A Benchmarking Study of Quantum Algorithms for Combinatorial Optimization

Krishanu Sankar,1 Artur Scherer,1 Satoshi Kako,2 Sam Reifenstein,2 Navid Ghadermarzy,1 Willem B. Krayenhoff,1 Yoshitaka Inui,2 Edwin Ng,2,3 Tatsuhiro Onodera,2,4 Pooya Ronagh,1,5,6,* and Yoshihisa Yamamoto2,∗

1QB Information Technologies (1QBit), Vancouver, BC, Canada
2Physics & Informatics Laboratories, NTT Research Inc., Sunnyvale, CA, USA
3E. L. Ginzton Laboratory, Stanford University, Stanford, CA, USA
4School of Applied and Engineering Physics, Cornell University, Ithaca, NY, USA
5Institute for Quantum Computing, University of Waterloo, Waterloo, ON, Canada
6Department of Physics & Astronomy, University of Waterloo, Waterloo, ON, Canada

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We study the performance scaling of three quantum algorithms for combinatorial optimization: measurement-feedback coherent Ising machines (MFB-CIM), discrete adiabatic quantum computation (DAQC), and the Dür–Hoyer algorithm for quantum minimum finding (DH-QMF) that is based on Grover’s search. We use MaxCut problems as a reference for comparison, and time-to-solution (TTS) as a practical measure of performance for these optimization algorithms. For each algorithm, we analyze its performance in solving two types of MaxCut problems: weighted graph instances with randomly generated edge weights attaining 21 equidistant values from −1 to 1; and randomly generated Sherrington–Kirkpatrick (SK) spin glass instances. We empirically find a significant performance advantage for the studied MFB-CIM in comparison to the other two algorithms. We empirically observe a sub-exponential scaling for the median TTS for the MFB-CIM, in comparison to the almost exponential scaling for DAQC and the proven $\mathcal{O}(\sqrt{2^n})$ scaling for DH-QMF. We conclude that the MFB-CIM outperforms DAQC and DH-QMF in solving MaxCut problems.

I. INTRODUCTION

Combinatorial optimization problems are ubiquitous in modern science, engineering, and medicine. These problems are often NP-hard, so the runtime of classical algorithms for solving them is expected to scale exponentially. One approach for tackling such hard optimization problems is to map them to the Ising spin glass model [1],

$$\mathcal{H} = -\sum_{i<j} J_{ij} S_i S_j - \sum_i h_i S_i.$$

Here, each $S_i$ represents a classical Ising spin attaining a value of ±1. [$J_{ij}$] is an Ising coupling matrix, and [$h_i$] is a vector of local field biases on the spin sites. When all $h_i$ are zero, the Ising model is equivalent to a (weighted) MaxCut problem on a graph with vertices corresponding to the spin sites and edge weights corresponding to the Ising couplings between the spin sites. Various mathematical programming problems, such as partitioning problems, binary integer linear programming, covering and packing problems, satisfiability problems, colouring problems, Hamiltonian cycles, tree problems, and graph isomorphisms can be formulated in the Ising model, with the required number of spins scaling at most cubically with respect to the problem size [2]. This has been a primary motivation for the recent extensive study of various Ising solvers. Several potential areas of industrial application of Ising solvers include drug discovery and biocatalyst development (e.g., in lead optimization or virtual screening), compressed sensing, deep learning (e.g., in the synaptic pruning of deep neural network), scheduling (e.g., resource allocation and traffic control), computational finance, and social networks (e.g., community detection).

Approximate algorithms and heuristics, such as semi-definite programming (SDP) [3], simulated annealing (SA) [4, 5] and its variants [6, 7], and breakout local search (BLS) [8] have been widely used as practical tools for solving MaxCut problems. However, even problem instances of moderate size require substantial computation time and, in the worst cases, solutions cannot be found with such approximate algorithms and heuristics. To overcome these shortcomings, a search for alternative solutions using various forms of quantum computing has been actively pursued. Adiabatic quantum computation [9], quantum annealing [10, 11], and the quantum approximate optimization algorithm (QAOA) [12] using circuit model quantum computers have been proposed. A coherent Ising machine (CIM) using networks of quantum optical oscillators has also been studied and implemented [13, 14].

Given that the present circuit model quantum computers suffer from short coherence times, gate errors, and limited connectivity among qubits, a fair comparison between them and modern heuristics is not yet possible [15–17]. This situation raises the important question of whether quantum devices can, even in principle, provide sensible solutions to combinatorial optimization problems, assuming all sources of noise and imperfections
can be overcome and ideal quantum processors are built in the future. In order to address this pressing question, we perform a comparative numerical study on three distinct quantum approaches, ignoring the effects of noise, gate errors, and decoherence, that is, we compare the ultimate theoretical limits of three quantum approaches.

The first approach is based on the effects of constructive and destructive quantum interference of amplitudes in a circuit model quantum computer that utilizes only unitary evolution of pure states and projective (exact) measurement of qubits. The approach uses Grover’s search algorithm [18, 19] as a key computational primitive. We call this approach “DH-QMF” in reference to Diirr and Hoyer’s “quantum minimum finding” algorithm [20]. Our scaling analysis of DH-QMF is presented in Section IV; additional details are provided in Appendix G. A review of related literature and a discussion of how our analysis differs from previous work are given in Appendix A.

The second approach is based on adiabatic quantum state preparation implemented on a circuit model quantum computer. The underlying concept, the quantum adiabatic theorem, goes as far back as the seminal work of Born and Fock [21]. Its application to quantum computing and solving optimization problems was introduced by Farhi et al. [9]. A Trotterized approximation to adiabatic evolution gives rise to a discrete implementation suitable for the circuit model. We refer to this approach as “discrete adiabatic quantum computation” (DAQC). This algorithm uses an iterative unitary evolution of pure states in a quantum circuit according to a mixing Hamiltonian and a problem Hamiltonian, which in the framework of adiabatic quantum computation correspond to the initial and final Hamiltonians of evolution, respectively. The coefficients in the exponents form the gate parameters, which can be treated as hyperparameters that follow a tuned schedule, and the overall number of Trotter steps directly pertains to the circuit depth of the algorithm. To attain the ultimate theoretical performance limit, we use pre-tuned DAQC schedules and allow for quantum circuits of arbitrary depth. Our scaling analysis of DAQC is presented in Section III; additional details are provided in Appendix D. In the presence of noise, the closely related NISQ-type “quantum approximate optimization algorithm” (QAOA) [12, 22] (see Appendix A) deviates from DAQC in its use of (a) shallow (i.e., short-depth) quantum circuits (hence, attempting to perform ground-state preparation diabatically as opposed to adiabatically) and (b) an outer classical optimization routine to variationally optimize the diabatic evolution. We do not include QAOA in this study in view of its poor and unstable scaling, which we empirically observed in comparison to that of DAQC. This poor performance is exacerbated especially if the overhead of the classical optimizer is taken into account. Our observations are consistent with the challenges of variational quantum algorithms in overcoming the barren plateau problem [22, 23]. Further details are provided in Appendix E.

The third approach is based on a measurement-feedback coherent Ising machine (MFB-CIM) [24, 25]. This algorithm utilizes a quantum-to-classical transition in an open-dissipative, non-equilibrium network of quantum oscillators. A critical phenomenon known as pitchfork bifurcation realizes the transition of squeezed vacuum states to coherent states in the optical parametric oscillator. The measurement-feedback circuit plays several important roles. It continually reduces entropy and sustains a quasi-pure state in the quantum oscillator network in a controlled manner using repeated approximate measurements. It, additionally, implements the Ising coupling matrix $J_{ij}$ and local field vector $[h_i]$ in an iterative fashion. Finally, it removes the amplitude heterogeneity among the oscillators and destablizes the machine state out of local minima. Table I summarizes the differences among the three approaches studied in this paper.

When studying quantum algorithms, it is important to consider the effects of noise and control errors, and the overhead needed to overcome them. Several previous studies have investigated these effects on the performance of QAOA (here viewed as a NISQ-type, diabatic counterpart to DAQC). In references [26, 27], various Pauli noise channels, namely the dephasing, bit-flip, and the depolarizing noise channels, are considered. These two papers report on the fidelity of the state prepared by a noisy QAOA circuit to the state prepared by an ideal QAOA circuit, for varying amounts of physical noise affecting the circuit. In contrast, [28] models noise via single-qubit rotations by an angle chosen from a Gaussian distribution with variance values of $T_G/T_2$, where $T_G$ is the gate time and $T_2$ is the decoherence time of the qubits. All three papers provide insight into how noise affects the expected energy of the prepared state. Note that arbitrary-depth circuits are permitted in our study of DAQC, and optimal circuit depths resulting in the best algorithmic performance can be substantially larger than the size of circuits suitable for NISQ devices.

DH-QMF circuits are much deeper than typical DAQC circuits; thus, their performance is significantly hampered by various sources of noise unless the algorithm is run on a fault-tolerant quantum computer with quantum error correction [29–36]. Different noise models have been used to study the sensitivity of Grover’s search by simulating small quantum circuits that apply it to simple functions. [29] introduces random Gaussian noise on each step of Grover’s search. [30] studies the effect of gate imperfections on the probability of success of the algorithm. [33] examines the effect of unbiased and isotropic unitary noise resulting from small perturbations of Hadamard gates. [31] models the effect of decoherence by introducing phase errors in each qubit and time step and using a perturbative method. [34] conducts a numerical analysis on the effects of single-qubit and two-qubit gate errors and memory errors, modeling decoherence using a depolarizing channel. The impact of using a noisy oracle is examined in [32], wherein noise is modelled by introducing
TABLE I: Three approaches studied for MaxCut problems: the Dürr–Høyer algorithm for quantum minimum finding (DH-QMF) based on Grover’s search, the discretized adiabatic quantum computation algorithm (DAQC), and the measurement-feedback coherent Ising machine (MFB-CIM).

|                      | DH-QMF          | DAQC            | MFB-CIM         |
|----------------------|-----------------|-----------------|-----------------|
| Quantum dynamics     | Closed-unitary  | Closed-unitary  | Open-dissipative |
| Operational principle| Amplitude amplification by quantum interference | Adiabatic quantum evolution | Quantum-to-classical transition |
| Information carrier  | Digital (spin-1/2 particle) | Digital (spin-1/2 particle) | Analog (harmonic oscillator) |
| Decoherence time      | $T_2 \to \infty$ | $T_2 \to \infty$ | $T_2 \to \infty$ |
| Dissipation time      | $T_1 \to \infty$ | $T_1 \to \infty$ | $T_1$: finite |
| Gate error            | None            | None            | Vacuum noise limited |
| Spin-spin coupling    | all-to-all      | all-to-all      | all-to-all      |

random phase errors. The effects of localized dephasing are studied in [36]. Finally, [35] investigates the effects of various noise channels using trace-preserving, completely positive maps applied to density matrices.

In our benchmark study, by “solving” an optimization problem we mean finding an actual optimal solution with high probability (as opposed to an approximate, suboptimal solution). For a fair comparison, this notion pertains to all three algorithms considered in this work. As a practical measure of the algorithms’ performance, we use the time-to-solution (TTS) metric, which refers to the time required to find an optimal solution with high confidence. For the MFB-CIM and DAQC, the TTS is computed as the number of “shots” (i.e., trials) that must be performed to ensure a high probability (specified by a target probability of success, often taken to be 0.99) of observing an optimal solution at least once, multiplied by the time required for the execution of a single shot. Similarly, for DH-QMF, the TTS is computed as the overall number of Grover iterations required to ensure a target probability of success of observing an actual optimal solution, multiplied by the time required to implement a single Grover iteration.

We have evaluated the wall-clock TTS of the three algorithms introduced above for solving MaxCut problems, and empirically found exponential scaling laws for them already in the relatively small problem size range of 4 to 800 spins. In order to elucidate the ultimate performance limits of these solvers, we assume no extrinsic noise, gate errors, or connectivity limitations exist in the hardware. That is, we assume that phase decoherence ($T_2$) and energy dissipation ($T_1$) times are infinite and gate errors are absent. Consequently, the overheads associated with performing quantum error correction and building fault-tolerant architectures and protocols are not included in our benchmarking study, as they would make the comparison less favourable for circuit model quantum algorithms against the MFB-CIM. We also assume that all spins (represented by qubits in the circuit model) can be coupled to each other via (non-local) spin–spin interaction with a universal gate time of 10 nanoseconds. Therefore, there is no need to implement expensive sequences of swap gates or other bus techniques for transferring quantum information across the hardware. However, since energy dissipation and stochastic noise both constitute important computational resources for the MFB-CIM, we allow a finite energy dissipation time $T_1$, as well as a finite gate error limited by vacuum noise, for the MFB-CIM.

We emphasize that we compare optimistic lower bounds on the TTS for the circuit-model quantum algorithms considered in this paper. It is for this reason that we do not include the overhead costs associated with quantum error correction and the realization of fault-tolerant quantum computation schemes that become necessary for deep circuits of DH-QMF and DAQC. The impact of such overhead costs, for instance, when using topological surface code built of error-prone physical qubits and gates for encoding logical qubits and logical operations, is estimated more precisely in other recent works, for example, in [37–39]. The asymptotic overhead introduced by fault-tolerant architectures can be inferred as follows. For DAQC, the circuit depth of each Trotter layer scales linearly with the problem size $n$; see Section III. Therefore, the error rate of each logical gate must scale inversely with $n$, necessitating a code distance logarithmic in $n$. Fault-tolerant operations on an encoding scheme of distance $d$ introduce at least a factor of $d$ in physical gate time overhead. Hence, we can expect the TTS for the DAQC algorithm to increase by an
\(\Omega(\log n)\) factor. Similarly, for DH-QMF, which is based on Grover’s search requiring circuits of depth \(\tilde{\Theta}(\sqrt{2^n})\) (see Section IV), the incurred overhead results in an increase in the TTS by a factor of \(\Omega(n)\). This rough estimate does not account for compilation overhead, which would typically further increase the TTS. In addition, it also does not account for overheads caused by decoding and active error correction.

