Trying to Understand Confinement
in the Schrödinger Picture

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Abstract

We study the gauge-invariant gaussian ansatz for the vacuum wave functional and show that it potentially possesses many desirable features of the Yang–Mills theory, like asymptotic freedom, mass generation through the transmutation of dimensions and a linear potential between static quarks. We point out that these (and other) features can be studied in a systematic way by combining perturbative and $1/n$ expansions. Contrary to the euclidean approach, confinement can be easily formulated and easily built in, if not derived, in the variational Schrödinger approach.

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1 Introduction

We shall consider the pure Yang–Mills theory, else called gluodynamics, with no dynamical quarks. It is widely believed that in this theory the potential between probe static quarks is infinitely rising linearly with the quark separation. If proved correct, it will be one of the most remarkable findings in physics, and one would need to explain it from different angles. The present paper is an attempt to understand the physics of confinement in the framework of the Schrödinger picture.

The use of the Schrödinger equation to discuss the ground state of gluodynamics as well as the two source quarks on top of the ground state, is not new. To mention a few early works, see refs. [1] [2] [3]. More recently, the interest to the Schrödinger equation has been revived by Kogan and Kovner who suggested a variational vacuum wave functional in a gaussian form averaged over all gauge transformations [4]. We shall use this form of the trial wave functional making a slight but essential generalization of the ansatz suggested in [4]. This work has been influenced also by a recent paper by Zarembo [5] though we differ from that reference in several aspects.

2 Schrödinger equation

The spatial components of the Yang–Mills potential $A^a_i(x)$ play the role of coordinates, the electric field strength $E^a_i(x) = A^a_i(x)$ plays the role of momenta. The classical energy of the Yang–Mills field is

$$H_{\text{class}} = \int d^3 x \left[ \frac{1}{2g^2} (E^a_i(x))^2 + \frac{1}{2g^2} (B^a_i(x))^2 \right]$$

where

$$B^a_i = \epsilon_{ijk} (\partial_j A^a_k + (1/2) f^{abc} A^b_j A^c_k)$$

is the magnetic field strength. Upon quantization the electric field is replaced by variational derivative, $E^a_i(x) \to -ig^2 \delta / \delta A^a_i(x)$, if one uses the ‘coordinate representation’ for the wave functional. The functional Schrödinger equation for the wave functional $\Psi[A^a_i(x)]$ takes the form

$$H_\Psi[A_i] = \int d^3 x \left\{ -\frac{g^2}{2} (\delta A_i^a(x))^2 + \frac{1}{2g^2} (B_i^a(x))^2 \right\} \Psi[A_i] = \mathcal{E} \Psi[A_i]$$

where $\mathcal{E}$ is the eigenenergy of the state in question.

The Schrödinger equation (3) should be supplemented by the gauge condition. For a state without external colour sources the condition is that the wave functional $\Psi[A_i]$ should be gauge-invariant,

$$\Psi[A^S_i] = \Psi[A_i],$$

where

$$A^S_i \equiv S^\dagger A_i S + i S^\dagger \partial_i S$$
is the gauge transformation. For a state with an external static quark at $z_1$ and an antiquark at $z_2$ the wave functional $\Psi^\prime_{\beta}[A_i; z_1, z_2]$ should satisfy the gauge condition

$$\Psi^\prime_{\beta}[A_i^S; z_1, z_2] = \left[ S^I(z_1) \right]_{\alpha}^{\alpha'} \Psi^\prime_{\beta'}[A_i; z_1, z_2] \left[ S(z_2) \right]_{\beta'}^{\beta}. \quad \text{(6)}$$

If the probe sources belong to another representation the gauge-transforming matrices $S(z_{1,2})$ should be taken in appropriate representation.

## 3 Ultraviolet regularization

Strictly speaking, the hamiltonian (3) is senseless since any calculations with it are plagued by ultraviolet divergences. In order to give meaning to the Schrödinger equation (3) one has to regularize both the kinetic and potential energy parts of the hamiltonian. Examples of ultraviolet regularization are provided by the lattice hamiltonian [7] or by the ‘heat kernel’ method [4].

The requirements on possible ultraviolet regularizations are rather stringent. In addition to the apparent requirement that the regularization should be gauge invariant, there is a less trivial requirement that the regularized hamiltonian, together with the momentum and angular momentum operators, should satisfy the Poincaré algebra. Otherwise, there would be various inconsistencies, like that the energy of a glueball state would not obey the relativistic formula $E^2 = m^2 + p^2$. To my knowledge, a consistent regularization scheme has not been yet introduced.

Our aim is, however, the vacuum state with and without static sources, therefore any of the two mentioned regularization schemes would suit us here. In fact, one has to solve the regularized Schrödinger equation

$$\mathcal{H}^{reg}(g, M)\Psi(g, M) = \mathcal{E}(g, M)\Psi(g, M) \quad \text{(7)}$$

where $g$ is the bare coupling constant given at certain large normalization scale $M$ which one should eventually put to infinity. [For example, $M$ can be viewed as the inverse lattice spacing in case of the lattice regularization].

Let us consider the ground state or vacuum of the theory. Its energy $\mathcal{E}$ is by far dominated by the energy of the normal zero-point oscillations of the gauge field, which diverges quartically and, being regularized, is proportional to $M^4$. From the perturbation theory one infers

$$\mathcal{E}^{pert}(g, M) = 2(N_c^2 - 1) \sum \left( \frac{\hbar\omega}{2} \right) + \text{perturbative corrections}$$

$$= 2(N_c^2 - 1)V \int \frac{d^3k}{(2\pi)^3} \frac{|k|}{2} + \text{perturbative corrections}$$

$$= VM^4(c_{00} + c_{01}g^2 + c_{02}g^4 + \ldots). \quad \text{(8)}$$

It is not the full truth, however. We expect also non-perturbative contributions to the vacuum energy determined by the dimensional parameter $\Lambda$ obtained through the transmutation of dimensions,
\[ \Lambda = M \left( \frac{bg^2}{16\pi^2} \right)^{-\frac{b'}{2\pi^2}} \exp \left( -\frac{8\pi^2}{bg^2} \right) \left( 1 + O(g^2) \right), \quad b = \frac{11}{3} N_c, \quad b' = \frac{34}{3} N_c^2, \quad (9) \]

where \( b \) is the one-loop and \( b' \) is the two-loop Gell-Mann–Low coefficient. Contributions proportional to powers of \( \Lambda \) cannot be obtained in any order of the perturbation theory. However, there are no reasons why such contributions should be absent. Therefore, we write for the vacuum energy a general double expansion:

\[ \frac{\mathcal{E}(g, M)}{V} = M^4(c_{00} + c_{01}g^2 + c_{02}g^4 + \ldots) + M^3\Lambda(c_{10} + c_{11}g^2 + \ldots) + M^2\Lambda^2c_{20} + \ldots + M\Lambda^3c_{30} + \ldots + \Lambda^4(c_{40} + c_{41}g^2 + \ldots). \quad (10) \]

Meanwhile, all physical observables, like the string tension in the static potential, glueball masses, gluon condensate, topological susceptibility, etc., should arise as renorm-invariant combinations of the bare coupling \( g \) and the ultraviolet cutoff \( M \), i.e. should be expressed trough \( \Lambda \) only, in appropriate powers. This makes a straightforward use of the variational principle to extract physical observables almost a hopeless task: before getting to physically interesting quantities encoded in the last term in eq. (10) one has to minimize the first four terms which are by far the larger!

