ABELIAN 1-CALABI-YAU CATEGORIES

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Abstract. In this paper, we show all \(k\)-linear abelian 1-Calabi-Yau categories over an algebraically closed field \(k\) are derived equivalent to either the category of coherent sheaves on an elliptic curve, or to the finite dimensional representations of \(k[[t]]\). Since all abelian categories derived equivalent with these two are known, we obtain a classification of all \(k\)-linear abelian 1-Calabi-Yau categories up to equivalence.

1. Introduction

In this paper, we classify abelian 1-Calabi-Yau categories over an algebraically closed field \(k\). Recall that an abelian 1-Calabi-Yau category is a \(k\)-linear Hom/Ext-finite abelian category together with natural isomorphisms \(\text{Hom}(X,Y) \cong \text{Ext}(Y,X)^*\) for \(X,Y \in A\).

Our main result (reformulated in the body of the text as Theorem 4.7) is the following.

**Theorem 1.1.** Let \(A\) be an indecomposable abelian 1-Calabi-Yau category. Then \(A\) is derived equivalent to one of the following two categories.

1. Finite dimensional representations of \(k[[t]]\).
2. The category of coherent sheaves on an elliptic curve.

There is a general interest in the classification of categories which are homologically small in some sense (see e.g. [6], [8], [12], [15]). The above theorem represents an enhancement of our knowledge in this area.

Besides this general motivation we mention the following particular application. Recently Polishchuk and Schwartz [11] constructed a category \(\mathcal{C}\) of holomorphic vector bundles on a non-commutative 2-torus. Polishchuk subsequently showed that \(\mathcal{C}\) is derived equivalent to the category of coherent sheaves on an elliptic curve [11]. Part of Polishchuk’s proof amounts to establishing the highly non-trivial fact that \(\mathcal{C}\) is 1-Calabi-Yau [11, Cor 2.12]. Once one knows this, one could now finish the proof by simply invoking Theorem 1.1 (with \(A\) being a suitable abelian hull of \(\mathcal{C}\)).

We briefly outline some steps in the proof of Theorem 1.1. Some of our tools come from representation theory of algebras and non-commutative algebraic geometry. Other tools were already employed by Polishchuk, but are now used in a more abstract setting.

Fix a connected abelian 1-Calabi-Yau category \(A\). First, we prove the existence of endo-simple objects in \(A\), i.e. objects \(X \in A\) such that \(\text{End} X \cong k\). Associated to such objects there are twist functors \([14] T_A, T_A^*\). These functors are mutually inverse auto-equivalences of \(D^b(A)\) which on objects take the values \(T_X Y = \text{cone}(X \otimes \text{RHom}(X,Y) \to Y)\) and \(T_X^* Y = \text{cone}(Y[-1] \to X[-1] \otimes \text{RHom}(Y,X)^*)\).

Using twist functors we establish various useful facts. Most notably, we prove that the subcategory of endo-simple objects in \(A\) has no cycles of non-zero maps (Proposition 3.3) and hence is ordered. We also show that all Auslander-Reiten components of \(A\) are homogeneous tubes based on endo-simple objects (Proposition 3.5).

We may assume that \(A\) has at least two non-isomorphic endo-simple objects as the remaining case is easily disposed with. Using connectedness and the results mentioned in the previous paragraph we may in fact select non-isomorphic endo-simple objects \(E\) and \(B\) such that \(\text{Hom}(E,B) \neq 0\). After doing so we consider the sequence of objects \(\mathcal{E} = (T^n_B E)_{n \in \mathbb{Z}}\) in \(D^b A\). We construct a certain associated \(t\)-structure on \(D^b(A)\) with heart \(\mathcal{H}\) such that \(\mathcal{E}\) is an ample sequence in the sense of [10] in \(\mathcal{H}\). Hence \(\mathcal{E}\) defines a finitely presented graded coherent algebra \(A\) such that \(\mathcal{H}\) is equivalent to the category \(\text{qgr}(A)\) of finitely presented graded \(A\)-modules modulo finite dimensional ones.
We then show that $A$ is a domain of Gelfand-Kirillov dimension two and we invoke the celebrated Artin and Stafford classification theorem \cite{ArtinStafford} which shows that $\text{qgr}(A)$ is of the form $\text{coh}(X)$ for a projective curve $X$. Since $\mathcal{H}$ is 1-Calabi-Yau this implies that $X$ must be an elliptic curve, finishing the proof.

It is not hard to describe the abelian 1-Calabi-Yau categories that occur within the derived equivalence classes in Theorem \cite{1Calabi} (see e.g. \cite{Calabi}). We discuss this using the language of this paper in \cite{1Calabi}

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2. Preliminaries

Throughout this paper, fix an algebraically closed field $k$ of arbitrary characteristic. All algebras and categories are assumed to be $k$-linear.

We will also assume all abelian categories are connected in the sense that between two indecomposable objects there is an unoriented path of non-zero maps between indecomposables.

If $\mathcal{A}$ is abelian, we write $D^b \mathcal{A}$ for the bounded derived category of $\mathcal{A}$. The category $D^b \mathcal{A}$ has the structure of a triangulated category. Whenever we use the word "triangle" we mean "distinguished triangle".

An abelian or triangulated category $\mathcal{A}$ is Ext-finite if for all objects $X, Y \in \text{Ob}(\mathcal{A})$ one has that $\dim_k \text{Ext}^i(X, Y) < \infty$ for all $i \in \mathbb{N}$. We say that $\mathcal{A}$ is hereditary if $\text{Ext}^i(X, Y) = 0$ for all $i \geq 2$.

2.1. Serre duality. If $\mathcal{C}$ is triangulated category, we will say that $\mathcal{C}$ satisfies Serre duality if there exists an auto-equivalence $F : \mathcal{C} \to \mathcal{C}$, called the Serre functor, such that, for all $X, Y \in \text{Ob}\mathcal{C}$, there is an isomorphism

$$\text{Hom}_\mathcal{C}(X, Y) \cong \text{Hom}_\mathcal{C}(Y, FX)^*$$

which is natural in $X$ and $Y$ and where $(-)^*$ denotes the vector space dual.

