Convergence analysis of one-point large deviations rate functions of numerical discretizations for stochastic wave equations with small noise

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In this work, we present the convergence analysis of one-point large deviations rate functions (LDRFs) of the spatial finite difference method (FDM) for stochastic wave equations with small noise, which is essentially about the asymptotical limit of minimization problems and not a trivial task for the nonlinear cases. In order to overcome the difficulty that objective functions for the original equation and the spatial FDM have different effective domains, we propose a new technical route for analyzing the pointwise convergence of the one-point LDRFs of the spatial FDM, based on the $\Gamma$-convergence of objective functions. Based on the new technical route, the intractable convergence analysis of one-point LDRFs boils down to the qualitative analysis of skeleton equations of the original equation and its numerical discretizations.

Keywords: one-point large deviations rate functions, numerical discretizations, convergence analysis, $\Gamma$-convergence, stochastic wave equations

AMS subject classifications: 60F10, 60H35, 49J45

1. Introduction

The asymptotics of large deviations rate functions (LDRFs) of numerical discretizations for stochastic differential equations (SDEs) has received increasing attention, which is devoted to revealing the relationship between the probabilities of rare events associated with numerical discretizations and those associated with the underlying SDEs. The existing literature analyzing the asymptotics of LDRFs of numerical discretizations mainly focuses on two aspects. On one hand, some of them study the ability of numerical discretizations to preserve the large deviations principles (LDPs) of the original equations. For instance, [5] (resp. [6]) shows that a large class of stochastic symplectic discretizations can asymptotically (resp. weakly asymptotically) preserve the LDPs of certain long-time observables of a linear stochastic oscillator (resp. stochastic linear Schrödinger equations), but many nonsymplectic ones do not share this property. These reveal the superiority of stochastic symplectic discretizations from the perspective of LDPs. [4] investigates the asymptotical preservation of numerical discretizations for the LDPs of sample paths and invariant measures of parabolic stochastic partial differential equations (SPDEs). On the other hand, some of the existing work gives the error estimate or convergence of LDRFs of numerical discretizations for SDEs, which provides the theoretical foundation for numerical approximations of LDRFs associated with SDEs. For example, [3] gives an error estimate between one-point LDRFs of the midpoint method and that of linear stochastic Maxwell equations with small noise. In addition, [14] gives the locally uniform convergence orders of one-point LDRFs of the stochastic $\theta$-method for nonlinear stochastic ordinary differential equations (SODEs) with small noise.

In this paper, we study the convergence of one-point LDRFs of numerical discretizations for the nonlinear stochastic wave equation driven by the space-time white noise:

$$\frac{\partial^2}{\partial t^2} u^\varepsilon(t,x) = \frac{\partial^2}{\partial x^2} u^\varepsilon(t,x) + b(u^\varepsilon(t,x)) + \sqrt{\varepsilon} \sigma(u^\varepsilon(t,x)) \dot{W}(t,x), \quad (t,x) \in (0,T] \times [0,1],$$

(1.1)

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\[ u^\varepsilon(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u^\varepsilon(0, x) = v_0(x), \quad x \in [0, 1], \]
\[ u^\varepsilon(t, 0) = u^\varepsilon(t, 1) = 0, \quad t \in (0, T]. \]

Here, \( T > 0 \) is a given positive number, the small parameter \( \varepsilon \in (0, 1] \) denotes the noise intensity, and \( \{W(t, x), (t, x) \in \mathcal{O}_T\} \) is a Brownian sheet defined on a complete filtered probability space \( (\mathcal{Q}, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathcal{P}) \) with \( \{\mathcal{F}_t\}_{t \in [0, T]} \) satisfying the usual conditions, where \( \mathcal{O}_T := [0, T] \times [0, 1] \). Conditions on functions \( b, \sigma : \mathbb{R} \to \mathbb{R} \) and the initial values \( u_0, v_0 \) will be specified in Section 2. Using the weak convergence method (see e.g., [2]), it can be shown that \( \{u^\varepsilon\}_{\varepsilon > 0} \) satisfies the LDP on \( \mathcal{C}(\mathcal{O}_T; \mathbb{R}) \). Further, the LDP of sample paths of (1.1) and the contraction principle immediately indicate that for any \( x_0 \in (0, 1) \), \( \{u^\varepsilon(T, x_0)\}_{\varepsilon > 0} \) satisfies the LDP on \( \mathbb{R} \) with a good rate function \( I \) (see Theorem 2.6), which is called the one-point LDP of (1.1) (see e.g., [18] for the one-point LDP of the KPZ equation). This means that for a Borel measurable set \( \mathcal{A} \subseteq \mathbb{R} \), the hitting probability \( \mathcal{P}(u^\varepsilon(T, x_0) \in \mathcal{A}) \approx e^{-\frac{1}{\varepsilon} \inf I(A)} \) for sufficiently small \( \varepsilon \). As an important quantity, the one-point LDRF \( I \) characterizes the exponential decay speed of the probability of \( u^\varepsilon(T, x_0) \) deviating from \( u^0(T, x_0) \), with \( u^0 \) being the solution of (1.1) provided \( \varepsilon = 0 \).

Generally, the one-point LDRF \( I \) is determined by a minimization problem and lacks for explicit expression. Thus it is necessary to resort to numerical discretizations in order to approximate \( I \). In this process, one is mainly faced with two problems:

(P1) Which kind of numerical discretizations for (1.1) can satisfy the one-point LDP, for a given discretization parameter?

(P2) For a numerical discretization satisfying the one-point LDP, how to analyze the convergence of its one-point LDRF?

Concerning (P1), we discrete (1.1) by the spatial finite difference method (FDM), and show that the continuified spatial FDM also satisfies the one-point LDP with a good rate function \( I^\varepsilon \), with \( n \) being the spatial discretization parameter (see Theorem 2.7). As for (P2), to the best of our knowledge, there have been no any results on the convergence analysis of one-point LDRFs of numerical discretizations for SPDEs with small noise. Aiming at filling the gap, this work focuses on presenting the pointwise convergence of \( I^\varepsilon \) to \( I \) as \( n \) tends to infinity.

The one-point LDRFs \( I \) and \( I^\varepsilon \) are implicitly determined by minimizations problems (see (3.1)):

\[ I(y) = \inf_{\{f \in C(\mathcal{O}_T; \mathbb{R})\}} J_y(f), \quad I^\varepsilon(y) = \inf_{\{f \in C(\mathcal{O}_T; \mathbb{R})\}} J^\varepsilon_y(f), \quad y \in \mathbb{R}, \]

where the objective function \( J_y \) (resp. \( J^\varepsilon_y \)) is the restriction of the LDRF of sample paths of (1.1) (resp. the spatial FDM) on \( \{f \in C(\mathcal{O}_T; \mathbb{R}) : f(T, x_0) = y\} \). \( J_y \) and \( J^\varepsilon_y \) are closely related to the skeleton equations of (1.1) and the spatial FDM, respectively. And they are given by

\[ J_y(f) := \begin{cases} \inf_{\{h \in L^2(\mathcal{O}_T; \mathbb{R}) : \chi(h) = f\}} \frac{1}{2} \|h\|^2_{L^2(\mathcal{O}_T)}, & \text{if } f(T, x_0) = y, \ f \in \text{Im}(Y), \\ +\infty, & \text{otherwise}, \end{cases} \]

and

\[ J^\varepsilon_y(f) := \begin{cases} \inf_{\{h \in L^2(\mathcal{O}_T; \mathbb{R}) : \chi^\varepsilon(h) = f\}} \frac{1}{2} \|h\|^2_{L^2(\mathcal{O}_T)}, & \text{if } f(T, x_0) = y, \ f \in \text{Im}(Y^\varepsilon), \\ +\infty, & \text{otherwise}, \end{cases} \]

respectively. Here, \( Y \) (resp. \( Y^\varepsilon \)) is the solution mapping of the skeleton equation of (1.1) (resp. the spatial FDM). Generally speaking, it is not trivial to analyze the convergence of \( I^\varepsilon \), since this problem is essentially tackling the asymptotical limit of minimization problems. Concerning the convergence analysis of \( I^\varepsilon \), we are faced with two difficulties: One is that objective functions \( J_y \) and \( J^\varepsilon_y \) have no explicit expressions, which are also determined by minimization problems; The other is that \( J_y \) and \( J^\varepsilon_y \) have different effective domains, which hinders us from approximating \( I \) and \( I^\varepsilon \) using same minimization sequences. Notice that for SODEs with non-degenerate noises, [14] gives the effective domains and explicit expressions of \( J_y \) and \( J^\varepsilon_y \), and then obtains the
Our approach to proving the convergence of $I^n$ is based on the basic method in the theory of $\Gamma$-convergence (see Appendix A for the basic introduction to $\Gamma$-convergence). To clarify our ideas, we present our technical route for the convergence analysis of $I^n$ in Fig. 1. As is shown in Fig. 1, we prove that $I^n$ pointwise converges to $I$, by showing that $\{J^n_y\}_{n \in \mathbb{N}^+}$ $\Gamma$-converges to $J$ and is equi-coercive on $C(\partial_T; \mathbb{R})$. On one hand, the equicoerciveness of $\{J^n_y\}_{n \in \mathbb{N}^+}$ comes from the equi-continuity and uniform boundedness of $I^n(S_u)$ with $S_u$ being the closed ball of $L^2(\partial_T; \mathbb{R})$ with the radius $a$. On the other hand, the $\Gamma$-convergence of $\{J^n_y\}_{n \in \mathbb{N}^+}$ is derived via the $\Gamma$-liminf inequality of $\{J^n_y\}_{n \in \mathbb{N}^+}$ and $\Gamma$-limsup inequality of any subsequences of $\{J^n_y\}_{n \in \mathbb{N}^+}$. More precisely, we obtain the $\Gamma$-liminf inequality of $\{J^n_y\}_{n \in \mathbb{N}^+}$ by establishing the compactness of $\Upsilon$ and the locally uniform convergence of $I^n$ to $\Upsilon$. Further, by giving the locally Lipschitz property of $\Upsilon$ and properties of the inverse operator of $I^n$ besides the locally uniform convergence of $I^n$ to $\Upsilon$, we obtain the $\Gamma$-limsup inequality of any subsequences of $\{J^n_y\}_{n \in \mathbb{N}^+}$. Based on our technical route, the pointwise convergence of $I^n$ boils down to qualitative properties for the solutions to skeleton equations of both (1.1) and its spatial FDM. These qualitative properties (the compactness of $\Upsilon$, the locally Lipschitz property of $\Upsilon$, the locally uniform convergence of $I^n$ and so on) may be obtained for many usual numerical discretizations. Therefore, the technical route proposed in this paper is very promising in analyzing the pointwise convergence of one-point LDRFs of other numerical discretizations for SPDEs.

In summary, this work makes two main contributions: (1) We give the convergence analysis of one-point LDRFs of the spatial FDM for stochastic wave equations, which is first obtained for the nonlinear SPDEs. (2) We propose a new technical route for proving the pointwise convergence of one-point LDRFs of numerical discretizations for SPDEs with small noise. The rest of this paper is organized as follows. Section 2 gives some preliminaries and one-point LDPs of stochastic wave equations and its spatial FDM. Section 3 establishes the pointwise convergence of one-point LDRFs of the spatial FDM based on our technical route. We recall the main results of this work and refer to some future work in Section 4. The appendix collects some basic knowledge of $\Gamma$-convergence and gives the proof of Proposition 3.2.

