0. Introduction

0.1. The aim of this paper is to generalize to the singular case the following statements which in the nonsingular case are easy and well-known:

(1) The class of smooth surfaces with ample anticanonical divisor \(-K\), also known as Del Pezzo surfaces, is bounded.

(2) The class of smooth surfaces with ample canonical divisor \(K\) with \(K^2 \leq C\) is bounded.

(3) The class of smooth surfaces with ample canonical divisor \(K\) with \(K^2 = C\) is bounded.

(4) The set \(\{K_X^2\}\), where \(X\) is a smooth surface with ample \(K_X\), is the set \(\mathbb{N}\) of positive integer numbers.

0.2. Of course, the last of these statements is trivial, and the second and the third ones are equivalent. In the singular case, however, the set \(\{K_X^2\}\) a priori is just a certain subset in the set of positive rationals. We shall prove that under natural and rather weak conditions this set has a rigid structure: it satisfies the descending chain condition, abbreviated below to D.C.C. for short. The boundedness for surfaces with constant \(K^2\) will be
proved under much weaker conditions on singularities than for surfaces with $K^2$ only bounded.

0.3. In order to be useful, the generalizations should come from interesting real-life examples. The examples should also suggest what conditions on singularities are most natural. The section 2 contains several applications, due to J.Kollár, G.Xiao and others, that provide the motivation for our boundedness theorems. The most important application is the projectiveness of the coarse moduli space of stable surfaces. Two others are a formula bounding the automorphism group of a (possibly, singular) surface of general type by $cK^2$, and a theorem on the uniform plurigenera of elliptic threefolds.

0.4. These are the precise statements of the results proved in this paper:

1. Fix $\varepsilon > 0$. Consider all projective surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_j B_j$ such that $K_X + B$ is MR $\varepsilon$-log terminal and $-(K_X + B)$ is nef excluding only those for which at the same time $K_X$ is numerically trivial, $B$ is zero and $X$ has at worst Du Val singularities. Then the class $\{X\}$ is bounded (theorem 6.9).

2. Fix $\varepsilon > 0$, a constant $C$ and a D.C.C. set $\mathcal{C}$. Consider all surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_j B_j$ such that $K_X + B$ is MR $\varepsilon$-log terminal, $K_X + B$ is big and nef, $b_j \in \mathcal{C}$ and $(K + B)^2 \leq C$. Then the class $\{(X, \text{supp} B)\}$ is bounded (theorem 7.7).

3. Fix a constant $C$ and a D.C.C. set $\mathcal{C}$. Consider all surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_j B_j$ such that $K_X + B$ is MR semi-log canonical, $K_X + B$ is ample, $b_j \in \mathcal{C}$ and $(K + B)^2 = C$. Then the class $\{(X, \sum b_j B_j)\}$ is bounded (theorem 9.2).

4. Fix a D.C.C. set $\mathcal{C}$. Consider all surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_j B_j$ such that $K_X + B$ is MR semi-log canonical, $K_X + B$ is ample and $b_j \in \mathcal{C}$. Then the set $\{(K_X + B)^2\}$ is a D.C.C. set (theorem 8.2).

0.5. The necessary definitions will be given in the section 1. As the first approximation, the reader can look at the results obtained by dropping $B$. It is interesting to note that none of the conditions above can be weakened, see the examples in section 2. (Of course, it is always possible to choose an entirely different way of generalizing the nonsingular case and pick a set of conditions which is orthogonal to ours.)

0.6. I became interested in the subject reading [20] by Kollár, and that is where most of the motivation for the present paper comes from. The statements (3) and (4) above solve and generalize a Kollár’s conjecture formulated there. A hope for proving this conjecture came to me when I looked at the preprint [36] by Xiao, and especially at his proof of Proposition 5. I am most indebted to G.Xiao for answering many of my questions concerning his preprint. I am also thankful to V.V.Shokurov and K.Matsuki for useful discussions.

0.7. Throughout, we work with projective algebraic schemes over an algebraically closed field of arbitrary characteristic.
1. Standard definitions

1.1. For a normal variety $X$, $K_X$ or simply $K$ will always denote the class of linear equivalence of the canonical Weil divisor.

**Definition 1.2.** An $\mathbb{R}$-divisor $D = \sum d_i D_i$ is a linear combination of prime Weil divisors with real coefficients, i.e. an element of $N^1 \otimes \mathbb{R}$. An $\mathbb{R}$-divisor is said to be $\mathbb{R}$-Cartier if it is a combination of Cartier divisors with real coefficients, i.e. if it belongs to $\text{Div}(X) \otimes \mathbb{R} \subset N^1(X) \otimes \mathbb{R}$. The $\mathbb{Q}$-divisors and $\mathbb{Q}$-Cartier divisors are defined in a similar fashion.

**Definition 1.3.** Consider an $\mathbb{R}$-divisor $K + B = K_X + \sum b_j B_j$ and assume that

1. $K + B$ is $\mathbb{R}$-Cartier
2. $0 \leq b_j \leq 1$

For any resolution $f : Y \to X$ look at the natural formula

1. $K_Y + B^Y = f^*(K_X + \sum b_j B_j) = K_Y + \sum b_j f^{-1} B_j + \sum b_i F_i$

or, equivalently,

2. $K_Y + \sum b_j f^{-1} B_j + \sum F_i = f^*(K_X + \sum b_j B_j) + \sum f_i F_i$

Here $f^{-1} B_j$ are the proper preimages of $B_j$ and $F_i$ are the exceptional divisors of $f : Y \to X$. The coefficients $b_i, b_j$ are called codiscrepancies, the coefficients $f_i = 1 - b_i, f_j = 1 - b_j - \log$ discrepancies.

**Remark 1.4.** In fact, $K + B$ is not a usual $\mathbb{R}$-divisor but rather a special gadget consisting of a linear class of a Weil divisor $K$ (or a corresponding reflexive sheaf) and an honest $\mathbb{R}$-divisor $B$. This, however, does not cause any confusion.

**Definition 1.5.** An $\mathbb{R}$-Cartier divisor $K + B$ (or a pair $(X, B)$) is called

1. log canonical, if the log discrepancies $f_k \geq 0$
2. Kawamata log terminal, if $f_k > 0$
3. canonical, if $f_k \geq 0$
4. terminal, if $f_k > 0$
5. $\varepsilon$-log canonical, if $f_k \geq \varepsilon$
6. $\varepsilon$-log terminal, if $f_k > \varepsilon$

for every resolution $f : Y \to X$.

1.6. The name $\varepsilon$-log terminal was suggested to me by V.V.Shokurov.

**Definition 1.7.** In the two-dimensional case we shall say that $K + B$ is MR log canonical, MR Kawamata log terminal etc. if we require the previous inequalities to hold not for all resolutions $f : Y \to X$ but only for a distinguished one, the minimal desingularization.
1.8. In the surface case all the standard theorems of the log Minimal Model Program are valid under very weak assumptions, see the section 10. In particular, assuming $K + B$ to be MR Kawamata log terminal or MR log canonical is sufficient for all applications.

**Definition 1.9.** Let $X$ be a reduced scheme satisfying the Serre’s condition $S_2$ and $B = \sum b_j B_j$, $0 \leq b_j \leq 1$ be an $\mathbb{R}$-divisor. An $\mathbb{R}$-Cartier divisor $K_X + B$ (or a pair $(X, B)$) is called semi-log canonical if $f_k \geq 0$ in the above formula for every semiresolution $f : Y \to X$ (see [10] 12.2.1).

1.10. In dimension 2 and characteristic 0 the semi-log canonical singularities are classified in [22]. This is the complete list (modulo analytic isomorphism): nonsingular points, cones over nonsingular elliptic curves, double normal crossing points $xy = 0$, pinch points $x^2 = y^2 z$; and all finite quotients of above. In codimension one the semi-log canonical schemes have only normal crossing points.

1.11. Let $\nu : X^\nu \to X$ be a normalization of $X$, $X^\nu = \bigcup X_m$ be a decomposition into irreducible components and define $B_m$ on $X_m$ to be $\nu^{-1}(B)$ plus the double intersection locus. Then

$$K_{X^\nu} + \sum B_m = \nu^*(K_X + B)$$

and $K_{X_m} + B_m$ are log canonical ([10] 12.2.2,4).

**Definition 1.12.** We shall say that $K_X + B$ is MR semi-log canonical if $K_X$ is semi-log canonical and all $K_{X_m} + B_m$ are MR log canonical.

2. Examples

2.1. I include the following three examples from [20], in a generalized form. They help to understand what kind of boundedness results for singular surfaces are desirable. After the theorems of this paper are proved, these applications are no longer conjectural.

**Example 2.2** (Moduli of stable surfaces of general type). It is well known that the G.I.T. construction of a complete and projective moduli space $\overline{M}_g$ of stable curves does not work for surfaces of general type. As a bare minimum, the complete moduli space should parameterize surfaces appearing as semistable degenerations of nonsingular ones, and these have semi-log canonical singularities. In particular, they can have singularities of arbitrarily large multiplicities. On the other hand, by [26] 3.19 a normal surface with a singularity of multiplicity $\geq 7$ is not asymptotically Chow-(Hilbert-)stable, so the usual G.I.T. construction does not go through. The following is a rather general theorem which allows to circumvent this difficulty.

**Theorem 2.3** (Kollár, [19], 2.6). Let $C$ be an open class of $\mathbb{Q}$-polarized varieties with Hilbert function $H(t)$. Assume that the corresponding moduli functor $\mathcal{M}C$ is separated, functorially polarized, semipositive, bounded, complete and has tame automorphisms. Then $\mathcal{M}C$ is coarsely represented by a projective scheme $\mathcal{M}$.
2.4. Applying this theorem to surfaces of general type one considers stable surfaces, introduced for this purpose by Kollár and Shepperd-Barron in [22]. These are defined as surfaces with semi-log canonical singularities (in particular, they are reduced but not necessarily irreducible) such that \((\omega^N_X)^{**}\) is ample for some \(N > 0\). Some of the conditions in the theorem are easy to check. Separateness follows directly from the uniqueness of the canonical model (in any dimension). Stable surfaces have finite automorphism groups by Iitaka [14] 11.12 (and this is also true in higher dimensions). Completeness follows from the semi-stable reduction theorem and the log Minimal Model Theory (in dimension \(\dim X + 1\), or, as some say, \(\dim X + 1/2\)). Semipositiveness is harder but it is proved in [19]. The boundedness is exactly what we are concerned with in this paper. So if we want to include this application, the boundedness theorem for surfaces with positive \(K\) and constant \(K^2\) should be powerful enough as to include the case of semi-log canonical singularities. And it is indeed, see 9.2 with empty \(C\).

2.5. In fact, 9.2 is strong enough so that we are able to prove the projectiveness of the moduli space \(\overline{M}_{(K+B)^2}\) of surfaces with semi-log canonical and ample \(K + B\) where \(B\) is a reduced divisor (or, more generally, an \(\mathbb{R}\)-divisor with coefficients in a D.C.C. set \(C\)) and \((K + B)^2 = C\). This is a direct generalization of the moduli space \(\overline{M}_{g,k}\) of pointed stable curves (see [17]). This includes, for example, a projective moduli space, an open subset of which parameterizes smooth K3 surfaces \(X\) with ample and reduced divisors \(B\) with normal crossings. It is quite interesting to see what is the relation between this moduli space and the usual moduli space of polarized K3 surfaces. This will be carried out in detail elsewhere.

