Lefschetz classes of simple factors of Fermat Jacobian of prime degree over finite fields

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Abstract

We give a necessary and sufficient condition in terms of a matrix for which all Tate classes are Lefschetz for simple abelian varieties over an algebraic closure of a finite field. As an application, we prove under an assumption that all Tate classes are Lefschetz for simple factors of Fermat Jacobian of prime degree.

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1 Introduction

Let $p$ be a prime number. Let $\mathbb{F}_p$ be a finite field with $p$-elements and let $\bar{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}_p$. Let $\ell$ be a prime number different from $p$. Let $A_0$ be an abelian variety over a finite subfield $\mathbb{F}_q$ of $\bar{\mathbb{F}}$ and let $A$ be the abelian variety $A_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}$ over $\bar{\mathbb{F}}$. There is the cycle map

$$cl^i_A : CH^i(A) \otimes \mathbb{Q}_\ell \longrightarrow H^{2i}(A, \mathbb{Q}_\ell(i)),$$

where $CH^i(A)$ is the Chow group of algebraic cycles on $A$ of codimension $i$ modulo rational equivalence, and $H^{2i}(A, \mathbb{Q}_\ell(i))$ is the $\ell$-adic étale cohomology of $A$. Then we know that the image of the map $cl^i_A$ is contained in the space $T^i_\ell(A)$ of $\ell$-adic Tate classes of degree $i$ on $A$ which is defined as follows:

$$T^i_\ell(A) := \lim_{\substack{\longrightarrow \\ L/\mathbb{F}_q}} H^{2i}(A, \mathbb{Q}_\ell(i))^{Gal(\bar{\mathbb{F}}/L)}.$$

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Here $L/F_q$ runs over all finite extensions of $F_q$. We call the elements of the image of $cl^i_A$ the algebraic classes of degree $i$.

**Conjecture.** For all $i \geq 0$, $\text{Im}(cl^i_A) = T^i_\ell(A)$.

This is conjectured by Tate [14, Conjecture 1]. In this paper, we call the conjecture the Tate conjecture. The Tate conjecture for $A$ implies the Tate conjecture for $A_0/F_q$, that is, the cycle map $cl^i_{A_0}: \text{CH}^i(A_0) \otimes \mathbb{Q}_\ell \to H^{2i}(A, \mathbb{Q}_\ell(i))^{\text{Gal}(F/F_q)}$ is surjective. On the other hand, Tate [15] proved that $cl^1_{A_0}$ is surjective for all abelian varieties over finite fields. Therefore all Tate classes of degree 1 are divisor classes on $A$. The elements of the $\mathbb{Q}_\ell$-subalgebra of $T^i_\ell(A) := \bigoplus_{i \geq 0} T^i_\ell(A)$ generated by $T^1_\ell(A)$ are called the Lefschetz classes on $A$. If all $\ell$-adic Tate classes on $A$ are Lefschetz, then the Tate conjecture holds for $A$. But there are examples where not all Tate classes are Lefschetz and the Tate conjecture holds ([9, Example 1.8]). If $A$ is a product of elliptic curves, then all $\ell$-adic Tate classes on $A$ are Lefschetz by a result of Spiess [13]. Zarhin [19], Lenstra and Zarhin [6] gave other example (c.f. [9, A.7]). Kowalski [5] proved that for certain simple ordinary abelian varieties, all $\ell$-adic Tate classes are Lefschetz. In this paper, for a simple factor of the Jacobian of Fermat curve of prime degree, we give a necessary and sufficient condition for which all $\ell$-adic Tate classes are Lefschetz.

Let $m$ be a positive integer prime to $p$. Let $J_m$ be the Jacobian of Fermat curve of degree $m$ defined by the equation

$$x_0^m + x_1^m + x_2^m = 0 \subset \mathbb{P}^2_{F_q}.$$ 

Shioda-Katsura [10, Proposition 3.10] proved that $J_m$ is isogenous to a product of supersingular elliptic curves if and only if $p^\nu \equiv -1 \mod m$ for some $\nu$. If $m$ is a prime $l$ different from $p$, then the condition that $p^\nu \equiv -1 \mod l$ for some $\nu$ is equivalent to the condition that the residual degree $f$ of $p$ in $\mathbb{Q}((\mu_l))$ is even. Hence if $f$ is even, then a simple factor $A$ of $J_l$ is isogenous to a supersingular elliptic curve over $F$. In this case, all $\ell$-adic Tate classes on all powers of $A$ are Lefschetz. In this paper, we consider in case that $f$ is odd.

To state main result, we recall Jacobi sum. Let $\alpha$ be an element of the set

$$A^1_l := \{(a_0, a_1, a_2) \mid a_i \in \mathbb{Z}/l, a_i \neq 0, a_0 + a_1 + a_2 \equiv 0\}.$$ 

Then Jacobi sum $j(\alpha)$ is defined by

$$j(\alpha) = -\sum_{\begin{subarray}{c}
1+v_1+v_2=0 \\
v_i \in \mathbb{P}^\times_{p^f}
\end{subarray}} \psi(v_1)^{a_1} \psi(v_2)^{a_2}.$$
Here $\psi$ is a fixed character of order $l$ of $\mathbb{F}_p^\times$. By definition, $j(\alpha) \in \mathbb{Q}(\mu_l)$ where $\mu_l$ is the set of $l$-th root of unity. We identify $(\mathbb{Z}/l)^\times$ with the Galois group of $\mathbb{Q}(\mu_l)/\mathbb{Q}$.