We will say an abelian category $\mathcal{A}$ has Serre duality if the category $D^b \mathcal{A}$ has a Serre functor. It has been proven in \cite{Calabi} that an abelian category $\mathcal{A}$ without non-zero projectives has a Serre functor if and only if the category $D^b \mathcal{A}$ has Auslander-Reiten triangles. In this case the action of the Serre functor on objects coincides with $\tau[1]$, where $\tau$ is the Auslander-Reiten translation.

2.2. Calabi-Yau categories. Let $\mathcal{A}$ be an Ext-finite abelian category with Serre duality. We will say $\mathcal{A}$ is Calabi-Yau of dimension $n$ or shorter that $\mathcal{A}$ is $n$-Calabi-Yau if $F \cong [n]$ for a certain $n \in \mathbb{N}$, thus if the $n^{th}$ shift is a Serre functor. We write $\text{CYdim} \mathcal{A} = n$.

The following well-known property relates the Calabi-Yau dimension and the homological dimension.

**Proposition 2.1.** Let $\mathcal{A}$ be an abelian Calabi-Yau category. Then $\text{CYdim} \mathcal{A} = \text{gl dim} \mathcal{A}$.

**Proof.** Let $n = \text{CYdim} \mathcal{A}$, then for every $X \in \text{Ob} \mathcal{A}$ we have $\text{Hom}(X, X) \cong \text{Ext}^n(X, X)^*$. Since the former is non-zero, we see $\text{CYdim} \mathcal{A} \leq \text{gl dim} \mathcal{A}$.

Let $i \in \mathbb{N}$ and $X, Y \in \text{Ob} \mathcal{A}$ be chosen such that $\text{Ext}^i(X, Y) \neq 0$. Using the Calabi-Yau property, we find $\text{Ext}^i(X, Y) \cong \text{Ext}^{n-i}(Y, X)$, hence $n \geq i$. We find $\text{CYdim} \mathcal{A} \geq \text{gl dim} \mathcal{A}$.

In particular, if $\mathcal{A}$ is a 1-Calabi-Yau category, then $\mathcal{A}$ is hereditary. Since $F \cong [1]$ and $F$ coincides with $\tau[1]$ on indecomposables of $\mathcal{A}$, it follows that $\tau$ is naturally isomorphic to the identity functor on $\mathcal{A}$ and hence $\text{Hom}_\mathcal{A}(X, Y) \cong \text{Ext}_\mathcal{A}(Y, X)^*$, for all objects $X, Y \in \mathcal{A}$.

2.3. Twist functors. Let $\mathcal{A}$ be an abelian 1-Calabi-Yau category. For an object $A \in D^b \mathcal{A}$, we may consider the twist functors, $T_A$ and $T^*_A$, in $D^b \mathcal{A}$ whose values on objects are up to isomorphism characterized by the following triangles

$$T_A X[-1] \to A \otimes \text{RHom}(A, X) \overset{\epsilon}{\to} X \to T_A X$$

and

$$T^*_A X \to X \overset{\epsilon^*}{\to} A \otimes \text{RHom}(X, A)^* \to T^*_A X[1]$$

where $\epsilon : A \otimes \text{RHom}(A, X) \to X$ and $\epsilon^* : X \to A \otimes \text{RHom}(X, A)^*$ are the canonical morphisms.
Let $S$ be an endo-simple object, i.e. $\text{End}(S) \cong k$. Since $\mathcal{A}$ is 1-Calabi-Yau, we know from [14, Proposition 2.10] that $T_S$ and $T_S^*$ are inverses. In particular, they are autoequivalences.

2.4. Ample sequences. For the benefit of the reader, we will recall some definitions and results from [10] which will be used in the rest of this paper. Throughout, let $\mathcal{A}$ be a Hom-finite abelian category.

We begin with the definition of ample sequences.

(1) A sequence $\mathcal{E} = (E_i)_{i \in \mathbb{Z}}$ is called projective if for every epimorphism $X \to Y$ in $\mathcal{A}$ there is an $n \in \mathbb{Z}$ such that $\text{Hom}(E_i, X) \to \text{Hom}(E_i, Y)$ is surjective for $i < n$.

(2) A projective sequence $\mathcal{E} = (E_i)_{i \in \mathbb{Z}}$ is called coherent if for every $X \in \text{Ob}\mathcal{A}$ and $n \in \mathbb{Z}$, there are integers $i_1, \ldots, i_s \leq n$ such that the canonical map

$$\bigoplus_{j=1}^s \text{Hom}(E_{i_j}, \text{Hom}(E_{i_j}, X)) \to \text{Hom}(E_i, X)$$

is surjective for $i < 0$.

(3) A coherent sequence $\mathcal{E} = (E_i)_{i \in \mathbb{Z}}$ is ample if for all $X \in \mathcal{A}$ the map $\text{Hom}(E_i, X) \neq 0$ for $i < 0$.

Let $A_{ij} = \text{Hom}(E_i, E_j)$ for $i \leq j$. We may define an algebra $A = A(\mathcal{E}) = \oplus_{i \leq j} A_{ij}$ in a natural way. If $A_{ii} \cong k$, then $A$ is a coherent $\mathbb{Z}$-algebra in the sense of [10] (see [10, Proposition 2.3]).

We will refer to the right $A$-modules having a resolution by finitely generated projectives as coherent modules. These modules form an abelian category, $\text{coh} A$, and the finite dimensional modules form a Serre subcategory denoted by $\text{coh}^b A$. We define the quotient

$$\text{cohproj} A \cong \text{coh} A/\text{coh}^b A.$$ 

We may use this to give a description of Ext-finite abelian categories with an ample sequence.

Theorem 2.2. [10, Theorem 2.4] Let $\mathcal{E} = (E_i)$ be an ample sequence, $A = A(\mathcal{E})$ the corresponding $\mathbb{Z}$-algebra, then there is an equivalence of categories $\mathcal{A} \cong \text{cohproj} A$.

We will be interested in the special case where there is an automorphism $t : D^b \mathcal{A} \to D^b \mathcal{A}$ such that $E_i \cong t^i E$. We let $R = R(\mathcal{E}) = \oplus_{i \in \mathbb{N}} R_i$ where $R_i = \text{Hom}(E_i, t^i E)$ and make it into a $\mathbb{Z}$-graded algebra in an obvious way.

If $R$ is noetherian then the coherent $R$-modules correspond to the finitely generated ones and $\text{cohproj} R$ corresponds to $\text{qgr} R$, the finitely generated modules modulo the finite dimensional ones.

