Phase Transitions and Pairing Signature in Strongly Attractive Fermi Atomic Gases

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(Dated: August 18, 2021)

We investigate pairing and quantum phase transitions in the one-dimensional two-component Fermi atomic gas in an external field. The phase diagram, critical fields, magnetization and local pairing correlation are obtained analytically via the exact thermodynamic Bethe ansatz solution. At zero temperature, bound pairs of fermions with opposite spin states form a singlet ground state when the external field $H < H_{c1}$. A completely ferromagnetic phase without pairing occurs when the external field $H > H_{c2}$. In the region $H_{c1} < H < H_{c2}$ we observe a mixed phase of matter in which paired and unpaired atoms coexist. The phase diagram is reminiscent of that of type II superconductors. For temperatures below the degenerate temperature and in the absence of an external field, the bound pairs of fermions form hard-core bosons obeying generalized exclusion statistics.

PACS numbers: 03.75.Ss, 03.75.Hh, 05.30.Pr, 71.10.Pm

I. INTRODUCTION

Recent achievements in manipulating quantum gases of ultracold atoms have opened up exciting possibilities for the experimental study of many-body quantum effects in low-dimensional systems. Experimental observation of superfluidity and phase separation in imbalanced Fermi atomic gases have stimulated great interest in exploring exotic quantum phases of matter with two mismatched Fermi surfaces. The pairing of fermionic atoms with mismatched Fermi surfaces may lead to a breached pairing phase and a nonzero momentum pairing phase of Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states. In general the nature of pairing and superfluidity in strongly interacting systems is both subtle and intriguing.

Pairing is well known to be a momentum space phenomenon, in which two fermions with opposite spin states form a bound pair which behaves like a boson. The bound pairs form a superfluid, while the unpaired fermions remain as a separated gas phase in momentum space. Such superfluid states with gapless excitations in ultracold atomic gases provide an exciting insight into the superfluid regime in quantum many-body physics. Fermi gases of ultracold atoms with population imbalance have been predicted to exhibit a quantum phase transition between the normal and superfluid states. Mismatched Fermi surfaces can appear in different quantum systems, such as type II superconductors in an external magnetic field, a mixture of two species of fermionic atoms with different densities or masses, and charge neutral quark matter.

These exotic phases have attracted newfound interest in the one-dimensional (1D) integrable two-component Fermi gas, which was used to study BCS-BEC crossover and quantum phase separation in a trapping potential. The 1D Fermi gases can be experimentally realized by applying strongly transverse confinement to the Fermi atomic clouds. In the 1D interacting Fermi gas, the Fermi surface is reduced to the Fermi points. The lowest excitation destroys a bound pair close to the Fermi surface. Charge and spin propagate with different velocities due to the pair-wise interaction. The external magnetic field triggers energy level crossing such that the Fermi surfaces of paired fermions and unpaired fermions vary smoothly with respect to the external field. As we shall see in this paper, the presence of the external field at zero temperature has an important bearing on the nature of quantum phase transitions in 1D interacting fermions.

In general the exact Bethe ansatz (BA) solution of any model provides reliable physics beyond mean field theory. The thermodynamic Bethe ansatz (TBA) provides a way to obtain the ground state signature and finite temperature behaviour of integrable 1D quantum many-body systems. At zero temperature, the TBA equations naturally reduce to dressed energy equations in which the external field is explicitly involved. Thus the band fillings are subsequently varied with respect to the external field. This gives an elegant way to analyze quantum phase transitions in the presence of an external field by means of the dressed energy formalism. Our aim here is to obtain new exact results from this formalism for characteristics of pairing phases and quantum phase transitions in the 1D two-component strongly attractive Fermi gas of cold atoms. We present a systematic way to obtain the critical fields and magnetic properties at zero temperature for strongly interacting fermions. We find that the bound pairs of fermions with opposite spin states form a singlet ground state when the external field $H < H_{c1}$. A completely ferromagnetic phase without pairing occurs when the external...
field $H > H_{c2}$. In the region $H_{c1} < H < H_{c2}$ we observe a mixed phase of matter in which paired and unpaired atoms coexist. However, in the absence of the external magnetic field, we show that the bound pairs of fermions behave like hard-core diatoms obeying nonmutual generalized exclusion statistics (GES) at temperatures much less than the binding energy.

This paper is set out as follows. In section II, we present the BA solution of the 1D two-component interacting Fermi gas. The ground state properties are also analysed. In section III, we introduce the TBA in order to set up the dressed energy formalism. The quantum phase transitions and magnetic properties are studied by means of the dressed energy formalism in section IV. We discuss the distribution profiles and the thermodynamics of the 1D strongly attractive Fermi gas of atoms at low temperatures in section V, along with the connection to GES. Section VI is devoted to concluding remarks.