Let $A_0$ be an absolute simple factor of the Jacobian of $V_l$ over $\mathbb{F}_p$, and let $A := A_0 \otimes \mathbb{F}$. We denote by $C(A)$ the center of $\text{End}_{\mathbb{F}}(A) \otimes \mathbb{Q}$. Let $\pi_0$ be the Frobenius endomorphism of $A_0$. Then $\pi_0$ can be given by Jacobi sum (cf. [20]): $\pi_0 = j(\alpha)$ for some $\alpha \in A_0$. Therefore $C(A) = \mathbb{Q}(j(\alpha))$ and $C(A) \subset \mathbb{Q}(\mu_l)$.

Our main result is the following:

**Theorem 1.1.** Let $l$ be an odd prime number different from $p$. Let $A$ be a simple factor of the Jacobian of the Fermat curve of degree $l$ over $\mathbb{F}$. Let $\alpha = (a_0, a_1, a_2)$ be an element of $A_0$ such that $C(A) = \mathbb{Q}(j(\alpha))$. Let $H_\alpha$ be the Galois group of $\mathbb{Q}(\mu_l)/C(A)$. Then all $\ell$-adic Tate classes on all powers of $A$ are Lefschetz if and only if for any odd Dirichlet character $\chi$ of $(\mathbb{Z}/l)^\times$ with $\chi|_{H_\alpha} = 1$, $\sum_{i=0}^{2} \chi(a_i) \neq 0$.

In particular, all $\ell$-adic Tate classes on all powers of $A$ are Lefschetz in the following cases:

1. $[C(A) : \mathbb{Q}] \not\equiv 0 \pmod{6}$,
2. $\alpha$ is equal to an element of the form $(a, a, b)$ up to permutation,
3. $[C(A) : \mathbb{Q}] = 2^{s+1} \cdot 3$ ($s \geq 0$).

**Corollary 1.2** (Corollary 4.1). Let $l$ be an odd prime number different from $p$. Let $J(C_l)$ be the Jacobian variety of the hyperelliptic curve

$$C_l : y^2 = x^l - 1.$$  

Then all $\ell$-adic Tate classes on all powers of $J(C_l)$ are Lefschetz.

This corollary is an analogue of Shioda’s result on Hodge conjecture for $J(C_l)/C$ ([12 Corollary 5.3]). In case that $p \equiv 1 \pmod{l}$, Shioda’s argument in [12] works over finite fields and shows that the above corollary holds. The argument also shows that not all $\ell$-adic Tate classes are Lefschetz and the Tate conjecture holds for $J(C_9)$. However in case that $p \not\equiv 1 \pmod{l}$, the argument needs a similar result to the key lemma in Shioda’s argument which is not proven (cf. [11, p. 181, Lemma]). In this paper, we use other argument. More precisely we use Milne’s result on the Tate conjecture (see §2).

The key of the proof of Theorem [11] is to give a necessary and sufficient condition for which all $\ell$-adic Tate classes are Lefschetz for simple abelian varieties in terms of a matrix as follows: for a simple abelian variety $A$ over
In $\mathbb{F}$, we define a matrix $T_A$ whose rank depends only on $A$ (see §3). Using Milne’s result, we prove the following:

**Theorem 1.3 (Theorem 3.1).** Let $A$ be a simple abelian variety over $\mathbb{F}$ of dimension $\geq 2$. Then all $\ell$-adic Tate classes on all powers of $A$ are Lefschetz if and only if $\text{rank } T_A = r$. Here $r$ is the half of the number of all distinct embeddings $C(A) \rightarrow \mathbb{C}$.

This paper is organized as follows: in §2, we recall Milne’s result on the Tate conjecture. In §3, we prove Theorem 1.3 and we calculate the matrix $T_A$ in a special case (Theorem 3.4). In the last section, we prove Theorem 1.1 using the necessary and sufficient condition.

**Notation**

Throughout this paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For a finite étale $\mathbb{Q}$-algebra $E$, $\Sigma_E := \text{Hom}(E, \overline{\mathbb{Q}})$. If $E$ is a field Galois over $\mathbb{Q}$, we identify $\Sigma_E$ with the Galois group $\text{Gal}(E/\mathbb{Q})$.

For a finite set $S$, $\mathbb{Z}^S$ denotes the set of functions $f : S \rightarrow \mathbb{Z}$.

An affine algebraic group is of multiplicative type if it is commutative and its identity component is a torus. For such a group $W$ over $\mathbb{Q}$, $\chi(W) := \text{Hom}(W_{\overline{\mathbb{Q}}}, \mathbb{G}_m)$ denotes the group of characters of $W$.

For a finite étale $\mathbb{Q}$-algebra $E$, $(\mathbb{G}_m)_E/\mathbb{Q}$ denotes the Weil restriction $\text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ which is characterized by $\chi((\mathbb{G}_m)_E/\mathbb{Q}) = \mathbb{Z}^{\Sigma_E}$.

For an abelian variety $A$ over $\mathbb{F}$, $\text{End}_{\mathbb{F}}^0(A)$ denotes $\text{End}_{\overline{\mathbb{F}}}(A) \otimes \mathbb{Q}$, and $C(A)$ denotes the center of $\text{End}_{\mathbb{F}}^0(A)$.

## 2 Milne’s result on the Tate conjecture

We recall Milne’s result on the Tate conjecture. For an abelian variety $A$ over $\mathbb{F}$, there are three important groups of multiplicative type $L(A)$, $M(A)$ and $P(A)$ which act on $H^{2*}(A) := \bigoplus_{i \geq 0} H^{2i}(A, \mathbb{Q}_\ell(i))$. These groups are introduced by Milne [7], [8], [9], who proved that these groups are characterized by the following properties (see [9] p. 14, Lemma): for all $r$,

(a) $H^{2*}(A^r)^{L(A)}$ is the space of Lefschetz classes.

