PULLBACK RANDOM ATTRACTORS FOR FRACTIONAL STOCHASTIC $p$-LAPLACIAN EQUATION WITH DELAY AND MULTIPLICATIVE NOISE

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Abstract. This paper is concerned with the pullback random attractors of nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay driven by multiplicative white noise defined on bounded domain. We first prove the existence of a continuous nonautonomous random dynamical system for the equations as well as the uniform estimates of solutions with respect to the delay time and noise. We then show pullback asymptotical compactness of solutions and the existence of tempered random attractors by utilizing the Arzela-Ascoli theorem and appropriate uniform estimates of the solutions. Finally, we establish the upper semicontinuity of the random attractors when time delay approaches zero.

1. Introduction. In this paper, we investigate the pullback random attractors of the following nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay driven by multiplicative white noise

$$
\frac{\partial u}{\partial t} + (-\Delta)^\alpha_p u + \delta u = F(x,u(t,x)) + f(x,u(t-\rho,x)) + g(t,x) + u \circ dW \frac{dt}{dt}, \quad x \in \mathcal{O}, \quad t > \tau,
$$

(1)

with boundary condition

$$
u(t,x) = 0, \quad x \in \partial \mathcal{O}, \quad t > \tau
$$

(2)

and initial condition

$$
u_\tau(s,x) := u(\tau+s,x) = \varphi(s,x), \quad x \in \mathcal{O}, \quad s \in [-\rho,0],
$$

(3)

where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \mathcal{O}$, $\alpha \in (0,1)$, $(-\Delta)^\alpha_p$ is the non-local, nonlinear, fractional $p$-Laplacian with $2 \leq p < \infty$, $\delta > 0$ is a constant, $\tau \in \mathbb{R}$, $\rho > 0$ is the delay time of the system, the superlinear term $F : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a dissipative condition as specified later, $f : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity capturing the time delay,
$g \in L_{loc}^2(\mathbb{R}, L^2(\mathcal{O}))$ is a deterministic time-dependent forcing and $W$ is a two-side real-valued Wiener process on a probability space. The symbol $\circ$ in (1) indicates that the stochastic equation is understood in the sense of Stratonovich’s integration.

If $0 < \alpha < 1$ and $p = 2$, then the fractional $p$-Laplacian $(-\Delta)^\alpha_p$ reduces to the standard linear fractional Laplace operator $(-\Delta)^\alpha$. The attractors of partial differential equations with standard linear fractional Laplacian have been extensively investigated in many publications, see, e.g., [15, 17, 23, 25, 26, 27, 30, 35, 37, 39, 40] and the references therein. If $\alpha = 1$ and $p > 2$, then the fractional $p$-Laplacian $(-\Delta)^\alpha_p$ becomes the standard $p$-Laplace operator given by $-\text{div}(|\nabla u|^{p-2}\nabla u)$ which is a model equation arising from porous media mechanics. The attractors of systems with standard $p$-Laplacian have been investigated in [11, 12, 18] for the deterministic case and in [13, 14, 21, 22, 49, 50] for the stochastic case. If $\alpha = 1$ and $p = 2$, then the fractional $p$-Laplacian $(-\Delta)^\alpha_p$ reduces to the standard Laplace operator. In this special case, random attractors of stochastic PDEs have been extensively studied in [2, 5, 10, 32] for the autonomous equations and [3, 14, 33, 34, 36] for the nonautonomous equations. We should mention that existence of random attractors for stochastic PDEs with fractional $p$-Laplacian $(-\Delta)^\alpha_p$ have also been studied in [41, 42] recently.

On the other hand, evolution equations with a time delay arise from evolution phenomena in physics, biology and engineering when after-effect, time lag or pre-history influence are taken into account in the model of the systems, in this case, the response of a system depends not only on the current state of the system, but also on the past history of the system, we refer the readers to [16, 28, 47] for more comments and citations. It is worth mentioning that the existence of attractors for partial differential equations with delays have become a large and growing interdisciplinary area of research in recent years, see [6, 19, 20, 31, 38] for deterministic differential equations with delays and [5, 7, 24, 43, 44, 46, 48] for stochastic differential equations with delays.

However, to the best of our knowledge, the existence and upper semicontinuity of attractors of PDEs with nonlinear fractional $p$-Laplacian $(-\Delta)^\alpha_p$ and time delay for $0 < \alpha < 1$ and $p > 2$ remains open even in the deterministic case. Therefore, inspired by above-mentioned aspects, the main object of the present paper is to close this gap and prove the existence of tempered pullback random attractor as well as the limit behavior of long term dynamics as the time delay goes to zero for nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay (1)-(3) for all $0 < \alpha < 1$ and $2 \leq p < \infty$. We want to point out that since the fractional $p$-Laplacian $(-\Delta)^\alpha_p$ is a nonlinear nonlocal operator defined on the whole space $\mathbb{R}^n$, we should interpret the homogeneous Dirichlet boundary (2) as $u = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$ instead of $u = 0$ only on $\partial \mathcal{O}$, and then extend $F(\cdot, u(t))$ and $f(\cdot, u(t-\rho))$ to the entire space $\mathbb{R}^n$ by setting $F(x, u(t)) = 0$ and $f(x, u(t-\rho)) = 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{O}$.

2. Preliminaries. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the standard probability space where $\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}$, $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. Given $t \in \mathbb{R}$, denote by $\theta_t : \Omega \to \Omega$ the measure-preserving transformation

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$ 

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a parametric dynamical system. In addition, we denote $(X, d)$ be a complete separable metric space with Borel $\sigma$-algebra $\mathcal{B}(X)$. Suppose
\( \mathcal{D} \) is a collection of some families of nonempty subsets of \( X \). The following result for nonautonomous random dynamical systems can be found in [33] and [34].

**Proposition 1.** Let \( \mathcal{D} \) be an inclusion closed collection of some families of nonempty subsets of \( X \), and \( \Phi \) be a continuous nonautonomous random dynamical system on \( X \) over \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \). Then \( \Phi \) has a \( \mathcal{D} \)-pullback attractor \( \mathcal{A} \) in \( \mathcal{D} \) if \( \Phi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( X \) and \( \Phi \) has a closed measurable \( \mathcal{D} \)-pullback absorbing set \( K \) in \( \mathcal{D} \). The \( \mathcal{D} \)-pullback attractor \( \mathcal{A} \) is unique and is given by, for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\mathcal{A}(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)).
\]

If, in addition, both \( \Phi \) and \( K \) are \( T \)-periodic for \( T > 0 \), then so is the attractor \( \mathcal{A} \).

Next, we recall the nonlocal, fractional \( p \)-Laplace operator \((-\Delta)_p^\alpha\) with \( 0 < \alpha < 1 \) and \( 2 \leq p < \infty \). Throughout this paper, we write the norm of \( L^2(\mathbb{R}^n) \) and \( L^r(\mathbb{R}^n) \) for \( r > 2 \) as \( \| \cdot \| \) and \( \| \cdot \|_{L^r(\mathbb{R}^n)} \), respectively. Denote by

\[
\mathcal{L}_n^{-1}(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R}^n \text{ is measurable, } \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{(1 + |x|)^{n+\alpha}} dx < \infty \right\}.
\]

For \( u \in \mathcal{L}_n^{-1}(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \) and \( \epsilon > 0 \), define \((-\Delta)_{p,\epsilon}^\alpha u(x)\) by

\[
(-\Delta)_{p,\epsilon}^\alpha u(x) = C(n, p, \alpha) \int_{y \in \mathbb{R}^n, |y-x| > \epsilon \atop |x-y|^{n+\alpha}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+\alpha}} dy,
\]

where \( C(n, p, \alpha) \) is a normalized constant given by

\[
C(n, p, \alpha) = \frac{\alpha 4^n \Gamma(\frac{p\alpha + n + 2}{2})}{\pi^{n/2} \Gamma(1 - \alpha)} \quad \text{with } \Gamma \text{ being the usual Gamma function.}
\]

Then we define the fractional \( p \)-Laplace operator \((-\Delta)_p^\alpha\) with \( 0 < \alpha < 1 \) and \( 2 \leq p < \infty \) by

\[
(-\Delta)_p^\alpha u(x) = \lim_{\epsilon \downarrow 0} (-\Delta)_{p,\epsilon}^\alpha u(x) = C(n, p, \alpha) \text{ P.V. } \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+\alpha}} dy, \quad x \in \mathbb{R}^n,
\]

if the limit exists, where P.V. means the principal value of the integral. Furthermore, the fractional Sobolev space \( W^{\alpha,p}(\mathbb{R}^n) \) with \( 0 < \alpha < 1 \) and \( 2 \leq p < \infty \) is defined by

\[
W^{\alpha,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha}} dxdy < \infty \right\},
\]

which is endowed with the norm

\[
\|u\|_{W^{\alpha,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x)|^p dx + \|u\|_{W^{\alpha,p}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}, \quad \forall u \in W^{\alpha,p}(\mathbb{R}^n),
\]

where

\[
\|u\|_{W^{\alpha,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha}} dxdy, \quad u \in W^{\alpha,p}(\mathbb{R}^n)
\]
is the so-called Gagliardo seminorm of $W^{\alpha,p}(\mathbb{R}^n)$. Furthermore, it follows from [8, Theorems 6.5 and 6.9] that

$$
\begin{align*}
\|u\|_{L^r(\mathbb{R}^n)} & \leq C\|u\|_{W^{\alpha,r}(\mathbb{R}^n)} , & 2 \leq r \leq 2p - 2 \quad \text{and} \quad 0 < \alpha < \min\{1, n/2\}, \\
\|u\|_{L^r(\mathbb{R}^n)} & \leq C\|u\|_{W^{\alpha,r}(\mathbb{R}^n)} , & 2 \leq r < \infty , \quad \alpha = n/2,
\end{align*}
$$

(7)

where $C = C(n, p, \alpha) > 0$ is a constant and $p$ satisfies

$$
\begin{align*}
2 \leq p & \leq \frac{2n-2\alpha}{n-2}, & \alpha < n/2, \\
2 \leq p & < \infty , & \alpha = n/2.
\end{align*}
$$

(8)

The reader is referred to [4, 8, 29, 30] for more details about fractional operators and spaces.

Note that the fractional Laplacian $(-\Delta)^{\alpha}$ given by (4) is a nonlocal operator defined on $\mathbb{R}^n$, so it is a natural way to interpret the homogeneous Dirichlet boundary (2) as $u = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$ instead of $u = 0$ only on $\partial\mathcal{O}$. In addition, note that $F(\cdot, u(t)) : \mathcal{O} \to \mathbb{R}$ and $f(\cdot, u(t-\rho)) : \mathcal{O} \to \mathbb{R}$ are only defined on $\mathcal{O}$ for any $t, u \in \mathbb{R}$. Based on the above interpretation, we can extend $F(\cdot, u(t))$ and $f(\cdot, u(t-\rho))$ to the entire space $\mathbb{R}^n$ by setting $F(x, u(t)) = 0$ and $f(x, u(t-\rho)) = 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{O}$. Such an extension is always used in this paper. In other words, any function defined on $\mathcal{O}$ will be identified with its trivial extension to $\mathbb{R}^n$. Furthermore, based on this interpretation, we introduce the space $H$ and $V$ defined by

$$
H = \{u \in L^2(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \mathcal{O}\},
$$

$$
V = \{u \in W^{\alpha,p}(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \mathcal{O}\},
$$

(9)

whose norms are denoted by $\| \cdot \|_H$ and $\| \cdot \|_V$, respectively. Then $V$ is a closed subset of $W^{\alpha,p}(\mathbb{R}^n)$ and $(V, \| \cdot \|_{W^{\alpha,p}(\mathbb{R}^n)})$ is a separable, uniformly convex and reflexive Banach space. The dual space of $V$ is denoted by $V^*$. Denote by $C^H_\rho := C([-\rho, 0], H)$ with $\rho > 0$ the space of all continuous functions from $[-\rho, 0]$ to $H$ with norm

$$
\|u\|_{C^H_\rho} = \sup_{s \in [-\rho, 0]} \|u(s)\|, \quad u \in C([-\rho, 0], H).
$$

(10)

Throughout this paper, the letter $c > 0$ denotes a generic constant which may change its values from line to line or even in the same line.

