Publishing Community-Preserving Attributed Social Graphs with a Differential Privacy Guarantee

Xihui Chen¹, Sjouke Mauw¹,² and Yuniorm Ramírez-Cruz¹

¹SnT, ²CSC, University of Luxembourg
6, av. de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg
{xihui.chen, sjouke.mauw, yunior.ramirez}@uni.lu

September 4, 2019

Abstract

We present a novel method for publishing differentially private synthetic attributed graphs. Unlike preceding approaches, our method is able to preserve the community structure of the original graph without sacrificing the ability to capture global structural properties. Our proposal relies on C-AGM, a new community-preserving generative model for attributed graphs. We equip C-AGM with efficient methods for attributed graph sampling and parameter estimation. For the latter, we introduce differentially private computation methods, which allow us to release community-preserving synthetic attributed social graphs with a strong formal privacy guarantee. Through comprehensive experiments, we show that our new model outperforms its most relevant counterparts in synthesising differentially private attributed social graphs that preserve the community structure of the original graph, as well as degree sequences and clustering coefficients.

Keywords: attributed social graphs, generative models, differential privacy, community detection

1 Introduction

The use of online social networks (OSNs) has grown steadily during the last years, and is expected to continue growing in the future. Billions of people share many aspects of their lives on OSNs and use these systems to interact with each other on a regular basis. The ubiquity of OSNs has turned them into one of the most important sources of data for the analysis of social phenomena. Such analyses have led to significant findings used in a wide range of applications, from efficient epidemic disease control [22, 5] to information diffusion [44, 13].
Despite the undeniable social benefits that can be obtained from social network analysis, access to such data by third parties such as researchers and companies should understandably be limited due to the sensitivity of the information stored in OSNs, e.g. personal relationships, political preferences and religious affiliations. In addition, the increase of public awareness about privacy and the entry into effect of strong privacy regulations such as GDPR [1] strengthen the reluctance of OSN owners from releasing their data. Therefore, it is of critical importance to provide mechanisms for privacy-preserving data publication to encourage OSN owners to release data for analysis.

Social graphs are a natural representation of social networks, with nodes corresponding to participants and edges to connections between participants. In view of the privacy discussion, social network owners should only release sanitised sample of the underlying social graphs. However, it has been shown that even social graphs containing only structural information remain vulnerable to privacy attacks leveraging knowledge from public sources [29], deploying sybil accounts [2, 24], etc. In order to prevent such attacks, a large number of graph anonymisation methods have been devised. Initially, the proposed methods focused on editing the original graph via vertex/edge additions and deletions until obtaining a graph satisfying some privacy property. A critical limitation of graph editing methods is their reliance on assumptions about the adversary knowledge, which determine the information that needs to be anonymised and thus the manner in which privacy is enforced. To avoid this type of assumptions, an increasingly popular trend is that of using semantic privacy notions, which place formal privacy guarantees on the data processing algorithms rather than the dataset. Among semantic privacy notions, differential privacy [8] has become the de facto standard due to its strong privacy guarantees.

According to the type of published data, we can divide differentially private mechanisms for social graphs into two classes. The methods in the first category directly release specific statistics of the underlying social graph, e.g. the degree sequence [16, 9] or the number of specific subgraphs (triangles, stars, etc.) [43]. The second family of methods focuses on publishing synthetic social graphs as a replacement of real social networks in a two-step process [27, 35, 36, 39]. In the first step, differentially private methods are used to compute the parameters of a generative graph model that accurately captures the original graph properties. Then, in the second step, this model is sampled for synthetic graphs, profiting from the fact that the result of post-processing the output of differentially private algorithms remains differentially private [17].

Differential privacy requires one to define a privacy budget in advance, which determines the amount of perturbation that will be applied to the outputs of algorithms. In consequence, the methods in the first family need to either limit in advance the number of queries that will be answered or deliver increasingly lower quality answers. On the contrary, the methods in the second family can devote the entire privacy budget to the model parameter estimation, without further degradation of the privacy of the sampled graphs. For this reason, in this paper we focus on the second type of methods.

For analysts, the utility of synthetic graphs is determined by the ability of the graph models to capture relevant properties of the original graph. To satisfy this need, several graph models have been proposed to accurately capture global structural properties such
as degree distributions and clustering coefficients, as well as heterogeneous attributes of
the users such as gender, education or marital status. A common limitation of the afore-
mentioned approaches is their inability to represent an important type of information: the
community structure. Informally, a community is a set of users who are substantially more
interrelated among themselves than to other users of the network. This interrelation may,
e.g., stem from the explicit existence of relations between the users. An example of such
a community is a group of Gmail users who frequently e-mail each other, as represented
by the occurrence of a large number of edges connecting the user nodes from the group.
Alternatively, interrelations may stem from the co-occurrence of relevant features, such as
users working at the same company or alumni from the same university. The emergence of
communities has been documented to be an inherent property of social networks [34, 41].
For analysts, the availability of synthetic attributed graphs that preserve the community
structure of the original graph represents an opportunity to improve existing applications.
For example, they may be able to improve online shopping recommendations based on the
common purchases of users belonging to the same community. Current models and meth-
ods are insufficient for enabling such an analysis, as they either lack information about the
community structure or they lack vertex features.

In this paper, we address the problem discussed in the previous paragraph by intro-
ducing a new generative attributed graph model, C-AGM (short for Community-Preserving
Attributed Graph Model), which in addition to global structural properties, is also capa-
bile of preserving the community structure of the original graph. C-AGM is based on the
attributed graph model AGM [11], and improves on it by incorporating the capability of
preserving the number and sizes of the communities of the original graph, as well as the
densities of intra- and inter-community connections (that is, connections between nodes
belonging to the same community or to different communities, respectively). C-AGM also
preserves a number of statistics describing the correlations between the feature vectors that
describe the users and the existence of connections between pairs of users, as well as their
community co-affiliation. We equip C-AGM with efficient parameter estimation and graph
sampling methods, and provide differentially private variants of the former, which allow us
to release synthetic attributed social graphs with a strong privacy guarantee and increased
utility with respect to preceding approaches.

Summary of contributions:

- We propose a new generative attributed graph model, C-AGM, which captures a num-
  ber of properties of the community structure, as discussed in the previous paragraph,
  along with global structural properties.

- We present efficient methods for learning an instance of our model from an input
  graph and sampling community-preserving synthetic attributed graphs from this in-
  stance. We show, via a number of experiments on real-world social networks, that the
  community structures of synthetic graphs sampled from our model are more similar
to those of the original graphs than those of the graphs sampled from previously exist-
ing models. Additionally, we show that this behaviour is obtained without sacrificing
the ability to preserve global structural features.

- We devise differentially private methods for computing the parameters of the new model. We demonstrate that our methods are practical in terms of efficiency and accuracy. To support the latter claim, we empirically show that differentially private synthetic attributed graphs generated by our model suffer a reasonably low degradation with respect to their counterparts, in terms of their ability to capture the community structure and structural features of the original graphs.

2 Related Work

**Private graph synthesis.** The key to synthesising social graphs is the model which determines both the information embedded in the published graphs and the properties preserved. Mir et al. [27] used the Kronecker graph generative model [20] to generate differentially private graphs. As the Kronecker model cannot accurately capture structural properties, Sala et al. [35] proposed an alternative approach which makes use of the $dK$-graph model. Wang et al. [36] further improved the work of Sala et al. by considering global sensitivity instead of local sensitivity (refer to Section 3 for the definition of sensitivity). Xiao et al. [39] introduced the HRG-graph model [7] and found that it can further reduce the amount of added noise and thus increase the accuracy.

The approaches described so far work on unlabelled graphs. Pfiffer et al. [11] introduced a new model called AGM, which attaches binary attributes to nodes and captures the correlations between shared attributes and the existence of connections. Jorgensen et al. [12] adopted this model and proposed differentially private methods to accurately estimate the model parameters. They also designed a new graph generation algorithm based on the TCL model [10], which enables the model to sample attributed graphs preserving the clustering coefficient. As discussed previously, C-AGM, the model introduced in this paper, is comparable to this model in preserving global structural properties of the original graphs, but it outperforms it by also capturing the community structure.

**Private statistics publishing.** Degree sequences and degree correlations are two types of the statistics frequently studied in the literature. The general trend in publishing these statistics under differential privacy consists in adding noise to the original sequences and then post-processing the perturbed sequences to enforce or restore certain properties, such as graphicality [16], vertex order in terms of degrees [9], etc. Subgraph count queries, e.g. the number of triangles or $k$-stars, have also received considerable attention. Among the approaches to accurately compute such queries, we have the so-called ladder functions [43] and smooth sensitivity [15, 37].

**Community-preserving graph generation models.** A number of existing random graph models claim to capture community structure, e.g., BTER [18], ILFR [34], SBM [38] and its variants (e.g., DCSBM [14] and DCPMM [31]). BTER generates community-preserving social graphs given expected node degrees and, for every degree value $\sigma$, the average of the clustering coefficients of the nodes of degree $\sigma$. The model assumes that every community
is a set of $\sigma$ nodes with degree $\sigma$. On the contrary, C-AGM makes no assumptions on the community partition received. Finally, ILFR and the variants of SBM preserve edge densities at the community level but, unlike our new model, they do not preserve the clustering coefficients of the original graph.

3 Preliminaries

3.1 Notation

An attributed graph is represented as a triple $G = (\mathcal{V}, \mathcal{E}, X)$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $X$ is a binary matrix called the attribute matrix. The $i$-th row of $X$ is the attribute vector of $v_i$, which is individually denoted by $\tau(v_i)$. Every column of $X$ represents a binary feature, which is set to 1 (true), or 0 (false), for each user. For example, if the $j$-th column represents the attribute “owning a car”, $X_{ij} = 1$ means that the user represented by $v_i$ owns a car. Non-binary real-life attributes are assumed to be binarised. For example, a binarisation of the integer-valued attribute “age” is $\{\text{age} \leq 16”, “17 \leq \text{age} \leq 26”, “27 \leq \text{age} \leq 64”, “\text{age} \geq 65”\}$. The order of the columns of $X$ is fixed, but arbitrary, and has no impact on the results described hereafter. Throughout the paper, we deal with undirected graphs. That is, if $(v_i, v_j) \in \mathcal{E}$, then $(v_j, v_i) \in \mathcal{E}$. Additionally, we use $A$ to denote the adjacency matrix of the graph.

We use $C = \{C_0, C_1, \ldots, C_p\}$, with $C_i \subseteq \mathcal{V}$ for every $i \in \{0, 1, \ldots, p\}$, to represent a community partition of the attributed graph. As the term suggests, in this paper we assume that $C_i \cap C_j = \emptyset$, with $0 \leq i < j \leq p$, and $\cup_{C_i \in C} C_i = \mathcal{V}$. The community $C_0$ has a special interpretation. Since some community detection algorithms assign no community to some vertices, we will use $C_0$ as a “discard” community of unassigned vertices. We do so to avoid having a potentially large number of singleton communities, for which no meaningful co-affiliation statistics can be computed. We use $\psi_C(v_i)$ to denote the community to which the node $v_i$ belongs in the community partition $C$. We will use $\psi(v_i)$ for short in cases where the partition is clear from the context.

3.2 Differential Privacy

Differential privacy [8] is a well studied statistical notion of privacy. The intuition behind it is to randomise the output of an algorithm in such a way that the presence of any individual element in the input dataset has a negligible impact on the probability of observing any particular output. In other words, a mechanism is $\varepsilon$-differentially private if for any pair of neighbouring datasets, i.e. datasets that only differ by one element, the probabilities of obtaining any output are measurably similar. The amount of similarity is determined by the parameter $\varepsilon$, which is commonly called the privacy budget. In what follows, we will use the notation $\mathcal{D}$ for the set of possible datasets, $\mathcal{O}$ for the set of possible outputs, and $D \sim D'$ for a pair of neighbouring datasets.

