Calculation of vertical bearing capacity factor \( N_{\gamma} \) of strip footing by FEM

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**Abstract.** This paper presents the calculation of vertical bearing capacity factor \( N_{\gamma} \) of strip footings, which is generally representative of the footing of the buildings, by using the finite element algorithm based elasto-viscoplastic solution. The soil behaviour is idealized as elastic perfectly plastic satisfying the Mohr Coulomb yield criterion with only non-associated flow rules because it tends to treat the actual behaviour of soil in which value of its frictional angle is always different from that of dilative angle. The total bearing resistance is computed using the superposition assumption in Terzaghi classical equation of the three basic components. The comparison of the present solutions with the previous published works including experimental and numerical studies have been presented and discussed. The results reveal in close agreement with the existing published data. In the nutshell, by applying the non-associated soil flow rule representing the real behaviour of soil, the conservation of the superposition assumption is confirmed. Therefore, it could conclude that the results obtaining from these techniques may be confidently applied to a wide range of soil bearing resistance problems.

1. **Introduction**

One amongst the most highly interesting areas in geotechnical engineering for researchers and practicing engineers is the computation of bearing capacity of various footings such as strip, circular, ring and conical footings. Generally, the calculation of bearing capacity is based on the superposition assumption proposed by Terzaghi, in which the contribution of different loading and soil strength parameter such as cohesion, friction angle, surface surcharge and self-weight, expressed in the form of non-dimensional bearing capacity factors; \( N_c \), \( N_q \) and \( N_{\gamma} \) are added [1]. Based on the used solution techniques, both analytical and numerical solutions to bearing capacity problem can be classified into the following categories: the limit equilibrium method, the upper bound plastic limit analysis method, method of characteristics, finite difference method and finite element method. In recent year, finite element method has been widely used to compute the bearing capacity of strip, circular, ring and conical footings and its pre- failure behaviour because it is capable of combining all the parameters into a single problem. Even though analytical expressions derived from the theory of plasticity provide the bearing capacity factors; \( N_c \) and \( N_q \) that remain unchanged from the variation in the roughness of the footing-soil interface [2,3,4,5], there is a considerable difference in the magnitudes of \( N_{\gamma} \) between the smooth and rough footing [2,3,4,5,6]. Consequently, there are several methods have been used to calculate \( N_{\gamma} \), including the stress field method [7], the partial plastic stress field method [8], the limit equilibrium method [1,2,9], the method of characteristics [10,11], the method of stress...
characteristics [5], the limit analysis method [12,13,14], analytical upper-bound solution [15], the finite difference method [16,17] and the finite element method [4,18]. Different methods lead to different assumptions and procedures and consequently lead to discrepancies in their results.

Finite elements algorithm based elasto-viscoplastic solutions have been successfully employed to evaluate the bearing capacity [4,18]. The current note presents the computation of bearing capacity factor \( N_\gamma \) of strip footing, which is generally representative of the footings of buildings, using finite element algorithm based elasto-viscoplastic solution. The soil behavior is idealized as elastic perfectly plastic satisfying the Mohr Coulomb yield criterion with only non-associated flow rules because it tends to treat the actual behavior of soil as the value of its frictional angle is always different from that of dilative angle. The total bearing resistance is computed using the superposition assumption in Terzaghi classical equation of the three basic components. The comparison of the authors’ results with previous available published works, including experimental and numerical studies, have been presented and discussed. The results reveal a close agreement with existing published data.

2. Solution techniques

The solution technique of the solution is generally divided into two: analytical technique and numerical technique. Moreover, there are several methods are embedded inside these two techniques. In this paper, finite element algorithm based viscoplastic technique has proven to be an efficient way of solving the plasticity problem in geomechanics is employed. Likewise, this technique has been suggested by Zienkiewicz and Cormeau[19]; Zienkiewicz and Humpheson [20] and subsequently successfully used by Griffith[4] and Manoharan and Dasgupta [18]. This method is also known as the initial strain solution techniques. The plasticity problem is solved by iterating using the equivalent elastic solutions until any stresses which temporarily violated yield have returned back to the failure surface by obeying the fairly strict tolerances. To verify the change in body’s forces from iteration to one another, the convergence criterion plays a vital role in doing so. As the stresses violate the yield, the balancing forces of the body (self-equilibrating body forces) are incremented at each iteration by the amount related to the magnitudes of violated stresses. As the increment in body forces diminished, the stresses also return to the yield surface. For this present work, convergence is allowed as long as the change in body forces with respect to the maximum absolute value, nowhere is more than 0.1%. Yet, Zienkiewicz et al. has also been reported a failure of the algorithm to converge at collapse [19,20]. Throughout this study, 500 is set to be the maximum restricted value for the number of steps of time integration. The displacement control finite element is employed satisfying the equilibrium and continuity. Eight-node quadrilateral isoparametric element with reduction of integration technique using a (2x2) Gaussian quadrature is used for all types of foundations to compute their stiffness. To define the bearing pressure, the average of the vertical stress component in the first row of integration points below the displaced nodes is calculated.

