Petrov type I silent universes with G3 isometry group: the uniqueness result recovered

Lode Wylleman and Norbert Van den Bergh

Faculty of Applied Sciences TW16, Gent University, Galglaan 2, 9000 Gent, Belgium

E-mail: lwyllema@cage.ugent.be

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Abstract

Irrotational dust spacetimes with vanishing magnetic Weyl curvature are called silent universes (Matarrese et al 1994 Phys. Rev. D 72 320). The silent universe conjecture (Sopuerta 1997 Phys. Rev. D 55 5936, van Elst et al 1997 Class. Quantum Grav. 14 1151) states that the only algebraically general silent universes are the orthogonally spatially homogeneous Bianchi I models. In the same paper by Sopuerta, this was confirmed for the subcase where the spacetime also admits a group G3 of isometries. However, the proof contains a conceptual mistake. We recover the result in a different way.

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1. Introduction

Irrotational dust ($\omega^a = 0, p = 0$) solutions of the Einstein perfect fluid field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \mu u_a u_b$$

with vanishing magnetic part $H_{ab}$ of the Weyl tensor [1] were called silent universes for the first time by Matarrese, Pantano and Saez [2]. The reason for this nomenclature was that the signal exchange in GR can occur only via sound and gravitational waves, none of which modes are allowed when $p = H_{ab} = 0$. From a more mathematical point of view, the evolution equations for the matter density $\mu$, the three non-zero eigenvalues $\theta_1, \theta_2, \theta_3$ of the expansion tensor $\theta_{ab}$ and the two independent eigenvalues (e.g., $E_1, E_2$) of the remaining electric part $E_{ab} = C_{abcd} u^c u^d$ of the Weyl tensor, form an autonomous system of ordinary differential equations (see also below); no spatial gradients appear, such that each fluid element evolves indeed as a separate or ‘silent’ universe, once the constraint equations are satisfied by the initial data. This setting looked very appealing towards numerical schemes and simulations in an astrophysical and cosmological context, e.g. for the description of structure formation in the universe and the study of the gravitational instability mechanism in general relativity, where a clear motivation for taking $p = \omega^a = 0, H_{ab} \approx 0$ was given in [3].
The only allowed Petrov types for silent universes are 0, D and I: the Friedmann–Robertson–Walker dust metrics exhaust the type 0 class, while all Petrov type D solutions are known explicitly too. They are characterized [4] by the fact that the Weyl tensor is degenerate in the same plane as the shear and are given by the Szekeres [5] family, including the subcase of the Ellis [6] LRS class II dust models and such well-known examples as the Lemaitre–Tolman–Bondi model and the orthogonally spatially homogeneous Kantowski–Sachs model. For the algebraically general case (Petrov type I), the situation is rather different, the only known silent universe being the orthogonally spatially homogeneous Bianchi I model.

In two independent papers by Sopuerta [7] and van Elst et al [8], the propagation of the constraint \( H_{ab} = 0 \) was shown to give rise to a triplet of (in principle) infinite chains of equations, identically satisfied for Petrov type 0, D and spatially homogeneous silent models, but not in general. What happens is that for non-spatially homogeneous Petrov type I silent universes the initial values for \( \theta_\alpha, E_\alpha \) and \( \mu \) must lie on a (low-dimensional) algebraic variety, i.e. there exist severe polynomial relations between these variables. As was conjectured by the authors of [7, 8], there are good reasons to believe that the variety is actually empty, i.e. there exist no non-spatially homogeneous Petrov type I silent models at all, at least for zero cosmological constant. This is the so-called ‘silent universe conjecture’, no definitive proof of which has been given until now, mainly because the degree and the number of terms of the polynomials are massive. However, we mention here that in the case of strictly positive cosmological constant, the above variety has been shown to contain three (analogous) two-dimensional components, corresponding to two explicit new families of metrics each [9].

In the present paper, we correct a reasoning in [7] and reconfirm the silent universe conjecture (for vanishing cosmological constant) in the subcase where the spacetime admits a group G3 of isometries.

For silent universes, one can deduce in general (proposition 3 of [7]) that there always exists an orthonormal basis \( \{ e_0, e_\alpha \} \) (with \( e_0 \equiv u \)) of common eigenvector fields of the expansion tensor \( \theta_{ab} \) and the remaining electric part \( E_{ab} = C_{acbd}u^cu^d \) of the Weyl tensor, which are all parallelly transported along the matter flow lines and are hypersurface orthogonal, such that a local coordinate system \( (t, x^\alpha) \) exists for which \( e_0 \equiv \frac{\partial}{\partial t} \) and \( e_\alpha \propto \frac{\partial}{\partial x^\alpha} \). Below we will use the notation \( \partial_\alpha \) for \( e_\alpha \) whenever the frame vector fields act as differential operators.

