ON THE LARGE TIME ASYMPOTOTICS OF KLEIN-GORDON TYPE EQUATIONS WITH GENERAL DATA

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ABSTRACT. We study the Klein-Gordon equation with general interaction terms, which may be linear or nonlinear, and space-time dependent. We initiate the study of such equations with large (non-radial) data. We prove that global solutions are asymptotically given by a free wave and a weakly localized part. The proof is based on constructing in a new way the Free Channel Wave Operator, and further tools from the recent works [30, 31, 46, 47]. This work generalizes the results of part of [30, 31] on the Schrödinger equation to arbitrary dimension, and non-radial data.

1. Introduction

The analysis of dispersive and hyperbolic wave equations and systems is of critical importance in the study of evolution equations in Physics and Geometry. It is well known that the asymptotic solutions of such equations, if they exist, show a dizzying zoo of possible solutions. Besides the “free wave”, which corresponds to a solution of the equation without interaction terms, a multitude or other solutions may appear. Such solutions are localized around possibly moving center of mass. They include nonlinear bound states, solitons, breathers, hedgehogs, vortices etc... The analysis of such equations is usually done on a case by case basis, due to this complexity. [44]

A natural question then follows: is it true that the solutions of dispersive/hyperbolic equations converge in appropriate norm ($L^2$ or $H^1$) to a free wave and independently moving localized parts?

In fact this is precisely the statement of Asymptotic Completeness in the case of N-body Scattering [39, 11, 8, 7, 40, 42, 18, 19]. In the N-body case the possible outgoing clusters are clearly identified, as bound states of subsystems.

Another situation in which recently a major progress was achieved involves nonlinear, completely integrable equations in one dimension [4, 22]

But when the interaction term includes time dependent potentials (even localized in space) and more general nonlinear terms, we do not have an a-priory knowledge of the possible asymptotic states.

In the case of time independent interaction terms, one can use spectral theory. The scattering states evolve from the continuous spectrum, and the localized part is formed by the point spectrum. Once the interaction is time dependent/nonlinear, that decomposition is not possible. In recent works on Schrödinger type problems, it was possible to obtain general results for time dependent and nonlinear interaction terms.[30, 31, 47]. In this work we initiate the study of hyperbolic equations based on the above new approach. We will focus here on the Klein-Gordon(KG for short) equation in arbitrary dimension, and with

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general interaction terms, including semi-linear interactions. These are the first results on large data multichannel scattering for nonlinear KG equations, which are not integrable.

For earlier works on time dependent potentials we mention: charge transfer Hamiltonians [53, 12, 52, 33, 35, 36, 6], decaying in time potentials and small potentials [17, 37], time periodic potentials [54, 17] and random (in time) potentials [1]. See also [2, 3]. For potentials with asymptotic energy distribution more could be done [43].

A recent progress for more general localized potentials without smallness assumptions is obtained in [46]. Some tools from this work will be used in this paper.

Turning to the nonlinear case, Tao [49, 50, 51] has shown that the asymptotic decomposition holds for NLS with inter-critical nonlinearities, in 3 or higher dimensions, in the case of radial initial data. In particular, in a very high dimension, and with an interaction that is a sum of smooth compactly supported potential and repulsive nonlinearity, Tao was able to show that the localized part is smooth and localized.

In other cases, Tao showed the localized part is only weakly localized and smooth.

Tao’s work uses direct estimates of the incoming and outgoing parts of the solution to control the nonlinear term, via Duhamel representation. In a certain sense, it is in the spirit of Enss work. See also [34].

For the critical power wave equations and wave map problems there has been a great progress in understanding the large time behavior (with large data). See e.g. [10, 9, 23, 5] and cited references.

In the case of nonlinear KG equation, there are no results on multichannel scattering with large data.

There are many major works on the stability of coherent structures e.g. [32, 15, 24, 45] and large literature on NLS, KdV and more. For the case of small data and long range type interactions, see [13, 14, 28, 29, 27, 21, 26, 25].

In contrast, the new approach of Liu-Soffer [30, 31] is based on proving a-priori estimates on the full dynamics, which hold in a suitably defined domains of the extended phase-space. That is, one proves propagation estimates in domains exterior to the support of the interaction. Similar propagation observables were used in many other works, mostly linear problems, with time independent potentials. See e.g. [20, 11, 8] and cited references. In this way it was possible to show the asymptotic decomposition for general localized interactions, including time and space dependent ones. Radial initial data is assumed, to ensure the localization of the nonlinear part of the interaction terms.

More detailed information can be obtained on the localized part of the solution. Besides being smooth, its expanding part (if it exists) can grow at most like $|x| \leq \sqrt{t}$, and furthermore, is concentrated in a thin set of the extended phase-space.

The free part of the solution concentrates on the propagation set where $x = vt, v = 2P$, and $P$ being the dual to the space variable, the momentum, is given by the operator $-i\nabla_x$.

The weakly localized part is found to be localized in the regions where

$$ |x|/t^\alpha \sim 1 \quad \text{and} \quad |P| \sim t^{-\alpha}, \quad \forall \ 0 \leq \alpha \leq 1/2. $$

It therefore shows that the spreading part follows a self similar pattern. See [30, 31].

The method of proof is based on three main parts: first, construct the Free Channel Wave Operator. Then prove localization of the remainder localized part, and use it to prove the smoothness of the localized part. Finally, by using further propagation estimates which
are adapted to localized solutions, prove the concentration on thin sets of the phase-space corresponding to self-similar solutions. In this work we will mainly do the first part, and some of the second part of [30, 31].

It should be emphasized that the spreading localized solutions, if they exist, were shown to have a non-small nuclei part around the origin. This is true for both the results of Tao and Liu-Soffer.

Therefore, these are not pure self-similar solutions, as they appear in the special cases of critical nonlineairities. See e.g. [48, 10]. We expect a similar behavior of the weakly localized part of the solutions of KG equations.

We will follow here this point of view. It was generalized in [47] to include non-radial data and interactions, and with localized interactions to arbitrary dimension, in the case of the Schrödinger problem. This generalization is based on refined localization of the channel wave operators. By localizing around the phase-space support of the free wave, we get a sharper decomposition of the localized and scattering parts. Therefore, we can avoid the need for localization of the interactions in some cases. The idea of sharp localization was used in other ways in the study of long-range scattering theory, e.g. in [38, 41].

Here we follow these constructions also for the KG case, mainly by viewing the dynamics of the KG equation as generated by a couple of Schrödinger type equations, with dispersion relation given by $\sqrt{p^2 + m^2}$ and group velocity $v = p / \sqrt{p^2 + m^2}$.

The extension to this case of the previous methods proceeds along similar ideas, at least when the interaction terms are localized in space.

When the interaction terms are not localized around a given point, but only satisfy $L^p$ decay conditions (with $1 \leq p \leq 2$), the situation is more complicated.

The problem comes from the fact that $L^p$ decay estimates for the KG equation require control of derivatives of the initial data. This is due to the poor dispersion for hyperbolic equations for frequencies near infinity.

We deal with this problem by introducing an extra cutoff of high frequencies into the construction of the channel wave operators, as we explain next.

The key tool from scattering theory that is used to study multichannel scattering is the notion of channel wave operator, which we denote by

$$\Omega^*_a \equiv s - \lim_{t \to \infty} U_a(-t)U(t)u(0).$$

Here the limit is taken in the strong sense in a suitable Hilbert space. $U(t)u(0)$ is the solution of the KG equation with initial data $u(0), \dot{u}(0)$ and dynamics (linear or nonlinear) $U(t) = U(t, 0)$ generated by a hamiltonian $H(t)$ and the equation:

$$u_{tt} + Hu = 0.$$

Typically, for KG equations, we choose

$$H = -\Delta + 1 + N_0.$$

$N_0$ may depend on $u, t, x$.

The asymptotic dynamics $U_a$ is generated by a Hamiltonian $H_a$ for a given channel denoted by $a$. In this work we will only construct the free channel, where $H_a = -\Delta + 1$

A crucial observation is that one can modify the definition of the Channel wave operators to

$$\Omega^*_a \equiv s - \lim_{t \to \infty} U_a(-t)J_a U(t)\psi(0).$$
1.1. Problem and Results. We consider a general class of Klein-Gordon type equations of the form:

\begin{equation}
(\Box + 1)u = -N(u, x, t) = -V(x, t)u - N_0(u)u \\
\tilde{u}(0) := (u(x, 0), \dot{u}(x, 0)) = (u_0(x), \dot{u}_0(x)) \in \mathcal{S}, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},
\end{equation}

for a Hilbert space $\mathcal{S} = H^1 \oplus L^2$, with space dimension $n \geq 1$.

Here $\Box := \partial_t^2 - \Delta_x$ and $N_0(u)$ is real.

The term $N(u, x, t)$ includes a combination of the following cases:

1. (Theorem 1.5, $n \geq 1$) **Local time-dependent potential** $N(u, x, t) = V(x, t)u$, such that either $V(x, t) \in L^{\infty}_t L^{\delta, \infty}_x(\mathbb{R}^n \times \mathbb{R})$ or $V(x, t) \in L^{\infty}_t L^{3, \infty}_x(\mathbb{R}^n \times \mathbb{R})$ for some $\delta > 1$. In addition,

\begin{equation}
\|V(x, t)u(t)\|_{L^{\infty}_t L^{\delta, \infty}_x} \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{\mathcal{H}^1}.
\end{equation}

Furthermore $\tilde{u}(0) \in \mathcal{S}$ and such that it leads to a global uniformly bounded solution:

\begin{equation}
C(\|\tilde{u}(0)\|_{\mathcal{S}}) := \sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{\mathcal{S}} \lesssim \|\tilde{u}(0)\|_{\mathcal{S}} 1.
\end{equation}

2. (Theorem 1.3 and Theorem 1.4) When $n \geq 1$, $N(u, x, t) = V(x, t)u$, such that $V(x, t) \in L^{\infty}_t L^{3, \infty}_x(\mathbb{R}^n \times \mathbb{R})$. And $\tilde{u}(0) \in H^1 \oplus L^2$ leads to a global solution with

\begin{equation}
C(\|\tilde{u}(0)\|_{\mathcal{S}}) := \sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{\mathcal{S}} \lesssim \|\tilde{u}(0)\|_{\mathcal{S}} 1.
\end{equation}

Here $(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \mathbb{R}$,

\begin{equation}
L^p_{\delta, \infty} := \{f(x) : \langle x \rangle^\delta f(x) \in L^p_x\}, \quad \text{for } 1 \leq p \leq \infty
\end{equation}

and denote weighted Agmon-Sobolev space

\begin{equation}
\mathcal{H}^\sigma_0 := \{f(x) : \|\langle x \rangle^\sigma f(x)\|_{L^2_{\delta, \infty}} < \infty\}.
\end{equation}

Let $\mathcal{S}_\delta$ denote the complex Hilbert space $\mathcal{H}_0^\delta \oplus \mathcal{H}_0^\delta$ of vector functions $\vec{v} = (v_1, v_2)$ with the norm

\begin{equation}
\|\vec{v}\|_{\mathcal{S}_\delta} = \|v_1\|_{\mathcal{H}^\delta_0} + \|v_2\|_{\mathcal{H}^\delta_0} < \infty.
\end{equation}

We use $\mathcal{H}^\sigma$, $\mathcal{S}$ to denote, respectively, $\mathcal{H}^\sigma_0$ and $\mathcal{S}_0$ for simplicity.

Throughout the paper, we always assume that there is a global $\mathcal{S}$ solution $(u(t), \dot{u}(t))$ to (1.3).

**Remark 1.1.** When there exists a nonlinearity $N(u) = N_0(u)u$, we regard $N_0(u)$ as a linear time-dependent perturbation.

**Remark 1.2.** Typical example for (1) is

\begin{equation}
N(u, x, t) = V(x, t)u + a(x)u^2 + b(x)u^3, \quad \text{in 1 dimension}
\end{equation}

provided that we have global existence in $\mathcal{H}^1 \oplus \mathcal{H}^0$ (for notation, see (1.8)). See Theorem 4.3 for more details.

**Typical examples for (2) are**

\begin{equation}
N(u, x, t) = V(x, t)u + \lambda u^3 + \lambda' u^4, \quad \text{in 3 or higher dimensions.}
\end{equation}
More generally, one can control
\[(\Box + 1 + V(x, t))u = f(u)u, \quad 1 + V(x, t) \geq v_0 > 0,\]
with
\[\sup_t \|f(u)\|_{L^2_x} < \infty.\]

Here \(V(x, t)\) can be of general charge transfer type, that is, \(V(x, t) = \sum_{j=1}^N V_j(x - g_j(t)v_j, t)\).

See Theorem 4.4 for more details.

We write \(X \lesssim Y, Y \gtrsim X\) to indicate \(X \leq CY\) for some constant \(C > 0\) and \(X \lesssim_a Y\) to indicate \(X \leq CY\) for some \(C = C(a) > 0\).

