Some properties of graded comultiplication modules

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Abstract: Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper we will obtain some results concerning the graded comultiplication modules over a commutative graded ring.

Keywords: Graded comultiplication module, Graded multiplication module, Graded submodule

1 Introduction and preliminaries

Graded multiplication modules ($gr$-multiplication modules) over commutative graded ring have been studied by many authors extensively (see [1–7]). As a dual concept of $gr$-multiplication modules, graded comultiplication modules ($gr$-comultiplication modules) were introduced and studied by Ansari-Toroghy and Farshadifar [8]. A graded $R$-module $M$ is said to be graded multiplication module ($gr$-multiplication module) if for every graded submodule $N$ of $M$ there exists a graded ideal $I$ of $R$ such that $N = IM$ (see [3]). A graded $R$-module $M$ is said to be graded comultiplication module ($gr$-comultiplication module) if for every graded submodule $N$ of $M$ there exists a graded ideal $I$ of $R$ such that $N = (0 : M)I$, where $(0 : M)I = \{m \in M : mI = 0\}$ (see [8]). Also it was shown that $M$ is a $gr$-comultiplication module if and only if for each graded submodule $N$ of $M$, $N = (0 : M \text{ Ann}_R(N))$. Here we will study the class of graded comultiplication modules and obtain some further results which are dual to classical results on graded multiplication modules (see Section 2).

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [9] and [10] for these basic properties and more information on graded rings and modules. Let $G$ be a group with identity $e$ and $R$ be a commutative ring with identity $1_R$. Then $R$ is a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called to be homogeneous of degree $g$ where the $R_g$’s are additive subgroups of $R$ indexed by the elements $g \in G$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Also we write, $h(R) = \bigcup_{g \in G} R_g$. Moreover, $R_e$ is a subring of $R$ and $1_R \in R_e$. Let $I$ be an ideal of $R$. Then $I$ is called a graded ideal of $(R, G)$ if $I = \bigoplus_{g \in G} (I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a $G$-graded ring need not be $G$-graded.

Let $R$ be a $G$-graded ring and $M$ an $R$-module. We say that $M$ is a $G$-graded $R$-module (or graded $R$-module) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $M = \bigoplus_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_g s_h$ with...
Let \( M = \bigoplus_{g \in G} M_g \) be a graded \( R \)-module and \( N \) a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus_{g \in G} N_g \) where \( N_g = N \cap M_g \) for \( g \in G \). In this case, \( N_g \) is called the \( g \)-component of \( N \).

Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. A proper graded ideal \( I \) of \( R \) is said to be graded maximal ideal (or \( gr \)-maximal ideal) of \( R \) if \( I \) is a graded ideal of \( R \) such that \( I \subseteq J \subseteq R \), then \( I = J \) or \( J = R \).

A non-zero (resp. a proper) graded ideal \( I \) of a \( G \)-graded ring \( R \) is said to be \( gr \)-large (resp. \( gr \)-small) if for every non-zero (resp. proper) graded ideal \( J \) of \( R \), we have \( I \cap J \neq 0 \) (resp. \( I + J \neq R \)).

A graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded minimal (\( gr \)-minimal) if it is minimal in the lattice of graded submodules of \( M \).

A non-zero (resp. a proper) graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be \( gr \)-large (resp. \( gr \)-small) if for every non-zero (resp. proper) graded submodule \( L \) of \( M \), we have \( N \cap L \neq 0 \) (resp. \( L + N \neq M \)).

A graded \( R \)-module \( M \) is said to be \( gr \)-uniform (resp. \( gr \)-hollow) if each of its proper graded submodules is \( gr \)-large (resp. \( gr \)-small).

A graded \( R \)-module \( M \) is said to be \( gr \)-simple if \( 0 \) and \( M \) are its only graded submodules.

A graded \( R \)-module \( M \) is said to be \( gr \)-faithful if \( aM = 0 \) implies \( a = 0 \) for \( a \in h(R) \).

