Gibbs–Tolman approach to the curved interface effects in asymmetric nuclei

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We redefine the surface tension coefficient and the symmetry energy for an asymmetric nuclear Fermi-liquid drop with a finite diffuse layer. Considering two-component charged Fermi-liquid drop and following Gibbs-Tolman concept, we introduce the equimolar radius $R_e$ of sharp surface droplet at which the surface tension is applied and the radius of tension surface $R_s$ (Laplace radius) which provides the minimum of the surface tension coefficient $\sigma$. We have shown that the nuclear Tolman length $\xi$ is negative and the modulus of $\xi$ growth quadratically with asymmetry parameter $X = (N - Z)/(N + Z)$.

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I. INTRODUCTION

The nucleus is a two component, charged system with a finite diffuse layer. This fact specifies a number of various peculiarities of the nuclear surface and symmetry energies: dependency on the density profile function, non-zero contribution to the surface symmetry energy, connection to the nuclear incompressibility, etc. The additional refinements appear due to the quantum effects arising from the smallness of nucleus. In particular, the curved interface creates the curvature correction to the surface energy $E_S$ and the surface part of symmetry energy $E_{sym}$ of order $A^{1/3}$ and can play the appreciable role in small nuclei as well as in neck region of fissionable nuclei.

The presence of the finite diffuse layer in nuclei creates the problem of the correct definition of the radius and the surface of tension for a small drop with a diffuse interface. Two different radii have to be introduced in this case: the equimolar radius $R_e$, which gives the actual size of the corresponding sharp-surface droplet, and the radius of tension $R_s$, which derives, in particular, the capillary pressure. Below we will address this problem to the case of two-component nuclear drop. In general, the presence of the curved interface affects both the bulk and the surface properties. The curvature correction is usually negligible in heavy nuclei. However, this correction can be important in some nuclear processes. For example the yield of fragments at the nuclear multifragmentation or the probability of clusterization of nuclei from the freeze-out volume in heavy ion collisions. In both above mentioned processes, small nuclei necessarily occur and the exponential dependence of the yield on the surface tension should cause a sensitivity of both processes to the curvature correction. Moreover the dependency of the curvature interface effects on the isotopic asymmetry of small fragments can significantly enhance (or suppress) the yields of neutron rich isotopes.

In the present paper, we analyze of the interface effects in an asymmetric nuclear Fermi-liquid drop with a finite diffuse layer. We follow the ideology of the extended Thomas-Fermi approximation (ETFA) with effective Skyrme-like forces combining the ETFA and the direct variational method with respect to the nucleon densities, see Ref. [6]. The proton and neutron densities $\rho_p(r)$ and $\rho_n(r)$ are generated by the diffuse-layer profile functions which are eliminated by the requirement that the energy of the nucleus should be stationary with respect to variations of these profiles. In order to formulate proper definition for the drop radius, we use the concept of the dividing surface, originally introduced by Gibbs [4]. Following the Gibbs method, which is applied to the case of two component system, we introduce the superficial (surface) density as the difference (per unit area of dividing surface) between actual number of particles $A$ and the number of bulk, $A_V$, and neutron excess, $A_{-\nu}$, particles which a drop would contain if the particle densities were uniform.

The plan of the paper is the following. In Sect. II we discuss the Gibbs’s derivation of equimolar radius in the case of two-component system with diffuse layer. We then derive in Sect. III the surface energy and the surface contribution to symmetry energy. The relation of the leptodermous $A^{-1/3}$-expansions for finite nuclei to the nuclear matter equation of state is discussed in Sect. IV. Our conclusions are given in Sect. V.

II. DIVIDING SURFACE AND EQUIMOLAR RADIUS IN ASYMMETRIC NUCLEI

We consider first the spherical nucleus at zero temperature, having the mass number $A = N + Z$, the neutron excess $A_\nu = N - Z$ and the asymmetry parameter $X = A_\nu/A$. The total binding energy of nucleus is $E$. An actual nucleus has the finite diffuse layer of particle density distribution. Thereby, the nuclear size is badly specified. In order to formulate proper definition for the nuclear radius, we will use the concept of dividing surface of radius $R$, originally introduced by Gibbs [4]. Following Refs. [4, 5], we introduce the formal dividing surface of radius $R$, the corresponding volume $V = 4\pi R^3/3$ and the surface area $S = 4\pi R^2$. Note that the dividing surface is arbitrary but it should be located within the nuclear diffuse layer.

The energy of a nucleus $E$, as well as the mass number
$A$ and the neutron excess $A_-$, are split into the volume and surface parts,

$$E = E_V + E_S + E_C,$$  \hspace{1cm} (1)

$$A = A_V + A_S, \quad A_- = A_-V + A_-S.$$  \hspace{1cm} (2)

Here the Coulomb energy $E_C$ is fixed and does not depend on the dividing radius $R$. The bulk energy $E_V$ and the surface energies $E_S$ can be written as

$$E_V = (-P_V + \lambda \rho_V + \lambda_- \rho_-V) V$$  \hspace{1cm} (3)

and

$$E_S = (\sigma + \lambda \rho_S + \lambda_- \rho_-S) S.$$  \hspace{1cm} (4)

Here $P_V$ is the bulk pressure

$$P_V = -\frac{\partial E_V}{\partial V} |_{A_V},$$  \hspace{1cm} (5)

