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Statistic behaviors of gauge-invariance-dominated 1D chiral current random model

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Abstract

By considering energy flow, we construct the one-dimensional (1d) model consisting of the quasiparticles caused by asymmetric hopping (in carrier position space) or the complex bosonic potential whose varying gradient with a chiral ordering plays the role of ingredient of quasiparticles. A bosonic potential can be generated and the chaotic dynamics of chiral excitations after disorder average can be investigated in the presence of gauge invariance. This feature is also shared by the well-known non-Hermitian systems.

1. Introduction

The quantum critical behaviors can be found in an itinerant electron system due to the quantum-fluctuation-induced phase transitions between disordered phase and the Fermi liquid phase. The emergence of Sachdev-Ye-Kitaev (SYK) physics can be observed by the chaotic non-Fermi liquid behaviors as well as the conformal symmetry in large-N limit. In perspective of the materials and related applications, the long-range order will be suppressed by the lowered dimension, like the thermal fluctuation in two-dimensional (2D) crystals, or the conformational-entropy-induced random end-to-end interaction in 1D quantum systems, like the 1D polymers [1]. Thus the low-dimensional system would exhibits more rich statistical physics, as well as the instability under external local or global effects. Thst is also why the SYK system is usually built as zero-dimensional to remove the momentum-dependence. There are several experimental protocols to realizing this, e.g. the graphene under strong magnetic field [2] with flat Landau levels, or the artificial Kagome-type optical lattice [3] as well as the topological superconducting wire coupled with quantum dots [4]. Similar implementation has also been discussed in other topological flat band systems. The generation of maximal thermalization in SYK model relies on the randomness of coupling where the antisymmetry exchange interactions play the key role in a disordered system. Such random couplings have also been studied in Thirring model [5], Falicov-Kimball model [6], random impedance network [7], and the Anderson localized integer spin Hall systems [8]. The projection onto the zero-energy states in a flat band is an essential procedure during these implementations, which connecting the Coulomb phase in real space under perturbations to the many-body localization (MBL) protected quantum orders [9] like the spin glass order in flat-band localization system [10].

In conformal perturbation theory, the random coupling could be marginally irrelevant in large-N limit in a translationally invariant system, e.g. the $1+1$ dimensional Thirring model in the absence of chirality [5]. Thus it is important to get further insight into the problem about the random couplings regarding their marginal relevance or irrelevance. Also, in the presence of replica symmetry the dynamics of such random coupling system under a fluctuating U(1) gauge field is rarely studied. This is due to the complexity originates from the gauge fixing restrictions, e.g. in [11], the gauge fixing restrictions are ignored for the saddle-point results in large-N limit. However, for systems in the absence of replica symmetry or in the presence of spin glass order due to the strong localization effect, the resulting system will lost global symmetry due to the emergent nonlocal symmetry sectors. For this reason, we consider the complex coupling problem in a modified Wishart-SYK form which is dominated by a U(1) gauge-field. The most outstanding feature is the emergence of three-degrees-of-freedom characteristic. We start by considering a conserving current in momentum space with a finite energy-gradient-
induced frustrated hopping, which can be treated as an energy current in real space. Then a bosonic potential can be generated and the chaos of chiral excitation after the disorder average can be investigated in the presence of gauge invariance.

Considering the energy and particle flows, the 1D model consisting of the quasiparticles is generated by the asymmetric hopping (in carrier position space) or the complex bosonic potential whose varying gradient with a chiral ordering generate the quasiparticles. This feature is also shared by the well-known non-Hermitian systems [12]. For such a 1D system, since all the extended eigenstates will be localized once an infinitesimal disorder is introduced, the statistical behaviors of the system depend more on the related (topological) symmetries, instead of the disorder strength. A typical example is the nonlinear-response-induced 1D current to \( n \)-th order

\[
J_n^{\alpha} = \sigma_{\alpha\beta} (E_n^\beta)^n,
\]

where \( \alpha = (x, y) \) is the planar component of the electric field. The effective electric field term can be represented as the gradient of potential in real space, \( E(r) = -\nabla \phi(r) \), or in momentum space, \( E_q = -i q \phi_q \) where \( \phi_q \) is the scalar potential. For the latter case, as \( q \) is the in–plane component of the electron scattering momentum, in long-wavelength limit the summation over \( q \) vanishes [13]. For second-order response with \( n = 2 \), the density matrix \( \rho \) should be expanded in power of electric field, thus we have \( \rho \propto (-\phi(r))^2 \). Similar to the screened Coulomb interaction whose spacial dimension can be modulated by the projectional operation, we consider a 1D current with the distribution of quasiparticles at different positions created by the corresponding eigenstates. In terms of a 1D chiral quasiparticle system, we develop a useful tool in investigating the many-body quantum chaos or ergodicity in large-\( N \) limit that allows the extension to higher dimensions, and involving more (pseudo–) degrees-of-freedom that arise by the ensemble average. Here we briefly introduce some developments on this topic. First one is the zero-spacial-dimensional system in non-Fermi liquid picture [14] as well as the Thirring or Gross-Neveu models with non-random interaction term in 1D spacial dimension where the left and right moving fermions are exactly diagonal. In this case the leading term is quadratic in the four-fermion coupling (also the four-point interaction), and the interaction is marginally relevant for one sign of it which leads to a transition from chaotic phase to non-chaotic phase. A common character for systems of this case is the existence of conserved quantity, like the conserved fermion number [15], or other quantities that commutes with the Hamiltonian. Note that here time-reversal symmetry (of BDI symmetry class) will be broken after the phase transition, while in our article we impose the time-reversal-invariance Majorana chain of BDI symmetry class unless specifically mentioned. The second one is the nonchiral homogeneus systems where the random couplings are marginally irrelevant in large \( N \) limit [5] as the translational invariance in space smoothing out the phase transitions related to the chiral symmetry breaking (after the ensemble average is performed), and thus the left and right moving fermions exhibit scale invariance. Different to this case, in our model, the chiral symmetry breaking is guaranteed (down to the IR) by adding some restriction on the statistical rules which leads to the scale invariance breaking. Such restrictions are realized through the gauge field as well as the induced intrinsic fluctuations to the (effective) field operators, and the explicit breaking of scale invariance can be seem from the equation of motion of the effective field operators, as we illustrate in detail in sections 2.3–2.7. In the calculations therein, we consider the intrinsic property of the gauge invariance and the commutation relations between the effective field operator and a functional of it, which exhibits great convenience in the statistical-mechanical calculations with the ensemble average. The third one is the chiral SYK model in the extended \( (1 + 1) \) d [16], but different to this case where the conformal symmetry (and Lorentz symmetry) breaking are intrinsic due to the spin structure of chiral Majorana fermions, the scaling invariance broken in our model more likes the explicit one, which is due to the gauge field after the disorder (ensemble) average, instead of the spontaneous or intrinsic one. In this work, the numerical evidences verify the emergent Wigner–Dyson statistic of different ensembles: GOE (Gaussian orthogonal ensemble), GUE (Gaussian unitary ensemble) and GSE (Gaussian symplectic ensemble).

2. Hopping-induced current model with gauge-invariance

For this system, we set \( z < z' \) in position space, where \( z = 1 \) denotes the surface layer while \( z = N \) denotes the bottom layer. In such a 1D chiral system, it is easy to know that the current carriers (hopping fermions governed by a U(1) gauge field) has momenta \( q_x > q_y \). Instead of considering the decay of the current energy with depth into the calculations, we set a conserved current energy, while the energy density is variable. Such an energy density \( \mathcal{E} \) of current could be the effective velocity \( v_x = \mathcal{E}/q_x \) for carriers in momentum space or the quasiparticle number operator (will also be called ’current density’ in this paper) \( \rho_x = \mathcal{E}/z \) for quasiparticles in position space. Throughout this paper, we set a chiral-dependent constant ratio \( n_0 \gtrsim 1 \), which is the quasiparticle number at initial position, while the quasiparticle number operators in any other positions are written as \( n_{0z} \). Thus the carrier number (which is a quantity in position space) has \( n_z < n_{0z} \). However, since the quantity which is really one-dimensional is the carrier random hopping-induced energy gradient (which is nonzero only in the \( z \)-direction), like a effective electric field originates from the gradient of a boson field. Thus in order to
investigating the dynamics of 1D randomly interacting model under gauge field, we need to using the quasiparticle which can be really one-dimensional insteads of the carrier which indeed has a three-dimensional momentum although only its $q_z$-component takes effect during a current generation. We will define the quasiparticle in this paper as the derivation of carrier-number-dependent boson field, where the boson field here is defined in spite of the Jordan-Wigner representation. The number operator of quasiparticle defined in this way will be inversely proportional to the carrier number (both in position space), and it can be regarded as the 1D current density, which follows the chiral structure in the distribution.

2.1. Current density

Note that these fermions should indeed have three-dimensional momentum, but we just ignore the in-plane components by taking the long-wavelength limit to the effective electric field term, while the summation over in-plane components of the momentum are self-cancelled. Thus the current can be expressed by the single carrier random hopping term in $z$-direction of momentum space which is coupled to a U(1) gauge field

$$J = \sum_{i<k_{q_z}>q_z} \sum_{\alpha, \beta} \psi^{\dagger}_{\alpha}(q_z) e^{iA_{\alpha,\beta}} t^{\delta q_z}_{ik} c_{\beta}(q_z),$$

(2)

where we assume $c_{\beta}^{\dagger}$ creates a fermion in eigenstate $\psi_{\beta}(q_z)$ in momentum space. The fluctuating U(1) gauge field reads $A_{\alpha,\beta} = \frac{1}{2\pi} \delta_{\alpha,\beta} \mathbf{A}(q) \cdot \mathbf{q}$. The vector potential $\mathbf{A}(q) = (0, 0, A_{q_z})$ can be regarded has only the $z$-component due to the in-plane cancellation effects as mentioned above, the gradient of this vector potential in momentum space is the original of the effective electrical field. Due to the chiral structure, there is another gauge constrain on the vector potential $\mathbf{V} \cdot \mathbf{A} = \sum_{q_z} A_{q_z, q_z} + \sum_{q_z} (A_{q_z, q_z} + A_{q_z, q_z}) = 0$. The phase factor may be changed by the fluctuation of gauge field, but the phase accumulated on a close circuit will be gauge invariant, $\sum_{j} A_{q_{j}, q_{j}} = 2\pi N$ where $N$ is the number of flux quanta. For fermions considered here, the random hopping term $t^{\delta q_z}_{ik}$ should be an antisymmetry tensor, and produces the eigenvalues when acted by the corresponding eigenstates. That means the indices of $t^{\delta q_z}_{ik}$ should be summed antisymmetrically due to the intrinsic chiral property

$$t^{\delta q_z}_{ik} = t_0 (\delta_{ik}, \delta_{kq_z} - \delta_{iq_z} \delta_{kq_z}),$$

(3)

where we use the notations of fermion indices with an overline to represent their $q_z$-momentum-related information carried by them. $t_0$ is a constant in unit of energy.

By decomposing the gauge field to two transformation operators $e^{iA_{\alpha,\beta}} = e^{i\theta_{\alpha}} e^{-i\theta_{\beta}}$ which acting on the coherent states separately, we have

$$\langle \psi^{\dagger}_{\alpha}(q_z) c_{\beta} \rangle e^{i\theta_{\alpha}} = \langle c_{\beta} \rangle e^{i\theta_{\alpha}},$$

$$e^{-i\theta_{\beta}} \psi_{\beta}(q_z) c_{\beta} = \langle c_{\beta} \rangle e^{-i\theta_{\beta}}.$$  

(4)

Here $\theta_{\alpha} = \phi_{\alpha}(q_z) \sum_{q_z} q_z$, where $\sum_{q_z} q_z = \sum_{q_z} c_{q_z}^{\dagger} c_{q_z}$ is a conserved quantity for fermions in $q_z$ species (although it is no more conserved for the fluctuating gauge field).

Then through the complicated but straightforward process as presented in appendix A, the above hopping current in terms of 3D carriers can be transformed to that in terms of 1D Majorana fermions,

$$J = \sum_{i<k_{\alpha,\beta}} \chi_{\alpha}(A_{\beta}) t^{\delta q_z}_{ik} \chi_{\beta}(A_{\beta}),$$

(5)

where we use the notations $\alpha$ and $\beta$ to index the chiral current densities (i.e. the quasiparticle number $\rho_{\alpha} \propto n_{\alpha}^{-1}$ which is inverse proportional to the carrier number as we shown in appendix A).

The chiral current density, which originates from the random hopping in a fermion current, is the only original degrees-of-freedom (DOF) in this system, and we will using the ordering $\alpha < \beta$ throughout this paper, which enforcing in any cases $q_{\alpha} < q_{\beta}$ (see equations (98)–(100) and section 3 for a detailed proof). As will be proved below, the subgroup $\{K\}$ has the same size with that of $\alpha \lor \beta$, which is order $O(M)$, origin from the infinitely fractionalized current-density, as discussed in next subsection.

Then we consider an interacting model between two carriers of different species (before and after the random hopping, respectively) as described above, under the effect of gauge field.

$$H = \sum_{\alpha, \beta} \sum_{\gamma, \delta} c_\gamma^\dagger e^{i\phi_{\delta}(q_\gamma)} e^{i\phi_{\beta}(q_\delta)} c_\delta T_{\alpha\beta}^{\gamma\delta} c_\delta^\dagger e^{i\phi_{\beta}(q_\delta)} e^{i\phi_{\alpha}(q_\gamma)} c_\gamma,$$

$$= - \sum_{i < k < j < \alpha < \beta} \chi_i(\alpha) \chi_j(\alpha) \chi_k(\beta) \chi_j(\beta),$$

(6)

where gauge fields are decomposed into transformation operators acting on the carrier fermions of different species, and transform them into complex fermions as we shown in appendix A. Here we define \( \chi_i(\alpha) = \varphi_i^\dagger(\alpha) \chi_i \) as the Majorana fermion operator that can create a particle in eigenstate \( \varphi_i(\alpha) \). Note that here \( i = 0, 1 \) are the mutually independent variable groups. The anyons generated here by the boson field could also be a hard core boson or the chiral Majorana fermion defined under a normal ordering, or even the ones defined through Dirac-\( \gamma \) matrix which satisfy the Clifford algebra similar to the Klein factor. The antisymmetry tensor \( T_{\alpha\beta}^{\gamma\delta} \) is important in diagonalizing the current matrix for each species, which reads

$$T_{\alpha\beta}^{\gamma\delta} = (\delta_{\pi\beta} \delta_{\gamma\tau} - \delta_{\pi\alpha} \delta_{\gamma\tau} - \delta_{\pi\alpha} \delta_{\gamma\tau} + \delta_{\pi\beta} \delta_{\gamma\tau}).$$

(7)

2.2. Restrictions from gauge invariance

The invariant gauge field here is induced solely by the chiral structure of the current density. (or inverse quasiparticle number) distribution. The gauge field \( A_{\alpha\beta} \) can be expressed in terms of the one-dimensional boson fields \( \Phi(\alpha) \) defined above antisymmetrically

$$A_{\alpha\beta} = \frac{1}{2} (\overline{A}_{\alpha\beta} - \overline{A}_{\beta\alpha}) = \theta_\alpha - \theta_\beta,$$

$$\overline{A}_{\alpha\beta} = \theta_\alpha - \theta_\beta,$$

$$\overline{A}_{\beta\alpha} = \theta_\beta - \theta_\alpha,$$

(8)

Similar to the gauge field coupled to fermion system which satisfies the Coulomb gauge, the above gauge field satisfies \( \nabla A_{\alpha\beta} = \frac{1}{2} (\nabla \overline{A}_{\alpha\beta} + \nabla \overline{A}_{\beta\alpha}) = 0 \), where \( \nabla \) here denotes the gradient with respect to the quasiparticle number \( n_\alpha = n_\beta \) and it follow the chiral character: \( \nabla \overline{A}_{\alpha\beta} \approx \nabla \overline{A}_{\beta\alpha} \), \( \nabla \overline{A}_{\beta\alpha} \approx - \overline{A}_{\alpha\beta} \), and this chirality-dependent behavior is similar to the derivative of the \( z \)-positions in real space, \( \partial_z = - \partial_{\bar{z}} \) (see appendix A), or the derivative of imaginary time, \( \partial_t \psi(\tau) = \psi(\tau) \) where \( \tau' = \tau + i0^+ \).

Combining the discussions in section 2.1 and appendix A, the phases \( \theta_i \) can be written as

$$\theta_i = \Phi(\alpha) \rho_i,$$

(9)

where current density (consisted by the quasiparticles in position space) can be expressed in terms of 1D quasiparticle boson field \( \Phi(\alpha) \),

$$\rho_i = \partial_\alpha \frac{\Phi(\alpha)}{2\pi} = \partial_\alpha \sum_{\gamma} \rho_\gamma = \partial_\alpha \sum_{\gamma} \rho_\gamma = \partial_\gamma \frac{i}{2\pi} \ln \frac{\phi(z)}{2\pi} = \partial_\gamma \frac{i}{2\pi} \ln \frac{n_\gamma}{n_\gamma - 1}$$

(10)

is always a conserved quantity for a part of supercharge product (with gauge fixing restriction) in a single sector \( \alpha \) (as will be clarified in the next section), by matching the notations of anyons (can be the complex fermions or the hard core bosons) with \( O(\pi) \) at level one, but it is never a conserved quantity for the fluctuating gauge field \( A_{\alpha\beta} \) itself as it can changes all the time but keeping the gauge invariance. For example, if we replace the summing indices of the current density by the Majorana fermion indices, the resulting term \( \sum_{i < j} \rho_i \) will not be a conserved quantity for each species (or sectors). This is due to the gauge fixing requirement which will be clarified below.

Thus the \( U(1) \) gauge field can be written in the form which is similar to the derivative 'kinetic term' of a Lagrangian describing the noninteracting potential (but here for the current density, which can be viewed as the quasiparticle charge)

$$A_{\alpha\beta} = - i \rho_i \sum_{\gamma} \rho_\gamma - i \rho_j \sum_{\gamma} \rho_\gamma = - i \frac{\Phi(\alpha)}{2\pi} \partial_\alpha \Phi(\alpha) = - i \frac{\Phi(\beta)}{2\pi} \partial_\beta \Phi(\beta).$$

(11)

According to the initial definition of the hopping matrix which induces the current by connecting the four fermions of two species, the disorder average process will be gauge invariant under the following transformations [11]

$$A_{\alpha\beta} \rightarrow A_{\alpha\beta} + \Phi_\alpha - \Phi_\beta,$$

$$c_\gamma^\dagger(\alpha) \rightarrow c_\gamma^\dagger(\alpha) e^{i\Phi(\alpha)},$$

$$c_\gamma(\alpha) \rightarrow c_\gamma(\alpha) e^{-i\Phi(\alpha)},$$

$$c_\gamma^\dagger(\beta) \rightarrow c_\gamma^\dagger(\beta) e^{i\Phi(\beta)},$$

$$c_\gamma(\beta) \rightarrow c_\gamma(\beta) e^{-i\Phi(\beta)},$$

(12)
where the last four lines are the U(1) symmetry acting on fermions through a constraining variable and such intrinsic gauge invariance will be incorporated by performing a Jordan-Wigner transformation (equation (138)). Base on the gauge properties, we can not simply integrating out all the four fermion indices when we try to integrating out the $O(M^2)$ gauge field $A_{\alpha\beta}$ (we consider $M \approx N$ here in large limit), since it requires gauge fixing to avoiding overcounting redundant configurations [11]. This is similar to the problem of an intercluster-coupling model where the coupling behaviors are dominated by a fluctuating gauge field as reported in [11]. Specifically in the system described in this paper, the above problem can be understood in this way: a pair of degrees-of-freedom $(\alpha, \beta)$ with size $O(M^2)$ is initially fully shared by a part of fermions $c^\dagger_{\alpha}(\alpha)$ and $c^\dagger_{\beta}(\beta)$ with degrees-of-freedom $(i, l)$, then after the above gauge invariant transformations (we consider no replica symmetry breaking here), the $(\alpha, \beta)$ must also be shared by another part of fermions $c^\dagger_{\beta}(\beta)$ and $c^\dagger_{\alpha}(\alpha)$, and in this process $A_{\alpha\beta}$ becomes $A_{\alpha\beta} + \Phi_{\beta}(\beta) - \Phi_{\alpha}(\alpha)$, then the $O(M^2)$ gauge field $A_{\alpha\beta}$ must be constrained by only $O(N)$ constraining fermion degrees-of-freedom $\Phi_{\beta}(\beta)$ or $\Phi_{\alpha}(\alpha)$. In other word, the space of gauge configurations consist of $\Phi_{\beta}(\beta)$ and $\Phi_{\alpha}(\alpha)$ can only be occupied by $O(M)$ gauge configurations of $A_{\alpha\beta}$. Thus the unique gauge configurations over the four fermion indices should be of order $O(N^2)$ (or $O(N^3M)$ instead of order $O(N^2)$). Note that in the above disorder average process, the Lagrangian multipliers in replica diagonal configurations change as $G_{\alpha} \rightarrow G_{\alpha} e^{i \Phi_{\alpha}(\alpha)}$, $\Sigma_{\alpha} \rightarrow \Sigma_{\alpha} e^{i \Phi_{\alpha}(\alpha)}$. In [11], this mismatching of gauge configurations is ignored in large-$N$ limit, which will not affect the conformal results, and their resulting model is the SYK$_4$ one. While in this article, we will consider the essential effects of the invariant gauge field $A_{\alpha\beta}$ brought initially by the antisymmetry hopping tensor in a 1D chiral current.

As we mentioned above, the chiral gradient operator acts on the gauge field is similar to the imaginary time derivative of the Majorana fermions, thus to representing such a restriction (gauge fixing)-induced suppression to the fermion degrees-of-freedom, we can give each Majorana fermion operator an imaginary time component, and then defining a bosonic supercharge (bilocal field), each one containing both the $\alpha$ and $\beta$ degrees-of-freedoms. This additional time-component of each Majorana fermion (which is for more conveninently sign their intrinsic degrees-of-freedom properties) should be distinguished from the ones relating them and their replicas (and combine them to form the Lagrangian multiplier field). The additional time-component thus will not be summed. The details of this expression will be clarified in the next section, where we show that this antisymmetry algebra guarantees the formation of well-defined weight functions of a single bosonic supercharge. The four Majorana fermions are also classified by four mutually independent degrees-of-freedom $i, j, k, l$, which is similar to the normal four-point interacting SYK model. However, as the statistical behaviors in our model originate from the current density (that being classified into two distinct sectors), to considering the intrinsic physical property of the current density in 1D, the restriction from the gauge invariance is required, which is related to the self-similarity between the two original degrees-of-freedom, i.e. the current densities $\alpha$ and $\beta$. This can also be revealed by imaginary-time-dependence of the Majorana fermions. We will see that, the restriction brought by a finite-size time mapping group will reduce the four degrees-of-freedom of the fermion indices $ijkl$ to simply three degrees-of-freedom.

### 2.3. Gauge-invariance and the effective overlap between fermion bilinear and interacting fermion in $\alpha$-sector

Treating the phase $\Phi(\tau)$ as a continuously differentiable function of the imaginary time, we have (in addition to the equation (12)),

$$A_{\alpha}(\tau) \rightarrow A_{\alpha}(\tau) - \partial_{\tau} \Phi_{\alpha}(\tau).$$

From equation (12), by introducing another positional coordinate $r$, we can represent the fermion field in Heisenberg representation as $\psi_{i} = e^{\nu \psi_{i}(r)} e^{-\nu L}$ where $s = i, j$ is the gauge-field induced DOF. $L$ is the effective Gorâkov Hamiltonian

$$L = \int d\tau d\tau' \psi^\dagger (\tau) L_{\psi} \psi (\tau) + \frac{1}{2} \int d\tau d\tau' \psi^\dagger (\tau) \psi (\tau) \psi (\tau') \psi (\tau'),$$

where the kinetic term $L_{\psi}$ can be described by the kinetic energy operator after minimal substitution $L_{\psi} = (i \partial_{\tau} - A_{\alpha}(\tau) )^2$, $L_{\psi}^* = (-i \partial_{\tau} - A_{\alpha}(\tau) )^2$.

The shift of the fermion operators during the gauge transformation can be described by the continuously differential result

$$c_{\alpha}^\dagger (\tau) = c_{\alpha}^\dagger (\tau) e^{i \Phi_{\alpha}(\tau)},$$

$$c_{\alpha} (\tau) = c_{\alpha} (\tau) e^{-i \Phi_{\alpha}(\tau)},$$

where we have the following relation due to the variance of gauge phase

$$L_{\tau}[\Phi_{\alpha}] e^{i \Phi_{\alpha}} = e^{i \Phi_{\alpha}} L_{\tau} e^{i \Phi_{\alpha}} \Phi_{\alpha},$$

$$L_{\tau}^*[\Phi_{\alpha}] e^{-i \Phi_{\alpha}} = e^{-i \Phi_{\alpha}} L_{\tau}^* e^{-i \Phi_{\alpha}} \Phi_{\alpha},$$

(16)
with the functional containing the variation of \((\Phi_0' - \Phi_0): \mathcal{L} \equiv [\Phi_0'] = (-i\partial_\tau - (A_\alpha - \partial_\tau \Phi_0(\tau)))^2, \mathcal{L}' \equiv [\Phi_0'] = (i\partial_\tau - (A_\alpha - \partial_\tau \Phi_0(\tau)))^2.

