On a phase field model of Cahn–Hilliard type for tumour growth with mechanical effects

Harald Garcke * Kei Fong Lam † Andrea Signori ‡

December 5, 2019

Abstract
Mechanical effects have mostly been neglected so far in phase field tumour models that are based on a Cahn–Hilliard approach. In this paper we study a macroscopic mechanical model for tumour growth in which cell-cell adhesion effects are taken into account with the help of a Ginzburg–Landau type energy. In the overall model an equation of Cahn–Hilliard type is coupled to the system of linear elasticity and a reaction-diffusion equation for a nutrient concentration. The highly non-linear coupling between a fourth-order Cahn–Hilliard equation and the quasi-static elasticity system lead to new challenges which cannot be dealt within a gradient flow setting which was the method of choice for other elastic Cahn–Hilliard systems. We show existence, uniqueness and regularity results. In addition, several continuous dependence results with respect to different topologies are shown. Some of these results give uniqueness for weak solutions and other results will be helpful for optimal control problems.

Key words. Tumour growth, Cahn–Hilliard equation, mechanical effects, linear elasticity, elliptic-parabolic system, existence and uniqueness, continuous dependence.

AMS subject classification. 35K25, 35K57, 74B05.

1 Introduction
Modelling of tumour growth is one of the challenging frontiers of applied mathematics. In the last years phase field models for tumour growth have been studied intensively. Alike classical free boundary models they use a continuum approach to describe the growth of tumours. However, an advantage to free boundary models is that phase field models allow for topology changes like break up and coalescence. In phase field models an order parameter is introduced to describe the tumour fraction locally in space. In this paper the order parameter is denoted by $\varphi$ and it will take the value $+1$ in regions occupied solely by tumour cells and $-1$ in regions occupied solely by healthy cells. As summarised in Lima et al. [37, 38] stress effects resulting from tumour growth severely affect the growth itself. Experimental studies, see [6, 30, 45], show that stresses can inhibit tumour growth. In this
paper we will consider the effect of stresses on the mobility and on the proliferation rate. In particular, the mobility and the proliferation rate will decrease with increasing stresses. Often mechanically-coupled models for tumour growth use reaction-diffusion systems to model proliferation and nutrient diffusion with specific body force fields that take elastic effects into account, see \[31, 38, 44\]. However, in the recent work of Lima et al. \[37\] on selection, calibration and validation of different models of tumour growth, it turned out that phase field methods taking elastic effects into account are the best modelling approach and in particular superior to reaction-diffusion models, see the conclusion section of \[37\]. It is the goal of this paper to generalise the model studied in \[37, 38\] and in particular also take nutrient diffusion into account as the latter definitely will have an important effect on tumour growth.

We will consider balance equations for the tumour and nutrient concentrations which will be of parabolic type. As diffusion and growth take place on a timescale much larger than that associated with inertia, we disregard inertial terms and consider instead a quasi-static approximation. We hence consider the following extension of the phase field tumour model proposed by Lima et al. \[37, 38\]. Let \(\Omega \subset \mathbb{R}^d, d = 2, 3\), denote a \(C^{1,1}\) or convex bounded domain, with boundary \(\Gamma := \partial \Omega\) and let \(\Gamma_D, \Gamma_N \subset \Gamma\) are relatively open such that

\[
\Gamma = \Gamma_D \cup \Gamma_N, \quad \Gamma_D \cap \Gamma_N = \emptyset, \quad |\Gamma_D| > 0,
\]

where \(|\Gamma_D|\) stands for the \((d - 1)\)-dimensional Hausdorff measure of \(\Gamma_D\). For a fixed but arbitrary time \(T > 0\), we consider the system

\[
\begin{align*}
\varphi_t &= \text{div}(m(\varphi, \sigma, u, E(u))\nabla \mu) + U(\varphi, \sigma, E(u)) \quad \text{in } Q := \Omega \times (0, T), \quad (1.1a) \\
\mu &= -\varepsilon \Delta \varphi + \varepsilon^{-1} \psi'(\varphi) - \chi \sigma + W_{\varphi}(\varphi, E(u)) \quad \text{in } Q, \quad (1.1b) \\
W_{\varphi} &= \frac{1}{2}(E(u) - \bar{E}(\varphi)) : C'(\varphi)(E(u) - \bar{E}(\varphi)) - C(\varphi)(E(u) - \bar{E}(\varphi)) : \bar{E}'(\varphi), \quad (1.1c) \\
\beta \sigma_t &= \Delta \sigma + S(\varphi, \sigma) \quad \text{in } Q, \quad (1.1d) \\
0 &= \text{div}(W_{\varphi}(\varphi, E(u))) \quad \text{in } Q, \quad (1.1e) \\
W_{\varphi} &= C(\varphi)(E(u) - \bar{E}(\varphi)), \quad (1.1f) \\
\varphi(., 0) &= \varphi_0(.,) \quad \text{in } \Omega, \quad (1.1g) \\
0 &= \partial_n \varphi = \partial_n \mu, \quad \partial_n \sigma + \kappa(\sigma - \sigma_B) = 0 \quad \text{on } \Sigma := \Gamma \times (0, T), \quad (1.1h) \\
u &= 0 \quad \text{on } \Sigma_D := \Gamma_D \times (0, T), \quad (1.1i) \\
W_{\varphi} n &= g \quad \text{on } \Sigma_N := \Gamma_N \times (0, T). \quad (1.1j)
\end{align*}
\]

In the above equations, the primary variables of the model are \(\varphi\) (the difference in volume fractions between the tumour and healthy cells), \(\mu\) (the associated chemical potential), \(\sigma\) (the nutrient concentration) and \(u\) (the displacement). Moreover, \(n\) indicates the outward unit normal of \(\Gamma\) and \(\partial_n\) stands for the outward normal derivative. The quantity \(E(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)\) is the symmetric strain tensor, \(\psi'\) is the derivative of a double-well function \(\psi\) (with the classical example being \(\psi(s) = (s^2 - 1)^2\)), \(W_{\varphi}, W_{\varphi}E\) denote the partial derivatives of the elastic energy \(W(\varphi, E(u))\) with respect to its arguments, and \(\varepsilon > 0\) is a positive parameter associated to the thickness of the interfacial layer. It is worth noting that, in the phenomena we are interested in, the strain is usually small so that the linearised strain tensor is used.

Equations (1.1a)-(1.1j) comprises of a Cahn–Hilliard system for \((\varphi, \mu)\) with a positive mobility \(m(\varphi, \sigma, u, E(u))\) and a source term \(U(\varphi, \sigma, E(u))\) modelling the growth and death...
of cells. As one biologically relevant example for the source term $U$ we suggest

$$U(\varphi, \sigma, \mathcal{E}(u)) = \frac{\lambda_p f(\varphi)\sigma}{1 + |W_\mathcal{E}(\varphi, \mathcal{E}(u))|} - \lambda_\alpha k(\varphi),$$

(1.2)

for some bounded functions $f(\varphi)$ and $k(\varphi)$. The coefficients $\lambda_p \geq 0$ and $\lambda_\alpha \geq 0$ have the meaning of proliferation and apoptosis rates, respectively. As discussed in [3, p. 353], one should also account for the effects of mechanical interactions in cell growth, such as the stress exerted on the replicating cells by the surrounding environments which leads to a strong dependence of cellular proliferation on mechanical stresses. The above choice (1.2) ensures that as the magnitude of the stress $S := W_\mathcal{E}(\varphi, \mathcal{E}(u))$ increases, the effects of proliferation are reduced. Other possible forms of $U$ include the von Mises stress as a stress measure, see [37], and could also be used in the theory stated later.

In (1.1d), directed movement of the tumour cells by chemotaxis is captured in the term $-\chi \sigma$, so that $\chi \geq 0$ is a chemotactic sensitivity, and the effects of elastic deformation on the movement of the tumour cells is prescribed by the term $W_\mathcal{E}(\varphi, \mathcal{E}(u))$, whose full expression is given in (1.1e).

In (1.1c), $C(\varphi)$ is a symmetric and positive definite elasticity tensor depending on $\varphi$ and $\mathcal{E}(\varphi)$ is the stress free strain (strain due to growth), see e.g. [18, 28, 37, 38]. We assume that the evolution of the nutrient can be described by a reaction-diffusion equation (1.1d), where $S(\varphi, \sigma)$ is a term accounting for sources and sinks in the nutrient density. One example is

$$S(\varphi, \sigma) = -\lambda_c h(\varphi)\sigma + B(\sigma_c - \sigma)$$

(1.3)

for some non-negative and bounded function $h(\varphi)$, and the coefficient $\lambda_c \geq 0$ has the meaning of a consumption rate. Meanwhile, the term $B(\sigma_c - \sigma)$ models the supply of nutrients from nearby capillaries, so that $B \geq 0$ is a constant supply rate, and $\sigma_c$ is the nutrient concentration from the capillaries. Furthermore, after non-dimensionalization, the prefactor $\beta > 0$ in front of the time derivative can be interpreted as the ratio between the nutrient diffusion timescale and the tumour doubling timescale. In many instances, $\beta$ is small, and it makes sense to consider $\beta = 0$ to obtain a quasi-static approximation. In fact, the same is done for the equation of mechanical stress, which we assume there is an instantaneous relaxation into mechanical equilibrium, leading to the equation (1.1f).

For boundary conditions, we consider the no-flux condition $\partial_\nu \mu = 0$, and $\partial_\nu \varphi = 0$ for the Cahn–Hilliard component. For $\kappa > 0$, we have a Robin boundary condition for $\sigma$, where $\sigma_B$ can be seen as the nutrient concentration supplied on the boundary, and in the case $\kappa = 0$ we return to a no-flux condition for the nutrient. For the deformation $u$, we postulate a zero Dirichlet condition on $\Gamma_D$ to take into account the possible presence of a rigid part of the body such as a bone which prevents variations of the displacement, and a Neumann condition on $\Gamma_N$ so that the normal component of the stress $S = W_\mathcal{E}$ on $\Gamma_N$ is equal to some load given by a fixed source $g$.

Let us comment that the dependence of the mobility $m$ on the displacement $u$ and the stress $\mathcal{E}(u)$ is based on the observation that a tumour induces significant mechanical stresses on the surrounding tissue during growth, and thus can lead to an inhibition of further growth [32]. A possible volume free energy of the system is the following:

$$E(\varphi, \sigma, u) := \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} |\psi(\varphi)|^2 + \frac{\beta}{2} |\sigma|^2 + W(\varphi, \mathcal{E}(u)) dx,$$

(1.4)

$$W(\varphi, \mathcal{E}(u)) = \frac{1}{2}(\mathcal{E}(u) - \mathcal{E}(\varphi)) : C(\varphi)(\mathcal{E}(u) - \mathcal{E}(\varphi)).$$
where the first two terms of $E$ yields the Ginzburg–Landau energy, the third term is a nutrient free energy, and the last term is the elastic energy. We note that in the case $\beta = 0$, the nutrient evolves quasi-statically and thus the nutrient free energy is not present in (1.4).

For more information on modelling for tumour growth models in the context of Cahn–Hilliard type models we refer to the book of Cristini and Lowengrub [11] and to the articles [22, 23, 32]. Analytical aspects for tumour models based on a Cahn–Hilliard equation coupled to a nutrient reaction-diffusion equation have been studied in [9, 10, 15, 20, 41]. Well-posedness result for extended models of Cahn–Hilliard systems coupled to a flow field have been studied in [14, 16, 21, 39]. Pioneering numerical simulations showing in particular that Cahn–Hilliard type models can describe the invasive behaviour of tumours are due to Cristini, Lowengrub, Wise and coworkers [12, 13, 47]. For recent numerical computations for extended models we refer to [1, 22, 29]. Cahn–Hilliard models with mechanical effects have been first introduced by Cahn and Larché [34] and Onuki [43] and later derived systematically from thermodynamic principles by Gurtin [28]. Analytical results for this so-called Cahn–Larché system are due to [3, 5, 18, 19] and for a numerical treatment of Cahn–Larché systems we refer to [24, 26, 36].

Following this Introduction, Section 2 gives an account of the main results of this paper (existence, uniqueness, regularity and continuous dependence results). In Section 3 we prove the existence result with a non-standard Galerkin approach. Section 4 studies a quasi-static limit and in Section 5 further regularity is shown. Under the assumption that the elasticity tensor is constant we show continuous dependence and uniqueness results in the final Section 6.

2 Main results

We denote the standard Lebesgue and Sobolev spaces over $\Omega$ by $L^p := L^p(\Omega)$ and $W^{k,p} := W^{k,p}(\Omega)$, for $p \in [1, \infty]$ and $k > 0$, and denote the corresponding norms by $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^{k,p}}$. In the case $p = 2$, we use the notation $H^k := H^k(\Omega) = W^{k,2}(\Omega)$ and the norm $\| \cdot \|_H^k$. For any Banach space $Z$, we denote its dual by $Z'$, and the corresponding duality pairing by $\langle \cdot, \cdot \rangle_Z$. When $Z = H^1(\Omega)$, we use the notation $\langle \cdot, \cdot \rangle_H^1$. The $L^2(\Omega)$-inner product is denoted by $\langle \cdot, \cdot \rangle$, while the $L^2(\Gamma)$ and $L^2(\Gamma_N)$-inner products are denoted by $\langle \cdot, \cdot \rangle_\Gamma$ and $\langle \cdot, \cdot \rangle_{\Gamma_N}$, respectively. For Lebesgue spaces and Sobolev spaces over $\Gamma$, we use the notation $L^p_\Gamma := L^p(\Gamma)$ and $W^{k,p}_\Gamma := W^{k,p}(\Gamma)$, respectively, along with the norms $\| \cdot \|_{L^p_\Gamma}$ and $\| \cdot \|_{W^{k,p}_\Gamma}$. We define the Sobolev space $H^1_0(\Omega)$ as the set $\{ f \in H^1(\Omega) : \partial_n f = 0 \text{ on } \Gamma \}$, and for the displacement $u$, we introduce the following function space:

$$X(\Omega) := \{ f \in H^1(\Omega)^d : f|_{\Gamma_D} = 0 \}. $$

Notice that by [7, Thm. 6.15-4, pp. 409–410], a Korn-type inequality is valid in $X(\Omega)$: there exists a constant $C_K > 0$ such that

$$\| u \|_{H^1} \leq C_K \| \mathcal{E}(u) \|_{L^2} \quad \forall u \in X(\Omega). \tag{2.1}$$

A weak solution to (1.1) is defined as follows:

**Definition 2.1 (Weak solution).** We say that $(\varphi, \mu, \sigma, u)$ is a weak solution to (1.1) if

$$\varphi, \sigma \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^1(\Omega)'), \quad \mu \in L^2(0,T; H^1(\Omega)), \quad u \in L^2(0,T; X(\Omega)), $$

4
with \( \varphi(0) = \varphi_0, \sigma(0) = \sigma_0 \) in \( L^2(\Omega) \) and

\[
0 = \int_0^T \langle \varphi_t, \zeta \rangle + (m(\varphi, \sigma, \mathcal{E}(\mathbf{u})) \nabla \mu, \nabla \zeta) - (U(\varphi, \sigma, \mathcal{E}(\mathbf{u})), \zeta) \, dt,
\]

\[
0 = \int_0^T (\mu, \xi) - \varepsilon(\nabla \varphi, \nabla \xi) - \varepsilon^{-1}(\psi'(\varphi), \xi) + \chi(\sigma, \xi) - (W(\psi(\varphi), \mathcal{E}(\mathbf{u})), \xi) \, dt,
\]

\[
0 = \int_0^T \beta(\sigma_t, \zeta) + (\nabla \sigma, \nabla \zeta) + \kappa(\sigma - \sigma_B, \zeta) - (S(\varphi, \sigma), \zeta) \, dt,
\]

\[
0 = \int_0^T (C(\varphi)(\mathcal{E}(\mathbf{u}) - \hat{\mathcal{E}}(\varphi)), \nabla \eta) - (g, \eta)_{\Gamma_N} \, dt
\]

for all \( \zeta \in L^2(0, T; \mathcal{H}^1(\Omega)), \xi \in L^2(0, T; \mathcal{H}^1(\Omega)) \cap L^\infty(Q) \) and \( \eta \in L^2(0, T; X(\Omega)) \).

The first main result of this work concerns the existence of weak solutions to (1.1) and is formulated as follows.

**Theorem 1 (Existence).** Let \( \Omega \subset \mathbb{R}^d, d = 2, 3 \) be a bounded domain with either has a \( C^{1,1} \)-boundary or \( \Omega \) is convex. Suppose

(A1) \( g \in L^2(\Gamma_N)^d, \varepsilon \) is a positive constant and \( \beta, B, \kappa, \chi \) are non-negative constants such that at least one of \( \{B, \kappa\} \) is non-zero if \( \beta = 0 \).

