Abstract

Rare kaon processes appear to be particularly suitable to study the extensions of the standard model, especially if the possibility for eventual direct evidence becomes unlikely. In this review, we discuss processes that are important as a test of either the standard model or supergravity. Moreover, some of these are important even for both the standard model and for supergravity.

Particular attention is paid to the reduction of uncertainties in the calculation, especially the ones coming from the confinement effects. Recent approaches, such as chiral perturbation theory, the large $N_c$-expansion, QCD sum rules and lattice QCD, are discussed. This is found to be the best strategy in view of the fact that supersymmetric effects are rather tiny.
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1 Standard Model and Supersymmetry

1.1 Introduction

Three basic interactions (electromagnetic, weak, and strong) seem to be reasonably (even very well) described by a theory known as the standard model\cite{1}-\cite{10}. The fourth known interaction, gravitation, is not included. Its classical version, the general theory of relativity, describes macroscopic phenomena rather well; the quantized theory of gravity, in spite of many attempts, still does not exist as a self-consistent theory. Fortunately, gravitational interactions at energies of present experiments do not play any important role. Therefore, the standard model may be tested to a high degree of accuracy.

Up to now there has been no confirmed disagreement between the standard model and experiment. The standard model describes many phenomena very well; actually, the description is too good for the taste of many theoretical physicists. This ‘unhappiness’ arises from the answer to our question about the standard model: the standard model is not a satisfactory theory for the reasons to be discussed later.

Theoretically, one tries to construct theories which go beyond the standard model and improve the drawbacks of the standard model in some aspects. This includes a desirable reducing of the number of parameters, improvement in convergence properties and, particularly, the understanding of the Higgs sector which, in itself, at least in the minimal version of the standard model, shows unpleasant behavior at the quantum level. Clearly, theories which go beyond the standard model\cite{11}, bring new physics into play - usually a plethora of new particles. Naively, this looks very promising because new particles may, at least in principle, be detected in one or another way. However, almost perfect agreement between the standard model and experiment requires that new physics should have very little impact on low-energy phenomenology. This means that new particles are rather heavy. Therefore, their direct production and detection at large accelerators might finally appear very difficult, especially if the present upper limits on masses of new particles become considerably larger.

An alternative but complementary way of searching for new physics are low-energy experiments with a high degree of accuracy. This way is based on the following points.

(i) There are processes that appear at higher order in electroweak interactions and which are very sensitive to ‘impurities’ caused by new physics. A special role is here played by rare kaon processes. These include rare kaon decays, CP violation in the kaon system, $\epsilon'/\epsilon$ parameter, etc.

(ii) Significant progress has been made in reducing the uncertainties in theoretical predictions in the standard model. These uncertainties are usually connected with the treating of hadrons, i.e., with the problem of QCD confinement. A few new, rather sophisticated approaches have been addressed to this problem: QCD lattice calculations\cite{12, 13}, the large-$N_c$ approach\cite{14}, and the QCD-duality approach\cite{15}.

Since the effects of new physics appear to be rather tiny, the importance of reducing the uncertainties in calculations and/or errors in experiments is conditio sine qua non for the future progress.

A number of experiments are presently running at BNL, KEK, CERN and FNAL. Kaon beams are usually produced from high-energy proton beams colliding on fixed targets. Exception is the CP-LEAR project at CERN where low-energy $p\bar{p}$ collisions are used to study
kaons in the final state, especially CP violation and CPT tests.

A clean, intense source of K-mesons is expected in \( \phi \)-factories, like DAΦNE in Frascati, which is to be built in the next few years\(^{[16]} \).

A few proposals for K-factories have been considered in the last few years, like European Hadron Facility\(^{[17]} \) (abandoned) and TRIUMPH\(^{[18]} \) with extremely intense beams (planned after 1995).

**Rare Kaon Decays**

The kaon system has proved to be the graveyard of many wrong theories. The history goes from the \( \tau - \theta \) puzzle, CP violation, the \( \Delta I = 1/2 \) rule, the absence of \( \Delta S = 0 \) neutral currents to the present search for lepton-flavor violation and the existence of the fourth generation. Last but not least, new ideas, e.g., composite models and/or supersymmetry have to pass hard tests in kaon physics.

The standard model (SM), based on the gauged \( SU(3) \times SU(2) \times U(1) \) theory with the minimal Higgs sector, can be proved/disproved in the following ways:

(i) Going to higher energies and looking whether the SM still works; this is basically the approach of collider physics.

(ii) Precision measurements: measuring the parameters of the SM highly accurate and looking for discrepancies.

(iii) Searching for processes which are suppressed or forbidden in the standard model. This basically defines the physics of rare processes, which are the subject of this paper.

An example of forbidden process is

\[
K_L \to \mu e,
\]

and, clearly, its observation at any level would be a clean signal for new physics. The present experimental limit is\(^{[19]} \)

\[
BR(K_L \to \mu e) < 0.94 \times 10^{-10}.
\]

Other examples of \( K \) decays that are forbidden in the SM, but are predicted in new models are

\[
K^+ \to \pi^+ \mu^+ e^-, \\
K_L \to \pi^0 \mu e, \\
K^+ \to \pi^+ X^0,
\]

where \( X^0 \) is a light scalar or pseudoscalar.

There are many processes in kaon physics that are strongly suppressed in the standard model, but are not totally forbidden. We list typical examples.

The first process, \( K_L \to e^+ e^- \), is especially sensitive to the Higgs sector of new physics since, unlike in the SM, there would be no large helicity suppression.

The \( K_L \to \pi^0 e^+ e^- \) process is suppressed by a factor \( \sim 10^{-6} \) owing to the GIM mechanism and, in addition, by a factor of \( \sim 10^{-5} \) owing to CP conservation.
The $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ is probably the cleanest test of higher-order electroweak interactions in the standard model. This particular process belongs to the class of more general $K$ decays of the type

$$K^+ \rightarrow \pi^+ \text{ nothing},$$

(1.4)

where ‘nothing’ denotes any light neutral particle(s) that cannot be detected.

Particularly interesting is the process

$$K^+ \rightarrow \pi^+ \nu \bar{\nu}?$$

(1.5)

This channel receives contributions from standard decays (i.e., $K^+ \rightarrow \pi^+ \nu \bar{\nu}$) and from the new physics beyond the standard model. Being practically a short-distance phenomenon, it has the advantage with respect to processes such as $K_L \rightarrow \mu \bar{\mu}$ or $K^+ \rightarrow \pi^+ e^+ e^-$, where long-distance contributions enter the game. Also, any mechanism for lepton flavor violation (horizontal gauge bosons, leptoquarks, etc.) would lead to $K^+ \rightarrow \pi^+ \nu_e \bar{\nu}_\mu$. In addition, flavor-changing neutral currents in N=1 supergravity would give a contribution to $K_L \rightarrow \pi^+ \nu \bar{\nu}$ in addition to the SM contribution.

Finally, the direct evidence for superparticles would be possible in processes such as

$$K^+ \rightarrow \pi^+ \tilde{\gamma} \tilde{\gamma},$$

(1.6)

provided the photinos $\tilde{\gamma}$ are light enough to be produced. Last but not least, the existence of the fourth generation would certainly influence this rare decay.

### 1.2 Limits of the Standard Model and Supersymmetry

The standard model is a theory with a lot of parameters which are not explained by the model itself. These include lepton and quark masses, Yukawa couplings, mixing angles, etc., whose pattern is not understood. Why have left-handed fermions so far appeared in $SU(2)$ doublets and right-handed ones in $SU(2)$ singlets? Why are there three colors, and why is the electric charge quantized? The family problem is not understood: why three generations of quarks and leptons?, etc.

Searching for answers one is forced to go beyond the standard model. Among different theories, supersymmetry is particularly interesting [20]-[47]; it is the symmetry of bosons and fermions, whose theoretical motivation we discuss in the following.

(i) Experience teaches us that nature obviously prefers gauge theories and supersymmetry is the next logical gauge theory.

(ii) The spin degree of freedom is naturally contained only in supersymmetry.

(iii) Supersymmetric theories are much better behaved mathematically, and are even finite in some cases.

(iv) Locally gauged supersymmetry relates the SUSY generators to the generators of space-time transformations leading to supergravity, i.e., to a natural coupling of SUSY to gravity.

The standard model can hardly be considered as a fundamental theory since it contains, e.g., QED, which is not an asymptotically free theory. Its interactions must become strong
at some higher energy scale. This is considered as a hint that one has to treat the standard model rather as a low-energy effective theory of a more fundamental one.

The Higgs sector

The Higgs sector gives additional hints which cast doubt upon the fundamental character of the standard model. The minimal Higgs sector needed to construct the standard model contains one complex Higgs doublet $\Phi$ with the potential

$$V(\Phi) = \lambda |\Phi^\dagger \Phi - \frac{1}{2\sqrt{2}G_F}|^2. \quad (1.7)$$

After spontaneous symmetry breaking, this leads to one physical Higgs field, with the mass

$$M_H^2 = \frac{\sqrt{2}}{G_F} \lambda, \quad (1.8)$$

with an unknown coupling constant $\lambda$.

As far as the Higgs mass is concerned, it cannot be taken arbitrarily large. Radiative corrections lead to a lower bound:

$$M_H \geq 6.6 \text{ GeV}. \quad (1.9)$$

The various analyses show that the upper limit on the Higgs mass is

$$M_H \leq 1 \text{ TeV}. \quad (1.10)$$

Once the upper limit, $M_H \leq$ a few TeV, has been set, the huge radiative corrections to the mass of the Higgs boson provide an additional hint that one needs a new physics beyond the standard model. These corrections arise, e.g., from different loops, which turn out to be quadratically divergent, as is typical of scalar theory, i.e., $\delta M_H^2 \sim \Lambda^2$. Since it would be unnatural to expect the corrections to be larger than the upper limit on the Higgs mass, one expects the new physics to give an effective cutoff scale below a few TeV.

The idea is to use a higher symmetry to eliminate quadratic divergences in the Higgs mass corrections. Here supersymmetry appears to be the proper theory because it can in principle eliminate the problem of quadratic divergences. The mechanism is rather simple: In supersymmetry the loops of normal particles are always accompanied by the loops of superpartners. The extra minus sign, appearing because of the fermion loop, leads basically to the cancellation of divergences. This nice property persists as long as the imposed supersymmetry is exact. Since supersymmetry necessarily has to be broken, the requirement of approximate cancellation of divergences imposes a constraint on the masses of superparticles.

The Higgs mechanism brings in the vacuum expectation value $v$ which determines $M_W$ and fermion masses: it is $v \simeq 250 \text{ GeV}$. This value is not derived in the standard model itself and needs a fundamental explanation. Any attempt made so far to explain $v$ in terms of higher symmetries brings us to the fine-tuning and the hierarchy problem.

Let us call $\mu_{\text{weak}}$ a scale at which $SU(2) \times U(1)$ breaking takes place, and $M$ a scale where a fundamental theory becomes relevant. Typically, this is a very high scale, e.g.,
$M_{\text{GUT}} \sim 10^{14} \text{ GeV}, M_{\text{Planck}} \sim 10^{19} \text{ GeV}$, etc. Now, in the fundamental theory, the mass of the Higgs particle is evaluated at the fundamental scale $M$, and $M_H^2(M)$ has to be scaled down to the weak scale $\mu_{\text{weak}}$. Typically, the expression is

$$M_H^2(\mu_{\text{weak}}) = M_H^2(M) + \text{const.} \ g^2 \int_{\mu_{\text{weak}}}^M dp^2 + \Re(M)g^2 + \mathcal{O}(g^4), \quad (1.11)$$

where $\Re(M)$ grows almost as $\ln M$ when $M \to \infty$, and $g$ is a coupling constant. The second term in (1.11) diverges quadratically as $M \to \infty$.

In order to have $M_H^2(\mu_{\text{weak}}) \ll M^2$, one has to fine-tune the parameter $M_H^2(M)$ extremely accurately to cancel the second term in (1.11) which is if order $M^2$. This is known as the fine-tuning problem or the naturalness problem. Suppose, for example, that the standard model is valid up to a GUT scale of $10^{15} \text{ GeV}$ or even up to the Planck scale of $10^{19} \text{ GeV}$. If these two scales plus the weak scale were input into the theory, the Higgs mass would have to be chosen with an accuracy of $10^{-34}$ compared with $M_{\text{Planck}}$. Clearly, the natural value for $M_H^2(\mu_{\text{weak}})$ is $\sim \mathcal{O}(M^2)$. Therefore, the fundamental question is actually the hierarchy problem: why is $\mu_{\text{weak}} \ll M^2$? Even if the question of fine-tuning were solved in a satisfactory way, the hierarchy problem would still have to be solved.

As we have already mentioned, one has to impose constraints on possible supersymmetry breaking to preserve the cancellation of divergences. Such a symmetry breaking is called soft. An example of soft symmetry breakdown is a spontaneous symmetry breaking. Clearly, with softly broken SUSY, the cancellation among diagrams is partial. The finite results obtained are related to the SUSY breaking scale $M_{\text{susy}}$.

A naive expectation would be that $M_{\text{susy}}$ should be of the order of $\mu_{\text{weak}}$. This really has to be the case if SUSY is broken explicitly. However, if the supersymmetry breakdown is soft, then the splittings among supermultiplets are proportional to $g M_{\text{susy}}$, $g$ being the coupling constant. This happens in supergravity: a soft breakdown may appear at $M_{\text{susy}} \sim 10^{11} \text{ GeV}$, and mass splittings and $M_W$ could still be of the order of $100 \text{ GeV}$.

However, the 'small' scale, $M_{\text{susy}} \sim 10^{11} \text{ GeV}$, is still very large so that gravity cannot be neglected. Clearly, at such a scale all particles would have a gravitational interaction and the mass splitting would be of the order

$$\frac{M_{\text{susy}}^2}{M_{\text{Planck}}} \sim 10^3 \text{ GeV}, \quad (1.12)$$

i.e., comparable with the scale of electroweak symmetry breaking. The inverse Planck mass is basically the gravitational constant, $\kappa \sim M^{-1}_{\text{Planck}}$. This leads naturally to local supersymmetry or supergravity.

In addition to graviton (spin 2) there also appears gravitino, its superpartner (spin 3/2), with the mass of the order

$$m_{3/2} \sim \frac{M_{\text{susy}}}{M_{\text{Planck}}}, \quad (1.13)$$

and the breakdown of supergravity induces the electroweak breakdown with $M_W$ typically of the order of $m_{3/2}$. 

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2 Low-Energy $N = 1$ Supergravity

2.1 Soft Supersymmetry Breaking

After the spontaneous breaking of a local $N = 1$ supergravity one is left with a low-energy theory which is an explicitly broken global supersymmetry plus some ‘soft-breaking’ terms. The theory is the supersymmetrized minimally extended standard model.

The full lagrangian of the SM contains superpartners as well as the standard particles. In the supersymmetric limit, the minimal model of supersymmetry has the gauge group $SU(3) \times SU(2) \times U(1)$, with three generations of left chiral matter superfields

$$Q(3, 2, 1/3), \ U^c(3, 1, -4/3), \ D^c(3, 1, 2/3), \ L(1, 2, -1), \ E^c(1, 1, 2)$$

(2.1)

and two Higgs superfields

$$H(1, 2, 1) \ and \ H'(1, 2, -1).$$

(2.2)

The superpotential contains the most general gauge-invariant couplings which preserve matter parity and contains only the fields required by supersymmetrization:

$$W = h^{ij}_U H U^c_i Q_j + h^{ij}_D H'^c_i D^c_j + h^{ij}_E H'^c_i L_j + \mu H H'.$$

(2.3)

where $F$ is the fermion number. This symmetry is introduced to enforce baryon and lepton number conservation. With R-parity conserved, normal particles are R even, while all superparticles are R odd. As a consequence, the lightest superparticle (LSP) is stable.

The important new discrete symmetry introduced is $R$-parity, defined as

$$R = (-)^{3(B-L)} (-)^F,$$

(2.4)

where $F$ is the fermion number. This symmetry is introduced to enforce baryon and lepton number conservation. With R-parity conserved, normal particles are R even, while all superparticles are R odd. As a consequence, the lightest superparticle (LSP) is stable.

A global $R$ invariance is achieved in the limit $\tilde{m}_i \to 0$ and $h_U = h_D = h_E = 0$, i.e., a chiral symmetry protects gauginos from getting mass in all orders in perturbation theory.

As we discussed earlier, the most convenient way of breaking supersymmetry softly is to couple it to a hidden sector of $N = 1$ supergravity. The situation is much simplified by the assumption of having a flat Kähler metric. There are a few types of soft SUSY breaking.

