Quantum causality determines the arrow of time

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In the philosophy of science, the origin of an arrow of time is viewed as problematic. We describe here how this arrow really follows from the causal structure of quantum physics. This connection is not really new - it is just overlooked in most discussions of the arrow of time.

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I. QUANTUM CAUSALITY

Eddington introduced the concept of the arrow of time - the one way flow of time as events develop and our perceptions evolve. He pointed out that the origin of such an arrow appears to be a mystery in that the underlying laws of physics (at least at the time of Eddington) are time symmetric and would work equally well if run in the reverse time direction. The laws of classical physics follow from the minimization of the action and are indeed time symmetric. This view has been beautifully captured by Carlo Rovelli [1], who writes: “The difference between past and future, between cause and effect, between memory and hope, between regret and intention…in the elementary laws that describe the mechanisms of the world, there is no such difference.” But what picks out only those solutions running forward in time?

By now, there is a large literature on the arrow of time [1–10]. Essentially all of the literature accepts the proposition that the fundamental laws of physics do not distinguish between past and future and could equally well be run backwards. There is also a recognition that the second law of thermodynamics does distinguish between these directions as it states that entropy cannot decrease in what we refer to as the future. This leads to the idea of a thermodynamic arrow of time. Many view this thermodynamic arrow as the origin of the passage of time, or at least of our consciousness of that passage.

Our point in this paper is that the basic premise of such reasoning is not valid in quantum theory. Quantum physics in its usual form has a definite arrow of causality - the time direction that causal quantum processes occur. This can be concealed by conventions for how we count the flow of time, and associated conventions in quantum foundations. However when the phrase “fundamental laws of physics” includes the rules for quantization, there is always only one time direction whose flow is compatible with quantum processes.

The connection is most obvious in quantum field theory where causality has long played a foundational role [1]–[15]. This is expressed compactly in Sec. II. In standard treatments of quantum mechanics causality is less obvious. But quantum field theory and quantum mechanics are clearly different facets of the same theory and one can also find evidence of causality in the latter. While none of the physics which we are discussing is new, to the best of our knowledge the disentangling of real causal physics from conventions has not been presented before in connection with the arrow of time. The discussion here is an elaboration of the comments which we presented in Ref. [16].

An example will illustrate our main point. Consider a comparison of reactions of a style which are often mentioned as a test of time-reversal invariance, which we will imagine proceeding through a long-lived resonance,

\[
A + B \rightarrow R \rightarrow C + D \quad \text{vs} \quad C + D \rightarrow R \rightarrow A + B . \quad (1)
\]

Here A, B, C, D are particles and R is a resonance. These reactions are related by time reversal, as their matrix elements are complex conjugates of each other. It is often colloquially said that the second reaction is the backwards-in-time version of the first one. However, in reality both proceed forwards in time and the final state products emerge from the long-lived resonance only after a positive time. The equality of the transition probabilities is a consequence of the time reversal invariance of the underlying Lagrangian, but the fact that reactions run forward in time is a property of the arrow of causality of quantum physics. That this example invokes a long-lived resonance is not essential - it is done only to make the time ordering of the process manifestly evident. In reality all scattering reactions carry the same causal ordering.

This occurs because quantum physics is more than just the Lagrangian, but also involves the quantization procedure. The causal ordering can be traced back to factors of i which appear in the quantization rules. These then tell us what we mean by positive energy for wavefunctions, and how positive energy propagates. Since the initial state particles in both of these reactions carry positive energy, their behavior is governed by these propagators. Because time reversal symmetry is anti-unitary (that is, it involves complex conjugation) it reverses all factors of i. The Lagrangian can be invariant under this operation, but the quantization procedure is not. If we change the
factors of $i$, then the causal direction would shift. However, the key feature is that there is one causal direction singled out.

Conventions play a role in all such discussions. In what might initially seem like a joke, we can appear to make these reactions “run backwards in time” by using a clock which counts down instead of up. We can make a clock which runs backwards. Or when using an hourglass the lower chamber reads time by an increasing amount of sand while the upper chamber reads time by a decreasing amount of sand. If we do this, then our resonance reactions will proceed in negative directions of this new time. This is less of a joke than it seems because the reversing of time is mathematically described by a substitution of a reversed time variable, i.e. $\tau = -t$, which is the substitution highlighted by the classical arrow of time puzzle. We need to be able to differentiate between having this substitution corresponding to just running our clocks backwards, and having physical processes running both directions in time.