Definition 1 ($\varepsilon$-differential privacy [8]). A randomised mechanism $\mathcal{M} : \mathcal{D} \rightarrow \mathcal{O}$ satisfies
ε-differential privacy if for every pair of neighbouring datasets \( D, D' \in \mathcal{D}, \) \( D \sim D' \), and for every \( S \subseteq \mathcal{O} \), we have

\[
\Pr(\mathcal{M}(D) \in S) \leq e^\varepsilon \Pr(\mathcal{M}(D') \in S).
\]

A number of differentially private mechanisms have been proposed. For queries of the form \( q : \mathcal{D} \rightarrow \mathbb{R}^n \), the most widely used mechanism to enforce differential privacy is the so-called Laplace mechanism, which consists in obtaining the (non-private) output of \( q \) and adding to every component a carefully chosen amount of random noise, which is drawn from the Laplace distribution

\[
\text{Lap}(\lambda) : f(y | \lambda) = \frac{1}{2\lambda} \exp\left(\frac{-|y|}{\lambda}\right),
\]

where \( y \) is a real-valued variable indicating the noise to be added, \( \lambda = \frac{\Delta_q}{\varepsilon} \) and \( \Delta_q \) is a property of the original function \( q \) called *global sensitivity*. This property is defined as the largest difference between the outputs of \( q \) for any pair of neighbouring datasets, that is

\[
\Delta_q = \max_{D \sim D'} \| q(D) - q(D') \|_1,
\]

where \( \| \cdot \|_1 \) is the \( L_1 \) norm. For categorical (non-numerical) queries of the form \( q : \mathcal{D} \rightarrow \mathcal{O} \), where \( \mathcal{O} \) is a finite set of categories, the so-called exponential mechanism [26] is the most commonly used. In this case, for each value \( o \in \mathcal{O} \), a score is assigned by a function (usually called *scoring function*) quantifying the value’s utility, denoted by \( u(o, D) \). The global sensitivity of \( u \) is

\[
\Delta_u = \max_{o \in \mathcal{O}, D \sim D'} | u(o, D) - u(o, D') |,
\]

and the randomised output is drawn with probability

\[
\frac{\exp\left(\frac{\varepsilon \cdot u(o, D)}{2\Delta_u}\right)}{\sum_{o' \in \mathcal{O}} \exp\left(\frac{\varepsilon \cdot u(o', D)}{2\Delta_u}\right)}.
\]

Differentially private methods are composable [25]. That is, given a set of algorithms \( \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n\} \) such that \( \mathcal{M}_i \) (1 ≤ \( i \) ≤ \( n \)) satisfies \( \varepsilon_i \)-differential privacy, if the algorithms are applied sequentially and the results combined by a deterministic method, then the final result satisfies \( \sum_i \varepsilon_i \)-differential privacy. If the algorithms are applied independently on disjoint subsets of the input, then \( \max_i \{\varepsilon_i\} \)-differential privacy is satisfied. Moreover, post-processing on the output of an \( \varepsilon \)-differentially private algorithm also satisfies \( \varepsilon \)-differential privacy if the post-processing is deterministic or randomised with a source of randomness independent from the noise added to the original algorithm [17]. These properties allow us to divide a complex computation, such as the set of model parameters in our case, into a sequence of sub-tasks for which differentially private methods exist or can be more easily developed.

In addition to the global sensitivity, a dataset-dependent notion, called *local sensitivity* [33], has been enunciated. The local sensitivity of query \( q \) on a dataset \( D \) is defined as

\[
LS_q(D) = \max_{D \sim D'} \| q(D) - q(D') \|_1,
\]

that is, the maximum difference between the output of \( q \) on \( D \) and those on its neighbouring datasets. It is simple to see that \( \Delta_q = \max_D LS_q(D) \).
4 The C-AGM Model

In this section we give the formal definition of C-AGM. We introduce the methods for sampling synthetic graphs from the model, and describe the methods for learning the model parameters from an attributed graph.

4.1 Overview

Algorithm 1 summarises the process by which C-AGM is used for publishing synthetic attributed graphs. As discussed in [9, 17, 16], synthetic graph generation is done as a post-processing step of the differentially private computation, so the synthetic graphs are also differentially private.

Algorithm 1: Given \( G = (V, E, X) \), obtain \( t \) differentially private attributed synthetic graphs.

1. Split privacy budget;
2. Obtain differentially private community partition;
3. Differentially-privately estimate C-AGM parameters;
4. for \( i \in \{1, 2, \ldots, t\} \) do
   5. Sample \( X_i \) from C-AGM;
   6. Sample \( E_i \) from C-AGM;
   7. \( G_i \leftarrow (V, E_i, X_i) \)
5. end

The manner in which the privacy budget is split among the different computations (step 1) is discussed in Section 5. For the differentially private community partition (step 2), we introduce in this paper an extension of the algorithm ModDivisive [32]. The purpose of this extension is to incorporate information from node attributes into the objective function optimised by ModDivisive. We discuss the community partition method in detail in Section 5.1. A thorough description of the parameters of C-AGM is given in Section 4.2, and parameter estimation is discussed in Section 4.4.

Once the model parameters have been estimated, we can sample any number of synthetic attributed graphs from the model, as described in steps 4 to 8 of Algorithm 1. The differentially private parameter estimation methods introduced in this paper use the notion of \textit{neighbouring attributed graphs} [12], which is discussed in detail in the preamble of Section 5. Under this notion, the existence of relations (edges) and personal characteristics of the network users (feature vectors) are treated as sensitive, but vertex identities are not. Thus, the synthetic graphs generated by Algorithm 1 have the same vertex set as the original graph, whereas the attribute matrix and the edge set are sampled from the model (step 7). For every new synthetic attributed graph, we first sample the attribute matrix, and then this matrix is used, in combination with an edge generation model (Section 4.3.1), to generate the edge set of the synthetic graph. There are two reasons for dividing this
process into two steps. The first one is to make the sampling process efficient. The second reason is to profit from the two-step process to enforce the intuition that users with similar features are more likely to be connected in the social network. The attributed graph sampling procedure is discussed in detail in Section 4.3.

4.2 Model Parameters

As we discussed in Section 1, given an attributed graph $G$ and a community partition $C$ of $G$, the purpose of C-AGM is to capture a number of properties of $C$ that are overlooked by previously defined models, without sacrificing the ability to capture global structural properties such as degree distributions and clustering coefficients. To that end, C-AGM models the following properties of the community partition:

1. the number and sizes of communities;
2. the number of intra-community edges in every community;
3. the number of inter-community edges;
4. the distributions of attribute vectors in every community;
5. the distributions of the so-called attribute-edge correlations [12], for the set of inter-community edges and for the set of intra-community edges in every community.

Graphs generated by C-AGM will have the same number of vertices as the original graph, as well as the same number of communities. Moreover, every community will have the same cardinality as in the original graph, and the same number of intra-community edges. The number of inter-community edges of the generated graph will also be the same as that of the original graph. Notice that the model preserves the total number, but not necessarily the pairwise numbers of inter-community edges for every pair of communities.

Attribute-edge correlations were defined in [12] as heuristic values for characterising the relation between the feature vectors labelling a pair of vertices and the likelihood that these vertices are connected. They encode the intuition that, for example, co-workers who attended the same university and live near to each other are more likely to be friends than persons with fewer features in common, whereas friends are more likely to support the same sports teams or go to the same bars than unrelated persons. In [12], attribute-edge correlations are considered to behave uniformly over the entire graph. Here, we introduce the rationale that they behave differently within different communities, as well as across communities.

A key element in the representation of attribute-edge correlations is the notion of aggregator functions. An aggregator function $\beta: \{0,1\}^k \times \{0,1\}^k \rightarrow \mathcal{B}$ maps a pair of attribute vectors $x, x'$ of dimensionality $k$ into a value in a discrete range $\mathcal{B}$, which is used as a descriptor, also called aggregated feature, of the pair $(x, x')$. For example, $\mathcal{B}$ can contain a set of similarity levels for pairs of feature vectors, such as $\{\text{low, medium, high}\}$, and $\beta$ can map
a pair of vectors whose cosine similarity is in the interval $[0, 0.33]$ to low, a pair of vectors whose cosine similarity is in the interval $[0.67, 1]$ to high, etc. Attribute-edge correlations, along with the community-wise distributions of attribute vectors, are useful for analysts, as they allow to characterise the members of a community in terms of frequently shared features, hypothesise explanations for the emergence of a community, etc.

Formally, a C-AGM model is defined as a quintuple $\langle V, C, \Theta^c_M, \Theta^c_X, \Theta^c_F \rangle$, where:

- $V$ is a set of vertices.
- $C$ is a community partition of $V$.
- $\Theta^c_M$ is an instance of an edge set generative model that preserves properties 1 to 3 of the community partition $C$, as well as degree distributions and clustering coefficients. The model introduced in this paper is called CPGM, and is described in detail in Section 4.3.1.
- $\Theta^c_X$ is an instance of an attribute vector generative model, which aims to preserve property 4. The model defines, for every community $C \in C$ and every attribute vector $x$, the probability $\Pr(\tau(v) = x \mid v \in C, \Theta^c_X)$ that a vertex in $C_i$ is labelled with $x$. The model introduced in this paper is described in detail in Section 4.4.2.
- $\Theta^c_F$ is an instance of a generative model for attribute-edge correlations, which aims to preserve property 5. This model defines:
  - The discrete range $B$ and an aggregator function $\beta$.
  - The probability $\Pr(\beta(\tau(v_i), \tau(v_j)) = s \mid \Theta^c_F, \psi_C(v_i) = \psi_C(v_j) = C, A_{i,j} = 1)$ for every community $C \in C$ and every value $s \in B$.
  - The probability $\Pr(\beta(\tau(v_i), \tau(v_j)) = s \mid \Theta^c_F, \psi_C(v_i) \neq \psi_C(v_j), A_{i,j} = 1)$ for every value $s \in B$.

The instantiations that we propose for these three components are described in detail in Section 4.4.3.

### 4.3 Sampling Attributed Graphs from an Instance of C-AGM

Given a C-AGM model $G = \langle V, C, \Theta^c_M, \Theta^c_X, \Theta^c_F \rangle$, with $V = \{v_1, v_2, \ldots, v_n\}$, an attributed graph $G = (V, E, X)$ is sampled from $G$ with probability $\Pr(G \mid G) = \Pr(E, X \mid G)$ which, for the sake of tractability, is approximated as

$$\Pr(E, X \mid \Theta^c_F, \Theta^c_X, C, \Theta^c_M) = \Pr(E \mid \Theta^c_F, \Theta^c_M, X, C) \cdot \Pr(X \mid \Theta^c_X, C).$$
That is, we first sample from $\Theta_X^c$ the attribute vectors labelling each vertex and then use them in sampling the edge set. Again, to keep the sampling process tractable, we introduce an additional independence assumption, according to which

$$\Pr(X | \Theta_X^c, \mathcal{C}) = \prod_{v \in V} \Pr(\tau(v) | \psi_C(v)).$$

The computation of the probabilities of the form $\Pr(x | \psi_C(v))$ will be discussed in Section 4.4.2. Introducing the assumption that edges are sampled independently from each other, the probability of generating $\mathcal{E}$ given $\Theta_F^c, \Theta_M^c, X,$ and $\mathcal{C}$ is

$$\Pr(\mathcal{E} | \Theta_F^c, \Theta_M^c, X, \mathcal{C}) = \prod_{v_i, v_j \in V} \Pr(A_{i,j} | \Theta_F^c, \Theta_M^c, \beta(\tau(i), \tau(j)), \mathcal{C}).$$

As it is inefficient to sample edges directly from this distribution, we adapt the sampling method introduced in [11] to account for the computation of community-wise separated counts. Thus, edges are drawn from the distribution

$$Q(i,j) \propto Q_M'(i,j) \cdot \Gamma(\beta(\tau(v_i), \tau(v_j)), \mathcal{C}),$$

where $Q_M'(i,j)$ is the probability that $(v_i, v_j)$ is drawn from the edge generation model $\Theta_M^c$, given $\mathcal{C}$, as a candidate edge; while $\Gamma(\beta(\tau(v_i), \tau(v_j)), \mathcal{C})$ is the probability that it is accepted by $\Theta_F$, given $\mathcal{C}$. We split the computation of $\Gamma(\beta(\tau(v_i), \tau(v_j)), \mathcal{C})$ into two cases: $\Gamma_{intra}(\beta(\tau(v_i), \tau(v_j)), \mathcal{C})$, for every $C \in \mathcal{C}$ and every $i,j$ such that $\psi_C(v_i) = \psi_C(v_j) = C$; and $\Gamma_{inter}(\beta(\tau(v_i), \tau(v_j)))$, for every $i,j$ such that $\psi_C(v_i) \neq \psi_C(v_j)$. Formally, we have

$$Q_M'(i,j) = \frac{\Pr(A_{i,j} = 1 | \Theta_M^c, \mathcal{C})}{\sum_{v_i, v_j \in V} \Pr(A_{p,q} = 1 | \Theta_M^c, \mathcal{C})},$$

$$\Gamma_{intra}(\beta(\tau(v_i), \tau(v_j)), \mathcal{C}) = \frac{R_{intra}(\beta(\tau(v_i), \tau(v_j)), \mathcal{C})}{\text{SupR}},$$

and $\Gamma_{inter}(\beta(\tau(v_i), \tau(v_j))) = \frac{R_{inter}(\beta(\tau(v_i), \tau(v_j)))}{\text{SupR}}$, where

$$R_{intra}(\beta(\tau(v_i), \tau(v_j)), \mathcal{C}) = \frac{\Pr(\beta(\tau(v_i), \tau(v_j)) | \Theta_F^c, \psi_C(v_i) = \psi_C(v_j) = C, A_{i,j} = 1)}{\Pr(\beta(\tau(v_i), \tau(v_j)) | \Theta_M^c, \psi_C(v_i) = \psi_C(v_j) = C, A_{i,j} = 1)},$$

$$R_{inter}(\beta(\tau(v_i), \tau(v_j))) = \frac{\Pr(\beta(\tau(v_i), \tau(v_j)) | \Theta_F^c, \psi_C(v_i) \neq \psi_C(v_j), A_{i,j} = 1)}{\Pr(\beta(\tau(v_i), \tau(v_j)) | \Theta_M^c, \psi_C(v_i) \neq \psi_C(v_j), A_{i,j} = 1)},$$

and $\text{SupR} = \sup \bigcup_{s \in \mathcal{B}, C \in \mathcal{C}} (R_{intra}(s, C) \cup R_{inter}(s))$.