The most essential relation between the fermion field \(\psi(\tau)\) and the \(r\) operator for the following discussion is
\[ [\psi(\tau), r] = 0. \]  
(17)

Next we start by the commutation relation between the imaginary-time-dependent fermion field \(\psi(\tau)\) with the quantity \(r\) in positional space as appears in the traditional functional derivative (note that the powers of \(r\) could be of arbitrary high-order for a continuously differentiable functional),
\[
[r \psi(\tau), r^m] = [\psi(\tau), r] r^{m-1} + r [\psi(\tau), r^m] = [\psi(\tau), r] r^{m-1} + r^2 \psi(\tau), r] r^{m-2} + r^3 \psi(\tau), r] r^{m-3} \ldots + r^{m-1} [\psi(\tau), r] = \sum_{\ell=0}^{m-1} r^{\ell} [\psi(\tau), r] r^{m-1-\ell}. 
\]  
(18)

Due to the critical commutation relation mentioned above, \([\psi(\tau), r] = 0\), the above equation becomes
\[
[\psi(\tau), r^m] = m [\psi(\tau), r] r^{m-1} = mr^{m-1}[\psi(\tau), r],
\]  
(19)

or equivalently,
\[
[\psi(\tau), r^m] = mr^{m-1}\psi(\tau)r - mr^m\psi(\tau) = m\psi(\tau)r^m - mr\psi(\tau)r^{m-1}.
\]  
(20)

Then we have two important relations between \(\psi(\tau)r^m\) and \(r^m\psi(\tau)\)
\[
\psi(\tau)r^m + r^m\psi(\tau) = r^{m-1}\psi(\tau)r + r\psi(\tau)r^{m-1},
\]
\[
\psi(\tau)r^m - r^m\psi(\tau) = mr^{m-1}\psi(\tau)r - mr\psi(\tau)r^{m-1},
\]  
(21)

where we can obtain the following expressions for the \(\psi(\tau)r^m\) and \(r^m\psi(\tau)\)
\[
\psi(\tau)r^m = mr^{m-1}\psi(\tau)r - (m - 1)r^m\psi(\tau) = m \frac{r^m\psi(\tau)}{m - 1} - \frac{1}{m - 1} r^{m-1}\psi(\tau),
\]
\[
r^m\psi(\tau) = mr\psi(\tau)r^{m-1} - (m - 1)\psi(\tau)r^m = \frac{m}{m - 1} r^{m-1}\psi(\tau)r - \frac{1}{m - 1} \psi(\tau)r^m.
\]  
(22)

There is a useful list,
\[
\frac{1}{m} [\psi(\tau), r^m] = \frac{[\psi(\tau), r] r^{m-1}}{m} = \frac{r [\psi(\tau), r] r^{m-2}}{m} = \frac{r^2 [\psi(\tau), r] r^{m-3}}{m} \ldots = \frac{r^{m-1} [\psi(\tau), r]}{m} = r^{m-1}\psi(\tau)r - r^m\psi(\tau),
\]  
(23)

where we can obtain
\[
r^{m-k}[\psi(\tau), r] r^k = \frac{k}{m} [\psi(\tau), r^m] + r^m\psi(\tau).
\]  
(24)

For example, we would verify by \(r^{m-1}[\psi(\tau), r] r\) which satisfies the equality,
\[
r^{m-1}[\psi(\tau), r] r = \frac{1}{m} \psi(\tau)r^m - \frac{1}{m - 1} r^{m-1}\psi(\tau).
\]
\[
= \frac{1}{2} r^m\psi(\tau) + \frac{1}{4} r^{m-1}\psi(\tau)r + \frac{1}{8} r^{m-2}\psi(\tau)r^2 + \ldots + \frac{1}{2m - 1} r^2\psi(\tau)r^{m-2} - \frac{1}{2m - 1} \psi(\tau)r^m
\]
\[
= \frac{1}{2} r^m\psi(\tau) + \left(\frac{1}{2} - \frac{1}{2m - 1}\right) r^{m-1}\psi(\tau) + \left(1 - \frac{m}{2m - 1}\right) \frac{1}{m} [\psi(\tau), r^m] - \frac{1}{2m - 1} \psi(\tau)r^m.
\]  
(25)

Then since \(\frac{1}{m} [\psi(\tau), r^m]\) can be expressed in terms of the following general form
\[
r^k[\psi(\tau), r] r^{m-k-1} = r^k\psi(\tau)r^{m-k} - r^{k+1}\psi(\tau)r^{m-k-1},
\]  
(26)
which represent the same quantity for \( k = 0, \ldots, m - 1 \), we have
\[
2^k \psi(\tau) r^{m-k} = r^{-k} \psi(\tau) r^{m-(k-n)} + r^{k-n} \psi(\tau) r^{m-(k+n)},
\] (27)
which represent the same quantity for integers \( k = 0, \ldots, m - 1 \) and \( n = 0, \ldots, \min[k, m-k] \), and we have
\[
[[r^{-k} \psi(\tau) r^{m-(k-n)}, r^n], r^m] = 0,
[[r^{-k} \psi(\tau) r^{m-(k-n)}, r^n, r^m] = 0,
[[r^{k-n} \psi(\tau) r^{m-(k+n)}, r^n, r^m] = 0,
\]
\[
r^{n'} \psi(\tau) r^{m-n} - r^n \psi(\tau) r^{m-n'} = [r^n \psi(\tau) r^{m-n'}, r^{m-n}] = \frac{n' - n}{m} [\psi(\tau), r^m]
\]
\[
= (n' - n) [\psi(\tau), r] r^{m-1} = [r^{m-n'}, \psi(\tau)] r^{m-(m-n')},
\] (28)
for \( n' > n \).

To related to the topological character as described by equation (114), we rewrite the commutator \([\psi(\tau), r^m]\) in belowing form
\[
[\psi(\tau), r^m] = h[\psi(\tau), r] r^{m-1} + r^h [\psi(\tau), r^{m-h}]
\]
\[
= h[\psi(\tau), r] r^{m-1} + \sum_{j=2}^{h} (r^j - r^{j-1}) [\psi(\tau), r^{m-h}] + r [\psi(\tau), r^{m-h}]
\]
\[
= h[\psi(\tau), r] r^{m-1} + \sum_{j=2}^{h} (r^j - r^{j-1}) [\psi(\tau), r] + r [\psi(\tau), r^{m-h}] + \sum_{j=m-h+1}^{m-1} (r^j - r^{j-1}) [\psi(\tau), r] (m - h - 1).
\] (29)

The (correlation) pattern of the topological modes \( \chi_2 \chi_3 \) can be described by terms \([\psi(\tau), r^m] r^{m-1}\) while the terms \([\psi(\tau), r] r^{m-1} + (r^j - r^{j-1}) [\psi(\tau), r]\) (for \( j = m - h + 1, \ldots, m - 1 \)) correspond to the mode \( \chi_2 \chi_{m-j} \). The edge modes \( \chi_1 \chi_2 + k_m \) corresponds to the term \( r[\psi(\tau), r^{m-h}] + \sum_{j=m-h+1}^{m-1} (r^j - r^{j-1}) [\psi(\tau), r] (m - h - 1) \).

Using the relations equations (26)–(28), the mode not in the edges can be expressed as
\[
\chi_2 \chi_{3+m-j} = [\psi(\tau), r] r^{m-1} + (r^j - r^{j-1}) [\psi(\tau), r]
\]
\[
= (r^j - r^{j-1} + r^{m-h}) \psi(\tau) r - (r^{j+1} - r^j + r^m) \psi(\tau),
\] (30)
and the powers of \( r \) act on commutator \([\psi(\tau), r]\) can be expressed in terms of the modes (for \( j = 1, \ldots, m - 1 \))
\[
r^{m-j}[\psi(\tau), r] = \int \chi_2 \chi_3 - \chi_2 \chi_4 - \chi_2 \chi_5 - \cdots - \chi_2 \chi_{j+2}.
\] (31)

2.4. exchanging rule between \([\psi(\tau), r]\) and powers of \( r \)

Next we discuss the exchanging rule for between the fermion field \( \psi(\tau) \) and the powers of \( r \) operator, which leads to the results of above section. Firstly, according to the original relation equation (17), we have
\[
r^k (\psi(\tau)) r^k = (r^k \psi(\tau)) r^k,
\] (32)
where \( k \) is an arbitrary positive integer. According to above subsection, we using the functional \( Q \) to replacing the \( r \) operator, we have
\[
[[\psi, Q], Q] = 0,
\] (33)
by define the \( Q \) in the form of supercharge
\[
Q = \frac{1}{2} \int \! \! d^n \tau_2 C_{\tau_1 \tau_2} \psi(\tau_1) \psi(\tau_2),
\] (34)
where \( C_{\tau_1 \tau_2} \) is an antisymmetry tensor satisfies \( C_{\tau_1 \tau_2} = -C_{\tau_2 \tau_1} \). Then the above relation equation (36) reads
\[
Q^k (\psi(\tau) Q^k) = (r^k \psi(\tau)) Q^k,
\] (35)
which directly leads to due to the relation equation (36),
\[
(\psi(\tau) Q) Q = 2(Q(\psi(\tau)) Q - Q(Q)(\psi(\tau))).
\] (36)

We picking \( k = 1, 2 \) to prove this relation.
Firstly note that
\[ [\psi(\tau), Q] = \left[ \frac{1}{2} \int d\tau_1 \tau_1 C_{\tau_1, \tau_2, \tau_3} \psi^\dagger (\tau_1) \psi^\dagger (\tau_2) \right] \psi(\tau) \]
\[ + \frac{1}{2} \int d\tau_2 \tau_2 C_{\tau_2, \tau_3, \tau_4} \psi^\dagger (\tau_2) \psi^\dagger (\tau_3) \psi(\tau) \]
\[ - \frac{1}{2} \int d\tau_4 C_{\tau_4, \tau_3, \tau_2} \psi^\dagger (\tau_4) \psi^\dagger (\tau_3) \psi^\dagger (\tau_2) \]
\[ = \frac{1}{2} \int d\tau_2 \tau_2 C_{\tau_2, \tau_3, \tau_4} \psi^\dagger (\tau_2) \psi^\dagger (\tau_3) \psi^\dagger (\tau_4) \]
\[ = \frac{1}{2} \int d\tau_2 \tau_2 C_{\tau_2, \tau_3, \tau_4} \psi^\dagger (\tau_2) \psi^\dagger (\tau_3) \psi^\dagger (\tau_4) \]  \(\text{(37)}\)

The same result can be obtained by constructing two anticommutators for the product of both orders \((\psi(\tau)Q)\) and \((Q\psi(\tau))\), but we use the method that handling the order of operators in left-ordered product only, and subtracted by the right-ordered one. Thus we have
\[
\psi(\tau)Q^k = Q^k\psi(\tau) + C_{12} \psi(\tau)(\psi^\dagger (\tau_2) \psi^\dagger (\tau_1)) \\
+ C_{23} \psi(\tau)(\psi^\dagger (\tau_2) \psi^\dagger (\tau_1)) + C_{34} \psi(\tau)(\psi^\dagger (\tau_2) \psi^\dagger (\tau_1)) + \ldots \\
+ C_{2k-1} \psi(\tau)(\psi^\dagger (\tau_2) \psi^\dagger (\tau_1)) + \ldots \\
+ C_{2k} \psi(\tau)(\psi^\dagger (\tau_2) \psi^\dagger (\tau_1)) \]  \(\text{(38)}\)
where the combination formula reads \(C_{nk} = \frac{(2k)!}{n!(2k-n)!}\).

For \(k = 1\),
\[
Q(\psi(\tau)Q) = Q[\int d\tau_2 \tau_2 C_{\tau_2, \tau_3} \psi^\dagger (\tau_2) + Q\psi(\tau)] \\
= Q[\int d\tau_2 C_{\tau_2, \tau_3} \psi^\dagger (\tau_2)] + Q\psi(\tau) = (Q\psi(\tau))Q, \]  \(\text{(39)}\)

For \(k = 2\),
\[
Q^2(\psi(\tau)Q) = Q^2[\int \int d\tau_3 d\tau_4 \tau_3 \tau_4 C_{\tau_3, \tau_4, \tau_5, \tau_6} \psi^\dagger (\tau_3) \psi^\dagger (\tau_5) \psi^\dagger (\tau_4)] \\
+ Q[\int d\tau_4 \tau_4 C_{\tau_4, \tau_5, \tau_6} \psi^\dagger (\tau_4) \psi^\dagger (\tau_5) \psi^\dagger (\tau_6)] \\
- Q^2[\int \int d\tau_3 d\tau_4 \tau_3 \tau_4 C_{\tau_3, \tau_4, \tau_5, \tau_6} \psi^\dagger (\tau_3) \psi^\dagger (\tau_5) \psi^\dagger (\tau_4)] \\
+ Q[\int d\tau_4 \tau_4 C_{\tau_4, \tau_5, \tau_6} \psi^\dagger (\tau_4) \psi^\dagger (\tau_5) \psi^\dagger (\tau_6)] \\
+ (Q^2\psi(\tau))Q^2 \\
= (Q^2\psi(\tau))Q^2. \]  \(\text{(40)}\)

Similarly, for \(k = 3\), where \(Q\) contains six \(\psi^\dagger(\tau)\) operators, two sets of products each of whose amount is \((C_{12}^3 + C_{23}^3 + C_{34}^3)\) will be cancelled out and leads to \(Q^3(\psi(\tau)Q^3) = (Q^3\psi(\tau))Q^3\).

Specifically, for \(Q[\psi(\tau), Q] = Q^2[\psi(\tau), Q] = [\psi(\tau), Q]Q^2\) (we remind the reader that the \(Q\) can be replaced by \(r\) to connect to the content of above subsection), according to above results, we have
\[
Q[\psi(\tau), Q] = Q[\psi(\tau)Q] - Q(Q\psi(\tau)) = (Q\psi(\tau))Q^2 - Q^2(\psi(\tau))Q, \]  \(\text{(41)}\)

where we use \(Q(Q\psi(\tau))Q = Q((Q\psi(\tau))Q) = Q((Q\psi(\tau))Q)\) (which can be obtained through the above procedure), and thus (see also equation (36))
\[
(Q\psi(\tau))Q^2 = 2Q^2(\psi(\tau)Q) - Q^2(Q\psi(\tau)), \\
(\psi(\tau)Q)Q^2 = 2(Q\psi(\tau))Q^2 - Q^2(\psi(\tau)Q) = 3Q^2(\psi(\tau)Q) - Q^2(Q\psi(\tau)), \\
Q^2(\psi(\tau)Q) = 2(Q\psi(\tau))Q^2 - (\psi(\tau)Q)Q^2, \\
Q^2(\psi(\tau)) = 2Q^2(\psi(\tau)Q) - (Q\psi(\tau))Q^2 = 3(Q\psi(\tau))Q^2 - 2(\psi(\tau)Q)Q^2, \]  \(\text{(42)}\)
which leads to
\[
(Q\psi(\tau))Q^2 = \frac{1}{3} Q^3 \psi(\tau) + \frac{2}{3} \psi(\tau) Q^3,
\]
\[
Q^2 \psi(\tau) Q = \frac{1}{3} \psi(\tau) Q^3 + \frac{2}{3} Q^3 \psi(\tau).
\]

Similarly, we can further obtain
\[
(Q\psi(\tau))Q^3 = \frac{1}{4} Q^4 \psi(\tau) + \frac{3}{4} \psi(\tau) Q^4,
\]
\[
Q^3 \psi(\tau) Q = \frac{1}{4} \psi(\tau) Q^4 + \frac{3}{4} Q^4 \psi(\tau),
\]

which can be cast into more general form
\[
(Q\psi(\tau))Q^k = Q^{k-1} \psi(\tau) + \frac{k-1}{k} \left[ \psi(\tau), Q^{k+1} \right] = [\psi(\tau), Q(k-1)Q^k].
\]

2.5. effective field operator in terms of the functional and the two-fold pseudo-DOF
Base on the functional \(Q\), we rewrite the commutator \([\psi(\tau), r^m]\) as \([\psi, (r), L[Q]]\) by introducing the functional \(L[Q]\) (effective field operator) as
\[
L[Q] = L[0] + \sum_{m=1}^{\infty} \frac{Q^m d(m)L[Q]}{m!} \bigg|_{Q=0},
\]
where \(Q\) is a functional which also plays the role of test function here. This expression can be rewritten as
\[
L[Q] = e^{QH}[L][0] e^{-QH} = L[0] + \sum_{m=1}^{\infty} \frac{Q^m}{m!} \left[ \mathcal{H} \left( \cdots \left[ \mathcal{H} \left( \mathcal{H} \left( \mathcal{H} \left( L[0] \right) \right) \right] \right) \right]
\]
\[
= L[0] + \sum_{m=1}^{\infty} \int d\tau_1 \cdots d\tau_m \frac{\delta^n L[0]}{\delta \psi_{\alpha}^{(\tau_1)} \cdots \delta \psi^{(\tau_m)}},
\]
where \(\mathcal{H} = \int d\tau \psi_{\alpha}^{(\tau)} \mathcal{H}(\tau)\) can be expressed in the form of Gorâkov equation [17] by endows the unshifted functional \(L[0]\) a imaginary variable \(\tau_2 \neq \tau_1\), i.e. \(L[0; \tau_2]\)
\[
\mathcal{H} = \int d\tau L[\gamma] \mathcal{H}(\gamma) L[\gamma] + \frac{1}{2} \int d\tau \int d\tau' V_{\tau-\tau'} L[\gamma] L[\gamma] L[\gamma] L[\gamma],
\]
where \(\mathcal{H}(\gamma)\) can be expressed in the form of kinetic operator \((-i \partial_\gamma - A_2)^2\).

Then we have
\[
[\psi_{\alpha}, L[Q]] = [\psi_{\alpha}, L[0]] + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{d(m)L[Q]}{dQ^{(m)}} \bigg|_{Q=0} [\psi_{\alpha}, Q^m]
\]
\[
= [\psi_{\alpha}, L[0]] + [\psi_{\alpha}, Q] \frac{\partial L[Q]}{\partial Q}
\]
\[
= [\psi_{\alpha}, L[0]] + \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \frac{d(m-1)L[Q]}{dQ^{(m-1)}} \bigg|_{Q=0} [\psi_{\alpha}, Q] Q^{m-1},
\]
where
\[
\frac{\partial L[Q]}{\partial Q} = e^{QH} \left[ \frac{dL[Q]}{dQ} \right] \bigg|_{Q=0} e^{-QH}
\]
\[
= e^{QH} \left[ \mathcal{H} \right. \left. L[0; \tau_2] \right] e^{-QH} = \frac{1}{2} \int d\tau V_{\gamma-\gamma} L[\gamma] L[\gamma] L[\gamma] L[\gamma] \bigg|_{Q=0} [\psi_{\alpha}, Q] Q^{m-1},
\]
\[
= e^{QH} \left[ \mathcal{H}(\tau_2) L[0; \tau_2] - \int d\tau V_{\gamma-\gamma} L[\gamma] L[\gamma] L[\gamma] L[\gamma] \bigg|_{Q=0} [\psi_{\alpha}, Q] Q^{m-1},
\]
where \(e^{QH} L[0; \tau_2] e^{-QH} = L[Q; \tau_2]\), with \(\tau_2\) the parameter generated by shifted \(\tau_1\) within the \(\mathcal{H}\), and thus \(\partial_\gamma \mathcal{H} = 0\) as \(\tau_2\) here is a 'gradient'-dependent quantity now like the \(V_{\gamma-\gamma}\).
For a comparison, the above derivative \( \frac{\partial L[Q]}{\partial Q} \) can also be expressed by

\[
\frac{\partial L[Q]}{\partial Q} = \left[ \frac{dL[Q]}{dQ} \right]_{Q=0} e^{-QH} + \sum_{m=1}^{\infty} \frac{Q^m}{m!} \left[ \frac{\partial L[Q]}{\partial Q} \right]_{Q=0} e^{-QH} \]

\[
= e^{QH}[H, L[0; \tau_2]] e^{-QH} = \left[ H, L[0; \tau_2] \right] + \sum_{m=1}^{\infty} \frac{Q^m}{m!} \left[ \frac{\partial L[Q]}{\partial Q} \right]_{Q=0} e^{-QH} \]

Thus using equation (47), we also have

\[
\frac{\partial e^{QH}L[0]}{\partial Q} e^{-QH} = \frac{dQ}{dQ} \left[ L[0] + \sum_{m=1}^{\infty} \frac{Q^m}{m!} \frac{d^{(m+1)}L[Q]}{dQ^{(m+1)}} \right]_{Q=0} e^{-QH}.
\]

Using the relation

\[
[H, L[0; \tau_2]] = -\frac{\delta H}{\delta L[0; \tau_2]} = -\left[ \hat{H}(\tau) L[0; \tau_2] \right] = -\frac{1}{2} \int_{\gamma \tau = \tau_2} d\gamma \hat{V}_{\gamma - \tau_2} L'_0[\gamma L[0; \tau_2]] = \frac{1}{2} \int_{\gamma \tau = \tau_2} d\gamma \hat{V}_{\gamma - \tau_2} L'_0[\gamma L[0; \tau_2]],
\]

we can express \( H \) as

\[
\hat{H} = \int d\tau_2 L\prime[0; \tau_2] - \frac{\delta H}{\delta L[0; \tau_2]} = \int d\tau_2 L\prime[0; \tau_2] H(\tau_2) L[0; \tau_2] + \int_{\gamma \tau < \tau_2} d\gamma \int_{\gamma \tau = \tau_2} d\gamma' \hat{V}_{\gamma - \tau_2} L\prime[0; \gamma L[0; \tau_2]],
\]

which is equivalent to equation (48).

Similar to equation (50), for \( L\prime[Q] \) we have

\[
\frac{\partial L\prime[Q]}{\partial Q} = \left[ \frac{dL\prime[Q]}{dQ} \right]_{Q=0} e^{-QH} \]

\[
= \frac{dQ}{dQ} \left[ L[0; \tau_2] H(\tau_2) + \frac{1}{2} \int d\gamma \hat{V}_{\gamma - \tau_2} L\prime[0; \tau_2 L[0; \gamma]] e^{-QH} \right]
\]

\[
= \left[ L[0; \tau_2] H(\tau_2) + \int d\gamma \hat{V}_{\gamma - \tau_2} L\prime[0; \tau_2 L[0; \gamma]] L[0; \gamma] \right] e^{-QH}
\]

\[
= \left[ H(\tau_2) L[0; \tau_2] + \int d\gamma \hat{V}_{\gamma - \tau_2} L\prime[0; \tau_2 L[0; \gamma]] L[0; \gamma] \right] e^{-QH},
\]

where using integration by parts we have

\[
\int d\tau_2 (L\prime[0; \tau_2] H(\tau_2) - H(\tau_2) L\prime[0; \tau_2]) L[0; \tau_2] = 0.
\]

Then the two-fold sublattice-like (pseudo) degrees-of-freedom can be obtained by combining equation (50) and equation (55),

\[
\frac{\partial L\prime[Q]}{\partial Q} = (-1)^a \left[ H(\tau) L\prime[Q; \tau] + \int d\gamma' \hat{V}_{\gamma - \tau_2} L\prime[Q; \tau L[0; \gamma']] L[0; \gamma'] \right],
\]

where \( a = 2 \) corresponds to the creation operator while \( a = 1 \) corresponds to anihilation operator. That is to say, the creation or anihilation operators correspond to the fermion fields of different sites, respectively (in the sublattice).

Thus, in terms of the functional \( L[Q] \), which plays the role of effective field operator in the Gorâkov equation, the effect of topological modes can be counted by considering their effective particle-number
operator. And the two-fold pseudo-DOF can be shown by the equation of motion of the effective field operator $L$ [Q]. Further, although it originates from the statistical properties (of scalar products) in bosonic chain, it follows some pattern of symmetry like the $Z_2$-fermionic parity.

### 2.6. Calculations using the two-fold DOF character and the Jacobian

As will be shown in the next subsection, the functional determined here can be related to the original commutator between fermion field $\psi(\tau)$ and the operator $r$. This can be related to the effective overlap (replacement) between fermion bilinear and interacting fermion in $\alpha$-sector as we mention below, where we introduce the fermionic field matrix constituted by the fermionic indices $n = 1, \cdots, N$ and the sublattice (or pseudospin) indices $s = a, b$.

$$\Psi = \begin{pmatrix} \psi_1^a & \psi_1^b \\ \vdots & \vdots \\ \psi_N^a & \psi_N^b \end{pmatrix}. \quad (58)$$

Then according to Weinsteinâ"Aronszajn identity, we obtain a relation similar to equation (57)

$$\text{Det}[:\Psi^\dagger:] + I_N = (-1)^{N-2} \text{Det}[:\Psi:] - I_1. \quad (59)$$

Next, according to the matrix determinant lemma, we know the trace of a certain matrix can be seen as the directional derivative of the determinant, and as long as $\Psi$ has two columns $(s = a, b)$, we have

$$\begin{align*}
\text{Det}[:\Psi^\dagger:] + I_N &= 1 + \text{Tr}[:\Psi^\dagger:\] + \text{Tr}[:\Psi:] \cdot \text{Tr}[:\Psi^\dagger:\], \\
\text{Det}[:\Psi^\dagger:] - I_N &= \text{Det}[:\Psi:] + 1 - \text{Tr}[:\Psi^\dagger:],
\end{align*} \quad (60)$$

When the DOF representing the flavors of fermion bilinear terms are omitted to conserving the total number of DOFs, in which case $\psi_n = \psi_1$, we have

$$\begin{align*}
\text{Det}[:\Psi^\dagger:] + I_N &= 1 + \text{Tr}[:\Psi:] \cdot \text{Tr}[:\Psi^\dagger:], \\
\text{Det}[:\Psi:] - I_N &= 1 - \text{Tr}[:\Psi:] \cdot \text{Tr}[:\Psi^\dagger:], \\
\text{Det}[:\Psi^\dagger:] &= \text{Det}[:\Psi:] = 0.
\end{align*} \quad (61)$$

Now we know that the DOF of current densities represented by two sectors $\alpha$ and $\beta$ results in the sublattice (or pseudospin) DOF in the local Majorana fermions of the sector $\alpha$. That is the reason why we must using the definition of $\chi_{ij}$ instead of $\gamma_{ij}$ in the $\alpha$-sector, when considering the global behaviors of the whole system. In terms of this, due to the effect of gauge field-modified single particle operator (equation (15)), the distinguishable creation and annihilation operators can be replaced by the the DOF described by sublattice indices $s = i, j$, i.e., $\psi_i^a \psi_j^b \rightarrow \psi_i^1 \psi_j^1$.

We can using the Lagrange multipliers in replica theory to describe such additional DOF, in terms of a partition function

$$Z_r = \prod_{i}^{N} \prod_{s, t = a, b} D[\psi_{i, s}^a, \psi_{i, t}^b] \text{Exp}[-\int dr \int dr' \psi_{i, s}^a(\tau) H_{i, s}^a(\tau, \tau') \psi_{i, t}^b(\tau)]$$

$$= \prod_{i}^{N} \prod_{s, t = a, b} D[\psi_{i, s}^a, \psi_{i, t}^b] \text{Exp}[-\int dr \int dr' \psi_{i, s}^a(\tau) H_{i, s}^a(\tau, \tau') \psi_{i, t}^b(\tau)]]^{\frac{N}{2}}$$

$$= \pi^N \text{Det}[H_{i, s}^a(\tau, \tau')]^{-N/2}, \quad (62)$$

where $H_{i, s}^a(\tau, \tau')$ here denotes a $2 \times 2$ imaginary symmetric matrix, and in last step we using the formula of Gaussian integral. The above partition function can be symmetric in terms of the bosonic fluctuation as a function of the coordinate $\phi^F(r)$, $Z_r = \int \text{Exp}[-\int \frac{1}{2} (\partial_r \phi^F(r))^2]$, where the action can be normalized by adding another term $\frac{1}{2} (\phi^F(r))^2$ into the exponential part [18]. The field $\phi^F(r)$ can be written in terms of an orthonormal basis $E_m(r)$ and the saddle point $r_0$ [18, 19]

$$\phi^F(r) = \psi_{i, s}(r - r_0) + \sum_{m=1}^\infty \phi_m^F E_m(r), \quad (63)$$

where the variation of $\phi^F(r)$ with respect to $r_0$ is orthogonal to that with respect to $\phi_m^F$, and in to make two sublattice flavors be mutually independent (to conserve the number of degrees of freedom), we have

$$\int dr E_m(r) \partial_r \psi_{i, s}(r) = 0. \quad (64)$$

In terms of the functional Taylor expansion [20], we can know that the $r$-independent term $\phi_m^F$ is indeed a Lagrangian $\mathcal{L}[\phi^F(r), \phi^F(r)]$, and the result of equation (64) is due to
\[
\phi^F(r) = \sum_{m=1}^{\infty} \int_0^r d\eta_1 \cdots \int_0^r d\eta_m \frac{\partial^m \phi^F(r)}{\partial u(r-\eta_1) \cdots \partial u(r-\eta_m)} u(r-\eta) \cdots u(r-\eta_m),
\]

(65)

where \( \eta \) is a function of \( u(r - r') \), and that results in the transition of dependence on \( r \) to \( r' \) for the \( \phi^F_{\eta} \) term.

