Local laws for polynomials of Wigner matrices

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Abstract

We consider general self-adjoint polynomials in several independent random matrices whose entries are centered and have the same variance. We show that under certain conditions the local law holds up to the optimal scale, i.e., the eigenvalue density on scales just above the eigenvalue spacing follows the global density of states which is determined by free probability theory. We prove that these conditions hold for general homogeneous polynomials of degree two and for symmetrized products of independent matrices with i.i.d. entries, thus establishing the optimal bulk local law for these classes of ensembles. In particular, we generalize a similar result of Anderson for anticommutator. For more general polynomials our conditions are effectively checkable numerically.

Keywords: Polynomials of random matrices, local law, generalized resolvent, linearization, Dyson equation

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1 Introduction

Polynomials of random matrices have been subject of intensive research in the last thirty years. In the 1980’s Voiculescu realized that random matrices and their polynomials can be used to solve some basic problems in operator algebras of free groups, which gave birth to free probability theory. Roughly speaking, large independent random matrices serve as concrete approximants to free elements in abstract noncommutative probability spaces, i.e. unital C*-algebras with a tracial state. In other words, freeness is the appropriate operator algebraic analogue of independence in classical probability. A classical example for such result is Theorem 2.2 from [55] showing that the trace of a self-adjoint polynomial $p(X_1,\ldots,X_k)$ in $k$ independent $N \times N$ standard complex Gaussian (GUE) matrices converges in expectation and almost surely, as the size of the matrices goes to infinity, to the trace of the polynomial $p(s_1,\ldots,s_k)$ in free semicircular variables.

Voiculescu’s pioneering result has since been extended in many directions. Convergence in operator norm was proved in [38], while convergence of the spectrum, in particular absence of outliers, was established in [37]. Another direction of generalizations was to replace Gaussian matrices with Wigner matrices, i.e. retain independence of the matrix elements while dropping the special distribution; for the first such result see [27], followed by many others, e.g. [5, 18, 19, 25, 50] and references therein. Yet another line of research concerns certain qualitative properties of the limiting spectral measure. For example, the limiting spectral measure for self-adjoint polynomials does not contain atoms [18, 53] and for monomials it is even absolutely continuous [26]. Very recently, the Hölder continuity of the cumulative distribution function was studied for polynomials [14] and rational functions [49] of random matrices.

A common feature of all these results, as well as the scope of the underlying methods, is that they describe the spectrum of $p(X_1,\ldots,X_k)$ on the global scale, which is typically by a factor $N$ larger than the scale of the eigenvalue spacing. What happens on scales in between? Recent developments revealed that the eigenvalue density of Wigner and related matrices on mesoscopic scales, i.e., scales...
involving $\sim N^\gamma$ eigenvalues for $0 < \gamma < 1$, also becomes deterministic in the large $N$ limit. Such results are commonly called local laws and they have been established in increasing generality for Hermitian matrices; with independent entries, see e.g. \cite{15, 28, 39, 54}, with general short range correlation structure for their matrix elements \cite{2, 29}, as well as for adjacency matrices for random regular graphs \cite{16, 17, 18, 29, 57}. Local laws beyond mean field models, in particular for band matrices are especially challenging \cite{16, 17, 18, 29}.

One of the main motivations for local laws is their key role in the proof of the Wigner-Dyson-Mehta conjecture on the local spectral universality, see \cite{32}. Recent developments on the local ergodicity of the Dyson Brownian motion (DBM) have demonstrated that local laws are the only model-dependent inputs for the universality proofs using the DBM, see \cite{33} for an overview and newer results in \cite{31, 35, 46}.

In this paper we prove optimal local laws for self-adjoint polynomial models, thus connecting two large areas of recent research in random matrices. We will combine methods from free probability theory, most importantly the concept of linearization, with techniques developed for local laws, such as large deviations and fluctuation averaging phenomenon. We point out that mesoscopic spectral properties for general polynomials have not been studied before. Local laws have only been established for a very few specific polynomials such as (i) the anticommutator, $X_1 X_2 + X_2 X_1$, of two independent Wigner matrices in \cite{6} and (ii) the (non-Hermitian) product $Y_1 Y_2 \cdots Y_k$ of several independent i.i.d. matrices in \cite{5, 51}.

We now explain the method and some difficulties. The first major obstacle is that the entries of a general polynomial $P := p(X_1, \ldots, X_k)$ of, say, independent $N \times N$ Wigner matrices, have a very complex non-local correlation structure. This makes it impossible to apply the tools developed in \cite{2} or \cite{8} directly in the polynomial setting. However, the well-known linearization trick, originally developed in the context of automata theory \cite{14, 52} and revived for use in random matrix theory \cite{37, 38}, transforms the polynomial model into a much larger random matrix $H$ with a transparent correlation structure. In fact, the linearized matrix is a tensor linear combination of the independent Wigner matrices with matrix coefficients whose dimension $m \times m$ depends only on the polynomial $p$ and is independent of $N$. This structure exactly corresponds to certain block matrices and more generally Kronecker random matrices introduced in \cite{4}. We remark that the linearization technique has been widely used in the free probability community to study polynomials of random matrices on the global scale, see e.g. \cite{5, 20, 38, 41, 42} and \cite{7, Chapter 5} for a pedagogical introduction.

Local laws for Kronecker matrix $H$ have been studied in detail in \cite{4} by proving concentration of its resolvent $(H - z I_N \otimes I_N)^{-1}$ around the solution of corresponding matrix Dyson equation for spectral parameter $z$ in complex upper half-plane. In contrast to the Kronecker case, to study the resolvent $(P - z)^{-1}$ of our polynomial, we have to consider the generalized resolvent of the linearized matrix $H$, i.e., $(H - z J \otimes I_N)^{-1}$, where $J$ is a rank-one $m \times m$ matrix. Thus the results on the Kronecker matrices cannot be directly applied, in fact a priori it is unclear whether the generalized resolvent is stable. This is the second major obstacle in the study of polynomial models, and we overcome it by simultaneously considering the generalized resolvent of $H$ and its usual regularized version. It turns out that a certain nilpotency structure inherent for linearizations of polynomials yields the boundedness of the generalized resolvents even after the regularization is removed.

After these two key obstacles cleared, we can essentially use the local law established for general Kronecker matrices in \cite{4} under two basic conditions: (i) the solution of the underlying Dyson equation is bounded and (ii) the stability operator is invertible. These conditions are verified for homogeneous polynomials of degree two in Wigner matrices, substantially generalizing the case of anticommutator studied by Anderson in \cite{6}. We also verify them for the symmetrized product $Y_1 \cdots Y_k (Y_1 \cdots Y_k)^*$ of independent matrices with i.i.d. entries. For more general polynomials, the validity of these conditions depends on the structure of the linearization but they are independent of $N$, so they are numerically checkable. Notice that the linearization of a polynomial is not unique. In fact, any of the standard linearizations, obtained via a simple recursive procedure, typically has unnecessarily large dimension. It is much more effective to use the so-called minimal linearization, which is canonical \cite{21, 32, 11, 12, 13}, and we present numerical examples to demonstrate its advantages. Since both linearizations are nilpotent, our theory equally applies to them. We expect that for any self-adjoint polynomial there
exists a linearization for which the conditions (i) and (ii) above hold everywhere in the bulk, i.e., where
the density of states is bounded and bounded away from zero, and in fact the minimal linearization is
a natural candidate.

The task of dealing with the generalized resolvent of the linearization is inherent in other works
on polynomials of random matrices that use the resolvent method, see [1, 25, 38, 50]. In the most
general setup, Anderson in [5] used one of the explicit standard linearizations to prove the global law
and the convergence of the norm for polynomials in Wigner matrices. The structure of the standard
linearization allowed him to control the generalized resolvent directly from the resolvent of \( P \) via Schur
complement formula. A simpler version of this idea was presented in [7, Chapter 5.5]. For the canonical
minimal linearization such simple a priori bound is not available. From algebraic point of view, the
main novelty of our work is to identify a nilpotency structure in the minimal linearization and show
that this structure is sufficient to control the generalized resolvent. From the analytic point of view, we
advocate the method of the stability analysis of the Dyson equation combined with large deviation and
fluctuation averaging estimates as presented in [4], which itself is a natural extension of many previous
works on local laws for Wigner and Wigner-type matrices. This approach substitutes the Poincaré
inequality used in [25] and the \( L^p \) bounds used in [5] whose analogue for Wigner and Wishart matrices
go back to Bai and Silverstein [9, 10].

We close this introduction with a remark on local spectral universality. Our local law is optimal
and it provides the necessary input for the customary proofs via the Dyson Brownian motion (DBM)
as mentioned above. Thus we could easily prove bulk universality for polynomials that already have a
small additive GUE component. We cannot, however, apply the usual DBM argument to the linearized
matrix since it would need to assume that a small global Gaussian component is present in \( H \), but
\( H \) has many zero blocks by construction. This fundamental difficulty has been overcome for certain
band matrices [23, 24] which also has many zero entries. However, the specific band structure was
essential in those proofs. For the local spectral universality for a polynomial \( P \) the structure of the
linearized matrix needs to be exploited in a similar fashion. We note that apart from the trivial case
of Hermitian polynomials of a single random matrix, currently the only nontrivial universality results
for polynomials are obtained for very special cases and only for Gaussian matrices by exploiting their
determinantal structure, see, e.g., the survey [3] on products of large Gaussian random matrices.

In Section 2 we introduce the concept of nilpotent linearization, the corresponding Dyson equation
and we present our main result together with the conditions expressed in terms of the solution to the
Dyson equation. Section 3 is devoted to control the generalized resolvent by exploiting the nilpotent
structure. In Section 4 we present the existence and uniqueness of the solution to the Dyson equation
by using semicircular variables. In Section 5 we give a proof of the local law. Finally, as an application,
In Section 6 we show the optimal bulk local law for general homogeneous polynomials of degree two in
Wigner matrices and for symmetrized products of matrices with i.i.d. entries. Additional information
on two different linearizations, as well as their numerical comparison are deferred to Appendix A,
while in Appendix B we collected some basic information on semicircular variables for the reader’s
convenience.

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2 Main results

2.1 Linearization and Dyson equation in \( C^* \)-algebras

Fix \( \alpha, \beta \in \mathbb{N} \). Let \( \mathcal{A} \) be a unital \( C^* \)-algebra with norm \( \| \cdot \|_{\mathcal{A}} \) and identity element \( 1_{\mathcal{A}} \), and let
\( x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta \in \mathcal{A} \) with \( x_i^* = x_i \) for \( 1 \leq i \leq \alpha \). For any \( n \in \mathbb{N} \) and \( r = (r_1, \ldots, r_n) \in \mathcal{A}^n \) we
define \( \| r \| := \max_{1 \leq i \leq n} |r_i|_{\mathcal{A}} \). Denote by
\[ \mathbb{C}(x, y, y^*) := \mathbb{C}(x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta, y_1^*, \ldots, y_\beta^*) \]
the set of polynomials with complex coefficients in noncommutative elements \( \{x_\alpha, y_\beta, y_\beta^* \} \), where \( 1 \leq \alpha \leq \alpha_s, 1 \leq \beta \leq \beta_s \). Let \( p := p(x, y, y^*) \in \mathbb{C}(x, y, y^*) \) and assume that \( p \) is self-adjoint, i.e.,

\[
(p(x, y, y^*))^* = p(x, y, y^*).
\]

It is a common and convenient practice to study the polynomials via their linearizations. Linearization allows to transform polynomial model into a linear one, which is typically easier to analyze. The price for doing this is the increased dimension of the model, which can quickly become prohibitive for more complicated polynomials.

**Definition 2.1 (Self-adjoint linearization).** Let \( m \in \mathbb{N} \) and let \( L \in (\mathbb{C}(x, y, y^*))^{m \times m} \) be a matrix, whose matrix elements are polynomials of degree at most 1. Suppose that

\[
L = \begin{pmatrix} \lambda & \ell^* \\ \ell & \hat{L} \end{pmatrix},
\]

where \( \hat{L} \) is the \((m-1) \times (m-1)\) submatrix of \( L \). We call \( L \) a self-adjoint linearization (or simply linearization) of \( p \in \mathbb{C}(x, y, y^*) \) if \( L^* = L \) and there exists \( \varepsilon > 0 \) such that for all \( \|x\| < \varepsilon, \|y\| < \varepsilon \), the matrix \( \hat{L} \) is invertible and satisfies

\[
p = \lambda - \ell^* \hat{L}^{-1} \ell.
\]

We will refer to \( m \) as the dimension of the linearization \( L \).

Note that due to the property \( L^* = L \) a self-adjoint linearization \( L \) can be written as

\[
L = K_0 \otimes 1_\beta - \sum_{\alpha=1}^{\alpha_s} K_\alpha \otimes x_\alpha - \sum_{\beta=1}^{\beta_s} (L_\beta \otimes y_\beta + L_\beta^* \otimes y_\beta^*),
\]

where \( K_\alpha, L_\beta \in \mathbb{C}^{m \times m} \) and \( K_0^* = K_0, K_\alpha^* = K_\alpha \). In this paper all linearizations are self-adjoint, so we will not stress self-adjointness all the times.

For each polynomial one can write many different linearizations. In the related literature [4, 21, 38, 42] one can distinguish two groups of methods used for constructing the linearizations of polynomial (and more generally rational) functions. One group uses very explicit algorithms to build linearizations first for monomials, and then extending them to linear combinations of monomials. These algorithms are well-known, but for the sake of completeness we will give in Appendix A.1 a version of such an explicit linearization. This is a standard construction that typically yields a linearization in very high dimension. For many practical reasons it is better to work with smaller linearizations, which naturally leads to the notion of minimal linearization.

**Definition 2.2 (Minimal linearization).** A linearization of a polynomial is called minimal if it has the smallest dimension among all linearizations.

Minimal linearization can be obtained by reducing the dimension of some previously constructed linearization (see e.g. [21, Chapter 2.3]) and then using the symmetrization trick if needed to restore self-adjointness [42, Lemma 4.1 (3)]. For completeness, as well as for the reader’s convenience, in Appendix A.2 we present a somewhat different algorithmic procedure that directly yields a minimal (self-adjoint) linearization from any (self-adjoint) linearization.

Typically the dimension of a minimal linearization is significantly smaller compared to the standard linearization constructed in Appendix A.1 (see Appendix A.3 for comparison), which makes it much more convenient to work with if we want to study the model numerically.

In order to use linearizations for studying the resolvents of polynomials of random matrices it will be convenient to work with a special class of nilpotent linearizations that we introduce now.

**Definition 2.3 (Nilpotent family).** A family of matrices \( \{R_i \in \mathbb{C}^{m \times m} : i \in I \} \) is called nilpotent if there exists an integer \( n \) such that \( R_{i_1} R_{i_2} \ldots R_{i_n} = 0 \) for any \( n \)-tuple of indices \( (i_1, i_2, \ldots, i_n) \in I^n \).
Define the matrix \( J : = e_1 e_1^t \in \mathbb{C}^{m \times m} \), where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^m \), i.e. \( J \) is an \( m \times m \) matrix having the \((1,1)\)-entry equal to 1 and all the other entries equal to zero. Let \( \langle \cdot, \cdot \rangle : \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C} \) be the usual scalar product in \( \mathbb{C}^m \) linear in the second variable. For brevity we denote \([n] := \{1, \ldots, n\}\) for any \( n \in \mathbb{N}\).

**Definition 2.4 (Nilpotent linearization).** A linearization of a polynomial \( p \) of the form (2.3) is called nilpotent if

(i) \( K_0 \) is invertible;
(ii) \( \langle e_1, K_0^{-1} e_1 \rangle = 1 \);
(iii) The family of matrices

\[
\left\{ \pi K_0 \pi^{-1} \pi', \pi' L_\beta K_0^{-1} \pi', \pi' L_\beta^* K_0^{-1} \pi' : \alpha \in [\alpha_*], \beta \in [\beta_*] \right\}
\]

is nilpotent, where we set \( \pi := J K_0^{-1} \) and \( \pi' := I - \pi \).

One can easily see that (ii) is equivalent to \( p(0,0,0) = 1_{\mathcal{A}} \). Indeed, let \( \langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathbb{C}^m \otimes \mathcal{A} \times \mathbb{C}^m \otimes \mathcal{A} \rightarrow \mathcal{A} \) be an operator given by

\[
\langle l, r \rangle_{\mathcal{A}} := \sum_{k=1}^m l_k^* r_k
\]

where \( r, l \in \mathbb{C}^m \otimes \mathcal{A} \), \( r = \sum_{k=1}^m e_k \otimes r_k \), \( l = \sum_{k=1}^m e_k \otimes l_k \) and \( e_k = (\delta_{ik})_{i=1}^m \). Then by the Schur complement formula and (2.2) we have that

\[
\langle e_1 \otimes 1_{\mathcal{A}}, L^{-1} e_1 \otimes 1_{\mathcal{A}} \rangle_{\mathcal{A}} = p^{-1}.
\]

If we now take \( x = y = 0 \), then \( L^{-1} = K_0^{-1} \otimes 1_{\mathcal{A}} \) and thus \( \langle e_1, K_0^{-1} e_1 \rangle_{\mathcal{A}} = (p(0,0,0))^{-1} \). Shifting the polynomial by a constant, without loss of generality, we may and will assume in the rest of the paper that the constant term of the polynomial is 1, i.e. we write \( p(x,y,y^*) = 1_{\mathcal{A}} - q(x,y,y^*) \) for some polynomial \( q(x,y,y^*) \) with \( q(0,0,0) = 0 \). Furthermore, note that \( \pi \) is a projection by (ii) and \( J = e_1 e_1^t \), but in general it is not an orthogonal projection.

We will show in Section 3.1 that for any polynomial of the form \( p = 1_{\mathcal{A}} - q \) both linearizations constructed in Appendix A belong to the class of nilpotent linearizations. This property will be used to obtain an a priori bound for the generalized resolvent of the linearization, that we define below.

Denote by \( \mathbb{C}^{m \times m} \otimes \mathcal{A} \) the set of \( m \times m \) matrices with elements from \( \mathcal{A} \). We can look at \( L \) as an operator on \( \mathbb{C}^{m \times m} \otimes \mathcal{A} \) equipped with the Banach space structure from \( \mathcal{A} \). For any \( z \in \mathbb{C}_+ \) we will consider the generalized resolvent of \( L \) defined as \( (L - zJ \otimes 1_{\mathcal{A}})^{-1} \).

From the Schur complement formula

\[
\langle e_1 \otimes 1_{\mathcal{A}}, (L - zJ \otimes 1_{\mathcal{A}})^{-1} e_1 \otimes 1_{\mathcal{A}} \rangle_{\mathcal{A}} = \langle \lambda - z1_{\mathcal{A}} - \ell^* \tilde{L}^{-1} \ell \rangle_{\mathcal{A}} = (p - z1_{\mathcal{A}})^{-1}
\]

i.e. the \((1,1)\)-component of the generalized resolvent is the resolvent of \( p \), viewed as an element of \( \mathcal{A} \).

In particular, if we take \( \mathcal{A} \) to be \( \mathbb{C}^{N \times N} \), then the resolvent of a polynomial \( p \) of matrices of size \( N \times N \) is given by the upper left \( N \times N \) block of the generalized resolvent of the corresponding linearization.

In Section 3.2 we show that generalized resolvent of a nilpotent linearization is well defined for all \( z \in \mathbb{C}_+ \). More precisely, define a norm \( \| \cdot \| \) on \( \mathbb{C}^{m \times m} \otimes \mathcal{A} \) by

\[
\| R \| := \max_{1 \leq k,l \leq m} \| R_{kl} \|_{\mathcal{A}},
\]

where \( R = \sum_{k,l=1}^m E_{kl} \otimes R_{kl} \) and \( E_{kl} = (\delta_{ik} \delta_{lj})_{1 \leq i,j \leq m} \) is the standard basis in \( \mathbb{C}^{m \times m} \). Then the following lemma holds.

**Lemma 2.5.** Let \( q \in \mathbb{C}(x,y,y^*) \) be a self-adjoint polynomial with \( q(0,0,0) = 0 \). Let \( L \in \mathbb{C}^{m \times m} \otimes \mathcal{A} \) be a nilpotent linearization of \( 1_{\mathcal{A}} - q(x,y,y^*) \). Then there exist \( C_1 > 0 \) and \( n_1 \in \mathbb{N} \), depending on \( L \), such that for all \( z \in \mathbb{C}_+ \)

\[
\| (L - zJ \otimes 1_{\mathcal{A}})^{-1} \| \leq C_1 \left( 1 + \frac{1}{\text{Im } z} \right) \left( 1 + \max_{1 \leq \alpha \leq n} \| x_\alpha \|_{\mathcal{A}} + \max_{1 \leq \beta \leq \beta_*} \| y_\beta \|_{\mathcal{A}} \right).
\]
Suppose now that we have a nilpotent linearization $L$ in the form $L_0$. Define the linear map 
$\Gamma : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}$ by 
$$\Gamma[R] = \sum_{\alpha=1}^{\alpha} K_\alpha RK_\alpha + \sum_{\beta=1}^{\beta} (L_\beta RL_\beta^* + L_\beta^* RL_\beta).$$  
(2.6)
For any $z \in \mathbb{C}_+$ (spectral parameter) we consider the equation 
$$-M^{-1} = zJ - K_0 + \Gamma[M]$$  
(2.7)
for the unknown matrix $M \in \mathbb{C}^{m \times m}$. We will always consider solutions with the side condition that $\text{Im} M \geq 0$ where $\text{Im} M = \frac{1}{2i}(M - M^*)$. We call equation (2.7) the Dyson equation for linearization (DEL).

Note that (2.7) is very similar to the matrix Dyson equations (MDE) extensively studied in the literature in connection with large random matrices (see e.g. [10] and [2]). Their solutions typically give the deterministic part of the resolvent of a random matrix. The main difference between (2.7) and the MDE in [2] is that instead of the identity matrix, the spectral parameter $z$ appears with a coefficient matrix $J$ of smaller rank. This makes (2.7) much harder to analyse, in particular basic boundedness and stability properties do not follow directly from the structure of (2.7) alone. Nevertheless, the fact that (2.7) comes from the linearization of a polynomial, especially that it is nilpotent still ensures its good properties.

For any matrix $R \in \mathbb{C}^{m \times m}$ we denote by $\|R\|$ the operator norm induced by the Euclidean norm in $\mathbb{C}^m$. The next lemma states the existence and uniqueness of the solution to (2.7), in particular we may denote the solution $M = M(z)$, indicating its dependence on the spectral parameter.

**Lemma 2.6 (Existence and uniqueness of solution of DEL).** Let $L$ be a nilpotent linearization of the self-adjoint polynomial $1_L - q(x, y, y^*)$ with $q(0, 0, 0) = 0$ and let $\Gamma : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}$ be defined as in (2.6). There exists a matrix-valued function $M : \mathbb{C}_+ \to \mathbb{C}^{m \times m}$ such that for all $z \in \mathbb{C}_+$

(i) $\|M(z)\| \leq C (1 + 1/\text{Im} z)$ for some $C > 0$ independent of $z$;
(ii) $M(z)$ depends analytically on $z$;
(iii) $\text{Im} M(z) \geq 0$;
(iv) $M(z)$ satisfies the DEL (2.7).

