Kinetics of the Bose-Einstein Condensation

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Abstract. We study the bosonic Boltzmann-Nordheim kinetic equation, which describes the kinetic regime of weakly interacting bosons with s-wave scattering only. We consider a spatially homogeneous fluid with an isotropic momentum distribution. The issue of the dynamical formation of a Bose-Einstein condensate has been studied extensively. We supply here the completed equations of motion for the coupled system, the energy density distribution of the normal fluid and the density of the condensate. With this information the post-nucleation self-similar solution is investigated in more detail than before.
1 Introduction

The dynamics of weakly interacting quantum fluids is governed, in approximation, by the Boltzmann-Nordheim kinetic equation. There is no a priori restriction on either the density or the temperature of the quantum fluid. Since, as explained in every textbook on Statistical Mechanics, an ideal Bose fluid undergoes a transition from a normal fluid to one with a condensate component, one would expect that the kinetic equation retains some information on how the condensate is formed dynamically.

In general, the appropriate kinetic equation is fairly complex and to analyse the details of the transition will be a difficult task. There is however one particular case which has been investigated in considerable depth. It is assumed that the Bose fluid is spatially homogeneous and the momentum distribution is isotropic. The Bose particles are massive and interact only through s-wave scattering. The distribution function $f$, as governed by the kinetic equation, depends then only on the energy $\epsilon, \epsilon \geq 0$, and on time $t, t \geq 0$. Based on theoretical considerations and numerical simulations the following scenario has been developed for the kinetics of the condensation process. One starts with an initial distribution $f(\epsilon,0) = f(\epsilon)$ which has the density $\rho = \int_0^\infty d\epsilon \sqrt{\epsilon} f(\epsilon)$. $\rho$ is assumed to be supercritical, $\rho > \rho_c$.

Then the solution $f(\epsilon,t)$ of the kinetic equation has a piece which concentrates near $\epsilon = 0$ and nucleates the condensate at some finite time $t_*$. More precisely,

$$\sqrt{\epsilon} f(\epsilon,t) = \sqrt{\epsilon} f_{\text{reg}}(\epsilon,t) + n(t)\delta(\epsilon)$$

(1.1)

with $n(t) = 0$ for $t < t_*$ and $n(t) > 0$ for $t > t_*$. Here $f_{\text{reg}}$ is a density without any delta functions. In the limit $t \to \infty$, $n(t)$ tends to $\rho - \rho_c$ while the regular piece tends to the critical Bose-Einstein distribution $(e^{\beta\epsilon} - 1)^{-1}$ with the inverse temperature $\beta$ determined through the initial energy $e = \int_0^\infty d\epsilon \sqrt{\epsilon} \epsilon f(\epsilon)$. Near $t_*$ the solution has a self-similar structure, which will be explained below.

In our contribution we investigate, in more detail than previous studies, the precise mechanism of how the condensate is generated and annihilated (on the level of the Boltzmann-Nordheim kinetic equation). In particular we obtain additional information on the self-similar structure of the solution for $t > t_*$. A numerical check of these predictions would help to further elucidate the kinetics of the Bose-Einstein transition.

2 The Boltzmann-Nordheim equation for the energy distribution

We use throughout dimensionless variables in units which minimize the number of prefactors. The physically correct dimensions are supplied, e.g., in [1]. Instead of denoting the energy by $\epsilon$ it will be more convenient to use the neutral $x, y, z$. The distribution function is denoted by $f(x), f \geq 0$, with volume element $\sqrt{x} dx, x \geq 0$,
since for $\varepsilon = p^2/2$ one has $d^3p = d\Omega p^2|d|p| = \sqrt{2\varepsilon}d\varepsilon$ in three dimensions. To $f$ we associate the density $\rho$ and energy $e$ as

$$\rho(f) = \int_0^\infty dx \sqrt{x}f(x), \quad e(f) = \int_0^\infty dx \sqrt{x}x f(x).$$

(2.1)

Physically $\rho$ is the mass and $e$ the energy per unit volume of the spatially homogeneous Bose fluid, but we stick to the more colloquial expressions.

$f(t)$ is governed by the kinetic equation

$$\frac{d}{dt}f(t) = C_4(f(t)).$$

(2.2)

$C_4$ is the collision operator which describes the scattering of two incoming to two outgoing particles, the 4-wave scattering in wave turbulence. For bosons, and under the assumptions stated in the introduction, one obtains

$$C_4(f)(x) = \frac{1}{\sqrt{x}} \int_{D(x)} dy dz I(x, y, z)(\tilde{f}(x)\tilde{f}(w)f(y)f(z) - f(x)f(w)\tilde{f}(y)\tilde{f}(z)).$$

(2.3)

Here $\tilde{f}(x) = 1 + f(x)$, $w = y + z - x$, and the domain $D(x) = \{y, z|y \geq 0, z \geq 0, x \leq y + z\}$ for $x \geq 0$. The integral kernel $I$ results from working out the $\delta$-functions for energy and momentum conservation of the full 3D collision operator, where $x, w$ are the energies of the incoming and $(y, z)$ the energies of the outgoing particles, see [1, 2] for details. $I$ is defined by

$$I(x, y, z) = \begin{cases} \sqrt{w} & \text{for } y \leq x, z \leq x, 0 \leq w, \\ \sqrt{x} & \text{for } x \leq y, x \leq z, \\ \sqrt{z} & \text{for } x \leq y, z \leq x, \\ \sqrt{y} & \text{for } y \leq x, x \leq z. \end{cases}$$

(2.4)

If $\rho(f) < \infty$, $e(f) < \infty$ and if $f(t)$ is bounded, then one can easily work out that

$$\frac{d}{dt}\rho(f(t)) = \int_0^\infty dx \sqrt{x}C_4(f(t))(x) = 0,$n

(2.5)

$$\frac{d}{dt}e(f(t)) = \int_0^\infty dx \sqrt{x}x C_4(f(t))(x) = 0.$n

(2.6)

Thus mass and energy are conserved. As we will see, if $f$ which diverges at $x = 0$, the conservation laws may break down.

