Modified HPM for high-order linear fractional integro-differential equations of Fredholm-Volterra type

Z K Eshkuvatov, M H Khadijah and B M Taib
Faculty of Science and Technology, University Sains Islam Malaysia (USIM), Negeri Sembilan, Malaysia
E-mail: zainidin@usim.edu.my

Abstract. In this study, we consider high-order (m-th order) linear fractional integro-differential equations (FracIDEs) of Fredholm-Volterra type with boundary conditions. At first we use auxiliary function to transform nonhomogeneous boundary condition into homogeneous boundary condition and reduce FracIDEs with homogeneous boundary conditions into Fredholm-Volterra fractional integral equations (FracIEs) of the second kind. Then, modified homotopy perturbation method (HPM) is applied to solve the FracIEs. Suitable choices of unknown parameters together with two step iteration lead to the higher accurate approximate solution. Existence of inverse of fractional differentiation allows us to find the solution of original FracIDEs. Finally, two numerical example with comparisons other methods are presented to show the validity and the efficiency of the method presented.

1. Introduction

Fractional derivative is a derivative of any arbitrary order, real or complex. The first appearance of the concept of a fractional derivative is found in a letter written to Guillaume de l'Hôpital (1661 –1704, French Mathematician) by the famous mathematician Gottfried Wilhelm (von) Leibniz (1646 – 1716, German polymath and philosopher) in 30 September 1695. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Joseph Liouville (1809-1882, French Mathematician) in a paper from 1832. Leonhard Euler (1707-1783, Swiss Mathematician) made the first step in the right direction in 1729 by establishing the Gamma function. Euler published some ideas for fractional calculus using the Gamma function in a natural way

$$\Gamma(s) = \int_{0}^{\infty} t^{s-1} e^{-t} dt,$$  \hspace{1cm} (1)

which is defined for all $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, i.e. Gamma function is defined to all complex numbers except the non-positive integers (where the function has simple poles). It is easy to get that

$$\Gamma(s+1) = s\Gamma(s)$$

Therefore
\( \Gamma(1) = 1, \ \Gamma(n) = (n-1)!, \ n \in \mathbb{N} \)

On the other hand

\[
\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)} \tag{2}
\]

From (2) it follows that

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{n+1}{2}\right) = \frac{(2n-1)!}{2^n} \cdot \sqrt{\pi}.
\]

Baron Augustin-Louis Cauchy (1789 – 1857, French Mathematician and Physicist) showed that repeated application of the integral operator or \(n\)-fold integral operator of a function can be expressed with a single integral i.e.

\[
I_a^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)dt \tag{3}
\]

Detail proof of (3) can be found in [1]. The natural extension of such a definition to real order \(\alpha > 0\) is

\[
I_a^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)dt,
\]

This is called the Left Riemann-Liouville Fractional Integral of order \(\alpha\).

In the recent years, fractional integro-differential equations (FracIDEs) has taken the interest of researchers for their wide application in various fields. FracIDEs appears in many modelling problems such as economics [2], signal processing [3], control theory [4] and so on. It is known that solving FracIDEs analytically is possible most of the cases, therefore finding approximate solution of FracIDEs is the important topic in numerical analysis. There are numerous techniques that have been used to solve FracIDEs approximately for instance Adomian decomposition method [5], vibrational iteration method [6], homotopy perturbation method [7] and discrete Galerkin method [8].

In this note, we consider high-order (\(m\)th order) Fredholm-Volterra IDEs of fractional order

\[
0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p, \ m - 1 < \alpha_p \leq m, \ m, p \in \mathbb{N},
\]

\[
D_a^\alpha y(t) + \sum_{i=0}^{p-1} c_i(t) D_a^\alpha y(t) = f(t) + \lambda_1 \int_a^b K_1(t,s) y(s)ds + \lambda_2 \int_a^t K_2(t,s) y(s)ds, \tag{4}
\]

under boundary conditions

\[
\begin{cases}
 y^{(i)}(a) = a_i, \quad 0 \leq i \leq L, \\
 y^{(j)}(b) = b_j, \quad L + 1 \leq j \leq m - 1.
\end{cases} \tag{5}
\]

where the function \(c_i(t), i = 0, \ldots, p - 1\) and \(f(t)\) are continuous functions on \([a, b]\), the kernels \(K_i, i \in \{1, 2\}\) are continuous functions from \(D = \{(t,s): a \leq s \leq t \leq b\}\) to \(\mathbb{R}\), and \(D_a^\alpha\) denotes the fractional derivative in the Caputo sense.

