PRINCIPAL BUNDLES ON COMPACT COMPLEX MANIFOLDS
WITH TRIVIAL TANGENT BUNDLE

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Abstract. Let $G$ be a connected complex Lie group and $\Gamma \subset G$ a cocompact lattice. Let $H$ be a complex Lie group. We prove that a holomorphic principal $H$–bundle $E_H$ over $G/\Gamma$ admits a holomorphic connection if and only if $E_H$ is invariant. If $G$ is simply connected, we show that a holomorphic principal $H$–bundle $E_H$ over $G/\Gamma$ admits a flat holomorphic connection if and only if $E_H$ is homogeneous.

1. Introduction

Let $T = \mathbb{C}^n/\Gamma$ be a complex torus, so $\Gamma$ is a lattice of $\mathbb{C}^n$ of maximal rank. For any $x \in T$, let $\tau_x : T \to T$ be the holomorphic automorphism defined by $z \mapsto z + x$. Let $H$ be a connected linear algebraic group defined over $\mathbb{C}$. A holomorphic principal $H$–bundle $E_H$ over $T$ admits a holomorphic connection if and only if $\tau_x^* E_H$ is holomorphically isomorphic to $E_H$ for every $x \in T$; also, if $E_H$ admits a holomorphic connection, then it admits a flat holomorphic connection [3, p. 41, Theorem 4.1].

If $\Gamma$ is a cocompact lattice in a connected complex Lie group $G$, then $G/\Gamma$ is clearly a compact connected complex manifold with trivial tangent bundle. Let $M$ be a connected compact complex manifold such that the holomorphic tangent bundle $TM$ is holomorphically trivial. Then there is a connected complex Lie group $G$ and a cocompact lattice $\Gamma \subset G$ such that $G/\Gamma$ is biholomorphic to $M$. The manifold $M$ is Kähler if and only if $M$ is a torus. Our aim here is to investigate principal bundles on $M$ admitting a (flat) holomorphic connection.

Let $G$ be a connected complex Lie group and $\Gamma \subset G$ a cocompact lattice. For any $g \in G$, let

$$\beta_g : M := G/\Gamma \to G/\Gamma$$

be the automorphism defined by $x \mapsto gx$. Let $H$ be a connected complex Lie group.

A holomorphic principal $H$–bundle $E_H$ over $M$ is called invariant if for each $g \in G$, the pulled back bundle $\beta_g^* E_H$ is isomorphic to $E_H$. A homogeneous holomorphic principal $H$–bundle on $M$ is a pair $(E_H, \rho)$, where $f : E_H \to M$ is a holomorphic principal $H$–bundle, and

$$\rho : G \times E_H \to E_H$$

is a holomorphic left–action on the total space of $E_H$, such that the following two conditions hold:

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(1) \((f \circ \rho)(g, z) = \beta_g(f(z))\) for all \((g, z) \in G \times E_H\), and
(2) the actions of \(G\) and \(H\) on \(E_G\) commute.

If \((E_H, \rho)\) is homogeneous, then \(E_H\) is invariant.

We prove the following theorem (see Theorem \ref{thm:1.1} and Proposition \ref{prop:3.2}):\n
**Theorem 1.1.** A holomorphic principal \(H\)-bundle \(E_H\) over \(M\) admits a holomorphic connection if and only if \(E_H\) is invariant.

Assume that the group \(G\) is simply connected. A holomorphic principal \(H\)-bundle \(E_H\) over \(M\) admits a flat holomorphic connection if and only if \(E_H\) is homogeneous.

If \(\text{Lie}(G)\) is semisimple and \(G\) is simply connected, then we prove that given any invariant principal \(H\)-bundle \(E_H\), there is a holomorphic action \(\rho : G \times E_H \to E_H\) such that \((E_H, \rho)\) is homogeneous (see Lemma \ref{lem:4.1}). This gives the following corollary (see Corollary \ref{cor:4.2}):

**Corollary 1.2.** Assume that \(\text{Lie}(G)\) is semisimple and \(G\) is simply connected. If a holomorphic principal \(H\)-bundle \(E_H \to M\) admits a holomorphic connection, then it admits a flat holomorphic connection.

