LIMIT THEOREMS FOR A LOCALIZATION MODEL OF 2-STATE QUANTUM WALKS

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We consider 2-state quantum walks (QWs) on the line, which are defined by two matrices. One of the matrices operates the walk at only half-time. In the usual QWs, localization does not occur at all. However, our walk can be localized around the origin. In this paper, we present two limit theorems, that is, one is a stationary distribution and the other is a convergence theorem in distribution.

Keywords: limit distribution; localization; 2-state quantum walk.

1. Introduction

The 2-state quantum walk (QW) on the line \( Z = \{ 0, \pm 1, \pm 2, \ldots \} \) has been intensively studied, and the limit theorems are obtained.\cite{1,2,3,4,5} For example, the limit distribution of the usual walks was calculated.\cite{6,7} In the present paper, we consider a localization model of 2-state QWs. The motivation is the analysis of the time-inhomogeneous QWs. Our walks are determined by two matrices, one of which operates the walk at only half-time. The walk can be considered as one of the time-dependent models, for which there are some results.\cite{8,9,10} Particularly, Ref.\cite{8} and\cite{9} discuss localization. We present the two limit theorems that show the localization of the probability distribution. One is calculation of the limit value for the probability \( P(X_t = x) \) which walker is at position \( x \in Z \) starting from the origin, where \( t \in \{ 0, 1, 2, \ldots \} \) is time, and the other is the convergence in distribution. In the usual walks, localization can not occurs. However, if we change the matrix at only half-time, then we find that the localization occurs from the results in this paper. The localization of QWs, which can be applied to quantum search,\cite{11} is often investigated.\cite{12,13,14,15,16,17} If \( \limsup_{t \to \infty} P(X_t = x) > 0 \) for a position \( x \), we call that the localization occurs. Therefore, our result insists that the localization occurs for any initial state. Moreover, we obtain the convergence in distribution of \( X_t / t \) as \( t \to \infty \). This limit distribution is described by both \( \delta \)-function and a density function. For 3-state Grover walk, similar limit theorems were shown.\cite{18} The
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limit distribution of a 4-state walk corresponding to the 2-state walk with memory was also computed. The present paper is organized as follows. In Section 2, we define our walk. We present the limit theorems as our main result in Section 3. Section 4 is devoted to the proofs of the theorems. By using the Fourier analysis, we obtain the limit distribution. Summary is given in the final section.

2. Definition of a localization model of 2-state QWs

In this section, we define a localization model of 2-state QWs on the line. Let \( |x \rangle \ (x \in \mathbb{Z}) \) be an infinite components vector which denotes the position of the walker. Here, \( x \)-th component of \(|x\rangle\) is 1 and the other is 0. Let \( |\psi_t(x)\rangle \in \mathbb{C}^2 \) be the amplitude of the walker at position \( x \) at time \( t \). The walk at time \( t \) is expressed by

\[
|\Psi_t\rangle = \sum_{x \in \mathbb{Z}} |x\rangle \otimes |\psi_t(x)\rangle.
\]

The time evolution of our walk is depicted with the following two unitary matrices:

\[
U = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix},
\]

\[
H = \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \\ \sin \theta_1 & -\cos \theta_1 \end{bmatrix} = \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix},
\]

where \( c = \cos \theta, s = \sin \theta (\theta \in [0, 2\pi)), \theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) and \( c_1 = \cos \theta_1, s_1 = \sin \theta_1 (\theta_1 \in [0, 2\pi)) \). Moreover, we introduce four matrices:

\[
P = \begin{bmatrix} c & s \\ 0 & 0 \end{bmatrix},
\quad Q = \begin{bmatrix} 0 & 0 \\ 0 & -s \end{bmatrix},
\quad P_1 = \begin{bmatrix} c_1 & s_1 \\ 0 & 0 \end{bmatrix},
\quad Q_1 = \begin{bmatrix} 0 & 0 \\ s_1 & -c_1 \end{bmatrix}.
\]

