An Interconnection-Based Interpretation of the Loewner Matrices*

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Abstract—We introduce a novel interpretation of the Loewner and shifted Loewner matrices constructed from tangential interpolation data. This interpretation requires the development of new objects: the left- and right-Loewner matrices. Once the interpretation is introduced, a method for the on-line estimation of the Loewner and shifted Loewner matrices is presented.

I. INTRODUCTION

The generalized realization problem [1] for linear systems involves the construction of a minimal state-space model of an underlying system consistent with data obtained by sampling the transfer matrix at points of the complex plane [1]. The problem is closely related to the rational interpolation problem, which seeks to construct a rational matrix function consistent with given interpolation data [1],[2],[3]. A particular general case of the rational interpolation problem involves using tangential data, that is matrix data sampled in specific directions. This has been used for the model reduction of multi-variable systems in [4] and generalized state-space (descriptor systems) in [1]. A relaxed version of the rational interpolation problem is the so-called problem of rational interpolation without a specific area of analyticity. This has been widely studied and solution approaches include the generating system approach, which produces a matrix function that parameterizes the set of all interpolants [3],[5],[6], and the method of using the Hankel matrix to directly construct state-space models [7],[8], which has been used in the first treatments of the partial realization problem [9],[10].

Finally, another important tool to solve the generalized realization problem is the Loewner matrix, also known as the divided-difference matrix [1]. This matrix is closely related to the Hankel matrix [11],[12], and has been first used to solve rational interpolation problems in [13]. The Loewner matrix can be factored into generalized controllability and observability matrices, which can then be used to construct state-space models [14]. A recent result shows that the Loewner framework of [1] can be interpreted as a special case of a two-sided moment-matching procedure for descriptor systems [15], following the development of time-domain definitions of moments in [16]. In the literature there are multiple methods for constructing state-space realizations from the knowledge of the Loewner matrices, which vary depending on the rank and size of such matrices [1]. While interesting, the actual construction of the realization using the Loewner matrices is not the objective of this paper: the interested reader should refer to Sections 4 and 5 of [1] or to [3].

One of our goals is to provide an intuitive and efficient method for determining the Loewner matrix by sampling the input and output signals of a linear system. The entries of the matrix are collected by probing the system and filtering its output with two auxiliary systems. This provides a system theoretic interpretation (interconnection based) of the Loewner matrix as the input and output "gains" of a transformed system. Such an interpretation, which only relies on an interconnection perspective, has the potential to allow defining the "Loewner matrix" for more general classes of systems, and hence to extend the tangential interpolation methods of [1] and [3] to (classes of) nonlinear systems and time-varying systems.

The structure of this paper is as follows. In Section 2 notation and terminology are presented, along with preliminary results and a discussion of the problem. In Section 3 a few new objects related to the Loewner and shifted Loewner matrices are introduced. In Section 4 a conceptual experimental setup yielding an interconnection-based interpretation of the Loewner and shifted Loewner matrices is introduced. Finally, in Section 5 we present results on estimating the entries of the Loewner and shifted Loewner matrices and provide bounds on the estimation error as a function of time.

II. NOTATION, PRELIMINARY RESULTS, AND PROBLEM SETUP

In what follows we use standard notation. \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{Z} \) the set of integers, \( \mathbb{C} \) the set of complex numbers, \( \mathbb{C}^n \) the set of complex vectors having \( n \) rows, and \( \mathbb{C}^{n \times m} \) the set of complex matrices having \( n \) rows and \( m \) columns. \( \mathbb{C}^{-} \) denotes the elements of \( \mathbb{C} \) having negative real part. \( \sigma(A) \) denotes the spectrum of a square matrix \( A \). \( e_i \) denotes a vector with all entries equal to 0 except for the \( i \)-th entry which is equal to 1. \( \delta(t) \) denotes the Dirac delta function.

The paper [1] has provided methods for constructing generalized state-space representations described by equations...
of the form

\[ E \dot{x}(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t), \]

with state \( x(t) \in \mathbb{C}^n \), input \( u(t) \in \mathbb{C}^m \), output \( y(t) \in \mathbb{C}^p \), and matrices \( E, A, B, C, \) and \( D \) of appropriate dimensions, consistent with a set of so-called tangential interpolation data, or matrix data sampled in specific directions. In what follows we specialize the definitions and results of [1] considering the special case in which \( E = I \) and \( D = 0 \), therefore we consider systems of the form

\[ \dot{x}(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t), \]

(1)

(2)

The following assumptions hold throughout the paper.