2.6. The following easy trick allows to reduce the boundedness of semi-log canonical surfaces that are \(a\,\text{priori}\) not irreducible to the boundedness of irreducible normal log canonical surfaces (see [20]). Let \(X^r = \cup X_m\) be a decomposition into irreducible components as in 1.11. Then

\[
K^2_X = \sum (K_{X_m} + B_m)^2
\]

So if we have enough information about possible values of \((K + B)^2\) and know the boundedness for normal surfaces with log canonical singularities, this should help us in the general situation. This shows that it is natural to consider not only the canonical divisor \(K\) but also the canonical divisor with a “boundary” \(K + B\). The following example explains the importance of coefficients \(b_j\) in \(B = \sum b_jB_j\) other than 1.

**Example 2.7** (Automorphisms of log surfaces of general type). The characteristic of the base field is assumed to be zero for this application.

Consider a nonsingular surface with ample \(K_X\). A general fact is that the group of biregular automorphisms of \(X\) \(G = \text{Aut}(X)\) is finite. Let \(\pi : X \rightarrow Y\) denote a quotient morphism. Then by the Hurwitz formula one has
where $B_j$ are the branching divisors. It follows that $K_Y + B = K_Y + \sum (1 - 1/n_j) B_j$ is log terminal and that

$$|\text{Aut}(X)| = \frac{K_X^2}{(K_Y + B)^2},$$

so if $(K_Y + B)^2 \geq 1/c$ then $|\text{Aut}(X)| \leq cK_X^2$. The reader will certainly recognize that for curves this is the original construction of Hurwitz which gives $\text{Aut}(X) \leq 42(2g - 2)$ because $2g - 2 + \sum (1 - 1/n_j) \geq 1/42$ if it is $> 0$. It was Xiao’s idea to use the same construction for surfaces of general type in [36]. Using this and other methods he proves that for nonsingular surfaces with ample canonical class one has $|\text{Aut}(X)| \leq 42^2 K_X^2$. As an application of 8.2 we have the following theorem.

**Theorem 2.8.** Fix a D.C.C. set $C$. Then for every surface $X$ with $K + B$ ample, semi-log canonical, and with $b_j \in C$ one has the following bound for the automorphism group interchanging components of $B$ with the same coefficients

$$|\text{Aut}(X, B)| \leq c(C)(K + B)^2$$

where the constant $c(C)$ depends only on the set $C$.

**Proof.** The group $G = \text{Aut}(X, B)$ is known to be finite, cf. [14], 11.12. Consider $\pi : X \to Y = X/G$. Now use the same formulas

$$K + B = \pi^*(K_Y + D) \quad \text{and} \quad |\text{Aut}(X, B)| = \frac{(K + B)^2}{(K_Y + D)^2}$$

It easily follows that $K_Y + D$ is also semi-log canonical. The coefficients of $D$ belong to the set

$$\{0 \leq 1 - (1 - \sum n_j b_j)/m_i \leq 1 \mid b_j \in C, m_i, n_j \in \mathbb{N}\}$$

which is obviously also a D.C.C. set. Now we only need to see that a D.C.C. set of positive numbers is bounded from below by a positive constant. □

2.9. Here is one more application of 8.2 due to J.Kollár [20].

**Corollary 2.10** (Uniform plurigenera of elliptic 3-folds). For a smooth 3-fold of Kodaira dimension 2 in characteristic 0, there exists an absolute constant $N$ such that $h^0(NK) \neq 0$.

**2.11.** This was previously known for Kodaira dimension 0 (Kawamata, [16]), dimension 1 (Mori, [25]) and for dimension 3 and $\chi(\mathcal{O}_X) \leq M$ (Fletcher, [11]).

The D.C.C. set used in this application is

$$C = \left\{\frac{1}{12}, \ldots, \frac{11}{12}, 1 - \frac{1}{k} \mid k \in \mathbb{N}\right\}$$
2.12. The following series of examples show that the restrictions of 0.4 are in a sense the weakest possible, and that none of them can be weakened further.

**Example 2.13.** Consider the cone over an elliptic curve embedded by a complete linear system of arbitrary degree in some projective space. The family of these cones is, evidently, not bounded. The singularities are simple elliptic and they are log canonical, $-K$ is ample.

**Example 2.14.** There are infinitely many, I would say, hopelessly many types of log Del Pezzo surfaces, i.e. surfaces with ample $-K$ and log terminal (=quotient in dimension 2 and characteristic 0) singularities. These include all surfaces $\mathbb{P}^2/G$, $G$ a finite subgroup in $PGL(3)$, for example. So, for the ample $-K$ the condition on singularities should be stronger than just log canonical or log terminal. The $\varepsilon$-log terminal condition seems to be the best substitute.

**Example 2.15.** The next example shows that the condition $b_j \leq 1 - \varepsilon$ is necessary even for smooth surfaces. Consider $\mathbb{P}^2$ with two lines $B_1$ and $B_2$ and the surface obtained by blowing up the point of the intersection of these lines, then several times the point of the intersection of the first line and the exceptional divisor. Choosing $b_1$ very close to 1 and taking for $K + B$ the full preimage $f^*(K_{\mathbb{P}^2} + b_1B_1 + b_2B_2)$ changed a little bit in the exceptional divisors, one easily obtains an infinite sequence of smooth surfaces with ample $-(K + B)$ and a strictly increasing Picard number. The same construction works for surfaces with ample $K + B$ as well.

**Example 2.16.** This example shows that the set $\mathcal{C}$ in 0.4 has to satisfy the D.C.C. To see this, take a rational ruled surface $F_e$, $e \geq 2$ and $B = (2 + a)/4(B_1 + B_2 + B_3 + B_4)$, where $B_i$ are general elements in the linear system $|s_e + ef|$, $s_e$ is the exceptional section, $f$ is a fiber. If $a$ is the positive root of the quadratic equation $ea^2 + (e - 2)a = 1$ then $K + B$ is ample and $(K + B)^2 = 1$. Note that as $e \to \infty$, $(2 + a)/4$ approaches its limit $1/2$ from above.

**Remark 2.17.** It was conjectured by V.V. Shokurov in [34] that all sets naturally appearing in conjunction with log Minimal Model Program have to satisfy either the descending or the ascending chain conditions. The first example of this general phenomenon was given in [1] for the set of Fano indices of log Del Pezzo surfaces. See [2], [4], [12] for further examples.

### 3. Some methods for proving boundedness

**Definition 3.1.** One says that a certain class of schemes $\mathcal{B}$ is bounded if there exists a morphism $f : \mathcal{X} \to \mathcal{S}$ between two schemes of finite type such that every scheme in $\mathcal{B}$ appears as one of the geometric fibers of $f$, not necessarily in a one-to-one way. We do not require that every geometric fiber of $f$ belongs to $\mathcal{B}$. Usually, though, the class is defined by a combination of
algebraic conditions some of which are open and others are closed. In this case one can find a constructible subset in $S$, points of which parameterize exactly elements of the class $B$.

**Definition 3.2.** In the same way, we say that a class $B$ of schemes with closed subschemes $\{(X, Z)\}$ is bounded if there exist three schemes of finite type $-\mathcal{X}$, a closed subscheme $Z \subset \mathcal{X}$, and $\mathcal{S}$, and a morphism $f : \mathcal{X} \to \mathcal{S}$ such that every element of $B$ appears as a fiber of $f$.

**Definition 3.3.** Finally, we can assign certain coefficients to subschemes of $X$ and then by boundedness of $\{(X, \sum b_j B_j)\}$ we mean that all $\{(X, B_j)\}$ are bounded in the previous sense and, in addition, that there are only finitely many possibilities for the sets of coefficients $\{b_j\}$.

**Definition 3.4.** A polarization on a scheme $H$ is a class of an ample Cartier divisor. A $\mathbb{Q}$-polarization on a normal variety is a $\mathbb{Q}$–Cartier divisor $a$, positive multiple of which is a polarization. It is possible to define an $\mathbb{R}$-polarization on some nonnormal varieties too when there is a suitable notion of an $\mathbb{R}$-divisor. A semi-log canonical scheme with an ample $\mathbb{R}$-Cartier divisor $K + B$ would be an example.

3.5. Consider a class of polarized reduced schemes over an algebraically closed field $k$ $(X, H)$ with a fixed Hilbert function $\mathcal{H}(t) = \chi(tH)$. It is known that this class is bounded provided any of the following conditions are satisfied:

1. $\dim X = 2$ (Matsusaka [23] for normal surfaces, Kollár [18] for the general case).
2. $\dim X = 3, X$ are normal and $\text{char } k = 0$ ([18]).
3. $X$ are nonsingular and $\text{char } k = 0$ (Matsusaka’s Big Theorem, see [23]).

3.6. Moreover, if $\text{char } k = 0$ and $X$ are normal, then only the first two coefficients of $\mathcal{H}(t)$, i.e., up to constants, $H^{\dim X}$ and $H^{\dim X-1}K_X$, are important, by the Riemann-Roch inequalities of Kollár-Matsusaka [21]. In dimension two this is also true in arbitrary characteristic, see Lemma 2.5.2 [18].

**Lemma 3.7.**  
(1) Fix $C > 0$ and $N \in \mathbb{N}$, and let $B = \{(X, H)\}$ be a class of normal $\mathbb{Q}$-polarized surfaces such that $NH$ is Cartier, $H^2 \leq C$. Then the class $B$ is bounded.

(2) Fix $C, C' > 0$ and $N \in \mathbb{N}$. Let $B = \{(X, H, B)\}$ be a class of normal $\mathbb{Q}$-polarized surfaces with subschemes, where $B$ is either a divisor with $HB \leq C'$ or a 0-dimensional subscheme of length $\leq C'$. Then the class $B$ is bounded.

**Proof.** The first part is just a reformulation of the above remarks. In (2), since the class $\{(X, H)\}$ is bounded, all the surfaces can be embedded by a uniform multiple of $H$ in the same projective space $\mathbb{P}$. Then the subschemes
Lemma 3.8. Let \( \{X\} \) be a certain class of schemes and assume that every scheme \( X \) is isomorphic to a blowup \( Bl_Z Y \) of a scheme \( Y \) at a subscheme \( Z \). If the class \( \{(Y,Z)\} \) is bounded then the class \( \{X\} \) is bounded.

Proof. After subdividing \( S \) we can assume that both \( Y \) and \( Z \) are flat over \( S \). Then \( Bl_Z Y \) will give a required family. □

3.9. I shall use one more method for proving boundedness given below, and a similar method for proving the descending chain condition. They sound almost trivial but I still would like to formulate them explicitly.

Theorem 3.10. Let \( B \) be a certain class of schemes. Assume that for every infinite sequence \( \{X_s \in B\} \) there exists an infinite subsequence \( \{X_{s_k}\} \) which is bounded. Then the class \( B \) is bounded.