Let \(F_c(\lambda \leq a) := 1 - \tilde{F}_c(\lambda/a), \; F_j(\lambda > a) := \tilde{F}_j(\lambda/a), \; j = 1, 2,\)
\[(\tilde{F}_c(\lambda \leq a) := \tilde{F}_c(\lambda/a), \; \tilde{F}_j(\lambda \leq a) := 1 - \tilde{F}_j(\lambda/a)).\]

Let \(U(t, 0)\) denote the dynamical group of KG equation (1.3), that is, for \(\vec{u} \in S,\)
\[(1.16) \quad U(t, 0)\vec{u} = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = : (u(t), \dot{u}(t))^T\]
and \(U_0(t, 0)\), the dynamical group of the free KG equation. Let \(P := -i\nabla_x\). Throughout the paper,
\[(1.17) \quad F_l\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} F_l u(t) \\ F_l \dot{u}(t) \end{pmatrix}, \quad l := 1, 2, c.\]

For \(\vec{v} = (v_1(x), v_2(x))^T\), let
\[(1.18) \quad \langle P \rangle^\delta_1\vec{v} := \begin{pmatrix} \langle P \rangle^\delta v_1(x) \\ v_2(x) \end{pmatrix}\]
and
\[(1.19) \quad \langle P \rangle^\delta_2\vec{v} := \begin{pmatrix} v_1(x) \\ \langle P \rangle^\delta v_2(x) \end{pmatrix}.

Let
\[(1.20) \quad \Omega^*(t) := \begin{pmatrix} u_{\Omega}(t) \\ \dot{u}_{\Omega}(t) \end{pmatrix} := U_0(0, t)U(t, 0)\vec{u}(0),\]
\[\Omega_\alpha^*(t) := \begin{pmatrix} \frac{|x|}{r^\alpha} \leq 1 \end{pmatrix} \tilde{F}_1(|P| \leq \ell^\delta)\Omega(t)^* \]
\[(1.21) \quad \Omega_\alpha^*(t) := \begin{pmatrix} \frac{|x|}{r^\alpha} \leq 1 \end{pmatrix} \tilde{F}_1(|P| \leq \ell^\delta)U_0(0, t)U(t, 0)\]
in 3 or higher dimensions. Here are our main results:
Theorem 1.3. Let \((u(t), \dot{u}(t))\) be a global solution to equation (1.3) in \(S\) and \(H_0 := -\Delta_x\). For \(n \geq 3\), given \(\alpha, \beta > 0\) satisfying
\[
\frac{n(1 - \alpha - \beta) - 3\beta}{2} > 1,
\]
if \(V(x, t) \in L_t^\infty L_x^2(\mathbb{R}^n \times \mathbb{R})\), the channel wave operator
\[
\Omega_\alpha^{\beta, \gamma} := s^{-\lim}_{t \to \infty} \Omega_\alpha^{\beta, \gamma}(t)
\]
exists from \(S \to L_x^2 \oplus L_x^2\), that is,
\[
u := s^{-\lim}_{t \to \infty} \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \mathcal{F}_1(|P| \leq t^\beta) u_\Omega(t)
\]
exists in \(L_x^2\) and
\[
\dot{u}_+ := s^{-\lim}_{t \to \infty} \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \mathcal{F}_1(|P| \leq t^\beta) \dot{u}_\Omega(t)
\]
exists in \(L_x^2\).

Based on Theorem 1.3, we have that as \(t \to \infty\), for all \(a \in [0, 1)\),
\[
\|\mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 u_\Omega(t) - \dot{u}_+\|_{\mathcal{H}^a} \to 0,
\]
and
\[
\|\langle P \rangle^{-\frac{1}{2}} \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \dot{u}_\Omega(t) - \dot{u}_+\|_{\mathcal{H}^a} \to 0
\]
for some \(\dot{u}_+, \dot{u}_+ \in \mathcal{H}^a\):

Theorem 1.4. Let \((u(t), \dot{u}(t))\), \(V\) be as in Theorem 1.3. Then given \(\alpha > 0\) satisfying
\[
\frac{n(1 - \alpha - \beta) - 3\beta}{2} > 1,
\]
for some \(\beta > 0\), for all \(a \in [0, 1)\),
\[
\|\mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 u_\Omega(t) - \dot{u}_+\|_{\mathcal{H}^a} \to 0,
\]
and
\[
\|\langle P \rangle^{-\frac{1}{2}} \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \dot{u}_\Omega(t) - \dot{u}_+\|_{\mathcal{H}^a} \to 0
\]
for some \(\dot{u}_+, \dot{u}_+ \in \mathcal{H}^a\).

Proof. Based on Theorem 1.3, for (1.29), it follows from that
\[
\mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 u_\Omega(t) = \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \mathcal{F}_1(|P| \leq t^\beta) u_\Omega(t) + \mathcal{F}_c \left( \frac{x}{t^\alpha} \right) \leq 1 \mathcal{F}_1(|P| > t^\beta) u_\Omega(t)
\]
with
\[
\|\mathcal{F}_1(|P| > t^\beta) u_\Omega(t)\|_{\mathcal{H}^a} = \|\langle P \rangle^{-\frac{1}{2}} \mathcal{F}_1(|P| > t^\beta) u_\Omega(t)\|_{\mathcal{H}^a} \leq \frac{1}{t^\beta(1 - a)} \|u(t)\|_{\mathcal{H}^a} + \|\dot{u}(t)\|_{\mathcal{L}_x^2} \leq \frac{1}{t^\beta(1 - a)} C(||u(0)||_{\mathcal{S}}) \to 0
\]
as $t \to \infty$. For (1.30), it follows from that

\[
\iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \dot{\Omega}(t) = \iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \iota \Gamma_1(\{P \leq \ell^3\}) \dot{\Omega}(t) + \iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \iota \Gamma_1(\{P > \ell^3\}) \dot{\Omega}(t)
\]

with

\[
\|\langle P \rangle^{-1} \iota \Gamma_1(\{P \leq \ell^3\}) \dot{\Omega}(t)\|_{\mathcal{H}^0} = \frac{1}{\langle P \rangle} \iota \Gamma_1(\{P \leq \ell^3\}) \langle P \rangle^{-1-a} \iota \dot{\Omega}(t)\|_{\mathcal{H}^0} \leq \frac{1}{\beta(1-a)} \|u(t)\|_{\mathcal{H}^0} + \|\dot{u}(t)\|_{L^2_x} \leq \frac{1}{\beta(1-a)} C(\|\dot{u}(0)\|_{L^2_x}) \to 0
\]
as $t \to \infty$. $\ddot{u}_+ \in \mathcal{H}^\alpha$ since $\dot{u}_+(t) \in \mathcal{H}_x^1$ and $\ddot{u}_+(t) \in L^2_x$.

**Theorem 1.5.** Let $(u(t), \dot{u}(t))$ be a global solution to equation (1.3) in $\mathcal{S}$. If $u(t), V(x, t)$ are as in 1 (Localized time-dependent potential), then for $n \geq 1, b \in [0, 1/2), \alpha \in (b, \min(1 - b, 1 - \frac{2\alpha}{n}))$,

1. the free channel wave operator

\[
\Omega_{a,b}^* := \lim_{t \to \infty} \frac{|t|}{t^\beta} \iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \iota \Gamma_1(\{P \leq \ell^3\}) \dot{\Omega}(t)
\]

exists from $\mathcal{S} \to \mathcal{S}$. In particular,

\[
u_+ := \lim_{t \to \infty} \frac{|t|}{t^\beta} \iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \iota \Gamma_1(\{P \leq \ell^3\}) \dot{\Omega}(t)
\]

exists in $\mathcal{H}_x^1$ and

\[
\ddot{u}_+ := \lim_{t \to \infty} \frac{|t|}{t^\beta} \iota \Gamma_c(\frac{|x|}{t^\alpha} \leq 1) \iota \Gamma_1(\{P \leq \ell^3\}) \dot{\Omega}(t)
\]

exists in $L^2_x$.

2. furthermore, given $e > 0$, if $\delta > 2$ and if $\alpha, b$ also satisfy

\[
e > 1 - b > \alpha > b \geq 0,
\]

there exist $u^1_{+, b, 0}, u^1_{+, b, b} \in \mathcal{H}_x^1, \ddot{u}^1_{+, b, b} \in L^2_x$ such that we have the following asymptotic decomposition

\[
\lim_{t \to \infty} \|u(t) - \cos(t \sqrt{H_0 + 1}) u^1_{+, b, b} - \sin(t \sqrt{H_0 + 1}) \ddot{u}^1_{+, b, b} - u_{+, b, 0}(t)\|_{\mathcal{H}_x^1} = 0
\]

and

\[
\lim_{t \to \infty} \|\dot{u}(t) + \sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} \ddot{u}^1_{+, b, b} - \cos(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} \ddot{u}^1_{+, b, b} - v_{+, b, 0}(t)\|_{L^2_x} = 0
\]

where $u_{+, b, 0}, v_{+, b, 0}$ are the weakly localized parts of the solution, with the following property: It is weakly localized in the region $|x| \leq t^\alpha$, in the following sense

\[
\langle P \rangle u_{+, b, 0}(t), |x| \langle P \rangle u_{+, b, 0}(t) \rangle_{L^2_x} \lesssim t^\alpha C(\|\dot{u}(0)\|_{L^2_x})^2,
\]

and

\[
\langle v_{+, b, 0}(t), |x| v_{+, b, 0}(t) \rangle_{L^2_x} \lesssim t^\alpha C(\|\dot{u}(0)\|_{L^2_x})^2.
\]

**Remark 1.6.** $e = 1/2 + 0$ if we choose $\alpha, b$ wisely.
Remark 1.7. When $n = 1$, (1.4) is satisfied when either $V(x, t) \in L^2_t L^2_0 (\mathbb{R}^n \times \mathbb{R})$ or $V(x, t) \in L^\infty_t L^2_{\delta, 0} (\mathbb{R}^n \times \mathbb{R})$ for some $\delta > 0$ and

\begin{equation}
\sup_t \|u(t)\|_{H^1} \leq \|\bar{u}(0)\|_S.
\end{equation}

2. Preliminaries

2.1. Free KG equations. Let $\bar{u}_0(t) := (u_0(t), \dot{u}_0(t))$ be the solution to a free KG equation

\begin{equation}
\begin{cases}
(\Box + 1)u_0(t) = 0 \\
\dot{u}_0(0) = \bar{u}(0) = (u(x, 0), \dot{u}(x, 0)) \in S, 
\end{cases}
\end{equation}

Let $H_0 := -\Delta_x$. $u_0(t)$ and $\dot{u}_0(t)$ have following representation

\begin{equation}
u_0(t) = \cos(t \sqrt{H_0 + 1})u(0) + \frac{\sin(t \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}} \dot{u}(0)
\end{equation}

and

\begin{equation}\dot{u}_0(t) = -\sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1}u(0) + \cos(t \sqrt{H_0 + 1})\dot{u}(0).
\end{equation}

Let

\begin{equation}A_0 := \begin{pmatrix} 0 & -1 \\ H_0 + 1 & 0 \end{pmatrix}.
\end{equation}

(2.1) is equivalent to

\begin{equation}\partial_t [\bar{u}_0(t)] = -A_0 \bar{u}_0(t).
\end{equation}

So $\bar{u}_0(t)$ has another representation

\begin{equation}\bar{u}_0(t) = e^{-tA_0} \bar{u}(0),
\end{equation}

that is,

\begin{equation}U_0(t, 0) = e^{-tA_0}.
\end{equation}

In the following context, we need following standard dispersive decay estimate for the KG propagator, see for instance Hörmander [16](Corollary 7.2.4) for a proof.