A graded \( R \)-module \( M \) is said to be graded finitely generated if there exist \( x_{g_1}, x_{g_2}, \ldots, x_{g_n} \in h(M) \) such that \( M = Rx_{g_1} + \cdots + Rx_{g_n} \).

A graded \( R \)-module \( M \) is said to be \( gr \)-Artinian if satisfies the descending chain condition for graded submodules.

A proper graded ideal \( P \) of a \( G \)-graded ring \( R \) is said to be graded prime ideal (\( gr \)-prime ideal) if whenever \( r, s \in h(R) \) with \( rs \in P \), then either \( r \in P \) or \( s \in P \) (see [11]).

A non-zero graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be a graded second (\( gr \)-second) if for each homogeneous element \( a \) of \( R \), the endomorphism of \( M \) given by multiplication by \( a \) is either surjective or zero (see [8]).

### 2 Results

The following lemma is known (see [12] and [6]), but we write it here for the sake of references.

**Lemma 2.1.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then the following hold:

(i) If \( I \) and \( J \) are graded ideals of \( R \), then \( I + J \) and \( I \cap J \) are graded ideals.

(ii) If \( N \) is a graded submodule of \( M \), \( r \in h(R) \), \( x \in h(M) \) and \( I \) is a graded ideal of \( R \), then \( Rx, IN \) and \( rN \) are graded submodules of \( M \).

(iii) If \( N \) and \( K \) are graded submodules of \( M \), then \( N + K \) and \( N \cap K \) are also graded submodules of \( M \) and \( (N :_R M) = \{ r \in R : rM \subseteq N \} \) is a graded ideal of \( R \).

(iv) Let \( \{ N_\lambda \} \) be a collection of graded submodules of \( M \). Then \( \bigcap_\lambda N_\lambda \) and \( \bigcup_\lambda N_\lambda \) are graded submodules of \( M \).

Recall that a non-zero graded \( R \)-module \( M \) over a \( G \)-graded ring \( R \) is said to be \( gr \)-prime if \( Ann_M(R) = Ann_M(K) \) for every non-zero graded submodule \( K \) of \( M \) (see [10]).

**Theorem 2.2.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. If \( M \) is a \( gr \)-comultiplication \( gr \)-prime \( R \)-module, then \( M \) is a \( gr \)-simple module.

**Proof.** Let \( K \) be a non-zero graded submodule of \( M \). Since \( M \) is \( gr \)-prime, we have \( Ann_R(K) = Ann_R(M) \). By [8, Lemma 3.5], we have \( K = M \). Therefore \( M \) is a \( gr \)-simple module. \( \square \)

Recall that a graded \( R \)-module \( M \) is called finitely \( gr \)-cogenerated if for every non-empty family of graded submodules \( K_r (r \in L) \) of \( M \) such that \( \cap_{r \in L} K_r = 0 \), there exists a finite subset \( F \subseteq L \) verifying \( \cap_{r \in F} K_r = 0 \) (see [5]).
Theorem 2.3. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. If $M$ is finitely $gr$-cogenerated $gr$-comultiplication $R$-module such that for each graded ideal $J$ of $R$ and for each collection $\{K_\alpha\}_{\alpha \in \Lambda}$ of graded submodules of $M$, we have $(\bigcap_{\alpha \in \Lambda} K_\alpha)J = \bigcap_{\alpha \in \Lambda}(K_\alpha J)$, then $M$ is gr-Artinian module.

Proof. Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a descending chain of graded submodules of $M$. Set $I = Ann_R(\bigcap_{i=1}^\infty K_i)$. By assumption, $J(\bigcap_{i=1}^\infty K_i) = (\bigcap_{i=1}^\infty JK_i)$. Hence $(\bigcap_{i=1}^\infty JK_i) = 0$. Since $M$ is finitely $gr$-cogenerated $R$-module, there exists a positive integer $j$ such that $JK_j = 0$. Since $M$ is a $gr$-comultiplication $R$-module, $K_j \subseteq (\bigcap_{i=1}^\infty K_i)$. This completes the proof because the reverse inclusion is clear.

Recall that a $G$-graded ring $R$ is said to be a $gr$-comultiplication ring if it is a $gr$-comultiplication $R$-module (see [8]).