(i) A cubic gauge-invariant polynomial in complex scalar fields (trilinear scalar couplings):

$$[\xi_U H U^c Q + \xi_D H'^c D^c Q + \xi_E H'^c E^c L + \mu B H H']_A + h.c.,$$

(2.5)

where $\xi$ are $3 \times 3$ flavor matrices. This part of soft breaking contributes to squark and slepton mass matrices.
(ii) Gaugino Majorana terms:

\[ \frac{1}{2} \sum_\alpha \tilde{m}_\alpha \lambda_\alpha \lambda_\alpha + h.c. \]  

(2.6)

These terms obviously give masses to all gauginos, i.e., to wino, zino, and gluino.

(iii) Mass terms of the scalar fields \( z_i \) of the chiral superfields:

\[ M^2_{ij} z_i^* z_j + h.c. \]  

(2.7)

These terms give masses to all scalar superparticles, i.e., squarks, sleptons, and higgs.

(iv) The last term

\[ \mu BHH' \]  

(2.8)

enters the Higgs potential.

In addition, if one assumes that the low-energy theory is a Grand Unified Theory, it would lead to a set of relations among the soft couplings and masses. Under the assumption that a grand unification appears at some scale \( M_X \), a natural assumption is the universal gauge coupling. All soft-breaking terms are assumed to be parametrized at \( M_X \) by a universal gaugino mass \( m_{1/2} \), all scalar masses are given by \( m_0 \) at \( M_X \), and there is a universal trilinear scalar coupling \( A \).

The above scalar and gaugino mass terms are allowed since gauge invariance is preserved. However, gauge invariance forbids mass terms for quarks, leptons, and gauge bosons - they will acquire their masses through spontaneous gauge symmetry breaking. Incidentally, only such particles have been seen until now.

The only peculiar behavior as far as the origin of mass is concerned shows higgsino. While the higgs particle acquires mass from the usual weak spontaneous breakdown, the higgsino gets mass both from the \( \mu \)-term in \( L_{susy} \) and from the weak \( SU(2) \times U(1) \) symmetry breaking.

### 2.2 Flavor-Changing Neutral Currents

The existence of two scalar doublets in the supersymmetric extension of the standard model does not lead to flavor-changing neutral currents since the underlying supersymmetry forbids it at the tree level.

However, one-loop radiative corrections induced by the charged scalar would generally lead to the couplings between squarks, quarks, and gluinos containing flavor-changing parts. In addition, the hidden sector of supergravity, needed for the super-Higgs mechanism will bring new phase into the theory. If a typical SUSY breaking scale were of the order of \( 10^{10} \text{ GeV} \), one might expect large renormalization effects.

After \( SU(2) \times U(1) \) symmetry is spontaneously broken, the potential invariant under global SUSY brings into play squark mass terms \( M_Q^1 M_Q \), \( M_Q \) being the squark mass matrix.
and with mass terms in $L$ typically of the form

$$q_L^\dagger q_L + (L \rightarrow R), \quad (2.9)$$

$q_{L,R}$ being the squarks (L- and R- handed, respectively). In the minimal N=1 supergravity extension of the SM, the soft-breaking terms contribute in such a way that squark masses are degenerate for different generations

$$\mu_L^2 q_L^\dagger q_L + \mu_R^2 q_R^\dagger q_R, \quad (2.10)$$

and $\mu_{L,R}^2$ are of the order of the global SUSY breaking scale, $\mu \sim 10^2 \text{ GeV}$, and usually taken to be equal to the gravitino mass $m_3/2$.

Trilinear interactions between the Higgs field and two squark fields (cf. Eq.(2.5)) contribute with a strength $A$ and are proportional to $\mu$. The parameter $A$ may, in principle, be calculable from the hidden sector of supergravity.

Left-right squark mixings, i.e., the mass terms of the type $q_L^\dagger q_R + h.c.$ are induced through the electroweak spontaneous breakdown and are proportional to $M_Q$.

Finally, radiative corrections to the down-squark masses are induced by the charged Higgs and the higgsino loop and are proportional to $\hat{M}_d^2$.

The down-squark mass term becomes (taking into account quantum corrections)

$$M_D^2 \simeq \begin{pmatrix} \mu_L^2 & M_d M_d^\dagger + c M_u M_u^\dagger & A \cdot m_{3/2} \hat{M}_d \\ M_d M_d^\dagger & \mu_R^2 & \hat{M}_d^\dagger \cdot \hat{M}_d \\ c V^\dagger \hat{M}_u V & \hat{M}_u^\dagger \cdot \hat{M}_u & \mu_R^2 \end{pmatrix}, \quad (2.11)$$

The parameter $c$ is negative and is usually assumed to be $O(1)$. The parameter $A$ is generally a complex parameter; it can be written as

$$A = |A| e^{-2i\phi_A}. \quad (2.12)$$

The matrices $M_D^2$ and $M_d$ cannot be diagonalized simultaneously by the same transformation. Let us introduce 2 unitary matrices, $\tilde{U}_1$ and $\tilde{U}_2$, such that $\tilde{U} = \tilde{U}_1 \tilde{U}_2$ and

$$\tilde{U}_1 = \begin{pmatrix} e^{i\phi_A} U_L^d & 0 \\ 0 & e^{-i\phi_A} U_R^d \end{pmatrix}, \quad (2.13)$$

where $U_L^d, U_R^d$ diagonalize the down-quark matrix. Then the matrix $\tilde{U}$ diagonalizes the down-squark matrix:

$$\tilde{U}^\dagger M_D^2 \tilde{U} = \tilde{U}_2^\dagger \begin{pmatrix} \mu_L^2 & \hat{M}_d^2 + c V^\dagger \tilde{M}_u V & |A| m_{3/2} \hat{M}_d \\ |A| m_{3/2} \hat{M}_d & \mu_R^2 & \hat{M}_d^\dagger \cdot \hat{M}_d \end{pmatrix} \tilde{U}_2. \quad (2.14)$$

$\hat{M}_{u,d}$ are diagonalized $M_{u,d}$ matrices and

$$V = U_L^{\dagger} U_L \quad (2.15)$$

is the usual Kobayashi-Maskawa matrix.
If one neglects the left-right squark mixing, and since $\hat{M}_u^2 \gg \hat{M}_d^2$, a reasonable approximation for $\tilde{U}_2$ is

$$\tilde{U}_2 \simeq \begin{pmatrix} V^\dagger & 0 \\ 0 & 1 \end{pmatrix},$$ (2.16)

which leads to

$$\hat{M}_D^2 \simeq \begin{pmatrix} \mu^2 \mathbf{1} + c \hat{M}_u^2 & |A|m_{3/2}V\hat{M}_d^\dagger \\ |A|m_{3/2}V\hat{M}_d^\dagger & \mu^2 \mathbf{1} + \hat{M}_d^2 \end{pmatrix},$$ (2.17)

The current- and mass- eigenstates are connected via

$$\tilde{d} = \tilde{U} \begin{pmatrix} \tilde{d}_L^{(0)} \\ \tilde{d}_R^{(0)} \end{pmatrix}$$ (2.18)

and the latter induce flavor-changing couplings.
3 Recent Approaches in Calculating Hadronic Transition Amplitudes

3.1 Chiral Perturbation Theory - Basics

Chiral perturbation theory (ChPT) represents a viable alternative low energy theory of strong and electroweak interactions\[48\]-\[83\]. Its importance as an alternative approach to the usual formulation of the standard model is especially evident in hadronic processes where the lack of knowledge of QCD confinement is blurring our view of electroweak interactions. This is even more pronounced in kaon physics, since kaons and pions, being pseudo-Goldstone bosons, are even less reliably described in phenomenological quark models.

This theory has recently become important in connection with some other alternative approaches, such as QCD hadronic duality sum rules\[56\]-\[58\], QCD simulation on lattice\[54\] and in the large-$N_c$ approach of Bardeen et al.\[71\]. In all of these approaches, ChPT is used in some way or another: in duality sum rules, the tree level chiral realization of hadronic currents and/or operators is used in the parametrization of the hadronic side of the sum rule. In the large-$N_c$ approach of ref.\[71\], the chiral representation of currents is used to construct weak transition operators at zero momentum, which is needed in order to describe the operator evolution at large distances. Finally, the lattice calculation of weak matrix elements is (presently) performed for off-shell transitions, such as the $K \to \pi$ transition. The relation with physical amplitudes ($K \to \pi\pi$) is achieved using the tree level ChPT relation between the above transitions.

Chiral perturbation theory is based on our knowledge of the fundamental symmetries of the QCD lagrangian, e.g., the softly broken chiral $SU(3)_L \times SU(3)_R$ symmetry. The formulation of ChPT is based\[51, 52\] on the following Ansatz: in any given order in perturbation theory, the most general lagrangian consistent with a given symmetry and field theory conditions on analiticity, perturbative unitarity, etc. Formulated in this way, ChPT becomes a quantum field theory.

The strong lagrangian at order $p^2$ (with minimal number of derivatives) is given by the nonlinear $\sigma$-model

$$L^{(2)} = \frac{f^2}{8} tr(\partial_\mu U \partial^\mu U^\dagger) = \frac{f^2}{8} g_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b$$

for a massless field, with the invariant metric

$$g_{ab}(\phi) = tr(\partial_a U \partial_b U^\dagger).$$

$U$ is the unitary matrix field

$$U(\phi) = \exp \frac{i}{f} \Phi.$$

$\Phi = \frac{1}{\sqrt{2}} \lambda^a \phi^a$ is given by

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & K^0 & -\sqrt{3} \eta_8 \end{pmatrix}.$$
If the quark mass matrix $M$ is different from zero, all terms in $\mathcal{L}_{\text{strong}}$ would pick up additional terms. Then, the strong lagrangian to the lowest order, reads

$$\mathcal{L}^{(2)}_{\text{strong}} = \frac{f^2}{8} tr(\partial_\mu U \partial^\mu U^\dagger) + v tr(M U + U^\dagger M),$$

(3.5)

with

$$\frac{4v}{f^2} = \frac{m^2_{\pi^+}}{m_u + m_d} = \frac{m^2_{K^+}}{m_s + m_u} = \frac{m^2_{K^0}}{m_s + m_d} = B_0,$$

(3.6)

and $v$ is proportional to the quark condensate, $v = -\frac{1}{4} < \bar{q}q |0>$. The lowest order $\mathcal{L}^{(2)}_{\text{strong}}$ at the tree level has basically two couplings, $f_\pi$ and $v$, which are expected to be calculable in QCD.

The lagrangian $\mathcal{L}^{(2)}_{\text{strong}}$ of the nonlinear $\sigma$-model is nonrenormalizable and agrees with QCD only at the tree level (leading behavior). In order to get the full strong lagrangian in ChPT one has to include all possible terms in the lagrangian and take account of all graphs in perturbation theory. The classical theory is equivalent to tree graphs of quantum field theory. However, a consistent quantum field theory requires the presence of loop graphs, since without loops, the theory violates unitarity. For example, the effective lagrangian $\mathcal{L}^{(2)}$ describes the scattering of pseudoscalar mesons to order $p^2$ (tree graphs), the parameters being the quark masses, $f_\pi$ and $v$. The unitarity demands that at order $p^4$ the $T$-matrix involves cuts with discontinuities, the contribution being determined up to a polynomial in the external momenta. In field theory, these discontinuities appear in the one-loop graphs from $\mathcal{L}^{(2)}$.

The inclusion of loops would generally lead to infinities. To get rid of them, the theory should allow for counterterms. Using the regularization that preserves chiral symmetry (e.g., dimensional regularization), one finds that one needs counterterms of order $p^4$, since loop-graphs coming from the $p^2$-lagrangian are of that order. Furthermore, one finds that all counterterms needed are contained in $\mathcal{L}^{(4)}$. This is intuitively clear, since we demand that regularization preserves chiral symmetry. Using the terms in $\mathcal{L}^{(4)}$ as counterterms leads to finite results for all Green’s functions to one-loop order. However, the renormalized graphs are unambiguous only up to a polynomial in the external momenta.

In order to include the electromagnetic interaction, one has to couple the quarks to hermitian external fields, as proposed by Gasser and Leutwyler [52].

The lagrangian $\mathcal{L}^{(2)}$ becomes

$$\mathcal{L}^{(2)} = \frac{f^2}{8} tr(D_\mu U D^\mu U^\dagger) + v tr(M U + U^\dagger M),$$

(3.7)

where $D_\mu$ is the covariant derivative

$$D_\mu = \partial_\mu U - ie\mathcal{A}_\mu[Q,U],$$

(3.8)

and $\mathcal{A}_\mu$ is an external electromagnetic field.

---

3 The choice of the effective lagrangian is not unique. At the tree level, both linear and nonlinear lagrangians lead to the proper value of the two constants $f_\pi$ and $v$. At the one loop level, however, they disagree. The linear (renormalizable) $\sigma$ model gives relations among the couplings which are in contradiction with experiment.
3.2 Chiral Realization of Hadronic Weak Interactions

The extension of ChPT to weak processes is straightforward. However, the predictive power is often spoiled by a large number of unknown coupling constants.

In weak $\Delta S = 1$ kaon decays, CP violation, etc., one has only a few channels and this makes it difficult to go beyond the tree-level lagrangian. Nevertheless, the chiral realization of weak lagrangians has been found to be very useful as it ‘supports’ other approaches, such as QCD lattice calculations, QCD hadronic duality sum rules, and the large-$N_c$ expansion.

For example, to the order $p^2$, the octet part of the $\Delta S = 1$ weak lagrangian is given by

$$L^{(2)}_{\Delta S=1} = g_8 \tilde{\mathcal{L}}_8 + h_8 \Theta,$$

where $g_8$ and $h_8$ are unknown coupling constants, not fixed by ChPT alone. The operators $\tilde{\mathcal{L}}_8$ and $\Theta$ have the following realization to the lowest order in derivatives and masses

$$\tilde{\mathcal{L}} = \frac{G_F}{2\sqrt{2}} s_1 c_1 c_3 \, \text{tr}(\Lambda \partial_\mu U \partial^\mu U^\dagger)$$
$$\Theta = \frac{G_F}{2\sqrt{2}} s_1 c_1 c_3 \frac{8v}{f^2} \, \text{tr}(\Lambda U \mathcal{M} + \Lambda (U \mathcal{M})^\dagger).$$

The operator $\Theta$ is the tadpole operator which in general contributes to the off-shell Green’s functions, but does not contribute to the $S$ matrix. For example, the operator $\Theta$ contributes to the amplitudes $\mathcal{A}(K \to \pi\pi)$, $\mathcal{A}(K \to \pi)$, and $\mathcal{A}(K \to 0)$. The first graph in Fig. 1-I gives $\sim \frac{4|m_s - m_d|}{f^2} h_8$, which is, however, exactly canceled by the contribution of the second graph in Fig. 1-I. On the other hand, if one writes the amplitude $\mathcal{A}(K \to \pi\pi)$ in terms of the amplitudes $\mathcal{A}(K \to \pi)$ and $\mathcal{A}(K \to 0)$ (the PCAC relation), one finds again that the tadpole contributions in the last two amplitudes cancel each other and $\mathcal{A}(K \to \pi\pi)$ is tadpole-free.

Recently, Shabalin has suggested an interesting and intriguing mechanism that turns a quadratic GIM suppression of the $d - s$ self-energy graph into a logarithmic one, showing a clear enhancement of the $\Delta I = 1/2$ transition. The effect comes basically from the leading logarithmic one-gluon corrections to the bare graph. However, it has been shown by Guberina, Peccei, and Picek that the full QCD correction in the leading logarithmic approximation (LLA) reduces the tadpole contribution to a negligible amount, which vanishes in the chiral limit.

The physical amplitude $\mathcal{A}(K \to \pi\pi)$ is therefore proportional to $g_8$, which can be extracted from experiment. Numerically, $|g_8| \approx 5.1$. Next-to-leading corrections to $K \to 2\pi$ and $K \to 3\pi$ have been calculated by Kambor et al. The counterterms were fitted from the decay rates and slope parameters. The lowest-order value of $g_8$ reduces by 30%, whereas the analogous 27-plet constant remains practically unchanged.