II. THE MODEL

The model we consider has interacting atoms in two hyperfine levels [1] and [2], which are coherently coupled with laser or Rabi frequency fields. Under strong transverse confinement, the system is effectively described along the axial direction by the 1D Hamiltonian $H = H_0 + H_{int} + H_c$. The first term

$$H_0 = \sum_{j=1}^{2} \int \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \psi_j(x) dx$$

contains the kinetic energy and the trapping potential $V(x)$. The second term

$$H_{int} = g_{1D} \int \psi_1^\dagger(x) \psi_2^\dagger(x) \psi_2(x) \psi_1(x) dx$$

(2)

describes s-wave interaction and

$$H_c = \frac{1}{2} \Omega \int \left( \psi_2^\dagger(x) \psi_1(x) + \psi_1^\dagger(x) \psi_2(x) \right) dx$$

(3)

is the coupling term. Here $\psi_1(x)$ and $\psi_2(x)$ are the atomic field creation operators, $g_{1D}$ is the 1D interaction strength and $\Omega$ is the Rabi frequency of coupling fields.

Defining $\psi_1 = (\phi_\uparrow + \phi_\downarrow)/\sqrt{2}$ and $\psi_2 = (\phi_\uparrow - \phi_\downarrow)/\sqrt{2}$, the Hamiltonian becomes

$$H = \sum_{j=\uparrow,\downarrow} \int \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \phi_j(x) dx$$

$$+ g_{1D} \int \phi_\uparrow^\dagger(x) \phi_\downarrow^\dagger(x) \phi_\downarrow(x) \phi_\uparrow(x) dx$$

$$- \frac{1}{2} \Omega \int \left( \phi_\uparrow^\dagger(x) \phi_\uparrow(x) - \phi_\downarrow^\dagger(x) \phi_\downarrow(x) \right) dx.$$

(4)

The new field operators $\phi_\downarrow$ and $\phi_\uparrow$ describe the atoms in the states $|\downarrow\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$ and $|\uparrow\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$.

This Hamiltonian also describes the 1D $\delta$-interacting spin-$\frac{1}{2}$ Fermi gas with an external magnetic field $H = \Omega$. Here we consider the homogeneous case $V(x) = 0$ with periodic boundary conditions for a line of length $L_{\text{int}}$. Unless specifically indicated, we use units of $\hbar = 2m = 1$.

We define $E = m g_{1D}/\hbar^2$ and a dimensionless interaction strength $\gamma = c/n$ for the physical analysis, with linear density $n = N/L$, where $N$ is the number of fermions. The inter-component interaction can be tuned from strongly attractive ($g_{1D} \to -\infty$) to strongly repulsive ($g_{1D} \to +\infty$) via Feshbach resonances.

The model was solved by nested BA equations \cite{16,17}, for the energy eigenspectrum

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2$$

(5)

in terms of the $N$ BA wave numbers $\{k_j\}$, which are the quasimomenta of the fermions. They satisfy the BA equations \cite{16,17},

$$\exp(ik_j L) = \prod_{\ell=1}^{M} k_j - \Lambda_\ell + ic/2,$$

$$\prod_{\ell=1}^{N} \Lambda_\alpha - k_j - ic/2 = - \prod_{\beta=1}^{M} \Lambda_\alpha - \Lambda_\beta + ic. \quad (6)$$

Here $j = 1, \ldots, N$ and $\alpha = 1, \ldots, M$, with $M$ the number of spin-down fermions. The additional parameters $\{\Lambda_\alpha\}$ are the rapidities for the internal spin degrees of freedom.

The distribution of the quasimomenta in the complex plane was studied recently \cite{30}. For weakly attractive interaction, the system describes weakly bound Cooper pairs where the quasimomenta are distributed in a BCS-like manner \cite{30} (Figure II(a)). In this limit, the ground state energy per unit length is given by

$$E \approx \frac{\hbar^2}{2m} n^3 \left( \frac{\gamma^2}{2} (1 - P^2) + \frac{\pi^2}{12} + \frac{\pi^2}{4} P^2 \right),$$

(7)

with the polarization $P = (N - 2M)/N$. The bound state has a small binding energy $\epsilon_b = \hbar^2 n |\gamma|/m$ and is therefore unstable against thermal fluctuations. For strongly attractive interaction, the bound pairs form hard-core bosons (Figure II(b)). The energy per unit length derived directly from Eqs (6) is

$$E \approx \frac{\hbar^2}{2m} \left\{ \frac{\pi^2}{4} (1 - P^2) \right\} \left( 1 + \frac{4(1 - P)}{|\gamma|} \right)$$

(8)

with binding energy $\epsilon_b = \hbar^2 n^2 |\gamma|^2/(4m)$. Generally, the total momentum for bound pairs and that for unpaired fermions are both zero. Thus the BA roots for the model with population imbalance do not show sufficient evidence for the existence of a FFLO state which might exist in the asymmetric BCS pairing model \cite{30,12}. In FFLO
states the unpaired fermions have an asymmetric distribution at the Fermi surface resulting in a net total momentum for bound pairs.\[12\]

\[18\]

\(\sigma\) paired fermions, unpaired fermions and the spin two Fermi seas: one contains bound pairs, another comprised by the real spin parameter \(\Lambda\) complex strings \(\Lambda\) - 2 at finite temperature, spin quasimomenta form as unpaired fermions sit in the outer wings of the distribution. (b) For strongly attractive interaction, they can penetrate into the central region.