There is an interesting possibility to circumvent this difficulty, at least for the vacuum state. Let us assume that we have succeeded in building the regularized operators for the full stress-energy tensor, \( \theta_{\mu\nu} = \left[ F^a_{\mu\alpha}F^a_{\nu\alpha} - (1/4)g_{\mu\nu}F^a_{\gamma\alpha}F^a_{\gamma\alpha} \right]/g^2 \), in accordance with the Poincarè algebra, the hamiltonian being \( \mathcal{H} = \int d^3x \theta_{00} = \int d^3x \left( E^2 + B^2 \right)/2g^2 \). The vacuum energy is \( \mathcal{E} = V\langle \theta_{00} \rangle \) where \( \langle \ldots \rangle \) denotes averaging over the vacuum wave functional.

Because of the Lorentz invariance of the vacuum one expects \( \langle \theta_{00} \rangle = -\langle \theta_{11} \rangle = -\langle \theta_{22} \rangle = -\langle \theta_{33} \rangle \) and hence \( \langle \theta_{00} \rangle = (1/4)\langle \theta_{\mu\nu} \rangle \). The last quantity is naively zero (following from the above definition of \( \theta_{\mu\nu} \)), however under proper regularization it should be equal, through trace anomaly, to

\[ \theta_{\mu\nu} = \frac{\beta(g^2)(F^a_{\mu\nu})^2}{4g^4} = -b\frac{B^2 - E^2}{16\pi^2}. \quad (11) \]

[We have neglected higher-loop contributions to the Gell-Mann–Low function \( \beta(g^2) \) which can always be done if the ultraviolet cutoff \( M \) is chosen large enough and hence the bare coupling \( g \) small enough.] The quantity \( \langle (F^a_{\mu\nu})^2 \rangle \) is known as the gluon condensate appearing in the ITEP sum rules and has to be renorm-invariant and hence proportional to \( \Lambda^4 \).

Thus if one wants to extract directly the non-perturbative physics from the variational approach one should rather maximize the gluon condensate \( \langle B^2 - E^2 \rangle \) than minimize the energy density \( (B^2 + E^2)/2g^2 \).

In the latter quantity the kinetic and potential energies of the zero-point gluon oscillations add up and diverge quartically, see eq. (8); in the former quantity they cancel each other. In the zero order of the perturbation theory this cancellation is exact due to the

\[ \text{Footnote 2: This is what one actually does in building the euclidean instanton vacuum from the variational principle}, \]

where one is also interested in finding the shift of the vacuum energy in respect to the perturbative one.
virial theorem for harmonic oscillators. Under a proper regularization of the operators, respecting the $SO(4,2)$ generalization of the Poincaré algebra (including the dilatation and conformal operators), this cancellation should hold to any order of the perturbation theory, so that the properly defined gluon condensate gets a non-zero contribution only from the non-perturbative physics which is characterized by being non-analytic in the coupling constant, see eq. (9).

Unfortunately, the prescription to maximize $\langle B^2 - E^2 \rangle$ is a priori not well-defined either as one can always invent a crazy trial functional making the quantity as large as one likes. Despite these pessimistic remarks concerning the foundations for the use of the variational principle, it may still serve as a useful guide to understand non-perturbative physics in a language different from the usual euclidean approach. Being unable to formulate the problem in a mathematically unequivocal way, we shall henceforth proceed in a somewhat sloppy manner: we shall not regularize the hamiltonian but simply cut the divergent momenta integrals ‘by hands’ at certain large momentum $M \gg \Lambda$. We shall not go beyond the one-loop calculations, so this barbarous procedure will be sufficient for our needs. In fact, we shall be able to separate the perturbative and nonperturbative physics using a specific variational ansatz, and minimizing the nonperturbative vacuum energy will make sense, see section 8.

4 Kogan–Kovner variational ansatz and its generalization

Kogan and Kovner have suggested a gaussian trial wave functional for the vacuum state, averaged over all gauge transformations of the ‘coordinates’ $A_i$:

$$\Psi[A] = \int DU(x) \exp \left( -\frac{1}{g^2} Q[A^U] \right)$$

(12)

where $A_i^U \equiv U^\dagger A_i U + i U^\dagger \partial_i U$ is the gauge-transformed Yang–Mills potential and $Q[A]$ is a quadratic functional (hence the notation):

$$Q[A] = \int \int d^3 x \, d^3 y \, K_{ij}(x-y) \, \text{Tr} \, A_i(x) A_j(y).$$

(13)

The kernel $K_{ij}(x-y)$ is the only variational (or trial) function of the ansatz. Its Fourier transform has the general structure compatible with space isotropy:

$$K_{ij}(p) = a(|p|) \cdot \delta_{ij} + b(|p|) \cdot \frac{p_i p_j}{p^2}.$$  

(14)

The two scalar functions $a(p)$ and $b(p)$ are the variational ‘parameters’ of the ansatz. As we shall see below, minimization of the vacuum energy in the leading order of the perturbation theory (the coefficient $c_{00}$ in the notations of eq. (10)) leads to the following result:

$$a_{\text{pert}}(p) = |p|, \quad b_{\text{pert}}(p) = \text{arbitrary}.$$  

(15)

In ref. [3] the longitudinal part of the kernel $b(p)$ has been put to zero (without compelling reasons). In ref. [6] it has been shown that, because of the averaging over gauges in the wave function, the kernel $b(p)$ is not necessary in the vacuum expectation value.
functional (12), certain forms of “actions” in the exponent of Ψ[A] are physically equivalent; in particular, the longitudinal part of the quadratic form Q[A] can be removed by choosing another “action” in the exponent, containing higher powers of A_i. Having no objections to this statement in general we note, however, that if one restricts oneself to purely quadratic “actions” of the form (13) the longitudinal part of Q[A_i] becomes physically meaningful. Moreover, we shall show below that by suitably choosing the longitudinal function b(p) in (14) one can reproduce the correct (11/3)N_c coefficient in the one-loop Gell-Mann–Low function. In ref. [9] the value (12/3)N_c has been obtained instead for that coefficient, with b(p) set to zero.

According to the variational principle the vacuum energy satisfies the inequality

\[ \mathcal{E} \geq \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \] (16)

One has to minimize the r.h.s. in the free parameters / functions of the chosen ansatz, in our case, in the two scalar functions a(p) and b(p).

5 Evaluation of the norm \( \mathcal{N} = \langle \Psi | \Psi \rangle \)

The norm \( \mathcal{N} \) standing in the denominator of eq. (16) is understood as the functional integral,

\[ \mathcal{N} = \langle \Psi | \Psi \rangle = \int DA \Psi^*[A] \Psi[A] = \int DA \int DU_1 \int DU_2 \exp \left( -\frac{1}{g^2} Q[A_{U_1}] - \frac{1}{g^2} Q[A_{U_2}] \right). \] (17)

One first integrates over the gauge potential \( A \) by saddle-point method; since \( Q[A_{U_1}, U_2] \) are quadratic in \( A \) the saddle-point integration is exact.

