ON HOMALOIDAL POLYNOMIAL FUNCTIONS OF DEGREE 3 AND PREHOMOGENEOUS VECTOR SPACES

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Abstract. In this paper we consider homaloidal polynomial functions $f$ such that their multiplicative Legendre transform $f_*$, defined as in [EKP02, Section 3.2], is again polynomial. Following Dolgachev [Dol00], we call such polynomials EKP-homaloidal. We prove that every EKP-homaloidal polynomial function of degree three is a relative invariant of a symmetric prehomogeneous vector space. This provides a complete proof of [EKP02, Theorem 3.10, p. 39]. With respect to the original argument of Etingof, Kazhdan and Polischuk our argument focuses more on prehomogeneous vector spaces and, in principle, it may suggest a way to attack the more general problem raised in [EKP02, Section 3.4] of classification of EKP-homaloidal polynomials of arbitrary degree.

Let $V$ be a complex vector space of dimension greater or equal to two and denote by $V^*$ its dual. If $f$ is a function on $V$ and $x \in V$, we denote by $f'(x) \in V^*$ the differential of $f$ at $x$. Let $f$ be a homogeneous function defined on $V$ such that the Hessian of $\ln f$ is generically invertible. There always exists a homogeneous function $f_*$ defined on $V^*$ satisfying

$$f_* \left( \frac{f'(x)}{f(x)} \right) = \frac{1}{f(x)}$$

on an open subset of $V$ ([Dol00, p. 193, Lemma 1]). Following [EKP02] we call $f_*$ the multiplicative Legendre transform of $f$. Assume further that $f$ is polynomial. It turns out that $f$ is homaloidal, i.e. its polar map $x \mapsto f'(x)$ induces a birational map $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V^*)$, if and only if $f_*$ is a rational function, see [EKP02, p. 37, Proposition 3.6] or [Dol00, p. 193, Theorem 2]. By [EKP02, Proposition 3.5], we have, under the canonical identification $V = V^{**}$:

$$f_{**} = f,$$

$$f_*' \circ f_*' \circ f_* = \text{id}_V.$$

Definition 1. Following [Dol00], if $f$ is a homogeneous polynomial function such that $\det \text{Hess}(\ln f) \neq 0$, we call it EKP-homaloidal if its multiplicative Legendre transform $f_*$ is again polynomial. In this case $f_*$ is a homogeneous polynomial function too and $\deg(f) = \deg(f_*)$.

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In particular every EKP-homaloidal polynomial function is homaloidal as it follows from the above discussion.

Notations. Let $f$ be an EKP-homaloidal polynomial function. We introduce the following notations

$$P(V) \supseteq V(f) = X \supseteq \text{Sing}(X) = Z$$

and

$$P(V^*) \supseteq V(f_*) = X_* \supseteq \text{Sing}(X_*) = Z_*$$

Where by $V(f)$ we denote the hypersurface defined by $f$ in the projective space. Moreover in what follows we are going to denote by $\phi_f$ and $\phi_{f_*}$ the polar maps associated to $f$ and $f_*$ respectively.

Whenever $W \subset P(V)$ is a projective variety, we denote by $W^\vee$ its dual variety, namely the closure of the set of hyperplanes which are tangent at a point in $W$. This is a subvariety of $P(V^*)$. We denote by $Q_f$ the polarization of $f$, that is the only three-linear symmetric form on $V$ such that $Q_f(A, A, A) = f(A)$, for every $A \in V$. Note in particular that

$$3Q_f(A, A, -) = f'(A) \in V^*$$

and moreover

$$\frac{\partial^2}{\partial A \partial B} f(x) = 6Q_f(A, B, x)$$

for every $A, B, x \in V$.

For every $A \in V$ such that $f(A) \neq 0$, denote by

$$\tau_{f, A} := d_A \left( \frac{f'}{f} \right) : V \rightarrow V^*$$

the linear map given by the differential at $A$ of the rational map $\frac{f'}{f} : V \dashrightarrow V^*$, or in other words the second logarithmic differential of $f$ computed at $A$.

We refer to [Kim03] for the definition of prehomogeneous vector space and to [Ber00] for the definition of prehomogeneous symmetric space.

Theorem 2. Let $f$ be an irreducible EKP-homaloidal polynomial function of degree three. Then $f$ is the relative polynomial invariant of an irreducible prehomogeneous symmetric space.

