Limit distribution of a quantum walk driven by a CMV matrix

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Abstract Quantum walks with CMV matrices, five-diagonal unitary matrices, started to be studied in 2003 [1]. The spectral analysis for CMV matrices told us that the quantum walks could localize in distribution depending on the Verblunsky parameters of the matrices. In this paper, we work on a quantum walk whose system is manipulated by a CMV matrix with homogeneous Verblunsky parameters, and present long-time limit distributions. One can understand from the theory that the quantum walk does not localize and how it approximately distributes after the long-time evolution has been executed on the walk.

Keywords CMV matrix · Quantum walk · Limit distribution

1 Introduction

Coined quantum walks, specifically the Hadamard walk, were introduced in [2] and their limit law in the case of translationally invariant walks with an arbitrary unitary coin in $U(2)$ are well known, see for instance [3]. There are different methods to derive these results, and a natural one, in view of the assumed translational invariance is the Fourier method, see [4].

The purpose of this short paper is to obtain limit laws when the unitary evolution is given by a CMV matrix. These were introduced in the context of orthogonal polynomials on the unit circle in [5], for which there is a review paper [6], and were first used to study quantum walks in [1]. Then, a quantum walk driven by a CMV matrix was numerically studied and some probability distributions for the walk were visualized [7]. We consider two cases: in the first case the unitary evolution is given by a CMV matrix and in the second case it is given by the product of a CMV matrix and a simple permutation matrix.
In the first case the limit law is already given in [3] and the result is given here in an appendix. In the second case we obtain a new form of a limit law. In all cases this is studied by means of Fourier analysis and some numerical examples are included to show the agreement between our theoretical result, namely Theorem 1, is section 4.2 and numerical simulations.

2 Definition of a quantum walk

A quantum walker with two coin states $|0\rangle$ and $|1\rangle$ is located at points in $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. The state of the system is described by a normalized vector on the tensor Hilbert space $\mathcal{H}_p \otimes \mathcal{H}_s$. The Hilbert space $\mathcal{H}_p$ encodes the integer points and it is spanned by the orthogonal normalized basis $\{|x\rangle : x \in \mathbb{Z}\}$. The Hilbert space $\mathcal{H}_s$ represents the coin states and it is spanned by the orthogonal normalized basis $\{|0\rangle, |1\rangle\}$. The state of the quantum walk at time $t (= 0, 1, 2, \ldots)$, represented by $|\Psi_t\rangle \in \mathcal{H}_p \otimes \mathcal{H}_s$, updates with a unitary operation,

$$|\Psi_{t+1}\rangle = U |\Psi_t\rangle,$$

where, given parameters $\rho \in (0, 1)$ and $\nu \in \mathbb{R}$,

$$U = \sum_{x \in \mathbb{Z}} |x - 1\rangle \langle x| \otimes \left( \rho_0^2 |1\rangle \langle 0| + \rho_0 \alpha_0 |1\rangle \langle 1| \right)$$

$$+ |x\rangle \langle x| \otimes \left\{ -\alpha_0 \rho_0 |0\rangle \langle 0| - (1 - \rho_0^2) |0\rangle \langle 1| - (1 - \rho_0^2) |1\rangle \langle 0| + \rho_0 \alpha_0 |1\rangle \langle 1| \right\}$$

$$+ |x + 1\rangle \langle x| \otimes \left( -\alpha_0 \rho_0 |0\rangle \langle 0| + \rho_0^2 |0\rangle \langle 1| \right),$$

with $\alpha_0 = \rho e^{i\nu}$ and $\rho_0 = \sqrt{1 - \rho^2}$. The notation $i$ denotes the imaginary unit in complex numbers. We assume $|\Psi_0\rangle = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) = (|0\rangle \otimes |\phi\rangle)$ with $|\alpha|^2 + |\beta|^2 = 1$.

