CONVERGENCE RATE OF DISTRIBUTED DYKSTRA’S ALGORITHM WITH SETS DEFINED AS LEVEL SETS OF CONVEX FUNCTIONS

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Abstract. We investigate the convergence rate of the distributed Dykstra’s algorithm when some of the sets are defined as the level sets of convex functions. We carry out numerical experiments to compare with the theoretical results obtained.

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1. INTRODUCTION

Let $X$ be a finite dimensional Hilbert space. For a finite set $V$, consider the problem

$$\min_{x \in X} \sum_{i \in V} \delta_{C_i}(x) + \frac{1}{2} \|x - \bar{x}\|^2,$$

(1.1)

where $\delta_{C_i}()$ is the indicator function of the set $C_i$ defined by

$$C_i := \{x : g_i(x) \leq 0\}$$

(1.2)

for some closed convex subdifferentiable function $g_i : X \to \mathbb{R}$ with full domain. If $C_i$ were sets that are easy to project onto rather than through (1.2), then Dykstra’s algorithm [Dyk83] is one way to solve problem (1.1). It was recognized in [Han88] that Dykstra’s algorithm is block coordinate ascent on the dual. We prefer to call it Dykstra’s algorithm because the Boyle-Dykstra theorem [BD85] shows the convergence to the primal minimizer even if a dual maximizer is absent. (In [Han88] and most other papers on block coordinate methods, a dual maximizer is assumed to exist, with a constraint qualification or otherwise.) The proof in [BD85] is rewritten in the form of mathematical programming in [GM89].

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Solving \( (1.1) \) directly may not be easy to do if only function values and a subgradient of \( g(\cdot) \) is easy to obtain in each iteration. As was discussed in [Com00, BCRZM03], an iterative method to find the minimizer of \( (1.1) \) is to project onto outer approximates

\[
\{ x : g_i(\tilde{x}) + \langle \tilde{s}, x - \tilde{x} \rangle \leq 0 \}
\]

(1.3)

of \( C_i \), where \( \tilde{x} \) is some point in \( X \). Halfspaces like (1.3) are easier to project onto than \( C_i \) itself. The method proposed in [Com00] shares more similarity with Haugazeau’s algorithm [Hau68].

In [BCRZM03], the authors extend Dykstra’s algorithm while projecting onto supersets of \( C_i \) (not necessarily of the form (1.3)), showing the convergence to a primal minimizer under a constraint qualification.

In a series of papers [Pan17, Pan18a, Pan18b], we showed that extending Dykstra’s algorithm leads to a distributed optimization algorithm for problems of the form

\[
\min_{x \in X_i} \sum_{i \in V} \left[ f_i(x) + \frac{1}{2} \| x - \bar{x}_i \|^2 \right],
\]

where \( X_i \) are finite dimensional Hilbert space, and \( f_i : X_i \rightarrow \mathbb{R} \cup \{\infty\} \) are closed convex functions that are either proximable, or subdifferentiable with full domain. The algorithm, which we call the distributed Dykstra’s algorithm, has many favorable properties. Such properties include being distributed, asynchronous, decentralized (similar to the special case of the averaged consensus problem), and having deterministic convergence rates. Other properties include being applicable for time-varying graphs, allow partial communication of data (so that computations are not limited by communication speeds), and having convergence rates compatible with various first order algorithms for various special cases.

1.1. **Contributions of this paper.** It appears that the convergence rates of Dykstra’s algorithm for (1.1) has not been studied. In this paper, we study the convergence rate of the distributed Dykstra’s algorithm when the outer approximates of the form (1.3) are used. We show that our algorithm has \( O(1/k) \) convergence (for the dual objective value) for the case when \( |V| = 1 \) in (1.1), and \( O(1/k^{1/3}) \) convergence for the distributed Dykstra’s algorithm. We also perform numerical experiments to compare the theoretical rates obtained.

1.2. **Notation.** Throughout this paper, we assume that the Hilbert spaces are finite dimensional. We denote \( P_C(x) \) to be the projection of \( x \) onto the set \( C \). Other notations are standard in convex analysis.

2. **THE CASE OF ONE SET**

In this section, we work on the case (1.1) when \( |V| = 1 \). The primal problem and its (Fenchel) dual are

\[
(P) \min_{x \in X} \frac{1}{2}\| x - \bar{x} \|^2 + f(x), \quad \text{and} \quad (D) \max_{z \in X} \frac{1}{2}\| z - \bar{x} \|^2 + f^*(z), \quad (2.1)
\]

where \( f(\cdot) = \delta_{\{g(x) \leq 0\}}(\cdot) \) and \( g : X \rightarrow \mathbb{R} \) is a subdifferentiable function with full domain. Strong duality holds for (2.1). We now look at the basic algorithm in Algorithm (2.1). Note the similarities of the Algorithm (2.1) to Haugazeau’s algorithm [Hau68]; See also [BC11]. We make the following assumption on \( g(\cdot) \).
Algorithm 2.1. This algorithm finds iterates \( \{x_k\}_k \) that converges to the solution of (2.1).

Set \( x_0 = \bar{x} \).

Set \( H_0 = \{ x : c_0^T x \leq b \} \) so that \( C \subset H_0 \). (Note: \( c_0 \) and \( b \) can be chosen to be 0)

For \( k = 0, 1, \ldots \)

Find \( \tilde{C}_k \) such that \( C \subset \tilde{C}_k \) and \( x_k \notin \tilde{C}_k \). A typical choice is

\[
\tilde{C}_k = \{ x : g(x_k) + \langle x - x_k, s_k \rangle \leq 0 \}
\]

for some \( s_k \in \partial g(x_k) \).

Let \( x_{k+1} = P_{H_k \cap \tilde{C}_k}(\bar{x}) \).

Let \( H_{k+1} \) be the halfspace such that \( x_{k+1} = P_{H_{k+1}}(\bar{x}) \).

End For

Assumption 2.2. Suppose that \( g : X \to \mathbb{R} \) is a subdifferentiable function with full domain and \( \min_{x \in X} g(x) < 0 \).

If \( \min_x g(x) > 0 \), then the problem is infeasible. If \( \min_x g(x) = 0 \), then note that a slight perturbation of \( g(\cdot) \) would render the problem infeasible. See [RW98 Theorem 9.41(b)] for more connections to stability. So this assumption ensures the stability of the problem.

Proposition 2.3. Suppose Assumption 2.2 holds and let \( R \) be a bounded set. Then there is some \( c > 0 \) such that if \( x \in R \), \( g(x) \geq 0 \) and \( s \in \partial g(x) \), then \( ||s|| > c \).

Proof. Seeking a contradiction, suppose \( (x_i, s_i) \) satisfies the conditions and \( \lim_{i \to \infty} ||s_i|| = 0 \), \( g(x_i) \geq 0 \). Let \( \bar{x} \) be \( \lim_{i \to \infty} x_i \). By the outer semicontinuity of the subgradient mapping, \( 0 \in \partial g(\bar{x}) \) and \( g(\bar{x}) \geq 0 \), which contradicts Assumption 2.2.

Lemma 2.4. Let \( X \) be a finite dimensional Hilbert space, and let \( g : X \to \mathbb{R} \) be a subdifferentiable function with full domain satisfying Assumption 2.2. Let \( C := \{ x : g(x) \leq 0 \} \), and let \( R \) be a bounded set. Let \( \bar{x} \in R \). Let \( \hat{x} = P_C(\bar{x}) \), let \( H \) be a halfspace such that \( C \subset H \), and let \( x = P_H(\hat{x}) \). Then the following hold:

1. \( \bar{x} - \hat{x} \in N_C(\hat{x}) \).
2. There is some constant \( \gamma > 0 \) such that if \( g(x) \geq 0 \) and \( s \in \partial g(x) \), then \( ||s|| \geq \gamma \).
3. Let \( \hat{H} \) be the halfspace \( \{ x : \langle \hat{x} - x, x - \hat{x} \rangle \geq 0 \} \). Then for the constant \( \gamma > 0 \) in (2), \( g(x) \geq \gamma d(x, \bar{H}) \).
4. Let \( \bar{R} \) be a compact set. Let \( \bar{\gamma} := \sup \{ ||\bar{s}|| : \bar{s} \in \partial g(\bar{x}), \bar{x} \in \bar{R} \} \), which is finite from the fact that \( \text{dom}(g) = X \). For \( \bar{x} \) such that \( g(\bar{x}) > 0 \), let \( \hat{H} \) be the halfspace \( \hat{H} := \{ x : g(\bar{x}) + \langle x - \bar{x}, \bar{s} \rangle \leq 0 \} \). Then \( d(\bar{x}, \hat{H}) = g(\bar{x})/||\bar{s}|| \).

Moreover, if \( \bar{x} \in \bar{R} \), we have \( d(\bar{x}, \hat{H}) \geq g(\bar{x})/\bar{\gamma} > 0 \).

