Chiral Random Matrix Theory and Chiral Perturbation Theory

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Abstract. Spontaneous breaking of chiral symmetry in QCD has traditionally been inferred indirectly through low-energy theorems and comparison with experiments. Thanks to the understanding of an unexpected connection between chiral Random Matrix Theory and chiral Perturbation Theory, the spontaneous breaking of chiral symmetry in QCD can now be shown unequivocally from first principles and lattice simulations. In these lectures I give an introduction to the subject, starting with an elementary discussion of spontaneous breaking of global symmetries.

1. Introduction
Many tend to think of spontaneous breaking of global symmetries in quantum field theory as something rather simple, almost trivial. To illustrate that this is certainly not so, let us first briefly review the standard textbook treatment of this phenomenon. Typically, one considers a field theory of a real scalar field with a potential that, appropriately for this school, can be dubbed a “sombrero potential”,

\[ V(\phi) = -\frac{1}{2}m^2\phi(x)^2 + \frac{\lambda}{4!} \phi(x)^4. \] (1)

From a perspective of classical physics this looks unstable. Certainly a constant field configuration of \( \phi(x) = 0 \) cannot possibly be a good starting point for a perturbative expansion. Instead, one considers the two minima of the potential at

\[ \phi = \pm v = \pm \sqrt{\frac{6}{\lambda}m} \] (2)

and says that a consistent treatment of such a field theory must be based on one vacuum that is either at \( \phi = v \) or at \( \phi = -v \), which one being undecidable. Here a comparison is often made to a classical picture, like a pencil standing exactly on its tip: such a system is rotationally symmetric, but any small perturbation will evidently make the pencil fall in the direction of the perturbation. Spontaneous symmetry breaking! Indeed, the pencil will now lie flat on the table, and it will have picked just one random direction, the one that was induced by the perturbation, a perturbation we can make as small as we like. What about the potential energy of the pencil? It was first converted to kinetic energy, and then, on impact with the table, to heat. Such a
system is dissipative: If it were not, the pencil would simply bounce back up. There may be issues such as accuracy of initial conditions and so on, but this last point illustrates the difficulty with a simple picture of spontaneous symmetry breaking.

To investigate this a little more closely, let us proceed to quantum mechanics and glue two harmonic oscillator potentials together, separated by a distance \( 2a \). The potential is thus

\[
V(x) = \frac{1}{2} m \omega^2 (|x| - a)^2
\]

The ground state of this quantum mechanical system must have no nodes, and we can almost trivially identify the correct ground state by drawing two Gaussian bumps centered at \( x = \pm a \),

\[
\psi_{\pm}(x) \sim \exp \left[ -\frac{1}{2} m \omega (x \mp a)^2 \right]
\]

which are, individually, ground states of either of the two harmonic oscillators. If we smoothly join the two at \( x = 0 \) so that we construct, roughly,

\[
\psi_0(x) \sim \psi_+(x) + \psi_-(x)
\]

this will be a very good approximation to the true ground state. What about the 1st excited state? It should have one node, and again we can easily guess its form. Suppose we glue the two wave functions \( \psi_{\pm}(x) \) together, but with a twist: we flip the sign of one of them to get

\[
\psi_1(x) \sim \psi_+(x) - \psi_-(x)
\]

This state will have a slightly higher energy than \( \psi_0(x) \), but it is clear from this simple construction that \( \psi_0(x) \) and \( \psi_1(x) \) are almost degenerate. The higher excited states can be visualized similarly, by gluing two wave functions together around \( x = 0 \), more and more excited states, and alternatingly adding and subtracting combinations.

Does such a system display spontaneous symmetry breaking? It is well known that in this quantum mechanical case one of the two apparently distinct vacua will never be selected. Although the first excited state is almost degenerate with the ground state, there will always be a finite energy difference \( E_1 - E_0 \). If we prepare a state that initially is localized in only one of the two wells, it will tunnel to the other side. The tunneling time goes like \( (E_1 - E_0)^{-1} \). Only when the two minima are infinitely far from each other does that tunneling time go to infinity. So our naive picture of spontaneous symmetry breaking can clearly not be valid for such a simple quantum mechanical system of just one degree of freedom. One cannot contain the probability density on just one side of \( x = 0 \).

These two examples, one classical and one quantum mechanical, illustrate that the phenomenon of spontaneous symmetry breaking in quantum field theory cannot be quite as simple as the intuition based on the sombrero potential seems to suggest. Nevertheless, as is so often the case in physics: the argument may not be quite right, but the answer is correct. There is no spontaneous breaking of symmetry in quantum mechanical systems of a finite number of degrees of freedom. But there can be spontaneous breaking of symmetries in quantum field theory. The distinction here comes precisely from the infinite number of degrees of freedom, the infinite number of quantum field theory modes. Although each of them individually can prevent spontaneous breaking of symmetry by tunneling, this mechanism can be blocked when an infinite number of modes have to tunnel. Subtleties in the arguments have been discussed on and off in the literature, see for instance refs. [2, 3]. For a more recent discussion, with plenty of references to earlier literature, see ref. [4].

1 This is also a standard textbook example, see e.g., ref. [1] for a nice and clear discussion.
To make the confusion complete, let us also point out another related phenomenon in quantum field theory: tunneling from one vacuum to another. We have just argued that in quantum field theory the tunneling between degenerate vacua can be prohibited due to the fact that an infinite number of degrees of freedom have to simultaneously align. Yet, if we tilt the sombrero potential slightly so that one vacuum really becomes energetically favored it is also common knowledge that in quantum field theory we do have tunneling to the true vacuum. How can this be? It turns out to be another example of a superficial argument that is basically correct nevertheless, and there is an endless amount of literature on this subject. Suffice it to say here that when there is a genuine non-degeneracy of vacua, one can have bubble formation and a first order phase transition where the true vacuum gobbles up the false vacuum. For a nice discussion of the essential physics of this situation see, e.g., ref. [5].

Of particular importance is the phenomenon of spontaneous breaking of continuous global symmetries because it is linked to what is known as Goldstone’s Theorem. This theorem implies the appearance of one massless mode associated with each broken generator of the continuous group of symmetries. Again there are caveats, and there are several. One is space-time dimensionality: a massless scalar in two space-time dimensions is not well defined in an infinite volume due to infrared singularities, and indeed there is no Goldstone phenomenon in two space-time dimensions [6]. In higher dimensions the theorem is incredibly strong, as it says that there is no mass gap, an otherwise forbiddingly difficult question to tackle. The Goldstone Theorem is most concisely stated in the operator language. Since the symmetry is continuous, let us consider the associated conserved current operator. Ordinarily the vacuum $|0\rangle$ is assumed to be annihilated by the corresponding conserved charge operator $Q$ (the vacuum is not charged under $Q$). If the vacuum is not annihilated by $Q$, the symmetry is spontaneously broken. Remarkably, this implies the existence of an associated massless mode. This is the content of Goldstone’s Theorem.

In QCD, the spontaneous breaking of chiral symmetry is a profound phenomenon that has been inferred indirectly over a long period of both theoretical and experimental development, starting from the days when the fundamental theory of strong interactions was not even known. In hindsight, it is incredible that it could be gotten at without a correct understanding of the underlying microscopic mechanism. Today we’re much better off: we know the fundamental Lagrangian of QCD and it is “just” a matter of checking whether chiral symmetries are broken spontaneously or not. In practice, this is not simple at all. In fact, the question of showing spontaneous breaking of chiral symmetry in QCD and QCD-like theories is a challenge at the level of showing confinement and the existence of a mass gap in pure Yang-Mills theory. It is one of the remarkable successes of the numerical non-perturbative approach to QCD – lattice QCD – that this now has been shown beyond any doubt. An essential ingredient in this comes from the new understanding of chiral symmetry breaking through Random Matrix Theory.

The purpose of these very elementary lectures was to acquaint students with the new developments in our understanding of chiral symmetry breaking in QCD. The amount of time was insufficient to give a complete review and this is reflected in these notes which have focused on only a few of the many interesting aspects. That chiral Random Matrix Theory can say something exact about a complicated quantum field theory such as QCD with light quarks sounds like a wild idea, and it was initially met by a lot of skepticism in the lattice QCD community. All of this skepticism turned out to be unwarranted. There is now complete understanding of how and why chiral Random Matrix Theory represents an exact limit of the light quark partition function of QCD in a specific finite-volume regime. A precise mapping can be made between observables computed in QCD in that regime, the standard chiral Lagrangian of light-quark QCD and chiral Random Matrix Theory. This is a fantastic achievement, a most surprising and deep relation between a quantum field theory as complicated as QCD and a sequence of universal phenomena that follow from the spontaneous breaking of chiral symmetry alone. One
example suffices to illustrate the enormous progress that has been achieved. Let us consider the eigenvalues of the QCD Dirac operator $D$:

$$D\psi_n = \lambda_n \psi_n \quad (7)$$

To make the discussing well-defined, let us consider this eigenvalue problem in a theory with a finite ultraviolet cut-off $\Lambda$ and a finite infrared cut-off $L$. As will be discussed further down in these lectures, spontaneous breaking of chiral symmetry requires that the smallest non-zero eigenvalues $\lambda_n$ accumulate towards zero at a rate proportional to $1/L^4$ in four dimensions. This is thus a simple test: do they or do they not? Before the developments described here, the strongest statement that could be made on basic principles was a proof that the $\lambda_n$’s accumulate at least as fast as $1/L$ [7]. But this is a very weak condition. Even in a free theory, eigenvalues accumulate as $1/L$. All the theorem says, then, is that Dirac eigenvalues in QCD do not accumulate with a rate that is slower than that of free fermions, i.e., as if there were no gauge interactions whatsoever. Can one not do better? With the advent of chiral Random Matrix Theory this problem has been solved, and it is now known that in QCD with two light flavors (the $u$ and $d$ quarks), Dirac operator eigenvalues do accumulate towards the origin at a much faster rate that goes exactly as $1/L^4$. Not only that, the precise probability distributions of single individual Dirac eigenvalues, $\lambda_n$, $i = 1, 2, 3, \ldots$ ordered according to their distance from the origin can be computed exactly. These distributions, as well as all spectral correlation functions of these smallest Dirac operator eigenvalues follow universal scaling laws whose exact analytical forms are known.

