BOUNDEDNESS AND ASYMPTOTIC STABILITY IN A TWO-SPECIES CHEMOTAXIS-COMPETITION MODEL WITH SIGNAL-DEPENDENT SENSITIVITY

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ABSTRACT. This paper deals with the two-species chemotaxis-competition system
\[
\begin{align*}
    u_t &= d_1 \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u - a_1 v) \quad \text{in } \Omega \times (0, \infty), \\
    v_t &= d_2 \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - a_2 u - v) \quad \text{in } \Omega \times (0, \infty), \\
    w_t &= d_3 \Delta w + h(u, v, w) \quad \text{in } \Omega \times (0, \infty),
\end{align*}
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\), \(n \in \mathbb{N}\); \(h, \chi_i\) are functions satisfying some conditions. In the case that \(\chi_i(w) = \chi_i\), Bai–Winkler [1] proved asymptotic behavior of solutions to the above system under some conditions which roughly mean largeness of \(\mu_1, \mu_2\). The main purpose of this paper is to extend the previous method for obtaining asymptotic stability. As a result, the present paper improves the conditions assumed in [1], i.e., the ranges of \(\mu_1, \mu_2\) are extended.

1. Introduction. Nowadays, mathematics is useful in many things, for example, physics, chemistry, biology, computer, medical, architecture, and so on. Here we focus on biology. One of the famous and basic models in biology is the Lotka–Volterra competition system. On the other hand, many mathematicians study a chemotaxis system lately, which describes a part of the life cycle of cellular slime molds with chemotaxis. After the pioneering work of Keller–Segel [8], a number of variations of the chemotaxis system are proposed and investigated (see e.g., [2], [4] and [5]). Also, multi-species chemotaxis systems have been studied by e.g., [7] and [13]. In this paper we focus on a two-species chemotaxis system which describes a situation in which multi populations react on a single chemoattractant. Moreover, we assume that both populations reproduce themselves, and mutually compete with the other, according to the classical Lotka-Volterra kinetics, i.e., with coupling coefficients \(a_1, a_2 > 0\) in
\[
    u_t = u(1 - u - a_1 v), \quad v_t = v(1 - a_2 u - v).
\]

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We consider the two-species chemotaxis system

\[
\begin{align*}
  u_t &= d_1 u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u (1 - u - a_1 v), \quad x \in \Omega, \ t > 0, \\
  v_t &= d_2 v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v (1 - a_2 u - v), \quad x \in \Omega, \ t > 0, \\
  w_t &= d_3 w + h(u, v, w), \quad x \in \Omega, \ t > 0, \quad \nabla u \cdot v = \nabla v \cdot u = \nabla w \cdot v = 0, \quad x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align*}
\]  

(1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \in \mathbb{N} \)) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward normal vector to \( \partial \Omega \). The initial data \( u_0, v_0 \) and \( w_0 \) are assumed to be nonnegative functions. The unknown functions \( u(x, t) \) and \( v(x, t) \) represent the population densities of two species and \( w(x, t) \) shows the concentration of the substance at place \( x \) and time \( t \).

The problem (1) is an interesting problem on account of the influence of chemotaxis, diffusion, and the Lotka–Volterra kinetics. In mathematical view, global existence and behavior of solutions are fundamental theme. In the case \( \chi_i(w) = \chi_i \) and \( h(u, v, w) = \alpha u + \beta v - \gamma w \), Bai–Winkler [1] showed global existence of solutions to (1) when \( n = 2 \). Moreover, they considered asymptotic behavior of solutions to (1). When \( a_1, a_2 \in (0, 1) \), they proved that the solution \( (u, v, w) \) satisfies \( u(t) \to u^*, v(t) \to v^*, w(t) \to \frac{\alpha u^* + \beta v^*}{\gamma} \) in \( L^\infty(\Omega) \) as \( t \to \infty \), where \( u^* = \frac{1-a_1}{1-a_1a_2} \) and \( v^* = \frac{1-a_2}{1-a_1a_2} \), under the conditions

\[
\begin{align*}
  \mu_1 &> \frac{d_2 \chi_1^2 u^*}{4a_1\gamma(1-a_1a_2)d_1d_2d_3} - \frac{d_1a_1^2 \chi_1^2 v^*}{4\mu_2 a_2}, \\
  \mu_2 &> \frac{\chi_2^2 v^*(a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)}{16d_2d_3a_2\gamma(1-a_1a_2)}.
\end{align*}
\]  

(2)

These conditions are not natural because they are not symmetric. When \( a_1 \geq 1 \geq a_2 > 0 \), they obtained that \( u(t) \to 0, v(t) \to 1, w(t) \to \frac{1}{16} \) in \( L^\infty(\Omega) \) as \( t \to \infty \) under the condition that there exists \( a'_1 \in [1, a_1] \) such that \( a'_1a_2 < 1 \) and

\[
\mu_2 > \frac{\chi_2^2(a'_1\alpha^2 + a_2\beta^2 - 2a'_1a_2\alpha\beta)}{16d_2d_3a_2\gamma(1-a'_1a_2)}.
\]  

(3)

In the non-competitive case \( (a_1 = a_2 = 0) \), global existence and asymptotic stability were established under some conditions for \( \chi_i(w) \) ([10]). In the case that \( d_1 = d_2 = d_3 = 1 \) and \( h(u, v, w) = u + v - w \), Zhang–Li [14] proved global existence of solutions to (1) under the assumption that \( \mu_i > 0 \) is small and \( \chi_i(w) \leq \frac{K_i}{(1 + \sigma_i w)^{\frac{1}{\gamma}}} \) for \( \sigma_i > 1 \) and \( K_i > 0 \) being small enough.

The purpose of the present paper is to improve the methods in [1], [10] for obtaining global existence and asymptotic stability of solutions to (1) under a more general and sharp condition for the sensitivity function \( \chi_i(w) \). We shall suppose throughout this paper that \( h, \chi_i (i = 1, 2) \) satisfy the following conditions:

\[
\begin{align*}
  \chi_i &\in C^1([0, \infty)) \cap L^1([0, \infty)) \quad (0 < \exists \theta < 1), \quad \chi_i > 0 \quad (i = 1, 2), \\
  h &\in C^1([0, \infty) \times [0, \infty) \times [0, \infty)), \quad h(0, 0, 0) = 0, \\
  \exists \gamma > 0: \frac{\partial h}{\partial u}(u, v, w) &\geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma, \\
  \exists \delta > 0, \exists M > 0: |h(u, v, w) + \delta w| &\leq M(u + v + 1), \\
  \exists k_i > 0: -\chi_i(w)h(0, 0, w) &\leq k_i \quad (i = 1, 2).
\end{align*}
\]  

(4) (5) (6) (7) (8)
We also assume that there exists \( p > n \) such that
\[
2d_1d_3 \chi_i'(w) + \left( (d_3 - d_1)p + \sqrt{(d_3 - d_1)^2p^2 + 4d_1d_3p} \right) [\chi_i(w)]^2 \leq 0 \quad (i = 1, 2). \tag{9}
\]
The above conditions cover the prototypical example \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \) (\( K_i > 0, \sigma_i > 1 \)), \( h(u,v,w) = u + v - w \). We assume that the initial data \( u_0, v_0, w_0 \) satisfy
\[
0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1,q}(\Omega) \quad (\exists q > n). \tag{10}
\]

Now the main results read as follows. The first one is concerned with global existence and boundedness in (1).

**Theorem 1.1.** Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0, a_1, a_2 \geq 0 \). Assume that \( h, \chi_1, \chi_2 \) satisfy (4)–(9). Then for any \( u_0, v_0, w_0 \) satisfying (10) for some \( q > n \), there exists an exactly one pair \( (u, v, w) \) of nonnegative functions
\[
u, v, w \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
\]
which satisfy (1). Moreover, the solutions \( u, v, w \) are uniformly bounded, i.e., there exists a constant \( C_1 > 0 \) such that
\[
\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \leq C_1
\]
for all \( t \geq 0 \), and the solutions \( u, v, w \) are the Hölder continuous functions, i.e., there exist \( \alpha \in (0, 1) \) and \( C_2 > 0 \) such that
\[
\|u\|_{C^{2+\alpha,1+\frac{2}{3}}(\overline{\Omega} \times [1, t])} + \|v\|_{C^{2+\alpha,1+\frac{2}{3}}(\overline{\Omega} \times [1, t])} + \|w\|_{C^{2+\alpha,1+\frac{2}{3}}(\overline{\Omega} \times [1, t])} \leq C_2
\]
for all \( t \geq 1 \).

