Stacks in Representation Theory

What is a continuous representation of an algebraic group?

Joseph Bernstein
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1 Introduction.

In this note I would like to introduce a new approach to (or rather a new language for) representation theory of groups. Namely, I propose to consider a (complex) representation of a group $G$ as a sheaf on some geometric object. This point of view necessarily leads to a conclusion that the standard approach to (continuous) representations of algebraic groups should be modified.

Let us start with a $p$-adic or finite field $F$ and fix an algebraic group $G$ defined over $F$. In the standard approach we consider the set $G = G(F)$ of $F$-points of $G$ as a topological group and study an appropriate category $Rep(G)$ of continuous representations of $G$.

The main goals of this note is to explain that this approach is ideologically inconsistent. In fact I will describe how to extend the category $Rep(G)$ to some larger category $M(G, F)$ that better corresponds to our intuitive understanding of representations of $G$.

We will see that this category can be naturally described as a product of categories $Rep(G_i)$ over all pure inner forms of the group $G$. On the level of simple objects this means that $Irr(M(G, F)) \approx \bigsqcup Irr(G_i)$. This agrees with observation by several mathematicians (e.g. by D. Vogan [V]) that when we classify irreducible representations it is better to work with the union of sets $Irr(G_i)$ for several forms of the group $G$ than with one set $Irr(G)$.

1.1 Representations and sheaves on Stacks.

In order to describe the category $M(G)$ I propose to consider representations as sheaves on algebraic stacks.

The stacks play more and more important role in contemporary Mathematics. Since they are not yet the common language in representation theory I will recall some basic notions related to stacks.
Informally stack is a “space” $X$ such that every point $x \in X$ is endowed with a group $G_x$ of automorphisms of inner degrees of freedom at this point.

We see that in order to consider stacks we should first fix a Geometric Environment, i.e. a category $\mathcal{S}$ of spaces on which we model our stacks. In fact the category $\mathcal{S}$ should be considered together with some Grothendieck topology. The standard term for such category $\mathcal{S}$ is “site”.

Usually one works with the following sites:

(i) Category of schemes over a field $F$
(ii) Category of smooth manifolds
(iii) Category of topological spaces (e.g. totally disconnected)
(iv) Category $\text{Sets}$ of sets.

### 1.2 Groupoids

Stacks in the category $\mathcal{S} = \text{Sets}$ are **groupoids**. Let me remind that by definition groupoid is a category in which all morphisms are isomorphisms.

Groupoids represent rather elementary examples of stacks. However they exhibit many features of the general case. For this reason I would like to discuss them in some detail.

#### 1.2.1 Examples of groupoids

To every discrete group $G$ we assign the **basic groupoid** $BG = pt/G$ as follows:

An object of the category $BG$ is a $G$-torsor $T$ and morphisms in this category are morphisms of $G$-sets.

More generally, given an action of the group $G$ on a set $Z$ we define the **quotient groupoid** $BG(Z) = Z/G$ as follows:

Object of $BG(Z)$ is a $G$-torsor $T$ equipped with a $G$-morphism $\nu : T \to Z$. Morphisms are morphisms of $G$-sets over $Z$.

#### 1.2.2 Representations as sheaves on groupoids.

Given a groupoid $\mathcal{X}$ it is natural to think about it as a geometric object (some kind of a space). We define a **sheaf** $R$ (of complex vector spaces) on a groupoid $\mathcal{X}$ to be a functor $R : \mathcal{X} \to \text{Vect}$ (we will see later why this notion is natural). We denote by $Sh(\mathcal{X})$ the category of sheaves on $\mathcal{X}$.

**Claim.** Category $Sh(BG)$ is naturally equivalent to the category $Rep(G)$.

This gives us a ”geometric” description of the category $Rep(G)$.

**Remark.** If $G$ is a totally disconnected topological group one can consider the basic groupoid $BG$ as a topological groupoid. One can then consider sheaves on a topological groupoid $\mathcal{X}$ taking this topology into account.

Technically the work with topological groupoids is a little involved. Later we describe another way to define the category $Sh(\mathcal{X})$ for topological groupoids that is technically simpler.
1.3 Algebraic groups and stacks.

Now let us consider the case of an algebraic group $G$ over $F$. By analogy with the discrete case one can define the basic stack $B\mathcal{G} = pt/\mathcal{G}$. This is an algebraic stack modeled on the category $\mathcal{S}$ of schemes over $F$ (more details later).

For any algebraic stack $\mathcal{X}$ over $F$ its $F$-points $\mathcal{X}(F)$ form a groupoid. We define an $F$-sheaf on the stack $\mathcal{X}$ to be a sheaf on the groupoid $\mathcal{X}(F)$. The category of these sheaves we denote $Sh_F(\mathcal{X})$.