2. Chiral symmetry in QCD

If we look at the Lagrangian density of QCD with massless quarks, we are struck by the fact that the left-handed and right-handed fields decouple. If we had exactly $N_f$ massless quarks in QCD, the global symmetry group would be $U(N_f) \times U(N_f)$. Of this a subgroup is $U(1)$ baryon number, which remains unbroken in QCD. The singlet chiral $U(1)$ symmetry is apparent only: it is broken by the chiral anomaly, and is therefore not a symmetry of the quantum theory. What remain are two independent flavor rotations $SU(N_f)$ for the left and right handed quarks. If this symmetry is broken as expected, $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)$ due to the formation of a flavor-independent chiral condensate ($\bar{\psi}\psi$), we have $N_f^2 - 1$ broken generators and hence, by Goldstone’s Theorem, $N_f^2 - 1$ massless bosons. This is a profound statement! How this slowly came to be realized before quarks had even been introduced is a story in itself. One consequence of this is that a low-energy representation of the QCD partition function had been inferred long before QCD was considered. In the next subsection we will give a lightning review of the moderne viewpoint on this.

2.1. The Chiral Lagrangian

Because of Goldstone’s Theorem, the low-energy degrees of freedom of massless QCD are those of the Nambu-Goldstone bosons. In reality, even the $u$ and $d$ quarks are not exactly massless. But on the QCD scale of $\sim 1$ GeV they look nearly massless; their masses are on the order a few MeV. At some point it must be possible to treat the quark masses as small perturbations on top of a theory of genuine Goldstone bosons. The chiral Lagrangian does exactly this. Consider first a truly massless $N_f$-flavor theory where chiral symmetry is spontaneously broken as discussed above. To describe the coset of chiral symmetry breaking it is convenient to introduce an unusual non-linear field representation in terms of group elements $U(x) = \exp[i\sqrt{2}\Phi(x)/F]$, where $F$ is the pion decay constant and $\Phi(x) = \lambda^a \phi^a(x)/\sqrt{2}$ represents the collection of $N_f^2 - 1$ pion fields. A field theory based on this should have the following ingredients: (1) kinetic energy terms for the pions with canonical normalization, (2) vanishing interactions at zero energy (as follows
from Goldstone’s Theorem) and (3) invariance under the chiral rotations. In the absence of any other principle that could fix terms in the Lagrangian the only solution will be to write down all terms that comply with the two conditions above. This gives us an endless series of invariants:

$$\mathcal{L} = \frac{F^2}{4} \text{Tr}[\partial_{\mu} U^\dagger \partial^\mu U] + L_1 (\text{Tr}[\partial_{\mu} U^\dagger \partial^\mu U])^2 + L_2 \text{Tr}[\partial_{\mu} U^\dagger \partial_\nu U] \text{Tr}[\partial^\nu U^\dagger \partial^\mu U] + \ldots$$

(8)

with an infinite series of couplings (we display only the first two, \(L_1\) and \(L_2\) here). Values of all these couplings are left totally unspecified. They must depend on details of the dynamics (which gauge group is responsible for the symmetry breaking and so on). This is the chiral Lagrangian.

This cannot be the full story because we also want to treat quark masses, at least as a specified perturbation to the above Lagrangian. There is a beautifully simple way to do this, which goes under the name of the spurion technique. The chiral Lagrangian (8) is by construction invariant under global transformations \(U \rightarrow U_R U_L^\dagger\). A mass term in the QCD Lagrangian,

$$\bar{\psi}_L m \psi_R + \bar{\psi}_R m^\dagger \psi_L$$

(9)

is of course not invariant under \(\psi_L \rightarrow U_L \psi_L, \psi_R \rightarrow U_R \psi_R\). But if the mass (matrix) were a field transforming like \(m \rightarrow U_L m U_R^\dagger\) then the QCD Lagrangian with this mass term would be invariant. Then we know what the corresponding terms must look like in the chiral Lagrangian. The leading term will be

$$\mathcal{L} = \Sigma \text{Tr}[mU + m^\dagger U^\dagger]$$

(10)

and there will of course be higher order terms involving higher powers of \(m\) and derivatives.

Such a chiral Lagrangian will be invariant under the combined transformation

$$m \rightarrow U_L m U_R^\dagger, \quad U \rightarrow U_R U_L^\dagger.$$  

(11)

A Lagrangian constructed in this way will thus explicitly break chiral symmetries in precisely the same way as the underlying theory of QCD.

2.2. Different counting schemes

So far we have defined the chiral Lagrangian by the totality of all terms that transform properly under chiral transformations. This looks rather hopeless, since there are infinitely many terms. Does such a theory have any predictive power at all? Weinberg [8] showed that it indeed does. First, since all possible terms are already included in the Lagrangian, it is by construction renormalizable since quantum corrections cannot produce terms not already present in the tree-level Lagrangian. This goes counter to everybody’s intuition about renormalizable field theories, but it is correct. The chiral Lagrangian is an effective field theory and as such perfectly renormalizable. The theory, however, is not well-defined at all energy scales, and this is what we should consider next. The trouble is the appearance of higher and higher derivatives in the Lagrangian. Normally, this indicates that theory will not be a truly local theory and this may hold here as well. Certainly, the more derivatives, the more there will be contributions from the high energy scale. But the theory is not even supposed to hold at the high energy scale – it is precisely a low-energy effective theory. The way to deal with this was explained by Weinberg: consider an ordering of the Lagrangian where one expands in the number of derivatives. If this is to make sense, the Lagrangian must necessarily be viewed as a theory with a cut-off. In QCD, one would expect the cut-off \(\Lambda\) to be around 1 GeV; in detail this shows up in the form of the combination \(4\pi F^2\), which indeed is of that order. Dependence on this cut-off can be removed for observables below this energy scale.

The expansion in terms of the number of derivatives can be made precise, see e.g. ref. [9]. A counting is introduced which makes it systematic. Since it is in terms of derivatives (or
momenta), this is referred to as the $p$-expansion. The pion mass will also enter explicitly in the chiral Lagrangian, due to non-vanishing quark masses that explicitly break chiral symmetry in QCD. So an ordering with respect to $m^2_\pi$ needs to be introduced as well. The natural expectation works: one can systematically treat $m_\pi$ as being of order momentum $p$. Then chiral perturbation theory has only one expansion parameter, $p$. The scale is given by the expected ultraviolet cut-off $4\pi F$ so that the dimensionless expansion parameter is $\sim p/(4\pi F)$. This expansion works quite well phenomenologically, at least in the sector of the two lightest quarks (and we shall only be concerned with this light quark sector here).

The $p$-expansion breaks down eventually if one takes the massless limit at finite four-volume $V$. This is most easily seen by considering the pion propagator

$$\Delta(p^2) = \frac{1}{V} \frac{1}{p^2 + m^2_\pi} .$$

(12)

For simplicity, let us take the finite volume $V$ to be a symmetric four-torus of length $L$. The smallest (quantized) momentum is thus of order $1/L$. In a massless theory this makes the propagator $\Delta(p^2)$ vanish for large $L$ like $L^{-2}$. But for the momentum zero mode the cut-off is entirely given by the mass. Clearly, for $m_\pi \gg 1/L$ the propagator is still protected by the mass, but as $m_\pi L \sim 1$ or much smaller we enter a new regime. From this point and onward the usual perturbative expansion cannot make sense in the zero-mode sector. Is this of importance? Can we not just ignore the zero mode? The answer is unfortunately no. Although it concerns only one mode, this one mode potentially overwhelms all other contributions from the perturbative expansion. The resolution of this problem is to treat the zero-mode sector exactly in a sense that will be more clear below [10, 11]. Gasser and Leutwyler developed a modified perturbation theory that includes this feature, while retaining all other properties of the usual perturbative expansion of chiral Lagrangians. This has become known as an $\epsilon$-expansion (not to be confused with the expansion of similar name that expands in dimensionality $4 - d$). The non-zero momentum modes still retain their usual pertubative expansion since the propagator for these modes will go at least as $L^{-2}$, even in the chiral limit.

A systematic chiral counting for this so-called $\epsilon$-regime is as follows [11]. Let $\epsilon \sim 1/L$. Then

$$m_\pi \sim \epsilon^2 , \quad p \sim \epsilon .$$

(13)

It is more instructive to think of this in terms of the microscopic degrees of freedom. Let us restrict ourselves to two degenerate light quarks of mass $m$. Due to the Gell-Mann–Oakes–Renner relation

$$F^2m^2_\pi = 2m\Sigma$$

(14)

where $\Sigma$ is the chiral condensate, we see that $m \sim \epsilon^4$. In the $\epsilon$-regime the quark masses scale as inverse powers of the volume. In fact, we can think of this as a regime where the extreme chiral limit $m \rightarrow 0$ is taken in a way that correlates with the way the four-volume $V$ is sent to infinity. The relevant proportionality factor must have dimension 3, and indeed the right way to think of it is that the combination $m\Sigma V$ is kept of order unity. Here the condensate $\Sigma$ provides the missing constant of proportionality, which is directly related to the fact that the combination $m\bar{\psi}\psi$ is a renormalization group invariant, in fact it is the explicit chiral symmetry breaking term in the QCD Lagrangian.