**Assumption 1:** The triple of matrices \( (A, B, C) \) is a minimal realization of the system (1)-(2), i.e. the system (1)-(2) is reachable and observable.

To pose any interpolation, or realization, problem, and, in particular, to define the Loewner and shifted Loewner matrices one has to introduce tangential data (see [1]). Tangential data are composed of right and left interpolation data. The right interpolation data are described by the set

\[ \{ (\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C}, \ r_i \in \mathbb{C}^m, \ w_i \in \mathbb{C}^p, \ i = 1, \ldots, \rho \}. \]

These can be defined, equivalently, by the matrices

\[ \Lambda = \text{diag}[\lambda_1, \ldots, \lambda_\rho] \in \mathbb{C}^{\rho \times \rho}, \]
\[ R = [r_1 \ \ldots \ r_\rho] \in \mathbb{C}^{m \times \rho}, \]
\[ W = [w_1 \ \ldots \ w_\rho] \in \mathbb{C}^{p \times \rho}. \]

The left interpolation data are described by the set

\[ \{ (\mu_j, \ell_j, v_j) \mid \mu_j \in \mathbb{C}, \ \ell_j^T \in \mathbb{C}^p, \ v_j^T \in \mathbb{C}^m, \ j = 1, \ldots, \upsilon \}, \]

or, equivalently, by the matrices

\[ M = \text{diag}[\mu_1, \ldots, \mu_\upsilon] \in \mathbb{C}^{\upsilon \times \upsilon}, \]
\[ L = [\ell_1 \ \ldots \ \ell_\upsilon] \in \mathbb{C}^{p \times \upsilon}, \quad V = [v_1 \ \ldots \ v_\upsilon] \in \mathbb{C}^{m \times \upsilon}. \]

The next assumption is required to guarantee the existence and uniqueness of matrices introduced in what follows.

**Assumption 2:** The matrices \( A, \Lambda, \) and \( M \) have no common eigenvalues, that is \( \sigma(A) \cap \sigma(\Lambda) = \emptyset, \sigma(A) \cap \sigma(M) = \emptyset, \sigma(M) \cap \sigma(\Lambda) = \emptyset. \)

**Assumption 3:** The matrix \( R \) is full rank and the matrix \( L \) is full rank.

The objective of the generalized realization problem is to determine a state-space realization of the form (1)-(2) such that the associated transfer matrix, namely the rational matrix function \( H(s) = C(sI - A)^{-1}B \), satisfies the interpolation conditions

\[ H(\lambda_i)r_i = w_i, \ i = 1, \ldots, \rho, \]

(3)

\[ \ell_jH(\mu_j) = v_j, \ j = 1, \ldots, \upsilon. \]

The Loewner matrix, \( L \), and the shifted Loewner matrix, \( \sigma L \), are defined in terms of the tangential data (3) and (4) as (see [14])

\[ L = \begin{bmatrix} v_1r_1 - \ell_1w_1 & \cdots & v_\rho r_\rho - \ell_1w_\rho \\ \mu_1 - \lambda_1 & \cdots & \mu_\rho - \lambda_\rho \\ \vdots & \ddots & \vdots \\ v_1r_1 - \ell_\upsilon w_1 & \cdots & v_\rho r_\rho - \ell_\upsilon w_\rho \\ \mu_1 - \lambda_1 & \cdots & \mu_\rho - \lambda_\rho \end{bmatrix}, \]

(4)

\[ \sigma L = \begin{bmatrix} \mu_1v_1r_1 - \lambda_1\ell_1w_1 & \cdots & \mu_\rho v_\rho r_\rho - \lambda_1\ell_1w_\rho \\ \mu_1 - \lambda_1 & \cdots & \mu_\rho - \lambda_\rho \\ \vdots & \ddots & \vdots \\ \mu_1v_1r_1 - \lambda_\upsilon\ell_\upsilon w_\upsilon & \cdots & \mu_\rho v_\rho r_\rho - \lambda_\upsilon\ell_\upsilon w_\upsilon \\ \mu_1 - \lambda_1 & \cdots & \mu_\rho - \lambda_\rho \end{bmatrix}, \]

(5)

(6)

respectively. Note that, since the rational transfer matrix \( H(s) \) generates the data, the shifted Loewner matrix is the Loewner matrix associated to the transfer matrix \( sH(s) \). Furthermore, the Loewner matrix is the unique (by Assumption 2) solution of the Sylvester equation

\[ LA - ML = LW - VR, \]

(7)

and the shifted Loewner matrix is the unique (again by Assumption 2) solution of the Sylvester equation

\[ \sigma L \Lambda - M \sigma L = LW \Lambda - MVR. \]