Proof. Property (1) is obvious. We now prove (2) by contradiction. Since \( \bar{x} \) lies in the bounded set \( R \), \( P_C(\bar{x}) \) also lies in a bounded set. Since \( x \) lies in the ball with center \( \bar{x} \) and radius \( ||\bar{x} - \hat{x}|| \), \( x \) lies in a bounded set as well. Apply Proposition 2.3.

Next, we prove (3). By the optimality conditions on \( \hat{x} \), there is some subgradient \( \bar{s} \in \partial g(\bar{x}) \) that is a positive multiple of \( \bar{x} - \hat{x} \). Then

\[
g(x) \geq g(\hat{x}) + \langle \bar{s}, x - \hat{x} \rangle \overset{g(\hat{x})=0}{=} ||\bar{s}|| d(x, \hat{H}) \overset{(2)}{=} \gamma d(x, \bar{H}).
\]

Lastly, (4) is elementary.
Lemma 2.5. Let $X$ be a finite dimensional Hilbert space, $\bar{x} \in X$, $H_1$ be a half-space, and $x_1 = P_{H_1}(\bar{x})$. Let $H_2$ be a halfspace, and let $d = d(x_1, H_2)$. Let $x_2 = P_{H_1 \cap H_2}(\bar{x})$. Then $\|\bar{x} - x_2\|^2 \geq \|\bar{x} - x_1\|^2 + d^2$.

Proof. Since $x_2 \in H_1$ and $x_1 = P_{H_1}(\bar{x})$, we have $\langle \bar{x} - x_1, x_2 - x_1 \rangle \leq 0$. Also, since $x_2 \in H_2$, we have $\|x_1 - x_2\| \geq d$. Hence

$$\|\bar{x} - x_2\|^2 = \|\bar{x} - x_1\|^2 + \|x_1 - x_2\|^2 + 2\langle \bar{x} - x_1, x_1 - x_2 \rangle \geq \|\bar{x} - x_1\|^2 + d^2.$$ 

□

The following result gives convergence rates for sequences defined by recurrences.

Lemma 2.6. Let $\{a_k\}_{k=1}^\infty$ be a nonnegative sequence.

1. Suppose $\{a_k\}_{k=1}^\infty$ has the recurrence $a_k \geq a_{k+1} + \theta a_k^2$ for some $\theta > 0$. Then $\{a_k\}$ has a $O(1/k)$ rate of convergence.

2. Suppose $\{a_k\}_{k=1}^\infty$ has the recurrence $a_k \geq a_{k+1} + \theta a_k^4$ for some $\theta > 0$. Then $a_k \leq \left(\frac{1}{\theta^2} + (k-1)3\theta (3\theta a_1^2 + 1)^{-1}\right)^{-1/3}$ for all $k \geq 1$, which means that $\{a_k\}$ has a $O(1/k^{1/3})$ rate of convergence.

Proof. Part (1) was addressed in \cite[Lemma 6.2]{BT13} and \cite[Lemma 3.8]{Bec15}. Part (2) was addressed in \cite[Pan18c]{Pan18c}.

We now turn to the problem we try to prove. Let $\bar{d}$ be the distance $d(\bar{x}, C)$, $\hat{x} := P_C(\bar{x})$ so that $\bar{d} = \|\hat{x} - \bar{x}\|$, and $d_k := \|\bar{x} - x_k\|$. The objective value of (2.1) is $\frac{1}{2}\bar{d}^2$. Making use of Lemma 2.4(3), we observe that

$$g(x_k) \geq \gamma d(x_k, \hat{H}) \geq \gamma (\|\hat{x} - \bar{x}\| - \|x_k - \bar{x}\|) = \gamma (\bar{d} - d_k).$$

Moreover, by Lemma 2.4(4), $d(x_k, \hat{C}_k) \geq g(x_k)/\gamma$. We have

$$d_{k+1}^2 \geq d_k^2 + d(x_k, \hat{C}_k)^2 \geq d_k^2 + \left[\frac{\gamma^2}{\gamma}\right]^2 (\bar{d} - d_k)^2.$$ 

Let $v_k = d_k^2 - d_k^2$. Note that the objective value of $\min_{x \in X} \frac{1}{2}\|\bar{x} - x\|^2 + \delta_{H_k}(x)$ is $\frac{1}{2}d_k^2$, while objective value of (1.1) with $|V| = 1$ is $\frac{1}{2}\bar{d}^2$. In other words, $v_k$ is twice the gap between the actual and estimated objective values. We have

$$v_{k+1} \leq v_k - \frac{\gamma^2}{\gamma^2} (\bar{d} - d_k)^2 = v_k - \frac{\gamma^2}{\gamma^2} \frac{\gamma^2}{(d + d_k)^4} \leq v_k - \frac{\gamma^2}{\gamma^2} \frac{\gamma^2}{2\bar{d}^4}.$$ 

By Lemma 2.6(1), $v_k$ converges to zero at a $O(1/k)$ rate.

3. Preliminaries from \cite{Pan18c}

In this section, we list down the preliminaries and description of the distributed Dykstra’s algorithm studied in \cite{Pan18a, Pan18b}. We do not claim originality in this section, and we recall some results useful for the subsequent proofs.

Let $V$ and $E$ be finite sets. Define the set $X := X_1 \times \cdots \times X_{|V|}$, where each $X_i$ is a finite dimensional Hilbert space. For each $i \in V$, let $f_i : X_i \to \mathbb{R} \cup \{\infty\}$ be a closed convex function, and let $f_i : X \to \mathbb{R} \cup \{\infty\}$ be defined by

$$f_i(x) = f_i([x_i]).$$

(3.1)
Let $\delta_C(\cdot)$ be the indicator function for a closed convex set $C$. For each $\alpha \in \bar{E}$, let $H_\alpha \subset X$ be a linear subspace, and define $f_\alpha : X \to \mathbb{R}$ by $f_\alpha(x) = \delta_{H_\alpha}(x)$. The setting for the distributed Dykstra’s algorithm is

$$\min_{x \in X} \frac{1}{2} \|x - \bar{x}\|^2 + \sum_{i \in V} f_i(x) + \sum_{\alpha \in \bar{E}} \delta_{H_\alpha}(x). \quad (3.2)$$

Note that the last two sums in (3.2) can be written as $\sum_{\alpha \in \bar{V} \cup \bar{E}} f_\alpha(x)$. Typically, the hyperplanes $\{H_\alpha\}_{\alpha \in \bar{E}}$ are overdetermined (see Definition 3.2 later). Partition the set $V$ as the disjoint union $V = \bigcup_{i=1}^m V_5$ so that

- $f_i(\cdot)$ are proximable functions for all $i \in V_1$.
- $f_i(\cdot)$ are indicator functions of closed convex sets for all $i \in V_2$.
- $f_i(\cdot)$ are subdifferentiable functions such that $\text{dom}(f_i) = X_i$ for all $i \in V_3$.
- $f_i(\cdot)$ are indicator functions $\delta_{C_i}(\cdot)$, where $C_i := \{x : g_i(x) \leq 0\}$, and $g_i : X_i \to \mathbb{R}$ is a closed convex subdifferentiable function with $\text{dom}(g_i) = X_i$ for all $i \in V_5$.

The (Fenchel) dual of (3.2) can be found to be

$$\max_{z_\alpha \in X, \alpha \in \bar{V} \cup \bar{E}} F(\{z_\alpha\}_{\alpha \in \bar{V} \cup \bar{E}}), \quad (3.3)$$

where

$$F(\{z_\alpha\}_{\alpha \in \bar{V} \cup \bar{E}}) := -\frac{1}{2} \|\bar{x} - \sum_{\alpha \in \bar{V} \cup \bar{E}} z_\alpha\|^2 + \frac{1}{2} \|\bar{x}\|^2 - \sum_{\alpha \in \bar{V} \cup \bar{E}} f'_\alpha(z_\alpha). \quad (3.4)$$

We now explain that the problem (3.2) includes the general case of the distributed Dykstra’s algorithm in [Pan18a, Pan18b].