(b) $H^{2*}(A^r)^{M(A)}$ is the space of algebraic classes, provided numerical equivalence coincides with $\ell$-adic homological equivalence.

(c) $H^{2*}(A^r)^{P(A)}$ is the space of $\ell$-adic Tate classes on $A^r$. 

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Statement (a) is proved by Milne [7]. Statement (b) is proved by using a result of Jannsen [4]. Clozel [1] proved that for an abelian variety $A$ over $\mathbb{F}$, there is a set of primes $\ell$ of positive density for which $\ell$-adic homological equivalence and numerical equivalence coincide. Deligne [2] gives a more precise version of Clozel’s result. Statement (c) is almost by definition of $P(A)$. The following theorem is due to Milne [9, p. 14, Theorem]:

**Theorem 2.1.** Let $A$ be an abelian variety over $\mathbb{F}$. Then $P(A) \subset M(A) \subset L(A)$, and

(i) the Tate conjecture holds for $A$ if and only if $P(A) = M(A)$;

(ii) all $\ell$-adic Tate classes on all powers of $A$ are Lefschetz if and only if $P(A) = L(A)$.

We now recall the definitions of the Lefschetz group and the group $P$ associated to an abelian variety over $\mathbb{F}$, and recall a description of the character group of these groups. For properties and results on the groups, we refer to Milne [7], [8] and [9].

Let $A$ be an abelian variety over $\mathbb{F}$. A polarization $\lambda : A \to A^\vee$ of $A$ determines an involution of $\text{End}_\mathbb{F}(A)$ which stabilizes $C(A)$. The restriction of the involution to $C(A)$ is independent of the choice of $\lambda$. By $\dagger$, we denote this restriction on $C(A)$.

**Definition 2.2 ([7, 4.3, 4.4], [8, pp.52-53], [9, A.3]).** The Lefschetz group $L(A)$ of $A$ is the algebraic group over $\mathbb{Q}$ such that

$$L(A)(R) = \{ \alpha \in (C(A) \otimes R)^\times \mid \alpha \alpha^\dagger \in R^\times \}$$

for all $\mathbb{Q}$-algebras $R$.

We can describe $L(A)$ as a subgroup of $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$ in terms of characters as follows ([9, A.7]): $L(A)$ is a subgroup of $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_{C(A)}}}{\{ g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0 \}}.$$

(2.1)

Here $\iota g$ is a function sending an element $\sigma$ of $\Sigma_{C(A)}$ to $g(\iota \sigma)$, and $\sum g(\sigma)$ denotes $\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma)$.

**Definition 2.3 ([8, §4], [9, A.7]).** Let $A_0$ be a model of $A$ over a finite field $\mathbb{F}_q \subset \mathbb{F}$, and let $\pi_0$ be the Frobenius of $A_0$. Then the group $P(A)$ of $A$ is the smallest algebraic subgroup of $L(A)$ containing some power of $\pi_0$. It is independent of the choice of $A_0$.
To state Milne’s result on the character group of $P$, we introduce Weil numbers and some notion which is related to Weil numbers. A Weil $q$-number of weight $i$ is an algebraic number $\pi$ such that $q^N \pi$ is an algebraic integer for some $N$ and the complex absolute value $|\sigma(\pi)|$ is $q^{i/2}$, for all embeddings $\sigma : \mathbb{Q}[\pi] \to \mathbb{C}$. Then $\pi$ is a unit at all primes of $\mathbb{Q}[\pi]$ not dividing $p$. We define the slope function $s_\pi$ of $\pi$ as follows: for any prime $p$ dividing $p$ of a field containing $\pi$,

$$s_\pi(p) = \frac{\text{ord}_p(\pi)}{\text{ord}_p(q)}.$$  \hspace{1cm} (2.2)

For the definition of Weil numbers, $s_\pi(p) + s_\pi(i_p) = i (= \text{wt}(\pi))$. Here $i$ is complex conjugation on $\mathbb{C}$. The slope function determines a Weil $q$-number up to a root of unity.

Let $\pi$ be a Weil $p^{f}$-number and let $\pi'$ be a Weil $p^{f'}$-number. We say $\pi$ and $\pi'$ are equivalent if $\pi'^f = \pi^{f'} \cdot \zeta$ for some root of unity $\zeta$. We define a Weil germ to be an equivalent class of Weil numbers. For a Weil germ $\pi$, the slope function and weight of $\pi$ are the slope function (see (2.2)) and weight of any representative of $\pi$, and $\mathbb{Q}[\pi]$ is defined to be the smallest subfield of $\mathbb{Q}^{al}$ containing a representative of $\pi$.

Now assume that $A$ is simple and that $\text{End}_F^0(A) = \text{End}_F^0(A_0)$. Then the Frobenius endomorphism $\pi_0$ of $A_0$ generates $C(A)$ over $\mathbb{Q}$, i.e. $C(A) = \mathbb{Q}[\pi_0]$. We know that the Frobenius endomorphism $\pi_0$ of $A_0$ is a Weil $q$-number of weight 1. Let $\pi_A$ denote the germ represented by $\pi_0$. Milne’s result on the character of $P(A)$ is the following ([9, A.7]): let $g$ be a character of $L(A)$. Then $g$ is trivial on $P(A)$ if and only if for all primes $v$ dividing $p$ of a field containing all conjugates $\sigma(\pi_0)$,

$$\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma) s_{\sigma\pi_A}(v) = 0.$$ \hspace{1cm} (2.3)

Note that $s_{\sigma\pi_A}(v) = s_{\pi_A}(\sigma^{-1}v)$.