2. Preliminaries

In this section, we present the one-point LDPs of the stochastic wave equation and its spatial FDM. We begin with some notations. Throughout this paper, let $\mathbb{N}^+$ be the set of all positive integers. Denote by $| \cdot |$ the 2-norm of a vector or matrix. For a given non-empty set $U \subseteq \mathbb{R}^d$, denote by $C(U; \mathbb{R})$ the space of all real-valued continuous functions defined on $U$, endowed with the supremum norm $||f||_{C(U)} := \sup_{x \in U} |f(x)|$, $f \in C(U; \mathbb{R})$. For $\alpha \in (0, 1)$, denote by $C^\alpha(U; \mathbb{R})$ the space of all $\alpha$-Hölder continuous functions from $U$ to $\mathbb{R}$, endowed with
the norm \( \|f\|_{C^0(U)} := \|f\|_{C(U)} + |f|_{C^0(U)} \), where the semi-norm \( |f|_{C^0(U)} := \sup \{|f(x) - f(y)| : x, y \in U, x \neq y\} \). In addition, \( C^k(U; \mathbb{R}) \) denotes the space of \( k \)th continuously differentiable functions from \( U \) to \( \mathbb{R} \), and \( C^\infty(U; \mathbb{R}) = \cap_{k \in \mathbb{N}^+} C^k(U; \mathbb{R}) \). Let \( L^2(U; \mathbb{R}) \) stand for the space of all square integrable functions from \( U \) to \( \mathbb{R} \) with the norm \( \|f\|_{L^2(U)}^2 := \int_U |f(x)|^2 \, dx \). Let \( E \) and \( F \) be given topological vector spaces. For a mapping \( T : E \to F \) and \( K \subseteq E \), denote by \( D_T := \{x \in E : T(x) < +\infty\} \) the effective domain of \( T \). The infimum of an empty set is always interpreted as \(+\infty\). Let \( K(a_1, a_2, \ldots, a_m) \) denote some generic constant dependent on the parameters \( a_1, a_2, \ldots, a_m \), which may vary from one place to another.

The LDP deals with the exponential decay of probabilities of rare events, where the decay rate is characterized in terms of the LDRF. Throughout this section, let \( \mathcal{X} \) be a Polish space, i.e., complete and separable metric space. A real-valued function \( I : \mathcal{X} \to [0, \infty] \) is called a rate function if it is lower semicontinuous, i.e., for each \( a \in [0, \infty) \), the level set \( I^{-1}([0, a]) \) is a closed subset of \( \mathcal{X} \). If all level sets \( I^{-1}([a, b]) \), for \( a, b \in [0, \infty) \), are compact, then \( I \) is called a good rate function. Let \( I \) be a rate function and \( \{\mu_\varepsilon\}_{\varepsilon > 0} \) be a family of probability measures on \( \mathcal{X} \). We say that \( \{\mu_\varepsilon\}_{\varepsilon > 0} \) satisfies an LDP on \( \mathcal{X} \) with the rate function \( I \) if

\[
(\text{LDP1}) \quad \liminf_{\varepsilon \to 0} \varepsilon \ln \mu_\varepsilon(U) \geq - I(U) \quad \text{for every open } U \subseteq \mathcal{X},
\]

\[
(\text{LDP2}) \quad \limsup_{\varepsilon \to 0} \varepsilon \ln \mu_\varepsilon(C) \leq - I(C) \quad \text{for every closed } C \subseteq \mathcal{X}.
\]

We also say that a family of random variables \( \{Z_\varepsilon\}_{\varepsilon > 0} \) valued on \( \mathcal{X} \) satisfies an LDP with the rate function \( I \), if its distribution satisfies (LDP1) and (LDP2). We refer readers to [12] for more details about the LDP.

### 2.1 Finite difference method

Stochastic wave equations have a wide application in many fields, which can describe the motion of a strand of DNA floating in a liquid, the wave propagation through the atmosphere or the ocean with random media and so on; see e.g., [1, 7]. For (1.1), the Green function associated to \( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \) with homogeneous Dirichlet boundary conditions is given by

\[
G_t(x,y) = \sum_{j=1}^{\infty} \frac{\sin(j\pi t)}{j\pi} \varphi_j(x) \varphi_j(y), \quad t \in [0, T], \, x, y \in [0, 1],
\]

where \( \varphi_j(x) = \sqrt{2} \sin(j\pi x), \, j \geq 1 \). Throughout this paper, we always assume that \( u_0 \in C^2([0, 1]; \mathbb{R}) \) with \( u_0(0) = u_0(1) = 0, \, v_0 \in C^1([0, 1]; \mathbb{R}) \) and that \( b, \sigma : \mathbb{R} \to \mathbb{R} \) are globally Lipschitz continuous:

\[
|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y| \quad \forall \, x, y \in \mathbb{R}
\]

for some \( L > 0 \), which implies the linear growth of \( b \) and \( \sigma \), i.e., \( |b(x)| + |\sigma(x)| \leq L_1(1 + |x|), \, x \in \mathbb{R} \) for some \( L_1 > 0 \). In this situation, the exact solution of (1.1) is given by (see, e.g., [19])

\[
\begin{align*}
&u^\varepsilon(t, x) \\
&= \int_0^1 G_t(x, z) v_0(z) \, dz + \frac{\sigma}{\varepsilon} \left( \int_0^1 G_t(x, z) u_0(z) \, dz + \int_0^{\varepsilon} \int_0^1 G_t(x, z) b(u^\varepsilon(s, z)) \, dz \, ds \right) \\
&\quad + \sqrt{\varepsilon} \int_0^1 \int_0^1 G_t(x, z) \sigma(u^\varepsilon(s, z)) \, dW(s, z), \quad (t, x) \in \mathcal{O}_T.
\end{align*}
\]

In this paper, we are interested in the one-point LDPs of both (1.1) and its spatial FDM. The spatial FDM for stochastic wave equation is first studied by [19], where the authors prove that the strong convergence order of the spatial FDM is almost 1/3. Given a function \( w \) on the mesh \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \), we define the difference operator

\[
\delta_n w_i := \frac{n^2}{2} (w_{i-1} - 2w_i + w_{i+1}), \quad \text{for } i \in \mathbb{Z}_n := \{1, 2, \ldots, n - 1\},
\]
For the numerical study on stochastic wave equations, we refer to [19].

One-point LDPs

Denote \( \kappa_n(z) := \lfloor zn \rfloor \) and \( n \in \mathbb{N}^+ \), where \( \lfloor \cdot \rfloor \) is the floor function. Hereafter, for any \( n \in \mathbb{N}^+ \) and measurable function \( w : [0,1] \rightarrow \mathbb{R} \), denote by \( \Pi_n(w) \) the linear interpolation of \( w \) with nodes \( 0, \frac{1}{n}, \ldots, 1 \), i.e., \( \Pi_n(w)(x) = w(\kappa_n(x)) + (nx - \lfloor nx \rfloor)(w(\kappa_n(x)) + \frac{1}{n} - w(\kappa_n(x))) \), \( x \in [0,1] \). Further, introduce the discrete Green function

\[
G^n_t(x,y) = \sum_{j=1}^{n-1} \frac{\sin(j\pi t\sqrt{c_j^2})}{j\pi \sqrt{c_j^2}} \phi_j(\kappa_n(y)) \phi_j(\kappa_n(x)), \quad t \in [0,T], \ x, y \in [0,1], \tag{2.2}
\]

where \( c_j^2 := \sin^2\left(\frac{j\pi}{2n}\right) / \left(\frac{j\pi}{2n}\right)^2 \) with \( 4/\pi^2 \leq c_j^2 \leq 1, n \in \mathbb{N}^+ \), \( j = 1, 2, \ldots, n-1 \), and \( \phi_j, n = \Pi_n(\phi_j) \). By [19, Eq. (12)], one has

\[
u^{n}(t,x) = \int_0^1 G^n_t(x,z)w_0(\kappa_n(z))dz + \int_0^1 \frac{\partial}{\partial t}G^n_t(x,z)u_0(\kappa_n(z))dz + \int_0^1 \int_0^1 G^n_{t-s}(x,z)b(u^{n}(s,\kappa_n(z)))dzds + \sqrt{\varepsilon} \int_0^1 \int_0^1 G^n_{t-s}(x,z)\sigma(u^{n}(s,\kappa_n(z)))dW(s,z), \quad (t,x) \in \mathcal{O}_T.
\]

For the numerical study on stochastic wave equations, we refer to [1, 8, 9, 10, 17] and their references.

2.2 One-point LDPs

In this part, we show that for any \( x_0 \in (0,1) \) and \( n \in \mathbb{N}^+ \), both \( \{u^n(T,x_0)\}_{n \in \mathbb{N}} \) and \( \{u^{n}(T,x_0)\}_{n \in \mathbb{N}} \) satisfy the LDPs on \( \mathbb{R} \). These can be done by the contraction principle and the LDP of \( \{u^n\}_{n \in \mathbb{N}} \) and \( \{u^{n}(\cdot, x_0)\}_{n \in \mathbb{N}} \) on \( C(\mathcal{O}_T; \mathbb{R}) \). And the LDPs of \( \{u^n\}_{n \in \mathbb{N}} \) and \( \{u^{n}(\cdot, x_0)\}_{n \in \mathbb{N}} \) can be established based on the weak convergence method, which are standard and similar to [2, Theorem 9]. Thus we only give the sketch of proofs for above results. Before that we give some properties of the Green functions \( G \) and \( G^n \).

Proposition 2.1 There is \( K > 0 \) dependent on \( T \) such that for any \( t, s \in [0,T] \) and \( x, y \in [0,1] \),

1. \( |G_t(x,y)| \leq K \).
2. \( \int_0^1 |G_t(x,z) - G_t(y,z)|^2 dz \leq K|x-y| \).
3. \( \int_0^1 |G_t(x,z) - G_s(x,z)|^2 dz \leq K|t-s| \).
4. \( \left| \int_0^1 G_t(x,y) \phi(y)dy \right| \leq K||\phi||_{C([0,1])} \) and \( \left| \int_0^1 G_t(x,y) \psi(y)dy \right| \leq K||\psi||_{C([0,1])} \) for any \( \phi \in C([0,1]) \) and \( \psi \in C^1([0,1]) \).

Proof. It follows from [19, Eq. (3)] that (1) holds. As shown in the proof of Proposition 1 of [19], (4) holds. As for (2), since \( \{\phi_j, j \in \mathbb{N}^+\} \) is an orthonormal basis of \( L^2([0,1]; \mathbb{R}) \) and \( |\phi_j(x) - \phi_j(y)| \leq \sqrt{2}(j\pi|x-y|) \), it follows from (2.1) that for any \( t \in [0,T] \) and \( x, y \in [0,1] \) with \( x \neq y \),

\[
\int_0^1 |G_t(x,z) - G_t(y,z)|^2 dz \leq \sum_{j=1}^{n} \frac{\sin^2(j\pi t)}{j^2 \pi^2} |\phi_j(x) - \phi_j(y)|^2 \leq K \sum_{j=1}^{n} \frac{1}{j^2} (1 \wedge j^2|x-y|^2) \tag{2.3}
\]
\[ \leq K \sum_{j < |x-y|^{-1}} |x-y|^2 + K \sum_{j \geq |x-y|^{-1}+1} \frac{1}{j^2} \leq K|x-y|, \]

which proves (2). The inequality (3) can be proved similar to (2).

\[ \square \]

**Proposition 2.2** There is \( K > 0 \) dependent on \( T \) such that for all \( n \in \mathbb{N}^+, t, s \in [0,T] \) and \( x, y \in [0,1] \),

1. \( \int_0^1 |G^n_t(x, z)|^2 dz \leq K. \)
2. \( \int_0^1 |G^n_t(x, z) - G^n_t(y, z)|^2 dz \leq K|x-y|. \)
3. \( \int_0^1 |G^n_t(x, z) - G^n_t(x, t)|^2 dz \leq K|t-s|. \)
4. \(| \int_0^1 G^n_t(x, y) \varphi(\kappa_n(y))dy | \leq K\| \varphi \|_{C((0,1))} \) for \( \varphi \in C([0,1]) \) and \(| \int_0^1 G^n_t(x, y) \psi(\kappa_n(y))dy | \leq K\| \psi \|_{C((0,1))} \) for \( \psi \in C^1([0,1]) \).

**Proof.** It follows from (20) and the estimates for \( A_1 \) and \( A_2 \) in the proof of [19, Proposition 3] that (1) and (4) hold. Further, by the facts \( \int_0^1 \varphi_j(\kappa_n(y)) \varphi_j(\kappa_n(y))dy = \delta_{ij} \) (see [19, Eq. (15)]) and \(| \varphi_j(x) - \varphi_j(x) | \leq \sqrt{2} (j-1)|x-y|J 2 \), and using same arguments in (2.3), we get (2) and (3).

