A quantum description of bubble growth in a superheated fluid

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We discuss a quantum description of bubble growth in a superheated liquid helium by addressing the problem of operator ordering ambiguities that arise due to the presence of position dependent mass (PDM) in this system. Using a supersymmetric quantum mechanics formalism along with the Weyl quantization rule, we are able to identify specific operator orderings for this problem. This is a general method which should be applicable to other PDM systems.

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I. INTRODUCTION

The study of bubble nucleation and growth is a broad field of study with rich physics [1]. Some of the more recent examples include experimental and theoretical studies of superfluid helium in a superheated state [2], and the theoretical study of nucleation of magnetic bubbles [3]. In this paper we look at the problem of the quantum description of bubble growth in superheated liquid helium. This is just one example of many systems whose classical nature is well known but whose quantum representation is not. The fact that not all functions can be quantized uniquely into operators in the Hilbert space has been known for some time. The Groenewald-van Hove theorem has explicitly shown that Dirac’s program of assigning operators to all functions by having the Poisson brackets change into commutators works only for polynomials of quadratic form or less [4].

In the case of superheated liquid helium, the difficulty in quantization appears in the form of a position dependent mass (PDM), whereby the kinetic term in the classical Hamiltonian is space dependent. In fact, many systems in nature have the PDM structure; for instance, free carriers, such as electrons, in semiconductors of non-uniform chemical composition are often described by a PDM Hamiltonian [5, 6]. Other examples of PDM appear in various instances of nuclear many-body problems, quantum dots, and nano-mechanical systems just to name a few [7]. As a result there have been a number of attempts to address the issue of operator ordering ambiguities due to PDM using the Galileian transformation and other methods [8].

We aim in the present paper to provide insight into several existing techniques in quantization, their convergence and possible future usage. In particular, we show in this paper how to apply the Supersymmetric quantum mechanics (SUSY QM) formalism [9] which have been previously applied to solve simpler PDM problems [10–12]. The main difference is that, unlike the usual examples of SUSY QM, our Hamiltonian contains a specific potential which is not easily factorizable; the method presented in this paper shows how one may apply SUSY QM to a more general situation.

On the other hand, we apply to our system a more formal quantization procedure which takes the Heisenberg commutation relations as primary, the Weyl quantization otherwise referred to as the “Weyl Transform” [4, 13, 14]. This allows one to quantize a PDM Hamiltonian but does not provide explicit operator ordering. By combining the two approaches namely the SUSY QM method and the Weyl Transform, we are able to identify specifically a couple of correct operator orderings out of infinitely many possible orderings.

The paper is divided as follows: In Section 2, we give a brief description of nucleation in superheated liquid helium, which makes the problem of PDM obvious. In Section 3, we describe the details of using the SUSY QM formalism, to address the PDM problem. In Section 4, we introduce Weyl quantization and use it to obtain an operator form for our classical Hamiltonian. This resulting Weyl quantum Hamiltonian is converted to a point mass representation to compare with the Hamiltonian obtained through SUSY methods. In Section 5 we study the effective potential using realistic parameters, and provide a conclusion.

II. BUBBLE NUCLEATION IN SUPERHEATED LIQUID HELIUM

A superheated (or supercooled) fluid [15] is typically metastable since it cannot make a direct, uniform transition to the stable phase throughout its volume. The transition to the stable phase occurs via the nucleation of a droplet of stable fluid, such that once the nucleus reaches a critical size it grows quickly, ultimately converting the entire liquid from metastable to stable phase. Nucleation can occur due to fluctuations in pressure and volume within a liquid, whereby small quantities of a new phase (“bubbles”) form within a homogeneous phase. If the liquid were stable these bubbles would disappear; but superheated liquid helium is metastable, so these bubbles can grow and eventually transform the system from the old liquid phase onto the new vapor phase.