On the kinetic level the entropy per unit volume is the one of a non-interacting Bose fluid and hence given by

$$s(f) = \int_0^\infty dx \sqrt{x}(\tilde{f}(x)\log \tilde{f}(x) - f(x)\log f(x)).$$

(2.7)
From (2.2) it follows that
\[ \frac{d}{dt}s(f(t)) = \sigma(f(t)) \] (2.8)
with the entropy production
\[ \sigma(f) = \int_{0 \leq x, y, z < \infty, x \leq y + z} dxdydz I(x, y, z)(A - B) \log(A/B), \] (2.9)
\[ A = \tilde{f}(x)f(y)f(z), \quad B = f(x)f(w)\tilde{f}(y)\tilde{f}(z). \] (2.10)
Clearly \( \sigma(f) \geq 0 \).

For a stationary solution \( \sigma(f) = 0 \). Introducing \( \psi = \log(\tilde{f}/f) \), the condition \( \sigma(f) = 0 \) is equivalent to \( \psi \) being a collisional invariant, to say
\[ \psi(x) + \psi(y + z - x) = \psi(y) + \psi(z) \] (2.11)
for all \( x, y, z \geq 0 \) such that \( x \leq y + z \). (2.11) admits as only solutions the affine functions \( \psi(x) = c_0 + c_1 x \).

At given \( \beta \) the maximal density is
\[ \rho_c = \int_0^\infty dx \sqrt{\frac{x}{e^\beta x - 1}} = \beta^{-3/2} \rho_0, \quad \rho_0 = \int_0^\infty dx \sqrt{\frac{x}{e^x - 1}} \] (2.13)
and correspondingly the maximal energy is
\[ e_c = \int_0^\infty dx \sqrt{\frac{x}{e^\beta x - 1}} x = \beta^{-1/2} e_0, \quad e_0 = \int_0^\infty dx \sqrt{\frac{x}{e^x - 1}} x. \] (2.14)
Hence there is the critical line
\[ \left( \frac{e_c}{e_0} \right) = \left( \frac{\rho_c}{\rho_0} \right)^{5/3} \] (2.15)
which divides the \((\rho, e)\) quadrant into the domain \( D_{\text{nor}} \) of normal fluid and the domain \( D_{\text{con}} \) with some fraction of condensate. If, for given initial \( f \), \((\rho(f), e(f)) \in D_{\text{con}}\), then \( \rho(f) > \rho_c \) and \( f(t) \) has no stationary solution to approach in the limit as \( t \to \infty \). Based on equilibrium statistical mechanics, it is thus natural to extend the class of stationary solutions to
\[ \sqrt{x}f_{\beta, \mu}(x) + n_{\text{con}} \delta(x), \quad n_{\text{con}} \geq 0, \] (2.16)
with the condition that for \( n_{\text{con}} > 0 \) necessarily \( \mu = 0 \). The density of this distribution is denoted by \( \rho(\beta, \mu) = \rho(f_{\beta, \mu}) \) for \( \mu < 0 \) and by \( \rho(\beta, 0, n_{\text{con}}) = \rho_c + n_{\text{con}} \) for
\( \mu = 0 \). The condensate has zero energy. With this extension there is a one-to-one relation \( \Phi \) between \( (\rho, e) \) and \( (\beta, \mu, n_{\text{con}}) \).

If in equilibrium there is a \( \delta \)-function at \( x = 0 \), it seems natural to allow for such a singular contribution also in the dynamics. The precise formulation will be the task of the next section. In fact, Lu \cite{3} proves that, for any \( f \) with \( \rho(f) < \infty, e(f) < \infty \), in the long time limit the solution \( f(t) \) converges to \( (2.16) \) with the parameters \( (\beta, \mu, n_{\text{con}}) \) determined through the map \( \Phi \) from the initial data. The precise statement requires some preparations and is therefore deferred to Appendix C.

3 Coupled equations for normal fluid and condensate

If there is a condensate at time \( t \), the full distribution function is

\[
\sqrt{x} f_{\text{tot}}(x, t) = \sqrt{x} f(x, t) + n(t) \delta(x).
\]

(3.1)

We adopt the convention that \( f \) always denotes a continuous function on \( \mathbb{R}_+ = (0, \infty) \) such that \( \rho(f) < \infty, e(f) < \infty \) and with a possible divergence at \( x = 0 \). We postulate that \( (2.2), (2.3) \) remain valid when \( f \) is substituted by \( f_{\text{tot}} \). Inserting \( f_{\text{tot}} \) in \( (2.3) \) yields products of the form \( fff, \delta ff, \delta \delta f, \delta \delta \delta, ff, \delta f, \) and \( \delta \delta \). Using the continuity of the integral kernel \( I \), it can be shown that the contribution of products with more than one \( \delta \) vanishes \cite{1,4}. Therefore \( (2.2) \) turns into a coupled system of equations for \( f \) and \( n \), apparently first derived in \cite{5}, which reads

\[
\frac{d}{dt} f = C_4(f) + nC_3(f),
\]

(3.2)

\[
\frac{d}{dt} n = -n \int_0^\infty dx \sqrt{x} C_3(f)(x).
\]

(3.3)

The collision operator \( C_3 \) is familiar from 3-wave interactions. Here it arises since one of the collision partners is at rest. Using isotropy and integrating over the momentum and energy delta functions, one obtains,

\[
C_3(f)(x) = \frac{2}{\sqrt{x}} \int_0^x dy \left( \tilde{f}(x) f(x-y) f(y) - f(x) \tilde{f}(x) f(y) \right) + \frac{4}{\sqrt{x}} \int_x^\infty dy \left( \tilde{f}(y) f(y-x) f(y) - f(x) f(y-x) \tilde{f}(y) \right),
\]

(3.4)

see \cite{1} for details.

Of course, to properly justify \( (3.2), (3.3) \), one has to go back to the microscopic model of a fluid of weakly interacting bosons. One imposes an initial quasifree state, \( \langle \cdot \rangle \), such that

\[
\langle a(k) \rangle = \alpha \delta(k), \quad \langle a(k)a(k') \rangle = \delta(k) \delta(k') \alpha^2,
\]

\[
\langle a(k)^*a(k') \rangle = \delta(k-k') \left( |\alpha|^2 \delta(k) + f(k^2) \right),
\]

(3.5)
where \(a(k), a(k)^*\) are the bosonic annihilation and creation operators labeled by momentum \(k \in \mathbb{R}^3\). \(\alpha \in \mathbb{C}\) and \(\alpha^2\) is the condensate density. The state \(\langle \cdot \rangle\) is invariant under spatial translations. If the BBGKY hierarchy is quasifreely truncated at the sixth order, then at least formally one arrives at the coupled system (3.2), (3.3).