Lemma 2.1. We have uniformly for all $t \in \mathbb{R}$ that

\begin{equation}
\|e^{itP} f\|_{L^\infty_t (\mathbb{R}^n)} \leq \frac{1}{\langle t \rangle^{n/2}} \|P^{n/2} f\|_{L^1(\mathbb{R}^n)}.
\end{equation}

2.2. Perturbed KG and its Duhamel’s formulas. Let $(u(t), \dot{u}(t))$ be the solution to a perturbed KG equation

\begin{equation}
\begin{cases}
(\Box + 1)u(t) = -V(x, t)u(t) \\
\dot{u}(0) = (u(x, 0), \dot{u}(x, 0)) \in S, 
\end{cases}
\end{equation}

Let

\begin{equation}V(x, t) := \begin{pmatrix} 0 & 0 \\ V(x, t) & 0 \end{pmatrix}.
\end{equation}

(2.9) implies

\begin{equation}\partial_t [\bar{u}(t)] = -(A_0 + V(x, t))\bar{u}(t),
\end{equation}

\begin{equation}V(x, t) := \begin{pmatrix} 0 & 0 \\ V(x, t) & 0 \end{pmatrix}.
\end{equation}
that is,

\[ (2.12) \quad \partial_t[U(t,0)\vec{u}(0)] = -(\mathcal{A}_0 + \mathcal{V}(x,t))U(t,0)\vec{u}(0). \]

Based on (2.7) and (2.12), we derive a Duhamel’s formula for \( \vec{u}(t) \)

\[ (2.13) \quad \vec{u}(t) = U_0(t,0)\vec{u}(0) - \int_0^t ds U_0(t,s)\mathcal{V}(x,s)\vec{u}(s) \]

where we use

\[ (2.14) \quad \frac{d}{ds}[U_0(t,s)U(s,0)] = U_0(t,0)\frac{d}{ds}[U_0(0,s)U(s,0)] = -U_0(t,s)\mathcal{V}(x,s)U(s,0) \]

and

\[ (2.15) \quad \vec{u}(t) = U_0(t,0)\vec{u}(0) + \int_0^t ds \frac{d}{ds}[U_0(t,s)\vec{u}(s)], \]

that is,

\[ (2.16) \quad u(t) = \left( \cos(t \sqrt{H_0 + 1})u(0) + \frac{\sin(t \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}}\vec{u}(0) \right) + \int_0^t ds \frac{e^{-i(t-s)\sqrt{H_0+1}}}{2i \sqrt{H_0 + 1}}V(s)u(s) - \int_0^t ds \frac{e^{i(t-s)\sqrt{H_0+1}}}{2i \sqrt{H_0 + 1}}V(s)u(s) \]

and

\[ (2.17) \quad \dot{u}(t) = -\sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1}u(0) + \cos(t \sqrt{H_0 + 1})\vec{u}(0) - \int_0^t ds \frac{e^{i(t-s)\sqrt{H_0+1}}}{2}V(s)u(s) - \int_0^t ds \frac{e^{-i(t-s)\sqrt{H_0+1}}}{2}V(s)u(s). \]

Recall that

\[ (2.18) \quad \Omega(t)^*\vec{u}(0) = U_0(0,t)\vec{u}(t) = \begin{pmatrix} u_\Omega(t) \\ \dot{u}_\Omega(t) \end{pmatrix}. \]

Based on (2.13), a Duhamel’s formula for \( \Omega(t)^*\vec{u}(0) \) is given by

\[ (2.19) \quad \Omega(t)^*\vec{u}(0) = \vec{u}(0) - \int_0^t ds U_0(0,s)\mathcal{V}(x,s)\vec{u}(s), \]

that is,

\[ (2.20) \quad u_\Omega(t) = u(0) + \int_0^t ds \frac{\sin(s \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}}V(s)u(s) \]

and

\[ (2.21) \quad \dot{u}_\Omega(t) = \dot{u}(0) - \int_0^t ds \cos(s \sqrt{H_0 + 1})V(s)u(s). \]
2.3. Estimates for interaction terms.

Lemma 2.2. If \( V(x, t) \in L_\infty^\alpha L_2^\beta(\mathbb{R}^n \times \mathbb{R}) \), we have that for \( t \geq 1, n \geq 3, \) and for \( \alpha \in (0, 1 - \frac{2}{n}) \), \( \beta > 0. \)

\[
(2.22) \quad \| \mathcal{F} c \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [u_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s),
\]

\[
(2.23) \quad \| \mathcal{F} c \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s),
\]

\[
(2.24) \quad \| \mathcal{F} c \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [u_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s),
\]

\[
(2.25) \quad \| \mathcal{F} c \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s),
\]

\[
(2.26) \quad \| (\mathcal{F} c [u_{\Omega}(t), \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [u_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s)^2,
\]

\[
(2.27) \quad \| (\mathcal{F} c [\tilde{u}_{\Omega}(t), \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s)^2,
\]

and

\[
(2.28) \quad \| (\mathcal{F} c [\tilde{u}_{\Omega}(t), \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s)^2.
\]

Proof. Let

\[
(2.30) \quad a(t) := (\mathcal{F} c [\tilde{u}_{\Omega}(t), \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [u_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(t) u(t) \|_{L_\infty^\alpha L_2^\beta},
\]

\[
(2.31) \quad (\mathcal{F} c [\tilde{u}_{\Omega}(t), \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \leq t^\alpha \| \partial_t [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(t) u(t) \|_{L_\infty^\alpha L_2^\beta}.
\]

Using Cauchy-Schwartz’s inequality, Hölder’s inequality and Lemma 2.1 in this order, we have that for \( |t| \geq 1, \)

\[
(2.32) \quad |a(t)| \lesssim_n \| \mathcal{F} c u_{\Omega}(t) \|_L_2^n \times \| \mathcal{F} c \mathcal{F} c [\tilde{u}_{\Omega}(t)] \|_L_2^n \lesssim_n \frac{1}{|t|} \| V(t) u(t) \|_{L_\infty^\alpha L_2^\beta} \times \| \mathcal{F} c \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| \mathcal{F} c \|_L_2^n \times \| \mathcal{F} c \|_L_2^n \|_t^{|x|} \leq 1) \mathcal{F} c \|_t^{|P|} \frac{1}{|t|^{(1-\alpha-\beta)/2}} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} \lesssim_n \frac{1}{|t|} \| V(x, t) \|_{L_\infty^\alpha L_2^\beta} C(\| \tilde{u}(0) \|_s)^2.
\]

Here we also use

\[
(2.33) \quad \| \mathcal{F} c u_{\Omega}(t) \|_L_2^n \leq \| \tilde{u}(t) \|_s \leq C(\| \tilde{u}(0) \|_s)
\]
and
\begin{equation}
\|\langle P \rangle^{\alpha\delta} \tilde{F}_1(|P| \leq t^\beta)\|_{L^1_t \rightarrow L^1_x} \leq \langle t \rangle^{(n+1)\beta/2}.
\end{equation}

Here actually, we have
\begin{equation}
\|\mathcal{F}_c \tilde{F}_1(|P| \leq t^\beta) \partial_i [u_{\Omega}(t)]\|_{L^2_t} \leq \frac{1}{|t|^{\frac{1}{2}-\frac{n\beta}{2}}} \|V(x, t)\|_{L^\infty_t L^2_x} C\langle \tilde{u}(0)\rangle_S.
\end{equation}

and we get (2.22). Since
\begin{equation}
\langle \tilde{F}_1 u_{\Omega}(t), \mathcal{F}_c \tilde{F}_1 \partial_i [u_{\Omega}(t)] \rangle_{L^2_t} = \langle \tilde{F}_1 \partial_i [u_{\Omega}(t)], \mathcal{F}_c \tilde{F}_1 u_{\Omega}(t) \rangle_{L^2_t},
\end{equation}
we get (2.27). For (2.28), let
\begin{equation}
\dot{u}(t) := \langle \tilde{F}_1 \dot{u}_{\Omega}(t), \mathcal{F}_c \tilde{F}_1 \partial_i [u_{\Omega}(t)] \rangle_{L^2_t}
\end{equation}
\begin{equation}
= - \langle \tilde{F}_1 \dot{u}_{\Omega}(t), \mathcal{F}_c \tilde{F}_1 \cos(t \sqrt{H_0 + 1}) V(t) u(t) \rangle_{L^2_t}.
\end{equation}

According to (2.21), Cauchy-Schwarz inequality, Hölder’s inequality and Lemma 2.1 in this order, we have that for \( |t| \geq 1 \),
\begin{equation}
|\dot{u}(t)| \leq \|\tilde{F}_1 \dot{u}_{\Omega}(t)\|_{L^2_t} \times \|\mathcal{F}_c \tilde{F}_1 \cos(t \sqrt{H_0 + 1}) V(t) u(t)\|_{L^2_t}
\end{equation}
\begin{equation}
\leq \|\tilde{F}_1 \dot{u}_{\Omega}(t)\|_{L^2_t} \times \|\mathcal{F}_c\|_{L^2_t \rightarrow L^2_t} \|\langle P \rangle^{\alpha\delta} V(x, t) u(t)\|_{L^\infty_t L^2_x}
\end{equation}
\begin{equation}
\leq \frac{1}{|t|^{\frac{1}{2}-\frac{n\beta}{2}}} \|V(x, t)\|_{L^\infty_t L^2_x} C\langle \tilde{u}(0)\rangle_S^2.
\end{equation}

Here we use
\begin{equation}
\|\langle P \rangle^{\alpha\delta} \tilde{F}_1(|P| \leq t^\beta)\|_{L^1_t \rightarrow L^1_x} \leq \langle t \rangle^{(n+3)\beta/2}.
\end{equation}

Here actually, we have
\begin{equation}
\|\mathcal{F}_c \tilde{F}_1(|P| \leq t^\beta) \partial_i [u_{\Omega}(t)]\|_{L^2_t} \leq \frac{1}{|t|^{\frac{1}{2}-\frac{n\beta}{2}}} \|V(x, t)\|_{L^\infty_t L^2_x} C\langle \tilde{u}(0)\rangle_S.
\end{equation}

and we get (2.23). Since
\begin{equation}
\langle \tilde{F}_1 \dot{u}_{\Omega}(t), \mathcal{F}_c \tilde{F}_1 \partial_i [u_{\Omega}(t)] \rangle_{L^2_t} = \langle \tilde{F}_1 \partial_i [u_{\Omega}(t)], \mathcal{F}_c \tilde{F}_1 u_{\Omega}(t) \rangle_{L^2_t},
\end{equation}
we get (2.29). (2.24), (2.25) follow by using (2.22), (2.23) and by using
\begin{equation}
\tilde{F}_1(|P| \leq t^\beta) \mathcal{F}_c \left( \frac{|x|}{t^\rho} \right) \leq 1) \mathcal{F}_1(|P| > t^\beta) \mathcal{F}_c \left( \frac{|x|}{t^\rho} \right) \leq 1) \mathcal{F}_1
\end{equation}
and
\begin{equation}
\|\mathcal{F}_1(|P| > t^\beta) \mathcal{F}_c \left( \frac{|x|}{t^\rho} \right) \leq 1) \|_{L^2_t \rightarrow L^2_t} \sim N \left( \frac{1}{t^\rho} \right).
\end{equation}

**Lemma 2.3.** If either \( V(x, t) \in L^\infty_t L^2_{\delta, x}(\mathbb{R}^n \times \mathbb{R}) \) or \( V(x, t) \in L^\infty_t L^\infty_{\delta+n/2, x}(\mathbb{R}^n \times \mathbb{R}) \) for some \( \delta > 1 \) and if
\begin{equation}
\|V(x, t) u(t)\|_{L^\infty_t L^2_x} \leq \sup_{t \in \mathbb{R}} \|u(t)\|_{L^1_t},
\end{equation}
then for \( b \in [0, 1/2) \), \( \alpha \in (0, \min(1 - b, 1 - \frac{2-\delta}{n})) \), \( t \geq 1, n \geq 1 \),
\begin{equation}
\|\mathcal{F}_c \mathcal{F}_1 P u_{\Omega}(t)\|_{L^2_t} \leq \frac{1}{t^{1+\beta}} \|V(x, t)\|_{L^\infty_t L^\infty_{\delta, x}} C\langle \tilde{u}(0)\rangle_S.
\end{equation}
\begin{align}
(2.47) & \quad \| \mathcal{F}_c \mathcal{F}_i \dot{u}_\Omega(t) \|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S), \\
(2.48) & \quad \| \mathcal{F}_c \mathcal{F}_i(P) u_\theta(t) \|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S), \\
(2.49) & \quad \| \mathcal{F}_c \mathcal{F}_i \dot{u}_\Omega(t) \|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S), \\
(2.50) & \quad \left| (\mathcal{F}_i \mathcal{F}_c \mathcal{F}_i(P) \partial_t [u_\Omega(t)], (P) u_\Omega(t)) \right|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S)^2, \\
(2.51) & \quad \left| (\langle P \rangle u_\Omega(t), \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i(P) \partial_t [u_\Omega(t)]) \right|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S)^2, \\
(2.52) & \quad \left| (\mathcal{F}_c \mathcal{F}_i \mathcal{F}_i \partial_t [u_\Omega(t)], \dot{u}_\Omega(t)) \right|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S)^2, \\
\text{and} \\
(2.53) & \quad \left| (\dot{u}_\Omega(t), \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i \partial_t [u_\Omega(t)]) \right|_{L^2_t} \lesssim n, b, \alpha, \delta \frac{1}{t^{1+\beta}} \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S)^2, \\
\text{with} \\
(2.54) & \quad \beta := \min \left\{ \frac{(1 - \alpha)n}{2}, \frac{\delta}{2}, 1, \tilde{\delta} - 1 \right\} > 0 \\
\text{for } \tilde{\delta} = \frac{1+\delta}{2}.
\end{align}

**Proof.** Write \( \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i(P) \partial_t [u_\Omega(t)] \) as
\begin{align}
(2.55) & \quad \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i(P) \partial_t [u_\Omega(t)] = \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i(P) \sin t \frac{\sqrt{H_0 + 1}}{\sqrt{H_0 + 1}} V(t) u(t) \\
(2.56) & \quad = \mathcal{F}_i \mathcal{F}_c \mathcal{F}_i \sin t \frac{\sqrt{H_0 + 1}}{\sqrt{H_0 + 1}} V(t) u(t).
\end{align}

Based on assumption (2.45),
\begin{align}
(2.57) & \quad \| V(x, t) u(t) \|_{L^{\infty}_{t, x} \cap L^{1}_{t, x}} \lesssim \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S) \\
\text{when } V(x, t) \in L^{\infty}_{t} L^{2}_{x} \text{ and} \\
(2.58) & \quad \| V(x, t) u(t) \|_{L^{\infty}_{t, x} \cap L^{1}_{t, x}} \lesssim \| V(x, t) \|_{L^{\infty}_{t, x}} C(\| \tilde{u}(0) \|_S) \\
\text{when } V(x, t) \in L^{\infty}_{t} L^{2}_{\delta+n/2, x}.
\end{align}