Here we note that the vanishing result of equation (66) can be related to the orthonormality of the saddle points around \( \tau_r \). If we represent the commutator \([H, L[0; \tau_r]]\) in terms of an orthonormal set of the function \( f(\tau_r) \),

\[
[H, L[0; \tau_r]] = \sum_{n} F_n f_n(\tau_r) = F(\tau_r) - F(\tau_r - \tau^0_r),
\]

(66)

where \( \tau^0_r \) denotes all the degenerate saddle points around \( \tau_r \). The functional \( F(\tau_r) \) is a result of summation over all possible fluctuation directions, and \( \tau_r \) denotes all the possible fluctuation directions other than the one to which the saddle point \( \tau^0_r \) belongs. Then we can divide \( F(\tau_r) \) into two parts by making sure \( f_n(\tau_r) \) forms an orthonormal basis

\[
F(\tau_r) = F(\tau_r - \tau^0_r) + \sum_{n} F_n f_n(\tau_r),
\]

(67)

which means the function \( F(\tau_r) \) is a summation of components (functional) in different directions that modified by the mutually independent weights (i.e. the saddle points whose amount corresponds to the dimension of Euclidean space-time). Thus the variation of \( F(\tau_r) \) with respect to \( \tau^0_r \) and \( F_n \) (saddle points in all possible fluctuation directions) are nonzero. In terms of the mechanism, this is similar to the transition of area-law entanglement (the summation boundary of fluctuation directions here) to the volume-law entanglement (weight distribution for each direction here) in a multipartite system with non-Abelian symmetry [21].

The term \( F(\tau_r - \tau^0_r) \) describes the fluctuation of target function \( F(\tau_r) \) around the saddle-point \( \tau^0_r \), while the summation (commutator) \( \sum_{n} \) \( F_n f_n(\tau_r) \) corresponds to all the other possible fluctuations of target function \( F(\tau_r) \) around the saddle-points \( \tau_r = \tau^0_r \). Due to the mutual independence between saddle points, the variation of \( F(\tau_r) \) with respect to \( \tau^0_r \) are orthogonal to that with respect to \( F_n \) [22], we have

\[
\int d\tau_r f_n(\tau_r) \partial_{\tau^0_r} F(\tau_r - \tau^0_r) = f_n(\tau_r) F(\tau_r - \tau^0) = 0,
\]

(68)

which shows that the product consists of the normalized basis and the fluctuating function around a saddle-point \( \tau^0_r \) is an even function of \( \tau_r \) (in the interval where \( f_n(\tau_r) \) is a set of orthonormal functions). As \( F_n \) represents the group made up by the saddle-points that each one of them from a fluctuation direction distinct from all the others, and the summation range of fluctuation directions in the term \( \sum_{n} F_n f_n(\tau_r) \) is related to the direction to which the saddle point \( \tau^0_r \) belongs (\( \tau^0_r \)). Thus the critical term \( F(\tau_r - \tau^0_r) \) equivalents to \( \sum_{n} F_n f_n(\tau_r) \) above expression can be rewritten as

\[
\int d\tau_r f_n(\tau_r) f_{\eta'}(\tau_r) = 0,
\]

(69)

Using Jacobian the integral over all possible (shifted) \( F(\tau_r) \) can be written as

\[
dF(\tau_r) = J d\tau^0_r \prod \limits_{\tau_r} dF_n,
\]

(70)

with

\[
J = \left[ \int d\tau_r (\partial_{\tau^0_r} F(\tau_r - \tau^0_r))^2 \right]^{1/2} \left[ \prod_{\tau_r = \tau^0_r} \int d\tau_r (\partial_{\tau^0_r} F(\tau_r - \tau))^2 \right]^{1/2},
\]

(71)

where the integral over \( \tau_r \) after each squared derivatives of \( F_n \) with respect to saddle points guarantees the single-saddle-point-dependence and make sure the Jacobian is diagonal, and the second term in above expression reduces to one due to the orthonormality of functions \( f_n(\tau_r) \), \( \int d\tau_r f_n(\tau_r) f_{\eta'}(\tau_r) = \delta_{\eta_n, \eta'} \). Note that in equation (68), the range of integral over \( \tau_r \) is restricted by the saddle-point within the function \( F_n \). Equation (70) turns the integral from the functional measurement \( dF(\tau_r) \) to the saddle points (each one of them of different fluctuation directions).

If we consider \( d \) values of the saddle point (denoted as \( \tau^0_{2 \mu} \) with \( \mu = 1, \cdots, m \) in the \( \tau^0_r \) fluctuation direction, which corresponds to the \( d \)-dimensional (Euclidean) space-time in field theory, we have
Here the nonzero commutation relation between $[\tau_0^{\mu}, \psi_\alpha] = \lambda_\mu \psi_\alpha$, where we can choose

$$\lambda_\mu = \left( \prod_{\mu=1}^{m-1} \left( \frac{n}{\sqrt{\pi}} \right)^{n-1} \right)^{m/2} \left( \frac{n}{\pi} \right)^{3/2} \frac{1}{2^{m-1}} \frac{1}{\lambda_\mu} \theta_m \int d^n \tau \left[ \frac{\partial F_\lambda(\tau)}{\partial \tau_0^{\mu}} \right]_m^{m/2},$$

where $d^n \tau = \prod_{\mu=1}^{m-1} d(\tau_0^{\mu})$ and $\theta_m = m^{m+2} \pi^{-1/2}$ for odd $m$, and $\lambda_\mu$ is the corresponding weight for each eigenvalue (next we simply denote $\tau_0^{\mu}$ as $\gamma_\mu$). The function $F_\lambda(\tau)$ represented in terms of the Gaussian distribution,

$$F_\lambda(\tau) = \sum_{\mu=1}^{m} e^{-\gamma_\mu^2 \lambda_\mu},$$

where $\lambda_\mu$ satisfies

$$\partial_{\gamma_\mu} F_\lambda(\tau) = 2F_\lambda(\tau) \lambda_\mu \gamma_\mu.$$

Similar results can be obtained also by using the joint probability distribution instead of the individual weights, where we will find that the eigenvalue density reads

$$\rho(\gamma_\mu) = e^{-\int e^{-\int e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2}} \prod_{\mu=1}^{m} \left( \gamma_\mu^2 \right)^{1/2} \prod_{i<j} \left( \gamma_i - \gamma_j \right)^2}$$

$$= (m-1)! e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2} \det \left( \begin{array}{cccc}
\rho_{0,0} & \rho_{0,1} & \cdots & \rho_{0,m-1} \\
\rho_{1,0} & \rho_{1,1} & \cdots & \rho_{1,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{m-2,0} & \rho_{m-2,1} & \cdots & \rho_{m-2,m-2}
\end{array} \right),$$

where $\rho_{ij} = \int d\gamma^{ij} \gamma_1^i \gamma_2^j e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2}$.

2.7. functional in terms of the topological modes

The functional defined here can be related to the topological modes through the following relation

$$\psi_\alpha L[Q] = h \chi_2 \chi_3 + \chi_4 \chi_{m+2} + \sum_{j=m-h+1}^{m-1} r^j \psi(\tau), r] = \left( m + \frac{h(h+1)}{2} - 1 \right) \chi_2 \chi_3 = \sum_{j=1}^{m} \psi_{m-j} \chi_2 \chi_{3-j},$$

$$L[Q] \psi_\alpha = \psi_\alpha(\tau) L^*= \sum_{j=1}^{m} \psi_{m-j} \chi_2 \chi_{3-j}.$$ (76)

Through equation (31), the above relation can be rewritten as

$$\psi_\alpha L[Q] = h \chi_2 \chi_3 + \chi_4 \chi_{m+2} + \sum_{j=m-h+1}^{m-1} r^j \psi(\tau), r] = \left( m + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{h-1}} \right) \chi_2 \chi_3$$

$$= \left( m + \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{h-1}} \right) e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2} \psi(\tau), r]$$

$$L[Q] \psi_\alpha = \psi_\alpha(\tau) L^*[Q] = \left( \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{h-1}} \right) e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2} \psi(\tau), r],$$

where we can choose

$$\psi_\alpha = \left( \frac{1}{r} + \frac{1}{r^2} + \cdots + \frac{1}{r^{h-1}} \right)$$

$$L[Q] = e^{-\frac{1}{2} \sum_{\mu=1}^{m} \gamma_\mu^2} \psi(\tau), r].$$ (77)

Here the nonzero commutation relation between $\psi_\alpha$ and $L[Q]$ can be understood by treating $r$ with negative powers as operators, e.g. from equation (31) we have
\[
\left[ \frac{1}{r} , r^{m-1}[\psi(\tau), r] \right] = r^{m-2}[\psi(\tau), r] - r^{m-1}\psi(\tau) + r^m\psi(\tau) \frac{1}{r},
\]

and thus
\[
[\psi_0, L[Q]] = m r^{m-1}[\psi(\tau), r] = \psi_0(L[Q] - L^k[Q])
= \sum_{j=2}^{\infty} \left( r^{m-j}[\psi(\tau), r] - r^{m-1}\psi(\tau) \frac{1}{r^{j-2}} + r^m\psi(\tau) \frac{1}{r^{j-1}} \right).
\]

From equation (31), we know the the definition of the operator \( r \) has nonzero commutator with itself in different powers. That results in the commutator \([\psi(\tau), r]\) acted by \( r \) with higher powers can be obtained by the one with lower powers through adding a \( r \) operator in the left-side or right-side. For example, in the following expression,
\[
r^{m-j}[\psi(\tau), r]r = [\psi(\tau), r]r^{m-j+1},
\]
the operator \( r \) is defined according to the term \( r^{m-j}[\psi(\tau), r] \),
\[
r = \frac{r^{m-j}[\psi(\tau), r]}{r^{m-j}[\psi(\tau), r]} - \chi_2\chi_3 + \chi_2\chi_{j+2}.
\]

More generally, we have
\[
\chi_2\chi_{4+k} = 1 + r^{-k} - r^{-(k+1)}.
\]

This is guaranteed by the fact that the term \( r^{m-j}[\psi(\tau), r] \) must contain all the modes appear in \( r^{m-j+k}[\psi(\tau), r] \) with \( k \) the positive integer, which can be verified by equation (31). While the opposite case is not true. That also results in
\[
[\psi(\tau), r]r^{m-j} = [\psi(\tau), r]r^{m-j+k}r^k,
\]
as long as \((m-j-k) \geq 0\), which can also be expanded as
\[
(\psi(\tau)r)^m = (\psi(\tau)r^{m-j+k})r^k, \quad (r\psi(\tau))^m = (r\psi(\tau)r^{m-j+k})r^k.
\]

But it is false to write
\[
[\psi(\tau), r]r^{m-j} = [\psi(\tau), r]r^{m-j+k} \frac{1}{r^k},
\]
since unlike the \( r \) operators in positive powers, it has \([r^{m-j+k}, \frac{1}{r}] = 0\) which leads to
\[
[\psi(\tau), r]r^{m-j+k} \frac{1}{r^k} = [\psi(\tau), r] \left( r^{m-j+k} \frac{1}{r^k} \right).
\]

### 2.8. Randomness and the translational invariance

As we illustrated in the above subsections, the effective two-fold DOF represented by the equation of motion of the appropriately constructed functional \( L[Q] \), equivalents to the functional derivative of the Gor’kov equation \( H \) in mean field approximation (which satisfies the translational invariance before the functional derivatives with respect to effective field operator).

As can be seen from section 2.5, the two-fold DOF is represented in terms of the equation-of-motion of the functional \( L[Q] \) which can be expressed in terms of the effective field operator \( L[0] \) by \( L[Q] = e^{QH}L[0]e^{-QH} \), where the functional derivative with respect to the effective field operator breaks the translational invariance of the Gor’kov equation \( H \) by forcing a distinction between creation and anihilation operators. If without the functional derivative, it is obvious from equation (48) that the first term of Gor’kov equation \( H \) (realistic [23, 24] kinetic term) follows the scaling invariance (nonchiral) and the effective potential in second term depends only on the relative shift between each field operators (and thus is nonlocal). Note that here the realistic kinetic term means the usual linear kinetic term, which may give rise to stronger nonlocal effect, and results in, e.g. much more distinct symmetry sectors of the system Hamiltonian, as we know the (off-diagonal) eigenstate thermalization hypothesis (ETH) may be violated when treating the system as a whole where the level repulsion is absent between different sectors. Thus usually an unlinear kinetic term is needed for the relevant interactions, unless there are some sort of local effects, like the local chiral interactions, or the functional behaviors as introduced here, to enhance the local-type correlation.

Thus similar to the generalization in terms of the tensor models [25] that can realize the large-\( N \) physics without including the randomness, e.g. the gauge invariant four point function can generates an analogue to the maximally chaotic one [24, 25], an explicit breaking of translational invariance (in word, the chirality) can be evidenced by comparing the equation (84) and equation (86), which can be known (from below discussions).
that originates from the distinctions between $\alpha$ and $\beta$ sectors of current density. Also, such breaking of translational invariance cannot be restored by the ensemble (or disorder) average over the different sites along the 1d spacial direction, but we note that it should be accessible to restoring the translational invariance by constructing proper filter for each operator that accommodates with the gauge invariance, but this beyond the topic of this article and we refer readers to [5, 23] for more informations about this method.

3. Theoretical preparation: IR cutoff in imaginary time component of the propagator

For fermionic fields the anticommutation relation is strictly obeyed in terms of the time order product $\langle t|\cdot|t\rangle$, but its no always the case for a normal ordering product $\langle \cdot |\cdot \rangle$ when consider the Feynman propagator which need to satisfies the Lorentz invariance,

$$\mathcal{T}\psi_i(\tau)\psi^\dagger_j(\tau') = \psi_i(\tau)\psi^\dagger_j(\tau') + \langle 0|\mathcal{T}\psi_i(\tau)\psi^\dagger_j(\tau')|0\rangle,$$  \hspace{1cm} (88)

where $\langle 0|\cdot|0\rangle$ measures the vacuum expectation value, and this term becomes zero when the product inside it has already be in the normal ordering, in which case $\mathcal{T}|1\rangle = \mathcal{T}|\cdot\rangle = |\cdot\rangle$. Due to the restriction of Lorentz invariance, the Feynman propagator reads $\langle 0|\mathcal{T}\psi_i(\tau)\psi^\dagger_j(\tau')|0\rangle = \langle 0|\psi_i(\tau)\psi^\dagger_j(\tau')|0\rangle$ when $\tau > \tau'$, and $\langle 0|\mathcal{T}\psi_i(\tau)\psi^\dagger_j(\tau')|0\rangle = \langle 0|\psi_i(\tau')\psi^\dagger_j(\tau)|0\rangle$ when $\tau < \tau'$. Thus we can rewrite the above relation as

$$\mathcal{T}\psi_i(\tau)\psi^\dagger_j(\tau^-) = \mathcal{T}[\psi_i(\tau)\psi^\dagger_j(\tau^-)] = \psi_i(\tau)\psi^\dagger_j(\tau^-) + \langle 0|\psi_i(\tau)\psi^\dagger_j(\tau^-)|0\rangle,$$

$$\mathcal{T}\psi_i(\tau^-)\psi^\dagger_j(\tau) = \mathcal{T}[\psi_i(\tau^-)\psi^\dagger_j(\tau)] = \psi_i(\tau^-)\psi^\dagger_j(\tau) + \langle 0|\psi_i(\tau^-)\psi^\dagger_j(\tau)|0\rangle,$$  \hspace{1cm} (89)

where the field product are anticommuting within time ordering bracket, and

$$\psi_i(\tau)\psi^\dagger_j(\tau^-) = \psi^\dagger_j(\tau)\psi_i(\tau^-),$$

$$\psi_i(\tau^-)\psi^\dagger_j(\tau) = \psi^\dagger_j(\tau^-)\psi_i(\tau),$$  \hspace{1cm} (90)

which results in nonzero vacuum expectation values.

Next we introduce the most essential basement for this article, the two-particle self-consistency theory [26] which is of the nonperturbative regime. By assuming $\tau$ close enough to the $\tau'$, i.e. setting a IR cutoff in the time domain, we have the following results for the fermionic Green’s function in Nambu representation (we set the expectation value of number operators have here $(i,j)$ can be treated as pseudospin indices, $(n_i + n_j) = 1$, and the subscripts can be replaced by the up- and down-spin, i.e. $n_1 + n_{\bar{1}} = 1$): For the diagonal elements of Nambu Green’s function matrix, we have

$$G_n(n_1, \tau_1) = n_1 = -\langle c_1(\tau)c^{\dagger}_{n_1}(\tau)\rangle,$$

$$G_{n_1}(\tau_1, \tau_1) = n_{\bar{1}} = \langle c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau)\rangle = G_{n_1}(\tau, \tau^-) = -\langle c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau^-)\rangle = \langle c_{n_1}(\tau^-)c^{\dagger}_{\bar{1}}(\tau)\rangle,$$  \hspace{1cm} (91)

where we conclude that, within a product, the exchange of fermion indices $n \leftrightarrow \bar{n}$ equivalents to the exchange of imaginary time $\tau \leftrightarrow \tau'$ (which are infinitely close to each other).

Then, instead of the general anticommutation relation for $\tau' = \tau$, which is $\{c_n, c^{\dagger}_{n'}\} = 1$, we have (according to $n_1 - (n_{\bar{1}} = 1)$; and we omit the expectation notation hereafter)

$$-c_1(\tau)c^{\dagger}_{n_1}(\tau') + c_1(\tau)c^{\dagger}_{n_1}(\tau^-) = -c_1(\tau)c^{\dagger}_{n_1}(\tau') - c_1(\tau)c^{\dagger}_{n_1}(\tau),$$

$$-c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau') + c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau^-) = -c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau') - c_{n_1}(\tau)c^{\dagger}_{\bar{1}}(\tau),$$  \hspace{1cm} (92)

which is equivalent to

$$n_1 = -c_1(\tau)c^{\dagger}_{n_1}(\tau') = c_1(\tau)c^{\dagger}_{n_1}(\tau),$$

$$n_{\bar{1}} = 1 - n_1 = -c_{n_1}(\tau)c^{\dagger}_{n_1}(\tau) = c_{n_1}(\tau)c^{\dagger}_{n_1}(\tau^-).$$  \hspace{1cm} (93)

Next we define

$$\{c_1(\tau), c^{\dagger}_{n_1}(\tau')\} = \delta_{\tau, \tau'},$$

$$\{c_1(\tau'), c^{\dagger}_{n_1}(\tau)\} = \delta_{\tau', \tau},$$

$$\{c_{n_1}(\tau), c^{\dagger}_{\bar{1}}(\tau^-)\} = \delta_{\tau, \tau^-},$$

$$\{c_{n_1}(\tau^-), c^{\dagger}_{\bar{1}}(\tau)\} = \delta_{\tau^-, \tau}.$$  \hspace{1cm} (94)

It can be verified that all those delta functions are neither zero or one (by, e.g. inserting $c_i(\tau)c^{\dagger}_{n_1}(\tau') = -c^{\dagger}_{n_1}(\tau')c_i(\tau)$ with assumption $\tau' = \tau$ or $c_{n_1}(\tau)c^{\dagger}_{n_1}(\tau') = 1 - c^{\dagger}_{n_1}(\tau')c_{n_1}(\tau)$ with assumption $\tau^+ = \tau$ into equation (92)). We also have
\[ c_{i}^{\dagger}(\tau) c_{i}(\tau) = \delta_{\tau,\tau'} + n_{i}, \]
\[ c_{i}^{\dagger}(\tau) c_{i}(\tau') = \delta_{\tau,\tau'} + n_{i}, \]
\[ c_{i}^{\dagger}(\tau) c_{i}(\tau) = \delta_{\tau,\tau'} - n_{i}, \]
\[ c_{i}^{\dagger}(\tau) c_{i}(\tau') = \delta_{\tau,\tau'} - n_{i}, \]

where the conservation law enforces
\[ \delta_{\tau,\tau'} + \delta_{\tau',\tau} = -2, \]
\[ \delta_{\tau,\tau'} + \delta_{\tau',\tau} = 2. \]

Next we expressing the above relations in terms of the Majorana fermions. We introduce the Majorana operators \( \gamma_{2j} = \sigma_{1}^{\dagger} \) and \( \gamma_{2j-1} = \sigma_{1}^{\dagger} \). But in the following section, we will using another definition of Majorana operators \( \chi_{k} \) which can better describe the phase feature of the system. We note that, two sectors \( \alpha \) and \( \beta \) are defined in this system. In \( \alpha \)-sector, there is no imaginary time difference between the two Majorana operators, and due to the gauge-invariance restriction, their topological phase-dependent behaviors follows
\[ \chi_{i} \chi_{j} = \chi_{i} \chi_{i+n} = \gamma_{2(i-1)} \gamma_{2(i-1+n)} \]
for \( i \geq 2 \) and \( n \geq 1 \) is an positive integer, that means the system would always of the bosonic Majorana chain (and in the nontrivial phase) instead of the fermion one until \( i = 1 \) (thus in the trivial phase). While in the \( \beta \)-sector where finite time difference exist, we have the definition just follows the above one \( \chi_{k} \chi_{l}(\tau') = \gamma_{2j-1} \gamma_{2j} \) (preserves the fermionic parity symmetry). But indeed, the unusual definition for the Majorana product in \( \alpha \)-sector is only due to the gauge conservation-enforced restriction of the system, i.e. their topological phase exhibit unusual features due to the intrinsic DOF of the system, and thus the resulting topological phase also affected by the inner DOFs correlations (like the DOFs in \( \beta \)-sector). Thus for \( \beta \)-sector which can be considered as fermionic Majorana chain, we obtain a conclusion which will be used below,
\[ \chi_{k}(\tau') = \chi_{l}(\tau - 1) \chi_{l}(\tau + 1) \]
\[ \chi_{k}(\tau') = -i[c_{i}^{\dagger}(\tau) c_{i}(\tau') - c_{i}(\tau) c_{i}^{\dagger}(\tau')] = -i[\delta_{\tau,\tau'} + 1] \]
\[ \chi_{k}(\tau') = -i[c_{i}(\tau') c_{i}^{\dagger}(\tau) - c_{i}^{\dagger}(\tau') c_{i}(\tau)] = -i[-1 - \delta_{\tau,\tau'}], \]

which are consistent with equations (94)–(96). It can also be verified that
\[ \chi_{k}(\tau') = -\chi_{l}(\tau') \chi_{k}(\tau'), \]

while the effect for exchange of imaginary time cannot be easily identified
\[ \chi_{k}(\tau') \chi_{l}(\tau) = -i(\delta_{\tau,\tau'} - 1), \]
\[ \chi_{k}(\tau') \chi_{l}(\tau) = -i(\delta_{\tau,\tau'} - 1), \]

but this relation is not needed in the following.

4. System

4.1. Bosonic supercharge

To illustrating the restriction effect from the gauge field, we firstly define the bosonic supercharge and temporarily incorporating the real wave functions (which describe the eigenstates created by Majorana fermions) into each Majorana fermion operator,
\[ Q_{\theta} = i \sum_{i < k < l} \chi_{k}(\alpha, \eta_{i}) \chi_{l}(\beta), \]

where only the Majorana fermion operators of the same sector satisfy the anticommutation relation, i.e. \( \{ \chi_{k}(\alpha), \chi_{l}(\beta) \} = \delta_{\theta} \) \( \chi_{k}(\beta), \chi_{l}(\beta) \) = \( \delta_{k} \). And the Majorana fermions of different sectors satisfy \( i < k, j < l \). While it always commutes with each other between operators in \( \alpha \)-sector and \( \beta \)-sector. Note that the following discussions will still be valid if we set \( \tau_{1} = \tau_{2} \) but \( \tau \neq \tau' \).

In the absence of gauge invariance (both charges are of the same sector), this supercharge commutes with the Hamiltonian
\[ H = Q_b^2 \]
\[ = (i \sum_{i \neq k} \sum_{\alpha} \chi_i(\alpha, \tau) \chi_k(\alpha', \tau)) (i \sum_{j \neq l} \sum_{\alpha} \chi_j(\alpha, \tau) \chi_l(\alpha', \tau)), \]  
\[ (103) \]
and the system becomes of the usual four-DOF regime. Note that the cases \( i = k \) (or \( j = l \)) are precluded by the nonzero correlations between \( i \) and \( j \) (or \( k \) and \( l \)) which are of the same sector and thus required to be antisymmetric with respect to each other. This also requires two supercharges satisfying such correlating configuration should not be the replica part like \( Q_b^2 Q_b \), instead, their correlation should be the same type with that between two \( U(1) \) gauge fields:
\[ [A_{\alpha\beta}, A_{\alpha'\beta'}] = \sum_{\gamma < \nu} [\delta_{\alpha\beta} \delta_{\beta'\gamma} - \delta_{\beta\gamma} \delta_{\gamma\nu}] = \delta_{\alpha\beta} \delta_{\alpha'\gamma} - \delta_{\alpha\gamma} \delta_{\alpha'\nu}] A_{\beta\gamma}. \]  
\[ (104) \]
Note that we set \( \alpha(\alpha') < \beta(\beta') \) to make \( A_{\alpha\beta} \) and \( A_{\alpha'\beta'} \) of the same sign. The antisymmetry property of gauge field also signals the broken time-reversal symmetry. The summation over all gauge fields with different \( (\alpha, \beta) \) indices is invariant.

If we restrict the group of imaginary time evolution \( \mathcal{M}_{\tau \rightarrow \tau'} \) has a fixed number of elements, this will impose some restrictions to the selection of the indices \( i, j, k, l \). For example, by looking at the part with degree-of-freedom \( \beta \), the evolution from \( \tau \) to \( \tau' \), has \( M^2 \) elements given by the random combinations of \( \chi_\beta(\beta, \tau) \) and \( \chi_\beta(\beta, \tau') \), and we assume this is the maximal size of the group \( \mathcal{M}_{\tau \rightarrow \tau'} \), which consists \( M^2 \) elements \( m_{\tau \rightarrow \tau'} \), and cannot be enlarged in the \( \alpha \) subspace. Every single mapping in \( \beta \) subspace \( m_{\tau \rightarrow \tau'} \) should mostly has \( M \) “copies” in \( \alpha \) subspace, which follow the same mapping. Thus to make sure the group \( \mathcal{M}_{\tau \rightarrow \tau'} \) still has \( M^2 \) irreducible elements, there can only be \( M \) elements in \( \alpha \) subspace, which can be represented by \( m_{\tau \rightarrow \tau'} \), or equivalently, \( m_{\tau \rightarrow \tau'} \). According to the above definition of current density with gauge-fixing restrictions, the time evolution group \( \mathcal{M}_{\tau \rightarrow \tau'} \) with fixed number of elements is a direct result of independence between the operator evolution speed \( \partial_\tau \chi(\tau) = \chi(\tau) \) and the variance after each time step \( \chi(\tau) \).