Recall that if $W \subset P(V)$ is a projective variety, its secant variety is by definition the closure of the union of all the secant lines in $P(V)$ passing through two distinct points in $W$. Recall also [Zak93, Chapter IV] the definition and classification of Severi varieties.

Remark 3. The following Corollary is stated in [EKP02], but its proof in the form provided there, does not seem to be complete. In particular in the proof of Proposition 3.16, last two lines of the proof, the argument based on the rank of the hessian matrix alone it is not enough to conclude in general that $Z$ is smooth.

Our proof of Corollary 4 follows a completely different approach than that in [EKP02]. We show how to put on $V$ a structure of prehomogeneous vector space in which $f$ is a polynomial invariant, this is the content of our
Theorem 2. As a consequence we will get Corollary 4. In our opinion this may be helpful in attacking the more general question raised in [EKP02, Section 3.4] of classification of EKP-homaloidal polynomial functions.

Corollary 4. [EKP02, Theorem 3.10] If \( f \) satisfies the hypothesis of the above Theorem then the singular locus of \( V(f) \) is one of the four classical Severi varieties and its secant variety is \( V(f) \).

Before proving Theorem 2 and then Corollary 4 we are going to establish a number of preliminary lemmas.

Lemma 5. If \( f \) is an irreducible EKP-homaloidal polynomial function of degree three then

\[
\phi_f(X \setminus Z) \subseteq Z_* \quad \phi_{f_*}(X_* \setminus Z_*) \subseteq Z
\]

in particular \( Z, Z_* \neq \emptyset \).

Proof. Since \( f_* \) resp. \( f'_* \) is homogeneous of degree 3 resp. 2, (3) amounts to the equality

\[
f'_*(f'(x)) = \frac{x}{f(x)}.
\]

By (1), \( f_*(f'(x)) = f(x)^2 \). Thus,

\[
f'_*(f'(x)) = f(x) \cdot x.
\]

Our claim follows immediately. \( \square \)

Remark 6. Under the hypotheses of Theorem 2, \( f_* \) is irreducible too. Indeed by definition of the Legendre transform, we have \( f_*(f'(x)) = f(x)^2 \) and in case \( f_* \) is a reducible polynomial the preceding equality implies that \( f \) is reducible too.

Lemma 7. The variety \( Z \), as a set, is defined by the equations:

\[
z \in Z \iff \forall A \in V, Q_f(A, z, z) = 0
\]

Proof. In fact, \( Z \) is by definition the singular locus of \( V(f) \), so that (recall (4)) \( z \) is in \( Z \) if and only if \( f'(z) = 0 \). \( \square \)

Lemma 8. For every \( A \in V \) such that \( f(A) \neq 0 \) the map \( \tau_{f,A} \) induces a linear isomorphism between \( \mathbb{P}(V) \) and \( \mathbb{P}(V^*) \) which maps \( Z \) into \( X^\vee \subseteq Z_* \).

An analogous statement holds true for \( f_* \).

Proof. Let \( \mathcal{V} \) be the open subset of \( V \) whose elements are vectors \( A \in V \) such that \( f(A) \neq 0 \) and \( Q(A, A, z) \neq 0 \) for at least one \( z \) in every irreducible component of \( Z \). We claim that \( \mathcal{V} \) is not empty. If it were so, let \( Z' \) be an irreducible component of \( Z \) such that \( Q(A, A, z') = 0 \) for every \( A \in V \) and every \( z' \in Z' \). As a consequence \( Q(A, B, z') = 0 \) for every \( A, B \in V \) and \( z' \in Z' \). By (5) this would imply that every point of \( Z' \) is a singular point of \( X \) of order three, and \( X \) would be a cone. But a cone is never homaloidal, a contradiction.

