MODELING OF CONSERVATIVE FIELDS IN PIECEWISE INFINITE PLATE WITH A THIN INCLUSION

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ABSTRACT. In the article the formulas for the modeling of conservative fields in piecewise infinite plate with a thin inclusion found. The accuracy of the found formulas is of order equal to the thickness of the outer layer. The problem for higher accuracy solved by means of asymptotic formulas.

Keywords: conservative field, internal coupling conditions, problem of Robin, Dirichlet problem

Mathematics Subject Classification 2010: 35Jxx Elliptic equations and systems; 35A22 Transform methods

1. Introduction

Problems about structure of the conservative field of a two-layer flat plate leads to a separate system of the Laplace’s equation with a boundary condition to Dirichlet and internal coupling conditions [1], [2]. In the work the method of transformation operators is applied. Yaremko O. E. developed this method [1], [2], [3], [4], [5]. As a result the conservative field for the two-layer piecewise plate can be interpreted as a deformation of the conservative field of a homogeneous piecewise plate. Thus the deforming transformation operator is written out. The deforming transformation operator is recursive, and convenient for practical realization on the computer. In this case, use the analytic representation of a field by means of the deforming transformation operator is difficult.

We consider the case of small thickness l cross-border layer of the plate. Let’s consider that the physical properties of the layers very different, that is the ratio of the thermal conductivity of the layers close to zero.

The method of the deforming transformation operator gives algorithm inconvenient for realization. Formulas which have no these defect are found based on the Euler–Maclaurin formula.

Numerical methods of research lead to ill-conditioned [8] systems of the linear algebraic equations.

1.1. The half-plane with internal coupling condition. Modelling of potential fields for a semi-limited infinite plate with thin inclusion leads to the Dirichlet problem for two-layer half-plane with coupling conditions on the direct. Let’s consider a boundary problem about the solving of separate system of the Laplace’s equation

\[ a_1 \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad (x, y); \quad 0 < x < l, \quad -\infty < y < \infty \]

\[ a_2 \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad (x, y); \quad l < x, \quad -\infty < y < \infty \]

by the boundary condition for direct \( x = l \):

\[ u_1(x, y) = \hat{u}(x, y), \quad x = 0 \]
by the internal coupling conditions [1], [2], [3], [4], [5] for direct $x = l$:

$u_1(x, y) = u_2(x, y), \frac{\partial u_1}{\partial x}(x, y) = \frac{\partial u_2}{\partial x}(x, y), k > 0,$

Here function $\hat{u} = \hat{u}(x, y)$- is harmonic in the right half-plane, $H = \{(x, y) : 0 < x, y \in R\}$ and continuous in $\{(x, y) : 0 \leq x, y \in R\}$ and the conditions are met

$\hat{u} = 0(\sqrt{x^2 + y^2}),
\int_{-\infty}^{\infty} \frac{\hat{u}(0, y)}{1 + y^2} dy < \infty.$

The problem (1)-(4) is ill-posed [8] in the case of a thin inclusion, that is, small value $l$, and in the case of a strong heterogeneity, that is, large differences in the parameters $a_1, a_2$. The solution of this problem by known methods is accompanied by considerable computing difficulties. New analytical methods to solve this problem it is necessary to develop.

1.2. Thin strip. Function $\hat{u} = \hat{u}(x, y)$ is a harmonic function in the right half-plane $H = \{(x, y) : 0 < x, y \in R\}$ continuous in $\{(x, y) : 0 \leq x, y \in R\}$ and the conditions are met

$\hat{u} = 0(\sqrt{x^2 + y^2}),
\int_{-\infty}^{\infty} \frac{\hat{u}(0, y)}{1 + y^2} dy < \infty.$

Let’s consider the Dirichlet problem for Laplace’s equation in the strip $H_l = \{(x, y) : 0 < x < l, y \in R\}$

$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, (x, y); 0 < x < l, -\infty < y < \infty$

with boundary conditions

$u(0, y) = \hat{u}(0, y), u(l, y) = 0.$

The problem (5)-(6) is ill-posed [8] at small values $1 \approx 1$.