Theorem 3.11. Let \( C \) be an ordered set. Assume that for every infinite sequence \( \{x_s \in C\} \) there exists an infinite nondecreasing subsequence \( \{x_{s_k}\} \). Then the set \( C \) satisfies the descending chain condition.

4. ADDITIONAL DEFINITIONS AND EASY TECHNICAL RESULTS

**Definition 4.1.** For two \( \mathbb{R} \)-divisors we write \( D_1 \geq D_2 \) (resp. \( D_1 > D_2 \)) if \( D_1 - D_2 \) is effective, i.e. the coefficients for all prime components are nonnegative (resp. is effective and nonzero).

**Definition 4.2.** For two \( \mathbb{R} \)-divisors on possibly different normal varieties of the same dimension we write \( D_1 \geq c \cdot D_2 \) (resp. \( D_1 > c \cdot D_2 \)) if

\[
\lim_{n \to \infty} \frac{h^0(nD_1) - h^0(nD_2)}{n^{\dim}} \geq 0
\]

(resp. is strictly positive). Here \( n \) is assumed to be divisible enough.

**Definition 4.3.** One says that an \( \mathbb{R} \)-Cartier divisor \( D \) is nef if \( DC \geq 0 \) for any effective curve \( C \).

**Definition 4.4.** The Iitaka dimension or the \( D \)-dimension of a Weil divisor \( D \) on a normal variety, denoted by \( \nu(D) \), is the number in the formula \( h^0(nD) \sim n^{\nu(D)} \) as \( n \to \infty \), unless \( h^0(nD) = 0 \) for all \( n > 0 \), in which case \( \nu(D) \) is defined to be equal to \(-\infty\). A divisor with \( \nu(D) = \dim X \) is called big. The Kodaira dimension of a variety \( X \) is defined as \( \nu(K_Y) \) where \( Y \) is a resolution of singularities of \( X \).

**Lemma 4.5.** (1) \( D_1 \geq D_2 \) implies \( D_1 \geq c \cdot D_2 \),

(2) if \( H^i(nD_1) = H^i(nD_2) = 0 \) for \( i > 0 \), \( n \gg 0 \), then \( D_1 \geq D_2 \) (resp. \( D_1 > D_2 \)) is equivalent to \( D_1^{\dim} \geq c \cdot D_2^{\dim} \) (resp. \( D_1^{\dim} > c \cdot D_2^{\dim} \)),
(3) in particular, (2) is applicable when \( D_1 \) and \( D_2 \) are ample.

Proof. (1) is clear, (2) and (3) trivially follow from the Riemann–Roch formula and the Serre vanishing theorem.

Lemma 4.6. Let \( D_1 \) be an ample \( \mathbb{R} \)-divisor on a normal variety \( X \), \( f : Y \to X \) be a birational projective morphism with \( Y \) normal, and write

\[ D_2 = f^*(D_1) + \sum e_j E_j + \sum f_i F_i \]

where the divisors \( F_i \) are exceptional for \( f \) and \( E_j \) are not. Then

1. if \( e_j \leq 0 \) then \( D_2 \leq c D_1 \)
2. if, moreover, for some index \( e_j < 0 \) or \( f_i < 0 \) then \( D_2 < D_1 \)
3. assume that \( e_j \leq 0 \), \( D_2 \) is nef; then all \( f_j \leq 0 \) i.e. \( D_2 \leq f^*(D_1) \)
4. assume that \( e_j \leq 0 \), \( D_2 \) is nef and that \( D_1 = f^*(D_2) \); then \( D_2 = D_1 \)

i.e. all \( e_j = f_i = 0 \)

Proof. Most of the statements are elementary, others follow from the Negativity of Contractions Lemma, see [33], 1.1 or [10], 2.19.

Lemma 4.7. Let \( D_1 \) be a divisor on a nonsingular surface \( X \) such that \( H^i(nD_1) = 0 \) for \( i > 0 \), \( n \gg 0 \). Let \( D_2 \) be a nonzero divisor such that \( -D_2 \) is not quasieffective (i.e. there exists an ample divisor \( H \) such that \( -D_2 H < 0 \)). Assume that \( D_2 \leq D_1 \). Then \( D_2^2 \leq D_1^2 \).

Proof. Use the Riemann–Roch theorem for \( nD_1, nD_2 \) and the fact that \( h^2(nD_2) = h^0(K - nD_2) = 0 \) for \( n \gg 0 \).

5. The diagram method

5.1. In proving the boundedness results for surfaces with nef \( -(K + B) \) the main part will be to bound rank of the Picard group. A way for doing this, called the diagram method, comes from the theory of reflection groups in hyperbolic spaces. Given a convex polyhedron in a hyperbolic space it is possible to bound the dimension of this space purely combinatorially, provided subpolyhedra of the main polyhedron satisfy certain arithmetic properties. The diagram method was successfully applied to surfaces in the works of V.V.Nikulin [29], [28], [30], [27] and of the author [1], and also to Fano 3-folds in [31], [32]. The vector space in question is \( \text{Pic}(X) \otimes \mathbb{R} \) which is hyperbolic by the Hodge Index theorem or some linear subspace of it, the polyhedron is generated by exceptional curves.

In the next section I shall give one more application of the diagram method. But first we need a few definitions and facts.

Definition 5.2. An exceptional curve on a surface is an irreducible curve \( F \) with \( F^2 < 0 \). The set of all exceptional curves on the surface \( X \) will be denoted by \( \text{Exc}(X) \). A \((-n)\)-curve is a smooth rational curve \( F \) with \( F^2 = -n \).
5.3. To a set of exceptional curves we associate a weighted graph. Each curve \( F \) corresponds to a vertex of weight \(-F^2\) and two vertices \( F_1 \) and \( F_2 \) are connected by an edge of weight \( F_1 F_2 \). An edge of weight 1 will be called simple. We also assign to every vertex a nonnegative number, the arithmetical genus of the corresponding curve. However, in the situation of interest to us all exceptional curves will have genus zero, as 5.20 shows. The (possibly infinite) graph corresponding to all exceptional curves on a surface \( X \) will be denoted by \( \Gamma(\text{Exc}(X)) \).

Definition 5.4. A finite set of exceptional curves \( \{F_i| i = 1 \ldots r\} \) (and the corresponding weighted graph) is called elliptic (resp. parabolic, hyperbolic) if the matrix \((F_{ij}, F_{ij})\) has signature \((0, r)\) (resp. \((0, r-1), (1, r-1)\)).

Definition 5.5. A finite set of exceptional curves (and the corresponding weighted graph) is called Lanner if it is hyperbolic but any proper subset of it is not.

5.6. The following theorem will be the most important for our purposes, see \[27\], 3.4–6 for the proof.

Theorem 5.7. Let \( X \) be a nonsingular surface such that the Iitaka dimension of the anticanonical divisor is nonnegative. Let us assume that for certain constants \( d, c_1, c_2 \) the following conditions hold:

1. The diameter of any Lanner subgraph \( \mathcal{L} \subset \Gamma(\text{Exc}(X)) \) does not exceed \( d \);
2. If \( \nu(-K) = 2 \) then for any connected elliptic subgraph \( \mathcal{E} \subset \Gamma(\text{Exc}(X)) \) with \( n \) vertices, the number of (unordered) pairs of its vertices on distance \( \rho \), where \( 1 \leq \rho \leq d-1 \), does not exceed \( c_1 n \), and the number of pairs on distance \( \rho \), where \( d \leq \rho \leq 2d-1 \), does not exceed \( c_2 n \);
3. If \( \nu(-K) = 1 \) then for any \((-1)\)-curve \( E \) for which \( EP > 0 \) (\( P \) is the positive part of the Zariski decomposition for \(-K\) and won't be used later) and for any connected elliptic subgraph \( \mathcal{E} \subset \Gamma(\text{Exc}(X)) \) with \( n+1 \) vertices containing \( E \), the number of pairs of its vertices different from \( E \) and on distance \( \rho \), where \( 1 \leq \rho \leq d-1 \), does not exceed \( c_1 n \), and the number of pairs on distance \( \rho \), where \( d \leq \rho \leq 2d-1 \), does not exceed \( c_2 n \);
4. If \( \nu(-K) = 0 \), \(-K = \sum_{b_j>0} b_j B_j\) then for any connected elliptic subgraph \( \mathcal{E} \subset \Gamma(\text{Exc}(X)) \) with \( n+m \) vertices, \( m \) of which \( E_1 \ldots E_m \) correspond to \((-1)\)- or \((-2)\)-curves different from \( B_j \), the number of pairs of its vertices different from \( E_1 \ldots E_m \) and on distance \( \rho \), where \( 1 \leq \rho \leq d-1 \), does not exceed \( c_1 n \), and the number of pairs on distance \( \rho \), where \( d \leq \rho \leq 2d-1 \), does not exceed \( c_2 n \).

Then

1. If \( \nu(-K) = 2 \) then \( \text{rk Pic}(X) \leq 96(c_1 + c_2/3) + 69 \);
2. If \( \nu(-K) = 1 \) then \( \text{rk Pic}(X) \leq 96(c_1 + c_2/3) + 70 \);
3. If \( \nu(-K) = 0 \) then \( \#(B_j) \leq 96(c_1 + c_2/3) + 68 \).
5.8. We need a few definitions for blowing up and down weighted graphs. They are merely reformulations on the language of graphs of usual operations of blowing up points on a nonsingular surface.

**Definition 5.9.** A weighted graph is said to be *minimal* if it does not contain vertices of weight 1 and arithmetical genus 0.

**Definition 5.10.** *Blowing up* a vertex $F$ is the operation on a weighted graph consisting of adding a new vertex $E$ of weight 1 and of arithmetical genus 0, connected only with the vertex $F$ by a simple edge and increasing weight of $F$ by 1.

**Definition 5.11.** *Blowing up* a simple edge $F_1 F_2$ is the operation on a weighted graph consisting of adding a new vertex $E$ of weight 1 and of arithmetical genus 0, connected only with the vertices $F_1$ and $F_2$ by simple edges, removing the edge between them and increasing weights of $F_1$ and $F_2$ by 1.

5.12. Blowing up can be easily defined in a more general situation but we won’t need it.

**Definition 5.13.** *Blowing down* is the inverse operation to blowing up.

**Definition 5.14.** The canonical class $K = K(\Gamma)$ of a weighted graph $\Gamma$ is the function on vertices defined by the formula

$$KF_i = -F_i^2 - 2 + 2p_a(F_i)$$

**Definition 5.15.** The log discrepancies $f_i$ for a finite weighted graph $\Gamma$ are defined as solutions of the following system of linear equations (if exist):

$$(K + \sum (1 - f_i)F_i)F_j = 0 \text{ for all } j$$

Note that this system has a unique solution if the matrix $(F_iF_j)$ is invertible (for example if the graph $\Gamma$ is elliptic or hyperbolic).

**Definition 5.16.** A weighted graph $\Gamma$ is said to be *log terminal* if for every elliptic subgraph $\Gamma' \subset \Gamma$ all log discrepancies of $\Gamma'$ are positive.

