SOLVING NON LINEAR DIFFERENTIAL EQUATIONS BY USING A G - HOMOTOPY ANALYSIS METHOD

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Abstract. This paper proposes an efficient hybrid analytical method named the Generalized Homotopy Analysis Method (GHAM) for solving nonlinear differential equations. The proposed hybrid method consists of topology based Homotopy Analysis Method (HAM) and Laplace typed integral transform (G-transform). Compared with HAM, GHAM does not require the effort to perform repeated integration and differentiation, when it is applied to solve higher order differential equations. Compared with G-transform, GHAM serves as an effective approach to tackle higher-order nonlinear differential equations. The effectiveness of GHAM is illustrated through the analysis of numerical examples, and the obtained results are graphically depicted. Also, GHAM is effectively utilized to analyze the density of the forest cover incorporating various parameters, which include the seed reproduction, seed deposition, seed establishment rates, old trees due to aging and the coefficients of mortality due to space variables.

1. Introduction

Last few decades, a number of analytical methods have been introduced to solve nonlinear differential equations in different fields. The list of analytical methods include the Homotopy Perturbation Method (HPM), Non-Perturbation Method as well as the Homotopy Analysis Method (HAM) which provides the solutions in the form of a convergent series. Though these methods are efficient in solving nonlinear differential equations, it possess inherent limitations. The use of HPM is subject to the existence of physical parameters involved in the differential equations. The convergence region yield by the solutions is also hard to be controlled.

Liao (1992) introduced the HAM as a general analytical methodology to produce the solutions in a series form pertaining to a variety of linear as well as nonlinear equations. Liao (1997) proposed a strong solution methodology using HAM for analyzing nonlinear heat transfer problems that arise in the study of plate heating with a microwave. It is interesting to mention that, the HAM controls the convergence region and the rate of approximation of the series can also be handled effectively. In contrast to other analytical techniques, the HAM offers the
advantage of ensuring the convergence of the series solution pertaining to nonlinear problems. This is achieved by appropriately determining the convergence-control parameter $c_0$.

Liao (2004) studied the fundamental concepts related to HAM and applied to study a variety of nonlinear equations. They encompass ordinary differential equations, differential-integral equations, differential-difference equations, partial differential equations, as well as coupled equations. The HAM is effective in approximating the solution of nonlinear problems. This is achieved through determining suitable sets of base functions as well as suitable initial conditions.

In Abbasbandy (2007), a generalized Hirota-Satsuma coupled KdV equation was tackled by using HAM. This method is able to yield more accurate results when compared with HPM as well as ADM (i.e. Adomian Decomposition Method). The HAM is implemented to handle the discrete KdV differential-difference equations by Zou et al. (2007). On the other hand, the authors Nasabzadeh and Toutounian (2013), Lu and Liu (2014), Brociek et al (2016), Hariharan (2017), applied the HAM to undertake engineering related problems that involve variety of linear and nonlinear differential and integral equations.

Kim (2017) developed a Laplace typed integral transform that generalizes variety of transformation methods, e.g. Sumudu and Elzaki as well as Laplace methods. Recently, Abazari and Ganji (2011) extended two-dimensional differential transform method and their reduced form, by presenting and proving some theorems, to obtain the solution of partial differential equations (PDEs) with proportional delay. Ragab et al (2012) proposed HAM to solve the Navier-Stokes equation. Khan et al (2014) proposed HPM and VIM to solve the Navier-Stokes equation. Sakar et al (2016) applied HAM to solve fractional partial differential equations (PDEs) with proportional delay.

Delgado et al. (2016) proposed a hybrid method that yields new analytical solutions for fractional partial differential equations subject to Liouville-Caputo and Caputo-Fabrizio conditions. The devised method is based on the integration of homotopy and the Laplace transform methodology. Arshad et al. (2017) proposed a new method known as HANTM (i.e., Homotopy Analysis Natural Transform Method) to carry out the study of different types of the linear and nonlinear Fokker-Plank equations. Nemah (2020) introduced Homotopy Transforms of Analysis Method (HTAM) to solve the Navier-Stokes equation. On the other hand, Bagyalakshmi and Krishnan (2020) devised a method known as TPDTM (i.e. Tarig Projected Differential Transform Method) to analyze the heat transfer problems through fractional nonlinear partial differential equations.

