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Technical appendix to “Adaptive estimation of stationary Gaussian fields”

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Abstract
This is a technical appendix to “Adaptive estimation of stationary Gaussian fields” [6]. We present several proofs that have been skipped in the main paper. These proofs are organised as in Section 8 of [6].

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1. Proof of Proposition 8.1

Proof of Proposition 8.1. First, we recall the notations introduced in [3]. Let $N$ be a positive integer. Then, $I_N$ stands for the family of subsets of $\{1, \ldots, N\}$ of size less than 2. Let $T$ be a set of vectors indexed by $I_N$. In the sequel, $T$ is assumed to be a compact subset of $\mathbb{R}^{(N(N+1)/2)+1}$. The following lemma states a slightly modified version of the upper bound in remark 7 in [3].

Lemma 1.1. Let $T$ be a supremum of Rademacher chaos indexed by $I_N$ of the form

$$T := \sup_{t \in T} \left| \sum_{\{i,j\}} U_i U_j t_{\{i,j\}} + \sum_{i=1}^N t_{\{i\}} + t \varnothing \right|,$$

where $U_1, \ldots, U_N$ are independent Rademacher random variables. Then for any $x > 0$,

$$\mathbb{P} \{ T \geq \mathbb{E}[T] + x \} \leq 4 \exp \left( -\frac{x^2}{D^2} \land \frac{x}{L_2 E} \right),$$

where $D$ and $E$ are defined by:

$$D := \sup_{t \in T} \sup_{\|\alpha\|_2 \leq 1} \left| \sum_{i=1}^N U_i \sum_{j \neq i} \alpha_j t_{\{i,j\}} \right|,$$

$$E := \sup_{t \in T} \sup_{\|\alpha^{(1)}, \alpha^{(2)}\|_2 \leq 1} \left| \sum_{i=1}^N \sum_{j \neq i} t_{\{i,j\}} \alpha^{(1)}_i \alpha^{(2)}_j \right|.$$
Contrary to the original result of [3], the chaos are not assumed to be homogeneous. Besides, the \( t_{i,j} \) are redundant with \( t_\varnothing \). In fact, we introduced this family in order to emphasize the connection with Gaussian chaos in the next result.

A suitable application of the central limit theorem enables to obtain a corresponding bound for Gaussian chaos of order 2.

**Lemma 1.2.** Let \( T \) be a supremum of Gaussian chaos of order 2.

\[
T := \sup_{t \in T} \left| \sum_{i,j} t_{i,j} Y_i Y_j + \sum_i t_i Y_i^2 + t_\varnothing \right| ,
\]

where \( Y_1, \ldots, Y_N \) are independent standard Gaussian random variable. Then, for any \( x > 0 \),

\[
P \{ T \geq E[T] + x \} \leq \exp \left( -\frac{x^2}{E[D]^2 L_1} \wedge \frac{x}{E L_2} \right),
\]

where

\[ D := \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \sum_{i,j} Y_i (1 + \delta_{i,j}) \alpha_j t_{i,j} , \]

\[ E := \sup_{t \in T} \sup_{\alpha_1, \|\alpha_1\|_2 \leq 1} \sup_{\alpha_2, \|\alpha_2\|_2 \leq 1} \sum_{i,j} \alpha_1, \alpha_2, t_{i,j} (1 + \delta_{i,j}) . \]

The proof of this Lemma is postponed to the end of this section. To conclude, we derive the result of Proposition 8.1 from this last lemma. For any matrix \( R \in F \), we define the vector \( t^R \in \mathbb{R}^{nr(nr+1)/2+1} \) indexed by \( \mathcal{I}_{nr} \) as follows

\[
t^R_{\{i,k,(j,l)\}} := \delta_{k,l}(2 - \delta_{i,j}) \frac{R_{i,j}}{n}, \quad t^R_{\{i,(k,l)\}} := \frac{R_{i,j}}{n}, \quad \text{and} \quad t^R_\varnothing := -tr(R) ,
\]

where \( \delta_{i,j} \) is the indicator function of \( i = j \). In order to apply Lemma 1.2 with \( N = nr \) and \( T = \{ t^R | R \in F \} \), we have to work out the quantities \( D \) and \( E \).
Applying Cauchy-Schwarz identity yields
\[ D^2 = \frac{4}{n^2} \sup_{R \in F} \left\{ \sum_{k=1}^{n} \sum_{j=1}^{r} \left( \sum_{i=1}^{r} Y_{[i,k]} R_{[i,j]} \right)^2 \right\} = \frac{4}{n} \sup_{R \in F} \text{tr}(R Y Y^* R^*) . \] (4)

Let us now turn the constant \( E \)
\[ E = \sup_{t \in T} \sup_{\alpha_1, \alpha_2 \in \mathbb{R}^{nr}} \sum_{1 \leq i,j \leq r} \sum_{1 \leq k,l \leq n} (1 + \delta_{ij} \delta_{k,l}) t_{i,j} R_{i,j} \alpha_1^k \alpha_2^l \]
\[ \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1 \]

From this last expression, it follows that \( E \) is a supremum of \( L_2 \) operator norms
\[ E = \frac{2}{n} \sup_{R \in F} \varphi_{\text{max}} \left( \text{Diag}^{(n)}(R) \right) , \]
where \( \text{Diag}^{(n)}(R) \) is the \((nr \times nr)\) block diagonal matrix such that each diagonal block is made of the matrix \( R \). Since the largest eigenvalue of \( \text{Diag}^{(n)}(R) \) is exactly the largest eigenvalue of \( R \), we get
\[ E = \frac{2}{n} \text{sup}_{R \in F} \varphi_{\text{max}}(R) . \] (5)

Applying Proposition 1.2 and gathering identities (4) and (5) yields
\[ \mathbb{P}(Z \geq \mathbb{E}(Z) + t) \leq \exp \left[ - \left( \frac{t^2}{L_1 \mathbb{E}(V)} \wedge \frac{t}{L_2 B} \right) \right] , \]
where \( B = E \) and \( V = D^2 \).

**Proof of Lemma 1.1.** This result is an extension of Corollary 4 in [3]. We shall closely follow the sketch of their proof adapting a few arguments. First, we upper bound the moments of \((T - \mathbb{E}(T))_+\). Then, we derive the deviation inequality from it. Here, \( x_+ = \max(x, 0) \).

**Lemma 1.3.** For all real numbers \( q \geq 2 \),
\[ \|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{L q \mathbb{E}(D)} + L q E , \] (6)
where \( \|T\|_q^q \) stands for the \( q \)-th moment of the random variable \( T \). The quantities \( D \) and \( E \) are defined in Lemma 1.1.
By Lemma 1.3, for any $t \geq 0$ and any $q \geq 2$,
\[
P(T \geq \mathbb{E}(T) + t) \leq \frac{\mathbb{E}[(T - \mathbb{E}(T))^q]}{t^q} \leq \left( \frac{\sqrt{tqE(D)} + LqE}{t} \right)^q .
\]

The right-hand side is at most $2^{-q}$ if $\sqrt{tqE(D)} \leq t/4$ and $LqE \leq t/4$. Let us set
\[
q_0 := \frac{t^2}{16LE(D)^2} \wedge \frac{t}{4LE}.
\]

If $q_0 \geq 2$, then $\mathbb{P}(T \geq \mathbb{E}(T) + t) \leq 2^{-q_0}$. On the other hand if $q_0 < 2$, then
\[
4 \times 2^{1-q_0} \geq 1.
\]
It follows that
\[
P(T \geq \mathbb{E}(T) + t) \leq 4 \exp \left( -\log(2) \left[ \frac{t^2}{4E(D)^2} \wedge \frac{t}{E} \right] \right).
\]

\begin{proof}[Proof of Lemma 1.3] This result is based on the entropy method developed in [3]. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function such that $T = f(U_1, \ldots, U_N)$. In the sequel, $U_1', \ldots, U_N'$ denote independent copies of $U_1, \ldots, U_N$. The random variable $T_i'$ and $V^+$ are defined by

\[
T_i' := f(U_1, \ldots, U_{i-1}, U_i', U_{i+1}, \ldots, U_N),
\]
\[
V^+ := \mathbb{E} \left[ \sum_{i=1}^N (T - T_i')^2 | U_1^N \right],
\]

where $U_1^N$ refers to the set $\{U_1, \ldots, U_N\}$. Theorem 2 in [3] states that for any real $q \geq 2$,
\[
\| (T - \mathbb{E}(T))_+ \|_q \leq \sqrt{Lq} \| \sqrt{V^+} \|_q.
\]

To conclude, we only have bound the moments of $\sqrt{V^+}$. By definition,
\[
T = \sup_{t \in \mathcal{T}} \left| \sum_{(i,j)} U_iU_jt_{(i,j)} + \sum_{i=1}^N t_{(i)} + t_Z \right| .
\]

Since the set $\mathcal{T}$ is compact, this supremum is achieved almost surely at an element $t^0$ of $\mathcal{T}$. For any $1 \leq i \leq N$,
\[
(T - T_i')^2 \leq \left( (U_i - U_i') \sum_{j \neq i} U_jt^0_{\{i,j\}} \right)^2 .
\]
Gathering this bound for any $i$ between 1 and $N$, we get
\[
V^+ \leq \sum_{i=1}^{N} \mathbb{E} \left[ \left( (U_i - U'_i) \sum_{j \neq i} U_j t_0^{0} \{i,j\} \right)^2 \right] U_1^N
\]
\[
\leq 2 \sum_{i=1}^{N} \left( \sum_{j \neq i} U_j t_0^{0} \{i,j\} \right)^2
\]
\[
\leq 2 \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left[ \sum_{i=1}^{N} \alpha_i \left( \sum_{j \neq i} t_0^{0} \{i,j\} U_j \right)^2 \right]
\]
\[
\leq 2 \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left[ \sum_{i=1}^{N} U_i \left( \sum_{j \neq i} \alpha_j t_{(i,j)} \right) \right]^2 = 2D^2 .
\]
Combining this last bound with (7) yields
\[
\|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{Lq\sqrt{2}} \|D\|_q
\]
\[
\leq \sqrt{Lq} \left[ \mathbb{E}(D) + \|D - \mathbb{E}(D)\|_q \right] .
\]
(8)