There is another way in which we can understand the appearance of the scale $m\Sigma V$. In two-flavor QCD, the chiral condensate $\langle \bar{\psi}\psi \rangle$ at finite quark mass $m$ and finite volume $V$ is the trace of the quark propagator,

$$\langle \bar{\psi}\psi \rangle = \frac{1}{2V} \partial_m \ln Z$$

(13)
\[
\begin{align*}
&= \frac{1}{V} \left\langle \text{Tr} \frac{1}{D + m} \right\rangle \\
&= \frac{1}{V} \left\langle \sum_j \frac{1}{i\lambda_j + m} \right\rangle \\
&= \frac{1}{V} \left\langle \sum_{j > 0} \left( \frac{1}{i\lambda_j + m} + \frac{1}{-i\lambda_j + m} \right) \right\rangle \\
&= \frac{1}{V} \left\langle \sum_{j > 0} \frac{2m}{\lambda_j^2 + m^2} \right\rangle
\end{align*}
\]

where we have used that each quark contributes to \( \langle \bar{\psi} \psi \rangle \) with the same amount. Use has also been made of the fact that every non-vanishing eigenvalue \( \lambda_j \) can be matched by its opposite-sign counterpart. This follows directly from the spectrum being chiral, \( \{D, \gamma^5\} = 0 \). The above spectral representation can be used to deduce the so-called Banks-Casher relation for the infinite-volume chiral condensate:

\[
\Sigma = \lim_{m \to 0} \lim_{V \to \infty} 2m \int_0^\infty d\lambda \frac{\rho(\lambda)}{\lambda^2 + m^2} = \pi \rho(0)
\]

This is a formal expression that should be understood only in terms of a cut-off theory. The chiral condensate \( \langle \bar{\psi} \psi \rangle \) is ill-defined in the ultraviolet, and indeed one expects the spectral density \( \rho(\lambda) \) to behave like \( \sim \lambda^3 \) (as in a free theory) for very large momenta, and this would lead to a quadratic divergence \( \langle \bar{\psi} \psi \rangle \sim m\Lambda^2 \). A term like \( \sim m \ln(\Lambda) \) is also expected. However, the important point here is not the behavior at the ultraviolet end of the spectrum, but near the origin. Whether chiral symmetry is spontaneously broken or not is determined by whether or not the spectral density \( \rho(\lambda) \) vanishes there. If \( \rho(0) \) is to be non-vanishing, the discrete eigenvalues \( \lambda_j \) must accumulate there at a rate inversely proportional to the volume [12]:

\[
\Delta \lambda \sim \frac{1}{V} \quad \text{near} \quad \lambda \sim 0
\]

so as to yield a finite density at the origin. The inverse of the proportionality factor is precisely the condensate \( \Sigma \) on account of the Banks-Casher relation. We thus find that \( \lambda \Sigma V \) will be of order unity for the smallest eigenvalues if we are to generate a chiral condensate. As will become clear shortly, there is an \( \epsilon \)-regime of a partially quenched chiral Lagrangian hidden in this statement.

Before defining the partially quenched theory, let us return to the integral representation of \( \langle \bar{\psi} \psi \rangle \). If we know how this condensate behaves as a function of the mass \( m \), can we not invert the relation to get \( \rho(\lambda) \)? The trouble is that \( \rho(\lambda) \) is also a function of quark masses \( m \). If it were not, we could indeed simply invert the relation. The trick to achieve this is to do partial quenching: one introduces new unphysical quarks that are used to produce an intermediate generating function, but which are removed at the end. The spectral density will then be independent of the masses of these physical quarks, but if we know the analogue of their condensates \( \langle \bar{\psi} \psi \rangle \), we can invert the integral representation and compute the spectral density. There are two ways of introducing such partially quenched quarks: one is by means of a graded structure [13], the other by means of replicas [14, 15]. Both methods work beautifully in perturbation theory, and for the same reason: observables computed in perturbation theory will depend polynomially on the number of additional fermionic \( N_f \) and bosonic \( N_b \) species. If both fermions and bosons are kept, the anticommutation sign from fermions just cancel the corresponding contribution from the fermions. So an equal number of degenerate fermions and bosons do not contribute to the partition function, a statement that looks entirely trivial if phrased formally in terms of functional determinants. In the replica method, one keeps only the fermions (or, equivalently,
only the bosons) and instead removes them from the partition function by taking the replica limit \( N_f \to 0 \). Since the behavior is polynomial in perturbation theory, it is unambiguous how to do so in that setting.

In the \( \epsilon \)-regime, integrals have to be done exactly, beyond perturbation theory. Curiously, then, both the graded method and the replica method run into technical difficulties. In the graded method it turns out that the naive extension of the chiral Lagrangian to a graded coset fails to yield convergent integrals, a problem that never surfaces in perturbation theory. This problem can be overcome by a judicious choice of bosonic integration \([16]\): The graded integration domain must be a combination of a compact and non-compact regions. In the replica framework the difficulty arises from the precise specification of how to analytically continue in the integer \( N_f \). In perturbation theory this is trivial (because dependence on \( N_f \) is polynomial only), but beyond perturbation theory one needs more structure as guidance – knowing a function at all integers is of course not sufficient for defining an unambiguous analytic continuation into the real line. Fortunately, the required structure is precisely present in the cases of interest \([17]\), in fact there is a remarkable connection to the theory of integrable systems and the so-called Toda lattice hierarchy. It is beyond the scope of these simple lectures to go into that fascinating subject, to which we here only refer to the literature \([18]\). It was based on this structure, and its relation to the graded analog, that the partially quenched partition function of the chiral Lagrangian was first computed in the \( \epsilon \)-regime \([17, 19]\).

### 3. What is Random Matrix Theory?

What is a “random matrix”? It is perhaps not obvious, but with the conventional definition we must fix its size. Let us first, for simplicity, consider square matrices of size \( N \times N \). Elements in this matrix can now be chosen at random, according to our chosen distribution. But this cannot be the whole story, because some matrices have particular symmetries and particular transformation properties under conjugation. Indeed, a sensible definition of Random Matrix Theory is based on a division into certain classes of random matrices, a classification that dates back to Dyson \([20]\). The three main classes are labelled by an index \( \beta \), which can take values 1, 2 or 4. Here we will focus the \( \beta = 2 \) class where one picks the random matrices to be Hermitian, and thus having real eigenvalues.

To produce a Random Matrix Theory one does as in statistical mechanics and *sums over an ensemble* of a given set of random matrices \( M \). This produces a partition function,

\[
Z = \int dM \exp \left[ -N \text{Tr}(M^2) \right] ,
\]

where, for simplicity, we have here restricted ourselves to a Gaussian distribution. We can associate with this a probability distribution

\[
P(M) = \frac{1}{Z} \exp \left[ -N \text{Tr}(M^2) \right]
\]

and for Hermitian matrices the measure is

\[
dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d(\text{Re}M_{ij})d(\text{Im}M_{ij}) .
\]

We are not interested in the matrices themselves, but rather in their real eigenvalues \( \lambda_i \). It is therefore advantageous to go to an eigenvalue representation of the partition function by means of diagonalization. This is achieved by a unitary transformation:

\[
M = U^\dagger DU , \quad D = \text{diag}(\lambda_1, \ldots, \lambda_N)
\]
with $U^\dagger U = 1$.

We note that the Hermitian matrix $M$ is described by $N^2$ coefficients, while the diagonal matrix $D$ has $N$ coefficients and $U$ has $N^2 - N$. So we can indeed make the change of variables

$$M \rightarrow D, U.$$  

Since the Haar measure on $U(N)$ is left and right invariant, the associated Jacobian $J$ of this change of variables can only depend on $D$, i.e., $J = J(\lambda_1, \ldots, \lambda_N)$ and

$$dM = \left( \prod_{i=1}^{N} d\lambda_i \right) J(\lambda_1, \ldots, \lambda_N) dU.$$  

To compute $J$, it suffices to go near the identity in $U(N)$:

$$U = 1 + i\varepsilon$$  

where, for unitarity, $\varepsilon^\dagger = \varepsilon$. Then

$$M = U^\dagger DU = D - i[\varepsilon, D]$$

or, in components,

$$M_{ij} = \lambda_i \delta_{ij} + i\varepsilon_{ij}(\lambda_i - \lambda_j)$$  

[no sum on $i, j$].  