$\sigma$ is the surface tension and $\rho_V = A_V/V$ and $\rho_-V = A_-V/V$ are, respectively, the total (isoscalar) and the neutron excess (isovector) volume densities, $\rho_S = A_S/S$ and $\rho_-S = A_-S/S$ are the corresponding surface densities. We have used the isoscalar $\lambda = (\lambda_n + \lambda_p)/2$ and isovector $\lambda_- = (\lambda_n - \lambda_p)/2$ chemical potentials, where $\lambda_n$ and $\lambda_p$ are the chemical potentials of neutron and proton, respectively. The Coulomb energy $E_C$ must be excluded from the chemical potentials $\lambda$ and $\lambda_-$ because of Eqs. (1), (6) and (7). Namely,

$$\lambda_n = \frac{\partial E}{\partial N} |_Z, \quad \lambda_p = \frac{\partial E}{\partial Z} |_N - \lambda_C,$$  \hspace{1cm} (6)

where

$$\lambda_C = \frac{\partial E_C}{\partial Z} |_N.$$  \hspace{1cm} (7)

Generally, the realistic (experimental) chemical potentials $\lambda_{tot,n}$ and $\lambda_{tot,p}$ contain the contributions of the volume, $\lambda_{vol}$, surface, $\lambda_{surf}$, symmetry, $\lambda_{sym}$, and Coulomb, $\lambda_C$, parts

$$\lambda_{tot,n} = \frac{\partial E}{\partial N} |_Z = \lambda_{vol} + \lambda_{surf} + \lambda_{sym}$$

and

$$\lambda_{tot,p} = \frac{\partial E}{\partial Z} |_N = \lambda_{vol} + \lambda_{surf} - \lambda_{sym} + \lambda_C,$$  \hspace{1cm} (7)

where

$$\lambda_{sym} = 2b_{sym}X$$

and $b_{sym}$ is the symmetry energy. The knowledge of the chemical potentials $\lambda_{tot,n}$ and $\lambda_{tot,p}$ allows us to evaluate the Coulomb shift $\lambda_C$. On the $\beta$-stability line, the following condition should be satisfied

$$\lambda_{tot,n} - \lambda_{tot,p} |_{X = X^* (A)} = 0,$$  \hspace{1cm} (8)

and Eq. (7) provides the relation

$$\lambda_C = 4b_{sym}X^*.$$  \hspace{1cm} (9)

Here $X^* = X^*(A)$ indicates the $\beta$-stability line.

Notation $E_V$ stands for the nuclear matter energy of the uniform densities $\rho_V, \rho_-V$ within the volume $V$. The state of the nuclear matter inside the specified volume $V$ is chosen to have the chemical potentials $\mu$ and $\mu_-$ equal to that of the actual droplet. In more detail, from the equation of state for the nuclear matter one has chemical potentials $\mu(\rho, \rho_-)$ and $\mu_-(\rho, \rho_-)$ as functions of the isoscalar, $\rho$, and isovector, $\rho_-$, densities. Then, the following conditions should be fulfilled:

$$\mu(\rho = \rho_V, \rho_- = \rho_-V) = \lambda,$$

$$\mu_-(\rho = \rho_V, \rho_- = \rho_-V) = \lambda_-$$  \hspace{1cm} (10)

to derive the specific values of densities $\rho_V$ and $\rho_-V$.

The surface part of the energy $E_S$ as well as the surface particle number $A_S$ and the surface neutron excess $A_-S$ are considered as the excess quantities responsible for “edge” effects with respect to the corresponding volume quantities. Using Eqs. (1) – (4) one obtains

$$\sigma = \frac{E - \lambda A - \lambda_- A_-}{S} + \frac{P_V V}{S} - \frac{E_C}{S} = \frac{\Omega - \Omega_V}{S}.$$  \hspace{1cm} (11)

Here the grand potential $\Omega = E - \lambda A - \lambda_- A_- - E_C$ and its volume part $\Omega_V = -P_V V = E_V - \lambda A_V - \lambda_- A_- V$ were introduced. From Eq. (11) one can see how the value of the surface tension depends on the choice of the dividing radius $R$,

$$\sigma [R] = \frac{\Omega}{4\pi R^2} + \frac{1}{3} P_V R.$$  \hspace{1cm} (12)

Taking the derivative from Eq. (12) with respect to the formal dividing radius $R$ and using the fact that observables $E, \lambda, \lambda_-$ and $P$ should not depend on the choice of the dividing radius, one can rewrite Eq. (12) as

$$P_V = 2 \frac{\sigma [R]}{R} + \frac{\partial}{\partial R} \sigma [R],$$  \hspace{1cm} (13)

which is the generalized Laplace equation. The formal values of surface densities $\varrho_{0,S}$ and $\varrho_-S$ can be found from (2) as

$$\varrho_S[R] = \frac{A}{4\pi R^2} - \frac{1}{3} \varrho_V R,$$

$$\varrho_-S[R] = \frac{A_-}{4\pi R^2} - \frac{1}{3} \varrho_-V R.$$  \hspace{1cm} (14)

In Eqs. (12) – (14) square brackets denote a formal dependence on the dividing radius $R$ which is still arbitrary and may not correspond to the actual physical size of the nucleus. To derive the physical size quantity an additional condition should be imposed on the location of
the dividing surface. In general, the surface energy $E_S$ for the arbitrary dividing surface includes the contributions from the surface tension $\sigma$ and from the binding energy of particles within the surface layer. The latter contribution can be excluded for the special choice of dividing (equimolar) radius $R = R_e$ which satisfy the condition

$$\langle \varphi S \lambda + \varphi_+ S \lambda_- \rangle_{R=R_e} = 0 \ .$$

Here we use the notation $R_e$ by the analogy with the equimolar dividing surface for the case of the one-component liquid [2, 5]. For the dividing radius defined by Eq. (15) the surface energy reads

$$E_S = \sigma_e S_e \ ,$$

where $\sigma_e \equiv \sigma(R_e)$ and $S_e = 4\pi R_e^2$. Using Eqs. (14), (15), the corresponding volume $V_e = 4\pi R_e^2/3$ is written as

$$V_e = \frac{\lambda A + \lambda_- A}{\lambda \varphi \nu + \lambda_- \varphi_- \nu_+} \ .$$

