Secondary Quantum Hamiltonian Reductions

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Abstract

Recently, it has been shown how to perform the quantum hamiltonian reduction in the case of general $\mathfrak{sl}(2)$ embeddings into Lie (super)algebras, and in the case of general $\mathfrak{osp}(1|2)$ embeddings into Lie superalgebras. In another development it has been shown that when $\mathcal{H}$ and $\mathcal{H}'$ are both subalgebras of a Lie algebra $\mathcal{G}$ with $\mathcal{H}' \subset \mathcal{H}$, then classically the $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebra can be obtained by performing a secondary hamiltonian reduction on $\mathcal{W}(\mathcal{G}, \mathcal{H}')$. In this paper we show that the corresponding statement is true also for quantum hamiltonian reduction when the simple roots of $\mathcal{H}'$ can be chosen as a subset of the simple roots of $\mathcal{H}$.

As an application, we show that the quantum secondary reductions provide a natural framework to study and explain the linearization of the $\mathcal{W}$ algebras, as well as a great number of new realizations of $\mathcal{W}$ algebras.

\texttt{ENSLAPP-A-507/95}
\texttt{hep-th-9503042}
February 1995

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1 Introduction

In recent years, we have seen a great activity in the area of extended conformal algebras, i.e. algebras that contain the Virasoro algebra as a subalgebra, see e.g. [1]. Examples of this are the well known Kac-Moody algebras and superconformal algebras, but also the more complicated \( \mathcal{W} \) algebras, introduced by Zamolodchikov in 1985, [2]. Apart from their inherent interest as mathematical objects, such algebras are of interest in many different fields of physics, such as the study of integrable hierarchies [3], string theory [4], Toda theories [5], quantum Hall effect [6], etc.

Different methods have been developed for the construction of extended conformal algebras. One method is the direct construction, essentially the solution, using algebraic computation, of a set of consistency equations for a prescribed set of fields, see e.g. [7]. Another method for constructing \( \mathcal{W} \)-algebras is the coset method, the generalization of the well-known coset construction in conformal field theory, for a review see e.g. [1]. However, one of the most powerful methods of constructing extended conformal algebras is the hamiltonian reduction. The starting point here is an affine Lie (super)algebra, on which one imposes a suitably chosen set of constraints. The reduced algebra is then an extended conformal algebra [8, 9, 10, 11]. The quantum version of this reduction has recently been done using BRST techniques [12, 13, 14, 15, 16, 17, 18].

It has recently been shown [19] that in certain cases one can impose extra constraints on a \( \mathcal{W} \) algebra obtained by hamiltonian reduction, and get yet another \( \mathcal{W} \) algebra; this procedure is called secondary hamiltonian reduction. In this paper we show how to quantize this procedure, thus performing the secondary quantum hamiltonian reduction.

As an application of the technique developed, we find the generalized quantum Miura transformation corresponding to the secondary reduction. This secondary quantum Miura transformation leads to a large number of new realizations of \( \mathcal{W} \) algebras.

As another application, we show that the secondary quantum hamiltonian reduction yields a general method of constructing linearizations of \( \mathcal{W} \) algebras; For a large class of \( \mathcal{W} \) algebras, we find that we can use the secondary quantum hamiltonian reduction to construct linearizations in a systematic way. Recall that a linearization of a \( \mathcal{W} \) algebra, introduced in [20] is the embedding of that \( \mathcal{W} \) algebra into a larger algebra which is equivalent to a linear algebra.

This paper is organized in the following way: In section 2, we briefly remind the reader of the primary hamiltonian reduction; section 2.1 is a description of the classical reduction, while section 2.2 gives a brief account of the quantum reduction. In section 2.3 we briefly recall the classical secondary reduction. These sections are included with the purpose of keeping the paper reasonably self-contained.

Sections 3 and 4 are the two main sections of the paper. In these sections we find the cohomology corresponding to the secondary quantum hamiltonian reduction. In section 3 we use the theory of spectral sequences to show that for triples \( \mathcal{G}, \mathcal{H}' \) and \( \mathcal{H} \) with \( \mathcal{H}' \subset \mathcal{H} \), and satisfying certain supplementary conditions, the secondary reduction of \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) gives as a result \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \). In section 4 we give a method to find explicitly the cohomology corresponding to \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \), i.e. to find expressions for the generators of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) in terms of the generators of \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \). Furthermore we describe the generalized quantum Miura transformation corresponding to the secondary quantum hamiltonian reduction, a transformation which gives us numerous new realizations of \( \mathcal{W} \) algebras. Section 4.4 is an example of the secondary quantum hamiltonian reduction.

The main results of these sections are collected in theorem 2 page 10, theorem 3 page 15, and theorem 4 page 17.

In section 5 we show how to use the technique, developed in the preceding sections, to linearize \( \mathcal{W} \) algebras. Using the secondary quantum hamiltonian reduction we show that we can embed many
\( \mathcal{W} \) algebras into larger algebras, which are equivalent to linear algebras.

Finally we have included two appendices, one on spectral sequences and one on the use of modified gradings in the hamiltonian reduction.

For all explicit calculations, we have used the OPE mathematica package of K. Thielemans [21].

2 Hamiltonian Reductions: a Reminder

2.1 The Classical Case

Let us briefly recall the construction of classical \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebras as they appear in the context of Hamiltonian reduction [9]. We start with a Lie algebra \( \mathcal{G} \) with generators \( t^a \) and inner product \( g^{ab} = \langle t^a t^b \rangle \). Furthermore we consider a regular subalgebra \( \mathcal{H} \subset \mathcal{G} \), and we denote the generators of the principal \( \mathfrak{s}(2) \) of \( \mathcal{H} \) by \( \{ M^+, M^0, M^- \} \). \( M_0 \) defines a grading \( \text{gr}(\cdot) \) of \( \mathcal{G} \):

\[
\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_- = \sum_m \mathcal{G}_m
\]

where \( m \) is the eigenvalue under the operator \( \text{ad}(M_0) \). We denote the affine Lie algebra corresponding to \( \mathcal{G} \) as \( \mathcal{G}^{(1)} \), and we write the affine current \( J \) in the form

\[
J(z) = J^a(z) \ t_a.
\]

\( t_a = g_{ab} t^b \) is an element of the dual algebra, \( g_{ab} \) is defined by \( g_{ab} g^{bc} = \delta^c_a \). We will use greek letters as indices for currents with negative grades, and barred greek letters for currents with non-negative grades, i.e. \( J^\alpha : t_\alpha \in \mathcal{G}_- \) and \( \bar{J}^\alpha : \bar{t}_\alpha \in \mathcal{G}_0 \cup \mathcal{G}_+ \). The constraints that we want to impose are

\[
(J^\alpha(z) - \chi^\alpha) = 0 \quad (t^\alpha \in \mathcal{G}_-)
\]

\( \chi^\alpha = \chi(J^\alpha) \) define the set of constraints and \( \mathcal{M}_- = \chi^\alpha t_\alpha \). These constraints are chosen such that they are first class. This means that the Poisson bracket of two constraints is weakly zero (i.e. one finds zero when imposing the constraints after computation of the Poisson bracket). We can then apply the general technique developed by Dirac to take care of these constraints:

The first class constraints generate gauge transformations, i.e. they are associated to a group of symmetries of the physical states of the theory. To eliminate this symmetry, one imposes new constraints (gauge fixing constraints) in such a way that the set of all constraints becomes second class (i.e. it is no more first class), and the matrix formed by the Poisson brackets of two constraints \( C_{ij}(t_1, t_2) = \{ \Phi_i(t_1), \Phi_j(t_2) \} \) is invertible. As these constraints must not interfere with the physical contents of the theory, one constructs new brackets (called Dirac brackets) with the help of the Poisson brackets and the inverse \( C^{ij} \) of \( C_{ij} \):

\[
\{ X(z), Y(w) \}_D = \{ X(z), Y(w) \} - \int dt_1 dt_2 \{ X(z), \Phi_i(t_1) \} C^{ij}(t_1, t_2) \{ \Phi_j(t_2), Y(w) \}
\]

where \( \Phi_i(t) \) are the (second class) constraints. These Dirac brackets are defined such that any quantity has (strongly) zero Dirac brackets with any of the constraints. In other words, we have decoupled the constraints from the other physical quantities.

In the case of the constraints (2.1), it can be shown that one can choose gauge-fixing constraints such that the remaining generators correspond to the highest weight components with respect to the embedded \( \mathfrak{s}(2) \): The gauge-fixed current \( J_{gf} \) is of the form \( J_{gf} = \chi^\alpha t_\alpha + J^t_i t_i \) with \( [M_+, t_i] = 0 \). As
the constraints are decoupled from the other physical quantities, it is clear that the Dirac bracket of two \( J^\alpha \)'s will close (polynomially) on the \( J^\alpha \)'s. These Dirac brackets realize the \( W \) algebra \( W(G, H) \).

To get a realization of the \( W \) generators as polynomials of the currents \( J^a \), one uses the gauge transformations generated by the first class constraints

\[
[J^a(w)]^g = J^a(w) + \int dz \epsilon_\alpha(z) \{ J^\alpha(z), J^a(w) \} + \frac{1}{2} \int dz dx \epsilon_\alpha(z) \epsilon_\beta(x) \{ J^\beta(x), \{ J^\alpha(z), J^a(w) \} \} + \ldots
\]

(2.3)

to fix \( [J(w)]^g \) to be of the form

\[
[J(z)]^g = \chi^\alpha t_\alpha + W^i(z) t_i
\]

(2.4)

where the \( W^i(z) \) are polynomials in the \( J^a \)'s and realize also the \( W \) algebra when using the Poisson brackets.

It can also be shown that one can realize the \( W \) algebra by using the zero grade generators only: starting with \( J(z) = \chi^\alpha t_\alpha + J^\alpha_0(z) t_\alpha \) where \([M_0, t_\alpha] = 0\) and doing the gauge transformations as above, one gets the \( W^i(z) \) as polynomials in the \( J^\alpha_0 \)'s. This transformation is called the (classical) Miura transformation.

Finally, let us note that the choice (2.1) of constraints is not unique. We introduce a new grading operator \( H = M_0 + U \), where \( U \) is an element of the Cartan subalgebra. \( H \) defines a grading which we write as

\[
\mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+ = \sum_n \mathcal{G}^n_+.
\]

In [22, 11] it was shown that the constraints obtained by replacing \( \mathcal{G}_- \) by \( \mathcal{G}^n_- \) in equation (2.1) leads to the same classical algebra, \( W(G, H) \), if \( U \) commutes with the \( s\ell(2) \) algebra and ”respects” the highest weights, i.e. satisfies the non-degeneracy condition

\[
\ker ad(M_+) \cap \mathcal{G}_- = 0
\]

2.2 Primary Quantum Hamiltonian Reduction

This section is intended as a brief recapitulation of the method developed in [16] of quantum hamiltonian reduction. We want to quantize the Hamiltonian reduction which we have presented in the previous section. To do this, we will use the BRST formalism, which is a standard procedure in the framework of constrained systems, see e.g. [12].

For each constraint we introduce a ghost-antighost pair \((c_\alpha, b^\alpha)\). Corresponding to these constraints we define a BRST operator \( s \):

\[
s(\phi)(w) = \oint_w dz j(z)\phi(w)
\]

\[
j(z) = (J^\alpha(z) - \chi^\alpha)c_\alpha(z) + \frac{1}{2} \epsilon^{\alpha\beta\gamma} b^\gamma(z)c_\beta(z)c_\alpha(z)
\]

(2.5)

The quantized \( W \) algebra \( W(G, H) \) is then

\[
W(G, H) = H^\alpha(\Omega; s)
\]

where \( \Omega \) is the operator product algebra generated by the affine currents, the ghosts and anti-ghosts, and their derivatives and normal ordered products. As usual when using BRST quantization, one defines a set of modified generators

\[
\hat{J}^\alpha(z) = s(b^\alpha)(z) + \chi^\alpha = J^\alpha(z) + f^{\alpha\beta\gamma} b^\gamma(z)c_\beta(z)
\]
and it turns out that it is useful to modify in a similar way the non-constrained generators:

\[ \hat{J}^\alpha(z) = J^\alpha(z) + f^{\alpha\beta\gamma}(b^\gamma c_\beta)_0(z) \]

where we have introduced the normal ordered product of fields \( A_j(z) \) \( (j = 1, 2, \ldots) \):

\[ (A_1 A_2)_0(w) = \oint \frac{dz}{z - w} A_1(z) A_2(w) \quad \text{and} \quad (A_n \ldots A_1)_0(w) = (A_n (A_{n-1} \ldots A_1)_0)_0(w) \]

(2.6)

With these definitions the space \( \Omega^\alpha \) generated by \( \hat{J}^\alpha \) and \( b^\alpha \) is actually a subcomplex (i.e. \( s(\Omega^\alpha) \subseteq \Omega^\alpha \)) with trivial cohomology \( H^n(\Omega^\alpha; s) = \delta_{n,0} C \). The space \( \Omega_{\text{red}} \) generated by the “hatted” non-constrained generators \( \hat{J}^\alpha \) and the ghosts \( c_\alpha \) is also a subcomplex, and in fact using the version of the Künneth formula which was shown in [16], one can show that we need not consider the trivial subcomplexes \( \Omega^\alpha \) since

\[ H^* (\Omega; s) \cong H^* (\Omega_{\text{red}} \otimes (\otimes_\alpha \Omega^\alpha); s) \cong H^* (\Omega_{\text{red}}; s) \otimes (\otimes_\alpha H^* (\Omega^\alpha; s)) \]

(2.7)

In order to actually calculate this cohomology and find explicit expressions for the generators of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \), one splits the BRST operator \( s \) into two nilpotent anticommuting operators \( s_0 \) and \( s_1 \) defined by the currents

\[ j_0(z) = -\chi^\alpha c_\alpha(z) \]
\[ j_1(z) = J^\alpha(z) c_\alpha(z) + \frac{1}{2} f^{\alpha\beta\gamma} b^\gamma(z) c_\beta(z) c_\alpha(z) \]

(2.8)

Corresponding to these two operators, we can define a bigrading of the complex \( \Omega \) as a combination of the ghostnumber and the grading \( gr(\cdot) \):

\[ J^\alpha : (m, -m) \]
\[ b^\alpha : (m, -m - 1) \]
\[ c_\alpha : (-m, m + 1) \]

(2.9)

where \( m \) is the grade of \( t_a \) respectively \( t_\alpha \). With this definition \( s_0 \) has bigrade \((1, 0)\) and \( s_1 \) has bigrade \((0, 1)\).