4.2. Supersymmetries of gauge invariant system

For a superproduct satisfying the gauge invariance introduced by the \( U(1) \) gauge field \( A_{\alpha\beta} \) (see equation (104)), there should be only three mutually independent Majorana indices, which is also equivalent to the case with four mutually independent Majorana indices but only one sector owns the full evolution group \( \tau \rightarrow \tau' \) whose size is constrained by the gauge invariance. Now the supercharge product reads
\[ Q_b^2 = (i \sum_{i \neq k} \sum_{\alpha} \chi_i(\alpha, \tau) \chi_k(\beta, \tau)) (i \sum_{j \neq l} \sum_{\alpha} \chi_j(\alpha, \tau) \chi_l(\beta, \tau)) \]
\[ = - \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \sum_{\{\tilde{I}\}} \frac{1}{|\{\tilde{I}\}|} \sum_{I \in \{\tilde{I}\}} \sum_{\{\tilde{J}\}} \frac{1}{|\{\tilde{J}\}|} \sum_{J \in \{\tilde{J}\}} \chi_{\tilde{I}}(\alpha, \tau) \chi_{\tilde{J}}(\beta, \tau) \chi_{I}(\beta, \tau) \chi_{J}(\beta', \tau) \]
\[ = \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \Phi(\beta, \beta', \alpha, \alpha') \]
\[ = \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \frac{1}{2\pi} \phi(\beta, \beta', \alpha, \alpha') \chi_\beta(\beta, \tau) \chi_\alpha(\alpha, \tau) \]
\[ = i \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \phi(\beta, \beta', \alpha, \alpha') \chi_\beta(\beta, \tau) \chi_\alpha(\alpha, \tau). \]  
\[ (105) \]
where \( \{\tilde{I}\} \) denotes the selected chiral Majorana fermion fields which form the multiple boson field. We define the summations
\[ \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \sum_{\{\tilde{I}\}} \frac{1}{|\{\tilde{I}\}|} \sum_{I \in \{\tilde{I}\}} \sum_{\{\tilde{J}\}} \frac{1}{|\{\tilde{J}\}|} \sum_{J \in \{\tilde{J}\}} \phi(\beta, \beta', \alpha, \alpha') \]
\[ = \sum_{\beta \neq \beta'} \sum_{\alpha \neq \alpha'} \sum_{\{\tilde{I}\}} \frac{1}{|\{\tilde{I}\}|} \sum_{I \in \{\tilde{I}\}} \sum_{\{\tilde{J}\}} \frac{1}{|\{\tilde{J}\}|} \sum_{J \in \{\tilde{J}\}} \phi(\beta, \beta', \alpha, \alpha'). \]  
\[ (106) \]
For a chiral current, the Majorana fermions in \( \alpha \)-sector has only the reduced degrees-of-freedom (thus the summation over \( \alpha \)) can be incorporated into the summation over \( \tilde{I} \) and the boundary of summation over \( \alpha \), \( \tilde{I} \)-states is also solely determined by \( \{\tilde{I}\} \). The size of group \( |\{\tilde{I}\}| \) is \( O(N^2 M) \) while that of \( |\{\tilde{I}\}| \) is \( O(M) \), which means, although the above \( Q_b^2 \) term is defined initially in a similar way with the Wishart SYK model, but under the restricts of gauge invariance, it finally turns to be the one similar to the single supercharge defined in \( \mathcal{N} = 1 \) supersymmetry SYK model [27] but with a \( N \times N \times \sqrt{N} \) antisymmetry tensor instead of the \( N \times N \times N \) one.
Also, as can be seen from above expression (cf equation (7) as well as equations (98)–(100)), the Hermiticity (permutations under antisymmetry exchanges) only exist between \(\alpha\) and \(\beta\) sectors (i.e. \(\bar{f}\) and \(\bar{f}\)).

Here the multiple boson field is defined in terms of the hybridizing Majorana fermions through the 1D quadratic coupling in bilinear form

\[
\Phi(\beta) = \frac{\beta}{2\pi} = i\sum_{\alpha}^{\beta-1} \rho_{\alpha} = i\sum_{\{\beta\}} \varphi_{\alpha}(\alpha, \tau) \varphi_{\tau}(\alpha, \tau) \chi_{\alpha} \chi_{\tau}. \tag{107}
\]

Due to the Hermiticity, this boson field plays the role of antisymmetry tensor within the expression of supercharge product, due to the antisymmetry result under exchange \(\varphi_{\alpha}(\alpha, \tau) \varphi_{\tau}(\alpha, \tau) = -\varphi_{\tau}(\alpha, \tau) \varphi_{\alpha}(\alpha, \tau)\). That makes the boson field a perfect probability amplitude factor within the product and it sums to be zero by its own (in absent of the selection effect of the latter terms related to \(\beta\), \(\Phi(\beta) = 0\)).

According to the current density defined above (section 2.2 and appendix A), for a certain sector (\(\beta\) here), this boson field, which is the summation over current densities, can be transformed into a single current density located at the upper boundary of previous summations above a half filling which is a conserved quantity for a certain sector. Thus the above antisymmetry-induced cancellation means the current density cannot exists in the absence of gauge invariance. If we recollect the \(\chi_{\beta}(\alpha, \tau)\) and \(\chi_{\alpha}(\beta, \tau)\) to the same sector \(\gamma\), i.e. \\
\(\chi_{\gamma}(\alpha, \tau) \chi_{\gamma}(\beta, \tau) \delta_{\gamma} = \chi_{\tau}(\gamma, \tau) \chi_{\gamma}(\gamma, \tau') \delta_{\gamma} (\tau = \tau')\). While the operators \(\chi_{\alpha}(\alpha, \tau)\) and \(\chi_{\beta}(\beta, \tau)\) will also be classified into the same sector automatically (denoted as \(\eta\)). This is because classification of sectors simply depends on the different kinds of interacting relations (symmetry/antisymmetry) within each sector, and now we have \(\chi_{\eta}(\alpha, \tau) \chi_{\eta}(\beta, \tau') \rightarrow \chi_{\eta}(\eta, \tau) \chi_{\eta}(\eta, \tau')\). It can be verified that this will be still the three-DOF configuration like the previous one. This can be verified by checking the commutation/anticommutation relations between arbitrarily two of those four Majorana operators (see appendix D), where we can find three antisymmetry parts and three symmetry parts, in contrast to the four-DOF one where there will be four antisymmetry parts and two symmetry parts.

After the selection effect due to the relations equations (98)–(100), there will be only the permutations meet \((i < k < j < l)\) or \((j < l < i < k)\) survive. It can be easily understood why other permutations all compensate with each other. Note that here we temporarily omit the cases with \(i = j\) and \(k = l\), which will results in the product in form of

\[
\chi_{i}(\tau) \chi_{j}(\tau) \chi_{k}(\tau) \chi_{l}(\tau^{\pm}) = \delta_{\tau, \tau^{\pm}} + 1 - 2n_{t}. \tag{108}
\]

For all permutations meets \(i = j\) but \(k \neq l\) or \(k = l\) but \(i \neq j\), the cancellation happen as

\[
\chi_{i = a}(\tau) \chi_{j = b}(\tau) \chi_{j = d}(\tau^{\prime}) + \chi_{i = a}(\tau) \chi_{k = d}(\tau) \chi_{j = a}(\tau^{\prime}) = 0,
\]

\[
\chi_{i = a}(\tau) \chi_{k = b}(\tau) \chi_{j = c}(\tau) \chi_{i = d}(\tau^{\prime}) + \chi_{i = c}(\tau) \chi_{k = a}(\tau) \chi_{j = b}(\tau^{\prime}) = 0, \tag{109}
\]

respectively. For all permutations meets \(i > j\) and \(l > i\) the cancellation happen as

\[
\chi_{i = a}(\tau) \chi_{k = b}(\tau) \chi_{j = c}(\tau) \chi_{l = d}(\tau^{\prime}) + \chi_{i = a}(\tau) \chi_{k = d}(\tau) \chi_{j = c}(\tau) \chi_{l = b}(\tau^{\prime}) = 0, \tag{110}
\]

respectively. For all permutations meets \(k > j\) and \(l > i\) the cancellation happen as

\[
\chi_{i = a}(\tau) \chi_{k = b}(\tau) \chi_{j = c}(\tau) \chi_{l = d}(\tau^{\prime}) + \chi_{i = a}(\tau) \chi_{k = d}(\tau) \chi_{j = c}(\tau) \chi_{l = b}(\tau^{\prime}) = 0. \tag{111}
\]

respectively. Note that due to the absence of anticommutation relation between the two sectors, we have

\[
\chi_{i = a}(\tau) \chi_{k = b}(\tau^{\prime}) = \chi_{i = a}(\tau) \chi_{k = a}(\tau^{\prime}).
\]

As only the permutations of the forms \((i < k < l < j)\) and \((j < l < i < k)\) survive, and the corresponding \((k,l)\)-states would also cancel each other, we will consider only the \((i < k < j < l)\) case here.

Then the summation over couplings has the following relation. In above gauge-invariant charge product, the coupling is an antisymmetry tensor and the summation over couplings has the following relation

\[
\Phi(\beta) = \frac{\beta}{2\pi} = i\sum_{\alpha}^{\beta-1} \rho_{\alpha} = i\sum_{\{\beta\}} \varphi_{\alpha}(\alpha, \tau) \varphi_{\tau}(\alpha, \tau) \chi_{\alpha} \chi_{\tau}. \tag{107}
\]
which reduces to half of a single supercharge owns the full time evolution. As can be seen, at the special points where $\bar{\tau} = \bar{k}$, these two operators forms a noninteracting fermion,

$$\lim_{\tau \to \tau'} \chi_j(\gamma, \tau) \chi_k(\gamma, \tau') \delta_{jk} = \frac{1}{2}$$

Here the instantaneous $\tau \to \tau'$ is guaranteed by fail of low-energy approximation here due to the absence of strong interacting boson self-energy and disorder. That is also why $\bar{\tau} = \bar{k}$ in the $\eta$-sector where the bosonic self-energy dominates over the Matsubara frequency.
4.3. Numerical results

It can be calculated that the number of all survival \((i, k, l)\) combinations is \(\sum_{i=2}^{N-i-1} (i - 1) [\sum_{j=0}^{N-i-1} (N - i - j)] = \frac{1}{24} N (2 - N - 2N^2 + N^3)\) and the number of distinguishing survival \((k, l)\)-states is \(\frac{1}{24} (N - 2)(N - 1) \approx N^2\). If we further classify those survival \((k, l)\)-states by the value \(|k - l|\), we will see that the number of each \((k, l)\)-state (i.e. the number of corresponding bosonic Majorana \((i, j)\)-states) is always \(|k - l|(k - 1)\), this can be seen as a weight for each distinguishing \((k, l)\)-state.

The states number for each distinguishing \((k, l)\) state are shown in figure 1(a), and the many-body level spacing distribution is shown in figure 1(c), the averaged spacing ratio is \(\langle r \rangle \sim 0.66\), which is of the GSE. After we classify them into groups each with distinguishing values of \(|k - l|\) (of the order of \(N\)), the corresponding number density distribution is shown in figure 1(b). For each \((k, l)\)-state, the distributions of the value of \(|k - l|\) are shown in figure 2, where we can see the many-body level spacing distribution follows the GUE with \(\langle r \rangle \sim 0.59\) (figure 2(b)) when we ignore the weight \(|k - l|(k - 1)\) on each state, and follows GOE with \(\langle r \rangle \sim 0.52\) (figure 2(d)) when the weighted distributions are considered.

As shown in figure 1(d), through the statistical properties, for each \((k, l)\)-state of the class \(|k - l| = 1\), their corresponding \((i, j)\) states should exhibit the same topological feature. Using the above Majorana operator of \(\alpha\)-sector in equation (97), we can explain the classification criterion of \((i, j)\)-states for each \((k, l)\)-state of the same difference \(|k - l|\). In \(\alpha\)-sector, we have

\[
\begin{align*}
\chi_{1} & = \gamma_{1(i+1-n)} \chi_{i+1} + \gamma_{1(i+1-n)} \chi_{i+2}, \\
\chi_{i} & = \gamma_{i(i-1-n)} \chi_{i+1} + \gamma_{i(i-1-n)} \chi_{i+2},
\end{align*}
\]

(114)

where integers satisfy \(i \geq 2, j \geq 3\) and \(n \geq 1\), e.g. for \(n = |k - l| = 1\), we have

\[
\begin{align*}
\chi_{2} & = \sigma_{1}^{z} \sigma_{2}^{z} = \sigma_{1}^{z} \sigma_{2}^{z} = \gamma_{2} \gamma_{3}, \\
\chi_{4} & = \sigma_{2}^{z} \sigma_{4}^{z} = \sigma_{2}^{z} \sigma_{4}^{z} = \gamma_{4} \gamma_{5}, \\
\chi_{4} & = \sigma_{2}^{z} \sigma_{4}^{z} = \sigma_{2}^{z} \sigma_{4}^{z} = \gamma_{4} \gamma_{5},
\end{align*}
\]

(115)

and for \(n = |k - l| = 2\), we have

\[
\begin{align*}
\chi_{2} & = \sigma_{1}^{z} \sigma_{2}^{z} = \sigma_{1}^{z} \sigma_{2}^{z} = \gamma_{2} \gamma_{3}, \\
\chi_{3} & = \sigma_{2}^{z} \sigma_{3}^{z} = \sigma_{2}^{z} \sigma_{3}^{z} = \gamma_{4} \gamma_{5}, \\
\chi_{4} & = \sigma_{2}^{z} \sigma_{4}^{z} = \sigma_{2}^{z} \sigma_{4}^{z} = \gamma_{6} \gamma_{9}.
\end{align*}
\]

(116)

For \(j > i \geq 2\), all \((i, j)\)-states with the same value of \(|i - j|\) can be classified into the same group. While the states with \(i = 1\) corresponds to the trivial phase with conserved fermionic charge, and without the degenerated group
state. For example, as shown in figure 1(d), for \(|k - l| = 1\), we have
\[
\chi_1 \chi_2 = \chi_1 \chi_3 + \chi_2 \chi_3,
\]
(117)
as topologically \(\chi_2 \chi_3 = \chi_3 \chi_4\), we can further have
\[
\chi_1 \chi_2 = \chi_1 \chi_4 + \chi_2 \chi_4 + \chi_3 \chi_4.
\]
(118)
We also notice that, for a certain class, the rule for \((i,j)\) parts exhibits an ergodic feature: no matter how many the nodes is, the \((i,j)\) parts always making the maximal value of \(\sum |i - j|\) under the restriction that each node can only be go throughed once. While for \(|k - l| > 1\), there will be more \((i,j)\) parts with \(i = 1\). A explanation for the replacement between \(\chi\) and \(\gamma\) are presented in section 2.3.

5. Current-density flavor: the \(M\)-dependent statistics

5.1. Three-DOF configuration in BDI symmetry class: \(O(M) = O(N)\)

Firstly we discuss the case in symmetry class BDI, where the many-body localization-induced symmetry protected topological states exist as we illustrated in section 2.2. This is a symmetry class different to the AIII or CII classes, the bulk states Base on current density flavors defined above (equation (178)), as long as the Majorana chain is dominated by many-body localization, the chiral current densities can be classified by \(M = N - 2\) categories. Each one of these \(M\) categories of current density with \(\beta = 1, 2, \cdots, M\) can be expressed in terms of
\[
\Phi(\beta) = \sum_{j \geq 2} \chi_{i-1} \chi_{j},
\]
(119)
which is the relation equation (114).

For charge product \(\chi_i \chi_j \chi_{kl}\), since initially \((i,j)\) and \((k,l)\) are of different sectors, the delta-function \(\delta_{[i], [k]}\) in second line should means the upper summing boundary of \(\bar{I}\) is the same with that of \(l\) (or the size of group \(\{I\}\) is the same with that of \(\{l\}\)). This is how such gauge-invariant-type correlation happen and relating two supercharges to generate a 3-DOF configuration. As the groups \(\{\bar{I}\}\) and \(\{I\}\) are of different sectors \((\alpha, \beta)\), and have not additional extension in size (unlike the group of \(\{k, l\}\)), then there differences (relative degrees-of-freedom) can be regarded as degenerated into a single one which is of that between two the sectors \(|(\alpha, \beta) = ([\bar{I}], [I]) \otimes \{\bar{K}, \bar{I}\} = O(M)O(M) = O(M^2)\)). Once \(\delta_{[\bar{I}], [I]} = 1\), the gauge invariance reduce the size of subsystem consisting of \([i]\) and \([l]\) from the \(O(M^2)\) to \(O(M)\). In this case, there will be a three-DOF configuration which corresponds to completely localized Majorana modes in the bulk part of the 1D chain as we discussed in section 2.3, and the \((N - 1)(N - 2)/2\) elements in the group of \(\{\bar{K}, \bar{I}\}\) are occupied by the \(M \approx (N - 2)\) flavors of current density, unevenly. This nonuniform occupation (with a gradient) results from the gauge invariant restriction. In terms of the probability distribution (see appendix B), this can be explained more clearly in the following way. As now each \((k, l)\)-state has the same probability to be occupied by each one of the \(M\) current flavor, the probabilities for each flavor is
\[
P_{\beta=1,2, \cdots, M} = \frac{1}{M} \sum_{i=1}^{M} \varphi_{[i]} \varphi_{[\bar{I}]} = 1/M,
\]
(120)
which is a constant. While by performing the Schmidt decomposition to \((\bar{K}, \bar{I})\)-states, the corresponding probability for each ‘Schmidt vector’ is variant
\[
P_{\alpha=2,3, \cdots, N-1} = \frac{1}{N - 2} \sum_{n=1}^{N-2} \frac{1}{n}, \sum_{n=2}^{N-2} \frac{1}{n}, \cdots, \frac{1}{N - 2},
\]
(121)
In this case, each part of \((\bar{K}, \bar{I})\) has different probability \((\varphi_{[\bar{K}]} \varphi_{[\bar{I}]})\) which depends on the value of \(|\bar{K} - \bar{I}|\). This probability is not commutate with the number of \((\bar{K}, \bar{I})\)-states, since \(P_{\beta=1,2, \cdots, M}\) is a constant and the probabilities have to sum up to one.

It can be verified that only the combinations satisfy \(i < k \leq j < l\) survival (the another possible case \(j < l \leq i < k\) will not be considered here). For the 3-DOF configuration, as each flavor of current density owns equal weight (represented by \((i,j)\) part here) with respect to each part of Majorana index (represented by \((k,l)\) here), it requires the probability reads
\[
P_{k<l} = \frac{1}{M + \partial_{\Delta_{kl}}} \Delta_{kl} - \Delta_{kl},
\]
(122)
where $1_M = 1/M$ denotes the sum over all probabilities of current densities (or $i,j$)-part of the same sector, and that ensures the equal weight for each sector of current density. $\Delta_{kl} = l - k$ denotes the distance between two Majorana indices (in terms of the unit step in space of Majorana DOF).

From the expression of above probability we can obtain

$$\partial \Delta_{kl} \Delta_{kl} = \lim_{\Delta_{kl} \to -\infty} 1 = \frac{1_M P_{k,l}}{M + \partial \Delta_{kl} \Delta_{kl} - \Delta_{kl}},$$

(123)

where $\partial \Delta_{kl} \Delta_{kl} = -1$ is the only $\Delta_{kl}$-independent quantity here, and, as we discuss in section 7.3, this can be explained by introducing another framework consist of new unit which is consistent with the relation

$$\partial \Delta_{kl} \Delta_{kl} = \Delta_{kl} - \Delta_{kl},$$

(124)

which is consistent with the relation (obtained from $\partial \Delta_{kl} (\partial \Delta_{kl} \Delta_{kl}) = \partial \Delta_{kl} (\Delta_{kl} - \Delta_{kl}) = 0$)

$$\frac{\partial \Delta_{kl} (\Delta_{kl} - \Delta_{kl})}{\Delta_{kl} - \Delta_{kl}} = \partial \Delta_{kl} \ln \frac{1}{\partial \Delta_{kl} (\Delta_{kl} - \Delta_{kl})} = -1$$

(125)

Thus the dependence of $M$ on the Majorana-index-difference $\Delta_{kl}$ can be described by

$$\partial \Delta_{kl} M = -1 - (\Delta_{kl} - \Delta_{kl}) = -1 - \frac{1 - \Delta_{kl} - \Delta_{kl}}{1 - (\Delta_{kl} - \Delta_{kl})},$$

(126)

where the condition for DOFs $O(N) \sim O(M)$ enforce $\lim_{\Delta_{kl} \to -\infty} M = \max \Delta_{kl} = -1 - \frac{\Delta_{kl} - \Delta_{kl}}{1 - (\Delta_{kl} - \Delta_{kl})}$ here.

5.2. Numerical simulations

The system exhibits GSE statistics when it meets the condition $O(M) \sim O(N)$. Since now the size of $\{k, l\}$ group is of order $O(N^2)$ and much larger than that of current densities $O(M)$, the many body localization inside the Majorana chain will naturally leads to thermalization in the statistical behaviors of the chiral current densities which are randomly occupied by each $(k,l)$-state. As shown in figure 3(a), the current density flavor distribution versus $(k,l)$-states plot exhibits a nearly constant slope, and the many-body level statistics (the probability $P_{k,l}$ exhibits GSE distribution. The average value of spacing ratio is $(r) \sim 0.701$.

When $O(M) \sim O(N^2)$, the size of group $\{K, I\}$ is then nearly the same with that of current density, and this results in a constant probability distribution in both the current density group and $\{K, I\}$ group:

$$P_{k,l} = P_{l,k} = 1/M = 1/N^2.$$ This will results in a four-DOF configuration with $\binom{N^4}{4} \sim N^4$ mutually independent terms whose distribution follows the SYK$_4$ model. The corresponding numerical simulations are shown in figure 4, where we set $O(M) \sim O(N^2)$, the corresponding many-body level distribution of current densities in this configuration still follows the Wigner–Dyson statistic and thus the DOF of chiral current density is still thermalized (quantum chaotically), and we found $(r) \sim 0.603$.

When the magnitude of $M$ exceeds $N^2$, which is, when the flavor number of current density beyonds the critical chaotic condition for the four-DOF configuration, the spectrum tends to Poisson distribution (completely dominated by Poisson distribution when $O(M) \sim O(N^3)$) which signals the Anderson-localized insulating state for current densities in contrast to the disordered metal state. The simulation for cases with $O(M) > O(N^2)$ can be realized by adding a randomness factor to the previous GUE case. For each index $k = 2, \ldots, N - 1$, the randomness factors could be $\delta_1 = 1, \ldots, N - \Delta_{kl} / N - \Delta_{kl} = 1, \ldots, N - k$, $\delta_2 = 1, \ldots, N - k$, $\delta_3 = 1, \ldots, N - k$. Only the $\delta_1$ and $\delta_3$ factors correspond to the Poisson case. While $\delta_1$ and $\delta_3$ factors correspond to a intermediate state between GSE and Poisson statistics. As should in figure 5, the $\delta_1$ and $\delta_3$ factors exhibit the same effect and both realizing the $O(M) = O(N^3)$ simulation. This is also evidenced by the averaged level spacing ratios obtained for these for randomness factors: $(r)_{b_1} = 0.4705$, $(r)_{b_2} = 0.4535$, $(r)_{b_3} = 0.3954$, $(r)_{b_4} = 0.3853$. For these two case, we also perform the Many-body level statistics in term of $P(\ln r)$ in figure 6, where the average level spacing are close to the results of figure 5, $\langle r \rangle_{b_1} = 0.4715$, $(r)_{b_2} = 0.4532$, $(r)_{b_3} = 0.3975$, $(r)_{b_4} = 0.387$.

5.3. Automatically realized distribution beyond Poisson distribution in AIII or CII symmetry classes

The Poisson distribution for current density spectrum here corresponds to the Anderson-localized 1D bulk in the AII or CII symmetry classes. In the mean time, as the current density flavor number $M$ increase to of order $O(N^3)$, the many-body localization inside the Majorana chains is overwhelmed by the thermalization effect, and the characters of Majorana chain are no longer simply determined by the boundary zero modes. As we stated above, in the AIII or CII symmetry classes, the many-body localization inside the Majorana chains is overwhelmed by the thermalization effect, and the characters of Majorana chain are no longer simply determined by the boundary zero modes, but all the effective modes, which can be viewed as edges of several parallel chains (which can describe the, e.g. Anderson-localized 1D current densities where the interactions only
exist at the 0D boundaries. This also leads to the absence of translational invariance and degenerate ground state in the 1D Majorana chain. Then the equation (119) is no more valid, and the Majorana modes in bulk are no more the frustration-free localized eigenstates of $Q_b^2$, i.e. each one of them it becomes depending on their summation upper boundary. For example, now the fermion bilinear term should be $\chi_1 \chi_2 = \chi_2 \chi_3$ (in $\alpha$-sector), which indicates a phase transition from the topological Haldane-type nontrivial one to the trivial one.

The AIII or CII symmetry classes correspond to the noninteracting Anderson-localized bulk for the 1D system with a $\mathbb{Z}$ invariance and, respectively, processing the $\mathbb{Z}_4$ and $\mathbb{Z}_2$ classifications at the thermalized edges [9]. Now we still treating each Majorana chain as the edges of parallel chains at a certain number. As we explained above, the Anderson localized current densities indeed equivalent to the thermalized Majorana chain which is far away from the critical point of quantum chaos. And now the translational invariance along the Majorana chain is absent. In this case, the category of current density can no more be distinguished simply by the number of edge zero modes, but the number of generated edge states. Thus the relation equation (178, 179) should be replaced by a more general one, $\sigma_i^\alpha \sigma_j^\beta = \sigma_i^\beta \sigma_j^\alpha (i, j = 1, 2, 3, \ldots)$. For example, now the fermion bilinear term should becomes $\chi_1 \chi_2 = \chi_2 \chi_3$ (in $\alpha$-sector), which indicates a phase transition from the topological Haldane-type nontrivial one to the trivial one. In this case, the corresponding probability for each current density flavor is determined by the value of $\Delta_{1\alpha}(|k - 1|)$.

