if \( \dim T^k_{\Phi_I, \Phi_{II}}(X) \neq 0 \) then \( \dim H^k_{\Phi_I, \Phi_{II}}(X) = \infty \).

If \( \Phi_I = \Phi_{II} = \Phi \) then we use the notations \( \Omega^k_\Phi(X) \), \( H^k_\Phi(X) \), and \( \overline{T}^k_\Phi(X) \) instead of \( \Omega^k_{\Phi_I, \Phi_{II}}(X) \), \( H^k_{\Phi_I, \Phi_{II}}(X) \), and \( \overline{T}^k_{\Phi_I, \Phi_{II}}(X) \) respectively. Thus, the \( \Lambda \)-cohomology \( H^k_\Phi(X) \) (respectively, the reduced \( \Lambda \)-cohomology \( \overline{T}^k_\Phi(X) \)) is the \( k \)-th cohomology (respectively, the \( k \)-th reduced cohomology) of the cochain complex \( \{\Omega^k_\Phi(X), d\} \).

In [3], Gol’dshhtein and Troyanov realized the \( k \)-th \( L^{\Phi_I, \Phi_{II}} \)-cohomology as the \( k \)-th cohomology of some Banach complex. Here we apply this approach to \( L^{\Phi_I, \Phi_{II}} \)-cohomology.

Fix an \((n+1)\)-tuple of \( N \)-functions \( \mathcal{F} = \{\Phi_0, \Phi_1, \ldots, \Phi_n\} \) and put

\[
\Omega^k_\mathcal{F}(X) = \Omega^k_{\Phi_{\mathcal{F}, \mathcal{F}_{n+1}}}(X);
\]

Since the weak exterior differential is a bounded operator \( d : \Omega^k_\mathcal{F}(X) \rightarrow \Omega^{k+1}_\mathcal{F}(X) \), we obtain a Banach complex

\[
0 \rightarrow \Omega^0_\mathcal{F}(X) \rightarrow \Omega^1_\mathcal{F}(X) \rightarrow \cdots \rightarrow \Omega^k_\mathcal{F}(X) \rightarrow \cdots \rightarrow \Omega^n_\mathcal{F}(X) \rightarrow 0.
\]

The \( \Lambda \)-cohomology \( H^k_\mathcal{F}(X) \) (respectively, the reduced \( \Lambda \)-cohomology \( \overline{T}^k_\mathcal{F}(X) \)) of \( X \) is the \( k \)-th cohomology (respectively, the \( k \)-th reduced cohomology) of the Banach complex \( \{\Omega^*_\mathcal{F}, d\} \).
The above-defined cohomology spaces $H^k_F(X)$ and $\overline{H}^k_F(X)$ in fact depend only on $\Phi_{k-1}$ and $\Phi_k$:

$$H^k_F(X) = H^k_{\Phi_{k-1}, \Phi_k}(X) = Z^k_{\Phi_k}(X) / B^k_{\Phi_{k-1}, \Phi_k};$$

$$\overline{H}^k_F(X) = \overline{H}^k_{\Phi_{k-1}, \Phi_k}(X) = Z^k_{\Phi_k}(X) / B^k_{\Phi_{k-1}, \Phi_k}.$$

The results on abstract Banach complexes by Gol’dstein and Troyanov enable us to interpret $\text{Orlicz}$ cohomology in terms of a Poincaré–Sobolev–Orlicz type inequality for differential forms on a Riemannian manifold $X$:

**Theorem 3.3.** $H^k_{\Phi_I, \Phi_{II}}(X) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed differential form $\omega \in L^{\Phi_{II}}(X, \Lambda^k)$ there exists a differential form $\theta \in L^{\Phi_I}(X, \Lambda^{k-1})$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^{\Phi_I}} \leq C\|\omega\|_{L^{(\phi_{II})}}.$$ 

This result is an immediate consequence of Lemma 2.4.

