Modified scattering for the one-dimensional Schrödinger equation with a dissipative nonlinearity for arbitrary large initial data

Xuan Liu∗, Ting Zhang†
School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

Abstract

We study the asymptotic behavior in time of solutions to the one-dimensional nonlinear Schrödinger equation with a dissipative nonlinearity $\lambda|u|^\alpha u$, where $0 < \alpha \leq 2$, and $\lambda$ is a complex constant satisfying $\text{Im}\lambda > \frac{\alpha|\text{Re}\lambda|}{2\alpha+1}$. For arbitrary large initial data, we present the time decay estimates when $4/3 < \alpha \leq 2$, and the large time asymptotics of the solution when $\frac{\alpha}{2} < \alpha \leq 2$. The proof is based on the vector fields method and on a semiclassical analysis method.

Keywords: Schrödinger equation, Decay estimates, Modified scattering, Semiclassical Analysis.

1 Introduction.

We consider the large time behavior of solutions to the Cauchy problem of the one-dimensional Schrödinger equation

$$
\begin{align*}
\begin{cases}
i\partial_t u + \frac{1}{2}\partial_x^2 u + \lambda|u|^\alpha u = 0, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
$$

(1.1)

where $\alpha > 0$, $\lambda \in \mathbb{C}$, and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function. The equation (1.1) is a particular case of the more general complex Ginzburg-Landau equation on $\mathbb{R}^N$

$$\partial_t u = e^{i\theta} \Delta u + |\zeta|^\alpha u$$

where $|\theta| \leq \frac{\pi}{2}$ and $\zeta \in \mathbb{C}$, which in fact is a generic modulation equation that describes the nonlinear evolution of patterns at near-critical conditions. See for instance [8, 10, 22].

There are many papers that studied the global well-posedness problem, decay and the asymptotic behavior of the solution (see [2, 5, 7, 9, 13, 14, 17, 18, 20, 27] and references therein). We use the following classification with respect to the value $\alpha$: the values $\alpha > 2$ we call the super-critical in the scattering problem, the value $\alpha = 2$ is the critical one and $0 < \alpha < 2$ we refer as the sub-critical.

Let us first recall some known large time asymptotics of (1.1) in the real coefficient case. Concerning the super-critical case $\alpha > 2$, $\lambda \in \mathbb{R}$, it is well-known that the solution

∗E-mail address: lxmath@zju.edu.cn
†E-mail address: zhangting79@zju.edu.cn
$u(t)$ behaves like a free solution $\exp \left((it/2)\partial^2_x\right) \phi$ for $t$ sufficiently large \cite{3,10,25,26}. The strategy for this free asymptotic profile largely relies on the rapid decay of the nonlinearity. More precisely, since $\int_1^\infty |u(t)|^\alpha dt \approx \int_1^\infty t^{-\alpha/2} dt < \infty$ by expecting that $u(t)$ decays like a free solution, the nonlinearity can be regarded as negligible in the long time dynamics. On the other hand, in the sub-critical and critical case, the nonexistence of usual scattering states was obtained for $0 < \alpha \leq 1$ in \cite{24}, and in \cite{1} for $1 < \alpha \leq 2$, $\lambda < 0$ by making use of the time decay estimate of solutions obtained from pseudo-conformal conservation law. In this case, some modification is required to expect large time asymptotics of the solution. In fact, in the critical case $\alpha = 2$, $\lambda \in \mathbb{R}$, the time decay estimates and the modified wave operators to \cite{1} for the small initial data were established in \cite{2,13,14,15,20}.

Next, we review some results on the complex coefficient case. When $\text{Im}\lambda < 0$, the solutions to \cite{1} may blow up in finite time. In fact, for any given compact set $M \subset \mathbb{R}^N$, it was proved in \cite{6} that, for $0 < \alpha \leq \frac{4}{N-2}$, there exists a class of solutions to \cite{1}, which blow up exactly on $M$. Moreover, it was proved in \cite{1} that for $0 < \alpha < \frac{4}{N}$, $\text{Im}\lambda < 0$, every $H^1$ solution of \cite{1} blows up, in finite or infinite time. The question of finite time blowup is still open.

In this paper, we concentrate our attention on the sub-critical and critical dissipative case: $0 < \alpha < 2$, $\text{Im}\lambda > 0$ (For the super-critical case, we refer to the aforementioned works \cite{3,10,25,26}, where a super-critical real $\lambda$ were studied, and the ideals are still applicable to a complex $\lambda$). The critical case $\alpha = 2$ has been studied in \cite{21}, in which the positivity of $\text{Im}\lambda$ visibly affects the decay rate of $\|u(t, x)\|_{L^\infty}$ and, actually, it decays like $(t \log t)^{-1/2}$. Later, Kita-Shimomura \cite{17,18} have studied the asymptotic behavior of solutions to \cite{1} in the subcritical dissipative case. In \cite{17}, they established the time decay estimates

$$\|u(t, x)\|_{L^\infty} \lesssim t^{-1/\alpha}, \quad \text{for } t \geq 1$$

and the asymptotic formula of the solutions when $0 < \alpha < 2$ with $\alpha$ sufficiently close to 2 and $u_0$ sufficiently small. In \cite{18}, for arbitrary $u_0 \in H^{1,0} \cap H^{0,1}$, they obtained the same decay estimates as stated above when $\frac{1+\sqrt{17}}{4} < \alpha < 2$, and the large time asymptotic when $\frac{1+\sqrt{17}}{4} < \alpha < 2$, under the large dissipative assumption

$$\lambda_2 \geq \frac{\alpha |\lambda_1|}{2\sqrt{\alpha} + 1}, \quad \lambda = \lambda_1 + i\lambda_2. \quad (1.2)$$

To derive the decaying properties in \cite{14,15,17,18,21}, they wrote $u(t, x)$ as

$$u(t, x) = (it)^{-1/2} \exp(i x^2/(2t)) \mathcal{F}v(t, x/t) + (\text{error term}), \quad (1.3)$$

with $v(t) = e^{-it/2\Delta} u(t)$, and then estimated $\mathcal{F}v(t)$ by applying certain gauge transform. Recently, for a class of special nonvanishing initial values, the large time asymptotic behaviors of the solutions were established in \cite{3,21} for $\frac{1+\sqrt{17}}{2} < \alpha < \frac{4}{3}$, $\text{Im}\lambda > 0$ in any space dimension $N \geq 1$.

In this paper, we establish the time decay estimate and the large time asymptotic behavior of the system

$$\begin{cases}
(D_t - F(D))u = \lambda |u|^\alpha u, & t > 0, x \in \mathbb{R} \\
\quad u(x, 0) = u_0(x),
\end{cases} \quad (1.4)$$

in the sub-critical and critical dissipative case, where $D_t = \frac{\partial}{\partial t}$, $D = \frac{\partial}{\partial x}$, $u$ is a complex valued function, and $F(\xi)$ is a second order constant coefficients classical elliptic symbol,
which has an expansion

\[ F(\xi) = c_2 \xi^2 + c_1 \xi + c_0 \]  

(1.5)

with \( c_2 > 0 \), \( c_1, c_0 \in \mathbb{R} \). For the classical one-dimensional Schrödinger equation (1.1), there is a vector filed \( L = x + it\partial_x \), which is the generator of the Galilean group of symmetries, satisfying

\[ e^{it\partial^2_x} x = L e^{it\partial^2_x}, \]

and

\[ [i\partial_t + \frac{1}{2} \partial^2_x, L] = 0. \]

For (1.3), we can also define a vector field

\[ \mathcal{L} = x + tF'(D), \]  

(1.6)

which satisfies the commutation relation \([D_t - F(D), \mathcal{L}] = 0\). As for the complex \( \lambda \), we assume the large dissipative condition (1.2), which contributes to removing the smallness assumption on the initial data.

It will be convenient to use the Hilbert space \((m \in \mathbb{N})\)

\[ H^{0,m} = L^2(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2m} dx) = \{ u \in L^2(\mathbb{R}) : |\cdot|^m u(\cdot) \in L^2(\mathbb{R}) \} \]

equipped with the norm

\[ \| u \|_{H^{0,m}}^2 = \| u \|_{L^2}^2 + \| x^m u \|_{L^2}^2. \]

We now state our results in this paper.

**Theorem 1.1.** Assume that \( u_0 \in H^{0,1} \), \( 0 < \alpha \leq 2 \), \( \lambda \) satisfies the condition (1.2) and \( \mathcal{L} \) is the vector filed defined in (1.6). Then there exists a unique global solution \( u \) to the system (1.4) satisfying

\[ u \in C ([0, \infty), L^2), \quad \mathcal{L}u \in C ([0, \infty), L^2), \]  

(1.7)

and

\[ \| u(t,x) \|_{L^2}^2 + \| \mathcal{L}u(t,x) \|_{L^2}^2 \leq 2 \| u_0 \|_{H^{0,1}} \]  

(1.8)

for any \( t \geq 0 \). Furthermore, if \( u_0 \in H^{0,2} \) and \( \alpha \geq 1 \), then \( \mathcal{L}^2 u \in C ([0, \infty), L^2) \) and there exists a constant \( C > 0 \) such that for all \( t \geq 1 \),

\[ \| \mathcal{L}^2 u(t,x) \|_{L^2} \leq C(\| u_0 \|_{H^{0,2}} + \| u_0 \|_{H^{0,2}}^{2\alpha+1} t^{2-\alpha}). \]  

(1.9)

Next, we derive the time decay rate of the global solution obtained in Theorem 1.1.

**Theorem 1.2.** Assume that \( u_0 \in H^{0,1} \), \( 4/3 < \alpha \leq 2 \), \( \lambda \) satisfies the condition (1.2) and \( u \) is the global solution obtained in Theorem 1.1. There exists a constant \( C > 0 \) such that for all \( t \geq 1 \),

\[ \| u(t,x) \|_{L^\infty} \leq \begin{cases} C(t \log t)^{-1/2}, & \text{when } \alpha = 2, \\ C t^{-1/\alpha}, & \text{when } 4/3 < \alpha < 2. \end{cases} \]  

(1.10)
Remark 1.1. The similar time decay estimate as (1.10) was first established in [18] under the assumptions $\frac{1+\sqrt{33}}{12} < \alpha \leq 2$ and (1.2), which was then extended to the case $\frac{7+\sqrt{145}}{12} < \alpha \leq 2$ in [16]. We note that $\frac{4}{3} < \frac{1+\sqrt{33}}{12} \approx 1.686$ and $\frac{4}{3} < \frac{7+\sqrt{145}}{12} \approx 1.586$. Therefore, Theorem 1.2 is an improvement of the corresponding results in [16, 18].

Remark 1.2. Theorem 1.2 is valid without any smallness conditions on the initial data. Moreover, the solution decays faster than the free solution. Recall that in one dimension, the free solution decays like $t^{-\frac{1}{2}}$.