\(\mathcal{C}_3\) conserves energy, but not mass. By considering the H-Theorem as in (2.8), one concludes that \(\sigma(f) = 0\) is equivalent to \(\psi = \log(f/f)\) being a collisional invariant, in the sense that
\[
\psi(x) + \psi(y) = \psi(x + y) \quad (3.6)
\]
for all \(x, y \geq 0\). Clearly, \(\psi\) must be linear. Hence the stationary solutions for \(\mathcal{C}_3\) are given by \(f_{\beta, 0}\) with \(\beta > 0\), where \(\beta\) is determined by the initial energy \(e(f)\).

The coupled system (3.2), (3.3) has an obvious defect. If \(n(0) = 0\), then \(n(t) = 0\) for all \(t\), no condensate is nucleated. A second defect lies in the definite sign of the term on the right hand side of (3.3). If \(f\) is bounded, then \(\int_0^\infty dx \sqrt{x} \mathcal{C}_3(f)(x) \geq 0\), as will be shown below. Hence, if present initially, the condensate density is monotone decreasing. The derivation of (3.2), (3.3) provides no immediate indication of how the condensate is generated. Only, since physically the total mass is conserved, the mass lost by the normal component has to be identified with the mass of the condensate.

In view of (2.5) valid for bounded \(f\)'s a natural guess is that the loss of mass for (3.2) is linked to the divergence of \(f\) at \(x = 0\). The relevant properties are stated in two propositions, which will be proved in Appendix A.

**Proposition 1.** Let \(f\) be continuous on \(\mathbb{R}_+\), \(\rho(f) < \infty\), \(e(f) < \infty\), and let
\[
\lim_{x \to 0} x^{7/6} f(x) = b, \quad b \geq 0. \quad (3.7)
\]
Then
\[
\lim_{\delta \to 0} \int_\delta^\infty dx \sqrt{x} \mathcal{C}_4(f)(x) = -\Gamma_4 b^3. \quad (3.8)
\]
Here \(\Gamma_4 \cong 3.05\) and the defining integral is provided in (A.13).

**Proposition 2.** Let \(f\) be continuous on \(\mathbb{R}_+\), \(\rho(f) < \infty\), \(e(f) < \infty\), and let
\[
\lim_{x \to 0} xf(x) = a, \quad a \geq 0. \quad (3.9)
\]
Then
\[
\lim_{\delta \to 0} \int_\delta^\infty dx \sqrt{x} \mathcal{C}_3(f)(x) = -\Gamma_3 a^2 + 2 \int_0^\infty dx xf(x). \quad (3.10)
\]
Here \(\Gamma_3 = -2 \int_0^1 dx x^{-1} \log(1 - x) = \pi^2/3\).

**Remark:** In agreement with the fact that the condensate has zero energy, under the condition (3.7) energy is still conserved in the sense that
\[
\lim_{\delta \to 0} \int_\delta^\infty dx \sqrt{x} \mathcal{C}_j(f)(x) = 0 \quad \text{for } j = 3, 4. \quad (3.11)
\]
In (3.10) there is a loss of mass as $\Gamma_3 a^2$ and a gain given by the integral expression. We interpret this as the corresponding loss/gain of the condensate. Therefore (3.2) and (3.3) are improved to

$$\frac{d}{dt}f(t) = C_4(f(t)) + n(t)C_3(f(t)),$$

$$\frac{d}{dt}n(t) = n(t)(\Gamma_3 a(t)^2 - 2 \int_0^\infty dxxf(x,t)).$$

Note that for $f_{\beta,0}$ one has

$$a = \beta^{-1}, \quad 2 \int_0^\infty dxxf_{\beta,0}(x) = \beta^{-2}\Gamma_3.$$ 

Thus the Bose-Einstein distributions of (2.16) are indeed stationary solutions for (3.12), (3.13).

In (3.13) we did not include the term $\Gamma_4 b(t)^3$, since $b > 0$ implies $a = \infty$. Mass is transferred between normal fluid and condensate only through $nC_3$. Thus it seems that the coupled system (3.12), (3.13) still has the defect that, in case $n(t) = 0$, no condensate can be nucleated. We will see in the next section, how this objection is met.

Tentatively we assume that a solution $f(t), n(t)$ to (3.12), (3.13) has the properties:

(i) $f(t) \geq 0, n(t) \geq 0$ are continuous in $t$, $\rho(f(t)) < \infty, e(f(t)) < \infty$.

(ii) $f(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and $\lim_{x \to 0} xf(x,t) = a(t)$ exists with $a(t) < \infty$.

4 Scaling theory close to nucleation of condensate

We note that if $n(t_0) > 0$ for some $t_0$, then $n(t) > 0$ for all $t > t_0$. Thus, if $n(0) = 0$, there must be a first time, $t_*$, such that $n(t) = 0$ for $t < t_*$ and $n(t) > 0$ for $t > t_*$.

Of course, depending on the initial $f$, one could have $t_* = \infty$. In this section we plan to study the solution close $t_*$, the time of nucleation of the condensate. We mostly follow [6], but the post-nucleation seems to be novel.

(i) $n(0) = 0, t_* < \infty$, pre-nucleation of the condensate.

For the scaling theory the quadratic terms of $C_4$ can be neglected. Physically this corresponds to the semiclassical approximation. Since $n(t) = 0$ for $t < t_*$, we have to study

$$\frac{d}{dt}f = C_{4,sc}(f)$$

with the semiclassical collision operator

$$C_{4,sc}(f)(x) = \frac{1}{\sqrt{x}} \int_{D(x)} dydzI(x,y,z)(f(x)f(y)f(z) + f(w)f(y)f(z) - f(x)f(w)f(y) - f(x)f(w)f(z)).$$

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Close to $x = 0$ the scaling ansatz is

$$f(x, t^* - t) = \frac{1}{\tau(t)^\nu} \phi_-\left(\frac{x}{\tau(t)}\right), \quad 0 < t \ll 1.$$  \tag{4.3}

Inserting in (4.1), (4.2) yields

$$\tau(t)^{-\nu-1} \frac{d}{dt} \tau(t) (\nu \phi_- (x) + x \phi'_-(x)) = \tau^{-3\nu+2} C_{\text{4,sc}}(\phi_-)(x).$$ \tag{4.4}

We fix the time scale by setting $\tau^{2\nu-3} \dot{\tau} = 1$. Then, for $\nu > 1$,

$$\tau(t) = (2(\nu - 1)t)^{1/(\nu - 1)}.$$ \tag{4.5}

The scaling function $\phi_-$ becomes then the solution of the nonlinear eigenvalue problem

$$\nu \phi_-(x) + x \frac{d}{dx} \phi_-(x) = C_{\text{4,sc}}(\phi_-)(x).$$ \tag{4.6}