The computation of $Q_M'(i,j)$ will be discussed in Section 4.3.1, whereas that of $\Gamma_{inter}(\beta(\tau(v_i), \tau(v_j)))$, and every $\Gamma_{intra}(\beta(\tau(v_i), \tau(v_j)), \mathcal{C})$ will be discussed in Section 4.4.3.
Algorithm 2 describes the procedure to sample an attributed graph from C-AGM. The method first generates the attribute vectors (line 1). Then, it pre-computes the acceptance probabilities (lines 2 to 11). In line 3, the call to SampleEdgeSet consists in the sequential execution of Algs. 3 and 4, which will be described in detail in Section 4.3.1. Finally, the loop in lines 12 to 20 repeatedly draws candidate edges from the edge generation model and adds to the graph those that are accepted according to the pre-computed probabilities (lines 17 and 18). The method stops when the required number of edges is added.

Algorithm 2: SampleFromCAGM(\(V, C, \Theta^c_M, \Theta^c_X, \Theta^c_F\))

1. \(X' \leftarrow \text{SampleAttributeVectors}(\Theta^c_X)\);
2. \(Q'_M \leftarrow \text{ComputeQM}(\Theta^c_M, C)\);
3. \(E' \leftarrow \text{SampleEdgeSet}(Q'_M)\);
4. for \(s \in B\) do
   5. \(\Gamma_{\text{inter}}(s)\);
   6. for \(C \in C\) do
      7. \(\Gamma_{\text{intra}}(s, C)\);
   8. end
9. end
10. \(E' \leftarrow \emptyset\);
11. while \(|E'| < |E|\) do
12. \((v, w) \leftarrow \text{SampleEdge}(Q'_M)\);
13. \(s \leftarrow \beta(\tau(v), \tau(w))\);
14. \(u \leftarrow \text{Uniform}(0, 1)\);
15. if \((\psi_C(v) = \psi_C(w) \land u \leq \Gamma_{\text{intra}}(s, \psi_C(v)) \text{ or } \psi_C(v) \neq \psi_C(w) \land u \leq \Gamma_{\text{inter}}(s))\) then
16. \(E' \leftarrow E' \cup \{(v, w)\} \);\n17. end
18. end
19. return \(X', E'\);

4.3.1 Edge generation model

As we discussed in Section 4.2, the component \(\Theta^c_M\) of C-AGM is an edge generation model which preserves several properties of the community partition of the original graph (properties 1 to 3 listed in Section 4.2), in addition to the degree distribution and clustering coefficients. We call this model CPGM, and describe it in what follows.

The model takes as input the set of vertices, as well as the expected number of neighbours of every vertex \(v\) within its community (that is, its intra-community degree, denoted by \(d_{\text{intra}}(v)\)) and the expected number of neighbours outside its community (that is, the inter-community degree, denoted by \(d_{\text{inter}}(v)\)). These values are used to enforce the expected densities within every community and between communities. Additionally, adapting to our setting a heuristics introduced in [12], the model also requires the number of triangles having all vertices in one community (which we call intra-community triangles and denote by \(n_{\text{intra}}^\triangle\)), as well as the number of triangles spanning more than one community (inter-community triangles, denoted by \(n_{\text{inter}}^\triangle\)). As shown empirically in [12], synthetic
graphs that preserve the number of triangles of the original graph are more likely to approximate the clustering coefficient of the original graph. We adopt this intuition as well, but unlike [12], we separate the counts of intra- and inter-community triangles. As we will discuss in Section 5, \( \Delta_{intra} \) and \( \Delta_{inter} \) can be efficiently and accurately computed under differential privacy.

According to our model, the edge sampling process consists of two steps. The first step generates a graph that preserves the intra- and inter-community degrees, but not the number of intra- and inter-community triangles. Then, the second step iteratively edits the original edge set until \( \Delta_{intra} \) and \( \Delta_{inter} \) are enforced.

At the first step, we follow the idea of the CL model [6]. For every pair of vertices \( v \) and \( w \) satisfying \( \psi_C(v) = \psi_C(w) = C \), the intra-community edge \((v, w)\) is added with probability \( \pi_{intra} C(v,w) = \frac{d_{intra}(v)d_{intra}(w)}{2m^{intra}_C} \), where \( m^{intra}_C \) is the original number of intra-community edges in \( C \). That is, intra-community edges are added with a probability proportional to product of the intra-community degrees of the linked vertices. If \( \psi_C(v) \neq \psi_C(w) \), then the inter-community edge \((v, w)\) is added with probability \( \pi_{inter} (v,w) = \frac{d_{inter}(v)d_{inter}(w)}{2m^{inter}} \), where \( m^{inter} \) is the total number of inter-community edges in the original graph. Algorithm 3 describes the first step of the generation process.

### Algorithm 3: GenInitialEdgeSet\((d_{intra}, d_{inter}, C)\)

| Line | Description |
|------|-------------|
| 1    | \( E \leftarrow \emptyset; \) |
| 2    | for \( C \in \mathcal{C} \) do |
| 3    | \( m^{intra}_C \leftarrow \frac{1}{2} \sum_{v \in C} d_{intra}(v); \) |
| 4    | \( m \leftarrow 0; \) |
| 5    | while \( m \leq m^{intra}_C \) do |
| 6    | \( (v, w) \leftarrow \text{Sample}(\pi_{intra} C); \) |
| 7    | if \( (v, w) \notin E \) then |
| 8    | \( E \leftarrow E \cup \{(v, w)\}; \) |
| 9    | \( m \leftarrow m + 1; \) |
| 10   | end |
| 11   | end |
| 12   | \( m^{inter} \leftarrow \frac{1}{2} \sum_{v \in V} d_{inter}(v); \) |
| 13   | while \( m \leq \sum_{C \in \mathcal{C}} m^{intra}_C + m^{inter} \) do |
| 14   | \( (v, w) \leftarrow \text{Sample}(\pi_{inter}); \) |
| 15   | if \( (v, w) \notin E \) then |
| 16   | \( E \leftarrow E \cup \{(v, w)\}; \) |
| 17   | \( m \leftarrow m + 1; \) |
| 18   | end |
| 19   | end |
| 20   | end |

At the second step, we use the intuition that the clustering behaviour in social networks stems from the higher likelihood of users with common friends to connect [10], thus creating triangles. Algorithm 4 enforces the values of \( \Delta_{intra} \) and \( \Delta_{inter} \) of the original graph on the graph synthesised by Algorithm 3. In Algorithm 4, we denote by \( \mathcal{N}_{intra}(v) \) the set of neighbours of \( v \) in its community, that is \( \mathcal{N}_{intra}(v) = \{w | \psi_C(v) = \psi_C(w) \land (v, w) \in E\}. \)
Algorithm 4: GetFinalEdgeSet\( (d_{intra}, d_{inter}, n_{intra}^{\Delta}, n_{inter}^{\Delta}, C) \)

\[
\begin{align*}
1 & \mu_{intra}^{\Delta} \leftarrow \text{CountIntraCommTriangles}(\mathcal{E}) ; \\
2 & \textbf{while } \mu_{intra}^{\Delta} < n_{intra}^{\Delta} \textbf{ do} \\
3 & \quad \text{Uniformly sample } C \text{ from } C ; \\
4 & \quad \text{Sample } v_1 \text{ from } C \text{ with probability } d_{intra}(v_1) ; \\
5 & \quad \text{Uniformly sample } v_2 \text{ from } N_{intra}(v_1) ; \\
6 & \quad \text{Uniformly sample } v_3 \text{ from } N_{intra}(v_2) ; \\
7 & \quad \text{if } (v_1, v_3) \notin \mathcal{E} \land v_3 \neq v_1 \text{ then} \\
8 & \quad \quad (v'_1, v'_2) \leftarrow \text{GetOldestIntraCommEdge}(\mathcal{E}, C) ; \\
9 & \quad \quad n_{cn}^{prev} \leftarrow \text{GetCommonNeighbour}(v'_1, v'_2) ; \\
10 & \quad \quad \mathcal{E} \leftarrow \mathcal{E}/\{(v'_1, v'_2)\} ; \\
11 & \quad \quad n_{cn}^{new} \leftarrow \text{GetCommonNeighbour}(v_1, v_3) ; \\
12 & \quad \quad \text{if } n_{cn}^{prev} < n_{cn}^{new} \text{ then} \\
13 & \quad \quad \quad \mu_{intra}^{\Delta} \leftarrow \mu_{intra}^{\Delta} - n_{cn}^{prev} + n_{cn}^{new} ; \\
14 & \quad \quad \text{else} \\
15 & \quad \quad \quad \mathcal{E} \leftarrow \mathcal{E} \cup \{(v'_1, v'_2)\} ; \\
16 & \quad \end{align*}
\]

\[
\begin{align*}
17 & \text{end} \\
18 & \text{end} \\
19 & \mu_{inter}^{\Delta} \leftarrow \text{CountInterCommTriangles}(\mathcal{E}) ; \\
20 & \textbf{while } \mu_{inter}^{\Delta} < n_{inter}^{\Delta} \textbf{ do} \\
21 & \quad \text{Sample } v_1 \text{ from } \mathcal{V} \text{ with probability } d_{inter}(v_1) ; \\
22 & \quad \text{Uniformly sample } v_2 \text{ from } N_{inter}(v_1) ; \\
23 & \quad \text{Uniformly sample } v_3 \text{ from } N_{inter}(v_2) ; \\
24 & \quad (v'_1, v'_2) \leftarrow \text{GetOldestInterCommEdge}(\mathcal{E}, C) ; \\
25 & \quad n_{cn}^{prev} \leftarrow \text{GetCommonNeighbour}(v'_1, v'_2) ; \\
26 & \quad \mathcal{E} \leftarrow \mathcal{E}/\{(v'_1, v'_2)\} ; \\
27 & \quad n_{cn}^{new} \leftarrow \text{GetCommonNeighbour}(v_1, v_3) ; \\
28 & \quad \text{if } n_{cn}^{prev} < n_{cn}^{new} \text{ then} \\
29 & \quad \quad \mathcal{E} \leftarrow \mathcal{E} \cup \{(v_1, v_3)\} \quad \mu_{inter}^{\Delta} \leftarrow \mu_{inter}^{\Delta} - n_{cn}^{prev} + n_{cn}^{new} ; \\
30 & \quad \text{else} \\
31 & \quad \quad \mathcal{E} \leftarrow \mathcal{E} \cup \{(v'_1, v'_2)\} ; \\
32 & \quad \end{align*}
\]

\[
\begin{align*}
33 & \text{end} \\
34 & \text{end} \\
\end{align*}
\]

Likewise, we denote by \( N_{inter}(v) \) the set of neighbours of \( v \) in different communities, that is \( N_{inter}(v) = \{ w \mid \psi_C(v) \neq \psi_C(w) \land (v, w) \in \mathcal{E} \} \). In Algorithm 4, \( n_{intra}^{\Delta} \) is enforced first because adding or removing an intra-community edge may change the number of inter-community triangles as well, whereas inter-community triangles can be created without modifying the number of intra-community triangles. At every iteration, we sample a new edge. If replacing the oldest intra-community edge (in terms of the order of creation by Algorithm 3) with the newly sampled edge causes the number of intra-community triangles to increase, we make the edge exchange permanent. Otherwise, we do not add the newly sampled edge and set the oldest edge to be the youngest, keeping it in the graph. The iteration stops when the number of intra-community triangles is greater than or equal to
that of the original graph. Then, we proceed to enforce the number of inter-community triangles by adding inter-community edges. In this case the idea is to find open “wedges” composed of one intra-community edge \((u, v)\) and one inter-community edge \((v, w)\) such that the edge \((u, w)\) has not been added to the graph. This ensures that newly added edges will not affect the number of intra-community triangles. Let \((v', w')\) be the oldest inter-community edge. If the graph obtained by removing \((v', w')\) and adding \((v, w)\) contains more triangles than the current version of the synthetic graph, then \((v, w)\) is added and \((v', w')\) is removed. The iteration stops when the number of inter-community triangles is greater than or equal to that of the original graph.

