I. INTRODUCTION

Stochastic lattice gas models have been extensively studied recently as they are among the simplest examples of non-equilibrium systems. A powerful approach to understand their steady state was developed by Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim, as a macroscopic fluctuation theory (MFT) which gives, for large diffusive systems, the probability distribution of trajectories in the space of density profiles [1, 2]. The MFT relies on the hydrodynamic large deviation theory [3–5] which provides estimates for the probability of observing atypical space/time density profiles. It gives a framework to calculate a large number of properties of stochastic lattice gas, such as the large deviation functional of the density profiles. Recent developments of the hydrodynamic large deviation theory [6, 7] enabled to estimate also the large deviations of the current through the system. What the MFT provide are the equations of the time evolution of the most likely density profile responsible of a given fluctuation. What it does not provide, in general, is the solution of these equations which would give quantitative predictions for the distribution of the fluctuations. So far, for the large deviation functional of the density profiles in the steady state, the equations could only be solved in a few cases of non-equilibrium systems with open boundaries (the symmetric exclusion process [2], the Kipnis, Marchioro, and Presutti model [8, 9]). For the SSEP the results of the MFT were in full agreement with the results obtained [10, 11, 12] from the exact knowledge of the weights of the microscopic configurations in the steady state.

In our previous work [7], we developed a theory to calculate the large deviation function of the current through a long one dimensional diffusive lattice gas in contact at its two ends with two reservoirs at unequal densities. Our approach was based on an assumption, the additivity principle, which relates the large deviation function (LDF) of the current of a system to the LDF’s of subsystems, when one breaks a large system into large subsystems. This assumption is in fact equivalent to the hypothesis, within the hydrodynamic large deviation framework, that to observe, for a very long time period $T$, an average current $q = Q_T/T$, the system adopts a profile with a shape, fixed in time, but of course depending on $q$ (here $Q_T$ is the total number of particles transferred, say from the left reservoir to the system during time $T$). The additivity principle allows one to obtain explicit expressions [7] for all the cumulants of the integrated current $Q_T$. The predictions of our theory were tested in a few cases [7] and the results were found in complete agreement with what was already known or what could be derived by alternative approaches [13, 14, 15].

Recently, it was pointed out [7] that even if our predictions [7] are valid for some diffusive lattice gas, it might happen that, to produce an average current $q$ over a long period of time, the best profile is time-dependent. One of the goals of the present work is to show that, for a simple example, the weakly asymmetric exclusion process on a ring, this indeed happens for some range of parameters.

Let us consider, as we shall do it in the rest of this paper, the time evolution of a one dimensional stochastic lattice gas on a lattice of $N$ sites. According to the hydrodynamic formalism [7], a given lattice gas can be characterized by two functions $D(\rho)$ and $\sigma(\rho)$ of its density $\rho$. One way to define them [7] is to consider a one dimensional system of length $\sigma(\rho)$.
where \( N \) connected to reservoirs at its two ends. For such a lattice gas, the variance of the total charge \( Q_T \) transferred during a long time \( T \) from one reservoir to the other is given, for large \( N \), by definition of \( \sigma(\rho) \) by

\[
\frac{(Q_T^2)}{T} = \frac{\sigma(\rho)}{N} \tag{1}
\]

when both reservoirs are at the same density \( \rho \). On the other hand if the left reservoir is at density \( \rho + \Delta \rho \) and the right reservoir at density \( \rho \), the average current is given, for small \( \Delta \rho \), by

\[
\frac{\langle Q_T \rangle}{T} = \frac{D(\rho)\Delta \rho}{N} \tag{2}
\]

which is simply Fick's law and defines the function \( D(\rho) \). In the symmetric simple exclusion process \( \sigma(\rho) = \rho(1 - \rho) \) and \( D(\rho) = 1/2 \) \[4\], whereas in the Kipnis, Marchioro, Presutti model \[8, 9\] \( \sigma(\rho) = \rho^2 \) and \( D(\rho) = 1/2 \). The effect of a uniform weak electric field of strength \( \nu/(2N) \) acting from left to right on the particles is to modify \( \boxed{2} \) into

\[
\frac{\langle Q_T \rangle}{T} = \frac{D(\rho)\Delta \rho}{N} + \frac{\nu \sigma(\rho)}{N} \tag{3}
\]

This equation follows from the linear response theory \[4\].