1.3. The circle with the internal coupling conditions. Let the function $\hat{u} = \hat{u}(x, y)$- is a harmonic function in the unit circle, $B = \{(x, y) : x^2 + y^2 < 1\}$ and continuous in its closure $\bar{B}$. The boundary problem about the solving a separate system of the Laplace’s equation consider in the circle $B$

$\Delta u_1 = 0, (x, y) \in K_R,$

$\Delta u_2 = 0, (x, y) \in B_R$

Here $K_R$ - annulus with the smaller radius R and larger radius 1, $B_R$ - the circle of radius R. Let the boundary condition on the circle satisfied S:

$u_1(x, y) = \hat{u}(x, y), (x, y) \in S$

internal coupling condition on a circle of radius R:

$u_1(x, y) = u_2(x, y),
\frac{kL_0}{\partial x} u_1(x, y) = \frac{\partial L_0}{\partial x} u_2(x, y), k > 0, (x, y) \in S_R.$

Let’s notice that the problem (7)-(9) is ill-posed [8] at values $R \approx 1$. 
1.4. **Thin annulus.** Let the function \( \hat{u} = \hat{u}(x, y) \) - is a harmonic function in the unit circle, \( B = \{(x, y) : x^2 + y^2 < 1\} \) and continuous in its closure \( \bar{B} \). In annulus \( K_R \) with the smaller radius \( R \) and larger radius \( 1 \), consider the Dirichlet problem about the solution of the Laplace’s equation

\[
\Delta u_1 = 0, (x, y) \in K_R,
\]

by the boundary conditions on the circumferences \( S, S_R \):

\[
u = \hat{u}, (x, y) \in S; u = 0, (x, y) \in S_R.
\]

The problem (10) is ill-posed [8] by the values \( R \approx 1 \).

2. **Transformation operators**

In the works of the author [1], [2], [3], [4] and [5] the concept of operator transformation \( J, J : \hat{u} \rightarrow u \), allowing a known solution \( \hat{u} \) of the model problem to solve \( u \) any of the four above-listed problems determined.

2.1. **The half-plane with the internal coupling conditions.** Transformation operator \( J \) has the form

\[
J : \hat{u} \rightarrow u, u = \chi(H_1)u_1 + \chi(H_2)u_2,
\]

where:

\[
u_1(x, y) = \sum_{j=0}^{\infty} \left( \frac{1 - k}{1 + k} \right)^j \left( \hat{u}(x + 2lj, y) - \frac{1 - k}{1 + k} \hat{u}(2l - x + 2lj, y) \right), \quad 0 < x < l,
\]

\[
u_2(x, y) = \frac{2k}{k + 1} \sum_{j=0}^{\infty} \left( \frac{1 - k}{1 + k} \right)^j \hat{u} \left( \frac{a_1}{a_2} (x - l) + l + 2lj, y \right), \quad l < x,
\]

\[
k = \frac{\lambda_1}{\lambda_2 a_1}.
\]

\( \chi \) - is the characteristic function of the set [6].

2.2. **Thin strip.** Transformation operator \( J : \hat{u} \rightarrow u \), can be determined by the formula

\[
u(x, y) = \sum_{j=0}^{\infty} (\hat{u}(x + 2lj, y) - \hat{u}(2l - x + 2lj, y)) ; \quad 0 < x < l.
\]

2.3. **Circle with the internal coupling conditions.** Transformation operator is expressed by the formula:

\[
J : \hat{u} \rightarrow u, u = \chi(K_R)u_1 + \chi(B_R)u_2,
\]

where:

\[
u_1(x, y) = \sum_{j=0}^{\infty} \left( \frac{1 - k}{1 + k} \right)^j \left( \hat{u}(xR^{2j}, y) - \frac{1 - k}{1 + k} \hat{u} \left( \frac{R^{2j+2}}{x}, \frac{R^{2j+2}}{y} \right) \right),
\]

\[
u_2(x, y) = \frac{2k}{k + 1} \sum_{j=0}^{\infty} \left( \frac{1 - k}{1 + k} \right)^j \hat{u}(xR^{2j}, yR^{2j}),
\]

