A tale of two skyrmions: the nucleon’s strange quark content in different large $N_c$ limits

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The nucleon’s strange quark content comes from closed quark loops, and hence should vanish at leading order in the traditional large $N_c$ (TLNC) limit. Quark loops are not suppressed in the recently proposed orientifold large $N_c$ (OLNC) limit, and thus the strange quark content should be non-vanishing at leading order. The Skyrme model is supposed to encode the large $N_c$ behavior of baryons, and can be formulated for both of these large $N_c$ limits. There is an apparent paradox associated with the large $N_c$ behavior of strange quark matrix elements in the Skyrme model. The model only distinguishes between the two large $N_c$ limits via the $N_c$ scaling of the couplings and the Witten-Wess-Zumino term, so that a vanishing leading order strange matrix element in the TLNC limit implies that it also vanishes at leading order in the OLNC limit, contrary to the expectations based on the suppression/non-suppression of quark loops. The resolution of this paradox is that the Skyrme model does not include the most general type of meson-meson interaction and, in fact, contains no meson-meson interactions which vanish for the TLNC limit but not the OLNC. The inclusion of such terms in the model yields the expected scaling for strange quark matrix elements.

During the past two decades there has been an extensive experimental program to study strange quark matrix elements of the nucleon [1]. They are of interest in large measure because they are sensitive to physics clearly beyond the naive quark model — they are nonzero only due to closed strange quark loops. Thus they are an ideal way to explore an important theoretical issue: the distinction between two variants of the large $N_c$ limit of QCD. In this paper we focus on strange matrix elements in Skyrme models [2], which are chiral soliton models often justified by appeals to large $N_c$ QCD [3, 4]. Attempting to understand the $N_c$ scaling of strange matrix elements in the context of Skyrme models raises an apparent paradox which this paper resolves.

The traditional method for generalizing QCD to many colors [5, 6] treats the quark as being in the fundamental representation of $SU(N)$. We will refer to this approach as the ‘t Hooft (or “traditional”) large $N_c$ (TLNC). Recently, an alternative method — dubbed the “orientifold large $N_c$” (OLNC) limit [7, 8] — for generalizing to large $N_c$ has been proposed, where quarks are taken to be in a two-index representation of color. The principal theoretical motivation for studying this limit was the connection of one flavor QCD in this limit to large $N_c$ supersymmetric Yang-Mills theory; this allows one to exploit powerful mathematical tools in the analysis of one-flavor QCD. However, there is an important connection to phenomenology: for $N_c = 3$ the antisymmetric representation is isomorphic to the fundamental representation.

The fundamental difference between the two approaches is that the TLNC limit suppresses quark loop effects while the OLNC does not. Quarks are double-color-indexed objects in the OLNC limit and scale in essentially the same was as gluons; all planar diagrams are leading order. Thus, mesons in the OLNC limit scale with $N_c$ in the same way as glueballs [11] which is distinct from the scaling in the TLNC limit:

$$\Gamma_n \sim N_c^{2-n} \quad \text{(OLNC)},$$

$$\Gamma_n \sim N_c^{1-n/2} \quad \text{(TLNC)},$$

(1)

where $\Gamma_n$ is a generic $n$-meson vertex. In effect, there is a rule to convert the generic scaling from the TLNC limit to the OLNC limit, namely, the substitution $N_c^k \rightarrow N_c^{2k}$.

An obvious consequence of the scaling of Eq. (1) is on Skyrme models. The $N_c$ scaling in such models is the result of the $N_c$ scaling of the parameters in a model. If one alters the scaling of the parameters of a Skyrmion in the TLNC limit through the generic replacement $N_c^k \rightarrow N_c^{2k}$ one finds that the mass of the Skyrmions in the OLNC limit scales as $M \sim N_c^2$. As shown in refs. [12, 13] the $N_c$ scaling of all generic properties of the baryon (mass, couplings, cross-sections, etc.) in the OLNC limit is consistent with the nucleon behaving as a Skyrmion. The consistency of this description is made even stronger due to Bolognesi’s observation [12] that the coefficient of the Witten-Wess-Zumino term in the OLNC limit is $N_c(N_c-1)/2 \sim N_c^2$, while in the TLNC limit it is $N_c$ [10].

The consistency of the Skyrme model with large $N_c$ QCD is deeper than merely showing that all of the generic $N_c$ scaling rules apply; spin and flavor play an essential role. The hedgehog structure of the classical solution to the Skyrme model imposes correlations between spatial directions and isospin. These correlations impose relations between certain observables computed at leading order in the collectively quantized Skyrmions which are independent of the details of the Skyrme Lagrangian [14]. These relations in
all Skyrme-type models encode an emergent symmetry of QCD — a contracted SU(2 Nf) symmetry where Nf is the number of flavors. These rules follow solely from the fact that the pion-nucleon coupling constant diverges at large Nc, while the pion-nucleon scattering amplitude is finite due to unitarity[13, 14, 15]. Since this condition holds for both the TLNC limit (gπNN ∼ Nc1/2) and OLNC limit (gπNN ∼ Nc) the contracted SU(2 Nf) spin-flavor symmetry must emerge in both variants of the large Nc limit of QCD.