Once \( D(\rho) \) and \( \sigma(\rho) \) are known for a given diffusive system, the probability of observing the evolution of a density profile \( \rho(x, s) \) and a rescaled current \( j(x, s) \) for \( 0 < s < T \) during a time \( T \sim N^2 \) is given, according to the hydrodynamic large deviation theory \[6\], by

\[
\text{Pro}(j(x, s), \rho(x, s)) \sim \exp \left[ - \frac{T^\nu_{[0,T]}(j, \rho)}{N} \right] \tag{4}
\]

where \( T^\nu_{[0,T]} \) is defined by

\[
T^\nu_{[0,T]}(j, \rho) = \int_0^T ds \int_0^1 dx \left[ j(x, s) + D(\rho(x, s))\rho'(x, s) - \nu \sigma(\rho(x, s)) \right]^2 \tag{5}
\]

with \( \rho' = \partial \rho/\partial x \) and where the rescaled current \( j(x, s) \) is related to the density profile \( \rho(x, s) \) by the conservation law

\[
\frac{d\rho(x, s)}{ds} = -\frac{dj(x, s)}{dx} \tag{6}
\]

A formalism equivalent to this hydrodynamic large deviation theory was developed independently \[10, 17\] in the context of the full counting statistics of the transport of free fermions through disordered wires. A simple derivation of \( \boxed{5} \) and \( \boxed{6} \) is given in the appendix.

The large deviation function \( G(j_0) \) of the current is then defined as

\[
\text{Pro} \left( \frac{Q_T}{T} = \frac{j_0}{N} \right) \sim \exp \left( -\frac{T}{N} G(j_0) \right) \tag{7}
\]

for large \( T \) and \( N \)

(In \[4\], one has first to take the limit \( T \to \infty \) and then make \( N \) large; in practice \[7\] should hold when \( T \gg N^2 \) as \( N^2 \) is the characteristic time of a diffusive system of size \( N \).

Now according to \[6\], the large deviation function \( G(j_0) \) is given by

\[
G(j_0) = \lim_{T \to \infty} \left[ -\frac{1}{T} \min_{\rho(x, s)} T^\nu_{[0,T]}(j, \rho) \right] \tag{8}
\]

where the current \( j(x, s) \) satisfies for large \( T \) and all \( x \) the constraint

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T j(x, s) ds = j_0 \tag{9}
\]

with the profile \( \rho(x, s) \) and the current \( j(x, s) \) connected by \[6\].

In the following, we will often consider, instead of \[7\], the generating function of the current:

\[
\langle e^{\lambda Q_T} \rangle \sim e^{T\mu(\lambda)} \text{ for large } T \tag{10}
\]

and then, according to \[7\], \[8\], \[9\], \( \mu(\lambda) \) is given by

\[
\mu(\lambda) = \frac{1}{N} \max_{j_0} [\lambda j_0 + G(j_0)] = \frac{1}{N} \lim_{T \to \infty} \frac{1}{T} \max_{\rho(x, s)} \left[ \lambda \int_0^T j(x, s) ds - T^\nu_{[0,T]}(j, \rho) \right] \tag{11}
\]

For a system of length \( N \) connected to two reservoirs at densities \( \rho_a \) and \( \rho_b \) at its two ends, the calculation of the large deviation function \( G(j_0) \) of the current is therefore reduced to finding the time-dependent profile \( \rho(x, s) \) which optimizes \[9\] under the constraints \[8\] and \[9\] and with the additional boundary conditions \( \rho(0, s) = \rho_a \) and \( \rho(1, s) = \rho_b \). All the results of our previous work \[7\] follow then from the assumption that this optimal profile does not vary with time (except from boundary effects near time \( 0 \) and time \( T \) which do not contribute in the large \( T \) limit).

