The simplest map with three-frequency quasi-periodicity and quasi-periodic bifurcations

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Abstract

We propose a new three-dimensional map that demonstrates the two- and three-frequency quasi-periodicity. For this map all basic quasi-periodic bifurcations are possible. The study was realized by using method of Lyapunov charts completed by plots of Lyapunov exponents, phase portraits and bifurcation trees illustrating the quasi-periodic bifurcations. The features of the three-parameter structure associated with quasi-periodic Hopf bifurcation are discussed. The comparison with non-autonomous model is carried out.

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1 Introduction

Quasi-periodic oscillations are widespread in nature and technology [1] [2] [3] [4]. Examples of quasi-periodic behavior are found in electronics [5], neurodynamics [6], astrophysics [7], physics of lasers [8], geology [9] and other areas of science.

In the simplest case the quasi-periodic oscillations are characterized by the presence of two incommensurable frequencies. In the phase space attractors
corresponding to such oscillations have the form of invariant tori \[10, 11, 12\].
There are more complicated cases of greater number of incommensurable
frequencies - then multi-frequency oscillations are observed which correspond
to invariant tori of higher dimension. Invariant tori may undergo a variety
of bifurcations. In this case we speak about quasi-periodic bifurcations \[13,
14, 15\]. The basic ones are the next three bifurcations:

- the saddle-node bifurcation. Collision of stable and unstable tori leads
to the abrupt birth of higher dimensional torus.

- the quasi-periodic Hopf bifurcation. Torus of higher dimension is born
softly.

- the torus-doubling bifurcation.

Currently there are no reliable identification algorithms for such bifurcations
and their search is executed by means of the characteristic dependence of
Lyapunov exponents on parameter \[13, 14, 15\].

While examining the quasi-periodic bifurcations the selection of the model
for studying plays a crucial role. Some aspects can be studied based on non-
autonomous systems. However, such systems form separate, special class. As
concerning the known autonomous models, the number of low-dimensional
variants is not very large. Several examples of suitable generators are pro-
posed recently. The first example, obviously, is a Chua’s circuit, which is
described by specific piecewise linear characteristic \[16\]. In \[17\] a generator
based on the modified Bonhoffer - van der Pol system is offered. Also
we know a modification of climate Lorenz model – the Lorenz-84 low-order
atmospheric circulation model \[18\]. Another radiophysical example imple-
mented as a four-dimensional system is a modified Anishchenko-Astakhov
generator \[19, 20, 21\]. In \[22, 23, 24, 25\] a family of simple generators of
quasi-periodic oscillations described by three-dimensional models was sug-
gested and studied (including experiment). There have been studied cases
of autonomous dynamics as well as generator under external forcing and
dynamics of coupled oscillators \[20, 21, 26, 27\].

It is known that flow systems (differential equations) are quite difficult
for research, especially it concerns the study of so delicate effects like quasi-
periodic bifurcations. Therefore it is natural to deal with a simpler model
such as discrete maps. In this case the image of two-frequency oscillations in
the phase space is an invariant curve. As for flows the same object emerges
in Poincaré section of torus. (Thus frequently such invariant curves are also referred to as tori.) The higher is the map dimension the higher may be dimension of the torus. Most recently there have been undertaken appropriate investigations of model maps. In paper [28] the four-dimensional system of two coupled universal maps with Neimark-Sacker bifurcation is examined. In [29, 30] there was studied four-dimensional map representing two logistic maps with delay. The authors of the papers [31, 32] undertook analogous study of six-dimensional system in the form of three logistic maps with delay. In [33] the six-dimensional model which represents two coupled discrete versions of Rössler system is examined.

However, the dimension of above-mentioned systems is too large compared to the dimension which is formally necessary to observe two- and three-frequency quasi-periodicity. An exception is the paper [34] that study the three-dimensional system in the form of three coupled in a ring logistic maps with a specific coupling. Nevertheless, analysis of the problem in this paper seems insufficient. The noteworthy model from [13, 14, 15] was used for studying the quasi-periodic bifurcations, but its construction is quite complex formal procedure. Although the model describes the very important points, the discussed picture is still not quite complete. (The authors carried out only two-parameter analysis, at that they focused mainly on the description of resonance 1:5.)

