THE CORRELATIONS OF FAREY FRACTIONS

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Abstract

It is proved that all correlations of the sequence of Farey fractions exist and explicit formulas are provided for the correlation measures.

1. Introduction

By the classical contributions of Franel and Landau ([7, 13], see also [6]), the existence of zero-free regions $1 - \delta_0 < \text{Re } s < 1$ for the Riemann zeta function, and in particular the Riemann hypothesis, are known to be equivalent to quantitative statements about the uniform distribution of Farey fractions. Although these problems remain widely open, the study of the distribution of Farey fractions is of independent interest. Ideas and techniques from this area, especially Weil–Salié type estimates on Kloosterman sums, turned out to be useful in proving sharp asymptotic formulas in problems from various areas of mathematics [2, 4, 5, 11, 16].

In the study of the spacing statistics of a sequence, correlations play an important role. There are very few sequences of interest for which one could establish the existence of correlation measures, and many of them are conditional, as in the important case of the zeros of the Riemann zeta function, or more general $L$-functions [10, 12, 14, 15]. In this paper we study the correlations of the sequence $\mathcal{F}_Q$ of Farey fractions of order $Q$ in $[0,1]$ as $Q \to \infty$.

Let $\nu \geq 1$ be an integer and let $F$ be a finite set of cardinality $N$ in $[0,1]$. The $\nu$-level correlation measure $\mathcal{R}^{(\nu)}_F(B)$ of a box $B \subset \mathbb{R}^{\nu-1}$ is defined as

$$\frac{1}{N} \# \left\{ (x_i)_{i=1}^{\nu} \in F^\nu : x_i \text{ distinct}, (x_1 - x_2, x_2 - x_3, \ldots, x_{\nu-1} - x_\nu) \in \frac{1}{N} B + \mathbb{Z}^{\nu-1} \right\}.$$

When $\nu = 2$, the pair correlation measure of an interval $I \subset \mathbb{R}$ is

$$\mathcal{R}^{(2)}_F(I) = \frac{1}{N} \# \left\{ (x, y) \in F^2 : x \neq y \text{ and } x - y \in \frac{1}{N} I + \mathbb{Z} \right\}.$$

Suppose that $(F_n)_n$ is an increasing sequence of finite subsets of $[0,1]$ and that

$$\mathcal{R}^{(\nu)}(B) = \lim_n \mathcal{R}^{(\nu)}_{F_n}(B)$$

exists for every box $B \subset \mathcal{R}^{\nu-1}$. Then $\mathcal{R}^{(\nu)}$ is called the $\nu$-level correlation measure of $(F_n)_n$. The measure $\mathcal{R}^{(2)}$ is called the pair correlation measure of $(F_n)_n$. If

$$\mathcal{R}^{(\nu)}(B) = \int_B g(\nu)(x_1, \ldots, x_{\nu-1}) \, dx_1 \ldots dx_{\nu-1},$$

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then $g_\nu$ is called the $\nu$-level correlation function of $(F_n)_n$. We denote

$$R_F^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1}) = 2^{-\nu+1}R_F^{(\nu)}\left(\prod_{j=1}^{\nu-1} [-\lambda_j, \lambda_j]\right).$$

Given $\Lambda > 0$ and two $(\nu - 1)$-tuples $A = (A_1, \ldots, A_{\nu-1})$, $B = (B_1, \ldots, B_{\nu-1}) \in \mathbb{Z}^{\nu-1}_+ = \{1, 2, \ldots\}^{\nu-1}$, we consider the map defined by

$$T_{A,B}(x, y) = \frac{3}{\pi^2} \left( \frac{B_1}{y(A_1 y - B_1 x)}, \ldots, \frac{B_{\nu-1}}{y(A_{\nu-1} y - B_{\nu-1} x)} \right)$$

(1.1) and the set

$$\Omega_{A,B,\Lambda} = \left\{ (x, y) : 0 < x \leq y \leq 1, y \geq \frac{3}{\pi^2 \Lambda}, 0 < A_j y - B_j x \leq 1 \right\}.$$  

(1.2)

We also consider the linear transformation on $\mathbb{R}^{\nu-1}$ defined by

$$T(x_1, x_2, \ldots, x_{\nu-1}) = (x_1 - x_2, x_2 - x_3, \ldots, x_{\nu-2} - x_{\nu-1}, x_{\nu-1}),$$

(1.3)

whose inverse is

$$T^{-1}(y_1, \ldots, y_{\nu-1}) = (y_1 + \ldots + y_{\nu-1}, y_2 + \ldots + y_{\nu-1}, \ldots, y_{\nu-2} + y_{\nu-1}, y_{\nu-1}).$$

The main result of this paper is the existence of correlation measures for the sequence $(F_Q)_Q$ of Farey fractions. This complements previous results [1, 9] on the existence and computation of consecutive spacings between Farey fractions.

**Theorem 1.** All $\nu$-level correlation measures of the sequence $(F_Q)_Q$ exist. Moreover, for any box $B \subset (0, \Lambda)^{\nu-1}$,$$
R^{(\nu)}(B) = 2 \sum_{A, B \in \mathbb{Z}^{\nu-1}_+}^{(A_j, B_j) = 1} \text{area}(\Omega_{A,B,\Lambda} \cap T_{A,B}^{-1}(T^{-1}B)).$$

(1.4)

For such a compact box $B$, $T^{-1}B$ is a compact set which does not contain the origin. Since $0 < y(A_j y - B_j x) \leq 1$ for all $(x, y) \in \Omega_{A,B,\Lambda}$, all denominators in (1.1) are smaller than or equal to 1; thus the sum in (1.4) has finitely many nonzero terms.

When $\nu = 2$, one gets a more explicit expression of the pair correlation.