The compact complex manifolds \(G/\Gamma\) of the above type with \(G\) non-commutative are the key examples of non-Kähler compact complex manifolds with trivial canonical bundle. Recently, these manifolds have started to play important role in string theory of theoretical physics (see \cite{5}, \cite{2}, \cite{6}). They have also become a topic of investigation in complex differential geometry (see \cite{8}, \cite{7}).

## 2. Homogeneous bundles and holomorphic connection

### 2.1. Holomorphic connection

Let \(G\) be a connected complex Lie group. Let \(\Gamma \subset G\) be a cocompact lattice. So
\[
M := G/\Gamma
\]
is a compact complex manifold. Let \(\mathfrak{g}\) be the Lie algebra of \(G\). Using the right-invariant vector fields, the holomorphic tangent bundle \(TM\) is identified with the trivial vector bundle \(M \times \mathfrak{g}\) with fiber \(\mathfrak{g}\), so
\[
TM = M \times \mathfrak{g} \to M.
\]
Let
\[
\beta : G \times M \to M
\]
be the left translation action. The map \(\beta\) is holomorphic. For any \(g \in G\), let
\[
\beta_g : M \to M
\]
be the automorphism defined by \(x \mapsto \beta(g, x)\).
Let $H$ be a connected complex Lie group. The Lie algebra of $H$ will be denoted by $\mathfrak{h}$. We recall that a holomorphic principal $H$–bundle over $M$ is a complex manifold $E_H$, a surjective holomorphic submersion $f : E_H \rightarrow M$ and a right holomorphic action of $H$ on $E_H$

$$\varphi : E_H \times H \rightarrow E_H$$

(so $\varphi$ is a holomorphic map), such that the following two conditions hold:

1. $f \circ \varphi = f \circ p_1$, where $p_1$ is the projection of $E_H \times H$ to the first factor, and
2. the action of $H$ on each fiber of $f$ is free and transitive.

Let $f : E_H \rightarrow M$ be a holomorphic principal $H$–bundle. Let $TE_H$ be the holomorphic tangent bundle of $E_H$. The group $H$ acts on the direct image $f^*TE_H$. The invariant part

$$At(E_H) := (f^*TE_H)^H \subset f^*TE_H$$

defines a holomorphic vector bundle on $M$, which is called the Atiyah bundle for $E_H$. Let

$$K := \ker(df)$$

be the kernel of the differential $df : TE_H \rightarrow f^*TM$ of $f$. The invariant direct image $(f_K)^H$ coincides with the sheaf of sections of the adjoint vector bundle $\text{ad}(E_H)$. We recall that $\text{ad}(E_H) \rightarrow M$ is the vector bundle associated to $E_H$ for the adjoint action of $H$ on $\mathfrak{h}$. Using the inclusion of $(f_K)^H$ in $(f^*TE_H)^H$, we get a short exact sequence of holomorphic vector bundles on $M$

\begin{equation}
0 \rightarrow \text{ad}(E_H) \rightarrow At(E_H) \overset{df}{\rightarrow} TM \rightarrow 0.
\end{equation}

It is known as the Atiyah exact sequence for $E_H$. (See [1].)

A holomorphic connection on $E_H$ is a holomorphic splitting of the short exact sequence in (5). In other words, a holomorphic connection on $E_H$ is a holomorphic homomorphism

$$D : TM \rightarrow At(E_H)$$

such that $(df) \circ D = \text{Id}_{TM}$, where $df$ is the homomorphism in (5) (see [1]).

Let $D$ be a holomorphic connection connection on $E_H$. The curvature of $D$ is the obstruction of $D$ to be Lie algebra structure preserving (the Lie algebra structure of sheaves of sections of $TM$ and $At(E_H)$ is given by the Lie bracket of vector fields). The curvature of $D$ is a holomorphic section of $\text{ad}(E_H) \otimes \bigwedge^2(TM)^*$. (See [1] for the details.)

A flat holomorphic connection is a holomorphic connection whose curvature vanishes identically.

2.2. Invariant and homogeneous bundles. We will now define invariant holomorphic principal bundles and homogeneous principal bundles.

