EXISTENCE AND INSTABILITY OF SOME NONTRIVIAL STEADY STATES FOR THE SKT COMPETITION MODEL WITH LARGE CROSS DIFFUSION

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Dedicated to Professor Wei-Ming Ni on the occasion of his 70th birthday

Abstract. This paper is concerned with the existence and stability of nontrivial positive steady states of Shigesada-Kawasaki-Teramoto competition model with cross diffusion under zero Neumann boundary condition. By applying the special perturbation argument based on the Lyapunov-Schmidt reduction method, we obtain the existence and the detailed asymptotic behavior of two branches of nontrivial large positive steady states for the specific shadow system when the random diffusion rate of one species is near some critical value. Further by applying the detailed spectral analysis with the special perturbation argument, we prove the spectral instability of the two local branches of nontrivial positive steady states for the limiting system. Finally, we prove the existence and instability of the two branches of nontrivial positive steady states for the original SKT cross-diffusion system when both the cross diffusion rate and random diffusion rate of one species are large enough, while the random diffusion rate of another species is near some critical value.

1. Introduction and statement of main results. In this paper we investigate the following quasilinear reaction diffusion model with cross-diffusion, which was first proposed by Shigesada-Kawasaki-Teramoto [16] for describing the segregation of two competing species under the intra- and inter-specific population pressure,

\[
\begin{align*}
  u_t &= \Delta [(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), \quad x \in \Omega, t > 0, \\
  v_t &= \Delta [(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2u - c_2v), \quad x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0, \quad x \in \partial \Omega, t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{align*}
\]

Here \(u(x, t)\) and \(v(x, t)\) represent the densities of two competing species at the location \(x\) and time \(t\), \(\Omega\) is a bounded region in \(\mathbb{R}^N\) with the smooth boundary.
and deeply investigated in \cite{8} and \cite{9}. In \cite{9} the authors also proved the related states for SKT model (1.1) with cross-diffusion in multidimensional case were widely investigated in \cite{SLEP} method. The stability/instability of such steady states was investigated in \cite{3} by applying several types of positive steady states with interior or boundary transition layers; while for the case $B < A < C$ the existence of global solution in time, the existence and the stability/instability of positive nontrivial steady states for the simplified SKT competition system \eqref{1.3} have been widely and deeply investigated by many mathematicians and ecologists. Especially for the simplified SKT competition model \eqref{1.1} with $\rho_{11} = \rho_{22} = \rho_{21} = 0$, i.e.

$$
\begin{align*}
 u_t &= d_1 \Delta u + u(a_1 - b_1 u - c_1 v), \quad x \in \Omega, \ t > 0, \\
 v_t &= d_2 \Delta v + v(a_2 - b_2 u - c_2 v), \quad x \in \Omega, \ t > 0, \\
 \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
$$

\begin{equation}
\tag{1.2}
\end{equation}

It is well known that for any fixed $d_1, d_2 > 0$ and any given nonnegative initial data, there exists a unique uniformly bounded global solution to system \eqref{1.2} and the asymptotic behaviour of the solution is nearly the same (except the case $B < A < C$) as that for Lotka-Volterra ODE competition model (tending to some constant steady state eventually). While for the case $B < A < C$, it follows from \cite{4} that if $\Omega$ is convex then system \eqref{1.2} has no stable nontrivial positive steady state.

The SKT competition model with cross diffusion (if $\rho_{12} \neq 0$ or $\rho_{21} \neq 0$) is a strongly coupled quasi-linear parabolic system, in the past three decades the SKT competition model with cross diffusion and some biological models with the SKT type of cross diffusion have attracted tremendous attention of both mathematicians and ecologists. Especially for the simplified SKT competition model \eqref{1.1} with $\rho_{11} = \rho_{22} = \rho_{21} = 0$, i.e.

$$
\begin{align*}
 u_t &= \Delta [(d_1 + \rho_{12} v)u] + u(a_1 - b_1 u - c_1 v), \quad x \in \Omega, \ t > 0, \\
 v_t &= d_2 \Delta v + v(a_2 - b_2 u - c_2 v), \quad x \in \Omega, \ t > 0, \\
 \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
$$

\begin{equation}
\tag{1.3}
\end{equation}

the existence of global solution in time, the existence and the stability/instability of steady states have been widely and deeply investigated by many mathematicians (see \cite{1}, \cite{3}, \cite{5}-\cite{15}, \cite{17}-\cite{21} and the references therein).

In this paper we shall be more interested in existence and stability analysis of some nontrivial positive steady states for the simplified SKT competition system \eqref{1.3}. Before stating our work we shall first give a brief survey on some known work on the existence and stability/instability of positive nontrivial steady states for the SKT model \eqref{1.1} or \eqref{1.3} especially when $\rho_{12}/d_1$ is large enough.

The first important theoretical work on the existence of nontrivial steady states for SKT model is due to \cite{13}, in which it was shown that in some strong competition case $B < C$ when $d_1$ and $\frac{\rho_{12}}{d_1}$ are large enough but $d_2$ is small, \eqref{1.3} admits several types of positive steady states with interior or boundary transition layers; the stability/instability of such steady states was investigated in \cite{3} by applying SLEP method.

The existence/non-existence and the priori estimates of nontrivial positive steady states for SKT model \eqref{1.1} with cross-diffusion in multidimensional case were widely and deeply investigated in \cite{8} and \cite{9}. In \cite{9} the authors also proved the related
uniform boundedness of nontrivial positive steady states and proposed three types of limiting stationary problems of (1.1) to classify all the possible asymptotic behavior of positive steady states as one of cross-diffusion, say $\rho_{12}$, tends to infinity (See Theorem 1.4 and Theorem 4.1 in [9]). By investigating the existence of steady states for some limiting systems in one dimensional case, Lou and Ni [9] also did some pioneering work on the existence of several types of spiky steady states for some related limiting systems and the original SKT competition system (1.3) when $\rho_{12}/d_1$ is large enough but $d_2$ is small.

For convenience of our later use, we shall restate the related results in [9] (See Theorem 1.4 and Theorem 4.1 in [9]) about all the possible limiting systems of the simplified SKT model (1.3) as $\rho_{12} \to \infty$ as follows.

**Theorem 1.9** Suppose that $N \leq 3$, $A \neq B$, $A \neq C$ and $\frac{a_2}{a_1}$ is not equal to any eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial \Omega$. Let $(u_i, v_i)$ be a sequence of positive nontrivial stationary solutions of (1.3) with $(d_1, \rho_{12}) = (d_{1,i}, \rho_{12,i})$ and the fixed $d_2 > 0$, then the following conclusions hold.

1. If $\rho_{12}/d_{1,i} \to \infty$ and $d_{1,i} \to d_1 \in (0, \infty)$ as $i \to +\infty$, then by passing to a subsequence if necessary, the sequence of $\{(u_i, v_i)\}$ must satisfy either (i) or (ii) below.

   (i) $(u_i, v_i)$ converges uniformly to positive $(\frac{a_1}{a_2}, \psi)$ as $i \to +\infty$, where $\psi$ is a positive constant and $(\zeta, \psi(x))$ satisfies

   $$\begin{cases}
   \int_{\Omega} \frac{1}{\psi(x)} \left( a_1 - \frac{b_1}{\psi(x)} - c_1 \psi(x) \right) dx = 0, \\
   d_2 \Delta \psi(x) + \psi(x) (a_2 - c_2 \psi(x)) - b_2 \zeta = 0, & x \in \Omega, \\
   \frac{\partial \psi}{\partial \nu} = 0, & x \in \partial \Omega.
   \end{cases}$$

   (ii) If $\|v_i\|_{L_\infty(\Omega)} \to 0$ as $i \to +\infty$, then $(u_i, \rho_{12,i}/d_{1,i})$ converges uniformly to positive $(u, w)$ as $i \to +\infty$, where $(u(x), w(x))$ is a positive solution of

   $$\begin{cases}
   d_1 \Delta [(1 + w(x))u(x)] + u(x) (a_1 - b_1 u(x)) = 0, & x \in \Omega, \\
   d_2 \Delta w(x) + w(x) (a_2 - b_2 u(x)) = 0, & x \in \Omega, \\
   \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega.
   \end{cases}$$

2. If $\rho_{12}/d_{1,i} \to \infty$ and $d_{1,i} \to \infty$ as $i \to +\infty$, then by passing to a subsequence if necessary, the sequence $\{(u_i, v_i)\}$ must satisfy either (i) or (iii) below.

   (i) $(u_i, v_i)$ converges uniformly to positive $(\frac{a_1}{a_2}, \psi)$ as $i \to +\infty$, where $\psi$ is a positive constant and $(\zeta, \psi(x))$ satisfies

   $$\begin{cases}
   \int_{\Omega} \frac{a_1}{1 + u(x)} dx - \int_{\Omega} \frac{b_1 \tau}{(1 + u(x))^2} dx = 0, \\
   d_2 \Delta w(x) + w(x) \left( a_2 - \frac{b_2 \tau}{1 + w(x)} \right) = 0, & x \in \Omega, \\
   \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega.
   \end{cases}$$

   (iii) If $\|v_i\|_{L_\infty(\Omega)} \to 0$ as $i \to +\infty$, then $(u_i, \rho_{12,i}/d_{1,i})$ converges uniformly to positive $(\frac{\tau}{1 + w(x)}, w)$, where $\tau$ is a positive constant and $(\tau, w(x))$ satisfies

   $$\begin{cases}
   \int_{\Omega} \frac{a_1}{1 + u(x)} dx - \int_{\Omega} \frac{b_1 \tau}{(1 + w(x))^2} dx = 0, \\
   d_2 \Delta w(x) + w(x) \left( a_2 - \frac{b_2 \tau}{1 + w(x)} \right) = 0, & x \in \Omega, \\
   \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega.
   \end{cases}$$

Since the work of Lou and Ni [9], the first limiting system (1.4) (a shadow system of (1.3)) has been deeply and widely investigated by some mathematicians. For the one dimensional case with $\Omega = (0, 1)$, Lou, Ni and Yotsutani [11] obtained some nearly optimal results on the existence and non-existence of positive steady states.
for the first limiting system (1.4), which can be briefly summarized as follows (see Figure 1):

If \( d_2 \geq \frac{a_2}{\pi^2} \), then the limiting system (1.4) does not have any non-constant solutions; while for \( 0 < d_2 < \frac{a_2}{\pi^2} \), if \( A < B < C \), or if \( C < B \) and \( A < \frac{B + 3C}{4} \), then the limiting system (1.4) has no non-trivial positive steady states; see also the region filled with slashes in Figure 1 in which the existence of positive monotone steady states are proved.

**Figure 1.** (a): \( B < C \) i.e. strong competition; (b): \( B > C \) i.e. weak competition.

For the case 1 in Figure 1, it was shown in [9] that there exist positive steady states with boundary spike layer near the positive constant steady state \((u^*, v^*)\) when \( d_2 \) is small enough (see Figure 2(a)). The instability of such spiky steady states was proved in [18].

For the case 2 in Figure 1, the existence and the spiky structure of another type of positive steady states with boundary spike layers \((\zeta_{d_2}, \psi_{d_2}(x))\) for the limiting system (1.4) were investigated in [11] when \( d_2 \) is small enough. In [19] for any \( A > \frac{B + 3C}{4} \) by applying different approach the authors also proved the existence
and the detailed structure of the spiky steady states \((\zeta_d, \psi_d(x))\) for the limiting system (1.4) and existence of the corresponding large spiky steady states \((u(x), v(x))\) perturbed from \((\zeta_d, \psi_d(x), \psi_d(x))\) for the original SKT competition model with large \(\rho_{12}\) and small \(d_2\) (see Figure 2(b)).

For the case \(3\) in Figure 1, in [11] it was proved that for any \(A > B\) the limiting system (1.4) has a type of nontrivial positive steady states \((\zeta_d, \psi_d(x))\) with singular bifurcation structure when \(d_2\) is near \(a_2/\pi^2\), where \((\zeta_d, \psi_d(x)) \rightarrow (0, 0)\) and \(\frac{\xi_d}{\psi_d(x)} \rightarrow \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - B/A \cos(\pi x)}}\) as \(d_2 \rightarrow a_2/\pi^2\). The existence and the stability of the steady states perturbed from \((\xi_d, \psi_d(x), \psi_d(x))\) for the original SKT competition system were proved in [15] (see Figure 2(c)). Similar existence and stability results for the shadow system (1.4) and for the original SKT model in multidimensional case were proved in [12].

As far as we know, there are only a few works on the limiting systems (1.5) and (1.6), where the \(v\) component of the steady state to the original SKT system (1.1) is assumed to tend to zero but \(\rho_{12} v(x)\) is positive and bounded as \(\rho_{12}\) tends to infinity. The first theoretical work based on the investigation of the third limiting system (1.6) is due to Lou and Ni [9], in which for any \(A > B\) and small enough \(d_2 > 0\) the authors proved the existence of positive spiky steady states for the limiting system (1.6) and the existence of the corresponding spiky steady states near \((a_1, b_1, 0)\) for the SKT model when both \(d_1\) and \(\rho_{12}/d_1\) are large enough but \(d_2\) is small enough. Such spiky steady states to the limiting system and the original SKT model were proved to be unstable in [17].

