FIBERED COMMENSURABILITY ON $\text{Out}(F_n)$

HIDETOSHI MASAI AND RYOSUKE MINEYAMA

Abstract. We define and discuss fibered commensurability of outer automorphisms of the free groups, which lets us study symmetries of outer automorphisms. The notion of fibered commensurability is first defined by Calegari-Sun-Wang on mapping class groups. The Nielsen-Thurston type of mapping classes is a commensurability invariant. One of the important facts of fibered commensurability on mapping class groups is for the case of pseudo-Anosovs, there is a unique minimal element in each fibered commensurability class. In this paper, we first show that being ageometric and fully irreducible is a commensurability invariant. Then for such outer automorphisms, we prove that there is a unique minimal element in each fibered commensurability class, under asymmetry assumption on the ideal Whitehead graphs.

1. Introduction

We define fibered commensurability of outer automorphisms of the free groups. Fibered commensurability is defined by a natural covering relation between outer automorphisms, which allows us to discuss symmetries of outer automorphisms. First, we recall the definition of fibered commensurability of mapping classes on surfaces introduced by Calegari-Sun-Wang [3]. A mapping class $\phi_1$ on a compact surface $S_1$ is said to cover another mapping class $\phi_2$ on a compact surface $S_2$ if $\phi_1$ is a lift of a power $\phi_2^n$ with respect to a finite covering $p : S_1 \to S_2$. Two mapping classes are (fibered) commensurable if there is a third mapping class which covers both. We may rephrase this covering relation in terms of the action on the fundamental groups: $\phi_1$ covers $\phi_2$ if we can find a finite index copy of $\pi_1(S_1)$ in $\pi_2(S_2)$ and representatives $\Phi_1, \Phi_2$ of $\phi_1, \phi_2$ respectively so that $(\Phi_2)^n|_{\pi_1(S_1)} = (\Phi_1)$ for some $n \in \mathbb{N}$. This equivalent definition of coverings can be adopted for the case of outer automorphisms of free groups of rank $\geq 2$. One subtle difference is that we need to pass to powers to define commensurability as an equivalence relation, see Section 3 for a detail. In Section 3 we define an equivalence relation called covering equivalence in order to make each commensurability class an partially ordered set. For the case of surfaces, it can be readily seen that the Nielsen-Thurston type is a commensurability invariant, see [3]. It is well-known that there are several similarities between pseudo-Anosov mapping classes and fully irreducible outer automorphisms. However, in the case of free groups, fully irreducible elements may be covered by reducible elements. Indeed, given any geometric fully irreducible outer automorphism $\phi$, any lift of $\phi$ corresponding to a mapping class of a surface with more than one boundary components would be reducible. This

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is because such a lift fixes the conjugacy class of a free factor subgroup corresponding to the boundary. The first aim of this paper is to show that being non-geometric is a commensurability invariant and for the non-geometric case, fully irreducibility is also a commensurability invariant property.

It is proved in [3] and [8] that for the case of pseudo-Anosov mapping classes, each commensurability class contains a unique minimal (orbifold) element. Especially, in [8], it is proved that if we consider pseudo-Anosov with all the singularity of the (un)stable foliation punctured, minimal element must be defined on a surface (not an orbifold). Existence of the unique minimal element is used for example, to prove that cusped random mapping tori are non-arithmetic in [10]. In this paper, instead of defining “orbifolds”, we consider the case where fully irreducible element is ageometric and each component of the ideal Whitehead graph admits no symmetry. The definition of ageometric outer automorphisms and the ideal Whitehead graphs are recalled in Section 2. Since every element \( \phi \in \text{Out}(F_2) \) is geometric, we focus on \( F_n \) with \( n \geq 3 \). Then our main theorem is the following.

**Theorem 1.1.** Let \( \phi \) be an outer automorphism of a free group of rank \( \geq 3 \). Suppose \( \phi \) is

- ageometric, fully irreducible, and
- every component of the ideal Whitehead graph admits no symmetry.

Then the commensurability class of \( \phi \) contains a unique minimal element (or more precisely, covering equivalent class) which is an outer automorphism of a free group of rank \( \geq 3 \).

The key idea is from [9], in which we give a proof of the existence of the minimal element for pseudo-Anosov case by using quadratic differentials. The paper is organised as follows. In section 2, we prepare basic facts of \( \text{Out}(F_n) \). We define fibered commensurability in section 3 and there, we prove that it is an equivalence relation. Several basic properties of fibered commensurability on \( \text{Out}(F_n) \) are also discussed in section 3. Then in section 4 we prove the main theorem.

2. Preliminaries

We refer the reader to the book of Handel and Mosher [5] for the topics discussed in this section. Most of our terminologies are defined along the chapter 2 of the book.

2.1. Marked graphs. In this article, unless otherwise stated, a graph \( G \) is always a finite cell complex of dimension 1 with all vertices having valence greater than or equal to 2. We denote by \( V(G) \), \( E(G) \) the set of vertices and the set of edges respectively. A metric graph is a graph \( G \) equipped with a path metric defined by a function \( \ell : E(G) \to (0, \infty) \). Here, a path \( \gamma \) is a concatenation of edges \( \gamma = e_1 e_2 \cdots e_n \) (\( e_i \in E(G) \)). If the initial point of a path \( \gamma \) coincides with the endpoint of \( \gamma \) then it is called a loop. A geodesic on a graph is a locally shortest path parametrised by arc length. Any finite path \( \gamma \) is homotopic rel endpoints to a unique geodesic; such a geodesic is said to be obtained from \( \gamma \) by tightening and denoted by \( \gamma_2 \). An \( \mathbb{R} \)-tree is a metric space \( T \) such that for any distinct two points \( x, y \in T \) there exists
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a unique embedded topological arc, denoted $[x, y] \subset T$, with endpoints $x, y$ and length $d(x, y)$. We call such an embedded arc a geodesic on an $\mathbb{R}$-tree. An $\mathbb{R}$-tree is called simplicial if the set of vertices forms a discrete set. If an $\mathbb{R}$-tree admits an isometric action by a free group $F$, we call it an $F$-tree. The universal covering $\hat{G}$ of a finite metric graph $G$ is a simplicial $F$-tree where $F \cong \pi_1(G)$.

The $n$-rose $R_n$ is a graph defined as the wedge of $n$ circles. We call a homotopy equivalence $\rho : R_n \rightarrow G$ from the $n$-rose to a metric graph $G$ a marking. A marked graph is a pair $(G, \rho)$ of a metric graph and a marking.

Let $G$ be a marked graph. Take distinct oriented edges $e, e' \in E(G)$ emanating from $v \in V(G)$. Assume that $e$ and $e'$ are parametrized by arc length from $v$ and we denote by $e(t)$ the point on $e$ distance $t$ from $v$. Define $G_s$ to be the graph obtained by identifying the points $e(t)$ and $e'(t)$ for all $t \in [0, s]$ if $e([0, s]) \cap e'([0, s]) = \{v\}$. The quotient map $G \rightarrow G_s$ is a homotopy equivalence. This $G_s$ can be regarded as a marked metric graph with push forward metric by the quotient map. When $s > 0$ we say that $G_s$ is obtained from $G$ by the length $s$ fold of $e$ and $e'$.

Given graphs $G, G'$, a continuous map $p : G' \rightarrow G$ is called a covering if $p$ satisfies following two conditions:

(i) Induced map $V(p) : V(G') \rightarrow V(G)$ on the vertex set is surjective.

(ii) Let $E(p) : E(G') \rightarrow E(G)$ be the induced map on the set of edges. For each vertex $v' \in V(G')$, the restriction $E(p)|_{E(v')}$ on the set of edges whose one endpoint is $v'$ is bijective.

If $G$ and $G'$ are metric graphs, we further require coverings to be local isometries.

2.2. Train track representatives. Let $F$ be a free group of rank $n \geq 2$, $\text{Aut}(F)$ the group of automorphisms of $F$. We denote by $i_\gamma : F \rightarrow F$ the inner automorphism of $\gamma \in F$, that is, $i_\gamma(\delta) = \gamma \delta \gamma^{-1}$ for $\delta \in F$. Let $\text{Inn}(F)$ be the inner automorphism group. The quotient

$$\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$$

is called the outer automorphism group of $F$. In what follows, by an outer automorphism, we mean an outer automorphism of a free group of rank $\geq 2$ unless otherwise stated. For an outer automorphism $\phi$, we use the notation $F(\phi)$ to denote the free group on which $\phi$ is defined.