Then, the evolution is determined by

\[
|\psi_{t+1}(x)\rangle = \begin{cases} 
P \psi_t(x+1) + Q \psi_t(x-1) & (t \neq \tau) \\
P_1 \psi_t(x+1) + Q_1 \psi_t(x-1) & (t = \tau) \end{cases},
\]

where \( \tau \in \{1, 2, \ldots\} \). Note that \( P + Q = U \) and \( P_1 + Q_1 = H \). The probability that the quantum walker \( X_t \) is at position \( x \) at time \( t \), \( P(X_t = x) \), is defined by

\[
P(X_t = x) = \langle \psi_t(x)|\psi_t(x)\rangle.
\]

In our main results, we focus on the probability distribution at time \( 2\tau + 1, 2\tau + 2 \). So, time \( \tau \) is called half-time in our walk.

The Fourier transform \(|\hat{\psi}_t(k)\rangle \ (k \in [-\pi, \pi]) \) of \(|\psi_t(x)\rangle \) is given by

\[
|\hat{\psi}_t(k)\rangle = \sum_{x \in \mathbb{Z}} e^{-ikx} |\psi_t(x)\rangle.
\]

By the inverse Fourier transform, we have

\[
|\psi_t(x)\rangle = \int_{-\pi}^{\pi} e^{ikx} |\hat{\psi}_t(k)\rangle \frac{dk}{2\pi}.
\]
From (5) and (7), the time evolution of $|\hat{\Psi}(k)\rangle$ becomes

$$
|\hat{\Psi}_{t+1}(k)\rangle = \begin{cases} 
\hat{U}(k) |\hat{\Psi}_t(k)\rangle & (t \neq \tau) \\
\hat{H}(k) |\hat{\Psi}_t(k)\rangle & (t = \tau)
\end{cases},
$$

(9)

where $\hat{U}(k) = R(k)U$, $\hat{H}(k) = R(k)H$ and $R(k) = \begin{bmatrix} e^{ik} & 0 \\
0 & e^{-ik} \end{bmatrix}$. Particularly, we see that

$$
|\hat{\Psi}_{2\tau+1}(k)\rangle = \hat{U}(k)^T \hat{H}(k) \hat{U}(k)^T |\hat{\Psi}_0(k)\rangle,
$$

(10)

$$
|\hat{\Psi}_{2\tau+2}(k)\rangle = \hat{U}(k)^T \hat{H}(k)^{\dagger} \hat{U}(k)^T |\hat{\Psi}_0(k)\rangle.
$$

(11)

In the present paper, we take the initial state as $|\psi_0(x)\rangle = \begin{bmatrix} T[\alpha, \beta] \ (x = 0) \\
T[0, 0] \ (x \neq 0) \end{bmatrix}$, where $|\alpha|^2 + |\beta|^2 = 1$ and $T$ is the transposed operator. We should note that $|\hat{\Psi}_0(k)\rangle = |\psi_0(0)\rangle$. Figures 1 and 2 depict probability distributions of the walk under the condition $\theta = \pi/4, \theta_1 = 0$. In Figure 1 (a), the probabilities at the position $x = 0, \pm 2$ are higher than other positions. Figure 2 shows the time evolution of probability distribution as $\tau = 24$. The walk evolves with the matrix $H$ at only time $\tau$. So, the walk is a usual QW till time $\tau$.

In comparison with both Figures 1 and 2 we show the probability distributions of a usual QW with $\theta = \theta_1 = \pi/4$ in Figures 3 and 4.

![Fig. 1. The probability distributions at time $t = 500$ as $\tau = 249$ and $\theta = \pi/4, \theta_1 = 0$.](image-url)
Fig. 2. The evolution of the probability distributions for time $t$ by density plot as $\tau = 24$ and $\theta = \pi/4, \theta_1 = 0$.

Fig. 3. The probability distributions at time $t = 500$ with $\theta = \theta_1 = \pi/4$.

Fig. 4. The evolution of the probability distributions for time $t$ by density plot with $\theta = \theta_1 = \pi/4$. 
3. Limit theorems for the walk

In this section, we show our main results. For our localization model of 2-state QWs, we obtain the two following limit theorems.

Theorem 1.