For any \( M \geq 0 \), denote \( S_M := \{ \phi \in L^2(\mathcal{O}_T; \mathbb{R}) : \| \phi \|_{L^2(\mathcal{O}_T)} \leq M \} \) and \( \mathcal{A}_M := \{ \phi : \Omega \times \mathcal{O}_T \rightarrow \mathbb{R}, \phi \text{ is predictable and } \phi(\omega) \in S_M, \ P-a.s. \}. \) In order to prove the LDP of \( \{ u^n_t \}_{n=1}^{+\infty} \) on \( C(\mathcal{O}_T; \mathbb{R}) \) based on the weak convergence method, we need to study the asymptotics of the controlled equation of (1.1), whose solution is

\[
\begin{align*}
\dot{u}^{v, \nu}_t(t, x) &= \int_{0}^{t} G_t(x, z)u_0(z)dz + \frac{\partial}{\partial t} \left( \int_{0}^{t} G_t(x, z)u_0(z)dz \right) dt + \int_{0}^{t} \int_{0}^{t} H_{t-s}(z, \xi)A^{v, \nu}(s, z)dzds \\
&+ \sqrt{\varepsilon} \int_{0}^{t} \int_{0}^{t} H_{t-s}(z, \xi)\sigma(u^{v, \nu}(s, z))dW(s, z) + \int_{0}^{t} \int_{0}^{t} G_{t-s}(z, \xi)\sigma(u^{v, \nu}(s, z))v(s, z)dzds,
\end{align*}
\]

for \( (t, x) \in \mathcal{O}_T \) and \( \nu \in \mathcal{A}_M \). Next, we present the uniform boundedness of \( \eta \)th moment of \( u^{v, \nu} \), \( \eta \geq 1 \).

**Proposition 2.3** For any \( M \in (0, +\infty) \) and \( \eta \geq 1 \),

\[
\sup_{v \in (0,1)} \sup_{\nu \in \mathcal{A}_M} \sup_{(t, x) \in \mathcal{O}_T} E|u^{v, \nu}(t, x)|^{\eta} < +\infty.
\]

The conclusion of Proposition 2.3 follows from the Burkholder inequality, the Gronwall inequality, the Hölder inequality and Proposition 2.1, whose proof is standard and thus is omitted. Similar proofs can be found in [2, Lemma 1] or [19, Proposition 1].

**Proposition 2.4** Let \( \mathcal{B} \subseteq \{ \phi : \Omega \times \mathcal{O}_T \rightarrow \mathbb{R} : \phi \text{ is predictable and } \| \phi \|_{L^2(\mathcal{O}_T)} < +\infty, P-a.s. \} \) be a family such that for all \( \eta \geq 2 \), \( \sup_{f \in \mathcal{B}} \sup_{(t, x) \in \mathcal{O}_T} E|f(t, x)|^\eta < +\infty \). For any \( f \in \mathcal{B} \) and \( \nu \in \mathcal{A}_M, M \in (0, +\infty) \), define

\[
\begin{align*}
\Phi_1(t, x) := & \int_{0}^{t} \int_{0}^{t} G_{t-s}(x, z)f(r, z)dW(s, z), \\
\Phi_2(t, x) := & \int_{0}^{t} \int_{0}^{t} G_{t-s}(x, z)f(r, z)v(r, z)d\nu dr,
\end{align*}
\]

for \( (t, x) \in \mathcal{O}_T \). Then for \( \alpha \in \left( 0, \frac{1}{2} \right) \) and \( i = 1, 2 \), \( \sup_{f \in \mathcal{B}, \nu \in \mathcal{A}_M} E\| \Phi_i \|_{C^\alpha(\mathcal{O}_T)} < +\infty \).

**Proof.** We only prove the conclusion for \( i = 2 \), the case \( i = 1 \) is almost same by an additional application of the Burkholder inequality. It follows from the Hölder inequality, the Minkowski inequality, Proposition 2.1 and \( \| v \|_{L^2(\mathcal{O}_T)} \leq M, P-a.s. \) that for any \( s \leq t \leq T, x, y \in [0,1] \) and \( \eta \geq 1 \),

\[
\begin{align*}
E|\Phi_2(t, x) - \Phi_2(s, y)|^\eta & \leq K(p)E|\Phi_2(t, x) - \Phi_2(t, y)|^\eta + K(p)E|\Phi_2(t, y) - \Phi_2(s, y)|^\eta \\
& \leq K(p)E|\Phi_2(t, x) - \Phi_2(t, y)|^\eta + K(p)E|\Phi_2(t, y) - \Phi_2(s, y)|^\eta.
\end{align*}
\]
The continuity of $(\Phi_t)_{t \geq 0}$ is given by

$\Phi_t = \Phi_0 + \int_0^t \Phi_s f(s, \Phi_s)ds$.

Based on Propositions 2.3 and 2.4, further using the Kolmogorov continuity theorem [16, Theorem C.6] yields that for any $p > 4$ and $q \in (0, 1 - \frac{1}{p})$,

$$\sup_{f \in \mathscr{B}, \nu \in \mathscr{A}_M} E \left[ \sup_{(t,x), (s,y) \in \mathcal{O}_T} \left| \Phi_2(t,x) - \Phi_2(s,y) \right|^p \right] < +\infty, \quad (2.4)$$

where $r((t,x), (s,y)) = (|t-s|^2 + |x-y|^2)^{1/2}$. For any $\alpha \in (0, \frac{1}{2})$, choosing $q = 2\alpha$ for $p > 1$ and using (2.4), we have $\sup_{f \in \mathscr{B}, \nu \in \mathscr{A}_M} E[\Phi_2|C(\mathcal{O}_T)] < +\infty$ for $\alpha \in (0, \frac{1}{4})$. Again by (2.4) and $\Phi_2(0,0) = 0$, it holds that $\sup_{f \in \mathscr{B}, \nu \in \mathscr{A}_M} E[\Phi_2|C(\mathcal{O}_T)] < +\infty$. In this way, we complete the proof.

Proposition 2.5 Let $Z^\epsilon_T(t,x) := \sqrt{\epsilon} \int_0^t G_{t-s}(x,y)f(s,y)dW(s,y)$, $(t,x) \in \mathcal{O}_T$ and $\mathcal{B}$ be as in Proposition 2.4. Then for every family $\{f^x\}_{x \geq 0} \subseteq \mathcal{B}$, $Z^\epsilon_T$ converges to 0 in $C(\mathcal{O}_T; \mathbb{R})$ in probability as $\epsilon \to 0$.

Proof. By Proposition 2.4, $E[\|Z^\epsilon_T\|_{C(\mathcal{O}_T)}] \leq \sqrt{\epsilon} \sup_{f \in \mathcal{B}} E[\|\Phi_1\|_{C(\mathcal{O}_T)}] \leq K\sqrt{\epsilon}$, which yields the desired result.

Theorem 2.6 For any $x_0 \in (0,1)$, $\{u^\epsilon(T,x_0)\}_{\epsilon > 0}$ satisfies the LDP on $\mathbb{R}$ with the good rate function $I$ given by

$$I(y) := \inf_{f \in C(\mathcal{O}_T; \mathbb{R}) \mid f(T,x_0) = y} J(f), \quad y \in \mathbb{R}.$$  

Here $J : C(\mathcal{O}_T; \mathbb{R}) \to \mathbb{R}$ is defined by

$$J(f) := \inf_{h \in L^2(\mathcal{O}_T; \mathbb{R}) \mid I(h) = f} \frac{1}{2} \|h\|^2_{L^2(\mathcal{O}_T)}, \quad f \in C(\mathcal{O}_T; \mathbb{R}),$$

where $\mathcal{O}$ is the solution mapping which takes $h \in L^2(\mathcal{O}_T; \mathbb{R})$ to the solution of the following stochastic equation

$$f(t,x) = \int_0^1 G_t(x,z)v_0(z)dz + \frac{\partial}{\partial x} \left( \int_0^1 G_t(x,z)u_0(z)dz \right) + \int_0^t \int_0^1 G_{t-s}(x,z)b(f(s,z))dzds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,z)\sigma(f(s,z))h(s,z)dzds.$$

Proof. Based on Propositions 2.3-2.5, one can use the same procedure as in the proof of [2, Theorem 9] to prove that $\{u^\epsilon\}_{\epsilon > 0}$ satisfies the LDP on $C(\mathcal{O}_T; \mathbb{R})$ with the good rate function given by (2.5). For fixed $x_0 \in (0,1)$, define the coordinate map $\tilde{\xi}_{(T,x_0)} : C(\mathcal{O}_T; \mathbb{R}) \to \mathbb{R}$ by $\tilde{\xi}_{(T,x_0)}(f) = f(T,x_0)$ for $f \in C(\mathcal{O}_T; \mathbb{R})$. It follows from the continuity of $\tilde{\xi}_{(T,x_0)}$ and the contraction principle [12, Theorem 4.1.2] that $\{u^\epsilon(T,x_0)\}_{\epsilon > 0}$ satisfies an LDP with the good rate function $I$. \hfill $\Box$
Theorem 2.7  For any $x_0 \in (0,1)$ and $n \in \mathbb{N}^+$, $\{u^{E,n}(T,x_0)\}_{E>0}$ satisfies the LDP on $\mathbb{R}$ with the good rate function $J^n$ given by

$$J^n(y) := \inf_{f \in C(\partial T;\mathbb{R})} \{ f(T,x_0)=y \}$$

Here $J^n : C(\partial T;\mathbb{R}) \to \mathbb{R}$ is defined by

$$J^n(f) := \inf_{h \in L^2(\partial T;\mathbb{R}) : \gamma^n(h)=f} \frac{1}{2} \|h\|^2_{L^2(\partial T)}, \quad f \in C(\partial T;\mathbb{R}),$$

where $\gamma^n$ is the solution mapping which takes $h \in L^2(\partial T;\mathbb{R})$ to the solution of the discrete skeleton equation

$$f(t,x) = \int_0^1 G^n_t(x,z)v_0(\kappa_n(z))dz \quad \text{and} \quad \int_0^1 \frac{\partial}{\partial t}G^n_t(x,z) u_0(\kappa_n(z))dz$$

$$\quad + \int_0^1 \int_0^1 G^n_{t-s}(x,z)b(f(s,\kappa_n(z)))dzds + \int_0^1 \int_0^1 G^n_{t-s}(x,z)\sigma(f(s,\kappa_n(z)))h(s,z)dzds.$$  \hspace{1cm} (2.6)

Proof. Notice that the discrete mean function $G^n$ has the similar integrability and Hölder continuity as $G$ due to Propositions 2.1 and 2.2. Accordingly, we can give the counterparts of Propositions 2.3-2.5 for the controlled equation of the spatial FDM, and show that for any $n \in \mathbb{N}^+$, $\{u^{E,n}\}_{E>0}$ satisfies the LDP on $C(\partial T;\mathbb{R})$ with the good rate function $J^n(f) = \inf_{h \in L^2(\partial T;\mathbb{R})} \{ f(h)=f \} \frac{1}{2} \|h\|^2_{L^2(\partial T)}, \quad f \in C(\partial T;\mathbb{R})$ based on the weak convergence method. Finally similar to the proof of Theorem 2.6, the contraction principle and the continuity of $\xi(T,x_0)$ finish the proof. \hfill $\square$

3. Pointwise convergence of one-point LDRFs of spatial FDM

For any $n \in \mathbb{N}^+$, $y \in \mathbb{R}$ and $f \in C(\partial T;\mathbb{R})$, denote

$$J_y(f) := \begin{cases} \inf_{\{h \in L^2(\partial T;\mathbb{R}) : \gamma^n(h)=f\}} \frac{1}{2} \|h\|^2_{L^2(\partial T)}, & \text{if } f(T,x_0)=y, \ f \in \text{Im}(\gamma^n), \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$I^n_y(f) := \begin{cases} \inf_{\{h \in L^2(\partial T;\mathbb{R}) : \gamma^n(h)=f\}} \frac{1}{2} \|h\|^2_{L^2(\partial T)}, & \text{if } f(T,x_0)=y, \ f \in \text{Im}(\gamma^n), \\ +\infty, & \text{otherwise}. \end{cases}$$

Then we rewrite rate functions $I^n$ and $I$ as, respectively,

$$I(y) = \inf_{f \in C(\partial T;\mathbb{R})} J_y(f), \quad I^n(y) = \inf_{f \in C(\partial T;\mathbb{R})} I^n_y(f), \quad y \in \mathbb{R}.$$  \hspace{1cm} (3.1)

As is mentioned in the introduction, we will give the pointwise convergence of $I^n$ to $I$ based on the technical route in Fig. 1. We present how to establish the $\Gamma$-convergence and equi-coerciveness of $\{J^n_y\}_{n \in \mathbb{N}^+}$ based on the qualitative analysis for skeleton equations of (1.1) and its spatial FDM. For this end, we need the following assumption.

