Incoherent control and entanglement for two-dimensional coupled systems

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Abstract

We investigate accessibility and controllability of a quantum system $S$ coupled to a quantum probe $P$, both described by two-dimensional Hilbert spaces, under the hypothesis that the external control affects only $P$. In this context accessibility and controllability properties describe to what extent it is possible to drive the state of the system $S$ by acting on $P$ and using the interaction between the two systems. We give necessary and sufficient conditions for these properties and we discuss the relation with the entangling capability of the interaction between $S$ and $P$. In particular, we show that controllability can be expressed in terms of the SWAP and $\sqrt{\text{SWAP}}$ operators acting on the composite system.

Introduction

Control theoretical methods and concepts provide powerful tools for the manipulation of the state of quantum systems as well as for the analysis of their dynamics [1, 2, 3, 4]. They are particularly relevant in view of the application of quantum systems in information processing algorithms [5, 6]. This paper is a study of the controllability properties of quantum systems, namely of the extent to which quantum systems can be manipulated by an external control. In most studies on the controllability of quantum systems, one assumes that the controls enter the model as appropriate functions usually modeling electro-magnetic fields in a semiclassical approximation. In these cases, the control $u$ is coherent, that is it directly affects the dynamics of the system to be controlled. In this area several Lie algebraic tools have been developed to test the controllability of both closed and open quantum systems [7, 8, 9, 10, 11, 12, 13]. There are several physical situations where it is not possible or very difficult to control the state of a system $S$ directly but it is easy to manipulate the state of an ancilla system (the probe) and then modify the state of $S$ via interaction with $P$. We call the control scheme incoherent control and investigate the controllability properties of $S$ in this context. Our study is in the spirit of the recent work in [14, 15] where controllability properties of alternative control schemes (e.g. control combined with measurement) were investigated.

We describe the state of a quantum system $S$ by a density matrix $\rho_S$, that is a positive, unit trace operator acting on the Hilbert space of the system $\mathcal{H}_S$. The convex set of all
possible states is denoted by $P_S$. Its boundary $\partial P_S$ is given by pure states, that is one-dimensional projectors in $H_S$, characterized by $\rho_S^2 = \rho_S$. The remaining states, called mixtures, are (not uniquely defined) convex superpositions of pure states. In a control theoretic framework, it is assumed that the time evolution of $\rho_S$ can be externally modified by means of a set of controls denoted by $u \in U$, where $U$ is a suitable parameter space, that is

$$\rho_S(t, u) = \gamma(t, u)[\rho_S]$$

where $\rho_S$ is the initial state in $S$ and $\{\gamma(t, u)|t \geq 0, u \in U\}$ is a multi-parameter (time and controls) family of time evolutions preserving the positivity of $\rho_S$ and its trace. The form of $\gamma(t, u)$ depends on the physical setup considered, whether $S$ is a closed system or rather it interacts with another system (as for example an external probe or the environment).

A typical control problem is that of arbitrarily driving $\rho_S$ in $P_S$ by means of the external controls $u$. The following definitions are standard in geometric control theory [16].

We say that $\rho'_S \in P_S$ can be reached from $\rho_S \in P_S$ at time $t$ if there exist some controls $u$ such that the time evolution $[\mathbb{1}]$ steers $\rho_S$ to $\rho'_S$ at time $t$: $\rho'_S = \rho_S(t, u)$. The set of all $\rho'_S$ which are attainable from $\rho_S$ at time $t$ is denoted by $\mathcal{R}(\rho_S, t)$; the reachable set from $\rho_S$ until time $T$ for the system $S$ is defined as

$$\mathcal{R}_T(\rho_S) = \bigcup_{0 \leq t \leq T} \mathcal{R}(\rho_S, t)$$

and it depends on the initial state $\rho_S$. The reachable set from $\rho_S$ is given by

$$\mathcal{R}(\rho_S) = \bigcup_{t \geq 0} \mathcal{R}(\rho_S, t) = \lim_{T \to +\infty} \mathcal{R}_T(\rho_S).$$

**Definition 0.1** The system $S$ is said to be **controllable** if and only if for all pairs $(\rho_S, \rho'_S) \in P_S \times P_S$ there is a set of controls $u$ such that $\rho_S(0) = \rho_S$ and $\rho_S(t, u) = \rho'_S$ for some $t \geq 0$.

Equivalently, we have controllability if and only if $\mathcal{R}(\rho_S) = P_S$ for all initial states $\rho_S$. The following definition refers to transfers between pure states of $S$.

**Definition 0.2** The system $S$ is said **pure-state controllable** if and only if for all pairs $(\rho_S, \rho'_S) \in \partial P_S \times P_S$ there is a set of controls $u$ such that $\rho_S(0) = \rho_S$ and $\rho_S(t, u) = \rho'_S$ for some $t \geq 0$.

A weaker property is accessibility.

**Definition 0.3** The system $S$ is said to be **accessible** if and only if $\mathcal{R}_T(\rho_S)$ contains a nonempty open set of $P_S$ for all $T > 0$ and for all $\rho_S \in P_S$.