3. **Nonautonomous random dynamical system to fractional stochastic $p$-Laplacian equations with delay.** In this section, based on the fact that the fractional $p$-Laplacian $(-\Delta)^{\alpha}_p$ is a nonlocal operator defined on $\mathbb{R}^n$, we establish the existence of a continuous nonautonomous random dynamical system to the fractional nonautonomous stochastic $p$-Laplacian equation with delay (1)-(3) when the boundary condition (2) is replaced by $u(t, x) = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$ for every $t > \tau$. Precisely, we consider the following nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay

$$
\frac{\partial u}{\partial t} + (-\Delta)^{\alpha}_pu + \delta u

= F(x, u(t, x)) + f(x, u(t-\rho, x)) + g(t, x) + u \circ \frac{dW}{dt}, \quad x \in \mathcal{O}, \ t > \tau,
$$

(11)

with boundary condition

$$
u(t, x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}, \ t > \tau
$$

(12)
and initial condition
\[ u_\tau(s, x) := u(\tau + s, x) = \varphi(s, x), \quad x \in \mathcal{O}, \ s \in [-\rho, 0]. \quad (13) \]

Throughout this paper, we assume that the nonlinear terms \( F \) and \( f \) satisfy the following conditions: for all \( u, u_1, u_2 \in \mathbb{R} \) and \( x, y \in \mathcal{O} \),
\[ F(x, u)u \leq -\delta_1 |u|^p + \varphi_1(x), \quad \varphi_1 \in L^1(\mathcal{O}) \cap L^{2\gamma/p}(\mathcal{O}), \quad (14) \]
\[ |F(x, u)| \leq \varphi_2(x)|u|^{p-1} + \varphi_3(x), \quad \varphi_2 \in L^\infty(\mathcal{O}), \varphi_3 \in L^2(\mathcal{O}), \quad (15) \]
\[ \frac{\partial}{\partial u} F(x, u) \leq \varphi_4(x), \quad \varphi_4 \in L^\infty(\mathcal{O}), \quad (16) \]
\[ |F(x, u) - F(y, u)| \leq |\varphi_5(x) - \varphi_5(y)|, \quad \varphi_5 \in V, \quad (17) \]
\[ f(x, 0) = 0 \quad \text{and} \quad |f(x, u_1) - f(x, u_2)| \leq \varphi_f(x)|u_1 - u_2|, \quad \varphi_f \in L^\infty(\mathcal{O}), \quad (18) \]
where \( \delta \) and \( p \) are positive constants and \( p \) satisfies (8). The following inequality (see [45]) is also needed:
\[ (|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq \beta|a - b|^p \quad \text{for all} \quad p \in [2, \infty), \ a, b \in \mathbb{R}, \quad (19) \]

where \( \beta \) is a positive constant depending only on \( p \).

Let \( z : \Omega \to \mathbb{R} \) be a random variable given by
\[ z(\omega) = -\int_0^\infty e^s \omega(s) ds. \]

Then \( y(t) = z(\theta_t) \) is the unique stationary solution of the one-dimensional stochastic equation \( dy + y dt = dW \). It follows from [1] and [9] that there exists a \( \theta_t \)-invariant set of full measure \( \Omega_0 \) such that \( z(\theta_t \omega) \) is pathwise continuous for every \( \omega \in \Omega_0 \) and
\[ \lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{t} = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_t^0 z(\theta_r \omega) dr = 0. \quad (20) \]

In addition, the continuity of \( z(\theta_t \omega) \) in \( t \) implies that there exists \( \tilde{C} = \tilde{C}(\tau, \omega, T) > 0 \) such that for all \( \tau \in \mathbb{R}, \ \omega \in \Omega_0, \ T > 0 \) and \( t \in [\tau, \tau + T] \),
\[ |z(\theta_t \omega)| \leq \tilde{C}. \quad (21) \]

For convenience, in the sequel, we will not distinguish \( \Omega_0 \) and \( \Omega \) and use the same notation \( \Omega \) for both \( \Omega_0 \) and \( \Omega \).

For our purpose, we need to convert the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11) into a deterministic one parametrized by \( \omega \in \Omega \). To that end, we introduce a new variable \( v \) by
\[ v(t) = e^{-z(\theta_t \omega)} u(t), \quad (22) \]

where \( u \) is a solution of (11)-(13). Substituting (22) into (11)-(13), we obtain for \( t > \tau \),
\[ \frac{dv}{dt} + e^{(p-2)z(\theta_t \omega)}(-\Delta)^p v + \delta v = z(\theta_t \omega) v + e^{-z(\theta_t \omega)} F(x, e^{z(\theta_t \omega)} v(t)) \]
\[ + e^{-z(\theta_{t-\rho} \omega)} f(x, e^{z(\theta_{t-\rho} \omega)} v(t - \rho)) + e^{-z(\theta_t \omega)} g(t, x), \quad x \in \mathcal{O}, \quad (23) \]

with boundary condition
\[ v(t, x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}, \ t > \tau \quad (24) \]
and initial condition
\[ u(t, s, x) := v(t + s, x) = e^{-z(\theta_t + \psi)} \varphi(s, x) \quad \text{and} \quad v(t, s, x) := \psi(s, x), \quad x \in \mathcal{O}, \quad s \in [-\rho, 0]. \]  

Moreover, we obtain a solution \( u \) and \( v \)

To define a continuous nonautonomous random dynamical system for the non-local fractional \( p \)-Laplacian equation with delay, we first need to prove the existence and uniqueness of solutions of problem (23)-(25). By a solution \( v \) of problem (23)-(25), we mean \( v \) satisfies the equation in the following sense.

**Definition 3.1.** Given \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( \psi \in C^H_{\rho} \), a continuous function \( v(\cdot, \tau, \omega, \psi) : [\tau - \rho, \infty) \to H \) is called a solution of problem (23)-(25) if for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( \psi \in C^H_{\rho} \), the following holds:

\[ v(t, \tau, \omega, \psi) = \psi(t) + \frac{d}{dt} \int_\Omega F(x, e^{z(\theta_t \omega)} v(t)) \xi(x) dx \quad \text{for} \quad t \geq \tau. \]

For the deterministic problem (23)-(25), under assumptions (14)-(16) and (18), by the Galerkin method as in [41, Theorem 2.2] one can prove that for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( \psi \in C^H_{\rho} \), it has a unique continuous solution \( v(\cdot, \tau, \omega, \psi) : [\tau - \rho, \infty) \to H \) in the sense of Definition 3.1 such that \( v(\cdot, \tau, \omega, \psi) \) is continuous in \( \psi \in C^H_{\rho} \) and is \( (\mathcal{F}, \mathcal{B}(C^H_{\rho})) \)-measurable in \( \omega \). Moreover, \( v \) satisfies the energy equation:

\[
\frac{d}{dt} \| v(t) \|_p^2 + C(n, p, \alpha) \| e^{z(\theta_t \omega)} v(t) \|_p^2 + \delta \| v(t) \|^2 = z(\theta_t \omega) \| v(t) \|^2 + e^{-z(\theta_t \omega)} \int_\Omega F(x, e^{z(\theta_t \omega)} v(t)) \xi(x) dx + e^{-z(\theta_t \omega)} \int_\Omega g(t, x) \xi(x) dx.
\]

By the unique solution \( v \) of deterministic problem (23)-(25) and the transform (22), we obtain a solution \( u \) of the stochastic \( p \)-Laplacian equation (11)-(13) which is given by

\[ u(t, \tau, \omega, \psi) = e^{z(\theta_t \omega)} v(t, \tau, \omega, \psi). \]

Moreover, \( u(\cdot, \tau, \omega, \psi) \) is continuous in \( \psi \in C^H_{\rho} \) and is \( (\mathcal{F}, \mathcal{B}(C^H_{\rho})) \)-measurable in \( \omega \). Define a mapping \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C^H_{\rho} \to C^H_{\rho} \) given by

\[ \Phi(t, \tau, \omega, \psi)(\cdot) = u_{t, \tau}(\cdot, \tau, \theta_{-\tau} \omega, \psi), \]
where \( u_{t+s}(s, \tau, \theta_{-\tau}, \omega) = u(t + s, \tau, \theta_{-\tau}, \omega) \) for \( s \in [-\rho, 0] \). Then \( \Phi \) is a continuous nonautonomous random dynamical system on \( C^{H}_\rho \) over \((\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\).

Let \( D = \{D(\tau, \omega) \subseteq C^{H}_\rho : \tau \in \mathbb{R}, \omega \in \Omega\} \) be a family of bounded nonempty subsets of \( C^{H}_\rho \). Such a family \( D \) is called tempered if for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{t \to +\infty} e^{-\delta t} \|D(\tau - t, \theta_{-\tau}, \omega)\|_{C^{H}_\rho}^2 = 0,
\]

where \( \|D(\tau - t, \theta_{-\tau}, \omega)\|_{C^{H}_\rho} = \sup_{u \in D(\tau - t, \theta_{-\tau}, \omega)} \|u\|_{C^{H}_\rho} \). Throughout the rest of this paper, we always use \( D \) to denote the collection of all tempered families \( D = \{D(\tau, \omega) \subseteq C^{H}_\rho : \tau \in \mathbb{R}, \omega \in \Omega\} \) which satisfies (30), that is,

\[
D = \{D = \{D(\tau, \omega) \subseteq C^{H}_\rho : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (30)}\}.
\]

In the next section, we will devoted to the \( D \)-uniform estimates of the solutions to (23)-(25). For this purpose, we assume that \( \delta \) satisfies

\[
\delta \geq 8\sqrt{2} \|\varphi\|_{L^\infty(\mathbb{C})}.
\]

We also make the following assumptions on the function \( g \):

\[
\begin{align*}
\int_{-\infty}^{0} e^{\delta \tau} \|g(r + \tau, \cdot)\|^2 \,dr &< \infty, \quad \forall \tau \in \mathbb{R}, \quad (32) \\
\int_{-\infty}^{0} e^{\hat{p} \delta \tau} \|g(r + \tau, \cdot)\|^2 \,dr &< \infty, \quad \forall \tau \in \mathbb{R}, \quad (33)
\end{align*}
\]

where

\[
\hat{p} = \frac{p}{C(n, p, \alpha)}. \quad (34)
\]

When constructing \( D \)-pullback attractors, we will assume that for every \( \gamma > 0 \),

\[
\lim_{t \to +\infty} e^{-\gamma t} \int_{-\infty}^{0} e^{\delta \tau} \|g(r - t, \cdot)\|^2 \,dr = 0. \quad (35)
\]

4. Uniform estimates of solutions. In order to construct \( D \)-pullback random absorbing sets for the nonautonomous random dynamical system \( \Phi \), we first derive the \( D \)-uniform estimates of the solutions to (23)-(25) in this section.

Lemma 4.1. Suppose that the assumptions (14), (18), (31) and (32) hold. Then for every \( \sigma, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), there exists \( T := T(\tau, \omega, D, \sigma) > 0 \) such that for all \( t \geq T \), the solution \( v \) of problem (23)-(25) with \( e^{z(\theta_{-\tau}, \omega)}\psi(\cdot) \in D(\tau - t, \theta_{-\tau}, \omega) \) satisfies

\[
\|v_{\sigma}(\cdot, \tau - t, \theta_{-\tau}, \omega, \psi)\|_{C^{H}_\rho}^2 \leq M \left(1 + R(\tau, \omega, \sigma)\right), \quad (36)
\]

\[
\int_{-\tau}^{\sigma - \tau} e^{rac{\sigma}{2} \delta (r - \sigma + \tau) + 2 \int_{\sigma - \tau}^{\tau} \zeta(\theta_{-\tau}, \omega) \,d\mu + (p - 2)z(\theta_{-\tau}, \omega)} \|v(r + \tau, \tau - t, \theta_{-\tau}, \omega, \psi)\|^2 \|f^\sigma_r \|_{W^p_{\alpha, p}(\mathbb{R}^n)} \,dr
\]

\[
\leq M \left(1 + R(\tau, \omega, \sigma)\right), \quad (37)
\]

where \( M > 0 \) is a constant independent of \( \tau, \omega \) and \( D \), and \( R(\tau, \omega, \sigma) \) is given by