**Remark 1.** Theorem 1.1 improves the result in [14] when \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \). Indeed, we do not need any condition for \( \mu_4 \). Moreover, the condition in [14] is
\[
\frac{r}{r+1} \frac{\sigma_i}{\sqrt{2(p-1)}}
\]
with some \( r > 0 \), while the condition (9) becomes \( K_i \leq \frac{\sigma_i}{\sqrt{2}} \) when \( \chi_i(w) = \frac{K_i}{(1+w)^{\sigma_i}} \).

Since Theorem 1.1 guarantees that \( u, v \) and \( w \) exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) by
\[
\begin{align*}
\alpha_1 &:= \min_{(u,v,w) \in I} h_u(u,v,w), & \quad \alpha_2 &:= \max_{(u,v,w) \in I} h_u(u,v,w), \\
\beta_1 &:= \min_{(u,v,w) \in I} h_v(u,v,w), & \quad \beta_2 &:= \max_{(u,v,w) \in I} h_v(u,v,w),
\end{align*}
\tag{11}
\]
where \( I = (0, C_1)^3 \) and \( C_1 \) is defined in Theorem 1.1.

In the case \( \alpha_1, \alpha_2 \in (0, 1) \) asymptotic behavior of solutions to (1) will be discussed under the following additional conditions: there exists \( \delta_1 > 0 \) such that
\[
4\delta_1 - a_1a_2(1 + \delta_1)^2 > 0 \tag{12}
\]
and
\[
\begin{align*}
\mu_1 &> \frac{\chi_1(0)^2u^*(1 + \delta_1)(\alpha_2^2a_1^2\alpha_1 + \beta_2^2a_2^2 - \alpha_1\beta_1a_1a_2(1 + \delta_1))}{4a_1d_1d_3\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}, \tag{13} \\
\mu_2 &> \frac{\chi_2(0)^2v^*(1 + \delta_1)(\alpha_2^2a_1^2\alpha_1 + \beta_2^2a_2^2 - \alpha_1\beta_1a_1a_2(1 + \delta_1))}{4a_2d_2d_3\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}.
\end{align*}
\tag{14}
\]
The second theorem gives asymptotic behavior in (1) in the case \( a_1, a_2 \in (0, 1) \).
Theorem 1.2. Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$ and $a_1, a_2 \in (0, 1)$. Under the conditions (4)–(10) and (12)–(14), the unique global solution $(u, v, w)$ of (1) satisfies that there exist $C > 0$ and $\lambda > 0$ such that
\[ \|u(t) - u^*\|_{L^\infty(\Omega)} + \|v(t) - v^*\|_{L^\infty(\Omega)} + \|w(t) - w^*\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0), \]
where
\[ u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2} \]
and $w^* \geq 0$ such that
\[ h(u^*, v^*, w^*) = 0. \]

Remark 2. The methods in the proof of Theorem 1.2 (and Theorem 1.4, see below) can be applied to the case $\chi_i(w) = \chi_i$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$. Then the conditions (12)–(14) have symmetry and relax the condition (2) assumed in [1]. Indeed, the conditions (2) are stronger than (12)–(14) when $\delta_1 = 1$. Moreover, in view of considering the function
\[ f(x) = \frac{a_1(\alpha^2 - \alpha \beta a_2)x^2 + (\beta^2 a_2 - \alpha^2 a_1)x}{-a_1 a_2 x^2 + 4x - 4} \]
(we put $x = 1 + \delta_1$), $x = 2 (\delta_1 = 1)$ is not a minimizer of the right-hand sides of (13) and (14) except the case $\beta^2 a_2 = \alpha^2 a_1$. Thus the conditions (12)–(14) relax (2).

This remark implies the following result which improves the previous work in [1].

Theorem 1.3. Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$ and $a_1, a_2 \in (0, 1)$. Assume that there exists a unique global solution $(u, v, w)$ of (1) satisfying that there exists $C > 0$ such that
\[ \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [1, t])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [1, t])} + \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [1, t])} \leq C \]
for all $t \geq 1$ and $\chi_1, \chi_2$ satisfy that there exist the positive constants $M_1, M_2 > 0$ and $\delta_1 > 0$ such that
\[ \chi_i(w) \leq M_i \quad \text{for all } w \geq 0 \quad (i = 1, 2), \]
\[ 4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0, \]
and
\[ \mu_i > \frac{M_i^2 (1 + \delta_1) (1 - a_1) (a_2^2 a_1 \delta_1 + \beta_2 a_2 - \alpha_1 \beta_1 a_1 a_2 (1 + \delta_1))}{4a_1 d_1 d_3 \gamma (1 - a_1 a_2) (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \quad (i = 1, 2). \]
Then the same conclusion as in Theorem 1.2 holds.

In the case $a_1 \geq 1 > a_2 > 0$ asymptotic behavior of solutions to (1) will be discussed under the following additional conditions: there exist $\delta_1 > 0$ and $a'_1 \in [1, a_1]$ such that
\[ 4\delta_1 - a'_1 a_2 (1 + \delta_1)^2 > 0, \]
\[ \mu_2 > \frac{\chi_2(0)^2 \delta_1 (a_2^2 a'_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a'_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a'_1 a_2 (1 + \delta_1)^2)}, \]
The third one is concerned with asymptotic behavior of solutions to (1) in the case $a_1 \geq 1 > a_2 > 0$. 
Theorem 1.4. Let \(d_1,d_2,d_3 > 0\), \(\mu_1,\mu_2 > 0\) and \(a_1 \geq 1\), \(a_2 \in (0,1)\). Under the conditions (4)–(9) and (15)–(16), the unique global solution \((u,v,w)\) of (1) has the following properties:

(i) Let \(a_1 > 1\) and take \(a'_1 \in (1,a_1]\) in (15)–(16). Then there exist \(C > 0\) and \(\lambda > 0\) satisfying
\[
\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t}
\]
for all \(t > 0\), where \(\bar{w} \geq 0\) such that \(h(0,1,\bar{w}) = 0\).

(ii) Let \(a_1 = 1\). Then there exist \(C > 0\) and \(\lambda > 0\) satisfying
\[
\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\lambda}
\]
for all \(t > 0\), where \(\bar{w} \geq 0\) such that \(h(0,1,\bar{w}) = 0\).

Remark 3. \(\delta_1 = 1\) and \(a'_1 = \frac{1+\min(a_1,a_2^{-1})}{2}\) satisfy (15). Thus there exist some \(\delta_1, a'_1, \mu_2\) satisfying (15)–(16). Moreover, from the same argument as in Remark 2 the conditions (15)–(16) relax (3).

From this remark we can also established the following result which improves the previous work in [1].

Theorem 1.5. Let \(d_1,d_2,d_3 > 0\), \(\mu_1,\mu_2 > 0\) and \(a_1 \geq 1\), \(a_2 \in (0,1)\). Assume that there exists a unique global solution \((u,v,w)\) of (1) satisfying that there exists \(C > 0\) such that
\[
\|u\|_{C^{2+a_1,1+a_2}\over\partial[\Omega \times [1,t]]} + \|v\|_{C^{2+a_1,1+a_2}\over\partial[\Omega \times [1,t]]} + \|w\|_{C^{2+a_1,1+a_2}\over\partial[\Omega \times [1,t]]} \leq C
\]
for all \(t \geq 1\) and \(\chi_1,\chi_2\) satisfies that there exist the positive constants \(M_1, M_2 > 0\), \(a'_1 \in [1,a_1]\) and \(\delta_1 > 0\) such that
\[
\chi_i(w) \leq M_i \quad \text{for all } w \geq 0 \quad (i = 1,2),
\]
\[
4\delta_1 - a'_1 a_2(1 + \delta_1)^2 > 0,
\]
and
\[
\mu_2 > \frac{M^2_2\delta_1(a_2 a'_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a'_1 a_2(1 + \delta_1))}{4a_2 a_1 d_3(4\delta_1 - a'_1 a_2(1 + \delta_1)^2)}.
\]
Then the same conclusion as in Theorem 1.4 holds.