To begin, let us focus on the Skyrme model, i.e. Skyrme’s original model[2], but generalized to three flavors so that the question of strangeness is relevant. The action for the model is

\[ S = \int d^4x \left( \frac{f_\tau^2}{4} \text{Tr}(L_\mu L_\mu) + \frac{2}{4} \text{Tr}([L_\mu, L_\nu]^2) \right) + S_{WWZ} \]  \tag{2}

where the left chiral current L_\mu is given by L_\mu = U^\dagger \partial_\mu U, with U ∈ SU(3) [2, 4]. S_{WWZ} is the well-known Witten-Wess-Zumino (WWZ) term, the addition of which is necessary for the Skyrme model to respect the symmetries of QCD[13, 14]. The U field can be written as \( U = \exp(i\bar{\tau} \cdot \vec{\pi}/f_\pi) \) where \( \bar{\tau} \) is the pseudoscalar meson field, and \( \vec{\pi} \) is a vector composed of the first three Gell-Mann matrices, \( \bar{\tau} \equiv (\lambda_1, \lambda_2, \lambda_3) \). From the scaling rules in Eq. (1), it is apparent that \( f_\pi \sim \epsilon \sim N_c^{1/2} \) for the TLNC limit, while for the OLNC limit the scaling is \( f_\pi \sim \epsilon \sim N_c \). The only way that \( N_c \) enters is through the parameters \( f_\pi \) and \( \epsilon \), and through the Witten-Wess-Zumino term[12, 13]. To show the \( N_c \) dependence of the parameters in an explicit form, we can write

\[ f_\pi = \sqrt{N_c} f_\pi \quad \epsilon = \sqrt{N_c} \bar{\tau} \]  \tag{TLNC}

\[ f_\pi = \frac{N_c(N_c-1)}{2} f_\pi \quad \epsilon = \frac{N_c(N_c-1)}{2} \bar{\tau} \]  \tag{OLNC}

where the barred quantities do not depend on \( N_c \). This implies that the action can be written as

\[ S = N_c \bar{S} \]  \tag{TLNC}

\[ S = \frac{N_c(N_c-1)}{2} \bar{S} \]  \tag{OLNC}

with \( \bar{S} \) independent of \( N_c \) and of the same form for both the TLNC limit and OLNC limit. The choice of the form \( \sqrt{N_c(N_c-1)/2} \) rather than \( N_c \) for the scaling of the parameters ensures that the Witten-Wess-Zumino term scales in the same way as the rest of the system, and is related to the fact that the baryon consists of \( N_c(N_c-1)/2 \) quarks in the OLNC limit[12].

This leads to an apparent paradox. In general, when a system is in the semi-classical regime, the size of a prefactor multiplying the action plays two roles: i) It controls the convergence of the semi-classical expansion, and ii) specific powers of the prefactor act as multiplicative factors for particular observables. Thus, when \( N_c \) is large enough to justify the neglect of subleading effects in both 1/\( N_c \) expansions, the only effect of going from the TLNC limit to the OLNC limit for the Skyrmon is to make the replacement \( N_c \rightarrow N_c(N_c-1)/2 \) in multiplicative factors for the various observables.

This is a surprising result, because it appears to leave no room for the effects of the different behaviors of quark loops in the two large \( N_c \) limits. At leading order, quark loops are suppressed in the TLNC limit, while not being suppressed in the OLNC limit. Thus, one would generically expect that strange quark matrix elements should scale as \( N_c^2 \) in the OLNC limit (that is, with leading order scaling), while in the TLNC limit they should be zero at leading order (that is, they should scale as \( N_c^0 \), one order below leading). However, given the simple replacement rule above, it appears that the Skyrme model must predict strange quark matrix elements from a priori quark loop effect considerations with the apparent Skyrme model results.

The resolution would be trivial if the TLNC limit of the Skyrme model had a leading order contribution to strange quark matrix elements. While this is counter to our expectations, calculations of strange quark matrix elements of the nucleon in assorted variants of Skyrme models have larger typical values than for other models on the market[1]. Since the calculations do not include any explicit 1/\( N_c \) corrections, the very fact that the results are non-zero seems to suggest that the leading order term does survive. However, a careful analysis shows that the strange quark matrix elements of the nucleon in the Skyrme model are zero at leading order in a systematic expansion around the TLNC limit. While there are no explicit 1/\( N_c \) corrections in the existing calculations based on collective quantization, there are implicit effects which are subleading in 1/\( N_c \) and which account for the entire result.