It may be marked also the paper [35] where the coupled logistic maps with quasi-periodic forcing are studied. The research concerns bifurcations and mechanisms of transition to chaos through the destruction of three-dimensional torus. However, not all the essential points have been investigated in detail. Moreover, this model belongs to the class of non-autonomous systems, i.e. systems with external forcing.

Thus, there is a problem to construct autonomous models in the form of maps with the minimum necessary dimension equal to three that allows to study the properties of quasi periodic dynamics and quasi periodic bifurcations. In the present paper such model is represented by a discrete analogue of quasi-periodic generator [23, 24]. We undertake research of the structure of the parameter space focusing on the phenomena associated with quasi-periodic Hopf bifurcation.
2 Torus map construction

Let us consider the generator of quasiperiodic oscillations [23, 24]:
\[
\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = 0,
\]
\[
\dot{z} = b(\varepsilon - z) - k\dot{x}^2.
\]
(1)

The multiplier ahead of the derivative \( \dot{x} \) contains the parameter \( \lambda \) which characterizes the depth of a positive feedback in the oscillator, the nonlinear term \( x^2 \) stimulating the excitation of oscillations and the term \( x^4 \) responsible for the saturation of the oscillations at large amplitudes. The dynamics of the variable \( z \) can occur linearly at a speed \( b \) or undergo the nonlinear saturation due to the term \( k\dot{x}^2 \).

Figure 1: Lyapunov chart for generator of quasi-periodic oscillations (1), \( b = 1, \varepsilon = 4, k = 0.02, \omega_0 = 2\pi \).

The model (1) has an equilibrium \( x_0 = y_0 = 0, z_0 = \varepsilon \). It is easily verified that this equilibrium undergoes Andronov-Hopf bifurcation AH at \( \lambda = -\varepsilon \) leading to the birth of a limit cycle. With increasing of parameter \( \lambda \) the Neimark-Sacker bifurcation NS consisting in the birth of a two-dimensional torus is possible. In Fig.1 we demonstrate the Lyapunov chart for system (1) where different colors indicate the areas of periodic regimes \( P \), the two-frequency quasi-periodicity \( T_2 \) and chaos \( C \). We can see the curve of Neimark-Sacker bifurcation NS with the adjoined set of Arnold tongues immersed in the region of quasi-periodic oscillations.
Let us rewrite equations (1) in the form of first-order system
\[
\dot{x} = y, \\
\dot{y} = (\lambda + z + x^2 - \beta x^4)y - \omega^2_0 x, \\
\dot{z} = b(\varepsilon - z) - ky^2,
\]
and construct a discrete analog of equations (2). For this purpose we use a substitution for time derivatives by finite differences similarly to [36, 37, 38]. The transition to finite differences will provide some additional characteristic time scale – a discretization step, which usually leads to new types of dynamics. The result is a model that we call a torus map:
\[
x_{n+1} = x_n + h \cdot y_{n+1}, \\
y_{n+1} = y_n + h \cdot ((\lambda + z_n + x_n^2 - \beta x_n^4)y_n - \omega^2_0 x_n), \\
z_{n+1} = z_n + h \cdot (b(\varepsilon - z_n) - ky^2_n).
\]
Here \(h\) is discrete time step. Note that for the first equation we use semi-explicit Euler scheme, i.e. we take the value of the variable \(y\) in \((n+1)\)-th moment. This discretization usually leads to more physically correct models [39].