The distinction highlighted by this “joke” illustrates the difference between the arrow of causality and the arrow of time. There is a convention associated with the measurement of time, which is not a relevant distinction. However, what we really observe is that all elementary processes run in one direction only - that is causality. One can always choose to align one’s clock in the direction of the causal action. But one does not find elementary processes running in both causal directions. It is for this reason that it is better to refer to an arrow of causality rather than an arrow of time.

This analysis will tell us that thermodynamics is not the origin of the arrow of time. The elementary processes of scattering and decay in isolation have no thermodynamic description, yet intrinsically display an arrow of causality. Indeed, the logic is the reverse. If all the elementary processes themselves carry an intrinsic arrow of causality, then the increase in entropy will also carry this arrow. The quantum causal arrow will necessarily induce the thermodynamic arrow. It is interesting that the macroscopic and classical arrow of time follows from the underlying causal quantum processes.

II. THE LAWS OF PHYSICS

When we talk about the fundamental “laws of physics”, what do we mean? For classical physics, it is best to describe these by the Principle of Least Action and the use of Lagrangians defining the action. This is because Noether has taught us that invariances of the action reveal symmetries of the world. The Lagrangians of the fundamental interactions are (mostly) invariant under time reversal $^1$. Simply from this fact alone, we know that solutions to the resulting classical equations of motion will then be equally possible in both time directions.

However in quantum physics the “laws of physics” not only include the Lagrangians and their subsequent equations of motion, but also the quantization procedures. There are various ways to describe quantization which have evolved over the years and which are taught at different levels of physics instruction. In describing the fundamental interactions in quantum field theory, the most advanced of these is that of path integral quantization. In path integral treatments, the action and the quantization are combined into a single object, which roughly is

$$\text{Sum over all fields/paths } \times e^{i\hbar S} \quad (2)$$

The factor of $i/\hbar$ in the exponential will prove to be important. Path integrals also provide the simplest formulation of the origin of the arrow of causality. The action defining the fundamental interactions may be time-reversal invariant, but the full path integral defining the quantum theory is not. The reactions that we find in the world will then exhibit a particular time ordering.

In the canonical quantization method using commutation relations, the fundamental action and the quantization procedures seem to be two separate things. We get equations of motion from the action and use the commutation rules to define the states. But the phrase “laws of physics” must include both. We will see that factors of $ih$ are also important in this regard. Here the statement is: The equations of motion may be time-reversal invariant but the quantization procedure is not. Probably the focus on the equations of motion alone and not on the quantization procedures is what has led astray many of the discussions of the arrow of time.

As we proceed, it is important to separate what is convention from what is the intrinsic physics. The bedrock convention needed here is that kinetic energy is positive. There is no point to entertaining any other convention. The rest mass and total energy of particles is then also positive. In our sample reactions of Eq. [1] both initial states come with positive energy and the resonance $R$ has an energy which is sum of the initial energies. We will be interested in following what happens to positive energy states.

III. PATH INTEGRAL QUANTIZATION

We start with the path integral treatment of quantum field theory, because this is the most direct illustration of causality. The reader who is not comfortable with

$^1$ There can be small violations of time reversal symmetry in the Standard Model Lagrangian, but this has nothing to do with the arrow of time.
quantum field theory may read this section by focussing on what happens to the flow of positive energy solutions. We will subsequently turn to canonical quantization and a discussion of conventions in order to extract the same physics.

Causality has been a major theme of quantum field theory from its early days [11][15]. The causal structure was used to formulate dispersion relations, and the analytic properties of amplitudes were heavily explored. One of the conditions for causal behavior is that commutators of field operators must vanish for spacelike separations, such that causal influences are restricted to the past lightcone. However, this also requires the differentiation of past and future. The lesson of past work is that the direction of causality is encoded in a subtle feature of the particle propagation, namely the factors of $i\epsilon$ (with $\epsilon$ being an infinitesimal positive quantity) which determine the analytic structure of amplitudes.