In other words,

$$i = j : \quad dM_{ii} = d\lambda_i$$

$$i < j : \quad d(\text{Re}M_{ij}) = (\lambda_i - \lambda_j)d(\text{Re}(i\varepsilon_{ij}))$$

$$d(\text{Im}M_{ij}) = (\lambda_i - \lambda_j)d(\text{Im}(i\varepsilon_{ij}))$$

so that

$$dM = \left( \prod_{i=1}^{N} d\lambda_i \right) \prod_{i<j} d(\text{Re}M_{ij})d(\text{Im}M_{ij})$$

$$= \left( \prod_{i=1}^{N} d\lambda_i \right) \prod_{i<j} (\lambda_i - \lambda_j)^2$$

up to the group volume of $U(N)$. Note also that

$$\text{Tr}(M^2) = \text{Tr}(U^\dagger DUU^\dagger DU) = \text{Tr}(D^2) = \sum_{i=1}^{N} \lambda_i^2$$

so that also the probability measure can written entirely in terms of eigenvalues and the partition is then

$$Z = \int \left( \prod_{i=1}^{N} d\lambda_i \right) \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N\sum_{i=1}^{N} \lambda_i^2}.$$  

There is an easy mnemonic to see that this Random Matrix Theory is in the $\beta = 2$ class since it can be read off from the power "2" in

$$\prod_{i<j}(\lambda_j - \lambda_i)^2.$$
This particular combination of eigenvalues can be written in the form of a so-called Vandermonde determinant raised to the second power:

\[ \prod_{i<j}(\lambda_j - \lambda_i)^2 = \Delta(\{\lambda_i\})^2 \]  

where

\[ \Delta(\{\lambda_i\}) = \det(\lambda_j^i) \quad j = 1, \ldots n, \quad i = 0 \ldots n - 1. \]  

The Random Matrix Theory (29) is of course well defined for all \( N \). When \( N \) is finite, the \( N \) eigenvalues will jump around in a rather random manner. But the presence of the Vandermonde determinant (squared) in front of the Gaussian damping factor has an important consequence: level repulsion. The probability of two eigenvalues approaching each other goes to zero. For finite \( N \), there could be two ways to get around this: 1) all eigenvalues could spread all over the real line or 2) eigenvalues could be confined to a finite interval, but level repulsion would force their distributions to lock in an essentially equidistant form. This will clearly depend on how we rescale eigenvalues. It turns out that the Gaussian damping factor in the probability distribution (29) is so strong that eigenvalues are forced to lie on a finite interval, and it is thus the 2nd realization of eigenvalue distributions we are seeing with such measures. The factor of \( N \) in the exponent is precisely inserted so as to ensure this compact distribution without further rescalings of the eigenvalues.

The distribution of eigenvalues becomes beautifully simple in the limit of very large \( N \): we get the famous Wigner semicircle distribution.

3.1. Chiral Random Matrix Theory

We are now ready to consider a Random Matrix Theory that is chiral. The identification of this class of theories to be the one relevant for low-energy QCD was first made by Verbaarschot in a remarkable series of papers about 15 years ago [21, 22]. The idea was to base the Random Matrix Theory on transformation properties of the Dirac operator \( D \) in QCD. In particular, since \( D \) anticommutes with \( \gamma^5 \), \( \{D, \gamma^5\} = 0 \), a random matrix \( M \) is introduced that has a similar structure. Moreover, gauge field configurations can be assigned to different classes depending on their topology, which is classified according to the invariant

\[ \nu = \frac{g^2}{16\pi} \int d^4x \text{Tr}(\tilde{F}_{\mu\nu} F^{\mu\nu}) \]  

where \( \tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \) is the dual QCD field strength tensor. In the QCD partition function one sums over all these different classes that have \( \nu = 0, \pm 1, \pm 2, \pm 3, \ldots \), but it can be useful to consider also sectors of fixed \( \nu \). A well known mathematical theorem ensures that in such sectors there are exact zero modes of the Dirac operator, all of definite chirality. The number of positive minus negative chirality zero modes is precisely the index \( \nu \). Leutwyler and Smilga [12] suggested to study properties of the Dirac operator in such sectors of fixed topology by means of the relationship to the chiral Lagrangian. This gives an exciting new opportunity for cross-checking how low-energy observables in the full theory (QCD) are in one-to-one correspondence with observables in the effective large-distance theory of the chiral Lagrangian.

Verbaarschot and Shuryak [21] introduced the following chiral Random Matrix Theory:

\[ Z = \int dM \left( \det(M) \right)^{N_f} \exp \left[ -\frac{N}{2} \text{Tr} V(M^2) \right] \]  

where \( V(M^2) \) will be discussed below, and where

\[ M = \begin{pmatrix} 0 & W \dagger \\ -W & 0 \end{pmatrix} \]
is a \((2N + \nu) \times (2N + \nu)\) block Hermitian matrix composed of the complex \(N \times (n + \nu)\) rectangular matrix \(W\). The integral is over the elements of this matrix. We note that both measure and integrand is then invariant under

\[
W \rightarrow V^\dagger W U, \quad U \in U(N), \quad V \in U(N + \nu)
\]

and for this reason this Random Matrix Theory is called chiral unitary. But why is it ‘chiral’? Consider the \((2N + \nu) \times (2N + \nu)\) matrix

\[
\gamma^5 \equiv \begin{pmatrix}
1_N & 0 \\
0 & -1_{N+\nu}
\end{pmatrix}
\]

This matrix clearly anticommutes with the matrix \(M\),

\[
\{ M, \gamma^5 \} = 0,
\]

which implies that all non-zero eigenvalues of \(M\) come paired \(\pm \lambda_j\). So the matrix \(M\) shares the \(U(1)\) chirality property of the full Dirac operator of QCD. Moreover, due to its rectangular block decomposition, there are \(\nu\) zero modes. The spectrum of \(M\) therefore has two parts: there are \(2N\) non-zero eigenvalues that are chirally paired and \(\nu\) zero modes.

Actually, the chiral Random Matrix Theory (33) was originally introduced with just a Gaussian measure, but it was believed from the beginning that the so-called ‘microscopic’ results that could be derived from it would be \textit{universal}, i.e. to a very large extent independent of the chosen potential. Indeed, it was proven in ref. [23] that all results to be discussed below hold for an arbitrary potential

\[
V(M^2) = \sum_{k \geq 1} \frac{g_k}{k} M^{2k}
\]

with, essentially, only one single constraint: the spectrum of eigenvalues of \(M\) must have support at the origin: \(\rho(0) \neq 0\). This is in beautiful accord with the intuition, to be made more precise below, that a non-vanishing chRMT spectral density at the origin is required for this theory to describe aspects of chiral symmetry breaking – in heuristic analogy with the Banks-Casher relation in QCD. The universality and robustness of this result is crucial for the understanding of why chiral Random Matrix Theory can describe chiral symmetry breaking in a full-fledged quantum field theory such as QCD.

Interestingly, one can also consider a chiral Random Matrix Theory where the spectral density at the origin \(\rho(0)\) is carefully tuned so as to produce a zero: \(\rho(0) = 0\) [24]. One might hope that this could represent chiral symmetry restoration, one way or another. This, however, has never been established. There are still universality classes in this case, one for each order of zero of the spectral density. Each class can be reached, successively, by tuning more and more of the parameters \(g_k\) in (36). This is an example of universality in what can be dubbed multicity (while the more general condition \(\rho(0) \neq 0\) corresponds to the non-critical case). A pity that there seems to be few instances of quantum field theory where these more sophisticated universality classes can be of relevance! One possibility could be two space-time dimensions, where spontaneous symmetry breaking is prohibited, and the spectral properties of the Dirac operator can be studied in, say, the Schwinger model [25]. In any case, the mere existence of these extended classes shows that the universality domain of the non-critical ensemble is restricted and by no means trivially implied for any chiral Random Matrix Theory potential \(V(M^2)\).

Very similar to the way we could introduce an eigenvalue representation (29) for the non-chiral (Gaussian) ensemble, one can also find an eigenvalue representation

\[
Z = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left( dz^2_i z_i^{2(N_f+\nu)} e^{-N V(z^2_i)} \right) \left( \det z^{2i(i-1)} \right)^2
\]
\[
= \int_0^\infty \prod_{i=1}^N d\lambda_i \, \lambda_i^{N_f} e^{-N V(\lambda_i)} \left( \det_{ij} \lambda_j^{i-1} \right)^2
\]

where the \(z^2\)’s are the real (and positive) non-vanishing eigenvalues of \(W^\dagger W\), and where in the last line we have just transformed to more convenient variables \(\lambda_i\). We have also used the fact that diagonalization of \(W^\dagger W\) automatically guarantees factorization of the potential part in the manner shown. It is interesting to note how the dependence on \(N_f\) and \(\nu\) has merged into the combination \(N_f + \nu\) only. This is a simplifying property that holds only in the massless case. The case of massive quarks will be discussed below. The square of the Vandermonde determinant has appeared just as in the non-chiral case. Also here it will result in an eigenvalue repulsion.

There are standard methods for computing the spectral correlation functions of a Random Matrix Theory with an eigenvalue representation such as (37). A convenient method is based on orthogonal polynomials, using a beautiful formalism that dates back to Dyson. Limitations of space does not permit a detailed discussion of this, and here we will restrict ourselves to explaining the main results that follow from that kind of analysis.