Set,
\[ Q_{d_k}(t) = Q_{a^k}(t) = \begin{cases} \sum_{i=0}^{L} \frac{a_i}{i!} (t-a)^i, & k \leq L, \\ -\sum_{j=k+1}^{m-1} \frac{b_j}{j!} (b-t)^j, & k > L. \end{cases} \] (6)

Since \( m-1 < \alpha_p \leq m \), then \( D^{\alpha_p} (Q_{d_k}) = 0 \) it is not hard to proof that the function 
\( \tilde{y}(t) = y(t) - Q_L(t) \) satisfies the equation

\[ D^{\alpha_p} \tilde{y}(t) + \sum_{i=0}^{p-1} c_i(t) D^{\alpha_p} \tilde{y}(t) = \tilde{f}(t) + \lambda_1 \int_a^b K_1(t,s) \tilde{y}(s) ds + \lambda_2 \int_a^b K_2(t,s) \tilde{y}(s) ds, \]

\[ \tilde{f}(t) = f(t) - \sum_{i=0}^{p-1} c_i(t) D^{\alpha_p} Q(t) + \lambda_1 \int_a^b K_1(t,s) Q(s) ds + \lambda_2 \int_a^b K_2(t,s) Q(s) ds. \] (7)

with the homogeneous initial conditions

\[ \tilde{y}^{(i)}(a) = 0, \quad i = 0, \ldots, L, \quad \tilde{y}^{(i)}(b) = 0, \quad j = L+1, \ldots, m-1, \] (8)

Set \( z = D^{\alpha_p} \tilde{y} \), then because of (8) we have

\[ \tilde{y}(t) - \sum_{i=0}^{m-1} \frac{\tilde{y}^{(i)}(a)}{i!} (b-t)^i = \tilde{y}(t) - \sum_{i=1}^{m-1} \frac{\tilde{y}^{(i-1)}(a)}{(L+i)!} (b-t)^{i+j} = J^{\alpha_p} z(t) \] (9)

and \( D^{\alpha_p} z = D^{\alpha_p} \tilde{y} \). Then problem (7) due to (9) can be rewritten as

\[ z(t) = -\sum_{i=0}^{p-1} c_i(t) J^{\alpha_p} z(t) + \tilde{f}(t) + \lambda_1 \int_a^b K_1(t,s) \left( J^{\alpha_p} z(s) + l_{m-L}(t) \right) ds \]

\[ + \lambda_2 \int_a^b K_2(t,s) \left( J^{\alpha_p} z(s) + l_{m-L}(t) \right) ds, \] (10)

\[ \tilde{f}(t) = f(t) - \sum_{i=0}^{p-1} c_i(t) D^{\alpha_p} Q(t) + \lambda_1 \int_a^b K_1(t,s) Q(s) ds + \lambda_2 \int_a^b K_2(t,s) Q(s) ds, \]

where

\[ l_{m-L}(t) = \sum_{i=1}^{m-L} \frac{\tilde{y}^{(i-1)}(a)}{(L+i)!} (b-t)^{i+j}. \]

Equation (10) is Fredholm-Volterra FracIEs of the 2nd kind and it can be solved by many methods such as spectral collocation method [9], Taylor methods [10] and MHPM (in the case \( p = 1 \)) [13] so on.

In this note, we solve equation (10) by using modified HPM and using (9) together with boundary condition (8) the solution of \( \tilde{y}(t) \) can be defined. The solution of equation (1) and (2) can be found by the transformation \( y(t) = \tilde{y}(t) + Q_{d_k}(t) \).

The modified homotopy perturbation method (MHPM), proposed by Ghorbani and Saberi-Nadjafi [11] for nonlinear integral equations. They have shown that MHPM is semi-analytical method and in
most of the cases lead to analytical solution. Through examples they have shown that the method is reliable and give better results than the standard homotopy perturbation method.