**Theorem 2.** The pair correlation function of $(F_Q)_Q$ is given by

$$g_2(\lambda) = \frac{6}{\pi^2 \lambda^2} \sum_{1 \leq k < \pi^2 \lambda/3} \varphi(k) \log \frac{\pi^2 \lambda}{3k}.$$  

(1.5)

Moreover, as $\lambda \to \infty$,

$$g_2(\lambda) = 1 + O(\lambda^{-1}).$$  

(1.6)

The graph of $g_2$ is compared with that of $g_{\text{GUE}}$ in Figure 1. Both functions vanish at zero, showing repulsion between the elements of the sequence. The strongest repulsion occurs for $g_2$ and is reflected by its vanishing on the whole interval $[0, 3/\pi^2]$. 
2. An expression of the smooth correlation sums

It is well known that the cardinality of $\mathcal{F}_Q$ is

$$N = N_Q = \sum_{k=1}^{Q} \varphi(k) = \frac{3Q^2}{\pi^2} + O(Q \log Q). \quad (2.1)$$

An application of the Korobov–Vinogradov exponential sum estimates gives the better estimate (see [17])

$$N_Q = \frac{3Q^2}{\pi^2} + O(Q(\log Q)^{2/3}(\log \log Q)^{4/3}). \quad (2.2)$$

If $\gamma = a/q$ and $\gamma' = a'/q'$ are two distinct elements in $\mathcal{F}_Q$, then $|\gamma' - \gamma| \geq 1/qq' \geq 1/Q^2$. It now follows from (2.1) that if $\min |\lambda_j| < 3/\pi^2$, then $\mathcal{R}^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1}) = 0$.

We also have for any box $B \subset \mathcal{R}^n-1$,

$$\mathcal{R}^{(\nu)}(B) = \mathcal{R}^{(\nu)}(B \setminus \{ (\lambda_1, \ldots, \lambda_{\nu-1}) : \min |\lambda_j| < 3/\pi^2 \}).$$

We denote the M"{o}bius function by $\mu$ and the least common multiple $d_1d_2/(d_1, d_2)$ of two positive integers $d_1$ and $d_2$ by $[d_1, d_2]$. We also set

$$e(t) = \exp(2\pi it), \quad t \in \mathbb{R},$$

and

$$x \cdot y = x_1y_1 + \ldots + x_ny_n, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

Exponential sums of the form

$$\sum_{\gamma \in \mathcal{F}_Q} e(r\gamma), \quad r \in \mathbb{Z},$$

play an important role in our computations. They are connected with the function

$$M(x) = \sum_{n \leq x} \mu(n)$$
by the formula (cf. \cite[formula (1), at p. 264]{f1})
\[
\sum_{\gamma \in \mathcal{F}_Q} e(r\gamma) = \sum_{d \geq 1 \atop d \mid r} dM\left(\frac{Q}{d}\right), \quad r, Q \in \mathbb{Z}, \; Q \geq 1. \tag{2.3}
\]

We fix an integer \( \nu \geq 2 \), a smooth real-valued function \( H \) on \( \mathbb{R}^{\nu-1} \) such that \( \text{supp}(H) \subset (0, \Lambda)^{\nu-1} \), and \( 0 < \Lambda' < \Lambda \) such that \( \text{supp}(H) \subset (0, \Lambda')^{\nu-1} \). The Fourier transform of \( H \) is defined by
\[
\hat{H}(x) = \int_{\mathbb{R}^{\nu-1}} H(y)e(-x \cdot y) \, dy, \quad x \in \mathbb{R}^{\nu-1}.
\]

We consider the \( \mathbb{Z}^{\nu-1} \)-periodic function \( f \) given by
\[
f(y) = f_Q(y) = \sum_{r \in \mathbb{Z}^{\nu-1}} H(N(y + r)), \quad y \in \mathbb{R}^{\nu-1}, \tag{2.4}
\]
and the smooth \( \nu \)-level correlation sum defined by
\[
S_\nu = R^{(\nu)}(Q, H) = \frac{1}{N} \sum_{\gamma_i \in \mathcal{F}_Q \text{ distinct}} f(\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \ldots, \gamma_{\nu-1} - \gamma_\nu). \tag{2.5}
\]

We will show that there exists a continuous function \( g_\nu : \mathbb{R}^{\nu-1} \to [0, \infty) \) such that
\[
\lim_{Q \to \infty} R^{(\nu)}(Q, H) = \int_{\mathbb{R}^{\nu-1}} H(x)g_\nu(x) \, dx. \tag{2.6}
\]

The Fourier coefficients in the Fourier series
\[
f(y) = \sum_{r \in \mathbb{Z}^{\nu-1}} c_r e(r \cdot y) \tag{2.7}
\]
of \( f \) are given by
\[
c_r = \int_{[0,1)^{\nu-1}} f(y)e(-r \cdot y) \, dy = \int_{[0,1)^{\nu-1}} e(-r \cdot y) \sum_{n \in \mathbb{Z}^{\nu-1}} H(N(y + n))
\]
\[
= \sum_{n \in \mathbb{Z}^{\nu-1}} \int_{[0,1)^{\nu-1}} e(-r \cdot y)H(N(y + n)) \, dy
\]
\[
= \sum_{n \in \mathbb{Z}^{\nu-1}} \int_{n + [0,1)^{\nu-1}} e(-r \cdot (u - n))H(Nu) \, du
\]
\[
= \int_{\mathbb{R}^{\nu-1}} e(-r \cdot u)H(Nu) \, du = \frac{1}{N^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e\left(-\frac{r \cdot y}{N}\right)H(y) \, dy
\]
\[
= \frac{1}{N^{\nu-1}} \hat{H}\left(\frac{1}{N}r\right).
\]
Since $H$ is supported on $(0, \Lambda)^{\nu-1}$, we can remove in (2.5) the condition that $\gamma_1, \ldots, \gamma_\nu$ are distinct, and gather for $Q$ large

\[
S_\nu = \frac{1}{N} \sum_{\gamma_1, \ldots, \gamma_\nu \in \mathcal{F}_Q} H(N(r_1 + \gamma_1 - \gamma_2, r_2 + \gamma_2 - \gamma_3, \ldots, r_{\nu-1} + \gamma_{\nu-1} - \gamma_\nu))
\]

\[
= \frac{1}{N} \sum_{\gamma_1, \ldots, \gamma_\nu \in \mathcal{F}_Q} f(\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \ldots, \gamma_{\nu-1} - \gamma_\nu)
\]

\[
= \frac{1}{N} \sum_{\gamma_1, \ldots, \gamma_\nu \in \mathcal{F}_Q} c_r e(r \cdot (\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \ldots, \gamma_{\nu-1} - \gamma_\nu))
\]

\[
= \frac{1}{N} \sum_{\gamma_1, \ldots, \gamma_\nu \in \mathcal{F}_Q} c_r e(r_1 \gamma_1 e((r_2 - r_1) \gamma_2) \ldots e((r_{\nu-1} - r_{\nu-2}) \gamma_{\nu-1}) e(r_{\nu-1} \gamma_\nu)).
\]

