A REMARK ON EXTENDED KIM’S CONJECTURE AND HYPO- LIE ALGEBRA

KIM YANGGON, WANG MOONOK

ABSTRACT. We have already conjectured 2 important guesses regarding Hypo- Lie algebra and modular simple Lie algebra. We would like to attach 2 important guesses more to this conjecture. Such new guesses are related to the Steinberg module.

1. INTRODUCTION

Let \( L \) be any modular simple Lie algebra of \( A_l \)-type or \( C_l \)-type over any algebraically closed field \( F \) of characteristic \( p \geq 7 \).

We proved that Kim’s conjecture is right for these simple Lie algebras. We still believe that Kim’s conjecture is right for other simple Lie algebras of classical type.

Furthermore 2 more guesses are plausible attached to the conjecture.

In section 2 we recollect some definitions and related facts to the conjecture. We shall give additional guesses to this conjecture in section 3.

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2. SOME DEFINITIONS AND RELATED FACTS

Now let $L$ be any simple Lie algebra of classical type with a CSA $H$ over an algebraically closed field $F$ of characteristic $p \geq 7$.

It is well known that any simple $L$-module is isomorphic to some quotient $L$-module of $V^\lambda(L)$, which is called a Verma module. Here $\lambda$ is a weight $\lambda : H \to F$ which is related to $\chi$ as a linear form in $H^*$.

Such a quotient $L$-module is called a Weyl module, denoted by $W^\lambda(L)$.

We posed a conjecture in [5] and proved completely in [2] that the conjecture is right for modular simple Lie algebras of $A_l$-type or $C_l$-type.

We still believe that our conjecture is also right for other modular simple Lie algebras of classical type. Furthermore we would like to announce to the interested readers the additional guesses attached to the conjecture.

**Definition 2.1.** Suppose that $V_0^\lambda(L) = W_0^\lambda(L)$ and $\rho_0^\lambda$ is its associated representation of $U(L)$; then this simple module is said to be a Steinberg module, while $U(L)/\ker \rho_0^\lambda$ is called a Steinberg algebra, which is isomorphic to $\rho_0^\lambda(U(L))$ associated with the Steinberg module.
For a root $\alpha$ in a root system $\Phi$ of $L$ we put $w_{\alpha} := (h_\alpha + 1)^2 + x_{-\alpha}x_\alpha$ and $g_{\alpha} := x_{\alpha}^{p-1} - x_{-\alpha}$ as in [0],[2],[3],[4],[5],[6].

By virtue of [8], we know that $w_{\alpha}$ belongs to the center of $U(\mathfrak{sl}_2(F))$, and that $g_{\alpha}$ is invertible in $\rho_0^\lambda(U(L))$ whenever $x_{\alpha}^p = x_{-\alpha}^p \equiv 0$ but $h_\alpha^p - h_\alpha \not\equiv 0$ modulo $\ker\rho_\lambda$.

**Proposition 2.2.** Let $L$ be any modular simple Lie algebra of $A_l$- type or of $C_l$- type over an algebraically closed field $F$ of characteristic $p \geq 7$.

Suppose that $\chi = 0$ and there exists a root $\alpha$ such that $\lambda(h_\alpha) = -1$; then we have a Steinberg module $V_0^\lambda(L) = W_0^\lambda(L)$, where $\lambda(h_\alpha^p) - \lambda(h_\alpha) = \chi(h_\alpha)^p$ holds.

**Proof.** We know that $w_{\alpha}$ acts on maximal vectors as a constant zero and so on factor $S_\alpha$-modules relative to a composition series in $V_0^\lambda(L)$, where $S_\alpha = Fx_\alpha + Fh_\alpha + Fx_{-\alpha} \simeq \mathfrak{sl}_2(F)$.

In such cases $g_{\alpha}$ becomes invertible in $U(L)/\ker\rho_0^\lambda$. Hence the proofs required along with Lee’s bases are exactly the same as those in [2].

For $A_l$-type, refer to theorem2.2, and for $C_l$-type refer to proposition3.2 respectively in the reference [2].

□

3. **Conjecture extended from [5]**
In the main proposition of the reference [0] we made use of a Steinberg module for simple Lie algebra $L$ of $B_l$-type, which arises seemingly unique.

For the proof of simplicity of Steinberg module we gave directly Lee’s base of Steinberg algebra $U(L)/\ker \rho_0^\lambda$, where $\lambda$ is a Weyl weight.