Using the technique of spectral sequences (see section [3] and appendix [A]), one can show that there is a vector space isomorphism

\[ H^n(\Omega_{\text{red}}; s) \cong H^n(\Omega_{\text{red}}; s_0) \cong \delta_{n,0} \Omega_{\text{hw}}, \]

where \( \Omega_{\text{hw}} \) is generated by the hatted generators that are highest weight under the embedded \( s\ell(2) \), i.e. \( \hat{J}^\alpha \in \Omega_{\text{hw}} \) iff \([M_+, t_\alpha] = 0\).

In order to actually find the elements in \( H^0(\Omega_{\text{red}}; s) \), i.e. the generators of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \), one can use the tic-tac-toe construction with \( \Omega_{\text{hw}} \) as starting point. For every highest weight generator \( \hat{J}^\alpha_{\text{hw}} \) we can construct the generator \( W^\alpha(z) \):

\[ W^\alpha(z) = \sum_{\ell=0}^m W^\alpha_\ell(z) \]
where \( m \) is the grade of \( \hat{J}_{hw}^\alpha, W_0^\alpha(z) = \hat{J}_{hw}^\alpha(z), \) and \( s_1(W_\ell^\alpha) + s_0(W_{\ell+1}^\alpha) = 0. \) We find that the bi-grade of \( W_\ell^\alpha \) is \( (m - \ell, \ell - m) \), and since \( s_1 \) vanishes on terms with bigrade \((0,0)\) we see that the sequence stops at \( \ell = m. \) It is easy to verify that \( s(W^\alpha) = 0. \)

In principle the operator product algebra of the \( W \)'s close only modulo \( s \)-exact terms. However, there are no elements in \( \Omega_{\text{red}} \) with negative ghostnumber, thus there are no \( s \)-exact terms with zero ghostnumber, and therefore the operator products of the \( W \)'s close exactly.

The operator product expansion (ope) preserves the grading, which implies that the operator product expansions of the zero grade part of the generators must give the same algebra as the ope’s of the full generators, i.e. the map \( W^\alpha = \sum_{\ell=0}^m W_\ell^\alpha \rightarrow W_m^\alpha \) must be an algebra isomorphism\(^1\). This defines a realization of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) in terms of the algebra \( \hat{G}_0^{(1)} \), the algebra generated by the “hatted” grade zero affine currents. This is known as the quantum Miura transformation, and can be used to define a free field realization of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) by using the Wakimoto construction \([23]\) to write the generators of \( \hat{G}_0^{(1)} \) in terms of free fields.

Just as in the classical hamiltonian reduction, we can modify the grading operator \( M_0 \) by adding a \( U(1) \) current obeying the non-degeneracy condition. In that case, the modified grading operator \( H = M_0 + U \) will lead to a modification of the BRST operator \((2.5)\). One finds, that the calculation of the cohomology leads to an equivalent but different (“twisted”) realization of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \).

### 2.3 Classical Secondary Reductions

First, we briefly recall the framework of secondary reductions as they appear in the classical case. We start with a \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra (defined as in section 1), with \( \mathcal{H}' \) a regular subalgebra of \( \mathcal{G} \). We suppose now that there is another regular subalgebra \( \mathcal{H} \) such that \( \mathcal{H}' \subset \mathcal{H} \). Since \( \mathcal{H}' \) is embedded in \( \mathcal{H} \), it is natural to wonder whether the \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra can be related to \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \).

In fact, considering the constraints associated to both \( \mathcal{W} \)-algebras, it is clear that we have to impose more constraints on \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) to get \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \); for instance, the number of primary fields (which is directly related to the number of constraints) is lower in \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) than in \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \). These (further) constraints will be imposed on \( W \) fields themselves, so that we will gauge a part of the \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra. In \([19]\), it has been proved:

**Theorem 1** Let \( \mathcal{G} = sl(N) \) and let \( \mathcal{H}' \) and \( \mathcal{H} \) be two regular subalgebras of \( \mathcal{G} \) such that

\[
\mathcal{H}' \subset \mathcal{H} \tag{2.10}
\]

Then, there is a set of constraints on the \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra such that the (associated) Hamiltonian reduction of this algebra leads to the \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra. We will represent this secondary reduction as

\[
\mathcal{W}(\mathcal{G}, \mathcal{H}') \rightarrow \mathcal{W}(\mathcal{G}, \mathcal{H}) \tag{2.11}
\]

The proof of this theorem relies on a general property of the Dirac brackets, which can be stated as follows:

We start with a Hamiltonian theory on which we impose constraints. Instead of considering directly the complete set of second class constraints, we can divide this set into several subsets (of second class constraints) and compute the Dirac brackets at each steps (using the Dirac bracket of the previous steps as initial Poisson brackets). Then the last Dirac brackets do not depend on the partition of the second class constraints set we have used.

\(^1\) actually this argument shows only that the map is an algebra homomorphism, but one can prove that the map is also injective.
Thus, coming back to our $W$-algebras, it is sufficient to find a gauge fixing for the $W(G, H')$ algebra such that the corresponding set of second class constraints is embedded into the set of second class constraints for $W(G, H)$ as soon as $H' \subset H$. Indeed, with such an embedding, it is clear that the constraints one will impose on the $W(G, H')$ generators will just be the constraints related to $H$ that are not in the subset associated to $H'$. Such a gauge has been explicitly constructed in [19] for $G = \mathfrak{sl}(N)$. Because of the generality of the property of Dirac brackets, and considering the construction of orthogonal and symplectic algebras from the (folding of) unitary ones [24], it is clear that the theorem is also true for the other classical Lie algebras.

We present below a quantization of the secondary reduction using the BRST formalism. Note that the BRST operator involves only first class constraints, so that the quantization is not straightforward: in the classical case, we have to embed the sets of second class constraints one into the other, while in the quantized version it is the sets of first class constraints that we will embed.

3 Quantum Secondary Reductions: Algebra Isomorphism

In order to show that $W(G, H)$ can be obtained from a secondary hamiltonian reduction of $W(G, H')$, we will use the theory of spectral sequences. For a good introduction see e.g. [25]; in appendix A we give a brief description of some main points in the theory.

Assume that we have a Lie algebra $G$ and two regular subalgebras $H'$ and $H$ with $H' \subset H$, as in the previous section. The principal $\mathfrak{sl}(2)$ subalgebra of $H'$ is denoted by $\{M'_-, M'_0, M'_+\}$, while the principal $\mathfrak{sl}(2)$ subalgebra of $H$ is $\{M_-, M_0, M_+\}$. The eigenvalues of the operator $\text{ad}(M_0)$ defines a natural grading of $G$:

$$G = G_- + G_0 + G_+ = \sum_m G_m. \quad (3.1)$$

A second grading is defined by $M'_0$, but as described in appendix B we can use the more general, but equivalent, grading operator $\text{ad}(H') = \text{ad}(M'_0 + U)$. We write the corresponding grading as:

$$G = G'_- + G'_0 + G'_+ = \sum_n G'_n. \quad (3.2)$$

We assume the gradings to be integer, and call $H$ the corresponding grading operator\footnote{For the algebras which we consider, it is always possible to choose an integer grading.}. We wish to constrain the negative grade parts of the two algebras, $G'_-$ and $G_-$ respectively. To each generator $t^\alpha \in G_-$ corresponds one of the first class constraints of the type $\phi^\alpha(z) = J^\alpha(z) - \chi^\alpha = 0$ where the $\chi^\alpha$ are constants. We denote by $\Phi$ the set of these first class constraints, and similarly $\Phi'$ denotes the set of constraints corresponding to $G'_-$. We assume that we can choose the constraints in such a way that $\Phi' \subset \Phi$, and we note that this implies that the following two conditions are satisfied:

1) the set of simple roots of $H'$ can be chosen to be a subset of the set of simple roots of $H$, and

2) $G'_- \subset G_-.$

The classification of triples $G$, $H'$, and $H$ satisfying these conditions are given in appendix B. Note that although the “usual” constraints (defined by the grading $\text{ad}(M_0)$) obeying $\Phi' \subset \Phi$ are very few, the use of modified gradings as described in the appendix gives us a large class of triples that satisfy $\Phi' \subset \Phi$. 
Let us introduce the notation for the indices:

\[
\begin{align*}
t_A & \in \mathcal{G}_- & t_A & \in \mathcal{G}_0 \cup \mathcal{G}_+ \\
t_a & \in \mathcal{G}'_- & t_a & \in \mathcal{G}'_0 \cup \mathcal{G}'_+ \\
t_\alpha & \in \mathcal{G}_- \setminus \mathcal{G}'_- & t_\alpha & \in \mathcal{G}_0 \setminus \mathcal{G}'_0 \\
\end{align*}
\]

(3.3)

Note that the generators \( t_\alpha \) must have grade zero with respect to \( H' \): \( t_\alpha \in \mathcal{G}'_0 \) (if \( t_\alpha \in \mathcal{G}'_+ \) then we can find a \( t_\alpha \in \mathcal{G}'_- \), corresponding to a generator in \( \mathcal{G}'_0 \setminus \mathcal{G}'_+ \); but this is in contradiction with the condition 2) above). Note also that \( t_\alpha \) is a highest weight generator under the embedding of \( \{ M'_-, M'_0, M'_+ \} \). To show this, assume a root basis, and let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) denote the simple roots of \( \mathcal{H}' \), and let \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m \) denote the simple roots of \( \mathcal{H} \). We can write \( M'_+ = \sum a^i t_{\alpha_i} \).

On the other hand, since \( t_\alpha \) has grade zero under \( H' \) we find that we can write \( \alpha = -\sum n_j \beta_j \). This shows that \( [M'_+, t_\alpha] = 0 \).

We can write the constraints in the form:

\[
\begin{align*}
\phi^a(z) &= J^a(z) - \chi^a = 0, & \phi^a & \in \Phi' \\
\phi^A(z) &= J^A(z) - \chi^A = 0, & \phi^A & \in \Phi \\
\end{align*}
\]

(3.4) (3.5)

where \( M'_+ = \chi^a t_\alpha \) and \( M'_- = \chi^A t_A \).

Corresponding to the constraints (3.4) we introduce ghosts \( c_a \) and anti-ghosts \( b^a \), and we define the BRST current \( j' \) by (see eq. (2.5))

\[
\dot{j'}(z) = (J^a(z) - \chi^a)c_a(z) + \frac{1}{2} f^{ab} b^c(z)c_b(z)c_a(z).
\]

Similarly we introduce ghosts \( c_A \) and anti-ghosts \( b^A \) corresponding to (3.3). The set of ghosts \( c_a \) is a subset of the set \( c_A \) and \( b^a \) is a subset of \( b^A \); in fact the set \( c_A \) is the union of the set \( c_a \) and the set \( c_A \), and the set \( b^A \) is the union of the set \( b^a \) and the set \( b^a \). The BRST current \( j \) is defined by

\[
j(z) = (J^A(z) - \chi^A)c_A(z) + \frac{1}{2} f^{AB} b^C(z)c_B(z)c_A(z).
\]

We define the current \( j'' \) by \( j = j' + j'' \), and we find

\[
j''(z) = j(z) - j'(z) = (J^a(z) - \chi^a)c_a(z) + \frac{1}{2} f^{AB} b^C(z)c_B(z)c_A(z) - f^{ab} b^c(z)c_b(z)c_a(z).
\]

Corresponding to the currents \( j, j' \), and \( j'' \) we define operators \( s, s' \), and \( s'' \) by

\[
s \phi(w) = \oint_w dz j(z) \phi(w),
\]

and similarly for \( s' \) and \( s'' \).

Using \( t_a \in \mathcal{G}'_0 \) and \( t_a \in \mathcal{G}_- \) we can show that terms of the form \( f^{ab} b^c c_a c_a, f^{a\beta} c_b c_{\beta} c_a \), and \( f^{a\beta} c_b c_{\beta} c_{\alpha} \) vanish. This means that we can write:

\[
\dot{j''}(z) = (J^a(z) - \chi^a)c_a(z) + \frac{1}{2} f^{ab} b^c(z)c_b(z)c_a(z) + f^{a\beta} c^{\beta} c_{\beta} c_{\alpha}(z) + f^{a\beta} c_b c_{\beta} c_{\alpha}(z)
\]

\[
= (J^a(z) - \chi^a)c_a(z) + \frac{1}{2} f^{a\beta} b^c(z)c_{\beta} c_{\alpha}(z)
\]

(3.6)
where in the last line $\tilde{J}^\alpha$ is defined by $\tilde{J}^\alpha(z) = s'(b^\alpha)(z) + \chi^\alpha = J^\alpha(z) + f^{\alpha\beta} c^\beta(z) c_\alpha(z)$. Note that since $J^\alpha$ is a highest weight generator with grade zero under the grading operator $H'$, $\tilde{J}^\alpha$ is actually a generator $W^\alpha$ of the algebra $W(G, H')$, so we can alternatively write

$$j''(z) = (W^\alpha(z) - \chi^\alpha)c_\alpha(z) + \frac{1}{2} f^{\alpha\beta\gamma} b^\gamma(z) c_\beta(z) c_\alpha(z)$$

(3.7)

Let $A$ denote the algebra generated by the currents as well as their derivatives and normal ordered products. $\Lambda'$ is the algebra generated by the ghosts $c_a$ and the anti-ghosts $b_a$ (and their derivatives and normal ordered products), $\Lambda''$ is generated by $c_\alpha$ and $b^\alpha$, and $\Lambda$ is generated by $c_A$ and $b^A$. Note that $\Lambda = \Lambda' \otimes \Lambda''$. The algebras $A \otimes \Lambda'$ and $A \otimes \Lambda$ are graded by ghost numbers, and we know $[12, 13, 16]$ that

$$W(G, H) \cong H^0(A \otimes \Lambda; s),$$
$$W(G, H') \cong H^0(A \otimes \Lambda'; s').$$

(3.8)

We can define a bigrading on the algebra $\Omega = A \otimes \Lambda : \Omega = \sum_{p,q} \Omega_{p,q}$ such that $s'$ has bigrading $(1, 0)$ and $s''$ has bigrading $(0, 1)$, namely:

$$J : (0, 0)$$
$$c_a : (1, 0)$$
$$c_\alpha : (0, 1)$$
$$b^a : (-1, 0)$$
$$b^\alpha : (0, -1)$$

(3.9)

Be careful that this bigrading is not the bigrading used in the previous section: it is based on two ghost numbers, while for primary reductions, the bigrading is based on the gradation of $G$, and on one ghost number (see eq. (2.9)). Define

$$\tilde{J}^A = s(b^A) + \chi^A = J^A + f^{AB} c^B c_B,$$
$$\tilde{J}^\alpha = J^\alpha + f^{\alpha\beta} c^\beta c_B,$$

