Quantum general invariance and loop gravity

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Abstract

A quantum physical projector is proposed for generally covariant theories which are derivable from a Lagrangian. The projector is the quantum analogue of the integral over the generators of finite one-parameter subgroups of the gauge symmetry transformations which are connected to the identity. Gauge variables are retained in this formalism, thus permitting the construction of spacetime area and volume operators in a tentative spacetime loop formulation of quantum general relativity.

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I. INTRODUCTION

In a recent series of papers I and my collaborators Josep Pons and Larry Shepley have analyzed the classical gauge symmetries in the phase space (cotangent bundle) formulation of a wide class of generally covariant dynamical systems which are derivable from a Lagrangian [1–4].

Our point of departure is the requirement that symmetries present in the configuration-velocity (tangent bundle) approach must be projectable onto the cotangent bundle. This view was suggested to us by pioneering work of Lee and Wald [5] which with some modifications provides a theoretical framework for symmetry explorations which were initiated by Bergmann and Komar [6], and extended by myself and Sundermeyer [7,8]. The outcome of our analysis is that in general a projectable gauge symmetry group exists, and it is a transformation group on the configuration-velocity variables. The group transformations arise from spacetime diffeomorphisms which contain a compulsory dependence on the lapse and shift gauge functions. For those dynamical models which possess additional symmetries, such as Ashtekar’s formulation of general relativity, one of these additional gauge transformations must be added to the diffeomorphism-induced transformations to achieve projectability. We have constructed the complete symmetry generators in the constrained phase space which contains all of the dynamical variables, including the gauge variables.

Two features of the resulting phase space formulation are especially significant, and both were noted almost thirty years ago by Bergmann and Komar [6]: First, rigid time translation, i.e., evolution in time, is not a gauge symmetry. Second, because of the fact that the gauge algebra contains derivatives of the metric, true symmetry invariants will be non-local. This occurs since nested commutators of generators contain derivatives of arbitrarily high order. The decoupling of symmetry and time evolution has profound implications. In this essay I shall further elaborate on a recent proposal which exploits the true symmetry in constructing quantum invariants in loop gravity [9]. The non-local nature of the invariants will be manifest. The basic idea is to retain gauge variables as quantum variables, but in a novel fashion: Since the classical gauge variables are arbitrary functions of time, we shall interpret both spatial and time coordinates of gauge variables as indices. In other words, gauge field variables each constitute a $4 \times \infty$ set.

In Section II I will give a physical motivation for retaining the gauge variables. Section III is an overview of the classical symmetry structure, culminating in a presentation of the finite classical generator of one-parameter subgroups of symmetry transformations. This object is employed in Section IV in the construction of a physical quantum projector. Finally, in Section V I turn to the loop formulation of quantum gravity. I propose a larger kinematical Hilbert Space formed not only from spatial Wilson loops with their associated holonomies, but also SU(2) gauge invariant loops containing legs in the new parameter time directions. The resulting structure permits the construction of true quantum spacetime invariants.

II. PHYSICAL MOTIVATION

One might be tempted to think that a foliation of spacetime into fixed time slices would irretrievably destroy four-dimensional spacetime symmetry. Indeed, most approaches to
quantum gravity are content with exploring the consequences of the residual spatial diffeomorphism and internal gauge group symmetries. The so-called scalar or Hamiltonian constraint is recognized as a generator of time evolution, although time evolution is generalized to incorporate the notion of advance of a “multifingered time”, an idea that has been promoted by Kuchař and others. (See [10] for a review of the problem of time in general relativity.) Multifingered time advances along directions perpendicular to the constant time hypersurfaces. Our work supplies an explanation - first noted by Lee and Wald [5] - for this strange dependence of diffeomorphisms on the spacetime metric; it is required to achieve projectability under the Legendre map to phase space. The earlier work by myself and Sundermeyer provides an equivalent explanation, as explained in [1]. Multifingered advance in time is in fact a gauge group symmetry, and it is this recognition that a spacetime diffeomorphism-induced group symmetry remains in the phase space formulation of general relativity that has neither been recognized, nor exploited in quantum general relativity.