\( \chi \) - is the characteristic function of the set [6].
2.4. Thin annulus. Transformation operator $J : \hat{u} \rightarrow u$, can be determined by the formula

$$u(x, y) = \sum_{j=0}^{\infty} \left( u \left( xR^{2j}, yR^{2j} \right) - u \left( \frac{R^{2j+2}x}{x}, \frac{R^{2j+2}y}{y} \right) \right).$$

Formulas (12) and (14) are useful only if the values $k \approx 1$. In the case of the two-layer body, which components are sharply different properties, that is at values $k$ close to zero, and for large values $k$, transformation operators cannot be used, in view of the slow convergence of the series at $\frac{1-k}{1+k} \approx 1$.

Formulas (13) and (15) cannot be used in view of the slow convergence of the series at $l \approx 1$ and $R \approx 1$, respectively.

The purpose of our research is to consist in creation of a design of transformation operators comfortable in these cases.

3. Auxiliary result

**Lemma 1.** If the function $y = f(x)$ is defined on the line segment $[0, 1]$ and has limited variation $[10]$ on the line segment $[0, 1]$, then for each value $R$ the following assessment for the difference of integral and its integral sum is carried

$$\left| \int_0^1 x^{-1}f(x)dx - \ln \frac{1}{R^2} \sum_{j=0}^{\infty} f(R^{2j}) \right| \leq \ln \frac{1}{R^2} \hat{V}(f),$$

**Proof.** Let’s spread integral into the sum $(n + 1)$ of the summand:

$$\int_0^1 x^{-1}f(x)dx = \sum_{j=0}^{n-1} \int_0^{R^{2j}} x^{-1}f(R^{2j})dx + \int_0^{R^{2n}} x^{-1}f(x)dx.$$

Variable substitution in the last integral results in equality:

$$\int_0^1 x^{-1}f(x)dx = \sum_{j=0}^{n-1} \int_0^{R^{2j}} \frac{f(R^{2j})}{x}dx + \int_0^{R^{2n}} x^{-1}f(x)dx.$$

Considering that

$$\left| f(R^{2j}) - f(R^{2j}) \right| \leq \frac{1}{0} V(f),$$

We obtain the required assessment:

$$\left| \int_0^1 x^{-1}f(x)dx - \sum_{j=0}^{\infty} \int_0^{R^{2j}} \frac{f(R^{2j})}{x}dx \right| \leq \sum_{j=0}^{\infty} \int_0^{R^{2j}} \left| f(R^{2j}) - f(R^{2j}) \right| \frac{dx}{x} \leq \frac{1}{0} V(f) \ln \frac{1}{R^2}.$$

$\hat{V}(f) - \text{variation of function [10] } f \text{ on the line segment } [0, 1]$. The flat analog of the Lemma is well known [8].

**Lemma 2.** If the function $y = f(x)$ is defined on the ray $[0; \infty)$ and has limited variation $[10]$ on the ray $[0; \infty)$; then for each value $l$ the following assessment for the difference of integral and its integral sum is carried

$$\left| \int_0^\infty f(x)dx - 2l \sum_{j=0}^{\infty} f(2lj) \right| \leq 2l \hat{V}(f).$$
The operator \( L \) and the value \( k \) is small. For component
\[
\text{the Euler–Maclaurin formula applied to the function } f(2lx) \text{ on the ray } [0; \infty) \text{ results in equality}
\]
\[
\sum_{j=0}^{\infty} f(2lj) \cong \frac{1}{2l} \int_0^{\infty} f(x)dx + \frac{f(0)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} x^{2k-1}(0).
\]
\[
\sum_{j=0}^{\infty} f(R^{2j}) \cong \frac{1}{\ln \pi} \int_0^{\infty} f(x)dx + \frac{f(1)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} L_0^{2k-1} f(1)
\]