Even though the HAM has many advantages, it is unavoidable to reduce the number of iterative differentiations and integrations when it is used to solve higher-order differential equations. In order to reduce the complexity, G-transform is incorporated with HAM and developed a computationally efficient hybrid method for solving linear as well as nonlinear differential equations.

This paper contributes a new and computationally efficient analytical method, namely GHAM. This new method is able to produce the solutions with rapid convergence series. GHAM is useful for tackling the computable terms in various differential equations, even though the governing equations contain high degree of nonlinear terms. The results obtained by GHAM are more accurate than the results of HAM.

The following are the merits of the proposed Generalised Homotopy Analysis Method:

- GHAM avoids multiple differentiation and integration that arise in HAM while solving higher order fractional differential equations.
- The proposed hybrid technique GHAM successfully overcomes the limitation of Laplace typed integral transform and can be effectively used to solve the complicated non linear fractional differential equations.
- The GHAM can be applied to solve various types of engineering problems and to obtain the
solutions by the suitable selection of auxiliary parameter.

2. Preliminaries
This section provides a comprehensive description of HAM and detail explanation of the Laplace Typed Integral Transform, i.e., G-transform.

2.1. Homotopy Analysis Method
The concept and principle of ‘homotopy’ was initiated by Sen (1983). Later, the HAM was suggested by Liao (2004) to obtain an exact solution of a given differential equation. The HAM establishes a continuous mapping that involves an initial approximation based on the best guess, where the rules of solution expression and coefficient ergodicity are incorporated. As a result, the procedure for tackling physical problems can be simplified.

Given the nonlinear differential equation

\[ N[u(t)] = 0, \quad t > 0 \quad (2.1) \]

where \( N' \) is the nonlinear operator, the unknown function with an independent variable \( t \) is denoted as \( u(t) \).

The zeroth order deformation must be utilized to yield the higher order deformation. At the same time, the solution vector \( u_0(t), u_1(t), \ldots, u_n(t) \) can be obtained as follows.

The \( n \)-th order deformation equation is

\[ L[u_n(t) - \chi_n u_{n-1}(t)] = hH(t)R_n(u_{n-1}, t), \quad (2.2) \]

where

\[ \chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & \text{otherwise} \end{cases} \quad (2.3) \]

and

\[ R_n(u_{n-1}, t) = \frac{1}{(n-1)!} \frac{\partial^{n-1}(N[\phi(t,s)])}{\partial s^{n-1}} |_{s=0}. \quad (2.4) \]

Given a nonlinear operator \( N \), Equation (2.4) to represent the term \( R_n(u_{n-1}, t) \). This leads us to obtain the following solution,

\[ u(t) = \sum_{k=0}^{n} u_k(t). \quad (2.5) \]

The next subsection presents the fundamental idea pertaining to the proposed HAM model for undertaking nonlinear partial differential equations.

2.2. Laplace Typed Integral Transform (G-transform)
The G-transform converts any time domain function \( u(t) \) into the associated frequency domain function \( G(\omega) \) is defined as

\[ G[u(t)] = \omega^\alpha \int_{a}^{b} e^{-t/\omega} u(t) dt. \quad (2.6) \]

where \( \alpha \) is a suitable integer, and \( \omega \) is the frequency variable to be selected. The \( n \)-th derivative property of G-transform implies

\[ G[u^n] = \frac{1}{\omega^n} G(u) - \frac{1}{\omega^{n-1}} u(0) \omega^\alpha - \frac{1}{\omega^{n-2}} u'(0) \omega^\alpha - \ldots - u^{(n-1)}(0) \omega^\alpha \quad (2.7) \]
Readers are referred to Kim(2017) for the fundamentals of G-transform along with the standard functions.

3. Generalized Homotopy Analysis method (GHAM)
A systematic procedure for solving the nonlinear partial differential equation

\[ D^\alpha v(x, t) + Rv(x, t) + Nv(x, t) = g(x, t) \]

is illustrated in Saratha et al.(2020).

In the following section, ordinary differential equation, linear and nonlinear partial differential equations are solved using GHAM. The results show an excellent agreement with those from some of the existing methods.

4. Numerical Examples
This sub-section illustrates the usefulness and computational efficiency of GHAM through the discussion of standard numerical examples.

