ON SMOOTHNESS OF CARRYING SIMPLICES

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Abstract. We consider dissipative strongly competitive systems \( \dot{x}_i = x_i f_i(x) \) of ordinary differential equations. It is known that for a wide class of such systems there exists an invariant attracting hypersurface \( \Sigma \), called the carrying simplex. In this note we give an amenable condition for \( \Sigma \) to be a \( C^1 \) submanifold-with-corners. We also provide conditions, based on a recent work of M. Benaïm ([On invariant hypersurfaces of strongly monotone maps, J. Differential Equations 136 (1997), 302–319]), guaranteeing that \( \Sigma \) is of class \( C^k + 1 \).

1. Introduction

We consider systems of ordinary differential equations (ODE’s) of class (at least) \( C^1 \)
\[ \dot{x}_i = x_i f_i(x) \] on the nonnegative orthant \( C := \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n \} \), \( n \geq 3 \).

We write \( F_i(x) = x_i f_i(x) \), \( F = (F_1, \ldots, F_n) \). The symbol \( DF = [\partial F_i/\partial x_j]_{i,j=1}^n \) stands for the derivative matrix of the vector field \( F \). The local flow generated by (E) on \( C \) will be denoted by \( \phi = \{ \phi_t \} \). A subset \( B \subset C \) is invariant [resp. forward invariant] if \( \phi_t x \in B \) for all \( (t, x) \in \mathbb{R} \times B \) [resp. for all \( (t, x) \in [0, \infty) \times B \)] for which \( \phi_t x \) is defined. For \( x \in C \), \( B \subset C \) the symbols \( \omega(x), \alpha(x), \omega(B), \alpha(B) \) have their usual meanings (see e.g. Hale [3]). A point \( x \in C \) is a rest point if \( \phi_t x = x \) for each \( t \in \mathbb{R} \) (alternatively, if \( F(x) = 0 \)). An invariant subset \( B \) of a compact invariant set \( S \) is called an attractor (resp. a repeller) relative to \( S \) if there is a relative neighborhood \( U \) of \( B \) in \( S \) such that \( \omega(U) = B \) (resp. \( \alpha(U) = B \)). For an attractor \( B \) relative to \( S \), by the repeller complementary to \( B \) we understand the set \( \{ x \in S : \omega(x) \cap B = \emptyset \} \). The attractor complementary to a repeller \( R \) is defined in an analogous way.

System (E) is dissipative if there is a compact set \( B \subset C \) such that for each bounded \( D \subset C \) its \( \omega \)-limit set \( \omega(D) \) is a nonempty subset of \( B \). By standard results on global attractors (see [3]), for a dissipative system (E) there exists a compact invariant set \( \Gamma \subset C \) (the global attractor for (E)) such that \( \omega(D) \subset \Gamma \) for each bounded \( D \subset C \). Evidently, \( 0 \in \Gamma \).

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For $I \subset \{1, \ldots, n\}$ denote
\[
C_I := \{ x \in C : x_i = 0 \text{ for } i \in I \},
\]
\[
C_I^0 := \{ x \in C_I : x_j > 0 \text{ for } j \notin I \},
\]
\[
\partial C_I := C_I \setminus C_I^0.
\]

From the form of (E) it follows readily that any $C_I$, as well as $\partial C_I$ and $C_I^0$, is invariant. We denote by (E)$_I$ the restriction of system (E) to $C_I$. Instead of $C_I^0$, $\partial C_I$, we write $C^\circ$, $\partial C$. $I'$ means $\{1, \ldots, n\} \setminus I$.

If system (E) is dissipative, so are all of its subsystems (E)$_I$. For each $I \subset \{1, \ldots, n\}$, the global attractor $\Gamma_I$ for (E)$_I$ equals $\Gamma \cap C_I$.

System (E) is called strongly competitive if $(\partial f_i / \partial x_j)(x) < 0$ for each $1 \leq i, j \leq n$, $i \neq j$, $x \in C$. A strongly competitive system is called totally competitive if $(\partial f_i / \partial x_j)(x) < 0$ for $1 \leq i \leq n$, $x \in C$. Such systems describe a community of $n$ interacting species where the growth of each species inhibits the growth of any other.

