A SHARP UNIQUENESS RESULT FOR A CLASS OF VARIATIONAL PROBLEMS SOLVED BY A DISTANCE FUNCTION

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(Dedicated to Arrigo Cellina in the occasion of his 65-th birthday)

Abstract. We consider the minimization problem for an integral functional \( J \), possibly non-convex and non-coercive in \( W^{1,1}_0(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded smooth set. We prove sufficient conditions in order to guarantee that a suitable Minkowski distance is a minimizer of \( J \). The main result is a necessary and sufficient condition in order to have the uniqueness of the minimizer. We show some application to the uniqueness of solution of a system of PDEs of Monge-Kantorovich type arising in problems of mass transfer theory.

1. Introduction

In this paper we consider the minimization problem

\[
\min_{u \in W^{1,1}_0(\Omega)} J(u), \quad J(u) = \int_{\Omega} [h(Du) - fu] \, dx,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded smooth (i.e. \( C^2 \)) open set, \( h: \mathbb{R}^n \to [0, +\infty] \) is a (possibly non-convex) Borel function, and \( f \in L^\infty(\Omega) \) is a non-negative function. We are interested in existence and uniqueness results.

The main assumptions on \( h \) involve the convex hull \( K \) of its zero-level set

\[
K := \text{co } Z, \quad Z := \{ \xi \in \mathbb{R}^n; h(\xi) = 0 \},
\]

and the rate of growth of \( h \) outside \( K \), defined by

\[
\Lambda := \sup \{ \lambda \geq 0; h(\xi) \geq \lambda(\rho(\xi) - 1) \ \forall \xi \in \mathbb{R}^n \},
\]

where \( \rho: \mathbb{R}^n \to \mathbb{R} \) is the gauge function of the convex set \( K \) (see Section 2.2). More precisely, the assumptions on \( h \) are the following:

(H1) \( h: \mathbb{R}^N \to [0, +\infty] \) is a Borel function;

(H2) \( K \) is a compact convex set containing 0 as an interior point, with boundary \( \partial K \) of class \( C^2 \), and strictly positive principal curvatures.

Notice that (H2) implies that \( K \) is a strictly convex set. Hence \( \partial K \subseteq Z \), and \( h(\xi) = 0 = \min h \) for every \( \xi \in \partial K \).

The main tool needed in our investigation is the Minkowski distance \( d \) from \( \partial \Omega \) associated to \( K \) (see Section 2.3). It is well known that \( d \) is a Lipschitz continuous function vanishing on \( \partial \Omega \), with \( Dd \in \partial K \) a.e. in \( \Omega \). In fact, \( d \) is the maximal element of the family \( \text{Lip}_0^1(\Omega, \rho) \) of all Lipschitz functions \( u: \overline{\Omega} \to \mathbb{R} \) satisfying \( Du \in K \) a.e. in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \).

Since \( f \geq 0 \), a direct inspection of \( J \) suggests that it is reasonable to expect that \( d \) is
a minimizer of $J$, provided that $h$ grows fast enough outside $K$. This guess will be proved in Theorem 3.2 below.

The aim of this paper is to give necessary and sufficient conditions in order to guarantee that $d$ is the unique minimizer of $J$. The leading idea of our techniques is to go deeply into the connection between these problems of Calculus of Variations and some aspects of the mass transport theory. In order to explain this connection, let us consider the model functional

$$J(u) = \int_{\Omega} [h(|Du|) - u] \, dx,$$

studied by A. Cellina in the seminal paper [10]. Here $\Omega$ is a convex subset of $\mathbb{R}^2$, and the Lagrangian is radially symmetric, so that $K$ is a ball. Assume, without loss of generality, that $K = B_1(0)$. In this case $\rho(\xi) = |\xi|$, and $d$ is the Euclidean distance function from the boundary of $\Omega$. In [10] it was proved that $d$ is a solution to (3), provided that the growth condition $\Lambda \geq r_\Omega$ holds (here $r_\Omega$ is the inradius of the set $\Omega$). The key point in the proof of the minimality of $d$ is to observe that for every $v \in L^\infty(\Omega)$, satisfying $0 \leq v(x) \leq \Lambda$ a.e. in $\Omega$, one has

$$J(u) - J(d) \geq \int_{\Omega} [v \max\{|Du| - 1, 0\} - u + d] \, dx$$

$$\geq \int_{\Omega} [v\langle Dd, Du - Dd \rangle - (u - d)] \, dx.$$  

Hence the conclusion follows once one exhibits a function $v$ as above which solves

$$-\operatorname{div}(vDd) = 1 \text{ in } \Omega,$$

in the sense of distributions. In fact, the hard part of the proof of the existence result in [10] is the construction of such a function $v$. Moreover it is shown, by examples, that the growth condition $\Lambda \geq r_\Omega$ cannot be improved. The result in [10] has been extended to convex domains in $\mathbb{R}^n$ and to more general functionals in subsequent works (see [8, 9, 14, 20, 21]).

Recently in [5, 6] it was proved that for every given non-negative continuous function $f$ there exists a unique non-negative continuous function $v_f$ solving

$$-\operatorname{div}(v_f Dd) = f \text{ in } \Omega,$$

without the requirement of $\Omega$ to be a convex set. In dealing with non-convex domains, the growth condition has to be modified, in order to take into account the presence of points on $\partial \Omega$ with negative curvatures. Nevertheless it can be proved that, if $\|f\|_\infty$ is small enough (see (H3) below), then the function $v_f$ satisfies $0 \leq v_f < \Lambda$ and hence $d$ is a solution to the minimum problem

$$\min_{u \in W^{1,1}_0(\Omega)} \int_{\Omega} [h(|Du|) - f u] \, dx.$$  

Going further in the study of Cellina’s minimization problem, one easily get that the estimates $0 \leq v_f < \Lambda$ imply an $a \text{ priori}$ bound on the gradient of minimizers. Namely, as a consequence of the analogous of (4), with $f$ instead of $1$, every solution $u \in W^{1,1}_0(\Omega)$ of (3) has to belong in fact to $W^{1,\infty}_0(\Omega)$, and $|Du| \leq 1$. Hence some information on the solutions to (3) can be obtained by studying the ancillary minimization problem with constraints

$$\min_{u \in W^{1,1}_0(\Omega), |Du| \leq 1} \int_{\Omega} f u \, dx,$$
which fits into a branch of the optimal mass transfer theory. It is plain that $d$ is always a solution of (1) for every bounded set and for every $f \in L^1(\Omega)$, $f \geq 0$ a.e. in $\Omega$ (and even for non-negative bounded measures). Clearly, if the (essential) support $\text{supp}(f)$ of $f$ coincides with $\Omega$, then $d$ is the unique minimizer. Moreover, it is well known that in the optimal mass transfer problems a long–range effect occurs, i.e. $d$ is the unique solution to (6) if the mass displacement spreads over the whole $\Omega$ (see [5, 6]). We shall show that, in fact, $d$ is the unique solution to (1) also in many cases where the mass transfer spreads only a part of $\Omega$.

The arguments above remain valid also in the anisotropic case. A key point in the analysis of problem (1) is the study of the constrained minimization problem

$$\min_{u \in W^{1,1}_0(\Omega)} \int_{\Omega} f u \, dx.$$

We shall show that the Minkowski distance $d$ associated to $K$ is the unique solution to (7) if and only if $\text{supp}(f)$ contains the singular set $\Sigma$ of those points where $d$ is not differentiable. Furthermore, we exhibit an explicit solution $u_f$ to (7) with $Du_f \in \partial K$ a.e. in $\Omega$, that coincides with $d$ if and only if $\Sigma \subseteq \text{supp}(f)$ (see Theorem 4.12 below). These results lead to the fact that problem (1) has either $d$ as unique solution, if $\Sigma \subseteq \text{supp}(f)$, or it has at least two distinct solutions $u_f$ and $d$, if $\Sigma \setminus \text{supp}(f) \neq \emptyset$.