Remark 1.3. Compared with the assumptions made in [14, 16, 17, 18, 21], Theorem 1.2 does not require the regularity condition $u_0 \in H^{\gamma}$ ($\gamma > \frac{1}{2}$). To derive the large time asymptotics of the solution, the main difficulty is to prove that the $L^{\infty}$ decay estimate (1.10) holds. To this end, we apply the semiclassical analysis method introduced by Delort [11] (see also [23, 27]) to deduce an ODE from (1.4). More precisely, we perform a semiclassical change of variables $u(t,x) = \frac{1}{\sqrt{t}} v\left(\frac{t}{x}, x\right)$, for some new unknown function $v$, and rewrite the system (1.4) as

$$\left(D_t - G^w_h(x \xi + F(\xi))\right)v = \lambda h^{\alpha/2} |v|^{\alpha - 1}v, \quad (1.11)$$

where the semiclassical parameter $h = \frac{1}{t}$, and the Weyl quantization of a symbol $a$, $G^w_h(a)$, will be given in (2.2). Next, we use the operators whose symbols are localized in a neighbourhood of $M := \{(x,\xi) \in \mathbb{R}^2 : x + F'(\xi) = 0\}$ of size $O(\sqrt{h})$ and set

$$v_\Lambda = G^w_h(\gamma(\frac{x + F'(\xi)}{\sqrt{h}}))v,$$

where $\gamma \in C_0^\infty(\mathbb{R})$ satisfying $\gamma = 1$ in a neighbourhood of zero. Then we can write

$$u(t,x) = \frac{1}{\sqrt{t}} v_\Lambda\left(\frac{t}{x}, x\right) + \text{(error term)} \quad (1.12)$$

and deduce an ODE for $v_\Lambda$ from (1.11) by developing the symbol $x \xi + F(\xi)$ on $M$:

$$D_t v_\Lambda = \left(-\left(x + c_1\right)^2/(4c_2) + c_0\right) v_\Lambda + \lambda h^{\alpha/2} |v_\Lambda|^{\alpha} v_\Lambda + \text{(remainder of high order in } h) .$$

Different with [14, 17, 18, 21], it is difficult to get an uniform $L^\infty$ control on $v_\Lambda$ directly from the above ODE, since the initial datum $u_0$ is not small and the estimates of the remainder terms depend on $\|v_\Lambda\|_{L^\infty}$ (see (4.23)–(4.24)). To resolve this, we combine the bootstrap and the contradiction argument to derive a rough $L^\infty$ estimate for $v_\Lambda$, and then in the solution $u$. Furthermore, this rough estimate for $v_\Lambda$ will be refined in Section 5 to establish the large time asymptotic formula of the solutions.

We note that the error term in (1.12) decay faster than (1.3) (see Lemma 4.1 below and Lemma 3.1 in [18]). This formally shows that the semiclassical analysis method can be used to extend the lower bound of $\alpha$. Our method is not applicable to study the time decay estimates of (1.4) in $\alpha \leq \frac{4}{3}$, since the remainder term decays more slowly than the principle part $v_\Lambda$ and therefore can not be considered as a small perturbation.

Finally, we give the large time asymptotic formula for the solutions and show the existence of modified scattering states for a certain range of the exponent in the nonlinear term.
Theorem 1.3. Assume \( \lambda \) satisfies the condition \( (1.2) \), and

\[
u_0 \in H^{0,1}, \quad \frac{1 + \sqrt{33}}{4} < \alpha \leq 2,
\]
or

\[
u_0 \in H^{0,2}, \quad \frac{7 + \sqrt{145}}{12} < \alpha \leq 2,
\]

then the followings hold:

(a) Let

\[
\Phi(t, x) = \int_1^t s^{-\alpha/2} |v_\Lambda(s, x)|^\alpha ds
\]

there exists a unique complex valued function \( z_+(x) \in L^\infty_x \cap L^2_x \) such that for some \( \gamma > 0 \),

\[
\|v_\Lambda(t, x) \exp \left( -i \left( -c_1^2/(4c_2) + c_0 t + \lambda \Phi(t, x) \right) \right) - z_+(x) \|_{L^\infty_x \cap L^2_x} = O(t^{-\gamma})
\]

holds as \( t \to \infty \).

(b) Let

\[
K(t, x) = \begin{cases} 
1 + 2\lambda_2 |z_+(x)|^2 \log t, & \text{when } \alpha = 2, \\
1 + \frac{2\lambda_2}{\alpha} |z_+(x)|^\alpha (t^{(2-\alpha)/2} - 1), & \text{when } 0 < \alpha < 2,
\end{cases}
\]

\[
\psi_+(x) = \alpha \lambda_2 \int \int s^{-\alpha/2} (|v_\Lambda(s, x)|^\alpha \exp (\alpha \lambda_2 \Phi(s, x)) - |z_+(x)|^\alpha) ds,
\]

and

\[
S(t, x) = \frac{1}{\alpha \lambda_2} \log (K(t, x) + \psi_+(x)).
\]

The asymptotic formula

\[
\left\| u(t, x) - \frac{1}{\sqrt{t}} e^{i\lambda \Phi(t, x)/t} z_+(x) \right\|_{L^\infty_x \cap L^2_x} = O(t^{-\gamma})
\]

holds as \( t \to \infty \), where \( \gamma \) is the same constant as in part (a).

(c) Let \( u_+(x) = \frac{1}{2\pi c_2} e^{-i\alpha \lambda_2} \tilde{z}_+(\frac{x}{\alpha \lambda_2}) \), we have the modified linear scattering

\[
\lim_{t \to \infty} \left\| u(t, x) - e^{i\lambda S(t, x)} e^{iF(D)t} u_+(x) \right\|_{L^2_x} = 0.
\]

(d) If \( u_0 \neq 0 \), then the limits

\[
\lim_{t \to \infty} (t \log t)^{\frac{\alpha}{2}} \| u(t, x) \|_{L^\infty_x} = \frac{1}{\sqrt{2\lambda_2}}, \quad \text{when } \alpha = 2,
\]

\[
\lim_{t \to \infty} t^{\frac{\alpha}{2}} \| u(t, x) \|_{L^2_x} = \left( \frac{2 - \alpha}{2 \alpha \lambda_2} \right)^{\frac{1}{\alpha}}, \quad \text{when } \alpha < 2,
\]

exist and are independent of the initial value.
Remark 1.4. In the case $u_0 \in H^{0,2}$, the range of $\alpha$ can be extended, since the error term in (1.12) will gain an additional $t^{3/2-\alpha}$ decay (see Lemmas 1.12, 1.13) by applying the operator $\mathcal{L}^2$.

Remark 1.5. Under the assumptions $\frac{9+\sqrt{177}}{12} < \alpha \leq 2$ and (1.12), Kita–Shimomura [18] established a similar large time asymptotic formula for the solutions. Since $\frac{7+\sqrt{135}}{12} \approx 1.587 < \frac{9+\sqrt{177}}{12} < 1.686 < \frac{9+\sqrt{177}}{12} \approx 1.859$, we see that Theorem 1.3 generates the result of [18] in the range of $\alpha$.

Remark 1.6. The limit in part (d) was first established in [7] for a class of special nonvanishing initial values. Theorem 1.3 shows that the same conclusion is also valid for general $u_0 \in H^{0,1}$.

Remark 1.7. By the definition (1.10) of $S(t, x)$, we can write the modification factor $e^{i\lambda S(t, x)}$ explicitly:

$$e^{i\lambda S(t, x)} = \frac{\exp\left\{ \frac{i\lambda}{\alpha\lambda_2} \log \left\{ 1 + 2\lambda_2|z_+(x)|^2 \log t + \psi_+(x) \right\} \right\}}{(1 + 2\lambda_2|z_+(x)|^2 \log t + \psi_+(x))^{1/2}} , \text{ when } \alpha = 2, \quad (1.21)$$

$$e^{i\lambda S(t, x)} = \frac{\exp\left\{ \frac{i\lambda}{\alpha\lambda_2} \log \left\{ 1 + \frac{2\alpha\lambda_2}{2-\alpha}|z_+(x)|^{\alpha((2-\alpha)/2) - 1} + \psi_+(x) \right\} \right\}}{(1 + \frac{2\alpha\lambda_2}{2-\alpha}|z_+(x)|^{\alpha((2-\alpha)/2) - 1} + \psi_+(x))^{1/\alpha}} , \text{ when } \alpha < 2. \quad (1.22)$$

To derive the limits (1.19)–(1.20), we first show that the limit function $z_+ \neq 0$ whenever $u_0 \neq 0$. In fact, $z_+ = 0$ would accelerate the decay rate of the solution, which together with the Duhamel’s formula (5.31) forces $u \equiv 0$. Using $z_+ \neq 0$, the asymptotic formula (1.17) and the explicit expression (1.21)–(1.22) for $e^{i\lambda S(t, x)}$, we deduce the limits (1.19)–(1.20).

The framework of this paper is organized as follows. In Section 2 we will present the definitions and properties of Semiclassical pseudo-differential operators. In Section 3 using the classical energy estimate method and Strichartz’s estimates, we will get the global existence and uniqueness of the solution to the system (1.4). Then in Section 4 we will prove the decay estimates as stated in Theorem 1.2 by the bootstrap argument. Finally, we prove Theorem 1.3 in Section 5.

2 Semiclassical pseudo-differential operators

The proof of the main theorem will rely on the use of a semiclassical formulation of the equation. We give in this section the definitions and properties of the classes of symbols and operators we shall use. A general reference is Chapter 7 of the book of Dimassi-Sjöstrand [12] or Chapter 4 of the book of Zworski [28] (see also [23, 27]).

Definition 2.1. An order function on $\mathbb{R} \times \mathbb{R}$ is a smooth map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}_+$: $(x, \xi) \mapsto M(x, \xi)$ such that there are $N_0 \in \mathbb{N}$ and $C > 0$, for any $(x, \xi), (y, \eta) \in \mathbb{R} \times \mathbb{R}$,

$$M(y, \eta) \leq C < x - y >^{N_0} < \xi - \eta >^{N_0} M(x, \xi),$$

where $< x > = \sqrt{1 + x^2}$.

Definition 2.2. Let $M$ be an order function on $\mathbb{R} \times \mathbb{R}$, and $\delta \geq 0$. One denotes by $S_\delta(M)$ the space of smooth functions $a(x, \xi, h) : \mathbb{R} \times \mathbb{R} \times (0, 1] \rightarrow \mathbb{C}$, satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha, \beta} M(x, \xi) h^{-(\alpha+\beta)\delta}, \forall \alpha, \beta \in \mathbb{N}. \quad (2.1)$$
A key role in this paper will be played by symbols $a$ verifying (2.1) with $M(x, \xi) = \frac{x + F'(\xi)}{\sqrt{\hbar}} > -N$, for $N \in \mathbb{N}$. Although this function $M$ is no longer an order function because of the term $\frac{M}{\sqrt{\hbar}}$, it is useful to keep the notation $a \in S_0(M)$ whenever $a, M$ verify (2.1). In effect, this generalized notation has been used throughout [23, 27].

In the rest of this section, we assume $F(\xi)$ is the symbol defined in (1.5).

**Definition 2.3.** Define the Weyl quantization to be the operator $G_h^w(a) = a^w(x, hD)$ acting on $u \in \mathcal{S}(\mathbb{R})$ by the formula

$$G_h^w(a)u = a^w(x, hD)u = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{(x-y)\xi}{\hbar}} a(\frac{x+y}{2}, \xi)u(y)dyd\xi. \quad (2.2)$$

**Proposition 2.4** (Composition for Weyl quantization, Theorem 7.3 in [12]). Suppose that $a, b \in \mathcal{S}$. Then

$$G_h^w(ab) = G_h^w(a) \circ G_h^w(b),$$

where

$$a^w_{\ast}b(x, \xi) := \frac{1}{(\pi\hbar)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{(x-y+z,\eta;\xi)}{\hbar}} a(x+z, \xi+\eta)b(x+y, \xi+\eta)dyd\eta dzd\xi,$$

and

$$\sigma(y, \eta, z, \xi) = \eta - y\xi.$$

**Proposition 2.5** (Proposition 2.4 in [27]). If $m_1 = \frac{x + F'(\xi)}{\sqrt{\hbar}} > -n$, $n \in \mathbb{N}$, $m_2$ is order function or $m_2 = \frac{x + F'(\xi)}{\sqrt{\hbar}} > l$, $l \in \mathbb{Z}$, $a \in S_0(m_1)$ and $b \in S_0(m_2)$, $\delta_1, \delta_2 \in [0, \frac{1}{2}]$, then

$$a^w_{\ast}b \in S_0(m_1m_2), \quad \delta = \max\{\delta_1, \delta_2\}.$$

Furthermore

$$a^w_{\ast}b = ab + \frac{i\hbar}{2} (\partial_xa\partial_\xi b - \partial_\xi a\partial_x b) + R,$$

where $R \in h^{2(1-\delta_1-\delta_2)}S_0(m_1m_2)$, and

$$R = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{(x+y+z,\eta;\xi)}{\hbar}} \left\{ - \int_0^1 \partial_x^2 a(x+tz, \xi)(1-t)d\eta \partial_\eta^2 b(x+y, \xi+\eta) + \int_0^1 \int_0^1 \partial_x^2 \partial_\xi a(x+sz, \xi+t\xi)d\eta d\eta \partial_\eta^2 b(x+y, \xi+\eta) - \int_0^1 \partial_\xi^2 a(x, \xi + t\xi)(1-t)d\eta \partial_\eta^2 b(x+y, \xi+\eta) \right\} dyd\eta dzd\xi.$$