As seen from Eqs. (10), (17), the droplet radius $R_e$ is determined by the equation of state for the nuclear matter through the values of the droplet chemical potentials $\lambda$ and $\lambda_-$. The surface tension $\sigma[R]$ depends on the location of the dividing surface. Function $\sigma[R]$ has a minimum at certain radius $R = R_s$ (radius of the surface of tension [5]) which usually does not coincide with the equimolar radius $R_e$. The radius $R_s$ (Laplace radius) denotes the location within the interface. Note that for $R = R_s$ the capillary pressure of Eq. (13) satisfies the classical Laplace relation

$$P_v = 2 \frac{\sigma[R]}{R} \bigg|_{R=R_s} \ .$$

The dependence of the surface tension $\sigma[R]$ of Eq. (12) on the location of the dividing surface for the nuclei $^{120}$Sn and $^{208}$Pb is shown in Fig. 1.

Following Gibbs and Tolman [1, 2], we will assume that the physical (measurable) value of the surface tension is that taken at the equimolar dividing surface. We assume, see also Ref. [3], that the surface tension $\sigma \equiv \sigma(R_e)$ approaches the planar limit $\sigma_\infty$ as

$$\sigma(R_e) = \sigma_\infty \left(1 - \frac{2\xi}{R_e} + O(R_e^{-2}) \right) \ ,$$

where $\xi$ is the Tolman’s length [2]. Note that the expression (19) can be considered as a particular case of expansion of any observable $W$ in a finite saturated Fermi-system over the dimensionless small parameter $r_0/R_e$, where $r_0 = (4\pi \rho_0^3)^{-1/3}$ and $\rho_0$ is the bulk particle density. Namely,

$$W = W_\infty + W_1 \frac{r_0}{R_e} + W_2 \left( \frac{r_0}{R_e} \right)^2 + \ldots \ .$$

FIG. 1. Surface tension $\sigma$ as a function of the dividing radius $R$ for nuclei $^{120}$Sn and $^{208}$Pb. The calculation was performed using energy $E$ from Eq. (22) and the SkM force. The Laplace radius $R_s$ denotes the dividing radius where $\sigma$ approaches the minimum value, i.e., the Laplace condition of Eq. (13) is satisfied.

Taking Eq. (13) for $R = R_s$ and comparing with analogous one for $R = R_e$, one can establish the following important relation (see Eq. (A9) in Appendix A)

$$\xi = \lim_{A \to \infty} (R_e - R_s) + O(X^2) \ .$$

This result leads to the conclusion that to obtain the non-zero value of Tolman length $\xi$, and, consequently, the curvature correction $\Delta \sigma_{\text{curv}} \neq 0$ for a curved surface, the nucleus must have a finite diffuse surface layer.

III. MICROSCOPIC CONSIDERATION

We will perform the numerical calculations using Skyrme type of the effective nucleon-nucleon interaction. The energy and the chemical potential for actual droplets can be calculated using a direct variational method within the extended Thomas-Fermi approximation [4]. The energy $E$ of the nucleus is given by the following functional

$$E = \int \! dr \left\{ \epsilon_{\text{kin}}[\rho_n, \rho_p; \nabla \rho_n, \nabla \rho_p] + \epsilon_{\text{Sk}}[\rho_n, \rho_p; \nabla \rho_n, \nabla \rho_p] + \epsilon_C[\rho_p] \right\} \ ,$$

where $\epsilon_{\text{kin}}[\rho_n, \rho_p; \nabla \rho_n, \nabla \rho_p]$ is the kinetic energy density, $\epsilon_{\text{Sk}}[\rho_n, \rho_p; \nabla \rho_n, \nabla \rho_p]$ is the potential energy density of Skyrme nucleon-nucleon interaction and $\epsilon_C[\rho_p]$ is the Coulomb energy density. The equilibrium condition can be written as a Lagrange variational problem. Namely,

$$\delta (E - \lambda_{\text{tot, n}} N - \lambda_{\text{tot, p}} Z) = 0 \ ,$$
where the variation with respect to all possible small changes of $\rho_n$ and $\rho_p$ is assumed.

Using the trial parameter profile for the neutron $\rho_n(r)$ and proton $\rho_p(r)$ densities and performing the direct variational procedure, we can evaluate the equilibrium particle densities $\rho(r) = \rho_n(r) + \rho_p(r)$ and $\rho_-(r) = \rho_n(r) - \rho_p(r)$, the total energy per particle $E/A$ and the chemical potentials $\lambda_{\text{tot},n}$ and $\lambda_{\text{tot},p}$ for a fixed asymmetry parameter $X$, see Ref. [2] for details. We will also consider the asymmetric nuclear matter where the energy $E_{\infty}$ is given by

$$E_{\infty} = \int d\mathbf{r} \left\{ \epsilon_{\text{kin}}[\rho_n, \rho_p] + \epsilon_{\text{Sk}}[\rho_n, \rho_p] \right\} .$$

(24)

