EQUIVARIANTLY UNIFORMLY RATIONAL VARIETIES

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Abstract. We introduce equivariant versions of uniform rationality: given an algebraic group $G$, a $G$-variety is called $G$-uniformly rational (resp. $G$-linearly uniformly rational) if every point has a $G$-invariant open neighborhood equivariantly isomorphic to a $G$-invariant open subset of the affine space endowed with a $G$-action (resp. linear $G$-action). We establish a criterion for $\mathbb{G}_m$-uniform rationality of smooth affine varieties equipped with hyperbolic $\mathbb{G}_m$-actions with a unique fixed point, formulated in terms of their Altmann-Hausen presentation. We prove the $\mathbb{G}_m$-uniform rationality of Koras-Russell threefolds of the first kind and we also give an example of a non $\mathbb{G}_m$-uniformly rational but smooth rational $\mathbb{G}_m$-threefold associated to pairs of plane rational curves birationally non equivalent to a union of lines.

Introduction

A uniformly rational variety is a variety for which every point has a Zariski open neighborhood isomorphic to an open subset of an affine space. A uniformly rational variety is in particular a smooth rational variety, but the converse is an open question [13, p.885].

In this article, we introduce stronger equivariant versions of this notion, in which we require in addition that the open subsets are stable under certain algebraic group actions. The main motivation is that for such varieties uniform rationality, equivariant or not, can essentially be reduced to rationality questions at the quotient level. We construct examples of smooth rational but not equivariantly uniformly rational varieties, the question of their uniform rationality is still open. We also establish equivariant uniform rationality of large families of affine threefolds.

We focus mainly on actions of algebraic tori $T$. The complexity of a $T$-action on a variety is the codimension of a general orbit, in the case of a faithful action, the complexity is thus simply $\dim(X) - \dim(T)$. Complexity zero corresponds to toric varieties, which are well-known to be uniformly rational when smooth. In fact they are even $T$-linearly uniformly rational in the sense of Definition 4 below. The same conclusion holds for smooth rational $T$-varieties of complexity one by a result of [21, Chapter 4]. In addition, by [3, Theorem 5] any smooth complete rational $T$-variety of complexity one admits a covering by finitely many open charts isomorphic to the affine space.

In this article, as a step toward the understanding of $T$-varieties of higher complexity, we study the situation of affine threefolds equipped with hyperbolic $\mathbb{G}_m$-actions. We use the general description developed by Altmann, Hausen and Süss (see [1, 2]) in terms of pairs $(Y, D)$, where $Y$ is a variety of dimension $\dim(X) - \dim(T)$ and $D$ a so-called polyhedral divisor on $Y$. In our situation, $Y$ is a rational surface and our main result, Theorem 16, allows us to translate equivariant uniform rationality into a question of birational geometry of curves on rational surfaces.

The article is organized as follows. In the first section we introduce equivariant versions of uniform rationality and summarize A-H presentations of affine $\mathbb{G}_m$-varieties. The second section explains how to use these presentations for the study of uniform rationality of these varieties. In the third section, we focus on families of $\mathbb{G}_m$-rational threefolds, we show, for example, that all Koras-Russell threefolds of the first kind, and certain ones of the second kind (see [16, 17]) are equivariantly uniformly rational and therefore uniformly rational. It is not known if these varieties are uniformly rational, without any group action. In a fourth section we find examples of smooth rational $\mathbb{G}_m$-threefolds including other Koras-Russell threefolds which are not equivariantly uniformly rational. It is not known if these varieties are uniformly rational, without any group action. In the last section, we introduce a weaker notion of equivariant uniform rationality and we illustrate differences between all these notions.

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1. Preliminaries

1.1. Basic examples of uniformly rational varieties. Recall that a variety of dimension $n$ is called uniformly rational if every point has a Zariski open neighborhood isomorphic to an open subset of $\mathbb{A}^n$. Some partial results are known, for instance every smooth complete rational surface is uniformly rational. In fact it follows from [5, 6] that the blowup of a uniformly rational variety along a smooth subvariety is again uniformly rational. Since open subset of uniformly rational varieties are uniformly rational, it follows that every open subsets of the blowup of a uniformly rational variety along a smooth subvariety is again uniformly rational. In particular this holds for affine modifications of uniformly rational varieties along smooth subvarieties.

Definition 1. [18, 9] Let $(X,D,Z)$ be a triple consisting of a variety $X$, an effective Cartier divisor $D$ on $X$ and a closed sub-scheme $Z$ with ideal sheaf $I_Z \subset O_X(-D)$. The affine modification of the variety $X$ along $D$ with center $Z$ is the scheme $X' = \tilde{X}_Z \setminus D'$ where $D'$ is the proper transform of $D$ in the blow-up $\tilde{X}_Z \to X$ of $X$ along $Z$.

A particular type of affine modification is the hyperbolic modification of a variety $X$ with center at a closed sub-scheme $Z \subset X$ (see [27]): It is defined as the affine modification of $X \times \mathbb{A}^1$ with center $Z \times \{0\}$, and divisor $X \times \{0\}$. As an immediate corollary of [5, Proposition 2.6], we obtain the following result:

Proposition 2. Affine modifications and hyperbolic modifications of uniformly rational varieties along smooth centers are again uniformly rational.

Example 3. Let $\mathbb{A}^n = \text{Spec}(\mathbb{C}[x_1, \ldots, x_n])$ and let $I = (f,g)$ be the defining ideal of a smooth subvariety in $\mathbb{A}^n$. Then the affine modification of $\mathbb{A}^n$ with center $I = (f,g)$ and divisor $D = \{f = 0\}$ is isomorphic to the subvariety $X' \subset \mathbb{A}^{n+1}$ defined by the equation:

$$\{g(x_1, \ldots, x_n) - yf(x_1, \ldots, x_n) = 0\} \subset \mathbb{A}^{n+1} = \text{Spec}(\mathbb{C}[x_1, \ldots, x_n, y]).$$

It is a uniformly rational variety.

1.2. Equivariantly uniformly rational varieties. Let $G$ be an affine algebraic group and let $X$ be a $G$-variety, that is an algebraic variety endowed with a $G$-action. We introduce equivariant versions, of uniform rationality.

Definition 4. Let $X$ be a $G$-variety and $x \in X$.

i) We say that $X$ is $G$-linearly rational at the point $x$ if there exists a $G$-stable open neighborhood $U_x$ of $x$, a linear representation of $G \to GL_n(V)$ and a $G$-stable open subset $U' \subset V \simeq \mathbb{A}^n$ such that $U_x$ is equivariantly isomorphic to $U'$.

ii) We say that $X$ is $G$-rational at the point $x$ if there exists an open $G$-stable neighborhood $U_x$ of $x$, an action of $G$ on $\mathbb{A}^n$ and an open $G$-stable subset $U' \subset \mathbb{A}^n$ such that $U_x$ is equivariantly isomorphic to $U'$.

iii) A $G$-variety that is $G$-linearly rational (respectively $G$-rational) at each point is called $G$-linearly uniformly rational (respectively $G$-uniformly rational).

iv) A $G$-variety that admits a unique fixed point $x_0$ by the $G$-action is called $G$-linearly rational (respectively $G$-rational) if it is $G$-linearly rational (respectively $G$-rational) at $x_0$.

$G$-linearly uniformly rational or just $G$-uniformly rational varieties are always uniformly rational. The converse is trivially false: for instance the point $[1:0]$ in $\mathbb{P}^1$ does not admit any $G_a$-invariant affine open neighborhood for the action defined by $t \cdot [u:v] \to [u+tv:v]$.

For algebraic tori $T$, as already mentioned in the introduction, it is a classical fact that smooth toric varieties are $T$-linearly uniformly rational. Moreover, it is known that every effective $T$-action on $\mathbb{A}^n$ is linearisable for $\dim(T) \geq n-1$ (see [14] for $n = 2$ and [4] for the general case), and in another direction every algebraic $G_m$-action on $\mathbb{A}^3$ is linearisable [17]. As a consequence, we obtain the following:

Theorem 5. For $T$-varieties of complexity 0, 1 and for $G_m$-threefolds the properties of being $T$-linearly uniformly rational and $T$-uniformly rational are equivalent.