What might we hope to gain from this larger symmetry? First of all, since rigid time evolution is not a symmetry we can reasonably expect that time will not be “frozen”. Different times are in principle distinguishable, even in the context of vacuum gravity. Second, although the classical gauge functions are freely prescribable functions of time, they do undergo quite specific, known symmetry transformations. Spacetime invariants, obtained by applying these transformations to functionals containing them will depend on them in a highly nontrivial manner. This has important implications for quantum gravity both in the conventional Wheeler-DeWitt treatment, and for newer loop versions employing the Ashtekar connection. The gauge variables must be retained as quantum operators. In fact, we ought to insist that the lapse and shift be retained; a quantum “fuzziness” only in spatial proper distances manifestly breaks the underlying symmetry of the theory. In retaining the gauge variables it becomes possible to construct operators representing true spacetime distances, areas and volumes. The absence of such objects in the loop approach is especially problematical, given the emergence of discreteness in spatial areas and volumes [11].

III. CLASSICAL GAUGE SYMMETRY

My collaborators and I have investigated conditions that must be fulfilled by infinitesimal gauge symmetry transformations in the original classical Lagrangian formalism of a wide variety of generally covariant theories in order that these variations can be mapped under the Legendre map to phase space. The theories include the relativistic particle, the relativistic string, conventional general relativity [1], Einstein-Yang-Mills [2], a real triad version [3], and the Ashtekar formulation, of general relativity [4]. The resulting projectable infinitesimal symmetry generators $G[\xi; t]$ all have the following structure:

$$G[\xi; t] = P_A \dot{\xi}^A + (\mathcal{H}_A + P_C N^{B'} C_{A'B'}^C) \xi^A,$$  

(3.1)

where the structure functions are obtained from the closed Poisson bracket algebra

$$\{\mathcal{H}_A, \mathcal{H}_{B'}\} =: C_{A'B'}^{C''} \mathcal{H}_{C''},$$

(3.2)
and where spatial integrations at time $t$ over corresponding repeated capital indices are assumed. The $N^A$ are the gauge functions. Their canonical momenta $P_A$ are primary constraints. The physical phase space is further constrained by secondary constraints $\mathcal{H}_A$. These constraints generate symmetry variations of the non-gauge variables. The “descriptors” $\xi^A$ are arbitrary spacetime functions.

If there is no symmetry in the Lagrangian description beyond general covariance, the indices $A$ range from zero to three. The corresponding gauge functions are the lapse $N$ and the shift $N^a$, so $N^A = \{N, N^a\}$. (My index convention is that spatial indices are lowercase latin letters from the beginning of the alphabet.) The lapse and shift appear in the spacetime metric

$$
(g_{\mu\nu}) = \begin{pmatrix}
-N^2 + N^c N^d g_{cd} & g_{ac} N^c \\
 g_{sd} N^d & g_{ab}
\end{pmatrix}.
$$

(3.3)

The projectable infinitesimal symmetries are induced by spacetime infinitesimal diffeomorphisms of the form

$$
x'^\mu = x^\mu - \delta^\mu_a \xi^a - n^\mu \xi^0.
$$

(3.4)

The normal $n^\mu$ to the fixed time hypersurface is expressed as follows in terms of the lapse and shift:

$$
n^\mu = (N^{-1}, -N^{-1} N^a).
$$

(3.5)

Projectable configuration-velocity functions may not depend on time derivatives of the lapse and shift; equation (3.4) represents the most general infinitesimal diffeomorphism producing variations which satisfy this requirement.