Throughout the rest of the paper the standing assumption will be:

(E) is a $C^1$ dissipative strongly competitive system of ODE’s satisfying the following:

1. $\{0\}$ is a repeller relative to $\Gamma$.
2. At each rest point $x \in C \setminus \{0\}$ one has $(\partial f_i / \partial x_i)(x) < 0$ for $1 \leq i \leq n$.

The following important result was established by M. W. Hirsch ([4]).

**Proposition 1.1.** The attractor $\Sigma \subset \Gamma$ complementary to the repeller $\{0\}$ is homeomorphic via radial projection to the standard $(n - 1)$-simplex $\Delta := \{ x \in C : x_1 + \cdots + x_n = 1 \}$. Moreover, the global attractor $\Gamma$ equals the convex hull of $\Sigma \cup \{0\}$.

Following M. L. Zeeman [15], the invariant compact set $\Sigma$ is referred to as the carrying simplex for (E). In the ecological interpretation, the carrying simplex can be thought of as expressing the balance between the growth of small populations ((0) is a repeller) and the competition of large populations (dissipativity).

M. W. Hirsch in [4] asked about sufficient conditions for the carrying simplex $\Sigma$ to be of class $C^1$. The time reverse flow $\{\phi_{-t}\}_{t \geq 0}$ restricted to the invariant set $C^\circ$ is strongly monotone and its derivative flow is strongly positive (for these terms see H. L. Smith’s monograph [12]). Therefore, when (E) possesses a repeller $R \subset \Sigma \cap C^\circ$ relative to $\Sigma$ we can utilize a powerful recent result of I. Tereščák [13] on nonmonotone manifolds to conclude that the repulsion basin $B(R) := \{ x \in \Sigma^\circ : \alpha(x) \subset R \}$ is a $C^1$ hypersurface. However, even in that case Tereščák’s theorem does not apply to the whole of $\Sigma$, for the time reverse flow fails to be strongly monotone on the boundary $\partial C$. Moreover, if we assume that (E) is permanent (a natural assumption from the applied viewpoint) then there is an attractor $A$ having the whole $C^\circ$ as its attraction basin, hence its repulsion basin (relative to $\Sigma$) equals $A$. In his paper [10] the present author gave a fairly weak condition implying the $C^1$ smoothness of $\Sigma$. It was done, however, at the expense of making use (for $n \geq 5$) of Pesin’s theory of invariant measurable families of embedded manifolds, which compels one to assume that $f$ has Hölder continuous derivatives.

In this note we show that a well-known, robust, and readily testable condition (see (A)) is enough to conclude that $\Sigma$ is $C^1$. Because our proofs exploit Oseledets’ theory of Lyapunov exponents, it suffices to assume $f$ is $C^1$ to get $C^1$ smoothness.
of \( \Sigma \). Next, conditions are given, based on recent results of M. Benaim [1], for the carrying simplex to possess higher order smoothness.

I would like to thank Michel Benaim for sending me a preprint of [1].

2. Statement of main results

For \( I \subset \{1, \ldots, n\} \) put

\[
\Sigma_I := C_I \cap \Sigma, \quad \Sigma_I^c := C_I^c \cap \Sigma, \quad \partial \Sigma_I := \partial C_I \cap \Sigma.
\]

We will call \( \Sigma_I \) a \( k \)-dimensional face of \( \Sigma \), where \( k = n - 1 - \text{card} \, I \). Evidently all \( \Sigma_I \), as well as \( \Sigma_I^c \) and \( \partial \Sigma_I \), are invariant. For \( I \subset \{1, \ldots, n\} \), the face \( \Sigma_I \) is the carrying simplex for subsystem \((E)_I\). The 0-dimensional face \( \Sigma_I^c \) consists of a single rest point \( x^{(i)} = (0, \ldots, 0, x_i^{(i)}, 0, \ldots, 0) \) with \( x_i^{(i)} > 0 \) (called the \( i \)-th axial rest point).