**III. THERMODYNAMIC BETHE ANSATZ**

The thermodynamic Bethe ansatz (TBA) has been well established in quantum integrable systems.\[18,25-26,27,28,29,30\] For the sake of completeness, we sketch the main idea and results of the TBA for the fermion model in this section.

At zero temperature, all quasimomenta \(k_j\) of \(N\) atoms form two-body bound states, i.e., \(k_j = \Lambda_j + \frac{i \pi}{2} c\), accompanied by the real spin parameter \(\Lambda\). Here \(j = 1, \ldots, M\). However, at finite temperature, spin quasimomenta form complex strings \(\Lambda_{n,j} = \Lambda_n + \frac{i \pi}{2} (n + 1 - 2j)c\) with \(j = 1, \ldots, n\). Here the number of strings \(\alpha\) is the position of the center for the length-\(n\) string on the real axis. The number of \(n\)-strings \(N_n\) satisfies the relation \(M = M' + \sum_n n N_n\). There are \(M'\) real \(\Lambda_j\) and there are \(N - 2M'\) real \(k_j\) for unpaired fermions. In the presence of a magnetic field, the ground state consists of two Fermi seas: one contains bound pairs, another contains unpaired fermions.

In the thermodynamic limit, i.e., \(N, L \to \infty\) with \(N/L\) finite, it is assumed that the distributions of Bethe roots is sufficiently dense along the real axis. After introducing the root distribution functions \(\sigma(k), \rho(k)\) and \(\xi_n(k)\) for paired fermions, unpaired fermions and the spin \(n\)-string as well as their hole densities \(\sigma^h(k), \rho^h(k)\) and \(\xi^h_n(k)\), the TBA equations (9) can be transformed into the form\[18\]

\[\sigma(k) + \sigma^h(k) = \frac{1}{\pi} - a_2 \ast \sigma(k) - a_1 \ast \rho(k),\]

\[\rho(k) + \rho^h(k) = \frac{1}{2\pi} - a_2 \ast \sigma(k) - \sum_{n=1}^{\infty} a_n \ast \xi_n(k),\]

\[\xi_n(\lambda) + \xi^h_n(\lambda) = a_n \ast \rho(\lambda) - \sum_{n=1}^{\infty} T_{nm} \ast \xi_n(\lambda).\] (9)

Here \(\ast\) denotes the convolution integral \((f \ast g)(\lambda) = \int f(\lambda - \lambda')g(\lambda')d\lambda')\) and\[12\]

\[a_m(\lambda) = \frac{1}{2\pi} \frac{m|c|}{(mc/2)^2 + \lambda^2}.\] (10)

The function \(T_{nm}(\lambda)\) can be found in Takahashi’s book.\[12\]

The equilibrium states at finite temperature \(T\) are described by the equilibrium quasiparticle and hole densities. The partition function \(Z = tr(e^{-\mathcal{H}/T})\) is defined as

\[Z = \sum_{\sigma,\sigma^h,\rho,\rho^h,\xi_n,\xi^h_n} W^\sigma_{\sigma^h,\rho,\rho^h,\xi_n,\xi^h_n}/T,\] (11)

where the densities satisfy the BA equations \[12\], and \(W := \mathcal{W}(\sigma, \sigma^h, \rho, \rho^h, \xi_n, \xi^h_n)\) is the number of states corresponding to the given densities. By introducing the combinatorial entropy \(S = \ln W\) the grand partition function can be presented as \(Z = e^{-G/T}\), where the Gibbs free energy \(G = E - \mu N - H M^2 - TS\). Here \(\mu\) is the chemical potential. The entropy and Gibbs free energy are given in terms of the BA root distribution functions of particles and holes for bound pairs and unpaired fermions as well as spin degrees of freedom.

The energy per unit length is defined by

\[E = \int_{-\infty}^{\infty} \left( k^2 \rho(k) + 2(k^2 - \frac{c^2}{4}) \sigma(k) \right) dk - M^2 H.\] (12)

Here \(H\) is the external magnetic field and \(M^2 = (N - 2M)/2L\) denotes the atomic magnetic momentum per unit length (where the Bohr magneton \(\mu_B\) and the Landé factor are absorbed into the magnetic field \(H\)).

The entropy per unit length is given by\[18\]

\[S = \int_{-\infty}^{\infty} \left( (\sigma(k) + \sigma^h(k)) \ln(\sigma(k) + \sigma^h(k))\right) - \sigma(k) \ln(\sigma(k) - \sigma^h(k)) \ln(\sigma^h(k)) \right) dk + \int_{-\infty}^{\infty} ((\rho(k) + \rho^h(k)) \ln(\rho(k) + \rho^h(k)) \right) \rho(k) \ln(\rho(k) - \rho^h(k)) \ln(\rho^h(k)) \right) dk + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \left( (\xi_n(\lambda) + \xi^h_n(\lambda)) \ln(\xi_n(\lambda) + \xi^h_n(\lambda)) \right) - \xi_n(\lambda) \ln(\xi_n(\lambda) - \xi^h_n(\lambda)) \ln(\xi^h_n(\lambda)) \right) d\lambda.\] (13)