In [7], algebraically general silent universes were investigated in such coordinates, and in section 5 restrictions on the coordinate components of a generic Killing vector field \( K \) were obtained. However, in the discussion of case (iii) a time dependence via \( K^z \) in (108) was overlooked and therefore the conclusion that \( K^t \) is constant and \( K^x_\alpha \) only depends on \( x^\alpha \) does not necessarily hold. But even if this conclusion were valid, a coordinate transformation \( x^\alpha \rightarrow x'^\alpha \) which transforms the components of a fixed Killing field \( K_0 \) into constant functions would not necessarily do the same for another Killing field \( K \), such that coordinates w.r.t. which every Killing field has constant coefficients (which was the content of theorem 1 in [7]) do not necessarily exist. Herewith the reasoning leading to the uniqueness of G3 Petrov type I silent universes (corollary 1 of [7]) breaks down.

Fortunately, there is a related but essentially different way to prove the G3 uniqueness result. The key point is that, contrary to gauge-dependent quantities such as the metric components, invariently-defined geometric objects must be invariant under the isometries¹. For example, scalar invariants such as the matter density and the eigenvalues of the Weyl and expansion tensor must by annihilated by any Killing field, and this yields sufficient extra information to prove the key lemma 3. In particular for Petrov type I, we note that the

¹ See [1], section 8.4 for an account.
eigenvector fields \( e_\alpha \) of the Weyl tensor and hence the corresponding rotation coefficients \( \Gamma_{\alpha bc} := e_\alpha \cdot \nabla_c e_b \) are examples of invariantly-defined geometric objects and hence are preserved under local isometries.

2. Mathematical setting

In what follows, \( U \) will always denote an open subset of a spacetime for which the conditions in the definition of a silent universe are satisfied everywhere. For such a subset, we will say the spacetime is BI if it is an orthogonally spatially homogeneous Bianchi I model on \( U \). The set of smooth functions on \( U \) will be denoted as \( \mathcal{F}(U) \), and all reasonings will only involve a finite number of polynomial combinations \( F_i \) (including 0) of such functions. Because of this, we can always assume \( U \) small enough such that for all couples \( (F_i, F_j) \) either \( F_i(p) = F_j(p), \forall p \in U \) (further denoted by \( F_i = F_j \)) or \( F_i(p) \neq F_j(p), \forall p \in U \) (denoted by \( F_i \neq F_j \)).

For silent universes, the only remaining dynamical variables are the matter density \( \mu \), the expansion eigenvalues \( \theta_\alpha \), the Weyl eigenvalues \( E_\alpha \), the purely spatial Ricci rotation coefficients \( 2q_\alpha \equiv -e_\alpha \cdot \nabla_\alpha e_\alpha - 1 \) and \( 2r_\alpha \equiv e_\alpha \cdot \nabla_{\alpha+1} e_{\alpha+1} \), and \( m_\alpha := \partial_\mu (\mu) \). In these variables, the first contracted Bianchi identity reads

\[
\partial_0 \mu = -\mu \theta \tag{2}
\]

for the derivatives of the expansion and the Weyl eigenvalues, resp. \( \theta_\alpha \), one has [9]

\[
\partial_0 \theta_\alpha = -E_\alpha - \frac{\theta_\alpha^2}{6} - \frac{\mu}{2} \sigma_\alpha, \tag{3}
\]

\[
x_\alpha \partial_0 \theta_\alpha = 6E_\alpha (-h_{\alpha+1} q_\alpha + h_{\alpha-1} r_\alpha) - \frac{h_\alpha}{2} m_\alpha, \tag{4}
\]

\[
\partial_\alpha \theta_\alpha + 1 = -4h_\alpha - r_\alpha, \quad \partial_\alpha \theta_\alpha - 1 = -4h_{\alpha+1} q_\alpha, \tag{5}
\]