1.3. Hyperbolic $G_m$-actions on smooth varieties. By a Theorem of Sumihiro (see [25]) every normal $G_m$-variety $X$ admits a cover by affine $G_m$-stable open subsets. This reduces the study of $G_m$-linearly uniformly rational varieties to the affine case. Recall that the coordinate ring $A$ of an affine $G_m$-variety $X$ is $\mathbb{Z}$-graded in a natural way by its subspaces $A_n := \{f \in A / f(\lambda \cdot x) = \lambda^n f(x), \forall \lambda \in G_m\}$ of semi-invariants of weight $n$. In particular $A_0$ is the ring of invariant functions on $X$. If $X$ is smooth with positively graded coordinate ring, then by [20], $X$ has the structure of a vector bundle over its fixed point locus $X^{G_m}$, and hence the question whether $X$ is $G_m$-linearly uniformly rational becomes intimately related to the uniform rationality of $X^{G_m}$. In this subsection, we consequently focus on hyperbolic $G_m$-actions. We summarize the correspondence between smooth affine varieties $X$
endowed with an effective hyperbolic \(G_m\)-action and pairs \((Y, D)\) where \(Y\) is a variety, that we call A-H quotient, and \(D\) is a so-called segmental divisor on \(Y\). All the definitions and constructions are adapted from [1].

**Definition 6.** A \(G_m\)-action is said to be hyperbolic if there is at least one \(n_1 < 0\) and one \(n_2 > 0\) such that \(A_{n_1}\) and \(A_{n_2}\) are nonzero.

**Definition 7.** Let \(X = \text{Spec}(A)\) be a smooth affine variety equipped with a hyperbolic \(G_m\)-action.

i) We denote by \(q : X \rightarrow Y_0(X) := X//G_m = \text{Spec}(A_0)\) the categorical quotient of \(X\).

ii) The A-H quotient \(Y(X)\) of \(X\) is the blow-up \(\pi : Y(X) \rightarrow Y_0(X)\) of \(Y_0(X)\) with center at the closed subscheme defined by the ideal \(I = \langle A_d \cdot A_{-d} \rangle\), where \(d > 0\) is chosen such that \(\bigoplus_{n \in \mathbb{Z}} A_{dn}\) is generated by \(A_0\) and \(A_{\pm d}\). It is a normal semi-projective variety (see [1]). By virtue of [26, Theorem 1.9, proposition 1.4], \(Y(X)\) is isomorphic to the fiber product of the schemes \(Y_{\pm}(X) = \text{Proj}_{A_0}(\bigoplus_{n \geq 0} A_{\pm n})\) over \(Y_0(X)\).

In the remainder of the article, we use the notation \(\pi : \hat{V} \rightarrow V\) to refer to the blow-up of an affine variety \(V\) with center at the closed sub-scheme defined by the ideal \(I \subset \Gamma(V, \mathcal{O}_V)\).

**Definition 8.** A segmental divisor \(D\) on a normal algebraic variety \(Y\) is a formal finite sum \(D = \sum [a_i, b_i] \otimes D_i\), where \(D_i\) are prime Weil divisors on \(Y\) and \([a_i, b_i]\) are closed intervals with rational bounds \(a_i \leq b_i\).

The set of all closed intervals with rational bounds, admits a structure of abelian semigroup for the Minkowski sum, the Minkowski sum of two intervals \([a_i, b_i]\) and \([a_j, b_j]\) being the interval \([a_i + a_j, b_i + b_j]\).

Every element \(n \in \mathbb{Z}\) determines a map from segmental divisors to the group of Weil \(\mathbb{Q}\)-divisors on \(Y\):

\[
D = \sum [a_i, b_i] \otimes D_i \rightarrow D(n) = \sum q_i D_i,
\]

where for all \(i, q_i \in \mathbb{Q}\) is the minimum of \(na_i\) and \(nb_i\).

**Definition 9.** A proper-segmental divisor (ps-divisor) \(D\) on a variety \(Y\) is a segmental divisor on \(Y\) such that for every \(n \in \mathbb{Z}\), \(D(n)\) satisfies the following properties:

1) \(D(n)\) is a \(\mathbb{Q}\)-Cartier divisor on \(Y\).
2) \(D(n)\) is semi-ample, that is, for some \(p \in \mathbb{Z}_{>0}\), \(Y\) is covered by complements of supports of effective divisors linearly equivalent to \(D(pm)\).
3) \(D(n)\) is big, that is, for some \(p \in \mathbb{Z}_{>0}\), there exists an effective divisor \(D\) linearly equivalent to \(D(pm)\) such that \(Y \setminus \text{Supp}(D)\) is affine.

In the particular case of hyperbolic \(G_m\)-action, the main Theorem of [1] can be reformulated as follows:

**Theorem 10.** For any ps-divisor \(D\) on a normal semi-projective variety \(Y\) the scheme

\[
\mathbb{S}(Y, D) = \text{Spec}(\bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(D(n))))
\]

is a normal affine variety of dimension \(\text{dim}(Y) + 1\) endowed with an effective hyperbolic \(G_m\)-action, whose A-H quotient \(\mathbb{S}(Y, D)\) is birationally isomorphic to \(Y\). Conversely any normal affine variety \(X\) endowed with an effective hyperbolic \(G_m\)-action is isomorphic to \(\mathbb{S}(Y(X), D)\) for a suitable ps-divisor \(D\) on \(Y(X)\).

**Remark 11.** Alternatively, see [8, 10], any finitely generated \(\mathbb{Z}\)-graded algebra \(A\) can be written in the form

\[
A = \bigoplus_{n < 0} \Gamma(Y, \mathcal{O}_Y(nD_-)) \oplus \Gamma(Y, \mathcal{O}_Y) \oplus \bigoplus_{n > 0} \Gamma(Y, \mathcal{O}_Y(nD_+))
\]

where \((Y, D_+, D_-)\) is a triple consisting in a normal variety \(Y\) and suitable \(\mathbb{Q}\)-divisors \(D_+\) and \(D_-\) on it. These two presentations are obtained from each other by setting \(D_- = D(-1)\), \(D_+ = D(1)\) and conversely \(D = \{1\} D_+ + [0, 1](-D_- - D_+)\).

**Remark 12.** A method to determine a possible ps-divisor \(D\) such that \(X \simeq \mathbb{S}(Y, D)\) is to embed \(X\) as a \(G_m\)-stable subvariety of an affine toric variety (see [1, section 11]). The calculation is then reduced to the toric case by considering an embedding in \(\mathbb{A}^n\) endowed with a linear action of a torus \(T\) of sufficiently large dimension \(n\). The inclusion of \(G_m \rightarrow T\) corresponds to an inclusion of the lattice \(\mathbb{Z}\) of one parameter subgroups of \(G_m\) in the lattice \(\mathbb{Z}^n = N\) of one parameter subgroups of \(T\). We obtain the exact sequence:

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{s} \mathbb{P} N = \mathbb{Z}^n \xrightarrow{p} \mathbb{P} N = \mathbb{Z}^n / \mathbb{Z} \longrightarrow 0,
\]
where \( F \) is given by the induced action of \( \mathbb{G}_m \) on \( \mathbb{A}^n \) and \( s \) is a section of \( F \). Let \( v_i \), for \( i = 1, \ldots, n \), be the first integral vectors of the unidimensional cones generated by the \( i \)-th column vectors of \( P \) considered as rays in the lattice \( \mathbb{N} \simeq \mathbb{Z}^{n-1} \). Let \( Z \) be the toric variety of dimension \( \dim(\mathbb{A}^n) - \dim(\mathbb{T}) \), determined by the fan in \( \mathbb{N} \) whose cones are generated the \( v_i \) for \( i = 1, \ldots, n \). Then each \( v_i \) corresponds to a \( \mathbb{T} \)-equivariant divisor where \( \mathbb{T}' = \text{Spec}(\mathbb{C}[\mathbb{N}']) \). By [1, section 11] \( Z \) contains the A-H quotient of \( X \) as a closed subset, and the support of \( D_i \) is obtained by restricting the \( \mathbb{T}' \)-invariant divisor corresponding to \( v_i \) to \( Y \). If \( X \) is the affine space endowed with a linear action of \( \mathbb{G}_m \), then \( Z \) is itself the A-H quotient of \( \mathbb{A}^n \). The segment associated to the divisor \( D_i \) is equal to \( s(\mathbb{R}_+^n \cap P^{-1}(v_i)) \). The section \( s \) can further be chosen so that the number of non zero coefficients in the associated matrix is minimal. The ps-divisor \( D \) from a such section will be called minimal. We would like to point out that this notion is more restrictive than that given in [1], in particular every minimal ps-divisor in our sense is also in the sense of [1].

2. Algebro-combinatorial criteria for \( \mathbb{G}_m \)-linear rationality

Given a smooth rational variety \( X \) endowed with a hyperbolic \( \mathbb{G}_m \)-action which admits a unique fixed point \( x_0 \), we develop in this section a method to test whether \( X \) is \( \mathbb{G}_m \)-rational.

**Definition 13.** [1, Definition 8.3] Let \( Y \) and \( Y' \) be normal semi-projective varieties and let \( D' = \sum [a'_i, b'_i] \otimes D'_i \) and \( D = \sum [a_i, b_i] \otimes D_i \) be ps-divisors on \( Y' \) and \( Y \) respectively.

