Anomaly free effective action for the elementary $M5$–brane

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Abstract

We construct an effective action describing an elementary $M5$–brane interacting with dynamical eleven–dimensional supergravity, which is free from gravitational anomalies. The current associated to the elementary brane is taken as a distribution valued $\delta$–function on the support of the 5–brane itself. Crucial ingredients of the construction are the consistent inclusion of the dynamics of the chiral two–form on the 5–brane, and the use of an invariant Chern–kernel allowing to introduce a $D = 11$ three–form potential which is well defined on the worldvolume of the 5–brane.

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1 Introduction

The until now only conjectured M–theory is supposed to be a unifying consistent theory in eleven dimensions whose low energy limit is $D = 11$ supergravity. Its elementary excitations are 2–branes and 5–branes which are “electromagnetically” dual to each other. These two excitations can coexist if their charges $e$ and $g$ satisfy the Dirac–quantization condition

$$eg = 2\pi n G,$$  

where $G$ is the eleven–dimensional Newton’s constant, usually written as $G = 2\kappa^2$, and $n$ is an integer.

The dynamics of the bosonic sector of the $M2$–brane is described by the coordinates $x^\mu(\sigma)$, $\mu = 0, \cdots, 10$, and the worldvolume swept out during its time evolution is three–dimensional. The bosonic sector of an $M5$–brane is described by the coordinates $x^\mu(\sigma)$ and by the self–interacting chiral two–form $b_{ij}(\sigma)$, whereas its worldvolume is six–dimensional. Thus, the main differences between the two excitations are the presence of the two–form $b_2$ and the possibility of gravitational anomalies on the 5–brane, while 2–branes are trivially anomaly free.

As shown in [1] the gravitational anomaly generated by $b_2$ and by the two complex chiral fermions on the 5–brane is represented by the anomaly polynomial $2\pi(X_8^{(0)} + \frac{1}{24}P_8)$, with

$$X_8 = \frac{1}{192(2\pi)^4} \left( tr R^4 - \frac{1}{4}(tr R^2)^2 \right)$$

$$P_8 = \frac{1}{8(2\pi)^4} \left( (tr F^2)^2 - 2 tr F^4 \right),$$

where $R$ is the target space $SO(1,10)$–curvature and $F$ the $SO(5)$–curvature of the normal bundle of the 5–brane. With $X_8^{(0)}$ we denote the pullback of the target space polynomial $X_8$ on the $M5$–brane worldvolume. The target space anomaly, associated to $X_8$, can be cancelled à la Green–Schwarz modifying the equation of motion of the $D = 11$ four–form curvature $H_4$, while $P_8$, the second Pontrjagin form, represents the residual anomaly whose cancellation requires (some sort of) the inflow mechanism. The anomaly itself, as variation of the quantum effective action $\Gamma_q$, is obtained through the descent formalism,

$$\mathcal{A} = \delta \Gamma_q = 2\pi \int_{M_6} \left( X_6^{(0)} + \frac{1}{24}P_6 \right),$$  

where $M_6$ is the 5–brane worldvolume. Our notation for descent equations is $X_8 = dX_7$, $\delta X_7 = dX_6$, and similarly for $P_8$. $X_6^{(0)}$ denotes again the pullback of $X_6$ on $M_6$.

The fundamental equation which describes the coupling of a 5–brane with charge $g$ to eleven–dimensional supergravity is

$$dH_4 = gJ_5,$$
where the 5–form $J_5$ is essentially the Dirac $\delta$–function on the 5–brane worldvolume (see below for a precise definition); we refer to such branes carrying a current with $\delta$–like support as *elementary* branes. It is eventually this equation which should induce the cancellation of the residual $SO(5)$–anomaly through inflow. In pure supergravity one has $dH_4 = 0$, and this allows to introduce a potential through $H_4 = dB_3$. If on the other hand $g \neq 0$, the first problem one has to face is how to introduce a potential $B_3$ in a consistent way. Since, moreover, the action for pure supergravity is cubic in $B_3$, the presence of a current $J_5$ with $\delta$–like support leads in the action to cubic products of terms with at least inverse–power–like short distance singularities; the second problem one has to face is related with an accurate treatment of these singularities.