Equalities (2.9) and (2.3) further yield

\[
S_\nu = \frac{1}{N} \sum_{r=(r_1, \ldots, r_{\nu-1}) \in \mathbb{Z}^{\nu-1}} c_r \sum_{d_1 | r_1 \atop d_2 | r_2 - r_1 \atop \ldots \atop d_{\nu-1} | r_{\nu-1} - r_{\nu-2} \atop d_\nu | r_{\nu-1}} d_1 \ldots d_\nu M\left(\frac{Q}{d_1}\right) \ldots M\left(\frac{Q}{d_\nu}\right)
\]

\[
= \frac{1}{N} \sum_{1 \leq d_1, \ldots, d_\nu \leq Q} d_1 \ldots d_\nu M\left(\frac{Q}{d_1}\right) \ldots M\left(\frac{Q}{d_\nu}\right) \sum_{r \in \mathbb{Z}^{\nu-1}} c_r.
\]

The divisibility conditions $d_1 | r_1, d_2 | r_2 - r_1, d_3 | r_3 - r_2, \ldots, d_{\nu-1} | r_{\nu-1} - r_{\nu-2}, d_\nu | r_{\nu-1}$, read as

\[
r_1 = \ell_1 d_1,
\]

\[
r_2 = \ell_1 d_1 + \ell_2 d_2,
\]

\[
\ldots
\]

\[
r_{\nu-1} = \ell_1 d_1 + \ldots + \ell_{\nu-1} d_{\nu-1} = \ell_\nu d_\nu,
\]

for some $\ell_1, \ldots, \ell_\nu \in \mathbb{Z}$. Thus, putting $d = (d_1, \ldots, d_{\nu-1}) \in \square_Q^{\nu-1} := [1, Q]^{\nu-1} \cap \mathbb{Z}^{\nu-1}$, $\ell = (\ell_1, \ldots, \ell_{\nu-1})$, we can further write

\[
S_\nu = \frac{1}{N} \sum_{d \in \square_Q^{\nu-1}} d_1 \ldots d_{\nu-1} M\left(\frac{Q}{d_1}\right) \ldots M\left(\frac{Q}{d_{\nu-1}}\right) \sum_{\ell \in \mathbb{Z}^{\nu-1}} c(d_1 \ell_1, d_1 \ell_1 + d_2 \ell_2, \ldots, d_1 \ell_1 + \ldots + d_{\nu-1} \ell_{\nu-1}) \sum_{d_\nu | d_1 \ell_1 + \ldots + d_{\nu-1} \ell_{\nu-1}} d_\nu M\left(\frac{Q}{d_\nu}\right).
\]
When $\nu = 2$, we simply get

$$S_2 = \frac{1}{N} \sum_{r \in \mathbb{Z}} c_r \sum_{\gamma_1, \gamma_2 \in \mathbb{F}_Q} e(r(\gamma_2 - \gamma_1)) = \frac{1}{N} \sum_{r \in \mathbb{Z}} c_r \left| \sum_{\gamma \in \mathbb{F}_Q} e(r \gamma) \right|^2$$

$$= \frac{1}{N} \sum_{r \in \mathbb{Z}} c_r \left( \sum_{1 \leq d \leq Q} d M \left( \frac{Q}{d} \right) \right)^2$$

$$= \frac{1}{N} \sum_{r \in \mathbb{Z}} c_r \left( \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M \left( \frac{Q}{d_1} \right) M \left( \frac{Q}{d_2} \right) \right)$$

$$= \frac{1}{N} \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M \left( \frac{Q}{d_1} \right) M \left( \frac{Q}{d_2} \right) \sum_{\ell \in \mathbb{Z}} c_{[d_1, d_2]}.$$ 

(2.12)

3. Proof of Theorem 1

Making use of (2.3) and of the fact that $\gamma \mapsto -\gamma$ is a bijection on $\mathbb{F}_Q$, we can write the inner sum in (2.11) as

$$\sum_{\gamma \in \mathbb{F}_Q} e(\gamma d \cdot \ell) = \sum_{\gamma \in \mathbb{F}_Q} e(-\gamma d \cdot \ell).$$

Also taking into account (2.8), we see that the contribution of the two inner sums in (2.11) equals

$$\sum_{\ell \in \mathbb{Z}^{\nu-1}} c(d_1 \ell_1, d_2 \ell_2, \ldots, d_1 \ell_1 + \ldots + d_{\nu-1} \ell_{\nu-1}) e(-\gamma d \cdot \ell)$$

$$= \sum_{\ell \in \mathbb{Z}^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e \left( -\gamma \sum_{i=1}^{\nu-1} d_i \ell_i - \sum_{j=1}^{\nu-1} x_j (d_1 \ell_1 + \ldots + d_j \ell_j) \right) H(Nx) \, dx$$

$$= \sum_{\ell \in \mathbb{Z}^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e \left( -\sum_{i=1}^{\nu-1} d_i \ell_i (x_i + \ldots + x_{\nu-1} + \gamma) \right) H(Nx) \, dx$$

$$= \sum_{\ell \in \mathbb{Z}^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e \left( -\sum_{i=1}^{\nu-1} d_i \ell_i (x_i + \ldots + x_{\nu-1}) \right) H(N(x_1, \ldots, x_{\nu-2}, x_{\nu-1} - \gamma)) \, dx.$$ 