To illustrate this, consider the nucleon’s strange scalar matrix element at zero momentum transfer for a Skyrmon in the exact SU(3) flavor limit. It is convenient to analyze this matrix element as a fraction, denoted \( X_s \), of the total scalar matrix elements of the three light flavors:

\[ X_s \equiv \frac{\langle N | \bar{s}s - \langle \bar{s}s \rangle_{\text{vac}} | N \rangle}{\langle N | \bar{u}u + \bar{d}d + \bar{s}s - \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle_{\text{vac}} | N \rangle} \]  \tag{4}

where \( |N\rangle \) represents the nucleon state and the quantities with subscript “vac” indicating a vacuum subtraction. For the exact SU(3) limit, \( X_s \) can be computed via collective quantization, with collective quantum variables specified by an SU(3) rotation \( A \) on the standard classical static hedgehog. That is, \( U \) is given by \( U = A^\dagger U_h A \) with the hedgehog
Skyrmion defined as \( U_h \equiv \exp(i\tau \cdot \vec{r} f(r)) \); the profile function \( f(r) \) is determined by minimizing the energy subject to the condition that the system has unit winding number. A standard calculation for \( X_s \) in the Skyrme model \[21\] gives

\[
X_s = \frac{1}{3} \langle N|1 - D_{ss}|N \rangle = \frac{1}{3} \int dA \psi^\ast_N(A) (1 - D_{ss}) \psi_N(A)
\]

where \( dA \) stands for the Haar measure for SU(3), \( D_{ss} = \frac{1}{2} \text{Tr} [\lambda_8 A \lambda_8 A^\dagger] \) (which is an SU(3) Wigner D-matrix), and \( \psi_N(A) \) is the collective wave function for the nucleon—i.e., an appropriately normalized SU(3) Wigner D-matrix.

As was discussed in another context, tracing the \( N_c \) dependence cleanly requires that the calculation be done with the coefficient of the WWZ term having an arbitrary explicit \( N_c \) dependence. This implies that the nucleon lies in a representation that is the generalization of the octet for arbitrary \( N_c \). The generalized representation “8”, which at \( N_c = 3 \) corresponds to the familiar octet, is specified by \((p,q) = (1, \frac{N_c - 1}{2})\) for the TLNC limit. The evaluation of Eq. \[21\] for arbitrary \( N_c \) can be done straightforwardly with the aid of the SU(3) Clebsch-Gordan coefficients appropriate for the “8” representation \[21\,22\]. The result is

\[
X_s = \frac{2(N_c + 4)}{N_c^2 + 10N_c + 21} = \frac{2}{N_c} + \mathcal{O}(1/N_c^2) \quad . \tag{6}
\]

Thus, \( X_s \) goes to zero as \( N_c^{-1} \) as \( N_c \to \infty \). \( X_s \) is subleading in a formal \( 1/N_c \) expansion around the TLNC limit exactly as expected on general grounds.

We note in passing that phenomenological calculations of strange quark matrix elements are typically done with \( N_c = 3 \) at the outset in the WWZ term. This builds in some subleading effects in \( 1/N_c \). For example, calculations of \( X_s \) for the exact SU(3) limit \[20\] gave 7/30, which is in agreement with Eq. \[6\] for \( N_c = 3 \).

Thus, it appears that despite our expectations that strange quark matrix elements of the nucleon in the OLNC limit are of leading order (i.e., \( \mathcal{O}(N_c^2) \)), the Skyrme model of Eq. \[2\] has strange quark matrix elements which are of order \( N_c^0 \) regardless of whether one is in the OLNC limit or the TLNC limit. As it happens, this conclusion is correct, but fortunately it is not the entire story. The fault lies not in our expectations for the OLNC limit but with the model: while the direct SU(3) generalization of Skyrme’s original model given in Eq. \[2\] does indeed have strange matrix elements of order \( N_c^0 \) in the OLNC limit, more general Skyrme-type models have matrix elements of order \( N_c^2 \).

We must recall that the general arguments from large \( N_c \) QCD do not justify the Skyrme model in the sense of Skyrme’s original model. The Skyrme model does manage to capture the generic large \( N_c \) scaling rules for all observables, as well as the model-independent relations of the contracted SU(2\( N_f \)) symmetry required from large \( N_c \) consistency rules. Beyond this, however, values of various couplings predicted by the Skyrme model should be viewed as model-dependent and thus essentially arbitrary from the point of view of large \( N_c \) QCD. Indeed, we know \textit{a priori} that the model does not capture all of the physics at leading order in \( 1/N_c \). For example, the Skyrme model only has one kind of meson field, the light pseudoscalar meson field, whereas large \( N_c \) in fact has an infinite number of meson fields. Even for the light pseudoscalar meson interactions, terms which are allowed in large \( N_c \) QCD are set to zero to make the calculations tractable; indeed, an infinite number of such terms are neglected.