The Feynman propagator governs the exchanges of particles in scattering amplitudes. With standard conventions, in momentum space it has the form (with $\hbar = c = 1$)

$$i D_F(q) = \frac{i}{q^2 - m^2 + i \epsilon}$$  \hspace{1cm} (3)

where our metric for four-vectors is $(+, -, -, -)$. The $i\epsilon$ is important. When we Fourier transform the propagator in time

$$i D_F(x) = D_F^{\text{for}}(x)\theta(t) + D_F^{\text{back}}(x)\theta(-t)$$  \hspace{1cm} (4)

with

$$D_F^{\text{for}}(x) = \int \frac{d^3q}{(2\pi)^3 2E_q} e^{-i(E_q t - \mathbf{q} \cdot \mathbf{x})}$$  \hspace{1cm} (5)

with $E_q = \sqrt{\mathbf{q}^2 + m^2}$ and $D_F^{\text{back}}(x) = (D_F^{\text{for}}(x))^*$. For our purposes here the most important aspect is that positive energy is, i.e. $e^{-iE_q t}$, is propagated forward in time. The scattering of positive-energy particles then carries this directionality.

If the propagator describes an unstable resonance, as in our example of Sec. 1, there is a finite imaginary part, $i\epsilon \rightarrow i\gamma$ with $\gamma > 0$, such that

$$i D_F(q) = \frac{i}{q^2 - m^2 + i \gamma} = \frac{i}{q^2 - (M - i\Gamma/2)^2}$$  \hspace{1cm} (6)

where $\gamma = M \Gamma$. The propagation forward in time exhibits exponential decay

$$D_F^{\text{for}}(x) = \int \frac{d^3q}{(2\pi)^3 2E_q} e^{-i(E_q t - \mathbf{q} \cdot \mathbf{x})} e^{-\gamma t/2E_q}$$  \hspace{1cm} (7)

Again it is positive energy which is propagated forward in time.

The propagator demonstrated the existence of an arrow of causality. If we look at the reactions given in Eq. 1 both initial states carry positive energy. The scattering reaction then has a causal direction for the flow of that energy. When there is a long lived resonance involved, the resonance excitation, as in Fig. 1, propagate the positive energy forward in time until it decays. This is the physical origin of the time asymmetry which we described in the introduction.

However, we need to understand where this propagator came from, and why there is this directionality. In a path integral treatment we quantize using $\exp i S$ where $S$ is the action. For a scalar field in the presence of a source $J(x)$ we use

$$Z_+[J] = \int [d\phi] e^{iS(\phi, J)} = \int [d\phi] e^{i \int d^4x \left( \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) + J \phi \right)}.$$  \hspace{1cm} (8)

It will be seen that this is the origin of the usual propagator. However, we could also consider the case with the...
opposite factor of $i$

$$Z_-[J] = \int [d\phi] e^{-i\mathcal{S}(\phi, J)} \tag{2}.$$  

(9)

What happens when we compare these two choices?

In these path integrals, we are confronted with Gaussian integrals, but with an imaginary argument. In order to make this well-defined we add a term $\pm i\phi^2/2$ to the Lagrangian density. When combined with the overall factor of $\pm i$, this provides a damping factor $\sim e^{-\epsilon \int d^4x \phi^2/2}$ in the path integral. The two path integrals can be solved by defining

$$\phi'(x) = \phi(x) + \int d^4y \Delta_{\pm F}(x-y) J(y)$$

$$i\Delta_{\pm F}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-i\mathbf{k}(x-y)} \frac{\pm i}{k^2 - m^2 \mp i\epsilon}$$

$$\left(\Box_x + m^2\right) \Delta_{\pm F}(x-y) = -\delta^{(4)}(x-y). \tag{10}$$

The path integral over all $\phi'$ is the same as that over $\phi$,

$$\int [d\phi'] = \int [d\phi]. \tag{11}$$

Using this one obtains

$$Z_{\pm}[J] = Z[0] \exp \left\{ -\frac{1}{2} \int d^4xd^4y J(x) iD_{\pm F}(x-y)J(y) \right\} \tag{12}$$

with

$$iD_{\pm F}(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{\pm i}{q^2 - m^2 \pm i\epsilon}. \tag{13}$$

