Long time asymptotic behavior for the derivative Schrödinger equation with nonzero boundary conditions

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Abstract

In this paper, we apply $\overline{\partial}$ steepest descent method to study the Cauchy problem for the derivative nonlinear Schrödinger equation with nonzero boundary conditions

$$i q_t + q_{xx} + i(|q|^2 q)_x = 0,$$

$$q(x, 0) = q_0(x),$$

where $\lim_{x \to \pm \infty} q_0(x) = q_{\pm}, \quad |q_{\pm}| = 1$. Based on the spectral analysis of the Lax pair, we express the solution of the derivative Schrödinger equation in terms of solutions of a Riemann-Hilbert problem. In a fixed space-time solitonic region $-3 < x/t < -1$, we compute the long time asymptotic expansion of the solution $q(x, t)$, which implies soliton resolution conjecture and can be characterized with an $N(\Lambda)$-soliton whose parameters are modulated by a sum of localized soliton-soliton interactions as one moves through the region; the residual error order $O(t^{-3/4})$ from a $\overline{\partial}$ equation.

Keywords: Derivative Schrödinger equation, Riemann-Hilbert problem, $\overline{\partial}$ steepest descent method, soliton resolution, asymptotic stability.

AMS: 35Q51; 35Q15; 37K15; 35C20.
1 Introduction

The study on the long-time behavior of nonlinear wave equations which is solvable by the inverse scattering method was first carried out by Manakov in 1974 [1]. By using this method, Zakharov and Manakov give the first result for large-time asymptotic of solutions for the NLS equation with decaying initial data [2]. The inverse scattering method also worked for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system [3–5]. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotics behavior of the solution for the MKdV equation by deforming contours to reduce the original Riemann-Hilbert (RH) problem to a model one whose solution is calculated in terms of parabolic cylinder functions [6]. Since then this method has been widely applied...
to the focusing NLS equation, KdV equation, Fokas-Lenells equation, short-pulse equation and Camassa-Holm equation etc. [7–12].

In recent years, McLaughlin and Miller further presented a $\bar{\partial}$ steepest descent method which combine steepest descent with $\bar{\partial}$-problem rather than the asymptotic analysis of singular integrals on contours to analyze asymptotic of orthogonal polynomials with non-analytical weights [13-14]. When it is applied to integrable systems, the $\bar{\partial}$ steepest descent method also has displayed some advantages, such as avoiding delicate estimates involving $L^p$ estimates of Cauchy projection operators, and leading the non-analyticity in the RH problem reductions to a $\bar{\partial}$-problem in some sectors of the complex plane which can be solved by being recast into an integral equation and by using Neumann series. Dieng and McLaughlin used it to study the defocusing NLS equation under essentially minimal regularity assumptions on finite mass initial data [15]; This $\bar{\partial}$ steepest descent method was also successfully applied to prove asymptotic stability of N-soliton solutions to focusing NLS equation [16]; Jenkins et.al studied soliton resolution for the derivative nonlinear NLS equation for generic initial data in a weighted Sobolev space [17]. Their work provided the soliton resolution property for derivative NLS equation, which decomposes the solution into the sum of a finite number of separated solitons and a radiative parts when $t \to \infty$. And the dispersive part contains two components, one coming from the continuous spectrum and another from the interaction of the discrete and continuous spectrum. For finite density initial data, Cussagna and Jenkins studied the defocusing NLS equation [18].

In this paper, we study the long time asymptotic behavior for the derivative nonlinear Schrödinger (DNLS) equation with nonzero boundary conditions

\begin{align}
  iq_t + q_{xx} + i\sigma(|q|^2q)_x &= 0, \\
  q(x,0) &= q_0(x),
\end{align}

where $\lim_{x \to \pm\infty} q_0(x) = q_\pm$, $|q_\pm| = 1$. Since the solution space of the equation (1.1) with $\sigma = 1$ and $\sigma = -1$ by the simple mapping $q(x,t) \to q(-x,t)$, we
only need to consider the case $\sigma = -1$ in our paper. The DNLS equation as a completely integrable system was first proposed by Kaup and Newell [19].

The DNLS equation is often used to describe various nonlinear waves. For instance, DNLS equation governs the evolution of small but finite amplitude nonlinear Alfvén waves which propagates quasi-parallel to the magnetic field in space plasma physics [20–24], sub-picosecond pulses in single mode optical fibers [25–27]. Moreover, DNLS equation also describe weak nonlinear electromagnetic waves in ferromagnetic [28], dielectric [29] and anti-ferromagnetic systems under external magnetic fields [30]. Either zero boundary conditions or nonzero boundary conditions for the DNLS equation have well physically significant. For problems of nonlinear Alfvén waves, weak nonlinear electromagnetic waves in magnetic and dielectric media, waves propagating strictly parallel to the ambient magnetic fields are modeled by zero boundary conditions, while those oblique waves are modeled by the nonzero boundary conditions. In optical fibers, pulses under bright background waves are modeled by the zero boundary conditions. Much work on the DNLS equation were also developed in [31–35].

Zhang and Yan presented the inverse scattering transform of the DNLS equation (1.1) for both zero/nonzero boundary conditions in terms of the matrix RH problems [36]. For Schwartz initial value $q_0 \in S(\mathbb{R})$, Xu and Fan derived the long-time asymptotic for (1.1) without soliton [37]

\[
q(x, t) = t^{-\frac{1}{2}} \alpha(\lambda_0) e^{\frac{ix^2}{4t} - iv(\lambda_0) \log t} + O(t^{-1} \log t). \tag{1.3}
\]

The long-time asymptotic for (1.1) with step-like initial data was further investigated [38]. Recently for generic initial data in $H^{2,2}(\mathbb{R})$, applying $\bar{\partial}$ steepest descent method, Jenkins et al obtained the following asymptotics for the equation (1.1) [17]

\[
q(x, t) = q_{sol}(x, t; D_I) + t^{-\frac{1}{2}} f(x, t) + O(t^{-\frac{3}{4}}), \tag{1.4}
\]

where $q_{sol}(x, t; D_I)$ is the soliton solutions of the equation (1.1) with modulating reflectionless scattering data. In our paper, for finite density initial data $q_0 - q_{\pm} \in H^{1,1}(\mathbb{R})$, we apply $\bar{\partial}$ steepest descent method to obtain the following long-time
asymptotic of the DNLS equation (1.1)

\[ q(x, t) = \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} |q_\Lambda(x, t)|^2 - 1 \right\} T(\infty)^{-2} q_\Lambda^r(x, t) + O(t^{-3/4}) \]  

(1.5)

This paper is arranged as follows. In section 2, we recall some main results on the construction process of the RH problem with respect to the initial problem of the DNLS equation (1.1) obtained in [33, 36], which will be used to analyze long-time asymptotics of the DNLS equation in our paper. In section 3, we introduce a function \( T(z) \) to define a new RH problem for \( M^{(1)}(z) \), which admits a regular discrete spectrum and two triangular decompositions of the jump matrix. In section 4, by introducing a matrix-valued function \( R(z) \), we obtain a mixed \( \bar{\partial} \)-RH problem for \( M^{(2)}(z) \) by continuous extension of \( M^{(1)}(z) \). In section 5, we decompose \( M^{(2)}(z) \) into a model RH problem for \( M^{(r)}(z) \) and a pure \( \bar{\partial} \) Problem for \( M^{(3)}(z) \). The \( M^{(r)}(z) \) can be obtained via a modified reflectionless RH problem \( M_\Lambda^{(r)}(z) \) for the soliton components which is solved in Section 6. In section 7, the error function \( E(z) \) between \( M^{(r)}(z) \) and \( M_\Lambda^{(r)}(z) \) can be computed with a small-norm RH problem. In Section 8, we analyze the \( \bar{\partial} \)-problem for \( M^{(3)}(z) \). Finally, in Section 9, based on the result obtained above, a relation formula is found

\[ M(z) = T(\infty)^{\sigma_3} M^{(3)}(z) E(z) M_\Lambda^{(r)}(z) R^{(2)}(z)^{-1} T(z)^{-\sigma_3}, \]

from which we then obtain the long-time asymptotic behavior for the DNLS equation (1.1) via a reconstruction formula.

2 The spectral analysis and a RH problem

The DNLS equation (1.1) is completely integrable and admits the Lax pair [19]

\[ \Phi_x = X \Phi, \quad \Phi_t = T \Phi, \]  

(2.1)

while

\[ X = ik^2 \sigma_3 + kQ, \]

\[ T = -(2k^2 + Q^2)X - iQ_3 \sigma_3, \]

\[ \]
where $k \in \mathbb{C}$ is a spectral parameter and

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.2)$$

By using the boundary condition (1.1), the Lax pair (2.1) becomes

$$\Phi_{\pm,x} \sim X_{\pm} \Phi_{\pm}, \quad \Phi_{\pm,t} \sim T_{\pm} \Phi_{\pm}, \quad x \to \pm \infty, \quad (2.3)$$

where

$$X_{\pm} = ik^2 \sigma_3 + kQ_{\pm}, \quad T_{\pm} = -(2k^2 - 1) X_{\pm}, \quad (2.4)$$

and

$$Q_{\pm} = \begin{pmatrix} 0 & q_{\pm} \\ -\bar{q}_{\pm} & 0 \end{pmatrix}.$$  

The eigenvalues of the matrix $X_{\pm}$ are $\pm ik \lambda$, which satisfy

$$\lambda^2 = k^2 + 1. \quad (2.5)$$

To avoid multi-valued case of eigenvalue $\lambda$, we introduce a uniformization variable

$$z = k + \lambda, \quad (2.6)$$

and obtain two single-valued functions

$$k(z) = \frac{1}{2}(z - \frac{1}{z}), \quad \lambda(z) = \frac{1}{2}(z + \frac{1}{z}). \quad (2.7)$$

Define two domains $D^+, D^-$ and their boundary $\Sigma$ on $z$-plane by

$$D^- = \{ z : \text{Re} z \text{Im} z < 0 \}, \quad D^+ = \{ z : \text{Re} z \text{Im} z > 0 \},$$

$$\Sigma = \{ z : \text{Re} z \text{Im} z = 0 \} = \mathbb{R} \cup i\mathbb{R}\{0\},$$

which are shown in Figure [1]
Figure 1: The domains $D^-$, $D^+$ and boundary $\Sigma = \mathbb{R} \cup i\mathbb{R} \setminus \{0\}$.

We derive the solution of the asymptotic spectral problem (2.3)

$$\Phi_{\pm} \sim Y_{\pm} e^{ik(z) \lambda(x) \sigma_3},$$

(2.8)

where

$$Y_{\pm} = \left( \begin{array}{c} 1 \\ iq_{\pm} \frac{iq_{\pm}}{z} \end{array} \right).$$

By making transformation

$$\mu_{\pm} = \Phi_{\pm} e^{-ik\lambda x \sigma_3},$$

(2.9)

then we have

$$\mu_{\pm} \sim Y_{\pm}, \quad x \to \pm \infty,$$

$$\det[\Phi_{\pm}] = \det[\mu_{\pm}] = \det[Y_{\pm}] = 1 + z^{-2},$$

and $\mu_{\pm}$ satisfy the Volterra integral equations

$$\mu_{\pm}(z) = Y_{\pm} + \int_{-\infty}^{\infty} Y_{\pm} e^{ik\lambda(x-y)\sigma_3} [Y_{\pm}^{-1} \Delta X_{\pm} \mu_{\pm}(z)] dy, \quad z \neq \pm i,$$

(2.10)

$$\mu_{\pm}(z) = Y_{\pm} + \int_{-\infty}^{\infty} [I + (x-y)X_{\pm}(z)] \Delta X_{\pm} \mu_{\pm}(z) dy, \quad z = \pm i,$$

(2.11)

where $\Delta X_{\pm} = k (Q - Q_{\pm})$.