The group $\text{Out}(F)$ acts on the set of conjugacy classes by the quotient of $\text{Aut}(F)$ action on $F$. An element $\phi \in \text{Out}(F)$ is reducible if there exists a free decomposition $F = A_1 \ast A_2 \ast \ldots \ast A_k \ast B$ $(k \geq 1)$ such that $\phi$ permutes the non-trivial conjugacy classes of $A_1, \ldots, A_k$. An outer automorphism $\phi$ is irreducible if it is not reducible. If, in addition, $\phi^k$ $(k \geq 1)$ are all irreducible then $\phi$ is called fully irreducible.

We fix an isomorphism $\pi_1(R_n) \cong F$. Then each loop in $R_n$ is labeled by an element of $F$. Any homotopy equivalence $g$ between marked graphs determines a homotopy equivalence between roses. By considering the induced action on $\pi_1(R_n) \cong F$, we get an outer automorphism $\phi$ of $F$ via the marking. Thus the map $g$ on a metric graph represents an outer automorphism $\phi$. We always assume that $g$ is an immersion on each edge. In addition, if $g$ maps vertices to vertices, we say that $g$ is a topological representative of $\phi$. 
A topological representative \( g : G \to G \) on a metric graph \( G \) is an irreducible train track map if

- for every edge \( e \in E(G) \), \( g|_e \) uniformly expands length by the factor \( \lambda > 1 \),
- every power \( g^k \) \((k \geq 1)\) is an immersion on each edge, and
- for each pair of edges \( e, e' \in E(G) \), we have \( g^k(e) \cap e' \) for some \( k \geq 1 \).

It is known that every fully irreducible outer automorphism \( \phi \) can be represented by an irreducible train track map \((\text{[11, Theorem 1.7]})\). We call such a representative an irreducible train track representative. For a fully irreducible \( \phi \), the stretch factor \( \lambda > 1 \) can also be characterised as the exponential growth rate of the word length of any element in \( F \) under iteration of any representative of \( \phi \). Hence the stretch factor \( \lambda \) is independent of the choice of train track representatatives, and denoted \( \lambda(\phi) \). Also it can be readily seen that \( \lambda(\phi) = \lambda(\psi \phi \psi^{-1}) \) for any \( \psi \in \text{Out}(F) \). See also Proposition 3.10 below. Let \( g : G \to G \) be an irreducible train track map. We say that a path \( \sigma \) in \( G \) is legal if \( g(\sigma) = g(\sigma) \). In contrast, if \( g^p(\sigma) = \sigma \) for some \( p \geq 1 \) a path \( \sigma \) is called a periodic Nielsen path. A Nielsen path is a periodic Nielsen path with \( p = 1 \). A periodic Nielsen path is indivisible if it is not a concatenation of periodic Nielsen paths. For a point \( \tilde{x} \in \tilde{G} \), two geodesics \( \gamma \) and \( \gamma' \) emanating from \( \tilde{x} \) determines the same direction if \( \gamma \land \gamma' \neq \{\tilde{x}\} \). This defines an equivalence relation on the set of arcs with an endpoint \( \tilde{x} \) and an equivalence class is called a direction at \( \tilde{x} \). We denote by \( D_{\tilde{x}} \) the set of directions at \( \tilde{x} \). Any set of directions \( D_{\tilde{x}} \) projects down to a set of directions of \( G \) denoted by \( D_x \). The sets of directions play a similar role as tangent spaces. Any continuous map \( h : G \to G \) which maps vertices to vertices naturally induce a map, denoted \( Dh \), on the set of directions. A turn is an unordered pair of directions with the same endpoint.

2.3. Periodicity of outer automorphisms. The action of \( \text{Aut}(F) \) on \( F \) can be extended to a continuous action on the Gromov boundary \( \partial F \) of \( F \). For \( \Phi \in \text{Aut}(F) \), we denote such extension by \( \hat{\Phi} : \partial F \to \partial F' \). Let \( \phi \in \text{Out}(F) \) be fully irreducible and \( \Phi \in \text{Aut}(F) \) its representative. We denote the fixed point set of the boundary extension by \( \text{Fix}(\hat{\Phi}) \), and denote the subset of non-repelling fixed points by \( \text{Fix}_N(\hat{\Phi}) \). We say that \( \Phi \) is a principal automorphism if \( \text{Fix}_N(\hat{\Phi}) \) contains at least three points. The set of principal automorphisms representing \( \phi \) is denoted by \( \text{PA}(\phi) \). Let \( g : G \to G \) be a train track representative of \( \phi \) and \( \tilde{G} \) a universal covering of \( G \). There is a bijection between representatives of \( \phi \) and the set of lifts of \( g \) to \( \tilde{G} \) given by the twisted equivariance; let \( \tilde{g} \) be a lift of \( g \), then for any \( \gamma \in F \), \( \tilde{g} \circ \gamma = \Phi(\gamma) \circ \tilde{g} \) holds for some \( \Phi \in \phi \). A lift \( \tilde{g} : \tilde{G} \to \tilde{G} \) is called a principal lift if the corresponding automorphism is a principal automorphism. A vertex \( v \in V(G) \) is called a principal vertex if \( v \) has at least three periodic directions or \( v \) is an endpoint of an indivisible periodic Nielsen path. We call a lift \( \tilde{v} \) of a principal vertex \( v \) a principal vertex on the universal covering. One can see that there exists at least one principal vertex \((\text{[4, Lemma 3.18]})\).

A fully irreducible \( \phi \in \text{Out}(F_0) \) is called rotationless if for each \( k > 0 \) and each \( \Phi_k \in \text{PA}(\phi^k) \), there is \( \Phi \in \text{PA}(\phi) \) such that \( \Phi^k = \Phi_k \). This is equivalent to the following condition: for every train track representative
Let \( g : G \rightarrow G \), every principal vertex is fixed and every periodic direction at a principal vertex is fixed. Any fully irreducible outer automorphism turns into rotationless after suitable iterations.

### 2.4. Attracting tree, stable lamination and ideal Whitehead graph.

For each fully irreducible outer automorphism \( \phi \in \Out(F) \), there exists an associated \( \mathbb{R} \)-tree \( T_+(\phi) \) called the \textit{attracting tree} for \( \phi \). We recall the construction of \( T_+(\phi) \). Let \( g : G \rightarrow G \) be an irreducible train track representative of \( \phi \in \Out(F) \). Here the graph \( G \) is marked by \( \rho_G : R_n \rightarrow G \). For each \( i \geq 0 \), the marking \( g^i \circ \rho_G : R_n \rightarrow G \) gives a new marked graph which we denote by \( G_i \). Each \( G_i \) is equipped with a metric so that each arc \( c \) has length

\[
l_{G_i}(c) = l_G(g^i(c)) / \lambda^i.
\]

The map \( g : G \rightarrow G \) induces a homotopy equivalence denoted \( g_{i+1,i} : G_i \rightarrow G_{i+1} \) that preserves marking, namely, \( g_{i+1,i} \circ \rho_G \) is homotopic to \( g^i \circ \rho_G \). We fix a base point on \( G_i \) so that the marking \( g^i \circ \rho_G : R_n \rightarrow G_i \) preserves the base point. Then the maps \( g_{j,i} : G_i \rightarrow G_j \) also preserve the base point. We lift the base point of each \( G_i \) to the tree \( \tilde{G}_i \) and choose a lift \( \tilde{g}_{j,i} \) of \( g_{j,i} \) via the base point. Thus we get a direct system

\[
\tilde{G}_i = \tilde{G}_0 \xrightarrow{\tilde{g}_{1,0}} \tilde{G}_1 \xrightarrow{\tilde{g}_{2,1}} \cdots.
\]

Now we define the \textit{attracting tree} \( T_+(\phi) \) for \( \phi \) as the \( F \)-equivariant direct limit of the sequence \( \{\tilde{G}_i\} \). The attracting tree is an \( \mathbb{R} \)-tree and well-defined up to isometric conjugacy. In particular, it does not depend on the choice of the train track representatives. If there is no confusion we simply denote \( T_+(\phi) \) by \( T_+ \). It is known that we have the direct limit map \( f_\phi : G \rightarrow T_+ \) which is surjective, \( F \)-equivariant and isometry on each legal path with respect to \( g \). Below we recall such basic properties of \( T_+ \) and \( f_\phi \) from \([5]\).

#### Proposition 2.1 ([5] Corollary 2.14])

Assume that an irreducible train track representative \( g : G \rightarrow G \) is rotationless and that \( \tilde{G} \) is a simplicial \( F \)-tree obtained from the universal cover of \( G \) by choosing a marking and a lifting of the base point. Then there is a surjective equivariant map \( f_\phi : \tilde{G} \rightarrow T_+ \) such that for all \( \tilde{x}, \tilde{y} \in \tilde{G} \), letting \( \sigma = [\tilde{x}, \tilde{y}] \), the following are equivalent.