This growth process is driven by pressure gradients across the surface of the bubble. A recent study [10] provided a comprehensive classical description of such a bubble nucleation process that takes into account the fluctuations in both the radius and the pressure. The Helmholtz
free energy corresponding to this process is:

$$
\Delta F = \frac{2\pi}{3} R^3 \kappa_1 (P_t - P_e)^2 - (P - P_e) \frac{4\pi R^3}{3} + 4\pi \sigma R^2
$$

(1)

where $R$ is the radius of the bubble and $P$ the pressure outside of the bubble. $P_t = P + 2\sigma/R$ is the pressure inside the bubble where $\sigma$ is the surface tension, $P_e$ is the internal equilibrium pressure of the bubble, and $\kappa_1$ is the isothermal compressibility of the stable phase at the equilibrium pressure. The first term does not affect the rate of nucleation since it describes the pressure fluctuations which average out to a constant. The second and third terms of Eq. (1) provide the barrier, with re-

nuations which average out to a constant. The second term in order to expand and fill the volume with the stable phase. The critical radius for vapor nucleation is then $R_c = 2\sigma/(P_e - P)$.

To establish the (classical) Hamiltonian for bubble nucleation, we note that the kinetic energy of a growing nucleus is given by

$$
E_k = \frac{1}{2} M(R) \left( \frac{dR}{dt} \right)^2
$$

(2)

where the variable mass is given by:

$$
M(R) = 4\pi \left( 1 - \frac{\rho_0}{\rho_L} \right)^2 \rho_L R^3
$$

(3)

with $\rho_L$ being the density of the liquid and $\rho_v \ll \rho_L$ is the vapor density. The potential term based on Eq. (1) leads to the Hamiltonian for the bubble nucleation problem [17]

$$
H_{\text{Class.}} = \frac{\rho^2}{2M_0x^3} + U_0x^2(1 - x),
$$

(4)

where $x = R/R_c$ is the radius of the bubble scaled to the critical radius and $U_0 = 4\pi \sigma R_c^2$ and $M_0 = 4\pi \rho_L R_c^3$. Returning to our particular case in question, we note that if we were to quantize our classical PDM Hamiltonian naively by replacing $x$ and $\rho$ by the corresponding position and momentum operators $\hat{x}$ and $\hat{\rho}$, an infinite set of operator forms are possible. For example, $\frac{\hat{\rho}^2}{x^2} \rightarrow \hat{\rho} \frac{\hat{\rho}^2}{x^2}$, $\hat{x} \rightarrow \frac{1}{\sqrt{2}} \hat{\rho} \frac{\hat{\rho} \hat{\rho}^2}{\sqrt{x^3}}$, $\frac{\hat{\rho}}{x^2} \rightarrow \frac{1}{\sqrt{2}} \hat{\rho} \frac{\hat{\rho} \hat{\rho}^2}{\sqrt{x^3}}$, ..., to list just some of the simplest cases; we show how to identify the correct specific operator ordering below.

III. APPLICATION OF SUSY TO PDM OF NON-SYMMETRIC POTENTIAL

While our potential is not super-symmetric, the super-symmetric quantum mechanics (SUSY QM) formalism permits the re-writing of the Hamiltonian in terms of the familiar creation-annihilation operators by introducing an effective potential (“superpotential”). The kinetic part of of a general PDM Hamiltonian (with $\hbar = 1$, and without any confining potential) can be written

$$
H = -\frac{1}{2}m(x)^a \frac{d}{dx} m(x)^b \frac{d}{dx} m(x)^a,
$$

(5)

where $a$ (and $b$) can take any value as long as $a + b = -\frac{1}{2}$. One can then write the creation-annihilation operators $A^\pm_a$ as

$$
A^+_a = \frac{1}{\sqrt{2}} m(x)^b \frac{d}{dx} m(x)^a + W_a(x)
$$

(6)

$$
A^-_a = -\frac{1}{\sqrt{2}} m(x)^a \frac{d}{dx} m(x)^b + W_a(x)
$$

(7)

where $W_a(x)$ is known as superpotential and the corresponding harmonic Hamiltonian is given by

$$
H^\pm_a = A^\pm_a A^\mp_a = T^\pm_a + V^\pm_a
$$

(8)

where

$$
T^{(+)} = -\frac{1}{2}m(x)^a \frac{d}{dx} m(x)^b \frac{d}{dx} m(x)^a
$$

(9)

$$
T^{(-)} = -\frac{1}{2}m(x)^b \frac{d}{dx} m(x)^a \frac{d}{dx} m(x)^b
$$

(10)

are the kinetic terms and $V^\pm_a$ are the corresponding potential terms that effectively play the role of the quadratic confining potential of a standard harmonic oscillator. The form of $W_a(x)$ is fixed by demanding that $A^\pm_a$ obey the Heisenberg algebra i.e. $[A^\pm_a, A^\mp_b] = 1$. This condition leads to

$$
W_a = \frac{1}{2} \int \sqrt{2m(x)} dx + \frac{4a + 1}{2} \left( \frac{1}{\sqrt{2m(x)}} \right)',
$$