$J^A_{\text{ghost}} \equiv f^{AB} c^B c_B$ is the ghost realization of the constrained part of the algebra. We will use as basis of $\Omega$ the set of “hatted” currents and the ghosts and anti-ghosts $\{\tilde{J}, c, b\}$. For each index $A$ the algebra $\Omega^A$ generated by $\tilde{J}^A$ and $b^A$ is an $s$-subcomplex with trivial cohomology $[16]$

$$H^n(\Omega^A; s) \cong \delta_{n,0} C.$$

Define $\tilde{\Omega}_{\text{red}}$ to be the algebra generated by $\{\tilde{J}^A, c_A\}$. As in eq. (2.7) we find that

$$H^*(\Omega; s) \cong H^*\left(\bigotimes_A \Omega^A; s\right) \cong H^*(\tilde{\Omega}_{\text{red}}; s) \otimes \left(\bigotimes_A H^*(\Omega^A; s)\right)$$
$$\cong H^*(\tilde{\Omega}_{\text{red}}; s) \otimes \left(\bigotimes_A C\right) \cong H^*(\tilde{\Omega}_{\text{red}}; s)$$

(3.10)

i.e. we can reduce the problem to finding the cohomology of $\tilde{\Omega}_{\text{red}}$, ignoring the trivial subcomplexes $\Omega^A$. In our case it will turn out to be convenient to perform this reduction only partly, in the sense that we will use the Künneth formula only to extract the subcomplexes $\Omega^a$. We will therefore define
\( \Omega_{\text{red}} \) to be the subcomplex generated by \( \{ \bar{J}^a, c_a \} \) and \( \Lambda'' \). The full complex \( \Omega \) can be written in the form

\[
\Omega \cong \Omega_{\text{red}} \otimes \left( \bigotimes_a \Omega^a \right)
\]

and we use the Künneth formula to find

\[
H^n(\Omega; s) \cong H^n(\Omega_{\text{red}}; s)
\]

(3.11)

Now we make a change of basis. As new basis we choose the currents \( \bar{J}^a = J^a + f^{ab} c_b \), the ghosts \( c_a \) and \( c_a \), and and anti-ghosts \( b^a \). We denote the space generated by \( \bar{J}^a \) and \( c_a \) by \( \Gamma \), so we have \( \Omega_{\text{red}} = \Gamma \otimes \Lambda'' \). Note that the cohomology of \( \Gamma \) with respect to the operator \( s' \) is the \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \) algebra:

\[
H^n(\Gamma; s') \cong \delta_{\alpha,0} \mathcal{W}(\mathcal{G}, \mathcal{H}')
\]

Note also that we have \( (\Gamma \otimes \Lambda'')^{p,q} = \Gamma^{p,0} \otimes (\Lambda'')^{0,q} \).

We can now consider the spectral sequence corresponding to the double complex \( (\Omega_{\text{red}}; s'; s'') \). The spectral sequence is a sequence of complexes \( (E^r_p; s_r) \), such that

\[
E^0_{p,q} = (\Omega_{\text{red}})^{p,q}
\]

\[
E^r_{p,q} = H^{p,q}(E^r; s_r) = \frac{E^p_{r,q} \cap \ker (s_r)}{E^p_{r,q} \cap \im (s_r)}
\]

(3.12)

where \( s_r \) is a nilpotent operator of bigrade \( (1-r, r) \), \( s_0 = s' \) and \( s_1 = [s''] \). The operators \( s_r \) for \( r \geq 2 \) are defined in appendix A. The notation \( s_1 = [s''] \), is to be interpreted as \( s_1([x]) = [s''(x)] \) of a given \( [x] \in E_1 \). This is well-defined because

\[
[s''(x + s'(y))] = [s''(x) + s''(s'(y))] = [s''(x) - s'(s''(y))] = [s''(x)].
\]

It is now possible to show that if the spectral sequence collapses, i.e. if there exists \( R \) such that \( E_r = E_R \) for \( r \geq R \), then we have

\[
E^\infty_{p,q} \cong F^q H^{p+q} / F^{q+1} H^{p+q}
\]

(3.13)

where \( E_\infty = E_R \) and \( F^q H \) is a filtration on the cohomology \( H(\Omega_{\text{red}}; s) \) defined by

\[
F^q H^{p+q} = H^{p+q}(\bigoplus_{i \geq 0} (\Omega_{\text{red}})^{p-i,q+i}; s) = \frac{(\bigoplus_{i \geq 0} (\Omega_{\text{red}})^{p-i,q+i}) \cap \ker s}{(\bigoplus_{i \geq 0} (\Omega_{\text{red}})^{p-i,q+i}) \cap \im s}
\]

(3.14)

thus we can in principle reconstruct the cohomology \( H(\Omega_{\text{red}}; s) \) from the spectral sequence, on the condition that we can reconstruct \( H(\Omega_{\text{red}}; s) \) from the quotient spaces \( F^q H^{p+q} / F^{q+1} H^{p+q} \).

The first element in the spectral sequence is

\[
E^0_{p,q} = (\Omega_{\text{red}})^{p,q} = (\Gamma \otimes \Lambda'')^{p,q}.
\]

For the second element we find

\[
E^1_{p,q} = H^{p,q}(E_0; s_0) \cong H^{p}(\Gamma; s') \otimes (\Lambda'')^{0,q} \cong \delta_{p,0} \mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes (\Lambda'')^{0,q}.
\]

(3.15)

---

3 Actually this condition is sufficient but not necessary; it can in fact be relaxed considerably, see appendix A.
The third element in the spectral sequence is \( E_2 = H(E_1, s_1) \). we find

\[
E_2^{p,q} = H^{p,q}(E_1; s_1) = \delta_{p,0} H^{p,q}(\mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes \Lambda''; [s''])
\]  

(3.16)

The spectral sequence collapses here, i.e. \([s'']\) is the last non-trivial operator in the sequence. In fact \( E_2^{p,q} \) is nontrivial only for \( p = 0 \), and since \( s_r \) has bigrade \((1 - r, r)\) it is clear that \( s_r \) is trivial for \( r \geq 2 \). We conclude that \( E_r = E_2 \) for any \( r \geq 2 \), and so \( E_\infty = E_2 \). Note that from equation (3.7) it follows that if we restrict \( s'' \) to \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes \Lambda'' \) then it maps into \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes \Lambda'' \), which means that we can replace \([s'']\) by \( s'' \) in equation (3.16).

From equation (3.13) it follows that we have

\[
F^q H^{p+q}/F^{q+1} H^{p+q} \cong E_2^{p,q} \cong \delta_{p,0} H^{p,q}(\mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes \Lambda''; s''),
\]

where \( F^q H^{p+q} \) is defined in equation (3.14).

\((\Omega_{red})^{p,q}\) is trivial for \( p < 0 \), and therefore \( \bigoplus_{i>0} (\Omega_{red})^{p-i,q+i} \) is trivial for \( p < 0 \). Using lemma 2 of appendix A one can verify that this implies that \( H(\mathcal{A} \otimes \Lambda; s) \cong E_2 \), and that this isomorphism is in fact an algebra isomorphism.

Let us collect the results of this section in the following

**Theorem 2** Given two \( \mathcal{W} \) algebras \( \mathcal{W}' = \mathcal{W}(\mathcal{G}, \mathcal{H}') \) and \( \mathcal{W} = \mathcal{W}(\mathcal{G}, \mathcal{H}) \) with \( \mathcal{H}' \subset \mathcal{H} \). If we can find sets of first class constraints \( \Phi' \) and \( \Phi \) (where \( \mathcal{W}' \) is the result of imposing the set of constraints \( \Phi' \) on \( \mathcal{G}^{(1)} \) and \( \mathcal{W} \) is the result of imposing \( \Phi \) on \( \mathcal{G}^{(1)} \) ) such that \( \Phi' \subset \Phi \), then:

1) It is possible to perform a secondary quantum hamiltonian reduction on \( \mathcal{W}' \). This secondary reduction consists of imposing a set \( \Phi'' \) of first class constraints on \( \mathcal{W}' \). There is a simple one-to-one correspondence between the constraints \( \Phi'' \) imposed on \( \mathcal{W}' \), and the “missing” constraints \( \Phi \setminus \Phi' \).

2) Let \( \mathcal{A} \) be the algebra generated by currents in \( \mathcal{G}^{(1)} \) and their derivatives and normal ordered products, and let \( \Lambda' \) be the algebra generated by the ghosts and anti-ghosts corresponding to the constraints \( \Phi' \). If we denote by \( s' \) and \( s \) the BRST operators corresponding to the quantum hamiltonian reduction leading to \( \mathcal{W}' \) and \( \mathcal{W} \) respectively, then the BRST operator that corresponds to the secondary quantum hamiltonian reduction of \( \mathcal{W}' \) is \([s - s'] = [s'']\). Considering \( \mathcal{W}' \) as the cohomology \( H^0(\mathcal{A} \otimes \Lambda'; s') \), \([s'']\) on an element \([x] \in \mathcal{W}' \) is defined by \([s''([x]) = [s''(x)]\].

3) Let \( \Lambda \) and \( \Lambda'' \) be the algebras generated by the ghosts corresponding to \( \Phi \) and \( \Phi'' \) respectively. The result of the secondary hamiltonian reduction of \( \mathcal{W}' \) is

\[
H^0(\mathcal{W}(\mathcal{G}, \mathcal{H}') \otimes \Lambda''; [s'']) \cong H^0(\mathcal{W}(\mathcal{G}, \mathcal{H}); s'') \cong H^0(\mathcal{A} \otimes \Lambda; s) \cong \mathcal{W}(\mathcal{G}, \mathcal{H}).
\]

4 Quantum Secondary Reduction: Direct Calculation

In section 2.2 we explained briefly the primary quantum hamiltonian reduction of the affine Lie algebra \( \mathcal{G}^{(1)} \) that results in the \( \mathcal{W} \) algebra \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \). Let us recall some of the main points of the procedure:

1) The cohomology \( H^0(\mathcal{A} \otimes \Lambda; s) \equiv H^0(\Omega; s) \) is isomorphic to the cohomology \( H^0(\Omega_{red}; s) \), where \( \Omega_{red} \) is the space generated by the “hatted” unconstrained generators \( \hat{J}^\alpha \) and the ghosts \( c_\alpha \).
2) There is a vector space isomorphism between the space $\Omega_{hw}$, generated by the hatted highest weight generators, and $W(G, H)$.

3) We can use the tic-tac-toe construction to construct explicit realizations of the generators of $W$ in terms of the hatted unconstrained generators. The starting points for the tic-tac-toe construction are the elements of $\Omega_{hw}$.

4) We can show that there exists an algebra isomorphism between the algebra $W(G, H)$, and the algebra generated by the zero-grade part of the $W$-generators as constructed by the tic-tac-toe method. This is the generalized quantum Miura transformation.

In this section we will take the corresponding steps for the secondary quantum hamiltonian reduction. Among the consequences will be the secondary quantum Miura transformation.

Note that we have already found the BRST cohomology $H^0(W' \otimes \Lambda''; s'')$ to be identical to the algebra $W(G, H)$; the aim of this section is to construct concrete realizations of $W(G, H')$-generators from the generators of $W(G, H')$.

### 4.1 Isomorphism between $W(G, H)$ and $H_0(\Gamma_{red}, s)$

The reduction of the $W'$ algebra is defined in terms of the grading $(H - H')$. The fact that this is actually a well-defined grading of the algebra follows from the fact that the simple roots of $H'$ has grade 1 both under $H$ and $H'$, which implies that $[M'_+, (H - H')] = 0$ and therefore $\tilde{H} - \tilde{H}'$ is a generator of $W'$. The generators to be constrained are $W^\alpha = \tilde{J}^\alpha$, which are just the generators with negative grade.

Define $\Gamma = W(G, H') \otimes \Lambda''$. Just as in the case of the primary reduction, we define “hatted” constrained generators by

$$\hat{W}^\alpha(z) = s''(b^\alpha)(z) + \chi^\alpha$$

and we find that in fact $\hat{W}^\alpha = \hat{J}^\alpha$. For each $\alpha$, define $\Gamma^\alpha$ to be space generated by $\hat{W}^\alpha$ and $b^\alpha$. We note that $\Gamma^\alpha$ is a subcomplex with trivial cohomology:

$$H^n(\Gamma^\alpha; s'') = \delta_{n,0} C$$

In the primary quantum hamiltonian reduction, we saw in section 2.2 that it was possible to split the complex $\Omega$ into a product of subcomplexes

$$\Omega = \Omega_{red} \otimes (\otimes_{\alpha} \Omega^\alpha),$$

where the subcomplex $\Omega_{red}$ was generated by the “hatted” unconstrained generators $\hat{J}^\alpha$ and the ghosts $c_\alpha$. We want to show that we can split the complex $\Gamma$ in a similar way:

**Property 1** It is possible to define a subcomplex $\Gamma_{red}$ generated by modified unconstrained generators $\hat{W}^A$ and ghosts $c_\alpha$, such that $\Gamma_{red}$ is a subcomplex (i.e. $s''(\Gamma_{red}) \subset \Gamma_{red}$).

We will do the proof by a double induction, using the conformal dimension and the $(H - H')$-grade of the generators as induction parameters.

We consider the “twisted” algebra, i.e. the algebra where the conformal dimensions of generators without derivatives are given by the $H'$-grade + 1, and the $c$ ghosts have conformal dimension 0 (alternatively we can think of it as simply a modified grade, defined by the $H'$-grade + 1 + the
degree of derivatives). In this case the conformal dimensions of all the constrained generators is 1 and the \((H - H')\)-grade of the constrained generators is less than zero.

We will need a lemma:

**Lemma 1** Consider an unconstrained generator \(W^\alpha\) with conformal dimension \(h\) and grade \(n\): all unconstrained generators occurring in \(s''(W^\alpha)\) has either conformal dimension strictly less than \(h\) or conformal dimension \(h\) and grade less than \(n\).

In the expression \(s''(W^\alpha)\), all generators are the result of OPEs between a constrained generator in \(j''\), and \(W^\alpha\). Thus all monomials\(^4\) of generators occurring in \(s''(W^\alpha)\) must have conformal dimension \(h\) and \((H - H')\)-grade less than \(n\). We can write:

\[
s''(W^\alpha) = P_\beta(c)W^\beta + Q_{\alpha\gamma}(c)W^\alpha W^\gamma + \cdots,
\]

where \(\cdots\) denote terms that are of higher order in the generators (constrained or unconstrained). The conformal dimension of \(P_\beta(c)\) is greater than or equal to zero, so the conformal dimension of \(W^\beta\) is less than or equal to \(h\), and \((H - H')\)-grade less than \(n\). Similarly, the conformal dimension of \(W^\gamma\) is less than or equal to \(h - 1\), and we see that even stronger inequalities hold for the higher order terms. This proves the lemma.