**Definition 5.17.** We say that a finite graph $\Gamma = \{F_i\}$ satisfies the condition $\ast(\varepsilon)$ if there exist constants $0 \leq b_i \leq 1 - \varepsilon < 1$ such that $(K + \sum b_i F_i)F_j \leq 0$ for all vertices $F_j$.

**Lemma 5.18.** If $\Gamma$ satisfies $\ast(\varepsilon)$ then every subgraph $\Gamma_1 \subset \Gamma$ and every graph $\Gamma_2$ obtained from $\Gamma$ by blowing down several vertices of weight 1 also satisfy $\ast(\varepsilon)$.

*Proof.* Evident. \hfill \square

**Lemma 5.19.** If an elliptic graph $\Gamma$ satisfies $\ast(\varepsilon)$ then for all the log discrepancies $f_i \geq 1 - b_i \geq \varepsilon$.

*Proof.* This is well known and follows easily from the negative definiteness of $(F_iF_j)$, see for example [3], 3.1.3. \hfill \square
Lemma 5.20. If an elliptic graph $\Gamma$ satisfies $*(\varepsilon)$ then every vertex $F$ has arithmetical genus 0 and its weight does not exceed $2/\varepsilon$.

Proof. Follows from

$$-2 \leq 2p_a(F) - 2 = (K + F)F = (K + (1 - \varepsilon)F)F + \varepsilon F^2 \leq (K + \sum b_j B_j)F + \varepsilon F^2 \leq \varepsilon F^2 < 0$$

Theorem 5.21. Every minimal elliptic log terminal graph is a tree with at most one fork and of type $A_n$, $D_n$ or $E_6$, $7$, or $8$. There exists a constant $S_1(\varepsilon)$ depending only on $\varepsilon$ such that $\sum (-F_i^2 - 2) \leq S_1(\varepsilon)$ if such a graph satisfies $*(\varepsilon)$.

Proof. The first part of the statement is well known, see for example [3]. The second part follows from the explicit description of elliptic graphs with all log discrepancies $\geq \varepsilon > 0$ given in [4], 3.3. □

Theorem 5.22. There exist $\leq 14(2/\varepsilon) + 29$ Lanner graphs with simple edges such that every Lanner graph with more than 5 vertices satisfying $*(\varepsilon)$ can be obtained from one of them by blowing up several vertices and edges. Each of these graphs is a tree or a cycle or a cycle and one more vertex. Each of these graphs has only simple edges and every vertex has at most 3 neighbors.

Proof. It follows from the theorems of V.V.Nikulin, [27], 4.4.18,19,21 which are valid for arbitrary log terminal Lanner graphs. Some of the graphs in [27] have a vertex of arbitrary positive weight $b$ but in our situation $b \leq 2/\varepsilon$ by 5.18 and 5.20. □

5.23. A typical example of a Lanner graph in the above statement is the chain containing three vertices of weights 1, 1 and $b \geq 1$. We won’t need explicit description of these graphs, just knowing that for every $\varepsilon$ there are finitely many of them will be sufficient.

6. BOUNDEDNESS FOR SURFACES WITH NEF $-(K + B)$

Lemma 6.1. Let $X$ be a nonsingular surface and assume that $-(K + \sum b_j B_j)$ is nef, where $0 \leq b_j \leq 1 - \varepsilon < 1$. Then every finite subgraph $\Gamma \subset \Gamma(\text{Exc}(X))$ satisfies the condition $*(\varepsilon)$.

Proof. Evident. □

Lemma 6.2. Let $X$ be a nonsingular surface and assume that $-(K + \sum b_j B_j)$ is nef, where $0 \leq b_j \leq 1 - \varepsilon < 1$. Then one of the following is true:

1. all $b_j = 0$ and $K$ is numerically trivial
2. $X$ is a rational surface, obtained by blowing up several points from $\mathbb{P}^2$ or $\mathbb{F}_n$ with $n \leq 2/\varepsilon$
(3) $X$ is an elliptic ruled surface without exceptional curves, i.e. a projectivization of a rank 2 locally free sheaf $\mathcal{E}$ on an elliptic curve $C$ and $\mathcal{E}$ is isomorphic to $\mathcal{O} \oplus \mathcal{F}$, $\mathcal{F} \in \text{Pic}^0(C)$ or to one of the only two, up to tensoring with an invertible sheaf, nonsplittable rank 2 bundles on $C$.

Proof. This lemma is practically proved in [27], 4.2.1 under weaker assumptions on $b_j$ and in arbitrary characteristic. We only need to see that in the case (2) one has $n \leq 2/\varepsilon$ by 5.20. □

**Theorem 6.3.** Let $X$ be a nonsingular surface and assume that $-(K + \sum b_j B_j)$ is nef, and $0 \leq b_j \leq 1 - \varepsilon < 1$. Then there exists a constant $A_1(\varepsilon)$ which depends only on $\varepsilon$ such that

$$\text{rk Pic}(X) \leq A_1(\varepsilon)$$

Proof. In order to prove this theorem we have to check the conditions (1) and (2–4) for graphs satisfying $*(\varepsilon)$. Among them, (1) is the hardest. Given an arbitrary graph satisfying $*(\varepsilon)$ and containing a vertex of weight 1, we can blow this vertex down and the new graph will satisfy the same condition by 5.18. After blowing down several vertices of an elliptic graph we get a minimal graph described in 5.21. It is well known that for an elliptic graph blowing down vertices of weights 1 in any order yields the same graph, so we can backtrack the situation by blowing up edges and vertices on the minimal elliptic graph in arbitrary order. All intermediate graphs again should satisfy $*(\varepsilon)$.

For a Lanner graph, removing any two vertices gives an elliptic graph. So, basically, we can do the same thing. Contracting several vertices we obtain one of the finitely many graphs of 5.22, and we can backtrack the situation blowing up, in any order, edges and vertices that do not affect two arbitrarily chosen vertices. Again, all intermediate graphs should satisfy $*(\varepsilon)$, moreover, they all should be Lanner.

Using these considerations, the proof easily follows from the lemmas 6.4, 6.5, 6.6, 6.7 below.

**Lemma 6.4.** Fix a weighted graph $\Gamma$ and pick one of its vertices $F$. Blow it up to get the vertex $E_1$, then blow up $E_1$ to get $E_2$, and so on. Call the intermediate graphs $\Gamma_1, \Gamma_2, \ldots$. Then for $k \gg 0$ the graph $\Gamma_k$ is not Lanner.

Proof. It is easy to see, using only elementary linear algebra, that there exists a fractional linear function $f(k)$ (which is an appropriately normalized determinant) so that the condition for the graph $\Gamma - E_k$ not to be hyperbolic is equivalent to $f(k) \geq 0$. But then the condition for $\Gamma$ to be hyperbolic is equivalent to $f(\infty) < 0$ and hence $f(k) < 0$ for $k \gg 0$. □

**Lemma 6.5.** In any satisfying $*(\varepsilon)$ graph which is elliptic or is Lanner with more than 5 vertices, each vertex has at most $2/\varepsilon - 2$ neighbors.

Proof. Indeed, in each of the initial graphs in 5.21 and 5.22 every vertex has at most 3 neighbors, so to produce a vertex with $d$ neighbors one has

to blow up some vertex $\geq d - 3$ times, which means that the weight of this vertex will be $\geq d - 2$. Now use 5.20.

**Lemma 6.6.** Fix an elliptic weighted graph $\Gamma$ consisting of two vertices connected by a simple edge. Then among all graphs $\Gamma'$ obtained from $\Gamma$ by blowing up only edges, there exist $\leq S_2(\varepsilon)$ graphs satisfying $\ast(\varepsilon)$, where the last function $S_2(\varepsilon)$ depends only on $\varepsilon$.

**Proof.** For any vertex $F$ in any such graph $\Gamma'$ let us introduce the height $h(F)$ as the minimal number of blowups needed to obtain this vertex from $\Gamma$. Consider this minimal sequence of blowups. Note that it is unique. On each step we get two chains of vertices of weight $\geq 2$ and a single vertex of weight 1 between them. After every blowup the sum $\sum (-F^2 - 2)$ increases by 1. Therefore by 5.21 $h(F) \leq S_1(\varepsilon)/2$ and there are $\leq 2^{S_1(\varepsilon)/2}$ such vertices. Finally, there exist only finitely many graphs $\Gamma'$ that contain only vertices of bounded height. □

**Lemma 6.7.** Fix an elliptic weighted graph $\Gamma$ consisting of two vertices connected by a simple edge. Blow up the edge to get the vertex $E_1$, then blow up $E_1$ to get $E_2$, and so on. Call the intermediate graphs $\Gamma_1, \Gamma_2, \ldots$. Then for $k > 5$ the graph $\Gamma_k$ is not log terminal, i.e. it does not satisfy $\ast(\varepsilon)$ for any $\varepsilon > 0$.

**Proof.** Indeed, $\Gamma_6 - E_6$ has type $E_9$ or worse. □

**End of the proof of 6.3.** The conditions (2–4) of 5.7 follow easily from 6.5. Indeed, the number of pairs in (2–4) on distance $\rho$ is bounded by $n/2(2/\varepsilon - 2)^\rho$.

To prove that the condition (1) of 5.7 is satisfied, consider any of the finitely many Lanner graphs of 5.22. We can assume that this graph has at least 6 vertices. We shall prove that there is a bound, in terms of $\varepsilon$, on the number of possible blowups that can be done preserving the condition $\ast(\varepsilon)$ and the property of the whole graph to remain Lanner. As was mentioned before, fixing any two vertices, the order of blowups not affecting these vertices is unimportant.

First, by 6.4, vertices can be blown up only finitely many times (again, there is a bound in terms of $\varepsilon$). Then an edge between some two vertices should be blown up. By 6.6, different edges between these two vertices can be blown up only finitely many times. Then some of the vertices may acquire more neighbors, but again their number is limited by 6.5. After that one of the new branches may grow longer, but its length is bounded by 6.7. Then, again there may be a few edges blown up, and then a few vertices may acquire some new neighbors. Finally, on this stage if some of the branches grow longer, a minimal elliptic intermediate subgraph should appear which has more than one fork. But this is impossible by 5.21.

Therefore, all conditions (1) and (2–4) of 5.7 are satisfied with constants $d(\varepsilon), c_1(\varepsilon), c_2(\varepsilon)$ that depend only on $\varepsilon$. In the cases $\nu(-K) = 2$ or 1
we immediately get the upper bound on the Picard number. In the case
\(\nu(-K) = 0\), 
\(-K = \sum_{b_j > 0} b_j B_j\) one has
\[K^2 \geq -\left(\#B_j\right)\left(2/\varepsilon\right)\]
by 5.20 and if \(X\) is rational we have \(\text{rk Pic}(X) = 10 - K^2\) by Noether’s formula. In the remaining two cases of 6.2 one certainly has \(\text{rk Pic}(X) \leq 20\). □

**Theorem 6.8.** Fix \(\varepsilon > 0\). Consider all nonsingular surfaces \(X\) with an \(\mathbb{R}\)-divisor \(B = \sum b_j B_j\) such that \(0 \leq b_j \leq 1 - \varepsilon < 1\) and 
\(-(K + B)\) nef excluding only those for which at the same time \(K_X\) is numerically trivial and \(B\) is zero. Then the class \(\{X\}\) is bounded.