(11)

which then implies

$$
V^\pm_a = \frac{1}{2} \left( \int \sqrt{m(x)} dx \right)^2 + \frac{4a + 1}{2} \frac{1}{4} \sqrt{m(x)} \left( \frac{1}{\sqrt{m(x)}} \right)^2 - \frac{(4a + 1)}{8} \frac{1}{\sqrt{m(x)}} \left( \frac{d}{dx} \frac{1}{\sqrt{m(x)}} \right)^2 \pm \frac{1}{2}.
$$

(12)

We emphasize here that the procedure is independent of the form of the mass – the mass $m(x)$ is completely general. The essence of the SUSY procedure is therefore the introduction of an extra potential $V_a$ which now contains all the effect of operator ordering via the parameter $a$. For our specific PDM corresponding to the case of superheated liquid Helium, $m(x) = M_0x^3$ so that:

$$
W_a = -\frac{3(4a + 1)}{\sqrt{32 M_0 x^5}} + \frac{1}{5} \sqrt{2 M_0 x^5}
$$

(13)

and

$$
V^\pm_a = \frac{21 + 48a - 144a^2}{32 M_0 x^5} + \frac{2}{25} M_0 x^5 + \frac{1}{2}.
$$

(14)
It is noted that when \( a = b = -\frac{1}{4} \)

\[
W_\alpha = \frac{1}{2} \int \sqrt{2m(x)} dx, \quad \text{and} \quad V_\alpha^\pm = W_\alpha^2 \mp \frac{1}{2}, \quad (15)
\]

which are identical, respectively, to the equivalent superpotential and potential obtained in the classical formalism using Poisson brackets in place of commutators.[10]

To solve our problem, we need to write the Hamiltonian corresponding to our situation in terms of the operators \( A_\alpha^+ \). Writing out the “momentum” part of \( A_\alpha^+ \) using our PDM \( m(x) = M_0 x^n \), one can show that \( A_\alpha^+ + A_\alpha^- = \frac{2}{\sqrt{2M_0 x^n}} \) which does not depend on the parameter \( a \) i.e. independent of ordering. Using this result, one is able to write down the potentials \( V_\alpha^\pm \) and \( U_0 x^2 (1 - x) \) in terms of \( A_\alpha^+ + A_\alpha^- \) in a straightforward manner. In terms of these operators, the Hamiltonian for our system is given by subtracting \( m(x) \) multiplying throughout by the mass of the bubble. Scaling our Hamiltonian by multiplying operators with the usual commutation relation, \( [\hat{A}, \hat{B}] = i \) independent of ordering. Using this relation, one is able to write down the potentials \( V_\alpha^\pm \) and \( U_0 x^2 (1 - x) \) in terms of \( A_\alpha^+ + A_\alpha^- \) in a straightforward manner. In terms of these operators, the Hamiltonian is modified to take into account the scaling of variables. The effective potential, \( V_\alpha(z) + V_{sys}(z) \), is fairly complicated compared to the original potential; in particular it contains a singularity at \( z = 0 \) such is the “price” of removing the PDM from the kinetic term. The energy spectrum of the effective Hamiltonian of Eq. (17) is identical to that of the original PDM Hamiltonian (up to a constant factor). On the other hand, because the position variable \( z \) is now in a different space, the eigenstate of Eq. (17) is not the eigenstate of the original Hamiltonian, although there are ways to map the eigenstate of Eq. (17) back onto the eigenstate of the original PDM Hamiltonian. Once the parameter \( a \) is determined, the unique operator ordering is decided; we now suggest a way to select the value of this parameter.

We note here that the steps involved to derive Eq. (17) is that of “picking a particular basis” for the Hamiltonian of Eq. (16) rather than a coordinate transformation from one position coordinate system to another position coordinate system. In particular, by choosing ansatz based on the well-known quantum harmonic oscillator, the resulting Hamiltonian is assured of being of unit measure and Hermitian. These requirements of unit measure and Hermiticity is explained in more detail in the next section, where a coordinate transformation is directly involved.