Let us from now on work entirely in rescaled variables \(\zeta = 2\pi N \rho(0) \lambda\). For fixed \(\zeta\) this implies smaller and smaller eigenvalues – those near the origin. A central object is the kernel \(K(\zeta_1, \zeta_2)\). If we know this kernel we know all spectral correlation functions

\[
\rho(\zeta_1, \ldots, \zeta_k) = \det_{ij} K(\zeta_i, \zeta_j)
\]

and in particular the spectral density itself,

\[
\rho(\zeta) = K(\zeta, \zeta) .
\]

The kernel appropriate for QCD is the Bessel kernel derived by Verbaarschot and Zahed [22],

\[
K(\zeta_1, \zeta_2) = \sqrt{\zeta_1 \zeta_2} \frac{\zeta_1 J_{\alpha+1}(\zeta_1) J_\alpha(\zeta_2) - \zeta_2 J_\alpha(\zeta_1) J_{\alpha+1}(\zeta_2)}{\zeta_1^2 - \zeta_2^2} ,
\]

where \(\alpha = N_f + \nu\) (we consider only positive \(\nu\) here). Despite appearance, this is regular at coincident points where it gives the microscopic spectral density

\[
\rho(\zeta) = \frac{\zeta}{2} \left( J_\alpha(\zeta)^2 - J_{\alpha+1}(\zeta) J_{\alpha-1}(\zeta) \right) .
\]

This is the most simple and remarkable result of the chiral Random Matrix Theory approach. The claim is that this gives the microscopic spectral density of QCD with \(N_f\) massless quarks in the finite-volume regime we consider. Below we shall see why this is so, and how to include massive quarks as well. Amazingly, if we boldly set \(N_f = 0\) this should give the spectral density of the Dirac operator in pure Yang-Mills theory! When we see the connection with the partially quenched chiral Lagrangian this makes it clear why this should be possible (with some caveats). Another immediate check on the above spectral density comes from the fact that it reproduces exact spectral sum rules derived by Leutwyler and Smilga [12] on the basis of the chiral Lagrangian alone.

### 3.2. Spectral properties of the Dirac operator

The pioneering paper that showed how these detailed predictions for the spectral density of the Dirac operator can be tested in lattice gauge theory simulation was ref. [26]. Quenched staggered fermions with gauge group \(SU(2)\) was used there. Because the chiral symmetries of staggered fermions away from the continuum limit are different from those of continuum fermions
when the fermions carry real or pseudo-real representations of the gauge group (pseudo-real in the case of fundamental fermions and gauge group $SU(2)$), the detailed predictions were not those of eq. (41). But also for this case the analytical prediction had been worked out [27] and although it differs in detail, it shares many of the properties with that of (41). The analytical expressions for continuum fermions based on this gauge group and fermions in the fundamental representation had been worked out as well [28]. We shall not go into great detail with these different predictions, but it is an amazing fact, first recognized by Verbaarschot [29], that there is a one-to-one correspondence between the Dyson classification of (chiral) Random Matrix Theory and the three chiral symmetry breaking patterns in vector-like gauge theories:

- The fermion representation $r$ is pseudo-real: Chiral symmetries are enhanced from $SU(N_f) \times SU(N_f)$ to $SU(2N_f)$, and the expected symmetry breaking pattern is $SU(2N_f) \rightarrow Sp(2N_f)$.
- The fermion representation $r$ is complex: The expected symmetry breaking pattern is $SU(N_f) \times SU(N_f) \rightarrow SU(N_f)$.
- The fermion representation $r$ is real: Chiral symmetries are again enhanced to $SU(2N_f)$, and the expected chiral symmetry breaking pattern is $SU(2N_f) \rightarrow SO(2N_f)$.

That there is indeed a one-to-one matching with the three Dyson classes, and that the chiral symmetry breaking patterns really do follow the above classification, has been tested on the lattice in great detail for a variety of different fermion representations and it works in all cases [30].

There is one further general peculiarity of simulations with staggered fermions on the lattice which in the beginning caused some confusion, but which really is just one further confirmation of the above classification scheme. This arises from the fact that staggered fermions away from the continuum limit have an additional $U(1)$ factor in the coset of chiral symmetry breaking (this holds for all three classes). As we will show below, this means that all results for staggered fermions away from the continuum are equivalent to those we would get if we had projected on the $\nu = 0$ sector. Any attempt at a further projection on sectors with $\nu \neq 0$ leads to the same partition function. So for staggered fermions away from the continuum it will look as if only the sector of $\nu = 0$ appears. This has been confirmed by a high-statistics study where gauge field configurations were even selected according to their (approximative) topological indices: all spectral data of the staggered Dirac operator look as if they have been obtained from the $\nu = 0$ sector [31]. This should of course change as the continuum limit is reached. By using highly improved actions as well as improved Dirac operators that smear both configurations and observables it has been checked that predictions of continuum fermions do match those of staggered fermions eventually [32].

While most initial studies of the spectral properties of the Dirac operator were restricted to gauge group $SU(2)$, results were soon available for the QCD gauge group of $SU(3)$ [33], nicely confirming the prediction (41). A far more exciting possibility was to check all of these predictions with ‘chirally good’ fermions based on the overlap operator. This was first done in ref. [34]. Here the effect of topology can be clearly seen, and this works well even for approximate realizations of the overlap operator [35, 36]. A high-statistics study with several eigenvalues included has been made in ref. [37], showing clear agreement with the predictions. It is also interesting to see how, with an infinite number of colors, chiral symmetry can break even at finite four-volume. Results still fall right on the chiral Random Matrix Theory predictions, now as a result of scaling in the number of colors $N_c$ [38]. Here one sees clearly how color and volume acquire complementary roles at large-$N_c$, the corresponding scaling variable being $\lambda \Sigma N_c V$.

So far our discussion has been restricted to the massless limit, which is easily taken in lattice gauge theory if one restrict oneself to the quenched theory. An interesting question is how to include quark masses in all of these predictions. If we consider the chiral Random Matrix Theory
(33), it is clear that masses \( m \) must be rescaled at the same rate as eigenvalues \( \lambda_i \). Let us re-emphasize that \( m \) in (33) is simply a dimensionless parameter in the matrix integral. It is not a quark mass. Just as the universal predictions for rescaled eigenvalues \( N \rho(0) \lambda \) correspond to volume-rescaled eigenvalues \( V \Sigma \lambda \) on the QCD side, so \( N \rho(0) m \) will correspond to the scaling variable \( V \Sigma m \) in QCD. The precise predictions were worked out in ref. [40]. The clue to the solution was there found in the theory of orthogonal polynomials, which is known to solve the spectral problem of eigenvalue representations like those of (33). Intuitively, it comes about as follows: if we know a set of orthogonal polynomials corresponding to (33) without knowing about as follows: if we know a set of orthogonal polynomials corresponding to (33), it is clear that masses \( m \) must be rescaled at the same rate as eigenvalues \( \lambda_i \). Let us re-emphasize that \( m \) in (33) is simply a dimensionless parameter in the matrix integral. It is not a quark mass. Just as the universal predictions for rescaled eigenvalues \( N \rho(0) \lambda \) correspond to volume-rescaled eigenvalues \( V \Sigma \lambda \) on the QCD side, so \( N \rho(0) m \) will correspond to the scaling variable \( V \Sigma m \) in QCD. The precise predictions were worked out in ref. [40].

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In this way of writing it, we nicely illustrate an essential feature:

\[
\lim_{\mu \to \infty} \rho(\zeta, \mu) = \rho(\zeta)
\]

In other words: when we take the (rescaled) quark mass to infinity we recover the quenched spectral density. This is as we would intuitively expect, since the quark should then decouple and the eigenvalues should distribute themselves according to the quenched distributions.

All of this was generalized in ref. [40] to any number of flavors, and the universality proof was there extended to this generalized case as well. Moreover, the explicit expressions for the massive kernels were provided there. From these, all spectral correlation functions with massive quarks follow. However, the expressions can become unwieldy, and there is actually a much more compact way of understanding these results. It turns out that, quite surprisingly, one can express all microscopic spectral correlators compactly in terms of finite volume QCD effective partition functions. The crucial ingredient is a powerful identity derived by Zinn-Justin [41] in the case of the ordinary unitary ensemble, and which is readily generalized to the chiral case.

Let us consider the \( \nu = 0 \) case in detail [42]. The kernel can then be expressed as

\[
K_N(z, z'; m_1, \ldots, m_{N_f}) = \frac{e^{-\frac{N}{2} (V(z^2) + V(z'^2))} \sqrt{zz'} \prod_{f} \sqrt{(z^2 + m_f^2)(z'^2 + m_f^2)}}{\tilde{Z}_0^{(N_f)}(m_1, \ldots, m_{N_f})} \times \\
\int_0^{\infty} \prod_{i=1}^{N-1} \left( d\lambda_i \lambda_i' (\lambda_i - z^2) (\lambda_i - z'^2) \prod_{f=1}^{N_f} (\lambda_i + m_f^2) e^{-NV(\lambda_i)} \right) \left| \det \lambda_i^{-1} \right|^2
\]

The last integral is over \( (N-1) \) eigenvalues only. However, in the large-\( N \) limit we shall consider below, this distinction can be ignored. Thus, in the large-\( N \) limit we have

\[
K_N(z, z'; m_1, \ldots, m_{N_f}) = e^{-\frac{N}{2} (V(z^2) + V(z'^2))} \sqrt{zz'} \prod_{f} \sqrt{(z^2 + m_f^2)(z'^2 + m_f^2)} \\
\frac{\tilde{Z}_0^{(N_f+2)}(m_1, \ldots, m_{N_f}, i z, i z')}{\tilde{Z}_0^{(N_f)}(m_1, \ldots, m_{N_f})} ,
\]

where the matrix model partition function in the numerator is evaluated for a theory corresponding to \( (N_f + 2) \) fermions, of which two have imaginary mass. By means of the usual
factorization property, all higher $n$-point spectral correlation functions are then also explicitly expressed in terms of the two matrix model partition functions $\tilde{Z}_\nu^{(N_f)}$ and $\tilde{Z}_\nu^{(N_f+2)}$. The spectral density corresponds to the two additional (imaginary) masses being equal:

$$\rho^{(N_f)}(z; m_1, \ldots, m_{N_f}) = \lim_{N \to \infty} K_N(z, z; m_1, \ldots, m_{N_f}) .$$

(46)