(A2) \( \psi = \psi_1 + \psi_2 \) is non-negative with \( \psi_1, \psi_2 \in C^2(\mathbb{R}) \), \( \psi_1 \) is convex, \( \psi'(0) = 0 \), and

\[
\forall \delta > 0 \exists C_\delta > 0 : |\psi_1'(s)| \leq \delta \psi_1(s) + C_\delta \text{ for all } s \in \mathbb{R},
\]

\[
\exists C_1 > 0 : |\psi_2''(s)| \leq C_1 \text{ for all } s \in \mathbb{R}.
\]

(A3) \( m \in C^0(\mathbb{R}, \mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d \times d}), f, h, k \in C^1(\mathbb{R}) \) and there exist positive constants \( C_2, C_3 \) such that

\[
0 \leq h(s) \leq 1, \quad |f(s)| \leq 1, \quad |k(s)| \leq 1, \quad C_2 \leq m(s, t, x, A) \leq C_3
\]

for all \( s, t \in \mathbb{R}, x \in \mathbb{R}^d, \text{ and } A \in \mathbb{R}^{d \times d} \).

(A4) For the elastic energy \( W \), we postulate

\[
W(s, \mathcal{E}) = \frac{1}{2} (\mathcal{E} - \hat{\mathcal{E}}(s)) : C(s)(\mathcal{E} - \hat{\mathcal{E}}(s)), \quad s \in \mathbb{R}, \mathcal{E} \in \mathbb{R}^{d \times d}
\]

where the elasticity tensor \( C(s) \) is bounded, Lipschitz continuous and differentiable, while the stress-free strain \( \hat{\mathcal{E}}(s) \) is Lipschitz continuous and differentiable. In addition, we require that \( C(s) \) fulfills the usual symmetry conditions of linear elasticity and that there exists a \( C_4 > 0 \) such that for all \( s \in \mathbb{R}, \mathcal{E} \in \mathbb{R}_{\text{sym}}^{d \times d} \)

\[
C_4 |\mathcal{E}|^2 \leq \mathcal{E} : C(s)\mathcal{E}.
\]

(A5) \( \lambda_p, \lambda_o, \lambda_c : [0, T] \to \mathbb{R} \) are non-negative continuous functions, while \( \sigma_c : \Omega \times (0, T) \to \mathbb{R} \) and \( \sigma_B : \Gamma \times (0, T) \to \mathbb{R} \) are non-negative measurable and bounded functions.

(A6) \( \varphi_0 \in H^1(\Omega) \) with \( \psi(\varphi_0) \in L^1(\Omega) \) and \( \sigma_0 \in L^2(\Omega) \) with

\[
0 \leq \sigma_0 \leq \max \left( \| \sigma_c \|_{L^\infty(Q)}, \| \sigma_B \|_{L^\infty(\Sigma)} \right) =: M.
\]
Then, there exists at least one weak solution to (1.1) in the sense of Definition 2.1 such that

\[ 0 \leq \sigma(x,t) \leq M \quad \forall (x,t) \in Q, \tag{2.4} \]

and there exists a positive constant \( C = C(\|\varphi_0\|_{H^1}, \|\psi(\varphi_0)\|_{L^1}, \|g\|_{L^2(\Gamma_N)}) \), but not on \( \|\sigma_0\|_{L^2} \) such that for a.e. \( s \in (0,T) \), the following energy inequality holds

\[
\sup_{s \in (0,T)} \left( \|\varphi(t)\|_{H^1}^2 + \|\psi(\varphi(t))\|_{L^1} + \beta\|\sigma(t)\|_{L^2}^2 + \|u(t)\|_{X(\Omega)}^2 \right)
+ \|\varphi\|_{H^1(0,T;H^1(\Omega)^\prime)}^2 + \beta\|\sigma\|_{H^1(0,T;H^1(\Omega)^\prime)}^2 + \|\mu\|_{L^2(0,T;H^1)}^2
+ \|\sigma\|_{L^2(0,T;H^1)}^2 \leq C(1 + \beta\|\sigma_0\|_{L^2}^2). \tag{2.5} \]

It is worth noting that assumption (A4) implies that \( W \in C^1(\mathbb{R} \times \mathbb{R}^{d \times d}, \mathbb{R}) \) is non-negative with \( W(s,M) = W(s,M^t) \) for all \( s \in \mathbb{R} \) and \( M \in \mathbb{R}^{d \times d} \), and there exist positive constants \( C_4, C_5 \) such that for all \( s \in \mathbb{R}, M_1, M_2 \in \mathbb{R}^{d \times d} \) and \( E \in \mathbb{R}_{\text{sym}} \),

\[
(W,e(s,M_1) - W,e(s,M_2)) \cdot (M_1 - M_2) \geq C_4 |M_1 - M_2|^2, \tag{2.6} \]

\[
|W(e,E)| + |W(e,E)\varphi| \leq C_5(1 + \rho|^2 + |E|^2), \tag{2.7} \]

\[
|W(e,E)| \leq C_5(1 + \rho + |E|). \tag{2.8} \]

Moreover, assumption (A2) postulates that the derivative of the convex part \( \psi_1 \) can be bounded by \( \psi \). This requirement covers the case of regular and polynomial growth potentials, so, for instance, the standard choice \( \psi(s) = (s^2 - 1)^2 \) is allowed.

**Remark 2.1.** Let us remark that the requirement \( u = 0 \) on \( \Sigma_D \) is just to avoid non-necessary technicalities. In fact, the same procedure presented here will be enough to handle the case \( u = f \) for some given source \( f \neq 0 \). Indeed, it suffices to set \( w := u - f \) and solve the problem for this auxiliary variable \( w \) which now enjoys the same condition as (1.1). Moreover, let us claim that also the case in which \( |\Sigma_D| = 0 \) can be handled by arguing as in [17], [18].

It will turn out that the estimates for the solutions are uniform in \( \beta \in (0,1) \). This allows us to deduce the quasi-static limit \( \beta \to 0 \) which is formulated as follows.

**Theorem 2 (Quasi-static limit).** For each \( \beta \in (0,1) \), let \( (\varphi_\beta, \mu_\beta, \sigma_\beta, u_\beta) \) denote a weak solution to (1.1) obtained from Theorem 7 with corresponding initial data \( (\varphi_0, \sigma_0) \). Then, there exist limit functions \( (\varphi_*, \mu_*, \sigma_*, u_*) \) such that, along a non-relabelled subsequence,

\[
\varphi_\beta \to \varphi_* \text{ weakly* in } L^\infty(0,T;H^1) \cap H^1(0,T;H^1)^\prime, \]

and strongly in \( C^0([0,T];L^r(\Omega)) \) and a.e. in \( Q \),

\[
\mu_\beta \to \mu_* \text{ weakly in } L^2(0,T;H^1), \]

\[
u_* \to \nu_* \text{ weakly* in } L^\infty(0,T;\mathcal{X}(\Omega)), \]

and strongly in \( L^2(0,T;\mathcal{X}(\Omega)) \) and a.e. in \( Q \),

\[
\beta \sigma_\beta \to 0 \text{ weakly in } L^2(0,T;H^1(\Omega)^\prime), \]

\[
\sigma_* \to \sigma_* \text{ strongly in } L^2(0,T;H^1) \text{ and a.e. in } Q, \tag{2.9} \]

for any \( r < \infty \) in two spatial dimensions and any \( r < 6 \) in three spatial dimensions. Furthermore, \( (\varphi_*, \mu_*, \sigma_*, u_*) \) satisfies (2.2a), (2.2b), (2.2c) and

\[
0 = \int_0^T (\nabla \sigma_*, \nabla \zeta) + \kappa(\sigma_* - \sigma_B, \zeta)_\Gamma - (S(\varphi_*, \sigma_*), \zeta) dt, \tag{2.10} \]
for all \(\zeta \in L^2(0, T; H^1(\Omega)), \xi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)\) and \(\eta \in L^2(0, T; X(\Omega))\), the initial condition \(\varphi_\ast(0) = \varphi_0\), the boundedness property (2.4), as well as the energy inequality (2.5) with \(\beta = 0\).

Next, we present a series of regularity assertions for the weak solutions to (1.1).

**Theorem 3 (Regularity).** We assume (A1)-(A6) and denote by \((\varphi, \mu, \sigma, u)\) a weak solution to (1.1) obtained from Theorem 1. Then, there exists an exponent \(p > 2\) such that
\[
\nabla u \in L^\infty(0, T; L^p(\Omega)^{d\times d}).
\]

Moreover, we have the following:

1. If \(\sigma_0 \in H^1(\Omega)\) and \(\sigma_B \in H^1(0, T; L^2(\Gamma))\), then
   \[
   \sigma \in L^\infty(0, T; H^1(\Omega)) \quad \forall \beta \in [0, \infty), \quad \sigma \in H^1(0, T; L^2(\Omega)) \quad \forall \beta > 0.
   \]
   Furthermore, if \(\Omega\) has a \(C^{1,1}\)-boundary and \(\sigma_B\) also belongs to \(L^2(0, T; H^{1/2}(\Gamma))\), then
   \[
   \sigma \in L^2(0, T; H^2(\Omega)) \quad \forall \beta \in [0, \infty).
   \]

2. Suppose 
   (B1) The stress-free strain \(\bar{E}(\varphi)\) satisfies the affine linear ansatz (Vegard’s law)
   \[
   \bar{E}(\varphi) = \hat{E} + \mathcal{E}^* \varphi,
   \]
   where \(\hat{E}\) and \(\mathcal{E}^*\) are constant symmetric tensors.
   (B2) The elasticity tensor \(C(\varphi) = C\) is a constant, positive definite and symmetric tensor,
   hold, then
   \[
   \psi_1'(\varphi) \in L^2(Q), \quad \varphi \in L^2(0, T; H^2_n(\Omega)).
   \]

**Remark 2.2.** The regularity assertion (2.11) on the displacement field \(u\) follows from the proof of [44, Theorem 1.1] by choosing \(A_{ijkl} = C(\varphi)_{ijkl}, f_i = 0, f_{ij} = C(\varphi)_{ijkl}(\mathcal{E}(\varphi))_{kl}\) and \(\tau = g + C(\varphi)\mathcal{E}(\varphi)n\). Hence, we omit the details of the proof.

Our last results state the continuous dependence of the weak solutions to (1.1) on the initial conditions and the data, and subsequently leads to the uniqueness of solutions.

**Theorem 4 (Continuous dependence).** Further to (A1)-(A6), (B1)-(B2), we assume

(C1) The mobility \(m(\varphi, \sigma, u, \mathcal{E}(u))\) is taken to be a constant (w.l.o.g. we set it to be 1).

(C2) The functions \(f, h\) and \(k\) are Lipschitz continuous, whose Lipschitz constants we shall denote by a common notation \(L > 0\).

(C3) The convex part \(\psi_1\) of the potential \(\psi\) satisfies
   \[
   |\psi_1'(s) - \psi_1'(r)| \leq C(1 + |s|^q + |r|^q)|s - r| \quad \text{for all } s, r \in \mathbb{R},
   \]
   and the derivative of the non-convex part \(\psi_2\) is Lipschitz continuous (again we denote the Lipschitz constant by \(L\)). The exponent \(q \in \{2, 4\}\) is specified below depending on the norms involved.
Let
\[
X_q := \begin{cases} 
  L^2 & \text{if } q = 2, \\
  H^1(\Omega)' & \text{if } q = 4, \\
\end{cases} \quad \mathcal{P}_q := \begin{cases} 
  L^\infty & \text{if } q = 2, \\
  L^2 & \text{if } q = 4. \\
\end{cases}
\]

Then, for any pair \(\{(\varphi_i, \mu_i, \sigma_i, u_i)\}_{i=1,2}\) of weak solutions to \(\text{(1.1)}\) corresponding to data \(\{(\varphi_{0,i}, \sigma_{0,i}, g_i, \sigma_{c,i}, \sigma_{B,i})\}_{i=1,2}\), there exists a positive constant \(K_1\) not depending on the differences \(\varphi_1 - \varphi_2, \mu_1 - \mu_2, \sigma_1 - \sigma_2\) and \(u_1 - u_2\), as well as \(\beta\), such that
\[
\begin{align*}
\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;X_q)}^2 &+ \|\mu_1 - \mu_2\|_{L^2(0,T;H^1)}^2 + \beta\|\sigma_1 - \sigma_2\|_{L^\infty(0,T;L^2)}^2 \\
&+ \|\sigma_1 - \sigma_2\|_{L^2(0,T;H^1)}^2 + \|u_1 - u_2\|_{\mathcal{P}_q(0,T;X(\Omega))}^2 + \|\mu_1 - \mu_2\|_{L^2(0,T;X_q)}^2 \\
&\leq K_1 \left(\|\varphi_{0,1} - \varphi_{0,2}\|_{X_q}^2 + \beta\|\sigma_{0,1} - \sigma_{0,2}\|_{L^2}^2\right) \\
&\quad + K_1 \left(\|g_1 - g_2\|_{L^2(\Gamma_N)}^2 + \|\sigma_{B,1} - \sigma_{B,2}\|_{L^2(\Sigma)}^2 + \|\sigma_{c,1} - \sigma_{c,2}\|_{L^2(Q)}^2\right). \\
\end{align*}
\] (2.12)

In particular, the weak solutions to both \(\text{(1.1)}\) and its quasistatic variant are unique.

Under further assumptions on the convex part \(\psi_1\), we obtain the following continuous dependence in stronger norms with a time discretisation approach.

**Theorem 5.** Further to \((\text{A1}) - \text{A6}, \text{B1} - \text{B2}, \text{C1} - \text{C3}\) with exponent \(q = 2\), we assume that \(\Omega\) has a \(C^{1,1}\)-boundary and

(C4) The convex part \(\psi_1\) of the potential \(\psi\) satisfies
\[
|\psi''_1(s) - \psi''_1(r)| \leq C(1 + |s| + |r|)|s - r|, \quad \text{for all } s, r \in \mathbb{R}.
\]

Then, for any pair \(\{(\varphi_i, \mu_i, \sigma_i, u_i)\}_{i=1,2}\) of weak solutions to \(\text{(1.1)}\) corresponding to data \(\{\varphi_{0,i}, \sigma_{0,i}, g_i, \sigma_{c,i}, \sigma_{B,i}\}_{i=1,2}\), there exists a positive constant \(K_2\) not depending on the differences \(\varphi_1 - \varphi_2, \mu_1 - \mu_2, \sigma_1 - \sigma_2\) and \(u_1 - u_2\), as well as \(\beta\), such that
\[
\begin{align*}
\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H^1)}^2 &+ \|\mu_1 - \mu_2\|_{L^2(0,T;H^1)}^2 \\
&\leq K_2 \left(\|\varphi_{0,1} - \varphi_{0,2}\|_{H^1}^2 + \beta\|\sigma_{0,1} - \sigma_{0,2}\|_{L^2}^2\right) \\
&\quad + K_2 \left(\|g_1 - g_2\|_{L^2(\Gamma_N)}^2 + \|\sigma_{B,1} - \sigma_{B,2}\|_{L^2(\Sigma)}^2 + \|\sigma_{c,1} - \sigma_{c,2}\|_{L^2(Q)}^2\right).
\end{align*}
\]

Moreover, for any \(\beta > 0\) and data \(\{\sigma_{0,i}, \sigma_{B,i}\}_{i=1,2}\) satisfying \(\sigma_{0,i} \in H^1(\Omega)\) and \(\sigma_{B,i} \in H^1(0,T;L^2(\Gamma))\), there is a constant \(K_3\) independent of the differences \(\varphi_1 - \varphi_2, \mu_1 - \mu_2, \sigma_1 - \sigma_2\) and \(u_1 - u_2\), such that
\[
\|\sigma_1 - \sigma_2\|_{H^1(0,T;L^2) \cap L^\infty(0,T;H^1)}^2 \leq K_3 \left(\|\varphi_{0,1} - \varphi_{0,2}\|_{H^1}^2 + \|\sigma_{0,1} - \sigma_{0,2}\|_{H^1}^2 + \|g_1 - g_2\|_{L^2(\Gamma_N)}^2\right) \\
&\quad + K_3 \left(\|\sigma_{B,1} - \sigma_{B,2}\|_{H^1(0,T;L^2)}^2 + \|\sigma_{c,1} - \sigma_{c,2}\|_{L^2(Q)}^2\right).
\]

Lastly, if \(\sigma_{B,1}, \sigma_{B,2}\) also belong to \(L^2(0,T;H^{1/2}(\Gamma))\), we also have
\[
\|\sigma_1 - \sigma_2\|_{L^2(0,T;H^1)}^2 \leq K_3 \left(\|\varphi_{0,1} - \varphi_{0,2}\|_{H^1}^2 + \|\sigma_{0,1} - \sigma_{0,2}\|_{H^1}^2 + \|g_1 - g_2\|_{L^2(\Gamma_N)}^2\right) \\
&\quad + K_3 \left(\|\sigma_{B,1} - \sigma_{B,2}\|_{H^1(0,T;L^2)}^2 + \|\sigma_{c,1} - \sigma_{c,2}\|_{L^2(Q)}^2\right).
\]

Notice that conditions \((\text{C3})\) and \((\text{C4})\) still comply for the classical quartic potential \(\psi(s) = (s^2 - 1)^2\).
3 Existence

Due to the presence of the source terms $U$ and $S$, the system (1.1) does not admit a variational structure, and so an implicit time discretisation as the one used in [17] [18] may no longer be applicable. Hence, we consider a Faedo–Galerkin approximation to establish a weak solution to system (1.1).