Remark 4. In Theorems 1.2 and 1.4 we can find \(w^* \geq 0\) satisfying \(h(u^*, v^*, w^*) = 0\) and \(\bar{w} \geq 0\) satisfying \(h(0,1,\bar{w}) = 0\). Indeed, from (5)–(7) for every \(a, b \geq 0\) there exists \(\bar{w}\) such that \(h(a,b,\bar{w}) = 0\). Indeed, if we choose \(w_1 \geq M(a_1 + b + 1)\), then (7) yields \(h(a,b,w_1) \leq M(a + b + 1) - \delta w_1 \leq 0\). On the other hand, (5) and (6) imply that \(h(a,b,0) \geq h(0,0,0) \geq 0\). Hence, by the intermediate value theorem there exists \(\bar{w} \geq 0\) such that \(h(a,b,\bar{w}) = 0\).

The strategy for the proof of Theorem 1.1 is to construct estimates for \(\int\Omega u^p\) and \(\int\Omega v^p\) by modifying a method in [10]. One of the keys for this strategy is to derive inequality
\[
\frac{d}{dt} \int\Omega u^p[f_1(w)]^{-r} \leq a \int\Omega u^p[f_1(w)]^{-r} - b \left(\int\Omega u^p[f_1(w)]^{-r}\right)^{\frac{p+1}{p}}
\]
for some positive constants \(a, b\), where \(f_1(w) := \exp\left\{\frac{1}{\alpha_1} \int_0^w \chi_1(s) \, ds\right\}\). Applying the new method in [10] for obtaining the above inequality, we can improve the result in [14]. On the other hand, the strategy for the proof of Theorems 1.2 and 1.4 is to
modify an argument in [1]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt} E(t) \leq -\varepsilon \left( \int_{\Omega} (u - \overline{u})^2 + \int_{\Omega} (v - \overline{v})^2 + \int_{\Omega} (w - \overline{w})^2 + \int_{\Omega} |\nabla w|^2 \right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$, where $(\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3$ is a solution of (1). For finding the above inequality we apply more “suitable” estimates for

$$\int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \quad \text{and} \quad \int_{\Omega} \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w.$$

These enable us to improve the conditions (2) and (3).

This paper is organized as follows. In Section 2 we prove global existence and boundedness (Theorem 1.1). Sections 3 and 4 are devoted to the proof of asymptotic stability (Theorems 1.2, 1.4).

2. Global existence and boundedness. In this section we shall show global existence and boundedness in (1). Firstly we will recall the well-known result about local existence of solutions to (1).

**Lemma 2.1.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 \geq 0$ and $a_1, a_2 \geq 0$. Assume that $h, \chi_1, \chi_2$ satisfy (4), (5), (7). Then for any $u_0, v_0, w_0$ satisfying (10) for some $q > n$, there exist $T_{\text{max}} \in (0, \infty)$ and an exactly one pair $(u, v, w)$ of nonnegative functions

$$u, v, w \in C(\Omega \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}}))$$

which satisfy (1). Moreover, either $T_{\text{max}} = \infty$ or

$$\lim_{t \to T_{\text{max}}} \left( \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \right) = \infty.$$

**Proof.** The proof of local existence of classical solutions to (1) is based on a standard contraction mapping argument, which can be found in [11, 12]. Finally the maximum principle is applied to yield $u > 0, v > 0, w \geq 0$ in $\Omega \times (0, T_{\text{max}})$.

Let $(u, v, w)$ be the solution of (1) on $[0, T_{\text{max}}]$ as in Lemma 2.1. We introduce the functions $f_1 = f_1(w)$ and $f_2 = f_2(w)$ by

$$f_i(w) := \exp \left\{ \frac{1}{d_i} \int_{0}^{w} \chi_i(s) \, ds \right\} \quad \text{for } i = 1, 2$$

to prove the following lemma.

**Lemma 2.2.** Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$ and $a_1, a_2 > 0$. Assume that $\chi_1, \chi_2$ satisfy (4) and (9) with some $p > n$. Then there exist positive constants $r_1 = r_1(d_1, d_3, p)$ and $r_2 = r_2(d_2, d_3, p)$ such that

$$\frac{d}{dt} \int_{\Omega} u^p f_1^{-r_1} \leq \mu_1 \int_{\Omega} u^p f_1^{-r_1}(1 - u) - \frac{r_1}{d_1} \int_{\Omega} u^p f_1^{-r_1} \chi_1(w) h(u, v, w) \quad (17)$$

and

$$\frac{d}{dt} \int_{\Omega} v^p f_2^{-r_2} \leq \mu_2 \int_{\Omega} v^p f_2^{-r_2}(1 - v) - \frac{r_2}{d_2} \int_{\Omega} v^p f_2^{-r_2} \chi_2(w) h(u, v, w). \quad (18)$$
Proof. We let \( p \geq 1 \) and \( r > 0 \) be fixed later. From the first and third equations in (1) we have

\[
\frac{d}{dt} \int_\Omega u^p f_{1}^{r} = p \int_\Omega u^{p-1} f_{1}^{r} \nabla \cdot (d_1 \nabla u - u \chi_1(w) \nabla w) \\
+ p \mu_1 \int_\Omega u^p f_{1}^{r-1} (1 - u - a_1 v) \\
- \frac{d_3 r}{d_1} \int_\Omega u^p f_{1}^{r-1} \chi_1(w) \Delta w - \frac{r}{d_1} \int_\Omega u^p f_{1}^{r-1} \chi_1(w) h(u, v, w).
\]

Denoting by \( I_1 \) and \( I_2 \) the first and third terms on the right-hand side as

\[
I_1 := p \int_\Omega u^{p-1} f_{1}^{r} \nabla \cdot (d_1 \nabla u - u \chi_1(w) \nabla w), \\
I_2 := -\frac{d_3 r}{d_1} \int_\Omega u^p f_{1}^{r-1} \chi_1(w) \Delta w,
\]

we can write as

\[
\frac{d}{dt} \int_\Omega u^p f_{1}^{r} = I_1 + I_2 + p \mu_1 \int_\Omega u^p f_{1}^{r-1} (1 - u - a_1 v) \\
- \frac{r}{d_1} \int_\Omega u^p f_{1}^{r-1} \chi_1(w) h(u, v, w).
\] (19)

We shall show the following inequality:

\[
\exists p > n, \exists r > 0; I_1 + I_2 \leq 0.
\]

Noting that

\[
d_1 f_1 \nabla \left( \frac{u}{f_1} \right) = d_1 \nabla u - u \chi_1(w) \nabla w,
\]

we obtain

\[
I_1 = d_1 p \int_\Omega u^{p-1} f_{1}^{r} \nabla \cdot (f_1 \nabla \left( \frac{u}{f_1} \right)) \\
= d_1 p \int_\Omega \left( \frac{u}{f_1} \right)^{p-1} f_{1}^{r+p-1} \nabla \cdot f_1 \nabla \left( \frac{u}{f_1} \right) \\
= -d_1 p(p - 1) \int_\Omega \left( \frac{u}{f_1} \right)^{p-2} f_{1}^{r-p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2 \\
- p(-r + p - 1) \int_\Omega \left( \frac{u}{f_1} \right)^{p-1} f_{1}^{r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w.
\]