The unitary operation $U$ can be decomposed to the product of two unitary operations, $U = U_{CMV} U_f$, with

$$U_{CMV} = \sum_{x \in \mathbb{Z}} |x - 1\rangle \langle x| \otimes \left( \rho_0 \alpha_0 |1\rangle \langle 0| + \rho_0^2 |1\rangle \langle 1| \right)$$

$$+ |x\rangle \langle x| \otimes \left\{ -(1 - \rho_0^2) |0\rangle \langle 0| - \alpha_0 \rho_0 |0\rangle \langle 1| + \rho_0 \alpha_0 |1\rangle \langle 0| - (1 - \rho_0^2) |1\rangle \langle 1| \right\}$$

$$+ |x + 1\rangle \langle x| \otimes \left( \rho_0^2 |0\rangle \langle 0| - \alpha_0 \rho_0 |0\rangle \langle 1| \right),$$

$$U_f = \sum_{x \in \mathbb{Z}} |x\rangle \langle x| \otimes \left( |0\rangle \langle 1| + |1\rangle \langle 0| \right).$$
Once standard bases are given to the Hilbert spaces $\mathcal{H}_p$ and $\mathcal{H}_c$, we get a matrix representation of $U_{\text{CMV}},$

$$U_{\text{CMV}} = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \rho_0 \alpha_0 & \rho_0 & 0 & 0 & 0 \\
\cdots & -(1 - \rho_0^2) & -\alpha_0 \rho_0 & 0 & 0 & 0 \\
\cdots & \rho_0 \alpha_0 & -(1 - \rho_0^2) & \rho_0 \alpha_0 & \rho_0^2 & 0 \\
\cdots & 0 & 0 & \rho_0 \alpha_0 & -(1 - \rho_0^2) & -\alpha_0 \rho_0 \\
\cdots & 0 & 0 & 0 & \rho_0 \alpha_0 & -(1 - \rho_0^2) \\
\cdots & 0 & 0 & 0 & 0 & \rho_0^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix},$$

which is also known as a CMV matrix. The operation $U_f$ is expressed as

$$U_f = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 1 & 0 & 0 & 0 \\
\cdots & 1 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 1 & 0 & 0 \\
\cdots & 0 & 0 & 1 & 0 & 0 \\
\cdots & 0 & 0 & 1 & 0 & 0 \\
\cdots & 0 & 0 & 1 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},$$

and the unitary operation $U$ becomes

$$U = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 & 0 \\
\cdots & \rho_0^2 & \rho_0 \alpha_0 & 0 & 0 & 0 \\
\cdots & -(1 - \rho_0^2) & -\rho_0 \alpha_0 & -(1 - \rho_0^2) & 0 & 0 \\
\cdots & -\alpha_0 \rho_0 & \rho_0 & -\alpha_0 \rho_0 & -(1 - \rho_0^2) & 0 \\
\cdots & 0 & 0 & -\alpha_0 \rho_0 & \rho_0 \alpha_0 & \rho_0^2 \\
\cdots & 0 & 0 & 0 & \rho_0 \alpha_0 & -(1 - \rho_0^2) \\
\cdots & 0 & 0 & 0 & 0 & -\alpha_0 \rho_0 \\
\cdots & 0 & 0 & 0 & 0 & \rho_0^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}.$$
With $||\psi|| = \sqrt{\langle \psi|\psi \rangle} \ (|\psi\rangle \in \mathcal{H}_s)$, the quantum walker is observed at position $x \in \mathbb{Z}$ at time $t \in \{0, 1, 2, \ldots\}$ with probability

$$P(X_t = x) = \langle \Psi_t | \{ |x\rangle \otimes (|0\rangle + |1\rangle) \} |\Psi_t \rangle = \left| \left\{ |x\rangle \otimes (|0\rangle + |1\rangle) \} |\Psi_t \rangle \right|^2. \quad (8)$$

### 3 Fourier transform

The Fourier analysis has been used for discovering the interesting behavior of quantum walks [8]. Let $|\hat{\psi}_t(k)\rangle \in \mathbb{C}^2 \ (k \in [-\pi, \pi])$ be the Fourier transform of the quantum walk at time $t$,

$$|\hat{\psi}_t(k)\rangle = \sum_{x \in \mathbb{Z}} e^{-ikx} \left\{ |x\rangle \otimes (|0\rangle + |1\rangle) \} |\Psi_t \rangle, \quad (9)$$

from which the Fourier inverse transform reproduces the system of quantum walk,

$$|\Psi_t \rangle = \sum_{x \in \mathbb{Z}} |x\rangle \otimes \int_{-\pi}^{\pi} e^{ikx} |\hat{\psi}_t(k)\rangle \frac{dk}{2\pi}. \quad (10)$$

