BERGMAN METRIC AND CAPACITY DENSITIES ON PLANAR DOMAINS

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ABSTRACT. We give quantitative estimates of the Bergman distance through positivity of capacity densities.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $0 \in \partial \Omega$. We set $\mathbb{D}_r := \{ z : |z| < r \}$ and $K_r := \overline{\mathbb{D}}_r - \Omega$. The famous Wiener’s criterion states that $0$ is a regular point if and only if
\[
\sum_{k=1}^{\infty} \frac{k}{\log[1 / C_t(K_{2-k})]} = \infty,
\]
where $C_t(K_{2-k})$ denotes the logarithmic capacity of $K_{2-k}$. In a similar manner, Carleson-Totik [4] characterized the Hölder continuity of the Green function $g_{\Omega}(\cdot, w)$ at $0$ through positivity of certain capacity density, under some mild restrictions on $\partial \Omega$ near $0$. It is also known that the regularity of $0$ implies that the Bergman kernel is exhaustive at $0$ (cf. [12]) and the Bergman metric is complete at $0$ (cf. [2], [11]). Zwonek [16] showed that the Bergman kernel is exhaustive at $0$ if and only if
\[
\sum_{k=1}^{\infty} 2^{2k} \log[1 / C_t(K_{2-k})] = \infty.
\]
On the other hand, a similar characterization for the Bergman completeness at $0$ is still missing, although some partial results exist (cf. [14]).

The goal of this paper is to get quantitative estimates of the Bergman distance through positivity of certain capacity densities.

Motivated by the work of Carleson-Totik [4], we give the following

Definition 1.1. Let $\varepsilon > 0$ and $0 < \lambda < 1$ be fixed. For every $a \in \partial \Omega$ we set
\[
K_t(a) := \overline{\mathbb{D}}_t(a) - \Omega; \quad \mathbb{D}_t(a) := \{ z : |z - a| < t \}
\]
\[
N_a(\varepsilon, \lambda) := \{ n \in \mathbb{Z}^+ : C_t(K_{\lambda^n}(a)) \geq \varepsilon \lambda^n \}
\]
\[
N^n_a(\varepsilon, \lambda) := N_a(\varepsilon, \lambda) \cap \{ 1, 2, \ldots, n \}.
\]
We define the $(\varepsilon, \lambda)$–capacity density of $\partial \Omega$ at $a$ by
\[
D_a(\varepsilon, \lambda) := \lim inf_{n \to \infty} \frac{|N^n_a(\varepsilon, \lambda)|}{n}.
\]

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We define the weak and strong $(\varepsilon, \lambda) -$capacity density of $\partial \Omega$ by

$$D_W(\varepsilon, \lambda) := \liminf_{n \to \infty} \frac{\inf_{a \in \partial \Omega} |N_a^n(\varepsilon, \lambda)|}{n}$$
and

$$D_S(\varepsilon, \lambda) := \liminf_{n \to \infty} \frac{|\bigcap_{a \in \partial \Omega} N_a^n(\varepsilon, \lambda)|}{n}$$
respectively.

It is easy to see that $D_W(\varepsilon, \lambda) \geq D_S(\varepsilon, \lambda)$, and $C_t(K_t(a)) \geq \varepsilon t$ implies $D_a(\varepsilon, \lambda) = 1$. Recall that $\partial \Omega$ is said to be uniformly perfect if $\inf_{a \in \partial \Omega} C_t(K_t(a)) \geq \varepsilon t$ (cf. [15]). Thus if $\partial \Omega$ is uniformly perfect then $D_W(\varepsilon, \lambda) = D_S(\varepsilon, \lambda) = 1$ for some $\varepsilon > 0$. On the other hand, it was pointed out in [4] that the domain $\Omega = \mathbb{D} - \{0\} - \bigcup_{k=1}^{\infty} \left[2^{-2k+1}, 2^{-2k}\right]$ satisfies $D_W(\varepsilon, 1/2) > 0$ for some $\varepsilon > 0$ while $\partial \Omega$ is non-uniformly perfect. Actually, one may verify that $D_S(\varepsilon, 1/2) > 0$.

Let $\delta(z)$ denote the euclidean distance from $z$ to $\partial \Omega$ and $d_B(z_0, z)$ the Bergman distance from a fixed point $z_0$ to $z$. We have

**Theorem 1.1.** (1) If $D_S(\varepsilon, \lambda) > 0$ for some $\varepsilon, \lambda$, then

$$d_B(z_0, z) \gtrsim |\log \delta(z)|, \quad \forall z \in \Omega.$$  

(2) If $D_W(\varepsilon, \lambda) > 0$ for some $\varepsilon, \lambda$, then

$$d_B(z_0, z) \gtrsim \frac{|\log \delta(z)|}{\log |\log \delta(z)|}$$

for all $z$ sufficiently close to $\partial \Omega$.

In [5], (1.1) was verified by a different method in case $\partial \Omega$ is uniformly perfect. Estimate of type (1.2) was first obtained by Blocki [1] for bounded pseudoconvex domains with Lipschitz boundaries in $\mathbb{C}^n$ (see also [6] and [8] for related results).