Given an algebraic group $\mathcal{G}$ over $F$ we define the category $\mathcal{M} = \mathcal{M}(\mathcal{G}, F)$ as the category of $F$-sheaves on the algebraic stack $B\mathcal{G}$. I call the objects of this category stacky $G$-modules.

Now starting with the algebraic group $\mathcal{G}$ we can consider two competing definitions of a representation.

**Definition 1.** Category $\text{Rep}(G)$ obtained from $G$ by a chain of constructions

$\mathcal{G} \implies \text{group } G = \mathcal{G}(F) \implies \text{groupoid } \mathcal{Y} = BG \implies Sh(\mathcal{Y})$

**Definition 2.** Category $\mathcal{M}(G, F)$ obtained from $G$ by a chain of constructions

$\mathcal{G} \implies \text{stack } \mathcal{X} = B\mathcal{G} \implies \text{groupoid } \mathcal{X}(F) \implies Sh(\mathcal{X}(F))$

The subtle point is that the groupoids $\mathcal{Y}$ and $\mathcal{X}(F)$ are not always equivalent. So the category $\mathcal{M}(\mathcal{G}, F) = Sh(\mathcal{X}(F))$ might be not equivalent to the category $\text{Rep}(G) = Sh(\mathcal{Y})$.

The standard notion of a continuous representation of $G$ is based on definition 1. However in my opinion the definition 2 is more appropriate.

1.4 About this note.

In section 2 I describe a striking example that shows that the standard definition of representations is not a good one. In section 3 I discuss the case of groupoids. In section 4 I shortly describe how to give a technical definition of a stack. In section 5 I discuss a technical definition of an $F$-sheaf on an algebraic stack and explain how to describe $F$-sheaves using equivariant sheaves.

It is well known that the structure of the category $\text{Rep}(G)$ to a large extent is determined by the structure of the algebraic group $\mathcal{G}$ (Harish-Chandra’s Lefschetz principle). This should be related to conjectural description of irreducible representations in terms of Langlands’ dual group. It seems clear that this will work better if we replace the set $\text{Irr}(G)$ by the set of simple stacky $G$-modules.

In section 6 I explain the algebra-geometric structure of the groupoids $B\mathcal{G}(F)$ that probably is related to this phenomenon.

This note is an expanded version of the lecture that I gave at the Fourth Conference of Tsinghua Sanya International Mathematics Forum (TSIMF) in December 2013. I would like to thank TSIMF organizers for the invitation.
Main ideas about stacks in representations theory came out of my research in the framework of the ERC grant 291612. Much of my work on this subject was done during my visits to MPIM, Bonn. I would like to thank MPIM for very creative atmosphere.

2 An example.

First I describe a striking example that illustrates what is wrong with the standard approach. In this example, and later in the note, I freely use the notion of an equivariant sheaf that should be one of the central notions in representation theory. Here are some remarks about this notion.

2.1 Equivariant sheaves.

Let a topological group $G$ act on a topological space $Z$ (action $a : G \times Z \to Z$). For simplicity we assume that all spaces are locally compact and totally disconnected. We denote by $\mathcal{SH}_G(Z)$ the category of $G$-equivariant sheaves (of complex vector spaces) on $Z$. This category will play the central role in what follows.

Recall, that a $G$-equivariant sheaf is a sheaf $F$ on $Z$ equipped with an isomorphism $\alpha : a^*(F) \to pr_Z^*(F)$ of two liftings of $F$ to the space $G \times Z$ satisfying some natural conditions.

Let me remind two standard facts about equivariant sheaves.

**Fact 1.** The category $\mathcal{SH}_G(pt)$ is equivalent to the category $\text{Rep}(G)$ of smooth representations of $G$.

**Fact 2.** Suppose that $Z$ is a quotient space of the group $G$. Fix a point $z \in Z$ and denote by $H$ its stabilizer in $G$. Then we have an equivalence of categories $\mathcal{SH}_G(Z) \approx \mathcal{SH}_H(z) \approx \text{Rep}(H)$.

2.2 Heuristics

My example is based on the following heuristic geometric principle.

Let $a : G \times Z \to Z$ be a transitive action of an algebraic group $G$ on an algebraic variety $Z$. Passing to $F$-points we get a continuous action $a : G \times Z \to Z$.

This action is usually not transitive and we can write $Z$ as a union of open orbits $Z = \bigsqcup Z_i, i = 1, \ldots, n$.