Let \( V = \{ v = (v_1, \ldots, v_n) : v_i \in \mathbb{R} \} \) stand for the vector space of all free \( n \)-dimensional vectors (in particular, we write the tangent bundle of the orthant \( C \) as \( TC = C \times V \)). Depending on the context, \( \| \cdot \| \) may mean the Euclidean norm of a vector, or the operator norm of a matrix, associated with the Euclidean norm.

For \( I \subset \{1, \ldots, n\} \), we denote

\[
V_I := \{ v \in V : v_i = 0 \text{ for } i \in I \}.
\]

For any two points \( x, y \in C_I \), we write \( x \leq_I y \) if \( x_i \leq y_i \) for all \( i \in I' \), and \( x <_I y \) if \( x_i < y_i \) and \( x \neq y \). Moreover, \( x \ll_I y \) if \( x_i < y_i \) for all \( i \in I' \). For \( I = \emptyset \) we write simply \( \leq, <, \ll \). The reversed symbols are used in the obvious way. As each \((C_I, \leq_I)\) is a lattice, we can define, for \( I \subset \{1, \ldots, n\} \) with card \( I \leq n - 1 \)

\[
x^{[I]} := \bigvee_{i \in I'} x^{(i)},
\]

where it is easy to see that \( x^{[I]} \ll_I x^{[J]} \) for \( I \subset J \).

The following result probably belongs to the folklore in the theory of competitive systems, but I have not been able to locate its proof.

Lemma 2.1. For each \( I \subset \{1, \ldots, n\} \) with \( 1 \leq \text{card} \, I \leq n - 2 \) we have \( y \ll_I x^{[I]} \) for all \( y \in \Sigma_I \).

Proof. Suppose to the contrary that there is \( y \in \Sigma_I \) not in the \( \ll_I \) relation to \( x^{[I]} \). Assume first that \( y = x^{[I]} \), that is, \( x^{[I]} \in \Sigma_I \). For \( i \in I', j \in I', i \neq j \), we have \( x^{[I]}_j > x^{(i)}_j = 0 \). As \( f_i(x^{(i)}) = 0 \), it follows by strong competitiveness that \( f_i(x^{[I]}) < 0 \) for \( i \in I' \). Therefore we have \( F_i(x^{[I]}) = x^{[I]}_i f_i(x^{[I]}) < 0 \) for all \( i \in I' \). Consequently, \( \phi_t x^{[I]} \ll_I x^{[I]} \) for \( t > 0 \) sufficiently small. But \( \Sigma_I \) is invariant, so \( \phi_t x^{[I]} \in \Sigma_I \) for all \( t > 0 \). We have thus obtained two points in \( \Sigma_I \) related by \( \ll_I \), which contradicts Lemma 2.5 in Hirsch [4]. Assume that \( y \in \Sigma_I \) is not in the \( \leq_I \) relation to \( x^{[I]} \). Take an index \( k \) for which \( y_k > x_k^{[I]} \). Let \( J \subset \{1, \ldots, n\} \) stand for the set of those indices \( j \) for which \( y_j = 0 \). Evidently \( k \in J' \) and \( I \subset J \). We have \( y \in \Sigma_I \cap C_J^c = \Sigma \cap C_J^c \cap C_J^c = \Sigma \cap C_J^c = \Sigma_J^c \). As a consequence, \( y_j > x_j^{(k)} = 0 \) for \( j \in J', j \neq k \), and \( y_k > x_k^{[I]} = x_k^{(k)} \) (since \( k \notin I \)). But this means that \( y \gg_J x^{(k)} \). As both these points are in \( \Sigma_J \), this again is in contradiction to Lemma 2.5 in [4]. □

We say \((E)\) satisfies hypothesis \((A)\) if

For each \( 1 \leq i \leq n \) one has \( f_i(x^{[I]}) \geq 0 \).
In light of the strong competitiveness, (A) can be equivalently formulated as:

For each \( I \subset \{1, \ldots, n\} \) with \( 1 \leq \text{card } I \leq n - 1 \) one has \( f_i(x^{[I]}) \geq 0 \) for \( i \in I \).