Since the random variable $D$ defined in Lemma 1.1 is a measurable function $f_2$ of the variables $U_1, \ldots, U_N$, we apply again Theorem 2 in [3].
\[
\|(D - \mathbb{E}(D))_+\|_q \leq \sqrt{Lq} \left[ \sqrt{V_2^+} \right]_q ,
\]
where $V_2^+$ is defined by
\[
V_2^+ := \mathbb{E} \left[ \sum_{i=1}^{N} (D - D'_i)^2 \right] U_i^N ,
\]
and $D'_i := f_2(U_1, \ldots, U_{i-1}, U'_i, U_{i+1}, \ldots, U_N)$. As previously, the supremum in $D$ is achieved at some random parameter $(t^0, \alpha^0)$. We therefore upper bound $V_2^+$ as previously.
\[
V_2^+ \leq \sum_{i=1}^{N} \mathbb{E} \left[ \left( (U_i - U'_i) \left( \sum_{j \neq i} \alpha_j^0 t_{(i,j)} \right) \right)^2 \right] U_i^N
\]
\[
\leq 2 \sum_{i=1}^{N} \left( \sum_{j \neq i} \alpha_j^0 t_{(i,j)} \right)^2
\]
\[
\leq 2 \sup_{\alpha^0 \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left( \sum_{i=1}^{N} \sum_{j \neq i} \alpha_j^0 t_{(i,j)} \right)^2 = 2E^2 .
\]
Gathering this upper bound with (8) yields
\[
\|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{Lq\mathbb{E}(D) + LqE} .
\]
Proof of Lemma 1.2. We shall apply the central limit theorem in order to transfer results for Rademacher chaos to Gaussian chaos. Let \( f \) be the unique function satisfying \( T = f(y_1, \ldots, y_N) \) for any \((y_1, \ldots, y_N) \in \mathbb{R}^N\). As the set \( T \) is compact, the function \( f \) is known to be continuous. Let \((U^{(j)}_i)_{1 \leq i \leq N, j \geq 0}\) an i.i.d. family of Rademacher variables. For any integer \( n > 0 \), the random variables \( Y^{(n)} \) and \( T^{(n)} \) are defined by

\[
Y^{(n)} := \left( \sum_{j=1}^{n} \frac{U^{(j)}_i}{\sqrt{n}}, \ldots, \sum_{j=1}^{n} \frac{U^{(j)}_N}{\sqrt{n}} \right),
\]

\[
T^{(n)} := f \left( Y^{(n)} \right).
\]

Clearly, \( T^{(n)} \) is a supremum of Rademacher chaos of order 2 with \( nN \) variables and a constant term. By the central limit theorem, \( T^{(n)} \) converges in distribution as \( n \) tends to infinity. Consequently, deviation inequalities for the variables \( T^{(n)} \) transfer to \( T \) as long as the quantities \( E \left[ D^{(n)}(T) \right], E^{(n)}, \) and \( E[T^{(n)}] \) converge.

We first prove that the sequence \( T^{(n)} \) converges in expectation towards \( T \). As \( T^{(n)} \) converges in distribution, it is sufficient to show that the sequence \( T^{(n)} \) is asymptotically uniformly integrable. The set \( T \) is compact, thus there exists a positive number \( t_\infty \) such that

\[
T^{(n)} \leq t_\infty \left[ \sum_{i,j} |Y^{(n)}_i Y^{(n)}_j| + 1 \right]
\]

\[
\leq t_\infty \left[ 1 + (N + 1)/2 \sum_{i=1}^{N} \left( Y^{(n)}_i \right)^2 \right].
\]

It follows that

\[
\left( T^{(n)} \right)^2 \leq t_\infty^2 \left( \frac{N + 1}{2} \right)^2 \frac{N + 2}{2} \left[ 1 + \sum_{i=1}^{N} \left( Y^{(n)}_i \right)^4 \right]. \tag{9}
\]

The sequence \( Y^{(n)}_i \) does not only converge in distribution to a standard normal distribution but also in moments (see for instance [1] p.391). It follows that \( \lim_{n \to \infty} E \left[ (T^{(n)})^2 \right] \leq \infty \) and the sequence \( f \left( Y^{(n)} \right) \) is asymptotically uniformly integrable. As a consequence,

\[
\lim_{n \to \infty} E \left[ T^{(n)} \right] = E[T].
\]

Let us turn to the limit of \( E \left[ D^{(n)} \right] \). As the variable \( T^{(n)} \) equals

\[
T^{(n)} = \sup_{t \in T} \left| \sum_{i,j} t_{i,j} \sum_{1 \leq k, l \leq n} \frac{U^{(k)}_i U^{(l)}_j}{n} + \sum_{i} t_i \sum_{1 \leq k \leq n} \frac{U^{(k)}_i}{\sqrt{n}} \sum_{l \neq k} \frac{U^{(l)}_i}{\sqrt{n}} + t_0 + \sum_{i} t_i \right|,
\]
it follows that

\[ D^{(n)} = \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \left| \sum_{1 \leq i \leq N} \sum_{1 \leq k \leq n} U_i^{(k)} \left\{ \sum_{j \neq i} \frac{t(i,j)}{n} \sum_{1 \leq l \leq n} \alpha_{j,l}^{(l)} + 2 \sum_{l \neq k} \frac{t(i,l)}{n} \alpha_{i,k}^{(l)} \right\} \right| \]

\[ \leq \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \left\{ \sum_{i} \frac{U_i^{(i)}}{\sqrt{n}} \sum_{j} (1 + \delta_{i,j}) \frac{t(i,j)}{\sqrt{n}} \sum_{1 \leq l \leq n} \alpha_{j,j}^{(l)} \right\} + A^{(n)}, \tag{10} \]

where the random variable \( A^{(n)} \) is defined by

\[ A^{(n)} := \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \sum_{i=1}^{N} \sum_{j=1}^{n} t(i) \frac{U_i^{(j)}}{n} \alpha_{i,j}^2. \]

Straightforwardly, one upper bounds \( A^{(n)} \) by \( t_\infty / n \sqrt{\sum_{i=1}^{nN} \sum_{j=1}^{n} \left( U_i^{(j)} \right)^2} \) and its expectation satisfies

\[ \mathbb{E} \left( \left| A^{(n)} \right| \right) \leq t_\infty \sqrt{\frac{N}{n}}, \]

which goes to 0 when \( n \) goes to infinity. Thus, we only have to upper bound the expectation of the first term in (10). Clearly, the supremum is achieved only when for all \( 1 \leq j \leq N \), the sequence \( (\alpha_{j,j}^{(1)})_{1 \leq l \leq n} \) is constant. In such a case, the sequence \( (\alpha_{j,j}^{(1)})_{1 \leq j \leq N} \) satisfies \( \|\alpha^{(1)}\|_2 \leq 1 / \sqrt{n} \). It follows that

\[ \mathbb{E} \left[ D^{(n)} \right] = \mathbb{E} \left\{ \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \mathbb{E} \left[ \sum_{i} Y_i^{(n)} \sum_{j} (1 + \delta_{i,j}) \alpha_{j,j} \right] \right\} + O \left( \frac{1}{\sqrt{n}} \right). \]

Let \( g \) be the function defined by

\[ g(y_1, \ldots, y_N) = \sup_{t \in T} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left[ \sum_{i} y_i \sum_{j} (1 + \delta_{i,j}) \alpha_{j,j} \right], \]

for any \((y_1, \ldots, y_N) \in \mathbb{R}^N\). The function \( g(\cdot) \) is measurable and continuous as the supremum is taken over a compact set. As a consequence, \( g(Y^{(n)}) \) converges in distribution towards \( g(Y) \). As previously, the sequence is asymptotically uniformly integrable since its moment of order 2 is uniformly upper bounded. It follows that \( \lim \mathbb{E} \left[ D^{(n)} \right] = \mathbb{E} \left[ D \right] \).

Third, we compute the limit of \( E^{(n)} \). By definition,

\[ E^{(n)} = \sup_{t \in T} \sup_{\alpha_1, \alpha_2 \in \mathbb{R},, \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1} \sum_{i=1}^{n} \sum_{k=1}^{N} \alpha_{i,k}^{(1)} \left[ \sum_{j \neq i} \sum_{l=1}^{n} \alpha_{j,l}^{(l)} \frac{t(i,j)}{n} + 2 \sum_{l \neq k} \alpha_{i,k}^{(l)} \frac{t(i,l)}{n} \right] \]

\[ = \sup_{t \in T} \sup_{\alpha_1, \alpha_2, \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1} \sum_{i=1}^{n} \sum_{j=1}^{N} (1 + \delta_{i,j}) \frac{t(i,j)}{n} \left[ \sum_{k=1}^{N} \sum_{l=1}^{n} \alpha_{1,k}^{(k)} \alpha_{2,j}^{(l)} \right] + O \left( \frac{1}{n} \right). \]
As for the computation of $D^{(n)}$, the supremum is achieved when the sequences
\((\alpha_{1,k}^1)_{1 \leq k \leq n}\) and \((\alpha_{2,l}^2)_{1 \leq l \leq n}\) are constant for any \(i \in \{1, \ldots, N\}\). Thus, we only have to consider the supremum over the vectors $\alpha_1$ and $\alpha_2$ in $\mathbb{R}^N$.

$$E^{(n)} = \sup_{t \in T} \sup_{\alpha_1, \alpha_2 \in \mathbb{R}^N} \|\alpha_i\|_2 \leq 1 \cdot \frac{1}{n} \sum_{i=1}^{N} \sum_{j=1}^{N} (1 + \delta_{ij}) t_{i,j} \alpha_{1,i} \alpha_{2,j} + O\left(\frac{1}{n}\right).$$

It follows that $E^{(n)}$ converges towards $E$ when $n$ tends to infinity.

The random variable $T^{(n)} - E(T^{(n)})$ converges in distribution towards $T - E(T)$.

By Lemma 1.1,

$$P(T - E(T) \geq x) \leq \lim \exp \left(-\frac{x^2}{E[D^{(n)}]^2 L_1} \wedge \frac{x}{E^{(n)} L_2}\right),$$

for any $x > 0$. Combining this upper bound with the convergence of the sequences $D^{(n)}$ and $E^{(n)}$ allows to conclude. \(\square\)

2. Proof of Theorem 3.1

**Proof of Lemma 8.3.** We only consider here the anisotropic case, since the isotropic case is analogous. This result is based on the deviation inequality for suprema of Gaussian chaos of order 2 stated in Proposition 8.1. For any model $m'$ belonging to $\mathcal{M}$, we shall upper bound the quantities $\mathbb{E}(Z_{m'})$, $B_{m'}$, and $\mathbb{E}(W_{m'})$ defined in (42) in [6].