The paper is organized as follows: A brief review of the theory of fractional calculus is given in Section 2. Application of modified HPM for FracIEs (10) is described in Section 3. In Section 4, we will present examples to show the efficiency of the proposed method and end up with conclusion and discussion.

2. Preliminaries

Definition 2.1 A real function \( f(x), x > 0 \) is said to be in space \( C_{\mu}, \mu \in R \) if there exists a real number \( p > \mu \), such that \( f(x) = x^p f_1(x) \) where \( f_1(x) \in C(0, \infty), \) and it is said to be in space \( C_{\mu}^n \) if \( f^{(n)} \in C_{\mu}, n \in N \).

Definition 2.2 The Riemann-Liouville fractional integral operator of order \( \alpha \) of a function \( f \in C_{\mu}, \mu \geq -1 \) is defined as

\[
\begin{align*}
J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0, \\
J^0 f(x) &= f(x)
\end{align*}
\]

For \( f \in C_{\mu}, \mu \geq -1 \) and \( \alpha, \beta \geq 0 \) with \( \gamma \geq -1 \), some properties of the operator \( J^\alpha \), which are needed here, are as follows:

(i) \( J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \)
(ii) \( J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \)
(iii) \( J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \)

Definition 2.3 The fractional derivative of \( f \) in the Caputo sense is defined as

\[
D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt,
\]

for \( f \in C_{\mu}^m, \mu \geq -1 \) and \( m-1 < \alpha \leq m \).

Lemma 2.1 [10] If the fractional derivative \( D^\alpha f(x), m-1 < \alpha \leq m \) of a function \( f(x) \) is integrable, then the following properties hold:

1) \( D^\alpha J^\alpha f(x) = f(x) \)
2) \( J^\alpha \left[D^\alpha f(x)\right] = f(x) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)}{j!} (x-a)^j \).

3. Modified Homotopy Perturbation Method (MHPM)

Let us solve Fredholm-Volterra IEs (10) by using MHPM. To illustrate the basic concepts of modified HPM for equation (10) we re-write it as operator form,

\[
Lz = f + Nz
\]
where
\[ Lz = z, \quad f = \tilde{f}, \]
\[ Nz = -\sum_{i=0}^{n-1} a_i(t)J_x^{a_i}z(t) + \lambda_i \int_a^b K_i(t,s)J_x^{a_i}z(s)ds \]
\[ + \lambda_1 \int_a^b K_1(t,s)J_x^{a_1}z(s)ds \]  
(13)

Following [11], homotopy function constructed as follows:
\[ H(v, \beta, p) = (1 - p)[L(v) - L(y_0)] + p[L(v) - f - N(v)], \quad y_0 = \sum_{j=0}^{N} \beta_j g_j(x), \]  
(14)
where \( g_j(x), j = 0, \ldots, n \) are the selective functions and \( \beta_j, j = 0, \ldots, n. \) are unknown coefficients to be determined later. In equation (14), parameter \( p \in [0,1] \) is an imbedding parameter, and hence, it is obvious that
\[ H(v, \beta, 0) = L(v) - L(y_0), \]
\[ H(v, \beta, 1) = L(v) - f - N(v). \]  
(15)

and as \( p \) increases from 0 to 1, \( H(v, \beta, p) \) varies continuously from \( L(v) - L(y_0) \) to \( L(v) - f - N(v) \).
In topology, such continuous variation is called deformation, and \( L(v) - L(y_0) \) and \( L(v) - f - N(v) \) are called homotopic. Equating equation (14) to be zero, we have
\[ L(v) = \sum_{j=0}^{N} \beta_j g_j(x) + p\left(f + N(v) - \sum_{j=0}^{N} \beta_j g_j(x)\right) \]  
(16)

In view of basic assumption of homotopy perturbation method, solution of equation (16) can be expressed as a power series
\[ v = \sum_{k=0}^{\infty} p^k v_k \]  
(17)
when \( p \to 1 \), then equation (17) corresponding to equation (16) becomes the approximate solution of equation (12) i.e.,
\[ v(x) = v_0(x) + v_1(x) + v_2(x) + \cdots \]  
(18)
The series (18) is convergent for most cases, and also the rate of convergence depends on \( N(v) \). The proof can be found in He, J. H. [12].