We have

\[ \frac{\delta Q[A]}{\delta A_i^a(x)} = \int d^3y \ K_{ij}(x-y) 2 \text{Tr} \ U_1^\dagger(x) t^a U_1(x) \left[ U_1^\dagger(y) A_j(y) U(y) + i U_1^\dagger(y) \partial_j U(y) \right] \] (18)

where \( t^a \) are \( SU(N_c) \) generators in the fundamental representation normalized to \( \text{Tr} t^a t^b = (1/2) \delta_{ab} \). The saddle-point equation for \( A_i \) is

\[ \frac{\delta Q[A_{U_1}]}{\delta A_i^a(x)} + \frac{\delta Q[A_{U_2}]}{\delta A_i^a(x)} = 0. \] (19)

The solution of this equation gives the saddle-point value of \( A_i \):

\[ \bar{A}_i = \frac{i}{2} \left( U_1 \partial_i U_1^\dagger + U_2 \partial_i U_2^\dagger \right) + \text{corrections}. \] (20)

In the one-loop approximation it is sufficient to neglect corrections to this value as they prove to be actually of the order of \( g^2 \), see below. This saddle-point value becomes exact if the kernel \( K(x-y) \) is a delta function considered as an example in ref. [6]. In ref. [5] more
complicated technique has been involved to solve the saddle-point equation than seems to be necessary for our goals here.

It is amusing that ‘on the diagonal’ where the gauge transformations in \( \Psi \) and the conjugate \( \Psi^* \) coincide (\( U_1 = U_2 \)) the saddle-point value \( \bar{A}_i \) (that is where \(|\Psi[A]|^2 \) is mostly concentrated) is a pure gauge potential. That is consistent with the euclidean instanton-vacuum picture where the system spends much of the ‘time’ in a pure gauge state.

Actually, the introduction of two gauge transformations \( U_1 \) and \( U_2 \) is abundant as one can always absorb one of them into a shift of \( A_i \) since the integration measure \( DA_i \) is gauge invariant. Renaming the integration variable \( A_i^{U_1} \rightarrow A_i \) so that \( A_i^{U_2} \) becomes \( A_i^U \) where \( U \) is the ‘relative’ gauge transformation, \( U = U_1^† U_2 \), one can rewrite the norm as

\[
\mathcal{N} = \int DU \int DA \exp \left(-\frac{1}{g^2}Q[A] - \frac{1}{g^2}Q[A^U] \right) \tag{21}
\]

and the saddle-point field as

\[
\bar{A}_i = \frac{i}{2} U \partial_i U^† + \text{ corrections.} \tag{22}
\]

Generally speaking, this is not a pure gauge potential; it becomes a pure gauge only when the relative gauge transformation \( U \) does not differ much from unity.

Let us introduce the shifted field \( A' \) counted from the saddle-point value \((22)\), i.e. let us put \( A_i = (i/2) U \partial_i U^† + A'_i \) and expand \( Q[A] + Q[A^U] \) in eq. (21) in powers of \( A' \).

The term quadratic in \( A' \) is

\[
\int d^3 x d^3 y \; K_{ij}(x - y) \left\{ 2 \text{Tr} \; A'_i(x) A'_j(y) + \text{Tr} \left[ U(y) U^†(x) - 1 \right] A'_i(x) \left[ U(x) U^†(y) - 1 \right] A'_j(y) \right\}.
\tag{23}
\]

The term linear in \( A' \) is

\[
\int d^3 x d^3 y \; K_{ij}(x - y)(-i) \text{Tr} \; A'_i(x) \left\{ \left[ U(x) U^†(y) - 1 \right] \left[ U(y) \partial_j U^†(y) \right] \left[ U(y) U^†(x) - 1 \right] + \left[ U(x) U^†(y) - 1 \right] \left[ U(y) \partial_j U^†(y) \right] \left[ U(y) U^†(x) - 1 \right] \right\}.
\tag{24}
\]

Finally, the term not containing at all \( A' \) is

\[
\int d^3 x d^3 y \; K_{ij}(x - y) \left\{ \frac{1}{2} \text{Tr} \; \partial_i U^†(x) \partial_j U(y) + \frac{1}{4} \text{Tr} \; \partial_i U^†(x) \partial_j U^†(x) \left[ U^†(y) U(x) - 1 \right] + \frac{1}{4} \text{Tr} \; \partial_i U(y) \partial_j U^†(x) \left[ U(x) U^†(y) - 1 \right] \right\}.
\tag{25}
\]

We see that had \( K_{ij}(x - y) \) been a \( \delta \)-function one could replace \( U(x) U^†(y) \) by unity matrix, so that the linear term in \( A' \) (24) would vanish. In the same approximation one would be left only with the first terms in eqs. (23, 25). In fact this approximation to \( Q[A] + Q[A^U] \) is correct for any function \( K_{ij}(x - y) \) if we restrict ourselves to the one-loop approximation, i.e. neglect \( O(g^2) \) terms in calculating the norm \( \mathcal{N} \).

To show that we first notice that if we, indeed, neglect the linear term (24) and leave only the first term in eq. (23) and the first term in eq. (25), the propagator of the \( A'_i \) field
is of the order of \( g^2 \), and the propagator of the \( U \) field is also of the order of \( g^2 \). The last statement needs to be made more precise as we, generally speaking, deal with a non-linear field: that will be done below. What can immediately be seen is that the fluctuations of the gradients of the \( U \) field are of the order of \( g^2 \), hence the characteristic square bracket \([U(x)U^\dagger(y) - 1]\) is of that order, too, since it is non-zero only when one expands it in the gradients of the \( U \) field.

To check that we have correctly neglected all the terms we expand \( \exp(-Q[A]/g^2 - Q[A^U]/g^2) \) in powers of the presumably small terms. For example, let us consider the third and fourth terms of eq. (23) expanded to the linear order. There is a factor \( 1/g^2 \) coming with the expansion, a factor \( g^2 \) arising from the contraction of the two \( A' \) fields and an extra factor \( g^2 \) owing to the fluctuations of the square brackets, as explained above. We obtain thus a \( g^2 \) correction, as compared to the main contribution to the norm which is \( O(1) \).

Let us consider the expansion in the linear term (24). Expanding to the first order we get an identical zero (because it is odd in \( A' \)), so we have to consider the second term in the expansion of the exponent. We have \( (1/g^2)^2 \) from the expansion to the second order, \( g^2 \) from the \( A' \) propagator and at least \( g^4 \) from averaging over the \( U \)'s, that is again a \( O(g^2) \) correction. The same is true for the expansion of the last two terms in eq. (25).

We conclude that if we are willing to neglect terms of the order of \( g^2 \) (i.e. restricting ourselves to the one-loop approximation) we can use the saddle-point field given by eqs.(20, 22) and approximate the norm by a factorized expression:

\[
N \approx N_A \cdot N_U, \tag{26}
\]

\[
N_A = \int DA' \exp \left\{ -\frac{2}{g^2} \int d^3 x d^3 y K_{ij}(x - y) \, \text{Tr} \, A'_i(x) A'_j(y) \right\}, \tag{27}
\]

\[
N_U = \int DU \exp \left\{ -\frac{1}{2 g^2} \int d^3 x d^3 y K_{ij}(x - y) \, \text{Tr} \, \partial_i U^\dagger(x) \partial_j U(y) \right\}. \tag{28}
\]

We see that the norm is factorized into independent integrals over \( A' \) and over \( U \). The former is a trivial gaussian integral while the latter is non-trivial since the field \( U \) is subject to the unitarity constraint.