1. Let us first consider the expectation of $Z_{m'}$. Let $U'_{m,m'}$ be the new vector space defined by

$$U'_{m,m'} := U_{m,m'} \sqrt{D \Sigma} \frac{1}{\sqrt{p}},$$

where $U_{m,m'}$ is introduced in the proof of Lemma 8.2 in [6]. This new space allows to handle the computation with the canonical inner product in the space of matrices. Let $B^{(2)}_{m',m',m',m'^2}$ be the unit ball of $U'_{m,m'}$ with respect to the canonical inner product. If $R$ belongs to $U_{m,m'}$, then $\|R\|_{H'} = \|R\sqrt{D \Sigma}/p\|_F$, where $\|.|_F$ stands for the Frobenius norm.

$$Z_{m'} = \sup_{R \in B^{(2)}_{m',m'^2}} \frac{1}{\sqrt{p}} \text{tr} \left[RD \Sigma (YY^* - I_{p^2})\right]$$

$$= \sup_{R \in B^{(2)}_{m',m'^2}} \text{tr} \left[R \sqrt{D \Sigma} \frac{1}{p} (YY^* - I_{p^2})\right] \quad (11)$$

$$= \left\|PU'_{m,m'} \sqrt{D \Sigma} \frac{1}{p} (YY^* - I_{p^2})\right\|_F,$$
where \( \Pi_{m',m'} \) refers to the orthogonal projection with respect to the canonical inner product onto the space \( U_{m',m'} \). Let \( F_1, \ldots, F_{d_{m^2,m^2}} \) denote an orthonormal basis of \( U_{m',m'} \).

\[
E(Z_m') = \sum_{i=1}^{d_{m^2,m^2}} \mathbb{E} \left[ tr^2 \left( F_i \sqrt{\frac{D_\Sigma}{p^2}} (YY^* - I_{p^2}) \right) \right]
\]

\[
= \sum_{i=1}^{d_{m^2,m^2}} \mathbb{E} \left[ \sum_{j=1}^{p^2} F_i(j,j) \frac{\sqrt{D_\Sigma(j,j)}}{p} (YY^*_{j,j} - 1) \right]^2
\]

\[
= \sum_{i=1}^{d_{m^2,m^2}} \frac{2}{np^2} tr(F_i D_\Sigma F_i)
\]

\[
\leq \sum_{i=1}^{d_{m^2,m^2}} \frac{2 \varphi_{\max}(D_\Sigma)}{np^2} = \frac{2 d_{m^2,m^2} \varphi_{\max}(\Sigma)}{np^2}.
\]

Applying Cauchy-Schwarz inequality, it follows that

\[
E(Z_m') \leq \sqrt{\frac{2d_{m^2,m^2} \varphi_{\max}(\Sigma)}{np^2}}.
\]

(12)

2. Using the identity (11), the quantity \( B_m' \) equals

\[
B_m' = \frac{2}{n} \sup_{R \in B^{(2)}_{m^2,m^2}} \varphi_{\max} \left( R \frac{\sqrt{D_\Sigma}}{p} \right).
\]

As the operator norm is under-multiplicative and as it dominates the Frobenius norm, we get the following bound

\[
B_m' \leq \frac{2 \sqrt{\varphi_{\max}(\Sigma)}}{np}.
\]

(13)

3. Let us turn to bounding the quantity \( E(W_m') \). Again, by introducing the ball \( B^{(2)}_{m^2,m^2} \), we get

\[
W_m' = \frac{4}{n} \sup_{R \in B^{(2)}_{m^2,m^2}} \frac{1}{p^2} tr \left[ R \overline{YY}^* D_\Sigma R \right]
\]

\[
\leq \frac{4 \varphi_{\max}(\Sigma)}{np^2} \sup_{R \in B^{(2)}_{m^2,m^2}} tr \left[ R \overline{YY}^* R \right]
\]

\[
\leq \frac{4 \varphi_{\max}(\Sigma)}{np^2} \left( 1 + \sup_{R \in B^{(2)}_{m^2,m^2}} tr \left[ R (YY^* - I_{p^2}) R \right] \right).
\]
Let $F_1, \ldots, F_{d_{m^2,m'^2}}$ an orthonormal basis of $U_{m,m'}$, and let $\lambda$ be a vector in $\mathbb{R}^{d_{m^2,m'^2}}$. We write $\|\lambda\|_2$ for its $L_2$ norm.

$$
E\left( \sup_{R \in B_{m^2,m'^2}(\mathbb{R})} \operatorname{tr} \left[ R (Y Y^* - I_{p^2}) R^* \right] \right)
= E\left( \sup_{\|\lambda\|_2 \leq 1} \sum_{i,j=1}^{d_{m^2,m'^2}} \lambda_i \lambda_j \operatorname{tr} \left[ F_i F_j (Y Y^*/n - I_{p^2}) \right] \right)^2
\leq \sum_{i,j=1}^{d_{m^2,m'^2}} E \left( \operatorname{tr} \left[ F_i F_j (Y Y^*/n - I_{p^2}) \right] \right)^2.
$$

The second inequality is a consequence of Cauchy-Schwarz inequality in $\mathbb{R}^{d_{m^2,m'^2}^2}$ since the $L_2$ norm of the vector $(\lambda_i \lambda_j)_{1 \leq i,j \leq d_{m^2,m'^2}} \in \mathbb{R}^{d_{m^2,m'^2}}$ is bounded by $1$. Since the matrices $F_i$ are diagonal, we get

$$
E\left( \sup_{R \in B_{m^2,m'^2}(\mathbb{R})} \operatorname{tr} \left[ R (Y Y^*/n - I_{p^2}) R^* \right] \right) \leq \frac{1}{n} \sum_{i,j=1}^{d_{m^2,m'^2}} \|F_i F_j\|_2^2.
$$

It remains to bound the norm of the products $F_i F_j$ for any $i, j$ between 1 and $d_{m^2,m'^2}$.

$$
\sum_{i,j=1}^{d_{m^2,m'^2}} \|F_i F_j\|_2^2 = \sum_{i,j=1}^{d_{m^2,m'^2}} \sum_{k=1}^{p^2} F_i[k,k]^2 F_j[k,k]^2 = \sum_{k=1}^{p^2} \left( \sum_{i=1}^{d_{m^2,m'^2}} F_i[k,k]^2 \right)^2.
$$

For any $k \in \{1, \ldots, p^2\}$, $\sum_{j=1}^{d_{m^2,m'^2}} F_i[k,k]^2 \leq 1$ since $(F_1, \ldots, F_{d_{m^2,m'^2}})$ form an orthonormal family. Hence, we get

$$
\sum_{i,j=1}^{d_{m^2,m'^2}} \|F_i F_j\|_2^2 \leq \sum_{k=1}^{p^2} \sum_{i=1}^{d_{m^2,m'^2}} F_i[k,k]^2 = d_{m^2,m'^2}.
$$

All in all, we have proved that

$$
E(W_{m'}) \leq \frac{4\varphi_{\max}(\Sigma)}{np^2} \left[ 1 + \sqrt{\frac{2d_{m^2,m'^2}}{n}} \right]. \quad \text{(14)}
$$

Gathering these three bounds and applying Proposition 8.1 allows to obtain the following deviation inequality:

$$
P\left( Z_{m'} \geq \sqrt{\frac{2\varphi_{\max}(\Sigma)}{n}} \left\{ \sqrt{1 + \alpha/2} \sqrt{d_{m^2,m'^2}} + \xi \right\} \right)
\leq \exp \left\{ - \frac{F_{\varphi_{\max}(\Sigma)}(1 + \sqrt{2d_{m^2,m'^2}/n})}{2L_1(1 + \sqrt{2d_{m^2,m'^2}/n})} \right\}
\leq \exp \left\{ - \frac{\rho_{\varphi_{\max}(\Sigma)}(\sqrt{d_{m^2,m'^2}}, m')}{2L_1(1 + \sqrt{2d_{m^2,m'^2}/n})} \right\} \left[ \frac{\xi_{\varphi_{\max}(\Sigma)}}{\sqrt{2L_2}} \right],
$$

where

$$
\rho_{\varphi_{\max}(\Sigma)}(\sqrt{d_{m^2,m'^2}}, m') = \frac{\varphi_{\max}(\Sigma)}{\sqrt{2L_2}} \left\{ \sqrt{1 + \alpha/2} \sqrt{d_{m^2,m'^2}} + \xi \right\}.
$$
where \( \omega_{m,m'} = \left( \sqrt{1 + \alpha/2} - 1 \right) \sqrt{d_{m^2,m'^2}} \). As \( n \) and \( d_{m^2,m'^2} \) are larger than one, there exists a universal constant \( L'_2 \) such that

\[
\left( \frac{\sqrt{1 + \alpha/2} - 1}{2L_1 (1 + \sqrt{2d_{m^2,m'^2}/n})} \right)^{\sqrt{n}/d_{m^2,m'^2}} \sqrt{2L_2} \geq 4L'_2 \sqrt{d_{m^2,m'^2}} \left( \frac{\sqrt{1 + \alpha/2} - 1}{\sqrt{1 + \alpha/2}} \right)^{d_{m^2,m'^2}} \left( \sqrt{1 + \alpha/2} - 1 \right).
\]

Since the vector space \( U_{m,m'} \) contains all the matrices \( D(\theta') \) with \( \theta' \) belonging to \( m', d_{m^2,m'^2} \) is larger than \( d_{m'} \). Besides, by concavity of the square root function, it holds that \( \sqrt{1 + \alpha/2} - 1 \geq \alpha \sqrt{1 + \alpha/2} - 1 \). Setting \( L'_1 := \left[ 4L_1 (1 + \sqrt{2}) \right]^{-1} \wedge \sqrt{2L_2}^{-1} \) and arguing as previously leads to

\[
\frac{\xi(\sqrt{1 + \alpha/2} - 1)}{L_1 (1 + \sqrt{2d_{m^2,m'^2}/n})} \geq L'_1 \xi \left[ \frac{\alpha}{\sqrt{1 + \alpha/2}} \wedge \alpha \sqrt{n} \right] \cdot
\]

Gathering these two inequalities allows us to conclude that

\[
\mathbb{P} \left( Z_{m'} \geq \frac{2\varphi_{\max}(\Sigma)}{n} \left\{ (1 + \alpha/2) d_{m^2,m'^2} + \xi \right\} \right) \leq \exp \left\{ -L'_2 \sqrt{d_{m'}} \left( \frac{\alpha}{\sqrt{1 + \alpha/2}} \wedge \frac{\alpha^2}{1 + \alpha/2} \right) - L'_1 \xi \left[ \frac{\alpha}{\sqrt{1 + \alpha/2}} \wedge \sqrt{n} \right] \right\}.
\]

Proof of Lemma 8.4 in \([6]\). The approach falls in two parts. First, we relate the dimensions \( d_m \) and \( d_{m^2} \) to the number of nodes of the torus \( \Lambda \) that are closer than \( r_m \) or \( 2r_m \) to the origin \((0,0)\). We recall that the quantity \( r_m \) is introduced in Definition 2.1 of \([6]\). Second, we compute a nonasymptotic upper bound of the number of points in \( \mathbb{Z}^2 \) that lie in the disc of radius \( r \). This second step is quite tedious and will only give the main arguments.

Let \( m \) be a model of the collection \( M_1 \). By definition, \( m \) is the set of points lying in the disc of radius \( r_m \) centered on \((0,0)\). Hence,

\[
\Theta_m = \text{vect} \left\{ \Psi_{i,j}, \ (i,j) \in m \right\},
\]

where the matrices \( \Psi_{i,j} \) are defined by Eq. (14) in \([6]\). As \( \Psi_{i,j} = \Psi_{-i,-j} \), the dimension \( d_m \) of \( \Theta_m \) is exactly the number of orbits of \( m \) under the action of the central symmetry \( s \).