In the former case, as $O(M)$ is relatively small (for a whole system), we have $[\partial_\alpha \Phi(\alpha)/2\pi, \partial_\beta \Phi(\beta)/2\pi] = \delta_{\alpha,\beta} = 0$, and thus each current density in a certain sector is independet of their summation upper boundary (the next sector), which results in the relations $[\rho_0, \beta] = 0$, $[\sum_{\alpha=1}^{\beta-1} \rho_0, \beta] = 0$ (or $[\rho_0, \beta'] = 0$, $[Q_b^2, \beta'] = 0$). While for the latter case, the fractionally-induced distance between two adjacent current densities $|\alpha - \beta| \sim 1/M$ is vanishingly small, we have $[\partial_\alpha \Phi(\alpha)/2\pi, \partial_\beta \Phi(\beta)/2\pi] = -\frac{1}{2\pi} \delta_0 \delta(\alpha - \beta)$. In this case, the states

![Figure 3. Current density spectrum (a,c) and GUE distribution of the level spacing ratios (b,d) for four-DOF configuration. We set $N = 20$ and $N = 200$ in (a,b) and (c,d), respectively. The many-body level statistics in term of $P(\ln r)$ which corresponds to the panel (d) is shown in (e).](image-url)
corresponding to each flavor of current density are no more a frustration-free localized eigenstates of the system $Q_b$.

As shown in figure 7, in this case the level statistic follows the behaviors beyond Poisson distribution, and the averaged level spacing ratio is $\langle r \rangle = 0.3623$, which is below the typical value of Poisson one (0.38).
5.4. Statistic tool

Here we illustrate the statistic tool use in the above simulations. For current density spectrum in terms of the \((k, l)\)-states, by arranging the current density levels (eigenenergies obtained from the charge product term dominated by gauge field) in ascending order (repeated values due to the degeneracy are removed), we use the definition of the level spacing ratio

\[ r_i = \frac{E_{i+1} - E_i}{\text{min}(E_{i+1} - E_i, E_{i+2} - E_i, \ldots)} \]

here, which will be half of the results using the another definition

\[ r_i = \frac{\text{min}(E_{i+1} - E_i, E_{i+2} - E_i, \ldots)}{\text{max}(E_{i+1} - E_i, E_{i+2} - E_i, \ldots)} \]

The corresponding probabilities are calculated in terms of \( P(r) \) or \( P(\ln r) \). The results are compared with the Wigner formulas

\[ P_w(r) = \frac{1}{Z_3} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{\beta + 3/2}}, \]

\[ P_w(\ln r) = P(r)r, \]  

(127)

with \( \beta = 1, 2, 4 \) (\( Z_3 = \frac{8}{27}, \frac{4}{81} \sqrt{3}, \frac{4}{729} \sqrt{3} \)) for GOE, GUE, and GSE, respectively. This method works well for Poisson case and the one beyond Poisson distribution for both the small-\( r (P_w \sim r^{-\beta}) \) and large-\( r (P_w \sim r^{-(2+\beta)}) \) regions. While for GOE and GUE cases, it is valid only for the small-\( r (P_w \sim r^{-\beta}) \) region.

6. Conclusions

In contrast with the 1D non-Hermitian quasiperiodic model [12] where most symmetries and topological orders are well preserved, we considering a Hermitian one, where the chiral current induced by the antisymmetry tensors plays a key role in the emergence of conformal field effect as well as the topological-phase-dependent statistical variance of the gauge-field-dominated conserved extensive quantity, which describes the product of two bosonic supercharge here. We also reveals the relation between antisymmetry-relation-dependent topological phase transitions/breaking and the transitions between different statistical behaviors, which is, equivalently, the transition between the quantum localized (non-Hermitian) and thermalized (Hermitian) phases. This is also meaningful in exploring the measurement-induced uncertainties in the many-body entangled systems [28]. The local unit quantities used in this article, which are defined only by the independence on the centroid variable of a whole system, is similar in spirit with the definition of Fibonacci numbers, and appear in the form of the constant ratio between two neighboring values of a variable whose continuous variation is replaced by the fractionated steps, and the system size always depend only on the larger one. That also corresponds to the key reason for the existence of hierarchy gaps of Hermitian system. In an integrability-chaos transition, the Fibonacci numbers as well as the inverse participation ratio (IPR) is important in identifying the degree of thermalization, where the Fibonacci numbers in numerator and denominator of a constant ratio
usually correspond to the two adjacent periodicities of the primary and secondary system symmetry sectors (for example, the GOE of the periodicity described by the mode number eigen and the GUE of the periodicity described by the mode number four) [29-31]. Such concept that using a rational (and constant) ratio, like the Golden ratio in the thermodynamic limit is used in this article.

Data availability statement

No new data were created or analysed in this study.

Appendix A. Current density obtained using the Jordan-Wigner transformation

We define the boson fields here in position space as

$$\phi(z) = 2\pi \sum_{i=1}^{z-1} c_i^\dagger c_i.$$  \hfill (128)

Then using the Jordan-Wigner transformation, we can define the hard core bosons as

$$b_i^\dagger = c_i^\dagger e^{i\delta(z)},$$

$$b_i = e^{-i\delta(z)} c_i.$$  \hfill (129)
It is easy to know that
\[ \partial_z \phi(z) = 2\pi \epsilon^z_1 c_z = 2\pi b^z_1 b_z = 2\pi n_z, \]  

(130)

where the boson operators here are locally anticommuting. Next we will denote the number of quasiparticles (participating the formation of current) by \( n_z \), which is a dimensionless quantity.

By choosing a certain ordering, we consider the boson number exhibit a power law relation with the position \( z \), basing on the truth that the perpendicular current often shows a power-law decay with the depth of a layered material. For convenient, we choose the bottom layer as the zeroth layer, with the smallest boson number \( b^z_0 b_0 = n_0 \lesssim 1 \), while the boson number in other layers are \( n_i = n_i^0 \) where \( i > 0 \) is the layer index. In long wavelength limit with \( q_z \to 0 \) and \( z \to \infty \), we found the following relations which are valid in long-wavelength limit,
\[
e^{iq_z z} = \frac{\phi(z)}{2\pi},
\]

\[
\lim_{q_z \to 0} (-i q_z) = \lim_{z \to \infty} \frac{\partial \phi(z)}{2\pi} = \lim_{z \to \infty} \frac{n_0^z}{n_0 - 1} \approx n_0 - 1 (\approx \ln n_0) \equiv \partial_z,
\]

(131)

which indicates a strongly nonlinear relation between the boson number and \( z \). In the following we define \( b^z_0 b_z = n_z \) as the boson number operator at position \( z \), which, for a chirally one-dimensional system (not the translational invariant one), is the largest boson number. We will see that this is the precondition for the projection from the fermion tunneling in three-dimension space to the boson interaction in one-dimensional space. That is to say, the chirality of bosons here are necessary, but due to the anticommuting relation between boson operators, different orderings can be directly sum up, i.e. the symmetry permutations.

Note that the key point to generate \((1 + 1)\)-D SYK model is by changing the fermions in 3D momentum space, which carrier the original degrees-of-freedom (induced by the hopping matrix related to the conductivity tensor), to the 1D modes in position space, which are the quasiparticles related to the survival perpendicular current under (linear or nonlinear) responses. Thus all the position notation-\( z \) in this paper corresponds to the quasiparticle in position space, like the above-defined boson field \( \phi(z) \), is indeed equivalents to the Fourier transformation to the real fermions in momentum space (with notation \( q_z \)). Specifically, by virtue of Jordan-Wigner transformation in 1D systems, this Fourier transformation is be carried out by a summation over large number of steps (after infinitely fractionation). Thus the derivation operator \( \partial_z \) with positions of quasiparticles is in fact equivalents to the Fourier transformation to change the representation of fermions in momentum space to the real space,
\[ \partial_z = \int dq_z e^{iq_z z} = \frac{\phi(z)}{2\pi} \int dq_z = \lim_{q_z \to 0} (-i q_z), \]

\[ \int dq_z = i q_z \frac{2\pi}{\phi(z)} = i q_z e^{-i q_z z}, \]

(132)

where the second line is consistent with the third line of equation (131). Thus in long-wavelength limit, the operators \( i q_z \), together with \( \partial_z \), are of the position space. Note that here we artificially define \( \int dz = 2\pi \) instead of \( \int dz = 1 \) since although \( z \) is defined in position space for quasiparticle, but The real fermion number operator in momentum space (1D now) is simply the inverse position of its corresponding quasiparticle
\[ n_{q_z} = n_z e^{-i q_z z} = 2\pi (n_0 - 1) = 2\pi \partial_z = \frac{1}{z}, \]

(133)

which are also be obtained by performing the Fourier transformation to the real fermion number operator in position space
\[ n_{q_z} = \int e^{-i q_z z} dz = \int \partial_z \ln \frac{\phi(z)}{2\pi} dz = i q_z \int dz = -2\pi (i q_z). \]

(134)

Similarly, the derivation with the fermion number operator in 1D position space obtained is such way is equivalents to the Fourier transformation to change the quasiparticles in momentum space representation to the real space representation, which is the current density, the 1D mode we need.

Base on the above results which are valid in the long-wavelength limit, the 1D quasiparticle number (e.g. the current-induced spin-wave excitations) operator in real space and momentum space read as
which are related by the Fourier transformation $n_z = \frac{1}{2\pi} \int e^{i\eta_z^2} n_q dq_z$.

The boson field in momentum space can be written as

$$\phi(q_z) = \int e^{-i\eta_z^2} \phi(z) d\eta_z = \int 2\pi \phi^{-1}(z) \phi(z) d\eta_z = 2\pi.$$  \hfill (136)

By performing the Fourier transformation to the effective electric field term $E_{q_z}$ in momentum space,

$$\int E_{q_z} e^{i\eta_z^2} dq_z = \int (-i\eta_z^2 \phi(q_z)) e^{i\eta_z^2} dq_z = \lim_{\eta_z \to 0} (-i\eta_z^2) \phi(z) = \partial_z \phi(z).$$  \hfill (137)

Thus the above Fourier transform in fact turn the system in momentum space to the boson-number space.

The boson number also directly related to energies of the corresponding modes, and these boson modes can be considered as excitations coupled by the SYK-type random all-to-all interactions. But such one-dimensional boson mode can be obtained only after the Fourier transform to the states of fermions in three-dimensional momentum space. We can look into the inner process by taking $c_i^+$ as an example,

$$\int c_i^+ e^{i\eta^z} c_i^z dq_z = c_i^+ e^{i\eta^z} \frac{\phi(z)}{2\pi} = c_i^+ e^{i\eta^z} \frac{n_i^2}{n_0 - 1} = c_i^+ e^{i\phi(n_i)} = B_i^+,$$  \hfill (138)

where there is a new one-dimensional boson field defined as

$$\Phi(n_z) = -i \ln \frac{\phi(z)}{2\pi} = -i \ln \frac{n_i^2}{n_0 - 1}.$$  \hfill (139)

This boson field reflects the variance with boson number at position $z$, instead of simply the variance with position. Next we will replace the boson number indices $n_z(n_z)$ simply by $\alpha(1, \cdots, M)$. The above boson field will results in the one-dimensional boson operators with new definition, like $B_i^+$ (to distinguish with the one $b_i^+$ defined in momentum space) whose index $\alpha$ indicates the corresponding boson number $n_i$ at position $z$ while $B_i^+B_\alpha$ corresponds to the current density $\rho_\alpha \sim 1/n_0^\alpha$ which is suppressed by the fermion number $n_0^\alpha$. Thus we obtain the simple expression

$$\Phi(\alpha) = \frac{\alpha - 1}{2\pi} \sum \rho_\gamma.$$  \hfill (140)

Then the gauge-invariant current density is obtained as the derivation of this boson field,

$$\frac{\partial\Phi(\alpha)}{2\pi} = \partial_n \frac{-i}{2\pi} \ln \frac{\phi(z)}{2\pi} = -i \frac{1}{2\pi} \frac{1}{n_0^\alpha} = \rho_\alpha.$$  \hfill (141)

The last step is valid here due to the long wavelength limit in momentum space.

**Appendix B. definitions about the gauge invariant chiral current density**

Firstly we consider a continuous SU(2) symmetry in complex fermion system, where

$$c_{ik} = \frac{1}{\sqrt{2}} (c_{i\beta} \pm ic_{i\bar{\beta}}) = e^{\pm i\eta_i},$$

$$c_{ik}^+ = \frac{1}{\sqrt{2}} (c_{i\bar{\beta}}^+ \mp ic_{i\beta}^+) = e^{\mp i\bar{\eta}_i},$$

which can be obtained from the usual Majorana system by adding the restrictions like U(1) charge conservation. Then we remove such restriction and turn back to the Majorana system. Then according to sections 2.5–2.6, we can replace the DOF as (using the same notation in [16])

$$\frac{1}{\sqrt{2}} (\chi_i(\tau) + i\chi_j(\tau)) = \phi(\tau) \equiv e^{i\phi(\tau)},$$

$$\frac{1}{\sqrt{2}} (\chi_i(\tau) - i\chi_j(\tau)) = \phi^+(\tau) \equiv e^{-i\phi(\tau)}.$$  \hfill (143)
where the boson field $\phi(\tau)$ satisfies
\begin{equation}
[\phi(\tau), \phi(\tau')] = i\pi \text{sgn}(\tau - \tau') = \ln(\tau - \tau' - i\eta) - \ln(\tau' - \tau - i\eta),
\end{equation}
\begin{equation}
[\partial_\tau \phi(\tau'), \phi(\tau)] = [\partial_\tau \phi(\tau), \phi(\tau')] = 2\pi i \delta(\tau - \tau') = \frac{1}{\tau - \tau' - i\eta} + \frac{1}{\tau' - \tau - i\eta}.
\end{equation}

But dividing the boson field into $\phi(\tau) = \psi(\tau) + \psi^\dagger(\tau)$ [16], we have
\begin{equation}
[\psi(\tau), \psi^\dagger(\tau')] = -\ln 2\pi (\tau' - \tau - i\eta),
\end{equation}
\begin{equation}
[\partial_\tau \psi(\tau), \psi^\dagger(\tau')] = [\partial_\tau \psi(\tau'), \psi^\dagger(\tau)] = \frac{1}{\tau' - \tau - i\eta},
\end{equation}
\begin{equation}
[\partial_\tau \psi^\dagger(\tau), \psi^\dagger(\tau')] = [\partial_\tau \psi^\dagger(\tau'), \psi^\dagger(\tau)] = \frac{1}{\tau - \tau' - i\eta},
\end{equation}
\begin{equation}
[\partial_\tau \psi(\tau'), \psi^\dagger(\tau')] = -[\partial_\tau \psi(\tau), \psi^\dagger(\tau'), \phi(\tau') - \psi^\dagger(\tau')],
\end{equation}
where from the last formula we can further obtain
\begin{equation}
[\partial_\tau \phi(\tau), \phi(\tau') - \psi^\dagger(\tau')] = [\partial_\tau \psi(\tau), \phi(\tau') - \psi^\dagger(\tau')].
\end{equation}

Considering the time ordering product [16]
\begin{equation}
\mathcal{T} c(\tau)c^\dagger(\tau) = e^{-i\phi(\tau)} : e^{i\psi(\tau')},
\end{equation}
\begin{equation}
= \frac{1}{2\pi i (\tau' - \tau - i\eta)} e^{-i[\phi(\tau') - \psi^\dagger(\tau') + \psi(\tau') - \psi(\tau)]},
\end{equation}
\begin{equation}
= \frac{1}{2\pi i (\tau' - \tau - i\eta)} \left[ e^{-i[\phi(\tau') - \psi^\dagger(\tau')]} - \left( \frac{1}{2\pi i (\tau' - \tau - i\eta)} \right) e^{-i[\phi(\tau') - \psi^\dagger(\tau') + \psi(\tau') - \psi(\tau)]} \right],
\end{equation}
\begin{equation}
= \frac{1}{2\pi i (\tau' - \tau - i\eta)} + \frac{\partial_\tau \phi(\tau)}{2\pi} - O(\tau - \tau'),
\end{equation}
where in the third line we using the commutation
\begin{equation}
[e^{-i\psi(\tau)}, e^{i\psi(\tau')} = \left( \frac{1}{2\pi i (\tau' - \tau - i\eta)} - 1 \right) e^{i\psi(\tau')} e^{-i\psi(\tau)},
\end{equation}
with the vacuum expectation term reads
\begin{equation}
\frac{1}{2\pi i (\tau' - \tau - i\eta)} e^{i\phi(\tau')}, \text{and for the first term within the bracket of third line, we expand it as}
\end{equation}
\begin{equation}
e^{-i[\phi(\tau') - \psi^\dagger(\tau')]} = 1 - i(\phi(\tau') - \phi(\tau')) - \frac{1}{2}(\phi(\tau') - \phi(\tau'))^2
\end{equation}
\begin{equation}
= 1 - i(\partial_\tau \phi(\tau') - \tau') + O((\tau - \tau')^m + O((\tau - \tau')^2),
\end{equation}
where we can safely approximate (with $m$ depends on $\phi(\tau)$)
\begin{equation}
\frac{\phi(\tau) - \phi(\tau')}{\tau - \tau'} = \partial_\tau \phi(\tau) + O((\tau - \tau')^m),
\end{equation}
since we will be have the end $(\tau - \tau') \to 0$. Thus we obtain the normal ordered product as
\begin{equation}
c(\tau)c^\dagger(\tau) = \mathcal{T} c(\tau)c^\dagger(\tau) - \langle 0 | \mathcal{T} c(\tau)c^\dagger(\tau) | 0 \rangle
\end{equation}
\begin{equation}
= \mathcal{T} c(\tau)c^\dagger(\tau) - \frac{1}{2\pi i (\tau' - \tau - i\eta)}
\end{equation}
\begin{equation}
= \frac{\partial_\tau \phi(\tau)}{2\pi}.
\end{equation}

Then we have the commutation relation
\begin{equation}
[\psi(\tau), \psi^\dagger(\tau')] = : [-i \psi(\tau)c(\tau') : \psi^\dagger(\tau)c(\tau') :]
\end{equation}
\begin{equation}
= -\frac{\partial_\tau \phi(\tau)}{2\pi} \cdot \frac{\partial_\tau \phi(\tau')}{2\pi}
\end{equation}
\begin{equation}
= -\frac{1}{2\pi i} \partial_\tau \left( \frac{1}{\tau - \tau' - i\eta} + \frac{1}{\tau' - \tau - i\eta} \right)
\end{equation}
which also leads to
\begin{equation}
[\partial_\tau \phi(\tau'), \partial_\tau \phi(\tau)] = [\partial_\tau^{(1)} \phi(\tau), \phi(\tau')] = [\phi(\tau), \partial_\tau^{(2)} \phi(\tau')].
\end{equation}

Thus in the following, we can treating the term $\frac{i}{\pi \Delta}$, i.e. the vacuum expectation term in $(\tau - \tau') \to 0$ limit, as a variable, and fix the value of time ordering product, then the charge depending on the normal ordered fermion.
product is related to the vacuum term, and vanishes in the limit of $\frac{1}{2\pi} \to \infty$, which will appears below (equation (171)).

In the above expression of antisymmetry tensor $T^p_{ij}$, any pairs of two Majorana fermion indices (each with an overline) enforced by a nonzero delta function carrier the same information about the chiral current density but not means they are the same index (just able to combined into a multiple boson field). For example, for the subgroup $\mathcal{S}$ of the $\beta$-sector, which is the part carriering the informations about the chiral current density in group $\{k, l\}$, we have $\delta_{\mathcal{S}} = 1$, and in the mean time we are able to have

$$\sum_{p} (\epsilon^p_j \epsilon^p_j - \frac{1}{2})$$

where we define the unit quantity (for sector $\beta$) in a projection form

$$\bar{1} := \frac{2}{\beta' - 1} \left( \sum_{i} c_{\gamma}^p c_{\gamma} - c_{\beta'-1}^p c_{\beta'-1} \right)$$

where $\gamma$ is an arbitrary index and during the summation we can choose initial step $\gamma = 1$ and $\beta' - 1 = M$ to make sure the size of group $\{\beta\}$ is of order $O(M)$. This unit quantity is a conserved $U(1)$ quantity only for a group of variables of the sector $\beta$, but it is not a conserved quantity for a group containing the elements of different sectors, like a gauge field $A_{\beta}$. For example, in sector $\beta' - 2$, we have

$$\sum_{p} c_{\gamma}^p c_{\gamma} - \frac{11}{2}(\beta' - 2) = c_{\gamma}^p c_{\beta'-2}$$

with

$$\gamma' := \frac{2}{\beta' - 2} \left( \sum_{i} c_{\gamma}^p c_{\gamma} - c_{\beta'-2}^p c_{\beta'-2} \right)$$

The above definition of unit quantity is different from the usual definition of normal ordered $U(1)$ charge, which is in the form $i\chi(\chi) := \epsilon^p_j c_{\beta} - 1/2\pi$, but we will see that, the definition of such locally-conserved unit quantity is essensial for the gauge invariance of the system investigated in this paper. According to above definitions, we can also obtain these relations (between neighboring sectors)

$$\frac{1}{2} = \sum_{p} c_{\gamma}^p c_{\gamma} - \frac{1}{2}(\beta' - 2) = \sum_{p} c_{\gamma}^p c_{\gamma} - c_{\beta'-1}^p c_{\beta'-1} - (\beta' - 2)\frac{1}{2},$$

$$\partial_{\beta} \bar{1} = \frac{2}{\beta' - 1} (c_{\gamma}^p c_{\gamma} - \frac{1}{2}),$$

$$(\beta' - 2)\left( \frac{1}{2} - \frac{\gamma'}{2} \right) = c_{\gamma}^p c_{\gamma} - \frac{1}{2}.$$  

As shown in the section 4, the distinct statistic behaviors will be related to the specific limiting conditions. We choose the fermionic term $Q_{j,k} := \chi(\tau)\chi(\tau'\delta_{\tau_{K}} = \frac{2\pi}{\beta} \delta(\tau - \tau')$ as an example, and note that it can be rewritten as $Q_{j,k} := \chi(\tau)\chi(\tau'\delta_{\tau})$ in perspective of statistical behavior, where we have the derivative

$$\partial_{\tau} \delta(\tau - \tau') = \partial_{\tau} 2\chi(\tau)\chi(\tau'\delta_{\tau} = -2i\pi [Q_{j,k}(\tau), Q_{j,k}(\tau')].$$

This derivative term is equivalent to

$$\partial_{\tau} \delta(\tau - \tau') = 1 - \lim_{\tau \to \infty} \delta(\tau - \tau').$$
where the limiting term \( \lim_{\tau \to -\infty} \delta(\tau - \tau') \) is independent with \( \tau \) and thus satisfies
\[
\partial_\tau \lim_{\tau \to -\infty} \delta(\tau - \tau') = \partial_\tau \frac{\delta(\tau - \tau')}{1 + 2\pi [Q, Q']} = 0, \tag{161}
\]
then we obtain
\[
\frac{\partial_\tau \delta(\tau - \tau')}{1 - \partial_\tau \delta(\tau - \tau')} = -\delta(\tau - \tau') \frac{1}{1 - \partial_\tau \delta(\tau - \tau')}. \tag{162}
\]
Considering the independence with \( \tau \), this limiting term can also be replaced by \( \lim_{\tau \to -\tau'} \delta(\tau - \tau') = 1_{\tau'} \), where we define \( 1_{\tau'} \) here as a \( \tau' \)-dependent local unit quantity (independent of \( \tau \)). Thus we approximate result
\[
1_{\tau'} = \frac{\delta(\tau - \tau')}{1 - \partial_\tau \delta(\tau - \tau')},
\]
\[
\lim_{\tau \to -\tau'} \delta(\tau - \tau') = \lim_{\tau \to -\tau'} 1_{\tau'} = \frac{1_{\tau'}}{1 - \partial_\tau 1_{\tau'}}. \tag{163}
\]
Similarly, the independence with \( \tau' \) of this limiting expression results in
\[
\frac{\partial_{\tau'} 1_{\tau'}}{1 - \partial_{\tau'} 1_{\tau'}} = -1_{\tau'} \frac{1}{1 - \partial_{\tau'} 1_{\tau'}}, \tag{164}
\]
thus the derivative of \( 1_{\tau'} \) with \( \tau' \) reads
\[
\partial_{\tau'} 1_{\tau'} = \delta(\tau - \tau') \partial_{\tau'} \frac{1}{1 - \partial_{\tau'} \delta(\tau - \tau')} \left( \frac{\partial_{\tau'} 1_{\tau'}}{1 - \partial_{\tau'} 1_{\tau'}} - 1 \right) = \partial_{\tau'} \ln 1_{\tau'} \frac{1}{1 - \partial_{\tau'} 1_{\tau'}} + 1, \tag{165}
\]
where we can also know \( \frac{\partial_{\tau'} 1_{\tau'}}{1 - \partial_{\tau'} 1_{\tau'}} = -\frac{1_{\tau'}}{\lim_{\tau' \to -\infty} 1_{\tau'}} \).

As a mathematical trick widely used in section 7, the expression equation (166) can be regarded as a limiting result of a certain variable \( k \),
\[
-\lim_{k \to -\infty} \sum_{m=0}^{k-1} \left[ \partial_{\tau'} 1_{\tau'} \right]^{-m} = -\lim_{k \to -\infty} \sum_{m=0}^{k-1} \left[ \partial_{\tau'} 1_{\tau'} \right]^{-m} = \frac{-1}{1 - \partial_{\tau'} 1_{\tau'}} = \frac{-1}{1 - \partial_{\tau'} 1_{\tau'}} \tag{166}
\]
then the detailed form of expression related to \( m \) and \( k \) can be obtained by relating this equation to equations (166), (165).

As all the DOF’s of this system originate from the different categories of interaction in different sectors, and the distinct Majorana fermions in each sector, the time difference \((\tau - \tau')\) is indeed correlated to the Majorana fermionic DOF. To see this, we using the SO(\(N\)) Kac–Moody algebra and write the commutator with \( j = k \) as
\[
[\rho_\tau(\tau', \gamma), \chi_l(\tau', \eta)] = -\delta(\tau - \tau') \chi_l \chi_l \tag{167}
\]
Note that here arbitrarily two operators at the same time are symmetrical with each other, e.g.
\[
[\chi_l(\tau', \gamma), \chi_{\rho}(\tau', \eta)] = 0. \]
By treating the Majorana indices as countable quantities, we can taking the limit \( I \to i \) and then the above commutator transforms to the previous one appearing in equation (159). Similar to above procedure, in perspective of removing the dependence of Majorana-\( l \), the limit \( I \to i \) is equivalents to \( I \to \infty \) during the calculation. Thus we have the limiting result
\[
\lim_{I \to \infty} (\rho_\tau(\tau', \gamma)) = \frac{-\delta(\tau - \tau') \chi_l \chi_l}{1 - \partial_\tau (\rho_{\tau} \chi_l \chi_l)} = \frac{i}{2\pi} \partial_\tau \delta(\tau - \tau'). \tag{168}
\]
Then we obtain
\[
\partial_\tau \delta(\tau - \tau') = -\frac{\partial \left[ \frac{\partial_\tau \delta(\tau - \tau')}{2\pi} \right]} {\partial l} (1 - \lim_{l \to -\infty} \partial_\tau \delta(\tau - \tau')), \tag{169}
\]
where we obtain another operator transformation under limiting condition,
\[
\lim_{l \to -\infty} \left[ \frac{\partial \left[ \frac{\partial_\tau \delta(\tau - \tau')}{2\pi} \right]} {\partial l} \right] = \partial_\tau \left[ \frac{\partial \left[ \rho_{\tau} \chi_l \chi_l \right]} {\partial l} \right] = \frac{-\delta(\tau - \tau')}{1 - \partial_\tau (\rho_{\tau} \chi_l \chi_l)}. \tag{170}
\]
Then combined with the \( \left[ \frac{i}{2\pi} \partial_\tau \delta(\tau - \tau') , l \right] = 0 \), we obtain the following expressions of \( l \)-dependent identity

\[
I_l \equiv \partial_\tau \delta(\tau - \tau') \left( \lim_{\ln \frac{\tau}{\tau'} \to -\infty} \frac{1}{\partial_\tau \delta(\tau - \tau')} - \left( \frac{\partial l}{\partial \tau} \right)^{-1} \right) \\
\equiv \partial_\tau \delta(\tau - \tau') \left( \lim_{l \to -\infty} \frac{1}{\partial_\tau \delta(\tau - \tau')} - \partial_l \ln \left( \frac{i}{2\pi} \right) \right),
\]

which can be rewritten as

\[
I_l \equiv 1 - \partial_l \ln \left( \frac{i}{2\pi} \right) \left( \partial_\tau \delta(\tau - \tau') - \partial_\tau \delta(\tau - \tau') \right) \\
\equiv 1 - \partial_l \partial_\tau \delta(\tau - \tau') - \partial_l \ln \left( \frac{i}{2\pi} \right) \partial_\tau \delta(\tau - \tau').
\]

These two expressions of identity can be related by the relation

\[
\lim_{\partial l \to \partial \tau} \left[ - \left( \frac{\partial \delta}{\partial \tau} \right)^{-1} \right] = -\partial_l \ln \left( \frac{i}{2\pi} \right),
\]

which is independent with \( \partial \ln \left( \frac{i}{2\pi} \right) \). By substituting equation (170) into the above identity, we have \( \lim_{l \to -\infty} I_l = 1 \).