This function is the unique solution of (2.7) in the class of matrix-valued functions with $\text{Im} M(z) \geq 0$ that are analytic in the upper half-plane, i.e. if $M' : \mathbb{C}_+ \to \mathbb{C}^{m \times m}$ and $M'$ satisfies (i) – (iv), then $M' = M$.

Lemma 2.6 will be proven in Section 4. In the rest of the paper, $M = M(z)$ will always denote the unique solution to (2.7) obtained in Lemma 2.6.

**Lemma 2.7 (Stieltjes transform representation).** Let $M(z)$ be the unique solution to DEL (2.7) constructed in Lemma 2.6. We then have the following:

(i) For any $z \in \mathbb{C}_+$

$$M(z) = M^\infty + \int_{\mathbb{R}} V(dx) \frac{1}{x - z},$$  
(2.8)
where $M^\infty \in \mathbb{C}^{m \times m}$ and $V(dx)$ is a (positive semidefinite) matrix-valued measure on $\mathbb{R}$ with compact support;
(ii) For almost every $x \in \mathbb{R}$ there exists the limit $\lim_{y \to 0^+} \pi^{-1} \text{Im} M(x + iy) = V(x) \in \mathbb{C}^{m \times m}$; if the limit is finite on some interval $I \subset \mathbb{R}$ everywhere, then $V(dx)$ is absolutely continuous on $I$ and $V(dx) = V(x)dx$;
(iii) There exists $C > 0$ such that for any $z \in \mathbb{C}_+$

$$\text{Tr} \text{Im} M(z) \leq C \langle e_1, \text{Im} M(z) e_1 \rangle.$$  

In particular, we have that $\text{supp}(V_{11}) = \text{supp}(\text{Tr} V)$.

This lemma will be proven in Section 4.
2.2 Polynomials and linearization of random matrices

In this section we specialize the setup from Section 2.1 to the matrix setup, i.e. to the case when \( \mathcal{A} = \mathbb{C}^{N \times N} \) for some \( N \in \mathbb{N} \) equipped with the usual matrix operator norm, induced by the Euclidean norm on \( \mathbb{C}^N \), and Hermitian conjugation to define the \( C^* \)-algebra structure. To indicate this special case in the notation, instead of \( x_1, x_2, \ldots, y_1, y_2, \ldots \) we will use capital letters, \( X_1, X_2, \ldots, Y_1, Y_2, \ldots \) for the \( N \times N \) matrices. Moreover, we assume that these matrices are random and independent. The self-adjoint matrices \( X_\alpha \) will be Wigner-type matrices, i.e. they have independent elements up to Hermitian symmetry, while the matrices \( Y_\beta \) will have independent entries without any restriction. We assume the matrix elements are centered and their variances are 1/N. We collect these assumptions in the following list:

**Assumption 2.8.** Let \( X^{(N)} := \{ X^{(N)}_\alpha, \alpha \in [\alpha_+] \} \) and \( Y^{(N)} := \{ Y^{(N)}_\beta, \beta \in [\beta_+] \} \) be two families of \( N \times N \) random matrices such that

(\( H_1 \)) the joint family \( X^{(N)} \cup Y^{(N)} \) is independent;

(\( H_2 \)) \( X^{(N)}_\alpha \) are Hermitian random matrices having independent centered entries with variance \( N^{-1} \);

(\( H_3 \)) \( Y^{(N)}_\beta \) are (non-Hermitian) random matrices having independent centered entries with variance \( N^{-1} \);

(\( H_4 \)) entries of \( X^{(N)}_\alpha \) and \( Y^{(N)}_\beta \) satisfy the moment bounds

\[
\max_{i,j \in [N]} \left( \max_{\alpha \in [\alpha_+]} \mathbb{E} \left[ |\sqrt{N} X^{(N)}_\alpha(i,j)|^p \right] + \max_{\beta \in [\beta_+]} \mathbb{E} \left[ |\sqrt{N} Y^{(N)}_\beta(i,j)|^p \right] \right) \leq C_p.
\]

Another set of assumptions concerns the properties of the solution of the Dyson equation for linearization (2.7). To this end we introduce the notions of the \( \kappa \)-bulk and the stability operator, which plays a crucial role in the analysis of the stability of the solution of (2.7).

**Definition 2.9** (Density of states). Let \( M(z) \) denote the unique solution of the DEL (2.7) given in Lemma 2.6. Define the function \( \rho : \mathbb{R} \to [0, +\infty] \)

\[
\rho(E) := \lim_{\eta \to 0_+} \frac{1}{\pi} \langle e_1, \text{Im} M(E + i\eta) e_1 \rangle,
\]

where the limit exists due to Lemma 2.7 almost everywhere. If the limit does not exist at \( E \), we set \( \rho(E) = \infty \) for convenience, to make the definition unambiguous. We will refer to \( \rho \) as the (absolutely continuous part of the) density of states of \( p \).

It will follow from the proof of Lemma 2.6 (see (4.17) and (4.21) below) that \( \rho(E) \) does not depend on the choice of linearization.

**Definition 2.10** (Bulk, \( \kappa \)-bulk). We say that \( E \in \mathbb{R} \) belongs to the bulk if \( 0 < \rho(E) < \infty \). For any \( \kappa > 0 \) we define the set \( B_\kappa := \{ E \in \mathbb{R} : \kappa < \rho(E) < \kappa^{-1} \} \), which we will call the \( \kappa \)-bulk.

We remark that Definition 2.9 slightly differs from the standard definition used for the matrix Dyson equation in [2], where the density of states was defined via the trace of \( \text{Im} M \) as \( \tilde{\rho}(E) := \lim_{\eta \to 0_+} \frac{1}{\pi \eta} \text{Tr} \text{Im} M(E + i\eta) \) and not only its \( (1,1) \)-component. The current definition is justified since our main object is the polynomial \( p \) and not its linearization \( L \). Note, that it follows from (iii) in Lemma 2.7 that \( \rho(E) \) and \( \tilde{\rho}(E) \) are comparable, i.e., a posteriori the bulk could have been defined using \( \rho \) instead of \( \tilde{\rho} \).

From now on we fix \( \kappa > 0 \).

**Definition 2.11** (Stability operator). Let \( \Gamma \) be defined as in (2.6) and \( M \) obtained in Lemma 2.6. Then the operator

\[
\mathcal{L} : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}, \quad \mathcal{L}[R] := R - M \Gamma[R] M
\]

is called the stability operator corresponding to the DEL (2.7).
Assumption 2.12. There exists a constant $C_3$, depending only on $\kappa$ and the polynomial $p$, such that for any $z \in \mathbb{C}_+$ with $\text{Re } z \in B_\kappa$ and $0 < \text{Im } z < \infty$ we have

(M1) $\|M(z)\| \leq C_3$;

(M2) $\|\mathcal{L}^{-1}(z)\| \leq C_3$.

The local law is formulated using the following the notion of stochastic domination.

Definition 2.13 (Stochastic domination). Let $\mathcal{D} \subset \mathbb{C}$ and let $(\Phi^{(N)}_w)_{N \in \mathbb{N}}$ and $(\Psi^{(N)}_w)_{N \in \mathbb{N}}$, $w \in \mathcal{D}$, be two sequences of nonnegative random variables. Then we say that $\Phi$ is stochastically dominated by $\Psi$ uniformly on $\mathcal{D}$ if for all $\varepsilon, D > 0$ there exists $C(\varepsilon, D) > 0$ such that for all $N \in \mathbb{N}$

$$
\mathbb{P}\left[\Phi^{(N)}(\lambda) \geq N^\varepsilon \Psi^{(N)}(\lambda)\right] \leq \frac{C(\varepsilon, D)}{ND}
$$

with $C(\varepsilon, D)$ independent of $N$ and $w$. In this case we write $\Phi \prec \Psi$.

We are now ready to state our main result.

Theorem 2.14 (Local law for polynomials). Let $p \in \mathbb{C}[x, y, y^*]$ be a self-adjoint polynomial with $p(0, 0, 0) = 1_{\mathcal{S}^2}$ and let $L$ be a nilpotent linearization of $p$ be defined as in [23]. Let $M(z)$ be a solution of the corresponding DEL (2.7) constructed as in Lemma 2.4. Suppose that the families of random matrices $X^{(N)}, Y^{(N)}$ satisfy conditions (H1)-(H4) and that $M(z)$ satisfies (M1)-(M2) for some fixed $\kappa > 0$. Then the local law holds for $p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ in the $\kappa$-bulk up to the optimal scale, i.e., for any $\gamma > 0$

$$
\max_{i,j \in [N]} \|g_{ij}(z) - \langle e_1, M(z)e_1 \rangle \delta_{ij}\| \ll \sqrt{\frac{1}{N \text{Im } z}}, \quad \left\| \frac{1}{N} \sum_{i=1}^{N} g_{ii}(z) - \langle e_1, M(z)e_1 \rangle \right\| \ll \frac{1}{N \text{Im } z} \quad (2.9)
$$

uniformly for $z \in D_{\kappa, \gamma}$ with $D_{\kappa, \gamma} := \{z \in \mathbb{C} : \text{Re } z \in B_\kappa, \ N^{-1+\gamma} \leq \text{Im } z \leq 1\}$, where $g(z)$ is the resolvent matrix of the polynomial

$$
g(z) := \left(p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*) - z \otimes I_N\right)^{-1}.
$$

Note that the typical distance between two adjacent eigenvalues in the bulk is of order $N^{-1}$. Thus the exponent in the bound $\text{Im } z \geq N^{-1+\gamma}$ is the lowest possible that allows for a deterministic limit of the resolvent. In [23] we formulated the local law in the entrywise and in the tracial sense, but it is easy to extend the first result to a more general anisotropic sense that approximates $\langle u, g(z)v \rangle$ for any deterministic vectors $u, v \in \mathbb{C}^N$ by adapting the method from [22] Section 7 or [11] Section 6.1 to Kronecker random matrices in the spirit of [4].

We now comment on the assumptions (M1)-(M2). We expect that these hold for an appropriate linearization for any self-adjoint polynomial, but this remains an open question in full generality. However, in Section 6.1 we prove (M1)-(M2) for a general homogeneous polynomial of degree two in Wigner matrices. For other polynomials we remark that these two assumptions can be checked numerically since they require the solution $M(z)$ of the Dyson equation for a linearization (2.7) that can be computed by an effective fixed point iteration. The numerics can be speeded up by reducing the dimension of the DEL, e.g. by considering the minimal linearization instead of the standard one, see Appendix A.3 for some examples.

Local laws provide information that can be used to estimate with relatively high precision the locations of individual eigenvalues of the corresponding random matrix, as well as to show the delocalization of its eigenvectors. These results have been obtained many times in the literature, therefore we state them without proofs and refer the interested reader to, e.g., [11] Section 5.

Corollary 2.15 (Bulk rigidity). Let $\lambda_i, 1 \leq i \leq N$, be the eigenvalues of $p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ in the increasing order. For each $E \in B_\kappa$ denote by $\iota(E)$ the index of the eigenvalue that is typically close to $E$, i.e.,

$$
\iota(E) := \left[N \int_{-\infty}^{E} \rho(dx)\right]. \quad (2.10)
$$
Then

$$\sup\{|\lambda_{i(E)} - E| : E \in B_{\kappa}\} \sim \frac{1}{N}.$$ 

**Corollary 2.16** (Delocalization of bulk eigenvectors). For $1 \leq i \leq N$ denote by $u_i \in \mathbb{C}^N$ the normalized eigenvector of $p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ that corresponds to the eigenvalue $\lambda_i$. Then for any deterministic unit vector $b \in \mathbb{C}^N$ and $E \in B_\kappa$ we have

$$|b \cdot u_{i(E)}| \sim \frac{1}{\sqrt{N}},$$

where $i(E)$ is defined as in (2.10).

Although the main focus of this paper is the local law and its consequences, we remark that our method also gives an optimal $1/N$ speed of convergence of the empirical spectral distribution of any self-adjoint polynomial $p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ to its limiting density on the global scale. More precisely, we have the following:

**Proposition 2.17** (Speed of convergence). Let $p \in \mathbb{C}(x, y, y^*)$ be a self-adjoint polynomial with $p(0, 0, 0) = 1$ and let $\rho$ be the density of states. Suppose that the families of random matrices $X^{(N)}, Y^{(N)}$ satisfy conditions (H1)-(H4). Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ and let $f$ be a smooth function on $\mathbb{R}$. Then

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) - \int_{\mathbb{R}} f(x) \rho(dx) \right| \leq \frac{1}{N}.$$  (2.11)

In particular, we have

$$\left| \frac{1}{N} \text{Tr} p(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*) - \int_{\mathbb{R}} x \rho(dx) \right| \leq \frac{1}{N}. $$  (2.12)

Note that this result does not assume the conditions (M1)-(M2). In fact, (2.12) shows that the speed of convergence in the customary definition of asymptotic freeness of the random variables $(X^{(N)}, Y^{(N)}, [Y^{(N)}]^*)$ is of order $1/N$.

In the rest of the paper, whenever this does not cause any confusion, we will suppress the $N$-dependence in $X$, $Y$ and other $N$-dependent objects.

### 3 Linearizations: nilpotency and a priori bound

In this section we prove that the linearizations of polynomials constructed in Appendix A possess some nice properties. More precisely, we show in Lemmas 3.1 and 3.2 that both the standard and the minimal linearizations are nilpotent, and then, in Section 3.2, we prove that the bound (2.5) holds for the generalized resolvents of any nilpotent linearization.

Note, that in Appendix A we consider linearizations of noncommutative polynomials in self-adjoint variables only. We start this section with a short remark explaining why this is indeed enough.

Define the real and imaginary parts of an element $a \in \mathfrak{a}$ as

$$\text{Re} \, a := \frac{a + a^*}{2}, \quad \text{Im} \, a := \frac{a - a^*}{2i},$$

so that $\text{Re} \, a$ and $\text{Im} \, a$ are self-adjoint and $a = \text{Re} \, a + i \, \text{Im} \, a$. Then (2.3) can be rewritten as

$$L = K_0 \otimes 1_\mathfrak{a} - \sum_{\gamma=1}^{\gamma_s} K_\gamma \otimes x_\gamma,$$  (3.1)

where $\gamma_s := \alpha_s + 2 \beta_s$ and we defined for $\beta \in [[\beta_* ]]$

$$x_{\alpha_* + \beta} := \sqrt{2} \text{Re} \, x_\beta, \quad x_{\alpha_* + \beta_* + \beta} := \sqrt{2} \text{Im} \, x_\beta, \quad K_{\alpha_* + \beta} := \sqrt{2} \text{Re} \, L_\beta, \quad K_{\alpha_* + \beta_* + \beta} := -\sqrt{2} \text{Im} \, L_\beta.$$  (3.2)
We can now use formulas (2.3), (3.1) and (3.2) to switch between linearizations of \( q \in \mathbb{C}(x, y, y^*) \) and \( \tilde{q} \in \mathbb{C}(x, \Re y, \Im y) \) with \( \tilde{q}(x, \Re x, \Im x) = q(x, y, y^*) \). Clearly \( q(0, 0, 0) = 0 \) is equivalent to \( \tilde{q}(0, 0, 0) = 0 \) which we will assume in the sequel. Therefore, in the current section and Section 4 with a slight abuse of notation, by defining \( x = (x_\gamma, \gamma \in [\gamma_s]) \), it will be enough to consider only self-adjoint polynomials only of the form \( \tilde{q}(x) \) with \( \tilde{q}(0) = 0 \) and linearizations \( L \) of \( \tilde{q}(x) \) of the form (3.1). In Section 5 we will go back to the linearization (2.3).

### 3.1 Joint nilpotency

In the next lemma we show that the standard linearization constructed in Appendix A.1 is nilpotent.

**Lemma 3.1 (Nilpotency of the standard linearization).** Let \( \tilde{q} \in \mathbb{C}(x) \) be a self-adjoint polynomial satisfying \( \tilde{q}(0) = 0 \). Let

\[
L = K_0 \otimes \mathbb{I}_{2^m} - \sum_{\gamma=1}^{\gamma_s} K_\gamma \otimes x_\gamma
\]

be a linearization of \( \mathbb{I}_{2^m} - \tilde{q} \) constructed in Appendix A.1. Then \( L \) is nilpotent.

**Proof.** First of all, note that (i) and (ii) in the definition of the nilpotent linearization follow directly from (A.2). Thus, in order to finish the proof we need to show that the family

\[
\left\{ \pi' K_\gamma K_0^{-1} \pi', : \gamma \in [\gamma_s] \right\}
\]

is nilpotent, where, we recall,

\[
\pi := J K_0^{-1}, \quad \pi' := I - \pi.
\]

From the representation (A.2) we have that \( \pi = J K_0^{-1} = J = K_0^{-1} J \), hence \( \pi \) commutes with \( K_0^{-1} \). This implies that for any \( \gamma \in [\gamma_s] \)

\[
\pi' K_\gamma K_0^{-1} \pi' = \pi' K_\gamma (\pi + \pi') K_0^{-1} \pi' = \pi' K_\gamma \pi' K_0^{-1} \pi' = \Theta \tilde{K}_\gamma \tilde{K}_0^{-1} \Theta^{-1},
\]

where we recall the structure of \( K_0 \) and \( K_\gamma \), explicitly indicating their minors after separating the first row and column:

\[
K_0 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \Theta \tilde{K}_0 & \\
\vdots & & & \\
0 & & & \\
\end{pmatrix}, \quad K_\gamma = \begin{pmatrix}
* & * & \cdots & * \\
* & & & \\
\vdots & & & \\
* & & & \\
\end{pmatrix}
\]

with \( \tilde{K}_0, \tilde{K}_\gamma \in \{0, 1\}^{(m-1) \times (m-1)} \) and \( \tilde{K}_0 \) being a permutation matrix. Stars indicate arbitrary unspecified matrix elements.

The key observation is the following relation between the location of nonzero matrix elements of \( \tilde{K}_0 \) and \( \tilde{K}_\gamma \)

\[
\text{if } \tilde{K}_0 = (e_{\tau_1}, e_{\tau_2}, \ldots, e_{\tau_{m-1}})^t \text{ then } \tilde{K}_\gamma = (c_1^* e_{\tau_2}, c_2^* e_{\tau_3}, \ldots, c_{m-2}^* e_{\tau_{m-1}}, 0)^t
\]

where \( \tau \) is the permutation on \([m-1]\) determined by the permutation matrix \( \tilde{K}_0 \), \( e_\tau \) is the \( \tau \)-th coordinate vector in \( \mathbb{C}^{m-1} \) and \( c_\tau \in \{0, 1\} \) are some constants. In other words (3.3) says that an entry of \( \tilde{L} \) may contain \( x_\gamma \) only if the entries just below it and on the right side contain \( \mathbb{I}_{2^m} \). For the proof, notice that this rule is immediate for the basic block of \( L \) for monomials (A.1) and it remains valid after taking the conjugate transpose or applying any of the rules (R1)-(R3).

Next, the fact that \( \tilde{K}_0 \) is a symmetric permutation matrix implies that \( \tilde{K}_0^{-1} = \tilde{K}_0 \), which means that \( \tilde{K}_0^{-1} = (e_{\tau_1}, e_{\tau_2}, \ldots, e_{\tau_{m-1}}) \). Therefore,

\[
\tilde{K}_\gamma \tilde{K}_0^{-1} = (c_1^* e_{\tau_2}, c_2^* e_{\tau_3}, \ldots, c_{m-2}^* e_{\tau_{m-1}}, 0)^t(e_{\tau_1}, e_{\tau_2}, \ldots, e_{\tau_{m-1}}) = \sum_{i=1}^{m-2} c_i^* E_{i,i+1}
\]

is strictly upper-triangular. A family of strictly upper-triangular matrices is obviously nilpotent. This finishes the proof of the lemma.
Lemma 3.2 (Nilpotency of the minimal linearization). Let \( \tilde{q} \in \mathbb{C}(\mathbf{x}) \) be a self-adjoint polynomial satisfying \( \tilde{q}(0) = 0 \). Let

\[
L = K_0 \otimes 1_{\mathcal{A}} - \sum_{\gamma=1}^{\gamma_*} K_\gamma \otimes x_\gamma
\]

be a minimal linearization of \( 1_{\mathcal{A}} - \tilde{q} \). Then \( L \) is nilpotent.