For a system on a ring of \( N \) sites, as we shall consider here, the optimization problem is the same except for the boundary condition which becomes
\[ \rho(0,s) = \rho(1,s) \] and the fact that the total density \( \rho_0 \) on the ring becomes an additional conserved quantity

\[ \int_0^1 \rho(x,s)dx = \rho_0 . \]

Our paper is organized as follows: in section II, we consider a general lattice gas on a ring and show under what conditions the flat profile becomes unstable. In sections III and IV, we write the large deviation function of the current, when the optimal profile has a fixed shape moving at a constant velocity. In section V, we present exact numerical results on the weakly asymmetric exclusion process for small system sizes which give evidence that for some range of parameters, consistent with the results of section III, the optimal profile is no longer flat but becomes space-time dependent (for \( \rho_0 = 1/2 \) it is only space dependent). In section VI, we analyze the limit of a strong asymmetry and obtain a simple expression for the large deviation function of the current, under the assumption that the optimal profile becomes in this limit a step function. In section VII, we show that, even for a strongly asymmetric case such as the totally asymmetric exclusion process, one can exhibit time-dependent profiles determined by the Jensen Varadhan functional which give, for large system size, the exact large deviation function of the current previously calculated by the Bethe ansatz.

II. CONDITIONS FOR THE STABILITY OF A FLAT PROFILE

Consider a lattice gas on a ring of \( N \) sites with total density \( \rho_0 \). Under the assumption that the optimal profile \( \rho(x,s) \) is flat, that is

\[ \rho(x,s) = \rho_0 \]

the large deviation function \( G(j_0) \) is given by

\[ G_{\text{flat}}(j_0) = -\frac{[j_0 - \nu \sigma(\rho_0)]^2}{2\sigma(\rho_0)} \]  \[ (12) \]

which, by (11), gives for \( \mu(\lambda) \)

\[ \mu_{\text{flat}}(\lambda) \approx \frac{\lambda(\lambda + 2\nu)\sigma(\rho_0)}{2N}. \]  \[ (13) \]

A natural question is whether one could increase \( G(j_0) \) (and \( \mu(\lambda) \)) by adding to this flat profile some small \( (\epsilon \ll 1) \) space and time dependent perturbation of the form

\[ j(x,s) = j_0 + \epsilon[j_1(x)\cos(\omega s) + j_2(x)\sin(\omega s)] \]

in which case due to (5)

\[ \rho(x,s) = \rho_0 + \frac{\epsilon}{\omega}[j_1(x)\sin(\omega s) + j_2(x)\cos(\omega s)] \]

where \( j_1(x) \) and \( j_2(x) \) are periodic functions of period 1. The resulting expression for \( G(j_0) \) to second order in \( \epsilon \) is

\[ G(j_0) = -\frac{(j_0 - \nu \sigma(\rho_0))^2}{2\sigma(\rho_0)} + \epsilon^2 \left[ \frac{j_0^2}{4\omega^2\sigma(\rho_0)} + \frac{j_0^2}{2\omega^2\sigma(\rho_0)} \right] + \frac{j_0^2}{8\omega^2\sigma(\rho_0)} \left[ \frac{\nu^2\sigma''(\rho_0)}{4\omega^2} - \frac{j_0^2\sigma'^2(\rho_0)}{4\omega^2\sigma^2(\rho_0)} \right] \]

As this expression is quadratic in the currents \( j_1 \) and \( j_2 \), the various Fourier modes are not coupled. Choosing for \( j_1(x) \) and \( j_2(x) \)

\[ j_1(x) = a\cos(2\pi x) + b\sin(2\pi x) \]

\[ j_2(x) = c\cos(2\pi x) + d\sin(2\pi x) \]  \[ (14) \]

one gets

\[ G(j_0) = -\frac{(j_0 - \nu \sigma(\rho_0))^2}{2\sigma(\rho_0)} + \epsilon^2 \left[ \frac{ad - bc\pi\sigma''(\rho_0)\pi}{\omega^2\sigma(\rho_0)} \right] + \frac{\pi^2\sigma'^2(\rho_0)j_0^2}{2\omega^2\sigma^2(\rho_0)} + \frac{\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2} \]  \[ (15) \]

