Abstract

We show that an invariant surface allows to construct the Jacobi vector field along a geodesic and construct the formula for the normal component of the Jacobi field. If a geodesic is the transversal intersection of two invariant surfaces (such situation we have, for example, if the geodesic is hyperbolic), then we can construct a fundamental solution of the the Jacobi-Hill equation $\ddot{u} = -K(u)u$. This is done for quadratically integrable geodesic flows.

§1. Introduction.

1.1. Definitions. Suppose $G = (g_{ij})$ is a Riemannian metric on a surface $P^2$, a curve $\gamma : [a, b] \rightarrow P^2$ is a geodesic. We will assume that the parameter $t \in [a, b]$ of the geodesic $\gamma$ is natural or natural, multiplied by a constant.

Definition 1 Geodesic variation of a geodesic $\gamma$ is called the smooth mapping $\Gamma : [-\varepsilon, \varepsilon] \times [a, b] \rightarrow P^2$ such that

1) for any fixed $s_0 \in [-\varepsilon, \varepsilon]$ the curve $\Gamma(s_0, t) : [a, b] \rightarrow P^2$ (as the curve of parameter $t \in [a, b]$) is a geodesic,

2) for any $t \in [a, b]$ $\Gamma(0, t) = \gamma(t)$.

Definition 2 Jacobi vector field along the geodesic $\gamma$ is the vector field $J = \frac{d\Gamma}{ds}|_{s=0}$, where $\Gamma$ is a geodesic variation of the geodesic $\gamma$.

By definition, Jacobi vector field is a smooth vector field along the geodesic.

Definition 3 Jacobi vector field $J$ is called normal if it is orthogonal to the geodesic at every point of the geodesic.

It is known that the projection of a Jacobi field $J$ to the vector field of normals to the geodesic is a normal Jacobi vector field.

The length of a normal Jacobi vector field $J$ satisfies the Jacobi-Hills equation for the normal component $\ddot{x} + K(\gamma(t))x = 0$, where $K$ is the Gauss curvature and $t$ is the natural parameter (see, for example, [1] or [2]).

Consider real numbers $a \neq b$. Denote by $A$ the point $\gamma(a) \in P^2$, denote by $B$ the point $\gamma(b) \in P^2$.

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Definition 4 The points $A$ and $B$ are called conjugate along the geodesic $\gamma$ if there exists a non-zero Jacobi vector field $J$ along the geodesic $\gamma$ such that $J(a) = J(b) = 0$. (Figure 1)

Figure 1:

The point $A$ can coincide with the point $B$. It happens if the geodesic $\gamma$ is closed or self-intersecting. In the first case the point $A$ is called self-conjugate along the geodesic $\gamma$.

1.2. Jacobi vector fields as the projection of invariant vector fields from the co-tangent space. The metric allows to identify canonically the tangent and co-tangent bundles of the surface $P^2$. Therefore we have a scalar product and a norm on every co-tangent plane. For example, suppose $G$ in coordinates $(x, y)$ reads $\lambda(x, y)(dx^2 + dy^2)$. Then the scalar product on $T^*P^2$ is given by the formula $<(p_x, p_y), (\dot{p}_x, \dot{p}_y)> = \frac{p_x \dot{p}_x + p_y \dot{p}_y}{\lambda(x, y)}$.

Definition 5 Geodesic flow of the metric $G$ is called the Hamiltonian system on $T^*P^2$ with the Hamiltonian $H \overset{def}{=} \frac{1}{2} |p|^2$, where $p$ is momentum and $|.|$ is the norm.
In particular, the Hamiltonian $H$ of the geodesic flow of the metric $\lambda(x, y)(dx^2 + dy^2)$ is given by the formula $H(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2\lambda(x, y)}$.

It is known that the trajectories of the geodesic flow projects onto the geodesics.

**Definition 6** An imbeded into $T^*P^2$ surface $I^2$ is called invariant if the vector field of the geodesic flow is tangent to $I^2$.

**Definition 7** Let $\gamma: [a, b] \to T^*P^2$ be the solution of the geodesic flow. The vector field $V$ along the curve $\gamma$ is called invariant if it is invariant with respect to the geodesic flow.

In other words, consider the one parametric family of mappings $S_\tau: T^*P^2 \to T^*P^2$. The mapping $S_\tau$ moves a point $x \in \gamma \subset T^*P^2$ along the trajectory $\gamma$ of the geodesic flow during the time $\tau$. The vector field $V$ is called invariant if for any $\tau$ the differential $dS_\tau|_x$ takes the vector field $V$ to itself.

In §2 we will show that a geodesic variation allows to canonically construct an invariant surface; a Jacobi vector field allows to canonically construct an invariant vector field; the projection of the invariant vector field is the Jacobi vector field and the composition of the projection and an imbedding of the invariant surface is the geodesic variation.

**1.3. Jacobi vector fields of integrable geodesic flows.** A geodesic flow is called integrable if it is integrable as a Hamiltonian system. That is there exists a smooth function $F: T^*P^2 \to R$ such that

1) $F$ is constant on the trajectories of the geodesic flow

2) the differentials $dH$ and $dF$ are linear independent almost everywhere.

