METRIC PROPERTIES OF BAUMSLAG–SOLITAR GROUPS

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Abstract. We compute estimates for the word metric of Baumslag–Solitar groups in terms of the Britton's lemma normal form. As a corollary, we find lower bounds for the growth rate for the groups BS(p, q), with 1 < p ≤ q.

1. Introduction

In this article we investigate the word length of elements in the groups BS(p, q), presented by

$$\langle a, t \mid ta^p t^{-1} = a^q \rangle.$$ 

We use the Britton normal form to obtain an estimate for the word length, and use it to compute a lower bound for the growth rate.

Recall that a function $f : G \to \mathbb{R}$ is a metric estimate for a group $G$ with finite symmetric generating set $S$ if there exist constants $C_1, D_1, C_2, D_2 > 0$ so that for every element $g \in G$, we have

$$C_1 f(x) - D_1 \leq ||x||_S \leq C_2 f(x) + D_2,$$

where $|| \cdot ||_S$ is the word metric with respect to $S$.

Collins, Edjvet and Gill and independently Brazil [2, 3] showed that the groups BS(1, q) have rational growth series, with explicit closed form series given in [3]. Edjvet and Johnson also gave rational growth series for the groups BS(q, q) [5] (see also [13] Section 2.8). Finding an expression for the growth series or the growth rate of BS(p, q) for 1 < p < q has proven to be a stubbornly difficult problem. Freden et al. have made some progress in a series of papers [1, 9, 10]. Tom Wong has also made some progress [15], finding various estimates for the growth rate. In each case the authors have focussed on the (difficult) problem of computing the growth of just elements equal to a power of the generator $a$, the so-called horocyclic elements. In [10] Freden and Knudson prove that the growth series of the horocyclic subgroup is rational when $p \mid q$, and conjecture that when $p \nmid q$ it is not D-finite.

Computing geodesics in these groups is an equally non-trivial problem. The second author gave a linear time algorithm for the case BS(1, q) [7], and Diekert and Laun gave a quadratic time algorithm for the case BS(p, q) when $p \mid q$ [4, 13].

There has also been interest in the language of geodesics for these groups. Groves showed that no set of geodesics surjecting to BS(1, q) can be regular [11]. The second author constructed a context-free and 1-counter language of geodesics for BS(1, 2) [6]. Freden and Adams give a context-sensitive combing in the case of BS(2, 7) [11].

The paper is organised as follows. In Section 2 we compute the metric estimate for BS(1, q), which we extend to the general case in Section 3. In Sections 4–5 we

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use the estimate to compute lower bounds for the growth rate. In Section 0 we compare the bounds obtained with some known exact values for the growth rate.

2. The groups $BS(1, q)$

We are assuming that $q > 1$. The group $BS(1, q)$ admits the presentation

$$(a, t \mid tat^{-1} = a^q),$$

so observe that by rewriting

$$ta = a^qt \quad ta^{-1} = a^{-q}t \quad at^{-1} = t^{-1}a^q \quad a^{-1}t^{-1} = t^{-1}a^{-q}$$

we see that every element admits an expression of the type

$$t^{-m}a^nt^n$$

with $m, n \geq 0$, and $N$ can only be multiple of $q$ if one of the $m, n$ are zero. Under these conditions, it is easy to see that this expression is unique.

From here we can find the expression of the estimate of the metric.

**Proposition 2.1.** There exist constants $C_1, C_2, D_1, D_2 > 0$ such that for every element $x = t^{-m}a^nt^n$ of $BS(1, q)$, $N \neq 0$, we have

$$C_1(m + n + \log |N|) - D_1 \leq ||x|| \leq C_2(m + n + \log |N|) + D_2,$$

where $||x||$ is the word metric with respect to the generators $a, t$.

Observe that the base of the logarithm is irrelevant, since a change of base would only imply an adjustment of the constants. So we will work with the base that is most convenient in each case. Also observe that when $N = 0$ the normal form word $t^k$ for $k \in \mathbb{Z}$ is a geodesic.