Hypothesis (A) is well known in the literature on mathematical ecology. Consider the Lotka–Volterra competitive system

\[
\dot{x}_i = x_i(b_i - \sum_{j=1}^{n} a_{ij}x_j),
\]  

(2.1)

with \( b_i > 0, \ a_{ij} > 0 \). For (2.1) the \( i \)-th axial rest point is given by \( x^{(i)}_i = b_i/a_{ii} \). It is easy to see that (A) is now equivalent to

\[
b_i \geq \sum_{j=1}^{n} a_{ij} \frac{b_j}{a_{jj}} \quad \text{for each } 1 \leq i \leq n.
\]

We are now in a position to state our main result.

**Theorem A.** Assume that (E) satisfies (A). Then the carrying simplex \( \Sigma \) is a \( C^1 \) submanifold-with-corners neatly embedded in \( C \).

For submanifolds-with-corners their neat embeddings, see [10].

We now state some consequences of hypothesis (A). System (E) is permanent if there is \( \epsilon > 0 \) such that \( \lim \inf_{t \to \infty} \rho(\phi_t x, \partial C_{\Gamma}) \geq \epsilon \) for each \( x \in C_{\Gamma} \), where \( \rho \) stands for the Euclidean distance between a point and a set.

**Proposition 2.2.** If (A) is satisfied, then each of the subsystems \( (E)_I \) is permanent.

**Proof.** In order not to encumber our presentation with too many subscripts, we prove the assertion for \( I = \emptyset \), that is, for system (E) only. For each \( i, 1 \leq i \leq n \), we have as a result of strong competitiveness and Lemma 2.1 that \( f_i(x) > 0 \) for all \( x \in \Sigma_I \). Now take a neighborhood \( U_i \) of \( \Sigma_I \) in \( C \) of the form

\[
U_i = \{(x_1, \ldots, x_n) : 0 \leq x_i < \epsilon_i, (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \tilde{U}_i \},
\]

where \( \epsilon_i > 0 \) and a relative neighborhood \( \tilde{U}_i \) of \( \Sigma_I \) in \( C_{\Gamma} \) are so small that \( f_i(x) > 0 \) for all \( x \in U_i \). As \( \Gamma \) is the global attractor for (E) and \( \Sigma \) is the attractor relative to \( \Gamma \) complementary to \( \{0\} \), there exists a forward invariant neighborhood \( U \) of \( \Sigma \) in \( C \) with the property that \( \phi_t x \in U \) for \( x \in C \setminus \{0\} \) and sufficiently large \( t \). Also, \( U \) can be taken so small that all the sets \( \{x \in U : x_i < \epsilon_i\} \) are contained in \( U_i \). Now observe that for \( t \) so large that \( \phi_t x \) belongs to \( U \) one has

\[
\frac{d(\phi_t x)_i}{dt} = F_i(\phi_t x) = x_i f_i(\phi_t x) > 0
\]

as long as \( (\phi_t x)_i < \epsilon_i \). From this it readily follows that \( \lim \inf_{t \to \infty} \rho(\phi_t x, \Sigma_I) \geq \epsilon_i \) for any \( x \in C^\circ \).

In view of results on attractors contained in Hale [3] we have the following.

**Lemma 2.3.** Under the assumptions of Proposition 2.2, for each \( I \subset \{1, \ldots, n\} \) the invariant compact set \( \partial \Sigma_I \) is a repeller relative to \( \Sigma_{\Gamma} \).

For \( I \subset \{1, \ldots, n\} \) denote by \( A_I \) the attractor (relative to \( \Sigma_I \)) complementary to \( \partial \Sigma_I \). As \( A_I \) can be viewed as the global attractor for the semiflow \( \{\phi_t\}_{t \geq 0} \) restricted to the connected metric space \( \Sigma_I \), a result of Gobbino and Sardella (Thm. 3.1 in [2]) yields that \( A_I \) is connected.
The ecological interpretation of the property described in Proposition 2.2 is as follows. In each subcommunity none of the species goes extinct, and invasion of a proper subcommunity by others causes the populations of the previously present species to shrink due to the larger amount of competition.