The role of the singular set $\Sigma$ in the uniqueness result can be understood from a mass transfer theory viewpoint. Namely, if $f$ is a non-negative continuous function, it can be proved that $u$ is a solution of the constrained minimization problem (1) if and only if there exists a non-negative continuous function $v$ such that the pair $(u, v)$ is a solution to the system of PDEs

$$\begin{cases}
- \text{div}(v \, D\rho(Du)) = f & \text{in } \Omega, \\
\rho(Du) \leq 1 & \text{in } \Omega, \\
\rho(Du) = 1 & \text{in } \{v > 0\}
\end{cases}$$

complemented with the conditions

$$\begin{cases}
u \geq 0, v \geq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(see Section 5.2). In [12, 13] we have constructed a non-negative continuous function $v_f$ such that $(d, v_f)$ is a solution of (8)–(9). Moreover we have proved that if $(u, v)$ solves (8)–(9), then $v = v_f$, and $u = d$ in $\{v > 0\}$. In terms of optimal transport problems, the conclusions above state that, for every given mass density $f \geq 0$, the transport density $v_f$ is uniquely determined, while the transport potential $u$ may differ from $d$ only in the region $\{v_f = 0\}$ where no mass transfer occurs (long range effect).

For what concerns the uniqueness of the solution, we start from the fact that $d$ is the unique element $u$ in the family $\text{Lip}_1^0(\Omega, \rho)$ matching the condition $u = d$ on $\Sigma$.

The results in [13] imply that if $u$ is a solution of (1), then $u = d$ in the transport set $\{v_f > 0\}$. Furthermore, due to its structure, the closure of the transport set contains $\Sigma$ if and only if $\Sigma \subseteq \text{supp}(f)$. Then, whenever $\Sigma \subseteq \text{supp}(f)$, $d$ is the unique solution to (1).

The paper is organized as follows. In Section 2 we introduce some notation and review some preliminary result. In Section 3 we recall the properties of the transport density $v_f$, associated to the Monge-Kantorovich system (8)–(9), and we prove that the Minkowski distance $d$ is a solution to the minimization problem (1), provided that $\|f\|_\infty$ is small enough.
Section 2 is devoted to the proof of the uniqueness result. More precisely, in the first part of the section we prove some fine property of functions defined by a max-convolution of cone-shaped functions. These properties will be essential in order to obtain necessary and sufficient conditions for the uniqueness of the minimizer in problem (8). Finally, in Section 2 we give some extension to more general functionals, and we prove a uniqueness result for the Monge-Kantorovich system (8)-(9). We also show some connection with the convergence of solutions to the anisotropic $p$-Laplace equation as $p \to \infty$.

2. Notation and preliminaries

2.1. Basic notation. The standard scalar product of $x$ and $y \in \mathbb{R}^n$ will be denoted by $\langle x, y \rangle$, while $|x|$ will denote the Euclidean norm of $x$. Concerning the segment jointing $x$ with $y$, we set

$$[x, y] := \{tx + (1 - t)y; \ t \in [0, 1]\}, \quad \|x, y\| := [x, y] \setminus \{x, y\}.$$ 

As is customary, $B_r(x_0)$ and $\overline{B}_r(x_0)$ are respectively the open and the closed ball in $\mathbb{R}^n$ centered at $x_0$ and with radius $r > 0$.

Given a set $A \subseteq \mathbb{R}^n$ its closure and its boundary will be denoted by $\overline{A}$ and $\partial A$ respectively.

A bounded open set $O \subset \mathbb{R}^n$ (or, equivalently, $\overline{O}$ or $\partial O$) is of class $C^k$, $k \in \mathbb{N}$, if for every point $x_0 \in \partial O$ there exists a ball $B = B_r(x_0)$ and a one-to-one mapping $\psi: B \to D$ such that $\psi \in C^k(B)$, $\psi^{-1} \in C^k(D)$, $\psi(B \cap O) \subseteq \{x \in \mathbb{R}^n; \ x_n > 0\}$, $\psi(B \cap \partial O) \subseteq \{x \in \mathbb{R}^n; \ x_n = 0\}$.

If $f: \overline{O} \to \mathbb{R}$ is measurable, we define $\text{supp}(f)$ as the intersection of all closed sets $C \subseteq \overline{O}$ such that $f \neq 0$ a.e. in $C$.

2.2. Convex geometry. By $K^n_0$ we shall denote the class of all nonempty, compact, convex subsets of $\mathbb{R}^n$ with the origin as an interior point. The polar set of $K \in K^n_0$ is defined by

$$K^0 = \{p \in \mathbb{R}^n; \ \langle p, x \rangle \leq 1 \ \forall x \in K\}.$$ 

We recall that, if $K \in K^n_0$, then $K^0 \in K^n_0$ and $K^{(0)} = (K^0)^0$ (see [13, Thm. 1.6.1]).

Given $K \in K^n_0$, its gauge function is

$$\rho_K(\xi) = \inf\{t \geq 0; \ \xi \in tK\}, \quad \xi \in \mathbb{R}^n.$$ 

Let $0 < c_1 \leq c_2$ be such that $\overline{B}_{c_2^{-1}}(0) \subseteq K \subseteq \overline{B}_{c_1^{-1}}(0)$. Upon observing that $\xi/\rho_K(\xi) \in K$ for every $\xi \neq 0$, we get

$$c_1|\xi| \leq \rho_K(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^n. \quad (10)$$

We say that $K \in K^n_0$ is of class $C^2_\downarrow$ if $\partial K$ is of class $C^2$ and all the principal curvatures are strictly positive functions on $\partial K$. We recall that, if $K$ is of class $C^2_\downarrow$, then $K^0$ is also of class $C^2_\downarrow$ (see [13, p. 111]).

From now on we shall always assume that

$$K \in K^n_0 \text{ is of class } C^2_\downarrow. \quad (11)$$

Since $K$ will be kept fixed, from now on we shall use the notation $\rho = \rho_K$ and $\rho^0 = \rho_{K^0}$.

We collect here some known properties of $\rho$ and $\rho^0$ that will be used in the sequel (see e.g. [13], Theorem 2.1).
Theorem 2.1. Let $K$ satisfy (11). Then the following hold:
(i) The functions $\rho$ and $\rho^0$ are convex, positively 1-homogeneous in $\mathbb{R}^n$, and of class $C^2$ in $\mathbb{R}^n \setminus \{0\}$.
(ii) For every $\xi, \eta \in \mathbb{R}^n$, we have
\begin{equation}
\rho(\xi + \eta) \leq \rho(\xi) + \rho(\eta), \quad \rho^0(\xi + \eta) \leq \rho^0(\xi) + \rho^0(\eta),
\end{equation}
and equality holds if and only if $\xi = \lambda \eta$ or $\eta = \lambda \xi$ for some $\lambda \geq 0$.
(iii) For every $\xi \neq 0$, $D\rho(\xi)$ belongs to $\partial K^0$ (i.e. $\rho^0(D\rho(\xi)) = 1$), while $D\rho^0(\xi)$ belongs to $\partial K$ (i.e. $\rho(D\rho^0(\xi)) = 1$). More precisely, $D\rho(\xi)$ is the unique point of $\partial K^0$ such that
\begin{equation}
\langle D\rho(\xi), \eta \rangle = \rho(\xi), \text{ and } \langle p, \xi \rangle < \rho(\xi) \forall p \in K^0, \ p \neq D\rho(\xi).
\end{equation}
Symmetrically, the gradient of $D\rho^0(\xi)$ is the unique point of $\partial K$ such that
\begin{equation}
\langle D\rho^0(\xi), \eta \rangle = \rho^0(\xi), \text{ and } \langle x, \xi \rangle < \rho^0(\xi) \forall x \in K, \ x \neq D\rho^0(\xi).
\end{equation}

2.3. The distance function. Throughout the paper, we shall assume that
\begin{equation}
\Omega \subset \mathbb{R}^n \text{ is a nonempty, bounded, open, connected set of class } C^2.
\end{equation}

Definition 2.2. Let $K \subset \mathbb{R}^n$ be a convex set fulfilling (11). The Minkowski distance from the boundary of $\Omega$ associated to the convex body $K$ is the function $d: \overline{\Omega} \to \mathbb{R}$ defined by
\begin{equation}
d(x) = \min_{y \in \partial \Omega} \rho^0(x - y), \quad x \in \overline{\Omega}.
\end{equation}

Let us define the following spaces:
\begin{equation}
\operatorname{Lip}^1(\Omega, \rho) := \left\{ u \in C(\overline{\Omega}); \ u(x) - u(y) \leq \rho^0(x - y) \forall [x, y] \subset \overline{\Omega} \right\}
\end{equation}
\begin{equation}
= \left\{ u \in \operatorname{Lip}(\overline{\Omega}); \ Du \in K \text{ a.e. in } \Omega \right\},
\end{equation}
\begin{equation}
\operatorname{Lip}_0^1(\Omega, \rho) := \left\{ u \in \operatorname{Lip}^1(\Omega, \rho); \ u(y) = 0 \forall y \in \partial \Omega \right\}.
\end{equation}
It is well known that $d \in \operatorname{Lip}_0^1(\Omega, \rho)$, and that $d \geq u$ for every $u \in \operatorname{Lip}_0^1(\Omega, \rho)$ (see e.g. [4, 13]).