**Heuristic Geometric Principle.**

1. The space $Z$ is ”good”, i.e. it is easy to describe
2. Every individual orbit $Z_i$ might be a ”bad” space, that means that it is difficult to describe.

2.3 An example – representations of orthogonal groups.

Fix an $n$-dimensional vector space $V$ over $F$. 
The group \( G = GL(V) \) acts on the space \( Z \) of non-degenerate quadratic forms. Fix a form \( Q \in Z \), denote by \( H \) the corresponding orthogonal group \( O(Q) \) and by \( Z_0 \) the \( G \)-orbit of \( Q \) in \( Z \). Then we have an equivalence of categories

\[
Rep(H) \approx Sh_G(Z_0)
\]

According to heuristic geometric principle the space \( Z_0 \) might be (and often is) a ”bad” space. This means that the category \( Rep(H) \approx Sh_G(Z_0) \) might be a ”bad” category.

However we see that this category can be naturally extended to a ”good” category \( \mathcal{M} := Sh_G(Z) \) of \( G \)-equivariant sheaves on good algebraic space \( Z \).

We have natural decomposition \( \mathcal{M} \approx \prod Sh_G(Z_i) \), where \( Z_i \) are \( G \)-orbits in \( Z \). In particular the set \( Irr(\mathcal{M}) \) is a disjoint union of sets \( Irr(Sh_G(Z_i)) \).

It seems reasonable to assume that the classification of simple objects of the category \( \mathcal{M} \) might be relatively simple problem, but then to sort out which of them correspond to the orbit \( Z_0 \) might be a more difficult problem (and it is not clear whether it is a meaningful one).

**Remark.** This example indicates that for a general group \( G \) it would be natural to include the category \( Rep(G) \) into some larger category \( \mathcal{M}(G) \). I had this example for some time until I realized how one can define this larger category using sheaves on stacks.

### 3 Some remarks about groupoids

#### 3.1 Equivalence of groupoids

We know that if two objects of some category are isomorphic then we can consider them as two realizations of the same geometric structure.

Similarly, if two groupoids \( \mathcal{X} \) and \( \mathcal{Y} \) are equivalent (as categories) we can assume that they represent two realizations of the same geometric structure.

A subtle point here is that the equivalences between these groupoids form a groupoid. This means that if we fix an equivalence \( Q : \mathcal{X} \to \mathcal{Y} \) then this equivalence itself has automorphisms, and it is not clear how we should think about them.

**Example.** Consider an action \( a : G \times Z \to Z \).

Let us define a groupoid \( BG_0(Z) \) as follows:

Objects of \( BG_0(Z) \) are points \( z \in Z \) and morphisms are defined by \( Mor(z, z') := \{ g \in G | gz = z' \} \).

We also denote the groupoid \( BG_0(pt) \) by \( BG_0 \).

**Claim.** The groupoid \( BG_0(Z) \) is canonically equivalent to the groupoid \( BG(Z) \).

The groupoid \( BG_0(Z) \) might be considered as a ”matrix” version of the groupoid \( BG(Z) \). It is better suited for computations.
3.2 Theory of groups and theory of groupoids.

I would like to explain that the theories describing groups and groupoids are essentially equivalent.

**Proposition.** 1. Every groupoid $\mathcal{X}$ is canonically decomposed as a disjoint union of connected groupoids.

2. A connected groupoid $\mathcal{Y}$ is equivalent to the basic groupoid of some group $G$.

This result shows that any question about groupoids can be reduced to a question in group theory.

In other words the difference between theories of groups and groupoids is in their emphasis. In my opinion the relation between the theory of groupoids and the group theory is very similar to the relation between linear algebra and matrix calculus.

While these two theories are basically equivalent clearly linear algebra is much more intuitive. So I expect that the stacky approach will become a standard tool in representation theory.

3.3 Equivalence between groups and connected groupoids.

The group $G$ corresponding to a connected groupoid $\mathcal{Y}$ is defined not canonically. It depends on a choice of an object $Y \in \mathcal{Y}$.

If we choose an object $Y$ then we get a canonical equivalence of categories $Q = Q_Y : \mathcal{Y} \to BG$, where $G := Aut(Y)$. If we pick another object $Y'$ we get a different equivalence $Q' : \mathcal{Y} \to BG'$.

Note that any choice of an isomorphism between $Y$ and $Y'$ defines natural isomorphisms $G \cong G'$ and $Q \cong Q'$. However there is no preferred choice for such an isomorphism.

Next three constructions show that usually in Mathematics we encounter groupoids and not groups.

3.3.1 Multiplicative groupoid of a category.

**Construction I.** Starting with any category $C$ we construct the multiplicative groupoid $C^* = Iso(C)$ that has the same collection of objects as category $C$ and isomorphisms of $C$ as morphisms.