We found out that Steinberg modules may still arise for any modular simple Lie algebra $L$ of classical type over $F$ even if $\lambda$ is not a Weyl weight.

The reason goes as follows.

We are well aware that $U(L)$ is integral over its center $Z(U(L))$, so that $w_\alpha$ has an irreducible integral equation of degree $p$ over $Z(U(L))$.

If $Fx_\alpha + Fh_\alpha + Fx_{-\alpha} = \mathfrak{sl}_2(F)$, then this irreducible polynomial over the center of $U(\mathfrak{sl}_2(F))$ has the form

$$\prod_{k=0}^{p-1}(t - k^2) - z^2 - 4xy,$$

where

$$x = x_{-\alpha}^p;$$

$$y = x_{\alpha}^p;$$

$$z = h_\alpha^p - h_\alpha,$$
\[ t = (h_\alpha + 1)^2 - 4x_\alpha x_\alpha, \]

\[ k = 0, 1, 2, \cdots, p - 1 \text{ (refer to [8]).} \]

Let \( \{\lambda_i|1 \leq i \leq l\} \) denote the set of fundamental dominant weights and let \( \lambda \) be the Weyl weight for the time being. In other words \( \lambda = \sum_{i=1}^{l} \lambda_i = \sum_{\alpha > 0} \alpha. \)

Then for any root \( \alpha_j \) with \( 1 \leq j \leq l \) we have

\[ \lambda(h_{\alpha_j}) = \langle \lambda, \alpha_j \rangle = \frac{\sum_i 2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_i \delta_{ij} = 1. \]

We thus obtain \( (-\lambda)(h_{\alpha_j}) = -1. \)

Putting \( S_{\alpha_j} = Fx_{\alpha_j} + h_{\alpha_j} + Fx_{-\alpha_j}, \) which is isomorphic to \( \mathfrak{s}\mathfrak{l}_2(F), \) we know that \( w_{\alpha_j} \) acts as 0 on factor \( S_{\alpha_j} \)-modules relative to a composition series in \( W_0^{-\lambda}(L) \) since every Weyl module has a maximal vector.

In other words, \( h_{\alpha_j} \) acts on a maximal vector as -1. According to [8], \( g_{\alpha_j} \) for \( B_l \)-type \( L \) is invertible in \( W_0^{-\lambda}(L) \) because \( w_{\alpha_j} \) is nilpotent in this Steinberg algebra. Note that \( w_\alpha \) becomes either nilpotent or invertible in this Steinberg algebra. So we could make in the reference [0] a Lee’s basis of its associated Steinberg algebra by way of \( g_\alpha. \)

For the purpose of making a Lee’s basis for \( \chi = 0, \) it is necessary to get \( g_\alpha \) invertible in \( W_0^{-\lambda}(L). \) Hence we might as well justify our extended conjecture in general motivated by this idea.
[extended CONJECTURE ]

In addition to guesses (i),(ii) in the conjecture in [5], we might give 2 more guesses (iii) and (iv) explained below.

(iii)Let $L$ be any modular simple Lie algebra of classical type over an algebraically closed field $F$ of characteristic $p \geq 7$; then we conjectured in [5] that $V_\chi^\lambda(L) = W_\chi^\lambda(L)$ and now we conjecture even more that there exists a Lee’s basis whenever $\chi \neq 0$, where $\lambda$ is a weight $\lambda : H \rightarrow F$ relating to a maximal vector of the Weyl module.

However even if $\chi = 0$, we also conjecture that we might give Lee’s basis to the Steinberg algebra associated with $W_0^{-\lambda}(L)$ for a Weyl weight $\lambda$.

(iv)We conjecture that A simple $L$-module over $F$ of dimension less than $p^m$ arises with $2m + \text{rank}(L) = \text{dim}(L)$ if and only if $\chi = 0$ and $\lambda(h_\alpha) \neq -1$ for any root $\alpha \in \text{the base } \Delta$ of the root system $\Phi$ of $L$.

So combining (iii) and (iv) we guess that there might be only a finite number of simple $L$ modules of dimension $< p^m$ up to isomorphism and hence $L$ is a Hypo- Lie algebra.

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Emeritus professor, Department of Mathematics, Jeonbuk National University, 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.

Email address: kyk1.chonbuk@hanmail.net wang@hanyang.ac.kr