On disjunction of equations in the semigroup language with no constants

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Abstract

A semigroup $S$ is an equational domain if any finite union of algebraic sets over $S$ is algebraic. We prove that every nontrivial semigroup in the standard language $\{\cdot\}$ is not an equational domain.

Introduction

It follows from commutative algebra that the union $Y$ of two algebraic sets $Y_1, Y_2$ over a field $k$ is algebraic again (i.e. $Y$ is a solution of a system of algebraic equations). In the category of groups there also exist examples, where any finite union of algebraic sets over a group $G$ is algebraic. Following [1], such groups are called equational domains.

Notice that any group $G$ (as an algebraic structure) can be considered in three different languages:

1. $L_1 = \{\cdot, -1, 1\}$;
2. $L_G = \{\cdot, -1, 1\} \cup \{g|g \in G\}$;
3. $L_H = \{\cdot, -1, 1\} \cup \{h|h \in H \leq G\}$,

where the last two languages are the extensions of the first one by constants. An equation over a group is an expression $w(X) = 1$, where $w(X)$ is a product of the letters $X \cup X^{-1}$ and language constants. Hence, any group language $L \in \{L_1, L_G, L_H\}$ defines its class of algebraic sets $AS(L, G)$ over a group $G$. Obviously, the next inclusions holds $AS(L_1, G) \subseteq AS(L_H, G) \subseteq AS(L_G, G)$.

It is possible that a group $G$ is not an equational domain in the languages $L_1$ or $L_H$, however, $G$ is an equational domain in the rich language $L_G$.

Indeed, if we consider groups in the language with no constants $L_1$, the search of equational domains among such groups is vain, since the next holds.

Statement. [1] Any nontrivial group of the language $L_1$ is not an equational domain.

In the current paper we prove the similar result for semigroups, however its proof is more complicated than in group case (Theorem 3.4).

1 Definitions of semigroup theory

Let us give the main definitions of semigroup theory, for more details one can recommend [2].
A semigroup is a non-empty set with an associative binary operation $\cdot$ which is called a multiplication.

Let $a$ be an arbitrary element of a semigroup $S$. A period $n$ of $a$ is the number of elements in the subsemigroup generated by $a$. For an element $a$ with a period $n$ one can define an index as the minimal number $m$ such that $a^m = a^n$.

The elements $a, b \in S$ commutes if $ab = ba$. A semigroup is trivial if it contains a single element.

An element $a \in S$ is an idempotent if $aa = a$. A semigroup $S$ is a band (or idempotent) if all its elements are idempotents. A band $S$ is called rectangular if there holds the identity $xyz = xz$ for all $x, y, z \in S$.

A semigroup is nowhere commutative if any pair of its distinct elements does not commute. The next theorem describes nowhere commutative semigroups.

**Theorem 1.1.** A semigroup $S$ is nowhere commutative iff $S$ is a rectangular band.

2 Algebraic geometry over semigroups

We shall consider semigroups in the language $\mathcal{L} = \{\cdot\}$.

Denote by $X$ the finite set of variables $x_1, x_2, \ldots, x_n$. A term of the language $\mathcal{L}$ in variables $X$ is a finite product of variables from the set $X$.

An equation over the language $\mathcal{L}$ is an equality of two terms $\tau(X) = \sigma(X)$. For example, the expressions $x_1 x_2^2 = x_3 x_1, x_1^2 x_2 x_3 = x_1 x_3$ are equations over $\mathcal{L}$. A system of equations (a system for shortness) is an arbitrary set of equations.

The solution set of a system $S$ in the semigroup $S$ is naturally defined and denoted by $V_S(S)$. A set $Y \subseteq S^n$ is algebraic over a semigroup $S$ if there exists a system $S$ in variables $x_1, x_2, \ldots, x_n$ with the solution set $Y$.

Following [1], let us give the main definition of the paper. A semigroup $S$ is called an equational domain (e.d.) if any finite union $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ of algebraic sets $Y_i$ is algebraic.