Corollary 1. When \( k \in (0, 1) \) completed asymptotically

\[
\sum_{j=0}^{\infty} \left( \frac{1-k}{1+k} \right)^j f(x + 2lj) \cong \frac{1}{2l} \int_0^{\infty} e^{\ln f(x + \epsilon)dx} + \frac{f(x)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} L_h^{2k-1} f(x).
\]

When \( k \in (1, \infty) \) completed asymptotically

\[
\sum_{j=0}^{\infty} \left( \frac{1-k}{1+k} \right)^j f(x + 2lj) \cong \frac{f(x)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} (2^{2k} - 1) L_h^{2k-1} f(x).
\]

The operator \( L_{2h} \) has the form \( L_{2h} = 2h + \frac{d}{dx} \).

Corollary 2. When \( k \in (0, 1) \) completed asymptotically

\[
\sum_{j=0}^{\infty} \left( \frac{1-k}{1+k} \right)^j f(r R^{2j}) \cong \frac{1}{\ln \pi} \int_0^{1} e^{\ln f(rz)dz} + \frac{f(r)}{2} + \sum_{k=1}^{\infty} \frac{(\ln \frac{1}{2r})^{2k-1} B_{2k}}{k!} L_{2h}^{2k-1} f(r).
\]

When \( k \in (1, \infty) \) completed asymptotically

\[
\sum_{j=0}^{\infty} \left( \frac{1-k}{1+k} \right)^j f(r R^{2j}) \cong \frac{f(r)}{2} + \sum_{k=1}^{\infty} \frac{(\ln \frac{1}{2r})^{2k-1} B_{2k}}{k!} (2^{2k} - 1) L_{2h}^{2k-1} f(r).
\]

The operator \( L_{2h} \) has the form \( L_{2h} = 2h + \frac{d}{dx} \).

4. Main result

4.1. The half-plane with the internal coupling conditions. Study the case of a thin shell, that is the case of a small thickness of the final layer \( l \). Physically this means that the coefficients of heat capacity layers are very different.

Theorem 1. Let the condition is fulfilled

\[
\frac{1-k}{1+k} = e^{2hl},
\]

and the value \( k \) is small. For component \( u_1 \) and \( u_2 \) of the problem solution \([1]-[3]\) approximate formulas are valid

\[
u_2 \approx \frac{1 - e^{2hl}}{2l} u_3,
\]
The following theorem is obtained - variation of function \( u \) taken from a variable \( \varepsilon \) be found on the approximate formula

\[
2h\hat{u}_3 + \frac{\partial}{\partial n}\hat{u}_3 = \hat{u}, (x, y), x = 0.
\]

The assessment is fair

\[
\left| \frac{1 - e^{2hl}}{2l} \int_0^\infty e^{\varepsilon h}\hat{u}(x + \varepsilon, y) d\varepsilon - u_2(x, y) \right| \leq (1 - e^{2hl}) \frac{\infty}{0} (e^{\varepsilon h}\hat{u}(x + \varepsilon, y)).
\]

**Proof.** Let’s apply the lemma 2. As a result we obtain the formula:

\[
\frac{2k}{k + 1} \frac{1}{2l} \int_0^\infty e^{\varepsilon h}\hat{u}(x + \varepsilon, y) d\varepsilon - u_2(x, y) \leq \frac{2k}{k + 1} \frac{\infty}{0} (e^{\varepsilon h}\hat{u}(x + \varepsilon, y)).
\]

We will express \( k \) through \( h \). Get the assessment, which proved for \( u_2 \). The formula links solutions the Dirichlet problem and the Robin problem from [12]

\[
u_2 = - \int_0^\infty e^{\varepsilon h}\hat{u}(x + \varepsilon, y) d\varepsilon.
\]

This theorem allows the physical interpretation: component of the solution \( u_2 \) with the accuracy to a numerical multiplier it is approximately equal to the solution of the third homogeneous boundary problem with boundary condition

\[2h\hat{u}_3 + \frac{\partial}{\partial n}\hat{u}_3 = \hat{u}, (x, y), x = 0.\]