Before formulating sufficient conditions for \( \Sigma \) to be of class \( C^{k+1} \) we need to introduce some notation (we follow Benaîm’s paper [1]). For \( x \in A_I, I \subset \{1, \ldots, n\} \) with \( \text{card} \ I \leq n-2 \), we denote by \( \lambda(x) \) the largest eigenvalue of the symmetrization of the matrix \((-DF^I(x))\), where \( DF^I := [\partial F_i/\partial x_j]_{(i,j) \in I' \times I'}. \) Further, \( d(x) \) stands for the square root of

\[
\min_{i,j \not\in I} \frac{\partial F_i(x)}{\partial x_j(x)} \frac{\partial F_j(x)}{\partial x_i(x)}.
\]

Put \( \lambda_I := \sup\{\lambda(x) : x \in A_I\} \) and \( d_I := \inf\{d(x) : x \in A_I\} \).

We say that \( (E) \) satisfying (A) fulfills (C) if for each \( I \) with \( 0 \leq \text{card} \ I \leq n-2 \) any one of the conditions (C1) or (C2) holds:

- (C1) \( k \sup\{\|DF^I(x)\| : x \in A_I\} < 2(k+1)d_I \).
- (C2) \( k\lambda_I < 2(k+1)d_I \).

Theorem B. Assume that a \( C^{k+1} \) system \( (E) \) satisfies (A) and (C). Then the carrying simplex \( \Sigma \) is a \( C^{k+1} \) submanifold-with-corners.

3. Proof of Theorem A

Let \( S \) be the \((n-1)\)-dimensional sphere \( \{v \in V : \|v\| = 1\} \). For a vector subspace \( W \) of \( V \) and \( 0 \leq k \leq \dim W \), the symbol \( \mathbb{G}_kW \) denotes the compact metrizable space of all \( k \)-dimensional vector subspaces of \( W \), endowed with the standard topology: for any two \( Z_1, Z_2 \in \mathbb{G}_kW \), their distance is defined as the Hausdorff distance between \( Z_1 \cap S \) and \( Z_2 \cap S \).

The linearization of \( (E) \) generates on \( TC \) a linear skew-product (local) flow \( (\phi_t x, D\phi_t(x)v) \), where \( D\phi_t_0(x)v_0 \) is the value at time \( t_0 \) of the solution of the variational equation \( \xi = DF(\phi_0 x)\xi \) with initial condition \( \xi(0) = v_0 \).

For a linear subset \( \mathcal{C} \) of the product bundle \( B \times W \), where \( B \subset \Sigma \) and \( W \) is a vector subspace of \( V \), we will denote by \( \mathcal{C}_x \) the set of all those \( v \in W \) such that \( (x,v) \in \mathcal{C} \) (in other words, \( \{x\} \times \mathcal{C}_x \) is the fiber of \( \mathcal{C} \) over \( x \)). A linear subset \( \mathcal{C} \) of \( B \times W \) is called invariant if for each \( (x,v) \in \mathcal{C} \) and each \( t \in \mathbb{R} \) one has \( (\phi_t x, D\phi_t(x)v) \in \mathcal{C} \).

Denote the set of all ergodic measures supported on a compact invariant \( B \subset \Sigma \) by \( \mathbb{M}_{\text{erg}}(B) \). The multiplicative ergodic theorem of Oseledec (see e.g. Mané [8]) assures us that if \( B \times W \) is an invariant bundle, then for each \( m \in \mathbb{M}_{\text{erg}}(B) \) there exist an invariant \( m \)-measurable set \( B_{\text{reg}} \subset B \) (the set of regular points), a collection \( \mathcal{C}_1, \ldots, \mathcal{C}_l \) of invariant linear subsets given by \( m \)-measurable maps \( B_{\text{reg}} \ni x \mapsto (\mathcal{C}_k)_x \in \mathbb{G}_dW \) (the Oseledec decomposition) and a collection \( \Lambda_1 < \cdots < \Lambda_l \) of reals (Lyapunov exponents) such that

1. \( W = \bigoplus_{k=1}^l (\mathcal{C}_k)_x \) for \( x \in B_{\text{reg}} \),
2. \[
\lim_{t \to \pm \infty} \frac{\log \|D\phi_t(x)v\|}{t} = \Lambda_k
\]

for \( 1 \leq k \leq l, x \in B_{\text{reg}} \) and \( v \in (\mathcal{C}_k)_x \).
Lemma 3.1. For each $m \in M_{\text{erg}}(\Sigma)$ there is $I = I(m) \subset \{1, \ldots, n\}$ such that the support $\text{supp} \ m$ of $m$ is contained in $A_I$.