Here, the kinetic energy density $\epsilon_{\text{kin}}[\rho_n, \rho_p]$ and the potential energy density $\epsilon_{\text{Sk}}[\rho_n, \rho_p]$ do not include the terms which depend on the gradients of nucleon density providing the bulk particle density $\rho_0 = \text{const}$. Note also that the Coulomb energy density $\epsilon_C[\rho_0]$ does not contribute to the energy $E_{\infty}$. We will derive the volume (bulk) part of energy $E_V$ as

$$E_V = E_{\infty} \quad \text{and} \quad \varrho_V = \rho_0 .$$

(25)

Using the energy $E_V$ from Eq. (25), the above obtained values of the chemical potentials $\lambda_n$ and $\lambda_p$ and the relations

$$\left. \frac{\partial E_V}{\partial A} \right|_{V,A} = \lambda_n, \quad \left. \frac{\partial E_V}{\partial A} \right|_{V,A} = \lambda_p,$$

(26)

we will evaluate the equilibrium bulk densities $\varrho_V = \rho_0$ and $\varrho_- = \rho_- = \rho_0$.

The nuclear beta-stability requires the fulfillment of the condition (8). In Fig. 2 we compare the results for the beta-stability line $Z = Z^*(N)$ obtained from Eqs. (1), (7) and (8) with the experimental data (solid dots). One can see that the solid line gives the acceptable description for the experimental data. Note that the bulk neutron-proton ratio obtained within the Gibbs-Tolman method might slightly differ from that of an actual drop. The dashed line in Fig. 2 represents function $Z_V(N_V)$ which corresponds to $Z^*(N)$, where the number of protons $Z_V$ and neutrons $N_V$ are taken for the nuclear matter within the equimolar volume (17). We can see that for nuclei along the beta-stability line one has $X_V = (N_V - Z_V)/(N_V + Z_V) < X^*$. That is because the part of nucleons (mainly neutrons) are located near the nuclear surface and do not contribute to the volume ratio $N_V/Z_V$.

For arbitrary dividing radius $R$ and fixed asymmetry parameter $X$ we evaluate then the volume, $A_V = 4\pi\varrho_0R^3/3$ and $A_\sim = 4\pi\varrho_-R^3/3$, the surface, $A_S = 4\pi\varrho_0R^2$ and $A_- = 4\pi\varrho_-R^2$, particle numbers and the volume part of equilibrium energy $E_V$. All evaluated values of $E_V[R]$, the bulk densities $\varrho_V$ and $\varrho_-$ and the surface particle densities $\varrho_S[R]$ and $\varrho_- S[R]$ depend on the radius $R$ of dividing surface and asymmetry parameter $X$. The actual physical radius $R_e$ of the droplet can be derived by the condition (15), i.e., by the requirements that the contribution to $E_S$ from the bulk binding energy (term $\sim (\varrho_S \lambda + \varrho_- \lambda_-)$ in Eq. (11)) should be excluded from the surface energy $E_S$. In Fig. 3 we represent the calculation of the specific surface particle density $\varrho_S \lambda + \varrho_- \lambda_-$ as a function of the radius $R$ of dividing surface. Equimolar dividing radius $R_e$ in Fig. 3 defines the physical size of the sharp surface droplet and the surface at which the surface tension is applied, i.e., the equimolar surface where Eq. (16) is fulfilled.
dependence of the equimolar dividing radius $R_e$ on the asymmetry parameter $X$ is shown in Fig. 4. Note that the value of equimolar radius $R_e$, which is derived by Eq. (17), is not considerably affected by the Coulomb interaction. We have also evaluated the values of $R_e$ neglecting the Coulomb term in Eq. (22), i.e., assuming $E_C = \lambda C = 0$. The difference as compared with data presented in Fig. 4 does not exceed 0.5%. Omitting the Coulomb energy contribution to the total energy $E$ of Eq. (22) and evaluating the bulk energy $E_V$ of Eq. (25), one can obtain the surface part of energy $E_S = E - E_V$ and the surface tension coefficient $\sigma [R_e]$ on the equimolar dividing surface for nuclei with different mass number $A \sim R_e^3$ and asymmetry parameter $X$. The dependence of the surface tension coefficient $\sigma [R_e]$ on the doubled inverse equimolar radius $2/R_e$ (see Eq. (22)) is shown in Fig. 5. The surface tension $\sigma [R_e, X]$ approaches the planar limit $\sigma_\infty (X)$ in the limit of zero curvature $2/R_e \to 0$. As seen from Fig. 5, the planar limit $\sigma_\infty (X)$ depends on the asymmetry parameter. This dependence reflects the fact that the symmetry energy $b$ in mass formula contains both the volume $b_V$ and surface $b_S$ contributions, see Refs. [7, 8]

\[ b(A) = b_V + b_S \, A^{-1/3}. \]  

(27)

In Fig. 6 we show the $X$-dependence of the surface tension $\sigma_\infty (X)$. This dependence can be approximated by

\[ \sigma_\infty (X) = \sigma_0 + \sigma_- X^2. \]  

(28)

The dependence of parameters $\sigma_0$ and $\sigma_-$ on the Skyrme force parametrization is shown in Table I. The isovector term $\sigma_-$ in the surface tension (28) is related to the surface contribution $b_S$ in Eq. (27) to the symmetry energy as

\[ b_S \approx 4\pi r_0^2 \sigma_-. \]  

(29)

see Appendix A, Eq. (A5). The numerical calculation of the volume symmetry energy gives for SkM force $b_V = 26.5$ MeV. Using Eq. (29), we evaluate the surface-to-volume ratio $r_{SV} = |b_S/b_V| = 1.17 \div 1.47$ for Skyrme force parametrizations from Table I. Note that in the previous theoretical calculations, the value of surface-to-volume ratio $r_{SV}$ varies strongly within the interval $1.6 \leq r_{SV} \leq 2.8$, see Refs. [7, 9].