**Lemma 2.1.** Assume $\Gamma_0(\xi)$ satisfy $|\partial^\alpha \Gamma_0(\xi)| \lesssim C_\alpha < \xi >^{-1-\alpha}$ for any $\alpha \in \mathbb{N}$, then we have

$$\Gamma_0(\frac{x + F'(\xi)}{\sqrt{\hbar}})^2 \frac{x + F'(\xi)}{\sqrt{\hbar}} = \frac{x + F'(\xi)}{\sqrt{\hbar}} \Gamma_0(\frac{x + F'(\xi)}{\sqrt{\hbar}}) = \Gamma_0(\frac{x + F'(\xi)}{\sqrt{\hbar}})^2 \frac{x + F'(\xi)}{\sqrt{\hbar}}, \quad (2.3)$$

In addition, if $|\partial^\alpha (\xi \Gamma_0(\xi))| \lesssim C_\alpha < \xi >^{-1-\alpha}$ for any $\alpha \in \mathbb{N}$, then we have

$$\Gamma_0(\frac{x + F'(\xi)}{\sqrt{\hbar}})^2 \frac{x + F'(\xi)}{\sqrt{\hbar}} = \Gamma_0(\frac{x + F'(\xi)}{\sqrt{\hbar}})^2 \frac{x + F'(\xi)}{\sqrt{\hbar}}, \quad (2.4)$$
Proof. Since $\Gamma_0(\frac{x+\xi(z)}{\sqrt{h}}) \in S_\frac{1}{2}(<\frac{x+\xi(z)}{\sqrt{h}} >^{-1})$, \( \frac{x+\xi(z)}{\sqrt{h}} \in S_\frac{1}{2}(<\frac{x+\xi(z)}{\sqrt{h}} >) \), it follows from Proposition 2.3 that
\[
\Gamma_0(\frac{x+\xi(z)}{\sqrt{h}})^2 \frac{x+\xi(z)}{\sqrt{h}} = \Gamma_0(\frac{x+\xi(z)}{\sqrt{h}})^2 x + \xi(z) - \frac{x+\xi(z)}{\sqrt{h}} \frac{\partial_t \varphi_0(x+\xi(z))}{\sqrt{h}} + R
\]
where we use $R = 0$, since $\partial_t \varphi(x+\xi(z)) = \partial_t x + \partial_t x \xi(z) = 0$. Similarly, we can show that
\[
\frac{x+\xi(z)}{\sqrt{h}} \partial_t \varphi_0(x+\xi(z)) = \partial_t x + \partial_t x \xi(z).
\]
This completes the proof of (2.3). Finally, (2.4) follows easily from (2.3):
\[
\Gamma_0(\frac{x+\xi(z)}{\sqrt{h}})^2 \frac{x+\xi(z)}{\sqrt{h}} = (\Gamma_0(\frac{x+\xi(z)}{\sqrt{h}})^2 \frac{x+\xi(z)}{\sqrt{h}})^2
\]

\[= \Gamma_0(\frac{x+\xi(z)}{\sqrt{h}})^2 \frac{x+\xi(z)}{\sqrt{h}} \frac{x+\xi(z)}{\sqrt{h}}.
\]

\[\square\]

**Proposition 2.6** (Proposition 2.7 in [27]). Let $a(\xi)$ be a smooth function satisfying $|\partial_\xi a(\xi)| \leq C_\alpha < \xi >^{-\alpha}$ and $h \in (0, 1]$. Then
\[
\|G_h(a(\frac{x+\xi(z)}{\sqrt{h}}))\|_{L^2, L^\infty} = O(h^{-\frac{1}{2}}), \|G_h(a(\frac{x+\xi(z)}{\sqrt{h}}))\|_{L^2, L^2} = O(1).
\]

### 3 Proof of the global existence and uniqueness.

To establish the global existence and uniqueness of (1.4) in the following context, we need some lemmas.

**Lemma 3.1.** Assume $f : [1, T] \times \mathbb{R} \rightarrow C$, $T > 1$ is a smooth function, there exists a positive constant $C$ independent of $T, f$ such that for all $t \in [1, T]$
\[
\|f(t, x)\|_{L^\infty} \leq C T^{-\frac{1}{2}} \|f(t, x)\|_{L^2}^{1/2} \|\nabla f(t, x)\|_{L^2}^{1/2}, \tag{3.1}
\]
\[
\|\nabla f(t, x)\|_{L^1} \leq C \|f(t, x)\|_{L^2}^{1/2} \|\nabla^2 f(t, x)\|_{L^2}^{1/2}. \tag{3.2}
\]

**Proof.** To begin with, it is useful to introduce a certain phase function. Let
\[
\phi(t, x) = \frac{x^2 + 2c_1tx}{4c_2t}.
\]
For the initial datum

\[ L = x + tF'(D) = x + c_1 t - 2c_2 t \partial_x, \]  

(3.3)

it is straightforward to check that

\[ -2c_2 t \partial_x (f(t, x)e^{i\phi(t, x)}) = e^{i\phi(t, x)}L f(t, x), \]  

(3.4)

and

\[ -4c_2^2 t^2 \partial_{xx} (f(t, x)e^{i\phi(t, x)}) = e^{i\phi(t, x)}L^2 f(t, x). \]  

(3.5)

From (3.4), we have, for \( t \in [1, T] \),

\[ \left\| \partial_x (f(t, x)e^{i\phi(t, x)}) \right\|_{L^2_x} \leq \frac{C}{t} \left\| L f(t, x) \right\|_{L^2_x}. \]  

(3.6)

On the other hand, using Gagliardo-Nirenberg’s inequality, we obtain

\[ \left\| f(t, x) \right\|_{L^\infty_x} = \left\| f(t, x)e^{i\phi(t, x)} \right\|_{L^\infty_x} \leq C \left\| f(t, x)e^{i\phi(t, x)} \right\|_{L^t_x}^{1/2} \left\| \partial_x (f(t, x)e^{i\phi(t, x)}) \right\|_{L^2_x}^{1/2}, \]

which together with (3.6) yields (3.1).

Next, we prove (3.2). Using (3.4) and Gagliardo-Nirenberg’s inequality, we get

\[ \left\| L f(t, x) \right\|_{L^1_t} \leq Ct \left\| \partial_x (f(t, x)e^{i\phi(t, x)}) \right\|_{L^t_x} \leq Ct \left\| f(t, x)e^{i\phi(t, x)} \right\|_{L^\infty_x}^{1/2} \left\| \partial_{xx} (f(t, x)e^{i\phi(t, x)}) \right\|_{L^2_t}^{1/2}, \]

which together with (3.5) yields (3.2).

Using the classical energy estimate method, we can obtain the following lemma easily and omit the details.

**Lemma 3.2.** Assume \( \text{Im}\lambda > 0 \), and \( u \in C([0, T]; L^2) \) is a solution of (1.4), then we have

\[ \|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2}, \quad t \in [0, T]. \]

Then, using the Strichartz estimates and Lemma 3.2, we can obtain the following global existence and uniqueness result.

**Theorem 3.3.** Assume (1.2), (1.3) and \( 0 < \alpha \leq 2 \). For any \( u_0 \in H^{0,1} \), the system (1.4) has a unique global solution \( u \) satisfying (1.7)-(1.8). Furthermore, if \( u_0 \in H^{0,2} \) and \( \alpha \geq 1 \), then (1.9) holds.

**Proof.** Since the proof is classical (see, e.g. Chapter 4 in [3]), we only give a brief here. For the initial datum \( u_0 \in H^{0,1} \), the existence and uniqueness of a local strong \( L^2 \) solution to the Cauchy problem (1.4) easily follow from the Strichartz’s estimate

\[ \|u\|_{L^\infty L^2 T} \leq C \|u_0\|_{L^2} + C \|u^\alpha u\|_{L^1 T}, \]

and the standard contraction argument. Moreover, using Lemma 3.2 we can extend this local solution to be global in positive time.

Next, we derive the estimate of \( \|Lu\|_{L^2} \). For \( u_0 \in H^{0,1} \), let \( u \in C([0, \infty); L^2) \) be a solution to (1.4). Applying the operator \( \mathcal{L} \) to (1.4), and using the fundamental commutation property \([D_t - F(D), \mathcal{L}] = 0\), we get

\[ (D_t - F(D))\mathcal{L}u = \lambda \mathcal{L}(|u|^\alpha u). \]
Therefore, for all \( t \) which together with Lemma 3.2 yields (1.8). Moreover, one can prove that

\[
\mathcal{L}(|u|^\alpha u) = \frac{\alpha + 2}{2} |u|^\alpha \mathcal{L}u - \frac{\alpha}{2} |u|^{\alpha - 2} u^2 \overline{\mathcal{L}u},
\]

and that for \( \alpha \geq 1 \),

\[
\mathcal{L}^2(|u|^\alpha u) = \frac{\alpha + 2}{2} |u|^\alpha \mathcal{L}^2 u + \frac{\alpha}{2} |u|^{\alpha - 2} u^2 \overline{\mathcal{L}^2 u} + \frac{(\alpha + 2)\alpha}{2} |u|^{\alpha - 2} \text{Im}(\mathcal{L}u \overline{\mathcal{L}u}) \mathcal{L}u
\]

\[ - \frac{\alpha(\alpha - 2)}{2} |u|^{\alpha - 4} u^2 \text{Im}(\mathcal{L}u \overline{\mathcal{L}u}) - \alpha |u|^{\alpha - 2} \mathcal{L}u^2. \]  

(3.8)

From (3.7) and (1.2), we have for all \( t \geq 0 \),

\[
- \text{Im} \left( \lambda \mathcal{L}(|u|^\alpha u) \overline{\mathcal{L}u} \right) = - \text{Im} \left( \frac{\alpha + 2}{2} |u|^\alpha |\mathcal{L}u|^2 + \text{Im}(\lambda \frac{\alpha}{2} |u|^{\alpha - 2} u^2 \overline{\mathcal{L}u}^2) \right) \leq \left( -\frac{\alpha + 2}{2} \lambda_2 + \frac{\alpha}{2} |\lambda| \right) |u|^\alpha |\mathcal{L}u|^2 \leq 0.
\]

Therefore, for all \( t \geq 0 \), we have

\[
\| \mathcal{L}u(t) \|_{L^2} \leq \| \mathcal{L}u(0) \|_{L^2} = \| xu_0 \|_{L^2},
\]

(3.9)

which together with Lemma 3.2 yields (1.8). Moreover, one can prove that \( u, \mathcal{L}u \in C \left( (0, \infty), L^2 \right) \) in the spirit of Proposition 6.5.1 in [3].