Similar to the discussion above equation (226), by letting

\[
\partial_l \ln \left( \frac{i}{2\pi} \right) \left( \partial_\tau \delta(\tau - \tau') \right) + \frac{\partial l \delta(\tau - \tau')}{\lim_{\ln \frac{\tau}{\tau'} \to -\infty} \partial_\tau \delta(\tau - \tau')} = \partial_l \ln \left( \frac{i}{2\pi} \right) = 0,
\]

we can obtain more detailed form of the local unit quantity \(- \left( \frac{\partial \delta}{\partial \tau} \right)^{-1}\),

\[
- \left( \frac{\partial \delta}{\partial \tau} \right)^{-1} = \partial_l \ln \left( \frac{i}{2\pi} \right) \partial_\tau \delta(\tau - \tau') \\
= \frac{1}{\partial_l \ln \left( \frac{i}{2\pi} \right)} - \lim_{\ln \frac{\tau}{\tau'} \to -\infty} \partial_\tau \delta(\tau - \tau') \\
= -\partial_l \ln \left( \frac{i}{2\pi} \right) \lim_{\ln \frac{\tau}{\tau'} \to -\infty} \left[ \frac{1}{\partial_\tau \delta(\tau - \tau')} \right].
\]

Due to the same reason as we discuss in section 7.4, the negative one should be \( \ln \left( \frac{i}{2\pi} \right) \)-dependent (unlike the positive one), in the mean time, the chiral symmetry will not be necessarily preserved due to the absence of symmetry-protected-topological phases here (which usually appears in 1D noninteracting systems, where an even number of \( U(1) \) conserved fermionic charge (instead of bosonic like this article) guarantees the chiral symmetry), and thus we have a phase-shift-dependent chiral transformation result \( C_x^\dagger C_y C_y^{-1} = (-1)^y + i C_y^\dagger C_x \) where the raising and lowering operators here are all complex and obtained by using the Jordan-Wigner type transformation. Due to this reason, the derivative \( \partial_l \ln \left( \frac{i}{2\pi} \right) \ln \left( \frac{i}{2\pi} \right) \) could be possible to contains arbitrary number of \((-1)\)-terms (as long as it \( \mod 2 = 0 \)), thus to uniquely determines the term \(- \left( \frac{\partial \delta}{\partial \tau} \right)^{-1}\), the only option is by taking the limit of vanishing dependence of \((-1)\) with \( \ln \left( \frac{i}{2\pi} \right) \), and this will spontaneously happen as the dependence of \( \partial_\tau \delta(\tau - \tau') \) with \( \ln \left( \frac{i}{2\pi} \right) \) tends to vanish. For example, the last line of above expression can be reads

\[
-\partial_l \ln \left( \frac{i}{2\pi} \right) \left[ \frac{1}{\lim_{\ln \frac{\tau}{\tau'} \to -\infty} \partial_\tau \delta(\tau - \tau')} \right] = \partial_l \ln \left( \frac{i}{2\pi} \right) \lim_{\ln \frac{\tau}{\tau'} \to -\infty} \partial_\tau \delta(\tau - \tau') \\
= m \partial_l \ln \left( \frac{i}{2\pi} \right) \frac{1}{\lim_{\ln \frac{\tau}{\tau'} \to -\infty} \partial_\tau \delta(\tau - \tau')},
\]

where \( m \) is an nonzero integer and it equals to zero only in the \( \partial_l \ln \left( \frac{i}{2\pi} \right) = 0 \) limit.

**Appendix C. Flavors of chiral current density in BDI symmetry class: topological phases**

By now we have introduce the procedure to obtain the 1D quasiparticle representation through the Jordan-Wigner transformation. As it is well-known, the Majorana fermion is a powerful tool in built the models for...
many novel physical phenomena, especially when a phase in gauge theory is been considered just like our case. Using the Majorana-fermion zero modes, the 1D system can exhibits distinct phases recognized by counting the number of zero modes localized at the edge of the chain, as long as the number of flavors of Majorana fermion modes \(O(N^2)\) is large enough (square of that of chiral current density \(O(M)\)).

Here we illustrate more details about the flavors of the chiral current density, which is of order \(O(M)\), and it is an independent DOF relative to each part of Majorana fermion indices. Firstly, as shown in appendix A, by using Jordan-Wigner transformation, we obtain the hard core bosons as well as the complex fermions though the bosonization where the gauge invariant boson field is generated by the energy gradient of chiral current. In this way, the current densities, as the quasiparticle excitations, can be expressed by the product of Majorana fermion operators or the raising/lowering operators (hard core bosons) which are all assigned by the local structure. Defining the Majorana fermion operators in the following way \((i = 1, 2, 3, \cdots)\),

\[
\chi_{2i-1} = \sigma^x_i = -i(c_i - c_i^\dagger) / \sqrt{2}, \\
\chi_{2i} = \sigma^y_i = c_i + c_i^\dagger / \sqrt{2}, \\
\chi_{2i+1} = \sigma^z_i = i\sigma^x_i = (c_i^\dagger c_i - \frac{1}{2}).
\]

(176)

The \(\mathbb{Z}_2\) fermion parity operator is then given by

\[
P = \prod_i i\chi_{2i-1}\chi_{2i}.
\]

(177)

As illustrated in [32, 33], the Majorana modes in 1D chain could be in trivial phase or nontrivial phase in the absence of time-reversal symmetry (TRS), which are \(\sigma^x_i \sigma^x_j \sigma^x_k (i = 1, 2, 3, \cdots)\) and \(\sigma^x_i \sigma^x_j \sigma^x_k (i = 1, 2, 3, \cdots)\), respectively. Here the latter one describes the bosonic spin ground state. While in the presence of TRS, we consider the BDI symmetry class where the quadratic couplings (here is to reflecting the intrinsic character of current densities in fermionic bilinear form) between Majorana fermions reduce the \(\mathbb{Z}\) invariance (like what happen in the Anderson-localized bulk) into the \(\mathbb{Z}_2\) invariance.

The TRS will prevent the formation of fermion bilinear terms, but cannot gap out the topological-protected degeneracies, like the local single-particle state (constituted by the Majorana modes of the same fermion parity) or the nonlocal single-particle state (superposition of two Majorana end modes of opposite fermion parities) [33]. In the presence of TRS, there would be infinite symmetry protected topological (SPT) phases instead of just two: the trivial and nontrivial Haldane phases, which can be described by bosonic Majorana modes: \(\sigma^x_i \sigma^x_{i+k}\) (\(i = 1, 2, 3, \cdots\)), indicated by the topological integer index \(k = 2, 3, 4, \cdots\). Here the topological index \(k\) also correspond to the number of edge gapless Majorana modes, and odd \(k\) corresponds to even superposition of bosonic ground states which will gap out the dangling Majorana modes \((\sigma^x_i \sigma^x_j)\) and breaks symmetry of \(P\) [32]. This also corresponds to the \(N \mod 4 = 0\) case in the SYK\(_4\) model in the edge of Majorana chain [9, 34], in which case the particle-hole symmetry is anomalous. While an even number \(k\) corresponds to the odd superpositions of bosonic ground states. In this case, the particle-holer asymmetry is not anomalous \((N \mod 8 = 0;\) Note that here \(N\) is the number of chains of the same type but can be simply replaced by \(N\) since the current density of a certain phase can be equivalently considered as a collection of boundary fermions of \(k\) parallel chains), and there will be nondegenerate ground states which will be asymptotically degenerate in large-\(N\) limit due to the TRS-protected topological order. The robust MBL-induced nontrivial character inside the bulk avoids the zero modes being fully gapped out and suppress the reansonances induced by disorders as well as the thermalization effects at the edge. The resulting stable degenerate ground state is an edge zero mode localized within the gap of topological phase. During this process, the symmetry \(P\) is unbroken, and the formation of nonzero fermion bilinear expectation value \(\sigma^x_i \sigma^x_j\) leads to \(\mathbb{Z}_2\) spontaneous symmetry breaking. We also note that, the trivial phase in conformal (low energy) limit, where all the anomalous components are absent, preserves both the gauge invariance and discrete \(\mathbb{Z}_2\) symmetry (due to the absence of degenerate group state).

For chiral current densities expressed in terms of summation of Majorana modes, it is straightforward to conclude that

\[
\sigma^x_i \sigma^x_{j+m} = \sigma^x_j \sigma^x_{j+m} = \sigma^x_k \sigma^x_{j+m} = \sigma^x_i \sigma^x_{j+m} (i, j, m = 1, 2, 3, \cdots; k, l = 2, 3, \cdots),
\]

\[
\sigma^y_i \sigma^y_{j+m} = \sigma^y_j \sigma^y_{j+m} = \sigma^y_k \sigma^y_{j+m} = \sigma^y_i \sigma^y_{j+m} (i, j, m = 1, 2, 3, \cdots; k, l = 2, 3, \cdots).
\]

(178)
Base on these relations, the Majorana modes of different categories are
\[ \alpha = 1: \quad \sigma_1^x \sigma_1^x = \sigma_1^x \sigma_2^x + \sigma_1^x \sigma_2^x = \sigma_1^x \sigma_2^x + \sigma_1^x \sigma_2^x = \cdots \]
\[ = \sum_{i=1,2,3} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^x \sigma_{i+1}^x), \]
\[ \alpha = 2: \quad \sigma_1^x \sigma_1^y + \sigma_1^x \sigma_2^y, \]
\[ \alpha = 3: \quad \sigma_1^y \sigma_1^x + \sigma_1^y \sigma_2^x + \sigma_1^y \sigma_2^x, \]
\[ \cdots, \]
\[ \alpha = M: \quad \sum_{i=1}^{(M+1)/2} (\sigma_i^x \sigma_i^x + \sigma_i^x \sigma_{i+1}^x + \sigma_i^x \sigma_{i+1}^x), \] (179)

where we assume \( M \) is an odd number in the last line, otherwise the summation upper boundary should be \((M + 2)/2\) if \( M \) is an even number.

**Appendix D. Relevant interaction effects**

To make sure the theory be conformal in the large-\( M \)-limit, we need to recognize which perturbations are marginally relevant (or irrelevant). It was shown that, for a non-chiral system, for example, for a system consider both the fermions with left and right flavors [5], the non-random interactions, which are diagonal between these two flavors, will be relevant only in terms of the quadratic form and for one sign of the four-fermion coupling (treated as a perturbation during the renormalization analysis). In this kind of models, the relevant component of the four-fermion interactions will leads to SYK\(_4\) behaviors and exhibiting unstable non-fermi-liquid features. This model is in fact equivalents to our model in this paper after removing the constraints given by the imaginary time-evolution mapping between the two initial degrees-of-freedom (which are described by two subspaces as shown in above section).

**D.1. Without time-mapping (gauge fixing) restriction**

For a clear comparison, we firstly represent some results of [9], which is the case without the time-mapping restriction. In terms of the four antisymmetry random tensors \( g_{ij} \), there are three kinds of the random coupling diagrams (with the indices describing the mutually independent variables not being summed)

\[
h_1 = \sum_{mn} g_{ij} g_{mn} g_{kl} \chi_i \chi_j \chi_k \chi_l \]
\[ h_2 = \sum_{mn} g_{im} g_{mn} g_{kl} \chi_i \chi_j \chi_k \chi_l \]
\[ h_3 = \sum_{mn} g_{im} g_{m} g_{nl} \chi_i \chi_j \chi_k \chi_l \] (180)

Only the pattern of \( h_1 \) survival in large-\( N \)-limit, and thus be marginally relevant. While \( h_2 \) and \( h_3 \) are subleading in large-\( N \)-limit. During this process, each antisymmetry random tensor can describes a product of two coupling Majorana fermions. To further understand this we show the following examples of the tensors in each pattern: For \( h_1 \):

\[ g_{ij} = \sum_{\alpha} \varphi_i(\alpha) \varphi_j(\alpha), \]
\[ g_{mn} = \sum_{\alpha, \beta} \varphi_m(\alpha) \varphi_n(\beta), \]
\[ g_{im} = \sum_{\alpha, \beta} \varphi_m(\beta) \varphi_n(\alpha), \]
\[ g_{il} = \sum_{\beta} \varphi_i(\beta) \varphi_l(\beta), \] (181)

where only the tensors \( g_{ij} \) and \( g_{il} \) are completely independent with other tensors, and the commutation relation between these two tensors is exactly like that between two products \( \varphi_i(\alpha) \varphi_j(\alpha) \) and \( \varphi_i(\beta) \varphi_j(\beta) \) before the summation over quasiparticle-number components. For \( h_2 \):
\[ g_{im} = \sum_\alpha \varphi_i(\alpha) \varphi_m(\alpha), \]
\[ g_{mu} = \sum_\alpha \varphi_m(\alpha) \varphi_u(\beta), \]
\[ g_{mj} = \sum_\beta \varphi_u(\beta) \varphi_j(\beta), \]
\[ g_{kl} = \sum_{\alpha, \beta} \varphi_k(\alpha) \varphi_l(\beta), \quad (k = i, m; l = n, j). \]  

(182)

For \( h_3 \):

\[ g_{im} = \sum_{\alpha, \beta} \varphi_i(\alpha) \varphi_m(\beta), \]
\[ g_{mj} = \sum_{\alpha, \beta} \varphi_m(\beta) \varphi_j(\alpha), \]
\[ g_{kn} = \sum_{\alpha, \beta} \varphi_k(\beta) \varphi_n(\alpha), \quad (k = m, l; n = i, j) \]
\[ g_{nl} = \sum_{\alpha, \beta} \varphi_n(\alpha) \varphi_l(\beta), \quad (n = i, j; l = k, m). \]  

(183)

D.2. With time-mapping restriction

As we discuss in the above section, for one of the subspaces which has all the elements \((N^2)\) of the time mapping group \( \mathcal{M}_{\tau \to \tau'} \), it will restrict the two Majorana fermions in another subspace own only the \( N \) elements of the time mapping group, which means the product of these two Majorana fermions can only be of the forms \( \chi_i(\tau) \chi_j(\tau') \) or \( \chi(\tau) \chi_i(\tau) \). We will next examining the relevance of these two cases in a statistical ensemble are indeed the same. We choosing the subspace of \( \alpha \) as the one own only the \( N \) elements. Then for the case of different times, \( \chi_i(\tau) \chi_j(\tau') \), the relevant coupling pattern is

\[ h_1' = \sum_{mn} g_{ij} g_{mn} g_{jk} \chi_i \chi_j \chi_k \chi_l, \]
\[ g_{ij} \rightarrow g_{\tau \tau'} = \sum_{\alpha, \beta} \varphi_i(\alpha, \tau') \varphi_j(\alpha, \tau'), \]
\[ g_{mn} = \sum_{\alpha, \beta} \varphi_m(\alpha) \varphi_n(\beta), \]
\[ g_{nm} = \sum_{\alpha, \beta} \varphi_n(\beta) \varphi_m(\alpha), \]
\[ g_{jk} \rightarrow g_{\tau \tau'} = \sum_{\beta} \varphi_j(\beta, \tau_2) \varphi_k(\beta, \tau_2'). \]  

(184)

For the case of the same times, \( \chi_i(\tau) \chi_j(\tau) \), which allows the two Majorana fermions have independent indices, then the relevant coupling pattern is of the one with three degrees-of-freedom as we stated above. This pattern reads

\[ h_2' = \sum_{mn} g_{ij} g_{mn} g_{jk} \chi_i \chi_j \chi_k \chi_l, \]
\[ g_{ij} = \sum_{\alpha} \varphi_i(\alpha, \tau) \varphi_j(\alpha, \tau), \]
\[ g_{mn} = \sum_{\alpha, \beta} \varphi_m(\alpha, \tau) \varphi_n(\beta, \tau'), \]
\[ g_{nm} = \sum_{\alpha, \beta} \varphi_n(\beta, \tau) \varphi_m(\alpha, \tau'), \]
\[ g_{jk} = \sum_{\beta} \varphi_j(\beta, \tau') \varphi_k(\beta, \tau'). \]  

(185)

The mechanism producing the three degrees-of-freedom can also be understood in this way: The Majorana wave functions (random variables) within the tensors \( g_{ij} \) and \( g_{jk} \) are respectively, of the same time, thus only \( N^3 \) elements for the evolution from \( \tau \) to \( \tau' \) are allowed over these two terms.

D.3. Gauge fixing restriction in terms of anticommutation relations of Majorana operators

The above expression of \( Q^2 \) is restricted by the finite-size of time-mapping group. This can also be seen from the relations between the four coupled Majorana fermions, \( \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau_1), \chi_i(\beta, \tau_2), \chi_j(\beta, \tau_2) \), where we can at most find four zero commutators among them (here we focus only on the relationships between two Majorana fermions within each bosonic charge and that of fermion-products between two charges).
The four coupled Majorana fermions appear in the expression of $Q_E$ also share this character:

\[ \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau'_1), \chi_k(\beta, \tau_2), \chi_l(\beta, \tau'_2), \]

where we can still at most find four commutating relations among them,

\[ \{ \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau'_1) \} = 0, \]
\[ \{ \chi_i(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_l(\beta, \tau'_2) \} = 0. \]

(186)

This is in contrast with the case without time-mapping restriction, where the random couplings are marginally relevant at SYK$_4$ fixed point, where the corresponding four Majorana fermions are

\[ \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau'_1), \chi_k(\beta, \tau_2), \chi_l(\beta, \tau'_2), \]

and we can find five commutating relations among them,

\[ \{ \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau'_1) \} = 0, \]
\[ \{ \chi_i(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_l(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_l(\beta, \tau'_2) \} = 0. \]

(187)

### Appendix E. Relation to the Riemann zeta function and the related mathematical deductions of this article

For our system, the magnitude of current density flavor $M$ and that of Majorana fermions $N$ dominate the phase transitions between chaotic to Anderson-localized phases and the statistical distribution of many-body level spectrum. Here we illustrate that the distinct statistics can also reflected by the nontrivial zeros of the Riemann zeta function. We note that, for a physical system described in this article, we using the special definition about the local unit quantity to reflecting the gauge-invariance, in the mean time, as such local unit quantities are indeed complex in the carrier representation in position space.

In the region of GSE, the gauge symmery is indeed preserved by the UV cutoff on the quasiparticle position space, which is valid in both the position or energy spaces. In the region of GSE, the gauge symmery is indeed preserved by the UV cutoff on the quasiparticle position space, which is valid in both the position or energy spaces, respectively, are connected by a ratio

\[ \delta z = \frac{n_0}{\sum_{\gamma} n_0^\gamma}, \]

where we can still at most find four commutating relations among them,

\[ \{ \chi_i(\alpha, \tau_1), \chi_j(\alpha, \tau'_1) \} = 0, \]
\[ \{ \chi_i(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_k(\beta, \tau'_2) \} = 0, \]
\[ \{ \chi_i(\alpha, \tau_1)\chi_j(\beta, \tau_2), \chi_l(\beta, \tau'_2) \} = 0. \]

(188)

If we turn to the derivative with respect to the quasiparticle positions $\gamma(< z - 1)$ for the essential term $\frac{\phi(z)}{2\pi}$ which validating to mixing the momentum space and position spaces in certain limiting cases,

\[ \frac{\phi(z)}{2\pi} = \frac{\phi(z)}{2\pi} \ln \frac{\phi(z)}{2\pi} = \frac{\phi(z)}{2\pi} \ln \sum_{\gamma} n_0^\gamma = \frac{\phi(z)}{2\pi} \ln \sum_{\gamma} n_0^\gamma, \]

and its derivative with the $n_0$ is related to the single current density.

As a result of this cutoff which is valid in both the position or energy spaces, we have $e^{1/z} = 1 + \frac{1}{z}$, and thus we can obtain the gauge-invariant shift $\delta z$ in the position space, which is the Fourier transformed version of the above invariant quantity $\ln n_0$ which corresponding to the Luttinger–Ward identity $\partial_z \ln \phi(z)$,

\[ \partial_z \ln \frac{\phi(z)}{2\pi} = \ln(1 + \delta z) = \delta z + O(z^{-2}), \]

(190)

These two shifts $\delta z$ and $\ln n_0$ in position and energy spaces, respectively, are connected by a ratio $\delta z / \ln n_0 = \frac{1}{\epsilon - 1} \approx 1.58$, which is correct as long as $O(M) = O(N)$. Thus we can define the local unit quantities in
terms of positions $z$ as

$$
I_z := e^{1/z} = 1 + \sum_{\gamma} \frac{n_0^z}{\gamma},
$$

$$
I_{z+1} := e^{1/(z+1)} = 1 + \sum_{\gamma} \frac{n_0^{z+1}}{\gamma},
$$

$$
\cdots.
$$

(191)

As long as the gauge symmetry is not broken, the unit quantities satisfy

$$
\partial_z I_z + \frac{1}{z} = \frac{1}{z^2 I_{z+1}},
$$

(192)

The term $-\ln z$ appears in last line is useful in distinguishing the GSE from GUE. As we stated above, in terms of quasiparticle positions, GSE and GUE domains correspond to $O(M) = O(N)\,(n_0 = 1 + 1/z$ in quasiparticle position space) and $O(M) = O(N^2)\,(n_0 = 1 + (z - 1/2)^{-1}$ in quasiparticle position space), respectively. Simple simulation implies

$$
\lim_{z \to \infty} \frac{\ln z}{z^2} = \lim_{z \to \infty} \frac{1}{z},
$$

(193)

in GSE case which leads to $I_z \to I_{z+1}$ due to second line of equation (192); and

$$
\lim_{z \to \infty} \frac{\ln z}{z^2} = \lim_{z \to \infty} \frac{1}{z^2} + \frac{1}{z} - \frac{1}{z^2},
$$

(194)

in GUE case which corresponds to the complex argument of Riemann Hypothesis $\zeta(\nu + i\eta)$ with $n \to \infty$.

More detailed mathematical discussion for GUE in random matrix theory in this limit, with mean asymptotic spacing $2\pi/\ln t_{\infty}$, are studied in [33].

In GUE limit, where the gauge-invariance is broken due to the unbounded edge of Zeta function, the global unit quantity is $\frac{1}{2\pi} \zeta^{-1}(1)$. With unbounded edge, we have $\frac{\partial_{\phi(z)}}{\partial(z)} = \frac{n_0^z}{\sum_{\gamma} n_0^z}$, $I_m$ in this limit, which corresponds to the complex argument in Riemann hypothesis with the nontrivial zeros $I_m = \frac{1}{z} + i\eta$ with $n \to \infty$.

E.1. GSE: $n_0 = 1 + 1/z$

Now we already known that, for three-DOF configuration where $O(M) \sim O(N)$, the current density follows GUE statistics. As we discuss in appendix A, the flavor number of current density and Majorana fermions are $O(M) \sim |\alpha - \beta|^{-1}(\sim |z - z'|^{-1})$ and $O(N) \sim |n_0 - 1|^{-1}$, which are expressed in terms of the quasiparticles in position space or fermionic carriers in position space, respectively. Next we simply denote the maximal step number in terms of the current density flavors, i.e. $z_m = |z - z'|^{-1} \sim O(M)$ (for quasiparticles in the position space). Here We found that, as long as $O(N) \geq O(M)$, equation (141) is correct and the system exactly meets the three-DOF configuration as we discussed above.

In GSE regime, we can consider each step of quasiparticle position $\delta z = 1/z$ as a constant which is independent of $z$, and $\partial_z \delta z = \frac{1}{z^2} \sim 0$ due to the IR cutoff at order $O(z^{-1})$ (i.e. terms of order $O(z^{-2})$ or higher are omitted) in the position space. Then the quasiparticle number $n_0^z$ Boson phase $\phi(z)/2\pi$, its derivative with $z$ can be written as

$$
\partial_z n_0^{z-\delta z} = \frac{n_0^z - n_0^{z-\delta z}}{\delta z} = n_0^{z-\delta z} \ln n_0^z,
$$

(195)

where $\ln n_0 = 1/z$ under the IR cutoff, and $n_0 = 1 + 1/z$ as required by GSE condition. Then the derivative with $z$ for the boson field $\phi(z)$ is
\[
\partial_z \frac{\phi(z)}{2\pi} = \partial_z \sum_{\gamma} n_0^\gamma = \partial_z n_0^{z-\delta z} + \partial_z n_0^{-2\delta z} + \cdots + \partial_z n_0
\]
\[
= \frac{n_0^2 - n_0^{z-\delta z}}{2\delta z} + \frac{n_0^3 - n_0^{-2\delta z}}{2\delta z} + \cdots + 0
\]
\[
= \frac{n_0^2}{\delta z} \left[ \sum_{\gamma=1}^{(z-1)/\delta z} \frac{1}{\gamma} - \sum_{\gamma=1}^{(z-1)/\delta z} \frac{n_0^{-\gamma}}{\gamma} \right]
\]
\[
= \frac{n_0^2}{\delta z} \left[ \sum_{\gamma=1}^{(z-1)/\delta z} \frac{1}{\gamma} - \sum_{\gamma=1}^{(z-1)/\delta z} \frac{n_0^{-\gamma}}{\gamma} \right],
\]
(196)

The above expression reduces to \(n_0^2/\delta z\) in the large \(O(M) = O(N)\) limit. We note that, for the exponential part of the quasiparticle operate at the edge, \((z - \delta z)\), its derivative with \(z\) will be \(\partial_z (z - \delta z) = \delta z/\delta z = 1\) as long as the gauge symmetry is preserved, and this situation changes when the system enters the GUE regime.