Once assigning a standard basis to the Hilbert space $\mathcal{H}_s$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (11)$$

we can tell from Eq. (11) that the Fourier transform gets updated with a $2 \times 2$ unitary matrix $\hat{U}(k)$,

$$|\hat{\psi}_{t+1}(k)\rangle = \hat{U}(k) |\hat{\psi}_t(k)\rangle, \quad (12)$$

where

$$\hat{U}(k) = R \left( \frac{-\nu}{2} \right) H(k) R \left( \frac{\nu}{2} \right), \quad (13)$$

with

$$R(\varphi) = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix} \ (\varphi \in \mathbb{R}),$$

$$H(k) = \begin{bmatrix} -\rho_0 e^{i\nu} + e^{-i(k-\nu)} - \rho^2 e^{i\nu} + \rho_0^2 e^{i(k-\nu)} & \rho_0 (e^{i\nu} + e^{i(k-\nu)}) \\ -\rho^2 e^{-i\nu} + \rho_0^2 e^{-i(k-\nu)} & \rho_0 (e^{-i\nu} + e^{-i(k-\nu)}) \end{bmatrix}. \quad (15)$$

$$|\hat{\psi}_t(k)\rangle = \hat{U}(k)^\dagger |\hat{\psi}_0(k)\rangle = R \left( \frac{-\nu}{2} \right) H(k)^\dagger R \left( \frac{\nu}{2} \right) |\hat{\psi}_0(k)\rangle = R \left( \frac{-\nu}{2} \right) H(k)^\dagger |\hat{\phi}\rangle, \quad (16)$$
where, remembering $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$, we get

$$|\tilde{\phi}\rangle = R\left(\frac{\nu}{2}\right) |\hat{\psi}(k)\rangle = R\left(\frac{\nu}{2}\right) |\phi\rangle = \begin{bmatrix} e^{i\nu/2} \alpha \\ e^{-i\nu/2} \beta \end{bmatrix}.$$  \hspace{1cm} (17)

Note that $|e^{i\nu/2}\alpha|^2 + |e^{-i\nu/2}\beta|^2 = |\alpha|^2 + |\beta|^2 = 1$.

The Fourier inverse transform

$$|\Psi_t\rangle = \sum_{x \in \mathbb{Z}} |x\rangle \otimes R\left(-\frac{\nu}{2}\right) \int_{-\pi}^{\pi} e^{ikx} H(k)^t |\tilde{\phi}\rangle \frac{dk}{2\pi},$$  \hspace{1cm} (18)

gives another representation of the probability distribution on Fourier space,

$$P(X_t = x) = \left| R\left(-\frac{\nu}{2}\right) \int_{-\pi}^{\pi} e^{ikx} H(k)^t |\tilde{\phi}\rangle \frac{dk}{2\pi} \right|^2 = \left| \int_{-\pi}^{\pi} e^{ikx} H(k)^t |\tilde{\phi}\rangle \frac{dk}{2\pi} \right|^2 = \left| \int_{-\pi}^{\pi} e^{i(k+\nu)x} H(k+\nu)^t |\tilde{\phi}\rangle \frac{dk}{2\pi} \right|^2 = \left| \int_{-\pi}^{\pi} e^{ikx} \tilde{H}(k)^t |\tilde{\phi}\rangle \frac{dk}{2\pi} \right|^2,$$  \hspace{1cm} (19)

with

$$\tilde{H}(k) = H(k + \nu) = \begin{bmatrix} -\rho^2 e^{i\nu} + e^{-ik} & -\rho e^{i\nu} + \rho^2 e^{-ik} \\ -\rho^2 e^{-i\nu} + \rho^2 e^{ik} & \rho^2 e^{i\nu} + \rho^2 e^{-ik} \end{bmatrix}.$$  \hspace{1cm} (20)

We are, hence, allowed to analyze the quantum walk defined by

$$|\hat{\psi}_t(k)\rangle = \tilde{H}(k)^t |\tilde{\phi}\rangle,$$  \hspace{1cm} (21)

as long as we focus on the probability distribution. After this point, we are going to concentrate on Eq. (21) instead of Eq. (12).

