Local distributions for eigenfunctions and for perfect colorings of q-ary hypercube

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Abstract. Under study are the eigenfunctions and perfect colorings of the graph of n-dimensional q-ary Hamming space. We obtain the interdependence of local distributions of an eigenfunction in two orthogonal faces. We prove also an analogous result for perfect colorings.

1 Introduction

We study the eigenfunctions and perfect colorings of n-dimensional q-ary hypercube. The particular case of perfect colorings, which is extensively investigated now, corresponds to the completely regular codes. The aim of the paper is to provide a connection between the local distributions in two orthogonal faces. Earlier this question was considered in [2,4–6] for the 1-error correcting perfect codes and perfect colorings in binary case (q = 2). In case q > 2 the question is investigated in [1] for the 1-error-correcting codes. In [3] a more general case of the direct product of graphs is studied; however, the formula is not extended for the classes of graphs.

The paper is organised as follows: In Section 2 we give some necessary notations and propositions. In Section 3 we establish a formula for local weight enumerators of an eigenfunction in a pair of orthogonal faces. Using this, we obtain in Section 4 the formula for local weight enumerators of a perfect coloring in a pair of orthogonal faces. Both derived formulas are symmetric under choice of the face from the pair.

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2 Preliminaries

Consider the set \( F_q = \{0, 1, \ldots, q - 1\} \) as the group modulo \( q \) and \( F_q^n \) as the abelian group \( F_q \times \cdots \times F_q \). We investigate functions and the colorings on the graph of \( F_q^n \) of q-ary n-dimensional hypercube; in this graph two vertices are adjacent if they differ in exactly one position.

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Let $\alpha \in \mathbb{F}_q^n$ be an arbitrary vertex. Here and elsewhere $I$ denotes a subset of $\{1, \ldots, n\}$ and $\overline{I} = \{1, \ldots, n\} \setminus I$. We denote the support of the vertex $\alpha$ by $s(\alpha)$ (i.e., the set of its nonzero positions); the cardinality of the support is the Hamming weight of $\alpha$ and is denoted by $wt(\alpha)$; the Hamming distance between two vertices $\alpha$ and $\beta$ that equals the Hamming weight of $\alpha - \beta$ is denoted by $\rho(\alpha, \beta)$. We write $W_i(\alpha)$ for the sphere of radius $i$ centered at the vertex $\alpha$ (i.e., the set of all vertices with distance $i$ from $\alpha$) and we write $B_i(\alpha)$ for the ball of radius $i$ centered at the vertex $\alpha$ (i.e., the set of all vertices with distance at most $i$ from $\alpha$). By definition, put

$$\Gamma_I(\alpha) = \{ \beta \in \mathbb{F}_q^n : \beta_i = \alpha_i \ \forall \ i \notin I \},$$

then $\Gamma_I(\alpha)$ is an $|I|$-dimensional face, it has the structure of $\mathbb{F}_q^{|I|}$. We write simply $W_I$ and $\Gamma_I$ instead of $W_i(\alpha)$ and $\Gamma_I(\alpha)$ in the case of the all-zero vertex $\alpha$. Two faces $\Gamma_I(\alpha)$ and $\Gamma_J(\beta)$ are orthogonal if $J = \overline{I}$. Obviously, two orthogonal faces have exactly one common vertex. Given $\alpha, \beta \in \mathbb{F}_q^n$, we denote $\langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \ldots + \alpha_n \beta_n \mod q$.

Let us consider the set of all functions $f : \mathbb{F}_q^n \to \mathbb{C}$ as $q^n$-dimensional vector space $V$ over the complex field $\mathbb{C}$. Let $\xi = e^{2\pi \sqrt{-1}/q}$. For $\beta \in \mathbb{F}_q^n$, the function $\varphi^\beta \in V$, where

$$\varphi^\beta(\alpha) = \xi^{\langle \alpha, \beta \rangle}, \quad \alpha \in \mathbb{F}_q^n,$$

is called the character. All characters $\varphi^\beta$, $\beta \in \mathbb{F}_q^n$, form the orthogonal basis of the vector space $V$ with respect to the inner product $\langle , \rangle$ defined as follows:

$$\langle f, g \rangle = \sum_{\beta \in \mathbb{F}_q^n} f(\beta) g(\beta).$$

The Fourier transform $\hat{f}$ of the function $f$ is defined as the inner product with the characters:

$$\hat{f}(\alpha) = \langle f, \varphi^\alpha \rangle = \sum_{\beta \in \mathbb{F}_q^n} f(\beta) \overline{\xi^{\langle \alpha, \beta \rangle}}, \quad \alpha \in \mathbb{F}_q^n. \tag{1}$$