1. \( f_\phi(\tilde{x}) = f_\phi(\tilde{y}) \).
2. For some \( k \geq 0 \), the path \( g_\phi^k(\sigma) \) is either trivial or Nielsen.

In particular, \( f_\phi \) restricts to an isometry on all legal paths.

#### Theorem 2.2 ([5] Theorem 2.15])

Let \( \phi \in \Out(F_n) \) be fully irreducible, and denote \( T_+ = T_+(\phi) \). To each \( \Phi \in \Aut(F) \) representing \( \phi \) there is associated a homothety \( \Phi_+ : T_+ \rightarrow T_+ \) with stretch factor \( \lambda(\phi) \) satisfying the following properties:

1. For each irreducible train track representative \( g : G \rightarrow G \) of \( \phi \), letting \( \tilde{g} : \tilde{G} \rightarrow \tilde{G} \) be the lift corresponding to \( \Phi \), we have \( \Phi_+ \circ f_\phi = f_\phi \circ \tilde{g} \).
2. For each \( \Phi, \Phi' \) representing \( \phi \), if \( \gamma \in F \) is the unique element such that \( \Phi = i_\gamma \circ \Phi \) then \( \Phi_+ = t_\gamma \circ \Phi_+ \).

#### Lemma 2.3 ([5] Lemma 2.16])

Suppose that \( \phi \in \Out(F) \) is fully irreducible and rotationless, \( g : G \rightarrow G \) is a train track representative, \( \Phi \) is a
principal automorphism representing $\phi$ and $\tilde{\phi} : \tilde{G} \to \tilde{G}$ is the principle lift corresponding to $\Phi$, and $\Phi_+ : T_+ \to T_+$ is as in Theorem 2.14 and Corollary 2.15. Then

1. $f_\tilde{g}(\text{Fix}(\tilde{g}))$ is a branch point $b$ of $T_+$, and $\text{Fix}(\Phi_+) = \{b\}$.
2. Every direction based at $b$ is fixed by $D\Phi_+$ and has the form $Df_\tilde{g}(d)$ where $d$ is a fixed direction based at some $\tilde{v} \in \text{Fix}(\tilde{g})$.
3. The assignment $\Phi \to b = \text{Fix}(\Phi_+)$ of (1) defines a bijection between $PA(\phi)$ and the set of branch points of $T_+$.

Let $g : G \to G$ be a train track representative for $\phi \in \text{Out}(F_r)$. The stable lamination $\Lambda(g)$ of $g$ is the set of all the bi-infinite edge paths

$$\gamma = \cdots e_{-1}e_0e_1e_2\cdots$$

in the graph $G$ which satisfy: for any $i, j \in \mathbb{Z}$ with $i \leq j$, there exist $n \in \mathbb{N}$ and an edge $e \in E(G)$ such that the path $e_i \cdots e_j$ is a subpath of $g^n(e)$. A path $\gamma \in \Lambda(g)$ is called a leaf. A leaf can be regarded as an unordered pair of distinct points in the boundary $\partial\tilde{G}$ of the universal cover of $G$. Since $\partial\tilde{G}$ is naturally identified with $\partial F(\phi)$, we may regard $\Lambda(g)$ as a subset of $\partial F(\phi) \times \partial F(\phi) \setminus \{\text{diagonal}\}$. It turns out $\Lambda(g)$ is independent of the choice of $g$, and hence we denote by $\Lambda(\phi)$ the stable lamination of $\phi$. The following proposition due to Bestvina-Feighn-Handel plays a key role in our discussion.

**Proposition 2.4** ([2, Theorem 2.14 and Corollary 2.15]). Let $\phi$ be a fully irreducible outer automorphism. Then the stabilizer $\text{Stab}(\Lambda(\phi))$ is virtually cyclic. Furthermore, if $\lambda(\psi) \neq 1$ for some $\psi \in \text{Stab}(\Lambda(\phi))$, then $\phi$ and $\psi$ have common nonzero powers.

For every principal automorphism $\Phi$, let $L(\Phi)$ be the set of leaves $\{P_1, P_2\}$ of $\Lambda(g)$ with $P_1, P_2 \in \text{Fix}_N(\Phi)$ the elements of $L(\Phi)$ are the singular leaves associated to $\Phi$. We define the component of the ideal Whitehead graph determined by $\Phi$ to be the graph with one vertex for each point in $\text{Fix}_N(\Phi)$, and an edge connecting $P_1, P_2 \in \text{Fix}_N(\Phi)$ if there is a leaf $\{P_1, P_2\} \in L(\Phi)$. The ideal Whitehead graph $\tilde{\mathcal{W}}(\phi)$ is defined as the disjoint union of components $\tilde{W}(\Phi)$, one for each principal automorphism $\Phi$ representing $\phi$. $F(\phi)$ naturally act on $\tilde{\mathcal{W}}(\phi)$ isometrically cofinitely. We denote by $\mathcal{W}(\phi)$ the quotient $\tilde{\mathcal{W}}(\phi)/F(\phi)$. We remark that a component of $\tilde{\mathcal{W}}(\phi)$ may have vertex of valence 1. For a component $W$ of $\tilde{\mathcal{W}}(\phi)$, we call a non-trivial graph automorphism on $W$ symmetry.

Given $H < F(\phi)$ a finitely generated subgroup, let $G_H$ denote the covering of $G$ corresponding to $H$. Then a finitely generated subgroup $H < F(\phi)$ carries a leaf of $\Lambda(g)$ if there exists a leaf $\gamma \in \Lambda(g)$ which lifts to a bi-infinite path in $G_H$.

Suppose that $b \in T_+$ is a branch point, $\Phi \in PA(\phi)$ is the principal automorphism corresponding to $b$ by Lemma 2.3, and $\tilde{g} : \tilde{G} \to \tilde{G}$ is the corresponding principal lift. Each singular leaf $\ell$ in $L(\tilde{g})$ passes through some $x \in \text{Fix}(\tilde{g})$, and so $f_\tilde{g}(\ell)$, the realization of $\ell$ in $T_+$, passes through the point $f_\tilde{g}(x) = b$; we call these the singular leaves at $b$. Each singular leaf at $b$ is divided by $b$ into singular rays at $b$ which are the images under $f_\tilde{g}$ of the singular rays at some $x \in \text{Fix}(\tilde{g})$. Define the local Whitehead graph at a point $z \in T_+$, denoted $W(z; T_+)$, to be the graph with one vertex for each
Definition 3.1. An outer automorphism $\phi_1$ covers an outer automorphism $\phi_2$ if there exist

- a finite index subgroup $H < F(\phi_2)$ which is isomorphic to $F(\phi_1)$,
- representatives $\Phi_1 \in \phi_1$, $\Phi_2 \in \phi_2$, and
- $k \in \mathbb{N}$

such that $\Phi_2^k(H) = H$ and $\Phi_2^k|_H = \Phi_1$.

Remark 3.2. The equality $\Phi_2^k|_H = \Phi_1$ must pass through the isomorphism between $H$ and $F(\phi_1)$, however for notational simplicity we omit to write the isomorphism.

Remark 3.3. If $\psi$ covers $\phi$ then there exist train track representatives $g^l : G' \to G$, $g : G \to G$ of $\psi$ and $\phi$ respectively such that $G'$ covers $G$ and $g^l$ is a lift of $g^k$ for some $k \in \mathbb{N}$.

By using this covering relation, we define the following relation.

Definition 3.4. Let $\phi_1$ and $\phi_2$ be outer automorphisms. Then we say $\phi_1 > \phi_2$ if there exists $k \in \mathbb{N}$ so that $\phi_1^k$ covers $\phi_2^k$. Two outer automorphisms $\phi_1, \phi_2$ are said to be covering equivalent if $\phi_1 > \phi_2$ and $\phi_2 > \phi_1$.

We first show that the relation $>$ is transitive and hence a total order on each commensurability class.

Proposition 3.5. Let $\phi_1$, $\phi_2$ and $\phi_3$ be outer automorphisms. Suppose $\phi_1 > \phi_2$ and $\phi_2 > \phi_3$, then $\phi_1 > \phi_3$.

Proof. Since $\phi_2 > \phi_3$, there are positive integers $k_2$ and $l_2$, representatives $\Phi_2$ and $\Phi_3$ respectively of $\phi_2^{k_2}$ and $\phi_3^{k_3}$, and a subgroup $H_3 < F(\phi_3)$ which is isomorphic to $F(\phi_2)$ such that $\Phi_3^k(H_3) = H_3$ and $\Phi_3^k|_{H_3} = \Phi_2$. Furthermore, there are $k_2 \in \mathbb{N}$ and a representative $\Phi_2'$ of $\phi_2^{k_2}$ whose restriction of some power $(\Phi_2')^{l_2}$ on some subgroup $H_2 < F(\phi_2)$ coincides with $\Phi_1$, a representative of $\phi_1^{k_2}$.