- i) Let \( \varphi : Y \to Y' \) be a morphism such that \( \varphi(Y) \) is not contained in \( \text{Supp}(D') \) for any \( i \). The **polyhedral pull-back** of \( D' \) is defined by \( \varphi^*(D') := \sum [a'_i, b'_i] \otimes \varphi^*(D'_i) \), where \( \varphi^*(D'_i) \) is the usual pull-back of \( D'_i \).

- ii) Let \( \varphi : Y \to Y' \) be a proper dominant map. The **polyhedral push-forward** of \( D \) is defined by \( \varphi_* (D) := \sum [a_i, b_i] \otimes \varphi_* (D_i) \), where \( \varphi_* (D_i) \) is the usual push-forward of \( D_i \).

Let \( \varphi : Y \to Y' \) be a birational morphism and let \( D' \) be a divisor on \( Y' \), then we decompose the pull-back of \( D' \) by \( \varphi \) as follows: \( \varphi^*(D') = (\varphi^{-1})_* (D') + R \) where \( (\varphi^{-1})_* (D') \) is the strict transform of \( D' \) and \( R \) is supported in the exceptional locus of \( \varphi \).

**Definition 14.** Two pairs \((Y_i, D_i)\), \( i = 1, 2 \) consisting of a variety \( Y_i \) and a Cartier divisor \( D_i \) on \( Y_i \) are called **birationally equivalent** if there exist a variety \( Z \), and two proper birational morphisms \( \varphi_i : Z \to Y_i \) such that the strict transforms \( (\varphi^{-1}_1)_* (D_1) \) and \( (\varphi^{-1}_2)_* (D_2) \) of \( D_1 \) and \( D_2 \) respectively are equal. For ps-divisors, we extend this notion in the natural way to pairs \((Y_i, D_i)\) consisting of a semi-projective variety \( Y_i \) and a ps-divisor \( D_i \) on \( Y_i \) using the polyhedral pull-back defined above.

Since we consider hyperbolic \( \mathbb{G}_m \)-actions with a unique fixed point, the construction of the A-H quotient \( Y(X) \) as in Definition 7 ensures that \( Y(X) \) has only one exceptional divisor \( E \) over \( Y_0(X) \). We denote by \( \mathcal{D} \) the segmental divisor obtain from the ps-divisor \( D \) corresponding to \( X \) by removing all irreducible components whose support does not intersect \( E \). The following example illustrate a situation for which \( \mathcal{D} \neq D \).

**Example 15.** Let \( S \) be the affine surface defined by \( \{x^2y + x = z^2\} \subset \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \) and let \( X := \mathbb{A}^1 \) be the cylinder over \( S \), endowed with the hyperbolic \( \mathbb{G}_m \)-action induced by the linear one \( \lambda(x, y, z, t) \to (\lambda^2x, \lambda^{-6}y, \lambda^3z, \lambda^4t) \) on \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]) \). Using the method described in Remark 12, we find that \( X \) is equivariantly isomorphic to \( \mathbb{S}(\hat{\mathbb{A}}^2_{(u,v)}, \mathcal{D}) \) with:

\[
\mathcal{D} = \left\{ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right\} D_1 + \left\{ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right\} D_2 - \left\{ \begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array} \right\} D_3 + \left[ \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} \right] E,
\]

where \( E \) is the exceptional divisor of the blow-up \( \pi : \hat{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2 \simeq \text{Spec}(\mathbb{C}[u, v]) \simeq \text{Spec}(\mathbb{C}[yt^2, xy]) \), and where \( D_1, D_2 \) and \( D_3 \) are the strict transforms of the curves \( L_1 = \{v = 0\}, L_2 = \{1 + v = 0\} \) \( L_3 = \{u = 0\} \) in \( \hat{\mathbb{A}}^2 = \text{Spec}(\mathbb{C}[u, v]) \). The divisor \( D_2 \) does not intersect the exceptional divisor \( E \), so:

\[
\mathcal{D} = \left\{ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right\} D_1 - \left\{ \begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array} \right\} D_3 + \left[ \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \end{array} \right] E.
\]

**Theorem 16.** Let \( X \) be a smooth affine rational variety endowed with a hyperbolic \( \mathbb{G}_m \)-action with a unique fixed point \( x_0 \). Then \( X \) is \( \mathbb{G}_m \)-rational if and only if the following holds:

1) There exists pairs \((Y, \mathcal{D})\) and \((Y', \mathcal{D}')\) such that \( \mathbb{S}(Y, \mathcal{D}) \) is equivariantly isomorphic to \( X \) and \( \mathbb{S}(Y', \mathcal{D}') \) is equivariantly isomorphic to \( \mathbb{A}^n \) endowed with a hyperbolic \( \mathbb{G}_m \)-action.

2) The pairs \((Y, \mathcal{D})\) and \((Y', \mathcal{D}')\) are birationally equivalent.
Proof. Suppose that $X$ is $G_m$-rational so that there exists an open $G_m$-stable neighborhood $U_{x_0}$ of $x_0$, an action of $G_m$ on $\mathbb{A}^n$, an open $G_m$-stable subvariety $U' \subset \mathbb{A}^n$, and an equivariant isomorphism $\varphi : U_{x_0} \to U'$, We can always reduce to the case where $U_{x_0}$ and $U'$ are principal open sets. Indeed $U_{x_0}$ is the complement of a closed stable subvariety of $X$ determined by an ideal $I = (f_0, \ldots, f_k)$ where each $f_i \in I(X, \mathcal{O}_X)$ is semi-invariant. As $U_{x_0}$ contains $x_0$, at least one of the $f_i$ does not vanish at $x_0$. Denoting this function by $f$, the principal open subset $X_f := X \setminus V(f)$ is contained in $U_{x_0}$. The restriction of $\varphi$ to $X_f$ induces an isomorphism between $X_f$ and $\varphi(X_f)$. This yields a divisor $\hat{\varphi} : U' \setminus \varphi(X_f)$ on $U'$. Since $\mathbb{A}^n$ is factorial, this divisor is the restriction of a principal divisor $\text{Div}(\hat{f}')$ on $\mathbb{A}^n$ for a certain regular function $\hat{f}'$. By construction $\varphi$ induces an equivariant isomorphism between $X_f$ and $\mathbb{A}^n$.

Note that any non-constant semi-invariant function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(x_0) \neq 0$ is actually invariant. Indeed, letting $w$ be the weight of $f$, we have $\lambda \cdot f(x_0) = \lambda^w f(x_0) = f(\lambda^{-1} \cdot x_0) = f(x_0)$ for all $\lambda \in G_m$, and so $w = 0$.

Let $(Y, D)$ be the pair corresponding to $X$ with $D$ minimal in the sense defined in Remark 12. We can identify every invariant function $f$ on $X$ non-vanishing at $x_0$ with an element $f$ of $\Gamma(Y, \mathcal{O}_Y)$ such that $V(f) \subset Y$ does not contain any irreducible component of $\hat{\varphi}$. Let $\hat{\varphi}$ be the pair corresponding to $(\varphi, D)$, and let $\hat{\varphi}$ be the restriction of $\varphi$ to $X$. Then $\hat{\varphi}$ is an isomorphism, and the canonical map $\hat{\varphi} : D \to D$ describes the open embedding $X_f \to X$.

Denoting $i : Y_f \hookrightarrow Y$ the canonical open embedding, we said that the pair $(Y_f, D_f = i^*(\hat{\varphi}))$ describes the equivariant open embedding $j : X_f \simeq S(Y_f, i^*(\hat{\varphi})) \hookrightarrow X$, and we have the following diagram:

\[
\begin{array}{ccc}
X_f & \xrightarrow{j} & X = S(Y, D) \\
\downarrow{q} & & \downarrow{q} \\
X_f//\mathbb{G}_m = Y_{0, f} & \xrightarrow{\pi} & Y_0 = X//\mathbb{G}_m \\
\downarrow{\pi_{Y_f}} & & \downarrow{\pi} \\
Y_f & \xrightarrow{i} & Y = BI(Y_0).
\end{array}
\]

A similar description holds for the principal invariant open subset $\mathbb{A}^n_{y^0}$ of $\mathbb{A}^n$ endowed with a hyperbolic $\mathbb{G}_m$-action. We denote the A-H quotient $\mathbb{A}^n_{y^0}$ of $\mathbb{A}^n$ simply by $Y_{f, y^0}$, and the corresponding ps-divisor by $D_{y^0}$.

By [1, Corollary 8.12] $X_f$ and $\mathbb{A}^n_{y^0}$ are equivariantly isomorphic if and only if there exist a normal semi-projective variety $Y''$, birational morphisms $\sigma_1 : Y_f \to Y''$ and $\sigma_2 : Y_{f, y^0} \to Y''$ and a ps-divisor $D''$ on $Y''$ such that $D \simeq \sigma_1^*(D'')$ and $D_{y^0} \simeq \sigma_2^*(D'')$. Since $\sigma_1$ is projective and birational, it either contracts the unique exceptional divisor $E$ of $Y_f$ over $Y_{0, f}$, or it is an isomorphism. But if $\sigma_1$ contracts $E$ then $S(Y'', D'')$ is not equivariantly isomorphic to $X_f$ by Definition 7. Therefore $\sigma_1$ is an isomorphism. The same holds for $\sigma_2$.