Let
\begin{align}
(2.59) & \quad \mathcal{F}^-_i := \mathcal{F}_i(|\cdot| > 1) \mathcal{F}_i(|\cdot| \leq 100) \\
\text{and} \\
(2.60) & \quad \mathcal{F}^+_i := \mathcal{F}_i(|\cdot| > 100).
\end{align}

(2.46) to (2.53) follow from the following two estimates. One is that for \( t \geq 1 \),
\begin{align}
(2.61) & \quad \| \mathcal{F}_c \left( \frac{|x|}{t^a} \leq 1 \right) \mathcal{F}^-_i e^{\pm i t \sqrt{H_0 + 1}} \langle \cdot \rangle^{-\tilde{\delta}} \|_{L^1_t \rightarrow L^2_x} \lesssim \\
& \quad \| \mathcal{F}_c \left( \frac{|x|}{10 t^a} \leq 1 \right) \mathcal{F}(\frac{|x|}{t^a} \leq 1) \mathcal{F}_c e^{\pm i t \sqrt{H_0 + 1}} \langle \cdot \rangle^{-\tilde{\delta}} \chi(|x| \geq \sqrt{t}) \|_{L^1_t \rightarrow L^2_x +}
\end{align}
\[ \| \mathcal{F}_c(\frac{|x|}{t^\alpha}) \|_{L^2_{t,x}} \leq 1 \| \mathcal{F}_1 e^{it\sqrt{\mathcal{H}_0+1}}(x)^{-\delta} \chi(|x| < \sqrt{t}) \|_{L^1_{t} \to L^\infty_x} \lesssim_{\alpha} \frac{t^{\alpha/2} + 1}{t^N}, \]

where we use that for \( l = 0, 1 \), Lemma 2.1 implies that

\[ \| \mathcal{F}_c(\frac{|x|}{t^\alpha}) \|_{L^2_{t,x}} \leq 1 \| \mathcal{F}_1 e^{it\sqrt{\mathcal{H}_0+1}}(x)^{-\delta} \chi(|x| \geq \sqrt{t}) \|_{L^1_{t} \to L^\infty_x} \lesssim_{\alpha} \frac{t^{\alpha/2} + 1}{t^N}, \]

\[ \| \mathcal{F}_1(\mathbb{P}) \|_{L^2_{t,x}} \leq 100 \| \mathcal{F}_1 e^{it\sqrt{\mathcal{H}_0+1}} \|_{L^1_{t} \to L^\infty_x} \lesssim_{\alpha} \frac{1}{t^N}, \]

which follows from the method of non-stationary phase since

\[ \mathcal{F}_c(\frac{|x|}{t^\alpha}) \leq 1 e^{i|x|q} e^{iq^2} e^{-iqy} \chi(|y| < \sqrt{t}) = \frac{1}{i(x \cdot \hat{q} + 2t|q| - y \cdot \hat{q})} \partial_{|q|} \mathcal{F}_c(\frac{|x|}{t^\alpha}) \leq 1 e^{i|x|q} e^{iq^2} e^{-iqy} \chi(|y| < \sqrt{t}) \]

with

\[ |x \cdot \hat{q} + 2t|q| - y \cdot \hat{q}| \geq t^{1-b} \]

due to factors (Recall that \( \alpha \in (0, \min(1 - \frac{2\delta}{n}, 1 - b)) \))

\[ \mathcal{F}_c(\frac{|x|}{t^\alpha}) \leq 1, \chi(|y| < \sqrt{t}), \mathcal{F}_1(t^\alpha|q| > 1). \]

The other one is that

\[ \| \mathcal{F}_c(\frac{|x|}{t^\alpha}) \|_{L^2_{t,x}} \leq 1 \mathcal{F}_1 e^{it\sqrt{\mathcal{H}_0+1}}(x)^{-\delta} \chi(|x| \geq |t|/100) \|_{L^2_{t,x}} \]

\[ \| \mathcal{F}_c(\frac{|x|}{t^\alpha}) \|_{L^2_{t,x}} \leq 1 \mathcal{F}_1 e^{it\sqrt{\mathcal{H}_0+1}}(x)^{-\delta} \chi(|x| < |t|/100) \|_{L^2_{t,x}} \]

\[ \lesssim_{\alpha} \frac{1}{t^\alpha} + \frac{1}{t^N}, \]

where we use

\[ \| \chi^{-\delta} \chi(|x| \geq |t|/100) \|_{L^2_{t,x}} \leq \frac{1}{|t|^\delta} \]

and the method of non-stationary phase for the part with \( \chi(|x| < |t|/100) \). We finish the proof.
2.4. Commutator estimate.

**Lemma 2.4.** For $t \geq 1$, $b < \alpha \leq 1, \beta > 0$, $l = 0, 1$,

\[(2.69) \quad \|\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1^{(l)}(t^\beta|P| > 1)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha - b},\]

\[(2.70) \quad \|\mathcal{F}_c^{(l)}(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1(t^\beta|P| > 1)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha - b},\]

\[(2.71) \quad \|\mathcal{F}_c^{(l)}(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1^{(l)}(|P| \leq t^\beta)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha + \beta},\]

where

\[(2.72) \quad \|\mathcal{F}_c^{(l)}(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1(|P| \leq t^\beta)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha + \beta},\]

In particular,

\[(2.73) \quad \mathcal{F}_1^{(l)}(k) := \frac{d^l}{dk^l}[\mathcal{F}_1],\]

and

\[(2.74) \quad \mathcal{F}_c^{(l)}(k) := \frac{d^l}{dk^l}[\mathcal{F}_c].\]

Proof. Let

\[(2.75) \quad \mathcal{F}_c := \mathcal{F}_1^{(l)}.\]

Write $[\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}]$ as

\[(2.76) \quad [\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1(t^\beta|P| > 1)]\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha - b},\]

\[(2.77) \quad [\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1(|P| \leq t^\beta)]\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha + \beta},\]

and

\[(2.78) \quad [\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F}_1(|P| \leq t^\beta)]\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha + \beta}.\]

Since

\[(2.79) \quad \|\mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1) - \mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha - b},\]

\[(2.80) \quad \mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1), \mathcal{F} = c_n \int d^d \xi \tilde{\mathcal{F}}(\xi) e^{it^\beta \xi} \times \left[ e^{-it^\beta \xi} \mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1) e^{it^\beta \xi} - \mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1) \right],\]

\[(2.81) \quad \|\mathcal{F}_c^l(\frac{|x|}{t^\alpha} \leq 1) - \mathcal{F}_c(\frac{|x|}{t^\alpha} \leq 1)\|_{L^2_t(L^r_x)} \lesssim n \frac{1}{r^\alpha - b}.\]
we have that for each $\psi \in L^2$, 
\[
\|\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \mathcal{F}_1^{(f)}(t^b|P| > 1)|\psi\|_{L^2(\mathbb{R}^n)} \lesssim_n \frac{1}{\rho^{a-b}} \int d^n\xi \|\hat{\mathcal{F}}(\xi)\|\|\psi(x)\|_{L^2(\mathbb{R}^n)} 
\]
\[
\lesssim_n \frac{1}{\rho^{a-b}}\|\psi\|_{L^2(\mathbb{R}^n)}.
\]
Similarly, we have (2.70). For (2.71), since 
\[
(\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \mathcal{F}_1^{(f)}(|P| \leq t^b)|P| > 1)) \lesssim \frac{1}{\rho^{a-b+1}},
\]
(2.71) follows by taking $b = -\beta$ in (2.69). Similarly, we have (2.72). We finish the proof. 

\[\Box\]

**Corollary 2.5.** For $t \geq 1, 0 \leq b < \alpha \leq 1, \beta > 0$,
\[
\|\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \partial_1[\mathcal{F}_1(|P| \leq t^b)|F_1(\bar{t}^b|P| > 1)]\|_{L^2(\mathbb{R}^n)} \lesssim \frac{1}{\rho^{a-b+1}}.
\]

**Proof.** Since 
\[
[\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \partial_1[\mathcal{F}_1(|P| \leq t^b)|F_1(\bar{t}^b|P| > 1)] = 
[\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \partial_1[\mathcal{F}_1(|P| \leq t^b)]F_1(\bar{t}^b|P| > 1) + \partial_1[\mathcal{F}_1(|P| \leq t^b)]F_c(\frac{|x|}{\rho^a} \leq 1), F_1(\bar{t}^b|P| > 1),
\]
and since 
\[
[\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \mathcal{F}_1(|P| \leq t^b)]\partial_1[\mathcal{F}_1(\bar{t}^b|P| > 1)] = 
[\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \mathcal{F}_1(|P| \leq t^b)]\partial_1[\mathcal{F}_1(\bar{t}^b|P| > 1)] + \mathcal{F}_1(|P| \leq t^b)[\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1), \partial_1[\mathcal{F}_1(\bar{t}^b|P| > 1)],
\]
(2.84) follows by using Lemma 2.4. Similarly, we have (2.85) by using Lemma 2.4. 

\[\Box\]

3. **Proof of the existence of Channel Wave Operator and properties of Weakly Localized part**

**Proof of Theorem 1.3.** For $\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1)(|P| \leq \beta)u_\Omega(t)$, according to (2.20), use Duhamel’s formula to expand it
\[
\mathcal{F}_c(\frac{|x|}{\rho^a} \leq 1)(|P| \leq \beta)u_\Omega(t) = \mathcal{F}_c(|x| \leq 1)(|P| \leq 1)u_\Omega(1) + 
\int_1^t ds \partial_1[\mathcal{F}_c(\frac{|x|}{s^a} \leq 1)|\mathcal{F}_1(|P| \leq s^b)|u_\Omega(s) + 
\int_1^t ds \mathcal{F}_c(\frac{|x|}{s^a} \leq 1)|\mathcal{F}_1(|P| \leq s^b)]\partial_1[\mathcal{F}_1(|P| \leq s^b)]u_\Omega(s) + 
\int_1^t ds \mathcal{F}_c(\frac{|x|}{s^a} \leq 1)|\mathcal{F}_1(|P| \leq s^b)]\sin(s\sqrt{H_0 + 1})V(s)u(s)
\]
\[
= \mathcal{F}_c(|x| \leq 1)(|P| \leq 1)u_\Omega(1) + u_{\Omega,1}(t) + u_{\Omega,2}(t) + u_{\Omega,3}(t).
\]
For $u_{\Omega,3}(t)$, by using Lemma 2.2, we have
\begin{equation}
(3.2) \quad \lim_{t \to \infty} u_{\Omega,3}(t) \text{ exists in } L^2_{\alpha}.
\end{equation}

**Estimate for $u_{\Omega,1}(t)$:** For $u_{\Omega,1}(t)$, if we can show
\begin{equation}
(3.3) \quad \int_1^\infty ds \left\| \hat{\partial}_x [\mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1] |\overline{\mathcal{F}} \hat{u}(|P| \leq s^\beta) u_{\Omega}(s)| \right\|^2_{L^2_{\beta}} \leq C(\|\tilde{u}(0)\|_S)^2,
\end{equation}
then for $t_2 \geq t_1 > 1$, using Hölder’s inequality in $s$ variable and then Fubini’s theorem, one has
\begin{equation}
(3.4) \quad \|u_{\Omega,1}(t_2) - u_{\Omega,1}(t_1)\|_{L^2_{\alpha}} = \left\| \int_{t_1}^{t_2} d\tilde{s} \hat{\partial}_x [\mathcal{F}_c \left( \frac{|x|}{s^\alpha} \leq 1 \right) \overline{\mathcal{F}} \hat{u}(|P| \leq s^\beta) u_{\Omega}(s)] \right\|^2_{L^2_{\beta}} \leq \left( \int_{t_1}^{t_2} d\tilde{s} \left\| \hat{\partial}_x [\mathcal{F}_c \left( \frac{|x|}{s^\alpha} \leq 1 \right) \overline{\mathcal{F}} \hat{u}(|P| \leq s^\beta) u_{\Omega}(s)] \right\|^2_{L^2_{\beta}} \right)^{1/2} \leq g(t_1) \to 0 \text{ as } t_1 \to \infty,
\end{equation}
where
\begin{equation}
(3.5) \quad g(t) := \sqrt{\int_{t}^\infty d\tilde{s} \left\| \hat{\partial}_x [\mathcal{F}_c \left( \frac{|x|}{s^\alpha} \leq 1 \right) \overline{\mathcal{F}} \hat{u}(|P| \leq s^\beta) u_{\Omega}(s)] \right\|^2_{L^2_{\beta}}}
\end{equation}
and $g(t) \to 0$ as $t \to \infty$ due to (3.3). Then we get the existence of $u_{\Omega,1}(\infty)$ in $L^2_{\alpha}$. Now we prove (3.3) by using propagation estimates (for Propagation estimates, see [47]). To be precise, choose
\begin{equation}
(3.6) \quad B_1(t) := \mathcal{F}_1(|P| \leq \beta^\alpha) \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \leq 1 \right) \overline{\mathcal{F}} \hat{u}(|P| \leq \beta^\alpha).
\end{equation}
Let
\begin{equation}
(3.7) \quad \langle B_1(t) : u_{\Omega}(t) \rangle := (u_{\Omega}(t), B_1(t) u_{\Omega}(t))_{L^2_{\beta}}.
\end{equation}
Then
\begin{equation}
(3.8) \quad |\langle B_1(t) : u_{\Omega}(t) \rangle| \leq C(\|\tilde{u}(0)\|_S)^2.
\end{equation}
Let
\begin{equation}
(3.9) \quad R(t) := (u_{\Omega}(t), \hat{\partial}_t [\mathcal{F}_1(|P| \leq \beta^\alpha) \mathcal{F}_c \hat{u}(|P| \leq \beta^\alpha) u_{\Omega}(t)])_{L^2_{\alpha}} + (u_{\Omega}(t), \mathcal{F}_1(|P| \leq \beta^\alpha) \mathcal{F}_c \hat{u}(|P| \leq \beta^\alpha) \hat{\partial}_t [\mathcal{F}_1(|P| \leq \beta^\alpha) u_{\Omega}(t)])_{L^2_{\alpha}} - 2(u_{\Omega}(t), \sqrt{\mathcal{F}_c} \hat{u}(|P| \leq \beta^\alpha) \hat{\partial}_t [\mathcal{F}_1(|P| \leq \beta^\alpha) \mathcal{F}_c u_{\Omega}(t)])_{L^2_{\alpha}}.
\end{equation}
Compute $\hat{\partial}_t \langle B_1(t) : u_{\Omega}(t) \rangle$
\begin{equation}
(3.10) \quad \hat{\partial}_t \langle B_1(t) : u_{\Omega}(t) \rangle = (u_{\Omega}(t), \mathcal{F}_1 \hat{\partial}_t [\mathcal{F}_c] \mathcal{F}_c u_{\Omega}(t))_{L^2_{\alpha}} + (\hat{\partial}_t u_{\Omega}(t), \mathcal{F}_1 \mathcal{F}_c \hat{u}(|P| \leq \beta^\alpha) u_{\Omega}(t))_{L^2_{\alpha}} + (u_{\Omega}(t), \mathcal{F}_1 \mathcal{F}_c \hat{u}(|P| \leq \beta^\alpha) \hat{\partial}_t [\mathcal{F}_1(|P| \leq \beta^\alpha) u_{\Omega}(t)])_{L^2_{\alpha}} + R(t) + 2(u_{\Omega}(t), \sqrt{\mathcal{F}_c} \mathcal{F}_1(|P| \leq \beta^\alpha) \hat{\partial}_t [\mathcal{F}_1(|P| \leq \beta^\alpha) \mathcal{F}_c u_{\Omega}(t)])_{L^2_{\alpha}} =: A_1(t) + A_2(t) + A_3(t) + R(t) + A_4(t).
\end{equation}
Here $A_1(t), A_4(t) \geq 0$ for all $t$ and $A_2(t), A_3(t), R(t) \in L^1_{\alpha}[1, \infty)$ due to Lemma 2.4 and our assumption on $\alpha$, that is,
\begin{equation}
(3.11) \quad \frac{n(1 - \alpha - \beta) - 3\beta}{2} > 1.
\end{equation}
Hence,