Effective Lagrangian for Radiative Kaon Decays

The electromagnetic field may be introduced into $L^{(2)}_{\Delta S=1}$ through a covariant derivative

$$L^{(2)}_{\Delta S=1} = \bar{g}_8 tr (\Lambda D_\mu UD^\mu U^\dagger),$$
In addition, one has to add to it all possible terms of fourth order in derivatives and/or external fields allowed by chiral symmetry. Finally one finds\[53\]

\[ L^{(4)}_{\Delta S=1, em} = -\frac{ie}{f_\pi^2} 2\tilde{g}_8 \{ w_1 \text{tr}(Q\Lambda\hat{L}_\mu \hat{L}_\nu) + w_2 \text{tr}(Q\hat{L}_\mu \Lambda\hat{L}_\nu) \} + \frac{e^2 f_\pi^2 \tilde{g}_8}{2} w_4 F^{\mu\nu} F_{\mu\nu} \text{tr}(\Lambda QUQU^\dagger) + h.c., \] (3.12)

and \( \hat{L}_\mu \) is a left-handed current with covariant derivatives. Only the terms relevant for radiative \( K \) decays with one pion in the final state are kept. Again, \( w_1, w_2, \) and \( w_4 \) are undetermined couplings. To this order, one has to add the contributing part of the strong + electromagnetic lagrangian

\[ L^{(4)}_{\text{strong+em}} = -ie L_9 F^{\mu\nu} tr(QD_\mu UD_\nu U^\dagger + QD_\mu U^\dagger D_\nu U) + e^2 L_{10} F^{\mu\nu} F_{\mu\nu} tr(QUQU^\dagger), \] (3.13)

and the anomalous Wess-Zumino lagrangian

\[ L^{(4)}_{WZ} = \frac{\alpha}{4\sqrt{2}\pi f^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} (\pi^0 + \frac{\eta}{\sqrt{3}}) + \mathcal{O}(\phi^3, \phi^5), \] (3.14)

linear in meson fields. This completes the tools necessary to calculate radiative \( K \) decays, which have been discussed in Chapter 4.

### 3.3 Large-\( N_c \) Expansion in ChPT and Beyond

**Large-\( N_c \) Expansion in the Weak Sector in ChPT**

The large-\( N_c \) limit can be taken consistently for all currents and/or operators appearing in the weak sector of ChPT. For example, the coupling \( g_8 \) in (3.9) is not determined in ChPT. The value of \( g_8 \) is known only in the large-\( N_c \) limit, in which the weak lagrangian (3.9) is reduced to the product of bare currents. With \( g_8 \) determined in the large-\( N_c \) limit, the amplitude \( \mathcal{A}(K \to \pi\pi) \) coincides with the large-\( N_c \) vacuum saturation approximation for the matrix elements of local 4-quark operators.

The \( p^4 \)-level lagrangian introduces many unknown coupling constants. Typically, the amplitude receives contributions from one loop graphs calculated with the \( p^2 \)-lagrangian and from tree graphs stemming from the \( p^4 \)-lagrangian. The unknown couplings in \( p^4 \)-lagrangians are to be used as counterterms needed to renormalize possible divergences in loop integrals. The finite part of counterterms depends on the renormalization point \( \mu \); this dependence cancels the logarithmic \( \mu \) dependence of the renormalized loop-graph contribution. If one uses dimensional regularization, one encounters only logarithmic divergences. Working instead with cutoff regularization, one also encounters quadratic divergences. The latter, however, disappear after renormalization, as we shall discuss later.

\[ \tilde{g}_8 = \frac{G_F}{2\sqrt{2}} s_1 c_1 c_3 g_8, \] (3.11)
The role of meson loops and/or counterterms is very important, as can be seen by studying rare kaon decays (cf. Chapter 4). The physical meaning of the counterterms (couplings) is the following: they are remnants of the original quarks and gluons after the latter are integrated out in the functional integral. Therefore, they contain both short- and long-distance physics, i.e., the effects of, e.g., vector mesons (long distance) and hard gluons (short distance). Of course, in some cases, such as the $p^4$ couplings $L_1, \ldots, L_{10}$ in the strong+electromagnetic lagrangian, they are perfectly saturated by low-lying meson resonances. The situation is, however, quite different for weak couplings, e.g., the constants $w_i$ in (3.13). Once determined, the renormalized couplings depend on the renormalization point $\mu$; however, their natural order of magnitude is $(4\pi)^{-2}$.

The large-$N_c$ expansion turns out to be a very useful tool in treating many problems in strong interactions. In the weak-interaction sector the situation is somewhat different. The idea of the large-$N_c$ expansion is based on the expectation that the true expansion parameter is not $1/N_c = 1/3$, but rather something like $1/4\pi N_c$ or even $1/4\pi N_c^2$, as it happens in QED, where an expansion in the coupling constant $e$ becomes an expansion in $\alpha = e^2/4\pi$. Unfortunately, in the weak-interaction sector the subleading $1/N_c$ corrections are often large, even huge in some cases.

If one works with an explicit cutoff, the cutoff dependence comes in different ways: (i) A possible $\Lambda^4$ contribution is absent because of chiral symmetry, (ii) quadratic cutoff dependence is, by power counting and chiral symmetry, of the form $\Lambda^2 \times$ tree-level $\mathcal{L}^{(2)}$, (iii) logarithmic terms are of the form $\ln \Lambda^2 \times$ tree-level $\mathcal{L}^{(4)}$. It is then possible to absorb the cutoff dependence in the redefinition of coupling constants. It has been shown by Bijnens and Guberina that the results obtained coincide with the ones obtained using dimensional regularization.

As an example, the $K^0 - \bar{K}^0$ mixing transition amplitude appears to be

$$< K^0 | \mathcal{L}_{sdsd} | \bar{K}^0 > = \left( \frac{f_0^2}{2} + \frac{8g_1}{f_0^2} \right) m_K^2 - \frac{\Lambda^2}{16\pi^2} 5 \cdot m_K^2 + \cdots$$

The leading term is of the order $N_c^2 \sim f_0^2$, i.e., it is the leading factorizable contribution. The new unknown coupling $g_1$ is of the order $N_c$. The cutoff dependence can be partially absorbed in the definition of the renormalized coupling $f^{ren}$, and the rest enters the renormalized coupling $g_1^{ren}$. The last subtraction is consistent with the large-$N_c$ behavior, since $g_1$ and the quadratic divergence are of the same order in $N_c$. The logarithmic divergences are handled in the same way.

At the end, one can remove all quadratic and logarithmic divergences in the renormalization procedure. The logarithmic $\mu$ dependence is cancelled by the corresponding $\mu$ dependence of the counterterms. Both, the cutoff regularization and the dimensional regularization lead to the same result.

In the next subsection, we discuss the Bardeen-Buras-Gérard approach to the large-$N_c$ expansion, and QCD hadronic duality sum rules.

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4The leading terms in the large-$N_c$ expansion of the $K$-decay amplitudes lead to the ratio $A(K^0 \rightarrow \pi^+ \pi^-)/A(K^+ \rightarrow \pi^+\pi^0)$ of the order 1, which is by an order of magnitude smaller than the experimental result.
Bardeen-Buras-Gérard Approach versus ChPT

The Bardeen-Buras-Gérard (BBG) approach[71] is based on the large-$N_c$ expansion of QCD (with quarks and gluons), which implies the existence of an equivalent dual-meson representation. The effective $1/N_c$ expansion looks like a string theory. The leading-$N_c$ theory contains infinite trajectories of stable meson resonances. Since it is expected that the low-energy theory is only sensitive to low-lying states, one may use a proper truncation of the full theory. The proposal of Bardeen et al. is to use a nonlinear $\sigma$ model, including loop effects, as a first approximation. The strong lagrangian (3.7) in ChPT, discussed in preceding sections, now appears as the low-energy truncation of QCD.

The correct infrared behavior of low-energy theory is guaranteed by low-mass and low-momentum loop contribution. The truncation is achieved by introducing a physical momentum cutoff in loop integrals. As expected, the main contribution comes from the quadratic cutoff $\Lambda^2$. This is the crucial difference with respect to ChPT, where the cutoff dependence is absorbed in the renormalized quantities. Formally, the typical next-to-leading order amplitude in the BBG approach looks like

$$c^{-1}(\mu_{QCD})A \sim \Lambda^2 + b \ln \frac{m_K^2}{\Lambda^2} + \cdots,$$

(3.16)

which is to be compared with a typical amplitude in ChPT

$$A \sim w(\mu^2) + b \ln \frac{m_K^2}{\mu^2} + \cdots.$$

(3.17)

For $\Lambda \sim \mu$, the logarithmic terms are the same, and the finite counterterms in (3.17) are traded for the quadratic cutoff in (3.16). However, this difference is only the formal one since the amplitude (3.17) is the full amplitude, whereas the BBG result (3.16) is the matrix element of the local operator, i.e., the short-distance behavior is factorized out in the form of the Wilson coefficient. This makes the comparison between the BBG approach and ChPT by no means trivial. In addition, the physical cutoff in the BBG approach plays the role of a scale that has to match the corresponding renormalization scale of QCD. In the chiral limit, for example, one has to match the quadratic $\Lambda^2$ behavior with the logarithmic $\ln \mu_{QCD}$ one, which appears difficult[61]. However, higher resonances are expected to improve matching, smoothing the quadratic behavior of the scale $\Lambda$ and forcing the approximate logarithmic behavior\(^5\).

The second difference with respect to ChPT, i.e., that no counterterms were allowed in [71] means that, e.g., vector mesons and higher resonances have to be added separately. In ChPT, the latter are contained in counterterms.

\(^5\) It is interesting to note[55] that one can exactly control the $\mu$ dependence if one calculates the leading $1/N_c$ behavior of the penguin operator $Q_b = -8 \sum_q (\bar{s}_L q_R)(\bar{q}_R d_L)$. The leading term has the following chiral realization

$$Q_b^{ChPT} = -16v^2 \frac{8L_5}{f^2} \text{tr} (\Lambda \partial_{\mu} U \partial^{\mu} U^\dagger).$$

(3.18)

Now, $v^2$ is proportional to $(\overline{m}_q)^{-2}$, where $\overline{m}_q$ is the running quark mass, whose QCD behavior is $\overline{m}_q^2 \sim m^2 \alpha_s(\mu_{QCD}^2)^{9/11}$. On the other hand, the leading $1/N_c$ behavior of the Wilson coefficient is given by $c \sim \alpha_s(\mu_{QCD}^2)^{9/11}$. Obviously, doing a systematic $1/N_c$ expansion, one achieves an exact cancellation of the $\mu$ dependence.
As we mentioned, one would naively guess that the quadratic cutoff would have to mimic counterterms if both theories were equivalent. This, however, is not true for the following reasons: (i) contrary to the cutoff-terms, the counterterms contain long-distance effects as well as short-distance ones. In the approach of [71], the latter are factorized in the form of Wilson coefficients; (ii) the counterterms in ChPT also contain the effects of higher resonances (vector mesons, etc.). As given in [71], they have to be added separately.

In addition, ChPT would implicitly require the importance of counterterms in order to keep the whole correction in weak amplitudes moderate. Otherwise, the perturbative expansion would break since in some cases loop corrections contain huge logarithms [76]. This implies the importance of missing corrections due to vector mesons, etc., in the BBG approach.

From the above points it appears difficult to achieve one-to-one correspondence between ChPT and the large-$N_c$ approach of Bardeen et al. However, we would like to point out that in spite of our inability to achieve it, this genuine approach of Bardeen et al. leads to reasonable results. For example, the result for the $B$ parameter, $B \sim 0.7$ is rather stable as well as the result for the $K^0 \to \pi^+\pi^-$ amplitude. The latter is actually close to the experimental value, i.e., the $\Delta I = 1/2$ rule seems to be explained in the BBG approach. The $\Delta I = 3/2$ amplitude, however, is sensitive to the choice of the cutoff which induces a large uncertainty.

Recently, an exciting project has been started [74] in an attempt to derive the low-energy chiral realization of the $\Delta S = 1, 2$ operators by integrating out the quark fields in a gluonic background. The spontaneous breaking of chiral symmetry is triggered by a source term, which is added to the QCD lagrangian. The bosonization of the QCD lagrangians generates the effective lagrangian of ChPT but with explicit values for different couplings! Further work in this direction is in progress.

### 3.4 QCD Hadronic Duality Sum Rules

In the standard model, the effective hadronic weak lagrangian is usually given in terms of local 4-quark operators, e.g., the $\Delta S = 1$ lagrangian is given as

$$\mathcal{L}_{eff} = \sum_i c_i(\mu)O_i,$$

where $c_i(\mu)$ are the Wilson coefficients and $O_i$ are local operators. As we have discussed in the preceding sections, there is a complementary representation in the form of effective chiral lagrangian (3.9). Both pictures are equivalent. From the technical point of view these are associated with definite problems: using the lagrangian (3.19), one cannot reliably calculate the matrix elements of the composite local operators and the use of the complementary in ChPT gives rise to a large number of undetermined couplings. The duality proposed in the QCD hadronic duality sum-rule approach [56] spells out the consistency of both representations; indeed, there is a ‘window’ where both pictures reliably describe the same quantity. This is achieved by writing down a system of finite-energy sum rules (FESR’s) which relate the integrals of the hadronic spectral functions to the corresponding integrals calculated in QCD.
The starting point is the two-point function

\[ \psi(q^2) = i \int d^4 x e^{i q \cdot x} < 0|T(O(x)O^\dagger(0))|0>, \tag{3.20} \]

whose imaginary part enters the sum rules

\[ \Re_n(s_0) = \int_0^{s_0} dt \frac{t^n}{\pi} \Im \psi(t)_{\text{hadronic}} = |g'(\mu)|^2 \int_0^{s_0} dt \frac{t^n}{\pi} \sum_{\Gamma} |<0|\tilde{\mathcal{L}}|\Gamma>|^2 \]

\[ = \frac{s_0^{n+5}}{(n+5)(16\pi^2)^3} \left( \frac{\alpha_s(s_0)}{\alpha_s(\mu^2)} \right)^\gamma \left\{ a + d' \frac{\alpha_s(s_0)}{\pi} + b \frac{m_s^2(s_0)}{s_0} \right\} \]

\[ + c \frac{m_s^2}{s_0^2} \frac{<0|\bar{q}q|0>}{s_0^2} + d \frac{<0|F_{\mu\nu}^a F^{a\mu\nu}|0>}{s_0^2} + \ldots, \tag{3.21} \]

where \( g' = e^{-1}(\mu)g \). As can be seen from the l.h.s. of (3.21), the spectral function \( \frac{1}{\pi} \Im \psi(t) \) derived from the correlator (3.20) describes how the operator \( O \) couples the vacuum to physical states. In its ChPT version, it enables one to calculate the threshold behavior of the intermediate states (\( \Gamma = K\pi, K\pi\pi \), etc.). The unknown constant \( g' \) is factorized out as an overall normalization. The upper limit of integration, \( s_0 \), is the onset of the QCD continuum. It should be high enough, so that the QCD expansion on the r.h.s. of (3.21) has sense. The window we are talking about is roughly of the range of \( s_0 \) where the two pictures are dual. Clearly, this window cannot be of large range; for too low values of \( s_0 \), the QCD expansion breaks, and for too high values, ChPT cannot reliably parametrize the hadronic spectral function.

The parameters \( a \) and \( a' \) on the r.h.s. of (3.21) are calculable in perturbative QCD; \( a \) gives the asymptotic behavior in the chiral limit, and \( a' \) is a coefficient of finite \( \alpha_s \) corrections. Mass corrections are taken into account by the second term, and the third and fourth terms are nonperturbative corrections coming from quark and gluon condensates, respectively. Leading logarithmic corrections are summed up by making use of the renormalization group equation, and they are factorized out as a ratio of coupling constants to the power of \( \gamma \), the latter being proportional to the anomalous dimension of the local operator. For the multiplicatively renormalizable operators, \( c^2(\mu^2) \) is of the form

\[ c^2(\mu^2) = \left( \frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)} \right)^{-\gamma}, \tag{3.22} \]

i.e., the \( \mu \) dependence of the Wilson coefficient exactly cancels the \( \mu \) dependence of the matrix element.