\[
R(\tau, \omega, \sigma) = \int_{-\infty}^{\sigma - \tau} e^{rac{\sigma}{2} \delta (r - \sigma + \tau) + 2 \int_{\sigma - \tau}^{\tau} \zeta(\theta_{-\tau}, \omega) \,d\mu - 2z(\theta_{-\tau}, \omega)} (1 + \|g(r + \tau)\|^2) \,dr. \quad (38)
\]
Proof. We start with the energy equation (27) and first deal with the second term on the right-hand side of (27). We get by (14) that
\[
e^{-z(\theta, \omega)} \int_{\mathcal{O}} F(x, e^{z(\theta, \omega)} v(t)) v(t) \, dx 
\leq -\delta t e^{(p-2)z(\theta, \omega)} \|v\|_{L_p}^p + e^{-2z(\theta, \omega)} \|\varphi_1\|_{L^1(\mathcal{O})}.
\]
By (18) and the Young inequality, we know that the third term on the right-hand side of (27) satisfies
\[
e^{-z(\theta, \omega)} \int_{\mathcal{O}} f(x, e^{z(\theta, \rho)} v(t-\rho)) v(t) \, dx \leq \frac{\delta}{32} \|v(t)\|^2 + 8\delta e^{-2z(\theta, \omega) - z(\theta, \rho)}
\]
\[
\int_{\mathcal{O}} \varphi_1^2(x) \|v(t-\rho)\|^2 \, dx \frac{\delta}{32} \|v(t)\|^2 + \frac{8\|\varphi_1\|^2_{L^\infty(\mathcal{O})}}{\delta} e^{-2z(\theta, \omega) - z(\theta, \rho)} \|v(t-\rho)\|^2.
\]
For the last term on the right-hand side of (27), by the Young inequality, we have
\[
e^{-z(\theta, \omega)} \int_{\mathcal{O}} g(t, x) v(x) \, dx \leq \frac{\delta}{32} \|v\|^2 + 8\delta e^{-2z(\theta, \omega)} \|g(t)\|^2.
\]
Therefore, it follows from (5) and (39)-(41) that
\[
\frac{d}{dt} \|v(t, \tau, \omega, \psi)\|^2 + \frac{7}{4} \frac{\delta}{2} \|v(t)\|^2 + C_1 e^{(p-2)z(\theta, \omega)} \|v\|_{W^{n, p}(\mathbb{R}^n)}^p 
\leq -\frac{\delta}{8} \|v(t, \tau, \omega, \psi)\|^2
\]
\[
+ \frac{16\|\varphi_1\|_{L^\infty(\mathcal{O})}}{\delta} e^{-2z(\theta, \omega) - z(\theta, \rho)} \|v(t-\rho, \tau, \omega, \psi)\|^2
\]
\[
+ C_2 e^{-2z(\theta, \omega)} \left[ \|g(t)\|^2 + \|\varphi_1\|_{L^1(\mathcal{O})} \right] ,
\]
where
\[
C_1 = \min\{C(n, p, \alpha), 2\delta_1\} \quad C_2 = \max\left\{ \frac{16}{\delta}, 2 \right\}.
\]
Multiplying (42) by \(e^{7\delta t-2f^s_{\tau-} \theta(t, \omega) d\mu}\) and then integrating the inequality on \((\tau-t, \sigma+s)\) for any fixed \(s \in [-\rho, 0]\) with \(\sigma > \tau - t + \rho, \) and then replacing \(\omega\) by \(\theta_{\tau-}\), we get that
\[
\|v(\sigma+s, \tau-t, \theta_{\tau-}, \omega, \psi)\|^2 
\leq e^{7\delta t-2f^s_{\tau-} \theta(t, \omega) d\mu} \|v(r, \tau-t, \theta_{\tau-}, \omega, \psi)\|_{W^{n, p}(\mathbb{R}^n)}^p 
\leq e^{7\delta t-2f^s_{\tau-} \theta(t, \omega) d\mu} \|v(t, \tau-t, \omega, \psi)\|^2 \|C_{n+1}^p\|
\]
\[
- \frac{\delta}{8} \int_{\tau-t}^{\sigma+s} e^{7\delta (r-\sigma-s) + 2f^s_{r-} \theta(t, \omega) d\mu} \|v(r, \tau-t, \theta_{\tau-}, \omega, \psi)\|^2 \|C_{n+1}^p\| 
\]
\[
+ e^{7\delta (\tau-t-\sigma-s) + 2f^s_{\tau-t} \theta(t, \omega) d\mu} \|v(r, \tau-t, \theta_{\tau-}, \omega, \psi)\|^2 \|C_{n+1}^p\| 
\]
\[
+ \frac{16\|\varphi_1\|_{L^\infty(\mathcal{O})}}{\delta} \int_{\tau-t}^{\sigma+s} e^{7\delta (r-\sigma-s) + 2f^s_{r-} \theta(t, \omega) d\mu} e^{-2z(\theta, \omega) - z(\theta, \rho)} \|v(r-\rho, \tau-t, \theta_{\tau-}, \omega, \psi)\|^2 dr.
\]
\[ + C_2 \int_{-t}^{\sigma+s-t} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) + 2 f_{r+\tau}^{s+\tau} z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \left[ \| g(r) \|^2 + \| \phi_1 \|_{L^1(\Omega)} \right] dr. \]

Changing variables in above inequality, we have
\[
\| v(\sigma + s, \tau - t, \theta_{-\tau}\omega, \psi) \|^2
\]
\[
\quad + C_1 \int_{-t}^{\sigma+s-t} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) + 2 f_{r+\tau}^{s+\tau} z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| v(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi) \|^2 dr
\]
\[
\leq e^{\frac{2}{7}(\tau-t(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| \psi \|^2_{C^2_H}
\]
\[
- \frac{\delta}{8} \int_{-t}^{\sigma+s-t} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| v(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi) \|^2 dr
\]
\[
\quad + C_2 \int_{-t}^{\sigma+s-t} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \left[ \| g(r + \tau) \|^2 + \| \phi_1 \|_{L^1(\Omega)} \right] dr. \quad (44)
\]

Next, we compute the third term on the right-hand side of (44),
\[
\int_{-t}^{\sigma+s-t} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| v(r + \tau - \rho, \tau - t, \theta_{-\tau}\omega, \psi) \|^2 dr
\]
\[
= \int_{-t-\rho}^{\sigma+s-t-\rho} e^{\frac{2}{7}(r(\sigma-s)+2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| v(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi) \|^2 dr
\]
\[
\leq e^{\frac{2}{7}(\tau-(\sigma+s+\tau)) + 2 f_{r+\tau}^s z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} \| \psi \|^2_{C^2_H} dr
\]
\[
\quad \cdot \| v(r + \tau, \tau - t, \theta_{-\tau}\omega, \psi) \|^2 dr. \quad (45)
\]

We get from (20) that for \( \epsilon_1 = \frac{7\delta}{16} \) and \( \omega \in \Omega \), when \( r \to -\infty \),
\[
e^{\frac{2}{7}(r+\tau) z(\theta_{r+\tau}) d\mu - 2 z(\theta_{r+\tau})} + 2 z(\theta_{r+\tau}) - z(\theta_{r+\tau})\]
\[
\leq e^{\frac{2\delta}{7} + 4\rho} = 1. \quad (46)
\]
Therefore, by (20), (45), (46) and a simple calculation we know that there is $T_1(\omega) > 0$ such that for all $t \geq T_1$,

$$\int_{-t}^{\sigma+t-}\epsilon^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \geq e^{2}[z(\theta,\omega)-z(\theta_{-\tau},\omega)]$$

$$\|v(r + \tau - \rho, \tau - t, \theta_{-\tau}, \omega, \psi)\|^{2}dr$$

$$\leq e^{\frac{7}{2}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu}\|v\|_{C_{\mu}^{H}}^{2}$$

$$+\int_{-t}^{\sigma+s-}\epsilon^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v(r + \tau - t, \theta_{-\tau}, \omega, \psi)\|^{2}dr. \quad (47)$$

Therefore, using a similar method with the proof of (47), it follows from (21), (31), (44), (47) and the fact that $s \in [-\rho, 0]$ that for all $t \geq T_1$,

$$\|v(\sigma + s, \tau - t, \theta_{-\tau}, \omega, \psi)\|^{2}$$

$$+C_1\int_{-t}^{\sigma+s+\tau}e^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \geq e^{2}[z(\theta,\omega)-z(\theta_{-\tau},\omega)]$$

$$\leq e^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2}$$

$$+\left(\frac{16}{\delta}\|\varphi\|_{L^{\infty}(\mathcal{O})}^{2}-\frac{\delta}{8}\right)$$

$$\int_{-t}^{\sigma+s-}\epsilon^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v(r + \tau - t, \theta_{-\tau}, \omega, \psi)\|^{2}dr$$

$$+C_2\epsilon^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\int_{-t}^{\sigma+s+\tau}e^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2}$$

$$\left[\|g(r + \tau)\|^{2} + \|\varphi\|_{L^{1}(\mathcal{O})}\right]dr \leq e^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2}$$

$$+C\int_{-t}^{\sigma+s-}\epsilon^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2}$$

$$+\left(\frac{16}{\delta}\|\varphi\|_{L^{\infty}(\mathcal{O})}^{2}-\frac{\delta}{8}\right)\int_{-t}^{\sigma+s-}\epsilon^{7}(r-\sigma-s+\tau)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v(r + \tau - t, \theta_{-\tau}, \omega, \psi)\|^{2}dr. \quad (48)$$

Since $e^{7}(\theta_{-\tau},\psi) \in D(\tau - t, \theta_{-\tau}, \omega)$ with $D \in D$, by (20), (21) and (30) one can verify that for $c_2 = \frac{34}{16}$,

$$\lim_{t \to +\infty} e^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2}$$

$$\leq \lim_{t \to +\infty} e^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \geq e^{2}[z(\theta,\omega)-z(\theta_{-\tau},\omega)]$$

$$\|D(\tau - t, \theta_{-\tau}, \omega)\|_{C_{\mu}^{H}}^{2}$$

$$= e^{7}(\tau-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \lim_{t \to +\infty} e^{-7}(\tau-t+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \geq e^{2}[z(\theta,\omega)-z(\theta_{-\tau},\omega)]$$

$$\|D(\tau - t, \theta_{-\tau}, \omega)\|_{C_{\mu}^{H}}^{2} = c \lim_{t \to +\infty} e^{-7}(\tau-t+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu \geq e^{2}[z(\theta,\omega)-z(\theta_{-\tau},\omega)]$$

$$\|D(\tau - t, \theta_{-\tau}, \omega)\|_{C_{\mu}^{H}}^{2} = 0. \quad (49)$$

Therefore, by (49), we find that there exists $T_2 = T_2(\tau, \omega, D, \sigma, \tau) > T_1$ such that for all $t \geq T_2$,

$$e^{7}(\tau-t-\sigma)+2\int_{r}^{\sigma+s+\tau}z(\theta,\omega)d\mu\|v\|_{C_{\mu}^{H}}^{2} \leq 1. \quad (50)$$
Similarly, by (20) and (32) we find that for all \(\sigma > t + \rho\),
\[
\int_{-t}^{\sigma - \tau} e^{\frac{1}{2} \delta (r - \sigma + \tau) + 2 \int_{r}^{r_0} z(\theta, \omega) d\mu - 2z(\theta, \omega)} \left[ \left\| g(r + \tau) \right\|^2 + \| \varphi_1 \|_{L^1(\Omega)} \right] dr
\]
\[
\leq \int_{-\infty}^{0} e^{\frac{1}{2} \delta r} (1 + \| g(r + \tau) \|^2) dr < \infty.
\] (51)

It follows from (48)-(51) that there exists \(T = T(\tau, \omega, D, \sigma) \geq T_2\) such that for all \(t \geq T\),
\[
\| v_\sigma(s, \tau - t, \theta_{-\tau}, \psi) \|^2 + C_1 \int_{-t}^{\sigma + s - \tau} e^{\frac{1}{2} \delta (r - \sigma - s + \tau) + 2 \int_{r}^{r_0} z(\theta, \omega) d\mu + (p-2)z(\theta, \omega)} \left\| v(r + \tau - t, \theta_{-\tau}, \psi) \right\|_{W^{\alpha,p}(\mathbb{R}^n)}^p dr
\]
\[
\leq c + c \int_{-\infty}^{\sigma} e^{\frac{1}{2} \delta (r - \sigma + \tau) + 2 \int_{r}^{r_0} z(\theta, \omega) d\mu - 2z(\theta, \omega)} (1 + \| g(r + \tau) \|^2) dr.
\] (52)

Then the desired results (36) and (37) follows from (52) immediately. This completes the proof of Lemma 4.1.

The following estimate in \(W^{\alpha,p}(\mathbb{R}^n)\) plays an important role in the proof of the asymptotic compactness of the nonautonomous random dynamical system associated with the stochastic \(p\)-Laplacian equation (11)-(13).