The function $F$ is called an integral. Note that the geodesic flow preserves the vector field $\text{sgrad}(F)$. Since that, the projection of the vector field $\text{sgrad}(F)$ is a Jacobi vector field. Using this, we can construct a number of pairs of conjugate points. Let $L^2$ be a Louville torus of an integrable geodesic flow. Restrict the natural projection to the torus $L^2$. A connected component of the set of critical points of $\pi|_{L^2}$ is called a caustic. It is known (see, for example, [3]) that a caustic is a smooth simple curve and can not intersect other caustic.

**Remark** Sometimes caustics are called the projection of the set of critical points of the mapping $\pi|_{L^2}$. According to our definition, caustics are curves in the phase space $T^*P^2$.

Suppose the intersection of the trajectory $\hat{\gamma}$ with the caustics includes points $\hat{A} \in L^2 \subset T^*P^2$ and $\hat{B} \in L^2 \subset T^*P^2$. We prove that the projections $\pi(\hat{A})$ and $\pi(\hat{B})$ are conjugate along the geodesic $\gamma = \pi(\hat{\gamma})$. Indeed, consider the restriction of the vector field $\text{sgrad}(F)$ to the Liouville torus $L^2$. Consider the projection $d\pi(\text{sgrad}(F))$. Since the rank of the projection $d\pi(\text{sgrad}(F))$ is less than 2 in the points $\pi(\hat{A})$ and $\pi(\hat{B})$, we see that the projections of the vector fields $\text{sgrad}(H)$ and $\text{sgrad}(F)$ are parallel in the points. Therefore the normal component of the
vector field $\pi \text{grad}(F)$ equals zero in the points $\pi(\hat{A})$ and $\pi(\hat{B})$. Thus points $\pi(\hat{A})$ and $\pi(\hat{B})$ are conjugate.

1.4. Jacobi vector fields for a hyperbolic geodesic.

**Definition 8** A closed geodesic $\gamma$ is called em hyperbolic if the corresponding trajectory $\hat{\gamma}$ of the geodesic flow is hyperbolic.

That is if we restrict the geodesic flow to the isoenergy surface (we denote isoenergy surface by $Q^3 \{ H = \text{const} \}$) then the multipliers of the trajectory not lie on the unit circle.

It is known that in a regular neighborhood $U(\hat{\gamma}) \subset Q^3$ of a hyperbolic trajectory there exists a pair of invariant two-dimensional surfaces ($L_+$ and $L_-$). The intersection of the surfaces coincides with $\hat{\gamma}$ (See, for example, [4]).

An invariant surface allows to construct a solution of the Jacobi-Hill equation. In §3 we will show that the solutions that correspond to the surfaces $L_+$ and $L_-$ are not proportional. Therefore we have a fundamental solution of the Jacobi-Hill equation $\ddot{x} + K(\gamma(t))x = 0$.

Assume that the geodesic flow is integrable, and the integral is a Bott function. That is the restriction of the integral to the isoenergy surface satisfies the following properties (we denote the restriction by $F$):

1. The critical manifolds of $F$ are compact sets.
2. If $D$ is an arbitrary 2-disk that is transversal with critical manifolds, then the restriction $F|_D$ is a Morse function.

Suppose a connected component of the set of critical points is homeomorphic to the circle. Consider a transversal disk. The dimension of the transversal disk equals 2. The restriction of the function $F$ to the transversal disk has Morse singularity of index 0, 2, or 1. In the last case the connected component of the critical set is called a saddle circle. Hyperbolic trajectories of a Bott integrable geodesic flow are saddle circles.

**Remark** Let a trajectory of a Bott integrable Hamiltonian system be a hyperbolic trajectory. Then it is a saddle circle. The reverse statement is not true. There exist saddle circles that are not hyperbolic trajectories.

Consider the Liouville fiber that contains a saddle circle. In a neighborhood of a point of the saddle circle the Liouville fiber is homeomorphic to a pair of intersecting surfaces.

Recall that a saddle circle is called nonorientable if the intersection of the Liouville fiber with a regular neighborhood of the saddle circle is homeomorphic to the self-intersecting Möbius band. A saddle circle is called orientable the intersection of the Liouville fiber with a saddle neighborhood is homeomorphic to two intersecting annuli.

Consider a closed geodesic $\gamma$. Let a point $B \in \gamma$ be self-conjugate along the geodesic. Consider the set of points conjugate to $B$. Denote by $N(\gamma)$ the number of elements in the set. It is known that $N(\gamma)$ does not depend on the choice of the initial point $B \in \gamma$. 

4
If for the point $C \in \gamma_1$ there are no conjugate points along $\gamma_1$, then by definition put $N(\gamma_1) = 0$.

**Theorem 1** Suppose $P^2$ is an orientable surface, the geodesic flow of a metric $G$ is integrable, and $\gamma$ is a saddle circle. Then

- if the circle $\gamma$ is nonorientable, then $N(\pi(\gamma))$ is odd number,
- if the circle $\gamma$ is orientable, then $N(\pi(\gamma))$ is even number.

The statement of the result for nonorientable saddle circles were proved by A. Wittek.