**Definition 1.2.** For $\varepsilon > 0$, $0 < \lambda < 1$ and $\gamma > 1$ we set

$$N_a(\varepsilon, \lambda) := \{n \in \mathbb{Z}^+ : C_t(K_t(a)) \geq \varepsilon \lambda^n \}$$

$$N_a^n(\varepsilon, \lambda, \gamma) := N_a(\varepsilon, \lambda, \gamma) \cap \{1, 2, \ldots, n\}.$$

We define the $(\varepsilon, \lambda, \gamma) -$capacity density of $\partial \Omega$ at $a$ by

$$D_a(\varepsilon, \lambda, \gamma) := \liminf_{n \to \infty} \frac{\sum_{k \in N_a^n(\varepsilon, \lambda, \gamma)} k^{-1}}{\log n}$$

and the weak and strong $(\varepsilon, \lambda, \gamma) -$capacity densities of $\partial \Omega$ by

$$D_W(\varepsilon, \lambda, \gamma) := \liminf_{n \to \infty} \frac{\inf_{a \in \partial \Omega} |N_a^n(\varepsilon, \lambda, \gamma)|}{\log n}$$
and

\[ D_S(\varepsilon, \lambda, \gamma) := \liminf_{n \to \infty} \frac{|\bigcap_{a \in \partial \Omega} N_a^n(\varepsilon, \lambda, \gamma)|}{\log n} \]

respectively.

Note that \( D_W(\varepsilon, \lambda, \gamma) \geq D_S(\varepsilon, \lambda, \gamma) \). If \( C_l(K_t(a)) \geq \varepsilon t^\gamma \) for some \( \varepsilon > 0 \) and \( \gamma > 1 \), then \( D_a(\varepsilon, \lambda, \gamma) = 1 \) in view of the following well-known formula

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) = \text{Euler constant}. \]

**Theorem 1.2.** If \( D_W(\varepsilon, \lambda, \gamma) > 0 \) for some \( \varepsilon, \lambda, \gamma \), then

\[ d_B(z_0, z) \gtrsim \log \log |\log \delta(z)| \]

for all \( z \) sufficiently close to \( \partial \Omega \).

The proofs of the theorems depend on precise estimates of the Green function. We also develop a general method to obtain quantitative results on hyperconvexity of bounded planar domains through the logarithmic capacity, which is of independent interest.

Some interesting questions arise.

**Problem 1.** Does there exist an example with \( D_W(\varepsilon, \lambda) > 0 \) (resp. \( D_W(\varepsilon, \lambda, \gamma) > 0 \)) for some \( \varepsilon, \lambda \) (resp. \( \varepsilon, \lambda, \gamma \)) while \( D_S(\varepsilon, \lambda) = 0 \) (resp. \( D_S(\varepsilon, \lambda, \gamma) = 0 \))?

**Problem 2.** Suppose \( D_S(\varepsilon, \lambda, \gamma) > 0 \) for some \( \varepsilon, \lambda, \gamma \). Does one have

\[ d_B(z_0, z) \gtrsim \log |\log \delta(z)| \]

for \( z \) sufficiently close to \( \partial \Omega \)?

2. Capacities

In this section we shall review different notions of capacities and present some basic properties of them. Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) and \( K \subset \Omega \) a compact (non-polar) set in \( \Omega \). We define the Dirichlet capacity \( C_d(K, \Omega) \) of \( K \) relative to \( \Omega \) by

\[ C_d(K, \Omega) = \inf_{\phi \in L(K, \Omega)} \int_\Omega |\nabla \phi|^2 \]

where \( L(K, \Omega) \) is the set of all locally Lipschitz functions \( \phi \) on \( \Omega \) with a compact support in \( \Omega \) such that \( 0 \leq \phi \leq 1 \) and \( \phi|_K = 1 \). If \( \Omega = \mathbb{C} \) then we write \( C_d(K) \) for \( C_d(K, \Omega) \). By the definition we have

\[ K_1 \subseteq K_2 \text{ and } \Omega_1 \supseteq \Omega_2 \Rightarrow C_d(K_1, \Omega_1) \leq C_d(K_2, \Omega_2). \]

In view of Dirichlet’s principle, the infimum in (2.1) is attained at the function \( \phi_{\min} \) which is exactly the Perron solution to the following (generalized) Dirichlet problem in \( \Omega \setminus K \):

\[ \Delta u = 0; \quad u = 0 \text{ n.e. on } \partial \Omega; \quad u = 1 \text{ n.e. on } \partial K. \]
We call $\phi_{\min}$ the capacity potential of $K$ relative to $\Omega$. In case $\partial \Omega$ and $\partial K$ are both $C^1$—smooth, integration by parts gives

$$
C_d(K, \Omega) = \int_{\Omega} |\nabla \phi_{\min}|^2 = \int_{\Omega \setminus K} |\nabla \phi_{\min}|^2
= - \int_{\Omega \setminus K} \phi_{\min} \Delta \phi_{\min} + \int_{\partial(\Omega \setminus K)} \phi_{\min} \frac{\partial \phi_{\min}}{\partial \nu} d\sigma
= \int_{\partial K} \frac{\partial \phi_{\min}}{\partial \nu} d\sigma =: \text{flux}_{\partial K} \phi_{\min}
$$

(2.4)

where $\nu$ is the outward unit normal vector fields on $\partial(\Omega \setminus K)$. By the maximum principle we conclude that $\partial \phi_{\min} / \partial \nu \geq 0$ holds on $\partial K$.

