Model-Free Robust Reinforcement Learning with Linear Function Approximation

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Abstract

This paper addresses the problem of model-free reinforcement learning for Robust Markov Decision Process (RMDP) with large state spaces. The goal of the RMDPs framework is to find a policy that is robust against the parameter uncertainties due to the mismatch between the simulator model and real-world settings. We first propose Robust Least Squares Policy Evaluation algorithm, which is a multi-step online model-free learning algorithm for policy evaluation. We prove the convergence of this algorithm using stochastic approximation techniques. We then propose Robust Least Squares Policy Iteration (RLSPI) algorithm for learning the optimal robust policy. We also give a general weighted Euclidean norm bound on the error (closeness to optimality) of the resulting policy. Finally, we demonstrate the performance of our RLSPI algorithm on some benchmark problems from OpenAI Gym [1].

1 Introduction

Model-free Reinforcement Learning (RL) algorithms typically learn a policy by training on a simulator. In the RL literature, it is nominally assumed that the testing environment is identical to the training environment (simulator model). However, in reality, the parameters of the simulator model can be different from the real-world setting. This can be due to the approximation errors incurred while modeling, due to the changes in the real-world parameters over time, and can even be due to the possible adversarial disturbances in the real-world. For example, in many robotics applications, the standard simulator parameter settings (of the mass, friction, wind conditions, sensor noise, action delays) can be different from that of the actual robot in the real-world. This mismatch between the training and testing environment parameters can significantly degrade the real-world performance of the model-free learning algorithms trained on a simulator model.

The RMDP framework [2][3] addresses the (planning) problem of computing the optimal policy that is robust against parameter uncertainties. The RMDP formulation considers a set of model parameters (uncertainty set) under the assumption that the actual parameters lie in this uncertainty set, and the learning algorithm computes a robust policy that performs best under the worst model. RMDP problem have been analyzed extensively in the tabular case [2][6] and under linear function approximation [7]. Model-based learning [5] and model-free learning [9] have been studied in the tabular setting. [9] also studied the simulation based policy evaluation using function approximation.

In this paper, we address the problem of learning a policy that is robust against parameter uncertainties, which encapsulates the mismatch between the training and testing environments, in RMDPs with very large state spaces. We focus on the approach of least squares based online model-free reinforcement learning with linear function approximation. Similar to the framework in [9], we assume that a nominal simulator is available for model-free learning and the parameter uncertainty set is characterized with respect to this nominal but unknown simulator model parameters. We develop the robust versions of classical least squares based model-free policy

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evaluation and policy iteration algorithms. Our algorithmic and technical contributions are as follows.

(i) Robust Least Squares Policy Evaluation (RLSPE($\lambda$)) algorithm: We first propose RLSPE($\lambda$) algorithm, a multi-step online model-free policy evaluation algorithm with linear function approximation. This can be thought as the robust version of classical least squares based RL algorithms for policy evaluation, like LSTD($\lambda$) and LSPE($\lambda$). We prove the convergence of this algorithm using stochastic approximation techniques, and also characterize its approximation error due to the linear architecture.

(ii) Robust Least Squares Policy Iteration (RLSPI) algorithm: We then propose the RLSPI algorithm for learning the optimal robust policy. We also give a general $L_2$-norm bound on the error (closeness to optimality) of the resulting policy. To the best of our knowledge, this is the first work that presents a least squares based policy iteration algorithm for robust reinforcement learning with such guarantees.

(iii) Finally, we demonstrate the performance of the RLSPI algorithm on various RL test environments from OpenAI Gym [1], like, CartPole, MountainCar, Acrobat environments.

1.1 Related Work

RMDP formulation to address the parameter uncertainty problem was first proposed by [2] and [3]. [2] showed that the optimal robust value function and policy can be computed using the robust counterparts of the standard value iteration and policy iteration. To tackle the parameter uncertainty problem, other works considered distributionally robust setting [5], modified policy iteration [10], and more general uncertainty set [4]. These works mainly focused on the tabular setting. Linear function approximation method to solve large robust MDPs was proposed in [7]. Though this paper suggested a sampling based approach, a general model-free learning algorithm and analysis was not included. [9] proposed the robust versions of the classical model-free reinforcement learning algorithms such as Q-learning, SARSA, and TD-learning. They also proposed function approximation based algorithms with linear and non-linear architectures, focusing on the robust policy evaluation. [11] proposed a kernel-based batch RL algorithm for finding the robust value function, [12] introduced a soft-robust actor-critic algorithm, [13] proposed a probabilistic action robust framework and policy iteration approach based solution, and [14] employed an entropy-regularized policy optimization algorithm for continuous control.

Our work differs from the above in two significant ways. Firstly, we develop a new multi-step model-free reinforcement learning algorithm, RLSPE($\lambda$), for policy evaluation. Extending the classical least squares based policy evaluation algorithms, like LSPE($\lambda$) and LSTD($\lambda$), to the robust case is very challenging due to the nonlinearity of the robust TD($\lambda$) operator. We overcome this issue by a cleverly defined approximate robust TD($\lambda$) operator that is amenable to online learning using least squares approaches. Secondly, we develop a new robust policy iteration algorithm with provable guarantees on the performance. In particular, we give a general weighted Euclidean norm bound on the error of the resulting policy. While similar bounds are available for the non-robust settings, this is the first work to provide such a characterization in the robust setting.

2 Robust Markov Decision Processes

A Markov Decision Process is a tuple $M = (S, A, r, P, \alpha)$ where $S$ is the state space, $A$ is the action space, $r : S \times A \to \mathbb{R}$ is the reward function, and $\alpha \in (0, 1)$ is the discount factor. The transition probability matrix $P_{s,a}(s')$ represents the probability of transitioning to state $s'$ when action $a$ is taken at state $s$. We consider a finite MDP setting where the cardinality of state and action spaces are finite. A policy $\pi$ maps each state to an action. The value of a policy $\pi$ evaluated at state $s$ is given by

$$V_{\pi,P}(s) = \mathbb{E}_{\pi,P} \left[ \sum_{t=0}^{\infty} \alpha^t r(s_t, a_t) \mid s_0 = s \right], \text{ where, } a_t \sim \pi(s_t), \ s_{t+1} \sim P_{s_t,a_t}(\cdot).$$
The optimal value function and the optimal policy of an MDP with the transition probability $P$ is defined as $V^*_P = \max_{\pi} V_{\pi,P}$ and $\pi^*_P = \arg \max_{\pi} V_{\pi,P}$.

In RMDP framework, instead of a fixed transition probability matrix $P$, we consider a set of transition probability matrices $\mathcal{P}$. We assume that the set $\mathcal{P}$ satisfies the standard rectangularity condition \cite{2}. In RMDPs, the objective is to find a policy that maximizes the worst-case performance. Formally, the robust value function $V_{\pi}$ corresponding to a policy $\pi$ and the optimal robust value function $V^*$ are defined as \cite{2,3}

$$V_{\pi} = \inf_{P \in \mathcal{P}} V_{\pi,P}, \quad V^* = \sup_{\pi} \inf_{P \in \mathcal{P}} V_{\pi,P}. \quad (1)$$

The optimal robust policy $\pi^*$ is such that the robust value function corresponding to it matches the optimal robust value function, that is, $V_{\pi^*} = V^*$.

A generic characterization of the set $\mathcal{P}$ makes the RMDPs problems intractable to solve by model-free methods. In the standard model-free methods, the algorithm has access to a simulator that can simulate the next state given the current state and current action, according to a fixed transition probability matrix (that is unknown to the algorithm). However, generating samples according to each and every transition probability matrix from the set $\mathcal{P}$ is clearly infeasible. To overcome this difficulty, we use the characterization of the uncertainty set used in \cite{9}.

**Assumption 1** (Uncertainty Set). Each $P \in \mathcal{P}$ can be represented as $P_{s,a}(\cdot) = P_{s,a}^0(\cdot) + U_{s,a}(\cdot)$ for some $U_{s,a} \in \mathcal{U}_{s,a}$, where $P_{s,a}^0(\cdot)$ is the unknown transition probability matrix corresponding to the simulator and $\mathcal{U}_{s,a}$ is a confidence region around it.

Using the above characterization, we can write $\mathcal{P} = \{P^0 + U : U \in \mathcal{U}\}$ in the product space $S \times A$, where $\mathcal{U} = \bigcup_{s,a} \mathcal{U}_{s,a}$. So, $\mathcal{U}$ is the set of all possible perturbations to the simulator model $P^0$.

We consider robust Bellman operator for policy evaluation, defined as \cite{2}

$$T_{\pi}(V)(s) = r(s, \pi(s)) + \alpha \inf_{P \in \mathcal{P}} \sum_{s'} P_{s,\pi(s)}(s') V(s'), \quad (2)$$

a popular approach to solve \cite{1}. Using our characterization of the uncertainty set, we rewrite as

$$T_{\pi}(V)(s) = r(s, \pi(s)) + \alpha \sum_{s'} P_{s,\pi(s)}^0(s') V(s') + \alpha \inf_{U \in \mathcal{U}_{\pi}(s)} \sum_{s'} U_{s,\pi(s)}(s') V(s'). \quad (3)$$

For any set $\mathcal{B}$ and a vector $v$, define $\sigma_{\mathcal{B}}(v) = \inf \{u^T v : u \in \mathcal{B}\}$. We denote $|S|$ as the cardinality of the set $S$. Let $\sigma_{\mathcal{U}_{\pi}(s)}(v)$ and $r_{\pi}$ be the $|S|$ dimensional column vectors defined as $(\sigma_{\mathcal{U}_{\pi}(s)}(v) : s \in S)^T$ and $(r(s, \pi(s)) : s \in S)^T$, respectively. Let $P_{s,a}^0$ be the stochastic matrix corresponding to the policy $\pi$ where for any $s, s' \in S$, $P_{s,a}^0(s,s') = P^0(s'|s, \pi(s))$. Then, \eqref{eq:robust-bellman} can be written in the matrix form as

$$T_{\pi}(V) = r_{\pi} + \alpha P_{\pi}^0 V + \alpha \sigma_{\mathcal{U}_{\pi}}(V). \quad (4)$$

It is known that \cite{2} $T_{\pi}$ is a contraction in sup norm and the robust value function $V_{\pi}$ is the unique fixed point of $T_{\pi}$. The robust Bellman operator $T$ can also be defined in the same way as in the non-robust setting.

$$T(V) = \max_{\pi} T_{\pi}(V). \quad (5)$$

It is also known that \cite{2} $T$ is a contraction in sup norm, and the optimal robust value function $V^*$ is its unique fixed point.

### 3 Robust Least Squares Policy Evaluation

In this section, we develop the RLSPE($\lambda$) algorithm for learning the robust value function.
3.1 Robust TD(\(\lambda\)) Operator and the Challenges

In RL, a very useful approach for analyzing the multi-step learning algorithms like TD(\(\lambda\)), LSTD(\(\lambda\)), and LSPE(\(\lambda\)) is to define a multi-step Bellman operator called TD(\(\lambda\)) operator \([15,16]\). Following the same approach, we can define the robust TD(\(\lambda\)) operator as well. For a given policy \(\pi\), and a parameter \(\lambda \in [0, 1]\), the robust TD(\(\lambda\)) operator denoted by \(\lambda^{(\pi)} : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}\) is defined as

\[
T^{(\lambda)}_\pi(V) = (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m T^{m+1}_\pi(V).
\]

Note that for \(\lambda = 0\), we recover \(T_\pi\) assuming \(0^0 = 1\). The following result is straightforward.

**Proposition 1 (informal).** \(T^{(\lambda)}_\pi\) is a contraction in sup norm and the robust value function \(V_\pi\) is its unique fixed point, for any \(\alpha \in (0, 1), \lambda \in [0, 1]\).