**Lemma 4.2.** Suppose that the assumptions (16)-(18), (32) and (33) hold. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D\), there exists \(T := T(\tau, \omega, D) \geq \rho + 1\) such that for all \(t \geq T\), the solution \(v\) of problem (23)-(25) with \(e^{z(\theta_{-\tau})} \psi \in D(\tau - t, \theta_{-\tau}, \omega)\) satisfies
\[
\sup_{-\rho \leq s \leq 0} \left\| v_\tau(s, \tau - t, \theta_{-\tau}, \omega, \psi) \right\|_{W^{\alpha,p}(\mathbb{R}^n)}^p
\]
\[
\leq M_1 + M_1 \int_{-\infty}^{0} \left( e^{\frac{1}{4} \delta r + 2 \int_{r}^{r_0} z(\theta, \omega) d\mu - 2z(\theta, \omega)} + e^{\frac{1}{2} \delta r + 2 \int_{r}^{r_0} z(\theta, \omega) d\mu - 2z(\theta, \omega)} \right)
\]
\[
\left( \| g(r + \tau) \|^2 + 1 \right) dr,
\] (53)

where \(M_1 > 0\) is a constant independent of \(\tau, \omega\) and \(D\).

**Proof.** Multiplying (23) by \((-\Delta)^p \alpha v\) and integrating over \(\mathbb{R}^n\), we obtain
\[
\frac{C(n, p, \alpha)}{2p} \frac{d}{dt} \left\| v(t) \right\|_{W^{\alpha,p}(\mathbb{R}^n)}^p + c^{(p-2)z(\theta, \omega)} \left\| (-\Delta)^p v \right\|^2 + (\delta - z(\theta, \omega)) \left\| v(t) \right\|_{W^{\alpha,p}(\mathbb{R}^n)}^p
\]
\[
= e^{-z(\theta, \omega)} \int_{\mathbb{R}^n} F(x, e^{z(\theta, \omega)} v)(-\Delta)^p v dx + e^{-z(\theta, \omega)} \int_{\mathbb{R}^n} f(x, e^{z(\theta, \omega)} v(t - \rho))(-\Delta)^p v dx
\]
\[
+ e^{-z(\theta, \omega)} \int_{\mathbb{R}^n} g(t, x)(-\Delta)^p v dx.
\] (54)

We now estimate all items on the right-hand side of (54). For the first term on the right-hand side of (54), by (16) and (17) we have
For the last term on the right-hand side of (54), by the Young inequality, we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(F(x, e^{z(\theta_1 \omega)}v(x)) - F(y, e^{z(\theta_1 \omega)}v(y)))}{|x - y|^{n+p\alpha}} dxdy
\]

\[
e^{-z(\theta_1 \omega)} C(n, p, \alpha)
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(F(x, e^{z(\theta_1 \omega)}v(x)) - F(y, e^{z(\theta_1 \omega)}v(y)))}{|x - y|^{n+p\alpha}} dxdy
\]

\[
+ e^{-z(\theta_1 \omega)} C(n, p, \alpha)
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(F(y, e^{z(\theta_1 \omega)}v(x)) - F(y, e^{z(\theta_1 \omega)}v(y)))}{|x - y|^{n+p\alpha}} dxdy
\]

\[
\leq e^{-z(\theta_1 \omega)} C(n, p, \alpha) \frac{p-1}{p} \int \int |v(x) - v(y)|^p dxdy
\]

\[
+ e^{-z(\theta_1 \omega)} C(n, p, \alpha) \int \int \frac{\varphi_5(x) - \varphi_5(y)}{|x - y|^{n+2\alpha}} dxdy
\]

\[
+ C(n, p, \alpha) \int \int \frac{\varphi_4(y)|v(x) - v(y)|^p}{|x - y|^{n+2\alpha}} dxdy
\]

\[
\leq e^{-z(\theta_1 \omega)} \left( \frac{C(n, p, \alpha)}{p} \|\varphi_5\|_V^p
\]

\[
+ C(n, p, \alpha) \left( e^{-2z(\theta_1 \omega)} \frac{p-1}{p} + \|\varphi_4\|_{L^\infty(\mathcal{O})} \right) \|v\|_{W^{\infty,p}(\mathbb{R}^n)}. \tag{55}
\]

For the second term on the right-hand side of (54), we get from (18) that

\[
e^{-z(\theta_1 \omega)} \int_{\mathbb{R}^n} f(x, e^{z(\theta_1 \omega)}v(t - \rho)) (-\Delta)^{\alpha} v dx
\]

\[
\leq \frac{1}{4} e^{(p-2)z(\theta_1 \omega)} \|(-\Delta)^{\alpha} v\|^2 + e^{-z(\theta_1 \omega)} e^{-2z(\theta_1 \omega)} \|f(x, e^{z(\theta_1 \omega)}v(t - \rho))\|^2
\]

\[
\leq \frac{1}{4} e^{(p-2)z(\theta_1 \omega)} \|(-\Delta)^{\alpha} v\|^2
\]

\[
+ e^{-(p-2)z(\theta_1 \omega)} \|\varphi f\|_{L^\infty(\mathcal{O})} e^{-2z(\theta_1 \omega) - z(\theta_1 \omega)} \|v(t - \rho)\|^2. \tag{56}
\]

For the last term on the right-hand side of (54), by the Young inequality, we have

\[
e^{-z(\theta_1 \omega)} \int_{\mathbb{R}^n} g(t, x) (-\Delta)^{\alpha} v dx
\]

\[
\leq \frac{1}{4} e^{(p-2)z(\theta_1 \omega)} \|(-\Delta)^{\alpha} v\|^2 + e^{-2z(\theta_1 \omega)} e^{-(p-2)z(\theta_1 \omega)} \|g(t)\|^2. \tag{57}
\]

Consequently, it follows from (54)-(57) that

\[
\frac{d}{dt} \|v\|_{W^{\infty,p}(\mathbb{R}^n)}^p + \tilde{\rho} e^{(p-2)z(\theta_1 \omega)} \|(-\Delta)^{\alpha} v\|^2 + \rho(2\delta - 2z(\theta_1 \omega)) \|v\|_{W^{\infty,p}(\mathbb{R}^n)}^p
\]

\[
\leq ce^{-2z(\theta_1 \omega)} \|v\|_{W^{\infty,p}(\mathbb{R}^n)}^p
\]

\[
+ c e^{-2z(\theta_1 \omega) - z(\theta_1 \omega) - (p-2)z(\theta_1 \omega)} \|v(t - \rho)\|^2
\]

\[
+ ce^{-2z(\theta_1 \omega) - (p-2)z(\theta_1 \omega)} \|g(t)\|^2 + ce^{-z(\theta_1 \omega)}. \tag{58}
\]

Let $\tau \in \mathbb{R}$, $t \geq \rho + 1$, $\omega \in \Omega$, and $\sigma \in (\tau + s - 1, \tau + s)$ for $s \in [-\rho, 0]$. Similar with the proof the Lemma 4.1, multiplying (58) by $e^{\tau \rho \delta - 2\delta} \int_0^\tau \sigma e^{z(\theta_1 \omega) d\mu}$ and then
integrating the inequality on \((\sigma, \tau + s)\), we infer that

\[
\begin{align*}
& e^{\frac{1}{p} \rho \delta(s - \tau)} \int_0^{\tau + s} z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p d\mu \\
& \leq e^{\frac{1}{p} \rho \delta(\sigma - \tau) - \frac{2}{p} \int_0^{\tau + s} z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p d\mu} \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta r - 2 \rho \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta r - 2 \rho \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} \|f\|_{W^{\alpha, p}(\mathbb{R}^n)} dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta r - 2 \rho \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} \|g\|_{W^{\alpha, p}(\mathbb{R}^n)}^2 dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta r - 2 \rho \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} dr,
\end{align*}
\]

from which one gets that

\[
\begin{align*}
& \|v(\sigma, \tau - t, \omega, \psi)\|_{W^{\alpha, p}(\mathbb{R}^n)} \leq e^{\frac{1}{p} \rho \delta(\sigma - \tau)) - \frac{2}{p} \int_0^{\tau + s} z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p d\mu} \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta(r - s) - \frac{2}{p} \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta(r - s) - \frac{2}{p} \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} \|f\|_{W^{\alpha, p}(\mathbb{R}^n)} dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta(r - s) - \frac{2}{p} \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} \|g\|_{W^{\alpha, p}(\mathbb{R}^n)}^2 dr \\
& + c \int_0^{\tau + s} e^{\frac{1}{p} \rho \delta(r - s) - \frac{2}{p} \int_0^r z(\theta, \omega) e^{-2 z(\theta, \omega)} \|v\|_{W^{\alpha, p}(\mathbb{R}^n)}^p} dr.
\end{align*}
\]
We now estimate the second term on the right-hand side of (60), by (20), (36) and using a similar method with the proof of (47) and (51), we know that there exists $T_3 = T_3(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq T_3$,

$$
\begin{align*}
\int_{\tau - \rho - 1}^\tau e^{\frac{\tau}{2} \tilde{\rho}(r - s)} + 2 \tilde{p} \int_{s}^{r} z(\theta_{r - \tau} \omega) d\mu - 2z(\theta_{r - \tau} \omega) - (p - 2)z(\theta_{r - \tau} \omega) dr 
\leq & \int_{\tau - \rho - 1}^0 e^{\frac{\tau}{2} \tilde{\rho}(s - r)} + 2 \tilde{p} \int_{r}^{s} z(\theta_{s - \tau} \omega) d\mu - 2z(\theta_{s - \tau} \omega) - (p - 2)z(\theta_{s - \tau} \omega) dr \\
& \sup_{\tau - \rho - 1 \leq r \leq \tau} \|v(r, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p(\mathbb{R}^n)}^2 dr \\
& \leq c + c \int_{-\infty}^0 e^{\frac{\tau}{2} \delta r + 2 \tilde{p} \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - 2z(\theta_{\tau} \omega)} \left(\|g(r + \tau)\|^2 + 1\right) dr.
\end{align*}
$$

(61)

By (60) and (61), we get for $t \geq T_3$,

$$
\begin{align*}
\|v(r + s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p(\mathbb{R}^n)}^p 
& \leq c \int_{\tau - \rho - 1}^\tau e^{\frac{\tau}{2} \tilde{\rho}(\tau - s)} + 2 \tilde{p} \int_{s}^{\tau} z(\theta_{\tau - \tau} \omega) d\mu \|v(r, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p(\mathbb{R}^n)}^p dr \\
& + c + c \int_{-\infty}^0 \left(e^{\frac{\tau}{2} \delta r + 2 \tilde{p} \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - 2z(\theta_{\tau} \omega)} + e^{\frac{\tau}{2} \tilde{\rho} r + 2 \tilde{p} \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - 2z(\theta_{\tau} \omega)}\right) \\
& \left(\|g(r + \tau)\|^2 + 1\right) dr.
\end{align*}
$$

which combined with (32), (33) and the Gronwall-Bellman inequality gets that,

$$
\begin{align*}
\|v(r + s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p(\mathbb{R}^n)}^p 
& \leq c \\
& + c \int_{-\infty}^0 \left(e^{\frac{\tau}{2} \delta r + 2 \tilde{p} \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - 2z(\theta_{\tau} \omega)} + e^{\frac{\tau}{2} \tilde{\rho} r + 2 \tilde{p} \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - 2z(\theta_{\tau} \omega)}\right) \\
& \left(\|g(r + \tau)\|^2 + 1\right) dr.
\end{align*}
$$

(62)

Then (62) implies the desired result (53). This completes the proof of Lemma 4.2.

In what follows, we derive the estimate of solutions of problem (11)-(13) in $L^p(\mathbb{R}^n)$.