For $\tau \ll x \ll 1$ the solution in (4.3) is assumed to be frozen at a definite power law. This yields the right boundary condition

$$\phi_-(x) \sim x^{-\nu} \quad \text{for} \quad x \to \infty.$$ \tag{4.7}

For small $x$ the, conventional, left boundary condition is

$$\phi_-(x) \to \phi_0 \quad \text{for} \quad x \to 0.$$ \tag{4.8}

In the limit $x \to 0$, (4.6) then yields, noting that the terms in $C_{\text{4,sc}}(\phi_+)$ proportional to $\phi_0$ cancel each other,

$$\nu \phi_0 = \int_0^\infty dy \int_0^\infty dz \phi_-(y) \phi_-(z) \phi_-(y + z),$$ \tag{4.9}

which is consistent in the sense that both sides have the same sign.

$\nu$ and $\phi_0$ are free parameters. To have condensation at all, by Proposition 1 necessarily

$$\nu \geq 7/6.$$ \tag{4.10}

On the other hand to have a finite mass in the interval $0 \leq x \leq \tau(t)$ yields the upper bound

$$\nu < 3/2.$$ \tag{4.11}

Because of slow decay, the scaling function $\phi_-$ has infinite mass and energy. It would be nice, if (4.6) would determine a unique value of $\nu$. But very little is known in this direction.

Numerically one finds that

$$\nu = 1.234$$ \tag{4.12}
by fitting the power law of \( f(x, t_s - t) \) for \( x > \tau \), see [3, 7]. One also finds \( f(0, t_s - t) \approx \tau(t)^{-\nu} \phi_0 \) consistent with the right boundary condition (4.8). We refer to [6] for additional information on the eigenvalue equation (4.6).

At the onset of nucleation one has \( f(x, t_s) = x^{-\nu} \), which is outside the space of solutions. As we will argue next, for any \( t > t_s \) the solution falls back into its proper space.

(ii) \( n(0) = 0, t_s < \infty \), post-nucleation of the condensate.

As for \( t < t_s \) we can work with the semiclassical approximation, which amounts to

\[
\frac{d}{dt} f = C_{4,sc}(f) + nC_{3,sc}(f),
\]

\[
\frac{d}{dt} n = n\Gamma_3 a^2.
\]

The 3-wave semiclassical collision operator reads

\[
C_{3,sc}(f)(x) = \frac{2}{\sqrt{x}} \int_0^x dy \{ f(x-y)f(y) - f(x)f(x-y) - f(x)f(y) \} + \frac{4}{\sqrt{x}} \int_x^\infty dy \{ f(x)f(y) + f(y)f(y-x) - f(x)f(y-x) \}.
\]

We impose the scaling ansatz for \( x \) close to 0 as

\[
f(x, t_s + t) = \frac{1}{\tau(t)^{\tilde{\nu}}} \phi_+ \left( \frac{x}{\tau(t)} \right), \quad 0 < t \ll 1.
\]

The boundary condition at infinity is

\[
\phi_+(x) = x^{-\tilde{\nu}} \quad \text{for} \quad x \to \infty.
\]

If \( f \) denotes the continuation of the scaling solution from (i), then one should set

\[
\tilde{\nu} = \nu,
\]

since for a very short time span and away from \( x = 0 \) the distribution function is frozen. On the other hand a general \( \tilde{\nu} \) is of interest, since one can set \( t_s = 0 \) and impose by hand some power law for the intial distribution, i.e. one sets

\[
f(x) = x^{-\tilde{\nu}} h(x) \quad \text{for} \quad t = 0
\]

with a cutoff function \( h \) so to have bounded \( \rho(f), e(f) \). Here necessarily

\[
1 \leq \tilde{\nu} < 3/2,
\]

the lower bound being required for nucleation, the upper bound for \( \rho(f) < \infty \).

The condensate density is assumed to scale as

\[
n(t) = (t/t_0)^\gamma.
\]
Inserting in (4.14) yields
\[ \gamma t^{-1} = \Gamma_3 a(t)^2. \] (4.22)

To ensure \( a > 0 \), we have to impose the left boundary condition
\[ \phi_+(x) = \frac{a_0}{x} \quad \text{for} \quad x \to 0, \ a_0 > 0. \] (4.23)

Then
\[ a(t) = \lim_{x \to 0} x^\gamma \phi_+(\frac{x}{\tau}) = \tau^{-\gamma+1} a_0, \] (4.24)

which implies
\[ \tau(t)^{-\gamma+1} = \frac{1}{a_0} (\frac{\gamma}{\Gamma_3})^{1/2} t^{-1/2}. \] (4.25)

Next we insert the scaling ansatz (4.16), (4.21) in (4.13). Multiplying both sides by \( \tau^{-\gamma-3} \) one arrives at
\[ - \tau^{-\gamma-3} \dot{\phi}_+(\bar{\nu} \phi_+ + x \phi'_+) = C_{4,sc}(\phi_+) + (t/t_0)^\gamma \tau^{-3/2} C_{3,sc}(\phi_+). \] (4.26)

The prefactor on the left turns out to be independent of \( t \). Balancing the prefactor of \( C_{3,sc} \) yields
\[ \gamma = \frac{3 - 2 \bar{\nu}}{4(\bar{\nu} - 1)}. \] (4.27)

\( \gamma > 0 \) provided \( 1 < \bar{\nu} < 3/2 \) consistent with (4.20). We conclude that the scaling function \( \phi_+ \) satisfies
\[ - K_1(\bar{\nu} \phi_+ + x \phi'_+) = C_{4,sc}(\phi_+) + K_2 C_{3,sc}(\phi_+) \] (4.28)

with the boundary conditions (4.17) and (4.23) and the two positive constants
\[ K_1 = \frac{a_0^2 T_3 (3/2 - \bar{\nu})^{-1}, \quad K_2 = \left( \frac{1}{a_0^2 t_0 \Gamma_3} \right) \gamma. } \] (4.29)

\( a_0 \) and \( t_0 \) are free parameters.