Definition 2.3. The inradius of $\Omega$ is defined by $r_\Omega := \max\{d(x); \ x \in \Omega\}$.

We shall denote by $\Pi(x)$ the set of projections of $x$ on $\partial \Omega$, that is
\begin{equation}
\Pi(x) := \{ y \in \partial \Omega; \ d(x) = \rho^0(x - y) \}, \quad x \in \overline{\Omega}.
\end{equation}
We recall that $x \mapsto \Pi(x)$ is a sequentially upper semicontinuos multifunction, i.e.
\begin{equation}
x_k \in \overline{\Omega}, \ y_k \in \Pi(x_k), \ k \in \mathbb{N}; \ x_k \to x, \ y_k \to y \Rightarrow y \in \Pi(x).
\end{equation}

In some situations it will be convenient to consider an extension $d^\ast$ of $d$ to $\mathbb{R}^n$ by setting
\begin{equation}
d^\ast(x) = -\min_{z \in \overline{\Omega}} \rho^0(z - x), \quad x \in \mathbb{R}^n \setminus \Omega.
\end{equation}
This extension is the Minkowski signed distance from $\partial \Omega$. Under the assumption (13), we have that $d^\ast$ is of class $C^2$ in a tubular neighborhood $U$ of $\partial \Omega$ (see [12, Thm. 4.16]). In this neighborhood we can define the Cahn-Hoffman vector field
\begin{equation}
n_\rho(x) := -D\rho(Dd^\ast(x)), \quad x \in U.
\end{equation}
For every $y \in \partial \Omega$, the restriction of $Dn_\rho$ to the tangent space $T_y$ to $\partial \Omega$ at $y$ is a linear application from $T_y$ to $T_y$ having $n - 1$ real eigenvalues $\hat{\kappa}_1(y), \ldots, \hat{\kappa}_{n-1}(y)$, called the
principal $\rho$-curvatures or anisotropic principal curvatures of $\partial \Omega$ at $y$ (see [12, Def. 5.5]). The anisotropic mean curvature is defined by
\begin{equation}
H(y) := \frac{1}{n-1} \sum_{i=1}^{n-1} \tilde{k}_i(y) = \text{div} \ n_\rho(y) \quad \forall y \in \partial \Omega.
\end{equation}

A relevant quantity for the subsequent subjects will be
\begin{equation}
H_0 := \min \{ H(y); \ y \in \partial \Omega \}.
\end{equation}

It can be shown that for every $x_0 \in \Omega$ and $y \in \Pi(x_0)$ the function $d$ is differentiable at every $x \in ]y, x_0[$, and
\begin{equation}
Dd(x) = \frac{\nu(y)}{\rho(\nu(y))} = D\rho^0(x_0 - y), \quad \forall x \in ]y, x_0[, \ \ y \in \Pi(x_0),
\end{equation}
where $\nu(y)$ is the Euclidean inward normal unit vector of $\partial \Omega$ at $y$. Moreover one has
\begin{equation}
[y, x_0) = \{ y + tD\rho(\nu(y)), \ t \in [0, d(x_0)] \}, \ y \in \Pi(x_0)
\end{equation}
(see [12, Proposition 4.4]).

**Definition 2.4.** The singular set $\Sigma \subset \Omega$ of $d$ is the set of all points in $\Omega$ where $d$ is not differentiable.

It is known that $x \in \Omega \setminus \Sigma$ if and only if $x$ has a unique projection. Moreover $\Sigma$ has Lebesgue measure zero, and, since $d$ is of class $C^2$ near the boundary, $\Sigma \subset \Omega$ (see [12, Corollary 6.9 and Theorem 4.16]).

Notice that, thanks to (20) and the positive 0-homogeneity of $D\rho$, we infer that
\begin{equation}
D\rho(Dd(x)) = D\rho(\nu(y)) \quad \forall x \in \Omega \setminus \Sigma, \ y \in \Pi(x).
\end{equation}

**Definition 2.5.** The normal distance of a point $x \in \overline{\Omega}$ to the cut locus is defined by
\begin{equation}
\tau(x) := \begin{cases} 
\min \{ t \geq 0; \ x + tD\rho(Dd(x)) \in \Sigma \}, & \text{if } x \in \overline{\Omega} \setminus \Sigma, \\
0, & \text{if } x \in \Sigma.
\end{cases}
\end{equation}
The cut point $m(x)$ of $x \in \overline{\Omega} \setminus \Sigma$ is defined by $m(x) := x + \tau(x)D\rho(Dd(x))$.

We recall that $\tau$ is a continuous function in $\overline{\Omega}$. Furthermore, there exists $\mu > 0$ such that $\tau(y) \geq \mu$ for every $y \in \partial \Omega$ (see [12, Lemma 4.1 and Theorem 6.7]), and we have
\begin{equation}
\tau(y) = \sup \{ t \geq 0; \ y \in \Pi(y + tD\rho(\nu(y))) \}, \quad \forall y \in \partial \Omega
\end{equation}
(see [12, Corollary 6.8]).

### 3. Existence

In what follows we shall assume that
\begin{equation}
f \in L^\infty(\Omega) \ \text{is a nonnegative function.}
\end{equation}

Let us define the function
\begin{equation}
\nu_f(x) := \begin{cases} 
\int_0^{\tau(x)} f(\Phi(x, t)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t) \tilde{k}_i(x)}{1 - d(x) \tilde{k}_i(x)} \ dt, & \text{if } x \in \overline{\Omega} \setminus \Sigma, \\
0, & \text{if } x \in \Sigma,
\end{cases}
\end{equation}
where, for $x \in \overline{\Omega} \setminus \Sigma$ and $\Pi(x) = \{ y \}$, we have set
\begin{equation}
\Phi(x, t) := x + tD\rho(Dd(x)), \quad \tilde{k}_i(x) := \tilde{k}_i(y), \quad i = 1, \ldots, n - 1.
\end{equation}
Since the maps $\tau$ and $\tilde{\kappa}_i$, $i = 1, \ldots, n - 1$, are continuous in $\overline{\Omega} \setminus \Sigma$, and the map $\Phi$ is continuous in $(\Omega \setminus \Sigma) \times \mathbb{R}$ (see [12 Section 7]), the function $v_f$ is well defined and bounded in $\Omega$.

Let $c: \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be the function defined by

$$c(t, r) := \begin{cases} \frac{1-(1-tr)^n}{n}, & \text{if } t \neq 0, \\ \frac{1}{r}, & \text{if } t = 0. \end{cases}$$

(25)

It is straightforward to check that $t \mapsto c(t, r)$ is a strictly monotone decreasing function for $t \leq \frac{1}{r}$.

The main properties of the function $v_f$ defined in (24) are the following.

Proposition 3.1. The function $v_f$ belongs to $L^\infty(\Omega)$, $v_f \geq 0$ a.e. in $\Omega$, and

$$\|v_f\|_{L^\infty(Q)} < \|f\|_\infty c(H_0, r_\Omega),$$

(26)

for every compact set $Q \subset \Omega$, where $H_0$ is the constant defined in (14). In particular

$$\|v_f\|_\infty \leq \|f\|_\infty c(H_0, r_\Omega).$$

(27)

Moreover $v_f$ satisfies

$$\int_\Omega v_f(D\rho(Dd), D\varphi) \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in W^{1,1}_0(\Omega).$$

(28)

Proof. The facts that $v_f \in L^\infty(\Omega)$ and $v_f$ is a solution to (28) are proved in [12 Section 7] in the case $f \in C(\overline{\Omega})$. The very same proof also works for $f \in L^\infty(\Omega)$. The estimates (26) and (27) follows from a straightforward adaptation of Proposition 5.9 in [8].