Thus, while all terms in the Skyrme model correctly encode the leading order large \( N_c \) scaling laws for both large \( N_c \) limits (provided the coefficients are scaled properly), the converse is not true: all terms with the correct leading order scaling behavior are not included in the Skyrme model. The paradox we are considering is resolved provided there exist terms in Skyrme-type models which, while absent in the Skyrme model, are allowed at leading-order in the OLNC limit and which give rise to strange quark matrix elements. Such terms cannot contribute at leading order in the TLNC limit, as they represent quark loop effects.

It is not hard to see how this can happen. Recall that the principal difference between the two limits was the suppression of quark loop effects in the TLNC limit and not in the OLNC limit. One consequence of this at the level of meson-meson interactions is that all terms for an underlying SU(3) symmetric theory which require more than one summation over flavor indices are suppressed in the TLNC limit by one factor of \( 1/N_c \) for each summation beyond the first. The reason for this is simple: each distinct sum over flavors for mesons corresponds to distinct quark loops at the quark level. Each additional quark loop is suppressed by a factor of \( 1/N_c \) in the TLNC limit, but not in the OLNC limit. Terms with more than one flavor trace that are suppressed in the TLNC limit are known to exist in chiral perturbation theory \[23\]. If terms of this sort contribute to strange quark matrix elements of the nucleon when included in a Skyrme-type model, the paradox would be resolved.

To illustrate this idea, consider the effect of the inclusion of one such term from chiral perturbation theory,

\[
S' = \int d^4x L_4 \text{Tr} (L_m L^n) \text{Tr} (\chi^1 U + \chi U^1) . \tag{7}
\]

The coefficient \( L_4 \) is one of the standard constants in chiral perturbation theory at order \( p^4 \), and the scalar source \( \chi \) is taken to be proportional to the quark masses:

\[
\chi = 2B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} , \tag{8}
\]
where \( B_0 \) is a constant of proportionality which is order \( N_c^0 \). From the discussion above, it is evident that

\[
L_A \sim N_c^0 \quad \text{(TLNC)} \quad L_A \sim N_c^2 \quad \text{(OLNC)}
\]  

(9)

Consider a model with an action given by \( S_{SK} + S' \). We can probe the strangeness content of this model by considering the strange scalar matrix element at zero momentum transfer, \( i.e., \) the strange sigma term:

\[
\sigma_s \equiv \langle N | m_s (\langle \bar{s}s \rangle - \langle \bar{s}s \rangle_{\text{vac}}) | N \rangle = m_s \frac{\partial M_N}{\partial m_s}.
\]  

(10)

The second form for \( \sigma_s \) is obtained via the Feynman-Hellmann theorem [24]. Note that this quantity is intimately related to \( X_s \) of Eq. (5) and contains the same information.

As with the usual Skyrmion, the mass of the nucleon is dominated by the mass of the classical hedgehog Skyrmion. The profile function \( f(r) \) is obtained by varying the action subject to the hedgehog ansatz and imposing a unit winding number.

By standard large \( N_c \) rules, the profile function is independent of \( N_c \) at large \( N_c \), regardless of whether one studies the TLNC limit or the OLNC limit. However, the detailed form of \( f(r) \) is different in the two limits: \( S' \) contributes to the leading order action and hence to the variational equations at leading order in the OLNC limit, but not in the TLNC limit.

The contribution of the \( S' \) term to the mass of the nucleon may be computed straightforwardly, and from this the Feynman-Hellmann theorem can be used to compute its contribution to \( \sigma_s \), which we denote \( \sigma_s' \):

\[
\sigma_s' = L_A (32\pi m_s B_0) \int_0^\infty dr r^2 \left( f''(r) + \frac{2 \sin^2(f)}{r^2} \right)
\]

\[
\sigma_s' \sim N_c^0 \quad \text{(TLNC)} \quad \sigma_s' \sim N_c^2 \quad \text{(OLNC)}
\]  

(11)

where the scaling with \( N_c \) follows since everything on the righthand side of Eq. (11) scales as \( N_c^0 \) except for \( L_A \); the scaling of \( L_A \) with \( N_c \) is \( N_c^2 \), as given in Eq. (9).

The scaling of \( \sigma_s' \) in Eq. (11) is precisely as one would have expected from general arguments involving quark loops in the two limits. Moreover, this behavior is generic. In Skyrme-type models, the inclusion of meson-meson interaction terms the coefficients of which vanish at leading order in the TLNC limit, but not in the OLNC limit, can give rise to strange quark matrix elements of order \( N_c^2 \) in the OLNC limit. In the TLNC limit, however, such terms make only subleading (order \( N_c^0 \)) contributions by construction. This resolves the paradox.

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