IV. COMPARISON WITH WEYL QUANTIZATION

There exist in the literature several ways to formally convert classical function into operator form. Weyl quantization otherwise referred to as the “Weyl Transform” is a well-known way of doing so. There are many qualities of the Weyl transform that make it an optimum way to perform this conversion.[4 13 14]. The Weyl transform is defined as:

\[
W[FT\phi](x) = \frac{1}{2\pi} \int \int (FT)(\alpha, \beta) e^{i\alpha x/2} e^{i\beta x} \times \phi(x + \beta) \Omega(\alpha, \beta) d\alpha d\beta. \quad (20)
\]
Here

\[ [FT](\alpha, \beta) = \int T(p, x) \exp[-i(p\alpha + x\beta)] dp dx \tag{21} \]
denotes the Fourier transform of the classical function, \( T(x, p) \) and \( \phi(x) \) is the wave function upon which the Weyl operator acts. The weighting factor, \( \Omega(\alpha, \beta) \), characterizes the type of quantization to be utilized. Weyl quantization, in particular, is defined when \( \Omega(\alpha, \beta) = 1 \). Therefore, the Weyl transform of the kinetic part of our classical Hamiltonian is

\[
W \left[ \frac{p^2}{2M_0x^3} \phi \right](x) = \frac{1}{2\pi} \int \int \left( \frac{p^2}{2M_0x^3} \right)
\times \exp[i\alpha\text{sgn}x/2 \phi(x + \alpha \text{dod}x)] d\alpha d\beta. \tag{22}
\]

Using the result

\[
F \left( \frac{\pi}{x^2} \right) = [-\sqrt{2\pi}\delta''(\alpha)] \left[ -i\sqrt{\frac{\pi}{2}} \left( -\frac{\beta^2}{2} \text{sgn}\beta \right) \right], \tag{23}
\]
the Weyl transform for our case is

\[
W \left[ \frac{p^2}{2M_0x^3} \phi \right](x) = -\frac{\hbar^2}{2M_0R_z^2}
\times \left[ \frac{1}{x^3} \frac{\partial^2}{\partial x^2} \phi(x) - \frac{3}{x^4} \frac{\partial}{\partial x} \phi(x) + \frac{3}{x^5} \phi(x) \right]. \tag{24}
\]

We note here that the Hermiticity of the Hamiltonian must be ensured. One can show that, given a general Hamiltonian of the form \( \hat{H} = A(x)\frac{\partial^2}{\partial x^2} + B(x)\frac{\partial}{\partial x} + C(x) \), the Hermiticity condition,

\[
\int \psi^* \hat{H} \phi(x) dx = \int \left[ \hat{H} \psi \right]^* \phi(x) dx \tag{25}
\]

imposes the following conditions after integrating various terms by parts: (i) \( A^*(x) = A(x) \) i.e. \( A(x) \) is real; (ii) \( A'(x) = \text{Re}[B(x)] \); (iii) \( \text{Im}[C(x)] = \frac{i}{2} \text{Im}[B'(x)] \) where \( \text{Re}[\cdot] \) and \( \text{Im}[\cdot] \) denote real and imaginary parts respectively. This general result shows that the Weyl transform such as the result of Eq. (24) preserves the Hermiticity of the Hamiltonian.

In order to compare this result with the Hamiltonian obtained in SUSY, we begin with Eq. (24) and note that the associated kinetic term in the corresponding Lagrangian is

\[
L_K(x, \dot{x}, t) = \frac{1}{2} M_0 R_z^2 x^3 \dot{x}^2. \tag{26}
\]

We introduce a coordinate transformation \( x \rightarrow c z^a \) where \( c \) and \( \alpha \) are real numbers transforming \( L_K(x, \dot{x}, t) \) to \( L_K(z, \dot{z}, t) = \frac{1}{2} M_0 R_z^2 c^{2a} z^{5a - 2} \dot{z}^2 \). In order for the 1/\( x^3 \) factor in the first term to disappear, one needs to choose \( \alpha = \frac{2}{5} \) and \( c = \left( \frac{2}{5} \right) \) i.e. \( x \rightarrow \left( \frac{2}{5} \right)^{3/5} \), so that the mass term becomes a constant, \( m(x) \rightarrow 1 \) and Eq. (26) is transformed to

\[
L_K(z, \dot{z}, t) = \frac{1}{2} M_0 R_z^2 \dot{z}^2. \tag{27}
\]