Proof. By (A.12), (A.18) and (A.23) properties (i) and (ii) from the definition of the nilpotent linearization are satisfied. Thus it is left to show that the family of matrices

\[
\left\{ \pi' K_\gamma K_0^{-1} \pi', \quad \gamma \in [\gamma_*] \right\}
\]

is nilpotent. Define for brevity \( A_\gamma := K_\gamma K_0^{-1} \). Assume that \( \|x\| \leq \varepsilon \) for \( \varepsilon > 0 \) small enough, so that

\[
\left\langle K_0^{-1} e_1 \otimes 1_{\mathcal{A}_f} \left( I \otimes 1_{\mathcal{A}_f} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right)^{-1} e_1 \otimes 1_{\mathcal{A}_f} \right\rangle_{\mathcal{A}_f} = \frac{1}{1_{\mathcal{A}_f} - p(x)} \quad (3.4)
\]

and the objects on both sides can be expanded into a convergent geometric series. Using the notation

\[
\tilde{\pi} := \pi \otimes 1_{\mathcal{A}_f}, \quad \tilde{\pi}' := I \otimes 1_{\mathcal{A}_f} - \tilde{\pi},
\]

and defining the trace operator \( \langle \cdot \rangle_{\mathcal{A}_f} : \mathbb{C}^{m \times m} \otimes \mathcal{A}_f \to \mathcal{A}_f \) by

\[
\langle R \rangle_{\mathcal{A}_f} := \sum_{k=1}^{m} R_{kk} \quad \text{for } R \in \mathbb{C}^{m \times m} \otimes \mathcal{A}_f,
\]

equality (3.4) can be rewritten as

\[
\left\langle \tilde{\pi} \left( I \otimes 1_{\mathcal{A}_f} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right)^{-1} \right\rangle_{\mathcal{A}_f} = \frac{1}{1_{\mathcal{A}_f} - \tilde{q}(x)}
\]

Now using the geometric series expansion for \( (I \otimes 1_{\mathcal{A}_f} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma)^{-1} \) we have

\[
\frac{1}{1_{\mathcal{A}_f} - \tilde{q}(x)} = \left\langle \tilde{\pi} \left( I \otimes 1_{\mathcal{A}_f} + \sum_{k=1}^{\infty} \sum_{(a_1, \ldots, a_k) \in [\gamma_\star]^k} A_{a_1} \cdots A_{a_k} \otimes x_{a_1} \cdots x_{a_k} \right) \right\rangle_{\mathcal{A}_f}
\]

\[
= \langle \pi \rangle \otimes 1_{\mathcal{A}_f} + \sum_{k=1}^{\infty} \sum_{(a_1, \ldots, a_k) \in [\gamma_\star]^k} \langle \pi A_{a_1} \cdots A_{a_k} \rangle \otimes x_{a_1} \cdots x_{a_k}, \quad (3.5)
\]

where \( \langle \cdot \rangle \) denotes the usual trace operator, i.e., \( \langle R \rangle = \text{Tr} R \) for \( R \in \mathbb{C}^{m \times m} \). Since the polynomial \( \tilde{q} \) has no constant term, we can write it as \( \tilde{q}(x) = \sum_{\beta=1}^{\infty} \tilde{q}_\beta(x) \), where \( \tilde{q}_\beta \) is a homogeneous polynomial of degree \( \beta \). Clearly this summation is finite since \( \tilde{q}_\beta \equiv 0 \) whenever \( \beta \) is larger than the degree of \( \tilde{q} \). In other words, \( \tilde{q}_1 \) denotes the linear part of \( \tilde{q} \), \( \tilde{q}_2 \) the quadratic part, etc. Then \( (1_{\mathcal{A}_f} - \tilde{q}(x))^{-1} \) can be expanded as

\[
\frac{1}{1_{\mathcal{A}_f} - \tilde{q}(x)} = 1_{\mathcal{A}_f} + \sum_{\ell=1}^{\infty} \tilde{q}^\ell(x) = 1_{\mathcal{A}_f} + \sum_{\ell=1}^{\infty} \sum_{\beta_1, \ldots, \beta_\ell=1}^{\deg(\tilde{q})} \tilde{q}_{\beta_1} \cdots \tilde{q}_{\beta_\ell}. \quad (3.6)
\]

By construction we know that \( \langle \pi \rangle = 1 \). If we now compare (3.5) and (3.6) recursively degree by degree, then from degree one terms we get that

\[
\tilde{q}_1 = \sum_{\gamma=1}^{\gamma_*} \langle \pi A_\gamma \rangle x_\gamma.
\]

Similarly, from comparing the degree two terms we have

\[
\tilde{q}_2 + \tilde{q}_1 \tilde{q}_1 = \sum_{\alpha_1, \alpha_2=1}^{\gamma_*} \langle \pi A_{\alpha_1} A_{\alpha_2} \rangle x_{\alpha_1} x_{\alpha_2},
\]
so that
\[
\tilde{q}_2 = \sum_{a_1, a_2=1}^{\gamma_\pi} \langle \pi A_{a_1} A_{a_2} \rangle x_{a_1} x_{a_2} - \sum_{a_1=1}^{\gamma_\pi} \langle \pi A_{a_1} \rangle x_{a_1} \sum_{a_2=1}^{\gamma_\pi} \langle \pi A_{a_2} \rangle x_{a_2} = \sum_{a_1, a_2=1}^{\gamma_\pi} \langle \pi A_{a_1} A'_{a_2} \rangle x_{a_1} x_{a_2},
\]
where we used the following factorization property, based upon \( J = e_1^* e_1 \): for any \( B_1, B_2 \in \mathbb{C}^{m \times m} \)
\[
\langle \pi B_1 \rangle \langle \pi B_2 \rangle = \langle e_1, K_0^{-1} B_1 e_1 \rangle \langle e_1, K_0^{-1} B_2 e_1 \rangle = \langle \pi B_1 \pi B_2 \rangle.
\] (3.7)

Next, from comparing the degree three terms in (3.5) and (3.6) we get
\[
\tilde{q}_3 + \tilde{q}_2 \tilde{q}_1 + \tilde{q}_1 \tilde{q}_2 + \tilde{q}_1 \tilde{q}_1 \tilde{q}_1 = \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} A_{a_2} A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3}
\]
and thus
\[
\tilde{q}_3 = \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} (\pi + \pi') A_{a_2} (\pi + \pi') A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3} - \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} \pi' A_{a_2} \rangle \langle \pi A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3}
- \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} \rangle \langle \pi A_{a_2} \rangle \langle \pi A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3}
- \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} \rangle \langle \pi A_{a_2} \rangle \langle \pi A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3}
- \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} \rangle \langle \pi A_{a_2} \rangle \langle \pi A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3}
= \sum_{a_1, a_2, a_3=1}^{\gamma_\pi} \langle \pi A_{a_1} \rangle \langle \pi A_{a_2} \rangle \langle \pi A_{a_3} \rangle x_{a_1} x_{a_2} x_{a_3},
\]
where again, similarly as for quadratic terms, we used (3.7) to change the order of multiplication and taking trace.

Now we prove the general formula for \( \tilde{q}_\ell \) by induction on the degree \( \ell \). Suppose that for any \( k < \ell \)
\[
\tilde{q}_k = \sum_{a_1, \ldots, a_k=1}^{\gamma_\pi} \langle \pi A_{a_1} \pi' A_{a_2} \pi' \cdots \pi' A_{a_{k-1}} \pi' A_{a_k} \rangle x_{a_1} \cdots x_{a_k}.
\] (3.8)

Then from comparing the degree \( \ell \) terms in (3.5) and (3.6) we get
\[
\sum_{a_1, \ldots, a_\ell=1}^{\gamma_\pi} \langle \pi A_{a_1} A_{a_2} \cdots A_{a_{\ell-1}} A_{a_\ell} \rangle x_{a_1} \cdots x_{a_\ell}
= \sum_{\ell-1 \geq j_\ell > j_{\ell-1} \geq \cdots \geq j_1 \in \{0,1\}} \sum_{a_1, \ldots, a_\ell=1}^{\gamma_\pi} \langle \pi A_{a_1} \kappa_{j_1} A_{a_2} \kappa_{j_2} \cdots A_{a_{\ell-1}} \kappa_{j_{\ell-1}} A_{a_\ell} \rangle x_{a_1} \cdots x_{a_\ell}
= \tilde{q}_\ell + \tilde{q}_{\ell-1} \tilde{q}_1 + \tilde{q}_{\ell-2} \tilde{q}_2 + \tilde{q}_{\ell-2} \tilde{q}_1 \tilde{q}_1 + \cdots + \tilde{q}_1 \tilde{q}_1 \cdots \tilde{q}_1
\]
where \( \kappa_0 = \pi' \) and \( \kappa_1 = \pi \). Using the factorization property (3.7) and the induction hypothesis (3.8) one can see that for the terms in the last sum can be written as
\[
\tilde{q}_{i_1} \cdots \tilde{q}_{i_s} = \sum_{a_1, \ldots, a_\ell=1}^{\gamma_\pi} \langle \pi A_{a_1} \kappa_{j_1} A_{a_2} \kappa_{j_2} \cdots A_{a_{\ell-1}} \kappa_{j_{\ell-1}} A_{a_\ell} \rangle x_{a_1} \cdots x_{a_\ell}
\]
with
\[
\ell_s = \begin{cases} 1, & s \in \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{\ell-1}\}, \\ 0, & \text{otherwise} \end{cases}
\]
and $s < m$. Therefore, we deduce by induction that for all $\ell \in \mathbb{N}$

$$\bar{q}_\ell = \sum_{\alpha_1, \ldots, \alpha_\ell = 1}^{\gamma_*} \langle \pi A_{\alpha_1} \pi' A_{\alpha_2} \pi' \cdots A_{\alpha_{\ell-1}} \pi' A_{\alpha_\ell} \rangle x_{\alpha_1} \cdots x_{\alpha_\ell}.$$  

In particular, if $\bar{q}$ is a polynomial of degree $\ell^*$, then for any $\ell > \ell^*$

$$(\pi A_{\alpha_1} \pi' A_{\alpha_2} \pi' \cdots A_{\alpha_{\ell-1}} \pi' A_{\alpha_\ell}) = 0. \quad (3.9)$$

Since $L$ is a minimal linearization and $K_0, K_\gamma \in \mathbb{C}^{m \times m}$, by Proposition [A.6] we have that

$$\text{span} \left( \bigcup_{\pi \in \mathcal{I}} A_{\pi} e_1 \right) = \mathbb{C}^m, \quad \text{span} \left( \bigcup_{\pi \in \mathcal{I}} A_{\pi}^{*} K_0^{-1} e_1 \right) = \mathbb{C}^m,$$

where $\mathcal{I}$, $A_{\pi}$ and $A_{\pi}^{*}$ are defined in (A.8) and (A.9). For $\overline{\pi} = (\alpha_1, \ldots, \alpha_k) \in \mathcal{I}$ denote

$$\bar{r}_{\overline{\pi}} := A_{\alpha_1} \pi' \cdots \pi' A_{\alpha_k} e_1, \quad \tilde{r}_{\overline{\pi}} := A_{\alpha_1}^{*} \pi'^{*} \cdots \pi'^{*} A_{\alpha_k}^{*} K_0^{-1} e_1.$$  

Then we can show that in fact

$$\text{span} \left( \bigcup_{\pi \in \mathcal{I}} \bar{r}_{\overline{\pi}} \right) = \mathbb{C}^m, \quad \text{span} \left( \bigcup_{\pi \in \mathcal{I}} \tilde{r}_{\overline{\pi}} \right) = \mathbb{C}^m. \quad (3.10)$$

Indeed, using the fact that for any $(\alpha_1, \ldots, \alpha_\ell) \in [\gamma_*]^\ell$ and any $B \in \mathbb{C}^{m \times m}$

$$A_{\alpha_1} \pi' \cdots \pi' A_{\alpha_{\ell-1}} \pi B e_1 = A_{\alpha_1} \pi' \cdots \pi' A_{\alpha_{\ell-1}} e_1 \langle \pi B \rangle$$

it can be easily seen that

$$A_{\pi} e_1 = A_{\alpha_1} (\pi + \pi') A_{\alpha_2} \cdots (\pi + \pi') A_{\alpha_k} e_1 = \bar{r}_{\overline{\pi}} + u,$$

where

$$u \in \text{span} \left( \{ \bar{r}_{\emptyset} \} \cup \bigcup_{k=1}^{\ell-1} \bigcup_{\overline{\pi} \in [\gamma_*]^k} \bar{r}_{\overline{\pi}} \right),$$

This means that for any $\ell \in \mathbb{N}$

$$\text{span} \left( \{ e_1 \} \cup \bigcup_{k=1}^{\ell} \bigcup_{\overline{\pi} \in [\gamma_*]^k} A_{\pi} e_1 \right) \subset \text{span} \left( \{ \bar{r}_{\emptyset} \} \cup \bigcup_{k=1}^{\ell} \bigcup_{\overline{\pi} \in [\gamma_*]^k} \bar{r}_{\overline{\pi}} \right)$$

which implies the first equality in (3.10). The second equality can be shown similarly.

After all these preparations, we are ready to prove the nilpotency. Fix $\ell > \ell^*$ and $(\gamma_1, \ldots, \gamma_\ell) \in [\gamma_*]^\ell$, where $\ell^*$ denotes the degree of $\bar{q}$. Then for any $\overline{\pi}, \overline{\beta} \in \mathcal{I}$ of lengths $k_\alpha$ and $k_\beta$ correspondingly, by (3.9) we have

$$\langle \bar{r}_{\overline{\pi}} \pi A_{\gamma_1} \pi' \cdots A_{\gamma_\ell} \pi' \tilde{r}_{\overline{\beta}} \rangle = \langle \pi A_{\alpha_{k_\alpha}} \pi' A_{\alpha_{k_{\alpha-1}}} \pi' \cdots A_{\alpha_1} \pi' A_{\gamma_1} \pi' \cdots A_{\gamma_\ell} \pi' A_{\beta_1} \pi' A_{\beta_2} \pi' \cdots A_{\beta_{k_\beta}} \rangle = 0,$$

which together with (3.10) implies that $\pi' A_{\gamma_1} \pi' \cdots A_{\gamma_\ell} \pi' = 0$. This completes the proof of the lemma. \hfill \Box
3.2 A priori bound

The a priori bound on the generalized resolvent of any nilpotent linearization was formulated in Lemma 3.2. Now we give its proof using the Schur complement formula.

Proof of Lemma 3.2 First of all, with the definition

\[ T(z) := \left( I \otimes 1_{\mathcal{G}} - z \pi \otimes 1_{\mathcal{G}} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right)^{-1}, \quad (3.11) \]

we can rewrite the generalized resolvent as

\[ \left( (K_0 - zJ) \otimes 1_{\mathcal{G}} - \sum_{\gamma=1}^{\gamma_*} K_\gamma \otimes x_\gamma \right)^{-1} = (K_0^{-1} \otimes 1_{\mathcal{G}}) T(z). \quad (3.12) \]

Taking the quadratic form of this identity at \( e_1 \), we have

\[ \langle K_0^{-1} e_1 \otimes 1_{\mathcal{G}}, T(z) e_1 \otimes 1_{\mathcal{G}} \rangle_{\mathcal{G}} = \left( e_1 \otimes 1_{\mathcal{G}}, \left( (K_0 - zJ) \otimes 1_{\mathcal{G}} - \sum_{\gamma=1}^{\gamma_*} K_\gamma \otimes x_\gamma \right)^{-1} e_1 \otimes 1_{\mathcal{G}} \right)_{\mathcal{G}}. \]

From the definition of the linearization and (2.4), the right hand side is just the resolvent \( ((1 - z) 1_{\mathcal{G}} - \tilde{q}(x))^{-1} \), hence

\[ \langle K_0^{-1} e_1 \otimes 1_{\mathcal{G}}, T(z) e_1 \otimes 1_{\mathcal{G}} \rangle_{\mathcal{G}} = \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}}. \quad (3.13) \]

After multiplying this identity by \( e_1 \otimes 1_{\mathcal{G}} \) on the left and \( (K_0^{-1} e_1)^* \otimes 1_{\mathcal{G}} \) on the right we obtain

\[ \tilde{\pi} T(z) \tilde{\pi} = \pi \otimes \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}}, \quad (3.14) \]

recalling \( \pi = J K_0^{-1} \) and \( J = e_1^* e_1 \).

With \( \tilde{\pi} = \pi \otimes 1_{\mathcal{G}} \), and \( \tilde{\pi}' = I \otimes 1_{\mathcal{G}} - \tilde{\pi} \), we now define

\[ S := \tilde{\pi}' + \sum_{k=1}^{\infty} \left( \tilde{\pi}' \left( \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right) \tilde{\pi}' \right)^k = \tilde{\pi}' + \sum_{k=1}^{\infty} \left( \sum_{\gamma=1}^{\gamma_*} \pi' A_\gamma \pi' \otimes x_\gamma \right)^k \quad (3.15) \]

where the series are convergent, in fact finite, by the joint nilpotency of the family of matrices \( \{ \pi' A_\gamma \pi', 1 \leq \gamma \leq \gamma_* \} \) (see Lemma 3.2). In particular, there exists a \( k^* \in \mathbb{N} \) such that

\[ \| S \| \leq C (1 + \max_{\gamma} \| x_\gamma \|_{\mathcal{G}}^{k^*}) < \infty. \quad (3.16) \]

Notice that \( S \) is the inverse of \( \tilde{\pi}' T(z) \tilde{\pi}' \) on the range of \( \tilde{\pi} \), i.e.

\[ \tilde{\pi}' \left( I \otimes 1_{\mathcal{G}} - z \tilde{\pi} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right) \tilde{\pi}' S = S \tilde{\pi}' \left( I \otimes 1_{\mathcal{G}} - z \tilde{\pi} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right) \tilde{\pi}' = \tilde{\pi}'. \]

By the generalized Schur complement formula for \( T(z) = (\tilde{\pi} + \tilde{\pi}') T(z) (\tilde{\pi} + \tilde{\pi}') \) we have

\[ \tilde{\pi} T(z) \tilde{\pi}' = - \left( \pi \otimes \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}} \right) \left( I \otimes 1_{\mathcal{G}} - z \tilde{\pi} - \sum_{\gamma=1}^{\gamma_*} A_\gamma \otimes x_\gamma \right) \tilde{\pi}' S = - \left( \pi \otimes \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}} \right) \left( \sum_{\gamma=1}^{\gamma_*} \pi A_\gamma \pi' \otimes x_\gamma \right) S, \quad (3.17) \]

\[ \tilde{\pi}' T(z) \tilde{\pi} = - S \left( \sum_{\gamma=1}^{\gamma_*} \pi' A_\gamma \pi \otimes x_\gamma \right) \left( \pi \otimes \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}} \right) \left( \sum_{\gamma=1}^{\gamma_*} \pi A_\gamma \pi' \otimes x_\gamma \right) S, \quad (3.18) \]

\[ \tilde{\pi}' T(z) \tilde{\pi}' = S + S \left( \sum_{\gamma=1}^{\gamma_*} \pi' A_\gamma \pi \otimes x_\gamma \right) \left( \pi \otimes \frac{1}{1_{\mathcal{G}} - \tilde{q}(x) - z 1_{\mathcal{G}}} \right) \left( \sum_{\gamma=1}^{\gamma_*} \pi A_\gamma \pi' \otimes x_\gamma \right) S. \quad (3.19) \]
Since \( \tilde{q}(x) \) is self-adjoint, we have a bound on the inverse of \( 1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'} \)

\[
\left\| \frac{1}{1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'}} \right\|_{\mathcal{A}'} \leq \frac{1}{\eta}.
\]

(3.20)

Using now (3.20), the boundedness of \( S \) and formulas (3.14)-(3.19) it can be seen that there exists \( C > 0 \) such that \( ||T|| \leq C(1 + \eta^{-1}) \). The bound (2.5) now follows from (3.12) and (3.16). \( \square \)

Remark 3.3. Let \( \pi_{\#} \in \mathbb{C}^{m \times m} \) be an arbitrary rank 1 projection, \( \pi'_{\#} := I_m - \pi_{\#} \), \( \tilde{\pi}_{\#} := \pi_{\#} \otimes 1_{\mathcal{A}'} \), \( \tilde{\pi}'_{\#} := \pi'_{\#} \otimes 1_{\mathcal{A}'} \) and suppose that

\[
\tilde{\pi}_{\#}(L - zJ \otimes 1_{\mathcal{A}'})^{-1}\tilde{\pi}_{\#} = \pi_{\#} \otimes \frac{1}{1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'}}.
\]

(3.21)

where \( (L - zJ \otimes 1_{\mathcal{A}'})^{-1} \) is invertible due to (2.5), and the right-hand side is well-defined since \( z \in \mathbb{C}_+ \) and \( \tilde{q}(x) \) is self-adjoint. Using the invertibility of \( L - zJ \otimes 1_{\mathcal{A}'} \), we have trivially that

\[
(\tilde{\pi}_{\#} + \tilde{\pi}'_{\#})(L - zJ \otimes 1_{\mathcal{A}'}) (\tilde{\pi}_{\#} + \tilde{\pi}'_{\#})(L - zJ \otimes 1_{\mathcal{A}'})^{-1}(\tilde{\pi}_{\#} + \tilde{\pi}'_{\#}) = I_m \otimes 1_{\mathcal{A}'},
\]

(3.22)

and, in particular,

\[
\tilde{\pi}'_{\#}(L - zJ \otimes 1_{\mathcal{A}'}) (\tilde{\pi}_{\#} + \tilde{\pi}'_{\#}) (L - zJ \otimes 1_{\mathcal{A}'})^{-1}\tilde{\pi}_{\#} = 0_{m \times m} \otimes 1_{\mathcal{A}'},
\]

(3.23)

\[
\tilde{\pi}_{\#}(L - zJ \otimes 1_{\mathcal{A}'}) (\tilde{\pi}_{\#} + \tilde{\pi}'_{\#}) (L - zJ \otimes 1_{\mathcal{A}'})^{-1}\tilde{\pi}'_{\#} = \tilde{\pi}'_{\#}.
\]

(3.24)

It can be easily checked from (3.21) and (3.23)-(3.24) that

\[
\tilde{\pi}_{\#} (L - zJ \otimes 1_{\mathcal{A}'}) \tilde{\pi}'_{\#} \left( (L - zJ \otimes 1_{\mathcal{A}'})^{-1} - (L - zJ \otimes 1_{\mathcal{A}'})^{-1} (\pi_{\#} \otimes (1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'}) (L - zJ \otimes 1_{\mathcal{A}'})^{-1} \right) \tilde{\pi}'_{\#} = \tilde{\pi}'_{\#},
\]

(3.25)

so that \( \tilde{\pi}'_{\#} (L - zJ \otimes 1_{\mathcal{A}'}) \tilde{\pi}'_{\#} \) is invertible on Ran \( \tilde{\pi}'_{\#} \) and, moreover, its inverse satisfies the bound

\[
\left\| (\tilde{\pi}'_{\#} (L - zJ \otimes 1_{\mathcal{A}'}) \tilde{\pi}'_{\#})^{-1} \right\| \leq \left\| (L - zJ \otimes 1_{\mathcal{A}'})^{-1} \left( 1 + (1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'}) \right) (L - zJ \otimes 1_{\mathcal{A}'})^{-1} \right\|.
\]

(3.26)

In particular, if we take \( \pi_{\#} = J \), then by the definition of the linearization and the Schur complement formula

\[
(J \otimes 1_{\mathcal{A}'}) (L - zJ \otimes 1_{\mathcal{A}'})^{-1} (J \otimes 1_{\mathcal{A}'}) = J \otimes \frac{1}{1_{\mathcal{A}'} - \tilde{q}(x) - z 1_{\mathcal{A}'}}.
\]

(3.27)

and, therefore, \( \tilde{L} \), defined as in (2.1), is invertible and satisfies the bound (3.26).

Notice also, that \( \tilde{L} \) is independent of \( z \), so by taking \( C_2 \) to be equal to the evaluation of the bound on the right-hand side of (3.26) at, say, \( z = 1 \), we conclude that

\[
\left\| \tilde{L}^{-1} \right\| \leq C_2
\]

(3.28)

with \( C_2 \) depending only on \( \|x_\gamma\|_{\mathcal{A}'} \), \( 1 \leq \gamma \leq \gamma_s \) and the linearization \( L \).

4 Solution to the polynomial Dyson equation

Before starting the proof of Lemma 2.6 we observe that the linear map \( \Gamma \) can be written using only self-adjoint matrices. Indeed, if we define (compare with (3.22))

\[
K_{\alpha+\beta} := \sqrt{2} \text{Re} L_{\beta}, \quad K_{\alpha+\beta_+} := -\sqrt{2} \text{Im} L_{\beta}, \quad 1 \leq \beta \leq \beta_s,
\]

then for any \( R \in \mathbb{C}^{m \times m} \)

\[
\Gamma[R] = \sum_{\alpha=1}^{\alpha_s+2\beta_s} K_{\alpha} R K_{\alpha}.
\]
Therefore, the Dyson equation for linearization (2.7) can also be written as
\[-M^{-1} = zJ - K_0 + \sum_{\alpha=1}^{\gamma} K_{\alpha} MK_{\alpha}.\] (4.2)
where we introduced \(\gamma_s := \alpha_s + 2\beta_s\) for brevity.