Similarly, we see that

\[
I_2 = -\frac{d_3 r}{d_1} \int_\Omega \left( \frac{u}{f_1} \right)^{p} f_{1}^{r+p} \chi_1(w) \Delta w \\
= \frac{d_3 p r}{d_1} \int_\Omega \left( \frac{u}{f_1} \right)^{p-1} f_{1}^{r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w \\
+ \frac{d_3 r}{d_1} \int_\Omega \left( \frac{u}{f_1} \right)^{p} f_{1}^{r+p} \left( \frac{p - r}{d_1} \left[ \chi_1(w) \right]^2 + \chi_1'(w) \right) |\nabla w|^2.
\]
Therefore it follows that
\[ I_1 + I_2 \]
\[ = -d_1 p(p - 1) \int_\Omega \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) \right|^2 \]
\[ - \left( p(p - 1) - \left( 1 + \frac{d_3}{d_1} \right) p r \right) \int_\Omega \left( \frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left( \frac{u}{f_1} \right) \cdot \nabla w \]
\[ + \int_\Omega \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left[ \frac{d_3 r(-r + p)}{d_1} \right] \left( \chi_1(w) \right)^2 + \frac{d_3 r}{d_1} \chi_1'(w) \right] \nabla w^2 \]
\[ = -d_1 p(p - 1) \int_\Omega \left( \frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left( \frac{u}{f_1} \right) + \frac{p(p - 1) - \left( 1 + \frac{d_3}{d_1} \right) p r}{2d_1 p(p - 1)} \chi_1(w) \frac{u}{f_1} \nabla w \right|^2 \]
\[ + \int_\Omega \left( \frac{u}{f_1} \right)^p f_1^{-r+p} \left[ \frac{a_1 r^2 + 2a_2 r + a_3}{4d_1(p - 1)} \right] \nabla w^2, \]
where \( a_1, a_2, a_3 \) are given by
\[ a_1 := \left( p \left( \frac{d_3}{d_1} - 1 \right)^2 + 4 \frac{d_3}{d_1} \right) \left[ \chi_1(w) \right]^2, \]
\[ a_2 := (p - 1) \left( p \left( \frac{d_3}{d_1} - 1 \right) \left[ \chi_1(w) \right]^2 + 2d_3 \chi_1'(w) \right), \]
\[ a_3 := p(p - 1)^2 \left[ \chi_1(w) \right]^2. \]
Then there exists \( p > n \) such that the discriminant of \( a_1 r^2 + 2a_2 r + a_3 \)
\[ D := 4(p - 1)^2 \left[ \left( p \chi_1^2 \left( \frac{d_3}{d_1} - 1 \right) + 2d_3 \chi_1' \right)^2 - p \chi_1^4 \left( p \left( \frac{d_3}{d_1} - 1 \right)^2 + 4 \frac{d_3}{d_1} \right) \right] \]
is nonnegative in view of (9). Therefore we have that there exists \( r > 0 \) such that \( a_1 r^2 + a_2 r + a_3 \leq 0 \) and hence
\[ I_1 + I_2 \leq 0. \]
On the other hand, we can see from the positivity of \( u \) and \( v \) that
\[ \mu_1 u(1 - u - a_1 v) \leq \mu_1 u(1 - u). \]
Hence (19) implies
\[ \frac{d}{dt} \int_\Omega u^p f_1^{-r} \leq p \mu_1 \int_\Omega u^p f_1^{-r} (1 - u) - \frac{r}{d_1} \int_\Omega u^p f_1^{-r} \chi_1 h(u, v, w). \]
This means that (17) holds. In the same way, we obtain (18).

**Lemma 2.3.** Let \( d_1, d_2, d_3 > 0, \mu_1, \mu_2 > 0 \) and \( a_1, a_2 > 0 \). Assume that \( h, \chi_i \) satisfy (4)–(6), (8), and (9) with some positive constants \( k_i \) \((i = 1, 2) \) and \( p > n \), then
\[ \| u(t) \|_{L^p(\Omega)} \leq \left( e^{\| \chi_1 \|_{L^1(0, \infty)}} \right) \frac{\chi_1}{\mu_1} \max \left\{ \| u_0 \|_{L^p(\Omega)}, \frac{d_1 p \mu_1 + r_1 k_1}{d_1 p \mu_1} \right\}, \tag{20} \]
\[ \| v(t) \|_{L^p(\Omega)} \leq \left( e^{\| \chi_2 \|_{L^1(0, \infty)}} \right) \frac{\chi_2}{\mu_2} \max \left\{ \| v_0 \|_{L^p(\Omega)}, \frac{d_2 p \mu_2 + r_2 k_2}{d_2 p \mu_2} \right\}. \]

**Proof.** The proof is same as in [10, Lemma 3.2].
For the proof of Theorem 1.1, we put \( k > 0 \) and let \( \Delta \) denote the realization of the Laplacian in \( L^s(\Omega) \) with domain \( \{ z \in W^{2,s}(\Omega) \mid \nabla z \cdot \nu = 0 \text{ on } \partial \Omega \} \) for \( s \in (1, \infty) \). Then it is known ([3], [6]) that the operator \(-\Delta + k\) is sectorial and possesses closed fractional powers \((-\Delta + k)^\eta\), \( \eta \in (0, 1) \), with dense domain \( D((-\Delta + k)^\eta) \). Moreover, there exist \( c_1, c_2 > 0 \) such that if \( m \in \{0, 1\}, p \in [1, \infty], q \in (1, \infty) \) satisfy \( m < 2\eta \) and \( m - n/p < 2\eta - n/q \), then

\[ \| z \|_{W^{m,p}(\Omega)} \leq c_1 \| (-\Delta + k)^\eta z \|_{L^q(\Omega)} \quad (21) \]

for every \( z \in D((-\Delta + k)^\eta) \), and if \( p \in [1, \infty), q \geq p, \) then there exists \( \lambda > 0 \) such that

\[ \| (-\Delta + k)^\eta e^{(\Delta - \lambda) t} z \|_{L^q(\Omega)} \leq c_2 t^{-\eta - \frac{2(\frac{1}{p} - \frac{1}{q})}{\lambda}} e^{-\lambda t} \| z \|_{L^p(\Omega)} \quad (t > 0) \quad (22) \]

for all \( z \in L^p(\Omega) \).

**Proof of Theorem 1.1.** We let \( \tau \in (0, T_{\max}) \). In view of Lemma 2.1 it is sufficient to make sure that

\[ \| u(t) \|_{L^\infty(\Omega)} + \| v(t) \|_{L^\infty(\Omega)} + \| w(t) \|_{L^\infty(\Omega)} \leq C(\tau), \quad t \in (\tau, T_{\max}) \]

holds with some \( C(\tau) > 0 \). We let \( \rho \in \left( \frac{2\rho_0}{2\rho_1}, 1 \right) \). This means \( 1 < 2\rho - \frac{n}{p} \). Writing

\[ w = d_3 (\Delta - \delta/d_3) w + h(u, v, w) + \delta w, \]

and applying the variation of constants formula for \( w \), we have

\[ w(t) = e^{d_3 (\Delta - \delta/d_3) t} w_0 + \int_0^t e^{d_3 (t-s)(\Delta - \delta/d_3)} (h(u(s), v(s), w(s)) + \delta w(s)) \, ds. \]

From (21), (22) and (7) we obtain that for all \( t \in (\tau, T_{\max}) \),

\[ \| w(t) \|_{W^{1,\infty}(\Omega)} \leq c_1 \| (-\Delta + \delta/d_3)^p w(t) \|_{L^p(\Omega)} \]

\[ \leq c_1 c_2 t^{-\frac{p}{\rho}} e^{-\lambda t} \| w_0 \|_{L^p(\Omega)} + \int_0^t (t-s)^{-\rho} e^{-\lambda (t-s)} \| h(u(s), v(s), w(s)) + \delta w(s) \|_{L^p(\Omega)} \, ds \]

\[ \leq c_1 c_2 \int_0^t \int_0^t (t-s)^{-\rho} e^{-\lambda (t-s)} \, ds + c_1 c_2 \int_0^t \int_0^t (t-s)^{-\rho} e^{-\lambda (t-s)} \, ds \]

\[ \| w(t) \|_{W^{1,\infty}(\Omega)} \leq c_1 c_2 \left( \int_0^t \int_0^t (t-s)^{-\rho} e^{-\lambda (t-s)} \, ds \right) \]

\[ =: C_w(\tau). \quad (23) \]