Theorem 2.4. Let $R$ be a $gr$-comultiplication ring and $M$ a graded $R$-module.

(i) If $M$ is a $gr$-faithful $R$-module, then for each proper graded ideal $J$ of $R$, $(0 :_M J) \neq 0$ and $JM \neq M$.

(ii) If $M$ is a $gr$-faithful $gr$-comultiplication $R$-module, then for each collection $\{J_\alpha\}_{\alpha \in \Lambda}$ of graded ideals of $R$, $(0 :_M \bigcap_{\alpha \in \Lambda} J_\alpha) = \sum_{\alpha \in \Lambda}(0 :_M J_\alpha)$.

Proof. (i) Let $J$ be a proper graded ideal of $R$. Suppose that $(0 :_M J) = 0$. If $Ann_R(J)M \neq 0$, there exists $r_\lambda \in h(Ann_R(J))$ and $m_\lambda \in h(M)$ such that $r_\lambda m_\lambda \neq 0$. Since $Jr_\lambda m_\lambda = 0m_\lambda = 0, r_\lambda m_\lambda \in (0 :_M J) = 0$, which is impossible. Consequently, $Ann_R(J)M = 0$ and so $Ann_R(J) \subseteq Ann_R(M)$. Thus $Ann_R(J) = 0$. Since $R$ is a $gr$-comultiplication ring, we conclude that $J = Ann_R(Ann_R(J)) = R$, which is a contradiction. Therefore $(0 :_M J) \neq 0$. Now assume that $JM = M$. Then $Ann_R(J) = Ann_R(M) = 0$. A similar argument yields a similar contradiction and thus completes the proof.

(ii) For each $\alpha \in \Lambda$ since $M$ is $gr$-faithful it follows that $Ann_R(Ann_R(J_\alpha)M) = Ann_R(Ann_R(J_\alpha))$. Since $R$ is a $gr$-comultiplication ring, we have $Ann_R(Ann_R(J_\alpha)) = J_\alpha$. By [8, Theorem 3.8(a)], we conclude that $(0 :_M \bigcap_{\alpha \in \Lambda} J_\alpha) = (0 :_M \bigcap_{\alpha \in \Lambda} Ann_R(Ann_R(J_\alpha)) = (0 :_M \bigcap_{\alpha \in \Lambda} Ann_R(Ann_R(J_\alpha)M)) = \sum_{\alpha \in \Lambda}(0 :_M Ann_R(Ann_R(J_\alpha))) = \sum_{\alpha \in \Lambda}(0 :_M Ann_R(Ann_R(J_\alpha))).$

Theorem 2.5. Let $R$ be a graded ring and $M$ a graded comultiplication $R$-module. If every gr-prime ideal of $R$ is contained in a unique gr-maximal ideal of $R$, then every gr-second submodule of $M$ contains a unique gr-minimal submodule of $M$.

Proof. Let $N$ be a gr-second submodule of $M$. By [8, Theorem 3.9(a)], $N$ contains a gr-minimal submodule of $M$. Now, assume that $K_1$ and $K_2$ are two gr-minimal submodules of $M$, such that $K_1 \subseteq N$ and $K_2 \subseteq N$. Hence $Ann_R(N) \subseteq Ann_R(K_1)$ and $Ann_R(N) \subseteq Ann_R(K_2)$. Since $N$ is a gr-second submodule of $M$, by [8, Proposition 3.15(a)], we have $Ann_R(N)$ is a gr-prime ideal of $R$. Now, as $Ann_R(K_2)$ and $Ann_R(K_1)$ are gr-maximal ideals of $R$, by assumption we must have $Ann_R(K_1) = Ann_R(K_2)$. Thus by [8, Lemma 3.5], $K_1 = K_2$. 

Theorem 2.6. Let $R$ be a $G$-graded ring, $M$ a $gr$-comultiplication $R$-module and $(0 :_M I) \subseteq (0 :_M J)$ for some graded ideals $I$ and $J$ of $R$. If there exists a $gr$-finitely generated $gr$-multiplication submodule $N$ of $M$ such that $Ann_R(N) \subseteq I$, then $J \subseteq I$.