As \( d_{m^2} \) is defined as the dimension of the space \( U_m \), it also corresponds to the dimension of the space

\[
\text{vect} \left\{ C(\theta), \theta \in \Theta_m \right\} + \text{vect} \left\{ C(\theta)^2, \theta \in \Theta_m \right\},
\]

(15)
which is clearly in one to one correspondence with \(U_m\). Straightforward computations lead to the following identity:

\[
C(\Psi_{i_1,j_1})C(\Psi_{i_2,j_2}) = C(\Psi_{i_1+i_2,j_1+j_2})\left[1 + s_{i_1+i_2,j_1+j_2}\right] + C(\Psi_{i_1-i_2,j_1-j_2})\left[1 + s_{i_1-i_2,j_1-j_2}\right],
\]

where \(s_{x,y}\) is the indicator function of \(x = -x\) and \(y = -y\) in the torus \(\Lambda\).

Combining this property with the definition of \(\Theta_m\), we embed the space \((15)\) in the space

\[
\text{vect} \{C(\Psi_{i_1+i_2,j_1+j_2}), (i_1,j_1), (i_2,j_2) \in m \cup \{(0,0)\}\},
\]

and this last space is in one to one correspondence with

\[
\text{vect} \{C(\Psi_{i_1+i_2,j_1+j_2}), (i_1,j_1), (i_2,j_2) \in m \cup \{(0,0)\}\}. \tag{16}
\]

In the sequel, \(\mathcal{N}(m)\) stands for the set

\[
\{(i_1+i_2,j_1+j_2), (i_1,j_1), (i_2,j_2) \in m \cup \{(0,0)\}\}.
\]

Thus, the dimension \(d_{m^2}\) is smaller or equal to the number of orbits of \(\mathcal{N}(m)\) under the action of the symmetry \(s\).

To conclude, we have to compare the number of orbits in \(m\) and the number of orbits in \(\mathcal{N}(m)\). We distinguish two cases depending whether \(2r_m + 1 \leq p\) or \(2r_m + 1 > p\). First, we assume that \(2r_m + 1 \leq p\). For such values the disc of radius \(r_m\) centered on the points \((0,0)\) in not overlapping itself on the torus except on a set of null Lebesgue measure. In the sequel, \([x]\) refers to the largest integer smaller than \(x\). We represent the orbit space of \(m\) as in Figure 1. To any of these points, we associate a square of size 1. If we add \(2 + 2[r_m]\) squares to

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The black dots represent the orbit space of \(m\) and the white dots represent the remaining points of the orbit space of \(\mathcal{N}(m)\).}
\end{figure}
the \( d_m \) first squares, we remark that the half disc centered on \((0, 0)\) and with
length \( r_m \) is contained in the reunion of these squares. Then, we get
\[
d_m + 2 + 2 |r_m| \geq \frac{\pi r_m^2}{2}. \tag{17}
\]

The points in \( \mathcal{N}(m) \) are closer than \( 2r_m \) from the origin. Consequently, all the
squares associated to representants of \( \mathcal{N}(m) \) are included in the disc of radius
\( 2r_m + \sqrt{2} \).
\[
d_m^2 + 2 + 2 |2r_m| \leq \frac{\pi}{2} \left( 2r_m + \sqrt{2} \right)^2.
\]

Combining these two inequalities, we are able to upper bound \( d_m^2 \)
\[
2 + 2 |2r_m| + d_m^2 \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 (d_m + 1 + 2 |r_m|),
\]
\[
d_m^2 \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 d_m + 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 (1 + 2 |r_m|).
\]

Applying again inequality (17), we upper bound \( r_m \):
\[
r_m \leq \frac{2}{\pi} \left[ 1 + \sqrt{1 + \frac{\pi}{2} (1 + d_m)} \right].
\]

Gathering these two last bounds yields
\[
d_m^2 \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 \left[ 1 + \frac{1}{d_m} \left( 1 + \frac{4}{\pi} \sqrt{1 + \frac{\pi}{2} (1 + d_m)} \right) \right] d_m.
\]

This upper bound is equivalent to \( 4d_m \), when \( d_m \) goes to infinity. Computing
the ratio \( d_m^2/d_m \) for every model \( m \) of small dimension allows to conclude.

Let us turn to the case \( 2r_m + 1 > p \). Suppose that \( p \) is larger or equal to
9. The lower bound (17) does not necessarily hold anymore. Indeed, the disc
is overlapping with itself because of toroidal effects. Nevertheless, we obtain a
similar lower bound by replacing \( r_m \) by \( (p-1)/2 \):
\[
d_m + 2 + 2 \left| \frac{p-1}{2} \right| \geq \frac{\pi (p-1)^2}{8}.
\]

The number of orbits of \( \Lambda \) under the action of the symmetry \( s \) is \((p^2 + 1)/2\)
if \( p \) is odd and \([(p+1)^2 - 1]/2\) if \( p \) is even. It follows that \( d_m^2 \leq [(p+1)^2 - 1]/2 \).

Gathering these two bounds, we get
\[
\frac{d_m^2}{d_m} \leq \frac{(p+1)^2}{\pi (p-1)^2/4 - 2(p+1)}.
\]

This last quantity is smaller than 4 for any \( p \geq 9 \). An exhaustive computation
of the ratios when \( p < 9 \) allows to conclude.
Let us turn to the isotropic case. Arguing as previously, we observe that the dimension $d_{iso}^m$ is the number of orbits of the set $m$ under the action of the group $G$ introduced in in [6] Sect.1.1 whereas $d_m^2$ is smaller or equal to the number of orbits of $\mathcal{N}^{iso}(m)$ under the action of $G$. As for anisotropic models, we choose represent these orbits on the torus and associate squares of size 1 (see Figure 2). Assuming that $r_m < (p - 1)/2$, we bound $d_m$ and $d_m^2$.

\[ d_m + 1 \geq \frac{1}{8} \pi r_m^2 + \frac{1}{2} \left\lfloor \frac{\sqrt{2} r_m}{2} \right\rfloor, \]

\[ d_m^2 \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2 r_m} \right\}^2 \left( \frac{1}{8} \pi r_m^2 + \frac{1}{2} \left\lfloor \sqrt{2} r_m \right\rfloor \right). \]

Gathering these two inequalities, we get

\[ d_m^2 \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2 r_m} \right\}^2 d_m. \]

As a consequence, $d_m^2$ is smaller than $4d_m$ when $d_m$ goes to infinity. As previously, computing the ratio $d_m^2/d_m$ for models $m$ of small dimension allows to conclude. The case $r_m > (p - 1)/2$ is handled as for the anisotropic case.

3. Proofs of the minimax bounds

Proof of Lemma 8.5 in [6]. This lower bound is based on an application of Fano’s approach. See [7] for a review of this method and comparisons with Le Cam’s and
Assouad’s Lemma. The proof follows three main steps: First, we upper bound the Kullback-Leibler entropy between distributions corresponding to \( \theta_1 \) and \( \theta_2 \) in the hypercube. Second, we find a set of points in the hypercube well separated with respect to the Hamming distance. Finally, we conclude by applying Birgè’s version of Fano’s lemma.

**Lemma 3.1.** The Kullback-Leibler entropy between two mean zero-Gaussian vectors of size \( p^2 \) with precision matrices \((I_{p^2} - C(\theta_1))/\sigma^2 \) and \((I_{p^2} - C(\theta_2))/\sigma^2 \) equals

\[
\mathcal{K}(\theta_1, \theta_2) = \frac{1}{2} \left[ \log \left( \frac{|I_{p^2} - C(\theta_1)|}{|I_{p^2} - C(\theta_2)|} \right) + \text{tr} \left( \left[ I_{p^2} - C(\theta_2) \right] \left[ I_{p^2} - C(\theta_1) \right]^{-1} - p^2 \right) \right],
\]

where for any square matrix \( A \), \( |A| \) refers to the determinant of \( A \).

This statement is classical and its proof is omitted. The matrices \((I_{p^2} - C(\theta_1))\) and \((I_{p^2} - C(\theta_2))\) are diagonalizable in the same basis since they are symmetric block circulant (Lemma A.1 in [6]). Transforming vectors of size \( p^2 \) into \( p \times p \) matrices, we respectively define \( \lambda_1 \) and \( \lambda_2 \) as the \( p \times p \) matrices of eigenvalues of \((I_{p^2} - C(\theta_1))\) and \((I_{p^2} - C(\theta_2))\). It follows that

\[
\mathcal{K}(\theta_1, \theta_2) = \frac{1}{2} \sum_{1 \leq i, j \leq p} \left( \frac{\lambda_2[i,j]}{\lambda_1[i,j]} - \log \left( \frac{\lambda_2[i,j]}{\lambda_1[i,j]} \right) - 1 \right).
\]

For any \( x > 0 \), the following inequality holds

\[
x - 1 - \log(x) \leq \frac{9}{64} \left( x - \frac{1}{x} \right)^2.
\]

It is easy to establish by studying the derivative of corresponding functions. As a consequence,

\[
\frac{\lambda_2[i,j]}{\lambda_1[i,j]} - \log \left( \frac{\lambda_2[i,j]}{\lambda_1[i,j]} \right) - 1 \leq \frac{9}{64} \left( \frac{\lambda_2[i,j]}{\lambda_1[i,j]} - \frac{\lambda_1[i,j]}{\lambda_2[i,j]} \right)^2 \\
\leq \frac{9}{64} \left( \frac{1}{\lambda_1[i,j]} + \frac{1}{\lambda_2[i,j]} \right)^2 \left( \lambda_1[i,j] - \lambda_2[i,j] \right)^2.
\]

Let us first consider the anisotropic case. Let \( m \) be a model in \( \mathcal{M}_1 \) and let \( \theta' \) belong \( \Theta_m \cap B_1(0_p, 1) \). We also consider a positive radius \( r \) such that \((1 - \|\theta'\|_1 - 2rd_m)\) is positive. For any \( \theta_1, \theta_2 \in C_m(\theta', r) \) the matrices \((I_{p^2} - C(\theta_1))\) and \((I_{p^2} - C(\theta_2))\) are diagonally dominant and their eigenvalues \( \lambda_1[i,j] \) and \( \lambda_2[i,j] \) are larger than \((1 - \|\theta'\|_1 - 2rd_m)\).

\[
\mathcal{K}(\theta_1, \theta_2) \leq \frac{9}{16(1 - \|\theta'\|_1 - 2rd_m)^2} \sum_{1 \leq i, j \leq p} (\lambda_1[i,j] - \lambda_2[i,j])^2 \\
\leq \frac{9}{16(1 - \|\theta'\|_1 - 2rd_m)^2} \|C(\theta_1) - C(\theta_2)\|^2_f \\
\leq \frac{9d_m^{1/2}p^2}{8(1 - \|\theta'\|_1 - 2rd_m)^2}.
\]
We recall that $\| \cdot \|_F$ refers to the Frobenius norm in the space of matrices.