Let $g_{\Omega}(z, w)$ be the (negative) Green function on $\Omega$. Let $z \in \Omega \setminus K$ be given. Since $\Delta g_{\Omega}(\cdot, z) = 2\pi \delta_z$, where $\delta_z$ stands for the Dirac measure at $z$, we infer from Green’s formula that

$$
2\pi \phi_{\min}(z) = \int_{\Omega \setminus K} \phi_{\min} \Delta g_{\Omega}(\cdot, z) = \int_{\Omega \setminus K} g_{\Omega}(\cdot, z) \Delta \phi_{\min}
+ \int_{\partial(\Omega \setminus K)} \phi_{\min} \frac{\partial g_{\Omega}(\cdot, z)}{\partial \nu} d\sigma - \int_{\partial(\Omega \setminus K)} g_{\Omega}(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma
= \int_{\partial K} \frac{\partial g_{\Omega}(\cdot, z)}{\partial \nu} d\sigma - \int_{\partial K} g_{\Omega}(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma
= - \int_{\partial K} g_{\Omega}(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma
$$

(2.5)

because $g_{\Omega}(\cdot, z)$ is harmonic on $K$. This equality combined with (2.4) gives the following fundamental inequality which connects the capacity, Green’s function and the capacity potential:

$$
\frac{C_d(K, \Omega)}{2\pi} \inf_{\partial K} (-g_{\Omega}(\cdot, z)) \leq \phi_{\min}(z) \leq \frac{C_d(K, \Omega)}{2\pi} \sup_{\partial K} (-g_{\Omega}(\cdot, z)), \quad z \in \Omega \setminus K.
$$

(2.6)

Since $\Omega \setminus K$ can be exhausted by bounded domains with smooth boundaries, we conclude by passing to a standard limit process that the same inequality holds for every compact set $K$.

For a finite Borel measure $\mu$ on $\mathbb{C}$ whose support is contained in $K$ we define its Green potential relative to $\Omega$ by

$$
p_{\mu}(z) = \int_{\Omega} g_{\Omega}(z, w) d\mu(w), \quad z \in \Omega.
$$
Clearly, $p_\mu$ is negative, subharmonic on $\Omega$, harmonic on $\Omega \setminus K$, and $p_\mu(z) \to 0$ as $z \to \partial \Omega$. Given $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega p_\mu \Delta \phi dV = \int_\Omega \left[ \int_\Omega g_\Omega(z, w) d\mu(w) \right] \Delta \phi(z) dV(z)$$

$$= \int_\Omega \left[ \int_\Omega g_\Omega(z, w) \Delta \phi(z) dV(z) \right] d\mu(w) \quad \text{(Fubini’s theorem)}$$

$$= \int_\Omega \left[ \int_\Omega \Delta g_\Omega(z, w) \phi(z) dV(z) \right] d\mu(w) \quad \text{(Green’s formula)}$$

$$= \int_\Omega 2\pi \phi(w) d\mu(w).$$

Thus we obtain

$$(2.7) \quad \Delta p_\mu = 2\pi \mu$$

in the sense of distributions. The Green energy $I(\mu)$ of $\mu$ is given by

$$I(\mu) := \int_\Omega p_\mu d\mu = \int_\Omega \int_\Omega g_\Omega(z, w) d\mu(z) d\mu(w).$$

By (2.7) we have

$$(2.8) \quad I(\mu) = \frac{1}{2\pi} \int_\Omega p_\mu \Delta p_\mu = -\frac{1}{2\pi} \int_\Omega |\nabla p_\mu|^2.$$

Every compact set $K$ has an **equilibrium measure** $\mu_{\text{max}}$, which maximizes $I(\mu)$ among all Borel probability measures $\mu$ on $K$. A fundamental theorem of Frostman states that

1. $p_{\mu_{\text{max}}} \geq I(\mu_{\text{max}})$ on $\Omega$;
2. $p_{\mu_{\text{max}}} = I(\mu_{\text{max}})$ on $K \setminus E$ for some $F_s$ pole set $E \subset \partial K$.

By the uniqueness of the solution of the (generalized) Dirichlet problem we have

$$(2.9) \quad \phi_{\text{min}} = p_{\mu_{\text{max}}}/I(\mu_{\text{max}}).$$

We define the Green capacity $C_g(K, \Omega)$ of $K$ relative to $\Omega$ by

$$C_g(K, \Omega) := e^{I(\mu_{\text{max}})}.$$

It follows from (2.8) and (2.9) that

$$(2.10) \quad \frac{C_g(K, \Omega)}{2\pi} = -\frac{1}{\log C_g(K, \Omega)}.$$

Analogously, we may define the logarithmic capacity $C_l(K)$ of $K$ by

$$\log C_l(K) := \sup_{\mu} \int_\C \int_\C \log |z - w| d\mu(z) d\mu(w)$$

where the supremum is taken over all Borel probability measures $\mu$ on $\C$ whose support is contained in $K$. Let $R$ be the diameter of $\Omega$ and set $d = d(K, \Omega)$. Since

$$\log |z - w|/R \leq g_\Omega(z, w) \leq \log |z - w|/d, \quad z, w \in K,$$
we have
\begin{equation}
\log C_l(K) - \log R \leq \log C_g(K, \Omega) \leq \log C_l(K) - \log d.
\end{equation}

3. Estimates of the capacity potential

We first give a basic lemma as follows.