Proof. Let $N$ be a $gr$-finitely generated $gr$-multiplication submodule of $M$. Since $(0 :_M I) \subseteq (0 :_M J), (0 :_N I) \subseteq (0 :_N J)$. By [8, Theorem 3.7(a)], $N$ is a $gr$-comultiplication $R$-module. By [8, Theorem 3.7(c)], $JN \subseteq IN$. Since $N$ is a $gr$-finitely generated $gr$-multiplication module, $J \subseteq I + Ann_R(N) = I$ by [1, Lemma 3.9].

Lemma 2.7. Let $R$ be a $G$-graded ring, $M$ a $gr$-comultiplication $R$-module and $K$ and $N$ a graded submodules of $M$. Then $(0 :_M (K :_R N)) = Ann_R(K)N$.

Proof. Note first that $(K :_R N)Ann_R(K)N = Ann_R(K)(K :_R N)N \subseteq Ann_R(K)K = 0$. Hence $Ann_R(K)N \subseteq (0 :_M (K :_R N))$. Since $M$ is a $gr$-comultiplication module, $Ann_R(K)N = (0 :_M I)$ for
some graded ideal \( I \) of \( R \) and so \( \text{Ann}_R(K)N = 0 \). It follows that \( IN \subseteq (0 :_M \text{Ann}_R(K)) \). Since \( M \) is a \( gr \)-comultiplication module, \( (0 :_M \text{Ann}_R(K)) = K \). Thus \( I \subseteq (K : R N) \) and hence \( (0 :_M (K : R N)) \subseteq (0 :_M I) = \text{Ann}_R(K)N \). Therefore \( \text{Ann}_R(K)N = (0 :_M (K : R N)) \). \( \square \)

Let \( M \) and \( M' \) be two graded \( R \)-modules. A homomorphism of graded \( R \)-modules \( \varphi : M \to M' \) is a homomorphism of \( R \)-modules verifying \( \varphi(M_g) \subseteq M'_g \) for every \( g \in G \) (see [10]).

**Theorem 2.8.** Let \( R \) be a \( G \)-graded ring and \( M \) a \( gr \)-comultiplication \( R \)-module. Then we have the following

(i) For each graded endomorphism \( \varphi \) of \( M \), we have \( \varphi(M) = \text{Ann}_R(\ker(\varphi))M \).

(ii) If \( N \) is a \( gr \)-minimal submodule of \( M \), and \( K_1 \) and \( K_2 \) are graded submodules of \( M \) with \( K_1 \cap N = K_2 \cap N = 0 \), then \( N \cap (K_1 + K_2) = 0 \).

(iii) If \( \{N_\lambda\}_{\lambda \in \Lambda} \) is a collection of graded submodules of \( M \) such that \( \cap_{\lambda \in \Lambda} N_\lambda = 0 \) and \( I = \sum_{\lambda \in \Lambda} \text{Ann}_R(M_\lambda) \), then \( R = \text{Ann}_R(K) + I \) for every \( gr \)-finitely generated graded submodule \( K \) of \( M \).

**Proof.** (i) Let \( \varphi \) be a graded endomorphism of \( M \). Since \( \varphi(M) \cong M / \ker(\varphi) \), we have \( \text{Ann}_R(\varphi(M)) = (\ker(\varphi) :_R M) \). Since \( M \) is a \( gr \)-comultiplication module and \( \varphi(M) \) is a graded submodule of \( M \), we have \( \varphi(M) = (0 :_M \text{Ann}_R(\varphi(M))) \). Hence \( \varphi(M) = (0 :_M (\ker(\varphi) :_R M)) = \text{Ann}_R(\ker(\varphi))M \), by Lemma 2.7.