Assumption 1  For any $x \in \mathbb{R}$, $\sigma(x) \neq 0$.

3.1 $\Gamma$-liminf inequality of $\{J^n_y\}_{n \in \mathbb{N}^+}$

In order to give the properties of $I^n$, we need to introduce the discrete Dirichlet Laplacian. Define the discrete Dirichlet Laplacian $\Delta_n$ by $\Delta_n w(z) = 0$ for $z \in [0, \frac{1}{n}) \cup \{1\}$ and

$$\Delta_n w(z) = n^2 \left( w(\kappa_n(z) + \frac{1}{n}) - 2w(\kappa_n(z)) + w(\kappa_n(z) - \frac{1}{n}) \right), \quad z \in [\frac{1}{n}, 1),$$

where $\kappa_n(z)$ is the solution mapping which takes $h \in L^2(\partial T;\mathbb{R})$ to the solution of the discrete skeleton equation

$$f(t,x) = \int_0^1 G^n_t(x,z)v_0(\kappa_n(z))dz \quad \text{and} \quad \int_0^1 \frac{\partial}{\partial t}G^n_t(x,z) u_0(\kappa_n(z))dz$$

$$\quad + \int_0^1 \int_0^1 G^n_{t-s}(x,z)b(f(s,\kappa_n(z)))dzds + \int_0^1 \int_0^1 G^n_{t-s}(x,z)\sigma(f(s,\kappa_n(z)))h(s,z)dzds.$$  \hspace{1cm} (2.6)
for any measurable function \( w : [0, 1] \to \mathbb{R} \). Then one immediately has \( \Delta_z w(z) = \Delta_z w(\kappa_n(z)), z \in [0, 1] \). It is verified that for any measurable functions \( u, v : [0, 1] \to \mathbb{R} \) with \( u(0) = u(1) = v(0) = v(1) = 0 \), the following integration by parts holds (see also the proof of [15, Lemma 3.2])

\[
\int_0^1 \Delta_n u(x) v(\kappa_n(x)) \, dx = \int_0^1 u(\kappa_n(x)) \Delta_n v(x) \, dx. 
\]  

(3.2)

In addition, a direct computation leads to \( \Delta_n \varphi_j(\kappa_n(y)) = -j^2 \pi^2 e_j^4 \varphi_j(\kappa_n(y)), y \in [0, 1], j = 1, \ldots, n - 1 \), which produces

\[
\frac{\partial^2}{\partial t^2} G^n_t(x, y) = \Delta_n, G^n_t(x, y), \quad \frac{\partial^2}{\partial t^2} G^n_t(\kappa_n(x), y) = \Delta_n, G^n_t(\kappa_n(x), y) \quad \tag{3.3}
\]

for \( (t, x) \in \vartheta_T \), where \( \Delta_n, \kappa_n(\cdot) \) mean that the operator \( \Delta_n \) is posed w.r.t. the variables \( x \) and \( y \), respectively.

Recall \( S_a = \{ h \in L^2(\vartheta_T, \mathbb{R}) : \| h \|_{L^2(\vartheta_T)} \leq a \}, a \in [0, +\infty) \). The following two propositions gives the uniform boundedness and Hölder continuity of \( Y \) and \( Y^n \) on bounded sets. The proof of Proposition 3.1 is standard and is thus omitted. We postpone the proof of Proposition 3.2 to Appendix B.

**Proposition 3.1** For any \( a > 0 \), we have

\[
\sup_{n \in \mathbb{N}^+} \sup_{h \in S_a(t, x) \in \vartheta_T} |Y^n(h)(t, x)| \leq K(a, T), \quad \sup_{h \in S_a(t, x) \in \vartheta_T} |Y(h)(t, x)| \leq K(a, T).
\]

**Proposition 3.2** For any \( a > 0, t \in [0, T] \) and \( x, y \in [0, 1] \),

\[
\sup_{n \in \mathbb{N}^+} \sup_{h \in S_a} \left| \left| Y^n(h)(t, x) - Y^n(h)(s, y) \right| \right| \leq K(a, T)(|x - y|^{1/2} + |t - s|^{1/2}), \tag{3.4}
\]

\[
\sup_{h \in S_a} \left| \left| Y(h)(t, x) - Y(h)(s, y) \right| \right| \leq K(a, T)(|x - y|^{1/2} + |t - s|^{1/2}). \tag{3.5}
\]

**Corollary 3.3** \( Y \) is a compact operator from \( L^2(\vartheta_T, \mathbb{R}) \) to \( C(\vartheta_T, \mathbb{R}) \).

**Proof.** By Propositions 3.1 and 3.2, the subset \( Y(S_a), a > 0 \) is uniformly bounded and equicontinuous in \( C(\vartheta_T, \mathbb{R}) \), which indicates that \( Y(S_a), a > 0 \) is precompact in \( C(\vartheta_T, \mathbb{R}) \) due to the Arzelà–Ascoli theorem. Thus the proof is complete. \( \square \)

According to Fig. 1, in order to obtain the \( \Gamma \)-liminf inequality of \( \{ J^n \}_{n \in \mathbb{N}^+} \), we need to prove the locally uniform convergence of \( Y^n \).

**Proposition 3.4** For any \( a > 0, \delta \in (0, \frac{1}{4}) \) there is \( K(a, \delta, T) > 0 \) such that

\[
\sup_{h \in S_a(t, x) \in \vartheta_T} \left| \left| Y^n(h)(t, x) - Y(h)(t, x) \right| \right| \leq K(a, \delta, T)n^{-\delta}.
\]

**Proof.** Denote \( f^n = Y^n(h) \) and \( f = Y(h) \) for \( h \in S_a \). It follows from Propositions 4 and 5 in [19] that

\[
\sup_{(t, x) \in \vartheta_T} |f^n(t, x) - f(t, x)| \leq K(a, \delta, T)n^{-\delta}, \quad \forall \delta \in (0, \frac{1}{4})
\]

and

\[
\sup_{(t, x) \in \vartheta_T} \left| \frac{\partial}{\partial t} \left( \int_0^1 G^n_t(x, y)u_0(y) \, dy - \frac{\partial}{\partial t} \left( \int_0^1 G^n_t(x, y)u_0(\kappa_n(y)) \, dy \right) \right) \right| \leq K(n^{-\eta})
\]

for any \( \eta \in (0, \frac{1}{4}) \), provided that \( u_0 \in C^2([0, 1]; \mathbb{R}) \) and \( v_0 \in C^1([0, 1]; \mathbb{R}) \). Consequently, one has \( |f^n(\cdot, x) - f(\cdot, x)| \leq Kn^{-\delta} + I_1 + I_2 \) for any \( \delta \in (0, \frac{1}{4}) \), where

\[
I_1 := \int_0^t \int_0^1 |G_{t-s}(x, y)||b(f(s, y)) - b(f(s, \kappa_n(y)))| \, dy \, ds
\]

\[
+ \int_0^t \int_0^1 |G_{t-s}(x, y)||b(f(s, \kappa_n(y))) - b(f(s, \kappa_n(y)))| \, dy \, ds
\]
\[ + \int_0^t \int_0^1 |G_{t-s}(x, y) - G^n_{t-s}(x, y)| b(f^n(x, \kappa_n(y))) \, dy \, ds, \]

\[ I_2 := \int_0^t \int_0^1 |G_{t-s}(x, y)| \| \sigma(f(s, y)) - \sigma(f(s, \kappa_n(y))) \| h(s, y) \, dy \, ds \]

\[ + \int_0^t \int_0^1 |G_{t-s}(x, y)| \| \sigma(f(s, \kappa_n(y))) - \sigma(f^n(s, \kappa_n(y))) \| h(s, y) \, dy \, ds \]

\[ + \int_0^t \int_0^1 |G_{t-s}(x, y) - G^n_{t-s}(x, y)| \| \sigma(f^n(s, \kappa_n(y))) \| h(s, y) \, dy \, ds, \quad (t, x) \in \mathcal{O}_T. \]

It is shown in [19, Lemma 1] that for any \( \delta \in (0, \frac{1}{3}) \), there is \( K(\delta) > 0 \) such that

\[ \sup_{(t, x) \in \mathcal{O}_T} \left( \int_0^1 |G_t(x, y) - G^n_t(x, y)|^2 \, dy \right)^{1/2} \leq K(\delta)n^{-\delta}. \]

It follows from the above formula, the Hölder inequality, Proposition 2.1(1), Proposition 3.1 and (3.5) that

\[ I_2 \leq K(a, T) \left( \int_0^t \int_0^1 |G_{t-s}(x, y)|^2 \, dy \, ds \right)^{1/2} \| h \|_{L^2(\mathcal{O}_T)} \frac{1}{\sqrt{n}} \]

\[ + K \left( \int_0^t \int_0^1 |G_{t-s}(x, y)|^2 \, dy \sup_{x \in [0, 1]} |f(s, x) - f^n(s, x)|^2 \, ds \right)^{1/2} \| h \|_{L^2(\mathcal{O}_T)} \]

\[ + K(a, T) \left( \int_0^t \int_0^1 |G_{t-s}(x, y) - G^n_{t-s}(x, y)|^2 \, dy \, ds \right)^{1/2} \| h \|_{L^2(\mathcal{O}_T)} \]

\[ \leq K(a, \delta, T)n^{-\delta} + K(a, T) \left( \int_0^t \sup_{x \in [0, 1]} |f(s, x) - f^n(s, x)|^2 \, ds \right)^{1/2} \forall \delta \in (0, \frac{1}{3}). \]

Similarly, one has that for any \( \delta \in (0, \frac{1}{3}) \) and \( h \in \mathcal{S}_a \),

\[ I_1 \leq K(a, \delta, T)n^{-\delta} + K(a, T) \left( \int_0^t \sup_{x \in [0, 1]} |f(s, x) - f^n(s, x)|^2 \, ds \right)^{1/2}, \quad (t, x) \in \mathcal{O}_T. \]

In this way, we get that for any \( \delta \in (0, \frac{1}{3}) \) and \( h \in \mathcal{S}_a \),

\[ \sup_{x \in [0, 1]} |f^n(t, x) - f(t, x)|^2 \leq K(a, \delta, T) \left( n^{-2\delta} + \int_0^t \sup_{x \in [0, 1]} |f(s, x) - f^n(s, x)|^2 \, ds \right). \]

Finally the Gronwall inequality finishes the proof. \( \square \)

**Lemma 3.5** Let \( y \in \mathbb{R} \) be arbitrarily fixed. Then for any \( f \in C(\mathcal{O}_T; \mathbb{R}) \) and any sequence \( \{f_n\}_{n \in \mathbb{N}^+} \) converging to \( f \) in \( C(\mathcal{O}_T; \mathbb{R}) \), it holds that

\[ \liminf_{n \to +\infty} J^n_y(f_n) \geq J_y(f). \quad (3.6) \]

