Uncertainty for spin systems

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A modified definition of quantum mechanical uncertainty \( \Delta \) for spin systems, which is invariant under the action of \( SU(2) \), is suggested. Its range is shown to be \( \hbar^2 j \leq \Delta \leq \hbar^2 j(j+1) \) within any irreducible representation \( j \) of \( SU(2) \) and its mean value in Hilbert space computed using the Fubini-Study metric is determined to be \( \text{mean}(\Delta) = \hbar^2 j(j+1)/2 \). The most used sets of coherent states in spin systems coincide with the set of minimum \( \Delta \) uncertainty states.

Coherent states are an important tool in the study of wave phenomena finding many relevant applications in Quantum physics [1,2]. The familiar Glauber states [3,4] can be equivalently defined as the elements of the orbit of the Heisenberg-Weyl group which contains the ground state, as the eigenstates of the annihilation operator or as the minimum uncertainty wave-packets. Following these different definitions there are different approaches to the generalization of the concept of coherent states, the one based on the first definition [3] being the most popular. The generalization procedure has been extended to include spin systems [6,7] and others [8–11]. A full account of applications of coherent states in different areas of Physics can be found in [12].

In the group theoretical approach to coherent states Hilbert space is decomposed into the union of disjoint sets of coherent states, the group orbits. For spin systems the orbit space (the set of orbits) is composed almost entirely of 3-dimensional orbits with the exception of a finite number of 2-dimensional orbits which consist of the eigenvectors of \( \vec{r} \cdot \vec{J} \), with \( \vec{J} \) the generators of the Lie algebra of \( SU(2) \) and \( \vec{r} \) any numeric vector [8,4]. In each irreducible representation of \( SU(2) \) there is one particular orbit which admits an analytic representation in the complex plane,

\[
|z> = \frac{1}{(1+|z|^2)^j} e^{z J_x} \frac{1}{j+1} |j>. 
\]

It turns out that this orbit is singled out by the structure of orbit space: it is the 2-dimensional orbit composed of the eigenvectors of \( \vec{r} \cdot \vec{J} \) with the highest absolute value of its eigenvalue [4]. However the states belonging to this orbit are not all minimum uncertainty states; they do not even have constant uncertainty.

Uncertainty is an important property of a physical state, and it would be desirable to keep it playing a major role in the definition of coherent states. Minimum uncertainty states have been studied in [4] and, in the context of spin systems, states saturating the equality in the Heisenberg relation have been studied in [16] and called intelligent states. States saturating the equality in the Robertson relation have also been studied [17,18]. One unsatisfactory feature of intelligent states and of the commonly used definition of uncertainty for spin systems is that they are not invariants under the action of \( SU(2) \). As a consequence sets of coherent states based on these definitions cannot be represented as orbits of \( SU(2) \). This is in contrast with the situation in particle mechanics where the Heisenberg inequality and the uncertainty function used are invariants under the action of the Heisenberg-Weyl group.

Here we propose a new definition of uncertainty for spin systems,

\[
\Delta = \Delta J_x^2 + \Delta J_y^2 + \Delta J_z^2 ,
\]

which is a positive increasing function of the variances and which is invariant under the action of \( SU(2) \). It obeys the following invariant inequalities:

\[
\hbar^2 j \leq \Delta \leq \hbar^2 j(j+1) ,
\]

which play the role of uncertainty relations. As an immediate application we show that the particular set of coherent states which admits an analytic representation in the complex plane coincides with the set of minimum uncertainty states for this inequality. We use the Fubini-Study metric to compute the mean value of the uncertainty \( \Delta \) in Hilbert space with the result:

\[
\text{mean}(\Delta) = \hbar^2 j \left( j + \frac{1}{2} \right) ,
\]

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for any irreducible representation $j$. This shows that in higher dimensional representation spaces of $SU(2)$ most of the states have high values of uncertainty. In particular one has

$$\lim_{j \to \infty} \frac{\text{mean}(\Delta)}{\text{max}(\Delta)} = 1. \quad (1)$$

The paper is organized as follows: In section I we review some mathematical definitions concerning group orbits and invariants, the Glauber coherent states and their generalization, and the construction of spin coherent states. In section II we discuss the issue of Heisenberg-like inequalities and uncertainty relations. We propose the new definition of uncertainty $\Delta$ for spin systems and we state and prove the statements about $\Delta$ made above. We include an appendix on how to average quantities in $CP^N$ using the Fubini-Study metric.