Proof. By ergodicity of $m$ and invariance of all $\Sigma_i^n$, there is precisely one $I \subset \{1, \ldots, n\}$ such that $m(\Sigma^+_i) = 1$ and $m(\partial \Sigma_i) = 0$. Further, as points from $\Sigma_i^+ \setminus A_I$ are wandering (relative to $\Sigma$), one has $m(\Sigma_i^+ \setminus A_I) = 0$.

Fix $m \in M_{\text{erg}}(\Sigma_I)$ with $m(\Sigma_i^n) = 1$, and put $\mathcal{B} := \Sigma_I \times V_I$, $\mathcal{B}^{(i)} := \Sigma_I \times V_{I \setminus i}$, $i \in I$. Evidently, $\mathcal{B}$ is a subbundle of $\mathcal{B}^{(i)}$ of codimension one. From the structure of system (E) it follows that the bundles $\mathcal{B}$, $\mathcal{B}^{(i)}$ are invariant. Denote by $\Lambda_1 < \Lambda_2 \cdots < \Lambda_I$ the Lyapunov exponents on $\mathcal{B}$ for the ergodic measure $m$ (we will call them the internal Lyapunov exponents for $m$). Among the Lyapunov exponents on $\mathcal{B}^{(i)}$ there is one (denoted by $\lambda^{(i)}(m)$) corresponding to the measurable linear set $C^{(i)}_k \subset B^{(i)}$ such that $(C^{(i)}_k)_x \subset V_I$ for $m$-a.e. $x \in \Sigma_I^n$. We will refer to $\lambda^{(i)}(m)$ as the $i$-th external Lyapunov exponent for $m$ (this terminology is modeled on Hofbauer’s [6]).

The following result was essentially proved in the author’s paper [10] (except for terminology).

Theorem 3.2. Assume that for each $m \in M_{\text{erg}}(\partial \Sigma)$ all its external Lyapunov exponents are nonnegative. Then the following hold:

1. The carrying simplex $\Sigma$ is a $C^1$ submanifold-with-corners neatly embedded in $C$.

2. There are $\mu > 0$ and an invariant one-dimensional subbundle $S$ of $\Sigma \times V$ such that for each $m \in M_{\text{erg}}(\Sigma)$ one has
   (a) $\Sigma \times V = T\Sigma \oplus S$, where $T\Sigma$ denotes the tangent bundle of $\Sigma$, and $(C_1)_x \subset S$ for $m$-a.e. $x \in \Sigma$.
   (b) $\Lambda_1$ is internal.
   (c) $\Lambda_1 \leq -\mu$.

In the present section we make use of part 1 of Theorem 3.2 only.

In view of the above result, we need to prove only the following.

Proposition 3.3. Under the assumptions of Theorem A, for each $m \in M_{\text{erg}}(\partial \Sigma)$ all its external Lyapunov exponents are nonnegative.

Proof. Fix a measure $m \in M_{\text{erg}}(\Sigma_I)$ with $m(\Sigma_i^n) = 1$, and an index $i \in I$. By Lemma 3.1, $\text{supp} \ m \subset A_I$. Take a regular point $x \in \text{supp} \ m$ and a vector $v \in (B^{(i)})_x \setminus V_I$ such that its $i$-th coordinate $v_i$ is positive. As $(\partial F_i/\partial x_j)(\phi_t x) = 0$ for $j \neq i$, and $(\partial F_i/\partial x_i)(\phi_t x) = f_i(\phi_t x)$, it follows that the $i$-th coordinate $(D\phi_t(x)v)_i$ is the solution of the (nonautonomous) scalar linear ODE $\dot{\eta} = f_i(\phi_t x)\eta$ with initial condition $\eta(0) = v_i$. By strong competitiveness and Lemma 2.1, $f_i$ is positive on the compact invariant set $A_I \subset \Sigma_I$, hence there is $M > 0$ such that $f_i(\phi_t x) \geq M$ for all $(t, x) \in \mathbb{R} \times A_I$. The standard theory of differential inequalities yields