We assume that $S$ interacts with an initially uncorrelated external system $P$, the probe, which is described by a density operator $\rho_P$ acting on the Hilbert space $H_P$. We denote by $P_P$ the convex set of all the states $\rho_P$ and by $\partial P_P$ its subset of pure states. We assume that the initial state of the probe can be modified by means of the control $u$, $\rho_P(u)$, and that, after the interaction, we eliminate the degrees of freedom of $P$. Therefore, in our setting, $[\mathbb{1}]$ becomes

$$\rho_S(t, u) = \text{Tr}_P\left(X(t)\rho_S \otimes \rho_P(u)X(t)^\dagger\right),$$

(4)
where $Tr_P$ is the partial trace over the degrees of freedom of the probe and $X(t) = e^{-iH_{tot}t}$ is the unitary propagator acting on $\mathcal{H}_S \otimes \mathcal{H}_P$. We denote by $H_{tot} = H_S + H_P + H_I$ the Hamiltonian of the composite system. Here $H_S$ and $H_P$ are the Hamiltonians describing the free evolutions of $S$ and $P$ whereas the interaction term, $H_I$, represents their coupling. Evolution (4) is completely positive since it is the composition of completely positive maps. The control affects the initial state of the probe $P$, not the dynamics of the system $S$. For this reason, we will call incoherent control this model. We will restrict ourselves to the case of two-dimensional system and probe.

The structure of the paper is as follows. In Section 1, using a Cartan decomposition of the dynamics, we study controllability and accessibility of $S$ for the incoherent control scheme. Necessary and sufficient conditions are derived. Since the controllability properties are related to the entangling capability of the time evolution, in Section 2, we discuss the connection between controllability and entanglement. In Section 3, we consider some specific examples of application of our results. In Section 4, we draw some conclusions.

1 Controllability and accessibility conditions

In what follows we consider two-dimensional Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_P$. The time evolution of $\rho_S$ is given by (4) and we assume that $H_{tot}$ is an arbitrary, known Hamiltonian. Using a Cartan decomposition of the dynamics, it is possible to write the operator $X(t) \in SU(4)$ as a product of local transformations (that is evolutions acting separately on the two systems, generated by $H_S$ and $H_P$) and a non-local one. The latter depends on $H_I$ and is the only term leading to entanglement between $S$ and $P$, hence it is the part responsible for the controllability of the state of $S$ through the state of $P$. In fact, if $H_I = 0$, there are never correlations between $S$ and $P$ and $\rho_S(t, u) = \rho_S(t)$ for any control $u$, and $S$ is not controllable, as the reachable set is a one dimensional manifold. The Cartan subalgebra of $\mathfrak{su}(4)$ is given by

$$a = i \text{ span}\{\sigma^S_x \otimes \sigma^P_x, \sigma^S_y \otimes \sigma^P_y, \sigma^S_z \otimes \sigma^P_z\}$$

(5)

where $\sigma^S_{x,y,z}$ and $\sigma^P_{x,y,z}$ are the Pauli matrices acting in $\mathcal{H}_S$ and $\mathcal{H}_P$ respectively. The corresponding $SU(4)$ decomposition is $X(t) = L_1 e^{at} L_2$, where $L_1, L_2$ are in $SU(2) \otimes SU(2)$ and $a \in \mathfrak{a}$. Both $L_1$ and $L_2$ are time-dependent, even if not explicitly shown to make lighter the notation. They can be written as tensor products of operators acting separately on $\mathcal{H}_S$ and $\mathcal{H}_P$, $L_1 = L^S_1 \otimes L^P_1$ and $L_2 = L^S_2 \otimes L^P_2$. Therefore (4) becomes

$$\rho_S(t, u) = L^S_1 Tr_P \left( e^{at} \tilde{\rho}_S(t) \otimes \tilde{\rho}_P(t, u) e^{at} \right) L^P_1$$

(6)

where $\tilde{\rho}_S(t) = L^S_2 \rho_S L^S_1$ and $\tilde{\rho}_P(t, u) = L^P_2 \rho_P(u) L^P_1$, and we used the fact that operators acting separately on $\mathcal{H}_S$ and $\mathcal{H}_P$ commute.

We want to study the controllability and accessibility properties of our incoherent control system. The structure of the family of transformations in (6) is rather complex. In fact, the partial trace removes the probe degrees of freedom, leading to an irreversible dynamics containing, in the general case, memory terms. Then this family of time evolutions is, in general, neither a group of transformations (since they do not admit an inverse) nor a semigroup (since they are not Markovian). Therefore it is not possible to use standard results of control theory to check for controllability, but it is necessary to directly compute
Lemma 1.1 The system \( S \) evolving under \( (6) \) is controllable (and pure-state controllable) if and only if it is controllable for \( L_1^S = L_2^S = 1 \), that is under the evolution

\[
\rho_S(t,u) = \gamma(t,u)[\rho_S] = Tr_P\left(e^{at} \rho_S \otimes \rho_P(u) e^{at} \right).
\]

Proof: Consider an arbitrary \((\rho_S, \rho_S')\) \( \in \mathcal{P}_S \times \mathcal{P}_S \) and assume that \( S \) is controllable under \( (6) \). Since \( L_2^S \rho_S L_2^S \in \mathcal{P}_S \) and \( L_1^S \rho_S' L_1^S \in \mathcal{P}_S \), there is a control \( u \in \mathcal{U} \) such that \( (6) \) steers \( L_2^S \rho_S L_2^S \) into \( L_2^S \rho_S L_2^S \) for some \( t \geq 0 \), but this means that \( (7) \) steers \( \rho_S \) into \( \rho_S' \) in the same time \( t \) and under the same control \( u \). Since \((\rho_S, \rho_S')\) is an arbitrary pair in \( \mathcal{P}_S \times \mathcal{P}_S \), \( S \) is controllable under \( (6) \). Now assume that \( S \) is controllable under the action of \( (6) \). Arguing as above and considering the initial state \( L_2^S \rho_S L_2^S \) and the final state \( L_1^S \rho_S' L_1^S \), we prove that \( S \) is controllable under \( (6) \). For pure-state controllability, the proof is completely analogous.

A similar fact holds true when dealing with accessibility.