Definition 2.1. A holomorphic principal $H$–bundle $E_H$ over $M$ will be called invariant if for each $g \in G$, the pulled back holomorphic principal $H$–bundle $\beta_g^*E_H$ is isomorphic to $E_H$, where $\beta_g$ is the map in (4).

Definition 2.2. A homogeneous holomorphic principal $H$–bundle on $M$ is defined to be a pair $(E_H, \rho)$, where
• $f : E_H \rightarrow M$ is a holomorphic principal $H$–bundle, and
• $\rho : G \times E_H \rightarrow E_H$ is a holomorphic left–action on the total space of $E_H$,
such that the following two conditions hold:

1. $(f \circ \rho)(g, z) = \beta_g(f(z))$ for all $(g, z) \in G \times E_H$, where $\beta_g$ is defined in (4), and
2. the actions of $G$ and $H$ on $E_G$ commute.

If $(E_H, \rho)$ is a homogeneous holomorphic principal $H$–bundle, then $E_H$ is invariant. Indeed, for any $g \in G$, the automorphism of $E_H$ defined by $z \mapsto \rho(g, z)$ produces an isomorphism of $E_H$ with $\beta_g^*E_H$.

3. AUTOMORPHISMS OF PRINCIPAL BUNDLES

We continue with the notation of the previous section. We will give a criterion for the existence of a (flat) holomorphic connection on a holomorphic principal $H$–bundle over $M$.

**Theorem 3.1.** A holomorphic principal $H$–bundle $E_H$ over $M$ admits a holomorphic connection if and only if $E_H$ is invariant.

**Proof.** Let $f : E_H \rightarrow M$ be a holomorphic principal $H$–bundle. Let $\mathcal{A}$ denote the space of all pairs of the form $(g, \phi)$, where $g \in G$, and

$$\phi : E_H \rightarrow E_H$$

is a biholomorphism satisfying the following two conditions:

1. $\phi$ commutes with the action of $H$ on $E_H$, and
2. $f \circ \phi = \beta_g \circ f$, where $\beta_g$ is defined in (4).

So $\phi$ gives a holomorphic isomorphism of the principal $H$–bundle $\beta_g^*E_H$ with $E_H$.

We note that $\mathcal{A}$ is a group using the composition rule

$$(g_1, \phi_1) \cdot (g_2, \phi_2) = (g_1g_2, \phi_1 \circ \phi_2).$$

In fact, $\mathcal{A}$ is a complex Lie group, and the Lie algebra

$$\mathfrak{a} := \text{Lie}(\mathcal{A})$$

is identified with $H^0(M, \text{At}(E_H))$. Note that from (2) it follows that all holomorphic vector fields on $M$ are given by the Lie algebra of $G$ using the action $\beta$ in (3). The compactness of $M$ ensures that $\mathcal{A}$ is of finite dimension. As noted before, using the Lie bracket of vector fields, the sheaf of holomorphic sections of $\text{At}(E_H)$ has the structure of a Lie algebra. Hence the space $H^0(M, \text{At}(E_H))$ of global holomorphic sections is a Lie algebra.

Let

$$(6) \quad q : \mathcal{A} \rightarrow G$$

be the projection defined by $(g, \phi) \mapsto g$. The homomorphism of Lie algebras

$$dq : \mathfrak{a} = H^0(M, \text{At}(E_H)) \rightarrow \mathfrak{g} := \text{Lie}(G) = H^0(M, TM)$$
associated to \( q \) in (6) coincides with the one given by the homomorphism \( df \) in (5).

First assume that \( E_H \) admits a holomorphic connection. Recall that a holomorphic connection on \( E_H \) is a holomorphic splitting of the exact sequence in (5). Using a holomorphic connection, the vector bundle \( \text{At}(E_H) \) gets identified with the direct sum \( \text{ad}(E_H) \oplus TM \).