There have been various attempts to deal consistently with equation (1.5), with the aim of cancelling the residual gravitational anomaly. To circumvent the second problem, the strategy adopted in Ref. [2] consists in smoothing out the singular source $J_5$ and to replace it with a specific regular one, $J_5^{\text{reg}}$, carrying the same total flux as $J_5$. With this choice for the current the authors of [2] were able to construct a modified Wess–Zumino term, replacing $\frac{1}{6} \int B_3 dB_3 dB_3$ of pure supergravity, whose variation cancels indeed the residual anomaly. A drawback of a regular current $J_5^{\text{reg}}$ is that it does not admit a consistent coupling to elementary $M2$–branes: since the 5–brane charge is now continuously distributed Dirac’s condition (1.1) is no longer sufficient to make the Dirac–brane associated to the $M2$–brane unobservable. A Dirac–brane associated to the $M2$–brane is a 3–brane whose boundary is the $M2$–brane; it represents a generalization of the Dirac–string of a four–dimensional monopole. If $M2$–branes and $M5$–branes are simultaneously present the introduction of at least one Dirac–brane is unavoidable, in complete analogy with the case of charges and monopoles in four dimensions, see e.g. ref. [3]. A part from this one should explain why the regular current associated to the 5–brane should have the particular form $J_5^{\text{reg}}$. The authors of [4] instead insist on a $\delta$–like current and argue, as a consequence of eq. (1.5), that the 5–brane $SO(5)$–normal bundle $N$ splits in a line bundle $L$ and an $SO(4)$–bundle $N'$. This allows them to consider in the residual anomaly polynomial only $SO(5)$–connections which are reducible to $SO(4)$–connections, and to construct a local counterterm which cancels the corresponding anomaly. However, there remains an unphysical dependence on the choice of the splitting. Notice also that both references do not worry about the dynamics of the $b_2$–field. Finally, the cancellation of the residual anomaly in the compactified theory, corresponding to an $NS5$–brane in $D = 10$, $IIA$–supergravity, has been realized in [1, 5].

Aim of this paper is the construction of the low energy dynamics of the bosonic elementary $M$–theory 5–brane, coupled to the bosonic sector of dynamical $D = 11$ supergravity; the cancellation of the residual anomaly will be an automatic output of our construction, rather than an a priori requirement. Our point of view is that if $M$–theory is a consistent theory, there should exist a consistent low energy dynamics describing the interaction of $M5$– and $M2$–branes with/through *dynamical* eleven–dimensional supergravity. In this
sense our approach goes beyond the \(\sigma\)-model approach where the target space fields are supposed to satisfy the equations of pure source-less supergravity. We will concentrate on the dynamics of the 5-brane, since it bears the major difficulties, and include the 2-brane only at the end. Crucial ingredients of the construction are the inclusion of the \(b_2\)-field dynamics, and a consistent solution of (1.3) in terms of a \(D = 11\) three-form potential which admits a well defined pullback on \(M_6\), i.e. which is regular in the vicinity of the 5-brane worldvolume. There is a standard approach \[3\] to solve such an equation, involving Dirac-branes. In the present case however, due to the cubic interactions in the action, we need an alternative approach in terms of Chern-kernels \[6, 7\], which are able to codify the physical singularities of \(H_4\) near the 5-brane in a universal way.

Since we insist on a \(\delta\)-like current our natural framework is the one of \(p\)-currents (rather than \(p\)-forms), i.e. of \(p\)-forms with distribution valued coefficients \[8\]; consequently the differential \(d\) acts always in the sense of distributions, otherwise an equation like (1.5) would never make sense. We suppose also that our eleven-dimensional target space \(M_{11}\) is topologically trivial, so every closed \(p\)-current is also exact. Henceforth we will call our “currents” again “forms”.

The present paper presents the main result, i.e. the anomaly free low energy effective action, eqs. (3.1), (3.6) and (3.7); detailed proofs and applications will be presented elsewhere \[9\].

2 Equations of motion

The bosonic fields of \(D = 11\) supergravity are the metric \(g_{\mu\nu}(x)\) and the three-form potential \(B_3\); the bosonic fields on the closed 5-brane are the coordinates \(x^\mu(\sigma)\), \(\sigma^i = (\sigma^0, \ldots, \sigma^5)\) and the two-form \(b_2\). The field \(B_3\) can also be dualized to a six-form \(B_6\), but since there exists no formulation of \(D = 11\) supergravity which involves only \(B_6\) it is preferable to use a formulation in terms of only \(B_3\). We indicate the curvatures associated to \(B_3\) and \(b_2\) respectively as \(H_4 = dB_3 + \cdots\) and \(h_3 = db_2 + \cdots\). With the upper index \((0)\) we will indicate the pullback of a target space form to the 5-brane worldvolume \(M_6\) whenever it exists, e.g. \(B_3^{(0)}\) indicates the pullback of \(B_3\) to the six-dimensional submanifold \(M_6\).

We propose, as starting point, the following set of classical equations of motion and Bianchi identities for \(H_4\) and \(h_3\),

\[
\begin{align*}
    h_3 & = \, db_2 + B_3^{(0)} \quad (2.1) \\
    h_{ij} & = \, -2 \frac{\delta L}{\delta h_{ij}} \quad (2.2) \\
    dH_4 & = \, g J_5 \quad (2.3) \\
    d \ast H_4 & = \, \frac{1}{2} H_4 H_4 + g h_3 \circ J_5 + \frac{2\pi G}{g} X_8. \quad (2.4)
\end{align*}
\]

In equation (2.1) we defined the curvature of the two-form potential \(b_2\), according to the \(\sigma\)-model approach, in terms of the pullback of the target space potential \(B_3\). This
implies first of all that this pullback has to be well defined. Equation (2.2) amounts then to the generalized self-duality equation of motion for $b_2$ which is induced by the Born–Infeld lagrangian $\mathcal{L}(\tilde{h}) = \sqrt{\text{det}(\delta^j_i + i \tilde{h}^j_i)}$ where, according to the PST–approach \cite{10,11}, $h_{ij} = v^k h_{ijk}$, $\tilde{h}_{ij} = v^k (\ast h)_{ijk}$, $v_k = \partial_k a / \sqrt{- (\partial a)^2}$, and $a(\sigma)$ is a non propagating scalar auxiliary field.