**Lemma 3.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( 0 \in \partial \Omega \). Let \( 0 \leq h \leq 1 \) be a harmonic function on \( \Omega \) such that \( h = 0 \) n.e. on \( \partial \Omega \cap \mathbb{D}_{r_0} \) for some \( r_0 < 1 \). For all \( 0 < \alpha < 1/16 \) we have
\begin{equation}
\sup_{\Omega \cap \mathbb{D}_r} h \leq \exp \left[ -\frac{\log 1/(16\alpha)}{\log 1/\alpha} \int_0^\alpha \left( t \log \frac{t/\alpha}{2C_l(K_t)} \right)^{-1} dt \right]
\end{equation}
where \( K_t := \mathbb{D}_t - \Omega \).

**Proof.** The idea of the proof comes from [10] (see also [7]). Let \( \mathbb{D} \) be the unit disc. For \( t < r_0 \) and \( |z| = t \) we have
\begin{equation}
\sup_{\partial K_{\alpha t}} (-g_{\mathbb{D}}(\cdot, z)) \leq \log 2 + \sup_{\partial K_{\alpha t}} (-\log |\cdot - z|) \leq \log 2 - \log |t - \alpha t| \leq \log 4/t.
\end{equation}

Let \( \phi_{\alpha t} \) be the capacity potential of \( K_{\alpha t} \) relative to \( \mathbb{D} \). By (2.6) and (3.2) we have
\begin{equation}
\phi_{\alpha t}(z) \leq (\log 4/t) \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \quad \text{for } |z| = t.
\end{equation}
It follows that for \( z \in \Omega \cap \partial \mathbb{D}_t \)
\begin{equation}
(1 - \phi_{\alpha t}(z)) \sup_{\Omega \cap \mathbb{D}_t} h \geq \left[ 1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \right] h(z),
\end{equation}
while the same inequality holds for \( z \in \partial \Omega \cap \mathbb{D}_t \), because \( \lim_{z \to \zeta} h = 0 \) for n.e. \( \zeta \in \partial \Omega \cap \mathbb{D}_t \). By the (generalized) maximum principle, (3.3) holds on \( \Omega \cap \mathbb{D}_t \). On the other hand, since for \( |z| = \alpha t \) we have
\begin{equation}
\inf_{\partial K_{\alpha t}} (-g_{\mathbb{D}}(\cdot, z)) \geq \log 1/2 + \inf_{\partial K_{\alpha t}} (-\log |\cdot - z|) \geq \log 1/2 - \log(2\alpha t) = \log \frac{1}{4\alpha t},
\end{equation}
it follows from (2.6) that
\begin{equation}
\phi_{\alpha t}(z) \geq \log \frac{1}{4\alpha t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \quad \text{for } |z| = \alpha t.
\end{equation}
Substituting (3.5) into (3.3) we have
\begin{equation}
\sup_{\Omega \cap \mathbb{D}_t} h \cdot \frac{1 - \log \frac{1}{4\alpha t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}}{1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}} \leq \sup_{\Omega \cap \mathbb{D}_t} h \cdot \left( 1 + \frac{\log(16\alpha) \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}}{1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}} \right)
\end{equation}
for \( z \in \Omega \cap D_{\alpha t} \). Set \( M(t) := \sup_{\Omega \cap D_t} h \). It follows from (3.6) and (2.10) that
\[
\frac{\log M(t)}{t} - \frac{\log M(\alpha t)}{t} \geq \frac{1}{16\alpha} \left( t \log \frac{t}{4C_g(K_{\alpha t}, D)} \right)^{-1} \geq \frac{1}{16\alpha} \left( t \log \frac{t}{2C_l(K_{\alpha t})} \right)^{-1} \text{ (by (2.11)).}
\]
Integration from \( r/\alpha \) to \( r_0 \) gives
\[
\log \frac{1}{16\alpha} \int_{r/\alpha}^{r_0} \left[ t \log \frac{t}{2C_l(K_{\alpha t})} \right]^{-1} dt \leq \int_{r/\alpha}^{r_0} \frac{\log M(t)}{t} dt - \int_{r/\alpha}^{r_0} \frac{\log M(\alpha t)}{t} dt \leq \int_{r/\alpha}^{r_0} \frac{\log M(t)}{t} dt - \int_{r}^{r/\alpha} \frac{\log M(t)}{t} dt \leq (\log M(r_0) - \log M(r)) \log 1/\alpha,
\]
because \( M(t) \) is nondecreasing. Thus (3.1) holds because \( M(r_0) \leq 1 \).

