THE DEFORMED VIRASORO ALGEBRA AT ROOTS OF UNITY

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ABSTRACT

We discuss some aspects of the representation theory of the deformed
Virasoro algebra Vir\(_{p,q}\). In particular, we give a proof of the formula
for the Kac determinant and then determine the center of Vir\(_{p,q}\)
for \(q\) a primitive \(N\)-th root of unity. We derive explicit expressions
for the generators of the center in the limit \(t = qp^{-1} \to \infty\) and
elucidate the connection to the Hall-Littlewood symmetric functions.
Furthermore, we argue that for \(q = \sqrt[2N]{1}\) the algebra describes ‘Gentile
statistics’ of order \(N - 1\), i.e., a situation in which at most \(N - 1\)
particles can occupy the same state.
1. Introduction

In recent years it has been realized (see, in particular, [7]) that the theory of off-critical integrable models of statistical mechanics is intimately connected to the theory of infinite-dimensional quantum algebras and that such theories can be studied in close parallel to their critical counterparts, i.e., conformal field theories.

Algebras of particular interest are the so-called deformed Virasoro algebra, \( \text{Vir}_{p,q} \), introduced in [11,34] (see [4] for a review), their higher rank generalization, the deformed \( \mathcal{W} \)-algebras [9,2,3,13], as well as their linearized versions [17,18]. In particular, it has been argued [25,26] that the deformed Virasoro algebra plays the role of the dynamical symmetry algebra in the Andrews-Baxter-Forrester RSOS models [1].

For generic values of the deformation parameters\(^1\) the representation theory of the deformed Virasoro algebra \( \text{Vir}_{p,q} \) closely parallels that of the undeformed Virasoro algebra, as manifested, e.g., by the Kac determinant formula [34], Drinfel’d-Sokolov reductions [14,33] and the existence of Felder type free field resolutions [26,21,10].

In this paper we will discuss some aspects of the representation theory of \( \text{Vir}_{p,q} \). First we complete the proof of the Kac determinant conjectured in [34], where also the hardest part of the proof, the construction of a sufficient number of vanishing lines, was already established. Then we proceed to discuss the representation theory for \( q \) a primitive \( N \)-th root of unity, i.e., a complex number \( q \) such that \( q^N = 1 \) and \( q^k \neq 1 \) for all \( 0 < k < N \). (Henceforth, we use the notation \( q = \sqrt[N]{1} \) for \( q \) a primitive \( N \)-th root of unity.) As was already observed in [34], it follows from the Kac determinant formula that, for \( q = \sqrt[N]{1} \), Verma modules contain many singular vectors irrespective of the highest weight. We first analyze the case \( q = -1 \) in detail and find a close relation of \( \text{Vir}_{p,q} \) to the free fermion algebra. For \( q = \sqrt[N]{1} \), \( N > 2 \), we give a generating series for all singular vectors. We derive explicit expressions of all singular vectors (for generic highest weight) in the limit \( t = q/p \to \infty \). For generic \( t \), the singular vectors are deformations of these as we illustrate in various examples. We show that the existence of these generic singular vectors is a consequence of the fact that for \( q = \sqrt[N]{1} \) (and \( t \to \infty \)) the algebra \( \text{Vir}_{p,q} \) has a large

\(^1\) Here, ‘generic values of the deformation parameters’ will always stand for ‘not a root of unity.’
center. We compute this center by exploiting the isometry from the Verma module to the Hall-Littlewood symmetric polynomials [20].

The paper is organized as follows. In Section 2 we recall the definition of the deformed Virasoro algebra \( \text{Vir}_{p,q} \) and its highest weight modules, prove some simple properties and discuss the free field realization. In Section 3 we prove the Kac determinant formula and in Section 4 we discuss the representation theory of \( \text{Vir}_{p,q} \) for \( q = \sqrt{\text{I}} \). This section is divided into three parts. First we discuss the case of \( q = -1 \) and then proceed to the general case of \( q = \sqrt{\text{I}} \) for all \( N \in \mathbb{N} \). Finally, we make the previous analysis more explicit in the limit \( t = q/p \to \infty \). We conclude with some general comments in Section 5. In particular, we point out an interesting relation between \( \text{Vir}_{p,q} \) at \( q = \sqrt{\text{I}} \), and so-called Gentile statistics of order \( N - 1 \). The latter is a generalization of Fermi statistics (\( N = 2 \)) in which at most \( N - 1 \) particles can occupy the same state [16].

Three appendices follow. In Appendix A we give basic definitions and summarize some results from the theory of symmetric functions that are used throughout the paper. In Appendix B we provide some explicit examples of singular vectors at \( q = \sqrt{\text{I}} \) for \( N = 3, 4 \), and in Appendix C we establish some elementary identities for the products of generators of \( \text{Vir}_{p,q} \) in the limit \( t \to \infty \) which, when specialized to \( q = \sqrt{\text{I}} \), yield another derivation of the center of \( \text{Vir}_{p,q} \).

2. The deformed Virasoro algebra \( \text{Vir}_{p,q} \)

2.1. Definition

The deformed Virasoro algebra \([34,4], \text{Vir}_{p,q} \) is defined to be the associative algebra generated by \( \{ T_n, \ n \in \mathbb{Z} \} \), with relations

\[
\sum_{l \geq 0} f_l \left( T_{m-l} T_{n+l} - T_{n-l} T_{m+l} \right) = c_m \delta_{m+n,0},
\]

(2.1)

where \( f_l \) is determined through

\[
f(x) \equiv \sum_{l \geq 0} f_l x^l = \exp \left( \sum_{n \geq 1} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} \frac{x^n}{n} \right)
= \frac{1}{1-x} \frac{(qx;p^2)_\infty(q^{-1}px;p^2)_\infty}{(q^{-1}p^2x;p^2)_\infty(q^{-1}p^2x;p^2)_\infty},
\]

(2.2)
and
\[ c_m = \zeta (p^m - p^{-m}) , \quad \zeta = \frac{(1 - q)(1 - t^{-1})}{1 - p} . \] (2.3)

Here, \( p, q \in \mathbb{C} \) with \( p \) not a root of \(-1\). The series (2.2) are to be understood as formal power series and the equality holds in the region where both converge. For convenience we have introduced a third parameter \( t \) by \( t = qp^{-1} \). Also

\[(x; q)_M = \prod_{k=1}^{M} (1 - xq^{k-1}) , \quad (q)_M = (q; q)_M . \] (2.4)

In terms of formal series

\[ T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n} , \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n , \] (2.5)

the relations (2.1) read

\[ f(\frac{w}{z})T(z)T(w) - f(\frac{z}{w})T(w)T(z) = \zeta \left( \delta(\frac{w}{z}) - \delta(\frac{p^{-1}w}{z}) \right) , \] (2.6)

For future convenience we also introduce a generating series for \( c_m = -c_{-m} \)

\[ c(x) \equiv \sum_{m \geq 0} c_m x^m = \zeta \left( \frac{1}{1 - px} - \frac{1}{1 - p^{-1}x} \right) . \] (2.7)

Note that the algebra \( \text{Vir}_{p, q} \) is invariant under \( (p, q) \leftrightarrow (p^{-1}, qp^{-1}) \) and \( (p, q) \leftrightarrow (p^{-1}, q^{-1}) \) and carries a \( \mathbb{C} \)-linear anti-involution \( \omega \) defined by

\[ \omega(T_n) = T_{-n} . \] (2.8)

From the second expression for \( f(x) \) in (2.2) one can easily derive the recurrence relation

\[ f(x)f(px) = \frac{(1 - qx)(1 - q^{-1}px)}{(1 - x)(1 - px)} = 1 + \zeta pq^{-1} \left( \frac{1}{1 - x} - \frac{1}{1 - px} \right) , \] (2.9)

which in turn uniquely determines \( f(x) \).

**Remark.** We have defined \( \text{Vir}_{p, q} \) for any \( p \in \mathbb{C} \) by means of (2.1) – (2.3) considered as formal power series, as long as \( p \) is not a root of \(-1\). For \( p \) a root of \(-1\), the expressions (2.2) are ill-defined. In this case one may still define \( \text{Vir}_{p,q} \) in terms of a solution of the recurrence
relation (2.9). However, such a solution is not unique as can be seen by taking different limits in (2.2) to obtain inequivalent solutions to (2.9). It is rather straightforward to show by iterating (2.9) that, for $p$ an $N$-th root of $-1$, a necessary and sufficient condition for the existence of a solution is $q^{2N} = 1$. We have not found a succinct way to classify all the solutions that arise then. In this paper we will focus on the case where $p$ is not a root of $-1$, except for brief remarks in Sections 2.2 and 4.1.

The algebra $\text{Vir}_{p,q}$ can be considered to be a deformation of the Virasoro algebra $\text{Vir}$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},$$

in the following sense:

**Theorem 2.1** [34]. In the limit $q = e^\hbar \to 1$ with $t = q^\beta$ ($\beta$ fixed) we have

$$T(z) = 2 + \beta \left( L(z) + \frac{(1-\beta)^2}{4\beta} \right) \hbar^2 + \mathcal{O}(\hbar^4),$$

where $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n}$ satisfies the Virasoro algebra (2.10) with

$$c = 1 - \frac{6(1-\beta)^2}{\beta}.$$  

2.2. Modules

In this paper we will consider the class of modules of $\text{Vir}_{p,q}$ that are analogues, e.g., deformations, of the highest weight modules of the undeformed algebra. In particular, we are interested in studying those modules by means of standard techniques based on characters and contravariant forms. For that reason we extend $\text{Vir}_{p,q}$ by a derivation $d$ satisfying

$$[d, T_n] = -nT_n,$$

and define the category $\mathcal{O}$ of $\text{Vir}_{p,q}$ modules as the set of $d$-diagonalizable modules $V = \Pi_{z \in \mathbb{C}} V(z)$, such that each $d$-eigenspace $V(z)$ is finite dimensional. Let $P(V) = \{ z \in \mathbb{C} | V(z) \neq 0 \}$. We also require that there exists a finite set $z_1, \ldots, z_s \in \mathbb{C}$ such that $P(V) \subset \bigcup_{i=1}^s \{ z_i + \mathbb{N} \}$. Then, the character of a $\text{Vir}_{p,q}$ module $V \in \mathcal{O}$ is defined by

$$\text{ch}_V(x) = \text{Tr} x^d = \sum_{z \in \mathbb{C}} \dim(V(z)) x^z.$$  

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One can show that so defined category $O$ contains, in particular, highest weight modules, that are defined in the usual manner and include, among others, Verma modules and their (irreducible) quotients.

The Verma module $M(h)$ [34,9], with highest weight state $|h\rangle$ satisfying $T_0|h\rangle = h|h\rangle$ and $T_n|h\rangle = 0$, $n > 0$, has a basis indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \ldots > 0$, i.e., a basis of $M(h)_{(n)}$ is given by the vectors

$$|\lambda; h\rangle \equiv T_{-\lambda_1} \cdots T_{-\lambda_2} |h\rangle = T_{-n}^{m_{n}(\lambda)} \cdots T_{-1}^{m_{1}(\lambda)} |h\rangle, \quad (2.15)$$

where $\lambda$ runs through all partitions of $n$. We use the notation $\lambda \vdash n$. Furthermore, $m_i(\lambda) = \#\{j : \lambda_j = i\}$ denotes the number of parts of length $i$ in $\lambda$. For $\lambda \vdash n$ we also use $|\lambda| = n$ and define the length of $\lambda$ by $\ell(\lambda) = \sum_{i \geq 1} m_i(\lambda)$. The following two orderings are used; the (reverse) lexicographic ordering, i.e., $\lambda > \mu$ if the first non-vanishing difference $\lambda_1 - \mu_1$, $\lambda_2 - \mu_2$, $\ldots$, is positive, and the natural (partial) ordering, i.e., $\lambda \geq \mu$ if $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j$ for all $i \geq 1$.

From the previous discussion it follows that the character of the Verma module $M(h)$ is given by

$$\text{ch}_M(x) = \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right) = \sum_{n \geq 0} p(n) x^n, \quad (2.16)$$

where $p(n)$ is the number of partitions of $n$.

The anti-involution $\omega$ of $\text{Vir}_{p,q}$ (see (2.8)) determines a bilinear contravariant form (‘Shapovalov form’) on $M(h)$, uniquely defined by

$$G_{\lambda\mu} \equiv \langle \lambda; h|\mu; h\rangle = \langle h|\omega(T_{-\lambda})T_{-\mu} |h\rangle, \quad (2.17)$$

and

$$\langle h|h\rangle = 1. \quad (2.18)$$

In particular we have $G_{\lambda\mu} = 0$ for $|\lambda| \neq |\mu|$. In Section 3 we compute the so-called Kac determinant $G^{(n)}$, i.e., the determinant of the form $G_{\lambda\mu}$ on $M(h)_{(n)}$.

As an aside, one might wonder whether the category $O$ contains any finite dimensional irreducible representations of $\text{Vir}_{p,q}$. Clearly, $\text{Vir}_{p,q}$ is abelian for $q = 1$ and/or $t = 1$ and therefore has a wealth of one dimensional irreducible representations in these cases. The complete result is:
**Theorem 2.2.** The only irreducible finite dimensional Vir\(_{p,q}\) modules in \(\mathcal{O}\) are one dimensional and occur for

(i) \(q = 1\) and/or \(t = 1\), and \(h \in \mathbb{C}\) arbitrary.

(ii) \(p = q^{\frac{1}{3}}\) or \(p = q^{-\frac{1}{3}}\) and \(h^2 = p^{-1}(1 + p)^2\).

**Remark.** In case (i) we have \(c(x) = 0\) and \(f(x) = 1\). Case (ii) corresponds to a \(c(x) \neq 0\) deformation of the \(c = 0\) Virasoro module with \(h^2 = h^2_{1,1}\) (cf. (2.12) and (3.6)).

**Proof:** Let \(|h\rangle\) be the highest weight state of a finite dimensional irreducible module in \(\mathcal{O}\). Define \(a_n = \langle h | T_n T_{-n} | h \rangle\), \(n \geq 0\). Then \(a_0 = h^2\), and, as follows from the commutation relations (2.1),

\[
  a_n + \sum_{l=1}^{n} a_{n-l} f_l = c_n, \quad n \geq 1.
\]

(2.19)

Since the module is finite dimensional, we must have \(a_n = 0\) for \(n\) sufficiently large, say \(n > n_0\). Multiplying (2.19) by \(x^n\) and summing over \(n \geq n_0 + 1\), we find

\[
  (\sum_{i=0}^{n_0} a_i x^i) f(x) = c(x) + a_0.
\]

(2.20)

In arriving at (2.20) we have also used (2.19) with \(n \leq n_0\) to simplify some intermediate expressions.

First consider the case \(c(x) = 0\). This can occur only for the following values of the deformation parameters: \(q = 1\), \(p\) arbitrary; \(p = q\), \(q\) arbitrary; \(p = -1\), \(q\) arbitrary. (In view of the remark in Section 2.1, the last case is in fact covered by the first two.) We conclude that for \(c(x) = 0\), Vir\(_{p,q}\) is an abelian algebra and its irreducible modules are one dimensional as described by case (i).