**Example 3.1.** [Pan18a, Pan18b] (Distributed Dykstra’s algorithm is a special case of (3.2)) Let $G = (V, E)$ be an undirected connected graph. Suppose each $X_i = \mathbb{R}^m$ for all $i \in V$, and let $\bar{E} := E \times \{1, \ldots, m\}$. For each $x \in X = [\mathbb{R}^m]^{\bar{V}}$, we let $[x]_i \in \mathbb{R}^m$ be the $i$-th component, and we let $[[x]_i]_k$ be the $k$-th component of $[x]_i$. For each $(i, j, k) \in \bar{E}$, let the linear subspace $H_{(i, j), k} \subset X$ of codimension 1 be defined to be

$$H_{(i, j), k} := \{x \in X : [x]_i = [x]_j\}. \quad (3.5)$$

Then the problem (3.2) is equivalent to

$$\min_{x \in \mathbb{R}^m} \sum_{\alpha \in \bar{E}} \left[\frac{1}{2} \|x - [x]_i\|^2 + f_i(x)\right]. \quad (3.6)$$

For all $n \geq 1$ and $w \in \{1, \ldots, w\}$, define $f_{\alpha, n, w} : X \to \mathbb{R} \cup \{\infty\}$ by

$$f_{\alpha, n, w}(\cdot) = f_\alpha(\cdot) \quad \text{for all } \alpha \in \bar{E} \cup \{V_4 \cup V_5\} \quad (3.7a)$$

and

$$f_{\alpha, n, w}(\cdot) \leq f_\alpha(\cdot) \quad \text{for all } \alpha \in V_4 \cup V_5. \quad (3.7b)$$

For $i \in V_4$, the $f_{i, n, w}(\cdot) \leq f_i(\cdot)$ are obtained by taking affine minorants of $f_i(\cdot)$, as discussed in [Pan18b, Pan18a], and then $f_{i, n, w}(x) = f_{i, n, w}([x]_i)$. For $i \in V_5$, define $g_{i, n, w} : X_i \to \mathbb{R}$ by taking affine minorants of $g_i(\cdot) \leq g_i(\cdot)$, and then $f_{i, n, w}(x) := \delta_{\{x : g_{i, n, w}(x) \leq 0\}}(\cdot)$ so that

$$f_{i, n, w}(x) = \delta_{\{x : g_{i, n, w}(x) \leq 0\}}(x) \leq \delta_{\{x : g_i(x) \leq 0\}}(x) = f_i(x) \quad \text{for all } i \in V_5, x \in X_i.$$
which leads to (3.7b). Define the function $F_{n,w} : X_{[V∪E]} → ℝ ∪ \{∞\}$ to be

$$F_{n,w}((z_α)_{α ∈ \bar{E}∪V}) := -\frac{1}{2} \|\bar{x} - \sum_{α ∈ \bar{E}∪V} z_α\|^2 + \frac{1}{2}\|\bar{x}\|^2 - \sum_{α ∈ \bar{E}∪V} f^*_{α,n,w}(z_α).$$

(3.8)

By (3.7), $F_{n,w}(\cdot)$ is a lower approximate of $F(\cdot)$ for the maximization problem (3.3).

Based on our original motivation in Example 3.1 from [Pan18a, Pan18b], we make the following definition.

**Definition 3.2.** Let \( D := \cap_{α ∈ \bar{E}} H_α \). We say that a subset \( E' ⊂ \bar{E} \) connects \( V \) if

\[ \cap_{α ∈ E'} H_α = D. \] (3.9)

Since \( H_α \) were assumed to be linear subspaces, it is clear that condition (3.9) on \( E' \) is equivalent to

\[ \sum_{α ∈ E'} H^⊥_α = D^⊥. \] (3.10)

To provide more intuition, note that the set \( D \) defined through (3.9) has the simplifications

\[ D = \{ x ∈ [ℝ^m]^{|V|} : x = (x,x,\ldots,x) \text{ for some } x ∈ ℝ^m \}, \] (3.11a)

and

\[ D^⊥ = \{ x ∈ [ℝ^m]^{|V|} : \sum_{i ∈ V} [x]_i = 0 \}, \] (3.11b)

which are consistent with the usual product space formulation.

The following simple result was needed in [Pan18a] in order to show that the distributed Dykstra’s algorithm works for time varying graphs, but it is not needed here. Nevertheless, this result explains line 5 of Algorithm 3.4. The proof is exactly the same as in [Pan18c].

**Lemma 3.3.** [Pan18c] There is a constant \( c_{reg} > 0 \) such that for any \( v ∈ D^⊥ \) and any \( E' ⊂ \bar{E} \) such that \( E' \) connects \( V \), we can write \( v = \sum_{α ∈ E'} v_α \) so that \( v_α ∈ H^⊥_α \) and \( \|v_α\| ≤ c_{reg}\|v\| \) for all \( α ∈ E' \).

To simplify calculations, we let the vectors \( v_A, v_H \) and \( x \) in \( X \) be denoted by

\[ v_H = \sum_{α ∈ \bar{E}} z_α \] (3.12a)

\[ v_A = v_H + \sum_{i ∈ V} z_i \] (3.12b)

\[ x = \bar{x} - v_A. \] (3.12c)

We now state Algorithm 3.4 on the next page. Algorithm 3.4 calls on Algorithm 3.5 on page 8 as a subalgorithm.

**Remark 3.6.** (Intuition behind Algorithms 3.4 and 3.5) We summarize the intuition behind Algorithms 3.4 and 3.5. Dykstra’s algorithm is block coordinate ascent on the dual (3.3), and this is reflected in lines 7-14 of Algorithm 3.4. That is, find \( z ∈ X_{[E∪V]} \) that tries to improve the objective value of (3.4). As explained in [Pan18a], one only needs to keep track of \( x_i \) and \([z_i]_i \), for all \( i ∈ V \), and not all the variables. Line 5 corrects \( \{z_α\}_{α ∈ \bar{E}} \) so that the dual objective value remains the same, and this consideration is needed when we try to prove that the algorithm works for time-varying graphs. What is new in this paper is the consideration for \( i ∈ V \).
Algorithm 3.4. (Distributed Dykstra’s algorithm) Consider the problem \((3.2)\) along with the associated dual problem \((3.3)\).

Let \(\bar{w}\) be a positive integer. Let \(c_{\text{reg}} > 0\) satisfy Lemma 3.3. For each \(\alpha \in [\bar{E} \cup V]\setminus [V_4 \cup V_5]\), \(n \geq 1\) and \(w \in \{1, \ldots, \bar{w}\}\), let \(f_{\alpha,n,w}: \mathbb{X} \to \mathbb{R}\) be as defined in \((3.7)\).

Our distributed Dykstra’s algorithm is as follows:

01 Let

\[ z_i^{1,0} \in \mathbb{X} \] for each \(i \in V\) so that \([z_i^{1,0}]_j = 0 \in X_j\) for all \(j \in V\setminus\{i\} \).

02 For each \(i \in V_4\), let \(f_{i,1,0}: \mathbb{X} \to \mathbb{R}\) be a function such that \(f_{i,1,0}(\cdot) \leq f_{\cdot}(\cdot)\).

03 For \(n = 1, 2, \ldots\)

04 Let \(E_n \subset \bar{E}\) be such that \(E_n\) connects \(V\) in the sense of Definition 3.2.

05 Define \(\{z_{n,0}\}_{\alpha \in \bar{E}}\) so that:

\[
\begin{align*}
 z_{n,0}^\alpha &= 0 \text{ for all } \alpha \notin \bar{E}_n & (3.13a) \\
 z_{n,0}^\alpha &\in \mathcal{H}_\alpha^\perp \text{ for all } \alpha \in \bar{E} & (3.13b) \\
 \|z_{n,0}^\alpha\| &\leq c_{\text{reg}} \|v_{H}^{n,0}\| \text{ for all } \alpha \in \bar{E} & (3.13c) \\
 \text{and } &\sum_{\alpha \in \bar{E}} z_{n,0}^\alpha = v_{H}^{n,0}. & (3.13d)
\end{align*}
\]

(This is possible by Lemma 3.3)

06 For \(w = 1, 2, \ldots, \bar{w}\)

07 Choose a set \(S_{n,w} \subset \bar{E}_n \cup V\) such that \(S_{n,w} \neq \emptyset\).

08 If \(S_{n,w} \subset V_4 \cup V_5\), then

09 Apply Algorithm 3.3.

10 else

11 Set \(f_{i,n,w}(\cdot) := f_{i,n,w-1}(\cdot)\) for all \(i \in V_4\).

12 Define \(\{z_{n,0}^\alpha\}_{\alpha \in S_{n,w}}\) by

\[
\{z_{n,0}^\alpha\}_{\alpha \in S_{n,w}} = \arg\min_{z_\alpha, \alpha \in S_{n,w}} \frac{1}{2} \left\| \bar{x} - \sum_{\alpha \in S_{n,w}} z_{n,w-1}^\alpha - \sum_{\alpha \in S_{n,w}} z_\alpha \right\|^2 + \sum_{\alpha \in S_{n,w}} f_{n,w}(z_\alpha). 
\]

13 end if

14 Set \(z_{n,0}^\alpha := z_{n,w-1}^\alpha\) for all \(\alpha \notin S_{n,w}\).

15 End For

16 Let \(z_i^{n+1,0} = z_{i,\bar{w}}\) for all \(i \in V\) and \(v_{H}^{n+1,0} = v_{H}^{\bar{w}} = \sum_{\alpha \in \bar{E}} z_{n,\bar{w}}^\alpha\).