Finally, we apply the same argument as in the proof of (3.9) to derive (1.9). We assume \( u_0 \in H^{0.2} \) and \( \alpha \geq 1 \). Using the commutation relation \([D_x - F(D), \mathcal{L}^2] = 0 \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \mathcal{L}^2 u \|_{L^2}^2 = -\text{Im}(\lambda \int_R \mathcal{L}^2(|u|^\alpha u) \cdot \overline{\mathcal{L}^2 u} dx).
\]

(3.10)

From (3.8) and (1.2), we have that for all \( t \geq 0 \),

\[
-\text{Im}(\lambda \int_R \mathcal{L}^2(|u|^\alpha u) \cdot \overline{\mathcal{L}^2 u} dx) \leq \lambda_2 \frac{\alpha + 2}{2} \int_R |u|^\alpha |\mathcal{L}^2 u|^2 dx + \text{Im}(\lambda \frac{\alpha}{2} \int_R |u|^{\alpha - 2} u^2 (\overline{\mathcal{L}^2 u})^2 dx)
\]

\[ + C \int_R |u|^{\alpha - 1} |\mathcal{L}u|^2 |\mathcal{L}^2 u|^2 dx \leq C \|u\|_{L^\infty}^{\alpha - 1} \|\mathcal{L}u\|_{L^2} \|\mathcal{L}u\|_{L^2} \|\mathcal{L}^2 u\|_{L^2}. \]  

(3.11)

By (3.11) and Lemma 3.3, we obtain

\[
-\text{Im}(\lambda \int_R \mathcal{L}^2(|u|^\alpha u) \cdot \overline{\mathcal{L}^2 u} dx) \leq C t^{-\alpha/2} \|u\|_{L^2}^{\alpha + 1/2} \|\mathcal{L}u\|_{L^2} \|\mathcal{L}^2 u\|_{L^2}^{3/2}
\]

\[ \leq C t^{-\alpha/2} \|u_0\|_{H^{0.2}}^{\alpha + 1/2} \|\mathcal{L}^2 u\|_{L^2}^{3/2}, \]  

(3.12)
where we used (3.9) and Lemma 3.2 in the last inequality. It now follows from (3.10) and (3.12) that
\[
\frac{d}{dt} \| \mathcal{L}^2 u \|_{L^2}^{1/2} \leq C t^{-\alpha/2} \| u_0 \|_{H^{0,1}}^{\alpha+1/2}.
\]
Hence
\[
\| \mathcal{L}^2 u \|_{L^2} \leq \| u_0 \|_{H^{0,2}} + C t^{2-\alpha} \| u_0 \|_{H^{0,1}}^{2\alpha+1},
\]
which yields (1.9). Moreover, we can check that \( \mathcal{L}^2 u \in C ([0, \infty), L^2) \) as before and omit the details.

Using Lemma 3.1, Lemma 3.2, and (3.9), we obtain the following decay estimate for \( u \).

**Lemma 3.4.** Assume \( u_0 \in H^{0,1} \), and \( u \) is the global solution obtained in Theorem 3.3. There exists a constant \( C > 0 \) such that, for all \( t \geq 1 \),
\[
\| u(t, x) \|_{L^\infty} \leq C t^{-1/2} \| u_0 \|_{H^{0,1}}.
\]

### 4 The proof of Theorem 1.2

#### 4.1 Semiclassical reduction of the problem

We rewrite the problem in the semiclassical framework. Set
\[
u(t, x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t}), \quad h = \frac{1}{t}.
\]
Then the system (1.4) is rewritten as
\[
(D_t - G^w_n (x \xi + F(\xi))) v = \lambda t^{-\alpha/2} |v|^{\alpha} v.
\]
At the same time, set
\[
\tilde{\mathcal{L}} = \frac{1}{h} G^w_n (x + F'(\xi)).
\]
Indeed, since \( F'(\xi) = 2c_2 \xi + c_1 \), we have \( \tilde{\mathcal{L}} = (x + c_1)t + 2c_2 D_x \). Then by direct calculation, one can check that
\[
\tilde{\mathcal{L}}(|v|^{\alpha} v) = \frac{\alpha + 2}{2} |v|^{\alpha} \tilde{\mathcal{L}} v - \frac{\alpha}{2} |v|^{\alpha-2} v^2 \overline{\tilde{\mathcal{L}} v};
\]
and that for \( \alpha \geq 1 \),
\[
\tilde{\mathcal{L}}^2 (|v|^{\alpha} v) = \frac{\alpha + 2}{2} |v|^{\alpha} \tilde{\mathcal{L}}^2 v + \frac{\alpha}{2} |v|^{\alpha-2} v^2 \overline{\tilde{\mathcal{L}}^2 v} + \frac{(\alpha + 2)\alpha}{2} |v|^{\alpha-2} \text{Im}(\tilde{\mathcal{L}} v \overline{\tilde{\mathcal{L}} v}) \tilde{\mathcal{L}} v - \frac{\alpha (\alpha - 2)}{2} |v|^{\alpha-4} v^2 \text{Im}(\tilde{\mathcal{L}} v \overline{\tilde{\mathcal{L}} v}) \tilde{\mathcal{L}} v - \alpha |v|^{\alpha-2} v |\tilde{\mathcal{L}} v|^2.
\]
Thus, we have
\[
\| \tilde{\mathcal{L}} (|v|^{\alpha} v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \tilde{\mathcal{L}} v \|_{L^2},
\]
\[
\| \tilde{\mathcal{L}}^2 (|v|^{\alpha} v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \tilde{\mathcal{L}}^2 v \|_{L^2} + C \| v \|_{L^\infty}^{-1} \| \tilde{\mathcal{L}} v \|_{L^4}^2.
\]
Since \( \| \tilde{\mathcal{L}} v \|_{L^4}^2 \leq C \| v \|_{L^\infty} \| \tilde{\mathcal{L}}^2 v \|_{L^2} \), by Lemma 3.1 we obtain
\[
\| \tilde{\mathcal{L}}^2 (|v|^{\alpha} v) \|_{L^2} \leq C \| v \|_{L^\infty} \| \tilde{\mathcal{L}}^2 v \|_{L^2}.
\]
Moreover, we have
\[ \mathcal{L}u(t, x) = \frac{1}{\sqrt{t}}(\bar{\mathcal{L}}v)(t, \frac{x}{t}), \quad \mathcal{L}^2u(t, x) = \frac{1}{\sqrt{t}}(\bar{\mathcal{L}}^2v)(t, \frac{x}{t}). \]
Therefore,
\[ \|u(t, \cdot)\|_{L^2} = \|v(t, \cdot)\|_{L^2}, \quad \|u(t, \cdot)\|_{L^\infty} = t^{-1/2}\|v(t, \cdot)\|_{L^\infty}, \]
and
\[ \|\mathcal{L}u(t, \cdot)\|_{L^2} = \|\bar{\mathcal{L}}v(t, \cdot)\|_{L^2}, \quad \|\mathcal{L}^2u(t, \cdot)\|_{L^2} = \|\bar{\mathcal{L}}^2v(t, \cdot)\|_{L^2}. \]
In addition, form (3.9), (3.13) and (4.9), we have that, for all \( t \geq 1, \)
\[ \|\bar{\mathcal{L}}v(t, \cdot)\|_{L^2} \leq \|u_0\|_{H^{0,1}}, \quad \|\bar{\mathcal{L}}^2v(t, \cdot)\|_{L^2} \leq C(\|u_0\|_{H^{0,2}} + \|u_0\|_{H^{0,1}}^{2\alpha+1})t^{2-\alpha}. \]
From (4.8), we see that the proof of Theorem 1.2 reduces to estimate \( \|v(t, \cdot)\|_{L^\infty}. \) To do so, we decompose \( v = v_\Lambda + v_{\Lambda^c} \) with
\[ v_\Lambda = G_h^w(\Gamma)v, \quad (4.12) \]
where \( \Gamma(x, \xi) = \gamma(\frac{x+F'(\xi)}{h}), \gamma \in C_0^\infty(\mathbb{R}) \) satisfying that \( \gamma \equiv 1 \) in a neighbourhood of zero.
It is easy to obtain the \( L^\infty- \)estimates for \( v_{\Lambda^c} \) as follows.

**Lemma 4.1.** Assume \( u \) is the solution given by Theorem 3.3 with \( u_0 \in H^{0,1} \), and \( v_\Lambda \) is the function defined in (4.12), we have that for all \( t \geq 1, \)
\[ \|v_\Lambda(t, \cdot)\|_{L^\infty} \leq C\|u_0\|_{H^{0,1}}t^{-1/4}, \quad \|v_{\Lambda^c}(t, \cdot)\|_{L^2} \leq C\|u_0\|_{H^{0,1}}t^{-1/2}, \]
where the positive constant \( C \) is independent of \( t \) and \( u_0. \)

**Proof.** Let \( \Gamma_{-1}(\xi) := \frac{1-\gamma(\xi)}{\xi}, \) which satisfies \( |\partial^\alpha \Gamma_{-1}(\xi)| \lesssim C_\alpha < \xi^{-1-\alpha} \) for any \( \alpha \in \mathbb{N}. \) Then we can write
\[ 1 - \Gamma(x, \xi) = \sqrt{h}\Gamma_{-1}(\frac{x+F'(\xi)}{\sqrt{h}})(\frac{x+F'(\xi)}{h}). \]
Using Lemma 2.1 and the definition of \( \bar{\mathcal{L}} \) in (4.13), one gets
\[ G_h^w(1 - \Gamma)v = \sqrt{h}G_h^w(\Gamma_{-1}(\frac{x+F'(\xi)}{\sqrt{h}})) \circ (\bar{\mathcal{L}}v). \]
From (4.13) and Proposition 2.6 we have that, for all \( t \geq 1, \)
\[ \|G_h^w(1 - \Gamma)v\|_{L^\infty} \leq C\|\tilde{\mathcal{L}}v\|_{L^2}h^{1/4}, \quad \|G_h^w(1 - \Gamma)v\|_{L^2} \leq C\|\tilde{\mathcal{L}}v\|_{L^2}h^{1/2}, \] (4.14)
which together with (4.10) completes the proof of Lemma 4.1.

To get the \( L^\infty- \)estimates for \( v_\Lambda \) in large time, we deduce an ODE from the PDE system (1.2):
\[ D_tv_\Lambda = (-\frac{x+c_1}{2} + c_0) v_\Lambda + \lambda t^{-\alpha/2} |v_\Lambda|^\alpha v_\Lambda + t^{-\alpha/2} (R_1(v) + R_2(v)) \] (4.15)
where

\[ R_1(v) = t^{\alpha/2} [D_t - G^w_h(x + F(t), G^w_h(\Gamma))] v + t^{\alpha/2} (G^w_h(x + F(t)) - (x + c_0)^2/(4c_2 + c_0)) v_0 \]

and

\[ R_2(v) = -\lambda G^w_h(1 - \Gamma)(|v|^\alpha v) + \lambda(|v|\alpha |v_\alpha|). \]

The estimates for \( R_1(v) \) and \( R_2(v) \) are collected in the following Lemmas.