For RMDPs with very large state space, exact dynamic programming methods which involve the evaluation of \(\Phi\) or \(d\) are intractable. A standard approach to overcome this issue is to approximate the value function using some function approximation architecture. Here we focus on linear approximation architectures \([16]\). In linear approximation architectures, the value function is represented as the weighted sum of features as, \(V(s) = \phi(s)^\top w, \forall s \in S\), where \(\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_L(s))^\top\) is an \(L\) dimensional feature vector with \(L < |S|\), and \(w = (w_1, \ldots, w_L)^\top\) is a weight vector. In the matrix form, this can be written as \(\hat{V} = \Phi w\) where \(\Phi\) is an \(|S| \times L\) dimensional feature matrix whose \(s^{th}\) row is \(\phi(s)^\top\). We assume that the columns of \(\Phi\) are linearly independent, i.e., \(\text{rank}(\Phi) = L\).

The standard approach to find an approximate (robust) value function is to solve for a \(w_\pi\), with \(\hat{V}_\pi = \Phi w_\pi\), such that \(\Phi w_\pi = \Pi T^{(\lambda)}_\pi \Phi w_\pi\) where \(\Pi\) is a projection onto the subspace spanned by the columns of \(\Phi\). The projection is with respect to a \(d\)-weighted Euclidean norm. This norm is defined as \(|V|_D^2 = V^\top D V\) where \(D\) is a diagonal matrix with non-negative diagonal entries \((d(s), s \in S)\), for any vector \(V\). Under suitable assumptions, \([7]\) showed that \(\Pi T_\pi\) is a contraction in a \(d\)-weighted Euclidean norm. We also use a similar assumption stated below.

**Assumption 2.** (i) For any given policy \(\pi\), there exists an exploration policy \(\pi_\varepsilon = \pi_\exp(\pi)\) and a \(\beta \in (0, 1)\) such that \(\alpha P_{s,\pi(s)}(s') < \beta P_{s,\pi_\varepsilon(s)}(s')\), for all transition probability matrices \(P \in \mathcal{P}\) and for all states \(s, s' \in S\).

(ii) There exists a steady state distribution \(d_\pi_\varepsilon = (d_{\pi_\varepsilon}(s), s \in S)\) for the Markov chain with transition probability \(P_{s,\pi_\varepsilon}\) with \(d_{\pi_\varepsilon}(s) > 0, \forall s \in S\).

In the following, we will simply use \(d\) instead of \(d_{\pi_\varepsilon}\). Though the above assumption appears restrictive, it is necessary to show that \(\Pi T_\pi\) is a contraction in the \(d\)-weighted Euclidean norm, as proved in \([7]\). Also, a similar assumption is used in proving the convergence of off-policy reinforcement learning algorithm \([17]\). In the robust case, we can expect a similar condition because we are learning a robust value function for a set of transition probability matrices instead of a single transition probability matrix. We can now show the following.

**Proposition 2 (informal).** Under Assumption 2, \(\Pi T^{(\lambda)}_\pi\) is a contraction mapping in the \(d\)-weighted Euclidean norm for any \(\lambda \in (0, 1)\).

The linear approximation based robust value function \(\hat{V}_\pi = \Phi w_\pi\) can be computed using the iteration, \(\Phi w_{k+1} = \Pi T^{(\lambda)}_\pi \Phi w_k\). Since \(\Pi T^{(\lambda)}_\pi\) is a contraction, \(w_k\) will converge to \(w^*\). A closed form solution for \(w_{k+1}\) given \(w_k\) can be found by least squares approach as \(w_{k+1} = \arg \min_w \|\Phi w - \Pi T^{(\lambda)}_\pi \Phi w_k\|_D^2\). It can be shown that (details are given in the supplementary material), we can get a closed form solution for \(w_{k+1}\) as

\[
w_{k+1} = w_k + (\Phi^\top D \Phi)^{-1} \Phi^\top D (T^{(\lambda)}_\pi \Phi w_k - \Phi w_k).
\]

This is similar to the projected equation approach \([16]\) in the non-robust setting. Even in the non-robust setting, iterations using the \([7]\) is intractable for MDPs with large state space. Moreover,
when the transition matrix is unknown, it is not feasible to use (7) exactly even for smaller MDPs. Simulation-based model-free learning algorithms are developed for addressing this problem in the non-robust case. In particular, LSPE(λ) algorithm [16, 18] is used to solve the iterations of the above form.

However, compared to the non-robust setting, there are two significant challenges in learning the robust value function by using simulation-based model-free approaches.

(i) Non-linearity of the robust TD(λ) operator: The non-robust $T_\pi$ operator and the TD(λ) operator do not involve any nonlinear operations. So, they can be estimated efficiently from simulation samples in a model-free way. However, the robust TD(λ) operator when expanded will have the following form (derivation is given in the supplementary material).

$$T_\pi^{(\lambda)}(V) = (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \sum_{k=0}^{m} (\alpha P_\pi^{o})^k r_\pi + (\alpha P_\pi^{o})^{m+1} V + \alpha \sum_{k=0}^{m} (\alpha P_\pi^{o})^{k} \sigma_\mu (T^{(m-k)}_\pi V).$$

The last term is very difficult to estimate using simulation-based model-free approaches due to the composition of operations $\sigma_\mu$ and $T_\pi$. In addition, nonlinearity of the $T_\pi$ operator by itself adds to the complexity.

(ii) Unknown uncertainty region $U$: In our formulation, we assumed that the transition probability uncertainty set $P$ is given by $P = P^o + U$. So, for each $U \subseteq U$, $P^o + U$ should be a valid transition probability matrix. However, in the model-free setting, we do not know the simulator transition probability $P^o$. So, it is not possible to know $U$ exactly a priori. One can only use an approximation $\hat{U}$ instead of $U$. This can possibly affect convergence of the learning algorithms.

### 3.2 Robust Least Squares Policy Evaluation (RLSPE(λ)) Algorithm

We overcome the challenges of learning the robust value function by defining an approximate robust TD(λ) operator, and by developing a robust least squares policy evaluation algorithm based on that.

Let $\hat{U}$ be the approximate uncertainty set we use instead of the actual uncertainty set $U$. For a given policy $\pi$ and a parameter $\lambda \in [0, 1)$, approximate robust TD(λ) operator denoted by $\hat{T}_\pi^{(\lambda)} : \mathbb{R}^{|S|} \rightarrow \mathbb{R}^{|S|}$, is defined as

$$\hat{T}_\pi^{(\lambda)}(V) = (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \left[ \sum_{t=0}^{m} (\alpha P_\pi^{o})^t r_\pi + \alpha \sum_{t=0}^{m} (\alpha P_\pi^{o})^t \sigma_\mu (V) + (\alpha P_\pi^{o})^{m+1} V \right].$$

Note that even with $\hat{U}_\pi = U_\pi$, (9) is different from (8). This is to overcome the challenges due to the nonliterary associated with (8).

However, we emphasize that (9) is not an arbitrary definition. Note that, for $\hat{U}_\pi = U_\pi$, with $\lambda = 0$, we recover the operator $T_\pi$. Moreover, the robust value function $V_\pi$ is a fixed point of $\hat{T}_\pi^{(\lambda)}$ when $\hat{U}_\pi = U_\pi$ for any $\lambda \in (0, 1)$.

**Proposition 3.** Suppose $\hat{U}_\pi = U_\pi$. Then, for any $\alpha \in (0, 1)$ and $\lambda \in [0, 1)$, the robust value function $V_\pi$ is a fixed point of $\hat{T}_\pi^{(\lambda)}$, i.e., $\hat{T}_\pi^{(\lambda)}(V_\pi) = V_\pi$.

Intuitively, the convergence of any learning algorithm using the approximate robust TD(λ) operator will depend on the difference between the actual uncertainty set $U$ and its approximation $\hat{U}$. To quantify this, we use a metric similar to the one used in [9]. Let $\rho = \max_{s,a \in A} \rho_{s,a}$ where

$$\rho_{s,a} = \max \{ \max_{x \in U_{s,a}} \max_{y \in U_{s,a}} \|x - y\|_d / d_{\min}, \max_{x \in U_{s,a}} \max_{y \in U_{s,a}} \|x - y\|_d / d_{\min} \}$$

and $d_{\min} := \min_{s \in S} d(s)$. By convention, we set $\rho_{s,a} = 0$ when $\hat{U}_{s,a} = U_{s,a}$ for all $(s, a) \in (S, A)$. So, $\rho = 0$ if $\hat{U} = U$. Using this characterization and under some additional assumptions on the discount factor, we show that the approximate robust TD(λ) operator is a contraction in the $d$-weighted Euclidean norm.
where $\gamma$ we do not know the transition probability $P$. This can be written in a more succinct matrix form as given below (see the supplementary material).

\[
\|\Pi_T^{(\lambda)} V_1 - \Pi T^{(\lambda)} V_2\|_d \leq c(\alpha, \beta, \rho, \lambda) \|V_1 - V_2\|_d, \tag{10}
\]

where $c(\alpha, \beta, \rho, \lambda) = (\beta(2 - \lambda) + \rho\alpha)/(1 - \beta\lambda)$. So, if $c(\alpha, \beta, \rho, \lambda) < 1$, $\Pi T^{(\lambda)}$ is a contraction in the $d$-weighted Euclidean norm. Moreover, there exists a unique $w_\pi$ such that $\Phi w_\pi = \Pi T^{(\lambda)}(\Phi w_\pi)$. Furthermore, for this $w_\pi$,

\[
\|V_\pi - \Phi w_\pi\|_d \leq \frac{1}{1 - c(\alpha, \beta, \rho, \lambda)} \left(\|V_\pi - \Pi V_\pi\|_d + \frac{\beta\rho\|V_\pi\|_d}{1 - \beta\lambda}\right). \tag{11}
\]

We note that despite the assumption on the discount factor, we empirically show in Section 5 that our learning algorithm converges to a robust policy even if this assumption is violated. We also note that the upper bound in (11) quantifies the error of approximating the robust value function $V_\pi$ with the approximate robust value function $\Phi w_\pi$. As in [16,19,20], the error bound helps in characterizing performance of policy iteration algorithms.

Using the contraction property of approximate robust TD($\lambda$) operator, the linear approximation based robust value function $\hat{V}_\pi = \Phi w_\pi$ can be computed using the iteration, $\Phi w_{k+1} = \Pi T^{(\lambda)}(\Phi w_k)$. Similar to (7), we can get a closed form solution for $w_{k+1}$ as

\[
w_{k+1} = w_k + (\Phi^T D\Phi)^{-1} \Phi^T D(T^{(\lambda)} - \Phi w_k). \tag{12}
\]

This can be written in a more succinct matrix form as given below (see the supplementary material).

\[
w_{k+1} = w_k + B^{-1}(Aw_k + C(w_k) + b), \tag{13}
\]

\[
A = \Phi^T D(\alpha P^\pi - I) \sum_{m=0}^{\infty} (\alpha \lambda P^\pi)^m \Phi, \quad B = \Phi^T D\Phi, \tag{14}
\]

\[
C(w) = \alpha \Phi^T D \sum_{t=0}^{\infty} (\alpha \lambda P^\pi)^t \sigma_{\hat{a}_t}(\Phi w), \quad b = \Phi^T D \sum_{t=0}^{\infty} (\alpha P^\pi)^t r. \tag{15}
\]

Iterations by evaluating (13) exactly is intractable for MDPs with large state space, and infeasible if we do not know the transition probability $P^\pi$. To address this issue, we propose a simulation-based model-free online reinforcement learning algorithm, which we call robust least squares policy evaluation (RLSPE($\lambda$)) algorithm, for learning the robust value function.