Let us state Birgé’s version of Fano’s lemma [2] and a combinatorial argument known under the name of Varshamov-Gilbert’s lemma. These two lemma are taken from [4] and respectively correspond to Corollary 2.18 and Lemma 4.7.

**Lemma 3.2. (Birgé’s lemma)** Let $(S, d)$ be some pseudo-metric space and $\{ P_s, s \in S \}$ be some statistical model. Let $\kappa$ denote some absolute constant smaller than one. Then for any estimator $\hat{s}$ and any finite subset $T$ of $S$, setting $\delta = \min_{s, t \in T, s \neq t} d(s, t)$, provided that $\max_{s, t \in T} K(P_s, P_t) \leq \kappa \log |T|$, the following lower bound holds for every $p \geq 1$,

$$
\sup_{s \in S} E_s [d^p(s, \hat{s})] \geq 2^{-p} \delta^p (1 - \kappa) .
$$

**Lemma 3.3. (Varshamov-Gilbert’s lemma)** Let $\{0, 1\}^d$ be equipped with Hamming distance $d_H$. There exists some subset $\Phi$ of $\{0, 1\}^d$ with the following properties

$$
d_H (\phi, \phi') > d/4 \text{ for every } (\phi, \phi') \in \Phi^2 \text{ with } \phi \neq \phi' \text{ and } \log |\Phi| \geq \frac{d}{8} .
$$

Applying Lemma 3.2 with Hamming distance $d_H$ and the set $\Phi$ introduced in Lemma 3.3 yields

$$
\sup_{\theta \in \mathcal{C}_n(\theta', r)} E_\theta \left[ d_H (\hat{\theta}, \theta) \right] \geq \frac{d_m}{8} (1 - \kappa) ,
$$

provided that

$$
\frac{9d_m r^2 p^2}{8(1 - \| \theta' \|_1 - 2rd_m)^2} \leq \frac{\kappa d_m}{8} .
$$

Let us express (20) in terms of the Frobenius $\| \cdot \|_F$ norm.

$$
\sup_{\theta \in \mathcal{C}_n(\theta', r)} E_\theta \left[ \| C(\hat{\theta}) - C(\theta) \|_F^2 \right] \geq \frac{d_m r^2 p^2}{4} (1 - \kappa) .
$$

Since for every $\theta$ in the hypercube, $\sigma^{-2}(I_{p^2} - C(\theta))$ is diagonally dominant, its largest eigenvalue is smaller than $2\sigma^{-2}$. The loss function $l(\hat{\theta}, \theta)$ equals $\sigma^2/p^2 \text{tr}\{[C(\hat{\theta}) - C(\theta)][I - C(\theta)]^{-1}[C(\hat{\theta}) - C(\theta)]\}$. It follows that

$$
\sup_{\theta \in \mathcal{C}_n(\theta', r)} E_\theta \left[ l(\hat{\theta}, \theta) \right] \geq \sigma^2 \frac{d_m r^2}{8} (1 - \kappa) .
$$

Condition (21) is equivalent to $r^2 (1 - \| \theta' \|_1 - 2rd_m)^{-2} \leq \kappa/(9p^2 n)$. If we assume that

$$
r^2 \leq \frac{\kappa (1 - \| \theta' \|_1)^2}{18p^2 n} ,
$$
then $1 - \|\theta'\|_1 - 2rd_m \geq (1 - \|\theta'\|_1) \left(1 - 2d_m \sqrt{\kappa/(18np^2)}\right)$. This last quantity is larger than $(1 - \|\theta'\|_1) / \sqrt{2}$ if $d_m$ is smaller than $1.5(\sqrt{2} - 1) \sqrt{np^2/\kappa}$. Gathering inequality (22) and condition (23), we get the lower bound

$$\inf_{\tilde{\theta}} \sup_{\theta \in \text{Col}(C_m(\theta', r))} \mathbb{E}_{\tilde{\theta}} \left[ l \left( \tilde{\theta}, \theta \right) \right] \geq \inf_{\tilde{\theta}} \sup_{\theta \in C_m} \left( \theta', r \wedge (1 - \|\theta\|_1) \sqrt{np^2} \right) \mathbb{E}_{\tilde{\theta}} \left[ l \left( \tilde{\theta}, \theta \right) \right] \geq L \left( r^2 \wedge \frac{(1 - \|\theta\|_1)^2}{np^2} \right) d_m \sigma^2.$$

One handles models of dimension $d_m$ between $1.5(\sqrt{2} - 1) \sqrt{np^2/\kappa}$ and $\sqrt{np}$ by changing the constant $L$ in the last lower bound.

Let us turn to sets of isotropic GMRFs. The proof is similar to the non-isotropic case, except for a few arguments. Let $m$ belongs to the collection $\mathcal{M}_1$ and let $\theta'$ be an element of $\Theta_m^{iso} \cap B_1(0, r, 1)$. Let $\theta$ be such that $1 - \|\theta\|_1 - 8d_m^{iso}$ is positive. If $\theta_1$ and $\theta_2$ belong to the hypercube $C_m^{iso}(\theta', r)$, then

$$\mathcal{K}(\theta_1, \theta_2) \leq \frac{9d_m r^2 p^2}{2(1 - \|\theta\|_1 - 8d_m^{iso})^2}.$$

Applying Lemma 3.2 and 3.3, it follows that

$$\inf_{\tilde{\theta}} \sup_{\theta \in C_m^{iso}(\theta', r)} \mathbb{E}_{\tilde{\theta}} \left[ d_H \left( \tilde{\theta}, \theta \right) \right] \geq \frac{d_m^{iso}}{8} (1 - \kappa),$$

provided that $4.5d_m r^2 p^2 n (1 - \|\theta\|_1 - 8d_m^{iso})^2 \leq \kappa d_m^{iso} / 8$. As a consequence,

$$\inf_{\tilde{\theta}} \sup_{\theta \in C_m^{iso}(\theta', r)} \mathbb{E}_{\tilde{\theta}} \left[ l \left( \tilde{\theta}, \theta \right) \right] \geq \frac{d_m^{iso} r^2}{8} (1 - \kappa),$$

if $r^2 (1 - \|\theta\|_1 - 8d_m^{iso})^2 \leq \kappa (36p^2 n)^{-1}$. We conclude by arguing as in the isotropic case.

**Proof of lemma 8.6 in [6].** Let $m$ be a model in $\mathcal{M}_1$, $r$ be a positive number smaller than $1/(4d_m)$, and $\theta$ be an element of the convex hull of $C_m(0, r, 1)$. The covariance matrix of the vector $X^v\theta$ is $\Sigma = \sigma^2 [I - C(\theta)]^{-1}$. Since the field $X$ is stationary, $\text{Var}_\theta(X|0, 0)$ equals any diagonal element of $\Sigma$. In particular, $\text{Var}_\theta(X|0, 0)$ corresponds to the mean of the eigenvalues of $\Sigma$. The matrix $(I - C(\theta))$ is block circulant. As in the proof of Lemma 20, we note $\lambda$ the $p \times p$ matrix of the eigenvalues of $(L_{x^2} - C(\theta))$. By Lemma A.1 in [6],

$$\lambda_{i,j} = 1 + \sum_{(k,l) \in \Lambda} \theta_{[k,l]} \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right],$$
for any 1 \leq i, j \leq p. Since \( \theta \) belongs to the convex hull of \( C_m(0, p, r) \), \( \theta[k,l] \) is zero if \((k, l) \notin m\) and \(|\theta[k,l]| \leq r\) if \((k, l) \in m\). Thus \( \sum_{(k,l) \in \Lambda} |\theta[k,l]| \) is smaller than 1/2. Applying Taylor-Lagrange inequality, we get

\[
\frac{1}{1+x} \leq 1 - x + \frac{x^2}{(1 - |x|)^3},
\]

for any \( x \) between -1 and 1. It follows that

\[
\lambda[i,j]^{-1} \leq 1 - \sum_{k,l \in \Lambda} \theta[k,l] \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right] + 8 \left\{ \sum_{k,l \in \Lambda} \theta[k,l] \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right] \right\}^2.
\]

Summing this inequality for all \((i, j) \in \{1, \ldots, p\}^2\), the first order term turns out to be \( t^2 |C(\theta)|/p^2 \) which is zero whereas the second term equals \( 8t^2 |C(\theta)^2|/p^2 \). Since there are less than \( 2d_m \) non-zero terms on each line of the matrix \( C(\theta) \), its Frobenius norm is smaller than \( 2d_m r^2 \). Consequently, we obtain

\[
\text{Var}_\theta (X[0,0]) \leq \sigma^2 \left( 1 + 16d_m r^2 \right).
\]

Proof of Lemma 8.7 in [6]. This property seems straightforward but the proof is a bit tedious. Let \( i \) be a positive integer smaller than \( \text{Card}(M_1) \). By definition of the radius \( r_m \) in Equation (10) in [6], the model \( m_{i+1} \) is the set of nodes in \( \Lambda \setminus \{(0,0)\} \) at a distance smaller or equal to \( r_{m_{i+1}} \) from \( (0,0) \), whereas the model \( m_i \) only contains the points in \( \Lambda \setminus \{(0,0)\} \) at a distance strictly smaller than \( r_{m_{i+1}} \) from the origin.

Let us first assume that 2\( r_{m_{i+1}} \leq p \). In such a case, the disc centered on \((0,0)\) with radius \( r_{m_{i+1}} \) does not overlap with itself on the torus \( \Lambda \). To any node in the neighborhood \( m_{i+1} \) and to the node \((0,0)\), we associate the square of size 1 centered on it. All these squares do not overlap and are included in the disc of radius \( r_{m_{i+1}} + \sqrt{2}/2 \). Hence, we get the upper bound

\[
d_{m_{i+1}} + 1 \leq \pi (r_{m_{i+1}} + \sqrt{2}/2)^2.
\]

Similarly, the disc of radius \( r_{m_{i+1}} - \sqrt{2}/2 \) is included in the union of the squares associated to the nodes \( m_i \cup \{(0,0)\} \). It follows that \( 2d_m + 1 \) is larger or equal to \( \pi (r_{m_{i+1}} - \sqrt{2}/2)^2 \). Gathering these two inequalities, we obtain

\[
\frac{d_{m_{i+1}}}{d_m} \leq \frac{(r_{m_{i+1}} + \sqrt{2}/2)^2 - 1}{(r_{m_{i+1}} - \sqrt{2}/2)^2 - 1},
\]

if \( r_{m_{i+1}} \) is larger than \( 1 + \sqrt{2}/2 \). If \( r_{m_{i+1}} \) larger than 5, this upper bound is smaller than two. An exhaustive computation for models of small dimension allows to conclude.
If \(2r_{m+1} \geq p\) and \(2r_{m_1} < p\), then the preceding lower bound of \(d_{m_1}\) and the preceding upper bound of \(d_{m+1}\) still hold. Finally, let us assume that \(2r_{m_1} \geq p\).