It can be shown that the eigenfunction $\mu_{\pm}$ admit symmetry [36].
Proposition 1. Jost functions admit two reduction conditions on the z-plane:

The first symmetry reduction:

\[ \mu_\pm(z) = \sigma_2 \mu_\pm(\bar{z}) \sigma_2 = \sigma_1 \mu_\pm(-\bar{z}) \sigma_1. \]  \hspace{1cm} (2.12)

The second symmetry reduction:

\[ \mu_\pm(z) = \frac{i}{z} \mu_\pm(-z^{-1}) \sigma_3 Q_\pm. \]  \hspace{1cm} (2.13)

And for \( z \in \Sigma^0 = \Sigma \setminus \{ \pm i \} \), there exist scattering matrix which is a linear relation between \( \Phi_+ \) and \( \Phi_- \)

\[ \Phi_+(x, t, z) = \Phi_-(x, t, z) S(z), \]  \hspace{1cm} (2.14)

where

\[ S(z) = \begin{pmatrix} a(z) & -b(\bar{z}) \\ b(z) & a(\bar{z}) \end{pmatrix}, \quad \det[S(z)] = 1 \]  \hspace{1cm} (2.15)

with symmetry reduction:

\[ S(z) = \sigma_1 \overline{S(-\bar{z})} \sigma_1 = (\sigma_3 Q_-)^{-1} S \left( -z^{-1} \right) \sigma_3 Q_+. \]  \hspace{1cm} (2.16)

And the reflection coefficients are defined by

\[ \rho(z) = \frac{b(z)}{a(z)}, \quad \tilde{\rho}(z) = -\overline{\rho(\bar{z})}, \]  \hspace{1cm} (2.17)

with symmetry reduction:

\[ \rho(z) = \tilde{\rho}(-\bar{z}) = \frac{q_+}{q_-} \tilde{\rho}(-z^{-1}). \]  \hspace{1cm} (2.18)

Then

\[ a(z) = \frac{\text{Wr}(\Phi_+^1, \Phi_-^2)}{1 + z^{-2}}, \quad b(z) = \frac{\text{Wr}(\Phi_-^1, \Phi_+^1)}{1 + z^{-2}}. \]  \hspace{1cm} (2.19)

Although \( a(z) \) and \( b(z) \) has singularities at points \( \pm i \), \( |\rho(\pm i)| = 1 \). The uniqueness and existences of Lax pair from 36:
Proposition 2. If \( q - q_\pm \in L^{1,1}(\mathbb{R}_\pm) \), the fundamental eigenfunctions \( \mu_\pm \) defined by (2.10) and (2.11) exist and is the unique. Define \( \mu_\pm = (\mu^1_\pm, \mu^2_\pm) \) with \( \mu^1_\pm \) and \( \mu^2_\pm \) denoting the first and second column of \( \mu_\pm \) respectively. Then \( \mu^1_\pm \) and \( \mu^2_\pm \) are analytical on the \( D^+ \), and continuous in \( D^+ \); \( \mu^1_- \) and \( \mu^2_- \) are analytical on the \( D^- \), and continuous in \( D^- \). Moreover, form (2.19), \( a(z) \) is analytical on the \( D^+ \), and continuous in \( D^+ \setminus \{\pm i\} \). Further, \( \lambda a(z) \) is analytical on the \( D^+ \), and \( \lambda b(z) \) and \( \lambda b(z) \) are continuous in \( \Sigma^0 \) and \( \Sigma \) respectively.

The zeros of \( a(z) \) on \( \Sigma \) are known to occur and they correspond to spectral singularities. They are excluded from our analysis in this paper. To deal with our following work, we assume our initial data satisfy this assumption.

Assumption 1. The initial data \( q - q_\pm(x) \in L^{1,1}(\mathbb{R}_\pm) \) and it generates generic scattering data which satisfy that

1. \( a(z) \) has no zeros on \( \Sigma \).
2. \( a(z) \) only has finite number of simple zeros.
3. \( \rho(z) \) and \( \tilde{\rho}(z) \) belong to \( W^{2,\infty}(\Sigma) \cap W^{1,2}(\Sigma) \).

Suppose that \( a(z) \) has \( N_1 \) simple zeros \( z_1, \ldots, z_{N_1} \) on \( D^+ \cap \{z \in \mathbb{C} : \text{Im} z > 0, |z| > 1\} \), and \( N_2 \) simple zeros \( w_1, \ldots, w_m \) on the circle \( \{z = e^{i\varphi} : 0 < \varphi < \pi/2\} \). The symmetries (2.16) imply that

\[
a(\pm z_n) = 0 \iff a(\pm \bar{z}_n) = 0 \iff a\left( \pm \frac{1}{z_n} \right) = 0
\]

\[
\iff a\left( \pm \frac{1}{\bar{z}_n} \right) = 0, \quad n = 1, \ldots, N_1,
\]

and on the circle

\[
a(\pm w_m) = 0 \iff a(\pm \bar{w}_m) = 0, \quad m = 1, \ldots, N_2.
\]

So the zeros of \( a(z) \) come in pairs. It is convenient to define \( \zeta_n = z_n, \zeta_{n+N_1} = -z_n, \zeta_{n+2N_1} = \bar{z}_n^{-1} \) and \( \zeta_{n+3N_1} = -\bar{z}_n^{-1} \) for \( n = 1, \ldots, N_1 \); \( \zeta_{m+4N_1} = w_m \) and \( \zeta_{m+4N_1+N_2} = -w_m \) for \( m = 1, \ldots, N_2 \). Therefore, the discrete spectrum is

\[
Z = \{ \zeta_n, \zeta_{n+3N_1} \}_{n=1}^{4N_1+2N_2}, \quad (2.20)
\]
with $\zeta_n \in D^+$ and $\bar{\zeta}_n \in D^-$. And the distribution of $\mathcal{Z}$ on the $z$-plane is shown in Figure 2.

![Figure 2: Distribution of the discrete spectrum $\mathcal{Z}$. The red one is unit circle.](image)

As shown in [33], denote norming constant $c_n = b_n/a'(z_n)$. Then we have residue conditions as

$$\text{Res}_{z = \pm z_n} \left[ \frac{\mu_1^+ (z)}{a(z)} \right] = c_n e^{-2ik(\pm z_n)\lambda(\pm z_n)x} \mu_2^+ (\pm z_n), \quad (2.21)$$

$$\text{Res}_{z = \pm \bar{z}_n} \left[ \frac{\mu_2^+ (z)}{a(z)} \right] = \pm \frac{q_+}{q_-} z_n^{-2} \bar{c}_n e^{-2ik(\pm \bar{z}_n)\lambda(\pm \bar{z}_n)x} \mu_2^+ (\pm \bar{z}_n), \quad (2.22)$$

$$\text{Res}_{z = \pm \bar{z}_n} \left[ \frac{\mu_1^+ (z)}{a(z)} \right] = -\bar{c}_n e^{-2ik(\pm \bar{z}_n)\lambda(\pm \bar{z}_n)x} \mu_1^+ (\pm \bar{z}_n), \quad (2.23)$$

$$\text{Res}_{z = \pm \bar{z}_n} \left[ \frac{\mu_2^+ (z)}{a(z)} \right] = \pm \frac{q_+}{q_-} z_n^{-2} c_n e^{-2ik(\pm z_n)\lambda(\pm z_n)x} \mu_2^+ (\pm z_n). \quad (2.24)$$
For brevity, we introduce a new constant $C_n$ as: for $n = 1, ..., N_1$, $C_n = C_{n+N_1} = c_n$, $C_{n+2N_1} = -C_{n+3N_1} = \frac{q}{q_n} z_n^{-2} c_n$, for $m = 1, ..., N_2$, $C_{m+4N_1} = C_{m+4N_1+N_2} = c_{m+N_1}$, and the collection $\sigma_d = \{\zeta_n, C_n\}_{n=1}^{4N_1+2N_2}$ is called the scattering data.

Now we are going to take into account the time evolution of scattering data. If $q$ also depends on $t$ (i.e. $q = q(x, t)$), we can obtain the functions $a$ and $b$ as above for all times $t \in R$. Taking account of the $t$-part in (2.1), the $t$- derivative of $a$ and $b$ comes to

$$a_t(z; t) = 0, \quad b_t(z; t) = -(2k^2 - 1)k\lambda b(z; t).$$

Then we can obtain time dependence of scattering data which can be expressed as the following replacement

$$C(\zeta_n) \rightarrow C(t, \zeta_n) = c(0, \zeta_n)e^{-(2k(\zeta_n)^2 - 1)k(\zeta_n)\lambda t},$$

$$r(z) \rightarrow r(t, z) = r(0, z)e^{-(2k^2 - 1)k\lambda t}$$

In particular, if at time $t = 0$ the initial function $q(x, 0)$ produces $4N_1+2N_2$ simple zeros $\zeta_1, ..., \zeta_{4N_1+2N_2}$ of $a(z; 0)$ and if $q$ evolves accordingly to the (1.1), then $q(x, t)$ will produce exactly the same $N$ simple zeros at any other time $t \in R$. The scattering data with time $t$ is given by

$$\left\{e^{-(2k^2 - 1)k\lambda t}r(z), \left\{\zeta_n, e^{-(2k(\zeta_n)^2 - 1)k(\zeta_n)\lambda t}C_n\right\}_{n=1}^{4N_1+2N_2}\right\},$$

where $\left\{r(z), \left\{\zeta_n, C_n\right\}_{n=1}^{4N_1+2N_2}\right\}$ are obtained from the initial data $q(x, 0) = q_0(x)$. Denote the phase function

$$\theta(z) = k(z)\lambda(z) \left[ x/t - (2k(z)^2 - 1) \right],$$

(2.30)
and for convenience we denote $\theta_n = \theta(\zeta_n)$.

To propose and solve the matrix RH problem in the following inverse problem, we finally give the asymptotic behaviors of the modified Jost solutions and scattering matrix as $z \to \infty$ and $z \to 0$.

**Proposition 3.** The Jost solutions possess the following asymptotic behaviors

$$
\mu_\pm(x, t, z) = e^{i\nu_\pm(x, t; q)\sigma_3} + O(z^{-1}), \quad z \to \infty, \quad (2.31)
$$

$$
\mu_\pm(x, t, z) = \frac{i}{z} e^{i\nu_\pm(x, t; q)\sigma_3\sigma_3} Q_\pm + O(1), \quad z \to 0, \quad (2.32)
$$

where

$$
\nu_\pm(x, t; q) = \frac{1}{2} \int_{\pm \infty}^{x} (|q|^2 - 1) dy. \quad (2.33)
$$

The scattering matrices admit asymptotic behaviors

$$
S(z) = e^{-i\nu_0(t; q)\sigma_3} + O(z^{-1}), \quad z \to \infty, \quad (2.34)
$$

$$
S(z) = \text{diag}\left(\frac{q_-}{q_+}, \frac{q_+}{q_-}\right) e^{i\nu_0(t; q)\sigma_3} + O(z), \quad z \to 0, \quad (2.35)
$$

where

$$
\nu_0(t; q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 - 1) dy. \quad (2.36)
$$

Further we have $\rho(0) = \tilde{\rho}(0) = 0$.

Moreover, from trace formulae we have

$$
a(z) = \prod_{j=1}^{4N_1+2N_2} \frac{z - \zeta_j}{z - \bar{\zeta}_j} \exp\left\{-\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho(s)\tilde{\rho}(s))}{s - z} ds\right\}. \quad (2.37)
$$

Then by taking $z \to 0$, theta condition is obtained:

$$
\arg \frac{q_-}{q_+} + 2\nu_0 = 8 \sum_{n=1}^{N_1} \arg(z_n) + 4 \sum_{m=1}^{N_2} \arg(w_m) + \frac{1}{2\pi} \int_{\Sigma} \frac{\log(1 - \rho(s)\tilde{\rho}(s))}{s} ds + 2j\pi, \quad (2.38)
$$

where $j$ is an integer.
Define a sectionally meromorphic matrix
\[ M(z; x, t) = \begin{cases} 
(a(z)^{-1} \mu^1_+, \mu^2_+), & \text{as } z \in D^+, \\
(\mu^1_-, a(\bar{z})^{-1} \mu^2_+), & \text{as } z \in D^-,
\end{cases} \tag{2.39} \]
which solves the following (time-dependent) RHP.

\textbf{RHP0.} Find a matrix-valued function \( M(z) \) which satisfies:

- **Analyticity:** \( M(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma \) and has single poles \( Z \);
- **Symmetry:** \( M(z) = \sigma_2 M(\bar{z}) \sigma_2 = \sigma_1 M(-\bar{z}) \sigma_1 = i^\frac{1}{2} M(-1/z) \sigma_3 Q_-; \)
- **Jump condition:** \( M(z) \) has continuous boundary values \( M^\pm(z) \) on \( \Sigma \) and
  \[ M^+(z) = M^-(z) V(z), \quad z \in \Sigma, \tag{2.40} \]
where
  \[ V(z) = \begin{pmatrix} 1 - \bar{\rho}(z) \rho(z) & -e^{2it\theta} \bar{\rho}(z) \\
e^{-2it\theta} \rho(z) & 1 \end{pmatrix}; \tag{2.41} \]
- **Asymptotic behaviors:**
  \[ M(z) = e^{i\nu(x,t)q} + O(z^{-1}), \quad z \to \infty, \tag{2.42} \]
  \[ M(z) = -i^z e^{i\nu(x,t)q} \sigma_3 Q_- + O(1), \quad z \to 0; \tag{2.43} \]
- **Residue conditions:** \( M \) has simple poles at each point in \( \mathcal{Z} \cup \mathcal{Z}^- \) with:
  \[ \text{Res} M(z) = \lim_{z \to \zeta_n} M(z) \begin{pmatrix} 0 & 0 \\
C_n e^{-2it\theta_n} & 0 \end{pmatrix}, \tag{2.44} \]
  \[ \text{Res} M(z) = \lim_{z \to \zeta_n} M(z) \begin{pmatrix} 0 & -\bar{C}_n e^{2it\theta_n} \\
0 & 0 \end{pmatrix}. \tag{2.45} \]

From the asymptotic behavior in Proposition 2, the reconstruction formula of \( q(x, t) \) is given by
\[ q(x, t) = \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} (|q(x, t)|^2 - 1) \, dy \right\} m(x, t), \tag{2.46} \]
where
\[ m(x, t) = \lim_{z \to \infty} [zM]_{12}. \tag{2.47} \]
Take modulus on both sides of (2.46) yields
\[ |q(x, t)| = |m(x, t)|, \]
which is substituted back into (2.46) leads to
\[ q(x, t) = \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} (|m(x, t)|^2 - 1)dy \right\} m(x, t). \]  

(2.48)

3 Deformation to a mixed $\bar{\partial}$-RH problem

We find that the long-time asymptotic of RHP0 is affected by the growth and decay of the exponential function $e^{\pm 2it\theta}$ appearing in both the jump relation and the residue conditions. Therefore, in this section, we introduce a new transform $M(z) \rightarrow M^{(1)}(z)$, which make that the $M^{(1)}(z)$ is well behaved as $t \rightarrow \infty$ along any characteristic line.