(ii) Let \( N \) be a \( gr \)-minimal submodule of \( M \) and let \( K_1, K_2 \) be two graded submodules of \( M \) such that \( N \cap K_2 = N \cap K_1 = 0 \). Since \( M \) is a \( gr \)-comultiplication module, \( K_1 = (0 :_M \text{Ann}_R(K_1)) \) and \( K_2 = (0 :_M \text{Ann}_R(K_2)) \). Now \((0 :_M \text{Ann}_R(K_1)\text{Ann}_R(K_2)) \cap N = N \) or \( (0 :_M \text{Ann}_R(K_1)\text{Ann}_R(K_2)) \cap N = 0 \) because \( N \) is a \( gr \)-minimal submodule of \( M \). In the first case, we have \( N = (0 :_N \text{Ann}_R(K_1)\text{Ann}_R(K_2)) = (0 :_N \text{Ann}_R(K_1)) \setminus \text{Ann}_R(K_2) \). It follows that \( N = (0 :_N \text{Ann}_R(K_1)) \setminus \text{Ann}_R(K_2) = (0 :_M \text{Ann}_R(K_1) \setminus N :_N \text{Ann}_R(K_2)) = N \cap (0 :_M \text{Ann}_R(K_2)) = N \cap K_2 = 0 \) which is a contradiction. In the second case, \( N \cap (0 :_M \text{Ann}_R(K_1)\text{Ann}_R(K_2)) = 0 \) and since \( (0 :_M \text{Ann}_R(K_1)\text{Ann}_R(K_2)) \cap \text{Ann}_R(K_1) \cap \text{Ann}_R(K_2) = K_1 + K_2 \), we have \( N \cap (K_1 + K_2) = 0 \).

(iii) Let \( K \) be a \( gr \)-finitely generated graded submodule of \( M \) and let \( \text{Ann}_R(K) + I \neq R \). Then there exists a \( gr \)-maximal ideal \( J \) of \( R \) such that \( \text{Ann}_R(K) + I \subseteq J \). Since \( I \subseteq J \), we have \( (0 :_M J) \subseteq (0 :_M I) \cap (\cap_{\lambda \in \Lambda} (0 :_M \text{Ann}_R(N_\lambda)) = \cap_{\lambda \in \Lambda} N_\lambda = 0 \). Thus, \( (0 :_M J) = 0 \) and hence \( (0 :_N J) = 0 \). By [1, Theorem 3.8(c)], there exists \( p \in J \) such that \( 1 - p \in \text{Ann}_R(K) \subseteq J \), a contradiction. Thus \( \text{Ann}_R(K) + I = R \). \( \square \)

Recall that a proper graded ideal \( I \) of a \( G \)-graded ring \( R \) is said to \( gr \)-irreducible if whenever \( I = I_1 \cap I_2 \) for graded ideals \( I_1 \) and \( I_2 \) of \( R \), then \( I = I_1 \) or \( I = I_2 \) (see [11]).

**Theorem 2.9.** Let \( R \) be a \( G \)-graded ring and \( M \) a \( gr \)-comultiplication \( R \)-module such that \( \text{Ann}_R(M) \) is \( gr \)-irreducible ideal of \( R \). Then \( M \) is \( gr \)-hollow module.

**Proof.** Let \( K_1 \) and \( K_2 \) be graded submodules of \( M \) with \( M = K_1 + K_2 \). Then \( \text{Ann}_R(M) = \text{Ann}_R(K_1 + K_2) = \text{Ann}_R(K_1) \cap \text{Ann}_R(K_2) \). Since \( \text{Ann}_R(M) \) is \( gr \)-irreducible ideal of \( R \), either \( \text{Ann}_R(K_1) = \text{Ann}_R(K_2) \) or \( \text{Ann}_R(M) = \text{Ann}_R(K_2) \) without loss of generality, we can suppose that \( \text{Ann}_R(M) = \text{Ann}_R(K_1) \). Since \( M \) is \( gr \)-comultiplication module, \( M = K_1 \) by [8, Lemma 3.5]. It follows that \( M \) is \( gr \)-hollow module. \( \square \)

**Theorem 2.10.** Let \( R \) be a \( G \)-graded ring and \( M \) a \( gr \)-faithful \( gr \)-comultiplication module with the property \((0 :_M I) + (0 :_M J) = (0 :_M (I \cap J)) \) for any two graded ideals \( I \) and \( J \) of \( R \). Then a graded submodule \( N \) of \( M \) is a \( gr \)-small if and only if there exists a \( gr \)-large ideal \( I \) of \( R \) such that \( N = (0 :_M I) \).