Equation (2.3) is a Bianchi identity for $H_4$ which has to be solved in terms of the potential $B_3$; after that (2.4) becomes an equation of motion for this field. The five–form $J_5$ in (2.3) is defined as the Poincaré dual in the space of currents \cite{8} of the five–brane worldvolume, i.e. $\int \Phi_6 J_5 = \int_{M_6} \Phi_6^{(0)}$ for every smooth target space six–form $\Phi_6$. In an arbitrary coordinate system a local expression for $J_5$ is

$$J_5 = \frac{1}{5!6!} dx^{\mu_1} \cdots dx^{\mu_6} \varepsilon_{\mu_1 \cdots \mu_5 \nu_1 \cdots \nu_6} \int_{M_6} E_{\nu_1} \cdots E_{\nu_6} \delta^{11}(x - x(\sigma)), \quad (2.5)$$

where $E^\mu(\sigma) = d\sigma^i E_i^\mu(\sigma)$, and the 6 vectors $E_i^\mu(\sigma) = \partial_i x^\mu(\sigma)$ form a basis for the tangent space on $M_6$ at $\sigma$.

A basic problem one has to solve is how to give a well–defined meaning to the r.h.s. of (2.4) as a closed target space eight–form. For $g = 0$ it reduces to the $B_3$–equation of motion of pure supergravity \cite{7}. The second and third term on its r.h.s. are dictated by the presence of the 5–brane; the term proportional to $X_8$, see \cite{12}, realizes the standard Green–Schwarz cancellation mechanism for the target space anomaly and is a closed form. The presence of the second term – indications for its appearance have been given first in \cite{13} – is needed to make the r.h.s. of (2.4) a closed form, see below. In the form (2.4) this formula has been proposed in \cite{4}.

We begin by specifying what we mean with the expression $h_3 \circ J_5$: technically this target space eight–form is defined as the canonical push–forward of the 5–brane field $h_3$ to the eleven–dimensional target space. In general the ordinary product between a form on the brane and a form on the target space defines neither a form on the brane, nor a form on the target space. However, a ”product” of the kind $h_n \circ J_p$, where $J_p$ is the Poincaré dual of a $(D - p)$–manifold and $h_n$ an $n$–form on that manifold $(n + p \leq D)$, is defined in the distributional sense as a target space $(n + p)$–form according to

$$\int_{R^D} \Phi_{D-n-p} (h_n \circ J_p) = \int_{M_{D-p}} \Phi_{D-n-p}^{(0)} h_n, \quad (2.6)$$

for every test form. A local expression, following from this definition, is

$$h_n \circ J_p = \frac{1}{(n+p)! (D - n - p)!} dx^{\mu_1} \cdots dx^{\mu_{n+p}} \varepsilon_{\mu_1 \cdots \mu_{n+p} \nu_1 \cdots \nu_{D-n-p}} \int_{M_{D-p}} E_{\nu_1} \cdots E_{\nu_{D-n-p}} h_n \delta^{D}(x - x(\sigma)). \quad (2.7)$$

This corresponds to a local expression for the push–forward of $h_n$, and it involves only the worldvolume field $h_n$ and none of its ”by hand” extensions. The product notation

\footnote{As we will see below, the 5–brane charge is related to Newton’s constant by $2 \pi G = g^3$.}
$h_n \circ J_p$ is useful because from the above definition it follows that the standard Leibnitz rule holds for push–forward forms "as if they were factorized",

$$d(h_n \circ J_p) = h_n \circ dJ_p + (-)^p dh_n \circ J_p.$$ 

The last property of the push–forward operation we need is

$$\Phi J_p = \Phi^{(0)} \circ J_p,$$

for every target space form $\Phi$ which admits pull back.

The next point concerns the term $\frac{1}{2} H_4 H_4$ at the r.h.s. of (2.4), and its differential. The problematic aspect of this eight–form is represented by the fact that, due to (2.3), $H_4$ exhibits necessarily singularities near $M_6$, meaning that $H_4^{(0)}$ does not exist; in particular the computation $d\left(\frac{1}{2} H_4 H_4\right) = gH_4 J_5 = gH_4^{(0)} \circ J_5$ makes no sense.