From a fundamental viewpoint, such a comparative study is of interest but the outcome is difficult to predict, because the three algorithms are based on completely different computational principles, as shown in Table I. The DH-QMF algorithm iteratively deploys Grover’s search, which uses a unitary evolution of a superposition of computation basis states in order to amplify the amplitude of a target state by successive constructive interference, while the amplitudes of all the other states are attenuated by destructive interference. The DAQC algorithm attempts to prepare a pure state that has a large overlap with the ground state of the optimization problem through an approximation of the adiabatic quantum evolution. Finally, the ground state search mechanism of the MFB-CIM employs a collective phase transition at the threshold of an optical parametric oscillator (OPO) network. The correlations formed among the squeezed vacuum states in OPOs below the threshold guide the network toward oscillating at a ground state.

It is worth noting that all the algorithms in our study in various ways rely on hybrid quantum–classical architectures for computation. In an MFB-CIM with self-diagnosis and dynamical feedback control, a classical processor plays an important role by detecting when the OPO network is trapped in local minima, and destabilizes it out of those states. The DH-QMF algorithm also relies on comparing the values of an objective function with a (classical) threshold value. This threshold value is updated in a classical coprocessor as DH-QMF proceeds. Finally, DAQC relies on tuning a set of parameters (e.g., the rotation angles of quantum gates). These parameters can be treated as hyperparameters of a predefined approximate adiabatic evolution and tuned for the problem type solved by the algorithm. Alternatively, the quantum circuit can be viewed as a variational ansatz, in which case the gate parameters are optimized using a classical optimizer. In the latter case, the algorithm can be considered as a variational quantum algorithm [40]. QAOA is commonly viewed as such an algorithm. In previous studies, the contribution of the variational optimization of DAQC parameters to the TTS has often been ignored. In fact, while both approaches (i.e., hyperparameter tuning and variational optimization) have been adopted for solving MAXCUT problems using QAOA [41, 42], our investigation makes it clear that the variational approach hurts the TTS scaling significantly. The optimization landscape for such a variational quantum algorithm is ill-behaved, which results in a poor and unstable scaling for TTS with respect to the size of the MAXCUT instances (refer to Appendix E). As a result, the TTS scalings reported in this paper rely on pre-tuned DAQC schedules rather than variational optimization.

II. SCALING OF THE MFB-CIM

A CIM is a non-equilibrium, open-dissipative computing system based on a network of degenerate OPOs to find a ground state of Ising problems [13, 43–46]. The Ising Hamiltonian is mapped to the loss landscape of the OPO network formed by the dissipative coupling rather than the standard Hamiltonian coupling. By providing a sufficient gain to compensate for the overall network loss, a ground state of the target Hamiltonian is expected to build up spontaneously as a single oscillation mode [14]. However, the mapping of the cost function to the OPO network loss landscape often fails in the case of a frustrated spin problem due to the OPO amplitude inhomogeneity [13, 24]. In addition, with an increasing number of local minima occurring as problem sizes become larger, the machine state is trapped in those minima for a substantial amount of time, thereby causing the machine to report suboptimal solutions [14, 25]. Recently, self-diagnosis and dynamical feedback mechanisms have been introduced by a measurement-feedback CIM (MFB-CIM) to overcome these problems [24, 25]. This is achieved by a mutual coupling field dynamically modulated for each OPO to suppress the amplitude inhomogeneity and simultaneously to destabilize the machine’s state out of local minima.

A. Principle of Operation

A schematic diagram of two MFB-CIMs with predefined feedback control (hereafter referred to as “open-loop CIM”) and with self-diagnosis and dynamical feedback control (hereafter referred to as “closed-loop CIM”), is shown in Fig. 1(a). If the fibre ring resonator has high finesse, both CIMs are modelled via the Gaussian quantum theory [47, 48]. The dynamics captured by the master equation for the density operator (i.e., the Liouville–von Neumann equation) is driven by the parametric interaction Hamiltonian, \(\hat{H} = i\hbar S_{2i} \sum_{i} \left(\hat{a}_i^2 - \hat{a}_i^2\right)\), the measurement-induced state reduction (the third term on the right-hand side in Eq. (1)), the coherent injection (the fourth term on the right-hand side in Eq. (1)), as well as three Liouvillians. The Liouvillians pertain to the linear loss due to measurement and injection couplings, \(\hat{L}_i = J \hat{a}_i\), two-photon absorption loss (i.e., parametric back conversion) in a degenerate parametric amplifying device, \(\hat{L}_2 = \sqrt{B/2} \hat{a}_i^2\), and background linear losses, \(\hat{L}_i = \sqrt{\gamma} \hat{a}_i\), respectively [48]. The master equation is thus given by
\[
\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] + \sum_{i=1}^{n} \sum_{k=1,2,c} \left( \left[ \hat{L}^{(i)}_k, \hat{\rho} \hat{L}^{(i)}_k \right] + h.c. \right) + \sqrt{J} \sum_{i=1}^{n} \left( a_i \hat{\rho} + \hat{\rho} a_i^\dagger - \langle a_i + a_i^\dagger \rangle \hat{\rho} \right) w_i + \frac{J}{2} \sum_{i,k=1}^{n} e_i(t) J_{ik} \left( \langle \hat{a}_k + \hat{a}_k^\dagger \rangle + \frac{w_k}{\sqrt{J}} \right) [a_i^\dagger - \hat{a}_i, \hat{\rho}] . \tag{1}
\]

In general, the numerical integration of Eq. (1) requires exponentially growing resources as the problem size \( n \) (i.e., the number of spins) increases. Generally speaking, the size of the density matrix scales as \( \mathcal{O}(n_0^n \times n_0^n) \), where \( n_0 \gg 1 \) is the maximum number of photons possible for each OPO pulse. In MFB-CIMs, however, there is no entanglement between the OPO pulses, that is, the OPO states are separable. Therefore, the simulation’s memory requirements reduce to \( \mathcal{O}(n \times n_0^2) \). However, this reduction still yields too many c-number differential equations due to the large upper bounds on the number of photons \( n_0 \lesssim 10^7 \) and the number of spins \( n \leq 1000 \). The Gaussian quantum model has been introduced to overcome this difficulty [25, 48].

In the case of a small saturation parameter, \( g^2 = B/\gamma_{\text{s}} \ll 1 \), we can split the \( i \)-th OPO’s pulse amplitude operator, \( \hat{a}_i = \frac{1}{\sqrt{2}}(\hat{X}_i + i\hat{P}_i) \), into the mean field and small fluctuation operators, \( \hat{X}_i = \langle \hat{X}_i \rangle + \Delta \hat{X}_i \) and \( \hat{P}_i = \langle \hat{P}_i \rangle + \Delta \hat{P}_i \). The saturation parameter \( g^2 \) corresponds to the inverse photon number at twice the threshold pump rate of a solitary OPO. With an appropriate choice of the pump phase, each OPO mean-field is generated only in an \( \hat{X} \)-quadrature, that is, \( \langle \hat{P}_i \rangle = 0 \). The equation of motion for the mean field \( \mu_i = \langle \hat{X}_i \rangle/\sqrt{2} \) and the variances \( \sigma_i = \langle \Delta \hat{X}_i^2 \rangle \) and \( \eta_i = \langle \Delta \hat{P}_i^2 \rangle \) obey the following equations [48]:

\[
\begin{align*}
\frac{d}{dt} \mu_i &= \left[ -(1+j) + p - g^2 \mu_i^2 \right] \mu_i + j \xi e_i(t) \sum_k J_{ik} \mu_k + \sqrt{j} \left( \sigma_i - 1/2 \right) w_i , \tag{2} \\
\frac{d}{dt} \sigma_i &= 2 \left[ -(1+j) + p - 3g^2 \mu_i^2 \right] \sigma_i - 2j \left( \sigma_i - 1/2 \right)^2 + \left( (1+j) + 2g^2 \mu_i^2 \right) , \tag{3} \\
\frac{d}{dt} \eta_i &= 2 \left[ -(1+j) - p - g^2 \mu_i^2 \right] \eta_i + \left( (1+j) + 2g^2 \mu_i^2 \right) . \tag{4}
\end{align*}
\]

Here, \( t = \gamma_{\text{s}} T \) refers to normalized and dimensionless time, where \( T \) is physical (or wall-clock) time, and \( \gamma_{\text{s}} \) is the background loss rate of the cavity. The time \( t \) is normalized so that the background linear loss (with a signal amplitude decay rate of \( 1/e \)) is 1. The term \(- (1+j)\) in Eqs. (2) to (4) represents a background linear loss (-1) and an out-coupling loss \((-j)\) for optical homodyne measurement and feedback injection, where \( j = J/\gamma_{\text{s}} \) is a normalized out-coupling rate (see Fig. 1(a)). The parameter \( p = S/\gamma_{\text{s}} \) is a normalized linear gain coefficient provided by the parametric device. The term \( g^2 \mu_i^2 \) represents two-photon absorption loss (i.e., back conversion from signal to pump fields). The second and third terms on the right-hand side of Eq. (2), respectively, represent the Ising coupling term and the measurement-induced shift of the mean-field \( \mu_i \). The inferred mean-field amplitude, \( \hat{\mu}_k = \mu_k + \sqrt{\frac{1}{2j} w_k} \), deviates from the internal mean-field amplitude \( \mu_k \) by a finite measurement uncertainty in the optical homodyne detection. The random variable \( w_k \sqrt{\Delta T} \) attains values drawn from the standard normal distribution, where \( \Delta T \) is a time step for the numerical integration of Eqs. (2) to (4). The \( k \)-th Ising spin \( S_k = \pm 1 \) is determined by the inferred mean-field amplitude, \( S_k = \mu_k/|\mu_k| \). \( J_{ik} \) is the Ising coupling coefficient and \( e_i(t) \) is a dynamically modulated feedback-field amplitude, while \( \xi = 1/\sqrt{\frac{1}{N} \sum_{i,j} |J_{ij}|} \) is a feedback-gain parameter. The second term on the right-hand side of Eq. (3) represents the measurement-induced partial state reduction of the OPO field. The last terms of Eqs. (3) and (4), respectively, represent the variance increase by the incident (fresh) vacuum field fluctuations via linear loss and the pump noise coupled to the OPO field via gain saturation.

The dynamically modulated feedback-field amplitude \( e_i(t) \) is introduced to reduce the amplitude inhomogeneity [24], which is determined by the inferred signal amplitude \( \hat{\mu}_i \):

\[
\frac{d}{dt} e_i(t) = -\beta \left[ g^2 \mu_i^2 - a \right] e_i(t) . \tag{5}
\]

Here, \( \beta \) is a positive constant representing the rate of
change for the exponentially growing or attenuating feedback amplitude $e_i(t)$, and $\alpha$ is a target squared amplitude. Both $a$ and the pump rate $p$ are dynamically determined by the difference of the current Ising energy $\mathcal{E}(t) = -\sum_{i<k} J_{ik} S_i S_k$ and the lowest Ising energy $\mathcal{E}_{\text{opt}}$ visited previously:

$$a(t) = \alpha + \rho_a \tanh \left( \frac{\mathcal{E}(t) - \mathcal{E}_{\text{opt}}}{\Delta} \right),$$  

$$p(t) = \pi - \rho_p \tanh \left( \frac{\mathcal{E}(t) - \mathcal{E}_{\text{opt}}}{\Delta} \right).$$

Here, $\pi$, $\alpha$, $\rho_a$, $\rho_p$, and $\Delta$ are predetermined positive parameters which characterize the self-diagnosis and dynamic feedback control.

The machine can distinguish the following three modes of operation from the energy measurements. When $\mathcal{E}(t) - \mathcal{E}_{\text{opt}} < -\Delta$, the machine is in a gradient descent mode and moving toward a local minimum, in which case the pump is set to a positive value of $\pi - \rho_p$ (leading to parametric amplification). When $|\mathcal{E}(t) - \mathcal{E}_{\text{opt}}| \ll \Delta$, the machine is close to, or trapped in, a local minimum, in which case the pump is switched off (i.e., there is no parametric amplification) so as to destabilize the current spin configuration. When $\mathcal{E}(t) - \mathcal{E}_{\text{opt}} > \Delta$, the machine is attempting to escape from a previously visited local minimum, in which case the pump is set to a negative value of $\pi - \rho_p$ (i.e., there is parametric de-amplification) to increase the rate of spin flips.

Fig. 1(b) shows the time evolution of a closed-loop CIM to demonstrate its inherent exploratory behaviour from one local minimum to another. We solve a MAXCUT problem with randomly generated discrete edge-weights $J_{ij} \in \{-1, -0.9, \ldots, 0.9, 1\}$ over $n = 30$ vertices, for which an exact solution is obtained by performing an exhaustive search. The dynamical behaviour of the inferred Ising energy measured from the ground state energy, $\Delta \mathcal{E}(t) = \mathcal{E}(t) - \mathcal{E}_G$, the mean amplitude, $\mu(t)$, the feedback-field amplitude, $e(t)$, and the target squared amplitude, $a(t)$, are shown in Fig. 1(b) and (c). The results shown in Fig. 1(b) are taken from a single trial for one particular problem instance and a particular set of noise amplitudes $\omega_i \sqrt{\Delta t}$. The feedback parameters are set to $\alpha = 1.0$, $\pi = 0.2$, $\rho_a = \rho_p = 1.0$, $\Delta = 1/3$, and $\beta = 1.0$ [25]. The saturation parameter and the out-coupling loss are chosen as $g^2 = 10^{-4}$ and $j = 1$, respectively. The time step $\Delta t$ for the numerical integration of Eqs. (2) to (4) is identical to the normalized round-trip time $\Delta t_c = \gamma_n \Delta T_c = 0.025$. This means the signal-field lifetime $1/\gamma_n$ is 40 times greater than the round-trip time.