Taking $y_i = d_i (x_i + \ldots + x_{\nu-1})$, $i = 1, \ldots, \nu - 1$, that is,

$$\begin{align*}
x_1 &= \frac{y_1}{d_1} - \frac{y_2}{d_2} \\
x_2 &= \frac{y_2}{d_2} - \frac{y_3}{d_3} \\
\vdots \\
x_{\nu-2} &= \frac{y_{\nu-2}}{d_{\nu-2}} - \frac{y_{\nu-1}}{d_{\nu-1}} \\
x_{\nu-1} &= \frac{y_{\nu-1}}{d_{\nu-1}}
\end{align*}$$
These inequalities first give
\[ H_{N;d,\gamma}(y) = H\left(N\left(\frac{y_1}{d_1} - \frac{y_2}{d_2}, \ldots, N\left(\frac{y_{\nu-2}}{d_{\nu-2}} - \frac{y_{\nu-1}}{d_{\nu-1}}; N\left(\frac{y_{\nu-1}}{d_{\nu-1}} - \gamma\right)\right)\right), \]
with \(d = (d_1, \ldots, d_{\nu-1}) \in \mathbb{N}^{\nu-1}\), \(y = (y_1, \ldots, y_{\nu-1}) \in \mathbb{R}^{\nu-1}\), \(\gamma \in \mathcal{F}_Q\), the previous expression becomes
\[ \frac{1}{d_1 \ldots d_{\nu-1}} \sum_{\gamma \in \mathcal{F}_Q} \sum_{t \in \mathbb{Z}^{\nu-1}} \int_{\mathbb{R}^{\nu-1}} e(-\ell \cdot y) H_{N;d,\gamma}(y) \, dy \]
\[ = \frac{1}{d_1 \ldots d_{\nu-1}} \sum_{\gamma \in \mathcal{F}_Q} \sum_{t \in \mathbb{Z}^{\nu-1}} \widehat{H}_{N;d,\gamma}(\ell). \]
Applying Poisson summation to the inner sum, this further equals
\[ \frac{1}{d_1 \ldots d_{\nu-1}} \sum_{\gamma \in \mathcal{F}_Q} \sum_{t \in \mathbb{Z}^{\nu-1}} H_{N;d,\gamma}(\ell), \]
which we insert back into (2.11) to get
\[ S_\nu = \frac{1}{N} \sum_{d \in \mathbb{N}^{\nu-1}} M\left(\frac{Q}{d_1}\right) \ldots M\left(\frac{Q}{d_{\nu-1}}\right) \sum_{\gamma \in \mathcal{F}_Q} \sum_{t \in \mathbb{Z}^{\nu-1}} H_{N;d,\gamma}(\ell). \]
The support of \(H\) is included in \((0, \Lambda')^{\nu-1}\); thus we necessarily have
\[ 0 < N\left(\frac{\ell_j}{d_j} - \frac{\ell_{j+1}}{d_{j+1}}\right) < \Lambda', \quad j = 1, \ldots, \nu - 2. \]
These inequalities first give \(\ell_j d_{j+1} - \ell_{j+1} d_j \geq 1\), and second,
\[ \Lambda' > \frac{N(\ell_j d_{j+1} - \ell_{j+1} d_j)}{d_j d_{j+1}} \geq \frac{N}{d_j d_{j+1}}. \]
Therefore, for all \(Q \geq Q_0(\Lambda')\), we find that
\[ \frac{Q^2}{d_j d_{j+1}} = \frac{Q^2}{N} \cdot \frac{N}{d_j d_{j+1}} < \frac{Q^2 \Lambda'}{N} < c_\Lambda = \frac{\pi^2 \Lambda}{3}. \]
Here both \(Q/d_j\) and \(Q/d_{j+1}\) are at least 1. Hence each of them is at most \(c_\Lambda\). It follows that, for \(Q \geq Q_0(\Lambda')\), we have
\[ 1 \leq \frac{Q}{d_j} \leq c_\Lambda, \quad j = 1, \ldots, \nu - 1. \quad (3.1) \]
The same reasoning for \(j = \nu - 1\) gives, for \(Q \geq Q_0(\Lambda')\),
\[ \frac{Q}{q} \leq c_\Lambda. \quad (3.2) \]
Therefore,
\[ S_\nu = \frac{1}{N} \sum_{d \in \mathbb{N}^{\nu-1}} \sum_{t \in \mathbb{Z}^{\nu-1}} \sum_{1 \leq r_j \leq Q/d_j} \mu(r_1) \ldots \mu(r_{\nu-1}) \sum_{a/q \in \mathcal{F}_Q} \sum_{q \geq Q / c_\Lambda} H_{N;d,a/q}(\ell) \]
\[ = \frac{1}{N} \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \ldots \mu(r_{\nu-1}) \sum_{1 \leq d_j \leq Q/r_j} \sum_{\ell \in \mathbb{Z}^{\nu-1}} \sum_{a/q \in \mathcal{F}_Q} \sum_{q \geq Q / c_\Lambda} H_{N;d,a/q}(\ell). \]
The inner sum above is given by
\[
\sum_{a/q \in \mathcal{F}_q, q \geq Q/c_\Lambda} H\left(N\left(\frac{\ell_1}{d_1} - \frac{\ell_2}{d_2}\right), \ldots, N\left(\frac{\ell_{\nu-2}}{d_{\nu-2}} - \frac{\ell_{\nu-1}}{d_{\nu-1}}\right), N\left(\frac{\ell_{\nu-1}}{d_{\nu-1}} - \frac{a}{q}\right)\right).
\]