**Proof.** To prove the upper bound, assume (taking the inverse if not) that $N > 0$, and write $N$ in base $q$ as

$$N = \sum_{i=0}^{r} k_i q^i$$

with $0 \leq k_i < q$ and $k_r \neq 0$. Observe that $r = \lfloor \log_q N \rfloor$, and that

$$(1) \quad t^{-m}a^nt^n = t^{-m} \left(a^{k_0}t^{k_1}a^{k_2} \cdots t^{k_r}t^{-r-1}\right) t^n$$

which has length at most $m + n + 2q(r+1)$, which gives the desired inequality (with $C_2 = D_2 = 2q$).

For the lower bound, let $b_1 \ldots b_k$ be a geodesic for $x$. Define a sequence of elements $x_{k+1}, x_k, \ldots, x_1$ by $x_{k+1} = x$ and $x_i = x b_{k-1}^{-1} \cdots b_i^{-1}$ for $1 \leq i \leq k$. Let $m_i, n_i \in \mathbb{N}, N_i \in \mathbb{Z}$ such that $x_i$ has normal form $t^{-m_i}a^{N_i}t^{n_i}$. Since $x_1$ is the identity, the integer sequence $\{m_i + n_i\}$ must go from $m + n$ to 0 in $k$ steps. Since multiplying $t^{-m}a^nt^n$ by a generator reduces the sum $m + n$ by at most 1, we must have that $m + n \leq k$. The sequence $|N_i|$ also goes to 0, with multiplication of $x_i$ by a generator changing $|N_i|$ as follows.

- For $x_i t$, we have that $|N_i| = |N_{i-1}|$.
- For $x_i t^{-1}$, we have that $|N_i| = |N_{i-1}|$, except in the case $n_i = 0$. In this case $x_i = t^{-m_i}a^N_i$ and $x_i t^{-1} = t^{-m_i-1}a^{N_i}$, and

$$\log |N_{i-1}| = \log |qN_i| = \log |N_i| + \log q > \log |N_i|,$$
Lemma 3.1. We make use of the following normal form for group elements, based on Britton’s lemma (see [14]).

We make use of the following normal form for group elements, based on Britton’s lemma. The word $w$ appearing in $x = w(a, t)a^N$ can be written uniquely as $x = w(a, t)a^N$ where $w(a, t) \in \{t, at, a^2t, \ldots, a^{q-1}t, t^{-1}, at^{-1}, a^2t^{-1}, \ldots, a^{p-1}t^{-1}\}^*$ and is freely reduced.

The proof is straightforward by performing the following rewritings, so that the only large power of $a$ appearing is on the right hand side:

1. remove canceling pairs $aa^{-1}, a^{-1}a, tt^{-1}, t^{-1}t$;
2. replace $a^{r \pm dq}t$ by $a^{\pm r} a^{\pm dq}$, where $0 \leq r < q$;
3. replace $a^{s \pm dp}t^{-1}$ by $a^{\pm s} a^{\pm dp}$, where $0 \leq s < p$.

Uniqueness is an easy exercise based on Britton’s lemma. The word $w(a, t)$ is of the form

$$t^{m_0} a^{r_1} t^{m_1} a^{r_2} t^{m_2} \cdots a^{r_k} t^{m_k}$$

where:

1. $m_i, r_i \in \mathbb{Z}$.
2. $m_k = 0$ only if the word $w$ is empty.
3. $m_i \neq 0$ for $i \geq 1$.
4. $0 < r_i < q$.
5. if $p \leq r_i < q$, then $m_i > 0$.

We now use this normal form to obtain metric estimates. We study first the case $p < q$.