**Lemma 4.2.** Assume \( u \) is the solution given by Theorem 3.3 with \( u_0 \in H^{0,1} \), and \( v_\alpha \) is the function defined in (4.1), we have that for all \( t \geq 1 \),

\[
\|D_t - G^w_h(x + F(t), G^w_h(\Gamma))v\|_{L^\infty} \leq C\|\tilde{L}v\|_{L^2} t^{-5/4},
\]

\[
\|D_t - G^w_h(x + F(t), G^w_h(\Gamma))v\|_{L^2} \leq C\|\tilde{L}v\|_{L^2} t^{-3/2},
\]

where the positive constant \( C \) is independent of \( t \) and \( u_0 \). Furthermore, if \( u_0 \in H^{0,2} \) and \( \alpha \geq 1 \), we have that, for all \( t \geq 1 \),

\[
\|D_t - G^w_h(x + F(t), G^w_h(\Gamma))v\|_{L^\infty} \leq C\|\tilde{L}^2v\|_{L^2} t^{-7/4},
\]

\[
\|D_t - G^w_h(x + F(t), G^w_h(\Gamma))v\|_{L^2} \leq C\|\tilde{L}^2v\|_{L^2} t^{-2}.
\]

**Proof.** Since \( h = t^{-1} \), by a direct computation, we have

\[
[D_t, G^w_h(\Gamma)] f = -hiG^w_h(\Gamma) f + \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} (x-y)\xi \gamma(\frac{x+y}{2\sqrt{h}} + F'(\xi)) f(t,y) dy d\xi
\]

\[
+ \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} \gamma'(\frac{x+y}{2\sqrt{h}} + F'(\xi))(x+y) + F'(\xi)\sqrt{h} \frac{F''(\xi)}{2h} f(t,y) dy d\xi = \]

\[
= ihG^w_h(\gamma'\frac{x+y}{\sqrt{h}})(\xi \gamma(\frac{F''(\xi)}{\sqrt{h}} - \frac{x+y}{\sqrt{h}})) f(t,y) dy d\xi.
\]

where we use the fact that

\[
\frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} (x-y)\xi \gamma(\frac{x+y}{2\sqrt{h}} + F'(\xi)) f(t,y) dy d\xi
\]

\[
= \frac{1}{2\pi h} \int \int e^{i(x-y)\frac{\xi}{h}} \gamma(\frac{x+y}{2\sqrt{h}} + F'(\xi))(x+y) + F'(\xi)\sqrt{h} \frac{F''(\xi)}{2h} f(t,y) dy d\xi
\]

\[
= hiG^w_h(\Gamma) f + \frac{1}{2\pi h} \int e^{i(x-y)\frac{\xi}{h}} \gamma'(\frac{x+y}{2\sqrt{h}} + F'(\xi))(F''(\xi)i\sqrt{h} f(t,y) dy d\xi.
\]

On the other hand, using Propositions 2.4 and 2.5, we obtain

\[
[G^w_h(x + F(t)), G^w_h(\Gamma)] = ihG^w_h(\gamma'\frac{x+y}{\sqrt{h}})(\xi \gamma(\frac{F''(\xi)}{\sqrt{h}} - \frac{x+y}{\sqrt{h}})) + r_1 - r_2,
\]

where

\[
r_1 = \frac{1}{(2\pi)^3} \int \int e^{i(x-y)\frac{\xi}{h}} \frac{1}{h} \gamma''(\frac{x+y+F'(\xi+\eta)}{\sqrt{h}}) \left( F''(\xi+\eta) \right) dy d\xi.
\]

\[
r_2 = \frac{1}{(2\pi)^3} \int \int e^{i(x-y)\frac{\xi}{h}} \frac{1}{h} \gamma''(\frac{x+y+F'(\xi+\eta)}{\sqrt{h}}) \left( F''(\xi+\eta) \right) dy d\xi.
\]
Proposition 2.6, one gets Lemma 4.2. The function defined in (4.12), we have that, for all

Furthermore, if \( c \), we can rewrite (4.18) as

Using \( F'' = 2c_2 \) and the facts \( \int e^{2\pi z} dz = \delta(\eta)\pi h, \int e^{-2\pi z} dz = \delta(y)\pi h \), we get \( r_1 = \frac{c}{4}h \gamma''(x + F'(\xi)) \). Similarly, we have \( r_2 = \frac{c}{4}h \gamma''(x + F'(\xi)) \). Thus by (4.16) and (4.17), we obtain

Moreover, applying Lemma 2.1 we can rewrite (4.18) as

where \( \Gamma_{-2}(\xi) = \gamma'(\xi) \) satisfies \( |\gamma' \Gamma_{-2}(\xi)| \lesssim C_\alpha < \xi >^{-1-\alpha} \) for any \( \alpha \in \mathbb{N} \). Then using Proposition 2.6, one gets Lemma 3.2.

Lemma 4.3. Assume \( u \) is the solution given by Theorem 3.3 with \( u_0 \in H^{0,1} \), and \( v_\Lambda \) is the function defined in (4.12), we have that, for all \( t \geq 1 \),

Furthermore, if \( u_0 \in H^{0,2} \) and \( \alpha \geq 1 \), we have that, for all \( t \geq 1 \),

Proof. From (4.5), we have that

so that by Lemma 2.1

\[
(G_h''(x\xi + F(\xi)) - ((x + c_1)^2/(4c_2)) + c_0) v_\Lambda = \frac{1}{4c_2} G_h''((x + F'(\xi))^2 \gamma(x + F'(\xi))/\sqrt{h})) v.
\]
Applying Lemma 2.1 again, we can write the above identity as

\[
(G_h^n(x_\xi + F(\xi)) - (- (x + c_1)/(4c_2) + c_0)) v_\Lambda = \frac{h^{3/2}}{4c_2} G_h^n(\Gamma_{-3}(\frac{x + F'(\xi)}{\sqrt{h}})) (\tilde{L} v), \quad u_0 \in H^{0.1}
\]

\[
(G_h^n(x_\xi + F(\xi)) - (- (x + c_1)/(4c_2) + c_0)) v_\Lambda = \frac{h^2}{4c_2} G_h^n(\gamma(\frac{x + F'(\xi)}{\sqrt{h}})) (\tilde{L}^2 v), \quad u_0 \in H^{0.2},
\]

where \( \Gamma_{-3}(\xi) = \xi \gamma(\xi) \) satisfies \( |\partial^\alpha \Gamma_{-3}(\xi)| \lesssim C_\alpha < \xi >^{-1-\alpha} \) for any \( \alpha \in \mathbb{N} \). Then using Proposition 2.6 one gets Lemma 4.3.

**Lemma 4.4.** Assume \( u \) is the solution given by Theorem 3.3 with \( u_0 \in H^{0.1} \), and \( v \) is the function defined in (4.21), we have that, for all \( t \geq 1 \),

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^\infty} \leq C \|v\|_{L^\infty}^2 \|\tilde{L} v\|_{L^2} t^{-1/4},
\]

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^2} \leq C \|v\|_{L^\infty}^2 \|\tilde{L} v\|_{L^2} t^{-1/2},
\]

where the positive constant \( C \) is independent of \( t \) and \( u_0 \). Furthermore, if \( u_0 \in H^{0.2} \) and \( \alpha \geq 1 \), we have that, for all \( t \geq 1 \),

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^\infty} \leq C \|v\|_{L^\infty}^2 \|\tilde{L}^2 v\|_{L^2} t^{-3/4},
\]

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^2} \leq C \|v\|_{L^\infty}^2 \|\tilde{L}^2 v\|_{L^2} t^{-1}.
\]

**Proof.** Using the same method as that used to derive (4.14), one gets

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^\infty} \leq C h^{1/4} \|\tilde{L} v\|_{L^2}
\]

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^2} \leq C h^{1/2} \|\tilde{L} v\|_{L^2},
\]

which together with (4.6) proves the first part of Lemma 4.4.

When \( u_0 \in H^{0.2} \), we write

\[
1 - \Gamma(x, \xi) = h \Gamma_{-4}(\frac{x + F'(\xi)}{\sqrt{h}}) = h \Gamma_{-4}(\frac{x + F'(\xi)}{\sqrt{h}})^2,
\]

where \( \Gamma_{-4}(\xi) := \frac{1 - \gamma(\xi)}{\xi^2} \) satisfies \( |\partial^\alpha \Gamma_{-4}(\xi)| \lesssim C_\alpha < \xi >^{-\alpha} \) for any \( \alpha \in \mathbb{N} \). Then using Lemma 2.1 and the definition of \( \tilde{L} \) in (4.3), one gets

\[
G_h^n(1 - \Gamma)(|v|^\alpha v) = h G_h^n(\Gamma_{-4}(\frac{x + F'(\xi)}{\sqrt{h}})) \circ (\tilde{L}^2 v).
\]  

Applying Proposition 2.6 to (4.20), we obtain, for all \( t \geq 1 \),

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^\infty} \leq C h^{3/4} \|\tilde{L}^2 v\|_{L^2},
\]

\[
\|G_h^n(1 - \Gamma)(|v|^\alpha v)\|_{L^2} \leq C h \|\tilde{L}^2 v\|_{L^2},
\]

which together with (4.7) finishes the proof of Lemma 4.4.

Using the similar argument as in the proof of Lemma 4.3, we can obtain the following lemma and omit the details.
Lemma 4.5. Assume \( u \) is the solution given in Theorem 4.3 with \( u_0 \in H^{0.1} \), and \( v_\Lambda \) is the function defined in (4.12), we have that, for all \( t \geq 1 \),

\[
\|v\|^a v - \|v\|^a v_\Lambda \|_{L^\infty} \leq C(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty})\|v\|_{L^2} t^{-1/4},
\]

\[
\|v\|^a v - \|v\|^a v_\Lambda \|_{L^2} \leq C(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty})\|v\|_{L^2} t^{-1/2},
\]

where the positive constant \( C \) is independent of \( t \) and \( u_0 \). Furthermore, if \( u_0 \in H^{0.2} \) and \( \alpha \geq 1 \), we have that, for all \( t \geq 1 \),

\[
\|v\|^a v - \|v\|^a v_\Lambda \|_{L^\infty} \leq C(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty})\|\tilde{v}\|_{L^2} t^{-3/4},
\]

\[
\|v\|^a v - \|v\|^a v_\Lambda \|_{L^2} \leq C(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty})\|\tilde{v}\|_{L^2} t^{-1}.
\]

Since \( \|\tilde{v}\|_{L^2} \leq \|u_0\|_{H^{0.1}} \) and \( \|\tilde{v}\|_{L^2} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}})^2 t^{2-\alpha} \) by (4.10) and (4.11), it follows from Lemmas 4.2–4.5 that for all \( t \geq 1 \),

(a) when \( u_0 \in H^{0.1} \):

\[
\|R_1(v)\|_{L^\infty} \leq C \|u_0\|_{H^{0.1}} t^{-5/4+\alpha/2}, \tag{4.21}
\]

\[
\|R_1(v)\|_{L^2} \leq C \|u_0\|_{H^{0.1}} t^{-3/2+\alpha/2}, \tag{4.22}
\]

\[
\|R_2(v)\|_{L^\infty} \leq C \|u_0\|_{H^{0.1}} (\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty}) t^{-1/4}, \tag{4.23}
\]

\[
\|R_2(v)\|_{L^2} \leq C \|u_0\|_{H^{0.1}} (\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty}) t^{-1/2}, \tag{4.24}
\]

(b) when \( u_0 \in H^{0.2} \):

\[
\|R_1(v)\|_{L^\infty} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}})^{1/4-\alpha/2}, \tag{4.25}
\]

\[
\|R_1(v)\|_{L^2} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}})^{-\alpha/2}, \tag{4.26}
\]

\[
\|R_2(v)\|_{L^\infty} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}})(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty})^{5/4-\alpha}, \tag{4.27}
\]

\[
\|R_2(v)\|_{L^2} \leq C(\|u_0\|_{H^{0.2}} + \|u_0\|_{H^{0.2}})(\|v\|^a_{L^\infty} + \|v_\Lambda\|^a_{L^\infty}) t^{-1-\alpha}. \tag{4.28}
\]

4.2 The rough \( L^\infty \) estimate for \( v_\Lambda \)

In this subsection, we use the ODE (4.12) to derive the \( L^\infty \) estimate for \( v_\Lambda \). We assume \( u_0 \in H^{0.1} \) and \( \lambda \) satisfies the large dissipative assumption (4.2).

Let \( K \) be a sufficient large constant such that

\[
K > \max \left\{ 2C_1 \|u_0\|_{H^{0.1}} \left( (\log 2)^{1/2} + 2^{1/\alpha-1/2} \right), \|u_0\|_{H^{0.1}} + 1 \right\} \tag{4.29}
\]

and

\[
\frac{2-\alpha}{2}K^{-\alpha} + C_2 \|u_0\|_{H^{0.1}} \alpha^{\alpha+2} K^{-1} < \alpha\lambda_2, \tag{4.30}
\]

where \( C_1, C_2 \) are the constants in (4.31), (4.37) respectively.

From Lemma 4.1, (4.8), and Lemma 3.3 we have that, for \( t \in [1, 2] \),

\[
\|v_\Lambda(t, x)\|_{L^\infty} \leq \|v(t, x)\|_{L^\infty} + \|v\Lambda(t, x)\|_{L^\infty}
\leq C_1 \|u_0\|_{H^{0.1}} (1 + t^{-1/4})
\]
From (4.31), we see that
\[
\text{We describe only the proof of this lemma for the subcritical case } 4.
\]

Proof. Under the assumptions (4.32) and Lemma 4.6.