Theorem 3.2. Let $\Omega \subset \mathbb{R}^n$ be a set fulfilling (13), and let $f$ satisfy (23). Assume that (H1)–(H2) hold, together with

(H3) there exists $H^* \leq H_0$ such that $c(H^*, r_\Omega) \|f\|_\infty \leq \Lambda$,

where $\Lambda \in [0, +\infty]$ is the constant defined in (2). Then the distance function $d$ is a minimizer of $J$ in $W^{1,1}_0(\Omega)$. Moreover, if $u$ is a minimizer of $J$ in $W^{1,1}_0(\Omega)$, then $u \in \text{Lip}_0(\Omega, \rho)$.

Proof. Since $H_0 \leq 1/r_\Omega$ (see [12], Lemma 5.4), we have that $c(H^*, r_\Omega) \geq c(H_0, r_\Omega)$. From (H3) and (27) we infer that $\|v_f\|_\infty \leq \Lambda$. Hence, if $u \in W^{1,1}_0(\Omega)$ we have that

$$J(u) - J(d) \geq \int_\Omega [\max\{\Lambda(\rho(Du) - 1), 0\} - f(u - d)] \, dx$$

(29)

$$\geq \int_\Omega [v_f(\rho(Du) - 1) - f(u - d)] \, dx$$

$$\geq \int_\Omega [v_f(D\rho(Dd), Du - Dd) - f(u - d)] \, dx = 0,$$

where the third inequality is a consequence of

$$\langle D\rho(p), q - p \rangle = \langle D\rho(p), q \rangle - 1 \leq \rho(q) - 1 \quad \forall p \in \partial K, \, q \in \mathbb{R}^n$$

(see Theorem 2.1 (iii)), while the last equality follows from (28).

Moreover, by (26) the strict inequality holds in (24) whenever $Du(x) \notin K$ on a set of positive Lebesgue measure. Hence $J(u) = J(d)$ implies $Du \in K$ a.e. in $\Omega$.

Remark 3.3. A more precise existence result, containing detailed information about the structure of minimizers, will be given in Theorem 4.12 below.
Proof. Let $\rho$ to the fact that the function $\rho$ is the unique minimizer of $J$ in $W^{1,1}_0(\Omega)$ if and only if $\Sigma \subset \text{supp}(f)$. Theorem 4.1. Let $u$ be differentiable at every point $x \in [x_0, x_1]$, and assume that
\[ u(x_0 + t(x_1 - x_0)) = u(x_0) + t \rho^0(x_1 - x_0) \quad \forall t \in [0, 1]. \]
Then $u$ is differentiable at every point $x \in [x_0, x_1]$, and $Du(x) = D\rho^0(p)$, where $p := (x_1 - x_0)/\rho^0(x_1 - x_0)$.

Proof. Let $\bar{x} \in [x_0, x_1]$ By (2) there exists $\varepsilon > 0$ such that $\bar{x} + \sigma p \in \Omega$ and $u(\bar{x} + \sigma p) = u(\bar{x}) + \sigma$ for every $\sigma \in (-\varepsilon, \varepsilon)$. It is not restrictive to assume that $\bar{x} = 0$ and $u(\bar{x}) = 0$. Hence we have
\[ u(x_0 + t(x_1 - x_0)) = u(x_0) + t \rho^0(x_1 - x_0) \quad \forall t \in [0, 1]. \]
Then $u$ is differentiable at every point $x \in [x_0, x_1]$, and $Du(x) = D\rho^0(p)$, where $p := (x_1 - x_0)/\rho^0(x_1 - x_0)$.

Theorem 4.1. Let $\Omega \subset \mathbb{R}^n$ be a set fulfilling (13). Assume that (H1)–(H3) hold. Then $d$ is the unique minimizer of $J$ in $W^{1,1}_0(\Omega)$ if and only if $\Sigma \subset \text{supp}(f)$.

Remark 4.2. Although hypothesis (13) is needed for proving the uniqueness result, the preliminary results in Lemma 1.3, Proposition 1.4, Lemma 1.6, and Proposition 1.7 having an interest by themselves, are proved without using the regularity assumption on $\Omega$.

The following result is essentially due to M.G. Crandall [1, Lemma 7.3] (see also [1, Prop. 4.2]). We have to make some minor changes with respect to the Euclidean case, due to the fact that the function $\rho^0$ need not be symmetric.

Lemma 4.3. Let $u \in \text{Lip}^1(\Omega, \rho)$, $[x_0, x_1] \subset \Omega$ ($x_0 \neq x_1$), and assume that
\[ u(x_0 + t(x_1 - x_0)) = u(x_0) + t \rho^0(x_1 - x_0) \quad \forall t \in [0, 1]. \]
Then $u$ is differentiable at every point $x \in [x_0, x_1]$, and $Du(x) = D\rho^0(p)$, where $p := (x_1 - x_0)/\rho^0(x_1 - x_0)$.

Proof. Let $\bar{x} \in [x_0, x_1]$. By (2) there exists $\varepsilon > 0$ such that $\bar{x} + \sigma p \in \Omega$ and $u(\bar{x} + \sigma p) = u(\bar{x}) + \sigma$ for every $\sigma \in (-\varepsilon, \varepsilon)$. It is not restrictive to assume that $\bar{x} = 0$ and $u(\bar{x}) = 0$. Hence we have
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4. Uniqueness

The aim of this section is to prove the following uniqueness result.

Theorem 4.1. Let $\Omega \subset \mathbb{R}^n$ be a set fulfilling (13). Assume that (H1)–(H3) hold. Then $d$ is the unique minimizer of $J$ in $W^{1,1}_0(\Omega)$ if and only if $\Sigma \subset \text{supp}(f)$.

Remark 4.2. Although hypothesis (13) is needed for proving the uniqueness result, the preliminary results in Lemma 1.3, Proposition 1.4, Lemma 1.6, and Proposition 1.7 having an interest by themselves, are proved without using the regularity assumption on $\Omega$.

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Lemma 4.3. Let $u \in \text{Lip}^1(\Omega, \rho)$, $[x_0, x_1] \subset \Omega$ ($x_0 \neq x_1$), and assume that
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Then $u$ is differentiable at every point $x \in [x_0, x_1]$, and $Du(x) = D\rho^0(p)$, where $p := (x_1 - x_0)/\rho^0(x_1 - x_0)$.

Proof. Let $\bar{x} \in [x_0, x_1]$. By (2) there exists $\varepsilon > 0$ such that $\bar{x} + \sigma p \in \Omega$ and $u(\bar{x} + \sigma p) = u(\bar{x}) + \sigma$ for every $\sigma \in (-\varepsilon, \varepsilon)$. It is not restrictive to assume that $\bar{x} = 0$ and $u(\bar{x}) = 0$. Hence we have
\[ u(x_0 + t(x_1 - x_0)) = u(x_0) + t \rho^0(x_1 - x_0) \quad \forall t \in [0, 1]. \]
Then $u$ is differentiable at every point $x \in [x_0, x_1]$, and $Du(x) = D\rho^0(p)$, where $p := (x_1 - x_0)/\rho^0(x_1 - x_0)$.