3 Properties of torus map with quasi-periodic dynamics

Let us study the obtained map. We use the same set of parameters as for Fig.1 and will gradually increase the discretization parameter \(h\). In the center of Fig.2 a numerically calculated Lyapunov chart for system (3) at the value \(h = 0.05\) is shown. Different colors in the chart denote the following regions defined in accordance with the spectrum of Lyapunov exponents \(\Lambda_1, \Lambda_2, \Lambda_3\):

a) \(P\) – periodic regimes (cycles), \(\Lambda_1 < 0, \Lambda_2 < 0, \Lambda_3 < 0\);
b) \(T_2\) – two-frequency quasi-periodicity, \(\Lambda_1 = 0, \Lambda_2 < 0, \Lambda_3 < 0\);
c) \(T_3\) – three-frequency quasi-periodicity, \(\Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 < 0\);
d) \(C\) – chaotic regimes, \(\Lambda_1 > 0, \Lambda_2 < 0, \Lambda_3 < 0\);
e) \(HC\) – hyperchaotic regimes, \(\Lambda_1 > \Lambda_2 > 0, \Lambda_3 < 0\);
f) \(D\) – a divergence of trajectories.

Due to the smallness of the discretization parameter \(h\) the structure of the chart is partly qualitatively similar to that of the original flow system (1). (The smaller \(h\) the better an approximation to the flow system.) However
the discretization leads to replacement of periodic regimes of flow system \[1\] by two-frequency regimes and two-frequency by three-frequency ones in the Fig.1. Thus in Fig.2 we can see a picture of tongues of two-frequency regimes that in configuration are similar to traditional Arnold tongues. The mentioned set of tongues immerses in a region of three-frequency tori. The tongues in Fig.2 correspond to resonant two-frequency tori lying on the surface of three-frequency torus.

In Fig.2 we show examples of phase portraits in the various points of the parameter plane. Below and right to the line QH torus looks like a simple oval. In the three-frequency region this oval is smeared. Inside the tongues of two-frequency regimes the attractors have the form of closed invariant curves. In different tongues on the plain \((x, y)\) these curves differ in the number of turns around the origin. Such curves are replacing the simple limit cycles in

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Figure 2: Lyapunov chart of the torus map \(3\) and typical phase portraits. \(QH\) is a line of quasi-periodic Hopf bifurcation. Discretization parameter value \(h = 0.05\).
the Poincaré section of the original model (1) and their shape indicates that discretization even with a small step $h$ leads to a specific modification of the observed structure.

In turn inside tongues of two-frequency tori the areas of periodic regimes arise. In this case on the invariant curve of complex shape there is a set of points of appropriate long-period cycle, i.e. such regimes are resonant with respect to the corresponding two-frequency torus.

![Figure 3: Three Lyapunov exponents of the model (3) versus parameter $\lambda$ (a) and the magnified fragment (b). It is showed the location of quasi-periodic Hopf bifurcation point QH and saddle-node torus bifurcations SNT. Discretization parameter $h = 0.05$, parameter $\beta = 0.057$.](image)

Let us now discuss bifurcations of quasi-periodic regimes. With this goal we turn to Lyapunov exponent plots in Fig.3 calculated along selected in Fig.2 line $\beta = 0.057$. Such line crosses three-frequency periodicity region from the bottom to the top. At the point $QH$ a two-frequency torus ($\Lambda_1 = 0$) undergoes bifurcation. As can be seen from Fig.3, the feature of this bifurcation is that below the threshold the exponents $\Lambda_2$ and $\Lambda_3$ are equal, $\Lambda_2 = \Lambda_3$. At the bifurcation point both these exponents vanish. Beyond the
bifurcation point exponents do not coincide: the second exponent is equal to zero, $\Lambda_2 = 0$, and the third one becomes negative, $\Lambda_3 < 0$. Exactly at the point of bifurcation the condition $\Lambda_1 = \Lambda_2 = 0$ is fulfilled and a three-frequency torus emerges. This bifurcation is called quasi-periodic Hopf bifurcation $QH$. Its distinguishing feature is the realization of criterion of coincidence of two exponents beyond the bifurcation point [13, 15].