Now, suppose \(c(x) \neq 0\). In this case not all \(a_n, n \geq 0\), vanish and \(f(x)\) is a rational function. It follows from the recurrence relation (2.9) that \(\lim_{x \to \infty} f(x) = \pm 1\). Since \(\lim_{x \to \infty} c(x) = 0\), we must have \(a_n = 0\) for \(n \geq 1\), so that (2.20) becomes

\[
  h^2(f(x) - 1) = c(x).
\]

(2.21)

In terms of modes this is simply \(h^2 f_n = c_n\). We also deduce from \(a_1 = 0\) that

\[
  h^2 = \frac{c_1}{f_1} = \frac{(1 + p)^2}{p}.
\]

(2.22)
After solving (2.21) for \( f(x) \), we find that (2.9) holds if and only if \( q \) and \( p \) satisfy one of the conditions in (i) or (ii). In particular, in case (ii),

\[
f(x) = \frac{(1 - p^{-2}x)(1 - p^2x)}{(1 - p^{-1}x)(1 - px)}.
\]

We must still show that the resulting module is one dimensional. Let \( m_0 \) be the smallest \( m > 0 \) for which \( T_{-m}|h\rangle \neq 0 \). We have shown already that \( T_{m_0}T_{-m_0}|h\rangle = 0 \). For \( n = 1, \ldots, m_0 - 1 \) we find

\[
T_nT_{-m_0}|h\rangle = - \sum_{l=1}^{m_0-1} T_{n-l}T_{-m_0+l}|h\rangle - h f_{m_0} T_{n-m_0}|h\rangle = 0.
\]

(2.24)

Thus \( T_{-m_0}|h\rangle \) is a singular vector and must vanish. \( \square \)

### 2.3. The \( q \)-Heisenberg algebra \( \mathcal{H}_{p,q} \) and the free field realization of \( \text{Vir}_{p,q} \)

The \( q \)-Heisenberg algebra, \( \mathcal{H}_{p,q} \), is the associative algebra with generators \( \{\alpha_n, \ n \in \mathbb{Z}\} \) and relations

\[
[\alpha_m, \alpha_n] = m \left( \frac{1 - q^m}{1 + p^m} \right) \delta_{m+n,0}.
\]

(2.25)

Let \( U(\mathcal{H}_{p,q})_{\text{loc}} \) denote the local completion of \( \mathcal{H}_{p,q} \) (see [8]). Furthermore, let \( \omega_{\mathcal{H}} \) denote the \( \mathbb{C} \)-linear anti-involution of \( U(\mathcal{H}_{p,q})_{\text{loc}} \) defined by

\[
\omega_{\mathcal{H}}(\alpha_m) = p^{-m}\alpha_{-m}, \quad m \neq 0,
\]

\[
\omega_{\mathcal{H}}(q^{\alpha_0}) = pq^{-\alpha_0}.
\]

(2.26)

Again, we can add a derivation \( d \) to \( \mathcal{H}_{p,q} \) defined by

\[
[d, \alpha_m] = -m\alpha_m, \quad m \in \mathbb{Z}.
\]

(2.27)

For any \( \alpha \in \mathbb{C} \) we have an \( \mathcal{H}_{p,q} \) module \( F(\alpha) \), the so-called Fock space, which is irreducible for generic \( p \) and \( q \), and decomposes as

\[
F(\alpha) = \Pi_{n \geq 0} F(\alpha)_{(n)},
\]

(2.28)

under the action of \( d \). The module \( F(\alpha)_{(n)} \) has a basis indexed by partitions \( \lambda \vdash n \)

\[
|\lambda, \alpha\rangle \equiv \alpha_{-\lambda} |\alpha\rangle \equiv \alpha_{-\lambda_1} \cdots \alpha_{-\lambda_\ell} |\alpha\rangle,
\]

(2.29)
and where the highest weight vector (the ‘vacuum’) satisfies

\[ \alpha_0|\alpha\rangle = \alpha|\alpha\rangle, \]
\[ \alpha_m|\alpha\rangle = 0, \quad m > 0. \tag{2.30} \]

The anti-involution \( \omega_H \) of (2.26) induces a unique contravariant bilinear form \( \langle -| - \rangle_F \) on \( F(\alpha') \times F(\alpha) \) such that \( \langle \alpha'|\alpha\rangle = 1 \), where \( \alpha' \) is determined by \( q^{\alpha'} = pq^{-\alpha} \), i.e.,

\[ g_{\lambda\mu} \equiv \langle \lambda; \alpha'|\mu;\alpha\rangle_F = \langle \alpha'|\omega_H(\alpha_{-\lambda})\alpha_{-\mu}|\alpha\rangle_F. \tag{2.31} \]

We compute \( g_{\lambda\mu} \) explicitly in Section 3.

We now recall the free field realization of \( \text{Vir}_{p,q} \).

**Theorem 2.3** \([9,34]\). We have a homomorphism of algebras \( \iota : \text{Vir}_{p,q} \rightarrow U(\mathcal{H}_{p,q})_{\text{loc}} \) defined by

\[ \iota(T(z)) = \Lambda^+(z) + \Lambda^-(z), \tag{2.32} \]

where

\[ \Lambda^+(z) = p^{-\frac{1}{2}}q^{\alpha_0}\exp \left( \sum_{n \geq 1} \frac{\alpha_{-n}}{n} z^n \right) \exp \left( - \sum_{n \geq 1} \frac{\alpha_n}{n} z^{-n} \right), \]
\[ \Lambda^-(z) = p^{\frac{1}{2}}q^{-\alpha_0}\exp \left( - \sum_{n \geq 1} \frac{\alpha_{-n}}{n} (p^{-1} z)^n \right) \exp \left( \sum_{n \geq 1} \frac{\alpha_n}{n} (p^{-1} z)^{-n} \right). \tag{2.33} \]

Note that we can write

\[ \Lambda^+(z) = :\Lambda(z):, \quad \Lambda^-(z) = :\Lambda(p^{-1} z)^{-1}:, \tag{2.34} \]

where

\[ \Lambda(z) = p^{-\frac{1}{2}}q^{\alpha_0}\exp \left( - \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \right). \tag{2.35} \]

Furthermore

\[ \iota \circ \omega = \omega_H \circ \iota. \tag{2.36} \]

We sketch the proof since some intermediate results will be useful in later sections.

**Proof:** By means of standard free field techniques we find for \( |z_1| \gg |z_2| \) and \( \epsilon_i \in \{\pm\} \)

\[ \Lambda^{\epsilon_1}(z_1) \Lambda^{\epsilon_2}(z_2) = f^{\epsilon_1\epsilon_2}(\frac{z_2}{z_1}) :\Lambda^{\epsilon_1}(z_1)\Lambda^{\epsilon_2}(z_2):, \tag{2.37} \]
with
\[
f^{++}(x) = f^{--}(x) = f(x)^{-1}, \quad f^{+-}(x) = f(p^{-1}x), \quad f^{-+}(x) = f(px),
\]
where \(f(x)\) is defined in (2.2). Thus we have
\[
f\left(\frac{z_2}{z_1}\right)T(z_1)T(z_2) = \sum_{\epsilon_1, \epsilon_2} F^{\epsilon_1 \epsilon_2}(\frac{z_2}{z_1}) : \Lambda^{\epsilon_1}(z_1) \Lambda^{\epsilon_2}(z_2):,
\]
with
\[
F^{++}(x) = F^{--}(x) = 1, \quad F^{+-}(x) = f(x)f(p^{-1}x), \quad F^{-+}(x) = f(x)f(px).
\]
Subtracting the term with \(z_1 \leftrightarrow z_2\) gives
\[
f\left(\frac{z_2}{z_1}\right)T(z_1)T(z_2) - f\left(\frac{z_1}{z_2}\right)T(z_2)T(z_1) = (F^{+-}(x) - F^{-+}(\frac{1}{x})) : \Lambda^{+}(z_1) \Lambda^{-}(z_2):
\]
\[
+ (F^{-+}(x) - F^{++}(\frac{1}{x})) : \Lambda^{-}(z_1) \Lambda^{+}(z_2):.
\]
Using (2.9) we find
\[
F^{+-}(x) - F^{-+}(\frac{1}{x}) = \zeta(\delta(p^{-1}x) - \delta(x)),
\]
\[
F^{-+}(x) - F^{++}(\frac{1}{x}) = \zeta(\delta(x) - \delta(px)),
\]
while (2.33) gives
\[
: \Lambda^{+}(z) \Lambda^{-}(pz): = 1.
\]
This completes the proof. \(\square\)

The free field realization \(\iota : \text{Vir}_{p,q} \to \mathcal{U}(\mathcal{H}_{p,q})_{\text{loc}}\) equips the \(\mathcal{U}(\mathcal{H}_{p,q})_{\text{loc}}\) module \(F(\alpha)\) with the structure of a \(\text{Vir}_{p,q}\) module. In fact, from Theorem 2.3 it follows

**Corollary 2.4.** Let \(\alpha \in \mathbb{C}\) and \(p, q \in \mathbb{C}\) arbitrary. Define
\[
h(\alpha) = p^{-\frac{1}{2}}q^{\alpha} + p^{\frac{1}{2}}q^{-\alpha}.
\]
There exists a unique homomorphism \(\iota\) of \(\text{Vir}_{p,q}\) modules, \(\iota : M(h(\alpha)) \to F(\alpha)\), such that \(\iota(|h(\alpha)|) = |\alpha|\). The homomorphism \(\iota\) is an isometry with respect to the bilinear contravariant forms defined on \(M(h)\) and \(F(\alpha)\).
Remark. Using (2.32) and (2.37) one can show that \( \iota(T(z)) \) satisfies, in the sense of meromorphically continued products of operators, the following exchange relation [9],

\[
i(T(z)) \iota(T(w)) = S_{TT}(\frac{w}{z}) \iota(T(w)) \iota(T(z)),
\]

(2.45)

where

\[
S_{TT}(x) = f(\frac{1}{x}) f(x)^{-1},
\]

(2.46)
is a solution to the Yang-Baxter equation. This observation is further developed in [12], where a proposal is made for the definition of a deformed chiral algebra (DCA). In this formalism the deformed Virasoro algebra, \( \text{Vir}_{p,q} \), is naturally defined as a subalgebra of the DCA corresponding to the \( q \)-deformed Heisenberg algebra, \( \mathcal{H}_{p,q} \). While some of our discussion can be naturally recast in the language of DCAs, for most of our purposes the algebraic setup of Section 2.1 is sufficient.

3. The Kac determinant

An explicit formula for the Kac determinant of the Verma modules of \( \text{Vir}_{p,q} \) was conjectured in [34]. In this section we present its proof by using the isometry \( \iota : M(h(\alpha)) \to F(\alpha) \).

Let us first consider the determinant of the bilinear form on \( F(\alpha) \). For partitions \( \lambda, \mu \vdash n \) we have

\[
g_{\lambda\mu}^{(n)} = \langle \alpha | \omega_{\mathcal{H}}(\alpha_{-\lambda}) \alpha_{-\mu} \rangle_F = \delta_{\lambda\mu} z_\lambda \ p^{\lambda} \prod_{i=1}^{\ell(\lambda)} \frac{(1 - q^{\lambda_i})(1 - t^{-\lambda_i})}{1 + p^{\lambda_i}},
\]

(3.1)

where

\[
z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!.
\]

(3.2)

Theorem 3.1.

\[
g^{(n)} = \det g_{\lambda\mu} = C_n \prod_{r,s \geq 1 \atop rs \leq n} \left( p^r \frac{(1 - q^r)(1 - t^{-r})}{1 + p^r} \right)^{p(n-rs)},
\]

(3.3)

where \( C_n = \prod_{\lambda \vdash n} z_\lambda \) is a constant independent of \( p, q \) and \( q^\alpha \).

In deriving Theorem 3.1 we have used the following elementary result.
Lemma 3.2. Let \( f \) be a (complex valued) function on \( \mathbb{N} \). Then, for all \( n \geq 0 \),

\[
\prod_{\lambda \vdash n} \prod_{i=1}^{\ell(\lambda)} f(\lambda_i) = \prod_{\lambda \vdash n} \prod_{j_i=1}^{m_i(\lambda)} f(j_i) = \prod_{r,s \geq 1 \atop rs \leq n} f(r)^{p(n-rs)}.
\] (3.4)

Proof: Consider the left hand side of (3.4). Fix \( r \geq 1 \). Consider all partitions \( \lambda \vdash n \) with a row of length \( r \). Since there are \( p(n-r) \) partitions with at least one row of length \( r \), the first such row contributes a factor of \( f(r)^{p(n-r)} \). There are \( p(n-2r) \) partitions with at least two rows of length \( r \), so the second row of length \( r \) contributes \( f(r)^{p(n-2r)} \). Iterating this we conclude that the rows of length \( r \) in all partitions of \( n \) contribute a factor \( \prod_{s} f(r)^{p(n-rs)} \).

This proves equality of the left hand side to the right hand side. The equality of the middle formula to the right hand side is proved similarly, but now considering rows of length \( s \). \( \square \)

We are now ready for the main result of this section.

Theorem 3.3. The Kac determinant of \( M(h)_{(n)} \) is given by

\[
G^{(n)} = C_n \prod_{r,s \geq 1 \atop rs \leq n} (h^2 - h_{r,s}^2)^{p(n-rs)} \left( \frac{(1-q^r)(1-t^{-r})}{1+p^r} \right)^{p(n-rs)},
\] (3.5)

where

\[
h_{r,s} = t^r q^s + t^{-r} q^{-s} = p^{-r} q^{r-s} + p^r q^{-s-r},
\] (3.6)

and \( C_n \) is a constant independent of \( p, q \) and \( h \).

Proof: Fix \( n \in \mathbb{N} \). For partitions \( \lambda, \mu \vdash n \), define a matrix \( \Pi_{\lambda\mu}^{(n)} = \Pi^{(n)}(p, q, q^\alpha) \) by

\[
\iota(T_{-\lambda})|\alpha\rangle = \sum_{\mu \vdash n} \Pi_{\lambda\mu}^{(n)} \alpha_\mu|\alpha\rangle.
\] (3.7)

and let

\[
\Pi^{(n)} = \det \Pi_{\lambda\mu}^{(n)}.
\] (3.8)

Clearly, \( \Pi^{(n)}(p, q, q^\alpha) \) is a Laurent polynomial in \( p, q \) and \( q^\alpha \). Using (2.26), we have

\[
G^{(n)} = \Pi^{(n)}(p, pq^{-\alpha}) g^{(n)} \Pi^{(n)}(p, q, q^\alpha),
\] (3.9)
where \( g^{(n)} \) is given in Theorem 3.1. Now, the crucial step is that, from the explicit construction of singular vectors in terms of Macdonald polynomials \([34]\), it is known that \( G^{(n)} \) has vanishing lines

\[
    h^2 - h_{r,s}^2 = \left( p^{-\frac{1}{2}} q^{\frac{1}{2} - r} q^{\alpha} - p^{-\frac{1}{2}} q^{\frac{1}{2} - s} q^{-\alpha} \right) \times \left( p^{\frac{1}{2} + r - \frac{1}{2}} q^{\frac{1}{2} - r} q^{\alpha} - p^{\frac{1}{2} + s - \frac{1}{2}} q^{\frac{1}{2} - s} q^{-\alpha} \right),
\]

i.e., it has a factor

\[
    \prod_{r,s \geq 1, rs \leq n} (h^2 - h_{r,s}^2)^{p(n-rs)}.
\]

Thus, \( G^{(n)} \) is given by (3.5), up to a Laurent polynomial \( C_n \) in \( p, q \) and \( q^\alpha \). The proof will now be completed if we can show that \( C_n \) is actually a constant. As a Laurent polynomial in \( q^\alpha \), the leading term of \( \Pi^{(n)}(p, q, q^\alpha) \) is easily computed. It arises from

\[
    \Pi^{(n)}(p, q, q^\alpha) = \det \Pi^{(n)+}_{\lambda\mu}(p, q, q^\alpha),
\]

where

\[
    \Lambda_{-\lambda}^+|\alpha\rangle = \sum_\mu \Pi^{(n)+}_{\lambda\mu}(p, q, q^\alpha)|\mu - \alpha\rangle.
\]

This determinant can be computed by first going to a basis of \( F(\alpha)_{(n)} \) given by the vectors

\[
    A_{-\lambda}^+|\alpha\rangle \equiv A_{-\lambda_1} \ldots A_{-\lambda_l}|\alpha\rangle,
\]

where \( \lambda \) runs over all partitions of \( n \) and \( A_{-m} \) is defined through

\[
    \exp\left( \sum_{m \geq 1} \frac{\alpha_{-m}}{m} z^m \right) = \sum_{m \geq 0} A_{-m} z^m.
\]

The transition matrix between the basis (2.29) and (3.12) is obviously independent of \( p, q \) and \( q^\alpha \), and non-degenerate. In the basis (3.12) the matrix \( \Pi^{(n)+}_{\lambda\mu}(p, q, q^\alpha) \) is upper triangular (i.e., \( \Pi^{(n)+}_{\lambda\mu}(p, q, q^\alpha) = 0 \) unless \( \lambda \succeq \mu \)), and the diagonal elements are easily computed. We find

\[
    \Pi^{(n)+}(p, q, q^\alpha) = \prod_{\lambda \vdash n} (p^{-\frac{1}{2}} q^{\alpha})^{\ell(\lambda)}
\]

\[
    = \prod_{r,s \geq 1, rs \leq n} (p^{-\frac{1}{2}} q^{\alpha})^{p(n-rs)}
\]

\[
    = \prod_{r,s \geq 1, rs \leq n} (p^{-\frac{1}{2}} q^{-\frac{1}{2}} q^{\alpha})^{p(n-rs)},
\]

\[-12-\]
where we have used Lemma 3.2. Similarly, the leading term in $q^\alpha$ of $\Pi^{(n)}(p, q, pq^{-\alpha})$ arises from $\Pi^{(n)\alpha}(p, q, q^\alpha) = \det \Pi^{(n)\alpha}(p, q, q^\alpha)$ where
\[
\Lambda_{-\lambda}\langle\alpha\rangle = \sum_{\mu} \Pi^{(n)\alpha}_{\lambda\mu} \alpha_{-\mu}\langle\alpha\rangle ,
\] (3.15)
and is given by
\[
\Pi^{(n)\alpha}(p, q, pq^{-\alpha}) = \prod_{r,s \geq 1 \atop rs \leq n} \left(p - \frac{r}{r+1} q^{\frac{r-s}{2}} q^{-\alpha} - p - \frac{r-1}{2} q^{\frac{r-s}{2}} q^{-\alpha}\right)^{p(n-rs)} .
\] (3.16)
This, together with the factorization (3.9), shows that the prefactor $C_n$ in (3.5) is actually independent of $p$, $q$ and $q^\alpha$. □

As a consequence of the previous proof we also have

**Corollary 3.4.**

\[
\Pi^{(n)}(p, q, q^\alpha) = D_n \prod_{r,s \geq 1 \atop rs \leq n} \left(p - \frac{r}{r+1} q^{\frac{r-s}{2}} q^\alpha - p - \frac{r-1}{2} q^{\frac{r-s}{2}} q^{-\alpha}\right)^{p(n-rs)} ,
\] (3.17)
where $D_n$ is a constant independent of $p$, $q$ and $q^\alpha$.