**RLSPE($\lambda$) algorithm**: Generate a sequence of states and rewards, $(s_t, r_t, t \geq 0)$, using the policy $\pi$. Update the parameters as

\[
w_{t+1} = w_t + \gamma_t B_t^{-1}(A_t w_t + b_t + C_t(w_t)), \tag{16}
\]

\[
A_t = \frac{1}{t+1} \sum_{\tau=0}^{t} z_{\tau} (\alpha \Phi^T(s_{\tau+1}) - \Phi^T(s_{\tau})), \quad B_t = \frac{1}{t+1} \sum_{\tau=0}^{t} \Phi(s_{\tau})\Phi^T(s_{\tau}), \tag{17}
\]

\[
C_t(w) = \frac{\alpha}{t+1} \sum_{\tau=0}^{t} z_{\tau} \sigma_{\hat{a}_t}(s_{\tau}) (\Phi w), \quad b_t = \frac{1}{t+1} \sum_{\tau=0}^{t} \Phi(s_{\tau})r(s_{\tau}, \pi(s_{\tau})), \tag{18}
\]

\[
z_{\tau} = \sum_{m=0}^{\tau} (\alpha \lambda)^{\tau-m} \phi(s_m), \tag{19}
\]

where $\gamma_t$ is a deterministic sequence of step sizes. We assume that the step size satisfies the standard Robbins-Munro stochastic conditions for stochastic approximation, i.e., $\sum_{t=0}^{\infty} \gamma_t = \infty, \sum_{t=0}^{\infty} \gamma_t^2 < \infty$. 

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We use the on-policy version of the RLSPE(\(\lambda\)) algorithm in the above description. So, we implicitly assume that the given policy \(\pi\) is an exploration policy according to the Assumption 2. This is mainly for the clarity of the presentation and notational convenience. Also, this simplifies the presentation of the policy iteration algorithm introduced in the next section. An off-policy version of the above algorithm can be implemented using the techniques given in [17]. We now give the convergence result of the RLSPE(\(\lambda\)) algorithm.

**Theorem 2.** Let Assumption 2 hold. Also, let \(c(\alpha, \beta, \rho, \lambda) < 1\) so that \(\Pi_T^{(\lambda)}\) is a contraction according to Theorem 1. Let \(\{w_t\}\) be the sequence generated by the RLSPE(\(\lambda\)) algorithm given in (16)-(19). Then, \(w_t\) converges to \(w_\pi\) with probability 1 where \(w_\pi\) satisfies the fixed point equation \(\Phi w_\pi = \Pi_T^{(\lambda)} \Phi w_\pi\).

### 4 Robust Least Squares Policy Iteration

In this section, we introduce a policy improvement algorithm, which we call the robust least squares policy iteration (RLSPI) algorithm, for finding the optimal robust policy. RLSPI algorithm can be thought as the robust version of the LSPI algorithm [21]. RLSPI algorithm uses the RLSPE(\(\lambda\)) algorithm for policy evaluation. However, model-free policy improvement is difficult when working with value functions. To overcome this, we first introduce the robust state-action value function (Q-function).

For any given policy \(\pi\) and state-action pair \((s, a)\), we define the robust Q-value as,

\[
Q_\pi(s, a) = \inf_{P \in \mathcal{P}} \mathbb{E}_P \sum_{t=0}^{\infty} \alpha^t r(s_t, a_t) \mid s_0 = s, a_0 = a.
\]  

Instead of learning the approximate robust value function \(\bar{V}_\pi\), we can learn the approximate robust Q-value function \(\bar{Q}_\pi\) using RLSPE(\(\lambda\)). This can be done by defining the feature vector \(\phi(s, a)\) where \(\phi(s, a) = (\phi_1(s, a), \ldots, \phi_L(s, a))^\top\) and the linear approximation of the form \(\bar{Q}_\pi(s, a) = w^\top \phi(s, a)\) where \(w\) is a weight vector. The results from the previous section on the convergence of the RLSPE(\(\lambda\)) algorithm applies for the case of learning Q-value function as well.

RLSPI is a policy iteration algorithm that uses RLSPE(\(\lambda\)) for policy evaluation at each iteration. It starts with an arbitrary initial policy \(\pi_0\). At the \(k\)th iteration, RLSPE(\(\lambda\)) returns a weight vector that represents the approximate Q-value function \(\bar{Q}_{\pi_k}\) corresponding to the policy \(\pi_k\). The next policy \(\pi_{k+1}\) is the greedy policy corresponding to \(\bar{Q}_{\pi_k}\), defined as \(\pi_{k+1}(s) = \arg \max_{a \in A} \bar{Q}_{\pi_k}(s, a)\). For empirical evaluation purposes, we terminate the policy iteration for some finite value \(K\). RLSPI algorithm is summarized below.

**Algorithm 1 RLSPI Algorithm**

1: Initialization: Policy iteration step \(k \leftarrow 0\), Initial policy \(\pi_0\).
2: for \(k = 0 \ldots K\) do
3: Initialize the policy weight vector \(w_0\). Initialize time step \(t \leftarrow 0\).
4: repeat
5: Observe the state \(s_t\), take action \(a_t = \pi_k(s_t)\), observe reward \(r_t\) and next state \(s_{t+1}\).
6: Update the weight vector \(w_t\) according to RLSPE(\(\lambda\)) algorithm (c.f. [16]-[19])
7: \(t \leftarrow t + 1\)
8: until \(\|w_t - w_{t-1}\| < \delta\)
9: \(w_{\pi_k} \leftarrow w_t\)
10: Update the policy \(\pi_{k+1}(s) = \arg \max_{a \in A} \phi(s, a)^\top w_{\pi_k}\)
11: end for

For notational convenience and consistency, we work with value functions instead of Q-value functions. We first make some necessary assumptions.

**Assumption 3.** (i) Each policy \(\pi_k\) is an exploration policy, i.e. \(\pi_{\text{exp}}(\pi_k) = \pi_k\).
(ii) The Markov chain \(P_{\pi_k}\) has a stationary distribution \(d_{\pi_k}\) such that \(d_{\pi_k}(s) > 0, \forall s \in \mathcal{S}\).
(iii) There exists a finite scalar $\delta$ such that $\|V_{\pi_k} - \Pi_{d_{\delta_k}} V_{\pi_k}\|_{d_{\delta_k}} < \delta$ for all $k$, where $\Pi_{d_{\delta_k}}$ is a projection onto the subspace spanned by the columns of $\Phi$ under the $d_{\delta_k}$-weighted Euclidean norm.

(iv) For any distribution $\mu$, define another distribution $\mu_k = \mu H_k$ where $H_k$ is a stochastic matrix defined with respect to $\pi_k$. Also assume that there exists a distribution $\bar{\mu}$ and finite positive scalars $C_1, C_2$ such that $\mu_k \leq C_1 \bar{\mu}$ and $d_{\delta_k} \geq \bar{\mu}/C_2$ for all $k$.

We note that part (iv) of the above assumption is similar to the assumption used in the non-robust approximate policy iterations result given in [19]. The specific form of the stochastic matrix $H_k$ is deferred to the proof of Theorem 3 due to page limitation.

We now give the asymptotic convergence result for the RLSPI algorithm. We only present the case where $\rho = 0$. The proof for the general case is straightforward, but involves much more detailed algebra. So, we omit those details for the clarity of presentation.

**Theorem 3.** Let Assumption 2 and Assumption 3 hold. Let $\{\pi_k\}$ be the sequence of the policies generated by the RLSPI algorithm. Let $V_{\pi_k}$ and $V_k = \Phi w_{\pi_k}$ be true robust value function and the approximate robust value function corresponding to the policy $\pi_k$. Also, let $V^*$ be the optimal robust value function. Then, with $c(\alpha, \beta, 0, \lambda) < 1$,

$$\limsup_{k \to \infty} \|V^* - V_{\pi_k}\|_{\mu} \leq \frac{2\sqrt{C_1 C_2}}{(1 - c(\alpha, \beta, 0, \lambda))^2} \limsup_{k \to \infty} \|V_{\pi_k} - \bar{V}_k\|_{d_{\delta_k}}. \quad (21)$$

Moreover, from Theorem 1 and Assumption 3(iii), we have

$$\limsup_{k \to \infty} \|V^* - V_{\pi_k}\|_{\mu} \leq \frac{2\sqrt{C_1 C_2}}{(1 - c(\alpha, \beta, 0, \lambda))^3} \delta. \quad (22)$$

**5 Experiments**

We implemented our RLSPI algorithm, and evaluated its performance against the non-robust versions of Q-learning algorithm with linear function approximation and LSPI algorithm [21]. We chose a spherical uncertainty set with a radius $r$. For such a set $\mathcal{U}$, a closed form solution of $\sigma^2(\Phi w)$ can be computed for faster simulation. We note that in all the figures shown below, the quantity in vertical axis is averaged over 100 runs, with the thick line showing the averaged value and the band around shows the ±0.5 standard deviation. These figures act as the performance criteria for comparing results. We provide more details and additional experiment results in the supplementary material.

We used the CartPole, MountainCar and Acrobot environments from OpenAI Gym [1]. We trained LSPI algorithm and our RLSPI algorithm on these environments with default parameters. We also trained Q-learning algorithm CartPole environment with default parameters. Then, to evaluate the robustness of the policies obtained, we changed the parameters of these environments and tested the performance of the learned policies on the perturbed environment. The performance of our RLSPI algorithm is consistently superior to that of the non-robust algorithms.

In Figures 1-3 we show the robustness against action perturbations. In real-world setting, due to model mismatch or noise in the environments, the resulting action can be different from the intended action. We model this by picking a random action with some probability at each time step. Figure 1 shows the change in the average episodic reward against the probability of picking a random action for the CartPole environment. Figure 2 shows the average number of time steps to reach the goal in the MountainCar environment. Figure 3 shows the average episodic reward in the Acrobot environment. In all the three cases, RLSPI algorithm shows robust performance against the perturbations.

Figures 4-6 shows the test performance on the CartPole environment, by changing the parameters force_mag (external force disturbance), gravity, length (length of the pole on the cart). The default values of these parameters are 10, 9, 8, and 0.5 respectively. RLSPI again exhibits robust performance.
6 Conclusion and Future Work

We have presented an online model-free reinforcement learning algorithm to learn control policies that are robust against parameter uncertainties of the model. We first proposed a model-free robust policy evaluation algorithm called RLSPE(λ) and rigorously proved its convergence. We then developed a robust policy iteration algorithm called RLSPI algorithm with provable guarantees on the performance. The performance of the proposed algorithms are evaluated on benchmark problems.

In future, we plan to explore more viable structures for tractable RMDP formulations. We also plan to address this problem using batch reinforcement learning techniques, for improving the data efficiency. Off-policy policy gradient methods for robust reinforcement is another exciting but challenging direction.

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Appendix

7 Proofs of the Results in Section 3.1

7.1 Proof of Proposition 1

We first restate Proposition 1 formally and then give the proof.

Proposition 4. (i) For any \( V_1, V_2 \in \mathbb{R}^{|S|} \) and \( \lambda \in [0, 1) \), \( \|T^{(\lambda)}_\pi V_1 - T^{(\lambda)}_\pi V_2\|_\infty \leq \frac{\alpha(1-\lambda)}{1-\alpha\lambda}\|V_1 - V_2\|_\infty \). So, \( T^{(\lambda)}_\pi \) is a contraction in sup norm for any \( \alpha \in (0, 1) \), \( \lambda \in (0, 1) \).

(ii) The robust value function \( V_\pi \) is the unique fixed point of \( T^{(\lambda)}_\pi \), i.e., \( T^{(\lambda)}_\pi V_\pi = V_\pi \), for all \( \alpha \in (0, 1) \) and \( \lambda \in (0, 1) \).