3.1 Galerkin approximation

Let us point out that, since [A2] allows for the potential $\psi$ to have arbitrary polynomial growth, in the first step we prove Theorem 1 with a more regular initial condition $\varphi_0 \in H^1_0(\Omega)$, and then in Section 3.5 show how to complete the proof for an initial condition satisfying just (A6). To this end, let us consider

- $\{z_i\}_{i \in \mathbb{N}}$ as the set of eigenfunctions of the Neumann-Laplacian operator that is orthonormal in $L^2(\Omega)$ and orthogonal in $H^1(\Omega)$ with $z_1$ is the constant function $(\frac{1}{|\Omega|})^{1/2}$ and $(z_i, 1) = 0$ for $i \geq 2$. In [21] §3 it is also shown that $\{z_i\}_{i \in \mathbb{N}}$ forms a basis of $H^1_0(\Omega)$;

- $\{y_i\}_{i \in \mathbb{N}}$ as a Schauder basis of $X(\Omega)$, see [2]. One can choose for example eigenfunctions of a corresponding boundary value problem for an elasticity system which leads to a basis orthogonal in $L^2(\Omega)^d$ (see [35] Thm. 3.12.1, pp. 219-220).

Next, we define finite-dimensional spaces $Z_k$ and $Y_k$ as the linear span of the first $k$ functions of $\{z_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$, respectively, and we denote by $\Pi_k$ the $L^2$-projection onto the space $Z_k$. Then, the Faedo–Galerkin approximation of (2.2a)-(2.2d) reads as: for any $k \in \mathbb{N}$ find $(\varphi_k, \mu_k, \sigma_k, \mathbf{u}_k)$ of the form

$$
\varphi_k = \sum_{i=1}^{k} a_i^{k}(t)z_i(x), \quad \mu_k = \sum_{i=1}^{k} b_i^{k}(t)z_i(x), \quad \sigma_k = \sum_{i=1}^{k} c_i^{k}(t)z_i(x), \quad \mathbf{u}_k = \sum_{i=1}^{k} d_i^{k}(t)y_i(x),
$$

satisfying for a.e. $t \in (0, T)$ and $j \in \{1, \ldots, k\}$,

$$(3.1a) \quad 0 = (\varphi'_{k}, z_j) + (m(\varphi_{k}, \sigma_{k}, \mathbf{u}_{k}) \nabla \mu_{k}, \nabla z_j) - (\bar{U}(\varphi_{k}, \sigma_{k}, \mathbf{u}_{k}), z_j),$$

$$(3.1b) \quad \bar{U} = \lambda_0 f(\varphi_{k})g(\sigma_{k})/(1 + |W_{\varphi}(\varphi_{k}, \mathbf{E}(\mathbf{u}_{k}))|) - \lambda_k k(\varphi_{k}),$$

$$(3.1c) \quad 0 = (\mu_k - \varepsilon^{-1}\psi'(\varphi_{k}) + \chi \sigma_{k} - W_{\varphi}(\varphi_{k}, \mathbf{E}(\mathbf{u}_{k})), z_j) - (\varepsilon \nabla \varphi_{k}, \nabla z_j),$$

$$(3.1d) \quad 0 = \beta(\sigma'_{k}, z_j) + (\nabla \sigma_{k}, \nabla z_j) + \kappa(\sigma_{k} - \sigma_{B}, z_j)\Gamma - (S(\varphi_{k}, \sigma_{k}), z_j),$$

$$(3.1e) \quad S = -\lambda_{k} h(\varphi_{k})\sigma_{k} + B(\sigma_{c} - \sigma_{k}),$$

$$(3.1f) \quad 0 = (W_{\varphi}(\varphi_{k}, \mathbf{E}(\mathbf{u}_{k})), \nabla y_{j}) - (g, y_{j})\Gamma_{N},$$

$$(3.1g) \quad \varphi_{k}(0) = \varphi_{k,0} := \Pi_k \varphi_0, \quad \sigma_{k}(0) = \sigma_{k,0} := \Pi_k \sigma_0,$$

where in the definition of $\bar{U}$, the function $g(s) = \max(0, \min(s, ||\sigma_B||_{L^\infty(\Sigma)}), ||\sigma_c||_{L^\infty(Q)})$ is a truncation. It will turn out that the nutrient equation satisfies a comparison principle, but this is not valid at the Galerkin level, and thus we introduce the truncation $g$ to first derive the necessary a priori estimates, and then remove it at the continuous level.

The orthogonality of $\{z_i\}_{i \in \mathbb{N}}$ with respect to the $L^2$-inner product allows us to express (3.1a) as a system of ordinary differential equations in the coefficient vectors $a := (a_1^k, \ldots, a_k^k), \ b := (b_1^k, \ldots, b_k^k), \ c := (c_1^k, \ldots, c_k^k)$ and $d := (d_1^k, \ldots, d_k^k)$. It is not hard to see that the continuity of $m, \psi', f, g, h, k, W_{\varphi}, W_{\mathbf{E}}$ with respect to their arguments, as well
as the continuity of $\lambda_p$, $\lambda_a$ and $\lambda_c$ with respect to time yield that the differential-algebraic system contains only contributions that are continuous in $a, b, c, d$.

Here, we are going to show that the above system can be expressed as a system of ODEs in terms of $a$ and $c$ only and that classical results ensure the existence of a regular solution. From an analysis of equation (3.1a), it is straightforward to realise that $b$ can be expressed as a function of $a, c,$ and $d$ in a continuously differentiable fashion. Moreover, let us claim that $d$ can be expressed as a function of $a$ only. In the direction of formalising this fact, let us fix for convenience the following notation:

$$
\varphi(a) := \varphi_k = \sum_{i=1}^{k} a_i^k z_i(x), \quad u(d) := u_k = \sum_{i=1}^{k} d_i^k y_i(x).
$$

Furthermore, we point out that equation (3.1f) can be written as $0 = F(a, d)$, for a function $F : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ defined by

$$
F(a, d)_i = (W, \mathcal{E}(\varphi(a), \mathcal{E}(u(d))), \nabla y_i) - (g, y_i)\Gamma_N
$$

$$
= \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\mathcal{E}(d_j^k y_j), \nabla y_i \right) - \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\bar{\mathcal{E}}(a_j^k z_j), \nabla y_i \right)
$$

$$
- (g, y_i)\Gamma_N, \quad \text{for } i = 1, \ldots, k.
$$

Moreover, by virtue of symmetry, we can replace $\nabla y_i$ with $\mathcal{E}(y_i)$ to infer that

$$
F(a, d)_i = \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\mathcal{E}(d_j^k y_j), \mathcal{E}(y_i) \right) - \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\bar{\mathcal{E}}(a_j^k z_j), \mathcal{E}(y_i) \right) - (g, y_i)\Gamma_N
$$

$$
=: (\mathbb{A}(a)d)_i - q(a)_i,
$$

where

$$
(\mathbb{A}(a)d)_i = \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\mathcal{E}(d_j^k y_j), \mathcal{E}(y_i) \right),
$$

$$
q(a)_i = \left( \sum_{j=1}^{k} \mathcal{C}(\varphi(a))\bar{\mathcal{E}}(a_j^k z_j), \mathcal{E}(y_i) \right) + (g, y_i)\Gamma_N.
$$

An easy calculation using the fact that the tensor $\mathcal{C}$ is positive definite shows that the matrix $\mathbb{A}(a)$ is positive definite and hence invertible so that we can uniquely solve the linear system

$$
F(a, d) = \mathbb{A}(a)d - q(a) = 0 \quad (3.2)
$$

and deduce that

$$
d = \mathbb{A}^{-1}(a)q(a),
$$

which in turn proves that the solution $d$ of (3.2) depends continuously on $a$. Recalling that $b$ can be expressed as a function of $a, c, d$, we can rephrase equations (3.1a) and (3.1d) as a system of ordinary differential equations:

$$
\begin{cases}
a' = H_1(a, c), \\
c' = H_2(a, c),
\end{cases}
$$

10
for suitable functions $H_i$, $i = 1, 2$, that are continuous with respect to their arguments.

In light of the above observations, we invoke the Cauchy–Peano theorem to obtain the existence of $T_k \in (0, T]$ and local solutions $a, c \in C^1([0, T_k), \mathbb{R}^k)$ solving (3.1a) and (3.1d), from which we also get $b, d \in C^1([0, T_k), \mathbb{R}^k)$. Moreover, we can endow an initial condition for the Galerkin approximation $u_k$ via the relation (3.2). Namely, we set

$$d(0) = \lambda^{-1}(a(0))q(a(0))$$

which implies that $u_{k,0} := u_k(0) = \sum_{i=1}^k d_i^k(0) y_i(x) \in X(\Omega)$ satisfies

$$(W, \varepsilon(\varphi_{k,0}, \mathcal{E}(u_{k,0})), \nabla y_j) = (g, y_j)_{\Gamma_N} \quad \forall 1 \leq j \leq k.$$ (3.3)

Then, multiplying the above by $d_j^k(0)$, summing from $j = 1$ to $k$, and employing (2.6), Korn’s inequality and the trace theorem yields

$$\frac{C_4}{C_k} \|u_{k,0}\|^2_{X(\Omega)} \leq C_4 \|\mathcal{E}(u_{k,0})\|^2_{L^2} \leq (C(\varphi_{k,0})\mathcal{E}(u_{k,0}), \mathcal{E}(u_{k,0}))$$

$$= (C(\varphi_{k,0})\mathcal{E}(\varphi_{k,0}), \mathcal{E}(u_{k,0})), \Gamma_N) + (g, u_{k,0})_{\Gamma_N}$$

$$\leq C \left(\|\varphi_{k,0}\|_{L^2} + \|g\|_{L^2(N_\Gamma)}\right) \|u_{k,0}\|_{X(\Omega)}$$

which implies the following estimate of $u_{k,0}$ in $X(\Omega)$:

$$\|u_{k,0}\|_{X(\Omega)} \leq C \left(1 + \|\varphi_{k,0}\|_{L^2}\right)$$ (3.4)

for a positive constant $C$ independent of $k$.

Next, our goal is to derive uniform estimates in $k$ to pass to the limit. In the sequel, we denote positive constants that are independent of $\beta$ and $k$, which may vary from line to line, by the symbol $C$.

### 3.2 A priori estimates

Let $R, K > 0$ be constants yet to be determined. Multiplying (3.1a) with $b_j^k$ and with $K a_j^k$, (3.1c) with $(a_j^k)'$, (3.1d) with $R e_j^k$ and (3.11) with $(d_j^k)'$. Summing from $j = 1$ to $k$ and using the symmetry of the elastic tensor $C$ yields

$$0 = (\varphi_j', \mu_k) + \|m^{1/2}\nabla \mu_k\|^2_{L^2} - (\mathcal{E}(u_k), \mu_k)$$

$$0 = \frac{d}{dt} \|\varphi_k\|^2_{L^2} + K(m \nabla \mu_k, \nabla \varphi_k) - K(\mathcal{E}(u_k), \varphi_k)$$

$$0 = (\mu_k + \chi \sigma_k - W_\varphi(\varphi_k, \mathcal{E}(u_k)), \varphi_k') - \frac{d}{dt} \int_{\Omega} \frac{1}{\varepsilon} \psi(\varphi_k) + \frac{\varepsilon}{2} |\nabla \varphi_k|^2 dx$$

$$0 = \frac{d}{dt} \|\sigma_k\|^2_{L^2} + R \|\nabla \sigma_k\|^2_{L^2} + R \|\sigma_k\|^2_{L^2} - R \|\sigma_k\|_{L^2} + R \|\sigma_k\|_{L^2}$$

$$0 = (W, \varepsilon(\varphi_k, \mathcal{E}(u_k)), \mathcal{E}(u_k)' - (g, u_k')_{\Gamma_N}$$

Taking note of the identity

$$(W, \varphi_k' + (W, \varepsilon(\varphi_k', \mathcal{E}(u_k'))) = \frac{d}{dt} \int_{\Omega} W(\varphi_k, \mathcal{E}(u_k)) dx,$$
we obtain the following after summing the equations

\[
\frac{d}{dt} \int_\Omega \frac{\varepsilon}{2} |\nabla \phi_k|^2 + \frac{1}{\varepsilon} \psi(\phi_k) + \frac{K}{2} |\phi_k|^2 + \frac{R\beta}{2} |\sigma_k|^2 \, dx \\
+ \frac{d}{dt} \int_\Omega W(\phi_k, \mathcal{E}(u_k)) \, dx - \frac{d}{dt} \int_{\Gamma_N} g \cdot u_k \, dA + \|m^{1/2} \nabla \mu_k\|^2_{L^2} \\
+ R |\nabla \sigma_k|^2_{L^2} + R\kappa \|\sigma_k\|^2_{L^2} + R\lambda_c \|h^{1/2} \sigma_k\|^2_{L^2} + RB \|\sigma_k\|^2_{L^2} \\
= (\hat{U}, \mu_k + K \phi_k) + (\chi \sigma_k, \chi_k) - K(m \nabla \mu_k, \nabla \phi_k) + RB(\sigma, \sigma_k) + R\kappa(\sigma_B, \sigma_k). 
\] 

(3.5)

The last two terms on the right-hand side can be handled using the Cauchy–Schwarz and Young inequalities as follows:

\[
B(\sigma, \sigma_k) \leq \frac{1}{2}\kappa(\|\sigma_k\|^2_{L^2} + \|\sigma_B\|^2_{L^2}) + d_1 \|\sigma_k\|^2_{L^2} + \frac{B^2}{d_1} \|\sigma\|^2_{L^2},
\]

where \(d_1\) is a positive constant to be determined later. Let us observe that for an arbitrary test function \(\zeta \in H^1(\Omega)\), we multiply (3.5) with the coefficients of \(\Pi_k\zeta\), leading to

\[
(\varphi_k', \zeta) = (\varphi_k', \Pi_k\zeta) = -(m \nabla \mu_k, \nabla \Pi_k\zeta) + (\hat{U}, \Pi_k\zeta).
\]

The boundedness of \(m\) and \(\hat{U}\), and the estimate \(\|\Pi_k\zeta\|_{H^1} \leq C \|\zeta\|_{H^1}\) implies that

\[
\|\varphi_k'\|_{H^1(\Omega)} \leq C_3 \|\nabla \mu_k\|_{L^2} + C_{\lambda,g} \tag{3.6}
\]

for some positive constant \(C_{\lambda,g}\) depending only on \(\max_{t \in [0, T]} \lambda_p(t), \max_{t \in [0, T]} \lambda_a(t)\) and \(\max_{s \in \mathbb{R}} g(s)\). We now estimate the remaining terms of the right-hand side as follows:

\[
(\hat{U}, \mu_k) - (\hat{U}, \mu_k - \langle \mu_k \rangle) + \langle \mu_k \rangle (\hat{U}, 1) \\
\leq C_{\lambda,g} \|\mu_k - \langle \mu_k \rangle\|_{L^1} + C_{\lambda,g} |\langle \mu_k \rangle| \\
\leq \eta \|\nabla \mu_k\|^2_{L^2} + C(\eta^{-1}, c_p, C_{\lambda,g}) + C_{\lambda,g} \left( |\langle \mu_k \rangle| - \|\sigma_k\|^2_{L^2} + \|\sigma_k\|^2_{L^2} \right),
\]

\[
(\hat{U}, K \phi_k) \leq C(K, C_{\lambda,g}) + \|\varphi_k\|^2_{L^2},
\]

\[
K(m \nabla \mu_k, \nabla \phi_k) \leq \eta \|\nabla \mu_k\|^2_{L^2} + C(\eta^{-1}, K C_{\lambda,g}) \|\nabla \phi_k\|^2_{L^2},
\]

\[
(\sigma_k, \phi_k) \leq \|\sigma_k\|_{H^1(\Omega)} \|\phi_k\|_{H^1(\Omega)} \leq \eta \|\nabla \mu_k\|^2_{L^2} + C(\eta^{-1}, C_{\lambda,g}) \|\sigma_k\|^2_{H^1},
\]

where \(c_p > 0\) is the constant from the Poincaré inequality and \(\eta\) is a constant yet to be determined and where in the last inequality we made use of (3.6). Putting everything together and using the lower bound for the mobility, we obtain from (3.5)

\[
\frac{d}{dt} \int_\Omega \frac{\varepsilon}{2} |\nabla \phi_k|^2 + \frac{1}{\varepsilon} \psi(\phi_k) + \frac{K}{2} |\phi_k|^2 + \frac{R\beta}{2} |\sigma_k|^2 \, dx \\
+ \frac{d}{dt} \int_\Omega W(\phi_k, \mathcal{E}(u_k)) \, dx - \frac{d}{dt} \int_{\Gamma_N} g \cdot u_k \, dA + \|m^{1/2} \nabla \mu_k\|^2_{L^2} \\
+ (C_2 - 3\eta) \|\nabla \mu_k\|^2_{L^2} + R \|\nabla \sigma_k\|^2_{L^2} + \frac{R\kappa}{2} \|\sigma_k\|^2_{L^2} \\
\leq C(K, \eta^{-1}, C_{\lambda,g}, \kappa, d_1^{-1}, \sigma_B, \sigma_c) + (Rd_1 + C(\eta^{-1}, C_{\lambda,g})) \|\sigma_k\|^2_{H^1} \\
+ C_{\lambda,g} \left( |\langle \mu_k \rangle| - \|\sigma_k\|^2_{L^2} + \|\varphi_k\|^2_{L^2} + C(\eta^{-1}, K) \|\nabla \phi_k\|^2_{L^2}. \right)
\]