Due to the removal of initially generated edges, the synthetic graph may become disconnected. In this case, we apply an edge-swapping post-processing step to reconnect every small connected component to the main component (the connected component with the most nodes). If the post-processing reduces the number of triangles, we recall Algorithm 4. The alternation between the post-processing and Algorithm 4 is not guaranteed to yield a graph having exactly the required number of triangles, so we stop the iteration when the total number of triangles in the synthetic graph is within a 98% tolerance window with respect to the original one.

### 4.4 Parameter Estimation for C-AGM

We now discuss the methods for estimating the parameters of a C-AGM model from a given attributed graph with a community partition \(C\).

#### 4.4.1 Estimating \(\Theta^c_M\)

The estimation of \(\Theta^c_M\) reduces to computing the community-wise counters that it relies on: intra- and inter-community degrees of every vertex, the number of intra-community triangles for each community and the number of inter-community triangles. As we mentioned in Section 4.3.1, degrees and triangle counts will be used to preserve global structural properties of the generated graphs such as degree distribution and clustering coefficients. They can be efficiently computed in the original graph both exactly and under differential privacy.

#### 4.4.2 Estimating \(\Theta^c_X\)

In order to keep the estimation procedure tractable, we introduce the assumption that attributes are independent. This assumption simplifies the estimation and handles the sparsity of the attribute vectors when the number of attributes is large. As seen in \([11, 12]\), not having such an assumption severely limits the number of features that can be practically handled. Furthermore, as we will see in Section 5, in addition to tractability, this assumption will also allow us to limit the amount of noise added by the differentially private computation.
We will denote by $x_\ell$ be the value for the $\ell$-th component of the attribute of vector $x$. Likewise, we will denote by $\tau_\ell(v)$ the value of the $\ell$-th component of the vector labelling vertex $v$. We estimate the probability that a node $v$ is labelled with an attribute vector $x$ by the following formula:

$$\Pr(x|v, \Theta^c_X, C) = \Pr(x|\psi_C(v), \Theta^c_X) = \prod_{\ell=1}^{k} \Pr_\ell(x_\ell|\Theta^c_X, \psi_C(v)),$$

where $k$ is the number of columns of $X$ (ergo the cardinality of all attribute vectors) and $\Pr_\ell(x_\ell|\Theta^c_X, \psi_C(v)) = \frac{|\{v'\in\psi_C(v) | \tau_\ell(v') = x_\ell\}|}{|\psi_C(v)|}$.

### 4.4.3 Estimating $\Theta_F$

As we discussed in Section 4.2, for defining $\Theta_F$ it is necessary to define an aggregator function for pairs of attribute vectors. Our aggregator function is based on the widely used cosine similarity, that is, the cosine of the angle between two vectors. Since the range $B$ of aggregator functions needs to be discrete, we split the range $[0,1]$ of the cosine similarity into a set of intervals, determined by a parameter $\delta$ satisfying $0 < \delta \leq 1$. Let $s_{\cos}(x,x')$ denote the similarity between vectors $x$ and $x'$. Our aggregator function is defined as $\beta(x,x') = \lfloor s_{\cos}(x,x')/\delta \rfloor$. Note that, according to this definition, $B = \{ \lfloor s/\delta \rfloor | s \in [0,1] \}$. Finally, the probability of the attribute vectors of a pair of connected vertices being described by an aggregated feature $u \in B$ is computed as

$$\Pr(\beta(\tau(v_i),\tau(v_j)) = u | \Theta^c_F, C, A_{i,j} = 1) = \begin{cases} \frac{|\{(v_p,v_q)\in E | \beta(\tau(v_p),\tau(v_q)) = u \land \psi(v_p) = \psi(v_q) = \psi(v_i)\}|}{|\{(v_p,v_q)\in E | \psi(v_p) = \psi(v_q) = \psi(v_i)\}|} & \text{if } \psi(v_i) = \psi(v_j); \\ \frac{|\{(v_p,v_q)\in E | \beta(\tau(v_p),\tau(v_q)) = u \land \psi(v_p) \neq \psi(v_q)\}|}{|\{(v_p,v_q)\in E | \psi(v_p) \neq \psi(v_q)\}|} & \text{if } \psi(v_i) \neq \psi(v_j). \end{cases}$$

Compared to the approach introduced in [11, 12], our method uses a coarser granularity for aggregated features. Thanks to that, it avoids the need to compute $2^k$ different values, which is not only inefficient, but also results in an excessive amount of noise injected when applying differential privacy.

### 5 Differentially Private C-AGM

In this section, we describe in detail our mechanisms for obtaining differentially private instances of the C-AGM model, as well as the necessary adaptations of the sampling methods when the model has been computed under differential privacy. As we discussed in Section 3, the difference between different instantiations of differential privacy for graphs lies in the definition of the pairs of graphs that are considered to be neighbouring datasets. Here, we adopt the following definition from [12].
Definition 2 (Neighbouring attributed graphs [12]). A pair of attributed graphs \( G = (V, E, X) \) and \( G' = (V, E', X') \) are neighbouring, denoted \( G \sim G' \), if and only if they differ in the presence of exactly one edge or the attribute vector of exactly one node. That is,

\[
G \sim G' \iff |E \Delta E'| = 1 \quad \lor \quad (\exists v \in V \tau_G(v) \neq \tau_{G'}(v) \land \forall v' \in V \setminus \{v\} \tau_G(v') = \tau_{G'}(v')).
\]

Definition 2 entails that the existence of relations, that is the occurrence of edges, and the attributes describing every particular user, are treated as sensitive. On the contrary, vertex identifiers are treated as non-private. These criteria are in line with the current privacy policies of most social networking sites, where the fact that a profile exists is public information, but users can keep their personal information and friends list private or hidden from the general public. With Definition 2 in mind, we describe in what follows the differentially private computation of every parameter of C-AGM.

5.1 Obtaining the Community Partition

Our differentially private community partition method extends the algorithm ModDivisive [32], in such a way that it takes node attributes into account. In its original formulation, ModDivisive searches for a community partition that maximises modularity, a structural parameter encoding the intuition that a user tends to be more connected to users in the same community than to users in other communities [30]. Modularity is defined as

\[
\sum_{C \in \mathcal{C}} \left( \frac{\ell_C}{m} - \left( \frac{d_C}{2m} \right)^2 \right),
\]

where \( \ell_C \) is the number of edges between the nodes in \( C \) and \( d_C \) is the sum of degrees of the nodes in \( C \). ModDivisive uses the exponential mechanism, considering the set of possible partitions as the categorical co-domain, and using modularity as the scoring function.

In order to integrate node features into ModDivisive, we introduce a new objective function that combines the original modularity with an attribute-based quality criterion. The new objective function is defined as

\[
Q(C) = w_s \cdot Q_s(C) + w_a \cdot Q_a(C),
\]

where \( w_s \in [0, 1] \), \( w_a = 1 - w_s \), \( Q_s(C) \) is the modularity of the original graph and \( Q_a(C) \) is the modularity of an auxiliary graph obtained from the original as follows. First, we take the vertex set of the original graph. Then, we compute all pairwise similarities between their associated feature vectors. Similarities are computed using the cosine measure (as done in Section 4.4.3 for computing aggregated attributes, but without applying the discretisation). Finally, we add to the auxiliary graph the edges corresponding to the \( \left\lceil \frac{n(n-1)}{20} \right\rceil \) most similar attributed node pairs.

It is proven in [32] that the global sensitivity of \( Q_s(C) \) is upper bounded by \( \frac{3}{m} \), where \( m \) is the minimum number of edges of all potential graphs to publish. In the worst case,
\(\Delta_{Q_a(C)} = 3\), considering that the original graph is an arbitrary non-empty graph. However, this is not the case for real-life social graphs, so introducing more realistic assumptions about the value of \(m\) allows us to use smaller values of \(\Delta_{Q_a(C)}\) and thus reduce the amount of noise added in differentially privately computing \(Q_a(C)\). Throughout this paper, we assume \(m = 10,000\), which leads to \(\Delta_{Q_a(C)} = 0.0003\). As we will see in Section 6, all datasets used in our experiments comply with this assumption. In what follows, we apply an analogous reasoning for bounding \(\Delta_{Q_a(C)}\).

**Proposition 1.** Every graph \(G\) of order \(n\) satisfies \(LS_{Q_a(C)}(G) \leq \frac{60}{n}\).

**Proof.** Let \(G \sim G'\) be two neighbouring attributed graphs and let \(G_a\) and \(G'_a\) be the auxiliary graphs obtained from \(G\) and \(G'\), respectively. If the difference between \(G\) and \(G'\) consists only in one edge, then \(G_a = G'_a\), so in what follows we will consider that \(G\) and \(G'\) differ in one attribute vector. Let \(v\) be the (sole) vertex such that \(\tau_G(v) \neq \tau_{G'}(v)\). In the worst case, we have that, for every \(w \in V \setminus \{v\}\), \((v, w) \in G_a\) and \((v, w) \notin G'_a\) (or vice versa). It was shown in [32] that the modularities of two graphs differing in one edge differ in up to \(\frac{3}{m}\), where \(m\) is the minimum number of edges. Then, in the worst case we have \(LS_{Q_a}(G) \leq \frac{3(n-1)}{m_a}\), where \(n\) is the order of \(G\) and \(G'\), and \(m_a\) is the minimum number of edges in auxiliary graphs. As we discussed in Sect. 5.1, \(m_a \geq \frac{n(n-1)}{20}\), so \(LS_{Q_a}(G) \leq \frac{60}{n}\). The proof is thus completed. \(\square\)

Combining the result in [32] with that of Proposition 1, we conclude that \(LS_{Q(C)}(G) \leq 0.0003 \cdot w_a + \frac{60}{|V(G)|} \cdot w_a\) for every \(G\) satisfying the aforementioned assumptions, and use this value as an upper bound for \(\Delta_{Q_a(C)}\).

### 5.2 Attribute Vector Distribution

As discussed in Section 4.4, given a community partition \(C\), in order to obtain the differentially private estimation of \(\Theta_X^C\) (denoted by \(\bar{\Theta}_X^C\)), we need to compute the probability distribution of each attribute for every community, i.e., \(Pr_\ell(\tau_C(v) | \bar{\Theta}_X^C, v \in C)\), for each \(\ell \leq k\) (where \(k\) is the number of attributes) and \(C \in C\). Computing this probability reduces to computing the number of nodes whose \(\ell\)-th attribute has value 1, which we denote by \(n^{\ell C}_X\). Let \(n^C_X\) be the sequence \((n^{1 C}_X, n^{2 C}_X, \ldots, n^{k C}_X)\). In order to obtain the differentially private sequence \(\bar{n}^C_X = (\bar{n}^{1 C}_X, \bar{n}^{2 C}_X, \ldots, \bar{n}^{k C}_X)\), we add to each element in \(n^C_X\) noise sampled from \(Lap\left(\frac{k}{\varepsilon_X}\right)\), where \(\varepsilon_X\) is the privacy budget reserved for this computation and \(k\) is the global sensitivity of \(n^C_X\), as shown in the next result.

**Proposition 2.** The global sensitivity of \(n^C_X = (n^{1 C}_X, n^{2 C}_X, \ldots, n^{k C}_X)\) is \(k\).

**Proof.** Let \(G \sim G'\) be two neighbouring attributed graphs, let \(C \subseteq V\) be a community and let \(n^C_X(G)\) and \(n^C_X(G')\) be the instances of \(n^C_X\) in \(G\) and \(G'\), respectively. If the difference between \(G\) and \(G'\) consists only in one edge, then \(n^C_X(G) = n^C_X(G')\), so in what follows we
will consider that \( G \) and \( G' \) differ in one attribute vector. Let \( v \) be the (sole) vertex such that \( \tau_{G}(v) \neq \tau_{G'}(v) \). If \( v \notin C \), then \( n^{C}_{\chi}(G) = n^{C}_{\chi}(G') \). On the contrary, if \( v \in C \), for every component \( \ell \in \{1, \ldots, k\} \) such that \( \tau_{\ell,G}(v) \neq \tau_{\ell,G'}(v) \), we have that \( |n^{C}_{\ell}(G) - n^{C}_{\ell}(G')| = 1 \). In consequence, we have \( \Delta n^{C}_{\chi} = \max_{G \sim_{\varepsilon,F} G'} \| n^{C}_{\chi}(G) - n^{C}_{\chi}(G') \|_{1} = k \).