$$\liminf_{t \to \infty} \frac{\log(D\phi_t(x)v)_i}{t} \geq M.$$ 

$\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, therefore for all $t \in \mathbb{R}$ we have $\|D\phi_t(x)v\| \geq (D\phi_t(x)v)_i$. By regularity of $x$ we derive

$$\lim_{t \to \infty} \frac{\log\|D\phi_t(x)v\|}{t} = \lambda^{(i)}(m) \geq M > 0.$$
4. Proof of Theorem B

We begin by stating a result which is an adaptation of a theorem of M. Bena¨ım.

**Theorem 4.1.** Assume that a $C^{k+1}$, $k = 1, \ldots, \infty$, system (E) satisfies the following:

1. For each $m \in M_{\text{erg}}(\partial \Sigma)$ all external Lyapunov exponents are nonnegative.
2. There is $\eta > 0$ such that for each $m \in M_{\text{erg}}(\Sigma)$ the inequality

\begin{equation}
\Lambda_1(m) - (k + 1)\Lambda_2(m) < -\eta
\end{equation}

holds, where $\Lambda_1(m)$ and $\Lambda_2(m)$ denote respectively the smallest and the second smallest Lyapunov exponents (on $\Sigma \times V$) for $m$.

Then the carrying simplex $\Sigma$ is a $C^{k+1}$ submanifold-with-corners.

**Indication of proof.** Theorem 3.2.2 asserts that the tangent bundle $TC$ restricted to $\Sigma$ invariantly decomposes as the Whitney sum $T\Sigma \oplus S$, and for each $m \in M_{\text{erg}}(\Sigma)$ the smallest Lyapunov exponent $\Lambda_1(m) \leq -\mu < 0$ is the exponential growth rate of a vector from $S$, while any of the remaining Lyapunov exponents is the exponential growth rate of a vector tangent to $\Sigma$. This, together with (4.1), gives, with the help of Prop. 3.3 in [1] (based on a result of S. Schreiber [11]), that there are $c \geq 1$, $\alpha > 0$ and $\beta > 0$ such that

$$\|D\phi_t(x)v\| \leq ce^{-\alpha t}\|v\| \quad \text{for } t \geq 0, (x, v) \in S,$$

and

$$\frac{\|D\phi_t(x)v\|}{\|D\phi_t(x)w\|^{k+1}} \leq ce^{-\beta t} \quad \text{for } t \geq 0, x \in \Sigma, v \in (S)_x, w \in T_x\Sigma \setminus \{0\}.$$

The rest of the proof consists in applying the $C^{k+1}$ section theorem of Hirsch, Pugh and Shub [5], as in the proof of Thm. 3.4 in [1].

As a consequence of the above theorem and Proposition 3.3, we will have Theo-rem B once we prove the following.

**Proposition 4.2.** Assume that a $C^{k+1}$ system (E) satisfies (A) and (C). Then there exists $\eta > 0$ such that for each $m \in M_{\text{erg}}(\Sigma)$ the inequality (4.1) holds.