Since $\Phi_2$ and $\Phi_2'$ are representatives of certain powers of $\phi_2$, there is an element $\gamma \in F(\phi_2)$ such that $\Phi_2^{|l_2} = i_\gamma(\Phi_2')^{l_2}$. Since $H_2 < F_0$ is of finite index, there exists $m \in \mathbb{N}$ such that $(\Phi_2^{|l_2})^m(H_2) = H_2$. For certain $\delta \in F_0$, we have $i_\delta(\Phi_2')^{l_2k_2m} = (\Phi_2^{l_2m})^m$. By taking further power if necessary, we may suppose that $\delta \in H_2$. Therefore, by regarding $H_2$ as a subgroup of $H_3$, the restriction of $\Phi_3^{|k_2m}$, a power of a representative of $\phi_3^{k_2m}$, on $H_2$ coincides with $i_\delta\Phi_3^{k_2m}$, a representative of $\phi_3^{k_2m}$. \qed
For rotationless fully irreducible case, the covering equivalence is no more than conjugate. Note that for two automorphisms \( \phi_1 > \phi_2 \) and \( \phi_1 < \phi_2 \) imply \( F(\phi_1) = F(\phi_2) \) by the rank of subgroups in the definition of coverings.

**Proposition 3.6.** Let \( \phi_1, \phi_2 \) be rotationless fully irreducible outer automorphisms with \( F(\phi_1) = F(\phi_2) =: F \). If \( \phi_1 \) and \( \phi_2 \) are covering equivalent, then there exist \( \Phi_1 \in \phi_1, \Phi_2 \in \phi_2 \) and \( \Phi_3 \in \text{Aut}(F) \) such that \( \Phi_3 \Phi_1 \Phi_3^{-1} = \Phi_2 \).

**Proof.** Since \( \phi_1^\ell \) covers \( \phi_2^\ell \) for some \( \ell \in \mathbb{N} \), we have representatives \( \Phi_{1,\ell} \in \phi_1^\ell, \Phi_{2,\ell} \in \phi_2^\ell \) and an automorphism \( \Psi \in \text{Aut}(F) \) so that \( \Phi_{2,\ell}^{k\ell}(\Psi(F)) = \Psi(F) \) and \( \Phi_{2,\ell}^{k\ell} = \Psi^{-1}\Phi_{1,\ell} \) for some \( k \in \mathbb{N} \). Let \( k' \in \mathbb{N} \) be the number in the definition that \( \phi_1^{k'} \) covers \( \phi_2^{k'} \) for some \( \ell \in \mathbb{N} \). Let \( \lambda_1 \) and \( \lambda_2 \) be stretch factors of \( \phi_1 \) and \( \phi_2 \) respectively. Then we have \( \lambda_1^{k'} = \lambda_2 \lambda_1^{k} \), hence \( k = k' = 1 \).

Since \( \phi_1 \) and \( \phi_2 \) are rotationless, we can take \( \Phi_1 \in \text{PA}(\phi_1) \) and \( \Phi_2 \in \text{PA}(\phi_2) \) so that \( \Phi_*^* = \Phi_*^{*\ell} \) and \( \text{Fix}_N(\Phi_*) = \text{Fix}_N(\Phi_*^{*\ell}) \) \((* = 1, 2)\). Thus we have

\[
(1) \quad \Phi_2^\ell = \Psi^{-1}\Phi_1^\ell \Psi \iff \Phi_2 = (\Psi^{-1}\Phi_1\Psi)^\ell.
\]

On the other hand, for a fully irreducible outer automorphism \( \phi \), two elements \( \Phi_1, \Phi_2 \in \text{PA}(\phi) \) are distinct if and only if \( \text{Fix}_N(\Phi_1) \cap \text{Fix}_N(\Phi_2) = \emptyset \) (see [5, Corollary 2.9]). Together with this, the equation (1) gives

\[
\Phi_2 = \Psi^{-1}\Phi_1\Psi.
\]

Letting \( \Phi_3 := \Psi \) we have the conclusion. \(\square\)

We use the following lemma at several places.

**Lemma 3.7.** Let \( \Phi, \Psi \) be automorphisms on a free group \( F \). If \( \Phi \) and \( \Psi \) coincide on a finite index subgroup \( H < F \) and \( \Phi(H) = \Psi(H) = H \) then \( \Phi = \Psi \).

**Proof.** Recall that any finite index subgroup of a group contains a normal subgroup of finite index. Let \( N \) be such a subgroup of \( H \). Then \( N \) is free group of rank \( \geq 2 \). We consider representatives \( a_1, \ldots, a_m \) of \( H \) in \( F \). It suffices to show that \( \Phi(a_i) = \Psi(a_i) \) for each \( i = 1, 2, \ldots, m \).

Fix \( i \in \{1, 2, \ldots, m\} \) and choose \( x \in N < H \) arbitrary. By our assumption \( \Phi(x) = \Psi(x) \in H \), we denote this element by \( y \). On the other hand, since \( N \) is normal, \( a_i x a_i^{-1} \in N \subset H \), hence

\[
\Psi(a_i)y\Psi(a_i)^{-1} = \Psi(a_i x a_i^{-1}) = \Phi(a_i) y \Phi(a_i)^{-1}.
\]

This implies that \( \Psi(a_i)^{-1} \Phi(a_i) \) commutes with any element in \( \Phi(N) = \Psi(N) \). Since \( F \) and \( N \) are free, we deduce that \( \Psi(a_i)^{-1} \Phi(a_i) \) is trivial. \(\square\)

**Definition 3.8.** Two outer automorphisms \( \phi_1 \) and \( \phi_2 \) are said to be *commensurable*, denoted \( \phi_1 \sim \phi_2 \), if there is a third outer automorphism \( \phi_3 \) such that \( \phi_3 > \phi_1 \) and \( \phi_3 > \phi_2 \).

In the next proposition, we justify the notation \( \phi_1 \sim \phi_2 \).

**Proposition 3.9.** Let \( \phi_1, \phi_2 \) and \( \phi_3 \) be outer automorphisms. Suppose \( \phi_1 \sim \phi_2 \) and \( \phi_2 \sim \phi_3 \). Then \( \phi_1 \sim \phi_3 \).
Proof. Let $\phi_{i,i+1}$ be an outer automorphism which satisfies $\phi_{i,i+1} > \phi_i$ and $\phi_{i,i+1} > \phi_{i+1}$ for $i = 1,2$. Then in $F(\phi_2)$, there are corresponding finite index subgroups $H_{1,2}$ and $H_{2,3}$ and representatives $\Phi_{1,2}$ and $\Phi_{2,3}$ of some power of $\phi_2$. By the Nielsen-Schreier theorem, $H := H_{1,2} \cap H_{2,3}$ is a finite index free subgroup. Hence, there exist $k_{1,2}, k_{2,3} \in \mathbb{N}$ such that

- there exists $\delta \in F_b$ such that $\Phi_{1,2}^{k_{1,2}} = \delta \Phi_{2,3}^{k_{2,3}}$, and
- $\Phi_{1,2}^{k_{1,2}}(H) = H$ and $\Phi_{2,3}^{k_{2,3}}(H) = H$.

Hence $\phi_{1,2,3} := [\Phi_{1,2}^{k_{1,2}}] = [\Phi_{2,3}^{k_{2,3}}] \in \text{Out}(H)$ satisfies $\phi_{1,2,3} > \phi_{1,2}, \phi_{2,3}$. By Proposition 3.5 we see that $\phi_{1,2,3} > \phi_1$ and $\phi_{1,2,3} > \phi_3$. \(\square\)

From now on we only consider covering equivalent classes and by abuse of notation, we simply denote by $\phi \in \text{Out}(F_n)$ the equivalent class.

Now we collect properties invariant under taking commensurability.

**Proposition 3.10.** Let $\phi$ and $\psi$ be fully irreducible outer automorphisms. If $\phi \sim \psi$ then,

1. $T_+(\phi) = T_+(\psi),$
2. $T\hat{W}(\phi) = T\hat{W}(\psi),$
3. $\Lambda(\phi) = \Lambda(\psi),$
4. $\log(\Lambda(\phi))/\log(\Lambda(\psi)) \in \mathbb{Q}.$

**Proof.** It suffices to consider the case where $\psi > \phi$. Let $g : G \to G$ and $h : G' \to G'$ be train track representatives $\phi$ and $\psi$ respectively such that there is a finite covering $p : G' \to G$ and $h : G' \to G'$ of $g$ a lift of $g^k$ for some $k \in \mathbb{N}$. Then we may identify universal covers of $G$ and $G'$ and after taking iterations, $g$ and $h$ determine the same map on the universal covering. Since attracting tree of $\phi$ is defined by the direct system constructed from the universal cover of $G$ and a lift of a power $g$, we see that $\phi$ and $\psi$ define the same attracting tree. Hence (1) follows and we let $T_+ := T_+(\phi) = T_+(\psi)$.