17 Let \(f_{i,n+1,0}(\cdot) = f_{i,n,\bar{w}}(\cdot)\) for all \(i \in V_4\).

18 Let \(g_{i,n+1,0}(\cdot) = g_{i,n,\bar{w}}(\cdot)\) for all \(i \in V_5\).

When \(i \in V_5\), we have \(f_i = \delta_{C_i}(\cdot)\), where \(C_i = \{x : g_i(x) \leq 0\}\). Since projection onto \(C_i\) may not be easy, we let \(f_{i,n,w}(\cdot) = \delta_{\{x : g_{i,n,w}(x) \leq 0\}}(\cdot)\), where \(g_{i,n,w}(\cdot)\) is affine. One can see that the projection onto the halfspace \(\{x : g_{i,n,w}(x) \leq 0\}\) is
Algorithm 3.5. (Subalgorithm for outer approximates of \( C_i \)) This algorithm is run when line 9 of Algorithm 3.4 is reached. Note that to get to this subalgorithm, \( S_{n,w} \subset V_4 \cup V_5 \). Suppose Assumption 3.8 holds.

00 For all \( i \in S_{n,w} \cap V_4 \), use steps in the corresponding algorithm in Pan18c.
01 For each \( i \in S_{n,w} \cap V_5 \)
02 For \( \tilde{g}_{i,n,w-1} : X_i \rightarrow \mathbb{R} \) defined by
\[
\tilde{g}_{i,n,w-1}(x) := g_i\left(\bar{x} - \mathbf{v}^{n,w-1}_H - \mathbf{z}^{n,w-1}_i \right)_i + \langle s, x - \bar{x} - \mathbf{v}^{n,w-1}_H - \mathbf{z}^{n,w-1}_i \rangle_i,
\] (3.15)
where \( s \in \partial g_i(\bar{x} - \mathbf{v}^{n,w-1}_H - \mathbf{z}^{n,w-1}_i)_i \), consider
\[
\min_{x \in X_i} \left[ \frac{1}{2} \|\bar{x} - \mathbf{v}^{n,w-1}_i \|_i - x\|_i \right] \text{ s.t. } g_{i,n,w-1}(x) \leq 0 \text{ and } \tilde{g}_{i,n,w-1}(x) \leq 0.
\] (3.16)
03 Let the primal of (3.16) be \( x^+_i \), and its dual solution \( \bar{x} - \mathbf{v}^{n,w-1}_H \|_i x^+_i = z^+_i \).
04 Define \( g_{i,n,w} : X_i \rightarrow \mathbb{R} \) to be the affine function
\[
g_{i,n,w}(x) := g_{i,n,w-1}(x^+_i) + \langle x - x^+_i, \bar{x} - \mathbf{v}^{n,w-1}_H \|_i x^+_i \rangle.
\] (3.17)
05 In other words, \( g_{i,n,w}(\cdot) \) is chosen such that the primal and dual optimizers to (3.16) coincide with that of
\[
\min_{x \in X_i} \left[ \frac{1}{2} \|\bar{x} - \mathbf{v}^{n,w-1}_i \|_i - x\|_i \right] \text{ s.t. } g_{i,n,w}(x) \leq 0.
\] (3.18)
06 Define the dual vector \( \mathbf{z}^{n,w}_i \in \mathbf{X} \) to be
\[
[\mathbf{z}^{n,w}_i]_j := \begin{cases} z^+_i, & \text{if } j = i \\ 0, & \text{if } j \neq i. \end{cases}
\] (3.19)
07 End for
08 For all \( i \in V_4 \setminus S_{n,w} \), \( f_{i,n,w}(\cdot) = f_{i,n,w-1}(\cdot) \).
09 For all \( i \in V_5 \setminus S_{n,w} \), \( g_{i,n,w}(\cdot) = g_{i,n,w-1}(\cdot) \).

Easier than that of \( C_i \). We have \( f_{i,n,w}(\cdot) \leq f_i(\cdot) \), which is also \( f^{*}_{i,n,w}(\cdot) \geq f^{*}_{i}(\cdot) \). We shall show that solving a sequence of problems involving \( f^{*}_{i,n,w}(\cdot) \) instead of \( f^{*}_{i}(\cdot) \) would minorize the objective value in (3.4) by (3.8), and allow the dual objective value to converge to its optimum value, which in turn leads to convergence of the primal to its minimizer.

The following result is essential for showing that the distributed Dykstra’s algorithm is asynchronous, and also in showing that the problems involving the nodes in \( i \in V \) are separable.

Proposition 3.7. (Sparsity of \( \mathbf{z}_n \)) We have \( [\mathbf{z}^{n,w}_i]_j = 0 \) for all \( j \in V \setminus \{i\} \), \( n \geq 1 \) and \( w \in \{0, 1, \ldots, \bar{w}\} \).

We state some notation necessary for further discussions. For any \( \alpha \in \bar{E} \cup V \) and \( n \in \{1, 2, \ldots, \} \), let \( p(n, \alpha) \) be
\[
p(n, \alpha) := \max\{w' : w' \leq \bar{w}, \alpha \in S_{n,w'} \}.
\] (3.20)
In other words, \( p(n, \alpha) \) is the index \( w' \) such that \( \alpha \in S_{n,w'} \) but \( \alpha \notin S_{n,k} \) for all \( k \in \{w' + 1, \ldots, \bar{w}\} \). We make three assumptions listed below.

Assumption 3.8. (Start of Algorithm 3.3) Recall that at the start of Algorithm 3.3 \( S_{n,w} \subset V_4 \cup V_5 \). We make three assumptions.
Whenever \((n,w)\) is such that \(w > 1\) and \(S_{n,w} \subset V_4 \cup V_5\) so that Algorithm \(3.5\) is invoked, each \(z_{i,n,w-1}^* \in \mathbf{X}_i\) where \(i \in V_4 \cup V_5\), is such that \([z_{i,n,w-1}^*]_i \in \mathbf{X}_i\) is the optimizer to the problem

\[
\min_{z \in \mathbf{X}_i} \frac{1}{2} \| (x - v_{H_i}^{n,w-1})_i - z \|^2 + f_{i,n,w-1}^*(z).
\]  

(3.21)

In other words, suppose \(w_i \geq 1\) is the largest \(w'\) such that \(i \in S_{n,w'}\) and \(i \notin S_{n,w}\) for all \(\bar{w} \in \{w' + 1, w' + 2, \ldots, w - 1\}\). Then for all \(\bar{w} \in \{w_i + 1, \ldots, w - 1\}\), and \(\alpha \in S_{n,\bar{w}}\), the condition \(v \in H_\alpha\) implies \([v]_i = 0\).

(2) Suppose that for all \(i \in V_4 \cup V_5\), \(\bar{w} \in \{p(n,i) + 1, \ldots, \bar{w}\}\) and \(\alpha \in S_{n,\bar{w}}\), the condition \(v \in H_\alpha\) implies \([v]_i = 0\). (This implies \(x_i^{n,p(n,i)} = x_i^{n,\bar{w}}\).)

(3) Suppose that \(S_{n,1} = V_4 \cup V_5\) for all \(n > 1\).

With these assumptions, we are able to prove the following. Even though the proof in [Pan18b] for the analogue of Theorem 3.9 below was for the case of Example 3.1, the proofs can be carried over in a straightforward manner.

**Theorem 3.9.** [Pan18b] (Convergence to primal minimizer) Consider Algorithm 3.4. Assume that the problem \((3.2)\) is feasible, and for all \(n \geq 1\), \(E_n = [\cup_{w=1}^{\bar{w}} S_{n,w}] \cap E\), and \([\cup_{w=1}^{\bar{w}} S_{n,w}] \supset V\). Suppose also that Assumption 3.8 holds.

For the sequence \(\{z_{i,n,w}^\alpha\}_{i \leq n < \infty} \in \mathbf{X}\) for each \(\alpha \in E \cup V\) generated by Algorithm 3.4 and the sequences \(\{v_{H_{i,n,w}}^\alpha\}_{i \leq n < \infty} \subset \mathbf{X}\) and \(\{v_{A_{i,n,w}}^\alpha\}_{i \leq n < \infty} \subset \mathbf{X}\) thus derived, we have:

(i) For all \(n \geq 1\) and \(w_1, w_2 \in \{1, \ldots, \bar{w}\}\) such that \(w_1 \leq w_2\),

\[
F_{n,w_1}(z_{i,n,w_2}) \geq F_{n,w_1}(z_{i,n,w_1}) + \frac{1}{2} \sum_{w' = w_1 + 1}^{w_2} \| v_{A_{i,n,w_2}} - v_{A_{i,n,w_1}} \|^2.
\]

Hence the sum \(\sum_{n=1}^{\infty} \sum_{w=1}^{\bar{w}} \| v_{A_{i,n,w}} - v_{A_{i,n,w-1}} \|^2\) is finite and \(\{F_{n,w}(\{z_{i,n,w}^\alpha\}_{\alpha \in E \cup V})\}_{n=1}^{\infty}\)

(ii) There is a constant \(C\) such that \(\|v_{A_{i,n,w}}\|^2 \leq C\) for all \(n \in \mathbb{N}\) and \(w \in \{1, \ldots, \bar{w}\}\).