I. INTRODUCTION

A. Group orbits and invariants

Let $U(g)$ be a representation of the Lie group $G$ on the Hilbert space $\mathcal{H}$. The $G$-orbit through $|\phi> \in \mathcal{H}$ is the subset of $\mathcal{H}$ given by

$$C_\phi = \{|\psi> \in \mathcal{H} : |\psi> = U(g)|\phi>, \ g \in G\} . \quad (2)$$

It follows that

$$\dim C_\phi \leq \dim \ G \ \text{and} \ \dim C_\phi \leq \dim \mathcal{H} . \quad (3)$$

The relation “$|\phi'>$ lies on the same orbit as $|\phi>$” is clearly an equivalence relation: reflexive, symmetric and transitive. As a consequence $\mathcal{H}$ can be partitioned into disjoint orbits

$$\mathcal{H} = \bigcup_{\phi} C_\phi \quad (4)$$

where the label $\phi$ runs over orbits (equivalence classes) and not over vectors. The quotient space $\mathcal{H}/G$ is called the orbit space. A function $f(|\psi>)$ in Hilbert space $\mathcal{H}$ is said to be $G$-invariant if

$$f(U(g)|\psi>) = f(|\psi>), \ \forall g \in G, \ \forall |\psi> \in \mathcal{H} . \quad (5)$$

It follows that $G$-invariant functions are also functions on orbit space $\mathcal{H}/G$.

For more information on these issues see for instance [19,20].

B. Glauber states

The familiar Glauber states $|q,p>$ in particle mechanics can be seen as the $G$-orbit of the Heisenberg-Weyl group through the vacuum state $|0>$,

$$|q,p> = U(q,p)|0>, \quad (6)$$

where $U(p,q)$ is the Weyl operator

$$U(q,p) = e^{(pQ - qP)/\hbar} . \quad (7)$$

They are eigenstates of the annihilation operator and they admit the useful analytic representation in the complex plane

$$|p,q> = e^{(z^a+\bar{z}^a)}|0> = e^{-|z|^2/2} \sum_n \frac{z^n}{\sqrt{n!}}|n> , \quad (8)$$

with $z = (q+ip)/\sqrt{2\hbar}$. It can be shown that the Glauber states are minimum uncertainty states since
\[ \Delta Q^2 = \Delta P^2 = \hbar/2, \] (9)
and the equality sign is satisfied in the Heisenberg uncertainty relation (sometimes the square root of this relation is used; here we prefer this form)

\[ \Delta Q^2 \Delta P^2 \geq \hbar^2/4. \] (10)

The remaining \( G \)-orbits of the Heisenberg-Weyl group can be seen as generalized coherent states but they are not eigenstates of any particularly simple operator, they do not admit an analytic representation in the complex plane, and they are not minimum uncertainty states. Nevertheless they have constant values of uncertainty since both factors \( \Delta Q^2 \) and \( \Delta P^2 \) are \( G \)-invariant functions.

C. Spin coherent states

The group \( SU(2) \) admits representations classified according to integer and semi-integer values \( j \) with the Casimir operator \( J^2 = j(j+1)\hbar^2 \). Let \( \mathcal{H} \) be a Hilbert space carrying one such representation. Sets of generalized coherent states can be generated as the orbits of \( SU(2) \) in \( \mathcal{H} \),

\[ C_\phi = \{ |\vec{r} > \in \mathcal{H} : |\vec{r} > = U(|\vec{r}>)\phi, \vec{r} \in (4\pi)^3 \} \] (11)

\[ U(|\vec{r}>) = e^{i\vec{r}.\hat{J}/\hbar}, \] (12)

where we used the so-called canonical group coordinates for generality.