With this substitution, Eq.(24) is transformed into

\[
\hat{H}(z) = -\frac{\hbar^2}{2M_0R_z^2} \left[ \frac{d^2}{dz^2} - \frac{3}{5z} \frac{\partial}{\partial z} + \frac{12}{25z^2} \right]. \tag{28}
\]

Once the coordinate transform is carried out, the measure (or the Jacobian of integration \( \frac{dx}{dz} \)) associated with the Hamiltonian of Eq.(28) is no longer unity i.e. coordinate transformation does not preserve the inner product of the Hilbert space, which is problematic. In general, unit measure can be restored through the comparison of the inner products between two Hamiltonians \( \hat{H} \) and \( \hat{H} \) where \( \hat{H} \) is the Hamiltonian for the inner-product space with unit measure and \( \hat{H} \) corresponds to that with non-unit measure \( \mu(x) \) so as to preserve the Hilbert space:

\[
\int \psi^* \hat{H} \phi(z) \mu(z) dz = \int \tilde{\psi}^* \tilde{\hat{H}} \tilde{\phi}(z) dz. \tag{29}
\]

Noting that \( f(z) = \tilde{f}(z)/\sqrt{\mu(z)} \) where \( f(z) = \phi(z) \) or \( \psi(z) \), Eq. (29) implies the relationship between the two Hamiltonians as \( \hat{H} = \tilde{\hat{H}} \). Writing a general unit measure Hamiltonian \( \hat{H} \) as

\[
\hat{H} = \hat{A}(z) \frac{\partial^2}{\partial z^2} + \hat{B}(z) \frac{\partial}{\partial z} + \hat{C}(z), \tag{30}
\]
and the non-unit measure Hamiltonian as one without the tildes, the above relation between \( \hat{H} \) and \( \hat{H} \) implies

\[
\hat{A}(z) = A(z), \tag{31}
\hat{B}(z) = B(z) - A(z) \frac{\mu'(z)}{\mu(z)}, \tag{32}
\hat{C}(z) = A(z) \left[ \frac{3 \mu'(z)^2}{4 \mu(z)^2} - \frac{1}{2} \frac{\mu''(z)}{\mu(z)} \right] - B(z) \frac{1}{2} \frac{\mu'(z)}{\mu(z)} + C(z). \tag{33}
\]

We note that these results are general results for any Hamiltonian of the form Eq. (30); it is not a property of a specific Hamiltonian, or any transforms involving differential operators, but is rather a self-consistent result following from preserving the inner product, and ensuring Hermiticity. A corollary of these results is that for a non unit measure Hamiltonian, the Hermiticity condition implies \( B(z) = A'(z) + \frac{\mu'(z)}{\mu(z)} A(z) \) i.e. the Hamiltonian of Eq. (28) is still Hermitian. This relation also means that a point-mass, unit measure Hamiltonian will always have \( B(z) = 0 \) due to Hermiticity condition i.e. the first order derivative term always vanishes, which is what we normally see in “ordinary” Hamiltonians. In our case, since \( x = \left( \frac{z}{2} \right)^{2/5} \), the measure is calculated to be \( \mu(z) = \frac{dz}{dx} = \left( \frac{z}{2} \right)^{3/5} \). The Hamiltonian with unit measure emanating from Eq.(28) is then

\[
\hat{H}(z) = -\frac{\hbar^2}{2M_0R_z^2} \left[ \frac{d^2}{dz^2} + \frac{9}{100z^2} \right] + U_0 z^{3/5}(1 - z^{2/5}). \tag{34}
\]
Equating this result of Eq. (34) with the Hamiltonian obtained through SUSY QM, Eq. (17), we require
\[
\frac{9}{100z^2} = \frac{21 + 48a - 144a^2}{100z^2},
\]
which implies that specific operator orderings that correspond to the Weyl quantization rule are given by \(a \to -\frac{1}{2}\) or \(a \to \frac{1}{2}\). Therefore, the quantum Hamiltonian for the bubble nucleation problem that obeys the Weyl quantization rule is either
\[
H = -\frac{\hbar^2}{2M_0R_c^2} \left[ \frac{1}{x^{1/2}} \frac{d}{dx} \frac{d}{dx} \frac{d}{dx} \frac{1}{x^{1/2}} \right] + U_0x^2(1 - x) \quad (36)
\]
or
\[
H = -\frac{\hbar^2}{2M_0R_c^2} \left[ x^{3/2} \frac{d}{dx} \frac{d}{dx} \frac{d}{dx} x^{3/2} \right] + U_0x^2(1 - x). \quad (37)
\]
We have therefore narrowed down the possible operator orderings to two (out of infinity). This is the central result of our paper.