The equilibrium states are determined by the minimization condition of the Gibbs free energy, which gives rise to a set of coupled nonlinear integral equations – the TBA equations.\[12\] In terms of the dressed energies \(e^{h}(k) := T \ln(\sigma(k)/\sigma^h(k))\) and \(e^{a}(k) := T \ln(\rho(k)/\rho^h(k))\).
for paired and unpaired fermions these are
\[
e^b(k) = 2(k^2 - \mu - \frac{c^2}{4}) + Ta_2 \ln(1 + e^{-e^b(k)/T}) + Ta_1 \ln(1 + e^{-e^b(k)/T})
\]
\[
e^u(k) = k^2 - \mu - \frac{1}{2}H + Ta_1 \ln(1 + e^{-e^b(k)/T}) - T \sum_{n=1}^{\infty} a_n \ln(1 + \eta_n^{-1}(k))
\]
\[
\ln \eta_n(\lambda) = \frac{nH}{T} + a_n \ln(1 + e^{-e^u(\lambda)/T}) + \sum_{n=1}^{\infty} T_{mn} \ln(1 + \eta_n^{-1}(\lambda)).
\]

The positive part of \( \eta \) mined by \( H \) characterize the Fermi surfaces for bound pairs and \( Q \) per unit length at zero temperature is given by
\[
\frac{\mathrm{d}G}{\mathrm{d}\mu, H} = \int_{-B}^{B} e^b(\lambda') d\lambda' - n \frac{H}{T}
\]
\[
\frac{\partial G}{\partial H} = \int_{-B}^{B} e^b(\lambda') d\lambda' - \frac{1}{2} \int_{-Q}^{Q} e^u(k) dk
\]

The ground state is antiferromagnetic, i.e., the number of the fermionic atoms with up-spin states and the number of the fermionic atoms with down-spin states are equal. In this case the integral limit for the unpaired Fermi sea \( Q = 0 \) and \( \rho(k) = 0 \). For strong coupling, i.e. \( L|c| > 1 \) the dressed energy equations (16) reduce to the form
\[
e^b(\Lambda) \approx 2 \left( \Lambda^2 - \mu - \frac{c^2}{4} \right) - \frac{1}{2\pi} \int_{-B}^{B} \frac{2e^u(\Lambda') d\lambda'}{c^2 + k^2}
\]

For convenience of notation we denote
\[
p^b = -\frac{1}{\pi} \int_{-B}^{B} e^b(\Lambda) d\Lambda,
\]
\[
p^u = -\frac{1}{2\pi} \int_{-B}^{B} e^u(\Lambda) d\Lambda,
\]

as the pressure for bound pairs and unpaired fermions. Substituting equation (19) into \( p^b \), we have
\[
\pi p^b \left( 1 + \frac{2B}{\pi|c|} \right) \approx 4B \left( \mu - \frac{1}{3} B^2 + \frac{c^2}{4} \right).
\]

Furthermore, from the Fermi points \( e^b(\pm B) = 0 \), we have
\[
B^2 \approx \mu + \frac{c^2}{4} - \frac{p^b}{2|c|}.
\]

From the relation (18) together with the equations (21) and (22), we obtain the pressure and the ground state energy per unit length as
\[
p^b \approx \frac{\hbar^2}{2m} \frac{\pi^2 n^3}{24} \left( 1 + \frac{3}{2|\gamma|} \right)
\]
\[
E_0 \approx \frac{\hbar^2 n^3}{2m} \left( \frac{\gamma^2}{4} + \frac{1}{48\pi^2} \left( 1 + \frac{1}{|\gamma|} \right) \right).
\]

For the strongly attractive 1D Fermi gas, the low-energy excitations split into collective excitations carrying charge and collective excitations carrying spin. This leads to the phenomenon of spin-charge separation. The spin excitation is gapped with a divergent spin velocity
\[
v_s = \frac{n|\gamma|}{\sqrt{2}} \left( 1 + \frac{2}{|\gamma|} \right).
\]

Therefore the spin sector cannot be described by a conformal field theory. However, the charge sector is still critical with central charge \( C = 1 \) and the charge velocity
\[
v_c = \frac{v_F}{4} \left( 1 + \frac{1}{|\gamma|} \right)
\]

for the fully paired ground state, where the bound pairs behave like hard-core bosons. In the above equation

IV. QUANTUM PHASE TRANSITIONS

The function \( \eta_n(\lambda) := \xi(\lambda)/\xi^b(\lambda) \) is the ratio of the string densities. The Gibbs free energy per unit length is given by
\[
G = \frac{T}{\pi} \int_{-\infty}^{\infty} dk \ln(1 + e^{-e^b(k)/T}) + \frac{T}{2\pi} \int_{-\infty}^{\infty} dk \ln(1 + e^{-e^u(k)/T}).
\]

The TBA equations provide a clear picture of band fillings with respect to the field \( H \) and the chemical potential \( \mu \) at arbitrary temperatures. However, it is a challenging problem to obtain analytic results for the thermodynamics at low temperatures from the TBA.