In the sequel we will use the following notations for comparison relations. Let \(\mathcal{D} \subset \mathbb{C}\) and let 
\((\phi_w^{(N)}), (\psi_w^{(N)})_{N \in \mathbb{N}}, w \in \mathcal{D},\) be two sequences of complex-valued functions on \(\mathcal{D}.\) We will write
\[\phi_w^{(N)} \lesssim \psi_w^{(N)}\] (or simply \(\phi \lesssim \psi\)) if there exists \(C > 0\) depending only on the polynomial \(p\) such that \(\phi_w^{(N)} \leq C\psi_w^{(N)}\) uniformly for \(w \in \mathcal{D}\) and \(N \in \mathbb{N}.\) If \(\phi \lesssim \psi\) and \(\psi \lesssim \phi\) then we will write \(\phi \sim \psi.\)

Also, from now on we will always denote the real and imaginary parts of the spectral parameter \(z\) by \(E\) and \(\eta\) correspondingly, i.e., \(z := E + i\eta.\)

**Proof of Lemma 2.6 Existence.** Let \(\{s_1, s_2, \ldots, s_{\gamma_s}\}\) be a family of free semicircular variables in a \(C^*\)-probability space \((\mathcal{S}, \tau)\) (see Appendix B). Define
\[L_{\text{sc}} := K_0 \otimes 1 - \sum_{\gamma=1}^{\gamma_s} K_{\gamma} \otimes s_{\gamma},\]
and for \(z \in \mathbb{C}_+\) define a function \(M_{\text{sc}}(z) : \mathbb{C}_+ \to \mathbb{C}^{m \times m}\) by
\[M_{\text{sc}}(z) := (id \otimes \tau)(L_{\text{sc}} - zJ \otimes 1)^{-1}.\] (4.3)
The subscript in \(L_{\text{sc}}\) and \(M_{\text{sc}}\) refers to the semicircular elements.

We will show that the function \(M_{\text{sc}}\) is well-defined on \(\mathbb{C}_+\) and satisfies (i)-(iv) of Lemma 2.6. We now introduce some notation that will be used throughout the proof. Let \(\pi\) and \(\pi'\) denote as before projections on \(\mathbb{C}^{m \times m}\) given by \(\pi = JK_0^{-1}, \pi' = I - \pi,\) and let \(\tilde{\pi}\) and \(\tilde{\pi}'\) be projections on \(\mathbb{C}^{m \times m} \otimes \mathcal{S}\) defined by
\[\tilde{\pi} := \pi \otimes 1, \quad \tilde{\pi}' := I \otimes 1 - \tilde{\pi} = \pi' \otimes 1.\]
Define also the matrices \(A_{\gamma} := K_{\gamma} K_0^{-1}, \gamma \in [\gamma_s].\) Notice that the nilpotency of \(L\) implies that \(\{\pi' A_{\gamma} \pi\}_{\gamma=1}^{\gamma_s}\) is a nilpotent family.

We first show that \(M_{\text{sc}}(z)\) is well-defined and properties (i)-(iii) hold. To see this, we apply Lemma 2.5 with \(\mathcal{J} = \mathcal{S}\) to \((L_{\text{sc}} - zJ \otimes 1)^{-1}.\) Then from (2.5) (assuming only self-adjoint variables) we obtain that for any \(z \in \mathbb{C}_+\)
\[\|(L_{\text{sc}} - zJ \otimes 1)^{-1}\| \lesssim 1 + \frac{1}{\eta},\] (4.4)
Moreover, simple computation shows that
\[\text{Im } M_{\text{sc}}(z) = \eta (id \otimes \tau) \left( (L_{\text{sc}} - zJ \otimes 1)^{-1}(J \otimes 1)(L_{\text{sc}} - zJ \otimes 1)^{-1}) \right),\]
which yields that \(\text{Im } M_{\text{sc}}(z)\) is positive semi-definite.

In order to prove that \(M_{\text{sc}}(z)\) satisfies the DEL (2.7), consider its regularizations, i.e., a family of matrix-valued functions \(\{M_{\text{sc}, i u_k}\}_{k \in \mathbb{N}}\) given by
\[M_{\text{sc}, i u_k}(z) := (id \otimes \tau)(L_{\text{sc}} - (zJ + i u_k I) \otimes 1)^{-1}\] (4.5)
with \(z \in \mathbb{C}_+\) and \(u_k = k^{-1}.\) For any fixed \(k \in \mathbb{N},\) the imaginary part of \(zJ + i u_k I\) is positive definite, therefore it follows from [35, Lemma 5.4] (see also [47, Proposition 4.1]) that the function \(M_{\text{sc}, i u_k}(z)\) is analytic in \(\mathbb{C}_+\) and satisfies the self-consistent (or Matrix Dyson) equation
\[- \left[ M_{\text{sc}, i u_k}(z) \right]^{-1} = K_0 - (zJ + i u_k I) + \Gamma[M_{\text{sc}, i u_k}(z)],\] (4.6)
which can be viewed as a regularized version of the DEL (2.7). Using the a priori bound (4.4), for any fixed \(\eta = \text{Im } z\) we have
\[|u_k| \|(L_{\text{sc}} - zJ \otimes 1)^{-1}\| < \frac{1}{2}\] (4.7)
if \( k \in \mathbb{N} \) is large enough. Therefore, the resolvent identity implies that for \( k \in \mathbb{N} \) large enough (depending on \( \eta = \Im z \))

\[
(L_{sc} - (zJ + i u_k) \otimes 1)^{-1} = (I \otimes 1 - i u_k (L_{sc} - zJ \otimes 1)^{-1})^{-1} (L_{sc} - zJ \otimes 1)^{-1},
\]

so that by definition (4.5) the function \( M_{sc, i u_k}(z) \) satisfies a \( u_k \)-independent bound

\[
\|M_{sc, i u_k}(z)\| \lesssim 1 + \frac{1}{\eta}, \quad \forall k \geq k_0(\eta).
\]

On the other hand, the trivial resolvent bound implies that

\[
\|M_{sc, i u_k}(z)\| \leq \frac{1}{u_k}.
\]

Therefore, it is easy to see that the family of function \( \{M_{sc, i u_k} : \mathbb{C}_+ \to \mathbb{C}^{m \times m}\}_{k \in \mathbb{N}} \) is locally uniformly bounded, and thus, by the Montel’s theorem, is normal. From the a priori bounds (4.4) and (4.9) and the resolvent identity we have that for \( k \in \mathbb{N} \) large enough

\[
\|M_{sc, i u_k}(z) - M_{sc}(z)\| \lesssim \frac{1}{k} \left( 1 + \frac{1}{\eta} \right)^{2},
\]

which yields the pointwise limit \( \lim_{k \to \infty} M_{sc, i u_k}(z) = M_{sc}(z) \) for \( z \in \mathbb{C}_+ \). The normality of \( \{M_{sc, i u_k}\}_{k \in \mathbb{N}} \) implies that \( M_{sc}(z) \) is analytic on \( z \in \mathbb{C}_+ \). By rewriting (4.6) as

\[
(K_0 - (zJ + i u_k I))M_{sc, i u_k}(z) + \Gamma[M_{sc, i u_k}(z)]M_{sc, i u_k}(z) + 1 = 0
\]

and taking the limit \( k \to \infty \), we obtain that \( M_{sc}(z) \) satisfies the DEL (2.7).

**Uniqueness.** Suppose that \( M_1, M_2 : \mathbb{C}_+ \to \mathbb{C}^{m \times m} \) are two analytic solutions of (4.2) satisfying \( \Im M_1(z) \geq 0 \) and \( \Im M_2(z) \geq 0 \). It is easy to see that both \( M_{1,2}(z) \) are solutions of (4.2) on \( \mathbb{C}_+ \) if and only if for all \( z \in \mathbb{C}_+ \) functions \( M_{1,2}^{ns}(z) := K_0 M_{1,2}(z) \) satisfy

\[
M_{1,2}^{ns}(z) = \frac{1}{1 - z} \pi + \pi' + M_{1,2}^{ns}(z) \sum_{\gamma=1}^{\gamma_+} A_\gamma M_{1,2}^{ns}(z) A_\gamma \left( \frac{1}{1 - z} \pi + \pi' \right).
\]

If we recursively replace \( M_{1,2}^{ns}(z) \) in the RHS by the expression given in the RHS of (4.13), we obtain a series which is convergent for large \( z \) due to nilpotency of the linearization. Indeed, if we assume for simplicity that \( \gamma_+ = 1 \), then \( M_{1,2}^{ns}(z) \) can be rewritten as

\[
M_{1,2}^{ns}(z) = \sum_{\ell=0}^{\infty} C_\ell \left( \left( \frac{1}{1 - z} \pi + \pi' \right) A_1 \right)^{2\ell} \left( \frac{1}{1 - z} \pi + \pi' \right)
\]

where \( C_\ell \) denotes the \( \ell \)th Catalan number. Since \( C_\ell = \frac{1}{\ell + 1} \binom{2\ell}{\ell} \leq 4^{\ell} \), we conclude that the RHS of (4.14) contains \( O\left(16^{\ell}\right) \) products of type

\[
\sigma_1 A_1 \cdots \sigma_2 \pi A_2 \sigma_{2\ell + 1}
\]

with \( \sigma_i \in \{ \frac{1}{1 - z} \pi, \pi' \} \). Now we collect all terms of type (4.16) that behave asymptotically like \((1 - z)^{-i}\) for some \( i \in \mathbb{N} \) as \( |z| \to \infty \). By the nilpotency of \( \{\pi' A_\gamma \pi', 1 \leq \gamma \leq \gamma_+\} \) there exists \( k \in \mathbb{N} \) such that \( \pi' A_\gamma \pi' \cdots \pi' A_\gamma \pi' = 0 \). Therefore, the maximum \( \ell \) for which there exists a product (4.16) that behaves asymptotically as \((1 - z)^{-i}\) is less than \( k(i + 1)/2 \). This implies that if \(|1 - z| > 2|A_1|\|^k\), then the RHS in (4.14) converges and thus \( M_{1,2}^{ns}(z) = M_2^{ns}(z) \) on \( \mathbb{C}_+ \) by analyticity. If \( \gamma_+ > 1 \), then it can be shown similarly that on the set \(|z| > 2|A_1| \max_{1 \leq \gamma \leq \gamma_+} \|A_\gamma\|^k \) functions \( M_{1,2}^{ns}(z) \) and \( M_2^{ns}(z) \) coincide, which again implies \( M_1^{ns}(z) = M_2^{ns}(z) \) on \( \mathbb{C}_+ \). This concludes the proof that \( M_{sc}(z) \) defined in (4.3) is the unique solution to the DEL (2.7), i.e. \( M(z) = M_{sc}(z) \).
Proof of Lemma 2.7. $M(z)$ is a matrix-valued Herglotz function, therefore, from [34 (1.1)-(1.3)] it has the following representation

$$M(z) = B_1 z + B_0 + \int_\mathbb{R} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) V(dx), \quad (4.16)$$

where $B_1, B_0 \in \mathbb{C}^{m \times m}$, $B_1 = \lim_{\eta \to \infty} \left( i \eta M(i \eta) \right)$ and $V(dx)$ is a matrix-valued measure satisfying

$$\int_\mathbb{R} \left( c, V(dx) c \right) 1 + x^2 < \infty \quad \text{and} \quad \int V(dx) \geq 0$$

for any $c \in \mathbb{C}^m$ and Borel $I \subset \mathbb{R}$.

By definition (4.3) and the conclusion of the existence part of the proof of Lemma 2.6 we know that $M(z)$ can be written as

$$M(z) = M_{sc}(z) = (\text{id} \otimes \tau) \left( L_{sc} - z J \otimes 1 \right)^{-1}. \quad (4.17)$$

Similarly as in (2.1), express $L_{sc}$ in a 2 by 2 block form

$$L_{sc} = \begin{pmatrix} L_{sc} & \ell_{sc}^* \\ \ell_{sc} & \widehat{L}_{sc} \end{pmatrix} \quad (4.18)$$

by separating its first row and column, so that $1 - \hat{q}(s) = \lambda_{sc} - \ell_{sc}^* \hat{L}_{sc}^{-1} \ell_{sc}$. Now apply to $(L_{sc} - z J \otimes 1)^{-1}$ the Schur complement formula with respect to its $(1,1)$ component in the form

$$\begin{pmatrix} \lambda_{sc} - z 1 & \ell_{sc}^* \\ \ell_{sc} & L_{sc} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{L}_{sc}^{-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -\hat{L}_{sc}^{-1} \ell_{sc} \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\lambda_{sc} - z 1} \frac{\hat{L}_{sc}}{\ell_{sc}} \right) \left( \begin{pmatrix} 1 & \ell_{sc}^* \hat{L}_{sc}^{-1} \\ 0 & \ell_{sc} \end{pmatrix} \right) \quad (4.19)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \hat{L}_{sc}^{-1} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -\hat{L}_{sc}^{-1} \ell_{sc} \end{pmatrix} \left( \sum_{i=0}^{\infty} \frac{(1-\hat{q}(s)) i}{i+1} \right) \left( \begin{pmatrix} 1 & \ell_{sc}^* \hat{L}_{sc}^{-1} \\ 0 & \ell_{sc} \end{pmatrix} \right), \quad (4.20)$$

where $\hat{L}_{sc}$ is invertible and satisfies the bound $\| \hat{L}_{sc}^{-1} \| \leq C_2$, for $C_2 > 0$ depending only on the linearization $L$ (see (3.28) in Remark 3.3). The series in (4.20) is clearly convergent for $|z| > \| 1 - \hat{q}(s) \|_{\mathcal{X}}$. The expansion (4.20) together with (4.17) immediately imply that $\| M(i \eta) \| \leq C$ for some $C > 0$ and all $\eta > 1$ large enough, from which it follows that $B_1 = 0$ in (4.16).

By Definition 2.1 of self-adjoint linearizations, the submatrix $\hat{L}_{sc}$ is self-adjoint, i.e., $\Im \hat{L}_{sc} = 0$. Therefore, (4.20) implies that

$$\Im (L_{sc} - z J \otimes 1)^{-1} = -\begin{pmatrix} 1 & 0 \\ -\hat{L}_{sc}^{-1} \ell_{sc} & 0 \end{pmatrix} \left( \Im \sum_{i=0}^{\infty} \frac{(1-\hat{q}(s)) i}{i+1} \right) \left( \begin{pmatrix} 1 & \ell_{sc}^* \hat{L}_{sc}^{-1} \\ 0 & \ell_{sc} \end{pmatrix} \right). \quad (4.21)$$

From the properties of scalar-valued Herglotz functions (formula S1.1.9 in [43]), polarization (as in the proof of Lemma 5.3 in [31]) and (4.17) we obtain that

$$\int \mathbb{R} V(dx) = \lim_{\eta \to \infty} \eta \Im M(i \eta)$$

$$= (\text{id} \otimes \tau) \left( \begin{pmatrix} 1 & 0 \\ -\hat{L}_{sc}^{-1} \ell_{sc} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 0 \\ 0 & \ell_{sc} \end{pmatrix} \right) \left( \begin{pmatrix} 1 & \ell_{sc}^* \hat{L}_{sc}^{-1} \\ 0 & \ell_{sc} \end{pmatrix} \right), \quad (4.22)$$

where the second line follows from the boundedness of $\hat{L}_{sc}^{-1}$ and the expansion in (4.21), for which only the term corresponding to $i = 0$ does not vanish in the limit. In particular, (4.22) implies that $\| \int \mathbb{R} V(dx) \| < \infty$ and $V_{11}(\mathbb{R}) = 1$, i.e., that $V_{11}(dx)$ is a probability measure.

Since $V(dx)$ is a positive semidefinite matrix-valued measure, the inequality $\| \int \mathbb{R} f(x) V(dx) \| \leq \max_{x \in \mathbb{R}} |f(x)| \| \int \mathbb{R} V(dx) \|$ holds for all measurable functions $f : \mathbb{R} \to \mathbb{C}$. Therefore, the boundedness
of \( \| \int_{\mathbb{R}} V(dx) \| \) implies that both integrands in (4.16) separately have finite integrals. By taking the limit \( z \to \infty \) in (4.19) with \( B_1 = 0 \), we obtain that

\[
\lim_{z \to \infty} M(z) = B_0 - \int_{\mathbb{R}} \frac{x V(dx)}{1 + x^2}.
\]

On the other hand, taking the same limit in (4.20) leads to

\[
\lim_{z \to \infty} M(z) = (id \otimes \tau) \begin{pmatrix} 0 & 0 \\ 0 & \hat{L}_{sc}^{-1} \end{pmatrix}.
\]

(4.23)

Combining the two limits above and putting \( M^\infty := (id \otimes \tau) \begin{pmatrix} 0 & 0 \\ 0 & \hat{L}_{sc}^{-1} \end{pmatrix} \) yields the representation (2.8).

Writing now \( \text{Im} M(z) \) as

\[
\text{Im} M(z) = (id \otimes \tau)(\text{Im} (L_{sc} - zJ \otimes 1)^{-1}) = (id \otimes \tau)(\eta(L_{sc} - zJ \otimes 1)^{-1} (J \otimes 1) (L_{sc} - zJ \otimes 1)^{-1})
\]

and using consequently (4.17) and (4.20) we get that

\[
\| \text{Im} M(E + i \eta) \| \lesssim \eta \frac{\eta}{(E - \| 1 - \tilde{q}(s) \| \tau )},
\]

(4.25)

and thus

\[
\lim_{\eta \to 0_+} \text{Im} M(E + i \eta) = 0
\]

(4.26)

if \(|E| > \| 1 - \tilde{q}(s) \| \tau \). Therefore, by dominated convergence, for any \( a < b < -\| 1 - \tilde{q}(s) \| \tau \) or \( \| 1 - \tilde{q}(s) \| \tau < a < b \)

\[
\limsup_{\eta \uparrow 0} \int_a^b \text{Im} M(E + i \eta) dE = 0.
\]

(4.27)

By the Stieltjes inversion formula (see, e.g., [34] formula (1.4)), we conclude that \( \text{supp} V \subset [-\| 1 - \tilde{q}(s) \| \tau , \| 1 - \tilde{q}(s) \| \tau ] \). This finishes the proof of part (i) of Lemma 2.7.

Part (ii) of Lemma 2.7 follows from, e.g., [34] Lemma 5.4 and the Stieltjes transform representation (2.8) of \( M(z) \).

In order to prove part (iii), we compute the diagonal entries of \( \text{Im} M(z) \) using (4.17) and the equality (4.19). More precisely, from (4.19) we have that

\[
\text{Im} (L_{sc} - zJ \otimes 1)^{-1} = \left( \begin{array}{ccc} 1 & 0 \\ -L_{sc}^{-1} \ell_{sc} & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 \\ 0 & 0 \\ 0 & \ell_{sc}^{-1} L_{sc}^{-1} \end{array} \right).
\]

(4.28)

The polynomial \( \tilde{q} \) is self-adjoint, thus

\[
\text{Im} \frac{1}{1 - \tilde{q}(s) - z1} = \eta \frac{1}{(1 - \tilde{q}(s) - z1)(1 - \tilde{q}(s) - \overline{z}1)}.
\]

(4.29)

Applying now (4.28) and (4.29) to the free probability representation of \( M(z) \) from (4.3), we get that

\[
\langle e_1, \text{Im} M(z) e_1 \rangle = \eta \tau \left( \frac{1}{(1 - \tilde{q}(s) - z1)(1 - \tilde{q}(s) - \overline{z}1)} \right)
\]

(4.30)

and

\[
\langle e_i, \text{Im} M(z) e_i \rangle = \eta \tau \left( \left( \hat{L}_{sc}^{-1} \ell_{sc} \right)_{i-1} \left( \frac{1}{1 - \tilde{q}(s) - z1}(1 - \tilde{q}(s) - \overline{z}1) \right) \left( \hat{L}_{sc}^{-1} \ell_{sc} \right)^*_{i-1} \right)
\]

(4.31)
for \(2 \leq i \leq m\). In particular, (1.30) shows that the imaginary part of the upper left entry of \(M(z)\) is independent of the linearization.

Now, applying (G.7) from [7] to (4.31), using submultiplicativity of the norm (see [13] and \(\|s_\alpha\|_\gamma = 2\), we get that for all \(2 \leq i \leq m\),

\[
\langle e_1, \text{Im } M(z) e_1 \rangle \leq \left\| \left( \hat{L}_{sc}^{-1} e_{sc} \right)_{i-1} \right\|_\gamma^2 \eta \tau \left( \frac{1}{(1 - \bar{q}(s) - z\bar{1})(1 - \bar{q}(s) - \bar{1})} \right).
\]

Combining (4.30), (4.31) and (4.32) we end up with the following bound for \(\text{Tr Im } M(z)\)

\[
\text{Tr Im } M(z) \leq \langle e_1, \text{Im } M(z) e_1 \rangle \left( 1 + \sum_{i=2}^m \left\| \left( \hat{L}_{sc}^{-1} e_{sc} \right)_{i-1} \right\|_\gamma^2 \right).
\]

Therefore, if \(\lim_{\eta \to 0} \langle e_1, \text{Im } M(E + i\eta) e_1 \rangle = 0\) for some \(E \in \mathbb{R}\), then \(\lim_{\eta \to 0} \text{Tr Im } M(E + i\eta) = 0\), and similarly if \(\lim_{\eta \to 0} \text{Tr Im } M(E + i\eta) = \infty\) then \(\lim_{\eta \to 0} \langle e_1, \text{Im } M(E + i\eta) e_1 \rangle = \infty\). We conclude that \(\text{supp}(V_{11}) = \text{supp}(\text{Tr } V)\).}

\(\square\)

Lemma 4.1 (Stability of the solution of the DEL). For fixed \(\kappa > 0\) and under assumptions (M1) and (M2), there exists \(\epsilon > 0\) such that uniformly for all \(z \in \mathbb{C}_+\) with \(\text{Re } z \in B_\kappa\) the following holds true:

(i) For any \(R \in \mathbb{C}^{m \times m}\), \(\|R\| < \epsilon\), the matrix equation

\[-M^{-1} = zJ - K_0 + \Gamma[M] + R\]

has a solution, which we denote by \(M(R)\);

(ii) For any \(R_1, R_2 \in \mathbb{C}^{m \times m}\), \(\|R_1\| < \epsilon\), \(\|R_2\| < \epsilon\), we have

\[\|M(R_1) - M(R_2)\| \leq C \|R_1 - R_2\|.
\]

Proof. This follows easily from (M1) and (M2) (see e.g. proof of the Corollary 3.8 in [4]).}

\(\square\)

5 Proof of the local law

In order to establish the local law for the polynomials we will rely heavily on the linearization technique described in the previous sections. More precisely, given a self-adjoint polynomial \(p = p(X, Y, Y^*)\) in the variables \(X, Y\) and \(Y^*\), we consider one of its nilpotent linearizations \(L\) as defined in Section 2.1 Its generalized resolvent will give the necessary information on the resolvent of \(p\) via (2.4). So from now on our main object of interest will be the linearized random matrix \(H\) defined by

\[
H = K_0 \otimes I_N - \sum_{\alpha=1}^{\alpha_*} K_\alpha \otimes X_\alpha - \sum_{\beta=1}^{\beta_*} (L_\beta \otimes Y_\beta + L^*_\beta \otimes Y^*_\beta).
\]

(5.1)

This matrix plays the role of \(L\) in Section 2.1 but we use a different letter to stress that we are in the random matrix setup. We denote by \(I_N\) the unit element of \(\mathcal{A} = \mathbb{C}^{N \times N}\). We remark that random matrices of the form (5.1), in particular their resolvents, have been extensively studied in [4] where they were called Kronecker matrices.