Then (2) follows from the characterisation of the ideal Whitehead graph in terms of $T_+$, see [5 Section 3.3]. Note that we may identify both $\partial F(\phi)$ and $\partial F(\psi)$ with $\partial T_+$. Now (3) follows as the stable laminations are subsets of $\partial T_+ \times \partial T_+ \backslash \{\text{diagonal}\}$ given by $\Phi_+$ and $\Psi_+$ in Theorem 2.2 [5 Section 2.8].

For a train track representative $g : G \to G$ of a fully irreducible $\phi \in \text{Out}(F)$, it is known that the logarithm of the expansion factor $\lambda(\phi)$ is equal to the exponential growth late of a non-periodic loop $\sigma$ (see Remark 1.8. of [1]):

$$\log \lambda(\phi) = \limsup_{n \to \infty} \frac{\log \ell_G(g^n_\sigma)}{n}$$

where $\ell_G(\star)$ is the length function on loops determined by the metric on $G$ and a loop $\sigma$ is non-periodic if $g^n_\sigma \neq \sigma$ for all $n \in \mathbb{N}$. Note that this equality does not depends on the choice of the train track representatives. The covering $p$ is induced by the inclusion $F(\psi) \to F(\phi)$ in the definition of the covering. The inverse image $p^{-1}(\sigma) =: \sigma'$ of a non-periodic loop $\sigma$ on $G$ is also non-periodic in $G'$. Since $p$ is a finite cover, there exists $m \in \mathbb{N}$ such that

$$\ell_G(\sigma) \leq \ell_{G'}(\sigma') \leq m \ell_G(\sigma),$$
and hence
\[ \ell_G(g_k^0 \sigma) \leq \ell_G(h_2 \sigma') \leq m \ell_G(g_k^0 \sigma). \]

We get
\[
\log \lambda(\phi) = \limsup_{n \to \infty} \frac{\log \ell_G(g_k^{kn} \sigma)}{kn} \leq \limsup_{n \to \infty} \frac{1}{k} \frac{\log \ell_G(h_2^{m} \sigma')}{n} = \frac{1}{k} \log \lambda(\psi),
\]
and
\[
\log \lambda(\psi) = \limsup_{n \to \infty} \frac{\log \ell_G(h_2^{m} \sigma')}{n}
\leq \limsup_{n \to \infty} k \left( \frac{\log \ell_G(g_k^{kn} \sigma)}{kn} + \frac{\log m}{kn} \right) = k \log \lambda(\phi).
\]
Thus \( k \log \lambda(\phi) = \log \lambda(\psi) \), we have (4).

3.2. Ageometricity and commensurability. An outer automorphism \( \phi \) is toroidal if there exists \( k \in \mathbb{N} \) and a conjugacy class \([w]\) such that \( \phi^k([w]) = [w] \). If \( \phi \) is not toroidal, then it is called atoroidal.

**Proposition 3.11.** For outer automorphisms \( \phi \) and \( \psi \) assume that \( \psi > \phi \). Then following holds:

1. \( \phi \) is atoroidal \( \iff \) \( \psi \) is atoroidal.
2. \( \phi \) is fully irreducible and atoroidal \( \iff \) \( \psi \) is fully irreducible and atoroidal.

**Proof.** Let \( H \) be a subgroup of \( F(\phi) \) associated with \( \psi \) which is isomorphic to \( F(\psi) \). We fix a representative \( g : G \to G \) of \( \phi \). Let \( G' \) be a covering of \( G \) corresponding to \( H \) and \( g' : G' \to G' \) a lift of \( g \).

(1) If \( \psi \) is toroidal, then some power of \( \phi \) fixes a conjugacy class \([w]\) in \( F(\psi) \). Since \( \psi \) coincides with \( \phi \) on a subgroup which includes \( w \) up to isomorphism, \([w]\) must be fixed by some power of \( \phi \) and hence \( \phi \) is toroidal.

Conversely, assume that \( \phi \) is toroidal then there exists a conjugacy class \([w]\) in \( F(\phi) \) which is fixed by some power of \( \phi \). We may assume that \( w^n \) for some \( n \) is included in \( H \) since \( H \) is of finite index. Since a power of a representative of \( \psi \) coincides with a representative of \( \phi \) on \( H \), we have \( \psi([w^n]) = \phi([w^n]) = [w^n] \). This completes the proof of (1).

(2) The “if” part follows from Kurosh Subgroup Theorem [7, Theorem 1.10, Chapter IV]. The theorem states that for a free product \( F = \ast_{i=1}^n A_i \ast B \) of a group \( F \), any subgroup \( F' < F \) is also represented as a free product \( F' = \ast_{i=1}^m C_i \ast D \) where each \( C_i \) is the intersection of \( F' \) with a conjugate of \( A_i \).

Thus, if there exists a proper free factor \( A \) of \( F(\phi) \) whose conjugacy class is preserved by \( \phi \) then there also exists a proper free factor of \( H \) (which is the intersection of \( H \) with a conjugate of \( A \)) whose conjugacy class is preserved by \( \psi \) because \( \psi \) covers \( \phi \) with the subgroup \( H \). Hence the irreducibility of \( \psi \) implies the irreducibility of \( \phi \).

The converse is proved by referring Bestvina-Feighn-Handel [2] and Kapovich [9]. Note that in [9], fully irreducible elements are called iwip. Suppose \( \phi \) is atoroidal and fully irreducible and \( g : G \to G \) an irreducible train track representative. In [9], a train track representative is called clean if its transition matrix is positive after iteration, and its Whitehead graph is connected (we
omit the definitions here, see [6] for a detail). In [6 Proposition 4.4], it is proved that for an atoroidal element, being fully irreducible and having clean train track representative is equivalent. Hence we may suppose $g : G \to G$ is clean. Also, by Lemma 2.1 of [2], any lift $g' : G' \to G'$ corresponding to $F(\psi) \leq F(\phi)$ is also clean. As we know that $\psi$ is atoroidal by (1), [6 Proposition 4.4] implies that $\psi$ is fully irreducible.

As a corollary, we have a characterisation of commensurability in terms of stable laminations.

**Corollary 3.12.** Let $\phi_1$ and $\phi_2$ be atoroidal fully irreducible outer automorphisms. Then the followings are equivalent

1. $\phi_1 \sim \phi_2$ or $\phi_1 \sim \phi_2^{-1}$
2. We can identify $T_+ (\phi_1)$ and $T_+ (\phi_2)$ so that we have $\Lambda (\phi_1) = \Lambda (\phi_2)$ in the boundary of $T_+$.

**Proof.** That (1) implies (2) is in Proposition 3.10. By the standard theory of free groups, we may find a free group $F$ of rank $\geq 2$ which is a finite index subgroup of both $F(\phi_1)$ and $F(\phi_2)$. Then by taking a power of some representatives of $\phi_i$, we may lift $\phi_i$ to $F$ for $i = 1, 2$, denoted $\tilde{\phi}_i$. By Proposition 3.10 $\tilde{\phi}_1$ and $\tilde{\phi}_2$ have the same stable laminations and attracting trees. Also by Proposition 3.11 we see that both $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are atoroidal and fully irreducible. Now by Proposition 2.4 we see that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ have common nonzero powers.

From now on, we will focus on ageometric outer automorphisms.

**Definition 3.13.** A fully irreducible outer automorphism $\phi \in \text{Out}(F_n)$ is **ageometric** if there exists a train track representative $g : G \to G$ which does not have any periodic Nielsen path. We also call such a train track representative without periodic Nielsen path **ageometric**.

We remark that if an outer automorphism is ageometric then it is atoroidal. Indeed, if a conjugacy class $[\gamma]$ is preserved by $\phi$, then for any train track representative $g : G \to G$ of $\phi$, any path written like $\gamma_0 \gamma \gamma_0^{-1}$ is mapped to a path written like $\sigma_1 \gamma \sigma_1^{-1}$. Thus, in particular, we have $g(\gamma) = \gamma$. This means that the path $\gamma$ is a Nielsen path and hence $\phi$ is not ageometric.