### 5.3 Attribute-Edge Correlations

Recall that the aggregator function \( \beta \) defined in Section 4.4.3 maps every pair of attribute vectors to a non-negative integer in \( \mathcal{B} = \{\lfloor \frac{s}{\delta} \rfloor \mid s \in [0, 1]\} \), for some \( \delta \in [0, 1] \). In order to estimate \( \Theta^{C}_{F} \), we need to count, for each possible output of \( \beta \), the number of edges whose end-nodes are mapped to this value. For every \( t \in \mathcal{B} \) and every \( C \in \mathcal{C} \), let \( n^{C}_{F,\text{intra}} \) be the number of intra-community edges \((v, v')\) in \( C \) such that \( \beta((v), (v')) = t \). Likewise, let \( n^{C}_{F,\text{inter}} \) be the number of inter-community edges \((v, v')\) such that \( \beta((v), (v')) = t \). Thus, in order to compute \( \Theta^{C}_{F,G} \), we need to differentially privately compute \( n^{C}_{F,\text{intra}} \) for every \( C \in \mathcal{C} \), as well as \( n^{C}_{F,\text{inter}} \) for every \( t \in \mathcal{B} \). We denote by \( \pi^{C}_{F,\text{intra}} \) and \( \pi^{C}_{F,\text{inter}} \) the corresponding differentially private values.

The global sensitivity of each of these sequences is \( 2(|\mathcal{V}| - 2) \), which is unbounded. To overcome this problem, we follow an approach analogous to the one used in [12] for counting attribute-edge correlations in the entire graph. The method, introduced in [3], consists in truncating the edge set of the graph to ensure that the degree of all nodes is at most \( p \), which in the case of attribute-edge correlations ensures that the global sensitivity is \( 2p \) [12, 3]. In consequence, for every \( C \in \mathcal{C} \), we obtain \( \pi^{C}_{F,\text{intra}} \) from \( n^{C}_{F,\text{intra}} \) by adding noise sampled from \( \text{Lap}(\frac{2k}{\varepsilon_{F}}) \), where \( \varepsilon_{F} \) is the privacy budget reserved for this computation. Likewise, we obtain \( \pi^{C}_{F,\text{inter}} \) from \( n^{C}_{F,\text{inter}} \) by adding noise sampled from \( \text{Lap}(\frac{2k}{\varepsilon_{F}}) \).

### 5.4 CPGM Parameters

In what follows we describe the computation of the parameters of CPGM, namely the set of intra-community and inter-community degrees and triangle counts.

**Intra- and inter-community degrees.** Following the trend of previous differentially private degree sequence computation methods [16, 9], we first add noise to the raw values and then apply a post-processing on the perturbed degree sequences to restore certain properties of the original sequence, namely graphically and the order of the nodes in terms of their degrees, as well as certain community-specific properties.

Let \( d^{C}_{\text{intra}} = (d^{1,C}_{\text{intra}}, d^{2,C}_{\text{intra}} \ldots, d^{m,C}_{\text{intra}}) \), where \( m = |C| \) and \( d^{i,C}_{\text{intra}} \leq d^{i+1,C}_{\text{intra}} \ (1 \leq i < |C|) \), be the list of non-decreasingly ordered original intra-community degrees in \( C \in \mathcal{C} \). Analogously, let \( d^{C}_{\text{inter}} = (d^{1,C}_{\text{inter}}, d^{2,C}_{\text{inter}} \ldots, d^{m,C}_{\text{inter}}) \) be the sequence of inter-community degrees of nodes in \( C \).

The global sensitivity of the degree sequence of the entire graph is \( 2 \), as adding or removing one edge changes the degrees of exactly two nodes by 1 [9]. The same is true for every \( d^{C}_{\text{intra}} \) and \( d^{C}_{\text{inter}} \), since the degrees of at most two intra-community nodes (or at most one node in \( C \) and one node outside of \( C \)) change by 1. Thus, for every \( C \in \overline{\mathcal{C}} \), we obtain
from \(d_{\text{intra}}^C\) the differentially private sequence \(\tilde{d}_C\) by adding noise sampled from \(\text{Lap}(\frac{2}{\varepsilon d})\) to every degree value. Similarly, we obtain from \(d_{\text{inter}}^C\) the differentially private sequence \(\tilde{d}_{\text{inter}}^C\). Afterwards, the noisy sequences are post-processed to restore three properties: (i) the non-decreasing order between the intra-community degrees inside every community, (ii) the graphicality of the intra-community degrees of every community, and (iii) the graphicality of the inter-community degrees of all nodes in the graph. Property (i) is enforced using the method proposed in [9], whereas properties (ii) and (iii) are enforced using the method proposed in [16].

**Numbers of intra- and inter-community triangles.** The global sensitivity of the number of triangles in a graph is proven in [33] to be \(n-2\), where \(n\) is the number of vertices. The following result characterises the global sensitivity of the number of intra-community triangles.

**Proposition 3.** The global sensitivity of the number of intra-community triangles of a graph \(G\) with a community partition \(\mathcal{C}\) is \(\Delta n_{\Delta}^{\text{intra}} = \max_{C \in \mathcal{C}} \{|C| - 2\}\).

**Proof.** Let \(G \sim_{\text{at}} G'\) be two neighbouring attributed graphs and let \(\mathcal{C}\) be a community partition of \(V\). Let \(n_{\Delta}^{\text{intra}}(G)\) and \(n_{\Delta}^{\text{intra}}(G')\) be the numbers of intra-community triangles of \(G\) and \(G'\), respectively. If the difference between \(G\) and \(G'\) consists only in one attribute vector, then \(n_{\Delta}^{\text{intra}}(G) = n_{\Delta}^{\text{intra}}(G')\), so in what follows we will consider that \(G\) and \(G'\) differ in one edge. We will assume, without loss of generality, that \(E' \setminus E = (v,v')\). Two cases are possible for \(\psi_C(v)\) and \(\psi_C(v')\):

(i) \(\psi_C(v) \neq \psi_C(v')\). In this case, since \((v,v')\) is an inter-community edge, \(n_{\Delta}^{\text{intra}}(G) = n_{\Delta}^{\text{intra}}(G')\).

(ii) \(\psi_C(v) = \psi_C(v') = C\). In this case, we have that \(n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G) = |C \cap \mathcal{N}_G(v) \cap \mathcal{N}_G(v')|\), that is the number of common neighbours of \(v\) and \(v'\) in the same community.

It is simple to see that every pair of vertices \(v\) and \(v'\) such that \(\psi_C(v) = \psi_C(v') = C\) satisfy \(|C \cap \mathcal{N}_G(v) \cap \mathcal{N}_G(v')| \leq |C| - 2\). Hence, \(\Delta n_{\Delta}^{\text{intra}} = \max_{C \in \mathcal{C}} \{|C| - 2\}\).

Since the global sensitivity of triangle count queries is unbounded, the Laplace mechanism cannot be applied in this case. An accurate differentially private method for counting the number of triangles of a graph is presented in [43]. This method uses the exponential mechanism. It interprets the triangle count query as a categorical query, whose co-domain is a partition \(\mathcal{O}\) of \(\mathbb{Z}^+\). One of the elements of \(\mathcal{O}\) is a singleton set composed exclusively of the correct output of the query (the correct number of triangles in this case), whereas every other element contains a set of incorrect values which are treated as equally useful. They define the notion of *ladder function*, which is used as a scoring function on the elements of \(\mathcal{O}\). A ladder function gives better scores to the sets of values that are closer to the correct query answer. In order to differentially privately compute the number of triangles of a graph \(G\), the ladder function approach starts by obtaining the correct number of triangles. Then, the ladder function is built, and a set \(O \in \mathcal{O}\) is sampled following the exponential
mechanism. Finally, a random element of \( O \) is given as the differentially private output of the query [43].

It is shown in [43] that the best ladder function, in the sense that it adds the minimum necessary amount of noise, is the so-called local sensitivity at distance \( t \) [33], which is denoted as \( LS_q(G,t) \), and is defined as the maximum local sensitivity of the query \( q \) among all the graphs at edge-edit distance at most \( t \) from \( G \). Formally, \( LS_q(G,t) = \max_{G' \mid \phi(G,G') \leq t} LS_q(G') \), where \( \phi(G,G') \) is the edge-edit distance between \( G \) and \( G' \), that is the minimum number of edge additions and removals that transform \( G \) into \( G' \). It is also shown in [33, 43] that \( LS_q(G,t) = \max_{1 \leq i < j \leq |V|} LS_{ij}^q(G,t) \), where \( LS_{ij}^q(G,t) = \max_{G',G'' \mid \phi(G,G') \leq t,G'' \sim_j G' |q(G') - q(G'')|} \) and \( G'' \sim_j G'' \) indicates that \( G'' \) and \( G'' \) differ in exactly the addition or removal of \((v_i, v_j)\).

Here, we apply the ladder function approach for computing the number of intra-community triangles. To that end, we characterise the function \( LS_{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) \) for every graph \( G \) with a community partition \( \mathcal{C} \).

**Proposition 4.** For every graph \( G \), every community partition \( \mathcal{C} \) of \( G \), and every positive integer \( t \geq 1 \),

\[
LS_{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) = \max_{\{i,j \mid \psi_C(v_i) = \psi_C(v_j)\}} \left\{ \min \left\{ a_{ij} + \left[ \frac{t + \min\{t, b_{ij}\}}{2} \right], |\psi_C(v_i)| - 2 \right\} \right\},
\]

where \( a_{ij} = |\{v_\ell \in \psi_C(v_i) \mid A_{i,\ell} = 1 \land A_{j,\ell} = 1\}| \) and \( b_{ij} = |\{v_\ell \in \psi_C(v_i) \mid A_{i,\ell} \oplus A_{j,\ell} = 1\}| \).

**Proof.** Consider a graph \( G \) with a community partition \( \mathcal{C} \), and a positive integer \( t \geq 1 \). As discussed in [33, 43],

\[
LS_{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) = \max_{1 \leq i < j \leq |V|} LS_{ij}^{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t).
\]

For every \( i \) and \( j \) such that \( \psi_C(v_i) \neq \psi_C(v_j) \), we have that no intra-community triangle is created (resp. destroyed) by the addition (resp. removal) of \((v_i, v_j)\), so \( LS_{ij}^{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) = 0 \). Thus,

\[
LS_{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) = \max_{\{i,j \mid \psi_C(v_i) = \psi_C(v_j)\}} LS_{ij}^{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t).
\]

We now focus on determining \( LS_{ij}^{n_{\Delta}}^{\text{intra}}((G, \mathcal{C}), t) \) for every \( i \) and \( j \) such that \( \psi_C(v_j) = \psi_C(v_j) = C \). Consider a pair of such values \( i \) and \( j \), and let \( S_1 = \{v_\ell \in C \mid A_{i,\ell} = 1 \land A_{j,\ell} = 1\} \) and \( S_2 = \{v_\ell \in C \mid A_{i,\ell} \oplus A_{j,\ell} = 1\} \).

Let \( G_t \) be the class of all graphs that can be obtained by modifying \( G \) as follows:

1. Add \( \min\{b_{ij}, t\} \) arbitrary edges of the form \((x, y)\), where \( x \in \{v_i, v_j\} \) and \( y \in S_2 \).
2. If \( t > b_{ij} \), take an arbitrary subset \( S_3 \) of vertices of \( C \setminus (S_1 \cup S_2 \cup \{v_i, v_j\}) \), with cardinality \( \min \left\{ \left[ \frac{t-\min\{t, b_{ij}\}}{2} \right], |C \setminus (S_1 \cup S_2 \cup \{v_i, v_j\})| \right\} \). For every \( x \in S_3 \), add the edges \((v_i, x)\) and \((v_j, x)\).