Arguing as previously, we conclude that \(2d_{m_1} + 1 \geq \pi(p/2 - \sqrt{2}/2)^2\). The largest dimension of a model \(m \in M_1\) is \((p^2 - 1)/2\) if \(p\) is odd and \((p + 1)^2/2\) if \(p\) is even. Thus, \(d_{m+1} \leq \lfloor (p + 1)^2 - 3/2 \rfloor\). Gathering these two bounds yields

\[
\frac{d_{m+1}}{d_{m_1}} \leq 4\frac{(p + 1)^2 - 3}{(p - \sqrt{2})^2},
\]

which is smaller than 2 if \(p\) is larger than 10. Exhaustive computations for small \(p\) allow to conclude.

\[\square\]

**Proof of Proposition 6.7 in [6]**. This result derives from the upper bound of the risk of \(\tilde{\theta}_{p_1}\) stated in Theorem 3.1 and the minimax lower bound stated in Proposition 6.6 in [6].

Let \(\mathcal{E}(a)\) be a pseudo-ellipsoid that satisfies Assumption (H\(_a\)) and such that \(a_1^2 \geq 1/(np^2)\). For any \(\theta\) in \(\mathcal{E}(a) \cap B_1(0, 1) \cap \mathcal{U}(2)\), the penalty term satisfies \(\text{pen}(m) = K \sigma^2 \rho_2^2 d_m / np^2\) is larger than \(K d_m \sigma_{\max}(\Sigma)/np^2\). Applying Theorem 3.1, we upper bound the risk \(\tilde{\theta}_{p_1}\)

\[
\mathbb{E}_\theta \left[ l \left( \tilde{\theta}_{p_1}, \theta \right) \right] \leq L_1(K) \inf_{m \in M_1} \left[ l(\theta_{m_1}, \theta) + \text{pen}(m) \right] + L_2(K) \rho_2 \sigma^2 \frac{d_m}{np^2},
\]

for any \(\theta \in \mathcal{E}(a) \cap B_1(0, 1) \cap \mathcal{U}(2)\). It follows that

\[
\sup_{\theta \in \mathcal{E}(a) \cap \mathcal{E}(a) \cap \mathcal{U}(2)} \mathbb{E}_\theta \left[ l \left( \tilde{\theta}_{p_1}, \theta \right) \right] \leq L(K) \inf_{m \in M \cap \theta, \theta} \left[ l(\theta_{m}, \theta) + \rho_1^2 \rho_2 \sigma^2 \frac{d_m}{np^2} \right].
\]

Let \(i\) be a positive integer smaller or equal than \(\text{Card}(M_1)\). We know from Section 4.1 in [6] that the bias \(l(\theta_{m_i}, \theta)\) of the model \(m_i\) equals \(\text{Var}(X_{[0, 0]}|X_{m_i}) - \sigma^2\). Since \(\theta\) belongs to the set \(\mathcal{E}(a) \cap \mathcal{B}_1(0, 1, \cup \mathcal{U}(2))\), the bias term is smaller or equal to \(a_{i+1}^2\) with the convention \(a_{\text{Card}(M_i)+1}^2 = 0\). Hence, the previous upper bound becomes

\[
\mathbb{E}_\theta \left[ l \left( \tilde{\theta}_{p_1}, \theta \right) \right] \leq L(K) \inf_{1 \leq i \leq \text{Card}(M_1)} \left[ a_{i+1}^2 + \rho_1^2 \rho_2 \sigma^2 \frac{d_m}{np^2} \right]
\]

\[
\leq L(K, \rho_1, \rho_2) \inf_{1 \leq i \leq \text{Card}(M_1)} \left[ a_{i+1}^2 + \rho_1^2 \rho_2 \sigma^2 \frac{d_m}{np^2} \right].
\]

(24)

Applying Proposition 6.6 in [6] to the set \(\mathcal{E}(a) \cap \mathcal{B}_1(0, 1) \cap \mathcal{U}(2)\), we get

\[
\inf_{\tilde{\theta}} \sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0, 1) \cap \mathcal{U}(2)} \mathbb{E}_\theta \left[ l \left( \tilde{\theta}, \theta \right) \right] \geq \inf_{\tilde{\theta}} \sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0, 1) \cap \mathcal{U}(2)} \mathbb{E}_\theta \left[ l \left( \tilde{\theta}, \theta \right) \right]
\]

\[
\geq L \sup_{1 \leq i \leq \text{Card}(M_1)} \left( a_{i+1}^2 + \sigma^2 \frac{d_m}{np^2} \right). 
\]
Let us define $i^*$ by
\[
i^* := \sup \left\{ 1 \leq i \leq \text{Card}(\mathcal{M}_1), \ a_i^2 \geq \frac{\sigma^2 d_m}{np^2} \right\},
\]
with the convention $\sup \emptyset = 0$. Since $a_{i^*}^2 \geq \frac{\sigma^2}{np^2}$, $i^*$ is larger or equal to one. It follows that
\[
\inf_{\hat{\theta}} \sup_{\theta \in \mathcal{E}(m) \cap B_1(0, \eta)} \mathbb{E}_\theta \left[ l \left( \hat{\theta}, \theta \right) \right] \geq L_2 \left( a_{i^*+1}^2 \vee \frac{\sigma^2 d_m}{np^2} \right),
\]
which allows to conclude.

4. Proof of the asymptotic risks bounds

Proof of Corollary 4.6 in [6]. For the sake of simplicity, we assume that for any node $(i, j) \in m$, the nodes $(i, j)$ and $(-i, -j)$ are different in $\Lambda$. If this is not the case, we only have to slightly modify the proof in order to take account that $\|\Psi_{i, j}\|_F^2$ may equal one. The matrix $V$ is the covariance of the vector of size $d_m$ 
\[
\left( X_{i_1, j_1} + X_{-i_1, -j_1}, \ldots, X_{i_{dm}, j_{dm}} + X_{-i_{dm}, -j_{dm}} \right).
\]
(25)
Since the matrix $\Sigma$ of $X^v$ is positive, $V$ is also positive. Moreover, its largest eigenvalue is larger than $2\varphi_{max}(\Sigma)$.

Let us assume first the $\theta$ belongs to $\Theta^+_m$ and that Assumption $(H_1)$ is fulfilled. By the first result of Proposition 4.4 in [6],
\[
\lim_{n \to +\infty} np^2 \mathbb{E}_\theta \left[ l \left( \hat{\theta}_{m, \rho_1}, \theta \right) \right] = 2\sigma^4 tr \left[ IL_m V^{-1} \right] \geq \frac{\sigma^4}{\varphi_{max}(\Sigma)} tr[IL_m] = 2\sigma^4 \frac{d_m}{\varphi_{max}(\Sigma)},
\]
which corresponds to the first lower bound (30) in [6].

Let us turn to the second result. We now assume that $\theta$ satisfies Assumption $(\mathbb{H}_2)$. By the identity (28) of Proposition 4.4 in [6], we only have to lower bound the quantity $tr \left[ VW^{-1} \right]$.
\[
tr \left[ V^{-1}W \right] \geq \varphi_{max}(V)^{-1} tr \left[ W \right] \geq \frac{1}{2\varphi_{max}(\Sigma)} tr[\Sigma].
\]
Since the matrix $\Sigma^{-1} = \sigma^{-2} \left[ I_{\rho^2} - C(\theta) \right]$ is diagonally dominant, its smallest eigenvalue is larger than $\sigma^{-2}(1-\|\theta\|_1)$. The matrix $\left( I_{\rho^2} - C(\theta_{m, \rho_1}) \right)^2 \left( I_{\rho^2} - C(\theta) \right)^{-2}$
is symmetric positive. It follows that $W$ is also symmetric positive definite.

Hence, we get

$$\operatorname{tr} [V^{-1}W] \geq \frac{\sigma^{-2}}{2} \left[ 1 - \|\theta\|_1 \right] \sum_{k=1}^{d_m} \frac{m \left[ C(\Psi_{i_k,j_k}) \right] [I_{p^2} - C(\theta)]^{-2} [I_{p^2} - C(\theta)]^{-2}}{p^2}.$$  \hspace{1cm} (26)

The largest eigenvalue of $(I_{p^2} - C(\theta))$ is smaller than 2 and the smallest eigenvalue of $(I_{p^2} - C(\theta_{m,\rho}))$ is larger than $1 - \|\theta_{m,\rho}\|_1$. By Lemma A.1 in [6], these two matrices are jointly diagonalizable and the smallest eigenvalue of

$$(I_{p^2} - C(\theta_{m,\rho}))^2 (I_{p^2} - C(\theta))^{-2}$$

is therefore larger than $(1 - \|\theta_{m,\rho}\|_1)^2 / 4$. Gathering this lower bound with (26) yields

$$\operatorname{tr} [V^{-1}W] \geq \frac{d_m \sigma^{-2}}{2} \left[ 1 - \|\theta\|_1 \right] \left[ 1 - \|\theta_{m,\rho}\|_1 \right]^2.$$  

Lemma 4.1 in [6] states that $\|\theta_{m,\rho}\|_1 \leq \|\theta\|_1$. Combining these two lower bounds enables to conclude.

\[\square\]

Proof of Example 4.8 in [6].

**Lemma 4.1.** For any $\theta$ is the space $\Theta_{m_1,\rho}^{+,\text{iso}}$, the asymptotic variance term of $\hat{\theta}_{m_1,\rho}^{\text{iso}}$ equals

$$\lim_{n \to +\infty} n p^2 \mathbb{E}_\theta \left[ l \left( \hat{\theta}_{m_1,\rho}^{\text{iso}}, \theta \right) \right] = 2 \sigma^4 \operatorname{tr} \left( H \right) \operatorname{tr} \left( H^{2} \Sigma \right).$$

If $\theta$ belongs to $\Theta_{m_1,\rho}^{+,\text{iso}}$ and also satisfies $(\mathbb{H}_2)$, then

$$\lim_{n \to +\infty} n p^2 \mathbb{E}_\theta \left[ l \left( \hat{\theta}_{m_1,\rho}^{\text{iso}}, \hat{\theta}_{m_1,\rho}^{\text{iso}} \right) \right] = \frac{2 \sigma^4 \operatorname{tr} \left( \left[ (I - \theta_{m_1,\rho}) \left[ 1, 0 \right] H \Sigma \right]^2 \right)}{\operatorname{tr} \left( H^{2} \Sigma \right)},$$

where the $p^2 \times p^2$ matrix $H$ is defined as $H := C \left( \Psi_{1,0}^{\text{iso}} \right)$.