Since $D_f$ and $D_{y^0}$ are minimal, the pairs $(Y_f, D_f)$ and $(Y_{f, y^0}, D_{y^0})$ are equivalent, that is, there exists an isomorphism $\Phi : Y_f \to Y_{f, y^0}$ such that $(\Phi^{-1})_* (D_{y^0}) = D_f$. This implies that the pairs $(Y, D)$ and $(Y', D')$ are birationally equivalent and we obtain a commutative diagram:

\[
\begin{array}{ccc}
S(Y', D') & \xrightarrow{j'} & \mathbb{A}^n_{y^0} \simeq X_f \xrightarrow{j} X = S(Y, D) \\
\downarrow{q} & & \downarrow{q} \\
\mathbb{A}^n//\mathbb{G}_m = Y_0' & \xrightarrow{\pi'} & Y_0 = X//\mathbb{G}_m = Y_0 \\
\downarrow{\pi_{Y'}} & & \downarrow{\pi} \\
Y' & \xrightarrow{i'} & Y \simeq Y_f \xrightarrow{i} Y.
\end{array}
\]

Conversely, assume that $X = S(Y, D)$ and $\mathbb{A}^n = S(Y', D')$ endowed with an hyperbolic $\mathbb{G}_m$-action are such that the pairs $(Y, D)$ and $(Y', D')$ are birationally equivalent. We can further assume that there exists a birational map...
between $Y$ and $Y'$ which restricts to an isomorphism $\phi : Y_g \to Y_g'$ between the principal open sets $Y_g$ of $Y$ and $Y_g'$ of $Y'$ corresponding to suitable functions $g \in \mathcal{F}(Y, \mathcal{O}_Y)$ and $g' \in \mathcal{F}(Y', \mathcal{O}_{Y'})$ whose zero loci do not intersect the exceptional divisors of $Y \to Y_0$ and $Y' \to Y_0'$ respectively. Similarly as above the function $g$ can be identified with an invariant function on $X$ which does not vanish at $x_0$. By virtue of [2, Proposition 3.3] the pair $(Y_g, \mathcal{D}_g)$ describes the equivariant open embedding $X_g \simeq S(Y_g, \mathcal{D}_g) \hookrightarrow X$. In the same way, $g'$ corresponds to an invariant function on $\mathbb{A}^n$ and the pair $(Y_{g'}, \mathcal{D}_{g'})$ describes the equivariant open embedding $\mathbb{A}^n_{g'} \simeq S(Y_{g'}, \mathcal{D}_{g'}) \hookrightarrow \mathbb{A}^n$. This gives the result.

\[\square\]

3. Examples of $\mathbb{G}_m$-uniformly rational threefolds

In the particular case of affine threefolds, $\mathbb{G}_m$-linear uniform rationality is reduced (by the previous section) to a problem of birational geometry in dimension 2. Indeed, using Theorem 16, the question may then be considered at the level of the A-H quotients which are rational semi-projective surfaces.

3.1. Hyperbolic $\mathbb{G}_m$-action on $\mathbb{A}^3$. Using this presentation and the fact that every algebraic $\mathbb{G}_m$-action on $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ is linearisable [17], we are able to characterize hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$ in terms of their A-H presentations. Indeed let $G \times \mathbb{A}^3 \to \mathbb{A}^3$ be an effective hyperbolic $\mathbb{G}_m$-action given by $\lambda \cdot (x, y, z) \to (\lambda x, \lambda^a y, \lambda^{-c} z)$ with $(a, b, c) \in \mathbb{Z}_{\geq 0}^3$. After a suitable $G \times \mathbb{A}^3$-invariant cyclic cover along coordinate axes, we can assume that $\mathbb{A}^3/\mathbb{G}_m \simeq \mathbb{A}^2$, and the relation between such cyclic covers and the A-H presentations of the T-varieties are controlled by [23]. Let $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ be such that $\alpha a + b \beta - c \gamma = 1$. Let $\rho(a, c)$ be the greatest common divisor of $a$ and $c$, and let $\rho(b, c)$ be the greatest common divisor of $b$ and $c$, and let $\delta$ be the greatest common divisor of $\frac{\alpha}{\rho(a, c)}$ and $\frac{b}{\rho(b, c)}$. Then we have:

**Proposition 17.** Up to equivariant cyclic covers, every $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action is equivariantly isomorphic to the $\mathbb{G}_m$-variety $S(Y, D)$ with $Y$ and $D$ defined as follows:

i) $Y$ is isomorphic to a blow-up $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ of $\mathbb{A}^2$ at the origin.

ii) $D$ is of the form:

$$D = \left\{ \frac{\alpha \rho(a, c)}{c} \right\} \otimes D_1 + \left\{ \frac{\beta \rho(b, c)}{c} \right\} \otimes D_2 + \left[ \frac{\gamma}{\delta} \frac{\gamma}{\delta} + \frac{1}{\delta c} \right] \otimes E,$$

with $D_1, D_2$ are strict transforms of the coordinate axes and $E$ is the exceptional divisor of $\pi$.

**Proof.** Let $\mathbb{A}^3$ be endowed with a linear action of $\mathbb{G}_m$, the A-H presentation is obtained from the following exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{F} \mathbb{Z}^3 \xrightarrow{p} \mathbb{Z}^2 \to 0,$$

by the method described in Remark 12, where $F = \iota(a, b, -c)$, $s = (\alpha, \beta, \gamma)$ and $P = \begin{pmatrix} \frac{c}{\rho(a, c)} & 0 & \frac{a}{\rho(a, c)} \\ 0 & \frac{c}{\rho(b, c)} & \frac{b}{\rho(b, c)} \end{pmatrix}$. The algebraic quotient of $\mathbb{A}^3$ for an hyperbolic $\mathbb{G}_m$-action is isomorphic to $\mathbb{A}^2/\mu$ where $\mu$ is a finite cyclic group [12], thus the A-H quotient $Y(\mathbb{A}^3)$ is by construction a blow-up of $\mathbb{A}_2^2/\mu$. In this case $Y(\mathbb{A}^3)$ is smooth and corresponds to the toric variety $Z$ defined in Remark 12 that is a blow-up of $\mathbb{A}^2$ whose center is supported at the origin.

Let now us consider each $v_i$, for $i = 1, \ldots, 3$, as in Remark 12, that is the first integral vectors of the unidimensional cones generated by the i-th column vectors of $P = \begin{pmatrix} \frac{c}{\rho(a, c)} & 0 & \frac{a}{\rho(a, c)} \\ 0 & \frac{c}{\rho(b, c)} & \frac{b}{\rho(b, c)} \end{pmatrix}$. The first two $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as rays defining a toric variety correspond to the generators of $\mathbb{A}^2$ and thus the associated divisors are the strict transforms of the coordinate axes and the last on $v_3$ corresponds to the exceptional divisor. To determine the associated coefficients, we used the formula $[a_i, b_i] = s(\mathbb{R}_{\geq 0} \cap P^{-1}(v_i))$ given in Remark 12.

\[\square\]
Example 18. [24, example 1.4.8] The presentation of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ equipped with the hyperbolic $\mathbb{G}_m$-action $\lambda \cdot (x, y, z) = (\lambda^2 x, \lambda^3 y, \lambda^{-6} z)$ is $\mathcal{S}(\hat{\mathbb{A}}^2_{(u, v)}; D)$ with $\pi : \hat{\mathbb{A}}^2_{(u, v)} \rightarrow \mathbb{A}^2$ the blow-up of $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ at the origin and

$$D = \left\{ -\frac{1}{3} D_1 + \frac{1}{2} D_2 + \left[ \frac{1}{6} \right] E, \right\}$$

where $E$ is the exceptional divisor of the blow-up, $D_1$ and $D_2$ are the strict transforms of the lines $\{u = 0\}$ and $\{v = 0\}$ in $\hat{\mathbb{A}}^2$ respectively. Indeed, $\hat{\mathbb{A}}^3/\mathbb{G}_m = \text{Spec}(\mathbb{C}[u, v])$ and $d > 0$ in Definition 7 has to be chosen so that $\bigoplus_{n \in \mathbb{Z}} A_{dn}$ is generated by $A_0$ and $A_{\pm d}$. This is the case if $d$ is the least common multiple of the weights of the $\mathbb{G}_m$-action on $\mathbb{A}^3$. Thus $d = 6$ and $Y(X)$ is the blow-up of $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ with center at the closed sub-scheme with intelligent our choice of coordinates.