Focusing on the second limiting system (1.5) with the fixed \(d_1\), recently K. Kuto [5] proved the existence of a local branch of positive steady states \((u(x, a_2), w(x, a_2))\) with bifurcating structure in multi-dimensional case when \(a_2\) is near \(a_1 b_2/b_1\); and proved the existence of a global branch of positive steady state \((u(x, a_2), w(x, a_2))\) in one dimensional case with \(\Omega = (0, 1)\) for any \(a_2 \in (d_2 \pi^2, a_1 b_2/b_1)\), and the global branch of steady states in one dimensional case is proved to having special blowing up structure as \(a_2 \rightarrow d_2 \pi^2\), precisely speaking \(w(x, a_2) \rightarrow +\infty\) but \(u(x, a_2) \rightarrow \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - B/A \cos(\pi x)}}\) as \(a_2 \rightarrow d_2 \pi^2\). Recently the local bifurcating steady states to the limiting system (1.5) were proved to be unstable in [7], in which it is also proved that there exists a local branch of the corresponding unstable positive steady states to the original SKT model (1.1) when \(\rho_{12}\) is large enough.

In [5] it is remarked that in one dimensional case with \(\Omega = (0, 1)\) as \(d_2 \rightarrow \frac{a_2}{\pi^2}\) the \(w\) component of the nontrivial positive solution \((u(x), w(x))\) to the limiting system (1.5) tends to infinity; while the \(u\) component tends to a bounded positive function, which is the same limit of \(u\) component of the another branch of steady states perturbed from the first limiting system (1.4) as \(d_2 \rightarrow \frac{a_2}{\pi^2}\). Motivated by the work of [5], [12] and [15] on the steady states of the limiting systems (1.4) and (1.5) in one dimensional or multidimensional cases when \(d_2\) is near \(a_2/\lambda_1\) \((\lambda_1\) is the second eigenvalue of \(-\Delta\) under the homogeneous Neumann boundary condition), in this paper we focus on existence and stability analysis of nontrivial positive steady states to the third limiting system (1.6) and the SKT competition system (1.3) when both \(d_1\) and \(\rho_{12}/d_1\) are large enough but \(d_2\) is near \(a_2/\lambda_1\). It is natural to guess that there may also exist some local branches of steady states \((\tau_d, w_d(x))\) for the third limiting system when \(d_2\) is near \(a_2/\lambda_1\) and which may have similar or different blowing up structure (or bifurcating from infinity) from the recent work in [5] on the second limiting system (1.5), and in this paper we shall apply different approach
to investigate the existence and the stability as well as the detailed asymptotic structure of two local branches of blowing up steady states to the limiting system (1.6) and to the original SKT model (1.3) when $d_2$ is near $a_2/\lambda_1$.

Before stating our main results, we shall first give a brief derivation of the third limiting system (1.6) and the corresponding evolutional system, which is the limiting system of (1.3) when the $v$ component tends to zero but the limit of $\frac{\partial v}{\partial t}$ is positive and bounded as both $\frac{\partial v}{\partial t}$ and $d_1$ approach the infinity.

Denote $\alpha = \frac{\rho_{12} d_1}{\alpha}$ and set $w = \alpha v$, $\phi = (1 + \alpha v)u$, then system (1.3) can be rewritten as

\[
\begin{cases}
\left(\frac{\phi}{1 + w}\right)_t = d_1 \Delta \phi + \phi \left(a_1 - b_1 \frac{\phi}{1 + w} - \frac{1}{\alpha} c_1 \phi\right), & x \in \Omega, \ t > 0, \\
w_t = d_2 \Delta w + w \left(a_2 - b_2 \frac{\phi}{1 + w} - \frac{1}{\alpha} c_2 \phi\right), & x \in \Omega, \ t > 0,
\end{cases}
\]

(1.7)

Introducing $s = \frac{1}{d_1}$ and $r = \frac{1}{\alpha}$, then (1.7) becomes the following system

\[
\begin{cases}
\left(\frac{\phi}{1 + w}\right)_t = \Delta \phi + s \frac{\phi}{1 + w} \left(a_1 - b_1 \frac{\phi}{1 + w} - rc_1 \phi\right), & x \in \Omega, \ t > 0, \\
w_t = d_2 \Delta w + w \left(a_2 - b_2 \frac{\phi}{1 + w} - \frac{1}{\alpha} c_2 \phi\right), & x \in \Omega, \ t > 0,
\end{cases}
\]

(1.8)

Let $r \to 0^+$, then system (1.8) can be reduced to the following system

\[
\begin{cases}
\left(\frac{\phi}{1 + w}\right)_t = \Delta \phi + s \frac{\phi}{1 + w} \left(a_1 - b_1 \frac{\phi}{1 + w}\right), & x \in \Omega, \ t > 0, \\
w_t = d_2 \Delta w + w \left(a_2 - b_2 \frac{\phi}{1 + w}\right), & x \in \Omega, \ t > 0,
\end{cases}
\]

(1.9)

Further we consider the limiting problem of (1.9) as $s \to 0^+$. If we assume that all the quantities in the first equation of (1.9) remain bounded as $s \to 0^+$, equivalently if $u(x), \alpha v(x)$ are bounded as $d_1, \alpha \to \infty$, then $\phi(x,t) \to \tau(t)$ as $s \to 0^+$ for $x \in \Omega$ due to the boundary condition. It is evident that $(\tau(t), w(x,t))$ satisfies the following limiting system

\[
\begin{cases}
\int_{\Omega} \left(\frac{\tau}{1 + w}\right)_t \mathrm{d}x = \int_{\Omega} \frac{\tau}{1 + w} \left(a_1 - b_1 \frac{\tau}{1 + w}\right) \mathrm{d}x, & t > 0, \\
w_t = d_2 \Delta w + w \left(a_2 - b_2 \frac{\tau}{1 + w}\right), & x \in \Omega, \ t > 0,
\end{cases}
\]

(1.10)
Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ be the eigenvalues of the following eigenvalue problem
\[
\begin{cases}
-\Delta \varphi(x) = \lambda \varphi(x), & x \in \Omega, \\
\frac{\partial \varphi}{\partial \nu} = 0, & x \in \partial \Omega,
\end{cases}
\]
and let $\varphi_0(x), \varphi_1(x), \cdots, \varphi_i(x), \cdots$ be the corresponding normalized eigenfunction with $\min_{\Omega} \varphi_i(x) = -1$.

In one dimensional case with $\Omega = (0, 1)$, it is well known that all the eigenvalues of (1.11) are simple, and $\lambda_1 = \pi^2$ with a sign changing eigenfunction $\varphi_1(x) = \cos(\pi x)$. However, in multidimensional case for $j \geq 1 \lambda_j$ may be not simple. In the following, we assume that $A > B$ and denote $\mu_{+}$ to be the roots of $g(\mu) \triangleq \int_\Omega (1 + \mu \varphi_1)^{-2} \, dx = \frac{A}{B} \mu_+^{2}$ with $\mu_+ > \mu_-$ and $1 + \mu_\pm \varphi_1(x) > 0$ for any $x \in \Omega$, for the 1-dim case $\Omega = (0, 1)$ by simple computation it can be verified that $\mu_\pm = \pm \sqrt{1 - \frac{B}{A}}$. For multi-dimensional case the existence of the $\mu_\pm$ and the properties of the functions $(1 + \mu_\pm \varphi_1(x))$ and $g(\mu)$ will be stated in Lemma 2.3 (see also Corollary 1 in [12]). The values $\mu_+ \text{ and } \mu_- \text{ enable us to precisely estimate the limiting profile of the ratio } \frac{\tau_i}{u_i}$ \text{ of the steady states to the limiting system (1.10)} as $d_2 \to a_2 / \lambda_1$, which with the non-degeneracy of $g(\mu)$ at $\mu = \mu_{\pm}$ are also useful in proving the existence and the instability of the perturbed nontrivial positive steady states of (1.10) when $d_2$ is near $\frac{a_2}{\lambda_1}$.

Now we state our main results on the existence and instability of nontrivial steady states to the limiting system (1.10).

**Theorem 1.1.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple, then there exists $\delta_0 > 0$ such that for each fixed $d_2 \in (a_2 / \lambda_1 - \delta_0, a_2 / \lambda_1)$, system (1.10) has two nontrivial positive solutions $(\tau_-^{d_2}, w_{\tau_-^{d_2}}^{d_2}(x))$ satisfying
\[
\lim_{d_2 \to \lambda_1 \delta_0} \left( \frac{a_2}{\lambda_1} - d_2 \right) \left( \frac{\tau_{d_2}^{\pm}}{\lambda_1} \right) = k_\pm \left( \frac{b_2}{a_2} (1 + \mu_\pm \varphi_1(x)) \right),
\]
with $k_\pm = \frac{a_2^2}{b_2^2} \frac{\int_\Omega \frac{w_-^{d_2}}{1 + \mu_\pm \varphi_1} \, dx}{\int_\Omega \frac{\varphi_1}{1 + \mu_\pm \varphi_1} \, dx} > 0$.

**Theorem 1.2.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple. If $\frac{a_2}{\lambda_1} - d_2 > 0$ is small enough, then the nontrivial positive steady states $(\tau_{d_2}^{\pm}, w_{\tau_{d_2}^{\pm}}^{d_2}(x))$ of the limiting system (1.10) obtained in Theorem 1.1 are spectrally unstable in $\mathbb{R} \times H^1(\Omega)$.

By virtue of the detailed asymptotic structure and the spectral results obtained for the limiting system (1.10), by applying perturbation argument we can further prove the existence and the instability of the perturbed steady states to the original SKT model (1.3) when both $d_1$ and $\rho_{12} / d_1$ are large enough.

Our results for the original SKT model (1.3) are stated as follows.

**Theorem 1.3.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple. Suppose that $\frac{a_2}{\lambda_1} - d_2 > 0$ is small enough. There exists large $\bar{d} > 0$ such that if $d_1 \geq \bar{d}$ and $\frac{a_2}{\lambda_1} \geq \bar{d}$, the SKT cross-diffusion system (1.3) has two types of nontrivial positive steady states $(u(x), v(x)) \text{ satisfying } (u(x), \frac{\partial}{\partial t} u(x)) \to \left( \frac{\tau_{d_2}^{\pm}}{1 + w_{\tau_{d_2}^{\pm}}^{d_2}(x)}, \frac{w_{\tau_{d_2}^{\pm}}^{d_2}(x)}{1 + w_{\tau_{d_2}^{\pm}}^{d_2}(x)} \right) \text{ (resp.) uniformly in } \bar{d} \text{ as } d_1, \rho_{12} / d_1 \to \infty$, with $(\tau_{d_2}^{\pm}, w_{\tau_{d_2}^{\pm}}^{d_2}(x))$ obtained in Theorem 1.1.
**Theorem 1.4.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple. Suppose that $\frac{a_2}{\lambda_1} - d_2 > 0$ is small enough. There exists a large $\bar{d} > 0$ such that if $d_1 \geq \bar{d}$ and $\frac{a_2}{d_1} \geq \bar{d}$, the nontrivial positive steady states $(u(x), v(x))$ of the SKT competition system (1.3) obtained in Theorem 1.3 are unstable in $H^1(\Omega) \times H^1(\Omega)$.

The present paper is organized as follows. In Section 2, we investigate the existence and the stability of two local branches of blowing up steady states to the limiting system (1.6) and give the detailed proofs of Theorems 1.1 and 1.2. The existence and instability of the nontrivial positive steady states to the original cross-diffusion system (1.3) will be investigated in Section 3.

2. Existence and instability of nontrivial positive steady states to the limiting system (1.6).

2.1. Existence of nontrivial positive steady states to the limiting system (1.6). In this subsection, we first state some preliminary results on the non-existence of nontrivial positive steady states to the limiting system (1.10).

**Lemma 2.1.** If $A < B$, then the limiting system (1.10) has no positive steady states.

**Lemma 2.2.** If $d_2 \geq \frac{a_2}{\lambda_1}$, then the limiting system (1.10) has no nontrivial positive steady states.

Lemma 2.1 can be proved by nearly the same argument as in [5] and Lemma 2.2 can be similarly proved as in [11], here we omit the details of the proofs.