Proof of Lemma 4.1. Apply Proposition 4.4 in [6] noting that $V = \operatorname{tr} [H \Sigma H] / p^2$ and

$$W = \frac{\operatorname{tr} \left[ \left[ (I - \theta_{m_1,\rho}) \left[ 1, 0 \right] H \Sigma \right]^2 \right]}{\sigma^4 p^2}.$$
We then combine these expressions. By symmetry between $i$ and $j$ we get
\[ \text{cov}_\theta (X[0,0], X[1,0]) = \frac{\sigma^2}{2p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\cos \left( \frac{2\pi i}{p} \right) \cos \left( \frac{2\pi j}{p} \right)}{1 - 2\theta[1,0] \left[ \cos \left( \frac{2\pi i}{p} \right) \cos \left( \frac{2\pi j}{p} \right) \right]} \].

This second part is based on the spectral representation of the field $X$ and follows arguments which come back to Moran [5]. We shall compute the limit of $\text{cov}_\theta (X[0,0], X[1,0])$ when the size of $\Lambda$ goes to infinity. As the field $X$ is stationary on $\Lambda$, we may diagonalize its covariance matrix $\Sigma$ applying Lemma A.1 in [6]. We note $D_\Sigma$ the corresponding diagonal matrix defined by
\[ D_\Sigma[(i-1)p+j,(i-1)p+j] = \sum_{k=1}^{p} \sum_{l=1}^{p} \text{cov}_\theta (X[0,0], X[k,l]) \cos \left( 2\pi \left( \frac{ki}{p} + \frac{lj}{p} \right) \right) \],

for any $1 \leq i, j \leq p$. Straightforwardly, we express $\text{cov}_\theta (X[0,0], X[1,0])$ as a linear combination of the eigenvectors
\[ \text{cov}_\theta (X[0,0], X[1,0]) = \frac{1}{p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \cos \left( \frac{2\pi i}{p} \right) D_\Sigma[(i-1)p+j,(i-1)p+j] \]

Applying Lemma A.1 in [6] to the matrix $\Sigma^{-1}$ and noting that $\theta \in \Theta^{iso,+}$ allows to get another expression of the eigenvalues of $\Sigma$
\[ D_{\Sigma[(i-1)p+j,(i-1)p+j]} = \frac{\sigma^2}{1 - 2\theta[1,0] \left[ \cos \left( \frac{2\pi i}{p} \right) \cos \left( \frac{2\pi j}{p} \right) \right]} \].

We then combine these expressions. By symmetry between $i$ and $j$ we get
\[ \text{cov}_\theta (X[0,0], X[1,0]) = \frac{\sigma^2}{2p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\cos \left( \frac{2\pi i}{p} \right) \cos \left( \frac{2\pi j}{p} \right)}{1 - 2\theta[1,0] \left[ \cos \left( \frac{2\pi i}{p} \right) \cos \left( \frac{2\pi j}{p} \right) \right]} \].
If we let \( p \) go to infinity, this sum converges to the following integral
\[
\lim_{p \to +\infty} \text{cov}_\theta (X_{[0,0]}, X_{[1,0]}) = \frac{\sigma^2}{2} \int_0^1 \int_0^1 \frac{\cos(2\pi x) + \cos(2\pi y)}{1 - 2\theta_{[1,0]}(\cos(2\pi x) + \cos(2\pi y))} \, dx \, dy
\]
\[
= \frac{\sigma^2}{2\theta_{[1,0]}} \left[ -1 + \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{1 - 2\theta_{[1,0]}(\cos(x) + \cos(y))} \, dx \, dy \right].
\]

This last elliptic integral is asymptotically equivalent to \( \log 16[4(1 - 4\theta_{[1,0]})]^{-1} \) when \( \theta_{[1,0]} \to 1/4 \) as observed for instance by Moran [5]. We conclude by substituting this limit in expression (33) in [6].

**Proof of Example 4.9 in [6].** First, we compute \([\theta(p)]_{m_1[1,0]}\). By Lemma 4.1 in [6], it minimizes the function \( \gamma(\cdot) \) defined in (19) in [6] over the whole space \( \Theta_{m_1} \). We therefore obtain
\[
[\theta(p)]_{m_1[1,0]} = \text{tr}(\Sigma H) = \text{tr}(\Sigma H^2).
\]

Once again, we apply Lemma A.1 in [6] to simultaneously diagonalize the matrices \( H \) and \( \Sigma^{-1} \). As previously, we note \( D \Sigma \) the corresponding diagonal matrix of \( \Sigma \).

\[
D\Sigma [(i-1)p+j, (i-1)p+j] = \frac{\sigma^2}{1 - 2\alpha \left[ \cos \left( 2\pi \left( \frac{p}{4} + \frac{j}{4p} \right) \right) + \cos \left( 2\pi \left( \frac{p}{4} + \frac{i}{4p} \right) \right) \right]}
\]
\[
= \frac{\sigma^2}{1 - 4\alpha \cos \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right)}.
\]

Analogously, we compute the diagonal matrix \( D \left( \Psi_{m_1[1,0]} \right) \)
\[
D \left( \Psi_{m_1[1,0]} \right) [(i-1)p+j, (i-1)p+j] = 2 \left[ \cos \left( \frac{2\pi}{p} \right) + \cos \left( \frac{2\pi}{p} \right) \right].
\]

Combining these two last expressions, we obtain
\[
\text{tr}(H\Sigma) = \sum_{i=1}^p \sum_{j=1}^p \sigma^2 \frac{2 \left[ \cos \left( \frac{2\pi}{p} \right) + \cos \left( \frac{2\pi}{p} \right) \right]}{1 - 4\alpha \cos \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right)}.
\]

Let us split this sum in 16 parts depending on the congruence of \( i \) and \( j \) modulo 4. As each if of these 16 sums is shown to be zero, we conclude that \( \text{tr}(H\Sigma) = [\theta(p)]_{m_1[1,0]} = 0 \). By Lemma 4.1, the asymptotic risk of \( \hat{\theta(p)}_{m_1} \) therefore equals
\[
\lim_{n \to +\infty} np^2 \mathbb{E}_{\theta(p)} \left[ \text{tr} \left( \hat{\theta(p)}_{m_1} \right), [\theta(p)]_{m_1} \right] = \frac{\text{tr}(H^4\Sigma^2)}{\text{tr}(H^2\Sigma)}.
\]
First, we lower bound the numerator
\[
tr(H^4 \Sigma^2) = \sigma^4 \sum_{i=1}^{p} \sum_{j=1}^{p} \left\{ \frac{2 \left[ \cos \left( \frac{2\pi i}{p} \right) + \cos \left( \frac{2\pi j}{p} \right) \right]}{1 - 4\alpha \cos \left( \frac{\pi i}{p} \right) \cos \left( \frac{\pi j}{p} \right)} \right\}^4 .
\]
As each term of this sum is non-negative, we may only consider the coefficients \( i \) and \( j \) which are congruent to 0 modulo 4.
\[
tr(H^4 \Sigma^2) \geq \sigma^4 \sum_{i=0}^{p/4-1} \sum_{j=0}^{p/4-1} \frac{16 \left[ \cos \left( \frac{2\pi i}{p/4} \right) + \cos \left( \frac{2\pi j}{p/4} \right) \right]}{(1 - 4\alpha)^2}^4.
\]
If we let go \( p \) to infinity, we get the lower bound
\[
\lim_{p \to +\infty} \frac{tr(H^4 \Sigma^2)}{p^2} \geq \frac{\sigma^4}{(1 - 4\alpha)^2} \int_0^1 \int_0^1 \left[ \cos(2\pi x) + \cos(2\pi y) \right]^4 dx \, dy .
\]
Similarly, we upper bound \( tr(H^2 \Sigma) \) and let \( p \) go to infinity
\[
\lim_{p \to +\infty} \frac{tr(H^2 \Sigma)}{p^2} \leq \frac{4\sigma^2}{1 - 4\alpha} \int_0^1 \int_0^1 \left[ \cos(2\pi x) + \cos(2\pi y) \right]^2 dx \, dy .
\]
Combining these two bounds allows to conclude
\[
\lim_{p \to +\infty} \lim_{n \to +\infty} np^2 R_{\theta(p)} \left( \text{\textstyle \frac{\theta(p)}{m_1}} \right) \geq \frac{L\sigma^2}{1 - 4\alpha} .
\]

5. Miscellaneous

\textit{Proof of Lemma 1.1 in [6].} Let \( \theta \) be a \( p \times p \) matrix that satisfies condition (3) in [6]. For any \( 1 \leq i_1, i_2 \leq p \), we define the \( p \times p \) submatrix \( C_{i_1, i_2} \) as
\[
C_{i_1, i_2}[j_1, j_2] := C(\theta)[(i_1-1)p+j_1, (i_2-1)p+j_2] ,
\]
for any \( 1 \leq j_1, j_2 \leq p \). For the sake of simplicity, the subscripts \( (i_1, i_2) \) are taken modulo \( p \). By definition of \( C(\theta) \), it holds that \( C_{i_1, i_2} = C_{0, i_2-i_1} \) for any \( 1 \leq i_1, i_2 \leq p \). Besides, the matrices \( C_{0,i} \) are circulant for any \( 1 \leq i \leq p \). In short, the matrix \( C(\theta) \) is of the form
\[
C(\theta) = \begin{pmatrix}
C_{0,1} & C_{0,2} & \cdots & C_{0,p} \\
\vdots & \vdots & \cdots & \vdots \\
C_{0,p} & C_{0,1} & \cdots & C_{0,p-1}
\end{pmatrix} ,
\]
where the matrices \( C_{0,i} \) are circulant. Let \((i_1, i_2, j_1, j_2)\) be in \( \{1, \ldots, p\}^4 \). By definition,
\[
C(\theta)[(i_1-1)p+j_1, (i_2-1)p+j_2] = \theta[i_2-i_1, j_2-j_1] .
\]
Since the matrix $\theta$ satisfies condition (3) in [6], $\theta_{[i_2-i_1,j_2-j_1]} = \theta_{[i_1,i_2,j_1,j_2]}$. As a consequence, $C(\theta)([i_1-1,p+j_1,\ldots,i_2-1,p+j_2]) = C(\theta)([i_1-1,p+j_2,\ldots,i_2-1,p+j_1])$ and $C(\theta)$ is symmetric.

Conversely, let $B$ be a $p^2 \times p^2$ symmetric block circulant matrix. Let us define the matrix $\theta$ of size $p$ by

$$\theta_{[i,j]} := B_{[1,(i-1)p+j]}$$

for any $1 \leq i,j \leq p$. Since the matrix $B$ is block circulant, it follows that $C(\theta) = B$. By definition, $\theta_{[i,j]} = C(\theta)_{[1,(i-1)p+j]}$ and $\theta_{[-i,-j]} = C(\theta)_{[(i-1)p+j,1]}$ for any integers $1 \leq i,j \leq p$. Since the matrix $B$ is symmetric, we conclude that $\theta_{[i,j]} = \theta_{[-i,-j]}$.