**Proof:** The Laurent polynomial
\[
\Pi^{(n)}(p, q, q^\alpha) = \left(\prod_{r,s \geq 1 \atop rs \leq n} \left(p - \frac{r}{r+1} q^{\frac{r-s}{2}} q^\alpha - p - \frac{r-1}{2} q^{\frac{r-s}{2}} q^{-\alpha}\right)^{p(n-rs)}\right) \Pi^{(n)}(p, q, q^\alpha) ,
\] (3.18)
inherits the following duality invariances from $\text{Vir}_{p,q}$
\[
\Pi^{(n)}(p, q, q^\alpha) = \Pi^{(n)}(p^{-1}, q^{-1}, q^{-\alpha}) ,
\]
\[
\Pi^{(n)}(p, q, q^\alpha) = \Pi^{(n)}(p^{-1}, p^{-1} q, q^{-\alpha}) .
\] (3.19)
This, together with the factorization (3.9), then uniquely determines $\Pi^{(n)}(p, q, q^\alpha)$. □

From the Kac determinant (3.5) it follows that for generic $p$ and $q$ the category $\mathcal{O}$ of $\text{Vir}_{p,q}$ modules is isomorphic to that of $\text{Vir}$. In particular, the construction of singular vectors [34] and a Felder type resolution of the irreducible modules [26,21,10] are $q$-deformations of the corresponding constructions for $\text{Vir}$. At $q = \sqrt{\frac{\lambda}{\tau}}$, however, the Kac
determinant displays a large number of additional vanishing lines irrespective of the highest weight \( h \). This indicates the existence of additional, \( h \)-independent, singular vectors, and suggests that \( \text{Vir}_{p,q} \) at \( q = \sqrt[3]{1} \) has a large center. It is important to note, though, that Corollary 3.4 implies that the homomorphism \( \iota : M(h(\alpha)) \to F(\alpha) \), for generic \( p \) and \( \alpha \), is a bijection even for \( q = \sqrt[3]{1} \). In the following section we use this fact to establish results for \( \text{Vir}_{p,q} \) at \( q = \sqrt[3]{1} \) by using the free field realization.

4. The center and representations of \( \text{Vir}_{p,q} \) at roots of unity

In this section we analyze \( \text{Vir}_{p,q} \) and its representations for \( q \) a primitive \( N \)-th root of unity. To illustrate the main features we first discuss the simplest case, \( N = 2 \), before we proceed to general \( N \). In the last part of this section we make the analysis more explicit in the limit \( t \to \infty \).

4.1. Representations of \( \text{Vir}_{p,q} \) for \( q = -1 \)

For \( q = -1 \) we have

\[
f(x) = \frac{1 + x}{1 - x},
\]

(4.1)

i.e., \( f_0 = 1 \) and \( f_l = 2 \) for all \( l \geq 1 \). Thus \( \text{Vir}_{p,q} \) at \( q = -1 \) is given by

\[
[T_m, T_n] + 2 \sum_{l>0} (T_{m-l} T_{n+l} - T_{n-l} T_{m+l}) = c_m \delta_{m+n,0},
\]

(4.2)

where (cf. (2.3))

\[
c_m = -2 \left( \frac{1 + p}{1 - p} \right) (p^m - p^{-m}).
\]

(4.3)

Note that, for all \( m \) and \( n \), the sum over \( l \) in (4.2) is actually finite.

**Theorem 4.1.** For \( q = -1 \), \( \text{Vir}_{p,q} \) is equivalent to the algebra defined by the relations

\[
\{ T_m, T_n \} = \begin{cases} 
2(-1)^{\frac{m-n}{2}} T_{\frac{m+n}{2}} T_{\frac{m-n}{2}} + \tilde{c}_m \delta_{m+n,0} & \text{for } m + n \in 2\mathbb{Z}, \\
0 & \text{otherwise},
\end{cases}
\]

(4.4)

where

\[
\tilde{c}_m = 2 \left( p^{\frac{m}{2}} - p^{-\frac{m}{2}} \right)^2.
\]

(4.5)
Proof: The proof is straightforward if one uses the identity
\[
\sum_{l=0}^{m} (-1)^lf_l c_{m-l} = \tilde{c}_m, \tag{4.6}
\]
which is proved by induction. □

Remark. It is possible to arrive at the commutators (4.4) directly, using the free field realization, by exploiting a different factorization of the exchange matrix \( S_{TT}(x) = f(x)^{-1} f(1/x) = -1 \), namely \( S_{TT}(x) = g_+(x)^{-1} g_-(1/x) \) with \( g_+(x) = -g_-(x) = 1 \). See [12] for details regarding this procedure.

Note that the commutation relations (4.2) can be considered as an equation for the symmetrization of the product of two generators. In Section 4.3 we generalize this, in the limit \( t \to \infty \), to the symmetrization of a product of \( N \) generators for arbitrary \( q \in \mathbb{C} \).

The next result is a simple consequence of the commutation relations (4.2).

**Theorem 4.2.** The elements \((T_n)^2, n \in \mathbb{Z}\), are in the center of Vir\(_{p,q}\) at \( q = -1 \). In particular, this implies that the vectors \((T_{-n})^2|\lambda; h\rangle\) are singular for all \( p, h \in \mathbb{C} \) and \( n \in \mathbb{N} \).

Remark. On the basis (2.15) of \( M(h) \), the action of \( T_0 \) is upper-triangular, i.e.,
\[
T_0|\lambda; h\rangle = (-1)^{\ell(h)} h|\lambda; h\rangle + \sum_{\mu < \lambda} c_{\lambda\mu}|\mu; h\rangle, \tag{4.7}
\]
e.g., at level 2 on the basis \( \{T_{-1}T_{-1}|h\rangle, T_{-2}|h\rangle\} \) we have
\[
T_0 = \begin{pmatrix} h & -2 \\ 0 & -h \end{pmatrix}. \tag{4.8}
\]
In particular note that \( T_0 \) in (4.8) is not diagonalizable for \( h = 0 \).

Let \( M'(h) \) be the module obtained from \( M(h) \) by dividing out the submodule generated by the singular vectors \((T_{-n})^2|\lambda; h\rangle, n \geq 1\).

**Theorem 4.3.** The Kac determinant \( G'(n) \) of \( M'(h)_{(n)} \) is given by
\[
G'(n) = C_n \prod_{\begin{array}{c} r \geq 1 \\ r \leq n \end{array}} (h^2 - h_{r,1}^2)^{q_2(r:n-r)}, \tag{4.9}
\]

where $C_n$ is a constant independent of $p$ and $h$, and the integers $q_2(r; n)$ are determined by
\[
\prod_{n \geq 1 \atop n \neq r} (1 + x^n) = \sum_{n \geq 0} q_2(r; n) x^n. \tag{4.10}
\]

Proof: Observe that
\[
\tilde{c}_m = 2 \left( p^m \frac{m}{2} - p^{-m} \frac{m}{2} \right)^2 = -2(-1)^m h^2_{m,1}, \tag{4.11}
\]
while the character of the submodule of $M'(h)$ generated by $T_{-r|h}$ is given by (4.10). \qed

**Theorem 4.4.** Let $M'(h)$ be the quotient module defined as above.

(i) $M'(h)$ is irreducible provided $h^2 \neq h^2_{r,1}$ for all $r \geq 1$.

(ii) The character of $M'(h)$ is given by
\[
\text{ch}_{M'}(x) = \prod_{n \geq 1} (1 + x^n) = \prod_{n \geq 1} \left( \frac{1 - x^{2n}}{1 - x^n} \right). \tag{4.12}
\]

(iii) The ‘Witten index’ of $M'(h)$ is given by
\[
\text{Tr}(T_0 x^d) = h \prod_{n \geq 1} (1 - x^n). \tag{4.13}
\]

Proof:

(i) Follows from Theorem 4.3.

(ii) In $M'(h)$ we have an (orthogonal) basis of monomials
\[
T_{-\lambda_1} \cdots T_{-\lambda_\ell}|h\rangle, \quad \lambda_1 > \ldots > \lambda_\ell > 0, \tag{4.14}
\]
indexed by partitions with no equal parts.

(iii) Observe that, in $M'(h)$,
\[
T_0 (T_{-\lambda_1} \cdots T_{-\lambda_\ell}|h\rangle) = (-1)^{l(\lambda)} h (T_{-\lambda_1} \cdots T_{-\lambda_\ell}|h\rangle) . \tag{4.15}
\]

This concludes the proof. \qed

**Remark.** Clearly, the module $M'(h)$ can be realized in terms of free (Ramond) fermions, and is isomorphic to the irreducible Vir module at $c = \frac{1}{2}$ and $\Delta = \frac{1}{16}$. We thus have an action of $\text{Vir}_{p,q}$, for $q = -1$, on a Virasoro minimal model module.
Now, if \( h = \pm h_{m,1} \) for some \( m \), then \( M'(h) \) is reducible. We leave the general analysis of this situation for further study. Here, let us just remark, that if in addition \( t \) is a primitive \( M \)-th root of unity\(^2\), e.g., \( t = \exp(\frac{2\pi i}{M}) \), then if the equation

\[
h = \pm h_{m,1} = \pm 2\cos\left(\pi \frac{m-M}{M} - \frac{1}{2}\right) = \pm 2\sin\left(\frac{\pi m}{M}\right),
\]

holds for one particular \( m = m_0 \), it holds for all \( m = m_0 \pmod{M} \) and \( m = M - m_0 \pmod{M} \). It is straightforward to verify the following theorem.

**Theorem 4.5.** Suppose \( t = \exp(\frac{2\pi i}{M}) \) and \( h = \pm 2\sin(\frac{\pi m}{M}) \) for some \( m_0 \in \{0,1,\ldots,M-1\} \). Then the vectors \( T_{-m}|h\rangle \) are singular in \( M'(h) \) for all \( m = m_0 \pmod{M} \) and \( m = M - m_0 \pmod{M} \). Let \( M''(h) \) be the module obtained from \( M'(h) \) by dividing out the ideal generated by these singular vectors. The module \( M''(h) \) is irreducible and has character

\[
\text{ch}_{M''}(x) = \prod_{n \geq 1} \left( \frac{1 + x^n}{1 + x^{m_0 + nM}} \right),
\]

for \( m_0 = 0 \) or \( m_0 = M/2 \), and

\[
\text{ch}_{M''}(x) = \prod_{n \geq 1} \left( \frac{1 + x^n}{(1 + x^{m_0 + nM})(1 + x^{(M-m_0) + nM})} \right),
\]

otherwise.

**Remark.** The irreducibility of \( M''(h) \) for, e.g., \( M = 3 \) and \( h = h_{1,1} \) is in apparent conflict with case (ii) of Theorem 2.2. However, this is precisely a manifestation of the fact that the algebra is not uniquely defined for \( p \) a root of \(-1\).

**Remark.** Note that the vectors \( T_{-m}|h\rangle \) do not have to be singular in \( M(h) \). For example, consider \( M = 3 \), \( h = 0 \) (\( m_0 = 0 \)). Then

\[
T_1 T_{-3}|h\rangle = 2 T_{-1} T_{-1}|h\rangle,
\]

\(^2\) Note that in this case, \( p \) is a root of \(-1\) and thus the algebra \( \text{Vir}_{p,q} \) is not uniquely defined (cf. the remark in Section 2.1). The particular algebra we are discussing here corresponds to first taking \( q \rightarrow -1 \) and then \( t \rightarrow \sqrt{-1} \), i.e., to the solution (4.1) of the recursion relations (2.9).
which means that $T_{-3}|h\rangle$ is primitive, but non-singular, in $M(h)$. As a consequence, the Verma module $M(h = 0)$ for $q^2 = t^3 = 1$ has a submodule that is not generated by singular vectors, namely, the submodule generated by $T_{-3}|h\rangle$ and $T_{-1}T_{-1}|h\rangle$. This situation does not occur for Verma modules of the Virasoro algebra, but is common in the case of $\mathcal{W}$-algebras [5].

4.2. Representations of $\text{Vir}_{p,q}$ for $q = \sqrt[N]{1}$

In this section we consider the case where $q$ is an arbitrary primitive $N$-th root of unity with $N > 2$. Our main goal is to construct explicitly the series of singular vectors in the Verma module $M(h)$ for generic $p$, that are independent of $h$, and to analyze the structure of the resulting quotient module $M'(h)$. The results, given in Theorems 4.8 and 4.12, generalize those from the previous section. However, unlike for $q = -1$, the commutation relations of $\text{Vir}_{p,q}$ are extremely cumbersome for $q = \sqrt[N]{1}$, $N > 2$, and it is difficult to establish any of these results directly. For that reason we resort to the free field realization of Section 2.3, which considerably simplifies the entire analysis, as we now show.

Lemma 4.6. For $q = \sqrt[N]{1}$, the oscillators $\alpha_m$ with $m = 0 \text{ mod } N$ generate the center of the $q$-Heisenberg algebra, $\mathcal{H}_{p,q}$. In particular, they therefore also commute with $i(T(z))$.

Proof: This result is an obvious consequence of the commutation relations (2.25) in $\mathcal{H}_{p,q}$ for $q = \sqrt[N]{1}$. □

Since polynomials in the oscillators $\alpha_{-nN}$, $n \in \mathbb{N}$, commute with $i(T(z))$, upon acting on the vacuum they give rise to singular vectors in the Fock space $F(\alpha)$. Given that $i : M(h) \rightarrow F(\alpha)$ is an isomorphism (cf. Section 3), we conclude that there are corresponding singular vectors in the Verma module $M(h)$, that are independent of a particular value of $h$. We will now proceed to compute those vectors explicitly.