Lemmas 2.1 and 2.2 imply that to get the existence of nontrivial positive solutions to the limiting system (1.10), it is necessary to consider the case when $0 < d_2 < \frac{a_2}{\lambda_1}$ and $A > B$. In the following of this subsection we shall investigate the existence and the detailed structure of some nontrivial positive steady states to the limiting system (1.10) when $a_2 - d_2 \lambda_1 > 0$ is small enough. Let $(\tau, w(x))$ be a nontrivial positive steady state of system (1.10), it evidently satisfies the limiting system (1.6), that is

\[
\begin{align*}
\int_{\Omega} \frac{a_1}{1 + w(x)} \, dx - \int_{\Omega} \frac{b_1 \tau}{(1 + w(x))^2} \, dx &= 0, \\
d_2 \Delta w(x) + w(x) \left( a_2 - b_2 \frac{\tau}{1 + w(x)} \right) &= 0, \quad x \in \Omega, \\
\partial w / \partial \nu &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

(2.1)

To locate the precise asymptotic behavior of blowing-up steady states to the limiting system (2.1) as $d_2 \to a_2 / \lambda_1$, we convert the blowing-up solutions of (2.1) to the bounded solutions by some suitable transformation.

For each fixed small $\varepsilon = \frac{a_2}{\lambda_1} - d_2 > 0$, let $(\tau, w(x))$ be a positive steady state of system (2.1), and denote

\[
\tilde{\tau} = \varepsilon \tau, \quad \tilde{w}(x) = \varepsilon w(x), \quad d_2 = \frac{a_2}{\lambda_1} - \varepsilon.
\]

Multiplying the first equation of (2.1) by $\tau$, then $(\tilde{\tau}, \tilde{w}(x))$ satisfies the system

\[
\mathcal{H}(\varepsilon, \tilde{\tau}, \tilde{w}) = 0,
\]

(2.3)
with \( \mathcal{H} : (0, \delta) \times \mathbb{R} \times H^2_\tau(\Omega) \to \mathbb{R} \times L^2_\tau(\Omega) \) \((H^2_\tau(\Omega)) \triangleq \{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0 \} \)
defined by

\[
\mathcal{H}(\varepsilon, \tilde{\tau}, \tilde{w}(x)) = \left( \int_\Omega \frac{a_1 \tilde{\tau}}{\varepsilon + \tilde{w}(x)} \, dx - \int_\Omega \frac{b_1 \tilde{\tau}^2}{(\varepsilon + \tilde{w}(x))^2} \, dx, \int_\Omega \left( \frac{a_2}{\lambda_1} - \varepsilon \right) \Delta \tilde{w}(x) + \tilde{w}(x) \left( a_2 \frac{\tilde{\tau}}{\varepsilon + \tilde{w}(x)} \right) \right),
\]
for small \( \delta > 0 \).

Let \( \varepsilon = 0 \), then system (2.3) becomes the following reduced system

\[
\begin{aligned}
&\int_\Omega \frac{a_1 \tilde{\tau}}{\tilde{w}(x)} \, dx - \int_\Omega \frac{b_1 \tilde{\tau}^2}{\tilde{w}^2(x)} \, dx = 0, \\
&\left( \frac{a_2}{\lambda_1} \Delta \tilde{w}(x) + a_2 \tilde{w}(x) - b_2 \tilde{\tau} = 0, \right. \\
&\left. \frac{\partial \tilde{w}}{\partial \nu} = 0, \right. \\
&\left. x \in \Omega, x \in \partial \Omega. \right.
\end{aligned}
\]

Solving the boundary problem of \( \tilde{w}(x) \) in (2.4), we have

\[
\tilde{w}(x) = \frac{b_2}{a_2} (1 + \mu \varphi_1(x)) \tilde{\tau},
\]
with constant \( \mu \neq 0 \) to be determined later.

Substituting (2.5) into the first equation of (2.4), then solving the limiting system (2.4) is deduced to finding a constant \( \mu \) satisfying

\[
\int_\Omega a_1 \tilde{\tau} \, dx - \int_\Omega b_1 a_2 \tilde{\tau} \frac{1}{(1 + \mu \varphi_1)^2} \, dx = 0,
\]
which is equivalent to solving \( g(\mu) = \frac{A}{B} \) with

\[
g(\mu) \triangleq \frac{\int_\Omega (1 + \mu \varphi_1)^{-2} \, dx}{\int_\Omega (1 + \mu \varphi_1)^{-1} \, dx} \quad \text{for} \quad \mu \in \left( \frac{1}{\max_{\Omega} \varphi_1}, \frac{1}{\min_{\Omega} \varphi_1} \right) = I_0.
\]

In one dimensional case e.g. \( \Omega = (0, 1) \), by some simple computation it can be proved that \( g(\mu) = \frac{A}{B} \) has exactly two roots \( \mu = \pm \sqrt{\frac{1}{4} - \frac{B}{A}} \), which also proves that the limiting system (2.4) has precisely two families of positive solutions \( \tilde{\tau}(1, l_\pm(x)) \) for any \( \tilde{\tau} > 0 \) with \( l_\pm(x) = 1 \pm \sqrt{1 - \frac{B}{A} \cos(\pi x)} > 0 \), where \( l_+(x) \) is a positive decreasing function in \((0, 1)\) and \( l_-(x) \) is a positive increasing function.

For multidimensional domain \( \Omega \), the existence of the roots of \( g(\mu) \) is deeply investigated and proved in [12], we restate some basic estimates as follows.

**Lemma 2.3.** [12] Define \( g(\mu) \) as in (2.7) and suppose that \( N \leq 4 \) and \( A > B \). Then \( g(\mu) > g(0) = 1 \) for \( \mu \neq 0 \) and \( g(\mu) = \frac{A}{B} \) has exactly one positive and one negative roots, denoted by \( \mu_+ \) and \( \mu_- \) respectively. Moreover, \( g'(\mu_+) > 0 \) and \( g'(\mu_-) < 0 \).

**Remark 1.** In [12] it is proved that \( \mu g'(\mu) > 0 \) for \( \mu \neq 0 \), due to the fact that

\[
\begin{aligned}
\mu g'(\mu) &= \left( \int_\Omega \frac{dx}{1 + \mu \varphi_1} \right)^{-2} \left[ - \left( \int_\Omega \frac{dx}{(1 + \mu \varphi_1)^2} \right)^2 + 2 \int_\Omega \frac{dx}{(1 + \mu \varphi_1)^2} \int_\Omega \frac{dx}{1 + \mu \varphi_1} \right. \\
&\quad \left. - \int_\Omega \frac{dx}{(1 + \mu \varphi_1)^2} \int_\Omega \frac{dx}{1 + \mu \varphi_1} \right] \\
&\geq \left( \int_\Omega \frac{dx}{1 + \mu \varphi_1} \right)^{-2} \left[ 2 \int_\Omega \frac{dx}{(1 + \mu \varphi_1)^2} \int_\Omega \frac{dx}{1 + \mu \varphi_1} - 2 \left( \int_\Omega \frac{dx}{(1 + \mu \varphi_1)^2} \right)^2 \right].
\end{aligned}
\]
Here it should be noted that
\[ g'(\mu_{\pm}) = \frac{b_2}{a_2^2} \left( \int_{\Omega} \frac{dx}{l(x)} \right)^{-2} \left[ -2 \int_{\Omega} \frac{\varphi_1(x)dx}{l^3(x)} \int_{\Omega} \frac{dx}{l(x)} + \int_{\Omega} \frac{\varphi_1(x)dx}{l^2(x)} \right]. \]

The existence of positive functions \( l(x) = \frac{b_2}{a_2^2} (1 + \mu_\pm \varphi_1(x)) \) and the nonzero of \( g'(\mu_{\pm}) \) will be useful in our later investigation of existence and stability of the nontrivial positive steady states.

The explicit value of \( \mu_{\pm} \) in one-dimensional case and the properties of \( g'(\mu) \) stated in Lemma 2.3 also imply that system (2.4) has precisely two families of positive nontrivial solutions \((k, kl(x))\) for any \( k > 0 \) with \( l(x) = \frac{b_2}{a_2} (1 + \mu_\pm \varphi_1(x)) > 0 \), thus
\[ \mathcal{H}(0, k, kl(x)) = 0, \; \text{for any} \; k > 0. \]

Denote \( \hat{\mathcal{L}}_{\alpha_1} \) be the linearized operator of (2.4) around \((k, kl(x))\) with
\[
\hat{\mathcal{L}}_{\alpha_1} \left( \frac{\xi}{\psi(x)} \right) = \left( -\frac{\xi}{\int_{\Omega} \frac{dx}{l(x)}} \int_{\Omega} \frac{\varphi_1(x)dx}{l^2(x)} \int_{\Omega} \frac{dx}{l(x)} + \int_{\Omega} \frac{2b_1 \varphi_1(x)dx}{l^3(x)} \right),
\]

it can be checked later that zero is an eigenvalue of \( \hat{\mathcal{L}}_{\alpha_1} \) for any \( k > 0 \) with an eigenvector \((1, l(x))\).

In the following, we shall prove that for small \( \frac{a_2}{\alpha_1} - d_2 > 0 \) there exist positive nontrivial steady states of (2.3) perturbed from \((k_0, k_0l(x))\) for some specific positive constant \( k_0 > 0 \). Before proving the existence of the positive steady states of (2.3) for small \( \frac{a_2}{\alpha_1} - d_2 > 0 \), we shall firstly prove that the zero eigenvalue of \( \hat{\mathcal{L}}_{\alpha_1} \) is simple, such that the Lyapunov-Schmidt reduction method can be applied in our later proof of existence and stability analysis.

**Lemma 2.4.** Assume \( A > B, N \leq 4 \) and \( \lambda_1 \) is simple, let \( l(x) = \frac{b_2}{a_2} (1 + \mu_+ \varphi_1(x)) \) or \( l(x) = \frac{b_2}{a_2} (1 + \mu_- \varphi_1(x)) \) and define linear operator \( \mathcal{L}_{\alpha_1} : \mathbb{R} \times H^2_0(\Omega) \to \mathbb{R} \times L_2(\Omega) \) by
\[
\mathcal{L}_{\alpha_1} \left( \frac{\xi}{\psi(x)} \right) = \left( -\frac{\xi}{\int_{\Omega} \frac{dx}{l(x)}} \int_{\Omega} \frac{\varphi_1(x)dx}{l^2(x)} \int_{\Omega} \frac{dx}{l(x)} + \int_{\Omega} \frac{2b_1 \varphi_1(x)dx}{l^3(x)} \right),
\]
and let \( \mathcal{L}_{\alpha_1}^* : \mathbb{R} \times H^2_0 \to \mathbb{R} \times L_2(\Omega) \) be the adjoint operator of \( \mathcal{L}_{\alpha_1} \) with
\[
\mathcal{L}_{\alpha_1}^* \left( \frac{\tau}{\phi(x)} \right) = \left( -\tau \int_{\Omega} \frac{b_1 dx}{l^2(x)} - b_2 \int_{\Omega} \varphi(x) dx \right), \; \left( -\frac{a_2}{\lambda_1} \Delta \varphi(x) + a_2 \varphi(x) - b_2 \xi \right),
\]
respectively, then zero is a simple eigenvalue of \( \mathcal{L}_{\alpha_1} \) and \( \mathcal{L}_{\alpha_1}^* \) with
\[
\text{Ker}(\mathcal{L}_{\alpha_1}) = \text{Ker}(\mathcal{L}_{\alpha_1}^*) = \text{span}\{(1, l(x))^{\top}\},
\]
and
\[
\text{Ker}(\mathcal{L}_{\alpha_1}^*) = \text{Ker}(\mathcal{L}_{\alpha_1}) = \text{span}\{(0, \varphi_1(x))^{\top}\},
\]
respectively.
Proof. Firstly, we prove that zero is an eigenvalue, where the eigenfunction \((\xi, \psi(x))\) satisfies
\[
\begin{cases}
- \int_\Omega a_1 \psi(x) \, dx + \int_\Omega \frac{2b_1 \psi(x)}{l^2(x)} \, dx - \xi \int_\Omega \frac{b_1}{l^2(x)} \, dx = 0, \\
a_2 \Delta \psi(x) + a_2 \psi(x) - b_2 \xi = 0, \\
\frac{\partial \psi}{\partial \nu} = 0,
\end{cases}
\]
x \in \Omega, \quad (2.9)
\]
Solving the boundary value problem in (2.9), we have
\[
\psi(x) = \frac{b_2}{a_2} \xi (K \varphi_1(x) + 1), \quad \text{with } K \neq 0 \text{ a constant to be determined.} \quad (2.10)
\]
Substituting (2.10) into the first equation of (2.9) yields
\[
- \frac{b_2 a_1}{a_2} \int_\Omega \frac{(K \varphi_1(x) + 1)}{l^2(x)} \, dx + \frac{2b_1 b_2}{a_2} \int_\Omega \frac{(K \varphi_1(x) + 1)}{l^3(x)} \, dx - \int_\Omega \frac{b_1}{l^2(x)} \, dx = 0. \quad (2.11)
\]
Define
\[
f(K) \triangleq - \frac{b_2 a_1}{a_2} \int_\Omega \frac{(K \varphi_1(x) + 1)}{l^2(x)} \, dx + \frac{2b_1 b_2}{a_2} \int_\Omega \frac{(K \varphi_1(x) + 1)}{l^3(x)} \, dx - \int_\Omega \frac{b_1}{l^2(x)} \, dx.
\]
Note that \(f(K)\) is linear in \(K\) and by Lemma 2.3 and (2.8), we have
\[
f(\mu_{\pm}) = -a_1 \int_\Omega \frac{b_2}{a_2} (1 + \mu_{\pm} \varphi_1(x)) \, dx + 2b_1 \int_\Omega \frac{b_2}{a_2} (1 + \mu_{\pm} \varphi_1(x)) \, dx - \int_\Omega \frac{b_1}{l^2(x)} \, dx
\]
with \(l(x) = \frac{b_2}{a_2} (1 + \mu_{\pm} \varphi_1(x))\) resp. and
\[
f'(\mu_{\pm}) = - \frac{b_2}{a_2} \left[ \int_\Omega \frac{a_1 \varphi_1(x)}{l^2(x)} \, dx - \int_\Omega \frac{2b_1 \varphi_1(x)}{l^3(x)} \, dx \right]
\]
\[
= - \frac{b_2}{a_2} \left( \int_\Omega \frac{dx}{l(x)} \right)^{-1} \left[ b_1 \int_\Omega \varphi_1(x) l^2(x) \, dx \int_\Omega \frac{1}{l^2(x)} \, dx - 2b_1 \int_\Omega \varphi_1(x) l^3(x) \, dx \int_\Omega \frac{1}{l(x)} \, dx \right]
\]
\[
= - \frac{b_1 a_2}{b_2} \left( \int_\Omega \frac{dx}{l(x)} \right) g'(\mu_{\pm}) \neq 0,
\]
which proves that \(f(K) = 0\) has precisely one root \(K = \mu_{\pm}\) resp. for \(l(x) = \frac{b_2}{a_2} (1 + \mu_{\pm} \varphi_1(x))\) resp. Hence, zero is an eigenvalue of \(\mathcal{L}_{x_1}^*\) and \(\tilde{\mathcal{L}}_{x_1}^*\) with one dimensional kernel spanned by \((1 , l(x))^T\).