There is one important consistency check. In (4.28) we take the limit \( x \to 0 \).
The left hand side behaves as \( -K_1 a_0(\bar{\nu} - 1)/x \). For \( C_{4,sc} \) we use, see Appendix A,
\[ \lim_{x \to 0} x C_{4,sc}(\phi_+(x) = (a_0)^3 \Gamma_{4,1}, \quad \Gamma_{4,1} = \int_1^\infty dy \int_1^\infty dz ((y + z - 1)yz)^{-1}. \] (4.30)

Numerically \( \Gamma_{4,1} = 1.645 \). Clearly, one cannot balance the negative \( -K_1 a_0(\bar{\nu} - 1) \).
Therefore, \( C_{3,sc} \) must come into play. For small arguments the leading terms in the integrand of \( C_{3,sc} \) cancel and, to achieve the \( 1/x \) singularity, we have to assume the subleading behavior
\[ \phi_+(x)^{-1} = a_0 x + a_1 x^{3/2} \quad \text{for} \quad x \to 0. \] (4.31)
Then, see Appendix A,

\[
\lim_{x \to 0} x C_{3,sc}(\phi_+)(x) = -(a_0)^3 a_1 \Gamma_{3,1},
\]

\[
\Gamma_{3,1} = - \int_0^1 du \frac{1}{u(1-u)} (2 - 4u^{-1/2})(1 - u^{3/2} - (1 - u)^{3/2}).
\]  \hspace{1cm} (4.32)

Numerically \( \Gamma_{3,1} = 5.56 \). We arrive at

\[
-K_1 a_0 (\tilde{\nu} - 1) = a_0^3 \Gamma_{4,1} - K_2 a_0^3 a_1 \Gamma_{3,1},
\]  \hspace{1cm} (4.33)

which determines \( a_1 \). Thus \( \phi_+ \) has a subleading behavior as \( -a_1 x^{-1/2} \) for \( x \to 0 \). By the same procedure one could, in principle, compute the next order corrections to \( \phi_+ \).

At first sight the nonlinear eigenvalue problems (4.6) and (4.28) look rather similar, except for a further collision operator and the change of sign on the left. On physical grounds we expect however a very different solution behavior. (4.6) should have a single eigenvalue, or possibly a discrete set of eigenvalues \( \mathbb{E} \), while (4.28) should have for any \( \tilde{\nu} \) with \( 1 < \tilde{\nu} < 3/2 \) a solution satisfying the boundary conditions (4.17) and (4.23).

Physically the most interesting, and accessible, prediction is the exponent \( \gamma \) of (4.27) which governs the initial increase in the condensate density. If in (4.19), one sets \( 7/6 \leq \tilde{\nu} < 3/2 \), then, according to Proposition 1, \( C_4 \) provides a mechanism for nucleation. Immediately, i.e. for any \( t > 0 \), the divergence at \( x = 0 \) drops to \( a(t)/x \), \( a(t) \) from (4.22). For \( \tilde{\nu} = \nu \cong 1.234 \) numerical simulations are available \( \mathbb{E} \) and yield \( \gamma = 0.571 \) in good agreement with the prediction (4.27). At \( \tilde{\nu} = 7/6 \), (4.27) results in \( \gamma = 1 \). Aspects of this case have been established mathematically by Escobedo et al. and we state their results in Appendix B. Also the numerical solutions in \( [15] \) clearly show a linear initial increase in the condensate density and the rapid switch from the initial divergence \( x^{-7/6} \) to the slower \( x^{-1} \). For \( 1 < \tilde{\nu} < 7/6 \) and \( n(0) = 0 \), no mechanism for nucleation at time \( t = 0 \) is available. (Of course, at some later time the scenario as described under (i) may set in). Thus to verify the prediction (4.27) one would have to start with some small \( n(0) \), so to set \( C_3 \) in action.

5 Conclusions

In the “naive” picture of the condensation process the distribution function \( f \) develops a \( \delta \)-like concentration close to the origin, either at some finite time or in the long time limit. The Boltzmann-Nordheim kinetic equation tells a different story, however: If there is sufficient mass accumulated near the origin, the solution develops explosively a \( 1/x \) singularity at \( 0 \) (which is integrable for \( \sqrt{x} dx \)). Once the singularity is formed, mass of the normal fluid can be channeled into the condensate. On the other hand the condensate is annihilated at a rate \( 2 \int_0^\infty dx x f(x,t) \). For long times both processes balance so to approach the equilibrium condensate density.
Appendix: Proofs of Propositions 1, 2 and Eqs. (4.30), (4.32).

We start with the slightly easier proof of Proposition 2.

Proof of Proposition 2: We have to study the limit $\delta \to 0$ of

$$2 \int_{\delta}^{\infty} dx \int_{0}^{x} dy \{ f(x-y)f(y) - f(x)f(x-y) - f(x)f(y) \} - f(x)$$

$$+ 4 \int_{\delta}^{\infty} dx \int_{x}^{\infty} dy \{ f(x)f(y) + f(y-x)f(y) - f(x)f(y-x) \} + f(y)$$

$$= A_1(\delta) + A_2(\delta) + A_3(\delta),$$

(A.1)

where $A_1(\delta)$ is the integral with the first curly bracket, $A_2(\delta)$ the one with the second curly bracket, and $A_3(\delta)$ is the sum of the two integrals linear in $f$.

We consider $A_1(\delta)$. Each summand is linearly transformed such that the integrand is of product form, $f(x)f(y)$. For $f(x-y)f(y)$ the new domain of integration is $\{ x, y \geq 0 | \delta \leq x + y \}$, for $f(x)f(y-x)$ the new domain of integration is $\{ x, y \geq 0 | x \leq y, \delta \leq y \}$, and for $f(x)f(y)$ the domain of integration is $\{ x, y \geq 0 | \delta \leq x, y \leq x \}$. Altogether

$$A_1(\delta) = 2 \int_{0}^{\delta} dx \int_{\delta-x}^{\delta} dy f(x)f(y).$$

(A.2)

By continuity one has the bounds $a_-(\delta)x^{-1} \leq f(x) \leq a_+(\delta)x^{-1}$ valid for $0 \leq x \leq \delta$ with $\lim_{\delta \to 0} a_{\pm}(\delta) = a$. Inserting in (A.2) yields

$$\Gamma_3 a_-(\delta)^2 \leq A_1(\delta) \leq \Gamma_3 a_+(\delta)^2,$$

(A.3)

hence

$$\lim_{\delta \to 0} A_1(\delta) = \Gamma_3 a^2.$$

(A.4)