Lemma 1.2 The system \( S \) evolving under \( (6) \) is accessible if and only if it is accessible under the evolution \( (7) \).

Proof: Since the accessibility property does not depend on the initial state in \( \mathcal{P}_S \), the action of the map \( L_2^S \cdot L_2^S \) is not relevant. Therefore, denoting by \( \mathcal{R}_T(\rho_S) \) the reachable set from \( \rho_S \) until time \( T \) under \( (6) \), the corresponding set for the evolution \( (7) \) is given by \( L_2^S \mathcal{R}_T(\rho_S) L_2^S \). Since the map \( L_2^S \cdot L_2^S \) is a diffeomorphism, it maps nonempty open sets of \( \mathcal{P}_P \) in nonempty open sets of \( \mathcal{P}_P \). The thesis follows.
We find convenient to use a coherence vector representation for the states of the systems $S$ and $P$, that is

$$
\rho_S(t, u) = \frac{1}{2} \left( 1 + \tilde{s}(t, u) \cdot \tilde{\sigma}^S \right), \quad \rho_P(u) = \frac{1}{2} \left( 1 + \tilde{p}(u) \cdot \tilde{\sigma}^P \right)
$$

where $\tilde{s}$ and $\tilde{p}$ are real vectors and we introduced the vectors of Pauli matrices $\tilde{\sigma}^{S,P}$. The sets $\mathcal{P}_S$, $\mathcal{P}_P$ are given by the two Bloch spheres $\mathcal{S}_S = \{ \tilde{s} \in \mathbb{R}^3 | \| \tilde{s} \| \leq 1 \}$ and $\mathcal{S}_P = \{ \tilde{p} \in \mathbb{R}^3 | \| \tilde{p} \| \leq 1 \}$. In this representation the dynamics (7) can be written as $\tilde{s}(t, u) = \Gamma(t, u)(\tilde{s}_0)$, where $\tilde{s}_0 = Tr(\rho_S \tilde{\sigma}^S)$. However we prefer to write it in the form

$$
\tilde{s}(t, u) = \Gamma'(t, \tilde{s}_0)(\tilde{p}(u)) := A(t, \tilde{s}_0)\tilde{p}(u) + \tilde{a}(t, \tilde{s}_0),
$$

where the real matrix $A(t, \tilde{s}_0)$ is given by

$$
A(t, \tilde{s}_0) = \begin{pmatrix}
\sin(2c_y t) \sin(2c_z t) & -s_z \sin(2c_y t) \cos(2c_z t) & s_y \cos(2c_y t) \sin(2c_z t) \\
-s_z \sin(2c_x t) \cos(2c_z t) & \sin(2c_x t) \sin(2c_z t) & -s_x \cos(2c_x t) \sin(2c_z t) \\
-s_y \cos(2c_y t) \sin(2c_x t) & s_x \cos(2c_x t) \sin(2c_y t) & \sin(2c_x t) \sin(2c_y t)
\end{pmatrix}
$$

and the inhomogeneous part is

$$
\tilde{a}(t, \tilde{s}_0) = \begin{pmatrix}
s_x \cos(2c_y t) \cos(2c_z t) \\
s_y \cos(2c_x t) \cos(2c_z t) \\
s_z \cos(2c_x t) \cos(2c_y t)
\end{pmatrix}.
$$

It is convenient to write the dynamics as in (12) since, in this representation, the reachable set from $\rho_S$ at time $t$ is given by $\mathcal{R}(\rho_S, t) = \Gamma'(t, \tilde{s}_0)(\mathcal{S}_P) \subseteq \mathcal{S}_S$. Therefore it is an ellipsoid centered at $\tilde{a}(t, \tilde{s}_0)$, whose semi-axes are given by the singular values of $A(t, \tilde{s}_0)$. This ellipsoid expands and shrinks in time, and its center moves along the curve $\{\tilde{a}(t, \tilde{s}_0) | t \geq 0 \}$. For some graphical representations, see Section 3. We are ready to derive some constraints on $c_x, c_y$ and $c_z$ that are equivalent to controllability of $S$.

**Theorem 1** The system $S$ evolving under (6) is controllable and pure-state controllable if and only if there are $k_1, k_2, k_3 \in \mathbb{Z}$ such that

$$
\frac{c_x}{c_y} = \frac{2k_1 + 1}{2k_2 + 1}, \quad \frac{c_x}{c_z} = \frac{2k_1 + 1}{2k_3 + 1}, \quad \frac{c_y}{c_z} = \frac{2k_2 + 1}{2k_3 + 1}.
$$

**Proof:** A necessary condition for controllability is $\mathcal{R}(\rho_S) = \mathcal{S}_S$ for $\rho_S = 1/2$, the maximally mixed state. The coherence vector representation of this state is $\tilde{s}_0 = (0, 0, 0)$, therefore

$$
A(t, \tilde{s}_0) = \begin{pmatrix}
\sin(2c_y t) \sin(2c_z t) & 0 & 0 \\
0 & \sin(2c_x t) \sin(2c_z t) & 0 \\
0 & 0 & \sin(2c_x t) \sin(2c_y t)
\end{pmatrix}
$$