### 4 Limit distribution

In this section we see a limit distribution which catches the features of quantum walk. It is separately introduced for $(\rho, \nu) = (1/\sqrt{2}, \pi/2 + n\pi) (n \in \mathbb{Z})$ and for $(\rho, \nu) \neq (1/\sqrt{2}, \pi/2 + n\pi) (n \in \mathbb{Z})$ because the quantum walk for the first case is equivalent to the standard coined quantum walk whose limit distribution was already proved [3]. The statement of the limit distribution for the second case, however, contained the one for the first case.
The quantum walker converges in distribution. For any real number $x$, we, hence, found the walker at position $x$ state of the form $|\psi\rangle$. We should note that this result is very similar to the usual limit theorem for a coined walk, and is contained in [3]. The fact is also visualized in Fig. 1. A more interesting result is obtained when the parameters take values different from the ones above.

4.1 $(\rho, \nu) = (1/\sqrt{2}, \pi/2 + n\pi)$ ($n \in \mathbb{Z}$)

The operation $\hat{H}(k)$ contains a 2-step evolution of a standard coined quantum walk,

$$
\hat{H}(k) = -i (-1)^n \left\{ R \left( -\frac{k}{2} \right) \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-(1)^n i \pi/4} & -e^{-(1)^n i \pi/4} \\ -e^{-(1)^n i \pi/4} & -e^{-(1)^n i \pi/4} \end{bmatrix} \right\}^2
$$

$$
= -i (-1)^n \left\{ R \left( -\frac{k}{2} \right) V \right\}^2, \tag{22}
$$

where

$$
V = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-(1)^n i \pi/4} & -e^{-(1)^n i \pi/4} \\ -e^{-(1)^n i \pi/4} & -e^{-(1)^n i \pi/4} \end{bmatrix}. \tag{23}
$$

We should note that

$$
\left\{ R \left( -\frac{k}{2} \right) V \right\}^2 = \frac{1}{2} \begin{bmatrix} 1 - (-1)^n i e^{-ik} & 1 + (-1)^n i e^{-ik} \\ -1 + (-1)^n i e^{ik} & 1 + (-1)^n i e^{ik} \end{bmatrix}. \tag{24}
$$

We, hence, found the walker at position $x$ at time $t$ with probability

$$
P(X_t = x) = \left| \int_{-\pi}^{\pi} e^{ikx} \cdot (-i)^t (-1)^n \left\{ R \left( -\frac{k}{2} \right) V \right\}^{2t} |\phi\rangle \frac{dk}{2\pi} \right|^2.
$$

The quantum walk can be analyzed in a similar way to the standard coined walk, and we get a limit distribution, whose proof is omitted here because the readers refer to the method to compute it by Fourier analysis in [3].

Assumed $(\rho, \nu) = (1/\sqrt{2}, \pi/2 + n\pi)$ ($n \in \mathbb{Z}$) and given the localized initial state of the form $|\psi_0\rangle = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle)$ ($\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$), the quantum walker converges in distribution. For any real number $x$, we have

$$
\lim_{t \to \infty} \frac{X_t}{t} \leq x = \int_{-\infty}^{x} \frac{1}{\pi(1 - y^2)\sqrt{1 - 2y^2}} \Theta_n(y) I_{(-1/\sqrt{2}, 1/\sqrt{2})}(y) \ dy, \tag{26}
$$

where

$$
\Theta_n(x) = 1 + \left\{ |\alpha|^2 + |\beta|^2 + (-1)^n \cdot 2\Im(\alpha\beta) \right\} x, \tag{27}
$$

$$
I_{(-1/\sqrt{2}, 1/\sqrt{2})}(x) = \begin{cases} 1 & (x \in (-1/\sqrt{2}, 1/\sqrt{2})) \\ 0 & (x \notin (-1/\sqrt{2}, 1/\sqrt{2})) \end{cases}. \tag{28}
$$

The notation $\Im(z)$ denotes the imaginary part of a complex number $z$.

This result is very similar to the usual limit theorem for a coined walk, and is contained in [3]. The fact is also visualized in Fig. 1. A more interesting result is obtained when the parameters take values different from the ones above.
Fig. 1 $\rho = 1/\sqrt{2}, \nu = \pi/2$: The blue lines represent the probability distribution $P(X_t = x)$ at time $t = 500$ ((a)--1, (b)--1) and the red lines represent the limit density function ((a)--2, (b)--2). In (a)--3 and (b)--3, we confirm that the approximation (red points) obtained from the limit density function reproduces the features of the probability distribution as time $t$ becomes large enough. The walker launches with the localized initial state at the origin, $|\Psi_0\rangle = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle)$. 
4.2 \((\rho, \nu) \neq (1/\sqrt{2}, \pi/2 + n\pi)\quad (n \in \mathbb{Z})\)