Lemma 4.7. For $q = \sqrt[N]{1}$ and $p$ generic

$$
\lim_{z_i \rightarrow q^{-i}} \left( \prod_{i < j} f\left( \frac{z_j}{z_i} \right) \right) i(T(z_1) \ldots T(z_N)) = :\Lambda^+(zq^{N-1}) \ldots \Lambda^+(z): + :\Lambda^-(zq^{N-1}) \ldots \Lambda^-(z):.
$$

(4.20)
Proof: Following the steps in the proof of Theorem 2.3, we find

\[
\left( \prod_{i<j} f(\frac{z_j}{z_i}) \right) n(T(z_1) \ldots T(z_N)) = \sum_{\epsilon_1, \ldots, \epsilon_N} \left( \prod_{i<j} F_{\epsilon_i \epsilon_j} \left( \frac{z_j}{z_i} \right) \right) : \Lambda_{\epsilon_1}(z_1) \ldots \Lambda_{\epsilon_N}(z_N) : .
\]

(4.21)

For generic \( p \), \( F_{\epsilon_i \epsilon_j}(x) \) has a well-defined limit for \( x \to q^k \) as long as \( q^k \neq 1 \). It follows that the right hand side of (4.21) has a well-defined limit for \( z_i \to zq^{N-i} \). In this limit we find (cf. [12], Section 6.1)

\[
(4.21) \quad \to \quad : \Lambda^+(zq^{N-1}) \ldots \Lambda^+(z) : + : \Lambda^-(zq^{N-1}) \ldots \Lambda^-(z) : \\
+ \sum_{k=1}^{N-1} b_{k,N}(q) : \Lambda^+(zq^{N-1}) \ldots \Lambda^+(zq^{N-k}) \Lambda^-(zq^{N-(k+1)}) \ldots \Lambda^-(z) :,
\]

(4.22)

where

\[
b_{k,N}(q) = \prod_{i=1}^{k} \prod_{j=k+1}^{N} F^{+-}(q^{i-j}) \\
= \prod_{i=1}^{k} \prod_{j=k+1}^{N} \frac{(1 - q^{i-j-1})(1 - p^{-1}q^{i-j+1})}{(1 - q^{i-j})(1 - p^{-1}q^{i-j})} .
\]

(4.23)

To establish (4.22), we first consider terms in (4.21) that have \( \Lambda^-(z_i) \) to the left of some \( \Lambda^+(z_j) \). Choosing the rightmost such \( \Lambda^-(z_i) \) we see that there is a factor

\[
\ldots F^{+-}(\frac{z_{i+1}}{z_i}) \ldots \Lambda^-(z_i) \Lambda^+(z_{i+1}) \ldots ,
\]

which vanishes in the limit \( z_i \to zq^{N-i} \) because \( F^{+-}(q^{-1}) = 0 \). The remaining terms yield (4.22). Finally, we note that for \( q = \sqrt[N]{1} \), we have

\[
b_{k,N}(q) = 0, \quad k = 1, \ldots, N-1 ,
\]

(4.24)

because of the factor \( (1 - q^{-N}) \) that arises by setting \( i = 1 \) and \( j = N \) in (4.23). \( \square \)

Now, it follows from the explicit expression (2.33) that for \( q = \sqrt[N]{1} \), the terms

\[
: \Lambda^\pm(zq^{N-1}) \ldots \Lambda^\pm(z) :
\]

in (4.20) have an expansion in terms of \( \{ \alpha_{n,N}, n \in \mathbb{Z} \} \). This observation, together with Lemma 4.7, yields the following theorem.
Theorem 4.8. For $q = \sqrt[2]{1}$ and $p$ generic

$$
\Psi(z) = \lim_{z_{i} \to z_{q}^{-1}} \left( \prod_{i<j} f(\frac{z_{j}}{z_{i}}) \right) T(z_{1}) \ldots T(z_{N}) |h\rangle,
$$

(4.25)
is a well-defined generating series of singular vectors in $M(h)$.

Proof: The result of Lemma 4.7 is that $i(\Psi(z))$ is a well-defined series of singular vectors in $F(\alpha)$. Since $i$ is an isomorphism, this also proves the theorem. □

Although (4.25) defines $\Psi(z)$ as a series in the products of modes of $T(z)$ acting on the vacuum, interpreting this result directly within the Verma module must be done with some caution. On the one hand, upon expanding $T(z_{i})$ into power series, we find that the resulting modes $T_{m}$ have both $m \leq 0$ and $m > 0$. In the latter case one can commute those modes to the right until they annihilate the vacuum. However, as one can see from (2.1), this yields additional infinite summations. Therefore, the product $T(z_{q}^{N-1}) \ldots T(z)$ turns out to be a divergent series, even when acting on the vacuum.

On the other hand, it follows directly from the second expression in (2.2) that, for $q = \sqrt[2]{1}$,

$$
f(x)f(xq) \ldots f(xq^{N-1}) = 1.
$$

(4.26)

Since the product of $f(z_{j}/z_{i})$ in (4.25) has a factor of $f(q) \ldots f(q^{N-1}) = f(1)^{-1} = 0$ in the limit, it vanishes.

Remark: Note that, by using (4.26) and the exchange relations (2.45), the operator $T(z_{q}^{N-1}) \ldots T(z)T(z)$ is formally in the center of $\text{Vir}_{p,q}$. However, as we have seen above, it diverges. The products of $f(z_{j}/z_{i})$ in (4.25) provide a convenient regularizing factor (see, also [12]).

Now, let us carefully expand (4.25) in modes. We have

$$
\left( \prod_{i<j} f(\frac{z_{j}}{z_{i}}) \right) T(z_{1}) \ldots T(z_{N}) |h\rangle = \sum_{m_{1}, \ldots, m_{N} \in \mathbb{Z}} z_{1}^{m_{1}} \ldots z_{N}^{m_{N}} \sum_{l_{ij} \geq 0} \left( \prod_{i<j} f_{l_{ij}} \right) \times T_{-m_{1}-l_{12}-\ldots-l_{1N}} T_{-m_{2}+l_{12}-l_{23}-\ldots-l_{2N}} \ldots T_{-m_{N}+l_{1N}+\ldots+l_{N-1N}} |h\rangle.
$$

(4.27)

Upon introducing Littlewood’s raising operators $R_{ij}$, $i < j$ [24,27], acting on monomials as

$$
R_{ij} T_{-m_{1}} \ldots T_{-m_{N}} |h\rangle = T_{-m_{1}} \ldots T_{-(m_{i}+1)} \ldots T_{-(m_{j}-1)} \ldots T_{-m_{N}} |h\rangle,
$$

(4.28)
we can write (4.27) more succinctly,

\[ (4.27) \rightarrow \sum_{m_1, \ldots, m_N \in \mathbb{Z}} z_1^{m_1} \cdots z_N^{m_N} \left( \prod_{i<j} f(R_{ij}) \right) T_{-m_1} \cdots T_{-m_N} |h\rangle, \tag{4.29} \]

where

\[ f(R_{ij}) = \sum_{l \geq 0} f_l (R_{ij})^l. \tag{4.30} \]

Note that in this notation the commutation relations (2.1) simply become

\[ f(R_{12}) T_m T_n = f(R_{12}) T_n T_m + c \delta_{m+n,0}. \tag{4.31} \]

Now consider (4.29). Clearly, at a given level in the Verma module, say level \( d \), only the terms satisfying \( \sum m_i = d \) contribute. There are only a finite number of such terms that have all \( m_i \geq 0 \). In all other terms we can commute the \( T_{-m_i} \) with \( m_i < 0 \) to the right, where they annihilate the vacuum. In fact, this would have been quite simple if there was no central term in (4.31), as in this case there would be a full symmetry in \( m_1, \ldots, m_N \). However, because of the central term in the commutation relations, we do obtain subleading terms, i.e., with the product of a smaller number of the \( T_{m_i} \). Moreover, those central terms result in infinite sums over \( m_i < 0 \). It is easy to see that for \( |p| \) sufficiently small, one can sum up those series, and the final expression is manifestly well-defined in the limit \( z_i \rightarrow q^{N-i} z \).

To illustrate this procedure let us consider the simpler case with \( N = 2 \). Here we find

\[ \sum_{m_1, m_2 \in \mathbb{Z}} z_1^{m_1} z_2^{m_2} f(R_{12}) T_{-m_1} T_{-m_2} |h\rangle = \left( \sum_{\lambda} m_{\lambda}(z_1, z_2) f(R_{12}) T_{-\lambda_1} T_{-\lambda_2} + c(z_2/z_1) \right) |h\rangle, \tag{4.32} \]

where \( m_{\lambda}(z_1, z_2) \) are the monomial symmetric polynomials and the sum is over all partitions.

For \( N > 2 \) it becomes considerably more difficult to carry out this calculation. In Appendix B we summarize the result for \( N = 3 \). Here let us only say that, as follows from the discussion above, the leading term in (4.25), i.e., the term with \( N \) generators \( T_n \), is given by

\[ \sum_{\ell(\lambda) \leq N} m_{\lambda}(z, zq, \ldots, zq^{N-1}) \left( \prod_{i<j} f(R_{ij}) \right) T_{-\lambda_1} \cdots T_{-\lambda_N} |h\rangle, \tag{4.33} \]
where the sum is over all partitions \( \lambda \).

In particular, it follows from (4.33) that the singular vectors \( \Psi_{-d|h} \) for \( d = mN, \ m \in \mathbb{N} \), are non-vanishing and independent, i.e., none of the \( \Psi_{-d|h} \) is in the submodule generated by the other ones. The vanishing of the leading term for \( d \neq 0 \mod N \) follows from part (i) of the following lemma.

**Lemma 4.9.** For \( q = \sqrt[\lambda]{T} \) we have

(i)

\[
m_{\lambda}(z, zq, \ldots, zq^{N-1}) = \begin{cases} q^{\frac{mN(N-1)}{2}} (K(1)^{-1}K(q))_{\lambda, (mN)} \ z^{\lambda} & \text{if } \lambda \vdash mN \text{ for some } m \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise}. \end{cases}
\] (4.34)

(ii)

\[
m_{(\lambda_1+n, \ldots, \lambda_N+n)}(z, zq, \ldots, zq^{N-1}) = m_{(\lambda_1, \ldots, \lambda_N)}(z, zq, \ldots, zq^{N-1}), \quad \forall n \in \mathbb{N}.
\] (4.35)

**Proof:** For (i), use the following expansion of \( m_{\lambda}(x) \) in terms of Hall-Littlewood polynomials which immediately follows from (A.11) and (A.12) in Appendix A,

\[
m_{\lambda}(x) = \sum_{\mu} (K(1)^{-1}K(q))_{\lambda, \mu} \ P_{\mu}(x; q).
\] (4.36)

Then use (cf. [27])

\[
P_{\lambda}(z, zq, \ldots, zq^{N-1}; q) = q^{n(\lambda)} \left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] \ z^{\lambda},
\] (4.37)

where

\[
n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i,
\] (4.38)

and

\[
\left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] = \left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] = \frac{(q)_{\lambda}}{(q)_{\mu}(q)_{\lambda - \mu}}.
\] (4.39)

Now, for \( q = \sqrt[\lambda]{T} \), the \( q \)-multinomial (4.39) vanishes unless \( m_i(\lambda) = N \) for some \( i \in \mathbb{Z}_{\geq 0} \), i.e., only the terms for which \( \mu = (m^N) \) for some \( m \in \mathbb{Z}_{\geq 0} \) contribute in (4.36). This proves (i). Part (ii) is obvious from the definition of \( m_{\lambda}(x) \). \( \Box \)
We have not succeeded in finding a more explicit, but still tractable, expression for the singular vectors of (4.25) for arbitrary $N$. From the explicit expressions of some singular vectors at $q = \sqrt[3]{N}$, $N = 3, 4$, in Appendix B, it is however clear that the expressions drastically simplify in the limit $t \to \infty$ (or, equivalently, $t \to 0$). Indeed, in all examples only the leading term $(T_{-n})^N \langle h \rangle$ survives. In section 4.3 we analyze this limit in more detail.

For $q = \sqrt[3]{N}$, let $H'_{p,q}$ denote the reduced $q$-Heisenberg algebra, i.e., $H'_{p,q}$ with the oscillators $\{\alpha_n \mid n = 0 \mod N\}$ removed, and denote by $F'(\alpha)$ the Fock space of $H'_{p,q}$.

**Theorem 4.10.** For $q = \sqrt[3]{N}$, and generic $p$, we have a realization of $\text{Vir}_{p,q}$ on the sub Fock space $F'(\alpha) \subset F(\alpha)$. This realization is irreducible for generic $\alpha, p \in \mathbb{C}$ and the character is given by

$$\text{ch}_{F'}(x) = \prod_{n \geq 1} \left( \frac{1 - x^{nN}}{1 - x^n} \right) = \sum_{n \geq 0} p_N(n) x^n,$$  \hspace{1cm} (4.40)

where $p_N(n)$ is the number of partitions of $n$ with parts not equal to a multiple of $N$.

The proof of this theorem is clear except for the irreducibility of $F'(\alpha)$. This will follow from the result of Theorem 4.12.

Let $M'(h)$ denote the module obtained from $M(h)$ by dividing out the submodule generated by the singular vectors of (4.25). To investigate the irreducibility of $M'(h)$ we make use of the following

**Theorem 4.11.** The Kac determinant of the module $M'(h)$ is given by

$$\tilde{G}'^{(n)} = \frac{C_n}{(h^2 - h^2_{r,s})^{q_N(r,s; n-rs)}} \prod_{r,s \leq n \mod N} \frac{1 - t^{-r}}{1 + p^r} p_N(n-rs),$$  \hspace{1cm} (4.41)

where $C_n$ is a constant independent of $p$ and $h$, $p_N(n)$ is defined in (4.40), and the integers $q_N(r, s; n)$ are determined by

$$(1 + x^r + \ldots + x^{r(N-1-s)}) \prod_{n \geq 1 \atop n \not\equiv r} (1 + x^n + \ldots + x^{n(N-1)}) = \sum_{n \geq 0} q_N(r, s; n) x^n.$$  \hspace{1cm} (4.42)
Proof: The proof is completely analogous to the proof of Theorem 3.3. In particular, the second term arises from the Kac determinant \( g'(n) \) of \( F'(\alpha) \) while the first term is a remnant of the vanishing lines (3.10). Note that \( h_{r,s} = \pm h_{r,s+N} \) for \( q = \sqrt{1} \). □

The generalization of Theorem 4.4 reads

**Theorem 4.12.** Let \( q = \sqrt{1} \) and let \( M'(h) \) be defined as above, then

(i) \( M'(h) \) is irreducible provided \( h^2 \neq h_{r,s}^2 \) for all \( r \geq 1 \) and \( 1 \leq s \leq N - 1 \).

(ii) The character of \( M'(h) \) is given by

\[
\text{ch}_{M'}(x) = \prod_{n \geq 1} (1 + x^n + \ldots + x^{n(N-1)}) .
\]

(4.43)

Proof of Theorem 4.10: It remains to prove the irreducibility of \( F'(\alpha) \) for generic \( \alpha \) and \( p \). This follows from the irreducibility of \( M'(h(\alpha)) \) (Theorem 4.12 (i)) and the equality of characters

\[
\prod_{n \geq 1} \left( \frac{1 - x^{nN}}{1 - x^n} \right) = \prod_{n \geq 1} (1 + x^n + \ldots + x^{n(N-1)}) ,
\]

(4.44)

which imply that \( \iota : M'(h(\alpha)) \to F'(\alpha) \) is an isomorphism. □

It is useful to have an explicit expression for the image of a Verma module monomial \( T_{-\lambda}|h\rangle \) under the map \( \iota : M(h(\alpha)) \to F(\alpha) \). To describe the result, let us identify \( F(\alpha) \) with the ring of symmetric functions \( \Lambda \otimes \mathbb{Q}(p, q) \) over \( \mathbb{Q}(p, q) \) through the isomorphism \( j \),

\[
j(\alpha_{-\lambda_1} \ldots \alpha_{-\lambda_n}|\alpha\rangle) = p_\lambda ,
\]

(4.45)

where \( p_\lambda \) are the power sum symmetric functions (see, Appendix A). We then have

**Theorem 4.13.** Let \( \lambda \) be a partition of length \( \ell(\lambda) = n \), then

\[
\iota(T_{-\lambda_1} \ldots T_{-\lambda_n}|h\rangle) = \sum_{\epsilon_1, \ldots, \epsilon_n} \Lambda_{-\lambda_1}^{\epsilon_1} \ldots \Lambda_{-\lambda_n}^{\epsilon_n}|\alpha\rangle ,
\]

(4.46)

while

\[
\Lambda_{-\lambda_1}^{\epsilon_1} \ldots \Lambda_{-\lambda_n}^{\epsilon_n}|\alpha\rangle = (p^{-\frac{1}{2}} q^\alpha)^{\sum \epsilon_i} \left( \prod_{i < j} f(R_{ij})^{-\epsilon_i \epsilon_j} \right) h_{\lambda_1}(x) \ldots h_{\lambda_n}(x) ,
\]

(4.47)
where
\[ h_n^+(x) = h_n(x), \quad h_n^-(x) = p^{-n} e_n(-x). \] (4.48)
and \( f(x) \) as in (2.2).