For \(j = 1, \ldots, \nu - 1\), we set
\[
\Delta_j = q\ell_j - ad_j. \tag{3.3}
\]
Since \(\text{supp}(H) \subset (0, \Lambda']^{\nu-1}\), we have
\[
0 < \frac{N\Delta_j}{qd_j} = N\left(\frac{\ell_j}{d_j} - \frac{a}{q}\right) = N\left(\frac{\ell_j}{d_j} - \frac{\ell_{j+1}}{d_{j+1}}\right) + \ldots + N\left(\frac{\ell_{\nu-1}}{d_{\nu-1}} - \frac{a}{q}\right) < (\nu - j)\Lambda'.
\]
Hence \(\Delta_j \geq 1\) and, for \(Q \geq Q_0(\Lambda')\),
\[
\Delta_j \leq \frac{qd_j(\nu - j)\Lambda'}{N} \leq \frac{Q^2(\nu - j)\Lambda'}{N} \leq (\nu - j)c_\Lambda;
\]
therefore
\[
1 \leq \Delta_1, \ldots, \Delta_{\nu-1} \leq (\nu - 1)c_\Lambda.
\]
Note also from (3.3) that \(\ell_j\) is uniquely determined as
\[
\ell_j = \frac{\Delta_j + ad_j}{q}.
\]
This gives in turn that
\[
\Delta_j = \frac{\ell_j}{d_j} - \frac{\ell_{j+1}}{d_{j+1}} = \frac{\Delta_j + ad_j}{qd_j} - \frac{\Delta_{j+1} + ad_{j+1}}{qd_{j+1}} = \frac{1}{q} \left(\frac{\Delta_j}{d_j} - \frac{\Delta_{j+1}}{d_{j+1}}\right), \quad j = 1, \ldots, \nu - 2.
\]
We also have
\[
\frac{\ell_{\nu-1}}{d_{\nu-1}} - \frac{a}{q} = \frac{\Delta_{\nu-1}}{qd_{\nu-1}}.
\]
Here \(d_j\) needs to satisfy the congruence
\[
d_j = -\bar{a}\Delta_j \pmod{q}, \quad j = 1, \ldots, \nu - 1,
\]
where \(\bar{a}\) denotes the integer between 1 and \(q\) which satisfies \(a\bar{a} = 1 \pmod{q}\).

In summary, we infer that
\[
\mathcal{S}_\nu = \frac{1}{N} \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \ldots \mu(r_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)c_\Lambda} \sum_{a/q \in \mathcal{F}_q, q \geq Q/c_\Lambda} \sum_{1 \leq d_j \leq Q/r_j, d_j = -\bar{a}\Delta_j \pmod{q}} H\left(N\left(\frac{\Delta_1}{d_1} - \frac{\Delta_2}{d_2}, \ldots, \frac{\Delta_{\nu-2}}{d_{\nu-2}} - \frac{\Delta_{\nu-1}}{d_{\nu-1}}\right)\right). \tag{3.4}
\]
To simplify this expression, we consider the linear transformation \(T\) defined by (1.3). The function \(\tilde{H} = H \circ T\) is smooth, \(\text{supp}(\tilde{H}) \subset (0, (\nu - 1)\Lambda'] \times \ldots \times (0, \Lambda']\), and the expression in (3.4) becomes
\[
\frac{1}{N} \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \ldots \mu(r_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)c_\Lambda} \sum_{a/q \in \mathcal{F}_q, q \geq Q/c_\Lambda} \sum_{1 \leq d_j \leq Q/r_j, d_j = -\bar{a}\Delta_j \pmod{q}} \tilde{H}\left(N\left(\frac{\Delta_1}{d_1}, \frac{\Delta_2}{d_2}, \ldots, \frac{\Delta_{\nu-1}}{d_{\nu-1}}\right)\right).
\]
When \( c_\Lambda < 1 \), \( S_\nu = 0 \). For \( j = 1, \ldots, \nu - 1 \), we define

\[
e_j = \frac{d_j + \bar{a}\Delta_j}{q}.
\]

The congruence \( d_j = -\bar{a}\Delta_j \pmod{q} \) shows that each \( e_j \) is an integer. Moreover, \( d_j, \bar{a}, \Delta_j \) are all greater than or equal to 1, so \( e_j \geq 1 \). On the other hand, using (3.2), we see that \( d_j/q \leq Q/q \leq Q/q \leq c_\Lambda \) and \( \bar{a}\Delta_j/q \leq \Delta_j \leq (\nu - 1)c_\Lambda \), leading to

\[
1 \leq e_j \leq \nu c_\Lambda, \quad j = 1, \ldots, \nu - 1.
\]

With \( q, a, \Delta_j \) fixed, each value of \( e_j \) uniquely determines a value of \( d_j \), precisely

\[
d_j = qe_j - \bar{a}\Delta_j.
\]

Moreover, with \( e_j \) and \( \Delta_j \) fixed and \( a/q \) variable in \( F_Q \), \( a \) and \( q \) need to satisfy some extra conditions in order for \( d_j \) to belong to the set \( \{1, \ldots, \lfloor Q/r_j \rfloor \} \). Using (3.1), we infer that \( a \) and \( q \) necessarily fulfil

\[
\frac{Q}{c_\Lambda r_j} \leq \frac{Q}{c_\Lambda} \leq qe_j - \bar{a}\Delta_j \leq \frac{Q}{r_j}, \quad j = 1, \ldots, \nu - 1.
\]

We consider the convex region

\[
\Omega_{r,e,\Delta} = \left\{(x, y) : 0 < x \leq y \leq 1, \ y \geq \frac{1}{c_\Lambda}, \ \frac{1}{c_\Lambda r_j} \leq e_jy - \Delta_jx \leq \frac{1}{r_j}\right\},
\]

and the functions \( f_{e,\Delta} \), \( f_{e,\Delta}^{(j)} \) defined on \( Q\Omega_{r,e,\Delta} \) by

\[
f_{e,\Delta}(b, q) = \bar{H}(f_{e,\Delta}^{(1)}(b, q), \ldots, f_{e,\Delta}^{(\nu-1)}(b, q)),
\]

\[
f_{e,\Delta}^{(j)}(b, q) = \frac{N\Delta_j}{q(qe_j - b\Delta_j)}, \quad j = 1, \ldots, \nu - 1.
\]

We write \( b = \bar{a} \), and remark that as \( a/q \) runs over \( F_Q \) with \( q \geq Q/c_\Lambda \), so does \( b/q \).