**Theorem 1** Assumed \((\rho, \nu) \neq (1/\sqrt{2}, \pi/2 + n\pi)\quad (n \in \mathbb{Z})\) and given the localized initial state of the form \(|\Psi_0\rangle = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle)\quad (\alpha, \beta \in \mathbb{C} \text{ such that } |\alpha|^2 + |\beta|^2 = 1)\), the quantum walker converges in distribution. For any real number \(x\), we have

\[
\lim_{t \to \infty} \mathbb{P}\left(\frac{X_t}{t} \leq x\right) = \int_{-\infty}^{x} \frac{\sqrt{\eta_+ (y)} + \sqrt{\eta_- (y)}}{2\pi (1 - y^2)^{3/2}} \gamma(y) I_{(-h^*, h^*)} (y) \, dy, \quad (29)
\]

where

\[
\xi(x) = (\rho^2 - x^2)(\rho_0^2 - x^2) - \rho^2 \rho_0^2 (\cos^2 \nu) x^2, \quad (30)
\]
\[
\eta_{\pm} (x) = 1 - \rho^2 \rho_0^2 (1 + \sin^2 \nu) - (1 - \rho^2 \rho_0^2 \cos^2 \nu) x^2 
\pm 2 \rho \rho_0 (\sin \nu) \sqrt{\xi(x)}, \quad (31)
\]
\[
\gamma(x) = 1 + \left\{ |\alpha|^2 - |\beta|^2 - \frac{2 \rho \rho_0}{\rho} \left( \Re(\alpha \beta) \cos \nu - \Im(\alpha \beta) \sin \nu \right) \right\} x, \quad (32)
\]

and

\[
I_{(-h^*, h^*)} (x) = \begin{cases} 1 & (x \in (-h^*, h^*)) \\ 0 & (x \notin (-h^*, h^*)) \end{cases}, \quad (33)
\]

with

\[
h^* = \frac{1}{2} \left\{ \sqrt{(1 + \rho \rho_0)^2 - \rho^2 \rho_0^2 \sin^2 \nu} - \sqrt{(1 - \rho \rho_0)^2 - \rho^2 \rho_0^2 \sin^2 \nu} \right\}. \quad (34)
\]

The notation \(\Re(z)\) denotes the real part of a complex number \(z\).

Looking at Fig. 2, we realize the difference from Fig. 1 which was for the case of \((\rho, \nu) = (1/\sqrt{2}, \pi/2 + n\pi)\quad (n \in \mathbb{Z})\). The probability distributions in the pictures (a)-1 and (b)-1 vibrate harder than the ones of the standard coined walk in Fig. 1.

Here we move on to the proof of the theorem, that is, convergence of \(X_t/t\) in distribution as \(t \to \infty\). The unitary matrix \(\tilde{H}(k)\) holds the eigenvalues

\[
\lambda_j (k) = i \rho \rho_0 (\sin k - \sin \nu) - (-1)^j \sqrt{1 - \rho^2 \rho_0^2 (\sin k - \sin \nu)^2} \quad (j = 1, 2), \quad (35)
\]

and the eigenvalue \(\lambda_j (k)\) is associated with the eigenvectors in a normalized expression

\[
|v_j (k)\rangle = \frac{1}{\sqrt{N_j (k)}} \left[ \rho \rho_0 (e^{i \nu} + e^{-i k}) - (-1)^j \sqrt{J(k)} \right]. \quad (36)
\]
(a) \((\alpha, \beta) = (1/\sqrt{2}, i/\sqrt{2})\) (b) \((\alpha, \beta) = (1, 0)\)

\[ N_j(k) = 2 \left\{ J(k) - (-1)^j \rho \rho_0 (\cos k + \cos \nu) \sqrt{J(k)} \right\}, \quad (37) \]

\[ J(k) = 1 - \rho^2 \rho_0^2 (\sin k - \sin \nu)^2. \quad (38) \]
Since $J(k)$ is larger than 0 for any $k \in [-\pi, \pi)$ as long as $(\rho, \nu) \neq (1/\sqrt{2}, \pi/2 + n\pi)$ ($n \in \mathbb{Z}$), the normalizing factors $N_j(k)$ ($j = 1, 2$) keep to be larger than 0.