**Theorem 3.2.** Fix a compact set \( E \) in \( \Omega \) with \( C_l(E) > 0 \). Let \( \phi_E \) be the capacity potential of \( E \) relative to \( \Omega \). Set \( d = d(E, \partial \Omega) \). Then for all \( 0 < \alpha < 1/16 \)
\begin{equation}
\sup_{\Omega \cap D_t} \phi_E \leq \exp \left[ -\frac{1}{\log 1/\alpha} \int_{r}^{\alpha r} \left( t \log \frac{t}{2C_l(K_t)} \right)^{-1} dt \right].
\end{equation}

**Proof.** The solution of the (generalized) Dirichlet problem gives \( \lim_{z \to \zeta} \phi_E(z) = 0 \) for n.e. \( \zeta \in \partial \Omega \), and the (generalized) maximum principle gives \( 0 \leq \phi_E \leq 1 \). Thus Lemma 3.1 applies. \( \square \)

We shall give a few interesting consequences of Theorem 3.2. Let \( 0 < \lambda < 1 \). For \( N \gg 1 \) we have
\begin{equation}
\int_0^{\alpha d} \left( t \log \frac{t/\alpha}{2C_l(K_t)} \right)^{-1} dt \geq \int_0^{d} \left( t \log \frac{d}{2C_l(K_{\lambda t})} \right)^{-1} dt = \log \frac{1}{\lambda} \int_0^\infty \left( \log \frac{d}{2C_l(K_{\lambda s})} \right)^{-1} ds \\
\geq \sum_{n=N}^{\infty} \int_{\lambda^{n+1}}^{\lambda^n} \left( \log \frac{1}{2C_l(K_{\lambda s})} \right)^{-1} ds \\
\geq \sum_{n=N}^{\infty} \log[1/C_l(K_{\lambda^{n+1}})].
\end{equation}
On the other hand, we have
\[
\sum_{k=\lambda^{-N}}^{\infty} \frac{k}{\log[1/C_l(K_{\lambda^k})]} = \sum_{n=N}^{\infty} \sum_{k=\lambda^{-n}}^{\lambda^{-n-1}} \frac{k}{\log[1/C_l(K_{\lambda^k})]} \leq \frac{1}{\lambda} \sum_{n=N}^{\infty} \frac{\lambda^{-n}}{\log[1/C_l(K_{\lambda^{-n}})]}.
\]
(3.9)

By (3.7) ~ (3.9) we see that if \(\sum_{k=1}^{\infty} \frac{k}{\log[1/C_l(K_{\lambda^k})]} = \infty\), then \(0\) is a regular point for \(\Omega\), which is exactly the sufficient part of Wiener’s criteria.

Theorem 3.2 also yields a new proof of the following result due to Carleson-Totik [4].

**Corollary 3.3.** If \(D_W(\varepsilon, \lambda) > 0\) for some \(\varepsilon, \lambda\), then there exists \(\beta > 0\) such that
\[
\phi_E(z) \leq \delta(z)^{\beta}
\]
where \(\delta\) denotes the boundary distance of \(\Omega\).

**Proof.** Since \(D_W(\varepsilon, \lambda) > 0\), there exist \(c > 0\) and \(n_0 \in \mathbb{Z}^+\) such that
\[
|N_\alpha^n(\varepsilon, \lambda)| \geq cn, \quad \forall n \geq n_0 \text{ and } a \in \partial\Omega.
\]
Since \(N_\alpha^n(\varepsilon, \lambda)\) is decreasing in \(\varepsilon\), we may assume that \(\varepsilon\) is as small as we want. Note that for \(n \gg N \gg 1\)
\[
\int_{\lambda^n}^{\lambda^N} \left( t \log \frac{t/\alpha}{2C_l(K_{\lambda^k}(a))} \right)^{-1} dt \geq \sum_{k \in N_\alpha^n(\varepsilon, \lambda) \setminus N_\alpha^N(\varepsilon, \lambda)} \int_{\lambda^k}^{\lambda^{k-1}} \left( t \log \frac{t/\alpha}{2C_l(K_{\lambda^k}(a))} \right)^{-1} dt \geq \sum_{k \in N_\alpha^n(\varepsilon, \lambda) \setminus N_\alpha^N(\varepsilon, \lambda)} \left( \log \frac{1}{2\lambda \varepsilon \alpha} \right)^{-1} \int_{\lambda^k}^{\lambda^{k-1}} \frac{dt}{t} \geq \log 1/\lambda \cdot \left( \log \frac{1}{2\lambda \varepsilon \alpha} \right)^{-1} |N_\alpha^n(\varepsilon, \lambda) \setminus N_\alpha^N(\varepsilon, \lambda)| \geq \log 1/\lambda \cdot \left( \log \frac{1}{2\lambda \varepsilon \alpha} \right)^{-1} \cdot \frac{cn}{2} = \frac{c}{2} \cdot \left( \log \frac{1}{2\lambda \varepsilon \alpha} \right)^{-1} \cdot \log 1/\lambda^n.
\]
(3.11)

Since for every \(z\) there exists \(n \in \mathbb{Z}^+\) such that \(\lambda^n \leq |z - a| \leq \lambda^{n-1}\), it follows from (3.7) and (3.11) that
\[
\phi_E(z) \leq |z - a|^{\beta}
\]
for suitable constant \(\beta > 0\) which is independent of \(a\), so that (3.10) holds. \qed
Remark. It is remarkable that the converse of Corollary 3.3 holds under the additional condition that $\Omega$ contains a fixed size cone with vertex at any $a \in \partial \Omega$ (cf. [4]).

