Decision-making under uncertainty: a quantum value operator approach

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Abstract: We propose a quantum expected value theory for decision-making under uncertainty. Quantum density operator as value operator is proposed to simulate people’s subjective beliefs. Value operator guides people to choose corresponding actions based on their subjective beliefs through objective world. The value operator can be constructed from quantum gates and logic operations as a quantum decision tree. The genetic programming is used to optimize and auto-generate quantum decision trees.

Keywords: quantum decision theory; quantum decision tree; quantum logic; information entropy; value operator; quantum genetic programming;

Introduction

Decision-making can be viewed as a two-phase process: evaluation process and selection process. Classic decision theory holds that rational-economic person knows the utility function as well as probability distribution through evaluation process then maximizes utility by an optimal selection¹-³. Information completeness and selection consistence are required for expected utility theory, however in the real world information is rarely complete and consistency of selection cannot be guaranteed due to complexity and people show irrational behaviors which cannot be explained by classic decision theory⁴-⁷. Classic decision theory is a “black box”; people do not know what really happens inside the box. Scientists are trying to apply quantum theory to reveal how people make decisions⁸-¹¹. We believe that the results of decisions are the unity of subjective beliefs and objective facts, and the value of observed results is the bridge between those two different worlds. The decision’s “black box” can be opened through the bridge of the value.

Quantum expected value decision theory

Usually people subjectively choose an action \( a_i \) \( \in \{a_1, \cdots, a_m\} \) where nature’s objective state is in \( q_j \in \{q_1, \cdots, q_n\} \) when decisions were made, and the result of the decision value matrix \( v_{ij} \) depends on both the state of the nature and choice of brain shown in table 1. Natural state describes the objective world; we hypothesize that an uncertain natural state can be represented by superposition of all possible states in terms of the Hilbert state space¹² as in (1). Strategy state describes the subjective world; we also
hypothesize that undecided decision state can be represented by superposition of all possible actions as in (2). Since the information of decision-making under uncertainty is incomplete; the result of decision can be represented by a mixed state’s density operator as a value operator as in (3). Value operator is a sum of projection operators which projects a person’s beliefs onto an action of choice based on nature states. Quantum expected value can be represented as in (5). The uncertainty of observed value can be represented by Shannon information entropy as in (6).

\[ |\psi\rangle = \sum_j c_j |q_j\rangle, \sum_j |c_j|^2 = 1 \] (1)

\[ |S\rangle = \sum_i |\mu_i a_i\rangle, \sum_i |\mu_i|^2 = 1 \] (2)

\[ \hat{V} = \sum_i p_i |a_i\rangle \langle a_i|, \sum_i p_i = 1 \] (3)

\[ <\hat{V}> = \langle \psi |\hat{V}|\psi\rangle = \sum_k c_k^* \langle q_k| \sum_i p_i |a_i\rangle \langle a_i| \sum_j c_j |q_j\rangle \] (4)

\[ (q_k|q_j) = \delta_{jk}, \ |q_j\rangle \] is a set of quantum orthonormal basis. We have:

\[ <\hat{V}> = \sum_i p_i \sum_j |c_j|^2 |\langle a_i|q_j\rangle|^2 = \sum_i p_i \sum_j \omega_j v_{ij} \] (5)

\[ \Delta V = \Delta V_{\text{subjective}} + \Delta V_{\text{objective}} = -\left( \sum_i p_i \log p_i + \sum_j \omega_j \log \omega_j \right) \] (6)

Where \( p_i = |\mu_i|^2 \) is a person’s subjective belief in choosing an action \( a_i \); \( \omega_j = |c_j|^2 \) is the objective frequency at which natural state is in \( q_j \); value matrix \( v_{ij} = |\langle a_i|q_j\rangle|^2 \) is the results of decision when a person chooses an action \( a_i \) based on natural state \( q_j \).

Quantum expected value decision theory suggests that a subjective and objective unified result is obtained through people’s beliefs which are based on natural states.

The decision process of a person can be simulated by the continuous evolution of the value operator according to the external information environment as in equation (7) and (8). A final decision is equivalent to a Von Neumann projection measurement performed on the strategy state by the brain which chooses an action \( a_1 \) with probability \( p_1 \) as in equation (9).
\[
\frac{d}{dt} \tilde{V}(t) = [\hat{H}, \tilde{V}(t)]
\]
(7)
\[
\hat{H}|\psi\rangle = E_i|q_i\rangle
\]
(8)
\[
D: \text{projection to } |S\rangle_p \rightarrow |a_i\rangle
\]
(9)

Where $\hat{H}$ is the information energy operator; $E_i$ is all the information that decision makers can get when natural state is in $q_i$. Information is the essence of people’s subjective beliefs just like energy is the essence of the objective world. Valuable information can reduce uncertainty.