Note that \( S_1 \) and \( S_2 \) are the sets whose cardinalities define \( a_{ij} \) and \( b_{ij} \), respectively.
From the definition of $G_t$, it follows that every $G' \in G_t$ satisfies $\phi(G, G') \leq t$ and the graph $G'' \sim_{ij} G'$ satisfies

$$|n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G'')| = \min \left\{ a_{ij} + \left\lfloor \frac{t + \min \{t, b_{ij}\}}{2} \right\rfloor, |C| - 2 \right\}.$$ 

Now, consider an arbitrary graph $G'$, obtained by modifying $G$, such that $\phi(G, G') \leq t$ and $G' \not\in G_t$. Also consider the graph $G'' \sim_{ij} G'$. According to the definition of $G_t$, the following situations are possible:

(i) $G'$ is the result of adding to $G$ a proper subset of the set of edges added by steps 1 and 2 of the procedure described above for obtaining an element of $G_t$. In this case, only a proper subset of the triangles created (resp. destroyed) by the addition (resp. removal) of $(x, y)$ is added (resp. removed). Thus,

$$|n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G'')| < \min \left\{ a_{ij} + \left\lfloor \frac{t + \min \{t, b_{ij}\}}{2} \right\rfloor, |C| - 2 \right\}.$$ 

(ii) $G'$ is the result of applying $t - t'$ additional modifications ($t' < t$) on an element $H$ of $G_t$. Note that, by the definition of edge-edit distance, the additional modifications do not consist in reverting any edge addition made in steps 1 and 2 of the procedure described above. In this case, none of the additional modifications can result in the addition of a par of edges of the form $(v_i, x)$ and $(v_j, x)$, with $x \in C$, so

$$|n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G'')| = |n_{\Delta}^{\text{intra}}(H) - n_{\Delta}^{\text{intra}}(H')|$$

$$= \min \left\{ a_{ij} + \left\lfloor \frac{t + \min \{t, b_{ij}\}}{2} \right\rfloor, |C| - 2 \right\},$$

where $H' \sim_{ij} H$.

(iii) In every other case, the transformation that allows to obtain $G'$ from $G$ can be divided into a set of edge additions as the ones described in (i) and a set of additional modifications as the ones described in (ii). Applying an analogous reasoning, we have that

$$|n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G'')| < \min \left\{ a_{ij} + \left\lfloor \frac{t + \min \{t, b_{ij}\}}{2} \right\rfloor, |C| - 2 \right\}.$$ 

Summing up the set of cases analysed above, we have that, for every $i$ and $j$ such that $\psi_C(v_i) = \psi_C(v_j)$,

$$LS_{ij}^{n_{\Delta}^{\text{intra}}}(G, t) = \max_{\{G', G'' \mid \phi(G, G') \leq t, G' \sim_{ij} G''\}} \left\{ |n_{\Delta}^{\text{intra}}(G') - n_{\Delta}^{\text{intra}}(G'')| \right\}$$

$$= \min \left\{ a_{ij} + \left\lfloor \frac{t + \min \{t, b_{ij}\}}{2} \right\rfloor, |\psi_C(v_i)| - 2 \right\}.$$
and, in consequence,

\[ \text{LS}_{n\Delta}((G,C),t) = \max_{\{i,j \mid \psi_C(v_i) = \psi_C(v_j)\}} \text{LS}_{ij}^{\Delta}((G,C),t) = \max_{\{i,j \mid \psi_C(v_i) = \psi_C(v_j)\}} \left\{ \min \left\{ a_{ij} + \left\lfloor \frac{t + \min\{t,b_{ij}\}}{2} \right\rfloor, |\psi_C(v_i)| - 2 \right\} \right\}. \]

The proof is thus completed. \(\square\)

In Proposition 4, the operator \(\oplus\) denotes exclusive or. Notice that \(\text{LS}_{n\Delta}((G,C),t)\) can be efficiently computed for small values of \(t\), and it converges to the efficiently computable global sensitivity \(\Delta_{n\Delta}\) for \(t \geq 2 \max_{C \in C} |C|\), so it can be used for efficiently and privately computing the number of intra-community triangles.

Finally, for computing the number of inter-community triangles, we use the method from [43] to compute the number of triangles of the entire graph, and subtract from it the number of intra-community triangles computed with the method described in this subsection.

### 5.5 Summary

The methods discussed in the previous subsections allow to compute a differentially private instance of \(C\)-AGM. In what follows, we will use the notation \(C\)-AGMDP to clearly distinguish differentially private instances of \(C\)-AGM. The privacy budget \(\varepsilon\) is split among the different computations as follows: \(\varepsilon_c = \frac{\varepsilon}{2}\) for the community partition method, \(\varepsilon_F = \frac{\varepsilon}{6}\) for the estimation of \(\Theta_F\), and \(\varepsilon_d = \varepsilon_{\Delta} = \varepsilon_{\text{intra}} = \varepsilon_X = \frac{\varepsilon}{12}\) for the estimation of degree distributions, triangle counts and \(\Theta_X\).

**Remark 1.** Parameter estimation for \(C\)-AGMDP satisfies \((\varepsilon_c + \varepsilon_X + \varepsilon_F + \varepsilon_d + \varepsilon_{\Delta} + \varepsilon_{\text{intra}})\)-differential privacy.

**Proof.** The result follows straightforwardly from the fact that \(\varepsilon_c + \varepsilon_X + \varepsilon_F + \varepsilon_d + \varepsilon_{\Delta} + \varepsilon_{\text{intra}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{12} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \varepsilon\) \(\square\)

### 6 Experiments

The purpose of our experiments is to empirically validate the following two claims: (i) our \(CPGM\) model outperforms existing models in generating graphs whose community structures are more similar to those of the input graphs without sacrificing the ability to preserve global structural properties, and (ii) differentially private instances of \(C\)-AGM also outperform preceding models in terms of the preservation of community structure, while remaining comparable in terms of the preservation of global structural properties.
6.1 Datasets

For our experiments, we use three real-world social networks with node attributes. The first one has been collected from Petster, a website for pet owners to communicate [19]. It is an undirected graph whose nodes represent hamster owners. Each node is labelled with attributes containing information about the user’s pet. We extracted 13 binary attributes from 8 categorical attributes such as favourite food, gender, colour, species, year of birth, etc. The second one is a subset of Facebook available via SNAP [21]. In this dataset, node attributes are already binary and are tagged with serial pseudonyms. For our experiments, we selected the first 50 attributes with the smallest serial numbers. Finally, the third dataset, Epinions, is a directed graph extracted from an online consumer reviews system, where every vertex represents a reviewer [23]. In the original dataset, a directed edge from node \( A \) to node \( B \) exists if user \( A \) trusts the reviews of \( B \). For our experiments, we derived an undirected graph from the original dataset by keeping the same vertex set and adding an undirected edge for every pair of mutually trusting users. Additionally, we selected the 50 most frequently rated products as node attributes. If the user rated the product, the value is set to 1, otherwise it is set to 0. Table 1 summarises the main statistics of the three datasets.

| Dataset   | #nodes | #edges   | \( \#\triangle \) | cl. coeff. | #attr. |
|-----------|--------|----------|------------------|------------|-------|
| Petster   | 1,898  | 12,534   | 16,750           | 0.14       | 13    |
| Facebook  | 3,953  | 84,070   | 1,526,985        | 0.54       | 50    |
| Epinions  | 29,515 | 106,147  | 235,790          | 0.13       | 50    |

Table 1: Datasets used for our experiments.

6.2 Evaluation Measures

For every pair \((G, G')\), where \( G \) is a real-life graph and \( G' \) is a synthetic graph sampled from a model learned from \( G \), we evaluate the extent to which \( G' \) preserves the following properties of \( G \).

**Numbers of edges and triangles:** Our evaluation measures in this case are the relative errors of the numbers of edges and triangles in \( G' \) with respect to those in \( G \). We define these measures as \( \rho_E = \frac{|E_{G'} - |E_G|}{|E_G|} \) and \( \rho_\triangle = \frac{|n_\triangle(G') - n_\triangle(G)|}{n_\triangle(G)} \), respectively.

**Global clustering coefficient:** The global clustering coefficient (GCC) of a graph measures the proportion of wedges, that is, paths of length 2, that are embedded in triangles. It is defined as \( \frac{3n_\triangle}{n_w} \), where \( n_w \) is the number of wedges and \( n_\triangle \) is the number of triangles. We compare \( G \) and \( G' \) in terms of the relative error of the GCC of \( G' \) with respect to that of \( G \). We denote this measure by \( \rho_c \).

**Degree distribution.** We compare \( G \) and \( G' \) in terms of the Hellinger distance between their degree distributions. The Hellinger distance has been deemed as the most
appropriate distance for comparing probability distributions in previous works on graph synthesising [12, 28]. Given two probability distributions \( p_1 \) and \( p_2 \) on a discrete domain \( W \), the Hellinger distance between \( p_1 \) and \( p_2 \) is defined as

\[
H(p_1, p_2) = \frac{1}{\sqrt{2}} \sqrt{\sum_{w \in W} (\sqrt{p_1(w)} - \sqrt{p_2(w)})^2}.
\]

The Hellinger distance yields values in the interval \([0, 1]\). The more similar two distributions are, the smaller the Hellinger distance between them. For the particular case of degree distributions, we compute \( p_d \) and \( p'_d \), which are defined on the domain \( W = \{0, 1, \ldots, n-1\} \), where \( n \) is the number of vertices in \( G \) and \( G' \). For every \( i \in W \), \( p_d(i) \) (resp. \( p'_d(i) \)) is the probability that a vertex of \( G \) (resp. \( G' \)) has degree \( i \). The final score used for comparing \( G \) and \( G' \) is \( H_d = H(p_d, p'_d) \).

**Local clustering coefficients.** In a graph \( G \), the local clustering coefficient (LCC) of a node \( v \) measures the proportion of pairs of mutual neighbours of \( v \) that are connected by an edge. In the context of social graphs, \( LCC(v) \) is an indicator of the likelihood of \( v \)'s mutual friends to also be friends. \( LCC(v) \) is defined as \( \frac{2 \sum_{u \in \mathcal{N}(v) \setminus \{v\}} A_{u,v}}{|\mathcal{N}(v)|(|\mathcal{N}(v)|-1)} \) where \( \mathcal{N}(v) \) is the set of \( v \)'s neighbours. For comparing \( G \) and \( G' \) in terms of local clustering coefficients, we compute the distributions \( p_{\text{lc}} \) and \( p'_{\text{lc}} \), which are defined in the domain \( W = \{c \mid \exists v \in \mathcal{V}(LCC_G(v) = c \lor LCC_{G'}(v) = c)\} \) in such a way that for every \( i \in W \), \( p_{\text{lc}}(i) \) (resp. \( p'_{\text{lc}}(i) \)) is the probability that a vertex belonging to \( C \) in \( G \) (ergo, in the context of this paper, also in \( G' \)) is labelled with the attribute vector \( i \) in \( G \) (resp. in \( G' \)). We compare \( G \) and \( G' \) in terms of the parameter \( \rho_a \), which is defined as \( \rho_a = \max_{C \in \mathcal{C}} \{ H(p_F^C, \tilde{p}_F^C) \} \).

**Distribution of attribute-edge correlations.** Recall that \( k \) represents the number of components of every attribute vector labelling the vertices of both \( G \) and \( G' \). Given a community partition \( \mathcal{C} \) of \( G \), we define for every \( C \in \mathcal{C} \) the distributions \( p_F^C \) and \( \tilde{p}_F^C \) in the domain \( W = \{0, 1\}^k \) in such a way that, for every \( i \in W \), \( p_F^C(i) \) (resp. \( \tilde{p}_F^C(i) \)) is the probability that a vertex belonging to \( C \) in \( G \) (ergo, in the context of this paper) also in \( G' \) is labelled with the attribute vector \( i \) in \( G \) (resp. in \( G' \)). We compare \( G \) and \( G' \) in terms of the parameter \( \rho_a \), which is defined as \( \rho_a = \max_{C \in \mathcal{C}} \{ H(p_F^C, \tilde{p}_F^C) \} \).

**Detectability of community partition.** We evaluate to what extent state-of-the-art community detection algorithms find similar communities in \( G \) and \( G' \). To that end, we use the averaged \( F_1 \) score, denoted Avg-\( F_1 \), of the community structures \( \mathcal{C} \) and \( \mathcal{C}' \) determined by the algorithm in \( G \) and \( G' \), respectively. The averaged \( F_1 \) score has been widely used for evaluating community detection algorithms [42, 40, 32]. Given two communities \( C_1 \) and \( C_2 \), the \( F_1 \) score between these two communities, denoted \( F_1(C_1, C_2) \) combines two auxiliary measures: precision and recall. Precision is defined as \( \text{prec}(C_1, C_2) = \frac{|C_1 \cap C_2|}{|C_1|} \), whereas recall is defined as \( \text{recall}(C_1, C_2) = \frac{|C_1 \cap C_2|}{|C_2|} \). Precision and recall are combined as \( F_1(C_1, C_2) = \frac{2 \cdot \text{prec}(C_1, C_2) \cdot \text{recall}(C_1, C_2)}{\text{prec}(C_1, C_2) + \text{recall}(C_1, C_2)} \). If both precision and recall are zero, \( F_1 \) is made zero by convention. Following the evaluation strategy introduced in [42, 40, 32], given two sets of communities \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), we first determine the average of the \( F_1 \) values between every community of \( \mathcal{C}_1 \) and its best match in \( \mathcal{C}_2 \) (in terms of \( F_1 \)), then the average of the \( F_1 \) values
between every community of $C_2$ and its best match in $C_1$, and finally these two values are also averaged. The average $F_1$-score is defined as

$$\frac{1}{2|C_1|} \sum_{c_i^1 \in C_1} \max_{c_j^2 \in C_2} F_1(c_i^1, c_j^2) + \frac{1}{2|C_2|} \sum_{c_i^2 \in C_2} \max_{c_j^1 \in C_1} F_1(c_i^2, c_j^1).$$

Avg-$F_1$ values are in the interval $[0, 1]$. The larger the value of Avg-$F_1$, the more similar the community structures of $C_1$ and $C_2$ are considered to be.