We will use following corollary of Theorem 2.2.

Corollary 2.3. Let $A$ be a Hom-finite abelian category, $t$ be an autoequivalence of $\mathcal{A}$ and $E$ an object of $\mathcal{A}$. If $\mathcal{E} = (t^i E)$ is an ample sequence and the corresponding graded algebra $R = R(\mathcal{E})$ is noetherian, then $\mathcal{A} \cong \text{qgr} R$.

2.5. t-structures. In order to find derived equivalent categories, we will use the theory of t-structures [3].

Definition 2.4. A t-structure on a triangulated category $\mathcal{C}$ is a pair $(D^{\geq 0}, D^{\leq 0})$ of non-zero full subcategories of $\mathcal{C}$ satisfying the following conditions, where we denote $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$

(1) $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 1} \subseteq D^{\geq 0}$

(2) $\text{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$

(3) $\forall Y \in \mathcal{C}$, there exists a triangle $X \to Y \to Z \to X[1]$ with $X \in D^{\leq 0}$ and $Z \in D^{\geq 1}$.

Furthermore, we will say the t-structure is bounded if $\bigcap_{n} D^{\leq n} = \bigcap_{n} D^{\geq n} = \{0\}$.

We will say a t-structure is split if all triangles in [3] are split, or equivalently, if $\text{ind} \mathcal{C} = \text{ind} D^{\geq 1} \cup \text{ind} D^{\leq 0}$. We have following result.

Theorem 2.5. [3] Let $\mathcal{A}$ be an abelian category and let $(D^{\geq 0}, D^{\leq 0})$ be a bounded t-structure on $D^b \mathcal{A}$. Then the heart $\mathcal{H}$ is hereditary if and only if $(D^{\geq 0}, D^{\leq 0})$ is a split t-structure. In this case, $\mathcal{A}$ and $\mathcal{H}$ are derived equivalent.
2.6. **Elliptic curves.** For the benefit of the reader, we recall certain properties of the category of coherent sheaves on an elliptic curve $X$. This category has first been described in [2].

An elliptic curve is a curve of genus 1 and thus, in particular, $\mathcal{A} = \text{coh}\, X$ is a 1-Calabi-Yau category.

Let $\mathcal{O}$ be the structure sheaf and, for a point $P$, let $k(P)$ be a torsion sheaf. For a coherent sheaf $\mathcal{E}$ the degree and rank may be defined as

$$
\text{deg} \, \mathcal{E} = \chi(\mathcal{O}, \mathcal{E})
$$

$$
\text{rk} \, \mathcal{E} = \chi(\mathcal{E}, k(P)),
$$

respectively. It follows from the Riemann-Roch theorem that

$$(1) \quad \chi(\mathcal{E}, \mathcal{F}) = \text{deg} \, \mathcal{F} \, \text{rk} \, \mathcal{E} - \text{deg} \, \mathcal{E} \, \text{rk} \, \mathcal{F}.$$  

Furthermore, the *slope* of $\mathcal{E}$ is defined as $\mu(\mathcal{E}) = \frac{\text{deg} \, \mathcal{E}}{\text{rk} \, \mathcal{E}} \in \mathbb{Q} \cup \{\infty\}$. A coherent sheaf $\mathcal{F}$ is called *stable* or *semi-stable* if for every short exact sequence $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ we have $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$ or $\mu(\mathcal{E}) < \mu(\mathcal{F})$, respectively.

It is well-known that all indecomposable coherent sheaves are semi-stable. For stable sheaves, we have the following equivalent conditions

(1) $\mathcal{E}$ is stable,

(2) $\mathcal{E}$ is endo-simple, i.e. $\text{End}(\mathcal{E}) \cong k$,

(3) $\text{rk} \, \mathcal{E}$ and $\text{deg} \, \mathcal{E}$ are coprime.

Every semi-stable sheaf is a finite extension of an endo-simple one with itself. We may visualise this via the Auslander-Reiten quiver of $\text{coh}\, X$. All Auslander-Reiten components are homogeneous tubes, i.e. components of the form $\mathbb{Z}A_\infty/(\tau)$, cfr. Figure 1, where the bottom element is a stable sheaf.

Every such tube corresponds to an abelian subcategory of $\text{coh}\, X$ equivalent to $\text{Mod}^{fd} k[[t]]$ and all indecomposable objects in the same homogeneous tube have the same slope. Thus the full subcategory of $\text{coh}\, X$ spanned by all indecomposable objects of a given slope is an abelian subcategory of $\text{coh}\, X$ and is of the form $\oplus \text{Mod}^{fd} k[[t]]$, where the sum is indexed over the stable objects with the given slope.

Finally, it follows directly from (1) that, for non-isomorphic stable sheaves, $\mathcal{E}$ and $\mathcal{F}$, we have $\text{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$ if and only if $\mu(\mathcal{E}) < \mu(\mathcal{F})$. Thus for semi-stable sheaves $\mathcal{E}'$ and $\mathcal{F}'$ we have $\text{Hom}(\mathcal{E}', \mathcal{F}') \neq 0$ if and only if $\mu(\mathcal{E}') < \mu(\mathcal{F}')$ or $\mathcal{E}'$ and $\mathcal{F}'$ lie in the same tube.

### 3. **Endo-simple objects**

Let $\mathcal{A}$ be a connected $k$-linear abelian 1-Calabi-Yau category. It will turn out that the endo-simple objects are the building blocks of $\mathcal{A}$. Therefore, in this section, we will give some properties of endo-simple objects. Recall that $X$ is an endo-simple object if $\text{End} \, X \cong k$. It follows from the Calabi-Yau property that every endo-simple object is 1-spherical in the sense of [14].
Proposition 3.1. Let $C$ be a Hom-finite abelian category. For every object $X \in \text{Ob}_C$ there exists an endo-simple object occurring both as subobject and quotient object of $X$. In particular, $C$ has an endo-simple object.

Proof. Assume $X$ is not endo-simple and let $f : X \to X$ be a non-invertible endomorphism. We show that $\dim \text{End} I < \dim \text{End} X$ where $I = \text{im} f$.