**Lemma 4.3.** Suppose that the assumptions (14), (18) and (32) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T := T(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq T$, the solution $v$ of problem (23)-(25) with $e^{z(\theta_{-\omega}) \psi} \in D(\tau - t, \theta_{-\omega})$ satisfies

$$
\begin{align*}
\sup_{-\rho \leq s \leq 0} \|v_T(s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p(\mathbb{R}^n)}^p 
& \leq M_2 \int_{-\infty}^0 e^{\frac{\tau}{2} \delta r + p \int_{\theta_{\tau} \omega} z(\theta_{\tau} \omega) d\mu - p z(\theta_{\tau} \omega)} \left(\|g(r + \tau)\|^2 + 1\right) dr,
\end{align*}
$$

(63)

where $M_2 > 0$ is a constant independent of $\tau$, $\omega$ and $D$. 
Proof. Multiplying (23) by $|v|^{p-2}v$ and integrating over $\mathbb{R}^n$, we obtain that
\[
\frac{1}{p} \frac{d}{dt} \|v(t)\|_{L_p(\mathbb{R}^n)}^p + (\delta - z(\theta_1)) \|v(t)\|_{L_p(\mathbb{R}^n)}^p + e^{(p-2)z(\theta_1)} \int_{\mathbb{R}^n} (-\Delta_p v)|v|^{p-2}v dx
\]
\[
eq e^{-z(\theta_1)} \int_{\mathbb{R}^n} F(x, e^{z(\theta_1)} v(t)) |v|^{p-2}v dx + e^{-z(\theta_1)} \int_{\mathbb{R}^n} f(x, e^{z(\theta_1-\omega)} v(t-\rho)) |v|^{p-2}v dx + e^{-z(\theta_1)} \int_{\mathbb{R}^n} g(t, x)|v|^{p-2}v dx.
\]

We get from (19) that the third term on the left-hand side of (64) satisfies
\[
eq e^{(p-2)z(\theta_1)} \int_{\mathbb{R}^n} (-\Delta_p v)|v|^{p-2}v dx = e^{(p-2)z(\theta_1)} C(n, p, \alpha) \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{2p-2}}{|x-y|^{n+\alpha}} dxdy
\]
\[
\geq e^{(p-2)z(\theta_1)} C(n, p, \alpha) \frac{\beta}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{2p-2}}{|x-y|^{n+\alpha}} dxdy \geq 0.
\]

By (14) and Young’s inequality, we get that the first term on the right-hand side of (64) satisfies
\[
eq e^{-z(\theta_1)} \int_{\mathbb{R}^n} F(x, e^{z(\theta_1)} v(t)) |v|^{p-2}v dx \leq -\frac{\delta_1}{2} e^{(p-2)z(\theta_1)} \|v(t)\|_{L_p(\mathbb{R}^n)}^{2p-2} + ce^{-pz(\theta_1)} \|\varphi_1\|_{L_p(\mathbb{R}^n)}^{2p-2}.
\]

By (18) and the Young inequality, we know that the second term on the right-hand side of (64) satisfies
\[
eq e^{-z(\theta_1)} \int_{\mathbb{R}^n} f(x, e^{z(\theta_1-\omega)} v(t-\rho)) |v|^{p-2}v dx
\]
\[
\leq e^{-[z(\theta_1)-z(\theta_1-\omega)]} \int_{\mathbb{R}^n} \|\varphi_f(x)\|_{L_p(\mathbb{R}^n)} \|v(t-\rho)\|_{L_p^p(\mathbb{R}^n)}
\]
\[
\leq \frac{p-1}{p} \|\varphi_f\|_{L_p^\infty(\mathbb{R}^n)} \left( \frac{p-1}{p-2} \right)^{p-1} e^{-pz(\theta_1-\omega)} \|v(t-\rho)\|_{L_p^p(\mathbb{R}^n)}.
\]

For the last term on the right-hand side of (27), by the Young inequality, we have
\[
eq e^{-z(\theta_1)} \int_{\mathbb{R}^n} g(t, x)|v|^{p-2}v dx \leq \frac{\delta_1}{2} e^{(p-2)z(\theta_1)} \|v(t)\|_{L_p(\mathbb{R}^n)}^{2p-2} + ce^{-pz(\theta_1)} \|g(t)\|^2.
\]

Therefore, it follows from (14) and (64)-(68) that
\[
\frac{d}{dt} \|v(t, \tau - t, \omega, \psi)\|_{L_p(\mathbb{R}^n)}^p + (2\delta - pz(\theta_1)) \|v(t, \tau - t, \omega, \psi)\|_{L_p(\mathbb{R}^n)}^p
\]
\[
\leq \frac{\|\varphi_f\|_{L_p^\infty(\mathbb{R}^n)} \left( \frac{p-1}{p-2} \right)^{p-1}}{\delta^p} e^{-pz(\theta_1-\omega)} \|v(t-\rho, \tau - t, \omega, \psi)\|_{L_p(\mathbb{R}^n)}^p + ce^{-pz(\theta_1)} \|g(t)\|^2 + 1.
\]
Let $\tau \in \mathbb{R}, t \geq 2\rho + 1, \omega \in \Omega$, and $\sigma \in (\tau + s - 1, \tau + s)$ for $s \in [-\rho, 0]$. Multiplying (69) by $e^{\int_{\theta - t}^{\theta - \tau} \delta(\theta, \omega) d\mu}$, then integrating the inequality on $(\sigma, \tau + s)$ and substituting $\theta - \tau$ for $\omega$, we get that

$$
|v(\tau + s, \tau - t, \theta - \tau, \psi)|^{P}_{L^p(\mathbb{R}^n)} \\
\leq e^{\int \delta(\sigma - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} |v(\sigma, \tau - t, \theta - \tau, \psi)|^{P}_{L^p(\mathbb{R}^n)} \\
+ \frac{\|\varphi f\|_{L^p(\mathbb{R}^n)}^p}{\delta - 1} \left( \frac{p - 1}{p - 2} \right) \int_{\theta - \tau}^{\theta - \tau + \delta} e^{\int \delta(\tau - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} \left| v(\sigma, \tau - t, \theta - \tau, \psi) \right| \frac{d\mu}{\delta}
$$

(70)

By a simple calculation, we know that the second term on the right-hand side of (70) satisfies

$$
\int_{\theta - \tau}^{\theta - \tau + \delta} e^{\int \delta(\tau - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} \left| v(\sigma, \tau - t, \theta - \tau, \psi) \right| \frac{d\mu}{\delta}
$$

(71)

Therefore, we get from (70) and (71) that

$$
|v(\tau + s, \tau - t, \theta - \tau, \psi)|^{P}_{L^p(\mathbb{R}^n)} \\
\leq e^{\int \delta(\sigma - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} |v(\sigma, \tau - t, \theta - \tau, \psi)|^{P}_{L^p(\mathbb{R}^n)} \\
+ c \int_{\theta - \tau}^{\theta - \tau + \delta} e^{\int \delta(\tau - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} \left| v(\sigma, \tau - t, \theta - \tau, \psi) \right| \frac{d\mu}{\delta}
$$

(72)

Integrating (72) with respect to $\sigma$ on $(\tau + s - 1, \tau + s)$, we get for $t \geq 2\rho + 1,

$$
|v(\tau + s, \tau - t, \theta - \tau, \psi)|^{P}_{L^p(\mathbb{R}^n)} \\
\leq c \int_{\theta - \tau}^{\theta - \tau + \delta} e^{\int \delta(\tau - \tau - s) + p \int_{\theta - t}^{\theta - \tau} \mu(\theta, \omega) d\mu} \left| v(\sigma, \tau - t, \theta - \tau, \psi) \right| \frac{d\mu}{\delta}
$$

(73)
Suppose by (29). pactness of the continuous nonautonomous random dynamical system $\Phi$ as defined equality, we get that 

\[
\int_{-\infty}^{t} e^{\frac{1}{2} \lambda r + p} f_{\omega}^{p}(\theta r, \omega) d\mu r \omega (\|g(r + \tau)\|^{2} + 1) dr,
\]

which implies the desired result (63). This completes the proof of Lemma 4.3. □

The following estimates are concerned with the derivatives of the solution $v$ in time of (23)-(25) in $L^{2}(\mathbb{R}^{n})$ which will be used to show the pullback asymptotic compactness of the continuous nonautonomous random dynamical system $\Phi$ as defined by (29).

Lemma 4.4. Suppose (15) and (18) hold. Then for every $t \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset D$, there exists $T := T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution $v$ of (23)-(25) satisfies, for $t > T$,

\[
\left\| \frac{d}{dt} v(t, \tau, \omega, \psi) \right\|^{2} \leq M_{3} \left( \left\| v(t, \tau, \omega, \psi) \right\|^{2} + \| v(t, \tau, \omega, \psi) \|^{2} + \| v(t - \rho, \tau, \omega, \psi) \|^{2} + \| g(t) \|^{2} + \| z(\theta \omega) \|^{2} + 1 \right),
\]

where $e^{\frac{1}{2}(\omega)} \psi \in D(\tau, \omega)$, $M_{3} > 0$ is a constant independent of $\tau$, $\omega$ and $D$.

Proof. Multiplying (23) by $\frac{dv}{dt}$ and integrating over $\mathbb{R}^{n}$ one gets that

\[
\int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} F(x, e^{z(\theta \omega)} v(t)) \frac{dv}{dt} dx + \int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} f(x, e^{z(\theta \omega)} v(t - \rho)) \frac{dv}{dt} dx
\]

\[
+ \int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} g(t, x) \frac{dv}{dt} dx + \int_{\mathbb{R}^{n}} z(\theta \omega) \frac{dv}{dt} dx.
\]

(75)

From the Sobolev embedding inequality (7), the assumption (15) and Young’s inequality, we get that

\[
\int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} F(x, e^{z(\theta \omega)} v(t)) \frac{dv}{dt} dx \leq e^{-z(\theta \omega)} \int_{\mathbb{R}^{n}} |F(x, e^{z(\theta \omega)} v(t))| \frac{dv}{dt} dx
\]

\[
\leq \int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} \left( \varphi_{2}(x) |e^{z(\theta \omega)} v(t)|^{p-1} + \varphi_{3}(x) \right) \frac{dv}{dt} dx
\]

\[
\leq \frac{1}{8} \| \frac{dv}{dt} \|^{2} + 4e^{2(p-4)z(\theta \omega)} \| \varphi_{2} \|^{2}_{L^{\infty}(\mathbb{R})} \| v(t) \|^{2p-2}_{L^{2p-2}(\mathbb{R}^{n})} + 4e^{-2z(\theta \omega)} \| \varphi_{3} \|^{2}_{L^{2}(\mathbb{R})}
\]

\[
\leq \frac{1}{8} \| \frac{dv}{dt} \|^{2} + 4e^{2(p-4)z(\theta \omega)} \| \varphi_{2} \|^{2}_{L^{\infty}(\mathbb{R})} \| v(t) \|^{2p-2}_{L^{2p-2}(\mathbb{R}^{n})} + 4e^{-2z(\theta \omega)} \| \varphi_{3} \|^{2}_{L^{2}(\mathbb{R})}.
\]

(76)

By (18) and Young’s inequality, we have

\[
\int_{\mathbb{R}^{n}} e^{-z(\theta \omega)} f(x, e^{z(\theta \omega)} v(t - \rho)) \frac{dv}{dt} dx
\]

\[
\leq e^{-z(\theta \omega)} \int_{\mathbb{R}^{n}} |f(x, e^{z(\theta \omega)} v(t - \rho))| \frac{dv}{dt} dx
\]

\[
\leq \int_{\mathbb{R}^{n}} \varphi_{f}(x) |v(t - \rho)| \frac{dv}{dt} dx \leq \frac{1}{8} \| \frac{dv}{dt} \|^{2} + 2\| \varphi_{f} \|^{2}_{L^{2}(\mathbb{R})} \| v(t - \rho) \|^{2}.
\]

(77)
5. Existence of pullback attractors. In this section, we prove the existence and uniqueness of tempered pullback attractors for the nonautonomous nonlocal fractional stochastic p-Laplacian equation with delay (11)-(13) in $C^H_\rho$. To that end, we need to establish the existence of tempered random absorbing sets and the pullback asymptotic compactness of the nonautonomous random dynamical system $\Phi$ defined by (29).

**Lemma 5.1.** Suppose that the assumptions (14), (18), (31), (32) and (35) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the nonautonomous random dynamical system $\Phi$ has a closed measurable tempered pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ as defined by

$$K(\tau, \omega) = \left\{u \in C^H_\rho : \|u\|_{\mathcal{C}^H_\rho}^2 \leq \sup_{s \in [-\rho,0]} e^{2z(\theta, \omega)} R(\tau, \omega) \right\}$$

with $R(\tau, \omega)$ being a positive number given by

$$R(\tau, \omega) = M + M \int_{-\infty}^0 e^{\frac{4}{1+2p} z(\theta, \omega) + 1} dr,$$

where $M > 0$ is the same constant as in Lemma 4.1.

