Asymptotics of some generalised sine-integrals

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Abstract

We obtain the asymptotic expansion for large integer \( n \) of a generalised sine-integral

\[
\int_0^\infty \left( \frac{\sin x}{x} \right)^n \, dx
\]

by utilising the saddle-point method. This expansion is shown to agree with recent results of J. Schlage-Puchta in *Commun. Korean Math. Soc.* 35 (2020) 1193–1202 who used a different approach.

An asymptotic estimate is obtained for another related sine-integral also involving a large power \( n \). Numerical results are given to illustrate the accuracy of this approximation.

We also revisit the asymptotics of Ball’s integral involving the Bessel function \( J_{\nu}(x) \), which reduces to the above integral when \( \nu = 1/2 \).

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1. Introduction

The expansion of the generalised sine-integral

\[
I_n = \int_0^\infty \left( \frac{\sin x}{x} \right)^n \, dx
\]

for integer \( n \to \infty \) has recently been considered by Schlage-Puchta [7]. However the method used seems to be unnecessarily involved and our aim here is to present a more direct computation using the well-known saddle-point method for Laplace-type integrals. The interest in the integral \( I_n \) stems from the fact that the intersection of the unit cube with a plane orthogonal to a diagonal and passing through the midpoint has \((n-1)\)-measure equal to \(2\sqrt{n} I_n/\pi\). These intersections arise naturally in certain probabilistic problems; see the references cited in [7].

The second related sine-integral we consider is given by

\[
K_n = \int_0^\infty e^{-ax} \left( 1 - \frac{\sin^2 x}{x^2} \right)^n \, dx \quad (a > 0)
\]
for \( n \to \infty \) when the parameter \( a = O(1) \). An integral of this type was communicated to the author by H. Kaiser [3]. We employ a two-term saddle-point approximation to estimate the growth of \( K_n \) for large \( n \) and present numerical calculations to verify the accuracy of the resulting formula. In the final section we revisit the expansion for large \( n \) of Ball’s integral involving the Bessel function \( J_\nu(x) \), which reduces to (1.1) when \( \nu = \frac{1}{2} \).

2. The asymptotic expansion of \( I_n \)

We begin by writing the integral in (1.1) as

\[
I_n = \int_0^{\pi} \left( \frac{\sin x}{x} \right)^n dx + R_n(x), \quad R_n(x) = \int_{\pi}^{\infty} \left( \frac{\sin x}{x} \right)^n dx.
\]

It is easily seen that

\[
|R_n(x)| < \int_{\pi}^{\infty} \frac{dx}{x^n} \frac{\pi^{1-n}}{n-1}.
\]

The remainder term \( R_n(x) \) is therefore bounded by \( O(n^{-1} \pi^{-n}) = O(n^{-1} e^{-n(\pi/e)^{-n}}) \) and so is exponentially small as \( n \to \infty \).

Let \( \psi(x) = \log \left( \frac{x}{\sin x} \right) \), where \( \psi(0) = 0 \) and \( \psi(\pi) = \infty \). Then the integral over \([0, \pi]\) becomes

\[
I_n = \int_{0}^{\pi} \left( \frac{\sin x}{x} \right)^n dx = \int_{0}^{\pi} e^{-n \psi(x)} dx.
\]

This integral has a saddle point at \( x = 0 \) and the integration path \([0, \pi]\) is the path of steepest descent through the saddle. If we now make the standard change of variable \( \psi(x) = \tau^2 \) discussed, for example, in [2, p. 66] we obtain

\[
I_n = \int_{0}^{\infty} e^{-n \tau^2} \frac{dx}{d\tau} d\tau.
\]

From the expansion

\[
\tau^2 = \log \left( \frac{x}{\sin x} \right) = \frac{1}{6} x^2 + \frac{1}{180} x^6 + \frac{1}{2835} x^6 + \frac{1}{3780} x^8 + \frac{1}{467775} x^{10} + \ldots
\]

valid for \( |x| < \pi \), we find by inversion of this series using Mathematica that

\[
x = \sqrt{6} \left\{ \tau - \frac{1}{10} \tau^3 - \frac{13}{4200} \tau^5 - \frac{9}{14000} \tau^7 + \frac{17597}{77616000} \tau^9 + \frac{4873}{218400000} \tau^{11} + \ldots \right\}
\]

whence

\[
\frac{dx}{d\tau} = \sqrt{6} \sum_{k=0}^{\infty} b_k \tau^{2k} \quad (|\tau| < \tau_0).
\]