(iii) For all \(i \in V_3 \cup V_4\), \(n \geq 1\) and \(w \in \{1, \ldots, \bar{w}\}\), the vectors \(z_{i,n,w}\) are bounded.

Recall that by the optimality of \(z_{i,n,p(n,i)}^*\) in \((3.14)\) and Proposition 3.7, we have

\[
[z_{i,n,p(n,i)}^*]_i = \arg\min_{z_i \in \mathbf{X}_i} \frac{1}{2} \| z_i - \left( [x_i]_i - \sum_{\beta \neq i} \alpha \beta \right) \|_2^2 + f_{i,n,p(n,i)}^*(z_i).
\]

(3.22)

We also have

\[
[x_{i,n,p(n,i)}^*]_i = \arg\min_{x_i \in \mathbf{X}_i} \frac{1}{2} \| x_i - x_i^{n,p(n,i)} \|^2 + f_{i,n,p(n,i)}(x_i).
\]

(3.23)

To see that \((3.23)\) holds, there are three cases to consider. The first case is when \((3.14)\) in Algorithm 3.4 is solved, in which case one can check that \((3.23)\) holds by looking at the \(i\)th component in \((3.14)\). The second case is when \((3.18)\) in Algorithm 3.5 holds (which is equivalent to \((3.23)\)), and the last case is the treatment for subdifferentials functions in the analogue of Algorithm 3.5 in [Pan18b].

Note that \((3.22)\) and \((3.23)\) can be considered primal-dual pairs. (The more accurate primal-dual pair is \((2.1)\), but it is clear that we can change the sign and add a constant to one of the problems to get the pair \((2.1)\).)
Let the prox center \( \hat{x}^{n,p(n,i)}_i \) be as marked in \([3.22]\), and let \( \hat{z}^{n,p(n,i)}_i \in X_i \) be
\[
\hat{z}^{n,p(n,i)}_i = \arg\min_{\hat{z}_i \in X_i} \frac{1}{2} \| \hat{z}_i - x_i^{n,p(n,i)} \|^2 + f_i^*(\hat{z}_i),
\] (3.24)
and let \( \tilde{z}^{n,p(n,i)}_i \in X \) be such that \( \hat{z}^{n,p(n,i)}_i = \hat{z}^{n,p(n,i)}_i \) and \( |\hat{z}^{n,p(n,i)}_i|_j = 0 \) if \( j \neq i \). Note that \( [3.24] \) is distinct from \( [3.22] \). If \( e \in \tilde{E} \), then \( \tilde{z}^{e,p(n,e)}_i = z^{e,p(n,e)}_i \). Let \( \tilde{x}^{n,p(n,i)}_i \in X_i \) be found by the dual to \( [3.24] \), i.e.,
\[
\tilde{x}^{n,p(n,i)}_i = \arg\min_{\tilde{x}_i \in X_i} \frac{1}{2} \| \tilde{x}_i - x_i^{n,p(n,i)} \|^2 + f_i(\tilde{x}_i),
\] (3.25)
The Moreau decomposition theorem can be used to prove that
\[
[x^{n,p(n,i)}]_i + [\hat{z}^{n,p(n,i)}]_i = \hat{x}_i^{n,p(n,i)} + \hat{z}_i^{n,p(n,i)} = \tilde{x}_i^{n,p(n,i)}.
\] (3.26)
Define \( \Delta x_i \in X_i \) by
\[
\Delta x_i := [x^{n,p(n,i)}]_i - \hat{x}_i^{n,p(n,i)} \quad \text{and} \quad [\hat{z}^{n,p(n,i)}]_i - [z^{n,p(n,i)}]_i \in X_i.
\] (3.27)
Note that this value was called \( \Delta z_i \) in \([Pan18c]\). We have
\[
\langle [x^{n,p(n,i)}]_i, [\hat{z}^{n,p(n,i)}]_i \rangle - \langle [z^{n,p(n,i)}]_i, [z^{n,p(n,i)}]_i \rangle = \langle [x^{n,p(n,i)}]_i, [\Delta z_i] \rangle + \frac{1}{2} \| \Delta x_i \|^2.
\] (3.27)
Define \( \hat{v}^{n,w}_A \in X \) like \( [3.12] \) to be
\[
\hat{v}^{n,w}_A := \sum_{\alpha \in V \cup E} \hat{z}^{n,w}_i.
\] (3.28)
Let \( h^{n,w} = -F^{n,w}(z^{n,w}) - (-F(z^*)) \), where \( z^* \in X^{V \cup \tilde{E}} \) is any optimal solution, and let \( h^n \) be defined by
\[
h^n := h^{n,0} = -F^{n,0}(z^{n,0}) - (-F(z^*)).
\] (3.29)
Note that \( h^n \geq 0 \). For the case \( i \in V_4 \), define \( \Delta f_i \in \mathbb{R} \) to be
\[
\Delta f_i := f_i([x^{n,p(n,i)}]_i) - f_i([x^{n,p(n,i)}]_i).
\] (3.30)
Formula \( [3.7b] \) implies \( \Delta f_i \geq 0 \) for all \( i \in V_4 \). We now recall some formulas proved in \([Pan18c]\).

**Proposition 3.10.** \([Pan18c]\) For each \( \alpha \in V \cup \tilde{E} \), the partial subdifferential of \( -F \) in the \( \alpha \)-th coordinate is, by \([Pan18c]\)
\[
\partial(-F)_{\alpha}(\hat{z}^{n,w}) = \sum_{\beta \neq \alpha} \hat{z}^{n,w}_\beta - \sum_{\beta \neq \alpha} z^{n,p(n,\alpha)}_{\beta}.
\] (3.31)
Let \( \hat{t}_\alpha \) and \( t_\alpha \) be as marked. Define \( \hat{t}_\alpha \in X \) to be
\[
\hat{t}_\alpha := \sum_{\beta \neq \alpha} \hat{z}^{n,w}_\beta.
\] (3.32)
Note the inequality \( \| t_\alpha - \hat{t}_\alpha \| \leq \sqrt{\hat{w}} \sqrt{h^n - h^{n+1}} \) proved in \([Pan18c]\). Define \( s \in X^{V \cap \tilde{E}} \) by \( s_\alpha = \hat{t}_\alpha - t_\alpha \). Let \( z^* \in X^{V \cup \tilde{E}} \) be any optimal solution to [3.3].
From (3.31), we have $s \in \partial(-F(\hat{z}^{n,\bar{w}}))$, and the techniques in Pan18c give

$$-F(\hat{z}^{n,\bar{w}}) - [-F(z^*)] \leq \sum_{\alpha \in V \cup E} \left[ \|t_\alpha - \hat{t}_\alpha\| + \|t_\alpha - \hat{t}_\alpha\| \|z^*_\alpha - \hat{z}^{n,\bar{w}}_\alpha\| \right].$$

$$\leq \sum_{\alpha \in V \cup E} \left[ \|t_\alpha - \hat{t}_\alpha\| \|z^*_\alpha - \hat{z}^{n,\bar{w}}_\alpha\| \right] + \sqrt{\bar{w}} \sqrt{n - h + 1} \sum_{\alpha \in V \cup E} \|z^*_\alpha - \hat{z}^{n,\bar{w}}_\alpha\| \tag{3.33}$$

From the fact that $\hat{z}^{n,\bar{w}}_i = z^{n,\bar{w}}_i$ for all $i \in V_1 \cup V_2 \cup V_3$, we have

$$-F^{n,\bar{w}}(z^{n,\bar{w}}) - F(\hat{z}^{n,\bar{w}}) \quad \tag{3.34}$$

$$\leq \sum_{i \in V_4 \cup V_5} \left[ \|x^{n,p(n,i)}_i - [z^{n,p(n,i)}_i]_i, \Delta x_i\| + \frac{1}{2} \|\Delta x_i\|^2 + \|f_i(\hat{x}^{n,p(n,i)}_i) - f_{i,n,p(n,i)}([x^{n,p(n,i)}_i]_i)\| \right]$$

$$-\langle \hat{x} - v^{n,\bar{w}}_A, (v^{n,\bar{w}}_A - \hat{v}^{n,\bar{w}}_A) \rangle - \frac{1}{2} \sum_{i \in V_4 \cup V_5} \|z^{n,\bar{w}}_i - \hat{z}^{n,\bar{w}}_i\|^2 \leq \sum_{i \in V_4 \cup V_5} \left[ \|x^{n,p(n,i)}_i - [z^{n,p(n,i)}_i]_i, \Delta x_i\| + \|f_i(\hat{x}^{n,p(n,i)}_i) - f_{i,n,p(n,i)}([x^{n,p(n,i)}_i]_i)\| \right]$$

$$+ \|\hat{x} - v^{n,\bar{w}}_A\| \|v^{n,\bar{w}}_A - \hat{v}^{n,\bar{w}}_A\|.$$

Also, the steps in Pan18c give

$$\|\Delta x_i\| \leq \sqrt{\Delta f_i} \quad \text{for all } i \in V_4. \tag{3.35}$$