and $\tilde{a}(t, \tilde{s}_0) = (0, 0, 0)^T$. In this case the ellipsoid $\mathcal{R}(\rho_S, t)$ is centered in the center of $\mathcal{S}_S$ and its semi-axes are given by the diagonal entries of $A(t, \tilde{s}_0)$. We have $\mathcal{R}(\rho_S) = \mathcal{S}_S$ if and only if $A(\hat{t}, \tilde{s}_0) = \pm 1$ at some time $\hat{t}$, therefore $\sin(2c_x \hat{t}) = \pm 1$, $\sin(2c_y \hat{t}) = \pm 1$ and $\sin(2c_z \hat{t}) = \pm 1$ and hence $c_x \hat{t} = (2k_1 + 1)\pi/4$, $c_y \hat{t} = (2k_2 + 1)\pi/4$ and $c_z \hat{t} = (2k_3 + 1)\pi/4$ with $k_1, k_2$ and $k_3$ integers. Then conditions (15) hold true. Vice versa, assuming (15) and choosing $\hat{t} = (2k_1 + 1)\pi/4c_x$, it follows $c_x \hat{t} = (2k_1 + 1)\pi/4$, $c_y \hat{t} = (2k_2 + 1)\pi/4$ and
\[ c_z \dot{t} = (2k_1 + 1)\pi/4 \] and these relations are sufficient for controllability, since they imply that for an arbitrary initial state \( \rho_S \), \( A(t, \tilde{s}_0) = \pm 1 \) and \( \tilde{a}(t, \tilde{s}_0) = (0, 0, 0)^T \), that is \( \mathcal{R}(\rho_S) = S_S \).

Assume now that the system is pure-state controllable. A necessary condition is that \( \mathcal{R}(\rho_S) = S_S \) for the initial state with \( s_x = s_y = 0 \) and \( s_z = 1 \). In this case

\[
A(t, \tilde{s}_0) = \begin{pmatrix}
\sin (2c_y t) \sin (2c_z t) & -\sin (2c_y t) \cos (2c_z t) & 0 \\
\sin (2c_x t) \cos (2c_z t) & \sin (2c_x t) \sin (2c_z t) & 0 \\
0 & 0 & \sin (2c_x t) \sin (2c_y t)
\end{pmatrix} \tag{17}
\]

and

\[
\tilde{a}(t, \tilde{s}_0) = \begin{pmatrix} 0 \\ 0 \\ \cos (2c_x t) \cos (2c_y t) \end{pmatrix}. \tag{18}
\]

Using a singular value decomposition we can write \( A(t, \tilde{s}_0) = O_1 D(t, \tilde{s}_0) O_2(t, \tilde{s}_0) \), where \( O_1 \) and \( O_2(t, \tilde{s}_0) \) are orthogonal matrices whereas \( D(t, \tilde{s}_0) \) is diagonal, positive definite. Explicitly, they are given by

\[
O_1 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad O_2(t, \tilde{s}_0) = \begin{pmatrix}
-\cos (2c_z t) & -\sin (2c_z t) & 0 \\
\sin (2c_z t) & -\cos (2c_z t) & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{19}
\]

and

\[
D(t, \tilde{s}_0) = \begin{pmatrix}
\sin (2c_x t) & 0 & 0 \\
0 & \sin (2c_y t) & 0 \\
0 & 0 & \sin (2c_x t) \sin (2c_y t)
\end{pmatrix}. \tag{20}
\]

Since \( O_1 \) and \( O_2(t, \tilde{s}_0) \) are rotations, the semi-axes of the ellipsoid \( \mathcal{R}(\rho_S, t) \) are given by the absolute value of the diagonal entries of \( D(t, \tilde{s}_0) \) and oriented along the \( x, y \) and \( z \) directions. Therefore \( \mathcal{R}(\rho_S) = S_S \) if and only if \( D(\hat{t}_1, \tilde{s}_0) = \pm 1 \) and \( \tilde{a}(\hat{t}_1, \tilde{s}_0) = (0, 0, 0)^T \) at some time \( \hat{t}_1 \), that is \( \sin (2c_x \hat{t}_1) = \pm 1 \) and \( \sin (2c_y \hat{t}_1) = \pm 1 \). These conditions in turn imply \( c_y \hat{t}_1 = (2k_1 + 1)\pi/4 \) and \( c_z \hat{t}_1 = (2k_2 + 1)\pi/4 \) with \( k_1, k_2 \in \mathbb{Z} \). Considering the initial state \( s_y = s_z = 0 \) and \( s_x = 1 \) and proceeding as before, we conclude that there exists a time \( \hat{t}_2 \) such that \( c_y \hat{t}_2 = (2k_1 + 1)\pi/4 \) and \( c_z \hat{t}_2 = (2k_2 + 1)\pi/4 \) with \( k_1, k_2 \in \mathbb{Z} \). The thesis follows with \( k_1 = k_a + k_c + 2k_d \), \( k_2 = k_b + k_c + 2k_d \), and \( k_3 = k_b + k_d + 2k_b \). Conversely, if we assume \( 15 \) then at \( \hat{t} = (2k_1 + 1)\pi/4 c_x \) we have \( c_x \hat{t} = (2k_1 + 1)\pi/4 \), \( c_y \hat{t} = (2k_2 + 1)\pi/4 \), and \( c_z \hat{t} = (2k_3 + 1)\pi/4 \). Therefore for an arbitrary initial pure state \( D(\hat{t}, \tilde{s}_0) = \pm 1 \) and \( \tilde{a}(\hat{t}, \tilde{s}_0) = (0, 0, 0)^T \), hence \( \mathcal{R}(\rho_S) = S_S \) and the system is pure-state controllable.