Assume that we have already found hatted generators for all generators with conformal dimension less than \(h\), and define \(\Gamma_{\text{red}}^{h-1}\) to be the space generated by these hatted generators and the \(c\)'s. Assume that \(W^\alpha\) is any generator with conformal dimension \(h\) and grade 0, we will show that we can define \(\hat{W}^\alpha\) such that \(s''(\hat{W}^\alpha)\) is in \(\Gamma_{\text{red}}^{h-1}\). Consider \(s''(W^\alpha)\). According to the lemma, all unconstrained generators occurring in \(s''(W^\alpha)\) must have conformal dimension \(h\) and \((H - H')\)-grade less than \(n\). We can therefore write

\[
s''(W^\alpha) = \sum_{i,j} A_{ij} B_j, \quad A_{ij} \in B = \otimes_\beta \Gamma^\beta, \quad B_j \in \Gamma_{\text{red}}^{h-1}
\]

where the \(B_j\)'s are chosen to be linearly independent. Since \(j''\) is linear in the constrained currents, each of the terms \(A_{ij}\) are monomials in the constrained currents, the \(W^\alpha\)'s. Let us consider only those terms that have the highest grade, considered as monomials in \(W^\alpha\).

\[
s''(W^\alpha) = \sum_{i,j} A_{ij}^m B_j + \text{ lower orders terms }, \quad A_{ij}^m \text{ is order } m \text{ in } W^\alpha
\]

Now apply \(s''\) once again. We get:

\[
0 = \sum_{i,j} \left( s''(A_{ij}^m) B_j \pm A_{ij}^m s''(B_j) \right)
\]

We know that \(s''(B_j) \in \Gamma_{\text{red}}^{h-1}\), and \(s''(A_{ij}^m) \in B\). We also know that \(s''(A_{ij}^m)\) is of order \(m + 1\) in the \(W^\alpha\)'s, and these are the only possible terms of order \(m + 1\); and since the expression must vanish order by order in the \(W^\alpha\)'s, we find

\[
0 = \sum_{i,j} s''(A_{ij}^m) B_j
\]

Since the \(B_j\)'s are linearly independent we find that

\[
0 = \sum_i s''(A_{ij}^m)
\]

\(^4\)We use the word “monomial”, even though what we have is actually a normal-ordered product.
Now we use the fact that $B$ has trivial cohomology: since $\sum_i A^m_{ij}$ is in the kernel of $s''$ it must be in the image of $s''$, so we can find $X_j$ (of grade $m-1$ in the $W^{\alpha}$’s) such that $s''(X_j) = \sum_i A^m_{ij}$. Define

$$W_{(1)} = W - \sum_j X_j B_j.$$ 

We find that:

$$s''(W_{(1)}) = \sum_{i,j} A^m_{ij} B_j + \text{ lower orders terms} - \sum_j (s''(X_j)B_j \pm X_j s''(B_j))$$

$$= \sum_{i,j} A^m_{ij} B_j + \text{ lower order terms} - \sum_{i,j} A^m_{ij} B_j \mp \sum_j X_j s''(B_j)$$

$$= \text{ lower order terms} \mp \sum_j X_j s''(B_j)$$

(the $\pm$ depends on the Grassman parity of $X_j$). All these terms are of order at most $m-1$ in the $W^{\alpha}$’s. By induction we see that we can define $\hat{W}^{\alpha}$ such that $s''(\hat{W}^{\alpha})$ is a polynomial of degree 0 in the constrained currents.

We want to show that in fact no b’s appear in $s''(\hat{W}^{\alpha})$ either. Actually this is quite simple: write

$$s''(\hat{W}^{\alpha}) = B + \sum_{\alpha} B_\alpha b^\alpha + \sum_{\alpha,\beta} B_{\alpha \beta} b^\alpha b^\beta + \cdots.$$ 

Apply $s''$ again to get

$$0 = s''(B) + \sum_{\alpha} s''(B_\alpha) b^\alpha \pm B_\alpha (\hat{\mathcal{J}}^\alpha - \chi^\alpha) + \cdots.$$ 

Since $s''(B_\alpha)$ does not contain any constrained currents, we must have $0 = \sum_{\alpha} B_\alpha \hat{\mathcal{J}}^\alpha$, but this can only be true if $B_\alpha = 0$ for all $\alpha$. We see that indeed $s''(\hat{W}^{\alpha}) \in \Gamma_{\text{red}}$.

Now assume that we have found hatted generators for all generators with conformal dimension less than $h$, and with conformal dimension $h$ and grade less than $n$, and define $\Gamma^h_{\text{red},n-1}$ to be the space generated by these hatted generators and the $c$’s. Assume that $\hat{W}^{\alpha}$ is any generator with conformal dimension $h$ and grade $n$, we want to show that we can define $\hat{W}^{\alpha}$ such that $s''(\hat{W}^{\alpha}) \in \Gamma^h_{\text{red},n-1}$. Consider $s''(W^{\alpha})$. According to the lemma, any unconstrained generator $W^{\bar{\beta}}$ that occurs in $s''(\hat{W}^{\alpha})$ has either have conformal dimension less than $h$ or conformal dimension $h$ and grade less than $n$. We can therefore write

$$s''(W^{\alpha}) = \sum_{i,j} A_{ij} B_j, \quad A_{ij} \in \mathcal{B}, \quad B_j \in \Gamma^h_{\text{red},n-1}.$$ 

We can therefore repeat the arguments from above word by word to define $\hat{W}^{\alpha}$ such that $s''(\hat{W}^{\alpha}) \in \Gamma_{\text{red}}$.

We have shown that to any generator $W^{\alpha}$ we can construct $\hat{W}^{\alpha}$ such that $s''(\hat{W}^{\alpha}) \in \Gamma_{\text{red}}$. We have therefore shown that $\Gamma_{\text{red}}$ is a sub-complex.

Thus, $s''(\Gamma_{\text{red}}) \subset \Gamma_{\text{red}}$, and we can then use the Künneth theorem (see eq. (2.7) to find

$$H^* (\Gamma; s'') \cong H^* \left( \Gamma_{\text{red}} \otimes \left( \bigotimes_{\alpha} \Gamma^\alpha \right); s'' \right)$$

$$\cong H^* (\Gamma_{\text{red}}; s'') \otimes \left( \bigotimes_{\alpha} H^* (\Gamma^\alpha; s'') \right)$$

$$\cong H^* (\Gamma_{\text{red}}; s'')$$

(4.1)
Thus in order to calculate the cohomology $H^\ast(\Gamma; s''')$ it is in fact enough to calculate $H^\ast(\Gamma_{\text{red}}; s'')$.

Next step is to split $s''$ into two anti-commuting nilpotent operators $s''_0$ and $s''_1$ defined by the currents $j''_0$ and $j''_1$ respectively,

\[
\begin{align*}
  j''_0(z) &= -\chi^\alpha c_\alpha(z) \\
  j''_1(z) &= W^\alpha(z)c_\alpha(z) + \frac{1}{2}f^\alpha\beta\gamma b^\gamma(z)c_\beta(z)c_\alpha(z)
\end{align*}
\] (4.2)

In order to verify that $s''_0$ and $s''_1$ are indeed nilpotent and anti-commuting one can either directly calculate the operator products $j''_0(z)j''_0(w)$ etc., or one can use the fact that $s''_0 = s_0 - s'_0$ and $s''_1 = s_1 - s'_1$, where $s_0$, $s'_0$, $s_1$, and $s'_1$ are all nilpotent and anti-commuting.

Corresponding to this split, we can define a bigrading of $\Gamma$:

\[
W^\alpha, W^i : (m, -m) \\
b^\alpha : (m, -m - 1) \\
c_\alpha : (-m, m + 1)
\] (4.3)

where $m$ is the grade of $W^\alpha$ or $W^i$ defined by the grading $(H - H')$; with these definitions $s''_0$ has bigrading $(1, 0)$, while $s''_1$ has bigrading $(0, 1)$. We can now define the spectral sequence corresponding to the double complex $(\Gamma_{\text{red}}; s''_0; s''_1)$.

The first element of the spectral sequence is

\[
E^{p,q}_0 = \Gamma^{p,q}_{\text{red}}
\]

while the second element is the cohomology of $s''_0$:

\[
E^{p,q}_1 = H^{p,q}(E_0; s''_0) = \frac{\Gamma^{p,q}_{\text{red}} \cap \ker (s''_0)}{\Gamma^{p,q}_{\text{red}} \cap \im (s''_0)}
\] (4.4)

In the primary hamiltonian reduction one can show that for each ghost $c_A$, we can find a linear combination of generators $a_{A\bar{A}}\bar{J}^\bar{A}$, such that

\[
s_0(a_{A\bar{A}}\bar{J}^\bar{A}(z)) = s_0\left(a_{A\bar{A}}f^{A\bar{B}}_{\bar{B}}(c^Cb_C)_0(z)\right) = -a_{A\bar{A}}f^{A\bar{B}}_{\bar{B}}c^C\chi c_B(z) = c_A(z)
\]

If we replace the index $A$ by $\alpha$ in this equation, then since the index $\alpha$ corresponds to a generator with $H'$-grade zero and $\bar{A}$ has non-negative $H'$-grade, then we find that also $B$ and $C$ has $H'$-grade zero. This implies that $s'_0(a_{\alpha\bar{A}}\bar{J}^\bar{A}) = 0$, and since $s''_0 = s_0 - s'_0$ we find

\[
s''_0(a_{\alpha\bar{A}}\bar{J}^\bar{A}(z)) = c_\alpha(z).
\]

This shows that the ghosts $c_\alpha$ are $s'_0$-exact, and the cohomology of $s''_0$ is only non-trivial at ghostnumber zero, i.e. we find

\[
E^{p,q}_1 = H^{p,q}(E_0; s''_0) \cong \delta_{p+q,0}\Gamma_0
\] (4.5)

Where $\Gamma_0$ is the cohomology of $s''_0$ at ghostnumber zero.

We recall that in the primary quantum hamiltonian reduction, the zeroth cohomology of $s_0$ is $\Omega_{hw}$. In the secondary hamiltonian reduction there is no notion of highest weights, but $\Gamma_0$ can be considered to be the secondary hamiltonian reduction analogue of $\Omega_{hw}$.

Equation (4.5), together with the fact that the bigrade of the operator $s_r$ is $(1 - r, r)$ implies that $s_r$ is trivial for $r \geq 1$. Thus the spectral sequence collapses already here, and we have

\[
E^{p,q}_\infty = E^{p,q}_1 = H^{p,q}(E_0; s''_0) \cong \delta_{p+q,0}\Gamma_0
\]
Using the equation (3.13) of the theory of spectral sequences, we find that

\[ F^q(H^{p+q})/F^{q+1}(H^{p+q}) = \delta_{p+q,0} \Gamma_0 \]

where \( F^q(H^{p+q}) \) is defined as in eq. (3.14).

Note that with the bi-gradings defined in (4.3), \( \Gamma^{p,q}_{\text{red}} \) is trivial for \( q > 0 \), so \( F^q \Gamma^{p,q}_{\text{red}} = \oplus_{i \geq 0} \Gamma^{p-i,q+i}_{\text{red}} \) is trivial for \( q > 0 \), so we can use lemma 2 page 27 and find

\[ H^n(\Gamma_{\text{red}}; s'') \cong \delta_{n,0} \Gamma_0; \]

however, this isomorphism is a vector space isomorphism but not an algebra isomorphism.

We have proven the following:

**Theorem 3** The BRST operator \( s'' \) defined by

\[
\begin{align*}
    s''(\phi)(w) &= \oint_w dz j''(z)\phi(w) \\
    j''(z) &= (W^\alpha(z) - \chi^\alpha)c_\alpha(z) + f^{\alpha\beta\gamma}(b^\gamma c^\beta c_\alpha)0(z)
\end{align*}
\]

(4.7)

corresponding to the secondary hamiltonian reduction \( \mathcal{W}(G, \mathcal{H}') \to \mathcal{W}(G, \mathcal{H}) \) can be split into two anticommuting, nilpotent operators \( s''_0 \) and \( s''_1 \) defined by

\[
\begin{align*}
    j''_0(z) &= -\chi^\alpha c_\alpha(z) \\
    j''_1(z) &= W^\alpha(z)c_\alpha(z) + f^{\alpha\beta\gamma}(b^\gamma c^\beta c_\alpha)0(z)
\end{align*}
\]

The cohomology of \( s'' \) is

\[ H^n(\Gamma_{\text{red}}; s'') \cong \delta_{n,0} \Gamma_0, \]

where \( \Gamma_0 = H^0(\Gamma_{\text{red}}; s''_0) \). This isomorphism is a vector space isomorphism, but not an algebra isomorphism.

Note that \( \Gamma_{\text{red}} \) does not contain any elements of negative ghostnumber, and consequently \( \Gamma_0 \cong \ker (s''_0) \). This, together with the fact that we know the number of generators of \( \Gamma_0 \) (it is equal to the number of generators of \( \mathcal{W} \)) considerably simplifies the problem of finding \( \Gamma_0 \) in concrete examples.
4.2 Explicit Construction of Generators.

Once we have found the generators $V_0^k$ of $\Gamma_0$, we can use the tic-tac-toe construction to find the generators of $H^0(\Gamma_{\text{red}}, s'')$, i.e. the generators of $\mathcal{W}$. These take the form

$$V^k(z) = \sum_{\ell=0}^{p} V^k_{\ell}(z)$$

where $V^k_{\ell}$ is defined inductively by $s''(V^k_{\ell}) + s''(V^k_{\ell+1}) = 0$ and $p$ is given by $V_0^k \in \Gamma_{\text{red}}^{p,-p}$ (i.e. it is the grade of $V_0^k$). Note that if $V_0^k$ has bigrade $(p,-p)$, $V_1^k$ has bigrade $(p-1, -p+1)$, etc. $V_p^k$ has bigrade $(0,0)$, and the construction stops here because $s''_1$ vanishes on generators with grade zero. It is easy to verify that with this construction, $s''(V^k)$ is indeed zero.

The resulting generators $V^k$ constitutes a basis of the algebra $\mathcal{W}$. In principle the operator product expansion of these generators close only modulo $s''$-exact terms; however, since there are no elements in $\Gamma_{\text{red}}$ with negative ghostnumber, there can be no $s''$-exact terms with zero ghostnumber, and therefore the algebra of the $V^k$ closes exactly.