Proof. Let a saddle circle $\gamma$ be orientable. Consider the Liouville fiber that contains $\gamma$. By definition, a regular neighborhood of $\gamma$ the Liouville fiber is homeomorphic to a pair of intersecting annuli. Denote by $L_+$ oe of the annuli. Consider a vector field $J$ that is invariant and that is tangent to $L_+$. Denote by $J$ the projection $\pi(\gamma)$ of the vector field $J$.

Using the theorem of the existence and uniqueness of the solution of a differential equation, we have that in a neighborhood of a zero point a normal Jacobi vector field behaves as it is shown on Diagramm 2(a) (the situation of Diagramm 2(b) is forbidden). That is, the frame (velocity vector of the geodesic, Jacobi vector field) has different orientation at different sides of the geodesic $\gamma$.

Since $J$ has no zero points, we see that the frame (the velocity vector of $\gamma$, $J$) has the same orientation with respect to the invariant surface $L$ in every point of the geodesic. Hence the vector field $J$ has the same direction after and before the circuit along the geodesic. Hence, the number of zeros of $J$ is an even number. Proved.

1.6. Jacobi vector fields along hyperbolic geodesics of quadratically integrable geodesic flows.

**Definition 9** A geodesic flow is called linear integrable if it admits an integral $F$ such that in a neighborhood of any point the integral $F$ is given by the formula $F(x, y, p_x, p_y) = A(x, y)p_x + B(x, y)p_y$, where $x, y$ are coordinates on the surface, $p_x, p_y$ are the corresponding momenta, and $A, B$ are smooth functions of two variables.

**Definition 10** A geodesic flow is called quadratically integrable, if it is not linear integrable and if it admits an integral $F$ such that in a neighborhood of any point the integral $F$ is given by the formula $F(x, y, p_x, p_y) = A(x, y)p_x^2 + B(x, y)p_xp_y + C(x, y)p_y^2$, where $A, B, C$ are smooth functions of two variables.

Quadratically and linear integrable geodesic flows on closed surfaces are completely described. In [3] V.V. Kozlov proved that there are no linear and quadratic integrable geodesic flows on the surfaces of genus $g > 1$. Linear and quadratically integrable geodesic flows on the sphere were described by V.N.
Kolokolzov in [8]. Quadratically integrable geodesic flows on the torus were described by I.K. Babenko and N.N. Nekhoroshev in [10] (see, also, [13]).

In §5 we describe hyperbolic trajectories and invariant surfaces of quadratically integrable geodesic flows and obtain fundamental solutions of the Jacobi-Hill equation \(\ddot{x} + K(\gamma(t))x = 0\) along hyperbolic geodesics.

\section*{2. Canonical frame on \(T^*T^2\).}

\subsection*{Commutative relations for it.}

Let \(G\) be a Riemannian metric on the oriented surface \(P^2\). Consider the tangent bundle \(TP^2\). The space of non-zero vectors of \(TP^2\) is denoted by \(T_0P^2\).

The aim of this section is to canonically construct a frame (the vectors of the frame will be denoted by \(D_\phi, D_1, D_2, A\)) in every point of \(T_0P^2\). Since the metric allows to identify the tangent and co-tangent bundles of the surface \(P^2\), we will have the canonic frame in every point of \(T_0^*P^2\).

\subsection*{2.1. The frame.}

Let \(x_0\) be an arbitrary point of the surface \(P^2\), let \(v\) be a tangent vector at the point \(x_0\). The vector \(v\) is a point of \(T^*P^2\).

Denote by \(\rho_\phi(v)\) the vector \(v_1\) at the point \(x_0\) such that \(|v_1| = |v|\), the angle between \(v\) and \(v_1\) is equal to \(\phi\), and the frame \((v, v_1)\) is positive. In other words,
ρ_φ rotates a vector by the angle φ. Consider the vector \( D_\phi \overset{def}{=} \frac{d}{d\phi}|_{\phi=0} \rho_\phi(v) \) at the point \( v \in TP^2 \).

Denote by \( D_1 \) the vector field \( s\text{grad}(H) \).

By definition, put \( D_2 \overset{def}{=} [D_\phi, D_1] \).

Consider the so-called ”Liouville vector field” \( A \). Recall, that the vector field \( A \) is defined in the following way. Consider the one-parameter group of self-diffeomorphisms \( g^t : TP^2 \to TP^2 \). \( g^t(v) \overset{def}{=} \exp(t)\vec{v} \). Put by definition \( A(v) \overset{def}{=} \frac{d}{dt}|_{t=0} g^t(v) \).

Direct calculations show that for \( v \in T_0P^2 \) the vectors \( D_\phi, D_1, D_2, A \) are linear independent. Therefore the quadruple \((D_\phi, D_1, D_2, A)\) is a frame.

2.2. Commutative relations for the vectors of the frame. Denote by \( r(v) \) the function \( \sqrt{G(v,v)} \). The function \( r(v) \) is a smooth function on \( T_0P^2 \).