Let $\xi = \frac{x}{t}$, to obtain asymptotic behavior of $e^{2it\theta}$ as $t \rightarrow \infty$, we consider the real part of $2it\theta$:
\[ \text{Re}(2it\theta) = -t\text{Im}z\text{Re}z \left[ (\xi + 2) (1 + |z|^{-4}) - (\text{Re}^2z - \text{Im}^2z) (1 + |z|^{-8}) \right]. \]  

(3.1)

The signature of $\text{Im}\theta$ are shown in Figure 3.
In these figure we take $\xi = -4, -3, -2.6, -1.5, -1, 0$ respectively to show all type of $\text{Im} \theta$. The green curve is unit circle. In the red region, $\text{Im} \theta > 0$ while $\text{Im} \theta = 0$ on the red curve. And $\text{Im} \theta < 0$ in the white region.

Figure 3: In these figure we take $\xi = -4, -3, -2.6, -1.5, -1, 0$ respectively to show all type of $\text{Im} \theta$. The green curve is unit circle. In the red region, $\text{Im} \theta > 0$ while $\text{Im} \theta = 0$ on the red curve. And $\text{Im} \theta < 0$ in the white region.

In our paper, we only consider the case $-3 < \xi < -1$ which is corresponding to Figure 3 (c), because of its well property.

For brevity, we introduce some notations with respect to subscripts

\[
\mathcal{N} \triangleq \{1, \ldots, 4N_1 + 2N_2\}, \quad \nabla = \{n \in \mathcal{N}|\text{Im} \theta_n \leq 0\}, \\
\Delta = \{n \in \mathcal{N}|\text{Im} \theta_n > 0\}, \quad \Lambda = \{n \in \mathcal{N}|\text{Im} \theta_n = 0\}. 
\tag{3.2}
\]

For $n \in \Delta$, the residue of $M(z)$ at $\zeta_n$ in (2.44) are unbounded as $t \to \infty$. Similarly, for $n \in \nabla$, the residue at $\zeta_n$ approach to be zero as $t \to \infty$. Define

\[
\rho_0 = \min_{n \in \Delta \cup \nabla \setminus \Lambda} |\text{Im} \theta_n| \neq 0. 
\tag{3.3}
\]
To distinguish different type of zeros, we further give
\[ \nabla_1 = \{ j \in \{1, \ldots, N_1\} | \text{Im} \theta(z_j) \leq 0 \}, \Delta_1 = \{ j \in \{1, \ldots, N_1\} | \text{Im} \theta(z_j) > 0 \}, \]
\[ \nabla_2 = \{ i \in \{1, \ldots, N_2\} | \text{Im} \theta(w_i) \leq 0 \}, \Delta_2 = \{ i \in \{1, \ldots, N_2\} | \text{Im} \theta(w_i) > 0 \}, \]
\[ \Lambda_1 = \{ j_0 \in \{1, \ldots, N_1\} | \text{Im} \theta(z_{j_0}) = 0 \}, \Lambda_2 = \{ i_0 \in \{1, \ldots, N_2\} | \text{Im} \theta(w_{i_0}) = 0 \}. \]

For the poles \( \zeta_n \) with \( n \notin \Lambda \), we want to trap them for jumps along small closed circles enclosing themselves respectively. The jump matrix in (2.41) also needs to be restricted. Recall the well known factorizations of \( V(z) \):
\[
V(z) = \begin{pmatrix}
1 & -\tilde{\rho} e^{2it\theta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\rho e^{-2it\theta} & 1
\end{pmatrix},
\]
\[
= \begin{pmatrix}
1 & -\tilde{\rho} \\
\rho e^{2it\theta} & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\tilde{\rho} e^{2it\theta} \\
0 & 1
\end{pmatrix}. \quad (3.4)
\]
We will use these factorizations to deform the jump contours so that exponentials \( e^{\pm 2it\theta} \) are decaying in corresponding regions respectively. Define functions
\[
\delta(z) = \exp \left( -\frac{1}{2\pi i} \int_{i\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{2s} \right) \log(1 - \rho(s)\tilde{\rho}(s)) ds \right); \quad (3.6)
\]
\[
T(z) = T(z, \xi) = \prod_{n \in \Delta} \frac{z - \zeta_n}{\tilde{\zeta}_n^{-1}z - 1} \delta(z) = \prod_{j \in \Delta_1} \frac{z^2 - \tilde{z}_j^2}{\tilde{z}_j^{-2}z^2 - 1} \prod_{j \in \Delta_2} \frac{z^2 - w_i^2}{w_i^2z^2 - 1} \delta(z). \quad (3.7)
\]
In the above formulas, we choose the principal branch of power and logarithm functions.

**Proposition 4.** The function defined by (3.7) has following properties:
(a) \( T \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \), and for each \( n \in \Delta \), \( T(z) \) has simple zeros \( \zeta_n \) and simple poles \( \tilde{\zeta}_n \);
(b) \( T(z) = T^{-1}(\bar{z}) = T^{-1}(-z^{-1}) \);
(c) For \( z \in \mathbb{R} \), as \( z \) approaching the real axis from above and below, \( T \) has boundary values \( T_{\pm} \), which satisfy:
\[
T_+(z) = (1 - \rho(z)\tilde{\rho}(z))T_-(z), \quad z \in i\mathbb{R}; \quad (3.8)
\]
\[ 16 \]
(d) \( \lim_{z \to \infty} T(z) \triangleq T(\infty) \), where

\[
T(\infty) = \prod_{j \in \Delta_1} z_j^2 z_j^{-2} \prod_{i \in \Delta_2} \bar{w}_i^2 \exp \left( \frac{1}{4\pi i} \int_{i\mathbb{R}} s^{-1} \log(1 - \rho(s)\tilde{\rho}(s))ds \right),
\]

with \( |T(\infty)| = 1 \);

(e) As \( |z| \to \infty \) with \( |\arg(z)| \leq c < \pi \),

\[
T(z) = T(\infty) \left( 1 + z^{-1} \frac{1}{2\pi i} \int_{i\mathbb{R}} \log(1 - \rho(s)\tilde{\rho}(s))ds + O(z^{-2}) \right);
\]

(f) \( T(z) \) is continuous at \( z = 0 \), and \( \lim_{z \to 0} T(z) = T(0) = T(\infty)^{-1} \);

(g) \( \frac{a(z)}{T(z)} \) is holomorphic in \( D^+ \). And its absolute value is bounded in \( D^+ \cap \{z \in \mathbb{C}|Rez > 0\} \). Additionally, the ratio extends as a continuous function on \( i\mathbb{R} \).

**Proof.** Properties (a), (b), (d) and (f) can be obtain by simple calculation. And (c) follows from the Plemelj formula. By the Laurent expansion (e) immediately. For brevity, we omit calculation. For (g), from (2.37) we have

\[
\frac{a(z)}{T(z)} = T(\infty)^{-1} \prod_{j \in \Delta_1} \frac{z_j^2 - \bar{z}_j^{-2}}{z_j^{-2}z_j^2 - 1} \prod_{i \in \Delta_2} \frac{z^2 - w_i^2}{w_i^2z^2 - 1} \exp \left\{ - \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{\log(1 - \rho(s)\tilde{\rho}(s))}{s - z} \right\}.
\]

So \( \frac{a(z)}{T(z)} \) is holomorphic in \( D^+ \). And in above expression, all factors except the last integral is bounded for \( z \in D^+ \). From (2.17), \( 1 - \rho(s)\tilde{\rho}(s) = 1 + |\rho(s)|^2 \). Let \( z = x + yi \), then the real part of the exponential is \( -\frac{y}{2\pi} \int_{i\mathbb{R}} \frac{\log(1 + |\rho(s)|^2)}{|s - z|^2} ds \) which can be bounded as follows:

\[
\left| \frac{y}{2\pi} \int_{i\mathbb{R}} \frac{\log(1 + |\rho(s)|^2)}{|s - z|^2} ds \right| \leq \frac{1}{2\pi} \| \log(1 + |\rho(s)|^2) \|_{L^\infty(\mathbb{R})} \| \frac{y}{(s - x)^2 + y^2} \|_{L^1(\mathbb{R})} \lesssim \| \rho(s) \|_{L^\infty(\mathbb{R})}.
\]

\( \square \)
Additionally, let \( \varrho \) be a positive constant stratifying
\[
\varrho = \frac{1}{2} \min \left\{ \min_{j \neq i \in \mathcal{N}} |\zeta_i - \zeta_j|, \min_{j \in \mathcal{N}} \{ |\text{Im} \zeta_j|, |\text{Re} \zeta_j| \}, \min_{j \in \Lambda \setminus \text{Im} \theta(z)=0} |\zeta_j - z| \right\}. \tag{3.12}
\]

By above definition, for every \( n \in \mathcal{N} \), we define disks \( \mathbb{D}(\zeta_n, \varrho) \), such that they pairwise disjoint, also disjoint with \( \{ z \in \mathbb{C} | \text{Im} \theta(z) = 0 \} \) and \( \Sigma \). Introduce a piecewise matrix function
\[
G(z) = \begin{cases} 
\frac{1}{1 - C_n(z - \zeta_n)}e^{-2it\theta_n} & \text{as } z \in \mathbb{D}(\zeta_n, \varrho), n \in \nabla \setminus \Lambda; \\
\frac{1}{1 - C_n^{-1}(z - \zeta_n)e^{2it\theta_n}} & \text{as } z \in \mathbb{D}(\zeta_n, \varrho), n \in \Delta; \\
\frac{1}{1 - \bar{C}_n(z - \bar{\zeta}_n)}e^{2it\theta_n} & \text{as } z \in \mathbb{D}(\zeta_n, \varrho), n \in \nabla \setminus \Lambda; \\
\frac{1}{1 - \bar{C}_n^{-1}(z - \bar{\zeta}_n)e^{-2it\theta_n}} & \text{as } z \in \mathbb{D}(\zeta_n, \varrho), n \in \Delta; \\
I & \text{as } z \text{ in elsewhere}; 
\end{cases} \tag{3.13}
\]

Now we use \( T(z) \) and \( G(z) \) to define a new matrix-valued function \( M^{(1)}(z) \).
\[
M^{(1)}(z) = T(\infty)^{-\sigma_3}M(z)G(z)T(z)^{\sigma_3}, \tag{3.14}
\]

which then satisfies the following RH problem.

**RHP1.** Find a matrix-valued function \( M^{(1)}(z) \) which satisfies:

- **Analyticity:** \( M^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \), where
  \[
  \Sigma^{(1)} = \mathbb{R} \cup i\mathbb{R} \cup \left( \bigcup_{n \in \mathcal{N} \setminus \Lambda} \left( \partial \mathbb{D}(\zeta_n, \varrho) \cup \partial \mathbb{D}(\xi_n, \varrho) \right) \right), \tag{3.15}
  \]

  is shown in Figure 4.