In order to make sense, the power corrections in (3.21) should not exceed \( 20 - 30\% \); otherwise, higher mass and/or condensate corrections become important. This basically determines the allowed range of \( s_0 \) from the point of view of perturbative QCD expansion. It turns out that in the \( K^0-\bar{K}^0 \) mixing (\( B \) parameter)\[^{[56]} \), and in the \( K^+ \rightarrow \pi^+\pi^0 \) decay (\( \Delta I = 3/2 \) transition)\[^{[57]} \), this range is rather high, \( s_0 \sim 7 - 11 \text{ GeV}^2 \). This implies that one has to correct the pure ChPT behavior of hadronic spectral functions for the formation of resonances. The proposal in \[^{[56]} \) is to modulate the final-state interaction through the Breit-Wigner form and normalize it in such a way that for \( t = 0 \) it reduces to the chiral limit value.
The onset of scaling is obtained by taking the ratio of the two sum rules

\[ r_n(s_0) \sim \frac{\mathcal{R}_{n+1}(s_0)}{\mathcal{R}_n(s_0)}, \]

so that the unknown constant \( g' \) drops out. The ratio \( r \) is usually normalized in such a way that it equals 1 for the asymptotic case (no QCD). Then one looks for the duality region, i.e., for the region where the function \( r_n(s_0) \), calculated in ChPT, agrees with the same function calculated in QCD. Once the duality range of \( s_0 \) is fixed, any of the sum rules \( \mathcal{R}_n \) leads to the value of \( g \), for any \( s_0 \) in the duality range. With \( g \) determined, one can easily get the decay amplitude using the ChPT lagrangian.

The \( B \) parameter plays the role of the constant \( g' \). It has been calculated by Pich and de Rafael\[56\], with the result

\[ |B| = (0.33 \pm 0.09)[\alpha_s(\mu^2)]^{2/9}. \] (3.24)

Compared with typical results obtained in the alternative approach (large- \( N_c \) expansion, lattice QCD) the result (3.24) appears smaller roughly by a factor of 2, and, obviously, the vacuum saturation larger by a factor of 3.

The same method has been applied by Guberina, Pich and de Rafael\[57\] to the \( K^+ \to \pi^+\pi^0 \) decay and the result (3.24) agrees with experiment within a few percent. This should be considered as a successful test of the duality approach. It should be noted that especially this amplitude has almost always been reproduced in different approaches within a factor of 2, but better accuracy is difficult to achieve.

Recently, Prades et al.\[56\] have improved the calculation of \( B \) by a careful analysis of the hadronic parametrization. The result is somewhat higher, \(|B| = (0.39 \pm 0.10)[\alpha_s(\mu^2)]^{2/9}\).\[55\]

Unfortunately, the success shown by the above results does not pertain to the \( \Delta I = 1/2 \) amplitude. The calculation by Pich et al.\[55\] failed to reproduce this amplitude by an order of magnitude. It was quite difficult to understand this failure. Recently, definite progress has been made\[55\] which points out the reason for the failure. Namely, in the calculations of the \( B \) parameter and the \( \Delta I = 3/2 \) amplitude, finite \( \alpha_s \) corrections were taken into account. They were found to be moderate, and because of technical complexity, they were not calculated for the \( \Delta I = 1/2 \) amplitude, i.e., it was assumed that they were also moderate in this case. However, Pich has recently shown\[52\] that this assumption is premature. The perturbative finite \( \alpha_s \) corrections to the \( \Delta I = 1/2 \) amplitude are so huge that the whole perturbative expansion breaks. This clearly shows that the problem is highly nonperturbative, and cannot be handled in a perturbative way.

The calculation performed in\[52\] includes two assumptions that significantly reduce the complexity of calculation: (i) The operators \( O_\pm \) in the bare weak lagrangian are handled without mixing, i.e., as they are multiplicatively renormalizable, (ii) the penguin operator \( Q_6 \) is taken in the large-\( N_c \) limit in order to be multiplicatively renormalizable. Then, the calculation shows that for \( O_+ \), finite \( \alpha_s \) corrections are moderate, and for \( O_- \) and \( Q_6 \), they appear with the coefficients 47/5 and 423/20, respectively. With the usual \( \mu^2 = t \) rescaling which eliminates all logarithms in the spectral function, the corrections exceed 100%, even
at $t = 10 \text{ GeV}^2$, showing a clear breakdown of the perturbation expansion \(^6\).

We would like to point out that the discussed breakdown of the perturbative QCD expansion for $\Delta S = 1$ transitions may have serious consequences. Clearly, it is not possible to use the QCD duality approach in calculating these transitions. However, both the large-$N_c$ approach of Bardeen et al. and the lattice calculation also start from the Wilson expansion with the $\mu$ scale set at hadronic level. The huge radiative corrections both in the Wilson coefficients and in the matrix elements cast serious doubt on the validity of the starting point in the above calculations.

3.5 Hadronic Matrix Elements on the Lattice

Lattice QCD offers a method for calculating hadronic matrix elements from first principles. The technical difficulties are however enormous. Besides the usual problems caused by insufficient lattice size, lattice spacing, and computer time, the matching of the lattice and continuum operators is complicated by a conflict between the lattice regularization and the chiral properties of the theory \(\text{[63]}\). Since the chiral symmetry is broken on the lattice with Wilson fermions and the flavor content of the standard model is broken on the lattice with staggered (Kogut-Susskind) fermions, one must take a linear combination of lattice operators in order to form the desired continuum operators. This ‘mixing problem’ is extensively discussed in ref. \(\text{[64]}\).

The lattice calculation of the matrix element of a local operator between meson states is based on ‘measuring’ the corresponding three-point correlation function on the lattice:

$$G(x, 0, y) = < P_3(x)O(0)P_5(y) > .$$

(3.26)

$P_3$ are diquark operators with the same flavor content as the corresponding mesons, and $O$ is a 4-quark operator, e. g., appearing in the effective lagrangian (3.19). Let us illustrate the method on a typical matrix element, for example $\langle \pi^+ | \bar{s} \Gamma u \bar{u} \Gamma d | K^+ \rangle$. By replacing $\pi^+$ and $K^+$ by their diquark operators we arrive at the correlation function

$$G(x, 0, y) = < \bar{d}(x) \gamma_5 u(x) \bar{s}(0) \Gamma u(0) \bar{u}(0) \Gamma d(0) \bar{u}(y) \gamma_5 s(y) > .$$

(3.27)

The contraction of the quark fields in (3.27) in all possible ways yields the ‘eight’ and the ‘eye’ diagrams \(\text{[65]}\) depicted in Fig. 2. In $K \to \pi \pi$, the eye diagrams are pure $\Delta I = 1/2$, whereas the eight diagrams may be either $\Delta I = 1/2$ or $3/2$. In the $\Delta S = 2$ transition, only the eight diagram contributes. The propagators in Fig. 2 are quark propagators averaged over gauge configurations on the lattice. The configurations which include internal fermion loops are absent if the so-called ‘quenched’ approximation is used. Technically, the fermion determinant in the averages is set equal to one. The approximation is justified due to a $1/N_c$ suppression of the fermion loops. In the eye diagram, however, the $u$-quark loop is not of

\(^6\) It is interesting to note that this behavior persists in in higher orders. In the large-$N_c$ limit, the $O(\alpha_s^4)$ corrections can be easily computed, leading to the spectral function of the penguin operator $Q_6$:

$$\psi(t)_{\text{peng}} \sim [\alpha_s(t)]^{18/11} \left\{ 1 + 24.28 \frac{\alpha_s(t)}{\pi} + 470.72 \left( \frac{\alpha_s(t)}{\pi} \right)^2 + \cdots \right\}.$$

(3.25)
higher order in $1/N_c$ and therefore the eye cannot be eliminated on the basis of the quenched approximation. Moreover, its contribution is believed to be responsible for a large part of the $\Delta I = 1/2$ enhancement.

As regards the $K \to \pi\pi$ decays, there are basically two methods of calculation, depending on the prescription used for lattice fermions. In the first method, the $K \to \pi\pi$ amplitude is calculated directly, with all particles at rest and all quarks degenerate. All particles are on-shell, so the amplitude is well defined but the weak Hamiltonian must insert momentum. Lowest-order ChPT is then used to relate the amplitude to the physical one. The method involves a relatively small chiral extrapolation and Wilson fermions can be applied. This method has been used by the Bernard and Sony group [64, 13, 66] and the European Lattice Collaboration (ELC) [63, 12, 67].

The second method is to measure the $K \to \pi$ and $K \to \text{vac}$ matrix elements and use ChPT to relate these to the physical $K \to \pi\pi$ amplitude [14]. In ChPT, the physical amplitude is proportional to the matrix element $\langle \pi | O_{\text{sub}} | K \rangle$, where the subtracted operator is defined as [68]

$$O_{\text{sub}} = O - \rho (m_s + m_d) \bar{s}d.$$  (3.28)

The coefficient $\rho$ is determined from the ChPT relation

$$<0|O|K> = \rho (m_s - m_d) <0|\bar{s}\gamma_5d|K>.$$  (3.29)

This method is appropriate for staggered fermions owing to their good chiral properties, which allow the use of equation (3.29). In addition to calculating the eight and the eye diagrams, one has to calculate the matrix elements of the two-quark operator subtraction and the $K \to \text{vac}$ diagram needed to fix the coefficient of the subtraction. This method has been mainly used by the staggered group [68, 69, 70].

If one computes $K \to \pi$ or $K \to \pi\pi$ on the lattice, one has to face up to dealing with the $\pi\pi$ interactions in the final state (FSI). The FSI may cause a significant enhancement of the $\Delta I = 1/2$ amplitudes, and suppression of the $\Delta I = 3/2$ amplitudes [69]. In addition, there is a final state phase to be determined in order to sum both amplitudes to get the physical $K \to \pi^+\pi^-$ rates.

In the last few years calculations of the hadronic matrix elements on the lattice have shown substantial improvement. The systematic errors and finite size effects are now under better control. Let us quote some of the recent results most of which, however, should still be considered as qualitative.

Relatively reliable data are obtained for the $\bar{K}K$ amplitude. The ELC collaboration quotes two values for the $B$ parameter at $a^{-1} = 1.34$ GeV, depending on what type of fit is used: $0.91 \pm 0.11$ (linear fit) and $0.64 \pm 0.11$ (quadratic fit) [67]. The staggered group observed a deviation from the asymptotic scaling in the region of $\beta = 6/g^2$ from 6 to 6.4. They find that if the lattice spacing is reduced by a factor of 2 (keeping the physical volume fixed), $B$ drops from $0.69 \pm 0.02$ to $0.54 \pm 0.05$, whereas the scaling requires a change of about a few percent. The extrapolation to $a = 0$ with the anomalous dimension included would give $B g^{-4/9} \approx 0.45$ [68]. Bernard and Sony observed a similar behavior. Going from the lattice size $16^3 \times 40$ to $24^3 \times 40$ at $\beta = 6$, they found that $B$ dropped from $0.86 \pm 0.24$ to $0.69 \pm 0.12$ [66]. A probable explanation for this scaling violation is that one still sees large $O(a)$ effects.
As regards the $K \to \pi$ and $K \to \pi\pi$ amplitudes the data are less reliable. The ELC group used two methods of calculation. The first one was to translate their $\bar{K}K$ data into the $K\pi$ amplitude and the second was a direct $K \to \pi\pi$ calculation. The results for the $\Delta I = 3/2$ amplitude obtained using the two methods are $A_{3/2} = (7.0 \pm 0.8)10^{-8}m_K$ and $(8.6 \pm 0.8)10^{-8}m_K$, respectively, to be compared with the experimental value $3.7 \cdot 10^{-8}m_K$. The ratio $R$ obtained using the $K \to \pi$ method is $12 \pm 5$ with 75 configurations, but unfortunately $12 \pm 12$ with 110 configurations. The direct method yielded a worse result: $R = 35 \pm 30$ [7].

Sharpe suggested that the $\Delta I = 1/2$ rule might be an accumulation of factors of 1.5 - 2 due to different mechanisms [69]. Taking into account an enhancement of $R$ arising from the FSI, the lattice calculation should aim at about 62 % of the experimental value, i. e., $R_{\text{eye}+\text{eight}} \approx 14$. The staggered group finds that the eight diagram alone gives $R_{\text{eight}} = 3.6 \pm 0.02$, but according to their last data, the eye contribution is unexpectedly consistent with zero [70].
4 Rare Kaon Decays

4.1 $K_L \to \mu \bar{\mu}$ and $K_{L,S} \to \gamma\gamma$ Decays

As far as weak interactions are concerned, the $K_L \to \mu \bar{\mu}$ proceeds partly via a box diagram (diagram (a), Fig. 3), and partly via an $sdZ$ vertex (black box in diagram (b), Fig. 3). The first diagram is similar to the process $K_L \to \gamma\gamma$, which proceeds via diagram (b), Fig. 3, where $q$ denotes up-type quarks, u, c, and t. In fact, a major contribution to the decay rate for $K_L \to \mu \bar{\mu}$ comes from the higher-order electromagnetic process (diagram (d), Fig. 3), i.e., via a two-photon intermediate state. This diagram actually dominates the imaginary part of the amplitude $A(K_L \to \mu \bar{\mu})$, relating it to the process $K_L \to \gamma\gamma$:

$$\Gamma_{absorptive}^{K_L \to \mu \bar{\mu}} \sim \Gamma_{K_L \to \gamma\gamma}$$  \hspace{1cm} (4.1)

We would like to stress that eq. (4.1) would, in principle, provide us with a lower limit for the partial width

$$\Gamma_{K_L \to \mu \bar{\mu}} \geq \Gamma_{absorptive}^{K_L \to \mu \bar{\mu}},$$  \hspace{1cm} (4.2)

with $\Gamma_{absorptive}^{K_L \to \mu \bar{\mu}}$ being the ‘unitary bound’.

Although, naively, the processes $K_L \to \mu \bar{\mu}$ (diagram (a)) and $K_L \to \gamma\gamma$ (diagram (b)) would both have amplitudes of comparable strength, $A \sim e^4/M_W^2 \sim G_F \alpha$, experimentally, they differ by a factor of $10^{-5}$, the reason being, of course, the GIM suppression. The respective branching ratios are given by [19]

$$BR(K_L \to \mu \bar{\mu}) = (7.3 \pm 0.4 \times 10^{-9}),$$  \hspace{1cm} (4.3)

$$BR(K_L \to \gamma\gamma) = (5.70 \pm 0.27) \times 10^{-4}.$$  \hspace{1cm} (4.4)

Closely related to the $K_L \to \mu \bar{\mu}$ decay is the process $K^+ \to \pi^+ \nu\bar{\nu}$. It receives contributions both from diagram (c) and diagram (e) in Fig. 3. Especially, the unitary bound derived for $K_L \to \mu \bar{\mu}$ would set constraints on the $K^+ \to \pi^+ \nu\bar{\nu}$ decay, provided the $K_L \to \mu \bar{\mu}$ decay rate were dominated by the absorptive part of $K_L \to \gamma\gamma \rightarrow \mu \bar{\mu}$ and the dispersive part were negligible [80].

The effective lagrangian for the $d \bar{s} \rightarrow \mu \bar{\mu}$ and $d \bar{s} \rightarrow \nu \bar{\nu}$ processes in zeroth order in strong interactions is [79]

$$\mathcal{L} = \frac{G_F^2 M_W^2}{\pi^2} s_L \gamma_{\mu} d_L (\bar{C} \mu L \gamma^\mu \mu_L - \sum_i \bar{D}_i \bar{\nu}_i^\dagger \gamma^\mu \nu_i^\dagger).$$  \hspace{1cm} (4.5)

The coefficients $\bar{C}$ and $\bar{D}_i$ ($i = \text{generation index}$) are related, via the unitarity relation $\sum_j V_{js}^* V_{jd} = 0$, to the Inami-Lim coefficients $\bar{C}$ and $\bar{D}_i$ [72], which are functions of the mass ratios $x_j = m_{q_j}^2/M_W^2$ and $y_j = m_{q_j}^2/M_W^2$.

As we discussed earlier, there are potentially large long-distance effects in the $K_L \to \mu \bar{\mu}$ decay.

It is usually assumed that the dispersive part of the amplitude for $K_L \to \gamma\gamma \rightarrow \mu \bar{\mu}$ is small compared with the absorptive one [80]. If this were true, the limit on the short-distance part of $K_L \to \mu \bar{\mu}$ would be obtained as

$$A(K_L \to \mu \bar{\mu})_{\text{short-dist.}} \leq A(K_L \to \mu \bar{\mu})_{\text{exp}} - A(K_L \to \gamma\gamma \rightarrow \mu \bar{\mu})_{\text{absorp.}}.$$  \hspace{1cm} (4.6)
However, if the dispersive part of $\mathcal{A}(K_L \rightarrow \gamma\gamma \rightarrow \mu\bar{\mu})$ were large, and with opposite sign to $\mathcal{A}(K_L \rightarrow \mu\bar{\mu})_{\text{short-dist.}}$, the above constraint would no longer be valid.