We recall that \( h_n(x) \) and \( e_n(x) \) are, respectively, the completely symmetric functions and the elementary symmetric functions. [The expression (4.47) is to be understood as follows: first replace \( h_n^i(x) \) by the appropriate expression in (4.48), then act with the \( f(R_{ij}) \) on any combination of \( h_n \)'s and \( e_n \)'s as in (4.28). In particular, the \( f(R_{ij}) \) act only on the subscript of the symmetric function, and not on the prefactor \( p^{-n} \).]

Proof: As in [20], Proposition 3.9. □

As an application of Theorem 4.13, consider the leading order term (4.33) of the singular vector \( \Psi(z) \). Applying the map \( i \) we see that the term of leading order in \( q^\alpha \) (i.e., the term of order \( O((q^\alpha)^N) \)) is proportional to
\[
\sum_{\lambda \vdash N} m_\lambda(z, zq, \ldots, zq^{N-1}) h_\lambda(x) = \sum_{m_1, \ldots, m_N \in \mathbb{Z}_{\geq 0}} h_{m_1}(x) h_{m_2}(xq) \cdots h_{m_N}(xq^{N-1}) \\
= \sum_{m \geq 0} h_m(x^N) z^{mN},
\] (4.49)

where, in the last step, we have used
\[
\prod_{i=1}^N H(xq^{i-1}; t) = H(x^N; t^N). \] (4.50)

Now,
\[
h_m(x^N) = \sum_{\lambda \vdash m} z_{\lambda}^{-1} p_\lambda(x^N) = \sum_{\lambda \vdash m} z_{\lambda}^{-1} p_{N\lambda}(x),
\] (4.51)
so that indeed (cf. (4.45)) the vector (4.33) gives rise to a singular vector at the leading order in \( q^\alpha \). A similar computation also shows that the leading order in \( q^{-\alpha} \) works out. The subleading terms in (4.25) are required to make the subleading orders in \( q^\alpha \) work.

As a second application of Theorem 4.13 one can verify that, for \( q = -1 \), the vectors \( (T_{-n})^2 |h \rangle \) are indeed singular (cf. Theorem 4.2).
4.3. Representations of Vir\(_{p,q}\) for \(q = \sqrt[n]{T}\) in the limit \(t \to \infty\)

In the previous section we have analyzed Vir\(_{p,q}\) and its representations for \(q = \sqrt[n]{1}\) and generic \(p\). In this section we will make the analysis even more explicit in the limit \(t \to \infty\) (or, equivalently, the limit \(t \to 0\)), where, as we have emphasized already, we expect some drastic simplifications. At the same time the limit \(t \to \infty\) is a generic point, in the sense that the structure of the algebra and the modules is similar to that at other generic values of \(t\), e.g., multiplicities of singular vectors at \(t = \infty\) are equal to the multiplicities at a generic \(t\). It turns out that the most efficient method of studying this case is to exploit an interesting connection to Hall-Littlewood polynomials, whose basic properties have been summarized in Appendix A.

The deformed Virasoro algebra Vir\(_{p,q}\) at \(t = \infty\), denoted by \(\tilde{\text{Vir}}_q\), is generated by \(\{\tilde{T}_n, n \in \mathbb{Z}\}\), where

\[
\tilde{T}_m = \lim_{t \to \infty} T_m p^{-\frac{|m|}{T}},
\]

satisfy the relations given by the following theorem (cf. [4]):

**Theorem 4.14.** In the limit \(t \to \infty\) we have

\[
[\tilde{T}_m, \tilde{T}_n]_q = (q - q^{-1}) \sum_{l \geq 1} q^l \tilde{T}_{n-l} \tilde{T}_{m+l} + (1 - q) \delta_{m+n,0}, \quad \text{if } m > 0 > n, \quad (4.53)
\]

\[
[\tilde{T}_m, \tilde{T}_n]_q = -(1 - q) \sum_{l=1}^{m-n-1} \tilde{T}_{m-l} \tilde{T}_{n+l}, \quad \text{if } m > n > 0 \text{ or } 0 > m > n, \quad (4.54)
\]

\[
[\tilde{T}_0, \tilde{T}_m]_q = -(1 - q) \sum_{l=1}^{m-1} \tilde{T}_{-l} \tilde{T}_{m+l} + (q - q^{-1}) \sum_{l \geq 1} q^l \tilde{T}_{m-l} \tilde{T}_l, \quad \text{if } 0 > m, \quad (4.55)
\]

\[
[\tilde{T}_m, \tilde{T}_0]_q = -(1 - q) \sum_{l=1}^{m-1} \tilde{T}_{m-l} \tilde{T}_l + (q - q^{-1}) \sum_{l \geq 1} q^l \tilde{T}_{-l} \tilde{T}_{m+l}, \quad \text{if } m > 0, \quad (4.56)
\]

where \([-, -, q]_q\) is the \(q\)-commutator,

\[
[x, y]_q = xy - qyx. \quad (4.57)
\]

**Proof:** By a direct expansion of the commutation relation (2.1) to the leading order in \(t\) using Lemma 4.15 below. \(\Box\)
Lemma 4.15. For $t \to \infty$ we have
\begin{align*}
f_0 &= 1, \\
f_l &= (1-q) + O(t^{-1}), \quad (4.58)
\end{align*}
and
\begin{align*}
f_l - f_{l+m} &= -q^{2l-1}(1-q^2)t^{-l} + O(t^{-l-1}), \quad (4.59)
\end{align*}
for all $m > l \geq 1$.

Remark. Note that (4.58) is equivalent to
\begin{equation}
f(x) = \frac{1-qx}{1-x} + O(t^{-1}). \quad (4.60)
\end{equation}

Proof: By an explicit expansion of (2.2). □

Let us point out some obvious simplifications of the relations (4.53) – (4.56) over (2.1). First, unlike in $\text{Vir}_{p,q}$, we have subalgebras $\tilde{\text{Vir}}^\pm_q$ of $\tilde{\text{Vir}}_q$ generated by the $\tilde{T}_m$ with $\pm m > 0$. The relation (4.54) that defines those subalgebras has only a finite number of terms on the right hand side. Moreover, (4.54) is invariant under the ‘shift transformation’ induced by $(m, n) \to (m+k, n+k)$, $k \in \mathbb{Z}$, as long as one remains within a given subalgebra. Secondly, the commutation relations between the positive and negative mode generators do not produce the $\tilde{T}_0$, and the right hand side of (4.53) is already in ordered form. Finally, the form of (4.55) and (4.56) suggests that one should be able to represent $\tilde{T}_0$ as a sum $\tilde{T}_0 = \tilde{T}_0^+ + \tilde{T}_0^-$, where $\tilde{T}_0^\pm$ extend $\tilde{\text{Vir}}^\pm_q$ by a zero mode generator.

Now, let us consider the Verma module $\tilde{M}(h) = \coprod_{n \geq 0} \tilde{M}(h)_{(n)}$, in which we introduce a basis
\begin{equation}
|\lambda; h\rangle = \tilde{T}_{-\lambda_1} \ldots \tilde{T}_{-\lambda_\ell} |h\rangle, \quad \lambda \vdash n, \quad (4.61)
\end{equation}
of $\tilde{M}(h)_{(n)}$ as in (2.15).

Lemma 4.16. (cf. [20], Proposition 2.20) Let $\lambda, \mu \vdash n$. Then $\tilde{G}^{(n)}_{\lambda\mu} \equiv \langle \lambda; h | \mu; h \rangle$ is given by
\begin{equation}
\tilde{G}^{(n)}_{\lambda\mu} = \delta_{\lambda\mu} b_{\lambda}(q), \quad (4.62)
\end{equation}
where $b_{\lambda}(q) = \prod_{i \geq 1} (q)_{m_i(\lambda)}$, and is independent of $h$. 

\begin{center}
- 27 -
\end{center}
Proof: Consider the reverse lexicographic ordering on the set of partitions of \( n \). It is clear that \( \langle \lambda; h|\mu; h \rangle = 0 \) for \( \lambda > \mu \). Indeed, start by moving \( T_{\lambda_1} \) to the right. This will produce terms with \( T_m, m > \lambda_1 \) and eventually give a vanishing contribution when acting on the vacuum; unless \( T_{\lambda_1} \) is killed by a corresponding \( T_{-\lambda_1} \), in which case we have \( \lambda_1 = \mu_1 \). And so on. Because of symmetry, we also have \( \langle \lambda; h|\mu; h \rangle = 0 \) for \( \lambda < \mu \). The diagonal terms are then easily calculated. The \( h \)-independence is a consequence of the second simplification as discussed above. □

It immediately follows from Lemma 4.16 that the Kac determinant at level \( n \) is given by

\[
\tilde{G}^{(n)} = \prod_{\lambda \vdash n} \prod_{i \geq 1} (q)_{m_i(\lambda)} \prod_{r,s \geq 1, rs \leq n} (1 - q^r)^p(n-rs),
\]

(4.63)

where we have used Lemma 3.2. One can verify that (4.63), indeed, agrees with the leading order behaviour of the Kac determinant of Theorem 3.3.

Theorem 4.17. Let \( q = \sqrt[2N]{1} \). The vectors

\[
| (n^N); h \rangle = (\tilde{T}_{-n})^N | h \rangle, \quad n \in \mathbb{N},
\]

(4.64)

are singular for any \( h \in \mathbb{C} \).

Proof: From Lemma 4.16 it follows that, for \( q = \sqrt[2N]{1} \), the vectors \( (\tilde{T}_{-n})^N | h \rangle \) are orthogonal to any vector in \( \tilde{M}(h) \), i.e., they are in the radical of \( \tilde{M}(h) \), and hence are primitive.\(^3\) However, from the explicit commutation relations (4.53) it easily follows that \( \tilde{T}_m(\tilde{T}_{-n})^N | h \rangle = 0 \) for \( m > 0 \). Indeed, commuting \( \tilde{T}_m \) to the right will not produce terms containing \( \tilde{T}_{-p} \) for \( p < n \), hence \( \tilde{T}_m(\tilde{T}_{-n})^N | h \rangle \) cannot be in the submodule generated by the \( (\tilde{T}_{-p})^N | h \rangle \) with \( p < n \). In other words, it has to vanish. □

As expected, the existence of \( h \)-independent singular vectors is a consequence of a large center of \( \tilde{\Vir}_q \), as described by the following main theorem of this section.

Theorem 4.18. For \( q = \sqrt[2N]{1}, N > 2 \), the center of \( \tilde{\Vir}_q \) is generated by the elements \( (\tilde{T}_n)^N, n \in \mathbb{Z}\setminus\{0\} \).

\(^3\) See, e.g., [5] for the definition and properties of primitive vectors.
One can prove this theorem directly using the relations (4.53) – (4.56). We have summarized this rather tedious argument in Appendix C. A more elegant proof will be given in due course using the free field realization and symmetric functions.

The free field realization of $\tilde{\mathcal{Vir}}_q$ is obtained by taking the limit $t \to \infty$ in the free field realization of Section 2.3, after rescaling the generators of $\mathcal{H}_{p,q}$:

$$q^{\beta_0} = p^{-\frac{1}{2}} q^{\alpha_0}, \quad \beta_m = -\alpha_m p^\frac{m}{2}, \quad m \neq 0.$$  \hspace{1cm} (4.65)

In terms of the $q$-Heisenberg algebra $\tilde{\mathcal{H}}_q$,

$$[\beta_m, \beta_n] = m (1 - q^{|m|}) \delta_{m+n,0},$$  \hspace{1cm} (4.66)

we have a realization $\tilde{\iota} : \tilde{\mathcal{Vir}}_q \to \tilde{\mathcal{H}}_q$ defined by

$$\tilde{\iota}(\tilde{T}_m) = \tilde{\Lambda}^+_m, \quad \tilde{\iota}(\tilde{T}_m) = \tilde{\Lambda}^-_m, \quad m > 0,$$

$$\tilde{\iota}(\tilde{T}_0) = \tilde{\Lambda}^+_0 + \tilde{\Lambda}^-_0,$$  \hspace{1cm} (4.67)

where

$$\tilde{\Lambda}^+_m(z) = \lim_{t \to \infty} \Lambda^+(z p^{-\frac{1}{2}}) = q^{\beta_0} : \exp\left( \sum_{n \neq 0} \frac{\beta_n}{n} z^{-n} \right) :,$$

$$\tilde{\Lambda}^-_m(z) = \lim_{t \to \infty} \Lambda^-(z p^{\frac{1}{2}}) = q^{-\beta_0} : \exp\left( -\sum_{n \neq 0} \frac{\beta_n}{n} z^{-n} \right) :.$$  \hspace{1cm} (4.68)

The contraction identities now read (cf. (2.37))

$$\tilde{\Lambda}^{\epsilon_1}(z_1) \tilde{\Lambda}^{\epsilon_2}(z_2) = \tilde{\mathcal{f}}^{\epsilon_1 \epsilon_2} \left( \frac{z_2}{z_1} \right) : \tilde{\Lambda}^{\epsilon_1}(z_1) \tilde{\Lambda}^{\epsilon_2}(z_2) :,$$  \hspace{1cm} (4.69)

with

$$\tilde{\mathcal{f}}^{++}(x) = \tilde{\mathcal{f}}^{--}(x) = \tilde{\mathcal{f}}(x)^{-1}, \quad \tilde{\mathcal{f}}^{+-}(x) = \tilde{\mathcal{f}}^{--}(x) = \tilde{\mathcal{f}}(x),$$  \hspace{1cm} (4.70)

where (cf. (4.60)),

$$\tilde{\mathcal{f}}(x) = \frac{1 - qx}{1 - x}.$$  \hspace{1cm} (4.71)

The modes of $\tilde{\Lambda}^+_m(z)$ and $\tilde{\Lambda}^-_m(z)$ satisfy commutation relations

$$[\tilde{\Lambda}^+_m, \tilde{\Lambda}^+_n]_q = -[\tilde{\Lambda}^+_n, \tilde{\Lambda}^+_m]_q,$$  \hspace{1cm} (4.72)

$$[\tilde{\Lambda}^+_m, \tilde{\Lambda}^-_n]_q = -[\tilde{\Lambda}^-_n, \tilde{\Lambda}^+_m]_q + (1 - q)^2 \delta_{m+n,0},$$  \hspace{1cm} (4.73)

$$[\tilde{\Lambda}^-_m, \tilde{\Lambda}^-_n]_q = -[\tilde{\Lambda}^-_n, \tilde{\Lambda}^-_m]_q.$$  \hspace{1cm} (4.74)
Note that (4.72) and (4.74) are both equivalent to (4.54) but with $m, n \in \mathbb{Z}$.

**Remark.** The algebra of the vertex operators $\tilde{\Lambda}^\pm(z)$ has been studied in [20] (and in [19] for $q = -1$) in the context of the Hall-Littlewood symmetric functions. In (4.66) we have chosen a slightly different normalization for $\tilde{\mathcal{H}}_q$, that is more suitable to discuss the case $q = \sqrt[3]{1}$ than the one in [20].

Let $F(\beta)$ be the Fock space of the Heisenberg algebra $\tilde{\mathcal{H}}_q$. Since the structure of $\tilde{\text{Vir}}_q$ modules is independent of $h$, we can set $\beta = 0$ without loss of generality. As in Section 4.2 we identify $F(0)$ with the space of symmetric functions $\Lambda[q]$ by means of

$$\tilde{j}(\beta_{-\lambda_1} \cdots \beta_{-\lambda_n}|0)) = p_\lambda(x). \quad (4.75)$$

Note that $\tilde{j}$ is an isometry in the scalar product on $\Lambda[q]$ defined by (A.7).