(i) For odd time $2\tau + 1$,

$$
\lim_{\tau \to \infty} P(X_{2\tau+1} = x) = \begin{cases}
\frac{(c_1 s - s_1 c)^2 (1-|s|^2)}{c^3} K_1(-1; \beta, \alpha) & (x = -1), \\
\frac{(c_1 s - s_1 c)^2 (1-|s|^2)}{c^3} K_1(1; \alpha, \beta) & (x = 1), \\
\frac{2(c_1 s - s_1 c)^2 s}{c^3 (1-|s|)} \left(\frac{1-|s|}{c}\right)^{2|x|} K_2(x; \beta, \alpha) & (x = -3, -5, -7, \ldots), \\
\frac{2(c_1 s - s_1 c)^2 s}{c^3 (1-|s|)} \left(\frac{1-|s|}{c}\right)^{2|x|} K_2(x; \alpha, \beta) & (x = 3, 5, 7, \ldots), \\
0 & (x = 0, \pm2, \pm4, \ldots)
\end{cases}
$$

(13)

where

$$
K_1(x; \alpha, \beta) = c^2 |\alpha|^2 + 2s^2 (1 - |s|)^2 |\beta|^2 + \text{sign}(x)c s(1 - |s|)(\alpha \overline{\beta} + \overline{\alpha} \beta),
$$

(14)

$$
K_2(x; \alpha, \beta) = c^2 |\alpha|^2 + s(1 - |s|)^2 |\beta|^2 - \text{sign}(x)c s(1 - |s|)(\alpha \overline{\beta} + \overline{\alpha} \beta),
$$

(15)

and $\text{sign}(x)$ means a sign of $x$.

(ii) For even time $2\tau + 2$,

$$
\lim_{\tau \to \infty} P(X_{2\tau+2} = x) = \begin{cases}
\frac{(c_1 s - s_1 c)^2 s^2 (1-|s|^2)}{c^4} & (x = 0), \\
\frac{(c_1 s - s_1 c)^2 (1-|s|^2)}{c^3} K_3(-2; \beta, \alpha) & (x = -2), \\
\frac{(c_1 s - s_1 c)^2 (1-|s|^2)}{c^3} K_3(2; \alpha, \beta) & (x = 2), \\
\frac{2(c_1 s - s_1 c)^2 s}{c^3} \left(\frac{1-|s|}{c}\right)^{2|x|} K_4(x; \beta, \alpha) & (x = -4, -6, -8, \ldots), \\
\frac{2(c_1 s - s_1 c)^2 s}{c^3} \left(\frac{1-|s|}{c}\right)^{2|x|} K_4(x; \alpha, \beta) & (x = 4, 6, 8, \ldots), \\
0 & (x = \pm1, \pm3, \ldots)
\end{cases}
$$

(16)
where
\[
K_3(x; \alpha, \beta) = c^2 \{1 - s^2|(2 - |s|)| \} |\alpha|^2 + 2s^2(1 - |s|)^3|\beta|^2 \\
+ \text{sign}(x)\cos(1 + s^2)(1 - |s|)^2(\alpha\overline{\beta} + \overline{\alpha}\beta),
\]
(17)
\[
K_4(x; \alpha, \beta) = s^2(|\alpha|^2 - |\beta|^2) - \text{sign}(x)\cos(\alpha\overline{\beta} + \overline{\alpha}\beta) + |s|.
\]
(18)

As a simple example of the case when the localization occurs in our walk, if \( \theta = \pi/4, \theta_1 = 0 \) and \( |\psi(0)\rangle = T[1/\sqrt{2}, i/\sqrt{2}] \), then we have
\[
\lim_{\tau \to \infty} P(X_{2\tau+1} = \pm 1) = \frac{13 - 9\sqrt{2}}{2} = 0.136039 \ldots,
\]
(19)
\[
\lim_{\tau \to \infty} P(X_{2\tau+2} = 0) = \frac{3 - 2\sqrt{2}}{2} = 0.0857864 \ldots,
\]
(20)
\[
\lim_{\tau \to \infty} P(X_{2\tau+2} = \pm 2) = \frac{139 - 98\sqrt{2}}{4} = 0.101768 \ldots.
\]
(21)

Figure 5 corresponds to the behavior of each probability in (19), (20) and (21).