Lemma 1

\[
\begin{align*}
[D_1, D_2] &= r^2 KD_\phi, \\
[D_\phi, D_1] &= D_2, \\
[D_\phi, D_2] &= -D_1, \\
[A, D_1] &= D_1, \\
[A, D_2] &= D_2, \\
[A, D_\phi] &= 0,
\end{align*}
\]

where \( K(v) \overset{def}{=} K(\pi(v)) \), \( K: P^2 \to \mathbb{R} \) is the Gaussian curvature of the metric \( G \) on the surface \( P^2 \).

The first three relations follows from [3]. The last three relations were proved in [3].

Consider a geodesic trajectory \( \hat{g}(t) \). Consider an invariant vector field \( \hat{J} \) along \( \hat{g}(t) \). Let \( \hat{J} = xD_2 + yD_\phi + I D_1 + a A \).

Since the vector field \( \hat{J} \) is invariant, we have \([D_1, \hat{J}] = 0\). Using lemma 1, we obtain the following system:

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} + r^2 K x &= 0, \\
\dot{a} &= 0, \\
\dot{I} &= a.
\end{align*}
\] (1)

If the length of the vector \( \pi(D_1) \) equals 1, then from the first two equations it follows Jacobi-Hills equation \( \dot{x} + K x = 0 \). Note, that in this case \( x \) is normal component of the vector field \( \pi(\hat{J}) \). Projection of the vector field \( I \) is the horizontal component of the vector field \( \hat{J} \). Hence the projection \( J \) of the vector field \( \hat{J} \) is equal to \( x\vec{n} + a\dot{\gamma} \), where \( \vec{n} \) is the normal vector (to the geodesic) of length 1.

We shall prove that for every Jacobi vector field \( J \) there exists an invariant vector field \( \hat{J} \) such that \( \pi(\hat{J}) = J \).
Consider a Jacobi vector field $J(t)$ along the geodesic $γ(t)$. Denote by $x(t)$ the normal component of the vector field $J$, denote by $I(t)$ the horizontal component of the vector field $J$. By definition, $J(t) = \frac{d}{dt}|_{s=0}Γ(s, t)$, where $Γ(s, t)$ is a geodesic variation of the geodesic $\dot{γ}$. Consider the vector field $\dot{J} = \frac{d}{ds}|_{s=0}\dot{Γ}(s, t)$ along the trajectory $\dot{γ}$. Evidently the vector field $\dot{J}$ is invariant. Indeed, $S^T(Γ(s, t)) = Γ(s, t + τ)$.

Moreover, we see that the vector field $\dot{J}$ equals

$$\dot{J} = x(t)D_2 + y(t)D_ϕ + I(t)D_1 + a(t)A,$$

where $y(t), a(t)$ are smooth functions, $x$ is the normal component of the Jacobi vector field $J$, and $I$ is the horizontal component of the Jacobi vector field $J$.

§3. Fundamental solution of the Jacobi-Hill equation for hyperbolic geodesics.

Consider a geodesic trajectory $\dot{γ}(t)$. Denote by $γ$ the geodesic $π(\dot{γ}(t))$.

Suppose we have an 1-dimension subspace in every tangent space at the points of the geodesic trajectory. Suppose the set of the subspaces $l_{inv}$ is invariant with respect to the geodesic flow.

Consider the projection of every subspace $l_{inv}$ along the plane $< A, D_1 >$ to the plane $< D_2, D_ϕ >$. Denote the projection by $l_{inv}'$.

Consider the mapping $dπ : T(TP^2) → TP^2$. Then $dπ(l_{inv}')$ is 1-dimensional subspace in the tangent planes at the points of the geodesic. Using Lemma 1, we see that the set of $(l_{inv}')$ is invariant with respect to the geodesic flow. Since the vector $D_ϕ$ projects to 0, and since the projection of $D_2$ is orthogonal to the geodesic $γ$, we have that $dπ(l_{inv})$ is orthogonal to the geodesic $γ$.

Consider a direction vector field $Y$ of the set $l_{inv}$. Let an invariant vector field $ξ$ be parallel to $Y$. Then $ξ(t)$ equals $κ(t)Y(t)$, where $κ(t)$ is a smooth function. Suppose the coordinates of the vector field $Y$ in the frame $< D_ϕ, D_1, D_2, A >$ are equal to $(α(t), 0, β(t), 0)$. (Since $Y$ lies in the plane $< D_ϕ, D_2 >$, we see that the second and the fourth coordinates must be equal to 0.)

We have $Y(t) = α(t)D_2 + β(t)D_ϕ$. Then, $ξ(t) = κ(t)α(t)D_2 + κ(t)β(t)D_ϕ$. Using (4), we obtain $\frac{dl}{dt}(κα) = κβ$. Therefore,

$$κ(t) = \exp \left[\int_{t_0}^{t} \frac{β(s) - α(s)}{α(s)} ds\right] = \frac{α(t_0)}{α(t)} \exp \left[\int_{t_0}^{t} \frac{β(s)}{α(s)} ds\right].$$

Thus the vector field

$$ξ = \frac{α(t_0)}{α(t)} \exp \left[\int_{t_0}^{t} \frac{β(s)}{α(s)} ds\right] Y$$

is invariant with respect to the geodesic flow, and the projection of it is a Jacobi vector field.