Theorem 3.2. There exist constants $C_1, C_2, D_1, D_2 > 0$ such that for every element $x \in BS(p, q)$ for $1 \leq p < q$ written as $w(a, t)a^N$, we have

$$C_1(|w| + \log(|N| + 1)) - D_1 \leq |x| \leq C_2(|w| + \log(|N| + 1)) + D_2.$$
Proof. We first prove the upper bound. If $N = 0$ we are done. Assume (taking the inverse of $a^N$ if not) that $N > 0$ and write $N = d_1 q + r_1$ with $0 \leq r_1 < q$. Then $a^N = a^{r_1} t a^{d_1 p t^{-1}}$. Note that

$$d_1 p = d_1 q \left( \frac{p}{q} \right) \leq (d_1 q + r_1) \frac{p}{q} = N \left( \frac{p}{q} \right).$$

Now write $d_1 p = d_2 q + r_2$ with $0 \leq r_2 < q$, where

$$d_2 p = d_2 q \frac{p}{q} \leq (d_2 q + r_2) \frac{p}{q} = d_1 p \frac{p}{q} \leq N \left( \frac{p}{q} \right)^2.$$

Repeat to obtain

$$a^N = a^{r_1} t a^{r_2} t \ldots a^{r_k} t a^{d_k p t^{-k}}$$

with $1 \leq d_k p < q$ when the process terminates, and observe that $d_k p$ cannot be zero, or else the process terminates in the previous step. We have $1 \leq d_k p \leq N \left( \frac{p}{q} \right)^k$, so we deduce that $k \leq \log_{q/p} N$. Our word has length at most $q k + q + k$ since each $r_i < q$ and $d_1 p < q$. It follows that the length obtained for the word $a^N$ is at most

$$(q + 1) \log_{q/p} N + q$$

which yields our upper bound.

Next, the lower bound. Let $x_1 x_2 \ldots x_n$ be a geodesic for $x$ with $n = ||x||$, and let $w_i a^{N_i}$ be the normal form for the prefix of length $i$. We have $w_0 = \epsilon$ and $N_0 = 0$.

If $x_{i+1} = a^{\pm 1}$ then $w_{i+1} = w_i$, and $|N_{i+1}| \leq |N_i| + 1$. If $x_{i+1} = t$, put $N_i = d p + r$ with $0 \leq r < p$ and $d$ an integer.

- If $r = 0$ and $w_i$ ends with $t$, write $w = u a^{c} t$ with $0 \leq c < q$ and $u$ empty or ending in $t^{\pm 1}$. Then $w_i a^{N_i t^{-1}} = u a^{c} t a^{d p t^{-1}} = u a^{c+d q}$. It follows that $|w_{i+1}| = |u| < |w_i|$ and

$$|N_{i+1}| \leq |N_i| \left( \frac{q}{p} \right) + q.$$ 

- Otherwise $w_i a^{r^{t^{-1}}}$ is freely reduced. In this case $w_i a^{N_i t^{-1}} = w_i a^{r^{t^{-1}} a^{d q}}$ so $|w_{i+1}| \leq |w_i| + p$ and

$$|N_{i+1}| \leq |N_i| \left( \frac{q}{p} \right).$$

If $x_{i+1} = t$, put $N_i = dq + s$ with $0 \leq s < q$ and $d$ an integer.

- If $s = 0$ and $w_i$ ends with $t^{-1}$, write $w = u a^{c} t^{-1}$ with $0 \leq c < p$ and $u$ empty or ending in $t^{\pm 1}$. Then $w_i a^{N_i t} = u a^{c} t^{-1} a^{d q t} = u a^{c+d p}$. It follows that $|w_{i+1}| = |u| < |w_i|$ and

$$|N_{i+1}| \leq |N_i| + p < |N_i| \left( \frac{q}{p} \right) + q.$$ 

- Otherwise $w_i a^{r^{t}}$ is freely reduced. In this case $w_i a^{N_i t} = w_i a^{r^{t} a^{d p}}$ so $|w_{i+1}| \leq |w_i| + q$ and

$$|N_{i+1}| \leq |N_i| \left( \frac{p}{q} \right).$$
It follows that for multiplication by any generator we have

\[ |w_{i+1}| \leq |w_i| + q \quad \text{and} \quad |N_{i+1}| \leq |N_i| \left( \frac{q}{p} \right) + q. \]