3 Main result

Our aim is to prove Theorem 3.4 following the next plan. We consequently prove the absence of equational domains in the next classes of nontrivial semigroups:

1. idempotent semigroups (Lemma 3.1);
2. semigroups with the identity $x^2 = x^3$ (Lemma 3.2);
3. semigroups with an element $a$ such that $a^2 \neq a^3$ (Lemma 3.3).

**Lemma 3.1.** Any nontrivial idempotent semigroup $S$ is not an e.d.

**Proof.** Denote $X = \{x_1, x_2, x_3\}$ and assume the set

$$\mathcal{M} = V_S(x_1 = x_2) \cup V_S(x_1 = x_3) = \{(x_1, x_2, x_3) | x_1 = x_2 \text{ or } x_1 = x_3\}$$

is algebraic, i.e. it coincides with the solution set of a system $S(X)$.

There are exactly two cases.

1. A semigroup $S$ is nowhere commutative. By Theorem 1.1 $S$ is an rectangular band. Let $a, b \in S$ be a pair of two distinct elements and $c = ab$. By the identities $xyz = xz xx = x$, we have $ac = c, ca = a$. 

As \((a, c, c) \notin \mathcal{M}\), there exists an equation \(\pi(X) = \rho(X) \notin \mathcal{S}(X)\) which does not satisfy this point. Since the set \(\{a, c\}\) is a subsemigroup in \(S\), the values of the terms \(\pi(X), \rho(X)\) at the point \((a, c, c)\) belong to \(\{a, c\}\). Without loss of generality one can state \(\pi(a, c, c) = a, \rho(a, c, c) = c\). Hence, \(\pi(X)\) ends by the variable \(x_1\), whereas the term \(\rho(X)\) ends either by \(x_2\) or \(x_3\).

Thus, \(\pi(a, a, c) = \pi(a, c, a) = a\), and either \(\rho(a, c, a) = c\) (if \(\rho(X)\) ends by \(x_2\)) or \(\rho(a, a, c) = c\) (if \(\rho(X)\) ends by \(x_3\)). Anyway, we came to the contradiction, since \((a, a, c), (c, a, a) \in \mathcal{M}\).

2. There exist elements \(a, b \in S\) with \(ab = ba\) and \(c = ab\). It is easy to prove that \(ac = ca = c\). As \((a, c, c) \notin \mathcal{M}\), there exists an equation \(\pi(X) = \rho(X) \notin \mathcal{S}(X)\) which does not satisfy this point. Since the set \(\{a, c\}\) is a subsemigroup in \(S\), the values of the terms \(\pi(X), \rho(X)\) at the point \((a, c, c)\) belong to \(\{a, c\}\). Without loss of generality one can state \(\pi(a, c, c) = a, \rho(a, c, c) = c\). Hence, \(\pi(X)\) contains only \(x_1\), whereas \(\rho(X)\) contains \(x_2\) or \(x_3\).

Thus, \(\pi(a, a, c) = \pi(a, c, a) = a\), and either \(\rho(a, c, a) = c\) (if \(x_2\) occurs in \(\rho(X)\)) or \(\rho(a, a, c) = c\) (if \(x_2\) occurs in \(\rho(X)\)). Anyway, we came to the contradiction, since \((a, a, c), (a, c, a) \in \mathcal{M}\).

\[\square\]

**Lemma 3.2.** Suppose the identity \(x^2 = x^3\) holds in a semigroup \(S\) and \(S\) is not idempotent. Hence, \(S\) is not an e.d.

**Proof.** Denote \(X = \{x_1, x_2, x_3\}\) and assume the set

\[\mathcal{M} = V_S(x_1 = x_2) \cup V_S(x_1 = x_3) = \{(x_1, x_2, x_3) | x_1 = x_2 \text{ or } x_1 = x_3\}\]

is algebraic, i.e. it coincides with the solution set of a system \(\mathcal{S}(X)\).