- **Symmetry:** \( M^{(1)}(z) = \sigma_2 M^{(1)}(\bar{z}) \sigma_2 = \sigma_1 M^{(1)}(-z) \sigma_1 = i \frac{1}{z} M^{(1)}(-1/z) \sigma_3 Q_- \);

- **Jump condition:** \( M^{(1)} \) has continuous boundary values \( M^{(1)}_{\pm} \) on \( \Sigma^{(1)} \) and
  \[
  M^{(1)}_{+}(z) = M^{(1)}_{-}(z)V^{(1)}(z), \quad z \in \Sigma^{(1)}, \tag{3.16}
  \]
where

\[
V^{(1)}(z) = \begin{cases}
\begin{pmatrix}
1 & -e^{2it\theta} \tilde{\rho}(z) T^{-2}(z) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
e^{-2i\theta} \rho(z) T^2(z) & 1
\end{pmatrix}, & \text{as } z \in \mathbb{R}; \\
\begin{pmatrix}
1 & 0 \\
e^{-2i\theta} \rho(z) T^2(z) & 1
\end{pmatrix}
\begin{pmatrix}
1 & -e^{2i\theta} \tilde{\rho}(z) T^{-2}(z) \\
1 - \tilde{\rho}(z) \rho(z) & 1
\end{pmatrix}, & \text{as } z \in i\mathbb{R}; \\
\begin{pmatrix}
1 & 0 \\
-C_n (z - \zeta_n)^{-1} T^2(z) e^{-2i\theta_n} & 1
\end{pmatrix}, & \text{as } z \in \partial D(\zeta_n, \varrho), n \in \Delta; \\
\begin{pmatrix}
1 & 0 \\
\tilde{C}_n (z - \zeta_n)^{-1} T^{-2}(z) e^{2i\theta_n} & 1
\end{pmatrix}, & \text{as } z \in \partial D(\zeta_n, \varrho), n \in \Delta; \\
\begin{pmatrix}
1 & 0 \\
\tilde{C}_n^{-1} (z - \bar{\zeta}_n) e^{-2i\theta_n} T^2(z) & 1
\end{pmatrix}, & \text{as } z \in \partial D(\bar{\zeta}_n, \varrho), n \in \Delta;
\end{cases}
\]  

(3.17)

▶ Asymptotic behaviors:

\[
M^{(1)}(z) = e^{i\nu - (x,t,\theta) \sigma_3} + \mathcal{O}(z^{-1}), \quad z \to \infty,
\]

\[
M^{(1)}(z) = \frac{i}{z} e^{i\nu - (x,t,\theta) \sigma_3} \sigma_3 Q_\nu + \mathcal{O}(1), \quad z \to 0;
\]

(3.18)  

(3.19)

▶ Residue conditions: \(M^{(1)}\) has simple poles at each point \(\zeta_n\) and \(\bar{\zeta}_n\) for \(n \in \Lambda\) with:

\[
\text{Res } M^{(1)}(z) = \lim_{z \to \zeta_n} M^{(1)}(z) \begin{pmatrix}
0 \\
C_n e^{-2i\theta_n} T^2(\zeta_n)
\end{pmatrix},
\]

\[
\text{Res } M^{(1)}(z) = \lim_{z \to \zeta_n} M^{(1)}(z) \begin{pmatrix}
0 \\
-C_n T^{-2}(\bar{\zeta}_n) e^{2i\theta_n}
\end{pmatrix}.
\]

(3.20)  

(3.21)

Proof. Note that the triangular factors \((3.13)\) trades poles \(\zeta_n\) and \(\bar{\zeta}_n\) to jumps on the disk boundaries \(\partial D(\zeta_n, \varrho)\) and \(\partial D(\bar{\zeta}_n, \varrho)\) respectively for \(n \in \mathcal{N} \setminus \Lambda\). Then by simple calculation we can obtain the residues condition and jump condition from \((2.44), (2.45), (2.41), (3.13)\) and \((3.14)\). The analyticity and symmetry of \(M^{(1)}(z)\) is directly from its definition, the Proposition \(4\) \((3.13)\) and the properties of \(M\). As for asymptotic behaviors, from \(\lim_{z \to 0} G(z) = \lim_{z \to \infty} G(z) = I\) and Proposition \(4\) \((f)\), we obtain that \(M^{(1)}(z)\) has same asymptotic behaviors as \(M(z)\).  

\(\square\
Figure 4: The blue curve, including $\mathbb{R}$, $i\mathbb{R}$ and the small circles constitute $\Sigma^{(1)}$. Because $\text{Im}\theta(w_m) = 0$, it remain the pole of $M^{(1)}$. And $\text{Im}\theta(z_n) \neq 0$, so we change it to jump on $\partial\mathbb{D}(\zeta_n, \varrho)$.

4 Mixed $\bar{\partial}$-RH Problem

In this section, we make continuous extension for the jump matrix $V^{(1)}$ to remove the jump from $\Sigma$. Besides, the new problem is hoped to takes advantage of the decay/growth of $e^{2it\theta(z)}$ for $z \notin \Sigma$. For this purpose, we introduce new eight regions:

$$\Omega_{2n+1} = \{ z \in \mathbb{C} | n\pi/2 \leq \arg z \leq n\pi/2 + \varphi \}, \quad (4.1)$$
$$\Omega_{2n+2} = \{ z \in \mathbb{C} | (n+1)\pi/2 - \varphi \leq \arg z \leq (n+1)\pi/2 \}, \quad (4.2)$$

where $n = 0, 1, 2, 3$ and $\varphi > 0$ is an fixed sufficiently small angle achieving following conditions:

1. $\frac{2|\xi+2|}{|\xi+2|+1} < \cos 2\varphi < 1$;
2. each $\Omega_i$ doesn’t intersect any of $\mathbb{D}(\zeta_n, \varrho)$ or $\mathbb{D}(\bar{\zeta}_n, \varrho)$.
Define new contours as follow:

\[ \Sigma_k = e^{(k-1)i\pi/4+\varphi} R_+, \quad k = 1, 3, 5, 7, \]  (4.3)

\[ \Sigma_k = e^{ki\pi/4-\varphi} R_+, \quad k = 2, 4, 6, 8, \]  (4.4)

\[ \tilde{\Sigma} = \Sigma_1 \cup \Sigma_2 \ldots \cup \Sigma_8, \]  (4.5)

which is the boundary of \( \Omega_k \) respectively. In addition, let

\[ \Omega = \Omega_1 \cup \ldots \cup \Omega_8, \]  (4.6)

\[ \Sigma^{(2)} = \cup_{n \in \mathbb{N}\setminus \Lambda} \left( \partial \mathbb{D}(\bar{\zeta}_n, \varrho) \cup \partial \mathbb{D}(\zeta_n, \varrho) \right), \]  (4.7)

which are shown in Figure 5.

Figure 5: The yellow region is \( \Omega \). The blue circle constitute \( \Sigma^{(2)} \) together.

\textbf{Lemma 1.} Let \( \xi = \frac{v}{t} \in (-3, -1) \), and \( F(r) = r^2 + \frac{1}{r^2} \) is a real-valued function. Then for \( z = re^{i\varphi} \), the imaginary part of phase function \([3.1]\) satisfies

\[ \text{Im} \theta(z) \leq \frac{1}{16} |\sin 2\varphi|(|\xi + 2| - 1) F(r)^2, \quad \text{as } z \in \Omega_1, \Omega_3, \Omega_5, \Omega_7; \]  (4.8)

\[ \text{Im} \theta(z) \geq \frac{1}{16} |\sin 2\varphi| (1 - |\xi + 2|) F(r)^2, \quad \text{as } z \in \Omega_2, \Omega_4, \Omega_6, \Omega_8. \]  (4.9)
Proof. We only prove the case \( z \in \Omega_1 \), and the other regions are similarly. From (3.1) we have

\[
\text{Im} \, \theta(z) = \frac{1}{2} \text{Im} z \text{Re} \left[ (\xi + 2) (1 + |z|^{-4}) - (\text{Re}^2 z - \text{Im}^2 z) (1 + |z|^{-8}) \right] \\
= \frac{1}{4} r^2 \sin 2\phi \left[ (\xi + 2) (1 + r^{-4}) - r^2 \cos 2\phi (1 + r^{-8}) \right] \\
= \frac{1}{4} \sin 2\phi \left[ (\xi + 2) F(r) - \cos 2\phi \left( F(r)^2 - 2 \right) \right].
\]

(4.10)

\( F(r) \geq 2 \) leads to \( 2 \leq \frac{F(r)^2}{2} \). For \( z \in \Omega_1, \frac{2|\xi + 2|}{|\xi + 2| + 1} < \cos 2\varphi < \cos 2\phi \), then we have

\[
\frac{|\xi + 2|}{\cos 2\phi} F(r) \leq \frac{|\xi + 2| + 1}{4} F(r)^2.
\]

(4.11)

Substitute above inequality into (4.10) we obtain the consequence immediately.

\[ \square \]

Introduce a small enough constant \( 1 > \epsilon_0 > 0 \) with \((1 - \epsilon_0) \cos \varphi > \frac{1}{2}\). Let \( X_1 \in C_0^\infty (\mathbb{R}, [0, 1]) \), which is support in \((1 - \epsilon_0, 1 + \epsilon_0)\). And \( X_0 \) has support in \((-\epsilon_0, \epsilon_0)\) with \( X_0(z) = X_1(1 + z) \). In addition, we denote following functions for brief:

\[
p_1(z) = p_5(z) = \rho(z), \quad p_2(z) = p_6(z) = \frac{\bar{\rho}(z)}{1 - \rho(z)\bar{\rho}(z)}, \\
p_3(z) = p_7(z) = \frac{\rho(z)}{1 - \rho(z)\bar{\rho}(z)}, \quad p_4(z) = p_8(z) = \bar{\rho}(z).
\]

(4.12) (4.13)

Then the next step is to construct a matrix function \( R^{(2)} \). We need to remove jump on \( \mathbb{R} \) and \( i\mathbb{R} \), and have some mild control on \( \bar{\partial}R^{(2)} \) sufficient to ensure that the \( \bar{\partial} \)-contribution to the long-time asymptotics of \( q(x,t) \) is negligible. So we choose \( R^{(2)}(z) \) as

\[
R^{(2)}(z) = \begin{cases} 
\left( \begin{array}{cc} 1 & R_j(z)e^{2it\theta} \\
0 & 1 \end{array} \right), & z \in \Omega_j, j = 2, 4, 6, 8; \\
\left( \begin{array}{cc} 1 & 0 \\
R_j(z)e^{-2it\theta} & 1 \end{array} \right), & z \in \Omega_j, j = 1, 3, 5, 7; \\
I, & \text{elsewhere};
\end{cases}
\]

(4.14)
where the functions $R_j$, $j = 1, 2, \ldots, 8$, is defined in following Proposition.

**Proposition 5.** $R_j : \bar{\Omega}_j \rightarrow C$, $j = 1, 2, \ldots, 8$ have boundary values as follow:

\[
R_1(z) = \begin{cases} 
-\rho(z)T(z)^2 & \text{if } z \in \mathbb{R}^+, \\
0 & \text{if } z \in \Sigma_1,
\end{cases}
R_2(z) = \begin{cases} 
0 & \text{if } z \in \Sigma_2, \\
\bar{\rho}(z)T_+(z)^2 & \text{if } z \in i\mathbb{R}^+,
\end{cases}
\]

(4.15)

\[
R_3(z) = \begin{cases} 
\rho(z)T_-(z)^2 & \text{if } z \in i\mathbb{R}^+, \\
0 & \text{if } z \in \Sigma_3,
\end{cases}
R_4(z) = \begin{cases} 
0 & \text{if } z \in \Sigma_4, \\
-\bar{\rho}(z)T(z)^2 & \text{if } z \in \mathbb{R}^-,
\end{cases}
\]

(4.16)

\[
R_5(z) = \begin{cases} 
-\rho(z)T(z)^2 & \text{if } z \in \mathbb{R}^-, \\
0 & \text{if } z \in \Sigma_5,
\end{cases}
R_6(z) = \begin{cases} 
0 & \text{if } z \in \Sigma_6, \\
\bar{\rho}(z)T_+(z)^2 & \text{if } z \in i\mathbb{R}^-,
\end{cases}
\]

(4.17)

\[
R_7(z) = \begin{cases} 
\rho(z)T_-(z)^2 & \text{if } z \in i\mathbb{R}^-, \\
0 & \text{if } z \in \Sigma_7,
\end{cases}
R_8(z) = \begin{cases} 
0 & \text{if } z \in \Sigma_8, \\
-\bar{\rho}(z)T(z)^2 & \text{if } z \in \mathbb{R}^+.
\end{cases}
\]

(4.18)

$R_j$ have following property: for $j = 1, 5, 4, 8$,

\[
|\bar{\partial}R_j(z)| \lesssim |p_j'(|z|)| + |z|^{-1/2}, \text{ for all } z \in \Omega_j;
\]

(4.19)

and for $j = 2, 3, 6, 7$,

\[
|\bar{\partial}R_j(z)| \lesssim |z - i|, \text{ for all } z \in \Omega_j \text{ in a small fixed neighborhood of } \pm i,
\]

(4.20)

\[
|\bar{\partial}R_j(z)| \lesssim |p_j'(i|z|)| + |z|^{-1/2} + |\bar{\partial}X_1(|z|)|, \text{ for all } z \in \Omega_j.
\]

(4.21)

And

\[
\bar{\partial}R_j(z) = 0, \quad \text{if } z \in \text{ elsewhere}.
\]

(4.22)

**Proof.** Case I: $z \in \bar{\Omega}_j$, $j = 1, 5, 4, 8$.

Take $R_1(z)$ as an example with extensions

\[
R_1(z) = p_1(|z|)T^2(z) \cos(k_0 \arg z), \quad k_0 = \frac{2\pi}{\varphi}.
\]

(4.23)
The other cases are easily inferred. \( p_1(|z|) = \rho(|z|) \) is bounded. Denote \( z = re^{i\phi} \), then we have \( \bar{\partial} = \frac{\epsilon}{2} \left( \partial_r + \frac{i}{r} \partial_\phi \right) \). So

\[
\bar{\partial} R_1(z) = \frac{e^{i\phi}}{2} T^2(z) \left( p'_1(r) \cos(k_0\phi) - \frac{i}{r} p_1(r) k_0 \sin(k_0\phi) \right). \tag{4.24}
\]

To bound second term we use Cauchy-Schwarz inequality and obtain

\[
|p_1(r)| = |\rho(r)| = |\rho(r) - \rho(0)| = |\int_0^r \rho'(s) ds| \leq \| \rho'(s) \|_{L^2} r^{1/2}. \tag{4.25}
\]

And note that \( T(z) \) is a bounded function in \( \bar{\Omega}_1 \). Then the boundedness of \( \text{(4.19)} \) follows immediately.