Substituting (17) into (16) yields:
\[ L\left(\sum_{k=0}^{\infty} p^k v_k\right) = \sum_{j=0}^{N} \beta_j g_j(x) + p\left(f + N(\sum_{k=0}^{\infty} p^k v_k) - \sum_{j=0}^{N} \beta_j g_j(x)\right), \]  
(19)
and equating the terms with having identical power of \( p \) in (19), the following sequences are obtained:
\[
p^0: v_0 = \sum_{j=0}^{N} \beta_j g_j(x),
\]
\[
p^1: v_1 = \tilde{f} - \sum_{i=0}^{n-1} a_i(t) J^{\nu_{i-j}} v_0(t) + \lambda_1 \int_a^b K_1(t, s) J^{\nu_1} v_0(s) ds - v_0,
\]
\[
p^2: v_2 = \sum_{i=0}^{n-1} a_i(t) J^{\nu_{i-j}} v_0(t) + \lambda_1 \int_a^b K_1(t, s) J^{\nu_1} v_1(s) ds
\]
\[
+ \lambda_2 \int_a^b K_2(t, s) J^{\nu_2} v_1(s) ds,
\]
\[
\vdots
\]
\[
p^k: v_k = \sum_{i=0}^{n-1} a_i(t) J^{\nu_{i-j}} v_{n-1}(t) + \lambda_1 \int_a^b K_1(t, s) J^{\nu_1} v_{n-1}(s) ds
\]
\[
+ \lambda_2 \int_a^b K_2(t, s) J^{\nu_2} v_{n-1}(s) ds \quad \text{for} \quad k \geq 2.
\]

In MHPM, first iteration \(v_1\) is forced to be equal to zero to find the unknown values of \(\beta_j, j = 0, 1, 2, \ldots, N\). If \(v_1 = 0\) then, \(v_2 = v_3 = \cdots = 0\) and the solution will be obtained as

\[
v(x) = v_0(x)
\]

The unknowns are found by imposing boundary conditions.

**Remark:** In fractional equations where non integer differentiation involve, forcing \(v_1 = 0\) does not give unique solution and hence some terms of \(v_1 = 0\) are not equal to zero. Fortunately these nonzero terms impact to the next solution very small, therefore the approximate solution of the problem is searched as two steps i.e.

\[
v(x) = v_0(x) + v_1(x)
\]

Equation (22) is used as approximate solution.

4. **Numerical Example**
In this section, we have applied modified homotopy perturbation method to the fourth order linear fractional integro-differential equation with a known exact solution at \(\alpha = 4\).

**Remark:** In the examples, the following notations are used

\[
E_1(x) = \max_{0 \leq x \leq 1} \left| \text{exact solution} - v(x) \right|_{x=3.25}
\]
\[
E_2(x) = \max_{0 \leq x \leq 1} \left| \text{exact solution} - v(x) \right|_{x=3.5}
\]
\[
E_3(x) = \max_{0 \leq x \leq 1} \left| \text{exact solution} - v(x) \right|_{x=3.75}
\]
\[
E(x) = \max_{0 \leq x \leq 1} \left| \text{exact solution} - v(x) \right|_{x=4}
\]
Example 1: Consider the following linear fourth-order linear fractional integro differential equation:

\[ D^\alpha_y(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4, \quad (22) \]

Subject to the following boundary conditions:

\[
\begin{align*}
    y(0) &= 1,
    y''(0) &= 2, \\
    y(1) &= 1 + e, \quad y''(1) = 3e.
\end{align*}
\]

For \( \alpha = 4 \), the exact solution of problem (22)-(23) is

\[ y(x) = 1 + xe^x \]

Firstly, equation (22) is reduced into integral equation (11). In the reduction two unknowns \( A, B \) are appeared and these unknowns are found from boundary conditions (23). Table 1 shows the results of \( A, B \) for different value of \( \alpha \).

| \( \alpha \) | \( A \)       | \( B \)       |
|-------------|--------------|--------------|
| 3.25        | 0.9910555316 | 1.377185311  |
| 3.5         | 0.9855569381 | 2.306799253  |
| 3.75        | 0.9580481262 | 3.118814576  |
| 4           | 0.9200006577 | 3.806600712  |