Indeed, since we have an epimorphism $X \to I$ and monomorphism $I \to X$, we get a composition of monomorphisms $\text{Hom}(I, I) \to \text{Hom}(X, I) \to \text{Hom}(X, X)$. Since the image of this composition has to be in $\text{rad}(X, X)$, we know $\dim \text{Hom}(I, I) < \dim \text{Hom}(X, X)$. Iteration finishes the proof. □

Proposition 3.2. Let $S$ be an endo-simple object and $X \in \text{ind} \mathcal{A}$. Each of the canonical maps $S \otimes \text{Hom}(S, X) \to X$ and $X \to S \otimes \text{Hom}(X, S)^*$ is either a monomorphism or an epimorphism. If $\text{Hom}(X, S) \neq 0$, then the first map is a monomorphism. If $\text{Hom}(S, X) \neq 0$, then the latter is an epimorphism.

Proof. Consider in the derived category $D^b \mathcal{A}$ the twist functor $T_S$ characterized by

$$T_S X[-1] \to S \otimes R\text{Hom}(S, X) \to X \to T_S X.$$ 

It is shown in [14] that this is an equivalence. Applying the homological functor $H^0$ gives the long exact sequence

$$0 \to H^{-1}(T_S X) \to S \otimes \text{Hom}(S, X) \xrightarrow{H^0} X \to H^0(T_S X) \to S \otimes \text{Ext}(S, X) \to 0.$$ 

Since $X$ is indecomposable and $T_S$ is an equivalence, either $H^{-1}(T_S X)$ or $H^0(T_S X)$ is zero, hence $H^0$ is a monomorphism or an epimorphism, respectively.

If we assume furthermore $\text{Hom}(X, S) \neq 0$, and hence by the Calabi-Yau property $\text{Ext}(S, X) \neq 0$, we find $H^0(T_S X) \neq 0$. Hence $H^{-1}(T_S X) = 0$ and the canonical map $S \otimes \text{Hom}(S, X) \to X$ is a monomorphism.

The other case is dual. □

Proposition 3.3. The subcategory of endo-simples is a directed category.

Proof. Let $S_0 \to S_1 \to \cdots \to S_n \to S_0$ be a cycle of non-zero non-isomorphisms between endo-simple objects. We will assume $n$ is minimal with the property that such a cycle exists.

By Proposition 3.2 we know the canonical map $\epsilon : S_0 \otimes \text{Hom}(S_0, S_1) \to S_1$ is either a monomorphism or an epimorphism. If $\epsilon$ is a monomorphism, then we know the composition

$$S_n \otimes \text{Hom}(S_0, S_1) \to S_0 \otimes \text{Hom}(S_0, S_1) \xrightarrow{\epsilon} S_1$$

is non-zero. This induces a non-zero morphism $f : S_n \to S_1$. Since $f$ factors through $S_0 \otimes \text{Hom}(S_0, S_1)$, we know $f$ is not invertible.

Likewise, if $\epsilon$ is an epimorphism, we find a non-zero non-invertible morphism $S_0 \to S_2$. In both cases we have found a shorter cycle, contradicting with the minimality of $n$. □

We now wish to show that every object has a composition series with endo-simple quotients. Even more so, every indecomposable object has a composition series in which only one isomorphism class of endo-simple objects occur. We start with a lemma.

Lemma 3.4. Let $X \in \text{ind} \mathcal{A}$ such that the endo-simple object $S$ occurs both as subobject and quotient object of $X$. If $C = \text{coker}(S \otimes \text{Hom}(S, X) \to X)$ is not zero, then $S$ occurs as both subobject and quotient object of every direct summand of $C$.

Proof. Assume $C \neq 0$. Consider the exact sequence

$$0 \to S \otimes \text{Hom}(S, X) \to X \to H^0(T_S X) \to S \otimes \text{Ext}(S, X) \to 0.$$ 

from the proof of Proposition 3.2. We may splice this as

$$0 \to S \otimes \text{Hom}(S, X) \to X \to C \to 0$$

and

$$0 \to C \to H^0(T_S X) \to S \otimes \text{Ext}(S, X) \to 0.$$
Since $T_S$ is an automorphism and $X$ is indecomposable, we know $H^0(T_S X)$ is indecomposable. It now follows directly from [13, Lemma 2*] that $\text{Hom}(S, C_1) \cong \text{Ext}(C_1, S)^* \neq 0$ and $\text{Hom}(C_1, S) \cong \text{Ext}(S, C_1)^* \neq 0$ for every direct summand $C_1$ of $C$. Proposition 3.2 now yields that $S$ is both a subobject and quotient object of every direct summand of $C$. \hfill $\Box$

**Proposition 3.5.** Every indecomposable object is obtained by repeatedly extending a given endo-simple with itself.

**Proof.** Let $S$ be an endo-simple object and denote by $A_S$ the full subcategory of $A$ spanned by the objects $Z$ which can be obtained from $S$ by taking a finite amount of extensions with itself. The number of such extensions needed, will be denoted by $l_S(Z)$, and we will refer to it as the length of $Z$.

Since $A_S$ is a hereditary category with a unique simple $S$ such that $\dim \text{Ext}(S, S) = 1$, it follows easily that $A_S$ is equivalent to the finite dimensional representations of $k[[t]]$.

We will prove that if $X$ is an indecomposable object of $A$ such that $S$ occurs both as quotient and subobject, then $X \in A_S$. Note that by Proposition 3.1 we may assume such an $S$ exists.

For every subobject $A$ of $X$ in $A_S$ and quotient object $B$ of $X$ in $A_S$, we have

$$\dim \text{End}_A X \geq \min(l_S(A), l_S(B)),$$

thus we may deduce either the length of such subobjects or the length of such quotient objects is bounded. Assume that the length of $A$ is bounded, the other case is dual.

We will now construct in $A_S$ an ascending sequence of subobjects of $X$. Let $A_0 = S \otimes \text{Hom}(S, X)$ and denote $C_0 = \text{coker}(S \otimes \text{Hom}(S, X) \rightarrow X)$. We will assume $C_0 \neq 0$.