**Theorem 3.4.** (A) If $T^k_{\Phi_I, \Phi_{II}}(X) = 0$ then there exists a constant $C'$ such that for any differential form $\theta \in \Omega^{k-1}_{\Phi_I, \Phi_{II}}(X)$ there exists a closed form $\zeta \in Z^{k-1}_{\Phi_I}(X)$ such that

$$\|\theta - \zeta\|_{L^{(\phi_{II})}} \leq C'\|d\theta\|_{L^{(\phi_{II})}}. \tag{3.1}$$

(B) Conversely, if $\Phi_{II} \in \Delta_2 \cap \nabla_2$ and there exists a constant $C'$ such that for any form $\theta \in \Omega^{k-1}_{\Phi_I, \Phi_{II}}(X)$ there exists $\zeta \in Z^{k-1}_{\Phi_I}(X)$ such that $(3.1)$ holds then $T^k_{\Phi_I, \Phi_{II}}(X) = 0$.

**Proof.** Considering the Banach complex $\Omega_F$ with $F = \{\Phi_{II}, \ldots, \Phi_{II}, \Phi_I, \ldots, \Phi_I\}$, where $\Phi_{II}$ changes to $\Phi_I$ at the $k$th position, we get

$$H^k_F(X) = H^k_{\Phi_{II}, \Phi_{II}}(X); \quad \overline{H}^k_F(X) = \overline{H}^k_{\Phi_{II}, \Phi_{II}}(X).$$

Since $\Phi_I \in \Delta_2 \cap \nabla_2$, the Banach space $\Omega^{k-1}_{\Phi_{II}, \Phi_{II}}(X)$ is reflexive. Theorem 3.4 now stems from Lemma 2.4.

4. THE $L^{\Phi_I, \Phi_{II}}$-COHOMOLOGY OF $\mathbb{R}$

Let $\Phi_I$ and $\Phi_{II}$ be $N$-functions.

**Proposition 4.1.** $T^1_{\Phi_I, \Phi_{II}}(\mathbb{R}) \neq 0$.

**Proof.** Suppose on the contrary that $T^1_{\Phi_I, \Phi_{II}}(\mathbb{R}) = 0$. In accordance with Theorem 3.4 then there is a Sobolev inequality for functions on $\mathbb{R}$

$$\inf_{z \in \mathbb{R}} \|f - z\|_{(\Phi_I)} \leq C\|f'\|_{(\Phi_{II})} \tag{4.1}$$

for some real positive constant $C$.

Consider the function

$$\theta(x) = \omega_{1/2}(x) = \begin{cases} C e^{-\frac{1}{4x^2}} & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| > 1/2. \end{cases}$$

Here the constant $C$ is chosen so that

$$\int_{-\infty}^{\infty} \theta(x) \, dx = \frac{c}{2} \int_{-1}^{1} e^{-\frac{1}{1-t^2}} \, dt = 1.$$
Now, consider the family of smooth functions with compact support \( \{ f_a : \mathbb{R} \to \mathbb{R} : a > 0 \} \), where
\[
f_a(x) = \int_{-\infty}^{x} \left( \theta \left( x + \frac{3}{2} \right) + \theta \left( -x + a + \frac{1}{2} \right) \right) \, dx
\]
(we owe this construction to [2, pp. 8–9]). Then \( f_a(x) = 1 \) if \( x \in [1, a] \), \( f_a(x) = 0 \) if \( x \not\in [0, a+1] \), and \( \| f_a \|_{L^\infty} =: L < \infty \). Clearly, \( \| f_a - z \|_{(\Phi_1)} \) is finite only for \( z = 0 \).