### 3 Necessary and sufficient condition

Let $A$ be a simple abelian variety over $\mathbb{F}$. Let $A_0$ be a model of $A$ over a finite subfield $\mathbb{F}_q \subset \mathbb{F}$ with property that $\text{End}_F^0(A_0) = \text{End}_F^0(A)$. We take a finite Galois extension $K$ of $\mathbb{Q}$ containing all conjugates of $C(A)$. Let $\{p_1, \ldots, p_d\}$ be the set of primes of $K$ dividing $p$. We assume that $\dim A \geq 2$. Then $C(A)$ is totally imaginary and CM field (cf. [10], p.97).
Let \( \{\sigma_1, \ldots, \sigma_r, \iota \sigma_1, \ldots, \iota \sigma_r\} \) be the set of all distinct embeddings \( C(A) \to \mathbb{C} \). We define a \( d \times r \) matrix \( T_A \) as follows:

\[
T_A = (a_{ij}), \quad a_{ij} := s_{\sigma_j \pi_0}(p_i) - \frac{1}{2} (1 \leq i \leq d, 1 \leq j \leq r).
\]

Then the matrix \( T_A \) is independent of the choice of the model \( A_0/\mathbb{F}_q \), but depends on the ordering of the \( p_i \) and the \( \sigma_j \). The rank of \( T_A \) depends only on \( A \). Using Milne’s result on the character group of \( L(A) \) and \( P(A) \), we prove the following:

**Theorem 3.1.** Let \( A \) be a simple abelian variety over \( \mathbb{F} \) of dimension \( \geq 2 \). Then \( P(A) = L(A) \) if and only if \( \text{rank } T_A = r \).

We first describe the kernel of the natural map \( \mathbb{Z}^{\Sigma_{C(A)}} \to \chi(L(A)) \) in terms of a matrix. Let \( J \) be the \((r + 1) \times 2r\) matrix

\[
\begin{pmatrix}
I_r & -I_r \\
0 & B
\end{pmatrix}, \quad B = (1 \cdots 1)
\]

where \( I_r \) is the \( r \times r \) identity matrix. We consider the set of character functions for \( \sigma_i, \iota \sigma_i \) as a basis of \( \mathbb{Z}^{\Sigma_{C(A)}} \). We then have the following:

**Lemma 3.2.** The kernel of the natural map \( \phi : \mathbb{Z}^{\Sigma_{C(A)}} \to \chi(L(A)) \) coincides with the kernel of the map \( \mathbb{Z}^{\Sigma_{C(A)}} \to \mathbb{Z}^{r+1} \) defined by \( J \).

**Proof.** From (2.1), the kernel of \( \phi \) is \( \{g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0\} \). Here \( \iota g \) is a function sending an element \( \sigma \) of \( \Sigma_{C(A)} \) to \( g(\iota \sigma) \), and \( \sum g(\sigma) \) denotes \( \sum_{\sigma \in \Sigma_{C(A)}} g(\sigma) \). Therefore the kernel of \( \phi \) is the kernel of the map \( \mathbb{Z}^{\Sigma_{C(A)}} \to \mathbb{Z}^{2r+1} \) defined by the \((2r + 1) \times 2r\) matrix

\[
J' := \begin{pmatrix}
I_r & -I_r \\
-I_r & I_r \\
0 & B
\end{pmatrix}.
\]

By row operations, \( J' \) is equivalent over \( \mathbb{Z} \) to

\[
\begin{pmatrix}
I_r & -I_r \\
0 & 2B \\
0_r & 0_r
\end{pmatrix}
\]

where \( 0_r \) is the \( r \times r \) matrix with all entries equal to 0. From this we have \( \text{Ker}(J) = \text{Ker}(J') \).
Next we describe the condition (2.3) in terms of a matrix. Let $T'$ be the $d \times 2r$ matrix $(U V)$, where $U$ is the $d \times r$ matrix $(s_{\sigma_j, \pi_0}(p_i))$ and $V$ is the $d \times r$ matrix $(s_{\sigma_j, \pi_0}(p_i)) (1 \leq i \leq d, 1 \leq j \leq r)$. From (2.3), a character $g$ of $L(A)$ is trivial on $P(A)$ if and only if $g$ belongs to the kernel of the map $\mathbb{Z}^{\Sigma C(A)} \to \mathbb{Z}^d$ defined by $T'$. By Lemma 3.2, $L(A) = P(A)$ if and only if $\ker(J) = \ker(T')$. Hence we deduce Theorem 3.1 from the following lemma:

**Lemma 3.3.** Let notation be as above.

(1) $\text{rank } T' = \text{rank } T_A + 1$.

(2) $\ker(J) = \ker(T')$ if and only if $\text{rank } T' = r + 1$.

**Proof.** (1) From the equation $s_{\sigma_j, \pi_0}(p_i) + s_{\sigma_j, \pi_0}(q_i) = 1$ for all $i, j$, we easily see that the matrix $T'$ is column equivalent to the $d \times 2r$ matrix

$$
\begin{pmatrix}
C & 0_{d \times (r-1)} \\
0 & U \\
\vdots & \\
1 & 
\end{pmatrix}
$$

where $0_{d \times (r-1)}$ is the $d \times (r-1)$ matrix with all entries equal to 0.