If gauge symmetries exist beyond those induced by diffeomorphisms one obtains additional projectability conditions. I will call these additional symmetries “internal symmetries”. In all such theories we have considered, a Yang-Mills type connection constitutes an additional configuration variable. The Lagrangian does not depend on the time derivative of the temporal component of this connection, hence the additional projectability requirement is that symmetry variations may not depend on time derivatives of this temporal component. In this case the index $A$ acquires an additional range, over the dimension of the Lie algebra of the additional gauge group.

Specifically, in Ashtekar’s formulation of general relativity the Ashtekar connection $A^i_\mu$ is an element of the Lie algebra $so(3, R)$ or $so(3, C)$, and the lower case latin indices $i$ range from one to three. It turns out that variations of the temporal component of the connection induced by the diffeomorphisms (3.4) depend on time derivatives of this component, and hence these variations are not projectable. Projectable infinitesimal variations are obtained by adding an internal gauge transformation constructed from the diffeomorphism descriptor $\xi^0$ and the connection form contracted with the hypersurface normal. In the complex Ashtekar case the required internal gauge descriptor is $A^i_\mu n^\mu \xi^0 - i N^{-1} T^{ai} N_a \xi^0$. ($T^{ai}$ are the triad fields, from which we obtain the contravariant spatial metric $e^{ab} = T^{ai} T^{bi}$). In all of the cases we have considered, the constraint $\mathcal{H}_0$ generates the corresponding projectable infinitesimal variation of the non-gauge variables.
Returning to the infinitesimal generator \( [\xi; t] \) in (3.1), let me complete the list of variables in the Ashtekar case. The gauge functions are \( \{ \bar{N}, N^a, -A^i_0 \} =: N^A \), with their canonical momenta, which are primary constraints: \( \{ \bar{P}, \bar{P}_a, -\bar{P}_i \} =: P_A \). \( \bar{P}_i \) is the momentum conjugate to \( A^i_0 \). The secondary constraints are \( \{ \bar{\mathcal{H}}_0, \mathcal{H}_a, \mathcal{H}_i \} =: \mathcal{H}_A \), where \( \mathcal{H}_i \) generates internal \( SO(3) \) rotations of the non-gauge variables. (As has now become conventional, densities of arbitrary positive weight under spatial diffeomorphisms are represented by an appropriate number of tildes over the symbol. For negative weights the tilde is placed below the symbol.) The momenta conjugate to \( A^a_i \) are the densitized triad field \( \bar{T}^a_i := det(t^i_a)T^a_i \) where \( t^i_a \) is the covariant triad, the inverse of \( T^a_i \): \( t^i_a T^a_j = \delta^i_j \). (In the complex case the canonical pair is actually \( \{ \bar{T}^a_i, iA^i_a \} \).

It must be stressed that the generator \( G[\xi; t] \) in (3.1) is actually a function of the time \( t \), and it is assumed that the canonical variables appearing in this expression are solutions of the equations of motion. The gauge functions are however almost arbitrary; the only condition on them is that the lapse must be strictly positive. Also, the primary constraints undergo a trivial evolution; they are always zero. The evolution of the non-gauge variables, on the other hand, is generated by the canonical Hamiltonian, where explicit choices are made for the gauge functions.

The canonical Hamiltonian is \( H_c = N^A \mathcal{H}_A \). It generates time evolution of the non-gauge variables. We do not alter either the equations of motion or gauge variations in recognizing that since the gauge variables \( N^A \) are arbitrary functions of both space and time we can add a term \( \int_{-\infty}^{\infty} d^4x P_A(x) \bar{N}^A(x) \) to the canonical Hamiltonian. The new Hamiltonian becomes

\[
H(t) = \int_{-\infty}^{\infty} dt' \int d^3x P_A(\bar{x}, t') \bar{N}^A(\bar{x}, t') + \int d^3x N^A(\bar{x}, t) \mathcal{H}_A. \tag{3.6}
\]