(8)

In addition, as shown in [1, Prop. 3.1], the Loewner and shifted Loewner matrices are such that \( \sigma L - \Lambda L = VR \) and \( \sigma L - ML = LW \). Finally, these matrices can be expressed as

\[ L = -YX, \quad \sigma L = -YAX, \]

(9)

where

\[ Y = \begin{bmatrix} \ell_1C(\mu_1I - A)^{-1} \\ \vdots \\ \ell_\upsilon C(\mu_\upsilon I - A)^{-1} \end{bmatrix}, \]

(10)

and

\[ X = [(\lambda_1I - A)^{-1}Br_1 \cdots (\lambda_\rhoI - A)^{-1}Br_\rho]. \]

The matrices \( X \) and \( Y \) are referred to as the tangential generalized observability and tangential generalized controllability matrices, respectively. These are the solutions of the Sylvester equations

\[ AX + BR = XA \]

(11)

and

\[ YA + LC = MY, \]

(12)

respectively.

For the purposes of this paper it is sufficient to know that in order to construct the realization we must obtain the matrices\(^2\) \( W, V, L, \) and \( \sigma L \), however, because of Assumption 3

\(^2\)In a typical application one could choose interpolation points and the direction in which the data are sampled on the basis of a desired model reduction or system identification goal, meaning that \( R, L, \Lambda, \) and \( M \) are user-selected matrices.
and because $\sigma L - L \Lambda = VR$ and $\sigma L - M \Lambda = LW$, we need only be concerned with determining $L$ and $\sigma L$. As a result, the primary focus of the paper is to develop a system theoretic interpretation of the Loewner matrices, $L$ and $\sigma L$, with the additional goal of developing a framework allowing the experimental determination of the Loewner matrices for the system (1)-(2) when $A$, $B$, and $C$ are unknown.

III. THE LEFT- AND RIGHT-LOEWNER MATRICES

Before presenting the main results of the paper we must define a few additional objects. These objects are all well-defined due to Assumption 2. First, define the left-Loewner matrix, $L^L$, and the right-Loewner matrix, $L^R$, as the unique solutions of the Sylvester equations

$$ML^L - L^L \Lambda = VR,$$  
(10)

$$L^R \Lambda - ML^R = LW.$$  
(11)

In a similar fashion, we define the left-shifted-Loewner matrix, $L^L\ell$, and the right-shifted-Loewner matrix, $L^R\ell$, as the unique solutions of the Sylvester equations

$$M\sigma L^L - \sigma L^L \Lambda = MVR,$$  
(12)

$$\sigma L^R \Lambda - M\sigma L^R = LW\Lambda.$$  
(13)

These definitions, along with (5)-(6), carry the implication that

$$L = L^L + L^R.$$  
(14)

and that

$$\sigma L = \sigma L^L + \sigma L^R.$$  

From (10)-(11), noting that

$$M(ML^L) - (ML^L) \Lambda = MVR,$$

and that

$$(L^R \Lambda) \Lambda - M(L^R \Lambda) = LW\Lambda,$$

we have, by uniqueness of the solutions to (12)-(13), that

$$\sigma L^L = ML^L,$$  
(15)

$$\sigma L^R = L^R \Lambda.$$  
(16)

IV. THE CONCEPTUAL EXPERIMENTAL SETUP

In this section we develop a conceptual experimental setup for the construction of a sampled Loewner matrix corresponding to tangential interpolation data of the form (3) and (4). To this end we use the tangential data in matrix form to construct the two systems\(^3\)

$$\dot{\zeta}_r(t) = \Delta \zeta_r(t) + \Delta(t),$$  
(17)

$$v(t) = R\zeta_r(t),$$  
(18)

and

$$\dot{\zeta}_e(t) = M\zeta_e(t) + L\chi(t),$$  
(19)

$$\eta(t) = \zeta_e(t).$$  
(20)

with states $\zeta_r(t) \in \mathbb{C}^p$ and $\zeta_e(t) \in \mathbb{C}^n$, inputs $\Delta(t) \in \mathbb{C}^p$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^n$. Consider now the interconnection equations $v = u$ and $\chi = y$, see Figure 1. The state-space realization of the resulting interconnected system is therefore

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{\zeta}_e \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ BR & A & 0 \\ 0 & LC & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_e \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta,$$  
(21)

$$\eta = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_e \end{bmatrix},$$  
(22)

that is a system with state $[\zeta_r^T \ x^T \ \zeta_e^T]^T$, input $\Delta$, and output $\eta$. Note that all information on the Loewner matrices and the generalized controllability and observability matrices are (somehow) encoded in the system (21)-(22), but it is not obvious how to retrieve such information, i.e. the matrices $L^L, L^R, X,$ and $Y$ do not appear explicitly in the realization (21)-(22).