As the semigroup \(S\) is not idempotent, there exists an element \(a \in S\) with \(a \neq a^2\).

As \((a, a^2, a^2) \notin \mathcal{M}\), there exists an equation \(\pi(X) = \rho(X) \notin \mathcal{S}(X)\) which does not satisfy this point. Since the set \(\{a, a^2\}\) is a subsemigroup in \(S\), the values of the terms \(\pi(X), \rho(X)\) at the point \((a, a^2, a^2)\) belong to \(\{a, a^2\}\). Without loss of generality one can state \(\pi(a, a^2, a^2) = a, \rho(a, a^2, a^2) = a^2\). Hence, \(\pi(X) = x_1\), whereas the term \(\rho(X)\) contains either the variables from the set \(\{x_2, x_3\}\) or \(x_1\) occurs in \(\rho(X)\) at least two times.

Thus, \(\pi(a, a, a^2) = \pi(a, a^2, a) = a\), and either \(\rho(a, a^2, a) = a^2\) or \(\rho(a, a, a^2) = a^2\).

Anyway, we came to the contradiction, since \((a, a, a^2), (a, a^2, a) \in \mathcal{M}\).

\[\square\]

**Lemma 3.3.** If a semigroup \(S\) does not satisfy the identity \(x^2 = x^3\) it is not an e.d.

**Proof.** Denote \(X = \{x_1, x_2, x_3, x_4\}\). Assume that the solution set of a system \(\mathcal{S}(X)\) equals

\[\mathcal{M} = V_S(x_1 = x_2) \cup V_S(x_3 = x_4) = \{(x_1, x_2, x_3, x_4) | x_1 = x_2 \text{ or } x_3 = x_4\},\]

and \(\tau(X) = \sigma(X)\) is an equation of \(S\). By the condition of the lemma, there exists an element \(a \in S\) with \(a^2 \neq a^3\) (and, moreover, \(a \neq a^2\)).

By the choice of the system \(\mathcal{S}\), the points

\(P_1 = (a^2, a, a, a), P_2 = (a, a^2, a)\)

satisfy the equation \(\tau(X) = \sigma(X)\).
Suppose the term $\tau(X)$ contains $n_i$ occurrences of the variable $x_i$. Similarly, $m_i$ is the number of occurrences of the variable $x_i$ in $\sigma(x)$. The equalities $\tau(P_i) = \sigma(P_i)$ imply

$$\begin{align*}
a^{2n_1+n_2+n_3+n_4} &= a^{2m_1+m_2+m_3+m_4}, \\
a^{n_1+n_2+2n_3+n_4} &= a^{m_1+m_2+2m_3+m_4},
\end{align*}$$

Let us multiply the equations of the system above and obtain the equality

$$a^{3n_1+2n_2+3n_3+2n_4} = a^{3m_1+2m_2+3m_3+2m_4}. \quad (1)$$

Consider a point $Q = (a^3, a^2, a^3, a^2) \notin M$. Compute

$$\begin{align*}
\tau(Q) &= a^{3n_1+2n_2+3n_3+2n_4}, \\
\sigma(Q) &= a^{3m_1+2m_2+3m_3+2m_4}.
\end{align*}$$

By (1), we have $\tau(Q) = \sigma(Q)$. As $\tau(X) = \sigma(X)$ was chosen as an arbitrary equation of $S$, $Q \in V_{S}(S)$ that contradicts with the choice of the system $S$. \hfill \square

The main result immediately follows from Lemmas 3.1, 3.2, 3.3.

**Theorem 3.4.** Any nontrivial semigroup in the language $\mathcal{L} = \{\cdot\}$ is not an equational domain.

**References**

[1] E. Daniyarova, A. Miasnikov, V. Remeslennikov, Algebraic geometry over algebraic structures IV: equational domains and co-domains, Algebra & Logic, 49, 6, 2010, 715–756.

[2] J.M. Howie, Fundamentals of Semigroup Theory. Oxford: Clarendon Press, 1995, 351 p.

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