4.3 Generalized Quantum Miura Transformation

Because the operator product expansion preserves the grading, it is clear that the grade zero part of the generators gives a copy of the $\mathcal{W}$-algebra, or more precisely: the map $V^k \to V^k_p (V^k = V_0^k + V_1^k + \cdots + V_p^k)$ is an algebra homomorphism. In order to show that this map is in fact an algebra isomorphism, we need to show that the map is an injection. To show this, one can consider the so-called “mirror spectral sequence”, the spectral sequence obtained by inverting the role of $s''$ and $s''_1$. Thus for the mirror spectral sequence, we define

$$E^p_0 = \Gamma_{\text{red}}^{p,q} (\equiv E^{p,q}_0)$$

$$E^p_1 = H^p(E_0; s''_1) = \frac{\Gamma_{\text{red}}^{p,q} \cap \ker (s''_1)}{\Gamma_{\text{red}}^{p,q} \cap \im (s''_1)}$$

$$E^p_2 = H^p(E_1; s''_0) = \frac{E^{p,q}_1 \cap \ker (s''_0)}{E^{p,q}_1 \cap \im (s''_0)}$$

(4.8)

etc. We already know that $H^*(\Gamma_{\text{red}}; s'')$ is nontrivial only at ghostnumber zero. This implies that also $E^{p,q}_0$ is nontrivial only at ghostnumber zero, i.e. at $q = -p$. We find that $s''_1(W^{\bar{A}}) = 0$ iff $W^{\bar{A}}$ has bi-grade $(0,0)$. To see this, note that $s''_1$ has bigrade $(0,1)$, and that $\Gamma_{\text{red}}^{0,1} = \{0\}$. This shows that:

$W^{\bar{A}}$ has bi-grade $(0,0) \Rightarrow s''_1(W^{\bar{A}}) = 0.$

To see that the opposite is also true, note that for each $W^{\bar{A}}$ with grade larger than zero, there is a $W^{\bar{A}}$ such that

$$W^{\bar{A}}(z)W^{\bar{A}}(w) = \frac{g^{\bar{A}}}{(z-w)^{(1+h_A)}} + \cdots$$

where $\cdots$ denotes less singular terms. This gives rise to a term proportional to $\partial^{h_A}c_\alpha$ in $s''_1(W^{\bar{A}})$; and one can show that this term will not be cancelled by other terms in $s''_1(W^{\bar{A}})$, thus showing that $s''_1(W^{\bar{A}}) \neq 0$.

It follows that

$$\bar{E}^{p,-p}_1 = \delta_{p,0} \Gamma_{\text{red}}^{0,0}$$
Therefore there is an isomorphism of vector spaces

\[ H^0(\Gamma_\text{red}; s'') \cong \hat{E}^0,0 \]

and therefore the map from \( H^0(\Gamma_\text{red}; s'') \) to its zero grade component is injective, and therefore indeed an isomorphism of algebras. This proof is essentially identical to the one given in \([10]\) for the case of the primary hamiltonian reduction.

We have shown:

**Theorem 4** For generators \( V^k \) of \( \mathcal{W} \), constructed using the tic-tac-toe construction defined above, the mapping

\[ V^k = V^k_0 + V^k_1 + \cdots + V^k_p \rightarrow V^k_p \]

of the generator to the zero grade part of the generator is an algebra isomorphism.

This mapping is the generalization of the quantum Miura transformation to the case of the secondary hamiltonian reduction.

This theorem means that we can realize the generators of the algebra \( \mathcal{W} \) in terms of the generators of the simpler algebra \( \hat{\mathcal{W}}_0' \), generated by the “hatted” grade zero generators of \( \mathcal{W}' \). \( \hat{\mathcal{W}}_0' \) always includes the energy-momentum tensor \( \hat{T} \), since \( T \) is always part of the grade zero subspace of \( \mathcal{W}' \).

This construction gives us an impressive variety of new realizations of \( \mathcal{W} \) algebras: for every possible secondary hamiltonian reduction, written in the form

\[ \mathcal{G}^{(1)} \rightarrow \mathcal{W}(\mathcal{G}, \mathcal{H}') \rightarrow \mathcal{W}(\mathcal{G}, \mathcal{H}) \]

we get a realization of the generators of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) in terms of the hatted generators of the grade zero subalgebra of \( \mathcal{W}(\mathcal{G}, \mathcal{H}') \).

Similar realizations of \( \mathcal{W} \) algebras in terms of simpler \( \mathcal{W} \) algebras have been constructed before, see e.g. \([26, 27]\); however, the present construction gives a systematic method for constructing a large number of such realizations.

### 4.4 Example: \( \mathcal{W}(s\ell(3), s\ell(2)) \rightarrow \mathcal{W}_3 \)

Let us consider the simplest possible example of the secondary quantum hamiltonian reduction, namely the reduction of the Bershadsky algebra \( \mathcal{W}(s\ell(3), s\ell(2)) \) to the \( \mathcal{W}_3 \) algebra. We consider the regular embedded \( s\ell(2) \) subalgebra \( \{E_{\alpha_1}, H_{\alpha_1}, E_{-\alpha_1}\} \). The corresponding standard grading of \( s\ell(3) \) is

\[
\begin{pmatrix}
0 & 1 & \frac{1}{2} \\
-1 & 0 & 0 \\
-\frac{1}{2} & \frac{3}{2} & 0
\end{pmatrix}
\]

As described in section 2.2, we can modify the grading operator \( H_{\alpha_1} \) by adding a \( U(1) \) current. If we choose

\[ U = \begin{pmatrix}
\frac{1}{6} & 0 & 0 \\
0 & \frac{1}{6} & 0 \\
0 & 0 & -\frac{1}{3}
\end{pmatrix}
\]

then we get the modified integer gradings

\[
\begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]
The constrained current corresponding to these gradings are

\[
J_{\text{red}} = \begin{pmatrix}
H_{\alpha_1} & J_{\alpha_1} & J_{\alpha_1+\alpha_2} \\
1 & H_{\alpha_2} - H_{\alpha_1} & J_{\alpha_2} \\
0 & J_{-\alpha_2} & -H_{\alpha_2}
\end{pmatrix}
\] (4.9)

(to simplify the notation we suppress the \(z\) dependence). The grading and constraints for the \(W_3\)-algebra are:

\[
J_{\text{red}} = \begin{pmatrix}
0 & 1 & 2 \\
-1 & 0 & 1 \\
-2 & -1 & 0
\end{pmatrix}
\]

We define “improved” generators

\[
\begin{align*}
\tilde{J}^\alpha_{-\alpha_1} &= J^{\alpha_1} \\
\tilde{J}^\alpha_{-\alpha_2} &= J^{\alpha_2} + b^{-\alpha_1-\alpha_2}c_{-\alpha_1-\alpha_2} \\
\tilde{J}^{\alpha_1+\alpha_2}_{-\alpha_1-\alpha_2} &= J^{\alpha_1-\alpha_2} \\
\tilde{H}^{\alpha_1} &= H^{\alpha_1} - 2b^{-\alpha_1}c_{-\alpha_1} - b^{-\alpha_1-\alpha_2}c_{-\alpha_1-\alpha_2} \\
\tilde{H}^{\alpha_2} &= H^{\alpha_2} + b^{-\alpha_1}c_{-\alpha_1} - b^{-\alpha_1-\alpha_2}c_{-\alpha_1-\alpha_2}
\end{align*}
\]

We find that \(\mathcal{W}(s\ell(3), s\ell(2)) = H^0(\mathcal{A} \otimes \Lambda'; s')\) is generated by

\[
\begin{align*}
J &= \tilde{H}^{\alpha_1} + 2\tilde{H}^{\alpha_2} \\
G^- &= \tilde{J}^{-\alpha_2} \\
G^+ &= \tilde{J}^{\alpha_1+\alpha_2} + (k + 2)\partial \tilde{J}^{\alpha_2} - (\tilde{H}^{\alpha_1} \tilde{J}^{\alpha_2})_0 - (\tilde{H}^{\alpha_2} \tilde{J}^{\alpha_2})_0 \\
T &= \frac{1}{k+3} \left[ \tilde{J}^{\alpha_1} - \frac{1+k}{2} \partial \tilde{H}^{\alpha_1} + (\tilde{J}^{\alpha_2} \tilde{J}^{-\alpha_2})_0 + \frac{1}{3} \left( (\tilde{H}^{\alpha_1} \tilde{H}^{\alpha_1})_0 + (\tilde{H}^{\alpha_1} \tilde{H}^{\alpha_2})_0 + (\tilde{H}^{\alpha_2} \tilde{H}^{\alpha_2})_0 \right) \right].
\end{align*}
\]

Here \(J\) is a \(U(1)\) field and \(G^\pm\) are primary bosonic spin \(\frac{3}{2}\) fields. With the normalizations chosen, we have

\[
\begin{align*}
J(z)J(w) &= \frac{9+6k}{(z-w)^2} + \cdots \quad (4.13) \\
J(z)G^\pm(w) &= \pm \frac{3G^\pm(w)}{z-w} + \cdots \\
G^+(z)G^-(w) &= -\frac{(k+1)(2k+3)}{(z-w)^3} - \frac{(k+1)J(w)}{(z-w)^2} + \frac{(k+3)T - k+1}{z-w} \partial J - \frac{1}{3} (JJ)_0 + \cdots
\end{align*}
\]

where \(\cdots\) denotes non-singular terms. The central charge is \(c = \frac{(2k+3)(3k+1)}{(k+3)}\).

The BRST current for the secondary hamiltonian reduction can now be written in the form:

\[
J'' = (J^{-\alpha_2} - 1)c_{-\alpha_2} + b^{-\alpha_1-\alpha_2}c_{-\alpha_1}c_{-\alpha_2} = (G^- - 1)c_{-\alpha_2}
\] (4.14)
The operator \( s'' \) is found to act on the fields as follows:

\[
\begin{align*}
    s''(T) &= \frac{3}{2} G^- c_{-\alpha_2} + \frac{1}{2} \partial G^- c_{-\alpha_2} \\
    s''(J) &= 3 G^- c_{-\alpha_2} \\
    s''(G^+) &= -(k + 3) T c_{-\alpha_2} - (k + 1) J \partial c_{-\alpha_2} - \frac{k + 1}{2} \partial J c_{-\alpha_2} + \frac{1}{3} (J J c_{-\alpha_2})_0 \\
    s''(b^{-\alpha_2}) &= G^- - 1
\end{align*}
\]

and \( s''(G^-) = s''(c_{-\alpha_2}) = 0 \). The “hatted” operators are:

\[
\begin{align*}
    \hat{T} &= T - \frac{3}{2} b^{-\alpha_2} \partial c_{-\alpha_2} + \frac{1}{2} c_{-\alpha_2} \partial b^{-\alpha_2} \\
    \hat{G}^- &= G^- \\
    \hat{G}^+ &= G^+ \\
    \hat{J} &= J - 3 b^{-\alpha_2} c_{-\alpha_2}
\end{align*}
\]

and we find that \( s''(\hat{T}) = \frac{3}{2} \partial c_{-\alpha_2} \) and \( s''(\hat{J}) = 3 c_{-\alpha_2} \). The operators \( s''_0 \) and \( s''_1 \) are given in terms of the currents \( j''_0 \) and \( j''_1 \) respectively:

\[
\begin{align*}
    j''_0 &= -c_{-\alpha_2} \\
    j''_1 &= G^- c_{-\alpha_2}
\end{align*}
\]

We find that the generators of \( \Gamma_0 = H^0(\Gamma_{\text{red}}; s''_0) \) are \( T_2 = \hat{T} - \frac{1}{2} \partial \hat{J} \) and \( G^+ \). \( T_2 \) is already in the cohomology of \( s'' \), and using the tic-tac-toe construction with \( G^+ \) as the starting point, we find \( W \):

\[
\begin{align*}
    T_2 &= \hat{T} - \frac{1}{2} \partial \hat{J} \\
    W &= G^+ - \frac{1}{27} (\hat{J} \hat{J} \hat{J})_0 + \frac{1 + k}{6} (\hat{J} \partial \hat{J})_0 + \frac{(k + 3)}{3} (\hat{T} \hat{J})_0 \\
    &\quad - \frac{(k + 3)(k + 2)}{2} \partial \hat{T} - \frac{(k + 3)k + 4}{12} \partial^2 \hat{J}
\end{align*}
\]

This gives us a realization of the \( \mathcal{W}_3 \) algebra in terms of the generators of \( (\hat{\mathcal{W}}^2_3)_{\geq 0} \), the “hatted” generators of \( \mathcal{W}_3^2 \) with non-negative grade.

Using the primary hamiltonian reduction, we can find expressions for \( G^+ \), \( T \), and \( J \) in terms of the currents of the affine algebra \( s\ell(3)^{(1)} \). Inserting these expressions into equation (4.17) gives us a realization of \( \mathcal{W}_3 \) in terms of the currents of the \( s\ell(3)^{(1)} \). Note, however, that this is not identical to the realization we would get by doing the hamiltonian reduction to \( \mathcal{W}_3 \) in one step, using the primary hamiltonian reduction.

### 5 Linearization of \( \mathcal{W} \)-algebras

Very recently, the construction of linearized \( \mathcal{W} \) algebras [20] have attracted some attention. The idea in this construction is to add some extra generators to an algebra \( \mathcal{W} \), such that the resulting larger algebra is equivalent to a linear algebra.

We will show that the secondary quantum hamiltonian reduction gives us a general method to find such linearizations of \( \mathcal{W} \) algebras. In the specific case of the linearization of \( \mathcal{W}_3 \), we find the same result as [20].
The basic idea of our construction is very simple. Define $\mathcal{W}'_\sim$ to be the subalgebra of $\mathcal{W}'$ with negative grading, i.e. the constrained subalgebra of $\mathcal{W}'$, and define $\mathcal{W}'_{\geq 0}$ to be the subalgebra with nonnegative grading. Define furthermore $\hat{\mathcal{W}}'_{\geq 0}$ to be the algebra generated by the “hatted” generators in $\mathcal{W}_{\geq 0}$. We have above shown (property [ ]) that we can construct a realization of $\mathcal{W}$ as differential polynomials in the generators of $\hat{\mathcal{W}}'_{\geq 0}$. Let us denote the number of generators of an algebra $\mathcal{A}$ by $|\mathcal{A}|$, and define

$$n = |\mathcal{W}| - |\hat{\mathcal{W}}'_{\geq 0}| = |\mathcal{W}'_\sim|.$$  

We will show that it is possible to add $n$ of the generators of $\hat{\mathcal{W}}'_{\geq 0}$ to $\mathcal{W}$, in such a way that there is an invertible transformation between the resulting algebra $\mathcal{W}_{\text{ext}}$ and $\hat{\mathcal{W}}'_{\geq 0}$.

Let us write $\hat{\mathcal{W}}'_{\geq 0}$ in the form

$$\hat{\mathcal{W}}'_{\geq 0} = \Gamma_0 \oplus V.$$  

($V$ is not uniquely defined). It follows from the tic-tac-toe construction that the generators of $\mathcal{W}$ have the form:

$$W = W_0 + W_1, \quad W_0 \in \Gamma_0, \quad W_1 \in V.$$  

It is now clear, that if we extend $\mathcal{W}$ with a basis of generators in $V$, then the transformation $\hat{\mathcal{W}}'_{\geq 0} \leftrightarrow \mathcal{W}_{\text{ext}}$ is invertible. We have

**Theorem 5** Write the algebra $\hat{\mathcal{W}}'_{\geq 0}$ in the form $\hat{\mathcal{W}}'_{\geq 0} = \Gamma_0 \oplus V$, where $\Gamma_0 = \ker (s''\cdot)$. Define $\mathcal{W}_{\text{ext}}$ to be the algebra $\mathcal{W}$ extended with a basis of generators in $V$. Then there is an invertible mapping

$$\phi : \hat{\mathcal{W}}'_{\geq 0} \rightarrow \mathcal{W}_{\text{ext}}$$

We see that every secondary hamiltonian reduction gives rise to an embedding $\mathcal{W} \hookrightarrow \mathcal{W}_{\text{ext}}$, where $\mathcal{W}_{\text{ext}}$ is equivalent to an algebra $\hat{\mathcal{W}}'_{\geq 0}$ that will in general be simpler than $\mathcal{W}$.