As shown in Fig. 1(b1), the inferred Ising energy $\mathcal{E}(t)$ fluctuates up and down during the search for a solution even after the machine finds one of the degenerate ground states. As shown in Fig. 1(b2), the measured squared amplitude $g^2 \mu_i^2$ is stabilized to the target squared amplitude $a(t)$ through the dynamically modulated feedback mean-field $e_i(t)$. Several OPO amplitudes, however, flipped their signs followed by an exponential increase in $e_i(t)$, while most other OPOs maintained a target amplitude. During this spin-flip process, the feedback-field amplitude $e_i(t)$ increases exponentially and then decreases exponentially after the OPO’s squared amplitude $g^2 \mu_i^2$ exceeds the target squared amplitude $a(t)$. The mutual coupling strength $\sum \delta J_{ik} \mu_k$ is adjusted in order to decrease the energy continuously by flipping the “wrong” spins and preserving the “correct” ones. If the machine reaches local minima, which may also include global minima (in which case there are degenerate ground states), the current Ising energy $\mathcal{E}(t) = -\sum_{i<k} J_{ik} S_i S_k$ is roughly equal to the minimum Ising energy $\mathcal{E}_{\text{opt}}$ previously visited ($\mathcal{E}(t) \approx \mathcal{E}_{\text{opt}}$). The machine then decreases the target squared amplitude $a$, which helps it to escape from the local minimum. During this escape, the current Ising energy $\mathcal{E}(t)$ becomes greater than the minimum Ising energy $\mathcal{E}_{\text{opt}}$. The machine then switches the pump rate $p$ to a negative value and deamplifies the signal amplitude, which results in further destabilization of the local minimum. As a consequence of such dynamical modulation of the pump rate $p$ and the target squared amplitude $a$, the machine continually escapes local min-
inn, migrating from one local minimum to another as the computation carries on.

As such, it is defined via

\[ t_s = R_{99} \cdot t_{\text{max}}, \tag{8} \]

where \( R_{99} = \frac{\log(0.01)}{\log(1 - \frac{1}{e})} \) is the number of trials required to achieve a 99% probability of success. We solve 1000 instances for each problem size \( n = 4, \ldots, 30 \) to evaluate the median \( P_s \) and TTS. Note that \( t_s \) refers to the normalized and dimensionless TTS, while the actual wall-clock TTS (in seconds) is denoted by \( T \). These two notions of TTS are related via the equation \( t_s = \gamma_n T \). The wall-clock time \( T \) is estimated by assuming a cavity round-trip time of \( \Delta T_c = 10 \) nanoseconds (all-to-all spin coupling is implemented in 10 nanoseconds), and a \( 1/e \) signal amplitude decay time of 400 nanoseconds (\( \gamma_n \Delta T_c = 0.025 \)). An important observation from Fig. 3(b) is that the optimal median TTS scales as an exponential function of the square root of the problem size, that is, an exponential of \( \sqrt{n} \) rather than \( n \). This unique trend was first noticed in [17].

**B. Time-to-Solution**

Figures 3(a) and (b) show the median of the success probability \( P_s \) and the TTS \( t_s \) of the closed-loop CIM as a function of problem size \( n = 4, 5, \ldots, 30 \) with varying runtime \( t_{\text{max}} \). We perform 1000 trials, with a trial considered successful if the machine finds an exact solution within \( t_{\text{max}} \). The success probability \( P_s \) decreases exponentially with respect to \( n \), especially for \( t_{\text{max}} \leq 5 \). For a greater value of \( t_{\text{max}} \), the slope of the decay improves as shown in Fig. 3(a). The TTS is defined as the expected computation time required to find a ground state for a particular problem instance with 99% confidence.

**Fig. 2**: Variances \( \langle \Delta X^2 \rangle \) and \( \langle \Delta P^2 \rangle \) for (a) a closed-loop CIM and (b) an open-loop CIM. The shaded areas show the quantum domains \( \langle \Delta X^2 \rangle < 1/2 \) or \( \langle \Delta P^2 \rangle < 1/2 \). Note that these are the results for one particular OPO, i.e., for one of the trajectories shown in Fig. 1(b) and (c).

**Fig. 3**: (a) Success probability \( P_s \) and (b) time-to-solution (in units of signal field decay time \( 1/\gamma_n \)) as a function of problem size \( n \) for various runtimes \( t_{\text{max}} \). The black dotted line shows the best-fit TTS curve of the form \( AB^{\sqrt{n}} \).

**Fig. 4(a)** and (b) show the optimum TTS of the closed-loop CIM and the open-loop CIM with respect to the problem size \( n \). We solve two types of MAXCUT problems. The first type are randomly generated instances with edge weights \( J_{ij} \in \{-1, -0.9, \ldots, 0.9, 1\} \). We refer to these instances as 21-weight MAXCUT problem instances. The second type are randomly generated Sherrington–Kirkpatrick (SK) spin glass instances with \( J_{ij} = \pm 1 \). We study the open-loop CIM with the same Gaussian quantum model without dynamical modulation of \( e_i(t) \), \( a_i(t) \), and \( p_i(t) \), but with measurement-induced
state reduction (the third term of Eq. (2) and the second term of Eq. (3)) [48]. We set the feedback parameters $\beta = 0$, $\rho_a = 0$, and $\rho_p = 0$ for the open-loop CIM in order to have a constant feedback field strength $e_i(t) = e_i(0) = 1.0$. The pump rate $p$ is linearly increased from $p = 0.5$ at $t = 0$ (below threshold) to $p = 1.0$ at $t_{\text{max}}$ (above threshold). As shown in Fig. 4(a) and (b), the performance of the closed-loop CIM is superior to that of the open-loop CIM for both types of MaxCut problems.

Table II summarizes the best-fitting parameters $A$ and $B$ for a function of the form $t_s = AB^{\sqrt{n}}$ in both the closed-loop and open-loop CIMs. The smaller coefficient values for $B$ for the closed-loop CIM than those for the open-loop CIM highlight the superior scaling of the closed-loop CIM compared to the open-loop variant. We note that $A$ is expressed in units of a normalized time $t_s = \gamma_0 T$, where $T$ is the wall-clock time. It is worth noticing that the scaling law of the sub-exponential function is not necessarily optimal for fitting the data within the problem size range $n \leq 30$. In Section V, we present results for a much wider range for the SK model. Additional considerations with respect to inferring the true scaling law are discussed in Appendix F.

### Table II: Parameters $A$ and $B$ found by regression of a function of the form $AB^{\sqrt{n}}$ to the TTS curves of the closed-loop and open-loop CIMs for the two types of MaxCut instances.

|                | 21-weight random $J_{ij}$ | Binary random $J_{ij}$ |
|----------------|---------------------------|------------------------|
| $n=4,\ldots,30$ | closed loop | open loop | closed loop | open loop |
| $A$            | 0.26                     | 0.32                   | 0.16        | 0.13       |
| $B$            | 2.32                     | 4.12                   | 2.33        | 3.92       |

**C. Discrete-Time Model**

Section II B presented the results of our study of the performance of closed-loop and open-loop CIMs with a high-finesse cavity. Nevertheless, it is obvious that a low-finesse cavity with a larger signal decay rate $\gamma_0$ is favorable in terms of the runtime of the algorithm. This is because the wall-clock $T$ scales as $T = t_s/\gamma_0$. However, it appears that the continuous-time Gaussian quantum theory based on the master equation [Eq. (1)] breaks down in the case of a low-finesse cavity. Here, we describe a new discrete-time Gaussian quantum model [49].

We treat the MFB-CIM as an $n$-mode bosonic system with $2n$ quadrature operators, $\hat{X}_i, \hat{P}_i, \ldots, \hat{X}_n, \hat{P}_n$, satisfying $[\hat{X}_k, \hat{P}_k] = i\hbar \delta_{kk}$. If the system is in a Gaussian state, it is fully characterized by a mean-field vector $\mu$ and a covariance matrix $\Sigma$. In other words, the density operator of each OPO pulse can be written as $\hat{\rho}_i(\mu_i, \Sigma_i)$, where

$$\mu_i = \langle \hat{X}_i \rangle, \langle \hat{P}_i \rangle,$$

$$\Sigma_i = \begin{pmatrix} \langle \hat{X}_i^2 \rangle & \langle \hat{X}_i \hat{P}_i \rangle & \frac{1}{2} \langle \Delta \hat{X}_i \Delta \hat{P}_i + \Delta \hat{P}_i \Delta \hat{X}_i \rangle \\ \langle \hat{X}_i \hat{P}_i \rangle & \langle \hat{P}_i^2 \rangle & \langle \hat{P}_i \hat{X}_i \rangle \\ \frac{1}{2} \langle \Delta \hat{X}_i \Delta \hat{P}_i + \Delta \hat{P}_i \Delta \hat{X}_i \rangle & \langle \hat{P}_i \hat{X}_i \rangle & \langle \hat{X}_i^2 \rangle \end{pmatrix}.$$

We let $\hat{\rho}(\mu_i(\ell), \Sigma_i(\ell))$ denote the state of the $i$-th OPO pulse just before it starts its $\ell$-th round trip through the cavity. To propagate the state of the $i$-th signal pulse from $\hat{\rho}(\mu_i(\ell), \Sigma_i(\ell))$ to $\hat{\rho}(\mu_i(\ell+1), \Sigma_i(\ell+1))$, we perform the following five discrete maps iteratively: the background linear-loss map $B$, the OPO crystal propagation map $\chi$, the out-coupling loss map $B_{\text{out}}$, the homodyne detection map $H$, and the feedback injection map $D$. These discrete maps are defined in Appendix B.

In order to see how the wall-clock TTS of the closed-loop and open-loop CIMs is decreased by increasing the total cavity loss rate $\gamma_0(1+j)$, we solve the 21-weight MAXCUT instances and the SK model instances for $n = 30$ to explore the TTS as a function of the normalized total loss rate $\gamma_0\Delta T(1+j)$. The results are

![FIG. 4: The optimal (median) time-to-solution of the closed-loop CIM and open-loop CIM on (a) 21-weight randomly generated $J_{ij}$, and (b) binary-weight randomly generated instances ($J_{ij} = \pm 1, \text{SK model}$). The shaded regions represent the interquartile range (IQR), showing the region between the 25th and 75th percentiles obtained from the 1000 instances. The dashed blue and red lines are fitted curves of the form $AB^{\sqrt{n}}$.](image-url)
shown in Fig. 5. The saturation parameter and the out-coupling loss are chosen as $g^2 = 10^{-4}$ and $j = 1$, respectively. The feedback parameters are set to $\alpha = 0.5$, $\pi = 0.2$, $\rho_a = \rho_p = 0$ (while we keep $a$ and $p$ constant), and $\beta = 0.2$.

As expected, the TTS (expressed in terms of the number of round trips) decreases monotonically for both problem types and for both the closed-loop and open-loop CIMs as long as $\gamma_s \Delta T_c(1 + j) \lesssim 0.1$ (i.e., in the case of a high-finesse cavity). However, if $\gamma_s \Delta T_c(1 + j) \gtrsim 1$ (i.e., in the case of a very-low-finesse cavity), the TTS increases for both the closed-loop and the open-loop CIMs. This is because one homodyne measurement per round-trip loss does not provide sufficiently accurate information about the internal OPO pulse state and, therefore, the measurement-feedback circuit fails to implement the Ising Hamiltonian and self-diagnosis feedback properly. At $n = 30$, the optimum normalized loss rate is $\gamma_s \Delta T_c(1 + j) \approx 1$ for both the closed-loop and the open-loop CIMs.

III. SCALING OF DAQC

We now analyze the efficacy of the DAQC algorithm in solving MAXCUT problems. In this paper, DAQC is associated with the first-order Suzuki–Trotter expansion of the adiabatic Hamiltonian evolution. This algorithm attempts to prepare the ground state of a target Hamiltonian $H_p$. A typical circuit for DAQC is shown in Fig. 6. The state $|\psi\rangle^{\otimes n}$ is prepared on $n$ qubits, and is evolved through a sequence of $p$ “layers”. Each layer consists of an evolution according to $H_p$ along a computational basis, here chosen to be the Pauli-Z eigenbasis, followed by an evolution under a mixing Hamiltonian $H_M = \sum_i X_i$. A vector of tunable parameters $\gamma = (\gamma_1, \ldots, \gamma_p)$ is chosen, where each entry $\gamma_i$ corresponds to the angle of rotation along $H_p$ in the $i$-th layer. Similarly, a vector $\beta = (\beta_1, \ldots, \beta_p)$ is chosen for the $H_M$ evolutions. Finally, the qubits undergo projective measurements in the computational basis, and the measurement results are used to compute the energy values of $H_p$.

A “shot” of the circuit with parameters $(\gamma, \beta)$ is defined as a single execution of the circuit from preparation to measurement, and returns a single energy measurement. Multiple shots performed with the same parameters $(\gamma, \beta)$ return different results, as they are taken from independent copies of the same prepared state $|\psi(\gamma, \beta)\rangle$. For the weighted MAXCUT problem, we use the target Hamiltonian $H_p = \sum_{i,j} J_{ij} Z_i Z_j$, which is diagonal in the computational basis and whose ground states correspond to the largest cuts of the complete $n$-vertex graph with edge weights $J_{ij}$.

We study two schemes for optimizing the gate parameters of the DAQC algorithm. The first scheme treats gate parameters as hyperparameters that follow a tuned DAQC schedule. The second scheme uses a variational hybrid quantum–classical protocol to optimize the gate parameters, similar to the method typically used for QAOA. In our numerical experiments, we observed a better TTS scaling for the first scheme compared to the second scheme (see Appendices D and E); therefore, we use the first scheme to conduct our scaling analysis.

A. Time-to-Solution Scaling of DAQC

To study the time-to-solution of the DAQC algorithm in solving MAXCUT problems, we analyze the algorithm using pre-tuned Trotterized adiabatic scheduling. We use randomly generated graphs of size $n \in \{10, \ldots, 20\}$. Our test set consists of 1000 graphs of each size, with edge weights $J_{ij} = \pm 0.1j$, where $j \in \{0, 1, \ldots, 10\}$.

Given a parameter vector $(\gamma, \beta)$, we evaluate the TTS of DAQC as a product of two terms [6],

$$TTS(\gamma, \beta) = R_{99}(\gamma, \beta) \cdot t_{ss}, \tag{11}$$

where $t_{ss}$ is the time taken for a single shot.