Since (9) implies \( \chi_1' < 0 \), it follows from (20) and (23) that for all \( t \in (\tau/2, T_{\max}) \),

\[ \| u(t) \chi_1(w(t)) \nabla w(t) \|_{L^p(\Omega)} \leq \chi_1(0) \| u(t) \|_{L^p(\Omega)} \| \nabla w(t) \|_{L^\infty(\Omega)} \]

\[ \leq \chi_1(0) \sup_{0 \leq t < T_{\max}} \| u(t) \|_{L^p(\Omega)} C_w(\tau/2) =: c_4. \quad (24) \]
Employing the variation of constants formula for $u$ yields
\[
u(t) = e^{d_1(t-\tau/2)(\Delta-1)}u \left( \frac{\tau}{2} \right) - \int_{\tau/2}^{t} e^{d_1(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s))
\]
\[+ \int_{\tau/2}^{t} e^{d_1(t-s)(\Delta-1)}[(\mu_1 + d_1)u(s) - \mu_1 u(s)^2 - \mu_1 u(s)v(s)]
\]
\[=: J_1 + J_2 + J_3, \quad t \in (\tau, T_{\max}).
\]

Let $\eta \in \left( \frac{n}{2p}, \frac{1}{2} \right)$ and $\varepsilon \in (0, \frac{1}{2} - \eta)$. Then we see that $0 < 2\eta - \frac{n}{p}$ and $\eta + \varepsilon + \frac{1}{2} < 1$. By (21), (22) and Lemma 2.3 we observe that for all $t \in (\tau, T_{\max})$,
\[
\|J_1\|_{L^\infty(\Omega)} = \left\|e^{d_1(t-\tau/2)(\Delta-1)}u \left( \frac{\tau}{2} \right) \right\|_{L^\infty(\Omega)}
\]
\[\leq c_1 \left\|(-\Delta + 1)^\eta e^{d_1(t-\tau/2)(\Delta-1)}u \left( \frac{\tau}{2} \right) \right\|_{L^p(\Omega)}
\]
\[\leq c_1 c_2 \left(t - \frac{\tau}{2}\right)^{-\eta} e^{-\lambda t} \left\|u \left( \frac{\tau}{2} \right) \right\|_{L^p(\Omega)}
\]
\[\leq 2^n c_1 c_2 \tau^{-\eta} e^{-\lambda t} \sup_{0 \leq s < T_{\max}} \|u(t)\|_{L^p(\Omega)}.
\]

Now we recall the well-known fact ([6]). Let $p \in (1, \infty)$, then there exists $\lambda > 0$ such that for every $\varepsilon > 0$ we can find $c_5 > 0$ satisfying
\[
\|(-\Delta + k)^\eta e^{\lambda t} \nabla \cdot w\|_{L^p(\Omega)} \leq c_5 t^{-\eta - \varepsilon - \frac{1}{2}} e^{-\lambda t} \|w\|_{L^p(\Omega)} \quad (t > 0)
\]
for all $\mathbb{R}^n$-valued $w \in L^p(\Omega)$. Using (21), (24) and (25), we obtain
\[
\|J_2\|_{L^\infty(\Omega)} \leq \int_{\tau/2}^{t} \left\|e^{d_1(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s)) \right\|_{L^\infty(\Omega)} ds
\]
\[\leq c_1 \int_{\tau/2}^{t} \left\|(-\Delta + 1)^\eta e^{d_1(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s)) \right\|_{L^p(\Omega)} ds
\]
\[\leq c_1 c_5 \int_{\tau/2}^{t} (t - s)^{-\eta - \varepsilon - 1/2} e^{-(\nu + d_1)(t-s)} \|u(s)\chi_1(w(s))\nabla w(s)\|_{L^p(\Omega)} ds
\]
\[\leq c_1 c_5 \tau^{-\eta - \varepsilon - 1/2} e^{-(\nu + d_1)\tau} d\tau.
\]

Since the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ has the order preserving property, we infer
\[
J_3 \leq \int_{\tau/2}^{t} e^{d_1(t-s)(\Delta-1)} \left[ -\mu_1 \left( u(s) - \frac{\mu_1 + d_1}{2\mu_1} \right)^2 + \frac{(\mu_1 + d_1)^2}{4\mu_1} \right] ds
\]
\[\leq \frac{(\mu_1 + d_1)^2}{4\mu_1} \int_{\tau/2}^{t} e^{-d_1(t-s)} ds,
\]
and moreover, by the maximum principle we have
\[
J_3 \leq \frac{(\mu_1 + d_1)^2}{4d_1\mu_1} \int_{\tau/2}^{t} e^{-d_1(t-s)} ds
\]
\[\leq \frac{(\mu_1 + d_1)^2}{4d_1\mu_1} (1 - e^{-\frac{d_1\tau}{4}}).
\]
Therefore we obtain that there exists $C_u(\tau) > 0$ such that
\[ u(t) \leq \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3 \leq C_u(\tau), \quad t \in (\tau, T_{\max}). \]
The positivity of $u$ yields that $\|u(t)\|_{L^\infty(\Omega)} \leq C_u(\tau)$ for all $t \in (\tau, T_{\max}).$

The same argument as for $u$ gives the $L^\infty(\Omega)$ bound for $v$. Finally, the Hölder continuity of the solution $(u, v, w)$ comes from standard parabolic regularity theory ([9]). This completes the proof of Theorem 1.1.

3. Asymptotic behavior. Case 1: $a_1, a_2 \in (0, 1)$. In this section we will establish asymptotic stability of solutions to (1) in the case $a_1, a_2 \in (0, 1)$. For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 3.1 (see [1, Lemma 3.1]). Suppose $f : (1, \infty) \to \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t) \, dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.

Lemma 3.2. Let $a, b, c, d, e, f \in \mathbb{R}$. Suppose that $a > 0$, $d - \frac{b^2}{4a} > 0$, $f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0$. Then
\[ ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0 \] holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain
\[ ax^2 + bxy + cxz + dy^2 + eyz + fz^2 = a \left( x + \frac{by + cz}{2a} \right)^2 + \left( d - \frac{b^2}{4a} \right) \left( y + \frac{2ae - bc}{4a(4ad - b^2)} \right)^2 + \left( f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} \right) z^2. \]
In view of the above equation, (26) leads to (27).

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 3.3. Let $a_1, a_2 \in (0, 1)$ and $(u, v, w)$ a solution to (1). Under the conditions (4)–(10) and (12)–(14), there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions $E_1$ and $F_1$ defined by
\[ E_1(t) := \int_\Omega \left( u - u^* - u^* \log \frac{u}{u^*} \right) + \delta_1 \frac{a_1 M_1}{a_2 M_2} \int_\Omega \left( v - v^* - v^* \log \frac{v}{v^*} \right) + \delta_2 \frac{\int \left( w - w^* \right)^2}{2} \]
and
\[ F_1(t) := \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 + \int_\Omega |\nabla w|^2 \]
satisfy
\[ \frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \quad (t > 0). \]
Proof. Thanks to (12)–(14) we can choose $\delta_1 > 0$ defined in (12)–(14) and $\delta_2 > 0$ satisfying

$$\frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_1d_3} < \delta_2 < \frac{a_1\mu_1\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}{\alpha_2\alpha_1\delta_1 + \beta_2^2a_2 - \alpha_1\beta_1a_1a_2(1 + \delta_1)}$$

and

$$\frac{a_1\mu_1\chi_2(0)^2 v^*(1 + \delta_1)}{4a_2\mu_2d_2d_3} < \delta_2 < \frac{a_1\mu_1\gamma(4\delta_1 - a_1a_2(1 + \delta_1)^2)}{\alpha_2\alpha_1\delta_1 + \beta_2^2a_2 - \alpha_1\beta_1a_1a_2(1 + \delta_1)}.$$  