(3.7)

Let us point out that if \(K = 0\) then the last term on the right-hand side involving \(\|\nabla \phi_k\|^2_{L^2}\) vanishes. Furthermore, we recall the generalised Poincaré inequality: There exists a positive constant \(C_p = C_p(\Omega)\) such that for all \(f \in H^1(\Omega)\),

\[
\|f\|^2_{H^1} \leq C_p \left( \|\nabla f\|^2_{L^2} + \|f\|^2_{L^2} \right).
\]

(3.8)
so that by choosing $d_1 < \min(1, \kappa) \frac{1}{2 \mathcal{C}_p}$ and $R > \frac{8 \mathcal{C}_p \mathcal{C}(\eta^{-1}, C_{\lambda, g})}{\min(1, \kappa)}$, we obtain

$$(Rd_1 + C(\eta^{-1}, C_{\lambda, g})) \| \sigma_k \|^2_{H^1} \leq R \min(1, \kappa) \| \sigma_k \|^2_{H^1} \leq \frac{R}{4} \| \nabla \sigma_k \|^2_{L^2} + \frac{R \kappa}{4} \| \sigma_k \|^2_{L^2}.$$ 

Then, in (3.7) we choose $\eta = C_1^d$

$$\frac{d}{dt} \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_k|^2 + \frac{1}{\varepsilon} \psi(\varphi_k) + \frac{K}{2} |\varphi_k|^2 + \frac{R\beta}{2} |\sigma_k|^2 \, dx$$

$$+ \frac{d}{dt} \int_{\Omega} W(\varphi_k, \mathcal{E}(u_k)) \, dx - \frac{d}{dt} \int_{\Gamma_N} g \cdot u_k \, dA$$

$$+ \frac{C_2}{4} \| \nabla \mu_k \|^2_{L^2} + \frac{3R}{4} \| \nabla \sigma_k \|^2_{L^2} + \frac{R \kappa}{4} \| \sigma_k \|^2_{L^2}$$

$$\leq C + C_{\lambda, g} \left( |\langle \mu_k \rangle| - \| \sigma_k \|^2_{L^2} \right) + \| \varphi_k \|^2_{L^2} + C(\eta^{-1}, K) \| \nabla \varphi_k \|^2_{L^2}.$$ 

We will use this expression to obtain an energy inequality for the limit solutions. Next, to estimate the term involving the mean value $\langle \mu_k \rangle$, we choose $j = 1$ in (3.1c) (recalling $z_1$ is constant), and use (A1)-(A3) altogether and (A4) to deduce that

$$|\langle \mu_k \rangle| \leq C \left( 1 + \| \psi(\varphi_k) \|_{L^1} + \| \varphi_k \|^2_{L^2} + \| \mathcal{E}(u_k) \|^2_{L^2} \right) + \| \sigma_k \|^2_{L^2}$$

$$\leq C \left( 1 + \| \psi(\varphi_k) \|_{L^1} + \| \varphi_k \|^2_{H^1} + \| \mathcal{E}(u_k) \|^2_{L^2} \right) + \| \sigma_k \|^2_{L^2},$$

and so, using (3.8), (3.9) simplifies to

$$\frac{d}{dt} \int_{\Omega} |\nabla \varphi_k|^2 + \psi(\varphi_k) + K |\varphi_k|^2 + \beta |\sigma_k|^2 \, dx$$

$$+ \frac{d}{dt} \int_{\Omega} W(\varphi_k, \mathcal{E}(u_k)) \, dx - \frac{d}{dt} \int_{\Gamma_N} g \cdot u_k \, dA + \| \nabla \mu_k \|^2_{L^2} + \| \sigma_k \|^2_{H^1}$$

$$\leq C \left( 1 + \| \psi(\varphi_k) \|_{L^1} + \| \varphi_k \|^2_{H^1} + \| \mathcal{E}(u_k) \|^2_{L^2} \right).$$

Using the strict monotonicity of $W_{\mathcal{E}}$ with respect to its second argument, we find that for any $s \in \mathbb{R}$ and $\mathcal{M} \in \mathbb{R}^{d \times d}$

$$W(s, \mathcal{M}) = W(s, 0) + \int_0^1 W_{\mathcal{E}}(s, t\mathcal{M}) : t\mathcal{M} \frac{1}{2} \, dt \geq \frac{C_1}{2} \| \mathcal{M} \|^2 - C(1 + |s|^2).$$

Hence, by Young’s inequality, the trace theorem, and Korn’s inequality we have for some positive constants $c_1$ and $c_2$ that

$$\int_{\Omega} W(\varphi_k, \mathcal{E}(u_k)) \, dx - \int_{\Gamma_N} g \cdot u_k \, dA \geq c_1 \| \mathcal{E}(u_k) \|^2_{L^2} - c_2 (1 + \| \varphi_k \|^2_{L^2}).$$

(3.12)

On the other hand, by combining (2.7) with (3.4), we deduce

$$\int_{\Omega} W(\varphi_k, \mathcal{E}(u_k)) \, dx - \int_{\Gamma_N} g \cdot u_k \, dA$$

$$\leq C \left( 1 + \| \varphi_k \|^2_{L^2} + \| u_{k,0} \|^2_{X(\Omega)} + \| g \|^2_{L^2(\Gamma_N)} \right) \leq C \left( 1 + \| \varphi_k \|^2_{L^2} \right).$$

(3.13)
Therefore, provided we choose $K > c_2$, integrating (3.11) in time and employing the above estimate leads to
\[
\left(\|\varphi_k\|_{H^1}^2 + \|\psi(\varphi_k)\|_{L^1} + \beta\|\sigma_k\|_{L^2}^2 + \|\mathcal{E}(u_k)\|_{L^2}^2\right)(t) \\
+ \|\nabla \mu_k\|_{L^2(0,t;L^2)}^2 + \|\sigma_k\|_{L^2(0,t;H^1)}^2 \\
\leq C\left(1 + \|\psi(\varphi_k)\|_{L^1(0,t;L^1)} + \|\varphi_k\|_{L^2(0,t;H^1)} + \|\mathcal{E}(u_k)\|_{L^2(0,t;L^2)}\right) \\
+ C\|\varphi_{k,0}\|_{H^1}^2 + \|\psi(\varphi_{k,0})\|_{L^1} + \beta\|\sigma_{k,0}\|_{L^2}^2 \quad \forall t \in (0,T).
\] (3.14)

Thanks to the fact that the $\{z_i\}_{i \in \mathbb{N}}$ are a basis in $H^2(\Omega)$ and are orthonormal in $L^2(\Omega)$, and recalling our assumptions on the set $\Omega$, there exists a positive constant $C$ such that $\|\varphi_{k,0}\|_{H^2} \leq C\|\varphi_0\|_{H^2}$ and $\|\sigma_{k,0}\|_{L^2} \leq \|\sigma_0\|_{L^2}$. Furthermore, since $\varphi_0 \in H^2_0(\Omega)$, by [21] §3, p. 329 we have $\varphi_{k,0} = \Pi_{k} \varphi_0 \rightarrow \varphi_0$ strongly in $H^2_0(\Omega) \subset L^\infty(\Omega)$ and a.e. in $\Omega$. This implies that $\varphi_{k,0}$ is bounded uniformly in $L^\infty(\Omega)$, and in turn we can deduce the existence of a positive constant $c_*$ such that
\[
\|\psi(\varphi_{k,0})\|_{L^\infty} \leq c_* \quad \text{for all } k \in \mathbb{N}.
\]

Then, by Lebesgue convergence theorem we obtain
\[
\int_{\Omega} \psi(\varphi_{k,0}) \, dx \rightarrow \int_{\Omega} \psi(\varphi_0) \, dx \quad \text{as } k \rightarrow \infty.
\] (3.15)

Hence, there exists a positive constant $C$ such that
\[
\|\psi(\varphi_{k,0})\|_{L^1} \leq C,
\]
and through the use of Gronwall’s inequality in integral form and Korn’s inequality (2.1) we obtain
\[
\sup_{t \in (0,T)} \left(\|\varphi_k(t)\|_{H^1}^2 + \|\psi(\varphi_k(t))\|_{L^1} + \beta\|\sigma_k(t)\|_{L^2}^2 + \|u_k(t)\|_{X(\Omega)}^2\right) \\
+ \|\nabla \mu_k\|_{L^2(0,T;L^2)}^2 + \|\sigma_k\|_{L^2(0,T;H^1)}^2 \leq C(1 + \beta\|\sigma_0\|_{L^2}^2).
\] (3.16)

Then, from (3.10) we have $\langle \mu_k \rangle(t)$ is bounded in $L^\infty(0,T)$, which leads to
\[
\|\mu_k\|_{L^2(0,T;H^1)} \leq C(1 + \beta\|\sigma_0\|_{L^2}^2).
\] (3.17)

Furthermore, from (3.6) and arguing similarly with (3.14), we also have
\[
\|\varphi'_k\|_{L^2(0,T;(H^1)')} + \beta\|\sigma'_k\|_{L^2(0,T;(H^1)')} \leq C(1 + \beta\|\sigma_0\|_{L^2}^2).
\] (3.18)

### 3.3 Compactness assertions and passing to the limit

From the above estimates we immediately deduce the existence of functions $(\varphi, \mu, u, \sigma)$ such that, for a non-relabelled subsequence, we have
\[
\varphi_k \rightarrow \varphi \text{ weakly* in } L^\infty(0,T;H^1) \cap H^1(0,T;H^1)', \\
\text{and strongly in } C^0([0,T];L^r) \text{ and a.e. in } Q, \\
\mu_k \rightarrow \mu \text{ weakly in } L^2(0,T;H^1), \\
u_k \rightarrow u \text{ weakly* in } L^\infty(0,T;X(\Omega)), \\
\sigma_k \rightarrow \sigma \text{ weakly* in } L^2(0,T;H^1) \cap L^\infty(0,T;L^2) \cap H^1(0,T;(H^1)'), \\
\text{and strongly in } L^2([0,T];L^r) \text{ and a.e. in } Q
\]
for any \( r < \infty \) in two spatial dimensions and any \( r < 6 \) in three spatial dimensions. Next, we deduce the strong convergence of \( u_k \) to \( u \), which can be argued as follows. Since the span of \( \{y_i\}_{i \in \mathbb{N}} \) is dense in \( X(\Omega) \), we can choose a sequence \( \{v_k\}_{k \in \mathbb{N}} \) such that for each \( k \in \mathbb{N} \), and a.e. \( t \in (0, T), v_k(t) \in Y_k \) and \( v_k \to u \) strongly in \( L^2(0, T; X(\Omega)) \). It follows that the difference \( u_k - v_k \) converges weakly to zero in \( L^2(0, T; X(\Omega)) \) as \( k \to \infty \). Moreover, we can consider \( \eta = (u_k - v_k) \) in (3.11) and obtain thanks to the coercivity property (2.6) that

\[
0 = (C(\varphi_k)(E(u_k) - \tilde{E}(\varphi_k)), E(u_k - v_k)) - (g, u_k - v_k)_{\Gamma_N} \\
= (W, E(\varphi_k), E(u_k) - W, E(\varphi_k), E(u_k - v_k)) + (C(\varphi_k)(E(v_k) - \tilde{E}(\varphi_k)), E(u_k - v_k)) - (g, u_k - v_k)_{\Gamma_N} \\
\geq C_1\|E(u_k - v_k)\|^2_{L^2(\Omega)} + (C(\varphi_k)(E(v_k) - \tilde{E}(\varphi_k)), E(u_k - v_k)) - (g, u_k - v_k)_{\Gamma_N}.
\]

Integrating the above inequality over \((0, T)\), and applying the strong convergence of \( v_k \) to \( u \), the convergence properties of \( \varphi_k \) to \( \varphi \), and the weak convergence of \( u_k - v_k \) to zero in \( L^2(0, T; X(\Omega)) \), we then obtain

\[
\|E(u_k - v_k)\|^2_{L^2(\Omega)} \to 0 \quad \text{as} \quad k \to \infty.
\]

By Korn’s inequality this shows that \( u_k - v_k \) converges strongly to zero in \( L^2(0, T; X(\Omega)) \) and hence \( u_k \to u \) strongly in \( L^2(0, T; X(\Omega)) \) and a.e. in \( Q \).

Now, we aim at passing to the limit. The standard procedure is to fix \( j \in \mathbb{N} \) in (3.1), multiply (2.14), (3.1c), (3.1d), and (3.1i) with an arbitrary \( \theta \in C_c^\infty(0, T) \), pass to the limit \( k \to \infty \) with the above compactness results, and use the density of \( \cup_{k \in \mathbb{N}} Z_k \) in \( H^1(\Omega) \), and the density of \( \cup_{k \in \mathbb{N}} Y_k \) in \( X(\Omega) \) to show that the limit \((\varphi, \mu, \sigma, u) \) satisfies (2.2a)-(2.2d) for all \( \zeta \in L^2(0, T; H^1(\Omega)), \xi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega) \) and \( \eta \in L^2(0, T; X(\Omega)) \). The first step is to employ the above compactness assertions to pass to the limit \( k \to \infty \) and recover (2.2d). This can be done since we have the a.e. convergence and strong convergence of \( \varphi_k \) to \( \varphi \) and the particular form of \( W, E(\sigma, E) \). For the other equations, a key point to pass to the limit is strong convergence of \( E(u_k) \) to \( E(u) \) in \( L^2(0, T; L^2(\Omega)^{d \times d}) \) which follows from the strong convergence of \( u_k \). We omit the easy details and sketch the less obvious points.

**Mobility term.** Continuity of \( m \) from (A3) and the a.e. convergence of \( \varphi_k \) (resp. \( \sigma_k, u_k \) and \( E(u_k) \)) to \( \varphi \) (resp. \( \sigma, u \) and \( E(u) \)) leads to \( m(\varphi_k, \sigma_k, u_k, E(u_k)) \to m(\varphi, \sigma, u, E(u)) \) a.e. in \( Q \). Boundedness of \( m \) from (A3) and the dominated convergence theorem yields that

\[
\theta(t)m(\varphi_k, \sigma_k, u_k, E(u_k))\nabla z_j \to \theta(t)m(\varphi, \sigma, u, E(u))\nabla z_j \quad \text{strongly in} \quad L^2(Q),
\]

so that together with the weak convergence of \( \nabla \mu_k \) to \( \nabla \mu \) in \( L^2(Q) \) we find that

\[
\int_0^T \theta(t)(m(\varphi_k, \sigma_k, u_k, E(u_k))\nabla \mu_k, \nabla z_j) dt \to \int_0^T \theta(t)(m(\varphi, \sigma, u, E(u))\nabla \mu, \nabla z_j) dt.
\]

A similar argument can be used to show the convergence involving \( \tilde{U}(\varphi_k, \sigma_k, E(u_k)) \) using the boundedness and continuity of \( \tilde{U} \) with respect to its arguments.
Potential term. Continuity of \( \psi' \) and the a.e. convergence of \( \varphi_k \) to \( \varphi \) in \( Q \) yields \( \psi'(\varphi_k) \to \psi'(\varphi) \) a.e. in \( Q \). The above compactness results, the sublinear growth of \( \psi'_2 \) and the generalised dominated convergence theorem lead to

\[
\psi'_2(\varphi_k) \to \psi'_2(\varphi) \quad \text{strongly in } L^2(Q).
\]

For the monotone part \( \psi'_1(\varphi_k) \), we show that the family \( \{\theta(t)\psi'_1(\varphi_k)z_j\}_{k \in \mathbb{N}} \) is uniformly integrable over \( Q \), so that together with the a.e. convergence \( \psi'_1(\varphi_k) \) to \( \psi'_1(\varphi) \) in \( Q \), we obtain via Vitali’s convergence theorem that

\[
\int_Q \theta(t)\psi'_1(\varphi_k)z_j dx \, dt \to \int_Q \theta(t)\psi'_1(\varphi)z_j dx \, dt.
\]

We now show the uniform integrability. Let \( \eta > 0 \) be arbitrary, then choosing \( \delta > 0 \) sufficiently small so that

\[
\delta \left( TC(1 + \beta) + C_\delta \right) \|\theta\|_{L^\infty(0,T)} \|z_j\|_{H^2(\Omega)} < \eta,
\]

where \( C \) is the constant in (3.16), and \( C_\delta \) is the constant in (A2) associated to \( \delta \), we obtain from (A2) and the fact \( z_j \in H^2(\Omega) \) that for any measurable subset \( E \subset Q \) with \( |E| < \delta \),

\[
\int_E |\theta(t)\psi'_1(\varphi_k)z_j| dx \, dt \leq \|\theta\|_{L^\infty(0,T)} \|z_j\|_{H^2(\Omega)} \int_E \left( \delta \psi'_1(\varphi_k) + C_\delta \right) dx \, dt
\]

\[
\leq \left( \delta TC(1 + \beta) + C_\delta |E| \right) \|\theta\|_{L^\infty(0,T)} \|z_j\|_{H^2(\Omega)} < \eta,
\]

which implies the uniform integrability of the family \( \{\theta(t)\psi'_1(\varphi_k)z_j\}_{k \in \mathbb{N}} \).