**Proof.** In this proof, let \( y \in \mathbb{R} \) be fixed. Let \( \{f_n\}_{n \in \mathbb{N}^+} \) be any sequence converging to \( f \) in \( C(\mathcal{O}_T; \mathbb{R}) \). We may assume that \( A_y := \liminf_{n \to +\infty} J^n_y(f_n) < +\infty \), otherwise (3.6) holds naturally. In this way, there exists a subsequence \( \{n_k\}_{k \in \mathbb{N}^+} \) such that

\[ \lim_{k \to +\infty} J^n_y(f_{n_k}) = \liminf_{n \to +\infty} J^n_y(f_n) = A_y \in [0, +\infty), \]

which implies that \( J^n_y(f_{n_k}) < A_y + 1 \) for \( k \gg 1 \). We may assume that \( J^n_y(f_{n_k}) < A_y + 1 \) for \( k \in \mathbb{N}^+ \). According to the definition of \( J^n_y \), for any \( k \in \mathbb{N}^+ \), we have that \( f_{n_k}(T, x_0) = y \), and there is \( g_{n_k} \in L^2(\mathcal{O}_T; \mathbb{R}) \) such that \( T_{n_k}(g_{n_k}) = f_{n_k} \) and

\[ \frac{1}{2} \| g_{n_k} \|_{L^2(\mathcal{O}_T)}^2 \leq J^n_y(f_{n_k}) + \frac{1}{n_k}. \quad (3.7) \]
This leads to \( f(T, x_0) = \lim_{k \to +\infty} f_{n_k}(T, x_0) = y \) and \( \{g_{n_k}\}_{k \in \mathbb{N}^+} \subseteq \mathbb{S}_{\sqrt{2 \delta}, \frac{1}{4}}. \)

Note that for any \( a > 0, S_a \) is a compact Polish space endowed with the weak topology of \( L^2(\mathcal{O}_T; \mathbb{R}). \)

Therefore, for arbitrarily subsequence \( \{h_{n_k}\}_{k \in \mathbb{N}^+} \subseteq \{g_{n_k}\}_{k \in \mathbb{N}^+}, \) there exist a subsequence \( \{h_{n_{k_j}}\}_{j \in \mathbb{N}^+} \subseteq \{h_{n_k}\}_{k \in \mathbb{N}^+} \)

and \( h \in L^2(\mathcal{O}_T; \mathbb{R}) \) such that \( \lim_{j \to +\infty} h_{n_{k_j}} = h \) w.r.t. the weak topology of \( L^2(\mathcal{O}_T; \mathbb{R}). \) By Corollary 3.3, \( \lim_{j \to +\infty} \mathcal{Y}(h_{n_{k_j}}) = \mathcal{Y}(h) \) in \( \mathcal{C}(\mathcal{O}_T; \mathbb{R}). \) Moreover, it follows from Proposition 3.4 that for any \( \delta \in (0, \frac{1}{1}), \)

\[
\|\mathcal{Y}^{n_k}(g_{n_k}) - \mathcal{Y}(g)\|_{C(\mathcal{O}_T)} \leq \sup_{g \in \mathbb{S}_{\sqrt{2 \delta}, \frac{1}{4}}} \|\mathcal{Y}^{n_k}(g) - \mathcal{Y}(g)\|_{C(\mathcal{O}_T)} \leq K(A_T, \delta, T) n_k^{-\delta},
\]

which yields \( \lim_{k \to +\infty} \mathcal{Y}(g_{n_k}) = \lim_{k \to +\infty} \mathcal{Y}^{n_k}(g_{n_k}) = \lim_{k \to +\infty} f_{n_k} = f \) in \( \mathcal{C}(\mathcal{O}_T; \mathbb{R}). \)

Since \( \{h_{n_{k_j}}\}_{j \in \mathbb{N}^+} \) is also a subsequence of \( \{g_{n_k}\}_{k \in \mathbb{N}^+}, \) \( \mathcal{Y}(h) = \lim_{j \to +\infty} \mathcal{Y}(h_{n_{k_j}}) = f. \) This combined with the definition of \( J_y \) gives

\[
J_y(f) \leq \frac{1}{2} \|h\|^2_{L^2(\mathcal{O}_T)} \leq \frac{1}{2} \liminf_{j \to +\infty} \|h_{n_{k_j}}\|^2_{L^2(\mathcal{O}_T)}, \tag{3.8}
\]

where we used the weakly lower semicontinuity of norms and the fact that \( \{h_{n_{k_j}}\}_{j \in \mathbb{N}^+} \) converges weakly to \( h. \)

Combining the above discussion, we deduce that for any subsequence \( \{h_k\}_{k \in \mathbb{N}^+} \) of \( \{g_{n_k}\}_{k \in \mathbb{N}^+}, \) there is a subsequence \( \{h_{n_{k_j}}\}_{j \in \mathbb{N}^+} \) satisfying (3.8). Thus one has

\[
J_y(f) \leq \frac{1}{2} \liminf_{k \to +\infty} \|g_{n_k}\|^2_{L^2(\mathcal{O}_T)} \leq \liminf_{n \to +\infty} J_y^{n_k}(f_n) = \liminf_{n \to +\infty} J_y^n(f_n),
\]

where we used (3.7). Thus the proof is complete. \( \square \)

### 3.2 \( \Gamma \)-limsup inequality of subsequences of \( \{J_y^n\}_{n \in \mathbb{N}^+} \)

We proceed to follow the technical route in Fig. 1 to give the \( \Gamma \)-limsup inequality of subsequences of \( \{J_y^n\}_{n \in \mathbb{N}^+}. \)

For this end, we show that \( \mathcal{Y} \) is locally Lipschitz continuous in Proposition 3.6 and that \( \mathcal{Y}^n \) restricted on a special set is invertible in Proposition 3.7.

**Proposition 3.6** For any \( a \geq 0 \) and \( h_1, h_2 \in S_a. \) We have

\[
\sup_{(t, x) \in \mathcal{O}_T} |\mathcal{Y}(h_1)(t, x) - \mathcal{Y}(h_2)(t, x)| \leq K(a, T) \|h_1 - h_2\|_{L^2(\mathcal{O}_T)}.
\]

The proof of Proposition 3.6 comes from Proposition 2.1(1) and Proposition 3.1, and thus we omit its details.

Denote \( \mathcal{M}_n(\mathcal{O}_T; \mathbb{R}) := \{w \in \mathcal{C}(\mathcal{O}_T; \mathbb{R}) : w(0, \cdot) = \Pi_n(u_0)(\cdot), w(\cdot, 0) = w(\cdot, 1) = 0, w(\cdot, x) \in C^1([0, T]; \mathbb{R}) \text{ for any } x \in [0, 1], \frac{\partial^2}{\partial t^2} w(t, x) \}

exists for any \( (t, x) \in \mathcal{O}_T, \frac{\partial^2}{\partial t^2} w(\cdot, \kappa_n(\cdot)) \in L^2(\mathcal{O}_T; \mathbb{R}), \frac{\partial}{\partial t} w(0, \cdot) = \Pi_n(u_0)(\cdot), \text{ and } w(t, x) = \Pi_n(w(t, \cdot))(x), (t, x) \in \mathcal{O}_T \}. \) Then we have the following result.

**Proposition 3.7** The following properties hold.

1. (For any given \( n \in \mathbb{N}^+, \mathcal{Y}^n(\mathcal{L}^2(\mathcal{O}_T; \mathbb{R})) \subseteq \mathcal{M}_n(\mathcal{O}_T; \mathbb{R}). \))
2. (Let Assumption 1 hold and \( f \in \mathcal{M}_n(\mathcal{O}_T; \mathbb{R}). \) Then there is a unique \( h \in \mathcal{N}_n(\mathcal{O}_T; \mathbb{R}) := \{g \in \mathcal{L}^2(\mathcal{O}_T; \mathbb{R}) : g(t, x) = g(t, \kappa_n(x)), (t, x) \in \mathcal{O}_T \} \) such that \( \mathcal{Y}^n(h) = f. \) And in this case, \( h \) can be represented as

\[
h(t, x) = \frac{1}{\sigma(f(t, \kappa_n(x)))} \left[ \frac{\partial^2}{\partial t^2} f(t, \kappa_n(x)) - \Delta_n f(t, \kappa_n(x)) - b(f(t, \kappa_n(x))) \right], (t, x) \in \mathcal{O}_T. \tag{3.9}
\]

*Proof.* (1) It suffices to show that \( f := \mathcal{Y}^n(g) \in \mathcal{M}_n(\mathcal{O}_T; \mathbb{R}) \) for any \( g \in \mathcal{L}^2(\mathcal{O}_T; \mathbb{R}) \) with \( n \in \mathbb{N}^+ \) being fixed. According to Proposition 3.2, \( f \in \mathcal{C}(\mathcal{O}_T; \mathbb{R}). \) By (2.2) and (2.6), we have

\[
f(t, 0) = f(t, 1) = 0, t \in [0, T] \text{ and } f(0, x) = \int_0^1 \frac{\partial}{\partial y} G_0^2(x, y) u_0(\kappa(y)) \text{dy}
\]
By the integration by parts, we have
\[
\frac{\partial}{\partial t} f(t,x) = \int_0^1 \frac{\partial}{\partial t} G^n_t(x,y) v_0(\kappa_n(y)) dy + \int_0^1 \frac{\partial^2}{\partial t^2} G^n_t(x,y) u_0(\kappa_n(y)) dy - \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_{t-s}(x,y) b(f(s, \kappa_n(y))) dy ds + \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_{t-s}(x,y) \sigma(f(s, \kappa_n(y))) g(x,z) dz ds,
\]
where we used \( G^n_0(x,y) = 0, \forall x,y \in [0,1] \). Thus for any \( x \in [0,1] \), \( \frac{\partial}{\partial t} f(t,x) \) is continuous w.r.t. \( t \). Similarly, one can verify that \( \frac{\partial^2}{\partial t^2} f(t,x) \) exists for \( (t,x) \in \mathcal{O}_T \) and \( \frac{\partial^2}{\partial t^2} f(\cdot, \kappa_n(\cdot)) \in L^2(\mathcal{O}_T;\mathbb{R}) \). Using (3.10), (B.1) and \( \frac{\partial^2}{\partial t^2} G^n_0(x,y) = 0, \forall x,y \in [0,1] \) gives \( \frac{\partial}{\partial t} f(0,x) = \int_0^1 \frac{\partial}{\partial t} G^n_0(x,y) v_0(\kappa_n(y)) dy = \Pi_n(v_0)(x) \). Finally, by the definition of \( G^n, f(t,x) = \Pi_n(f(f(\cdot))) \), which proves \( f \in \mathcal{M}(\mathcal{O}_T;\mathbb{R}) \).

(2) We divide the proof of the second conclusion into two steps.

**Step 1: We prove that \( h \) defined by (3.9) satisfies \( Y^n(h) = f \).** Denote
\[
\Phi(f,h)(t,x) := \int_0^1 G^n_t(x,y) v_0(\kappa_n(y)) dy + \int_0^1 \frac{\partial}{\partial t} G^n_t(x,y) u_0(\kappa_n(y)) dy
\]
\[
+ \int_0^t \int_0^1 G^n_{t-s}(x,y) b(f(s, \kappa_n(y))) dy ds
\]
\[
+ \int_0^t \int_0^1 G^n_{t-s}(x,y) \sigma(f(s, \kappa_n(y))) h(y,s) dy ds, \quad (t,x) \in \mathcal{O}_T.
\]

Repeating the proof of the first conclusion, one has \( \Phi(f,h) \in \mathcal{M}(\mathcal{O}_T;\mathbb{R}) \). Hence, it suffices to prove \( \Phi(f,h)(t,\kappa_n(x)) = f(t, \kappa_n(x)), \forall (t,x) \in \mathcal{O}_T \), due to \( f \in \mathcal{M}(\mathcal{O}_T;\mathbb{R}) \). Substituting (3.9) into \( \Phi(f,h) \) yields that for \( (t,x) \in \mathcal{O}_T \),
\[
\Phi(f,h)(t,\kappa_n(x)) = \int_0^1 G^n_t(\kappa_n(x),y) v_0(\kappa_n(y)) dy + \int_0^1 \frac{\partial}{\partial t} G^n_t(\kappa_n(x),y) u_0(\kappa_n(y)) dy
\]
\[
+ V_1(t,x) - V_2(t,x)
\]

with \( V_1(t,x) := \int_0^t \int_0^1 G^n_{t-s}(\kappa_n(x),y) \frac{\partial^2}{\partial s^2} f(s, \kappa_n(y)) dy ds \) and \( V_2(t,x) := \int_0^t \int_0^1 G^n_{t-s}(\kappa_n(x),y) \kappa_n f(s, \kappa_n(y)) dy ds \).