**Case II:** \( z \in \bar{\Omega}_j, j = 2, 3, 6, 7 \).

The details of the proof are only given for \( R_2 \). Unlike the vanishing boundary condition case in \[{16}\], the determinant of \( M(z) \) is \( 1 + z^{-2} \). So to bound the \( \bar{\partial} \)-derivative construct by \( R^{(2)} \) in following section, the property of \( R^{(2)} \) at \( \pm i \) needs to be control. For this purpose, we make small adjustments to the extensions of \( R_2 \) as

\[
R_2(z) = R_{21}(z) + R_{22}(z), \tag{4.26}
\]

with a constant \( \delta_0 \) stratifying \( \varphi > \delta_0 \epsilon_0 \) and

\[
R_{21}(z) = \left[ 1 - X_1(|z|) \right] p_2(\bar{z}|z|) T^{-2}(z) \cos[k_0 \left( \frac{\pi}{2} - \arg z \right)], \tag{4.27}
\]

\[
R_{22}(z) = f(|z|) g(z) \cos[k_0 \left( \frac{\pi}{2} - \arg z \right)]
- \frac{i}{k_0} X_0 (a(z) f'(|z|)g(z) \sin[k_0 \left( \frac{\pi}{2} - \arg z \right)]). \tag{4.28}
\]

Among above function,

\[
f(z) = X_1(z) \frac{\bar{b}(z)}{a(z)}, \quad g(z) = \left( \frac{a(z)}{T(z)} \right)^2. \tag{4.29}
\]

Then \( f(z) \in W^{2,\infty} \). Obviously, \( R_{21}(z) \equiv 0 \) with \( |z| \) in the support of \( X_1 \) and \( R_{22}(z) \equiv 0 \) out the support of \( X_1 \). Note that

\[
\left| p_2(z) \right| = \left| \frac{\bar{\rho}(z)}{1 - \rho(z) \bar{\rho}(z)} \right| = \left| \frac{\bar{\rho}(z)}{1 - |\rho(z)|^2} \right| \lesssim |\rho(z)|, \quad \text{for } z \text{ out of supp}(X_1). \tag{4.30}
\]
Similarly in case I, \( R_{21}(z) \) can be bounded as

\[
|\overline{\partial}R_{21}(z)| \lesssim (1 - X_1(|z|)) \left( |P_2'(i|z|)| + |z|^{-1/2} \right) + |\overline{\partial}X_1(|z|)|. \tag{4.31}
\]

As for \( R_{22}(z) \), \( z = re^{i\phi} \),

\[
\overline{\partial}R_{22}(z) = \frac{e^{i\phi}}{2}g(z) \cos[k_0(\frac{\pi}{2} - \varphi)]f'(ir) \left( 1 - X_0(\frac{\varphi}{\delta_0}) \right)
+ \sin[k_0(\frac{\pi}{2} - \varphi)] \left[ \frac{ik_0}{r}f(ir) + \frac{1}{\delta_0k_0}X'_0(\frac{\varphi}{\delta_0})f'(ir) \right]
- \frac{i}{k_0} \sin[k_0(\frac{\pi}{2} - \varphi)]X_0(\frac{\arg z}{\delta_0})(rf'(ir)'). \tag{4.32}
\]

So \( |\overline{\partial}R_{22}(z)| \) is bounded, and we can write \( |\overline{\partial}R_{22}(z)| \lesssim X_1(z)|z|^{-1/2} \). So (4.20) is obtained. In addition, for \( z \sim i \),

\[
|\overline{\partial}R_{22}(z)| \lesssim |\sin[k_0(\frac{\pi}{2} - \varphi)]| + |1 - X_0(\frac{\varphi}{\delta_0})| = O(\varphi), \tag{4.33}
\]

from which (4.20) follows immediately.

In addition, from Proposition [1], \( R^{(2)} \) achieve the symmetry:

\[
R^{(2)}(z) = \sigma_2 \overline{R^{(2)}(\bar{z})} \sigma_2 = \sigma_1 \overline{R^{(2)}(-\bar{z})} \sigma_1 = \sigma_3 Q_- R^{(2)}(-1/z) \sigma_3 Q_- \tag{4.34}
\]

We now use \( R^{(2)} \) to define the new transformation

\[
M^{(2)}(z) = M^{(1)}(z)R^{(2)}(z), \tag{4.35}
\]

which satisfies the following mixed \( \overline{\partial} \)-RH problem.

**RHP2.** Find a matrix valued function \( M^{(2)}(z; x, t) \) with following properties:

- **Analyticity:** \( M^{(2)}(z; x, t) \) is continuous in \( \mathbb{C} \), sectionally continuous first partial derivatives in \( \mathbb{C} \setminus \left( \Sigma^{(2)} \cup \{ \xi_n, \bar{\xi}_n \}_{n \in \Lambda} \right) \) and meromorphic out \( \hat{\Omega} \);
- **Symmetry:** \( M^{(2)}(z) = \sigma_2 \overline{M^{(2)}(\bar{z})} \sigma_2 = \sigma_1 \overline{M^{(2)}(-\bar{z})} \sigma_1 = \sigma_3 Q_- M^{(2)}(-1/z) \sigma_3 Q_- \);
- **Jump condition:** \( M^{(2)} \) has continuous boundary values \( M^{(2)}_\pm \) on \( \Sigma^{(2)} \) and

\[
M^{(2)}_+(z; x, t) = M^{(2)}_-(z; x, t)V^{(2)}(z), \quad z \in \Sigma^{(2)}, \tag{4.36}
\]
where

\[
V^{(2)}(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -C_n(z - \zeta_n)^{-1}T^2(z)e^{-2 it\theta_n} & 1 \end{pmatrix}, & \text{as } z \in \partial\mathbb{D}(\zeta_n, q), n \in \nabla \setminus \Lambda; \\
\begin{pmatrix} 0 & 1 \\ 1 & -C_n^{-1}(z - \zeta_n)T^{-2}(z)e^{2 it\theta_n} \end{pmatrix}, & \text{as } z \in \partial\mathbb{D}(\zeta_n, q), n \in \Delta; \\
\begin{pmatrix} 0 & 1 \\ 1 & \bar{C}_n(z - \bar{\zeta}_n)^{-1}T^{-2}(z)e^{2 it\bar{\theta}_n} \end{pmatrix}, & \text{as } z \in \partial\mathbb{D}(\bar{\zeta}_n, q), n \in \nabla \setminus \Lambda; \\
\begin{pmatrix} 1 & 0 \\ \bar{C}_n^{-1}(z - \bar{\zeta}_n)e^{-2 it\bar{\theta}_n}T^2(z) & 1 \end{pmatrix}, & \text{as } z \in \partial\mathbb{D}(\bar{\zeta}_n, q), n \in \Delta; 
\end{cases}
\]

(4.37)

▷ Asymptotic behaviors:

\[
M^{(2)}(z) = e^{i\nu - (x, t, q)_3} + O(z^{-1}), \quad z \to \infty; \quad (4.38)
\]

\[
M^{(2)}(z) = i z e^{i\nu - (x, t, q)_3} \sigma_3 Q + O(1), \quad z \to 0; \quad (4.39)
\]

▷ \(\bar{\partial}\)-Derivative: For \(z \in \mathbb{C}\) we have

\[
\bar{\partial}M^{(2)} = M^{(2)}\bar{\partial}R^{(2)}, \quad (4.40)
\]

where

\[
\bar{\partial}R^{(2)} = \begin{cases} 
\begin{pmatrix} 0 & \bar{\partial}R_j(z)e^{2 it\bar{\theta}} \\ 0 & 0 \end{pmatrix}, & \text{as } z \in \Omega_j, j = 1, 3, 5, 7; \\
\begin{pmatrix} 0 & \bar{\partial}R_j(z)e^{-2 it\bar{\theta}} \\ 0 & 0 \end{pmatrix}, & \text{as } z \in \Omega_j, j = 2, 4, 6, 8; \\
0 & \text{elsewhere};
\end{cases}
\]

(4.41)

▷ Residue conditions: \(M^{(2)}\) has simple poles at each point \(\zeta_n\) and \(\bar{\zeta}_n\) for \(n \in \Lambda\) with:

\[
\text{Res } M^{(2)}(z) = \lim_{z \to \zeta_n} M^{(2)}(z) \begin{pmatrix} 0 \\ C_n e^{-2 it\theta_n}T^2(\zeta_n) \end{pmatrix}, \quad (4.42)
\]

\[
\text{Res } M^{(2)}(z) = \lim_{z \to \bar{\zeta}_n} M^{(2)}(z) \begin{pmatrix} 0 \\ -\bar{C}_n T^{-2}(\bar{\zeta}_n)e^{2 it\bar{\theta}_n} \end{pmatrix}. \quad (4.43)
\]
5 Decomposition of the mixed $\bar{\partial}$-RH problem

To solve RHP2, we decompose it into a model RH problem for $M^{(r)}(z)$ with $\bar{\partial}R^{(2)} \equiv 0$ and a pure $\bar{\partial}$-Problem with nonzero $\bar{\partial}$-derivatives. For the first step, we establish a RH problem for the $M^{(r)}(z)$ as follows.

**RHP3.** Find a matrix-valued function $M^{(r)}(z)$ with following properties:
- Analyticity: $M^{(r)}(z)$ is meromorphic in $\mathbb{C} \setminus \Sigma^{(2)}$;
- Jump condition: $M^r$ has continuous boundary values $M^{(r)}_\pm$ on $\Sigma^{(2)}$ and
  \[ M^r_+(z) = M^r_-(z)V^2(z), \quad z \in \Sigma^{(2)}; \]  
  (5.1)
- Symmetry: $M^{(r)}(z) = \sigma_2 M^{(r)}(\bar{z})\sigma_2 = \sigma_1 M^{(r)}(-\bar{z})\sigma_1 = i\frac{z}{z}M^{(r)}(-1/z)\sigma_3 Q_-;
- $\bar{\partial}$-Derivative: $\bar{\partial}R^{(2)} = 0$, for $z \in \mathbb{C}$;
- Asymptotic behaviors:
  \[ M^{(r)}(z) = e^{i\nu-(x,t,q)\sigma_3} + \mathcal{O}(z^{-1}), \quad z \to \infty, \]  
  (5.2)
  \[ M^{(r)}(z) = \frac{i}{z} e^{i\nu-(x,t,q)\sigma_3} \sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0; \]  
  (5.3)
- Residue conditions: $M^{(r)}$ has simple poles at each point $\zeta_n$ and $\bar{\zeta}_n$ for $n \in \Lambda$ with:
  \[ \text{Res}_{z=\zeta_n} M^{(r)}(z) = \lim_{z \to \zeta_n} M^{(r)}(z) \begin{pmatrix} 0 \\ C_n e^{-2it\theta_n} T^2(\zeta_n) \\ 0 \end{pmatrix}, \]  
  (5.4)
  \[ \text{Res}_{z=\bar{\zeta}_n} M^{(r)}(z) = \lim_{z \to \bar{\zeta}_n} M^{(r)}(z) \begin{pmatrix} 0 \\ -\bar{C}_n T^{-2}(\bar{\zeta}_n) e^{2it\bar{\theta}_n} \\ 0 \end{pmatrix}. \]  
  (5.5)

The unique existence and asymptotic of $M^{(r)}(z)$ will shown in section[6]

We now use $M^{(r)}(z)$ to construct a new matrix function

\[ M^{(3)}(z) = M^{(2)}(z)M^{(r)}(z)^{-1}, \]  
(5.6)

which removes analytical component $M^{(r)}(z)$ to get a pure $\bar{\partial}$-problem.

**$\bar{\partial}$-problem4.** Find a matrix-valued function $M^{(3)}(z)$ with following properties:
- Analyticity: $M^{(3)}(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$.