The decay $K_L \rightarrow \mu\bar{\mu}$ has been studied in ChPT\textsuperscript{81}. The main contribution comes from the diagram (a), Fig. 3, i.e., the decay proceeds via the $\gamma\gamma$ intermediate state. The $P\gamma\gamma$ vertex is described by the Wess-Zumino term. The corresponding integral is logarithmically divergent and the theory requires counterterms in order to renormalize the divergences. The finite part of the counterterms unfortunately cannot be determined, and one relies only on the logarithmic terms. This is certainly not quite reliable, although one does not expect that this arbitrariness essentially changes the conclusions.

If one performs renormalization in the $\overline{\text{MS}}$-scheme, the two-photon intermediate-state contribution is\textsuperscript{81}

$$\Gamma_{K_L \rightarrow \mu\bar{\mu}} = \Gamma_{K_L \rightarrow \gamma\gamma} \frac{\alpha^2 \beta}{2\pi^2} \left( \frac{m_\mu}{m_K} \right)^2 |\mathcal{A}|^2,$$

(4.7)

where $Re \mathcal{A}$ contains the logarithmic term, which is, of course, $\mu$ dependent\textsuperscript{7}:

$$Re \mathcal{A} = 3 \ln \left( \frac{m_\mu}{m^2} \right) - 7 + \frac{1}{\beta} \left\{ \frac{1}{2} \ln^2 \left( \frac{1+\beta}{1-\beta} \right) - \frac{1}{3} \pi^2 + 2 \text{Li}_2 \left( \frac{1-\beta}{1+\beta} \right) \right\},$$

(4.8)

and $\beta = (1 - 4m^2_\mu/m^2_K)^{1/2}$. On the other hand, $Im \mathcal{A}$ has no $\mu$ dependence:

$$Im \mathcal{A} = -\frac{\pi}{\beta} \ln \left( \frac{1+\beta}{1-\beta} \right).$$

(4.9)

The amplitude $\mathcal{A}$ has been obtained by using a once-subtracted dispersion relation with $Im \mathcal{A}$ being known from the calculation performed in ref.\textsuperscript{82}.

Choosing $\mu^2 \approx m^2_K$, one gets

$$\frac{\Gamma_{K_L \rightarrow \mu\bar{\mu}}}{\Gamma_{K_L \rightarrow \gamma\gamma}} = 3.6 \times 10^{-5}. $$

(4.10)

Theoretical result is about twice as large as the experimental one. This, however, should be considered as reasonable agreement, since:

(i) the calculation is not complete; the finite part of the counterterms has not been determined, and

(ii) the $\mu$ dependence of $Re \mathcal{A}$ is pronounced. In addition to the rather strong $\mu$ dependence (a factor of 4 in the range $0.2 \leq \mu \leq 1$ GeV ), one finds that, starting from $\mu \sim 0.3$ GeV, a dispersive part dominates over an absorptive one, the former being a factor of 2 larger than the latter, for $\mu^2 = m^2_K$.

Having in mind points (i) and (ii), it would be premature to claim the real dominance of the dispersive part. However, contrary to the usual assumptions, this shows that it may be significant.

Before we compare this calculation with the calculations in different approaches, we should remember the following. In principle, if the counterterms were determined, this would be a complete calculation of $\Gamma_{K_L \rightarrow \mu\bar{\mu}}$ ; i.e., it would also contain short-distance contributions.

\footnote{If one were able to determine the finite part of the counterterms, the $\mu$ dependence would disappear, i.e., the counterterms are also $\mu$ dependent.}

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In addition, all effects of heavy particles appearing in the theories which go beyond the SM would also be included in the counterterms.

\[ K_S \to \gamma \gamma \text{ process in ChPT} \]

This process is described in ChPT by the lagrangians (3.11)-(3.14). In the calculation performed in ref. [81], only the \( \Delta I = 1/2 \) part in \( \mathcal{L}_{\Delta S=1} \) has been kept. Then, the process proceeds via the diagrams shown in Fig. 4. It is the simplest example of the predictive power of chiral perturbation theory because it can arise only from loop graphs involving \( \mathcal{L}^{(2)} \), but does not receive any contribution from the direct couplings \( w_1, w_2, w_4 \). These couplings play the role of counterterms, i.e., the loop divergences have to be absorbed in these couplings. Since couplings are absent, that means that divergences must cancel and loops have to be finite. The graphs displayed are expected to have both quadratic and logarithmic divergences. The former are cancelled by an \( SU(3) \) invariant counterterm, and the latter also disappear. This surprising result is a consequence of the requirement that the amplitude vanishes in the \( m_K^2 = m_{\pi}^2 \) limit. The final result is given as [81]

\[
\Gamma_{K_S \to \gamma \gamma} = \frac{\bar{g}_s^2 \alpha^2 m_{K}^2 f_{\pi}^2}{2 \pi^3} \left(1 - \frac{m_{\pi}^2}{m_K^2}\right)^2 |F\left(\frac{m_{K}^2}{m_{\pi}^2}\right)|^2, \tag{4.11}
\]

where

\[
F(z) = \left\{1 - \frac{1}{z}\left[\pi^2 - \ln^2 Q(z) - 2i\pi \ln Q(z)\right]\right\},
\]

\[
Q(z) = \frac{1 - (1 - 4/z)^{1/2}}{1 + (1 - 4/z)^{1/2}}. \tag{4.12}
\]

The amplitude entering (4.11) is dominated by the imaginary part, which is also given in ref. [82]. Numerically, eq.(4.11) gives

\[
\Gamma_{K_S \to \gamma \gamma} = 1.4 \times 10^{-20} \text{ GeV}. \tag{4.13}
\]

From (4.13) one infers that

\[
BR(K_S \to \gamma \gamma)_{\text{theory}} = 2.0 \times 10^{-6}. \tag{4.14}
\]

The measured branching ratio is [19]

\[
BR(K_S \to \gamma \gamma)_{\text{exp}} = (2.4 \pm 1.2) \times 10^{-6}, \tag{4.15}
\]

in excellent agreement with the theoretical prediction (4.14). This is a nice example of how the full ChPT completely accounts for the process in question.
This decay basically proceeds via the $\pi^0$, $\eta$, and $\eta'$ poles

$$A = \sum_i A(P_i \rightarrow \gamma\gamma) \frac{<P_i|\mathcal{H}|K_L>}{m_K^2 - m_{P_i}^2}.$$  \hfill (4.16)

The $\eta'$ contributes only through an octet weak transition. The vertices $P_{\gamma\gamma}$ are again described by the Wess-Zumino term. The amplitude (4.16) has been carefully calculated in [81], and by Goity [83]. There are two sources of uncertainties that enter that calculation. The first one is the $\eta - \eta'$ mixing angle $\theta$, whose previous value was about 10°. However, recent measurements of $\eta$ production in $\gamma\gamma$ collisions [84] give a higher $\eta \rightarrow \gamma\gamma$ rate, and consequently a larger mixing angle $\theta$. The $\eta$ and $\eta'$ decays are well described by taking the values $\theta = 24^o \pm 2^o$ and $f_\pi/f_{\eta'} = 1.04 \pm 0.05$. \hfill (4.17)

In spite of that, the result for $A(K_L \rightarrow \gamma\gamma)$ is very sensitive to the parameters $\delta = <K_L|\eta_8>$ and $\kappa = <K_L|\eta_1>$. Plotting the experimental BR($K_L \rightarrow \gamma\gamma$) in terms of $\delta$ and $\kappa$ shows [83] that for agreement with experiment, the parameter $\delta$ should deviate from its chiral limit value.

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### 4.2 Radiative Kaon Decays in ChPT

We have already discussed the decay $K_S \rightarrow \gamma\gamma$, which is a nice example of the self-consistency of ChPT. This decay belongs to class (i) which includes decays where the contributions from the counterterm lagrangian are forbidden by some symmetry. Class (ii) includes decays where loops are finite as in class (i), but there is an additional contribution of the counterterms that is now scale-independent. Finally, class (iii) includes decays where the loop amplitudes diverge. In this case, ChPT should allow for the counterterms.

#### Decays $K \rightarrow \pi\gamma\gamma$

The amplitude $K^0 \rightarrow \pi^0\gamma\gamma$ can be completely calculated in ChPT. It is uniquely determined since, as in the process $K_S \rightarrow \gamma\gamma$, the counterterms do not contribute, and the amplitude is given in terms of the loop amplitude. The spectrum is given by [53]

$$\frac{d\Gamma(K_L \rightarrow \pi^0\gamma\gamma)}{dy} = \frac{4\alpha^2 m_K^5 g_8^2}{(4\pi)^5} \lambda^{1/2} (1, y, z^{-2}) |(y - z^{-2})F(yz^2) + (1 - y + z^{-2})F(y)|^2$$

$$y = m_{\gamma\gamma}^2/m_K^2 \quad [0 \leq y \leq (1 - z)^{-2} = 0.52]$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca), \quad (4.18)$$

---

8 A large mixing angle, $\theta \approx 20^o$, is also predicted in the large-$N_c$ limit [52].

9 The good agreement with experiment, obtained for $\theta = 13^o$ in ref. [51], is, of course, spoiled with the present value of $\theta$. The same is true for the calculation of ref. [85].

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where $F(z)$ is the function given in (4.12). The spectrum has a characteristic shape, quite different from the phase space. The integrated rate (4.18) gives

$$BR(K_L \to \pi^0\gamma\gamma) = 6.8 \times 10^{-7}. \quad (4.19)$$

This branching ratio has recently been measured. The reported result \[19\] is: $(2.0\pm0.5)\times10^{-6}$.

The parameter free prediction is the ratio

$$\frac{\Gamma(K_L \to \pi^0\gamma\gamma)}{\Gamma(K_S \to \gamma\gamma)} = 5.9 \times 10^{-4}. \quad (4.20)$$

A similar process, $K^+ \to \pi^+\gamma\gamma$, belongs to class (ii): contributions come from a finite-loop amplitude, from scale-independent counterterms\[17\], and, in addition, from the anomaly. The lower bound on the branching ratio is predicted to be

$$BR(K^+ \to \pi^+\gamma\gamma) \geq 4 \times 10^{-7}, \quad (4.21)$$

and is expected to be measured soon, since the present upper limit is\[13\]

$$BR(K^+ \to \pi^+\gamma\gamma) < 10^{-6}. \quad (4.22)$$

The decays $K \to \pi\gamma\gamma$ are also interesting because CP violation appears owing to the interference between the absorptive amplitude and the counterterms (complex numbers). The charge asymmetry is given as

$$\Gamma(K^+ \to \pi^+\gamma\gamma) - \Gamma(K^- \to \pi^-\gamma\gamma) = Im \hat{c} 1.5 \times 10^{-23} GeV, \quad (4.23)$$

which crucially depends on the value of $\hat{c}$. There is also a charge asymmetry for the $K \to \pi e^+ e^-$ decays; it is given as

$$\Gamma(K^+ \to \pi^+ e^+ e^-) - \Gamma(K^- \to \pi^- e^+ e^-) = Im w_+ 1.6 \times 10^{-25} GeV, \quad (4.24)$$

where $w_+$ is the following combination of counterterms

$$w_+ = -\frac{16\pi^2}{3}(w_1 + 2w_2 - 12L_9). \quad (4.25)$$

The source of CP violation in the standard model is presumably a phase in $g_8$. Turning on the electromagnetic interaction induces the electromagnetic penguin as a new source of CP violation. The latter contributes only to the counterterm $w_1$ in the large-$N_c$ approximation. Since the counterterms $L_i$ are real, the following relation holds:

$$Im \hat{c} = 2 Im w_+ = -\frac{32\pi^2}{3}Im w_1. \quad (4.26)$$

This relation predicts the ratio of charge asymmetry (4.23) versus charge asymmetry (4.24) to be $\simeq 200$.

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\[10\] The counterterms enter the amplitude in the form $\hat{c} = 32\pi^2[4(L_9 + L_{10}) - \frac{1}{3}(w_1 + 2w_2 + 2w_4)]$, which is estimated to be $O(1)$\[53\].
Decays $K_L \to \pi^0 e^+ e^-$ and $\eta \to \pi^0 \gamma \gamma$

The one-photon exchange contribution to the amplitude for $K_L \to \pi^0 e^+ e^-$ is purely CP violating. The intrinsic CP violation is again related to $Im w_1$ and is comparable with the ‘normal’ CP violation. However, what makes this amplitude peculiar is the rate: the one-photon exchange leads to $BR \sim 10^{-12} - 10^{-11}$. Higher-order contributions in ChPT\cite{53} are by two orders of magnitude smaller than the one-photon exchange contribution\textsuperscript{11}.

The electromagnetic suppression of the CP-conserving amplitude makes it probable that this decay is dominated by the CP-violating contributions coming from the small CP-even $K^0_1$ component of the $K_L$ and through the direct CP-violating contribution in $K^0_2 \to \pi^0 e^+ e^-$. The latter is of the same order or even larger than the indirect one.

There is another candidate, the $\eta \to \pi^0 \gamma \gamma$ decay, which causes difficulties. It receives contributions only from the finite loops, and the unique prediction is

$$\Gamma(\eta \to \pi^0 \gamma \gamma)_{\text{one-loop}} = 0.35 \times 10^{-2} \text{ eV}. \quad (4.27)$$

The experimental value is larger by two orders of magnitude\textsuperscript{19}:

$$\Gamma(\eta \to \pi^0 \gamma \gamma)_{\text{exp}} = (0.85 \pm 0.19) \text{ eV}. \quad (4.28)$$

Higher-order contributions are estimated, and the result is still far from the experimental value. This decay really looks like an unexpected failure of ChPT; however, before claiming it, one should rather wait for confirmation of (4.28) in independent measurements.

\textsuperscript{11} As expected, correct introduction of vector mesons as nonlinear realizations of chiral $SU(3)_V$ leads basically to the same results as those obtained using counterterms. The blind use of vector-meson dominance produces, e.g., contributions which are not permitted by chiral symmetry\textsuperscript{53}. 

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5 Rare Decays $K^+ \to \pi^+ \nu \bar{\nu}$ and $K^+ \to \pi^+ \tilde{\gamma} \tilde{\gamma}$

5.1 Supergravity Effects in Rare Processes

Given the fact that the limits on superparticle masses are presently rather high, roughly of the order of $M_W$ or higher, it is clear that supersymmetric effects are expected to be very tiny. The strongest constraints seem to come from the $\mu \to e\gamma$ process in the lepton sector and the $K_L - K_S$ system in the quark sector. The constraints imposed by these two processes are in general respected in $K_L \to \mu\mu, K_L \to \mu e, \mu \to eee, \mu N \to eN$, etc. [20, 21]

Muon anomalous magnetic moment

No useful constraints are expected from the $g-2$ factor [44]. The agreement between theory and experiment for the anomalous magnetic moment of the muon is better than $2 \times 10^{-8}$. In the supersymmetrized self-energy graph of the muon, the photino and the smuon enter the graph. With $\mu_L$ and $\mu_R$ degenerate, the contribution is proportional to the square of the muon mass. Then, the effective vertex takes the form

$$
\frac{e}{2m_\mu} F(q^2) \bar{u} \sigma_{\mu \nu} q^\nu u,
$$

(5.1)

where $q$ is the momentum of the photon. The $g-2$ gets a contribution through $F(0)$ which is proportional to the muon mass. Besides, there are graphs with wino and zino, Fig. 5. A massless photino graph has been calculated by Fayet [27] with the result $a = 1/2(g-2) \approx 10^{-9}$, provided the smuon masses are larger than 15 GeV. Adding wino and zino graphs, Ellis, Hagelin and Nanopoulos [28], and Barbieri and Maiani [29] have calculated the full contribution in global SUSY, with essentially the same result. The extension to a more realistic supergravity model has been made by Romão et al. [30]. Contributions of neutral gauginos have terms quadratic and linear in $m_\mu$. The latter are potentially large, but they are suppressed owing to a kind of GIM mechanism [23]. At the end, the dominant contribution comes from the photino graph, which dominates by an order of magnitude. For a given photino mass, $a$ is a decreasing function of gravitino mass. For $m_{3/2} \leq 15$ GeV, the present agreement between theory and experiment is preserved. This means that supergravity effects on $g-2$ are really small.

$\mu \to e\gamma$ decay

In the standard model with massless neutrinos the decay $\mu \to e\gamma$ is strictly forbidden owing to lepton flavor conservation. This also remains true in the supergravity model. However, allowing for massive neutrinos compatible with present experimental upper limits on their masses, the branching ratio for this decay is of the order of $\sim 10^{-16}$ in the standard model. This is far below the experimental upper limit, which is $7.2 \times 10^{-11}$.  

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In the supergravity there is an additional diagram for the $\mu \rightarrow e\gamma$ decay, diagram (a) in Fig. 5.