Note that \(\mathcal{L}_{x_1}^*\) is a Fredholm operator with index zero on the Hilbert space \(\mathbb{R} \times H_0^2(\Omega),\) thus \(\mathcal{L}_{x_1}^*\) also has one dimensional kernel. It can be checked that
\[
\ker(\mathcal{L}_{x_1}^*) = \ker(\tilde{\mathcal{L}}_{x_1}^*) = \text{span}\{0 , \varphi_1(T)\},
\]
and
\[
\left\langle \begin{pmatrix} 1 \\ l(x) \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_1(x) \end{pmatrix} \right\rangle = \int_\Omega l(x) \varphi_1(x) \, dx \neq 0, \quad (2.12)
\]
which further implies that \(\ker(\mathcal{L}_{x_1}^*) = \ker(\tilde{\mathcal{L}}_{x_1}^*)\), thus zero is a simple eigenvalue of \(\mathcal{L}_{x_1}^*\) and \(\tilde{\mathcal{L}}_{x_1}^*\). \(\square\)
In the following of this subsection, we shall focus on the existence of the bounded positive solution $\left(\tilde{\tau}, \tilde{w}(x)\right)$ perturbed from some $\left(k, k(t)\right)$ for system \eqref{2.3}, which corresponds to the existence of two types of solutions $\left(\tau, w(x)\right)$ to the limiting system \eqref{2.1} for small $\varepsilon > 0$.

Lemma 2.4 and \eqref{2.12} imply that $X = \mathbb{R} \times H^2_\varepsilon$, and $Y = \mathbb{R} \times L^2$ have the following direct decomposition

$$Y = \text{Ker}(\mathcal{L}_{\frac{2}{\varepsilon}^1}) \oplus \text{Range}(\mathcal{L}_{\frac{2}{\varepsilon}^1}), \quad X = \text{Ker}(\mathcal{L}_{\frac{2}{\varepsilon}^1}) \oplus X_R,$$

and

$$X_R = \text{Range}(\mathcal{L}_{\frac{2}{\varepsilon}^1}) \cap X,$$  \quad \text{(2.13)}

with $\oplus$ a direct sum in $Y$.

Thus we can define a projection $Q$ on $Y$ satisfying

$$QY = \text{Range}(\mathcal{L}_{\frac{2}{\varepsilon}^1}) = \text{Ker}(\mathcal{L}_{\frac{2}{\varepsilon}^1})^\perp, \quad (I - Q)Y = \text{Ker}(\mathcal{L}_{\frac{2}{\varepsilon}^1}).$$

In particular, the projection $I - Q : Y \rightarrow \text{Ker}(\mathcal{L}_{\frac{2}{\varepsilon}^1})$ can be expressed as

$$\begin{pmatrix} (I - Q)(u) \\ v \end{pmatrix} = \frac{\left\langle (u(x), v(x))^\top, (0, \varphi_1(x))^\top \right\rangle}{\left\langle (1, l(x))^\top, (0, \varphi_1(x))^\top \right\rangle} \begin{pmatrix} 1 \\ l(x) \end{pmatrix}.$$  \quad \text{(2.14)}

**Theorem 2.5.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple, then there exists a small $\delta_0 > 0$ such that for any fixed $\varepsilon \in (0, \delta_0)$, system \eqref{2.3} has two nontrivial positive steady states $\left(\tilde{\tau}_\varepsilon, \tilde{w}_\varepsilon(x)\right)$ satisfying

$$\lim_{\varepsilon \rightarrow 0} \begin{pmatrix} \tilde{\tau}_\varepsilon \\ \tilde{w}_\varepsilon(x) \end{pmatrix} = k_0 \begin{pmatrix} 1 \\ l(x) \end{pmatrix}, \quad \text{with} \quad k_0 = \frac{a_1^2}{b_2 \lambda_1} \frac{\int_{\Omega} \varphi_1^2 \, dx}{\int_{\Omega} \phi_1^2 \, dx} \quad \text{resp.} \quad \text{(2.15)}$$

**Proof.** Let

$$\begin{pmatrix} \tilde{\tau}_\varepsilon \\ \tilde{w}_\varepsilon(x) \end{pmatrix} = k_\varepsilon \begin{pmatrix} 1 \\ l(x) \end{pmatrix} + \varepsilon \begin{pmatrix} \tilde{\tau}_{1,\varepsilon} \\ \tilde{w}_{1,\varepsilon}(x) \end{pmatrix},$$  \quad \text{(2.16)}

be the solution of \eqref{2.3} for small $\varepsilon > 0$, with $\left(\tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)\right) \in \text{Range}(\mathcal{L}_{\frac{2}{\varepsilon}^1}) = \left\{ (0, \varphi_1(x)) \right\}^\perp$.

Substituting \eqref{2.16} into \eqref{2.3} and using $\mathcal{H}(0, k_\varepsilon, k_\varepsilon l(x)) = 0, \forall k_\varepsilon \in \mathbb{C}$, it follows that

$$0 = \mathcal{H}(\varepsilon, \tilde{\tau}_\varepsilon, \tilde{w}_\varepsilon) = \mathcal{H}(\varepsilon, k_\varepsilon + \varepsilon \tilde{\tau}_{1,\varepsilon}, k_\varepsilon l(x) + \varepsilon \tilde{w}_{1,\varepsilon}(x))$$

$$= \mathcal{H}(\varepsilon, k_\varepsilon + \varepsilon \tilde{\tau}_{1,\varepsilon}, k_\varepsilon l(x) + \varepsilon \tilde{w}_{1,\varepsilon}(x)) - \mathcal{H}(0, k_\varepsilon, k_\varepsilon l(x))$$

$$= \varepsilon \mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)),$$

where for small $\varepsilon > 0$ it holds that

$$\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = \left( \begin{pmatrix} \int_{\Omega} \frac{a_1 \tilde{\tau}_{1,\varepsilon}(x) + a_1 \tilde{w}_{1,\varepsilon}(x)}{k_\varepsilon l(x)} \, dx - \int_{\Omega} \frac{2 b_1 \tilde{\tau}_{1,\varepsilon}(x) + 1 - \tilde{w}_{1,\varepsilon}(x)}{k_\varepsilon l(x)} \, dx + o(1) \\ \frac{\partial}{\partial x} \Delta \tilde{w}_{1,\varepsilon}(x) + a_2 \tilde{w}_1(x) - k \Delta(x) - b_2 \tilde{\tau}_{1,\varepsilon} + \frac{a_2}{b_2} \right).$$

Thus for $\varepsilon \neq 0$, $\left(k_\varepsilon + \varepsilon \tilde{\tau}_{1,\varepsilon}, k_\varepsilon l(x) + \varepsilon \tilde{w}_{1,\varepsilon}(x)\right)$ is a solution of \eqref{2.3} if and only if

$$\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = 0$$

and $\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = 0$ for any $\varepsilon \in (0, \delta)$.

In the following, we shall investigate the existence of $\left(k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)\right)$ satisfying

$$\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = 0$$

for any $\varepsilon \in (0, \delta)$.

From \eqref{2.13}, $\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = 0$ can be rewritten as

$$\begin{pmatrix} (I - Q)\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}) = 0, \\ Q\mathcal{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}) = 0 \end{pmatrix}$$  \quad \text{(2.18)}
i.e.

\[
0 = \left( Q \left( \int_{\Omega} \frac{a_1 \varphi_1(x) - a_{-1} \varphi_1(x)}{k_0 l'(x)} \, dx - \int_{\Omega} \frac{2b_1 (\tilde{\tau}_{1,\varepsilon}(l(x) - 1 - \tilde{w}_{1,\varepsilon}(x))) \, dx + o(1)}{k_0 l'(x)} \right) \right),
\]

which \( \tilde{\mathcal{F}}(\varepsilon, k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}). \)

It is easy to see that \( \tilde{\mathcal{F}}(\varepsilon, k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) \) is continuous in \( k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x) \) and \( \varepsilon \) for \( 0 \leq \varepsilon \leq \delta \), thus \( \tilde{\mathcal{F}}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x)) = 0 \) if

\[
\lim_{\varepsilon \to 0} (k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}(x)) = (k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x)). \tag{2.19}
\]

Our first goal is to find the value of \((k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x))\) if the limit (2.19) exists.

From the first equation of (2.18) and by some simple computation, we can obtain

\[
0 = (I - Q) \mathcal{F}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x)) = \frac{b_2}{a_2} \lambda_1 k_0 \mu_{\pm} \int_{\Omega} \varphi_1^2(x) \, dx + a_2 \int_{\Omega} \frac{\varphi_1(x)}{1 + \mu_{\pm} \varphi_1(x)} \, dx,
\]

thus

\[
k_0 = -\frac{a_2}{b_2} \int_{\Omega} \frac{\mu_{\pm} \varphi_1(x)}{1 + \mu_{\pm} \varphi_1(x)} \, dx, \tag{2.20}
\]

It is straightforward to show that

\[
\int_{\Omega} \frac{\mu_{\pm} \varphi_1(x)}{1 + \mu_{\pm} \varphi_1(x)} \, dx = \mu_{\pm} \left( \int_{\Omega} \varphi_1(x) \, dx - \int_{\Omega} \frac{\mu_{\pm} \varphi_1^2(x)}{1 + \mu_{\pm} \varphi_1(x)} \, dx \right)
\]

\[
- \int_{\Omega} \frac{\mu_{\pm} \varphi_1^2(x)}{1 + \mu_{\pm} \varphi_1(x)} \, dx < 0,
\]

thus \( k_0 > 0 \) and \((I - Q) \mathcal{F}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x)) = 0 \) for any \( \tilde{\tau}_{1,0} \) and \( \tilde{w}_{1,0}(x) \).

From the second system of (2.18), we have

\[
0 = Q \mathcal{F}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x))
\]

\[
= Q \mathcal{L}_{\varphi_{1/2,1}} \left( \tilde{\tau}_{1,0}, \tilde{w}_{1,0}(x) \right) + Q \left( \int_{\Omega} \frac{2b_1 \, dx}{k_0 l'(x)} - \int_{\Omega} \frac{a_1 \, dx}{k_0 l'(x)} \right).
\]

(2.21)

Note that \( Q \mathcal{L}_{\varphi_{1/2,1}} \big|_{X_R} \) is invertible, then (2.21) can be rewritten as

\[
\left( \begin{array}{c}
\tilde{\tau}_{1,0} \\
\tilde{w}_{1,0}(x)
\end{array} \right) = \left( Q \mathcal{L}_{\varphi_{1/2,1}} \right)^{-1} Q \left( \begin{array}{c}
\int_{\Omega} \frac{a_1 \, dx}{k_0 l'(x)} - \int_{\Omega} \frac{2b_1 \, dx}{k_0 l'(x)} \\
- k_0 \Delta l(x) + \frac{b_2}{l'(x)}
\end{array} \right). \tag{2.21}
\]

Therefore, \( \tilde{\mathcal{F}}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}) = 0 \).