**Proof of Lemma 2.2 in [6].** For any $\theta' \in \Theta^+$, $\gamma_{n,p}(\theta')$ is defined as

$$\gamma_{n,p}(\theta') = \frac{1}{p^2} tr \left[ (I_{p^2} - C(\theta')) \Sigma X_v X_v^\top (I_{p^2} - C(\theta')) \right].$$

Applying Lemma A.1 in [6], there exists an orthogonal matrix $P$ that simultaneously diagonalizes $\Sigma$ and any matrix $C(\theta')$. Let us define $Y^i := \sqrt{\Sigma}^{-1} X_i$ and $D_\Sigma := P \Sigma P^\top$. Gathering these new notations yields

$$\gamma_{n,p}(\theta') = \frac{1}{p^2} tr \left[ (I_{p^2} - D(\theta')) D_\Sigma YY^\top (I_{p^2} - D(\theta')) \right],$$

where the vectors $Y^i$ are independent standard Gaussian random vectors. Except $YY^\top$, every matrix involved in this last expression is diagonal. Besides, the diagonal matrix $D_\Sigma$ is positive since $\Sigma$ is non-singular. Thus,

$$tr \left[ (I_{p^2} - D(\theta')) D_\Sigma YY^\top (I_{p^2} - D(\theta')) \right]$$

is almost surely a positive quadratic form on the vector space generated by $I_{p^2}$ and $D(\Theta^+)$. Since the function $D(\cdot)$ is injective and linear on $\Theta^+$, it follows that $\gamma_{n,p}(\cdot)$ is almost surely strictly convex on $\Theta^+$.  

**Proof of Lemma 4.1 and Corollary 4.2 in [6].** The proof only uses the stationarity of the field $X$ on $\Lambda$ and the $l_1$ norm of $\theta$. However, the computations are a bit cumbersome. Let $\theta$ be an element of $\Theta^+$. By standard Gaussian properties, the expectation of $X_{[0,0]}$ given the remaining covariates is

$$E_\theta \left( X_{[0,0]} | X_{\Lambda \setminus \{0,0\}} \right) = \sum_{(i,j) \in \Lambda \setminus \{0,0\}} \theta_{[i,j]} X_{[i,j]}.$$
By assumption (H₂), the \( l_1 \) norm of \( \theta \) is smaller than one. We shall prove by backward induction that for any subset \( A \) of \( \Lambda \{0,0\} \) the matrix \( \theta^A \) uniquely defined by

\[
\mathbb{E}_\theta (X_{[0,0]}|X_A) = \sum_{(i,j) \in A} \theta_{[i,j]}^A X_{[i,j]} \quad \text{and} \quad \theta_{[i,j]}^A = 0 \text{ for any } (i,j) \notin A
\]

satisfies \( \|\theta^A\|_1 \leq \|\theta\|_1 \). The property is clearly true if \( A = \Lambda \{0,0\} \). Suppose we have proved it for any set of cardinality \( q \) larger than one. Let \( A \) be a subset of \( \Lambda \{0,0\} \) of cardinality \( q - 1 \) and \((i,j)\) be an element of \( \Lambda \{A \cup \{0,0\}\} \). Let us derive the expectation of \( X_{[0,0]} \) conditionally to \( X_A \) from the expectation of \( X_{[0,0]} \) conditionally to \( X_{A \cup \{i,j\}} \).

\[
\mathbb{E}_\theta (X_{[0,0]}|X_A) = \mathbb{E}_\theta \left[ \mathbb{E}(X_{[0,0]}|X_A)|X_{A \cup \{i,j\}} \right] = \sum_{(k,l) \in A} \theta_{[k,l]}^{A \cup \{i,j\}|(i,j)} X_{[k,l]} + \theta_{[i,j]}^{A \cup \{i,j\}|(i,j)} \mathbb{E}_\theta [X_{[i,j]}|X_A] \tag{28}
\]

Let us take the conditional expectation of \( X_{[i,j]} \) with respect to \( X_{A \cup \{0,0\}} \). Since the field \( X \) is stationary on \( \Lambda \) and by the induction hypothesis, the unique matrix \( \theta_{(i,j)}^{A \cup \{0,0\}} \) defined by

\[
\mathbb{E}_\theta (X_{[i,j]}|X_{A \cup \{0,0\}}) = \sum_{(k,l) \in A \cup \{0,0\}} \theta_{(i,j)}^{A \cup \{0,0\}|[k,l]} X_{[k,l]} \quad \text{and} \quad \theta_{(i,j)}^{A \cup \{0,0\}|[k,l]} = 0 \text{ for any } (k,l) \notin A \cup \{0,0\} \text{ satisfies } \|\theta_{(i,j)}^{A \cup \{0,0\}}\|_1 \leq \|\theta\|_1
\]

Taking the expectation conditionally to \( X_A \) of this previous expression leads to

\[
\mathbb{E}_\theta (X_{[i,j]}|X_A) = \sum_{(k,l) \in A} \theta_{(i,j)}^{A \cup \{0,0\}|[k,l]} X_{[k,l]} + \theta_{(i,j)}^{A \cup \{0,0\}|0,0} \mathbb{E}_\theta (X_{[0,0]}|X_A) \tag{29}
\]

Gathering identities (28) and (29) yields

\[
\mathbb{E}_\theta (X_{[0,0]}|X_A) = \sum_{(k,l) \in A} \frac{\theta_{[k,l]}^{A \cup \{0,0\}|(i,j)} + \theta_{[i,j]}^{A \cup \{i,j\}|[k,l]} \theta_{(i,j)}^{A \cup \{0,0\}|[0,0]} + \theta_{(i,j)}^{A \cup \{0,0\}|0,0}}{1 - \theta_{A \cup \{i,j\}|[i,j]} \theta_{(i,j)}^{A \cup \{0,0\}|0,0}} X_{[k,l]},
\]

since \( |\theta_{A \cup \{i,j\}|[i,j]} \theta_{A \cup \{0,0\}|0,0}| < 1 \). Then, we upper bound the \( l_1 \) norm of \( \theta^A \)
using that \(\|\theta^{A\cup\{(i,j)\}}\|_1\) and \(\|\theta^{A\cup\{(0,0)\}}\|_1\) are smaller or equal to \(\|\theta\|_1\).

\[\|\theta^A\|_1 \leq \frac{\sum_{(k,l)\in A} |\theta^{A\cup\{(i,j)\}}(k,l)| + \sum_{(k,l)\in A} |\theta^{A\cup\{(i,j)\}}(k,l)|\theta^{A\cup\{(0,0)\}}(k,l)|}{1 - |\theta^{A\cup\{(i,j)\}}(i,j)\theta^{A\cup\{(0,0)\}}(0,0)|}\]

\[\leq \frac{\|\theta\|_1 + |\theta^{A\cup\{(i,j)\}}(i,j)| \left(\sum_{(k,l)\in A\cup\{(0,0)\}} |\theta^{A\cup\{(0,0)\}}(k,l)| - 1 - |\theta^{A\cup\{(0,0)\}}(0,0)|\right)}{1 - |\theta^{A\cup\{(i,j)\}}(i,j)\theta^{A\cup\{(0,0)\}}(0,0)|}\]

\[\leq \frac{\|\theta\|_1 + \|\theta^{A\cup\{(i,j)\}}(i,j)|\cdot(\|\theta\|_1 - 1) \left(1 + |\theta^{A\cup\{(0,0)\}}(0,0)|\right)}{1 - |\theta^{A\cup\{(i,j)\}}(i,j)\theta^{A\cup\{(0,0)\}}(0,0)|}.

Since \(\|\theta\|_1\) is smaller than one, it follows that \(\|\theta^A\|_1 \leq \|\theta\|_1\).

Let \(m\) be a model in the collection \(M_1\). Since \(m\) stands for a set of neighbors of \((0,0)\), we may define \(\theta^m\) as above. It follows that \(\|\theta^m\|_1 \leq \|\theta\|_1\). Since the field \(X\) is stationary on the torus, \(X\) follows the same distribution as the field \(X^*\) defined by \(X^*[i,j] = X[-i,-j]\). By uniqueness of \(\theta^m\), we obtain that \(\theta^m[i,j] = \theta^m[-i,-j]\). Thus, \(\theta^m\) belongs to the space \(\Theta_m\). Moreover, \(\theta^m\) minimizes the function \(\gamma(\cdot)\) on \(\Theta_m\). Since the \(1\) norm of \(\theta^m\) is smaller than one, \(\theta^m\) belongs to \(\Theta_m^{+}\). The matrices \(\theta^m\) and \(\theta_{m,\rho}^1\) are therefore equal, which concludes the proof in the non-isotropic case.

Let us now turn to the isotropic case. Let \(\theta\) belong to \(\Theta^{iso,+}\) and let \(m\) be a model in \(M_1\). As previously, the matrix \(\theta^m\) satisfies \(\|\theta^m\|_1 \leq \|\theta\|_1\). Since the distribution of \(X\) is invariant under the action of the group \(G\), \(\theta^m\) belongs to \(\Theta_m^{iso}\). Since \(\|\theta^m\|_1 \leq \|\theta\|_1\), \(\theta^m\) lies in \(\Theta_m^{+,iso}\). It follows that \(\theta^m = \theta_{m,\rho}^{iso}\).

\[Proof\ of\ Corollary\ 4.3\ in\ [6].\] Let \(\theta\) be a matrix in \(\Theta^+\) such that \((H_2)\) holds and let \(m\) be a model in \(M_1\). We decompose \(\gamma(\theta_{m,\rho}^1)\) using the conditional
expectation of $X[0,0]$ given $X_m$.

$$\gamma(\hat{\theta}_{m,\rho_1}) = \mathbb{E}_\theta \left[ X[0,0] - \sum_{(i,j) \in m} \hat{\theta}_{m,\rho_1}[i,j]X[i,j] \right]^2$$

$$= \mathbb{E}_\theta \left[ X[0,0] - \mathbb{E}_\theta (X[0,0] | X_m) \right]^2$$

$$+ \mathbb{E}_\theta \left[ \mathbb{E}_\theta (X[0,0] | X_m) - \sum_{(i,j) \in m} \hat{\theta}_{m,\rho_1}[i,j]X[i,j] \right]^2.$$

By Corollary (11) in [6], we know that

$$\mathbb{E}_\theta (X[0,0] | X_m) = \sum_{(i,j) \in m} \theta_{m,\rho_1}[i,j]X[i,j].$$

Combining these two last identities yields

$$\gamma(\hat{\theta}_{m,\rho_1}) = \gamma(\theta_{m,\rho_1}) + \mathbb{E}_\theta \left[ \sum_{(i,j) \in \Lambda \setminus \{(0,0)\}} (\theta_{m,\rho_1} - \hat{\theta}_{m,\rho_1})[i,j]X[i,j] \right]^2.$$

Subtracting $\gamma(\theta)$, we obtain the first result. The proof is analogous in the isotropic case.

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