**Example 1.** $C = Finsets$ — the category of finite sets.

In this case the groupoid $Iso(C)$ is essentially the collection of all symmetric groups $S_n$.

**Example 2.** $C = Vect_F$ — the category of finite dimensional vector spaces over a field $F$.

In this case the groupoid $Iso(C)$ describes the collection of groups $GL(n, F)$ for all $n$. 

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3.3.2 Poincare groupoid.

Construction II. Poincare groupoid of a topological space $X$.

Objects of $Poin(X)$ are points of $X$. Morphisms — homotopy classes of paths.

If the space $X$ is path connected then the groupoid $Poin(X)$ is connected. For any point $x \in X$ the group $Aut_{Poin(X)}(x)$ is the fundamental group $\pi_1(X, x)$.

This shows that the Poincare groupoid is more basic notion than the fundamental group.

3.3.3 Galois groupoid.

Construction III. Galois groupoid $Gal(F)$ of a field $F$.

Objects of the groupoid $Gal(F)$ are field extensions $F \rightarrow \Omega$ such that $\Omega$ is an algebraic closure of $F$. Morphisms are morphisms of field extensions.

The groupoid $Gal(F)$ is connected. If we fix an algebraic closure $\Omega$ then by definition the group $Aut_{Gal(F)}(\Omega)$ is the absolute Galois group $Gal(\Omega/F)$.

Again we see that the notion of Galois groupoid is more basic than the notion of Galois group.

Note that Poincare and Galois groupoids are very similar objects.

4 What is a stack ?

Let us fix some site $S$. I would like to describe the notion of a stack $\mathcal{X}$ modeled on $S$. I assume two features of this notion.

1. For every two stacks $\mathcal{X}, \mathcal{Y}$ the collection of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ form a groupoid $Mor(\mathcal{X}, \mathcal{Y})$.
2. Every object $S \in S$ is a stack.

The natural idea is to characterize a stack $\mathcal{X}$ by collection of groupoids $\mathcal{X}(S) := Mor(S, \mathcal{X})$ for all objects $S \in S$.

In fact usually it is enough to know the groupoids $\mathcal{X}(S)$ for objects $S$ in some subcategory $\mathcal{B} \subset S$ provided it is large enough. For example, if $S$ is the category of schemes we can restrict everything to the subcategory $\mathcal{B}$ of affine schemes.

4.1 Technical definition of a stack.

Fix a large subcategory $\mathcal{B} \subset S$. We define a stack $\mathcal{X}$ over the site $S$ to be the following collection of data:

(i) To every object $S \in \mathcal{B}$ we assign a groupoid $\mathcal{X}(S)$

(ii) To every morphism $\nu : S \rightarrow S'$ in $\mathcal{B}$ we assign a functor $\mathcal{X}(S') \rightarrow \mathcal{X}(S)$

(iii) To every composition of morphisms in $\mathcal{B}$ we assign an isomorphism of appropriate functors.
This data should satisfy a variety of compatibility conditions and some finiteness conditions. For details see for example [F].

4.2 Basic stack and quotient stack.

Here are some important examples of stacks over the category $S$ of schemes over the field $F$. Fix an algebraic group $G \in S$.

**Example 1.** Basic stack $BG$.

For an affine $F$-scheme $S$ an object of the groupoid $BG(S)$ is a principal $G$-bundle $P$ over $S$.

**Example 2.** Quotient stack $B\mathcal{G}(Z)$

Let $G$ act on a scheme $Z \in S$. We define the quotient stack $X = B\mathcal{G}(Z) = Z/G$ as follows:

Object of $X(S)$ is a principal $G$-bundle $P$ over $S$ equipped with a $G$-morphism $\nu : P \to Z$. Morphisms are morphisms of $G$-bundles over $Z$.

In this note we are interested only in quotient stacks.

5 $F$-sheaves on algebraic stacks.

5.1 Technical definition of $F$-sheaves on stacks.

Let $F$ be a local non-Archimedean field (or a finite field).

I will describe a technical definition of an $F$-sheaf on an algebraic stack $\mathcal{X}$ that corresponds to the intuitive notion of a sheaf on the topological groupoid $\mathcal{X}(F)$.