Using the group parameterization

\[ U(z, \theta) = Ne^{zJ_–/\hbar}e^{-z^*J_+/\hbar}e^{-i\theta J_z/\hbar}, \] (13)

where \( J_\pm = J_x \pm iJ_y \), and choosing the fiducial state \( |\phi > \) to be an eigenstate of \( J_z \), \( |m > \) with \( m = -j, \ldots, j \), one has

\[ |z; m > = U(z)|m > = Ne^{zJ_–/\hbar}e^{-z^*J_+/\hbar}|m >, \] (14)

where the phase factor resulting from \( e^{-i\theta J_z/\hbar} \) has been ignored and \( N \) stands for a normalization factor. Further choosing \( |j > \) as the fiducial state one has \( e^{-z^*J_+/\hbar}|j > = |j > \) and

\[ |z > = \frac{1}{(1 + |z|^2)^{j}}e^{zJ_–}|j >, \] (15)

after determination of the normalization factor. This analytic representation is not available in general for the sets generated from arbitrary fiducial vectors.

The analogous relation for spin systems to the Heisenberg inequality for canonically conjugate operators (10) is

\[ \Delta J^2 \Delta J^2 \geq \hbar^2 4/4|z|^2. \] (16)

Notice the important difference with (10) that now the right hand side of the inequality is not a constant. Following we shall call the left hand side of (16) the uncertainty \( \Delta J^2 \Delta J^2 \). Then it is clear that the set of states for which the equality in (10) is saturated and the set of states of minimum uncertainty are not the same. Moreover none of them coincide with any set of coherent states (13). On the other hand in particle mechanics the Glauber states satisfy the Heisenberg inequality and they are states of minimum uncertainty. In the spin states satisfying the equality sign in (10) have been called intelligent states. They are given by

\[ |\tau, N > = \frac{A_N}{(1 + |\tau|^2)^j} \sum_{l=0}^{N} \binom{N}{l} (2j-l)! \times \]

\[ \times \left( -\frac{2}{\hbar}\tau J_z \right)^l e^{J_+/\hbar}|j >, \] (17)

where \( N \) is a discrete label satisfying \( 0 \leq N \leq 2j \) and \( \tau \) is a continuous label which can be either real or purely imaginary. \( A_N \) is a normalization factor.

Finally we comment that the space of physical states for the irreducible representation \( j \) of \( SU(2) \) is \( CP^N \) with \( N = 2j \) (see the appendix):

\[ j \to \dim \mathcal{H} = 2j + 1 \to \text{projective space} : CP^{2j}. \] (18)

Its real dimension is \( 4j \).
II. UNCERTAINTY

A. Uncertainty relations

We recall the inequality valid for hermitian operators $A$ and $B$ [21]

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} \left( \sigma_{AB}^2 - [A, B]^2 \right),$$  (19)

where $\Delta A$ and $\Delta B$ are the standard deviations of the operators $A$ and $B$

$$\Delta A^2 = A^2 - \langle A^2 \rangle = <\psi | A^2 | \psi > - < \psi | A | \psi >^2.$$  (20)

and where

$$\sigma_{AB} = \{A, B\} - 2A \overline{B} \geq 0$$  (21)

is the covariance of $A$ and $B$. Since for hermitian operators $\sigma_{AB}$ is real and $[A, B]$ is purely imaginary, both parcels on the right hand side of (19) are positive and one can state that

$$\Delta A^2 \Delta B^2 \geq - \frac{1}{4} [A, B]^2.$$  (22)

This is called the Heisenberg relation while (19) is often called the Robertson relation. For canonically conjugate operators $Q$ and $P$ one has $[Q, P] = i\hbar$ and the Heisenberg uncertainty relation (10) follows immediately from (22). For spin systems (16) follows from $[J_x, J_y] = i\hbar J_z$. Notice that the equality can hold only if $\sigma_{AB} = 0$.