The propagator with the plus sign $D_+F$ is just the usual Feynman propagator. The other propagator $D_-F$ is similar but with different analyticity properties, having poles in the complex plane being across the real axis from the usual case. Using these poles, it can be decomposed into time-ordered components. In this case the positive energy modes flow backwards in time

$$iD_-F(x) = D_{F\tau}^\text{forward}(x)\theta(t) + D_{F\tau}^\text{backward}(x)\theta(-t) \tag{14}$$

with

$$D_{F\tau}^\text{backward}(x) = \int \frac{d^3q}{(2\pi)^3} 2E_q e^{-iE_q(t-\bar{q}\cdot\bar{x})} \tag{15}$$

and negative energy forward in time $D_{F\tau}^\text{forward}(x) = (D_{F\tau}^\text{backward}(x))^*$. This is the time-reversed version of the usual propagator.

The propagator is used to generate the Feynman rules of the interacting theory. These rules describe the Feynman diagrams for scattering and decay amplitudes. If one uses $D_-F$ instead of the usual propagators, the resulting amplitudes are time reversed. The usual properties of field theory apply, but with a different arrow of causality. Both options for propagation are causal and have a direction of causality - they differ only on the convention chosen for the measurement of time.

We note that the choice of $\pm i\epsilon$ was not driven by a desire to impose a preferred direction of time or causality. The direction was an output rather than an input.

In the next section we will see that this difference is in accordance with the anti-unitary character of time-reversal symmetry. While the Lagrangian may be time-reversal invariant, the full path integral is not because we form it using $e^{i\mathcal{S}}$. Under time-reversal, $Z_+$ is turned into $Z_-$, reversing the direction of the arrow of causality.

Both versions of the quantum theory are equally causal, but with different directions of causality. There are however some conventions associate with this difference. Here we have used the standard conventions for positive energy solutions and for the flow of time. We will discuss these conventions below. However, the key lesson is independent of conventions. There is an intrinsic asymmetry in the elementary quantum reactions for scattering and decay of positive energy particles. These happen in one preferred direction, but not in the other. They carry an arrow of causality. The direction is ultimately tied to a factor of $i$ in the quantization condition.

## IV. TIME REVERSAL

Let us consider the Principle of Least Action. The action is defined via the Lagrangian

$$S = \int dt \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - V(x) \right] \tag{16}$$

where $m$ is the mass of the particle and $V(x)$ is the potential. We are interested in the direction of time, so let us consider two choices. We will use $t$ for the usual time which runs in an increasing direction, and use $\tau = -t$ for a time coordinate which runs in the reverse direction. The action is invariant under this change

$$S = \int dt \left[ \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 - V(x) \right]. \tag{17}$$

Likewise the equations of motion which follow from this Lagrangian are the same for either $t$ or $\tau$,

$$\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{v}} - \frac{\partial L}{\partial x} = 0 \tag{18}$$

where $v = dx/dt$ and $\dot{v} = dx/d\tau$. This is the origin of the classical puzzle about the arrow of time. For every solution using $t$, there is an identical solution using $\tau$. Classical physics does not contain any information distinguishing the direction of time.

The time reversal invariance of the action is also true for classical fields, although one needs to also transform the field variables. In classical electromagnetism the time-reversal transformation is $x^\mu \rightarrow -x^\mu$, $A_\mu(t, \mathbf{x}) \rightarrow$
\[ A^\mu(-t, \mathbf{x}), J_\mu(t, \mathbf{x}) \rightarrow J^\mu(-t, \mathbf{x}) \] note the interchange of covariant and contravariant indices. The action is invariant under this transformation. This also then allows solution in both directions of time.

As an example, consider an antenna in empty space, which receives a pulse of current near \( t = 0 \). The resulting electromagnetic fields can be solved using Green’s functions in a textbook fashion. However, in addition to the retarded solutions, there are advanced solutions where the signal propagates in the backwards time direction. Indeed, linear combinations of the retarded and advanced solutions all satisfy the classical equations of motion. We have to discard by hand the advanced solutions if we want only forward propagation of the signal.