If \( A \in \mathcal{V} \) then by (3) \( \tau_{f,A} \) is a (linear) isomorphism whose inverse is given by \( \tau_{f_*,f'/f(A)} \) (note that \( f_*(f'/f(A)) = 1/f(A) \neq 0 \) by definition of \( f_* \)). We are going to prove that \( \tau_{f,A}(Z) \subseteq X^\vee \) now by giving a geometric
interpretation of the map $\tau_{f,A}$ on an open subset of $Z$. Fix a vector $A \in V$, and consider the subset of $Z$ given by

$$U_A := \{ z \in Z \mid Q_f(A, A, z) \neq 0 \} \subset Z .$$

Since $A \in V$, $U_A$ is a dense open subset of $Z$. By Lemma 7,

$$Q_f(z, z, A) = 0 \text{ for every } z \in Z \text{ and every } A \in V .$$

Consider the line $\bar{z}, A \subset \mathbb{P}(V)$ passing through $z \in U_A$ and the point in $\mathbb{P}(V)$ corresponding to $A$. Using (6), it is easy to check that the intersection of $\bar{z}, A \subset \mathbb{P}(V)$ with $X = V(f)$ consists of the point $z$ (with multiplicity two) and the simple point

$$z' := A - \frac{Q_f(A, A, A)}{3Q_f(z, A, A)} \cdot z = A - \frac{f(A)}{f'(A)(z)} \cdot z \in X \setminus Z .$$

The embedded tangent hyperplane $T_{z'}(X) \subset \mathbb{P}(V^*)$ of $X$ at $z'$ is given by the equation $f'(z') = 0$. By the definition of $z'$, (4) and (6), we are able to rewrite the equation of $T_{z'}(X)$ as

$$Q_f(A, A, -) - \frac{2f(A)}{f'(A)(z)} Q_f(z, A, -) = 0 .$$

On the other hand, by computing explicitly $d_A \left( \frac{f}{A} \right)(z)$ in terms of the polarization $Q_f$, we get that

$$\tau_{f,A}(z) = 3 \frac{2f(A)Q_f(z, A, -) - f'(A)(z)Q_f(A, A, -)}{f(A)^2} .$$

By comparing equations (7) and (8), we get that $\tau_{f,A}(z) = T_{z'}(X)$, which gives the required geometric interpretation of $\tau_{A,f}$ on the open subset $U_A \subset Z$. It follows that $\tau_{f,A}(U_A) \subset X^\vee$ and hence $\tau_{f,A}(Z) \subset X^\vee$. Since this last condition is a closed one it holds true for every $A \in V$ such that $f(A) \neq 0$.

**Definition 9.** Let $G \subset GL(V)$ be the subgroup preserving $X$:

$$G := \{ g \in GL(V) : g(X) = X \} .$$

**Corollary 10.**

(i) For each $A \in V \setminus \{ f = 0 \}$ we have that $\tau_{A,f}$ is a linear isomorphism that sends $Z$ onto $Z_*$ and $X$ onto $X_*$. (ii) $\phi_f(X \setminus Z) = Z_* = X^\vee$ and $\phi_{f_*}(X_* \setminus Z_*) = Z = (X_*)^\vee$. (iii) For every $z_1, z_2 \in Z$ there exist $A_1, A_2 \in V \setminus \{ f = 0 \}$ such that

$$\tau_{f,A_2}^{-1} \circ \tau_{f,A_1}(z_1) = z_2,$$

hence $G$ acts transitively on $Z$, and $Z, Z_*$ are smooth. (iv) The fibers of the Gauss map $\phi_f : X \to Z_*$ are all linear spaces of dimension $\dim X - \dim Z_*$ and $G$ acts transitively on them i.e. for every couple of distinct fibers of $\phi_f$, $F_1$ and $F_2$, there exists an element $g \in G$ such that $g(F_1) = F_2$.

**Proof.** We start by proving (i). By Lemma 8 if $A \in V$ and $A_* \in V^*$ are such that $f(A) \neq 0$ and $f_*(A_*) \neq 0$ then $\tau_{f,A}(Z) \subset X^\vee \subset Z_*$ and $\tau_{f_*,A_*}(Z_*) \subset X^\vee$.
If we choose \( A \in V \setminus \{f = 0\} \) and \( A_s = f'(A)/f(A) \), then by definition of \( f_s \) we have \( A_s \in V^* \setminus \{f_s = 0\} \) and by (3)
\[
\tau_{f,A} \circ \tau_{f^*,A^*} = \text{Id}_V
\]
and similarly \( \tau_{A,s} \circ \tau_{f,A} = \text{Id}_V \). This implies \( \tau_{f,A}(Z) = Z_s \) and \( X^\vee = Z_s \).
Moreover \((X_s)^\vee = Z \) in view of the symmetry (2). Recalling the geometric interpretation of the maps \( \tau_{f,A} \) given in the proof of Lemma 8 this completes the proof of item (ii) too.