The slope of curves $\sigma [R_e]$ in Fig. 5 gives the Tolman...
length $\xi$, see Eq. 19. The value of the Tolman length $\xi$ depends significantly on the asymmetry parameter $X$. In Fig. 7 we show such kind of dependence obtained from results of Fig. 4.

As seen from Fig. 7, one can expect the enhancement of the curvature effects in neutron rich nuclei. The $X$-dependence of Tolman length $\xi$ can be approximated as

$$\xi(X) = \xi_0 + \xi_- X^2.$$  \hspace{1cm} (30)

Both parameters $\xi_0$ and $\xi_-$ as well as the surface tension parameter $\sigma_-$ are rather sensitive to the Skyrme force parametrization, see Table I.

**IV. NUCLEAR MATTER EQUATION OF STATE AND $\left(A^{-1/3}, X\right)$-EXPANSIONS FOR FINITE NUCLEI**

Below we will consider the relation of the nuclear macroscopic characteristics (surface and symmetry energies, Tolman length, incompressibility, etc.) to the bulk properties of nuclear matter. Assuming a small deviations from the equilibrium, the equation of state (EOS) for an asymmetric nuclear matter can be written in the form expansion around the saturation point. One has for the energy per particle (at zero temperature)

$$\mathcal{E}(\epsilon, x) = \frac{E_{\infty}}{A} = \mu_{\infty} + \frac{K_{\infty}}{18} \epsilon^2 + b_{\infty} x^2 + \ldots ,$$  \hspace{1cm} (31)

where

$$\epsilon = \frac{\rho - \rho_{\infty}}{\rho_{\infty}}, \quad x = \frac{\rho_\alpha}{\rho}, \quad \rho = \rho_\alpha + \rho_\beta, \quad \rho_- = \rho_\alpha - \rho_\beta,$$

$\rho_{\infty}$ is the matter saturation (equilibrium) density, $\mu_{\infty}$ is the chemical potential, $K_{\infty}$ is the nuclear matter incompressibility and $b_{\infty}$ is the symmetry energy coefficient (all values are taken at the saturation point $\epsilon = 0$ and $x = 0$). Coefficients of expansion (31) are determined through the derivatives of the energy per particle $\mathcal{E}(\epsilon, x)$ at the saturation point:

$$\mu_{\infty} = \left. \frac{E_{\infty}}{A} \right|_{\rho=\rho_{\infty}, x=0} = \mathcal{E}(0,0),$$

$$K_{\infty} = 9 \rho^2 \frac{\partial^2 E_{\infty}/A}{\partial \rho^2} \bigg|_{\rho=\rho_{\infty}, x=0} = 9\rho_{\infty}^2 \mathcal{E}^{(2,0)},$$

$$b_{\infty} = \frac{1}{2} \frac{\partial^2 E_{\infty}/A}{\partial x^2} \bigg|_{\rho=\rho_{\infty}, x=0} = \frac{1}{2} \mathcal{E}^{(0,2)}. \hspace{1cm} (33)$$

We use the short notation

$$\mathcal{E}^{(n,m)} = \frac{\partial^n \mathcal{E}}{\partial \epsilon^n \partial x^m} \bigg|_{\epsilon=0, x=0}.$$  

Some coefficients $\mathcal{E}^{(n,m)}$ are vanishing. From the condition of minimum of $\mathcal{E}(\epsilon, x)$ at the saturation point one has $\mathcal{E}^{(1,0)} = \mathcal{E}^{(0,1)} = 0$. Odd derivatives with respect to $x$, i.e., $\mathcal{E}^{(n,m)}$ for odd $m$, also vanish because of the charge symmetry of nuclear forces.

Using $\mathcal{E}(\epsilon, x)$, one can also evaluate chemical potentials $\mu$, $\mu_-$ and pressure $P$ of the nuclear matter beyond the saturation point. Namely,

$$\mu(\epsilon, x) = \left. \frac{\partial E_{\infty}}{\partial A} \right|_{A,\epsilon} = \frac{\partial}{\partial \epsilon} (1 + \epsilon) \mathcal{E} - x \frac{\partial \mathcal{E}}{\partial x},$$

$$\mu_-(\epsilon, x) = \left. \frac{\partial E_{\infty}}{\partial A} \right|_{A,\epsilon} = \frac{\partial \mathcal{E}}{\partial x}, \hspace{1cm} (34)$$

$$P(\epsilon, x) = - \left. \frac{\partial E_{\infty}}{\partial V} \right|_{A,\epsilon} = \rho_{\infty} (1 + \epsilon)^2 \frac{\partial \mathcal{E}}{\partial \epsilon}. \hspace{1cm} (35)$$

Similarly to Eq. (31), in a finite uncharged system the energy per particle $E/A$ (we use $A = N + Z, A_- = N - Z, X = A_-/A$) of the finite droplet is usually presented as $(A^{-1/3}, X)$-expansion around infinite matter using the leptodermous approximation

$$E \equiv E(X, A^{-1/3}) = a_V + X^2 b_V +$$

$$A^{-1/3}(a_S + X^2 b_S + a_c A^{-1/3} + X^2 b_c A^{-1/3})$$  \hspace{1cm} (36)

$$= a_V + a_S A^{-1/3} + a_c A^{-2/3} +$$

$$X^2(b_V + b_S A^{-1/3} + b_c A^{-2/3}) \hspace{1cm} (37)$$

where $a_V$, $a_S$ and $a_c$ are, respectively, the volume, surface and curvature energy coefficients, $b_V$, $b_S$ and $b_c$ are,
TABLE I. Nuclear bulk parameters for different Skyrme forces.  