(2.1)

The first few coefficients \( b_k \) are

\[
b_0 = 1, \quad b_1 = -\frac{3}{10}, \quad b_2 = -\frac{13}{840}, \quad b_3 = \frac{9}{2000}, \quad b_4 = \frac{17597}{862400},
\]

\[
b_5 = \frac{53603}{218400000}, \quad b_6 = -\frac{124996631}{1629936000000}, \quad b_7 = -\frac{159706933}{4366252800000}, \ldots
\]
The circle of convergence of the series (2.1) is determined by the nearest point in the mapping \( x \mapsto \tau \) where \( dx/d\tau \) is singular; that is, when \( x = 3\pi/2 \) (since the point \( x = \pi \) maps to \( \infty \) in the \( \tau \)-plane). This yields the value \( \tau_0 = |\log \frac{3}{2} + \pi i|^{1/2} \approx 1.8717 \). Then we have

\[
\hat{I}_n \sim \sqrt{6} \int_0^\infty e^{-n\tau^2} \sum_{k=0}^\infty b_k \tau^{2k} d\tau = \sqrt{\frac{3}{2n}} \sum_{k=0}^\infty \frac{b_k}{n^k} \int_0^\infty e^{-w^{1/2}} dw
\]

where the coefficients \( c_k \) are defined by

\[
c_k := b_k \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k + \frac{1}{2})}.
\]

It follows that since we have extended the integration path in (2.2) beyond the circle of convergence of (2.1) the resulting asymptotic series is divergent.

| \( k \) | \( c_k \) | \( c_k \) |
|---|---|---|
| 1 | \(-\frac{3}{8} \) | 2 | \(-\frac{13}{12} \) |
| 3 | \(+\frac{27}{3200} \) | 4 | \(+\frac{527}{9921} \) |
| 5 | \(+\frac{482427}{66560000} \) | 6 | \(-\frac{124996631}{1003520000} \) |
| 7 | \(-\frac{5270328789}{13647820000} \) | 8 | \(-\frac{2479963506161}{26846167040000} \) |
| 9 | \(+\frac{599710579537974189}{2655720414020400000000} \) | 10 | \(+\frac{10703530420192887741}{2385853794390000000000} \) |
| 11 | \(+\frac{599710579537974189}{2655720414020400000000} \) | 12 | \(-\frac{1338797420743736055939}{462978493749060000000000} \) |

Thus, neglecting exponentially small terms, we have the asymptotic expansion

\[
I_n \sim \sqrt{\frac{3\pi}{2n}} \sum_{k=0}^\infty \frac{c_k}{n^k} \quad (n \to \infty),
\]

where the coefficients \( c_k \) are listed in Table 1 for \( 0 \leq k \leq 12 \). This expansion agrees with that obtained in [7] by less direct means, except for the value of the coefficient \( c_{10} \).

An integral of a similar nature is

\[
J_n = \int_0^\infty \left( 1 - \frac{\cos x}{\frac{1}{2}x^2} \right)^n dx = \int_0^\infty \left( \frac{\sin \frac{1}{2}x}{\frac{1}{2}x} \right)^{2n} dx = 2I_{2n}.
\]

From (2.3) its asymptotic expansion is therefore (to within exponentially small terms)

\[
J_n \sim \sqrt{\frac{3\pi}{2n}} \sum_{k=0}^\infty \frac{c_k}{(2n)^k} \quad (n \to \infty).
\]
3. An asymptotic estimate of another sine-integral

In this section we consider the following integral

\[ K_n = \int_0^\infty e^{-ax} \left(1 - \frac{\sin^2 x}{x^2}\right)^n \, dx \quad (a > 0) \quad (3.1) \]

for \( n \to \infty \) (not necessarily an integer) when the parameter \( a = O(1) \). We express \( K_n \) as a Laplace-type integral in the form

\[ K_n = \int_0^\infty e^{-n\psi(x)} f(x) \, dx, \]

where

\[ \psi(x) = -\log \left(1 - \frac{\sin^2 x}{x^2}\right), \quad f(x) = e^{-ax}. \]