If $\hat{\mathcal{W}}'_{\geq 0}$ is linear, the result of this procedure is a linearization of $\mathcal{W}$. Generically, $\hat{\mathcal{W}}'_{\geq 0}$ is not linear, but we find that it is actually linear for a large class of reductions:

**Property 2** All algebras of the form

$$\mathcal{W}(\mathfrak{sl}(N), \oplus_{n=1}^{l} \mathfrak{sl}(p_n)),$$  

$p_1 > p_n + 1, \quad \forall \ n \geq 2$

can be linearized by the secondary hamiltonian reduction

$$\mathcal{W}(\mathfrak{sl}(N), \mathfrak{sl}(2)) \rightarrow \mathcal{W}(\mathfrak{sl}(N), \oplus_{n=1}^{l} \mathfrak{sl}(p_n)).$$

The tic-tac-toe construction gives an algorithmic method for the explicit construction of these linearizations.

Let us restrict ourselves to showing this in the case of $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(m))$ – the general case is a straightforward generalization. So we consider the secondary reduction $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(2)) \rightarrow \mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(m))$. The constraints and highest weight gauge corresponding to $\mathcal{W}(\mathfrak{sl}(n), \mathfrak{sl}(2))$ are

\[
\begin{pmatrix}
* & * & \cdots & * & * \\
1 & * & \cdots & * & * \\
0 & * & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & * & \cdots & * & *
\end{pmatrix}
\begin{pmatrix}
U & T \\
1 & U
\end{pmatrix}
\begin{pmatrix}
G_1 & G_2 & \cdots & G_{n-2} \\
0 & 0 & \cdots & 0
\end{pmatrix}

= \begin{pmatrix}
0 & G_1 \\
0 & G_2 \\
\vdots & \vdots \\
0 & G_{n-2}
\end{pmatrix} + \frac{2k}{n-2} \mathbb{1}
\]

\[5\] It remains to show that $\mathcal{A}$ is indeed an algebra.
The \( G \)'s are bosonic spin \( \frac{n}{2} \) fields. The \( U(1) \) operator \( U \) commutes with the \( s\ell(n-2) \) Kac-Moody subalgebra in \( \mathcal{W}(s\ell(n), s\ell(2)) \), while the \( G \)'s have positive and the \( G \)'s negative \( U(1) \)-charge. This shows that the only possible operator product expansions containing nonlinear terms are \( G_i(z) \bar{G}_j(w) \). The constraints corresponding to \( \mathcal{W}(s\ell(n), s\ell(m)) \) are

\[
\begin{pmatrix}
* & * & \cdots & * & \cdots & * \\
1 & * & \cdots & * & \cdots & * \\
0 & 1 & \cdots & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & * \\
0 & 0 & \cdots & 0 & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & * \\
\end{pmatrix}
\]

We find that the secondary reduction is made by constraining \( \bar{G}_1 = 1, \bar{G}_2 = 0, \ldots, \bar{G}_{n-2} = 0 \), in general in addition to constraining also a number of the Kac-Moody currents. Since all the fields \( \bar{G}_i \) are constrained, it follows that \( \mathcal{W}_{G_{>0}} \) is linear.

For the \( so(n) \) algebras, due to the few cases that allow the secondary reductions (in our framework), it is clear that we will not be able to linearize most of the corresponding \( \mathcal{W} \)-algebras. In fact, demanding that the starting \( \mathcal{W} \)-algebra is built on \( \mathcal{H} = s\ell(2) \) and reasoning as above, it is easy to see that only the algebras\(^6\) \( \mathcal{W}BC_2 \) and \( \mathcal{W}D_3 \) can be linearized (the last one being in fact identical with \( \mathcal{W}A_3 \)).

For \( sp(2n) \) algebras, the complete classification of linearizable \( \mathcal{W}(sp(2n), \mathcal{H}) \) algebras is quite heavy and beyond the scope of the present article: we refer to [28] for an exhaustive classification. Let us just remark that the secondary reduction \( \mathcal{W}(sp(2n), sp(2)) \to \mathcal{W}(sp(2n), \mathcal{H}) \) with \( \mathcal{H} \) simple always provide a linearization of the \( \mathcal{W}(sp(2n), \mathcal{H}) \) algebra.

Let us remark that the spin of the new fields we add to linearize the algebra are always positive, since we take the positive grade part of a given \( \mathcal{W} \)-algebra\(^7\).

The most popular \( \mathcal{W} \)-algebras are the \( \mathcal{W}(\mathcal{G}, \mathcal{G}) \equiv \mathcal{W} \mathcal{G} \) ones: it is natural to see whether one can linearize these algebras. From the above property, it is easy to deduce:

**Property 3** The \( \mathcal{W} \)-algebras \( \mathcal{W}A_n \) and \( \mathcal{W}C_n \) can be linearized by the secondary reductions through the schemes:

\[
\begin{align*}
\mathcal{W}(s\ell(n+1), s\ell(2)) & \to \mathcal{W}A_n \\
\mathcal{W}(sp(2n), sp(2)) & \to \mathcal{W}C_n
\end{align*}
\]

For the \( \mathcal{W}BC_n \) and \( \mathcal{W}D_n \) algebras, our techniques allows to linearize only the \( \mathcal{W}BC_2 \) and \( \mathcal{W}D_3 \) algebras through

\[
\begin{align*}
\mathcal{W}(so(5), so(3)) & \to \mathcal{W}BC_2 \\
\mathcal{W}(so(6), so(3)) & \to \mathcal{W}D_3
\end{align*}
\]

The above method of linearizing \( \mathcal{W} \) algebras is not limited to the secondary quantum hamiltonian reduction – it can also be used in the primary quantum hamiltonian reduction. Following the above

---

\(^6\)We denote by \( WC_n \) the algebra \( \mathcal{W}(B_n, B_n) \) obtained from Hamiltonian reduction of \( B_n \), to distinguish it from the Casimir algebra \( WB_n \) that contains a spin \( \frac{n+1}{2} \) field. \( WC_n \) as the same spin contents as \( WC_n \) but different structure constants.

\(^7\)Actually the spin is at least 1.
procedure in that case, we find any \( \mathcal{W} \) algebra \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) can be extended by adding the generators in \( \hat{\mathcal{G}}_{\geq 0}^{(1)} \) which are not highest weight, \( \hat{\mathcal{G}}_{\geq 0}^{(1)} \) are the hatted affine currents with non-negative grade:

\[
\mathcal{W}_{ext} = \mathcal{W} + \left( \hat{\mathcal{G}}_{\geq 0}^{(1)} \setminus \hat{\mathcal{G}}_{hw}^{(1)} \right)
\]

The algebra \( \mathcal{W}(\mathcal{G}, \mathcal{H})_{ext} \) is then equivalent to \( \hat{\mathcal{G}}_{\geq 0}^{(1)} \).

As explicit examples of the linearization using secondary hamiltonian reduction, we will give the two simplest: the linearization of \( \mathcal{W}_3 = \mathcal{W}(s\ell(3), s\ell(3)) \) using \( \hat{\mathcal{W}}_{s\ell(3), s\ell(3)}^{(2)} \geq 0 = \hat{\mathcal{W}}_{s\ell(3), s\ell(2)}^{(2)} \geq 0 \) (this linearization was already given in [20]), and the linearization of \( \mathcal{W}_4 = \mathcal{W}(s\ell(4), s\ell(4)) \) using \( \hat{\mathcal{W}}_{s\ell(3), s\ell(2)}^{(2)} \geq 0 \).

Note also that the secondary reduction \( \mathcal{W}(so(5), so(3)) \to \mathcal{W}(so(5)) \) will provide the linearization of the \( W_{2,4} \) algebra [20]. For the linearization of the \( \mathcal{W}B_2 \) algebra (containing a spin \( \frac{5}{2} \) field), a secondary reduction of super algebras will have to be performed [28].

### 5.1 Linearization of \( \mathcal{W}_3 \)

As we already saw in example in section [4.1], \( \mathcal{W}_3 \) can be realized in terms of the generators \( \hat{T}, \hat{G}^+, \) and \( \hat{J} \):

\[
T = \hat{T} - \frac{1}{2} \partial \hat{J}
\]

\[
W = \hat{G}^+ - \frac{1}{2\hat{T}} (\hat{J} \partial \hat{J})_0 + \frac{1 + k}{6} (\hat{J} \partial \hat{J})_0 + \frac{(k + 3)}{3} (\hat{T} \hat{J})_0 \\
- \frac{(k + 3)(k + 2)}{2} \partial \hat{T} - \frac{(k + 3)(k + 4)}{12} \partial^2 \hat{J}
\]

If we add the current \( J = \hat{J} \) to the \( \mathcal{W}_3 \) algebra, then it is clear that the transformation \( \{T, W, J\} \leftrightarrow \{\hat{T}, \hat{G}^+, \hat{J}\} \) is invertible. The new operator product expansions of the extended \( \mathcal{W}_3 \) algebra are:

\[
J(z)J(w) = \frac{18 + 6k}{(z - w)^2} + \cdots
\]

\[
T(z)J(w) = \frac{12 + 6k}{(z - w)^3} + \frac{J}{(z - w)^2} + \frac{\partial J}{z - w} + \cdots
\]

\[
J(z)W(w) = \frac{(k^2 + 5k + 6)J}{(z - w)^3} + \frac{(2k^2 + 12k + 18)T - \frac{1}{3} (k + 3)(J \partial J)_0 + \frac{1}{2} (3k^2 + 15k + 18) \partial J}{(z - w)^2} \\
+ 3W + \frac{\frac{1}{2} (2k^2 + 9k + 11) \partial^2 J - (k + 3)(T \partial J)_0}{z - w} \\
+ \frac{\frac{1}{3} (J \partial J)_0 - (k + 2)(J \partial J)_0}{z - w} + \cdots
\]

while the (equivalent) nontrivial operator product expansions of the linear algebra generated by \( \hat{T}, \hat{G}^+, \) and \( \hat{J} \) are

\[
\hat{T}(z)\hat{T}(w) = -\frac{3k^2 + 11k + 18}{(z - w)^4} + \frac{2\hat{T}}{(z - w)^2} + \frac{\partial \hat{T}}{z - w} + \cdots
\]

\[
\hat{T}(z)\hat{G}^+(w) = \frac{\frac{2}{3} \hat{G}^+}{(z - w)^2} + \frac{\partial \hat{G}^+}{z - w} + \cdots
\]
\[
\hat{T}(z) \hat{J}(w) = -\frac{6}{(z-w)^3} + \frac{\hat{J}}{(z-w)^2} + \frac{\partial \hat{J}}{z-w} + \cdots
\]
\[
\hat{J}(z) \hat{G}^+(w) = \frac{3 \hat{G}^+}{z-w} + \cdots
\]
and of course \( \hat{J}(z) \hat{J}(w) = J(z)J(w) \).

### 5.2 Linearization of \( \mathcal{W}_4 \)

In order to show an example where the linearization has not been done before, we take the linearization of the \( \mathcal{W}_4 \)-algebra. In this case, the algebra \( \mathcal{W}(s\ell(4), s\ell(2)) \) contains \( T \), a \( U(1) \) subalgebra generated by \( U \), an affine \( s\ell(2) \) algebra generated by \( K^0 \) and \( K^\pm \), and 4 spin \( \frac{3}{2} \) fields \( G^\sigma \), \( \epsilon = \pm, \sigma = \pm \). \( G^\sigma \) has \( U(1) \)-charge \( \epsilon = 1 \) and the eigenvalue under \( K^0 \) is \( \sigma \frac{1}{2} \).

In the secondary reduction we constrain \( G^\pm \) and \( K^- \), so the algebra \( \hat{\mathcal{W}}(s\ell(4), s\ell(2))_{>0} \) is generated by \( \hat{T}, \hat{G}^{\pm}, \hat{K}^0, \hat{K}^+ \), and \( \hat{U} \). \( \hat{G}^{\pm} \) are primary Virasoro and Kac-Moody fields, with spin \( \frac{3}{2} \), \( U(1) \)-charge 1, and eigenvalue \( \pm \frac{1}{2} \) under \( \hat{K}^0 \). \( \hat{K}^+ \) is a primary spin 1 field with eigenvalue 1 under \( \hat{K}^0 \) (and \( U(1) \)-charge 0). The central charge is \( \hat{c} = -\frac{3(2k^2 + 11k + 32)}{k^4} \), and the rest of the nontrivial operator product expansions are:

\[
\hat{T}(z) \hat{U}(w) = -\frac{4}{(z-w)^3} + \frac{\hat{U}}{(z-w)^2} + \frac{\partial \hat{U}}{z-w} + \cdots
\]
\[
\hat{T}(z) \hat{K}^0(w) = -\frac{1}{(z-w)^3} + \frac{\hat{K}^0}{(z-w)^2} + \frac{\partial \hat{K}^0}{z-w} + \cdots
\]
\[
\hat{U}(z) \hat{U}(w) = \frac{k + 4}{(z-w)^2} + \cdots
\]
\[
\hat{K}^0(z) \hat{K}^0(w) = \frac{k+4}{2}(z-w)^2 + \cdots
\]
\[
\hat{K}^+(z) \hat{G}^+(w) = -\frac{\hat{G}^++}{z-w} + \cdots
\]

The tic-tac-toe construction gives us the expressions for the generators of \( \mathcal{W}_4 \):

\[
T = \hat{T} - \partial \hat{K}^0 - 2\partial \hat{U}
\]
\[
W_3 = \hat{G}^+ - 2(\hat{K}^0 \hat{U})_0 + (2k + 6)\partial \hat{K}^+ + (4k)(\hat{T} \hat{U})_0 - \frac{1}{2}(\hat{U} \hat{U} \hat{U})_0
\]
\[
-2(\hat{U} \hat{K}^0 \hat{K}^0)_0 + (k + 1)(\hat{U} \partial \hat{U}^*)_0 + (k + 2)(\hat{U} \partial \hat{K}^0)_0 + 4(k + 3)(\hat{K}^0 \partial \hat{K}^0)_0
\]
\[
+ \frac{(3k + 8)}{2} \partial^2 \hat{U} - (k + 2)(k + 3)\partial^2 \hat{K}^0 - (k + 3)(k + 4)\partial \hat{T}
\]
\[
W_4 = \hat{G}^+ + (\hat{K}^0 \hat{K}^0)_0 + \frac{1}{2}(\hat{G}^+ \hat{U})_0 + (\hat{G}^+ \hat{K}^0)_0 - (k + 5)(\hat{T} \hat{K}^+_0)
\]
\[
+ \frac{1}{2}(\hat{U} \hat{U} \hat{K}^+_0) + 2(\hat{K}^0 \hat{K}^0 \hat{K}^+_0) - (k + 3)(\hat{U} \partial \hat{K}^+_0)
\]
\[
- (k + 1)(\partial \hat{U} \hat{K}^+_0) + 2(\hat{K}^0 \partial \hat{K}^+_0) - k(\partial \hat{K}^0 \hat{K}^+_0) + \frac{(22 + 13k + 2k^2)}{2} \partial^2 \hat{K}^+
\]
\[
+ \frac{(4 + k)(952 + 643k + 108k^2)}{4(2552 + 1763k + 300k^2)}(\hat{T} \hat{T})_0 + \frac{k + 4}{4}(\hat{T} \hat{U} \hat{U})_0 - (k + 4)(\hat{T} \hat{K}^0 \hat{K}^0)_0
\]
We define the algebra \((\mathcal{W}_4)_{\text{ext}}\) by adding the generators \(\hat{K}^+, \hat{K}^0, \) and \(\hat{U}\) to the \(\mathcal{W}_4\) algebra. It is obvious that there is an invertible transformation between this extended algebra, and the linear algebra generated by \(\hat{T}, \hat{G}^+\hat{G}^-\hat{K}^+, \hat{K}^0,\) and \(\hat{U}\).