The $R_{99}$ is the number of shots that must be performed to ensure a 99% probability of observing the ground state of $H_p$. It is a metric commonly used to benchmark the success of heuristic optimization algorithms. If the state $|\psi(\gamma, \beta)\rangle$ has a probability $p$ of being projected onto the ground state, then

$$R_{99}(\gamma, \beta) = \frac{\log(0.01)}{\log(1 - p)}. \tag{12}$$

We estimated the time required for a single shot using the following assumptions for an ideal, highly performant quantum computer with access to arbitrary-angle, single-qubit $X$-rotations and two-qubit $ZZ$-rotations.

Assumption 1. The preparation and measurements of qubits collectively take 1.0 microseconds. The processor performs any single-qubit or two-qubit gate operations in 10 nanoseconds. Gate operations may be performed simultaneously if they do not act on the same qubit. In addition, all components of the circuit are noise-free and, therefore, there is no overhead for quantum error correction or fault-tolerant quantum computation.

For each problem size varying from 10 to 20 vertices, Fig. 7 shows a plot of the median TTS, suggesting that the TTS scales exponentially with respect to problem size. With more layers, DAQC has a lower potential scaling was achieved with $p \approx 20$ layers. However, near-term hardware will suffer from various sources of noise, such as decoherence and control noise, which will restrict us to employing shallow DAQC circuits with only a few layers, for example, $p = 4$.

The DAQC parameters $(\gamma, \beta)$ used in Fig. 7 were produced using the formula explained in what follows. Recall the setup for quantum adiabatic evolution [50]. Given an
initial Hamiltonian $H_0$ and a target Hamiltonian $H_1$, we consider the time-dependent Hamiltonian

$$H(t) = s(t)H_1 + (1 - s(t))H_0, \quad t \in [0, T]$$

over a total annealing time $T$, where the function $s(t)$ is an increasing schedule satisfying $s(0) = 0$ and $s(T) = 1$. The time-dependent Hamiltonian $H(t)$ is then applied to the ground state of $H_0$. Let $\psi(t)$ denote the wavefunction at time $t$, so that $\psi(0)$ is the ground state of $H_0$ and $\psi$ evolves according to the Schrödinger equation

$$\dot{\psi} = -i\left(s(t)H_1 + (1 - s(t))H_0\right)\psi.$$

We use Trotterization to approximate the prepared state $\psi(T)$. Let

$$c_k := \int_{(k-1)T/p}^{kT/p} s(t) \, dt \quad \text{and} \quad b_k := \int_{(k-1)T/p}^{kT/p} (1 - s(t)) \, dt.$$

Then,

$$\psi(T) \approx e^{-ib_kH_0}e^{-ic_kH_1} \cdots e^{-ib_1H_0}e^{-ic_1H_1}\psi(0), \quad (13)$$

and this approximation becomes exact in the limit as $p \to \infty$.

The Hamiltonians $H_0$ and $H_1$ are both chosen to have a Frobenius norm equal to 1. We divide both $H_M$ and $H_P$ by their corresponding norms, which can easily be calculated, as each Hamiltonian is a sum of the orthogonal Pauli terms

$$H_0 = \frac{1}{\|H_M\|} H_M = -\frac{1}{\sqrt{n}} \sum_i X_i$$

and

$$H_1 = \frac{1}{\|H_P\|} H_P = \frac{1}{\sqrt{\sum_{i,j} J_{ij}^2}} \sum_{i,j} J_{ij} Z_i Z_j.$$

Thus,

$$\gamma_k = \int_{(k-1)T/p}^{kT/p} \frac{s(t)}{\|H_P\|} \, dt \quad \text{and} \quad \beta_k = \int_{(k-1)T/p}^{kT/p} \frac{1 - s(t)}{\|H_M\|} \, dt.$$

Empirically, we found that enforcing this Frobenius normalization has yielded a very well-performing schedule for DAQC for multiple problem types. The theoretical basis for this is yet to be fully understood.

The schedule $s(t)$ should have an “inverted S” shape [51, 52] in order to handle the squeezed energy gap in the middle. We take $s(t)$ to be a cubic function with the general form

$$s(t) = \frac{t}{T} + a \cdot \frac{t}{T} \left(1 - \frac{t}{T} \right)$$

for a free parameter $a$. When $a = 0$, $s(t)$ is a straight linear path. When $a = 4$, $s(t)$ is a curved path with a slope of 0 at $t = T/2$. We found by empirical means that $a = 4$ and $T = p(1.6 + 0.1n)$ are the best hyperparameters. See Appendix D for more details.
(a) TTS Scaling for 21-Weight Graphs for Selected Numbers of DAQC Layers
(b) TTS Scaling versus Number of DAQC Layers
(c) TTS Scaling for the SK Model for a 20-Layer DAQC

FIG. 7: Scaling of the DAQC algorithm in solving MaxCut problems. The TTS results are obtained by simulating DAQC, using pre-tuned adiabatic scheduling rather than optimizing its parameters variationally. The number of qubits required to implement the algorithm is $n$. (a) TTS scaling for a 4-, 10-, 20-, and 50-layer DAQC algorithm as the problem size grows from 10 to 20 vertices. A best-fit line (dashed) is drawn to the median of the TTSs of the 1000 instances of each size, whose IQR ranges are represented using coloured bars. The equation of this linear regression is given by $\ln(TTS) = mn + b$, where $n$ is the problem size. In Appendix F, we present the results of additional regression analysis for more-general scaling laws of the form $\ln(TTS) = mn + b$. The highest confidence with respect to the quality of the regression fit is indeed obtained at an exponent value close to $c = 1$, which supports our conjecture that DAQC scales exponentially. In actuality, the scaling is found to be slightly sub-exponential, at the value $c \approx 0.9$. (b) Slope of the linear regression for a range of layers. The best scaling for DAQC on these 21-weight MaxCut instances is observed at 20 layers. (c) TTS scaling for the SK model, when using a 20-layer DAQC. A best-fit linear-regression is drawn to the median of the TTSs of the 1000 instances for each size.

FIG. 8: Trotterization of adiabatic evolution into $p = 6$ layers. The integrals computing $b_k$ and $c_k$ yield the coefficients for $H_0$ and $H_1$, respectively.

We also compare the TTS for DAQC to the TTS for Breakout-Local Search (BLS), a classical search algorithm. For each graph instance, 20 runs of BLS were performed, and runtimes were averaged to obtain the TTS. The algorithm’s runtime for each run was capped at 0.1 seconds, although the minimum value was almost always found within that time. Fig. 9 demonstrates that the TTS for DAQC shows no significant correlation with the TTS for BLS.

FIG. 9: Scatter plot of DAQC-TTS versus BLS-TTS indicates there is no significant correlation between the difficulty of an instance for DAQC versus the difficulty of an instance for Breakout-Local Search.

B. Challenges Encountered when Using the Variational Approach

We have also explored using the variational quantum–classical protocol, which is typical for the approach known as QAOA. This protocol includes an optimization loop which learns better parameters ($\gamma, \beta$) by using the data from already-performed shots. However, we found that including an optimization step did not improve the total TTS for the following reasons, and therefore did not include the step in our analysis.
The $R_{99}$ is impossible to measure without knowledge of the ground state, and therefore any optimization routine must instead rely on energy measurements. A common approach is to use the expected energy, $\langle \psi(\gamma, \beta) | H \psi(\gamma, \beta) \rangle$, which is estimated by averaging over the multiple shots taken with the parameters $(\gamma, \beta)$. This approach suffers from two limitations. First, we must use a large number of shots to accurately estimate the expected energy, which makes the optimization step costly. This is consistent with the challenges encountered in overcoming the problem known as barren plateau phenomenon [22, 23]. Second, the expected energy is an imperfect stand-in for $R_{99}$, and therefore optimization typically offers little to no improvement upon the annealing-inspired parameter schedule. See Appendix E for more details.

### IV. SCALING OF DH-QMF

We now consider using Dürr and Høyer’s algorithm for quantum minimum finding (DH-QMF) [20] to find the ground state of an Ising Hamiltonian corresponding to a MaxCut problem. Given a real-valued function $E : S \to \mathbb{R}$ on a discrete domain $S$ of size $N = |S|$, DH-QMF finds a minimizer of $E$ (out of the possibly many) using $O(\sqrt{N})$ queries to $E$. In our case, the domain $S$ is the set of all spin configurations of a classical Ising Hamiltonian on $n$ sites $(N = 2^n)$, and the function $E$ maps each spin configuration to its energy. The DH-QMF algorithm is a randomized algorithm, that is, it succeeds in finding the optimal solution only up to a (high) probability. The probability of failure of DH-QMF can be made arbitrarily small without changing the mentioned complexity. A schematic illustration of DH-QMF is shown in Fig. 10, and additional technical details can be found in Appendix G.

Given an $n$-spin Ising Hamiltonian

$$H = - \sum_{0 \leq i < j \leq n - 1} J_{ij} Z_i Z_j \quad (15)$$

corresponding to an undirected weighted graph of size $n$, its $N = 2^n$ energy eigenstates can be labelled by the integer indices $0 \leq y \leq N - 1$, with the corresponding energy eigenvalues $E(y)$. The index $y$ associated with a computational basis state $|y\rangle = |\eta_0\rangle \otimes \cdots \otimes |\eta_{n-1}\rangle$ represented by the classical bits $\eta_j \in \{0, 1\}$ is the binary representation $y = \sum_{j=0}^{n-1} \eta_j 2^j$ of the bit string $(\eta_0, \ldots, \eta_{n-1})$.

The algorithm starts by choosing uniformly at random an index $y \in \{0, \ldots, N - 1\}$ as the initial “threshold index”. The threshold index is used to initiate a Grover’s search [19, 53]. The Grover subroutine searches for a label $y^*$ whose energy is strictly smaller than the threshold value $E(y)$. We measure the output of Grover’s search and (classically) ascertain whether the search has been successful, $E(y^*) < E(y)$, in which case we (classically) update the threshold index from $y$ to $y^*$, and then continue by performing the next Grover’s search using the new threshold. The threshold is not updated if Grover’s search fails to find a better threshold.

In this paper, we assume a priori knowledge of a hyperparameter we call the number of “Grover iterations” (see Section IV B) inside every Grover’s search subroutine that guarantees a sufficiently small failure probability. However, the practical scheme for using DH-QMF consists of multiple trials of Grover’s search and iterative updates to the threshold index. We terminate this loop when the Grover subroutine repeatedly fails to provide any further improvement to $y$ and the probability of the existence of undetected improvements drops below a sufficiently small value. Finally, we return the last threshold index as the solution. As shown in [20], the overall required number of Grover iterations needed to find the ground state with sufficiently high probability, say 1/2, is in $O(\sqrt{N})$.

#### A. Time-to-Solution Benchmark for DH-QMF

We investigate the scaling of the time required by DH-QMF to find a solution of weighted MaxCut instances with a 0.99 success probability, assuming an optimistic scenario that is explained in Section IV B. This runtime is analogous to the TTS measure defined in previous sections for the heuristic algorithms of the MFB-CIM and DAQC and we therefore call this runtime a TTS as well. For each instance of the problem we have estimated an optimistic lower bound on the runtime of

![FIG. 10: Schematic illustration of the Dürr–Høyer algorithm for quantum minimum finding (DH-QMF) applied to searching for a spin configuration corresponding to the energy minimum (ground state). The possible spin configurations are labelled by the indices $y \in \{0, \ldots, 2^n - 1\}$. The algorithm starts by choosing uniformly at random an initial guess for the “threshold index” $y$, whose energy $E(y)$ serves as a threshold: solutions to the problem cannot have an energy value larger than this threshold. The main step of the algorithm is a loop consisting of Grover’s search for a spin configuration with an energy value strictly smaller than the threshold energy, followed by a threshold-index update. This loop needs to be repeated many times until the threshold index eventually holds the solution with a probability of success higher than a given target lower bound, say, e.g., $P_{\text{succ}} = 0.99$. The final step returns the threshold index as output. A key element of the Grover’s search subroutine is an oracle which marks all states whose energies are strictly smaller than the threshold energy. Note that Grover’s search may fail to output a marked state.](image-url)
the quantum algorithm with numbers of Grover’s iterations in DH-QMF set (ahead of any trials) to achieve an at least 0.99 success probability. As this optimal number of Grover’s iterations is dependent on the specific MAXCUT instance, we consider this an optimistic bound on performance of DH-QMF. We use the same test set of randomly generated 21-weight MAXCUT instances as in previous sections.

Our results are illustrated in Fig. 11. The optimistic values for the TTS are in the range of orders of magnitude of 1.0 milliseconds – 1.0 seconds for the considered range of the number of vertices, 10 ≤ n ≤ 20, using the same set of assumptions for the quantum processor as in Assumption 1.

Our estimates for the runtime of the quantum algorithm are obtained as follows. We note that DH-QMF consists of a sequence of Grover’s search algorithms. The total runtime of DH-QMF is therefore the sum of the runtimes of the quantum circuits, each of which corresponds to a Grover’s search. The runtime of each such circuit is calculated using the depth of that circuit, which is the length of the longest sequence of native operations on the quantum processor (i.e., qubit preparations, single-qubit and two-qubit gates, and qubit measurements) in that circuit, assuming maximum parallelism between independent operations. This path is also known as the “critical path” of a circuit. The runtime of the quantum processor is therefore identical to the sum of the runtimes of the operations along the critical path, with a contribution of 1.0 microsecond in total for both qubit initialization and measurement, and 10 nanoseconds for any quantum gate operation along the critical path.

The asymptotic scaling of the TTS is identical to the scaling of the circuit depth, which is

\[ \Theta \left( \sqrt{2^n} \right) \left( n^2 \log \log n + (\log n)^2 + n \right), \]

as shown in Appendix G. Here the \( \Theta \left( \sqrt{2^n} \right) \) contribution is that of the number of Grover iterations (identical to the query complexity of Grover’s search), while the \( \text{poly}(n, \log n, \log \log n) \) factors are the contribution of each single Grover iteration consisting of an oracle query with implementation cost \( \Theta \left( n^2 \log \log n + (\log n)^2 \right) \) and the Grover diffusion with cost \( \Theta(n) \). A nonlinear least-squares regression toward this scaling is shown in Fig. 11 for both the 21-weight and the SK model problem instances, respectively. Note that the contributions of logarithmic terms are significant only for small problem sizes.