### 6 Evaluation of the energy numerator \( \langle \Psi | \mathcal{H} | \Psi \rangle \)

We now proceed to evaluating the numerator of the vacuum energy (16). We have

\[
\langle \Psi | \mathcal{H} | \Psi \rangle = \int DU \int DA \int d^3 x \, \frac{1}{2 g^2} \left\{ \frac{\delta Q[A]}{\delta A_i^a(x)} \frac{\delta Q[A^U]}{\delta A_i^a(x)} + (B^a_i[A])^2 \right\} \exp \left\{ -\frac{1}{g^2} (Q[A] + Q[A^U]) \right\}. \tag{29}
\]

The exponential factors are the same as for the norm integral, so we have to analyze the prefactor which is new. Using eq. (18) we get

\[
\int d^3 x \, \frac{\delta Q[A]}{\delta A_i^a(x)} \frac{\delta Q[A^U]}{\delta A_i^a(x)} = \int \int d^3 x d^3 y d^3 z K_{ij}(x - y) K_{ik}(x - z) 2 \, \text{Tr} \left[ U^\dagger(x) A_j(y) U(x) A_k^U(z) \right]. \tag{30}
\]
We next expand this expression in the shifted field \( A' \) around the approximate saddle point substituting \( A_i = (i/2)U_\partial U^\dagger + A'_i \). To save space, we write down the expansion of the last factor in eq. (30) only, the trace.

The term quadratic in \( A' \) is

\[
\text{Tr} A'_j(y)A'_k(z) + \text{Tr} [U(z)U^\dagger(x) - 1] A'_j(y) \left[ U(x)U^\dagger(z) - 1 \right] A'_k(z)
\]

\[
+ \text{Tr} [U(z)U^\dagger(x) - 1] A'_j(y)A'_k(z) + \text{Tr} A'_j(y) \left[ U(x)U^\dagger(z) - 1 \right] A'_k(z).
\]

The term linear in \( A' \) is

\[
\frac{i}{2} \text{Tr} A'_k(z) \left\{ U(z)U^\dagger(x)U(y)\partial_j U^\dagger(y)U(x)U^\dagger(z) - U(x)U^\dagger(y)\partial_j U^\dagger(y)U(y)U^\dagger(x) \right\}.
\]

At \( y = x = z \) this term is zero implying that in the general case it can be identically rewritten à la eq. (24) using expressions of the type \( [U(z)U^\dagger(x) - 1] \). However, it leads to a rather lengthy formula, and we do not present it here.

Finally, the term without the \( A' \) field is

\[
- \frac{1}{4} \left\{ \text{Tr} \partial_j U^\dagger(y)\partial_k U(z) + \text{Tr} [U^\dagger(x)U(y) - 1] \partial_j U^\dagger(y) \left[ U(x)U^\dagger(z) - 1 \right] \partial_k U(z) \right\}
\]

\[
+ \text{Tr} [U^\dagger(x)U(y) - 1] \partial_j U^\dagger(y)\partial_k U(z) + \text{Tr} \partial_j U^\dagger(y) \left[ U(x)U^\dagger(z) - 1 \right] \partial_k U(z) \right\}.
\]

We have now to integrate the sum of eqs. (31, 32, 33) over the fields \( A' \) and \( U \) with the exponential weight given by eqs. (27, 28). The leading \( O(1) \) contributions come from the first term in eq. (31) and from the first term in eq. (33). Eq. (32) can contribute only in combination with the linear term from the exponent (24) expanded to the first order: it gives a \( g^2 \) correction according to the power counting rules of the previous section. By the same counting rules one can neglect all other terms in eqs. (31, 33).

A similar scenario takes place when we insert \( A_i = (i/2)U_\partial U^\dagger + A'_i \) into the magnetic energy \( B^2/2g^2 \). The leading \( O(1) \) contribution arises only from the ‘abelian’ piece made of the \( A' \) field, namely from \( \text{Tr} (\partial_i A'_j - \partial_j A'_i)^2/2g^2 \). All other terms, including the Skyrme-type term made solely of the \( U \) field, give \( O(g^2) \) corrections to the vacuum energy. Disregarding systematically all such terms we finally obtain:

\[
\langle \Psi | H | \Psi \rangle \approx \int DU \int DA \exp \left\{ -\frac{1}{g^2} \int d^3x \int d^3y \, K_{ij}(x - y) \right\}
\]

\[
\times \left[ 2\text{Tr} A_i(x)A_j(y) + \frac{1}{2} \text{Tr} \partial_i U^\dagger(x)\partial_j U(y) \right]
\]

\[
\times \frac{1}{g^2} \left\{ \int \int d^3x d^3y d^3z K_{ij}(x - y)K_{ik}(x - z) \left[ \text{Tr} A_j(y)A_k(z) - \frac{1}{4} \text{Tr} \partial_j U^\dagger(y)\partial_k U(z) \right] \right.
\]

\[
+ \frac{1}{2} \int d^3x \, \text{Tr} (\partial_i A_j - \partial_j A_i)^2 \right\}
\]

(we have deleted the primes in the notation of the \( A \) field).
To get the variational estimate for the vacuum energy we need now to divide (34) by the norm (26). Since the norm has a factorized form the vacuum energy is a sum of two terms:

\[
E \geq E = \langle \Psi | H | \Psi \rangle \left( \frac{\text{contribution from } A \text{ field}}{\langle \Psi | \Psi \rangle} \right) + \langle \Psi | \Psi \rangle \left( \frac{\text{contribution from } U \text{ field}}{\langle \Psi | \Psi \rangle} \right)
\]

(35)

where

\[
\text{contribution from } A \text{ field} = \frac{1}{g^2} \int DA \exp \left[ -\frac{2}{g^2} \int d^3x \int \frac{K_{ij}(x - y) \text{Tr} A_i A_j(y)}{2} \right]
\]

\[
\cdot \left\{ \int d^3x d^3y d^3z K_{ij}(x - y) K_{ik}(x - z) \text{Tr} A_j(y) A_k(z) + \frac{1}{2} \int d^3x \text{Tr} (\partial_i A_j - \partial_j A_i)^2 \right\},
\]

(36)

\[
\text{contribution from } U \text{ field} = -\frac{1}{4g^2} \int DU \exp \left[ -\frac{1}{2g^2} \int d^3x d^3y \frac{K_{ij}(x - y) K_{ik}(x - z) \text{Tr} \partial_i U^\dagger(y) \partial_k U(z)}{2} \right].
\]

(37)

Here \( \exp[...] \) in both expressions mean the same as in the corresponding denominators. The first term in eq. (36) is the contribution of the \( A_i \) field to the electric energy while the second is the contribution to the magnetic energy. Notice that in the one-loop approximation the \( U \) field contributes only to the electric energy.