For large \( n \) the exponential factor in the integrand consists of a series of peaks situated at \( x = k\pi, \ (k = 1, 2, \ldots) \) of decreasing height controlled by the decay of \( f(x) \); see Fig. 1 for a typical example. This is in marked contrast to the situation pertaining to the integral \( I_n \) in (1.1), where the second and successive peaks are of height \( O((k\pi)^{-n}) \) \((k \geq 1)\) and so are exponentially smaller than the (half) peak in \([0, \pi]\). Routine calculations show that

\[ \psi''(k\pi) = \frac{2}{(k\pi)^2}, \quad \psi'''(k\pi) = -\frac{12}{(k\pi)^3}, \quad \psi^{iv}(k\pi) = \frac{82}{(k\pi)^4} - \frac{8}{(k\pi)^2}. \]

Application of the two-term saddle-point approximation to the \( k \)th peak then yields the approximate contribution [5, p. 48], [6, §1.2.3]

\[ 2 \sqrt{\frac{\pi}{2n\psi''(k\pi)}} \left\{ 1 + \frac{c_2}{n} \right\} e^{-k\pi a} = k\pi \sqrt{\frac{\pi}{n}} \left\{ 1 + \frac{c_2}{n} \right\} e^{-k\pi a}, \]

Figure 1: Plot of the integrand in (3.1) when \( n = 5000 \) and \( a = 1/6 \) with horizontal scale \( \pi x \). The dashed curve represents \( e^{-\pi ax} \).
Asymptotic expansion of an integral

where

\[ c_2 = \frac{1}{2\psi''} \left( \frac{2f''}{f} - 2\psi'' f' \left( f'' - \frac{5\psi''^2}{6\psi'^2} - \frac{\psi'^3}{2\psi'} \right) \right) \]

with all derivatives being evaluated at \( x = k\pi \). This yields

\[ c_2 = \frac{1}{4} \left\{ 2(1 + a^2)(k\pi)^2 - 12ak\pi + 9 \right\} \]

Summing over all the peaks we then obtain

\[ K_n \sim \pi \sqrt{\frac{\pi}{n}} \left\{ \sigma_1 + \frac{1}{8n} \left( 2\pi^2(1 + a^2)\sigma_3 - 12\pi a\sigma_2 + 9\sigma_1 \right) \right\} \]

where

\[ \sigma_m := \sum_{k=1}^{\infty} k^m e^{-k\pi a}. \]

We have

\[ \sigma_1 = \frac{1}{4\sinh^2 \frac{1}{2}\pi a}, \quad \sigma_2 = \frac{\cosh \frac{1}{2}\pi a}{4\sinh^3 \frac{1}{2}\pi a}, \quad \sigma_3 = \frac{2 + \cosh \frac{1}{2}\pi a}{8\sinh^4 \frac{1}{2}\pi a}. \]

Hence we obtain our final estimate in the form

\[ K_n \sim \pi^{3/2} \frac{3}{4n^{1/2}} \left( 1 + \frac{T_1}{8n} \right) \coth^2 \frac{1}{2}\pi a \quad (n \to \infty), \quad (3.2) \]

where

\[ T_1 = 9 - 12\pi a \coth \frac{1}{2}\pi a + \pi^2(1 + a^2) \left( \frac{2 + \cosh \frac{1}{2}\pi a}{\sinh^2 \frac{1}{2}\pi a} \right) \]

with \( a > 0 \) fixed and of \( O(1) \).

In Table 2 we show computed values of \( K_n \) compared with the asymptotic estimate \( (3.2) \) for different values of \( n \) and the parameter \( a \). It is seen that the agreement is quite good and improves with increasing \( n \). However, since \( \psi''(k\pi) \) scales like \( k^{-2} \), the peaks progressively broaden as \( k \) increases with the consequence that the saddle-point approximation eventually breaks down. In addition, the parameter \( a \) cannot be too small on account of the fact that the envelope of the minima of the integrand, given by \( e^{-ax}(1 - \cos^2 x/x^2)^n \), presents a maximum value at \( x \simeq (2n/a)^{1/3} \) equal to approximately \( \exp \left[ -\frac{1}{3}(2na^2)^{1/3} \right] \). We require this last quantity to be small for the satisfactory estimation of each peak. This results in the condition \( a \gg (2n)^{-1/2} \).