Next, we present the theorem of the convergence in distribution for \( X_t/t \), where \( t \in \{2\tau + 1, 2\tau + 2\} \). Some similar results corresponding to Theorem 2 were shown for a 3-state walk or a 4-state walk.\[13][18]\] Moreover, localization of multi-state walks was computed.\[14\] The limit distribution of the usual 2-state walk does not have the delta measure.\[16\] However, we find that the limit distribution of the 2-state walk defined in this paper has a delta-measure from Theorem 2.
Theorem 2. For our localization model of 2-state QWs, we have
\[ \lim_{t \to \infty} P(X_t/t \leq x) = \int_{-\infty}^{x} f(x) \, dx, \] (22)
where
\[ f(x) = \delta_0(x) + \frac{|s|}{\pi(1-x^2)\sqrt{c^2-x^2}} \left[ 1 - \left( |\alpha|^2 - |\beta|^2 + \frac{(\alpha\beta + \bar{\alpha}\bar{\beta})s}{c} \right) x \right] \times \frac{a_2x^4 + a_1x^2 + a_0}{c^2(1-x^2)} I_{[-c,c]}(x), \] (23)
and \( \delta_0(x) \) denotes the delta-measure at the origin and \( I_A(x) = 1 \) if \( x \in A \), \( I_A(x) = 0 \) if \( x \notin A \). The values \( \Delta, a_0, a_1, a_2 \) are independent on initial state \( |\psi_0(0)\rangle \) as follows:
\[ \Delta = \frac{(c_1s - s_1c)^2}{1 + |s|}, \] (24)
\[ a_0 = c^2, \] (25)
\[ a_1 = 2cs(c_1s - s_1c) - c_1^2, \] (26)
\[ a_2 = (c_1s - s_1c)^2. \] (27)

Particularly, if \( \theta_1 = \theta \), then we obtain the density function of the usual walk, that is,
\[ f(x) = \frac{|s|}{\pi(1-x^2)\sqrt{c^2-x^2}} \left[ 1 - \left( |\alpha|^2 - |\beta|^2 + \frac{(\alpha\beta + \bar{\alpha}\bar{\beta})s}{c} \right) x \right] I_{[-c,c]}(x). \] (28)

Figure 6 shows the density function \( f(x) \) with \( \theta = \pi/4, \theta_1 = 0 \). We have \( \Delta = \frac{\sqrt{2}+1}{\sqrt{2}} \) in Figure 6.

We should note that the following relation between Theorems 1 and 2:
\[ \sum_{x \in \mathbb{Z}} \lim_{t \to \infty} P(X_t = x) = \Delta. \] (29)

Therefore, \( \lim_{t \to \infty} P(X_t = x) \) is not a probability measure.

(a) \( |\psi_0(0)\rangle = \mathcal{T}[1/\sqrt{2}, i/\sqrt{2}] \)

(b) \( |\psi_0(0)\rangle = \mathcal{T}[1, 0] \)

Fig. 6. The limit density function as \( \theta = \pi/4, \theta_1 = 0 \).
4. Proofs of theorems
In this section, we will prove Theorems 1 and 2 in Section 3. Our approach is based on the Fourier analysis.

4.1. Proof of Theorem 1
At first, the eigenvalues $\lambda_j(k) (j = 1, 2)$ of $\hat{U}(k)$ can be computed as

$$
\lambda_1(k) = \sqrt{1 - c^2 \sin^2 k} + ic \sin k, \quad \lambda_2(k) = -\sqrt{1 - c^2 \sin^2 k} + ic \sin k. \quad (30)
$$

The normalized eigenvector $|v_j(k)\rangle$ corresponding to $\lambda_j(k)$ is

$$
|v_j(k)\rangle = \sqrt{\frac{1 - c^2 \sin^2 k \pm c \cos k}{2s^2} \pm \sqrt{c^2 \sin^2 k - c \cos k}} \left[ \begin{array}{c} 1 \\ i^{-1} \frac{c}{s} \end{array} \right]. \quad (31)
$$