The flat profile is stable against the perturbation (14), if this is a negative definite quadratic form in \( a, b, c, d \). This is achieved when for all \( \omega \)

\[ \frac{1}{8\sigma(\rho_0)} + \frac{2\pi^2D^2(\rho_0)}{\omega^2\sigma(\rho_0)} + \frac{\pi^2\sigma'^2(\rho_0)j_0^2}{2\omega^2\sigma^2(\rho_0)} + \frac{\nu^2\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2} - \frac{\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2} > \frac{\mu_{\text{flat}}(\lambda)\sigma''(\rho_0)}{2\omega^2\sigma^2(\rho_0)} \]

The flat profile becomes therefore unstable if

\[ \frac{1}{8\sigma(\rho_0)} + \frac{2\pi^2D^2(\rho_0)}{\omega^2\sigma(\rho_0)} + \frac{\pi^2\sigma'^2(\rho_0)j_0^2}{2\omega^2\sigma^2(\rho_0)} + \frac{\nu^2\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2} - \frac{\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2} < \frac{\mu_{\text{flat}}(\lambda)\sigma''(\rho_0)}{2\omega^2\sigma^2(\rho_0)} \]  \[ (16) \]

i.e.

\[ 8\pi^2D^2(\rho_0)\sigma(\rho_0) + (\nu^2\sigma^2(\rho_0) - j_0^2)\sigma''(\rho_0) < 0 \]

which, given (11), can be rewritten as

\[ 4\pi^2D^2(\rho_0) < N\mu_{\text{flat}}(\lambda)\sigma''(\rho_0) \]  \[ (17) \]
When the flat profile becomes unstable, according to (15), the current takes the form

\[ j(x, t) = j_0 + A \cos 2\pi \left( x - x_0 - \frac{j_0 \sigma'(\rho_0)}{\sigma(\rho_0)} t \right) \quad (18) \]

where the amplitude \( A \) would be determined by expanding \( G(j_0) \) to higher order in \( \epsilon \).

One could analyze in a similar way the stability of the flat profile against other modes by choosing \( j_1(x) = a \cos(2\pi nx) + b \sin(2\pi nx) \) and \( j_2(x) = c \cos(2\pi nx) + d \sin(2\pi nx) \) and the threshold \( j_0 \) would become

\[ 8\pi^2 D^2(\rho_0)n^2 \sigma(\rho_0) + (\nu^2 \sigma^2(\rho_0) - j_0^2) \sigma''(\rho_0) < 0 \]

This shows that the fundamental \( (n = 1) \) is the first mode to become unstable.

### III. A SIMPLE TIME DEPENDENT PROFILE

The form (18) suggests that beyond the instability the optimal profile is a fixed shape moving at a constant velocity \( v \)

\[ \rho = g(x - vt) \text{.} \]

Due to conservation law (6) the current is then

\[ j(x, t) = j_0 - v \rho_0 + v g(x - vt) \]

If such a profile is the optimal profile, then the variational principle (15) reduces to

\[ G(j_0) = - \min_{g(x), v} \left\{ \int_0^1 \frac{dx}{2\sigma(g(x))} \left[ j_0 - v \rho_0 + v g(x) + D(g(x)) g'(x) - \nu \sigma(g(x)) \right] \right\}^2 \quad (19) \]

This is of the form (the term linear in \( g' \) gives a null contribution due to the periodic boundary conditions)

\[ G(j_0) = - \inf_{g(x), v} \int_0^1 dx \left[ X(g) + g'^2 Y(g) \right] \quad (20) \]

where

\[ X(g) = \frac{(j_0 - v \rho_0 + v g - \nu \sigma(g))^2}{2\sigma(g)} \]

and

\[ Y(g) = \frac{D^2(g)}{2\sigma(g)} \]

The optimal \( v \) in (15) is then given by

\[ v = - \frac{\int dx \frac{(j_0 - v \rho_0)}{\sigma(g)}}{\int dx \frac{(j_0 - v \rho_0)^2}{\sigma(g)}} = - \frac{j_0}{\int dx \frac{(j_0 - \rho_0)^2}{\sigma(g)}} \quad (21) \]

this last simplification being due to the constraint \( \int g(x) dx = \rho_0 \). With this constraint and for a fixed \( v \), a variational calculation of the optimal \( g \) in (20) shows that \( g \) should satisfy

\[ X''(g) - 2Y(g) g'' - g'' Y'(g) = C_2 \]

Multiplying both sides by \( g' \) allows one to integrate once so that \( g \) satisfies

\[ X(g) - g'^2 Y(g) = C_1 + C_2 g \]

where \( C_1 \) and \( C_2 \) are constants (which is an extension of the equation (15) of (7) to the case of the ring).