A closely related integral is

\[ K_n = \int_1^{\infty} e^{-ax} \left( 1 - \frac{\cos^2 x}{x^2} \right)^n \, dx. \]

The peaks in the graph of the integrand are similar to those indicated in Fig. 1 but now occur at \( x = (k + \frac{1}{2})\pi, \, k = 0, 1, 2, \ldots \). The lower limit of integration is chosen and to lie in the interval \( (\delta, \frac{1}{2}\pi - \delta) \) (with \( \delta > 0 \)) so as to avoid the origin and \( \frac{1}{2}\pi \). With \( \psi(x) = -\log(1 - \cos^2 x/x^2) \) we find

\[ \psi''((k + \frac{1}{2})\pi) = \frac{2}{(k + \frac{1}{2})^2 \pi^2}. \]
Table 2: Values of $K_n$ compared with asymptotic estimate (3.2).

| $n$ | $K_n$ | Asymptotic | $K_n$ | Asymptotic |
|-----|-------|------------|-------|------------|
| 100 | 0.02707847 | 0.02689533 | 0.00523230 | 0.00521489 |
| 200 | 0.01884203 | 0.01880232 | 0.00364706 | 0.00364449 |
| 500 | 0.01181371 | 0.01180983 | 0.00228888 | 0.00228866 |
| 1000 | 0.00588457 | 0.00588447 | 0.00114026 | 0.00114025 |
| 4000 | 0.002921970 | 0.002921838 | 0.00080580 | 0.00080580 |

Then by similar arguments we obtain the leading asymptotic approximation

$$\tilde{K}_n \sim \pi e^{-\pi a/2} \sqrt{n} \sum_{k=0}^{\infty} (k + 1/2) e^{-k \pi a} = \frac{\pi^{3/2} \cosh \frac{1}{2} \pi a}{4n^{1/2} \sinh \frac{1}{2} \pi a} \quad (n \to \infty).$$

4. An asymptotic expansion for Ball’s integral

Ball’s integral is given by [1]

$$L(\nu; n) = \int_0^\infty \left( \frac{\Gamma(1 + \nu) J_\nu(x)}{x^{\nu}} \right)^n x^{2\nu - 1} dx, \quad n \geq 2, \ \nu \geq \frac{1}{2}, \quad (4.1)$$

where $J_\nu(x)$ is the Bessel function of the first kind and $n$ is not necessarily an integer. In the case $\nu = \frac{1}{2}$ and integer values of $n$ the integral (4.1) (without the modulus signs) reduces to that in [1], since $\sqrt{\pi/(2x)} J_{1/2}(x) = \sin x/x$. The expansion of $L(\nu; n)$ for $n \to \infty$ has been derived in [1]; here we revisit this result by means of the transformation used in Section 1.

We divide the integration path into the intervals $[0, j_{\nu,1}]$ and $[j_{\nu,1}, \infty)$, where $j_{\nu,1}$ denotes the first zero of $J_\nu(x)$. In $[0, j_{\nu,1}]$, we have $J_\nu(x) \geq 0$ and the modulus signs may be dropped in this interval, where

$$\sigma(x) := \frac{\Gamma(1 + \nu) J_\nu(x)}{(x/2)^\nu} = \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!(\nu + 1)_k}$$
Asymptotic expansion of an integral

with \((a)_k = \Gamma(a + k)/\Gamma(a)\) being Pochhammer’s symbol. We set \(\psi(x) = -\log \sigma(x)\) in the interval \([0, j_{\nu,1}]\), so that the integral \((4.1)\) becomes

\[
L(\nu; n) = \int_0^{j_{\nu,1}} e^{-n\psi(x)} x^{2\nu - 1} dx + R_n(x),
\]

where

\[
R_n(x) = \int_{j_{\nu,1}}^{\infty} \left( \frac{\Gamma(1 + \nu)|J_\nu(x)|}{(x/2)^\nu} \right)^n x^{2\nu - 1} dx.
\]