We denote by $A_1(t), B_1(t), C_1(t)$ the functions defined as

$$A_1(t) := \int_{\Omega} \left( u - u^* - u^* \log \frac{u}{u^*} \right), \quad B_1(t) = \int_{\Omega} \left( v - v^* - v^* \log \frac{v}{v^*} \right),$$

$$C_1(t) := \frac{1}{2} \int_{\Omega} (w - w^*)^2,$$

and we write as

$$E_1(t) = A_1(t) + \delta_1 \frac{a_1\mu_1}{a_2\mu_2} B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - u^* \log s$ ($s \geq 0$) yields $A_1(t) = \int_{\Omega}(H(u) - H(u^*))$ is a nonnegative function for $t > 0$ (more detail, see [1, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By the straightforward calculations we infer

$$\frac{d}{dt} A_1(t) = -\mu_1 \int_{\Omega} (u - u^*)^2 - a_1\mu_1 \int_{\Omega} (u - u^*)(v - v^*) - d_1u^* \int_{\Omega} \frac{\nabla u}{u^2}$$

$$+ u^* \int_{\Omega} \chi_1(w) \nabla u \cdot \nabla w,$$

$$\frac{d}{dt} B_1(t) = -\mu_2 \int_{\Omega} (v - v^*)^2 - a_2\mu_2 \int_{\Omega} (u - u^*)(v - v^*) - d_2v^* \int_{\Omega} \frac{\nabla v}{v^2}$$

$$+ v^* \int_{\Omega} \chi_2(w) \nabla v \cdot \nabla w,$$

$$\frac{d}{dt} C_1(t) = \int_{\Omega} h_u(u - u^*)(w - w^*) + \int_{\Omega} h_v(v - v^*)(w - w^*) + \int_{\Omega} h_w(w - w^*)^2$$

$$- d_3 \int_{\Omega} |\nabla w|^2$$

with some derivatives $h_u, h_v$ and $h_w$. Hence we have

$$\frac{d}{dt} E_1(t) = I_3(t) + I_4(t),$$

where

$$I_3(t) := -\mu_1 \int_{\Omega} (u - u^*)^2 - a_1\mu_1(1 + \delta_1) \int_{\Omega} (u - u^*)(v - v^*) - \delta_1 \frac{a_1\mu_1}{a_2} \int_{\Omega} (v - v^*)^2$$

$$+ \delta_2 \int_{\Omega} h_u(u - u^*)(w - w^*) + \delta_2 \int_{\Omega} h_v(v - v^*)(w - w^*)$$

$$+ \delta_2 \int_{\Omega} h_w(w - w^*)^2$$

and

$$I_4(t) := \frac{d}{dt} E_1(t) = I_3(t) + I_4(t).$$
and
\[
I_4(t) := -d_1 u^* \int_{\Omega} \frac{|\nabla u|^2}{u^2} + u^* \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - d_2 v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\
+ v^* \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d_3 \delta_2 \int_{\Omega} |\nabla w|^2.
\]

(30)

At first, we shall show from Lemma 3.2 that there exists \( \varepsilon_1 > 0 \) such that
\[
I_3(t) \leq -\varepsilon_1 \left( \int_{\Omega} (u-u^*)^2 + \int_{\Omega} (v-v^*)^2 + \int_{\Omega} (w-w^*)^2 \right).
\]

(31)

To see this, we put
\[
g_1(\varepsilon) := \mu_1 - \varepsilon, \\
g_2(\varepsilon) := \frac{a_1 \mu_1}{a_2} (1 + \delta_1) - a_2^2 \mu_1^2 (1 + \delta_1)^2, \\
g_3(\varepsilon) := (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1 - \varepsilon)} \delta_2^2 \\
- \frac{(2h_v(\mu_1 - \varepsilon) - h_u a_1 \mu_1 (1 + \delta_1))}{4(\mu_1 - \varepsilon)} a_1^2 \mu_1^2 (1 + \delta_1)^2 \delta_2^2.
\]

Since \( \mu_1 > 0 \), we have \( g_1(0) = \mu_1 > 0 \). Due to (12), we infer
\[
g_2(0) = \frac{a_1 \mu_1}{4a_2} (4\delta_1 - a_1 a_2 (1 + \delta_1)^2) > 0.
\]

In light of (6) and the definitions of \( \delta_2 > 0 \), \( \alpha_i, \beta_i \geq 0 \) (defined in (11)) we obtain
\[
g_3(0) = \delta_2 \left( -h_w - \left( \frac{h_u^2}{4\mu_1} + \frac{a_2 (2h_v - h_u a_1 (1 + \delta_1))^2}{4a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) \\
= \delta_2 \left( -h_w - \left( \frac{h_u^2}{4\mu_1} + \frac{a_2 (4h_v^2 - 4h_u a_1 (1 + \delta_1) + h_u^2 a_1^2 (1 + \delta_1)^2)}{4a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) \\
\geq \delta_2 \left( -\gamma - \left( \frac{\alpha_2^2}{4\mu_1} + \frac{a_2 (4\beta_2^2 - 4\gamma a_1 \mu_1 (1 + \delta_1))}{4a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) \\
= \delta_2 \left( -\gamma - \left( \frac{\alpha_2^2 a_1 \delta_1 + \beta_2^2 a_2 - \alpha_1 \beta_1 a_1 \mu_1 (1 + \delta_1)}{a_1 \mu_1 (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)} \right) \delta_2 \right) > 0.
\]

Combination of the above inequalities and the continuity argument yields that there exists \( \varepsilon_1 > 0 \) such that \( g_i(\varepsilon_1) > 0 \) hold for \( i = 1, 2, 3 \). Thanks to Lemma 3.2 with
\[
a = \mu_1 - \varepsilon_1, \quad b = a_1 \mu_1 (1 + \delta_1), \quad c = -\delta_2 h_u, \\
d = \delta_2 \frac{a_1 \mu_1}{a_2} - \varepsilon_1, \quad e = -\delta_2 h_v, \quad f = -\delta_2 h_w - \varepsilon_1, \\
x = u(t) - u^*, \quad y = v(t) - v^*, \quad z = w(t) - w^*,
\]
we obtain (31) with \( \varepsilon_1 > 0 \). Lastly we will find \( \varepsilon_2 > 0 \) satisfying
\[
I_4(t) \leq -\varepsilon_2 \int_{\Omega} |\nabla w|^2.
\]

(32)
By virtue of the definition of $\delta_2 > 0$, we can find $\delta_3 \in \left(\frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_3\delta_3}, 1\right)$. Noting that $\chi'_1 < 0$ (from (9)) and then using the Young inequality, we have

$$u^* \int_\Omega \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w \leq \chi_1(0) u^* \int_\Omega \frac{|\nabla u \cdot \nabla w|}{u} \leq \frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_3\delta_3} \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{d_3\delta_2\delta_3}{1 + \delta_1} \int_\Omega |\nabla w|^2$$

and

$$v^* \delta_1 \frac{a_1\mu_1}{a_2\mu_2} \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w \leq \chi_2(0) v^* \delta_1 \frac{a_1\mu_1}{a_2\mu_2} \int_\Omega \frac{|\nabla v \cdot \nabla w|}{v} \leq \frac{\chi_2(0)^2 v^* \delta_1(1 + \delta_1)}{4d_3\delta_2} \int_\Omega \frac{|\nabla v|^2}{v^2} + \frac{d_3\delta_1\delta_2}{1 + \delta_1} \int_\Omega |\nabla w|^2.$$}

Plugging these into (30) we infer

$$I_4(t) \leq -u^* \left( d_1 - \frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_3\delta_3} \right) \int_\Omega \frac{|\nabla u|^2}{u^2}
- v^* \frac{a_1\mu_1}{a_2\mu_2} \left( d_2 - \frac{a_1\mu_1\chi_2(0)^2 v^*(1 + \delta_1)}{4d_3a_2\mu_2\delta_2} \right) \int_\Omega \frac{|\nabla v|^2}{v^2}
- d_3 \delta_2 \left( 1 - \frac{\delta_1 + \delta_3}{1 + \delta_1} \right) \int_\Omega |\nabla w|^2.$$}

We note from the definitions of $\delta_2 > 0$ and $\delta_3 > 0$ that

$$d_1 - \frac{\chi_1(0)^2 u^*(1 + \delta_1)}{4d_3\delta_3} > 0,$$
$$d_2 - \frac{a_1\mu_1\chi_2(0)^2 v^*(1 + \delta_1)}{4d_3a_2\mu_2\delta_2} > 0$$

and

$$1 - \frac{\delta_1 + \delta_3}{1 + \delta_1} = \frac{1 - \delta_3}{1 + \delta_1} > 0.$$}

Therefore we obtain that there exists $\varepsilon_2 > 0$ such that (32) holds. Combination of (29), (31) and (32) implies the end of the proof. 