\[ |A_1(t)| = A_1(t) \leq A_1(t) + A_4(t) \]

\[ \leq 2\|u_\Omega(t)\|_{L_t^2}^2 + \|A_2(t)\|_{L_t^2[1,\infty)} + \|A_3(t)\|_{L_t^2[1,\infty)} + \|R(t)\|_{L_t^2[1,\infty)}, \]

which implies that \( A_1(\infty) \) exists and

\[ \int_1^\infty ds \|\partial_s [\mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1)] \mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1)u_\Omega(t)\|_{L_t^2}^2 = A_1(\infty) \leq C(\|\bar{u}(0)\|_S)^2. \]

Therefore,

\[ u_{\Omega,1}(\infty) := \lim_{t \to \infty} u_{\Omega,1}(t) \text{ exists in } L_t^2. \]

**Estimate for \( u_{\Omega,2}(t) \):** Write \( u_{\Omega,2}(t) \) as

\[ u_{\Omega,2}(t) = \int_1^\infty ds \partial_s [\mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1)] u_\Omega(t) - \]

\[ \int_1^\infty ds [\partial_s [\mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1)] u_\Omega(t) =: u_{\Omega,21}(t) + u_{\Omega,22}(t). \]

For \( u_{\Omega,22}(t), u_{\Omega,22}(\infty) \) exists in \( L_t^2 \) since by using Lemma 2.4 and \( \alpha, \beta > 0, \)

\[ \|\partial_s [\mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1)] u_\Omega(t)\|_{L_t^2} \leq \frac{1}{s^{1+\alpha+\beta}} \|u_\Omega(t)\|_{L_t^2} \leq \epsilon. \]

For \( u_{\Omega,21}(t) \), we use propagation estimates. Choose

\[ B_{11}(t) := \mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1) \mathcal{F}_c(\frac{|x|}{s^\alpha} \leq 1) \]

and let

\[ \langle B_{11}(t) : u_\Omega(t) \rangle := (u_{\Omega}(t), B_{11}(t)u_\Omega(t))_{L_t^2}. \]

Let

\[ R_1(t) := (u_\Omega(t), \mathcal{F}_c \bar{\mathcal{F}_1} \partial_s [\mathcal{F}_c] u_\Omega(t))_{L_t^2} + \]

\[ (u_\Omega(t), \partial_s [\mathcal{F}_c] \bar{\mathcal{F}_1} \mathcal{F}_c u_\Omega(t))_{L_t^2} - 2(u_\Omega(t), \sqrt{\bar{\mathcal{F}_1} \mathcal{F}_c \partial_s [\mathcal{F}_c]} \sqrt{\bar{\mathcal{F}_1} u_\Omega(t)})_{L_t^2}. \]

Compute \( \partial_s \langle B_{11}(t) : u_\Omega(t) \rangle \)

\[ \partial_t \langle B_{11}(t) : u_\Omega(t) \rangle = (u_\Omega(t), \mathcal{F}_c \partial_t [\bar{\mathcal{F}_1}] \mathcal{F}_c u_\Omega(t))_{L_t^2} + \]

\[ (\partial_t [u_\Omega(t)], \mathcal{F}_c \bar{\mathcal{F}_1} \mathcal{F}_c u_\Omega(t))_{L_t^2} + (u_\Omega(t), \mathcal{F}_c \bar{\mathcal{F}_1} \mathcal{F}_c, \partial_t [u_\Omega(t)])_{L_t^2} + R_1(t) + \]

\[ 2(u_\Omega(t), \sqrt{\bar{\mathcal{F}_1} \mathcal{F}_c \partial_s [\mathcal{F}_c]} \sqrt{\bar{\mathcal{F}_1} u_\Omega(t)})_{L_t^2} =: A_{11}(t) + A_{12}(t) + A_{13}(t) + R_1(t) + A_{14}(t). \]

\( A_{11}(t), A_{14}(t) \geq 0 \) and \( A_{12}(t), A_{13}(t), R_1(t) \in L_t^1[1, \infty) \) due to Lemma 2.2 and Lemma 2.4. Then by using propagation estimates, we have that \( A_{11}(\infty) \) exists which implies that \( u_{\Omega,21}(\infty) \) exists in \( L_t^2 \). Hence,

\[ u_{\Omega,2}(\infty) := \lim_{t \to \infty} u_{\Omega,2}(t) \text{ exists in } L_t^2. \]
Hence, based on (3.37), (3.44), (3.29), we have

\begin{equation}
(3.22) \quad u_\omega := s- \lim_{t \to \infty} \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq t^\beta) \overline{u}_\Omega(t)
\end{equation}

exists in $L^2_\omega$.

Similarly, for $\mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq t^\beta) \overline{u}_\Omega(t)$, use Duhamel’s formula to expand it

\begin{align}
(3.23) \quad & \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq t^\beta) \overline{u}_\Omega(t) = \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq s^\beta) \overline{u}_\Omega(s) + \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \partial_s (\mathcal{F}_1 (|P| \leq s^\beta) \overline{u}_\Omega(s)) - \\
& \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq s^\beta) \cos (s \sqrt{H_0 + 1}) V(s) u(s) =: \mathcal{F}_c \left( \frac{|x|}{t} \right) \leq 1 \mathcal{F}_1 (|P| \leq 1) \overline{u}_\Omega(1) + \overline{u}_{\Omega,1}(1) + \overline{u}_{\Omega,2}(1) + \overline{u}_{\Omega,3}(1).
\end{align}

$\overline{u}_{\Omega,3}(\infty)$ exists in $L^2_\omega$ due to Lemma 2.2. For $\overline{u}_{\Omega,3}(t)$, break it into two pieces

\begin{align}
(3.24) \quad & \overline{u}_{\Omega,2}(t) = \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq s^\beta) \overline{u}_\Omega(s) - \\
& \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq s^\beta) \overline{u}_\Omega(s) =: \overline{u}_{\Omega,21}(t) + \overline{u}_{\Omega,22}(t)
\end{align}

$\overline{u}_{\Omega,22}(\infty)$ exists in $L^2_\omega$ due to Lemma 2.4. Both $\overline{u}_{\Omega,1}(\infty)$ and $\overline{u}_{\Omega,21}(\infty)$ exist in $L^2_\omega$ by using Lemma 2.2, Lemma 2.4 and propagation estimates via observing

\begin{align}
(3.25) \quad \begin{cases} 
\langle B_2(t) : \overline{u}_{\Omega,1}(t) \rangle \\
B_2(t) = B_1(t)
\end{cases} \quad \text{and} \quad \begin{cases} 
\langle B_{21}(t) : \overline{u}_{\Omega,21}(t) \rangle \\
B_{21}(t) = B_{11}(t)
\end{cases}
\end{align}

respectively. Then

\begin{equation}
(3.26) \quad u_\omega := s- \lim_{t \to \infty} \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| \leq t^\beta) \overline{u}_\Omega(t)
\end{equation}

exists in $L^2_\omega$ and we finish the proof.

$\Box$

**Proof.** Proof of Theorem 1.5 It is equivalent to show the following free channel wave operator

\begin{equation}
(3.27) \quad \overline{\Omega}_{a,b}^* := s- \lim_{t \to \infty} \langle P \rangle_1^{-1} \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \langle P \rangle_1 \mathcal{F}_1 (|P| > 1) \overline{\Omega}(t)^*
\end{equation}

exists from $\mathcal{S}$ to $\mathcal{S}$. For $\mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| > 1) \langle P \rangle u_\Omega(t)$, use Duhamel’s formula to expand it

\begin{align}
(3.28) \quad & \mathcal{F}_c \left( \frac{|x|}{t^\alpha} \right) \leq 1 \mathcal{F}_1 (t^\beta |P| > 1) \langle P \rangle u_\Omega(t) = \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (|P| > 1) \langle P \rangle u_\Omega(1) + \\
& \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (s^\beta |P| > 1) u_\Omega(s) + \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \partial_s (\mathcal{F}_1 (s^\beta |P| > 1) u_\Omega(s)) + \\
& \int_1^t ds \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \right) \leq 1 \mathcal{F}_1 (s^\beta |P| > 1) \sin (s \sqrt{H_0 + 1}) V(s) u(s)
\end{align}
\[ =: \mathcal{F}_{c}(|x| \leq 1)\mathcal{F}_{l}(|P| \leq 1)u_{\Omega}(1) + u_{\Omega,1}(t) + u_{\Omega,2}(t) + u_{\Omega,3}(t). \]

For \( u_{\Omega,3}(t) \), by using Lemma 2.3, we have
\[ u_{\Omega,3}(\infty) := \lim_{t \to \infty} u_{\Omega,3}(t) \text{ exists in } L^{2}_{\alpha}. \]

**Estimate for \( u_{\Omega,1}(t) \):** Choose
\[ B_{1}(t) := \mathcal{F}_{1}(t^{\beta}|P| > 1)\mathcal{F}_{c}(|x| \leq 1/\alpha)\mathcal{F}_{l}(t^{\beta}|P| > 1). \]

Let
\[ \langle B_{1}(t) : \langle P \rangle u_{\Omega}(t) \rangle := \langle \langle P \rangle u_{\Omega}(t), B_{1}(t)\langle P \rangle u_{\Omega}(t) \rangle \rangle. \]

Then
\[ |\langle B_{1}(t) : \langle P \rangle u_{\Omega}(t) \rangle| \leq C(||\overline{u}(0)||_{S})^{2}. \]

Let
\[ R(t) := \langle \langle P \rangle u_{\Omega}(t), \partial_{t}[\mathcal{F}_{1}(t^{\beta}|P| > 1)]\mathcal{F}_{c}\mathcal{F}_{l}(t^{\beta}|P| > 1)\langle P \rangle u_{\Omega}(t) \rangle \rangle + \]
\[ \langle \langle P \rangle u_{\Omega}(t), \mathcal{F}_{1}\partial_{t}[\mathcal{F}_{1}(t^{\beta}|P| > 1)]\mathcal{F}_{c}\mathcal{F}_{l}(t^{\beta}|P| > 1)\langle P \rangle u_{\Omega}(t) \rangle \rangle - \]
\[ 2\langle \langle P \rangle u_{\Omega}(t), \sqrt{\mathcal{F}_{c}\mathcal{F}_{l}(t^{\beta}|P| > 1)}\partial_{t}[\mathcal{F}_{1}(t^{\beta}|P| > 1)]\sqrt{\mathcal{F}_{c}\langle P \rangle u_{\Omega}(t) \rangle} \rangle \]
\[ =: A_{1}(t) + A_{2}(t) + A_{3}(t) + R(t) + A_{4}(t). \]