Note that \( \tilde{\mathcal{F}} \) is a \( C^2 \)-continuous mapping from \([0, \delta) \times \mathbb{R} \times X_R \) to \( \mathbb{R} \times \text{Range}(\mathcal{L}_{\varphi_{1/2,1}}) \), we denote the Fréchet derivative of \( \tilde{\mathcal{F}} \) in \((k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon})\) by \( D(k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}) \tilde{\mathcal{F}} \). Thus,

\[
D(k_{\varepsilon}, \tilde{\tau}_{1,\varepsilon}, \tilde{w}_{1,\varepsilon}) \tilde{\mathcal{F}}(0, k_0, \tilde{\tau}_{1,0}, \tilde{w}_{1,0}) = \left( \begin{array}{c}
\frac{b_2}{a_2} \lambda_{\mu_{\pm}} \int_{\Omega} \varphi_1^2(x) \, dx \\
0
\end{array} \right) \frac{Q \mathcal{L}_{\varphi_{1/2,1}}}{Q \mathcal{L}_{\varphi_{1/2,1}}},
\]

(2.22)
with \( \hat{\psi} = (\tilde{\tau}_1, \hat{w}_1)^T \) and
\[
A = Q \left( \frac{-a_1 \tilde{\tau}_1 \hat{l}(x) - a_1 - a_1 \hat{w}_1(x)}{\kappa_0 \hat{l}^2(x)} dx + \int_{\Omega} \frac{2b_1 (\tilde{\tau}_1 \hat{l}(x) - 1 - \hat{w}_1(x))}{\kappa_0 \hat{l}^3(x)} dx \right).
\]

Since \( A : \mathbb{R} \to \mathbb{R} \times L^2(\Omega) \) is bounded and \( Q \hat{\xi}_0 \big|_{\hat{\nu}} \) is invertible, then it is easy to prove that \( D(k_\varepsilon, \tilde{\tau}_1, \hat{w}_1, x) \hat{F}(0, k_0, \tilde{\tau}_1, \hat{w}_1, (x)) \) is invertible. Thus by applying the Implicit Function Theorem, there exist a small \( \delta_0 > 0 \) such that for each fixed \( 0 < \varepsilon < \delta_0, \hat{F}(\varepsilon, k_\varepsilon, \tilde{\tau}_1, \hat{w}_1, (x)) = 0 \) has a solution \( (k_\varepsilon, \tilde{\tau}_1, \hat{w}_1, (x)) \) satisfying \( (k_\varepsilon, \tilde{\tau}_1, \hat{w}_1, (x)) \to (k_0, \tilde{\tau}_1, \hat{w}_1, (x)) \) uniformly in \( x \) as \( \varepsilon \to 0 \). Together with (2.17) and (2.18), it also implies \( H(\varepsilon, \tilde{\tau}_1, \hat{w}_1, (x)) = 0 \) where \( \tilde{\tau}(\varepsilon) = k_\varepsilon + \varepsilon \tilde{\tau}_1, \hat{w}(x, \varepsilon) = k_\varepsilon \hat{l}(x) + \varepsilon \hat{w}_1, (x) \).

**Proof of Theorem 1.1.** By Theorem 2.5 and (2.2), it follows that there exists a small \( \delta_0 > 0 \) such that for any fixed small \( \varepsilon = \frac{\alpha}{\lambda_1} \), the limiting system (1.10) has two nontrivial positive steady states \((\tau_{d_2}, w_{d_2}(x)) = \frac{1}{\varepsilon}(\tilde{\tau}_1, \hat{w}_1, (x)) \) satisfying (1.12).

**Remark 2.** For one dimensional case e.g. \( \Omega = (0, 1) \), by applying different transformation and by applying local and global bifurcation argument, we can further prove that the two local branches of positive monotone steady states stated in Theorem 1.1 can be extended to two global branches of positive monotone steady states for any \( d_2 \in (0, \frac{\alpha}{\lambda_1}) \). The results on the global branches of monotone positive steady states and the detailed proof will be given in our forthcoming paper.

It is worth mentioning that in one or multidimensional case when \( d_2 \to a_2/\lambda_1 \) the asymptotic behavior of the ratio \( \tau/w(x) \) of the large steady states to the first limiting system (1.4) is the same as that of \( \xi/\psi(x) \) of the small steady states to the first limiting system (1.4). However even in one dimensional case it is still unclear that whether the steady states on the global branch tend to the steady state having spike layer or having other types of singular structure as \( d_2 \to 0 \). It was proved in [9] that for the case \( A > B \) and small \( d_2 > 0 \) the SKT model with cross diffusion has a small spiky steady state near \((a_3/b_1, 0)\), which is perturbed from the spiky steady state of the limiting system (1.6), and it is known that such spiky steady state is unstable [17].

### 2.2. Spectral instability of nontrivial positive steady states to the limiting system

In this subsection, we shall investigate the spectral stability of the positive steady states \((\tau_{d_2}, w_{d_2}(x)) \) to the limiting system (1.10), which were constructed in previous subsection.

The linearized system of (1.10) around the steady state \((\tau_{d_2}, w_{d_2}(x)) \) is as follows
\[
\begin{align*}
\xi \int_{\Omega} \frac{dx}{1 + w_{d_2}^2} - \int_{\Omega} \frac{\tau_{d_2} \psi}{(1 + w_{d_2})^2} dx &= \int_{\Omega} \frac{-a_1 \tau_{d_2} \psi}{(1 + w_{d_2})^2} dx + \int_{\Omega} \frac{2b_1 \tau_{d_2}^2 \psi}{(1 + w_{d_2})^3} dx \\
&\quad - \xi \int_{\Omega} \frac{b_1 \tau_{d_2}}{(1 + w_{d_2})^2} dx, \quad t > 0,
\end{align*}
\]
(2.23)

\[\psi_t = d_2 \Delta \psi + \left( a_2 - \frac{b_2 \tau_{d_2}}{(1 + w_{d_2})^2} \right) \psi - \frac{b_2 w_{d_2}}{1 + w_{d_2}} \xi, \quad x \in \Omega, t > 0,\]
\[\frac{\partial \psi}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0.\]
The corresponding eigenvalue problem of the linearized system (2.23) with the eigenvalue $\lambda$ is as follows,

\[
\begin{aligned}
\int_{\Omega} \frac{-a_1 \tau \psi(x)}{(1 + w_2(x))^2} \, dx + \int_{\Omega} \frac{2b_1 \tau^2 \psi(x)}{(1 + w_2(x))^3} \, dx - \xi \int_{\Omega} \frac{b_1 \tau \psi(x)}{(1 + w_2(x))^2} \, dx \\
= \lambda \xi \int_{\Omega} \frac{1}{1 + w_2(x)} \, dx - \lambda \int_{\Omega} \frac{\tau \psi(x)}{(1 + w_2(x))^2} \, dx,
\end{aligned}
\]

\[
\begin{aligned}
d_2 \Delta \psi(x) + \left[ a_2 - \left( \frac{b_2 \tau d_2}{1 + w_2(x)^2} \right) \right] \psi(x) - \frac{b_2 w_2(x)}{1 + w_2(x)} \xi = \lambda \psi(x), & \quad x \in \Omega, \\
\frac{\partial \psi}{\partial v} = 0, & \quad x \in \partial \Omega.
\end{aligned}
\]

Multiplying the first equation of (2.24) by $\tau d_2 > 0$, system (2.24) becomes

\[
\mathcal{L}_{d_2} \left( \begin{array}{c} \xi \\ \psi \end{array} \right) = \lambda H_{d_2} \left( \begin{array}{c} \xi \\ \psi \end{array} \right),
\]

(2.25)

where

\[
\mathcal{L}_{d_2} \left( \begin{array}{c} \xi \\ \psi \end{array} \right) \triangleq \left( \begin{array}{c}
- \xi \int_{\Omega} \frac{b_1 \tau^2 \psi}{(1 + w_2(x))^2} \, dx - \int_{\Omega} \frac{a_1 \tau^2 \psi}{(1 + w_2(x))^2} \, dx + \int_{\Omega} \frac{2b_1 \tau \psi}{(1 + w_2(x))^3} \, dx \\
\tau d_2 \Delta \psi + \left[ a_2 - \left( \frac{b_2 \tau d_2}{1 + w_2(x)^2} \right) \right] \psi(x) - \frac{b_2 w_2(x)}{1 + w_2(x)} \xi
\end{array} \right),
\]

\[
H_{d_2} \left( \begin{array}{c} \xi \\ \psi \end{array} \right) \triangleq \left( \begin{array}{c}
\xi \int_{\Omega} \frac{\tau d_2}{1 + w_2} \, dx - \int_{\Omega} \frac{\tau \psi}{(1 + w_2(x))^2} \, dx
\end{array} \right).
\]

Obviously, the eigenvalue problem (2.24) is equivalent to the eigenvalue problem (2.25) for small $a_2/\lambda_1 - d_2 > 0$. Further by Theorem 1.1, it is easy to check that as $d_2 \rightarrow \frac{a_2}{\lambda_1}$ the eigenvalue problem (2.25) can be reduced to the following limiting eigenvalue problem

\[
\mathcal{L}_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} \xi \\ \psi(x) \end{array} \right) = \lambda H_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} \xi \\ \psi(x) \end{array} \right),
\]

(2.26)

where

\[
\mathcal{L}_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} \xi \\ \psi(x) \end{array} \right) \triangleq \left( \begin{array}{c}
- \xi \int_{\Omega} \frac{b_1}{l^2(x)} \, dx - \int_{\Omega} \frac{a_1 \psi(x)}{l^2(x)} \, dx + \int_{\Omega} \frac{2b_1 \psi(x)}{l(x)} \, dx \\
\frac{a_2}{\lambda_1} \Delta \psi(x) + a_2 \psi(x) - b_2 \xi
\end{array} \right),
\]

and

\[
H_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} \xi \\ \psi(x) \end{array} \right) \triangleq \left( \begin{array}{c}
\xi \int_{\Omega} \frac{1}{l(x)} \, dx - \int_{\Omega} \frac{\psi(x)}{l^2(x)} \, dx
\end{array} \right),
\]

with $l(x) = \frac{b_2}{a_2}(1 + \mu \varphi_1(x))$.

**Lemma 2.6.** Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple, then there exits a small $\delta > 0$ such that zero is the unique eigenvalue of the limiting eigenvalue problem (2.26) in $\{ \lambda \in \mathbb{C} | \text{Re} \lambda \geq -\delta \}$. Moreover, zero is a simple eigenvalue of $(H_{\frac{a_2}{\lambda_1}}^{-1}) \mathcal{L}_{\frac{a_2}{\lambda_1}}$ with the eigenspace spanned by $(1, l(x))^\top$. 
Proof. The proof of this Lemma is divided into three steps.

Firstly, we prove that $a_2$ is not an eigenvalue of system (2.26).

By contradiction, suppose $\lambda = a_2$ is an eigenvalue of system (2.26) with an eigenfunction $(\xi^*, \psi^*(x))$ satisfying

\[
\begin{cases}
  -\xi^* \int_\Omega \frac{b_1}{l^2(x)} \, dx - \int_\Omega \frac{a_1 \psi^*(x)}{l^2(x)} \, dx + \int_\Omega \frac{2b_1 \psi^*(x)}{l^3(x)} \, dx \\
  = a_2 \int_\Omega \frac{\xi^*}{l(x)} \, dx - a_2 \int_\Omega \psi^*(x) \, dx \\
  \frac{a_2}{\lambda_1} \Delta \psi^*(x) + a_2 \psi^*(x) - b_2 \xi^* = a_2 \psi^*(x), \quad x \in \Omega, \\
  \frac{\partial \psi^*}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{cases}
\]

Solving the boundary value problem of $\psi^*$ in (2.27), we have

\[
\xi^* = 0 \quad \text{and} \quad \psi^*(x) = C \quad \text{with} \quad C \quad \text{a nonzero constant.} \tag{2.28}
\]

Substituting (2.28) into the first equation of (2.27), it yields

\[- \int_\Omega \frac{a_1}{l^2(x)} \, dx + \int_\Omega \frac{2b_1}{l^3(x)} \, dx = - \int_\Omega \frac{a_2}{l^2(x)} \, dx < 0,
\]

which implies that

\[
\int_\Omega \frac{2b_1}{l^3(x)} \, dx < \int_\Omega \frac{a_1}{l^2(x)} \, dx. \tag{2.29}
\]

On the other hand, according to the Hölder inequality and the definition of $\mu_\pm$ stated in Lemma 2.3, we have

\[
\left( \int_\Omega \frac{1}{l^2(x)} \, dx \right)^2 \leq \int_\Omega \frac{1}{l^3(x)} \, dx \int_\Omega \frac{1}{l(x)} \, dx = \int_\Omega \frac{1}{l^3(x)} \, dx \int_\Omega \frac{1}{a_1 l^2(x)} \, dx,
\]

which implies that

\[
\int_\Omega \frac{a_1}{l^2(x)} \, dx \leq \int_\Omega \frac{b_1}{l^2(x)} \, dx. \tag{2.30}
\]

This is contrary to (2.29), which proves that $a_2$ is not an eigenvalue of (2.26).