As an example, we can represent future market’s states in terms of the Hilbert state space as in (10), Hilbert strategy space is used to represent trader’s strategies as in (11), quantum density operator as value operator which projects a trader’s beliefs onto an action of buying or selling a security according to market environment as in (12).

\[
|\psi\rangle = c_1|q_1\rangle + c_2|q_2\rangle
\]
(10)
\[
|S\rangle = \mu_1|a_1\rangle + \mu_2|a_2\rangle
\]
(11)
\[
\tilde{V} = p_1|a_1\rangle(a_1) + p_2|a_2\rangle(a_2)
\]
(12)

Where $|q_1\rangle$ indicates a state in which the market is rising and $|q_2\rangle$ indicates a state in which the market is falling; $|a_1\rangle$ represents trader’s action to buy and $|a_2\rangle$ represents trader’s action to sell; $p_1$ represents the subjective probability which a trader choose to buy and $p_2$ represents the subjective probability which a trader choose to sell.

Quantum expected value can be represented as in (13). The uncertainty of observed value can be represented by Shannon information entropy as in (14).

\[
< \tilde{V} > = \sum_{i=1,2} \sum_{j=1,2} p_i \omega_j v_{ij} = (2p - 1)(2\omega - 1)v
\]
(13)
\[
\Delta V = -\left( \sum_{i=1,2} p_i \log p_i + \sum_{j=1,2} \omega_j \log \omega_j \right)
= -\left[ \log p (1-p) \log(1-p) \right] + [\omega \log \omega (1-\omega) \log(1-\omega)]
\]
(14)

Where $p$ represents the subjective probability which a trader choose to buy; $\omega$ is the objective frequency at which market state is rising. Quantum expected value is between $-v$ and $+v$ while observed value’s uncertainty is between 0 and 2.

There are three different situations:

1. **Complete certainty**
   a) $(p = \omega = 1 \mid p = \omega = 0) \rightarrow (< \tilde{V} > = v, \Delta V = 0)$
   b) $(p = 1, \omega = 0 \mid p = 0, \omega = 1) \rightarrow (< \tilde{V} > = -v, \Delta V = 0)$
2. Complete uncertainty
\[
(p = \omega = \frac{1}{2}) \rightarrow (\bar{V} > 0, \Delta V = 2)
\]

3. Uncertainty of real market
\[(0 \leq p \leq 1, \ 0 \leq \omega \leq 1) \rightarrow (-\nu \leq \bar{V} \leq \nu, \ 0 \leq \Delta V \leq 2)\]

Quantum decision tree (qDT) is used to simulate people's subjective beliefs

Equation (7) is difficult to solve accurately, but we can approximately obtain a value operator by constructing a qDT which composed of quantum gates and logic operations. The qDT composes of different nodes and branches. There are two types of nodes, non-leaf nodes and leaf nodes. The non-leaf nodes are composed of the operation set F as in (15); the leaf nodes are composed of the data set T as in (16) and (17). The construction process of a qDT is to randomly select a logic symbol from the operation set F as the root of qDT, and then grows corresponding branches according to the nature of the operation symbol and so on until a leaf node is reached.

\[
F = \{+ (ADD), \ast (MULTIPLY), \ ( (OR)) \} \tag{15}
\]

\[
T = \{H, X, Y, Z, S, D, T, I\} \tag{16}
\]

\[
\left\{ \begin{array}{l}
H = \frac{1}{\sqrt{2}}[1 \ 1] X = [0 1 1] Y = [0 0 1] Z = [0 1 0] \\
S = [1 0 0] 0 = [1 0 0] T = [0 e^{i\pi/4} 1]
\end{array} \right. \tag{17}
\]

The qDT of a value operator is a 2x2 matrix, and the value matrix needs to be diagonalized first and then normalized to get probability \( p_1 \) and \( p_2 \) as in (18), (19) and (20).

\[
\tilde{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \xrightarrow{\text{diagonalization}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \xrightarrow{\text{normalization}} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_1|a_1\rangle\langle a_1| + p_2|a_2\rangle\langle a_2| \tag{18}
\]

\[
|a_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |a_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; |a_1\rangle\langle a_1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, |a_2\rangle\langle a_2| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{19}
\]

\[
|v_{11} - \lambda |v_{12} - \lambda | = 0 \tag{20}
\]

**Fig. 1.** A trader's subjective beliefs can be simulated by quantum decision trees.

(a) qDT = \( ((X + S) \ // D) \)   (b) qDT = \( (((D \ // Y) \ * H) \ * S) \)   (c) qDT = \( (H + (D \ // (I \ // H))) \)
A trader’s subjectively beliefs can be simulated by qDTs. For example, there are a few strategies with different subjective beliefs that the trader took on trading securities as shown in Figure 1.