However, in quantum physics time reversal is an anti-unitary operator. That is, it involves complex conjugation as well as a change of the time coordinate. While classical physics has no intrinsic factors of \( i \), they are crucial in quantum physics. The discussion of the previous section illustrates why quantum physics itself is not time reversal invariant. As demonstrated above, while the Lagrangian may be time-reversal invariant, the full path integral is not because we form it using \( e^{i \int S} \). Under time-reversal, \( Z_+ \) is turned into \( Z_- \), reversing the direction of the arrow of causality. However, this is not invariance, because the quantization changes under these transformations. In particular, this means that having solutions with only one causal direction for a given path integral is a consistent outcome. If the path integral was actually invariant, both causal directions would be possible within the same theory.

One can contrast the classical antenna example above with a quantum counterpart. If we had a pulse of light arriving near \( t = 0 \) to excite an atom

\[ \gamma + A \rightarrow A^* \rightarrow A + \gamma \]  

the subsequent decay happens at a later time. There cannot be both retarded and advanced solutions here. In contrast to the classical theory, the mathematics in the quantum formalism tells us that this causal behavior should occur, and it matches our experience.

V. CANONICAL QUANTIZATION IN QUANTUM MECHANICS

Now let us focus on quantum mechanics and apply canonical quantization using commutation rules. We are instructed to form the canonical momentum

\[ p_t = \frac{\partial L}{\partial \dot{v}} = m \frac{dx}{d\tau} \]  

and from there to form the commutation relation

\[ [x, p_t] = i\hbar \]  

When using the time variable \( \tau \), we would get the canonical momenta

\[ p_\tau = \frac{\partial L}{\partial \dot{v}} = m \frac{dx}{d\tau} \]  

which is the negative of \( p_t \)

\[ p_t = -p_\tau. \]  

If one uses the same quantization assumption, we find

\[ [x, p_\tau] = i\hbar \]  

However, because of Eq. 23, this is not the same condition as Eq. 21 but differs by a minus sign. For completeness, let us list the other possibilities

\[ [x, p_t] = -i\hbar \]  

and

\[ [x, p_\tau] = -i\hbar \]

If we consider the relation between \( p_t \) and \( p_\tau \), we see that condition 4 is equivalent to condition 1 and condition 3 is equivalent to condition 2. These differ only in a trivial way in a relabeling of the coordinate, in the same sense as our “joke” in section 1 about running the clock backwards. But condition 1 and condition 2 (or 3) are not equivalent. There remain two inequivalent possible choices, both of which are compatible with what we know about quantization.

The two choices differ in the naming of the coordinate, \( t \) vs \( \tau \). We know that the usual rule 1 leads to the usual description, then the alternate rules 2 or 3 will lead to the time reversed description. We can see how this arises by considering how the Hamiltonian drives time evolution in the two cases. In the usual way, the commutation rules imply that

\[ [x, H] = \frac{1}{2m} 2p_t \ i\hbar = i\hbar \frac{dx}{dt} \]  

This is the origin of the identification

\[ H = i\hbar \frac{d}{dt} \]  

in quantum mechanics, and the fact that wavefunctions with \( e^{-iEt/\hbar} \) represent positive energy solutions. With condition 2 we get instead the negative of this

\[ [x, H] = \frac{1}{2m} 2p_\tau \ i\hbar = -i\hbar \frac{dx}{d\tau} \]  

In this case, Hamiltonian evolution uses

\[ H = -i\hbar \frac{d}{d\tau} \]  

2 We will argue in Sec. VIII that the two choices have similar physics - with the arrow of causality giving a direction to each choice. In this sense, we will say that they are covariant although not invariant.
and \( e^{iEt/\hbar} \) represent positive energy solutions.

This discussion is consistent with the formal theory of time reversal. Most quantum textbooks discuss the fact that the time-reversal operation is anti-unitary. That is, it also involves a complex conjugation. As a useful example, the non-relativistic coordinate space propagator

\[
G(x, x', t - t') = \sum_n \psi_n^*(x') \psi_n(x) \exp \left[ -\frac{i}{\hbar} E_n (t - t') \right]
\]

satisfies the time reversal property

\[
G(x, x', t' - t) = G^*(x, x', t - t')
\]

This confirms that the two choices above, 1 vs 2 (or equivalently 1 vs 3) correspond to propagation in the opposite time directions. However, at this stage we have not yet shown how these choices influence the causal structure of the theory.