**Proof.** By Lemma 4.1, we know that for every $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T := T(\tau, \omega, D_\omega) > 0$ such that for all $t \geq T$, the solution $v$ of problem (23)-(25) with $e^{\frac{4}{1+2p} z(\theta, \omega)} \psi \in D(\tau - t, \theta, \omega)$ for $-\rho \leq s \leq 0$ satisfies

$$v_\tau(t, \tau - t, \theta, \omega, \psi) \in \left\{v \in C^H_\rho : \|v\|_{\mathcal{C}^H_\rho}^2 \leq R(\tau, \omega) \right\}.$$  

(80)

By (28) and (80), we know for $t \geq T$,

$$\sup_{s \in [-\rho,0]} \|u_\tau(s, \tau - t, \theta, \omega, \psi)\|^2 = \sup_{s \in [-\rho,0]} e^{2z(\theta, \omega)} \|v_\tau(s, \tau - t, \theta, \omega, \psi)\|^2 \leq \sup_{s \in [-\rho,0]} e^{2z(\theta, \omega)} R(\tau, \omega).$$  

(81)

In addition, by (29) we have

$$\Phi(t, \tau - t, \theta, \omega) = u_\tau(s, \tau - t, \theta, \omega, \phi),$$
which along with (81) shows that \( \Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \) for all \( t \geq T \), which means that \( K \) pullback attracts all elements of \( D \).

In what follows, we verify that \( K \) is tempered, i.e., \( K \in D \). For this purpose, we firstly show that for any \( \gamma > 0 \),
\[
\lim_{t \to +\infty} e^{-\gamma t} R(\tau - t, \theta_{-t}\omega) = 0.
\] (82)

From (79) we know that
\[
R(\tau - t, \theta_{-t}\omega) = M
\]
\[
+ M \int_{-\infty}^{0} e^{\frac{7}{2} \delta r} z(t, \omega) d\mu + 2 f_{r_{-t}} z(t, \omega) d\mu e^{-2z(t, \omega)} \|g(r + \tau - t)\|^2 + 1) dr,
\] (83)

By (83) and (20) we know that there exists a positive constant \( T_1 > 0 \) such that for all \( t \geq T_1 \) and \( \gamma > 0 \),
\[
R(\tau - t, \theta_{-t}\omega) \leq M + M e^{\frac{7}{2} \gamma t} \int_{-\infty}^{0} e^{\delta r} (1 + \|g(r + \tau - t)\|^2) dr,
\]
which along with (35) one gets that
\[
\lim sup_{t \to +\infty} e^{-\gamma t} R(\tau - t, \theta_{-t}\omega)
\]
\[
\leq \lim sup_{t \to +\infty} e^{-\gamma t} M + \lim sup_{t \to +\infty} M e^{\frac{7}{2} \gamma t} \int_{-\infty}^{0} e^{\delta r} (1 + \|g(r + \tau - t)\|^2) dr
\]
\[
\leq \lim sup_{t \to +\infty} M e^{\frac{7}{2} \gamma t} \int_{-\infty}^{0} e^{\delta r} dr + \lim sup_{t \to +\infty} M e^{\frac{7}{2} \gamma t} e^{-\frac{7}{2} \gamma t} \int_{-\infty}^{0} e^{\delta r} \|g(r - t)\|^2 dr = 0.
\]

Therefore, from (82) and (20) we get that for every \( \gamma > 0 \),
\[
\lim_{t \to +\infty} e^{-\gamma t} \|K(\tau - t, \theta_{-t}\omega)\|
\]
\[
= \lim_{t \to +\infty} e^{-\frac{7}{2} \gamma t} \lim_{t \to +\infty} \left( e^{-\gamma t} R(\tau - t, \theta_{-t}\omega) \right)^{\frac{1}{2}} = 0,
\]
which combined with (30) means that \( K \subseteq D \). Furthermore, note that for each \( \tau \in \mathbb{R} \), \( R(\tau, \cdot) : \Omega \to \mathbb{R} \) is \( (\mathcal{F}, B(\mathbb{R})) \)-measurable. This means that \( K(\tau, \omega) \) is also measurable in \( \omega \in \Omega \), and therefore \( K \) is a closed measurable tempered \( D \)-pullback absorbing set for \( \Phi \). This completes the proof Lemma 5.1. \( \square \)

Next, we discuss the asymptotic compactness of the nonautonomous random dynamical system associated with the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13).

**Lemma 5.2.** Suppose that the assumptions (14)-(18) and (31)-(33) hold. Then the nonautonomous random dynamical system \( \Phi \) associated with the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) is \( D \)-pullback asymptotically compact in \( C^H_\rho \).

**Proof.** For every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), \( s \in [-\rho, 0] \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subseteq D \), we need to prove that the sequence \( \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, \varphi_n) = u_\tau(s, \tau - t_n, \theta_{-t_n}\omega, \varphi_n) \) has a convergent subsequence in \( C^H_\rho \) whenever \( t_n \to \infty \) and \( \varphi_n \in D(\tau - t_n, \theta_{-t_n}\omega) \). Next, we show that \( u_\tau(s, \tau - t_n, \theta_{-t_n}\omega, \varphi_n) \) is precompact in \( C^H_\rho \) by using the Arzela-Ascoli theorem.
Firstly, we prove that $u_τ(\cdot, τ - t_n, θ_{-τ}ω, ϕ_n)$ is uniformly equicontinuous in $C^H_ρ$. It follows from Lemmas 4.1, 4.3 and 4.4 combined with the continuity of $z(θ_1ω)$ and the fact that $g \in L^2(\mathbb{R}, L^2(\Omega))$ that there exist constants $N = N(τ, ω, ) ≥ 1$ and $M_4 = M_4(τ, ω, ) > 0$ such that for all $n ≥ N$,

$$\int^{τ}_{τ-ρ} \left\| \frac{d}{dr} v(r, τ - t_n, θ_{-τ}ω, ψ_n) \right\|^2 dr ≤ M_4. \quad (84)$$

Therefore, for any $n ≥ N$ and $s_1$, $s_2 ∈ [-ρ, 0]$, we get from (84) that

$$\|u_τ(s_2, τ - t_n, θ_{-τ}ω, ϕ_n) - u_τ(s_1, τ - t_n, θ_{-τ}ω, ϕ_n)\|
= \left\| \int^{τ+s_2}_{τ+s_1} \frac{d}{dr} v(r, τ - t_n, θ_{-τ}ω, ψ_n) dr \right\|
\leq |s_2 - s_1|^\frac{1}{2} \left( \int^{τ+s_2}_{τ+s_1} \left\| \frac{d}{dr} v(r, τ - t_n, θ_{-τ}ω, ψ_n) \right\|^2 dr \right)^{\frac{1}{2}}
\leq |s_2 - s_1|^\frac{1}{2} \left( \int^{τ}_{τ-ρ} \left\| \frac{d}{dr} v(r, τ - t_n, θ_{-τ}ω, ψ_n) \right\|^2 dr \right)^{\frac{1}{2}} \leq \sqrt{M_4}|s_2 - s_1|^\frac{1}{2}. \quad (85)$$

In addition, by (28) we know that

$$u_τ(s, τ - t_n, θ_{-τ}ω, ϕ_n) = e^{z(θ_1ω)}v_τ(s, τ - t_n, θ_{-τ}ω, ψ_n), \quad s ∈ [-ρ, 0]. \quad (86)$$

Hence, by (85) and (86) together with the continuity of $z(θ_1ω)$ one gets that for any $n ≥ N$ and $s_1$, $s_2 ∈ [-ρ, 0]$,

$$\|u_τ(s_2, τ - t_n, θ_{-τ}ω, ϕ_n) - u_τ(s_1, τ - t_n, θ_{-τ}ω, ϕ_n)\|
= \|e^{z(θ_2ω)}v_τ(s_2, τ - t_n, θ_{-τ}ω, ψ_n) - e^{z(θ_1ω)}v_τ(s_1, τ - t_n, θ_{-τ}ω, ψ_n)\|
\leq e^{z(θ_2ω)}\|v_τ(s_2, τ - t_n, θ_{-τ}ω, ψ_n)\| \cdot |e^{z(θ_2ω)} - e^{z(θ_1ω)} - 1|
+ e^{z(θ_1ω)}\sqrt{M_4}|s_2 - s_1|^\frac{1}{2}
→ 0 \quad as \quad s_2 - s_1 → 0,$$

which implies that $u_τ(\cdot, τ - t_n, θ_{-τ}ω, ϕ_n)$ is uniformly equi-continuous in $C^H_ρ$.

Secondly, we show that $u(τ+s, τ - t_n, θ_{-τ}ω, ϕ_n)$ is precompact in $H$ for every fixed $s ∈ [-ρ, 0]$. By the fact that the embedding $V ↪ H$ is compact, Lemma 4.2, Lemma 4.3 and (5), one can easily prove that $v(τ+s, τ - t_n, θ_{-τ}ω, ψ_n)$ is precompact in $H$ for every fixed $s ∈ [-ρ, 0]$, and therefore $u(τ+s, τ - t_n, θ_{-τ}ω, ϕ_n)$ is precompact in $H$ for every fixed $s ∈ [-ρ, 0]$ due to (86). Hence, the nonautonomous random dynamical system $Φ$ associated with the nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay (11)-(13) is $D$-pullback asymptotically compact in $C^H_ρ$. This completes the proof of Lemma 5.2.\[\square\]

Now, we are in the position to demonstrate the existence of tempered $D$-pullback attractors of $Φ$ associated with the nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay (11)-(13).

**Theorem 5.3.** Suppose that the assumptions (14)-(18), (31)-(33) and (35) hold. Then the continuous nonautonomous random dynamical system $Φ$ associated with the nonautonomous nonlocal fractional stochastic $p$-Laplacian equation with delay (11)-(13) has a unique $D$-pullback random attractor $A = \{A(τ, ω) : τ ∈ \mathbb{R}, ω ∈ Ω\} ∈ D$ in $C^H_ρ$. If, in addition, there exists $T > 0$ such that for each fixed $x ∈ \mathcal{O}$, the
nonautonomous function \( g(t,x) \) is \( T \)-periodic in \( t \in \mathbb{R} \), then the attractor \( \mathcal{A} \) is also \( T \)-periodic, i.e., \( \mathcal{A}(t+T,\omega) = \mathcal{A}(t,\omega) \) for all \( t \in \mathbb{R} \) and \( \omega \in \Omega \).

**Proof.** The existence, uniqueness and characterization of \( \mathcal{D} \)-pullback random attractor \( \mathcal{A} \) of \( \Phi \) in \( C^H_\rho \) follow from Proposition 1 immediately, based on Lemmas 5.1 and 5.2. Furthermore, the continuous nonautonomous random dynamical system \( \Phi \) associated with the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) is \( T \)-periodic due to the \( T \)-periodicity of nonautonomous function \( g(t,x) \) about \( t \) i.e., \( \Phi(t,T+\omega,\varphi) = \Phi(t,\tau,\omega,\varphi) \) for every \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). In addition, by (79) we know that \( R(\tau+T) = R(\tau) \) if nonautonomous function \( g(t,x) \) is \( T \)-periodic in \( t \), which together with Lemma 5.1 implies that the tempered \( \mathcal{D} \)-pullback absorbing set \( K \) is also \( T \)-periodic, i.e., \( K(T+\tau,\omega) = K(\tau,\omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). Therefore, by Proposition 1 we know that the attractor \( \mathcal{A} \) is \( T \)-periodic. This completes the proof of Theorem 5.3. \( \Box \)

6. Upper semicontinuity of attractors as delay approaches zero. In this section, we investigate the upper semicontinuity of random attractors of the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) when the delay approaches zero. Hereafter, we assume \( \rho \in (0,1] \) and write the solution and nonautonomous random dynamical system of the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) as \( u^\rho \) and \( \Phi^\rho \) to indicate their dependence on \( \rho \), respectively. Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let

\[
K^\rho(\tau,\omega) = \{ u \in C^H_\rho : \|u\|^2_{C^H_\rho} \leq \sup_{s\in[-\rho,0]} e^{2z(0,\omega)} R^\rho(\tau,\omega) \},
\]