The specialization of Theorem 4.13 to the case $t \to \infty$ gives an explicit identification of $\tilde{\mathcal{I}}(\tilde{M}(h))$ with symmetric functions (cf. [20], Proposition 3.9).

**Theorem 4.19.** For any partition $\lambda$, we have

$$\tilde{j}(\tilde{\Lambda}_{-\lambda_1} \cdots \tilde{\Lambda}_{-\lambda_n}|0)) = Q'_\lambda(x;q), \quad (4.76)$$

where

$$Q'_\lambda(x;q) = \left( \prod_{i<j} \frac{1 - R_{ij}}{1 - qR_{ij}} \right) h_{\lambda_1}(x) \cdots h_{\lambda_n}(x), \quad (4.77)$$

is a Milne polynomial.

**Proof:** For a definition of Milne polynomials, see Appendix A. The proof is essentially the same as in [20] and can be found in [15]. \(\square\)

**Remark.** Note that (4.76) makes sense for any sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$, where the $\lambda_i$ are nonnegative, but not necessarily in descending order. Thus one can use (4.76) to define $Q'_\lambda(x;q)$ for such a general sequence $\lambda$. In Appendix A we have introduced Milne polynomials for arbitrary sequences using their relation to Hall-Littlewood polynomials. It is easy to see that the two extensions are exactly the same, since the reordering identity (A.15) is identical with the commutation relations (4.72) (cf. [27], Example 2, p. 213). Moreover, (4.77) holds in this more general situation.

A consequence of Theorem 4.19 is the following result, which explicitly relates two types of ordering of products of modes of the vertex operators $\tilde{\Lambda}^\pm(z)$. 

\(-30\)
Corollary 4.20.

\[
\tilde{\Lambda}^- (z_1) \ldots \tilde{\Lambda}^- (z_n) := \sum_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n} P_\lambda (z; q) \tilde{\Lambda}^-_{\lambda_1} \ldots \tilde{\Lambda}^-_{\lambda_n}, \tag{4.78}
\]

where the sum is over all ordered sequences \( \lambda \) (not necessarily positive).

Proof: Write

\[
\tilde{\Lambda}^- (z_1) \ldots \tilde{\Lambda}^- (z_n) := \sum_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n} a_\lambda (z; q) \tilde{\Lambda}^-_{\lambda_1} \ldots \tilde{\Lambda}^-_{\lambda_n}. \tag{4.79}
\]

Using (4.68) we have

\[
\tilde{\Lambda}^- (z_1) \ldots \tilde{\Lambda}^- (z_n) : |0 \rangle = \exp \left( \sum_{k \geq 1} \beta_k (z^k + \ldots + z^k_n) \right) |0 \rangle \quad \tilde{\Lambda}^- \quad \exp \left( \sum_{k \geq 1} \frac{1}{k} p_k (x) p_k (z) \right) = \sum_\lambda P_\lambda (z; q) Q_\lambda (x; q),
\]

where, in addition, we have used

\[
\exp \left( \sum_{k \geq 1} \frac{1}{k} p_k (x) p_k (z) \right) = \prod_{i,j} \left( \frac{1}{1 - x_i z_j} \right), \tag{4.81}
\]

and the completeness relation (A.26).

Now, on the one hand, Theorem 4.19 implies that, for partitions \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \) we have \( a_\lambda (z; q) = P_\lambda (z; q) \). On the other hand, from the ‘shift invariance’ of the commutators (4.72), it follows that \( a_{\lambda_1+k, \ldots, \lambda_n+k} (z; q) = (z \ldots z^k_n)^k a_{\lambda_1, \ldots, \lambda_n} (z; q) \) for all \( k \in \mathbb{Z} \). This proves the corollary. \( \square \)

We may now return to the main subject of this section, which is the center of \( \tilde{\text{Vir}}_q \) and the proof of Theorem 4.18.

Proof of Theorem 4.18: Let \( q = \sqrt[N]{1} \). Consider the expansion (4.78) in Corollary 4.20 for \( n = N \) and \( z_i = z q^{i-1}, i = 1, \ldots, N \). By the same argument as in the proof of Lemma 4.9, we conclude that the coefficients \( P_\lambda (z; q) vanish, unless m_i(\lambda) = N \) for some \( i \in \mathbb{Z}_{\geq 0} \), i.e., only the terms for which \( \lambda = (m^N) \) for some \( m \in \mathbb{Z} \) contribute to the expansion. Thus we find

\[
\tilde{\Lambda}^- (z q^{N-1}) \ldots \tilde{\Lambda}^- (z) \tilde{\Lambda}^- (z) := \sum_{m \in \mathbb{Z}} q^N m (N-1) (\tilde{\Lambda}^-_m)^N z^{mN}. \tag{4.82}
\]
Since $1 + q^n + \ldots + q^{n(N-1)} = 0$ for $n \not\equiv 0 \mod N$, we also have
\[
: \tilde{\Lambda}^-(zq^{N-1}) \ldots \tilde{\Lambda}^-(zq) \tilde{\Lambda}^-(z) : = \exp \left( \sum_{k \neq 0} \frac{\beta_{-kN} z^{-kN}}{k} \right) : .
\] (4.83)

This shows that $(\tilde{\Lambda}_m)^N, m \in \mathbb{Z}$, are in the center of $\tilde{H}_q$. Since the free field realization acts faithfully on $F(0)$, this also shows that (cf. (4.67)) $(T_m)^N, m < 0$, are in the center of $\tilde{\text{Vir}}_q$, while repeating the same analysis for $\tilde{\Lambda}^+(z)$ extends this claim to $(T_m)^N, m > 0$.

One can check explicitly that, except for $N = 2$, the $(T_0)^N$ are not in the center, see the example in Appendix C.

It remains to verify that we have identified the entire center. In fact this follows from the irreducibility of the Verma module $M'(h)$, which is proved by taking the $t \to \infty$ limit in Theorem 4.12, or directly from the Kac determinant in (4.63). This concludes the proof of the theorem. □

As a consequence of Theorems 4.18 and 4.19, we obtain the following interesting identity for Milne polynomials at $q = \sqrt[n]{1}$, which was first discussed in [22,23] (see also [27], p. 234):

**Corollary 4.21.** Let $q = \sqrt[n]{1}$ and $n \in \mathbb{N}$, then
\[
Q'_{\lambda \cup (nN)}(x; q) = Q'_\lambda(x; q) Q'_{(nN)}(x; q),
\] (4.84)

for any partition $\lambda$.

**Proof:** Suppose $\lambda_i \geq n$ for $i \leq k$ and $\lambda_i < n$ for $i \geq k + 1$. Then, by Theorem 4.18, we have
\[
\tilde{\Lambda}^-_{-\lambda_1} \ldots \tilde{\Lambda}^-_{-\lambda_k} (\tilde{\Lambda}^-_n)^N \tilde{\Lambda}^-_{-\lambda_{k+1}} \ldots \tilde{\Lambda}^-_{-\lambda_k} |0\rangle = \tilde{\Lambda}^-_{-\lambda_1} \ldots \tilde{\Lambda}^-_{-\lambda_k} (\tilde{\Lambda}^-_n)^N |0\rangle.
\] (4.85)

Since $(\tilde{\Lambda}^-_n)^N$ is in the center of $\tilde{H}_q$, the action of $\tilde{\Lambda}^-_{-\lambda_1} \ldots \tilde{\Lambda}^-_{-\lambda_k}$ on the right hand side is only through the creation operators, $\beta_{-m}, m > 0$, and as a result $\tilde{f}$ factorizes, which proves (4.84). □

Furthermore, note that in the context of the free field realization, we obtain another simple proof of Theorem 4.17 by using Theorem 4.19, the faithfulness of $\tilde{f}$ and the identity
\[
Q'_{(nN)}(x; q) = (-1)^{n(N-1)} h_n(x^N) = (-1)^{n(N-1)} \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(x^N)
\]
\[
= (-1)^{n(N-1)} \sum_{\lambda \vdash n} z_\lambda^{-1} p_{N\lambda}(x),
\] (4.86)
which holds for $q = \sqrt[N]{1}$ (cf. [27], p. 235).

In Section 4.1 we have found that the center of $\text{Vir}_{p,q}$ at $q = -1$ could be calculated in terms of symmetrized products of the generators as given in (4.4). We will now show that a generalization of this result holds for $\widetilde{\text{Vir}}_{q}$ at $q = \sqrt[N]{1}$. In fact, this is a simple consequence of the following symmetrization theorem which holds for an arbitrary $q$.

**Theorem 4.22.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$,

$$\sum_{\sigma \in S_n/S_n^\lambda} \tilde{T}_{-\sigma \lambda_1} \cdots \tilde{T}_{-\sigma \lambda_n} = \sum_{\mu} M_{\lambda \mu}(q) \left[ \frac{n}{m(\mu)} \right] \tilde{T}_{-\lambda_1} \cdots \tilde{T}_{-\lambda_n}, \quad (4.87)$$

where $\sigma$ runs over all inequivalent permutations of the sequence $(\lambda_1, \ldots, \lambda_n)$, $\mu$ over all partitions such that $|\mu| = |\lambda|$ and $\ell(\mu) = \ell(\lambda)$, and the matrix $M(q)$ is given by $M(q) = K(1)^{-1}K(q)$, where $K(q)$ is the Kostka-Foulkes matrix.

**Proof:** Clearly, it is sufficient to verify (4.87) when both sides act on the vacuum in a Verma module. Then, by mapping the Verma module to the symmetric functions, we find that in fact the present theorem is equivalent to (the symmetrization) Lemma A.1 for Milne polynomials in Appendix A. $\square$

**Remark.** There is an obvious counterpart of this result for $\widetilde{\text{Vir}}_{q}^+$. However, unlike in Section 4.1, there is no simple symmetrization identity that would mix the positive and negative mode generators of $\widetilde{\text{Vir}}_{q}$. This might be one reason why there seems to be no simple generalization of this theorem to generic $t$.

5. Discussion

In Section 4.2 (see equation (4.44)) we have seen that, for $q = \sqrt[3]{1}$, the character of the reduced Verma modules $M'(h)$ is given by

$$\prod_{n \geq 1} \left( \frac{1 - x^{nN}}{1 - x^n} \right) = \prod_{n \geq 1} (1 + x^n + \ldots + x^{n(N-1)}). \quad (5.1)$$

While the left hand side of this equation has a natural interpretation in terms of the character of a bosonic Fock space with the oscillators $\alpha_{nN}$, $n \in \mathbb{Z}$, removed, the right hand side has a natural interpretation in terms of the character of the Fock space of a so-called
Gentile parafermion of order \(N - 1\) [16], i.e., a generalization of a fermion \((N = 2)\) defined by the property that at most \(N - 1\) particles can occupy the same state, as embodied in the equation \((\tilde{T}_n)^N = 0, n \in \mathbb{Z}\), for \(t \to \infty\) or a deformation thereof for generic \(t\). In other words, the left hand side of this equation counts the number of partitions without parts of length \(0 \mod N\), while the right hand side counts the number of partitions with at most \(N - 1\) equal parts. This equality is of course well-known in combinatorics.

Free fermions, of course, are well-known to give rise to conformal field theories. Indeed, as we have noted in Section 4.2, we have a (non-canonical) action of \(\text{Vir}_{p,q}\) at \(q = \sqrt[T]{1} = -1\) on certain modules over the (undeformed) Virasoro algebra \(\text{Vir}\) at central charge \(c = \frac{1}{2}\). This rather remarkable feature, that we have an action of a quantum group on modules over an undeformed algebra, resembles the action of Yangians on affine Lie algebra modules (see [6] and references therein), and was one of our motivations to study the algebra \(\text{Vir}_{p,q}\) in the first place. Of course, there is a crucial difference between between the two results, in that the affine Lie algebra modules are fully reducible into finite dimensional irreducible representations of this Yangian, while the \(c = \frac{1}{2}\) \(\text{Vir}\) module carries an infinite dimensional irreducible representation of \(\text{Vir}_{p,q}\) at \(q = -1\). In fact, in Theorem 2.2 we have seen that \(\text{Vir}_{p,q}\) does not have any finite dimensional irreducible representations in the category \(\mathcal{O}\) except for some trivial ones. It is an interesting open question whether \(\text{Vir}_{p,q}\) possesses any other nontrivial finite dimensional irreducible representations (e.g., cyclic representations).

In addition, one may wonder whether, analogous to \(q = -1\), the algebra \(\text{Vir}_{p,q}\) at \(q = \sqrt[T]{1}\) can also be realized on modules of \(\text{Vir}\). Clearly, the character (5.1) corresponds to the character of a conformal field theory of central charge \(c = (N - 1)/N\) and is thus non-unitary (for most \(N\)), as is a well-known property of Gentile parafermions for \(N \neq 2\) (see, e.g., [31] and references therein). This is left for further study as well. Interestingly, a collection of \(N\) such Gentile parafermions does realize a unitary conformal field theory, namely \(\hat{\mathfrak{sl}}_N\) at level 1 [32].

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Appendix A. Symmetric functions

In this appendix we give basic definitions and summarize some results from the theory of symmetric functions that are used throughout the paper. For further details and omitted proofs the reader should consult the monograph [27] and/or the references cited below.

A.1. Symmetric functions

Let $\Lambda$ be the ring of symmetric functions in countably many independent variables $x = \{x_1, x_2, \ldots \}$. Throughout the paper we are using the following standard bases in $\Lambda$ indexed by partitions, $\lambda$:

(i) monomial symmetric functions, $m_\lambda$,
(ii) elementary symmetric functions, $e_\lambda$,
(iii) complete symmetric functions, $h_\lambda$,
(iv) Schur functions, $s_\lambda$.

One defines those functions as the limit, when the number of variables goes to infinity, of the corresponding symmetric polynomials that are stable with respect to the adjunction of variables. For instance, the monomial symmetric function, $m_\lambda$, is the limit, $m_\lambda(x) = \lim_{n \to \infty} m_\lambda(x_1, \ldots, x_n)$, of the monomial symmetric polynomials

$$m_\lambda(x_1, \ldots, x_n) = \sum x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad (A.1)$$

where the sum in (A.1) runs over all inequivalent permutations $\alpha = (\alpha_1, \ldots, \alpha_n)$ of the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$.

The elementary symmetric functions and the complete symmetric functions are defined by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots, \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots, \quad (A.2)$$

where $e_r$ and $h_r$ are determined from their generating series

$$E(x; t) = \sum_{n \geq 0} e_n(x) t^n = \prod_{i \geq 1} (1 + x_i t),$$
$$H(x; t) = \sum_{n \geq 0} h_n(x) t^n = \prod_{i \geq 1} (1 - x_i t)^{-1}. \quad (A.3)$$
Finally, the Schur functions, \( s_\lambda \), can be defined as polynomials in the elementary symmetric functions, \( e_r \), or, equivalently, as polynomials in the complete symmetric functions, \( h_r \), namely
\[
s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i, j \leq m}, \quad s_\lambda = \det(h_{\lambda'_i - i + j})_{1 \leq i, j \leq n},
\]
where \( m \geq \ell(\lambda') \) and \( n \geq \ell(\lambda) \), respectively, and \( \lambda' \) is the partition conjugate to \( \lambda \).

Upon extension of \( \Lambda \) to \( \Lambda_Q = \Lambda \otimes \mathbb{Z} \mathbb{Q} \), the ring of symmetric functions with rational coefficients, one can introduce yet another basis, namely the power sum symmetric functions, \( p_\lambda \). Those are defined by
\[
p_\lambda = p_{\lambda_1}p_{\lambda_2} \cdots, \quad p_r = \sum x_i^r = m_r.
\]

The standard scalar product on \( \Lambda_Q \) is defined by
\[
\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu},
\]
where \( z_\lambda \) is given in (3.2). In fact (A.6) induces a well-defined scalar product on \( \Lambda \), with respect to which the Schur functions, \( s_\lambda \), form an orthonormal basis.