VI. QUANTUM MECHANICS AND RESONANCES

Here we wish to show how the aforementioned conditions are related, that is 1 = [1] and 2 = [2]. Let us first consider condition 1. In this presentation, we follow closely the treatment by Merzbacher [19] in order that the reader may have a standard reference, although many other pathways are possible.

Suppose that a quantum system, perturbed by a constant potential \( V \), is initially in an unperturbed energy eigenstate \( i \) with energy \( E_i \). If this initial state is embedded in a continuum of final states \( f \), the time evolution of transition amplitudes can be derived from standard time-dependent perturbation theory:

\[
i\hbar \frac{d}{dt} \langle i | U(t, 0) | i \rangle = \sum_n e^{i(E_f - E_i) t/\hbar} \langle f | V | n \rangle \langle n | U(t, 0) | i \rangle
\]

where \( E_f > E_i \), \( U \) is the time-evolution operator in the interaction picture and we used that

\[
\hat{V}(t) = e^{\frac{i}{\hbar} H_0 t} V e^{-\frac{i}{\hbar} H_0 t}
\]

with \( V (\hat{V}) \) being the perturbation operator in the Schrödinger (interaction) picture and \( H_0 \) is the unperturbed (time-independent) contribution to the Hamiltonian. It is clear from Eq. (33) that, because of the presence of several different energy gaps \((E_f - E_n)\), contributions from transition amplitudes \( \langle n | U(t, 0) | i \rangle \) to the equations of motion for \( \langle i | U(t, 0) | i \rangle \) are all of different phases, which implies that, for a continuum of final states, all such contributions tend to cancel each other. As a consequence of this destructive interference, the decay of the initial discrete state \( i \) is irreversible and one cannot expect a corresponding regeneration. This reflects the presence of the arrow of causality in such quantum processes.

The equations of motion for \( \langle i | U(t, 0) | i \rangle \) reads

\[
i\hbar \frac{d}{dt} \langle i | U(t, 0) | i \rangle = \sum_{f \neq i} e^{i(E_f - E_i) t/\hbar} \langle f | V | i \rangle \langle i | U(t, 0) | i \rangle + \langle i | V | i \rangle \langle i | U(t, 0) | i \rangle.
\]

In order to derive the exponential decay law, we assume that \( V \) is constant and also transitions from a discrete initial state \( i \) to a quasi-continuum of final states \( f \). In addition, we neglect all other contributions to the change in \( \langle f | U(t, 0) | i \rangle \). For \( t > 0 \) one obtains

\[
i\hbar \frac{d}{dt} \langle f | U(t, 0) | i \rangle = e^{i(E_f - E_i) t/\hbar} \langle f | V | i \rangle \langle i | U(t, 0) | i \rangle, \quad f \neq i
\]

which has the integral form

\[
\langle f | U(t, 0) | i \rangle = -\frac{i}{\hbar} \langle f | V | i \rangle \int_0^t dt' e^{i(E_f - E_i) t'/\hbar} \langle i | U(t', 0) | i \rangle.
\]

If we substitute Eq. (34) in the equations of motion for \( \langle i | U(t, 0) | i \rangle \) we obtain the following amplitude rate

\[
\frac{d}{dt} \langle i | U(t, 0) | i \rangle = \left( -\frac{1}{\hbar^2} \sum_{f \neq i} |\langle f | V | i \rangle|^2 \int_0^t dt' e^{i(E_f - E_i) t'/\hbar} (t' - t) - \frac{i}{\hbar} \langle i | V | i \rangle \right) \langle i | U(t, 0) | i \rangle
\]

where we removed the slowly-varying amplitude \( \langle i | U(t, 0) | i \rangle \) from the integrand since one is usually interested in times \( t \) for which the phase factor produces rapid oscillations.
better defined for $t \to \infty$ by using a factor of $i\epsilon$, that is
\[ \int_0^t dt' \ e^{\frac{i\pi}{h}(E-\epsilon)t'} = -\frac{1}{E-\epsilon} \]
\[ = -i\epsilon \left[ P \frac{1}{E} + i\pi \delta(E) \right] \] (38)

where $P$ denotes the Cauchy principal value. This sign is connected with the choice $1$, i.e., positive energies being propagated forward in time.