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Asymptotic behavior:

\[ M^{(3)}(z) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty; \tag{5.7} \]

\[ \text{\(\overline{\partial}\)-Derivative: We have} \]

\[ \overline{\partial} M^{(3)}(z) = M^{(3)}(z)W, \quad z \in \mathbb{C}, \]

where

\[ W = M^{(r)}(z)\overline{\partial}R^{(2)}(z)M^{(r)}(z)^{-1}. \tag{5.8} \]

**Proof.** By using properties of the solutions \( M^{(2)}(z) \) and \( M^{(r)}(z) \) for RHP3 and \( \overline{\partial}\)-problem 4, the analyticity is obtained immediately. And for its Asymptotic behavior, from \( M^{(r)}(z) - 1 = (1 + z^{-2})\sigma_2 M^{(2)}(z)^T \sigma_2 \) we have

\[
\lim_{z \to 0} M^{(3)}(z) = \lim_{z \to 0} \left( z M^{(2)}(z) \sigma_2 (z M^{(r)}(z)^T) \sigma_2 \right) \frac{1}{1 + z^2} = i e^{\nu - (x,t,q)} \sigma_3 \sigma_2 (i e^{\nu - (x,t,q)} \sigma_3 \sigma_3)^T \sigma_2 = I. \tag{5.9} \]

Since \( M^{(2)}(z) \) and \( M^{(r)}(z) \) achieve same jump matrix, we have

\[
M_{-}^{(3)}(z)^{-1} M_{+}^{(3)}(z) = M_{-}^{(2)}(z)^{-1} M_{-}^{(r)}(z) M_{+}^{(r)}(z)^{-1} M_{+}^{(2)}(z)
= M_{-}^{(2)}(z)^{-1} V^{(2)}(z)^{-1} M_{+}^{(2)}(z) = I,
\]

which implies \( M^{(3)}(z) \) has no jumps and is everywhere continuous. We also can show that \( M^{(3)}(z) \) has no pole. For \( \lambda \in \{\zeta_n, \bar{\zeta}_n\}_{n \in \Lambda} \), let \( N_{\lambda} \) denote the nilpotent matrix which appears in the left side of the corresponding residue condition of RHP4 and RHP5, we have the Laurent expansions in \( z - \lambda \)

\[ M^{(2)}(z) = a(\lambda) \left[ \frac{N_{\lambda}}{z - \lambda} + I \right] + \mathcal{O}(z - \lambda), \]

\[ M^{(r)}(z) = A(\lambda) \left[ \frac{N_{\lambda}}{z - \lambda} + I \right] + \mathcal{O}(z - \lambda), \]

where \( a(\lambda) \) and \( A(\lambda) \) are the constant matrix in their respective expansions. Then

\[
M^{(3)} = \left\{ a(\lambda) \left[ \frac{N_{\lambda}}{z - \lambda} + I \right] \right\} \left\{ \left[ \frac{-N_{\lambda}}{z - \lambda} + I \right] \sigma_2 A(\lambda)^T \sigma_2 \right\} + \mathcal{O}(z - \lambda)
= \mathcal{O}(1) , \tag{5.10} \]
which implies that $M^{(3)}(z)$ has removable singularities at $\lambda$. And the $\bar{\partial}$-derivative of $M^{(3)}(z)$ come from $M^{(3)}(z)$ due to analyticity of $M^{(r)}(z)$. In addition, unlike the zero boundary case, we must check its property at $\pm i$. The symmetries of $M^{(2)}(z)$ and $M^{(r)}(z)$ imply that

$$M^{(2)}(z) = \left( \frac{\gamma}{\pm \bar{q} - \gamma} \right) + \mathcal{O}(z \mp i), \quad (5.11)$$

$$M^{(r)}(z) = \frac{\pm i}{2(z \mp i)} \left( \frac{i}{\mp \bar{q} - i} \right) + \mathcal{O}(1), \quad (5.12)$$

for two constants $\gamma$ and $\iota$. Then the singular part of $M^{(3)}(z)$ vanishes at $z = \pm i$ by simple calculation immediately.

The unique existence and asymptotic of $M^{(3)}(z)$ will shown in section [7].

6 Asymptotic of $\mathcal{N}(\Lambda)$-soliton solutions

In this section, we build a reflectionless RH problem and show that its solution can approximated with $M^{(r)}$.

First we show the existence and uniqueness of solution of the above RHP3 which is related with original RH problem 0.

**Proposition 6.** The solution $M^{(r)}(z)$ of the RH problem 3 with scattering data $\{r(z), \zeta_n, C_n\}_{n \in \Lambda}$ exists and is unique. By an explicit transformation, $M^{(r)}(z)$ is equivalent to a reflectionless solution of the original RHP0 with modified scattering data $\{0, \{\zeta_n, \hat{c}_n\}_{n \in \Lambda}\}$, where

$$\hat{c}_n(x, t) = C_n \exp \left\{ -\frac{1}{i\pi} \int_{\mathbb{R}} \log(1 - |\rho(s)|^2) \left( \frac{1}{s - \zeta_n} - \frac{1}{2s} \right) \right\}. \quad (6.1)$$

**Proof.** To transform $M^{(r)}(z)$ to the soliton-solution of RHP0, the jumps and poles need to be restored. We reverses the triangularity effected in [3.14] and [4.35]:

$$N(z) = \left( \prod_{n \in \Delta} \zeta_n \right)^{-\sigma_3} M^{(r)}(z) T^{-\sigma_3} G^{-1}(z) \left( \prod_{n \in \Delta} \frac{z - \zeta_n}{\zeta_n^{-1} z - 1} \right)^{-\sigma_3}, \quad (6.2)$$

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with $G(z)$ defined in (3.13). First we verify $N(z)$ satisfying RHP0. This transformation to $N(z)$ preserves the normalization conditions at the origin and infinity obviously. And comparing with (3.14), this transformation restore the jump on $D(\zeta_n, \varrho)$ and $D(\zeta_n, \varrho)$ to residue for $n \notin \Lambda$. As for $n \in \Lambda$, take $\zeta_n$ as an example. Substitute (5.5) into the transformation:

$$\text{Res}_{z=\zeta_n} N(z) = \left( \prod_{n \in \Delta} \zeta_n \right)^{-\sigma_3} \text{Res}_{z=\zeta_n} M^{(r)}(z) T^{-\sigma_3} G(z)^{-1} \left( \prod_{n \in \Delta} \frac{z - \zeta_n}{\zeta_n^{-1} z - 1} \right)^{-\sigma_3}$$

$$= \lim_{z \to \zeta_n} \left( \prod_{n \in \Delta} \zeta_n \right)^{-\sigma_3} M^{(r)}(z) \left( \begin{array}{cc} 0 & 0 \\ C_n e^{-2it\theta_n} T^2(\zeta_n) & 0 \end{array} \right) \left( \prod_{n \in \Delta} \frac{z - \zeta_n}{\zeta_n^{-1} z - 1} \right)^{-\sigma_3}$$

$$= \lim_{z \to \zeta_n} N(z) \left( \begin{array}{cc} 0 & 0 \\ \hat{c}_n e^{-2it\theta_n} & 0 \end{array} \right).$$

(6.3)

Its analyticity and symmetry follow from the Proposition of $M^{(r)}(z)$, $T(z)$ and $G(z)$ immediately. So $N(z)$ is solution of RHP0 with absence of reflection, whose unique exact solution exists and can be obtained as described similarly in [18]. So $M^{(r)}(z)$ unique exists.

Although $M^{(r)}(z)$ admits uniqueness and existence, we can’t give its explicit expression. The jump matrix is uniformly near identity and doesn’t contribute to the asymptotic behavior of the solution.

**Lemma 2.** The jump matrix $V^{(2)}(z)$ in (4.37) satisfies

$$\| V^{(2)} - I \|_{L^\infty(\Sigma^{(2)})} = \mathcal{O}(e^{-2\rho_0 t}),$$

(6.4)

where $\rho_0$ is defined by (3.3).

**Proof.** Take $z \in \partial \mathbb{D}(\zeta_n, \varrho)$, $n \in \nabla \setminus \Lambda$ as an example.

$$\| V^{(2)} - I \|_{L^\infty(\partial \mathbb{D}(\zeta_n, \varrho))} = |C_n (z - \zeta_n)^{-1} T^2(z) e^{-2it\theta_n}|$$

$$\lesssim \varrho^{-1} e^{-\text{Re}(2it\theta_n)} \lesssim e^{2t\text{Im}(\theta_n)} \leq e^{-2\rho_0 t}.$$  

(6.5)

The last step follows from that for $n \in \nabla \setminus \Lambda$, $\text{Im}\theta_n < 0$. 

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Corollary 1. For $1 \leq p \leq +\infty$, the jump matrix $V^{(2)}(z)$ satisfies
\[ \| V^{(2)} - I \|_{L^p(\Sigma^{(2)})} \leq K_p e^{-2\rho_0 t}, \] (6.6)
for some constant $K_p \geq 0$ depending on $p$.

This estimation of $V^{(2)}$ inspires us to consider to completely ignore the jump condition on $M^{(r)}(z)$, because there is only exponentially small error (in $t$). We decompose $M^{(r)}(z)$ as
\[ M^{(r)}(z) = E(z)M^{(r)}_\Lambda(z), \] (6.7)
where $E(z)$ is an error function, which is a solution of a small-norm RH problem and we discuss it in Section 8. $M^{(r)}_\Lambda(z)$ solves RHP3 with $V^{(2)} \equiv 0$.

RHP5. Find a matrix-valued function $M^{(r)}_\Lambda(z; x, t)$ with following properties:

- Analyticity: $M^{(r)}_\Lambda(z; x, t)$ is analytical in $\mathbb{C} \setminus \{ \zeta_n, \bar{\zeta}_n \}_{n \in \Lambda}$;
- Symmetry: $M^{(r)}_\Lambda(z) = \sigma_2 M^{(r)}_\Lambda(\bar{z}) \sigma_2 = \sigma_1 M^{(r)}_\Lambda(-\bar{z}) \sigma_1 = \frac{i}{z} M^{(r)}_\Lambda(-1/z) \sigma_3 Q_-$;
- Asymptotic behaviors:
  \[ M^{(r)}_\Lambda(z; x, t) = e^{i\nu_- (x, t; q) \sigma_3} + \mathcal{O}(z^{-1}), \quad z \to \infty; \] (6.8)
  \[ M^{(r)}_\Lambda(z; x, t) = \frac{i}{z} e^{i\nu_- (x, t; q) \sigma_3} \sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0; \] (6.9)
- Residue conditions: $M^{(r)}_\Lambda$ has simple poles at each point $\zeta_n$ and $\bar{\zeta}_n$ for $n \in \Lambda$ with:
  \[ \text{Res } M^{(r)}_\Lambda(z) = \lim_{z \to \zeta_n} M^{(r)}_\Lambda(z) \begin{pmatrix} 0 & 0 \\ C_n e^{-2it\theta_n} T^2(\zeta_n) & 0 \end{pmatrix}, \] (6.10)
  \[ \text{Res } M^{(r)}_\Lambda(z) = \lim_{z \to \bar{\zeta}_n} M^{(r)}_\Lambda(z) \begin{pmatrix} 0 & -\bar{C}_n T^{-2}(\bar{\zeta}_n) e^{2it\bar{\theta}_n} \\ 0 & 0 \end{pmatrix}. \] (6.11)

Proposition 7. The RHP5 exists an unique solution. Moreover, $M^{(r)}_\Lambda(z)$ is equivalent to a reflectionless solution of the original RHP0 with modified scattering data $\{0, \{\zeta_n, \bar{\zeta}_n\}_{n \in \Lambda}\}$ as follows:

Case I: if $\Lambda = \emptyset$, then
\[ M^{(r)}_\Lambda(z) = e^{i\nu_- (x, t; q') \sigma_3} + \frac{i}{z} e^{i\nu_- (x, t; q') \sigma_3} \sigma_3 Q_-; \] (6.12)
Case I: if $\Lambda \neq \emptyset$ with $\Lambda_1 = \{ z_{jk} \}_{k=1}^{n_1}$ and $\Lambda_2 = \{ w_{is} \}_{s=1}^{n_2}$, then

$$M^{(r)}_\Lambda(z) = e^{i\nu_-(x,t;q'_s)\sigma_3} + \frac{i}{z} e^{i\nu_-(x,t;q'_s)\sigma_3} \sigma_3 Q_-
+ \sum_{s=1}^{n_2} \left[ \left( \frac{\alpha_s}{z-w_{is}} \frac{\kappa_s}{z-w_{is}} \right) + \left( \frac{-\alpha_s}{z+w_{is}} \frac{-\kappa_s}{z+w_{is}} \right) \right]
+ \sum_{k=1}^{n_1} \left[ \left( \frac{\beta_k}{z-z_{jk}} \frac{\varsigma_k}{z-z_{jk}} \right) + \left( \frac{-\beta_k}{z+z_{jk}} \frac{-\varsigma_k}{z+z_{jk}} \right) \right]
+ \sum_{k=1}^{n_1} i \left[ \left( \frac{-q_s \beta_k}{z_{jk} z_{jk} - 1} \frac{-q_s \varsigma_k}{z_{jk} z_{jk} - 1} \right) + \left( \frac{q_s \beta_k}{z_{jk} + 1} \frac{q_s \varsigma_k}{z_{jk} + 1} \right) \right],$$

(6.13)

where $\beta_k = \beta_k(x,t)$, $\varsigma_k = \varsigma_k(x,t)$, $\alpha_s = \alpha_s(x,t)$ and $\kappa_s = \kappa_s(x,t)$ with linearly dependant equations:

$$c_j^{-1} T(z_{jk})^{-2} e^{-2i\theta(z_{jk})} \beta_k = \frac{i}{z_{jk}} e^{i\nu_-(x,t;q'_s)\sigma_3} q_- + \sum_{h=1}^{n_2} \left( \frac{-\bar{\kappa}_h}{z_{jk} - \bar{w}_{ih}} + \frac{-\bar{\kappa}_h}{z_{jk} + \bar{w}_{ih}} \right)
+ \sum_{l=1}^{n_1} \left( \frac{-\bar{\varsigma}_l}{z_{jk} - \bar{z}_{jl}} + \frac{-\bar{\varsigma}_l}{z_{jk} + \bar{z}_{jl}} - \frac{iq_- \bar{\varsigma}_l}{z_{jl} z_{jk} - 1} - \frac{iq_- \varsigma_l}{z_{jl} z_{jk} + 1} \right),$$

(6.14)

$$c_j^{-1} T(z_{jk})^{-2} e^{-2i\theta(z_{jk})} \varsigma_k = \frac{i}{z_{jk}} e^{-i\nu_-(x,t;q'_s)\sigma_3} \bar{q}_- + \sum_{h=1}^{n_2} \left( \frac{\bar{\alpha}_h}{z_{jk} - \bar{w}_{ih}} - \frac{\bar{\alpha}_h}{z_{jk} + \bar{w}_{ih}} \right)
+ \sum_{l=1}^{n_1} \left( \frac{-\bar{\beta} l}{z_{jk} - \bar{z}_{jl}} - \frac{-\bar{\beta} l}{z_{jk} + \bar{z}_{jl}} + \frac{iq_+ \beta_l}{z_{jl} z_{jk} - 1} + \frac{iq_+ \bar{\beta}_l}{z_{jl} z_{jk} + 1} \right),$$