Proof. From (6) we have

\[
\|T^{(\lambda)}_\pi (V_1) - T^{(\lambda)}_\pi (V_2)\|_\infty = \|(1-\lambda) \sum_{m=0}^{\infty} \lambda^m (T^{m+1}_\pi V_1 - T^{m+1}_\pi V_2)\|_\infty
\]

\[
\leq (1-\lambda) \sum_{m=0}^{\infty} \lambda^m \|T^{m+1}_\pi V_1 - T^{m+1}_\pi V_2\|_\infty
\]

\[
\leq (a) (1-\lambda) \sum_{m=0}^{\infty} \lambda^m \alpha^{m+1} \|V_1 - V_2\|_\infty = \frac{\alpha(1-\lambda)}{1-\alpha\lambda}\|V_1 - V_2\|_\infty
\]

where (a) follows since \( T^{(\lambda)}_\pi \) is a contraction operator with contraction modulus \( \alpha \). This proves (i).

Since \( V_\pi \) is the unique fixed point of \( T^{(\lambda)}_\pi \), (ii) directly follows from (i) and the Banach Fixed Point Theorem [22, Theorem 6.2.3].

7.2 Proof of Proposition 2

We first restate Proposition 2 formally and then give the proof.

Proposition 5. Under Assumption 2, for any \( V_1, V_2 \in \mathbb{R}^{|S|} \) and \( \lambda \in [0, 1) \),

\[ \|\Pi T^{(\lambda)}_\pi V_1 - \Pi T^{(\lambda)}_\pi V_2\|_d \leq \frac{\beta(1-\lambda)}{1-\beta\lambda}\|V_1 - V_2\|_d \]

So, \( \Pi T^{(\lambda)}_\pi \) is a contraction mapping in \( d \)-weighted Euclidean norm for any \( \beta \in (0, 1) \), \( \lambda \in (0, 1) \).

Proof. From (6) we have

\[
\|T^{(\lambda)}_\pi (V_1) - T^{(\lambda)}_\pi (V_2)\|_d = \|(1-\lambda) \sum_{m=0}^{\infty} \lambda^m (T^{m+1}_\pi V_1 - T^{m+1}_\pi V_2)\|_d
\]

\[
\leq (1-\lambda) \sum_{m=0}^{\infty} \lambda^m \|T^{m+1}_\pi V_1 - T^{m+1}_\pi V_2\|_d
\]

\[
\leq (a) (1-\lambda) \sum_{m=0}^{\infty} \lambda^m \beta^{m+1} \|V_1 - V_2\|_d = \frac{\beta(1-\lambda)}{1-\beta\lambda}\|V_1 - V_2\|_d
\]

where (a) follows since \( \Pi T^{(\lambda)}_\pi \) is a contraction in the \( d \)-weighted Euclidean norm with contraction modulus \( \beta \) [7, Corollary 4]. From [15], \( \Pi \) is a nonexpansive mapping in the \( d \)-weighted Euclidean norm. So, \( \Pi T^{(\lambda)}_\pi \) has the stated property.

7.3 Derivation of (7)

Given \( w_k, w_{k+1} \) which satisfies the equation \( \Phi w_{k+1} = \Pi T^{(\lambda)}_\pi \Phi w_k \) can be written as the solution of the minimization problem \( w_{k+1} = \arg \min_w \|\Phi w - T^{(\lambda)}_\pi \Phi w_k\|_2^2 \). Taking gradient w.r.t. \( w \) and equating
to zero, we get \( \Phi^T D(\Phi w - T_\pi^{(\lambda)} \Phi w_k) = 0 \), which implies \( w_{k+1} = (\Phi^T D\Phi)^{-1} \Phi^T D T_\pi^{(\lambda)} \Phi w_k \). This can be written as

\[
w_{k+1} = (\Phi^T D\Phi)^{-1} \Phi^T D T_\pi^{(\lambda)} \Phi w_k = w_k + (\Phi^T D\Phi)^{-1} (\Phi^T D T_\pi^{(\lambda)} \Phi w_k - \Phi^T D\Phi w_k)
\]

\[= w_k + (\Phi^T D\Phi)^{-1} \Phi^T D(T_\pi^{(\lambda)} \Phi w_k - \Phi w_k).\]

### 7.4 Derivation of (8)

We have

\[
TV = r_\pi + \alpha P_\pi^o V + \alpha \sigma_{U_\pi}(V)
\]

\[
T^2V = r_\pi + \alpha P_\pi^o (TV) + \alpha \sigma_{U_\pi}(TV)
\]

\[
= (1 + \alpha P)r_\pi + (\alpha P_\pi^o)^2 V + \alpha (\alpha P_\pi^o)\sigma_{U_\pi}(V) + \alpha \sigma_{U_\pi}(TV).
\]

Suppose we have an induction hypothesis for \( T^m V \), i.e.,

\[
T^m V = \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k r_\pi + (\alpha P_\pi^o)^m V + \alpha \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k \sigma_{U_\pi}(T^{(m-1-k)} V).
\]

Now, to verify an induction step, observe that,

\[
T^{m+1} V = \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k r_\pi + (\alpha P_\pi^o)^m TV + \alpha \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k \sigma_{U_\pi}(T^{(m-1-k)} TV)
\]

\[= (a) \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k r_\pi + (\alpha P_\pi^o)^m (r_\pi + \alpha P_\pi^o V + \alpha \sigma_{U_\pi}(V)) + \alpha \sum_{k=0}^{m-1} (\alpha P_\pi^o)^k \sigma_{U_\pi}(T^{(m-1-k)} TV)
\]

\[= \sum_{k=0}^{m} (\alpha P_\pi^o)^k r_\pi + (\alpha P_\pi^o)^{m+1} V + \alpha \sum_{k=0}^{m} (\alpha P_\pi^o)^k \sigma_{U_\pi}(T^{(m-k)} V)
\]

where \( (a) \) follows from (23). Now, (8) directly follows from (6).

### 8 Proofs of the Results in Section 3.2

#### 8.1 Proof of Proposition 3

**Proof.** With \( \hat{U}_\pi = U_\pi \), we have,

\[
\hat{T}_\pi^{(\lambda)}(V_\pi) = (1 - \lambda) \sum_{m=0}^\infty \lambda^m \left[ \sum_{t=0}^{m} (\alpha P_\pi^o)^t r_\pi + \alpha \sum_{t=0}^{m} (\alpha P_\pi^o)^t \sigma_{U_\pi}(V_\pi) + (\alpha P_\pi^o)^{m+1} V_\pi \right]
\]

\[= (1 - \lambda) \sum_{m=0}^\infty \lambda^m \left[ \sum_{t=0}^{m} (\alpha P_\pi^o)^t (r_\pi + \alpha \sigma_{U_\pi}(V_\pi)) + (\alpha P_\pi^o)^{m+1} V_\pi \right]
\]

\[= (1 - \lambda) \sum_{m=0}^\infty \lambda^m \left[ \sum_{t=0}^{m} (\alpha P_\pi^o)^t (I - \alpha P_\pi^o) V_\pi + (\alpha P_\pi^o)^{m+1} V_\pi \right]
\]

\[= (1 - \lambda) \sum_{m=0}^\infty \lambda^m V_\pi = V_\pi.
\]

Here, to get (a), we wrote \( r_\pi + \alpha \sigma_{U_\pi}(V_\pi) = (I - \alpha P_\pi^o) V_\pi \) since \( T_\pi V_\pi = V_\pi \). (b) is from the telescopic sum of the previous equation. \( \square \)
8.2 Proof of Theorem 1

We have the following lemma which is similar to Lemma 4.2 in [9].

Lemma 1. For any vector \( V \in \mathbb{R}^{|\mathcal{S}|} \) and for all \((s, a) \in \mathcal{S} \times \mathcal{A}\),
\[
|\sigma_{\tilde{U}_{s,a}}(V) - \sigma_{U_{s,a}}(V)| \leq \rho \|V\|_{d}.
\]

Proof. First note that, for any \( x, y \in \mathbb{R}^{|\mathcal{S}|} \) we have
\[
x^T y \leq \frac{(x^T D y)}{d_{\min}} \leq \frac{(\|x\|_{d} \|y\|_{d})}{d_{\min}},
\]
where \((a)\) follows from Cauchy-Schwarz inequality with respect to \( \| \cdot \|_{d} \) norm.

Consider any \( p \in \tilde{U}_{s,a} \) and \( q \in U_{s,a} \setminus \tilde{U}_{s,a} \). For any \( V \in \mathbb{R}^{|\mathcal{S}|} \), we have
\[
p^T V = q^T V + (p - q)^T V \geq \sigma_{U_{s,a}}(V) + \min_{x \in \tilde{U}_{s,a}} \min_{y \in U_{s,a} \setminus \tilde{U}_{s,a}} (x - y)^T V
\]
\[
= \sigma_{\tilde{U}_{s,a}}(V) - \rho_{s,a} \|V\|_{d},
\]
where \((b)\) follows from \((24)\). By taking infimum on both sides with respect to \( p \in \tilde{U}_{s,a} \), we get,
\[
\sigma_{\tilde{U}_{s,a}}(V) \geq \sigma_{U_{s,a}}(V) - \rho_{s,a} \|V\|_{d}.
\]
We can also get \( \sigma_{U_{s,a}}(V) \geq \sigma_{\tilde{U}_{s,a}}(V) - \rho_{s,a} \|V\|_{d} \) by similar arguments. Combining, we get,
\[
|\sigma_{\tilde{U}_{s,a}}(V) - \sigma_{U_{s,a}}(V)| \leq \rho_{s,a} \|V\|_{d}.
\]
Since \( \rho = \max_{s,a} \rho_{s,a} \), we get the desired result. \(\square\)

We will use the following result which follows directly from [15] Lemma1.

Lemma 2. Under Assumption 2, for any \( V \), we have \( \|P_{\pi}^V V\|_{d} \leq \|V\|_{d} \).

Remark 1. The inequality in the Assumption 2 can be written as, \( \alpha P_{\pi_{\pi}(s') \in S} + \alpha U_{s_{\pi}(s')} \leq \beta P_{\pi_{\pi}(s') \in S} \). From this, we can conclude that \( \alpha U_{s_{\pi}(s')} \leq \beta P_{\pi_{\pi}(s') \in S} \).

Next, we show the following.