We show that the ageometricity is a commensurability invariant. To do that we recall an equivalent definition of ageometricity. We first recall the notion of geometric index. Let $T_+ (\phi)$ be the attracting tree of a fully irreducible $\phi \in \text{Out}(F(\phi))$. For each branched point $b$ of $T_+ (\phi)$, the degree of $b$ denoted $\deg (b)$ is the number of components of $T_+ (\phi) \setminus \{b\}$. For each $b$, let $[b]$ denote the $F(\phi)$ equivalent class. Then the geometric index of $T_+ (\phi)$ is

$$\text{ind}_{\text{geo}} (T_+ (\phi)) := \sum_{[b] : \deg (b) \geq 3} (\deg (b) - 2).$$

Then it is well-known that $\phi$ is ageometric if and only if $\text{ind}_{\text{geo}} (T_+ (\phi)) < \text{rank}(F(\phi)) - 2$.

**Proposition 3.14.** Let $\phi$ and $\psi$ be atoroidal fully irreducible automorphisms. Suppose $\psi > \phi$. Then $\phi$ is ageometric if and only if $\psi$ is ageometric.
4. Minimal elements

4.1. Topologically minimal element. By Proposition 3.10, we see that being atoroidal and fully irreducible is a commensurability invariant. In this subsection, for atoroidal and fully irreducible case, we prove that there is a minimum of the rank of the free group in any commensurability class.

Lemma 4.1. Let \( \phi \) be an atoroidal fully irreducible outer automorphism and \([\phi]\) the fibered commensurability class of \( \phi \). Suppose that every component of the ideal Whitehead graph \( \tilde{\mathcal{W}}(\phi) \) admits no symmetry. Then there is an element \( \phi_m \in [\phi] \) such that for any \( \phi' \in [\phi] \), \( F(\phi') \) is a finite index subgroup of \( F(\phi_m) \).

Proof. By Proposition 3.10 for any \( \psi \in [\phi] \) we have \( \tilde{\mathcal{W}}(\phi) = \tilde{\mathcal{W}}(\psi) =: \tilde{\mathcal{W}}_T \) and \( T_+ (\phi) = T_+ (\psi) =: T_+ \). Let \( \psi \in [\phi] \). Both \( F(\phi) \) and \( F(\psi) \) act on \( T_+ \) isometrically. We first show that in the group of orientation preserving isometries \( \text{Isom}^+ (T_+) \), \( \langle F(\phi) \cup F(\psi) \rangle \) contains \( F(\phi) \) and \( F(\psi) \) as finite index subgroups. First we consider the action of \( \langle F(\psi) \cup F(\phi) \rangle \) on \( \tilde{\mathcal{W}}_T \). Note that \( F(\phi) \) and \( F(\psi) \) act on \( \tilde{\mathcal{W}}_T \) so that stabilizer of each component is trivial and their quotient have finitely many components. Suppose that there is \( \gamma \in \langle F(\psi) \cup F(\phi) \rangle \) which stabilizes a component. As we have assumed that each component of \( \tilde{\mathcal{W}}_T \) admits no symmetry, \( \gamma \) must be trivial on the component. But since \( \gamma \) acts isometrically on \( T_+ \), it means that \( \gamma \) is identity in \( \langle F(\psi) \cup F(\phi) \rangle \). Hence as both \( \tilde{\mathcal{W}}_T / F(\phi) \) and \( \tilde{\mathcal{W}}_T / F(\psi) \) have only finitely many components, we see that \( F(\phi) \) and \( F(\psi) \) are finite index subgroups of \( \langle F(\psi) \cup F(\phi) \rangle \).

Since every element of \( \langle F(\psi) \cup F(\phi) \rangle \) is orientation preserving, no element can fix any edge. Also we know that there is no non-trivial stabilizer of any vertex. Recall that any element acting on a \( \mathbb{R} \)-tree by a finite order must fix a point. Therefore we see that \( \langle F(\psi) \cup F(\phi) \rangle \) is torsion free and hence free by [11] Theorem 0.2. Since \( \phi \) and \( \psi \) are commensurable, we have representatives \( \Phi, \Psi \) of \( \phi, \psi \) respectively and a finite index subgroup \( H \) of \( F(\phi) \) and \( F(\psi) \) such that \( \Phi^n_{|H} = \Psi^m_{|H} \) for some \( n, m \in \mathbb{N} \). Let \( \Theta_+ : T_+ \to T_+ \) be a map induced from \( \Phi^n_{|H} = \Psi^m_{|H} \) and \( \Theta(\gamma) = \Phi^n(\gamma) \) if \( \gamma \in F(\phi) \) and \( \Theta(\gamma) = \Psi^m(\gamma) \) if \( \gamma \in F(\psi) \). Then \( \Theta_+ \circ \gamma = \Theta(\gamma) \circ \Theta_+ \) for any \( \gamma \in F(\phi) \cup F(\psi) \). Let \( \delta = \gamma_1 \cdots \gamma_n \) be a concatenation of elements where \( \gamma_i \in F(\phi) \) if \( i \) even and \( \gamma_i \in F(\psi) \) otherwise. We define \( \Theta(\delta) := \Theta(\gamma_1) \cdots \Theta(\gamma_n) \). Since any point stabilizer is identity, if two concatenation \( \delta_1, \delta_2 \) of elements of \( F(\phi) \) and \( F(\psi) \) represents the same element in \( \text{Isom}^+ (T_+) \), we see that \( \Theta(\delta_1) = \Theta(\delta_2) \) in \( \text{isom}^+ (T_+) \). Therefore we get an automorphism \( \Theta \) on the
whole \( \langle F(\psi) \cup F(\phi) \rangle \) so that \( \Theta|_{\langle F(\phi) \rangle} = \Phi^n \) and \( \Theta|_{\langle F(\psi) \rangle} = \Psi^m \). In particular, the outer automorphism which is represented by \( \Theta \) on \( \langle F(\psi) \cup F(\phi) \rangle \) is an element of \([\phi]\). Note that the rank of \( \langle F(\psi) \cup F(\phi) \rangle \) is strictly smaller than that of \( F(\phi) \) or \( F(\psi) \), unless \( \phi > \psi \) or \( \psi > \psi \). Therefore, by repeating the procedure above, we obtain an element \( \phi_m \in [\phi] \) such that \( F(\phi_m) \) contains \( F(\phi') \) for any \( \phi' \in [\phi] \).

**Corollary 4.2.** Suppose the same assumption as Lemma 4.1 and let \( \phi_m \) be an element given by Lemma 4.1. Then every element \( \psi \in [\phi] \) has a positive power such that \( \psi^n > \phi_m \).

**Proof.** Note that by Lemma 4.1, we may suppose \( F(\psi) \) is a finite index subgroup of \( F(\phi_m) \). Since \( \psi \sim \phi_m \), we see that there are representatives \( \Psi \) and \( \Phi_m \) of some powers of \( \psi \) and \( \phi_m \) which agree in a finite index subgroup of \( F(\psi) \). After taking higher powers if necessary, we may suppose that \( \Phi_m \) preserves \( F(\psi) \) in \( F(\phi_m) \). Then by Lemma 4.1, we see that \( \Phi_m|_{F(\psi)} = \Psi \).

### 4.2. Symmetric train track representatives

We prove that if \( \psi^n \) covers \( \phi \), then we can find a train track representative of \( \psi \) on a symmetric graph (Lemma 4.1). First we recall the following lemma which allows us to fold graphs keeping it a train track representative of some power.

**Lemma 4.3** ([5, Lemma 4.2]). Suppose that \( g : G \to G \) is an irreducible train track map and oriented edges \( e_1 \) and \( e_2 \) in \( G \) with a common terminal vertex so that \( g(e_1) = g(e_2) \). Let \( p : G \to G' \) be the Stallings fold, the quotient map that identifies \( e_1 \) to \( e_2 \). Then there is an induced irreducible train track map \( g' : G' \to G' \) with the following properties:

1. \( g'p = pg \).
2. If \( f_g : \tilde{G} \to T_+ \) and \( f_{g'} : \tilde{G}' \to T_+ \) are as in Section 2.4 then \( p \) lifts to \( \tilde{p} : \tilde{G} \to \tilde{G}' \) such that \( f_g = f_{g'}\tilde{p} \).
3. If \( g \) is a train track representative of \( \phi \in \text{Out}(F_n) \) via a marking \( \rho : R_n \to G \), then \( g' \) is also a train track representative of \( \phi \) via the marking \( p \circ \rho : R_n \to G' \).
4. The function \( p \) restricts to a bijection between the principal vertices of \( g \) and \( g' \) preserving Nielsen classes.

Then the goal of this subsection is the following lemma.

**Lemma 4.4.** Let \( \phi, \psi \) be ageometric fully irreducible outer automorphisms. Suppose that \( F(\psi) \) is a finite index subgroup of \( F(\phi) \), and there exist representatives \( \Psi \) and \( \Phi \) of \( \psi \) and \( \phi \) respectively satisfying

\[
\Phi|_{F(\psi)} = \Psi^n
\]

for some \( n \in \mathbb{N} \).