By the integration by parts, \( \frac{\partial}{\partial t} f(0,\cdot) = \Pi_n(v_0)(\cdot) \) and \( G^n_0 \equiv 0 \), we have
\[
V_1(t,x)
\]
\[
= \int_0^1 \left( G^n_0(\kappa_n(x),y) \frac{\partial}{\partial t} f(t, \kappa_n(y)) - G^n_t(\kappa_n(x),y) \frac{\partial}{\partial t} f(0, \kappa_n(y)) \right) dy
\]
\[
- \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_{t-s}(\kappa_n(x),y) \frac{\partial}{\partial s} f(s, \kappa_n(y)) dy ds
\]
\[
= - \int_0^1 G^n_t(\kappa_n(x),y) v_0(\kappa_n(y)) dy - \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_{t-s}(\kappa_n(x),y) \frac{\partial}{\partial s} f(s, \kappa_n(y)) dy ds.
\]

By (3.2), (3.3), (B.1) and \( f \in \mathcal{M}(\mathcal{O}_T;\mathbb{R}) \),
\[
V_2(t,x)
\]
\[
= \int_0^t \int_0^1 \frac{\partial^2}{\partial s^2} G^n_{t-s}(\kappa_n(x),y) f(s, \kappa_n(y)) dy ds
\]
\[
= - \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_0(\kappa_n(x),y) f(t, \kappa_n(y)) dy + \int_0^1 \frac{\partial}{\partial t} G^n_t(\kappa_n(x),y) f(0, \kappa_n(y)) dy
\]
\[
- \int_0^t \int_0^1 \frac{\partial}{\partial s} G^n_{t-s}(\kappa_n(x),y) \frac{\partial}{\partial s} f(s, \kappa_n(y)) dy ds
\]
\[
= \int_0^1 \frac{\partial}{\partial t} G^n_t(\kappa_n(x),y) u_0(\kappa_n(y)) dy - f(t, \kappa_n(x))
\]
$$- \int_0^t \int_0^1 \frac{d}{ds} G^n_{t-s}(\kappa_n(x),y) \frac{d}{ds} f(s,\kappa_n(y)) dy ds.$$ 

Substituting $V_1$ and $V_2$ into $\Phi(f,h)$ yields $\Phi(f,h)(t,\kappa_n(x)) = f(t,\kappa_n(x))$, which implies $\Gamma^n(h) = f$. 

**Step 2:** We prove that if $g \in \mathcal{N}_n(\mathcal{O}_T;\mathbb{R})$ satisfies $\Gamma^n(g) = f$, then $g$ is given by the right-hand side of (3.9). Let $\Gamma^n(g) = f$ and $g \in \mathcal{N}_n(\mathcal{O}_T;\mathbb{R})$. It follows from (3.3), (3.10) and (B.1) that

$$\frac{d^2}{dt^2} f(t,\kappa_n(x))$$

$$= \Delta_n \left[ \int_0^1 G^n_{t-s}(\kappa_n(x),z) v_0(\kappa_n(z)) dz + \int_0^1 \frac{d}{dt} G^n_{t-s}(\kappa_n(x),z) u_0(\kappa_n(z)) dz \right]$$

$$+ \int_0^1 \int_0^1 G^n_{t-s}(\kappa_n(x),z) b(f(s,\kappa_n(z))) dz ds + \int_0^1 \int_0^1 G^n_{t-s}(\kappa_n(x),z) \sigma(f(s,\kappa_n(z))) g(s,z) dz ds$$

$$+ \int_0^1 \frac{d}{dt} G^n_{t}(\kappa_n(x),z) b(f(t,\kappa_n(z))) dz + \int_0^1 \frac{d}{dt} G^n_{t}(\kappa_n(x),z) \sigma(f(t,\kappa_n(z))) g(t,\kappa_n(z)) dz$$

$$= \Delta_n f(t,\kappa_n(x)) + b(f(t,\kappa_n(x))) + \sigma(f(t,\kappa_n(x))) g(t,\kappa_n(x)), \ (t,x) \in \mathcal{O}_T.$$ 

Thus, $g$ is given by the right-hand side of (3.9) due to $g(t,x) = g(t,\kappa_n(x))$, which finishes the proof. 

**Lemma 3.8** Let Assumption 1 hold. Then for any subsequence $\{J^m_y \}_{k \in \mathbb{N}^+} \subseteq \{J^n_y \}_{n \in \mathbb{N}^+}$ with $y \in \mathbb{R}$ being fixed, there is a subsequence $\{J^m_y \}_{j \in \mathbb{N}^+} \subseteq \{J^n_y \}_{k \in \mathbb{N}^+}$ such that

1. For any $f \in C(\mathcal{O}_T;\mathbb{R})$, there exists a sequence $\{f_j \}_{j \in \mathbb{N}^+}$ converging to $f$ in $C(\mathcal{O}_T;\mathbb{R})$ and

$$\limsup_{j \to +\infty} J^m_y(f_j) \leq J_y(f).$$

2. For any $f \in C(\mathcal{O}_T;\mathbb{R})$, $(\Gamma^m \limsup_{j \to +\infty} J^m_y(f_j))(f) \leq J_y(f).$

**Proof.** (1) One only needs to prove (3.11) for the case $J_y(f) < +\infty$. In this case, $f(T,x_0) = y$ and $f \in \text{Im} \mathcal{Y}$. Hence, for any $j \in \mathbb{N}^+$ there is $g_j \in L^2(\mathcal{O}_T;\mathbb{R})$ such that $\mathcal{Y}(g_j) = f$ and

$$\frac{1}{2} \|g_j\|_{L^2(\mathcal{O}_T)}^2 \leq J_y(f) + \frac{1}{j}.$$ 

(3.12)

Since $C^\infty(\mathcal{O}_T;\mathbb{R})$ is dense in $L^2(\mathcal{O}_T;\mathbb{R})$, there is $\phi^j \in C^\infty(\mathcal{O}_T;\mathbb{R})$ such that $\|\phi^j - g_j\|_{L^2(\mathcal{O}_T)} < \frac{1}{j}$. Let $\{n_k \}_{k \in \mathbb{N}^+}$ be any given subsequence and define $\phi^j_{n_k}(t,x) := \phi^j(t,\kappa_{n_k}(x)), \ (t,x) \in \mathcal{O}_T, j, k \in \mathbb{N}^+$. Further, for any $j \in \mathbb{N}^+$, it holds that $\lim_{k \to +\infty} \|\phi^j_{n_k} - \phi^j\|_{L^2(\mathcal{O}_T)} = 0$, which yields that for any $j \in \mathbb{N}^+$, there is $k_j \in \mathbb{N}^+$ such that $\|\phi^j_{n_{k_j}} - \phi^j\|_{L^2(\mathcal{O}_T)} < \frac{1}{j}$. Denoting $\tilde{h}_j := \phi^j_{n_{k_j}}$, we obtain that $\tilde{h}_j \in \mathcal{N}_{n_{k_j}}(\mathcal{O}_T;\mathbb{R}), j \in \mathbb{N}^+$ and $\lim_{j \to +\infty} \|\tilde{h}_j - g_j\|_{L^2(\mathcal{O}_T)} = 0$. This combined with (3.12) gives

$$\limsup_{j \to +\infty} \|\tilde{h}_j\|_{L^2(\mathcal{O}_T)}^2 \leq \limsup_{j \to +\infty} \|g_j\|_{L^2(\mathcal{O}_T)}^2 \leq 2J_y(f).$$

(3.13)

Define $\tilde{f}_j = \Gamma^{n_{k_j}}(\tilde{h}_j), \ j \in \mathbb{N}^+$. By (3.13), $\sup_{j \in \mathbb{N}^+} \|\tilde{h}_j\|_{L^2(\mathcal{O}_T)} + \sup_{j \in \mathbb{N}^+} \|g_j\|_{L^2(\mathcal{O}_T)} \leq K(y,f)$. Thus, it follows from Propositions 3.4 and 3.6 that

$$\lim_{j \to +\infty} \|\tilde{f}_j - f\|_{C(\mathcal{O}_T)} = \lim_{j \to +\infty} \|\Gamma^{n_{k_j}}(\tilde{h}_j) - \Gamma_y(g_j)\|_{C(\mathcal{O}_T)}$$

$$\leq \lim_{j \to +\infty} \|\Gamma^{n_{k_j}}(\tilde{h}_j) - \Gamma(\tilde{h}_j)\|_{C(\mathcal{O}_T)} + \lim_{j \to +\infty} \|\Gamma(\tilde{h}_j) - \Gamma_y(g_j)\|_{C(\mathcal{O}_T)}$$

$$\leq \lim_{j \to +\infty} \sup_{h \in \mathbb{R}} \|\Gamma^{n_{k_j}}(h) - \Gamma(h)\|_{C(\mathcal{O}_T)} + K(y,f,T) \lim_{j \to +\infty} \|\tilde{h}_j - g_j\|_{L^2(\mathcal{O}_T)} = 0.$$
Accordingly, one immediately has \( \lim_{j \to +\infty} y_j = y \) with \( y_j := \tilde{f}_j(T, x_0) \) due to (3.14) and \( f(T, x_0) = y \). Since \( y_j \neq y \) generally, the sequence \( \{\tilde{f}_j\}_{j \in \mathbb{N}^+} \) could not be a candidate for (3.11). For this end, we need to construct so-called “modification functions” to modify \( \tilde{f}_j \) to get our goal sequence satisfying \( f_j(T, x_0) = y, j \in \mathbb{N}^+ \) and \( \{f_j\}_{j \in \mathbb{N}^+} \subseteq \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}) \).

Since \( x_0 \in (0, 1) \), there is \( J(x_0) \in \mathbb{N}^+ \) such that \( \frac{1}{n_j} \leq \kappa_{n_j} (x_0) \leq \frac{\eta_j - 2}{n_j} \) for \( j \geq J(x_0) \). For \( j \geq J(x_0) \), we define \( p_j : \mathcal{O} \to \mathbb{R} \) by

\[
\begin{align*}
p_j(x) &= \begin{cases} 
(y - y_j) \left[ 1 + \frac{x - \kappa_{n_j} (x_0)}{\kappa_{n_j} (x_0)} \right]^3, & x \in [0, \kappa_{n_j} (x_0)], \\
y - y_j, & x \in [\kappa_{n_j} (x_0), \kappa_{n_j} (x_0) + 1/n_j], \\
(y - y_j) \left[ 1 - \frac{x - (\kappa_{n_j} (x_0) + 1/n_j)}{1 - (\kappa_{n_j} (x_0) + 1/n_j)} \right]^3, & x \in [\kappa_{n_j} (x_0) + 1/n_j, 1].
\end{cases}
\end{align*}
\]

Then we define \( f_j(t, x) = \tilde{f}_j(t, x) + \frac{\partial^2}{\partial x^2} w_j(x), (t, x) \in \mathcal{O}_T, j \geq J(x_0) \), where the modification term \( w_j \) is the linear interpolation of \( p_j \) with interpolation nodes \( 0, \frac{1}{n_j}, \ldots, \frac{n_j - 1}{n_j}, 1 \), i.e., \( w_j = \Pi_{n_j} (p_j) \). Without loss of generality, we may assume \( J(x_0) = 1 \) since we only need to study the properties of \( f_j \) for sufficiently large \( j \) in order to prove (3.11). By Proposition 3.7(1) and \( \tilde{f}_j = \Psi^{n_j} (\tilde{h}_j) \), we have \( \tilde{h}_j \in \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}), j \in \mathbb{N}^+ \). Thus, it holds that \( f_j(T, x_0) = y \) and \( f_j \in \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}) \) due to \( w_j(x_0) = p_j(x_0) = y - y_j, w_j(0) = w_j(1) = 0 \) and \( \tilde{h}_j \in \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}) \), \( j \in \mathbb{N}^+ \).