The left hand side of the Heisenberg inequality (22) is sometimes called the uncertainty. It is invariant under the action of the Heisenberg-Weyl group. And the right hand side of (22) is a constant. It is therefore natural to assign a particular physical significance to $\Delta Q^2 \Delta P^2$ and to the states satisfying the equality sign in this inequality. But the left hand side of the analogous spin inequality (16) is not invariant under the action of $SU(2)$ neither is its right hand side a constant. Therefore there seems to be no reason why $\Delta J_x^2 \Delta J_y^2$ should play a role for spin systems similar to the one played by $\Delta Q^2 \Delta P^2$ in particle mechanics, nor why states saturating the equality in (22) or in (19) should be particularly distinguished. Such states (intelligent states) have been studied in [16] and in [17,18] respectively and may certainly be important for the study of spin systems with Hamiltonians that break the $SU(2)$ symmetry such as systems under the action of one particular magnetic field pointing in the $z$-direction, but in what concerns the study of $CP^N$ as the representation space for spin systems prior to the definition of the Hamiltonian one should look for a $G$-invariant definition of uncertainty. We look for an uncertainty function which is positive and which increases with increasing values of the variances of the elements of the Lie algebra. The following additive rather than multiplicative combination of variances does the job

$$\Delta = \Delta J_x^2 + \Delta J_y^2 + \Delta J_z^2.$$  (23)

The following results hold:

I - The uncertainty $\Delta$ is $G$-invariant and therefore it is constant within sets of coherent states generated as orbits of $SU(2)$ in $CP^N$.

II - The uncertainty $\Delta$ is bounded from below and from above

$$\hbar^2 j \leq \Delta \leq \hbar^2 j(j + 1).$$  (24)

All values within this range are present in Hilbert space except for the representation $j = 1/2$ where all states have the same uncertainty $\Delta = \hbar^2 j$.

III - The set

$$\{|\psi\rangle \in \mathcal{H} : \Delta(|\psi\rangle) = \hbar^2 j\}$$  (25)

of minimum uncertainty vectors in the irreducible representation $j$ of $SU(2)$ coincides with the set of coherent states

$$|z\rangle = (1 + |z|^2)^{-j/2} e^{zJ_z} |j\rangle$$  (26)
generated as an orbit of SU(2) in \( \mathcal{H} \) and admitting an analytic representation in the complex plane.  

\[ \text{IV.} \quad \text{The mean value evaluated with the volume element naturally associated to the Fubini-Study metric of uncertainty on the whole of Hilbert space is given by} \]

\[
\text{mean}(\Delta) = \hbar^2 j(j + 1/2)
\]

for any irreducible representation \( j \) of SU(2).

Notice that the last statement is consistent with the second one for the \( j = 1/2 \) representation.

B. Proof

I. We have

\[
U^+(\vec{r})J_i U(\vec{r}) = \Lambda_i^j(\vec{r}) J_j ,
\]

where \( \Lambda_i^j \) are the matrices of the adjoint representation of SU(2), satisfying

\[
\Lambda_i^j(\vec{r}) \Lambda_i^k(\vec{r}) = \delta^{jk}, \quad \forall \vec{r}. \tag{29}
\]

The mean values of \( J_i \) transforms, within an orbit, according to the adjoint representation too,

\[
\mathcal{J}_i = < \vec{r} | J_i | \vec{r}^\prime > = < \phi | U^+(\vec{r})J_i U(\vec{r}) | \phi > = \Lambda_i^j(\vec{r}) < \phi | J_j | \phi > . \tag{30}
\]

Then \( \mathcal{J}_i \mathcal{J}_i \) is a G-invariant function

\[
\mathcal{J}_i \mathcal{J}_i = \Lambda_i^j(\vec{r}) < \phi | J_j | \phi > \Lambda_i^k(\vec{r}) < \phi | J_k | \phi > = < \phi | J_i | \phi > . \tag{31}
\]

This is one example of a wider set of invariants defined in [14]. The Casimir operator \( J_i J_i \) is invariant within the whole representation and consequently \( \mathcal{J}_i \mathcal{J}_i \) is G-invariant. Then

\[
\Delta = \sum_{i=x,y,z} \Delta J_i^2 = \sum_{i=x,y,z} \mathcal{J}_i \mathcal{J}_i - \mathcal{J}_i \mathcal{J}_i \tag{32}
\]

is the difference between two G-invariant functions and is therefore G-invariant too.