We will denote the generalized resolvent of \(H\) by \(G(z) := (H - zJ \otimes I_N)^{-1}\). By \(G_{kl} \in \mathbb{C}^{N \times N}\) and \(G_{ij} \in \mathbb{C}^{m \times m}\) we will denote the coefficient of \(G\) in the standard bases of \(\mathbb{C}^{m \times m}\) and \(\mathbb{C}^{N \times N}\) correspondingly, i.e.,

\[
G = \sum_{k,l=1}^m E_{kl} \otimes G_{kl} = \sum_{i,j=1}^N G_{ij} \otimes E_{ij}
\]

with \(E_{ij} = E^{(n)}_{ij} := (\delta_{ki} \delta_{lj})_{k,l=1}^N \in \mathbb{C}^{n \times n}\) for corresponding \(n \in \mathbb{N}\). More generally, for any \(R \in \mathbb{C}^{m \times m} \otimes \mathbb{C}^{N \times N}\) we will denote its coefficients in the standard basis of \(\mathbb{C}^{N \times N}\) by \(P_{ij}(R), 1 \leq i, j \leq N\), so that

\[
R = \sum_{i,j=1}^N P_{ij}(R) \otimes E_{ij}.
\]

In particular, we have \(P_{ij}(G) = G_{ij}\). Here is our main technical result.
Theorem 5.1 (Local law for the linearization). Let \( p \in \mathbb{C}(x,y,y^*) \) be a self-adjoint polynomial with \( p(0,0,0) = 1 \) and let \( L \) be a nilpotent linearization of \( p \) be defined as in (2.7). Let \( M(z) \) be a solution of the corresponding DEL (2.7) constructed as in Lemma 2.6. Let \( H \) be defined as in (5.1). Suppose that the families of random matrices \( X, Y \) satisfy conditions (H1)-(H4) and that \( M(z) \) satisfies (M1)-(M2) for some fixed \( \kappa > 0 \). Then the local law holds for \( H \) in the \( \kappa \)-bulk up to the optimal scale, i.e., for any \( \gamma > 0 \) we have

\[
\max_{i,j \in [N]} \left\| G_{ij}(z) - M(z) \delta_{ij} \right\| < \sqrt{\frac{1}{N \Im z}}, \quad \left\| \frac{1}{N} \sum_{i=1}^N G_{ii}(z) - M(z) \right\| < \frac{1}{N \Im z} \tag{5.2}
\]

uniformly for \( z \in D_{\kappa,\gamma} \) with \( D_{\kappa,\gamma} := \{ z \in \mathbb{C} : \Re z \in B_\kappa, \, N^{-1+\gamma} \leq \Im z \leq 1 \} \).

Proof of Theorem 2.14. It follows immediately from Theorem 5.1 and the Schur complement formula (2.1).

The rest of this section is devoted to the proof of Theorem 5.1. Throughout this section we will use regularizations of \( G \) and \( M \). For \( z, \omega \in \mathbb{C}_+ \) define

\[
G_\omega(z) = (H - zJ \otimes I_N - \omega I \otimes I_N)^{-1}
\]

and let \( M_\omega(z) \) be the solution of the regularized DEL

\[
- (M_\omega(z))^{-1} = z J + \omega I - K_0 + \Gamma[M_\omega(z)]
\tag{5.3}
\]

that is analytic in \( z \) and \( \omega \) and has positive definite imaginary part. Note that the existence of a solution to (5.3) was shown earlier [46, 40] and analyticity in \( \omega \) and \( z \) can be inferred from this general theory as well. As an alternative, analyticity in \( \omega \) and \( z \) can also be seen directly from an application of the implicit function theorem by differentiating (5.3). This also demonstrates the role of the stability operator from assumption (M2) for the regularity of the solution \( M_\omega(z) \). In fact, differentiating yields

\[
- (M_\omega(z))^{-1} \partial M_\omega(z) (M_\omega(z))^{-1} = K + \Gamma[\partial M_\omega(z)],
\tag{5.4}
\]

where either \( \partial = \partial_\omega \) with \( K = I \) or \( \partial = \partial_z \) with \( K = J \), depending on the variable we are interested in. After rearranging the terms, the above equation can be rewritten as

\[
\mathcal{L}_\omega[\partial M_\omega(z)] = -M_\omega(z)KM_\omega(z),
\]

where

\[
\mathcal{L}_\omega : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}, \quad \mathcal{L}_\omega(R) := R - M_\omega(z) \Gamma[R] M_\omega(z).
\]

From [3, Lemma 3.7] and the trivial bound \( \| M_\omega(z) \| \leq (\Im \omega)^{-1} \) we have that \( \| \mathcal{L}_\omega^{-1} \| \leq (\Im \omega)^{-10} \). Using implicit function theorem we conclude that \( \| \partial M_\omega(z) \| \leq (\Im \omega)^{-12} \) and that \( M_\omega(z) \) is analytic in \( \omega \) and \( z \).

The next lemma collects some properties of the regularizations \( G_\omega(z) \) and \( M_\omega(z) \).

Lemma 5.2. There exists \( C > 0 \) such that

(i) uniformly on \( E \in \mathbb{R}, \, \eta > 0 \) and \( u \geq 0 \)

\[
\| G_{1u}(z) \| \leq C \left( 1 + \frac{1}{\eta} \right);
\tag{5.5}
\]

(ii) if (M1) holds, then uniformly on \( E \in B_\kappa, \, \eta \geq 0 \) and \( u \geq 0 \)

\[
\| M_{1u}(z) \| \leq C, \quad \| (M_{1u}(z))^{-1} \| \leq C(1 + |z| + u);
\tag{5.6}
\]

(iii) if additionally (M2) holds, then uniformly on \( E \in B_\kappa, \, 0 \leq \eta \leq 1 \) and \( u \geq 0 \)

\[
\| (\mathcal{L}_{1u}(z))^{-1} \| \leq C.
\tag{5.7}
\]
Proof. Firstly, by specializing Lemma 2.5 for $\mathcal{M} = \mathbb{C}^{m \times m} \otimes \mathbb{C}^{N \times N}$, $\mathbf{x} = \mathbf{X}$, $\mathbf{y} = \mathbf{Y}$ and $\mathbf{L} = \mathbf{H}$, we obtain that there exists $C_1 > 0$ such that

$$\|G(z)\| \leq C_1 \left(1 + \frac{1}{\eta}\right)$$

By the resolvent identity, for any $E \in B_{\kappa}$, $\eta \geq 0$ and $u \geq 0$ we have that

$$G_{1u}(z) = G(z) + iu G_{1u}(z) G(z),$$

therefore from the trivial bound $\|G_{1u}(z)\| \leq u^{-1}$ we obtain

$$\|G_{1u}(z)\| \leq 2\|G(z)\|.$$

By the stability of the solution of the DEL (4.34) and (M1), there exists $C_2 > 0$ such that for any $E \in B_{\kappa}$, $\eta \geq 0$ and $0 \leq u \leq 1$

$$\|M_{1u}(z)\| \leq C_2.$$

On the other hand, if we apply the trivial bound $M_{1u}(z) \leq u^{-1}$ for $u \geq 1$, we obtain that

$$\|M_{1u}(z)\| \leq \max\{1, C_2\} =: C_3 \quad (5.8)$$

for $E \in B_{\kappa}$, $\eta \geq 0$ and $u \geq 0$. Now, using (5.3) and (5.8), there exists $C_4 > 0$ such that for all $E \in B_{\kappa}$, $\eta \geq 0$ and $u \geq 0$

$$\|(M_{1u}(z))^{-1}\| \leq C_4(1 + |z| + u). \quad (5.9)$$

To obtain (5.7) note that

$$\|L_{1u} - L\| \leq \|M_{1u}(z) - M(z)\| \left(\|M_{1u}(z)\| + \|M(z)\|\right) \|\Gamma\| \leq C_5u \quad (5.10)$$

for some $C_5 > 0$. Therefore by (M2) there exists $\epsilon_1 > 0$ and $C_6 > 0$ such that for $0 \leq u \leq \epsilon_1$

$$\|(L_{1u})^{-1}\| = \|L^{-1}(I - (L - L_{1u})L^{-1})\| \leq 2\|L^{-1}\| \leq C_6.$$

By the definition of $L_{1u}(z)$ and the trivial bound $\|M_{1u}(z)\| \leq u^{-1}$, there exists $\epsilon_2 > 0$ such that for $u \geq \epsilon_2$

$$\|(L_{1u})^{-1}\| \leq \frac{1}{1 - \|M_{1u}(z)\|^2\|\Gamma\|} \leq 2.$$

Finally, by [4] Lemma 3.7, compactness of $B_{\kappa}$, (5.8) and (5.9) there exists $C_7 > 0$ such that for all $E \in B_{\kappa}$, $0 \leq \eta \leq 1$ and $\epsilon_1 \leq u \leq \epsilon_2$

$$\|(L_{1u})^{-1}\| \leq \frac{\|M_{1u}(z)\|^2\|(M_{1u}(z))^{-1}\|^9}{(\text{dist}(\text{supp}(\rho_x), iu))^8} \leq C_7,$$

where $\rho_x(x) := \lim_{u \downarrow 0}(\pi u)^{-1} \text{Tr} \text{Im} M_{x+iu}(z)$. To finish the proof, take $C > \max\{2, 2C_1, C_3, C_4, C_6, C_7\}$.

Now we state the local law for the regularized resolvent.

**Lemma 5.3.** Uniformly for $E \in B_{\kappa}$, $0 \leq \eta \leq 1$ and $u \geq N^{-1+\gamma}$

$$\max_{i,j \in [N]} \|P_{ij}(G_{1u}(E + i\eta)) - M_{1u}(E + i\eta) \delta_{ij}\| < \sqrt{\frac{1}{Nu}}. \quad (5.11)$$

**Proof.** Follows from Lemma B.1 in [4]. Indeed, by (5.6) and (5.7) for all $E \in B_{\kappa}$, $0 \leq \eta \leq 1$ and $u \geq N^{-1+\gamma}$ we have

$$\max_{i,j \in [N]} \|P_{ij}(G_{1u}(E + i\eta)) - M_{1u}(E + i\eta) \delta_{ij}\| < \frac{1}{1+u} \sqrt{\frac{\|M_{1u}(E + i\eta)\|}{Nu}} + \frac{1}{(1+u^2)N} + \frac{1}{(1+u^2)Nu}.$$

The fact that $M_{1u}(E + i\eta)$ is bounded by (5.6) yields (5.11).
We are ready to prove the main theorem.

Proof of Theorem 5.2

By [4] Lemma 4.4 and Lemma 5.2 for $E \in B_\kappa$, $0 \leq \eta \leq 1$ and $\tilde{\eta} \geq 0$

$$\max_{1 \leq i \leq N} \| P_i \left( G_{i,\tilde{\eta}}(z) \right) - M_{i,\tilde{\eta}}(z) \| \chi(\lambda \tilde{\eta} \leq \vartheta \tilde{\eta}) \leq \frac{1}{\sqrt{N}} + \Lambda_{\tilde{\eta}} + \| (M_{i,\tilde{\eta}}(z))^{-1} \| \left( \lambda \tilde{\eta} \right)^2,$$

where $\chi(A)$ denotes the indicator function of an event $A$ and we introduced

$$\Lambda_{\tilde{\eta}} := \frac{1}{N} \left( \text{Tr} G_{i,\tilde{\eta}}(z)^* G_{i,\tilde{\eta}}(z) \right)^{1/2},$$

$$\Lambda_{\tilde{\eta}} := \frac{1}{\sqrt{2N}} \max_{i} \left( \text{Tr} P_i \left[ G_{i,\tilde{\eta}}(z)^* G_{i,\tilde{\eta}}(z) + G_{i,\tilde{\eta}}(z) G_{i,\tilde{\eta}}(z)^* \right] \right)^{1/2},$$

$$\Lambda_{\tilde{\eta}} := \max_{i,j \in [N]} \| P_{ij} \left( G_{i,\tilde{\eta}}(z) \right) - M_{ij}(z) \delta_{ij} \|,$$

as well as

$$\vartheta \tilde{\eta} := \frac{1}{4 \| \mathcal{Z} \|^{-1} \| M_{i,\tilde{\eta}}(z) \| \| \Gamma \| \| (M_{i,\tilde{\eta}}(z))^{-1} \| \).

To estimate $\Lambda_{\tilde{\eta}}$ note that $\Lambda_{\tilde{\eta}} = N^{-1} \| G_{i,\tilde{\eta}}(z) \|$ where for any $n \in \mathbb{N}$ we denote by $\| \cdot \|_{\text{hs}} : \mathbb{C}^{n \times n} \rightarrow [0, +\infty)$ the usual Hilbert-Schmidt norm, i.e., for any $R \in \mathbb{C}^{n \times n}$

$$\| R \|_{\text{hs}}^2 = \text{Tr} R^* R.$$

By the resolvent identity $G_{i,\tilde{\eta}}(z) = G_{i,(\eta + \tilde{\eta})}(E) - i \eta G_{i,\tilde{\eta}}(z) ((I_m - J) \otimes I_N) G_{i,(\eta + \tilde{\eta})}(E)$, therefore

$$\| G_{i,\tilde{\eta}}(z) \|_{\text{hs}} \leq \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}} + \eta \| G_{i,\tilde{\eta}}(z) \| \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}}$$

$$\leq \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}} + C \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}}$$

$$\leq \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}}.$$

where we used (5.5) to obtain the bound $\eta \| G_{i,\tilde{\eta}}(z) \| \leq C$ for some $C > 0$ uniformly on $E \in B_\kappa$, $0 < \eta \leq 1$ and $\tilde{\eta} \geq 0$. Since $G_{i,(\eta + \tilde{\eta})}(E)$ is a resolvent with spectral parameter $i(\eta + \tilde{\eta})$, we can apply to it the Ward identity, which together with Lemma 5.3 gives

$$\frac{1}{N} \| G_{i,(\eta + \tilde{\eta})}(E) \|_{\text{hs}} = \left( \frac{\text{Tr} \text{Im} G_{i,(\eta + \tilde{\eta})}(E)}{N^2 (\eta + \tilde{\eta})} \right)^{1/2} \lesssim \sqrt{\frac{1}{N(\eta + \tilde{\eta})}}, (5.12)$$

uniformly for $E \in B_\kappa$, $N^{-1+\gamma} \leq \eta \leq 1$ and $\tilde{\eta} \geq 0$. In order to estimate $\Lambda_{\tilde{\eta}}$, we introduce the norm $\| \cdot \|_w : \mathbb{C}^{m \times m} \otimes \mathbb{C}^{N \times N} \rightarrow [0, +\infty)$ given by

$$\| R \|_w^2 = \max_{1 \leq i \leq N} \text{Tr} P_i (R R^*).$$

One can easily see that $\Lambda_{\tilde{\eta}} \sim N^{-1/2} \| G_{i,\tilde{\eta}}(z) \|_w$. Then similarly as for $\| \cdot \|_{\text{hs}}$,

$$\| G_{i,\tilde{\eta}}(z) \|_w \leq \| G_{i,(\eta + \tilde{\eta})}(E) \|_w + C \| G_{i,(\eta + \tilde{\eta})}(E) \|_w$$

$$\lesssim \| G_{i,(\eta + \tilde{\eta})}(E) \|_w.$$

By applying again the Ward identity and Lemma 5.3 we obtain that uniformly for $E \in B_\kappa$, $N^{-1+\gamma} \leq \eta \leq 1$ and $\tilde{\eta} \geq 0$

$$\frac{1}{N} \| G_{i,(\eta + \tilde{\eta})}(E) \|_w^2 \lesssim \max_{1 \leq i \leq N} \frac{\text{Im} \text{Tr} P_i (G_{i,(\eta + \tilde{\eta})}(E))}{N(\eta + \tilde{\eta})} \lesssim \frac{1}{N(\eta + \tilde{\eta})}.$$

Together with (5.12) and (5.6) this implies that

$$\max_{1 \leq i \leq N} \| P_i (G_{i,\tilde{\eta}}(z) - M_{i,\tilde{\eta}}(z)) \chi(\lambda \tilde{\eta} \leq \vartheta \tilde{\eta}) \leq \frac{1}{\sqrt{N}} + \sqrt{\frac{1}{N(\eta + \tilde{\eta})}}.$$ (5.13)

23
for $E \in B_\kappa$, $N^{-1+\gamma} \leq \eta \leq 1$ and $\tilde{\eta} \geq 0$.

For any $1 \leq i, j \leq N$, $i \neq j$, and $z \in \mathbb{C}_+$ we have that

$$\|P_{ij}(G_{i\tilde{\eta}}(z))\| \chi(\Lambda \tilde{\eta} \leq \vartheta \tilde{\eta}) < \|P_{ii}(G_{i\tilde{\eta}}(z))\| \Lambda^{\tilde{\eta}}.$$  \hspace{1cm} (5.14)

This statement was proven in the form $|G_{ij}| \chi < |G_{jj}| \chi \Lambda_\omega$ in the course of the proof of \[4, Lemma 4.3\] (see \[4, Eq. (4.23)] and the discussion after it). The right-hand side of (5.14) can be bounded by $\|M_{i(\eta+\tilde{\eta})}(E)\| \|G_{i(\eta+\tilde{\eta})}(E)\|_w + \Lambda^\tilde{\eta}\|G_{i(\eta+\tilde{\eta})}(E)\|_w$. The second term can be absorbed into $\Lambda^\tilde{\eta}$, so by using (5.13) and (5.6) we end up with the bound

$$\max_{1 \leq i,j \leq N} \|P_{ij}(G_{i\tilde{\eta}}(z)) - M_{ij}(z)\| \chi(\Lambda \tilde{\eta} \leq \vartheta \tilde{\eta}) < \frac{1}{\sqrt{N}} + \frac{1}{N(\eta + \tilde{\eta})}$$

uniformly on $E \in B_\kappa$, $N^{-1+\gamma} \leq \eta \leq 1$ and $\tilde{\eta} \geq 0$.

Since $\vartheta \tilde{\eta}(z) \geq \tilde{\eta}^{-1}$ by (5.6) and (5.7), and $\Lambda^\tilde{\eta}(z) < \tilde{\eta}^{-2}$ by \[4, Lemma 4.4, (i)]$, we can choose $\tilde{\eta}_1 > 0$ such that for all $E \in B_\kappa$

$$\Lambda^{\tilde{\eta}_1}(E+i) \leq \vartheta^\tilde{\eta}(E+i).$$ \hspace{1cm} (5.15)

Then by \[4, Lemma A.2\] (5.15) holds not only for $\tilde{\eta} = \tilde{\eta}_1$, but a.w.o.p. for all $0 \leq \tilde{\eta} \leq \tilde{\eta}_1$. In particular, we will have that a.w.o.p.

$$\Lambda(E+i) \leq \vartheta(E+i).$$

On the other hand, if we take $\tilde{\eta} = 0$ in (5.13) we will get that for $E \in B_\kappa$ and $0 \leq \eta \leq 1$

$$\max_{1 \leq i \leq N} \|G_{ii}(z) - M_{ii}(z)\| \chi(\Lambda \leq \vartheta) < \frac{1}{\sqrt{N}} + \frac{1}{N\eta}.$$  \hspace{1cm}

Applying \[4, Lemma A.2\] to $\Lambda(E+i\eta)$ and $\vartheta(E+i\eta)$ we get that for $E \in B_\kappa$ and $N^{-1+\gamma} \eta \leq 1$

$$\Lambda(E+i\eta) \leq \vartheta(E+i\eta),$$

which yields the first inequality in (5.2).