To settle the question of how to compute the differential of $H_4 H_4$ one must first specify the singularities near $M_6$ present in $H_4$. Since $J_5$ is closed we can always write $J_5 = dK_4$ for some four–form $K_4$, and then $H_4 = dB_3 + gK_4$; the singularities of $H_4$ are then the ones of $K_4$ because $B_3$ is regular. Since $J_5 = dK_4$ these singularities can be essentially of two types: the first type corresponds to $\delta$–like singularities induced by a Dirac–brane, i.e. a 6–brane whose boundary is the 5–brane. This would lead to $K_4 = C_4$, where $C_4$ is the delta–function on the Dirac–brane, i.e. its Poincaré–dual. But for such singularities the product $H_4 H_4$ would not even define a distribution since the square of a $\delta$–function is not defined. For this reason the Dirac–brane approach can not be applied to the $M5$–brane effective action, based on the system (2.1)–(2.4).

The second type of possible singularities is represented by inverse–power–like singularities, like the ones of a Coulomb–field whose divergence equals a $\delta$–function, supported on the position of the source. In this case the inverse–power–like singular behaviour of $K_4$ near the 5–brane should be universal. By definition this behaviour is realized by the Chern–kernel [6], see below, which is appropriately expressed in terms of normal coordinates. Since the Chern–kernel will lead also to a well–defined product $H_4 H_4$ and, eventually, to a r.h.s. of (2.4) which defines a closed target space eight–form, we will base our effective action on this kernel.

\section{2.1 Normal coordinates and Chern–kernels}

We regard the introduction of a system of normal coordinates as a $D = 11$ diffeomorphism from the coordinates $x^\mu$ to the coordinates $(\sigma^i, y^a)$, with $i = 0, \cdots, 5$ and $a = 1, \cdots, 5$, specified by the functions $x^\mu(\sigma, y)$. The coordinates $y^a$ are called "normal" in that we require that

$$x^\mu(\sigma, 0) = x^\mu(\sigma), \quad N^a_\mu E^\mu_i = 0, \quad N^a_\mu N^{\mu b} = \delta^{ab},$$

\section*{(2.9)\footnote{Viceversa, if you require that the combination $H_4 = dB_3 + gC_4$ does not exhibit $\delta$–like singularities then $B_3$ can not be regular near $M_6$ because $dB_3$ must cancel the $\delta$–function singularities in $C_4$.}}
\[ \dot{N}_\mu^a(\sigma) = \frac{\partial x^\mu(\sigma, y)}{\partial y^a} \bigg|_{y=0}. \]

As a power series in \( y \) we have therefore
\[ x^\mu(\sigma, y) = x^\mu(\sigma) + y^a N^\mu_a(\sigma) + o(y^2). \] (2.10)

Since the vectors \( N^\mu_a(\sigma) \) specify a basis for the normal fiber, \( SO(5) \)-connection and curvature on \( M_6 \) can be parametrized by
\[ A^{ab} = N^{\mu b} \left( dN^\alpha_\mu + \Gamma^\nu_\mu N^\alpha_\nu \right), \]
\[ F^{ab} = dA^{ab} + A^{ac} A^{cb}, \] (2.11)

where \( \Gamma \) is the pullback of the eleven–dimensional affine connection.

Notice that, for chosen \( N^{\mu a} \), the conditions (2.9) determine only the structure of the coordinate system near the 5–brane; away from the 5–brane the coordinate system is only required to be one to one. So there is a large freedom left, which is expressed by the \( o(y^2) \)-terms above. For simplicity we suppose here that the normal coordinate system is defined globally in target space; the adaptation of our construction to the general case, where it can be defined only locally, is sketched in section five.

The definition of a Chern–kernel with the correct fall–off at infinity requires also the introduction of an extended \( SO(5) \)-connection one–form \( A^{ab}(\sigma, y) \) on the whole target space, asymptotically flat in \( |y| \) and restricted by the boundary conditions
\[ A^{ab}(\sigma, 0) = A^{ab}(\sigma). \] (2.12)

This means that the pullback of \( A^{ab}(\sigma, y) \) on the 5–brane reduces to the \( SO(5) \)-connection defined in (2.11), and that its curvature goes to zero at infinity along all \( y \)-directions. Unless otherwise stated from now on we will always use this extended connection and the associated extended curvature \( F^{ab} \).