Elasticity terms. Thanks to the strong convergence of \( u_k \) in \( L^2(0,T;X(\Omega)) \) and of \( \varphi_k \) in \( L^2(0,T;L^2(\Omega)) \) and due to (2.7) and (2.8), we have

\[
C_5(1 + |\varphi_k|^2 + |E(u_k)|^2) \to C_5(1 + |\varphi|^2 + |E(u)|^2) \quad \text{strongly in } L^1(Q),
\]

\[
C_5(1 + |\varphi_k| + |E(u_k)|) \to C_5(1 + |\varphi| + |E(u)|) \quad \text{strongly in } L^2(Q),
\]

and so by (A1) and the generalised dominated convergence theorem, we infer that

\[
\int_Q \theta(t)W_{\varphi}(\varphi_k,E(u_k)) \cdot \nabla y_j dx \, dt \to \int_Q \theta(t)W_{\varphi}(\varphi,E(u)) \cdot \nabla y_j dx \, dt,
\]

\[
\int_Q \theta(t)W_{\varphi}(\varphi_k,E(u_k))z_j dx \, dt \to \int_Q \theta(t)W_{\varphi}(\varphi,E(u))z_j dx \, dt
\]

on account of the fact that \( z_j \in H^2(\Omega) \subset L^\infty(\Omega) \).

Comparison principle. To establish the boundedness property (2.4) for \( \sigma \), so that \( U(\varphi,\sigma,E(u)) = U(\varphi,\sigma,E(u)) \), i.e., \( g(\sigma) = \sigma \), we employ a comparison principle. We recall that the positive part \( f_+ \) and negative part \( f_- \) of a function \( f \) are defined as

\[
f_+(x) = \max(f(x),0), \quad f_-(x) = \max(-f(x),0),
\]

so that \( f(x) = f_+(x) - f_-(x) \). Testing (2.2a) with \( -\sigma_- \) and using the relations

\[
\langle \sigma_1, (\sigma)_- \rangle = -\frac{d}{dt} \| (\sigma)_- \|^2_{L^2}, \quad \langle \nabla \sigma, \nabla (\sigma)_- \rangle = -\| \nabla (\sigma)_- \|^2_{L^2},
\]

16
we obtain
\[
\frac{\beta}{2} \frac{d}{dt} ||(\sigma)_-||^2_{L^2} + ||\nabla(\sigma)_-||^2_{L^2} + \kappa ||(\sigma)_-||^2_{L^2} + \lambda_c ||h^{1/2}(\sigma)_-||^2_{L^2} + B ||(\sigma)_-||^2_{L^2} \\
= -B(\sigma_c, (\sigma)_-) - \kappa M^{1/2} (\sigma_0)_{-,-1} \leq 0
\]
on account of the fact that $\sigma_B, \sigma_c$ and $(\sigma)_-$ are all non-negative. Integrating the above inequality and using the fact that $\sigma_0$ is non-negative we obtain
\[
||((\sigma)_- (t))||^2_{L^2} \leq ||((\sigma)_0)||^2_{L^2} \leq 0 \quad \text{for all } t \in (0, T),
\]
so that $\sigma$ is non-negative a.e. in $Q$. On the other hand, testing the equation (2.2c) by $(\sigma - M)_+$, where we recall that $M = \max(||\sigma_c||_{L^{\infty}(Q)}, ||\sigma_B||_{L^{\infty}(\Sigma)})$, yields
\[
\frac{\beta}{2} \frac{d}{dt} ||(\sigma - M)_+||^2_{L^2} + ||\nabla(\sigma - M)_+||^2_{L^2} + \kappa ||(\sigma - M)_+||^2_{L^2} \\
+ \lambda_c ||h^{1/2}(\sigma - M)_+||^2_{L^2} + B ||(\sigma - M)_+||^2_{L^2} \\
= -\kappa (M - \sigma_B, (\sigma - M)_+) - \lambda_c (h(\varphi) M, (\sigma - M)_+) - B (M - \sigma_c, (\sigma - M)_+) \leq 0
\]
on account of the fact that $h$, $M - \sigma_c$, $M - \sigma_B$ and $(\sigma - M)_+$ all are non-negative. Integrating the above inequality and using the fact that $\sigma_0 \leq M$ a.e. in $\Omega$ leads to
\[
||(\sigma - M)_+ (t))||^2_{L^2} \leq ||(\sigma_0 - M)_+||^2_{L^2} = 0 \quad \text{for all } t \in (0, T)
\]
so that $\sigma \leq M$ a.e. in $Q$ as we claimed.

### 3.4 Energy inequality

Let us now combine inequalities (3.10), (3.17) and (3.18) to obtain that
\[
\sup_{t \in (0, T)} \left( ||\varphi_k(t)||^2_{H^1} + ||\psi (\varphi_k(t))||^2_{L^1} + \beta ||\sigma_k(t)||^2_{L^2} + ||u_k(t)||^2_{X(\Omega)} \right) \\
+ ||\mu_k||^2_{L^2(0,T;H^1)} + ||\sigma_k||^2_{L^2(0,T;H^1)} + ||\varphi_k||^2_{H^1(0,T;H^1)} + ||\sigma_k||^2_{H^1(0,T;H^1)} \\
\leq C \left( 1 + ||\psi (\varphi_k)||^2_{L^2(0,T;L^2)} + ||\varphi_k||^2_{L^2(0,T;H^1)} + ||\sigma_k||^2_{H^1(0,T;L^2)} \right) \\
+ C ||\varphi_k||^2_{L^2(0,T;L^2)} + ||\psi (\varphi_k)||^2_{L^1} + ||\sigma_k||^2_{L^2}.
\]

From the compactness assertions stated in Section 3.3, we infer, by Fatou’s lemma and the non-negativity of $\psi$ that, for $a.e.s \in (0, T)$, it holds
\[
\int_{\Omega} \psi (\varphi_k) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} \psi (\varphi_k) \, dx.
\]
Moreover, invoking the weak/weak* lower semicontinuity of the norms, using the properties $||\varphi_k||^2_{H^1} \leq C ||\varphi_0||^2_{H^1}$, $||\sigma_k||^2_{L^2} \leq ||\sigma_0||^2_{L^2}$ originating from the orthogonality in $H^1(\Omega)$ of the basis functions $\{\psi_i\}_{i \in \mathbb{N}}$, and recalling (3.15), allow us to pass to the limit as $k \to \infty$ in the above inequality to obtain (2.3).
3.5 More general initial conditions

To complete the proof of Theorem 1 we now assume that \( \varphi_0 \in H^1(\Omega) \) with \( \psi(\varphi_0) \in L^1(\Omega) \) and use ideas of [1]. For any \( \delta \in (0, 1] \), we denote by \( \varphi_{0,\delta} \in H^2_{n}(\Omega) \) the unique solution to the elliptic problem:

\[
\begin{align*}
-\delta \Delta \varphi_{0,\delta} + \varphi_{0,\delta} &= \varphi_0 & \text{in } \Omega, \\
\partial_n \varphi_{0,\delta} &= 0 & \text{on } \Gamma.
\end{align*}
\]

(3.21)

The well-posedness and regularity of \( \varphi_{0,\delta} \) follows from standard application of the Lax–Milgram theorem and elliptic regularity theory (see, e.g., [27] for the corresponding regularity theory for convex domains). Furthermore, testing the above equation by \( \Delta \varphi_{0,\delta} \) we infer that

\[
2\delta \| \nabla \varphi_{0,\delta} \|^2_{L^2} + \| \varphi_{0,\delta} \|^2_{L^2} \leq \| \varphi_0 \|^2_{L^2},
\]

(3.22)

This implies that, as \( \delta \) goes to zero, we have

\[
\begin{align*}
\varphi_{0,\delta} &\to \varphi_0 & \text{weakly in } H^1(\Omega), \\
\varphi_{0,\delta} &\to \varphi_0 & \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega.
\end{align*}
\]

From [A2], we see that the function

\[
G(s) := \psi(s) + \frac{1}{2} C_1 |s|^2
\]

is convex and non-negative, since

\[
G''(s) = \psi''(s) + \psi''(s) + C_1 \geq 0.
\]

Furthermore, by the assumption [A6] on \( \varphi_0 \) it holds that \( G(\varphi_0) \in L^1(\Omega) \). Then, testing the elliptic problem [3.21] with \( G'(\varphi_{0,\delta}) \) yields

\[
\int_{\Omega} (\varphi_{0,\delta} - \varphi_0) G'(\varphi_{0,\delta}) dx = -\int_{\Omega} \delta G''(\varphi_{0,\delta}) |\nabla \varphi_{0,\delta}|^2 dx \leq 0.
\]

Since \( \psi'(0) = 0 \), we see that \( G'(0) = 0 \), and using the convexity of \( G \) and the previous inequality we infer that

\[
\int_{\Omega} G(\varphi_{0,\delta}) dx \leq \int_{\Omega} G(\varphi_0) + G'(\varphi_{0,\delta})(\varphi_{0,\delta} - \varphi_0) dx \leq \int_{\Omega} G(\varphi_0) dx < \infty.
\]

In particular, by the strong convergence of \( \varphi_{0,\delta} \to \varphi_0 \) in \( L^2(\Omega) \) and the weak lower semi-continuity of the \( L^2(\Omega) \)-norm, it holds that

\[
\limsup_{\delta \to 0} \int_{\Omega} \psi(\varphi_{0,\delta}) dx \leq \limsup_{\delta \to 0} \int_{\Omega} G(\varphi_{0,\delta}) dx + \limsup_{\delta \to 0} \int_{\Omega} -\frac{C_1}{2} |\varphi_{0,\delta}|^2 dx \leq \int_{\Omega} G(\varphi_0) dx - \lim_{\delta \to 0} \int_{\Omega} \frac{C_1}{2} |\varphi_{0,\delta}|^2 dx \leq \int_{\Omega} \psi(\varphi_0) dx.
\]

(3.23)

Hence, for given \( \varphi_0 \in H^1(\Omega) \) satisfying \( \psi(\varphi_0) \in L^1(\Omega) \), we consider the sequence of solutions \( (\varphi_{\delta}, \mu_\delta, \sigma_\delta, u_\delta) \) to (1.1) with initial conditions \( (\varphi_{0,\delta}, \sigma_0) \) such that \( \varphi_{0,\delta} \in H^2_{n}(\Omega) \)
is the unique solution to \(\text{(3.21)}\). Then, \((\varphi_\delta, \mu_\delta, \sigma_\delta, u_\delta)\) satisfies the energy inequality \(\text{(2.6)}\) with a right-hand side given by
\[
C\left(1 + \|\psi(\varphi_\delta)\|_{L^1(0,T;L^1)} + \|\varphi_\delta\|_{L^2(0,T;H^1)}^2 + \|\mathcal{E}(\sigma_\delta)\|_{L^2(0,T;L^2)}^2\right)
+C\|\varphi_\delta,0\|_{H^1}^2 + \|\psi(\varphi_\delta,0)\|_{L^1} + \beta\|\sigma_0\|_{L^2}^2
\]
with a positive constant \(C\) independent of \(\delta \in (0,1]\). Then, a Gronwall argument yields the uniform in \(\delta\) estimate so that the solution \((\varphi_\delta, \mu_\delta, \sigma_\delta, u_\delta)\) satisfies the same compactness assertions listed in Section \(3.3\) and converges along a non-relabelled subsequence to limit functions \((\varphi, \mu, \sigma, u)\) in the limit \(\delta \to 0\). The strong convergence of \(\mathcal{E}(u_\delta) \to \mathcal{E}(u)\) follows from the monotonicity argument outlined in \([17, 18]\). We omit the rest of the details and infer that \((\varphi, \mu, \sigma, u)\) is a weak solution fulfilling the assertions of Theorem \(1\) with initial condition \(\varphi_0\) satisfying \(\text{(A0)}\).

4 Quasi-static limit

We now consider the limit \(\beta \to 0\) in \(\text{(1.1)}\), i.e. we consider the quasi-static limit of the nutrient diffusion equation. In order to prove Theorem \(2\) we denote the solutions to the weak formulation \(\text{(2.2a)}-\text{(2.2d)}\) constructed in Theorem \(1\) by \((\varphi_\beta, \mu_\beta, \sigma_\beta, u_\beta)\). The compactness assertions aside from \(\text{(2.9)}\) are consequences of the uniform estimates obtained from analogues of \(\text{(2.5)}\), as well as the monotonicity argument of \([17, 18]\). From the uniform estimates we also have \(\sigma_\beta \to \sigma_*\) weakly in \(L^2(0,T;H^1(\Omega))\), which is sufficient to pass to the limit in the source terms \(U(\varphi_\beta, \mu_\beta, \mathcal{E}(u_\beta))\) and \(S(\varphi_\beta, \sigma_\beta)\) due to their particular forms \(\text{(1.2)}\) and \(\text{(1.3)}\), as well as the strong convergence of \(\varphi_\beta \to \varphi_*\) in \(C^0([0,T];L^r(\Omega))\) for any \(r < \infty\) if \(d = 2\) or \(r < 6\) if \(d = 3\). However, since the mobility \(m\) depends (perhaps non-linearly) on \(\sigma\), we require the a.e. convergence of \(\sigma_\beta\) to \(\sigma_*\) in \(Q\) which is not available simply from the uniform \(L^2(0,T;H^1(\Omega))\) estimate for \(\{\sigma_\beta\}_{\beta \in (0,1]}\). Therefore, in the following we derive a strong convergence result for \(\sigma_\beta\) in \(L^2(0,T;L^2(\Omega))\). First, considering \(\zeta \in L^2(0,T;H^1(\Omega))\), and passing to the limit \(\beta \to 0\) in
\[
0 = \int_0^T \beta(\sigma_{\beta,t}, \zeta) + (\nabla \sigma_{\beta}, \nabla \zeta) + \kappa(\sigma_{\beta} - \sigma_B, \zeta)\Gamma - (S(\varphi_\beta, \sigma_\beta), \zeta)\ dt
\]
yields
\[
0 = \int_0^T (\nabla \sigma_*, \nabla \zeta) + \kappa(\sigma_* - \sigma_B, \zeta)\Gamma - (S(\varphi_*, \sigma_*), \zeta)\ dt.
\]
Then, denoting \(\hat{\sigma} := \sigma_{\beta} - \sigma_*\) and taking the difference of the two equations above gives
\[
\int_0^T (\nabla \hat{\sigma}, \nabla \zeta) + \kappa(\hat{\sigma}, \zeta)\Gamma + B(\hat{\sigma}, \zeta) + \lambda_c(h(\varphi_\beta)\hat{\sigma}, \zeta)\ dt
= -\int_0^T (\beta \sigma_{\beta,t}, \zeta) - \lambda_c([h(\varphi_\beta) - h(\varphi_*)]\sigma_{\beta}, \zeta)\ dt
\]
Choosing \(\zeta = \sigma_{\beta} - \sigma_*\) and observe that
\[
\left|\int_0^T (\beta \sigma_{\beta,t}, \sigma_{\beta} - \sigma_*)\ dt\right| = \left|\frac{\beta}{2}\|\sigma_{\beta}(T)\|_{L^2}^2 - \frac{\beta}{2}\|\sigma_0\|_{L^2}^2 - \int_0^T (\beta \sigma_{\beta,t}, \sigma_*)\ dt\right|
\leq \frac{\beta}{2}M + \frac{\beta}{2}\|\sigma_0\|_{L^2}^2 + \left|\int_0^T (\beta \sigma_{\beta,t}, \sigma_*)\ dt\right| \to 0.
\]
and
\[ \int_0^T \lambda_c([h(\varphi_\beta) - h(\varphi_*)] \sigma_\beta, \sigma_\beta - \sigma_*) \, dt \leq \max_{t \in [0,T]} \lambda_c(t) C \|h(\varphi_\beta) - h(\varphi_*)\|_{L^2(0,T;L^2)} \to 0, \]
on account of the weak convergence of $\beta \sigma_{\beta,t} \to 0$ in $L^2(0,T;H^1(\Omega))$, the strong convergence $h(\varphi_\beta) \to h(\varphi_*)$ in $L^2(0,T;L^2(\Omega))$ (which is a consequence of the boundedness of $h$ and the a.e. convergence of $\varphi_\beta \to \varphi$ in $Q$), as well as the boundedness $0 \leq \sigma_\beta, \sigma_* \leq M$ a.e. in $Q$.