We also showed in Pan18c that there is a bound $c'$ such that $\Delta f_i \leq c'$ for all $i \in V_4$, which implies that $\Delta f_i \leq \sqrt{c'} \sqrt{\Delta f_i}$ for all $i \in V_4$. Let $L$ be the Lipschitz constant of $f_i(\cdot)$. Thus

$$\left[ f_i(x^{n,p(n,i)}_i) - f_{i,n,p(n,i)}([x^{n,p(n,i)}_i]_i) \right] \quad \tag{3.30}$$

$$\Delta f_i + \left[ f_i(\hat{x}^{n,p(n,i)}_i) - f_{i,n,p(n,i)}([x^{n,p(n,i)}_i]_i) \right] \quad \tag{3.27}$$

$$\leq \Delta f_i + L \|\Delta x_i\| \leq \sqrt{c'} L \Delta f_i \leq (\sqrt{c'} + L) \sqrt{\Delta f_i} \quad \text{for all } i \in V_4.$$

So by the above inequality and (3.33), we have

$$\sum_{i \in V_4} \left[ \|x^{n,p(n,i)}_i - [z^{n,p(n,i)}_i]_i, \Delta x_i\| + \|f_i(\hat{x}^{n,p(n,i)}_i) - f_{i,n,p(n,i)}([x^{n,p(n,i)}_i]_i)\| \right]$$

$$\leq \sum_{i \in V_4} \left[ \|x^{n,p(n,i)}_i - [z^{n,p(n,i)}_i]_i, \Delta x_i\| + \sqrt{c'} + L \|\Delta x_i\| \right]. \tag{3.36}$$

4. PROOF OF CONVERGENCE

In this section, we present the proof of convergence rate for the distributed Dykstra’s algorithm.

We need a plane geometry result for our proof.

**Proposition 4.1.** Refer to Figure [4.1] for an illustration. Consider two points $\hat{x}$ and $\hat{x}$, and let $H$ be the halfspace $\{\hat{x} : \langle \hat{x} - \hat{x}, x - \hat{x} \rangle \leq 0\}$. Suppose that $x$ is such that $\hat{x}$ lies in the halfspace $H := \{x : \langle \hat{x} - x, x - \hat{x} \rangle \leq 0\}$. Then $\|x - \hat{x}\| \leq \|\hat{x} - \hat{x}\|$, and $d(x, H) \geq \frac{\|x - \hat{x}\|^2}{\|x - \hat{x}\|^2}$. 
Proof. It is clear that \( \hat{x} \in H \) implies \( \angle \hat{x} x \hat{x} \geq \pi/2 \), which is equivalent to \( x \) being in the sphere with diameter \( |\hat{x} - \hat{x}| \) and center \( \frac{1}{2}(\hat{x} + \hat{x}) \). Thus \( |x - \hat{x}| \leq |\hat{x} - \hat{x}| \).

For a fixed value of \( \|\hat{x} - x\|\), the smallest distance \( d(x, \hat{H}) \) occurs when \( x \) lies on the boundary of the sphere. Let the projection of \( x \) onto the line segment connecting \( \hat{x} \) and \( \hat{x} \) be \( x' \). The triangles \( \hat{x} \hat{x} x \) and \( \hat{x} x' \hat{x} \) are similar, which gives the lower bound \( \frac{\|x - \hat{x}\|}{\|\hat{x} - \hat{x}\|} \) for \( d \) as needed. \( \square \)

![Diagram for Proposition 4.1](image)

**Figure 4.1.** Diagram for Proposition 4.1

We write down the convergence result and its proof.

**Theorem 4.2.** Consider Algorithms 3.4 and 3.5. Suppose that Assumption 3.8 holds. Suppose the iterates \( \{z_{n,w}^{n,p(n,i)}\}_{n,w} \) are bounded for all \( \alpha \in V \cup E \), and that a minimizer \( z^* \in X^{V \cup E} \) exists. Suppose that the functions \( g_i : \beta_i \to \mathbb{R} \) are such that \( \min_{x \in \beta_i} g_i(x) < 0 \) for all \( \alpha \in V_5 \). Then the values \( \{h^n\}_n \) in (3.29), which measures the rate at which the dual objective value converges to its optimal value, converges to zero at an \( O(1/n^{1/3}) \) rate.

Proof. Recall the \( \bar{x}_{n,p(n,i)} \) in (3.22). We see that \( \bar{x}_{n,p(n,i)} \) is bounded. By Theorem 3.9(ii) and (3.12), and \( \{z_{n,p(n,i)}\}_n \) was assumed to be bounded, \( \bar{x}_{n,p(n,i)} \) is bounded. In view of (3.25), \( \bar{x}_{n,p(n,i)} \) is the projection of \( \bar{x}_{n,p(n,i)} \) onto \( C \). We then have \( \bar{x}_{n,p(n,i)} \) is bounded. In view of (3.25), \( \bar{x}_{n,p(n,i)} \) is the projection of \( \bar{x}_{n,p(n,i)} \) onto a superset of \( C \), and so Proposition 4.1 can be applied, with \( \bar{x} \) being \( \bar{x}_{n,p(n,i)} \), \( \hat{x} \) being \( \bar{x}_{n,p(n,i)} \) and \( x \) being \( \{x_{n,p(n,i)}\}_n \). This shows that \( \bar{x}_{n,p(n,i)} \) (defined in (3.27)) and \( z_{n,p(n,i)} \) are bounded for all \( \alpha \in V_5 \). From here, we deduce that \( \|z_{n,p(n,i)}\| \leq \|z_{n,p(n,i)}\| \) and \( \|\bar{x} - v_{A}^{n,w}\| \) are bounded for all \( \alpha \in V_5 \) and \( \alpha \in V \cup E \). Let \( z^* \in X^{V \cup E} \) be any optimal solution to (3.3). We then have \( \|z_{n,w} - z_{n,w}^{n,p(n,i)}\| \) being bounded for all \( \alpha \in V \cup E \).

The quantity \( \|v_{A}^{n,w} - v_{A}^{n,w}\| \) is bounded by a multiple of \( \sum_{i \in V_5} \|\Delta x_i\| + \sum_{i \in V_4} \|\Delta f_i\| \) by (3.27), (3.28), (3.12) and (3.35). We note that \( x_{n,p(n,i)} \) is bounded, and the discussion after (3.7), and \( \{x_{n,p(n,i)}\}_n \in \{x : g_i(x) \leq 0\} \) by (3.23) and the discussion after (3.7), so \( f_i(x_{n,p(n,i)}) = f_i(x_{n,p(n,i)}(x_{n,p(n,i)})) = 0 \) for all \( \alpha \in V_5 \), which gives

\[
\sum_{i \in V_5} \|f_i(x_{n,p(n,i)}) - f_i(x_{n,p(n,i)}(x_{n,p(n,i)}))\| \leq \sum_{i \in V_5} \|\Delta x_i\| + \|\Delta f_i\|
\]

(4.1)
Then there are constants $c_1$, $c_2$ and $c_3$ such that

$$h^{n+1} \leq [F^{n,w}(z^n,w) - F(z^n)] - [F(z^n) - F(z^n)]$$

$$\leq c_1 h^n + c_2 \sum_{i \in V_4} \sqrt{\Delta f_i} + \sum_{i \in V_5} c_3 \|\Delta x_i\|. \quad (4.2)$$

The next step is to show how $\|\Delta x_i\|$ can be related to the decrease in $\{h^{n,w}\}$. We want to show that

$$h^{n+1} \leq h^{n+1,0} - c_4 \sum_{i \in V_4} [\Delta f_i]^2 - \sum_{i \in V_5} c_5 \|\Delta x_i\|^2. \quad (4.3)$$