The systems of normal coordinates and of extended connections fall into \( SO(5) \)-equivalence classes, the representatives being related by local \( SO(5) \)-transformations \( \Lambda^{ab}(\sigma, y) \),
\[ \tilde{y}^a = \Lambda^{ab} y^b, \quad \tilde{A} = \Lambda A T - A d \Lambda T. \]

In terms of an arbitrary normal coordinate system the current \( J_5 \) admits the simple local expression
\[ J_5 = \frac{1}{5!} dy^{a_1} \cdots dy^{a_5} \varepsilon^{a_1\cdots a_5} \delta^5(y), \] (2.13)
and we can now ask if there exists an \( SO(5) \)-invariant four–form \( K_4 \), polynomial in \( \tilde{y}^a \equiv y^a / \sqrt{y^2} \) and \( A^{ab} \), satisfying
\[ J_5 = dK_4. \] (2.14)

Such a four–form exists, it is indeed uniquely determined, and it is expressed in terms of the above data by the Chern–kernel \[ K_4 = \frac{1}{16(2\pi)^2} \varepsilon^{a_1\cdots a_5} \tilde{y}^{a_1} K^{a_2 a_3} K^{a_4 a_5}, \] (2.15)
where
\[ K'^{ab} = F^{ab} + D\dot{y}^a D\dot{y}^b, \quad D\dot{y}^a = d\dot{y}^a + \dot{y}^b A^b_{\cdot a} . \]

Local $SO(5)$-invariance is manifest and to verify (2.14) one has to compute the differential of $K_4$ in the sense of distributions. The salient properties of this four-form are that far away from the 5–brane, $y^a \to \infty$, it exhibits a typical Coulomb–like behaviour
\[ K_4 \sim \frac{1}{16(2\pi)^2} \varepsilon^{a_1 \cdots a_5} \dot{y}^{a_1} d\dot{y}^{a_2} d\dot{y}^{a_3} d\dot{y}^{a_4} d\dot{y}^{a_5} , \]
while near the 5–brane, $y^a \to 0$, it exhibits a universal $SO(5)$–invariant behaviour, which is independent of the choice of normal coordinates and of the extension of $A$. Notice, however, that the pullback of $K_4$ on $M_6$ does not exist.

We must stress that, although $K_4$ depends only on the equivalence class of normal coordinate systems and extended $SO(5)$–connections, it changes if one chooses another equivalence class. Inequivalent systems of normal coordinates are related by a transformation $y^a \to y'^a(\sigma, y)$, such that
\[ y'^a(\sigma, 0) = 0 \quad \Rightarrow \quad \frac{\partial y'^a}{\partial y^b}(\sigma, y)|_{y=0} = \delta^{ab} . \quad (2.16) \]

Such a change corresponds precisely to the ambiguity associated to the $o(y^2)$–terms in (2.10), which, in turn, reflect the huge arbitrariness of the normal coordinate systems away from the 5–brane. Moreover, one can choose infinitely many different extensions of the $SO(5)$–connection $A(\sigma)$ from a form on $M_6$ to a target space form, compatible with $y$–asymptotic flatness and (2.12). Under both types of changes we obtain a different four–form $K'_4$ such that
\[ dK'_4 = J_5 = dK_4 ; \]
Poincaré’s lemma implies then that locally there exists a three–form $Q_3$ such that
\[ K'_4 = K_4 + dQ_3 . \quad (2.17) \]

Moreover, since $K'_4$ and $K_4$ carry the same singular behaviour near the 5–brane, $Q_3$ behaves regularly as $y^a \to 0$ and using (2.12) and (2.16) one can verify that it has vanishing pullback on $M_6$, \[ Q_3^{(0)} = 0 . \quad (2.18) \]

This piece of information will become important in a moment. Since $K_4$ is $SO(5)$–invariant, we can now introduce an $SO(5)$–invariant three–form potential $B_3$ according to
\[ H_4 = dB_3 + gK_4 . \quad (2.19) \]

Under a change of equivalence class (2.17) we must require
\[ B'_3 = B_3 - gQ_3 , \quad (2.20) \]

\[ ^6 \text{The unique non vanishing contribution in the differential of } K_4 \text{ comes entirely from } d \left( \frac{1}{16(2\pi)^2} \varepsilon^{a_1 \cdots a_5} \dot{y}^{a_1} d\dot{y}^{a_2} d\dot{y}^{a_3} d\dot{y}^{a_4} d\dot{y}^{a_5} \right) = J_5 . \]

\[ ^7 \text{The four–form } K_4 \text{ has been introduced, as } 1/2 e_4, \text{ also in } \| \text{ but there it was treated as a closed form as it is away from the 5–brane.} \]
such that $H_4$ is independent of the new structures that we have introduced to construct $K_4$, i.e. the particular normal coordinate system that we have chosen and the particular extension of the $SO(5)$–connection. Notice also that (2.18) ensures that $B_3^{(0)}$ as well as $h_3$, apart from being well–defined, are independent of the new structures, too. Equation (2.19) provides a splitting of $H_4$ into a regular part which is also closed, $dB_3$, and a singular part, $K_4$, with a universal behaviour near $M_6$, in view of (2.17) and (2.18).