The initial function $f$ can be presented in the basis of the characters:

$$f(\alpha) = q^{-n} \sum_{\beta \in \mathbb{F}_q^n} \hat{f}(\beta) \xi^{\langle \alpha, \beta \rangle}, \quad \alpha \in \mathbb{F}_q^n. \tag{2}$$

Lemma 1. Let $\beta \in \mathbb{F}_q^n$ and $I \subseteq \{1, \ldots, n\}$. Then

$$\sum_{\alpha \in \Gamma_I} \varphi^\beta(\alpha) x^{|I|-|s(\alpha)|} y^{|s(\alpha)|} = (x - y)^{|I| \cap s(\beta)}(x + (q - 1)y)^{|I| - |I \cap s(\beta)|}.$$
Proof. Let $|I| = k$. Without loss of generality assume that $I = \{1, \ldots, n\}$. By definition of the characters,

$$
\sum_{\alpha \in \Gamma_I} \varphi^\beta(\alpha)x^{|I| - |s(\alpha)|}y^{|s(\alpha)|} = \sum_{\alpha_1 = 0}^{q-1} \cdots \sum_{\alpha_k = 0}^{q-1} \prod_{i=1}^k \xi^{\alpha_i \beta_i x^{|s(\alpha_i)|}y^{|s(\alpha_i)|}}.
$$

(For $a \in \mathbb{F}_q$ it holds $|s(a)| = 0$ if $a = 0$ and $|s(a)| = 1$ if $a \neq 0$.) Then we change the order of summations and multiplication:

$$
\prod_{i=1}^k \sum_{\alpha_i = 0}^{q-1} \xi^{\alpha_i \beta_i x^{|s(\alpha_i)|}y^{|s(\alpha_i)|}}.
$$

(3)

Owing to the properties of the primitive root of unity, we have

$$
\sum_{a = 0}^{q-1} \xi^{ab} = \begin{cases} 
0, & b \neq 0, \\
q, & b = 0,
\end{cases}
$$

and therefore

$$
\sum_{a = 0}^{q-1} \xi^{ab}x^{|s(\alpha)|}y^{|s(\alpha)|} = \begin{cases} 
x - y, & b \neq 0, \\
x + (q - 1)y, & b = 0.
\end{cases}
$$

Applying this to (3), we finally obtain

$$
(1 - t)^{|I| \cap s(\beta)}(1 + (q - 1)t)^{|I| - |I| \cap s(\beta)}.
$$

Now we introduce the concept of a local distribution. By definition, put

$$
v^{I,f}_j(\alpha) = \sum_{\beta \in \Gamma_I(\alpha) \cap W_j(\alpha)} f(\beta),
$$

the vector $v^{I,f}(\alpha) = (v^{I,f}_0(\alpha), \ldots, v^{I,f}_{|I|}(\alpha))$ is called the local distribution of the function $f$ in the face $\Gamma_I(\alpha)$ with respect to the vertex $\alpha$ or shortly the $(I, \alpha)$-local distribution of $f$. We say that the polynomial

$$
g^{I,\alpha}_f(x, y) = \sum_{j=0}^{|I|} v^{I,f}_j(\alpha)y^j x^{|I| - j} = \sum_{\beta \in \Gamma_I(\alpha)} f(\beta)y^{|s(\beta - \alpha)|}x^{|I| - |s(\beta - \alpha)|}
$$

is a local weight enumerator of the function $f$ in the face $\Gamma_I(\alpha)$ with respect to the vertex $\alpha$ or shortly the $(I, \alpha)$-local weight enumerator of $f$. We omit $\alpha$ (in all notations) if $\alpha = (0, \ldots, 0)$.

Let us describe the local weight enumerator of an arbitrary function in terms of its Fourier coefficients:
Lemma 2. Let $f$ be an arbitrary function. Then
\[ g_f^I(x, y) = q^{-n} \sum_{\beta \in F_q^n} \hat{f}(\beta)(x + (q - 1)y)^{|I| - |I \cap s(\beta)|} (x - y)^{|I \cap s(\beta)|}. \]

Proof. By Lemma 1,
\[ g_f^I(x, y) = \sum_{\beta \in \Gamma_I} f(\beta) y^{|s(\beta)|} x^{|I| - |s(\beta)|} \]
\[ = q^{-n} \sum_{\delta \in F_q^n} \hat{f}(\delta) \sum_{\beta \in \Gamma_I} \xi^{(\beta, \delta)} x^{|I| - |s(\beta)|} y^{|s(\beta)|}. \]
Then we can apply Lemma 1 and obtain (4).