The tail of the integral \(R_n(x)\) satisfies the bound

\[
|R_n(x)| < (2\nu\Gamma(1 + \nu))^n \int_{j_{\nu,1}}^{\infty} \frac{dx}{x^{(n-2)\nu + 1}} = \left( \frac{2\nu\Gamma(1 + \nu)}{j_{\nu,1}'} \right)^n \frac{j_{\nu,1}^2}{(n - 2)\nu}
\]

since \(|J_\nu(x)| \leq |J_\nu(j_{\nu,2}')| < 1\) in \([j_{\nu,1}, \infty)\), where \(j_{\nu,2}'\) is the second zero of \(J_\nu'(x)\). Defining the quantity

\[
\xi(\nu) := \frac{2\nu\Gamma(1 + \nu)}{j_{\nu,1}'},
\]

and noting that \(j_{\nu,1} = \pi\) when \(\nu = \frac{1}{2}\), we see that \(\xi(\frac{1}{2}) = 2^{-1/2}\). Use of Stirling’s approximation for the gamma function and the fact that [8, p. 485] \(j_{\nu,1} > \nu\), shows that \(\xi(\nu) \sim (2/\pi)^\nu \sqrt{2\pi\nu} \to 0\) as \(\nu \to \infty\). A plot of \(\xi(\nu)\) for \(\nu \geq \frac{1}{2}\) is shown in Fig. 2 where it is seen that \(\xi(\nu)\) decreases monotonically with increasing \(\nu\). Thus as \(n \to \infty\), the bound on \(R_n(x)\) when \(\nu \geq \frac{1}{2}\) is of \(O(n^{-1/2})\) and so is exponentially small.

![Figure 2: Plot of \(\xi(\nu)\) against \(\nu \geq \frac{1}{2}\).](image)

We now deal with the integral in \((4.2)\), where we note that \(\psi(0) = 0\) and \(\psi(j_{\nu,1}) = \infty\). Making the substitution \(\tau^2 = \psi(x)\), we obtain

\[
L(\nu; n) = \int_0^{\infty} e^{-n\tau^2} x^{2\nu - 1} \frac{dx}{d\tau} d\tau + R_n(x).
\]

\(^1\)It is found that \(\xi(\nu)\) is monotonically decreasing on \([0, \infty)\) with \(\xi(\nu) < 1\) for \(\nu > 0\).
From the expansion
\[ \tau^2 = \psi(x) = \frac{x^2}{4(1 + \nu)} + \frac{x^4}{32(1 + \nu)^2(2 + \nu)} + \frac{x^6}{96(1 + \nu)^3(2 + \nu)(3 + \nu)} + \cdots \]
valid for \( x < j_{\nu,1} \), we find upon inversion
\[ x = 2(1 + \nu)^{1/2} \left\{ \tau - \frac{\tau^3}{4(2 + \nu)} - \frac{(1 + 11\nu)\tau^5}{96(2 + \nu)^2(3 + \nu)} - \frac{(17\nu^2 - 9\nu - 20)\tau^7}{128(2 + \nu)^3(3 + \nu)(4 + \nu)} + \cdots \right\}. \]

The last expansion holds in \( \tau < \tau_0 \), where \( \tau_0^2 = \log 1/\sigma(j_{\nu,2}) \), since \( x = j_{\nu,2} \) is the nearest point in the mapping \( x \mapsto t \) where \( dx/d\tau \) is singular (the point \( x = j_{\nu,1} \) maps to \( \infty \) in the \( \tau \)-plane). This then yields the expansions
\[
x^{2\nu - 1} = (2(1 + \nu)^{1/2}\tau)^{2\nu - 1} \left\{ 1 - \frac{(2\nu - 1)\tau^2}{4(2 + \nu)} + \frac{(2\nu - 1)(6\nu^2 + \nu - 19)\tau^4}{96(2 + \nu)^2(3 + \nu)} - \frac{(2\nu - 1)^2(2\nu^2 - \nu^2 - 17\nu - 20)\tau^6}{384(2 + \nu)^3(3 + \nu)(4 + \nu)} + \cdots \right\}
\]
and
\[
\frac{dx}{d\tau} = 2(1 + \nu)^{1/2} \left\{ 1 - \frac{3\tau^2}{2(2 + \nu)} - \frac{5(1 + 11\nu)\tau^4}{48(2 + \nu)^2(3 + \nu)} - \frac{7(17\nu^2 - 9\nu - 20)\tau^6}{128(2 + \nu)^3(3 + \nu)(4 + \nu)} + \cdots \right\}.
\]