**E.2. GUE:** \(n_0 = 1 + \frac{1}{z - 1/2}\)

**E.2.1. \(n_0\).** In GUE regime, the quasiparticle operator in initial position is defined as \(n_0 = 1 + \frac{1}{z - 1/2}\) (in position representation of quasiparticles), and unlike the GSE case, we can theoretically set the cutoff to \(O(z^{-n})\) with arbitrary large \(n\) due to the unbounded summation edge in the calculations of this case.

Using Fourier transformation, we firstly represent the \(n_0\) in terms of the current density (as a function of summation boundary \(k\) which is always an integer)

\[
n_0(k) = \frac{\sum_{\gamma=0}^{k-1} \left( \frac{1}{z} \right)^\gamma}{\sum_{\gamma=0}^{k-1} \left( \frac{1}{z} \right)^\gamma},
\]
(197)

which can be rewritten as

\[
n_0(k) = \frac{\exp \left[ \int_{-1}^{1} \left( \frac{1}{z} \right)^h \right]}{\exp \left[ \int_{-1}^{1} \left( \frac{1}{z} \right)^h \right]}
\]
\[
= \frac{\exp \left( \int_{-1}^{1} \frac{1}{z-1} \frac{1}{z-1} \right)}{\exp \left( \int_{-1}^{1} \frac{1}{z-1} \frac{1}{z-1} \right)}
\]
\[
= \frac{\exp \left[ -i \arg(z - 1) - i \arg(z^k - 1) + \ln \left( \frac{z^{1-k} - 1}{z+1} \right) \right]}{\exp \left[ -i \arg(z - 1) - i \arg(z^k - 1) + \ln \left( \frac{z^{1-k} - 1}{z+1} \right) \right]}
\]
\[
= \exp \left[ -i \arg((-1)^k z^k - 1) + \ln \left( \frac{z^{1-k} - 1}{z+1} \right) \right],
\]
(198)

where in the last line we use the following results: Firstly since \(z\) will be a real or complex quantity in different representations (in terms of quasiparticles or real fermions) after the necessary Fourier transformations (whose detailed process may be omitted in the calculations), we have

\[
i \arg(z \pm 1) = \ln \left( \frac{z \pm 1}{|z \pm 1|} \right) = \frac{z \pm 1}{z},
\]
\[
i \arg(z^k \pm 1) = \ln \left( \frac{z^k \pm 1}{|z^k \pm 1|} \right) = \frac{z^k \pm 1}{z^k},
\]
(199)

where

\[
|z - 1| = \frac{z - 1}{\exp[i \arg(z - 1)]} = z.
\]
(200)
Here \( \exp[i \text{Arg}(z-1)] = \frac{z}{z-1} \) is in fact a very important quantity and we will analyse in detailed in following, and the quite normal relation

\[
\exp[-i \text{Arg}(z-1)](z-1) = \frac{z}{z-1} = z,
\]

has indeed more profound explanation: in terms of the current density (in quasiparticle representation) \((-z)\) can be expressed as a function of \((-\ln(-z)) = -(\pi m + 1) + \ln z)\):

\[
-z := f(-\ln(-z)) = \left[ \lim_{-\ln(-z) \to \infty} f(-\ln(-z)) \right](1 - \partial_{-\ln(-z)} f(-\ln(-z))) = \frac{z}{z-1}(1 - z). \tag{202}
\]

The expressions in this form will appear frequently in this paper, e.g. in the above expression, \((-z)\) as a function of \((-\ln(-z))\) can be expressed as a summation over a certain uncontinuous variable with bounded summation edge \((-\ln(-z) - 1)\) (Note that here \(-1\) is also a local unit whose actual value depends on the value of each step in the summation), and equivalents to the product of a limiting term (independent of \((-\ln(-z) - 1)\) and the derivative term (dependent on \((-\ln(-z) - 1)\)). This product is indeed a Fourier transformation under limiting condition of GUE region (e.g. \(m \to -\infty\) here). Second, the term \(\ln[(-1)^k]\) can be rewritten as

\[
\ln[(-1)^k] = i\pi - i \text{Arg}((-1)^k z^k - 1) = \ln(-1)^{2k-1} - \ln(-1)^{k-1}, \tag{203}
\]

with

\[
i \text{Arg}((-1)^k z^k - 1) = \ln \left[ \frac{(-1)^k z^k}{(-1)^k z^k - 1} \right] = \ln(-1)^{k-1}. \tag{204}
\]

Then similar to above expression equation (202), by virtue of the limiting conditions in GUE region, the term \(\exp[i \text{Arg}(1 - (-1)^k z^k)]\) can be recognized as a function of discrete variable \((-1)^k\),

\[
\exp[i \text{Arg}((1 - (-1)^k z^k - 1)] = (-1)^{k-1} = \lim_{z \to \infty} f(z^{-k})
\]

\[
= \frac{\frac{z + 1}{z - 1}}{1 - \partial_{z^{-1}} f(z^{-k})}
\]

\[
= \frac{\partial_{z^{-1}} f(z^{-k})}{\partial_{z^{-1}} f(z^{-k})}
\]

\[
= \frac{\partial_{z^{-1}} f(z^{-k})}{(-1)^k z^{-k}}.
\]

Now we already know \(n_0 = \frac{z + 1/2}{z - 1/2}\) with complex \(z\) in terms of current density representation can be rewritten as a function of discrete variable \(k\) (integer), \(n_0(k) = \exp[i \text{Arg}(1 - (-1)^k z^k)] - \ln \frac{1 - (-1)^k z^k}{z + 1},\) and becomes \(z\)-independent. This can indeed be written as

\[
n_0(k) = (1 - \partial_k n_0(k)) \frac{z + 1}{z - 1}
\]

\[
= (1 - \partial_k n_0(k))(1 + \sum_{\gamma=1}^{\infty} \frac{1}{z^\gamma} \left( \sum_{\lambda=0}^{\gamma-1} \left( \frac{1}{2} \right)^\lambda \right) \frac{z + 1}{z - 1})
\]

\[
\sum_{\gamma=1}^{\infty} \frac{1}{z^\gamma} \left( \sum_{\lambda=0}^{\gamma-1} \left( \frac{1}{2} \right)^\lambda \right) = 1 - \frac{1}{z + \frac{1}{2} + \frac{1}{2z}}
\]

and it is interesting to see that, within this expression there is an unbounded summation (we still define it as \(f\) to signal this feature)

\[
f(k) := \gamma_{\lambda=0}^{\gamma-1} \left( \frac{1}{2} \right)^\lambda = \lim_{k \to \infty} f(k)(1 - \partial_k f(k))
\]

\[
= \frac{2}{3}(1 - (-1)^k z^k),
\]

whose limiting value is

\[
\lim f(k) = n_0 \exp[\ln(1 - \partial_{[\ln z/2]}^{-\gamma}(z - 1/2)) - \ln(1 - \partial_{[\ln z/2]}^{-\gamma}(z + 1/2))] = \frac{2}{3}.
\]

Thus in this representation, \(\partial_k n_0(k) = 0\), in contrast with the result of equation (214).
E.2.2. $n_0^z$. Let us turn back to the boson field which determines the quasiparticle number operator $n_0^z$. Within the expression of boson field $\phi(z) = \frac{n_0^z}{2\pi} = \sum_{k=1} z^{z-k} e^{i\phi(z)}$, considering the $n_0^z$ as an invariant quantity in quasiparticle space, the invariance can be reflected by the constant step in the exponential part of quasiparticle number operators during the summation, which is $\delta z$:

$$\phi(z) = \frac{z^{z-k} e^{i\phi(z)}}{2\pi} = \prod_{k=1}^{z-k} n_0^z = \sum_{\lambda=0}^{z-k} n_0^{z-\lambda} e^{i\phi(z)} = \frac{n_0^z - 1}{n_0^z - 1}. \quad (210)$$

Next we simplify the notion $\delta z$ as 1. The above result is still correct in this simplified case, which can be verified by

$$\phi(z) = \frac{z^{z-k} e^{i\phi(z)}}{2\pi} = \prod_{k=1}^{z-k} n_0^z = \sum_{\lambda=0}^{z-k} n_0^{z-\lambda} e^{i\phi(z)} = \frac{n_0^z - 1}{n_0^z - 1}. \quad (211)$$

Expressing the $\phi(z)/2\pi$ in terms of the limiting condition $z \to \infty$,

$$\phi(z) = \frac{n_0^z - 1}{n_0^z - 1} = \lim_{z \to \infty} \frac{\phi(z)}{2\pi} (1 - \frac{n_0^z - 1}{2\pi}). \quad (212)$$

Since the boson field in the limiting condition $z \to \infty$, we have $\partial_z \lim_{z \to \infty} \frac{\phi(z)}{2\pi} = \partial_z \frac{1}{n_0} = 0$.

Unlike the GSE region where $n_0^z$ is $z$-independent, $\partial_z n_0^z \neq 0$. The long-wavelength limit in momentum space with $q_z \to 0$ results in $-i n_0^z = \partial_z$. Thus in the limiting condition, we have

$$\lim_{q_z \to 0} \partial_z = -i 0 = i0^+ = \ln n_0,$$

In the mean time, the derivative term $\partial_z n_0^z$ can be rewritten as

$$\partial_z n_0 = \frac{\partial(z - 1)}{\partial z} n_0^z \ln n_0 + n_0^{z-1} \partial\ln n_0 = \frac{n_0^z - 1}{n_0^z - 1}. \quad (213)$$

so we obtain the expression in familiar form

$$n_0 \ln n_0 = \frac{\partial z n_0}{1 - \frac{\partial z - 1}{\partial z}}, \quad (215)$$

where the left-hand-side is the limiting result $\lim_{z \to 1} \partial_z n_0$, and

$$\frac{\partial(z - 1)}{\partial z} = \partial z - 1 \partial z n_0. \quad (216)$$

Similarly we have

$$n_0 \ln n_0 = \frac{\partial z - 1}{1 - \frac{\partial z - 1}{\partial z}}, \quad (217)$$

with

$$\lim_{z \to 1} \partial z - 1 n_0 = n_0 \ln n_0, \quad (218)$$

A Fourier transformation is contained in these processes,

$$0^+ = \lim_{q_z \to 0} i \partial_z = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \exp[iz0^+], \quad (219)$$

which provides the factor $\frac{i}{2\pi}$. This can be seem from the expression of current density

$$\rho = \frac{-i}{2\pi n_0^z} \frac{\partial z}{2\pi} \partial_z \ln \frac{\phi(z)}{2\pi}, \quad (220)$$
where \( \frac{\alpha(z)}{2\pi} = \ln \frac{n_0^2}{n_0 - 1} = i\zeta z \to -1 \) in his limit. In the mean time,
\[
\ln n_0(k) = \ln \frac{z + 1}{z - 1} (1 - n_0^{-1}(k)f_k k n_0(k))
\]
\[
= (1 - \frac{z - 1}{z} \frac{1}{z^k - 1}) - (1 - \frac{z + 1}{z} \frac{1}{z^k - 1}) \ln \frac{-z}{1 - z} + 1
\]
\[
= (1 - \frac{\partial f}{\partial L} \frac{1}{z} \frac{1}{z^k - 1} \ln \frac{1}{z^k - 1}) - (1 - \frac{\partial f}{\partial L} \frac{1}{z} \frac{1}{z^k - 1} \ln \frac{1}{z^k - 1}).
\]
\[
(221)
\]

E.3. Gauge invariance-dominated radial derivative and normalized Lebesgue measurement on complex space

Next we introduce some direct results due to the Fourier transformations under certain limiting conditions.

Firstly, as an example, we focus on the function \( f_k(z) = \sum_{\gamma=0}^{k-1} \gamma^1 \frac{1}{z^k - 1} \), which corresponds to the limiting results
\[
[\frac{z - 1}{z}, \frac{1}{z}] = 0 \text{ now. Using the above results, we can obtain}
\]
\[
\frac{\partial f}{\partial z} z - 1 = \frac{z^2}{z - 1},
\]
\[
(222)
\]

which is the Fourier transformation during this process is equivalents to the replacement of derivative
\( \partial_z \) by \( \partial_z \). This is a natural result of the characters of integral operator in complex space acting on some holomorphic functions base on the defined specific unit, and the normalized Lebesgue measure [36] are affected by the boundedness in such space. Another application is the Luttinger-Ward analysis on calculating the fermion charge operator [37].

For \( f_k(z) \), if we make the derivative with \( k \) again on the above-obtained expression \( \partial_z f_k = 1 - \frac{k}{\lim_{k \to \infty}} \), we obtain
\[
\partial_k^{(2)} f_k = \partial_k 1 - \frac{\partial f_k}{\lim_{k \to \infty}} - \frac{\partial f_k}{\lim_{k \to \infty}} \ln \frac{1}{z} = \frac{1}{z^k} \ln \frac{1}{z} + \frac{1}{z^k - 1} k \partial_k \frac{1}{z^k},
\]
\[
(224)
\]

where 1 here is defined as a global unit according to its independence of \( k \), and its following transformations under the limiting condition of vanishing \( k \)-dependence of the unit term 1,
\[
\lim_{\partial_z \to 0} \frac{1}{z} \ln \frac{1}{z} = -k \partial_z \frac{1}{z},
\]
\[
\lim_{\partial_z \to 0} \partial_z \frac{z}{z - 1} = (\frac{z}{z - 1})^2 \frac{1}{z^k - 1},
\]
\[
(225)
\]

where the second line leads to the result about the difference between inversed \( f_k \) and its limiting result,
\[
\frac{1}{\lim_{k \to \infty}} = \frac{1}{\lim_{k \to \infty}} = -\lim_{k \to \infty} f_k \partial_z \left( \lim_{k \to \infty} f_k \right)^{-1} = \partial_k \ln \lim_{k \to \infty} f_k = \partial_k \ln \frac{z}{z - 1}.
\]
\[
(226)
\]

Note that the above two limiting expressions equation (225) corresponds to the cutoff at second order of \( k \)-derivative, i.e. \( \partial_k^{(2)} f_k = \frac{1}{z^k} \ln \frac{1}{z} + \frac{1}{z^k - 1} k \partial_k \frac{1}{z^k} = 0 \).

By writing \( f_k \) in terms of the \( k \)-independent unit, as \( f_k(z) = \frac{z}{z - 1}(1 - \frac{1}{z^k}) \), the limiting expression \( \lim_{k \to \infty} f_k \) can be expanded again as
\[
\frac{z}{z - 1} = \frac{z - 1}{1' - \frac{z}{z - 1}}(1 - \frac{1}{z^2}).
\] (227)

then it is easy to see that the above limiting expressions equation (225) as well as the cutoff at second order of \(k\)-derivative corresponds to the scaling behaviors that \(\partial_k \lambda\) and \(\partial_k \frac{1}{z^2}\) tend to zero in a same speed and in the mean time \(1' \to \frac{\mu}{\lim_{k \to \infty}}\). This result means in the framework consist of unit \(1'\), the step of variable \(k\) is larger than the one in previous framework consist of unit \(1\), allout these all units are both independent with \(k\), and their step ratio can be simply denoted as \(\lim_{k \to \infty} (k - 1)\), i.e. the ratio between summation boundary and the scaled variable itself. While if we keep the framwork be invariant, i.e. \(1' = 1\), then \(\partial_k (1 - \frac{1}{z}) \to \frac{z - 1}{z^2}\).

In conclusion, a direct result for the scaled \(f_k\) as defined in the beginning of this subsection is \(\partial_k \frac{z^k}{z - 1} = \partial_k \sum_{j=0}^{\infty} \frac{1}{j!} = z^k - \frac{z^{k+1}}{z - 1} = \frac{z^k}{z - 1} + \frac{1}{z^2} + \cdots = 1 + \frac{1}{z^2} + (-1)\). Note that here the \(1\) and \((-1)\) in last line are not the units of the same representation. We will see that the three possible configurations correspond to, respectively, the \(k\)-dependence of the three terms with the above expanded \(k\)-scaled ratio. The first configuration corresponds to \(\partial_k (-1) = \frac{z - 1}{z^2} = 0\) while \(\partial_k \frac{1}{z^2} = \frac{1}{z^2} = 0\). This corresponds to the current density representation where \(z \to \infty\) and the unit quantity is \(z^{-k} \sim z^{-k} \to 0\)

The second configuration corresponds to \(\partial_k 1 = \frac{1}{z^2} = 0\) while \(\partial_k (-1) = \frac{1}{z^2} = 0\). This corresponds to the Majorana fermion representation where \(z \to 0^+\) and the unit quantity is \((-1) \sim z^k \to 0\). Specifically, the third configuration corresponds to the Bernoulli polynomials (as discuss below) where \(\ln(z) \sim \ln(-z) = (\ln(1 - \frac{1}{z}))\) with \(\ln(1 - \frac{1}{z}) \sim \ln 1 \to 0\), which means the imaginary part \(\ln(1) = \pi (2m + 1)\) and available the analytical continuation for complex argument larger than \(1\) (i.e. beyond convergence \(|z| = 1\)). This configuration consider the \(z\)-dependence in terms of the normalized Lebesgue measure. While both the frist and second configurations correspond to the above-mentioned second order cutoff at \(\partial_k\).

### E.4. Lebesgue measure

Next we focus on the independence between \(\binom{1}{z}^y\) and bounded summation result \(\frac{z}{z - 1} (1 - \frac{1}{z^2})\). This independence results in \(\partial_{\binom{1}{z}^y} \frac{z}{z - 1} (1 - \frac{1}{z^2}) = 0\). (229)

Note that, in the unbounded limit \((k \to \infty)\), the result \(\frac{z}{z - 1}\) in fact does not directly depends on \((1/z)^{\gamma}\) since they are indeed not the quantities of the same representation (the summation over \(\gamma\) can be treated as Fourier transformation here), thus the derivative in the left-hand-side of above equation requires normalized Lebesgue measure on the complex space, \(\mathcal{S} = \left\{ \binom{1}{z}^m \right\}_{m=0}^{\infty} \), which is made up with \(\gamma\) integer steps. The unbounded edge means the step \(\partial_{\gamma}\) is an invariant quantity, so the value of \(\frac{z}{z - 1}\) only depends on the variations of \((1/z)^{\gamma}\) after each move by the step \(\partial_{\gamma}\). We will see that this is also base one the gauge invariance, which defines a unit space made up of holomorphic functions, e.g. \(\lim_{k \to \infty} \frac{z}{z - 1}\). Thus the derivative \(\partial_{\binom{1}{z}^\gamma} \frac{z}{z - 1}\) can be written as

\[
\partial_{\binom{1}{z}^\gamma} \frac{z}{z - 1} = \frac{z}{z - 1} \partial_{\binom{1}{z}^\gamma} \ln \frac{z^k - 1}{z^k} = \frac{z}{z - 1} \frac{z^k}{z^k - 1} \partial_{\binom{1}{z}^\gamma} \frac{1}{z^k}.
\] (230)

Here \(\partial_{\binom{1}{z}^\gamma} 1 = 0\) under some limiting condition with \(1\) denotes the local unit, and for integer \(m < \gamma\), \(\partial_{\binom{1}{z}^\gamma} 1 = 0\). This result will be used below.

In the mean time, this derivative is equivalents to

\[
\partial_{\binom{1}{z}^\gamma} \frac{z}{z - 1} = \frac{z^\gamma}{\ln z} \partial_{\binom{1}{z}^\gamma} \frac{z}{z - 1} = \sum_{m=1}^{\gamma} \left( \frac{1}{z} \right)^m \partial_{\binom{1}{z}^\gamma} \frac{z}{z - 1},
\] (231)
where the last line is the radial derivative of holomorphic function $\frac{z}{z - 1}$, which overcomes the problem that $\frac{z}{z - 1}$ and $(\frac{1}{z})^\gamma$ are in the different spaces, and this summation over unit steps $\gamma$ can be replaced by a larger one $(\gamma + 1)$ whose steps $\delta(\gamma + 1)$ is no more an invariant quantity. In terms of steps indexed by real integer $m$, the $\gamma$ should no more be treated as a real integer any more when it anticipates the calculation with $\gamma$. Considering the complex argument of $\gamma$, only $|\gamma|$ is of the same representation with that in the configuration of $m$. Then the above summation over $m$ can be expanded as

$$\sum_{m=1}^\gamma \left(\frac{1}{z}\right)^m \partial_{\psi} \left[ e^{\sum_{\gamma=0}^\infty \left(\frac{1}{z}\right)^\gamma} \right]$$

$$= \left[ \frac{1}{z} \sum_{m=1}^\gamma \gamma! \left(\frac{1}{z}\right)^{\gamma - \gamma!} + \sum_{\gamma=0}^\infty \frac{1}{z^{\gamma+1}} \frac{-\gamma!}{\gamma!} \right] + \sum_{m=1}^\gamma \frac{1}{z} \ln^m z + \sum_{m=1}^\gamma \partial_{\psi} \left[ \frac{1}{z} \right] \sum_{\gamma=0}^\infty \left(\frac{1}{z}\right)^\gamma \right] \left(\frac{1}{z}\right)^{\gamma - \gamma!}$$

$$= \left[ \frac{1}{z} + \frac{1}{z} \ln^2 z + \sum_{\gamma=0}^\infty \frac{1}{z^{\gamma+1}} \frac{-\gamma!}{\gamma!} \right] + \sum_{m=1}^\gamma \partial_{\psi} \left[ \frac{1}{z} \right] \sum_{\gamma=0}^\infty \left(\frac{1}{z}\right)^\gamma \right] \left(\frac{1}{z}\right)^{\gamma - \gamma!}. \quad (232)$$

Note that the boundary of summation over $m$ here, $\gamma$, is not in the same representation with the integers $m = 1, \ldots, |\gamma|$. The factor $(\frac{1}{z})^{\gamma - \gamma!}$ is independent with both the $(\frac{1}{z})^m$ ($m = 1, \ldots, \gamma$) and $\gamma$ (or factor $(\frac{1}{z})^\gamma$), e.g. for $m = \gamma$, we have

$$\frac{\partial \left(\frac{1}{z}\right)^{\gamma - \gamma!}}{\partial \left(\frac{1}{z}\right)^{\gamma!}} = \frac{\partial \left(\frac{1}{z}\right)^{\gamma - \gamma!}}{\partial \left(\frac{1}{z}\right)^{\gamma!}} = 0, \quad (233)$$

which is due to the fact that $\frac{\partial \gamma!}{\partial \gamma} = 1$ but $\frac{\gamma!}{\gamma!} = 1$ (here $\frac{\partial \gamma}{\partial \gamma} = 1$ is a limiting result of $\frac{\gamma}{\gamma!}$), and thus we have

$$\frac{\partial \left(\frac{1}{z}\right)^{\gamma - \gamma!}}{\partial \left(\frac{1}{z}\right)^{\gamma!}} = -\left(\frac{1}{z}\right)^{\gamma - \gamma!} \frac{\partial \gamma}{\partial \gamma} = -\left(\frac{1}{z}\right)^{\gamma - \gamma!}. \quad (234)$$

Its independence with $(\frac{1}{z})^{\gamma - \gamma!}$ can also be understood by substituting the above result (equation (234)) into $\frac{\partial \left(\frac{1}{z}\right)^{\gamma - \gamma!}}{\partial \left(\frac{1}{z}\right)^{\gamma!}}$, which turns out to be zero. This $\gamma$-independent term has the following relation

$$\left(\frac{1}{z}\right)^{\gamma - \gamma!} = \frac{\gamma}{\gamma} - \frac{1}{\gamma!} \frac{1}{\gamma!} = \frac{\gamma}{1} - \frac{1}{\gamma!} \frac{1}{\gamma!}, \quad (235)$$

which means it is in fact a limiting result of the ratio $\frac{\gamma}{\gamma!}$

$$\left(\frac{1}{z}\right)^{\gamma - \gamma!} = \lim_{\gamma \to \infty} \frac{\gamma}{\gamma!}, \quad (236)$$

and we also have the derivative $\frac{\partial \gamma}{\partial \gamma} = \gamma \ln \frac{1}{z}$.

A Fourier transformation under limiting condition $\gamma \to - |\gamma|$ results in an essential relation which connecting different representations $\frac{1}{z}$ $= \partial_{\psi} \gamma = -\partial_{\psi}$, and this is similar to the relation $\lim_{\gamma \to \infty} = \partial_\psi$ longwavelength limit. The unit quantity here is obviously not the ones in both the representations of $\gamma$ and $|\gamma|$, and we have $\frac{\partial \gamma}{\partial \gamma} = 1$ $= -\frac{\partial \gamma}{\partial \gamma} = -\frac{\partial \gamma}{\partial \gamma} = 1$.

In this new representation, we have $\left(\frac{1}{z}\right)^{\gamma - \gamma!} = \frac{\partial \gamma}{\partial \gamma} \frac{1}{z}$, thus the derivative $\partial_{\psi} (\frac{1}{z})^\gamma \gamma - \gamma!$ can be rewritten as

$$\partial_{\psi} (\frac{1}{z})^\gamma \frac{z}{z - 1} = \partial_{\psi} \left(\frac{1}{z}\right)^\gamma \sum_{\gamma=0}^\infty \left(\frac{1}{z}\right)^\gamma \frac{1}{z - 1}. \quad (237)$$

Defining the function of $\gamma$ as $f_\gamma = \sum_{\gamma=0}^\infty \left(\frac{1}{z}\right)^\gamma$, we can obtain

$$\partial_{\psi} (\ln f_\gamma - \ln \frac{1}{\gamma}) = 0,$$

$$\partial_{\psi} (\ln f_\gamma - \ln \frac{1}{\gamma}) = 0,$$

(238)
and the function $f_i$ can indeed be represented as
$$f_i = -\gamma \partial \gamma f_i = -|\gamma| \partial |\gamma| f_i.$$  \hfill (239)

Similarly, the limiting condition $\gamma \rightarrow -|\gamma|$ corresponds to another limiting condition $z \rightarrow 0^+$, as can be shown by the derivative of polylogarithm
$$\partial z \Li_{-1}(\frac{1}{z}) = z \Li_{-1}(\frac{1}{z}) = -z \Li_{-1}(\frac{1}{z}),$$  \hfill (240)

where the last line is due to $\lim_{z \rightarrow 0^+} \Li_{-1}(\frac{1}{z}) = \lim_{z \rightarrow 0^+} z + 1 = -1$.