The gaussian integrals over \( A_i \) can be easily done in the momentum space using the propagator

\[
\langle A_i^a(p) A_j^b(-p) \rangle = \frac{g^2}{2} \delta^{ab} K_{ij}^{-1}(p).
\]

(38)

Inversing the general expression (14) for \( K_{ij}(p) \) we have

\[
K_{ij}^{-1}(p) = \frac{1}{a(p)} \left( \delta_{ij} - \frac{p_i p_j}{p^2} \frac{b(p)}{a(p) + b(p)} \right).
\]

(39)

We are now in a position to express the contribution of the \( A_i \) field to the electric and magnetic energy of the vacuum through the trial functions \( a(p) \) and \( b(p) \). Using eqs.(36, 38, 34) we get

\[
\left( \begin{array}{c}
\text{magnetic energy density} \\
\text{electric energy density}
\end{array} \right) = \frac{\langle B^2 \rangle}{2g^2} = \frac{N_c^2 - 1}{2g^2} \int \frac{d^3p}{(2\pi)^3} \left( p^2 \delta_{ij} - p_i p_j \right) \frac{g^2}{2} K_{ij}^{-1}(p)
\]

\[
= \frac{N_c^2 - 1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{a(p)},
\]

(40)

\[
\left( \begin{array}{c}
\text{magnetic energy density} \\
\text{electric energy density}
\end{array} \right) = \frac{\langle E^2 \rangle}{2g^2} = \frac{N_c^2 - 1}{2g^2} \int \frac{d^3p}{(2\pi)^3} K_{ij}(p) K_{ik}(p) \frac{g^2}{2} K_{jk}^{-1}(p) + \frac{3a(p) + b(p)}{2}.
\]

(41)
To write down the contribution of the $U$ field to the (electric) energy density in a more compact way let us introduce new kernels,

$$L(x-y) = \partial_x \partial_y K_{ij}(x-y), \quad L(p) = p_i p_j K_{ij}(p) = p^2 [a(p) + b(p)]$$

and

$$S(y-z) = \int d^3 x \partial_y K_{ij}(x-y) \partial_z K_{ik}(x-z), \quad S(p) = p^2 [a(p) + b(p)]^2 = \frac{L^2(p)}{p^2}.$$  \hfill (42)

Eq. (37) can be compactly rewritten through the variational derivative of the norm,

$$\left( \text{contribution from the } U \text{ field} \right) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} S(p) \frac{\delta \log N_U}{\delta L(p)} \frac{(2\pi)^3}{V},$$

where

$$N_U = \int DU \exp \left[ -\frac{1}{2g^2} \int \frac{d^3 p}{(2\pi)^3} L(p) \text{Tr} U^\dagger(p) U(-p) \right].$$  \hfill (45)

In the next two sections we shall study the contribution of the $U$ field to the electric energy density in two regimes: perturbative, which will serve as a pedagogical example, and non-perturbative.

### 7 Perturbative regime

In the perturbative regime one assumes that the integration over the unitary field $U$ in eqs. (28, 45) is concentrated in the vicinity of unity matrices. Hence we can expand $U$ as

$$U = \exp(i\phi^a t^a) \approx 1 + i\phi^a t^a, \quad U^\dagger \approx 1 - i\phi^a t^a, \quad DU = D\phi^a.$$ \hfill (46)

In this approximation the action for the $N_c^2 - 1$ fields $\phi^a$ becomes quadratic, leading to the propagator

$$\langle \phi^a(p)\phi^b(-p) \rangle = \frac{2g^2 \delta^{ab}}{L(p)} = \frac{2g^2 \delta^{ab}}{p^2 [a(p) + b(p)]}.$$ \hfill (47)

The addition to the electric energy density arising from the $U$ field is thus

$$\left( \text{perturbative contribution from the } U \text{ field} \right) = -\frac{N_c^2 - 1}{8g^2} \int \frac{d^3 p}{(2\pi)^3} S(p) \frac{2g^2}{L(p)}$$

$$= -\frac{N_c^2 - 1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{L(p)}{2p^2} = -\frac{N_c^2 - 1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{a(p) + b(p)}{2}.$$ \hfill (48)

Naturally, this result comes also from the general formula (44).

Adding this result to eq. (11) we obtain
\[
\left( \text{electric energy density} \right) = \frac{\langle E^2 \rangle}{2g^2} = \frac{N_e^2 - 1}{2} \int \frac{d^3p}{(2\pi)^3} a(p). \tag{49}
\]

Notice the cancellation of the longitudinal piece associated with the trial function \(b(p)\). This is as it should be in the perturbative regime.

We now add up the electric (49) and the magnetic (40) parts of the vacuum energy and get the variational estimate for the vacuum energy:

\[
\frac{\mathcal{E}}{V} \geq \frac{N_e^2 - 1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{a(p) + \frac{p^2}{a(p)}}{2} \right]. \tag{50}
\]

This expression should be varied in respect to the trial function \(a(p)\). The r.h.s. of eq. (50) has the minimum at

\[a(p) = |p|, \quad b(p) = \text{arbitrary}, \tag{51}\]

the vacuum energy at the minimum being

\[
\frac{\mathcal{E}}{V} = 2(N_e^2 - 1) \int \frac{d^3p}{(2\pi)^3} \frac{|p|}{2}. \tag{52}
\]

This is, of course, the correct energy of the zero-point oscillations of \(N_e^2 - 1\) free gluons with two physical polarizations, each mode with momentum \(p\) carrying the energy \(|p|\). Note that at the minimum the virial theorem for harmonic oscillators is satisfied, since \(\langle E^2 \rangle = \langle B^2 \rangle\).

These nice results do not come unexpected since the gaussian wave functional used here is exact for the free fields. We have only shown that its exact form given by eq. (51) can be found from the variational principle.

8 Key point: non-zero v.e.v. of the Lagrange multiplier

Let us consider the \(SU(2)\) gauge group. The unitary matrix \(U\) in eq. (37) can be parametrized by quaternions,

\[
U = u_\alpha \sigma^-_\alpha, \quad U^\dagger = u_\alpha \sigma^+_\alpha, \quad \sigma^\pm_\alpha = (\pm i\tau, 1), \quad u^2_\alpha = 1. \tag{53}
\]

The last condition imposed on four real fields \(u_\alpha\) ensures that \(U\) is a matrix from \(SU(2)\).

The functional integral over \(SU(2)\) matrices \(U\) can be written as

\[
\int DU = \int Du_\alpha \prod_x \delta \left( u^2_\alpha(x) - 1 \right) = \int Du_\alpha \int D\lambda \exp \left[ -\frac{1}{g^2} \int d^3x (\lambda u^2_\alpha - \lambda) \right] \tag{54}
\]

where we have introduced an auxiliary integration over ‘Lagrange multiplier’ field \(\lambda(x)\) to ensure that the unitarity condition is satisfied at all space points. The factor \(1/g^2\) is inserted in eq. (54) for future convenience. The integration measure (54) is invariant under left and right shifts along the group, therefore it belongs to the class of the Haar measures.

\(^3\)Actually, eq. (50) can be found in ref. 3.
The dependence of the energy numerator $\langle \Psi | H | \Psi \rangle$ and of the denominator $\langle \Psi | \Psi \rangle$ on the $U$ field enters through the expression (see eq. (37))

$$\text{Tr} \partial_i U^\dagger(x) \partial_j U(y) = 2 \partial_i u_\alpha(x) \partial_j u_\alpha(y).$$

(55)

Therefore, the norm (28) or (45) reads

$$\mathcal{N}_U = \int D\lambda \exp \left( \frac{1}{g^2} \int d^3x \, \lambda \right) \int Du_\alpha \exp \left\{ -\frac{1}{g^2} \left[ \int d^3xd^3y L(x - y) u_\alpha(x) u_\alpha(y) + \int d^3x \lambda u_\alpha^2 \right] \right\}$$

(56)

where we have used the new kernel $L$ as defined by eq. (42). The inner integral is a gaussian integral over four real fields $u_\alpha$ with the Lagrange multiplier $\lambda$ playing the role of the mass term.