Next we study $A_2(\delta)$ by the same technique. For $f(x)f(y)$ the domain of integration is $\{ x, y \geq 0 | \delta \leq x < \infty, x \leq y \}$, for $f(y-x)f(y)$ the new domain of integration is $\{ x, y \geq 0 | 0 \leq x < \infty, x + \delta \leq y \}$, and for $f(x)f(y-x)$ the new domain of integration is $\{ x, y \geq 0 | \delta \leq x < \infty \}$. Altogether, using the symmetry of the integrand, one obtains

$$A_2(\delta) = -4 \int_{\delta}^{\infty} dx \int_{x-\delta}^{x} dy f(x)f(y).$$

(A.5)

We split the $x$ integration into $\delta \leq x < \delta_1$ and $\delta_1 \leq x < \infty$, with $\delta < \delta_1$. For the second interval

$$\lim_{\delta \to 0} \int_{\delta_1}^{\infty} dx \int_{x-\delta}^{x} dy f(x)f(y) = 0$$

(A.6)
by dominated convergence, since \( \int_{\delta_1}^\infty dx f(x) < \infty \) and \( \int_{x-\delta}^x dy f(y) \to 0 \) for \( \delta \to 0 \) with \( \delta_1 \leq x \). For the interval \( \delta \leq x \leq \delta_1 \) we use the bounds \( a_- (\delta_1) x^{-1} \leq f(x) \leq a_+ (\delta_1) x^{-1} \) valid for \( 0 \leq x \leq \delta_1 \) with \( \lim_{\delta_1 \to 0} a_+ (\delta_1) = a \). Letting \( \delta \to 0 \) and subsequently \( \delta_1 \to 0 \), we conclude

\[
\lim_{\delta \to 0} A_2(\delta) = -2 \Gamma_2 a^2. \tag{A.7}
\]

Finally for the linear terms we use that

\[
\int_0^\infty dx f(x) x < \infty. \tag{A.8}
\]

Therefore

\[
\lim_{\delta \to 0} (A_1(\delta) + A_2(\delta) + A_3(\delta)) = -\Gamma_2 a^2 + 2 \int_0^\infty dx f(x). \tag{A.9}
\]

**Proof of Proposition 1:** We have to study the limit \( \delta \to 0 \) of

\[
\int_\delta^\infty dx \int_{D(x)} dy dz I(x, y, z)\{(f(x) f(y) f(z) + f(w) f(y) f(z) - f(x) f(w) f(y) - f(x) f(w) f(z) + f(y) f(z) - f(x) f(w))
= B_1(\delta) + B_2(\delta), \tag{A.10}
\]

where \( B_1(\delta) \) is integral with the curly brackets and \( B_2(\delta) \) is the one quadratic in \( f \).

We consider first \( B_1(\delta) \). For each summand we make a linear change of variables to the product form \( f(x)f(y)f(z) \). For \( f(x)f(y)f(z) \) the domain integration is \( \{x, y, z \geq 0 | \delta \leq x, x \leq y + z\} \). For \( f(w)f(y)f(z) \) the new domain of integration is \( \{x, y, z \geq 0 | x + \delta \leq y + z\} \). Using the symmetry of the integrand, for \( f(x)f(w)f(y) \) the new domain of integration is \( \{x, y, z \geq 0 | x < y + z, z > \delta\} \) and for \( f(x)f(w)f(z) \) the new one is \( \{x, y, z \geq 0 | x < y + z, y > \delta\} \). Under these transformations the integral kernel \( I \) is not altered. Adding the four terms yields

\[
B_1(\delta) = -2 \int_0^\delta dx \int_\delta^{x+\delta} dy \int_0^{x-y+\delta} dz \sqrt{z} f(x) f(y) f(z)
+ \int_0^\delta dx \int_\delta^x dy \int_{x-y+\delta}^\delta dz \sqrt{x} f(x) f(y) f(z)
- \int_0^\delta dx \int_\delta^\infty dy \int_\delta^\delta dz \sqrt{x} f(x) f(y) f(z)
- \int_\delta^\infty dx \int_\delta^x dy \int_{x-y+\delta}^{\max(x-y, \delta)} dz \sqrt{y + z - x} f(x) f(y) f(z)
+ 2 \int_\delta^\infty dx \int_0^\delta dy \int_{x-y+\delta}^\infty dz \sqrt{y} f(x) f(y) f(z). \tag{A.11}
\]
As in the proof of Proposition 2, one splits the domain of the $x$ integration into $0 \leq x \leq \delta_1$, $\delta_1 \leq x < \infty$ with $\delta < \delta_1$. By continuity one has the bounds $b_-(\delta_1)x^{-7/6} \leq f(x) \leq b_+(\delta_1)x^{-7/6}$ valid for $0 \leq x \leq \delta_1$ with $\lim_{\delta_1 \to 0} b_\pm(\delta_1) = b$. Since $\rho(f) + e(f) < \infty$, we conclude that in the limit $\delta \to 0$ the integral over the interval $\delta_1 \leq x < \infty$ vanishes. Inserting the bounds, the integral over $0 \leq x < \delta_1$ becomes

$$\Gamma_4 b_-(\delta_1)^3 \leq \lim_{\delta \to 0} B_1(\delta) \leq \Gamma_4 b_+(\delta_1)^3, \quad (A.12)$$

where $\Gamma_4$ is independent of $\delta_1$ and given by

$$-\Gamma_4 = -2 \int_0^1 dx \int_1^{1+x} dy \int_0^{x-y+1} dz \sqrt{z(xyz)^{-7/6}}$$

$$+ \int_0^1 dx \int_x^1 dy \int_0^{x-y+1} dz \sqrt{x(xyz)^{-7/6}}$$

$$- \int_0^1 dx \int_1^\infty dy \int_1^{x-y+1} dz \sqrt{x(xyz)^{-7/6}}$$

$$- \int_1^\infty dx \int_1^x dy \int_1^{x-y+1} dz \sqrt{y+z-x(xyz)^{-7/6}}$$

$$+ 2 \int_1^\infty dx \int_0^1 dy \int_0^{\max(x-y,1)} dz \sqrt{y(xyz)^{-7/6}}. \quad (A.13)$$

The result follows by taking $\delta_1 \to 0$.

Finally we discuss $B_2(\delta)$. Under our assumptions each one of the integrals is finite separately. Therefore

$$\lim_{\delta \to 0} B_2(\delta) = \int_0^\infty dx \int_{\mathcal{D}_x} dydz I(x, y, z)(f(y)f(z) - f(x)f(y + z - x)) = 0, \quad (A.14)$$

by a linear change of coordinates in the second summand.