The solution of the resulting differential equation is given by
\[ \langle i|U(t,0)|i \rangle = \exp \left[ \frac{it}{\hbar} \sum_{f \neq i} \frac{1}{E_i - E_f + i\epsilon} - \frac{it}{\hbar} \langle i|V|i \rangle \right] \] (39)

In this way the use of $38$ leads to
\[ \langle i|U(t,0)|i \rangle = \exp \left[ -\frac{\Gamma}{2} t - i\Delta E_i \right] \] (40)

where
\[ \Gamma = \frac{2\pi}{\hbar} \sum_{f \neq i} \frac{1}{E_i - E_f} \] (41)

and
\[ \Delta E_i = \langle i|V|i \rangle + P \sum_{f \neq i} \frac{|\langle f|V|i \rangle|^2}{E_i - E_f}. \] (42)

We see that the $i\epsilon$ prescription entering here is also connected to the $i$ in the quantization condition, through the connection of positive energy propagating forward in time. Hence this establishes the identification $1 = \frac{1}{\hbar}$.

This result is essentially what one would obtain if we substituted $E_n \to E_n - i\epsilon$ in the non-relativistic propagator of Eq. $31$. That $i\epsilon$ and the one in the relativistic QFT propagator are clearly related.

Now, suppose that we choose the convention associated with $2$. Following a similar reasoning as above, instead of Eq. $39$ we would have obtained
\[ \langle i|U(\tau,0)|i \rangle = \exp \left[ -\frac{\tau}{\hbar} \sum_{f \neq i} \frac{1}{E_i - E_f + i\epsilon'} \right] \] (43)

Now since $\tau = -t$, one must have $\epsilon' = -\epsilon$. Hence in this case one introduces a factor $-i\epsilon$ which is connected to condition $2$, i.e., positive energies being propagated backward in time. As a result we confirm that $2 = \frac{1}{\hbar}$.

VII. THE FERMI PROBLEM

A well known example involving causality in emission and absorption processes is related to the Fermi model for propagation of light in quantum electrodynamics $11$. The Fermi’s two-atom system has been discussed in detail in the literature $20, 29$. The Fermi problem comprises the study of causality by means of a thorough analysis of the energy transfer between a pair of atoms. More specifically, Fermi proposes the following experiment. Consider two two-level atoms separated by some spatial distance $r$. They are coupled with a common quantum field prepared in the Minkowski vacuum state. Suppose that, at an initial time $\tau_0$, atom 1 is in the excited state and atom 2 is in the ground state. Atom 1 subsequently decays by emitting a photon which may in turn be absorbed by atom 2. As a result the probability for atom 2 to be excited remains zero until a time $\tau$ is reached such that $\tau - \tau_0 \geq r$. For a recent discussion in the framework of disordered systems see Ref. $30$.

The original formulation of the Fermi problem can be given using the standard techniques from time-dependent perturbation theory (within the interaction picture). For simplicity, let us consider two identical two-level atoms at rest and interacting with a massless scalar field. The atom-field interaction Hamiltonian has two local contributions of the form $\lambda m_k (\tau) \varphi(x_k(\tau))$, where $\lambda$ is a small coupling constant and $m_k$ is the monopole moment operator of the $k$ atom. The time evolution of the system is described with respect to proper time $\tau$ of the atoms. Assume that, at the initial time $\tau_0$, the system is in the state $|\phi_i\rangle = |e_1g_2\rangle \otimes |0_M\rangle$, where $|0_M\rangle$ is the Minkowski vacuum state of the scalar field. The transition probability amplitude to the final atom-field state $|\phi_f\rangle = |g_1e_2\rangle \otimes |0_M\rangle$ is given by (up to second order in $\lambda$)
\[ A_{\phi_i \rightarrow \phi_f} = -\frac{\lambda^2}{4} \int_{\tau_0}^\tau d\tau' \int_{\tau_0}^\tau d\tau'' \ e^{-i\omega_0(\tau'-\tau'')} D_{+F}(\tau'-\tau'', r) \] (44)