Estimate the Orlicz norms involved in (4.1). We have
\[
\rho_{\Phi_1} \left( \frac{f_a}{K} \right) = \int_{-\infty}^{\infty} \Phi_1 \left( \frac{f_a(x)}{K} \right) \, dx \geq \int_{1}^{a} \Phi_1 \left( \frac{1}{K} \right) \, dx = (a - 1) \Phi_1 \left( \frac{1}{K} \right).
\]
If \( \rho_{\Phi_1} \left( \frac{f_a}{K} \right) \leq 1 \) then \( (a - 1) \Phi_1 \left( \frac{1}{K} \right) \leq 1 \), which is equivalent to
\[
K \geq \Phi_1^{-1} \left( \frac{1}{a-1} \right).
\]
Hence,
\[
\| f_a \|_{(\Phi_1)} = \inf \left\{ K : \rho_{\Phi_1} \left( \frac{f_a}{K} \right) \leq 1 \right\} \geq \frac{1}{\Phi_1^{-1} \left( \frac{1}{a-1} \right)}.
\]
On the other hand,
\[
\rho_{\Phi_2} \left( \frac{f'_a}{K} \right) = \int_{-\infty}^{\infty} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx
\]
\[
= \int_{0}^{1} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx + \int_{a}^{a+1} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx \leq 2 \Phi_2 \left( \frac{L}{K} \right).
\]
We have
\[
2 \Phi_2 \left( \frac{L}{K} \right) \leq 1 \iff K \geq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)}.
\]
Put \( \mathcal{M}_{f'_a,a} = \left\{ K : \rho_{\Phi_2} \left( \frac{f'_a}{K} \right) \leq 1 \right\} \). We have shown that if \( K \geq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)} \) then \( K \in \mathcal{M}_{f'_a,a} \). Therefore,
\[
\| f'_a \|_{(\Phi_2)} = \inf \mathcal{M}_{f'_a,a} \leq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)}.
\]
Thus,
\[
C \geq \frac{\Phi_2^{-1} \left( \frac{1}{2} \right)}{L \Phi_1^{-1} \left( \frac{1}{a-1} \right)} \to \infty \text{ as } a \to \infty.
\]
The obtained contradiction proves the proposition. \( \square \)

**Corollary 4.2.** If \( \Phi_1 \) and \( \Phi_2 \) are N-functions then the space \( H^1_{\Phi_1,\Phi_2}(\mathbb{R}) \) is not separated; in particular, \( H^1_{\Phi_1,\Phi_2}(\mathbb{R}) \neq 0 \).

**Proposition 4.3.** If \( \Phi_1 \) and \( \Phi_2 \) are N-functions and \( \Phi_2 \in \Delta_2 \) then \( \overline{H}^1_{\Phi_1,\Phi_2}(\mathbb{R}) = 0 \).
Proof. Let \( \omega = a(x)dx \in \L_2(\mathbb{R}) \). For each \( n \), put
\[
C_m = \int_{-m}^{m} a(x) dx.
\]
If \( C_m = 0 \) then put \( \lambda_m(x) \equiv 0 \) for all \( x \in \mathbb{R} \). If \( C_m \neq 0 \) then put
\[
\lambda_m(x) = \text{sign} C_m \varepsilon_m \chi \left[ -\frac{|C_m|}{2\varepsilon_m}, \frac{|C_m|}{2\varepsilon_m} \right],
\]
where \( \varepsilon_m = t_m/m \) and \( t_m \) is the only root of the equation
\[
\Phi_2 \left( \frac{t_m}{m|C_m|} \right) = 1.
\]
(The function \( t \mapsto \Phi_2(t)/t \) is strictly increasing; see, for example, [9]). We obviously have
\[
\int_{\mathbb{R}} \lambda_m(x) dx = C_m = \int_{-m}^{m} a(x) dx.
\]
Compute the norm \( \|\lambda_m\|_{\Phi_2} \). We have
\[
\rho_{\Phi_2} \left( \frac{\lambda_m}{K} \right) = \int_{-|C_m|/2\varepsilon_m}^{|C_m|/2\varepsilon_m} \Phi_2 \left( \frac{\varepsilon_m}{K} \right) dx = \frac{|C_m|}{\varepsilon_m} \Phi_2 \left( \frac{\varepsilon_m}{K} \right).
\]
Thus,
\[
\rho_{\Phi_2} \left( \frac{\lambda_m}{K} \right) \leq 1 \iff \frac{|C_m|}{\varepsilon_m} \Phi_2 \left( \frac{\varepsilon_m}{K} \right) \leq 1 \iff \Phi_2 \left( \frac{\varepsilon_m}{|C_m|} \right) \leq \frac{\varepsilon_m}{|C_m|} \iff K \geq \frac{\varepsilon_m}{\Phi_2^{-1} \left( \frac{\varepsilon_m}{|C_m|} \right)}.
\]
Here \( \Phi_2^{-1} \) stands for the inverse function to \( \Phi_2 : [0, \infty) \to [0, \infty) \). Hence, \( \|\lambda_m\|_{\Phi_2} = \frac{1}{\Phi_2^{-1} \left( \frac{\varepsilon_m}{|C_m|} \right)} \). By the choice of \( \varepsilon_m \),
\[
\frac{\Phi_2(m\varepsilon_m)}{m\varepsilon_m} = \frac{1}{m|C_m|},
\]
and so \( \|\lambda_m\|_{\Phi_2} = \frac{1}{m} \).