In particular, the function

$$\exp \left[\int_{t_0}^{t} \frac{β(s)}{α(s)} ds\right]$$

is
satisfies the Jacobi-Hill equation $\ddot{x} + r^2 K x = 0$.

Suppose a geodesic $\gamma$ is hyperbolic. Then we have two invariant intersecting surfaces in a regular neighborhood of $\gamma$. Consider the subspaces that are tangent to these surfaces. Since the intersecting surfaces are invariant then the tangent subspaces are invariant, too. Then the projection of the surfaces to the planes $\langle D_\phi, D_2 \rangle$ along the planes $\langle D_1, A \rangle$ is two families of invariant 1-dimension subspace. Arguing as above, we can construct two Jacobi vector fields. The Jacobi vector field are nonproportional. Hence they define a fundamental solution of the Jacobi-Hill equation.

**Remark** It is easy to find the coordinates $\alpha, \beta$. Let the metric $G$ has the form $\lambda(dx^2 + dy^2)$. Let a field $Y$ equals $k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + K_1 \frac{\partial}{\partial p_x} + K_2 \frac{\partial}{\partial p_y}$. The coefficients $k_1, k_2, K_1, K_2$ are functions of $t$. Then,

$$\alpha = \frac{\dot{k}_2 - y k_1}{r^2}, \quad (4)$$

$$\beta = \left( \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x} k_2 - \frac{1}{2\lambda} \frac{\partial \lambda}{\partial y} k_1 \right) + \frac{\dot{x} K_2 - \dot{y} K_1}{r^2}. \quad (5)$$

**Remark** Now let the projection $d\pi(l_{inv})$ of an invariant set of subspaces $l_{inv}$ is not orthogonal to the geodesic. Consider the projection of every subspace $l_{inv}$ along the plane $\langle A, D_1 \rangle$ to the plane $\langle D_2, D_6 \rangle$. Denote the projection by $l'_{inv}$. Let $Y$ be a direction vector field of the subspaces $l_{inv}$, denote by $Y'$ the projection of the vector field $Y$ along the plane $\langle A, D_1 \rangle$ to the plane $\langle D_2, D_6 \rangle$. Suppose that in the coordinate system $(x, y, p_x, p_y)$ vector fields $Y, Y'$ are equal to $(k_1, k_2, K_1, K_2)$, $(k'_1, k'_2, K'_1, K'_2)$, respectively. Then,

$$\lambda \frac{\dot{x} k_2 - \dot{y} k_1}{r^2} = \lambda \frac{\dot{x} k'_2 - \dot{y} k'_1}{r^2},$$

$$\left( \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x} k_2 - \frac{1}{2\lambda} \frac{\partial \lambda}{\partial y} k_1 \right) + \frac{\dot{x} K_2 - \dot{y} K_1}{r^2} = \left( \frac{1}{2\lambda} \frac{\partial \lambda}{\partial x} k'_2 - \frac{1}{2\lambda} \frac{\partial \lambda}{\partial y} k'_1 \right) + \frac{\dot{x} K'_2 - \dot{y} K'_1}{r^2}. \quad (5)$$

Now let the curve $\gamma(\tau) = (x(\tau), y(\tau))$ is a geodesic. We do not require the parameter $\tau$ to be natural or natural, multiplying by a constant. Let the coordinates $(x, y)$ be isothermal. That is the metric $G$ has the form $\lambda(x, y)(dx^2 + dy^2)$. Suppose a geodesic trajectory $\tilde{\gamma}$ projects into the geodesic $\gamma$. Let $r$ be the length of the velocity vector of the geodesic in the parameter $t$. Consider the functions $k_1, k_2, K_1, K_2$ as the functions of the new parameter $\tau$.

**Lemma 2** Let $\tau(t)$ be the connection between the parameter $\tau$ and the parameter $t$. Consider the function $u(\tau) = \exp \left[ \int_{\tau_0}^{\tau} \frac{\beta(s)}{\alpha(s)} |\frac{dz}{ds}| ds \right]$, where

$$\alpha(\tau) = \lambda(\tau) \frac{\frac{dx}{ds} k_2 - \frac{dy}{ds} k_1}{r^2 |\frac{dx}{ds}|}, \quad \beta(\tau) = \left( \frac{1}{2\lambda(\tau)} \frac{\partial \lambda(\tau)}{\partial x} k_2 - \frac{1}{2\lambda(\tau)} \frac{\partial \lambda(\tau)}{\partial y} k_1 \right) + \frac{\frac{dx}{ds} K_2 - \frac{dy}{ds} K_1}{r^2 |\frac{dx}{ds}|}. \quad (5)$$

Then the function $u(t) = u(\tau(t))$ is a solution of the Jacobi-Hill equation.
The proof is by direct calculation.

§4. Quadratically integrable geodesic flows on the torus and on the sphere.

4.1. Quadratically integrable geodesic flows on the torus.

Let $L$ be a positive number. Denote by $S_L$ the circle with a smooth parameter $t \in \mathbb{R} \mod L$.