We focus on quantum phase transitions in the 1D strongly attractive Fermi gas at \( T = 0 \) by analyzing the dressed energy equations
\[
e^b(\Lambda) = 2 \left( \Lambda^2 - \mu - \frac{c^2}{4} \right) - \int_{-B}^{B} a_2(\Lambda - \Lambda') e^b(\Lambda') d\Lambda' - \int_{-Q}^{Q} a_1(\Lambda - k) e^u(k) dk,
\]
\[
e^u(k) = \left( k^2 - \mu - \frac{H}{2} \right) - \int_{-B}^{B} a_1(\Lambda - k) e^b(\Lambda) d\Lambda
\]
which are obtained from the TBA equations in the limit \( T \to 0 \). The dressed energy \( e^b(\Lambda) \leq 0 \) (\( e^u(k) < 0 \)) for \( |\Lambda| \leq B \) (\( |k| \leq Q \)) correspond to the occupied states. The positive part of \( e^b \) (\( e^u \)) corresponds to the unoccupied states. The integration boundaries \( B \) and \( Q \) characterize the Fermi surfaces for bound pairs and unpaired fermions, respectively. The Gibbs free energy per unit length at zero temperature is given by
\[
G(\mu, H) = \frac{1}{\pi} \int_{-B}^{B} e^b(\Lambda) d\Lambda + \frac{1}{2\pi} \int_{-Q}^{Q} e^u(k) dk.
\]

The magnetization \( M^z = nP/2 \) per unit length is determined by \( H, \gamma D \) and \( n \) through the relations
\[
-\partial G(\mu, H)/\partial \mu = n, -\partial G(\mu, H)/\partial H = M^z.
\]
νF = ℏnπ/4m. The bound pairs can be broken by a strong enough external field or thermal fluctuations. In the strong interaction limit, it was demonstrated in a recent experiment that the nature of the pairing is likely to be molecular and the mismatched Fermi surfaces do not prevent pairing but indeed quench the superfluidity. The state with polarization can be viewed as an ideal mixture of bosonic pairs and fermionic quasiparticles.

With polarization 0 < P < 1, from the equations (16) we obtain

\[ p^b \approx -\frac{4B}{\pi} \left( \frac{B^2}{3} - \mu - \frac{c^2}{4} + \frac{p^b}{|c|} + \frac{2p^u}{|c|} \right), \]

\[ p^u \approx -\frac{Q}{\pi} \left( \frac{Q^2}{3} - \mu - \frac{H}{2} + \frac{2p^b}{|c|} \right), \]

(26)

From the Fermi points e^b(B) = 0 and e^u(Q) = 0 we have

\[ 2(B^2 - \mu - \frac{c^2}{4}) + \frac{p^b}{|c|} + \frac{4p^u}{|c|} \approx 0, \]

\[ Q^2 - \mu - \frac{H}{2} + \frac{2p^b}{|c|} \approx 0. \]

(27)

It follows that

\[ p^b \approx \frac{8}{3\pi} \left( \mu + \frac{c^2}{4} - \frac{p^b}{2|c|} - \frac{2p^u}{|c|} \right)^\frac{3}{2}, \]

\[ p^u \approx \frac{2}{3\pi} \left( \mu + \frac{H}{2} - \frac{p^b}{2|c|} \right)^\frac{3}{2}. \]

(28)

With the help of the relation (18) and by lengthy iteration we find the effective chemical potentials for pairs \( \mu^b = \mu + \epsilon_b/2 \) and unpaired fermions \( \mu^u = \mu + H/2 \) are given by

\[ \mu^b \approx \frac{\hbar^2\pi^2}{2m} \left\{ \frac{(1-P)^2}{16} \left( 1 + \frac{4(1-P)}{3|\gamma|} + \frac{4P}{|\gamma|} \right) + \frac{4P^3}{3|\gamma|} \right\}, \]

and

\[ \mu^u \approx \frac{\hbar^2\pi^2}{2m} \left\{ P^2 \left( \frac{1 + 4(1-P)}{|\gamma|} \right) + \frac{(1-P)^3}{12|\gamma|} \right\}. \]

(29)

(30)

These results can give explicit chemical potentials for the two different species:

\[ \mu_\uparrow = \mu + H/2, \quad \mu_\downarrow = \mu - H/2. \]

(31)

In addition we have the total chemical potential \( \mu = \partial E/\partial n - HP/2 \). Here the energy per unit length with polarization follows from the relation \( E = n\mu - G(\mu, H) + nHP/2 \). Indeed, the energy obtained from the TBA formalism is in agreement with the result derived from the BA. The integration boundaries

\[ B \approx \frac{n\pi(1-P)}{4} \left( 1 + \frac{(1-P)}{2|\gamma|} + \frac{2P}{|\gamma|} \right), \]

\[ Q \approx n\pi P \left( 1 + \frac{2(1-P)}{|\gamma|} \right). \]

(32)

are the largest quasimomentum for bound pairs and unpaired fermions.