After \( n \) multiplications the value of \( |w_n| \) can be at most \( qn \), while \( |N_n| \) is bounded as follows. We have \( N_0 = 0 \), \( |N_1| \leq q \), \( |N_2| \leq \left( \frac{q}{p} \right) q + q \), \( |N_3| \leq \left( \frac{q}{p} \right)^2 q + \left( \frac{q}{p} \right) q + q \) and so on, so

\[ |N_n| \leq \sum_{i=0}^{n-1} \left( \frac{q}{p} \right)^i q = q \left( \frac{\left( \frac{q}{p} \right)^n - 1}{\left( \frac{q}{p} \right) - 1} \right) < C \left( \frac{q}{p} \right)^n \]

where \( C = \frac{q}{(p-q)} > 1 \) as \( qp > q - p \). We then have

\[ |N_n| + 1 \leq C \left( \frac{q}{p} \right)^n + 1 \leq 2C \left( \frac{q}{p} \right)^n \]

since \( C > 1 \) and \( \frac{q}{p} > 1 \). Then \( \log_{q/p}(|N_n|+1) \leq \log_{q/p}(2C) + n = \log_{q/p}(2C) + ||x|| \).

The two lower bounds combine to give the result with \( D_1 = \log_{q/p}(2C) \) and \( C_1 = \frac{1}{q+p} \).

The case \( p = q \) is considerably easier.

**Lemma 3.3.** There exists a constant \( C_1 > 0 \) such that for every element \( x \in BS(p,p) \) for \( p \geq 1 \) written as \( w(a,t)a^N \), we have

\[ C_1(|w| + |N|) \leq ||x|| \leq |w| + |N|. \]

**Proof.** Since \( |w| + |N| \) is the word length of the normal form, the upper bound is immediate.

The lower bound follows the same argument as the \( p < q \) case. Let \( x_1 \ldots x_n \) be a geodesic for \( x \) with \( n = ||x|| \), and define \( w_i, N_i \) as before.

Multiplication by \( a^{\pm 1} \) gives \( |w_{i+1}| = |w_i| \) and \( |N_{i+1}| \leq |N_i| + 1 \).

For multiplication by \( t^{\pm 1} \), let \( N_i = dp + r \) for \( 0 \leq r < p \) and \( d \) an integer.

- If \( r = 0 \) and \( w_i \) ends in \( t^{\pm 1} \), we have \( w_i = uac^\tau t^\mp 1 \) (with \( c < p \) and \( \frac{\tau}{p} \)) and \( w_i^{a^N} = uac^\tau t^\mp 1 a^dp \) so \( |w_{i+1}| \leq |w_i| + |N_{i+1}| \leq |N_i| + 1 \).
- Otherwise we have \( |w_{i+1}| \leq |w_i| + |N_{i+1}| \leq |N_i| + p \).

Then \( |w_n| \leq p||x|| \) and \( |N_n| \leq p||x|| \) which gives our lower bound with \( C_1 = \frac{1}{q+p} \).

\[ \square \]

4. **Lower Bound for the Growth Rate for \( BS(p,q) \)**

For \( q > p > 1 \), the exact growth rate for \( BS(p,q) \) is not known. Here, we will make use of the bounds specified above to find lower bounds for these rates. The growth function is given by

\[ \gamma(n) = \#B(n) = \#\{ x \in BS(p,q) : ||x|| \leq n \}, \]

but observe that we can consider the alternate set using the upper bound given above:

\[ D(n) = \{ x = w(a,t)a^N \in BS(p,q) : |w| + (q+1) \log_{q/p} N + q \leq n \} \]
and the bound implies precisely that $D(n) \subset B(n)$. Hence, we have that $\#D(n)$ is a lower bound for $\gamma(n)$.

To estimate the number of elements with normal form $w(a, t)a^N$ which satisfy

$$|w| + (q + 1) \log_{q/p} N + q = k,$$

observe that if $(q + 1) \log_{q/p} N + q = k$, then the number $N$ is of the order of an exponential with base

$$\left(\frac{q}{p}\right)^{\frac{1}{q+1}}$$

which goes to 1 as $q$ grows. The conclusion one can deduce from this is that in the set

$$\{x = w(a, t)a^N : |w| + (q + 1) \log_{q/p} N + q = k\}$$

the dominant part will be the part of those elements satisfying $|w| = k$ because it will be an exponential with base larger than

$$\left(\frac{q}{p}\right)^{\frac{1}{q+1}}$$

at least asymptotically.