Similarly, **Theorem 2.** Let the condition is fulfilled

\[
1 - k \frac{1}{1 + k} = e^{2hl},
\]

and the value \( k \gg 1 \). For the problem solution \( \{1\}-\{4\} \) approximate formulas are valid

\[
u_1 \approx \frac{1 - e^{2hl}}{1 + e^{2hl}} \frac{1}{2l} \left( \left( \hat{u}_3(x, y) - e^{2hl}\hat{u}_3(x + 2l, y) \right) + \ldots \right),
\]

\[
\ldots + e^{2hl}(\hat{u}_3(2l - x, y) - e^{2hl}\hat{u}_3(4l - x, y)) \right),
\]

\[
u_2 \approx \frac{1 - e^{2hl}}{2l} \frac{1}{2l} \left( \hat{u}_3(x, y) - e^{2hl}\hat{u}_3(x + 2l, y) \right).
\]

**Theorem 2** is proved on the basis of the Lemma 2.

4.2. **Thin strip.** We apply Lemma 2. For the solution of the Dirichlet problem in the strip \( \{5\}-\{9\} \) assessment is fair

\[
\left| \int_0^\infty \hat{u}(x + \varepsilon, y) - \hat{u}(2l - x + \varepsilon, y) d\varepsilon - u(x, y) \right| \leq \frac{\infty}{0} V,
\]

here

\[
\frac{\infty}{0} = \frac{\infty}{0} (\hat{u}(x + \varepsilon, y) - \hat{u}(2l - x + \varepsilon, y))
\]

- variation of function \( [10] \) taken from a variable \( \varepsilon \) on the interval \([0;\infty)\). Thus, the following theorem is obtained

**Theorem 3.** The solution of the Dirichlet problem in the strip \( \{10\}-\{14\} \) in the strip can be found on the approximate formula

\[
u(x, y) \approx \frac{\hat{u}_2(x, y) - \hat{u}_2(2l - x, y)}{2l},
\]
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Where \( \hat{u}_2 \) - the solution of the Neumann problem for the equation Laplace’s in a circle with the boundary condition

\[
\frac{\partial \hat{u}_2}{\partial x} = \hat{u}, x = 0.
\]

4.3. The axial case. The circle with with the conditions internal coupling. Study the case of a thin shell, that is the case of a small thickness of the outer layer \( 1-R \). Let’s consider also magnitude \( k \) of the small, that is consider the case when the coefficients of heat capacity layers differ greatly. Assuming

\[
\frac{1 - k}{1 + k} = R^{2h}.
\]

Formula follow from Lemma 1

\[
\left| \frac{2k}{k + 1} \ln \frac{1}{\varepsilon} \int_0^1 \varepsilon^{h-1} \hat{u}(x\varepsilon, y\varepsilon)d\varepsilon - u_2(x, y) \right| \leq \frac{2k}{k + 1} V^1_0 \left( \varepsilon^h \hat{u}(x, y) \right),
\]

\[
\left| \ln \frac{1}{\varepsilon} \int_0^1 \varepsilon^{h-1} \hat{u}(x\varepsilon, y\varepsilon)d\varepsilon - u_2(x, y) \right| \leq V^1_0 \left( \varepsilon^h \hat{u}(x, y) \right) \left( 1 - R^{2h} \right),
\]

Therefore, we come to the next result.

**Theorem 4.** For the solution of the problem (7)-(9) \( u_1, u_2 \) approximate formulas are valid

\[
u_1 \approx \frac{1}{\ln \frac{1}{\varepsilon}} \hat{u}_3(x, y) - \frac{R^{2h}}{\ln \frac{1}{\varepsilon}} \hat{u}_3 \left( \frac{R^2 x}{x}, \frac{R^2 y}{y} \right), \quad u_2 \approx \frac{1 - R^{2h}}{\ln \frac{1}{\varepsilon}} \hat{u}_3.
\]

This theorem allows the physical interpretation: component of the solution \( u_2 \) with the accuracy to a numerical multiplier it is approximately equal to the solution of the third homogeneous boundary problem with boundary condition

\[
2h \hat{u}_3 + \frac{\partial}{\partial n} \hat{u}_3 = \hat{u}, (x, y) \in S.
\]