Thus

\[
S_\nu = \frac{1}{N} \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \cdots \mu(r_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)c_\Lambda} \sum_{(b, q) \in(Q\Omega_{r,e,\Delta})} f_{e,\Delta}(b, q).
\]

By [4, Corollary 1], the inner sum above can be written as

\[
\frac{6}{\pi^2} \int_{Q\Omega_{r,e,\Delta}} f_{e,\Delta} + O\left(\frac{\|f_{e,\Delta}\|_\infty Q^2 \log Q + \|f_{e,\Delta}\|_\infty Q \log Q}{N}\right). \tag{3.5}
\]

Since \( H = \bar{H} \circ T^{-1} \), it is clear that \( \|f_{e,\Delta}\|_\infty \leq \|H\|_\infty \). Using the definition of \( \Omega_{r,e,\Delta} \), we also find for every \( j = 1, \ldots, \nu - 1 \) that

\[
\|Df_{e,\Delta}\|_\infty = \sup_{(b, q) \in(Q\Omega_{r,e,\Delta})} \left(\left| \frac{\partial f_{e,\Delta}^{(j)}}{\partial b} \right| + \left| \frac{\partial f_{e,\Delta}^{(j)}}{\partial q} \right|\right) \ll \Lambda \sum_{(b, q) \in(Q\Omega_{r,e,\Delta})} \frac{1}{q(qe_j - b\Delta_j)^2} \ll \frac{1}{Q},
\]

showing that the error term in (3.5) is \( \ll_H (\log Q)/Q \). Rescaling to \( (u, v) = (Qx, Qy) \), we find that

\[
S_\nu = \frac{6Q^2}{\pi^2 N} \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \cdots \mu(r_{\nu-1}) \sum_{1 \leq \Delta_j \leq (\nu-1)c_\Lambda} I_{r,e,\Delta} + O_H\left(\frac{\log Q}{Q}\right), \tag{3.6}
\]
where this time we put
\[ I_{r,e,\Delta} = \iint_{\Omega_{r,e,\Delta}} g_{e,\Delta}, \]
\[ g_{e,\Delta}(x, y) = \tilde{H}(g_{e,\Delta}^{(1)}(x, y), \ldots, g_{e,\Delta}^{(\nu - 1)}(x, y)), \quad \text{(3.7)} \]
\[ g_{e,\Delta}^{(j)}(x, y) = \frac{N\Delta_j}{Q^2y(e_j y - \Delta_j x)}, \quad j = 1, \ldots, \nu - 1. \]

Using (2.1) and the inequality
\[ |\tilde{H}(v) - \tilde{H}(w)| \leq \|\tilde{H}'\| |v - w| \leq 2\|H'\| |v - w|, \]
we see that formula (3.6) holds true after \( g_{e,\Delta}^{(j)} \) is replaced by
\[ g_{e,\Delta}^{(j)}(x, y) = \frac{3\Delta_j}{\pi^2y(e_j y - \Delta_j x)}, \quad j = 1, \ldots, \nu - 1, \quad \text{(3.8)} \]
in the formula for \( g_{e,\Delta} \). Therefore we infer that
\[ S_\nu = 2 \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \ldots \mu(r_{\nu - 1}) \sum_{\substack{1 \leq \Delta_j \leq (\nu - 1)c_\Lambda \\leq \nu c_\Lambda}} I_{r,e,\Delta} + O_H\left(\frac{\log Q}{Q}\right), \quad \text{(3.9)} \]

where \( I_{r,e,\Delta} \) and \( g_{e,\Delta} \) are as in (3.7) and \( g_{e,\Delta}^{(j)} \) is as in (3.8).

Next, we notice that the region \( \Omega_{r,e,\Delta} \) can be extended to
\[ \tilde{\Omega}_{r,e,\Delta} = \left\{(x, y) : 0 \leq x \leq y \leq 1, \; y \geq \frac{1}{c_\Lambda}, \; 0 < e_j y - \Delta_j x \leq \frac{1}{r_j}\right\}, \]
without changing the terms \( I_{r,e,\Delta} \) in (3.9). Indeed, if \( (x, y) \in \tilde{\Omega}_{r,e,\Delta} \setminus \Omega_{r,e,\Delta} \), there is \( j \) for which \( |e_j y - \Delta_j x| < 1/c_\Lambda r_j \), and thus
\[ |g_{e,\Delta}^{(j)}(x, y)| \geq \frac{3\Delta_j}{\pi^2/c_\Lambda r_j} = \frac{3c_\Lambda \Delta_j r_j}{\pi^2} \geq \frac{3c_\Lambda}{\pi^2} = \Lambda. \]
This yields \( g_{e,\Delta} = 0 \) on \( \tilde{\Omega}_{r,e,\Delta} \setminus \Omega_{r,e,\Delta} \). Hence
\[ S_\nu = 2 \sum_{1 \leq r_j \leq c_\Lambda} \mu(r_1) \ldots \mu(r_{\nu - 1}) \sum_{\substack{1 \leq \Delta_j \leq (\nu - 1)c_\Lambda \\leq \nu c_\Lambda}} \iint_{\tilde{\Omega}_{r,e,\Delta}} g_{e,\Delta} + O_H\left(\frac{\log Q}{Q}\right). \quad \text{(3.10)} \]

We take \( A_j = e_j r_j, \; B_j = \Delta_j r_j, \; A = (A_1, \ldots, A_{\nu - 1}) \) and \( B = (B_1, \ldots, B_{\nu - 1}) \), and consider regions \( \Omega_{A,B,\Lambda} \) and maps \( T_{A,B} \) as defined in (1.2) and (1.1). We put
\[ I_{A,B,\Lambda} = \iint_{\Omega_{A,B,\Lambda}} \tilde{H} \circ T_{A,B}. \]
Then (3.10) yields
\[ S_\nu = 2 \sum_{\substack{1 \leq A_j \leq (\nu - 1)c_\Lambda^2 \\leq \nu c_\Lambda^2 \\leq 1 \leq B_j \leq \nu c_\Lambda^2}} I_{A,B,\Lambda} \sum_{r_j \mid (A_j, B_j)} \mu(r_1) \ldots \mu(r_{\nu - 1}) + O_H\left(\frac{\log Q}{Q}\right) \]
\[ = 2 \sum_{\substack{1 \leq A_j \leq (\nu - 1)c_\Lambda^2 \\leq \nu c_\Lambda^2 \\leq 1 \leq B_j \leq \nu c_\Lambda^2 \\leq (A_j, B_j) = 1}} I_{A,B,\Lambda} + O_H\left(\frac{\log Q}{Q}\right). \]
The Correlations of Farey Fractions