Therefore, the Fourier transform $|\hat{\Psi}_0(k)\rangle$ is expressed by $|v_j(k)\rangle$ as follows:

$$
|\hat{\Psi}_0(k)\rangle = \sum_{j=1}^{2} \langle v_j(k) | \hat{\Psi}_0(k) \rangle |v_j(k)\rangle. \quad (32)
$$

In the proof, we focus on even time $2\tau + 2$. From (11) and (32), the Fourier transform at time $2\tau + 2$ is given by

$$
|\hat{\Psi}_{2\tau+2}(k)\rangle = \hat{U}(k)^{\tau+1} \hat{H}(k) \hat{U}(k)^{\tau} |\hat{\Psi}_0(k)\rangle
$$

$$
= \hat{U}(k)^{\tau+1} \hat{H}(k) \sum_{j_1=1}^{2} \lambda_{j_1}^{\tau}(k) \langle v_{j_1}(k) | \hat{\Psi}_0(k) \rangle |v_{j_1}(k)\rangle
$$

$$
= \sum_{j_1=1}^{2} \lambda_{j_1}^{\tau}(k) \langle v_{j_1}(k) | \hat{\Psi}_0(k) \rangle \left( \hat{U}(k)^{\tau+1} \hat{H}(k) |v_{j_1}(k)\rangle \right). \quad (33)
$$

Moreover, rewriting $\hat{U}(k)^{\tau+1} \hat{H}(k) |v_{j_1}(k)\rangle$ as

$$
\hat{U}(k)^{\tau+1} \hat{H}(k) |v_{j_1}(k)\rangle = \hat{U}(k)^{\tau+1} \sum_{j_2=1}^{2} \langle v_{j_2}(k) | \hat{H}(k) |v_{j_1}(k)\rangle |v_{j_2}(k)\rangle
$$

$$
= \sum_{j_2=1}^{2} \lambda_{j_2}^{\tau+1}(k) \langle v_{j_2}(k) | \hat{H}(k) |v_{j_1}(k)\rangle |v_{j_2}(k)\rangle, \quad (34)
$$

we obtain

$$
|\hat{\Psi}_{2\tau+2}(k)\rangle = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \lambda_{j_1}(k)^{\tau} \lambda_{j_2}(k)^{\tau+1} \langle v_{j_1}(k) | \hat{\Psi}_0(k) \rangle
$$

$$
\times \langle v_{j_2}(k) | \hat{H}(k) |v_{j_1}(k)\rangle |v_{j_2}(k)\rangle. \quad (35)
$$
We should note $\lambda_1(k)\lambda_2(k) = -1$. Calculating the inverse Fourier transform, we have

$$\langle \psi_{2+2}(x) \rangle = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \int_{-\pi}^{\pi} \lambda_{j_1}(k)^{\tau} \lambda_{j_2}(k)^{\tau+1} \langle v_{j_1}(k)|\hat{\Psi}_0(k)\rangle \times \langle v_{j_2}(k)|\hat{H}(k)|v_{j_1}(k)\rangle |v_{j_2}(k)\rangle e^{ikx} \frac{dk}{2\pi}.$$  (36)

By using the Riemann-Lebesgue lemma we see

$$\langle \psi_{2+2}(x) \rangle \sim (-1)^\tau \int_{-\pi}^{\pi} \lambda_1(k) \langle v_2(k)|\hat{\Psi}_0(k)\rangle \langle v_1(k)|\hat{H}(k)|v_2(k)\rangle |v_1(k)\rangle e^{ikx} \frac{dk}{2\pi}$$

$$+ (-1)^\tau \int_{-\pi}^{\pi} \lambda_2(k) \langle v_1(k)|\hat{\Psi}_0(k)\rangle \langle v_2(k)|\hat{H}(k)|v_1(k)\rangle |v_2(k)\rangle e^{ikx} \frac{dk}{2\pi},$$  (37)

where $g(\tau) \sim h(\tau)$ denotes $\lim_{\tau \to \infty} g(\tau)/h(\tau) = 1$. From (37), we get