Analysis of the dressed energy equations (16) shows that the fully paired ground state with \( M^2 = 0 \) is stable when the field \( H < H_c1 \), where

\[ H_{c1} \approx \frac{\hbar^2\gamma^2}{2m} \left( \frac{\gamma^2}{2} - \frac{\pi^2}{8} \right). \]

(33)

This critical field makes the excitation gapless. If the external field \( H > H_{c1} \), the pairing gap, defined by \( \Delta = (H_{c1} - H)/2 \), is completely diminished by the external field. Slightly above the critical point \( H_{c1} \), the system has a linear field-dependent magnetization

\[ M^z \approx \frac{2(H - H_{c1})}{n\pi^2} \left( 1 + \frac{2}{|\gamma|} \right) \]

(34)

with a finite susceptibility

\[ \chi \approx \frac{2}{n\pi^2} \left( 1 + \frac{2}{|\gamma|} \right). \]

(35)

This behaviour differs from the Pokrovsky-Talapov-type phase transition occurring in a gapped spin liquid. We note that this smooth phase transition in the attractive Fermi gas is reminiscent of the transition from the Meissner phase to the mixed phase in type II superconductors.

On the other hand, if the external field \( H > H_{c2} \), where

\[ H_{c2} \approx \frac{\hbar^2\pi^2}{2m} \left( \frac{\gamma^2}{2} + 2\pi^2 \left( 1 - \frac{4}{3|\gamma|} \right) \right) \]

(36)

all bound pairs are broken and the ground state becomes a normal ferromagnetic state of fully polarized atoms. Slightly below \( H_{c2} \) the phase transition is determined by the linear field-dependent relation

\[ M^z \approx \frac{1}{2\gamma} \left( 1 - \frac{(H_{c2} - H)}{4n\pi^2} \right) \left( 1 + \frac{10}{3|\gamma|} \right) \]

(37)

with a finite susceptibility

\[ \chi \approx \frac{1}{8n\pi^2} \left( 1 + \frac{10}{3|\gamma|} \right). \]

(38)

A mixed phase occurs in the region \( H_{c1} < H < H_{c2} \), with coexistence of spin singlet bound pairs and unpaired fermions with ferromagnetic order. The external field-magnetization relation

\[ \frac{H}{2} \approx \frac{\hbar^2\pi^2}{2m} \left\{ \frac{\gamma^2}{4} + 4\pi^2(m^2)^2 \left( 1 + \frac{4(1-2m^2)}{|\gamma|} - \frac{8m^2}{3|\gamma|} \right) - \frac{\pi^2}{16} (1-2m^2)^2 \left( 1 + \frac{8m^2}{|\gamma|} \right) \right\}, \]

(39)

follows by solving equations (29) and (30). It indicates the energy transfer relation among the kinetic energy
variation $\Delta E_k = \mu^u - \mu^b$, the binding energy $\epsilon_b$ and the Zeeman energy $\mu_B H$:

$$\Delta E_k + \epsilon_b = \mu_B H$$

which qualitatively agrees with the relation identified in experiment. In the above equation we take the Bohr magneton $\mu = 1$ and $m^* = M^*/n$.

Figure 2(a) shows the magnetization and the susceptibility for $|c| = 10$ and $n = 0.5, 1, 1.5, 2$. The magnetization gradually increases from $M^2 = 0$ to $n/2$ as the field increases from $H_{c1}$ to $H_{c2}$. It is important to note that the points of intersection at $H = \epsilon_b$ indicate where the Fermi surface of unpaired fermions exceeds the one for the bound pairs. This point separates the mismatched pairing phase into different breached pairing phases. The susceptibility shows discontinuities at the critical points, with $\chi = 0$ for $H < H_{c1}$ and $H > H_{c2}$. However, $\chi$ is finite and quickly decreases in the vicinity of $H_{c1}$. For larger densities (Figure 2(c) illustrates the case $n = 2$), $\chi$ slowly increases as $H \to H_{c2}$. We note that the coexistence of pairing and magnetization in the 1D attractive Fermi gas is similar to the Shubnikov phase of superconductivity and magnetization in type II superconductors.

Figure 3 shows the phase diagram in the $n - H$ plane for the particular value $|c| = 10$. As $n \to 0$, the two critical fields approach the binding energy $\epsilon_b$. The two critical fields have opposite monotonicity: $H_{c1}$ decreases with increasing $n$ whereas $H_{c2}$ increases with $n$. For the 1D Fermi gas in a harmonic trapping potential, the density is position-dependent and decreases away from the trapping centre. Thus, for sufficiently large central-density, the system has subtle segments; the mixed phase lies in the centre and the fully paired phase (or the fully unpaired phase) sits in the two outer wings for $H < \epsilon_b$ (or $H > \epsilon_b$). Nevertheless, for sufficiently low central-density, the cloud is either a fully paired phase or a fully unpaired phase for $H < \epsilon_b$ or $H > \epsilon_b$, respectively.

For the 1D Fermi gas, the local pair correlation is defined by

$$g_p^{(1)} = \langle \phi_\uparrow^d(0)\phi_\uparrow^d(0)\phi_\uparrow(0)\phi_\uparrow(0) \rangle \approx \frac{1}{2} \frac{dE}{dc}.$$  

For weakly attractive interaction,

$$g_p^{(1)} \approx n^2(1 - P^2)/4$$

indicating a two-component free Fermi gas phase. For strongly attractive interaction the local pair correlation is given by

$$g_p^{(1)} \approx \frac{1}{n^2} \left[ \gamma + \frac{\pi^2(1 - P)^2(1 + 3P)}{24\gamma^2} + \frac{8\pi^2P^3}{3\gamma^2} \right].$$

This has maximum and minimum values corresponding to the fully paired phase for $H < H_{c1}$ and the fully unpaired phase for $H > H_{c2}$, respectively. Depairing weakens the pair correlation in the region $H_{c1} < H < H_{c2}$. The phase transitions in the vicinities of the critical points are of second order.