We have learned from the previous section that $a(p) = |p|$ at large momenta. The behaviour of the trial function $b(p)$ is not defined by the leading-order perturbation theory, however one can reasonably assume that it can be also $\sim |p|$. Therefore, the asymptotics of the ‘kinetic energy’ term in eq. (56) is

$$L(p) \rightarrow c|p|^3, \quad c = 1 + b,$$

(57)

where $b$ is the (unknown) numerical coefficient in $b(p) \rightarrow b|p|$ at large momenta.

Although the number of fields is only four ($n = 4$) we shall treat it as a formal parameter. The gaussian integral over $n$ real fields $u_\alpha$ in eq. (56) can be formally written as

$$\int Du_\alpha \exp \{ \ldots \} = \exp \left[ -\frac{n}{2} \text{Sp} \log(L + \lambda) \right]$$

(58)

where Sp stands for the functional trace. Let us divide the Lagrange multiplier field $\lambda(x)$ into a point-independent part $\mu^3$ and a space-variable part $\lambda'(x)$:

$$\lambda(x) = \mu^3 + \lambda'(x), \quad \int d^3x \lambda'(x) = 0. \quad (59)$$

The functional trace in (58) can be rewritten as

$$\text{Sp} \log(L + \lambda) = V \int \frac{d^3p}{(2\pi)^3} \log \left[ L(p) + \mu^3 \right] - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \lambda'(q) \Pi(q) \lambda'(-q) + \mathcal{O}(\lambda^3)$$

(60)

with $\Pi(q)$ being a loop made of the $u_\alpha$ propagators:

$$\Pi(q) = \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{L(p + q/2) + \mu^3[L(p - q/2) + \mu^3]} - \frac{1}{L(p) + \mu^3[L(p) + \mu^3]} \right]. \quad (61)$$

Eq. (56) defines a peculiar $O(n)$ sigma model in three dimensions at $n = 4$. Were $n$ a large parameter the fluctuations of the $\lambda'$ field would be suppressed as $1/n$. In that case one could neglect, to the first approximation in $1/n$, the second and higher terms in eq. (60). Though $n = 4$ cannot be considered as particularly ‘large’ we shall nevertheless leave out the
quadratic and higher-order terms in $\lambda'$. In principle, these terms can be taken into account systematically in the $1/n$ expansion similarly to the way it is done in two-dimensional sigma models [10, 11, 12].

The dependence on the constant part of the Lagrange multiplier $\mu^3$ is given by (see eqs. (56, 60))

$$V \cdot \left\{ \mu^3 \left[ \frac{n(=4)}{2} \right] \int \frac{d^3p}{(2\pi)^3} \log \left[ L(p) + \mu^3 \right] \right\}. \quad (62)$$

This expression has an extremum in $\mu$ determined by the ‘gap’ equation

$$\frac{1}{g^2} - 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{L(p) + \mu^3} = 0, \quad (63)$$

the integral here being logarithmically divergent since $L(p) \approx c|p|^3$ at large $|p|$ (see eq. (57)).

Introducing an ultraviolet cutoff $M$ we get from (63) the vacuum expectation value of the Lagrange multiplier $\bar{\lambda} = \mu^3$ where

$$\mu = \text{const} \exp \left[ - \frac{8\pi^2}{g^2 2 (4/c)} \right]. \quad (64)$$

Comparing it with the asymptotic-freedom law (9) we see that in order to get the correct one-loop Gell-Mann–Low coefficient (at $N_c = 2$) we need to put $\frac{4}{c} = \frac{11}{3}$ or $c = 1 + \frac{b(p)}{|p|} \bigg|_{p \to \infty} = \frac{12}{11}. \quad (65)$

We notice that the first Gell-Mann–Low coefficient has been recently calculated in ref. [9] in the Kogan–Kovner ansatz using a different method, and it has been found to be 12/3 instead of 11/3. This is exactly what we get from eqs. (64, 65) if we deliberately put (as it has been done in [4]) $b(p) = 0$. However we emphasize again that there are no reasons to bind oneself with such restriction. The actual value of the trial function $b(p)$ at large $|p|$ should come from minimizing the second-order $O(M^4 g^2)$ term in eq. (10). We have systematically neglected such terms above, however they can be systematically collected as well. Those terms contain, in particular, the non-abelian commutator terms for the magnetic energy, involving ‘transverse’ gluons. Therefore, it can well happen that the ‘best’ $b(p) \to |p|/11$, as it should be for getting the correct Gell-Mann–Low coefficient.

Eq. (64) demonstrates how transmutation of dimensions occurs in the Schrödinger approach. We see that it is related to the non-zero vacuum expectation value of the Lagrange multiplier field needed to get rid of the unitarity restriction for gauge transformations. This phenomenon is similar to what happens in the $O(n)$ sigma models in two dimensions. Similarly to those models one needs, strictly speaking, the large $n$ parameter to justify the use of the saddle-point method. However, one can hope that $n = 4$ is sufficiently ‘large’ so that the conclusion that the Lagrange multiplier field $\bar{\lambda}$ does get a non-zero v.e.v. is not altered by fluctuations of the $\lambda'$ field about the saddle-point. These fluctuations can be anyhow treated systematically in a formal $1/n$ expansion.

It is instructive to separate the perturbative and nonperturbative contributions to the vacuum energy by adding and subtracting the perturbative contribution [13]. We have
Integration over unitary matrices $U$ another method of circumventing the unitarity condition has been suggested. In these refs. Here one has $N$ difference between the $SU(N)$ integrating over $N$ being replaced by a mass parameter $g$ transmutation of dimensions takes place, the bare coupling $\sigma$ in $\bar{\lambda}$ should asymptotically go to $(12/\pi N)$ for arbitrary $N$ norm principle comes to minimizing this functional. To perform that one needs to evaluate the $u\alpha$ field by introducing a parametrization.

Concluding this section we would like to make the following remark. In refs. [5, 6] another method of circumventing the unitarity condition has been suggested. In these refs. integration over unitary matrices $U$ has been replaced by

$$
\int DU = \int DU^\alpha_\beta DU^\dagger_\beta \delta \left( \|U\|^2 - \delta^\alpha_\beta \right) = \int D\sigma^\alpha_\beta \int DU^\alpha_\beta DU^\dagger_\beta \exp \left\{ - \int dx \Tr \sigma (U^\dagger U - 1) \right\}.
$$

(68)

Here one has $N_c^2$ complex fields $U, U^\dagger$ with $2N_c^2$ real degrees of freedom (neglecting the difference between the $SU(N_c)$ and $U(N_c)$ groups), with $N_c^2$ restrictions taken care of by integrating over $N_c^2$ Lagrange multiplier fields $\sigma = \sigma^\dagger$. Taking $N_c \to \infty$ does not help to justify any saddle point in $\sigma$, as it is assumed in refs. [7, 8]. Indeed, the effective action in $\sigma$ resulting from integrating over the unconstrained fields $U, U^\dagger$ depends not only on $N_c$ eigenvalues of $\sigma^\dagger_\beta$ varying slowly but also on $N_c^2 - N_c = O(N_c^2)$ ‘angle’ variables of $\sigma^\dagger_\beta$ which fluctuate violently. Meanwhile, the fluctuations of the ‘angle’ part of $\sigma^\dagger_\beta$ influence the effective action for the eigenvalues of the $\sigma$ field, and taking $N_c \to \infty$ does not quench the ‘angle’ variables as their number is $O(N_c^2)$. In short, the method does not work.