Next, by applying similar argument as in the proof of Lemma 2.4, we can prove that zero is an eigenvalue of $(H_{a_2/\lambda_1}^{-1} L_{a_2/\lambda_1})$ with the eigenspace spanned by $\Psi = (l(x))^\top$, here we omit the detailed proof. Furthermore, it can be checked that $\Psi \notin \text{Range}(L_{a_2/\lambda_1})$, which means that zero is also a simple eigenvalue of $(H_{a_2/\lambda_1}^{-1} L_{a_2/\lambda_1})$.

Finally, it remains to prove that (2.26) has no eigenvalue with nonnegative real part. Suppose that there exists an eigenvalue $\text{Re} \lambda^* \geq 0$ and $\lambda^* \neq a_2$, with the associated eigenfunction denoted by $(\xi^*, \psi^*(x))$, in such case the boundary value problem of $\psi^*(x)$ in (2.26) has a solution if and only if

\[
\psi^*(x) = \frac{b_2 \xi^*}{a_2 - \lambda^*}. \tag{2.31}
\]

Substituting (2.31) into the first equation of (2.26), it yields

\[
\int_\Omega \frac{-a_1 b_2}{l^2(x)} \, dx + \int_\Omega \frac{2b_1 b_2}{l^3(x)} \, dx - \int_\Omega \frac{b_1 (a_2 - \lambda^*)}{l^2(x)} \, dx \\
= \lambda^* (a_2 - \lambda^*) \int_\Omega \frac{dx}{l(x)} - \lambda^* \int_\Omega \frac{b_2 dx}{l^2(x)}. \tag{2.32}
\]
Theorem 2.7. For sufficiently small constant 

\[ D(\lambda^*)^2 + E\lambda^* + F = 0, \]

(2.33)

where

\[
D = \int_\Omega \frac{1}{l(x)} dx > 0,
\]

\[
E = \int_\Omega \frac{b_1}{l^2(x)} dx - \int_\Omega \frac{a_2}{l(x)} dx + \int_\Omega \frac{b_2}{l^2(x)} dx
\]

\[
= \left( b_1 - \frac{a_2 b_1}{a_1} + b_2 \right) \int_\Omega \frac{1}{l^2(x)} dx > 0,
\]

\[
F = - \int_\Omega \frac{a_1 b_2}{l^2(x)} dx + 2 \int_\Omega \frac{b_1 b_2}{l^3(x)} dx - \int_\Omega \frac{a_2 b_2}{l^2(x)} dx.
\]

Meanwhile,

\[
F = \frac{a_2^2 B}{b_2} \left( - \frac{A}{B} \int_\Omega \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^2} dx + 2 \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^3} dx - \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^2} dx \right)
\]

\[
= \frac{a_2^2 B}{b_2} \left[ - \left( \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^2} dx \right)^2 + 2 \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^3} dx \int_\Omega \frac{1}{1 + \mu \pm \varphi_1} dx 
\]

\[
- \int_\Omega \frac{1}{(1 + \mu \pm \varphi_1)^2} dx \int_\Omega \frac{1}{1 + \mu \pm \varphi_1} dx \right] \left( \int_\Omega \frac{1}{1 + \mu \pm \varphi_1} dx \right)^{-1}
\]

\[
\geq \frac{a_2^2 B}{b_2} \left[ \int_\Omega \frac{2 dx}{(1 + \mu \pm \varphi_1)^3} \int_\Omega \frac{dx}{1 + \mu \pm \varphi_1} \right] - \left( \int_\Omega \frac{2 dx}{(1 + \mu \pm \varphi_1)^2} \right)^2 \int_\Omega \frac{dx}{1 + \mu \pm \varphi_1}
\]

\[
> 0.
\]

The above inequality follows from \( g(\mu_{\pm}) = \frac{A}{B} > 1 \) and the Cauchy-Schwartz inequality. Since \( E > 0 \), we can see that \( \text{Re}\lambda^* < 0 \) from (2.33). This leads to a contradiction which completes the proof.

By Lemma 2.6 and the spectral perturbation argument, it is suffices to investigate the location of the eigenvalue for the problem (2.25) near zero when \( \frac{a_2}{\lambda^*} - d_2 > 0 \) is small. This is the key point in the stability analysis of the steady states to the limiting system (1.10).

**Theorem 2.7.** For sufficiently small constant \( \varepsilon \triangleq \frac{a_2}{\lambda^*} - d_2 > 0 \), the eigenvalue problem (2.25) has a real eigenvalue \( \sigma_{d_2} \) near zero with an associated eigenfunction \( (\xi_{d_2}, \psi_{d_2}(x)) \) satisfying

\[
\lim_{d_2 \to \frac{a_2}{\lambda^*}} \left( \sigma_{d_2}, \xi_{d_2}, \psi_{d_2}(x) \right) = (0, 1, l(x)).
\]

**Proof.** Let \( \lambda \) be a real eigenvalue of (2.25) near zero for small \( \frac{a_2}{\lambda^*} - d_2 > 0 \). Applying the perturbation argument with the decomposition (2.13), we shall show that there exists a corresponding real eigenfunction \( (\xi(d_2), \psi(x, d_2)) \) which can be represented as follows

\[
\begin{pmatrix}
\xi(d_2) \\
\psi(x, d_2)
\end{pmatrix} = \begin{pmatrix}
1 \\
l(x)
\end{pmatrix} + \begin{pmatrix}
\xi_1(d_2) \\
\psi_1(x, d_2)
\end{pmatrix}, \quad (\xi_1(d_2), \psi_1(x, d_2))^\top \in X_R,
\]

(2.34)

and \( \left( \xi_1(\frac{a_2}{\lambda^*}), \psi_1(x, \frac{a_2}{\lambda^*}) \right) = 0 \).
Substituting (2.34) into (2.25), we have
\[
\begin{aligned}
&\left\{ \begin{array}{l}
-\int_{\Omega} \frac{b_1 \tau_{d_2}^2 (1 + \xi_1)}{(1 + w_{d_2})^3} dx - \int_{\Omega} \frac{a_1 \tau_{d_2}^2 (l + \psi_1)}{(1 + w_{d_2})^2} dx + \int_{\Omega} \frac{2b_1 \tau_{d_2}^2 (l + \psi_1)}{(1 + w_{d_2})^3} dx \\
= \lambda(1 + \xi_1) \int_{\Omega} \frac{\tau_{d_2}}{1 + w_{d_2}} dx - \lambda \int_{\Omega} \frac{\tau_{d_2}^2 (l + \psi_1)}{(1 + w_{d_2})^2} dx, \\
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
d_2 \Delta (l + \psi_1) + & \left[ a_2 \frac{\delta_{d_2}}{(1 + w_{d_2})^2} \right] (l + \psi_1) - \frac{b_2}{1 + w_{d_2}} (1 + \xi_1) \\
= & \lambda (l + \psi_1),
\end{aligned}
\]  
\[x \in \Omega,\]
\[\frac{\partial \psi_1}{\partial \nu} = 0,\]
\[x \in \partial \Omega.\]

For any fixed $\delta(0, \delta_0)$ with small enough $\delta_0 > 0$, we denote
\[
\Omega_\delta = \{ (\xi, \psi, \lambda) | (\xi, \psi) \in X_R, |\xi| + ||\psi||_{L_2} \leq \delta, |\lambda| \leq \delta \},
\]
and define mapping $K : (\frac{a_2}{\lambda_1} - \delta, \frac{a_2}{\lambda_1}) \times \Omega_\delta \to Y$ by
\[
K(d_2, \xi_1, \psi_1, \lambda) \triangleq L_{d_2} \left( \begin{array}{c} 1 + \xi_1 \\ l(x) + \psi_1(x) \end{array} \right) - \lambda H_{d_2} \left( \begin{array}{c} 1 + \xi_1 \\ l(x) + \psi_1(x) \end{array} \right),
\]
with $L_{d_2}$ and $H_{d_2}$ defined by (2.25). Obviously the eigenvalue problem (2.35) is equivalent to solving
\[
K(d_2, \xi_1, \psi_1, \lambda) = 0,
\]
which can be rewritten as the following system
\[
\begin{aligned}
&QK(d_2, \xi_1, \psi_1, \lambda) = 0, \\
&(I - Q)K(d_2, \xi_1, \psi_1, \lambda) = 0.
\end{aligned}
\]  
(2.37)

It is easy to check that
\[
K \left( \frac{a_2}{\lambda_1}, 0, 0, 0 \right) = L_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} 1 \\ l(x) \end{array} \right) = 0.
\]

Also note that $K$ is a $C^1 -$ continuous mapping from $(\frac{a_2}{\lambda_1} - \delta, \frac{a_2}{\lambda_1}) \times \Omega_\delta$ to $Y$ and
\[
QK(\xi_1, \psi_1) \left( \frac{a_2}{\lambda_1}, 0, 0, 0 \right) = L_{\frac{a_2}{\lambda_1}} |_{X_R} : X_R \to \text{Range}(L_{\frac{a_2}{\lambda_1}}),
\]
is invertible. By applying the Implicit Function Theorem for real $\lambda$ near zero, there exists $C^1-$continuous functions $(\xi_1(d_2, \lambda), \psi_1(x, d_2, \lambda))$ satisfying $\psi_1(x, a_2/\lambda_1, 0) = \xi_1(a_2/\lambda_1, 0) = 0$.

Substituting $(\xi_1(d_2, \lambda), \psi_1(x, d_2, \lambda))$ into the second equation of (2.37), it yields
\[
\tilde{K}(d_2, \lambda) \triangleq (I - Q)K(d_2, \xi_1(d_2, \lambda), \psi_1(x, d_2, \lambda), \lambda) = 0.
\]

Obviously, $\tilde{K}(a_2/\lambda_1, 0) = 0$ and
\[
\begin{aligned}
\frac{\partial \Delta}{\partial \lambda} K(a_2/\lambda_1, 0) = & \left( I - Q \right) L_{\frac{a_2}{\lambda_1}} \left( \frac{\partial \xi_1}{\partial \psi_1} \right) - \left( I - Q \right) H_{\frac{a_2}{\lambda_1}} \left( \begin{array}{c} 1 \\ l(x) \end{array} \right) \\
= & -\left( I - Q \right) \left( \begin{array}{c} 0 \\ l(x) \end{array} \right) = k(x) \left( \begin{array}{c} 1 \\ l(x) \end{array} \right), \quad k(x) \neq 0.
\end{aligned}
\]
Therefore by applying the Implicit Function Theorem, there exists a small \( \delta > 0 \) such that for each \( d_2 \in \left( \frac{\omega_2}{\lambda_1} - \delta, \frac{\omega_2}{\lambda_1} \right) \), equation \( \tilde{K}(d_2, \lambda) = 0 \) has a unique solution \( \sigma_{d_2} \) satisfying \( \tilde{K}(d_2, \sigma_{d_2}) = 0 \) and \( \lim_{d_2 \to \frac{\omega_2}{\lambda_1}} \sigma_{d_2} = 0 \).

Let \((\xi_1(d_2), \psi_1(x, d_2)) = (\xi_1(d_2, \sigma_{d_2}), \psi_1(x, d_2, \sigma_{d_2}))\), it is known that \( \sigma_{d_2}, \psi_1(x, d_2) \) and \( \xi_1(d_2) \) satisfy system (2.36) and \( \lim_{d_2 \to \sigma_{d_2}} (\xi_1(d_2), \psi_1(x, d_2)) = (0, 0, 0) \).

Therefore, the eigenvalue problem (2.25) has a real eigenvalue \( \sigma_{d_2} \) near zero with the associated eigenfunction \((\xi_{d_2}, \psi_{d_2}(x)) = (1 + \xi_1(d_2), l(x) + \psi_1(x, d_2))\) satisfying \( \lim_{d_2 \to \frac{\omega_2}{\lambda_1}} (\xi_{d_2}, \psi_{d_2}(x)) = (1, l(x)) \).

**Proof of Theorem 1.2.** By virtue of Lemma 2.6, Theorem 2.7 and by applying the classical spectral perturbation theories, to prove Theorem 1.2 it suffices to prove that the eigenvalue \( \sigma_{d_2} \) obtained in Theorem 2.7 is positive for small \( \frac{\omega_2}{\lambda_1} - d_2 > 0 \).