We choose a decomposition $C_0 \cong X_1 \oplus C_0'$ where $X_1$ is indecomposable, hence by Lemma 3.4 we know $S$ occurs both as subobject and as quotient object of $X_1$ and of every direct summand of $C_0'$. Consider the following diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
0 & \rightarrow & A_0 & \rightarrow & A_1 & \rightarrow & S \otimes \text{Hom}(S, X_1) & \rightarrow & 0 \\
0 & \rightarrow & X & \rightarrow & X_1 \oplus C_0' & \rightarrow & 0 \\
& & C_1 \oplus C_0' & & C_1 \oplus C_0' & & \\
& & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}$$

It follows from Lemma 3.4 that $S$ occurs both as subobject and quotient of every indecomposable of $C_1 \oplus C_0'$ where $C_1 = \text{coker}(S \otimes \text{Hom}(S, X_1) \rightarrow X_1)$. Hence, using $C_0 \neq 0$, we have found a subobject $A_1 \in A_S$ of $X$ such that $l_S(A_0) < l_S(A_1)$. Iteration and using that the length is bounded, we see that $X \in A_S$. \hfill $\Box$

**Remark 3.6.** It follows from previous proposition that all Auslander-Reiten components of $A$ are homogeneous tubes, i.e. they are of the form $\mathbb{Z}A_\infty/\langle \tau \rangle$, cfr. Figure 1 were the bottom element is endo-simple.

Finally, we will formulate a useful corollary.

**Corollary 3.7.** Every cycle $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow X_0$ of non-zero non-isomorphisms between indecomposable objects belongs to a single homogeneous tube.

**Proof.** Directly from Propositions 3.3 and 3.5 \hfill $\Box$
Remark 3.8.  It follows that the set of homogeneous tubes of the category $\mathcal{A}$ are directed, thus there can be no cycle containing two objects from different homogeneous tubes.

4. Classification

Let $\mathcal{A}$ be a connected $k$-linear abelian Ext-finite 1-Calabi-Yau category. In this section, we wish to classify all such categories up to derived equivalence. If every two endo-simples of $\mathcal{A}$ are isomorphic, then $\mathcal{A}$ is equivalent to the finite dimensional nilpotent representations of the one loop quiver.

So, assume there are at least two non-isomorphic endo-simples, $E$ and $B$. By connectedness and Proposition 3.5, we assume $\text{Hom}(E, B) \neq 0$. First, we will find a $t$-structure in $D^b \mathcal{A}$ such that the heart $\mathcal{H}$ admits an ample sequence $\mathcal{E}$. Then we will use Theorem 2.2 to show $\mathcal{A} \cong \text{qgr } R(\mathcal{E})$. A discussion of $R(\mathcal{E})$ will then complete the classification of abelian 1-Calabi-Yau categories up to derived equivalence.

From here on, we will always denote $\text{Hom}(E, B)$ by $V$ and its dimension by $d$.

4.1. The sequence $\mathcal{E}$ and a $t$-structure in $D^b \mathcal{A}$. With $E$ and $B$ as above, associate the autoequivalence $t = T_B : D^b \mathcal{A} \to D^b \mathcal{A}$ and the sequence $\mathcal{E} = (E_i)$ where $E_i = t^i E$.

The following will define a $t$-structure in $\mathcal{C}$ with a hereditary heart $\mathcal{H}$.

\[
\text{ind } D^{\leq 0} = \{X \in \text{ind } \mathcal{C} \mid \text{there is a path from } E_i \text{ to } X, \text{ for an } i \in \mathbb{Z}\}
\]
\[
\text{ind } D^{> 1} = \text{ind } \mathcal{C} \setminus \text{ind } D^{\leq 0}
\]

If it follows directly from this definition that $t$ restricts to an autoequivalence on $\mathcal{H}$, which we will also denote by $t$. Note that this implies $E_i \in \text{Ob } \mathcal{H}$, for all $i \in \mathbb{Z}$. Also, since $\text{Hom}(B[-1], E_i) \neq 0$, there is no path from $E_i$ to $B[-1]$ and hence we have $B \in \text{Ob } \mathcal{H}$.

It follows from Theorem 2.2 that $\mathcal{H}$ is hereditary and $D^b \mathcal{H} \cong D^b \mathcal{A}$. Since $\mathcal{H}$ is a 1-Calabi-Yau category, the results we have proved about $\mathcal{A}$ apply to $\mathcal{H}$ as well.

Note that, since $t^i B \cong B$, we find there is a natural isomorphism $\text{Hom}(E, B) \cong \text{Hom}(E_i, B)$ and as such, we get triangles of the form $B[-1] \otimes V^* \to E_{i-1} \to E_i \to B \otimes V^*$. Such a triangle in $D^b \mathcal{A}$ gives rise to an exact sequence

\[
0 \to E_{i-1} \to E_i \to B \otimes V^* \to 0
\]

in $\mathcal{H}$, which is the universal extension of $E_{i-1}$ with $B$ and all these exact sequences lie in the same $t$-orbit.

Using Proposition 3.3, we may prove following easy lemma.

Lemma 4.1. Let $\mathcal{E} = (E_i)_{i \in I}$ and $B$ as above, then

1. $\text{Hom}(E_i, E_j) = \text{Ext}(E_j, E_i) = 0$ for $i > j$,
2. $\text{Hom}(B, E_i) = \text{Ext}(B, E_i) = 0$ for all $i \in I$.

If $\mathcal{H}$ is of the form coh $X$ for an elliptic curve $X$ (which we will show below to be the case) one may verify that $E$ corresponds to a stable vector bundle of rank $\dim V$ and $B$ to the structure sheaf $k(P)$ of a point $P$. The $E_i$ are equal to $E(-iP)$.

4.2. $\mathcal{E}$ is an ample sequence in $\mathcal{H}$. We now wish to show the sequence $\mathcal{E} = (E_i)_{i \in \mathbb{Z}}$ is ample. The following lemma will be useful.

Lemma 4.2. If $\text{Hom}(E_i, X) \neq 0$, then $\text{Hom}(E_j, X) \neq 0$ for all $j \leq i$.