For every $x \in \mathbb{R}^n$ let us define $P x := \langle D\rho^0(p), x \rangle p$, and let $0 < \delta < \varepsilon$ be such that
\[ |\langle D\rho^0(p), x \rangle + s| < \varepsilon, \quad \forall x \in B_\delta(0) \subseteq \Omega, \forall s \in (-\delta, \delta). \]
Fixed $r \in (0, \delta)$, for every $x \in B_\delta(0)$, the point $y = Px + rp$ belongs to $\Omega$. Moreover, from (31) and (32) we obtain
\[ \langle D\rho^0(p), x \rangle + r - \rho^0(Px + rp - x) \]
so that
\[ -r \left[ \rho^0 \left( \frac{Px - x}{r} + p \right) - 1 \right] \leq u(x) - \langle D\rho^0(p), x \rangle. \]
Similarly, the point $z = Px - rp$ belongs to $\Omega$, and
\[ u(x) - \langle D\rho^0(p), x \rangle \leq r \left[ \rho^0 \left( \frac{x - Px}{r} + p \right) - 1 \right]. \]
From the differentiability of $\rho^0$ at $p$, and the fact that $\rho^0(p) = 1$, we have
\[ \rho^0(p + q) = \rho^0(p) + \langle D\rho^0(p), q \rangle + o(|q|) = 1 + \langle D\rho^0(p), q \rangle + o(|q|), \quad q \to 0. \]
Combining (32) and (33), and using the fact that $\langle D\rho^0(p), x - Px \rangle = 0$, we get $u(x) - \langle D\rho^0(p), x \rangle = o(|x|), \ x \to 0$. Hence $u$ is differentiable at $\bar{x} = 0$ and $Du(\bar{x}) = D\rho^0(p)$. □
In Section 3 we have proved that \( d \) is a minimizer of \( J \) in \( W^{1,1}(\Omega) \). Since every other minimizer has to belong to \( \operatorname{Lip}_0^1(\Omega, \rho) \), \( d \) turns out to be the maximal minimizer of \( J \). Now we want to construct the minimal minimizer of \( J \) (see (44) below). This minimizer will be obtained as the supremum of cone-shaped functions.

Assume that

\[
S \text{ is a nonempty closed subset of } \overline{\Omega},
\]

and define the function

\[
u_S(x) := \max_{z \in S \setminus \partial \Omega} [d(z) - \rho^0(z - x)], \quad x \in \overline{\Omega}.
\]

The transport set of \( S \) is defined by

\[
T_S := \bigcup_{\substack{z \in S \\cap \Omega \\setminus \partial \Omega \\setminus \overline{\Omega} \\cup S \\setminus \partial \Omega \\cup \Omega \\cup \Omega \\setminus S \\cup \partial \Omega \\setminus \Omega \\cup \Omega \\setminus \Sigma \\cup \partial \Omega}} y, z \subset \Omega.
\]

The main properties of the function \( u_S \) are collected in the following result.

**Proposition 4.4.** Let \( S \subseteq \overline{\Omega} \) satisfy (33) and let \( u_S \) be the function defined in (33). Then the following hold.

(i) \( u_S \in \operatorname{Lip}_0^1(\Omega, \rho) \).

(ii) \( u_S \leq d \) in \( \overline{\Omega} \), and \( u_S = d \) in \( T_S \). Moreover, for every \( z \in S \cap \Omega \) and \( y \in \Pi(z) \), \( u_S \) is differentiable at every \( x \in ]y, z[ \) and \( Du_S(x) = Dd(x) \).

(iii) If \( u \in \operatorname{Lip}_0^1(\Omega, \rho) \) and \( u = d \) in \( S \), then \( u \geq u_S \) in \( \Omega \).

(iv) \( \rho(Du_S(x)) = 1 \) at every \( x \in \Omega \setminus S \) where \( u_S \) is differentiable.

(v) \( u_S = d \) in \( \Omega \) if and only if \( \Sigma \subseteq S \).

**Proof.** (i) It is plain that \( u_S \) is a Lipschitz function vanishing on \( \partial \Omega \). It remains to prove that

\[
(37) \quad u_S(x_1) - u_S(x_2) \leq \rho^0(x_1 - x_2) \quad \forall x_1, x_2 \in \Omega.
\]

Let \( x_1, x_2 \in \Omega \), and let \( z \in S \cup \partial \Omega \) be such that

\[
(38) \quad u_S(x_1) = d(z) - \rho^0(z - x_1).
\]

By the very definition of \( u_S \) we obtain

\[
-u_S(x_2) \leq -d(z) + \rho^0(z - x_2) \leq -d(z) + \rho^0(z - x_1) + \rho^0(x_1 - x_2),
\]

which, combined with (38), gives (37).

(ii) The inequality \( u_S \leq d \) follows from the fact that \( u_S \in \operatorname{Lip}_0^1(\Omega, \rho) \). If \( z \in S \), we have that \( u_S(z) \geq d(z) \), hence \( u_S(z) = d(z) \).

In order to prove that \( u_S = d \) in \( T_S \), let us fix \( z \in S \cap \Omega, \ y \in \Pi(z) \), and let \( x \in ]y, z[ \), so that \( \Pi(x) = \{y\} \). From Theorem 3.3(iii) we infer that

\[
\rho^0(x - y) = d(x) \geq u_S(x) \geq d(z) - \rho^0(z - x) = \rho^0(z - y) - \rho^0(z - x) = \rho^0(z - y).
\]

Hence \( d = u_S \) in \( T_S \), and the equality extends by continuity to \( T_S \).

Finally, we have that

\[
u_S(y + t(z - y)) = d(y + t(z - y)) = t \rho^0(z - y), \quad \forall t \in [0, 1],
\]

From Lemma 4.3 we conclude that \( u_S \) is differentiable at every \( x \in ]y, z[ \), and \( Du_S(x) = D\rho^0(z - y) = Dd(x) \) (see (20)).
(iii) Observe that \( u = d \) on \( S \cup \partial \Omega \). Let \( x_0 \in \Omega \setminus S \), and let us prove that \( u(x_0) \geq u_S(x_0) \). Let \( z \in S \cup \partial \Omega \) be such that \( u_S(x_0) = d(z) - \rho^0(z-x_0) \). Then we have that
\[
u_S(x_0) = d(z) - \rho^0(z-x_0) = u(z) - \rho^0(z-x_0) \leq u(x_0),
\]
where the last inequality follows from the fact that \( u \in \text{Lip}^1(\Omega, \rho) \).

(iv) From the general theory of marginal functions we have that for every \( x \in \Omega \setminus S \) where \( u_S \) is differentiable there exists \( z \in S \cup \partial \Omega \) such that \( Du_S(x) = D\rho^0(z-x) \) (see e.g. [3], Theorem 3.4.4). Then the conclusion follows from Theorem [3](iii).

(v) Assume that \( \Sigma \subseteq S \). From (iii) we have that \( u_S(x) = d(x) \) for every \( x \in \Sigma \). Let \( x \in \Omega \setminus \Sigma \). Let \( y \) be the unique projection of \( x \) on \( \partial \Sigma \) and let \( z \in \Sigma \) be the cut point of \( x \). Then we have
\[
d(x) = \rho^0(x-y) = \rho^0(z-y) - \rho^0(z-x) \leq u_S(x) \leq d(x),
\]
hence we conclude that \( u_S(x) = d(x) \).

Conversely, assume that \( u_S = d \) in \( \Sigma \). By contradiction, suppose that there exists a point \( x_0 \in \Sigma \), \( x_0 \notin S \). Let \( z \in S \cup \partial \Omega \) be such that \( u_S(x_0) = d(z) - \rho^0(z-x_0) \). Since \( u_S(x_0) = d(x_0) > 0 \), it is plain that \( z \in S \cap \Omega \), for otherwise we would have \( u_S(x_0) \leq 0 \). Summarizing, we have that
\[
\exists \ z \in S \cap \Omega : \ d(x_0) = u_S(x_0) = d(z) - \rho^0(z-x_0).
\]
On the other hand, we are going to show that the assumptions \( x_0 \in \Sigma \), \( x_0 \notin z \), imply that
\[
d(z) < d(x_0) + \rho^0(z-x_0).
\]
Namely, let \( y_0 \in \Pi(x_0) \). If \( y_0 \in \Pi(z) \), then \( x_0 \notin \|y_0, z\| \), since otherwise \( d \) should be differentiable at \( x_0 \), in contrast with the assumption \( x_0 \in \Sigma \). Moreover, from (39), \( d(x_0) < d(z) \), so that \( z \notin \|y_0, x_0\| \). Hence the three points \( y_0, x_0, z \) do not lie on the same ray, so that
\[
d(z) = \rho^0(z-y_0) < \rho^0(z-x_0) + \rho^0(x_0-y_0) = \rho^0(z-x_0) + d(x_0),
\]
and (40) holds. On the other hand, if \( y_0 \notin \Pi(z) \), then
\[
d(z) < \rho^0(z-y_0) \leq \rho^0(z-x_0) + \rho^0(z_0-y_0) = \rho^0(z-x_0) + d(x_0),
\]
and again (40) holds. \( \square \)

**Definition 4.5.** The reduced set \( S^* \) is the set of points \( z^* \in S \) such that the following holds: if \( z \in S \), \( y \in \Pi(z) \) and \( z^* \in [y, z] \), then \( z = z^* \).