All of this is exact, and at the level of unscaled variables. We can now take the microscopic scaling limit in which $\zeta = z N 2 \pi n(0)$ and $\mu_i = m_i N 2 \pi n(0)$ are kept fixed as $N \to \infty$. In this limit the pre-factor $\exp[-(N/2)(V(z^2) + V(z^2))]$ becomes replaced by unity. Identifying $\Sigma = 2 \pi n(0)$, this is the limit in which we can compare with the finite-volume partition function of QCD. Proceeding in this way, we get, with $C$ being an overall constant,

$$\rho^{(N_f, \nu)}(\zeta; \mu_1, \ldots, \mu_{N_f}) = C |\zeta| \prod_f (\zeta^2 + \mu_f^2) \frac{Z_\nu^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta, i\zeta)}{Z_\nu^{(N_f)}(\mu_1, \ldots, \mu_{N_f})} ,$$

(47)

and all $n$-point correlation functions are given by one magical Master Formula [42]

$$\rho^{(N_f, \nu)}(\zeta_1, \ldots, \zeta_n; \mu_1, \ldots, \mu_{N_f}) = \det_{a,b} K(\zeta_a, \zeta_b; \mu_1, \ldots, \mu_{N_f}) .$$

(48)

This generalizes to any $\nu$ [43] and in fact leads to some surprising identities among the partition functions involved, identities that express the connection to an associated integrable hierarchy. The advantage is that the finite-volume partition functions are known in exact analytical forms [44]. Results have also been generalized to the other chiral symmetry breaking classes [45, 46].

One immediate check on these massive spectral densities is that they reproduce exact massive spectral sum rules [47] just as the massless spectral densities reproduce the massless spectral sum rules of Leutwyler and Smilga.

Amazingly, the same procedure generalizes to the computation of individual eigenvalue distributions. Such a notion makes clear sense in the chiral Random Matrix Theory framework since there is a natural starting point from where to start counting eigenvalues (the origin). After the lattice gauge theory community had accepted that chiral Random Matrix Theory gives exact results for the spectral density of the Dirac operator, and all higher-point spectral correlation functions, there remained a curious belief that even the very notion of ordered Dirac operator eigenvalues and their precise distributions could not be given field theoretic meaning. This is of course not so: ordered Dirac operator eigenvalues have precisely as much meaning in the field theory framework as the spectral density and all spectral correlation functions. This will become clear below. But as first step towards this, we should describe how individual eigenvalue distributions of the chiral Random Matrix Theory can be given explicitly in terms of finite volume partition functions for first the smallest Dirac operator eigenvalue [48] and then the general distribution for the $k$th Dirac operator eigenvalue [49], as counted from the origin. Let us denote the joint probability distribution of the smallest $k$ ordered eigenvalues by $\omega_k(\zeta_1, \ldots, \zeta_{k-1}, \zeta_k)$, where $\zeta_1 \leq \cdots \leq \zeta_k$. By manipulations very similar to the ones used above, one can show that it can be written in terms of finite volume partition functions [49]

$$\omega_k(\zeta_1, \ldots, \zeta_{k-1}, \zeta_k) = C e^{-\zeta_k^2/4} \prod_{i=1}^{k-1} \left( \zeta_i^2 + \mu_i^2 \right)^{N_f} \prod_{i>j} \left( \zeta_i^2 - \zeta_j^2 \right)^{N_f} \prod_{j=1}^{N_f} \mu_j^\nu \times \frac{Z_\nu(\{\mu_i\})}{Z_\nu(\{\mu_i\})} .$$

(49)
where the “masses” $\sqrt{\zeta_k^2 - \zeta_i^2}$ are doubly degenerate (and $1 \leq i \leq k - 1$). Similarly, the set of “masses” $\{\zeta_k\}$ is $\nu$-fold degenerate.

A very high-accurate Monte Carlo study of these individual eigenvalue distributions has been done in ref. [50], for both the quenched case and for theories with dynamical finite-mass fermions. The agreement is quite spectacular, with parameter-free fits that lie right on the analytical predictions. Of course, if one for some reason would fear these predictions, one could also just look at the chiral condensate which follows a very precise scaling law based on these analytical predictions for the chiral condensate [51, 52].

One important question is how individual eigenvalue distributions, ordered in the way we have discussed above, can be understood and derived from the quantum field theory point of view. How do we even define such distributions in quantum field theory? It turns out that there is a simple answer [54]. For simplicity, let us here restrict ourselves to the case of just the smallest Dirac operator eigenvalue, the distributions of the larger eigenvalues follow analogously [54]. Here one can show that the so-called gap probability has a convergent expansion of the form

$$E_0(s) = 1 - \int_0^s d\lambda_1 \rho(\lambda_1) + \frac{1}{2} \int_0^s d\lambda_1 d\lambda_2 \rho_2(\lambda_1, \lambda_2) + \ldots$$

(50)

involving higher and higher spectral correlation functions. All of these spectral correlation functions can be computed in quantum field theory by means of suitable sources. The distribution of the smallest eigenvalue now follows by differentiation:

$$p_1(s) = -\frac{\partial}{\partial s} E_0(s).$$

(51)

For the $k$th eigenvalue this follows by a simple iterative scheme that involves increasingly higher order correlation function as the first term in the expansion [54]. But the convergence is fast in the appropriate region of $\lambda$, and this scheme is in fact a quite practical way of computing these distributions if one is only interested in high numerical accuracy.

Another issue concerns gauge field topology. It is clear from the chiral Random Matrix Theory construction that topology is intimately tied together with the whole construction. While a summation over topology is straightforward in QCD, there seems to be no way to perform such a summation in the context of chiral Random Matrix Theory. Every sector corresponds to matrices of different sizes. However, all physical observables can be decomposed into their counterparts in distinct topological sectors, and a summation over topology is therefore indirectly possible through [53]

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\nu} Z_{\nu} \langle \mathcal{O} \rangle_{\nu}$$

(52)

where $Z$ is the full partition function and subscript $\nu$ indicates when the average is taken in a sector of fixed topological charge. Note how averages are weighted by the partition function in fixed sectors. Using this relationship and the Master Formula derived above, it is possible to perform the sum analytically and derive an analogous formula for the spectral density of the full theory [53]. It is also particularly illuminating to see analytically how the triviality of results in the $N_f = 1$ theory are recovered from the highly non-trivial predictions in sectors of fixed topological charge.

3.3. Matching chiral Random Matrix Theory with the chiral Lagrangian

The Master Formula (48) gives a strong indication that all these chiral Random Matrix Theory results can be derived entirely from chiral Lagrangian framework. This is indeed the case [55, 16]. The framework is partially quenched chiral perturbation theory, which we have already discussed above. Through it, one can derive a one-to-one correspondence between the spectrum derived
from the chiral Lagrangian and the spectrum derived from chiral Random matrix Theory. The final steps showing that this holds for all spectral correlation functions were taken by Akemann and Basile in ref. [56].

What has been achieved is thus a precise and detailed understanding of how chiral Random Matrix Theory can yield exact results. It provides an exact representation of the leading-order chiral Lagrangian in the extreme finite-volume \( \epsilon \)-regime. The proof goes through the partially quenched theory which here serves two purposes at one time: (1) it provides predictions for genuine partially quenched lattice gauge theories in this finite-volume regime, and (2) it serves as a generating function of spectral correlation functions. In chiral Random Matrix Theory one does often not make use of a partially quenched (graded) version of the theory because other simpler techniques (such as the one based on orthogonal polynomials) are available. It is precisely by going through the graded version that the full equivalence has finally been established for all correlation functions.

An interesting question is whether one can extract information about the other low-energy constants \( F \) as well as all higher-order coupling constants of the chiral Lagrangian by means of spectral properties of the Dirac operator. In principle, this should be possible since the influence of these couplings on the distributions of the Dirac operator eigenvalues can computed. By a careful comparison to numerical lattice data, one should be able to extract these other low-energy parameters of QCD by considering only a few of the lowest Dirac operator eigenvalue – a remarkable thought. However, in practice it is not that easy. The simplest example is the leading correction to \( \Sigma \), which we already discussed above. In principle, by considering a variety of different volumes \( V \), and perhaps different geometries as well, one can extract the pion decay constant \( F \) from this.

Other chiral Random Matrix Theory approaches, aimed specifically at determining \( F \), have also been invented. One consists in introducing a new source in the chiral Lagrangian that couples directly to \( F \) to leading order in the chiral counting, while retaining anti-Hermiticity of the Dirac operator in that context. Although it is to be considered only as a source, it actually has a physical interpretation as imaginary isospin potential. With an external isospin potential the QCD measure is free of the sign problem that plagues QCD with baryon chemical potential. Making the potential purely imaginary ensures that the Dirac operator eigenvalues actually has a physical interpretation as imaginary isospin potential. With an external isospin potential the QCD measure is free of the sign problem that plagues QCD with baryon chemical potential. Making the potential purely imaginary ensures that the Dirac operator eigenvalues remain anti-Hermitian. This approach was first developed in the chiral Lagrangian framework [57], but inspired by a construction due to Osborn [58] who introduced a two-matrix chiral theory do describe baryon chemical potential, a chiral two-matrix theory was set up to describe imaginary isospin chemical potential [59]. Indeed, results from this theory, which we will describe next, agree precisely with those obtained from the chiral Lagrangian [57]. Because all the powerful machinery of the more usual chiral Random Matrix Theory is preserved, this formulation allows for an explicit calculation of all spectral correlation function.