**Definition 11** A metric $G$ on the torus $T^2$ is called Liouville if for an appropriate positive $L$ and for appropriate nonconstant positive functions $f : S_1 \to \mathbb{R}$, $h : S_L \to \mathbb{R}$ there exist a diffeomorphism $\chi : T^2 \to S_1 \times S_L$ that takes the metric $G$ to the metric $(f(x) + h(y))(dx^2 + dy^2)$, where $x, y$ are parameters on $S_1, S_L$, respectively.

**Definition 12** A metric $G$ on the torus $T^2$ is called pseudo-liouville if there exists a Liouville metric $G_{\text{liuv}}$ on the torus $T^2$ and a covering $\rho : T^2 \to T^2$ such that $\rho^*(G_{\text{liuv}}) = G$.

**Remark** Since the identity mapping $id : T^2 \to T^2$ is a 1-sheet covering a Liouville metric is a pseudo-liouville metric.

**Theorem 2** (10, 13) The geodesic flow of a metric $G$ on the torus $T^2$ is quadratically integrable iff the metric $G$ is pseudo-liouville.

4.2. Quadratically integrable geodesic flows on the sphere.

Consider the torus $S_1 \times S_L$. Let for smooth functions $f : S_1 \to \mathbb{R}$, $h : S_L \to \mathbb{R}$ the following conditions hold

1. The functions $f$, $h$ are nonnegative. The function $f$ equals zero only in the points $0$ and $\frac{1}{2}$.
   The function $h$ equals zero only in the points $0$ and $\frac{L}{2}$.

2. $f''(0) = f''(\frac{1}{2}) = h''(0) = h''(\frac{L}{2}) \neq 0$.

3. For any $x, y$ $f(x) = f(-x)$ and $h(y) = h(-y)$.

Consider the involution $\sigma : S_1 \times S_L \to S_1 \times S_L$, $\sigma(x, y) \overset{\text{def}}{=} (-x, -y)$. The fixed points of $\sigma$ are: $(0, 0)$, $(0, \frac{L}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{L}{2})$.

Consider the factorspace $\tilde{S} \overset{\text{def}}{=} S_1 \times S_L / \sigma$. $\tilde{S}$ is homeomorphic to the sphere. Consider the following structure of a smooth 2-dimensional manifold on $\tilde{S}$.

Denote by $\chi : S_1 \times S_L \to \tilde{S}$ the dual to the involution $\tilde{S}$ mapping. In other words, the mapping $\chi$ takes a point $(x, y) \in S_1 \times S_L$ to the point $((x, y), (-x, -y)) \in \tilde{S}$.

Consider smooth structure on $\tilde{S}$ such that the mapping $\chi$ is a smooth branched covering with branch points $\chi(0,0)$, $\chi(0, \frac{L}{2})$, $\chi(\frac{1}{2}, 0)$, $\chi(\frac{1}{2}, \frac{L}{2})$, $\chi(\frac{1}{2}, 0)$, $\chi(\frac{1}{2}, \frac{L}{2})$ of the branch index 1.
Obviously, the smooth structure exists.

Note, that the mapping $\chi$ in the appropriate coordinates is the Weierstrass $\wp$-function with half-periods $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

Consider the degenerate metric $((f(x) + h(y))(dx^2 + dy^2))$ on the torus $S_1 \times S_L$. The form $((f(x) + h(y))(dx^2 + dy^2))$ is positive definite everywhere except at the points $(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$. Since the function $f(x) + h(y)$ is preserved by the involution $\sigma$, we see that the metric $((f(x) + h(y))(dx^2 + dy^2))$ induces a metric on the sphere $\tilde{S}$ without branch points. The conditions on the functions $f$, $h$ allow to complement the metric in the branch points. Denote by $G_{L,f,h}$ the complemented metric.

**Definition 13** A metric $G$ on the sphere $S^2$ is called a Kolokoltzov metric if for an appropriate positive number $L > 0$ and for appropriate functions $f$, $h$ there exists a diffeomorphism $\tilde{S} \to S^2$ that takes the metric $G_{L,f,h}$ to the metric $G$.

**Theorem 3** (9) The geodesic flow of a metric $G$ on the sphere $S^2$ is quadratically integrable iff the metric $G$ is a Kolokoltzov metric.

**Remark** If the functions $f$ and $h$ are smooth functions, then the metric $G_{L,f,h}$ is smooth on $\tilde{S}$ without the branch points. The metric $G_{L,f,h}$ is $C^k$-smooth in a branch point iff the following condition holds. In the branch point Taylor series of the function $f$ as the function of $x$ coincides till $(k + 2)$-member with Taylor series of the function $h$ as the function of $-x$ (see [9]). In other words, the metric $G_{L,f,h}$ is $C^k$-smooth in a branch point $(0, 0)$ if for any natural $m \leq k + 2$, $\frac{d^m f}{dx^m} |_{0} = (-1)^m \frac{d^m h}{dy^m} |_{0}$.

4.3. Hyperbolic trajectories of the quadratically integrable geodesic flows on the torus.

Consider the isoenergy surface $\{ H = 1 \} = \{ \frac{p_x^2 + p_y^2}{f(x) + h(y)} = 1 \}$.

Following the paper [11] we describe the set of critical circles of the geodesic flow. For simplicity suppose $f$, $h$ are Morse functions.