Alongside the optimistic runtime, we have also computed lower bounds on the number of quantum gates, including concrete counts for the overall number of single-qubit gates, two-qubit CNOT gates, and T gates (see Fig. 12). Our circuit analysis in Appendix G yields the gate complexity

\[ \Theta \left( \sqrt{2^n} \right) \left( n^2 \log n \log \log n + (\log n)^2 + n \right). \]  

Our resource estimates have been generated using ProjectQ [54].

\[ O_{QMF} : |x\rangle |z\rangle \rightarrow |x\rangle |z \oplus f(x)\rangle, \]

where \( f(x) = 1 \) if, and only if, \( E(x) < E(y) \), and \( f(x) = 0 \) otherwise. Here, \( \oplus \) represents a bitwise XOR. The QMF oracle is constructed from multiple uses of the energy oracle \( O_E \) and an operator that compares the values held by two registers. Details of this construction can be found in Appendix G.2. The combined effect of querying \( O_{QMF} \) followed by the Grover diffusion (together forming the Grover iteration to be repeated \( O(\sqrt{2^n}) \) times) results in constructively amplifying the amplitudes of the marked items while diminishing the amplitudes of the unmarked ones.

When there are multiple solutions to a search problem, as is frequently the case in the Grover subroutine of QMF, the optimal number of Grover iterations needed to maximize the success probability depends on the number of marked items as well. Indeed, suppose we were to have knowledge of the number of marked items \( t \) ahead of time. Then, the optimal number of Grover iterations could be obtained from the closed formulæ provided in [53]:

\[ \varphi_{\text{succ}} = \sin^2 \left( (2m + 1) \theta \right), \]

\[ \varphi_{\text{fail}} = \cos^2 \left( (2m + 1) \theta \right). \]

Here, \( m \) is the number of Grover iterations, and \( \theta \) is defined by \( \sin^2 \theta = t/N \). Hence, the success probability is maximized for the optimal number of Grover iterations \( m_{\text{opt}} = \lfloor \pi/4\theta \rfloor \). We also observe that after exactly \( m_{\text{opt}} \) iterations the failure probability obeys

\[ \varphi_{\text{fail}} \leq \sin^2 \theta = t/N, \]

which is negligible when \( t \ll N \).

In practice, \( t \) and, consequently, \( m_{\text{opt}} \) are often unknown. Nevertheless, [53, Sec. 4 and Theorem 3] propose a method to find a marked item with query complexity \( O \left( \sqrt{N/t} \right) \) even when no knowledge of the number of solutions is assumed.

To simplify the analysis for our benchmark in this paper, we examine each MAXCUT instance and assume \( t \)
is known every time Grover’s search is invoked. This assumption provides a lower bound on the performance of DH-QMF. In view of the previous discussion, having knowledge of $t$ allows us to compute $m_{\text{opt}}, \varphi_{\text{succ}},$ and $\varphi_{\text{fail}}$.

We then boost the overall success probability of Grover’s search to any target success probability $p_{\text{opt}}$ by repeating it $K$ times, where $K$ satisfies

$$p_{\text{opt}} \leq 1 - \varphi_{\text{fail}}^K. \quad (20)$$

Moreover, if DH-QMF requires $J$ non-trivial threshold index updates in total, we must succeed in every boosted Grover search (each including $K$ Grover searches). The probability of this event is thus at least $p_{\text{opt}}^J$. Finally, let us denote the target lower bound for the probability of success of the overall DH-QMF algorithm by $p_{\text{succ}}$. We then must have

$$p_{\text{succ}} \leq p_{\text{opt}}^J. \quad (21)$$

We achieve a lower bound for $K$ using Eqs. (20) and (21):

$$K \geq \log \left( \frac{1 - p_{\text{succ}}^{1/J}}{\log \varphi_{\text{fail}}} \right). \quad (22)$$

Note that this number still depends on the optimal number $m_{\text{opt}}$ of Grover iterations. The remainder of this section explains how the latter number is sampled for each MAXCUT instance via Monte Carlo simulation.

Given a weighted graph, we first generate the histogram of the sizes of all cuts in the graph. Examples of such histograms are provided in Fig. 13. This cut-size
histogram allows us to perform a Monte Carlo simulation of the progression of DH-QMF as follows. The DH-QMF algorithm starts by choosing uniformly at random an initial cut $C$ as the threshold index. The resulting energy threshold is therefore sampled according to the cut-size histogram. Grover search then attempts to find a larger cut. The number of these cuts is $t$ in the notation above, and can be found if the cut-size histogram is known. Using Eq. (19), we can also compute the optimal number $m_{\text{opt}}$ of Grover iterations needed to achieve the highest possible success rate $\varphi_{\text{max}}$ in that search. We furthermore can now use Eq. (22) to predict the number $K$ of Grover searches needed to boost the success probability to at least $p_0$. The cut $C$ is now replaced with a larger cut also selected at random using the cut-size histogram, and this simulation is repeated for the next iteration in DH-QMF.

We repeatedly sample and update the threshold until we find a maximum cut (i.e., at an iteration where $t = 0$). At this point, we stop our Monte Carlo simulation (even though in practice it will not be known that $t$ has become zero). For each sampling step $j$, we count the total number $t_j$ of states contributing to strictly greater cuts and use it to calculate the optimal number $m_{\text{opt}}[j]$ of Grover iterations as well as the number of boosting iterations $K_j$ via Eq. (22).

We now obtain an optimistic TTS as well as an optimistic gate count estimate using the formulae

$$\text{TTS} = \sum_{j=1}^{J} K_j m_{\text{opt}}[j] \times \text{Runtime},$$

$$\text{# gates} = \sum_{j=1}^{J} K_j m_{\text{opt}}[j] \times \text{GateCount}. $$

Here, Runtime denotes the running time and GateCount indicates the gate count for a single Grover iteration. In Section IV A we provided optimistic estimates for the number of single-qubit gates, CNOT gates, and $T$ gates. The quantum circuit implementation of a single Grover iteration is presented in Appendix G.

V. COMPARISON OF THE THREE ALGORITHMS

A direct comparison of the three algorithms for solving MAXCUT problems is illustrated in Fig. 14. In Fig. 14 (a), the median wall-clock TTS of DH-QMF, DAQC, and the closed-loop MFB-CIM are plotted as a function of problem size $n$ for randomly generated 21-weight MAXCUT instances. The solid blue line indicates a best-fitting curve, $f_{\text{CIM}}(n) = AB^{\sqrt{n}}$, for the closed-loop MFB-CIM, where $A = 121$ nanoseconds and $B = 2.21$; the solid orange line represents a best-fitting curve, $f_{\text{DAQC}}(n) = A'B'^{n^{0.9}}$, for a 20-layer DAQC, where $A' = 3.56$ microseconds and $B' = 1.26$; and the solid green curve represents a best-fitting curve, $f_{\text{QMF}}(n) = \left(An^2 \log \log n + Cn^2 + Dn\right)B^n$, for DH-QMF, where $\tilde{B} = \sqrt{2}$, and $\tilde{A}$, $\tilde{C}$, and $\tilde{D}$ are equal to 3.9, 5.25 × 10^2, and −2.97 × 10^2 microseconds, respectively.

In order to see how the performance of a closed-loop MFB-CIM scales with increasing problem size, we solved MAXCUT problems with SK instances of problem sizes $n = 100, 200, \ldots, 800$. A total of 100 instances of the SK model for each problem size were randomly generated. Using a closed-loop MFB-CIM, we solved each instance 100 times to evaluate the success probability $P_s$ of finding a ground state and compute a wall-clock time to achieve a success probability of $\geq 0.99$. It is assumed that all-to-all spin coupling is implemented in 10 nanoseconds, which corresponds to a cavity round-trip time. The signal field lifetime is 100 nanoseconds, that is, $\gamma_s\Delta T_c = 0.1$. We use the continuous-time Gaussian model as described in Section II A. The results are shown in Fig. 14(b), along with the predicted performance of DAQC and DH-QMF for the SK model instances. The minimum wall-clock TTS for the closed-loop MFB-CIM at the optimized runtime $t_{\text{max}}$ scales as an exponential function of $\sqrt{n}$, while those for DH-QMF and DAQC scale as exponential functions of $n$. At a problem size of $n = 800$, the wall-clock TTS for the closed-loop MFB-CIM is $\sim 10$ milliseconds, while those for DH-QMF and DAQC are $\sim 10^{120}$ seconds and $\sim 10^{50}$ seconds, respectively.

For the bimodal SK model, which is known to be “easy” for many algorithms, and for a limited problem-size range of $100 \leq n \leq 500$, we empirically observe a sub-exponential scaling of $\Theta(2^{\sqrt{n}})$ for the closed-loop MFB-CIM’s TTS. Such a sub-exponential scaling in solving the SK model instances using CIM-based algorithms has also been reported in other recent studies [17, 55]. For the 21-weight problem instances, due to the limited problem-size range $5 \leq n \leq 30$ of the data available, we cannot reliably infer the actual asymptotic scaling. While our results for the MFB-CIM seem to agree well with a sub-exponential scaling (with the same exponent $\sqrt{n}$), extrapolations from numerical findings based on small-sized problem instances can potentially be misleading.

In contrast, the scaling of DAQC appears to be ex-
ponental. In the absence of empirical data for large problem sizes (even for the SK instances), we perform a careful regression analysis on our data, which we report in Appendix F. Our analysis suggests an exponential scaling for solving the SK model problem instances and a slightly sub-exponential scaling with the exponent $n^{0.9}$ for the 21-weight problem instances. Nevertheless, we remain reluctant to extrapolate any exponential scaling laws from this investigation.

As for the TTS scaling of the DH-QMF algorithm, an exponential law of $O\left(\sqrt{n^2}\right)$ for the query complexity is supported by rigorous proofs [20, 53]. Our benchmarking study reveals that this exponential scaling is not improved for problem instances based on the SK model. In addition, the query complexity does not account for the cost of a single query to the oracle. Our benchmarking results, shown in Fig. 11, are based on a regression towards the scaling given in Eq. (16), which includes an additional poly($n, \log n$) factor to account for the scaling of the circuit depth of our oracle implementation.

VI. CONCLUSION

In this paper, we have presented the results of our study of the scaling of two types of measurement-feedback coherent Ising machines (MFB-CIM) and compared this scaling to that of discrete adiabatic quantum computation (DAQC) and the Dürr–Höyer algorithm for quantum minimum finding (DH-QMF). We performed this comparative study by testing numerical simulations of these algorithms on 21-weight MAXCUT problems, that is, weighted MAXCUT problems with randomly generated edge weights attaining 21 equidistant values from $-1$ to $1$. We emphasize that our study was a numerical analysis; its results depend on the experimental choices we have empirically made to the best of our abilities.

The MFB-CIM of the first type is an open-loop MFB-CIM with predefined feedback control parameters and the second is a closed-loop MFB-CIM with self-diagnosis and dynamically modulated feedback control parameters. The open-loop MFB-CIM utilizes the anti-squeezed $X$ amplitude near threshold under a positive pump amplitude for finding a ground state but at larger problem sizes the machine is often trapped in local minima. The closed-loop MFB-CIM employs the squeezed $X$ amplitude under a negative pump amplitude, in which a finite internal energy is sustained through an external feedback injection signal rather than through parametric amplification. This second machine self-diagnoses its current state by performing Ising energy measurement and comparison with the previously attained minimum energy. The machine continues to explore local minima without getting trapped even in a ground state. We observed that for both the 21-weight MAXCUT problems and the SK Ising model, the closed-loop MFB-CIM outperforms the open-loop MFB-CIM. One remarkable result is that a low-finesse cavity machine realizes a shorter TTS than a high-finesse one. This fact clearly demonstrates that the dissipative coupling of the machine to external reservoirs is a crucial computational resource for MFB-CIMs. The wall-clock TTS of the closed-loop MFB-CIM closely follows TTS $\approx 4.32 \times (1.34)^{n^2}$ microseconds for the SK model instances of size $n$ ranging from 100 to 800, assuming a cavity round-trip time of 10 nanoseconds and a $1/e$ signal amplitude decay time of 100 nanoseconds ($\gamma_\Delta T_c = 0.1$). The performance of the MFB-CIM shown in Fig. 14 is already competitive against various heuristic solvers implemented on advanced digital platforms such as CPUs, GPUs, and FPGAs, in which massive parallel computation is performed over many billions of transistors [6, 7, 55–58]. Note that the results shown in Fig. 14 are based on the assumption of a MFB-CIM architecture that employs only a single OPO as an active element (i.e., it involves only a single optical resonator, along with a nonlinear optical crystal, pumped by a laser) for processing information encoded in time-multiplexed oscillations of the resonator. It is anticipated that advanced on-chip coherent network computing technologies (see, e.g., [59]) will allow the design of highly parallelized MFB-CIM architectures involving multiple OPO components operated in parallel, with the potential for massively parallel computation that would further enhance performance.