The dual objective function that we have at iteration $(n,w)$ is (3.8). Note that $S_{n+1,1}$ satisfies $S_{n+1,1} \subset V$ by Assumption [3.8](3), and $\delta_{H^n}(z^{n+1})$ is already finite for all $e \in E$, and by the sparsity of the $z_i$'s (Proposition [3.7](3)), maximizing (3.8) is equivalent to minimizing

$$\min_{|z_i| \in X_i, j \in V_4 \cup V_5} \sum_{i \in S_{n,w}} \left[\frac{1}{2} \|\bar{x}_i^{n+1,0} - [z_i]_i\|^2 + f_i^{n+1,0}(|z_i|)\right]. \quad (4.4)$$

where $\bar{x}_i^{n+1,0}$ be $|z_i| - \sum_{j \neq i} z_j^{n+1,0} = x_i^{n+1,0} + [z_i]_i$. We can look separately at the dual problems underbraced in the above problem. The dual of these problems are, up to a sign change and a constant,

$$\min_{x_i \in X_i} \frac{1}{2} \|\bar{x}_i^{n+1,0} - x_i\|^2 + f_i^{n+1,0}(x_i).$$

We now treat the case of $i \in V_5$. Recall that

$$[x_i^{n+1,0}]_i = \arg \min_{x_i \in X_i} \frac{1}{2} \|x_i - \bar{x}_i^{n+1,0}\|^2 + f_i^{n+1,0}(x_i), \quad (4.5a)$$

$$\dot{x}_i^{n+1,0} = \arg \min_{x_i \in X_i} \frac{1}{2} \|\dot{x}_i - \bar{x}_i^{n+1,0}\|^2 + f_i(\dot{x}_i), \quad (4.5b)$$

$$[x_i^{n+1,1}]_i = \arg \min_{x_i \in X_i} \frac{1}{2} \|x_i - \dot{x}_i^{n+1,0}\|^2 + f_i^{n+1,1}(x_i). \quad (4.5c)$$

Since $f_i^{n+1,0}$, $f_i$ and $f_i^{n+1,1}$ are indicator functions, the primal objective values of the problems are $\frac{1}{2} \|\dot{x}_i^{n+1,0} - [x_i^{n+1,0}]_i\|^2$, $\frac{1}{2} \|\dot{x}_i^{n+1,0} - \dot{x}_i^{n+1,0}\|^2$ and $\frac{1}{2} \|\dot{x}_i^{n+1,0} - [x_i^{n+1,1}]_i\|^2$ respectively. Let the halfspace $\dot{H}_i^{n+1}$ be

$$\dot{H}_i^{n+1} = \{x \in X_i : \langle x^{n+1,0}_i - \dot{x}_i^{n+1,0}, x - x^{n+1,0}_i \rangle \geq 0\}.$$ 

In view of Assumption [3.8](2), we have $\Delta x_i \leq [x_i^{n+1,0}]_i - \dot{x}_i^{n+1,0}$. Define $H_i^{n+1}$ to be the halfspace separating $[x_i^{n+1,0}]_i$ from the set $C_i$ formed in Algorithm 3.5. Recall that $\dot{x}_i^{n+1,0}$ and $[x_i^{n+1,0}]_i$ are bounded inside some set for all $i \in V$, say $\hat{R}$. Let $\gamma = \sup \{\|s\| : s \in \partial g_i(x), x \in V\}$ and $\hat{r} = \inf \{\|s\| : s \in \partial g_i(x), x \in V\}$, which both have to be finite numbers by arguments in Lemma 2.3. The boundedness of $\|\dot{x}_i^{n+1,0} - \dot{x}_i^{n+1,0}\|$ as mentioned earlier shows that there is a constant $c_5$ such that for all $n \geq 0$, $c_5 > 0$ and $\frac{\gamma}{\|x^{n+1,0}_i - \dot{x}_i^{n+1,0}\|} \geq c_5$. We have

$$d([x_i^{n+1,0}]_i, H_i^{n+1}) \geq \frac{\gamma}{\|x_i^{n+1,0} - \dot{x}_i^{n+1,0}\|} \geq c_5 \|\Delta x_i\|^2. \quad (4.6)$$
By Lemma 2.5, we have
\[ \|\bar{x}_{i}^{n+1,0} - [x_{n+1}^{1.0}]_{i}\|^2 \geq \|\bar{x}_{i}^{n+1,0} - [x_{n+1}^{1.0}]_{i}\|^2 + d([x_{n+1}^{1.0}]_{i}, H^{n+1})^2 \]
\[ \geq \|\bar{x}_{i}^{n+1,0} - [x_{n+1}^{1.0}]_{i}\|^2 + c_5 \|\Delta x_i\|^4. \] (4.7)

Note that \( f_{i,n+1,1}([x_{n+1}^{1.1}]_{i}) = f_{i,n+1,0}([x_{n+1}^{1.0}]_{i}) = 0 \), which gives the values of the optimization problems in (4.5). The strong duality between the primal problems of the type (4.5) and its dual (of the type (2.1)) implies that
\[ \frac{1}{2} \|\bar{x}_{i}^{n+1,0} - [x_{n+1}^{1.1}]_{i}\|^2 \]
\[ = \frac{1}{2} \|\bar{x}_{i}^{n+1,0} - [x_{n+1}^{1.0}]_{i}\|^2 + f_{i,n+1,1}([x_{n+1}^{1.1}]_{i}) \]
\[ = \frac{1}{2} \|\bar{x}_{i}^{n+1,0}\|^2 - \frac{1}{2} \|\bar{x}_{i}^{n+1,0} - [z_{i}^{n+1,1}]_{i}\|^2 - f_{i,n+1,1}([z_{i}^{n+1,1}]_{i}). \] (4.8)

There is a similar equation for \([z_{i}^{n+1,0}]_{i}\). Combining (4.7) and (4.8) gives us
\[ \frac{1}{2} \|\bar{x}_{i}^{n+1,0} - [z_{i}^{n+1,1}]_{i}\|^2 + f_{i,n+1,1}([z_{i}^{n+1,1}]_{i}) \]
\[ \leq \frac{1}{2} \|\bar{x}_{i}^{n+1,0} - [z_{i}^{n+1,0}]_{i}\|^2 + f_{i,n+1,0}([z_{i}^{n+1,0}]_{i}) - c_5 \|\Delta x_i\|^4 \] (4.9)
for the case when \( i \in V_1 \), an inequality similar to (4.9) was obtained in [Pan18c], but the last term would be replaced by \( c_4 |\Delta f_i|^2 \) instead, where \( c_4 > 0 \) is some constant. Summing up the inequalities of the form (4.9) over all \( i \) (note that if \( i \in V_1 \cup V_2 \cup V_3 \), \([z_{i}^{n+1,0}]_{i}\) = \([z_{i}^{n+1,1}]_{i}\)) and that the dual (3.8) can be written as the sum (4.4), we have (4.3) as needed.

We now consider 2 cases:

**Case 1:** If \( h^{n+1} \leq 2c_1 \sqrt{h^n - h^{n+1}} \), then choose \( \bar{c} > 0 \) such that \( h^n \leq \bar{c} \) for all \( n \geq 1 \). So
\[ h^n \geq h^{n+1} + \frac{1}{4c_1^2} [h^{n+1}]^2 \geq h^{n+2} + \frac{1}{4c_1^2} [h^{n+2}]^2 \geq h^{n+2} + \frac{1}{4c_1^2} [h^{n+2}]^4. \] (4.10)

**Case 2:** If \( h^{n+1} \geq 2c_1 \sqrt{h^n - h^{n+1}} \), then
\[ \sum_{i \in V_4} c_2 \sqrt{\Delta f_i} + \sum_{i \in V_5} c_3 \|\Delta x_i\| \geq h^{n+1} - c_1 \sqrt{h^n - h^{n+1}} \geq \frac{1}{2} h^{n+1}. \] (4.11)

By the power means inequality, we have
\[ c_4 \sum_{i \in V_4} |\Delta f_i|^2 + c_5 \sum_{i \in V_5} \|\Delta x_i\|^4 \geq \frac{1}{(|V_4| + |V_5|)^{1/4}} \left( \sum_{i \in V_4} |\Delta f_i|^{1/4} + \sum_{i \in V_5} \|\Delta x_i\|^{1/4} \right)^4 \] (4.12)

Incorporating (4.11) into (4.12) shows that there is some \( c_6 > 0 \) such that
\[ c_4 \sum_{i \in V_4} |\Delta f_i|^2 + c_5 \sum_{i \in V_5} \|\Delta x_i\|^4 \geq c_6 (h^{n+1})^4 \] (4.13)

Hence
\[ h^{n+2} \leq h^{n+1,0} \leq h^{n+1} - c_4 \sum_{i \in V_4} |\Delta f_i|^2 - c_5 \sum_{i \in V_5} \|\Delta x_i\|^4 \leq h^{n+1} - c_6 (h^{n+1})^4 \] (4.14)

Combining with (4.10), we have \( h^{n+2} \leq h^n - \min\{c_5, 4c_1^2 c_4^2\} [h^{n+2}]^4 \). By Lemma 2.6, this recurrence guarantees a \( O(1/n^{1/3}) \) convergence of \( \{h^n\}_n \). \( \square \)
5. Numerical experiments

In this section, we present the results of our numerical experiments to verify the effectiveness of Algorithm 3.3.