II. It is always possible to choose a representative \( |\psi> = \sum_{m=-j}^m c_m |m> \) within each orbit such that \( <\psi|\vec{J}|\psi> = \mathcal{J}_z \in \mathbb{C} \). Then \( \mathcal{J}_i \mathcal{J}_i = \mathcal{J}_z^2 \). But

\[
\mathcal{J}_z = \sum_{m=-j}^j m \hbar |c_m|^2 \Rightarrow \mathcal{J}_z \leq \hbar j . \tag{33}
\]

Therefore \( \mathcal{J}_i \mathcal{J}_i \leq \hbar^2 j^2 \), and this inequality is valid all over Hilbert space since it concerns a G-invariant function. On the other hand it is obvious that \( \mathcal{J}_i \mathcal{J}_i \geq 0 \). Since \( J_i J_i = \hbar^2 j(j+1) \) it follows that

\[
0 \leq \mathcal{J}_i \mathcal{J}_i \leq \hbar^2 j^2 \Leftrightarrow \hbar^2 j \leq \Delta \leq \hbar^2 j(j+1) . \tag{34}
\]

Now we consider the one-parameter set of vectors

\[
|\alpha> = \cos \alpha |j > + \sin \alpha | - j > \quad \text{with} \quad \alpha \in [0, \pi/2] . \tag{35}
\]

We have

\[
\mathcal{J}_x = \hbar \sqrt{j/2} \sin(2\alpha) \delta_{j,-j} , \quad \mathcal{J}_y = 0 , \quad \mathcal{J}_z = \hbar j \cos(2\alpha) \Rightarrow \mathcal{J}_i \mathcal{J}_i = \begin{cases} \hbar^2 j^2 & \text{for} \ j = 1/2 \\ \hbar^2 j^2 \cos^2(2\alpha) & \text{for} \ j \neq 1/2 \end{cases} . \tag{36}
\]

There is only one orbit in the \( j = 1/2 \) representation [14]; since \( \Delta \) is G-invariant it can only assume the value \( \hbar^2 j^2 \). On the other hand, for \( j \neq 1/2 \) it is clear that \( \mathcal{J}_i \mathcal{J}_i \) maps \( \alpha \) onto \( [0, \hbar^2 j^2] \), and the statements about the range of \( \mathcal{J}_i \mathcal{J}_i \) in Hilbert space are proven.
We notice from (33) that the maximum value of $J_i J_i$ is attained only at the vectors $|j>$ and $|-j>$ which we know to belong to the same orbit [14]. This single orbit coincides with the set (15) of coherent states $|z>$ since for $z = 0$ we have $|z> = |j>$.

We use the coordinates (A16) defined in the appendix to label physical states

$$|\psi> = \sum_{m=-j}^{j} c_m |m> = \sum_{n=0}^{N} Z_n(\theta_i, \beta_j) |n - N/2> = |\{\theta_i\}, \{\beta_j\}>.$$  \hfill (37)

Using the standard representation of the generator $J_z$ of the SU(2) Lie algebra [22] its mean value on a state $|\{\theta_i\}, \{\beta_j\}>$ is

$$\mathcal{J}_z = \sum_{m=-j}^{j} |c_m|^2 \hbar m = \hbar \sum_{n=0}^{N} x_n^2 \left(n - \frac{N}{2}\right).$$  \hfill (38)

The mean value of $\mathcal{J}_z^2$ in the whole of Hilbert space is thus (see (A24) in the appendix)

$$\text{mean}(\mathcal{J}_z^2) = \frac{\hbar^2}{V_N} \int_{CP^N} dv \mathcal{J}_z^2 = \frac{\hbar^2}{V_N} \sum_{m,n=0}^{N} \left[(m - \frac{N}{2}) \left(n - \frac{N}{2}\right) \int_{CP^N} dv (x_m x_n)^2\right]$$  \hfill (39)

Now we compute

$$\int_{CP^N} dv (x_m x_n)^2 = \frac{\pi^N}{(N+2)!} (1 + \delta_{mn})$$  \hfill (40)

and

$$\sum_{m,n=0}^{N} \left(m - \frac{N}{2}\right) \left(n - \frac{N}{2}\right) (1 + \delta_{mn}) = \sum_{n=0}^{N} \left(n - \frac{N}{2}\right)^2 = \frac{N(N+1)(N+2)}{12}$$  \hfill (41)

to arrive at

$$\text{mean}(\mathcal{J}_z^2) = \frac{\hbar^2}{V_N} \frac{\pi^N}{(N+2)!} \frac{N(N+1)(N+2)}{12} = \frac{\hbar^2 N}{12}.$$  \hfill (42)