Here the expressions have to be read modulo 3 (\( \alpha \) running from 1 to 3), \( \sigma_\alpha \equiv \theta_\alpha - 1/3 \theta \) denote the shear eigenvalues and \( 2h_\alpha := \theta_{\alpha+1} - \theta_{\alpha-1}, 2x_\alpha := E_{\alpha+1} - E_{\alpha-1} \). We remark that a particular result stated in these variables actually yields a triple of results, by cyclic permutation of the indices. We will use this in lemmas 1(a) and 3. For most of our purposes, it will be favourable to work with the following linear combinations of \( \theta_\alpha \) and \( E_\alpha \):

\[
x_1, \quad h_1, \quad \epsilon_1 := \frac{3}{2} E_1, \quad s_1 := -\frac{3}{2} \sigma_1, \quad b_1 := \frac{\theta_2 + \theta_3}{2}. \tag{7}
\]

The expressions for the derivatives of these variables (in terms of themselves and \( q_\alpha, r_\alpha, m_\alpha \)) can be readily deduced from (2) to (6). We shall frequently make use of the following three results:

(r1) \( \theta_\alpha = \theta_\beta (h_1 = 0, s_1 = h_1, s_1 = -h_1) \) implies \( E_\alpha = E_\beta (x_1 = 0, \epsilon_1 = x_1, \epsilon_1 = -x_1) \), see [4];

(r2) if all \( q_\alpha \) and \( r_\alpha \) vanish then the spacetime is BI, see [8, 10];

(r3) the only equilibrium point of the autonomous dynamical system formed by \( \partial_0 \mu = -\mu \theta, \) (2) and (3), is \( (E_\alpha = 0, \theta_\alpha = 0) \), see [7, 11].
3. Recovering the uniqueness result for G3 silent universes

**Lemma 1** (special cases). If (a) \( \theta_0 = 0 \) \((b_1 = s_1, b_1 = -h_1, b_1 = h_1)\), (b) \( b_1 = 0 \) or (c) \( x_1 s_1 = \epsilon_1 h_1 \) on \( U \), then the Petrov type is 0 or \( D \).

**Proof.** (a) It suffices to prove this for \( \alpha = 1 \) (cf above). Consecutive time derivatives of \( b_1 = s_1 \), substituting previously obtained equations in each step, yield \( \mu = 4 \epsilon_1 - \epsilon_1 s_1 + x_1 h_1 = 0 \) and \( (\epsilon_1 - x_1) (\epsilon_1 + x_1) = 0 \), i.e. \( E_1 = E_2 \) or \( E_1 = E_2 \). (b) The same procedure for \( b_1 = 0 \) yields \( \mu = -6 h_1^2 - 2 \epsilon_1 \) and \( x_1 h_1 = 0 \), hence \( E_2 = E_3 \) by (1). (c) For \( x_1 s_1 = \epsilon_1 h_1 \), this procedure immediately yields \( x_1 (s_1 - h_1) (s_1 + h_1) = 0 \), so again the Petrov type is 0 or \( D \) by (r1).

**Lemma 2.** An algebraically general silent universe \( U \) for which \( s_1 = ch_1 \), with \( c \) a constant function, is necessarily BI.

**Proof.** The same procedure as in lemma 1, applied on \( s_1 = ch_1 \), yields \( \epsilon_1 = (c^2 - 1) h_1^2 + c x_1 \), \( h_1 (c - 1) (c + 1) (h_1^2 - b_1 h_1 + x_1) = 0 \). By (r1), the last equation implies \( x_1 = h_1 (b_1 - ch_1) \), and hence \( \epsilon_1 = h_1 (c b_1 - h_1) \) by substitution. Two further time evolutions yield \( \mu = -h_1^2 - 2 c h_1 b_1 + 3 b_1^2 \) and an identity. Substituting the obtained expressions for \( s_1, \epsilon_1, x_1, \mu \) into (4)–(6) yields three homogeneous linear systems of five equations in the variables \( \partial_0 b_1, \partial_0 h_1, q_1, r_1, m_1 \) (for each \( \alpha \) separately), the coefficients of which depend polynomially on \( b_1, h_1 \). The product of the three corresponding determinants is constantly proportional to \( h_1^2 (c - 1) (c + 1) (h_1 - b_1) (b_1 + h_1)^2 \), and this is non-zero by (r1) and lemma 1(a). Hence the linear systems only allow the trivial solution; the result follows by (r2).

**Lemma 3.** When a vector field of the form \( \partial_0 + g_0 \partial_0 \), defined on an algebraically general silent universe \( U (g_0 \in F(U)) \) annihilates the scalars \( \mu \) and (7), then \( g_0 = 0 \).