We conduct 4 different sets of numerical experiments, and we now explain their common features. Just like in Pan18c, we look at the graph where \(|V| = 5\) and \(|E| = \{(1,2), (1,3), (1,4), (1,5)\}\). We look at the setting of Example 3.1 where \(X_i = \mathbb{R}^m\) and \(m = 10\) for all \(i \in V\), and look at hyperplanes of the form

\[
H((i,j)) = \{x \in X : x_i = x_j\}
\]

instead of the hyperplanes \(H((i,j),k)\) defined in (3.5) to simplify computations. Our code is equivalent to \(\tilde{w} = 8\) with

\[
S_{n,1} = \{(1,2)\}, S_{n,2} = \{(1,2)\}, S_{n,3} = \{(1,3)\}, S_{n,4} = \{(1,3)\},
S_{n,5} = \{(1,4)\}, S_{n,6} = \{(1,4)\}, S_{n,7} = \{(1,5)\}, \text{ and } S_{n,8} = \{(1,5)\}.
\]

Let \(e\) be \(\text{ones}(m,1)\). First, we find \(\{v_i\}_{i \in V}\) and \(\bar{x}\) such that \(\sum_{i \in V} v_i + |V| (e - \bar{x}) = 0\). We then find closed convex functions \(f_i(\cdot)\) such that \(v_i \in \partial f_i(e)\). It is clear from the KKT conditions that \(e\) is the primal optimum solution to (3.6) if [\(x_i\)] are all equal to \(\bar{x}\) for all \(i \in V\).

The \(f_i(\cdot)\) can be defined as either smooth or nonsmooth functions, or as the indicator functions of level sets of smooth or nonsmooth functions. They are described using some Matlab functions below.

(F-S) \(f_i(x) := \frac{1}{2} x^T A_i x + b_i^T x + c_i\), where \(A_i\) is of the form \(v v^T + rI\), where \(v\) is generated by \(\text{rand}(m,1)\), \(r\) is generated by \(\text{rand}(1)\). \(b_i\) is chosen to be such that \(v_i = \nabla f_i(e)\), and \(c_i = 0\).

(F-NS) \(f_i(x) := \max\{f_{i,1}(x), f_{i,2}(x)\}\), where \(f_{i,j}(x) := \frac{1}{2} x^T A_i x + b_i^T x + c_{i,j}\) for \(j \in \{1,2\}\). \(A_i\) is of the form \(v v^T + rI\), where \(v\) is generated by \(\text{rand}(m,1)\), \(r\) is generated by \(\text{rand}(1)\). \(b_{i,1}\) and \(b_{i,2}\) are chosen such that \(v_i = \frac{1}{2} \nabla f_{i,1}(e) + \nabla f_{i,2}(e)\) but \(v_i\) is neither \(\nabla f_{i,1}(e)\) nor \(\nabla f_{i,2}(e)\), and \(c_{i,1}\) and \(c_{i,2}\) are chosen such that \(f_{i,1}(e) = f_{i,2}(e)\).

(LS-S) \(f_i(x) := \delta_{\{g_i(x) \leq 0\}}(\cdot)\), where \(g_i(x) := \frac{1}{2} x^T A_i x + b_i^T x + c_i\), \(A_i\) is of the form \(v v^T + rI\), where \(v\) is generated by \(\text{rand}(m,1)\), \(r\) is generated by \(\text{rand}(1)\), and \(b_i\) and \(c_i\) are chosen such that \(g_i(e) = 0\) and \(v_i = \nabla g_i(e)\).

(LS-NS) \(f_i(x) := \delta_{\{g_i(x) \leq 0\}}(\cdot)\), where \(g_i(x) := \max\{g_{i,1}(x), g_{i,2}(x)\}\), \(g_{i,j}(x) := \frac{1}{2} x^T A_{i,j} x + b_{i,j}^T x + c_{i,j}\) for \(j \in \{1,2\}\). \(A_{i,1}\) and \(A_{i,2}\) are of the form \(v v^T + rI\), where \(v\) is generated by \(\text{rand}(m,1)\), \(r\) is generated by \(\text{rand}(1)\). \(b_{i,1}\) and \(b_{i,2}\) are chosen such that \(v_i = \frac{1}{2} (\nabla g_{i,1}(e) + \nabla g_{i,2}(e))\) but \(v_i\) is neither \(\nabla g_{i,1}(e)\) nor \(\nabla g_{i,2}(e)\), and \(g_{i,1}(e) = g_{i,2}(e) = 0\).

Note that in (F-S) and (LS-NS), the \(b_{i,1}\) and \(b_{i,2}\), as well as \(c_{i,1}\) and \(c_{i,2}\), are not uniquely defined. We refer to the source code to see how they are defined. For all the experiments, we investigate the convergence behavior of \(\frac{1}{2} ||x - x^*||^2\) and the duality gap defined by

\[
\frac{1}{2} ||x^{n,w}||^2 + \sum_{i \in V} f_i^{n,w}(\mathbf{x}_i^{n,w}) - \left[\frac{1}{2} ||x^*||^2 + \sum_{i \in V} f_i^*(\mathbf{x}_i^*)\right],
\]

where \((x^*, z^*)\) is a dual optimal solution. It is known that the duality gap is an upper bound for \(\frac{1}{2} ||x - x^*||^2\), something which is verified in all our experiments.

For the first set of experiments, we choose \(f_i(\cdot)\) such that \(f_i(\cdot)\) are of the form (F-S) for all \(i \in V\).
For the second set of experiments, we choose $f_i(\cdot)$ to be of the form (LS-S) for $i \in \{2, 3\}$, and $f_i(\cdot)$ to be of the form (F-S) for $i \in \{1, 4, 5\}$.

For the third set of experiments, we choose $f_i(\cdot)$ to be of the form (LS-NS) for $i \in \{2, 3\}$, and $f_i(\cdot)$ to be of the form (F-S) for $i \in \{1, 4, 5\}$.

For the last set of experiments, we choose $f_i(\cdot)$ to be of the form (LS-NS) for $i \in \{2, 3\}$, and $f_i(\cdot)$ to be of the form (F-NS) for $i \in \{1, 4, 5\}$.

In all the sets of experiments, we experiment over the cases when all the $f_i(\cdot)$ marked to be in (F-S) or (F-NS) are either all treated as subdifferentiable functions, or all treated as proximable functions (i.e., either $V = V_4$, or $V = V_1$), and investigate the behavior of both the duality gap and $\frac{1}{2}\|x - x^*\|^2$.

We now elaborate on Figure 5.1. The two diagrams in Figure 5.1 show semi-log plots for the values of the duality gap and $\frac{1}{2}\|x - x^*\|^2$ when the functions are either all treated as subdifferentiable functions, or all treated as proximable functions, with the first diagram corresponding to experiment 1 and the second diagram corresponding to experiment 2. There is a (relatively fast) linear convergence of all values for the first set of experiments, and a (relatively slow) linear convergence for the second set of experiments. The former is consistent with the theory in [Pan18c], while the latter is much better than the $O(1/k^{1/3})$ rate that this paper suggests.

We now elaborate on Figure 5.2 which describes a typical output from the third set of experiments. In the second diagram, a plot of the reciprocal of the duality gaps for the cases when we treat the smooth functions as proximable and subdifferentiable functions gives straight lines, which shows an $O(1/k)$ convergence of the duality gap. This is better than the $O(1/k^{1/3})$ rate proved in this paper. In the third and fourth diagrams, the plots of $\left[\frac{1}{2}\|x - x^*\|^2\right]^{-1/2}$ look like a union of straight lines, which shows the $O(1/k^2)$ convergence of $\frac{1}{2}\|x - x^*\|^2$. This cannot
yet be explained by the theory in both this paper and [Pan18c], where the upper bound on the convergence rate we have is $O(1/k^{1/3})$. There is also no noticeable performance improvement if we treat the smooth functions as a proximable function instead of a subdifferentiable function for both the duality gap and $\frac{1}{2}\|x - x^*\|^2$.

Figure 5.2. Plots for Experiment 3.

We now elaborate on Figure 5.3, which describes a typical output from the fourth set of experiments. Similar $O(1/k)$ and $O(1/k^2)$ rates for the convergence of the duality gap and $\frac{1}{2}\|x - x^*\|^2$ are observed, though our theory so far gives only $O(1/k^{1/3})$ for both quantities, just like what we saw for experiment 3. In the second figure, the straight line and dashed line correspond to the case when we treat the nonsmooth functions as proximable and subdifferentiable functions respectively. Now that the functions $f_i(\cdot)$ are nonsmooth functions, it is now noticeable that if the nonsmooth functions were treated as proximable functions, the convergence of the duality gap and $\frac{1}{2}\|x - x^*\|^2$ to zero is faster than if the nonsmooth functions were treated as subdifferentiable functions.

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Fig. 5.3. Plots for Experiment 4.

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