3 Eigenfunctions

The first object of our consideration is the set of all eigenfunctions of the $n$-dimensional $q$-ary hypercube $F_q^n$. As usual, we refer to as the eigenvalue of a graph the eigenvalue of its adjacency matrix. It is known that the eigenvalues $\lambda$ of the graph of $n$-dimensional $q$-ary hypercube are equal to
\[ \lambda_h = (q - 1)n - qh, \quad h = 0, 1, \ldots, n, \]
where $h$ is called the number of the eigenvalue $\lambda_h$. Obviously, an eigenvalue $\lambda$ has the number $h = h(\lambda) = \frac{(q - 1)n - \lambda}{q}$. The corresponding eigenfunctions (we call them $\lambda$-functions) satisfy the equations
\[ \sum_{\beta \in W_1(\alpha)} f(\beta) = \lambda_h f(\alpha), \quad \alpha \in F_q^n, \]
or in the matrix form:
\[ Df = \lambda_h f, \]
where $D$ is the adjacency matrix of $F_q^n$ and $f$ is a vector of the function $f$ values. It is easy to see that the Fourier coefficients $\hat{f}(\alpha)$ of a $\lambda$-function $f$ equal zero apart from the case, where the Hamming weight of $\alpha$ is equal to the number of $\lambda$.

We are going to derive the interdependence between the local weight enumerators for an eigenfunction in two orthogonal faces.

Theorem 1. Let $\lambda$ be an eigenvalue of $F_q^n$ with the number $h = \frac{(q - 1)n - \lambda}{q}$, let $f$ be an arbitrary $\lambda$-function, and let $\alpha \in F_q^n$. Then
\[ (x + (q - 1)y)^{h - |I|} g_f^\alpha(x, y) = (x' + (q - 1)y')^{h - |I|} g_f^\alpha(x', y'), \]
where $x' = x + (q - 2)y$, $y' = -y$. 


Proof. The faces $\Gamma_f(\alpha)$ and $\Gamma_f(\alpha)$ are orthogonal. Without loss of generality assume that $\alpha$ is the all-zero vertex. Using Lemma 2, we can express the $(I, 0)$-local weight enumerator of the $\lambda$-function $f$ in terms of the Fourier coefficients:

$$g^f_I(x, y) = q^{-n} \sum_{\beta \in \mathbb{F}_q^n} \hat{f}(\beta)(x + (q - 1)y)^{n - |I| - |s(\beta)| + |I \cap s(\beta)|}(x - y)^{|s(\beta)| - |I \cap s(\beta)|}.$$ 

Since $\hat{f}(\beta) = 0$ for every $\beta \notin W_h$, the summation can be taken over all vertices of weight $h$ instead of all vertices of $\mathbb{F}_q^n$. This implies

$$g^f_I(x, y) = q^{-n} (x + (q - 1)y)^{n - |I| - h(x - y)^{h - |I|} \times \sum_{\beta \in W_h} \hat{f}(\beta)(x + (q - 1)y)^{|I \cap s(\beta)|}(x - y)^{|I| - |I \cap s(\beta)|}.$$ 

We choose new variables $x'$ and $y'$ such that

$$\begin{cases} x' + (q - 1)y' = x - y, \\ x' - y' = x + (q - 1)y, \end{cases} \quad \text{or} \quad \begin{cases} x' = x + (q - 2)y, \\ y' = -y. \end{cases}$$

Hence,

$$g^f_I(x, y) = q^{-n} (x + (q - 1)y)^{n - |I| - h(x - y)^{h - |I|} \times \sum_{\beta \in W_h} \hat{f}(\beta)(x' - y')^{|I \cap s(\beta)|}(x + (q - 1)y')^{h - |I|}.$$ 

Comparing with Lemma 2, we finally have

$$g^f_I(x, y) = (x + (q - 1)y)^{n - |I| - h(x + (q - 1)y)^{h - |I|} g^f_I(x', y').$$

\[ \square \]

4 Perfect colorings

In this section we prove an analog of Theorem 1 for perfect colorings.

The partition $C = (C_1, \ldots, C_r)$ of $\mathbb{F}_q^n$ is called a perfect $r$-coloring (or an equitable partition, or a partition design) with the parameter matrix $S = (s_{ij})_{i,j=1,\ldots,r}$ if for every $i, j \in \{1, \ldots, r\}$ and each vertex $\alpha \in C_i$ the number of vertices $\beta \in C_j$ at distance 1 from $\alpha$ is equal to $s_{ij}$. Present a perfect $r$-coloring by $(0, 1)$-matrix $C$ of size $q^n \times r$ with the rows corresponding to the vertices of $\mathbb{F}_q^n$ and the columns corresponding to the colors $\{1, \ldots, r\}$. The matrix $C$ is
defined as follows: each row has only one nonzero position that marks the color of the corresponding vertex. In these terms the coloring is perfect if

\[ DC = CS, \]  

where \( D \) is the adjacency matrix of the hypercube \( F^q_n \).