Since the complex conjugation $\iota \in G$ acts on the set $\{p_1, \ldots, p_d\}$, after renumbering the $p_i$ if necessary, there is a positive integer $t$ such that

$$
\begin{align*}
\iota p_i &= p_{i+t} & \text{for } 1 \leq i \leq t, \\
\iota p_i &= p_i & \text{for } 2t + 1 \leq i \leq d.
\end{align*}
$$

From this, we obtain that for each $j$,

$$
\begin{align*}
s_{\sigma_j, \pi_0}(p_i) + s_{\sigma_j, \pi_0}(q_i) &= 1 & \text{for } 1 \leq i \leq t, \\
s_{\sigma_j, \pi_0}(p_i) &= \frac{1}{2} & \text{for } 2t + 1 \leq i \leq d.
\end{align*}
$$

These equations show that the matrix $C$ is row equivalent to the matrix

$$
\begin{pmatrix}
U' & 0_{(t+1) \times (r-1)} \\
0_{(d-t+1) \times (r+1)} & 0_{(d-t+1) \times (r-1)}
\end{pmatrix},
$$

where $U'$ is the $t \times r$ matrix $(s_{\sigma_j, \pi_0}(p_i) - \frac{1}{2}) (1 \leq i \leq t, 1 \leq j \leq r)$.

On the other hand, from (3.1) we also obtain that the matrix $T_A$ is row equivalent to the matrix

$$
\begin{pmatrix}
U'' & 0_{(d-t) \times r}
\end{pmatrix},
$$

where $U''$ is the $t \times r$ matrix $(s_{\sigma_j, \pi_0}(p_i) - \frac{1}{2}) (1 \leq i \leq t, 1 \leq j \leq r)$.
Hence
\[
\text{rank } T' = \text{rank } C = \text{rank } U'' + 1 = \text{rank } T_A + 1.
\]

(2) If \( \text{Ker}(J) = \text{Ker}(T') \), then clearly \( \text{rank } T' = r + 1 \).

Conversely assume that \( \text{rank } T' = r + 1 \). Then \( T' \) is row equivalent to a matrix of the following form
\[
\begin{pmatrix}
I_{r+1} & D \\
0_{(d-r-1)\times(r+1)} & 0_{(d-r-1)\times(r-1)}
\end{pmatrix}.
\]

Put \( D = (b_{ij}) \). From the equation \( s_{\sigma_i\pi_0}(p_i) + s_{\sigma_j\pi_0}(p_i) = 1 \), we have
\[
1 + b_{ii} = 1 \text{ if } i \neq 1, r + 1,
0 + b_{ij} = 1 \text{ otherwise}.
\]

Hence the matrix \((I_{r+1} D)\) is row equivalent to the matrix \( J \), which implies \( \text{Ker}(T') = \text{Ker}((I_{r+1} D)) = \text{Ker}(J) \).

\[
3.1 \text{ Calculation in special case}
\]

Applying Theorem 2.1 and Theorem 3.1, we have the following:

**Theorem 3.4.** Let \( A \) be a simple abelian variety over \( F \) of dimension \( \geq 2 \). Assume that \( C(A)/\mathbb{Q} \) is abelian with Galois group \( G \). Then all \( \ell \)-adic Tate classes on all powers of \( A \) are Lefschetz if and only if for any character \( \varphi \) of \( G \) with \( \varphi(\iota) = -1 \),
\[
\sum_{\sigma \in G} e(\sigma)\varphi(\sigma) \neq 0.
\]

Here \( e(\sigma) = s_{\pi_0}(\sigma p) - \frac{1}{2} \) where \( p \) is a prime of \( C(A) \) dividing \( p \).

In particular, all \( \ell \)-adic Tate classes on all powers of \( A \) are Lefschetz if one of the following condition holds:

1. \( G \) is cyclic of order \( 2^{s+1} \) \( (s \geq 0) \),
2. \( G \) is cyclic of order \( 2^{s+1}l \) with \( l \) odd prime and the order of \( \text{End}_{\mathbb{F}}(A) \) in the Brauer group \( \text{Br}(C(A)) \) of \( C(A) \) is odd.

To prove this theorem, we need the following proposition:

**Proposition 3.5.** Let \( A \) be a simple abelian variety over \( F \). Assume that \( C(A) \) is Galois over \( \mathbb{Q} \). Let \( G \) be the Galois group of \( C(A) \) over \( \mathbb{Q} \). Let \( p \) be a prime ideal of \( C(A) \) lying over \( p \). If the decomposition group \( G_p \) of \( p \) in \( C(A) \) is a normal subgroup of \( G \), then \( G_p = 1 \), namely \( p \) is completely decomposed in \( C(A) \).
Proof. Let a prime decomposition of $\pi_0$ in $C(A)$ be as follows:

$$\pi_0 = \prod_{\sigma \in G/G_p} \sigma(p)^e_\sigma.$$ 

Let $\tau$ be an element of $G_p$. Since $G_p$ is normal, we have $(\pi_0) = (\tau \pi_0)$ as ideals. From Lemma 3.6 below, we obtain that $\tau = 1$ and that $G_p = 1$. \qed

Lemma 3.6. Let $\pi$ be a Weil germ. Let $\pi_0 \in \mathbb{Q}[\pi]$ be a representative of $\pi$. Let $K$ be a Galois extension of $\mathbb{Q}$ such that $\mathbb{Q}[\pi] \subset K$. Then there is no elements $\sigma \in \text{Gal}(K/\mathbb{Q})$ satisfying the following conditions:

1. $\sigma$ fix the ideal $(\pi_0)$,
2. $\sigma$ is not trivial on $\mathbb{Q}[\pi]$.