A.2. Hall-Littlewood and Milne symmetric functions

The Hall-Littlewood (HL) symmetric functions, \( P_\lambda(x; q) \), form an orthogonal basis in the space of one parameter symmetric functions \( \Lambda[q] = \Lambda \otimes \mathbb{Z} \mathbb{Z}[q] \) with the scalar product defined by
\[
\langle p_\lambda, p_\mu \rangle_q = z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} (1 - q^{\lambda_i}).
\]
They can be calculated by applying the Gramm-Schmidt orthogonalization algorithm to the basis of Schur functions and are given explicitly as the limit of the corresponding HL polynomials [24]
\[
P_\lambda(x_1, \ldots, x_n; q) = \frac{1}{v_\lambda(q)} \sum_{w \in S_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right),
\]
where
\[
v_m(q) = \frac{(q)_m}{(1-q)_m}, \quad v_\lambda(q) = \prod_{i \geq 0} v_{m_i}(q).
\]
It is understood in (A.8) that the permutations \( w \) act on the variables \( x_1, \ldots, x_n \).

The functions \( P_\lambda(x; q) \) interpolate between the Schur functions, \( s_\lambda(x) \), and the monomial symmetric functions, \( m_\lambda(x) \), namely

\[
P_\lambda(x; 0) = s_\lambda(x), \quad P_\lambda(x; 1) = m_\lambda(x).
\]

The transition matrix \( K(q) \) defined by

\[
s_\lambda(x) = \sum_{\mu} K_{\lambda\mu}(q) P_\mu(x; q),
\]

is strictly upper unitriangular with respect to the natural order of partitions, i.e., \( K_{\lambda\mu}(q) = 0 \) unless \( |\lambda| = |\mu| \) and \( \lambda \succeq \mu \), and \( K_{\lambda\lambda}(q) = 1 \). The entries \( K_{\lambda\mu}(q) \in \mathbb{Z}[q] \) are called the Kostka-Foulkes polynomials. One can show that their coefficients are non-negative integers.

It follows from (A.10) that \( K(1) \) is the transition matrix between the monomial symmetric functions and the Schur functions,

\[
s_\lambda(x) = \sum_{\mu} K_{\lambda\mu}(1) m_\mu(x).
\]

Its entries are called the Kostka numbers.

Another family of symmetric functions, \( Q_\lambda(x; q) \), also referred to as HL symmetric functions, are scalar multiples of the \( P_\lambda(x; q) \), defined by

\[
Q_\lambda(x; q) = b_\lambda(q) P_\lambda(x, q),
\]

where \( b_\lambda(q) = \prod_{i \geq 1} (q)_{m_i(\lambda)} \), so that

\[
\langle P_\lambda, Q_\mu \rangle_q = \delta_{\lambda\mu}.
\]

An obvious consequence of the definition (A.13) (see, also (A.18)) is that for \( q = N \sqrt{1} \) the functions \( Q_{(kN)}(x; q) \) vanish.

It has already been observed in [24] that the definition of the \( Q_\lambda(x; q) \) in (A.8) and (A.13) makes sense for any sequence of integers \( (\lambda_1, \ldots, \lambda_n) \) that are not necessarily in a descending order and/or are not positive. For such generalized \( Q_\lambda \) one can prove [24] a reordering identity that allows to reduce \( Q_\lambda \), when \( \lambda_i \) are not in descending order, to a
linear combination of the $Q_\mu$, where $\mu_i$ are in descending order. For a two-term sequence the formula is (see, [27] p. 214)

$$Q_{(m,n)} - q Q_{(n,m)} = -Q_{(n-1,m+1)} + q Q_{(m+1,n-1)}.$$  \hfill (A.15)

The same equality holds within sequences $\lambda$ of length greater than two.

Now, let us consider the symmetric polynomials corresponding to $Q_\lambda$. It is clear that a particularly simple polynomial arises if we set the number of variables equal to the length of the partition, namely

$$Q_\lambda(x_1, \ldots, x_n; q) = (1 - q)^n \sum_{w \in S_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \right), \quad \text{ (A.16)}$$

where $\ell(\lambda) = n$. The main result of this section is a symmetrization lemma for such polynomials (cf. [30], Theorem 1):

**Lemma A.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $S_n/S_\lambda$ the subgroup of distinct permutations of the sequence $\lambda$. Then

$$\sum_{\sigma \in S_n/S_\lambda} Q_{\sigma \lambda}(x_1, \ldots, x_n; q) = \sum_\mu M_\lambda(q) \left[ \frac{n}{m(\mu)} \right] Q_\mu(x_1, \ldots, x_n; q), \quad \text{ (A.17)}$$

where $M(q) = K(1)^{-1}K(q)$, and the sum on the r.h.s. in (A.17) runs only over permutations $\mu$ such that $|\mu| = |\lambda|$ and $\ell(\mu) = \ell(\lambda)$.

**Proof:** By applying the symmetrization in $\lambda_1, \ldots, \lambda_n$ to the right hand side of (A.16), we obtain

$$\sum_{\sigma \in S_n/S_\lambda} Q_{\sigma \lambda}(x_1, \ldots, x_n; q) = (1 - q)^n \sum_{\sigma \in S_n/S_\lambda} \sum_{w \in S_n} w \left( x_1^{\sigma \lambda_1} \cdots x_n^{\sigma \lambda_n} \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \right)$$

$$= (1 - q)^n \sum_{\sigma \in S_n/S_\lambda} x_1^{\sigma \lambda_1} \cdots x_n^{\sigma \lambda_n} \sum_{w \in S_n} w \left( \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \right)$$

$$= (q)_n m_\lambda(x_1, \ldots, x_n), \quad \text{ (A.18)}$$

where we used (A.1) and the identity (see, [27] p. 207)

$$\sum_{w \in S_n} w \left( \prod_{i<j} \frac{x_i - q x_j}{x_i - x_j} \right) = v_n(q). \quad \text{ (A.19)}$$
Now use the fact that
\[
m_\lambda(x_1, \ldots, x_n) = m_\lambda(x_1, \ldots, x_m)|_{x_{n+1}=\ldots=x_m=0},
\] (A.20)
and
\[
Q_\mu(x_1, \ldots, x_m)|_{x_{n+1}=\ldots=x_m=0} = \begin{cases} Q_\mu(x_1, \ldots, x_n) & \text{if } \ell(\mu) \leq n, \\ 0 & \text{if } \ell(\mu) > n, \end{cases}
\] (A.21)
together with (A.11), (A.12) and (A.13) to find
\[
m_\lambda(x_1, \ldots, x_n) = \sum_\mu M_\lambda\mu(q) \frac{1}{b_\mu(q)} Q_\mu(x_1, \ldots, x_n; q),
\] (A.22)
where
\[
M(q) = K(1)^{-1}K(q).
\] (A.23)
The restriction on the range of the sum over \( \mu \) is then a straightforward consequence of the strict upper unitriangularity of \( K(q) \). \( \Box \)

We also need to introduce another family of symmetric functions, \( Q'_\lambda(x;q) \), called Milne’s symmetric functions [28,29] (see, also [15,22,23]). They are related to the HL functions, \( Q_\lambda(x;q) \), by a change of variables
\[
Q'_\lambda(x;q) = Q_\lambda(x_1 - q; q),
\] (A.24)
which has to be understood the sense of the \( \lambda \)-ring notation. This means that \( Q'_\lambda(x;q) \) is the image of \( Q_\lambda(x;q) \) by the ring homomorphism of \( \Lambda[q] \) that sends \( p_r(x) \) to \( (1-q^r)^{-1}p_r(x) \).

A defining property of Milne’s functions is their orthogonality to the HL functions,
\[
\langle P_\lambda, Q'_\mu \rangle = \delta_{\lambda\mu},
\] (A.25)
with respect to the product (A.6). This is equivalent to the completeness relation
\[
\sum_\lambda P_\lambda(x;q) Q'_\lambda(y;q) = \prod_{i,j} \left( \frac{1}{1-x_i y_j} \right).
\] (A.26)
Finally, an expansion in terms of Schur functions yields
\[
Q'_\lambda(x;q) = \sum_\mu K_{\mu\lambda}(q) s_\mu(x).
\] (A.27)

One can use (A.24) to extend the definition of \( Q'_\lambda(x;q) \) to arbitrary sequences \( \lambda \). Given that the transformation defined in (A.24) is a ring homomorphism, it is clear that so defined \( Q'_\lambda(x;q) \) will satisfy both the reordering identity (A.15) as well as (A.17) of the symmetrization lemma.
Appendix B. Explicit singular vectors for \( q = \sqrt{\nu} \)

In Theorem 4.8 of Section 4.2 we have seen that

\[
\Psi(z) = \lim_{z_i \to \sqrt{\nu}^{-i}} \left( \prod_{i<j} f(\frac{z_i}{z_j}) \right) T(z_1) \ldots T(z_N)|h\rangle, \tag{B.1}
\]

is a (well-defined) generating series of singular vectors in \( M(h) \) for \( q = \sqrt{\nu} \). We have also outlined how to make sense out of this expression. Here we carry out the procedure for \( N = 3 \). First,

**Theorem B.1.** We have

\[
\left( \prod_{i<j} f(\frac{z_i}{z_j}) \right) T(z_1)T(z_2)T(z_3)|h\rangle = \sum_{m_1,m_2,m_3 \geq 0} z_1^{m_1} z_2^{m_2} z_3^{m_3} \left( \prod_{i<j} f(R_{ij}) \right) T_{-m_1} T_{-m_2} T_{-m_3}|h\rangle \tag{B.2}
\]

\[
+ \zeta \left( \frac{p(z_3/z_2)}{1 - p(z_3/z_2)} F_{-+}^+(\frac{z_3}{z_2}) - \frac{p^{-1}(z_3/z_2)}{1 - p^{-1}(z_3/z_2)} F_{+-}^+(\frac{z_3}{z_2}) \right) T(z_1)|h\rangle
+ \zeta \left( \frac{p(z_3/z_1)}{1 - p(z_3/z_1)} F_{-+}^+(\frac{z_3}{z_1}) - \frac{p^{-1}(z_3/z_1)}{1 - p^{-1}(z_3/z_1)} F_{+-}^+(\frac{z_3}{z_1}) \right) T(z_2)|h\rangle
+ \zeta \left( \frac{1}{1 - p(z_2/z_1)} F_{-+}^+(\frac{z_3}{z_1}) - \frac{1}{1 - p^{-1}(z_2/z_1)} F_{+-}^+(\frac{z_3}{z_1}) \right) T(z_3)|h\rangle.
\]

where

\[
F_{(d)}^{\epsilon_1 \epsilon_2}(x) = \sum_{0 \leq m \leq d} F_m^{\epsilon_1 \epsilon_2} x^m. \tag{B.3}
\]

We sketch the proof, which is based on the following

**Lemma B.2.** Let

\[
c_{m,n} = \zeta (p^m - p^{-m}) \delta_{m+n,0}, \tag{B.4}
\]

then we have

\[
\sum_{l_1,l_2 \geq 0} f_{l_1}f_{l_2} T_{-m_1-l_1-l_2} c_{-m_2+l_1,-m_3+l_2} = \zeta \left( p^{-m_2} F_{m_2-m_3}^{+-} - p^{m_2} F_{m_2+m_3}^{+-} \right) T_{-m_1-m_2-m_3},
\]

\[
\sum_{l_1,l_2 \geq 0} f_{l_1}f_{l_2} c_{-m_1-l_1,-m_2-l_2} T_{-m_3+l_1+l_2} = \zeta \left( p^{-m_1} F_{-m_1-m_2}^{+-} - p^{m_1} F_{-m_1-m_2}^{+-} \right) T_{-m_1-m_2-m_3}, \tag{B.5}
\]

- 40 -
where $F^{\pm \mp}(x) = \sum_m F_m^{\pm \mp} x^m$ is defined in (2.40).

**Proof:** We have

$$\sum_{l_1, l_2 \geq 0} f_1 f_2 T_{-m_1-l_1-l_2} c_{-m_2+l_1,-m_3+l_2}$$

$$= \zeta \sum_{p,q \geq 0} f_1 f_2 (p^{-m_2+l_1} - p^{m_2-l_1}) \delta_{-m_2-m_3+l_1+l_2} T_{-m_1-l_1-l_2}$$

$$= \zeta (p^{-m_2} f(px) f(x) |_{x^{m_2+m_3}} - p^{m_2} f(p^{-1} x) f(x) |_{x^{m_2+m_3}}) T_{-m_1-m_2-m_3} .$$

(B.6)

The other identity is shown similarly. □

**Proof of Theorem B.1:** Writing out the left hand side of (B.2) in modes we distinguish three cases

(i) $m_1 \geq 0, m_2 \geq 0, m_3 \geq 0$,  
(ii) $m_2 < 0, m_3 \geq 0$,  
(iii) $m_1 < 0, m_2 \geq 0, m_3 \geq 0$.

In case (i) only a finite number of terms contribute. This gives the first term on the right hand side of (B.2). In case (ii) we move the term $T_{-m_2+l_{12}-l_{23}}$ to the right using the commutator (2.1). Then use Lemma B.2 to find

$$\zeta \sum_{m_1} \sum_{m_2 \leq -1} \sum_{m_3 \geq 0} z_1^{m_1} z_2^{m_2} z_3^{m_3} (p^{-m_2} F_{m_2+m_3}^{--} - p^{m_2} F_{m_2+m_3}^{++}) T_{-m_1-m_2-m_3} |h\rangle .$$

The sum over $m_1$ is unrestricted. The nonvanishing terms must have $m_2 + m_3 \geq 0$ because of the moding of $F^{+-}$ and $F^{-+}$. Thus, with the restriction on $m_2$ in place, we may let the sum on $m_3$ run over all integers. This allows for a change in the summation

$$m' = m_2 + m_3, \quad m'' = m_1 + m_2 + m_3,$$

after which all sums can be performed and we obtain

$$\zeta \left( \frac{p(z_3/z_2)}{1 - p(z_3/z_2)} F^{-+} \left( \frac{z_3}{z_1} \right) - \frac{p^{-1}(z_3/z_2)}{1 - p^{-1}(z_3/z_2)} F^{++} \left( \frac{z_3}{z_1} \right) \right) T(z_1) |h\rangle .$$

The remaining case (iii) is analyzed similarly. □

By taking the limit $z_i \to z q^{N-i}$ in Theorem B.1 and using $F^{-+}(q^{-1}) = F^{-+}(p^{-1} q^{-2}) = F^{+-}(q^{-2}) = 0$, we obtain
Corollary B.3. The following is an explicit form for the level $d$ singular vector at $q = \sqrt[3]{1}$ and arbitrary $h \in \mathbb{C}$

$$
\Psi_d = \sum_{m_1, m_2, m_3 \geq 0} q^{2m_1 + m_2} \left( \prod_{i<j} f(R_{ij}) \right) T_{-m_1} T_{-m_2} T_{-m_3} |h\rangle 
+ \zeta \left[ q^{2d} \left( \frac{1}{qp-1} - 1 \right) F^+ (q^{-2}) \right] T_{-d} 
+ q^d \left( \frac{1}{q^2 p-1} - 1 \right) F^+ (q^{-1}) - \frac{1}{q^2 p-1} F^+ (q^{-1}) \right] T_{-d} 
+ \left( \frac{qp^{-1}}{qp-1} - 1 \right) F^+ (p^{-2}) \right] T_{-d} |h\rangle .
$$
(B.8)

The expression $\Psi_d$ vanishes for $d \neq 0 \mod 3$ (we have explicitly verified this for small values of $d$).