Thus the classical evolution of all dynamical variables is effected by the time-ordered evolution operator

\[
\{ -, U[t, t_0] \} = \mathcal{T} \exp \left( \int_{t_0}^{t} dt' \{ -, H(t') \} \right), \tag{3.7}
\]

where \( \{, \} \) represents the Poisson Bracket which I here generalize to include an integral over the time indices of the gauge variables;

\[
\{ N_A(\bar{x}, t), P_B(\bar{x}', t') \} = \delta^3(\bar{x}, \bar{x}')\delta(t, t'). \tag{3.8}
\]

Let me demonstrate that \( U[t, t_0] \) does indeed correctly rigidly translate the gauge functions \( N^A \) in time. We have

\[
\{ N^A(t_0), U[t, t_0] \} = N^A(t_0) + \int_{t_0}^{t} dt_1 \{ N^A(t_0), \int dt' P_B(t') \bar{N}^B(t') \} + \int_{t_0}^{t} dt_1 \int_{t_0}^{t_2} dt_2 \{ N^A(t_0), \int dt' P_B(t') \bar{N}^B(t') \} + \ldots \\
= N^A(t_0) + \bar{N}^A(t_0)(t - t_0) + \frac{1}{2}\bar{N}^A(t_0)(t - t_0)^2 + \ldots
\]

5
\[
N(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n N^A(t_0)}{dt^n} (t - t_0)^n = N^A(t). \tag{3.9}
\]

(In the first equality we used the fact that the time-dependent contribution to the second term in the generator (3.6) yields a vanishing Poisson bracket. Therefore the integration over the times \(t_i\) is trivial.) Also \(U(t, t_0)\) maintains the primary constraint \(P_A \approx 0\).

Our next task is to find the expanded infinitesimal generator, expressed in terms of the larger set of canonical gauge variables, which effects the correct variations of the canonical variables. The following object does the job:

\[
G_v[\xi; t] = \int dt' P_A(t') \dot{\xi}^A(t') + H_A(t) \xi^A(t) + \int dt' P_C^{c'}(t')(N^{B'}(t')C^{c''}_{AB'}(t') \xi^A(t')). \tag{3.10}
\]

We turn now to the construction of finite one-parameter subgroups of the full gauge transformation group. We will build them up as usual from infinitesimal gauge transformations. We begin with the more familiar three-dimensional spatial diffeomorphism group. For this purpose it is instructive to display explicitly the general infinitesimal symmetry variations of the lapse and shift gauge variables:

\[
\begin{align*}
\delta N &= \xi^0 + \xi^a N_a - N^a \xi^0_{,a}, \quad (3.11a) \\
\delta N^a &= \xi^a - N e^{ab} \xi^0_{,b} + N_s e^{ab} \xi^0 + N_b^a \xi^b - N^b \xi^0_{,b}. \quad (3.11b)
\end{align*}
\]

If the descriptor \(\xi^0\) vanishes the \(\xi^a\) represent, for a fixed time, vector fields on the three-dimensional spatial manifold. They may be interpreted as tangents to a one parameter family of spatial manifold maps. We build up the finite maps by solving the set of ordinary differential equations which follow from (3.11), taking the spatial descriptors to be \(ds \xi^a\) and setting \(\xi^0 = 0\),

\[
\frac{d\xi^a}{ds} = -\xi^a(x). \tag{3.12}
\]

Using (3.12) it is straightforward to build up a formal power series in \(s\), since \(\ddot{\xi}^a = \xi^a_{,b} \xi^b\), etc. The result is

\[
x^a(s) = \sum_n \frac{(-1)^n s^n}{n!} \left( \prod_{i=1}^{n-1} \xi^a_{,i} \partial_{a_i} \right) \xi^a. \tag{3.13}
\]