Approximating pointwise the characteristic function of a box \( B \subset (0, \Lambda)^{\nu-1} \) from above and from below by smooth functions with compact support in \((0, \Lambda)^{\nu-1}\), we conclude that

\[
\mathcal{R}^{(\nu)}(B) = \sum_{A, B \in \mathbb{Z}_{+}^{\nu-1}} \int_{\Omega_{A, B, \Lambda}} \chi_B \circ T \circ T_{A,B} = \sum_{A, B \in \mathbb{Z}_{+}^{\nu-1}} \text{area}(\Omega_{A, B, \Lambda} \cap T^{-1}_{A,B}(T^{-1}B)).
\]

4. Proof of Theorem 2

To prove (1.5), we return to (2.12) and consider for each \( y > 0 \) the function

\[
H_y(x) = \frac{1}{y} H\left(\frac{x}{y}\right), \quad x \in \mathbb{R}.
\]

Then

\[
\widehat{H}_y(z) = \widehat{H}(yz),
\]

and from (2.8) we find that the inner sum in (2.12) can be written as

\[
\frac{1}{N} \sum_{\ell \in \mathbb{Z}} \hat{H}\left(\frac{\ell[d_1, d_2]}{N}\right) = \frac{1}{N} \sum_{\ell \in \mathbb{Z}} H_{[d_1, d_2]}(\ell N) / N(\ell).
\]

By Poisson’s summation formula, we have

\[
\sum_{\ell \in \mathbb{Z}} H_{[d_1, d_2]}(\ell N) / N(\ell) = \sum_{\ell \in \mathbb{Z}} H_{[d_1, d_2]}(\ell N).
\]

Combining (2.12), (4.1) and (4.2), we find that

\[
S_2 = \frac{1}{N^2} \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M\left(\frac{Q}{d_1}\right) M\left(\frac{Q}{d_2}\right) \sum_{\ell \in \mathbb{Z}} H_{[d_1, d_2]}(\ell N).
\]

Using the definition of \( M \) and \( H_y \), we can rewrite (4.3) as

\[
S_2 = \frac{1}{N^2} \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 \sum_{1 \leq r_1 \leq Q/d_1} \sum_{1 \leq r_2 \leq Q/d_2} \mu(r_1) \mu(r_2) \sum_{\ell \in \mathbb{Z}} \frac{N}{[d_1, d_2]} H\left(\frac{\ell N}{[d_1, d_2]}\right)
\]

\[
= \frac{1}{N} \sum_{1 \leq r_1 \leq Q} \sum_{1 \leq r_2 \leq Q} \mu(r_1) \mu(r_2) (d_1, d_2) \sum_{\ell \in \mathbb{Z}} H\left(\frac{\ell N}{[d_1, d_2]}\right)
\]

\[
= \frac{1}{N} \sum_{1 \leq r_1 \leq Q} \sum_{1 \leq r_2 \leq Q} \mu(r_1) \mu(r_2) \sum_{1 \leq d_1 \leq Q/r_1} \sum_{1 \leq d_2 \leq Q/r_2} (d_1, d_2) \sum_{\ell \in \mathbb{Z}} H\left(\frac{\ell N}{[d_1, d_2]}\right).
\]

Denote \( \delta = (d_1, d_2) \), so that \( d_1 = \delta q_1 \) and \( d_2 = \delta q_2 \) with \( (q_1, q_2) = 1 \). Then (4.4) becomes

\[
S_2 = \frac{1}{N} \sum_{1 \leq r_1, r_2 \leq Q} \sum_{1 \leq \delta \leq Q/(\max\{r_1, r_2\})} \mu(r_1) \mu(r_2) \delta \sum_{1 \leq q_1 \leq Q/\delta r_1} \sum_{1 \leq q_2 \leq Q/\delta r_2} H\left(\frac{\ell N}{\delta q_1 q_2}\right).
\]

Only values of \( \ell \) with \( \ell < \delta q_1 q_2 \Lambda'/N \) may produce a non-zero contribution in the inner sum in (4.5). For \( Q \) larger than some \( Q_0(\Lambda') \), this yields, in conjunction
As a result of (4.6)–(4.8) and of (2.1), we infer that
\[ \varepsilon \leq \frac{Q^2}{N} < c_\Lambda = \frac{\pi^2}{3}. \]
Therefore each of \( \delta, r_1, r_2, \ell \) should be at most \( c_\Lambda \), and in (4.5) we are left with
\[ S_2 = \frac{1}{N} \sum_{\ell, \delta, r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) \delta \sum_{1 \leq q_1 \leq Q/\delta r_1, 1 \leq q_2 \leq Q/\delta r_2} H \left( \frac{\ell N}{\delta q_1 q_2} \right). \]

Here
\[ H \left( \frac{\ell N}{\delta q_1 q_2} \right) = H \left( \frac{3\ell Q^2}{\pi^2 \delta q_1 q_2} \right) + O_\Lambda \left( \|H\|_\infty Q \log Q \right); \]
thus
\[ S_2 = \frac{1}{N} \sum_{\ell, \delta, r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) \delta \sum_{1 \leq q_1 \leq Q/\delta r_1, 1 \leq q_2 \leq Q/\delta r_2} H \left( \frac{3\ell Q^2}{\pi^2 \delta q_1 q_2} \right) + O_H \left( \log Q \right). \]