It is necessary to consider the formula from [3]. The formula links solutions of the first and third homogeneous boundary problems:

\[
\hat{u}_3 = \int_0^1 \varepsilon^{h-1} \hat{u}(x\varepsilon, y\varepsilon)d\varepsilon.
\]

Study the case of small values of thickness of the external layer \( 1-R \approx 0 \), thus we consider coefficient \( k >> 1 \). If the expression component \( u_2 \) rewritten in the form:

\[
u_2(x, y) = \frac{2k}{k + 1} \sum_{j=0}^{\infty} \left( \frac{k - 1}{1 + k} \right)^j \hat{u}(x R^{4j}, y R^{4j}) - \frac{2k}{k + 1} \sum_{j=0}^{\infty} \left( \frac{k - 1}{1 + k} \right)^j \hat{u}(x R^2 R^{4j}, y R^2 R^{4j}),
\]

define the number \( h \) using the formula

\[
\frac{k - 1}{1 + k} = R^{2h}.
\]

and lemma 1. We get the approximate formulas for component of the problem solution (7)-(9):

\[
u_1 \approx \frac{1 - R^{2h}}{1 + R^{2h}} \frac{1}{2 \ln \frac{1}{\varepsilon}} \left( \hat{u}_3(x, y) - \frac{R^{2h}}{2 \ln \frac{1}{\varepsilon}} \left( \hat{u}_3(x, y) - \frac{R^{2h}}{2 \ln \frac{1}{\varepsilon}} \right) \right),
\]
\[ u_2 \approx 1 - \frac{R^2 h}{2 \ln \frac{1}{R^2}} \left( \hat{u}_3(x, y) - R^2 \hat{u}_3(R^2 x, R^2 y) \right). \]

4.4. Thin ring. Let \( \hat{u} = \hat{u}(x, y) \) be harmonic in the unit circle, \( B = \{(x, y) : x^2 + y^2 < 1\} \) continuous in its closure \( \bar{B} \). In the ring \( K_{R^2} \) with the smaller radius \( R \) and larger radius 1, consider the Dirichlet problem \( (10)-(11) \).

We apply Lemma 1, we obtain the following assessment of solutions of the Dirichlet problem in the annulus \( (10)-(11) \)

\[ \left| \int_0^1 \left( \hat{u}(x, y, \varepsilon) - \hat{u} \left( \frac{R^2 x}{\varepsilon}, \frac{R^2 y}{\varepsilon} \right) \right) d\varepsilon - u(x, y) \right| \leq \frac{\infty}{\partial \varepsilon}, \]

here

\[ \frac{\infty}{\partial \varepsilon} = \frac{1}{0} \left( \hat{u}(x, y, \varepsilon) - \hat{u} \left( \frac{R^2 x}{\varepsilon}, \frac{R^2 y}{\varepsilon} \right) \right) - \]

variation of function \( [10] \) taken from a variable \( \varepsilon \) on the line segment \([0; 1]\). Thus, the following theorem is valid.

**Theorem 5.** For the solutions of the Dirichlet problem in the annulus \( (10)-(11) \) approximate formula is valid

\[ u(x, y) \approx \hat{u}_2(x, y) - \hat{u}_2 \left( \frac{R^2 x}{\varepsilon}, \frac{R^2 y}{\varepsilon} \right), \]

where \( \hat{u}_2 \) - the solution of the Neumann problem for the equation Laplace’s in a circle with the boundary condition

\[ \frac{\partial \hat{u}}{\partial n} = \hat{u} \left( x, y \right) \in \mathcal{S}. \]

5. Conclusion

From the results given in article follows, that the accuracy of the found formula is order equal to the thickness of the outer layer. Therefore, obtaining the formulas of higher accuracy class represents an important interest. This problem is solved by using asymptotic formulas presented in corollary 1, 2. The problem of distribution of the results on the multilayered plates and also on the boundary problems with the more general boundary conditions is actual.

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