In Theorem 1 we explicitly expressed the conditions of controllability in terms of the interaction between \( S \) and \( P \), that is as conditions involving the constants \( c_x, c_y \) and \( c_z \) in 13. Using these relations and the time \( \hat{t} \) defined in the proof of Theorem 1 we can compute \( \alpha_j(\hat{t}) = \pm e^{i\varphi}/2 \) in 19, with \( j = 0, x, y, z \) and \( \varphi \) a phase independent of \( j \). All cases are locally equivalent to

\[
e^{ai} = \frac{1}{2}(1 + \sigma_x^S \otimes \sigma_x^P + \sigma_y^S \otimes \sigma_y^P + \sigma_z^S \otimes \sigma_z^P). \tag{21}
\]

This is the SWAP operator \( X_{sw} \) satisfying \( X_{sw} \rho_S \otimes \rho_P X_{sw}^\dagger = \rho_P \otimes \rho_S \) (see also 15). Therefore it is possible to rewrite the result of Theorem 1 as follows.
Corollary 1.3 The system $S$ evolving under (4) is controllable and pure-state controllable if and only if there is a time $\hat{t} > 0$ for which $X(\hat{t})$ is locally equivalent to the SWAP operator:

$$X(\hat{t}) = L_1^S(t)X_{sw}L_2^S(t), \quad X_{sw} = e^{a\hat{t}}. \quad (22)$$

Remark 1.4 The controllability conditions are unchanged if we restrict the set of initial states in $P$ to pure states, that is $\{\rho_P(u) | u \in \mathcal{U}\} = \partial P$. In other terms, restricting the possible states for the (driving) probe to pure states does not restrict the controllability properties of the scheme. To see this, notice that the considerations before Lemma 1.1 are still valid for pure states, because unitary similarity transformations change pure states into pure states. Moreover, under the conditions of Theorem 1 the reachable set $\mathcal{R}(\rho_S,t)$ varies with continuity from $\mathcal{R}(\rho_S,0) = \rho_S$ to $\mathcal{R}(\rho_S,\hat{t}) = \partial P$, where $\hat{t}$ has been defined in Theorem 1 and $\rho_S$ is an arbitrary state. At every $t$, $\partial \mathcal{R}(\rho_S,t)$ is the set reachable by varying $\rho_P$ in the set of pure states and we have $\cup_{t \geq 0} \partial \mathcal{R}(\rho_S,t) = \mathcal{P}_S$ for every initial state $\rho_S$.

Accessibility is characterized by the following theorem.

Theorem 2 The system $S$ evolving under (4) is accessible if and only if $c_x \neq 0$, $c_y \neq 0$ and $c_z \neq 0$.

Proof: Assume that $S$ is accessible. If $c_x = 0$ were possible, starting with the initial state $\tilde{s}_0 = (0,0,1)$ we would have $s_y(t) = 0$ for all $t$, using (12) with (13) and (14). But this contradicts the accessibility assumption, then $c_x \neq 0$. In the same way we can prove that $c_y \neq 0$ and $c_z \neq 0$.

Conversely, if $c_x \neq 0$, $c_y \neq 0$ and $c_z \neq 0$ it follows that $det A(t, \tilde{s}_0) \neq 0$ almost everywhere in $[0,T]$ for every initial state $\tilde{s}_0$, since

$$det A(t, \tilde{s}_0) = s_x^2 \sin^2(2c_xt) \sin^2(2c_yt) + s_y^2 \sin^2(2c_xt) \sin^2(2c_yt) + s_z^2 \sin^2(2c_xt) \sin^2(2c_yt) + (1 - s_x^2 - s_y^2 - s_z^2) \sin^2(2c_xt) \sin^2(2c_yt) \sin^2(2c_zt). \quad (23)$$

This in turn implies that the set $\mathcal{R}_T(\rho_S)$ contains a nonempty open set in $\mathcal{S}_S$ for all $T$, for all initial state $\rho_S$. 

\square

2 Controllability and entanglement

In the previous section, we found controllability and accessibility conditions for the incoherent control model. These were given in Theorems 1 and 2. The system $S$ can be driven by $P$ because the interaction couples them and we can transfer into $S$ the ability of changing the states of $P$. At the end of the procedure, the induced entanglement between $S$ and $P$ is lost because we get rid of the degrees of freedom of $P$. Nevertheless, the entanglement itself is the key factor in the control procedure, since non entangling evolutions are necessarily neither controllable nor accessible. In this section, we investigate the relation between entanglement and controllability. For simplicity, we limit our attention to initial pure states $\rho_S \in \partial P_S$ and further consider $\rho_P \in \partial P$, since we have seen in Remark 1.4...
that controllability and pure state controllability are not changed if we consider only pure states in $P$.

Given a pure state $\rho$ in $\mathcal{H}_S \otimes \mathcal{H}_P$, we choose as a measure of the entanglement between $S$ and $P$ embodied in $\rho$ (i.e. as entanglement monotone) the concurrency defined as $\varepsilon(\rho) = \sqrt{\lambda_1 \lambda_2}$, where $\lambda_1, \lambda_2$ are the eigenvalues of the reduced matrix $\rho_S = \text{Tr}_P \rho$. It is possible to prove the following properties of $\varepsilon$: 1. $\varepsilon(\rho)$ is invariant under local operations; 2. $0 \leq \varepsilon(\rho) \leq 1/2$; 3. $\varepsilon(\rho) = 0$ if and only if $\rho$ is a factorized state; 4. $\varepsilon(\rho) = 1/2$ if and only if $\rho$ is a maximally entangled state. In the coherence vector representation

$$\varepsilon(\rho) = \frac{1}{2} \sqrt{1 - \| \vec{s} \|^2},$$

(24)

where $\vec{s} = \text{Tr} (\rho \vec{\sigma} S)$. Therefore $\| \vec{s} \|= 1$ if and only if $\rho$ is separable, $\| \vec{s} \|= 0$ if and only if $\rho$ is a maximally entangled state.