**Proof.** Take $m \in M_{\text{erg}}(\Sigma)$, and let $I \subset \{1, \ldots, n\}$ be such that $m(\Sigma_I^c) = 1$. From Lemma 3.1 we have supp $m \subset A_I$. By results contained in Sections 3 and 4 of [1], it follows that under assumption (C) there is $\eta_I > 0$ such that

$$\Lambda_1^I(m) - (k + 1)\Lambda_2^I(m) < -\eta_I$$

for all $m$ supported on $A_I$, where $\Lambda_1^I(m)$ [resp. $\Lambda_2^I(m)$] stands for the smallest [resp. second smallest] internal Lyapunov exponent for $m$. Theorem 3.2.2 gives $\Lambda_1^I(m) = \Lambda_1(m)$. Denote by $\lambda_{\text{min}}$ the smallest external Lyapunov exponent for $m$. If $\lambda_{\text{min}} \geq \Lambda_2^I(m)$, then $\Lambda_2^I(m) = \Lambda_2(m)$ and the inequality (4.1) is satisfied with $\eta_I$. Assume that $\lambda_{\text{min}} < \Lambda_2^I(m)$. Applying Theorem 3.2.2 and Proposition 3.3, we obtain $\Lambda_1(m) \leq -\mu < 0 \leq \lambda_{\text{min}} = \Lambda_2(m)$. Consequently, $\Lambda_1(m) \leq -\mu < 0 \leq (k + 1)\Lambda_2(m)$. It suffices to put

$$\eta := \min\{\mu, \eta_I : I \subset \{1, \ldots, n\}, \text{card } I \leq n - 1\}.$$

\[\square\]
5. Discussion

Remark 5.1. In formulating our results, we preferred that the assumptions be easily tractable rather than the weakest possible or that they cover a wide range of applications. In fact, they can be substantially weakened, as the following example shows.

A celebrated Lotka–Volterra system due to May and Leonard [9] has the form

\[ \begin{align*}
\dot{x}_1 &= x_1 (1 - x_1 - \alpha x_2 - \beta x_3), \\
\dot{x}_2 &= x_2 (1 - \beta x_1 - x_2 - \alpha x_3), \\
\dot{x}_3 &= x_3 (1 - \alpha x_1 - \beta x_2 - x_3),
\end{align*} \]

(5.1)

with \(0 < \beta < 1 < \alpha \) and \( \alpha + \beta > 2 \). It is easily verified that (5.1) is dissipative, totally competitive and has five rest points on \( C \): 0 (repelling), \( y \) with \( y_i = \frac{1}{1 + \alpha + \beta} \) and three axial ones \( x^{(i)} \) with \( x^{(i)}_1 = 1 \). Furthermore, \( \partial \Sigma \) is an attractor relative to \( \Sigma \) with \( \{ y \} \) as its complementary repeller (see pp. 67–68 in the book [7] by Hofbauer and Sigmund). As a consequence, \( \mathbf{M}_{\text{erg}}(\Sigma) = \{ \delta y, \delta x^{(1)}, \delta x^{(2)}, \delta x^{(3)} \} \).

The Lyapunov exponents for the Dirac delta on a rest point \( x \) are simply the real parts of the eigenvalues of \( DF(x) \). At \( y \) the smallest Lyapunov exponent is negative and the remaining ones are positive (see [7]). A simple calculation shows that at an axial rest point the unique internal Lyapunov exponent equals \(-1\) and the external exponents are \( 1 - \beta > 0 \) and \( 1 - \alpha < 0 \).

Assume now that \( \alpha < 2 \). Then for each ergodic measure on \( \Sigma \) the smallest Lyapunov exponent \(-1\) is internal. Applying ideas of the author’s earlier paper [10] we prove that the carrying simplex \( \Sigma \) for (5.1) is a \( C^1 \) submanifold-with-corners.

Remark 5.2. The systems (E) satisfying (A) [resp. (A) and (C)] are robust in the sense that if we perturb \( f \) in a neighborhood of \( \Sigma \) in the \( C^1 \) [resp. in the \( C^{k+1} \) topology], then the perturbed system possesses a carrying simplex of class \( C^1 \) [resp. \( C^{k+1} \)] (and each of its subsystems (E)$_I$ is permanent). This can be proved by reasoning similar to that in the proof of Theorem 4.1 to conclude that \( \Sigma \) is of class \( C^1 \).

Remark 5.3. In principle, results contained in Section 3 should carry over to the case where we allow \( f \) to depend periodically on \( t \), although finding an analog of (A) might be tricky (for time-periodic Lotka–Volterra strongly competitive systems, compare e.g. [14]).

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