For fixed \( j_0 \), \( \rho_0 \) and \( v \), if one denotes by \( g_1 \) and \( g_2 \) the two extrema of the profile \( g \) (generically, the profile \( g(x) \) is a periodic function of period 1 with a single minimum \( g_1 \) and a single maximum \( g_2 \)), one can determine the constants \( C_1 \) and \( C_2 \) by (22) in terms of \( g_1 \) and \( g_2 \) (as \( X(g_1) = C_1 + C_2 g_1 \) and \( X(g_2) = C_1 + C_2 g_2 \)). The differential equation (22) determines the whole profile (up to a translation on the ring) and the constants \( g_1 \) and \( g_2 \) are then fixed by the fact that

\[ \frac{1}{2} = \int_{x(g_1)}^{x(g_2)} dx = \int_{g_1}^{g_2} \frac{dg}{g} \]

and

\[ \frac{\rho_0}{2} = \int_{g_1}^{g_2} g \sqrt{\frac{Y(g)}{X(g) - C_1 - C_2 g}} \frac{dg}{g} \]

### IV. EXACT NUMERICS FOR THE WEAKLY ASYMMETRIC EXCLUSION PROCESS ON A RING

We wrote a program to calculate exactly \( \mu(\lambda) \) for the weakly asymmetric exclusion process (WASEP) on a ring of \( N \) sites with \( P = N \rho \) particles. In the simple symmetric exclusion process (SSEP), each particle jumps to its right at rate \( \frac{1}{2} \) and to its left at rate \( \frac{1}{2} \) and the functions \( D(\rho) \) and \( \sigma(\rho) \) are given by

\[ D_{\text{SSEP}} = \frac{1}{2} \quad (23) \]
WASEP: asymmetry $\nu = 10$ and density $1/2$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$\mu(\lambda)$ as defined by (10) for the weakly asymmetric exclusion process on a ring of $N = 6, 10, 14, 18, 22$ sites when the asymmetry parameter $\nu = 10$ and the density $1/2$ (thin lines). As $N$ increases, the data increase and seem to accumulate to the value predicted (dashed line) by assuming that the optimal profile is the one discussed in section III. If the optimal profile had been flat, the curve would have been predicted by (13) (thick line). The horizontal dotted line gives the value of $\mu$ below which the flat profile becomes unstable.}
\end{figure}

\[ \sigma_{\text{SEP}} = \rho(1 - \rho). \]  
\hspace{1cm} (24)

If one introduces a weak electric field to the right, the model becomes the WASEP and the rates become $\frac{1}{2} + \frac{\nu}{2N}$ to the right and $\frac{1}{2} - \frac{\nu}{2N}$ to the left.

As the evolution is a Markov process, one can build, as explained in [18, 19], from the Markov matrix, a $\lambda$-dependent matrix, the largest eigenvalue of which is $\mu(\lambda)$ defined by (10). According to the linear stability analysis, the flat profile becomes unstable (10) for

\[ j_0^2 < \rho (1 - \rho) [\nu^2 \rho (1 - \rho) - \pi^2] \]

or by (17) for

\[ N \mu_{\text{flat}}(\lambda) < \frac{\pi^2}{2}. \]  
\hspace{1cm} (25)

We have calculated the exact eigenvalue $\mu(\lambda)$ for lattice sizes from $N = 6$ to $22$, at density $1/2$ for an asymmetry $\nu = 10$ (in order to avoid negative rates for small system sizes, we have replaced in our programs the rates $\frac{1}{2} \pm \frac{\nu}{2N}$ by $\exp[\pm \nu/N]/2$. The results show a rather quick convergence with increasing $N$ towards the value corresponding to a shape determined by (19). Clearly the flat profile gives a value too low, incompatible with the numerical data.