We have also studied the scaling of the DAQC algorithm in solving 21-weight and SK model MAXCUT problem instances. We considered two schemes for optimizing the quantum gate parameters of DAQC, denoted in the paper as ($\gamma, \beta$). In the first scheme, we treat $\gamma$ and $\beta$ as hyperparameters that follow a schedule inspired by the adiabatic theorem. In this case, DAQC can be viewed as a Trotterization of an adiabatic evolution from the ground state of a mixing Hamiltonian to the ground state of a problem Hamiltonian. The second scheme is a variational hybrid quantum–classical algorithm (similar to the QAOA approach) wherein a classical optimizer is tasked with optimizing the gate parameters $\gamma$ and $\beta$. The variational scheme must perform repeated state preparation and projection measurements to estimate the ensemble averaged energy, which makes the optimization step not only costly but vulnerable to the shot noise of these measurements. Another disadvantage of the variational scheme is that optimizing the ensemble average energy does not necessarily improve the TTS, which is the more practical measure of performance for the algorithm (see Appendix E for more details). As shown in Fig. 17, the adiabatic schedules achieve very low $R_{99}$ values, suggesting a challenging bound on the allowed number of shots for the variational scheme to outperform the adiabatic scheme for this problem. Given these considerations, we used a pre-tuned adiabatic scheme to assess the performance limits of DAQC. In contrast, we note that the quantum state in an MFB-CIM survives against various heuristic solvers implemented on advanced digital platforms such as CPUs, GPUs, and FPGAs, in which massive parallel computation is performed over many billions of transistors [6, 7, 55–58]. Note that the results shown in Fig. 14 are based on the assumption of a MFB-CIM architecture that employs only a single OPO as an active element (i.e., it involves only a single optical resonator, along with a nonlinear optical crystal, pumped by a laser) for processing information encoded in time-multiplexed oscillations of the resonator. It is anticipated that advanced on-chip coherent network computing technologies (see, e.g., [59]) will allow the design of highly parallelized MFB-CIM architectures involving multiple OPO components operated in parallel, with the potential for massively parallel computation that would further enhance performance.

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These measurements perturb the internal quantum state of the OPO network but do not completely destroy it. As a result, the above drawback of a variational scheme for DAQC does not apply to the closed-loop MFB-CIM. The wall-clock TTS of DAQC with hypertuned adiabatic schedules is well-represented by the TTS ≈ 4.6 × (1.17)^n microseconds. As shown in Fig. 14, extrapolating this trend suggests that DAQC will perform poorly compared to MFB-CIM as the problem sizes increase due to an exponential dependence on the number, n, of vertices in the MaxCut problem compared to an exponential growth with a \( \sqrt{n} \) exponent in the case of MFB-CIM.

Finally, we have also studied the scaling of DH-QMF for solving 21-weight and SK model MaxCut problems. As this algorithm is based on Grover’s search, it performs \( O(\sqrt{2^n}) \) Grover iterations, implying it makes a number of queries, of the same order, to its oracle. The algorithm also iterates on multiple values of a classical threshold index; however, this does not change the dominating factors in the scaling of the algorithm. We have shown that the wall-clock TTS of DH-QMF is well-approximated by the TTS ≈ 17.3 \times 2^{n/2} n^2 \log \log n \text{ microseconds when extrapolated to larger problem sizes.} As shown in Fig. 14, DH-QMF requires a computation time that is many orders of magnitude larger than that for either DAQC or MFB-CIM. This comparatively poor performance of DH-QMF can be traced back to the linear amplitude amplification in the Grover iteration in contrast to the exponential amplitude amplification at the threshold of the OPO network. Our study thus leaves open the question of whether there exist optimization tasks for which Grover-type speedups are of practical significance.

**METHODS**

The methods used to obtain the benchmark results for each of the analyzed algorithms are provided in the corresponding sections, respectively. Additional details are provided as appendices in the Supplementary Material document.

**DATA AVAILABILITY**

The datasets generated and analyzed as part of our study are available from the corresponding author on reasonable request.

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**AUTHOR CONTRIBUTIONS**

K. S., A. S., and S. K. contributed equally to this research. With P. R.’s guidance, K. S., N. G., and W. B. K. performed numerical experiments pertaining to the analysis for discrete adiabatic quantum computation and the DAQC algorithm, and A. S. developed and implemented the benchmark analysis for Dürr–Høyer’s quantum minimum finding algorithm. With Y. Y.’s guidance, S. K., S. R., Y. L., E. N., and T. O. conducted the benchmark study for the coherent Ising machines. K. S. and A. S., with support from P. R., derived and formulated the details provided in Section III and Section IV and their corresponding appendices, respectively. S. K., with assistance from Y. Y., determined and wrote the content for Section II and its corresponding appendix. The remaining parts of the paper were written by P. R. and Y. Y., aided by K. S., A. S., and S. K. All authors contributed to ideation throughout this work. P. R. and Y. Y. led the overall efforts of this benchmark study.

**COMPETING INTERESTS**

The authors declare no competing interests.

**APPENDICES**

Appendix A: Review of Related Work

The motivation for our work is to benchmark coherent Ising machines (CIM) against circuit model quantum algorithms. In this appendix, we review related literature and discuss what makes our analysis differ from previous work.

*Coherent Ising Machine* — The CIM based on a network of degenerate optical parametric oscillators (DOPO) was originally proposed in [13]. Various schemes have been developed since then, based on optical delay line (ODL) coupling [60, 61] and measurement-feedback (MFB) coupling [47, 48]. The quantum master equations for ODL-CIM and MFB-CIM can be cast into c-number stochastic differential equations using a standard procedure based on representations over the phase space (e.g., the Wigner function, the Sudarshan–Glauber “P representation”, or the Husimi “Q representation”), the Fokker–Planck equation, and Itô calculus. In our work, we use the Gaussian approximation of MFB-CIM [48, 49, 62].

The standard CIM, which is based on a linear mutual coupling scheme, suffers from amplitude heterogeneity among the constituent DOPOs [13], resulting in a performance degradation due to an incorrect mapping of the Ising Hamiltonian to the DOPO network loss. To overcome this drawback, a self-diagnosis scheme, along with dynamical feedback control, has been devised [24, 25, 63]. Our work demonstrates the first analysis of such a scheme using the Gaussian approximation of MFB-CIM, and provides for the first time a benchmark against the circuit model quantum algorithms suitable for solving Ising problems. Our benchmarking analysis complements previous benchmarking studies against other solvers, including quantum annealing, digital annealing, simulated annealing, breakout local search, parallel tempering, and simulated bifurcation machines [17, 24, 55, 63].

*DH-QMF algorithm* — The quantum minimum finding algorithm (QMF) was originally proposed by Dürr and Høyer [20] shortly after Grover invented his algorithm for unstructured search [18, 19]. This randomized algorithm uses the quantum exponential searching algorithm by Boyer et al. [53, Sec. 4 and Theorem 3], which can be viewed as a generalization of Grover’s search algorithm. Quantum exponential searching is a randomized method for finding a solution to a search problem that has potentially multiple solutions even when their number is not known ahead of time (which is the typical scenario at each threshold-index update step during the progression of the DH-QMF algorithm). Denoting this unknown number of solutions by $t$, the algorithm returns one of the solutions with equal probability after an expected number $O(\sqrt{N/t})$ of Grover iterations if $t \geq 1$, and runs forever if $t = 0$. Assuming there is no time out in the execution of the DH-QMF algorithm, any state whose energy is strictly smaller than the energy corresponding to the threshold index may be chosen with some probability as the new threshold index. Dürr and Høyer [20] show that this probability is given by the inverse of the rank of the state (with respect to its energy value), but is independent of the dimension of the search space. This fact is then used to show that the algorithm finds the state of the minimum energy value with a success probability of $O(1)$ using at most $O(\sqrt{N})$ Grover iterations.

Our work presents for the first time a Monte Carlo simulation of the progression of the DH-QMF algorithm aimed at the inference of *concrete* lower bounds on the optimistic performance of the algorithm (in terms of the TTS). Our Monte Carlo analysis (see Section IV B) is based on sampling according to energy–frequency (equivalently, cut-size–frequency) histograms associated with weighted graphs. In particular, the energy (or cut-size) histogram generated for each weighted graph instance allows us to infer the number of marked states, which in turn allows us to determine the success and failure probabilities, as well as to predict both the optimal number of Grover iterations at each threshold-index update step and the number of Grover searches needed to boost the success probability beyond a target threshold. Our approach thus allows us to emulate the progression of the DH-QMF algorithm and hence to infer a concrete optimistic lower bound on the TTS for each given problem instance. In our studies, we have not encountered any
previous work that uses a similar approach to simulating the progression of the DH-QMF algorithm.

DAQC algorithm — The second circuit-based algorithm analyzed in our benchmarking study is discrete adiabatic quantum computation (DAQC). Its circuit ansatz is derived from implementing Hamiltonian simulation for a given adiabatic evolution using the well-established first-order Suzuki–Trotter expansion. Its application to solving various optimization problems has been extensively explored since quantum adiabatic evolution was first introduced to the solving of NP-complete problems by Farhi et al. [9]. A related concept is the quantum approximate optimization algorithm (QAOA) [12, 28], which can be viewed as a diabatic counterpart to DAQC. Indeed, the circuit ansatz of QAOA is very much akin to a Trotterized analogue of quantum adiabatic evolution: it uses an iterative unitary evolution of pure states in a quantum circuit according to a mixing Hamiltonian and a problem Hamiltonian, which in the framework of adiabatic quantum computation correspond to the initial and final Hamiltonians of evolution, respectively. QAOA is considered a promising candidate for solving combinatorial optimization problems on noisy, intermediate-scale quantum (NISQ) devices. Although its formulation is not restricted to implementations on such devices, QAOA has become associated with NISQ algorithms, due to its use of shallow (i.e., low-depth) quantum circuits, along with a variational method for optimizing the set of parameters specifying the unitary gates in those circuits. Its performance as a NISQ algorithm has been extensively studied in recent work [28, 41, 42, 64].

We have studied two schemes for optimizing Trotterization parameters. The first scheme treats gate parameters as hyperparameters that follow a tuned schedule for Trotterized adiabatic evolution. The second scheme uses a variational hybrid quantum-classical protocol to optimize the gate parameters. We found a performance advantage for the first scheme over the second. The challenges of the latter scheme are discussed in Appendix E. For this reason, we have used pre-tuned DAQC schedules for our benchmarking analysis. Moreover, to obtain the theoretical performance limit, we dropped the requirement of having to use only low-depth circuits that are necessary in the case of NISQ devices. That is, our results pertain to the implementation of DAQC using quantum circuits of arbitrary depth. Our new insights with respect to the optimization of quantum gate parameters, and our results on the concrete lower bounds on the optimistic performance of the DAQC algorithm (in terms of the TTS) in solving MaxCut problem instances, complement previous studies on the performance of QAOA. In particular, our extensive analysis of hyperparameter tuning for DAQC parameter schedules does not resemble any previous numerical results in the literature, and may serve as a new baseline for future benchmarking studies based on DAQC.

Appendix B: Discrete-Map Gaussian Model of the CIM

In this Appendix, we summarize the discrete-map Gaussian model of the CIM presented in [49], and we adapt the feedback step to include the dynamic feedback control used for the closed-loop MFB-CIM. This discrete-map model is used to study the optimization performance of the MFB-CIM in Section II C. In the discrete Gaussian quantum model of MFB-CIM, the density operator of the i-th OPO pulse is fully characterized by the mean amplitude \( \mu_i \) and covariance matrix \( \Sigma_i \) defined by Eqs. (9) and (10). The total density operator before all pulses start their \( \ell \)-th roundtrip is expressed by \( \otimes_{i=1}^{n_p} \hat{\rho}(\mu_i(\ell), \Sigma_i(\ell)) \). Propagation of the state of the \( i \)-th pulse through \( \ell \)-th roundtrip from \( \hat{\rho}(\mu_i(\ell), \Sigma_i(\ell)) \) to \( \hat{\rho}(\mu_i(\ell + 1), \Sigma_i(\ell + 1)) \) is described by performing the following discrete maps consecutively.

1. Background linear loss: The lumped background linear loss transforms the density operator as

\[
\hat{\rho}(\mu_i, \Sigma_i) \mapsto \text{tr}_c \left( B \left[ \hat{\rho}(\mu_i, \Sigma_i) \otimes \hat{\rho}(0_c, \Sigma_c^0) \right] \right),
\]

where \( \Sigma_c^0 = \text{diag}(1/2, 1/2) \) is the covariance of a coherent state. The beamsplitter map \( B \) is defined by

\[
B[\hat{\rho}(\mu, \Sigma)] = \hat{\rho}(S\mu, S\Sigma S^T),
\]

\[
S = \begin{pmatrix} t & 0 & -r & 0 \\ 0 & t & 0 & -r \\ r & 0 & t & 0 \\ 0 & r & 0 & t \end{pmatrix}.
\]

Here, \( t = \sqrt{1 - r^2} \) is the amplitude transmission coefficient of a fictitious beamsplitter which represents background linear loss. Physically, \( \hat{\rho}(0_c, \Sigma_c^0) \) is a reservoir vacuum state and it is traced out after mixing with the signal pulse at the beamsplitter.