|       | SkM  | SkM* | SLy230b | T6  |
|-------|------|------|---------|-----|
| $\mu_\infty$ (MeV) | -15.77 | -15.77 | -15.97 | -15.96 |
| $\rho_\infty$ (fm$^{-3}$) | 0.1603 | 0.1603 | 0.1595 | 0.1609 |
| $K_\infty$ (MeV) | 216.6 | 216.6 | 229.9 | 235.9 |
| $K_3$ (MeV) | 913.5 | 913.5 | 1016 | 1032 |
| $K_{\text{sym}}$ (MeV) | -148.8 | -155.9 | -119.7 | -211.5 |
| $b_\infty$ (MeV) | 30.75 | 30.03 | 32.01 | 29.97 |
| $L_\infty$ (MeV) | 49.34 | 45.78 | 45.97 | 30.86 |
| $\sigma_0$ (MeV-fm$^{-2}$) | 0.9176 | 0.9601 | 1.006 | 1.021 |
| $\xi_0$ (fm) | -0.3565 | -0.3703 | -0.3677 | -0.3593 |
| $\sigma_-$ (MeV-fm$^{-2}$) | -3.118 | -3.094 | -3.131 | -2.413 |
| $\xi_-$ (fm) | -5.373 | -5.163 | -4.590 | -2.944 |

respective, the volume, surface and curvature symmetry coefficients. The nuclear chemical potentials $\lambda$ and $\lambda_-$ are derived as

$$
\lambda(X, A^{-1/3}) = E/A - \frac{1}{3} \frac{\partial E/A}{\partial A^{-1/3}} X^{1/3} \frac{\partial E/A}{\partial X},
$$

$$
\lambda_-(X, A^{-1/3}) = \frac{\partial E/A}{\partial X}.
$$

(38)

Following Gibbs-Tolman method, one can derive the actual nuclear matter densities $\rho$ and $\rho_-$ from the conditions

$$
\mu(\epsilon, x) = \lambda(X, A^{-1/3}),
$$

$$
\mu_-(\epsilon, x) = \lambda_-(X, A^{-1/3}).
$$

(39)

Using Eq. (39), one can establish the relation of the macroscopic energy coefficients in the liquid drop model expansion Eq. (39) to the nuclear matter parameters in EOS (41), see Eqs. (A1) - (A9) of Appendix A. The results of numerical calculations of relevant quantities are represented in Tables I and II.

The value of the Tolman length $\xi_0$ can be related to the nuclear matter incompressibility $K_\infty$ and the surface tension coefficient $\sigma$ (10). Let us consider the expansion like (20) around the equilibrium state of the symmetric nuclear matter for the bulk density and the chemical potential:

$$
\varphi_\nu = \rho_\infty + \rho_1 \frac{r_0}{R_0} + \rho_2 \left( \frac{r_0}{R_0} \right)^2 + \ldots,
$$

$$
\lambda = \lambda_\infty + \lambda_1 \frac{r_0}{R_0} + \lambda_2 \left( \frac{r_0}{R_0} \right)^2 + \ldots.
$$

(40)

TABLE II. Mass formula coefficients for finite nuclei.  

|       | SkM  | SkM* | SLy230b | T6  |
|-------|------|------|---------|-----|
| $a_\nu$ (MeV) | -15.8 | -15.8 | -16.0 | -16.0 |
| $a_\sigma$ (MeV) | 15.0 | 15.7 | 16.5 | 16.7 |
| $a_c$ (MeV) | 7.30 | 7.92 | 8.26 | 8.16 |
| $b_\nu$ (MeV) | 30.8 | 30.0 | 32.0 | 30.0 |
| $b_s$ (MeV) | -44.2 | -44.1 | -44.9 | -35.1 |
| $b_c$ (MeV) | 35.7 | 35.1 | 28.6 | 17.3 |

$r_{s/\nu} = |b_s/b_\nu| = 1.44, 1.47, 1.40, 1.17$

where $\lambda_\infty \equiv \mu_\infty$ is the equilibrium chemical potential for the infinite nuclear matter. We will apply the Gibbs – Duhem relation

$$
dP_\nu = \nu \sigma d\lambda.
$$

(41)

Using the generalized Laplace equation (13) and Eqs. (19) and (40), we rewrite Eq. (41) as

$$
d \left( \frac{2\sigma_\infty}{R_\epsilon} - \frac{2\sigma_\infty \xi}{R_\epsilon^2} + \ldots \right) = \left( \rho_\infty + \rho_1 \frac{r_0}{R_\epsilon} + \rho_2 \frac{r_0^2}{R_\epsilon^2} + \ldots \right)
$$

$$
\times d \left( \lambda_\infty + \lambda_1 \frac{r_0}{R_\epsilon} + \lambda_2 \frac{r_0^2}{R_\epsilon^2} + \ldots \right).
$$

(42)

Nuclear incompressibility $K_\infty$ in terms of expansion (40) reads

$$
K_\infty = 9 \left. \frac{\partial P_\nu}{\partial \nu} \right|_{\rho_\nu=\rho_\infty} = 9 \left. \frac{\partial \lambda}{\partial \nu} \right|_{\nu=\rho_\infty} = 9 \rho_\infty \frac{\lambda_1}{\rho_1}.
$$

(43)