The first step consists in considering the eigenvalues of two different Dirac operators in QCD,

\[
\begin{align*}
D_1\psi_1^{(n)} &= [D(A) + i\mu_1\gamma_0]\psi_1^{(n)} = i\Lambda_1^{(n)}\psi_1^{(n)} \\
D_2\psi_2^{(n)} &= [D(A) + i\mu_2\gamma_0]\psi_2^{(n)} = i\Lambda_2^{(n)}\psi_2^{(n)} .
\end{align*}
\]

(53)

Imaginary isospin potential corresponds to \( \mu \equiv \mu_1 = -\mu_2 \). Such a source couples directly to \( F \) is the chiral Lagrangian for which the leading-order terms in a suitably defined \( \epsilon \)-regime read:

\[
Z^{(N_f)}_\nu = \int_{U(N_f)} dU (\det U)^\nu e^{\frac{1}{2}V F^2 Tr[U,B][U^\dagger,B]+\frac{1}{2} \Sigma V Tr(M^\dagger U+MU^\dagger)} .
\]

(54)

where the matrix \( B = \text{diag} (\{\mu_1\}, \{\mu_2\}) \) contains the two chemical potentials introduced above, and \( M = \text{diag}(m_1, \ldots, m_{N_f}) \) is the mass matrix. Chemical potential \( \mu_1 \) has been assigned to \( N_1 \) of the quarks and chemical potential \( \mu_2 \) to the remaining \( N_f - N_1 = N_2 \) quarks.
A two-matrix chiral theory that corresponds to this is

\[ Z^{(N_f)}_\nu = \int d\Phi d\Psi \ e^{-NTr(\Phi^\dagger \Phi + \Psi^\dagger \Psi)} \prod_{f=1}^{N_1} \det[D_1 + m_{f1}] \prod_{f=2}^{N_2} \det[D_2 + m_{f2}] \]  

(55)

where \( D \) is defined by

\[ D_f = \begin{pmatrix} 0 & i\Phi + i\mu_f \Psi \\ i\Phi^\dagger + i\mu_f \Psi^\dagger & 0 \end{pmatrix}, \quad f = 1, 2. \]

The matrices \( \Phi \) and \( \Psi \) are complex rectangular matrices of size \( N \times (N + \nu) \) in the same manner as in the usual chiral ensemble.

Without going into the technical details, an eigenvalue representation can also be found for this theory. Let us define

\[ c_1 = (1 + \mu_2^2)/\delta^2, \quad c_2 = (1 + \mu_1^2)/\delta^2, \quad d = (1 + \mu_1\mu_2)/\delta^2, \quad 1 - \tau = d^2/(c_1c_2), \]

(56)

Then,

\[ Z^{(N_f)}_\nu = \int_0^\infty \prod_i \left( dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f=2}^{N_2} (y_i^2 + m_{f2}^2) \right) \]

\[ \times \Delta(\{x^2\})\Delta(\{y^2\}) \det[I\nu(2dNx_iy_j)] e^{-N\sum_i c_1x_i^2 + c_2y_i^2}. \]

(57)

The two sets of eigenvalues \( x_i \) and \( y_i \) are obtained after diagonalizing

\[ \Phi_1 \equiv \Phi + \mu_1 \Psi, \quad \Phi_2 \equiv \Phi + \mu_2 \Psi, \]

(58)

and due to this redefinition the original matrices now become coupled in the exponent. This looks horribly complicated, but it actually has some recognizable structure. First of all, the parameter \( \delta \) is what measures the strength of the external source, the imaginary chemical potential. Next, by a stroke of luck, a formalism of bi-orthogonal polynomials had recently been invented for precisely this kind of problem by Eynard and Mehta [60]. Analytical miracles occur, and one ends with very simple analytical expressions for all spectral correlation functions. Particularly interesting spectral observables are those that vanish if \( \delta = 0 \). This leads to a strong signal with which the pion decay constant can be measured (since \( \delta \) couples to \( F \) in the chiral Lagrangian). All results derived from the chiral Lagrangian in [57] are reproduced in this Random Matrix Theory approach. Moreover, all spectral correlation functions can now be expressed in closed analytical forms [59]. Also individual eigenvalue distributions can be computed analytically in this two-matrix theory [61]. Detailed comparisons with lattice gauge theory data, for a variety of different volumes and including the finite-volume corrections [62], have been made recently [63].

3.4. Beyond chiral Random Matrix Theory
Once one has realized the connection between the leading-order chiral Lagrangian in the \( \epsilon \)-regime and chiral Random Matrix Theory, it becomes obvious how to combine the two. The most difficult part of the \( \epsilon \)-expansion is precisely the non-perturbative contribution from the momentum zero modes of the pseudo-Goldstone bosons. We have seen how a partially quenched
chiral Lagrangian had to be understood in this context just in order to derive properties of the spectral density. This is extremely useful for lattice gauge simulations since there one is often interested in doing partial quenching as an approximation to a real simulation. For example, one might wish to scan more parameter values with the same lattice configurations, or one has difficulty simulating with light enough fermions. Here partial quenching can provide additional information.

The first issue concerns pion-loop contributions to the leading-order chiral Lagrangian. This was investigated in the original paper by Gasser and Leutwyler [11]. Considering the contribution to the chiral condensate, they found that the Lagrangian remains form invariant except for a rescaling of the Σ-parameter,

\[ \Sigma \rightarrow \Sigma_{eff} = \Sigma \left[ 1 - \frac{N_f^2 - 1}{N_f} \frac{\beta_1}{\sqrt{V}} + \mathcal{O}(1/V) \right] \]

for a four-dimensional torus of volume \( V \). The coefficient is one of an infinite sequence of shape coefficients that appear in the \( \epsilon \)-expansion [64]. This rescaled low-energy constant \( \Sigma_{eff} \) appears naturally in the expansion of any observable in the \( \epsilon \)-regime. We see that the correction term disappears for large volume as \( 1/\sqrt{V} \), which is of order \( \epsilon^2 \), as expected. The expansion thus seems to work well. In the fully quenched case this is not the case. There a new scale, set by the \( \eta' \)-mass in physical terms, enters and there is no longer a simple one-parameter expansion. As usual, when things can go wrong, they will go wrong. In this case one ends up with a new term that grows logarithmically with volume – a “quenched finite volume logarithm” [65]. Although such a correction term can be kept under control if one only compares two different volumes (at least to that order), it is just one more indication of the fundamental difficulty of quenched theories. In a physical theory with dynamical quarks such a logarithm still shows up at higher orders, but always suppressed by additional inverse powers of \( L \), and therefore harmless.

Hansen [66] was the first to explore the \( \epsilon \)-regime beyond leading order for mesonic correlation functions in chiral perturbation theory. This was before Leutwyler and Smilga [12] had shown the usefulness of working in sectors of fixed topological charge \( \nu \), and a re-analysis of Hansen’s results in such sectors has been done [67]. The generalization of Hansen’s results to the quenched case was also done there. What is interesting about mesonic correlation functions in the \( \epsilon \)-regime is their polynomial dependence on the euclidean length scale. In the \( p \)-regime correlators have the conventional exponential fall-off, but in the \( \epsilon \)-regime, where one is essentially “inside the pion”, the correlation function is almost flat. One can understand this very simply. Consider the zero-momentum projection of the standard massive pion propagator,

\[ \int d^3x \, \Delta(x) = \cosh \left( \frac{m_\pi(T/2-t)}{2m_\pi \sinh(m_\pi T/2)} \right), \]

where \( T \) is the (arbitrarily chosen) euclidean time extent. Apart from the overall factor of \( 1/m_\pi^2 \), we can Taylor expand this in powers of \( m_\pi^2 \) to find

\[ \int d^3x \, \Delta(x) = \frac{1}{T m_\pi^2} + \frac{T}{2} \left( \tau - \frac{1}{2} \right)^2 - \frac{1}{12} + \frac{T^3}{24} \left[ \tau^2 (\tau - 1)^2 - \frac{1}{30} \right] m_\pi^2 + \ldots \]

The finite-order polynomials of this expansion are precisely what in the \( \epsilon \)-regime replace the ordinarily exponentially decaying correlation functions. The pole-term at \( m_\pi = 0 \) is the contribution from the momentum zero-mode, which indeed blows up in the simple expansion above (but which is treated exactly in the \( \epsilon \)-expansion). This is the way in which the \( \epsilon \)-regime expansion of correlation functions merge smoothly with the standard \( p \)-regime correlation functions. Such an approach has also been developed further, see, for instance, ref. [68].
An interesting question is whether one can combine the $\epsilon$-regime with the $p$-regime, either partially quenched or not. Similarly, one can consider the question of so-called ‘mixed actions’ in this context (partial quenching where the light quark limit is reached by using configurations generated with different types of fermions, for example). Ideally, one would like to be able to do only one calculation which would be valid in a domain that stretches between the two different counting regimes. This is an active area of research at the moment [69, 70, 71] and results are currently being compared to large-scale simulations based on overlap fermions [72].

4. Outlook
Here we have reviewed some of the exact relations that have been established. The fact that chiral Random Matrix Theory provides the leading contribution to the $\epsilon$-expansion of chiral perturbation theory has turned out to give an analytically very powerful new view on low-energy QCD.