For any $F$-scheme $Z$ we consider the topological space $Z = Z(F)$ and define an $F$-sheaf $R$ on $Z$ to be a sheaf on the topological space $Z$. The category of these sheaves we denote by $\text{Sh}_F(Z)$. Any morphism $\nu : Z \to W$ defines a functor $\nu^* : \text{Sh}_F(W) \to \text{Sh}_F(Z)$

**Natural question.** How to extend these categories to stacks? How to define the category $\text{Sh}_F(\mathcal{X})$?

Suppose we have some notion of $F$-sheaves on stacks. Fix an $F$-sheaf $R$ on a stack $\mathcal{X}$. Then for any affine scheme $S$ and any point $p \in \mathcal{X}(S) = \text{Mor}(S, \mathcal{X})$ we get an $F$-sheaf $R_p = p^*(F)$ on $S$. We also get a family of isomorphisms connecting these sheaves. Now we can try to use these sheaves and isomorphisms to characterize the $F$-sheaf $R$.

**Definition.** An $F$-sheaf $R$ on the stack $\mathcal{X}$ is a collection of $F$-sheaves $R_p$ for all morphisms $p : S \to \mathcal{X}$ and a collection of isomorphisms satisfying correct compatibility relations.

It is not difficult to check that this definition is compatible with informal definition of $F$-sheaves on an algebraic stack described before.
5.2 How to describe $F$-sheaves on an algebraic stack.

Let $\mathcal{X}$ be an algebraic stack over $F$. I would like to give a convenient description of $F$-sheaves on the stack $\mathcal{X}$ in terms of equivariant sheaves. I will do this for the case of a quotient stack $\mathcal{X} \approx \mathcal{Z}/G$.

Let $T_1, \ldots, T_r$ be representatives of isomorphism classes of $G$-torsors. They are described by elements of $H^1(\text{Gal}(F), G)$.

For every index $i$ consider the group $G_i = \text{Aut}(T_i)$ – this gives us the collection of all pure inner forms of the group $G$. We also consider the topological $G_i$-space $Z_i = \text{Mor}(T_i, \mathcal{Z})$.

**Claim.** Category $\text{Sh}_F(\mathcal{X})$ of $F$-sheaves on $\mathcal{X}$ is equivalent to the product of categories $\prod \text{Sh}_{G_i}(Z_i)$

5.2.1 Vogan’s picture.

In particular we see that the collection of simple objects of the category $\text{Sh}_F(\mathcal{X})$ is a disjoint union of collections of simple $G_i$-equivariant sheaves on $Z_i$.

In case when $\mathcal{Z} = \text{pt}$ we see that $\text{Irr}(\mathcal{M}(G, F)) = \bigcoprod \text{Irr}(G_i)$. If we postulate that the category $\mathcal{M}(G, F)$ is the "correct" category describing representations of the algebraic group $G$ then this bijection would explain the Vogan’s picture.

5.3 Reduction to the case of the group $GL(n)$.

Let me present one more description of the category of $F$-sheaves on a quotient stack that is often convenient in computations.

**Construction.** Suppose that $G$ is a linear algebraic group. Then we can imbed it into a group $\mathcal{P}$ isomorphic to $GL(n)$.

Using this we can realize our quotient stack $\mathcal{X} = \mathcal{Z}/G$ as the quotient $\mathcal{W}/\mathcal{P}$, where $\mathcal{W} = \mathcal{P} \times_G \mathcal{Z}$.

The group $\mathcal{P}$ has only one pure inner form (this is Hilbert 90 theorem). This implies that the category $\text{Sh}_F(\mathcal{X})$ can be realized as the category $\text{Sh}_P(\mathcal{W})$ of $P$-equivariant sheaves on $\mathcal{W}$, where $P = \mathcal{P}(F)$ and $\mathcal{W} = \mathcal{W}(F)$.

6 Algebra-geometric structure of the groupoids $\mathcal{X}(F)$.

Let $\mathcal{X}$ be an algebraic stack. Important role in the study of $F$-sheaves on $\mathcal{X}$ plays the fact that the collection of groupoids $\mathcal{X}(F)$ for different fields $F$ has an algebra-geometric structure.

**Proposition.** Let $L \supset F$ be a finite Galois field extension and $\Gamma = \text{Gal}(L/F)$ its Galois group. Then the group $\Gamma$ acts on the groupoid $\mathcal{X}(L)$ and the fixed point groupoid $\mathcal{X}(L)^\Gamma$ is naturally equivalent to the groupoid $\mathcal{X}(F)$. 
Here the fixed point groupoid $\mathcal{X}(L)^\Gamma$ is defined in a standard categorical manner. Its object is an object $X$ of the groupoid $\mathcal{X}(L)$ equipped with a collection of isomorphisms $\alpha_\gamma : X \to \gamma(X)$ satisfying natural compatibility conditions.

This proposition is just a reformulation of the descent property for the stack $\mathcal{X}$.

References

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