Here \( A_{1}(t), A_{4}(t) \geq 0 \) for all \( t \) and \( A_{2}(t), A_{3}(t), R(t) \in L^{1}_{\alpha}[1, \infty) \) due to Lemma 2.3, Lemma 2.4 and our assumption on \( \alpha \). Hence,
\[ |A_{1}(t)| = A_{1}(t) \leq A_{1}(t) + A_{4}(t) \leq 2||\langle P \rangle u_{\Omega}(t)||_{L^{2}_{\alpha}}^{2} + ||A_{2}(t)||_{L^{1}_{\alpha}(1, \infty)} + ||A_{3}(t)||_{L^{1}_{\alpha}(1, \infty)} + ||R(t)||_{L^{1}_{\alpha}(1, \infty)}, \]

which implies that \( A_{1}(\infty) \) exists and
\[ \int_{1}^{\infty} ds \sqrt{\|\partial_{s}[\mathcal{F}_{c}(|x|/s^{\alpha} \leq 1)]\mathcal{F}_{1}(s^{\beta}|P| > 1)\langle P \rangle u_{\Omega}(s) \rangle}^{2} = A_{1}(\infty) \leq C(||\overline{u}(0)||_{S})^{2}. \]

Therefore,
\[ u_{\Omega,1}(\infty) := \lim_{t \to \infty} u_{\Omega,1}(t) \text{ exists in } L^{2}_{\alpha}. \]

**Estimate for \( u_{\Omega,2}(t) \):** Write \( u_{\Omega,2}(t) \) as
\[ u_{\Omega,2}(t) = \int_{1}^{t} ds \partial_{s}[\mathcal{F}_{c}(s^{\beta}|P| > 1)]\mathcal{F}_{c}(|x|/s^{\alpha} \leq 1)\langle P \rangle u_{\Omega}(s) - \]
\[ \int_{1}^{t} ds \partial_{s}[\mathcal{F}_{1}(s^{\beta}|P| > 1)]\mathcal{F}_{c}(|x|/s^{\alpha} \leq 1)\langle P \rangle u_{\Omega}(s) =: u_{\Omega,21}(t) + u_{\Omega,22}(t). \]
For $u_{\Omega,22}(t)$, $u_{\Omega,22}(\infty)$ exists in $L^2_\lambda$ since by using Lemma 2.4 and $\alpha > b$,
\begin{equation}
\|\partial_t[F]\|_{L^2_\lambda} \leq \frac{1}{s^{1+\alpha-\delta}}\|\partial_t[F]\|_{L^2_\lambda} \in L^1_{\mu}[1, \infty).
\end{equation}

For $u_{\Omega,21}(t)$, we use propagation estimates. Choose
\begin{equation}
B_{11}(t) := \mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\} \mathcal{F}_c(t^b|P| > 1)\mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\}
\end{equation}
and let
\begin{equation}
\langle B_{11}(t) : (P)u_{\Omega}(t) \rangle := \langle (P)u_{\Omega}(t), B_{11}(t)(P)u_{\Omega}(t) \rangle_{L^2_\lambda}.
\end{equation}

Let
\begin{equation}
R_1(t) := \langle (P)u_{\Omega}(t), \mathcal{F}_c\mathcal{F}_c\partial_t[F]\rangle_{L^2_\lambda} +
\langle (P)\partial_t[u_{\Omega}(t)], \mathcal{F}_c\mathcal{F}_c\partial_t[F](P)u_{\Omega}(t) \rangle_{L^2_\lambda} +
2\langle (P)u_{\Omega}(t), \mathcal{F}_c\mathcal{F}_c\partial_t[F] \mathcal{F}_c(P)u_{\Omega}(t) \rangle_{L^2_\lambda} +
R_1(t) +
A_{11}(t) + A_{12}(t) + A_{13}(t) + R_1(t) + A_{14}(t).
\end{equation}

Then by using propagation estimates, we have that $A_{11}(\infty)$ exists which implies that $u_{\Omega,21}(\infty)$ exists in $L^2_\lambda$. Hence,
\begin{equation}
\lim_{t \to \infty} u_{\Omega,2}(t) \in L^2_\lambda.
\end{equation}

Hence, based on (3.37), (3.44), (3.29), we have
\begin{equation}
\tilde{u}_+ := s-\lim_{t \to \infty} (P)^{-1}\mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\} \mathcal{F}_c(t^b|P| > 1)(P)u_{\Omega}(t)
\end{equation}
exists in $\mathcal{H}^1$ which is equivalent to the existence of
\begin{equation}
u_+ := s-\lim_{t \to \infty} \mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\} \mathcal{F}_c(t^b|P| > 1)u_{\Omega}(t)
\end{equation}
in $\mathcal{H}^1$. Similarly, for $\mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\} \mathcal{F}_c(t^b|P| > 1)u_{\Omega}(t)$, use Duhamel’s formula to expand it
\begin{equation}
\mathcal{F}_c\left\{ \frac{|x|}{t^\alpha} \leq 1 \right\} \mathcal{F}_c(t^b|P| > 1)u_{\Omega}(1) +
\int_{1}^{t} ds \partial_s\left\{ \mathcal{F}_c\left\{ \frac{|x|}{s^\alpha} \leq 1 \right\} \mathcal{F}_c(s^b|P| > 1)u_{\Omega}(s) +
\int_{1}^{s} ds \mathcal{F}_c\left\{ \frac{|x|}{s^\alpha} \leq 1 \right\} \mathcal{F}_c(s^b|P| > 1)u_{\Omega}(s) -
\int_{1}^{s} ds \mathcal{F}_c\left\{ \frac{|x|}{s^\alpha} \leq 1 \right\} \mathcal{F}_c(s^b|P| > 1) \cos(s \sqrt{H_0 + 1})V(s)u(s)
\right\}
\end{equation}
$u_{\Omega,23}(\infty)$ exists in $L^2_\lambda$ due to Lemma 2.3. For $\tilde{u}_{\Omega,2}(t)$, break it into two pieces
\[
(3.48) \quad \dot{u}_{\Omega,2}(t) = \int_{1}^{t} ds \partial_s \left[ \mathcal{F}_1(s^b |P| > 1) \right] \mathcal{F}_c \left( \frac{|x|}{s^a} \leq 1 \right) \dot{u}_\Omega(s) - \int_{1}^{\infty} ds \partial_s \left[ \mathcal{F}_1(s^b |P| > 1) \right] \mathcal{F}_c \left( \frac{|x|}{s^a} \leq 1 \right) \dot{u}_\Omega(s) =: \dot{u}_{\Omega,21}(t) + \dot{u}_{\Omega,22}(t)
\]

\[\dot{u}_{\Omega,22}(\infty) \text{ exists in } L^2_x \text{ due to Lemma 2.4. Both } \dot{u}_{\Omega,1}(\infty) \text{ and } \dot{u}_{\Omega,21}(\infty) \text{ exist in } L^2_x \text{ by using Lemma 2.3, Lemma 2.4 and propagation estimates via observing}
\]

\[
(3.49) \quad \begin{cases}
\langle B_2(t) : \dot{u}_{\Omega,1}(t) \rangle \\
B_2(t) = B_1(t)
\end{cases}
\]
and

\[
\begin{cases}
\langle B_{21}(t) : \dot{u}_{\Omega,21}(t) \rangle \\
B_{21}(t) = B_{11}(t)
\end{cases}
\]

respectively. Then

\[
(3.50) \quad \dot{u}_+ := \lim_{t \to \infty} \frac{|x|}{t^a} < 1 \mathcal{F}_c \left( \frac{|x|}{s^a} \leq 1 \right) \mathcal{F}_1 (t^b |P| > 1) \dot{u}_\Omega(t)
\]
exists in \(L^2_x\). We finish the proof for (1). Before going to the proof of the second part of Theorem 1.5, let us remind you of the Duhamel’s formulas for \(u(t), \dot{u}(t)\),

\[
(3.51) \quad u(t) = \left( \cos(t \sqrt{H_0 + 1}) u(0) + \frac{\sin(t \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}} \dot{u}(0) \right) + \int_{0}^{t} ds \frac{e^{i(t-s) \sqrt{H_0 + 1}}}{2i \sqrt{H_0 + 1}} V(s)u(s) - \int_{0}^{t} ds \frac{e^{i(t-s) \sqrt{H_0 + 1}}}{2i \sqrt{H_0 + 1}} V(s)\dot{u}(s)
\]

\[=: u_f(t) + u_+(t) + u_-(t).
\]

and

\[
(3.52) \quad \dot{u}(t) = -\sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} u(0) + \cos(t \sqrt{H_0 + 1}) \dot{u}(0) - \int_{0}^{t} ds \frac{e^{i(t-s) \sqrt{H_0 + 1}}}{2} V(s)u(s) + \int_{0}^{t} ds \frac{e^{-i(t-s) \sqrt{H_0 + 1}}}{2} V(s)\dot{u}(s)
\]

\[=: \dot{u_f}(t) + \dot{u_+}(t) + \dot{u_-}(t).
\]

For the second part of Theorem 1.5, it follows from following theorem and we defer its proof to the end of this section:

**Theorem 3.1.** Let \(e, \alpha, b\) as in Theorem 1.5 and \(V(x, t)\) as in Theorem 1.5 for some \(\delta > 2\), there exist \(u_{e,\alpha,b}^\pm, v_{e,\alpha,b}^\pm \in \mathcal{H}_{21}^1, u_{e,\alpha,b}^\pm, v_{e,\alpha,b}^\pm \in L^2_x\) such that

\[
(3.53) \quad \lim_{t \to \infty} \|u_\pm(t) - \cos(t \sqrt{H_0 + 1}) u_{e,\alpha,b}^\pm - \frac{\sin(t \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}} \dot{u}_{e,\alpha,b}^\pm - u_{w,\alpha,b}^\pm(t)\|_{\mathcal{H}_{21}^1} = 0
\]

\[
(3.54) \quad \lim_{t \to \infty} \|u_\pm(t) + \sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} v_{e,\alpha,b}^\pm - \cos(t \sqrt{H_0 + 1}) \dot{v}_{e,\alpha,b}^\pm - v_{w,\alpha,b}^\pm(t)\|_{L^2_x} = 0
\]

where \(u_{w,\alpha,b}^\pm(t), v_{w,\alpha,b}^\pm(t)\) are weakly localized parts in the following sense

\[
(3.55) \quad (u_{w,\alpha,b}^\pm(t), |x| u_{w,\alpha,b}^\pm(t))_{L^2_x} \lesssim_b t^r C(|\bar{u}(0)|_{L^1_x})^2
\]

\[
(3.56) \quad (v_{w,\alpha,b}^\pm(t), |x| v_{w,\alpha,b}^\pm(t))_{L^2_x} \lesssim_b t^r C(|\bar{u}(0)|_{L^1_x})^2.
\]
Based on Theorem 3.1, we get the second part of Theorem 1.5 by setting
\[
(3.57) \quad u_{+e,\alpha,b}^1 := u_{+e,\alpha,b}^+ + u_{-e,\alpha,b}^-,
\]
\[
(3.58) \quad u_{-e,\alpha,b}^2 := v_{-e,\alpha,b}^- + \bar{v}_{-e,\alpha,b}^-,
\]
\[
(3.59) \quad \Omega_{\alpha,\pm}(t) := \Phi_{\alpha}(t) + \Phi_{-\alpha}(t),
\]
and finish the proof for Theorem 1.5.