### 6.3 Results and Discussion

We first evaluate the ability of our new edge generation model, CPGM, to synthesise graphs that preserve the community structures of the original graphs along with global structural properties. Then, we assess the overall quality of the differentially private C-AGMDP model.

#### 6.3.1 Evaluation of CPGM

We compare CPGM with the two most similar counterparts reported in the literature: TriCycle [12] and DCSBM [14]. TriCycle has been shown to preserve a number of global structural properties, but it does not aim to preserve the community structure; whereas DCSBM belongs to a family of models that has been shown to preserve the community structure, disregarding global structural properties. Figure 1 shows the extent to which the community structures found by the state-of-the-art community detection algorithm Louvain [4] in the synthetic graphs generated by each model are similar to those detected in the corresponding original graphs. Table 2 compares the behaviour of all three edge generation models in terms of global structural properties. In all cases, the values shown are averaged over 10 executions.

![Figure 1: Similarities of community structures found by Louvain in synthetic graphs to those found in the original graphs.](image)

From the analysis of these results, we extract three major observations. Firstly, as Figure 1 shows, the community structures of the graphs synthesised using our CPGM model are
consistently more similar to those of the original graphs, in comparison to those induced by DCSBM and TriCycle. This supports our claim that CPGM is able to preserve community structure to a larger extent. Note that, in several cases, our model performs almost twice as good as the second best, DCSBM. As expected, TriCycle shows the poorest results, corroborating the intuition that community structure needs to be explicitly included in the generative model if we want synthetic graphs to preserve it. The previous observations support our design choices of preserving (i) the community structure, and (ii) differentiated intra- and inter-community structural properties.

Secondly, the graphs generated by our CPGM model are consistently the most accurate in terms of the distributions of local clustering coefficients (see right-most column of Table 2), and are the most accurate in terms of global clustering coefficient in all but one dataset (see column labelled \( \rho_c \) in Table 2). An analogous observation can be made for the number of triangles (column labelled \( \rho_\Delta \)). We consider that these observations support our design choice of preserving separate intra- and inter-community edge densities and triangle counts. The comparably poorer performance of DCSBM in terms of global and local clustering coefficients also corroborates the need to explicitly model them, as do CPMG and TriCycle. A more detailed graphical description of the behaviour of the three models in terms of the distributions of local clustering coefficients is shown in Figure 3 (d)–(e), in Appendix A. The figure shows the complementary cumulative distribution functions of local clustering coefficients for the three models, and highlights the special ability of our CPGM model to capture the behaviour of the distribution for denser-than-normal graphs (the Facebook dataset in this case).

| Dataset  | Model   | \( \rho_E \) | \( \rho_\Delta \) | \( \rho_c \) | \( H_d \) | \( H_{\ell c} \) |
|---------|---------|--------------|-----------------|-------------|----------|-------------|
| Petster | CPGM    | 0.00         | 0.18            | 0.05        | 0.16     | 0.19        |
|         | DCSBM   | 0.00         | 0.12            | 0.49        | 0.17     | 0.27        |
|         | TriCycle| 0.00         | 0.00            | 0.19        | 0.18     | 0.21        |
| Facebook| CPGM    | 0.00         | 0.03            | 0.32        | 0.15     | 0.32        |
|         | DCSBM   | 0.00         | 0.25            | 0.71        | 0.25     | 0.64        |
|         | TriCycle| 0.00         | 0.04            | 0.56        | 0.37     | 0.60        |
| Epinions| CPGM    | 0.001        | 0.04            | 0.27        | 0.13     | 0.31        |
|         | DCSBM   | 0.002        | 0.60            | 0.83        | 0.14     | 0.26        |
|         | TriCycle| 0.001        | 0.04            | 0.22        | 0.10     | 0.31        |

Table 2: Comparison of edge generative models in terms of global structural properties.

Finally, we point out that all three models successfully preserve the properties of the degree distribution. Our CPGM model produces the most consistent results in all the datasets in terms of both Hellinger distances and the shape of the complementary cumulative distribution functions (see Figure 3 (a)–(c) in Appendix A). Again, CPGM is the one that better captures the cumulative distribution for the denser Facebook dataset.

Summing up, the results shown in this subsection support our claim that synthetic graphs sampled from our CPGM model preserve the community structure of the original
graph to a considerably larger extent than its closest counterparts, without sacrificing the ability to preserve global structural properties. Additionally, these results show that the manner in which CPGM computes intra- and inter-community parameters also helps it outperform competing models in preserving local and global clustering coefficients.

### 6.3.2 Evaluation of differentially private C-AGM

We compare C-AGMDP with two other models. The first one is the differentially private AGM model using TriCycle as edge set generator [12]. We refer to this model as AGMDP-Tri. It was shown in [12] that, despite the noise added to guarantee privacy, AGMDP-Tri still preserves to some extent TriCycle’s ability to capture global structural properties. As we saw in the previous section, TriCycle performs poorly in preserving the community structure of the original graph, so we additionally consider for our evaluation an additional model, which is a modification of C-AGMDP where CPGM is replaced by DCSBM as the edge generation model. We refer to this model as C-AGMDP-D.

In our experiments, we compare the behaviour of the three models under four privacy budgets. For the smaller Petster and Facebook datasets, we use the values 2.0, 3.0, 4.0 and 5.0. For the considerably larger Epinions dataset, using these values results in considerably noisy outputs, where pairwise differences between the models have been blurred away by the noise. Thus, in order to enable comparisons, we show for this dataset the results obtained using the values 6.0, 7.0, 8.0 and 9.0. Every particular comparison of the three models uses a common privacy budget. Since C-AGMDP-D and AGMDP-Tri each have less parameters than C-AGMDP, we re-allocate in each case the remaining privacy budget to other computations. In estimating C-AGMDP-D, we allocate to community partition the same budget as for C-AGMDP, that is \( \frac{\varepsilon}{12} \). C-AGMDP-D requires to compute the numbers of edges between every pair of communities. We assign to this computation the budget used in C-AGMDP for counting the number of intra-community triangles, i.e. \( \frac{\varepsilon}{12} \). Finally,
C-AGMDP-D is given for degree sequence computation the budget $\varepsilon'_d = \varepsilon_d + \varepsilon_\Delta = \frac{\varepsilon}{6}$, as it does not require to count the global number of triangles. Since AGMDP-Tri computes every parameter computed by C-AGMDP, except for the community partition (which takes half of the budget of C-AGMDP) we double the budget assigned to every other computation of AGMDP-Tri. Notice that these re-allocations give some advantages to C-AGMDP-D and AGMDP-Tri in their comparison with our C-AGMDP model, as they will be able to more accurately compute some of the parameters they have in common. We chose to allow this advantage considering that requiring a smaller number of computations is in fact a positive feature of a differentially private method, which should not be punished in the comparison. In C-AGMDP and C-AGMDP-D, ModDivisive is run with $w_s = 0.98$. Finally, in estimating the distribution of attribute-edge correlations (as discussed in Section 5.3), we set the maximum degree parameter $p$ to 100.

Figure 2 displays the behaviours of the three models in terms of community structure preservation, whereas Tables 3, 4 and 5 summarise their behaviours in terms of global structural properties and attribute-edge correlations on Petster, Facebook and Epinions, respectively. In what follows, we analyse these results from three different perspectives.

| $\varepsilon$ | Model     | $\rho_E$ | $\rho_\Delta$ | $\rho_c$ | $H_d$ | $H_{fc}$ | $\rho_a$ |
|--------------|-----------|----------|----------------|--------|------|--------|--------|
| 2.0          | C-AGMDP   | 0.56     | 0.09           | 0.43   | 0.42 | 0.39   | 0.14   |
|              | C-AGMDP-D | 0.22     | 0.55           | 0.69   | 0.30 | 0.44   | 0.23   |
|              | AGMDP-Tri | 0.25     | 0.09           | 0.25   | 0.23 | 0.30   | 0.17   |
| 3.0          | C-AGMDP   | 0.31     | 0.09           | 0.23   | 0.33 | 0.29   | 0.16   |
|              | C-AGMDP-D | 0.11     | 0.30           | 0.55   | 0.24 | 0.34   | 0.16   |
|              | AGMDP-Tri | 0.13     | 0.09           | 0.21   | 0.19 | 0.26   | 0.17   |
| 4.0          | C-AGMDP   | 0.19     | 0.08           | 0.20   | 0.29 | 0.26   | 0.13   |
|              | C-AGMDP-D | 0.06     | 0.29           | 0.54   | 0.22 | 0.32   | 0.14   |
|              | AGMDP-Tri | 0.09     | 0.08           | 0.19   | 0.19 | 0.25   | 0.17   |
| 5.0          | C-AGMDP   | 0.13     | 0.08           | 0.08   | 0.24 | 0.23   | 0.09   |
|              | C-AGMDP-D | 0.04     | 0.29           | 0.54   | 0.20 | 0.31   | 0.12   |
|              | AGMDP-Tri | 0.06     | 0.10           | 0.17   | 0.18 | 0.24   | 0.16   |

Table 3: Comparison of differentially private models on Petster.

**Community structure preservation.** In Figure 2, the four uppermost charts display the extent to which the community structures found by Louvain in the synthetic graphs generated by each differentially private model are similar to those detected in the corresponding original graphs. Two important features of the Louvain algorithm are shared by ModDivisive, the method used for obtaining community partitions in C-AGMDP and C-AGMDP-D. Both generate a community partition, and both operate by maximising modularity. In order to assess whether the community structures induced by our models in the synthetic graphs are also detectable by algorithms based on different criteria, we additionally obtained analogous results using the algorithm CESNA [42]. These results are shown in the lowermost four charts of Figure 2. Unlike Louvain, CESNA takes node at-
tributes into consideration for computing communities. However, CESNA tends to obtain substantially overlapping communities, whereas both C-AGMDP and C-AGMDP-D assume a partition. CESNA requires as a parameter the number of communities, which we set to 10.

Table 4: Comparison of differentially private models on Facebook.

| ε  | model       | $\rho_E$ | $\rho_\Delta$ | $\rho_c$ | $H_d$ | $H_{lc}$ | $\rho_a$ |
|----|-------------|----------|---------------|----------|-------|----------|---------|
| 2.0| C-AGMDP     | 0.10     | 0.01          | 0.59     | 0.25  | 0.54     | 0.13    |
|    | C-AGMDP-D   | 0.02     | 0.90          | 0.90     | 0.18  | 0.86     | 0.08    |
|    | AGMDP-Tri   | 0.06     | 0.13          | 0.58     | 0.31  | 0.59     | 0.19    |
| 3.0| C-AGMDP     | 0.05     | 0.01          | 0.51     | 0.22  | 0.47     | 0.09    |
|    | C-AGMDP-D   | 0.01     | 0.89          | 0.89     | 0.16  | 0.90     | 0.06    |
|    | AGMDP-Tri   | 0.03     | 0.15          | 0.59     | 0.31  | 0.59     | 0.19    |
| 4.0| C-AGMDP     | 0.03     | 0.01          | 0.50     | 0.21  | 0.46     | 0.07    |
|    | C-AGMDP-D   | 0.01     | 0.87          | 0.88     | 0.15  | 0.88     | 0.05    |
|    | AGMDP-Tri   | 0.02     | 0.15          | 0.60     | 0.32  | 0.19     | 0.08    |
| 5.0| C-AGMDP     | 0.02     | 0.01          | 0.48     | 0.21  | 0.43     | 0.06    |
|    | C-AGMDP-D   | 0.01     | 0.84          | 0.88     | 0.16  | 0.86     | 0.04    |
|    | AGMDP-Tri   | 0.01     | 0.15          | 0.60     | 0.33  | 0.59     | 0.19    |

From the analysis of these results, the most relevant observation is that, considering the communities detected by both Louvain and CESNA, the graphs sampled from our C-AGMDP model consistently rank as the ones whose community structure is most similar to that of the corresponding original graphs. In the particular cases of Petster and Facebook with the Louvain algorithm, the similarity values displayed by our C-AGMDP model are in some cases close to twice better than their counterparts for C-AGMDP-D, and considerably more than those for AGMDP-Tri. An additional important observation is that our model shows less variation for different amounts of noise, when compared to C-AGMDP-D and AGMDP-Tri.