Combination of these last two expansions then produces
\[
x^{2\nu - 1} \frac{dx}{d\tau} = 2^{2\nu}(1 + \nu)^{\nu} x^{2\nu - 1} \sum_{k=0}^{\infty} \left( \frac{-1}{k} b_k \tau^{2k} \right) \quad (\tau < \tau_0), \quad (4.4)
\]
where
\[
b_0 = 1, \quad b_1 = \frac{1 + \nu}{2(2 + \nu)}, \quad b_2 = \frac{3\nu^2 + 2\nu - 5}{24(2 + \nu)(3 + \nu)}, \quad b_3 = \frac{(1 + \nu)(\nu^3 - \nu^2 - 4\nu - 8)}{48(2 + \nu)^3(4 + \nu)},
\]
\[
b_4 = \frac{15\nu^7 + 15\nu^6 - 220\nu^5 - 918\nu^4 + 763\nu^3 + 15055\nu^2 + 26898\nu + 13688}{5760(2 + \nu)^3(3 + \nu)^2(5 + \nu)}, \ldots.
\]

Insertion of the expansion (4.4) into the integral in (4.3) yields
\[
2^{2\nu}(1 + \nu)^{\nu} \int_0^\infty e^{-\nu^2 \tau^{2\nu - 1}} \sum_{k=0}^{\infty} \left( \frac{-1}{k} b_k \tau^{2k} \right) d\tau \sim 2^{2\nu - 1}(1 + \nu)^{\nu} \Gamma(\nu) \sum_{k=0}^{\infty} \left( \frac{-1}{k} b_k \right)_k \frac{(-1)^k}{n^{k+\nu}}.
\]
As in Section 1, this will be a divergent expansion as we have integrated beyond the circle of convergence of \( x^{2\nu - 1} dx/d\tau \).

Thus, neglecting exponentially small terms we finally obtain the expansion
\[
L(\nu; n) \sim 2^{2\nu - 1}(1 + \nu)^{\nu} \Gamma(\nu) \sum_{k=0}^{\infty} \left( \frac{-1}{k} c_k \right)_k \frac{(-1)^k}{n^{k+\nu}} \quad (n \to \infty), \quad (4.5)
\]
where the first few coefficients \(c_k := b_k(n)\) are

\[
c_0 = 1, \quad c_1 = \frac{\nu(1 + \nu)\nu_1(\nu)}{2(2 + \nu)} , \quad c_2 = \frac{\nu(1 + \nu)\nu_2(\nu)}{24(2 + \nu)(3 + \nu)} , \quad c_3 = \frac{\nu(1 + \nu)\nu_3(\nu)}{48(2 + \nu)^2(4 + \nu)} ,
\]

\[
c_4 = \frac{\nu(1 + \nu)\nu_4(\nu)}{5760(2 + \nu)^3(3 + \nu)(5 + \nu)} , \quad c_5 = \frac{\nu(1 + \nu)\nu_5(\nu)}{11520(2 + \nu)^4(3 + \nu)(6 + \nu)} , \quad c_6 = \frac{\nu(1 + \nu)\nu_6(\nu)}{2903040(2 + \nu)^5(3 + \nu)^2(4 + \nu)(7 + \nu)} ,
\]

with the polynomials \(\nu_k(\nu)\) given by

\[
\begin{align*}
\nu_1(\nu) &= 1, \quad \nu_2(\nu) = 3\nu^2 + 2\nu - 5, \quad \nu_3(\nu) = (1 + \nu)(\nu^3 - \nu^2 - 4\nu - 8), \\
\nu_4(\nu) &= 15\nu^7 + 15\nu^6 - 220\nu^5 - 918\nu^4 + 763\nu^3 + 15055\nu^2 + 26898\nu + 13688, \\
\nu_5(\nu) &= 3\nu^9 - 7\nu^8 - 60\nu^7 - 246\nu^6 + 2307\nu^5 + 6825\nu^4 - 43668\nu^3 - 118508\nu^2 - 89904\nu - 19392, \\
\nu_6(\nu) &= 63\nu^{13} - 3276\nu^{11} + 16856\nu^{10} - 131726\nu^9 + 781856\nu^8 - 4685840\nu^7 - 14835768\nu^6 \\
&\quad + 104879595\nu^5 + 322760624\nu^4 - 328990364\nu^3 - 1748824256\nu^2 - 1801386304\nu - 590749440.
\end{align*}
\]

The first coefficients (with \(k \leq 3\)) agree with those found by Kerman et al. [4]. It is easily verified that when \(\nu = \frac{1}{2}\) the above coefficients agree with those listed in Table 1.