**Lemma 4.6.** Let \( z \in S \) and \( y \in \Pi(z) \). Then there exists a unique \( z^* \in S^* \) such that \( [y, z] \subseteq [y, z^*] \).

**Proof.** The uniqueness of \( z^* \) is a straightforward consequence of the definition of \( S^* \).

For what concerns the existence, assume first that \( z \in \Omega \), so that \( y \neq z \). Let us define
\[
A := \left\{ \lambda \geq 0; \ y_\lambda := y + \lambda \frac{z-y}{\rho^0(z-y)} \in S, \ y \in \Pi(y_\lambda) \right\}.
\]
We have that \( A \) is bounded, \( A \) is closed (see (17)), and \( d(z) \in A \), hence \( A \) admits a maximum \( \sigma \geq d(z) \).

We claim that the point \( z^* := y_\sigma \) belongs to \( S^* \). Namely, given \( z' \in S \), \( y' \in \Pi(z') \) such that \( z^* \in [y', z'] \), we have to prove that \( z^* = z^* \). If \( y' = y \), then the equality follows from the maximality of \( \sigma \). On the other hand, if \( y' \neq y \), we have that \( \Pi(z^*) \) contains two different points \( y \) and \( y' \), so that \( z^* \in \Sigma \). Since \( d \) is differentiable at every point in \( [y', z'] \), we infer that \( z^* = z^* \).
Consider now the case \( z \in \partial \Omega \cap S \), so that \( y = z \). If \( z \notin \Pi(z') \) for every \( z' \in \Omega \cap S \), then \( z \in S^* \). Otherwise let \( z' \in \Omega \cap S \) be such that \( y = z \in \Pi(z') \). From the first part of the proof, there exists \( z^* \in S^* \) such that \( \{z\} = [y, z] \subset \{y, z'\} \subset [y, z^*] \), and the proof is complete. \( \square \)

Each segment \([y, z]\), with \( z \in S^* \) and \( y \in \Pi(z) \), will be called a transport ray. We shall denote by \( T_S^y \) the union of the closures \([y, z]\) of all transport rays, that is

\[
T_S^y := \bigcup_{y \in \Pi(z)} [y, z] \subset \overline{\Pi}
\]

(To be precise, \( T_S^y \) also contains the points \( z \in S^* \cap \partial \Omega \), that are not closures of transport rays.)

From the definition it is plain that two different transport rays have empty intersection, and that transport rays are maximal, i.e. if \([y, z]\) is a transport ray, then \([y, y + t(z - y)]\) is not a transport ray for every \( t > 1 \).

**Proposition 4.7.** Let \( S \) satisfy (34). Then \( T_S^y \) is a closed subset of \( \overline{\Pi} \) and

\[
T_S := T_S^y \cup S = \bigcup_{y \in \Pi(z)} [y, z].
\]

**Proof.** Let \( T \) denote the set appearing in the right-hand side of (32). One easily obtain that \( T = T_S^y \). Namely, since \( S^* \subset S \), we have that \( T_S^y \subset T \), while the inclusion \( T \subset T_S^y \) follows from Lemma 4.4.

It remains to prove that \( T_S^y = \overline{T_S^y} \cup S \). Let us start by proving that \( T_S^y \) is a closed subset of \( \overline{\Pi} \). Let \( (x_j) \subset T_S^y \) be a sequence converging to a point \( x \). By definition of \( T_S^y \), for every \( j \in \mathbb{N} \) there exist \( z_j \in S^* \), \( y_j \in \Pi(z_j) \) and \( t_j \in [0, 1] \) such that \( x_j = y_j + t_j(z_j - y_j) \). We can extract a subsequence (not relabeled) such that \( z_j \to z \in S \), \( y_j \to y \in \partial \Omega \), \( t_j \to t \in [0, 1] \), so that \( x = y + t(z - y) \). By the upper semicontinuity of the multifunction \( \Pi \), we have that \( y \in \Pi(z) \), so that \( x = y + t(z - y) \in [y, z] \) with \( z \in S \) and \( y \in \Pi(z) \), i.e. \( x \in T = T_S^y \).

Finally, in order to prove that \( T_S^y = \overline{T_S^y} \cup S \) it is enough to observe that \( T_S \cup S \subset T_S^y \) and \( T_S^y = T \subset \overline{T_S^y} \cup S \). \( \square \)

Let us define the function

\[
\lambda^*(x) := \begin{cases} 
\rho^0(z - y), & \text{if } x \in [y, z] \text{ for some } z \in S^* \text{ and } y \in \Pi(z), \\
0, & \text{otherwise in } \overline{\Pi}.
\end{cases}
\]

**Proposition 4.8 (Upper semicontinuity of \( \lambda^* \)).** Let \( S \) satisfy (34). Then the function \( \lambda^* \) is upper semicontinuous in \( \overline{\Pi} \).

**Proof.** Let \( (x_j) \subset \overline{\Pi} \) be a sequence converging to a point \( x \in \overline{\Pi} \). We have to prove that \( \lambda^*(x) \geq \limsup_j \lambda^*(x_j) \). It is not restrictive to assume that \( \limsup_j \lambda^*(x_j) = \lim_j \lambda^*(x_j) \). If \( \lim_j \lambda^*(x_j) = 0 \) the conclusion is trivial, hence it is enough to consider only the case \( (x_j) \subset T_S^y \). By Proposition 4.7 we have that also \( x \in T_S^y \). By definition, for every \( j \in \mathbb{N} \) there exist \( z_j \in S^* \), \( y_j \in \Pi(z_j) \) and \( t_j \in [0, 1] \) such that \( x_j = y_j + t_j(z_j - y_j) \), so that \( \lambda^*(x_j) = \rho^0(z_j - y_j) \). We can pass to a subsequence (not relabeled) such that \( z_j \to z \in S \), \( y_j \to y \in \Pi(z) \), \( t_j \to t \in [0, 1] \), so that \( x = y + t(z - y) \). From Lemma 4.6 there exists \( z^* \in S^* \) such that \([y, z] \subset [y, z^*] \), hence

\[
\lambda^*(x) = \rho^0(z - y) = \lim_j \rho^0(z_j - y_j) = \lim_j \lambda^*(x_j),
\]

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concluding the proof.

The following proposition says that the measure of the set $S^*$ of the endpoints of transport rays has zero Lebesgue measure. This is a well known property in transport theory (see e.g., [13, Coroll. 6.1] or [16, Lemma 2.15]). We give a direct proof based on the techniques developed in [12] (see in particular Corollary 4.15 in [12]). Here $L^n$ denotes the $n$-dimensional Lebesgue measure.

Proposition 4.9. The reduced set $S^*$ has vanishing Lebesgue measure. As a consequence, $L^n(T^*_S \setminus T_S) = 0$.