Next we show that \( \{f_j\}_{j \in \mathbb{N}^+} \) converges to \( f \) and (3.11) holds. Noting that \( \lim_{j \to +\infty} \|w_j \|_{C([0, 1])} = \lim_{j \to +\infty} \|y - y_j \| = 0 \), we have \( \lim_{j \to +\infty} \|f_j - \tilde{f}_j \|_{C(\mathcal{O}_T)} = 0 \), which along with (3.14) leads to \( \lim_{j \to +\infty} f_j = f \) in \( C(\mathcal{O}_T; \mathbb{R}) \). It follows from \( \tilde{f}_j = \Psi^{n_j} (\tilde{h}_j) \), \( \tilde{h}_j \in \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}) \), and Proposition 3.7(2) that for \( (t, x) \in \mathcal{O}_T \) and \( j \in \mathbb{N}^+ \),

\[
\tilde{h}_j(t, x) = \frac{1}{\sigma(f_j(t, \kappa_{n_j} (x)))} \left[ \frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j} (x)) - \Delta_{n_j} f_j(t, \kappa_{n_j} (x)) - b(f_j(t, \kappa_{n_j} (x))) \right].
\]

Further, for \( (t, x) \in \mathcal{O}_T \) and \( j \in \mathbb{N}^+ \), define

\[
h_j(t, x) := \frac{1}{\sigma(f_j(t, \kappa_{n_j} (x)))} \left[ \frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j} (x)) - \Delta_{n_j} f_j(t, \kappa_{n_j} (x)) - b(f_j(t, \kappa_{n_j} (x))) \right].
\]

Again by Proposition 3.7(2), \( h_j \in \mathcal{M}_{\kappa_j}(\mathcal{O}_T; \mathbb{R}) \) and \( \Psi^{n_j} (h_j) = f_j \), \( j \in \mathbb{N}^+ \). We claim

\[
\lim_{j \to +\infty} \|h_j - \tilde{h}_j\|_{L^2(\mathcal{O}_T)} = 0.
\]

Using this claim, \( \lim_{j \to +\infty} \|h_j - g\|_{L^2(\mathcal{O}_T)} = 0 \), \( f_j \in \mathcal{O}_{\kappa_j} \) and (3.12), we derive

\[
\lim_{j \to +\infty} \|f_j\|_{L^2(\mathcal{O}_T)} \leq \frac{1}{2} \lim_{j \to +\infty} \|h_j\|_{L^2(\mathcal{O}_T)} \leq \frac{1}{2} \lim_{j \to +\infty} \|g\|_{L^2(\mathcal{O}_T)} \leq J_0(f),
\]

which proves (3.11). Thus, (3.11) is true once we justify the claim (3.17).

Next we prove (3.17). Noting that

\[
\frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j} (x)) - \Delta_{n_j} f_j(t, \kappa_{n_j} (x)) - \left( \frac{\partial^2}{\partial t^2} \tilde{f}_j(t, \kappa_{n_j} (x)) - \Delta_{n_j} \tilde{f}_j(t, \kappa_{n_j} (x)) \right) = \frac{2}{T^2} w_j(\kappa_{n_j} (x)) - \frac{\eta_j}{T^2} \Delta_{n_j} w_j(\kappa_{n_j} (x)), (t, x) \in \mathcal{O}_T,
\]

we get

\[
\frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j} (x)) - \Delta_{n_j} f_j(t, \kappa_{n_j} (x)) = \frac{2}{T^2} w_j(\kappa_{n_j} (x)), (t, x) \in \mathcal{O}_T.
\]
we have $h_j(t, x) = E_1(t, x) + E_2(t, x) + E_3(t, x)$, $(t, x) \in \mathcal{O}_T$ with

$$E_1(t, x) := \left( \frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j}(x)) - \Delta_{n_j} f_j(t, \kappa_{n_j}(x)) \right) \left( \frac{1}{\sigma(f_j(t, \kappa_{n_j}(x)))} - \frac{1}{\sigma(f_j(t, \kappa_{n_j}(x)))} \right).$$

$$E_2(t, x) := \frac{b_j(f_j(t, \kappa_{n_j}(x)))}{\sigma(f_j(t, \kappa_{n_j}(x)))} - \frac{b_j(f_j(t, \kappa_{n_j}(x)))}{\sigma(f_j(t, \kappa_{n_j}(x)))}, E_3(t, x) := \frac{2 w_j(\kappa_{n_j}(x)) - \tau^2 \Delta_{n_j} w_j(\kappa_{n_j}(x))}{\tau^2 \sigma(f_j(t, \kappa_{n_j}(x)))}.$$

Since $\lim_{j \to +\infty} f_j = \lim_{j \to +\infty} f_j = f$ in $C(\mathcal{O}_T; \mathbb{R})$, sup $\|f_j\|_{C(\mathcal{O}_T)} + \|\hat{f}_j\|_{C(\mathcal{O}_T)} < +\infty$, which yields that $\inf_{j \in \mathbb{N}^+} \inf_{(t, x) \in \mathcal{O}_T} |\sigma(f_j(t, x))| \geq c_1 > 0$ and $\inf_{j \in \mathbb{N}^+} \inf_{(t, x) \in \mathcal{O}_T} |\sigma(f_j(t, x))| \geq c_2 > 0$ for two constants $c_1$ and $c_2$ due to Assumption 1. This combined with $\lim_{j \to +\infty} (\sigma(f_j) - \sigma(f_j)) = 0$ in $C(\mathcal{O}_T; \mathbb{R})$ produces

$$\lim_{j \to +\infty} \sup_{(t, x) \in \mathcal{O}_T} \left| \frac{1}{\sigma(f_j(t, \kappa_{n_j}(x)))} - \frac{1}{\sigma(f_j(t, \kappa_{n_j}(x)))} \right| = 0. \quad (3.18)$$

Further, it follows from $\sup_{j \in \mathbb{N}^+} \|\hat{f}_j\|_{L^2(\mathcal{O}_T)} \leq K(y, f)$, $\sup_{j \in \mathbb{N}^+} (\|\sigma(f_j)\|_{C(\mathcal{O}_T)} + \|b(f_j)\|_{C(\mathcal{O}_T)}) < +\infty$ and (3.15) that

$$\sup_{j \in \mathbb{N}^+} \int_0^T \int_0^1 \left| \frac{\partial^2}{\partial t^2} f_j(t, \kappa_{n_j}(x)) - \Delta_{n_j} f_j(t, \kappa_{n_j}(x)) \right|^2 \, dx \, dt < +\infty. \quad (3.19)$$

Combining (3.18) and (3.19), one has that $E_1 \to 0$ in $L^2(\mathcal{O}_T; \mathbb{R})$ as $j \to +\infty$. It is easy to verify that $\lim_{j \to +\infty} \|b(f_j) - b(f_j)\|_{C(\mathcal{O}_T)} = 0$, which together with (3.18) and $\inf_{j \in \mathbb{N}^+} \inf_{(t, x) \in \mathcal{O}_T} |\sigma(f_j(t, x))| \geq c_1 > 0$ gives $E_2 \to 0$ in $L^2(\mathcal{O}_T; \mathbb{R})$ as $j \to +\infty$. A direct computation gives $p_j \in C^2([0, 1]; \mathbb{R})$ and $\|p_j\|_{C([0, 1])} \leq 6|y - y_j| \max \left\{ |\kappa_{n_j}(x_0)|^{-2}, 1 - (\kappa_{n_j}(x_0) + 1/n_j)^{-2} \right\}, j \in \mathbb{N}^+$. As a consequence, the Taylor formula yields for $x \in [0, 1]$.

$$|\Delta_{n_j} w_j(\kappa_{n_j}(x_0)| = n_j^2 |p_j(\kappa_{n_j}(x_0) + 1/n_j) - 2p_j(\kappa_{n_j}(x_0)) + p_j(\kappa_{n_j}(x_0) - 1/n_j)| \leq \|p_j\|_{C([0, 1])}. \quad (3.16)$$

Thus, we have $\lim_{j \to +\infty} |\Delta_{n_j} w_j(\kappa_{n_j}(x_0)| = 0$. The above formula, $\inf_{j \in \mathbb{N}^+} \inf_{(t, x) \in \mathcal{O}_T} |\sigma(f_j(t, x))| \geq c_1 > 0$ and

$$\lim_{j \to +\infty} \|w_j\|_{C([0, 1])} = 0 \text{ lead to } E_3 \to 0 \text{ in } L^2(\mathcal{O}_T; \mathbb{R}) \text{ as } j \to +\infty. \text{ In this way, we prove the claim (3.17) and thus finish the proof of (3.11).}$$

(2) Taking $\{f_j\}_{j \in \mathbb{N}^+}$ be the sequence satisfying (3.11), we obtain that $F := \inf_{j \to +\infty} \lim_{j \to +\infty} (F_j) = \lim_{j \to +\infty} F_j = f$ in $C(\mathcal{O}_T; \mathbb{R}) \leq \lim_{j \to +\infty} \sup_{j \to +\infty} (F_j) \leq J_y(f). \quad \square$

3.3 Pointwise convergence of $I^n$

In this part, we give the $I^\gamma$-convergence and equi-coerciveness of $\{J_y^n\}_{n \in \mathbb{N}^+}$, and then prove the pointwise convergence of $I^\gamma$, as shown in Fig. 1.

**Theorem 3.9** Let Assumption 1 hold and $y \in \mathbb{R}$ be fixed. Then for any subsequence $\{J_{y, k}^n\}_{k \in \mathbb{N}^+}$ of $\{J_{y, x}^n\}_{n \in \mathbb{N}^+}$, there is a subsequence $\{J_{y, j}^{n_j}\}_{j \in \mathbb{N}^+}$ which $I^\gamma$-converges to $J_y$ on $C(\mathcal{O}_T; \mathbb{R})$. Thus, $\{J_{y, x}^n\}_{n \in \mathbb{N}^+}$ $I^\gamma$-converges to $J_y$ on $C(\mathcal{O}_T; \mathbb{R})$.

**Proof.** It follows from Lemma 3.8(2), for any subsequence $\{J_{y, k}^n\}_{k \in \mathbb{N}^+}$ of $\{J_{y, x}^n\}_{n \in \mathbb{N}^+}$, there is a subsequence $\{J_{y, j}^{n_j}\}_{j \in \mathbb{N}^+}$ such that $(I^\gamma \limsup_{j \to +\infty} J_{y, j}^{n_j}) (f) \leq J_y(f)$ for any $f$ in $C(\mathcal{O}_T; \mathbb{R})$. Further, by Lemma 3.5, we have that
for any $f \in C(\partial_T; \mathbb{R})$,

$$(\Gamma\liminf_{n \to +\infty} J^n_y)(f) := \inf \{ \liminf_{n \to +\infty} J^n_y(f_n) : \lim_{n \to +\infty} f_n = f \text{ in } C(\partial_T; \mathbb{R}) \} \geq J_y(f),$$

which along with Proposition A.4 implies $(\Gamma\liminf_{n \to +\infty} J^n_y)(f) \geq J_y(f)$. Thus, it holds that $\{J^n_y\}_{j \in \mathbb{N}^+}$ $\Gamma$-converges to $J_y$ on $C(\partial_T; \mathbb{R})$. Finally, the proof is complete according to Proposition A.5.

Lemma 3.10 For any $y \in \mathbb{R}$, $\{J^n_y\}_{n \in \mathbb{N}^+}$ is equi-coercive on $C(\partial_T; \mathbb{R})$.

Proof. Fix $y \in \mathbb{R}$ and denote $\Psi_T^a(a) := \{ f \in C(\partial_T; \mathbb{R}) : J^n_y(f) \leq a \}$, $a \in \mathbb{R}$, $n \in \mathbb{N}^+$. It suffices to prove that for any $a \in \mathbb{R}$, there is a compact set $\mathbb{K}_a$ such that

$$\bigcup_{n \in \mathbb{N}^+} \Psi_T^a(a) \subseteq \mathbb{K}_a. \quad (3.20)$$

Note that (3.20) holds naturally for $a < 0$ due to the non-negativity of $J^n_y$. Thus it remains to prove (3.20) for $a \geq 0$.