**Example 4.1:** A fourth-order nonlinear differential equation is expressed [Abassy, 2010] as follows:

\[ \frac{d^4 u(t)}{dt^4} = 24(u(t))^5 + 40(u(t))^3 + 16u(t) \]  (4.1)

The initial conditions are

\[ u(0) = 0, \quad u_t(0) = 1, \quad u_{ttt}(0) = \frac{1}{3}. \]

The G-transform is applied to both sides of Equation (4.1). This leads to,

\[ G\left[\frac{d^4 u(t)}{dt^4}\right] = G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right]. \]  (4.2)

Using the properties of G-Transform, the following equation is obtained

\[ G[u(t)] = \omega^{\alpha+2} + \omega^{\alpha+4} + \omega^4 G\left[24(u(t))^5 + 40(u(t))^3 + 16u(t)\right]. \]  (4.3)

Applying GHAM,

\[ u_n(x, t) = \chi_n u_{n-1}(x, t) + G^{-1}[hH(x, t)R_n(u_{n-1}(x, t)) + hH(x, t)R_n(u_{n-1}(x, t))]. \]  (4.4)

where

\[ R_n(u_{n-1}) = G[u_{n-1}] - (1 - \chi_n)\omega^{\alpha+1}u(0) - (1 - \chi_n)\omega^{\alpha+2}u'(0) - (1 - \chi_n)\omega^{\alpha+3}u''(0) - (1 - \chi_n)\omega^{\alpha+4}u'''(0) + \omega^4 G[24(u_{n-1}(t))^5 + 40(u_{n-1}(t))^3 + 16u_{n-1}(t)]. \]

The above equation is solved with respect to \( n = 1, 2, 3, \ldots \), i.e.,

\[ u_0(t) = t + \frac{t^3}{3} \]

\[ u_1(t) = \frac{2}{15}ht^5 + \frac{17}{315}ht^7 + \frac{4}{189}ht^9 + \frac{2}{297}ht^{11} + \frac{19}{11583}ht^{13} + \frac{2}{7371}ht^{15} + \frac{1}{38556}ht^{17} + \frac{1}{941868}ht^{19} \]
Similarly, we can estimate $u_2, u_3, u_4, \ldots$. The series solution is expressed as

$$u(x, t) = \sum_{n=0}^{N} u_n(x, t). \quad (4.5)$$

Figure 1 illustrate the $h$-curves. It can be observed that, for $h \in [-1, 1]$, the method provides more accurate results. The solution curve $u(t)$ of equation (4.1) is depicted in Figure 2. Figure 3 shows the comparison among GHAM, ADM and IADM. It shows an excellent agreement with the other existing methods.

It is worth to mention that, as compared with ADM and Improved ADM [Abassy, 2010], GHAM does not require iterative differentiation and integration for solving higher-order differential equations.

**Example 4.2:** The Navier-Stoke’s equation is considered as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} \quad (4.6)$$

The initial condition is

$$u(x, 0) = f(x).$$

Now, $G$-transform method is applied to both sides of equation (4.6),

$$G\left[\frac{\partial u}{\partial t}\right] = G\left[\frac{\partial^2 u}{\partial x^2}\right] + G\left[\frac{1}{x} \frac{\partial u}{\partial x}\right] \quad (4.7)$$
\[ \frac{1}{\omega} G[u] - u(0) \omega^\alpha = G \left[ \frac{\partial^2 u}{\partial x^2} \right] + G \left[ \frac{1}{x} \frac{\partial u}{\partial x} \right] \]

\[ G[u] = \omega^{\alpha+1} f(x) + \omega \left[ G \left[ \frac{\partial^2 u}{\partial x^2} \right] + G \left[ \frac{1}{x} \frac{\partial u}{\partial x} \right] \right]. \]

Apply GHAM,

\[ u(x,t) = \chi_n u_{n-1}(x,t) + G^{-1}\left[h R_n(u_{n-1}, x, t)\right] \quad (4.8) \]

where

\[ R_n(u_{n-1}, x, t) = G[u(x,t)] - (1 - \chi_n) \omega^{\alpha+1} u(x,0) + \omega \left[ G \left[ \frac{\partial^2 u_{n-1}}{\partial x^2} \right] + G \left[ \frac{1}{x} \frac{\partial u_{n-1}}{\partial x} \right] \right]. \]