Lemma 3. For any \( V_1, V_2 \in \mathbb{R}^{|\mathcal{S}|} \),
\[
|\sigma_{\tilde{U}_{s,a}}(V_1) - \sigma_{\tilde{U}_{s,a}}(V_2)| \leq \left( \frac{\beta}{\alpha} + \rho \right) \|V_1 - V_2\|_{d}.
\]

Proof. For any \( s \in \mathcal{S} \) we have
\[
\sigma_{\tilde{U}_{s,a}}(V_2) - \sigma_{\tilde{U}_{s,a}}(V_1) = \inf_{q \in \tilde{U}_{s,a}} q^T V_2 - \inf_{q \in \tilde{U}_{s,a}} q^T V_1 = \inf_{q \in \tilde{U}_{s,a}} \sup_{q \in \tilde{U}_{s,a}} q^T V_2 - q^T V_1
\]
\[
\geq \inf_{q \in \tilde{U}_{s,a}} q^T (V_2 - V_1) = \sigma_{\tilde{U}_{s,a}}(V_2 - V_1) \geq \sigma_{U_{s,a}}(V_2 - V_1) - \rho \|V_1 - V_2\|_{d},
\]
where \((a)\) follows from Lemma 1. By definition, for any arbitrary \( \epsilon > 0 \), there exists a \( U_{s_{\pi}(s')} \in U_{s_{\pi}(s')} \) such that
\[
U_{s_{\pi}(s')}^T (V_2 - V_1) - \epsilon \leq \sigma_{U_{s,a}}(V_2 - V_1).
\]
Using (26) and (25),
\begin{align*}
\alpha(\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)) &\leq \alpha U^T_{s,\pi}(V_1 - V_2) + \alpha \epsilon + \rho \alpha \|V_1 - V_2\|_d \\
&\leq \alpha U^T_{s,\pi}(V_2 - V_1) + \alpha \epsilon + \rho \alpha \|V_1 - V_2\|_d \\
&\leq \beta(P^o_{s,\pi(s)})^T(V_1 - V_2) + \alpha \epsilon + \rho \alpha \|V_1 - V_2\|_d
\end{align*}
where (b) follows from Remark 1. Since \( \epsilon \) is arbitrary, we get,
\[\alpha(\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)) \leq \beta(P^o_{s,\pi(s)})^T(V_1 - V_2) + \rho \alpha \|V_1 - V_2\|_d.\]
By exchanging the roles of \( V_1 \) and \( V_2 \), we get \( \alpha(\sigma_{\hat{u}_s}(V_2) - \sigma_{\hat{u}_s}(V_1)) \leq \beta(P^o_{s,\pi(s)})^T(V_1 - V_2) + \rho \alpha \|V_1 - V_2\|_d \). Combining these and writing compactly in vector form, we get
\[\alpha|\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)| \leq \beta P^o_{s,\pi}(V_1 - V_2) + \rho \alpha \|V_1 - V_2\|_d \mathbf{1},\]
where \( \mathbf{1} = (1, 1, \ldots, 1)^T \), an \(|S|\)-dimensional unit vector. Since \( \alpha|\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)| \geq 0 \), by the property of the norm, we get
\[\alpha\|\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)\|_d \leq \|\beta P^o_{s,\pi}(V_1 - V_2) + \rho \alpha \|V_1 - V_2\|_d \mathbf{1}\|_d \]
\[\leq \beta \|P^o_{s,\pi}(V_1 - V_2)\|_d + \rho \alpha \|V_1 - V_2\|_d \mathbf{1}\|_d \]
\[\leq \beta \|V_1 - V_2\|_d + \rho \alpha \|V_1 - V_2\|_d \]
where (c) follows from triangle inequality and (d) from Lemma 2 and from the fact that and \( \|\mathbf{1}\|_d = 1 \). Dividing by \( \alpha \) and rearranging, we get the desired result.

**Proof of Proposition 1** We first observe
\[\|\alpha P^o_{s,\pi}(V_1)\|_d \leq \|\beta P^o_{s,\pi}(V_1)\|_d \leq \beta \|V_1\|_d,\]
where (a) follows from the Assumption 2 and (b) from the Lemma 2
Notice that for any finite \( t \) we have,
\[\|\alpha P^o_{s,\pi}(\alpha P^o_{s,\pi})^{t-1}(V_1)\|_d \leq \beta \|\alpha P^o_{s,\pi}(\alpha P^o_{s,\pi})^{t-1}(V_1)\|_d \]
where (c) follows from (28). Using this repeatedly, we get,
\[\|\alpha P^o_{s,\pi}(V_1)\|_d \leq \beta^t \|V_1\|_d.\]
Now,
\[\|\tilde{T}^{(s)}_{\pi}(V_1) - \tilde{T}^{(s)}_{\pi}(V_2)\|_d \]
\[= \|(1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \alpha \sum_{t=0}^{m} (\alpha P^o_{s,\pi})^t (\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2)) + (\alpha P^o_{s,\pi})^{m+1}(V_1 - V_2)\|_d \]
\[\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \alpha \sum_{t=0}^{m} \|(\alpha P^o_{s,\pi})^t(\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2))\|_d + \|(\alpha P^o_{s,\pi})^{m+1}(V_1 - V_2)\|_d \]
\[\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m (\beta^t - 1) \|(\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2))\|_d + \beta^{m+1} \|V_1 - V_2\|_d \]
\[\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [(\beta + \rho \alpha) \sum_{t=0}^{m} \|(\sigma_{\hat{u}_s}(V_1) - \sigma_{\hat{u}_s}(V_2))\|_d + \beta^{m+1} \|V_1 - V_2\|_d] \]
\[= \left[ (\beta + \rho \alpha) \frac{1 - \frac{\beta(1 - \lambda)}{1 - \beta \lambda}}{(1 - \beta \lambda)} \right] \|V_1 - V_2\|_d \]
\[= \frac{\beta (2 - \lambda) + \rho \alpha \alpha}{(1 - \beta \lambda)} \|V_1 - V_2\|_d, \quad (32)\]
where \( (d) \) follows from (30) and \((e) \) follows from Lemma 3.

From (16), \( T \) is a non-expansive operator in \( \| \cdot \|_d \). Thus,

\[
\| \Pi \tilde{T}_\pi^{(\lambda)} V_1 - \Pi \tilde{T}_\pi^{(\lambda)} V_2 \|_d \leq \| \tilde{T}_\pi^{(\lambda)} (V_1) - \tilde{T}_\pi^{(\lambda)} (V_2) \|_d \leq c(\alpha, \beta, \rho, \lambda) \| V_1 - V_2 \|_d.
\]

where \( c(\alpha, \beta, \rho, \lambda) = (\beta(2 - \lambda) + \rho\alpha)/(1 - \beta\lambda) \). This concludes the proof of getting (10).

For proving (11), first denote the operator \( \tilde{T}_\pi^{(\lambda)}(\lambda) \) by generating an infinitely long trajectory \( s_0, s_1, \ldots \) generated by \( \tilde{T}_\pi^{(\lambda)} \). Now, observe that

\[
\| \tilde{T}_\pi^{(\lambda)}(V) - \tilde{T}_\pi^{(\lambda)}(V) \|_d = \| (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\alpha \sum_{t=0}^{\infty} (\alpha P_\pi^m)^t \| \sigma_{\tilde{U}_\pi}(V) - \sigma_{\tilde{U}_\pi}(V) \|_d
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \sum_{t=0}^{\infty} (\alpha P_\pi^m)^t \| \sigma_{\tilde{U}_\pi}(V) - \sigma_{\tilde{U}_\pi}(V) \|_d
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \sum_{t=0}^{\infty} \| \sigma_{\tilde{U}_\pi}(V) - \sigma_{\tilde{U}_\pi}(V) \|_d
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \frac{(1 - \beta m + 1)}{1 - \beta \lambda} \alpha \rho \| V \|_d = \frac{\alpha \rho \| V \|_d}{1 - \beta \lambda}.
\]

where \( (f) \) follows (30) and \( (g) \) from Lemma 1.

Now,

\[
\| V_\pi - \Phi w_\pi \|_d \leq \| V_\pi - \Pi V_\pi \|_d + \| \Pi V_\pi - \Phi w_\pi \|_d
\]

\[
= (h) \| V_\pi - \Pi V_\pi \|_d + \| \Pi \tilde{T}_\pi^{(\lambda)} V_\pi - \Pi \tilde{T}_\pi^{(\lambda)} \Phi w_\pi \|_d
\]

\[
\leq (i) \| V_\pi - \Pi V_\pi \|_d + \| \Pi \tilde{T}_\pi^{(\lambda)} V_\pi - \Pi \tilde{T}_\pi^{(\lambda)} V_\pi \|_d + \| \Pi \tilde{T}_\pi^{(\lambda)} V_\pi - \Pi \tilde{T}_\pi^{(\lambda)} \Phi w_\pi \|_d
\]

\[
\leq \| V_\pi - \Pi V_\pi \|_d + \| \frac{\beta \rho \| V \|_d}{1 - \beta \lambda} + c(\alpha, \beta, \rho, \lambda) \| V_\pi - \Phi w_\pi \|_d
\]

We get (h) because \( \tilde{T}_\pi^{(\lambda)} V_\pi = V_\pi \) from Proposition 3 and \( \Pi \tilde{T}_\pi^{(\lambda)} \Phi w_\pi = \Phi w_\pi \) by the premise of the proposition. \((i)\) by triangle inequality. \((j) \) from (34) and (10). Rearranging, we get,

\[
\| V_\pi - \Phi w_\pi \|_d \leq \frac{1}{1 - c(\alpha, \beta, \rho, \lambda)} \left( \| V_\pi - \Pi V_\pi \|_d + \frac{\beta \rho \| V \|_d}{1 - \beta \lambda} \right).
\]

This completes the proof of Theorem 1.

\[\square\]

**8.3 Derivation of (13)**

For any bounded mapping \( W \), observe that

\[
\sum_{t=0}^{\infty} (\alpha P_\pi^m)^t W = (1 - \lambda) \sum_{m=0}^{\infty} \sum_{t=0}^{\infty} (\alpha P_\pi^m)^t W,
\]

yielded by exchanging summations. Using this observation with equations (9) and (12) we get (13).

**8.4 Motivation for (16)-(19)**

Consider approximating the optimization problem (computing the fixed point \( \Phi^*_w \) of the mapping \( \Pi \tilde{T}_\pi^{(\lambda)}(\lambda) \)) by generating an infinitely long trajectory \( s_0, s_1, \ldots \) generated by \( P_\pi^o \). Towards this, update \( w_t \) after each transition \( (s_t, s_{t+1}) \) according to

\[
w_{t+1} = \arg \min_{w} \sum_{\tau=0}^{t} \langle \phi(s_\tau)^T w - r(s_\tau, \pi(s_\tau)) - \alpha \sigma_{\pi, \pi(s_\tau)} (\Phi w_t) \rangle^2
\]
where the target value function at time \( \tau \) is considered to be \( r(s_\tau, \pi(s_\tau)) + \alpha \sigma_{P_{s_\tau, \pi(s_\tau)}}(\Phi w_t) \). Now, by the Uncertainty Set assumption, we can rewrite the above equation as

\[
  w_{t+1} = \arg \min_w \sum_{\tau=0}^{t} \left( \phi(s_\tau) w - r(s_\tau, \pi(s_\tau)) - \alpha \phi(i_{\tau+1})^\top w_t - \alpha \sigma_{\tilde{U}_{s_\tau, \pi(s_\tau)}}(\Phi w_t) \right)^2.
\]

Adding and subtracting \( \phi(s_\tau)^\top w_t \) we get

\[
  w_{t+1} = \arg \min_w \sum_{\tau=0}^{t} \left( \phi(s_\tau)^\top w - r(s_\tau, \pi(s_\tau)) - \alpha \phi(i_{\tau+1})^\top w_t - \phi(s_\tau)^\top w_t + \phi(s_\tau)^\top w_t - \alpha \sigma_{\tilde{U}_{s_\tau, \pi(s_\tau)}}(\Phi w_t) \right)^2.
\]

Now, by setting the gradient of the above equation in right hand side w.r.t \( w \) as 0 we get (16)-(19).

8.5 Proof of Theorem 2

To prove this theorem, we will use the following result from [18]. Let \( \| \cdot \| \) denote the standard Euclidean norm.

**Proposition 6.** [18, Proposition 4.1] Consider a sequence \( \{x_t\} \) generated by the update equation

\[
  x_{t+1} = x_t + \gamma_t (h_t(x_t) + e_t),
\]

where \( h_t : \mathbb{R}^n \to \mathbb{R}^n \), \( \gamma_t \) is a positive deterministic stepsize, and \( e_t \) is a random noise vector. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function. Assume the following:

(i) Function \( f \) is positive, i.e., \( f(x) \geq 0 \). Also, \( f \) has Lipschitz continuous gradient, i.e., there exists some scalar \( L > 0 \) such that

\[
  \| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|, \quad \forall x, y \in \mathbb{R}^n.
\]

(ii) Let \( F_t = \{x_0, x_1, \ldots, x_t\} \). There exists positive scalars \( c_1, c_2, \text{ and } c_3 \) such that

\[
  (\nabla f(x_t))^\top \mathbb{E}[h_t(x_t)|F_t] \leq -c_1 \| \nabla f(x_t) \|^2, \quad \forall t,
\]

\[
  \| \mathbb{E}[e_t|F_t] \| \leq c_2 \gamma_t (1 + \| \nabla f(x_t) \|), \quad \forall t,
\]

\[
  \mathbb{E}[\| h_t(x_t) + e_t \|^2 | F_t] \leq c_3 (1 + \| \nabla f(x_t) \|^2), \quad \forall t,
\]

where \( e_t \) is a positive deterministic scalar.