3. Definition of the problem

To complete the simulation in this paper, the plane strain solutions are raised to compute the bearing capacity factors for strip footing underlying the general cohesive-frictional soil. The ground surface is horizontal and besides the footing, the soil surface is subjected to the uniform surcharge pressure \( q \), while the footing is subjected to vertical downward load without any shear stress (eccentricity). Smooth footing base is considered assuming there is no mobilized shear stress over the base, while rough footing base has a perfect bounding with the ground underneath. Table 1 shows the summary of the calculation terms for the bearing capacity of strip footings. The footing is placed on a general cohesive-frictional soil with thickness \( h \). The ultimate bearing pressure \( (q_{ul}) \) is evaluated based on the equation (1). The notation \( Q_u \), \( p \), \( b \), used in equation (1) represent the ultimate applied load, the vertical pressure at any point along the footing base, a half-width of strip footing. The mesh of the present analysis is illustrated in Figure 1. It has 1307 and 408 representing the node number and the element number, respectively.
Table 1. A summary of the calculation terms for strip footing.

| Calculation terms                                      | Strip footing                                      |
|--------------------------------------------------------|---------------------------------------------------|
| Plane strain                                           | ✓                                                  |
| Element type: 8 node quadrilateral element             | ✓                                                  |
| Ultimate pressure load                                  | Equation (1)                                       |
|                                                       | $q_{ult} = \frac{Q_{ult}}{b}$, Where $Q_{ult} = \int_{0}^{b} pdy$ |
| Perfectly smooth and perfectly rough bases              | ✓                                                  |
| Soil friction angles ($\phi = 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ, 30^\circ, 35^\circ, 40^\circ, 45^\circ$) | ✓                                                  |

4. Soil constitutive relation

Soil behaviour satisfying the Mohr Coulomb yield criterion with only non-associated flow rules because it tends to treat the actual behaviour of soil as value of its frictional angle is always different from that of angle dilator. Regarding the determination of expansion angle, Vermeer and De Borst (1984) reported that soil dilation angle varies between 0° and 20° and is at least 20° lower than angle of soil friction [21]. Moreover, Cox (2008) utilized the light microscopy (LM) and scanning electron microscopy (SEM) imaging techniques for his experimental works to study the influence of grain shape on dilatancy using 4 different types of sand such as Silica 40 sand; Ottawa 20 sand; Rillito River sand and Santa Cruz River Sand. Melissa found that the limit of soil expansion angle is between 0° to 15° [22]. Hence, the present analysis is conducting by using the non-associated soil with the maximum dilation angle value of 15° if the internal friction angle of soil is greater than 35°.

The material properties of soil throughout the analysis in this paper are assumed such as Young’s modulus $E = 2 \times 10^5$ kN/m²; Poisson’s ratio $\nu = 0.3$, while cohesion $c = 20$ kN/m²; unit weight of soil $\gamma = 20$ kN/m³ and surface surcharge $q = 25$ kN/m². All these parameters are taken into account, if necessary. The friction angles of soil varied from 5° to 45° for the parametric study of bearing capacity factors.

The Mohr-Coulomb yield criterion of soil is described as a function of the three stress invariants $(\sigma_m, \bar{\sigma}, \theta)$ as [23]

$$F = \sigma_m \sin \varphi + \bar{\sigma}K(\theta) - c \cos \varphi$$

(2)

$$K(\theta) = \cos \theta - (\sin \varphi \sin \theta)/\sqrt{3}$$

(3)

is a function controlling the shape surface in the octahedral plane.
5. Results and discussion
The bearing capacity of shallow strip footing based Terzaghi’s superimpose assumption is expressed as:

\[ q_{ult} = c N_c + q_s N_q + \gamma b N_\gamma / 2 \]  

These three dimensionless bearing capacity factors \( N_c, N_q \) and \( N_\gamma \) are assumed to be a function of the soil friction angle. Finite element method has been utilized together with plasticity theory to predict bearing capacity factors. Griffiths used finite elements to compute each of the bearing capacity factors of strip footing in turn as a function of the soil friction angle by assuming (1) weightless soil with cohesion and friction to determine the \( N_c \), (2) weightless, cohesion-less soil with friction receiving a uniform surface surcharge to evaluate \( N_q \) and (3) cohesion-less soil with friction and self-weight to compute \( N_\gamma \) [4]. The present analysis has been conducted for both perfectly smooth and rough footings with the soil friction angle ranging from \( \theta \) to \( 45^\circ \). The vertical displacements applied to the footing are carried out continuously downwards with the constant rate of velocity until soil failure is reached. The soil resistance is based on parameters \( \varphi \) and \( c \) representing the angle of friction and cohesion of soil, respectively.