With increasing of control parameter $\lambda$ the route $\beta = \text{const}$ on the chart of Fig.1 crosses many tongues of two-frequency tori. On the plot of Fig.3a such tongues manifest themselves as dips in the graphs of the second exponent. The boundaries of these areas are formed by lines of saddle-node torus bifurcation SNT. One of the deepenings in the enlarged view is demonstrated in Fig.3b. Distinct characteristic of SNT bifurcation is that the second Lyapunov exponent $\Lambda_2$ vanishes, but values $\Lambda_2, \Lambda_3$ are not equal to each other [13, 15]. Herewith the third exponent $\Lambda_3$ remains always negative (see Fig.3b). On the other side of the tongue, such a bifurcation takes place in reverse order.

Accordingly we can indicate the curve of quasi-periodic Hopf bifurcation $QH$ in Fig.2 separating three-frequency and two-frequency regions. In the discrete model (3) such a line replaces the line of Neimark-Sacker bifurcation NS in the flow-prototype (1) displayed in Fig.1.

It should be noted that quasi-periodic Hopf bifurcation is essentially three-parameter phenomenon in contrast to the traditional two-parameter Neimark-Sacker bifurcation. The physical nature of this fact is explained by adding of parameter associated with additional frequency. In model (3) such a parameter can be a discretization step $h$, which is responsible for an additional time scale. Therefore, we will increase the parameter $h$ and trace the emerging changes in the structures of domains on the examined parameter plane.

Let us discuss the case $h = 0.1$, Fig.4. There are significant qualitative changes on the parameter plane. Distinctive set of Arnold-type tongues is disappeared. From the line of a quasi-periodic Hopf bifurcation $QH$ the bands of two-frequency regimes are issued. The crosswise bands of periodic regions (exact resonances) are built in these two-frequency areas. On the enlarged part of Lyapunov chart, Fig.4b, we see that mentioned resonances generate secondary sets of the fan-shaped two-frequency tongues, immersed in a three-frequency region.

To better visualize and distinguish periodic regimes we mark by different colors and numbers the cycles of various periods (Fig.4c). Gray color corre-
Figure 4: Lyapunov chart of torus map (3) for parameter of discretization $h = 0.1$ (a), its enlarged segment (b), the chart of regimes of torus map (c).

It corresponds to all the non-periodic regimes. It can be seen that the built-in areas of periodic regimes have different periods, at that their value is big enough.
One more representative fact in Fig.4 consists in approaching of period-10 tongue to discussed two-frequency band from the main two-frequency area as described in [14].

In Fig.5 we demonstrate the examples of phase portraits. There the cycle of period 10 can be seen and its transformations within corresponding two-frequency resonance region. An emergence of small isolated ovals may be observed around the elements of the period-10 cycle. Cycles of very high periods inside narrow regions of periodic regimes are very typical. Thus, the structure of parameter plane is different from Fig.2.

![Figure 5: Attractors of torus map (3). Parameter of discretization \( h = 0.1 \).](image)

Plots of Lyapunov exponents calculated along vertical line in Fig.5 are shown in Figure 6. In this case we observe a quasi-periodic Hopf bifurcation \( \text{QH} \). Another illustration of such bifurcation is a bifurcation tree presented in Fig.6b in appropriate scale. One can note "smeary" crown of the tree that signalized about quasiperiodic dynamics. At the point of quasi-periodic Hopf
bifurcation QH we observe the widening of bifurcation tree, and it occurs in soft way.

Figure 6: Dependence of three Lyapunov exponents on parameter $\lambda$ (a) and bifurcation tree (b), numerically calculated for the model (3). The point of quasi-periodic Hopf bifurcation QH is marked. Discretization step $h = 0.1$, parameter $\beta = 0.081$.

Figure 7 presents the enlarged fragments of Fig.6. We can see a strong irregularity of diagrams due to the increasing complexity of parameter plane. There are many alternating regions of two-frequency quasi-periodic and periodic regimes. Nevertheless, there are distinguished areas of resonance tori bounded by saddle-node bifurcation points SNT.

In Fig.7b we show the part of bifurcation tree to conclude that in contrast to the points QH at saddle-node torus bifurcation SNT points the expansion of the crown occurs abruptly. It happens due to the nature of the bifurcation -
Figure 7: Enlarged fragment of Fig.6. Points of saddle-node torus bifurcation SNT and typical phase portraits are indicated.

stable and saddle two-frequency tori collide and three-frequency torus occurs abruptly \[13, 15\]. On the bifurcation tree it is also clearly seen resonance window of periodic regime.