Proof. It suffices to show that $\text{Hom}(E_{i-1}, X) \neq 0$. Since $\text{Hom}(E_i, X) \neq 0$ and $t$ is an autoequivalence, we know $\text{Hom}(E_{i-1}, t^{-1} X) \neq 0$. Applying the functor $\text{Hom}(E_{i-1}, -)$ to the exact sequence

\[
0 \to t^{-1} X \to X \to B \otimes \text{Hom}(X, B)^* \to 0
\]

yields $\text{Hom}(E_{i-1}, X) \neq 0$. \hfill \Box

Proposition 4.3. In $\mathcal{H}$ the sequence $\mathcal{E} = (E_i)$ is ample.
Proof. First, we will show $\mathcal{E}$ is projective. Therefore, let $X \to Y$ be an epimorphism and let $K$ be the kernel. By the construction of $\mathcal{H}$ in [4.1] we know there are paths from the sequence $\mathcal{E}$ to every direct summand of $K$. Hence, by Corollary [3.7] we know $\text{Hom}(K, E_i) = 0$ for $i < 0$ and, by the Calabi-Yau property, $\text{Ext}(E_i, K) = 0$. Thus $\text{Hom}(E_i, X) \to \text{Hom}(E_i, Y)$ is surjective for $i < n$.

Next, we will show $\mathcal{E}$ is coherent. Thus we consider an object $X \in \mathcal{H}$ and we may assume there is a $j \in \mathbb{Z}$ such that $\text{Hom}(E_{j+2}, X) \neq 0$, and hence by Lemma [3.2] that $\text{Hom}(E_i, X) \neq 0$ for all $i < j + 2$. Fix an $i < j$, we will prove that $f : E_{i-1} \to X$ factors through $E_i \oplus E_j$. Iteration then implies $f$ factors through a number of copies of $E_{j-1} \oplus E_j$, and hence $\mathcal{E}$ is coherent.

To prove previous claim, it will be convenient to work in the derived category. The following two triangles in $\text{D}^b \mathcal{H}$ will be used

\begin{equation}
B \otimes V^*[-1] \xrightarrow{\theta} E_{i-1} \to E_i \to B \otimes V^*
\end{equation}

and

\begin{equation}
B \otimes V^*[-1] \xrightarrow{\varphi} E_j \to E_{j+1} \to B \otimes V^*
\end{equation}

where $V = \text{Hom}(E_i, B) \cong \text{Hom}(E_{j+1}, B)$. We may assume $f : E_{i-1} \to X$ does not factor though $E_i$, hence from triangle (2) it follows that the composition $f \circ \theta \neq 0$.

Note that, since $\text{Hom}(E_{j+1}, X) \neq 0$, we may use Corollary [3.7] to see $\text{Hom}(X, E_{j+1}) = 0$, and hence also $\text{Ext}(E_{j+1}, X) = 0$.

Applying the functor $\text{Hom}(-, X)$ on triangle (3) and using $\text{Ext}(E_{j+1}, X) = 0$, shows that every map $B \otimes V^*[-1] \to X$ factors though $\varphi$. Hence there is a morphism $g : E_j \to X$ such that the following diagram commutes.

\begin{equation}
\begin{array}{ccc}
B \otimes V^*[-1] & \xrightarrow{\theta} & E_{i-1} \\
\downarrow \varphi & & \downarrow f \\
E_j & \xrightarrow{g} & X
\end{array}
\end{equation}

Furthermore, applying $\text{Hom}(-, E_j)$ to triangle (2) yields that $\varphi$ factors through $\theta$, hence there is a map $h : E_{i-1} \to E_j$ such that $g \circ h \circ \theta = f \circ \theta$, or $(g \circ h - f) \circ \theta = 0$.

Summarizing, $f = g \circ h + f ' $, where $f ' : E_{i-1} \to X$ lies in $\text{ker}(\theta, X)$ and as such factors through $E_i$. The map $f$ factors though $E_i \oplus E_j$ and we may conclude the sequence $\mathcal{E}$ is coherent.

Finally, we show the sequence $\mathcal{E}$ is ample. Let $X$ be an indecomposable object. Due to the construction of $\mathcal{H}$, we know that there is an oriented path from $E_n$ to $X$, for a certain $n \in \mathbb{Z}$. Thus it suffices to prove that if $\text{Hom}(E_n, X) \neq 0$, then there is a finite set $I \subset \mathbb{Z}$ such that $\bigoplus_{i \in I} E_i \otimes \text{Hom}(E_i, X) \to X$ is an epimorphism.

Let $i_1, \ldots, i_m \in \mathbb{Z}$ be as in the definition of coherence. Consider the map

\begin{equation}
\theta : \bigoplus_{j=1}^m E_{i_j} \otimes \text{Hom}(E_{i_j}, X) \to X
\end{equation}

and let $C = \text{coker} \theta$. To ease notation, we will refer to the domain of $\theta$ by $\text{dom} \theta$.

There is an exact sequence $0 \to \text{im} \theta \to X \to C \to 0$. Using the Calabi-Yau property, we see $\text{Hom}(\text{im} \theta, C) \neq 0$, and since $\text{im} \theta$ is a quotient object of $\text{dom} \theta$, this yields $\text{Hom}(\text{dom} \theta, C) \neq 0$. Hence we may assume there is an $i_j$ such that $\text{Hom}(E_{i_j}, C) \neq 0$.

Since $\mathcal{E}$ is projective, there is an $l << 0$ such that the induced map in $\text{Hom}(E_l, C)$ lifts to a map in $\text{Hom}(E_l, X)$. Again using coherence, this map should factor through $\text{dom} \theta$. We may conclude $C = 0$, and hence $\theta$ is an epimorphism. \qed
4.3. Description of \( R = R(\mathcal{E}) \). Having shown in Proposition 4.3 that \( \mathcal{E} \) is an ample sequence, we may invoke Proposition 2.2 to see the that \( \mathcal{H} \cong \text{cohproj} A(\mathcal{E}) \).

We will now proceed to discuss the graded algebra \( R = R(\mathcal{E}) \). In particular, we wish to show \( R \) is a finitely generated domain of Gelfand-Kirillov dimension 2 which admits a Veronese subalgebra generated in degree one. It would then follow from [1] that \( R \) is noetherian and that \( qgr R \) is equivalent to \( \text{coh} X \) where \( X \) is a curve, while it would follow from Corollary 2.3 that \( \mathcal{H} \cong qgr R \).

We start by showing \( \text{GKdim} R = 2 \).

Lemma 4.4. Let \( \mathcal{E} = (E_i)_{i \in I} \) and \( B \) be as before. If \( j > i \), then
\[
\dim \text{Hom}(E_i, E_j) = (j - i)d^2
\]
where \( d = \dim \text{Hom}(E_0, B) \).