Now consider the set

$$E(n) = \{w(a, t) : |w| \leq n\}.$$  

Observe that if an element can be written as a word $w(a, t)$, then its length is bounded above by $|w|$, so we have that $E(n) \subset B(n)$. Hence the cardinality $\#E(n)$ is a genuine lower bound. And note also that this lower bound works as well for the case $BS(p, p)$.

So the rest of this paper will be dedicated to the computation of the cardinality of the sets $E(n)$.

5. The automaton for $E(n)$

Observe that as we have seen above, the word $w(a, t)$ can be written as a product of the words

$$t \at a^2t \ldots a^{q-1}t$$

$$t^{-1} \at^{-1} a^2t^{-1} \ldots a^{p-1}t^{-1}$$

so the words will be accepted by a finite state automaton. As an example, the case for $BS(2, 3)$ has words product of

$$t \at a^2t \t^{-1} \at^{-1}$$

so the words are the words accepted by the automaton in Figure 1 where the accept states are $S, 1$ and 2. Note that:

- The word starts at the state $S$ with a letter $a$, $t$ or $t^{-1}$.
- The word is in state 1 if the last letter was a $t$, and the next letters allowed are $a$ or $t$.
- The word is in state 2 if the last letter was a $t^{-1}$, and the next letters allowed are $a$ or $t^{-1}$.
- The word is in state 3 if the last letter was an $a$ but the last two were not $a^2$, and then the next letters allowed are $a$, $t$ or $t^{-1}$.
- The word is in state 4 if the last two letters were $a^2$, and the next letter allowed is only $t$. 
Figure 1. The automaton for the words $w$ in $BS(2,3)$

To see how many words have length $n$, we only need to find the largest eigenvalue of its adjacency matrix. Observe that there are no arrows arriving at the start state, so the matrix has a first row of zeros, so we will specify only the part of the matrix involving the states $1,2,3,4$. The matrix is

$$
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

We have:

**Proposition 5.1.** The growth rate for $BS(2,3)$ is bounded below by $2.1478990...$, the largest real zero of the polynomial

$$x^4 - 2x^3 - x^2 + x + 1.$$ 

This construction readily extends to the general case. The automaton just has more states going up of the same type as states $3$ and $4$, having $p-1$ states like $3$, with $t$ and $t^{-1}$ arrows, and $q-p$ states above those, with only an arrow $t$ going back to state $1$. The adjacency matrix (except the start state) is the following
$(q + 1) \times (q + 1)$-matrix

$$A_{pq} = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
& & & & & & 1 \\
& & & & & & \ddots \\
& & & & & & \vdots \\
& & & & & & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1
\end{pmatrix}$$

with the upper left block being $(p + 1) \times (p + 1)$. Its characteristic polynomial satisfies the recurrences:

$$P_{22}(x) = -x^3 + 2x^2 + x - 2 \quad P_{pp}(x) = -xP_{p-1,p-1}(x) + (-1)^{p-1}2(1 - x)$$

for $2 < p$, and

$$P_{pq}(x) = -xP_{p,q-1}(x) + (-1)^{q-1}(1 - x)$$

if $2 \leq p < q$. These recurrences allow the computation of $P_{pq}$, which is (up to a sign) the polynomial appearing in the following theorem:

**Theorem 5.2.** The growth rate for the group $BS(p, q)$ is bounded below by the largest real zero $\lambda_{pq}$ of the polynomial

$$P_{pq} = x^{q+1} - 2x^q - x^{q-1} + x^{q-p} + 1.$$ 

We have that $2 \leq \lambda_{pq} \leq 1 + \sqrt{2}$ for all $p, q$ with $2 \leq p \leq q$.