2. Parametric amplification/demultiplication during OPO crystal propagation: The propagation through a second-order nonlinear crystal with the pump pulse transforms the density operator as

\[
\hat{\rho}(\mu_i, \Sigma_i) \mapsto \text{tr}_b \left[ \chi \left( \hat{\rho}(\mu_i, \Sigma_i) \otimes \hat{\rho}(\mu_b, \Sigma_b^0) \right) \right],
\]

where \( \mu_b \) and \( \Sigma_b^0 = \text{diag}(1/2, 1/2) \) describe the initial condition of the (Gaussian) pump pulse, and the map \( \chi \) abstractly represents their joint propagation through the crystal, that is,

\[
\chi : \hat{\rho}(\mu_i, \Sigma_i) \otimes \hat{\rho}(\mu_b, \Sigma_b^0) \mapsto \hat{\rho}(\mu_{i,b}, \Sigma_{i,b}),
\]

where \( \hat{\rho}(\mu_{i,b}, \Sigma_{i,b}) \) is a joint two-mode Gaussian state of the signal and pump at the output. This joint output is determined by the equations of motion for the mean-field
and covariance matrix:

\[
\frac{d\langle \hat{X}_i \rangle}{dt} = \epsilon \langle \hat{X}_b \rangle \langle \hat{X}_i \rangle + \epsilon \langle \delta \hat{X}_b \delta \hat{X}_i + \delta \hat{P}_b \delta \hat{P}_i \rangle \tag{B6}
\]

\[
\frac{d\langle \hat{X}_b \rangle}{dt} = -\frac{\epsilon}{2} \langle \hat{X}_i^2 \rangle - \frac{\epsilon}{2} \langle \delta \hat{X}_i^2 \rangle - \frac{\epsilon}{2} \langle \delta \hat{P}_i^2 \rangle \tag{B7}
\]

\[
\frac{d\langle \delta \hat{X}_i^2 \rangle}{dt} = 2\epsilon \langle \hat{X}_b \rangle \langle \delta \hat{X}_i^2 \rangle + 2\epsilon \langle \hat{X}_i \rangle \langle \delta \hat{X}_b \delta \hat{X}_i \rangle \tag{B8}
\]

\[
\frac{d\langle \delta \hat{P}_i^2 \rangle}{dt} = -2\epsilon \langle \hat{X}_b \rangle \langle \delta \hat{P}_i^2 \rangle + 2\epsilon \langle \hat{X}_i \rangle \langle \delta \hat{P}_b \delta \hat{P}_i \rangle \tag{B9}
\]

\[
\frac{d\langle \delta \hat{X}_b \delta \hat{X}_i \rangle}{dt} = -2\epsilon \langle \hat{X}_b \rangle \langle \delta \hat{X}_b \delta \hat{X}_i \rangle \tag{B10}
\]

\[
\frac{d\langle \delta \hat{P}_b \delta \hat{P}_i \rangle}{dt} = -2\epsilon \langle \hat{X}_i \rangle \langle \delta \hat{P}_b \delta \hat{P}_i \rangle \tag{B11}
\]

\[
\frac{d\langle \delta \hat{X}_b \delta \hat{X}_i \rangle}{dt} = \epsilon \langle \hat{X}_b \rangle \langle \delta \hat{X}_b^2 \rangle - \delta \hat{X}_i^2 \rangle + \epsilon \langle \hat{X}_b \rangle \langle \delta \hat{X}_b \delta \hat{X}_i \rangle \tag{B12}
\]

\[
\frac{d\langle \delta \hat{P}_b \delta \hat{P}_i \rangle}{dt} = \epsilon \langle \hat{X}_b \rangle \langle \hat{P}_b^2 \rangle - \delta \hat{P}_i^2 \rangle - \epsilon \langle \hat{X}_b \rangle \langle \delta \hat{P}_b \delta \hat{P}_i \rangle \tag{B13}
\]

Here \( \epsilon \) is the parametric coupling rate defined by the Hamiltonian \( \mathcal{H} = i \frac{\hbar}{2} (\hat{a} \hat{b} \hat{a}^\dagger - \hat{b}^\dagger \hat{a}^2) \), where \( \hat{a} \) and \( \hat{b} \) are signal and pump annihilation operators. We assume that the input state into the crystal satisfies \( \langle \hat{P}_i \rangle = \langle \hat{P}_b \rangle = 0 \) (i.e., there is no coherent excitation along the quadrature-phase and \( \{ \langle \delta \hat{X}_b, \delta \hat{P}_i \rangle \} = \{ \langle \delta \hat{X}_b, \delta \hat{P}_b \rangle \} = 0 \) (both the signal and pump have no correlation between in-phase and quadrature-phase fluctuations). Note that \( \langle \hat{P}_i \rangle = \langle \hat{P}_b \rangle = 0 \) is satisfied at all times under the above conditions. The defined map \( \chi \) thus describes all such effects as linear parametric amplification/demixification, signal-pump entanglement formation, and back conversion from the pump to the signal.

3. Out-coupling and homodyne detection: The out-coupling of the internal signal pulse is described by the map

\[
\hat{\rho} (\mu_i, \Sigma_i) \rightarrow \hat{\rho} (\mu_{i,h}, \Sigma_{i,h}) = \mathcal{B}_{\text{out}} [\hat{\rho} (\mu_i, \Sigma_i) \otimes \hat{\rho} (0, \Sigma_h^0)] , \tag{B14}
\]

where the beamsplitter map \( \mathcal{B}_{\text{out}} \) is defined by Eqs. (B2) and (B3) with an out-coupling rate of \( r_{\text{out}} \). In Eq. (B14), a probe mode \( \hat{h} \) is prepared in a vacuum state and mixed with the signal pulse. This process creates a joint correlated state (entangled state) between the internal pulse and external (out-coupling) pulse. Suppose a homodyne measurement for the out-coupled pulse reports a result \( m_i (l) \) for the \( i \)-th signal pulse at the \( \ell \)-th round trip. Such an indirect approximate measurement projects the internal state to a new state by the map

\[
\hat{\rho} (\mu_{i,h}, \Sigma_{i,h}) \rightarrow \mathcal{H} [\hat{\rho} (\mu_{i,h}, \Sigma_{i,h})]
= \hat{\rho} (\mu_i (m_i), \Sigma_i (m_i)) , \tag{B15}
\]

where the homodyne detection map \( \mathcal{H} \) is defined by

\[
\mu_i (m_i) = \mu_i + \left( \frac{w_i - \mu_h}{\Sigma_{XX}} \right) v_X \tag{B16}
\]

\[
\Sigma_i (m_i) = \Sigma_i - \frac{v_X v_X^T}{\Sigma_{XX}} \tag{B17}
\]

Here, \( v_X \) is the \( X \) off-diagonal component (the degree of signal-pulse correlation) of the matrix \( \Sigma_{i,h} \) and \( \Sigma_{XX} \) is the \( X \) diagonal element of \( \Sigma_i \). The second terms of the right-hand sides of Eqs. (B16) and (B17) express the mean-field shift and variance reduction induced by the homodyne measurement.

4. Feedback injection: We implement the Ising coupling by applying the displacement operation for the internal pulse amplitude based on the measurement results of the \( \ell \)-th round trip, \( m_j (\ell) \), for all pulses except for the \( i \)-th pulse. The displacement magnitude is given by

\[
v_i (\ell) = J_0 e_i (\ell) \sum_{j \neq i} J_{ij} m_j (l) , \tag{B18}
\]

where \( e_i (\ell) \) is the feedback-field amplitude of the \( \ell \)-th round trip which is determined by the equation of motion, Eq. (5), for the closed-loop CIM. The feedback gain \( J_0 \) scales with the inverse of \( \sqrt{N_{\text{decay}} / N} \sum_{i,j} |J_{ij}| \), where \( N_{\text{decay}} \) is the number of round trips required for the signal field amplitude to attenuate by a factor of \( 1/e \). The feedback injection map is thus determined by

\[
\hat{\rho} (\mu_i, \Sigma_i) \rightarrow \mathcal{D}_{v_i} [\hat{\rho} (\mu_i, \Sigma_i)] = \hat{\rho} (\mu_i + v_i, \Sigma_i) . \tag{B19}
\]

The above four steps are applied to all pulses \( i = 1, \ldots, n \), completing one round trip through the CIM cavity.

Appendix C: Discussion of Optimal Loss Parameters for the MFB-CIM

In Fig. 5, the effect of changing the total loss rate \( \gamma_s \Delta T_c (1 + j) \) on the TTS by using the discrete-time model of the MFB-CIM is shown. There are various ways the total loss rate can be varied. For the results displayed in Fig. 5, we kept \( j \) constant at the value 1 (recall that \( j \) is a parameter that corresponds to the escape efficiency in [49], which is the ratio of the out-coupling loss associated with the optical homodyne measurement to the total cavity loss) and varied \( \gamma_s \Delta T_c \). There is a sweet spot around \( \gamma_s \Delta T_c (1 + j) \approx 1 \).

In Fig. 15(a) and Fig. 15(b), heat maps are shown of the TTS for a problem instance of size \( n = 30 \) are shown. Here, the \( x \)-axis represents the total loss rate \( \gamma_s \Delta T_c (1 + j) \) and the \( y \)-axis represents the out-coupling loss \( j \). In these plots, \( j = 1 \) on the \( y \)-axis corresponds to the TTS curves plotted in Fig. 5. The green contour lines correspond to fixed values for \( j = j' \gamma_s \). As evident from these plots, at least in the case of the open-loop CIM, an increase in the
value of the total loss rate, moving along the horizontal axis, results in the optimal region becoming larger, while moving along a green contour line, the optimal region becomes sharper. In the case of the closed-loop CIM, there appear to be two optimal regions. We believe that the more accurate optimal region in this case is the region along the vertical line given by $\gamma_0 \Delta T_c(1 + j) \approx 0.5$ or $(N_{\text{decay}} = 2)$, even though in this region the TTS is longer, because the total loss rate becomes sufficiently large, the nonlinearity increases in strength such that the error correction mechanism can no longer stabilize the amplitude to the desired target amplitude. The reason there is a short TTS in this region for $n = 30$ is that the problems are small enough that they can still be solved despite the unstable behaviour of the solver. However, in the case of the problem size $n = 100$, as shown in Fig. 15(c), this second region no longer has a short TTS, and the optimal TTS occurs in the region around the vertical line defined by $\gamma_0 \Delta T_c(1 + j) \approx 0.3$.

Appendix D: Hyperparameter Tuning for DAQC Parameter Schedules

As described in Section III.A, we have a recipe for generating DAQC parameter schedules for any problem Hamiltonian $H_P$ and number of layers $p$. We consider two hyperparameters for these schedules:

- The number $L = T/p$ is the evolution time in each Trotterized layer of the associated annealing schedule. A larger value of $L$ corresponds to a slower and therefore better associated annealing schedule, but also brings along a greater Trotterization error;
- The number $a$ is the coefficient of the cubic term in the adiabatic schedule. When $a = 0$ the schedule is linear, and when $a = 4$ the schedule is cubic, with $f'(T/2) = 0$. We therefore only consider $a \in [0, 4]$, because for $a > 4$ the schedule would be decreasing at $t = T/2$.

Here, we compile our results on the performance of DAQC with cubic schedules for various values of the hyperparameters $a$, $L$, and $p$. In Figs. 16 to 18, the horizontal axis displays the number of vertices for the problem instance, and the vertical axis displays the $R_{99}$ or TTS (in logarithmic scale). Each blue dot represents a single problem instance. All plots depict a total of 11,000 problem instances varying from 10 to 20 nodes in size. Each black point represents the geometric mean of all values of $R_{99}$ or TTS for problem instances of a given size. Finally, the red line indicates the best linear fit to the black points. The equation corresponding to the best-fit line is written in each subplot, where $n$ is the number of vertices.

We empirically found that a value of $L$ between 2.6 and 3.6 worked best. In Fig. 16, we plot the $R_{99}$ values of the good parameter schedule with hyperparameters $a \in \{0, 2, 4\}$ and $L \in \{2.8, 3.0, 3.2, 3.4, 3.6\}$. Note that $a = 4$ (a cubic schedule with a derivative of 0 at the inflection point) outperforms $a = 0$ (a linear schedule). We observed that, as the number of vertices $n$ increases, the optimal value of the scaling constant $L$ increases. Therefore, our tuned hyperparameter value used in Figs. 17 and 18 is $L = 1.6 + 0.1n$.

In Figs. 17 and 18, we present the scaling of a linear schedule opposite to that of a cubic schedule. As the number of layers increases, performance as measured by

![FIG. 15: Heat maps of the TTS for the Sherrington–Kirkpatrick model with the x-axis representing the total loss rate $\gamma_0 \Delta T_c(1 + j)$ and the y-axis representing the out-coupling loss $j$. (a) and (b) Heat maps for the closed-loop and open-loop CIMs for $n = 30$. (c) and (d) Heat maps for the closed-loop and open-loop CIMs for $n = 100$. The colours indicate the value of the TTS in terms of the number of round trips, where a darker colour represents a shorter TTS. The green contour lines correspond to fixed values for $J = jx$.](image)

![FIG. 16: $R_{99}$ of the good initial DAQC parameters at $p = 4$ layers for various values of $a$ and $L$, on all 1000 graph instances of each size ranging from 10 to 20.](image)
$R_{99}$ improves, as expected. However, with more layers, more time is required to perform a single circuit shot, and therefore the scaling of TTS is actually worse at 50 layers than it is at 20 layers. For large numbers of layers, the linear schedule and cubic schedule perform similarly, which is expected because both are Trotterizations of a very slow adiabatic schedule.

**Appendix E: Challenges Encountered with the Variational Optimization Protocol**

When DAQC parameter schedules are tuned variationally, the energy measurements from the quantum device are used to decide the next parameters to try via a hybrid quantum–classical process. A single “shot” with the parameters $(\gamma, \beta)$ consists of running the DAQC circuit once with parameters $(\gamma, \beta)$, and measuring the energy of the prepared state $|\psi(\gamma, \beta)\rangle$, which destroys the prepared state and returns a single measurement outcome. We perform a large number of shots using $(\gamma, \beta)$, and the results are averaged to estimate the expected energy

$$EE(\gamma, \beta) := \langle \psi(\gamma, \beta) | H_p | \psi(\gamma, \beta) \rangle.$$  \hfill (E1)

This expected energy is treated as a loss function which is minimized by a classical optimizer. This approach suffers from two major challenges.

Firstly, we want parameters $(\gamma, \beta)$ which minimize the $R_{99}$, rather than the expected energy. Although these two loss functions are related, they are not perfectly correlated, and this difference becomes more apparent as we move closer to the parameters which minimize $R_{99}$. Unfortunately, it is impossible to optimize the ansatz with respect to $R_{99}$, as this would require knowledge of the ground state.

Secondly, because projective measurements are stochastic, our estimate of the expected energy is approximate, and this makes parameter optimization difficult. To overcome this issue, we would need to use a large number of shots per point $(\gamma, \beta)$, which makes the variational algorithm costly.

In Fig. 19, we illustrate the implications of the first challenge. We consider a four-layer DAQC circuit on graphs of size 10, 15, and 20. For each graph $G$, the following analysis is performed. First, the cubic schedule $\theta_G$ (see Section III A) is found and its $R_{99}$ is calculated. The Nelder–Mead method is then used to optimize the expected energy, with its parameter schedule initialized as $\theta_G$ and given access to 100 perfect evaluations of expected energy (which ordinarily can only be approximated). The $R_{99}$ of the result is divided by the $R_{99}$ of the cubic schedule, and these ratios have been plotted in red. Finally, the Nelder–Mead method is used to optimize $R_{99}$, with a schedule initialized with $\theta_G$ and access to 100 perfect evaluations of $R_{99}$ (which is ordinarily impossible to calculate). The $R_{99}$ of the result is divided by the $R_{99}$ of the cubic schedule, and these ratios have been plotted in blue. For better visibility, the graph instances along the $x$-axis have been sorted by the $y$-values of the
red points. We observe that even with perfect estimation of the expected energy, optimization results in a worse final $R_{99}$ in 15 to 40 percent of graph instances. This is the case despite the fact that the cost (in shots) of performing this optimization has been discarded. The effect of including the cost would have been substantial.