In particular, the homomorphism

\[
H^0(M, \text{At}(E_H)) \longrightarrow H^0(M, TM)
\]

induced by \( df \) in (5) is surjective. Hence the homomorphism \( dq \) in (7) is surjective. Since \( G \) is connected, this implies that the homomorphism \( q \) in (6) is surjective. This immediately implies that \( E_H \) is invariant (recall that for any \((g, \phi) \in A\), the map \( \phi \) is a holomorphic isomorphism of the principal \( H \)–bundle \( \beta^* g E_H \) with \( E_H \)).

To prove the converse, assume that \( E_H \) is invariant. Therefore, the homomorphism \( q \) in (6) is surjective. Hence the homomorphism \( dq \) in (7) is surjective. Let \( d \) be the dimension of \( G \). Since \( dq \) is surjective, and \( TM \) is the trivial vector bundle of rank \( d \), there are \( d \) sections

\[
\sigma_1, \ldots, \sigma_d \in H^0(M, \text{At}(E_H))
\]

such that \( H^0(M, TM) = g \) is generated by \( \{dq(\sigma_1), \ldots, dq(\sigma_d)\} \).

We have a holomorphic homomorphism

\[
D : TM \longrightarrow \text{At}(E_H)
\]

defined by \( \sum_{i=1}^{d} c_i \cdot dq(\sigma_i)(x) \mapsto \sum_{i=1}^{d} c_i \cdot \sigma_i(x) \), where \( x \in M \), and \( c_i \in \mathbb{C} \). It is straightforward to check that \( (df) \circ D = \text{Id}_{TM} \), where \( df \) is the homomorphism in (5). Hence \( D \) defines a holomorphic connection on \( E_H \).

**Proposition 3.2.** Assume that the group \( G \) is simply connected. A holomorphic principal \( H \)–bundle \( E_H \) over \( M \) admits a flat holomorphic connection if and only if \( E_H \) is homogeneous.

**Proof.** Let \( f : E_H \longrightarrow M \) be a holomorphic principal \( H \)–bundle equipped with a flat holomorphic connection \( D \). Therefore, \( E_H \) is given by a homomorphism from the fundamental group \( \pi_1(M, (e, e\Gamma)) \) to \( H \). We will construct an action of \( G \) on \( E_H \).

Let \( p_2 : G \times M \longrightarrow M \) be the projection to the second factor. Since \( G \) is simply connected, the homomorphisms

\[
p_{2*}, \beta_* : \pi_1(G \times M, (e, e\Gamma)) \longrightarrow \pi_1(M, (e, e\Gamma))
\]

induced by \( p_2 \) and \( \beta \) (see (3)) coincide. Therefore, there is a canonical isomorphism of flat principal \( H \)–bundles

\[
\mu : p_{2*} E_H \longrightarrow \beta^* E_H
\]

which is the identity map over \( \{e\} \times M \).

This map \( \mu \) defines an action

\[
\rho : G \times E_H \longrightarrow E_H;
\]

for any \((g, x) \in G \times M\), the isomorphism

\[
\rho_{g,x} : (E_H)_x \longrightarrow (E_H)_{\beta g(x)}
\]
is the restriction of $\mu$ to $(g,x)$, where $\beta_g$ is the map in (4). This action $\rho$ makes $E_H$ a homogeneous bundle.

To prove the converse, take a homogeneous holomorphic principal $H$–bundle $(E_H, \rho)$. For any $g \in G$, let

$$\rho_g : E_H \longrightarrow E_H$$

be the map defined by $z \longmapsto \rho(g, z)$. Consider the group $A$ constructed in the proof of Theorem 3.1. Let

$$\delta : G \longrightarrow A$$

be the homomorphism defined by $g \longmapsto (g, \rho_g)$. It is easy to see that

$$q \circ \delta = \text{Id}_G,$$

where $q$ is the homomorphism in (6).

Let

$$d\delta : \mathfrak{g} \longrightarrow \text{Lie}(A) = H^0(M, \text{At}(E_H))$$

be the homomorphism of Lie algebras associated to the homomorphism $\delta$ in (8). From (9) it follows that

$$(dq) \circ d\delta = \text{Id}_g,$$

where $dq$ is the homomorphism in (7) (it is the homomorphism of Lie algebras corresponding to $q$).