The corresponding formal generator of finite variations of the dynamical variables is constructed with the aid of the infinitesimal generator [6]

\[
G : = G_v[\xi^a, \xi^0 = 0, \xi^i = 0; t] = \int dt' \tilde{P}_a(t') \dot{\xi}^a(t') + \dot{H}_a(t) \xi^a(t) + \int dt' P_C^{c'}(t')(N^{B'}(t')C^{c''}_{AB'}(t') \xi^A(t')).
\]

\[
= \tilde{H}_a(t) \xi^a(t) + \int dt' \tilde{P}_a(t') \left( N^a_{,a}(t') \xi^a(t') - N(t') \xi^a_{,a}(t') \right)
+ \int dt' \tilde{P}_a(t') \left( N_a^b(t') \xi^b(t') - N^b(t') \xi^b_{,a}(t') \right)
+ \int dt' \tilde{P}_i(t') \left( F_{ab}^{ij}(t') \xi^a(t) N^b(t') + i F_{ab}^{ij}(t') \tilde{T}^b_j(t') N(t') \xi^a(t') \right). \tag{3.14}
\]
The finite generator is

\[ 1 + s\{-, G\} + \frac{s^2}{2!}\{-, G\}, G\} + \ldots = \exp(s\{-, G\}). \quad (3.15) \]

We now consider the situation when the descriptors \( \xi^0 \) do not vanish. The analysis of the finite one-parameter subgroups of the full four-dimensional diffeomorphism-related gauge symmetry group is substantially complicated by the fact that the group depends on the dynamical variables. To simplify the discussion I will consider as an example a non-field theoretical model, the relativistic free point particle. Most convenient for our purposes is a classical formulation with an auxiliary gauge variable \([1]\).

The relativistic free particle with mass one is described by the Lagrangian

\[ L = \frac{1}{2N} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \frac{1}{2} N, \quad (3.16) \]

where \( x^\mu(\theta) \) is the vector variable in Minkowski spacetime, with metric \( (\eta_{\mu\nu}) = \text{diag}(-1,1,1,1) \), and \( N \) is an auxiliary variable whose equation of motion gives \( N = (-\dot{x}^\mu \dot{x}_\mu)^{1/2} \). \( N \) may be interpreted as a lapse, with corresponding metric \( g_{00} = -N^2 \) on the manifold parametrized by \( \theta \).

This is a generally covariant model as the dynamics does not change its form under arbitrary reparametrizations \( \theta' = \theta'(\theta) \). There exists a primary constraint \( \pi \simeq 0 \), and a secondary constraint \( H = \frac{N}{2}(p^\mu p_\mu + 1) \simeq 0 \). The canonical Hamiltonian is \( H_c = N^2(p^\mu p_\mu + 1) \).

The projectable infinitesimal reparametrizations are \( \theta' = \theta - N^{-1}\xi^0 \), and according to (3.1) the corresponding generator is

\[ G[\xi^0; \theta] = \int d\theta' \xi^0(\theta') \pi(\theta') + \xi^0(\theta)\frac{1}{2}(p_\mu(\theta)p^\mu(\theta) + 1). \quad (3.17) \]

Now referring again to (3.4) applied to the particle model, we deduce that

\[ \frac{d\theta}{ds}|_{s=0} = -\xi^0(\theta)N^{-1}(\theta)|_{s=0}. \quad (3.18) \]

Unfortunately, both \( \xi^0 \) and \( N \) will alter their functional dependence on \( \theta \) under the one-parameter group, so (3.18) is not particularly useful. We need to determine directly the one-parameter family of transformations of lapses \( N_s(\theta) \) which is generated by the lapse-dependent reparametrizations! First we note from (3.11a) that

\[ \frac{\partial N_s}{\partial s}(\theta)|_{s=0} = \frac{\partial \xi^0}{\partial \theta}(\theta)|_{s=0}. \quad (3.19) \]