To expose such matrices one has to select a specific set of coordinates, which allows rewriting the interconnected system (21)-(22) as a parallel interconnection (note that (21)-(22) has a natural series interconnection form).

**Theorem 1:** Consider the system (21)-(22). The coordinates transformation

$$\begin{bmatrix} \zeta_r \\ z_c \\ z_t \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ L^L & Y & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_e \end{bmatrix},$$

is such that the system in the new coordinates is described by the equations

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{z}_c \\ \dot{z}_t \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ z_c \\ z_t \end{bmatrix} + \begin{bmatrix} I \\ -X \end{bmatrix} \Delta,$$  
(23)

$$\eta = \begin{bmatrix} L^R & -Y & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ z_c \\ z_t \end{bmatrix}.$$  
(24)

**Proof:** The proof is obtained via a direct calculation. Differentiating with respect to time $z_c = x - X\zeta_r$ yields

$$\dot{z}_c = \dot{x} - X\dot{\zeta}_r = Ax + BR\zeta_r - X\Lambda\zeta_r - X\Delta = Az_c - (-AX + X\Lambda - BR)\zeta_r - X\Delta.$$

By Assumption 2, $X$ is the unique solution of (8), hence

$$\dot{z}_c = Az_c - X\Delta.$$  

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\(^3\)It is easy to see that similarity transformations of systems (17)-(18) and (19)-(20) produce the exact same interconnected system and Loewner matrix, meaning that the systems can be realized under modest assumptions.
Consider now \( z_\ell = \zeta_\ell + Yx + L_\ell \zeta_r \). Again, a direct differentiation yields
\[
\dot{z}_\ell = \dot{\zeta}_\ell + Y \dot{x} + L_\ell \dot{\zeta}_\ell
= Mz_\ell - (-YA + MY \Lambda) \zeta_r + L_\ell \Delta,
\]
which, by (9) and (10), yields
\[
\dot{z}_\ell = Mz_\ell + L_\ell \Delta.
\]

Finally,
\[
\eta = \zeta_\ell - L_\ell \zeta_r - Yx + z_\ell,
\]
which, utilizing (7) and (14), yields
\[
\eta = L_\ell \zeta_r - Yz_c + z_\ell,
\]
hence the claim.

Theorem 1 lends itself to a simple interpretation: the left-Loewner, right-Loewner, tangential generalized observability, and tangential generalized controllability matrices are the input and output “gains” of three systems interconnected in parallel, as illustrated in Figure 2, such that their overall input-output behaviour coincides with that of the system (21)-(22).

To obtain an analogous interpretation for the shifted Loewner matrices one has to perform time differentiation of the output of each of the systems (17)-(20). This yields the systems
\[
\begin{align*}
\dot{\zeta}_r(t) &= \Lambda \zeta_r(t) + \Delta(t), \\
v(t) &= R\dot{\zeta}_r(t) = RA\zeta_r(t) + RD(t),
\end{align*}
\]
and
\[
\begin{align*}
\dot{\zeta}_\ell(t) &= M\zeta_r(t) + D(t), \\
\eta(t) &= \dot{\zeta}_\ell(t) = M\zeta_r(t) + L\chi(t),
\end{align*}
\]
with states \( \zeta_r(t) \in \mathbb{C}^p \) and \( \zeta_\ell(t) \in \mathbb{C}^q \), inputs \( \Delta(t) \in \mathbb{C}^p \) and \( \chi(t) \in \mathbb{C}^n \), and outputs \( v(t) \in \mathbb{C}^m \) and \( \eta(t) \in \mathbb{C}^n \). Note that, similarly to (17)-(20), these are constructed only using interpolation data (in matrix form).

Consider again system (1)-(2) interconnected to (25)-(26) with the interconnection equations \( v = u \) and \( \chi = y \). This yields the overall system
\[
\begin{bmatrix}
\dot{\zeta}_r \\
\dot{\zeta}_\ell \\
\zeta_c \\
\zeta_\ell
\end{bmatrix}
= \begin{bmatrix}
\Lambda & 0 & 0 & 0 \\
BR & A & 0 & 0 \\
0 & LC & M & 0 \\
0 & LC & M & 0
\end{bmatrix}
\begin{bmatrix}
\zeta_r \\
x \\
\zeta_c \\
\zeta_\ell
\end{bmatrix}
+ \begin{bmatrix}
I \\
0 \\
0 \\
0
\end{bmatrix}
\Delta,
\]
and
\[
\eta = \begin{bmatrix}
0 & LC & M
\end{bmatrix}
\begin{bmatrix}
\zeta_r \\
x \\
\zeta_c \\
\zeta_\ell
\end{bmatrix}.
\]