Corollary 3.4. If $\mathcal{D}_W(\varepsilon, \lambda, \gamma) > 0$ for some $\varepsilon, \lambda, \gamma$, then there exists $\beta > 0$ such that

$$\phi_E(z) \leq (-\log \delta(z))^{-\beta}$$

for all $z$ sufficiently close to $\partial \Omega$.

Proof. Since $\mathcal{D}_W(\varepsilon, \lambda, \gamma) > 0$, there exist $c > 0$ and $n_0 \in \mathbb{Z}^+$ such that

$$|N_a^n(\varepsilon, \lambda, \gamma)| \geq c \log n, \quad \forall \ n \geq n_0 \text{ and } a \in \partial \Omega.$$

Note that for $n \gg N \gg 1$

$$\int_{\lambda^n}^{\lambda^N} \left( t \log \frac{t/\alpha}{2C_l(K_t(a))} \right)^{-1} dt \geq \sum_{k \in N_a^n(\varepsilon, \lambda, \gamma)} \left( \int_{\lambda^k}^{\lambda^{k-1}} \left( t \log \frac{t/\alpha}{2C_l(K_t(a))} \right)^{-1} dt \right)$$

$$\geq \sum_{k \in N_a^n(\varepsilon, \lambda, \gamma)} \left( \int_{\lambda^{k-1}}^{\lambda^k} \frac{dt}{t} \right)$$

$$\geq \sum_{k \in N_a^n(\varepsilon, \lambda, \gamma)} k^{-1} \geq \log n$$

where the implicit constants are independent of $a$. This combined with (3.7) gives

$$\phi_E(z) \leq (-\log |z - a|)^{-\beta}$$

for some constant $\beta > 0$ independent of $a$, which in turn implies (3.12). \qed

Corollary 3.5. Suppose $\inf_{a \in \partial \Omega} C_l(K_t(a)) \geq e \tau^\gamma$ for some $\varepsilon > 0$ and $\gamma > 1$. For every $\tau < \frac{1}{\gamma - 1}$ we have

$$\phi_E(z) \leq \text{const}_\tau (-\log \delta(z))^{-\tau}$$

for all $z$ sufficiently close to $\partial \Omega$.

Proof. By (3.7) we have for every $a \in \partial \Omega$

$$\sup_{t \in \mathbb{B}_r(a)} \phi_E \leq \exp \left[ -\frac{\log 1/(16\alpha)}{\log 1/\alpha} \int_0^{\alpha d} \left( t \log \frac{t/\alpha}{2C_l(K_t(a))} \right)^{-1} dt \right]$$

$$\leq \exp \left[ -\frac{\log 1/(16\alpha)}{\log 1/\alpha} \int_0^{\alpha d} \frac{dt}{t((\gamma - 1) \log 1/t + \log 1/(2\alpha \varepsilon))} \right]$$

$$\leq \text{const}_\tau (-\log r)^\tau$$

provided $\alpha$ sufficiently small, from which the assertion follows. \qed
4. Estimates of the Green function

Proposition 4.1. If $D_W(\varepsilon, \lambda, \gamma) > 0$ for some $\varepsilon, \lambda, \gamma$, then there exists $c \gg 1$ such that

\begin{equation}
\{g_\Omega(\cdot, w) \leq -1\} \subset \left\{ c^{-1} \phi_E(w)^{1+\beta} < \phi_E < c \phi_E(w)^{1+\beta} \right\}
\end{equation}

where $\beta$ is given as (3.12).

Proof. We shall first adopt a trick from [3]. Consider two points $z, w$ with $|z - w|$, $\delta(z)$ and $\delta(w)$ are sufficiently small. We want to show

\begin{equation}
|\phi_E(z) - \phi_E(w)| \leq c_0(-\log |z - w|)^{-\beta}
\end{equation}

for some numerical constant $c_0 > 0$. Without loss of generality, we assume $\delta(w) \geq \delta(z)$. If $|z - w| \geq \delta(w)/2$, this follows directly from (3.12). Since $\phi_E$ is a positive harmonic function on $\mathbb{D}(w, \delta(w))$, we see that if $|z - w| \leq \delta(w)/2$ then

\begin{align*}
|\phi_E(z) - \phi_E(w)| &\leq \sup_{\mathbb{D}(w, \delta(w)/2)} |\nabla \phi_E| |z - w| \\
&\leq c_1 \delta(w)^{-1} (-\log \delta(w))^{-\beta} |z - w| \quad \text{(by (3.12))} \\
&\leq c_1 (2|z - w|)^{-1} (-\log (2|z - w|))^{-\beta} |z - w| \\
&\leq c_0 (-\log |z - w|)^{-\beta}.
\end{align*}

The remaining argument is standard. Let $R$ be the diameter of $\Omega$. By (4.2) we conclude that if $\phi_E(z) = \phi_E(w)/2$ then

\[ \log \frac{|z - w|}{R} \geq - \left( \frac{2c_0}{\phi_E(w)} \right)^{1/\beta} - \log R \geq -c_2 \phi_E(w)^{-1/\beta} \]