Hence, choosing $\zeta = \sigma_\beta - \sigma_*$ in (4.1), neglecting the non-negative term $B \|\hat{\sigma}\|^2_{L^2} + \lambda_c \|h^{1/2}(\varphi_*)\hat{\sigma}\|^2_{L^2}$ on the left-hand side, and employing the generalised Poincaré inequality (3.8) yields
\[ \|\sigma_\beta - \sigma_*\|_{L^2(0,T;H^1)} \to 0 \quad \text{as} \quad \beta \to 0. \]

This yields the compactness assertion (2.9), and the rest of the proof follows similarly as described in the previous sections.

5 Regularity

5.1 Regularity for the nutrient

Suppose $\sigma_0 \in H^1(\Omega)$ and $\sigma_B \in H^1(0,T;L^2(\Gamma))$. The following formal estimates can be obtained rigorously at the level of the Galerkin approximation, and so we will only sketch the details. The nutrient system can be expressed as
\[
\begin{cases}
\beta \sigma_t - \Delta \sigma + B \sigma = -\lambda_c h(\varphi) \sigma + B \sigma_c =: f_\sigma & \text{in } Q, \\
\partial_n \sigma + \kappa (\sigma - \sigma_B) = 0 & \text{on } \Sigma, \\
\sigma(0) = \sigma_0 & \text{in } \Omega.
\end{cases}
\]

From Theorem 1 and from the assumption on the data, it easily follows that $f_\sigma \in L^2(Q)$. Hence, testing by $\sigma_t$ yields
\[
\frac{d}{dt} \frac{1}{2} \left( \|\nabla \sigma\|^2_{L^2} + B \|\sigma\|^2_{L^2} + \kappa \|\sigma\|^2_{L^2} \right) + \beta \|\sigma_t\|^2_{L^2} \leq \|f_\sigma\|_{L^2} \|\sigma_t\|_{L^2} + \kappa \int_\Gamma \sigma_B \sigma_t \, dA.
\]

In order to handle the last boundary term we integrate in time and by parts, and invoke the Young inequality and the trace theorem to obtain that
\[
\kappa \int_0^t \int_\Gamma \sigma_B \sigma_t \, dA
\]
\[
= -\kappa \int_0^t \int_\Gamma \sigma_{B,t} \sigma \, dA + \kappa \int_\Gamma \sigma_B(t) \sigma(t) \, dA - \kappa \int_\Gamma \sigma_B(0) \sigma_0 \, dA
\]
\[
\leq \frac{\delta}{4} \int_\Gamma |\sigma(t)|^2 + C(\|\sigma_{B,t}\|^2_{L^2(\Sigma)} + \|\sigma\|^2_{L^2(\Sigma)} + \|\sigma_B\|^2_{C^0(0,T;L^2(\Gamma))} + \|\sigma_0\|^2_{H^1}),
\]
for a positive $\delta$ to be chosen as
\[
\delta = \begin{cases}
\kappa & \text{if } \kappa > 0, \\
\epsilon B & \text{if } \kappa = 0,
\end{cases}
\]

20
where \( q \) denotes the inverse of the trace constant and we also owe to the improved smoothness of the data \( \sigma_B \in H^1(0,T; L^2(\Gamma)) \subset C^0(0,T; L^2(\Gamma)) \). Thus, integrating in time shows that
\[
\sigma \in L^\infty(0,T; H^1(\Omega)) \quad \forall \beta \in [0, \infty), \quad \sigma \in H^1(0,T; L^2(\Omega)) \quad \forall \beta > 0.
\]
Hence we can now absorb the possible term \( \beta \sigma_t \) in the source contribution \( f_\sigma \) which will still belong to \( L^2(Q) \). Then, provided we require \( \sigma_B \in L^2(0,T; H^{1/2}(\Gamma)) \), we can invoke elliptic regularity theory to conclude, independently of \( \beta \), that \( \sigma \in L^2(0,T; H^2(\Omega)) \) (see, e.g. [40, Thm. 4.18, pp. 137-138]).

5.2 Regularity under Vegard’s law and homogeneous elasticity

Under Vegard’s law [B1] and homogeneous elasticity [B2], the partial derivative \( W_{\varphi}(\varphi, \mathcal{E}(u)) \) assumes the following form
\[
W_{\varphi}(\varphi, \mathcal{E}(u)) = -C(\mathcal{E}(u) - \mathcal{E}^\ast) : \mathcal{E}^\ast,
\]
and by the regularities stated in Theorem [1] we see that \( W_{\varphi}(\varphi, \mathcal{E}(u)) \in L^\infty(0,T; L^2(\Omega)^{d \times d}) \).
Hence, it holds that \( f := \chi \varphi + \mu - W_{\varphi}(\varphi, \mathcal{E}(u)) - \varepsilon^{-1} \psi'_1(\varphi) \) belongs to \( L^2(Q) \). For \( N \in \mathbb{N} \), we introduce the truncation
\[
\varphi_N := \max(\mathcal{N}, \min(\mathcal{N}, \varphi)) \text{ a.e. in } Q.
\]
Then, it is clear that \( \varphi_N \to \varphi \) a.e. in \( Q \), and as \( \varphi_N \) is bounded it holds \( \psi'_1(\varphi_N) \in L^2(0,T; H^1(\Omega)) \cap L^\infty(Q) \). Testing (2.2b) with \( \psi'_1(\varphi_N) \) and using the convexity of \( \psi_1 \) leads to
\[
\int_\Omega \varepsilon^{-1} \psi'_1(\varphi) \psi'_1(\varphi_N) dx \leq \int_\Omega \frac{1}{2} \psi'_1(\varphi) \psi'_1(\varphi_N) + \varepsilon \psi''(\varphi_N) |\nabla \varphi_N|^2 dx \leq \|f\|_{L^2} \|\psi'_1(\varphi_N)\|_{L^2}.
\]
Using the facts that \( \psi'_1 \) is increasing and the monotonicity \( \psi'_1(s) \geq 0 \), we infer
\[
\|\psi'_1(\varphi_N)\|_{L^2} \leq \int_\Omega \psi'_1(\varphi) \psi'_1(\varphi_N) dx \leq \|f\|_{L^2} \|\psi'_1(\varphi_N)\|_{L^2}.
\]
This gives boundedness of \( \psi'_1(\varphi_N) \) in \( L^2(Q) \), and by Fatou’s lemma,
\[
\|\psi'_1(\varphi)\|_{L^2(Q)} \leq \liminf_{N \to \infty} \|\psi'_1(\varphi_N)\|_{L^2(Q)} \leq \|f\|_{L^2(Q)},
\]
which is the first assertion. In turn, (2.2b) is the weak formulation of the elliptic problem
\[
\begin{aligned}
&-\varepsilon \Delta \varphi = f - \varepsilon^{-1} \psi'_1(\varphi) \quad \text{in } \Omega, \\
&\partial_n \varphi = 0 \quad \text{on } \Gamma,
\end{aligned}
\]
and elliptic regularity then yields
\[
\int_0^T \|\varphi\|_{H^2}^2 dt \leq \int_0^T C \left( \|\varphi\|_{H^1}^2 + \|f\|_{L^2}^2 + \varepsilon^{-1} \|\psi'_1(\varphi)\|_{L^2}^2 \right) dt < \infty
\]
which completes the proof.

**Remark 5.1.** Having \( \varphi \in L^2(0,T; H^2) \) at disposal, it would be natural to hope to improve the regularity of the displacement \( u \) since, roughly speaking, equation [1.10] can be written as an elliptic equation for \( u \) whose source has a regularity which depends on \( \varphi \). Unfortunately, this is not in general possible due to the choice of the boundary conditions which prevent the regularity of \( u \) to be improved.
6 Continuous dependence under Vegard’s law and homogeneous elasticity

6.1 Preliminaries

The Riesz isomorphism $A : H^1(\Omega) \to H^1(\Omega)'$ is defined as

$$\langle Au, v \rangle := (u, v)_{H^1} = \int_\Omega (\nabla u \cdot \nabla v + uv) dx.$$  

It is well-known that the restriction of $A$ to $H_0^1(\Omega)$ yields an isomorphism from $H_0^1(\Omega)$ to $L^2(\Omega)$, so that its inverse $A^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is well-defined. Moreover, the following properties hold

$$\langle Au, A^{-1}w \rangle = \langle w, u \rangle \quad \text{for all } u \in H^1(\Omega), \ w \in H^1(\Omega)' ,$$

$$\langle w, A^{-1}z \rangle = (w, z)_* \quad \text{for all } w, z \in H^1(\Omega)' ,$$

where the symbol $(\cdot, \cdot)_*$ denotes the standard inner product in the dual of $H^1(\Omega)$, and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^1(\Omega)$ and its dual. By Lax–Milgram theorem, we have the estimate

$$\|A^{-1}z\|_{H^1}^2 = \langle z, A^{-1}z \rangle \leq \|z\|_* \|A^{-1}z\|_{H^1} \implies \|A^{-1}z\|_{H^1} \leq \|z\|_*$$

for all $z \in H^1(\Omega)'$. Furthermore, the continuous embedding $H^1(\Omega) \subset L^2(\Omega)$ yields

$$\langle u, v \rangle = \int_\Omega uv \quad \text{for all } u, v \in L^2(\Omega), \quad (6.1)$$

which implies, for any $f \in H^1(\Omega)$, the following interpolation inequality

$$\|f\|_{L^2}^2 = \langle f, f \rangle = \langle Af, A^{-1}f \rangle \leq \|f\|_{H^1} \|A^{-1}f\|_{H^1} \leq \|f\|_{H^1} \|f\|_* , \quad (6.2)$$

Moreover, for all $w \in H^1(0, T; H^1(\Omega))$, we also infer that

$$\langle w_t(t), A^{-1}w(t) \rangle = \frac{1}{2} \frac{d}{dt} \|w(t)\|_*^2 \quad \text{for a.e } t \in (0, T).$$

We also state here two special cases of the Gagliardo–Nirenberg inequality in three dimensions that we will use later:

$$\|f\|_{L^3} \leq C \|f\|_{L^2}^{1/2} \|f\|_{H^1}^{1/2} \quad \text{for all } f \in H^1(\Omega), \quad (6.3)$$

$$\|f\|_{L^\infty} \leq C \|f\|_{H^1}^{1/2} \|f\|_{H^2}^{1/2} \quad \text{for all } f \in H^2(\Omega). \quad (6.4)$$

We will prove the continuous dependence result for $d = 3$. The case $d = 2$ is easier due to better embedding properties and is omitted.

Taking the difference of system (1.1) between two sets of solutions $\{(\varphi_1, \mu_1, \sigma_1, \mathbf{u}_1)\}_{i=1,2}$, and denoting the differences as

$$\varphi := \varphi_1 - \varphi_2, \ \mu := \mu_1 - \mu_2, \ \sigma := \sigma_1 - \sigma_2, \ \mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2, \quad (6.5)$$

along with

$$f_i := f(\varphi_i), \ \ h_i := h(\varphi_i), \ \ k_i := k(\varphi_i), \ \ \psi' := \psi'(\varphi_i), \ \ W_{E,i} := W_{E}(\varphi_i, E(\mathbf{u}_i)),$$

$$\hat{f} := f_1 - f_2, \ \ \hat{h} := h_1 - h_2, \ \ \hat{k} := k_1 - k_2, \ \ \hat{\psi}' := \psi'_1 - \psi'_2, \ \ \sigma_{B} := \sigma_{B,1} - \sigma_{B,2}, \ \ \sigma_{c} := \sigma_{c,1} - \sigma_{c,2}, \ \ \mathbf{g} := \mathbf{g}_1 - \mathbf{g}_2,$$

$$\mathcal{W}_{\varphi} := W_{\varphi}(\varphi_1, E(\mathbf{u}_1)) - W_{\varphi}(\varphi_2, E(\mathbf{u}_2)) = -C(E(\mathbf{u}) - E^*(\varphi) : E^*), \quad (6.6)$$

22
for \( i = 1, 2 \), it holds that
\[
0 = \langle \varphi_t, \zeta \rangle + (\nabla \mu, \nabla \zeta) - \left( \frac{\lambda}{1 + |W, x, 1|} (f \sigma_1 + f_2 \sigma) - \lambda_0 \hat{k}, \zeta \right)
- \left( \frac{\lambda}{1 + |W, x, 1| (1 + |W, x, 2|)} ([W, x, 1] - |W, x, 2|), \zeta \right),
\]
\[
0 = (\mu, \xi) - \varepsilon (\nabla \varphi, \nabla \xi) - \varepsilon^{-1} (\varphi, \xi) + \chi(\sigma, \xi) + (C(E(u) - \mathcal{E}^* \varphi) : \mathcal{E}^*, \xi),
\]
\[
0 = \beta (\sigma_1, \zeta) + (\nabla \sigma, \nabla \zeta) + \kappa (\sigma - \sigma_B, \zeta) + (\lambda_0 h \sigma_1 + \lambda_0 h_2 \sigma, \zeta) + B(\sigma - \sigma_e, \zeta),
\]
\[
0 = (C(E(u) - \mathcal{E}^* \varphi), \nabla \eta) - (\hat{g}, \eta)_\Gamma_N,
\]
for a.e. \( t \in (0, T) \), and for all \( \zeta, \xi \in H^1(\Omega) \) and \( \eta \in X(\Omega) \).

### 6.2 Continuous dependence in weaker norms

To recover the operator \( A \) in the first two equations, we add to both sides of (6.7a) the term \((\mu, \zeta)\) and to both sides of (6.7b) the term \(-\varepsilon (\varphi, \xi)\). Moreover, we define a modified potential
\[
\Psi(s) := \psi(s) - \frac{\varepsilon}{2} s^2,
\]
which still fulfills (C3) as well as
\[
\Psi' := \Psi'(\varphi_1) - \Psi'(\varphi_2) = \hat{\psi}' - \varepsilon \varphi.
\]

In particular, (6.7a), and (6.7b) now assume the form
\[
0 = \langle \varphi_t, \zeta \rangle + (A\mu, \zeta) - (\mu, \zeta) - \left( \frac{\lambda}{1 + |W, x, 1|} (f \sigma_1 + f_2 \sigma) - \lambda_0 \hat{k}, \zeta \right)
- \left( \frac{\lambda}{1 + |W, x, 1| (1 + |W, x, 2|)} ([W, x, 1] - |W, x, 2|), \zeta \right),
\]
\[
0 = (\mu, \xi) - \varepsilon (\nabla \varphi, \nabla \xi) - \varepsilon^{-1} (\varphi, \xi) + \chi(\sigma, \xi) + (C(E(u) - \mathcal{E}^* \varphi) : \mathcal{E}^*, \xi),
\]
for a.e. \( t \in (0, T) \), and for all \( \zeta, \xi \in H^1(\Omega) \).