We must be careful in writing down the appropriate differential equation for arbitrary \( s \); the function \( \xi^0 \) also undergoes a variation under this reparametrization. This occurs because there is an essential difference between the metric-independent spatial coordinate transformation (3.4) when \( \xi^0 \) vanishes, and the spacetime transformation resulting from a nonvanishing \( \xi^0 \). In the former case the descriptors \( \xi^a \) are invariant; the Lie derivative of \( \xi^a \) with respect to itself is zero. In the later case it is \( n^\mu \xi^0 \) which is invariant. It follows from the variation of \( N \) that \( \xi^0 \) transforms as a scalar. In fact, \( \xi^0 \) acquires a dependence on
N. We notice that the infinitesimal variation of $\xi^0$ under the infinitesimal reparametrization $\theta' = \theta - ds \xi^0 N^{-1}$ is

$$\delta \xi^0 = \left. \frac{\partial \xi^0}{\partial \theta} \right|_{s=0} d\xi^0 N^{-1}$$

(3.20)

The one-parameter subgroup differential relations may now be generalized to arbitrary parameter value $s$:

$$\frac{\partial N_s}{\partial s}(\theta) = \frac{\partial \xi^0_s}{\partial \theta}(\theta),$$

(3.21)

and

$$\frac{\partial \xi^0_s}{\partial s}(\theta) = \frac{\partial \xi^0_s}{\partial \theta}(\theta) \xi^0_s N^{-1}.$$  

(3.22)

We now develop a formal power series solution in the parameter $s$ for $\xi^0_s(\theta)$ and $N_s(\theta)$. We deduce from (3.21) and (3.22) that

$$\frac{\partial}{\partial s}(\xi^0 N^{-1}) = 0.$$  

(3.23)

We find that

$$\xi^0_s := \left. \frac{\partial^N \xi^0_s}{\partial s^n} \right|_{s=0} = \xi^0 N^{-1} \frac{d}{d\theta} \left( \xi^0 N^{-1} \frac{d}{d\theta} \left( \xi^0 N^{-1} \frac{d}{d\theta} \left( \ldots \frac{d}{d\theta} \left( \xi^0 N^{-1} \frac{d\xi^0}{d\theta} \right) \ldots \right) \right) \right),$$

(3.24)

where $\xi^0 N^{-1}$ appears $n$ times. This leads to the following expansion in $s$:

$$\xi^0_s(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^0_s[n] s^n.$$  

(3.25)

The one-parameter family of lapses $N_s(\theta)$ follows almost immediately from (3.24) and (3.21).

$$N_s(\theta) = N(\theta) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{d}{d\theta} \xi^0_s[n](\theta) s^{n+1}.$$  

(3.26)

Let us also compute the one-parameter family of transformed particle positions. Since the $x^\mu$ are scalars these families will obey

$$\frac{\partial x^\mu_s}{\partial s} = \frac{\partial x^\mu}{\partial \theta} \xi^0_s N^{-1}.$$  

(3.27)

We find that

$$\frac{\partial^n x^\mu_s}{\partial s^n} \bigg|_{s=0} = \xi^0 N^{-1} \frac{d}{d\theta} \left( \xi^0 N^{-1} \frac{d}{d\theta} \left( \ldots \frac{d}{d\theta} \left( \xi^0 N^{-1} \frac{d x^\mu}{d\theta} \right) \ldots \right) \right) \right),$$

(3.28)

where in the last line we used the equation of motion $\dot{x}^\mu = N p^\mu$. The expression simplifies further using the equation of motion $\dot{p}^\mu = 0$ yielding the formal solution
\[ x_\mu^s(\theta) = x_\mu(\theta) + p^\mu \sum_{n=0}^{\infty} \frac{1}{n!} \xi_0^{[n]}(\theta) s^n. \] (3.28)