By solving the above equation for \( n = 1, 2, 3, \ldots \) provides

\[ u_0(x,t) = f(x) \]

\[ u_1(x,t) = \left[ f''(x) + \frac{1}{x} f'(x) \right] \frac{ht}{1!} \]

\[ u_2(x,t) = \left[ f^{IV}(x) + \frac{2x^2 f'''(x) - x f''(x) + f'(x)}{x^3} \right] \frac{h^2 t^2}{2!} \]

Similarly, we estimate \( u_3, u_4, \ldots \), and this leads to the series solution

\[ u(x,t) = \sum_{n=0}^{N} u_n(x,t). \]

Subject to the initial condition \( f(x) = x \), the following solution is obtained.

\[ u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \ldots \times (2n-3)^2}{x^{2n-1}} \frac{h^n t^n}{n!}. \quad (4.9) \]

Figure 4 and 5 represent the h-curve \( (h \in [-1,1]) \) and the solution curve of the Navier-Stoke equation (4.9) respectively. Figure 6 shows an excellent agreement with HAM (Ragab et al. 2012), HTAM (Nemah 2020) and VIM (Khan et al 2009).
Example 4.3: The generalized Burgers equation along with a proportional delay is given by

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u \left( \frac{x}{2}, \frac{t}{2} \right) \frac{\partial u(x, \frac{t}{2})}{\partial x} + \frac{1}{2} u(x,t)$$

(4.10)

The initial condition is

$$u(x,0) = 0.$$

Solving equation (4.10) for \(n = 1, 2, 3, \ldots\) leads to the following solution

$$u(x,t) = x(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots)$$

(4.11)

$$= xe^{ht}.$$

Figure 7 and 8 represent the h-curve \((h \in [-1,1])\) and the solution curve of the proportional delay equation (4.11) respectively. Figure 9 shows an excellent agreement with HPM (Sakar et al 2016) and DTM (Abazari and Ganji 2014).
Figure 7. The h-curve of Example 4.3.

Figure 8. The solution curve of Example 4.3.

Figure 9. Comparison among GHAM, HPM and DTM

Example 4.4: An initial value PDE with a proportional delay is constructed as follows

\[
\frac{\partial u}{\partial t} = u(x, t) \frac{\partial^2 u(x, \frac{t}{2})}{\partial x^2} - u(x, t)
\]

(4.12)

The corresponding initial condition is assumed as

\[
u(x, 0) = x^2.
\]

Solving equation (4.12) for \(n = 1, 2, 3, \ldots\) leads to the solution as shown below.

\[
u(x, t) = x^2 \left(1 + ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \ldots\right) = x^2 e^{ht}.
\]

(4.13)

Figure 10 and 11 depict the h-curve (\(h \in [-1, 1]\)) and the solution curve of (4.13), respectively. Figure 12 shows an excellent agreement with HPM (Sakar et al 2016) and DTM (Abazari and Ganji 2014).
From the above numerical examples, it can be observed that the solution of GHAM excellently agrees with the solutions obtained by the existing method. In addition, GHAM is also computationally efficient.

5. Application

This sub-section illustrates an application of GHAM to a reaction-diffusion model pertaining to forest boundary. Consider the following seed dynamics model,

\[ u_t = \delta \beta w - \frac{k_u^2 u^2}{1 + av^2} - fu^2 \]  
\[ v_t = fu - gv \]  
\[ w_t = \alpha v - \beta w + dw_{xx} \]  

where the density of a young class is \( u \) and the old class is \( v \); the density of airborne seed is \( w \); the coefficient of tree mortality and aging are \( g \) and \( f \) respectively; Other parameters include the rates of seed establishment (\( \delta \)), deposition (\( \beta \)), and reproduction (\( \alpha \)), as well as the mortality rate of young trees (\( \gamma(q) \)). In addition, \( d \) denotes the diffusion coefficient along with a space variable \( x \).