Proof. We assume that there is an element $\sigma \in \text{Gal}(K/\mathbb{Q})$ satisfying conditions (1) and (2). From condition (1), there is an unit $u$ of the integer ring of $K$ such that $\pi_0 = u \cdot \sigma \pi_0$. For any $\tau \in \text{Gal}(K/\mathbb{Q})$, we have $|\tau u| = 1$ since $|\tau \pi_0| = q^{1/2}$. Here we used that $\pi_0$ is a Weil $q$-number. Hence $u$ is a root of unity. Therefore we have $\pi_0^m = \sigma \pi_0^m$ for some $m > 0$. Since $\sigma$ acts on the subfield $\mathbb{Q}(\pi_0^m)$ of $\mathbb{Q}[\pi]$ trivially, $\mathbb{Q}(\pi_0^m)$ is not equal to $\mathbb{Q}[\pi]$ by condition (2).

On the other hand, since $\mathbb{Q}[\pi]$ is the smallest field containing a representative of $\pi$, we obtain that $\mathbb{Q}[\pi] \subset \mathbb{Q}(\pi_0^m)$. Hence $\mathbb{Q}(\pi_0^m) = \mathbb{Q}[\pi]$ which is a contradiction. \qed

Proof of Theorem 3.4. By Theorem 2.1 and Theorem 3.1, our task is reduced to calculate the matrix $T_A$. Since $C(A)$ is Galois over $\mathbb{Q}$, we may take $K = C(A)$. Let $G = \{\sigma_1, \ldots, \sigma_r, \iota \sigma_1, \ldots, \iota \sigma_r\}$. Here $\iota \in G$ is the complex conjugate on $C(A)$. Let $p$ be a prime of $C(A)$ dividing $p$. By Proposition 3.5, the set $\{\sigma p \mid \sigma \in G\}$ is the set of all primes of $C(A)$ dividing $p$. Let $e$ be the function $G \longrightarrow \mathbb{C}$ defined as $e(\sigma) = s_{\pi_0}(\sigma p) - \frac{1}{2}$ for $\sigma \in G$. From the proof of Theorem 3.1, the matrix $T_A$ is row equivalent to the matrix

$$\begin{pmatrix} U'' \\ 0_{r \times r} \end{pmatrix}$$

where $U''$ is the $r \times r$ matrix $(e(\sigma_i \sigma_j^{-1}))$ ($1 \leq i, j \leq r$). Since rank $T_A = \text{rank } U''$, we obtain that rank $T_A = r$ if and only if det$(U'') \neq 0$.

Now we calculate det$(U'')$. Let $\phi$ be the fixed character of $G$ with $\phi(\iota) = -1$. Let $f : G \longrightarrow \mathbb{C}$ be the function defined as $f(\sigma) = \phi(\sigma)e(\sigma)$ for $\sigma \in G$. Then we have

$$\text{det}(U'') = \text{det}\left( \left( f(\sigma_i \sigma_j^{-1}) \right) \right).$$
For any $\sigma \in G$, we have
\[
f(\iota \sigma) = \phi(\iota \sigma)e(\iota \sigma) = -\phi(\sigma)(-e(\sigma)) = f(\sigma).
\]
Hence $f$ is a function on $G' := G/\{1, \iota\}$. Now we need the following lemma:

**Lemma 3.7.** Let $G$ be a finite abelian group and let $f$ be a function on $G$ with values in some field of characteristic $0$. Then
\[
\det \left( f(\sigma \tau^{-1}) \right)_{\sigma, \tau \in G} = \prod_{\psi} \sum_{\sigma \in G} f(\sigma)\psi(\sigma).
\]
Here $\psi$ runs over all character of $G$.

For the proof of this lemma, see [18, Lemma 5.26].

From this lemma, we have
\[
\det \left( f(\sigma \tau^{-1}) \right)_{\sigma, \tau \in G'} = \prod_{\psi} \sum_{\sigma \in G'} f(\sigma)\psi(\sigma).
\] (3.2)
Here $\psi$ runs over all character of $G'$. Furthermore, by elementary calculation, we have
\[
\text{RHS of (3.2)} = \frac{1}{2} \prod_{\varphi} \sum_{\sigma \in G} e(\sigma)\varphi(\sigma).
\]
Here $\varphi$ runs over all character of $G$ with $\varphi(\iota) = -1$. By the above argument, rank $T_A = r$ if and only if for any such $\varphi$,
\[
\sum_{\sigma \in G} e(\sigma)\varphi(\sigma) \neq 0.
\]
This completes the proof of the first assertion of Theorem 3.4.

Now we put $E(\varphi) := \sum_{\sigma \in G} e(\sigma)\varphi(\sigma)$ and prove that $E(\varphi) \neq 0$ in some cases. Let $g$ be a generator of $G$. In case that condition (1) holds, then
\[
E(\varphi) = 2^{s-1} \sum_{i=0}^{2^s-1} e(g^i)\varphi(g^i). \quad \text{Since } \varphi(g) \text{ is a primitive } 2^{s+1}\text{-th root of unity, we have } [\mathbb{Q}(\zeta) : \mathbb{Q}] = 2^s. \text{ Therefore the set } \{1, \zeta, \zeta^2, \ldots, \zeta^{2^s-1}\} \text{ is a base of } \mathbb{Q}(\zeta) \text{ over } \mathbb{Q}. \text{ Therefore if } E(\varphi) = 0, \text{ then we have }
\[
eq e(1) = e(g) = \cdots = e(g^{2^s-1}) = 0.
\]
From this, we have $(\pi_0) = (p^n)$ as ideal in $C(A)$ which is fixed by the action of $\iota \in G$. Here $n := \text{ord}_p(\pi_0)$. By Lemma 3.6 we have $\iota = 1$, which is a contradiction. Therefore $E(\varphi) \neq 0$. 