Controllability means that the vector $\vec{s}$ can reach all points of the Bloch sphere from every initial $\vec{s}_0$. For this reason, the set of unitary propagators $\{X(t) \vert t \geq 0\}$ appearing in $\mathcal{F}$ must contain operators that create an arbitrary amount of entanglement as well as destroy it. In Corollary $[13]$ we stated that this set must contain the SWAP operator. This operator is non-entangling since it maps separable states into each other, and therefore, this characterization of controllability is not amenable of a direct interpretation in terms of entanglement. However, we observe that, if the set of unitary operators in $\mathcal{F}$ contains an operator locally equivalent to the SWAP operator, it also contains operators locally equivalent to $\sqrt{\text{SWAP}}$ operator and its inverse and these latter operators have important properties in terms of entanglement. They not only are perfect entanglers (see Definition $2.1$ below) but have a stronger entangling property which we are going to define and study below (see Lemma $2.3$).

**Definition 2.1** An operator $X \in U(4)$ is said to be a perfect entangler if and only if there exists a factorized state $\rho = \rho_S \otimes \rho_P$ such that $X \rho X^\dagger$ is a maximally entangled state: $\varepsilon(\rho) = 0$ and $\varepsilon(X \rho X^\dagger) = 1/2$.

The definition of perfect entangler is independent of the initial factorized state over which the operator acts. We define an entanglement property of the operator which is dependent of the initial state.

**Definition 2.2** An operator $X \in U(4)$ is called a perfect entangler for the set $\mathcal{F} \subseteq \partial \mathcal{P}_S$ if and only if, for all $\rho_S \in \mathcal{F}$, there exists a state $\rho_P$ such that $X \rho_S \otimes \rho_P X^\dagger$ is a maximally entangled state.

**Lemma 2.3** The family of perfect entanglers for the set $\partial \mathcal{P}_S$ is the local equivalence class of the $\sqrt{\text{SWAP}}$ operator and of its inverse$^1$.

**Proof:** Every operator $X \in U(4)$ can be written in the form $X = L_1 \epsilon^a L_2$. Moreover, we shall use the coherence vector representation introduced in the previous section. Assuming that $X$ is a perfect entangler for the set $\mathcal{P}_S$, it is possible to neglect the local contributions $L_2$, since it does not affect the set $\mathcal{F} = \partial \mathcal{P}_S$, and $L_1$, since $\varepsilon(\rho)$ is invariant under local operations. Consider the initial state $\rho_S = (1 + \sigma_z^2)/2$ and use the evolution equation (12) with (13) and (14), where $\vec{s}_0 = (0,0,1)$ represents the initial $\rho_S$, $\vec{p} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

$^1$the local equivalence class for an operator $Y$ is defined as $\{L_1 Y L_2 \vert L_1, L_2 \in SU(2) \otimes SU(2)\}$
represents the arbitrary $\rho_P$ (which exists by the assumption on $X$) and $\theta$, $\phi$ are the polar coordinates on $\partial P_P$. According to the discussion following (21), the conditions of maximal entanglement are given by \( \|\vec{s}\| = 0 \), that is
\[
\begin{align*}
\sin (2c_y) \sin (2c_z - \phi) \sin \theta &= 0 \\
\sin (2c_x) \cos (2c_z - \phi) \sin \theta &= 0 \\
\sin (2c_x) \sin (2c_y) \cos \theta + \cos (2c_x) \cos (2c_y) &= 0
\end{align*}
\]
whose solutions are
\[
\begin{align}
\cos 2(c_x \pm c_y) &= 0 \\
\sin \theta &= 0
\end{align}
\]
\[
\begin{align}
\sin (2c_y) &= 0 \\
\cos (2c_x) &= 0 \\
\cos (2c_z - \phi) &= 0
\end{align}
\]
\[
\begin{align}
\sin (2c_x) &= 0 \\
\cos (2c_y) &= 0 \\
\sin (2c_z - \phi) &= 0
\end{align}
\]
that is \( c_x \pm c_y = (2k_3 + 1)\pi/4 \), with \( k_3 \in \mathbb{Z} \). Since \( \theta \) and \( \phi \) can be arbitrarily chosen, there are no constraints on \( c_z \). Following an analogous procedure for \( \rho_S = (1 + \sigma_x^S)/2 \) and \( \rho_S = (1 + \sigma_y^S)/2 \), we obtain \( c_y = (2k_1 + 1)\pi/4 \) and \( c_x = (2k_3 + 1)\pi/4 \) respectively, with \( k_1, k_3 \in \mathbb{Z} \). Combining these relations we conclude that \( c_x = (2k_1 + 1)\pi/8 \), \( c_y = (2k_2 + 1)\pi/8 \) and \( c_z = (2k_3 + 1)\pi/8 \) with \( k_1, k_2, k_3 \in \mathbb{Z} \). Depending on their values, these parameters define the $\sqrt{\text{SWAP}}$ operator or its inverse, thus the $X$ operator is locally equivalent to $\sqrt{\text{SWAP}}$ or its inverse.

On the other hand, assume that $X$ is locally equivalent to the $\sqrt{\text{SWAP}}$ operator or its inverse (see [18, 20] for more analysis on the role of this operator in entanglement theory). Therefore its coefficients in the element of the Cartan subalgebra (8) are given by \( c_x = (2k_1 + 1)\pi/8 \), \( c_y = (2k_2 + 1)\pi/8 \) and \( c_z = (2k_3 + 1)\pi/8 \) with \( k_1, k_2, k_3 \in \mathbb{Z} \), and the condition of maximal entanglement for the initial state $\rho_S$, obtained specializing (12), (13), (14), is
\[
\vec{s} = \begin{pmatrix} 1 & p_z & -p_y \\ -p_z & 1 & p_x \\ p_y & -p_x & 1 \end{pmatrix} \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = 0
\]
where \( (s_x, s_y, s_z) := \vec{s}_0 \) and \( (p_x, p_y, p_z) := \vec{p} \). Condition (27) is fulfilled for every initial $\vec{s}_0$ by the choice $\vec{p} = -\vec{s}_0 = -(s_x, s_y, s_z)$, therefore the operator $X$ is a perfect entangler for the set $\partial P_S$ and the thesis is proved.