Proof. We apply \( \text{Hom}(E_i, -) \) to the short exact sequence
\[
0 \rightarrow E_{j-1} \rightarrow E_j \rightarrow B \otimes \text{Hom}(E_0, B)^* \rightarrow 0.
\]
We will proceed by induction on \( j > i \). Note that \( \dim \text{Hom}(E_i, B) = \dim \text{Hom}(E_0, B)^* = d \) and Lemma 4.1 implies that \( \text{Ext}(E_i, E_i) = 0 \).

If \( j = i+1 \), then it follows from \( \dim \text{Hom}(E_i, E_i) = \dim \text{Ext}(E_i, E_i) = 1 \) that \( \dim \text{Hom}(E_i, E_j) = d^2 \). For higher \( j \), we find by induction \( \dim \text{Hom}(E_i, E_j) = (j - i)d^2 \).

Lemma 4.5. Assume \( E \) and \( B \) are non-isomorphic endo-simple objects of \( \text{D}^b A \) chosen such that \( d = \dim \text{Hom}_{\text{D}^b A}(E, B) \) is minimal and \( d \neq 0 \). Then \( R \) is a domain.

Proof. It suffices to show every non-zero non-isomorphism \( f : E_0 \rightarrow E_i \) is a monomorphism. We will prove this by induction on \( i \). The case \( i = 0 \) is trivial. So let \( i \geq 1 \).

Since \( f \) is a quotient object of \( E_0 \) and \( \dim \text{Hom}(E, B) = d \), we see that \( \dim \text{Hom}(\text{im} f, B) \leq d \), and due to the minimality of \( d \), we know that either \( \dim \text{Hom}(\text{im} f, B) = 0 \), or \( \dim \text{Hom}(\text{im} f, B) = d \) and \( \text{im} f \) is an endo-simple object.

If \( \dim \text{Hom}(\text{im} f, B) = 0 \), the inclusion \( \text{im} f \hookrightarrow E_i \) has to factor through a map \( j : \text{im} f \rightarrow E_{i-1} \).

Composition gives a non-zero map \( E_0 \rightarrow E_{i-1} \) which is a monomorphism by the induction hypothesis. We conclude that \( f \) is a monomorphism.

We are left with \( \dim \text{Hom}(\text{im} f, B) = d \), and hence \( \dim \text{Hom}(K, B) = 0 \) where \( K = \ker f \). With \( \mathcal{E} \) being ample, we may assume there is a \( k \in \mathbb{Z} \), such that \( E_k \) maps non-zero to every direct summand of \( K \). Using the exact sequence \( 0 \rightarrow E_k \rightarrow E_{k+1} \rightarrow B \otimes \text{Hom}(E_{k+1}, B)^* \rightarrow 0 \), we find that for every \( l \in \mathbb{Z}, E_l \) maps non-zero to every direct summand of \( K \). Hence \( \text{Hom}(K, E_l) = 0 \) and thus \( K = 0 \). We conclude that \( f \) is a monomorphism.

In general, however, \( R \) will not be generated in degree 1. We show that the Veronese subalgebra \( R(3) = \oplus_k R_{3k} \) of \( R \) is generated in degree 1.

Lemma 4.6. The sequence \( \mathcal{E}^{(3)} = (E_{3k})_{k \in \mathbb{Z}} \) is an ample sequence. Furthermore \( R(3) = R(\mathcal{E}^{(3)}) \) is generated in degree 1.

Proof. The sequence \( \mathcal{E}^{(3)} \) is projective and ample since \( \mathcal{E} \) is. Coherence of \( \mathcal{E}^{(3)} \) may be shown as in the proof of Proposition 4.3.
Next, we prove $R^{(3)}$ is generated in degree one. Therefore, it suffices to show that for every $k > 1$ every map $E_0 \to E_{3k}$ factors through the canonical map $\theta : E_0 \to E_3 \otimes \Hom(E_0, E_3)^*$. We write $V = \Hom(E_0, E_3)$ and we consider the triangle

$$\begin{array}{ccc}
C & \xrightarrow{a} & E_3 \otimes V^* \\
\downarrow & & \downarrow \\
E_0 & \xrightarrow{\theta} & C[1]
\end{array}$$

where $C = T_{E_3}E_0$ is an endo-simple object since $T_{E_3}$ is an automorphism. Applying the functor $\Hom(-, E_{3k})$ to this triangle gives the exact sequence

$$0 \to \Hom(C[1], E_{3k}) \to \Hom(E_3 \otimes V^*, E_{3k}) \to \Hom(E_0, E_{3k}) \to \Hom(C, E_{3k}) \to 0.$$ 

We now consider the dimensions of these vector spaces. Since

$$\dim \Hom(E_0, E_{3k}) = (3k)d^2 < \dim \Hom(E_3 \otimes V^*, E_{3k}) = 9(k-1)d^4$$

we may see $\Hom(C[1], E_{3k}) \neq 0$ and $\dim \Hom(C, E_{3k}) \neq \dim \Hom(C[1], E_{3k})$, hence $E_{3k} \neq C[1]$.

Using Proposition 3.3 we obtain $\Hom(C, E_{3k}) = 0$, hence every map $E_0 \to E_{3k}$ lifts through $\theta$ and the algebra $R^{(3)}$ is generated in degree one.

4.4. **Classification up to derived equivalence.** We are now ready to prove the main result of this article.

**Theorem 4.7.** Let $\mathcal{A}$ be a connected $k$-linear abelian Ext-finite 1-Calabi-Yau category, then $\mathcal{A}$ is derived equivalent to either

1. the category of finite dimensional representations of $k[[t]]$, or
2. the category of coherent sheaves on an elliptic curve $X$.

**Proof.** By Proposition 3.1 we know there are endo-simple objects. First, assume all endo-simple objects are isomorphic. Using Proposition 3.3 we easily see that $\mathcal{A}$ is equivalent to $\Mod_k k[[t]]$.

Next, assume there are at least two non-isomorphic endo-simples, $E$ and $B$, such that $\Hom(E, B) \neq 0$, yet with a minimal dimension. Let $\mathcal{H}$ be the abelian category constructed in 3.4. By Lemmas 4.3 and 4.6 we know $R^{(3)} = R(\mathcal{E}^{(3)})$ is a domain of GK-dimension 2 which is finitely generated by elements of degree one, hence by [1] we find that $R^{(3)}$ is noetherian and $\qgr R^{(3)}$ is equivalent to the coherent sheaves on a curve $X$.