Let \( b_m(x) := \int_{-\infty}^{x} (\chi_{[-m,m]}(t)a(t) - \lambda_m(t)) dt \). Since \( b_m \) has compact support, \( b_m \in \L_1(\mathbb{R}) \) for each \( m \). Furthermore, \( \|db_m - \omega\|_{\Phi_2} \leq \|a\|_{\L_2(\mathbb{R})} + \|\lambda_m\|_{\L_2(\mathbb{R})} \to 0 \) as \( m \to \infty \) since for \( \Phi_2 \in \Delta_2 \) all functions in \( \L_1(\mathbb{R}) \) have absolutely continuous norm ([9 Theorem 10.3]). Thus, \( \H^1_{\Phi_1,\Phi_2}(\mathbb{R}) = 0 \).

All the results of this section are also valid for the half-line \( \mathbb{R}_+ \) (with similar proofs).

5. THE \( \L_{\Phi_1,\Phi_2} \)-COHOMOLOGY OF THE HYPERBOLIC PLANE

We will need the following Orlicz versions of Propositions 8.3 and 8.4 in [4], which are proved in absolutely the same manner:
Proposition 5.1. Let $M$ be a complete manifold of dimension $n$ and let $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form $\gamma$ such that $\gamma \in Z^{n-k}_{\Psi_1}(X)$, $\gamma \wedge \alpha \in L^1(X, \Lambda^n)$, and
\[
\int_M \gamma \wedge \alpha \neq 0,
\]
then $\alpha \notin B^k_{\Phi_1, \Phi_2}(X)$. In particular, $H^k_{\Phi_1, \Phi_2}(X) \neq 0$.

Proposition 5.2. Let $M$ be a complete manifold of dimension $n$ and let $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form $\gamma \in Z^{n-k}_{\Psi_1}(X) \cap Z^{n-k}_{\Psi_2}(X)$ such that
\[
\int_M \gamma \wedge \alpha \neq 0,
\]
then $\alpha \notin B^k_{\Phi_1, \Phi_2}(X)$. In particular, $H^k_{\Phi_1, \Phi_2}(X) \neq 0$.

The hyperbolic plane $\mathbb{H}^2$ is the Riemannian manifold that can be modelled as the space $\mathbb{H}^2$ endowed with the Riemannian metric
\[
ds^2 = e^{2z} dy^2 + dz^2.
\]

For an $N$-function $\Phi$, introduce the condition
\[
\int_0^1 \frac{\Phi(v)}{v^2} \, dv < \infty.
\]
(A)

(The upper integration limit 1 can be replaced by any positive number.)

Theorem 5.3. If $\Phi_1$ and $\Phi_2$ are $N$-functions such that their complementary $N$-functions $\Psi_1$ and $\Psi_2$ and the function $\Phi_2$ satisfy condition (A) then
\[
\dim(H^1_{\Phi_1, \Phi_2}(\mathbb{H}^2)) = \infty.
\]

We will need the following lemma, which is in fact Lemma 10.2 in [4]:

Lemma 5.4. There exist two smooth functions $f$ and $g$ on $\mathbb{H}^2$ such that

1. $f$ and $g$ are nonnegative;
2. $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;
3. $df$ and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;
4. the support of $df \wedge dg$ is contained in $\{(y, z) : |y| \leq 1, 0 \leq z \leq 1\}$;
5. $df \wedge dg \geq 0$;
6. $\int_{\mathbb{H}^2} df \wedge dg = 1$;
7. $\frac{\partial f}{\partial z}$ and $\frac{\partial g}{\partial z} \in L^\infty(\mathbb{H}^2)$;
8. $\frac{\partial f}{\partial z}$ and $\frac{\partial g}{\partial z}$ have compact support.