V. EXCLUSION STATISTICS

Strong thermal fluctuations can destroy the magnetically ordered phases, delineated by the two critical fields, above a critical temperature $T_c$, which in principle can be also calculated from the TBA equations. On the other hand, at temperatures much lower than the degeneracy temperature $T_2 := \frac{\hbar^2}{2m} n^2 \ll \epsilon_b$, the bound pairs are stable against weak thermal fluctuations. However, the individual pair wavefunctions do not overlap coherently, i.e., the existence of bound pairs does not lead to long range order at finite temperatures. This can be seen from the
finite temperature TBA equations \([14]\) in which the unpaired band is empty due to a large negative chemical potential. This behaviour is similar to that of 3D attractive fermions.\(^{32}\) In 1D the dynamical interaction and the statistical interaction in the pairing scattering process are inextricably related.\(^{33,34,35,36}\) This means that one bound pair excitation may cause a fractional number of holes below the Fermi surface due to the collective signature. This is the key point in understanding GES for 1D interacting many-body systems. We shall show that the bound pairs can be viewed as ideal particles obeying GES. GES has recently been applied to the 3D unitary Fermi gas.\(^{37}\)

In the absence of the magnetic field and at low temperatures, the unpaired dressed energy is positive due to a large negative chemical potential. It follows that the BA and TBA equations can be written as

\[
\begin{align*}
\sigma(k) + \epsilon^b(k) & = \frac{1}{\pi} - \int_{-\infty}^{\infty} a_2(k - \Lambda)\sigma(\Lambda)d\Lambda, \\
\epsilon^b(k) & = 2k^2 - \mu - \frac{1}{4}c^2 + T\epsilon_2 \ln \left(1 + e^{-\frac{\epsilon(k)}{\tau_B}}\right).
\end{align*}
\]

After neglecting exponentially small terms in \([20]\), the pressure for the bound pairs is given by

\[
p^b \approx \frac{2}{2\pi^2 h^2} \int_0^\infty \frac{\sqrt{\epsilon} d\epsilon}{1 + e^{-\tau_B/\epsilon}}
\]

with the function \(A(T) := \hbar^2 B^2/(2m) = (\mu + \frac{1}{4}c^2 - p^b/2\epsilon)\). Furthermore, using Sommerfeld expansion and iterating the pressure \(p^b\) with equation \([15]\), we obtain the cut-off energy \(A(\tau)\) in the form

\[
A(\tau) \approx A_0 \left[1 + \frac{16\tau^2}{3\pi^2} \left(1 - \frac{2}{|\gamma|}\right) + \frac{1024\tau^4}{9\pi^4} \left(1 - \frac{4}{|\gamma|}\right)\right],
\]

where

\[
A_0 = \frac{\hbar^2 n^2 \pi^2}{2m} \left(1 + \frac{1}{|\gamma|}\right).
\]

Here \(\tau = K_B T/T_d\) is the degenerate temperature. The pair distribution function \(n(\epsilon) := \pi\sigma(\epsilon)\) is given by

\[
n(\epsilon) = \frac{1}{\alpha(1 - e^{-2A(\epsilon)/K_B T})},
\]

where \(\alpha = 1 + 1/|2\gamma|\).

The chemical potential follows as

\[
\mu \approx \mu_0 \left[1 + \frac{16\tau^2}{3\pi^2} \left(1 - \frac{4}{3|\gamma|}\right) + \frac{1024\tau^4}{9\pi^4} \left(1 - \frac{56}{15|\gamma|}\right)\right]
\]

where

\[
\mu_0 \approx \frac{\hbar^2 n^2 \pi^2}{2m} \left(1 + \frac{4}{3|\gamma|}\right),
\]

which is consistent with \([29]\). The total energy per unit length and the free energy per unit length in the strong coupling regime are

\[
E \approx E_0 \left[1 + \frac{16\tau^2}{\pi^2} \left(1 - \frac{2}{|\gamma|}\right) + \frac{1024\tau^4}{5\pi^4} \left(1 - \frac{4}{|\gamma|}\right)\right]
\]

\[
- \frac{1}{2} n\epsilon_B,
\]

\[
F \approx E_0 \left[1 + \frac{16\tau^2}{\pi^2} \left(1 - \frac{2}{|\gamma|}\right) - \frac{1024\tau^4}{15\pi^4} \left(1 - \frac{4}{|\gamma|}\right)\right]
\]

\[
- \frac{1}{2} n\epsilon_B,
\]

respectively. Here

\[
E_0 = \frac{\hbar^2 n^3 \pi^2}{2m} \left(1 + \frac{1}{|\gamma|}\right)
\]

is consistent with \([23]\) obtained from \([10]\).