$$\left\{ \begin{array}{ll}
\frac{(-1)^\tau (\epsilon_2 s - s_1 c)\alpha}{c^2} \begin{bmatrix} -\beta \\ \alpha \end{bmatrix} & (x = 0), \\
\frac{(-1)^\tau (\epsilon_2 s - s_1 c)}{c^2} \begin{bmatrix} -csI_2\alpha - (1 - |s|)\beta \\ I_2 \{ |s| (1 - |s|)\alpha + cs\beta \} \end{bmatrix} & (x = -2), \\
\frac{(-1)^\tau (\epsilon_2 s - s_1 c)}{c^2} \begin{bmatrix} I_2 \{ cs\alpha - |s| (1 - |s|)\beta \} \\ (1 - |s|)\alpha - csI_2\beta \end{bmatrix} & (x = 2), \\
\frac{(-1)^\tau (\epsilon_2 s - s_1 c)L}{c^2} \begin{bmatrix} \pm cs\alpha - |s| (1 \mp |s|)\beta \\ |s| (1 \pm |s|)\alpha \mp cs\beta \end{bmatrix} & (x = \pm 4, \pm 6, \ldots), \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} & (x = \pm 1, \pm 3, \ldots),
\end{array} \right.$$  (38)
where \( L_x = \left\{ \frac{i(1-|s|)}{|s|} \right\}^{[x]} \). Similarly we can compute \(|\psi_{2r+1}(x)| \) as follows:

\[
|\psi_{2r+1}(x)| \sim \begin{cases}
(1-1)^{(e_{1}s-s_{2}s_{1})|1-|s|}} \left[ \frac{c(s\alpha - c\beta)}{-|s|(1-|s|)\alpha - c\beta} \right] & (x = -1), \\
(1-1)^{(e_{1}s-s_{2}s_{1})|1-|s|}} \left[ \frac{cs\alpha - |s|(1-|s|)\beta}{-c(\alpha + s\beta)} \right] & (x = 1), \\
(1-1)^{(e_{1}s-s_{2}s_{1})|1-|s|}} \left[ \frac{-c^2 \{ s(1-|s|)\alpha + c|s|\beta \}}{(1-|s|) \{ c|s|(1-|s|)\alpha + c^2 s\beta \}} \right] & (x = -3, -5, \ldots), \\
(1-1)^{(e_{1}s-s_{2}s_{1})|1-|s|}} \left[ \frac{-c^2 s\alpha + c|s|(1-|s|)\beta}{-c^2 \{ c|s|\alpha - s(1-|s|)\beta \}} \right] & (x = 3, 5, \ldots), \\
(0) & (x = 0, \pm 2, \pm 4, \ldots),
\end{cases}
\] (39)

where \( J_x = \frac{i(e_{1}s-s_{2}s_{1})L_x}{c^4|c(1-|s|)} \). From (6), (38) and (39), the proof is completed.

\( \blacksquare \)

4.2. Proof of Theorem 2

We calculate the characteristic function \( E(e^{ixX_1/t}) \) as \( t \to \infty \), where \( E(X) \) denotes the expected value of \( X \). At first, (35) can be written as

\[
|\hat{W}_{2r+2}(k)| = \hat{U}(k)^{r+1} \hat{H}(k) \hat{U}(k)^{r} |\hat{W}_0(k)| \\
= \lambda_1(k)^{2r+2} A_1(k) |v_1(k)| + \lambda_2(k)^{2r+2} A_2(k) |v_2(k)| - \lambda_3(k)^{2r+2} A_3(k) |v_1(k)| - \lambda_4(k)^{2r+2} A_4(k) |v_2(k)|, \quad (40)
\]

where \( \lambda_3(k) = \lambda_4(k) = i \)

\[
A_1(k) = \lambda_1(k) \langle v_1(k) | \hat{W}_0(k) \rangle \langle v_1(k) | \hat{H}(k) | v_1(k) \rangle, \quad (41)
\]

\[
A_2(k) = \lambda_2(k) \langle v_2(k) | \hat{W}_0(k) \rangle \langle v_2(k) | \hat{H}(k) | v_2(k) \rangle, \quad (42)
\]

\[
A_3(k) = \lambda_1(k) \langle v_2(k) | \hat{W}_0(k) \rangle \langle v_1(k) | \hat{H}(k) | v_2(k) \rangle, \quad (43)
\]