Proof. By the very definition of $S^*$ we have that

$$S^* \subset Z := \{y + \lambda^*(y) D\rho(\nu(y)); \ y \in \partial\Omega\}.$$ 

Let $Y_k: \mathcal{U}_k \to \mathbb{R}^n$, $\mathcal{U}_k \subset \mathbb{R}^{n-1}$ open, $k = 1, \ldots, N$, be local parameterizations of $\partial\Omega$ of class $C^2$, such that $\bigcup_{k=1}^N Y_k(\mathcal{U}_k) = \partial\Omega$. For every $k = 1, \ldots, N$, let $\Psi_k: \mathcal{U}_k \times \mathbb{R} \to \mathbb{R}^n$ be the map

$$\Psi_k(y,t) = Y_k(y) + tD\rho(\nu(Y_k(y))), \ (y,t) \in \mathcal{U}_k \times \mathbb{R}.$$ 

For every $k = 1, \ldots, n$ let $U_k \subset \mathcal{U}_k$ be a compact set such that $\bigcup_k Y_k(U_k)$ covers $\partial\Omega$, and let

$$A_k = \{(y,t); \ y \in U_k, t \in [0,\lambda^*(Y_k(y))]\}.$$ 

From Proposition 4.8, $\lambda^*$ is an upper semicontinuous function, hence for every $k = 1, \ldots, n$, $A_k$ is a compact set and the Lebesgue measure of the graph

$$\Psi_k^{-1}(Z) \cap A_k = \{(y,t) \in U_k \times \mathbb{R}; \ t = \lambda^*(Y_k(y))\}$$ 

vanishes. Moreover $\Psi_k \in C^1(\mathcal{U}_k \times \mathbb{R})$ for every $k = 1, \ldots, N$ (see [12, Theorem 4.13]), hence $\Psi_k$ is Lipschitz continuous on the compact set $A_k$. Let $L$ be the maximum of the Lipschitz constants of the functions $\Psi_1, \ldots, \Psi_N$. Since $\bigcup_{k=1}^N \Psi_k(A_k) = \overline{\Omega}$, and hence $S^* \subseteq Z \subseteq \bigcup_{k=1}^N \Psi_k(\Psi_k^{-1}(Z) \cap A_k)$, we finally get

$$L^n(S^*) \leq \sum_{k=1}^N L^n[\Psi_k(\Psi_k^{-1}(Z) \cap A_k)] \leq L^n \sum_{k=1}^N L^n[\Psi_k^{-1}(Z) \cap A_k] = 0.$$ 

Finally, the last assertion follows from the inclusion $T^*_S \setminus T_S \subseteq \partial\Omega \cup S^*$ (see Lemma 4.4).

Remark 4.10. In general $S^*$ is not closed. For example, let $\Omega = B_R(0) \subset \mathbb{R}^2$, with $R > 2\pi + 1$, and let

$$S := \{(r \cos \theta, r \sin \theta); \ \theta \in [0, 2\pi), \ \theta + 1 \leq r \leq R\}.$$ 

It is easy to verify that $S$ is a closed set. On the other hand, if we consider the Euclidean metric $(K = \overline{B}_1(0))$, we have that the reduced set is

$$S^* = \{((\theta + 1) \cos \theta, (\theta + 1) \sin \theta); \ \theta \in [0, 2\pi]\},$$ 

which is not closed since $S^* \setminus S^* = \{(2\pi + 1, 0)\}$. We can also easily compute the function $\lambda^*$,

$$\lambda^*(r \cos \theta, r \sin \theta) = R - \theta - 1, \ \ \theta + 1 \leq r \leq R, \ 0 \leq \theta < 2\pi, \ \ \lambda^* = 0 \ otherwise \ in \ \Omega,$$

which is upper semicontinuous but not continuous in $\overline{\Omega}$.

For future reference, we collect here the properties of the function $u_S$ defined in [12].

Corollary 4.11. Let $\Omega$ and $S$ satisfy (13) and (24) respectively. Then the function $u_S$ belongs to $\text{Lip}_0^0(\Omega, \rho)$ and satisfies the following properties.
(i) \( u_S = d \) on the closed set \( T_S^c \supseteq S \), and \( Du_S(x) = Dd(x) \) for a.e. \( x \in T_S^c \).
(ii) \( \rho(Du_S) = 1 \) a.e. in \( \Omega \).
(iii) If \( u \in \text{Lip}_0^1(\Omega, \rho) \) coincides with \( d \) on \( S \), then \( u_S \leq u \leq d \) in \( \overline{\Omega} \), and \( Du(x) = Dd(x) \) for a.e. \( x \in T_S^c \).
(iv) \( u_S = d \) if and only if \( \Sigma \subseteq S \).

**Proof.** (i) From Proposition 4.7 we have that \( T_S^c = \overline{T_S} \cup S \). Hence, from Proposition 4.4(ii), we deduce that \( u_S = d \) on \( T_S^c \) and \( u_S \) is differentiable at every point of \( T_S \), with \( Du_S = Dd \).

Since, by Proposition 4.9, \( \mathcal{L}^n(T_S^c \setminus T_S) = 0 \), we conclude that \( u_S \) is differentiable a.e. on \( T_S^c \) with \( Du_S = Dd \).

(ii) From Proposition 4.4(iv) we have that \( u_S \) is differentiable and \( \rho(Du_S) = 1 \) a.e. on \( \Omega \setminus S \). On the other hand, by (i) we have that \( u_S \) is differentiable for a.e. \( x \in T_S^c \supseteq S \), and \( \rho(Du_S(x)) = \rho(Dd(x)) = 1 \).

(iii) and (iv) are proved in Proposition 4.4(iii) and (v) respectively, with the exception of the equality \( Du = Dd \) a.e. on \( T_S^c \), which follows from Lemma 4.3 upon observing that \( u \) coincides with \( d \) along transport rays. \( \square \)

**Theorem 4.12.** Let the assumptions of Theorem 3.2 hold. Then the function \( u_f(x) := \max_{z \in \text{supp}(f) \cup \partial \Omega} [d(z) - \rho^0(z - x)] \), \( x \in \overline{\Omega} \) is a minimizer of \( J \) in \( W_0^{1,1}(\Omega) \). Moreover, any other minimizer \( u \) of \( J \) in \( W_0^{1,1}(\Omega) \) belongs to \( \text{Lip}_0^1(\Omega, \rho) \) and satisfies \( u_f \leq u \leq d \). In particular, \( u = d \) on the set \( T_f := T_{\text{supp}(f)}^e \) defined in (24) with \( S = \text{supp}(f) \).

**Proof.** Observe that \( u_f \) is the function defined by (23) with \( S = \text{supp}(f) \). Let \( v_f \) be the function defined in (24).

We claim that
\[
(v_f(x) = 0, \quad \forall x \in \Omega \setminus T_f).
\]

Property (13) is plain if \( x \in \overline{\Omega} \setminus T_f \). Assume now that \( x \in \Omega \setminus T_f, \ x \notin \Sigma \), and let \( z \in \Sigma \) be its cut point. Since \( \text{supp}(f) \subseteq T_f \), in order to show that \( v_f(x) = 0 \) it is enough to prove that \( [x,z] \) does not intersect \( T_f \). Assume, by contradiction, that there exists \( x_0 \in [x,z] \cap T_f \). Then \( x_0 \notin \Sigma \), and \( \Pi(x_0) = \Pi(x) =: \{y\} \). Moreover, since \( x_0 \in T_f \), then \( [y,x_0] \subseteq T_f \), which contradicts the fact that \( x \notin T_f \).

By Corollary 4.11(i) and (12), we obtain that \( u_f = d \) and \( Du_f = Dd \) a.e. in the set \( \{v_f \neq 0\} \). Hence, by (23), (27), (H3), and Corollary 4.11(ii), we conclude that
\[
J(u) - J(u_f) \geq \int_{\Omega} [v_f(D\rho(Dd), Du - Du_f) - (u-u_f)f] \, dx
\]
for every \( u \in W_0^{1,1}(\Omega) \) (see (29)). Then, by (28a), we get \( J(u) \geq J(u_f) \), i.e. \( u_f \) is a minimizer of \( J \).

Assume now that \( u \) is a minimizer of \( J \) in \( W_0^{1,1}(\Omega) \). Since \( d \) is also a minimizer, and \( u \in \text{Lip}_0^1(\Omega, \rho) \) (see Theorem 3.2), we have that
\[
0 = J(u) - J(d) = \int_{\Omega} [h(Du) + (d-u)f] \, dx.
\]

Since \( h \) and \( d - u \) are non-negative functions, it follows that \( h(Du) = 0 \) a.e. in \( \Omega \), and \( u = d \) in \( \text{supp}(f) \). From Corollary 4.11(iii) it follows that \( u_f \leq u \leq d \) in \( \Omega \) and \( u = d \) on \( T_f \). \( \square \)
Proof. See the proof Theorem 5.3 in [6].

Remark 4.13. It is not difficult to see that the conclusions of Theorem 4.12 continue to hold under the following assumptions: $\Omega$ and $K$ satisfy (13) and (11) respectively, $f \in L^1(\Omega)$ is a non-negative function, and $h$ is the indicator function of the set $K$.