We are going to prove (iii) now. Let \( z_1, z_2 \in Z \); by item (ii) there exist points \( p_1, p_2 \in X_s \setminus Z_s \) such that \( z_i = T_{p_i}(X_s) \). Given \( z \in X_s \) we denote by \( K_z X_s \subset \mathbb{P}(V^*) \) the tangent cone to \( X_s \) at \( z \), defined by the equivalence \( u \in K_z X_s \Leftrightarrow Q_{f_s}(z, u, u) = 0 \). Consider the following closed subsets of \( Z_s \),
\[
K_i = \{ z \in Z_s \mid z, p_i \subset K_z X_s \}
\]
for \( i = 1, 2 \). Observe that \( K_i \neq Z_s \). Indeed if it were so, the cone with vertex \( p_i \) and base \( Z_s \) would be contained in \( T_{p_i}(X_s) \) and \( Z_s \) would be degenerate. Then \( X \) would be a cone, but a cone is never homaloidal. Put \( U_i = Z_s \setminus K_i \), it follows that there are points \( \xi \in U_1 \cap U_2 \neq \emptyset \) and \( B_i \in \mathbb{P}(V^*) \setminus X_s \) such that by the geometric interpretation of the maps \( \tau_{f_s,-} \) given in the proof of Lemma 8 we have
\[
\tau_{f_s,B_i}(\xi) = z_i
\]
and hence
\[
\tau_{f_s,B_2} \circ \tau_{f_s,B_1}^{-1}(z_1) = z_2
\]
or in other words
\[
\tau_{f_s,A_2}^{-1} \circ \tau_{f_s,A_1}(z_1) = z_2
\]
where \( A_1 = (f'_s/f_s)(B_1) \) and \( A_2 = (f'_s/f_s)(B_2) \). It follows that \( G \) acts transitively on \( Z \) and \( Z_s \) is smooth. By (iii) this implies that \( Z_s \) is smooth too.

Concerning item (iv), by the duality theorem, a general fiber of \( \phi_f \) is a linear space of dimension \( \dim X - \dim X^\vee \). Since \( X^\vee = Z_s \) is homogeneous under the action of \( G \), this is true for any fiber.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** We may now argue as in [Cha03] to prove that \( G \) acts transitively on \( V \setminus \{f = 0\} \) (see [Cha03, Lemma 3.1]). Fix an element \( I \in V \) such that \( f(I) \neq 0 \). For any \( A \in V \) such that \( f(A) \neq 0 \), consider the map \( (\tau_{f,I})^{-1} \circ \tau_{f,A} \in GL(V) \). By Lemma 8, we get that \( (\tau_{f,I})^{-1} \circ \tau_{f,A} \in G \). Therefore the \( G \)-orbit of \( I \) contains the image of the map
\[
\varphi_I : V \setminus \{f = 0\} \longrightarrow V
A \mapsto (\tau_{f,I})^{-1}(\tau_{f,A}(I)).
\]
We are going now to compute the differential of \( \varphi_I \). First of all, observe that the map \( (\tau_{f,I})^{-1}(-) \) is linear, hence we only need to compute \( d_I(A \mapsto \tau_{f,A}(I)) \). By (8) and (3), a straightforward computation shows that
\[
\tau_{f,I+tB}(I) - \tau_{f,I}(I) = -2\tau_{f,I}(B)t + o(t).
\]
Moreover observe that $d_I(A \mapsto \tau_{f,A}(I))$ is equal to the directional derivative of the map $A \mapsto \tau_{f,A}(I)$ with respect to $B$ computed in $I$. It follows that

$$d_I(A \mapsto \tau_{f,A}(I))(B) = -2\tau_{f,I}(B)$$

and then

$$d_I(\varphi_I) = (\tau_{f,I})^{-1} \circ d_I(A \mapsto \tau_{f,A}(I)) = -2 \cdot \text{Id}.$$  

Therefore the image of the map $\varphi_I$, and hence the $G$-orbit of $I$, is a Zariski dense open subset of $V$ (see [Kim03] pg. 23 Lemma 2.1). Since we can repeat the same argument for all the elements $I \in V$ such that $f(I) \neq 0$, we conclude that $G$ acts transitively on $V \setminus \{f = 0\}$.