By Theorem 2.7, it is known that the eigenfunction \((\xi_{d_2}, \psi_{d_2}(x))\) associated with the eigenvalue \( \sigma_{d_2} \) satisfies the following boundary problem

\[
\begin{align*}
- \xi_{d_2} \int_{\Omega} \frac{b_1 \tau_2^x}{(1 + w_{d_2}(x))^2} dx - \int_{\Omega} \frac{a_1 \tau_2^x \psi_{d_2}(x)}{(1 + w_{d_2}(x))^2} dx + \int_{\Omega} \frac{2b_1 \tau_2^x \psi_{d_2}(x)}{(1 + w_{d_2}(x))^3} dx \\
= \sigma_{d_2} \int_{\Omega} \frac{\tau_2^x}{1 + w_{d_2}(x)} dx - \sigma_{d_2} \int_{\Omega} \frac{\tau_2^x \psi_{d_2}(x)}{(1 + w_{d_2}(x))^2} dx, \\
\int_{\Omega} \frac{d_2 \Delta \psi_{d_2} + \left(a_2 - \frac{b_2 \tau_2}{1 + w_{d_2}(x)}\right) \psi_{d_2} - \frac{b_2 w_{d_2}(x)}{1 + w_{d_2}(x)} \xi_{d_2}}{1 + w_{d_2}(x)} dx = \sigma_{d_2} \int_{\Omega} (l(x) + o(1)) \phi_{d_2}(x) dx \\
\frac{\partial \psi_{d_2}}{\partial \nu} = 0, \quad x \in \partial \Omega,
\end{align*}
\]

with \((\xi_{d_2}, \psi_{d_2}(x)) \to (1, l(x))\) as \( d_2 \to \sigma_{d_2} \).

Multiplying the second equation of (2.38) by \( \phi_1(x) \) and integrating on the \( \Omega \), we obtain

\[
(a_2 - d_2 \lambda_1) \int_{\Omega} (l(x) + o(1)) \phi_1(x) dx - \int_{\Omega} \frac{b_2 \tau_2 \phi_1(x)}{(1 + w_{d_2}(x))^2} dx - b_2 (1 + o(1)) \int_{\Omega} \frac{w_{d_2}(x)}{1 + w_{d_2}(x)} \phi_1(x) dx = \sigma_{d_2} \int_{\Omega} (l(x) + o(1)) \phi_1(x) dx.
\]

Recalling (2.27) and Theorem 1.1, we have

\[
a_2 - d_2 \lambda_1 = \varepsilon \lambda_1, \quad \tau_2 = \frac{1}{\varepsilon} (k_\pm + o(1)), \quad w_{d_2}(x) = \frac{1}{\varepsilon} (k_\pm l(x) + o(1)), \quad \text{for small } \varepsilon > 0.
\]

Substituting (2.40) into (2.39), it yields

\[
\varepsilon \lambda_1 \int_{\Omega} \left( \frac{b_2}{a_2} \mu_{\pm} \phi_1^2(x) + o(1) \right) dx - \varepsilon \int_{\Omega} \frac{b_2 (k_\pm l(x) + o(1)) \phi_1(x)}{(\varepsilon + k_\pm l(x) + o(1))^2} dx + \varepsilon b_2 (1 + o(1)) \int_{\Omega} \frac{\phi_1(x)}{\varepsilon + k_\pm l(x) + o(1)} dx = \sigma_{d_2} \int_{\Omega} \left( \frac{b_2}{a_2} \mu_{\pm} \phi_1^2(x) + o(1) \right) dx,
\]

which can be simplified as follows

\[
\varepsilon \lambda_1 \int_{\Omega} \left( \frac{b_2}{a_2} \mu_{\pm} \phi_1^2(x) + o(1) \right) dx - \int_{\Omega} \frac{\varepsilon b_2 \phi_1(x)}{k_0 l(x)} dx + \int_{\Omega} \frac{\varepsilon b_2 \phi_1(x)}{k_0 l(x)} dx + o(\varepsilon)
\]

\[
= \sigma_{d_2} \int_{\Omega} \left( \frac{b_2}{a_2} \mu_{\pm} \phi_1^2(x) + o(1) \right) dx.
\]
Thus we have
\[ \sigma_{d_2} = \varepsilon \lambda_1 + o(\varepsilon) > 0 \quad \text{for small} \quad \varepsilon > 0, \]
which completes the proof.

3. Existence and instability of nontrivial positive steady states to the original cross-diffusion system. In this section, we investigate the existence and stability of the nontrivial positive steady states to the original cross-diffusion system (1.3) when both \( d_1 \) and \( \frac{d_2}{d_1} \) are large enough.

Let \( r = \frac{1}{a}, s = \frac{1}{d_1}, w = \alpha v \) and \( \phi = (1 + \alpha v)u \), then the original cross-diffusion system (1.3) can be written as the following system

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \phi \\ w \end{pmatrix} &= \begin{pmatrix} \Delta \phi + s \frac{\phi}{1 + w} \left( a_1 - b_1 \frac{\phi}{1 + w} - rc_1 w \right) \\ d_2 \Delta w + w \left( a_2 - b_2 \frac{\phi}{1 + w} - rc_2 w \right) \end{pmatrix}, & x \in \Omega, t > 0, \\
\partial \phi / \partial \nu &= \partial w / \partial \nu = 0, & x \in \partial \Omega, t > 0.
\end{align*}
\]

(3.1)

To prove the existence of the steady state for the cross-diffusion system (1.3) when both \( d_1 \) and \( s \) are large enough, it is equivalent to prove the corresponding existence result for system (3.1) when both \( r \) and \( s \) are small enough.

3.1. Existence of nontrivial positive steady states for the perturbed system (3.1). Let \( (\phi(x), w(x)) \) be the steady state of (3.1) with positive \( s \) and \( r \), which satisfies the following boundary value problem

\[
\begin{align*}
\Delta \phi(x) + s \frac{\phi(x)}{1 + w(x)} \left( a_1 - b_1 \frac{\phi(x)}{1 + w(x)} - rc_1 w(x) \right) &= 0, & x \in \Omega, \\
d_2 \Delta w(x) + w(x) \left( a_2 - b_2 \frac{\phi(x)}{1 + w(x)} - rc_2 w(x) \right) &= 0, & x \in \Omega, \\
\partial \phi / \partial \nu &= \partial w / \partial \nu = 0, & x \in \partial \Omega.
\end{align*}
\]

(3.2)

Define the Banach space \( Y_0 = \{ u \in L_2(\Omega) | \int_\Omega u(x)dx = 0 \} \) and introduce the projection \( \mathcal{P} : L_2(\Omega) \rightarrow Y_0 \) by

\[ \mathcal{P} u(x) = u(x) - \frac{1}{|\Omega|} \int_\Omega u(x)dx. \]

(3.3)

For each fixed \( \phi \in H_0^2(\Omega) \), let \( \hat{\phi}(x) = \mathcal{P} \phi(x) \) and \( \tau = \frac{1}{|\Omega|} \int_\Omega \phi(x)dx \), such that

\[ \phi(x) = \tau + \hat{\phi}(x), \quad \hat{\phi}(x) \in Y_0 \cap H_0^2(\Omega) \triangleq \tilde{Y}_0. \]

(3.4)

By (3.1)-(3.4), if \( \tau \neq 0 \) and \( s > 0 \) then system (3.2) becomes the following boundary value problem

\[
\begin{align*}
\Delta \hat{\phi}(x) + s \mathcal{P} \left\{ \frac{\tau + \hat{\phi}(x)}{1 + w(x)} \left( a_1 - b_1 \frac{\tau + \hat{\phi}(x)}{1 + w(x)} - rc_1 w(x) \right) \right\} &= 0, & x \in \Omega, \\
\tau \int_\Omega \frac{\tau + \hat{\phi}(x)}{1 + w(x)} \left( a_1 - b_1 \frac{\tau + \hat{\phi}(x)}{1 + w(x)} - rc_1 w(x) \right) dx &= 0, \\
d_2 \Delta w(x) + w(x) \left( a_2 - b_2 \frac{\tau + \hat{\phi}(x)}{1 + w(x)} - rc_2 w(x) \right) &= 0, & x \in \Omega, \\
\partial \hat{\phi} / \partial \nu &= \partial w / \partial \nu = 0, & x \in \partial \Omega.
\end{align*}
\]

(3.5)
Theorem 3.1. Assume $A > B$, $N \leq 4$ and $\lambda_1$ is simple. For each fixed $d_2 \in (a_2/\lambda_1 - \delta, a_2/\lambda_1)$ with small $\delta > 0$, if $s > 0$ and $r > 0$ are small enough, then system (3.5) has two types of positive solutions $(\hat{\phi}^{\pm}_{s,r}(x,d_2), \tau^{\pm}_{s,r}(d_2), w^{\pm}_{s,r}(x,d_2)) \in \tilde{Y}_0 \times \mathbb{R} \times H^2_0(\Omega)$ satisfying $(\hat{\phi}^{\pm}_{s,r}(x,d_2), \tau^{\pm}_{s,r}(d_2), w^{\pm}_{s,r}(x,d_2)) \to (0, \tau^{\pm}_{d_2}, w^{\pm}_{d_2}(x))$ uniformly in $x \in \Omega$, as $r$ and $s \to 0^+$, where $(\tau^{\pm}_{d_2}, w^{\pm}_{d_2}(x))$ are two types of positive steady states obtained in Theorem 1.1 for small $a_2/\lambda_1 - d_2 > 0$.

Proof. Let $\delta_1$ and $\delta_2$ be small positive constants, and define $U^\pm_{\delta_1} := \left\{ (\hat{\phi}, \tau, w) \in \tilde{Y}_0 \times \mathbb{R} \times H^2_0(\Omega) \left| \|\hat{\phi}\|_{H^2(\Omega)} + |\tau - \tau^\pm_{d_2}| + \|w - w^\pm_{d_2}\|_{H^2(\Omega)} \leq \delta_1 \right. \right\}$.

For any fixed small $\frac{a_2}{\lambda_1} - d_2 > 0$ and small enough positive $\delta_1$ and $\delta_2$, define the mapping $\mathfrak{F}_{d_2} : [0, \delta_2] \times [0, \delta_2] \times U^\pm_{\delta_1} \to \tilde{Y}_0 \times \mathbb{R} \times L_2(\Omega)$ by

$$\mathfrak{F}_{d_2}(s, r, \hat{\phi}, \tau, w) = \begin{pmatrix} \Delta \hat{\phi} + s P \begin{pmatrix} \frac{\tau + \hat{\phi}}{1 + w} \\ a_1 - b_1 \frac{\tau + \hat{\phi}}{1 + w} - rc_1 w \end{pmatrix} \\ \tau \int_\Omega \frac{\tau + \hat{\phi}}{1 + w} \begin{pmatrix} a_1 - b_1 \frac{\tau + \hat{\phi}}{1 + w} - rc_1 w \end{pmatrix} dx \\ d_2 \Delta w + w \begin{pmatrix} a_2 - b_2 \frac{\tau + \hat{\phi}}{1 + w} - rc_2 w \end{pmatrix} \end{pmatrix}. \quad (3.6)$$

In the following, we denote $(\tau^{\pm}_{d_2}, w^{\pm}_{d_2}(x))$ simply by $(\tau_{d_2}, w_{d_2}(x))$. Obviously, $\mathfrak{F}_{d_2}(0, 0, 0, \tau_{d_2}, w_{d_2}(x)) = 0$ owing to Theorem 1.1. Moreover, the Fréchet derivative of $\mathfrak{F}_{d_2}$ with respect to $(\hat{\phi}, \tau, w)$ at $(0, 0, 0, \tau_{d_2}, w_{d_2}(x))$ is given by

$$D_{(\phi, \tau, w)} \mathfrak{F}_{d_2}(0, 0, 0, \tau_{d_2}, w_{d_2}(x)) \begin{pmatrix} \phi(x) \\ \xi \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \Delta \phi \\ \int_\Omega \begin{pmatrix} a_1 \tau_{d_2} \phi \frac{2b_1 \tau_{d_2}^2 \phi + b_1 \tau_{d_2}^2 \xi + a_1 \tau_{d_2}^2 \psi}{1 + w_{d_2}} + \frac{2b_1 \tau_{d_2}^3 \psi}{(1 + w_{d_2})^3} \\ d_2 \Delta \psi - b_2 w_{d_2} \phi + b_2 w_{d_2} \xi + b_2 w_{d_2} \psi - \frac{b_2 \tau_{d_2}}{1 + w_{d_2}} \psi \end{pmatrix} dx \\ \frac{\Delta}{\mathfrak{H}} \mathbf{L}_{d_2} \begin{pmatrix} \phi(x) \\ \xi \\ \psi(x) \end{pmatrix} \end{pmatrix}, \quad (3.7)$$

with the operator $\mathbf{L}_{d_2}$ defined by (2.25) and

$$\mathfrak{H} \phi(x) = \begin{pmatrix} \int_\Omega \begin{pmatrix} a_1 \tau_{d_2} \phi(x) \frac{2b_1 \tau_{d_2}^2 \phi(x) + a_1 \tau_{d_2}^2 \psi(x)}{1 + w_{d_2}(x)} + \frac{2b_1 \tau_{d_2}^3 \psi(x)}{(1 + w_{d_2}(x))^3} dx \\ \frac{b_2 w_{d_2}(x)}{1 + w_{d_2}(x)} \phi(x) \end{pmatrix} \\ \frac{b_2 w_{d_2}(x)}{1 + w_{d_2}(x)} \phi(x) \end{pmatrix}. \quad (3.8)$$