In order to get use of the large $N_c$ parameter one has to parametrize the $SU(N_c)$ group by introducing $O(N_c^2)$ ‘flat’ variables subject to $O(N_c)$ constraints. In the case of $N_c = 2$ it has been explicitly performed above. For larger $N_c$ we are, unfortunately, unaware of such a parametrization.

Finally, it should be mentioned that the second derivative of the effective potential (62) in $\lambda$ is positive-definite, which may seem to invalidate the use of the saddle-point method.
In fact, it is exactly the needed sign of the curvature, since integration over the Lagrange multiplier field $\lambda$ in eq. (56) actually goes along the *imaginary* axis. The integration path climbs to the maximal value of the integrand (which happens to be exactly on the real axis!) and then descends, as it should be in the saddle-point integration.

9 A queer way to get the linear potential

Let us now consider two static colour charges in the fundamental representation on top of the vacuum state. We shall use the trial wave functional suggested by Zarembo [6]. Apart from being a functional of the $A_i$ field, it is a function of where we put the probe quark ($z_1$) and antiquark ($z_2$); it is also a colour matrix:

$$\Psi_\beta^\alpha[A; z_1, z_2] = \int DU U_\gamma^\alpha(z_1)U_\beta^\gamma(z_2) \int DA \exp \left( -\frac{Q[A]}{g^2} \right).$$ (69)

One can immediately check that under gauge transformations the wave functional (69) satisfies the needed condition (6). As correctly stressed in ref. [6] the static potential between probe quarks is anyhow a property of the vacuum state and it does not matter much what precisely trial functional do we use to ‘measure’ it, provided it satisfies the Gauss condition (6). Eq. (69) is a simple and natural choice.

By definition, the potential between static sources is the hamiltonian averaged with the wave functional (69) minus the vacuum energy:

$$V(z_1 - z_2) = \langle \Psi(z_1, z_2) | H | \Psi(z_1, z_2) \rangle - \langle \Psi_0 | H | \Psi_0 \rangle.$$ (70)

The norm of the $\Psi(z_1, z_2)$ state is [6]

$$\mathcal{N}(z_1 - z_2) = \langle \Psi(z_1, z_2) | \Psi(z_1, z_2) \rangle = \int DU \int DA \text{Tr} \left[ U(z_1)U^\dagger(z_2) \right] \exp \left( -\frac{Q[A]}{g^2} - \frac{Q[A^U]}{g^2} \right).$$ (71)

where we have again, as in the vacuum case, introduced the relative gauge transformation $U = U_1^U U_2$. Performing the same one-loop approximations as in sections 5 and 6, it is easy to see that the contributions from the $A_i$ field get cancelled in eq. (71), and we are left with the correlation functions of the sigma model. Let us introduce a short-hand notation for these correlation functions:

$$\langle \langle O \rangle \rangle = \frac{\int DU \int dx \int dy L(x-y) \text{Tr} U^\dagger(x)U(y)}{\int DU \int dx \int dy L(x-y) \text{Tr} U^\dagger(x)U(y)},$$ (72)

The static potential can be written as

$$V(z_1 - z_2) = -\frac{4}{g^2} \left[ \frac{\int dx \int dy S(x-y) \langle \langle \text{Tr} U(z_1)U^\dagger(z_2) \text{Tr} U^\dagger(x)U(y) \rangle \rangle}{\langle \langle \text{Tr} U(z_1)U^\dagger(z_2) \rangle \rangle} - \frac{\int dx \int dy S(x-y) \langle \langle \text{Tr} U^\dagger(x)U(y) \rangle \rangle}{\langle \langle 1 \rangle \rangle} \right].$$ (73)
In the leading order of the $1/n$ expansion the integrals in eqs.\eqref{72, 73} become gaussian with the propagator given by eq. \eqref{67}. There are three possible contractions of the $u_\alpha$ fields in the numerator of eq. \eqref{73}: one of them gets cancelled by vacuum subtraction in \eqref{73}; the other two are equivalent. Using the $u_\alpha$ propagator \eqref{67} we obtain

$$V(r) = \frac{E(r)}{N(r)}; \quad N(r) = 2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{ipr}}{L(p) + \mu^3} = \frac{2}{2\pi^2} \int_0^\infty dp \frac{\sin(pr)}{p} \frac{1}{L(p) + \mu^3};$$

$$E(r) = -\int \frac{d^3p}{(2\pi)^3} \frac{L^2(p)}{p^2[L(p) + \mu^3]^2} = -\frac{1}{2\pi^2} \int_0^\infty dp \frac{\sin(pr)}{p} \left(\frac{L(p)}{L(p) + \mu^3}\right)^2,$$ \hspace{1cm} (74)

Let us first investigate the short-distance behaviour of $V(r)$. Small $r$ correspond to large $p$ where $L(p) = (12/11)|p|^3$ according to eq. \eqref{72}. We have at small $r$

$$E(r) \rightarrow -\frac{1}{4\pi r}; \quad N(r) \rightarrow \frac{11}{12} \ln \frac{1}{r}, \quad V(r) \rightarrow -\frac{1}{4\pi r} \frac{8\pi^2}{27} \ln \frac{1}{r}.$$ \hspace{1cm} (75)

This is almost the correct Coulomb potential modified by the running of the gauge coupling constant. The only difference with the true asymptotics at small $r$ is the absence of the $(N_c^2 - 1)/(2N_c) = 3/4$ colour factor in eq. \eqref{73}. The difference $1 - 3/4 = 1/4$ is due to our use of the leading order of the $1/n$ expansion at $n = 4$; simultaneously it gives the idea of the numerical error.

The behaviour of $V(r)$ at large separations depends crucially on the \textit{analytic properties} of $L(p)$. The asymptotics of the Fourier transforms \eqref{74} are defined by the singularities of the integrands in $p^2$. If $L(p, \mu) + \mu^3$ has a singularity at $p^2 = 0$ (for example, is odd in $|p|$) the large-$r$ asymptotics of both $E(r)$ and $N(r)$ are determined by the integration origin at $|p| = 0$: both quantities exhibit a power decay at large $r$. Examples we have considered all show that their ratio $V(r)$ is also decreasing as a power of $r$. This is not what we want for confinement.

If $L(p, \mu) + \mu^3$ is even in $|p|$ one can expand the integration range in $|p|$ in eq. \eqref{74} from $(0, \infty)$ to $(-\infty, \infty)$. The asymptotics will be then determined by the positions of the singularities of the integrands in the $|p|$ plane. If these singularities have a real part both $E(r)$ and $N(r)$ and consequently $V(r)$ are oscillating. Probably, this is also unacceptable.

If $L(p, \mu) + \mu^3$ has zeros on the imaginary axis of $|p|$, say at $|p| = \pm i\kappa$, $E(r)$ and $N(r)$ decay exponentially as $\sim \exp(-\kappa r)$. The key point is that if the integrand of $N(r)$ has a single pole at $|p| = \pm i\kappa$, the integrand of $E(r)$ has a \textit{double} pole. Taking the residue of the double pole one can differentiate $\exp(i|p|r)$ and get an extra factor of $r$ in $E(r)$ as compared to $N(r)$. The same mathematical mechanism works if $L(p, \mu) + \mu^3$ has a cut at some imaginary value of $|p|$. This is the way one obtains an infinitely rising linear potential.