In lattice gauge theory, where most of the results discussed have very direct applicability, one is dealing with a discretized version of the continuum field theory. This induces lattice artifacts that are not physical. In a first approximation such discretization errors can be neglected, but in a more careful analysis they must be taken into account. In fact, measuring the quantitative effects of such lattice artifacts can be done in a systematic way following what is known as the Symanzik program of effective lattice field theory. There is then a modified chiral Lagrangian that includes discretization effects, but phrased in the continuum language. One can then also ask for the effect of discretization errors on the Dirac operator spectrum [73]. Remarkably, chiral Random Matrix Theory can be extended to include such effects as well [74, 75]. It is particular interesting to see how the special case of $N_f = 1$, where there is no spontaneous breaking of chiral symmetries, looks in this context. The full partition function is, to leading order, a trivial exponential of the free energy since there is a mass gap even in the chiral limit. But it contains now two terms: one proportional to the quark mass $m$, and one proportional to the leading lattice-spacing artifact $a^2$. These two terms compete trivially in the full theory for which the leading term in the effective partition function simply reads

$$Z = \exp \left[ m \Sigma V - 2 W_8 V a^2 \right]$$

(62)

where $W_8$ is a new low-energy constant of Wilson fermions in this curious $N_f = 1$ theory (the constant of 2 in front is a convention inherited from the general-$N_f$ theory where one has a genuine chiral Lagrangian). If one splits the partition function into sectors of fixed index $\nu$, there is a non-trivial interplay between the two terms above due to the fixing of the index [76]. It is quite amazing how much carries over from the case of spontaneous chiral symmetry breaking once the index $\nu$ has been fixed. For the more physical case of a larger number of dynamical quarks also other observables, such as space-time correlation functions, can be computed for lattice gauge theory observables in the $\epsilon$-regime with explicit $a^2$ corrections. A variety of different scaling regimes can be considered. Such a program is presently being carried out for the case of Wilson fermions [77]. This is one of the clearly identifiable directions in which one can envision future research within this field.

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References
[1] E. Merzbacher, Quantum Mechanics, John Wiley & Sons, Inc. (New York) 1998.
[2] T. Banks, C. M. Bender and T. T. Wu, Phys. Rev. D 8 (1973) 3346.
[3] M. M. Nieto, V. P. Gutschick, F. Cooper, D. Strottman and C. M. Bender, Phys. Lett. B 163 (1985) 336.
[4] A. Perez and D. Sudarsky, arXiv:0811.3181 [hep-th].
J. M. Verbaarschot, Phys. Lett. B 287 (1996) 355 [arXiv:hep-th/9605183].

[45] G. Akemann and E. Kanzieper, Phys. Rev. Lett. 85 (2000) 1174 [arXiv:hep-th/0001188].

[46] T. Nagao and S. M. Nishigaki, Phys. Rev. D 62 (2000) 065006 [arXiv:hep-th/0001137]; Phys. Rev. D 62 (2000) 065007 [arXiv:hep-th/0003009].

[47] P. H. Damgaard, Phys. Rev. B 425 (1998) 151 [arXiv:hep-th/9711047].

[48] S. M. Nishigaki, P. H. Damgaard and T. Wettig, Phys. Rev. D 58 (1998) 087704 [arXiv:hep-th/9803007].

[49] P. H. Damgaard and S. M. Nishigaki, Phys. Rev. D 63 (2001) 045012 [arXiv:hep-th/0006111].

[50] P. H. Damgaard, U. M. Heller, R. Nielsen and K. Rummukainen, Phys. Lett. B 495 (2000) 263 [arXiv:hep-lat/0007041].

[51] J. M. Verbaarschot, Phys. Lett. B 368 (1996) 137 [arXiv:hep-ph/9509369].

[52] P. H. Damgaard, R. G. Edwards, U. M. Heller and R. Narayanan, Phys. Rev. D 61 (2000) 094503 [arXiv:hep-lat/9907016].

[53] P. H. Damgaard, Nucl. Phys. B 556 (1999) 327 [arXiv:hep-th/9903096].

[54] G. Akemann and P. H. Damgaard, Phys. Lett. B 583 (2004) 199 [arXiv:hep-th/0311171].

[55] J. C. Osborn, D. Toubolan and J. M. Verbaarschot, Nucl. Phys. B 540 (1999) 317 [arXiv:hep-th/9806110].

[56] F. Basile and G. Akemann, JHEP 0712 (2007) 043 [arXiv:0710.0376 [hep-th]].

[57] P. H. Damgaard, U. M. Heller, K. Shtitoff and B. Svetitsky, Phys. Rev. D 72 (2005) 091501 [arXiv:hep-lat/0508029]; Phys. Rev. D 73 (2006) 074023 [arXiv:hep-lat/0602030]. P. H. Damgaard, U. M. Heller, K. Shtitoff, B. Svetitsky and D. Toublan, Phys. Rev. D 73 (2006) 105016 [arXiv:hep-lat/0604054].

[58] J. C. Osborn, Phys. Rev. Lett. 93 (2004) 22001 [arXiv:hep-th/0403131].

[59] G. Akemann, P. H. Damgaard, J. C. Osborn and K. Splittorff, Nucl. Phys. B 766 (2007) 34 [Erratum-ibid. B 800 (2008) 406] [arXiv:hep-th/0609059].

[60] B. Eynard and M.L. Mehta, J. Phys. A 31 (1998) 4449 [cond-mat/9710230].

[61] G. Akemann and P. H. Damgaard, JHEP 0803 (2008) 073 [arXiv:0803.1171 [hep-th]].

[62] G. Akemann, F. Basile and L. Lellouch, JHEP 0812 (2008) 069 [arXiv:0804.3809 [hep-lat]]; P. H. Damgaard, T. Degranda and H. Fukushima, JHEP 0712 (2007) 060 [arXiv:0711.0167 [hep-lat]]; C. Lehner and T. Wettig, JHEP 0911 (2009) 005 [arXiv:0909.1489 [hep-lat]].

[63] C. Lehner, S. Hashimoto and T. Wettig, JHEP 1006 (2010) 028 [arXiv:1004.5584 [hep-lat]]; C. Lehner, J. Bloch, S. Hashimoto and T. Wettig, arXiv:1101.5576 [hep-lat].

[64] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B 343 (1990) 241.

[65] P. H. Damgaard, Nucl. Phys. B 608 (2001) 162 [arXiv:hep-lat/0105010].

[66] F. C. Hansen, Nucl. Phys. B 345 (1990) 685. F. C. Hansen and H. Leutwyler, Nucl. Phys. B 350 (1991) 201.

[67] P. H. Damgaard, M. C. Diamantini, P. Hernandez and K. Jansen, Nucl. Phys. B 629 (2002) 445 [arXiv:hep-lat/0112016]; P. H. Damgaard, P. Hernandez, K. Jansen, M. Laine and L. Lellouch, Nucl. Phys. B 656 (2003) 226 [arXiv:hep-lat/0211020].

[68] L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP 0401 (2004) 003 [arXiv:hep-lat/0312012]. L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP 0404 (2004) 013 [arXiv:hep-lat/0402002]. W. Bietenholz, T. Chiarappa, K. Jansen, K. I. Nagai and S. Scherer, JHEP 0402 (2004) 023 [arXiv:hep-lat/0311012].

[69] P. H. Damgaard and H. Fukushima, Nucl. Phys. B 793 (2008) 160 [arXiv:0707.3740 [hep-lat]].

[70] F. Bernardoni and P. Hernandez, JHEP 0710 (2007) 033 [arXiv:0707.3887 [hep-lat]]; F. Bernardoni, P. H. Damgaard, H. Fukushima and P. Hernandez, JHEP 0810 (2008) 008 [arXiv:0808.1986 [hep-lat]]; F. Bernardoni, P. Hernandez and S. Necco, JHEP 1001 (2010) 070 [arXiv:0910.2537 [hep-lat]]; F. Bernardoni, P. Hernandez, N. Garrison, S. Necco and C. Penas, arXiv:1008.1870 [hep-lat].

[71] P. H. Damgaard and H. Fukushima, JHEP 0901 (2009) 052 [arXiv:0812.2797 [hep-lat]].

[72] H. Fukushima, S. Aoki, S. Hashimoto, T. Kaneko, J. Noaki, T. Onogi and N. Yamada [JIQCD collaboration], Phys. Rev. Lett. 104 (2010) 122002 [Erratum-ibid. 105 (2010) 159901] [arXiv:0911.5555 [hep-lat]]; H. Fukushima et al. [JIQCD and TWQCD collaborations], arXiv:1012.4052 [hep-lat].

[73] S. R. Sharpe, Phys. Rev. D 74 (2006) 014512 [arXiv:hep-lat/0606002].

[74] P. H. Damgaard, K. Shtitoff and J. M. Verbaarschot, Phys. Rev. Lett. 105 (2010) 162002 [arXiv:1001.2937 [hep-th]]; G. Akemann, P. H. Damgaard, K. Shtitoff and J. M. Verbaarschot, arXiv:1012.0752 [hep-lat]; arXiv:1012.0752 [hep-lat].

[75] J. C. Osborn, arXiv:1012.4837 [hep-lat].

[76] G. Akemann, P. H. Damgaard, K. Shtitoff and J. M. Verbaarschot, PoS LATTICE2010 (2010) 079 [arXiv:1011.5121 [hep-lat]].

[77] A. Shindler, Phys. Lett. B 672 (2009) 82 [arXiv:0812.2251 [hep-lat]]; O. Bar, S. Necco and S. Schaefer, JHEP 0903 (2009) 006 [arXiv:0812.2403 [hep-lat]]; O. Bar, S. Necco and A. Shindler, JHEP 1004 (2010) 053 [arXiv:1002.1582 [hep-lat]]; S. Necco and A. Shindler, arXiv:1101.1778 [hep-lat].