The form $K_4$ satisfies the following chain of relations

$$dK_4 = J_5$$
$$d(K_4K_4) = 0$$
$$d(K_4K_4K_4) = \frac{1}{4} P_8 J_5$$
$$d(K_4K_4K_4K_4) = 0,$$

where $P_8$ is the second Pontrjagin form. These relations follow from an identity whose proof we will present in [9] (see however also [13] and [17]):

$$K_4K_4 = \frac{1}{4} df_7, \quad f_7 = P_7 + Y_7,$$

where $P_7$ is the Chern–Simons form associated to the Pontrjagin form $dP_7 = P_8$, and $Y_7$ is an $SO(5)$–invariant seven–form given by

$$Y_7 = \frac{1}{(2\pi)^4} \left[ \hat{y}^a D\hat{y}^b (F^3)^{ba} + \left( \frac{1}{2} \text{tr } F^2 - D\hat{y}^c D\hat{y}^d F^{cd} \right) \hat{y}^a D\hat{y}^b F^{ab} \right].$$

This proves immediately (2.22). To prove (2.23) one has also to use that in the sense of distributions

$$d(Y_7K_4) = dY_7K_4.$$

Notice that, due to the singular behaviour of $K_4$ near the 5–brane, one is not allowed to use Leibnitz’s rule for differentiation; otherwise in the above formulae one would obtain some meaningless expressions like $K_4J_5$ and $Y_7J_5$.

Formula (2.25) means that the inverse–power–like singularities of $K_4$ which give rise to the $\delta$–function in $dK_4$, cancel in the product $K_4K_4$ due to antisymmetry reasons, and that $K_4K_4$ amounts to a closed eight–current. Using this formula it is finally easy to verify that the r.h.s. of (2.4) is a well–defined closed form. It suffices to notice that

$$d\left( \frac{1}{2} H_4H_4 \right) = \frac{1}{2} d \left( dB_3dB_3 + 2gdB_3K_4 + g^2K_4K_4 \right)$$
$$= gdB_3J_5 = gdB_3^{(0)} \circ J_5,$$

which cancels against $d(gh_3 \circ J_5) = -gdh_3 \circ J_5 = -gdB_3^{(0)} \circ J_5$.

Since we have now a well defined system of equations of motion we can search for an action which gives rise to it. This is the aim of the last section.
3 The effective action

We write the bosonic effective action $\Gamma$ for an $M_5$–brane with charge $g$ interacting with $D = 11$ supergravity as the sum of a local classical action, which should reproduce the equations of motion for $b_2$ and $B_3$, resp. (2.2) and (2.4), and of the quantum effective action,

$$\Gamma = \frac{1}{G} (S_{\text{kin}} + S_{wz}) + \Gamma_q,$$

(3.1)

where we separated the classical action in kinetic terms and in a Wess–Zumino action. The invariant curvatures are given in (2.1) and in (2.19), so the reconstruction of the classical action is, indeed, a merely technical point. Actually, the field equations for $B_3$ and $b_2$ fix the classical action modulo terms which are independent of these fields; these terms are, in turn, fixed by invariance requirements, in the present case independence of the action of the choice of normal coordinates and of the extension of the $SO(5)$–connection. More precisely, according to the previous section we have to require invariance under

$$K_4' = K_4 + dQ_3$$

(3.2)

$$B_3' = B_3 - gQ_3$$

(3.3)

$$f_7' = f_7 + 8K_4Q_3 + 4Q_3dQ_3 + dQ_6$$

(3.4)

$$Q_3^{(0)} = 0 = Q_6^{(0)}.$$  

(3.5)

The relation (3.4) follows from the definition of $f_7$ in (2.25) and from the relation $K_4'K_4' = \frac{1}{4}df_7'$. It determines the seven–form $f_7' \equiv P_7' + Y_7'$ modulo a closed form $dQ_6$. The pullback of $Q_6$ vanishes for the same reasons as the pullback of $Q_3$.

Clearly, in the absence of the 5–brane we want to get back the action of pure $D = 11$ supergravity. Employing for the two–form field equation (2.2) the covariant PST–approach [11], the invariant kinetic terms for the space–time metric, for $B_3$, $b_2$ and $x^\mu(\sigma)$ are given by

$$S_{\text{kin}} = \int_{M_{11}} d^{11}x \sqrt{g_{11}} R - \frac{1}{2} \int_{M_{11}} H_4 \ast H_4 - g \int_{M_6} d^6\sigma \sqrt{g_6} \left( L(\tilde{h}) + \frac{1}{4} \tilde{h}^{ij}h_{ij} \right),$$

(3.6)

where $g_6$ is the determinant of the induced metric on the 5–brane. Notice that $H_4$ as well as $h_3$ are manifestly invariant under (3.2)–(3.3).