Chart of Lyapunov exponents for greater value of parameter \( h \) is shown in Fig.8. Now the band of period 10 is not visible. The bottom of the largest two-frequency tongue is limited by the bifurcation line, at that the corre-
sponding Lyapunov exponent vanishes. On the chart this line appears as a thin blue strip. Its type can be determined via studying the phase portraits: this is a torus-doubling bifurcation \[10, 15\]. Indeed, instead of a single oval we have two overlapping ovals. They are obtained by projection of two isolated closed invariant curves laying in three-dimensional space \((x, y, z)\) on the surface of torus. The phase portrait in Fig.8 illustrates another such bifurcation for already doubled torus. Thus, at increasing of the discretization parameter a new transformation occurs in comparison with Fig.4.

Figure 8: Chart of Lyapunov exponents and phase portraits of torus map \(3\) at \(h = 0.16\).

4 Non-autonomous model with quasi-periodic dynamics

It is interesting to compare the studied dynamics with the case of non-autonomous systems. It is necessary to choose such a system to investigate
that demonstrates two-frequency quasi-periodicity and Neimark-Sacker bifurcation. An appropriate example is a universal map \[28\] for which there are discovered all main bifurcation scenarios of two-dimensional maps:

\[
\begin{align*}
    x_{n+1} &= Sx_n - y_n - (x_n^2 + y_n^2), \\
    y_{n+1} &= J + x_n - \frac{1}{5}(x_n^2 + y_n^2).
\end{align*}
\]

In Fig.9 we demonstrate a diagram for the universal map \[28\], which combines the properties of Lyapunov exponent chart and chart of periodic regimes. It contains Neimark-Sacker bifurcation line NS, \(J = 1\), and a set of Arnold regular tongues.

Let us use the next form of external driving:

\[
\begin{align*}
    x_{n+1} &= Sx_n - y_n - (x_n^2 + y_n^2), \\
    y_{n+1} &= J + x_n - \frac{1}{5}(x_n^2 + y_n^2) + \varepsilon \cos 2\pi \theta_n, \\
    \theta_{n+1} &= w + \theta_n \mod 1.
\end{align*}
\]

A parameter determining the frequency of driving force is equal to the golden mean, \(w = \frac{\sqrt{5} - 1}{2}\), that provide the quasi-periodic type of observed dynamics.
Lyapunov chart of model \((5)\) is shown in Fig.10. Compared with Fig.9 the periodic regimes are replaced by two-frequency regimes, and two-frequency regimes becomes three-frequency ones in Fig.10. Also we can see bands of two-frequency modes replacing the tongues as well as typical ovals in bottoms of some of the wide tongues corresponding to torus-doubling bifurcations. However, a significant difference from the dynamics discussed above is the lack of periodic resonances within tongues. Accordingly, there are no secondary two-frequency resonances in their neighborhood.

Thus, the non-autonomous systems exhibit some characteristics that are typical for autonomous models with quasi-periodicity, but fundamental differences are inevitable.

![Figure 10: Lyapunov chart of universal two-dimensional map \((5)\) with quasi-periodic external force. Parameter \(\epsilon = 0.1\).](image)

5 Conclusion

Thus, the substitution of time derivatives by finite differences in the equation of generator of quasi-periodic oscillations \((1)\) provides a new convenient model in the form of three-dimensional map. This map demonstrates the regimes of two-frequency and three-frequency quasi-periodicity and all basic
quasi-periodic bifurcations. Typical Lyapunov exponent plots and bifurcation trees are presented. Such map allows to study three-parameter structure of quasi-periodic Hopf bifurcation. With variation of third parameter there are significant changes in the structure of parameter plane and the form of phase portraits. The fundamental difference from non-autonomous systems with quasi-periodic forcing is the possibility of various periodic resonances and associated with them secondary tongues of two-frequency regimes.

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