Keeping the same values of neutrino masses and mixing angles as in the SM, and varying gravitino and photino masses in the range $20 - 250$ GeV, one finds that for a given $m_{3/2}$ the branching ratio increases with $m_\gamma$, the reason being that the increase of the photino mass results in the decrease of the wino mass. Since the branching ratio contains a term proportional to $(M_W/M_\tilde{W})^2$, it produces an enhancement for light winos. This enhancement, however, weakens with increasing gravitino mass\cite{6}. Even for a very light gravitino, $m_{3/2} = 20$ GeV, the SUSY branching ratio is smaller than $10^{-13}$.

In view of the present experimental limits on SUSY particles, these effects are also very tiny.

$K_L - K_S$ System

In the standard model, the $K_L - K_S$ system is described by box graphs with $u, c, t$ quarks entering the loops. For degenerate quarks, these contributions cancel. The same remains true for supersymmetrized graphs since the partners of quarks and $W$ bosons, squarks and gauginos, have the same quantum numbers.

This cannot lead to the limit on squark masses, but gives limits on mass differences between the families. The bounds obtained are even strenghtened if one includes graphs with strong interacting gluinos. The condition for this is that superscalars have an off-diagonal mass matrix in the basis where the quark mass matrix is diagonal. Thus, the bounds on masses of $\tilde{d}, \tilde{s}, \tilde{b}$ superscalars are obtained. These bounds, coming from gluino exchange, are generally by an order of magnitude stronger than the corresponding ones coming from simple supersymmetrized graphs.

The usefulness of bounds is often spoiled by uncertainties in the QCD calculation of matrix elements of the effective operators once the heavy (super)fields are integrated out. We discuss these problems in Chapter 6.

5.2 The decay $K^+ \rightarrow \pi^+\nu\bar{\nu}$ in the standard model

At zeroth order in strong interactions the decay $K^+ \rightarrow \pi^+\nu\bar{\nu}$ proceeds through diagrams in Fig. 6. The first group of graphs, group (a), are box graphs, which, calculated in the $R_\xi$-gauge, contain a gauge-dependent part $\gamma(\xi)$. The same function, $\gamma(\xi)$, also appears in group (b) of diagrams, but with an opposite sign. Therefore, the whole set of graphs in Fig. 6 is gauge independent, as it should be. The whole set of graphs in Fig. 6 leads\cite{58, 70, 87} to the result\cite{43} for the effective lagrangian.

The branching ratio $BR(K^+ \rightarrow \pi^+\nu\bar{\nu})$ can be related to the $BR(K^+ \rightarrow \pi^0 e^+\nu)$ by using an isospin relation\cite{79}

$$BR(K^+ \rightarrow \pi^+\nu_i\bar{\nu}_i) = \frac{G_F^2 M_W^4}{4\pi^4} \frac{|\tilde{D}_i|^2}{V_{us}^2} BR(K^+ \rightarrow \pi^0 e^+\nu_e).$$

(5.2)
Using the experimental values \( BR(K^+ \to \pi^0 e^+ \nu_e) = (4.82 \pm 0.06)\%, M_W = (80.22 \pm 0.26) \) GeV, and \( G_F = 1.16637 \times 10^{-5} \) GeV\(^{-2}\), one gets

\[
BR(K^+ \to \pi^0 \nu_i \bar{\nu}_i) = 0.72 \times 10^{-6} \frac{\bar{D}_i^2}{V_{us}^2}.
\] (5.3)

The function \( \bar{D}_i \) is proportional to the mass correction function \( \bar{D}_i \), which is a function of mass ratios \( x_j = m_{q_j}^2/M_W^2 \) and \( y_i = m_{t_i}^2/M_W^2 \). Neglecting the lepton mass in \( y_i \), the value \( \bar{D}_c \) for \( m_c = 1.5 \) GeV is about \( 4 \times 10^{-3} \). This value decreases with increasing lepton mass: for the \( \tau \) lepton with \( m_\tau = 1.784 \) GeV, \( \bar{D}_c \) is reduced to \( 3.2 \times 10^{-3} \). However, the value of \( \bar{D}_t \) increases almost linearly with \( m_t \). In the range \( 80 \) GeV \( \leq m_t \leq 150 \) GeV, \( D_t \) is in the range \( 1.38 \leq \bar{D}_t \leq 2.72 \). Taking a lower limit, \( m_t \geq 90 \) GeV, one gains a factor of at least 400 in the ratio

\[
\frac{\bar{D}_t}{\bar{D}_c} \geq 400.
\] (5.4)

With neglect of the QCD corrections, (5.3) becomes

\[
BR(K^+ \to \pi^0 \nu_i \bar{\nu}_i) = 0.72 \times 10^{-6} |V_{ud}|^2 - \bar{D}_c(x_c) + \frac{V_{ts}^* V_{td}}{V_{us}^* V_{sd}} \bar{D}_t(x_t)^2,
\] (5.5)

where the approximation \( V_{ts}^* V_{td} \approx -V_{us}^* V_{ud} \) has been used. Neglecting lepton masses, one obtains the function \( \bar{D}(x) \) as

\[
\bar{D}(x) = \frac{x}{4} - \frac{3}{4} \frac{x}{1 - x} + \frac{1}{8} \left( 1 + \frac{3}{(1 - x)^2} - \frac{(4 - x)^2}{(1 - x)^2} \right) x \ln x.
\] (5.6)

The weak contribution to the similar process \( K_L \to \mu \bar{\nu} \) is given by

\[
BR(K_L \to \mu \bar{\nu})_{weak} \simeq 7.7 \times 10^{-5} |V_{us}|^2 |Re \sum_i V_{ts}^* V_{td} \bar{C}_i(x_i)|^2,
\] (5.7)

where \( \bar{C}_i \) is defined by

\[
\bar{C}(x) = x + \frac{3}{4} \frac{x^2}{1 - x} + \frac{3 x^2 \ln x}{4(1 - x)^2}.
\] (5.8)

As we have discussed in Chapter 4, the dominant contribution to the \( K_L \to \mu \bar{\nu} \) decay comes from the two-photon intermediate state. Taking the absorbive part of the amplitude as calculated in ChPT, eq. (4.9), and assuming no interference between the real part of the same amplitude and the weak contribution as given in (5.7), one obtains

\[
BR(K_L \to \mu \bar{\nu})_{disp} \simeq (2 \pm 2) \times 10^{-9}.
\] (5.9)

The above assumption then yields the constraint

\[
|Re V_{ts}^* V_{td} \bar{C}_i(x_t)|^2 \leq 1.7 \times 10^{-3}.
\] (5.10)

The constraint in (5.10) can be used keeping in mind that the possible interference in the dispersive part of the amplitude would weaken it.
5.3 QCD Corrections to the $K_L \rightarrow \mu \bar{\mu}$ and $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decays

QCD corrections to the process $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ have been discussed in a number of papers\cite{29, 87, 89-93}. A controversy existed for some time, and has been resolved by Novikov et al. in \cite{87}.

As far as the box diagram is concerned, there are three types of gluon contributions (diagrams (a) in Fig. 7): (i) gluons might be exchanged between $d$ and $s$ quarks, (ii) there is a quark-quark-$W$ vertex correction, and (iii) there is a self-energy graph (mass renormalization) to an intermediate quark.

It turns out that, working out QCD corrections in the Landau gauge, only the self-energy graph in Fig. 7 gives a logarithmic factor.

The loop integral in the box diagram is of the form
\[
I = \frac{1}{2} m_c^2 M_W^4 \int \frac{dp^2}{(p^2 + M_W^2)^2 p^2 + m_c^2} F(p^2) \approx \frac{1}{2} m_c^2 \int_{m_c^2}^{M_W^2} \frac{dp^2}{p^2} F(p^2).
\] (5.11)
The last approximation is valid only in the LLA.

An important point made by Novikov et al.\cite{86} is that the lower limit in (5.11) is not $\mu^2$ (arbitrary), but $m_c^2$. For $\mu^2 \leq m_c^2$, there are no logarithmic gluon corrections to the mass operator. This also solves the controversy with respect to previous calculations\cite{89-91}.

As is obvious from the graphs (a) in Fig. 7, the QCD corrections to the box graphs for $K_L \rightarrow \mu \bar{\mu}$ and $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ are the same. This is not the case for triangle graph corrections, to which we turn next.

The triangle graph includes the $T$ product of three hadronic currents. In contrast to the box graph, there exist now two independent distances. The calculation is simplified because there exists the Ward identity for the $sdZ$ vertex. Therefore, one can reduce the calculation of the matrix elements of $K_L \rightarrow \mu \bar{\mu}$ to the $T$ product of two currents and the matrix element of the $T$ product of two currents and the pseudoscalar density $\bar{c}\gamma_5 c$. Finally, one gets that the bare result for $K_L \rightarrow \mu \bar{\mu}$ is multiplied by the Wilson coefficient
\[
C_{\mu \bar{\mu}}^c = \frac{1}{2} \frac{4\pi}{\alpha_s(m_c^2)} \left[ \frac{3}{11} (\kappa_1^{12/25} - \kappa_1^{1/25}) + 3(\kappa_1^{1/25} - 1) + \frac{6}{7}(\kappa_1^{-6/25} - \kappa_1^{1/25}) \right]
+ \frac{1}{2} \left[ -\kappa_1^{12/25} + 2\kappa_1^{-6/25} + \kappa_1^{-24/25} \right],
\] (5.12)
where $\kappa_1 = \alpha_s(m_c^2)/\alpha_s(M_W^2)$. The expression (5.12) is given for two families. Its generalization to three families is straightforward\cite{29}. The following comments are in order: If $\alpha_s/4\pi \ln(M_W^2/m_e^2) \sim 1$, then the second term in (5.12) is of the same order as the neglected contributions in the first term. In the SVZ calculation\cite{86} the second term has been kept for the following reasons: (i) its extrapolation to the free-quark limit is smooth, (ii) the leading term is not so large numerically since in the free-quark limit it vanishes, and (iii) this remains true for the three families as long as the $t$-quark is reasonably smaller than $M_W$.

Generalization to three families is straightforward.

The corresponding $c$-quark contribution is given by $C_{\nu \bar{\nu}}^c$
\[
C_{\nu \bar{\nu}}^c = \frac{1}{2} \frac{4\pi}{\alpha_s(m_c^2)} \left[ \frac{3}{11} (\kappa_1^{12/25} - \kappa_1^{1/25}) + 12(\kappa_1^{1/25} - 1) \right]
+ \frac{6}{7}(\kappa_1^{-6/25} - \kappa_1^{1/25}) + \frac{1}{2}(\kappa_1^{12/25} + 2\kappa_1^{-6/25} - 3\kappa_1^{-24/25}).
\] (5.13)
QCD Corrections for $m_t \geq M_W$

The QCD corrections for the case $m_t \geq M_W$ are difficult to handle, since each of the subleading mass terms gets differently ‘renormalized’. However, the leading term for large $m_t$ is powerlike, i.e., the Wilson coefficients are simply

$$C_{\mu\bar{\mu}} = C_{\nu\bar{\nu}} = \frac{1}{4},$$

the dominant contribution coming from the $sdZ$ graph. This is, however, true for very large $m_t$; in practice, we use $m_t$ in the range $80 - 200$ GeV. Therefore, for large $m_t$ it is better to take the bare function $\bar{D}_t$ for $C_{\nu\bar{\nu}}$ and similarly for $C_{\mu\bar{\mu}}$.

The final expression then becomes

$$BR(K^+ \to \pi^+ \nu \bar{\nu}) = 0.72 \times 10^{-6}|V_{ud}|^2 - C_{\nu\bar{\nu}}^c + \frac{V_{ts}^* V_{td}}{V_{ts}^* V_{td}} C_{\nu\bar{\nu}}^t|^{2}. \quad (5.15)$$

The quantity $|V_{ts}^* V_{td}|^2$ is a very important factor entering the calculations of rare decays. It appears as a ‘weight’ to the $t$-quark loop contributions. Using the measured K-M matrix elements, one infers from unitarity that $0.9985 \leq |V_{tb}| \leq 0.9993$ and $|V_{ts}| \leq 0.054$, $|V_{td}| \leq 0.024$, (5.16) which gives the constraint

$$|V_{ts}^* V_{td}| < 0.0013. \quad (5.17)$$

It is important to note that any violation of the relations (5.16) would be a signal of new physics beyond the standard model. The derived constraint in (5.17) will be used to put an upper bound on $BR(K^+ \to \pi^+ \nu \bar{\nu})$.

In Fig. 8 we present the branching ratio for the decay $K^+ \to \pi^+ \nu \bar{\nu}$ as a function of $m_t$. Since we have used the upper limit (5.17) on K-M matrix elements, the result shown can be considered as an upper limit. One sees that varying $m_t$ in the range $90$ GeV $\leq m_t \leq 200$ GeV gives the branching ratio in the range

$$BR(K^+ \to \pi^+ \nu \bar{\nu}) = (1 - 4) \times 10^{-10}. \quad (5.18)$$

The last Particle Data limit is

$$BR(K^+ \to \pi^+ \nu \bar{\nu}) < 3.4 \times 10^{-8}. \quad (5.19)$$

The following comments are in order:

(i) If one includes only the $c$-quark contribution (without the QCD corrections), one gets $BR(K^+ \to \pi^+ \nu \bar{\nu}) = 1.1 \times 10^{-11}$, for $m_c = 1.5$ GeV. Adding the QCD corrections lowers the branching ratio to $3.1 \times 10^{-12}$. Clearly, both numbers are below the sensitivity of the BNL experiment E787 (a recent preliminary result is $BR \leq 5 \times 10^{-9}$, the expected sensitivity is $2 \times 10^{-10}$).
For larger values of $m_t$, the $t$-quark contribution dominates and the results presented in Fig. 8 can be used in different scenarios. If the experiment measures considerably larger values than predicted, it is unlikely that it could be explained by pushing $m_t$ too high, since that would violate the $K_L \to \mu \bar{\mu}$ bound \cite{4,3}. However, it would be possible to violate the unitarity constraint in (5.17) because, for example, of the existence of the fourth family. Therefore, any clear result beyond the predicted branching ratio should be considered as a serious signal of new physics. Clearly, further improvements of the upper limits on $m_t$ and/or $|V_{ts}V_{td}|$ would further constrain the range (5.18).

In the case that $m_t$ is determined from experiment, the eq. (5.15) may be used to test the standard-model prediction for the branching ratio. On the other hand, the measured branching ratio (4.3) may be used to derive an upper limit on $m_t$. Both possibilities are, of course, valid provided no significant impact of ‘new physics’ is present.

We have kept QCD corrections for the $c$-quark contribution only. Since the latter is not dominant, the errors entering through the LLA, the charmed quark mass, etc., are not significant for the branching ratio. For example, the threshold effects of quarks have recently been calculated\cite{93} for $C_{70}$ and they slightly modify the resulting branching ratio.

Recently, a penguin-box expansion has been performed systematically\cite{94}; cf. also \cite{95}. It enables one to perform a very detailed analysis of $K \to \mu \bar{\mu}$ and $K^+ \to \pi^+\nu\bar{\nu}$ with constraints coming from the $\epsilon, \epsilon'$ analysis included, yielding $BR(K^+ \to \pi^+\nu\bar{\nu}$ to be around $1 \times 10^{-10}$.