Theorem 2: Consider the system (27)-(28). The coordinates transformation
\[
\begin{bmatrix}
\zeta_r \\
\zeta_c \\
z_\ell
\end{bmatrix}
= \begin{bmatrix}
I & 0 & 0 \\
-XA & I & 0 \\
L^\ell A & Y & I
\end{bmatrix}
\begin{bmatrix}
\zeta_r \\
x \\
\zeta_\ell
\end{bmatrix}
\]
is such that the system in the new coordinates is described by the equations
\[
\begin{bmatrix}
\dot{\zeta}_r \\
\dot{z}_c \\
\dot{z}_\ell
\end{bmatrix}
= \begin{bmatrix}
\Lambda & 0 & 0 \\
0 & A & 0 \\
0 & 0 & M
\end{bmatrix}
\begin{bmatrix}
\zeta_r \\
z_c \\
z_\ell
\end{bmatrix}
+ \begin{bmatrix}
I \\
0 \\
0
\end{bmatrix}
\Delta,
\]
and
\[
\eta = \begin{bmatrix}
\sigma L^\ell & -YA & M
\end{bmatrix}
\begin{bmatrix}
\zeta_r \\
z_c \\
z_\ell
\end{bmatrix}.
\]

Proof: Similarly to Theorem 1, the proof is obtained via direct calculation. Differentiating with respect to time \( z_c = x - XA \zeta_r \) yields
\[
\dot{z}_c = \dot{x} - XA \dot{\zeta}_r
= AX + (AX - XA + BR) A \zeta_r + (BR - XA) \Delta,
\]
and, by (8),
\[
\dot{z}_c = A z_c - AX \Delta.
\]

Next, consider \( z_\ell = \zeta_\ell + Yx + L_\ell \Lambda \zeta_r \). Again, a differentiation yields
\[
\dot{z}_\ell = \dot{\zeta}_\ell + Y \dot{x} + L_\ell \Delta \dot{\zeta}_\ell
= Mz_\ell + (LC - MY + YA)x
+ (L_\ell \Lambda - MYX + YBR) A \zeta_r + (YBR + L_\ell \Lambda) \Delta,
\]
which, by (9), (10), and (15), yields
\[
\dot{z}_\ell = Mz_\ell + \sigma L^\ell \Delta.
\]

Finally,
\[
\eta = M \zeta_\ell + LCx
= (-ML^\ell - MYX + LCX) A \zeta_r
+ (LC - MY) z_c + Mz_\ell,
\]
which, by (7), (9), (11), (14), and (16), yields
\[
\eta = \sigma L^\ell \Lambda \zeta_r - YA z_c + Mz_\ell,
\]
hence the claim.

V. ONLINE ESTIMATION OF THE LOEWEuler MATRICES

In light of Theorem 1 it is now straightforward to devise a procedure which, under some additional assumptions, allows determining the entries of the Loewner matrix associated with the system (1)-(2) and the systems (17)-(20) (strictly speaking the Loewner matrix is associated to (1)-(2) and the interpolation data, however with some abuse of terminology.
we can associate the Loewner matrix to (1)-(2) and (17)-(20)).

**Proposition 1:** Consider the system (21)-(22), with initial condition \( x(0) = 0 \). Let \( \eta_i(t) \) denote the \( i \)-th row of \( \eta(t) \) at time \( t \), and let \( t_0 \in \mathbb{R} \) and \( t_1 \in \mathbb{R} \) be such that \( t_1 - t_0 \neq j \frac{2\pi k}{\lambda_j - \mu} \), for all \( k \in \mathbb{Z} \). Let \( \Delta(t) = e_j \delta(t) \), for \( j \in \{1, \ldots, \rho\} \). Then

\[
\mathbb{L}_{ij} = \begin{bmatrix}
\left( e^{\lambda_j t_1} - e^{\mu_i t_1} & e^{\mu_i t_0} - e^{\lambda_j t_0}
\end{bmatrix} \times \\
\begin{bmatrix}
\eta_i(t_0) - e^\top Y e^{A t_0} X e_j \\
\eta_i(t_1) - e^\top Y e^{A t_1} X e_j
\end{bmatrix},
\]

where \( e^{\lambda_j t} \) and \( e^{\mu_i t} \) denote the exponential matrices with respect to \( \lambda_j \) and \( \mu_i \), respectively.