Before proving Theorem 3.1, we have to introduce some lemmas. Based on the proof of the first part of Theorem 1.5, we deduce following lemma:

**Lemma 3.2.** Let \((u(t), \bar{u}(t))\) be a global solution to equation (1.3) in \(S\). If \(V(x, t), u(t)\) satisfy (1), then for \(n \geq 1, b \in (0, 1/2), \alpha \in (b, \min(1 - b, 1 - \frac{2\delta}{n}))\),
\[
(3.60) \quad \lim_{t \to \infty} (1 - \mathcal{F}_{\alpha} \mathcal{F}_1) \int_0^t ds e^{\pm is \sqrt{b^2 + 1}} V(s) u(s) = 0.
\]

**Proof.** The proof of Lemma 3.2 follows from a similar argument for Theorem 1. (3.60) follows from that for each \(\phi(x) \in L^2_x\),
\[
(3.61) \quad \| (1 - \mathcal{F}_c (\frac{|x|}{t^2}) \leq 1) \mathcal{F}_1 (t^{1/2 - \epsilon} |P| > 1) \| \phi(x) \|_{L^2_x} \to 0, \text{ as } t \to \infty.
\]

Before we prove Theorem 3.1, we need following lemma:

**Lemma 3.3 (Minimal and Maximal velocity bounds).** For \(a > 0, c \in (0, t), e > 1 - b > \alpha > b > 0, t \geq 1, j = 1, \cdots, n\),
\[
(3.62) \quad \| (\mathcal{F}_1 (\frac{x_j}{t^c}) > 1) \| \mathcal{F}_1 (t^b P_j > 1/10) e^{\pm i \sqrt{b^2 + 1}} \langle x_j \rangle^{-\delta} \|_{L^2_{\ell} (\mathbb{R}^n)} \lesssim b \frac{1}{|t^c + \sqrt{a}|^\delta},
\]
\[
(3.63) \quad \| (\mathcal{F}_2 (\frac{x_j}{t^c}) > 1) \| \mathcal{F}_1 (t^b P_j \leq 1/10) e^{\pm i \sqrt{b^2 + 1}} \langle x_j \rangle^{-\delta} \|_{L^2_{\ell} (\mathbb{R}^n)} \lesssim b \frac{1}{|t^c + \sqrt{c}|^\delta},
\]
\[
(3.64) \quad \| (\mathcal{F}_2 (\frac{-x_j}{t^c}) > 1) \| \mathcal{F}_1 (-t^b P_j > 1/10) e^{\pm i \sqrt{b^2 + 1}} \langle x_j \rangle^{-\delta} \|_{L^2_{\ell} (\mathbb{R}^n)} \lesssim b \frac{1}{|t^c + \sqrt{a}|^\delta},
\]
\[
(3.65) \quad \| (\mathcal{F}_2 (\frac{-x_j}{t^c}) > 1) \| \mathcal{F}_1 (-t^b P_j \leq 1/10) e^{\pm i \sqrt{b^2 + 1}} \langle x_j \rangle^{-\delta} \|_{L^2_{\ell} (\mathbb{R}^n)} \lesssim b \frac{1}{|t^c + \sqrt{c}|^\delta},
\]
\[
(3.66) \quad \| (\mathcal{F}_2 (\pm x_j > t^c) \mathcal{F}_1 (\pm t^b P_j \leq 1/10) e^{\pm i \sqrt{b^2 + 1}} e^{\pm i \sqrt{b^2 + 1} \frac{|x|}{t^c}} > 1) e^{\pm i \sqrt{b^2 + 1} \frac{|x|}{t^c}} e^{\pm i \sqrt{b^2 + 1} \frac{x}{t^c}} \|_{L^2_{\ell} (\mathbb{R}^n)} \lesssim b \frac{1}{|t^c + \sqrt{c}|^\delta}.
\]

**Remark 3.4.** When we use Lemma 3.3, we need \(\delta > 2\) in order to make it integrable in \(a\) or \(b\) when \(|a|, |b| \geq 1\).
Proof of Lemma 3.3. It is sufficient to check the case when \( j = 1 \). Break the LHS of (3.62) into two pieces

\[
(F_2(\frac{X_1}{t^c} > 1)F_1(t^b P_1 > 1/10)e^{ja \sqrt{H_0+1}}(x_1) =
\]

\[
(F_2(\frac{X_1}{t^c} > 1)F_1(t^b P_1 > 1/10)e^{ja \sqrt{H_0+1}}(x_1) \chi(|x_1| \geq (t^c + \sqrt{a})/1000) +
\]

\[
(F_2(\frac{X_1}{t^c} > 1)F_1(t^b P_1 > 1/10)e^{ja \sqrt{H_0+1}}(x_1) \chi(|x_1| < (t^c + \sqrt{a})/1000)
\]

=: A_1 + A_2.

For \( A_1 \),

\[
\|A_1\|_{L_2(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \leq \| (F_2(\frac{X_1}{t^c} > 1))F_1(t^b P_1 > 1/10)e^{ja \sqrt{H_0+1}}\|_{L_2(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \times \frac{1}{|t^c + \sqrt{a}|^d} \leq \frac{1}{|t^c + \sqrt{a}|^d}.
\]

For \( A_2 \), since by using factor \( (F_2(\frac{X_1}{t^c} > 1))F_1(t^b q_1 > 1/10) \) and factor \( \chi(|y_1| < (t^c + \sqrt{a})/1000) \),

\[
e^{jaq_1}e^{ja \sqrt{|q|^2+1}}e^{-jaq_1}\frac{1}{l(x_1 + \frac{aq_1}{\sqrt{|q|^2+1}} - y_1)} \partial_{q_1}[e^{jaq_1}e^{ja \sqrt{|q|^2+1}}e^{-jaq_1}] \]

with

\[
|x_1 + \frac{aq_1}{\sqrt{|q|^2+1}}| \geq t^c \chi(|a| \leq t^c) + \sqrt{a} \chi(a > t^c) \geq |t^c + \sqrt{a}|
\]

we have

\[
\|A_2\|_{L_2(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \leq \frac{1}{|t^c + \sqrt{a}|^d}
\]

via taking integration by parts in \( q_1 \) for enough times. Thus, we get (3.62). Similarly, we get (3.63), (3.64) and (3.65). For (3.66),

\[
\text{LHS of (3.66)} \leq \text{LHS of (3.63) or LHS of (3.65) + R}
\]

with

\[
R := \|F_2(\pm x_j > t^c)\|_{L_2(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \times e^{-ia \sqrt{H_0+1}}F_1(\frac{|x|}{t^a} \leq 1)e^{ia \sqrt{H_0+1}}e^{-ia \sqrt{H_0+1}}\langle x \rangle \delta \|_{L_2^0(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)}.
\]

Since

\[
R \leq \|F_2(\pm x_j > t^c)\|_{L_2(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \times e^{-ia \sqrt{H_0+1}}F_1(\frac{|x|}{t^a} \leq 1)\langle x \rangle \delta \|_{L_2^0(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)}
\]

\[
\|\langle x \rangle \delta \|_{L_2^0(\mathbb{R}^n) \rightarrow L_2^0(\mathbb{R}^n)} \leq \frac{1}{t^N} \frac{1}{\langle t - c \rangle^d}
\]

(choose \( N \) sufficiently large) \( \leq \frac{1}{\langle |t| + |c| \rangle^d} \).
by using (3.63), (3.65), we get (3.66) and finish the proof.

**Proof of Theorem 3.1.** First of all, we consider \( u_+(t) \). Let

\[
(3.75) \quad u_+(t) := \int_0^t ds \langle P \rangle^{-1} e^{-it \sqrt{H_0+V}} \mathcal{F}_c \left( \left| x \right| \leq 1 \right) \mathcal{F}_1 \left( \left| P \right| > 1 \right) \frac{e^{is \sqrt{H_0+V}}}{2i} V(s) u(s)
\]

and

\[
(3.76) \quad u_w(t) := \int_0^t ds \langle P \rangle^{-1} e^{-it \sqrt{H_0+V}} \mathcal{F}_{c,1} \left( \left| x \right| \leq 1 \right) \mathcal{F}_1 \left( \left| P \right| > 1 \right)
\]

where

\[
(3.77) \quad \mathcal{F}_{c,1} := 1 - \mathcal{F}_c \left( \left| x \right| \leq 1 \right) \mathcal{F}_1 \left( \left| P \right| > 1 \right).
\]

Then

\[
(3.78) \quad u_+(t) = u_+(t) + u_w(t).
\]

Based on Lemma 3.2, we know that

\[
(3.79) \quad \| u_+(t) - e^{-it \sqrt{H_0+V}} u_+(\infty) \|_{\mathcal{H}_1} \to 0, \text{ as } t \to \infty.
\]

Then for \( u_+(t) \), it is sufficient to show that \( u_w(t) \) is equal to a sum of a localized part and a part which will go to 0 as \( t \to \infty \). In the following context, we will prove such decomposition. Let

\[
(3.80) \quad u_{j+}(t) := \langle P \rangle^{-1} \mathcal{F}_{2,j} \left( x_j > t^f \right) \langle P \rangle u_w(t)
\]

and

\[
(3.81) \quad u_{j-}(t) := \langle P \rangle^{-1} \mathcal{F}_{2,j} \left( -x_j > t^f \right) \langle P \rangle u_w(t)
\]

where

\[
(3.82) \quad \mathcal{F}_{2,j} \left( x_j > t^f \right) := \left( \Pi_{l=1}^{j-1} \mathcal{F}_{2,j} \left( |x_l| \leq t^f \right) \right) \mathcal{F}_{2,j} \left( x_j > t^f \right),
\]

and

\[
(3.83) \quad \mathcal{F}_{2,j} \left( -x_j > t^f \right) := \left( \Pi_{l=1}^{j-1} \mathcal{F}_{2,j} \left( |x_l| \leq t^f \right) \right) \mathcal{F}_{2,j} \left( -x_j > t^f \right).
\]

Then

\[
(3.84) \quad u_w(t) = \langle P \rangle^{-1} \left( \Pi_{l=1}^{n} \mathcal{F}_{2,j} \left( |x_l| \leq t^f \right) \right) \langle P \rangle u_w(t) + \sum_{j=1}^{n} \left( u_{j+}(t) + u_{j-}(t) \right).
\]

Set

\[
(3.85) \quad u_{w,\epsilon}(t) := \langle P \rangle^{-1} \left( \Pi_{l=1}^{n} \mathcal{F}_{2,j} \left( |x_l| \leq t^f \right) \right) \langle P \rangle u_w(t).
\]

In the following, we will show

\[
(3.86) \quad \| u_{j,\pm}(t) \|_{\mathcal{L}_2} \to 0, \text{ as } t \to \infty.
\]

Break \( u_{j,\pm}(t) \) into three pieces

\[
(3.87) \quad u_{j,\pm}(t) = \langle P \rangle^{-1} \mathcal{F}_{2,j} \left( \pm x_j > t^f \right) \mathcal{F}_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle u_w(t) - \langle P \rangle^{-1} \mathcal{F}_{2,j} \left( \pm x_j > t^f \right) \mathcal{F}_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle u_w(t) + \langle P \rangle^{-1} \mathcal{F}_{2,j} \left( \pm x_j > t^f \right) \mathcal{F}_1 \left( \pm t^b P_j \leq 1/10 \right) \langle P \rangle u_w(t) =: u_{j,+,+} + u_{j,+,2} + u_{j,+,r}.
\]
According Lemma 3.3, we have
\[(3.88)\quad \|u_{j,\pm,1}\|_{L^2(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty.\]

For $u_{j,\pm,1} + u_{j,\pm,2}$, write it as
\[(3.89)\quad u_{j,\pm,1} + u_{j,\pm,2} = \langle P \rangle^{-1} \left( \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle e^{-it \sqrt{H_0 + 1}} u_s - \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle u_s(t) \right) - \langle P \rangle^{-1} \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle e^{-it \sqrt{H_0 + 1}} (u_s - u_s(t)) =: u_{j,\pm,1,1} + u_{j,\pm,1,2}\]

where
\[(3.90)\quad u_s := \int_0^\infty ds \frac{e^{is \sqrt{H_0 + 1}}}{2i \sqrt{H_0 + 1}} e^{-is \sqrt{H_0 + 1}} V(s)u(s).\]

For $u_{j,\pm,1,1}$,
\[(3.91)\quad u_{j,\pm,1,2} = -\langle P \rangle^{-1} \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) x e^{-it \sqrt{H_0 + 1}} \int_0^\infty ds \frac{e^{is \sqrt{H_0 + 1}}}{2i} V(s)u(s).\]

Due to Lemma 3.3,
\[(3.92)\quad \|u_{j,\pm,1,1}(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty.\]

For $u_{j,\pm,1,1}(t)$, due to (3.92), we have $u_{j,\pm,1,1}(t) \in \mathcal{H}^1$, which means
\[(3.93)\quad \langle P \rangle^{-1} \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle e^{-it \sqrt{H_0 + 1}} u_s \in \mathcal{H}^1.\]

If we can show that in $L^2_s$,
\[(3.94)\quad u_s = u_s(\infty), \quad \text{in the weak sense,}\]

then due to (3.93),
\[(3.95)\quad \langle P \rangle^{-1} \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle e^{-it \sqrt{H_0 + 1}} u_s = \langle P \rangle^{-1} \mathcal{F}_{j} \left( \pm x_j > t^r \right) \mathcal{F}^j_1 \left( \pm t^b P_j > 1/10 \right) \langle P \rangle e^{-it \sqrt{H_0 + 1}} u_s(\infty)\]

and therefore by using Lemma 3.2,
\[(3.96)\quad \|u_{j,\pm,1,1}(t)\|_{L^2(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty.\]

Now let us prove that in $L^2_s$,
\[(3.97)\quad u_s = u_s(\infty) \quad \text{in the weak sense.}\]

(3.97) is true due to (3.60). So we have
\[(3.98)\quad \|u_{j,\pm,1,1}(t)\|_{L^2(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty.\]

Similarly, we get the same result for $u_{-}(t)$.