If \( \min\{q_1, q_2\} < Q^{1-\varepsilon} \) for some \( \varepsilon > 0 \), then for sufficiently large \( Q \) (with respect to \( \varepsilon \) and \( \Lambda \)), one has \( 3\ell Q^2/\pi^2 \delta q_1 q_2 > \Lambda \), and thus \( H(3\ell Q^2/\pi^2 \delta q_1 q_2) = 0 \). On the other hand, we have (see [4, Corollary 1])
\[
\sum_{\min\{q_1, q_2\} > Q^{1-\varepsilon}} H \left( \frac{3\ell Q^2}{\pi^2 \delta q_1 q_2} \right)
= \frac{6}{\pi^2} \int x \leq Q/\delta r_1 \int y \leq Q/\delta r_2 \quad H \left( \frac{3\ell Q^2}{\pi^2 \delta x y} \right) \, dx \, dy + E_{H, \varepsilon}(Q),
\] (4.6)
where
\[ E_{H, \varepsilon}(Q) \ll H, \varepsilon \frac{Q^2 \|H'\|_\infty}{Q^{3(1-\varepsilon)}} Q^2 \log Q + \|H\|_\infty Q \log Q \ll H \frac{Q^{1+4\varepsilon}}{\log Q}. \]
The change of variables \((x, y) = (Q u, Q v)\) gives us that the main term in (4.6) can be expressed as
\[
\frac{6Q^2}{\pi^2} \int_{\min\{u, v\} \geq Q^{-\varepsilon}} H \left( \frac{3\ell}{\pi^2 \delta uv} \right) \, du \, dv.
\] (4.7)
For \( u, v \) as in (4.7) and \( Q \geq Q_0(\Lambda) \), we have \( \pi^2 \delta uv/3\ell \leq (\pi^2/3)Q^{-\varepsilon} < 1/\Lambda \) (since \( r_1, r_2, \ell \geq 1 \)); thus
\[
\int \int_{\min\{u, v\} \leq Q^{-\varepsilon}} H \left( \frac{3\ell}{\pi^2 \delta uv} \right) \, du \, dv = 0.
\] (4.8)
As a result of (4.6)–(4.8) and of (2.1), we infer that
\[
S_2 = 2 \sum_{\ell, \delta, r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) \delta \int_{0}^{1/\delta r_1} \int_{0}^{1/\delta r_2} H \left( \frac{3\ell}{\pi^2 \delta uv} \right) \, dv \, du
+ O_{H, \varepsilon}(Q^{-1+\varepsilon} \log Q).
\] (4.9)
Next, we put \( \lambda = \lambda_u(v) = 3\ell/\pi^2 \delta u \) and change the order of integration to express the double integral in (4.9) as

\[
\int_0^{1/\delta r_1} \int_{3\ell r_2/\pi^2 u}^\Lambda H(\lambda) \frac{3\ell}{\pi^2 \delta u \lambda^2} \, d\lambda \, du = \frac{3\ell}{\pi^2 \delta} \int_{3\ell r_2/\pi^2}^\Lambda \int_0^{1/\delta r_1} \frac{H(\lambda)}{\lambda^2} \, du \, d\lambda
\]

Inserting this back into (4.9), we get

\[
S_2 = \frac{6}{\pi^2} \sum_{\delta r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) \ell \int_{3\ell r_2/\pi^2}^\Lambda \frac{H(\lambda)}{\lambda^2} \log \frac{\pi^2 \delta}{3\ell r_2} \, d\lambda + O_{H,\varepsilon}(Q^{-1+\varepsilon}).
\]

At this point we take \( K = \ell \delta r_1 r_2 \) and rewrite (4.10) as

\[
S_2 = \frac{6}{\pi^2} \sum_{1 \leq K < c_A} \int_{3K/\pi^2}^\Lambda \frac{H(\lambda)}{\lambda^2} \log \frac{\pi^2 \delta}{3K} \, d\lambda \sum_{\delta r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) + O_{H,\varepsilon}(Q^{-1+\varepsilon}).
\]

Next, we put the inner sum on the right-hand side of (4.11) in the form

\[
\sum_{\delta r_1, r_2 \geq 1} \mu(r_1) \mu(r_2) \ell = \sum_{\delta r_1 \geq 1} \mu(r_1) \ell \sum_{r_2 | \ell r_1} \mu(r_2) = \sum_{r_1 | K} \mu(r_1) \ell = K \sum_{r_1 | K} \ell r_1 = K \frac{\varphi(K)}{K} = \varphi(K).
\]

Combining (4.11) and (4.12), we find that

\[
S_2 = \frac{6}{\pi^2} \sum_{1 \leq K < c_A} \varphi(K) \int_{3K/\pi^2}^\Lambda \frac{H(\lambda)}{\lambda^2} \log \frac{\pi^2 \delta}{3K} \, d\lambda + O_{H,\varepsilon}(Q^{-1+\varepsilon})
\]

A standard approximation argument with smooth functions as before completes the proof of (1.5).

To prove (1.6), we start with the well-known equalities

\[
\sum_{q=1}^\infty \frac{\varphi(q)}{q^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\text{Re } s > 2),
\]

and

\[
\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s^2} \, ds = \begin{cases} 0 & 0 \leq y \leq 1 \\ \log y & y > 1 \end{cases} (\sigma_0 > 2).
\]

From (13) and (14), we infer for fixed \( \sigma_0 > 2 \) that

\[
\sum_{1 \leq q < x} \varphi(q) \log \frac{x}{q} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{x^s}{s^2} \, ds.
\]
The integrand above has a simple pole at $s = 2$ with residue $1/\zeta(2) \cdot x^2/4 = 3x^2/2\pi^2$. Thus, by moving the integration contour to $\Re s = 1$ and using the trivial estimate
$$\int_1^{\sigma_0} \frac{|x^{s+it}|}{|s+iR|^2} ds \ll \frac{x^{\sigma_0}}{R},$$
we infer from (4.15) the equality
$$\sum_{1 \leq q < x} \varphi(q) \log \frac{x}{q} = \frac{3x^2}{\pi^2} + \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\zeta(s-1)}{\zeta(s)} \cdot \frac{x^s}{s^2} ds.$$ (4.16)

Utilizing the functional equation
$$\frac{\zeta(it)}{\zeta(1+it)} = \chi(it) \cdot \frac{\zeta(1-it)}{\zeta(1+it)}, \quad \chi(z) = \frac{(2\pi)^z}{2\Gamma(z) \cos(\pi z/2)},$$
and the equalities $\zeta(1-it) = \zeta(1+it)$ and (see [8, relation 8.332])
$$|\Gamma(it)|^2 = \frac{\pi}{t \sinh \pi t} (t > 0),$$
we find that the expression in (4.16) is given by
$$\frac{3x^2}{2\pi^2} + O\left(\int_0^\infty \frac{x\sqrt{t}}{(1+t^2)} dt\right) = \frac{3x^2}{2\pi^2} + O(x),$$
which completes the proof of (1.6).

Note that one gets a better error term than in (2.2) as a result of the presence of the factor $\log(x/q)$ instead of 1.

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