**Proof:** By Theorem 1 one has, trivially,

\[
\zeta_i(t) = \int_0^t e^{A(t-r)} \Delta(r) \, dr,
\]

\[
z_i(t) = -\int_0^t e^{A(t-r)} X \Delta(r) \, dr,
\]

and

\[
z_i(t) = \int_0^t e^{A(t-r)} L \Delta(r) \, dr.
\]

This produces the output

\[
\eta(t) = \int_0^t \left( e^{A(t-r)} L i + L e^\top \right) \Delta(r) \, dr
\]

\[
+ \int_0^t Y e^{A(t-r)} X \Delta(r) \, dr.
\]

Substituting \( \Delta(t) = e_j \delta(t) \), with \( j \in \{1, \ldots, \rho\} \), yields

\[
\eta(t) = \left( e^{A t} I \right) L i e_j + Y e^{A t} X e_j.
\]

Selecting the \( i \)-th row of \( \eta(t) \) yields

\[
\eta_i(t) = \left[ e^{\mu_i t} \ \ e^{\lambda_j t} \right] \begin{bmatrix}
L_{ij} \\
L_{ij}
\end{bmatrix} + e_i^\top Y e^{A t} X e_j.
\]

Consider now two samples of \( \eta_i(t) \) at times \( t_0 \) and \( t_1 \) and note that one could also arrange the samples as

\[
\begin{bmatrix}
\eta_i(t_0) - e_i^\top Y e^{A t_0} X e_j \\
\eta_i(t_1) - e_i^\top Y e^{A t_1} X e_j
\end{bmatrix} = \begin{bmatrix}
\left[ e^{\mu_i t_0} \ \ e^{\lambda_j t_0} \right] \left[ L_{ij} \right] \\
\left[ e^{\mu_i t_1} \ \ e^{\lambda_j t_1} \right] \left[ L_{ij} \right]
\end{bmatrix},
\]

Since, by Assumption 2 \( \lambda_j \neq \mu_i \) and by hypothesis \( t_1 - t_0 \neq j \frac{2\pi k}{\lambda_j - \mu} \), for all \( k \in \mathbb{Z} \), the matrix of exponentials can be inverted, yielding

\[
\mathbb{L}_{ij} = \begin{bmatrix}
\left( e^{\lambda_j t_1} - e^{\mu_i t_1} & e^{\mu_i t_0} - e^{\lambda_j t_0}
\end{bmatrix} \times \\
\begin{bmatrix}
\eta_i(t_0) - e^\top Y e^{A t_0} X e_j \\
\eta_i(t_1) - e^\top Y e^{A t_1} X e_j
\end{bmatrix},
\]

which proves the claim.

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4Since \( t_0 \in \mathbb{R} \) and \( t_1 \in \mathbb{R} \), this condition needs only be considered when \( \text{Re}\{\lambda_j - \mu_i\} = 0 \), for some \( i \) and \( j \).

The result in Proposition 1 can be exploited to provide asymptotically converging estimates of the entries of the Loewner matrix. To this end, define the estimate

\[
\hat{\mathbb{L}}_{ij} := \begin{bmatrix}
\left[ e^{\lambda_j t_1} - e^{\mu_i t_1} \right] \left[ e^{\mu_i t_0} - e^{\lambda_j t_0}
\end{bmatrix} \begin{bmatrix}
\eta_i(t_0) \\
\eta_i(t_1)
\end{bmatrix}, \tag{29}
\]

and note that this depends only upon available signals, i.e. \( \eta(t) \), \( e^{\lambda_j t} \), and \( e^{\mu_i t} \). The estimation error is therefore given by

\[
\mathbb{L}_{ij} - \hat{\mathbb{L}}_{ij} = -e_i^\top Y \begin{bmatrix}
\left( e^{\lambda_j t_1} - e^{\mu_i t_1} \right) \\
\left( e^{\mu_i t_0} + e^{\lambda_j t_0} \right)
\end{bmatrix} \begin{bmatrix}
\eta_i(t_0) \\
\eta_i(t_1)
\end{bmatrix} \left( e^{\lambda_j t_0} - e^{\mu_i t_0} \right) e^{A(t_0-t_0)} X e_j
\]

\[
+ e_i^\top Y \begin{bmatrix}
\left( e^{\lambda_j t_0} - e^{\mu_i t_0} \right) e^{A(t_0-t_0)} X e_j
\end{bmatrix}.
\]