(6.15)
and

\[
c_{i+s+N_1}^{-1} T(w_i) - 2 e^{-2i\theta(w_i)} e^{i\nu-(x,t;q^r_\Lambda)} \alpha_k = \frac{i}{w_{is}} e^{i\nu-(x,t;q^r_\Lambda)} q_- + \sum_{h=1}^{n_2} \left( \frac{\bar{\kappa}_h}{w_{is} - \bar{w}_{ih}} + \frac{\kappa_h}{w_{is} + \bar{w}_{ih}} \right) \\
+ \sum_{l=1}^{n_1} \left( \frac{\bar{\nu}_l}{w_{is} - \bar{z}_{ji}} + \frac{\nu_l}{w_{is} + \bar{z}_{ji}} - \frac{iq_-\bar{\nu}_l}{z_j w_{is} - 1} - \frac{iq_-\nu_l}{z_j w_{is} + 1} \right),
\]

(6.16)

\[
c_{i+s+N_1}^{-1} T(w_i) - 2 e^{-2i\theta(w_i)} e^{-i\nu-(x,t;q^r_\Lambda)} \kappa_k = \frac{i}{w_{is}} e^{-i\nu-(x,t;q^r_\Lambda)} q_- + \sum_{h=1}^{n_2} \left( -\frac{\bar{\alpha}_h}{w_{is} - \bar{w}_{ih}} + \frac{\alpha_h}{w_{is} + \bar{w}_{ih}} \right) \\
+ \sum_{l=1}^{n_1} \left( \frac{\bar{\beta}_l}{w_{is} - \bar{z}_{ji}} - \frac{\beta_l}{w_{is} + \bar{z}_{ji}} + \frac{iq_-\bar{\beta}_l}{z_j w_{is} - 1} - \frac{iq_-\beta_l}{z_j w_{is} + 1} \right),
\]

(6.17)

for \(k = 1, \ldots, n_1, s = 1, \ldots, n_2\) respectively.

**Proof.** The uniqueness of solution follows from the Liouville’s theorem. Case I can be simple obtain. As for Case II, the symmetries of \(M(r)(z)\) means that it admits a partial fraction expansion of following form as above. And to obtain \(\beta_k, \varsigma_k, \alpha_s\) and \(\kappa_s\), we substitute (6.13) into (6.11) and obtain four linearly dependant equations set above.

\[\square\]

**Corollary 2.** When \(\rho(s) \equiv 0\), the scattering matrices \(S(z) \equiv I\), which means \(q_- = q_+\). Denote \(q^r_\Lambda(x, t)\) is the \(N(\Lambda)\)-soliton with scattering data \(\{0, \{\zeta_n, \bar{\zeta}_n\}_{n \in \Lambda}\}\). By the reconstruction formula (2.47) and (2.48), the solution \(q^r_\Lambda(x, t)\) of (1.1) with scattering data \(\{0, \{\zeta_n, \bar{\zeta}_n\}_{n \in \Lambda}\}\) is given by:

\[
q^r_\Lambda(x, t) = e^{i\nu-(x,t;q^r_\Lambda)} \lim_{z \to \infty} z \left[ M^r_\Lambda \right]_{12}.
\]

(6.18)

Then in case I,

\[
u_{-}(x, t; q^r_\Lambda) = 0 \text{ and } q^r_\Lambda(x, t) = q_-.
\]

(6.20)
And in case II,

\[ u_\Lambda^r(x, t) = \lim_{z \to \infty} z \left[ M_\Lambda^{(r)} \right]_{12} \]

\[ = |i e^{i \nu_-(x, t; q_\Lambda^r)} q_- + 2 \sum_{s=1}^{n_2} \bar{\kappa}_k + 2 \sum_{k=1}^{n_1} (\bar{\zeta}_k - i q_- \varsigma_k)|, \quad (6.21) \]

which leads to \( \nu_-(x, t; q_\Lambda^r) = \frac{1}{2} \int_{-\infty}^{x} (|u_\Lambda^r(y, t)|^2 - 1) dy \) and

\[ q_\Lambda^r(x, t) = \lim_{z \to \infty} e^{i \nu_-(x, t; q_\Lambda^r)} z \left[ M_\Lambda^{(r)} \right]_{12} \]

\[ = e^{2i \nu_-(x, t; q_\Lambda^r)} \left( i e^{i \nu_-(x, t; q_\Lambda^r)} q_- + 2 \sum_{s=1}^{n_2} \bar{\kappa}_k + 2 \sum_{k=1}^{n_1} (\bar{\zeta}_k - i q_- \varsigma_k) \right). \quad (6.22) \]

7 The small norm RH problem for error function

In this section, we consider the error matrix-function \( E(z) \) and show that for large times, the error function \( E(z) \) solves a small norm RH problem which can be expanded asymptotically. From the definition (6.7), we can obtain a RH problem for the matrix function \( E(z) \).

**RHP6** Find a matrix-valued function \( E(z) \) with following properties:

- **Analyticity**: \( E(z) \) is analytical in \( \mathbb{C} \setminus \Sigma^{(2)} \);
- **Asymptotic behaviors**:

\[ E(z) \sim I + \mathcal{O}(z^{-1}), \quad |z| \to \infty; \quad (7.1) \]

- **Jump condition**: \( E \) has continuous boundary values \( E_\pm \) on \( \Sigma^{(2)} \) satisfying

\[ E_+(z) = E_-(z)V^E, \]

where the jump matrix \( V^E \) is given by

\[ V^E(z) = M_\Lambda^{(r)}(z)V^{(2)}(z)M_\Lambda^{(r)}(z)^{-1}. \quad (7.2) \]

Proposition 7 implies that \( M_\Lambda^{(r)}(z) \) is bound on \( \Sigma^{(2)} \). By using Lemma 2 and Corollary 1, we have the following estimates

\[ \| V^E - I \|_{L^p} \lesssim \| V^{(2)} - I \|_{L^p} = \mathcal{O}(e^{-2\rho_0 t}), \quad (7.3) \]
for $1 \leq p \leq +\infty$. This uniformly vanishing bound $\| V^E - I \|$ establishes RHP6 as a small-norm RH problem. Therefore, the existence and uniqueness of the RHP6 can shown by using a small-norm RH problem

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma(2)} \frac{(I + \eta(s))(V^E - I)}{s - z} ds,$$

(7.4)

where the $\eta \in L^2(\Sigma(2))$ is the unique solution of following equation

$$(1 - C_E)\eta = C_E(I),$$

(7.5)

here $C_E: L^2(\Sigma(2)) \rightarrow L^2(\Sigma(2))$ is a integral operator defined by

$$C_E(f)(z) = C_-(f(V^E - I)).$$

(7.6)

The Cauchy projection operator $C_-$ on $\Sigma(2)$ is

$$C_-(f)(s) = \lim_{z \rightarrow \Sigma(2)} \frac{1}{2\pi i} \int_{\Sigma(2)} \frac{f(s)}{s - z} ds.$$

(7.7)

Then by (7.2) we have

$$\| C_E \| \leq \| C_- \| \| V^E - I \|_{L^\infty} \lesssim O(e^{-2\rho_0 t}),$$

(7.8)

which means $\| C_E \| < 1$ for sufficiently large $t$, therefore $1 - C_E$ is invertible, and $\eta$ exists and is unique. Moreover,

$$\| \eta \|_{L^2(\Sigma(2))} \lesssim \frac{\| C_E \|}{1 - \| C_E \|} \lesssim O(e^{-2\rho_0 t}).$$

(7.9)

Then we have the existence and boundedness of $E(z)$. In order to reconstruct the solution $q(x, t)$ of (1.1), we need the asymptotic behavior of $E(z)$ as $z \rightarrow \infty$.

**Proposition 8.** For $E(z)$ defined in (7.4), it stratifies

$$|E(z) - I| \lesssim O(e^{-2\rho_0 t}).$$

(7.10)

As $z \rightarrow \infty$, the large $z$ expansion of $E$ is

$$E(z) = I + E_1z^{-1} + O(z^{-2}),$$

(7.11)
where
\[
E_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(2)}} (I + \eta(s)) (V^E - I) ds,
\] (7.12)
satisfying long time asymptotic behavior condition
\[
E_1 \lesssim O(e^{-2\rho_0 t}).
\] (7.13)

Proof. By combining (7.9) and (7.3), we obtain
\[
|E(z) - I| \leq |(1 - C_E)(\eta)| + |C_E(\eta)| \lesssim O(e^{-2\rho_0 t}).
\] (7.14)
As \(z \to \infty\), geometrically expanding \((s - z)^{-1}\) for \(z\) large in (7.4) leads to (7.11).
Finally for \(E_1\),
\[
|E_1| \lesssim \|V^E - I\|_{L^1} + \|\eta\|_{L^2} 2 \|V^E - I\|_{L^2} \lesssim O(e^{-2\rho_0 t}).
\] (7.15)

8 Analysis on the pure \(\bar{\partial}\)-Problem

Now we consider the asymptotics behavior of \(M^{(3)}(z)\). The \(\bar{\partial}\)-problem 4 of \(M^{(3)}(z)\) is equivalent to the integral equation
\[
M^{(3)}(z) = I + \frac{1}{\pi} \int_C \frac{M^{(3)}(s) W^{(3)}(s)}{z - s} dm(s),
\] (8.1)
where \(m(s)\) is the Lebesgue measure on the \(\mathbb{C}\). Denote \(C_z\) as the left Cauchy-Green integral operator defined by
\[
fC_z(z) = \frac{1}{\pi} \int_C \frac{f(s) W^{(3)}(s)}{z - s} dm(s).
\]
Then above equation (8.1) can be rewritten as
\[
M^{(3)}(z) = I \cdot (I - C_z)^{-1}.
\] (8.2)
The existence of operator \((I - C_z)^{-1}\) is given by the following Lemma.
Lemma 3. The norm of the integral operator $C_z$ decay to zero as $t \to \infty$:

$$\| C_z \|_{L^\infty \to L^\infty} \lesssim t^{-1/2},$$

which implies that $(I - C_z)^{-1}$ exists.

Proof. For any $f \in L^\infty$,

$$\| fC_z \|_{L^\infty} \leq \| f \|_{L^\infty} \frac{1}{\pi} \int_{C} \frac{|W^{(3)}(s)|}{|z - s|} dm(s),$$

where $W(s) = M^{(r)}(z)\bar{\partial}R^{(2)}(z)M^{(r)}(z)^{-1}$. So we only need to estimate the integral

$$\frac{1}{\pi} \int_{C} \frac{|W(s)|}{|z - s|} dm(s).$$

Since $W(s) \equiv 0$ out of $\bar{\Omega}$, we only need to focus on the estimate

$$\frac{1}{\pi} \int_{\Omega} \frac{|W(s)|}{|z - s|} dm(s).$$

Unlike the zero boundary case in [16], here $\det M^{(r)}(z) = 1 + z^{-2}$, and Proposition 8 implies that $|M^{(r)}(z)| \lesssim \sqrt{1 + |z|^{-2}}$. So

$$\frac{1}{\pi} \int_{\Omega} \frac{|W(s)|}{|z - s|} dm(s) \lesssim \frac{1}{\pi} \int_{\Omega} \frac{|\bar{\partial}R^{(2)}(s)|}{|z - s|} \frac{1 + |s|^{-2}}{|1 + s^{-2}|} dm(s). \quad (8.4)$$

Then for $j = 1, 4, 5, 8$, $|M^{(r)}(z)|$ is bounded in $\Omega_j$. But when $z \in \Omega_j$ for $j = 2, 3, 6, 7$, the singularity at $z = \pm i$ need to be treat more carefully. So in following calculation, we take $\Omega_2$ in the second case as an example, because it is more elaborate than $\Omega_j$ for $j = 1, 4, 5, 8$. Denote three sub-region of $\Omega_2$ as

$$D_1 = \mathbb{D}(0, 1 - \epsilon_0) \cap \Omega_2, \quad D_2 = \mathbb{D}(0, 1 + \epsilon_0) \setminus \mathbb{D}(0, 1 - \epsilon_0) \cap \Omega_2, \quad D_3 = \Omega_2 \setminus \mathbb{D}(0, 1 + \epsilon_0). \quad (8.5)$$