From the structure of $f_i$, and the duality symmetry $(p, q) \rightarrow (p^{-1}, q^{-1})$, it follows that the singular vectors at $q = \sqrt[3]{1}$ for $d = 0 \mod N$ can be written in terms of the following set of invariants

$$
\Delta_{m_1...m_k}^{rs} = a_{rs} \frac{p^r q^s (1 + p^{n-2r} q^{N-2s})}{\prod_{i=1}^{k} (1 + p^{m_i})}, \quad n = \sum_i m_i ,
$$
(B.9)

where the normalization factor

$$
a_{rs} = \begin{cases} 
\frac{1}{2} & \text{for } 2r = n \text{ and } 2s = 0 \mod N , \\
1 & \text{otherwise} .
\end{cases}
$$
(B.10)

Using (B.8) we now find the following singular vectors at $q = \sqrt[3]{1}$ for arbitrary $h \in \mathbb{C}$. At level $d = 3$

$$
\Psi_3 = (T_{-1} T_{-1} T_{-1} + a_{210} T_{-2} T_{-1} T_0 + a_{300} T_{-3} T_0 T_0 + a_{3} T_{-3}) |h\rangle ,
$$
(B.11)

with

$$
a_{210} = -3\Delta_{2}^{10} , 
$$
$$
a_{300} = -3\Delta_{3}^{21} ,
$$
$$
a_{3} = 3\Delta_{11}^{11} ,
$$
(B.12)

and, at level $d = 6$

$$
\Psi_6 = \left( T_{-2} T_{-2} T_{-2} + a_{321} T_{-3} T_{-2} T_{-1} + a_{411} T_{-4} T_{-1} T_{-1} + a_{330} T_{-3} T_{-3} T_0 
+ a_{420} T_{-4} T_{-2} T_0 + a_{510} T_{-5} T_{-1} T_0 + a_{600} T_{-6} T_0 T_0 + a_{6} T_{-6} \right) |h\rangle ,
$$
(B.13)
The equality of the coefficients $a_{210} = a_{321}$ and $a_{300} = a_{411}$ in the singular vectors at $d = 3$ and $d = 6$ is a direct consequence of Lemma 4.9 (ii) and the expression (4.33).

In addition we have computed the level $d = 4$ singular vector at $q = \sqrt{T}$

$$\Psi_4 = \left( T_{-1} T_{-1} T_{-1} T_{-1} + a_{2110} T_{-2} T_{-1} T_{-1} T_{0} + a_{2200} T_{-2} T_{-2} T_{0} T_{0} + a_{3100} T_{-3} T_{-1} T_{0} T_{0} \\
+ a_{4000} T_{-4} T_{0} T_{0} T_{0} + a_{22} T_{-2} T_{-2} + a_{31} T_{-3} T_{-1} + a_{40} T_{-4} T_{0} \right) |h\rangle,$$

where

$$a_{2110} = -4\Delta_{2}^{10} - 4\Delta_{3}^{11},$$

$$a_{2200} = 8\Delta_{4}^{20},$$

$$a_{3100} = 12\Delta_{4}^{20} + 4\Delta_{5}^{21},$$

$$a_{4000} = -8\Delta_{501}^{30} - 8\Delta_{501}^{31},$$

$$a_{22} = -8\Delta_{2}^{10},$$

$$a_{31} = -4\Delta_{2}^{10} + 4\Delta_{3}^{11},$$

$$a_{40} = 8\Delta_{301}^{20} - 8\Delta_{301}^{21}.$$

Observe that for $t \to \infty$ we have $\Delta_{m_1...m_k}^{r,s} \to 0$ for $0 < r \leq 2n$. Thus, in all the examples above, in the limit $t \to \infty$ only the leading term $(T_{-n})^N|h\rangle$ survives. Furthermore, note that, both for $N = 3$ and $N = 4$, all the factors in front of the fundamental invariants $\Delta_{m_1...m_k}^{r,s}$ are a multiple of $N$ (except for the one of the leading term). Thus, it appears that calculating modulo $N$, in the appropriate sense, is somehow equivalent to considering the $t \to \infty$ limit.
Appendix C. The center revisited

In this appendix we establish in a direct way some elementary identities for the products of generators of $\tilde{\text{Vir}}_q^\pm$ for a generic $q$. Specialization of those results to $q = \sqrt[3]{T}$ yields a direct proof of Theorem 4.14.

C.1. Preliminaries

The defining relations (4.54) of $\tilde{\text{Vir}}_q^\pm$ are invariant under the rescaling $\tilde{T}_m \to a^{|m|}\tilde{T}_m$, $a \in \mathbb{C}$, while the remaining relations in (4.53) – (4.56) can be further simplified by a judicious choice of $a$. In the following we will find it convenient to work with the generators

$$t_m = q^{|m|} \tilde{T}_m, \quad m \in \mathbb{Z}. \quad (C.1)$$

Then the relations involving the generators of $\tilde{\text{Vir}}_q^+$ are given by

$$t_m t_n = qt_n t_m - (1 - q) \sum_{l=1}^{m-n-1} t_{m-l} t_{n+l}, \quad m > n \geq 1, \quad (C.2)$$

$$t_m t_0 = qt_0 t_m - (1 - q) \sum_{l=1}^{m-1} t_{m-l} t_l + (q - q^{-1}) \sum_{l=1}^{\infty} t_{-l} t_{m+l}, \quad m \geq 1, \quad (C.3)$$

$$t_m t_{-n} = qt_{-n} t_m + (q - q^{-1}) \sum_{l=1}^{\infty} t_{-n-l} t_{m+l} + (1 - q)q^m \delta_{m,n}, \quad m, n \geq 1. \quad (C.4)$$

while the remaining ones have a similar form and can easily be worked out.

C.2. Identities in $\tilde{\text{Vir}}_q^\pm$

Now let us consider the subalgebras $\tilde{\text{Vir}}_q^\pm$ in more detail. We will discuss explicitly only the case of $\tilde{\text{Vir}}_q^+$ with the extension to $\tilde{\text{Vir}}_q^-$ being obvious.

The relation (C.2) implies that two subsequent generators satisfy

$$t_{m+1} t_m = qt_m t_{m+1}, \quad (C.5)$$

For the generators $t_{m+k}$ and $t_m$ with $k \geq 2$, there are additional terms, though it is still possible to rewrite (C.2) in a symmetric form

$$\sum_{j=1}^{k} t_{m+j} t_{m+k-j} = q \sum_{j=1}^{k} t_{m+k-j} t_{m+j}. \quad (C.6)$$
In view of (C.6) it is then natural to consider the following sums of generators\(^4\)

\[ s_m = t_m + t_{m+1} + \ldots , \quad (C.7) \]

in term of which (C.2) is equivalent to

\[ s_{m+1} t_m = qt_m s_{m+1} - (1 - q) s_{m+1} s_{m+1} . \quad (C.8) \]

The infinite sums and their products here and below should be understood in the graded sense. By iterating (C.8) we prove

**Lemma C.1.** For \( n \geq 1, \)

\[ (s_{m+1})^n t_m = q^n t_m (s_{m+1})^n - (1 - q^n)(s_{m+1})^{n+1} , \quad (C.9) \]

\[ s_{m+1} (t_m)^n = \sum_{j=0}^{n} (-1)^j q^{n-j} (q) j \left[ n \atop j \right] (t_m)^{n-j} (s_{m+1})^j . \quad (C.10) \]

Then, a straightforward induction yields

**Lemma C.2.** For \( n \geq 1, \)

\[ (t_m + s_{m+1})^n = \sum_{j=0}^{n} q^{j(j-1)/2} \left[ n \atop j \right] (t_m)^{n-j} (s_{m+1})^j , \quad (C.11) \]

\[ (qt_m + (q - q^{-1}) s_{m+1})^n = \sum_{j=0}^{n} (-1)^j \frac{q^{n-2j} (q)_{j+1}}{1-q} \left[ n \atop j \right] (t_m)^{n-j} (s_{m+1})^j . \quad (C.12) \]

For a partition \( \lambda \) of length \( \ell(\lambda) = n, \) let \( \lambda^{\text{op}} = (\lambda_1^{\text{op}}, \ldots, \lambda_n^{\text{op}}) \) be the increasing sequence of positive integers, \( \lambda_i^{\text{op}} = \lambda_{n-i}, \ i = 1, \ldots, n. \) For such a sequence we define

\[ \text{ht}(\lambda^{\text{op}}) = \sum_{i=1}^{\ell(\lambda)} (\ell(\lambda) - i)(\lambda_i^{\text{op}} - 1) \]

\[ = n(\lambda) - \frac{1}{2} \ell(\lambda)(\ell(\lambda) - 1) . \quad (C.13) \]

Finally, in terms of multiplicities, we can write \( (\lambda_1, \ldots, \lambda_n) = (1^{m_1} 2^{m_2} \ldots) , \) where \( m_i = m_i(\lambda). \)

Using Lemma C.2 we will now establish the main result of this section, which gives the expansion of powers of \( s_m \) into ordered products of generators.

\(^4\) Since (C.2) is invariant under rescaling, there seems to be no advantage in working with the generating series \( t^+(z) = \sum_{n \geq 1} t_n z^{-n} . \)
Theorem C.3. For \( n \geq 1 \),

\[
(s_m)^n = \sum_{\{\lambda | \ell(\lambda) = n\}} q^{\text{ht}(\lambda^{op})} \left[ \frac{n}{m(\lambda)} \right] t_{m+\lambda^{op}_1-1} \cdots t_{m+\lambda^{op}_n-1}.
\]  

(C.14)

Proof: We write \( s_m = t_m + s_{m+1} \), and expand \((s_m)^n = (t_m + s_{m+1})^n\) using (C.12),

\[
(s_m)^n = \sum_{m_1+n_2=n} q^{\text{ht}(1^{m_1} 2^{n_2})} \left[ \frac{n}{m_1, n_2} \right] (t_m)^{m_1} (s_{m+1})^{n_2}.
\]  

(C.15)

Since at a given level only products of a finite number of generators, \( t_m, t_{m+1}, \ldots, t_{m+s-1} \), can appear, after repeating this expansion \( s - 1 \) times we obtain a multiple sum

\[
(s_m)^n = \sum_{m_1+\ldots+m_s=n} q^{s-1 \text{ht}(1^{m_1} 2^{n_{i+1}})} \left[ \frac{n}{m_1, n_2} \right] \cdots \left[ \frac{n_{s-1}}{m_{s-1}, m_s} \right] (t_m)^{m_1} \cdots (t_{m+s-1})^{m_s},
\]  

where \( n_i = m_i + \ldots + m_s \). It follows from definition (C.13) that

\[
\sum_{i=1}^{s-1} \text{ht}(1^{m_i} 2^{n_{i+1}}) = \text{ht}(1^{m_1} 2^{m_2} \ldots s^{m_s}),
\]  

(C.17)

and it is obvious that

\[
\left[ \frac{n}{m_1, n_2} \right] \left[ \frac{n_2}{m_2, n_3} \right] \cdots \left[ \frac{n_{k-1}}{m_{s-1}, m_s} \right] = \left[ \frac{n}{m_1, \ldots, m_s} \right].
\]  

(C.18)

This completes the proof of the theorem. \( \Box \)

C.3. Proof of Theorem 4.18

Theorem C.4. Let \( q = \sqrt[2N]{1} \). Then \((t_m)^N, m \geq 1, \) belong to the center of \( \hat{\text{Vir}}_q \).

Proof: We will show separately that

\[
(t_m)^N t_n = t_n (t_m)^N, \quad m \geq 1,
\]  

(C.19)

for \( \pm n > 0 \) and \( n = 0 \).

Case 1: \( n > 0 \)

From Lemma C.1 we find

\[
(s_{m+1})^N t_m = t_m (s_{m+1})^N,
\]  

(C.20)
and
\[ s_{m+1}(t_m)^N = (t_m)^Ns_{m+1}, \]  
(C.21)
for arbitrary \( m \geq 1 \). Since, by Theorem C.3
\[ (s_m)^N = \sum_{k=0}^{\infty} q^{\frac{1}{2}kN(N-1)}(t_{m+k})^N, \]  
(C.22)
where each term in the sum is at a different level, this implies (C.19).

Case 2: \(-n > 0\)

First rewrite (C.4) in a more convenient form,
\[ t_m t_{-n} = \sum_{l=1}^{\infty} a_l t_{-n-l} t_{m+l} + c_n \delta_{m,n}, \]  
(C.23)
where \( a_0 = q, a_l = q - q^{-1}, l \geq 1 \), and \( c_n = (1 - q)q^{-n} \). By repeated use of (C.23) we obtain
\[ (t_m)^N t_{-n} = c_m(t_m)^{N-1} \delta_{m,n} \]
\[ + \sum_{k=1}^{N-1} (t_m)^{N-k-1} \sum_{l_1, \ldots, l_k = 0}^{\infty} a_{l_1} \cdots a_{l_k} c_{n+l_1+\ldots+l_k} t_m^{l_1} \cdots t_m^{l_k} \delta_{m,n+l_1+\ldots+l_k} \]
\[ + \sum_{l_1, \ldots, l_N = 0}^{\infty} a_{l_1} \cdots a_{l_N} t_{-n-l_1-\ldots-l_N} t_m^{l_1} \cdots t_m^{l_N}. \]  
(C.24)
This may be recognized as
\[ (t_m)^N t_{-n} = c_m(t_m)^{N-1} \delta_{m,n} \]
\[ + c_m \sum_{k=1}^{N-1} (t_m)^{N-k-1} \left( qt_m + (q - q^{-1})s_{m+1}\right)^k_{(k+1)m-n} \]  
(C.25)
\[ + \sum_{l \geq 0} t_{-n-l} \left( qt_m + (q - q^{-1})s_{m+1}\right)^N_{Nm+l}, \]
where the subscripts on the brackets indicate the level.

For \( m < n \), we find using (C.12) that only the last term in (C.25) with \( l = 0 \) contributes giving \( t_{-n}(t_m)^N \).

For \( m = n \), all terms in (C.25) contribute, however, we may set \( s_{m+1} = 0 \) in the second term. Thus the central charge term has an overall factor of \( 1 + q + \ldots + q^{N-1} = 0 \), while the last term is the same as above.
For \( m > n \), the first term does not contribute. Let us first consider the second term proportional to the central charge. Using (C.12) we can rewrite it as

\[
c_m \sum_{k=1}^{N-1} \sum_{j=0}^{k} (-1)^j \frac{q^{k-j}(q)_{j+1}}{1-q} \left[ \begin{array}{c} k \\ j \end{array} \right] (t_m)^{N-j-1}(s_{m+1})^j \bigg|_{Nm-n},
\]

where only the terms at the level \( Nm - n \) contribute. In particular, at this level we must have \( j \geq 1 \). Changing the order of summation we obtain

\[
c_m \sum_{j=1}^{N-1} (-1)^j \frac{q^{-j}(q)_{j+1}}{1-q} \left( \sum_{k=j}^{N-1} q^{k-j} \left[ \begin{array}{c} k \\ j \end{array} \right] \right) (t_m)^{N-j-1}(s_{m+1})^j \bigg|_{Nm-n}.
\]

Clearly all terms in the sum vanish. Thus once more the only contribution arises from the last term in (C.25) and it is the same as above.

Case 3: \( n = 0 \)

Using (C.3) we find

\[
(t_m)^n t_0 = q^N t_0 (t_m)^N
\]

\[
+ \sum_{k=1}^{N} q^{k-1}(t_m)^{N-k} \left[ (q - 1) \sum_{l=1}^{m-1} t_{m-l} t_l \right] (t_m)^{k-1} + \sum_{k=1}^{N} q^{k-1}(t_m)^{N-k} \left[ (q - q^{-1}) \sum_{l=1}^{\infty} t_{-l} t_{m+l} \right] (t_m)^{k-1}.
\]

This expression is clearly the same as the one obtained by setting \( t_0 = t_0^+ + t_0^- \), where \( t_0^+ \) satisfies (C.2) with \( n = 0 \) and \( t_0^- \) satisfies (C.4) with \( n = 0 \). Since the proof of the two cases above does not depend on the specific value of \( n \), by exactly the same algebra we show that \( t_0 \) commutes with \((t_m)^N\). \( \square \)

An obvious modification of the above argument proves that \((T_m)^N\) for \( m < 0 \) lies in the center. This then concludes the proof of Theorem 4.18.

Remark. Note that \((t_0)^N\), \((N \geq 3)\), is not in the center as the following example for \( N = 3 \) shows,

\[
(t_0)^3 t_{-2} = q^3 t_{-2} (t_0)^3 - (1-q)q^2(1+q+q^2)(t_{-1})^2(t_0)^2 + (1-q)^3 q(2+q) t_{-2} t_0 - (1-q)^4 (1+q)(t_{-1})^2 + \ldots,
\]

\[\text{(C.28)}\]
where the dots stand for terms with strictly positive modes on the right.

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