### 5.4 $K^+ \to \pi^+\nu\bar{\nu}$ Decay in the Minimal Supergravity Model

We first supersymmetrize the box diagram, i.e., the internal quarks and W-bosons become squarks and gauginos (Fig. 9a). This has to be compared with the box diagram in the standard model which is given as

$$A_{SM}^{\text{box}}(K^+ \to \pi^+\nu\bar{\nu}) = -\frac{1}{M_W^2 16\pi^2} V_{ts}^* V_{td} g(x, y) = g(x, y) = \frac{1}{y-x} \{ -(\frac{x}{x-1})^2 \ln x + (\frac{y}{y-1})^2 \ln y + \frac{1}{x-1} - \frac{1}{y-1} \}.$$ \hspace{1cm} (5.21)

The supersymmetrization leads to the following expression:

$$A_{suss}^{\text{box}}(K^+ \to \pi^+\nu\bar{\nu}) \simeq \frac{1}{m_\tilde{W}^2 64\pi^2} V_{ts}^* V_{td} [g(\tilde{x}, \tilde{y}) - g(x, y)] (\tilde{s}_L \gamma_\mu d_L)(\tilde{\nu} \gamma^\mu \nu_L).$$ \hspace{1cm} (5.22)

Numerically, the ratio

$$\Re = |A_{suss}^{\text{box}} / A_{SM}^{\text{box}}|$$ \hspace{1cm} (5.23)

is of $\mathcal{O}(1)$. We have used $m_\tilde{t}^2 - m_\tilde{u}^2 \simeq m_t^2$ and $m_\tilde{W} = 40$ GeV, $m_t = 40$ GeV. If one fixes $m_\tilde{g} = 70$ GeV, then $\Re$ grows with $m_t$; for $m_t = 90$ GeV, $\Re = 0.4$, and for $m_t = 160$ GeV, $\Re$ reaches 1. For very large values of $m_t$, e.g., $m_t = 200$ GeV, $\Re = 1.4$. With increasing $m_\tilde{a}$, $\Re$ becomes somewhat smaller: for $m_\tilde{a} = 100$ GeV, $m_t = 60$ GeV, $\Re = 0.2$, and reaches 1 for $m_t = 200$ GeV.
Although the SUSY contribution may, in principle, be a significant correction for large values of \( m_t \), it is unlikely, as noticed by Bertolini and Masiero[47], that this really would happen. In fact, the large \( \Re \) would mean the large SUSY contribution to the \( K_L \rightarrow \mu \bar{\mu} \) box. This is even more pronounced since the Dirac algebra suppression in the \( K_L \rightarrow \mu \bar{\mu} \) box by factor of 4 does not appear in the SUSY version and, moreover, sneutrinos that enter the \( K_L \rightarrow \mu \bar{\mu} \) box further increase the amplitude.

The promising contributions appear to be spenguin graphs (b) and (c) of Fig. 9. Graphs (b) with the external photon contribute, for example, to the \( K^+ \rightarrow \pi^+ e^+ e^- \) decay. With the external \( Z \)-boson, graphs (b) and (c) contribute to the \( K_L \rightarrow \mu \bar{\mu} \) and \( K^+ \rightarrow \pi^+ \nu \bar{\nu} \) decays.

For set (b) it turns out that the self-energy graphs exactly cancel with the vertex correction graphs. Therefore, one has to include two L-R insertions in the squark propagator[47]. The resulting mixing \( \bar{d}_L - d_R \) is given by \( A_{\text{3/2}} M_d \) (cf. (2.14)):

\[
\begin{pmatrix}
  m_{3/2}^2 + m_b^2 + cm_t^2 & Am_{3/2} m_b \\
  Am_{3/2} m_b & m_{3/2}^2 + m_b^2
\end{pmatrix}.
\]

The standard-model value for \( K_L \rightarrow \mu \bar{\mu} \) is given by

\[
\mathcal{A}_{\text{SM}}(K_L \rightarrow \mu \bar{\mu}) = \frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi \sin^2 \theta_w} V^*_{ts} V_{td} \bar{C}(x_t)(\bar{s}_L \gamma_\mu d_L)(\bar{\mu}_L \gamma^\mu \mu_L),
\]

with \( \bar{C}(x_t) \) given by (5.8). The SUSY contributes as[47]

\[
\mathcal{A}_{\text{susy}}(K_L \rightarrow \mu \bar{\mu}) \simeq -\frac{G_F}{\sqrt{2}} \frac{2 \alpha_s}{3 \pi} V^*_{ts} V_{td} (Am_{3/2} m_b / m_\tilde{g})^2 [C(\bar{x}_b) - (m_d / m_b)^2 C(\bar{x}_d)] (\bar{s}_L \gamma_\mu d_L)(\bar{\mu}_L \gamma^\mu \mu_L),
\]

with \( C(x) \) defined as

\[
C(x) = \frac{1}{(x-1)^2} \left\{ \frac{1}{2} - \frac{2}{x-1} + \frac{3}{(x-1)^2} \ln x - \frac{1}{x(x-1)} \right\}.
\]

This leads to

\[
\Re(K_L \rightarrow \mu \bar{\mu}) \leq \frac{2 \alpha_s}{3 \alpha} \sin^2 \theta_w (Am_{3/2} m_b / m_\tilde{g})^2 |C(\bar{x}_b)| / C(\bar{x}_t).
\]

The corresponding expressions for \( \mathcal{A}(K^+ \rightarrow \pi^+ \nu \bar{\nu}) \) can be obtained by changing \( \bar{C}(x_t) \) to \( \bar{D}(x_t) \), i.e.

\[
\Re(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = |\bar{C}(x_t) / \bar{D}(x_t)| \Re(K_L \rightarrow \mu \bar{\mu}).
\]

Numerically, (5.29) presents a sizeable correction only for small values of the supergravity parameters. For \( m_{3/2} = 100 \text{ GeV}, m_\tilde{g} = 70 \text{ GeV}, m_b = 40 \text{ GeV} \), and choosing \( m_t = 90 \text{ GeV} \), \( \Re(K^+ \rightarrow \pi^+ \nu \bar{\nu}) \simeq 0.3 \). \( \Re \) decreases with increasing \( m_t \); for \( m_t = 150 \text{ GeV} \), \( \Re = 0.14 \). On the other hand, larger values of \( m_\tilde{b} \) suppress \( \Re \) significantly: for \( m_\tilde{b} = 60 \text{ GeV}, m_t = 60 \text{ GeV} \), \( \Re \) is lowered to 0.1.

Comparison with the supersymmetrized box shows that for \( m_t = 90 \text{ GeV} \), both supersymmetric contributions are of the same order, \( 30 - 40\% \). With increasing \( m_t \), \( \Re_{\text{spenguin}} \)}
diminishes and $\Re \sigma_{box}$ increases. Although a precise prediction is difficult because of the arbitrariness of SUSY parameters, it seems that in the mass range $90 \text{ GeV} \leq m_t \leq 150 \text{ GeV}$ the SUSY effects\(^{12}\) may reach $50 - 100\%$, leading to the enhancement in the branching ratio:

$$BR(SM + SUSY) \sim (2 - 4)BR(SM).$$ (5.30)

Prospects to detect the supersymmetry in this rare decay are presently not very bright. However, once $m_t$ and/or K-M matrix elements are determined, the prediction (5.15) for the branching ratio in the SM is rather reliable. The precise measurements may be able to disentangle the possible departures from the standard model. For example, the branching ratio enhanced by a factor of 4 may be due to supersymmetry, but also to the fourth generation; the precise determination, however, of the K-M matrix elements plus unitarity could eliminate the latter possibility.

### 5.5 $K^+ \to \pi^+ \tilde{\gamma} \tilde{\gamma}$ decay

The $K^+$-decay into a pion and a pair of the lightest superparticles is a very interesting decay mode since it belongs to the ‘direct’-decay type: the superparticles are produced as real particles. However, in the minimal $N = 1$ supergravity model we are discussing, it is not likely that this mode will appear for the simple reason: this decay is not allowed kinematically. The underlying assumptions are: that the $\tilde{\gamma}$ is the lightest SUSY particle and $R$-parity is conserved. The existing limits are the following\(^{19}\): $m_{\tilde{\gamma}} > 15 \text{ GeV}$ for $m_{\tilde{f}} = 100 \text{ GeV}$, $m_{\tilde{\gamma}} > 5 \text{ GeV}$ for $m_{\tilde{q}} = 55 \text{ GeV}$, and there is no lower limit for $m_{\tilde{e}} > 58 \text{ GeV}$. Although for the heavy selectron there is no lower limit on $m_{\tilde{\gamma}}$, there is a cosmological limit\(^{19}\) that requires the photino to be heavier than $300 - 500 \text{ MeV}$ or very light, $m_{\tilde{\gamma}} \leq 100 \text{ eV}$. Actually, there is a new allowed region\(^{19}\) $m_{\tilde{\gamma}} = 4 - 20 \text{ MeV}$ provided the bound $m_{3/2} < 40 \text{ TeV}$ is satisfied.

The second underlying assumption (R-parity) may also be relaxed. The SUSY models with $R$-breaking (which allows the lightest superpartner to decay) have been considered for a long time and recently discussed again\(^{47}\). Therefore, it looks meaningful to discuss the possibility of the $K^+ \to \pi^+ \tilde{\gamma} \tilde{\gamma}$ decay both from the experimental and theoretical point of view.

The largest contribution to the decay comes from the presence of large effects of flavor-changing quark-squark-photino couplings, which simply leads to a tree-graph $\bar{s}_L d_L \to \tilde{\gamma} \tilde{\gamma}$ via a $\tilde{d}_L$ exchange. The result is very simple\(^{47}\)

$$\mathcal{A}(K^+ \to \pi^+ \tilde{\gamma} \tilde{\gamma}) = \frac{4\pi\alpha}{9} [cm_t^2/(m_{3/2}^4 + cm_{3/2}^2 m_{3/2}^2)] V_{ts}^* V_{td} (\bar{s}_L \gamma_\mu d_L)(\bar{\tilde{\gamma}} \gamma_\mu \tilde{\gamma}),$$ (5.31)

and has to be compared with the dominant part of the $K^+ \to \pi^+ \nu \bar{\nu}$ decay:

$$\mathcal{A}(K^+ \to \pi^+ \nu \bar{\nu}) = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi \sin^2 \theta_w} V_{ts}^* V_{td} \bar{D}_t(x_t)(\bar{s}_L \gamma_\mu d_L)(\bar{\nu}_L \gamma_\mu \nu_L).$$ (5.32)

\(^{12}\)There is a contribution of the graph where a neutral higgsino plays the role of gluino; this contribution is, however, suppressed by small Yukawa couplings. This is not the case for charged higgsinos: although enhanced, this contribution is still smaller than the contribution of the standard model\(^{47}\).
It is easily seen, that the former decay may well dominate over the latter. Therefore, the experimental upper limit on $K^+ \rightarrow \pi^+ + 'nothing'$ may be converted to the lower limit on the squark mass which appears as $m_{\tilde{q}}^{-2}$ in the expression (5.31). Using the present experimental bound in (5.19) one derives the following constraint on the squark mass:

$$m_{\tilde{q}} > 50 \text{ GeV.}$$

(5.33)

As we have said, this limit is valid provided, that the decay $K^+ \rightarrow \pi^+ \tilde{\gamma} \tilde{\gamma}$ is allowed kinematically.

---

13 In order to compare both amplitudes, (5.32) has to be multiplied by $\sqrt{3}$ for 3 neutrino species (the Majorana nature of the photino is already taken into account in (5.31)).

14 The tree-graph decay into neutral higgsinos is suppressed for the same reason as in the spenguin contribution involving higgsinos.
6 Constraints on Supergravity Parameters from CP Violation and $K \to \pi \pi$ Decays

6.1 $K^0 - \bar{K}^0$ Mixing

In this section we discuss how the $K_L - K_S$ mass difference and the $\epsilon$ parameter in the $K^0 - \bar{K}^0$ system constrain a general version of the minimal $N = 1$ supergravity model\[42\]. The supersymmetric contributions to $\Delta m_K$ and $\epsilon$ are box graphs depicted in Figs. 10 and 11. The graphs in Fig. 10 represent the mere supersymmetrization of the 'standard box': the internal quarks are replaced by their scalar partners, the squarks $\tilde{u}$, whereas the $W$ bosons and the Higgs scalars are substituted by their fermionic partners, $\tilde{W}$ and $\tilde{H}$, respectively. The gluon exchange diagrams in Fig. 11, proportional to the strong coupling, are expected to give the dominant contribution\[35, 40\]. One usually assumes no great cancellation between different contributions to $\Delta m_K$ and $\epsilon$, and requires each contribution to be less than the experimental values\[15\].

Diagrams (a) in Fig. 10 yield a constraint\[42\]
\[
(V^\dagger \Delta M_Q^2 V)_{12} \leq \frac{1}{\frac{1}{100} M_W},
\]
which implies near degeneracy amongst the squarks. This is consistent with the usual assumption about masses of scalar fields of chiral superfields
\[
M_{ab}^2 = m_{3/2}^2 \delta_{ab}
\]
at a renormalization point of the Planck mass.

The contributions of the diagrams (b) in Fig. 10 place a constraint on the flavor matrices $\xi_U$, and $\xi_D$
\[
\xi_{U,D} = m_{3/2} A \lambda_{U,D} + \tilde{\xi}_{U,D}
\]
appearing in the trilinear scalar couplings (2.5). The $\tilde{\xi}_{U,D}$ is an arbitrary small matrix. However, more stringent constraints on $\xi_{U,D}$ are set by diagrams (a) in Fig. 11, leading to effective operators with mixed L-R helicities, i.e., with the structure typical of penguin graphs in the SM. The local $\Delta S = 2$ hamiltonian generated by this graph is given by
\[
H_{eff} = \frac{1}{120} \alpha_s \frac{M_{SD}^2 M_{DS}^2}{m_{3/2}^2} f(\frac{m_2^2}{m_{3/2}^2}) (\theta_1 - 3 \theta_2),
\]
where
\[
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix} = \begin{bmatrix}
\tilde{s}_L \gamma_\mu d_L & \tilde{s}_R \gamma^\mu d_R
\end{bmatrix}
\begin{bmatrix}
\delta^{ik} \delta_{kl}
\end{bmatrix}
\]
and
\[
f(x) = 20 \int_0^1 d\zeta \zeta^3 \frac{[\zeta + x(1 - \zeta)]^{-3}}{[\zeta + (1 - \zeta)]^{-3}}
\]
\[
f(1) = 1.
\]

\[\text{Franco and Mangano}\[34\] gave the general constraint assuming a coherent contribution of diagrams (a) and (b) in Fig. 11.\]
The notation for the mass parameters in (6.4) is explained in [42].

In order to estimate the supergravity contribution in $\Delta m_K$ and $\epsilon$, we need an estimate of the matrix elements of the operators $\theta_1$ and $\theta_2$. Using the vacuum saturation as a starting point, we define [46]

$$<\bar{K}_0|\theta_i|K_0> = \frac{1}{2} f_K^2 m_K^2 (1 + \frac{1}{N_c}) B_{\theta_i}.$$ (6.7)

With this definition, the vacuum saturation values are $B_{\theta_1} = -9.6$ and $B_{\theta_2} = -4$. As usual, the L-R helicity structure of our operators disqualifies the reliability of the vacuum saturation estimate. A better estimate is obtained by making use of the QCD duality approach [56]. We have to study the behavior of the two-point function

$$\psi_{\theta_2}(q^2) = i \int d^4 x e^{i q \cdot x} <0|T(\theta_2(x)\theta_2^\dagger(0))|0>.$$ (6.8)

The first step is to calculate $\frac{1}{\pi} Im \psi_{\theta_2}(t)$ both in QCD using the representation (6.5) of $\theta_2$ and in ChPT using the chiral representation

$$\theta_{\text{chiral}}^2 = B_{\theta_2} \frac{1}{3} (f_K^2) : (i \frac{f_2^2}{2} U^\mu U^\dagger U^\mu)^{23} (i \frac{f_2^2}{2} U^\dagger U^\mu U)^{23} :.$$ (6.9)

Next, we establish the duality region in terms of FESR’s. One needs two sum rules to fix $s_0$, which is the onset of the asymptotic QCD continuum. These are

$$\Re_0 = \int_{4m_K^2}^{s_0} dt \frac{1}{\pi} Im \psi_{\theta_2}(t),$$

$$\Re_1 = \int_{4m_K^2}^{s_0} dt \frac{1}{\pi} t Im \psi_{\theta_2}(t).$$ (6.10)

Once $s_0$ is fixed, any of the above sum rules leads to the value of $B_{\theta_2}$. The ratio of the sum rules

$$r = \frac{6}{5s_0} \frac{\Re_1}{\Re_0}$$ (6.11)

is a very sensitive test of duality. It does not depend on $B_{\theta_2}$ and may be used to fix $s_0$.

Fig. 12 shows the results plotted in both in ChPT and in QCD. The departure from the asymptotic QCD prediction (the dashed line) is due to mass and condensate corrections. The dots are the values of $r$ in ChPT, and the solid line is the full QCD result. The duality region is clearly established in the range $8 \leq s_0 \leq 11.5$ GeV$^2$. Any of the sum rules in (6.10) leads to the value of $B_{\theta_2}$ once $s_0$ has been determined.