- **qDT** = \((X + S) / D\)

1) \((X + S) \rightarrow \tilde{V} = 0.69|a_1\rangle(a_1) + 0.31|a_2\rangle(a_2)\) (69% belief to buy, 31% belief to sell)
2) \(D \rightarrow \tilde{V} = 0.5|a_1\rangle(a_1) + 0.5|a_2\rangle(a_2)\) (50% belief to buy, 50% belief to sell)

- **qDT** = \(((D / Y) + H) + S\)

1) \(((D * H) + S) \rightarrow \tilde{V} = 0.55|a_1\rangle(a_1) + 0.45|a_2\rangle(a_2)\) (55% belief to buy, 45% belief to sell)
2) \(((Y * H) + S) \rightarrow \tilde{V} = 0.45|a_1\rangle(a_1) + 0.55|a_2\rangle(a_2)\) (45% belief to buy, 55% belief to sell)

- **qDT** = \((H + (D / 0 / H))\)

1) \((H + D) \rightarrow \tilde{V} = 0.5|a_1\rangle(a_1) + 0.5|a_2\rangle(a_2)\) (50% belief to buy, 50% belief to sell)
2) \((H + H) \rightarrow \tilde{V} = 0.5|a_1\rangle(a_1) + 0.5|a_2\rangle(a_2)\) (50% belief to buy, 50% belief to sell)
3) \((H + I) \rightarrow \tilde{V} = |a_1\rangle(a_1)\) (100% belief to buy)

![Fig. 2. The evolution of the quantum value operator.](image)

The qDT can be optimized by the genetic programming (GP)\(^{16}\) as in (21). A tree structure is used for encoding by GP, which is particularly appropriate to solve hierarchical and structured complex problems. Value of measured results is used as fitness function in GP to optimize qDTs. If market is rising and a trader chooses to buy, the value \(V_k\) is positive and the trader would make a profit, otherwise value \(V_k\) is negative and the trader would lose money; if market is falling and a trader chooses to sell, the value \(V_k\) is positive and the trader would make a profit, otherwise value \(V_k\) is negative and the trader would lose money. The purpose of GP iterative evolution is to find a satisfactory qDT through learning historical data as shown in Figure 2. By using selection, crossover and mutation, GP continually evolve and auto-generate the optimized qDTs.

\[
qDT = \arg\max_{qDT \in \mathcal{P}(UT)} \left( \frac{\sum_k V_k}{\max\{V_k\}} \right) \quad \text{max}\{V_k\} \text{ is the biggest loss of trading} \quad (21)
\]
Algorithm: the quantum decision tree evolution

**Input:**
- Training data set \( \{d_k = (q_j, v_i)\} \) which includes \( N \) samples, each sample consists of natural state and value.

**Setting**
1) Operation set \( F = \{+, \ast, /\} \)
2) Data set \( T = \{H, X, Y, Z, S, D, T, I\} \), eight basic quantum gates
3) Crossover probability = 70%; Mutation probability = 5%.

**Initialization:**
- Population: randomly create 100 ~ 500 qDTs.

**Evolution:**
- for \( k = 0 \) to \( N \) or a satisfactory generation is evolved:
  a) Calculate fitness for each qDT.
  b) According to the quality of fitness:
    i. Selection: selecting parent qDTs.
    ii. Crossover: generate a new offspring using the roulette algorithm based on crossover probability.
    iii. Mutation: randomly modify parent qDT based on mutation probability.

**Output:**
- A qDT of the best fitness.

The \( k \)-data of rebar (transaction cycle is day) traded on the Shanghai Futures Exchange is used as the historical data for learning and optimization. Based on the quantum decision tree (Fig. 3 (c)), there are a few strategies with different subjective beliefs that the trader took (Fig. 3 (d)).

\[
q_{\text{DT}} = \left( X/\left( T + ((H/(H*H)) + (X+X))/((I*Y + Z)/(I*H))/D)/I\right) \right)
\]