$\Gamma_4$ cannot be computed explicitly. Numerically one finds $\Gamma_4 = 3.05$.

We supply the proofs of (4.30) and (4.32).

**Lemma 3.** Let $f$ be continuously differentiable on $\mathbb{R}_+$, $f \geq 0$, $\int_1^\infty dx f(x) < \infty$, and

$$\lim_{x \to 0} x f(x) = a_0 \geq 0. \quad (A.15)$$

Then

$$\lim_{x \to 0} x \mathcal{C}_{4,sc}(f)(x) = (a_0)^3 \int_1^\infty dy \int_1^\infty dz ((y + z - 1)yz)^{-1}. \quad (A.16)$$

**Proof:** The domains in the definition of $\mathcal{D}(x)$ are labelled by I, II, III, IV in the order of (2.4). Each domain will be discussed separately.
**ad I:** All arguments are small and one can use that $f(x) = a_0/x$ for small $x$. Then one has

\[
x f(x) \left( \frac{1}{\sqrt{x}} \int_0^x dy \int_{x-y}^x dz \sqrt{w(f(y)f(z) - f(w)f(y) - f(w)f(z))} \right.
+ \left. \frac{1}{\sqrt{x}} \int_0^x dy \int_{x-y}^x dz \sqrt{w}(f(y)f(z) - f(w)f(y)f(z)) \right)
\]

valid for small $x$, where we use that in the last line the integral is well defined.

**ad II:** We first note that

\[
x f(x) \int_x^\infty dy \int_x^\infty dz (f(y)f(z) - f(y+z-x)f(y) - f(y+z-x)f(z)) = 0, \quad (A.18)
\]

since each term of the integrand is integrable. For the remaining integral we choose a fixed $\delta > 0$ and split the $y,z$ integration into $[x, \delta]$ and $[\delta, \infty)$. The integral over $[\delta, \infty) \times [\delta, \infty)$ is proportional to $x$ and the off-diagonal part is proportional to $x \log x$. Thus it remains to study the domain $[x, \delta]$ and $[x, \delta]$. Choosing $\delta$ small and approximating $f(x)$ by $a_0/x$, one obtains

\[
(a_0)^3 x \int_x^\delta dy \int_x^\delta dz ((y+z-x)yz)^{-1} = (a_0)^3 x \int_1^{\delta/x} dy \int_1^{\delta/x} dz ((y+z-1)yz)^{-1}
\]

(A.19)

which tends to the expression in (A.16) as $x \to 0$.

**ad III** (and ad IV by symmetry): We fix $\delta > 0$ such that $(xf(x) - a_0) \leq c(\delta)$ with $c(\delta) \to 0$ as $\delta \to 0$. We split the integral over $y$ in $[0, \delta]$ and $[\delta, \infty)$. For the first domain the argument in ad I shows that the limit $x \to 0$ vanishes. The remaining integral is

\[
\sqrt{x} \int_\delta^\infty dy \int_0^x dz \sqrt{z}(f(x)f(z)(f(y) - f(y+z-x))
+ f(w)f(y)f(z) - f(x)f(w)f(y))
\]

\[
\approx \sqrt{x} \int_\delta^\infty dy \int_0^x dz \sqrt{z}(a_0x^{-1}f(z)f'(y)(x-z) + f(y)^2a_0z^{-1} - a_0x^{-1}f(y)^2),
\]

(A.20)

which is of order $x$ and vanishes as $x \to 0$.

**Lemma 4.** Let $f$ be once continuously differentiable on $\mathbb{R}_+$, $f \geq 0$, $\int_1^\infty dx f(x) < \infty$ and there exists constants $\varepsilon, \delta, \epsilon > 0$, $\delta > 0$, $a_0, b_0, c_0$, such that

\[
|f(x)^{-1} - a_0x - b_0x^{3/2}| \leq c_0x^{3/2+\varepsilon}
\]

(A.21)
for $x \in [0, \delta]$. Then
\[
\lim_{x \to 0} x C_{3,sc}(f)(x) = (a_0)^3 b_0 \int_0^1 du \frac{1}{u(1-u)} (2 - 4u^{-1/2})(1 - u^{3/2} - (1 - u)^{3/2}) . \tag{A.22}
\]

**Proof:** We study both terms of $C_{3,sc}$ separately.

**Term 1.** Since $x$ is small we can use the approximation (A.21) to obtain
\[
x^{2} \sqrt{x} \int_{0}^{x} dy f(x-y) f(x) (f(x)^{-1} - f(y)^{-1} - f(x-y)^{-1})
\approx (a_0)^3 b_0 x^{2} \sqrt{x} \int_{0}^{x} dy ((x-y)yx)^{-1} (x^{3/2} - y^{3/2} - (x-y)^{3/2})
= 2(a_0)^3 b_0 \int_{0}^{1} dy ((1-y)y)^{-1} (1 - y^{3/2} - (1 - y)^{3/2}) . \tag{A.23}
\]

**Term 2.** As before we split the $y$ integration into $[x, \delta]$ and $[\delta, \infty)$. For the first integral we follow (A.23) and arrive at
\[
x^{2} \sqrt{x} \int_{x}^{\delta} dy f(x) f(y) f(y-x) (f(y-x)^{-1} + f(x)^{-1} - f(y)^{-1})
\approx 4(a_0)^3 b_0 \int_{x}^{\delta} dy (y(y-1))^{-1} ((y-1)^{3/2} + 1 - y^{3/2}) , \tag{A.24}
\]
which tends to the expression in (A.22) as $x \to 0$.

For the second integral one has
\[
4 \sqrt{x} \int_{\delta}^{\infty} dy f(y-x) f(y) + 4 f(x) \sqrt{x} (\int_{\delta}^{\infty} dy f(y) - \int_{\delta}^{\infty} dy f(y-x))
= 4 \sqrt{x} \int_{\delta}^{\infty} dy f(y)^2 + 4 f(x) \sqrt{xf(\delta)} , \tag{A.25}
\]
which is proportional to $\sqrt{x}$ and thus subleading.