Proof of Theorem 5.1. From Theorem 4.12 we have that $u_f$ and $d$ are both minimizers of $J$ in $W^{1,1}_0(\Omega)$, and that any other minimizer $u$ of $J$ satisfies $u_f \leq u \leq d$. Hence $d$ is the unique minimizer of $J$ if and only if $u_f = d$. The conclusion now follows from Corollary 4.11(iv).

Remark 4.14. If $f$ is a continuous function such that $f > 0$ everywhere in $\Sigma$, by Theorem 4.1 $d$ is the unique minimizer of $J$. In this case we have

$$v_f > 0 \text{ in } \overline{\Omega} \setminus \Sigma,$$

and the uniqueness result also follows from Theorem 6.1 in [13]. Nevertheless, condition (H3) is only necessary for the uniqueness. As an example, let $\Omega \subset \mathbb{R}^2$ be an ellipse centered at the origin and with a focus in $(1,0)$. Let $f \in C(\Omega)$ satisfy $f > 0$ in $\Omega \cap ((-1,1) \times \mathbb{R}^+)$, $f = 0$ otherwise in $\Omega$. Let $h$ be the indicator function of the ball $B_1(0)$, so that $\Lambda_h = +\infty$. In this case $\Sigma$ is the interval jointing $(-1,0)$ with $(1,0)$, without endpoints, so that $\Sigma \subseteq \text{supp}(f)$, and hence, by Theorem 4.1 $d$ is the unique minimizer of $J$. On the other hand it can be easily checked that in this case $v_f > 0$ only in $\Omega \cap ((-1,1) \times \mathbb{R}^+)$.}

5. Extensions and applications

5.1. Some extension. The existence result can be generalized without any effort to minimum problems of the form

$$\min_{u \in W^{1,1}_0(\Omega)} J(u), \quad J(u) := \int_{\Omega} [h(Du) - g(x,u)] \, dx,$$

where

(H4) $g: \Omega \times \mathbb{R} \to [0,\infty)$ is a measurable function, Lipschitz continuous with respect to the second variable, satisfying $g(\cdot,0) \in L^1(\Omega)$ and

$$0 \leq Du(x,u) \leq L, \quad \text{a.e. } (x,u) \in \Omega \times \mathbb{R}.$$

Theorem 5.1. Let $\Omega \subset \mathbb{R}^n$ be a set fulfilling (11). Assume that (H1), (H2) and (H4) hold, together with

(H3') there exists $H^* \leq H_0$ such that

$$L c(H^*, r_\Omega) \leq \Lambda.$$

Then the distance function $d$ is a minimizer of $J$ in $W^{1,1}_0(\Omega)$. Moreover, if $u \in W^{1,1}_0(\Omega)$ is a minimizer of $J$ in $W^{1,1}_0(\Omega)$, then $Du(x) \in K$ for a.e. $x \in \Omega$.

Proof. See the proof Theorem 5.3 in [3].

Remark 5.2. If $g(x,\cdot)$ is concave for a.e. $x \in \Omega$, assumption (H4) can be relaxed with the following requirement:

$$g'_+(x,0) \leq L, \quad g'_-(x,d(x)) \geq 0, \quad \text{a.e. } x \in \Omega.$$

Namely, let us define

$$\tilde{g}(x,u) := \begin{cases} g(x,u), & \text{if } 0 \leq u \leq d(x), \\ g(x,0) + g'_+(x,u) u, & \text{if } u < 0, \\ g(x,d(x)) + g'_-(x,d(x)) (u - d(x)), & \text{if } u > d(x). \end{cases}$$
Let $\tilde{J}$ denote the functional in (17) with $\tilde{g}$ instead of $g$. It is plain that, if $g$ satisfies (19), then $\tilde{g}$ satisfies (H4), so that by Theorem 5.2, $\tilde{d}$ is a minimizer of $\tilde{J}$. Moreover, $\tilde{g} \geq g$, so that $\tilde{J} \leq J$, and $J(d) = \tilde{J}(d)$, hence $d$ is a minimizer of $J$.

Concerning the uniqueness of the minimizer, the result is the following.

**Theorem 5.3.** Assume that (H1), (H2), (H3’) and (H4) hold, and let

$$A := \{ x \in \Omega : g(x, d(x)) > g(x, u) \wedge u < d(x) \}.$$ 

If $\Sigma \subset \overline{A}$, then $d$ is the unique minimizer of $J$.

**Proof.** Let $u_A$ be the function defined by (23) with $S = \overline{A}$. Let $u \in \text{Lip}_0^1(\Omega, \rho)$ be a minimizer of $J$. It is clear that we must have $u = d$ in $A$, for otherwise we would have $J(u) > J(d)$. From Proposition 4.4(iii) we have that $u \geq u_A$. Moreover, from Proposition 4.4(v) we conclude that $d = u_A \leq u \leq d$, that is $u = d$. $\square$

**Remark 5.4.** The analogous of Theorems 5.1 and 5.3 can be proved if in (H4) we require $p \geq 1$.

### 5.2. A system of PDEs of Monge-Kantorovich type.

Let us read the results of Sections 3 and 4 in terms of properties of solutions to the Monge-Kantorovich system of PDEs (8)-(9).

Assume that $f : \Omega \to \mathbb{R}$ is a bounded continuous non-negative function, and that $K$ satisfies (11). Let $v$ satisfy (11). Let $u \in \text{Lip}_0^1(\Omega, \rho)$ be a solution to (7) (see (29)).

Finally, Theorem 4.1 states that the system (8)-(9) admits the unique solution $(d, v_f)$ if and only if $\Sigma \subset \text{supp}(f)$.

#### 5.3. Convergence of solutions to p-Laplace equation.

Let $f \in L^\infty(\Omega)$, $f \geq 0$. In [13] it is proved that the functionals

$$J_p(u) := \begin{cases} \int_\Omega \frac{1}{p}(Du)^p - fu \, dx & u \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

$\Gamma$–converge in $L^1(\Omega)$ as $p \to \infty$ to the functional

$$J(u) := \begin{cases} -\int_\Omega fu \, dx, & u \in \text{Lip}_0^1(\Omega, \rho), \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

and that the sequence $(u_p)$ of the minimizers of $J_p$ is bounded in $W_0^{1,q}(\Omega)$ for every $q > 1$. As a consequence, any converging subsequence of $(u_p)$ converges to a minimizer of $J$. We remark that, for every $p > 1$, $u_p$ is the unique distributional solution of the anisotropic p-Laplace equation

$$\begin{cases} -\text{div}(A_p(Du_p)) = f & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases}$$
where \( A_p(0) = 0 \) and \( A_p(\xi) = \rho(\xi)^{p-1}D\rho(\xi) \) for every \( \xi \neq 0 \). By Theorem 4.1, if \( \Sigma \subset \text{supp}(f) \), then \( d \) is the unique minimizer of \( J \). In this case we can conclude that the whole sequence \( (u_p) \) converges weakly in \( W^{1,q}_0(\Omega) \) to \( d \).

In the special case \( \rho(\xi) = |\xi| \), the minimizer \( u_p \) of \( J_p \) is the unique distributional solution in \( W^{1,p}_0(\Omega) \) of the \( p \)-Laplace equation

\[
\begin{cases}
-\Delta_p u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

In this case it is known (see [4]) that \( (u_p) \) converges in \( C(\Omega) \) to the function

\[ U(x) := \begin{cases} 
  d(x), & \text{if } x \in \Omega \setminus A, \\
  w(x), & \text{if } x \in A,
\end{cases} \]

where \( A := \text{int}\{x \in \Omega; \ f(x) = 0\} \), and \( w \in C(\overline{A}) \) is the unique viscosity solution of the \( \infty \)-Laplace equation

\[
\begin{cases}
-\Delta_\infty w = 0 & \text{in } A, \\
w = d & \text{on } \partial A
\end{cases}
\]

(see [17, 3]). If \( \Sigma \subset \text{supp}(f) \), then our results state that, for every \( q > 1 \), the sequence \( (u_p) \) converges to \( d \) as \( p \to \infty \) in \( W^{1,q}_0 \). As a consequence, we have \( w = d \) in \( A \).

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