In what follows, we assume that \( v \) satisfies a bootstrap hypotheses on \( t \in [1, T_1] \):
\[
\|v_\Lambda(t, x)\|_{L^\infty} \leq \begin{cases} 
K (\log t)^{-1/2}, & \text{when } \alpha = 2, \\
K t^{1/2 - 1/\alpha}, & \text{when } 4/3 < \alpha < 2.
\end{cases}
\]

From (4.31), we see that \( T_1 > 2 \). Moreover, under the bootstrap hypotheses (4.32), one can check that \( \lambda t^{-\alpha/2}|v_\Lambda|^\alpha v_\Lambda + t^{-\alpha/2}(R_1(v) + R_2(v)) \) is integrable on \((1, T_1)\); so that \( v_\Lambda(t, x) \in C([1, T_1); L^\infty) \) by the equation (4.15).

Lemma 4.6. Under the assumptions (4.32) and \( 4/3 < \alpha < 2 \), we have that, for all \( t \in (1, T_1) \),
\[
\|v_\Lambda(t, x)\|_{L^\infty} \leq \begin{cases} 
K (\log t)^{-1/2}, & \text{when } \alpha = 2, \\
K t^{1/2 - 1/\alpha}, & \text{when } 4/3 < \alpha < 2.
\end{cases}
\]

Proof. We describe only the proof of this lemma for the subcritical case \( 4/3 < \alpha < 2 \). For the critical case \( \alpha = 2 \), this lemma is proved in the same way.

We prove it by contradiction argument. Assume there exists some \((t_0, \xi_0) \in (1, T_1) \times \mathbb{R}\) such that \( |v_\Lambda(t_0, \xi_0)| > K t_0^{1/2 - 1/\alpha} \). From (4.31) and the continuity of \( v_\Lambda(t, \xi_0) \), we can find \( t^* \in (1, t_0) \) such that
\[
|v_\Lambda(t, \xi_0)| > K t_0^{1/2 - 1/\alpha}, \quad \text{holds for all } t^* < t \leq t_0
\]
and furthermore \( |v_\Lambda(t^*, \xi_0)| = K t_0^{1/2 - 1/\alpha} \). Multiplying \( |v_\Lambda(t, \xi_0)|^{-(\alpha+2)} v_\Lambda(t, \xi_0) \) on both hand sides of (4.15) and taking the imaginary part, we obtain
\[
- \frac{1}{\alpha} \frac{d}{dt} |v_\Lambda|^{-\alpha} = -\lambda_2 t^{-\alpha/2} - \text{Im} \left( t^{-\alpha/2} (R_1(v) + R_2(v)) \frac{v_\Lambda}{v_\Lambda} \right) |v_\Lambda|^{-(\alpha+2)}. \tag{4.35}
\]

On the other hand, from (4.21) and (4.23), we have
\[
\|R_1(v) + R_2(v)\|_{L^\infty} \leq C \|u_0\|_{H^{0.1}} t^{-5/4 + \alpha/2} + (\|v_\Lambda\|^2_{L^\infty} + \|v_\Lambda\|^2_{L^\infty}) t^{-1/4}
\leq C \|u_0\|_{H^{0.1}} 2^{\alpha+2} K^{\alpha} t^{-5/4 + \alpha/2}, \tag{4.36}
\]
for all \( t \in (t_*, t_0) \), where we used (4.29), Lemma 4.1 and the bootstrap hypotheses (4.32) in the second inequality. It then follows from (4.34)–(4.36) that there exists \( C_2 > 0 \) such that,
\[
- \frac{1}{\alpha} \frac{d}{dt} |v_\Lambda|^{-\alpha} \leq -\lambda_2 t^{-\alpha/2} + C_2 \|u_0\|_{H^{0.1}} 2^{\alpha+2} K^{\alpha} t^{-3/4 + \alpha/2 + 1/\alpha}. \tag{4.37}
\]

Integrating (4.37) from \( t_* \) to \( t \), we get
\[
|v_\Lambda(t, \xi_0)|^{-\alpha} - |v_\Lambda(t_*, \xi_0)|^{-\alpha} \geq \frac{2 \alpha \lambda_2}{2 - \alpha} \left( t^{1-\alpha/2} - t_*^{1-\alpha/2} \right) - C_2 \|u_0\|_{H^{0.1}} \frac{2^{\alpha+2} K^{\alpha} t^{-3/4 + \alpha/2 + 1/\alpha}}{1 - \alpha/2 + 1/\alpha}.
\]
This inequality together with \( |v_\Lambda(t_*, \xi_0)| = K t_*^{1/2-1/\alpha} \) implies that
\[
\left( \frac{t_*^{1/2-1/\alpha}}{v_\Lambda(t, \xi_0)} \right)^{-\alpha}
\geq \left( \frac{t_*}{t} \right)^{1-\alpha/2} K^{-\alpha} + \frac{2\alpha \lambda_2}{2-\alpha} \left( 1 - \left( \frac{t_*}{t} \right)^{1-\alpha/2} \right)
\]
\[-C_2 \|u_0\|_{H^{\alpha,1}} \alpha^{2\alpha+2} K^{-1} t_*^{-3/4+1/\alpha} 1 - \left( \frac{t_*}{t} \right)^{1/4-\alpha/2+1/\alpha} \frac{1}{1/4 - \alpha/2 + 1/\alpha} \]
\[= f(t). \] (4.38)
Note that \( f(t_*) = K^{-\alpha} \) and \( f(t) \) is monotone increasing around \( t = t_* \). Indeed, by (4.30)
\[ f'(t_*) = \left( \frac{\alpha-2}{2} K^{-\alpha} + \alpha \lambda_2 - C_2 \|u_0\|_{H^{\alpha,1}} \alpha^{2\alpha+2} K^{-1} t_*^{-3/4+1/\alpha} \right) t_*^{-1} > 0. \]
Thus, if \( t \) is slightly larger than \( t_* \), then \( \left( t^{1/\alpha-1/2} |v_\Lambda(t, \xi_0)| \right)^{-\alpha} > K^{-\alpha} \), which contradicts (4.34). This completes the proof of Lemma 4.5.

From the bootstrap hypotheses (4.32), Lemma 4.6 and the standard continuation argument, we get \( T_1 = \infty \). Thus we obtain that, for all \( t \geq 1 \),
\[
\|v_\Lambda(t,x)\|_{L^\infty, t} \leq \begin{cases} 2K (\log t)^{-1/2}, & \text{when } \alpha = 2, \\ 2K t^{1-2/1-\alpha}, & \text{when } 4/3 < \alpha < 2. \end{cases} \] (4.39)
Finally, from (4.39), (4.8) and Lemma 4.1, we obtain the time decay estimates in Theorem 1.2.

5 The proof of Theorem 1.3

In this section, we prove Theorem 1.3. We consider only the nontrivial case \( u_0 \neq 0 \). To establish the asymptotic formula for the solutions for a wider range of \( \alpha \), we first need to refine the \( L^\infty \) estimate of \( v_\Lambda \).

5.1 The refined \( L^\infty \) estimate for \( v_\Lambda \)

From (4.15), (4.21)–(4.28), (4.39) and Lemma 4.1, we see that \( v \) satisfies
\[
D_t v_\Lambda = \left( -(x + c_1)^2/(4c_2) + c_0 \right) v_\Lambda + \lambda t^{-\alpha/2} |v_\Lambda|^\alpha v_\Lambda + t^{-\alpha/2} R(v), \] (5.1)
where \( R(v) \) satisfies, for all \( t \geq 1 \),
(a) \( u_0 \in H^{\alpha,1} \), there is a positive constant \( C(\|u_0\|_{H^{\alpha,1}}) \) independent of \( t \) such that
\[
\|R(v)\|_{L^\infty} \leq C(\|u_0\|_{H^{\alpha,1}}) t^{-\frac{3}{2}+\alpha/2}; \] (5.2)
\[
\|R(v)\|_{L^2} \leq C(\|u_0\|_{H^{\alpha,1}}) t^{-3/2+\alpha/2}. \] (5.3)
(b) \( u_0 \in H^{\alpha,2} \), there is a positive constant \( C(\|u_0\|_{H^{\alpha,2}}) \) independent of \( t \) such that
\[
\|R(v)\|_{L^\infty} \leq C(\|u_0\|_{H^{\alpha,2}}) t^{1/4-\alpha/2}, \] (5.4)
Lemma 5.1. Assume $\alpha > \frac{1 + \sqrt{33}}{4}$ when $u_0 \in H^{0.1}$ and $\alpha > \frac{7 + \sqrt{145}}{12}$ when $u_0 \in H^{0.2}$, we have
\[
\left\{ \begin{array}{l}
h_1(\varepsilon) := \frac{3}{4} - \frac{\alpha}{2} + \lambda_2 \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon}, \text{ when } u_0 \in H^{0.1}, \frac{1 + \sqrt{33}}{4} < \alpha \leq 2, \\
h_2(\varepsilon) := \frac{9}{4} - \frac{3\alpha}{2} + \lambda_2 \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon}, \text{ when } u_0 \in H^{0.2}, \frac{7 + \sqrt{145}}{12} < \alpha \leq 2.
\end{array} \right.
\]

Since $\alpha \geq \frac{1 + \sqrt{33}}{4}$ when $u_0 \in H^{0.1}$ and $\alpha \geq \frac{7 + \sqrt{145}}{12}$ when $u_0 \in H^{0.2}$, we have
\[
\left\{ \begin{array}{l}
h_1(0) = \frac{3}{4} - \frac{\alpha}{2} + \frac{2 - \alpha}{2\alpha}, < 0, \\
h_2(0) = \frac{9}{4} - \frac{3\alpha}{2} + \frac{2 - \alpha}{2\alpha}, < 0.
\end{array} \right.
\]

Thus by the continuity of $h_1$, $h_2$, we can find $0 < \varepsilon_1 < 2\alpha\lambda_2$ such that for any $0 < \varepsilon < \varepsilon_1$
\[
\frac{3}{4} - \frac{\alpha}{2} + \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon_0} < 0, \text{ when } u_0 \in H^{0.1}, \frac{1 + \sqrt{33}}{4} < \alpha \leq 2, \quad \left(5.7\right)
\]
\[
\frac{9}{4} - \frac{3\alpha}{2} + \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon_0} < 0, \text{ when } u_0 \in H^{0.2}, \frac{7 + \sqrt{145}}{12} < \alpha \leq 2. \quad \left(5.8\right)
\]

We then define
\[
K_0 = \left( \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon_0} \right)^{\frac{1}{\alpha}}, \quad T_0 = \max \left\{ \left( \frac{K_0^{\alpha+1}\varepsilon_0}{2C_3\alpha A} \right)^{-\frac{1}{\alpha}}, e \right\}, \quad \left(5.9\right)
\]
where $C_3$ is the constant in (5.14) and
\[
A = \left\{ \begin{array}{l} 
C(||u_0||_{H^{0.1}}), \quad \text{when } u_0 \in H^{0.1}, \\
C(||u_0||_{H^{0.2}}), \quad \text{when } u_0 \in H^{0.2}.
\end{array} \right.
\]

Next, for $t$ sufficiently large, we prove the following refined $L^\infty$ estimate for $v_A$.

Lemma 5.1. Assume $u_0 \in H^{0.1}$, $\frac{1 + \sqrt{33}}{4} < \alpha \leq 2$ or $u_0 \in H^{0.2}$, $\frac{7 + \sqrt{145}}{12} < \alpha \leq 2$. There exists $T^* > T_0$, such that for all $t > T^*$, we have
\[
\|v_A(t, x)\|_{L^\infty} \leq \left\{ \begin{array}{ll}
K_0(\log t)^{-1/2}, & \text{when } \alpha = 2, \\
K_0 t^{1/2 - 1/\alpha}, & \text{when } \alpha < 2.
\end{array} \right.
\]

Proof. We describe only the proof of this lemma for the subcritical case $4/3 < \alpha < 2$. For the critical case $\alpha = 2$, this lemma is proved in the same way.