### 6 Conclusion

In this paper, we have considered secondary quantum hamiltonian reductions, i.e. hamiltonian reductions that can be described by the diagram:

$$
\begin{aligned}
\mathcal{G}^{(1)} &\xrightarrow{\mathcal{W}(\mathcal{G}, \mathcal{H}')} \mathcal{W}(\mathcal{G}, \mathcal{H}) \\
\mathcal{G}^{(1)} &\xrightarrow{\mathcal{W}(\mathcal{G}, \mathcal{H})} \mathcal{W}(\mathcal{G}, \mathcal{H})
\end{aligned}
$$

or in words: starting with a Lie algebra \(\mathcal{G}\), and two regular subalgebras \(\mathcal{H}'\) and \(\mathcal{H}\) with \(\mathcal{H}' \subset \mathcal{H}\) satisfying certain conditions as described in appendix \([3]\) we carry out the hamiltonian reduction of the \(\mathcal{W}\) algebra \(\mathcal{W}(\mathcal{G}, \mathcal{H}')\) with suitable constraints, and show that the result is the \(\mathcal{W}\) algebra \(\mathcal{W}(\mathcal{G}, \mathcal{H})\).
Note that for $G = \mathfrak{s\ell}(N)$, the conditions that we impose on $H'$ and $H$ in order to perform the secondary quantum Hamiltonian reduction are more restrictive than the conditions necessary for the classical secondary Hamiltonian reduction, see [19]. This should not be taken as a sign that not all classical secondary Hamiltonian reductions can be quantized; it simply reflects the fact that the method that we have used for the quantum secondary reduction in this paper cannot be applied to all possible secondary reductions. On the other hand, we have been able to explicitly do some quantum reductions when $G = \mathfrak{so}(N)$ or $G = \mathfrak{sp}(2N)$, while the techniques has not been developped for the classical case.

The quantum secondary reductions show that the $\mathcal{W}$ algebras $\mathcal{W}(G, H)$ that can be obtained by the Hamiltonian reduction of a certain affine Lie algebra $G^{(1)}$ are not only related by their common “ancestor” $G^{(1)}$, but that they are mutually directly connected by the Hamiltonian reduction. As a simple example, consider this diagram of the possible Hamiltonian reductions connecting the algebras $\mathcal{W}(\mathfrak{s\ell}(4), H)$; the simple lines symbolize the quantum reductions we have been able to perform, the double lines symbolize secondary reductions that gives rise to linearizations, and the dashed line the classical secondary reduction that is not quantized by our method:

There are two important consequences that follow from the secondary quantum Hamiltonian reduction. One of these is the secondary quantum Miura transformation. The usual quantum Miura transformation can be used to find free field realizations of the $\mathcal{W}$ algebras, and in a similar way the secondary quantum Miura transformation can be used to find realizations of $\mathcal{W}$ algebras in terms of subalgebras of other $\mathcal{W}$ algebras. For example, in the diagram above there are 4 (5 if the dashed line is included) possible secondary reductions, and the secondary quantum Miura transformation corresponding to these gives us realizations of $\mathcal{W}_4$ in terms of $\mathcal{W}(\mathfrak{s\ell}(4), \mathfrak{s\ell}(2))$ or $\mathcal{W}(\mathfrak{s\ell}(4), \mathfrak{s\ell}(3))$, and of $\mathcal{W}(\mathfrak{s\ell}(4), \mathfrak{s\ell}(3))$ and $\mathcal{W}(\mathfrak{s\ell}(4), 2 \mathfrak{s\ell}(2))$ in terms of $\mathcal{W}(\mathfrak{s\ell}(4), \mathfrak{s\ell}(2))$ (and $\mathcal{W}_4$ in terms of $\mathcal{W}(\mathfrak{s\ell}(4), 2 \mathfrak{s\ell}(2))$ if the dashed line is included).

The other consequence that follows from the secondary quantum Hamiltonian reduction is the linearization of $\mathcal{W}$ algebras. For a large class of algebras $\mathcal{W}(G, \bigoplus_{n=1}^{l} \mathcal{H}_n)$, where the possible $\mathcal{H}_n$'s are given in section 5, we can find a secondary Hamiltonian reduction and a corresponding extended algebra $\mathcal{W}(G, \bigoplus_{n=1}^{l} \mathcal{H}_n)_{ext}$ which is equivalent to a linear algebra with new generators of positive spin. In particular, we are able to linearize the $\mathcal{WA}_n$, $\mathcal{WC}_n$ and $\mathcal{WBC}_2$ algebras. To take once again the diagram above as example, this procedure can give us linearizations of $\mathcal{W}_4$ and $\mathcal{W}(\mathfrak{s\ell}(4), \mathfrak{s\ell}(3))$. This linearization of $\mathcal{W}$ algebras could be very useful in study of the representation theory of $\mathcal{W}$ algebras. In fact one could use the linearization to reduce the representation theory of the non-linear $\mathcal{W}$ algebras.
Note that, as mentioned above, we have not in this paper exhausted the possible secondary reductions; a number of classical secondary reductions cannot be quantized using the present methods, and it would be of interest to find a method to quantize these remaining secondary reductions.

Besides these problems, there are other open questions about the secondary quantum hamiltonian reduction. It would be interesting to generalize the procedure to supersymmetric $\mathcal{W}$ algebras, and to do the secondary quantum Miura transformation and the linearization also in that case [28]. Another interesting possibility is to study in more detail the linearization of $\mathcal{W}$ algebras, and to what extent we can actually reduce the analysis of the non-linear $\mathcal{W}$ algebras to the analysis of the matching linear algebra.

Acknowledgements: The authors would like to thank R. Stora for stimulating discussions. One of the authors (JOM) would like to thank the Niels Bohr Institute, where this work was started, and the Danish Natural Science Research Council for financial support. Finally the authors would like to thank the referee for a very thorough referee report, and for making valuable suggestions for improvements of the paper.

Appendices

A Spectral Sequences

In this appendix, we will give a few key definitions that are used in the theory of spectral sequences. For a good introduction to the theory of spectral sequences see e.g. [25].

We assume that we have a complex $(\Omega, s)$, i.e. a graded space $\Omega = \sum_n \Omega^n$ and a nilpotent derivation $s : \Omega^n \to \Omega^{n+1}$. We assume furthermore that it is possible to define a filtration on the space, i.e. a sequence of subspaces $F^q\Omega$ such that

$$\{0\} \subset \cdots \subset F^q\Omega \subset F^q\Omega \subset F^q\Omega \subset \cdots \subset \Omega.$$  

We define a sequence of “generalized co-cycles” $Z^{p,q}_r$ by

$$Z^{p,q}_r = F^q\Omega^{p+q} \cap s^{-1}(F^{q+r}\Omega^{p+q+1}) = \{ x \in F^q\Omega^{p+q} | s(x) \in F^{q+r}\Omega^{p+q+1} \} \quad (A.1)$$

We note that it is natural to define

$$Z^{p,q}_\infty = F^q\Omega^{p+q} \cap \ker s.$$  

We also define a sequence of “generalized co-boundaries” $B^{p,q}_r$ by

$$B^{p,q}_r = F^q\Omega^{p+q} \cap s(F^{q-r}\Omega^{p+q-1})$$

$$B^{p,q}_\infty = F^q\Omega^{p+q} \cap \text{im } s, \quad (A.2)$$

and we see that (suppressing the $(p,q)$ indices)

$$\cdots \subset B_r \subset B_{r+1} \subset \cdots \subset B_\infty \subset \cdots \subset Z_r \subset \cdots.$$  

A Spectral Sequences

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$$B^{p,q}_\infty = F^q\Omega^{p+q} \cap \text{im } s, \quad (A.2)$$

and we see that (suppressing the $(p,q)$ indices)

$$\cdots \subset B_r \subset B_{r+1} \subset \cdots \subset B_\infty \subset \cdots \subset Z_r \subset \cdots.$$
From these generalized cocycles and coboundaries, we can now define a sequence of “generalized cohomologies” \( E^{p,q}_r \) by

\[
E^{p,q}_0 = \frac{F^q Q^{p+q}}{F^{q+1} Q^{p+q}} \\
E^{p,q}_r = \frac{Z^{p,q}_r}{(Z^{p-1,q+1}_{r-1} + B^{p,q}_{r-1})} \\
E^{p,q}_\infty = \frac{Z^{p,q}_\infty}{(Z^{p-1,q+1}_{\infty} + B^{p,q}_\infty)}
\]

(A.3)

It is now possible to show that for every space \( E_r \), we can define a nilpotent derivative \( s_r \). \( s_r \) is defined by the commutative diagram:

\[
\begin{array}{ccc}
Z^{p,q}_r & \xrightarrow{s} & Z^{p+1-r,q+r}_r \\
\eta \downarrow & & \downarrow \eta \\
E^{p,q}_r & \xrightarrow{s_r} & E^{p+1-r,q+r}_r
\end{array}
\]

(A.4)

where \( \eta \) is the canonical projection operator from \( Z_r \) onto \( E_r \). In other words, for \( [x] \in E^{p,q}_r \), \( s_r([x]) = [s(x)] \).

With all these definitions, we are now finally in a position to state the main theorems of the theory of spectral sequences:

The “generalized cohomologies” \( E_r \) that we have introduced are in fact cohomologies, namely

\[
E^{p,q}_{r+1} \cong H^{p,q}(E_r; s_r)
\]

(A.5)

If the filtration exhausts all of the space \( \Omega \), and if the generalized co-cycles \( Z^{p,q}_r \) converges to \( Z^{p,q}_\infty \), i.e. if \( \Omega = \bigcup_n F^n \Omega \) and \( Z^{p,q}_\infty = \cap_r Z^{p,q}_r \), then

\[
E^{p,q}_\infty \cong \frac{F^q H^{p+q}}{F^{q+1} H^{p+q}}
\]

(A.6)

where \( F^q H \) is the filtration on the cohomology \( H(\Omega; s) \) induced by the filtration on \( \Omega \):

\[
F^q H^{p+q} = H^{p+q}(F^q \Omega; s)
\]

This is the principal result of the theory of spectral sequences. It gives us a way to find the cohomology \( H(\Omega; s) \), supposing that we are able to use the knowledge of the spaces \( F^q H^{p+q}/F^{q+1} H^{p+q} \) to reconstruct \( H(\Omega; s) \). The usefulness of the spectral sequences rests on the fact that in practical application the spectral sequence often collapses after a few steps, i.e. \( s_r \) is identically zero for \( r > r_0 \) where \( r_0 \) is some low number.

Let us show the following

**Lemma 2** If \( F^q \Omega = 0 \) for \( q > 0 \), then \( H(\Omega; s) \cong E_\infty \)

Namely \( F^q \Omega = 0 \) for \( q > 0 \) implies that \( F^q H^{p+q} = 0 \) for \( q > 0 \). This means that

\[
E^{p,0}_\infty \cong \frac{F^0 H^p}{F^1 H^p} \cong F^0 H^p \\
E^{p+1,-1}_\infty \cong \frac{F^{-1} H^p}{F^0 H^p}
\]

(A.7) (A.8)
etc..., and we can use this to show that

\[ F^{-r} H^p \cong E^{p+r, -r}_\infty \oplus \cdots \oplus E^{p,0}_\infty \]

or equivalently (since \( E^{p,q}_\infty = 0 \) for \( q > 0 \))

\[ H^p(\Omega; s) \cong \sum_{r \in \mathbb{Z}} E^{p+r, -r}_\infty \]

which proves the lemma 2.

Let us mention here that the isomorphism \( E^{p,q}_\infty \cong F^q H^{p+q}/F^{q+1} H^{p+q} \) is in general a vector space isomorphism. If the space \( \Omega \) in addition to being a vector space is also a algebra (as in the case that we are interested in here) then if we can define algebras on all the cohomologies in the spectral sequence such that the algebra on \( E^{p,q}_0 = F^q \Omega^{p+q}/F^{q+1} \Omega^{p+q} \) is induced by the algebra on \( F^q \Omega^{p+q} \), i.e. \([a], [b] \in E_0 : [a] \circ [b] = [a \circ b] \) (where \( \circ \) denotes the algebra composition), and the algebra on \( E^{r+1} = H(E_r, s_r) \) is induced by the algebra on \( E_r \), then the the isomorphism \( E^{p,q}_\infty \cong F^q H^{p+q}/F^{q+1} H^{p+q} \) is an algebra isomorphism. However, even if \( E^{p,q}_\infty \cong F^q H^{p+q}/F^{q+1} H^{p+q} \) is an algebra isomorphism, it may be nontrivial to reconstruct the algebra of \( H^n(\Omega; s) \).