For any $n \in \mathbb{N}^+$ and any $f_n \in \Psi_T^a(a)$, $a \geq 0$, we have $J^n_y(f_n) \leq a < +\infty$. According to the definition of $J^n_y$, for any $n \in \mathbb{N}^+$, there is $h_n \in L^2(\partial_T; \mathbb{R})$ such that $J^n_y(h_n) = f_n$ and $\frac{1}{2}\|h_n\|_{L^2(\partial_T)} < J^n_y(f_n) + 1 \leq a + 1$. Thus one has $\{h_n\}_{n \in \mathbb{N}^+} \subseteq \mathbb{S}_{2a+2}^j$, which together with Proposition 3.1 and (3.4) implies that for any $n \in \mathbb{N}^+$,

$$\|f_n\|_{C(\partial_T)} \leq K(a, T), \quad |f_n(t,x) - f_n(s,y)| \leq K(a, T)(|x - y|^2 + |t - s|^2)^{1/4}.$$

Consequently, we obtain $f_n \in \{g \in C^{1/2}(\partial_T; \mathbb{R}) : \|g\|_{C^{1/2}(\partial_T)} \leq K(a, T)\} =: \mathbb{K}_a$, for any $n \in \mathbb{N}^+$, i.e., $\Psi_T^a(a) \subseteq \mathbb{K}_a$ for any $n \in \mathbb{N}^+$. By the Arzelà–Ascoli theorem, $\mathbb{K}_a$ is compact in $C(\partial_T; \mathbb{R})$ for $a \geq 0$. Thus the proof is complete due to Definition A.6.

Theorem 3.11 Let Assumption 1 hold. Then $\lim_{n \to +\infty} J^n_y(y) = I(y)$, $y \in \mathbb{R}$.

Proof. For any given $y \in \mathbb{R}$, it follows from Theorem 3.9 and Lemma 3.10 that $\{J^n_y\}_{n \in \mathbb{N}^+}$ is equi-coercive and $\Gamma$-converges to $J_y$ on $C(\partial_T; \mathbb{R})$. Thus Theorem A.7 and (3.1) finishes the proof.

4. Conclusions

In this work, we propose a new technical route for tackling the convergence of one-point LDRFs of the spatial FDMs for stochastic wave equations with small noise. The technical route mainly depends on the qualitative analysis of skeleton equations of the original equations and its numerical discretizations, which provides a promising approach to analyzing one-point LDRFs of numerical discretizations for other nonlinear SPDEs with small noise. For example, we believe that the new technical route also works for the spatial FDM of stochastic heat equations.

This paper deals with the case of spatial semi-discretizations for SPDEs. On this basis, we can further study the convergence of one-point LDRFs of full discretizations for SPDEs. On one hand, it is possible to use our technical route to show that one-point LDRFs of full discretizations converge to those of the corresponding spatial semi-discretizations. On the other hand, the semi-discretizations for the original equations are SODEs with small noise, and thus one can use the method in [14] to study the convergence of one-point LDRFs of full discretizations for the original equations. We also refer interested readers to the numerical experiments in [5, 14] for the numerical simulation of the one-point LDRFs.

It is interesting to investigate the convergence order of one-point LDRFs of the spatial FDM for stochastic wave equations, and to relate it to its strong or weak convergence order. In fact, there have been no results revealing the relationship between the convergence orders of LDRFs of numerical discretizations and the strong (weak) convergence orders of numerical discretizations, and we leave it as an open problem.
Appendix

A. $\Gamma$-convergence

In this part, we introduce some definitions and results in the theory of $\Gamma$-convergence. We refer the interested readers to [11, 20] for more details on $\Gamma$-convergence. Let $X$ be a metric space and $\mathbb{R}^\infty = \mathbb{R} \cup \{\pm\infty\}$ denote the set of extended real numbers. In this part, we always let $F_n, F : X \to \mathbb{R}, n \in \mathbb{N}^+$ be given functionals.

**Definition A.1** The sequence $\{F_n\}_{n \in \mathbb{N}^+}$ is said to $\Gamma$-converge to $F$, if

1. For all sequences $\{x_n\}_{n \in \mathbb{N}^+} \subseteq X$ with $\lim_{n \to +\infty} x_n = x$ in $X$, the liminf inequality holds:
   \[
   \liminf_{n \to +\infty} F_n(x_n) \geq F(x).
   \]
2. For any $x \in X$, there is a recovery sequence $\{x_n\}_{n \in \mathbb{N}^+}$ such that $\lim_{n \to +\infty} x_n = x$ in $X$ and
   \[
   \limsup_{n \to +\infty} F_n(x_n) \leq F(x).
   \]

**Remark A.2** Notice that under the first condition of Definition A.1, the second condition is equivalent to that for any $x \in X$, there is $\{x_n\}_{n \in \mathbb{N}^+}$ converging to $x$ in $X$ and $\lim_{n \to +\infty} F_n(x_n) = F(x)$.

**Definition A.3** The $\Gamma$-lower limit and the $\Gamma$-super limit of $\{F_n\}_{n \in \mathbb{N}^+}$ are, respectively,

\[
(\Gamma\text{-liminf}_{n \to +\infty} F_n)(x) = \inf \left\{ \liminf_{n \to +\infty} F_n(x_n) : x_n \to x \text{ in } X \right\}, \quad x \in X,
\]

\[
(\Gamma\text{-limsup}_{n \to +\infty} F_n)(x) = \inf \left\{ \limsup_{n \to +\infty} F_n(x_n) : x_n \to x \text{ in } X \right\}, \quad x \in X.
\]

If $\Gamma\text{-liminf}_{n \to +\infty} F_n = \Gamma\text{-limsup}_{n \to +\infty} F_n = F$, then we write $F = \Gamma\text{-lim}_{n \to +\infty} F_n$ and we say that the sequence $\{F_n\}_{n \in \mathbb{N}^+}$ $\Gamma$-converges to $F$ (on $X$) or that $F$ is the $\Gamma$-limit of $\{F_n\}_{n \in \mathbb{N}^+}$ (on $X$).

Readers can refer to [11, Definition 4.1, Proposition 8.1] and [20, Section 13.1] on the equivalence of Definitions A.1 and A.3.

The following give some relationships between the $\Gamma$-limit of a sequence of functionals and the $\Gamma$-limit of its subsequence.

**Proposition A.4** [11, Proposition 6.1] If $\{F_{n_k}\}_{k \in \mathbb{N}^+}$ is a subsequence of $\{F_n\}_{n \in \mathbb{N}^+}$, then

\[
\Gamma\text{-liminf}_{n \to +\infty} F_n \leq \Gamma\text{-liminf}_{k \to +\infty} F_{n_k}, \quad \Gamma\text{-limsup}_{n \to +\infty} F_n \geq \Gamma\text{-limsup}_{k \to +\infty} F_{n_k}.
\]

**Proposition A.5** [11, Proposition 8.3] $\{F_n\}_{n \in \mathbb{N}^+}$ $\Gamma$-converges to $F$ on $X$ if and only if every subsequence of $\{F_n\}_{n \in \mathbb{N}^+}$ contains a further subsequence which $\Gamma$-converges to $F$.

Next, we introduce the well-known result, concerning the variational calculus, in the theory of $\Gamma$-convergence.

**Definition A.6** [11, Definition 7.6] We say that the sequence $\{F_n\}_{n \in \mathbb{N}^+}$ is equi-coercive (on $X$), if for every $t \in \mathbb{R}$, there exists a compact subset $K_t$ of $X$ such that $\{F_n \leq t\} \subseteq K_t$ for every $n \in \mathbb{N}^+$.

**Theorem A.7** [11, Theorem 7.8] If $\{F_n\}_{n \in \mathbb{N}^+}$ is equi-coercive and $\Gamma$-converges to $F$ on $X$, then $\min F(x) = \varliminf_{n \to +\infty} F_n(x)$. 

B. Proof of Proposition 3.2

We first prove (3.4). Denote $\theta^2 := \tau^2(h)$ for $h \in L^2(\partial T; \mathbb{R})$. Then $f^0(t, x) = f_1^0(t, x) + f_2^0(t, x) + f_3^0(t, x) + f_4^0(t, x), (t, x) \in \partial T$ with $f_1^0(t, x) := \int_0^1 G_1^0(x, y) u_0(\kappa_n(y)) \, dy, f_2^0(t, x) := \int_0^1 \frac{\partial}{\partial y} G_2^0(x, y) u_0(\kappa_n(y)) \, dy, f_3^0(t, x) := \int_0^1 \frac{\partial}{\partial y} G_3^0(x, y) u_0(\kappa_n(y)) \, dy, f_4^0(t, x) := \int_0^1 \frac{\partial}{\partial y} G_4^0(x, y) u_0(\kappa_n(y)) \, dy$.\]
It follows from (2) and (3) of Proposition 2.2 and $v_0 \in C^1([0,1];\mathbb{R})$ that for any $s \leq t \leq T$ and $x, y \in [0,1]$,

$$|f_1^n(t,x) - f_1^n(s,y)| \leq \int_0^1 \left| G_1^n(x,z) - G_1^n(y,z) \right| v_0(\kappa_n(z))dz + \left( \int_0^1 \left| G_2^n(x,z) - G_2^n(y,z) \right|^2 dz \right)^{1/2} + \left( \int_0^1 \left| G_3^n(x,z) - G_3^n(y,z) \right|^2 dz \right)^{1/2}$$

$$\leq K(T)(|t-s|^{1/2} + |x-y|^{1/2}).$$

Noting that the $(n-1)$-dim vectors $(\frac{\partial}{\partial x} \phi_j(x), k = 1,\ldots,n-1)$, $j = 1,\ldots,n-1$ are an orthonormal basis of $\mathbb{R}^{n-1}$ we have that for any measurable function $w : [0,1] \to \mathbb{R}$,

$$\int_0^1 \frac{\partial}{\partial t} G_0^n(x,y)w(\kappa_n(y))dy = \sum_{j=1}^{n-1} \frac{1}{\sqrt{n}} \phi_j(x) \sum_{k=1}^{n-1} \frac{1}{\sqrt{n}} \phi_j\left(\frac{k}{n}\right)w\left(\frac{k}{n}\right) = \Pi_n(w)(x), \quad x \in [0,1]. \quad (B.1)$$

Combining the above formula, (3.2) and (3.3) leads to

$$f_2^n(t,x) = \int_0^t \int_0^1 \frac{\partial}{\partial s} G_0^n(s,y)dyu_0(\kappa_n(y))dy + \int_0^1 \frac{\partial}{\partial t} G_0^n(x,y)u_0(\kappa_n(y))dy + \Pi_n(u_0)(x).$$

It follows from the fact $\|\Delta_n u_0\|_{C([0,1])} \leq K \|u_0\|_{C([0,1])}$, Proposition 2.2 and the Hölder inequality that for any $s \leq t \leq T$ and $x \in [0,1],$

$$|f_2^n(t,x) - f_2^n(s,y)| \leq K \int_s^t (\int_0^1 |G_1^n(x,z)|^2 dz)^{1/2} dr + K \int_s^t (\int_0^1 |G_2^n(x,z)|^2 dz)^{1/2} dr + K|x-y|$$

$$\leq K(t-s)^{1/2} + |x-y|^{1/2}.$$
Further, the Hölder inequality, Proposition 2.2(2) and Proposition 3.1 lead to
\[
|f^a_n(s,x) - f^a_n(s,y)| \\
\leq K \left( \int_0^1 \int_0^1 \left( G^a_{n,r}(x,z) - G^a_{n,r}(y,z) \right)^2 |\sigma(f^a_n(r, \kappa_n(z)))|^2 \mathrm{d}z \mathrm{d}r \right)^{1/2} \|h\|_{L^2(\mathcal{F}_T)} \\
\leq K(a,T) |x-y|^{1/2}, \quad s \in [0,T], \ x, y \in [0,1], \ h \in S_a.
\]
Similar to the argument for tackling \( f^a_n \), one can obtain that for any \( s,t \in [0,T], \ x, y \in [0,1] \),
\[
\sup_{h \in S_a} |f^a_n(t,x) - f^a_n(s,y)| \leq K(a,T)(|x-y|^{1/2} + |t-s|^{1/2}).
\]
Combining the above estimates yields \((3.4)\). By a similar method, it is verified that \((3.5)\) holds, and this proof is complete. \(\square\)

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