**Proof.** Indeed, in the case (3) of 6.2 there are only three deformation types. All other surfaces, except \(\mathbb{P}^2\), are obtained from \(\mathbb{F}_n\), \(n \leq 2/\varepsilon\) by \(\leq A_1(\varepsilon) - 2\) blowups. Now use 3.8. □

**Theorem 6.9.** Fix \(\varepsilon > 0\). Consider all projective surfaces \(X\) with an \(\mathbb{R}\)-divisor \(B = \sum b_j B_j\) such that \(K_X + B\) is MR \(\varepsilon\)-log terminal and 
\(-(K_X + B)\) is nef excluding only those for which at the same time \(K_X\) is numerically trivial, \(B\) is zero and \(X\) has at worst Du Val singularities. Then the class \(\{X\}\) is bounded.

**Proof.** By 6.8 we already know that the class of minimal desingularizations \(\{Y\}\) of surfaces \(\{X\}\) is bounded. This, however, does not yet guarantee the boundedness of the class \(\{X\}\). To prove it we shall use a “sandwich” argument: we shall prove that there exist two birational morphisms \(Y \to X \to Z\) with \(\{Z\}\) also bounded.

By 6.2 we can assume that \(X\) is rational, moreover, by 6.3 the rank of the Picard group of the minimal resolution of \(X\) is effectively bounded.

We consider the following cases:

**Case 1.** \(B\) is nonempty or \(K\) is not numerically trivial.

We want to show that we can find a contraction \(X \to Z\) so that \(K_Z\) is ample or is relatively ample with respect to some \(\mathbb{P}^1\)-fibration. First, assume that there exists an \(\mathbb{R}\)-divisor \(D\) such that \(K + B + D\) is numerically trivial and satisfies the same condition 
\(*(\varepsilon)\). According to our conditions, 
\(B + D\) is nonempty. If we decrease one of the coefficients in \(K + B + D\), the result will be not nef. Therefore, according to the Minimal Model Program there exists an extremal ray and the corresponding contraction. If \(E\) is a component of \(B + D\) with \(E^2 \leq 0\) then \(K + B + D - eE\) for \(0 < e \ll 1\) has a nonnegative intersection with \(E\). Hence, there always exists an extremal contraction which does not contract \(E\), so the image of \(B + D\) will be again nonempty.

Performing several such contractions we arrive either at a surface \(Z\) with 
\(\text{rk Pic}(Z) = 1\) and ample \(-K_Z\) or at a surface \(Z\) with \(\text{rk Pic}(Z) = 2\) which has a \(\mathbb{P}^1\)-fibration. Because the rank of the Picard group of the minimal resolution is bounded and the self-intersection numbers of exceptional curves
are bounded from below, there are only finitely many types of singularities that $Z$ can have and all of them can be effectively described in terms of $\varepsilon$. Therefore, in the case of $\text{rk} \, \text{Pic}(Z) = 1$ a fixed multiple of $-K_Z$ is an ample Cartier divisor and $K_Z^2$ is effectively bounded from above. In the case of a $\mathbb{P}^1$-fibration, $-K_Z + (\lceil 2/\varepsilon \rceil - 1) \times \text{fiber}$ is ample, and the same argument applies. The class $\{Z\}$ is bounded by $3.7$.

Now, instead of proving that such a divisor $D$ as above exists, consider an arbitrary ample divisor $A$ on $X$ and observe that we can as well use $D'/N$ in the place of $D$, where $D'$ is a general element of a very ample linear system $|N(\delta A - K - B)|$ for $N \gg 0$, $0 < \delta \ll \varepsilon$.

The surface $Y$ is obtained from $Z$ by blowing up a certain closed subscheme. Again, from the boundedness of the Picard number of the resolution of $Y$ and the squares of exceptional curves, we see that the length of this subscheme is effectively bounded in terms of $\varepsilon$. Now use $3.8$.

**Case 2.** $B$ is empty, $K$ is numerically trivial but $X$ has worse than Du Val singularities.

Consider a partial resolution $f : Y \to X$ dominated by the minimal resolution of singularities of $X$ such that, letting $K_Y + B_Y = f^*(K_X)$, $B_Y$ has positive coefficients in all exceptional divisors of $f$. By the previous case, there exists a polarization $H$ on $Y$ with $H^2$ and $K_Y H$ bounded. Also, there exists a lower bound, in terms of $\varepsilon$, for $b_j^Y$. Since for any ample divisor $H$,

$$\sum b_j^Y B_j H = -K_Y H$$

we can bound all $B_j H$ and, therefore, for $H' = f_*(H)$ we can bound $H'^2$, $H'K_X$ and an integer $N$ such that $NH$ is Cartier. Then apply $3.7$ again.

**Corollary 6.10.** The class of projective complex surfaces $X$ with $K$ numerically trivial and log terminal and singularities worse than only Du Val, is bounded.

**Proof.** Indeed, by [6], Theorem C, $GK_X$ is Cartier for some $G \in \{1, 2 \ldots 21\}$, so every such surface is $1/21$-log terminal.

**Theorem 6.11.** Fix $\varepsilon > 0$. Then there exists a constant $A_2(\varepsilon)$ such that for any projective surface $X$ with $-(K + B)$ nef and $K + B$ MR $\varepsilon$-log terminal,

$$\sum b_j \leq A_2(\varepsilon)$$

**Proof.** Indeed, for an ample Cartier divisor $H$

$$\sum b_j \leq \sum b_j B_j H \leq -KH$$

and on each surface $X$ as above we can find $H$ with a bounded $-KH$. □

**6.12.** The following theorem, which applies only to the case when $B$ is empty and $-K$ is ample, but which in this case is stronger than ours, belongs to V.V.Nikulin.
Lemma 7.1. Assume that on a surface $X$ with ample $-K$, log terminal $K$ and singularities of multiplicities $\leq N$ is bounded.

7. Boundedness for surfaces with big and nef $K+B$

Theorem 6.13 (Nikulin [27] 4.7.2). Fix $N > 0$. Then the class of surfaces with ample $-K$, log terminal $K$ and singularities of multiplicities $\leq N$ is bounded.

Lemma 7.1. Assume that on a surface $X$, $K+B = K + b_0B_0 + \sum_{j>0} b_jB_j$ is big and MR log canonical. Then one of the following is true:

1. $K + \sum_{j>0} b_jB_j$ is big;
2. there exists $0 \leq b'_0 < b_0$ such that $K + xB_0 + \sum_{j>0} b_jB_j$ is big iff $x > b'_0$, and there exists a morphism $f : X \rightarrow X'$ such that the $\mathbb{R}$-divisor $D = f^*(K + b'_0B_0 + \sum_{j>0} b_jB_j)$ on $X'$ is nef but not numerically trivial, and $D^2 = 0$. Moreover, if $b'_0 > 0$ then the linear system $|ND|$, $N \gg 0$ and divisible, defines a $\mathbb{P}^1$-fibration $\pi : X' \rightarrow C$ to a nonsingular curve $C$;
3. there exists $0 \leq b'_0 < b_0$ such that $K + xB_0 + \sum_{j>0} b_jB_j$ is big iff $x > b'_0$, and there exists a morphism $f : X \rightarrow X'$ such that the $\mathbb{R}$-divisor $D = f^*(K + b'_0B_0 + \sum_{j>0} b_jB_j)$ is numerically trivial.

Proof. It is well known that a divisor on a projective variety is big if and only if it is a sum of an ample and an effective divisors. Hence, the property of being big is an open property. Similarly, the property of $(X, K+B)$ not to have a minimal model with the image of $K+B$ nef is also an open property, modulo the Cone and Contraction theorems of Minimal Model Program. Indeed, it is equivalent to existence of a covering family $\{C_t\}$ with $(K+B)C_t < 0$. Therefore, in the cases (2) and (3) $K + b'_0B_0 + \sum_{j>0} b_jB_j$ has a minimal model $X'$ with nef, but not big $D$. If $f^*(B_0)D < 0$ then the Minimal Model Program applied to $K + D - \delta B_0$, $0 < \delta \ll 1$, gives a $\mathbb{P}^1$-fibration. In the case (2) with $b'_0 = 0$, if the characteristic of the ground field is 0, then some multiple of $D$ is base point free and defines an elliptic fibration. We won’t need this fact, however. In positive characteristic the same is true but the proof requires using the classification theory, which we would like to avoid. \qed

7.2. The constants $A_1(\varepsilon)$, $A_2(\varepsilon)$ in the following theorem were defined in 6.3, 6.11.

Theorem 7.3. Assume that on a surface $X$, $K+B = K + \sum_{j \in J} b_jB_j$ is big and MR $\varepsilon$-log terminal. Then there exists a subset of indices $J' = J'_1 \cup J'_2 \subset J$ such that $K+B = K + \sum_{j \in J'} b_jB_j$ is big and

$$|J'_1| \leq A_1(\varepsilon) + 1, \sum_{j \in J'_2} b_j \leq A_2(\varepsilon)$$

Proof. Decrease the coefficient $b_0$. We have 3 cases as in 7.1.

Case 1. Pick another coefficient and continue.
Case 3. Find the maximal resolution $X''$ of $X'$ dominated by $X$:

$$X \xrightarrow{g} X'' \xrightarrow{h} X'$$

such that

$$K_{X''} + B'' = h^*(f_*(K + b'_0B_0 + \sum_{j>0} b_jB_j))$$

has all nonnegative coefficients. Let $J' \subset J$ be the set of indices for the nonnegative $b''_j$ in the latter divisor, including zeros. Then $K_X + \sum_{j \in J'} b_jB_j$ is big since it contains

$$c(K + B) + (1 - c)f^*(f_*(K + b'_0B_0 + \sum_{j>0} b_jB_j)) \equiv \text{big + nef}$$

for $0 < c \ll 1$. Part of the divisors $B_j$, $j \in J'$, lie in the exceptional set of $X'' \to X'$ so their number is bounded by $\rk \Pic(X'') - 1$, which is less than or equal to $A_1(\varepsilon) - 1$ by 6.3. The sum of the coefficients for the others is bounded by $A_2(\varepsilon)$ by 6.11. Also, we should not forget the divisor $B_0$ itself.

Case 2. Let $b'_0$ be as in 7.1, (2). To begin the proof, let us first assume that the linear system $|ND|$, $N \gg 0$ is base point free and defines a morphism with connected fibers $\pi \circ f : X \to C$ to a nonsingular curve $C$. For every fiber $F_k = \sum F_{kl}$ of $\pi \circ f$ consider the maximal rational number $f_k$ such that

$$K + b'_0B_0 - \sum f_kF_k$$

has at least one nonnegative coefficient in $F_{kl}$, say, $F_{k0}$. To present the idea of the proof more clearly, assume than we can contract for every $k$ all components of $F_k$ other than $F_{k0}$ to obtain a morphism

$$f''' : X \to X'''$$

Note that the coefficients of components of

$$K_{X'''} + B''' = f'''_*(K + b'_0B_0 - \sum f_kF_k)$$

are all nonnegative.