\[
A_4(k) = \lambda_2(k) \langle v_1(k) | \hat{W}_0(k) \rangle \langle v_2(k) | \hat{H}(k) | v_1(k) \rangle. \quad (44)
\]
Substituting $2\tau + 2 = t$, we can calculate the $r$-th moment of $X_t$ as follows:

$$E((X_t)^r) = \sum_{x \in \mathbb{Z}} x^r P(X_t = x)$$

$$= \int_{-\pi}^{\pi} \langle \hat{\Psi}_t(k) \rangle \left( D_x^r \hat{\Psi}_t(k) \right) \frac{dk}{2\pi}$$

$$= (t)_r \int_{-\pi}^{\pi} \left\{ \sum_{j=1}^{4} h_j(k) \right\}^r |A_j(k)|^2 - \lambda_1(k)^4 \lambda_3(k)^4 h_1(k)^2 A_3(k)\overline{A_3}(k)$$

$$- \overline{A_1}(k)^4 \lambda_3(k)^4 h_3(k)^2 A_3(k) - \lambda_2(k)^4 \lambda_4(k)^4 h_2(k)^2 A_2(k)\overline{A_2}(k)$$

$$- \lambda_3(k)^4 \lambda_4(k)^4 h_4(k)^2 A_4(k) \right\} \frac{dk}{2\pi} + O(t^{r-1}), \quad (45)$$

where $h_j(k) = D\lambda_j(k)/\lambda_j(k)$, $D = i(d/dk)$ and $(t)_r = t(t-1) \times \cdots \times (t-r+1)$.

By using the Riemann-Lebesgue lemma, we have

$$\lim_{t \to \infty} E((X_t/t)^r) = \int_{-\pi}^{\pi} \sum_{j=1}^{4} h_j^r(k) |A_j(k)|^2 \frac{dk}{2\pi}$$

$$= 0^r \Delta + \int_{-\pi}^{\pi} \sum_{j=1}^{2} h_j^r(k) |A_j(k)|^2 \frac{dk}{2\pi}, \quad (46)$$

where

$$\Delta = \int_{-\pi}^{\pi} \sum_{j=3}^{4} |A_j(k)|^2 \frac{dk}{2\pi} = \frac{(c_1 s - s_1 c)^2}{1 + |s|}. \quad (47)$$

Therefore, we obtain

$$\lim_{t \to \infty} E((X_t/t)^r) = 0^r \Delta + \int_{-\infty}^{\infty} x^r \frac{|s|}{\pi(1-x^2)^{1/2}} \left[ 1 - \left\{ |\alpha|^2 - |\beta|^2 + \frac{(\alpha\beta + \bar{\alpha}\beta)s}{c} \right\} x \right]$$

$$\times \frac{a_2 x^4 + a_1 x^2 + a_0}{c^2(1-x^2)} \mathcal{I}_{[-|c|,|c|]}(x) \ dx$$

$$= \int_{-\infty}^{\infty} \frac{x^r f(x) \ dx}{}, \quad (48)$$

where

$$a_0 = c^2, \quad (49)$$

$$a_1 = 2 s_1 c (c_1 s - s_1 c) - c_1^2, \quad (50)$$

$$a_2 = (c_1 s - s_1 c)^2. \quad (51)$$

We should remark

$$\int_{-\infty}^{\infty} g(x) \delta_0(x) \ dx = g(0). \quad (52)$$
By (35), we can compute the characteristic function $E(e^{izX_t}/t)$ as $t \to \infty$. Thus the proof of Theorem 2 is completed.

5. Summary

In the final section, we conclude and discuss the probability distribution of our walks. In the usual 2-state walk defined by the matrix $U$, the localization does not occur at all. However, if another matrix $H(\neq U)$ operates the walk at only half-time, then localization occurs. In Theorem 1, the behavior of the probability $P(X_t = x)$ was calculated as $t \to \infty$. Moreover, we found that the limit distribution of $X_t/t$ had both a delta measure and a density function from Theorem 2. The interesting problem is calculation of the limit distribution for the walk which the matrix $H$ operates more than twice.

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