Distributions of attribute-edge correlations. On each original graph, we compute the distributions of attribute-edge correlations in each community detected by CESNA. Then, we compute the equivalent distributions on each synthetic graph and compare it to that of the original graph in terms of $\rho_a$ (see right-most columns of Tables 3, 4 and 5).

From the analysis of these results, we can see that the synthetic graphs sampled from C-AGMDP and C-AGMDP-D, the two models that consider community structures, consistently outperform those sampled from AGMDP-Tri in terms of $\rho_a$. Another important observation is that the qualities, in terms of $\rho_a$, of synthetic graphs sampled from our C-AGMDP model and those sampled from its variant C-AGMDP-D are quite similar. This observation suggests that C-AGMDP can in some cases be seen as a meta-model, where several edge generation models can be used, e.g. CPGM and DCSBM in this experiment.

Global structural properties. From the analysis of Tables 3, 4 and 5, we can see that, as expected, our C-AGMDP model suffers a larger degradation than C-AGMDP-D and AGMDP-Tri in terms of the measures that depend on parameters for which the latter models were
allocated larger privacy budgets, most notably the numbers of edges. This problem was particularly serious on Petster, and became less important as the number of edges of the real graph increased. It is worth noting, however, that in the cases where a differentially private parameter underwent a post-processing, most notably regarding the number of triangles, our model obtained considerably better results. For example, the error rate dropped to 0.01 for the Facebook dataset. Also in the Facebook graph, despite the larger error rate in the number of edges, in some cases our model showed roughly the same or even better performance in preserving the degree sequence and clustering coefficients than the AGMDP-Tri model. Also note that, although C-AGMDP-D performs best in preserving the degree sequences, in general it failed to preserve the clustering coefficients.

| $\varepsilon$ | model       | $\rho_E$ | $\rho_\Delta$ | $\rho_c$ | $H_d$ | $H_{fc}$ | $\rho_a$ |
|-------------|-------------|----------|----------------|--------|------|---------|--------|
| 6.0         | C-AGMDP     | 0.08     | 0.20           | 0.53   | 0.23 | 0.21    | 0.09   |
|             | C-AGMDP-D   | 0.08     | 0.62           | 0.85   | 0.19 | 0.23    | 0.09   |
|             | AGMDP-Tri   | 0.13     | 0.31           | 0.16   | 0.13 | 0.27    | 0.13   |
| 7.0         | C-AGMDP     | 0.07     | 0.18           | 0.53   | 0.23 | 0.21    | 0.09   |
|             | C-AGMDP-D   | 0.07     | 0.69           | 0.91   | 0.20 | 0.24    | 0.09   |
|             | AGMDP-Tri   | 0.13     | 0.31           | 0.16   | 0.12 | 0.27    | 0.13   |
| 8.0         | C-AGMDP     | 0.05     | 0.16           | 0.54   | 0.20 | 0.20    | 0.09   |
|             | C-AGMDP-D   | 0.05     | 0.70           | 0.91   | 0.18 | 0.24    | 0.08   |
|             | AGMDP-Tri   | 0.14     | 0.32           | 0.18   | 0.11 | 0.26    | 0.13   |
| 9.0         | C-AGMDP     | 0.04     | 0.14           | 0.54   | 0.17 | 0.20    | 0.09   |
|             | C-AGMDP-D   | 0.04     | 0.75           | 0.93   | 0.18 | 0.25    | 0.08   |
|             | AGMDP-Tri   | 0.15     | 0.31           | 0.18   | 0.11 | 0.26    | 0.13   |

Table 5: Comparison of differentially private models on Epinions.

7 Conclusions

We have presented, to the best of our knowledge, the first community-preserving differentially private method for publishing synthetic attributed graphs. To devise this method, we developed C-AGM, a new community-preserving generative attributed graph model. We have equipped C-AGM with efficient parameter estimation and sampling methods, and have devised differentially private variants of the former. A comprehensive set of experiments on real-world datasets support the claim that our method is able to generate useful synthetic graphs satisfying a strong formal privacy guarantee. Our main direction for future work is to improve C-AGM by increasing the repertoire of community-related statistics captured by the model, and by equipping it with a new differentially private community partition method that integrates node attributes via a low-sensitivity objective function.
and/or differentially private maximum-likelihood estimation methods.

Acknowledgements: The work reported in this paper received funding from Luxembourg’s Fonds National de la Recherche (FNR), via grant C17/IS/11685812 (PrivDA).

References

[1] Regulation (EU) 2016/679 of the European parliament and of the council of 27 April 2016 on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing directive 95/46/ec (general data protection regulation). *OJ*, L 119:1–88, 4.5.2016.

[2] Lars Backstrom, Cynthia Dwork, and Jon M. Kleinberg. Wherefore art thou r3579x?: anonymized social networks, hidden patterns, and structural steganography. *Communications of the ACM*, 54(12):133–141, 2011.

[3] Jeremiah Blocki, Avrim Blum, Anupam Datta, and Or Sheffet. Differentially private data analysis of social networks via restricted sensitivity. In *Proc. 4th Innovations in Theoretical Computer Science (ITCS)*, pages 87–96. ACM Press, 2013.

[4] Vincent D Blondel, Jean-Loup Guillaume, Renaud Lambiotte, and Etienne Lefebvre. Fast unfolding of communities in large networks. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(10):P10008, 2008.

[5] Lauren E. Charles-Smith, Tera L. Reynolds, Mark A. Cameron, Mike Conway, Eric H. Y. Lau, Jennifer M. Olsen, Julie A. Pavlin, Mika Shigematsu, Laura C. Streichert, Katie J. Suda, and Courtney D. Corley. Using social media for actionable disease surveillance and outbreak management: A systematic literature review. *PLOS ONE*, 10(10):1–20, 10 2015.

[6] Fan Chung and Linyuan Lu. The average distances in random graphs with given expected degrees. *Proceedings of the National Academy of Sciences*, 99(25):15879–15882, 2002.

[7] Aaron Clauset, Cristopher Moore, and M. E. J. Newman. Hierarchical structure and the prediction of missing links in networks. *Nature*, 453:98–101, 2008.

[8] Cynthia Dwork. Differential privacy. In *Proc. 33rd International Colloquium on Automata, Languages and Programming (ICALP)*, volume 4052 of *Lecture Notes in Computer Science*, pages 1–12. Springer, 2006.

[9] Michael Hay, Chao Li, Gerome Miklau, and David D. Jensen. Accurate estimation of the degree distribution of private networks. In *Proc. 19th IEEE International Conference on Data Mining (ICDM)*, pages 169–178. IEEE Computer Society, 2009.
[10] Joseph J. Pfeiffer III, Timothy La Fond, Sebastián Moreno, and Jennifer Neville. Fast
generation of large scale social networks while incorporating transitive closures. In
Proc. 4th International Conference on Privacy, Security, Risk and Trust, (PASSAT),
pages 154–165. IEEE Computer Society, 2012.

[11] Joseph J. Pfeiffer III, Sebastián Moreno, Timothy La Fond, Jennifer Neville, and Brian
Gallagher. Attributed graph models: modeling network structure with correlated
attributes. In Proc. 23rd International World Wide Web Conference (WWW), pages
831–842. ACM Press, 2014.

[12] Zach Jorgensen, Ting Yu, and Graham Cormode. Publishing attributed social graphs
with formal privacy guarantees. In Proc. 2016 International Conference on Manage-
ment of Data (SIGMOD), pages 107–122. ACM Press, 2016.

[13] Kundan Kandhway and Joy Kuri. Using node centrality and optimal control to max-
imize information diffusion in social networks. IEEE Trans. Systems, Man, and Cy-
bernetics: Systems, 47(7):1099–1110, 2017.

[14] Brian Karrer and Mark EJ Newman. Stochastic blockmodels and community structure
in networks. Physical review. E, 83(1):016107, 2011.

[15] Vishesh Karwa, Sofya Raskhodnikova, Adam D. Smith, and Grigory Yaroslavtsev. Pri-
vate analysis of graph structure. ACM Transactions on Database Systems, 39(3):22:1–
22:33, 2014.

[16] Vishesh Karwa and Aleksandra B. Slavkovic. Differentially private graphical degree
sequences and synthetic graphs. In Proc. 2012 International Conference on Privacy in
Statistical Databases (PSD), volume 7556 of Lecture Notes in Computer Science,
pages 273–285. Springer, 2012.

[17] Daniel Kifer and Bing-Rong Lin. Towards an axiomatization of statistical privacy and
utility. In Proc. 29th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of
Database Systems (PODS), pages 147–158. ACM Press, 2010.

[18] Tamara G. Kolda, Ali Pinar, Todd D. Plantenga, and C. Seshadhri. A scalable gener-
ative graph model with community structure. SIAM J. Scientific Computing, 36(5),
2014.

[19] Jérôme Kunegis. KONECT: the koblenz network collection. In Proc. 22nd Interna-
tional World Wide Web Conference (WWW), pages 1343–1350. ACM Press, 2013.

[20] Jure Leskovec and Christos Faloutsos. Scalable modeling of real graphs using kronecker
multiplication. In Proc. 24th International Conference on Machine Learning (ICML),
pages 497–504. ACM Press, 2007.

[21] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset
collection. http://snap.stanford.edu/data, 2014.
[22] Nelly Marquetoux, Mark A. Stevenson, Peter Wilson, Anne Ridler, and Cord Heuer. Using social network analysis to inform disease control interventions. *Preventive Veterinary Medicine*, 126:94–104, 2016.

[23] Paolo Massa and Paolo Avesani. Trust-aware recommender systems. In *Proc. 2007 ACM Conference on Recommender Systems (RecSys)*, pages 17–24. ACM Press, 2007.

[24] Sjouke Mauw, Yunior Ramírez-Cruz, and Rolando Trujillo-Rasua. Robust active attacks on social graphs. *Data Mining and Knowledge Discovery*, 33(5):1357–1392, 2019.

[25] Frank McSherry. Privacy integrated queries: an extensible platform for privacy-preserving data analysis. *Communications of the ACM*, 53(9):89–97, 2010.

[26] Frank McSherry and Kunal Talwar. Mechanism design via differential privacy. In *Proc. 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 94–103. IEEE Computer Society, 2007.

[27] Darakhshan J. Mir and Rebecca N. Wright. A differentially private graph estimator. In *Proc. 2009 ICDM International Workshop on Privacy Aspects of Data Mining (ICDM)*, pages 122–129. IEEE Computer Society, 2009.

[28] Prateek Mittal, Charalampos Papamanthou, and Dawn Xiaodong Song. Preserving link privacy in social network based systems. In *Proc. 20th Annual Network and Distributed System Security Symposium (NDSS)*. The Internet Society, 2013.

[29] Arvind Narayanan and Vitaly Shmatikov. De-anonymizing social networks. In *Proc. 30th IEEE Symposium on Security and Privacy (S&P)*, pages 173–187. IEEE Computer Society, 2009.

[30] M. E. J. Newman and M. Girvan. Finding and evaluating community structure in networks. *Physical Review E*, 69(2):026113, 2004.

[31] Mark E. J. Newman. Community detection in networks: Modularity optimization and maximum likelihood are equivalent. *CoRR*, abs/1606.02319, 2016.

[32] Hiep H. Nguyen, Abdessamad Imine, and Michaël Rusinowitch. Detecting communities under differential privacy. In *Proc. 2016 ACM on Workshop on Privacy in the Electronic Society (WPES)*, pages 83–93. ACM Press, 2016.

[33] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In *Proc. 39th Annual ACM Symposium on Theory of Computing (STOC)*, pages 75–84. ACM Press, 2007.

[34] Liudmila Ostroumova Prokhorenkova and Alexey Tikhonov. Community detection through likelihood optimization: In search of a sound model. In *Proc. 30th World Wide Web Conference (WWW)*, pages 1498–1508. ACM Press, 2019.
Figure 3 shows the comparison of degree distributions (a–c) and distributions of local clustering coefficients (d–f) in terms of the complementary cumulative distribution functions (CCDF). Every degree or LCC value (x-axis) is mapped to the percentage of vertices having a larger value (y-axis).
Figure 3: Detailed comparison of edge generative models in terms of complementary cumulative distribution functions of degree and local clustering coefficient distributions.