The Wess–Zumino action, which appears to be the crucial ingredient of the effective action, is written as the integral of an eleven–form, $S_{wz} = \int_{M_{11}} L_{11}$, with

$$L_{11} = \frac{1}{6} B_3dB_3dB_3 - \frac{g}{2} \left( b_2 dB_3^{(0)} \right) \circ J_5 + \frac{g}{2} B_3dB_3K_4 +$$

$$+ \frac{g^2}{2} B_3K_4K_4 + \frac{g^3}{24} f_7K_4 + \frac{2\pi G}{g} X_7 H_4.$$  

(3.7)

We stress that all terms that involve $B_3$ or $b_2$ in this formula are fixed by their equations of motion (2.2) and (2.4); in particular the coefficient of the second term, which is the
unique one involving $b_2$, is fixed by the PST–symmetries. There are two terms in $L_{11}$ which are independent of $B_3$ and $b_2$ and which are not fixed by the equations of motion, but by the invariance requirements: $\frac{2\pi G}{g} X_7 K_4 (a)$, and $\frac{g^3}{24} f_7 K_4 (b)$. The term $(a)$ is related with the contribution $X_8$ at the r.h.s. of (2.4): to get this contribution it would have been sufficient to include only the term $\frac{2\pi G}{g} X_7 dB_3$ in $L_{11}$ which would have led to no $SO(1, 10)$–anomaly in $S_{wz}$, since $\int X_7 dB_3 = \int X_8 B_3$ is $SO(1, 10)$–invariant; but the point is that the term $\frac{2\pi G}{g} X_7 dB_3$ alone is not invariant under (3.2)–(3.3) and so one has to add the term $(a)$ (to obtain $\frac{2\pi G}{g} X_7 H_4$), which introduces in turn an $SO(1, 10)$–anomaly.

For the same reason one has to add the term $(b)$; without this term the first four terms in $L_{11}$ would not be invariant under (3.2)–(3.3). A straightforward calculation shows, indeed, that $L_{11}$ as given above is invariant under the transformations (3.2)–(3.4), as well as under the standard gauge transformations $\delta B_3 = d\Lambda_2$, $\delta b_2 = d\Lambda_1 - \Lambda_2^{(0)}$, up to a closed form.

A formal device to make all these invariances of $S_{wz}$ manifest consists in computing the differential of $L_{11}$. Using the formulae of the preceding section one obtains

$$L_{12} = dL_{11} = \frac{1}{6} H_4 H_4 H_4 + \frac{g}{2} (h_3 dB_3^{(0)}) \circ J_5 + \frac{g^3}{24} P_7 J_5 + \frac{2\pi G}{g} (X_8 H_4 + g X_7 J_5).$$

To give meaning to this formula one has to go to twelve dimensions; the closed 5–brane has to be extended to a closed 6–brane $M_{11} \equiv M_6$, in such a way that the restriction to $M_6$ of the normal bundle of $M_7$ w.r.t. $M_{11} \times R$ coincides with the normal bundle of $M_6$ w.r.t. $M_{11}$. The form $J_5$ is here then the Poincarè–dual of $M_7$ w.r.t. $M_{11} \times R$; restricted to $M_{11}$ it coincides with the eleven–dimensional $J_5$ appearing in $L_{11}$.

In $L_{12}$ the potentials appear only through their curvatures or through $dB_3^{(0)}$, which are all manifestly invariant under (3.2)–(3.3). The Chern–Simons form $P_7$ entering in $L_{12}$ is defined in terms of the extended $SO(5)$–connection $A^{ab}(\sigma, y)$, but since it appears multiplied by $J_5$ one gets back $A^{ab}(\sigma, 0) = A^{ab}(\sigma)$ and hence also the term $P_7 J_5$ is independent of the chosen extension. This means that under (3.2)–(3.3) we have $L'_{12} = L_{12}$, and therefore $L'_{11} = L_{11} + dL_{10}$ for some ten–form; this ensures that $S_{wz}$ is invariant.

From the twelve–dimensional point of view the term $\frac{g^3}{24} P_7 J_5$ is necessary to make $L_{12}$ a closed form, as can be seen using (2.21)–(2.23).

It is now easy to compute the gravitational anomalies carried by the classical action; the kinetic terms are invariant and in the Wess–Zumino action only the last two terms contribute, due to $\delta f_7 = dP_6$, $\delta X_7 = dX_6$, with

$$\delta \left( \frac{1}{G} S_{wz} \right) = -2\pi \int_{M_6} \left( X_6^{(0)} + \frac{g^3}{2\pi G} \frac{1}{24} P_6 \right).$$

This should cancel against the quantum anomaly $\delta \Gamma_q$ in (1.4). To see that this is indeed the case it suffices to remember that the 5–brane tension in $M$–theory is tied to Newton’s constant $12, 18$ by $T_5 = \left( \frac{2\pi}{G} \right)^\frac{1}{3}$. From (3.4) and (3.1) we see that in our framework the
5–brane tension amounts to \( T_5 = \frac{g^3}{\mathcal{G}} \). This means that the magnetic charge of the 5–brane is tied to Newton’s constant by

\[ g^3 = 2\pi G, \]

and the effective action is anomaly free. So anomaly cancellation confirms once more that there is only one fundamental scale in \( M \)–theory.