(iii) The deterministic sequences \( \{\gamma_t\} \) and \( \{\epsilon_t\} \) satisfy

\[
  \sum_{t=0}^{\infty} \gamma_t = \infty, \quad \sum_{t=0}^{\infty} \epsilon_t^2 < \infty, \quad \sum_{t=0}^{\infty} \gamma_t \epsilon_t^2 < \infty, \quad \lim_{t \to \infty} \epsilon_t = 0.
\]

Then, with probability 1:

(i) The sequence \( \{f(x_t)\} \) converges.

(ii) The sequence \( \{\nabla f(x_t)\} \) converges to zero.

(iii) Every limit point of \( \{x_t\} \) is a stationary point of \( f \).

Using the above result, we now prove Theorem 2.

**Proof of Theorem 2.** We rewrite (16) as \( w_{t+1} = w_t + \gamma_t (h_t(w_t) + e_t) \), where,

\[
  h_t(w_t) = B_t^{-1}(A_t w_t + b + C(w_t)),
\]

\[
  e_t = B_t^{-1}(A_t w_t + b_t + C_t(w_t)) - B_t^{-1}(A_t w_t + b + C(w_t)).
\]

Define

\[
  f(w) = \frac{1}{2}(w - w_\tau)^\top \Phi^\top D\Phi(w - w_\tau).
\]
We now verify the conditions given in Proposition 6.

Verifying (36): By definition $f(w) \geq 0$. Also, $\nabla f(w) = \Phi^T D \Phi (w - w_\pi)$. Hence,
\[
\| \nabla f(w) - \nabla f(w') \| = \| \Phi^T D \Phi (w - w') \| \leq \| \Phi^T D \Phi \|_{\text{op}} \| w - w' \|,
\]
where $\| \cdot \|_{\text{op}}$ is the operator norm corresponding to the Euclidean space. Set $L = \| \Phi^T D \Phi \|_{\text{op}}$. From (ii) in Assumption 2 and $\Phi$ being full rank, we know that $\Phi^T D \Phi$ is a positive definite matrix and hence $L$ is a positive scalar. This verifies (36).

Verifying (37): It is straightforward to verify that $Aw_t + b + C(w_t) = \Phi^T D(\bar{T}^\lambda) \Phi w_t - \Phi w_t$ by comparing (12) - (15). Then,
\[
(\nabla f(w_t))^\top \mathbb{E}[h_t(w_t) | F_t] = (w_t - w_\pi)^\top (Aw_t + b + C(w_t))
= (w_t - w_\pi)^\top \Phi^T D(\bar{T}^\lambda) \Phi w_t - \Phi w_t < 0,
\]
where the last inequality is by using Lemma 9 from [15] with the fact that, from Theorem 1, $\Pi(\bar{T}^\lambda)$ is a contraction. Since $(\nabla f(w_t))^\top \mathbb{E}[h_t | F_t]$ is strictly negative, we can find a positive scalar $c_2$ that satisfies (37).

Verifying (38): We can write $e_t = \Delta^1_t w_t + \Delta^2_t + \Delta^3_t (w_t)$, where,
\[
\Delta^1_t = B_t^{-1} A_t - B^{-1} A, \quad \Delta^2_t = B_t^{-1} b_t - B^{-1} b, \quad \Delta^3_t (w_t) = B_t^{-1} C_t (w_t) - B^{-1} C (w_t).
\]
From Proposition 2.1 of [18], we have
\[
\| \mathbb{E}[\Delta^1_t] \| \leq \frac{\bar{c}_1}{\sqrt{t+1}}, \quad \| \mathbb{E}[\Delta^2_t] \| \leq \frac{\bar{c}_2}{\sqrt{t+1}} \quad \forall t.
\]
(41)

So, we will now bound $\mathbb{E}[\Delta^3_t (w_t) | F_t]$.
\[
\begin{align*}
\mathbb{E}[\Delta^3_t (w_t) | F_t] &= \mathbb{E}[B_t^{-1} C_t (w_t) - B^{-1} C (w_t) | F_t] \\
&= \mathbb{E}[B_t^{-1} C_t (w_t) - B_t^{-1} C_t (w_t) + B^{-1} C_t (w_t) - B^{-1} C (w_t) | F_t] \\
&\leq \mathbb{E}[\| B_t^{-1} - B^{-1} \| \| C_t (w_t) \| | F_t] + \| B^{-1} \| \mathbb{E}[\| C_t (w_t) - C (w_t) \| | F_t].
\end{align*}
\]
(42)

First consider $\| C_t (w_t) \|$. We can bound
\[
\| z_r \| \leq \sum_{m=0}^\tau (\alpha \lambda)^{\tau-m} \| \phi(s_m) \| \leq \bar{c} \quad (a)
\]
\[
| \sigma_{\tilde{u}_{r,s(r)}} (\Phi w_t) | = \sup_{u \in \tilde{U}_{r,s(r)}} u^\top (\Phi w_t) \leq \sup_{u \in \tilde{U}_{r,s(r)}} \| u \| \| \Phi w_t \| \leq \bar{c}_2 \| w_t \| \quad (b)
\]
(33)

(43)

where $\bar{c}_3$ and $\bar{c}_3$ are finite positive scalars. For (a), we used the fact that $\sup_s \| \phi(s) \| < \infty$. For (b), we used the fact that $\sup_{u \in \tilde{U}_{r,s(r)}} \| u \| < \infty$ since $\tilde{U}_{r,s(r)}$ is a finite set for any $s \in S$ and $\| \Phi w_t \| \leq c' \| w_t \|$ for some finite positive scalar $c'$. Using (43) and (44).
\[
\| C_t (w_t) \| \leq \frac{\alpha}{(t+1)^{\frac{1}{2}}} \sum_{r=0}^t \| z_r \| \sigma_{\tilde{u}_{r,s(r)}} (\Phi w_t) \| \leq \frac{\alpha}{(t+1)^{\frac{1}{2}}} \sum_{r=0}^t \| z_r \| \sigma_{\tilde{u}_{r,s(r)}} (\Phi w_t) \| \leq \bar{c} \| w_t \|.
\]
(44)

From Lemma 4.3 of [18], we have $\mathbb{E}[\| B_t^{-1} - B^{-1} \|] \leq \bar{c} \sqrt{(t+1)}$. Using this and (45), we get,
\[
\mathbb{E}[\| B_t^{-1} - B^{-1} \| \| C_t (w_t) \|] \leq \frac{\bar{c}_5}{\sqrt{t+1}} \| w_t \|.
\]
(46)
For bounding $\mathbb{E}[\|C_t(w_t) - C(w_t)\|]$ in \((42)\), we define $V_t$ and $n_t$ as,

$$V_t = \frac{\alpha}{(t+1)} \sum_{m=0}^{t} \phi(s_m) \sum_{\tau=1}^{\infty} \left[ (\alpha \lambda P^\pi_\tau)^{-m} \sigma_\tau w_t \right](s_m), \quad n_t(s) = \sum_{\tau=0}^{t} I(s_\tau = s).$$

Note that $n_t(s)$ is the number of visits to state $s$ until time $t$. Then,

$$\mathbb{E}[C_t(w_t)|F_t] = \mathbb{E} \left[ \frac{\alpha}{(t+1)} \sum_{m=0}^{t} \sum_{\tau=0}^{\tau} (\alpha \lambda)^{\tau-m} \phi(s_m) \mathbb{E} \left[ \sigma_\tau w_t | F_m \right] | F_t \right]$$

\((a)\)

$$\mathbb{E} \left[ \frac{\alpha}{(t+1)} \sum_{m=0}^{t} \sum_{\tau=0}^{\tau} (\alpha \lambda)^{\tau-m} \phi(s_m) (P^\pi_\tau)^{\tau-m} \sigma_\tau w_t(s_m) | F_t \right]$$

\((b)\)

$$\mathbb{E} \left[ \frac{\alpha}{(t+1)} \sum_{m=0}^{t} \phi(s_m) \sum_{\tau=0}^{\tau} (\alpha \lambda P^\pi_\tau)^{\tau-m} \sigma_\tau w_t(s_m) | F_t \right] - \mathbb{E}[V_t | F_t]$$

\((c)\)

$$\mathbb{E} \left[ \frac{\alpha}{(t+1)} \sum_{s \in \mathcal{S}} n_t(s) \phi(s) \sum_{i=0}^{\infty} \left( (\alpha \lambda P^\pi_\tau)^i \sigma_\tau w_t(s) \right) | F_t \right] - \mathbb{E}[V_t | F_t], \quad (47)$$

where \((a)\) follows since the transition probability matrix governing along policy $\pi$ is $P^\pi_\tau$, \((b)\) by exchanging the order of summation, and \((c)\) by using the definition of $n_t(s)$. Note that,

$$\mathbb{E}[C(w_t)|F_t] = \mathbb{E}[\alpha \Phi^T D \sum_{i=0}^{\infty} (\alpha \lambda P^\pi_\tau)^i \sigma_\tau w_t | F_t]$$

$$= \mathbb{E}[\alpha \sum_{s \in \mathcal{S}} d_s \phi(s) \sum_{i=0}^{\infty} ((\alpha \lambda P^\pi_\tau)^i \sigma_\tau w_t(s) | F_t]. \quad (48)$$

So, using \((48)\) and \((47)\),

$$\| \mathbb{E} \left[ (C_t(w_t) - C(w_t)) | F_t \right] \|$$

\((d)\)

$$\leq \frac{\alpha}{(t+1)} \sum_{s \in \mathcal{S}} \left( \frac{\mathbb{E}[n_t(s)]}{t+1} - d_s \right) \phi(s) \sum_{i=0}^{\infty} ((\alpha \lambda P^\pi_\tau)^i \sigma_\tau w_t(s) - \mathbb{E}[V_t | F_t])$$

\((c)\)

$$\leq \frac{\bar{c}_{36}}{t+1} \| w_t \| + \mathbb{E}[\| V_t \| | F_t] \leq \frac{\bar{c}_{37}}{t+1} \| w_t \| + \frac{\bar{c}_{38}}{t+1} \| w_t \| \quad (49)$$

For getting \((d)\), note that $\| \phi(s) \sum_{i=0}^{\infty} ((\alpha \lambda P^\pi_\tau)^i \sigma_\tau w_t(s)) \| \leq c' \| w_t \|$ for some positive constant $c'$, using \((44)\) and the fact that the summation is bounded due to the discounted factor $(\alpha \lambda)$. For getting \((c)\), we use the result from Lemma 4.2 \([18]\) that $\left| \frac{\mathbb{E}[n_t(s)]}{t+1} - d_s \right| \leq \frac{c' \| w_t \|}{t+1}$ for some positive number $c'$. Also, it is straightforward to show that $\mathbb{E}[\| V_t \| | F_t] \leq c' \| w_t \| t+1$ for some positive number $c'$ using \((44)\) and the fact that the summation is bounded due to the discounted factor $(\alpha \lambda)$.