5.1. Computation of \( N_\gamma \)
Unlike the two previous soils bearing capacity factors \( N_c \) and \( N_q \), there is no closed form solution for \( N_\gamma \) that contributes to bearing capacity because soil self-weight is included. To define solely the contribution of the \( N_\gamma \) term to the total bearing capacity of a footing, the assumption of soil to be weighted, cohesionless with no surcharge acting is made by vanishing the first two terms of the equation (2). In addition, the initial shear strength of soil prior to loading is constant for weightless cohesive-frictional soils and weightless cohesion-less soils with a uniform constant surcharge, while it varies for weighted cohesion-less frictional soil. Consequently, Footing-soil interface has a significantly effect on \( N_\gamma \), however, it has a slight influence on \( N_c \) and \( N_q \). To analyse \( N_\gamma \), the weighted cohesion-less soil is taken into account and its initial vertical stresses are obtained by the product of the unit weight of soil and the distance of the element below the ground surface. The horizontal initial shear stresses in the others two horizontal directions are determined by the multiplication between its vertical stress and the coefficient of earth pressure at rest \( (k_0) \). \( k_0 \) is equal to unity in the present analysis. With the ultimate bearing capacity, \( N_\gamma \) value corresponding to the specified friction angle of soil calculated in the analysis is computed using [4] as:

\[ N_\gamma = 2 \times q_{ult} / (\gamma \times b) \]  

where \( q_{ult} \) is computed from equation (1).
Figure 2 compares the calculation of bearing capacity factor $N_\gamma$ for smooth strip footing from the current analysis with the available solution published in literature. At low and medium friction angles for $\varphi \leq 40^\circ$, the close agreement is found as compared the present solution with the available results in literature such as with the solutions obtained by Hansen and Christensen (1969) employing the method of stress characteristics[8], with results computed by Booker (1969) utilizing stress characteristics method[7], with solution calculated by Bolton and Lau (1993) [5]; Dongdong et al., (2016) [24] and Martin (2004) [25] using method of characteristics and with the mean value between lower and upper bound limit analysis produced by Ukritchon et al., (2003) [14]. However, their solutions are for high friction angles $\varphi > 40^\circ$ noticeably greater than the current solution. Likewise, it also found out that the current solution is considerably smaller than the existing published solution in literature such as Hill (1950) mechanism [4]; Sokolovskii (1965) [10] and Martin (2004) [25] used method of characteristics, Chen (1975) [12] employed upper bound analysis, Griffiths (1982) [4] solved by finite element method, Michalowski (1997) [6] utilized upper bound analysis and Frydman and Burd (1997) [16] used finite different method.

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The analogy of the computed bearing capacity factor $N_\gamma$ for rough strip footing of the present analysis with previously available solution is plotted in Figure 3. The excellent agreement is discovered for low and medium friction angles for $\varphi \leq 40^\circ$, when the current result is compared to available results which are produced by (1) Meyerhof (1963) using Semi-empirical method [14], (2) Hansen and Christensen (1969) employing the method of stress characteristics [8], (3) Booker (1969) utilizing the method of stress characteristics [7], (4) Griffiths (1982) applying finite element method [4], (5) Ukritchon et al., (2003) considering the mean value of $N_\gamma$ from lower and upper bound limit analysis [14], (6) Kumar (2003) employing limit equilibrium [11], (7) Martin (2004) using method of stress characteristics [26], (8) Hjiaj (2005) utilizing upper bound analysis [26] and (9) Snodi (2011) employing the method of characteristics [27]. Nevertheless, their solutions are noticeably greater than the current solution for high friction angles $\varphi > 40^\circ$. On the other hand, at low and medium friction angles for $\varphi \leq 30^\circ$, the present result is slight lower as analogized to the existing solutions calculated.
by Terzaghi (1943) (used limit equilibrium) [1], Caquot and Kerisel (1953) (employed the method of stress characteristics) [14], Vesic (1973) (utilized limit equilibrium) [14], Michalowski (1997) (applied upper bound analysis) [6], Frydman and Burd (1997) (based on finite difference method) [16] and Soubra (1999) (employed upper bound analysis) [15]. Though, the noticeable discrepancies appear for $\varphi > 30^\circ$. Furthermore, the overestimation of $N_\gamma$ over the current analysis for rough strip footing have presented such as Prandtl (1921) applying limit equilibrium [4], Chen (1975) using upper bound analysis [12] and Bolton and Lau (1993) [5] and Han et al., [24] employing the method of stress characteristics [5].

![Figure 3. Comparison of bearing capacity factor $N_\gamma$ of smooth strip footing from published solutions and present analysis.](image)

6. Conclusion

Bearing capacity factor $N_\gamma$ was computed for a strip footing using FEM. The analysis was executed by considering the actual properties of soil in which the dilation angle is generally less than $15^\circ$. The results reveal in close agreement with the existing published data. The assumption of additive contribution of soil cohesion, surcharge and weight was proposed by Terzaghi provides bearing capacity predictions sufficiently accurate. Bearing capacity factors depend on the angle of dilation, especially for higher values of the friction angle. The magnitudes of the factors for a rough footing base, especially with greater value of phi, were significantly higher than those for a smooth footing.

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