To prove the averaged local law we will use the fluctuations averaging mechanism, proof of which in a suitable form can be found in \[4, Proposition 4.6\], see also Section 10 of \[33\] related previous proofs. To this end, we introduce conditional expectation with respect to the $i$th rows and columns of the matrices $X_\alpha$ and $Y_\beta$

$$\mathbb{E}_i[\cdot] := \mathbb{E}\left[ \cdot \left| X_\alpha(k,l), Y_\beta(k,l) : \alpha \in \{\alpha_s\}, \beta \in \{\beta_s\}, k,l \in [N] \setminus \{i\} \right. \right],$$

and a family of operators

$$\mathcal{D}_i[\cdot] := \text{Id}[\cdot] - \mathbb{E}_i[\cdot].$$

By the Schur complement formula (see e.g. \[4, Section 4.1\]), for all $i \in [N]$ we have that

$$-\frac{1}{G_{ii}} = zJ - P_{ii}(H) + \sum_{k,l \neq i} P_{kk}(H)P_{kl}(G^{(i)})P_{li}(H),$$ \hspace{1cm} (5.16)

where

$$G^{(i)}(z) := \left( K_0 \otimes I_N - \sum_{\alpha=1}^{\alpha_s} K_\alpha \otimes X^{(i)}_\alpha - \sum_{\beta=1}^{\beta_s} (L_\beta \otimes Y^{(i)}_\beta + L^*_\beta \otimes Y^{(i)*}_\beta) - zJ \otimes I_N \right)^{-1}$$

and matrices $X^{(i)}_\alpha$ and $Y^{(i)}_\alpha$ are obtained from $X_\alpha$ and $Y_\alpha$, respectively, by replacing their $i$th rows and columns by zero. We use the $1/G_{ii}$ notation for the inverse of the $m \times m$ matrix $G_{ii}$. Taking the expectation $\mathbb{E}_i$ on both sides of (5.16), using (5.1), the assumptions (H1)-(H3) about the distribution of $X^{(N)}$ and $Y^{(N)}$, as well as the independence of $G^{(i)}$ from $P_{kk}(H)$ and $P_{kl}(H)$, yields the equation

$$-\frac{1}{G_{ii}} = -\mathbb{E}_i\frac{1}{G_{ii}} - \mathcal{D}_i\frac{1}{G_{ii}} = zJ - K_0 + \Gamma \left[ \frac{1}{N} \sum_{j=1}^{N} G_{jj} \right] + D_i, \hspace{1cm} (5.17)$$

\[24\]
for \((G_{ii})_{i=1}^N\). Here, the random error term \(D_i = D_i(z)\) takes the form

\[
D_i = -\partial_i \frac{1}{G_{ii}} + \Gamma \left[ \frac{1}{N} \sum_{j \neq i} (P_{jj} G^{(i)} - G_{jj}) \right] - \Gamma \left[ \frac{G_{ii}(z)}{N} \right].
\]

We subtract (5.17) from the DEL (2.7) and multiply the result from the left by its solution \(M\) and from the right by \(G_{ii}\) to see that the difference between \(G_{ii}(z)\) and \(M(z)\) can be written as

\[
G_{ii}(z) - M(z) = M(z) \Gamma \left[ \frac{1}{N} \sum_{j=1}^N (G_{jj}(z) - M(z)) \right] G_{ii}(z) - M(z) \partial_i G_{ii}(z)
, \quad \partial_i \left[ \frac{1}{G_{ii}(z)} \right],
\]

Note, that by using the large deviation bounds (see e.g. [4, Lemma 4.3]) we have

\[
\| \partial_i \left[ \frac{1}{G_{ii}(z)} \right] \| < \sqrt{\frac{1}{N\eta}}.
\]

After taking the average over \(i \in [N]\), rearranging the terms in (5.18) and using the entry-wise local law from (5.2), (M1), boundedness of \(\Gamma\) and the formula

\[
G_{jj}(z) - P_{jj}(G^{(i)}(z)) = G_{ji}(z) \frac{1}{G_{ii}(z)} G_{ij}(z), \quad j \neq i,
\]

we obtain

\[
\mathcal{L} \left[ \frac{1}{N} \sum_{i=1}^N G_{ii}(z) - M(z) \right] = -M(z) \frac{1}{N} \sum_{i=1}^N \partial_i \left[ \frac{1}{G_{ii}(z)} \right] M(z) + \mathcal{O}_\prec \left( \frac{1}{N} + \frac{1}{N\eta} \right), \quad (5.19)
\]

where \(\mathcal{O}_\prec(N^{-1} + (N\eta)^{-1})\) collects terms stochastically dominated by \(N^{-1} + (N\eta)^{-1}\). Applying again the entry-wise local law (5.2) and (5.6), we have that uniformly on \(E \in B_\kappa\) and \(N^{-1+\gamma} < \eta \leq 1\)

\[
\max_{i,j \in [N]} \left\| \frac{1}{M(z)} G_{ij}(z) - \delta_{ij} I_m \right\| < \sqrt{\frac{1}{N\eta}} \leq N^{-\gamma}. \quad (5.20)
\]

Inequality (5.20) allows us to improve a bound on the first term on the RHS of (5.19) by using the fluctuation averaging (see [4, Proposition 4.6]), which gives that for \(E \in B_\kappa\) and \(N^{-1+\gamma} < \eta \leq 1\)

\[
\frac{1}{N} \sum_{i=1}^N \partial_i \left[ \frac{1}{G_{ii}(z)} \right] M(z) < \frac{1}{N\eta}.
\]

Now the boundedness of \(\mathcal{L}^{-1}\) from (M2) yields the second inequality in (5.2). This completes the proof of Theorem 5.1.

Finally we prove the speed of convergence in the global law:

Proof of Proposition 2.17 First we prove (2.11) for bounded test functions. In this case, by the Helffer-Sjöstrand formula (see, e.g. Section 11.2 of [33]) it is sufficient to prove (2.11) for all functions of the form \(f(x) = (x - z)^{-1}\) with any fixed \(z \in \mathbb{C}_+\). Consider any nilpotent linearization of \(p\) and let \(M(z)\) be the solution of DEL (2.7). Notice that

\[
\| M(z) \| \leq C \left( 1 + \frac{1}{\text{Im} z} \right), \quad \| \mathcal{L}^{-1} \| \leq C (1 + |z|)^2 \left( 1 + \frac{1}{\text{Im} z} \right)^2, \quad (5.21)
\]
where the first bound was obtained in Lemma 5.6 (i). The second bound is a consequence of the identity
\[ \mathcal{L}^{-1}[R] = (\text{id} \otimes \tau) \left( \frac{1}{L - z J \otimes 1} \left( M^{-1} R M^{-1} \otimes 1 \right) \right) \left( \frac{1}{L - z J \otimes 1} \right), \]
the a priori bound \( \|M^{-1}\| \leq C(1 + |z|) \), see (5.9), and the bound (2.5) applied to semicircular elements as in (4.4). The identity (5.22) follows immediately by expressing the derivative of the function
\[ \Phi(A) := (\text{id} \otimes \tau)(L - (z J - A) \otimes 1)^{-1}, \]
at \( A = 0 \) in two different ways. The derivative on the free probability level gives the right hand side of (5.22), while the derivative on the level of the Dyson equation (2.7) perturbed with \( A \) gives the left hand side, see (5.21) for a similar calculation.

Thus (5.21) shows that the analogues of the conditions (M1)-(M2) hold away from the real axis, i.e. on any compact set \( \{ z \in \mathbb{C} : |z| \leq C^*, \text{Im} z \geq c^* \} \) with fixed positive thresholds \( c^*, C^* \). Inspecting the proof of the local law in Section 5, we see that the entire argument goes through under these modified assumptions. We leave the details to the reader.

Finally, using the boundedness of all \( X \) and \( Y \) matrices with very high probability, a standard cutoff argument yields (2.11) for arbitrary smooth function. \( \square \)

6 Examples

In this section we prove optimal bulk local law (in the sense of Theorem 2.11) for two concrete families of polynomials of random matrices, namely, for the eigenvalues of quadratic forms in Wigner matrices and for eigenvalues of symmetrized products (i.e. singular values of products) of matrices with \( i.i.d. \) entries.

6.1 Local law for homogeneous polynomials of degree two in Wigner matrices

Consider a family of noncommutative self-adjoint polynomials of degree 2 in \( \gamma_s \geq 2 \) variables given by
\[ q(x_1, \ldots, x_{\gamma_s}) = x^t \Xi x, \]
where \( x = (x_1, \ldots, x_{\gamma_s})^t \) and \( \Xi \) is a Hermitian \( \gamma_s \times \gamma_s \) matrix. We will assume that \( \Xi \) is invertible. Note that if we take \( \Xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), then \( q(x_1, x_2) = x_1 x_2 + x_2 x_1 \), the anticommutator, which was studied by Anderson in [6].

Suppose that \( L \) is the minimal linearization of \( 1_{s^d} - \tilde{q}(x) \) and suppose that for any \( \kappa > 0 \) the assumptions (M1) and (M2) hold for \( L \) and the corresponding solution of the DEL everywhere in the \( \kappa \)-bulk. This, together with Theorem 2.11, would imply that the optimal local law holds for the polynomial \( 1_{s^d} - \tilde{q} \) everywhere in the \( \kappa \)-bulk. Therefore, in order to prove the local law it is enough to find a minimal linearization of \( 1_{s^d} - \tilde{q} \) that satisfies (M1) and (M2).

Before proceeding to the proof, we fix a specific linearization of the polynomial \( 1_{s^d} - \tilde{q} \), which is particularly suitable for our computations. More precisely, let \( L(x) = K_0 - \sum_{\gamma=1}^{\gamma_s} K_\gamma x_\gamma \) with
\[ K_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Xi^{-1} \end{pmatrix}, \quad K_\gamma = \begin{pmatrix} 0 & \hat{e}_\gamma^t \\ \vdots & \vdots \\ \hat{e}_\gamma & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \]
where \( \{ \hat{e}_\gamma : 1 \leq \gamma \leq \gamma_s \} \) denotes the canonical basis of \( \mathbb{C}^{\gamma_s} \). In particular the operator \( \Gamma \) is given by the formula
\[ \Gamma \left( \begin{pmatrix} \alpha & a^t \\ b & A \end{pmatrix} \right) = \left( \frac{\text{tr} A}{a} \frac{b^t}{\alpha I_{\gamma_s}} \right), \]
for \( \alpha \in \mathbb{C}, a, b \in \mathbb{C}^{\gamma_s} \) and \( A \in \mathbb{C}^{\gamma_s \times \gamma_s} \). One can easily see that \( L(x) \) gives a linearization of \( 1_{s^d} - \tilde{q} \). Moreover, this linearization is in fact minimal. To show this latter property of (6.1), note that the
matrix representation of the series \((1 - \tilde{q})^{-1}\) corresponding to the linearization \((6.1)\) (see Remark A.5) is given by \((K_0^{-1}e_1, e_1, K_1K_0^{-1}, \ldots, K_\gamma K_0^{-1})\). From the special structure of \(K_0^{-1}\) and \(K_\gamma\) we see that for
\[K_\gamma K_0^{-1} e_1 = \begin{pmatrix} 0 \\ \hat{\gamma} \end{pmatrix}, \quad 1 \leq \gamma \leq \gamma_*.
\]
Therefore, with \(\{e_\gamma : 1 \leq \gamma \leq \gamma_* + 1\}\) being the canonical basis of \(\mathbb{C}^{\gamma_*+1}\), we have
\[
\text{span}\{e_1, K_1K_0^{-1}e_1, \ldots, K_\gamma K_0^{-1}e_1\} = \text{span}\{e_1, e_2, \ldots, e_{\gamma_*+1}\} = \mathbb{C}^{\gamma_*+1},
\]
which corresponds to condition \((A.6)\) in Proposition A.6. Similarly, one can show that the condition \((A.7)\) is satisfied as well, i.e.,
\[
\text{span}\{K_0^{-1}e_1, K_0^{-1}K_1K_0^{-1}e_1, \ldots, K_\gamma K_0^{-1}K_\eta K_0^{-1}e_1\} = \mathbb{C}^{\gamma_*+1}.
\]

We then conclude using Proposition A.6 that the linearization \((6.1)\) is indeed minimal.

Below we show that for this choice of linearization conditions \((M1)\) and \((M2)\) hold everywhere in the \(\kappa\)-bulk.

### 6.1.1 Boundedness of \(M\) (assumption \((M1)\))

First, we realize that the solution \(M(z)\) of the corresponding \(DEL\) has the following structure
\[
M(z) = \begin{pmatrix}
M_{11}(z) & 0 & \cdots & 0 \\
0 & \ddots & & \\
\vdots & & \ddots & \\
0 & & & \hat{M}(z)
\end{pmatrix},
\]
(6.3)
where by \(\hat{M}(z)\) we denote the \(\gamma_* \times \gamma_*\) submatrix of \(M(z)\). Indeed, by \((6.1)\) and \((6.2)\) the right hand side of \((5.3)\) as well as taking the inverse preserves the claimed block structure. Since the solution is obtained via a fixed point argument from an arbitrary starting point by iterating these operations and then taking the limit \(\omega \to 0\), it takes the form \((6.3)\).

Now, for \(M = M(z)\) of the form \((6.3)\) we write the \(DEL\)
\[
I + (zJ - K_0)M + \sum_{\gamma=1}^{\gamma_*} K_\gamma M K_\gamma M = 0,
\]
which can be split into two parts
\[
I + (z - 1)M_{11} + M_{11} \text{Tr} \hat{M} = 0, \tag{6.4}
\]
\[
I - \Xi^{-1}\hat{M} + M_{11} \hat{M} = 0.
\]

From \((6.4)\) we obtain that
\[
\hat{M} = (\Xi^{-1} - M_{11}I)^{-1}. \tag{6.5}
\]
Recall, that by the definition of the \(\kappa\)-bulk there exists \(\eta_0 > 0\) small enough such that for all \(\eta \in [0, \eta_0]\) and \(E \in B_\kappa\) we have \(\text{Im} M_{11}(E + i\eta) \geq \kappa/2\). Therefore, since \(\Xi\) is self-adjoint, \((6.5)\) implies that \(\|\hat{M}\| \leq 2/\kappa\) for all \(E \in B_\kappa\) and \(\eta \in [0, \eta_0]\). Moreover, by plugging \((6.5)\) into \((6.4)\) we derive an equation for \(M_{11}\)
\[
1 + (z - 1)M_{11} + \sum_{\gamma=1}^{\gamma_*} \frac{M_{11}}{\xi_\gamma - M_{11}} = 0,
\]
where by \(\xi_\gamma \in \mathbb{R}, 1 \leq \gamma \leq \gamma_*\), we denoted the eigenvalues of \(\Xi\). Note that if there exists an unbounded solution \(m\) of the equation
\[
1 + (z - 1)m + \sum_{\gamma=1}^{\gamma_*} \frac{m}{\xi_\gamma - m} = 0,
\]

27
then \( m(z) \) can be unbounded only near the point \( z = 1 \). In this case

\[
\lim_{z \to 1} (z - 1) m(z) = \gamma_* - 1,
\]

which implies that \( \text{Im} m(z) < 0 \) in the neighborhood of \( z = 1 \). Therefore, function \( M_{11}(z) \), whose imaginary part by Lemma 2.3 (iii) must be nonnegative on \( \mathbb{C}_+ \), has absolute value bounded by some \( C > 0 \) for all \( z \in \mathbb{C}_+ \). We conclude that the assumption (M1) holds for the linearization (6.1) of the polynomial \( 1 - \tilde{q}(x) \) with a constant \( C_3 = \max\{C,2/\kappa\} \) depending only on the model parameters \( \kappa \) and \( \Xi \).

### 6.1.2 Boundedness of \( \mathcal{L}^{-1} \) (assumption (M2))

In order to prove the stability assumption (M2), we will have to extract additional information from the \( \text{DEL} \) (2.7) by taking its imaginary part at \( \eta = 0 \)

\[
\text{Im} M = M^* \sum_{\gamma=1}^{\gamma_*} \text{Im} MK_{\gamma}M.
\]  

(6.6)

By using (6.1), (6.3) and (6.5) and comparing the \((1,1)\)-components of both sides of (6.6), we obtain that for all real \( z = E, E \in B_\kappa \),

\[
|M_{11}|^2 \text{Tr} \hat{M}^\dagger \hat{M} = 1.
\]  

(6.7)

Now, consider the space of \((\gamma_* + 1) \times (\gamma_* + 1)\) matrices with basis vectors \( \{E_{11}, E_{12}, \ldots\} \), on which the linear operator \( \mathcal{L} \) is acting, as \( \mathbb{C}^{(\gamma_*+1)^2} \) with the standard basis \( \{e_1, e_2, \ldots\} \). On this latter space \( \mathcal{L} \) can be represented by the matrix

\[
A_{\mathcal{L}} := I_{(\gamma_*+1)^2} - \sum_{\gamma=1}^{\gamma_*} MK_{\gamma} \otimes M^t K_{\gamma},
\]

or more explicitly

\[
A_{\mathcal{L}} = \begin{pmatrix} I_{\gamma_*+1} & -A_{12} \\ -A_{21} & I_{(\gamma_*+1)\gamma_*} \end{pmatrix},
\]

where

\[
A_{12} = M_{11} \sum_{\gamma=1}^{\gamma_*} e_{\gamma}^t \otimes M^t K_{\gamma}, \quad A_{21} = \sum_{\gamma=1}^{\gamma_*} \hat{M} e_{\gamma} \otimes M^t K_{\gamma}.
\]

Note, that

\[
\det(A_{\mathcal{L}}) = \det(I_{\gamma_*+1} - A_{12} A_{21}),
\]

therefore, in order to prove invertibility of \( \mathcal{L} \) it will be enough to show invertibility of

\[
I_{\gamma_*+1} - A_{12} A_{21} = \begin{pmatrix} 1 - M_{11}^2 \text{Tr}(\hat{M})^2 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \ddots & I_{\gamma_*} - M_{11}^2(\hat{M}^\dagger \hat{M}) \\ 0 & \cdots & 0 & 1 - M_{11}^2 \text{Tr}(\hat{M})^2 \end{pmatrix}.
\]  

(6.8)

Assume that the upper-left entry is not invertible, i.e.,

\[
1 - M_{11}^2 \text{Tr}(\hat{M})^2 = 1 - M_{11}^2 \sum_{\gamma=1}^{\gamma_*} \frac{1}{(\xi^{-1}_\gamma - M_{11})^2} = 0,
\]

where we used (6.5). In this case, from (6.7), we obtain that for all \( 1 \leq \gamma \leq \gamma_* \)

\[
(M_{11}(\xi^{-1}_\gamma - M_{11})^{-1})^2 = |M_{11}(\xi^{-1}_\gamma - M_{11})^{-1}|^2,
\]

28
so that $\mathrm{Im}(M_{11}(\xi^{-1} - M_{11})^{-1}) = \mathrm{Im}(M_{11})\xi^{-1}|\xi^{-1} - M_{11}|^{-2} = 0$ for all $1 \leq \gamma \leq \gamma_s$, which leads to a contradiction with $\mathrm{Im} M_{11} > \kappa$ for $z = E$, $E \in B_\kappa$.

Consider now the case when $\det(I_{\gamma_s} - M_{11}(\hat{M}^* \hat{M})) = 0$, so that the lower-right submatrix of (6.8) is singular. This implies that there exists $\omega \in \mathbb{C}^{\gamma_s}$, $\|\omega\| = 1$, such that

$$M_{11}^2 \omega^* \hat{M}^* \hat{M} \omega = 1. \quad (6.9)$$

We can rewrite the LHS of (6.9) as

$$\sum_{k=1}^{\gamma_s} \langle \omega, v_k \rangle (\overline{\omega}, v_k) = \sum_{k=1}^{\gamma_s} ((\mathrm{Re} \omega, v_k))^2 + ((\mathrm{Im} \omega, v_k))^2,$$

where $v_k := (M_{11} \hat{M}_{k1}, M_{11} \hat{M}_{k2}, \ldots, M_{11} \hat{M}_{k\gamma_s})^t$. Due to (6.7), using triangular and Cauchy-Schwarz inequalities, we have

$$\left| \sum_{k=1}^{\gamma_s} ((\mathrm{Re} \omega, v_k))^2 + ((\mathrm{Im} \omega, v_k))^2 \right| \leq \|\omega\|^2 \sum_{k=1}^{\gamma_s} \|v_k\|^2 = |M_{11}|^2 \mathrm{Tr} \hat{M}^* \hat{M} = 1. \quad (6.10)$$

Assumption (6.9) implies that the first inequality in (6.10) is in fact an equality and that

$$|\langle \mathrm{Re} \omega, v_k \rangle|^2 = |\mathrm{Re} \omega|^2 \|v_k\|^2, \quad |\langle \mathrm{Im} \omega, v_k \rangle|^2 = \|\mathrm{Im} \omega\|^2 \|v_k\|^2, \quad 1 \leq k \leq \gamma_s.$$ 

Thus there exist $c_1^{(1)}, \ldots, c_{\gamma_s}^{(1)}, c_1^{(2)}, \ldots, c_{\gamma_s}^{(2)} \in \mathbb{C}$ such that

$$v_k = c_k^{(1)} \mathrm{Re} \omega = c_k^{(2)} \mathrm{Im} \omega,$$

and we see that the rows of the matrix $M_{11} \hat{M}$ are linearly dependent. At the same time we know that since $\mathrm{Im} M_{11} > \kappa$ in the $\kappa$-bulk, by (6.5) matrix $\hat{M}$ must be invertible. From the obtained contradiction we conclude that $I_{\gamma_s+1} - A_{12} A_{21}$, $A_{\Sigma}$ and $\Sigma$ are all invertible for $z = E$ with $E \in B_\kappa$, so that there exists $C > 0$ depending only on $\kappa$ and $\Sigma$, such that $\|\Sigma^{-1}(E)\| \leq C$ for all $E \in B_\kappa$. Now a simple continuity argument, together with the a priori bound from Lemma 6.6 shows that the condition (M2) holds for the model given by (6.11) everywhere in the $\kappa$-bulk.

### 6.2 Local law for singular values of a product of independent non-Hermitian matrices

Consider $q(y_1, \ldots, y_{\beta_s}, y_1^*, \ldots, y_{\beta_s}^*) = y_1 \cdots y_{\beta_s} y_{\beta_s}^* \cdots y_1^*$. Then a minimal linearization of the polynomial $\mathbb{1}_W - q(y, y^*)$ is given by

$$L = \begin{pmatrix}
1 & y_1 & y_2 & \cdots & y_{\beta_s} \\
y_1^* & 1 & y_2^* & \cdots & y_{\beta_s}^* \\
y_2^* & 1 & y_2 & \cdots & y_{\beta_s} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
y_{\beta_s}^* & 1 & y_{\beta_s} & \cdots & 1
\end{pmatrix} = K_0 \otimes \mathbb{1} + Y, \quad (6.11)$$

or, using the representation (2.8) and the basis vectors $E_{ij} = e_i e_j^t$, by a set of matrices

$$K_0 = E_{1,1} + \sum_{j=2}^{2\beta_s} E_{j,2\beta_s+2-j}, \quad L_{\beta} = E_{\beta,2\beta_s+1-\beta}, \quad 1 \leq \beta \leq \beta_s. \quad (6.12)$$
Here, the corresponding operator $\Gamma$ has the simple form

$$\Gamma[R] = \text{diag}(r_{2\beta_\ast+1-i,2\beta_\ast+1-i})_{i=1}^{2\beta_\ast}, \quad R = (r_{ij})^{2\beta_\ast}_{i,j=1}, \quad (6.13)$$

where $\text{diag}(a) \in \mathbb{C}^{2\beta_\ast \times 2\beta_\ast}$ is the diagonal matrix with vector $a \in \mathbb{C}^{2\beta_\ast}$ along its diagonal.

Before proving that assumptions (M1) and (M2) hold for this model, we show first that the solution matrix $M(z)$ has the following structure

$$M(z) = \sum_{j=1}^{2\beta_\ast} m_j(z) E_{jj} + \sum_{j=2}^{2\beta_\ast} m_{\beta_\ast+1}(z) E_{jj,2\beta_\ast+2-j} - m_{\beta_\ast+1}(z) E_{\beta_\ast+1,\beta_\ast+1}, \quad (6.14)$$

for some $m_j : \mathbb{C} \to \mathbb{C}, 1 \leq j \leq 2\beta_\ast$. In order to do so we introduce an auxiliary parameter $\alpha > 0$ and consider the linearization $L_\alpha = K_0 \otimes 1 + \alpha Y$ of the polynomial $1_{J,J} - \alpha^{2\beta_\ast} q(y, y^*)$ with $\Gamma_\alpha = \alpha^2 \Gamma$. We use the representation (4.17) of the corresponding solution $M_\alpha$ to the DEL with $\Gamma_\alpha$ and expand into geometric series for any fixed $z$ and sufficiently small $\alpha$ as

$$M_\alpha(z) = (\text{id} \otimes \tau) \left[ \left( (K_0 - zJ)^{-1} \otimes 1 \right) \sum_{k=0}^{\infty} (-\alpha)^k X^k \right], \quad X := \tilde{Y} (K_0 - zJ)^{-1} \otimes 1,$$

where $\tilde{Y}$ is defined as all the $y_i$ inside $Y$ are replaced by free circular elements $\tilde{y}_i$. Due to its cyclic structure the only powers of $X$ with non-zero elements on the diagonal are integer multiples of $2\beta_\ast$ and since $X^{2\beta_\ast}$ has $\zeta^{-1} q(\tilde{y}, \tilde{y}^*)$ with $\zeta := 1 - z$ in all diagonal entries we conclude that $(K_0 - zJ)^{-1} M_\alpha(z)$ has a constant diagonal. Furthermore, the subalgebra of $\mathbb{C}^{2\beta_\ast \times 2\beta_\ast}$ of all matrices with the same non-zero entries as $M(z)$ from (6.14) is left invariant by matrix inversion, addition of $K_0 - zJ$ (cf. (6.12)) as well as application of the operator $\Gamma_\alpha$. Thus similarly to the argument we used in Section 6.1 the solution $M_\alpha$ also has only these non-zero entries and thus is of the form (6.14) for small enough $\alpha$. Since $M_\alpha$ is analytic in $\alpha > 0$ for every $z \in \mathbb{C}_+$, we conclude that (6.14) also holds at $\alpha = 1$.