Subcase 1. $K_{X'''} + B'''$ is nef but not numerically trivial.

Then $K_X + B''' + (b_0 - b'_0)B_0$ is big, so we can use $J'$ corresponding to $B'''$ plus $B_0$ itself, i.e. to all $B_j$ not in fibers of $\pi \circ f$. Evidently,

$$\sum_{j \in J' - 0} b_j \leq 2 < A_2(\varepsilon)$$

Subcase 2. The opposite to the subcase 1.

At the same time $K_{X'''} + B''' + \sum b_{k0}B_{k0}$ is nef but not big. Hence, there exist several $F_{k0}$, say, $F_{00}, F_{10} \ldots F_{s0}$ such that

$$K_{X'''} + B''' + b'_{00}F_{00} + \sum_{i=1}^s b_{i0}B_{i0}$$
is numerically trivial. Find the maximal partial resolution $X''$ of $X'''$ dominated by $X$:

$$X \xrightarrow{g} X'' \xrightarrow{h} X'''$$

such that

$$K_{X''} + B'' = h^*(K_{X'''} + B''') + b_0'F_0 + \sum_{i=1}^{s} b_i B_i$$

has all nonnegative coefficients. Then choose $J'$ as in the case 3. Again, we can divide $J'$ into two parts, $J'_1$ and $J'_2$ and count indices as above. We shouldn’t forget one more divisor, $F_0$, to arrive at the final estimate.

In this proof we assumed the existence of a fibration $\pi \circ f : X \to C$. If there is a component $B_k$ of $b_0'B_0 + \sum_{j>0} b_j B_j$ that intersects $D$ positively (i.e. “horizontal”) then applying the Cone and the Contraction theorems to $K + b_0'B_0 + \sum_{j>0} b_j B_j - \epsilon B_k$ for $0 < \epsilon \ll 1$ several times we obtain such a ($\mathbb{P}^1$)-fibration. The case when no such component $B_k$ exists corresponds to an elliptic fibration. We could still prove that a multiple of $D$ is base point free, i.e. the abundance theorem, (cf. [35]), but this would involve some classification theory. Instead, we can use the same argument as above with not actual fibers but fibers in the numerical sense. Consider a connected component of $\text{supp}(b_0'B_0 + \sum_{j>0} b_j B_j)$. Then by the Hodge Index theorem the corresponding graph is elliptic or parabolic. A parabolic graph corresponds to a fiber, an elliptic graph – to a part of a fiber, the case when several curves of the fiber have coefficients 0. Then argue as above.

As for the existence of a contraction of components of $F_k$ other than $F_{k0}$, note that all we need in this proof is the contraction $g : X \to X''$ and it can be constructed without obtaining $X'''$ first. Now, $g : X \to X''$ exists because the corresponding graph is log terminal, cf. section 10.

**Corollary 7.4.**

$$|J'| \leq A_1(\varepsilon) + A_2(\varepsilon)/\min(b_j) + 1$$

**Theorem 7.5.** Assume that on a surface $X$, $K + B = K + \sum b_j B_j$ is big and MR log canonical. Further assume that $b_j$ belong to a D.C.C. set $\mathcal{C}$. Then there exists a constant $\delta(\mathcal{C}) > 0$ depending only on the set $\mathcal{C}$ such that $K + \sum (b_j - \delta)B_j$ is big.

**Proof.** Assume the opposite. Then, similarly to the proof of 7.1, there exists an infinite sequence $\delta_n \to 0$ and a sequence of surfaces $X'_n$ such that the direct image of $K + \sum (b_j - \delta_n)B_j$, which we will denote by $K + B'_n$, is numerically trivial on a general fiber of a $\mathbb{P}^1$-fibration or simply is numerically trivial. In the former case for some nonnegative integers $k_j$ we get

$$\sum k_j(b_j - \delta_n) = 2$$

which is trivially impossible. In the latter case we get a contradiction as follows. If $B'_n$ is not empty then $K + B'_n - \alpha B_1$ is not nef and the corresponding
extremal ray does not contain $B_1$. Therefore, after several contractions we can assume that $\text{rk} \text{Pic } X'_n = 1$. Then the number of irreducible components in $B'_n$ is less than $3/\min(C) + 1$ and the coefficients of $B'_n$ give an infinite strictly increasing sequence of chains $\{b_{n,k}\}$. This is impossible by [4], 5.3. Analyzing the proof of 5.3 shows that the condition for $K + B'_n$ to be log canonical can be weakened to MR log canonical.

Theorem 7.6. Fix $C > 0$ and a D.C.C. set $\mathcal{C}$. Then there exist a bounded class of surfaces with divisors $(Z, D)$ such that for every surface $X$ with $K + B$ nef, big, MR log canonical, with $b_j \in \mathcal{C}$ and $(K + B)^2 \leq C$ there exists a diagram

$$
Y \xrightarrow{g} Z \\
f \downarrow \\
X
$$

in which

1. $Y$ is the minimal desingularization of $X$,
2. defining $K_Y + B^Y = f^*(K + B)$, $D = g(\text{supp } B^Y \cup \text{Exc}(f))$ where $\text{Exc}(f)$ is the union of exceptional divisors of $f$.

Proof. Start with a divisor $K_Y + f^{-1}B + \sum F_i$ on $Y$. By 7.5 we can decrease the coefficients of $B^Y$ to obtain a divisor $B'$ with the following properties:

1. $K_Y + B'$ is big,
2. coefficients of $B'$ belong to a finite set $\mathcal{C}'$ of rational numbers that depends only on the original set $\mathcal{C}$,
3. all the coefficients of $B'$ are less than $1 - \varepsilon < 1$, i.e. $K_Y + B'$ is MR $\varepsilon$-log terminal. Here $\varepsilon$ depends again only on $\mathcal{C}$.

Next use 7.3 to obtain a big divisor $K_Y + B''$. The coefficients of $B''$ belong to the same set $\mathcal{C}'$ and, moreover, the number of components of $B''$ is bounded in terms of $\mathcal{C}$.

Apply the log Minimal Model Program in a slightly more general than usual form described in section 10 (because $K_Y + B''$ is not necessarily log canonical) to obtain a log canonical model $g : Y \rightarrow Z$ of $K_Y + B''$. Then $H = g(K_Y + B'')$ is ample and gives a $\mathbb{Q}$-polarization on $Z$. From the classification of surface log terminal singularities (cf. [4] 3.3) it follows that $Z$ can have only finitely many types of singularities. Indeed, for each of these singularities number of exceptional curves on the minimal resolution is bounded, and all log discrepancies are $\geq \varepsilon > 0$, so the corresponding weights are less than $2/\varepsilon$. Therefore there exists an integer $N(\mathcal{C})$ such that $NH$ is a polarization of $Z$. Note that by construction $H \leq K + B$, hence

$$
H^2 \leq (K + B)^2 \leq C
$$

by 4.5. Also, $HK \leq H^2$ and

$$
p_a(NH) = 1/2(N^2H^2 + NHK) \geq 0.
$$
Hence, there are only finitely many possible values for $H^2$ and $HK$. Now we use 3.7 to conclude that the class of surfaces $\{Z\}$ is bounded. To prove that the class $\{(Z,D)\}$ is bounded we also have to show that $DH = g(\text{supp } B \cup \text{Exc}(f))H$ is bounded. This is clear for the components of $g(\text{supp } B''')$. So let us consider the rest of $g(\text{supp } B \cup \text{Exc}(f))$. First, consider divisors $D_k$ on $Y$ with $g(D_k) \neq \text{point}$ which are not $f$-exceptional $(-2)$-curves. Their coefficients $d_k$ in $K_Y + B_Y$ are bounded from below by min$(1/3, C)$. We have $g^*(H) \leq (K_Y + B_Y)^2 \leq C$ for $f$-exceptional $(-2)$-curves one has $E = g^*(H) + 1/2 \sum (Hg(D_k))D_k \leq K + B_Y$ by 4.6 because we add to $g^*(H)$ only $f$-exceptional divisors and $E^2 \geq H^2 + 1/2 \sum (Hg(D_k))^2$. On the other hand, by lemma 4.7 $E^2 \leq (K_Y + B_Y)^2 \leq C$.

This concludes the proof. □

**Theorem 7.7.** Fix $\varepsilon > 0$, a constant $C$ and a D.C.C. set $\mathcal{C}$. Consider all surfaces $X$ with an $\mathbb{R}$-divisor $B = \sum b_jB_j$ such that $K_X + B$ is MR $\varepsilon$-log terminal, $K_X + B$ is big and nef, $b_j \in \mathcal{C}$ and $(K + B)^2 \leq C$. Then the class $\{(X, \text{supp } B)\}$ is bounded

**Proof.** Consider a diagram as in 7.6 above. Because the family $(Z,D)$ is bounded, changing $Z$ and $D$ we can assume that $Z$ is nonsingular and that the only singularities of $D$ in the exceptional set for $Y \to Z$ are nodes. But then $Y$ is obtained from $Z$ by several blowups at these nodes of $D$ and their number depends only on $\varepsilon$. Indeed, since $K_Y + B_Y$ is nef, when blowing up the point of intersection of two curves with coefficients $b_1$ and $b_2$, the new coefficient should be $\leq b_1 + b_2 - 1$. On the other hand, all coefficients in $K_Y + B_Y$ are nonnegative. Since every $b_k \leq 1 - \varepsilon < 1$, only finitely many such blowups can be done.

By 3.8 we conclude that the family $(Y, \text{supp } B \cup \text{Exc}(f))$ is bounded. The surfaces $Y$ come with a polarization, call it $H_Y$ with bounded $H_Y^2$, $H_YK_Y$, $H_Y(\text{supp } B \cup \text{Exc}(f))$. Since there are only finitely many configurations for the exceptional curves $\text{Exc}(f)$ and they are all log terminal, hence rational, we can construct a polarization $H_X$ on $X$ with bounded $H_X^2$, $H_XK_X$, $H_X(\text{supp } B)$. At this point, we can use 3.7 one more time. □

### 8. Descending Chain Condition

**8.1.** The aim of this section is to prove the following
Theorem 8.2. Fix a D.C.C. set \( C \). Consider all surfaces \( X \) with an \( \mathbb{R} \)-divisor \( B = \sum b_j B_j \) such that \( K_X + B \) is MR semi-log canonical, \( K_X + B \) is ample and \( b_j \in C \). Then the set \( \{(K_X + B)^2\} \) is a D.C.C. set.

8.3. The theorem will be proved in several steps, under weaker and weaker conditions.

Theorem 8.4. Theorem 8.2 holds for \( K + B \) MR Kawamata log terminal.

Proof. We can certainly assume that there exists a constant \( C \) such that \((K + B)^2 \leq C\). By 7.6, we know that there exists a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow f & & \downarrow \\
X & & \\
\end{array}
\]

with \( Z \) and \( D = \text{supp} B \cup \text{Exc}(f) \) bounded.

Assume first that \( Z \) and \( D \) are actually fixed. Then the proof immediately follows from 3.11 and the following theorem, since for surfaces \( H^2 \leq (g^* g^* H)^2 \).