**B Appendix: Singular solutions for the Boltzmann-Nordheim equation**

We state in more detail the recent results of Escobedo, Mischler, and Velásquez [8]. They consider the Boltzmann-Nordheim equation (2.2) with initial $f$ which diverges as $x^{-7/6}$ for $x \to 0$. More precisely they assume that the initial $f$ is once continuously differentiable on $(0, \infty)$ and satisfies
\[
|f(x) - Ax^{-7/6}| \leq B x^{-(7/6-\delta)} , \quad 0 \leq x \leq 1 , \tag{B.1}
\]
\[ |f'(x) + \frac{7}{6}Ax^{-13/6}| \leq Bx^{-(13/6 - \delta)}, \quad 0 \leq x \leq 1, \quad (B.2) \]

\[ f(x) \leq Be^{-Dx}, \quad 1 \leq x \quad (B.3) \]

for some positive constants \( A, B, D, \) and \( \delta. \) Note that \( \rho(f) < \infty, e(f) < \infty \) and that there is no condition on their size. In particular \( \rho(f) \) could be much smaller than \( \rho_c. \)

**Theorem** (EMV). For any \( f \) satisfying (B.1) to (B.3), there exists a unique solution, \( f(x,t) \), to (2.2), continuously differentiable in \( t \) for \( 0 < t < \infty \) and continuous in \( x \) for \( 0 < x < \infty \), as well as a function \( b(t) \), satisfying

\[ 0 \leq f(x,t) \leq Lx^{-7/6}e^{-Dx}, \quad x > 0, \quad t \in (0,T), \quad (B.4) \]

\[ |f(x,t) - b(t)x^{-7/6}| \leq Lx^{-(7/6 - \delta/2)}, \quad 0 \leq x \leq 1, \quad t \in (0,T), \quad (B.5) \]

\[ 0 \leq b(t) \leq L, \quad t \in (0,T), \quad (B.6) \]

for some positive constant \( L \) and for some \( T > 0, T \) depending on \( A, B, \) and \( \delta. \)

Together with our Proposition 1 it then follows that, for \( 0 \leq t < T, \)

\[ e(f(t)) = e(f), \quad (B.7) \]

\[ \rho(f(t)) = \rho(f) - \Gamma_4 \int_0^t dsb(s)^3, \quad (B.8) \]

\[ n(t) = \Gamma_4 \int_0^t dsb(s)^3. \quad (B.9) \]

Thus the condensate density increases linearly in \( t \) for small \( t. \)

It would be of interest to extend such a result to the coupled system. As argued, we expect that \( C_3 \) dominates \( C_4 \) and therefore in (B.5) the divergence at \( x = 0 \) should drop from \( x^{-7/6} \) to \( x^{-1} \), as also reported in [1] for numerical solutions.

Currently, the restriction to bounded \( T \) is needed for the proof. Whether there is really such a restriction can only be speculated. One could argue that by the choice of initial conditions one has opened already a channel for the formation of condensate. If the system uses only that channel, then

\[ \Gamma_4 \int_0^\infty dtb(t)^3 < \rho(f) \quad (B.10) \]

and necessarily \( b(t) \to 0 \) as \( t \to \infty. \) Once \( f(t) \) is subcritical, the mass transfer could stop and \( f(t) \) should converge for large \( t \) to the corresponding unique equilibrium distribution.
C Appendix: Long time behavior

Lu [3] considers Equation (2.2) in the weak form,

\[ \frac{d}{dt} \int_0^\infty \varphi(x)f(x, t)\sqrt{x}dx = \int_0^\infty \varphi(x)C_4(f(t))(x)\sqrt{x}dx \]  
(C.1)

with test functions \( \varphi \in C_0^2(\mathbb{R}_+) \) and proves that (C.1) remains meaningful even if \( \sqrt{x}f(x, t)dx \) is substituted a positive measure on \( \mathbb{R}_+ \). We denote such a measure by \( f(dx, t) \). The possible roughness of \( f(dx, t) \) is balanced by rewriting the right hand side of (C.1) in such a way that the collisional difference \( \varphi(x) + \varphi(y + z - x) - \varphi(y) - \varphi(z) \) appears. For \( t = 0 \) we assume finite mass and energy, i.e. \( \int_0^\infty f(dx) < \infty \) and \( \int_0^\infty xf(dx) < \infty \). Then (C.1) has a solution \( f(dx, t) \), which conserves mass and energy. It is not known whether the solution constructed by Lu is identical to the solution of the coupled system (3.12), (3.13). The stationary measures of (C.1) are necessarily of the form

\[ f_{\text{tot}, \beta, \mu}(dx) = f_{\beta, \mu}(x)\sqrt{x}dx + n_{\text{con}}\delta(x)dx \]  
(C.2)

with either \( \mu \leq 0 \) and condensate density \( n_{\text{con}} = 0 \) or \( \mu = 0 \) and \( n_{\text{con}} > 0 \).

Let us assume that the initial data are given by a density, i.e. \( f(dx) = f_{\text{ac}}(x)\sqrt{x}dx \), such that

\[ \rho = \rho(f) = \int_0^\infty f_{\text{ac}}(x)\sqrt{x}dx < \infty, \quad e = e(f) = \int_0^\infty xf_{\text{ac}}(x)\sqrt{x}dx < \infty. \]  
(C.3)

In general, the solution to (C.1) is then a measure, \( f(dx, t) \). It can be uniquely decomposed into an absolutely continuous and a singular part,

\[ f(dx, t) = f_{\text{ac}}(x, t)\sqrt{x}dx + f_s(dx, t). \]  
(C.4)

Now let \( f_{\beta, \mu} \) be the equilibrium distribution corresponding to \( \rho(f), e(f) \), where \( \mu = 0 \) in case \( n_{\text{con}} = \rho - \rho_c > 0 \). Lu [3] proves that, for \( \rho \leq \rho_c \),

\[ \lim_{t \to \infty} \int_0^\infty f_s(dx, t) = 0, \quad \lim_{t \to \infty} \int_0^\infty |f_{\text{ac}}(x, t) - f_{\beta, \mu}(x)|\sqrt{x}dx = 0. \]  
(C.5)

On the other hand for \( \rho > \rho_c \) one has

\[ \lim_{t \to \infty} \int_0^\infty |f_{\text{ac}}(x, t) - f_{\beta, \mu}(x)|x\sqrt{x}dx = 0 \]  
(C.6)

and

\[ \lim_{t \to \infty} \int_{\{x \mid 0 \leq x^4 \leq r(t)\}} f(dx, t) = \rho - \rho_c. \]  
(C.7)

where \( r(t) \) is the integral in (C.6). Thus for large times the solution has a \( \delta \)-peak like concentration in the interval \([0, r(t)^{1/4}]\) with weight \( n_{\text{con}} \).
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