It is a straightforward computation to see that the numbers $\lambda_{pq}$ are ordered by the lexicographic order of the pair $(p, q)$. That is, they satisfy that:

- If $p < p'$, we have $\lambda_{pq} < \lambda_{p'q'}$ (regardless of $q$ and $q'$).
- For a given $p$, if $p \leq q < q'$, then we have $\lambda_{pq} < \lambda_{p'q'}$.

A table for some numbers $\lambda_{pq}$ is

| $q$ | 2 | 3 | 4 | 5 | ... | 10 | ... | 20 |
|-----|---|---|---|---|-----|----|----|----|
| $p = 2$ | 2 | 2.14790 | 2.20557 | 2.22919 | ... | 2.24668 | ... | 2.24698 |
| 3 | 2.26953 | 2.31651 | 2.33529 | ... | 2.34841 | ... | 2.34859 |
| 4 | 2.35930 | 2.37627 | ... | 2.38786 | ... | 2.38801 |
| 5 | 2.39246 | ... | 2.40345 | ... | 2.40358 |
| ... | ... | ... | ... | ... | ... | ... |
| 10 | ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |
| 20 | ... | ... | ... | ... | ... | ... | 2.41421 |
6. SOME UPPER BOUNDS AND SOME EXACT VALUES

In his Master’s thesis, Tom Wong computes the size of spheres of small radius in various Baumslag-Solitar groups (Table 5.1 in [15]). Using this data and Fekete’s Lemma (see page 63 of [15]) he obtains upper bounds for the spherical growth rates of the following groups. (Note that the spherical growth sequence is submultiplicative in any finitely generated group, so Fekete’s Lemma applies.)

- For $BS(2, 2)$ the sphere of radius 18 contains 3014654 elements, so an upper bound for the growth rate is $\sqrt[18]{3014654}$ which is approximately 2.290.
- For $BS(2, 3)$ the sphere of radius 18 contains 38595072 elements, so an upper bound for the growth rate is $\sqrt[18]{38595072}$ which is approximately 2.639.
- For $BS(3, 5)$ the sphere of radius 15 contains 11615210 elements, so an upper bound for the growth rate is $\sqrt[15]{11615210}$ which is approximately 2.958.

Recall that if $S(z)$ is the generating function for the spherical growth series and $B(z)$ is the generating function for the growth series, then

$$B(z) = \frac{S(z)}{1 - z}.$$

If the dominant singularity (radius of convergence) of $B(z)$ is $r$ then the exponential growth rate of the growth series is $\frac{1}{r}$. Since by Theorem 5.2 the growth rate of $BS(p, q)$ is bounded below by 2, the dominant singularity of $B(z)$ is at most $\frac{1}{2}$, so the factor $1 - z$ in the denominator does not affect the dominant singularity, that is, the spherical growth rate is the same as the growth rate for all $2 \leq p \leq q$.

Combining these bounds we have the following estimates.

- For $BS(2, 2)$ the growth rate is between 2 and 2.290.
- For $BS(2, 3)$ the growth rate is between 2.147 and 2.639.
- For $BS(3, 5)$ the growth rate is between 2.335 and 2.958.

In the case $p = q$ the exact growth rates can be obtained from the generating functions obtained by Edjvet and Johnson [5]. For $BS(2, 2)$ the generating function is

$$\frac{1 - z - 2z^3}{(1 - z)(1 - 2z)^2}$$

which has a dominant singularity of $\frac{1}{2}$, so the growth rate is exactly 2. It follows that the lower bound obtained in Proposition 5.1 is sharp. This is the only case where this is so, the case for $BS(p, p)$ for higher values of $p$ produces growth rates which are larger than the lower bounds obtained in Theorem 5.2. For instance, for $BS(3, 3)$, the generating function is

$$\frac{(1 + z)^2(1 - 2z)(1 + z + 2z^3)}{(1 - z)(1 - z - 4z^2)(1 - z - 2z^2 - 2z^3)}$$

which has singularities at 1, $-0.64039, 0.39039, 0.44062$ and $-0.72031 \pm 0.7848i$, so 0.39039 is the dominant singularity (radius of convergence) which gives a growth rate of approximately 2.56154, already significantly larger than the lower bound obtained in Theorem 5.2.
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