where $r = |x_1 - x_2|$ is the spatial separation between the atoms and $\omega_0$ is the atomic energy gap. Notice the presence of the usual Feynman propagator $D_{+F}$. Introducing the variables $\xi = \tau' - \tau''$ and $\eta = \tau + \tau'$, expression $44$ becomes
\[ A_{\phi_i \rightarrow \phi_f} = -\frac{\lambda^2}{4} \int_{\Delta\tau}^{\Delta\tau} d\xi (\Delta\tau - |\xi|) e^{-i\omega_0\xi} D_{+F}(\xi, r). \] (45)

When one inserts the expression for the Feynman propagator in Eq. $45$, one gets two contributions. One of them will produce a causal term, proportional to the Heaviside theta function $\theta(-r + \Delta\tau)$. However, the other contribution yields a finite term for $\Delta\tau > r$. The latter has the generic form $f(\omega_0 r)/(\omega_0^2 r^2)$, where $f(x)$ remains bounded as $x \to \infty$. As argued by Pauli, Stueckelberg and others $31, 32$, the concept of causality in relativistic quantum field theory has meaning only in the wave zone $\omega_0 r \gg 1$. Hence causality is preserved in the Fermi system.

This is the standard result. Observe that it tells us that atom 2 will only get excited after the decay of atom 1 (and the subsequent emission of a photon), and never
before. This is enforced by the presence of the usual Feynman propagator $D_{+F}$ in the amplitude which appears as a consequence of the usual time-evolution operator (connected with the conventional causal behavior in the direction of an increasing time coordinate). However, one could also consider the time-reversed process, i.e., atom 2 in the excited state and atom 1 in the ground state. The associated amplitude would be the complex conjugate of Eq. (44) since they would be related by time reversal. In this case one would employ the propagator $D_{-F}$ in the calculation. In any case, the conclusion would be the same: The ground-state atom would only get excited after the decay of the excited atom, and causality remains preserved. This simple example again clearly demonstrates the presence of an intrinsic arrow of causality in the quantum description of causal processes.

VIII. LIVING IN A CAUSAL WORLD

How would we know if we were living in a world where quantum propagation was derived using $\exp(-iS)$ instead of $\exp(+iS)$, or with the alternate canonical quantization rules? We have seen that the unconventional alternate corresponds to propagation backwards in time compared to the conventional choice. Does this mean that we would see resonances decaying before they are produced?

It is here that the distinction between causal direction and time direction becomes significant. In a causal world, reactions flow in a single time direction, but not in the reverse direction. However, it is our convention whether we label that time direction that of increasing entropy, and hence determines the arrow of thermodynamics and the arrow of time. The laws of physics give a causal direction of the increase of entropy. The arrow of causality determines the arrow of thermodynamics, and the arrow of time.

IX. CONCLUSION

The point of this paper is to document the way that the laws of quantum physics encodes an arrow of causality. This is contained in the factor $i$ associated with the quantization procedure. The fundamental action or Hamiltonian may be time symmetric, but the quantization rules are not. This is most clear in the path integral framework, where the causal direction is associated with the phase in $\exp(\pm iS)$. This causal arrow is what gives a direction to scattering and decay processes.

However, there can also be a convention in the choice of the time coordinate, which is basically a decision to run the clock with time increasing or decreasing in the direction of the causal direction. This implies that both possible causal directions have the same phenomenology. Either version is equally valid.

While we have not discussed it here, we note that it is possible for a theory to contain modes with both signs of $i$, i.e., $1$ and $2$ as long as one type of modes is unstable. This exception illuminates the usual causality rules, by understanding the conditions under which they can be circumvented. The choice of clock direction here would be given by the stable modes. Different types of modes will have different causal directions, which means propagating against that clock direction. However, this unusual situation is the exception. The rule for normal quantum theories is to have only one causal arrow.

These considerations have important implications for discussions of the arrow of time. The basic statement saying that the fundamental laws of physics do not differentiate an arrow of time is not correct. At the microscopic level, reactions run in one direction but not the other. The causal direction of these reactions determines the direction of increasing entropy, and hence determines the arrow of thermodynamics.

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