**Lemma 3.4.** Let $a_1, a_2 \in (0, 1)$ and let $(u, v, w)$ be a solution to (1). Under the conditions (4)–(10) and (12)–(14), $(u, v, w)$ has the following asymptotic behavior:

$$\|u(t) - u^*\|_{L^\infty(\Omega)} \to 0, \|v(t) - v^*\|_{L^\infty(\Omega)} \to 0, \|w(t) - w^*\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty).$$

**Proof.** We let $f_1(t) := \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 \geq 0$. We have $f_1(t)$ is a nonnegative function, and thanks to the regularity of $u, v, w$ (see Theorem 1.1) we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (28) over $(1, \infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^\infty f_1(t) \, dt \leq \frac{1}{\varepsilon} E_1(1) < \infty.$$}

Therefore we obtain from Lemma 3.1 that $f_1(t) \to 0$. 

□
In order to complete the proof of Theorem 1.2, we will prepare the following lemma.

**Lemma 3.5.** Let \((\overline{u}, \overline{v}, \overline{w}) \in \mathbb{R}^3\) be any solution of (1) and \((u, v, w)\) a global bounded classical solution to (1). Suppose that there exist two decreasing functions \(h_1, h_2\) on \((0, \infty)\) and \(t_0 > 0\) such that

\[
\int_\Omega (u - \overline{u})^2 + \int_\Omega (v - \overline{v})^2 + \int_\Omega (w - \overline{w})^2 \leq h_1(t),
\]

\[
\left( \int_{t-1}^{t} \int_\Omega |\nabla w|^2 \right)^{\frac{1}{n+2}} \leq h_2(t) \quad \text{for all } t > t_0.
\]

Then there exist \(C > 0\) and \(t_1 > 0\) such that

\[
\|u(t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(t) - \overline{v}\|_{L^\infty(\Omega)} + \|w(t) - \overline{w}\|_{L^\infty(\Omega)} \leq C([h_1(t-1)]^{\frac{1}{2}} + h_2(t))
\]

for all \(t > t_1\).

**Proof.** The arguments in [1, Lemma 3.6] and Theorem 1.1 lead to the proof of this lemma. \(\square\)

**Proof of Theorem 1.2.** From the L'Hôpital theorem applied to \(H_1(s) := s - u^* \log s\) we can see

\[
\lim_{s \to u^*} \frac{H_1(s) - H_1(u^*)}{(s - u^*)^2} = \lim_{s \to u^*} \frac{H_1''(s)}{2} = \frac{1}{2u^*}.
\]

(33)

In view of the combination of (33) and \(\|u - u^*\|_{L^\infty(\Omega)} \to 0\) from Lemma 3.4 we obtain that there exists \(t_0 > 0\) such that

\[
\frac{1}{4u^*} \int_\Omega (u - u^*)^2 \leq A_1(t) = \int_\Omega (H(u) - H(u^*)) \leq \frac{1}{u^*} \int_\Omega (u - u^*)^2 \quad (t > t_0).
\]

(34)

A similar argument yields that there exists \(t_1 > t_0\) such that

\[
\frac{1}{4v^*} \int_\Omega (v - v^*)^2 \leq B_1(t) \leq \frac{1}{v^*} \int_\Omega (v - v^*)^2 \quad (t > t_1).
\]

(35)

We infer from (34) and the definitions of \(E_1(t), F_1(t)\) that \(E_1(t) \leq c_6 F_1(t)\) for all \(t > t_1\) with some \(c_6 > 0\). Plugging this into (28), we have

\[
\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \leq -\frac{\varepsilon}{c_6} E_1(t) \quad (t > t_1),
\]

which implies that there exist \(c_7 > 0\) and \(\ell > 0\) such that

\[
E_1(t) \leq c_7 e^{-\ell t} \quad (t > t_1).
\]

Thus we obtain from (34) and (35) that

\[
\int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 \leq c_8 E_1(t) \leq c_7 e^{-\ell t}
\]

for all \(t > t_1\) with some \(c_8 > 0\). Moreover, there exists \(c_9 > 0\) such that

\[
\int_{t-1}^{t} \int_\Omega |\nabla w|^2 \leq \int_{t-1}^{t} F(t) \leq -\frac{1}{\varepsilon} \int_{t-1}^{t} \frac{d}{dt} E_1(t) \leq \frac{1}{\varepsilon} E(t-1) - c_9 e^{-\ell t}.
\]

Thanks to Lemma 3.5, we achieve that there exist \(C > 0\) and \(\lambda > 0\) such that

\[
\|u(t) - u^*\|_{L^\infty(\Omega)} + \|v(t) - v^*\|_{L^\infty(\Omega)} + \|w(t) - w^*\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0).
\]

This completes the proof of Theorem 1.2. \(\square\)
4. Asymptotic behavior. Case 2: \( a_1 \geq 1 > a_2 > 0 \). In this section we will prove asymptotic stability in the case \( a_1 \geq 1 > a_2 > 0 \).

Lemma 4.1. Let \( a_1 \geq 1 > a_2 > 0 \) and let \((u, v, w)\) be a solution to (1). Under the conditions (4)–(10) and (15)–(16), there exist \( \delta_1, \delta_2 > 0, \varepsilon > 0 \) and \( a'_1 \geq 1 \) such that the nonnegative functions \( E_2 \) and \( F_2 \) defined by

\[
E_2(t) := \int_\Omega u + \delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} \int_\Omega (v - 1 - \log v) + \frac{\delta_2}{2} \int_\Omega (w - \bar{w})^2
\]

and

\[
F_2(t) := \int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 + \int_\Omega |\nabla w|^2
\]

satisfy

\[
\frac{d}{dt} E_2(t) \leq -\varepsilon F_2(t) - \mu_1 (a'_1 - 1) \int_\Omega u \quad (t > 0).
\]  

Proof. Thanks to (15)–(16) we can take \( \delta_1 > 0, a'_1 \geq 1 \) defined in (15)–(16) and choose \( \delta_2 > 0 \) such that

\[
\frac{a'_1 \mu_1 \gamma (0)^2 \delta_1}{4a_2 \mu_2 a_3} < \delta_2 < \frac{a'_1 \mu_1 \gamma (4\delta_1 - a'_1 a_2 (1 + \delta_1)^2)}{a_2^2 a_1^2 \delta_1 + \beta_2^2 a_2 - a_1 \beta_1 a'_1 a_2 (1 + \delta_1)}.
\]

We denote by \( A_2(t), B_2(t), C_2(t) \) the nonnegative functions defined as

\[
A_2(t) := \int_\Omega u, \quad B_2(t) = \int_\Omega (v - 1 - \log v),
\]

\[
C_2(t) := \frac{1}{2} \int_\Omega (w - \bar{w})^2,
\]

and we write as

\[
E_2(t) = A_2(t) + \delta_1 \frac{a'_1 \mu_1}{a_2 \mu_2} B_2(t) + \delta_2 C_2(t).
\]

Then by the straightforward calculations we infer

\[
\frac{d}{dt} A_2(t) \leq -\mu_1 \int_\Omega u^2 - a'_1 \mu_1 \int_\Omega u(v - 1) - \mu_1 (a'_1 - 1) \int_\Omega u,
\]

\[
\frac{d}{dt} B_2(t) = -\mu_2 \int_\Omega (v - 1)^2 - a_2 \mu_2 \int_\Omega u(v - 1) - d_2 \int_\Omega \frac{|\nabla v|^2}{v^2}
\]

\[
+ \int_\Omega \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w,
\]

\[
\frac{d}{dt} C_2(t) = \int_\Omega h_u (w - \bar{w}) + \int_\Omega h_v (v - 1)(w - \bar{w}) + \int_\Omega h_w (w - \bar{w})^2
\]

\[
- d_3 \int_\Omega |\nabla w|^2
\]

with some derivatives \( h_u, h_v \) and \( h_w \). Hence we have

\[
\frac{d}{dt} E_2(t) \leq I_5(t) + I_6(t) - \mu_1 (a'_1 - 1) \int_\Omega u,
\]  