Then there exist \( k \in \mathbb{N} \) and train track representatives \( s : \Gamma' \to \Gamma' \) of \( \psi \), \( t : \Gamma' \to \Gamma' \) of \( \psi^{nk} \) and \( t' : \Gamma \to \Gamma \) of \( \phi^k \) satisfying the following.

(i) \( s^{nk} = t' \),
(ii) there is a covering \( pr : \Gamma' \to \Gamma \) with \( pr \circ t' = t \circ pr \), and
(iii) \( s, t, \) and \( t' \) are ageometric.
Proof. Note that by assumption, we see that $\psi^n > \phi$. We take an ageometric train track representative $g : G \to G$ of $\phi$. By assumption, there is a finite covering $p : G' \to G$ and an ageometric train track representative $g' : G' \to G'$ which is a lift of $g$. Let $\hat{G}$ be a universal covering of both $G$ and $G'$. As we observed in Proposition 3.10, we see that

- $T_+ (\phi) = T_+ (\psi) = T_+ (\psi^n) =: T_+$,
- $f_g = f_g'$.

Let $\Psi_+ : T_+ \to T_+$ be a homothety corresponding to a representative $\Psi \in \psi$ by Theorem 2.2. First note that $f_g$ is a bijection between the set of principal vertices of $\hat{G}$ and the set of the branched points of $T_+$. This is because $g$ is ageometric and hence there is no periodic Nielsen paths, see [5, Corollary 3.2].

- **Step 1.** We define a map $\tilde{h}' : V_p(G') \to V_p(\hat{G}')$ by $f_g^{-1} \circ \Psi_+ \circ f_g$, where $V_p$ denotes the set of principal vertices. As $f_g$ is $F(\psi)$-equivariant and $\Psi_+$ is twisted $F(\psi)$-equivariant, we see that $\tilde{h}' \circ \gamma = \Psi (\gamma) \circ \tilde{h}'$ for any $\gamma \in F(\psi)$. Hence we have an induced map $h' : V_p(G') \to V_p(G')$.

Then, we will extend $h'$ to the whole graph by extending it to edges. In general, it is difficult to extend $h'$, so we consider foldings. In order to keep the symmetry of $G'$, our strategy is to fold $G$ first, and then lift it to $G'$ whenever we need a folding. As taking powers of $\phi$ and $\psi^n$ does not change the situation, we are flexible to take powers of them. Note, however, that we cannot take any power of $\psi$ to get a desired conclusion. For a given edge or path $a$, we denote by $a_-$ (resp. $a_+$) the initial (resp. terminal) vertex of $a$.

- **Step 2.** We choose an edge $e \in E(G')$ so that on the initial point $e_-$, $h'$ is defined. Let $\tilde{e}$ be a lift of $e$ in $\hat{G}'$. Then $E_+ := f_g^{-1} \circ \Psi_+ \circ f_g (\tilde{e}_+)$ is a collection of points. Let $\tilde{E}$ denote the set of points in $E_+$ that attain the shortest distance from $h'(e_-)$. Since $\hat{G}'$ is simplicial, $\tilde{E}$ is finite. Let $\tilde{\epsilon}$ denote the union of geodesics in $\hat{G}'$ that connect $h'(e_-)$ and $\tilde{E}$. This $\tilde{\epsilon}$ is a branched path and after eliminating backtracks $f_g (\tilde{\epsilon})$ can be identified with $\Psi_+ \circ f_g (\tilde{\epsilon})$. Since edges identified by $f_g$ are identified by a power of $\tilde{g}$ (Proposition 2.1), by Lemma 4.3, we can $F(\phi)$-equivariantly fold $\hat{G}'$ so that $\tilde{e}$ becomes a legal path after eliminating backtracks. We remark that we fold $\hat{G}'$ not only $F(\psi)$-equivariantly but $F(\phi)$-equivariantly, in other words, we apply Lemma 4.3 to $g : G \to G$, not $g' : G' \to G'$. Let $p_1 : \hat{G}' \to \hat{G}_1$ denote the map obtained by above finite sequence of foldings. Let $\epsilon_1$ denote a path obtained from $p_1 (\tilde{e})$ by eliminating backtracks. Then $f_g (\epsilon_1)$ has no backtrack and on $\hat{G}_1/F(\phi)$, we have an ageometric train track representative $g_1$ of some power of $\phi$. Note that there is a bijection between the principal vertices of $G'$ and the set of principal vertices of $G_1$. Hence we can define $h'$ on the set of principal vertices of $G_1$ as well. We also define $h'(e)$ to be the legal path on $G'_1$ given by $\pi \circ p_1 (\epsilon_1)$ emanating from $h'(e_-)$. Thus we get a map, which again denoted by $h'$, which is defined on every principal vertex, edge $e$, and hence on $e_+$. By abuse of notations we let $G := G_1$, $g := g_1$ and $G' := G'_1$.

- **Step 3.** We will repeat Step 2 with a slight modification. Let $e \in E(G')$ be a new chosen edge so that $h'(e_-)$ has already been defined. Let $G_1$ be
as defined in Step 2. First, if we have defined images of some edges, then when we fold edges, we need to have edges identified by a fold have the same image by $h'$. This is true because if two edges are identified by a fold in Lemma 4.3 then they are mapped to the same path in $T_+$. Hence we can keep $h'$ well-defined after foldings. Next, if the image by $h'$ of the endpoint $e_+$ of $e$ has already been defined, new image of $e_+$ defined by using $e$ may be different. However as they are inverse images of a point in $T_+$ by $f_g^{-1}$, again by Proposition 2.1 and Lemma 4.3, we can find a finite folding sequence on $G_1$ which identifies such two images of $e_+$ and on the resulting graph $G_2$ there is still an ageometric train track representative $g_2$ of (a power of) $\phi$. Let $G'_2$ denote the covering of $G_2$ corresponding to $F(\psi) < \pi_1(G_2)$. By letting $G := G_2$, $g := g_2$ and $G' := G'_2$, we repeat the same procedure.

Since the number of edges of $G'$ is finite, this process would terminate. Thus we get a map $h' : G' \to G'$ which maps vertices to vertices. Since $f_g$ and $\Psi_+$ preserves legal turn structure, $h'$ also maps legal turns to legal turn. Furthermore, $\gamma \circ (h')^n$ for some $\gamma \in F(\psi)$ and $n \in \mathbb{N}$ coincides with some lift $\hat{g}'$ of $g'$. After taking more powers if necessary we may suppose that $\gamma \circ (h')^n$ and $\hat{g}'$ fixes every direction at each principal vertex. Since homotheties with the same stretch factor which fix every direction at every principal vertex must coincide, we see that $\gamma \circ (h')^n = \hat{g}'$. It follows that $h'$ is homotopy equivalence and some power of $h'$ is a lift of $g$. By letting $\Gamma = G$, $\Gamma' = G'$, $s = h'$, $t = g$, and $t' = g'$, we have a desired ageometric train track representatives.

\[\square\]

4.3. Dynamically minimal element. Let $\phi$ be atoroidal and fully irreducible. We have proved that after taking a power of $\phi$, we can find an element in $[\phi]$ whose supporting free group has the minimum rank. We now show that the dynamically minimal outer automorphism is also realised on such minimal rank free group. The key ingredient is the following lemma.

**Lemma 4.5.** Let $g : G \to G$ and $h : \Gamma \to \Gamma$ be ageometric train track representatives of $\psi$ and $\phi$ respectively. Suppose that

- there is a finite covering $p : G \to \Gamma$ such that for some $n \in \mathbb{N}$, $p \circ g^n = h \circ p$, and
- every component of $\widetilde{TW}(\psi) = \widetilde{TW}(\phi) =: \widetilde{W}$ admits no symmetry.

Then we can find an equivalence relation $\sim$ on $G$ so that

- the quotient map $\pi : G \to G/\sim$ factors through $p : G \to \Gamma$,
- the induced map $\pi_* : \pi_1(G) \to \pi_1(G/\sim)$ is injective and its image is of finite index, and
- there exists $\bar{g} : G/\sim \to G/\sim$ such that $\pi \circ g = \bar{g} \circ \pi$.

**Proof.** The idea of the proof is similar to the one in [9, Lemma 3.1] that is for pseudo-Anosov mapping classes. Indeed, if $g$ were a homeomorphism, then we defined $\sim$ as

\[x \sim y \iff \exists z \in \Gamma \text{ such that } x, y \in g^m(p^{-1}(z)) \text{ for some } m \in \mathbb{N}.
\]

However, since $g$ is not injective, we need to define $\sim$ step by step.
• **Step 1. Vertices.** We define ~ for vertices \( v_1, v_2 \in V(G) \) by
\[
v_1 \sim v_2 \iff \exists v' \in V(\Gamma) \text{ such that } v_1, v_2 \in g^m(p^{-1}(v')) \text{ for some } m \in \mathbb{N}.
\]
Since there are only finitely many vertices, the numbers of equivalent vertices are finite. Furthermore, by definition, we see that
\[
v_1 \sim v_2 \Rightarrow g(v_1) \sim g(v_2).
\]
Therefore \( g \) induces a map \( \bar{g} \) on the set of equivalence classes of vertices.