Equating in (12) the terms of the same order in curvature $R^{-1}_\epsilon$ and taking the incompressibility definition from Eq. (43), one obtains the following relations

$$
\rho_1 = 18 \frac{\sigma_\infty}{K_\infty r_0}, \quad \lambda_1 = 2 \frac{\sigma_\infty}{\rho_\infty r_0}
$$

(44)

and

$$
\xi = -9 \frac{\sigma_\infty}{K_\infty \rho_\infty} - \frac{\lambda_2}{\lambda_1} r_0.
$$

(45)

Equation (45) gives an idea how the Tolman length $\xi$ depends on the incompressibility $K_\infty$ and the surface tension coefficient $\sigma$. In particular, if the second order $\sim R^{-2}$ correction in the chemical potential $\lambda$ of Eq. (40) is negligible, namely,

$$
\lambda = \lambda_\infty + \lambda_1 \frac{r_0}{R_\epsilon},
$$

we obtain from Eq. (45) the following important relation

$$
\xi \approx -9 \frac{\sigma_\infty}{K_\infty \rho_\infty}.
$$

(46)
That means that the Tolman length disappears in the case of incompressible Fermi liquid with \( K_\infty \to \infty \). We note also the relation of the surface tension coefficient \( \sigma \) to the incompressibility \( K_\infty \) and the diffuseness parameter \( a \) of the nuclear surface layer \([1, 2]\)

\[
\sigma_\infty \approx \frac{1}{18} K_\infty \rho_\infty a . \tag{47}
\]

Comparing Eqs. (46) and (47) we conclude that

\[
\xi \approx -a/2 .
\]

This result leads to the conclusions that the nuclear Tolman length is negative and the non-zero value of \( \xi \) requires the finite diffuse layer.

V. CONCLUSIONS

Considering a small two-component, charged droplet with a finite diffuse layer, we have introduced a formal dividing surface of radius \( R \) which splits the droplet onto volume and surface parts. The corresponding splitting was also done for the binding energy \( E \). Assuming that the dividing surface is located close to the interface, we are then able to derive the surface energy \( E_S \). In general, the surface energy \( E_S \) includes the contributions from the surface tension \( \sigma \) and from the binding energy of \( A_S \) particles located within the surface layer. The equimolar surface and thereby the actual physical size of the droplet are derived by the condition \( g_S \lambda + g_- S \lambda_- = 0 \) which means that the latter contribution is excluded from the surface energy providing \( E_S \propto \sigma \).

In a small nucleus, the diffuse layer and the curved interface affect the surface properties significantly. In agreement with Gibbs-Tolman concept \([11, 12]\), two different radii have to be introduced in this case. The first radius, \( R_s \), is the surface tension radius (Laplace radius) which provides the minimum of the surface tension coefficient \( \sigma \) and the fulfillment of the Laplace relation \([13]\) for capillary pressure. The another one, \( R_e \), is the equimolar radius which corresponds to the equimolar dividing surface due to the condition \([15]\) and defines the physical size of the sharp surface droplet, i.e., the surface at which the surface tension is applied. The difference of two radii \( R_e - R_s \) in an asymptotic limit of large system \( A \to \infty \) derives the Tolman length \( \xi \). That means the presence of curved surface is not sufficient for the presence of the curvature correction in the surface tension. The finite diffuse layer in the particle distribution is also required. We point out that the Gibbs-Tolman theory allows to treat a liquid drop within thermodynamics with minimum assumptions. Once the binding energy and chemical potential of the nucleus are known its equimolar radius, radius of tension and surface energy can be evaluated using the equation of state for the infinite nuclear matter. For a symmetric liquid the value of Tolman length is about of half of the diffuseness parameter \( a \) for the nuclear surface layer. We have also established the relation of the macroscopic energy coefficients in the liquid drop model expansion Eq. (36) to the nuclear matter parameters.

The sign and the magnitude of the Tolman length \( \xi \) depend on the interparticle interaction. We have shown that the Tolman length is negative for a nuclear Fermi liquid drop. As a consequence, the curvature correction to the surface tension leads to the hindrance of the yield of light fragments at the nuclear multifragmentation in heavy ion collisions. We have also shown that the Tolman length is sensitive to the neutron excess and its absolute value growth significantly with growing asymmetry parameter \( X \).

Appendix A: Relation of nuclear matter EOS to the characteristics of finite nuclei

We will start from the nuclear matter EOS given by Eq. (31) and take into consideration the relations \([32, 33]\) and the following higher order coefficients

\[
K_3 = 6 K_\infty + 27 \rho^2 \frac{\partial^3 E_\infty}{\partial \rho^3} |_{\rho=\rho_\infty, x=0} ,
\]

\[
L_\infty = \frac{3}{2} \rho \frac{\partial^2 E_\infty}{\partial \rho \partial x^2} |_{\rho=\rho_\infty, x=0} , \tag{A1}
\]

\[
K_{\text{sym}} = \frac{9}{2} \rho^3 \frac{\partial^4 E_\infty}{\partial \rho^2 \partial x^2} |_{\rho=\rho_\infty, x=0} , \tag{A2}
\]

for the expansion \([31]\). Here \( K_3 \) is the bulk anharmonicity coefficient, \( L_\infty \) is the density-symmetry coefficient (symmetry energy slope parameter), \( K_{\text{sym}} \) is the symmetry energy curvature parameter. Using \([19]\), we write also

\[
\sigma \approx \sigma_\infty (1 - 2 \xi / R_e) ,
\]

\[
\sigma_\infty \approx \sigma_0 + \sigma_- X^2 , \quad \xi \approx \xi_0 + \xi_- X^2 \tag{A3}
\]

and

\[
a_\nu = \mu_\infty , \quad b_\nu = b_\infty . \tag{A4}
\]