Since $TM$ is the trivial vector bundle with fiber $\mathfrak{g}$, the homomorphism $d\delta$ in (10) produces a homomorphism of vector bundles

$$d\delta : TM \longrightarrow \text{At}(E_H);$$

$$\tilde{d}\delta(x,v) = (d\delta)(v)(x),$$

where $v \in \mathfrak{g}$, and $x \in M$. Combining (11) and the fact that the homomorphism $dq$ coincides with the one given by the homomorphism $df$ in (5), we conclude that $(df) \circ d\delta = \text{Id}_{TM}$. Therefore, $d\delta$ defines a holomorphic connection on $E_H$. The curvature of this holomorphic connection vanishes identically because the linear map $d\delta$ is Lie algebra structure preserving.

\[\square\]

Remark 3.3. In Proposition 3.2, it is essential to assume that $G$ is simply connected. To give examples, take $G$ to be an elliptic curve and $\Gamma$ to be the trivial group $e$. Take $H = \mathbb{C}^*$, and take $E_H = L$ to be a nontrivial holomorphic line bundle of degree zero. Then $L$ admits a flat holomorphic connection because $L$ is topologically trivial, while $L$ does not admit any homogeneous structure because $L$ is not trivial.

4. The semisimple case

In this section we assume that $\mathfrak{g}$ is semisimple, and $G$ is simply connected.

Let $f : E_H \longrightarrow M := G/\Gamma$ be an invariant holomorphic principal $H$–bundle.
Lemma 4.1. There is a holomorphic left-action of $G$
\[ \rho : G \times E_H \longrightarrow E_H \]
such that $(E_H, \rho)$ is homogeneous.

Proof. Consider the homomorphism
\[ dq : a := \text{Lie}(A) \longrightarrow g \]
in (7). It is surjective because $E_H$ is invariant implying that $q$ is surjective. Since $g$ is semisimple, there is a Lie algebra homomorphism
\[ \theta : g \longrightarrow a \]
such that $(dq) \circ \theta = \text{Id}_g$ (see [4, p. 91, Corollaire 3]). Fix such a homomorphism $\theta$.

Since $G$ is simply connected, there is a unique homomorphism of Lie groups
\[ \Theta : G \longrightarrow A \]
such that $\theta$ is the Lie algebra homomorphism corresponding to $\Theta$. Now we have an action
\[ \rho : G \times E_H \longrightarrow E_H \]
defined by $\Theta(g) = (g, \{ z \mapsto \rho(g, z) \})$. The pair $(E_H, \rho)$ is a homogeneous holomorphic principal $H$–bundle. \qed

Combining Lemma 4.1 with Theorem 3.1 and Proposition 3.2, we get the following:

Corollary 4.2. If a holomorphic principal $H$–bundle $E_H \longrightarrow G/\Gamma$ admits a holomorphic connection, then it admits a flat holomorphic connection.

References
[1] M. F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957), 181–207.
[2] K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng and S.-T. Yau, Anomaly cancellation and smooth non-Kähler solutions in heterotic string theory, Nuclear Phys. B 751 (2006), 108–128.
[3] I. Biswas and T. L. Gómez, Connections and Higgs fields on a principal bundle, Ann. Glob. Anal. Geom. 33 (2008), 19–46.
[4] N. Bourbaki, Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie, Actualités Sci. Ind. No. 1285, Hermann, Paris, 1960.
[5] M. Fernandez, S. Ivanov, L. Ugarte and R. Villacampa, Non-Kähler Heterotic String Compactifications with non-zero fluxes and constant dilaton, Comm. Math. Phys. 288 (2009), 677–697.
[6] E. Goldstein and S. Prokushkin, Geometric model for complex non-Kähler manifolds with SU(3) structure, Comm. Math. Phys. 251 (2004), 65–78.
[7] G. Grantcharov, Geometry of compact complex homogeneous spaces with vanishing first Chern class, Adv. Math. 226 (2011), 3136–3159.
[8] D. Grantcharov, G. Grantcharov and Y. Poon, Calabi-Yau connections with torsion on toric bundles, Jour. Diff. Geom. 78 (2008), 13–32.

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