First, testing (6.7b) with \( \sigma \) gives
\[
\frac{\beta}{2} \frac{d}{dt} \|\sigma\|_{L^2}^2 + \|\nabla \sigma\|_{L^2}^2 + \frac{B}{2} \|\sigma\|_{L^2}^2 + \frac{\kappa}{2} \|\sigma\|_{H^1}^2 \leq C \left( \|\varphi\|_{L^2}^2 + \|\hat{\sigma}_e\|_{L^2}^2 + \|\hat{\sigma}_B\|_{L^2}^2 \right),
\]
where we neglected the non-negative term \( \lambda_0 h_2 \sigma^2 \) and the constant \( C \) is independent of \( \beta \). To this, we add the equalities obtained from testing (6.8a) by \( A^{-1} \varphi \), (6.8b) with \( \varphi \) and (6.7a) with \( u \), whilst recalling the relation \( (A\mu, A^{-1} \varphi) = (\mu, \varphi) = (\mu, \varphi) \), and recalling the coercivity estimate from (2.6):
\[
C E(u) : E(u) \geq C u \|E(u)\|_{L^2}^2,
\]
we arrive at
\[
\frac{1}{2} \frac{d}{dt} \left( \|\varphi\|_{L^2}^2 + \beta \|\sigma\|_{L^2}^2 \right) + \varepsilon \|\varphi\|_{H^1}^2 + \|\nabla \sigma\|_{L^2}^2 + \frac{B}{2} \|\sigma\|_{L^2}^2 + \frac{\kappa}{2} \|\sigma\|_{H^1}^2 \leq C \left( \|\varphi\|_{L^2}^2 + \|\hat{\sigma}_e\|_{L^2}^2 + \|\hat{\sigma}_B\|_{L^2}^2 \right),
\]
\[
=: I_1 + I_2 + I_3 + I_4.
\]
Invoking the boundedness and Lipschitz continuity of \( f \) and \( k \), the boundedness of \( \sigma_1 \) and \( \sigma_2 \), the trace theorem, Young’s inequality, Korn’s inequality and the interpolation inequality \((6.2)\), we can estimate the terms on the right-hand side as follows:

\[
I_1 \leq \frac{\delta_1}{4}||\mathbf{E}(\mathbf{u})||^2_{L^2} + \frac{\varepsilon}{8}||\sigma||^2_{H^1} + \frac{\delta_2}{2}||\sigma||^2 + C||\varphi||^2 + (\mu, A^{-1}\varphi),
\]

\[
I_2 \leq C(||\varphi||_{L^2} + ||\sigma||_{L^2})||A^{-1}\varphi||_{L^2} \leq \frac{\delta_3}{2}||\sigma||^2_{L^2} + \frac{\varepsilon}{8}||\varphi||^2_{H^1} + C||\varphi||^2,
\]

\[
I_3 \leq C(||\mathbf{E}(\mathbf{u})||_{L^2} + ||\varphi||_{L^2})||A^{-1}\varphi||_{L^2} \leq \frac{\delta_3}{4}||\mathbf{E}(\mathbf{u})||^2_{L^2} + \frac{\varepsilon}{8}||\varphi||^2_{H^1} + C||\varphi||^2;
\]

\[
I_4 \leq \frac{\delta_1}{2}||\mathbf{E}(\mathbf{u})||^2_{L^2} + \frac{\varepsilon}{8}||\varphi||^2_{H^1} + C\left(||\varphi||^2 + ||\hat{g}||^2_{L^2(\Gamma_N)}\right)
\]

for \( \delta_1, \delta_2 > 0 \) yet to be determined. Choosing \( \delta_1 = \frac{C}{2} \) then gives

\[
\frac{d}{dt}\left(||\varphi||^2_{L^2} + \beta ||\sigma||^2_{L^2}\right) + \varepsilon ||\varphi||^2_{H^1} + 2||\nabla \sigma||^2_{L^2} + B||\sigma||^2_{L^2} + K||\sigma||^2_{L^2} + C_4||\mathbf{E}(\mathbf{u})||^2_{L^2} + 2\varepsilon^{-1}(\hat{\Psi}', \varphi) \leq C\left(||\varphi||^2 + ||\hat{\sigma}||^2_{L^2} + ||\hat{\sigma}||^2_{L^2} + ||\hat{g}||^2_{L^2(\Gamma_N)}\right) + 2\delta_2 ||\sigma||^2_{L^2} + 2(\mu, A^{-1}\varphi). \tag{6.11}
\]

Next, from testing \((6.7b)\) with \( A^{-1}\varphi \) we infer that

\[
2(\mu, A^{-1}\varphi) \leq 2||\varphi|| \left(\varepsilon ||\varphi||_{H^1} + \varepsilon^{-1}||\hat{\Psi}'|| + \chi||\sigma||_{L^2} + c_*=||\mathbf{E}(\mathbf{u})||_{L^2} + c_*||\varphi||_{L^2}\right),
\]

where

\[
c_* := \max_{1 \leq i,j,k,l \leq d} \left|C_{ijkl}\right|, \quad e_* := \max_{1 \leq i,j \leq d} \left|\mathbf{E}_{ij}\right|.
\]

By the continuous embedding \( L^{6/5}(\Omega) \subset H^1(\Omega)^d \) and \( H^1(\Omega) \subset L^6(\Omega) \), \((C3)\) with exponent \( q = 4 \), and Hölder’s inequality we obtain that

\[
||\hat{\Psi}'||_{L^{6/5}} \leq C ||\hat{\Psi}'||_{L^6} \leq C \left(1 + ||\varphi_1||_{L^6}^4 + ||\varphi_2||_{L^6}^4\right)||\varphi||_{L^6} \leq C ||\varphi||_{H^1}
\]

taking into account \( \varphi_1, \varphi_2 \in L^\infty(0, T; H^1(\Omega)). \) Then, by Young’s inequality

\[
2(\mu, A^{-1}\varphi) \leq \frac{\varepsilon}{4}||\varphi||^2_{H^1} + \delta_2 ||\sigma||^2_{L^2} + \frac{C_4}{2}||\mathbf{E}(\mathbf{u})||^2_{L^2} + C||\varphi||^2. \tag{6.12}
\]

Next, recalling the convex-concave decomposition of the potential \( \psi \), we deduce that

\[
2\varepsilon^{-1}(\hat{\Psi}', \varphi) \geq 2\varepsilon^{-1}(\psi_1'(\varphi_1) - \psi_2'(\varphi_2), \varphi) - 2\varepsilon||\varphi||^2_{L^2} \geq -C||\varphi||^2_{L^2} \geq -\frac{\varepsilon}{4}||\varphi||^2_{H^1} - C||\varphi||^2.
\]

Substituting the above and \((6.12)\) into \((6.11)\) leads to the differential inequality

\[
\frac{d}{dt}\left(||\varphi||^2 + \beta ||\sigma||^2_{L^2}\right) + \frac{\varepsilon}{2}||\varphi||^2_{H^1} + \frac{C_4}{2}||\mathbf{E}(\mathbf{u})||^2_{L^2} + 2||\nabla \sigma||^2_{L^2} + \frac{B}{2}||\sigma||^2_{L^2} + \frac{K}{2}||\sigma||^2_{L^2} - 3\delta_2||\sigma||^2_{L^2} \leq C\left(||\varphi||^2 + ||\hat{\sigma}||^2_{L^2} + ||\hat{\sigma}||^2_{L^2} + ||\hat{g}||^2_{L^2(\Gamma_N)}\right),
\]

with a constant \( C \) independent of \( \beta \). For the case \( \beta > 0 \), we can move the last term on the left-hand side to the right hand side and invoke Gronwall’s inequality and Korn’s
inequality to deduce (2.12) aside for the estimate of \( \mu \). For the case \( \beta = 0 \), at least one of \( \{B, k\} \) is non-zero, and so choosing \( \delta \) sufficiently small and possibly invoking the generalised Poincaré inequality (if \( B = 0 \)), we can absorb the contribution \( 3\delta \|\sigma\|_{L^2}^2 \) on the left hand side, we deduce via Gronwall’s inequality also an analogous estimate. Then, to complete the proof, from (6.7b) we infer
\[
\int_0^T \|\mu\|_{L^2}^2 \, dt \leq C \int_0^T \left( \|\varphi\|_{H^1}^2 + \|\mathcal{E}(u)\|_{L^2}^2 + \|\sigma\|_{L^2}^2 \right) \, dt,
\]
which yields the remaining \( L^2(0, T; H^1(\Omega')) \) estimate for \( \mu \).

### 6.3 Continuous dependence in stronger norms

Now, suppose the exponent \( q \) in (C3) is 2, testing (6.7a) with \( \varepsilon \varphi \) and (6.7b) with \( \mu \) yields upon summing
\[
\frac{\varepsilon}{2} \frac{d}{dt} \|\varphi\|_{L^2}^2 + \|\mu\|_{L^2}^2 = \varepsilon \left( \frac{\lambda_p}{1+|W,\varepsilon,1|}(f\sigma_1 + f_2\sigma) - \lambda_y k, \varphi \right) + \varepsilon \left( \frac{\lambda_p f_2 \sigma_2}{(1+|W,\varepsilon,1|)(1+|W,\varepsilon,2|)} ([W,\varepsilon,1] - [W,\varepsilon,2]), \varphi \right) + \varepsilon^{-1} (\dot{\psi}, \mu) - \chi(\sigma, \mu) - (C(\mathcal{E}(u) - \mathcal{E}(\varphi) : \mathcal{E}, \mu) =: J_1 + J_2 + J_3.
\]

Then, by Young’s inequality, the Lipschitz continuity of \( f \) and \( k \), as well as the boundedness of \( \sigma_1 \) and \( \sigma_2 \), we obtain that
\[
J_1 \leq C\|\varphi\|_{L^2}^2 + C\|\sigma\|_{L^2}^2,
\]
\[
J_2 \leq C\|\mathcal{E}(u)\|_{L^2}^2 + C\|\varphi\|_{L^2}^2,
\]
\[
J_3 \leq C \left( 1 + \|\varphi_1\|_{L^\infty}^2 + \|\varphi_2\|_{L^\infty}^2 \right) \|\varphi\|_{L^2}^2 + C\|\sigma\|_{L^2}^2 + C\|\mathcal{E}(u)\|_{L^2}^2 + \frac{1}{2}\|\mu\|_{L^2}^2.
\]

Employing the Gagliardo–Nirenberg inequality (6.4), as well as the regularities \( \varphi_1, \varphi_2 \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) we see that
\[
\varepsilon \frac{d}{dt} \|\varphi\|_{L^2}^2 + \|\mu\|_{L^2}^2 \leq C \left( 1 + \|\varphi_1\|_{H^1}^2 + \|\varphi_2\|_{H^2}^2 \right) \|\varphi\|_{L^2}^2 + C\|\mathcal{E}(u)\|_{L^2}^2 + C\|\sigma\|_{L^2}^2.
\]

Then, Gronwall’s inequality as well as the estimate (2.12) for \( u \) and \( \sigma \) yields the continuous dependence results for \( \varphi \) in \( L^\infty(0, T; L^2(\Omega)) \) and for \( \mu \) in \( L^2(0, T; L^2(\Omega)) \). Next, testing (6.7a) with \( \eta = u \) and using the coercivity estimate (6.10), as well as the continuous dependence result for \( \varphi \) in \( L^\infty(0, T; L^2(\Omega)) \), allows us to infer that
\[
\|\mathcal{E}(u)\|_{L^2}^2 \leq C\|\varphi\|_{L^2}^2 + C\|\tilde{g}\|_{L^2(\Gamma_N)}^2.
\]

Taking supremum in time and applying Korn’s inequality yields the continuous dependence result for \( u \) in \( L^\infty(0, T; X(\Omega)) \).

### 6.4 Continuous dependence via time discretisation approach

In this section, we assume (C3) with exponent \( q \) equal to 2 and (C3) to hold. Recalling the notation for the difference between two sets of weak solutions (6.5a–6.5b), for \( t \in (0, T) \) and \( h > 0 \), we define
\[
\delta_h \varphi(t) := \frac{\varphi(t) - \varphi(t-h)}{h}, \quad \delta_h \sigma(t) := \frac{\sigma(t) - \sigma(t-h)}{h}, \quad \delta_h u(t) := \frac{u(t) - u(t-h)}{h}.
\]
Moreover, we set
\[ \varphi(t) := \varphi_{t_1} - \varphi_{t_2}, \quad \sigma(t) := \sigma_{t_1} - \sigma_{t_2}, \quad u(t) := u_{t_1} - u_{t_2}, \quad \text{for all } t \leq 0, \]
where \( u_{t_1}, i \in \{1, 2\} \), is the unique solution of the following elliptic problem:
\[ (W, F(\varphi_{t_1}, E(u_{t_1})), \nabla \eta) = (C(E(u_{t_1}) - \hat{E} - E^* \varphi_{t_1}), \nabla \eta) = (g_i, \eta)_{\Gamma_N} \quad \forall \eta \in X(\Omega). \]
The coercivity estimate \( [6.10] \) and the Lax–Milgram theorem give the existence of a unique \( u_{t_1} \in X(\Omega) \), and by taking the difference of the elliptic equations for \( u_{t_1} \) and \( u_{t_2} \) we can infer from testing \( \eta = (u_{t_1} - u_{t_2}) \) and Korn’s inequality that
\[ \|u_{t_1} - u_{t_2}\|^2_{H^1} \leq C\|\varphi_{t_1} - \varphi_{t_2}\|^2_{L^2} + C\|\hat{g}\|^2_{L^2(\Gamma_N)}. \quad (6.13) \]

Let us point out that the difference \( \hat{U} := U(\varphi_1, \sigma_1, E(u_1)) - U(\varphi_2, \sigma_2, E(u_2)) \) is already bounded in \( L^2(Q) \) by previous results. Namely from \( [6.8a] \), we have
\[ \|\hat{U}\|_{L^2(Q)} \leq C(\|\varphi\|_{L^2(Q)} + \|\sigma\|_{L^2(Q)} + \|E(u)\|_{L^2(Q)}) \]
\[ \leq C(\|\varphi_{t_1} - \varphi_{t_2}\|_{L^2} + \beta\|\sigma_{t_1} - \sigma_{t_2}\|_{L^2} + \|\hat{g}\|_{L^2(\Gamma_N)} + \|\hat{\sigma}_B\|_{L^2(\Sigma)} + \|\hat{\sigma}_C\|_{L^2(\Omega)}) \]
\[ =: C \mathcal{V}, \]
with a constant \( C \) independent of \( \beta \). Similarly, for \( \hat{S} := S(\varphi_1, \sigma_1) - S(\varphi_2, \sigma_2) \), we have
\[ \|\hat{S}\|_{L^2(Q)} \leq C(\|\varphi\|_{L^2(Q)} + \|\sigma\|_{L^2(Q)}) \leq C \mathcal{V}, \]
with a constant \( C \) independent of \( \beta \).

Without loss of generality, we fix \( t \in (0, T) \), and take \( h \) sufficiently small so that \( t - h > 0 \). Then, integrating \( [6.7a] \) over time from \( t - h \) to \( t \), and dividing by \( h \), we have
\[ 0 = (\delta_h \varphi(t), \zeta) + \frac{1}{h} \left( \int_{t-h}^{t} \nabla \mu(t) \, d\tau, \nabla \zeta \right) - \frac{1}{h} \left( \int_{t-h}^{t} \hat{U}(\tau) \, d\tau, \zeta \right) \quad (6.14) \]
holding for all \( \zeta \in H^1(\Omega) \). Choosing \( \zeta = \mu(t) \), and combining the resulting equality with the one obtained from testing \( [6.7b] \) with \( \xi = \delta_h \varphi(t) \), we note that a cancellation occurs and obtain
\[ 0 = \varepsilon(\nabla \varphi(t), \nabla \delta_h \varphi(t)) + \varepsilon^{-1}(\hat{\psi}'(t), \delta_h \varphi(t)) - \chi(\sigma(t), \delta_h \varphi(t)) \]
\[ - (C(E(u(t)) - E^* \varphi(t)) : E^*, \delta_h \varphi(t)) + \frac{1}{h} \int_{t-h}^{t} (\nabla \mu(t), \nabla \mu(t)) - (\hat{U}(\tau), \mu(t)) \, d\tau. \]
To the above, we add the equality obtained from testing \( \eta = \delta_h u(t) \) in \( [6.12] \), as well as using the identity from testing \( \zeta = \varepsilon^{-1} \hat{\psi}'(t) - \chi \sigma(t) \) in \( [6.14] \), leading to
\[ 0 = \varepsilon(\nabla \varphi(t), \nabla \delta_h \varphi(t)) + (C(E(u(t)) - E^* \varphi(t)), \delta_h (E(u(t)) - E^* \varphi(t))) \]
\[ - (\hat{g}, \delta_h u(t))_{\Gamma_N} + \frac{1}{h} \int_{t-h}^{t} (\nabla \mu(t), \nabla \mu(t) - \varepsilon^{-1} \hat{\psi}'(t) + \chi \sigma(t)) \, d\tau \]
\[ - \frac{1}{h} \int_{t-h}^{t} (\hat{U}(\tau), \mu(t) - \varepsilon^{-1} \hat{\psi}'(t) + \chi \sigma(t)) \, d\tau. \quad (6.15) \]
We aim to send $h \to 0$ rigorously. By the identity $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$ and the Lebesgue differentiation theorem, we see that for any $s > 0$,

$$
\int_0^s \varepsilon (\nabla \varphi(t), \nabla \delta_h \varphi(t)) \, dt
= \frac{\varepsilon}{2h} \int_0^s \| \nabla \varphi(t) \|^2_{L^2} - \| \nabla \varphi(t-h) \|^2_{L^2} + \| \nabla (\varphi(t) - \varphi(t-h)) \|^2_{L^2} \, dt
\geq \frac{\varepsilon}{2h} \left( \int_{s-h}^s \| \nabla \varphi(t) \|^2_{L^2} dt - \int_{-h}^0 \| \nabla \varphi(t) \|^2_{L^2} \, dt \right)
\to \frac{\varepsilon}{2} \| \nabla \varphi(s) \|^2_{L^2} - \frac{\varepsilon}{2} \| \nabla \varphi(0) \|^2_{L^2}.
$$