We now formally record the following consequence of Theorem 1

**Corollary 2.4** The system $S$ evolving under (4) is controllable and pure-state controllable if and only if there is a time $\tilde{t} > 0$ for which $X(\tilde{t})$ is locally equivalent to the $\sqrt{\text{SWAP}}$ operator.

**Proof:** If we define $\tilde{t} = \bar{t}/2$ (where $\bar{t}$ has been defined in Corollary 1.3), we have $e^{a\tilde{t}} = \sqrt{e^{at}}$ and then
\[
X(\tilde{t}) = L_1^S(\tilde{t})\sqrt{X_{sw}L_2^S(\tilde{t})}, \quad \sqrt{X_{sw}} = e^{a\tilde{t}}.
\]

The following theorem establish the relation between incoherent controllability and the entanglement properties of the system.
Theorem 3 The system $S$ evolving under $[\tilde{\sigma}]$ is controllable (pure-state controllable) if and only if there is a time $\bar{t} > 0$ such that the operator $X(\bar{t})$ is a perfect entangler for the set $\partial P_S$.

Proof: The proof follows from Lemma 2.3 and Corollary 2.4. □

In the first part of the proof of Lemma 2.3 we used the fact that $X$ is a perfect entangler for three particular pure states to show that it has to be locally equivalent to the square root of the SWAP operator or its inverse. This in turns implies controllability and the vice versa is also true. A consequence of this is that controllability can be expressed in terms of specific transitions for three states. In particular, we can say that the system $S$ is (incoherent) controllable if and only if at some $\bar{t}$ we can realize the transformations $\vec{q}_x \rightarrow (0, 0, 0)$, $\vec{q}_y \rightarrow (0, 0, 0)$ and $\vec{q}_z \rightarrow (0, 0, 0)$ in the Bloch sphere $\mathcal{S}_S$, where $\vec{q}_i$, $i = x, y, z$ are three orthonormal vectors ($\vec{q}_i \cdot \vec{q}_j = \delta_{ij}$) such that

$$\vec{q}_i \cdot \dot{\vec{q}}^S = L^S_2(\bar{t}) \vec{q}_i, \quad i = x, y, z. \quad (29)$$

The choice of the states depends on the operator. We can summarize this in the following Theorem.

Theorem 4 The system $S$ is incoherent controllable if and only if it is possible to perform the state transfers $\vec{q}_{x,y,z} \rightarrow (0, 0, 0)$ all at the same time, for the three orthonormal states defined in $\mathcal{S}_S$.

There are other sets of transformations which alone characterize controllability other than the ones in Theorem 1. For example, if we do not require that they occur all at the same time, we can take (in appropriate coordinates determined by the local part of $X$) $(0, 0, 0) \rightarrow (1, 0, 0)$ at $t_1$ and $(0, 0, 0) \rightarrow (0, 1, 0)$ at $t_2$. In fact, the first transition requires

$$\begin{cases} \sin (2c_y t_1) \sin (2c_z t_1) \sin \theta \cos \phi = 1 \\
\sin (2c_y t_1) \sin (2c_z t_1) \sin \theta \sin \phi = 0 \\
\sin (2c_y t_1) \cos \theta = 0 \end{cases} \quad (30)$$

whose solution is $c_y t_1 = (2k_a + 1)\pi/4$, $c_z t_1 = (2k_b + 1)\pi/4$ with $k_a, k_b \in \mathbb{Z}$. Analogously the second transition leads to $c_z t_2 = (2k_c + 1)\pi/4$, $c_x t_2 = (2k_d + 1)\pi/4$ with $k_c, k_d \in \mathbb{Z}$. Combining these relations as we did in the proof of Theorem 1 we prove that $S$ is controllable.

3 Examples

In this section, we illustrate the incoherent control model and the results obtained in this paper by three examples covering all admitted cases and finally we summarize our results.

Case 1 - Ising Hamiltonian: $H_1 = \sigma_x^S \otimes \sigma_x^P$.

Since $c_y = c_z = 0$, the system is neither accessible nor controllable by Theorems 1 2. We can also obtain this result via a direct computation, evaluating the reachable sets and referring to the definitions of controllability and accessibility. In the coherence vector representation the time-evolution of the system is

$$\begin{pmatrix} s_x(t) \\ s_y(t) \\ s_z(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ s_x 2t & 0 & 0 \\ -s_y 2t & 0 & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} s_x \\ s_y \cos 2t \\ s_z \cos 2t \end{pmatrix} \quad (31)$$
Case 2 - Anisotropic Hamiltonian: $H_I = \sigma_x^S \otimes \sigma_x^P + \sigma_y^S \otimes \sigma_y^P + 2\sigma_z^S \otimes \sigma_z^P$.

The system is accessible but not controllable, since (15) are not satisfied. The evolution of the reachable set at time $t$ is represented in Figure 2. This set has a non-vanishing measure in the Bloch sphere for (almost) all time, however pure states are never attained, but $\rho_S = (1 \pm \sigma_z^S)/2$.