We define a local distribution of a coloring as a local distribution of characteristic functions of the colors. More precisely, a local distribution of the coloring \( C \) in the face \( \Gamma_I(\alpha) \) with respect to the vertex \( \alpha \) (or \((I, \alpha)\)-local distribution) is the \( r \times (|I| + 1) \)-matrix

\[
v^{I,C}(\alpha) = \begin{pmatrix}
v_0^{I,C_1}(\alpha) & \ldots & v_{|I|}^{I,C_1}(\alpha) \\
\vdots & \ddots & \vdots \\
v_0^{I,C_r}(\alpha) & \ldots & v_{|I|}^{I,C_r}(\alpha)
\end{pmatrix},
\]

where \( v_j^{I,C_i}(\alpha) = |C_i \cap W_j(\alpha) \cap \Gamma_I(\alpha)|, \ i = 1, \ldots, r, \) and \( j = 0, \ldots, |I| \). Let \( g^{I,\alpha}_{C_i}(x, y), \ i = 1, \ldots, r, \) be the \((I, \alpha)\)-local weight enumerator of the \( i \)th color \( C_i \); i.e.,

\[
g^{I,\alpha}_{C_i}(x, y) = \sum_{j=0}^{|I|} v_j^{I,C_i}(\alpha)y^jx^{|I|−j}.
\]

The vector-function

\[
g^{I,\alpha}_C(x, y) = (g^{I,\alpha}_{C_1}(x, y), \ldots, g^{I,\alpha}_{C_r}(x, y))
\]

is called the local weight enumerator of the coloring \( C \) in the face \( \Gamma_I(\alpha) \) with respect to the vertex \( \alpha \) (or the \((I, \alpha)\)-local weight enumerator).

The next theorem is an analog of Theorem 1 for perfect colorings.

**Theorem 2.** Let \( C = (C_1, \ldots, C_r) \) be an arbitrary perfect coloring of \( F^q_n \) with parameter matrix \( S \) and \( \alpha \in F^q_n \). Put \( h(S) = \frac{(q-1)hE − S}{q} \), where \( E \) is an identity matrix. Then

\[
g^{\alpha}_C(x, y)(x + (q-1)y)^{h(S)−|I|E} = g^{\alpha}_{C_i}(x', y')(x' + (q-1)y')^{h(S)−|I|E}. \tag{7}
\]

**Proof.** Without loss of generality assume that \( \alpha = (0, \ldots, 0) \).

Perfect colorings are closely related with eigenfunctions of the hypercube. Indeed, let \( \mu_1, \ldots, \mu_r \) be the all eigenvalues (not necessarily distinct) of the parameter matrix \( S \) and let \( T^1, \ldots, T^r \) be the linearly independent eigenvectors of \( S \) that corresponds to the eigenvalues; i.e.,

\[
ST^i = \mu_iT^i, \ i = 1, \ldots, r.
\]
Thus, for the matrices $T = [T^1, \ldots, T^r]$ and $M = \text{diag}\{\mu_1, \ldots, \mu_r\}$ it holds

$$ST = TM.$$ 

Multiplying both sides of (3) by $T$ and applying the last equation, we have for the matrix

$$F = CT$$

(8)

that

$$DF = DCT = CST = CTM = FM.$$ 

It means that the columns $F^1, \ldots, F^r$ of $F$ are the eigenfunctions of $D$ or $\lambda$-functions; i.e.,

$$DF^i = \mu_i F^i, \quad i = 1, \ldots, r.$$ 

Applying Theorem 1 to these $\lambda$-functions, we have

$$(x + (q-1)y)^{h_i-|I|} g_{F^i}(x, y) = (x' + (q-1)y')^{h_i-|I|} g_{F^i}(x', y'), \quad i = 1, \ldots, r, \quad (9)$$

where for $i = 1, \ldots, r$ the value $h_i$ is equal to the number of the eigenvalue $\mu_i$ of the hypercube $F^i_q$; i.e., $h_i = (q-1)n - \mu_i$. Put $g_F = (g_{F^1}, \ldots, g_{F^r})$ and

$$M_I(x, y) = \text{diag} \left\{ (x + (q-1)y)^{h_1-|I|}, \ldots, (x + (q-1)y)^{h_r-|I|} \right\}.$$ 

So we can rewrite the equations (9) in terms of these matrices:

$$g_{F^i}(x, y) M_I(x, y) = g_{F^i}(x', y') M_I(x', y').$$

It follows from (8) that

$$g_F = (g_{F^1}, \ldots, g_{F^r}) = (g_{C^1}, \ldots, g_{C^r})T = g_C T.$$ 

Therefore, we obtain

$$g^T_C(x, y) T M_I(x, y) = g^T_C(x', y') T M_I(x', y').$$

(10)

Then we multiply both sides of (10) by $T^{-1}$ and recall the definition of a matrix function:

$$(x + (q-1)y)^{\frac{(q-1)nE-E-|I|}{q}} = T M_I(x, y) T^{-1},$$

which gives (7) and concludes the proof.
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