Similarly, a short calculation using the symmetry of the constant elasticity tensor $C$ shows that

$$
\int_0^s (C(E(u(t)) - E^*(\varphi(t))), \delta_h(E(u(t)) - E^*(\varphi(t))) \, dt
= \frac{1}{h} \int_0^s \int_{\Omega} W(\varphi(t), E(u(t))) - W(\varphi(t-h), E(u(t-h))) \, dx \, dt
+ \frac{1}{2h} \int_0^s \| \sqrt{C}(E(u(t) - u(t-h)) - E^*(\varphi(t) - \varphi(t-h)) \|^2_{L^2} \, dt
\geq \frac{1}{h} \int_{s-h}^s \int_{\Omega} W(\varphi(t), E(u(t))) \, dx \, dt - \frac{1}{h} \int_{-h}^0 \int_{\Omega} W(\varphi(t), E(u(t))) \, dx \, dt
\to \int_{\Omega} W(\varphi(s), E(u(s))) \, dx - \int_{\Omega} W(\varphi(0), E(u(0))) \, dx.
$$

Meanwhile,

$$
\int_0^s (\tilde{g}, \delta_h u(t))_{\Gamma_N} \, dt
= \frac{1}{h} \int_{s-h}^s (\tilde{g}, u(t))_{\Gamma_N} \, dt - \frac{1}{h} \int_{-h}^0 (\tilde{g}, u(0))_{\Gamma_N} \, dt
\to (\tilde{g}, u(s))_{\Gamma_N} - (\tilde{g}, u(0))_{\Gamma_N},
$$

and

$$
\frac{1}{h} \int_{s-h}^s (\nabla \mu(t), \nabla \zeta) - (\tilde{U}(t), \zeta) \, dt
\to (\nabla \mu(s), \nabla \zeta) - (\tilde{U}(s), \zeta) \quad \forall \zeta \in H^1(\Omega).
$$

Hence, integrating (6.15) over $(0, s)$ and passing to the limit $h \to 0$ yields

$$
\frac{\varepsilon}{2} \| \nabla \varphi(s) \|^2_{L^2} + \int_{\Omega} W(\varphi(s), E(u(s))) \, dx + \int_0^s \| \nabla \mu(t) \|^2_{L^2} \, dt
\leq \frac{\varepsilon}{2} \| \nabla \varphi(0) \|^2_{L^2} + \| W(\varphi(0), E(u(0))) \|_{L^1} + (\tilde{g}, u(0) - u(s))_{\Gamma_N}
+ \int_0^s (\nabla \mu(t), \varepsilon^{-1} \nabla \tilde{\psi}'(t) - \chi \nabla \sigma(t) + (\tilde{U}(t), \mu(t) - \varepsilon^{-1} \tilde{\psi}'(t) + \chi \sigma(t)) \, dt.
$$

Then, by Young’s inequality, (2.12) and (6.13), the right-hand side can be bounded by

$$
\text{RHS} \leq C \left( \| \varphi(0) \|^2_{L^1} + \| u(0) \|^2_{L^1} + \| \tilde{g} \|^2_{L^2(\Gamma_N)} + \| u(s) \|^2_{H^1} \right) + \frac{1}{2} \int_0^s \| \nabla \mu(t) \|^2_{L^2} \, dt
+ C \| \sigma \|^2_{L^2(0,T;H^1)} + C \| \tilde{U} \|^2_{L^2(Q)} + C \| \mu \|^2_{L^2(Q)} + C \| \tilde{\psi}' \|^2_{L^2(0,T;H^1)}
\leq C \mathcal{Y}^2 + \frac{1}{2} \int_0^s \| \nabla \mu(t) \|^2_{L^2} \, dt + C \| \tilde{\psi}' \|^2_{L^2(0,T;H^1)}.
$$

27
As a consequence of the calculations in Section [6.13] and the continuous dependence estimate (2.12), we have

\[ \| \tilde{v}' \|^2_{L^2(Q)} \leq C \left( 1 + \| \varphi_1 \|^2_{L^2(0,T;H^2)} + \| \varphi_2 \|^2_{L^2(0,T;H^2)} \right) \| \varphi \|^2_{L^\infty(0,T;L^2)} \leq C \mathcal{Y}^2, \quad (6.17) \]

while invoking (6.5) and (6.4) we see that

\[ \| \nabla \tilde{v}' \|^2_{L^2(Q)} \leq C \| \psi''(\varphi_1) - \psi''(\varphi_2) \| \nabla \varphi_1 \|^2_{L^2(Q)} + C \| \psi''(\varphi_2) \| \nabla \varphi_2 \|^2_{L^2(Q)} \]

\[ \leq C \int_Q (1 + |\varphi_1|^2 + |\varphi_2|^2) \| \varphi \|^2 \nabla \varphi_1 \|^2_{L^2} + (1 + |\varphi_2|^2) \| \nabla \varphi_2 \|^2_{L^2} \, dx \, dt \]

\[ \leq C \int_0^T (1 + \| \varphi_1 \|^2_{H^2} + \| \varphi_2 \|^2_{H^2}) \| \varphi \|^2_{H^1} \]

\[ \leq C \mathcal{Y}^2 + C \int_0^T (1 + \| \varphi_1 \|^2_{H^2} + \| \varphi_2 \|^2_{H^2}) \| \nabla \varphi \|^2_{L^2}, \]

where we have used the second inequality of (6.17). Hence, from (6.16), we infer that

\[ \| \nabla \varphi(s) \|^2_{L^2} + \int_0^s \| \nabla \mu(t) \|^2_{L^2} \, dt \leq C \mathcal{Y}^2 + C \int_0^T (1 + \| \varphi_1 \|^2_{H^2} + \| \varphi_2 \|^2_{H^2}) \| \nabla \varphi \|^2_{L^2}, \]

and the result follows first from the application of the integral form of Gronwall’s inequality and then the observation that by elliptic regularity

\[ \| \varphi \|^2_{L^2(0,T;H^2)} \leq C \left( \| \varphi \|^2_{L^2(0,T;H^1)} + \| \mu \|^2_{L^2(Q)} + \| \tilde{v}' \|^2_{L^2(Q)} \right) \]

\[ + C \left( \| \sigma \|^2_{L^2(Q)} + \| \mathcal{E}(u) - \mathcal{E} \psi \|_{L^2} \right) \]

\[ \leq C \mathcal{Y}^2. \]

For the nutrient, under the hypothesis on the data \( \sigma_{B_1}, \sigma_{B_2} \in H^1(0,T;L^2(\Gamma)) \), we infer from Theorem [3] that \( \sigma_1, \sigma_2 \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \). Then, in (6.7c) the term \( \beta(\sigma_1, \zeta) \) can be written as \( \beta(\sigma_1, \zeta) \) and choosing \( \zeta = \delta_h \sigma(t) \) and integrating over \( (0,s) \) yields with the help of previous calculations

\[
0 = \int_0^s (\beta \sigma(t) + \dot{S}(t) + B(\sigma(t) - \dot{\sigma}_c(t), \delta_h \sigma(t)) + (\nabla \sigma(t), \nabla \delta_h \sigma(t)) \\
+ \kappa(\sigma(t) - \dot{\sigma}_B(t), \delta_h \sigma(t))) \, dt \\
\geq \frac{1}{2h} \int_0^s \left( \| \nabla \sigma(t) \|^2_{L^2} + B \| \sigma(t) \|^2_{L^2} + \kappa \| \sigma(t) \|^2_{L^2} \right) \\
- \| \nabla \sigma(t - h) \|^2_{L^2} - B \| \sigma(t - h) \|^2_{L^2} - \kappa \| \sigma(t - h) \|^2_{L^2} \, dt \\
+ \int_0^s (\beta \sigma(t) + \dot{S}(t) - B \dot{\sigma}_c(t), \delta_h \sigma(t)) - \kappa(\dot{\sigma}_B(t), \delta_h \sigma(t)) \, dt.
\]
By a change of variables we deduce that
\[
0 \geq \frac{1}{2h} \int_{s-h}^{s} \left( \|\nabla \sigma (t)\|_{L^2}^2 + B\|\sigma(t)\|_{L^2}^2 + \kappa\|\sigma(t)\|_{L^2}^2 \right) dt \\
- \frac{1}{2h} \int_{-h}^{0} \left( \|\nabla \sigma (t)\|_{L^2}^2 + B\|\sigma(t)\|_{L^2}^2 + \kappa\|\sigma(t)\|_{L^2}^2 \right) dt \\
+ \int_{0}^{s} (\beta \sigma(t) + \hat{S}(t) - B\hat{\sigma}_e(t), \delta \sigma(t)) dt + \int_{0}^{s} (\delta \hat{\sigma}_B(t), \sigma(t)) \Gamma dt \\
- \frac{1}{h} \int_{s-h}^{s} (\hat{\sigma}_B(t), \sigma(t)) \Gamma dt + \frac{1}{h} \int_{-h}^{0} (\hat{\sigma}_B(t + h), \sigma(t)) \Gamma dt.
\]
Setting \( \hat{\sigma}_B(t) = \hat{\sigma}_B(0) \) for \( t \leq 0 \), and applying the Lebesgue differentiation theorem, we find that in the limit \( h \to 0 \) it holds
\[
\int_{k}^{k+1} \left( \|\nabla \sigma(s)\|_{L^2}^2 + B\|\sigma(s)\|_{L^2}^2 + \kappa\|\sigma(s)\|_{L^2}^2 \right) \left( 1 + \frac{1}{2}\beta \int_{a}^{t} \|\sigma(t)\|_{L^2}^2 dt \right) \\
\leq \frac{1}{2} \left( \|\nabla \sigma(0)\|_{L^2}^2 + B\|\sigma(0)\|_{L^2}^2 + \kappa\|\sigma(0)\|_{L^2}^2 \right) + C\|\hat{\sigma}_c\|_{L^2(\Gamma)}^2 + C\|\hat{\sigma}_e\|_{L^2(\Gamma)}^2 \\
+ C\|\hat{\sigma}_B\|_{L^2(\Sigma)}^2 + C\|\sigma\|_{C^0([0,T];L^2(\Gamma))}^2 + C\|\sigma\|_{C^0([0,T];L^2(\Gamma))}^2 \\
\leq C \hat{\gamma}^2 + C\|\sigma(0)\|_{H^1}^2 + C\|\hat{\sigma}_B\|_{H^1([0,T];L^2(\Gamma))}^2.
\]
Furthermore, assuming \( \sigma_{B1}, \sigma_{B2} \in L^2(0,T;H^{1/2}(\Gamma)) \), elliptic regularity gives
\[
\|\sigma\|_{L^2(0,T;H^2)}^2 \\
\leq C \left( \|\sigma(t)\|_{L^2(\Sigma)}^2 + \|\sigma\|_{L^2(0,T;H^1)}^2 + \|\hat{\sigma}_c\|_{L^2(\Gamma)}^2 + \|\hat{\sigma}_e\|_{L^2(\Gamma)}^2 + \|\hat{\sigma}_B\|_{L^2(0,T;H^{1/2}(\Gamma))}^2 \right),
\]
and this completes the proof.

**Acknowledgement**

The third author gratefully acknowledges financial support from the LIA-COPDESC initiative and from the research training group 2339 “Interfaces, Complex Structures, and Singular Limits” of the German Science Foundation (DFG).

**References**

[1] A. Agosti, P.F. Antonietti, P. Ciarletta, M. Grasselli and M. Verani: A Cahn-Hilliard-type equation with application to tumor growth dynamics. Math. Methods Appl. Sci. 40(18) (2017), 7598-7626
[2] H.W. Alt: Linear Functional Analysis, an Application Oriented Introduction. Springer, London, (2016)
[3] E. Bonetti, P. Colli, W. Dreyer, G. Gilardi, G. Schimperna and J. Sprekels: On a model for phase separation in binary alloys driven by mechanical effects. Phys. D 165 (2002), 48-65
[4] H. Byrne and L. Preziosi: Modelling solid tumour growth using the theory of mixtures. Math. Medicine and Biol. 20 (2003), 341–366
[5] M. Carrive, A. Miranville and A. Piétrus: The Cahn-Hilliard equation for deformable elastic continua. Adv. Math. Sci. Appl. 10 (2000), 539–569
[6] G. Cheng, J. Tse, R.K. Jain and L.L. Munn: Micro-environmental mechanical stress controls tumour spheroid size and morphology by suppressing proliferation and inducing apoptosis in cancer cells. PLoS One 4 (2009), e4632
[32] D.A. Hormuth II, J.A. Weis, S.L. Barnes, M.I. Miga, E.C. Rericha, V. Quaranta and T.E. Yankeelov: A mechanically coupled reaction-diffusion model that incorporates intratumoral heterogeneity to predict in vivo glioma growth. J. R. Soc. Interface 14 (2017), 20161010

[33] A.F. Jones, H.M. Byrne, J.S. Gibson and J.W. Dold: A mathematical model of the stress induced during avascular tumour growth. J. Math. Biol. 40(6) (2000), 473–499

[34] F.C. Larché and J.W. Cahn: The effect of self-stress on diffusion in solids. Acta Metall. 30 (1982), 1835–1845

[35] L.P. Lebedev, I.I. Vorovich and M.J. Cloud: Functional analysis in mechanics. Second edition. Springer Monographs in Mathematics, Springer, New York (2013)

[36] P.H. Leo, J. Lowengrub and H.J. Jou: A diffuse interface model for microstructural evolution in elastically stressed solids. Acta Mater. 46 (1998), 2113–2130

[37] E.A.B.F. Lima, J.T. Oden, A. Shahmoradi, D.A. Hormuth II, T.E. Yankeelov and R.C. Almeida: Selection, calibration, and validation of models of tumor growth. Math. Models Methods Appl. Sci. 26(12) (2016) 2341–2368

[38] E.A.B.F. Lima, J.T. Oden, B. Wohlmuth, A. Shahmoradi, D.A. Hormuth II, T.E. Yankeelov, L. Scarabosio and T. Horger: Selection and validation of predictive models of radiation effects on tumor growth based on noninvasive imaging data. Comput. Methods Appl. Mech. Engrg 327 (2017), 277–305

[39] J. Lowengrub, E. Titi and K. Zhao: Analysis of a mixture model of tumor growth. European J. Appl. Math. 24(5) (2013), 691–734

[40] W. McLean: Strongly elliptic systems and boundary integral equations. Cambridge University Press, Cambridge, (2000)

[41] A. Miranville, E. Rocca and G. Schimperna: On the long time behavior of a tumor growth model. J. Differential Equations 267(4) (2019), 2616–2642

[42] J.T. Oden, A. Hawkins and S. Prudhomme: General diffuse-interface theories and an approach to predictive tumor growth modeling. Math. Models Methods Appl. Sci. 20(3) (2010) 477–517

[43] A. Onuki: Ginzburg-Landau approach to elastic effects in the phase separation of solids. J. Phys. Soc. Jpn. 58 (1989), 3065–3068

[44] P. Shi and S. Wright. Higher integrability of the gradient in linear elasticity. Math. Ann. 299 (1994) 435–448

[45] T. Stylianopoulos, J.D. Martin, V.P. Chauhan, S.R. Jain, B. Diop-Frimpong, N. Bardeesy, B.L. Smith, C.R. Ferrone, F.J. Hornicek, Y. Boucher, L.L. Muna and R.K. Jain. Causes, consequences, and remedies for growth-induced solid stress in murine and human tumors. Proc. Natl. Acad. Sci. 109 (2012) 15101–15108

[46] J.A. Weis, M.I. Miga1, L.R. Arlinghaus, X. Li, A.B. Chakravarthy, V. Abramson, J. Farley and T.E. Yankeelov: A mechanically coupled reaction-diffusion model for predicting the response of breast tumors to neoadjuvant chemotherapy. Physics in Medicine and Biology 58(17) (2013) 5851–5866

[47] S.M. Wise, J. Lowengrub, H.B. Frieboes and V. Cristini: Three-dimensional multispecies nonlinear tumor growth—I: Model and numerical method. J. Theoret. Biol. 253(3) (2008), 524–543