By symmetry one has

$$\text{mean}(\mathcal{J}_x^2) = \text{mean}(\mathcal{J}_y^2) = \text{mean}(\mathcal{J}_z^2).$$  \hfill (43)

and consequently

$$\text{mean}(\mathcal{J}_i \mathcal{J}_i) = 3 \text{mean}(\mathcal{J}_z^2) = \frac{\hbar^2 N}{4} = \frac{\hbar^2}{2} j.$$  \hfill (44)

The mean value of uncertainty (32) in Hilbert space is therefore

$$\text{mean}(\Delta) = \text{mean}(\mathcal{J}_i \mathcal{J}_i) - \text{mean}(\mathcal{J}_i \mathcal{J}_i) = \hbar^2 j(j+1) - \frac{\hbar^2}{2} j = \hbar^2 j \left(j + \frac{1}{2}\right).$$  \hfill (45)

**APPENDIX A: THE FUBINI-STUDY METRIC AND THE VOLUME ELEMENT IN $CP^N$**

Two vectors in Hilbert space $\mathcal{H}$ differing by a multiplicative non-zero complex constant $\alpha$ represent the same physical state,

$$|z'> \sim |z> \quad \text{if} \quad |z'> = \alpha |z>.$$  \hfill (A1)
Therefore the space of physical states is the space of rays in Hilbert space or projective space, that is the space of equivalence classes defined by \( |\psi \rangle \) and excluding the vector \( |\psi \rangle = 0 \). The projective spaces constructed from finite-dimensional Hilbert spaces are called \( CP^N \) and are well studied spaces \([23,24]\). The superscript \( N \) stands for their complex dimension which is one unit lower then the complex dimension of the Hilbert space from which they are constructed.

If \( |n\rangle \) is a basis for \((N+1)\)-dimensional Hilbert space any vector \( |\psi\rangle \) can be written as

\[
|\psi\rangle = \sum_{n=0}^{N} Z_n |n\rangle .
\]

The complex numbers \( Z_n \) are homogeneous coordinates in \( \mathcal{H} \) and they can also be used as coordinates in \( CP^N \) provided one makes the identifications

\[
Z'_n \sim Z_n \text{ if } \exists \alpha : \forall n, Z'_n = \alpha Z_n .
\]

We start by reminding the reader that the unit \( N \)-sphere can be defined as the hyper-surface in \((N+1)\)-dimensional Euclidean space with coordinates \( x_i , i = 0, .., N \) that satisfies

\[
\sum_{i=0}^{N} x_i^2 = 1 .
\]

Intrinsic coordinates \( \theta_i \) can be defined by

\[
x_i = \cos \theta_i \prod_{j=i+1}^{N} \sin \theta_j ,
\]

Their range is \((0, \pi)\) except for for \( \theta_1 \) with range \((0, 2\pi)\). \( \theta_0 = 0 \) is not a coordinate. The metric induced on the \( N \)-sphere by its embedding in \((N+1)\)-dimensional Euclidean space in this coordinates is diagonal with components

\[
g_{ii} = \left( \prod_{j=i+1}^{N} \sin \theta_j \right)^2 ,
\]

and the volume element is

\[
dv = \prod_{i=1}^{N} \sin^{(n-1)} \theta_i \ d\theta_i .
\]

Real projective space \( RP^N \) follows the same construction with the range of \( \theta_1 \) being \((0, \pi)\) too, plus the identifications

\[
(0, \theta_2, .., \theta_N) \equiv (\pi, \pi - \theta_2, .., \pi - \theta_N) .
\]

For quantum mechanical purposes the metric of interest in \( CP^N \) is the Fubini-Study metric \([24]\). Its line element in the homogeneous coordinates \( Z_i \) is

\[
ds^2 = \frac{1}{X^2} \sum_{i=0}^{N} dZ_i d\overline{Z}_i - \frac{1}{X^4} \sum_{i=0}^{N} dZ_i \overline{Z}_i \sum_{j=0}^{N} Z_j d\overline{Z}_j ,
\]

where we have defined

\[
X^2 = \sum_{i=0}^{N} Z_i \overline{Z}_i .
\]