Moreover the argument in [Cha03, Proposition 3.1] and computations in [Cha03, Proposition 3.2] show that $V$ is indeed a symmetric prehomogeneous space i.e. there exists an involution of the group $G$ such that the stabilizer of a point in the dense orbit is contained in the set of fixed points for the involution and moreover contains the connected component of the identity of this set.

We are going to show that the action of $G$ on $V$ is irreducible now. Let $W \subset V$ a proper subspace of $V$ invariant for the action of $G$. Since $G$ acts transitively on $V \setminus \{f = 0\}$ and $W$ is a proper subspace we must have $W \subset \{f = 0\}$. If $W \cap Z \neq \emptyset$, since $G$ acts transitively on $Z$ too (see Corollary 10 (ii)), then $Z \subset \mathbb{P}(W)$. But then its dual variety is a cone, so that $V(f_\ast)$ is a cone, and this contradicts the fact that $f_\ast$ is homaloidal. It follows that $W \subset \text{Sm}(X)$, and by Corollary 10 (iv), it intersects every fiber of the Gauss map. It follows that every tangent hyperplane to $X$ contains the subspace $W$, since every tangent hyperplane is tangent to $X$ in a point of $W$. But then $X$ has degenerate dual, a contradiction.

The proof of the theorem is now complete. ☐

Before proceeding to the proof of Corollary 4 we are going to recall a result of McCrimmon [McC65]. Let $F$ be an homogeneous polynomial of degree $d$ on the vector space $V$. Fix a point $I \in V$ such that, \(F(I) \neq 0\) and \(\tau_{F,I}\) is a non-degenerate bilinear form. There exists then a rational mapping

$$H : V \to \text{Hom}(V,V)$$

$$A \mapsto H_A$$

defined for all $A \in V$ with $F(A) \neq 0$, such that

$$\tau_{F,A}(B,C) = \tau_{F,I}(H_A(B),C)$$

for every $A, B, C \in V$.

**Theorem 11.** [McC65, Theorem 1.2] If $F$ satisfies the relation $F(H_A(B)) = h(A)Q(B)$ whenever both sides are defined, $h$ a suitable rational function, then the product

$$A * B = -\frac{1}{2}d_I(M \mapsto H_M(B))(A)$$

defines on the vector space $V$ a Jordan algebra structure. If moreover we choose $I$ such that $F(I) = 1$ then

$$F(R_A(B)) = F(A)^2F(B)$$

where $R_A$ denotes the quadratic representation of the Jordan algebra.
Recall that the quadratic representation of a Jordan algebra is defined in the following way. Denote by $L_A$ the linear map induced on $V$ by left multiplication by an element $A$ of the Jordan algebra. Then

$$R_A := L_A^2 - L_{A^2}.$$

**Remark 12.** An element $A$ in a Jordan algebra is invertible if and only if $R_A$ is an invertible linear map, see [FK94, Proposition II.3.1, p. 33]. It follows that in the Jordan algebra defined by (10) if (11) holds and $\{F = 0\}$ is not contained in an hyperplane then the set of invertible elements coincide with $V \setminus \{F = 0\}$.

Finally, we deduce the announced Corollary 4 from Theorem 2.

**Proof of Corollary 4.** By Theorem 2 above, $V$ is an irreducible prehomogeneous symmetric space. Fix an element $I \in V$ such that $f(I) = 1$. Observe that the map $H_A$ in Theorem 2 equals $\tau_f^{-1} \circ \tau_{f,I}$ and then $H_A \in G$. It follows that there exists a rational function $h$ such that $f(H_A(B)) = h(A)f(B)$ and then (10) defines on $V$ a structure of Jordan algebra in such a way that (11) is satisfied. A direct computation, see [Cha03, p. 179], shows that the above product coincides with the one defined in [Ber00, p. 43]. By [Ber00, Theorem V.4.6] $V$ is a simple Jordan algebra and, recall Remark 12, its determinant coincides with a scalar multiple of $f$. By the classification of simple Jordan algebras [Jac68, pp. 204–205, Corollary 2], the singular locus of $V(f)$ is a Severi variety (see [Zak93, Chapter IV, Theorem 4.8]). □

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