3.2. $\mathbb{G}_m$-linear uniform rationality. In this subsection we will prove that some hypersurfaces of $\mathbb{A}^4$ are $\mathbb{G}_m$-linearly uniformly rational. In particular, every Koras-Russell threefold of the first kind $X$ is $\mathbb{G}_m$-linearly uniformly rational. These varieties are defined by equations of the form:

$$\{x + x^d y + z^{a_2} + t^{a_3} = 0\} \subset \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]),$$

where $d \geq 2$, and $a_2$ and $a_3$ are coprime. They are smooth rational and endowed with hyperbolic $\mathbb{G}_m$-actions with algebraic quotients isomorphic to $\mathbb{A}^2/\mu$ where $\mu$ is a finite cyclic group. They have been classified by Koras and Russell, in the context of the linearization problem for $\mathbb{G}_m$-actions on $\mathbb{A}^3$ [17].

These threefolds can be viewed as affine modifications of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, z, t])$ along the principal divisor $D_f = \{f = 0\}$ with center $I = (f, g)$ where $f = -x^d$ and $g = x + z^{a_2} + t^{a_3}$. But since the center is supported on the cuspidal curve included in the plane $\{x = 0\}$ and given by the equation $C = \{x = z^{a_2} + t^{a_3} = 0\}$ (see [27]) their uniformly rationality does not follow directly from Corollary 2.

3.2.1. A general construction. Here we give a general criterion to decide the $\mathbb{G}_m$-uniform rationality of certain threefolds, arising as stable hypersurfaces of $\mathbb{A}^4$ endowed with a linear $\mathbb{G}_m$-action. Since $X$ is rational, its A-H quotient $Y(X)$ is also rational.

The aim is to use the notion of birational equivalence of ps-divisors to construct an isomorphism between a $\mathbb{G}_m$-stable open set of the variety $X$ with a corresponding stable open subset of $\mathbb{A}^3$. By Theorem 16 and Proposition 17, the technique is to consider a well chosen sequence of birational transformations $Y(X) \rightarrow \mathbb{A}^2$ which maps the support of the ps-divisor corresponding to the threefolds $X$ on to the strict transforms of the coordinate lines and the exceptional divisor in $\hat{\mathbb{A}}^2$. Moreover, since we look for $\mathbb{G}_m$-stable open subset of $X$ containing the fixed point, it is enough to consider birational map $Y_0(X) \rightarrow \mathbb{A}^2$ which send a pair of curves on the coordinates axes of $\mathbb{A}^2$.

Let $p \in \mathbb{C}[v]$ be a polynomial of degree $k \geq 1$ such that $p(0) = 0$, let $\alpha_2, \alpha_3$ and $d$ be integers such that $d \alpha_3$ and $\alpha_2$ are coprime. Let $X$ be a hypersurface in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ defined by one of the following equations:

$$X = \{y^{d-1} z^{a_2} + t^{a_3} + p(xy)/y = 0\} \subset \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]).$$

Every such $X$ is endowed with a hyperbolic $\mathbb{G}_m$-action induced by the linear action on $\mathbb{A}^4$ defined by $\lambda \cdot (x, y, z, t) = (\lambda^{a_2} x, \lambda^{-a_2} z, \lambda^{d \alpha_3} y, \lambda^{a_3} t)$. The unique fixed point for this action is the origin of $\mathbb{A}^4$ and is a point of $X$.

Theorem 19. With the notation above we have:

1) $X$ is equivariantly isomorphic to $\mathcal{S}(\hat{\mathbb{A}}^2_{(u, v)}; D)$ with

$$D = \left\{ \frac{a}{\alpha_2} D_1 + \frac{b}{\alpha_3} D_2 + \left[ \frac{1}{\alpha_2 \alpha_3} \right] E, \right\}$$

where $E$ is the exceptional divisor of the blow-up $\pi : \hat{\mathbb{A}}^2_{(u, v)} \rightarrow \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v])$, $D_1$ and $D_2$ are the strict transforms of the curves $L_1 = \{u = 0\}$ and $L_2 = \{u + p(v) = 0\}$ in $\mathbb{A}^2$ respectively, and $(a, b) \in \mathbb{Z}^2$ are chosen so that $a \alpha_3 + b \alpha_2 = 1$.

2) $X$ is smooth if and only if $L_1 + L_2$ is a SNC divisor in $\mathbb{A}^2$.

3) Under these conditions, $X$ is $\mathbb{G}_m$-linearly rational at $(0, 0, 0, 0)$.

Proof. 1) The A-H presentation is obtained from the following exact sequence:

$$0 \rightarrow \mathbb{Z} \xrightarrow{s} \mathbb{Z}^4 \xrightarrow{p} \mathbb{Z}^3 \rightarrow 0,$$
by the method described in Remark 12, where \( F = \iota(\alpha_2\alpha_3, -\alpha_2\alpha_3, d\alpha_3, \alpha_2) \), and using the Jacobian criterion, we conclude that \( X \) is smooth if and only if \( \alpha_i \neq \alpha_j \) for \( i \neq j \).

3) Let \( D = L_1 + L_2 \subset \mathbb{A}^2_{(u,v)} \) and let \( \mathbb{A}^2_{(u,v)} \hookrightarrow \mathbb{P}^2_{[u:v:w]} \) be the embedding of \( \mathbb{A}^2 \) as the complement of the line at the infinity \( L_\infty = \{w = 0\} \). We denote by \( \bar{D} = L_1 + L_2 \) the closure of \( D \) in \( \mathbb{P}^2_{(u:v:w)} \) (see Figure 3.1). The only singularity is at the intersection of \( L_2 \) and \( L_\infty \). After a sequence of elementary birational transformations we reach the \( k \)-th Hirzebruch surface \( \bar{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k)) \). The proper transform of \( L_2 \) is a smooth curve intersecting the section of negative self-intersection transversally (see Figure 3.2). The second step is the blow-up of all the intersection points between \( L_1 \) and \( L_2 \) except the point corresponding to the origin in \( \mathbb{A}^2 \), followed by the contraction of the proper transform of the fiber passing through each point of the blowup (see Figure 3.3). The final configuration is then the Hirzebruch surface \( F_1 \) in which the proper transforms of \( L_1 \) and \( L_2 \) have self-intersection 1 and intersect each other in a unique point. Then \( \mathbb{P}^2 \) and the desired divisor are obtained from \( F_1 \) by contracting the negative section (see Figure 3.4).
This resolution gives a birational map from the A-H quotient of $X$ to the A-H quotient of $\mathbb{A}^3$ which induces an isomorphism in a neighborhood of the origin of $\mathbb{A}^2$. By Theorem 16 this gives a $\mathbb{G}_m$-equivariant isomorphism between an open neighborhood of the origin in $X$ and an open neighborhood of the origin in $\mathbb{A}^3$.

Let $p(v) = v(1 + g(v))$ be the polynomial which appears in the statement, and let $\phi$ be the birational map defined by:

$$\phi : (u, v) \rightarrow (-u'(g(v' + u') + 1), v' + u').$$

Its inverse is defined by

$$\phi^{-1} : (u', v') \rightarrow \left(-\frac{u}{1 + g(v)}, v + \frac{u}{1 + g(v)}\right).$$

Then $\phi(u + p(v)) = v'(g(v' + u') + 1)$ and we obtain:

$$Y(\mathbb{A}^n) \xrightarrow{i} Y' = \hat{\mathbb{A}}^2_{(u, v', t)} \setminus V(1 + g(v)) \simeq \hat{\mathbb{A}}^2_{(u', v', t)} \setminus V(g(v' + u') + 1) \xrightarrow{\psi'} Y(\mathbb{A}^3)$$

and $i : Y' \hookrightarrow \hat{\mathbb{A}}^2_{(u, v', t)}$, then $S(Y', i^*(\mathcal{D})) = U$ is an equivariant open neighborhood of the fixed point in $X$, which is moreover equivariantly isomorphic to an open subset of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[Y, Z, T])$ endowed of the hyperbolic $\mathbb{G}_m$-action. The action on $\mathbb{A}^3$ is defined by $\lambda \cdot (Y, Z, T) = (\lambda^{-\alpha_2} Y, \lambda^{d\alpha_3} Z, \lambda^{\alpha_3} T)$ using Proposition 17.

Remark 20. In the particular case where $L_1 + L_2$ is not a smooth normal crossing divisor in $\mathbb{A}^2$, that is the point 2 of the Theorem 19 is not satisfied, but the crossing of $L_1$ and $L_2$ at the origin is transversal, then $S(Y, D)$ is equivariantly isomorphic to a normal but not smooth $\mathbb{G}_m$-variety $V$ with a unique fixed point contained in its regular locus. The same process as before can be applied, and the variety $V$ admits an open $\mathbb{G}_m$-stable neighborhood of the fixed point isomorphic to a $\mathbb{G}_m$-stable neighborhood of the fixed point of $\mathbb{A}^3$ endowed with a linear hyperbolic $\mathbb{G}_m$-action. In other words $V$ is $\mathbb{G}_m$-linearly rational, but not uniformly rational, since it is singular.