Denote by $K(f)$ ($K(h)$) the set of critical points of the function $f$ (respectively, $h$). In [11], E. Selivanova proved that the set of critical points of the metric $G$ is the union of the sets

\[ O^+_f = \{(x, y, p_x, p_y) : x \in K(f), p_x = 0, p_y = \sqrt{f(x) + h(y)}\}, \]

\[ O^-_f = \{(x, y, p_x, p_y) : x \in K(f), p_x = 0, p_y = -\sqrt{f(x) + h(y)}\}, \]

\[ O^+_h = \{(x, y, p_x, p_y) : y \in K(h), p_x = \sqrt{f(x) + h(y)}, p_y = 0\}, \]

\[ O^-_h = \{(x, y, p_x, p_y) : y \in K(h), p_x = -\sqrt{f(x) + h(y)}, p_y = 0\}. \]

Besides, the circle

\[ \{(x, y, p_x, p_y) : x = x_0 \in K(f), p_x = 0, p_y = -\sqrt{f(x) + h(y)}\} \]
is a saddle circle iff the point $x_0$ is a nodegenerate critical point of the Morse index 1.

It is possible to prove that the saddle circles of quadratically integrable geodesic flow on the torus are hyperbolic trajectories.

4.4. Saddle circles of the quadratically integrable geodesic flow on the sphere.

Suppose the functions $f$, $h$ satisfy the conditions 1, 2, 3 from section 4.2. Consider the torus $S_1 \times S_L$ with the (degenerate) metric $ds^2 = (f(x) + h(y))(dx^2 + dy^2)$. Consider the torus without the points $(0, 0)$, $(0, \frac{L}{2})$, $(\frac{L}{2}, \frac{L}{2})$, $(\frac{L}{2}, 0)$. Then $ds^2$ is a metric. Using the previous section, we have that the saddle circles of the metric are the circles

$\{(x, y, px, py) : x = x_0, px = 0, py = \pm \sqrt{f(x) + h(y)}, \text{where} \ x_0 \text{is a critical point of the Morse index 1}\}$

and

$\{(x, y, px, py) : y = y_0, py = 0, px = \pm \sqrt{f(x) + h(y)}, \text{where} \ y_0 \text{is a critical point of the Morse index 1}\}$.

The involution $\sigma$ rearrange pairs of circles. Since that, any pair factorize to a saddle circle. Such saddle circles are called simple. If the projection of the saddle circle of the geodesic flow does not contain the point from the set $\{(0, 0), (0, \frac{L}{2}), (\frac{L}{2}, \frac{L}{2}), (\frac{L}{2}, 0)\}$, then the circle is simple.

There exist two nonsimple saddle circles (we denote them by $\hat{\gamma}_1$, $\hat{\gamma}_2$) of the geodesic flow of a Kolokoltzov metric. Consider the segments

$I_f^+ = \{(x, y, px, py) : x \in (0, \frac{1}{2}), y = 0, py = 0, px = \sqrt{f(x) + h(y)}\}$,

$I_f^- = \{(x, y, px, py) : x \in (0, \frac{1}{2}), y = \frac{1}{2}, py = 0, px = -\sqrt{f(x) + h(y)}\}$,

$I_h^+ = \{(x, y, px, py) : x = 0, y \in (0, \frac{L}{2}), py = 0, py = \sqrt{f(x) + h(y)}\}$,

$I_h^- = \{(x, y, px, py) : x = 0, y \in (0, \frac{L}{2}), py = 0, py = -\sqrt{f(x) + h(y)}\}$.

The segments factorize to the circle (denoted by $\hat{\gamma}_1$). Since the restriction of the integral to the transversal disk to an intervals has a singularity of Morse index 1, we see that $\hat{\gamma}_1$ is a saddle circle.

Note, that the simple saddle circles of the geodesic flow of a Kolokoltzov metric are hyperbolic trajectories. There exist examples of the Kolokoltzov metric such that the saddle circles ($\hat{\gamma}_1$, $\hat{\gamma}_2$) are not hyperbolic trajectories.

§5. Fundamental solution of the Jacobi equation for the hyperbolic geodesics of quadratically integrable geodesic flows.

The formulas (3–5) allows to construct a fundamental solution of the Jacobi-Hill equation for saddle circles of the quadratically integrable geodesic flows.

5.1. Torus. Without loss of generality it can be assumed that a saddle circle as the set of points coincides with the set $\{(x, y, px, py) : y = 0, py = \pm \sqrt{f(x) + h(y)}\}$.
\( p_x = \sqrt{f(x) + h(y)} \), and that \( h(0) = 0 \). Consider the Liouville fiber, which contains the saddle circle. The Liouville fiber is the set \( \{ (x, y, p_x, p_y) : p_x = \sqrt{f(x)}, p_y = \pm \sqrt{h(y)} \} \). It is easy to see that the vectors \( Y_\pm = \partial_x \pm \frac{\sqrt{f(x)}}{\sqrt{2}} \partial_p \) are

1) tangential to the Liouville fiber and

2) lie in the planes \( < D^2 D_\phi > \).

Therefore the vectors \( Y_\pm = \partial_x \pm \frac{\sqrt{f(x)}}{\sqrt{2}} \partial_p \) are direction vectors of two families of invariant 1-subspaces.