- \( X \rightarrow \bar{\psi} = 0.5|a_1\rangle(a_1| + 0.5|a_2\rangle(a_2| \) (50% belief to buy, 50% belief to sell)
- \( Y \rightarrow \bar{\psi} = 0.5|a_1\rangle(a_1| + 0.5|a_2\rangle(a_2| \) (50% belief to buy, 50% belief to sell)
- \( I \rightarrow \bar{\psi} = 0.5|a_1\rangle(a_1| + 0.5|a_2\rangle(a_2| \) (50% belief to buy, 50% belief to sell)
- \( (T + (H + (X + X))) \rightarrow \bar{\psi} = 0.76|a_1\rangle(a_1| + 0.24|a_2\rangle(a_2| \) (76% belief to buy, 24% belief to sell)
- \( (T + ((H*H) + (X+X))) \rightarrow \bar{\psi} = 0.97|a_1\rangle(a_1| + 0.03|a_2\rangle(a_2| \) (97% belief to buy, 3% belief to sell)
- \( (X*D) \rightarrow \bar{\psi} = 0.5|a_1\rangle(a_1| + 0.5|a_2\rangle(a_2| \) (50% belief to buy, 50% belief to sell)
- \( (X*(D*(Y*Z))) \rightarrow \bar{\psi} = 0.5|a_1\rangle(a_1| + 0.5|a_2\rangle(a_2| \) (50% belief to buy, 50% belief to sell)
- \( (X*(I*H)) \rightarrow \bar{\psi} = |a_1\rangle(a_1| \) (100% belief to buy)
Fig. 3 Pictures of rebar trading simulated by quantum decision tree. (a) Price fluctuation of rebar traded on Shanghai Futures Exchange from 2009/3/27 to 2018/6/1 (b) Evolution of qDTs (c) An automatically generated qDT with the best fitness (d) A trader’s subjective beliefs simulated by a qDT (red bar: people’s beliefs to buy, green bar: people’s beliefs to sell).

Discussion

Human beings record a large amount of data through the observation of the world. It is through the study of the recorded data that human beings gradually understand the objective world and build a simplified subjective “world model” in the brain. People make decisions by considering both the world’s objectivity and the subjectivity of their beliefs. Observed value is a bridge between objective world and subjective beliefs.

We proposed a quantum decision theory based on quantum expected value. Instead of value function used in reinforcement learning, value operator is used to simulate brain’s beliefs under uncertainty. Classic decision theory asks the bit question, 0 or 1? There are only two possible answers - 0 or 1 (either-or). Quantum decision theory asks the qubit question, 0 and 1? Now, the answer could have infinite possibilities (both-and). Quantum decision theory has inherent uncertainty due to superposition of quantum states and an observable result is obtained from the “collapse” of the state. Scientists tend to think that the brain’s decision is computing rather than reasoning, a quantum value operator computes the probability of taking an action due to incomplete information and the result usually cannot be given by a definite cause, but can only be obtained probabilistically.
References

1. Von Neumann, J. and Morgenstern, O. Theory of Games and Economic Behavior, Princeton, NJ: Princeton University Press (1944).

2. Savage, L. J. The foundation of Statistics, New York, NY: Dover Publication Inc. (1954).

3. Binmore, K. Rational Decisions, Princeton, NJ: Princeton University Press (2009).

4. Allais, M. and Hagen, O. Expected Utility Hypothesis and the Allais Paradox. Dordrecht: Reidel Publishing Company, pp. 434-698, (1979).

5. Ellsberg, D. Risk, ambiguity and the savage axioms. Q. J. Economics 75, 643-669, (1961).

6. Kahneman, D. and Tversky, A. Prospect theory: an analysis of decision under risk. Econometrica 47, 263-292, (1979).

7. Simon, H.A. Reason in Human Affairs, Stanford, CA: Stanford University Press (1983).

8. La Mura, P. Projective expected utility, Journal of Mathematical Psychology, 53(5), 408–414, (2009).

9. Busemeyer, J.R. and Bruza, P.D. Quantum Models of Cognition and Decision, Cambridge University Press (2012).

10. Yukalov, V.I. Evolutionary Processes in Quantum Decision Theory, Entropy 2020, 22, 681.

11. Khrennikov, A. et al. Quantum probability in decision making from quantum information representation of neuronal states. Scientific Reports, 8, 1, (2018).

12. Dirac, P.A.M. The Principles of Quantum Mechanics, Oxford University Press (1958).

13. Shannon, C.E. A mathematical theory of communication. Bell System Tech. J., 27, 379-423, (1948).

14. Von Neumann, J. Mathematical Foundations of Quantum Theory, Princeton, NJ: Princeton University Press (1932).

15. Nielsen, M.A. and Chuang, I.L. Quantum computation and quantum information, Cambridge University Press (2000).

16. Koza, J.R. Genetic programming, on the programming of computers by means of natural selection, Cambridge, MA: MIT Press (1992).

17. Sutton, R.S. and Barto, A.G., Reinforcement Learning, Cambridge, MA: MIT Press (2018).