Theorem 8.5. Let \( \{Y^{(s)} \mid s \in \mathbb{N}\} \) be a sequence of surfaces as above. Then changing \((Z, D)\) and picking a subsequence one has for every \( s < t \)

\[
g(t)^* (g(s)^* (K + B^{Y^{(s)}})) \leq K + B^{Y^{(t)}}
\]

Proof. Let me outline the strategy first. After canceling \( K \) on both sides of the inequality 3, we have to compare the coefficients for all prime divisors. We are going to look at coefficients in \( B^{Y^{(t)}} \). Some of them come from \( B_X^{(t)} \) and hence belong to the set \( C \). Since this set satisfies a descending chain condition, we can hope that after passing to a subsequence all the required inequalities will be satisfied. Then, the other part of coefficients in \( B^{Y^{(t)}} \) comes from the exceptional divisors of \( f^{(t)} \). A priori, there is no information about these coefficients except that they are nonnegative and are less than 1. But, by lemma 4.6 as soon as we prove the inequalities for the first part of \( B^{Y^{(t)}} \), the inequalities for the exceptional divisors will follow automatically. Finally, the fact that \( K + B^{Y^{(s)}} \) is MR Kawamata log terminal will mean that for every fixed \( s \), after modifying \( Z \) and \( D \), there are only finitely many inequalities we have to check.

Consider \( g^{(s)} : Y^{(s)} \rightarrow Z \) and the divisors \( g_s(K_{Y^{(s)}} + B^{Y^{(s)}}) \) on \( Z \). We shall denote various prime divisors by \( D_{\ldots} \) and the corresponding sequences of coefficients in \( K + B^{Y^{(s)}} \) by \( x_{\ldots}^{(s)} \), where \( \ldots \) stands for a certain index. For any fixed divisor \( D_{\ldots} \), after picking a subsequence, we can assume that the sequence \( x_{\ldots}^{(s)} \) is either increasing or decreasing (not necessarily strictly) and has the limit \( c_{\ldots} \). Below we shall talk, for example, about a divisor \( D_{\ldots} \) that we get by blowing up a particular point \( P \) on \( Z \). Picking a subsequence, we can assume that this point is blown up for all the surfaces \( Y^{(s)} \) or is not
blown up at all and then the analysis of the coefficients $x^{(s)}$ is superfluous. The important fact that we shall use is that all the coefficients $x^\_\_\_$ are strictly less than 1.

**Step 1.** Changing $(Z,D)$ we can assume that the points being blown up are nonsingular on $Z$ and are at worst the nodes of $D$. Moreover, if we blow up a point $P$ which is better than a node then for any divisor $D_k$ over $P$ in $B^{Y(s)}$ the corresponding coefficient in $g^{(t)}(g_{s}^{(s)}(K + B^{Y(s)}))$ is negative, so we have the inequality 3.

Therefore, below we consider only the situation of two normally crossing divisors $D_1$ and $D_2$ and the corresponding numbers $c_1, c_2 \leq 1$.

**Step 2.** Points $P$ with $c_1 = c_2 = 1$.

Blow up $P$ to get $D_3$. If $c_3 < 1$, change $Z$ to this new blown up surface and go on to the next step. If $c_3 = 1$ then for any divisor $D_k$ over $P$ we have $c_k = 1$ by a very simple computation taking into account that all $K + B^{Y(s)}$ are nef. Since the coefficients $x_1^{(s)}$ and $x_2^{(s)}$ are both less than 1, for the inequality 3 to be true for $s = 1$ over $P$ we need to check only finitely many divisors and the inequalities are satisfied after picking a subsequence because $\lim_{s \to \infty} x_k^{(s)} = c_k = 1$. Then do the same for $s = 2$ and so on. For the new sequence the inequalities 3 are satisfied for all divisors over $P$.

**Step 3.** Points $P$ with $c_1 = 1, c_2 < 1$.

Let us change notation slightly in this step. Denote $D_{2; 1} = D_2$, and then $D_{2; k+1}$ will be the exceptional divisor of a blowup at $D_1 \cap D_{2; k}$, $k = 1, 2, 3 \ldots$. Then denote by $D_{3; k}$ the exceptional divisor of a blowup at the point $D_{2; k} \cap D_{2; k+1}$, by $D_{4; k}$ the exceptional divisor of a blowup at $D_{2; k} \cap D_{3; k}$ (or at $D_{3; k} \cap D_{2; k+1}$) and so on.

The claims below are elementary and follow immediately from the fact that $K + B^{Y(s)}$ are nef.

**Claim 8.6.** In this way we obtain finitely many sequences $\{x^{(s)}_{p; k} \mid k \in \mathbb{N}\}$ that are the only possible nonnegative coefficients in $B^{Y(s)}$.

Note that $c_{2; 1} \geq c_{2; 2} \geq \ldots$ and denote by $c = \lim_{k \to \infty} c_{2; k}$.

**Claim 8.7.** There exist naturally defined positive integers $n(p)$ such that $\lim_{k \to \infty} c_{p; k} = 1 - n(p)(1 - c)$

**Claim 8.8.** If for some $k, s$ $x^{(s)}_{2; k+2} < x^{(s)}_{2; k+1} > d$ then for the same $k, s$ one also has $x_{p; k}^{(s)} > 1 - n(p)(1 - d)$

Now consider two cases.

**Case 1.** There exists $k_0$ such that for infinitely many $s$

$$x_{2; k_0}^{(s)} \leq c$$
Pick a subsequence of $s$ with this property. Change $Z$ by its $(k_0 - 1)$-blowup. Then again for the inequality 3 to be true for $s = 1$ we have to check only finitely many divisors and we get the corresponding inequalities because $c_{p,k} \geq 1 - n(p)(1 - c)$ by 8.8 with $d = c$. Then proceed with $s = 2$ and so on.

Case 2. For any $k$ there exist only finitely many $s$ such that

$$x_{2;k}^{(s)} \leq c$$

Then 8.8 with $d = c$ implies that for any fixed $p, k$ there exist only finitely many $s$ such that

$$x_{p;k}^{(s)} \leq 1 - n(p)(1 - c)$$

Define a set

$$A_2 = \{x_{2;k}^{(s)} \mid x_{2;k}^{(s)} > c; k, s \in \mathbb{N}\}$$

Fix $k_0$ such that $c_{2;k_0} < \min A_2 \cap C$, which is possible because $C$ is a D.C.C. set. Then pick a subsequence of $s$ with the following property:

$$(4) \quad x \in A_2, x < x_{2;k_0}^{(s)} \text{ implies } x \notin C$$

Introducing

$$A_p = \{x_{p;k}^{(s)} \mid x_{p;k}^{(s)} > 1 - n(p)(1 - c); k, s \in \mathbb{N}\}$$

we can also assume that

$$(5) \quad x \in A_p, x < 1 - n(p)(1 - x_{2;k_0}^{(s)}) \text{ implies } x \notin C$$

Now, again, as above, we change $Z$ by its $(k_0 - 1)$-blowup and arrange the inequality 3 to be true for $s = 1$ picking a subsequence. The important thing here is that the relevant inequalities may fail only for $x \notin C$ which is OK by lemma 4.6.

Then proceed with the same procedure for $s = 2$ and so on.

Checking for a vicious circle, I emphasize that in this step we change $Z$ for each point $P$ only once.

Step 4. Points with $c_1 < 1, c_2 < 1$.

Change $Z$ by blowing it up several times so that for any future blowup all divisors should have negative coefficients in $B_{Y^{(s)}}$.

At this point we achieved that the inequality 3 is satisfied for any divisor which is exceptional for $g^{(s)}$. Finally, let us settle it for the divisors $D_k$ on $Z$ itself.

Step 5. $D_1$ is a divisor on $Z$.

If $\{x_1^{(s)}\}$ is an increasing sequence, we are done. Otherwise, omitting finitely many $s$, we have $x_1^{(s)} < \min\{x_1^{(s)}\} \cap C$, so the coefficients $x_1^{(s)}$ correspond to exceptional divisors and we are OK by lemma 4.6.
So far, we assumed that \((Z,D)\) is fixed. To prove the general case we use 3.11. Pick an arbitrary sequence of surfaces satisfying the initial conditions. Since by 7.6 the class \((Z,D)\) is bounded, there are only finitely many numerical possibilities for \(Z, D\) and the subsequent blowups so, after picking a subsequence, we can argue in the same way as we did for the fixed \((Z,D)\).

**Remark 8.9.** Several important ideas of the proof given here are taken directly from the Xiao’s proof of Proposition 5, [36]. Unfortunately, I do not understand the arguments of [36] completely and therefore cannot say precisely how close I follow them. However, it is clear that the Xiao’s proof may apply only to a set \(C\) with a single limit point at \(1\).

**Theorem 8.10.** Theorem 8.2 holds for \(K + B\) MR log canonical.

**Proof.** Let \(\{X^{(s)}, K + B^{(s)}\}\) be a sequence with strictly decreasing \((K + B)^2\). For each \(X^{(s)}\) consider the maximal log crepant (i.e. all new log codiscrepancies equal 1) partial resolution \(f^{(s)}\) which is dominated by the minimal desingularization. Let \(E_i\) be the divisors with coefficients 1 in \(f^*(K + B)\). Then for the appropriate choice of \(\epsilon_i \to 0\) the divisors \(f^*(K + B) - \sum \epsilon_i E_i\) will be ample and MR Kawamata log terminal, the coefficients will belong to a new, but again a D.C.C., set, and the sequence of the squares will still be strictly decreasing. So we get a contradiction by 3.11 and 8.4. \(\Box\)

*End of the proof of theorem 8.2.*

By 1.11,
\[(K + B)^2 = \sum (K_{X_m} + B_m)^2\]
and all \(K_{X_m} + B_m\) are log canonical. By 8.10 every summand in this formula belongs to a D.C.C. set, and so does the sum. \(\Box\)

**9. Boundedness for the constant \((K + B)^2\)**

**9.1.** In this section we shall prove the following theorem. Again, it will be done first for \(K + B\) MR Kawamata log terminal and MR log canonical.

**Theorem 9.2.** Fix a constant \(C\) and a D.C.C. set \(C\). Consider all surfaces \(X\) with an \(\mathbb{R}\)-divisor \(B = \sum b_j B_j\) such that \(K_X + B\) is MR semi-log canonical, \(K_X + B\) is ample, \(b_j \in C\) and \((K + B)^2 = C\). Then the class \(\{(X, \sum b_j B_j)\}\) is bounded.

**Theorem 9.3.** Theorem 9.2 holds for \(K + B\) MR Kawamata log terminal. Moreover, in this case \(K + B\) may be taken to be big and nef instead of ample.

**Proof.** Again, first assume that in the diagram 7.6 \((Z,D)\) is fixed. Then, for the general case we use 3.10 since \((Z,D)\) moves in a bounded family.