(37)
where

\[ I_5(t) := -\mu_1 \int_\Omega u^2 - a_1 \mu_1 (1 + \delta_1) \int_\Omega u(v - 1) - \delta_1 \frac{a_1 \mu_1}{a_2} \int_\Omega (v - 1)^2 \]

\[ + \delta_2 \int_\Omega h_u(w - \bar{w}) + \delta_2 \int_\Omega h_v(v - 1)(w - \bar{w}) + \delta_2 \int_\Omega h_w(w - \bar{w})^2 \]

and

\[ I_6(t) := -d_2 \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{\| \nabla v \|^2}{v^2} + \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{\chi_1(w)}{v} \nabla v \cdot \nabla w - d_3 \delta_2 \int_\Omega |\nabla w|^2. \quad (38) \]

From the same argument as in the proof of Lemma 3.3 we obtain that there exists \( \varepsilon_1 > 0 \) such that

\[ I_5(t) \leq -\varepsilon_1 \left( \int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \right). \quad (39) \]

On the other hand, thanks to \( \chi_2 < 0 \) and the Young inequality, we infer that

\[ \delta_1 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w \leq \delta_1 \chi_2(0) \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v \cdot \nabla w|}{v} \leq d_3 \delta_2 \frac{a_1 \mu_1}{a_2 \mu_2} \int_\Omega \frac{|\nabla v|^2}{v^2} + \frac{a_1 \mu_1 \chi_2(0)^2}{4a_2 \mu_2 d_2} \int_\Omega |\nabla w|^2. \]

Plugging this into (38), we have

\[ I_6(t) \leq - \left( d_3 \delta_2 - \frac{a_1 \mu_1 \chi_2(0)^2 \delta_1}{4a_2 \mu_2 d_2} \right) \int_\Omega |\nabla w|^2. \]

Noting by the definition of \( \delta_2 > 0 \) that

\[ d_3 \delta_2 - \frac{a_1 \mu_1 \chi_2(0)^2 \delta_1}{4a_2 \mu_2 d_2} > 0, \]

we obtain that there exists \( \varepsilon_2 > 0 \) such that

\[ I_6(t) \leq -\varepsilon_2 \int_\Omega |\nabla w|^2. \quad (40) \]

Combination of (37), (39) and (40) implies the end of the proof. \( \square \)

**Lemma 4.2.** Let \( a_1 \geq 1 > a_2 > 0 \) and let \( (u, v, w) \) be a solution to (1). Under the conditions (4)–(10) and (15)–(16), \((u, v, w)\) has the following asymptotic behavior:

\[ \|u(t)\|_{L^\infty(\Omega)} \to 0, \quad \|v(t) - 1\|_{L^\infty(\Omega)} \to 0, \quad \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \to 0 \quad (t \to \infty). \quad (41) \]

**Proof.** From the same argument as in Lemma 3.4 we can obtain (41). \( \square \)

**Lemma 4.3.** Let \( a_1 > 1, \ a_2 \in (0, 1) \) and let \( (u, v, w) \) be a solution to (1). Under the conditions (4)–(10) and (15)–(16), there exist \( C > 0 \) and \( \lambda > 0 \) such that

\[ \|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0). \]

**Proof.** Combination of \( u \to 0 \) in \( L^\infty(\Omega) \) as \( t \to \infty \) (from Lemma 3.4) and

\[ \lim_{s \to 0} \frac{s}{s^2 + (a_1' - 1)s} = \frac{1}{a_1' - 1} \]

implies that there exists \( t_0 > 0 \) such that

\[ \frac{1}{2(a_1' - 1)} \int_\Omega u^2 + \frac{1}{2} \int_\Omega u \leq A_2(t) \leq \frac{2}{a_1' - 1} \int_\Omega u^2 + 2 \int_\Omega u \quad (t > t_0). \quad (42) \]
In light of a similar argument to seeing (34) we obtain that there exists \( t_1 > t_0 \) such that
\[
\frac{1}{4} \int_\Omega (v - 1)^2 \leq B_2(t) \leq \int_\Omega (v - 1)^2 \quad (t > t_1).
\] (43)
The definitions of \( E_2(t) \), \( F_2(t) \) and (42), (43) yield that
\[
E_2(t) \leq c_{10} \left( F_2(t) + (a'_1 - 1) \int_\Omega u \right) \quad (t > t_1)
\]
with some \( c_{10} > 0 \). Plugging this into (36), we have
\[
\frac{d}{dt} E_2(t) \leq -\varepsilon F_2(t) - \mu_1 (a'_1 - 1) \int_\Omega u \leq -\frac{\varepsilon}{c_{10}} E_2(t) \quad (t > t_1),
\]
which implies that there exist \( c_{11} \) and \( \ell > 0 \) such that
\[
E_2(t) \leq c_{11} e^{-\ell t} \quad (t > t_1).
\]
Therefore from (42) and (43) we can find \( c_{12} > 0 \) satisfying
\[
\int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \leq c_{12} E_2(t) \leq c_{11} c_{12} e^{-\ell t} \quad (t > t_1).
\]
Moreover, there exists \( c_{13} > 0 \) such that
\[
\int_{t-1}^t [\nabla w]^2 \leq \int_{t-1}^t F(t) \leq -\frac{1}{\varepsilon} \int_{t-1}^t \frac{d}{dt} E_1(t) \leq \frac{1}{\varepsilon} \int_{t-1}^t E(t - 1) \leq c_{13} e^{-\ell t}.
\]
Thanks to Lemma 3.5, we achieve that there exist \( C > 0 \) and \( \lambda > 0 \) such that
\[
\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad (t > 0),
\]
which implies the end of the proof.

**Lemma 4.4.** Let \( a_1 = 1 \), \( a_2 \in (0, 1) \) and let \( (u, v, w) \) be a solution to (1). Under the conditions (4)–(10) and (15)–(16), there exist \( C > 0 \) and \( \lambda > 0 \) such that
\[
\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \bar{w}\|_{L^\infty(\Omega)} \leq C (t + 1)^{-\lambda} \quad (t > 0).
\]

**Proof.** We have already known that there exists \( t_0 > 0 \) such that (43) holds for all \( t > t_0 \). Hence the Cauchy–Schwarz inequality and the boundedness of \( (u, v, w) \) imply that there exists \( c_{14} > 0 \) satisfying
\[
E_2(t) \leq \int_\Omega u + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \leq c_{14} \left( \int_\Omega u^2 \right)^{\frac{1}{2}} + c_{14} \left( \int_\Omega (v - 1)^2 \right)^{\frac{1}{2}} + c_{14} \left( \int_\Omega (w - \bar{w})^2 \right)^{\frac{1}{2}} \leq \sqrt{3} c_{14} \left( \int_\Omega u^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \bar{w})^2 \right)^{\frac{1}{2}} = c_{14} \sqrt{3 F_2(t)}
\]
for all \( t > t_0 \). Thus from (36) we can find \( c_{15} > 0 \) such that
\[
\frac{d}{dt} E_2(t) \leq -c_{15} E_2(t) \quad (t > t_0),
\]
which implies that there exists \( c_{16} > 0 \) satisfying
\[
E_2(t) \leq \frac{c_{16}}{t + 1} \quad (t > t_0).
\]
Therefore we have
\[ \int_{\Omega} u^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \bar{w})^2 \leq c_{17} E_2(t) \leq \frac{c_{16} c_{17}}{t + 1} \quad (t > t_0) \]
with some \( c_{17} > 0 \). Moreover, (36) yields
\[ \int_{t-1}^t |\nabla w|^2 \leq \int_{t-1}^t F_2(t) \leq -\frac{1}{\varepsilon} \int_{t-1}^t \frac{d}{dt} E_2(t) \leq -\frac{1}{\varepsilon} E_2(t-1) \leq \frac{c_{16}}{\varepsilon(t+1)} \quad (t > t_0). \]
From Lemma 3.5 we obtain that there exist \( C > 0 \) and \( \lambda > 0 \) such that
\[ \|u\|_{L^\infty(\Omega)} + \|v - 1\|_{L^\infty(\Omega)} + \|w - \bar{w}\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\lambda} \quad (t > 0), \]
which means the end of the proof. □

\textbf{Proof of Theorem 1.4.} Part (i) follows from Lemma 4.3, while (ii) is contained in Lemma 4.4. □

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