Next we consider in case that the order of $G$ is $2^{s+1}l$ with $l$ an odd prime. Then $\varphi(g)$ is a primitive $2^{s+1}$-th root of unity or a primitive $2^{s+1}l$-th root of unity. If $\varphi(g)$ is a primitive $2^{s+1}$-th root of unity and if $E(\varphi) = 0$, then we have

$$
\sum_{i=0}^{2^{s+1}-1} e(g^i)x^i = h(x) \cdot (b_1x^{2^s-1} + b_2x^{2^{s-2}} + \cdots + b_{2^s-1}x + b_{2^s})
$$

for some $b_i \in \mathbb{Q}$. Here $h(x) := x^{2^s(l-1)} - x^{2^s(l-2)} + \cdots - x^{2^s} + 1$ is the minimal polynomial to $\varphi(g)$. From this equation, we obtain that for $0 \leq t \leq 2^s - 1$ and for $0 \leq i \leq l - 1$,

$$
e(g^{2^s i + t}) = (-1)^i e(g^i). \quad (3.3)
$$

Then the ideal $(\pi_0)$ is fixed by $g^l$ because equation (3.3) holds. This is a contradiction by Lemma 3.6. Hence $E(\varphi) \neq 0$.

If $\varphi(g)$ is a primitive $2^{s+1}$-th root of unity and if $E(\varphi) = 0$, then by a similar argument in case that $\varphi(g)$ is a primitive $2^{s+1}l$-th root of unity, for each $0 \leq t \leq 2^s - 1$ we have

$$
\sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) = 0.
$$

On the other hand, let $n$ be the order of $\text{End}_p^0(A)$ in the Brauer group $\text{Br}(C(A))$ of $C(A)$. By class field theory, $n$ is the smallest integer such that $n \cdot \text{inv}_v(\text{End}_p^0(A))$ belongs to $\mathbb{Z}$ for all prime $v$ of $C(A)$. We here have

$$
\text{inv}_v(\text{End}_p^0(A)) = s_n(v)[C(A)_v : \mathbb{Q}_p].
$$

By Proposition 3.5 $[C(A)_v : \mathbb{Q}_p] = 1$. Hence, for any $1 \leq t \leq 2^s$ we have

$$
n \cdot \sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) \equiv \frac{n}{2} \mod \mathbb{Z}.
$$

Since $n$ is odd from the assumption, $\sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) \neq 0$. Therefore $E(\varphi) \neq 0$.

4 Proof of Theorem 1.1

We here prove Theorem 1.1. We introduce some notation: For any $c \in (\mathbb{Z}/l)^\times$, we denote by $\langle c \rangle$ the least natural number such that $\langle c \rangle \equiv c \mod l$. We write $H$ for the subgroup of $(\mathbb{Z}/l)^\times$ generated by $p$. We identify $(\mathbb{Z}/l)^\times$ with the Galois group of $\mathbb{Q}(\mu_l)/\mathbb{Q}$.  

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Proof of Theorem 1.1. By a result of Shioda-Katsura mentioned in Introduction, we may assume that the residual degree $f$ of $p$ in $\mathbb{Q}(\mu_l)$ is odd and that $\dim A \geq 2$. By González’s result [3, Theorem 3.3], we have

$$H_\alpha = \left\{ c \in (\mathbb{Z}/l)^\times \mid \sum_{h \in H} \sum_{i=0}^2 \langle hca_i \rangle = \sum_{h \in H} \sum_{i=0}^2 \langle ha_i \rangle \right\}. \quad (4.1)$$

Let $G$ be the Galois group of $C(A)/\mathbb{Q}$. Then we have

$$G \simeq (\mathbb{Z}/l)^\times / H_\alpha.$$ 

Let $\iota \in G$ be the complex conjugation on $C(A)$. Then to give a character $\phi$ of $G$ with $\phi(\iota) = -1$ is equivalent to give a odd Dirichlet character $\chi$ of $(\mathbb{Z}/l)^\times$ with $\chi|_{H_\alpha} = 1$. By Theorem 3.4, it suffices to show that for any odd Dirichlet character $\chi$ of $(\mathbb{Z}/l)^\times$ with $\chi|_{H_\alpha} = 1$, $\sum_{\sigma \in G} e(\sigma) \chi(\sigma) = 0$ if and only if $\sum_{i=0}^2 \chi(a_i) = 0$. Here $e(\sigma) = s_{j(a)}(\sigma p) - \frac{1}{2}$ where $p$ is a fixed prime of $C(A)$ dividing $p$.

Therefore for a such Dirichlet character $\chi$, we calculate $\sum_{\sigma \in G} e(\sigma) \chi(\sigma)$. We first consider $e(\sigma)$. Let $q$ be a prime of $\mathbb{Q}(\mu_l)$ dividing $p$. Then we have

$$e(\sigma) = s_{j(a)}(cq) - \frac{1}{2}.$$ 

Here $c \in (\mathbb{Z}/l)^\times$ is a representative of $\sigma$. Hence we have

$$\sum_{\sigma \in G} e(\sigma) \chi(\sigma) = \frac{1}{d} \sum_{c \in (\mathbb{Z}/l)^\times} e(c) \chi(c), \quad (4.2)$$

where $d$ is the cardinality of $H_\alpha$ and $e(c) = s_{j(a)}(cq) - \frac{1}{2}$. By the ideal decomposition of $j(a)$ in $\mathbb{Q}(\mu_l)$ (cf. [10]), we have