In what follows, we prove Lemma 5.1 in the case $u_0 \in H^{0.1}$, $\frac{1 + \sqrt{33}}{4} < \alpha \leq 2$ by the contradiction argument. The case $u_0 \in H^{0.2}$, $\frac{7 + \sqrt{145}}{12} < \alpha \leq 2$ follows from a similar argument, but using (5.2)–(5.3), (5.8) instead of (5.4)–(5.5), (5.7).

Assume by contradiction that there exist $(t_n, \xi_n)_{n=1}^\infty \subset (T_0, \infty) \times \mathbb{R}$ with $t_n$ monotone increases to $\infty$ such that
\[
|v_A(t_n, \xi_n)| > K_0 t_n^{1/2 - 1/\alpha}, \quad \text{for all } n \geq 1. \quad \left(5.10\right)
\]
We first claim that, for every fixed $n \geq 1$, we have
\[
|v_A(t, \xi_n)| > K_0 t^{1/2 - 1/\alpha}, \quad \text{for all } t \in (T_0, t_n). \quad \left(5.11\right)
\]
In fact, if this claim is not true, there would exist some \( t^*_n \in (T_0, t_n) \) such that

\[
|v_A(t_n, \xi_n)| > K_0 t_n^{1/2-1/\alpha}, \quad \text{holds for all } t \in (t^*_n, t_n),
\]

and furthermore \( |v_A(t^*_n, \xi_n)| = K_0(t^*_n)^{1/2-1/\alpha} \). Multiplying \( |v_A(t, \xi_n)|^{-(\alpha+2)} v(t, \xi_n) \) on both hand sides of \((5.1)\) and taking the imaginary part, we obtain, for all \( t \in (t^*_n, t_n), \)

\[
-\frac{1}{\alpha} \frac{d}{dt} |v_A(t, \xi_n)|^{-\alpha} = -\lambda_2 t^{-\alpha/2} - \text{Im} \left( t^{-\alpha/2} R(\nu) \right) |v_A|^{-(\alpha+2)}.
\]

Then using \((5.2)\) and \((5.12)\), we deduce that there exists \( C_3 > 0 \) such that for all \( t \in (t^*_n, t_n), \)

\[
-\frac{1}{\alpha} \frac{d}{dt} |v_A(t, \xi_n)|^{-\alpha} \leq -\lambda_2 t^{-\alpha/2} + C_3 \|u_0\|_{H^0,1} K_0^{-3/4+1/\alpha}.
\]

Integrating the above inequality from \( t^*_n \) to \( t \) and using \( |v_A(t^*_n, \xi_0)| = K_0(t^*_n)^{1/2-1/\alpha} \), we obtain

\[
\left( t^{1/\alpha-1/2} |v_A(t, \xi_n)| \right)^{-\alpha} \geq \left( \frac{t^*_n}{t} \right)^{1-\alpha/2} K_0^{-\alpha} + \frac{2\alpha \lambda_2}{2-\alpha} \left( 1 - \left( \frac{t^*_n}{t} \right)^{1-\alpha/2} \right) \alpha K_0^{-3/4+1/\alpha} \frac{1 - \left( \frac{t^*_n}{t} \right)^{1/4-\alpha/2+1/\alpha}}{1/4 - \alpha/2 + 1/\alpha}.
\]

Thus, if \( t \) is slightly larger than \( t^*_n \), then \( |v_A(t^*_n, \xi_n)|^{-(\alpha+2)} \left| v_A(t, \xi_n) \right|^{-\alpha} > K_0^{-\alpha} \), which contradicts \((5.12)\). Thus we finish the proof of Claim \((5.11)\).

In what follows, we use Claim \((5.1)\) to derive a contradiction to \((5.10)\). From \((5.13)\), \((5.2)\) and \((5.11)\), we see that \((5.14)\) holds for all \( t \in (T_0, t_n) \). Integrating \((5.14)\) from \( T_0 \) to \( t_n \), using \((5.2)\) and Claim \((5.11)\) we have, for every \( n \geq 1, \)

\[
\left( \frac{t_n}{T_0} \right)^{1/\alpha} |v_A(T_0, \xi_n)|^{-\alpha} \geq \left( \frac{t_n}{T_0} \right)^{1-\alpha/2} + \frac{2\alpha \lambda_2}{2-\alpha} \left( 1 - \left( \frac{T_0}{t_n} \right)^{1-\alpha/2} \right) \alpha K_0^{-3/4+1/\alpha} \frac{1 - \left( \frac{T_0}{t_n} \right)^{1/4-\alpha/2+1/\alpha}}{1/4 - \alpha/2 + 1/\alpha}.
\]

Since \( K_0^{-\alpha} \geq \left( \frac{t_n}{T_0} \right)^{1/\alpha} |v_A(T_0, \xi_n)|^{-\alpha} \) by \((5.10)\), and \( \left| v_A(T_0, \xi_n) \right|^{-\alpha} \to 0 \) as \( n \to \infty \) by Claim \((5.11)\) we can let \( n \to \infty \) in \((5.15)\) to obtain

\[
K_0^{-\alpha} \geq \frac{2\alpha \lambda_2}{2-\alpha}.
\]

This contradicts the definition of \( K_0 \) in \((5.9)\), and thus completes the proof of Lemma \((5.1)\).
5.2 The proof of Theorem 1.3

We are now in a position to prove Theorem 1.3. We describe only the proof for \( u_0 \in H^{0,1} \), \( 1+\sqrt{2} < \alpha \leq 2 \). The case \( u_0 \in H^{0,2}, \frac{1}{2}+\sqrt{3} < \alpha \leq 2 \) follows from a similar argument, but using (5.4)–(5.5), (5.8) instead of (5.2)–(5.3), (5.7).

Assume \( T^* \) is the constant in Lemma 5.1. \( \Phi(t, x), K(t, x), \psi_+(x), S(t, x) \) are the functions defined in (1.13)–(1.16), respectively. From (1.13) and Lemma 5.1, we have

\[
\|\Phi(t, x)\|_{L^\infty_x} \leq \int_1^{T^*} s^{-\alpha/2}\|v_\Lambda(s, x)\|_{L^\infty_x}^2 ds + K_0^\alpha \int_{T^*}^t s^{-1} ds \leq C + K_0^\alpha \log t, \tag{5.16}
\]

for all \( t \geq 1 \), where we use \((\log s)^{-1} \leq (\log T^*)^{-1} \leq 1 \) for \( s \geq T^* \) in the critical case \( \alpha = 2 \). Set \( w(x) = -(x + c_1)^2/(4c_2) + c_0 \), and

\[
z(t, x) = v_\Lambda(t, x)e^{-i(w(x)t + \gamma \Phi(t, x))}, \quad t > 1. \tag{5.17}
\]

From the equation (5.1) and (5.17), we have that for \( t > 1 \),

\[
\partial_t z(t, x) = \frac{iR(v)}{t^{\alpha/2}} e^{-i(w(x)t + \gamma \Phi(t, x))};
\]

so that for all \( t_2 > t_1 > 1 \), we have

\[
z(t_2, x) - z(t_1, x) = \int_{t_1}^{t_2} s^{-\alpha/2} R(v)e^{-i(w(x)s + \gamma \Phi(s, x))} ds. \tag{5.18}
\]

Since \(-1/4 + \lambda_2 K_0^\alpha < 0 \) by (5.7) and (5.9), it follows from (5.2)–(5.3), (5.16) and (5.18) that

\[
||z(t_2, x) - z(t_1, x)||_{L^\infty_x \cap L^2_v} \lesssim \int_{t_1}^{t_2} s^{-\alpha/2} \|R(v)\|_{L^\infty_x \cap L^2_v} e^{\lambda_2 \|\Phi(s, x)\|_{L^\infty_x}} ds \\
\lesssim \int_{t_1}^{t_2} s^{-5/4 + \lambda_2 K_0^\alpha} ds \\
\lesssim t_1^{-1/4 + \lambda_2 K_0^\alpha}, \tag{5.19}
\]

for any \( t_2 > t_1 > 1 \). Thus there exists \( z_+(x) \in L^2_x \cap L^\infty_v \) such that

\[
||z(t, x) - z_+(x)||_{L^2_x \cap L^\infty_v} \lesssim t^{-1/4 + \lambda_2 K_0^\alpha}. \tag{5.20}
\]

This finishes the proof of part (a).

Next, we prove part (b). We first derive the asymptotic formula for \( \Phi(t, x) \). Note that

\[
\partial_t \Phi(t, x) = t^{-\alpha/2} |v_\Lambda(t, x)|^\alpha = t^{-\alpha/2} |z(t, x)|^\alpha e^{-\alpha \lambda_2 \Phi(t, x)}, \quad \text{for all } t > 1;
\]

so that

\[
\partial_t e^{\alpha \lambda_2 \Phi(t, x)} = \alpha \lambda_2 t^{-\alpha/2} |z(t, x)|^\alpha.
\]

Integrating the above equation from 1 to \( t \), we get

\[
e^{\alpha \lambda_2 \Phi(t, x)} = 1 + \alpha \lambda_2 \int_1^t s^{-\alpha/2} |z(s, x)|^\alpha ds. \tag{5.21}
\]
From (1.14), (1.15) and (5.21), we have
\[ e^{\alpha \lambda t \Phi(t,x)} - K(t,x) - \psi_+(x) = -\alpha \lambda_2 \int_t^\infty s^{-\alpha/2} (|z(s,x)|^\alpha - |z_+(x)|^\alpha) \, ds. \tag{5.22} \]
Since \(|u|^\alpha - |v|^\alpha| \lesssim (|u|^{\alpha-1} + |v|^{\alpha-1}) |u - v|, \) and \(z(s,x), z_+(x) \in L_x^\infty, \) we deduce from (5.20) and (5.22) that, for all \( t \geq 1, \)
\[ \left\| e^{\alpha \lambda t \Phi(t,x)} - K(t,x) - \psi_+(x) \right\|_{L_x^\infty \cap L_x^2} \lesssim \int_t^\infty s^{-1/4-\alpha/2+\lambda_2 K_0^\alpha} \, ds \lesssim t^{-\beta}, \tag{5.23} \]
where \( \beta = -(3/4 - \alpha/2 + \lambda_2 K_0^\alpha) > 0 \) by (5.7) and (5.9).