**Outline of proof.** It suffices to prove this for \( \alpha = 1 \) (cf above), and we set \( X := \partial_1 + g_1 \partial_0 \). By substituting the annihilation condition \( m_1 = g_1 \mu (3 h_1 - s_1) \) for \( \mu \) into \( X(x_1) = X(h_1) = X(\epsilon_1) = X(s_1) = x_1 X(s_1) = X(b_1) = 0 \), respectively, one gets five equations \( eq_1, \ldots, eq_5 \), which form a homogeneous linear system in the variables \( g_1, r_1, g_1, \) the coefficients of which depend polynomially on \( \mu, x_1, h_1, \epsilon_1, s_1, b_1 \).

Assume \( g_1 \neq 0 \). This implies \( det_1 = det_2 = det_3 = 0 \), where \( det_i \) is the determinant corresponding to the linear system \( (eq_1, eq_2, eq_3) \), \( i = 3, 4, 5 \). The combinations \( det_3 - \frac{4}{h_1} det_1 \) and \( 6 \epsilon_1 det_1 - 2 s_1 det_1 \) can be factorized as \( (x_1 s_1 - \epsilon_1 h_1) p_1, (x_1 s_1 - \epsilon_1 h_1) p_2 \), respectively, such that \( p_1 = p_2 = 0 \) by lemma 1(c). To complete the proof we will use three additional relations \( p_3 = p_4 = p_5 = 0 \), where \( p_3 \) is defined by \( \partial_0 p_3 + (6 b_1 - 2 s_1) p_1 := 4 h_1 p_3 \) and \( p_4, p_5 \) are the respective polynomials \( \partial_0 p_2 + (3 b_1 - s_1) p_2 \) and \( \partial_0 p_3 \).

We scale \( \mu, \epsilon_1, x_1 \) with \( h_1^2 \) and \( b_1, s_1 \) with \( h_1 \); this comes down to specializing \( h_1 = 1 \) in \( p_1, \ldots, p_5 \), and we will denote the resulting variables and polynomials with a bar. The respective results w.r.t. \( \bar{m} \) of the couples \( (\bar{p}_1, q) \) with \( q = \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5 \) yield four irreducible polynomial relations \( u_1 = u_2 = u_3 = u_4 = 0 \) for the variables \( \bar{\tau}, \bar{\tau}, \bar{b}, \bar{\bar{b}} \).

Next, the results w.r.t. \( \bar{\bar{\tau}} \) of the couples \( (u_1, u_i) \), \( i = 1, 2, 4, 5 \), are respectively of the form

\[
\bar{b}_1 \bar{d}_1 v_1 (\bar{x}, \bar{x}, \bar{b}, \bar{b}), \bar{b}_1 \bar{b}_1 v_2 (\bar{x}, \bar{x}, \bar{b}, \bar{b}) \text{ and } \bar{b}_1 \bar{b}_1 v_3 (\bar{x}, \bar{x}, \bar{b}, \bar{b}) \text{, such that } v_1 = v_2 = v_3 = 0 \text{ by lemma 1(b).}
\]

Finally, the results w.r.t. \( \bar{\bar{\tau}} \) of \( (v_1, v_2) \) and \( (v_1, v_3) \) yield

\[
(\bar{b}_1 - \bar{\bar{\tau}})^2 (\bar{b}_1 + \bar{\bar{\tau}})^2 P_1 P_2 = 0, \quad (\bar{b}_1 - \bar{\bar{\tau}})^2 Q_1 Q_2 = 0,
\]

respectively, where \( P_1, P_2, Q_1, Q_2 \) and \( \bar{b}_1 + \bar{\bar{\tau}} \) are different and irreducible polynomials in \( \bar{b}_1 \) and \( \bar{\bar{\tau}} \). By lemma 1(a) the factor \((\bar{b}_1 - \bar{\bar{\tau}})^2 \) in both left-hand sides of (8) can be striped out. Since the remaining expressions have no common factors, their resultant w.r.t. \( \bar{b}_1 \) must yield
a non-trivial polynomial relation $P(\pi_1) = 0$. Hence $\pi_1 = \frac{\partial_1}{k_1}$ is constant on $U$, such that the spacetime is BI by lemma 2. But then $\partial_1 \propto \frac{\partial_1}{k_1}$ annihilates all $E_{a\mu}$, $\theta_a$, and $\mu$, and hence the same is true for $g_1 \partial_0 = (\partial_1 + g_1 \partial_0) - \partial_1$, which leads to a contradiction with (r3). We conclude that our assumption $g_1 \neq 0$ was false and this finishes the proof.