$$s_{j(a)}(cq) - \frac{1}{2} = \frac{1}{f} \sum_{h \in H} \sum_{i=0}^2 \left( \frac{\langle ha_i e^{-1} \rangle}{l} - 1 \right) - \frac{1}{2}$$

$$= \frac{1}{fl} \sum_{i=0}^2 \sum_{h \in H} \left( \langle ha_i e^{-1} \rangle - \frac{l}{2} \right).$$
From this equation and (4.2),

\[
\frac{1}{d} \sum_{c \in (\mathbb{Z}/l)^\times} e(c)\chi(c) = \frac{1}{fld} \sum_{c \in (\mathbb{Z}/l)^\times} \sum_{i=0}^{2} \sum_{h \in H} \left( \langle ha_i c^{-1} \rangle - \frac{l}{2} \right)\chi(c)
\]

\[
= \frac{1}{fld} \sum_{i=0}^{2} \sum_{h \in H} \sum_{c \in (\mathbb{Z}/l)^\times} \left( \langle ha_i c^{-1} \rangle - \frac{l}{2} \right)\chi(c)
\]

\[
= \frac{1}{fld} \sum_{i=0}^{2} \sum_{h \in H} \sum_{c \in (\mathbb{Z}/l)^\times} \chi(a_i)^{-1} \sum_{c \in (\mathbb{Z}/l)^\times} \left( \langle c \rangle - \frac{l}{2} \right)\chi(c)^{-1}
\]

\[
= \frac{1}{ld} \sum_{i=0}^{2} \chi(a_i)^{-1} \sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle\chi(c)^{-1}.
\]

Now our task is reduced to show that \(\sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle\chi(c)^{-1} \neq 0\). Let \(L(s, \chi)\) be the \(L\)-series attached to \(\chi\). Then \(L(1, \chi) \neq 0\) by [18, Corollary 4.4]. Furthermore by [18, Theorem 4.9], we have

\[
\sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle\chi(c)^{-1} = b \cdot L(1, \chi), \quad (b \in \mathbb{C}^\times).
\]

Hence the proof of the first assertion of the theorem is complete.

Next we consider in case (1). Let \(g\) be a generator of \((\mathbb{Z}/l)^\times\). Let \(\chi\) be an odd Dirichlet character of \((\mathbb{Z}/l)^\times\) generated by \(g^m\) for some \(m\). Now \(l - 1 = mk\) for some \(k > 0\). If \(\sum_{i=0}^{2} \chi(a_i) = 0\), then \(1 + \chi(a_1 a_0^{-1}) + \chi(a_2 a_0^{-1}) = 0\). Since \(\chi(c)\) is a root of unity, by elementary computation \(\chi(a_1 a_0^{-1})\) is a primitive cubic root of unity. Hence \(m\) is divided by 3. Moreover since \(\chi(-1) = -1\) and \(-1 \equiv g^{\frac{l-1}{2}}\), we obtain that \(m\) does not divide \(\frac{l-1}{2}\). Hence \(m\) is even and \(k\) is odd. Therefore \(m \equiv 0 \mod 6\).

From the above argument, we see that if \(l - 1 \equiv 0 \mod 6\), then the order of \(\chi\) is not divided by 6. Therefore we may assume that \(l - 1 \equiv 0 \mod 6\). Let \(H'\) be the subgroup of \((\mathbb{Z}/l)^\times\) generated by \(g^6\). If \([C(A) : \mathbb{Q}] \neq 0 \mod 6\), then \(H_a \nsubseteq H'\). For any odd character \(\chi\) of \((\mathbb{Z}/l)^\times\) with \(\chi|_{H_a} = 1\), we obtain that \(\ker(\chi) \nsubseteq H'\). Therefore the order of \(\chi\) is not divided by 6.

From the above argument, we see that \(\sum_{i=0}^{2} \chi(a_i) \neq 0\) for any such \(\chi\). Hence the assertion follows from the first assertion of the theorem.
Case (2) is easy. Let $\chi$ be an odd Dirichlet character of $\left(\mathbb{Z}/l\right)^\times$. If $\alpha = (a_0, a_1, a_2) = (a, a, b)$ up to permutation and if $\sum_{i=0}^{2} \chi(a_i) = 0$, then $\chi(a^{-1}b) = -2$. This is a contradiction because $\phi(c)$ is a root of unity.

Lastly we consider in case (3). From Theorem 3.4, it suffices to show that the order $n$ of $\text{End}_F^0(A)$ in the Brauer group $\text{Br}(C(A))$ of $C(A)$ is odd. By the proof of Theorem 3.4, the order $n$ is the smallest integer $n$ such that $n \cdot s_{j(\alpha)}(v)$ belongs to $\mathbb{Z}$ for all prime $v$ of $C(A)$. Now $s_{j(\alpha)}(v) = \frac{m}{f}$ for some $m \in \mathbb{Z}$. Therefore $n$ is a divisor of $f$. Since $f$ is odd, $n$ is also odd.

**Corollary 4.1** (Corollary 1.2). Let $l$ be an odd prime number different from $p$. Let $J(C_l)$ be the Jacobian variety of the hyperelliptic curve

$$C_l : y^2 = x^l - 1.$$  

Then all $\ell$-adic Tate classes on all powers of $J(C_l)$ are Lefschetz.

**Proof.** The hyperelliptic curve $C_l$ is a quotient of the Fermat curve $V_l$ (cf. [12, §5]). By González’s result [3, Theorem 3.3], $J(C_l)$ is a power of a simple abelian variety $A$ such that $C(A) = \mathbb{Q}(j(\alpha))$ for some $\alpha = (a, a, b)$ (cf. [12]). Now the assertion follows from Theorem 1.1.

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