3.2.2. Applications. Specifying the coefficients of the polynomial $p \in \mathbb{C}[v]$ defined in the previous sub-section, we list below particular hypersurfaces of $\mathbb{A}^3$ which are $\mathbb{G}_m$-uniformly rational.

Proposition 21. The following hypersurfaces in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ are $\mathbb{G}_m$-linearly rational:

$$X_1 = \{x + x^k y^{k-1} + z^{\alpha_2} + t^{\alpha_3} = 0\},$$

$$X_2 = \{x + y^{d-1}(x^d + z^{\alpha_2}) + t^{\alpha_3} = 0\},$$

considering the equivariant isomorphisms $\psi_1$ and $\psi_2$, respectively, in the proof.
Proof. Applying Theorem 19, \( X_1 \) corresponds to the choice \( d = 1 \) and \( p(v) = v + v^k \), and \( X_2 \) corresponds to the choice \( d \geq 2 \) and \( p(v) = v + v^d \).

1) An explicit isomorphism \( \psi_1 : X_1 \setminus V(1 + (xy)^{d-1}) \to \mathbb{A}^3 \setminus V(1 + (YZ^{\alpha_2} + YT^{\alpha_3})^{d-1}) \) is given by:

\[
\psi_1 : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} = \left( \frac{-y}{1+(xy)^{d-1}} \frac{1}{z} \frac{1}{t} \right).
\]

Its inverse \( \psi_1^{-1} \) is given by:

\[
\psi_1^{-1} : \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \left( \frac{-1 + (YZ^{\alpha_2} + YT^{\alpha_3})^{d-1}}{Z} \frac{Z}{T} \right).
\]

2) An explicit isomorphism \( \psi_2 : X_2 \setminus V(1 + (xy)^{d-1}) \to \mathbb{A}^3 \setminus V(1 + (Y^d Z^{\alpha_2} + YT^{\alpha_3})^{d-1}) \) is given by:

\[
\psi_2 : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} = \left( \frac{-y}{1+(xy)^{d-1}} \frac{1}{z} \frac{1}{t} \right).
\]

Its inverse \( \psi_2^{-1} \) is given by:

\[
\psi_2^{-1} : \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \left( \frac{-Y^{d-1} Z^{\alpha_2} - \frac{T^{\alpha_3}}{1 + (Y^d Z^{\alpha_2} + YT^{\alpha_3})^{d-1}}} {Z} \frac{Z}{T} \right).
\]

Theorem 22. All Koras-Russell threefolds of the first kind \( \{ x + x^k y + z^{\alpha_2} + t^{\alpha_3} = 0 \} \) in \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x,y,z,t]) \) are \( \mathbb{G}_m \)-linearly uniformly rational, considering the equivariant isomorphism \( \psi \) in the proof.

Proof. Let \( X = \{ x + x^k y + z^{\alpha_2} + t^{\alpha_3} = 0 \} \) be a Koras-Russell threefold of the first kind, let \( \mathcal{U} \) be the principal open subset of \( X \) where \( x \) does not vanish and let \( \mathcal{V} \) be the principal open subset of \( X \) where \( 1 + yx^{d-1} \) does not vanish. The principal open subsets \( \mathcal{U} = X_\lambda \) and \( \mathcal{V} = X_{1+yx^{d-1}} \) form a covering of \( X \) by \( \mathbb{G}_m \)-stable open subsets.

Since \( \Gamma(\mathcal{U},\mathcal{O}_U) = \mathbb{C}[x,x^{-1},y,z,t]/(x + x^k y + z^{\alpha_2} + t^{\alpha_3}) \simeq \mathbb{C}[x,x^{-1},z,t] \), \( X \) is \( \mathbb{G}_m \)-linearly rational at every point of \( \mathcal{U} \).

By Proposition 21, we have an explicit \( \mathbb{G}_m \)-equivariant isomorphism between an open neighborhood of the fixed point in \( X_1 = \{ x + x^k y^{k-1} + z^{\alpha_2} + t^{\alpha_3} = 0 \} \) and an open subset of \( \mathbb{A}^3 \). Moreover \( X_1 \) admits an action of the cyclic group \( \mu_{k-1} \) given by \( \epsilon \cdot (x,y,z,t) \to (x,\epsilon y,\epsilon z,\epsilon t) \) such that the action of \( \mu_{k-1} \) factor through that of \( \mathbb{G}_m \). Thus the quotient for the action of the cyclic group and the isomorphism obtained in Proposition 21 commute. In this case the quotient of \( \mathbb{A}^3 \) for the action of \( \mu_{k-1} \) is still isomorphic to \( \mathbb{A}^3 \). Since \( X_1//_{\mu_{k-1}} \simeq X \), the \( \mathbb{G}_m \)-equivariant map \( \psi_1 \) given in Proposition 21 descends to a \( \mathbb{G}_m \)-equivariant isomorphism \( \psi \):

\[
X_1 \setminus V(1 + (xy)^{k-1}) \to \mathbb{A}^3 \setminus V(1 + (YZ^{\alpha_2} + YT^{\alpha_3})^{k-1}).
\]

\[
X \setminus V(1 + yx^{k-1}) \to \mathbb{A}^3 \setminus V(1 + (Y(Z^{\alpha_2} + T^{\alpha_3}))^{k-1}).
\]

Remark 23. The variety \( X \) is endowed with an hyperbolic \( \mathbb{G}_m \)-action, the \( \mathbb{G}_m \)-stable principal open subset \( \mathcal{V} = X_{1+yx^{d-1}} \) is isomorphic to a principal open subset of \( \mathbb{A}^3 \) endowed with an hyperbolic \( \mathbb{G}_m \)-action but the \( \mathbb{G}_m \)-stable principal open subset \( \mathcal{U} = X_x \) is isomorphic to a principal open subset of \( \mathbb{A}^3 \) endowed with a \( \mathbb{G}_m \)-action with positive weights only.
Proposition 24. The Koras-Russell threefolds of the second kind given by the equations

\[ X = \{ x + y(x^l + z^{m_l})^l + t^{m_3} = 0 \}, \]

in \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]) \) with \( l = 1 \) or \( l = 2 \) or \( d = 2 \) are \( \mathbb{G}_m \)-linearly uniformly rational.

Proof. In the case \( l = 1 \) we consider the \( \mathbb{G}_m \)-uniformly rational variety:

\[ X_2 = \{ x + y^{d-1}(x^d + z^{m_2}) + t^{m_3} = 0 \}, \]

given in Proposition 21. The cyclic group \( \mu_{d-1} \) on \( X_2 \) via \( \epsilon \cdot (x, y, z, t) \rightarrow (x, \epsilon y, z, t) \), and this action factors through that of \( \mathbb{G}_m \). Thus the quotient for the action of cyclic group and the isomorphism obtains in Proposition 21 commute. The conclusion follows by the same method as in the proof of Theorem 22.

Let \( X_{d-1} = \{ x + y^{d-1}(x^d + z^{m_2})^l + t^{m_3} = 0 \} \rightarrow X \) be the cyclic cover of order \( dl - 1 \) of \( X \) branched along the divisor \( \{ y = 0 \} \). The A-H presentation of \( X_{d-1} \) (see [23]) is \( \mathbb{S}(\mathbb{A}^2_{(u,v)}, D) \) with:

\[ D = \left\{ \frac{a}{\alpha_2} \right\} D_{\alpha_2} + \left\{ \frac{b}{\alpha_3} \right\} D_{\alpha_3} + \left\{ 0, \frac{1}{\alpha_2 \alpha_3} \right\} E, \]

where \( E \) is the exceptional divisor of the blow-up \( \pi: \mathbb{A}^2_{(u,v)} \rightarrow \mathbb{A}^2 \simeq \text{Spec}(\mathbb{C}[u, v]) \simeq \text{Spec}(\mathbb{C}[y^d z^{m_2}, yx]) \), and where \( D_{\alpha_2} \) and \( D_{\alpha_3} \) are the strict transforms of the curves \( L_1 = \{ v + (u + v^d)^l = 0 \} \) and \( L_2 = \{ u = 0 \} \) in \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) \) respectively, \( (a, b) \in \mathbb{Z}^2 \), being chosen so that \( ad\alpha_3 + bx_2 = 1 \).

First of all, variables \( l \) and \( d \) can be exchanged, just considering the automorphism of \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) \) which send \( u \) on \( u - (v - u^d)^d \) and \( v \) on \( v - u^d \). Then \( v + (u + v^d)^l \) is sent on \( v \). From now it will be assume that \( l = 2 \).