### Appendix F: Regression Analysis on the Scaling of DAQC

To investigate the validity of assuming an exponential scaling of the TTS for DAQC, we conducted a regression analysis for the more general scaling law

$$\log(\text{TTS}) = An^c + B.$$  \hfill (F1)

In Fig. 20, we display the minimum sum of squared residuals, which is a direct measure of how well the regression model fits the data, for the range of values $0.5 \leq c \leq 1.5$ of the exponent $c$. For each value of $c$ within this range, the method of stochastic gradient descent is used to find the optimum parameter values of $A^*$ and $B^*$ that minimize the sum of squared residuals. Our regression analysis includes additional Gaussian noise injection at each iteration of the update. For both the 21-weight problem instances and the bimodal SK model instances, we observe the highest confidence with respect to the quality of the regression fit for exponent values that are close to $c = 1$. This suggests that DAQC indeed scales exponentially in solving these problem instances. For the 21-weight problem instances, the scaling is actually slightly sub-exponential at the value $c \approx 0.9$.

### Appendix G: Grover’s Search as a Subroutine of DH-QMF

Grover’s search algorithm [18, 19] has been extensively studied and applied since its invention more than twenty years ago. This appendix provides more details with a focus on its implementation as a subroutine of DH-QMF. In Appendix G1, we start with a brief review of how Grover’s search algorithm works. In Appendix G2, we expand on the quantum circuits used to implement the QMF oracle, which is required when Grover’s search is employed as a subroutine of DH-QMF, and explain the contributions to its resource requirements.

#### 1. A Brief Review of Grover’s Search Algorithm

The circuit of Grover’s search algorithm is illustrated in Fig. 21. The quantum circuit takes as inputs an $n$-qubit register vertex and a single-qubit regis-
The **flag** register is used to encode the possible spin configurations (and any superpositions of them); it is initialized in the state \( |0\rangle^\otimes n \) and transformed into a uniform superposition \( \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \) by applying a Hadamard gate (denoted by \( H \)) to each qubit. The **flag** qubit is prepared in the state \( |\rangle \equiv \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = H |0\rangle \).

Grover’s search is implemented by repeatedly applying the “Grover iterations” a number of times specified by \( m \). After \( m \) Grover iterations, the register **vertex** is measured in the computational basis. The measurement result (\( n \) classical bits) is intended to yield a solution to the problem.

The effect of the Grover iteration is the combined effect of an oracle query followed by the Grover diffusion. The composition of the oracle followed by Grover diffusion forms the so-called Grover iteration, which is repeated \( m \) times. The effect of a single Grover iteration is the combined effect of an oracle query followed by the Grover diffusion.

The composition of the oracle followed by Grover diffusion forms the so-called Grover iteration, which is repeated \( m \) times. Here, \( H \) denotes the Hadamard gate.

The effect of the Grover iteration is the combined effect of an oracle query followed by the Grover diffusion. To explain the key role of the quantum oracle, it is useful to formulate the search problem as follows. Let \( \{x_1, \ldots, x_N\} \) denote the set of the \( N \) unordered items. We define a classical function \( f : \{x_1, \ldots, x_N\} \rightarrow \{0, 1\} \) such that \( f(x) = 1 \) if and only if \( x \) has the property we are looking for, and \( f(x) = 0 \) otherwise. The problem thus consists in finding an item \( x \in \{x_1, \ldots, x_N\} \) such that \( f(x) = 1 \). The quantum oracle \( O_f \) corresponding to the classical function \( f \) is a unitary implementation of \( f \).

It is commonly defined as

\[
O_f : |x\rangle_{\text{vertex}} |z\rangle_{\text{flag}} \mapsto |x\rangle_{\text{vertex}} |z \oplus f(x)\rangle_{\text{flag}},
\]

where \( z \in \{0, 1\} \) and \( \oplus \) represents a bitwise XOR. If we choose \( z = 0 \), the **flag** qubit outputs the value 1 if and only if \( x \) is a solution to the search problem. We say the oracle marks the solution states. The crucial property is that the oracle can be queried on a superposition of \( N \) input states, and to compute the corresponding function values it needs to be queried only once:

\[
\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle_{\text{vertex}} |0\rangle_{\text{flag}} \xrightarrow{O_f} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle_{\text{vertex}} |f(x)\rangle_{\text{flag}}.
\]

In Grover’s algorithm, we prepare the **flag** qubit in the \( |\rangle \) state. The resulting effect is a “phase kick-back”, which gives rise to a minus sign as a phase whenever the input is a solution state:

\[
|x\rangle_{\text{vertex}} |\rangle_{\text{flag}} \xrightarrow{O_f} (-1)^{f(x)} |x\rangle_{\text{vertex}} |\rangle_{\text{flag}}.
\]

Observe that the state of the **flag** qubit remains unaffected, and we effectively implement the transformation \( |x\rangle_{\text{vertex}} \mapsto (-1)^{f(x)} |x\rangle_{\text{vertex}} \), which is the definition of a “phase oracle”. However, the **flag** qubit plays a crucial role in inducing this transformation. While the factor \((-1)^{f(x)}\) seems like a global phase for a single term, it becomes a relative phase for a superposition of inputs:

\[
\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle_{\text{vertex}} |\rangle_{\text{flag}} \mapsto \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle_{\text{vertex}} |\rangle_{\text{flag}}.
\]

The following Grover diffusion implements a reflection about the mean amplitude. If \( \alpha_x \) denotes the amplitude of the \( |x\rangle \) component prior to applying the Grover diffusion, the effect of the latter is \( \alpha_x \mapsto 2\tilde{\alpha} - \alpha_x \), where \( \tilde{\alpha} := \frac{1}{N} \sum \alpha_x \). Observe that the amplitudes of the marked components (those that pick up a negative phase after the oracle query) are amplified while the amplitudes of all other components decrease. The combined effect of an oracle query followed by the Grover diffusion thus results in amplitude amplification of the solution states, while shrinking the amplitudes of all other states in the superposition. When repeated numerous times, the amplitudes of the solution states eventually become significantly larger than those of the non-solution states. The quadratic speedup with respect to classical search can be understood as coming about from adding amplitudes \( \Omega \left( \frac{1}{\sqrt{N}} \right) \) to the marked items with each query, which results in an \( \mathcal{O} \left( \sqrt{N} \right) \) convergence. This convergence rate was shown by Grover to be also optimal. Hence, the query complexity is actually \( \Theta \left( \sqrt{N} \right) \).

### 2. The QMF Oracle

The search for a ground state of an Ising Hamiltonian

\[
H = - \sum_{i<j} J_{ij} Z_i Z_j
\]

(corresponding to an undirected weighted graph with weights \( w_{ij} = -J_{ij} \)) requires an oracle which marks all states whose energies are strictly smaller than the energy corresponding to the latest updated threshold index value, respectively, which we refer to as the “QMF oracle” in this paper. Its quantum circuit implementation is shown in Fig. 22. Note that here, instead of using the weights \( w_{ij} = \pm 0.1 j \) \( \in [-1, 1] \) for \( j \in \{0, 1, \ldots, 10\} \), we take the weights to be the integers \(-10 \leq w_{ij} \leq 10\); this facilitates the quantum circuit implementation of arithmetic operations without altering the underlying MAXCUT problem.

In addition to the \( n \)-qubit register **vertex** for encoding the possible spin configurations and any superpositions of them and a single-qubit register **flag** for holding the result of the oracle, several other computational registers as well as ancillae are required to reversibly compute the energies \( E(x) \) and \( E(y) \) and compare their values. More concretely, we need another \( n \)-qubit register to encode the value \( y \) of the threshold index as a quantum state.

![Diagram of Grover's search](image-url)
These registers computed data related to the Hamiltonian. Both are initialized in such graphs is twice as large as this bound. Since generic weighted graphs have full connectivity, the total number of edges in such graphs is \( \binom{n}{2} = n(n-1)/2 \), where \( n \) is the number of vertices, while the maximum absolute edge weight in our analysis is \( \max_{x \in \mathcal{E}} |w_x| = 10 \). Hence, we may use \( E_0 := 10\binom{n}{2} = 5n(n-1) \) and choose the registers \( \text{data}(H) \) to be of size \( \lceil \log_2 (10n(n-1)) \rceil \) in \( \mathcal{O}(\log n) \).

The energy values \( E(x) \) and \( E(y) \) are computed using two separate energy oracles, whose quantum circuit implementation is provided in Fig. 22(b). For a given input \( |x\rangle = |\ell_0\rangle \otimes \cdots \otimes |\ell_{n-1}\rangle \) held in the \( \text{vertex} \) register, we serially execute the shown circuit template for every vertex pair \((i, \ell)\) in the graph whose edge \( e_{i\ell} \) is nonzero. Each such circuit subtracts or adds the value \( J_{i\ell} \) in the \( \text{data}(H) \) register, depending on whether \( \xi_{i\ell} = \xi_{\ell i} \) or \( \xi_{i\ell} \neq \xi_{\ell i} \), respectively, effectively contributing the term \((-1)^{\xi_{i\ell}}(-1)^{\xi_{\ell i}}(-1)^{J_{i\ell}}\) to the overall energy. The series for all pairs of vertices accumulates the sum \( \sum_{ij} (-1)^{\xi_{ij}}(-1)^{\xi_{ji}}(-1)^{J_{ij}} \), which together with the initial value \( E_0 \) results in the value \( E(x) = E_0 - \sum_{ij} (-1)^{\xi_{ij}}(-1)^{\xi_{ji}}(-1)^{J_{ij}} \) for the quantum state \( |y\rangle = |\eta_0\rangle \otimes \cdots \otimes |\eta_{n-1}\rangle \) representing the threshold index \( y \). For generic weighted graphs with full connectivity, this serial implementation contributes a factor \( \mathcal{O}(n^2) \) to the overall circuit depth scaling. Moreover, there is an additional contribution from the arithmetic operations needed to implement addition and subtraction of the constant integer \( J_{i\ell} \) within the \( \text{data}(H) \) register. Our circuit implementations and resource estimates have been obtained using projectQ [56].

The implementation of addition or subtraction of a constant \( c \), that is, \(|E\rangle \mapsto |E \pm c\rangle \), in projectQ [54] is based on Draper’s addition in Fourier space [66], which allows for optimization when executing several additions in sequence, which applies to our circuits. Due to cancellations of the quantum Fourier transform (QFT) and its inverse, QFT \( QFT^{-1} = 1 \), for consecutive additions or subtractions within the sequence given by the serial execution of circuits shown in Fig. 22(b), the overall sequence contributes a multiplicative factor scaling only as \( \mathcal{O}(\log \log n) \) to depth, and a multiplicative factor in \( \mathcal{O}(\log n \log \log n) \) to the gate complexity. To understand these contributions, recall that the registers \( \text{data}(H) \) are of size \( \mathcal{O}(\log n) \). The remaining initial QFT and the final inverse QFT, which transform into and out of the Fourier space in that scheme (cp. [54]), contribute an additional additive term \( \mathcal{O}((\log n)^2) \) to both the depth and the gate complexity of the overall sequence. Hence, the implementation of the energy oracle \( O_E \) contributes the factors \( \mathcal{O}(n^2 \log \log n + (\log n)^2) \) to the overall circuit depth and \( \mathcal{O}(n^2 \log n \log \log n + (\log n)^2) \) to the overall gate complexity.

The energy computation is followed by a unitary operation called “Compare”, which compares the energies \( E(x) \) and \( E(y) \). Using methods developed in [67], we can implement this comparison by a circuit with a depth only logarithmic in the number of qubits, that is, with a depth in \( \mathcal{O}(\log \log n) \), while its gate complexity is \( \mathcal{O}(\log n) \). An additional single-qubit ancilla is used to store the result.
of the comparison. Concretely, initialized in state $|0\rangle$, the ancilla is output in the state $|f(x, y)\rangle$, where

$$f(x, y) = \begin{cases} 0, & \text{if } E(x) \geq E(y) \\ 1, & \text{if } E(x) < E(y) \end{cases}. \quad (G5)$$

Using a CNOT gate, we copy out this result to the single-qubit flag (bottom wire) and reverse the whole circuit used to compute the result so as to uncompute the entanglement with the garbage generated along the way.

In summary, the QMF oracle is a quantum circuit of depth $O(n^2 \log \log n + (\log n)^2)$ and gate complexity $O(n^2 \log n \log \log n + (\log n)^2)$. The Grover diffusion requires an $n$-controlled NOT gate to implement the reflection, which is a circuit of depth and gate complexity both scaling as $O(n)$ in terms of elementary gates. Putting all contributions together, a single Grover iteration in our implementation has a circuit of depth in $O(n^2 \log \log n + (\log n)^2 + n)$, while its gate complexity is $O(n^2 \log n \log \log n + (\log n)^2 + n)$. While we have not explicitly shown it, we note that the growth rates of circuit depth and gate counts are lower-bounded by the same scalings, meaning that in the above expressions we may replace the $O(\cdot)$ notation by $\Theta(\cdot)$.

As an additional final remark, we note that it is possible to achieve a slightly better circuit depth scaling for the Grover iteration, namely as $O(n + (\log n)^3 + \log \log n)$, by a parallel (instead of serial) execution of the circuit components shown in Fig. 22(b) pertaining to each vertex pair $(i, f)$ in the graph. However, this parallelization would come at an unreasonably high additional space cost, as it would necessitate the use of $n(n-1)$ computational registers of size $O(\log n)$ instead of only two. The number of qubits required would scale as $O(n + n^2 \log n)$. In contrast, our serial implementation above requires only $O(n + \log n)$ qubits.

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