Using Lemma 3, we get the following equations for \( \alpha, \beta \).

\[
\alpha(x) = \pm \sqrt{f(x)} \quad (6)
\]

\[
\beta(x) = \frac{f'(x)}{2\sqrt{h}} - \sqrt{\frac{h''(0)}{2h}} \quad (7)
\]

Combining (6, 7) with the formula for \( u \), we get a fundamental solution of the Jacobi-Hill equation:

\[
u_+(x) = \sqrt{f(x)} \exp \left[ \sqrt{\frac{h''(0)}{2}} \int_{x_0}^x \frac{ds}{\sqrt{f(s)}} \right]
\]

\[
u_-(x) = \sqrt{f(x)} \exp \left[ -\sqrt{\frac{h''(0)}{2}} \int_{x_0}^x \frac{ds}{\sqrt{f(s)}} \right]
\]

**Remark** Since the function \( u_+ \) increase and since the function \( u_- \) decrease, we see that there are no conjugate points along the hyperbolic geodesics of a quadratically integrable geodesic flow on the torus.

**5.2. Sphere.** For a simple geodesic the answer coinsides with the answer for the torus.

Consider the nonsimple saddle circle \( \hat{\gamma}_1 \). The circle can be represented as four glued segments \( I^+_h, I^+_f, I^-_h, I^-_f \). Using (8), we see that the pair of functions

\[
u'^+ (x) = \sqrt{f(x)} \exp \left[ \sqrt{\frac{h''(0)}{2}} \int_{x_0}^x \frac{ds}{\sqrt{f(s)}} \right]
\]

\[
u'^- (x) = \sqrt{f(x)} \exp \left[ -\sqrt{\frac{h''(0)}{2}} \int_{x_0}^x \frac{ds}{\sqrt{f(s)}} \right]
\]

is a fundamental solution on the segments \( I^+_f, I^-_f \).

Arguing as above, the pair of functions

\[
u^h_+ (y) = \sqrt{h(y)} \exp \left[ \sqrt{\frac{h''(0)}{2}} \int_{y_0}^y \frac{ds}{\sqrt{h(s)}} \right]
\]

\[
u^h_- (y) = \sqrt{h(y)} \exp \left[ -\sqrt{\frac{h''(0)}{2}} \int_{y_0}^y \frac{ds}{\sqrt{h(s)}} \right]
\]
is a fundamental solution on the segments $I_h^+, I_h^-$. We shall glue the fundamental solutions of the Jacobi-Hill equation in the points $(0, 0), (0, \frac{L}{2}), (\frac{L}{2}, 0), (\frac{L}{2}, \frac{L}{2})$. Consider the point $(0, 0)$. We have to find constants $C_{11}, C_{12}, C_{21}, C_{22}$ such that

$$\lim_{x \to 0} C_{11} \left( \frac{u^f_+}{\dot{u}^f_+} + C_{12} \frac{u^f_-}{\dot{u}^f_-} \right) = \lim_{y \to 0} \left( \frac{u^h_+}{\dot{u}^h_+} \right),$$

$$\lim_{x \to 0} C_{21} \left( \frac{u^f_+}{\dot{u}^f_+} + C_{22} \frac{u^f_-}{\dot{u}^f_-} \right) = \lim_{y \to 0} \left( \frac{u^h_-}{\dot{u}^h_-} \right).$$

Therefore,

$$C_{11} = \lim_{y \to 0} \sqrt{h(y)} \exp \left[ \frac{\sqrt{f''(0)}}{2} \int_{y_0}^y \frac{ds}{\sqrt{h(s)}} \right], \quad C_{12} = 0 \quad (11)$$

$$C_{21} = 0, \quad C_{22} = \lim_{x \to 0} \sqrt{f(x)} \exp \left[ \frac{\sqrt{h''(0)}}{2} \int_{x_1}^x \frac{ds}{\sqrt{f(s)}} \right]. \quad (12)$$

Arguing as above, we can glue the fundamental solutions in the points $(0, 0) = A_1, \left( \frac{L}{2}, \frac{L}{2} \right) = A_2, \left( \frac{L}{2}, 0 \right) = B_1, \left( 0, \frac{L}{2} \right) = B_2$. We obtain a fundamental solution $(x_+, x_-)$ of the Jacobi-Hill equation along the geodesic line $\pi(\gamma_1)$.

Consider a point $x_1 \in I_h^f$. Using the Sturm-Liouville theorem, we see that there exists a point $(\frac{L}{2} - x_2) \in I_h$ that is conjugate to the point $x_1$. Using (11-12), we get the following equation for $x_2$:

$$\lim_{y \to 0} \sqrt{h(y)h(\frac{L}{2} - y)} \exp \left[ \frac{\sqrt{f''(0)}}{2} \int_{y_0}^y \frac{ds}{\sqrt{h(s)}} \right] =$$

$$= \lim_{x \to 0} \sqrt{f(x)} \exp \left[ \frac{\sqrt{h''(0)}}{2} \int_{x_1}^x \frac{ds}{\sqrt{f(s)}} \right]. \quad (13)$$

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