Since $R$ is noetherian, it follows from 2.2 that $\mathcal{H}$ is equivalent to $\qgr R^{(3)}$.

The structure sheaf $\mathcal{O}_X$ of $X$ is an endo-simple object. Since the genus of $X$ is $\dim H^1(\mathcal{O}_X) = \dim \Ext(\mathcal{O}_X, \mathcal{O}_X) = \dim \Hom(\mathcal{O}_X, \mathcal{O}_X) = 1$, we know $X$ is an elliptic curve. □

4.5. **Classification of abelian categories.** We will now combine Theorem 4.7 with [5] Proposition 5.1 to obtain a description of all abelian 1-Calabi-Yau categories. First, we recall some results from [7].

Let $\mathcal{A}$ be any hereditary abelian category. A torsion theory on $\mathcal{A}$, $(\mathcal{T}, \mathcal{F})$, is a pair of full additive subcategories of $\mathcal{A}$, such that $\Hom(\mathcal{T}, \mathcal{F}) = 0$ and having the additional property that for every $X \in \Ob \mathcal{A}$ there is a short exact sequence

$$0 \to T \to X \to F \to 0$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

We will say the torsion theory $(\mathcal{T}, \mathcal{F})$ is split if $\Ext(\mathcal{F}, \mathcal{T}) = 0$. In case of a split torsion theory we obtain, by tilting, a hereditary category $\mathcal{H}$ derived equivalent to $\mathcal{A}$ with an induced split torsion theory $(\mathcal{T}, \mathcal{F}[1])$. Furthermore, the category $\mathcal{H}$ will only be hereditary if and only if $(\mathcal{F}, \mathcal{T})$ is a split torsion theory.

We now discuss all possible torsion theories when $\mathcal{A}$ is equivalent to $\coh X$. Note that, since $\mathcal{H}$ will be 1-Calabi-Yau and hence hereditary, all torsion theories on $\mathcal{A}$ will be split.

Let $(\mathcal{F}, \mathcal{T})$ be a torsion theory on $\mathcal{A}$, and let $\mathcal{E}$ be an indecomposable of $\mathcal{T}$. Then every indecomposable $\mathcal{F}$ with slope strictly larger than $\mu(\mathcal{E})$ has to be in $\mathcal{T}$ since $\Hom(\mathcal{E}, \mathcal{F}) \neq 0$. Furthermore, if $\mathcal{E}$ is in $\mathcal{T}$ and there is a path from $\mathcal{E}$ to an indecomposable $\mathcal{E}'$, then $\mathcal{E}' \in \ind \mathcal{T}$.

We may now give a characterization of all possible torsion theories.
Theorem 4.8. [5] Let $X$ be an elliptic curve. Every category $\mathcal{H}$ derived equivalent to $\mathcal{A} = \text{coh} X$ may be obtained by tilting with respect to a torsion theory. Moreover, all torsion theories on $\text{coh} X$ are split and may be described as follows. Let $\theta \in \mathbb{R} \cup \{\infty\}$. Denote by $\mathcal{A}_{\geq \theta}$ and $\mathcal{A}_{> \theta}$ the subcategory of $\mathcal{A}$ generated by all indecomposables $E$ with $\mu(E) > \theta$ and $\mu(E) \geq \theta$, respectively. All full subcategories $\mathcal{T}$ of $\mathcal{A}$ with $\mathcal{A}_{\geq \theta} \subseteq \mathcal{T} \subseteq \mathcal{A}_{> \theta} \subseteq \mathcal{A}$ give rise to a torsion theory $(\mathcal{F}, \mathcal{T})$, with $\text{ind} \mathcal{F} = \text{ind} \mathcal{A} \setminus \text{ind} \mathcal{T}$.

Proof. That these are all possible torsion theories, follows from the above discussion. That all categories $\mathcal{H}$ may be obtained in this way, is shown in [5 Proposition 5.1]. Alternatively, it is straightforward to check these torsion theories generate all bounded $t$-structures on $D^b \mathcal{A}$ up to shifts.

Example 4.9. We give some examples of torsion theories. In here $\mathcal{H}$ always stands for the category tilted with respect to the described torsion theory.

1. If $\theta \in \mathbb{Q} \cup \{\infty\}$ and $\mathcal{T} = \mathcal{A}_{\geq \theta}$, then the tilted category $\mathcal{H}$ is equivalent to $\text{coh} X$. If $\mathcal{T} = \mathcal{A}_{> \theta}$, then $\mathcal{H}$ is dual to $\mathcal{A}$.
2. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathcal{T} = \mathcal{A}_{\geq \theta} = \mathcal{A}_{> \theta}$ then $\mathcal{H}$ is equivalent to the category of holomorphic bundles on a noncommutative two-torus [9].

Theorem 4.8 classifies all categories derived equivalent to $\text{coh} X$. We further need to classify all categories derived equivalent to $\mathcal{B} = \text{Mod}^{id} k[[t]]$.

Let $\mathcal{H}$ be such a category derived equivalent to $\mathcal{B}$. Then $\mathcal{H}$ induces a $t$-structure $(D^{>0}, D^{\geq 0})$ on $D^b \mathcal{B}$. Since this $t$-structure is split, we may assume the heart $\mathcal{H} = D^{\leq 0} \cap D^{>0}$ contains the endo-simple object $E$ of $\mathcal{B}[0]$ and, since $\mathcal{B}$ has only one endo-simple object, this is the unique endo-simple object of $\mathcal{H}$, up to isomorphism.

Moreover, for every $X \in \mathcal{B}$ we have $\text{Hom}(X, B) \neq 0$ and $\text{Hom}(B, X) \neq 0$, thus we have $\mathcal{B} \subseteq D^{\leq 0} \cap D^{>0} = \mathcal{H}$.

Since $\mathcal{B}$ has only one endo-simple object, $E$ is the unique endo-simple object of $\mathcal{H}$, up to isomorphism. From this we infer $\mathcal{B} = \mathcal{H}$ as subcategories of $D^b \mathcal{B}$.

We conclude that every category derived equivalent to $\text{Mod}^{id} k[[t]]$ is in fact equivalent to $\text{Mod}^{id} k[[t]]$.

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