We will also need the following generalization of item (3) above:

Lemma 5.5. If $\Phi$ is an $N$-function satisfying condition (A) then $df, dg \in L^\infty(\mathbb{H}^2, \Lambda^1)$.

Proof: Recall the construction of the functions $f$ and $g$ of [4] Lemma 10.2.

Choose smooth functions $h_1$, $h_2$, and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:

1. $h_1, h_2, k$ are nonnegative;
2. $h_i(y) = 0$ if $|y| \geq 1$;
3. $h'_1(y)h_2(y) \geq 0$ and $h_1(y)h'_2(y) \leq 0$ for all $y$;
(4) the support of the function $h'_1(y)h_2(y) - h_1(y)h'_2(y)$ is not empty;
(5) $k'(z) \geq 0$ for all $z$;
(6) $k(z) = 1$ if $z \geq 1$ and $k(z) = 0$ if $z \leq 0$.

Then $f$ and $g$ are defined as $f(y,z) := h_1(y)k(z)$ and $g(y,z) := h_2(y)k(z)$ respectively.

We will now prove that $df \in L^\Phi$ by modifying the argument of the proof of [4 Lemma 10.2].

Indeed,

$$df = h_1(y)k'(z)dz + k(z)h'_1(y)dy.$$ 

The first summand $h_1(y)k'(z)dz$ has compact support, and the second summand $k(z)h'_1(y)dy$ is zero outside the infinite rectangle $Q = \{|y| \leq 1; z \geq 0\}$.

Choose $D < \infty$ such that $|k(z)h'_1(y)| \leq D$ on $Q$. We have

$$|k(z)h'_1(y)dy| \leq D |dy| = De^{-z}.$$ 

Since the area element of $\mathbb{H}^2$ is $dA = e^z dydz$, for any $a > 0$ we infer

$$\int_{\mathbb{H}^2} \Phi(a|k(z)h'_1(y)dy|)dA \leq \int_Q \Phi(aDe^{-z})e^z dydz = aD \int_{\infty}^{1} \frac{\Phi(v)}{aDe^{-z}} dz.$$ 

Putting $aDe^{-z} = v$ in the last integral, we get

$$aD \int_{0}^{\infty} \frac{\Phi(v)}{v^2} dv = aD \int_{0}^{1} \frac{\Phi(v)}{v^2} dv < \infty.$$ 

Thus, $\rho_\Phi(ak(z)h'_1(y)) < \infty$ for any $a$. Consequently, $k(z)h'_1v \in L^\Phi(\mathbb{H}^2, \Lambda^1)$. Thus, $df = h_1(y)k'(z)dz + k(z)h'_1(y)dy$ also lies in $L^\Phi(\mathbb{H}^2, \Lambda^k)$.

The lemma is proved. 

\medskip

Proof of Theorem 5.3 Take the functions $f$ and $g$ on $\mathbb{H}^2$ defined in Lemma 5.3 and consider the 1-forms $\alpha = df$ and $\gamma = dg$ on $\mathbb{H}^2$. Obviously, $d\alpha = d\gamma = 0$.

By Lemmas 5.2 and 5.3, $\alpha \in L^\Phi$ for any $N$-function $\Phi$ such that $\int_0^1 \Phi(v)/v^2 dv < \infty$ and $\gamma \in L^{\Phi_1} \cap L^{\Phi_2}$ if $\int_0^1 \Psi_1(v)/v^2 dv < \infty$ and $\int_0^1 \Psi_2(v)/v^2 dv < \infty$.