6.2.1 Boundedness of $M$ (assumption (M1))

We now prove that $M$ is bounded everywhere in the $\kappa$-bulk. Using the structure of $M$ (6.14), the DEL (2.7) can be reduced to the following system of equations for $m_\beta, 1 \leq \beta \leq 2\beta_\ast$,

$$\begin{align*}
1 - \zeta m_1 + m_2 \beta m_1 &= 0, \\
1 - m_{\beta_\ast+1} + m_{2\beta_\ast+1-\beta} m_\beta &= 0, & 2 \leq \beta \leq 2\beta_\ast, \\
-m_{2\beta_\ast+2-\beta} + m_{2\beta_\ast+1-\beta} m_{\beta_\ast+1} &= 0, & 2 \leq \beta \leq 2\beta_\ast, i \neq \beta_\ast + 1.
\end{align*}$$

From these equations we obtain that all $m_\beta, \beta \geq 2$, can be expressed in terms of $m_1$:

$$\begin{align*}
m_{\beta_\ast+1} &= \zeta m_1, \\
m_{\beta_\ast+1+\beta} &= m_{\beta_\ast+1} = (\zeta m_1)^{\beta+1}, & 0 \leq \beta \leq \beta_\ast - 1, \\
m_\beta &= m_1 m_{\beta_\ast+1} = \zeta^{\beta-1} m_1, & 2 \leq \beta \leq \beta_\ast,
\end{align*}$$

and $m_1(z) = \langle e_1, M(z) e_1 \rangle$ satisfies the following polynomial equation

$$1 - \zeta m_1 + \zeta^{\beta_\ast} m_1^{\beta_\ast+1} = 0, \quad \zeta = 1 - z. \quad (6.16)$$

From (6.16) it is easy to see that $|m_1(z)|$ can be unbounded only in the neighborhood of $z = 1$. Moreover, we will show that there exists $c(\kappa) = c(\kappa, \beta_\ast) > 0$ small enough such that

$$B_\kappa \subset [1 - C(\beta_\ast), 1 - c(\kappa, \beta_\ast)], \quad (6.17)$$

where $C(\beta_\ast) \geq 2$ comes from the boundedness of the support of the density of states. In order to prove the upper bound in (6.17) we may, without loss of generality, consider only $z \in \mathbb{C}_+$ with
\(|\zeta| = |1 - z| \leq 4^{-\psi} + 1\) and \(\eta = \text{Im} z\) small. We will show that for such \(z\) the condition \(E = \text{Re} z \in B_\kappa\) implies \(|\zeta| \geq c(\kappa)\), where \(c(\kappa)\) will be specified below. Rewrite (6.10) as

\[
(\zeta m_1)^{\beta_1+1} = -\zeta (1 - \zeta m_1),
\]

from which it follows that \(|\zeta m_1|^{\beta_1+1} \leq |\zeta|(1 + |\zeta m_1|)\), so that \(|\zeta m_1| \leq 2|\zeta|^{-\beta_1+1}|\). This last bound implies that \(|\zeta m_1| < 1/2\), which, together with (6.18), yields

\[
|m_1| \sim |\zeta|^{-\frac{\beta_1}{\psi}}.
\]

Suppose that \(|\zeta|^{-\frac{\beta_1}{\psi}} \geq C' \kappa^{-1}\) with some large constant \(C'\). For \(E\) in the \(\kappa\)-bulk and \(\eta\) small we have \(\text{Im} m_1 \leq 2\kappa^{-1}\), which together with (6.19) gives \(|\text{Re} m_1| \sim |\zeta|^{-\frac{\beta_1}{\psi}}\). In the regime when \(\eta \ll |1 - E|\) and \(|\zeta| \sim |1 - E|\), by taking the imaginary part of the equation (6.16) and dividing it through by \(\text{Im} m_1(\text{Re} m_1)^{-1}\) we obtain

\[
0 = -(1 - E) \text{Re} m_1 + (\beta_1 + 1)(1 - E)^{\beta_1} (\text{Re} m_1)^{\beta_1+1} + O \left( \frac{|\text{Im} m_1|}{|\text{Re} m_1|} + \frac{\eta}{|\zeta|^{\beta_1}} \right). \tag{6.20}
\]

Choosing \(C'\) sufficiently large and \(\eta\) sufficiently small (depending on \(\kappa\)) the error term becomes negligible, using \(\text{Im} m_1 \geq \kappa\) since we are in \(B_\kappa\). Using the scaling of \(1 - E\) and \(\text{Re} m_1\) in \(\zeta\), we obtain \(|(1 - E) \text{Re} m_1| \sim |\zeta|^{-\frac{\beta_1}{\psi}}\) and \(|(\beta_1 + 1)(1 - E)^{\beta_1} (\text{Re} m_1)^{\beta_1+1}| \sim 1\). Since \(\zeta\) is small, this leads to a contradiction in (6.20), hence \(|\zeta|^{-\frac{\beta_1}{\psi}} \geq C' \kappa^{-1}\) cannot hold. This finishes the proof of (6.17) with \(c(\kappa) = (\kappa/C')^{-\frac{\beta_1}{\psi}}\). Now, for any \(E \in B_\kappa\) and \(\eta > 0\) small enough, (6.16) and (6.17) imply that \(|m_1| \leq (c(\kappa))^{-1} C\), which gives an effective bound on \(m_1\). Boundeness of \(|m_1|\) together with (6.15) implies that assumption (M1) holds everywhere in \(B_\kappa\) with \(C_3 = c(\kappa)^\beta_1\).

### 6.2.2 Boundedness of \(L^{-1}\) (assumption (M2))

In this section we show that assumption (M2) holds everywhere in the \(\kappa\)-bulk, which together with Theorem [2.14] implies optimal bulk local law for the singular values of a product of matrices \(Y_1 \cdots Y_\beta\), satisfying (H1)-(H4).

By (6.12), which, in particular, gives that \(M'(z) = M(z)\), and (6.14), matrix \(A_\mathcal{L} = I - \sum_{\beta=1}^{\beta_1} (ML_\beta \otimes ML_\beta)\) representing the stability operator \(\mathcal{L}\) in the standard basis of \(\mathbb{C}^{2 \beta_1}\) can be written as

\[
A_\mathcal{L} = I_{2\beta_1} - (m_1 E_{1,2\beta_1})^{\otimes 2} - \sum_{\beta=2}^{\beta_1} (m_1 E_{1,2\beta_1+1-\beta} + m_{1+1} E_{2,2\beta_1+2-\beta,2\beta_1+1-\beta})^{\otimes 2}
\]

\[
- \sum_{\beta=1}^{\beta_1-1} (m_{2\beta_1+1-\beta} E_{2,2\beta_1+1-\beta,2\beta_1+1-\beta} + m_{\beta+1} E_{2,2\beta_1+1-\beta})^{\otimes 2} - (m_{\beta+1} E_{2,2\beta_1+1-\beta})^{\otimes 2},
\]

where for any \(k \in \mathbb{N}\) and \(R \in \mathbb{C}^{k \times k}\) we denote \(R^{\otimes 2} := R \otimes R\). After removing from \(A_\mathcal{L}\) rows and columns for all indices such that either row or column of the corresponding index has only one non-zero entry equal to 1, we obtain that \(\text{det}(A_\mathcal{L}) = \text{det}(\hat{A}_\mathcal{L})\), where

\[
\hat{A}_\mathcal{L} = -I_{2\beta_1} + \sum_{\beta=1}^{2\beta_1} m_{1}^2 E_{1,2\beta_1+1-\beta} + m_{\beta+1}^2 \sum_{\beta=1}^{\beta_1-1} (E_{1,2\beta_1+1-\beta} + E_{\beta_1+1,2\beta_1+1-\beta,2\beta_1+1-\beta}).
\]

Divide \(\hat{A}_\mathcal{L}\) into four blocks of equal size \(A_{ij}, 1 \leq i, j \leq 2\), so that, e.g., \(A_{11}\) denotes the upper-left \(\beta_1 \times \beta_1\) submatrix of \(A\). Then by the lower-triangular structure of \(A_{11}\) and \(A_{22}\) having \(-I_{\beta_1}\) on their
diagonals, skew-diagonal shape of $A_{12}$ and $A_{21}$, and (6.15) we have that $\det(\tilde{A}_{\mathcal{F}})$ is equal to

$$
\det(A_{11} - A_{12}A_{22}^{-1}A_{21}) = \det \left( \begin{array}{cccc}
v - 1 & v & v & \cdots & v \\
\omega & v - 1 & v & \cdots & v \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\omega & v - 1 & v & \omega & v - 1 \\
\omega & v - 1 & v & \omega & v - 1 \\
\end{array} \right),
$$

with $v = \zeta^{2\beta_{s}m_{1}^{2}(\beta_{s}+1)}$ and $\omega = (\zeta m_{1})^{2}$. One can easily see that the above determinant is equal to the determinant of the following tridiagonal matrix

$$
\begin{pmatrix}
v - 1 & 1 & & & \\
\omega & v - 1 - \omega & 1 & & \\
\omega & \omega & v - 1 - \omega & 1 & \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\omega & \omega & \omega & v - 1 & 1 \\
\omega & \omega & \omega & \omega & v - 1 - \omega \\
\end{pmatrix},
$$

that is equal to

$$(v - 1)\det(T_{\beta_{s}^{-1}}(v - 1 - \omega, 1, \omega)) - \omega \det(T_{\beta_{s}^{-2}}(v - 1 - \omega, 1, \omega)), \quad (6.21)$$

where $T_{k}(a, b, c)$ denotes a $k \times k$ Toeplitz tridiagonal matrix with $a$ on the main diagonal, and $b$ and $c$ above and below the main diagonal respectively. From (6.10) we have

$$v - 1 - \omega = -2\zeta m_{1}, \quad (6.22)$$

and thus $(v - 1 - \omega)^{2} = 4\omega$. Note, that under the condition $a^{2} = 4bc$ the determinant of the Toeplitz tridiagonal matrix takes a particularly simple form

$$\det(T_{k}(a, b, c)) = (k + 1) \left(\frac{a}{2}\right)^{k}.$$

A simple calculation from (6.21) and (6.22) gives that

$$\det(A_{\mathcal{F}}) = (\beta_{s} + 1)(-\zeta m_{1})^{\beta_{s}} + (\zeta m_{1})^{2}\beta_{s}(-\zeta m_{1})^{\beta_{s}-1}.$$  

Hence, $\det(A_{\mathcal{F}}) = 0$ implies that (since $\zeta m_{1} \neq 0$ in the $\kappa$-bulk)

$$\zeta m_{1} = \frac{\beta_{s} + 1}{\beta_{s}}. \quad (6.23)$$

Now if we plug (6.23) into (6.14) we obtain

$$m_{1} = \left(\frac{\beta_{s}}{\beta_{s} + 1}\right)^{\beta_{s}} \frac{1}{\beta_{s}}, \quad \zeta = \frac{(\beta_{s} + 1)^{\beta_{s} + 1}}{\beta_{s}^{\beta_{s}}}, \quad z = 1 - \frac{(\beta_{s} + 1)^{\beta_{s} + 1}}{\beta_{s}^{\beta_{s}}}.$$  

Since at $z = 1 - (\beta_{s} + 1)^{\beta_{s} + 1}/\beta_{s}^{\beta_{s}}$ the imaginary part of $m_{1}(z)$ vanishes, this point does not belong to the $\kappa$-bulk. This argument, which can be made effective, yields that (M2) holds.

### A Linearizations of noncommutative polynomials: construction and minimization

Let $\mathcal{A}$ be a unital $C^{*}$-algebra and let $x = \{x_{1}, \ldots, x_{\gamma}\} \subset \mathcal{A}$ be a family of self-adjoint noncommutative variables. Let $\tilde{q} = \tilde{q}(x)$ be a self-adjoint polynomial such that $\tilde{q}(0) = 0$. In this section we present two methods to construct a (self-adjoint) linearization of $1 - \tilde{q}$. 

A.1 Standard algorithm for constructing a symmetric linearization

We present now a simple and fairly standard procedure for constructing a (self-adjoint) linearization. Several versions of this algorithm have appeared in the literature, see e.g. [5], but for definiteness we present it here in a setup the most convenient for us. For simplicity, we will call it standard linearization.

Suppose that for general (even not necessarily self-adjoint) polynomials \( a_1, \ldots, a_k \in \mathbb{C}(\mathbf{x}) \) we have matrices

\[
\begin{pmatrix}
0 & d_i \\
\frac{b_i}{U_i} & U_i
\end{pmatrix} \in \mathbb{C}(\mathbf{x})^{m_i \times m_i}, \quad U_i \in \mathbb{C}(\mathbf{x})^{(m_i-1) \times (m_i-1)}, \quad 1 \leq i \leq k,
\]

such that

\[-d_i U_i^{-1} b_i = a_i, \quad 1 \leq i \leq k.\]

We now show how to construct a linearization for a scalar multiple of \( a_i \), for sums of \( a_i \)'s and for the real part of \( a_i \). One can easily check that the following rules hold:

1. With \( \hat{q}_1 \) denoting the linear part of \( \hat{q} \), put \( \lambda = 1 - \hat{q}_1 \).

2. Write \( \hat{q} - \hat{q}_1 \) as the sum of monomials of type \( \zeta \alpha_1 \cdots \alpha_k x_{\alpha_1} \cdots x_{\alpha_k} \) of degree at least two with \( (\alpha_1, \ldots, \alpha_k) \in [\gamma]_1^k \) and \( \zeta \alpha_1 \cdots \alpha_k \in \mathbb{C} \); for definiteness order the terms in the sum with respect to their degree from lowest to highest, and within each group of monomials of the same degree order them lexicographically with respect to \( (\alpha_1, \ldots, \alpha_k) \).

3. For each such monomial construct a linearization of \( x_{\alpha_1} \cdots x_{\alpha_k} \) using the basic linearization rule

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & x_{\alpha_1} \\
0 & 0 & \cdots & x_{\alpha_2} & -\frac{1}{\lambda} \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & x_{\alpha_{k-1}} & \ddots & \ddots & \ddots \\
x_{\alpha_k} & -\frac{1}{\lambda} & 0 & \cdots & 0
\end{pmatrix}
\]

gives \(-d U^{-1} b = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \). (A.1)

4. Then use rule (R1) to multiply the monomials by non-zero coefficients \( \zeta \alpha_1 \cdots \alpha_k \).

5. Next, use rule (R2) to obtain a linearization (not Hermitian at this point) of the sum of monomials.

6. Finally, rule (R3) applied to the linearization obtained on the previous stage gives a symmetric linearization.
Note, that if we write \( L \) obtained along this procedure in the form (3.1), then

\[
K_0 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & & \\
0 & & \ddots & \\
0 & & & \Theta \hat{K}_0
\end{pmatrix},
\]

where \( \hat{K}_0 \) is a permutation matrix and \( \Theta = \text{Diag}(e^{i\theta_1}, \ldots, e^{i\theta_{m-1}}) \) for some \( \theta_1, \ldots, \theta_{m-1} \in \mathbb{R} \).

### A.2 Minimal linearization

The procedure described in the previous section is only one of many ways of constructing a (self-adjoint) linearization. For example, we could repeat (R3) as many times as we wish creating a linearization of higher dimension which would still have all the required properties. In this section we will show how one can reduce the dimension of the linearization, which can be particularly useful if we want to check numerically that the conditions (M1)-(M2) are satisfied for some particular polynomial. Symmetric minimal linearizations have been constructed before, see [42, Lemma 4.1 (3)]. We present a particularly direct construction here.

**Definition A.1** (Minimal linearization). We say that linearization \( L \) of a polynomial \( \mathbb{1}_{\mathcal{M}} - \tilde{q}(x) \in \mathbb{C}(x) \) is **minimal** if it has the smallest dimension among all linearizations of \( \mathbb{1}_{\mathcal{M}} - \tilde{q}(x) \).

In our construction of the minimal linearization we will need the notion of a **matrix representation of a series**.

**Definition A.2** (Matrix representation of a series). Let \( s = s(x_1, \ldots, x_{\gamma_s}) \) be a formal series in noncommutative variables \( x_1, \ldots, x_{\gamma_s} \):

\[
s = s_0 + \sum_{k=1}^{\infty} \sum_{(\gamma_1, \ldots, \gamma_k) \in [\gamma_s]^k} s_{\gamma_1 \ldots \gamma_k} x_{\gamma_1} \cdots x_{\gamma_k}.
\]

Let \( v_1, v_2 \in \mathbb{C}^m \) and \( V_1, \ldots, V_{\gamma_s} \in \mathbb{C}^{m \times m} \). We say that \( (v_1, v_2, V_1, \ldots, V_{\gamma_s}) \) is a **matrix representation** of \( s \) if for any \( k \in \mathbb{N} \) and any \( (\gamma_1, \ldots, \gamma_k) \in [\gamma_s]^k \)

\[
s_0 = \langle v_1, v_2 \rangle, \quad s_{\gamma_1 \ldots \gamma_k} = \langle v_1, V_{\gamma_1} \cdots V_{\gamma_k} v_2 \rangle.
\]

We call \( m \) the **dimension** of the linearization.

**Remark A.3**. The concept of matrix representation of a formal series introduced in Definition A.2 is also known in the literature (see e.g. Chapter 1, Section 5 in [21]) as linear representation or linearization. In order to avoid confusion in this paper we will reserve the term **linearization** only for objects introduced in Definition 2.1.

Similarly as for linearizations, we can define a **minimal matrix representation** of a series.

**Definition A.4** (Minimal matrix representation of a series). Matrix representation of a series is called **minimal** if it has the smallest dimension among all possible matrix representations of this series.

**Remark A.5**. The advantage of introducing the minimal matrix representation is the following. On one hand, it is very easy to see that

\[
L = K_0 \otimes \mathbb{1}_{\mathcal{M}} - \sum_{\gamma=1}^{\gamma_s} K_0 \otimes K_\gamma
\]

is a linearization of \( \mathbb{1}_{\mathcal{M}} - \tilde{q}(x) \in \mathbb{C}(x) \) if and only if

\[
(K_0^{-1} e_1, e_1, K_1 K_0^{-1}, \ldots, K_{\gamma_s} K_0^{-1})
\]

(A.4)
gives a matrix representation of \((1 - \tilde{q})^{-1} := 1 + \sum_{k=1}^{\infty} \tilde{q}^k\). Indeed, by the Schur complement formula (2.4) we have that
\[
\langle K_0^{-1} e_1 \otimes 1_{\mathcal{A}} - \sum_{\gamma=1}^{\gamma_m} K_\gamma K_0^{-1} \otimes x_\gamma \rangle^{-1} e_1 \otimes 1_{\mathcal{A}} = \frac{1}{1_{\mathcal{A}} - \tilde{q}(x)},
\]
and thus if we assume that \(\sum_{\gamma=1}^{\gamma_m} \|K_\gamma K_0^{-1}\| \|x_\gamma\|_{\mathcal{A}} \leq 1/2\) and expand both the LHS and the RHS of (A.5) into a power series with respect to \(x_\gamma\)'s, we will see that the coefficients in the expansion of \((1 - \tilde{q})^{-1}\) are given by the matrix representation (A.4).

On the other hand, there is a simple characterization of the minimal matrix representation of a series, which is stated in the following proposition. Therefore, one can use minimization of a special type of matrix representations (1 - \tilde{q})^{-1} in order to construct a minimal linearization of the polynomial \(1_{\mathcal{A}} - \tilde{q}(x)\).

**Proposition A.6** ([21], Proposition 2.1). Let \(s = s(x_1, \ldots, x_{\gamma_m})\) be a series in noncommutative variables \(x_1, \ldots, x_{\gamma_m}\) and let \((v_1, v_2, V_1, \ldots, V_{\gamma_m})\) with \(v_1, v_2 \in \mathbb{C}^n\) and \(V_1, \ldots, V_{\gamma_m} \in \mathbb{C}^{m \times m}\) be one of its matrix representations. The matrix representation \((v_1, v_2, V_1, \ldots, V_{\gamma_m})\) is minimal if and only if
\[
\text{span}\left(\{v_2\} \cup \bigcup_{k=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_k} V_{\alpha_1} \cdots V_{\alpha_k} q_2\right) = \mathbb{C}^n, \quad (A.6)
\]
\[
\text{span}\left(\{v_1\} \cup \bigcup_{k=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_k} V_{\alpha_1} \cdots V_{\alpha_k} q_1\right) = \mathbb{C}^m. \quad (A.7)
\]

In the next lemma we will show how to construct a linearization \(L\) of the form (A.3), such that the corresponding matrix representation of \((1 - \tilde{q})^{-1}\) satisfies (A.6)-(A.7). This would imply that this linearization is minimal, since otherwise it would be possible to construct a minimal representation of \((1 - \tilde{q})^{-1}\) with dimension smaller than minimal. The matrices \(K_\gamma\) below faithfully represent the collection of self-adjoint matrices \(K_\gamma\) on a smaller space \(\tilde{U}\), which is the natural smallest space.

Before stating the next lemma let us introduce some notation that will be used to describe the minimization algorithm. Denote by \(\mathcal{I}\) the set of multi-indices
\[
\mathcal{I} := \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \left\{ (\alpha_1, \ldots, \alpha_k) \in [\gamma_s]^k \right\}. \quad (A.8)
\]
For any \(k \in \mathbb{N}\), multi-index \(\pi := (\alpha_1, \ldots, \alpha_k) \in [\gamma_s]^k\) and a family of matrices \(\{R_\alpha : \alpha \in [\gamma_s]\}\) we will denote
\[
R_\emptyset := I, \quad R_\pi := R_{\alpha_1} \cdots R_{\alpha_k}. \quad (A.9)
\]
For any two multi-indices \(\pi, \pi' \in [\gamma_s]^k\) and \(\beta \in [\gamma_s]^{\ell}\) we will denote by \(\pi \beta\) the concatenation of \(\pi\) and \(\beta\), i.e., \(\pi \beta := (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)\), and by \(\pi^\alpha\) the multi-index taken in the reversed order, i.e., \(\pi^\alpha := (\alpha_k, \ldots, \alpha_1)\). Finally for any multi-index \(\pi\) of length \(k\) and a linearization \(L\) of the form (A.3) we will denote
\[
\xi_\pi := K_0^{-1} K_{\alpha_1} \cdots K_0^{-1} K_{\alpha_k} K_0^{-1} e_1.
\]

**Lemma A.7** (Minimization algorithm). Let \(\tilde{q} \in \mathbb{C}(x)\) be self-adjoint such that \(\tilde{q}(0) = 0\) and let \(L = K_0 \otimes 1_{\mathcal{A}} - \sum_{\gamma=1}^{\gamma_m} K_\gamma \otimes x_\gamma\) be an arbitrary \(n\)-dimensional (self-adjoint) linearization of \(1_{\mathcal{A}} - \tilde{q}(x)\) with \(K_0\) invertible. Denote \(A_\gamma := K_\gamma K_0^{-1}\).