One special case for \( \xi_0^0(\theta), N_s(\theta) \) and \( x_\mu^s(\theta) \) is especially worthy of note. Notice that when \( \xi_0^0(\theta) = N(\theta) \) we have \( \xi_0^0(\theta) = \xi^0(\theta + s), N_s(\theta) = N(\theta + s), \) and \( x_\mu^s(\theta) = x_\mu(\theta + s) \). The effect of this one-parameter family of diffeomorphism-induced transformations on the lapse and particle position is to advance them in \( \theta \) by the parameter value \( s \)! This is an illustration of a general property of the one-parameter time-like diffeomorphism-induced transformations on the dynamical variables, evident also for the explicit variations of lapse and shift exhibited in (3.11); although the gauge group elements do not in general generate rigid translations in time, when the group acts on solutions for which the lapse \( N \) and shift \( N^a \) are equal to the descriptors \( \xi^0 \) and \( \xi^a \), respectively, the result is to effect time evolution of the solutions. This is to be noted for the infinitesimal transformations in (3.11) since substitution of these choices for the descriptors yields \( \delta N = \dot{N} ds \) and \( \delta N^a = \dot{N}^a ds \).

We are finally able to write down the finite diffeomorphism-induced generator of variations of \( N, x_\mu, \) and \( p^\mu \) in the free relativistic particle case. Letting \( G(\theta, s) := \int d\theta' \pi(\theta') \xi_0^0(\theta') + \xi_0^2(\theta') \frac{1}{2}(\hat{p}_\mu \hat{p}^\mu + m^2) \), the result is the parameter-ordered exponential

\[
S \left( \exp \left[ \int_0^s ds' \left\{ -G(\theta, s') \right\} \right] \right) \\
1 + \int_0^s ds_1 \left\{ -G(\theta, s_1) \right\} + \int_0^s ds_2 \int_0^{s_2} ds_1 \left\{ -G(\theta, s_1) \right\} G(\theta, s_2) + \ldots
\] (3.29)

IV. THE QUANTUM PHYSICAL PROJECTOR

I shall elaborate here on a recent proposal \[9\] which is inspired by Carlo Rovelli’s introduction of an operator he calls a “physical projector” \[12\]. The fundamental idea is to average over the symmetry group. But before we can do this we must fix our Hilbert space.

I propose to enlarge this Hilbert space to include time-parametrized gauge functions in addition to the non-gauge variables. If we take the non-gauge variables to be the spatial metric, we obtain a generalization of the Wheeler-DeWitt approach to quantum gravity. In this essay, however, I will generalize the loop approach which employs the Ashtekar connection.

In part to explain the procedure, and also in part to check whether it yields plausible results in a well understood simple theory, we shall first address the group averaging question for the free relativistic particle.

We consider a mixed momentum/Schrödinger representation where \( \hat{p}^\mu \) and \( \hat{N}(\theta) \) are multiplicative operators. We interpret the argument \( \theta \) of \( \hat{N}(\theta) \) as a parameter, so our quantum relativistic particle model has been converted into a field theory. The momentum conjugate to \( \hat{N}(\theta) \) is thus also a field, as in our classical description above.

Our task is to calculate the parameter-ordered functional integral of the quantum version of the finite classical generator given in (3.29). The corresponding physical projector is

\[
\hat{P} := S \left( [D\xi^0] \exp \left( -i \int_0^s ds' \hat{G}(\theta, s') \right) \right).
\] (4.1)
Unfortunately, this is a highly non-trivial functional of the descriptor field $\xi^0$, and a reasonable approximation scheme has not yet been found for performing this functional integral.

V. LOOP QUANTUM GRAVITY

The conventional approach to loop quantum gravity takes as the kinematical arena a Hilbert space constructed from traces of holonomies of closed spatial loops. (See [13] for a recent review.) These traces are invariant under SU(2) gauge rotations of the holonomies. But as a consequence of identities satisfied by the traces the states constructed with them are linearly dependent. Rovelli and Smolin realized that the isolation of a linearly independent set corresponded to the notion of spin network that had been invented by Roger Penrose [14,15].

I have not yet worked out all of the implications of the following proposal, but it does seem to me to exhibit several attractive features, and I would anticipate that some variation of it will survive in a fully articulated quantum theory.