Once the initial state is expanded as $|\tilde{\phi}\rangle = \sum_{j=1}^{2} \langle v_j(k)|\tilde{\phi}\rangle |v_j(k)\rangle$ in the eigenspace of matrix $\tilde{H}(k)$, the Fourier transform gets the expression in the eigenspace,

$$|\hat{\psi}_t(k)\rangle = \sum_{j=1}^{2} \lambda_j(k)^t \langle v_j(k)|\tilde{\phi}\rangle |v_j(k)\rangle. \quad (39)$$

With $D = i \cdot d/dk$ and the Pochhammer notation $(t)_r = t \cdot (t-1) \times \cdots \times (t-r+1)$, one approaches the $r$-th moments ($r = 0, 1, 2, \ldots$)

$$\mathbb{E}[X_t^r] = \int_{-\pi}^{\pi} \langle \hat{\psi}_t(k) | \left(D^r | \hat{\psi}_t(k)\right) \rangle \frac{dk}{2\pi}$$

$$= (t)_r \left\{ \sum_{j=1}^{2} \int_{-\pi}^{\pi} \left( \frac{i \lambda_j(k)}{\lambda_j(k)} \right)^r \left| \langle v_j(k)|\tilde{\phi}\rangle \right|^2 \frac{dk}{2\pi} \right\} + O(t^{r-1}), \quad (40)$$

from which

$$\lim_{t \to \infty} \mathbb{E} \left[ \left( \frac{X_t}{t} \right)^r \right] = \sum_{j=1}^{2} \int_{-\pi}^{\pi} \left( \frac{i \lambda_j(k)}{\lambda_j(k)} \right)^r \left| \langle v_j(k)|\tilde{\phi}\rangle \right|^2 \frac{dk}{2\pi}, \quad (41)$$

where the functions $i \lambda_j(k)/\lambda_j(k)$ are computed to be of the form

$$\frac{i \lambda_j(k)}{\lambda_j(k)} = (-1)^j \frac{\rho \rho_0 \cos k}{\sqrt{1 - \rho^2 \rho_0^2 (\sin k - \sin \nu)^2}} \quad (j = 1, 2). \quad (41)$$

We change the variable in the integrals from $k$ to $x$ by putting $i \lambda_j(k)/\lambda_j(k) = x (j = 1, 2)$, and reach another integral representation of the limit

$$\lim_{t \to \infty} \mathbb{E} \left[ \left( \frac{X_t}{t} \right)^r \right] = \int_{-\infty}^{\infty} x^r \cdot \frac{\sqrt{\eta_+(x)} + \sqrt{\eta_-(x)}}{2\pi(1 - x^2)\sqrt{\xi(x)}} \gamma(x) I_{(-h^*, h^*)}(x) dx, \quad (42)$$

where the functions $\xi(x)$, $\eta_{\pm}(x)$, $\gamma(x)$, and $I_{(-h^*, h^*)}(x)$ are given in Eqs. (30)–(33). This convergence of the $r$-th moments shown in Eq. (42) guarantees the statement of Theorem [1].

Since the substitution $\rho = 1/\sqrt{2}$ and $\nu = \pi/2 + n\pi$ ($n \in \mathbb{Z}$) makes the functions $\xi(x) = (1 - 2x^2)^2/4$, $\sqrt{\eta_+(x)} + \sqrt{\eta_-(x)} = \sqrt{1 - 2x^2}$, $\gamma(x) = 1 + \{ |\alpha|^2 - |\beta|^2 + (-1)^n \cdot 23(\alpha\beta) \} x = \Theta_n(x)$, and $h^* = 1/\sqrt{2}$, we fulfill Eq. (28) again, which means that the limit distribution for the case $(\rho, \nu) = (1/\sqrt{2}, \pi/2 + n\pi)$ ($n \in \mathbb{Z}$) is allowed to be combined into Theorem [1].
5 Summary

We studied a quantum walk on the line which evolved with a CMV matrix, and demonstrated long-time limit distributions. The quantum walker did not localize at all and spread away as time $t$ became large. The limit distributions, moreover, approximately depicted the probability distributions of the quantum walk, shown as Figs. 2 and 3. While some special values of parameters $\theta$ and $\nu$ produced a standard quantum walk whose limit distributions were already derived in [3], the others gave a different type of limit distribution as we can expect from the bigger oscillation in probability distributions of the walk in Fig 2.