Using the conditions \([32]\) for the chemical potentials and both relations \([38, 34]\), we obtain
The value of neutron skin

\[
\frac{\rho - \rho_\infty}{\rho_\infty} \approx A^{-1/3} \frac{6a_s}{K_\infty} + X^2 \left[ \frac{3L_\infty}{K_\infty} + A^{-1/3} \left( \frac{6(b_S - 2a_SL_\infty/K_\infty)}{b_\infty} \right) \left( 1 - \frac{L_\infty}{b_\infty} \right) \frac{6a_s}{K_\infty} \left( L_\infty \left( 1 - \frac{K_3}{K_\infty} \right) + K_{sym} \right) \right]
\]

and

\[
a_s = 4\pi r_0^2 \sigma_0 \quad , \quad b_S = 4\pi r_0^2 \left( \sigma_- + \frac{2L_\infty}{K_\infty} \sigma_0 \right) \quad , \quad a_c = -8\pi r_0 \sigma_0 \left( \xi_0 + \frac{3\sigma_0}{K_\infty \rho_\infty} \right), \quad (A5)
\]

\[
b_c = -8\pi r_0 \sigma_0 \left\{ \frac{\xi_- + \left(L_\infty \sigma_- + \frac{3\sigma_0}{K_\infty \rho_\infty} \right) \xi_0 + \frac{3\sigma_0}{K_\infty \rho_\infty} \left[ L_\infty \left( 4 + \frac{K_3}{K_\infty} \right) - \frac{K_{sym}}{K_\infty} \right] \frac{3\sigma_0}{K_\infty \rho_\infty} \left( 2 + \frac{K_\infty \sigma_-}{2b_\infty \sigma_0} \right) \right\} . \quad (A6)
\]

Here we have assumed \( A^{-1/3} \ll 1 \). The equimolar, \( R_e \), and Laplace, \( R_s \), radii defined by Eqs. (17) and (18) read

\[
R_e \approx r_0 A^{1/3} \left[ 1 - A^{-1/3} \frac{8\pi r_0^2 \sigma_0}{K_\infty} \right]
+ X^2 \left[ \frac{L_\infty}{K_\infty} - A^{-1/3} \left( \frac{8\pi r_0^2 \sigma_-}{K_\infty} \left( 1 - \frac{L_\infty}{b_\infty} + \frac{K_\infty}{3\mu_\infty} \right) \right) + \frac{8\pi r_0^2 \sigma_0}{K_\infty} \left[ \frac{L_\infty}{K_\infty} \left( 3 + \frac{K_3}{K_\infty} \right) - \frac{K_{sym}}{K_\infty} \right] \right] . \quad (A7)
\]

\[
R_s \approx r_0 A^{1/3} \left[ 1 - A^{-1/3} \left( \frac{\xi_0}{r_0} + \frac{8\pi r_0^2 \sigma_0}{K_\infty} \right) \right]
+ X^2 \left[ \frac{L_\infty}{K_\infty} - A^{-1/3} \left( \frac{\xi_-}{r_0} + \frac{3\sigma_-}{b_\infty \rho_\infty} \left( \frac{\sigma_-}{\sigma_0} + \frac{2L_\infty}{K_\infty} \frac{2b_\infty}{3\mu_\infty} \right) \right) X^2 = \frac{3\sigma_-}{b_\infty \rho_\infty} \left( \frac{\sigma_- + 2L_\infty}{K_\infty} \frac{2b_\infty}{3\mu_\infty} \right) \right] . \quad (A8)
\]

Using the derivations of \( R_e \) and \( R_s \), one obtains

\[
R_e - R_s \approx \xi_0 + \left( \frac{\xi_-}{r_0} + \frac{3\sigma_-}{b_\infty \rho_\infty} \left( \frac{\sigma_-}{\sigma_0} + \frac{2L_\infty}{K_\infty} \frac{2b_\infty}{3\mu_\infty} \right) \right) X^2 \approx \frac{3\sigma_-}{b_\infty \rho_\infty} \left( \frac{\sigma_- + 2L_\infty}{K_\infty} \frac{2b_\infty}{3\mu_\infty} \right) X^2 . \quad (A9)
\]

To describe separately the neutron and proton density distributions we introduce the neutron radius, \( R_n \), and the proton radius, \( R_p \), as the dividing radii with zero value for the corresponding surface densities \( \varrho_n, S = (\varrho_S + \varrho_-, S)/2 \) and \( \varrho_p, S = (\varrho_S - \varrho_-, S)/2 \) :

\[
\varrho_n, S|_{R=R_n} = 0 \quad , \quad \varrho_p, S|_{R=R_p} = 0 .
\]

The value of neutron skin \( r_{np} = R_n - R_p \) is then written as

\[
r_{np} = R_n - R_p \approx X \left[ \frac{2\sigma_-}{b_\infty \rho_\infty} + A^{-1/3} \left( \frac{4\pi r_0^2 \sigma_0}{3b_\infty} \left( \xi_- + \frac{\xi_0}{\sigma_0} \sigma_- \right) + 4\pi r_0^2 \sigma_- \left( \frac{2\sigma_-}{b_\infty \rho_\infty} + \frac{4\sigma_0}{K_\infty} \left( \frac{L_\infty}{K_\infty} + \frac{3b_\infty}{K_\infty} \right) \right) \right) \right] . \quad (A10)
\]

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