1. Define a subspace \(U \subset \mathbb{C}^n\)
\[
U := \text{span}\left(\bigcup_{\pi \in \mathcal{I}} A_\pi e_1\right) \subset \mathbb{C}^n. \quad (A.10)
\]

2. Denote by \(P_U : \mathbb{C}^n \to U\) the orthogonal projection onto \(U\) and define a subspace \(\tilde{U} \subset \mathbb{C}^n\) by
\[
\tilde{U} := \text{span}\left(\bigcup_{\pi \in \mathcal{I}} (P_U A^* P_U) \pi K_0^{-1} e_1\right) \subset \mathbb{C}^n. \quad (A.11)
\]

Let \(m := \text{dim}\ \tilde{U}\) be the dimension of \(\tilde{U}\).
3. Choose a basis of $\tilde{U}$ in the form \{(P_\beta A^*P_U)\beta^{-1}e_1, 1 \leq i \leq m\} for some multi-indices $\beta_i$ with $\beta_1 = \emptyset$.

4. Define $K_0 = (K_0(i,j))_{i,j=1}^m \in \mathbb{C}^{m \times m}$ and $K_\gamma = (K_\gamma(i,j))_{i,j=1}^m \in \mathbb{C}^{m \times m}$ by

\[
K_0(i,j) = (\xi_\beta, K_0 \xi_\beta), \quad K_\gamma(i,j) = (\xi_\beta, K_\gamma \xi_\beta).
\]

5. Take an arbitrary unitary matrix $W \in \mathbb{C}^{m \times m}$ such that $We_1 = (K_0e_1)/\|K_0e_1\|_2$ and define

\[
\tilde{K}_0 := \frac{1}{\|K_0e_1\|_2}W^*K_0W, \quad \tilde{K}_\gamma := \frac{1}{\|K_0e_1\|_2}W^*K_\gamma W. \tag{A.12}
\]

Then $\tilde{L} := \tilde{K}_0 \otimes \mathbb{1}_m - \sum_{\gamma=1}^n \tilde{K}_\gamma \otimes x_\gamma$ gives a minimal (self-adjoint) linearization of $\mathbb{1}_m - \tilde{q}(x)$.

**Proof.** Let us take a matrix representation of $(1 - \tilde{q})^{-1}$ given by $(K_0^{-1}e_1, e_1, A_1, \ldots, A_\gamma)$ (see Remark [A.3]). Denote

\[
l_0 := P_\gamma K_0^{-1}e_1, \quad r_0 := P_\gamma e_1, \quad Q_\gamma := P_\gamma A_\gamma P_\gamma,
\]

where $P_\gamma : \mathbb{C}^n \to \tilde{U}$ is an orthogonal projection onto $\tilde{U}$. Then $(l_0, r_0, Q_1, \ldots, Q_\gamma)$ also gives a matrix representation of $(1 - q)^{-1}$. Indeed, for any $v \in \mathbb{C}^n$ and $\pi \in \mathcal{I}$ we have that

\[
(P_\gamma A^*P_U)\pi v = A_\pi P_\gamma v, \quad Q_\pi^* v = (P_\gamma A^*P_U)\pi v = (P_\gamma A^*P_U)\pi P_\gamma v, \quad \tag{A.13}
\]

where the first equality in (A.13) follows from the definition of $U$ (A.10) and the fact that $A_\gamma(U) \subset U$, whereas the second is due to the definition of $\tilde{U}$ (A.11) and $P_\gamma A^*P_U(U) \subset \tilde{U}$. In particular, we have that

\[
(P_\gamma A^*P_U)\pi e_1 = A_\pi e_1, \quad Q_\pi^* l_0 = (P_\gamma A^*P_U)\pi K_0^{-1}e_1, \quad \tag{A.14}
\]

which implies

\[
\langle l_0, Q_\pi r_0 \rangle = \langle (P_\gamma A^*P_U)\pi K_0^{-1}e_1, e_1 \rangle = \langle K_0^{-1}e_1, (P_\gamma A^*P_U)\pi e_1 \rangle = \langle K_0^{-1}e_1, A_\pi e_1 \rangle.
\]

This means that for any $\pi \in \mathcal{I}$

\[
\langle l_0, Q_\pi r_0 \rangle = \langle K_0^{-1}e_1, A_\pi e_1 \rangle \tag{A.15}
\]

and we conclude that $(l_0, r_0, Q_1, \ldots, Q_\gamma)$ gives a matrix representation of $(1 - \tilde{q})^{-1}$. In other words, we can still construct a matrix representation of $(1 - \tilde{q})^{-1}$ if we restrict $A_\gamma$ to an $m$-dimensional subspace $\tilde{U} \subset \mathbb{C}^n$.

Moreover,

\[
\text{span} \left( \bigcup_{\pi \in \mathcal{I}} Q_\pi r_0 \right) = \tilde{U}, \quad \text{span} \left( \bigcup_{\pi \in \mathcal{I}} Q_\pi^* l_0 \right) = \tilde{U}. \tag{A.16}
\]

To see this, assume that there exists $\tilde{u} \in \tilde{U}$ such that $\tilde{u} \perp Q_\pi r_0$ for all $\pi \in \mathcal{I}$. Then using (A.13) we get that

\[
0 = \langle \tilde{u}, Q_{\alpha_1} \cdots Q_{\alpha_k} r_0 \rangle = \langle \tilde{u}, A_{\alpha_1} \cdots A_{\alpha_k} e_1 \rangle,
\]

and since this holds for every multi-index, this implies that $\tilde{u} \perp U$ and thus $\tilde{u} = 0$. The second equality in (A.16) can be obtained similarly.

Now we will show that matrices $K_0$ and $K_\gamma$ represent $Q_\gamma|_\tilde{U} : \tilde{U} \to \tilde{U}$ and $Q_\gamma|_U : U \to \tilde{U}$ in a properly chosen basis. To this end, for any multi-index $\pi \in \mathcal{I}$ denote

\[
l_\pi := Q_\pi^* l_0, \quad r_\pi := Q_\pi r_0, \tag{A.14}
\]

and note that due to (A.14) \{l_{\beta_i} : 1 \leq i \leq m\} gives a basis of $\tilde{U}$ for some set of multi-indices \{\beta_i, 1 \leq i \leq m\} with $\beta_1 = \emptyset$.

Now we show that \{r_{\beta_i} : 1 \leq i \leq m\} is linearly independent, hence it also forms a basis of $\tilde{U}$.

Suppose there exist $c_1, \ldots, c_m \in \mathbb{C}$ such that $\sum_{j=1}^m c_j r_{\beta_j} = 0$. This means, that for all $\pi \in \mathcal{I}$

\[
\left\langle l_\pi, \sum_{j=1}^m c_j r_{\beta_j} \right\rangle = 0.
\]
Using (A.15) and the straightforward identity $A_\alpha K_0^{-1} = K_0^{-1} A_\alpha$ that is valid for all $\alpha \in \mathcal{I}$, it is easy to see that for all $\alpha, \beta \in \mathcal{I}$

$$0 = \left< l_{\alpha}, \sum_{j=1}^{m} c_j r_{\beta_j} \right> = \sum_{j=1}^{m} c_j \left< l_{\alpha}, r_{\beta_j} \right> = \left< \sum_{j=1}^{m} c_j l_{\beta_j}, r_{\alpha} \right> \quad \text{for all } \alpha \in \mathcal{I}. \quad (A.17)$$

Since $\{l_{\beta_j}, 1 \leq i \leq m\}$ is a basis of $\tilde{U}$ and $\tilde{U}$ is generated by $\{r_{\alpha}, \alpha \in \mathcal{I}\}$, we have from (A.17) that $\sum_{j=1}^{m} c_j l_{\beta_j} = 0$, which implies that $\sum_{j=1}^{m} c_j = 0$. This shows that $\{r_{\beta_j}, 1 \leq j \leq m\}$ is linearly independent and thus forms a basis of $\tilde{U}$.

Define now $n \times m$ matrices $B_L$ and $B_R$, whose columns are the basis vectors $l_{\beta_j}$ and $r_{\beta_j}$ correspondingly, i.e., $B_L := (l_{\beta_i} : 1 \leq i \leq m)$ and $B_R := (r_{\beta_i} : 1 \leq i \leq m)$. Then from (A.15) we have that

$$\mathbb{K}_0 = B_L^* B_R, \quad \mathbb{K}_\gamma = B_L^* Q_\gamma B_R. \quad (A.18)$$

Matrix $\mathbb{K}_0$ is obviously invertible from this construction, since the columns of $B_L$ and $B_R$ form two bases of $\tilde{U}$, thus $B_R = B_L T$ for some invertible $T \in \mathbb{C}^{m \times m}$. On the other hand, since $P_U^{-1} := B_R (B_L^* B_R)^{-1} B_L^*$ is a projection onto $\tilde{U}$, we have

$$\left< e_1, (\mathbb{K}_0^{-1})_\alpha \mathbb{K}_0 e_1 \right> = \left< B_L e_1, P_U^{-1} B_L^* Q_1 P_U^{-1} \cdots P_U^{-1} Q_{\alpha} P_U^{-1} B_R e_1 \right> = \left< l_0, Q_\alpha r_0 \right> \quad (A.19)$$

for any $\alpha \in \mathcal{I}$, which implies that $(\mathbb{K}_0 e_1, \mathbb{K}_1, \mathbb{K}_0^{-1}, \ldots, \mathbb{K}_\gamma, \mathbb{K}_0^{-1})$ is a matrix representation of $(1-\tilde{q})^{-1}$ of dimension $m$.

In the last step we make a change of basis that allows us to replace $\mathbb{K}_0 e_1$ by $e_1$. By the choice of $W$ we have that $\mathbb{W}^* \mathbb{K}_0 e_1 = \|\mathbb{K}_0 e_1\|_2 e_1$, so that

$$\left< e_1, (\mathbb{K}_0^{-1})_\alpha \mathbb{K}_0 e_1 \right> = \|\mathbb{K}_0 e_1\|_2^2 \left< e_1, \mathbb{W}^* \mathbb{K}_0^{-1} \mathbb{W} (\mathbb{W}^* \mathbb{K}_0 \mathbb{W}^* \mathbb{K}_0^{-1} \mathbb{W})_\alpha e_1 \right>, \quad (A.20)$$

and $\|\mathbb{K}_0 e_1\|_2^2$ will be absorbed by one of the $\tilde{K}_0^{-1}$ if we define $\tilde{K}_0$ and $\tilde{K}_\gamma$ via (A.12). Therefore, from the definition of $\tilde{K}$ (A.12), (A.20), (A.19) and (A.15) we obtain that

$$\left< \tilde{K}_0^{-1} e_1, (\tilde{K} \tilde{K}_0^{-1})_\alpha e_1 \right> = \left< K_0^{-1} e_1, A_\alpha e_1 \right>, \quad (A.21)$$

and we conclude that $(\tilde{K}_0^{-1} e_1, \tilde{K}_1 \tilde{K}_0^{-1}, \ldots, \tilde{K}_\gamma \tilde{K}_0^{-1})$ is a matrix representation of $(1-\tilde{q})^{-1}$.

Moreover,

$$\text{span} \left( \bigcup_{\alpha \in \mathcal{I}} (\tilde{K} \tilde{K}_0^{-1})_\alpha e_1 \right) = \mathbb{C}^m, \quad \text{span} \left( \bigcup_{\alpha \in \mathcal{I}} (\tilde{K} \tilde{K}_0^{-1})_\alpha \tilde{K}_0^{-1} e_1 \right) = \mathbb{C}^m. \quad (A.22)$$

Indeed, suppose that there exist $c_1, \ldots, c_m \in \mathbb{C}$ satisfying $\sum_{i=1}^{m} |c_i| > 0$, such that

$$\sum_{i=1}^{m} c_i (\tilde{K} \tilde{K}_0^{-1})_\beta e_1 = 0.$$

Then for any $\alpha \in \mathcal{I}$

$$\left< (\tilde{K} \tilde{K}_0^{-1})_\alpha \tilde{K}_0^{-1}, \sum_{i=1}^{m} c_i (\tilde{K} \tilde{K}_0^{-1})_\beta e_1 \right> = 0$$

which by (A.21) and (A.15) means that $\left< l_{\alpha}, \sum_{i=1}^{m} c_i r_{\beta_i} \right> = 0$ and contradicts to the fact that $\{r_{\beta_i}, 1 \leq i \leq m\}$ is a basis of $\tilde{U}$ and $\{r_{\alpha}: \alpha \in \mathcal{I}\} = \tilde{U}$. Therefore, $\{(\tilde{K} \tilde{K}_0^{-1})_\beta e_1, 1 \leq i \leq m\}$ is linearly independent, which implies the first equality in (A.22). The second equality in (A.22) can be shown using a similar argument.

Now we can finish the proof of Lemma A.7. By construction matrices $\tilde{K}_0$ and $\tilde{K}_\gamma$ are Hermitian. Moreover, by (A.18) and (A.12) $\tilde{K}_0$ is invertible and by (A.21)

$$\left< e_1, \tilde{K}_0^{-1} e_1 \right> = \left< e_1, K_0^{-1} e_1 \right> = 1. \quad (A.23)$$
It remains to show that \( L_m := \tilde{K}_0 \otimes \mathbb{1}_{\mathcal{I}} - \sum_{\gamma=1}^r \tilde{K}_\gamma \otimes x_\gamma \) is minimal and satisfies (2.2).

Similarly as in Remark \ref{rem:1}, if we assume that \( \sum_{\gamma=1}^r \| \tilde{K}_\gamma \tilde{K}_0^{-1} \|_{\mathcal{I}} \leq 1/2 \), then by (A.21) and (A.5)

\[
\left\langle e_1 \otimes \mathbb{1}_{\mathcal{I}}, \left( \tilde{K}_0 \otimes \mathbb{1}_{\mathcal{I}} - \sum_{\gamma=1}^r \tilde{K}_\gamma \otimes x_\gamma \right)^{-1} e_1 \otimes \mathbb{1}_{\mathcal{I}} \right\rangle_{\mathcal{I}} = \frac{1}{1_{\mathcal{I}} - \tilde{q}(x)}.
\]

On the other hand, if similarly to (2.1) we write

\[
L_m = (\lambda_m \ell^*_m), \ell_m
\]

then by the Schur complement formula

\[
\frac{1}{[L_m^{-1}]_{11}} = \lambda_m - \ell^*_m \ell_m,
\]

which together with (A.24) implies (2.2).

Finally, minimality of \( L_m \) follows from (A.22). Indeed, (A.22) implies that

\[
\left( \tilde{K}_0^{-1} e_1, e_1, \tilde{K}_1 \tilde{K}_0^{-1}, \ldots, \tilde{K}_r \tilde{K}_0^{-1} \right)
\]

is a matrix representation of \((1 - \tilde{q})^{-1}\) of the lowest possible dimension. If we assume that there exist a linearization \( L' = K'_0 \otimes \mathbb{1}_{\mathcal{I}} - \sum_{\gamma=1}^r K'_\gamma \otimes x_\gamma \) with dimension smaller than \( m \), then

\[
\left( (K'_0)^{-1} e_1, K'_1(K'_0)^{-1}, \ldots, K'_r(K'_0)^{-1} \right)
\]

would give a matrix representation of \((1 - \tilde{q})^{-1}\) of dimension smaller than \( m \), which would lead to a contradiction. We conclude that \( L_m \) is a minimal linearization of \( 1_{\mathcal{I}} - \tilde{q}(x) \).

**A.3 Numerical comparison of two linearizations**

In the next two tables we show how the dimension of the standard linearization from Appendix \ref{app:1} relates to the dimension the minimal linearization for polynomials having different degrees and structures.

The first table (Figure A.1) shows how the dimensions of the two different linearizations depend on the degree of the polynomial. For a given degree, we generated random samples of noncommutative polynomials in two noncommutative variables by choosing the coefficients of all possible monomials up to the given degree independently (up to symmetry constraints) and uniformly from an interval.

| degree | \( \gamma \) | \( \delta \) |
|--------|--------------|--------------|
| 1      | 1            | 1            |
| 2      | 9            | 3            |
| 3      | 41           | 5            |
| 4      | 137          | 9            |
| 5      | 393          | 13           |
| 6      | 1033         | 21           |
| 7      | 2563         | 29           |
| 8      | 6153         | 45           |

Figure A.1: For random polynomials with a given degree, \( \gamma \) and \( \delta \) are the average dimensions of the standard and the minimal linearizations, respectively.

The second table, Figure A.2, illustrates how the dimension of the minimal linearization may depend on the structure of the polynomial. We again generated samples of polynomials in two noncommutative variables. This time each sample is characterized by two given numbers, the lowest and highest degree

\[
\text{Figure A.2: For random polynomials with a given structure, \( \gamma \) and \( \delta \) are the average dimensions of the standard and the minimal linearizations, respectively.}
\]

\[
\text{Figure A.3: For random polynomials with a given degree and structure, \( \gamma \) and \( \delta \) are the average dimensions of the standard and the minimal linearizations, respectively.}
\]

38
of the monomials allowed in the polynomials. The coefficients of the monomials are given as a product of independent (up to symmetry constraints) random variable uniformly distributed on an interval and a 0–1 Bernoulli random variable with parameter chosen is such a way that the standard linearization for all four samples have approximately the same dimension around 2000. In other words, the Bernoulli variable picks an appropriate subset of the all possible monomials and then we further randomize its coefficient. This random preselection is necessary to keep the calculation at manageable length.

| Min degree | Max degree | \(\gamma\) | \(\delta\) |
|------------|------------|-------------|-------------|
| 1          | 7          | 2113        | 29          |
| 6          | 8          | 2076        | 44.9        |
| 9          | 10         | 2082.8      | 86.2        |
| 11         | 12         | 1930        | 150.9       |

Figure A.2: For random polynomials consisting of monomials with a given minimal and maximal degree, \(\gamma\) and \(\delta\) are the average dimensions of the standard and the minimal linearizations, respectively.

Above results suggest that the minimal linearization provides a substantial reduction in the size of the linearization for a typical polynomial with no restriction on its structure, and this reduction becomes less significant if we restrict the polynomial to have only monomials of higher degrees. In other words, minimal linearization is the most advantageous over the customary one if the polynomial is the sum of many monomials. Note that randomization excludes the polynomials of very special structure, for example, high powers of linear combinations of noncommutative variables, which may behave very differently.

## B Properties of semicircular noncommutative random variables

The aim of this section is to recall some basic definitions related to the \(C^*\)-probability spaces and semicircular random variables that are used throughout the paper. For a more complete introduction to the subject we refer the reader to [7, Section 5].

**Definition B.1** (\(C^*\)-algebra and \(C^*\)-probability space). We call \(\mathcal{A}\) a (unital) \(C^*\)-algebra, if

(i) \(\mathcal{A}\) is a (unital) algebra endowed with an involution \(\ast\) and norm \(\| \cdot \|_{\mathcal{A}}\) satisfying

\[\|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}, \quad \|a\ast a\|_{\mathcal{A}} = \|a\|^2_{\mathcal{A}}\]

for any \(a, b \in \mathcal{A}\);

(ii) \((\mathcal{A},\| \cdot \|_{\mathcal{A}})\) is a Banach space.

If \(\mathcal{A}\) is a unital \(C^*\)-algebra and \(\tau : \mathcal{A} \to \mathbb{C}\) is a linear complex-valued functional such that

\[\tau(a^*a) \geq 0, \quad \tau(1_{\mathcal{A}}) = 1\]

for any \(a \in \mathcal{A}\) and the unit element \(1_{\mathcal{A}} \in \mathcal{A}\), then we call \((\mathcal{A},\tau)\) a \(C^*\)-probability space. We will always assume that the state \(\tau\) is tracial \((\tau(ab) = \tau(ba)\) for all \(a, b \in \mathcal{A}\)\) and faithful \((\tau(a^*a) = 0\) implies that \(a = 0\)).

We call the elements of a \(C^*\)-probability space \(\mathcal{A}\) non-commutative random variables. A family \(\{a_1, \ldots, a_k\} \subset \mathcal{A}\) of non-commutative random variables is characterized by its non-commutative distribution, a map \(\mu_{a_1, \ldots, a_k} : \mathbb{C}\langle x_1, \ldots, x_k\rangle \to \mathbb{C}\) given by

\[\mu_{a_1, \ldots, a_k}(P) = \tau(P(a_1, \ldots, a_k)), \quad P \in \mathbb{C}\langle x_1, \ldots, x_k\rangle,\]

where we recall that \(\mathbb{C}\langle x_1, \ldots, x_k\rangle\) denotes the set of (noncommutative) polynomials in \(x_1, \ldots, x_k\).

A family \(\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{A}\) of subalgebras of \(\mathcal{A}\), each containing \(1_{\mathcal{A}}\), is called freely independent if

\[\tau(a_1a_2\cdots a_k) = 0\]
for any \((i_1, \ldots, i_k) \in \{1, \ldots, n\}^k\) and \(a_1 \in \mathcal{A}_{i_1}, \ldots, a_k \in \mathcal{A}_{i_k}\) with \(\tau(a_j) = 0\) and \(i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k\). Noncommutative variables \(a_1, \ldots, a_n\) are freely independent if the subalgebras generated by \(a_1, \ldots, a_n\) are freely independent.

A freely independent family of noncommutative variables \(s_1, \ldots, s_k\) from the \(C^*\)-probability space \((\mathcal{A}, \tau)\) is called a semicircular system if \(s_i^* = s_i\) and

\[
\tau(s_j^*) = \begin{cases} 
0, & j \text{ is odd}, \\
C_\ell, & j = 2\ell, \text{ even}, 
\end{cases}
\]  

(B.1)

where \(C_\ell\) is the \(\ell\)-th Catalan number, \(C_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell}\). We will denote by \(\mathcal{A}\) the unital (with unit element \(\mathbb{1}\)) \(C^*\)-algebra generated by \(\{s_1, \ldots, s_k\}\), and \((\mathcal{A}, \tau)\) will be the corresponding \(C^*\)-probability space. The spectrum of a semicircular element \(s\) is equal to the interval \([-2, 2]\), in particular we have that \(\|s\|_{\mathcal{A}} = 2\).

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