Since $\int_{\mathbb{H}^2} \alpha \wedge \gamma \neq 0$, Proposition 5.1 shows that $\alpha \not\in \overline{B}^{\Phi_1, \Phi_2}(\mathbb{H}^2)$.

Now, using the isometry group of $\mathbb{H}^2$, we obtain an infinite family of linearly independent classes in $\overline{\mathcal{T}}_{\Phi_1, \Phi_2}(\mathbb{H}^2)$. 

\medskip

6. The $L^\Phi$-Cohomology of the Ball

In this section, we prove the "$L^\Phi$-Poincaré lemma", i.e., the vanishing of the $L^\Phi$-cohomology of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$.

Since $\mathbb{B}^n$ has finite volume, $H^0_{\Phi_1, \Phi_2}(\mathbb{B}^n) = \overline{\mathcal{T}}^0_{\Phi_1, \Phi_2}(\mathbb{B}^n) = \mathbb{R}$ for any $N$-functions $\Phi_1$ and $\Phi_2$.

For the case of $L^p$ spaces, Gol’dstein, Kuz’minov, and Shvedov proved the vanishing of the $L^p$-cohomology of the ball in [3] Lemma 3.2; for $p \neq q$, Gol’dstein and Troyanov found necessary and sufficient conditions on $p$ and $q$ for the vanishing of the $L^{p,q}$-cohomology of $\mathbb{B}^n$.

Their proof is based on the following fact, established by Iwaniec and Lutoborski in [5].

Proposition 6.1. For any bounded convex domain $U \subset \mathbb{R}^n$ and any $k = 1, 2, \ldots, n$, there exists an operator $T = T_U : L^1_{loc}(U, \Lambda^k) \rightarrow L^1_{loc}(U, \Lambda^{k-1})$.
with the following properties:

(a) \( T(d\theta) + dT\theta = \theta \) (in the sense of currents);

(b) \( |T\theta(x)| \leq C \int_U \frac{|\theta(y)|}{|y - x|^{n-1}} dy. \)

We prove Corollary 6.2.

If \( \Phi \) is an \( N \)-function then the operator \( T \) maps \( L^\Phi(U, \Lambda^k) \) continuously into \( L^\Phi(U, \Lambda^{k-1}) \).

Proof. The following Orlicz space version of Young’s inequality for convolution holds (see the proof of Corollary 7 in [10, pp. 230–231]): If \( f \in L^\Phi \) and \( g \in L^1 \) then \( f \ast g \in L^\Phi \) and

\[ \|f \ast g\|_\Phi \leq \|f\|_\Phi \|g\|_1. \]

Applying this inequality to \( f = |\theta| \) and \( g(x) = |x|^{1-n} \), we obtain the corollary from Proposition 6.1. In the Orlicz norms, the norm of the operator \( T \) is bounded by \( \|g\|_1 \).

\[ \square \]

Corollary 6.3. The operator \( T : \Omega^k(U, \Lambda^k) \rightarrow \Omega^{k-1}(U, \Lambda^{k-1}) \) is bounded and \( Td\omega + dT\omega = \omega \) for any \( \omega \in \Omega^k(U) \).

Corollary 6.3 gives the following

Theorem 6.4. If \( \Phi \) is an \( N \)-function then \( H^k_\Phi(\mathbb{B}^n) = 0 \) for all \( k = 1, ..., n \).

Proof. Let \( \omega \in Z^k_\Phi(\mathbb{B}^n) \). By Corollary 6.3 \( T\omega \in L^\Phi(\mathbb{B}^n, \Lambda^{k+1}) \). Since \( \omega = dT\omega + Td\omega = d(T\omega) \), we conclude that \( [\omega] = [dT\omega] = 0 \in H^k_\Phi(\mathbb{B}^n) \) and so \( H^k_\Phi(\mathbb{B}^n) = 0 \).

\[ \square \]

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\[ ^1 \text{Though it is required in Corollary 7 in [10, pp. 230–231] that \( \Phi \in \Delta_2 \), the proof of Young’s inequality works for general \( N \)-functions \( \Phi \).} \]
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