To provide a bound on the estimation error, set \( t_1 = t_0 + \delta \), with \( \delta \neq j \frac{2\pi k}{\lambda_j - \mu} \), \( k \in \mathbb{Z} \), and note that

\[
\mathbb{L}_{ij} - \hat{\mathbb{L}}_{ij} = -e_i^\top Y \begin{bmatrix}
\left( \lambda_j - \mu_i \right) X e_j
\end{bmatrix}
\]

\[
+ e_i^\top Y \begin{bmatrix}
\left( \lambda_j - \mu_i \right) X e_j
\end{bmatrix}.
\]

Taking any \( p \)-norm of both sides, it is easy to see that there exist \( \gamma_1(\delta) > 0 \) and \( \gamma_2(\delta) > 0 \) such that

\[
\begin{bmatrix}
\mathbb{L}_{ij} - \hat{\mathbb{L}}_{ij}
\end{bmatrix} \leq \gamma_1(\delta) \left[ e^{\text{Re}(\lambda_j) \mu_i t_0} + e^{\text{Re}(\mu_i) t_0} \right] + \gamma_2(\delta) \left[ e^{\text{Re}(\lambda_j) \mu_i t_0} + e^{\text{Re}(\mu_i) t_0} \right]
\]

\[
= \gamma_1(\delta) e^{(A - \text{Re}(\mu_i) I) t_0} + \gamma_2(\delta) e^{(A - \text{Re}(\lambda_j) I) t_0},
\]

which lends itself to the following convergence result, the proof of which is straightforward, hence omitted.

**Proposition 2:** Consider the interconnected system (21)-(22). Let \( t \in \mathbb{R} \) and let \( \delta \in \mathbb{R} \) be such that \( \delta \neq j \frac{2\pi k}{\lambda_j - \mu} \), \( k \in \mathbb{Z} \). Let the estimate of \( \lambda_j \) be (see (29))

\[
\hat{\lambda}_j(t) = \begin{bmatrix}
\left( e^{\lambda_j(t+\delta)} - e^{\mu_i(t+\delta)} \right) \\
e^{\mu_i(t+\delta)} - e^{\lambda_j(t+\delta)}
\end{bmatrix} \begin{bmatrix}
\eta_i(t) \\
\eta_i(t + \delta)
\end{bmatrix},
\]

and assume \( \text{Re}(A - \mu_i I) \subset \mathbb{C}^- \) and \( \text{Re}(A - \lambda_j I) \subset \mathbb{C}^- \). Then

\[
\lim_{t \to \infty} \begin{bmatrix}
\mathbb{L}_{ij} - \hat{\mathbb{L}}_{ij}(t)
\end{bmatrix} = 0.
\]

Proposition 2 gives a region of convergence in the complex plane for the estimation error based on \( \lambda_j \), \( \mu_i \), and \( A \), i.e. if the eigenvalues of both \( A - \mu_i I \) and \( A - \lambda_j I \) have all negative real part then the estimation error converges to zero as \( t \to \infty \).

Similarly to Theorem 1, Theorem 2 can be exploited to devise a procedure which allows determining the entries of the shifted Loewner matrix associated with the system (1)-(2) and the systems (17)-(20).

**Proposition 3:** Consider the system (27)-(28), with initial condition \( x(0) = 0 \). Let \( \eta_i(t) \) denote the \( i \)-th row of \( \eta(t) \)

---

\( \text{If the initial condition of system (1)-(2) is non-zero then the bound has a constant offset proportional to the initial condition and we require a small modification to the argument in Proposition 1 and a slightly more complicated estimation procedure.} \)
at time $t$, and let $t_0 \in \mathbb{R}$ and $t_1 \in \mathbb{R}$ be such that $t_1 - t_0 \neq j \frac{2\pi k}{\lambda_j - \mu_i}$, for all $k \in \mathbb{Z}$. Let $\Delta(t) = e_j \delta(t)$, for $j \in \{1, \cdots, \rho\}$. Then

(i) if $\lambda_j \neq 0$ and $\mu_i \neq 0$:

$$\sigma_{L_{ij}} = \left[ \frac{\lambda_j e^{\lambda_j t_1} - \mu_i e^{\mu_i t_1}}{\mu_i \lambda_j (e^{\mu_i t_1} + \lambda_j t_1 - e^{\mu_i t_1} + \lambda_j t_1)} \right] \times \left[ \begin{array}{c}
\eta(t_0) - e_i^T Y A e^{A t_0} A X e_j \\
\eta(t_1) - e_i^T Y A e^{A t_1} A X e_j 
\end{array} \right];$$