**Proof.** Suppose first that \( N \) is a \( gr \)-small submodule of \( M \). Since \( M \) is a \( gr \)-comultiplication module, there exists graded ideal \( I \) of \( R \) such that \( N = (0 :_M I) \). Suppose \( I \cap J = 0 \) for some graded ideal \( J \) of \( R \). Then \( N + (0 :_M J) = (0 :_M I) + (0 :_M J) = (0 :_M (I \cap J)) \) as \( N \) is a \( gr \)-small submodule of \( M \). \( 0 :_M J = M \) since \( N \). Since \( M \) is a \( gr \)-small submodule of \( M \), \( (0 :_M J) = M \), and hence \( J \subseteq \text{Ann}_R(M) \). Thus \( I \) is a \( gr \)-large ideal of \( R \). Conversely, suppose that \( I \) is a \( gr \)-large ideal of \( R \) such that \( N = (0 :_M I) \) and \( K \) is a graded submodule of \( M \) such that \( N + K = (0 :_M I) + K = M \). Since \( M \) is a \( gr \)-comultiplication module, there exists a graded ideal \( J \) of \( R \) such that \( K = (0 :_M J) \). Then \( (0 :_M (I \cap J)) = (0 :_M I) + (0 :_M J) = (0 :_M I) + K = M \) and so \( I \cap J \subseteq \text{Ann}_R(M) \). Since \( I \)
is gr-large ideal of $R$, $J = 0$. Hence $K = \langle 0 :_M J \rangle = M$. Therefore $N = \langle 0 :_M I \rangle$ is a gr-small submodule of $M$.

**Theorem 2.11.** Let $R$ be a $G$-graded ring and $M$ a gr-comultiplication module with the property $I = \text{Ann}_R(0 :_M I)$ for each graded ideal $I$ of $R$. Then a graded submodule $N$ of $M$ is a gr-large if and only if there exists a gr-small ideal $I$ of $R$ such that $N = \langle 0 :_M I \rangle$.

*Proof.* Suppose first that $N$ is a gr-large submodule of $M$. Since $M$ is a gr-comultiplication module, there exists a graded ideal $I$ of $R$ such that $N = \langle 0 :_M I \rangle$. Suppose $I + J = R$ for some graded ideal $J$ of $R$. Then $N \cap \langle 0 :_M J \rangle = \langle 0 :_M I \rangle \cap \langle 0 :_M J \rangle = \langle 0 :_M I + J \rangle = \langle 0 :_M R \rangle = 0$. Since $N$ is a gr-large submodule of $M$, $\langle 0 :_M J \rangle = 0$. Hence $J = \text{Ann}_R(\langle 0 :_M J \rangle) = \text{Ann}_R(0) = R$. So $I$ is a gr-small ideal of $R$. Conversely, suppose that $I$ is a gr-small ideal of $R$ such that $N = \langle 0 :_M I \rangle$. Assume $K$ is a graded submodule of $M$ such that $K \cap N = K \cap \langle 0 :_M I \rangle = 0$. Since $M$ is gr-comultiplication module, there exists a graded ideal $J$ of $R$ such that $K = \langle 0 :_M J \rangle$. It follows that $\langle 0 :_M (I + J) \rangle = \langle 0 :_M I \rangle \cap \langle 0 :_M J \rangle = K \cap \langle 0 :_M I \rangle = 0$. Then $I + J = \text{Ann}_R(\langle 0 :_M (I + J) \rangle) = \text{Ann}_R(0) = R$. Since $I$ is a gr-small ideal of $R$, $J = R$. Hence $K = \langle 0 :_M J \rangle = \langle 0 :_M R \rangle = 0$ and so $N = \langle 0 :_M I \rangle$ is a gr-large submodule of $M$. 