Since $-\phi_E$ is subharmonic on $\Omega$, it follows that

\[ \psi(z) := \begin{cases} \\
\log \frac{|z - w|}{R} \\
\max \left\{ \log \frac{|z - w|}{R}, -2c_2 \phi_E(w)^{-1-1/\beta} \psi(z) \right\}
\end{cases} \]

if $\phi_E(z) \geq \phi_E(w)/2$

otherwise.

is a well-defined negative subharmonic function on $\Omega$ with a logarithmic pole at $w$, and if $\phi_E(z) \leq \phi_E(w)/2$ then we have

\[ g_\Omega(z, w) \geq \psi(z) \geq -2c_2 \phi_E(w)^{-1-1/\beta} \psi(z), \]

so that

\[ \{g_\Omega(\cdot, w) \leq -1\} \cap \{\phi_E \leq \phi_E(w)/2\} \subset \{\phi_E \geq (2c_2)^{-1} \phi_E(w)^{1+1/\beta}\}. \]

Since $\{\phi_E \geq \phi_E(w)/2\} \subset \{\phi_E > c^{-1} \phi_E(w)^{1+1/\beta}\}$ if $c \gg 1$, we have

\[ g_\Omega(\cdot, w) \leq -1 \subset \{\phi_E > c^{-1} \phi_E(w)^{1+1/\beta}\}. \]

By the symmetry of $g_\Omega$, we immediately get

\[ \{g_\Omega(\cdot, w) \leq -1\} \subset \{\phi_E < (c \phi_E(w))^{1+\beta}\}. \]

\[ \square \]
Lemma 4.2 (cf. [9]). Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) and \( U \) a relatively compact open set in \( \Omega \). For every \( w \in U \) we have

\[
\min_{\partial U}(-g_{\Omega}(\cdot, w)) \leq \frac{2\pi}{C_{d}(U, \Omega)} \leq \max_{\partial U}(-g_{\Omega}(\cdot, w)).
\]

Proof. Since \( g_{\Omega}(\cdot, w) \) is harmonic on \( \Omega \setminus \overline{U} \) and vanishes n.e on \( \partial \Omega \), it follows from the maximum principle that

\[
\sup_{\Omega \setminus U}(-g_{\Omega}(\cdot, w)) = \max_{\partial U}(-g_{\Omega}(\cdot, w)) \quad \text{and} \quad \inf_{\partial U}(-g_{\Omega}(\cdot, w)) = \min_{\partial U}(-g_{\Omega}(\cdot, w)).
\]

Then we have

\[
\left\{-g_{\Omega}(\cdot, w) \geq \max_{\partial U}(-g_{\Omega}(\cdot, w)) \right\} \subset \overline{U} \subset \left\{-g_{\Omega}(\cdot, w) \geq \min_{\partial U}(-g_{\Omega}(\cdot, w)) \right\}.
\]

Set \( F_c = \{-g_{\Omega}(\cdot, w) \geq c\} \). It suffices to show

\[
C_{d}(F_c, \Omega) = \frac{2\pi}{c}.
\]

Indeed, the function \( \phi_c := -c^{-1}g_{\Omega}(\cdot, w) \) is the capacity potential of \( F_c \) relative to \( \Omega \). Thus we have

\[
C_{d}(F_c, \Omega) = -\text{flux}_{\partial F_c} \phi_c = -\text{flux}_{\partial \Omega} \phi_c = c^{-1}\text{flux}_{\partial \Omega} g_{\Omega}(\cdot, w) = 2\pi/c.
\]

where the second and last equalities follow from Green’s formula.

Lemma 4.3. Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( 0 \in \partial \Omega \). Let \( \beta > \alpha > 0 \). Suppose \( C_l(K_r) \geq \varepsilon r \) for some \( \varepsilon, r > 0 \). There exists a positive number \( c \) depending only on \( \alpha, \beta, \varepsilon \) such that for every point \( w \) with \( |w| = \beta r \) and \( D_{2\alpha r}(w) \subset \Omega \) we have

\[
\{g_{\Omega}(\cdot, w) \leq -c\} \subset D_{\alpha r}(w).
\]

Proof. By (4.3) and Harnack’s inequality it suffices to show

\[
C_{d}(D_{\alpha r}(w), \Omega) \geq c'
\]

for some positive constant \( c' \) depending only on \( \alpha, \beta, \varepsilon \). By the definition we see that

\[
C_{d}(D_{\alpha r}(w), \Omega) = C_{d}(\Omega_r, C_\infty - D_{\alpha r}(w)) \geq C_{d}(K_r, C_\infty - D_{\alpha r}(w)).
\]

We consider the conformal map

\[
T : C_\infty - \overline{D_{\alpha r}(w)} \to \mathbb{D}, \quad z \mapsto \frac{\alpha r}{z - w}.
\]

Since the Dirichlet energy is invariant under conformal maps, it follows that

\[
C_{d}(K_r, C_\infty - \overline{D_{\alpha r}(w)}) = C_{d}(T(K_r), \mathbb{D}).
\]