I want to retain the gauge variables as operators; the lapse $N(\vec{x}, t)$, shift components $N^a(\vec{x}, t)$, and the time components of the Ashtekar connection $A^i_\mu(\vec{x}, t)$ each constitute a $4 \times \infty$ set of freely specifiable variables. We can use them to construct spacetime loops with associated holonomies. In particular I propose to attach timelike legs to finite open paths in space. We might try, for example, holonomies (parallel transport matrices) along timelike paths of the form

$$T \exp \left( \int_{t_1}^{t_2} dt' A_\mu n^\mu \right),$$

(5.1)

where it is understood that $A_\mu$ are to be taken as independent of $t$, and $A_\mu := A^i_\mu \tau_i$, the $\tau_i$ being the Pauli matrices. Then we construct closed spacetime loops, with associated holonomies, by first transporting along a spatial path with fixed initial and final points, say from $\vec{x}_1$ to $\vec{x}_2$, then forward in time from $t_1$ to $t_2$, back along (a generally distinct) spatial path from $\vec{x}_2$ to $\vec{x}_1$, and then finally backward in time from $t_2$ to $t_1$. The trace of this holonomy is invariant under internal $SU(2)$ rotations. Products of the traces associated with loops will satisfy the same spinor and retracing identities referred to above. We might reasonably expect, therefore, that a four-dimensional spin network will constitute a linearly independent set of kinematical states. This hypothesis is now being explored, as are the following associated problems: What is the relation of these states, after integrating over the spatial diffeomorphism group, to the knot states in the three-dimensional spin network formalism (see [13] for a review), and what is the appropriate measure in this space?

The next task will be to attempt to give some sense to the formal physical projector

$$\hat{P} := S \left( [D\xi] \exp \left( -i \int_0^s ds' \hat{G}(t, s') \right) \right).$$

(5.2)

This is a daunting challenge. On the one hand we do not yet have a general expression for the one-parameter descriptor families $\xi^\mu(t, s)$. Even worse, the products of operators appearing in $\hat{G}(t, s')$ are not well defined. However, regularization techniques are available, and they have been employed successfully in similar expansions, resulting in a structure
which has been called a “spin foam” [16,17]. (Incidentally, this regularization technique, and the construction of the measure, are achieved with a real Ashtekar connection. The symmetry generators in this formulation also have the form (3.1) [18].) The expansion and regularization of (5.2), and its relation to spin foams is the focus of current research. Since spacetime area and volume operators will very likely be well-defined in this formalism, we can reasonably anticipate that when acting on four-dimensional networks we will encounter eigenstates of these operators with discrete eigenvalues.

VI. CONCLUSIONS

In this essay I have reviewed our current understanding of classical gauge symmetries in Hamiltonian formulations of generally covariant theories which are derivable from a Lagrangian. These symmetries form an infinite dimensional transformation group, and I displayed explicitly the general form of the generator of finite one-parameter subgroups which are connected to the identity. Gauge functions are retained as dynamical variables, and although they undergo non-trivial variations under arbitrary symmetry transformations, their time evolution is completely arbitrary. I have argued that there is ample physical motivation for retaining these gauge variables in a quantum theory of gravity. But recognizing their arbitrary evolution, it is both reasonable and consistent with their symmetry variations to conceive of their quantum operator analogues as independent operators at distinct times.

True symmetry invariants can then be obtained in this formalism by integrating the finite quantum symmetry generator over the gauge group. I call the resulting operator the physical projector.

I have proposed a tentative implementation of this approach in a new loop approach to quantum gravity. Using the arbitrary gauge functions in the Ashtekar approach which are the lapse, shift, and the temporal component of the Ashtekar connection, we construct traces of holonomies around spacetime loops. I speculate that the resulting linearly independent states are four-dimensional spin networks. The physical projector may be expressed as an infinite expansion. Once the terms in this expansion are suitably regularized, it may be possible either truncate this expansion, or achieve partial infinite sums. We will then be in position to calculate spacetime areas and volumes.
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