Case 3 - Isotropic (Heisenberg) Hamiltonian: $H_I = \sigma_x^S \otimes \sigma_x^P + \sigma_y^S \otimes \sigma_y^P + \sigma_z^S \otimes \sigma_z^P$.

where $\vec{s}_0 = (s_x, s_y, s_z)$, and the reachable sets are given by

$$\mathcal{R}(\rho_S, t) = \{(r_x, r_y, r_z) \in \mathcal{S}_S | r_x = s_x, |r_y - s_y \cos 2t| \leq s_z \sin 2t, |r_z - s_z \cos 2t| \leq s_y \sin 2t\}$$

and

$$\mathcal{R}(\rho_S) = \{(r_x, r_y, r_z) \in \mathcal{S}_S | r_x = s_x, r_y^2 + r_z^2 \leq s_y^2 + s_z^2\}.$$ (32)

Then $\mathcal{R}_T(\rho_S)$ is a set of null measure in $\mathcal{S}_S$ and the system is not accessible. Moreover, $\mathcal{R}(\rho_S) \neq \mathcal{S}_S$ for every initial state $\rho_S$, therefore the system is not controllable.

The time evolution of $\mathcal{R}(\rho_S, t)$ is represented in Figure 1 as time evolves. At every time, this set collapses to a segment, contained in the plane with constant $s_x$ for all $t$.

Case 3 - Isotropic (Heisenberg) Hamiltonian: $H_I = \sigma_x^S \otimes \sigma_x^P + \sigma_y^S \otimes \sigma_y^P + \sigma_z^S \otimes \sigma_z^P$.

Figure 1: Evolution of $\mathcal{R}(\rho_S, t)$ for the Ising Hamiltonian $H_I = \sigma_x^S \otimes \sigma_x^P$. The initial state is $\vec{s}_0 = (0, 0, 1/2)$ and $t = \pi/12, \pi/8$ and $\pi/4$ in the three pictures. The reachable set collapses into a segment. The system is neither accessible nor controllable.

Figure 2: Evolution of $\mathcal{R}(\rho_S, t)$ for the Anisotropic Hamiltonian $H_I = \sigma_x^S \otimes \sigma_x^P + \sigma_y^S \otimes \sigma_y^P + 2\sigma_z^S \otimes \sigma_z^P$. The initial state is $\vec{s}_0 = (0, 0, 1/2)$ and $t = \pi/12, \pi/8$ and $\pi/4$ in the three pictures. The system is accessible but not controllable.
In this case the system is both accessible and controllable. The reachable sets $\mathcal{R}(\rho_S, t)$ grow and shrink in time, and $\mathcal{R}(\rho_S, t = (2k + 1)\pi/4) = \mathcal{S}_S$, with $k \in \mathbb{Z}$. See Figure 3 for a graphical representation of this evolution.

4 Conclusions

In this paper we have described a model of incoherent control for a system $S$ coupled to a probe $P$, that is a control that does not affect the Hamiltonian of $S$ but it is performed through control on the probe and interaction of the probe with the system. We have restricted our analysis to the simplest but important case of two dimensional probe and system and assumed that we have complete control on the probe. In fact we have proved that it is not restrictive to assume that the state of the probe is a pure state. We have derived necessary and sufficient conditions for accessibility (Theorem 2) and controllability (Theorem 1), and we have discussed the relation between this latter property and the entangling properties of the unitary evolution of the systems (Theorem 3). The SWAP and $\sqrt{\text{SWAP}}$ operators play a special role both in characterizing controllability and in its relation with the entanglement. Controllability and entanglement are meant to be in finite time and our analysis is completely deterministic.

This study is a first step in the investigation of control schemes via incoherent control. Natural extensions are to higher dimensional system and probe as well to cases where the probe is only partially controllable. The interplay between the (coherent) control of the probe and the incoherent controllability of the system is also of interest in practice as well as the study of mixed coherent-incoherent control schemes. Another direction for future research is the study of controllability for incoherent control schemes for open quantum systems.

References

[1] Information Complexity and Control in Quantum Physics, edited by A. Blaquiere, S. Dinerand and G. Lochak (Springer, New York, 1987)

[2] A. G. Butkovskiy and Yu. I. Samoilenko, Control of Quantum-mechanical Processes and Systems (Kluwer Academic, Dordrecht, 1990)
[3] W. S. Warren, H. Rabitz, and M. Dahleh, Science 259, 1581 (1993)

[4] S. Lloyd, Phys. Rev. A 62, 022108 (2000)

[5] J. Gruska, *Quantum Computing* (McGraw-Hill, 1999)

[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000)

[7] S. G. Schirmer, H. Fu and A. I. Solomon, Phys. Rev. A 63, 063410 (2001)

[8] F. Albertini and D. D’Alessandro, IEEE Transactions on Automatic Control 48, 1399 (2003)

[9] L. Viola, S. Lloyd and E. Knill, Phys. Rev. Lett. 83, 4888 (1999)

[10] L. Viola and S. Lloyd, Phys. Rev. A 65, 010101 (2002)

[11] G. M. Huang, T. J. Tarn and J. W. Clark, J. Math. Phys. 24, 2608 (1983)

[12] C. Altafini, J. Math. Phys. 44, 2357 (2003)

[13] V. Ramakrishna, M. Salapaka, M. Dahleh, H. Rabitz and A. Peirce, Phys. Rev. A 51, 960 (1995)

[14] R. Vilela Mendes and V. I. Manko, Phys. Rev. A 67, 053404 (2003)

[15] A. Mandilara and J. W. Clark, Phys. Rev. A 71, 013406 (2005)

[16] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997

[17] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, 1978)

[18] J. Zhang, J. Vala, K. B. Whaley and S. Sastry, Phys. Rev. A 67, 042313 (2003)

[19] S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997)

[20] A. T. Rezakhani, Phys. Rev. A 70, 052313 (2004)