(ii) if $\lambda_j = 0$:

$$\sigma_{L_{ij}} = (\mu_i e^{\mu_i t_1})^{-1} (\eta(t_0) - e_i^T Y A e^{A t_0} A X e_j);$$

(iii) if $\mu_i = 0$:

$$\sigma_{L_{ij}} = (\lambda_j e^{\lambda_j t_1})^{-1} (\eta(t_0) - e_i^T Y A e^{A t_0} A X e_j).$$

Proof: By Theorem 2 one has, trivially,

$$\eta(t) = \int_0^t (M e^{M(t-\tau)} \sigma L ) \, d\tau + \int_0^t Y A e^{A(t-\tau)} A X \Delta(t) \, d\tau.$$

Substituting $\Delta(t) = e_j \delta(t)$, with $j \in \{1, \cdots, \rho\}$, yields

$$\eta(t) = (M e^{M t} \sigma L + \sigma L e^{A t} ) e_j + Y A e^{A t} A X e_j.$$  

Selecting the $i$-th row of $\eta(t)$ yields

$$\eta_i(t) = \left[ \begin{array}{c} \mu_i e^{\mu_i t} \\ \lambda_j e^{\lambda_j t} \end{array} \right] \left[ \begin{array}{c} \sigma L_{fi} \\ \sigma L_{fj} \end{array} \right] + e_i^T Y A e^{A t} A X e_j.$$  

To prove case (ii), note that if $\lambda_j = 0$ then $\sigma L_{fj} = 0$, hence

$$\eta_i(t) = \mu_i e^{\mu_i t} \sigma L_{fi} + e_i^T Y A e^{A t} A X e_j$$  

and

$$\sigma_{L_{ij}} = (\mu_i e^{\mu_i t})^{-1} (\eta_i(t) - e_i^T Y A e^{A t_0} A X e_j).$$

To prove case (iii), note that if $\mu_i = 0$ then $\sigma L_{fi} = 0$, hence

$$\eta_i(t) = \lambda_j e^{\lambda_j t} \sigma L_{fj} + e_i^T Y A e^{A t} A X e_j$$  

and

$$\sigma_{L_{ij}} = (\lambda_j e^{\lambda_j t})^{-1} (\eta_i(t) - e_i^T Y A e^{A t_0} A X e_j).$$

For the case (i), consider two samples of $\eta_i(t)$ at times $t_0$ and $t_1$ and note that one could arrange the available samples as

$$\left[ \begin{array}{c}
\eta_i(t_0) - e_i^T Y A e^{A t_0} A X e_j \\
\eta_i(t_1) - e_i^T Y A e^{A t_1} A X e_j
\end{array} \right] = \left[ \begin{array}{c}
\mu_i e^{\mu_i t_0} \\ \mu_i e^{\mu_i t_1}
\end{array} \right] \left[ \begin{array}{c}
\lambda_j e^{\lambda_j t_0} \\ \lambda_j e^{\lambda_j t_1}
\end{array} \right] \left[ \begin{array}{c}
\sigma L_{fj} \\ \sigma L_{fi}
\end{array} \right].$$

Since, by Assumption 2, $\lambda_j \neq \mu_i$ and, by hypothesis, $t_1 - t_0 \neq j \frac{2\pi k}{\lambda_j - \mu_i}$, for all $k \in \mathbb{Z}$, the matrix of exponentials can be inverted, yielding

$$\sigma_{L_{ij}} = \left[ \frac{\lambda_j e^{\lambda_j t_1} - \mu_i e^{\mu_i t_1}}{\mu_i \lambda_j (e^{\mu_i t_1} + \lambda_j t_1 - e^{\mu_i t_1} + \lambda_j t_1)} \right] \times \left[ \begin{array}{c}
\eta_i(t_0) - e_i^T Y A e^{A t_0} A X e_j \\
\eta_i(t_1) - e_i^T Y A e^{A t_1} A X e_j
\end{array} \right].$$

which proves the result.

Similarly to Proposition 2, estimates of the entries of the shifted Loewner matrix can be defined for each of the cases (i), (ii), and (ii) discussed in Proposition 3.

VI. CONCLUSION

We have presented a new interpretation of the Loewner matrices. To this end we have defined new objects: the left- and right-Loewner matrices, the introduction of which is instrumental for the novel interpretation of the Loewner matrices constructed from tangential interpolation data. Using this interpretation we provide a method for online estimation of the Loewner matrices. Due to its systems theoretic nature, this interpretation has the potential to extend the definition of “Loewner matrices” to nonlinear systems and time-varying systems, and hence to extend previously studied tangential interpolation methods to more general classes of systems.

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