Remark 1. The purpose of taking the combinations $p_1, p_2, p_3$ instead of $\det_2, \det_3$ and $\partial_0 p_1$ from which they are derived, is a lowering of the degree from the start. This results in a reduction of the total computation time by a factor of $10^2$ to $10^3$.

Theorem. An algebraically general silent universe $U$ with a group $G3$ isometries is necessarily a BI spacetime.

Proof. Since the Petrov type is I, the orthonormal basis $\{e_0, e_\alpha\}$ is uniquely determined (the principal Weyl tetrad); for any point $p \in U$, the dimension of the linear isotropy group of $p$ is 0 and hence the dimension of the orbit $O_p$ under the supposed $G3$ isometry group is 3. This implies the existence of three functionally independent Killing vector fields $K_i$ on $U$, $i = 1, \ldots, 3$. Writing the expansions $K_i = K_i^0 \partial_0 + K_i^\alpha \partial_\alpha = K_i^a \partial_a$ w.r.t. the principal Weyl tetrad in block matrix form:

$$K = \begin{bmatrix} K^0 & K^1 \\ K^2 & K^3 \end{bmatrix} = K^0 \partial_0 + A \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} = M \begin{bmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \tag{9}$$

where $K^0 \in \mathcal{F}(U)^{3 \times 1}$, $A \in \mathcal{F}(U)^{3 \times 3}$ and $M \in \mathcal{F}(U)^{3 \times 4}$, we thus have $\text{rank}(M) = 3$ and hence $\text{rank}(A) = 2$ or 3 everywhere on $U$.

(i) $\text{rank}(A) = 2$. In this case, there exist functions $f_1, f_2, f_3 \in \mathcal{F}(U)$ such that $\sum_{i=1}^3 f_i A_{i\alpha} = 0$, where $A_{i\alpha}$ denotes the $i$th row of $A$. Applying the same linear combination to (9), we get

$$\psi \partial_0 \equiv \left( \sum_{i=1}^3 f_i K_i^0 \right) \partial_0 = \sum_{i=1}^3 f_i K_i,$$

where the coefficient $\psi$ nowhere vanishes on $U$ because $\text{rank}(M) = 3$. Hence, since $K_i$ annihilate all Weyl and expansion eigenvalues, the same is true for $\partial_0$, which leads again to a contradiction with (r3). Hence this case cannot occur.

(ii) $\text{rank}(A) = 3$. In this case $A$ is everywhere invertible on $U$. When applying the inverse, say $C$, on the left of (9), the $\alpha$-component of the resulting equations yields

$$\partial_\alpha + g_\alpha \partial_0 = C_{\alpha i} K_i \tag{10}$$

with $g_\alpha := C_{\alpha i} K_i^0$. Hence $\partial_\alpha + g_\alpha \partial_0$ ($\alpha = 1, 2, 3$) annihilate all scalar invariants, and this implies $g_\alpha \equiv 0$ by lemma 3. Thus all $\partial_\alpha$ annihilate (in particular) $\theta_{\alpha+1}$ and $\theta_{\alpha-1}$, which forces all $r_\alpha$ and $q_\alpha$ to vanish by (5) and (r1); the result follows by (r2).

2 See, e.g., [12], p 159. More explicitly, this can be checked by specializing, e.g., $s_1 = 2$ in the remaining expressions: the degrees of $b_1$ do not drop, and calculating the specialized resultant gives a non-zero number, such that the original resultant cannot be identically 0 (see also [10] for this argument).

3 See [1], sections 8.3, 8.4 and 9.2 for an account.
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References

[1] Kramer D, Stephani H, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)
[2] Matarrese S, Pantano O and Saez D 1994 Phys. Rev. D 72 320
[3] Matarrese S, Pantano O and Saez D 1993 Phys. Rev. D 47 1311
[4] Barnes A and Rowlingson R R 1989 Class. Quantum Grav. 6 949
[5] Szekeres P 1975 Commun. Math. Phys. 41 56
[6] Ellis G F R 1967 J. Math. Phys. 8 1171
[7] Sopuerta C F 1997 Phys. Rev. D 55 5936
[8] van Elst H, Uggla C, Lesame W M and Ellis G F R 1997 Class. Quantum Grav. 14 1151
[9] Van den Bergh N and Wylleman L 2004 Class. Quantum Grav. 21 2291
[10] Mars M 1999 Class. Quantum Grav. 16 3245
[11] Bruni M, Matarrese S and Pantano O 1995 Phys. Rev. Lett. 74 1916
[12] Cox D, Little J and O’Shea D 1992 Ideals, Varieties and Algorithms (Berlin: Springer)