The estimate for the tail of the integral over the interval \([2^{\nu}T(1 + \nu), \infty)\) in [4], however, is only \(O(n^{-1})\), which is not sufficiently sharp to justify the expansion of the main integral beyond its leading term. We have demonstrated that the tail of the integral [4] is exponentially small as \(n \to \infty\).

An obvious extension of [4] is the integral

\[
\mathcal{L}(\nu, a; n) = \int_0^\infty \left( \frac{\Gamma(1 + \nu)(J_{\nu}(x))}{(x/2)^{\nu}} \right)^n x^{a-1} dx, \quad a > 0,
\]

where \(a\) is fixed and it is assumed that \(n\) satisfies the condition \(n(\nu + \frac{1}{2}) > a\) to secure convergence at infinity. The procedure described above then produces the expansion (when exponentially small terms are neglected)

\[
\mathcal{L}(\nu, a; n) \sim 2^{a-1}(1 + \nu)^{a/2} \Gamma\left(\frac{1}{2}a\right) \sum_{k=0}^\infty \frac{(-1)^k d_k}{\eta^{k+a/2}} \quad (n \to \infty), \quad (4.6)
\]

where the first few coefficients \(d_k\) are given by

\[
d_0 = 1, \quad d_1 = \frac{\left(\frac{1}{2}a\right)}{2(2 + \nu)}, \quad d_2 = \frac{\left(\frac{1}{2}a\right)}{48(2 + \nu)^2(3 + \nu)}((3a - 14)\nu + 9a - 10),
\]

\[
d_3 = \frac{\left(\frac{1}{2}a\right)}{192(2 + \nu)^3(3 + \nu)(4 + \nu)} \left( (a^2 - 14a + 64)\nu^2 + (7a^2 - 66a + 32)\nu + 4(a - 4)(3a + 2) \right),
\]

\[
d_4 = \frac{\left(\frac{1}{2}a\right)}{46080(2 + \nu)^4(3 + \nu)^2(4 + \nu)(5 + \nu)} \left( 15a^3 - 420a^2 + 4820a - 23824\nu^4 \\
+ (225a^3 - 5340a^2 + 42860a - 65776)\nu^3 + (1245a^3 - 23340a^2 + 103740a + 100560)\nu^2 \right)
\]
Table 3: Values of the absolute relative error in the computation of $\mathcal{L}(\nu, a; n)$ against truncation index $k$ when $\nu = 4/3$ and $n = 100$.

| $k$ | $a = 8/3$ | $a = 2/3$ | $a = 10/3$ |
|-----|-----------|-----------|-----------|
| 0   | $4.664 \times 10^{-03}$ | $6.676 \times 10^{-04}$ | $6.565 \times 10^{-03}$ |
| 1   | $2.738 \times 10^{-06}$ | $8.987 \times 10^{-07}$ | $1.047 \times 10^{-05}$ |
| 2   | $3.307 \times 10^{-08}$ | $6.661 \times 10^{-10}$ | $6.041 \times 10^{-08}$ |
| 3   | $4.006 \times 10^{-10}$ | $2.405 \times 10^{-11}$ | $5.961 \times 10^{-10}$ |
| 4   | $2.914 \times 10^{-12}$ | $3.655 \times 10^{-13}$ | $2.743 \times 10^{-12}$ |

$$+(3015a^3 - 39300a^2 + 45940a + 252784)\nu + 2700a^3 - 18000a^2 - 18800a + 109504 \right)$$

When $a = 2\nu$, it is seen that the $d_k$ reduce to the coefficients $c_k$ ($k \leq 4$) appearing in the expansion (4.6).

In Table 2 we present the values of the absolute relative error in the evaluation of $\mathcal{L}(\nu, a; n)$ using the expansion (4.6) for different truncation index $k$. The first column shows the values $\nu = 4/3$, $a = 8/3$, which corresponds to the integral (4.1).

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