### 4 Coupling to \( M2 \)–branes

It is now simple to couple our action to a closed \( M2 \)–brane with charge \( e \) and worldvolume \( M_3 \). If we indicate the current associated to the 2–brane, i.e. the Poincaré dual of \( M_3 \), with \( J_8 \), it is only eq. (2.4) that gets modified to

\[ d*H_4 = \frac{1}{2} H_4 H_4 + g h_3 \circ J_5 + \frac{2\pi G}{g} X_8 + e J_8. \]

When 2–branes and 5–branes are simultaneously present to write an action we must introduce at least one Dirac–brane, see e.g. [3]. In the Chern–kernel approach, which avoids the Dirac–brane for the 5–brane, we must introduce a Dirac–3–brane, with worldvolume \( M_4 \), associated to the 2–brane: \( \partial M_4 = M_3 \). Calling the associated current \( C_7 \) we have

\[ J_8 = dC_7. \]

To take the new coupling into account it would be sufficient to modify the Wess–Zumino action by the term \( e \int_{M_3} B_3 = e \int_{M_{I1}} B_3 J_8 = e \int_{M_{I1}} dB_3 C_7 \); but again, to cope with (3.2)–(3.5), we have to set

\[ S_{wz}^{(e,g)} \equiv S_{wz} + e \int_{M_{I1}} H_4 C_7. \]

Under a change of Dirac–brane \( M_4 \to M_4 + \partial M_5 \), we have \( C_7 \to C_7 + dC_6 \), where \( C_6 \) is the Poincaré dual of \( M_5 \). Under such a change the Wess–Zumino action changes by

\[ \Delta S_{wz}^{(e,g)} = e \int_{M_{I1}} H_4 dC_6 = -eg \int_{M_{I1}} J_5 C_6 = -eg N, \]

where the integer \( N \) counts the number of intersections between \( M_5 \) and \( M_6 \). The effective action \( \Gamma^{(e,g)} \equiv \frac{1}{G} \left( S_{kin} + S_{wz}^{(e,g)} \right) + \Gamma_q \) changes accordingly by

\[ \Gamma^{(e,g)} \to \Gamma^{(e,g)} - \frac{eg}{G} N, \]

which is irrelevant if Dirac’s condition (1.1) holds.

This proves that the Dirac–brane is unobservable and that in \( M \)–theory elementary \( M2 \)–branes and elementary \( M5 \)–branes can consistently coexist, in compatibility with gravitational anomaly cancellation.

### 5 Discussion

The effective action we constructed incorporates \( M2 \)–branes and \( M5 \)–branes in a consistent way. It is based on the equations of motion (2.2) and (2.4), and on the definition
of the potentials $B_3$ and $b_2$ according to (2.1) and (2.19). The first step was a proof of the consistency of these equations of motion using a Chern–kernel which codifies the singularities of $H_4$ near the 5–brane in an invariant way. Next we wrote an action which gives rise to these equations of motion, requiring that the action does not depend on the structure of the Chern–kernel away from the 5–brane. This action is uniquely determined and cancels automatically the gravitational anomalies.

In the text we supposed that the system of normal coordinates can be defined globally. In general one is only guaranteed that it can be defined in a tubular neighborhood of the 5–brane, see e.g. [19]. In this situation one can define a $\tilde{K}_4$ in this neighborhood as in (2.15) – so there it satisfies $d\tilde{K}_4 = J_5$ – and try to extend it outside as a closed form. For 5–branes for which such a $\tilde{K}_4$ can be extended to the whole target space our construction holds true. In this case the eight–form $\tilde{K}_4 \tilde{K}_4$ is again closed and since the target space is supposed to be trivial we have $\tilde{K}_4 \tilde{K}_4 = \frac{1}{4} df_7$, for some globally defined seven–form. These ingredients are sufficient to write down the corresponding effective action, by replacing in (3.6) and (3.7) $K_4 \rightarrow \tilde{K}_4$, $f_7 \rightarrow \tilde{f}_7$. Notice that in a topologically trivial target space $J_5$ can always be written as the differential of some four–form; we ask here more, i.e. that this four–form shares with $K_4$ the singular behaviour near $M_6$.

One may ask which are the equations of motion for the coordinates $x^\mu(\sigma)$ produced by the classical part of our effective action. The derivation of these equations might show up some problematic aspects, due to our use of normal coordinates. Notice, however, that this question is somewhat academic in that only the total action (classical plus quantum) is anomaly–free. The question whether there exists a supersymmetric (or $\kappa$–invariant) version of our action encounters the same fate: since the classical action carries a gravitational anomaly, its (possible) supersymmetric extension carries also a supersymmetry anomaly, the so called “supersymmetric partner”; this means that also the problem of supersymmetry can be stated only for the total effective action.

Together with the proofs not reported here in [1] we will discuss in particular a duality–symmetric formulation, involving both the three–form $B_3$ and its dual $B_6$ [20], the coupling of our action to open membranes ending on 5–branes (which carry gravitational anomalies on their boundaries, too), and the reduction to ten dimensions.

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