**Theorem 2.12.** Let $R$ be a $G$-graded ring and $M$ a gr-comultiplication module with the property $I = \text{Ann}_R(0 :_M I)$ for each graded ideal $I$ of $R$. Then $M$ is gr-uniform if and only if $R$ is gr-hollow.

*Proof.* Suppose first that $M$ is gr-uniform and $I$ a graded proper ideal of $R$ such that $I + J = R$ for some graded ideal $J$ of $R$. Then $\langle 0 :_M I \rangle \cap \langle 0 :_M J \rangle = \langle 0 :_M (I + J) \rangle = \langle 0 :_M R \rangle = 0$. Since $M$ is gr-uniform, $\langle 0 :_M I \rangle = 0$ or $\langle 0 :_M J \rangle = 0$. Then $I = \text{Ann}_R(\langle 0 :_M I \rangle) = \text{Ann}_R(0) = R$ or $J = \text{Ann}_R(\langle 0 :_M J \rangle) = \text{Ann}_R(0) = R$. Since $I$ is graded proper ideal of $R$, $J = R$. Hence $I$ is a gr-small ideal of $R$. Therefore $R$ is gr-hollow. Conversely, let $0 \neq N$ and $K$ be two graded submodules of $M$ such that $N \cap K = 0$. Since $M$ is a gr-comultiplication module, there exist graded ideals $I$ and $J$ of $R$ such that $N = \langle 0 :_M I \rangle$ and $K = \langle 0 :_M J \rangle$. Hence $\langle 0 :_M (I + J) \rangle = \langle 0 :_M I \rangle \cap \langle 0 :_M J \rangle = N \cap K = 0$. Thus $I + J = \text{Ann}_R(\langle 0 :_M (I + J) \rangle) = \text{Ann}_R(0)$ and since $R$ is gr-hollow, $I = R$ or $J = R$. Hence $I = R$ or $J = R$. It follows that $K = \langle 0 :_M J \rangle = \langle 0 :_M R \rangle = 0$ and so $N$ is gr-large submodule of $M$. Therefore $M$ is gr-uniform.

**Definition 2.13.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $L$ a graded submodule of $M$.

1. A graded homomorphism $\varphi : L \to M$ is called gr-trivial if there exists $r \in h(R)$ such that $\varphi(x) = rx$ ($x \in L$).
2. A graded module $M$ is said to be a gr-strongly-self cogenerated provided that for each graded submodule $N$ of $M$ there exists a family $\varphi_{\lambda}$ ($\lambda \in \Lambda$) of gr-trivial endomorphisms of $M$, for some index set $\Lambda$, such that $N = \bigcap_{\lambda \in \Lambda} \text{ker} \varphi_{\lambda}$.

**Theorem 2.14.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then $M$ is a gr-comultiplication module if and only if $M$ is gr-strongly-self cogenerated.

*Proof.* Suppose first that $M$ is gr-comultiplication $R$-module and $N$ a graded submodule of $M$. Since $M$ is a gr-comultiplication module, there exists a graded ideal $I$ of $R$ such that $N = \langle 0 :_M I \rangle$. For each $i \in I \cap h(R)$, let $\varphi_{i}$ denote the gr-trivial graded endomorphism of $M$ defined by $\varphi_{i}(m) = im$ ($m \in M$). Thus $N = \bigcap_{i \in I} \text{ker} \varphi_{i}$. Conversely, let $N$ be a graded submodule of $M$. By hypothesis there exists an index set $\Lambda$ and gr-trivial graded homomorphism $\varphi_{\lambda}$ ($\lambda \in \Lambda$) such that $N = \bigcap_{\lambda \in \Lambda} \text{ker} \varphi_{\lambda}$. For each $\lambda \in \Lambda$ there exists $r_{\lambda} \in h(R)$ such that $\theta_{\lambda}(m) = r_{\lambda}m$ ($m \in M$). Let $J$ denote the graded ideal $\sum_{\lambda \in \Lambda} Rr_{\lambda}$. It is easy to show that $N = \langle 0 :_M J \rangle$. Therefore $M$ is a gr-comultiplication module.
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