(87)

where \( R^\rho(\tau,\omega) = R_\epsilon(\tau,\omega) \) is the number as in Lemma 5.1. From (87) and Lemma 5.1 we know that \( K^\rho = \{ K^\rho(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is a \( \mathcal{D} \)-pullback absorbing set of \( \Phi^\rho \) in \( C^H_\rho \) for all \( \rho \in (0,1] \) since the estimates of Lemma 5.1 are uniform with respect to \( \rho \) in \( (0,1] \). Then \( \Phi^\rho \) has a \( \mathcal{D} \)-pullback attractor \( \mathcal{A}^\rho \subset \mathcal{D} \) in \( C^H_\rho \) by Lemma 5.1. The invariance of \( \mathcal{A}^\rho \) implies for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\mathcal{A}^\rho(\tau,\omega) \subset K^\rho(\tau,\omega).
\]

(88)

When \( \rho = 0 \), then the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11) becomes a nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation without delay

\[
\frac{\partial u}{\partial t} + (-\Delta)^\rho u + \delta u = F(x,u(t,x)) + f(x,u(t,x)) + g(t,x) + u \circ \frac{dW}{dt}, \quad x \in \mathcal{O}, \; t > \tau,
\]

(89)

with boundary condition

\[
u(t,x) = 0, \quad x \in \mathbb{R}^n \setminus \mathcal{O}, \; t > \tau
\]

(90)

and initial condition

\[
u(\tau,x) = \varphi(x), \quad x \in \mathcal{O}.
\]

(91)

We use \( \Phi^0 \) to denote the nonautonomous random dynamical system generated by the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation without delay (89)-(91) in \( H \) and define a collection of all tempered families of nonempty subsets of \( H \) by

\[
\mathcal{D}^0 = \left\{ \{ D(\tau,\omega) \subset H : \tau \in \mathbb{R}, \omega \in \Omega \} : \lim_{t \to +\infty} e^{-\delta t} \| D(\tau - t, \theta_{-t}\omega) \|_H = 0 \right\}.
\]
Observe that all estimates and results in the previous sections are valid for \( \rho = 0 \). Therefore, we know that the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation without delay (89)-(91) generates a continuous nonautonomous random dynamical system \( \Phi^\rho \) in \( H \), \( \Phi^\rho \) has a unique tempered \( \mathcal{D}^\rho \)-pullback attractor \( \mathcal{A}^\rho = \{A^\rho(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}^\rho \) in \( H \) and a tempered \( \mathcal{D}^\rho \)-pullback absorbing set \( K^\rho = \{K^\rho(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) given by

\[
K^\rho(\tau, \omega) = \{u \in H : \|u\|^2 \leq e^{2z(\omega)}R^\rho(\tau, \omega)\},
\]

where \( R^\rho(\tau, \omega) \) is the same as in (87). It follows from (87) and (92) that for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\limsup_{\rho \rightarrow 0} \|K^\rho(\tau, \omega)\|_{C^p} = \|K^0(\tau, \omega)\|.
\]

In order to prove the upper semicontinuity of the \( \mathcal{D} \)-pullback attractor \( \mathcal{A}^\rho \), we need the following lemma concerned with convergence of solutions for the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) as \( \rho \rightarrow 0 \).

**Lemma 6.1.** Let \( u^\rho \) and \( u \) be the solutions of the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation with delay (11)-(13) and the nonautonomous nonlocal fractional stochastic \( p \)-Laplacian equation without delay (89)-(91) with initial data \( \varphi^\rho \) and \( \varphi \), respectively. Suppose that the assumptions (14), (16), (18) and (31) hold. If \( \lim_{\rho \rightarrow 0} \sup_{\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\| = 0 \), then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \), \( T > 0 \) and \( t \in [\tau, \tau + T] \),

\[
\lim_{\rho \rightarrow 0} \sup_{\rho \leq s \leq 0} \|u^\rho(t + s, \tau, \omega, \varphi^\rho) - u(t, \tau, \omega, \varphi)\| = 0.
\]

**Proof.** For any \( s \in [-\rho, 0] \) be fixed and \( t \in \mathbb{R} \), denote by

\[
\overline{v}(t) = v^\rho(t + s) - v(t),
\]

where \( v^\rho(t, \tau, \omega, \psi^\rho) = e^{-z(\omega)}v^\rho(t, \tau, \omega, \psi^\rho) \) and \( v(t, \tau, \omega, \psi) = e^{-z(\omega)}v(t, \tau, \omega, \psi) \) with \( \psi^\rho(s, x) = e^{-z(\omega)}\psi^\rho(s, x) \) for \( s \in [-\rho, 0] \) and \( \psi(x) = e^{-z(\omega)}\psi(x) \), from which combined with (89) one gets that

\[
\frac{dv}{dt} + e^{(p-2)z(\omega)}(-\Delta)_p v + \delta v = z(\omega)v + e^{-z(\omega)}F(x, e^{z(\omega)}v(t))
+ e^{-z(\omega)}f(x, e^{z(\omega)}v(t)) + e^{-z(\omega)}g(t, x), \quad x \in \mathcal{O}, \quad t > \tau.
\]

It follows from (23), (95) and (96) we get that for \( t > \tau - s \) and \( s \in [-\rho, 0] \),

\[
\frac{d\overline{v}}{dt} + e^{(p-2)z(\omega)}(-\Delta)_p \overline{v} + \delta \overline{v} = z(\omega+s)\overline{v} + [z(\omega+s) - z(\omega)]v(t)
+ e^{-z(\omega)}F(x, e^{z(\omega+s)}u^\rho(t + s)) - e^{-z(\omega)}F(x, e^{z(\omega)}v(t))
+ e^{-z(\omega)}f(x, e^{z(\omega+s)}u^\rho(t + s - \rho)) - e^{-z(\omega)}f(x, e^{z(\omega)}v(t))
+ e^{-z(\omega)}g(t + s, x) - e^{-z(\omega)}g(t, x), \quad x \in \mathcal{O}.
\]
Taking inner product of (97) with $\overline{v}$, we get that for $t > \tau - s$ and $s \in [-\rho, 0]$,
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \frac{C(n, p, \alpha)}{2} e^{(p-2)z(\theta_\mathcal{O})} \|v\|_{L^p(\mathbb{R}^n)}^p + \delta \|v\|^2 = z(\theta_\mathcal{O}) \|v\|^2
\]
\[
+ [z(\theta_{t+s}) - z(\theta_\mathcal{O})] \int_\mathcal{O} v(t,x) \overline{v}(t,x) dx
\]
\[
+ \int_\mathcal{O} \left[ e^{-z(\theta_{t+s})} F(x, u^\rho(t+s)) - e^{-z(\theta_\mathcal{O})} F(x, u(t)) \right] \overline{v}(t) dx
\]
\[
+ \int_\mathcal{O} \left[ e^{-z(\theta_{t+s})} f(x, u^\rho(t+s - \rho)) - e^{-z(\theta_\mathcal{O})} f(x, u(t)) \right] \overline{v}(t) dx
\]
\[
+ \int_\mathcal{O} \left[ e^{-z(\theta_{t+s})} g(t+s, x) - e^{-z(\theta_\mathcal{O})} g(t, x) \right] \overline{v}(t) dx. \tag{98}
\]
For the second term on the right-hand side of (98), we get from (20) and the Young inequality that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and every given $0 < \tau \leq 1$, there exists $0 < \rho_0 \leq 1$ such that for all $\rho \leq \rho_0$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T]$,
\[
[z(\theta_{t+s}) - z(\theta_\mathcal{O})] \int_\mathcal{O} v(t,x) \overline{v}(t,x) dx \leq c \|v(t)\|^2 + c\tau \|u(t)\|^2, \tag{99}
\]
where $c$ is a positive constant independent of $\rho$ and $\tau$. For the third term on the right-hand side of (98), we get from (14), (16), (20) and the Young inequality that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists $0 < \rho_1 \leq 1$ such that for all $\rho \leq \rho_1$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T]$,
\[
\int_\mathcal{O} \left[ e^{-z(\theta_{t+s})} F(x, u^\rho(t+s)) - e^{-z(\theta_\mathcal{O})} F(x, u(t)) \right] \overline{v}(t) dx
\]
\[
= \int_\mathcal{O} e^{-z(\theta_{t+s})} \left[ F(x, u^\rho(t+s)) - F(x, u(t)) \right] \overline{v}(t) dx
\]
\[
+ \int_\mathcal{O} \left( e^{-z(\theta_{t+s})} - e^{-z(\theta_\mathcal{O})} \right) F(x, u(t)) \overline{v}(t) dx
\]
\[
\leq c \|\overline{v}(t)\|^2 + c\tau + c\tau \left( \|u^\rho(t+s)\|^p_{L^p} + \|u(t)\|^p_{L^p} \right), \tag{100}
\]
where $c$ is a positive constant independent of $\rho$ and $\tau$. For the fourth term on the right-hand side of (98), we similarly get from (18), (20) and the Young inequality that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists $0 < \rho_2 \leq 1$ such that for all $\rho \leq \rho_2$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T]$,
\[
\int_\mathcal{O} \left[ e^{-z(\theta_{t+s})} f(x, u^\rho(t+s - \rho)) - e^{-z(\theta_\mathcal{O})} f(x, u(t)) \right] \overline{v}(t) dx
\]
\[
= \int_\mathcal{O} e^{-z(\theta_{t+s})} \left[ f(x, u^\rho(t+s - \rho)) - f(x, u(t)) \right] \overline{v}(t) dx
\]
\[
+ \int_\mathcal{O} \left( e^{-z(\theta_{t+s})} - e^{-z(\theta_\mathcal{O})} \right) f(x, u(t)) \overline{v}(t) dx
\]
\[
\leq c \|\overline{v}(t)\|^2 + c \|u^\rho(t+s - \rho) - u(t)\|^2 + c\tau \|u(t)\|^2, \tag{101}
\]
here $c$ is a positive constant independent of $\rho$ and $\tau$. By again the Young inequality one gets that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists $0 < \rho_3 \leq 1$ such that for all $\rho \leq \rho_3$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T]$,

$$
\begin{align*}
\int_{\Omega} \left[ e^{-z(\theta_{t+s} \omega)} g(t + s, x) - e^{-z(\theta_t \omega)} g(t, x) \right] \tau(t) dx \\
= \int_{\Omega} e^{-z(\theta_{t+s} \omega)} [g(t + s, x) - g(t, x)] \tau(t) dx \\
+ \int_{\Omega} \left( e^{-z(\theta_{t+s} \omega)} - e^{-z(\theta_t \omega)} \right) g(t, x) \tau(t) dx \\
\leq c \|\tau(t)\|^2 + c\|g(t + s) - g(t)\|^2 + c\|g(t)\|^2,
\end{align*}
$$

(102)

where $c$ is a positive constant independent of $\rho$ and $\tau$. Therefore, it follows from (98)-(102) that there exists $0 < \rho_4 \leq 1$ such that for all $\rho \leq \rho_4$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T],

$$
\frac{d}{dt} \|\tau(t)\|^2 \leq c\|\tau(t)\|^2 + c\|u^\rho(t + s - \rho) - u(t)\|^2 + c\|g(t + s) - g(t)\|^2 \\
+ c\|g(t)\|^2 + c\|u^\rho(t + s)\|_{L^p} + \|u(t)\|_{L^p} + 1).
$$

(103)

Integrating (103) over $\tau - s, t$ with $t \in [\tau, \tau + T]$ and $t \geq \tau - s$ combined with the fact that $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ we get that

$$
\|\tau(t)\|^2 \leq \|\tau(\tau - s)\|^2 + c \int_{\tau - s}^t \|\tau(r)\|^2 dr + c \int_{\tau - s}^t \|u^\rho(r + s - \rho) - u(r)\|^2 dr \\
+ c \int_{\tau - s}^t \|g(r + s) - g(r)\|^2 dr + c\|u^\rho(r + s)\|_{L^p} + \|u(r)\|_{L^p} + 1) dr.
$$

(104)