Fig. 13 shows the results obtained from the sum rule $\Re_0$. $B_{\theta_2}$ plotted as a function of $s_0$ shows a plateau behavior in the duality region. The results obtained from the sum rule $\Re_1$ are similar. The values for $B_{\theta_2}$ agree within 2% in both cases. Taking into account various theoretical uncertainties [46], we obtain the final estimate

$$|B_{\theta_2}| = 0.25 \pm 0.15.$$ (6.12)

There are some principal difficulties in determining the matrix element of the operator $\theta = \theta_1 - 3\theta_2$. The QCD duality approach determines only the absolute value of the $B$
parameter and a separate determination of \( B_{\theta_1} \) would not be very useful. However, the operator \( \theta_1 \) is of higher order in ChPT with respect to \( \theta_2 \) and is expected to be significantly smaller. In addition, the QCD spectral functions of the operators \( \theta \) and \( 3 \theta_2 \) show very similar \( t \)-dependence and are of about the same magnitude\(^{40}\). Therefore, we may neglect the \( \theta_1 \) contribution to a very good approximation.

Using our estimate (6.12) of the matrix element of the \( \theta_2 \) operator and the experimental values for \( \Delta m_K \) and \( \epsilon \), we obtain the following constraints on \( M_{SD}^2 \) and \( M_{DS}^* \):

\[
\text{Re} \left( \frac{M_{SD}^2 M_{DS}^*}{m_{3/2}^3} \right) \leq 4.5 \times 10^{-5} \frac{m_{3/2}^3}{M_W^2} \left| f \left( \frac{m_{\tilde{g}}}{m_{3/2}^2} \right) \right|^{-1},
\]

(6.13)

\[
\text{Im} \left( \frac{M_{SD}^2 M_{DS}^*}{m_{3/2}^3} \right) \leq 3 \times 10^{-7} \frac{m_{3/2}^3}{M_W^2} \left| f \left( \frac{m_{\tilde{g}}}{m_{3/2}^2} \right) \right|^{-1}.
\]

(6.14)

These constraints are rather sensitive to the variation of the ratio \( m_{\tilde{g}}/m_{3/2} \). If one takes \( m_{\tilde{g}} = m_{3/2} \), one finds that (6.13) and (6.14) are weaker by a factor of 5 than the corresponding constraints of Dugan et al.\(^{42}\).

The superbox diagrams (b) in Fig. 11 yields the effective hamiltonian\(^{39},^{38} -^{40}\)

\[
\mathcal{H}_{\text{eff}} = \frac{\alpha_s^2}{36 m_{\tilde{g}}^2} \sum_{i,j} S(x_i, x_j) V_{is}^* V_{id} V_{js}^* V_{jd} (\bar{s}_L \gamma_\mu d_L)^2
\]

(6.15)

where the sum runs over indices \( u, c, \) and \( t \), and

\[
x_1 = \frac{m_d^2}{m_{\tilde{g}}^2}, \quad x_2 = \frac{m_s^2}{m_{\tilde{g}}^2}, \quad x_3 = \frac{m_b^2}{m_{\tilde{g}}^2}.
\]

(6.16)

The down squark masses are related to each other through the equation (2.17). Thus, one may assume that in a good approximation \( m_d^2 = m_s^2 = m_b^2 + m_t^2 \). The function \( S(x, y) \) is given by

\[
S(x, y) = 11 \frac{K(x) - K(y)}{x - y} + 4 \frac{I(x) - I(y)}{x - y}
\]

(6.17)

\[
K(x) = \frac{x^2 \ln x}{(1 - x)^2} + \frac{1}{1 - x},
\]

\[
I(x) = \frac{x \ln x}{(1 - x)^2} + \frac{1}{1 - x}.
\]

(6.18)

Using the unitarity of K-M matrix one finds the super-box contribution to \( \Delta m_K \)

\[
(\Delta m_K)_{sbox} = \frac{\alpha_s^2}{54 m_{\tilde{g}}^2} B f^2 m_K |V_{is}^* V_{id}|^2 DS,
\]

(6.19)

where

\[
\Delta S = S(x_3, x_3) + S(x_1, x_1) - 2S(x_1, x_3),
\]

(6.20)

and \( B = 1 \) for the vacuum saturation estimate of the LL operator matrix element between the \( K^0 \) and \( \bar{K}^0 \) states. For \( m_t \ll m_{\tilde{g}} \), \( \Delta S \) can be replaced by the second derivative of \( S(x, y) \).
and (2.17) simplifies, yielding an $m_t^4$ dependence of $\Delta m_K$ and $\epsilon$. We shall, however, use the exact form (6.19).

Eq. (6.19) can be used to place lower bounds on the gluino mass and down-squark masses, provided one has sufficient information about the quark masses and K-M parameters. Unfortunately, the top-quark mass $m_t$ is still not known. The best one can do is to use the present experimental lower bound $m_t > 91$ GeV [19] and the upper limit $m_t \leq 180$ GeV from the standard model constraints [96]. As regards the K-M parameters, a rigorous upper bound given by (5.17) can be used. The lower bounds on $m_{\tilde{g}}$ and $m_{\tilde{d}}$ for two values of $m_t$ are as follows

\[
\begin{array}{ll}
  m_t & 100 \quad 150 \\
  m_{\tilde{g}} > & 40 \quad 50 \\
  m_{\tilde{d}} > & 105 \quad 160
\end{array}
\]

The imaginary part of the strong superbox diagram (b) in Fig. 11 yields a contribution to the $\epsilon$ parameter. Whereas for typical values $m_{\tilde{g}} \simeq m_{\tilde{d}} \simeq m_{3/2} = 100$ GeV ($\Delta m_K)^{sbox}$ is rather small (of the order of 0.1($\Delta m_K)^{exp}$), the contribution to $\epsilon$ can be quite large [35]. By making use of (6.13), one finds that

\[
|\epsilon_{sbox}| = \frac{\sqrt{2}}{108} \alpha_s^2 \frac{B f_K^2 m_K}{m_{\tilde{g}}^2 \Delta m_K} s_1^2 s_2^2 (s_2 s_3 s_\delta) \Delta S, \quad (6.21)
\]

which compared with $\epsilon_{exp} = 2.27 \times 10^{-3}$, and assuming $(\Delta m_K)^{sbox}/(\Delta m_K)^{exp} \approx 0.1$, leads to a stringent constraint on $s_2 s_3 s_\delta$:

\[
s_2 s_3 s_\delta \leq 2 \times 10^{-5} \frac{1}{s_2^2}. \quad (6.22)
\]

This bound has important consequences for the contribution of penguins and superpenguins to $\epsilon'$, which we discuss in the next section.

### 6.2 CP Violation in the $\Delta S = 1$ transition

Supersymmetric contributions to $\epsilon'$ are represented by the superpenguin operators depicted in Fig. 14. Again, the corresponding diagrams involving winos are neglected because they are proportional to $g^2$ rather than to $g_s^2$. The superpenguin (a) in Fig. 14 leads to a phase in the $K_L \to \pi\pi$ amplitude. $\xi_{s\text{pen}}$ compared with $\xi_{\text{pen}}$ from the standard model equals [12]

\[
\frac{\xi_{s\text{pen}}}{\xi_{\text{pen}}} = \frac{1}{5} \left(\frac{g_3}{g}\right)^2 \frac{(m_t/m_\tilde{d})^2}{\ln(m_t/m_\text{c})^2}. \quad (6.23)
\]

This could be larger than 1 if $m_t$ is very large. However, in this case, the superbox diagram dominates $\epsilon$ and yields a strong bound on $s_2 s_3 s_\delta$, (6.22), which in its turn makes the penguin
contribution very small. Thus, the conclusion is that the superpenguin (a) in Fig. 12 may give a significant contribution to $\epsilon'/\epsilon$ only in a very small region of parameter space\cite{12}.

The scalar mass insertions $M_{SD}^2$ and $M_{DS}^2$ which have been discussed in the preceding section, also appear in the $\Delta S = 1$ transition dipole moment operators\cite{12}, represented by the superpenguin diagram (b) in Fig. 14. The corresponding effective Hamiltonian is given by

$$H_{\text{eff}} = \frac{1}{32\pi} \frac{\alpha_3 g s}{m_{3/2}^2} F(x) \left[ \frac{M_{SD}^2}{m_{3/2}^2} \bar{s}_R \sigma_{\mu\nu} F^{\mu\nu} d_L + \frac{M_{DS}^2}{m_{3/2}^2} \bar{s}_\sigma \sigma_{\mu\nu} F^{\mu\nu} d_R \right], \quad (6.24)$$

with

$$x = \frac{M^2_{SD}}{m_{3/2}^2}$$

$$F(x) = \sqrt{x} [12 \Im(x) + \frac{4}{3} x^2 \Re(1)], \quad F(1) = 1 + \frac{1}{9},$$

$$\Im(x) = \int_0^1 d\zeta (1 - \zeta) \left[ \zeta + x(1 - \zeta) \right]^{-2}. \quad (6.25)$$

This effective Hamiltonian could give a significant contribution to $\epsilon'$ and to the $\Delta I = 1/2$ amplitude in the $K \to \pi\pi$ decay. Using, for example, the bag model estimate\cite{17} of the above transition moment operators\cite{17}, one finds the phase and the magnitude of the $\Delta I = 1/2$ amplitude

$$\xi_{\text{trans mom}} = 250 \left( \frac{100 \text{ GeV}}{m_{3/2}} \right) F(x) \Re \left( \frac{M_{SD}^2}{m_{3/2}^2} + \frac{M_{DS}^2}{m_{3/2}^2} \right), \quad (6.26)$$

$$a_{\text{trans mom}}^{1/2} = 1.2 \times 10^{-5} \left( \frac{100 \text{ GeV}}{m_{3/2}} \right) F(x) \Re \left( \frac{M_{SD}^2}{m_{3/2}^2} + \frac{M_{DS}^2}{m_{3/2}^2} \right). \quad (6.27)$$

The constraints (6.26) and (6.27) allow us to assume that the imaginary parts of $M_{SD}^2$ and $M_{DS}^2$ are much smaller than the real parts. Assuming further that $\Re M_{SD} \approx \Re M_{DS}$, one obtains

$$|\epsilon'/\epsilon|_{\text{trans mom}} \leq 2 \times 10^{-2} \Phi(x), \quad (6.28)$$

$$a_{\text{trans mom}}^{1/2} \leq 20 \times 10^{-8} \Phi(x), \quad (6.29)$$

with

$$\Phi(x) = F(x) [f(x)]^{-1/2} \quad \Phi(1) = 1 + \frac{1}{9}. \quad (6.30)$$

In view of the recently measured values $\Re(\epsilon'/\epsilon) = (2.2 \pm 1.2) \times 10^{-3}$ \cite{19}, the bound (6.28) is not satisfactory. On the other hand, the bound (6.29), compared with the experimental value $a_{1/2}^{1/2} = 27 \times 10^{-8} \text{ GeV}$, opens an interesting, although perhaps unrealistic, speculation that a great deal of yet unexplained $\Delta I = 1/2$ enhancement could be attributed to the transition dipole effective operators induced by the extended supergravity model.

\footnote{The bag model estimate may be rather crude for the operators containing gluon fields. The QCD duality approach is, however, very difficult to apply here because of the huge $\alpha_s$ corrections, typical of $\Delta I = 1/2$ processes\cite{62}.}
The mass ratio $m_\tilde{g}/m_{3/2}$ is yet unknown. Assuming $m_\tilde{g} = m_{3/2}$ and requiring
\[ |\epsilon'/\epsilon|_{\text{trans mom}} = 14\xi_{\text{trans mom}} \leq 3 \times 10^{-3}. \] (6.31)
o one finds the following constraint
\[ \text{Im} \left( \frac{M_{\tilde{g}D}^2}{m_{3/2}^2} + \frac{M_{D*}^2}{m_{3/2}^2} \right) \leq 8 \times 10^{-7} \left( \frac{m_{3/2}}{100 \text{ GeV}} \right). \] (6.32)

This constraint is more stringent than the one obtained from (6.26).
7 Conclusions

The processes discussed in this review have been selected in such a way that they are interesting from both the theoretical and experimental point of view. The latter means that rare processes we are discussing are expected to be detected in the near future. Of particular theoretical interest are processes which either have significant corrections (here ‘significant' means corrections of the order $O(1)$) owing to the supersymmetry, or the standard model predictions are clear and unambiguous. In the latter case, the departure may indicate physics beyond the standard model, and in the former case precise measurements again check the possible new physics.

The process $K^+ \to \pi^+ \nu \bar{\nu}$ receives supersymmetric corrections which are of the order 1. The resulting branching ratio is larger by a factor of $2^{-4}$ with respect to the standard model prediction. The latter is a function of the K-M parameters $V_{ts}^* V_{td}$ (only the upper limit is known), and of the top-quark mass, for which we know the lower limit. Partial knowledge of the above parameters leads to the prediction $BR(K^+ \to \pi^+ \nu \bar{\nu}) = (1 - 4) \times 10^{-10}$. The range given will become more narrow if the lower limit on $m_t$ increases. This decay is dominated by short-distance dynamics, and long-distance effects are expected to be small, and its standard model prediction is relatively free of usual uncertainties. Therefore, if the decay is measured, it can also be used as a way to determine $m_t$.

Direct production of superparticles (photinos) is possible in the decay $K^+ \to \pi^+ \tilde{\gamma} \tilde{\gamma}$, provided it would not appear to be kinematically forbidden. The contribution is large enough, so that the present upper bound on $K^+ \to \pi^+ \text{‘nothing'}$ leads to the constraint $m_{\tilde{d}} > 50 \text{GeV}$.

Supersymmetric contributions to $\Delta m_K$ and $\epsilon$ may be significant for large values of $m_t$. This yields constraints on gluino and squark masses as well as the L-R helicity mixing mass parameters $M_{\bar{D}S}$ and $M_{\bar{S}D}$. Further constraints on these parameters follow from the requirement that the contribution to $\epsilon'$ of the transition dipole moment operators in the general version of the supergravity model must not be too large. In the constrained version of the model the superpenguin contribution to $\epsilon'/\epsilon$ appears to be rather small.

Given the fact that supersymmetric effects are rather tiny, it is important to reduce uncertainties in the calculation, especially the ones coming from the nonperturbative (confinement) effects in QCD. Some recent approaches, such as chiral perturbation theory, the large-$N_c$ expansion, QCD hadronic duality sum rules, lattice QCD, are discussed. Particularly interesting appear predictions of ChPT in rare decays. Especially, the reachness of radiative $K$ decays clearly shows the predictive power of ChPT. Some of these predictions are to be tested in the forthcoming experiments in the near future.

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Figure Captions

**Figure 1.** Typical graphs in ChPT. (I) Tree-level $K^0$ decay, (II) pion decay constant and wave function renormalization, (III) loop corrections to the $K^0 - \bar{K}^0$ mixing, (IV) loop corrections to the $K^+$ decay. The square is an insertion of the weak lagrangian and the circle is a strong interaction vertex.

**Figure 2.** The ‘eight’(a) and the ‘eye’(b) diagram contribution to the correlation between two mesons and a four quark operator.

**Figure 3.** Rare-decays graphs at the quark level, (a) box graph for the $K_L \rightarrow \mu \bar{\mu}$ decay, (b) box graph for the $K_L \rightarrow \gamma \gamma$ decay, (c) sdZ-vertex graph for the $K_L \rightarrow \mu \bar{\mu}$ decay, (d) two-photon contribution to the $K_L \rightarrow \mu \bar{\mu}$ decay, (e) box graph for the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay.

**Figure 4.** Rare decays in ChPT. (a) two-photon contribution to the $K_L \rightarrow \mu \bar{\mu}$ decay, (b) loop graphs for the $K_S \rightarrow \gamma \gamma$ decay, (c) pole contribution to the $K_L \rightarrow \gamma \gamma$ decay.

**Figure 5.** Supersymmetric contributions to the $\mu \rightarrow e \gamma$ and $g - 2$. (a) $\mu \rightarrow e \gamma$ decay in SUSY. (a), (b), and (c): contribution to $g - 2$.

**Figure 6.** $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay. (a) box graphs, (b) sdZ-vertex contributions.

**Figure 7.** QCD corrections to the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay. (a) corrections to box graphs, (b) corrections to the sdZ-vertex graph.

**Figure 8.** Upper limit on $BR(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ in the standard model as a function of top-quark mass.

**Figure 9.** Supersymmetric contributions to the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay. (a) supersymmetrized box graph. (b) supersymmetrized $sd\gamma$ and $sdZ$ vertices. (c) supersymmetrized $sdZ$ vertex with mass insertions.

**Figure 10.** Supersymmetric box graphs proportional to weak couplings.

**Figure 11.** Supersymmetric box graphs proportional to strong couplings.

**Figure 12.** The ratio of the sum rules $r$ plotted versus $s_0$ in ChPT (dots) and in full QCD (solid line).

**Figure 13.** Results for $B_{\theta_2}$ as a function of $s_0$.

**Figure 14.** Supersymmetric penguin graphs.