Splitting the complex homogeneous coordinates into their absolute values and phases

\[
Z_i = X_i e^{i\alpha_i} ,
\]
the Fubini-Study metric splits into two blocks relative to the $X_i$ and to the $\alpha_i$,
\[ ds^2 = ds_X^2 + ds_\alpha^2 , \tag{A12} \]
with
\[ ds_X^2 = \frac{1}{X^2} \left( \sum_{i=0}^N dX_i^2 - dX^2 \right) \tag{A13} \]
\[ ds_\alpha^2 = \frac{1}{X^2} \sum_{i=0}^N X_i^2 d\alpha_i^2 - \frac{1}{X^2} \left( \sum_{i=0}^N x_i^2 d\alpha_i \right)^2 \tag{A14} \]

The intrinsic coordinates on the sphere (A3) and the phases relative to $\alpha_0$
\[ \beta_i = \alpha_i - \alpha_0 \, , \quad i = 1, \ldots, N \tag{A15} \]
can be used as intrinsic coordinates on $CP^N$. However we should remark that the ranges of all the coordinates $\theta_i$ are $(0, \pi/2)$ since the $X_i$ are absolute values and cannot therefore be negative. Moreover these coordinates are clearly singular whenever $\theta_i = \{0, \pi/2\}$. The relation of this coordinates with the homogeneous ones is
\[ Z_i = X e^{i\alpha_0 x_i(\theta_j)} e^{i\beta_i} . \tag{A16} \]
Plugging this expression into the previous formulas for the line elements (A13)-(A14) one gets
\[ ds_X^2 = \sum_{i=0}^N dx_i^2 = \sum_{i=1}^N g_{ii} d\theta_i^2 \tag{A17} \]
\[ ds_\alpha^2 = \sum_{i=1}^N x_i^2 d\beta_i^2 - \left( \sum_{i=1}^N x_i^2 d\beta_i \right)^2 = \sum_{i,j=1}^N h_{ij} d\beta_i d\beta_j \tag{A18} \]
The first is the line element in the unit sphere (A6) and in the phase line element $ds_\alpha^2$ we have defined the metric
\[ h_{ij} = x_i^2 (\delta_{ij} - x_j^2) \tag{A19} \]
with inverse
\[ h^{ij} = \frac{1}{x_i^2} + \frac{\delta_{ij}}{x_i^2} . \tag{A20} \]
The volume element for the phase coordinates is
\[ dv_\alpha = \sqrt{\det(h_{ij})} = \sqrt{\det(\delta_{ij} - x_j^2)} \prod_{k=1}^N x_k d\beta_k = \sqrt{1 - \sum_{i=1}^N x_i^2 \prod_{k=1}^N x_k d\beta_k} = \prod_{i=0}^N x_i \prod_{j=1}^N d\beta_j = \prod_{i=1}^N \cos \theta_i \sin^{i-1} \theta_i \, d\beta_i , \tag{A21} \]
where we used (A2) for $x_i$ in the last equality. Using (A7) for $dv_X$ the combined volume element is
\[ dv = dv_X dv_\alpha = \prod_{i=1}^N \cos \theta_i \sin^{2i-1} \theta_i \, d\theta_i d\beta_i . \tag{A22} \]
The total volume of $CP^N$ becomes easy to compute
\[ V_N = \prod_{i=1}^N \int_0^{\pi/2} d\theta_i \cos \theta_i \sin^{2i-1} \theta_i \int_0^{2\pi} d\beta_i = \prod_{i=1}^N \frac{1}{2i} 2\pi = \frac{\pi^N}{N!} . \tag{A23} \]
Now we are able to compute mean values of functions in Hilbert space as their integral in $\mathbb{C}P^N$ weighted with the Fubini-Study volume element $(A22)$ and divided by the volume $V_N$ of $\mathbb{C}P^N$ $(A23)$. Since the functions we are interested in are of the type $\langle \psi | A | \psi \rangle = A$ we shall write explicitly mean$(A)$ to emphasize that the mean value is not taken on quantum states but rather on the whole of $\mathbb{C}P^N$,

$$\text{mean}(A) = \frac{1}{V_N} \int_{\mathbb{C}P^N} d\nu A.$$  

(A24)

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