Using \((46)\) and \((49)\) in \((42)\), we get

$$\| \mathbb{E}[\Delta_t^2(w_t)] \| \leq \frac{\bar{c}_3}{\sqrt{t+1}} \| w_t \|. \quad (50)$$

Notice that

$$\| w_t \| \leq \| w_t - w_\pi \| + \| w_\pi \| \leq \| (\Phi^T D \Phi)^{-1} \| \| (\Phi^T D \Phi)(w_t - w_\pi) \| + \| w_\pi \|$$

$$\leq \bar{c}_{39}(1 + \| \nabla f(w_t) \|). \quad (51)$$
Now,

\[
\|E[c_i|F_t]\| = |E[\Delta_1^i]| \|w_t\| + |E[\Delta_2^i]| \|w_t\| + |E[\Delta_3^i(w_t)|F_t]\| \leq \frac{c_1}{\sqrt{t+1}} \|w_t\| + \frac{c_2}{\sqrt{t+1}} \|w_t\| \leq \frac{c_3}{\sqrt{t+1}} (1 + \|\nabla f(w_t)\|) \quad (52)
\]

where \((f)\) is by using \((41)\) and \((50)\) and \((g)\) is by using \((51)\). This completes the verification of the condition \((38)\).

Verifying \((39)\): From definition, we have \(h_t(w_t) + e_t = B_t^{-1}(A_t w_t + b_t + C_t(w_t))\), for all \(t\). Using similar steps as before, it is straightforward to show that \(\|B_t\| \leq c_{41}, |A_t| \leq c_{42}, |b_t| \leq c_{43}\) for some positive scalars \(c_{41}, c_{42}, c_{43}\). From \((45)\) we have \(\|c_t(w_t)\| \leq \tilde{c}_{33}\|w_t\|\). Combining these, we get

\[
\|h_t(w_t) + e_t\| \leq \tilde{c}_{44}(1 + \|w_t\|). \quad (53)
\]

Now, from \((51)\) and \((36)\), \(\|h_t(w_t) + e_t\|^2 \leq \tilde{c}_{45}(1 + \|\nabla f(w_t)\|^2)\) for all \(t\). So, finally we have,

\[
E[\|h_t(w_t) + e_t\|^2|F_t] = E[\|h_t(w_t) + e_t\|^2|w_t] \leq \tilde{c}_3(1 + \|\nabla f(w_t)\|^2) \quad \forall t,
\]

showing that \((39)\) is satisfied.

Verifying \((40)\): From \((52)\), \(e_t = 1/\sqrt{(t+1)}\). So, this condition is satisfied. So, all the assumption of Proposition \((6)\) are satisfied. Hence the result of that Proposition is true. In particular, \(\nabla f(w_t) = \Phi^\top D\Phi(w_t - w_\pi)\) converges to 0. Since \(\Phi^\top D\Phi\) is positive definite, this implies that \(w_t \to w_\pi\). \(\square\)

9 Proof of the Results in Section 4

Let \(\tilde{V}_k = \Phi w_{\pi_k}\) and \(V^*\) be the optimal robust value function. Define

\[
e_k = V_{\pi_k} - \tilde{V}_k, \quad l_k = V^* - V_{\pi_k}, \quad g_k = V_{\pi_{k+1}} - V_{\pi_k}. \quad (54)
\]

Interpretations of these expressions: Since the robust value function in the \(k\)th iteration \(\tilde{V}_k\) is used as a surrogate for the robust value function \(V_{\pi_k}\), \(e_k\) quantifies the approximation error. \(g_k\) signifies the gain of value functions between iterations \(k\) and \(k+1\). Finally, \(l_k\) encapsulates the loss in the value function because of using policy \(\pi_k\) instead of the optimal policy.

Let \(|x|\) denote element-wise absolute values of vector \(x \in \mathbb{R}^S\). We first prove the following result. This parallels to the result for the nonrobust setting in \((19)\).

**Lemma 4.** We have

\[
|l_{k+1}| \leq c(\alpha, \beta, 0, \lambda)\tilde{H}_k(|l_k| + |e_k|) + c(\alpha, \beta, 0, \lambda)\tilde{H}_k(|g_k| + |e_k|) \quad \text{and} \quad (55)
\]

\[
|g_k| \leq c(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\tilde{H}_{k+1})^{-1}(\tilde{H}_{k+1} + \tilde{H}_k)|e_k| \quad (56)
\]

where the stochastic matrices \(\tilde{H}_k, \tilde{H}_k\), and \(\tilde{H}_{k+1}\) are defined in \((59)-(60)\).

**Proof.** As before, denote the operator \(\tilde{T}^{(\lambda)}_\pi\) as \(\tilde{T}^{(\lambda)}_\pi\) when \(\tilde{U}_\pi = U_\pi\). Now, similar to \((31)\), for any policy \(\pi\) and \(V_1, V_2\), we get that

\[
|\tilde{T}^{(\lambda)}_\pi(V_1) - \tilde{T}^{(\lambda)}_\pi(V_2)| = |(1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\alpha \sum_{t=0}^{m} (\alpha P^o_\pi)^t (\sigma_{\tilde{U}_\pi}(V_1) - \sigma_{\tilde{U}_\pi}(V_2)) + (\alpha P^o_\pi)^{m+1}(V_1 - V_2)]|
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\alpha \sum_{t=0}^{m} |(\alpha P^o_\pi)^t|\sigma_{\tilde{U}_\pi}(V_1) - \sigma_{\tilde{U}_\pi}(V_2)| + |(\alpha P^o_\pi)^{m+1}|V_1 - V_2|] \quad (a)
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m \beta \sum_{t=0}^{m} (\beta P^o_\pi)^t|V_1 - V_2| + |(\beta P^o_\pi)^{m+1}|V_1 - V_2| \quad (b)
\]

(57)
where (a) follows from Assumption \(2\) and \(\pi_c\) (dependent on \(\pi\)) being the exploration policy. (b) follows from (27) in Lemma \([3]\).

Recall that the optimal robust value function \(V^*\) and the optimal robust policy \(\pi^*\) satisfy the equation \(T_{\pi^*}^\lambda V^* = V^*\). Using this,

\[
l_{k+1} = V^* - V_{\pi_{k+1}} = \bar{T}_{\pi}^\lambda V^* - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}}
\]

\[
\leq (\bar{T}_{\pi}^\lambda V^* - T_{\pi}^\lambda V_{\pi_k}) + (T_{\pi}^\lambda V_{\pi_k} - \bar{T}_{\pi}^\lambda V_{\pi_k}) + (T_{\pi}^\lambda \bar{V}_k - \bar{T}_{\pi}^\lambda \bar{V}_k)
\]

\[
+ (T_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_k}) + (T_{\pi_{k+1}}^\lambda V_{\pi_k} - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_k})
\]

\[
\leq (\bar{T}_{\pi}^\lambda V^* - T_{\pi}^\lambda V_{\pi_k}) + (T_{\pi}^\lambda V_{\pi_k} - \bar{T}_{\pi}^\lambda V_{\pi_k})
\]

\[
+ (T_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_k}) + (T_{\pi_{k+1}}^\lambda V_{\pi_k} - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_k})
\]

\[
\leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t (|l_k| + |e_k|)] + (\beta P_{\pi_{k+1}}^o)^{m+1} (|l_k| + |e_k|)]
\]

\[
+ (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t (|g_k| + |e_k|)] + (\beta P_{\pi_{k+1}}^o)^{m+1} (|g_k| + |e_k|)]
\]

\[
= c(\alpha, \beta, 0, \lambda) \bar{H}_k (|l_k| + |e_k|) + c(\alpha, \beta, 0, \lambda) \bar{H}_k (|l_k| + |e_k|)
\]

(58)

Here (c) follows because \(\pi_{k+1}\) is the greedy policy w.r.t. \(\bar{V}_k\) and hence \(\bar{T}_{\pi}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k \leq 0\). (d) follows from (57) noting (i) in Assumption [3]. Finally, (e) follows by taking

\[
\bar{H}_s = \frac{1}{c(\alpha, \beta, 0, \lambda)} (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t + (\beta P_{\pi_{k+1}}^o)^{m+1}], \quad \text{and}
\]

\[
\bar{H}_j = \frac{1}{c(\alpha, \beta, 0, \lambda)} (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t + (\beta P_{\pi_{k+1}}^o)^{m+1}], \quad \text{for } j \geq 1.
\]

(59)

(60)

Note that, matrices \(\bar{H}_s\) and \(\bar{H}_j\) are stochastic matrices. This follows easily by verifying \(\bar{H}_s \mathbf{1} = \mathbf{1}\). \(\bar{H}_j \mathbf{1} \neq \mathbf{1}\) using analysis in (32).

The same argument can be repeated to get

\[-l_{k+1} \leq c(\alpha, \beta, 0, \lambda) \bar{H}_s (|l_k| + |e_k|) + c(\alpha, \beta, 0, \lambda) \bar{H}_k (|g_k| + |e_k|).
\]

Combining, we get

\[|l_{k+1}| \leq c(\alpha, \beta, 0, \lambda) \bar{H}_s (|l_k| + |e_k|) + c(\alpha, \beta, 0, \lambda) \bar{H}_k (|g_k| + |e_k|).
\]

Now,

\[g_k = \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} - \bar{T}_{\pi_k}^\lambda V_{\pi_k}
\]

\[= \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} + \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} - \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k + (\bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k) + \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}}
\]

\[\geq \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} + \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} - \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k + \bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}}
\]

where the last inequality follows because \(\pi_{k+1}\) is the greedy policy w.r.t. \(\bar{V}_k\) and hence \(\bar{T}_{\pi_{k+1}}^\lambda \bar{V}_k - \bar{T}_{\pi_{k+1}}^\lambda V_{\pi_{k+1}} \geq 0\). From (57), noting (i) in Assumption (3) we have

\[-g_k \leq (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t (|g_k| + |e_k|)] + (\beta P_{\pi_{k+1}}^o)^{m+1} (|g_k| + |e_k|)]
\]

\[+ (1 - \lambda) \sum_{m=0}^{\infty} \lambda^m [\beta \sum_{t=0}^{m} (\beta P_{\pi_{k+1}}^o)^t (|e_k|)] + (\beta P_{\pi_{k+1}}^o)^{m+1} (|e_k|)]
\]

\[\leq c(\alpha, \beta, 0, \lambda) \bar{H}_k (|g_k| + |e_k|) + c(\alpha, \beta, 0, \lambda) \bar{H}_k (|e_k|).
\]
Repeating the above argument for \(-g_k = \hat{T}_{\pi_k}^{(\lambda)} V_{\pi_k} - \hat{T}_{\pi_{k+1}}^{(\lambda)} V_{\pi_{k+1}}\), we get
\[
|g_k| \leq c(\alpha, \beta, 0, \lambda) \hat{H}_{k+1} |(g_k| + |e_k|) + c(\alpha, \beta, 0, \lambda) \hat{H}_k |e_k|.
\]
So, \(|g_k| \leq c(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\hat{H}_{k+1})^{-1}(\hat{H}_{k+1} + \hat{H}_k)|e_k|\). Thus, proving the lemma.