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If we do not use the operation $U_{f}$, the operation onto the Fourier transform $|\hat{\psi}_{t}(k)\rangle$ contains a 2-step evolution of a standard coined quantum walk,

$$\hat{U}(k) = R\left(\frac{\nu}{2}\right) \left\{ R\left(\frac{\nu}{2}\right) \left[ \frac{\rho_{0} - \rho}{\rho \rho_{0}} \right] \right\}^{2} R\left(\frac{-\nu}{2}\right),$$

from which

$$|\hat{\psi}_{t}(k)\rangle = \hat{U}(k)|\hat{\psi}_{0}(k)\rangle = R\left(\frac{\nu}{2}\right) \left\{ R\left(\frac{-\nu}{2}\right) \right\}^{2} R\left(\frac{-\nu}{2}\right)|\phi\rangle.$$  \hspace{1cm} (44)

Note that

$$\left\{ R\left(\frac{-\nu}{2}\right) \left[ \frac{\rho_{0} - \rho}{\rho \rho_{0}} \right] \right\}^{2} = \left[ \frac{\rho_{0} e^{-ik} - \rho^{2} - \rho \rho_{0} e^{-ik} - \rho \rho_{0}}{\rho \rho_{0} e^{-ik} + \rho \rho_{0}} \right],$$

and $|e^{-i\nu/2}\alpha|^{2} + |e^{i\nu/2}\beta|^{2} = |\alpha|^{2} + |\beta|^{2} = 1$. The walker is, hence, observed at position $x$ at time $t$ with probability

$$P(X_{t} = x) = \left| R\left(\frac{\nu}{2}\right) \int_{-\pi}^{\pi} e^{ikx} \left\{ R\left(\frac{-\nu}{2}\right) \left[ \frac{\rho_{0} - \rho}{\rho \rho_{0}} \right] \right\}^{2} R\left(\frac{-\nu}{2}\right)|\phi\rangle \frac{dk}{2\pi} \right|^{2},$$

(47)

The initial state of the Fourier transform should be reconsidered as $R\left(-\nu/2\right)|\phi\rangle$ in Eq. (43).

By a similar Fourier analysis used for quantum walks in [8], the walker converges in distribution as $t \rightarrow \infty$.

For any real number $x$, we have

$$\lim_{t \rightarrow \infty} P\left(\frac{X_{t}}{t} \leq x\right) = \int_{-\infty}^{\infty} \frac{\rho}{\pi(1 - y^{2})}\sqrt{\rho_{0}^{2} - y^{2}} \Delta(y) I_{(-\rho_{0}, \rho_{0})}(y) \, dy,$$

(48)

where

$$\Delta(x) = 1 + \left\{ |\alpha|^{2} + |\beta|^{2} - \frac{2\rho}{\rho_{0}} \left( \Re(\alpha \beta) \cos \nu + \Im(\alpha \beta) \sin \nu \right) \right\} x,$$

(49)

$$I_{(-\rho_{0}, \rho_{0})}(x) = \begin{cases} 1 & (x \in (-\rho_{0}, \rho_{0})) \\ 0 & (x \notin (-\rho_{0}, \rho_{0})) \end{cases}.$$  \hspace{1cm} (50)

This result for the coined quantum walk is already contained in [8]. The proof for this limit theorem is omitted in this paper because of the similarity to the computation written in [8]. Figure 3 shows the behavior of the quantum walk when the operation lacks the unitary matrix $U_{f}$. 
Limit distribution of a quantum walk driven by a CMV matrix

\[ (a) \ (\alpha, \beta) = (1/\sqrt{2}, i/\sqrt{2}) \]

\[ (b) \ (\alpha, \beta) = (1, 0) \]

Fig. 3. \( \rho = 1/\sqrt{2}, \nu = \pi/4 \): The blue lines represent the probability distribution \( P(X_t = x) \) at time \( t = 500 \) ((a)–1, (b)–1) and the red lines represent the limit density function ((a)–2, (b)–2). In (a)–3 and (b)–3, we confirm that the approximation (red points) obtained from the limit density function reproduces the features of the probability distribution as time \( t \) becomes large enough. The walker launches with the localized initial state at the origin, \( |\Psi_0\rangle = |0\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) \).

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