• **Step 2. Directions.** Similarly to the case of vertices, we define ~ for directions as follows. Let \( d_i \in D(G) \) be directions at some vertices of \( G \) for \( i = 1, 2 \). Then
\[
d_1 \sim d_2 \iff \exists d' \in D(\Gamma) \text{ such that } d_1, d_2 \in (Dg)^m(Dp^{-1}(d')) \text{ for some } m \in \mathbb{N}.
\]
Note that we again have \( d_1 \sim d_2 \Rightarrow Dg(d_1) \sim Dg(d_2) \), and hence the map \( \bar{g} \) can be extended to the equivalence classes of directions. We also remark that if \( d_i \in D_v(G) \) for \( i = 1, 2 \) and \( d_1 \sim d_2 \), then \( v_1 \sim v_2 \). Recall that since \( g \) is atoroidal, each component of \( \bar{T}W \) corresponds to a principal vertex. By assumption, every component of \( \bar{T}W \) admits no symmetry. Hence we may label vertices of \( \bar{T}W \). We call each label an angle at the principal vertex. Since both \( g \) and \( p \) induce graph isomorphisms on the ideal Whitehead graph, we see that if \( d_1 \sim d_2 \), then \( d_1 \) and \( d_2 \) must have the same angle.

• **Step 3. Edges.** We define ~ on edges by using ~ on directions as follows. First, we choose a direction \( d \in D(G) \). We consider the equivalence class \([d]\). For each direction \( d' \) in \([d]\), there is a corresponding edge \( e'_d \). Since \([d]\) is a finite set, we can find the shortest edge \( e_s \) among \( e'_d \)'s. For each edge \( e'_d \) longer than \( e_s \), we take subarcs, denoted by \( e'_d \)'s, of the same length as \( e_s \) and put a new vertex at the new endpoint. By definition of ~ and the fact that \( p \circ g^h = h \circ p \), we see that new vertices are mapped to some vertices by some power of \( g \). New vertices have two directions, inward directions and outward directions. Since \( p, g, \) and \( h \) preserve angles, we see that at the new vertices, those inward (resp. outward) directions all have the same angle. Those \( e'_d \)'s are defined to be equivalent. We repeat this process for other edges. Since the number of edges on which we have not defined equivalence relation strictly decreases at each time, this process would terminate. We have subdivided \( G \), and by abuse of notation, we denote by \( \tilde{G} \) the subdivided graph. By the definition of ~, we see that for \( e_1, e_2 \in E(G) \), if \( e_1 \sim e_2 \), then \( g(e_1) \sim g(e_2) \). Thus we define ~ on \( G \), and by construction we see that \( \pi : G \to G/\sim \) is continuous and we have a continuous map \( \bar{g} : G/\sim \to G/\sim \) such that \( \pi \circ g = \bar{g} \circ \pi \).

Hence it suffices to prove that \( \pi_s : \pi_1(G) \to \pi_1(G/\sim) \) is injective and the image is of finite index. First we prove the injectivity. By \( \pi \circ g = \bar{g} \circ \pi \), we see that \( \ker(\pi_s) = g_s \) invariant. Suppose the contrary that there is a nontrivial loop \( \gamma \) in \( \ker(\pi_s) \). Then \( \pi(\gamma) \) is contractible and must have backtracks. Note that \( \gamma \) is almost like a covering map, except that \( \pi \) may fold edges with the same angles. Since \( \pi \) is locally a homeomorphism except on vertices, backtracks of \( \pi(\gamma) \) must turn back at vertices. By construction, on each vertex, directions which may be identified by \( \pi \) have the same angle. Therefore by Lemma 4.3, we may fold edges corresponding to those directions so that the resulting train track still represents a power of \( g \). Let \( \gamma \) still denote the loop.
that we get from $\gamma$ by deleting the folded part. Note that those edges are identified by $p$, and hence the quotient map $\pi$ naturally is inherited to the folded graph, which we again denote by $\pi$. We may repeat folding until the image of $\gamma$ by $\pi$ contains no backtrack. However this means that $\pi(\gamma)$ is a point and hence $\gamma$ is also a point, contradicting the fact that $\gamma$ is nontrivial. Now note that for any loop $\gamma$ in $G/\sim$, $\pi^{-1}(\gamma)$ is a non-trivial loop. Then since the quotient map is finite to one, we see that the index of the image $\pi_k(\pi_1(G))$ is finite. 

\textbf{Corollary 4.6.} Let $\phi \in \text{Out}(F)$ be an element satisfying the condition in Theorem 1.1 and $\phi_m \in [\phi]$ given by Lemma 4.7. Then for any $\psi \in [\phi]$, there is $\psi' \in [\phi]$ such that $F(\psi) = F(\phi_m)$ and $\lambda(\psi) = \lambda(\psi')$.

\textbf{Proof.} By Corollary 4.2, we see that some power of $\psi$ covers an $\phi_m$. Then by Lemma 4.5 if $\phi$ itself is not a lift, then we must be able to find $\phi'$ with $F(\phi_m) \leq F(\phi')$. But this contradicts the minimality of $F(\phi_m)$, and hence $\psi$ is a lift of some element $\psi'$ defined on $F(\phi_m)$. 

Now we are ready to prove the main theorem.

\textbf{Proof of Theorem 1.1.} Let $\phi_m \in [\phi]$ be an outer automorphism such that $F(\phi_m)$ is the minimal rank free group given by Lemma 4.1. Let $\phi_1, \phi_2 \in [\phi]$ with $F(\phi_1) \cong F(\phi_2) \cong F(\phi_m)$. By Proposition 3.10 we see that $\log(\lambda(\phi_1))/\log(\lambda(\phi_2)) \in \mathbb{Q}_+$. Suppose $\log(\lambda(\phi_1))/\log(\lambda(\phi_2)) \in \mathbb{Q}_+ \setminus \mathbb{N}$. Then there is some integers $m, n \in \mathbb{Z}$ such that $\lambda := \log(\lambda(\phi_1^m)) + \log(\lambda(\phi_2^n))$ is the greatest common divisor, namely $\log(\lambda(\phi_1^m))/\lambda \in \mathbb{N}$ for both $i = 1, 2$. Without loss of generality, we may assume $m > 0$ and $n < 0$. Let $\Lambda := \Lambda(\phi_1^m) = \Lambda(\phi_2^n)$ (Proposition 3.10) denote the stable lamination. Now since both $\phi_1$ and $\phi_2$ are in $	ext{stab}(\Lambda)$, $\phi_1^m \phi_2^n$ is in $	ext{stab}(\Lambda)$, and have stretch factor greater than 1. Hence by Proposition 2.4 we see that $\phi' := \phi_1^m \phi_2^n \in [\phi]$. Thus we can find an element in $[\phi]$ whose logarithm of the stretch factor is at most half of $\phi_1$ and $\phi_2$. Now suppose there are $\psi \in [\phi]$ with $\log(\lambda(\psi))/\log(\lambda(\phi')) \in \mathbb{Q}_+ \setminus \mathbb{N}$. By Corollary 4.6 we may find $\psi' \in [\phi]$ with the same dilation as $\psi$ and $F(\psi') \cong F(\phi_m)$. Then by repeating the argument above, we may find an element in $[\phi]$ whose logarithm of the stretch factor is at most half of $\phi'$ and $\psi$. Since there is a lower bound of the smallest dilation of fully irreducible elements in $\text{Out}(F(\phi_m))$, this process would terminate. Thus we can find $\phi_{\min}$ with $\log(\lambda(\varphi))/\log(\lambda(\phi_{\min})) \in \mathbb{N}$ for all $\varphi \in [\phi]$ and $F(\phi_{\min}) \cong F(\phi_m)$. If there is another element with the same property, again by Proposition 2.4 they are covering equivalent. Therefore $\phi_{\min}$ is the unique (covering equivalent class of) minimal element with respect to $\leq$. 

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**Mathematical Science Group of Advanced Institute for Materials Research, Tohoku University, 2-1-1, Katahira, Aoba-ku, Sendai, 980-8577, Japan**

*E-mail address*: masai@tohoku.ac.jp

**Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan**

*E-mail address*: r-mineyama@cr.math.sci.osaka-u.ac.jp