We now prove the asymptotic formula (1.17). Since \( z(t,x) = v_\Lambda(t,x) e^{-i(u(x)t + \lambda \Phi(t,x))} \)
and \(|e^{i\lambda_S(t,x)} - e^{-\lambda_2 S(t,x)} - e^{-\lambda_2 \Phi(t,x)}| \leq 1, \) we have
\[
\begin{align*}
\left\| e^{i(\lambda_S(t,x)\Phi(t,x))} z_+(x) - v_\Lambda(t,x) \right\|_{L_x^\infty \cap L_x^2}
&\leq \left\| e^{i\lambda_S(t,x)} e^{-\lambda_2 S(t,x)} z_+(x) \right\|_{L_x^\infty \cap L_x^2} + \left\| e^{i\lambda_S(t,x)} (e^{-\lambda_2 S(t,x)} - e^{-\lambda_2 \Phi(t,x)}) \right\|_{L_x^\infty \cap L_x^2} + \left\| z_+(x) - z(t,x) \right\|_{L_x^\infty \cap L_x^2}
&\leq \left\| e^{i\lambda_S(t,x)} z_+(x) - v_\Lambda(t,x) \right\|_{L_x^\infty \cap L_x^2} + \left\| z_+(x) - z(t,x) \right\|_{L_x^\infty \cap L_x^2}.
\end{align*}
\tag{5.24} \]

Note that \( K(t,x) + \psi_+(x) \geq 1/2 \) for \( t \) sufficiently large by (5.23); so that
\[
\begin{align*}
\left\| (e^{i\lambda_1 S(t,x)}(e^{-\lambda_2 S(t,x)} - e^{-\lambda_2 \Phi(t,x)})) \right\|_{L_x^\infty \cap L_x^2}
&\leq \left\| (K(t,x) + \psi_+(x))^{-1/\alpha} - e^{-\lambda_2 \Phi(t,x)} \right\|_{L_x^\infty \cap L_x^2}
\leq \left\| e^{\lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))^{1/\alpha} \right\|_{L_x^\infty \cap L_x^2}
\leq \left\| e^{\alpha \lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))^{1/\alpha} \right\|_{L_x^\infty \cap L_x^2} \lesssim t^{-\beta/\alpha}. \tag{5.25} \]
Moreover, from the proof argument in (5.25), we have
\[
\begin{align*}
\left\| (e^{i\lambda_1 S(t,x)} - e^{i\lambda_2 \Phi(t,x)}) e^{-\lambda_2 \Phi(t,x)} \right\|_{L_x^\infty \cap L_x^2}
&\lesssim \left\| S(t,x) - \Phi(t,x) \right\|_{L_x^\infty \cap L_x^2}
\leq \left\| e^{\lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))^{1/\alpha} \right\|_{L_x^\infty \cap L_x^2}
\leq \left\| e^{\alpha \lambda_2 \Phi(t,x)} - (K(t,x) + \psi_+(x))^{1/\alpha} \right\|_{L_x^\infty \cap L_x^2} \lesssim t^{-\beta/\alpha}. \tag{5.26} \]

Combining (5.19), (5.22) - (5.26), we get
\[
\left\| e^{i(\lambda_S(t,x)\Phi(t,x))} z_+(x) - v_\Lambda(t,x) \right\|_{L_x^\infty \cap L_x^2} \lesssim t^{-\gamma}, \tag{5.27} \]
for \( t \) sufficiently large, where \( 0 < \gamma := \min\{1/4 - \lambda_2 K_0^\alpha, \beta/\alpha\} < 1/4. \) It then follows from Lemma 1.10.1 and (5.27) that the asymptotic formula (1.17) holds.

Next, we prove part (c). From the asymptotic formula (1.17), we have
\[
\begin{align*}
e^{-iF(D) t} e^{-i\lambda S(t,x)} u(t,x)
&= e^{-iF(D) t} \frac{1}{\sqrt{t}} z_+(x) e^{i\alpha t \xi t + \xi^2 t^2/\alpha} + O_L(t^{-\gamma})
&= \frac{1}{2\pi} \int e^{i(x-y) \xi} e^{-iF(\xi) t} \frac{1}{\sqrt{t}} z_+(y) e^{i\alpha t \xi^2 + \xi^2 t^2/\alpha} \, dyd\xi + O_L(t^{-\gamma})
\end{align*}
\]
\[ = \frac{1}{2\pi} \int e^{i(x-y) \xi} e^{-iF(\xi) t} \frac{1}{\sqrt{t}} z_+(y) e^{i\alpha t \xi^2 + \xi^2 t^2/\alpha} \, dyd\xi + O_L(t^{-\gamma}) \]
Similarly, we have for $t > T$

Thus we deduce from (5.31), (5.32) and (5.2) that for $t > T$

The key observation is that the solution decays faster when

Proof. The above two inequalities together with (4.1) and Lemma 4.1 yields (5.30).

On the other hand, using (1.13) and Lemma 5.1, we have, for $s > T^*$

Finally, we prove part (d). We first claim that

Claim 5.1. If the limit function $z_+(x)$ in (5.20) satisfies $z_+(x) = 0$ for a.e. $x \in \mathbb{R}$, then we must have $u_0 = 0$.

Proof. The key observation is that the solution decays faster when $z_+ = 0$:

In fact, since $z_+ = 0$, it follows from (5.17)–(5.18) that

On the other hand, using (1.13) and Lemma 5.1 we have, for $s > T^*$

Thus we deduce from (5.31), (5.32) and (5.2) that for $t > T^*$,

Similarly, we have for $t > T^*$,

The above two inequalities together with (4.11) and Lemma 4.1 yields (5.30).

Next, we apply the decay estimates (5.30) to prove that $u_0 = 0$. Since $z_+ = 0$, we deduce from (4.1), (1.17) and Duhamel’s formula that

$$u(t, x) = \lambda \int_t^\infty e^{iF(D)(t-s)}(\|u^2u)(s)ds.$$
Moreover, it follows from Strichartz’s estimate and Hölder’s inequality that

\[
\|u\|_{L^4(T, \infty), L^\infty_x} \lesssim \int_T^\infty \|u(s, x)\|_{L^\infty_x}^2 \\|u(s, x)\|_{L^2_x} \, ds \\
\lesssim \|u\|_{L^4(T, \infty), L^\infty_x} \left( \int_T^\infty \left( \|u(s, x)\|_{L^\infty_x} \|u(s, x)\|_{L^2_x} \right)^{4/3} \, ds \right)^{3/4} \\
\leq C \|u\|_{L^4(T, \infty), L^\infty_x} T^{1-3\alpha/4},
\]

where we use (5.30) in the last step. Since \( \alpha > 4/3 \), we can choose \( T > T^* \) sufficiently large such that \( C T^{1-3\alpha/4} \leq \frac{1}{2} \); so that \( \|u\|_{L^4(T, \infty), L^\infty_x} = 0 \). This together with the uniqueness of the solutions implies \( u = 0 \), and thus \( u_0 = 0 \).

We now prove part (d). We consider only the subcritical case \( 4/3 < \alpha < 2 \). For the critical case \( \alpha = 2 \), this is proved in the same way. Since \( u_0 \neq 0 \), there exists \( x_0 \in \mathbb{R} \) such that \( z_+(x_0) \neq 0 \) by Claim 5.1. Then by (1.17) and (1.22), we have

\[
t^{1/\alpha} \|u(t, x)\|_{L^\infty_x} \geq \frac{t^{1/\alpha - 1/2} |z_+(x_0)|}{\left( 1 + \frac{2\alpha\lambda_2}{2\alpha} |z_+(x_0)|^{(2/\alpha)(2-\alpha) - 1} + \psi_+(x) \right)^{1/\alpha}} + O(t^{-\gamma})
\]

which implies

\[
\liminf_{t \to \infty} t^{1/\alpha} \|u(t, x)\|_{L^\infty_x} \geq \left( \frac{2 - \alpha}{2\alpha\lambda_2} \right)^{1/\alpha}.
\]

On the other hand, from (4.1), Lemma 4.1, Lemma 5.1 and (5.9), we get

\[
t^{1/\alpha} \|u(t, x)\|_{L^\infty_x} \leq \left( \frac{2 - \alpha}{2\alpha\lambda_2 - \varepsilon_0} \right)^{1/\alpha} + C t^{1/\alpha - 3/4}.
\]

Since \( \varepsilon_0 \) can be chosen to be arbitrary small, we have

\[
\limsup_{t \to \infty} t^{1/\alpha} \|u(t, x)\|_{L^\infty_x} \leq \left( \frac{2 - \alpha}{2\alpha\lambda_2} \right)^{1/\alpha}.
\]

The limit (1.20) is now an immediate consequence of (5.33) and (5.34). This completes the proof of Theorem 1.3.

**Acknowledgements**

This work is partially supported by NSF of China under Grants 11771389, 11931010 and 11621101.

**References**

[1] I. Barab, Nonexistence of asymptotically free solutions for nonlinear Schrödinger equations, J. Math. Phy. 25 (1984), 3270–3273.

[2] R. Carles, Geometric optics and long range scattering for one-dimensional nonlinear Schrödinger equations, Comm. Math. Phys. 220 (2001), 41–67.
[3] T. Cazenave, Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics. American Mathematical Society, New York, 2003.

[4] T. Cazenave, S. Correia, F. Dickstein and F. B. Weissler, A Fujita-type blowup result and low energy scattering for a nonlinear Schrödinger equation, São Paulo J. Math. Sci. 9 (2015), no. 2, 146–161.

[5] T. Cazenave, Z. Han, Asymptotic behavior for a Schrödinger equation with nonlinear subcritical dissipation. Nonlinear Anal. 205 (2021), 112243, 37pp.

[6] T. Cazenave, Z. Han and Y. Martel, Blowup on an arbitrary compact set for a Schrödinger equation with nonlinear source term. J. Dyn. Diff. Equat. (2020), https://doi.org/10.1007/s10884-020-09841-8.

[7] T. Cazenave, Y. Martel, Modified scattering for the critical nonlinear Schrödinger equation. J. Funct. Anal. 274 (2018), no. 2, 402–432.

[8] M.C. Cross and P.C. Hohenberg, Pattern formation outside of equilibrium. Rev. Mod. Phys. 65 (1993), 851–1112.

[9] P. A. Deift, X. Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. Comm. Pure Appl. Math. 56 (2003), no. 8, 1029–1077.

[10] J. Ginibre, G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations. J. Math. Pures Appl. 64 (1985), 363–401.

[11] J.M. Delort, Semiclassical microlocal normal forms and global solutions of modified one-dimensional KG equations. Ann. Inst. Fourier (Grenoble), 66 (2016), no. 4, 1451-1528.

[12] M. Dimassi, J. Sjöstrand, Spectral asymptotics in the semi-classical limit. London Mathematical Society Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999.

[13] J. Ginibre, T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, Comm. Math. Phys. 151 (1993), 619–645.

[14] N. Hayashi, P. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. Amer. J. Math. 120 (1998), no. 2, 369-389.

[15] N. Hayashi, P. Naumkin, Domain and range of the modified wave operator for Schrödinger equations with a critical nonlinearity, Comm. Math. Phys. 267 (2006), 477–492.

[16] G. Jin, Y. Jin, C. Li, The initial value problem for nonlinear Schrödinger equations with a dissipative nonlinearity in one space dimension. J. Evol. Equ. 16 (2016), 983-995.

[17] N. Kita, A. Shimomura, Asymptotic behavior of solutions to Schrödinger equations with a subcritical dissipative nonlinearity. J. Differential Equations. 242 (2007), 192-210.
[18] N. Kita, A. Shimomura, Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data. J. Math. Soc. Japan. 61 (2009), 39-64.

[19] A. Mielke, The Ginzburg-Landau equation in its role as a modulation equation, North-Holland. Amsterdam. in Handbook of dynamical systems. (2002), no. 2, 759–834.

[20] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension. Comm. Math. Phys. 139 (1991), 479–493.

[21] A. Shimomura, Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities. Comm. Partial Differential Equations. 31 (2006), 1407–1423.

[22] K. Stewartson and J.T. Stuart, A non-linear instability theory for a wave system in plane Poiseuille flow. J. Fluid Mech. 48 (1971), 529-545.

[23] A. Stingo, Global existence and asymptotics for quasi-linear one-dimensional Klein-Gordon equations with mildly decaying Cauchy data. Bull. Soc. Math. France, 146 (2018), no. 1, 155-213.

[24] W.A. Strauss, Nonlinear Scattering Theory, Scattering Theory in Mathematical Physics, Reidel, Dordrecht, Holland, 1974, pp. 53–78, edited by J.A. Lavita and J-P. Marchand.

[25] W.A. Strauss, Dispersion of low-energy waves for two conservative equations. Arch. Rat. Mech. Anal. 55 (1974), 86–92.

[26] Y. Tsutsumi, K. Yajima, The asymptotic behavior of nonlinear Schrödinger equations. Bull. Am. Math. Soc. 11 (1984), 186–188.

[27] T. Zhang, Global solutions of modified one-dimensional Schrödinger equation. Commun. Math. Res. (2021), in press, doi:10.4208/cmrr.2021-0015.

[28] M. Zworski, Semiclassical analysis, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI, 2012.