By showing that \( X_{d-1} \) is \( \mathbb{G}_m \)-linearly rational, one can explicit a birational map between \( X \) and \( \mathbb{A}^3 \). This map will be an equivalent isomorphism between an open subset of \( X \) containing the fixed point and an open subset of \( \mathbb{A}^3 \). The divisor \( D = L_1 + L_2 \) is birationally equivalent to \( D' = \{ uv = 0 \} \) by the birational endomorphism of \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) \) defined by \( u \rightarrow \frac{u^l + (v - u^d)^l}{1 - (v - u^d)^l} \) and \( v \rightarrow v - u^d \). Thus \( X_{d-1} \) is \( \mathbb{G}_m \)-linearly rational. Moreover the application \( \varphi \) is \( \mu_{2d-1} \)-equivariant, considering the action of \( \mu_{2d-1} \) given by \( \epsilon \cdot (u, v) \rightarrow (\epsilon^d u, \epsilon v) \). The desired result is now obtained by the same technique as in the proof of Theorem 22. □

4. Examples of non \( \mathbb{G}_m \)-rational varieties

Since the property to be \( G \)-uniformly rational is more restrictive than being only uniformly rational, it is not surprising that there are smooth and rational \( G \)-varieties which are not \( G \)-uniformly rational. In this section we will exhibit some \( \mathbb{G}_m \)-varieties which are smooth and rational but not \( \mathbb{G}_m \)-uniformly rational. However it is not known if these varieties are uniformly rational. This provides candidates to show that the uniform rationality conjecture has a negative answer.

Proposition 25. Let \( C \subset \mathbb{A}^2 \) be a smooth affine curve of positive genus passing through the origin with multiplicity one and let \( X \) be a \( \mathbb{G}_m \)-variety equivariantly isomorphic to \( \mathbb{S}(\mathbb{A}^2_{(u,v)}, D) \) with \( D = \{ \frac{1}{p} \} D + \{ 0, \frac{1}{p} \} E \), where \( E \) is the exceptional divisor of the blow-up and \( D \) is the strict transform of \( C \). Then \( X \) is a smooth rational \( \mathbb{G}_m \)-variety but not a \( \mathbb{G}_m \)-uniformly rational variety.

Proof. This is a direct consequence of the classification of hyperbolic \( \mathbb{G}_m \)-actions on \( \mathbb{A}^3 \) given in Proposition 17. In this case the irreducible components of the support of the ps-divisors are all rational. But the variety \( \mathbb{S}(\mathbb{A}^2_{(u,v)}, D) \) given in [23, proposition 3.1] admits the support of \( D \) in the support of its ps-divisors. As the support of \( D \) is not rational it follows that the varieties obtained by this construction are not \( \mathbb{G}_m \)-linearly rational and thus not \( \mathbb{G}_m \)-uniformly rational since the two properties are equivalent in the case of \( \mathbb{G}_m \)-varieties of complexity two (see Theorem 5). □

Example 26. Let \( V(h) \) be a smooth affine curve of positive genus passing through the origin with multiplicity one. Then the hypersurface \( \{ h(xy, zy)/y + t^p = 0 \} \) is stable in \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]) \), for the linear \( \mathbb{G}_m \)-action given by \( \lambda \cdot (x, y, z, t) = (\lambda^p x, \lambda^{-p} y, \lambda^p z, \lambda t) \). This variety is smooth using the Jacobian criterion and rational as its algebraic quotient is rational but not \( \mathbb{G}_m \)-uniformly rational.

4.1. Numerical obstruction for rectifiability of curves

For a \( \mathbb{G}_m \)-variety \( \mathbb{S}(Y, D) \), the non-rationality of the irreducible components of the support of \( D \) (see Proposition 25) is not the only obstruction to being \( \mathbb{G}_m \)-rational. There exist divisors \( D = L_1 + L_2 \) where \( L_1 \) is isomorphic to \( \mathbb{A}^1 \) for \( i = 1, 2 \) and such that \( D \) is not birationally equivalent to \( D' = \{ uv = 0 \} \). Such \( D \) can be used to construct \( \mathbb{G}_m \)-variety \( \mathbb{S}(Y, D) \) where the irreducible components of the support of the ps-divisors are all rational and such that \( \mathbb{S}(Y, D) \) is not \( \mathbb{G}_m \)-rational. To prove the existence of
such $D$, we will use an invariant, the Kumar-Murthy dimension (see [22]). Recall that a pair $(X, D)$ is said smooth if $X$ is a smooth projective surface and $D$ is a SNC divisor on $X$. For every divisor $D$ on a smooth projective variety, we define the Iitaka dimension, $\kappa(X, D) := \sup \dim \phi_{nD}|(X)$ in the case where $|D| \neq \emptyset$ for some $n$, and $\kappa(X, D) := -\infty$ otherwise, where $\phi_{|nD|} : X \to \mathbb{P}^N$ is the rational map associated to the linear system $|nD|$ on $X$.

**Lemma 27.** Let $D_0 = \sum_{i=1}^{k} D_i$ be a reduced divisor on a complete surface $X_0$, with $D_i$ irreducible for each $i \geq 0$. For any birational morphism $\pi : X \to X_0$ such that the pair $(X, D_X)$ is smooth, with $D_X$ the strict transform of $D$, the value $\kappa(X, 2K_X + D_X)$ does not depend on the choice of $\pi$.

**Proof.** By the Zariski strong factorization Theorem, it suffices to show that this dimension is invariant under blow-ups. Let $(X, D_X)$ be a resolution of the pair $(X_0, D_0)$ such that $X$ is smooth and $D_X$ is SNC. Let $\pi : \tilde{X} \to X$ be the blow-up of a point $p$ in $X$. Since $D_X$ is SNC, there are three possible cases: $p \notin D_X$, $p$ is contained in a unique irreducible component of $D_X$, or $p$ is a point of intersection of two irreducible components $D_X$. We have then for any integer $n$: $n(2K_\tilde{X} + D_\tilde{X}) = \pi^*(n(2K_X + D_X)) + n(2 - m)E$, $m = 2, 1, 0$ respectively. Therefore, $\Gamma(X, \mathcal{O}(n(2K_X + D_X))) = \Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X)) + (2 - m)E))) = \Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X))))$, and so, by the projection formula ([15, II.5]), $\Gamma(X, \mathcal{O}(\pi^*(n(2K_X + D_X))) \cong \Gamma(X, \mathcal{O}(n(2K_X + D_X)))$ for any integer $n$. \hfill \Box

**Definition 28.** The Kumar-Murthy dimension $k_M(X_0, D_0)$ of $(X_0, D_0)$ is the Iitaka dimension $\kappa(X, 2K_X + D_X)$ where $\pi : X \to X_0$ is any birational morphism such that the pair $(X, D_X)$ is smooth.

**Definition 29.** A pair $(Y, D)$ (as in definition 14) is birationally rectifiable if it is birationally equivalent to the union of $k \leq N = \dim(Y)$ general hyperplanes in $\mathbb{P}^N$. Note in particular that $Y$ is rational and that the irreducible components of $D$ are either rational or uniruled.

Since, the Kumar-Murthy dimension of the pair $(\mathbb{P}^2, D)$, where $D$ is a union of two distinct lines, is equal to $-\infty$, we obtain:

**Proposition 30.** If a reduced divisor $D = D_1 + D_2$ in $\mathbb{P}^2$ is birationally rectifiable then $k_M(\mathbb{P}^2, D) = -\infty$.

**Example 31.** Let $C = \{u + (v + u^2)^2 = 0\}$ and $C' = \{\alpha v - \beta + u = 0\}$ be two curves in $\mathbb{A}^2 = \text{Spec} (\mathbb{C}[u, v])$ where $(\alpha, \beta) \in \mathbb{C}^2$ are generic parameters chosen such that $C$ and $C'$ intersect normally. Let $\bar{C}$ and $\bar{C}'$ be the closures in $\mathbb{P}^2$ of $C$ and $C'$ respectively and let $D = \bar{C} + \bar{C}'$. Then:

i) $C$ and $C'$ are isomorphic to $\mathbb{A}^1$

ii) $k_M(\mathbb{P}^2, D) \neq -\infty$.

**Proof.** The curve $C'$ is clearly isomorphic to $\mathbb{A}^1$. In the case of $C$, consider the following two automorphisms:

$$\psi_1 : (u, v) \to (u, v + u^2)$$

$$\psi_2 : (u, v) \to (u + v^2, v)$$

then the composition $\psi_2 \circ \psi_1 : \begin{cases} u &\to u + (u + v^2)^2 \\ v &\to v + u^2 \end{cases}$ sends $C$ on a coordinate axe. A minimal log-resolution $\pi : S_7 \to \mathbb{P}^2$ of $\bar{C} \cup \bar{C}'$ is obtained by performing a sequence of seven blow-ups, five of them with centers lying over the singular point of $C$ and the remaining two over the singular point of $C'$.