V. A BRIDGE BETWEEN A WEAK AND A STRONG ASYMMETRY

Assuming that the optimal profile is the one discussed in section III if one tries to make $\nu$ large, and one writes

\[ j_0 = \nu i_0 \]  
\hspace{1cm} (26)

the moving profile which satisfies (22) becomes very steep in the regions where it varies, and it takes the form of a step function with two constant values $g_1$ and $g_2$ separated by two discontinuities

\[ g(x) = g_1 \quad \text{for} \quad 0 < x < y \]
\[ g(x) = g_2 \quad \text{for} \quad y < x < 1 \]  
\hspace{1cm} (27)

so that the parameters $g_1$, $g_2$ and $y$ are related to $\rho$ and $i_0$ by

\[ \rho_0 = yg_1 + (1 - y)g_2 \]  
\hspace{1cm} (28)
\[ i_0 = y\sigma(g_1) + (1 - y)\sigma(g_2). \]  
\hspace{1cm} (29)

The expression of the velocity (21) then becomes

\[ v = \nu \frac{\sigma(g_2) - \sigma(g_1)}{g_2 - g_1}. \]  
\hspace{1cm} (30)

Then using the fact that $g'$ vanishes when $g(x) = g_1$ or $g_2$ and replacing (26) into (22) implies that the constants $C_1, C_2$ (in (22)) vanish at order $\nu^2$. Thus asymptotically in $\nu$, one can rewrite (22) as

\[ g'' = \nu^2 \frac{1}{D(g)^2 (g_1 - g_2)^2} \left[ g_2 (\sigma(g_1) - \sigma(g)) \right. \]  
\hspace{1cm} (31)
\[ + g_1 (\sigma(g) - \sigma(g_2)) + g (\sigma(g_2) - \sigma(g_1)) \left. \right]^2 \]

and

\[ G(j_0) = -\nu^2 \int \frac{dx}{\sigma(g)(g_1 - g_2)^2} \left[ g_2 (\sigma(g_1) - \sigma(g)) \right. \]  
\hspace{1cm} (32)
\[ + g_1 (\sigma(g) - \sigma(g_2)) + g (\sigma(g_2) - \sigma(g_1)) \left. \right]^2 \]

If one replaces $g$ by its expression (21), one gets that the order $\nu^2$ of $G(j_0)$ vanishes. The next order in the large $\nu$ expansion is dominated by the rounding-off of the discontinuities in (22) as given by (31). As the profile $g(x)$ is composed of two monotonic parts one can then use (31) into (32) and obtain for $g_2 > g_1$

\[ G(j_0) = -2\nu \int_{g_1}^{g_2} dg \frac{D(g)}{(g_2 - g_1) \sigma(g)} \left[ \sigma(g)(g_2 - g_1) \right. \]  
\hspace{1cm} (33)
\[ - \sigma(g_1)(g_2 - g) - \sigma(g_2)(g - g_1) \left. \right| \]
It is remarkable that \( y \) is not present in this expression. In the case of the weakly asymmetric exclusion process on a ring, expressions \( [20, 24] \) of \( D(\rho) \) and \( \sigma(\rho) \) lead for large \( \nu \) to

\[
G(j_0) = -\nu \left[ g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} \right. \\
\left. - (1 - g_1)(1 - g_2) \ln \left( \frac{1 - g_1}{1 - g_2} \right) \right] \tag{33}
\]

In this case \( (33) \) becomes \( v = \nu (1 - g_1 - g_2) \).

If one takes formally \( \nu = N \), the hopping rates \( 1/2 \pm \nu/2N \) become 1 and 0, so that the model reduces to the totally asymmetric exclusion process and one gets from \( [7, 26, 35] \)

\[
\text{Pro} \left( \frac{Q_T}{T} = i_0 \right) \sim \exp \left( -T \left[ g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} \right. \\
\left. - (1 - g_1)(1 - g_2) \ln \left( \frac{1 - g_1}{1 - g_2} \right) \right] \right) \tag{34}
\]

As we will see it in section \( VI \) this is exactly the large deviation function predicted by the Jensen-Varadhan theory \( [20] \) to maintain a profile \( [27] \) formed of a shock and an antishock in the totally asymmetric exclusion process. Other aspects of the relation between the large deviation functional of the weakly asymmetric exclusion process and the Jensen-Varadhan functional in systems with open boundary conditions will be presented in \( [21] \).