We see that in the strongly attractive regime and in the absence of the magnetic field the bound pairs behave like hard-core bosons at low temperatures and have massless excitations, i.e.

\[
F(T) = F(0) - \frac{\pi C(K_BT)^2}{6h\epsilon_c} + O(T^4).
\]

Here the central charge \(C = 1\) and \(\epsilon_c\) is given by \([23]\). The specific heat is given by

\[
\epsilon_c = \frac{nK_B \tau}{3(1 + \frac{1}{|\gamma|})} \left(1 + \frac{128\tau^2}{5\pi^4} \left(1 - \frac{2}{|\gamma|}\right)\right).
\]

On the other hand, the statistical signature of the fully paired state can be described by GES.\(^{33,35}\) In this formalism the pair distribution function is given by

\[
n(\epsilon) \approx (\alpha + w(\epsilon))^{-1}
\]

where \(w(\epsilon)\) satisfies the GES relation

\[
w^\alpha(\epsilon) (1 + w(\epsilon))^{1-\alpha} = e^{-\frac{2A(\epsilon)}{\tau_B}}.
\]

Here \(\epsilon\) denotes the energy of pairs. For the strongly attractive Fermi gas, we find the GES parameter

\[
\alpha \approx 1 + 1/|2\gamma|.
\]

Now following Isakov et al.\(^{36}\), at low temperatures, i.e., for \(K_B T < T_d\), we find the cut-off energy

\[
A(T) \approx A_0 \left[1 + \frac{16\tau^2}{3\pi^2} \left(1 + O(\tau^3)\right)\right],
\]

which agrees with \([47]\) to leading order and next leading order in the strong coupling regime (for higher order terms, the reader is referred to Refs.\(^{36}\)). Figure 4 shows the close agreement between the TBA distribution function \([49]\) and the GES most probable distribution of fermion pairs \([50]\) for different values of interacting...
strength at low temperatures. We see clearly that the dynamical interaction $\gamma$ continuously varies the GES, with the most probable distribution of fermion pairs approaching that of hard-core bosonic molecules with an effective statistics parameter $\alpha = 1$ as the interaction increases. In this sense the dynamical attractive interaction makes the fermions more exclusive. In the GES formalism the total energy per unit length and the free energy per unit length follow as

$$E \approx E_0 \left[ 1 + \frac{16\tau^2}{\pi^2\alpha^4} + O(\tau^4) \right] - \frac{1}{2} n\epsilon_b,$$

$$F \approx E_0 \left[ 1 - \frac{16\tau^2}{\pi^2\alpha^4} + O(\tau^4) \right] - \frac{1}{2} n\epsilon_b,$$

(59)

which again agree well with the TBA results (52) for strong coupling, see Figure 5.

In conclusion, we have studied pairing and quantum phase transitions in the strongly attractive 1D Fermi gas with an external magnetic-like field. Analytic results have been obtained for the critical fields $H_{c1}$ and $H_{c2}$, magnetization, critical behaviour and local pair correlation. The pairing induced by an interior gap in the system differs from conventional BCS pairing and gapped spin liquids. The smooth pair breaking phase transitions seen in the attractive Fermi gas are reminiscent of the superconductivity breaking phase transitions in type II superconductors. At low temperatures we predict that the hard-core bound pairs of fermionic atoms obey GES. The thermodynamics of the hard-core pairs obey universal temperature dependent scaling.

VI. CONCLUSION

In conclusion, we have studied pairing and quantum phase transitions in the strongly attractive 1D Fermi gas with an external magnetic-like field. Analytic results have been obtained for the critical fields $H_{c1}$ and $H_{c2}$, magnetization, critical behaviour and local pair correlation. The pairing induced by an interior gap in the system differs from conventional BCS pairing and gapped spin liquids. The smooth pair breaking phase transitions seen in the attractive Fermi gas are reminiscent of the superconductivity breaking phase transitions in type II superconductors. At low temperatures we predict that the hard-core bound pairs of fermionic atoms obey GES. The thermodynamics of the hard-core pairs obey universal temperature dependent scaling.

We emphasize here that in the presence of an external magnetic field, pair breaking in the 1D two-component strongly attractive Fermi gas sheds light in understanding the pairing signature of the 3D strongly interacting Fermi gas of ultracold atoms in which superfluid and normal phases can coexist. Although there is no long range order in 1D quantum many-body physics, the mismatched Fermi surfaces do not prevent pairing. This pairing signature has also been observed in the 3D two-component atomic gas with high spin population imbalances. In addition, for the 1D Fermi gas, the magnetic field triggers spin imbalances when the external field is greater than the first critical field $H_{c1}$ and less than the second critical field $H_{c2}$. The energy
transfer relation [44] found for this model is also consistent with experimental observations in the 1D Fermi gas [14]. The phase diagram presented in Figure 3 clearly shows the phase separation and the pairing signature with changing magnetic field. It may be possible to experimentally test our theoretical predictions for the quantum phase transitions and critical fields in the 1D two-component strongly interacting Fermi gas via the experimental advances in trapping ultracold atoms.

This work has been supported by the Australian and German Research Councils. The authors thank Prof. M. Takahashi for helpful discussions and C.L. thanks Prof. Yu.S. Kivshar for support.

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