Note that $e^{-z(\theta_t \omega)}$ is uniformly continuous on $[\tau - 1, \tau + T]$ for any $\omega \in \Omega$, there exists $0 < \rho_5 \leq \min\{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4\}$ such that for all $\rho \leq \rho_5$, $s \in [-\rho, 0]$ and $t \in [\tau, \tau + T]$ with $t \geq \tau - s,

$$
\int_{\tau - s}^t \|u^\rho(r + s - \rho) - u(r)\|^2 dr \\
= \int_{\tau - s}^{\tau + \rho - s} \|u^\rho(r + s - \rho) - u(r)\|^2 dr + \int_{\tau + \rho - s}^t \|u^\rho(r + s - \rho) - u(r)\|^2 dr \\
\leq 2 \int_{\tau - s}^\tau \|u^\rho(r) - \varphi^\rho\|^2 dr + 2 \int_{\tau + \rho - s}^{\tau + \rho - s} \|u(r) - \varphi\|^2 dr \\
+ \int_{\tau - s}^t \|u^\rho(r + s) - u(r + \rho)\|^2 dr \\
\leq 2 \int_{\tau - s}^{\tau + \rho - s} \|u^\rho(r) - \varphi^\rho\|^2 dr + 2 \int_{\tau - s}^t \|u(r) - \varphi\|^2 dr \\
+ 2 \int_{\tau - s}^\tau \|u^\rho(r + s) - u(r)\|^2 dr + 2 \int_{\tau - s}^t \|u(r + \rho) - u(r)\|^2 dr \\
\leq 2\rho \|\varphi^\rho - \varphi\|_{L^p}^2 + 2 \int_{\tau - s}^{\tau + \rho - s} \|u(r) - \varphi\|^2 dr + c\int_{\tau - s}^t \|\tau(r)\|^2 dr
$$

(105)
where $c$ is a positive constant independent of $\rho$ and $\tau$. Therefore, for every $t \in [\tau, \tau + T]$ and $t \geq \tau - s$, it follows from (104) and (105) that

$$\|\varphi(t)\|^2 \leq \|\varphi(\tau - s)\|^2$$

$$+ c \int_{\tau-e}^{\tau} \|u(r)\|^2 dr + 2 \int_{\tau-e}^{\tau} \|u(r + \rho) - u(r)\|^2 dr,$$

(105)

We now estimate all the terms on the right-hand side of (107). By the limit

$$\lim_{\rho \to 0} \int_{\tau}^{\tau + 2\rho} \|u(r) - \varphi\|^2 dr = 0$$

we know that there exists $\rho_0 \leq \rho_5$ such that for all $\rho \leq \rho_0$,

$$\int_{\tau}^{\tau + 2\rho} \|u(r) - \varphi\|^2 dr \leq \epsilon.$$  

(107)

Due to the uniformly continuity of $u$ from $[\tau, \tau + 1 + T]$ to $H$ we know that there exists $\rho_7 \leq \rho_6$ such that for all $\rho \leq \rho_7$ and $r \in [\tau, \tau + T]$,

$$\|u(r + \rho) - u(r)\|^2 dr \leq \epsilon.$$  

(108)

By the fact that $g \in L^2_{t, \omega, \rho, \varphi}(\mathbb{R}, L^2(O))$ we get from the Lebesgue dominated convergence theorem that

$$\lim_{\epsilon \to 0} \int_{\tau}^{\tau + T} \|g(r + \rho) - g(r)\|^2 dr = 0,$$

which means that there exists $\rho_8 \leq \rho_7$ such that for all $\rho \leq \rho_8$ and $s \in [-\rho, 0]$,

$$\int_{\tau}^{\tau + T} \|g(r + \rho) - g(r)\|^2 dr \leq \epsilon.$$  

(109)

Furthermore, by (14), (27) and (31) combined with the proof process of Lemma 4.1 we know that there exists $c = c(\tau, \omega, T) > 0$ such that for all $\rho \in (0, 1], t \in [\tau, \tau + T]$ with $t \geq \tau - s$ and $s \in [-\rho, 0]$,

$$\|\varphi(t + s, \tau, \omega, \varphi^\rho)\|^2 + \int_{\tau}^{t + s} \|\varphi(t, \tau, \omega, \varphi^\rho)\|_p^p dr \leq c + \epsilon \|\varphi^\rho\|_{C^\rho}^2.$$  

(110)

Similarly, we can also prove that

$$\|u(t, \tau, \omega, \varphi)\|^2 + \int_{\tau}^{t} \|u(t, \tau, \omega, \varphi)\|_p^p dr \leq c + \epsilon \|\varphi\|^2.$$  

(111)

It follows from (107)-(112) we get that for all $\rho \leq \rho_8$, $t \in [\tau, \tau + T]$ with $t \geq \tau - s$ and $s \in [-\rho, 0]$,

$$\|\varphi(t)\|^2 \leq c \int_{\tau-e}^{\tau} \|\varphi(r)\|^2 dr$$

$$+ \|\varphi(\tau - s)\|^2 + c \rho \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + \epsilon \left(\|\varphi^\rho\|_{C^\rho}^2 + \|\varphi\|^2 + 1\right).$$
which along with Bellman’s lemma implies that for all for all \( \rho \leq \rho_T, t \in [\tau, \tau + T] \) with \( t \geq \tau - s \) and \( s \in [-\rho, 0] \),

\[
\|\tilde{\nu}(t)\|^2 \leq c\|\tilde{\nu}(\tau - s)\|^2 + c\rho \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + c\|\varphi\|^2 + 1.
\]

(112)

Note that

\[
\|\tilde{\nu}(\tau - s)\|^2 = \|\nu^\rho(\tau) - \nu(t - s)\|^2 \leq 3e^{-2z(\theta_T + \omega)}\|\nu^\rho(\tau) - \varphi\|^2 \\
+ 3\left(e^{-z(\theta_T + \omega)} - e^{-z(\theta_T - \omega)}\right)^2 \|\varphi\|^2 + 3e^{-2z(\theta_T - \omega)}\|\nu(t - s) - \varphi\|^2,
\]

which along with the continuity of \( u \) and \( e^{-z(\theta_T \omega)} \) at \( \tau \) shows that there exists \( \rho_0 \leq \rho_8 \) such that for all \( \rho \leq \rho_0 \) and \( s \in [-\rho, 0] \),

\[
\|\tilde{\nu}(\tau - s)\|^2 \leq c \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + c\|\varphi\|^2 + 1.
\]

(113)

It follows from (113) and (115) one gets that for all \( \rho \leq \rho_0, t \in [\tau, \tau + T] \) with \( t \geq \tau - s \) and \( s \in [-\rho, 0] \),

\[
\|\nu^\rho(t + s) - \nu(t)\|^2 \leq c \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + c\|\varphi\|^2 + 1.
\]

(114)

Therefore, from (115) and (112) combined with the continuity of \( e^{-z(\theta_T \omega)} \) one know that there exists \( \rho_{10} \leq \rho_0 \) such that for all \( \rho \leq \rho_{10} \), \( t \in [\tau, \tau + T] \) with \( t \geq \tau - s \) and \( s \in [-\rho, 0] \),

\[
\|u^\rho(t + s) - u(t)\|^2 = \|e^{z(\theta_T + \omega)}u^\rho(t + s) - e^{z(\theta_T \omega)}u(t)\|^2 \\
\leq 2c^2z(\theta_T + \omega)\|\nu^\rho(t + s) - \nu(t)\|^2 + 2\left[c^2z(\theta_T + \omega) - z(\theta_T \omega)\right] \|\nu(t)\|^2 \\
\leq c \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + c\|\varphi\|^2 + 1.
\]

(115)

On the other hand, for the case \( \tau \leq t \leq \tau - s \). Let \( r = t - \tau \). Then \( t = r + \tau \), \( 0 \leq t \leq \rho \) and one has

\[
\|u^\rho(t + s) - u(t)\|^2 \leq 2\|u^\rho(t + s) - \varphi\|^2 + 2\|u(t) - \varphi\|^2 \\
\leq 2 \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + 2\|u(r + \tau) - \varphi\|^2,
\]

which along with the continuity of \( u \) at \( \tau \) and the fact that \( 0 \leq r \leq \rho \) we know that there exists \( \rho_{11} \leq \rho_{10} \) such that for all \( \rho \leq \rho_{11} \) and \( \tau \leq t \leq \tau - s \) with \( s \in [-\rho, 0] \),

\[
\|u^\rho(t + s) - u(t)\|^2 \leq 2 \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + \|u(r + \tau) - \varphi\|^2.
\]

(116)

Hence, by (116) and (117) we get that for all \( \rho \leq \rho := \min\{\rho_{10}, \rho_{11}\} \leq 1, t \in [\tau, \tau + T] \) and \( s \in [-\rho, 0] \),

\[
\|u^\rho(t + s) - u(t)\|^2 \leq c \sup_{-\rho \leq s \leq 0} \|\varphi^\rho(s) - \varphi\|^2 + c\|\varphi\|^2 + 1,
\]

from which one can easily deduce the desired result (94). This completes the proof of Lemma 6.1.

Next, we are concerned with the uniform compactness of random attractors with respect to \( \rho \).
Lemma 6.2. Suppose that the assumptions (14)-(18), (31)-(33) and (35) hold. If \( \rho_n \to 0 \) and \( u_n \in A^\rho_n(\tau, \omega) \), then there exist a subsequence \( \{u_{n_m}\} \) of \( \{u_n\} \) and \( u \in H \) such that

\[
\lim_{m \to \infty} \sup_{-\rho_{n_m} \leq s \leq 0} \|u_{n_m}(s) - u\| = 0. \tag{117}
\]

Proof. Let \( \{t_n\}_{n=1}^\infty \) be a sequence of numbers with \( t_n \to \infty \) as \( n \to \infty \). By the invariance of \( A_{\rho_n} \), there exists \( \pi_n \in A_{\rho_n}(\tau - t_n, \theta - t_n, \omega) \) such that

\[
u_n = \Phi_{\rho_n}(t_n, \tau - t_n, \theta - t_n, \omega, \pi_n). \tag{118}
\]

From (88) one gets that \( \pi_n \in K^\rho(\tau - t_n, \theta - t_n, \omega) \). Since all estimates of solutions in Section 4 are uniform with respect to bounded \( \rho \), by repeating the arguments of Lemma 5.2 we have that \( \Phi_{\rho_n}(t_n, \tau - t_n, \theta - t_n, \omega, \pi_n)(0) \) is precompact in \( H \), from which one can verify that all the assumptions \( \rho \) of Lemma 6.2.

\[
\Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(0) = 0 \quad \text{as} \; m \to \infty. \tag{119}
\]

In addition, by the proof of Lemma 5.2, we get that the \( \Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(\cdot) \) is equicontinuous, which means that for any \( s \in [-\rho_{n_m}, 0] \),

\[
\Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(s) - \Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(0) \]

\[
\to 0 \quad \text{as} \; m \to \infty. \tag{120}
\]

It follows from (119)-(121) we get that for any \( s \in [-\rho_{n_m}, 0] \),

\[
\|u_{n_m}(s) - u\| = \|\Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(s) - u\|
\]

\[
\leq \|\Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(s) - \Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(0)\|
\]

\[
+ \|\Phi_{\rho_m}(t_{n_m}, \tau - t_{n_m}, \theta - t_{n_m}, \omega, \pi_{n_m})(0) - u\|
\]

\[
\to 0 \quad \text{as} \; m \to \infty. \tag{121}
\]

From (122) one can easily get the desired result (118). This completes the proof of Lemma 6.2.

Now, we are in the position to state the upper semicontinuity of the random attractors \( A^\rho \) as \( \rho \to 0 \).

Theorem 6.3. Suppose that the assumptions (14)-(18), (31)-(33) and (35) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{\rho \to 0} d_H(A^\rho(\tau, \omega), A^0(\tau)) = 0, \tag{122}
\]

where the distance \( d_H \) is defined for any subsets \( E \subseteq C^H_\rho \) and \( S \subseteq H \) by

\[
d_H(E, S) = \sup_{\phi \in E} \inf_{\psi \in S} \sup_{-\rho \leq s \leq 0} \|\phi(s) - \psi\|. \]

Proof. Let \( \rho_n \to 0 \) as \( n \to \infty \). Suppose \( \varphi^{\rho_n} \in C^H_\rho \) and \( \varphi \in H \) satisfying \( \sup_{-\rho_n \leq s \leq 0} \|\varphi^{\rho_n}(s) - \varphi\| \to 0 \) as \( n \to \infty \). Therefore, it follows from Lemma 6.1 that for any \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( t \geq \tau \),

\[
\sup_{-\rho_n \leq s \leq 0} \|\Phi_{\rho_n}(t, \tau, \omega, \varphi^{\rho_n})(0) - \Phi_{\rho_n}(t, \tau, \omega, \varphi)(0)\| \to 0 \quad \text{as} \; n \to \infty. \tag{123}
\]

Hence, by (92), (93), (124) and Lemma 6.2 one can verify that all the assumptions of Theorem 2.1 in [43] are satisfied. Therefore, we get (123) from [43, Theorem 2.1] immediately. This completes the proof of Theorem 6.3. \( \Box \)
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