In the case where the complex \( (\Omega; s) \) can be given the structure of a double complex structure \( (\Omega; s', s'') \) with two anti-commuting nilpotent operators \( s' \) and \( s'' \), and with a bigrading \( \Omega = \sum_{p,q} \Omega^{p,q} \), the spectral sequence simplifies somewhat. Define \( s = s' + s'' \). The filtration is defined in terms of the bi-grading as

\[ F^q \Omega = \bigoplus_{i,j} \Omega^{i,j}_{\geq q} \]

\[ F^q \Omega^{p,q} = \bigoplus_{i \geq 0} \Omega^{p-i,q+i} \]

(A.9)

The first element in the spectral sequence, \( E_0 \), is defined by

\[ E^{p,q}_0 = F^q \Omega^{p+q}/F^{q+1} \Omega^{p+q} \cong \Omega^{p,q} \]

(A.10)

and the nilpotent bigrade \((1, 0)\)-operator \( s_0 \) on \( E_0 \) is defined by the commutative diagram

\[
\begin{array}{ccc}
F^q \Omega^{p,q} & \xrightarrow{s} & F^q \Omega^{p+q+1} \\
\eta \downarrow & & \downarrow \eta \\
E^{p,q}_0 & \xrightarrow{s_0} & E^{p+1,q}_0 \\
\end{array}
\] (A.11)

where \( \eta \) is the canonical projection operator

\[ F^q \Omega^{p,q} \xrightarrow{\eta} F^q \Omega^{p+q}/F^{q+1} \Omega^{p+q} \cong \Omega^{p,q} \]

This means that if we identify \( x \in E^{p,q}_0 \) with \( x \in F^q \Omega^{p+q} \), then \( s_0(x) = \eta(s(x)) \) – and since \( \eta \) here is the projection operator on \( \Omega^{p+1,q-1} \), \( s_0 \) is simply the bigrade \((1, 0)\) part of \( s \): \( s_0 = s' \).
The second element in the spectral sequence is

\[ E_1^{p,q} = \frac{Z_1^{p,q}/(Z_0^{p-1,q+1} + B_0^{p,q})}{\Omega^{p,q} \cap s^{-1}(\Omega^{p,q+1})/\Omega^{p,q} \cap s(\Omega^{p-1,q})} \] (A.12)

Note that \( \Omega^{p,q} \cap s^{-1}(\Omega^{p,q+1}) = \Omega^{p,q} \cap \ker s_0 \) and \( \Omega^{p,q} \cap s(\Omega^{p-1,q}) = \Omega^{p,q} \cap \text{im } s_0 \); so, in agreement with equation (A.5), we can also write \( E_1 \) as

\[ E_1^{p,q} = H^{p,q}(E_0; s_0) \] (A.13)

The operator \( s_1 \) on \( E_1 \) is again defined by the commutative diagram

\[
\begin{array}{ccc}
Z_1^{p,q} & \xrightarrow{s} & Z_1^{p,q+1} \\
\eta \downarrow & & \downarrow \eta \\
E_1^{p,q} & \xrightarrow{s_1} & E_1^{p,q+1}
\end{array}
\] (A.14)

and \( \eta \) is again the canonical projection operator. Consider \( x \in \ker s_0 \) and let \([x] = \eta(x)\) be the corresponding equivalence class in \( E_1 \). Then \( s_1([x]) = \eta(s(x)) \) is just the bigrade \((0, 1)\)-part of \( s(x) \), projected on \( E_1 \), i.e.:

\[ s_1([x]) = [s'(x)]. \]

B  Shift of the Constraints Using a \( U(1) \) Generator

We are looking for couples of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebras such that the sets of first class constraints are embedded one into the other. We first consider the case \( \mathcal{G} = \mathfrak{sl}(N) \), and to clarify the presentation, we focus on the secondary reductions of type \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \to \mathcal{W}(\mathcal{G}, \mathcal{G}) \equiv \mathcal{W}(\mathcal{G}) \).

In \( \mathfrak{sl}(N) \), the regular subalgebras \( \mathcal{H} \) can always be chosen in such a way that the simple roots of \( \mathcal{H} \) are also simple roots of \( \mathcal{G} \). Let \( \mathcal{H} = \mathfrak{h}_n \) where \( \mathcal{H}_n \) are simple subalgebras of rank \( r_n = \text{rank}(\mathcal{H}_n) \), ordered in such a way that \( r_n \leq r_m \) if \( n > m \). We define as simple roots

Simple roots of \( \mathcal{G} \):
- \( \alpha_1, \ldots, \alpha_{r_1}; \alpha_{r_1+1}, \ldots, \alpha_{r_1+r_2}; \alpha_{r_1+r_2}, \ldots, \alpha_{r_1+r_2+r_3}; \alpha_{r_1+r_2+r_3}; \ldots \)
- \( \ldots; \alpha_{r_{l-1}}, \ldots, \alpha_{r_{l-1}+r_l}; \alpha_{r_{l-1}+r_l}; \alpha_{r_{l-1}+r_l+1}, \ldots, \alpha_{N-1} \)

Simple roots of \( \mathcal{H} \):
- \( \alpha_1, \ldots, \alpha_{r_1}; \alpha_{r_1+1}, \ldots, \alpha_{r_1+r_2}; \alpha_{r_1+r_2}, \ldots, \alpha_{r_1+r_2+r_3}; \ldots \)
- \( \ldots; \alpha_{r_{l-1}+r_l}; \alpha_{r_{l-1}+r_l+1}, \ldots, \alpha_{r_{l-1}+r_l+1} \)

with

\[ \rho_n = \sum_{i=1}^{n} (r_i + 1) \] (B.1)

In the fundamental representation of \( \mathfrak{sl}(N) \), this simply means that we have divided the \( N \times N \) matrix into \( r_j \times r_j \) blocks of decreasing size, plus (when it exists) a block \((N - r_l) \times (N - r_l)\).

The gradation associated to the Cartan generator of the principal \( \mathfrak{sl}(2) \) in \( \mathcal{H} \) attributes a grade 1 to each simple root of \( \mathcal{H} \), but the grade of the simple roots of type \( \alpha_{\rho_n} \) is

\[ \text{gr}(\alpha_{\rho_n}) = -\frac{r_n + r_{n+1}}{2} < 0 \] (B.2)
where we have set $r_{\ell+1} = 0$. This implies that the root generators $E_{\rho_n}$ are constrained in $W(G, H)$ although they are not in $W(G)$. Thus, it is clear that we have to introduce a new gradation such that

$$gr(\alpha_{\rho_n})' \geq 0 \quad \text{(B.3)}$$

while not changing the resulting $W$-algebra. Let $H$ be the gradation we are looking for. Then, if $M_0$ is the Cartan generator of the $sl(2)$ embedding we are considering (it has not been changed because we want the $W$-algebra to be the same), the new gradation is characterized by the generator $U = H - M_0$ which commutes with the $sl(2)$ algebra and which "respects" the highest weight gauge. Thus, classifying the different gradations $H$ is the same as classifying the different $U(1)$ generators submitted to the non-degeneracy condition

$$\text{ker } ad(M_+) \cap G' = 0 \quad \text{(B.4)}$$

where $G'$ denotes the subalgebra of $G$-generators which have negative grade w.r.t. $H$. This technique has been developed in [11, 22], where all the possible gradations leading to the same $W$-algebra have been classified. The procedure goes along the following lines.

We start with the decomposition of the fundamental of $sl(N)$ w.r.t. the principal $sl(2)$ in $H$:

$$N = \bigoplus_{\mu=1}^I n_{\mu} D_{j_{\mu}} \quad \text{with} \quad j_{\mu} \neq j_{\nu} \text{ when } \mu \neq \nu \quad \text{(B.5)}$$

and add the following $U(1)$ eigenvalues

$$\overline{N} = \bigoplus_{\mu=1}^I n_{\mu} D_{j_{\mu}}(y_{\mu}) \quad \text{(B.6)}$$

Then, computing the adjoint representation from this decomposition of the fundamental,

$$G = \bigoplus_k D_k(Y_k) \quad \text{(B.7)}$$

where the $Y_k$'s are differences of two $y_{\mu}$'s. The eigenvalues of the allowed $U(1)$ generators will be characterized by the equations

$$|Y_k| \leq k \quad \text{and} \quad Y_k \in \frac{1}{2} \mathbb{Z} \mathbb{Z} \quad \forall \ D_k(Y_k) \quad \text{(B.8)}$$

Then, the different gradations will be $M_0 + U$, with $M_0$ the Cartan generator of the $sl(2)$ under consideration, and $U$ one of the allowed $U(1)$ generators.

Now, to get a gradation satisfying both equations (B.3) and (B.8), we have to impose

$$|y_{\mu} - y_{\nu}| \leq |j_{\mu} - j_{\nu}| \quad \text{and} \quad y_{\mu} - y_{\nu} \geq j_{\mu} + j_{\nu} \quad \text{(B.9)}$$

which is clearly satisfied only if one of the two $j$'s is zero, ie if $H$ is simple. In that case, the $sl(2) \oplus U(1)$ decomposition

$$D_j(y) \oplus (N - 2j - 1)D_0(z) \quad \text{with} \quad y = \frac{j(N - 2j - 1)}{N} \quad \text{and} \quad z = -\frac{j(2j + 1)}{N} \quad \text{(B.10)}$$

indeed gives a gradation where all the simple roots of $sl(N)$ have positive grades, and whose associated $W$-algebra is $W(sl(N), H)$.

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8If $H$ is simple, there will be only one $D_j$ representation with $j \neq 0$ in the fundamental of $G$
For the general secondary reduction $\mathcal{W}(s\ell(N), \mathcal{H}') \rightarrow \mathcal{W}(s\ell(N), \mathcal{H})$, the reasoning follows along the same lines. We however have to look at the grade of all the roots (since some simple roots have negative grades in the general case). Then, one asks the gradations to satisfy the same lines. We however have to look at the grade of all the roots (since some simple roots or of the form $\mathcal{H}' = \bigoplus_{\alpha} m_{\alpha} s\ell(p_{\alpha})$ (B.11) the $U(1)$ generator exists iff $\mathcal{H}$ decomposes as:

$$\mathcal{H} = \bigoplus_{\alpha} m_{\alpha} s\ell(q_{\alpha}) \quad \text{with} \quad \begin{cases} 2 \leq p_{\alpha} \leq q_{\alpha} \quad \forall \alpha \\ |p_{\alpha} - p_{\beta}| \leq |q_{\alpha} - q_{\beta}| \quad \forall \alpha, \beta \end{cases}$$

(B.12)

Now, turning to the case of orthogonal and symplectic algebras, we can do the same calculation. However, for these algebras, the $U(1)$ generator is much more constrained (see [11], sections 5.2 and 5.3) so that there are less $U(1)$ generators satisfying both the non-degeneracy and the embedding conditions. Note that one has really to check in each case that the sets of currents constrained to 1 are also embedded one into the other, since the simple roots of $\mathcal{H}'$ are not always simple roots of $\mathcal{H}$.

Apart from these restrictions, the calculation is the same as for $s\ell(N)$ algebras, so that one is led to

**Property 4** In the case of secondary reductions of type $\mathcal{W}(s\ell(N), \mathcal{H}') \rightarrow \mathcal{W}(s\ell(N), \mathcal{H})$, we have the following necessary and sufficient condition for the existence of a $U(1)$ generator which satisfies both the non-degeneracy $[B, A]$ and the embedding of the set of constraints associated to $\mathcal{H}$ into the set of constraints associated to $\mathcal{H}'$:

If $\mathcal{H}'$ decomposes as $\mathcal{H}' = \bigoplus_{\alpha} m_{\alpha} s\ell(p_{\alpha})$ (B.11) the $U(1)$ generator exists iff $\mathcal{H}$ decomposes as:

$$\mathcal{H} = \bigoplus_{\alpha} m_{\alpha} s\ell(q_{\alpha}) \quad \text{with} \quad \begin{cases} 2 \leq p_{\alpha} \leq q_{\alpha} \quad \forall \alpha \\ |p_{\alpha} - p_{\beta}| \leq |q_{\alpha} - q_{\beta}| \quad \forall \alpha, \beta \end{cases}$$

(B.12)

Now, turning to the case of orthogonal and symplectic algebras, we can do the same calculation. However, for these algebras, the $U(1)$ generator is much more constrained (see [11], sections 5.2 and 5.3) so that there are less $U(1)$ generators satisfying both the non-degeneracy and the embedding conditions. Note that one has really to check in each case that the sets of currents constrained to 1 are also embedded one into the other, since the simple roots of $\mathcal{H}'$ are not always simple roots of $\mathcal{H}$.

Apart from these restrictions, the calculation is the same as for $s\ell(N)$ algebras, so that one is led to

**Property 5** In the case of secondary reductions of type $\mathcal{W}(G, \mathcal{H}') \rightarrow \mathcal{W}(G, \mathcal{H})$ with $G = so(N)$ or $sp(N)$, we have the following necessary and sufficient condition for the existence of a $U(1)$ generator which satisfies both the non-degeneracy $[B, A]$ and the embedding of the sets of constraints.

- For $so(N)$, $\mathcal{H}$ and $\mathcal{H}'$ must be of the form

$$\begin{cases} \mathcal{H}' = (n + 1) so(p) \\
\mathcal{H} = n so(p) \oplus so(p + 2) \quad \text{with} \quad \begin{cases} N = (n + 1)p + 2 \quad ; \quad n \geq 0 \\
N \equiv p \quad [\text{mod} \ 2] \end{cases} \end{cases}$$

- For $sp(N)$, $\mathcal{H}$ and $\mathcal{H}'$ must be of the form

$$\begin{cases} \mathcal{H}' = s\ell(2) \oplus s\ell(2p_{\mu}) \\
\mathcal{H} = sp(4) \oplus s\ell(2p_{\mu}) \quad \text{with} \quad p_{\mu} \in N \end{cases}$$

or of the form

$$\begin{cases} \mathcal{H}' = s\ell(2) \\
\mathcal{H} = \bigoplus_{j} sp(2q_{j}) \oplus s\ell(p_{\mu}) \quad \text{with} \quad \begin{cases} \text{either } p_{\mu} \in 2N, p_{\mu} \geq p_{\mu} + 1 \quad \forall \mu \geq 2 \quad \text{and} \quad p_{1} \geq 2q_{j} + 1 \quad \forall j \\
or \quad p_{1} \in (2N + 1), p_{1} \geq p_{\mu} + 2 \quad \forall \mu \geq 2 \quad \text{and} \quad p_{1} \geq 2q_{j} + 1 \quad \forall j \\
or \quad q_{1} \geq q_{1} + 1 \quad \forall j \geq 1 \quad \text{and} \quad q_{1} \geq \frac{1}{2}(p_{\mu} + 1) \quad \forall \mu \end{cases} \end{cases}$$

Let us remark that in the case $G = so(5)$, the $U(1)$ generator exists when considering the reduction $\mathcal{W}(so(5), so(3)) \rightarrow \mathcal{W}(so(5))$, while in the case $G = sp(4)$ the $U(1)$ generator exists for the reduction $\mathcal{W}(sp(4), s\ell(2)) \rightarrow \mathcal{W}(sp(4))$ which is in agreement with the isomorphism between the $so(5)$ and $sp(4)$ algebras.
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