Then the integral $\int_{\Omega_2} \frac{|W(s)|}{|z - s|} dm(s)$ is divide to three part:

$$I_i = \int_{D_i} \frac{|\bar{\partial}R^{(2)}(s)|}{|z - s|} \frac{1 + |s|^{-2}}{|1 + s^{-2}|} dm(s), \quad \text{for } i = 1, 2, 3. \quad (8.6)$$
Let \( s = u + vi = re^{i\theta} \), \( z = x + yi \). In the following calculation, we will use the inequality
\[
\| s - z \|^{-1}_{L^q(\mathbb{R}^+)} = \left\{ \int_0^{+\infty} \left( \frac{(v - y)^2}{u - x} + 1 \right)^{-\frac{q}{2}} d \left( \frac{v - y}{|u - x|} \right) \right\}^{\frac{1}{q}} |u - x|^{-\frac{1}{p}} \lesssim |u - x|^{-\frac{1}{p}},
\]
(8.7)
with \( 1 \leq q < +\infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). For \( s \in D_3, |s| > 1 + \epsilon_0 \), then
\[
\frac{1 + |s|^{-2}}{|1 + s^{-2}|} \lesssim \frac{1 + |s|^2}{|s|^2 - 1} < 1 + \frac{2}{\epsilon_0^2 + 2\epsilon_0} < \infty.
\]
(8.8)
Then together with (4.14), we have
\[
I_3 \lesssim \int_{\Omega_2} \left| \tilde{\rho} R(2)(s) \right| |z - s| dm(s) = \int_{\Omega_2} \left| \tilde{\rho} R_2(s) e^{2i\theta t} \right| |z - s| dm(s).
\]
(8.9)
Moreover, by Lemma 1,
\[
|e^{2i\theta t}| \leq e^{-c \sin 2\theta F(r)^2} \leq e^{-2cu} \leq e^{-2cu}
\]
(8.10)
where \( c \) is a positive constant, and the last step follows form
\[
v \geq \max \left\{ 1 + \epsilon_0, \frac{u}{\tan \varphi} \right\} \geq 1 + \epsilon_0 > 1.
\]
Substitute (4.19) and above inequality into (8.9) and obtain:
\[
I_3 \lesssim \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |p'_2(i\rho)| \frac{|e^{-4cut}|}{|z - s|} dvdu + \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} \frac{|r|^{-1/2} e^{-2cut}}{|z - s|} dvdu
\]
\[
+ \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} \frac{\tilde{\rho} X_1(r) e^{-4cut}}{|z - s|} dvdu.
\]
By Cauchy-Schwarz inequality, the first item have
\[
\int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} \frac{|p'_2(i\rho)| e^{-4cut}}{|z - s|} dvdu \leq \int_0^{+\infty} \left\| \tilde{\rho}' \right\|_{L^2(i\mathbb{R})} \|s - z\|^{-1} \left\| L^2(\mathbb{R}^+) \right\| e^{-2cut} du
\]
\[
\leq \int_0^{+\infty} e^{-2cut} |u - x|^{-\frac{1}{2}} du \lesssim t^{-\frac{1}{2}}.
\]
(8.11)
So does the last item. Before we estimating the second item, we consider for 
\( p > 2 \),

\[
\left( \int_{u \tan \varphi}^{+\infty} |\sqrt{u^2 + V(2)}|^{-\frac{p}{2}} dv \right)^{\frac{1}{p}} = \left( \int_{u \tan \varphi}^{+\infty} |r|^{-\frac{p}{2} + 1} v^{-1} dr \right)^{\frac{1}{p}} \lesssim u^{-\frac{1}{2} + \frac{1}{p}}. \tag{8.12}
\]

Then

\[
\int_0^{+\infty} \int_{u \tan \varphi}^{+\infty} \frac{|r|^{-1/2} e^{-2cut}}{|z - s|} dv du \leq \int_0^{+\infty} \| |r| \|_{L_p^p(\frac{u}{\tan \varphi}, +\infty)} \| |s - z|^{-1} \|_{L^p(\mathbb{R}^+)} e^{-2cut} du
\]

\[
\lesssim \int_0^{+\infty} e^{-2cut} |u - x|^{-\frac{1}{2}} u^{-\frac{1}{2} + \frac{1}{p}} du \lesssim t^{-\frac{1}{2}}. \tag{8.13}
\]

Combing above inequality we final have \( I_3 \lesssim t^{-\frac{1}{2}} \). As for \( I_2 \), the singularity at \( i \) can be balanced by (4.20), and recall that \( 1 > \epsilon_0 > 0 \) with \( (1 - \epsilon_0) \cos \varphi > \frac{1}{2} \)

\[
I_2 \lesssim \int_0^2 \int_{1/2}^2 \frac{e^{-2cut}}{|z - s|} \frac{1 + |s|^2}{|s + r|} dv du \lesssim \int_0^2 \int_{1/2}^2 \frac{e^{-2cut}}{|z - s|} dv du
\]

\[
\lesssim \int_0^2 |u - x|^{-1/2} e^{-2cut} du \lesssim |t|^{-1/2}. \tag{8.14}
\]

Finally, consider \( I_1 \), similarly we have

\[
I_1 \lesssim \int_0^{1-\epsilon_0} \int_u^{1-\epsilon_0} (|p'_2(ir)| + |r|^{-1/2} + |\overline{\partial}X_1(r)|) \frac{e^{-2cut}}{|z - s|} dv du, \tag{8.15}
\]

which can be estimated same as \( I_3 \). So the proof is completed. \( \square \)

As \( z \to \infty \), \( M^{(3)}(z) \) has asymptotic expansion:

\[
M^{(3)}(z) = I - M^{(3)}_1(x,t)z + O(z^{-2}), \tag{8.16}
\]

where \( M^{(3)}_1 \) is a \( z \)-independent coefficient. The asymptotic behavior of \( M^{(3)}_1 \) given by following Proposition.

**Proposition 9.** As \( z \to \infty \), the expansion above holds with

\[
M^{(3)}_1(x,t) = \frac{1}{\pi} \int_C M^{(3)}(s) W^{(3)}(s) dm(s). \tag{8.17}
\]

There exist constants \( T_1 \), such that for all \( t > T_1 \), \( M^{(3)}_1(x,t) \) satisfies

\[
|M^{(3)}_1(x,t)| \lesssim t^{-3/4}. \tag{8.18}
\]
Proof. Lemma 3 and (8.2) implies that for large $t$, \( \| M^{(3)}(z) \|_{L^\infty} \lesssim 1 \). The proof proceeds along the same lines as the proof of above Proposition. For same reason, we only estimate the integral on $\Omega_2$. Like in the above Proposition,

\[
\frac{1}{\pi} \int_{\Omega_2} M^{(3)}(s)W^{(3)}(s)dm(s) \lesssim \frac{1}{\pi} \int_{\Omega_2} |\bar{\partial}R_2(s)e^{2it\theta}| \frac{1 + |s|^{-2}}{|1 + s^{-2}|} dm(s). \tag{8.19}
\]

Let $s = u + vi = re^{i\theta}$. And we also divide right integral of above inequality to three parts

\[
I_{i+3} = \frac{1}{\pi} \int_{D_i} |\bar{\partial}R_2(s)e^{2it\theta}| \frac{1 + |s|^{-2}}{|1 + s^{-2}|} dm(s). \tag{8.20}
\]

For $I_4$, \( \frac{1 + |s|^{-2}}{|1 + s^{-2}|} < \infty \), so

\[
I_4 \lesssim \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |p_2'(ir)| e^{-2cut} dvdu + \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |r|^{-\frac{1}{q}} e^{-2cut} dvdu
\]
\[+ \int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |\bar{\partial}X_1(r)| e^{-2cut} dvdu. \tag{8.21}
\]

Note that

\[
\left( \int_{\frac{u}{\tan \varphi}}^{+\infty} e^{-2cut} dv \right)^{\frac{1}{q}} = \left( \int_{\frac{u}{\tan \varphi}}^{+\infty} e^{-2cutq} d(2cutq) \right)^{\frac{1}{q}} (2cutq)^{-\frac{1}{q}}
\]
\[\lesssim e^{-c't^2(ut)^{-\frac{1}{q}}}, \tag{8.22}\]

where $c'$ is a positive constant. Then the first integral in (8.21) have

\[
\int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |p_2'(ir)| e^{-2cut} dvdu
\]
\[\lesssim t^{-\frac{1}{2}} \int_0^{+\infty} \| \tilde{p}' \|_{L^2(\mathbb{R})} u^{-\frac{1}{q}} e^{-c'u^2t} du \lesssim t^{-\frac{3}{4}}.
\]

The last integral can be bounded in same way. To estimate the second item, we also use Cauchy-Schwarz inequality for $4 > p > 2$ and $\frac{1}{q} + \frac{1}{p} = 1$

\[
\int_0^{+\infty} \int_{\frac{u}{\tan \varphi}}^{+\infty} |r|^{-\frac{1}{2}} e^{-2cut} dvdu \lesssim t^{-\frac{1}{2}} \int_0^{+\infty} u^{\frac{3}{2} - \frac{3}{4}} e^{-c'u^2t} du \lesssim t^{-\frac{3}{4}}. \tag{8.23}
\]
The bound for $I_4$ follows in the same manner as for $I_6$. Turning to $I_5$, we also use $|\partial R_2(z)| \lesssim |z - i|$ and obtain

$$ I_5 \lesssim \int_{D_2} \frac{e^{-2cut}}{|z - s| |s + i|} \, dm(s) \lesssim \int_{1/2}^{2} \int_{u}^{2} e^{-2cut} \, dvdu $$

$$ = \int_{1/2}^{2} (cut)^{-1} \left( e^{-2cut} - e^{-4cut} \right) \, du \lesssim t^{-1}. \quad (8.24) $$

This estimate is strong enough to obtain the result.

9  Asymptotic for the DNLS equation

Now we begin to construct the long time asymptotics of the DNLS equation (1.1). Inverting the sequence of transformations (3.14), (4.35), (5.6) and (6.7), we have

$$ M(z) = T(\infty)^{\sigma_3} M^{(3)}(z) E(z) M^{(r)}(\Lambda)(z) R^{(2)}(z)^{-1} T(z)^{-\sigma_3}. \quad (9.1) $$

To reconstruct the solution $q(x, t)$ by using (2.46), we take $z \to \infty$ out of $\bar{\Omega}$. In this case, $R^{(2)}(z) = I$. Further using Propositions 4, 8 and 9, we can obtain that

$$ M(z) = T(\infty)^{\sigma_3} \left( I + M^{(3)}(z) z^{-1} \right) E(z) M^{(r)}(\Lambda)(z) $$

$$ T(\infty)^{-\sigma_3} \left( 1 + z^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \log(1 - \rho(s) \bar{\rho}(s)) ds \right)^{-\sigma_3} + O(z^{-2}), \quad (9.2) $$

whose admits long time asymptotics

$$ M(z) = T(\infty)^{\sigma_3} M^{(r)}(\Lambda)(z) T(\infty)^{-\sigma_3} \left( 1 + z^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \log(1 - \rho(s) \bar{\rho}(s)) ds \right)^{-\sigma_3} $$

$$ + O(z^{-2}) + O(t^{-3/4}). $$

From (2.47),

$$ |m(x, t)| = | \lim_{z \to \infty} z \left[ M(z) \right]_{12} | = | T(\infty)^{-2} \left| \lim_{z \to \infty} z \left[ M^{(r)}(\Lambda)(z) \right] \right|_{12} $$

$$ + \lim_{z \to \infty} \left[ M^{(r)}(\Lambda)(z) \right]_{12} \frac{1}{2\pi i} \int_{\mathbb{R}} \log(1 - \rho(s) \bar{\rho}(s)) ds | + O(t^{-3/4}) $$

$$ = | q^{(r)}_{\Lambda}(x, t) | + O(t^{-3/4}), \quad (9.3) $$

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where \( q_{r}^{\Lambda}(x, t) \) is given in Corollary 2. Then from (2.48),

\[
q(x, t) = \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} (|m(x, t)|^2 - 1) dy \right\} m(x, t)
\]

\[
= \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} (|q_{r}^{\Lambda}(x, t)|^2 - 1) dy \right\} T(\infty)^{-2} q_{r}^{\Lambda}(x, t) + O(t^{-3/4}).
\]  

(9.4)

Therefore, we achieve main result of this paper.

**Theorem 1.** Let \( q(x, t) \) be the solution for the initial-value problem (1.1) with generic data \( u_{0}(x) \in H^{1,1}(\mathbb{R}) \) and scattering data \( \{r(z), \{\zeta_{n}, C_{n}\}_{n=1}^{4N_{1}+2N_{2}}\} \). Let \( \xi = \frac{\kappa}{t} \) with \(-3 < \xi < -1\). Denote \( q_{r}^{\Lambda}(x, t) \) be the \( N(\Lambda) \)-soliton solution corresponding to scattering data \( \{0, \{\zeta_{n}, \hat{c}_{n}\}_{n \in \Lambda}\} \) shown in Corollary 2. And \( \Lambda \) is defined in (3.2). There exist a large constant \( T_{1} = T_{1}(\xi) \), for all \( T_{1} < t \to \infty \),

\[
q(x, t) = \exp \left\{ \frac{i}{2} \int_{-\infty}^{x} (|q_{r}^{\Lambda}(x, t)|^2 - 1) dy \right\} T(\infty)^{-2} q_{r}^{\Lambda}(x, t) + O(t^{-3/4}),
\]  

(9.5)

where \( u_{r}^{\Lambda}(x, t) \) and \( T(z) \) are show in Propositions 4 and Corollary 2 respectively.

The long time asymptotic expansion (9.5) shows the soliton resolution of for the initial value problem of the the derivative nonlinear schrödinger equation, which can be characterized with an \( N(\Lambda) \)-soliton whose parameters are modulated by a sum of localized soliton-soliton interactions. Our results also show that the poles on curve soliton solutions of short-pulse equation has dominant contribution to the solution as \( t \to \infty \).

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