Thus in this case, $(\frac{1}{z})^{-1|\gamma|} = \frac{1}{\ln(\gamma)} = \frac{1}{\ln(-\gamma)} = \frac{1}{\ln|\gamma|} \rightarrow \infty$, and the ratio between two representations can be rewritten as
$$\frac{\gamma}{|\gamma|} = \frac{\gamma - |\gamma|}{\ln(-\gamma) - \ln|\gamma|}(1 + \gamma \ln \frac{|\gamma|}{\gamma}),$$  \hfill (241)

and $f_i$ can be related to the polylogarithm function by
$$\frac{\partial f_i}{\partial \Li_{-1}(\frac{1}{z})} = \frac{f_i}{\Li_{-1}(\frac{1}{z})} \zeta \frac{\partial}{\partial z}$$  \hfill (242)

The relation between $f_i$ (in representation of $\gamma$ whose limiting condition $|\gamma| \rightarrow -\gamma$ corresponds to invariant quantity $\frac{1}{z} \rightarrow \infty$) and the polylogarithm functions (in representation of $\frac{1}{z}$ whose limiting condition is $|\gamma| \gg \gamma$ and results in $|\gamma|$-independent factor $(\frac{1}{z})^{-1|\gamma|}$), can be further understood by considering the following expressions in the representation of $\gamma$ (any terms with summation boundary depends on $\gamma$ are of this representation),

$$\partial |\gamma| \frac{\gamma - |\gamma|}{|\gamma|^2} = \frac{|\gamma| - \gamma}{|\gamma|^2},$$  \hfill (243)

which is consistent with the limiting result $\partial |\gamma| = -|\gamma|^{-1}$ as well as the relation $\left[ \left( \frac{1}{z} \right)^{-1|\gamma|} \right] = 0$ which is valid in both two representations. Base on the result $\partial |\gamma| |\gamma| = -1$, we have another expression,

$$\partial |\gamma| \Li_{0}(e^{-|\gamma|}) = -\Li_{-1}(e^{-|\gamma|}) = -e^{-|\gamma|} |\gamma| \Li_{0}(e^{-|\gamma|}),$$  \hfill (244)

thus we can obtain that the condition $|\gamma| \rightarrow -\gamma$ in representation of $\gamma$ also equivalents to the condition $z \sim \ln(-e^{i|\gamma|}) \sim i\pi (2m + 1) + |\gamma| \rightarrow 0^+$.

By rewriting equation (243) as $\partial |\gamma| \left[ \frac{\gamma - |\gamma|}{|\gamma|^2} \right)$, we can obtain another $\gamma$-independent ratio $\frac{\gamma - |\gamma|}{|\gamma|^2}$, which reflects a direct result of taking the limit $|\gamma| \rightarrow -\gamma$, i.e. $\partial |\gamma| = -1/|\gamma|$. Next we prove this in terms of polylogarithm function.

E.5. Invariant unit quantities in two representations

We can notice a fact which is a result of local invariance: the limiting result of the factors $\frac{\gamma}{|\gamma|}$ and $\frac{\gamma - |\gamma|}{|\gamma|^2}$ are in different representation, however, both can be related by the above-mentioned $\gamma$-independent ratio factor $\frac{\gamma - |\gamma|}{|\gamma|^2}$, then we have

$$\lim_{|\gamma| \rightarrow -\infty} \frac{\gamma - |\gamma|}{|\gamma|^2} = \left( \lim_{|\gamma| \rightarrow -\infty} \frac{\gamma}{|\gamma|} \right) \frac{\gamma - |\gamma|}{\gamma}.$$  \hfill (245)

The reason here is that although the ratio factor (just like the above $\frac{\gamma}{|\gamma|}$ in current density representation) here is invariant under the limiting transition of $|\gamma|$, it plays the role of exponential factor connecting different representations, and thus change the limiting result of $\frac{\gamma}{|\gamma|}$.

For the first factor $\frac{\gamma}{|\gamma|}$, the $\gamma$-independent unit reads

$$1 = \lim_{|\gamma| \rightarrow -\infty} \frac{\gamma}{|\gamma|} = \frac{1}{1 - \partial |\gamma|} \gamma,$$

$$= \lim_{|\gamma| \rightarrow -\infty} \sum_{m=0}^{\infty} \left[ \partial |\gamma| \frac{\gamma}{|\gamma|} \right]^m,$$  \hfill (246)
with \( \partial\gamma_{\gamma} \frac{\gamma}{|\gamma|} \) = \( \gamma \ln \frac{1}{z} \), while for the second factor \( \frac{-\ln |\gamma|}{|\gamma|} \), the \( \gamma \)-independent unit reads

\[
\begin{align*}
Y' &= \lim_{|\gamma| \to \infty} \frac{-\ln |\gamma|}{|\gamma|} = \frac{1}{1 - \partial\gamma \frac{\gamma}{|\gamma|}} \\
&= \lim_{|\gamma| \to \infty} \sum_{m=0}^{\infty} \partial\gamma \frac{\gamma}{|\gamma|} \frac{m}{|\gamma|}^{1/m} \tag{247}
\end{align*}
\]

with \( \partial\gamma \frac{\gamma}{|\gamma|} = (\gamma - |\gamma|) \ln \frac{1}{z} \). Just like the relation between \( \gamma \) and \( |\gamma| \), we have \( 1 = Y' \), but \( \partial\gamma |\gamma| = \partial\gamma Y' \).

That means, for these two factors, \( \frac{\gamma}{|\gamma|} \) and \( \frac{-\ln |\gamma|}{|\gamma|} \), the unit quantities are different due to the different extent of limiting transform of \( |\gamma| \), and both of them are larger than the unit quantity in representation of integer-\( m \), which is \( \frac{1}{|\gamma|} = 1 \). As a result the difference between the limiting form of these two factors reads

\[
\lim_{|\gamma| \to \infty} (-1) = \frac{-1}{1 - \partial\gamma \frac{\gamma}{|\gamma|}} = \frac{-1}{1 + |\gamma| \ln \frac{1}{z}}.
\]

By substituting the above results, we can further obtain the following expressions

\[
\begin{align*}
\frac{1}{Y'} &= \frac{1 - 1^{1}}{1 - 1^{1}} = \ln \left( \frac{1}{z} \right)^{1/|\gamma|}, \\
\frac{\partial\gamma \frac{\gamma}{|\gamma|}}{\partial\gamma |\gamma|} &= \gamma \left( \frac{1}{|\gamma|} \right).
\end{align*}
\]

By defining the factor

\[
\mathcal{X} := \frac{1 - \partial\gamma \frac{\gamma}{|\gamma|}}{\partial\gamma |\gamma|} = \frac{1}{Y'}, \tag{249}
\]

the second line of equation (248) can be rewritten as

\[
\frac{\partial - \text{Li}_0(\mathcal{X})}{\partial \text{Li}_0(\mathcal{X})} = \frac{\partial |\gamma| \mathcal{X}}{\partial |\gamma| |\gamma|}. \tag{250}
\]

In terms of Bernoulli polynomials, the factor in denominator can be expressed as

\[
\text{Li}_0\left( \frac{1}{\mathcal{X}} \right) = e^{\ln(-1) + \text{Li}_0(\mathcal{X})} = -\text{Li}_0(\mathcal{X}) - B_0 \left( \frac{1}{2} \right) \left( 1 - \frac{\ln \frac{1}{z}}{\ln \frac{1}{z}} \right), \tag{251}
\]

where the Bernoulli polynomials \( B_0 \) as a function of \( \frac{1}{2} \left( 1 - \frac{\ln \frac{1}{z}}{\ln \frac{1}{z}} \right) \) requires the parameter \( \mathcal{X} \) to satisfy

\(-\ln(-1) = \ln \frac{1}{z} \). We will see that this restriction indeed corresponds to, in representation of this Bernoulli polynomial, \( \ln \frac{1}{z} \), or equivalently, \( 2 \ln(-1) \to 0 = \partial\gamma |\gamma| = \partial\gamma Y' \), where zero here denotes a quantity shared by the two factors in \( \gamma \)-representation. The expression within the bracket of Bernoulli polynomial also implies that, this restriction also results in \( 1 = -\frac{1}{|\gamma|} = -1 \), which meets the above-mentioned condition \( \gamma = -|\gamma| \) as a special case of \( \gamma = \infty \), and also, now we have \( \frac{1}{z} \to 0 \) i.e. treating \( \frac{1}{z} = \text{Exp}[\partial\gamma Y'] \) as the unit quantity which cannot be further departed (or derivated), and in the mean time \( |\gamma| \to \frac{1}{z} \).

We will see that, in this case, the above-obtained result \( \frac{\partial |\gamma|}{\partial |\gamma| Y'} = -1 = \gamma \sqrt{|\gamma|} \) can be verified once more by the result \( \frac{\partial |\gamma|}{\partial |\gamma| Y'} = 1 \) which can be obtained by using the following limiting property of polylogarithm function,

\[
\lim_{|\gamma| \to 0} \frac{\text{Li}_0(Y)}{Y} = 1, \tag{252}
\]

where we define \( |Y| := \frac{\ln \frac{1}{z}}{\ln \frac{1}{z}} = \frac{Y}{\text{Exp}[\partial\gamma Y']} \), and since the left-hand-side of this equation can be replaced by \( \partial |Y| \text{Li}_0(Y) = 1 - \lim_{|Y| \to 0} \text{Li}_0(Y) \), we can have polylogarithm function \( \text{Li}_0(Y) = \frac{1}{m} = \ln(-1)^{-1} \) by choosing a proper order \( s' \), then \( |Y| \) can be departed as

\[
\lim_{|Y| \to 0} \frac{1}{Y} \left( -\ln(-1) - \ln \mathcal{X} \right) = \frac{\text{Li}_0(Y)}{\lim_{Y \to \infty} \text{Li}_0(Y)} = \frac{Y}{e^{\text{Exp}[\partial\gamma Y']}} = 1 - \partial |Y| \text{Li}_0(Y) = 0. \tag{253}
\]
Thus in this limit, the Bernoulli polynomial in equation (251) reduces to $B_0\left(\frac{1}{2}\right)$ where $\frac{1}{2}$ is the above-mentioned limiting result, as a ratio between $\ln(-1) (\gamma = -\frac{1}{2})$ is the negative unit in representation of $m$ and $1 = 0^+$ (i.e. zero in representation of $\gamma$, whose unit quantity are defined according to the independence with the variable $\gamma$),

\[
\frac{1}{2} = \frac{\ln(-1)}{\ln(-1)^2} = \frac{\ln(-1)}{\ln(-1)^2}.
\]

(254)

Here we present some relations derived from the results $\frac{\partial}{\partial |\gamma|} = 1 = \frac{1}{\ln|\gamma|}$, and $\frac{\partial|\gamma|}{\partial |\gamma|} = -1$. Firstly note that the representation of Bernoulli polynomial requires $\gamma = \frac{1}{2}$ instead of $-\frac{1}{2}$, then the identity $\frac{\partial|\gamma|}{\partial |\gamma|} = 1$ can be written in another form,

\[
\frac{\partial|\gamma|}{\partial |\gamma|} = 1 + \frac{|\gamma|}{|\gamma|} = 1 = \frac{1}{\ln|\gamma|} = 1
\]

(255)

where the $\gamma$-dependent identities follows $\frac{\partial|\gamma|}{\partial |\gamma|} = \frac{\ln(-1)}{\ln(-1)} = \frac{1}{\ln|\gamma|} = 1$, and the term $\frac{1}{\ln|\gamma|} = (-1)^{-1}$ satisfies

\[
|\gamma|\frac{\partial|\gamma|}{\partial |\gamma|} = \ln(-1) = (-1)^{-1} = 0,
\]

(256)

where $(-1)^{-1} = -\frac{\ln(-1)}{\ln(-1)}$, and thus $(-1)^{-1} = -\ln(-1) = -\ln(-1)$. Then it is easy to check that

\[
\lim_{|\gamma| \to \infty} |\gamma| = \frac{1}{2}, \text{ with } |\gamma|\frac{\partial|\gamma|}{\partial |\gamma|} = \frac{1}{2}.
\]

As we can see, in this representation, all quantities are defined according to their dependence between each other. We again by treated $\frac{1}{2}$ here as a polynomial function at a certain order $a$, then we have $\frac{1}{2} = \text{Li}_a(|\gamma|) = \frac{|\gamma|\partial|\gamma|}{\partial |\gamma|} = \text{Li}_a(|\gamma|)$, and obviously this can only happen in the limit $|\gamma| \to 0$ (and infinitely large order $a$). It can be verified that, in this limit, the $\gamma$-dependence of the factor $\frac{1}{\ln|\gamma|} = 2$ disappears as a result of $\lim_{|\gamma| \to \infty} |\gamma| = \frac{1}{2}$, the $\gamma$-dependence of the factor $\frac{1}{\ln|\gamma|} = 2$

Thus the restriction Bernoulli polynomial in terms of $-\ln(-X) \equiv \ln(-X)$ instead reflect another limiting transform $\lim_{|\gamma| \to \infty} |\gamma| = |\gamma|\frac{\partial|\gamma|}{\partial |\gamma|} = |\gamma|\frac{\ln(-1)}{\ln(-1)}$, which does not contrast with $\lim_{|\gamma| \to \infty} |\gamma|\frac{\partial|\gamma|}{\partial |\gamma|} = |\gamma|=1$, but just corresponds to another unit quantity, and we can use a simple expression to generalize the key feature in this representation, which is $-1 = \frac{\partial|\gamma|}{\partial |\gamma|} = |\gamma|\frac{\partial|\gamma|}{\partial |\gamma|}$.

For example, this limiting transform can be verified by substituting

\[
\frac{\partial|\gamma|}{\partial |\gamma|} = \frac{1}{\ln|\gamma|} = (-1)^{-1} = (-1)^{-1} = 0,
\]

(257)

into the derivative $\frac{\partial|\gamma|}{\partial |\gamma|} = \frac{1}{\ln|\gamma|}$, which still turns out to be $\frac{1}{\ln|\gamma|}$. Note that here we still use the notion $1$ to denotes the unit quantity ($\gamma$-independent) in this configuration, which satisfies $\gamma^{-1} = |\gamma|^{-1}$ and thus

\[
\gamma = |\gamma|\frac{\partial|\gamma|}{\partial |\gamma|},
\]

\[
\gamma + 1 = \frac{1}{1} = \ln|\gamma|\frac{\partial|\gamma|}{\partial |\gamma|} = \gamma\frac{\partial|\gamma|}{\partial |\gamma|} = |\gamma|^{-1},
\]

(258)

In this configuration, the relation between $\gamma$ and $|\gamma|$ can be described by the Lambert $\mathcal{W}$-function. We treating $\gamma$ as a complex quantity $|\gamma| = \mathcal{W}(\gamma)$ $e^{\mathcal{W}(\gamma)}$ where $\gamma = e^{-i\Omega \gamma}$ $\to e^{\gamma}$, and the $\gamma$ is defined as a Lambert $\mathcal{W}$-function of $|\gamma|$, which satisfies

\[
\mathcal{W}(\gamma) := \gamma = \ln(-1) - \ln|\gamma| = \ln|\gamma| - \ln = -\ln|\gamma| - 1,
\]

(259)

and $\ln \gamma = \frac{1}{\gamma} - \gamma$. Relating the formulas of Lambert function, we have more another expression of $\ln(-1)$,

\[
\ln(-1) = \ln|\gamma| - \ln|\gamma| + \sum_{m=1}^{\infty} \left(\prod_{k=1}^{m} \frac{(-\ln|\gamma|)^k}{k!} \right) S[m, m - k + 1] \left(\frac{1}{\ln|\gamma|}\right)^m,
\]

(260)

where $S[m, m - k + 1]$ is a stirling number. We will see that, the transition $\frac{\gamma}{\gamma} \to \ln|\gamma|$ is equivalents to

\[
-\ln \gamma \to \gamma, \text{ which is consistent with the identity of Lambert function } e^{\mathcal{W}(\gamma)} = \frac{1}{\mathcal{W}(\gamma)} = \frac{\gamma}{\gamma}, \text{ and the term } -\ln|\gamma| = \ln(-1) - \gamma \text{ can not be soly identified. By virtue of Lambert function, we have }
\]

\[
\ln = \ln|\gamma| + \ln|\gamma| + 1 = -e^{\gamma} - \ln|-\gamma| \text{ equals to the infinite power tower of } |\gamma|. \text{ Then the derivative }
\]

\[
\frac{\partial}{\partial |\gamma|} = \frac{-\gamma}{|\gamma|} \text{ is equivalents to}
\]

\[
\frac{\partial}{\partial |\gamma|} = \frac{-\gamma}{|\gamma|} \text{ is equivalents to}
\]
\[
\frac{\partial \gamma}{\partial |\gamma|} = \ln |\gamma| = \gamma + \ln \gamma = \frac{1}{e^{\gamma} (1 + \gamma)}. \quad (261)
\]

According to the integer formula of Lambert function
\[
\int \gamma d|\gamma| = |\gamma| (\gamma - 1 + \frac{1}{\gamma}) + \text{cont.}, \quad (262)
\]
which can be verified by the derivative
\[
\frac{\partial |\gamma| (\gamma - 1 + \frac{1}{\gamma})}{\partial |\gamma|} = |\gamma| \partial |\gamma| \ln(1 - \ln |\gamma|) = \gamma. \quad (263)
\]
This integer indeed corresponds to the limiting result of $|\gamma| \to 0$, where we have
\[
\lim_{|\gamma| \to 0} \frac{\ln |\gamma|}{|\gamma|} \to 1, \quad (264)
\]
which reveals the Lambert functions here are not the principal branch. This limiting result indeed equivalents to the integer formula
\[
\lim_{|\gamma| \to 0} \int_{-\gamma}^{-\frac{\gamma}{\gamma}} \frac{\gamma}{\gamma} = \lim_{|\gamma| \to 0} \left[ -\gamma - \frac{\gamma^2}{2} - \cdots \right] = \int \ln |d|\gamma| = 1, \quad (265)
\]
as $\gamma \to -\infty$ as $|\gamma| \to 0$. So we can see that this process is to remove the dependence on $|\gamma|$ for the unit $1 = \lim_{|\gamma| \to -\infty} |\gamma|$. This can be clarified by writing the integer about $\ln |\gamma|$ as a polylogarithm function of $\gamma$, $\text{Li}_{s-1}(\gamma)$, then $\partial_s \text{Li}_{s+1}(\gamma) = \ln |\gamma| \partial_s \text{Li}_{s} + f(\gamma) = \partial_s |\gamma| \text{Li}_{s+1}(\gamma)$. Relating this to equation (262), we obtain
\[
\lim_{|\gamma| \to 0} (\gamma + \frac{1}{\gamma}) = -\gamma, \quad \text{in which case } \partial |\gamma| \gamma - (1 - \frac{1}{\gamma}) = 1 - \frac{1}{\gamma},
\]
then using the limit
\[
\lim_{\gamma \to \infty} \left(1 - \frac{1}{\gamma}\right)^{-\gamma - 1} = e, \quad (266)
\]
we have
\[
\frac{\partial (-\gamma - 1)}{\partial (-\gamma)} \ln(1 - \frac{1}{\gamma}) = \frac{\partial |\gamma|}{\partial |\gamma|} = e^{\gamma}, \quad (267)
\]
where
\[
\frac{\partial (-\gamma - 1)}{\partial (1 - \frac{1}{\gamma})} = \frac{\partial (-\gamma)}{\partial (1 - \frac{1}{\gamma})} + \frac{\partial (-1)}{\partial (1 - \frac{1}{\gamma})} = -\gamma - \left(1 - \frac{1}{\gamma}\right)^{\gamma - 1}. \quad (268)
\]
Thus we can also obtain
\[
\frac{\partial (-1)}{\partial |\gamma|} = [(-\gamma - 1 - \left(1 - \frac{1}{\gamma}\right)^{\gamma - 1}) - (-\gamma - 1 - \left(1 - \frac{1}{\gamma}\right)^{\gamma - 1})][e^{-\gamma} - \frac{1}{|\gamma|} (1 - \frac{1}{\gamma})]. \quad (269)
\]
In the mean time, under representation of the Bernoulli polynomial, this limiting result has another identity,
\[
\frac{1}{2} = \lim_{|\gamma| \to 0} Y \partial_Y Y|Y|, \quad (270)
\]
and this is connected to the another familiar relation $\frac{\partial m}{\partial |\gamma|} = \frac{1}{|\gamma|} = 1$ (as the effect of gauge-invariance can be seen everywhere) by a similar relation
\[
\frac{\partial Y}{\partial |Y|} = \frac{1}{Y} \frac{1}{Y} + Y \partial_Y Y|Y| = \frac{1}{1 + \frac{1}{Y} \frac{1}{Y}}. \quad (271)
\]
Instituting $|Y| = e^{\text{Arg} Y}$, we can further obtain the relation (still using the form we often used in above)
\[
\frac{\partial_Y \ln \frac{Y}{|Y|}}{|Y|} = 1 - e^{-i \text{Arg} Y} = 1 - \frac{\ln |Y|}{\lim_{|Y| \to \infty} \ln |Y|} = \frac{Y}{|Y|} \left[1 - \frac{\ln |Y|}{\lim_{|Y| \to \infty} \ln |Y|}\right], \quad (272)
\]
where the limiting term as well as the derivative term in the last line can be obtained as

\[
\lim_{Y \to 0} \frac{Y}{|Y|} = \frac{y}{|y|} = \frac{y}{1 - \partial_{|y|} y} = \frac{y}{1 - (1/y^2 - |y|/|y|^2)}. \tag{273}
\]

Instituting the result \(\ln(-1) = \frac{1}{2} 0^+\) into the above limit expression \((|Y| \to 0)\ln \frac{1}{|X|} = -\ln(-X) = 0\) (and thus \(\ln(\frac{1}{X}) = 0^+)\), we have \(\ln(1) + \ln X = \frac{1}{2} 0^+ + \ln X = 0^+\), thus we can know \(\ln \frac{1}{X} = \frac{3}{2} 0^-\). This can be explained by the identity of \(X = \frac{i}{\ln(\frac{1}{|y|})} = \frac{i}{|y|\ln |y|}\) which is of the representation of \(\gamma\), where \(1(\frac{1}{|y|})\) should not be treated as an invariant unit. As a result, the actual value of \(\ln X\) cannot be uniquely identified, for example, \(\ln X \cdot \infty = X = 2 \ln(-1) + \frac{3}{2} 0^- = \frac{3}{2} 0^-\). We can comparing the equation (273) and the result \(\ln \frac{1}{X} = \frac{3}{2}\) with the equation (207) and equation (209), respectively, to see this connection.

Thus the expression with the bracket of Bernoulli polynomial can be explained in a more clear way in terms of the function 

\[
B_0 \left[ \frac{1}{2} \left( 1 - \frac{\ln(\frac{1}{|y|})}{|y|} \right) \right] = B_0 \left[ \frac{1}{2} \left( 1 - \ln(\frac{1}{|y|}) \right) \right] = B_0 \left[ f_Y(1 - \partial_{|y|} y f_Y) \right], \tag{274}
\]

where \(\lim_{|Y| \to 0}\) is equivalent to \(\lim_{\ln(-1)\lim_{|Y| \to 0} \ln \frac{1}{|Y|} \to \infty}\). Then as a special case of equation (251) at \(X = -\gamma \frac{1}{|y|} = -1\), we have

\[
-\ln |y| = -B_0 \left( \frac{1}{2} \right) = -\ln |y| - 1 = -e^{-\ln(|y|)} = -\frac{1}{2}, \tag{275}
\]

which also reveals the correctness of equation (251).

In this case, \(\frac{1}{z} = -\ln |y|\) is equivalent to the double partial derivatives \(\partial^{(2)}_{|y|} |y|\), and the limiting result

\[
\ln \left( \frac{1}{|y|} \right) \to 0^+\]

can be explained by the operator transform \(\ln \left( \frac{1}{|y|} \right) = \partial^{(2)}_{|y|} |y| = \partial^{(2)}_{|y|} (2m + 1) = \lim 2|\gamma| \to 0^+\). Thus the unit quantity in \(m\)-representation can be expressed as \(\gamma|\gamma|\). Also, the two representations in the limiting case can be related by the above-mentioned polylogarithm functions by constructing the Kummer’s function in proper form,

\[
\text{Li}_1 \left( \frac{1}{z} \right) = \text{Li}_1(e^p) \sum_{m=0}^{p-1} \text{Li}_1(e^{1+\gamma m}) \tag{276}
\]

where we define the limiting form of \(\ln \frac{1}{z}\) as

\[
P = \lim_{|y| \to \gamma} \ln \frac{1}{z} = \partial^{(2)}_{|y|} |y| = -\ln |y| \tag{277}
\]
[15] Davison R A et al 2017 Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography Phys. Rev. B 95 155131
[16] Lian B, Sondhi S L and Yang Z 2019 The chiral SYK model J. High Energy Phys. 2019 1–55
[17] Gor’kov T 2015 Eilenberger, and Ginzburg-Landau Equations. Statistical Mechanics of Superconductivity (Tokyo: Springer) pp 201–27
[18] Zinn-Justin J 1981 Perturbation series at large orders in quantum mechanics and field theories: application to the problem of resummation Phys. Rep. 70 109
[19] Friston K 2002 Functional integration and inference in the brain Prog. Neurobiol. 68 113–43
[20] Protopopov I V, Wen W H and Abanin D A 2017 Effect of SU(2) symmetry on many-body localization and thermalization Phys. Rev. B 96 041122
[21] Zinn-Justin J 1981 Perturbation series at large orders in quantum mechanics and field theories: application to the problem of resummation Phys. Rep. 70 109–67
[22] Berkoz M et al 2017 Higher dimensional generalizations of the SYK model J. High Energy Phys. 2017 1–24
[23] Hu Y and Biao L 2022 Chiral Sachdev-Ye model: Integrability and chaos of anyons in 1+ 1 d Phys. Rev. B 105 125109
[24] Narayan P and Junggi Y 2017 SYK-like tensor models on the lattice J. High Energy Phys. 2017 1–46
[25] Allen S and Tremblay A M S 2001 Nonperturbative approach to the attractive Hubbard model Phys. Rev. B 64 075115
[26] Fu W et al 2017 Supersymmetric sachdev-ye-kitaev models Phys. Rev. D 95 026009
[27] Pöyhönen K, Moghaddam A G and Ojanen T 2022 Many-body entanglement and topology from uncertainties and measurement-induced modes Physical Review Research 4 023200
[28] Li X, Li X and Das Sarma S 2017 Mobility edges in one-dimensional bichromatic incommensurate potentials Phys. Rev. B 96 085119
[29] Xia X et al 2022 Exact mobility edges in the non-Hermitian t1 -t2 model: Theory and possible experimental realizations Phys. Rev. B 105 014207
[30] You Y-Z, Andreas W W L and Cenke X 2017 Sachdev-Ye-Kitaev model and thermalization on the boundary of many-body localized fermionic symmetry-protected topological states Phys. Rev. B 95 115130
[31] Fidkowski L and Kitaev A 2011 Topological phases of fermions in one dimension Phys. Rev. B 83 075103
[32] Marr P 2022 Majorana nanowires for topological quantum computation: A tutorial} arXiv:2206.14828
[33] Kim J et al 2019 Symmetry breaking in coupled SYK or tensor models Phys. Rev. X 9 021043
[34] Keating J P and Snaith N C 2000 Random matrix theory and (1/2 + it) Commun. Math. Phys. 214 57–89
[35] Stevic S 2005 On an integral operator on the unit ball in} Journal of Inequalities and Applications 2005 1–8
[36] Gu Y et al 2020 Notes on the complex Sachdev-Ye-Kitaev model J. High Energy Phys. 2020 1–74