![Figure 4.1](res.jpg)

**Figure 4.1.** Resolution of $(\mathbb{P}^2, (\bar{C} + \bar{C}')$, the divisors $E_i$ and $E_i'$ are exceptional divisors obtained blowing-up $C \cap L_\infty$ and $C' \cap L_\infty$ numbered according to the order of their extraction.
Using the ramification formula for the successive blow-ups occurring in \( \pi \), we find that the canonical divisor of \( S_T \) is equal to \( K_{S_T} = -3l + E_1 + 2E_2 + 3E_3 + 6E_4 + 10E_5 + E'_1 + 2E'_2 \), where \( l \) denotes the proper transform of a general line in \( \mathbb{P}^2 \). The total transform of the divisor \( C + C' \) is equal to \( \pi^*(C + C') = C + 2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + C' + E'_1 + 2E'_2 \), where we have identified \( C \) and \( C' \) with their proper transforms in \( S_T \).

Since \( C \) is of degree 4 and \( C' \) is of degree 2, the proper transform of \( C + C' \) in \( S_T \) is linearly equivalent to 6l and we obtain

\[
2K_{S_T} + D = 2K_{S_T} + \pi^*(C + C') - (2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + E'_1 + 2E'_2) = E_4 + 2E_5 + E'_1 + 2E'_2,
\]

which is an effective divisor. Thus \( k_M(\mathbb{P}^2, D) \neq -\infty \), and by Proposition 30, \( D \) is not birationally rectifiable.

\( \square \)

4.2. Application. Let \( X \) be the subvariety of \( \mathbb{A}^5 = \text{Spec}(\mathbb{C}[w, x, y, z, t]) \) defined by the two equations \( \{ w + y(x + yw^2)^2 + t^{α_3} = 0 \} \) and \( \{ αx(yx - β) + w + z^{α_2} = 0 \} \), where \((α, β) \in \mathbb{C}^2 \) are the same parameters as in Example 31. This variety is endowed with a hyperbolic \( G_m \)-action induced by the linear one on \( \mathbb{A}^5 \). Moreover, it is equivariantly isomorphic to \( \mathbb{A}^5 \) with \((w, x, y, z, t) = (λ^{α_2α_3}w, λ^{α_2α_3}x, λ^{-α_2α_3}y, λ^{α_2}z, λ^{α_2}t) \). Moreover, \( X \) is the total transform of the divisor \( \mathcal{D}_\lambda \) in \( \mathbb{A}^5 = \text{Spec}(\mathbb{C}[x, y, z, t]) \) defined by \( \{ z^{α_2} - αx(yx - β) + y(x + y(z^{α_2} - αx(yx - β)))^2 + \bar{t}^{α_3} = 0 \} \).

Theorem 32. The threefold \( X \) is a smooth rational \( G_m \)-variety but not a \( G_m \)-uniformly rational variety.

Proof. The A-H presentation of \( X \) is given by \( \mathcal{S}(\hat{\mathbb{A}}^2_{(u, v)}, D) \) with

\[
D = \begin{cases}
\alpha \\ \frac{1}{α_3}
\end{cases} D_1 + \begin{cases}
b \\ \frac{1}{α_3}
\end{cases} D_2 + \begin{cases}
0 \\
\frac{1}{α_3}
\end{cases} E,
\]

where \( E \) is the exceptional divisor of the blow-up \( \pi : \hat{\mathbb{A}}^2_{(u, v)} \to \mathbb{A}^2 \), \( D_1 \) and \( D_2 \) are the strict transform of the curves \( C \) and \( C' \) of Example 31, and \((a, b) \in \mathbb{Z}^2 \) such that \( α_3 + ba_2 = 1 \). The presentation comes from the fact that \( D \) is endowed with an action of \( μ_α = μ_3 \) factoring through that of \( G_m \) and given by \( (ε, ξ) \cdot (x, y, z, t) \to (x, y, εz, ξt) \).

By [23, Example 3.1], \( X/μ_α \) is equivariantly isomorphic to \( \mathcal{S}(\hat{\mathbb{A}}^2_{(u, v)}, \begin{cases}
\frac{1}{α_3}
\end{cases} D_1 + \begin{cases}
0 \\
\frac{1}{α_3}
\end{cases} E) \), and \( X/μ_3 \) is equivariantly isomorphic to \( \mathcal{S}(\hat{\mathbb{A}}^2_{(u, v)}, \begin{cases}
\frac{1}{α_3}
\end{cases} D_2 + \begin{cases}
0 \\
\frac{1}{α_3}
\end{cases} E) \). In fact, \( X/μ_α \) is equivariantly isomorphic to \( \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, t]) \) with the \( G_m \)-action defined via \( λ \cdot (x, y, t) = (λ^{α_α}x, λ^{-α}y, λt) \) and \( X/μ_3 \) is equivariantly isomorphic to \( \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \) with the \( G_m \)-action defined via \( λ \cdot (x, y, z) = (λ^{α_2}x, λ^{-α}y, λz) \). In particular \( X \) is a Koras-Russell threefolds (see [16, 17, 23]). The result follows from Proposition 30 and Example 31.

\( \square \)

5. Weak equivariant rationality

The property to be \( G \)-linearly uniformly rational is very restrictive. We will now introduce a weaker notion:

Definition 33. A \( G \)-variety \( X \) is called \textit{weakly} \( G \)-rational at a point \( x \) if there exist an open \( G \)-stable neighborhood \( U_x \) of \( x \), an open subvariety \( V \) of \( \mathbb{A}^n \) equipped with a \( G \)-action and a \( G \)-equivariant isomorphism between \( U_x \) and \( V \). We said that \( X \) is \textit{weakly} \( G \)-uniformly rational if it is weakly \( G \)-rational at every point.

Note that, in contrast with Definition 4 ii) we only require that \( V \subset \mathbb{A}^n \) is endowed with a \( G \)-action, in particular it need not be the restriction of a \( G \)-action on \( \mathbb{A}^n \). In summary we have a sequence of implications between these different notions of \( G \)-rationality: \( G \)-linearly uniformly rational implies \( G \)-uniformly rational which implies \( G \)-weakly uniformly rational, which finally implies uniformly rational.

Theorem 34. Let \( S \subset \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \) be the surface defined by the equation \( z^2 + y^2 + z^3 - 1 = 0 \), equipped with the \( μ_2 \)-action \( τ \cdot (x, y, z) \to (x, y, -z) \) on \( \mathbb{A}^3 \), where \( τ \) is the non trivial element of \( μ_2 \). Then \( S \) is weakly \( μ_2 \)-uniformly rational but not \( μ_2 \)-uniformly rational.
Proof. The surface $S$ is the cyclic cover of $\mathbb{A}^2$ of order 2 branched along the smooth affine elliptic curve $C = \{ y^2 + x^3 - 1 = 0 \} \subset \mathbb{A}^2$. By construction, the inverse image of $C$ in $S$ is equal to the fixed points set of the involution. It follows that $S$ is not $\mu_2$ rational at the point $p = (1, 0, 0)$. Indeed, every $\mu_2$-action on $\mathbb{A}^2$ being linearisable (see [19, Theorem 4.3]), its fixed points set is rational. Therefore there is no $\mu_2$-stable open neighborhood of $p$ which is equivariantly isomorphic to a stable open subset of $\mathbb{A}^2$ endowed with a $\mu_2$-action. However, there is an open subset $U$ of $\mathbb{A}^2$ which can be endowed with a $\mu_2$-action such that $U$ is equivariantly isomorphic to a $\mu_2$-stable open neighborhood of $p$.

Indeed, letting $u = z + y$ and $v = z - y$, $S$ is isomorphic to the surface defined in $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[u, v, x])$ by the equation $\{ uv - x^3 + 1 = 0 \}$. The open subset $V_1 = S \setminus \{ 1 + x + x^2 = u = 0 \}$ is isomorphic to $\mathbb{A}^2$ with coordinates $u$ and $v/\left(1 + x + x^2\right) = (x - 1)/u = w$. Let $V = S \setminus \{ 1 + x + x^2 = 0 \}$ be an open subset in $V_1$, and let $x = uw + 1$, then $V$ has the following coordinate ring:

$$\mathbb{C}\left[ u, w, \frac{1}{(uw + 1)^2 + uw + 1 + 1} \right] = \mathbb{C}\left[ u, w, \frac{1}{(uw)^2 + 3uw + 3} \right].$$

The open subset $V$ contains the point $p$ and is stable by the action of $\mu_2$ defined by:

$$\tau \cdot (u, v) = \left( \frac{u((uw)^2 + 3uw + 3)}{uw + 3}, \frac{u((uw)^2 + 3uw + 3)^{-1}}{uw + 3} \right).$$

So $S$ is $\mu_2$-weakly rational but not $\mu_2$-rational.

\[\square\]
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