VI. LARGE DEVIATIONS OF THE CURRENT IN THE TOTALLY ASYMMETRIC EXCLUSION PROCESS

In the totally asymmetric process, each particle jumps to its neighboring site, on its right, at rate 1, if the target site is empty (and there is no other jump).

The large deviation function of the current of the totally asymmetric exclusion process on a ring of \( N \) sites, with \( P \) particles, has been calculated exactly \( [18, 22] \). If \( Q_T \) is the total number of jumps during time \( T \) over a given bond on the ring, one knows that for large \( T \),

\[
\langle e^{Q_T} \rangle \sim e^{\mu(\lambda) T} \tag{35}
\]

and explicit expressions of \( \mu(\lambda) \) has been obtained for all \( N \) and \( P \) by the Bethe ansatz \( [18, 22] \).

For large \( N \), it was shown in particular (equation (53) of \( [22] \) with the proper redefinition of the parameters) that for \( \lambda < 0 \)

\[
\mu(\lambda) = \frac{(1 - e^{\lambda \rho_0})(1 - e^{\lambda(1-\rho_0)})}{(1 - e^\lambda)}. \tag{36}
\]

We are going now to argue that this result can be understood, by assuming that \( (35) \) is dominated by configurations of the form \( [27] \) moving at a velocity \( v = 1 - g_1 - g_2 \). These density profiles are everywhere constant except for a shock (at some position \( z \) with \( g(z - 0) = g_1 \) and \( g(z + 0) = g_2 \) for \( g_2 > g_1 \)) and an antishock (at position \( z + y \) with \( g(z + y - 0) = g_2 \) and \( g(z + y + 0) = g_1 \)). From the Jensen-Varadhan theory \( [21] \) the probability of maintaining such a shape moving at this velocity on the ring over a very long period of time \( T \) reduces to the probability of maintaining an antishock between the densities \( g_2 \) and \( g_1 \) moving at velocity \( v \). The probability of the latter event, which we denote by \( P_T(g_1, g_2) \) is given \( [20] \) by

\[
P_T(g_1, g_2) \sim \exp \left( -T \left[ g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} \right. \\
\left. - (1 - g_1)(1 - g_2) \ln \left( \frac{1 - g_1}{1 - g_2} \right) \right] \right). \tag{37}
\]

The corresponding integrated current \( Q_T \) is

\[
Q_T = T[yg_1(1 - g_1) + (1 - y)g_2(1 - g_2)]
\]

since over a long period of time \( T \) a given bond spends a fraction \( y \) of the time at density \( g_1 \) and \( 1 - y \) at density \( g_2 \).

Therefore, if the configurations of the form \( [27] \) dominate the large deviations of the current, one expects

\[
\mu(\lambda) = \max_{y, g_1, g_2} \left\{ \lambda[yg_1(1 - g_1) + (1 - y)g_2(1 - g_2)] - \left[ g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} - (1 - g_1)(1 - g_2) \ln \left( \frac{1 - g_1}{1 - g_2} \right) \right] \right\} \tag{38}
\]

where the maximum has to satisfy the constraint

\[
\rho_0 = yg_1 + (1 - y)g_2 \tag{39}
\]

A calculation of the optimum in \( (38) \), with the constraint \( (39) \) leads to

\[
g_1 = \frac{e^\lambda - e^{\lambda(1-\rho_0)}}{e^\lambda - 1}; \quad g_2 = \frac{e^{\lambda \rho_0} - 1}{e^\lambda - 1}
\]

and \( (35) \) becomes \( (36) \).

This shows that the result of the Bethe ansatz \( (36) \) can be physically understood in terms of an optimal profile which takes the form of the step function \( (27) \). The probability of maintaining this profile is given by the Jensen-Varadhan expression \( (37) \) which in fact is identical to \( (35) \) obtained, in the large \( \nu \) limit, for the WASEP from the hydrodynamic large deviation theory.