Using GHAM, the above model of nonlinear equations (5.1), (5.2), (5.3) are solved. The solution of the forest model is obtained using GHAM is presented below.

\[ u_0 = k_1 e^{lx} \]  
\[ v_0 = k_2 e^{mx} \]  
\[ w_0 = k_3 e^{nx} \]  
\[ u_1 = hJ_1(x)t \]
\[ v_1 = hJ_2(x)t \]
\[ w_1 = hJ_3(x)t \]

\[ u_2 = h[\delta \beta hJ_3(x) \frac{t^2}{2} - kk_1 e^{lx} h^2 J_2(x)^2 \frac{t^3}{3}] \]
\[ - hkk_2^2 e^{2mx} J_1(x) \frac{t^2}{2} + kakk_1 e^{lx} h^4 J_2(x)^4 \frac{t^5}{5} \]
\[ + kakk_2^4 e^{4mx} hJ_2(x) \frac{t^2}{2} - kakk_2^4 e^{6mx} hJ_2(x) \frac{t^7}{7} \]
\[ - kakk_2^6 e^{6mx} hJ_2(x) \frac{t^2}{2} \]
\[ + kakk_1 e^{lx} h^8 J_2(x)^8 \frac{t^9}{9} \]
\[ - kakk_2^8 e^{8mx} hJ_1(x) \frac{t^2}{2} - fh^2 J_1(x)^2 \frac{t^3}{3} \]

\[ v_2 = h^2 [fJ_1(x) - gJ_2(x)] \frac{t^2}{2} \]

\[ w_2 = h^2 [\alpha J_2(x) \frac{t^2}{2} - \beta J_3(x) \frac{t^2}{2} + d\alpha k_2 n^2 e^{nx}] \]
\[ - \frac{t^2}{2} - d\beta k_3 n^2 e^{nx} \frac{t^2}{2} + d^2 k_3 n^4 e^{nx} \frac{t^2}{2} \]

where

\[ J_1(x) = \delta \beta k_3 e^{nx} - kk_1 k_2^2 e^{lx} e^{2mx} + kakk_1 k_2^4 e^{4mx} \]
\[ - kakk_2^6 e^{6mx} + kakk_1 k_2^8 e^{8mx} - f k_1 \frac{e^{2lx}}{2} \]
\[ J_2(x) = f k_1 e^{lx} - gk_2 e^{nx} \]
\[ J_3(x) = \alpha k_2 e^{nx} - \beta k_3 e^{nx} + dk_3 n^2 e^{nx} \]

Similarly, we can estimate \( u'_i, v'_i \) and \( w'_i \) and this leads to

\[ u(x, t) = \sum_{n=0}^{N} u_n(x, t) \quad (5.4) \]
\[ v(x, t) = \sum_{n=0}^{N} v_n(x, t) \quad (5.5) \]
\[ w(x, t) = \sum_{n=0}^{N} w_n(x, t) \quad (5.6) \]

Figures 13, 16 and 19 represent the h-curve of \( u, v, w \) and Figures 14, 17 and ?? show the solution curves of \( u, v, w \) for the forest model, respectively. Figures 15, 20 and 21 show an excellent agreement with HPM (Rajasingh et al 2015).
Figure 13. The h-curve for u.

Figure 14. The solution curve for u.

Figure 15. Comparison between GHAM and HPM

Figure 16. The h-curve for v.

Figure 17. The solution curve for v.

Figure 18. Comparison between GHAM and HPM
Comparison: Rajasingh et al. (2015) obtained the approximate solutions of this forest model with the use of iterative integrations. In this paper, more accurate solutions are obtained by applying the proposed GHAM approach. Furthermore, the $h$-curves corresponding to the solution curves of $u$, $v$, $w$ are plotted, and the convergence region can be easily identified.

6. Conclusion

In this paper, a new hybrid model known as GHAM has been introduced. GHAM is useful for analyzing various higher-order nonlinear ordinary and partial differential equations. The GHAM offers a number of benefits. The need of multiple differentiations and integrations in HAM can be totally avoided while solving higher-order differential equations. The limitation of Laplace typed integral transform can be eliminated by using GHAM, and it is effective in solving complicated nonlinear differential equations. From the application perspective, GHAM is useful for analyzing various scientific problems in the engineering domain. Using the present method, the accurate solution of differential equations can be achieved by the appropriate selection of the auxiliary parameters. The usefulness of GHAM in tackling forest modelling problem is also clearly demonstrated. The exact solutions are obtained in a series form and the solution curves are graphically depicted, ascertaining the applicability of GHAM in the forest modelling case study.

In the future, one can apply GHAM to Bagley-Torvik equation, complicated Navier-Stokes equation and epidemic model etc.

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