**Proof of Theorem 3.** From the Lemma 4 taking \(\limsup\) on both sides of (55) we have
\[
\limsup_{k \to \infty} |l_k| \leq \limsup_{k \to \infty} c(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\hat{H}_s)^{-1}(\hat{H}_s + \hat{H}_k)|e_k|
\]
\[
+ c(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\hat{H}_s)^{-1}\hat{H}_k|e_k|
\]
\[
\leq \limsup_{k \to \infty} c(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\hat{H}_s)^{-1}(\hat{H}_s + \hat{H}_k)|e_k|
\]
\[
+ c^2(\alpha, \beta, 0, \lambda)(I - c(\alpha, \beta, 0, \lambda)\hat{H}_s)^{-1}\hat{H}_k(I - c(\alpha, \beta, 0, \lambda)\bar{H}_{k+1})^{-1}(\bar{H}_{k+1} + \bar{H}_k)|e_k|
\]
\[
\leq \frac{2c(\alpha, \beta, 0, \lambda)}{(1 - c(\alpha, \beta, 0, \lambda)^2)} \limsup_{k \to \infty} H_k|e_k|
\]
(61)
where (a) follows from (56) in Lemma 4 (b) follows by taking
\[
H_k = (1 - c(\alpha, \beta, 0, \lambda))^2(I - c(\alpha, \beta, 0, \lambda)\hat{H}_s)^{-1}\left(\frac{\hat{H}_s + \hat{H}_k}{2}\right)
\]
\[
+ c(\alpha, \beta, 0, \lambda)\hat{H}_k(I - c(\alpha, \beta, 0, \lambda)\bar{H}_{k+1})^{-1}\left(\bar{H}_{k+1} + \bar{H}_k\right).\]
(62)
Notice that \(H_k\) is a stochastic matrix. To see this, we know that, if \(P_i, i = \{1, 2, 3, 4\}\) are stochastic matrices and \(c < 1\), then \(P_i P_j P_k P_i, (P_i + P_j)/2\), and \((1 - c)(I - cP_i)^{-1}\) are valid stochastic matrices as well. Now, it is easy to verify that \(H_k 1 = 1\). Then, \(\mu_k = \mu H_k\) is a valid probability distribution.

Let \(x^2\) denote element-wise squares of vector \(x \in \mathbb{R}^{|S|}\). Now, from (61) we have
\[
\limsup_{k \to \infty} \|l_k\|_2^2 = \limsup_{k \to \infty} \mu H_k|e_k|^2 \leq \frac{4c^2(\alpha, \beta, 0, \lambda)}{(1 - c(\alpha, \beta, 0, \lambda))^4} \limsup_{k \to \infty} \mu H_k|e_k|^2
\]
\[
\leq \frac{4c^2(\alpha, \beta, 0, \lambda)}{(1 - c(\alpha, \beta, 0, \lambda))^4} \limsup_{k \to \infty} \mu|e_k|^2 = \frac{4c^2(\alpha, \beta, 0, \lambda)}{(1 - c(\alpha, \beta, 0, \lambda))^4} \limsup_{k \to \infty} \|e_k\|^2_{\mu_k}
\]
\[
\leq \frac{4C_1 C_2 c^2(\alpha, \beta, 0, \lambda)}{(1 - c(\alpha, \beta, 0, \lambda))^4} \limsup_{k \to \infty} \|e_k\|^2_{\mu_k}.
\]
Here (c) follows from Jensen’s inequality. To see this, let
\[
H_k = \begin{pmatrix}
-q_1...
-q_2...
\vdots
-q_{|S|}
\end{pmatrix}
\]
where \(q_i \in \mathbb{R}^{|S|}\), for all \(i\), are probability vectors.

For any \(x \in \mathbb{R}^{|S|}\), let \(x(j)\) denote the \(j^{th}\) coordinate value in \(x\). Now, for each \(i \in \{1, 2, \ldots, |S|\}\), define \(|S|\)-discrete valued random variable \(X_i\) such that it takes value \(|e_k|\) with probability \(q_i(j)\) for all \(j \in \{1, 2, \ldots, |S|\}\). Thus, from Jensen’s inequality, we have
\[
[H_k|e_k|^2 = ((E[X_1]), (E[X_2]), \ldots, (E[X_\lambda]))^T \leq (E[X_1^2], E[X_2^2], \ldots, E[X_\lambda^2])^T = H_k|e_k|^2.
\]
(d) follows by noting that for any \(x \in \mathbb{R}^{|S|}\), from (iv) in Assumption 3, we have \(\|x\|_{\mu_k}^2 \leq c_1 |x|^2 \leq C_1 C_2 \|x\|_{\mu_k}^2 \) for all \(k\). Thus proving (21) of this theorem.

Now, since \(\rho = 0, 22\) of this theorem directly follows from (iii) in Assumption 3 and (11) in Theorem 1 This completes the proof of this theorem.
10 Experiments

In all the experiments reported, we use a spherical uncertainty set \( \{ x : \| x \|_2 \leq r \} \) where \( r \) is the radius parameter. For such a set, we can compute a closed form solution for \( \sigma_{\hat{w}, \pi(x)} (\Phi w) \) as \( \sqrt{r w^\top \Phi^\top \Phi w} \) \cite{21}. Note that, we can precompute \( \Phi^\top \Phi \) once and reuse it in every iteration of the RLSPI Algorithm, thus saving the computational overhead.

Chain MDP: We first consider a tabular MDP problem represented in the Figure 7 for verifying the convergence of RLSPI algorithm. This MDP consists of 10 states depicted by circles here. We have two actions, that is, move left or right. The actions fail to remain in a given direction with probability 0.1, depicted by the red arrows. Thus, with probability 0.9, actions succeed to be in a given direction, depicted by the blue (action left being unchanged) and green (action right being unchanged) arrows. Finally, visiting states of yellow color, that is 0 and 9, are rewarded 1, and visiting other states are rewarded 0.

\cite{21} observes that learning algorithms often attain sub-optimal policies under such MDPs due to the randomization of actions (as depicted by red arrows in the Figure 7). It is also straightforward that the optimal policy of this MDP is moving left for states 0 through 4 and moving right for states 5 through 9. We train RLSPI algorithm on this MDP with \( \alpha = 0.9, \lambda = 0.0 \). We use the space spanned by polynomials, degree up to 2, as the feature space and set \( \delta = 0.1 \) (error of weights as mentioned in Step 8 of the RLSPI algorithm). We select \( r \) as 0.01 times the constant \( \| \Phi^\top \Phi \|_F^{-1} \) where \( \| \cdot \|_F \) is the Frobenius norm.

Figure 8 shows how the Q-value functions in RLSPI algorithm training evolve as the iteration progress. From this, we note that RLSPI algorithm is able to find the optimal policy with relatively less number of iterations. From this figure, we also note that the Q-value functions corresponding to the optimal policy in RLSPI algorithm converges to the optimal robust value function.

Examples from OpenAI Gym \cite{1}: We now provide more details for the OpenAI Gym experiments demonstrated in Section 5. We use the radial basis functions (RBFs) for the purpose of feature spaces in our experiments. The general expression for RBFs is \( \psi(x) = \exp(-\|x - \mu\|^2/\sigma) \) where the RBF parameters \( \mu \) and \( \sigma \) are chosen before running the experiment. Here \( x \) is a concatenation of states and actions when both \( S, A \) are continuous spaces. In this case, the feature map is simply defined as \( \phi(s, a) = \psi((s, a)) \) where \((s, a)\) represents the concatenation operation. While working with experiments whose action space is discrete, we naturally choose \( \phi(s, a) \) to be the vector \((I(a = 1)\psi(s), \ldots, I(a = |A|)\psi(s))^\top \) where \( I(E) \) is the indicator function which produces value 1 if the event \( E \) is true, and 0 otherwise. After a few trials, we observed that using few (typically 3-8) uniformly spaced RBFs in each dimension of \( x \), here \( x \) is as described before, with approximate overlap percentage of 2.5 works suitably for getting the desired results shown here and in Section 5. Figure 9 illustrates this for the case of using two \((= n)\) uniformly spaced RBFs with one dimensional \( x \) variable. For this illustration, we have the low(\( l \)) and high(\( h \)) values...
of $x$ to be $-0.5$ and $0.5$ respectively. Thus, the centers (i.e., parameter $\mu$) of the two uniformly spaced RBFs in Figure 9 are $-0.5$ and $0.5$. We select the parameter $\sigma$ as $\frac{(h-l)^2}{n^3} = 0.125$.

We execute this idea on the OpenAI Gym environments. We also experiment on FrozenLake8x8 OpenAI Gym environment, for which we use the tabular feature space since the state space is discrete.

A short description of the OpenAI Gym tasks CartPole, MountainCar, Acrobot, and FrozenLake8x8 we used are as follows.

**CartPole:** By a hinge, a pole is attached to a cart, which moves along a one-dimensional path. The motion of the cart is controllable, which is either to move it left or right. The pole starts upright, and the goal is to prevent it from falling over. A reward of $+1$ is provided for every time-step that the pole remains upright. CartPole consists of a 4-dimension continuous state space with 2 discrete actions.

**MountainCar:** A car is placed in the valley and there exists a flag on top of the hill. The goal is to reach the flag. The control signal is the acceleration and deceleration in continuous domain. A reward of 0 is provided if the car reaches goal, otherwise it is provided $-1$. MountainCar consists of a 3-dimension $S \times A$ continuous space.

**Acrobot:** Two poles attached to each other by a free moving joint and one of the poles is attached to a hinge on a wall. Initially, the poles are hanging downwards. An action, positive and negative torques can be applied to the movable joint. A reward of $-1$ is provided every time-step until the end of the lower pole reaches a given height, at termination 0 reward is provided. The goal is to maximize the reward gathered. Acrobot consists of a 6-dimension continuous state space with 3 discrete actions.

**FrozenLake8x8:** A grid of size $8 \times 8$ consists of some tiles which lead to the agent falling into the water. The agent is rewarded 1 after reaching a goal tile without falling and rewarded 0 in every other timestep.

In Section 5, we provided performance evaluation curves in Figures 1-6. Here, we provide more results.

Figures 10 shows the average time steps to reach the goal in MountainCar environment as we change the parameter $\text{max\_speed}$. The default value of this parameter is 0.07. As the parameter deviates from the default value, the performance of the policy obtained by the LSPI algorithm degrades quickly whereas the performance of the policy obtained by the RLSPI algorithm is fairly robust. Figure 11 shows the average cumulative reward on the MountainCar environment as we
change the parameter \textit{power}. The default value of this parameter is $15 \times 10^{-4}$. We again note that the RLSPI algorithm showcases robust performance. Figure 12 shows the ratio of average time to reach the goal and the number of trajectories which actually reach the goal on the FrozenLake8x8 environment against probability of picking a random action. Note that for large values of this probability both algorithms take more time to reach the goal or often fall into the water. Here again, RLSPI shows robust performance.

In each policy iteration loop, in both LSPI and RLSPI algorithms, we generate $t$ trajectories of horizon length $h$ using the last updated policy (the initial policy $\pi_0$ is random.) We generally stop the simulation after 10-20 policy iteration loops. The details of the parameters are shown in Table 1 in addition to $\lambda$ being set to zero.

| OpenAI Gym Environment | Discount $\alpha$ | Weights error $\delta$ | $t$ | $h$ |
|------------------------|------------------|------------------------|-----|-----|
| CartPole               | 0.95             | 0.01                   | 150 | 200 |
| MountainCar            | 0.95             | 0.05                   | 1000| 20  |
| Acrobat                | 0.98             | 0.1                    | 100 | 200 |
| FrozenLake8x8          | 0.99             | 0.01                   | 3000| 200 |

Table 1: Details of hyper-parameters in the experiments.

For completeness, we also point out some weaknesses of the experiments we have done. Firstly, we are not optimizing over the parameter $r$ which is the radius of the spherical set associated with the uncertainty. We believe that performing a hyper-parameter search for the best $r$ will make the policy obtained by the RLSPI further robust. Secondly, since we are focusing on the linear approximation architecture for developing the theoretical understanding of model-free robust RL, the experiments may not be immediately scalable to very high dimensional OpenAI Gym environments which typically require nonlinear approximation architecture.

To end this section, we mention the software configurations used to generate these results: \textit{Python3.7 with OpenAI Gym [1]} and few basic libraries (non-exhaustive) like numpy, scipy, matplotlib. Also, the hardware configurations used was \textit{macOS High Sierra Version 10.13.6, 16 GB LPDDR3, Intel Core i7}.