Since

\[
K_r \subset D_{(1+\beta)r}(w) - D_{2\alpha r}(w),
\]
we have
\[ T(K_r) \subset \mathbb{D}_{1/2} - D_{\alpha/(1+\beta)}, \]
so that
\[
C_d(T(K_r), \mathbb{D}) = -\frac{2\pi}{\log C_g(T(K_r), \mathbb{D})} \geq \frac{2\pi}{\log 2 - \log C_l(T(K_r))}. \tag{4.8}
\]
Since
\[ |T^{-1}(z_1) - T^{-1}(z_2)| = \frac{\alpha r}{|z_1 z_2|} \cdot |z_1 - z_2| \leq \frac{(1 + \beta)^2 r}{\alpha} \cdot |z_1 - z_2| \]
for all \( z_1, z_2 \in T(K_r) \), we have
\[ C_l(T(K_r)) \geq \alpha (1 + \beta) \frac{2 r}{\alpha} \cdot C_l(K_r) \geq \frac{\alpha \varepsilon}{(1 + \beta)^2}. \]
This combined with \( (4.6) \sim (4.8) \) gives \( (4.5) \). \( \square \)

**Proposition 4.4.** Suppose \( D_S(\varepsilon, \lambda) > 0 \) for some \( \varepsilon, \lambda \). There exists \( c \gg 1 \) such that for every \( k \in \bigcap_{a \in \partial \Omega} N_a^n(\varepsilon, \lambda) \) and every \( w \) with \( \lambda^{k-1/3} \leq \delta(w) \leq \lambda^{k-1/2} \),
\[ \{ g_\Omega(\cdot, w) \leq -c \} \subset \{ \lambda^k < \delta < \lambda^{k-1} \}. \tag{4.9} \]

**Proof.** Take \( a(w) \in \partial \Omega \) such that \( |w - a(w)| = \delta(w) \). Note that
\[ C_l(K_{\lambda^k}(a(w))) \geq \varepsilon \lambda^k. \]
Thus Lemma 4.3 applies. \( \square \)

5. Proofs of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1**
(1) Let \( c \) be as Proposition 4.4. Let \( z \) be sufficiently close to \( \partial \Omega \). Take \( n \in \mathbb{Z}^+ \) such that \( \lambda^n \leq \delta(z) \leq \lambda^{n-1} \). Write
\[ \bigcap_{a \in \partial \Omega} N_a^n(\varepsilon, \lambda) = \{ k_1 < k_2 < \cdots < k_m \}. \]
We may choose a Bergman geodesic jointing \( z_0 \) to \( z \), and a finite number of points on this geodesic with the following order
\[ z_0 \rightarrow z_{k_1} \rightarrow z_{k_2} \rightarrow \cdots \rightarrow z, \]
such that
\[ \lambda^{k_j-1/3} \leq \delta(z_{k_j}) \leq \lambda^{k_j-1/2}. \]
By Proposition 4.4 we have
\[ \{ g_\Omega(\cdot, z_{k_j}) \leq -c \} \cap \{ g_\Omega(\cdot, z_{k_{j+1}}) \leq -c \} = \emptyset \]
so that \( d_B(z_{k_j}, z_{k_{j+1}}) \geq c_1 > 0 \) for all \( j \), in view of Theorem 1.1 in [1]. Since \( m_n \gg n \), we have
\[ d_B(z_0, z) \geq \sum_j d_B(z_{k_j}, z_{k_{j+1}}) \gg n \gg |\log \delta(z)|. \]

(2) The assertion follows directly from Corollary 3.3 and Corollary 1.8 in [6]. \( \square \)
Proof of Theorem 1.2. Let $c$ be as Proposition 4.1. Let $z$ be sufficiently close to $\partial \Omega$. We may choose a Bergman geodesic jointing $z_0$ to $z$, and a finite number of points $\{z_k\}_{k=1}^m$ on this geodesic with the following order

$$z_0 \to z_1 \to z_2 \to \ldots \to z_m \to z,$$

where

$$c \phi_E(z_{k+1})^{\frac{\beta}{1+\beta}} = c^{-1} \phi_E(z_k)^{\frac{1+\beta}{\beta}}$$

and

$$c^{-1} \phi_E(z_m)^{\frac{1+\beta}{\beta}} \leq \phi_E(z) \leq c \phi_E(z_m)^{\frac{\beta}{1+\beta}}.$$

By Proposition 4.1 we have

$$\{g_\Omega(\cdot, z_k) \leq -1\} \cap \{g_\Omega(\cdot, z_{k+1}) \leq -1\} = \emptyset$$

so that $d_B(z_k, z_{k+1}) \geq c_1 > 0$ for all $k$.

Note that

$$\log \phi_E(z_0) = \left(\frac{\beta}{1+\beta}\right)^2 \log \phi_E(z_1) + \frac{\beta}{1+\beta} \log c^2 = \ldots$$

$$= \left(\frac{\beta}{1+\beta}\right)^{2m} \log \phi_E(z_m) + \frac{\beta}{1+\beta} \frac{1 - \left(\frac{\beta}{1+\beta}\right)^2}{1 - \left(\frac{\beta}{1+\beta}\right)^2} \log c^2.$$

Thus we have

$$m \approx \log |\log \phi_E(z_m)| \approx \log |\log \phi_E(z)| \gtrsim \log \log |\log \delta(z)|,$n

so that

$$d_B(z_0, z) \geq \sum_{k=1}^{m-1} d_B(z_k, z_{k+1}) \geq c_1 (m - 1) \gtrsim \log \log |\log \delta(z)|.$$
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