To give an example we choose

$$L(p, \mu) + \mu^3 = \frac{12}{11} \left[ p^2 + \left( \frac{11}{12} \right)^{3/2} \mu^2 \right]^{3/2},$$ \hspace{1cm} (76)

satisfying the above considerations and also the asymptotics $(12/11)|p|^3$. The coefficient inside the square brackets has been chosen so that $b(p)$ is not singular at $p = 0$. The corresponding potential $V(r)$ is plotted in Fig.1.
Both the Coulomb part (75) and the linear part are clearly seen. In mathematical terms, both $E(r)$ and $N(r)$ decay exponentially as $\sim \exp(-\text{const} \mu r)$ but $N(r)$ is additionally suppressed as $1/r$. Being invered, it gives a linear rising potential.

Let us put it in a more general way: In the original formula for the potential (70) both the numerator and the denominator have to decay exponentially if there is a mass gap in the theory. However the denominator, that is the norm of the state with colour sources $\langle \Psi(z_1, z_2) | \Psi(z_1, z_2) \rangle$, has to fall off as a factor $1/r$ faster than the average of the Hamiltonian, $\langle \Psi(z_1, z_2) | H | \Psi(z_1, z_2) \rangle$. Of course, one can always artificially normalize the wave functional to unity but that does not seem to be a natural thing to do.

We have called this way of getting confinement ‘queer’ because it is very different from the usual Euclidean standpoint where one has to demonstrate a tremendously small $\exp(-\text{Area})$ behaviour of the Wilson loop or a similar behaviour of the Polyakov’s loops correlator. This is very difficult both theoretically and in practice. However in the Schrödinger approach getting the linear potential is not at all queer but rather quite natural.

10 Adjoint sources

Let us first of all introduce the gauge transformation in the adjoint representation:

$$O^{ab}(U) = 2\text{Tr} \left[ U^\dagger t^a U t^b \right], \quad O^{ac} O^{bc} = O^{ca} O^{cb} = \delta^{ab}.$$  \hspace{1cm} (77)

We take the wave functional for a state with two adjoint sources sitting at points $z_{1,2}$ in the form similar to eq. (78):

$$\Psi^{ab}[A; z_1, z_2] = \int D U \ O^{ac}(z_1) O^{bc}(z_2) \int D A \exp \left( - \frac{Q[A]}{g^2} \right).$$  \hspace{1cm} (78)

It satisfies the needed gauge transformation law,
\[
\Psi^{ab}[A^S; z_1, z_2] = \left[ O^{-1}(S(z_1)) \right]^{aa'}^{bb'} \Psi^{a'\beta'}[A; z_1, z_2] \left[ O(S(z_2)) \right]^{b'b}.
\] (79)

Proceeding in the same fashion as in the previous section we arrive to the following expression for the potential for static adjoint charges:

\[
V^{adj}(z_1 - z_2) = -\frac{4}{g^2} \left[ \frac{\iint dx \, dy \, S(x - y) \langle \langle \left[ \text{Tr} \, U^\dagger(z_1)U(z_2)\text{Tr} \, U(z_1)U^\dagger(z_2) - 1 \right] \text{Tr} \, U^\dagger(x)U(y) \rangle \rangle}{\langle \langle \text{Tr} \, U^\dagger(z_1)U(z_2)\text{Tr} \, U(z_1)U^\dagger(z_2) - 1 \rangle \rangle} - \frac{\iint dx \, dy \, S(x - y) \langle \langle \text{Tr} \, U^\dagger(x)U(y) \rangle \rangle}{\langle \langle 1 \rangle \rangle} \right].
\] (80)

Passing from integration over unitary matrices \(U\) to the flat variables \(u_\alpha\) and the Lagrange multiplier \(\lambda\) one observes that there are much more possible contractions of \(u_\alpha\) fields than in the case of fundamental sources. Not going into details here we mention that thanks to the ‘gap’ equation (63) contributions potentially possessing the confining behaviour get cancelled. There is no linear potential between static adjoint sources even if there is such for the fundamental charges.

11 What is to be done next

We believe that the variational approach may become a powerful alternative to the Euclidean study of gluodynamics. To put it on a more solid basis one needs:

- To compute within the variational ansatz given by eqs. (12, 13, 14) the first perturbative correction to the vacuum energy, which is of the order of \(g^2 M^4\). Only the asymptotics of the trial functions \(a(p), b(p)\) at large \(|p|\) enter this calculation. The function \(a(p) = |p|\) is determined from the first-order calculation (see above), the function \(b(p)\) is not. Computing the \(O(g^2)\) correction will fix this function. There is a good chance that the best \(b(p)\) will turn out to be \(|p|/11\) as required by asymptotic freedom, eq. (65).

- To compute \(1/n\) corrections to the saddle-point approximation to the sigma model (56). It will enable one to learn, by minimizing the r.h.s. of eq. (66), the nonperturbative propagator \(L(p, \mu) + \mu^3\).

Both computations seem to be feasible. There are also obvious applications of the variational approach:

- The glueball masses and wave functions
- The \(\theta\) dependence of the vacuum energy, or the topological susceptibility of the vacuum
- The area behaviour of the spatial Wilson loop (?)
12 Conclusions

The gaussian variational ansatz for the vacuum wave functional is, on one hand, sufficiently simple to allow its study by conventional field-theoretic methods but on the other hand, it is probably sufficiently ‘rich’ to incorporate many desired features of the Yang–Mills theory.

We have shown that by including the longitudinal structure in the gaussian kernel one can, in principle, get the correct one-loop $\beta$-function of the Yang–Mills theory within the variational approach.

The famous transmutation of dimensions arises very natural in the Schrödinger picture. It results from a phenomenon similar to what is well known in $O(n)$ sigma models in two dimensions, namely from the appearance of a nonzero vacuum expectation value of the Lagrange multiplier field needed to get rid of the unitarity constraint on the gauge transformations. We have performed it explicitly for the $SU(2)$ gauge group; unfortunately we are unable to generalize the method to higher $N_c$.

The appearance of the mass gap through the transmutation of dimensions is good news, however it is not sufficient by itself to get a linear potential between source quarks. The confinement requirement can be formulated as a simple analyticity property of the propagator in the sigma model, once the Lagrange multiplier gets a nonzero v.e.v. This propagator can be systematically studied within the $1/n$ expansion, again similar to what has been developed in $d = 2$ models.

The variational approach allows certain freedom in action. Instead of working hard trying to establish the best parameters / functions within a variational ansatz, one can choose them to be what one likes, for example, taking the propagator in the form of eq. (76) leading to a beautiful static potential shown in Fig.1. After all, the variational principle says that one gets an upper bound for the energy for any trial function. Using, for example, eq. (76) it is interesting to check if it leads to reasonable glueball states, reasonable topological susceptibility, and so on.

The most challenging theoretical question is whether the variational wave functional gives rise to the area law for the spatial Wilson loop. To check it, the representation of the Wilson loop via the nonabelian Stokes theorem of ref. [13] would be probably helpful.

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