By lemma 4.6 the inequalities 3 become equalities only when
\[g^{(s)}(K + B) = K + B^{Y^{(s)}}\]
for every \(s\) with a fixed \(B\).
Since the log canonical model of $K + B^{Y(s)}$ is the image of the linear system $|N(K + B^{Y(s)})|$, $N \gg 0$, all surfaces in the sequence have the same log canonical model. This means that the class $\{(Z, B)\}$ is bounded and since all coefficients in $B^{Y(s)}$ are less than 1, that $\{(Y, B)\}$ is bounded. Since

$$K_{Y(s)} + B^{Y(s)} = f^*(K_{X(s)} + B_{X(s)})$$

we get a diagram $Y(s) \to X(s) \to Z$ and the class $\{(X, B)\}$ is bounded by the “sandwich” principle. □

**Theorem 9.4.** Theorem 9.2 holds for $K + B$ MR log canonical.

**Proof.** As in the previous proof, by 7.6 we first assume existence of the diagram

$$\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{f} & & \\
X & & \\
\end{array}$$

with $Z$ and $D = \text{supp } B \cup \text{Exc}(f)$ fixed. Pick an infinite sequence of surfaces $\{X(s)\}$ and for $K_{Y(s)} + B^{Y(s)} = f(s)^*(K_{X(s)} + B(s))$ write

$$(K_{Y(s)} + B^{Y(s)})^2 = \text{Base}^{(s)} + \text{Defect}^{(s)}$$

where

$$\text{Base}^{(s)} = (g_s^*(K_{Y(s)} + B^{Y(s)}))^2$$

$$\text{Defect}^{(s)} = (K_{Y(s)} + B^{Y(s)})^2 - (g_s^*(K_{Y(s)} + B^{Y(s)}))^2$$

Note that $\text{Defect}^{(s)} \leq 0$ for all $s$.

Now if $K + B$ were MR Kawamata log terminal then from 8.5 we could conclude that, after picking a subsequence, $\{\text{Base}^{(s)}\}$ is a nondecreasing sequence and that $\text{Defect}^{(s)} \to 0$. But because every MR log canonical $K + B$ can be approximated by MR Kawamata log terminal divisors as in the proof of 8.10, we can conclude just the same in the present situation. But then $\text{Base}^{(s)} + \text{Defect}^{(s)} = C$ implies that $\text{Defect}^{(s)} = 0$ and $\text{Base}^{(s)} = C$, i.e.

$$g(s)^*(K + B) = K + B^{Y(s)} \text{ for every } s \text{ with a fixed } \mathcal{B}$$

Since $K + B^{X(s)}$ is the image of the linear system $|N(K + B^{Y(s)})|$, $N \gg 0$, we get $(X(s), B) = (Z, \mathcal{B})$, so all surfaces in the sequence are actually isomorphic to each other. In the general situation $(Z, D)$ is not fixed but the class $(Z, D)$ is bounded, so we get that the class $(X, B)$ is bounded. □

**End of the proof of 9.2.**

**Proof.** Let $X^\nu = \cup X_m$ be the normalization of $X$. Then as in 1.11

$$(K + B)^2 = \sum_m (K_{X_m} + B_m)^2,$$
the coefficients of $B_m$ belong to $\mathcal{C} \cup \{1\}$ and $K_{X_m} + B_m$ are log canonical. Applying 8.10 and 9.4 we see that there are only finitely many possibilities for $(K_{B_m})^2$ and that all $(X, \text{supp } B_k)$ belong to a bounded class. Now the class $\{X\}$ is bounded by the Conductor Principle [18] 2.3.5. It immediately follows that in fact $\{(X, \text{supp } B)\}$ is bounded. □

10. ON LOG MMP FOR SURFACES

10.1. The Log Minimal Model Program in dimension two is undoubtedly much easier than in higher dimensions. There are two main circumstances that distinguish the surface case.

The first difference is that the Log Minimal Model Program in dimension two is characteristic free, see [24] for nonsingular surfaces and [35] for open surfaces. The Cone Theorem follows from the nonsingular case by a simple argument as in [35]. The proofs of the Contraction Theorem and of the Log Abundance Theorem in higher dimension use the Kodaira vanishing theorem which fails in positive characteristic. However, in dimension two they can be proved using the contractibility conditions of Artin [5] because the corresponding configurations of curves are rational. In fact, in characteristic $p > 0$ contractibility conditions are even weaker than in characteristic 0.

The second difference is the existence of a unique minimal resolution of singularities.

The statements in this section and their proofs are elementary. I provide them here because I need them for this paper and they are slightly more general than those appearing elsewhere.

**Theorem 10.2 (The Cone Theorem).** Let $X$ be a normal surface and $K + B = K_X + \sum b_j B_j$ be an $\mathbb{R}$-Cartier divisor with $b_j \geq 0$. Let $A$ be an ample divisor on $X$. Then for any $\epsilon > 0$ the Mori-Kleiman cone of effective curves $\overline{NE}(X)$ in $NS(X) \otimes \mathbb{R}$ can be written as

$$\overline{NE}(X) = \overline{NE}_{K+B+\epsilon A}(X) + \sum R_k$$

where, as usual, the first part consists of cycles that have positive intersection with $K+B+\epsilon A$ and $R_k$ are finitely many extremal rays. Each of the extremal rays is generated by an effective curve.

**Proof.** It suffices to prove this theorem for the minimal desingularization $Y$ of $X$ with $K_Y + B^Y = f^*(K_B)$ instead of $K+B$ since every effective curve on $X$ is an image of an effective curve on $Y$, and so $\overline{NE}(X)$ is the image of $\overline{NE}(Y)$ under a linear map $f_* : N_1(Y) \to N_1(X)$. Then for a nonsingular surface the statement easily follows from the usual Cone Theorem [24] by the same argument as in [35], 2.5. The reason for this is, certainly, that for any curve $C \neq B_j$ one has $C(K + B) \geq CK$. The set of extremal rays for $K + B + \epsilon A$ is a subset of extremal rays for $K + \epsilon A$, possibly enlarged by some of the curves $B_j$. □
Theorem 10.3 (Contraction Theorem). Let $X$ be a projective surface with MR log canonical $K+B = K + \sum b_j B_j$. Let $R$ be an extremal ray for $K+B$. Then there exists a nontrivial projective morphism $\phi_R : X \to Z$ such that $\phi_R(C_X) = \mathcal{O}_Z$ and $\phi(C) = pt$ iff the class of $C$ belongs to $R$, and $K_Z$ is also log canonical. If $K_X$ is Kawamata log terminal then so is $K_Z$.

Proof. By the previous theorem $R$ is generated by an irreducible curve $C$. If $C^2 > 0$ then as in [24] 2.5, $f^*(C)$ on the minimal resolution (defined, according to Mumford, for every Weil divisor $C$) is in the interior of $\overline{NE}(Y)$, so $C$ is in the interior of $\overline{NE}(X)$. This is possible only if $\rho(X) = 1$ and then $\phi_R$ maps the whole $X$ to a point.

If $C^2 = 0$ then the graph corresponding to $\text{supp}(f^*(C))$ is parabolic and it is rational because $KC < 0$. By the Riemann–Roch formula $h^0(nf^*(C))$ grows linearly in $n$, so for some $N \gg 0$ the linear system $|N f^*(C)|$ is base point free and defines a projective morphism to a curve. Similarly to [24], 2.5.1, this is possible only if $\rho(X) = 2$.

Now let us assume that $C^2 < 0$. Consider the minimal resolution $f : Y \to X$ and let $F_i$ be the maximal number of exceptional curves of $f$ such that $C \cup_i F_i$ is connected. Then, since $(K+B)C < 0$ and the quadratic form $|F_i F_i|$ is negative definite, all the log discrepancies of the graph $C \cup_i F_i$ are strictly positive, so this graph is log terminal. Looking at the list of the minimal log terminal graphs (for example, in [3]) one easily observes that they are all rational in the sense of Artin [5]. So are all the nonminimal log terminal elliptic graphs obtained from them by simple blowups. Hence, by [5] the configuration of curves $C \cup_i F_i$ can be contracted to a normal point on a projective surface $Z$. By normality of $X$, this defines a morphism $\phi_R : X \to Z$ satisfying the required properties. \hfill $\square$

Theorem 10.4 (Easy Log Abundance Theorem). Let $X$ be a projective surface with MR Kawamata log terminal $K+B = K + \sum b_j B_j$. Assume that $K+B$ is big and nef. Then for some $N \gg 0$ the linear system $|N(K+B)|$ is base point free and defines a birational projective morphism $f : X \to Z$ with $N(K+B) = f^*(H)$ for some ample divisor $H$ on $Z$. $K_Z$ is also Kawamata log terminal.

Proof. We have $(K+B)^2 > 0$. If $K+B$ is not ample then by the Nakai-Moishezon criterion of ampleness there exists a curve $C$ such that $(K+B)C = 0$. By the Hodge Index theorem, $C^2 < 0$. Then, as in the previous theorem, $C$ can be contracted to a normal point on a projective surface $Z_1$, $K+B = g^*(K_{Z_1} + g_*(B))$ and the latter divisor is again MR Kawamata log terminal. Now the statement follows by induction on rank of the Picard group of $Z_1$. \hfill $\square$

Remark 10.5. It is elementary to generalize the above three theorems to a relative situation of an arbitrary projective morphism $X \to S$ to a variety $S$. 
11. Concluding remarks, generalizations, open questions

11.1. Following the logics of the theorems presented here one should expect that there exist infinitely many types of surfaces with ample canonical divisor $K$, $K^2 \leq C$ and empty $B$ if we allow singularities worse than $\varepsilon$-log terminal, for example arbitrary quotient singularities. It would be interesting to see the examples.

11.2 (Effectiveness). Note that it is quite straightforward to obtain effective formulas for $A_1$ in 6.3 and $S_1$ in 5.21. However, methods of this paper do not provide effective estimates for functions $A_2$ in 6.11 and $c(C)$ in 2.8.

11.3. I would go as far as to conjecture that the conditions for the boundedness formulated in 0.4 are the most natural ones and that direct analogs of all four our main theorems should be true in any dimension.

11.4. In the case of $K$ negative, i.e. Fano varieties, there are several results supporting this conjecture. Most importantly, it is true for toric log Fano varieties with empty $B$ by Borisovs [8]. Fano 3-folds with terminal singularities (i.e. 1-log terminal) are bounded (see Kawamata [15] for $Q$-factorial case, general case is due to Mori, unpublished). Log Fano 3-folds of fixed index $N$ and Picard number 1 (in particular, they are $(1/N)$-log terminal) are bounded by [7].

11.5. In the case of positive $K$ nothing is known in dimension greater than two.

11.6. Let me list the main points where dimension two is essential in the proofs: diagram method, lemma 4.7, lemma 7.5. It could also be argued that all proofs involving $D^2$ do not have a chance to be generalized to higher dimensions because only in dimension $n = 2$ $D^n$ behaves well under birational transformations. However, this obstacle can be avoided by using systematically $\leq_c$ instead of $\leq$ and then applying 4.5.

11.7. Theorem 6.8 has applications to singularities of curves on surfaces with positive $-K$ such as the projective plane or Del Pezzo surfaces. For example, it implies that an irreducible curve of degree $d$ for $d \gg 0$ cannot have tangents of high multiplicity as compared to $d$. It would be interesting to spell this connection out and compare with known results.

11.8. Most of the results of this paper remain valid if one works in the category of algebraic spaces of dimension two instead of the category of algebraic surfaces.

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