We note here that it seems on first sight that one could avoid the operator ordering ambiguity from the outset by picking a useful position representation related to well known quantum harmonic oscillator then ensures Hermiticity and unit measure owing to the Stone-von Neumann theorem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(a) \(V_{sys}(z)\) for pressure \(P = 0.8P_c\) (Dashed line) and \(P = 0.95P_c\) (Solid line). (b) Same as (a) but for \(V_v(z)\) instead. (c) \(V_v(z) + V_{sys}(z)\) for the same values of pressure. (d) \(V_v(z) + V_{sys}(z)\) for several values of pressure \(P \in [0.85P_c, 0.95P_c]\). In this figure, the position \(z\) is in scaled units (in units of \(R_c\)) and the potential is in eV.}
\end{figure}

V. CONCLUDING REMARKS

The goal of this paper was in finding the correct operator ordering for the bubble nucleation problem. A full quantum mechanical treatment of the bubble nucleation process that addresses tunneling through the barrier will be presented elsewhere [19]. In this section, before we conclude the paper, we attempt to gain some physical insight into this problem by briefly examining the effective Hamiltonian of Eq. (17) and Eq. (34) with realistic parameters. We consider physically reasonable parameters for this system so that we can get an order of magnitude estimate for the parameters. The typical values for superfluid Helium at temperature \(T = 4K\) is \(\sigma = 0.12 \times 10^{-3}N/m\), \(P_c = 8.1445 \times 10^4N/m^2\), \(\rho_L = 140kg/m^3\). For zero applied pressure, the classical critical radius is \(R_c = 29.5 \times 10^{-10}m\). At \(T = 4K\), the thermal de Broglie wavelength of a single Helium atom is \(\Lambda = h/\sqrt{2\pi mK}\approx 4.36 \times 10^{-10}m\), and hence thermal momentum of \(p_{Th} = h/\Lambda = 1.52 \times 10^{-24}gms^{-1} \gg \sqrt{U_0R_c}\) over all range of applied pressure \(P\). Typically a single bubble contains an order of 100 Helium atoms, or momentum of roughly 10\(p_{Th}\).

With these realistic numbers, we have calculated and plotted the shape of \(V_v(z) = -\frac{\hbar^2}{2M_0R_c^2} \frac{9}{100z^2}\) and \(V_{sys}(z) = U_0z^{1/5}(1 - z^{2/5})\) in Fig. 1 for various values of applied pressure \(P/P_c\). The non-confining shape of the potential makes it clear why the numerical solution to the eigenvalue problem is difficult and non-convergent. The changing shape of \(V_v(z)\) near \(z = 0\) due to changes in pressure and subsequently \(R_c\) is also shown in Fig. 1. On the other hand, the potential for larger \(z\) corresponding
to $V_{sys}(z)$ looks simpler. It is clear from Fig. [1] demonstrating unbounded potential that the energy spectrum is expected to be continuous, and the bubble growth is similar to a ball rolling downhill with the “gradient” dependent on the applied pressure. We note that had we not transformed the Hamiltonian and left the PDM in the kinetic energy term we would not be able to come up with such an intuitive understanding of the system.

To conclude, we have studied the bubble nucleation problem in a superheated liquid Helium and provided a quantum mechanical description. Although the system seems rather complicated at the outset and furthermore involves an exotic Hamiltonian that diverges at the origin due to the specific form of the PDM, we were able to extract useful information through the methods of SUSY QM and the Weyl transform. We have presented a convergence between two methods by converting to a point mass description of the quantum Hamiltonian. Further analysis will need to be performed in regards to comparison to the microscopic description of the problem at hand, and these will be presented elsewhere[19].

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