For $\dot{u}_s(t)$, let
\[(3.99)\quad \dot{u}_s(t) := - \int_0^\infty ds e^{-it \sqrt{H_0 + 1}} \mathcal{F}_c \left( \frac{|x|}{t^r} \leq 1 \right) \mathcal{F}_1(t^b |P| > 1) \frac{e^{is \sqrt{H_0 + 1}}}{2} V(s)u(s).\]
\[ (3.100) \quad \dot{u}_w(t) := -\int_0^t ds e^{-is\sqrt{\varepsilon + 1}} \mathcal{F}_{\varepsilon,1} e^{is\sqrt{\varepsilon + 1} \overline{V}(s)u(s)}. \]

Via a similar argument as what we did for \( u_+(t) \), we get the same result for \( \dot{u}_+(t) \) by setting
\[ (3.101) \quad \dot{v}_{w,a,b,c,d}^+(t) := \left( \Pi_{n=1}^n \mathcal{F}_\varepsilon(\|x\| \leq t^r) \right) \dot{u}_w(t). \]
Similarly, we get the same result for \( \dot{u}_-(t) \). We finish the proof. \( \Box \)

4. Applications

4.1. Estimates for free radiation. Let \( W^1_x \) denote the \( L^1_x \) Sobolev space. In this section, we show that if
\[ (4.1) \quad \tilde{u}(0) = \Omega^\beta \alpha \tilde{v}, \quad n \geq 3 \]
for some \( \tilde{v} \in L^2_x(\mathbb{R}^n) \oplus L^2_x(\mathbb{R}^n) \) satisfying
\[ (4.2) \quad \langle P \rangle^{\frac{n+1}{2}} \tilde{v} \in W^1_x(\mathbb{R}^n) \oplus L^1_x(\mathbb{R}^n), \]
then for \( t \geq 1, \)
\[ (4.3) \quad \|\langle x \rangle^{-(n+1)/2} \tilde{u}(t)\|_{L^2_x \oplus L^1_x} \leq \frac{1}{\langle t \rangle^{n/2-1}} \left( 1 + \|V(x,t)\|_{L^\infty_x L^1_t} \right) \|\langle P \rangle^{\frac{n+1}{2}} \tilde{v}\|_{W^1 \oplus L^1}. \]

Here
\[ (4.4) \quad \Omega^\beta \alpha := (\Omega^\beta \alpha)^*. \]

**Theorem 4.1.** When \( n \geq 3 \), let \( u(t), \tilde{u}(t) \) be as in (1.3) and \( \tilde{u}(0) \) be as in (4.1). If
\[ (4.5) \quad \sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{S} \leq \tilde{u}(0) 1 \]
and if \( V(x,t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R}) \), \( \langle P \rangle^{\frac{n+1}{2}} \tilde{v} \in W^1_x(\mathbb{R}^n) \oplus L^1_x(\mathbb{R}^n) \), then for \( t \geq 1, \)
\[ (4.6) \quad \|\langle x \rangle^{-(n+1)/2} \tilde{u}(t)\|_{L^2_x \oplus L^1_x} \leq \frac{1}{\langle t \rangle^{n/2-1}} \left( 1 + \|V(x,t)\|_{L^\infty_x L^1_t} \right) \|\langle P \rangle^{\frac{n+1}{2}} \tilde{v}\|_{W^1 \oplus L^1}. \]

**Proof.** Compute
\[ (4.7) \quad \tilde{u}(t) = U(t,0)\Omega^\beta \alpha \tilde{v} = s- \lim_{s \to \infty} U(t,s)U_0(s,0)\mathcal{F}_1(|P| \leq s^\beta) \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \leq 1 \right) \tilde{v} \]
\[ = s- \lim_{s \to \infty} U(t,s)U_0(s,t)U_0(t,0)\mathcal{F}_1(|P| \leq s^\beta) \mathcal{F}_c \left( \frac{|x|}{s^\alpha} \leq 1 \right) \tilde{v} \]
\[ = s- \lim_{s \to \infty} U(t,s)U_0(t,s+t)U_0(t,0)\mathcal{F}_1(|P| \leq (t+s)^\beta) \mathcal{F}_c \left( \frac{|x|}{(t+s)^\alpha} \leq 1 \right) \tilde{v} \]

By using Duhamel’s formula to expand \( U(t,s+t)U_0(t+s,t) \) in the expression of \( \tilde{u}(t) \), provided that \( \tilde{u}(t) \) exists in \( S \),
\[ (4.10) \quad \tilde{u}(t) = s- \lim_{s \to \infty} \left( U_0(t,0)\mathcal{F}_1(|P| \leq (t+s)^\beta) \mathcal{F}_c \left( \frac{|x|}{(t+s)^\alpha} \leq 1 \right) \tilde{v} \right) + \]
\[ \int_0^t du U(t,t+u)\mathcal{F}_c \left( \frac{|x|}{(t+s)^\alpha} \leq 1 \right) \tilde{v} \]
\[ U(t, 0) \tilde{v} + \int_0^\infty dU(t, t + u) V(t + u) U_0(u + t, 0) \tilde{v} = U_0(t, 0) \tilde{v} + \int_0^\infty dU(t, t + u) V(t + u) U_0(u + t, 0) \tilde{v}. \]

Let
\[(4.11) \quad \tilde{v} = \left( \begin{array}{c} v \\ \bar{v} \end{array} \right), \quad \bar{v}(t) = \left( \begin{array}{c} v(t) \\ \bar{v}(t) \end{array} \right). \]

Since using Lemma 2.1 and Hölder’s inequality,
\[(4.12) \quad \|V(x, t + u) \cos((t + u) \sqrt{H_0 + 1}) v\|_{L^2_x} \leq \frac{1}{(t + u)^{\alpha/2}} \|V(x, t)\|_{L^\infty_v L^2_x} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x}, \]

similarly since
\[(4.13) \quad \|V(x, t + u) \frac{\sin((t + u) \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}} v\|_{L^2_x} \leq \frac{1}{(t + u)^{\alpha/2}} \|V(x, t)\|_{L^\infty_v L^2_x} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x}, \]

and since
\[(4.14) \quad \|\langle \xi \rangle^{-\alpha/2} U_0(t, 0) \tilde{v}\|_{L^2_v L^2_x} \leq \frac{1}{(t)^{\alpha/2}} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x}, \]

we get
\[(4.15) \quad \|\langle \xi \rangle^{-\alpha/2} \bar{u}(t)\|_{L^2_v L^2_x} \leq \frac{1}{(t)^{\alpha/2}} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x} + \int_0^\infty du \|\langle 0, V(x, t + u) v(u + s) \rangle^T \|_S \]
\[\leq \frac{1}{(t)^{\alpha/2}} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x} + \int_0^\infty du \frac{1}{(t + u)^{\alpha/2}} \|V(x, t)\|_{L^\infty_v L^2_x} \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x} \]
\[\leq \frac{1}{(t)^{\alpha/2}} \left(1 + \|V(x, t)\|_{L^\infty_v L^2_x} \right) \|\langle P \rangle^{\alpha/2} \tilde{v}\|_{L^2_v L^2_x} \]

and finish the proof.

\[ \square \]

**Remark 4.2.** Indeed, such \( \frac{1}{(t)^{\alpha}} \) can be improved if both of the potential and \( \tilde{v} \) are localized in space and if \( \tilde{v} \) has frequency away from 0.

**Discussion**

The above result is a generalization of local decay estimates in the following sense: In the case of time independent linear interaction term that is also localized, the range of the wave operator above is equal to the range of the projection on the continuous spectral part of the Hamiltonian. In this case local decay holds for all localized initial data, and with rate of decay which is optimal. However, when the interaction term is time dependent and or nonlinear, there is no such decomposition. In this case the question arises as to what decay estimates hold for solutions which dipserse? That is solutions which asymptotically have no weakly localized part. The above estimate is in fact a decay estimate for such solutions.

**4.2. Application to typical nonlinear examples.** In this section, we show that for \( a(x), b(x) \in L^\infty_{0,0}(\mathbb{R}), \delta > 2, \) if there is a global solution \( \bar{u}(t) \) in \( S \) with a uniform \( S \) bound for following Nonlinear KG equation in one space dimension

\( (\Box + 1 + V(x))u = a(x)u^2 + b(x)u^3 \)

then the asymptotic behavior of the solution can be rewritten as the sum of a free part plus a weakly localized part:
Theorem 4.3. If \( a(x), b(x) \in L^r_0(\mathbb{R}) \) for some \( \delta > 2 \) if there is a global solution \( \vec{u}(t) \) to (4.16) in \( S \) satisfying
\[
(4.17) \quad C(||\vec{u}(0)||_S) := \sup_{t \in \mathbb{R}} ||\vec{u}(t)||_S \lesssim ||\vec{u}(0)||_S 1,
\]
where \( V(x) \geq 0 \) is a generic potential satisfying \( V(x) \in L^2_{\sigma,x} \) for some \( \sigma > 2 \), then for \( \alpha, b \) also satisfy
\[
(4.18) \quad e > 1 - b > \alpha > b \geq 0,
\]
there exist \( u^1_{+,e,a,b}, u^2_{+,e,a,b} \in \mathcal{H}_x^1, \hat{u}^1_{+,e,a,b}, \hat{u}^2_{+,e,a,b} \in L^2_x \) such that we have the following asymptotic decomposition
\[
(4.19) \quad \lim_{t \to \infty} ||u(t) - \cos(t \sqrt{H_0 + 1}) u^1_{+,e,a,b} - \frac{\sin(t \sqrt{H_0 + 1})}{\sqrt{H_0 + 1}} \hat{u}^1_{+,e,a,b} - \hat{u}_{w,e,a,b}(t)||_{\mathcal{H}_x^1} = 0
\]
and
\[
(4.20) \quad \lim_{t \to \infty} ||\vec{u}(t) + \sin(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} u^2_{+,e,a,b} - \cos(t \sqrt{H_0 + 1}) \sqrt{H_0 + 1} \hat{u}^2_{+,e,a,b} - v_{w,e,a,b}(t)||_{L^2_x} = 0
\]
where \( u_{w,e,a,b}, v_{w,e,a,b} \) are the weakly localized parts of the solution, with the following property: It is weakly localized in the region \( |x| \leq t^e \), in the following sense
\[
(4.21) \quad (\langle P \rangle u_{w,e,a,b}(t), |x| \langle P \rangle u_{w,e,a,b}(t))_{L^2_x} \lesssim t^e C(||\vec{u}(0)||_S)^2,
\]
and
\[
(4.22) \quad (v_{w,e,a,b}(t), |x| v_{w,e,a,b}(t))_{L^2_x} \lesssim t^e C(||\vec{u}(0)||_S)^2.
\]

Proof. Due to (4.17), we have
\[
(4.23) \quad ||a(x)u(x,t)||_{L^r_t L^2_{x}} \leq ||a(x)||_{L^p_{x}} ||u(x,t)||_{L^r_t L^2_{x}} \leq C(||\vec{u}(0)||_S)||a(x)||_{L^p_{x}},
\]
\[
(4.24) \quad ||a(x)u(x,t)||_{L^r_t L^2_{x}}^2 \leq ||a(x)||_{L^p_{x}} ||u(x,t)||_{L^r_t L^2_{x}} ||u(x,t)||_{L^r_t L^2_{x}} \leq C(||\vec{u}(0)||_S)||a(x)||_{L^p_{x}},
\]
\[
(4.25) \quad ||b(x)u(x,t)||_{L^r_t L^2_{x}} \leq ||b(x)||_{L^p_{x}} ||u(x,t)||_{L^r_t L^2_{x}} \leq C(||\vec{u}(0)||_S)||a(x)||_{L^p_{x}},
\]
and
\[
(4.26) \quad ||b(x)u(x,t)||_{L^r_t L^2_{x}}^3 \leq ||b(x)||_{L^p_{x}} ||u(x,t)||_{L^r_t L^2_{x}} ||u(x,t)||_{L^r_t L^2_{x}} \leq C(||\vec{u}(0)||_S)||a(x)||_{L^p_{x}}.
\]
Since \( V(x,t) = a(x)u + b(x)u^2 \in L^\infty_t L^2_{0,\delta}(\mathbb{R}) \) for some \( \delta > 2 \), so by using Theorem 1.5, we get desired conclusion and finish the proof.

When space dimension \( n \geq 3 \), if there is a global solution \( u(t) \) in \( \mathcal{H}^1 \), then when \( N(u,x,t) = V(x,t) + \sum_{j=1}^{N} \pm \lambda_j |u|^{p_j} \) for \( 1 \leq p_j \leq \frac{4}{n-2} \), the channel wave operator exists:

Theorem 4.4. Let \((u(t), \vec{u}(t)) \) be as in Theorem 1.3. If \( V(x,t) \in L^\infty_t L^2_x(\mathbb{R}^n \times \mathbb{R}) \) and if \( N(u,x,t) = V(x,t) + \sum_{j=1}^{N} \pm \lambda_j |u|^{p_j} \) for \( \lambda_j > 0, 1 \leq p_j \leq \frac{n}{n-2}, V(x,t) \) satisfying
\[
(4.27) \quad V(x,t) = \sum_{j=1}^{M} V_j(x-g_j(t)v_j,t), \quad V_j(x,t) \in L^\infty_t L^2_x, v_j \in \mathbb{R}^n, \text{ real functions } g_j(t),
\]
and if
\begin{equation}
\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} \leq \|\vec{u}(0)\|_S,
\end{equation}
then the channel wave operator exists.

Proof. When \(1 \leq p_j \leq \frac{n}{n-2}\),
\begin{equation}
\|\|u(t)\|_L^p_t \|_L^\infty_x \leq C(\|u(t)\|_{L^2_t}^{\frac{2}{p_j-2}}, \|u(t)\|_{L^2_x}) \leq C(\|u(t)\|_{L^2_t}^{\frac{2}{n-2}}, \|u(t)\|_{H^1}).
\end{equation}
\begin{equation}
\|V(x, t)\|_{L^\infty_t L^2_x} \leq \sum_{j=1}^M \|V_j(x, t)\|_{L^\infty_t L^2_x} < \infty.
\end{equation}
The assumptions of Theorem 1.3 is satisfied and we get the existence of channel wave operator. \(\Box\)

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