That the fluctuations are due, in the strong asymmetric case, to configurations formed by a gas of
shocks and antishocks has been already pointed out by Fogedby [21, 27, 28]. The calculation of this section shows that the large deviation of the current, in the range \( \lambda < 0 \), can be understood quantitatively in terms of a single pair of shock-antishock. Whether the Fogedby theory would allow to understand all the current fluctuations, including the range \( \lambda > 0 \) where the expression of \( \mu(\lambda) \) is more complicated [22] than [36], remains an interesting open question.

VII. CONCLUSION

In the present work we have determined the limit of stability [16, 17] of a flat profile for a diffusive lattice gas on a ring. This instability beyond which the optimal profile becomes modulated is of the same nature as the phase transition found for several other non-equilibrium systems [5, 23, 24, 27]. As the calculation is based on a local stability analysis, one cannot of course exclude first order transitions, i.e. that the flat profile might become globally unstable.

In section [11] we have obtained numerical evidence that the macroscopic fluctuation theory predicts correctly the large deviation function of the current for the weakly asymmetric exclusion process. The numerical results are consistent with the second order phase transition predicted in section [11] and with a modulated density profile moving at a constant velocity as suggested in section [11]. These results could in principle be confirmed by solving the Bethe ansatz equations for the WASEP, since \( \mu(\lambda) \) can be calculated exactly for the ring geometry [24, 37].

It would be interesting to extend the results of the present work to the case of open boundary conditions. One difficulty is that the time-independent profile, found in [7], is much more complicated than the flat profile for the ring geometry, and we did not succeed so far to obtain the condition which would generalize (10, 17) for this open geometry.

Lastly, we noticed that the large deviation function [4], obtained for the weakly asymmetric diffusion process in the large drift limit is identical to the one predicted for a strong asymmetry by the Jensen-Varadhan theory [37] (see also [21]).

Despite this bridge between the large deviation function of the current of weakly and strongly asymmetric systems, and some recent results on zero-range processes [18], a theory of current fluctuations for strongly asymmetric lattice gas such as the ASEP with open boundary conditions remains an open problem.

VIII. APPENDIX: A DERIVATION OF (45)

We present here an heuristic derivation of the hydrodynamic large deviations [4]. Let us consider a system of \( N \) sites and decompose it into \( N/l \) boxes of \( l \) sites each. Let us define the density \( \rho_i(t) \) in box \( i \) at time \( t \) and \( q_i(t) \) the total number of particles transferred from box \( i \) to box \( i+1 \) during a time interval \( t, t+\tau \) (this time \( \tau \) should be large enough for the \( q_i(t) \) to be a Gaussian characterized by its average and its variance as in [13], but short enough compared to the characteristic time of variation of the densities \( \rho_i(t) \)).

If one writes that the \( q_i(t) \) are Gaussian, one gets

\[
\text{Pro}(q_i(s), \rho_i(s)) \sim \exp \left[ -\sum_{k=1}^{T/\tau} \sum_{i=1}^{N/l} \frac{1}{2\sigma(\rho_i(s)) } \left( q_i(s) + \frac{\rho_i(s) - \rho_i(s)}{l} \frac{\tau}{\nu} \right)^2 \right]
\]

where \( k = s \tau \). The factor \( \nu/N \) comes from the weak asymmetry of the jumps. Clearly the conservation of the number of particles gives

\[
\rho_i(s + \tau) = \rho_i(s) + \frac{q_{i-1}(s) - q_i(s)}{l}
\]

Now if one takes a continuous limit by writing

\[
\rho_i(s) = \rho \left( \frac{i}{N}, \frac{s}{N^2} \right)
\]

and one defines a rescaled current by

\[
q_i(s) = \frac{\tau}{N} j \left( \frac{i}{N}, \frac{s}{N^2} \right)
\]

one gets

\[
\text{Pro}(j(x,s), \rho(x,s)) \sim \exp \left[ -N^{-1} \int_0^T ds \int_0^1 dx \right]
\]

\[
\left( j(x,s) + D(\rho(x,s)) \frac{d\rho(x,s)}{dx} - \nu \sigma(\rho(x,s)) \right)^2
\]

which is exactly (15). Furthermore [10] with the scaling [12] leads to [6].

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