Table 2. Comparisons with two methods for \( \alpha = 4 \).

| \( x \) | Exact sol | MHPM | MHAM [14] | Err_MHPM | Err_MHAM |
|---------|-----------|------|-----------|----------|----------|
| 0       | 1         | 1    | 1         | 0        | 0        |
| 0.1     | 1.110517092 | 1.102647336 | 1.107047479 | 0.0078698 | 0.003469613 |
| 0.2     | 1.244280552 | 1.229286556 | 1.237721115 | 0.0149940 | 0.006559437 |
| 0.3     | 1.404957642 | 1.384226779 | 1.395985741 | 0.0207309 | 0.008971901 |
| 0.4     | 1.596729879 | 1.572164823 | 1.586238703 | 0.0245651 | 0.010491177 |
| 0.5     | 1.824360635 | 1.798234919 | 1.813384918 | 0.0261257 | 0.010975718 |
| 0.6     | 2.093271280 | 2.068063429 | 2.082918367 | 0.0252079 | 0.010352913 |
| 0.7     | 2.409626895 | 2.387829229 | 2.401010221 | 0.0217977 | 0.008616674 |
| 0.8     | 2.780432743 | 2.784330528 | 2.774603867 | 0.0038977 | 0.005828876 |
| 0.9     | 3.213642800 | 3.205058944 | 3.211517152 | 0.0085839 | 0.002125648 |
| 1       | 3.718281828 | 3.718281829 | 3.718552210 | 5.410E-8  | 0.000227037 |
Table 2 shows the comparison of the proposed method with the HAM proposed in [14] for $\alpha = 4$. From this it follows that proposed method is comparable with other method. It is a god agreement with the exact solution. Figure 4.1 shows that the proposed method approaches to exact solution when value of $\alpha$ increases. It can be shown that the approximate solution can be more accurate with more number of terms.

5. Conclusion

In this work, we have used modified HPM to solve Fredholm-Volterra IDE of order $n-1<\alpha_p \leq n$ on the interval $[a, b]$. Auxiliary function is used to reduce nonhomogeneous Fredholm-Volterra IDEs (4)-(5) into homogeneous Fredholm-Volterra IDEs (7)-(8). Modified HPM is applied for Fredholm-Volterra IEs (10). When FracIEs is solved by modified HPM due to existence of inverse differential operator, we are able to find approximate solution of equation (4)-(5). As a numerical solution Table 4.1 shows that the proposed method is effective, comparable and reliable. Figure 4.1 shows that the proposed method approaches to exact solution when the value of $\alpha$ increases. All numerical calculations are made in Maple 17.

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References

[1] Kanwal R P 1971 Linear Integral Equations Theory and Technique (Academic Press, INC London, LTD)
[2] Baillie R T 1996 Journal of Econometrics 73(1) 5-59
[3] Panda R and Dash M 2006 Signal Processing 86(9) 2340-2350
[4] Bohannan G W 2008 Journal of Vibration and Control 14(9-10) 1487-1498
[5] Momani S and Qaralleh R 2006 Computers & Mathematics with Application 52(3) 459-470
[6] Sayevand K 2015 Applied Mathematical Modelling 39(15) 4330-4336
[7] Elbeleze A A, Kilicman A, Taib B M 2016 Journal of King Saud University-Science 28(1) 61-68
[8] Mokhtary P 2016 Acta Mathematica Scientia 36(2) 560-578
[9] Ma X and Huang C 2014 Applied Mathematical Modelling 38(4) 1434-1448
[10] Huang L, Li X F, Zhao Y and Duan X Y 2011 Computers and Mathematics with Applications (CAMWA) 62 1127–1134
[11] Ghorbani A and Saberi-Nadjafi J 2008 Computers & Mathematics with Applications (CAMWA) 56(4) 1032-1039
[12] He J H 1999 *Computer Methods In Applied Mechanics And Engineering* **178**(3) 257-262
[13] Hamson K M, Eshkuvatov Z K and Taib B M 2017 *Proceeding of Postgraduate Seminar on Science and Technology (KOSIST2016), Universiti Sains Islam Malaysia* 91-106
[14] Zhang X, Tang B and He Y 2011 *Computers and Mathematics with Applications* **62** 3194–3203