Note that the restricted Laplace operator $\Delta_+ \overset{\Delta}{=} \Delta|_{\tilde{Y}_0} : \tilde{Y}_0 \to Y_0$ is invertible and the operator $\mathfrak{H} : L_2(\Omega) \to \mathbb{R} \times L_2(\Omega)$ is bounded. Furthermore, by Theorem 1.2 it is known that for small $\frac{a_2}{\lambda_1} - d_2 > 0$ the operator $H^{-1}_{d_2} \mathbf{L}_{d_2}$ has precisely one eigenvalue $\sigma_{d_2} > 0$ near zero, which implies that the operator $\mathbf{L}_{d_2}$ is invertible for small $\frac{a_2}{\lambda_1} - d_2 > 0$.
Thus it is easy to prove that the operator \( D_{\phi, \tau, w, 0} \mathfrak{F}_{d_2}(0, 0, 0, \tau_{d_2}, w_{d_2}(x)) \) is invertible, then by applying the Implicit Function Theorem it follows that there exist \((\phi_{s,r}(x, d_2), \tau_{s,r}(d_2), w_{s,r}(x, d_2)) \in \tilde{Y}_0 \times \mathbb{R} \times H_0^2(\Omega), \) which are continuous in \(d_2, s\) and \(r,\) and satisfy \( \mathfrak{F}_{d_2}(s, r, \phi_{s,r}(x, d_2), \tau_{s,r}(d_2), w_{s,r}(x, d_2)) = 0 \) for small \( \frac{\alpha_2}{\alpha_1} - d_2 > 0 \) and small \( s, r > 0; \) furthermore, \((\phi_{s,r}(x, d_2), \tau_{s,r}(d_2), w_{s,r}(x, d_2)) \to (0, \tau_{d_2}, w_{d_2}(x)) \) uniformly in \( x, \) as \( s, r \to 0. \)

3.2. Instability of nontrivial positive steady states to the original cross diffusion system. For each fixed small \( \frac{\alpha_2}{\alpha_1} - d_2 > 0 \) and small enough \( s, r > 0, \) let \((\phi_{s,r}(x, d_2), w_{s,r}(x, d_2)) \) be one of the two types of nontrivial positive steady states of system (3.1) with \( \phi_{s,r}(x, d_2) = \tau_{s,r}(d_2) + \phi_{s,r}(x, d_2) \) obtained in Theorem 3.1.

The linearized system of (3.1) around \((\phi_{s,r}(x, d_2), w_{s,r}(x, d_2))\) is as follows

\[
\begin{aligned}
\psi_t &= d_2 \Delta \psi + a_{s,1}^r \psi + a_{s,2}^r \psi, \\
\partial \varphi &= \partial \psi = 0, \\
\end{aligned}
\]

with

\[
\begin{aligned}
\tilde{b}_{s,1}^r(x, d_2) &= \frac{1}{1 + w_{s,r}(x, d_2)}, \\
\tilde{b}_{s,2}^r(x, d_2) &= \frac{a_1}{1 + w_{s,r}(x, d_2)} - \frac{2b_1 \phi_{s,r}(x, d_2)}{(1 + w_{s,r}(x, d_2))^2}, \\
\tilde{a}_{s,1}^r(x, d_2) &= \frac{a_1 \phi_{s,r}(x, d_2)}{(1 + w_{s,r}(x, d_2))^2} + \frac{2b_1 (\phi_{s,r}(x, d_2))^2}{(1 + w_{s,r}(x, d_2))^3}, \\
\tilde{a}_{s,2}^r(x, d_2) &= \frac{b_2 w_{s,r}(x, d_2)}{1 + w_{s,r}(x, d_2)}, \\
\tilde{a}_{2,2}^r(x, d_2) &= a_2 - \frac{b_2 \phi_{s,r}(x, d_2)}{(1 + w_{s,r}(x, d_2))^2} = 2\tau_{d_2} w_{s,r}(x, d_2).
\end{aligned}
\]

In the following we shall investigate the following associated eigenvalue problem of (3.9),

\[
\begin{aligned}
s\lambda \tilde{b}_{s,1}^r(x, d_2) \varphi(x) + s\lambda \tilde{b}_{s,2}^r(x, d_2) \psi(x) \\
= \Delta \varphi(x) + s\lambda \tilde{a}_{s,1}^r(x, d_2) \varphi(x) + s\lambda \tilde{a}_{s,2}^r(x, d_2) \psi(x), \quad x \in \Omega, \\
\lambda \psi(x) = d_2 \Delta \psi(x) + a_{s,1}^r(x, d_2) \varphi(x) + a_{s,2}^r(x, d_2) \psi(x), \quad x \in \Omega, \\
\partial \varphi &= \partial \psi = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

By applying the more general abstract stability results and the analytic semigroup theories (see also [2]), to get the stability/instability of the nonnegative steady state for the quasilinear SKT cross-diffusion system, it suffices to prove that the steady state is spectrally stable/unstable, i.e. the linearized operator has no eigenvalue with nonnegative real part /has an eigenvalue with positive real part.

**Theorem 3.2.** Assume \( A > B, \) \( N \leq 4 \) and \( \lambda_1 \) is simple. For each fixed \( d_2 \in \left( \frac{\alpha_2}{\alpha_1} - \delta, \frac{\alpha_2}{\alpha_1} \right) \) with \( \delta > 0 \) small, then there exist small small positive \( \tilde{s}(d_2) \) and \( \tilde{r}(d_2) \) such that for \( 0 < s < \tilde{s}(d_2) \), \( 0 < r < \tilde{r}(d_2) \), system (3.10) has an eigenvalue \( \lambda(d_2, s, r) \) with positive real part satisfying \( \lim_{s, r \to 0^+} \lambda(d_2, s, r) = \sigma_{d_2} \).
Proof. Define the projection $P$ as in (3.3) and let
\begin{align}
\begin{cases}
\dot{\phi}(x) = P\phi(x) = \phi(x) - \frac{1}{|\Omega|} \int_{\Omega} \phi(x)dx,
\xi = (I - P)\phi(x) = \frac{1}{|\Omega|} \int_{\Omega} \phi(x)dx,
\end{cases}
\end{align}
then the eigenvalue problem (3.10) for $s > 0$ becomes the following eigenvalue problem with eigenfunction $(\xi, \phi(x), \psi(x)) \in \mathbb{C} \times \hat{Y}_0 \times H^2_0(\Omega)$
\begin{align}
\begin{cases}
\xi \int_{\Omega} a_{11}^{s,r} dx + \int_{\Omega} a_{12}^{s,r} \psi(x)dx = \lambda \xi \int_{\Omega} b_{11}^{s,r} dx + \lambda \int_{\Omega} b_{12}^{s,r} \psi(x)dx - \int_{\Omega} (a_{11}^{s,r} - \lambda b_{11}^{s,r})\dot{\phi}(x)dx,
\Delta \dot{\phi} = -sP\{a_{11}^{s,r}(\xi + \dot{\phi}) + a_{12}^{s,r} \psi\} + s\lambda P\{b_{11}^{s,r}(\xi + \dot{\phi}) + b_{12}^{s,r} \psi\},
x \in \Omega,
d_2 \Delta \psi + a_{22}^{s,r} \psi + a_{21}^{s,r} \xi = \lambda \psi - a_{21}^{s,r} \dot{\phi},
x \in \Omega,
\frac{\partial \dot{\phi}}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0,
x \in \partial \Omega.
\end{cases}
\end{align}
Denote $\Delta_+ \triangleq \Delta|_{\hat{Y}_0} : \hat{Y}_0 \to Y_0$, it is obvious that $0 \notin \sigma(\Delta_+)$ and there exists a constant $C_0$ such that
\begin{align}
||\Delta_+^{-1}||_{Y_0 \to \hat{Y}_0} \leq C_0.
\end{align}
Multiplying the first and second equations of (3.12) by $\tau_{d_2}$ where $(\tau_{d_2}, w_{d_2}(x))$ is obtained in Theorem 1.1 for each fixed small $\frac{s^2}{\lambda^2} - d_2 > 0$, then for any small $s > 0$, $\frac{s^2}{\lambda^2} - d_2 > 0$ the eigenvalue problem (3.12) can be rewritten as
\begin{align}
\begin{cases}
\xi \int_{\Omega} a_{11}^{s,r} dx + \int_{\Omega} a_{12}^{s,r} \psi(x)dx = \lambda \xi \int_{\Omega} b_{11}^{s,r} dx + \lambda \int_{\Omega} b_{12}^{s,r} \psi(x)dx - \int_{\Omega} (a_{11}^{s,r} - \lambda b_{11}^{s,r})\dot{\phi}(x)dx,
\dot{\phi}(x) = s\tau_{d_2}^{-1}(\Delta_+)^{-1}P\{-a_{11}^{s,r}(\xi + \dot{\phi}) + a_{12}^{s,r} \psi\} + s\lambda \tau_{d_2}^{-1}P\{b_{11}^{s,r}(\xi + \dot{\phi}) + b_{12}^{s,r} \psi\},
x \in \Omega,
d_2 \Delta \psi + a_{22}^{s,r} \psi + a_{21}^{s,r} \xi = \lambda \psi - a_{21}^{s,r} \dot{\phi},
x \in \Omega,
\frac{\partial \dot{\phi}}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0,
x \in \partial \Omega.
\end{cases}
\end{align}
Note that $(\phi_{s,r}(x, d_2), w_{s,r}(x, d_2)) \to (\tau_{d_2}, w_{d_2}(x))$ as $s, r \to 0^+$, thus
\begin{align}
\begin{cases}
a_{1j}^{s,r}(x, d_2) \triangleq \tau_{d_2} a_{1j}^{s,r}(x, d_2) \to a_{1j}^{0,0}(x, d_2),
b_{1j}^{s,r}(x, d_2) \triangleq \tau_{d_2} b_{1j}^{s,r}(x, d_2) \to b_{1j}^{0,0}(x, d_2),
a_{2j}^{s,r}(x, d_2) \to a_{2j}^{0,0}(x, d_2),
\end{cases}
\end{align}
with
\begin{align*}
b_{11}^{0,0}(x, d_2) &= \frac{\tau_{d_2}}{1 + w_{d_2}(x)}, b_{12}^{0,0}(x, d_2) = -\frac{\tau_{d_2}^2}{(1 + w_{d_2}(x))^2},
a_{11}^{0,0}(x, d_2) &= \frac{a_{11}^{\tau_{d_2}}}{1 + w_{d_2}(x)} - \frac{2b_1 \tau_{d_2}^2}{(1 + w_{d_2}(x))^2},
a_{12}^{0,0}(x, d_2) &= -\frac{a_{12}^{\tau_{d_2}}}{(1 + w_{d_2}(x))^2} + \frac{2b_1 (\tau_{d_2})^3}{(1 + w_{d_2}(x))^3},
a_{21}^{0,0}(x, d_2) &= -\frac{b_2 w_{d_2}(x)}{1 + w_{d_2}(x)}, a_{22}^{0,0}(x, d_2) = a_2 - \frac{b_2 \tau_{d_2}}{(1 + w_{d_2}(x))^2}.
\end{align*}
Also notice that

\[
\begin{align*}
\alpha_{11}^0(x, d_2) &\to \frac{1}{r(x)}, \quad \beta_{11}^0(x, d_2) \to -\frac{1}{r(x)}, \\
\alpha_{11}^0(x, d_2) &\to \frac{a_1}{r(x)} - \frac{b_2}{r(x)}, \\
\alpha_{21}^0(x, d_2) &\to -\frac{a_1}{r(x)} + \frac{b_2}{r(x)}, \\
\alpha_{21}^0(x, d_2) &\to -b_2 a_{22}^0(x, d_2) \to a_2.
\end{align*}
\]  

(3.16)

Recall that

\[\tau_{d_2}^{-1} = \varepsilon \hat{\tau}_0^{-1}, \quad \sigma_{d_2} = \varepsilon \lambda_1 + a(\varepsilon), \quad \text{with} \quad \hat{\tau}_0^{-1} \to (k_0)^{-1}, \quad \text{as} \quad \varepsilon \to 0^+,
\]

we can rewrite the second equation of (3.14) with \(\varepsilon, s > 0\) as

\[
\hat{\varphi} = s \varepsilon G(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda),
\]

(3.18)

with

\[
G(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda) = \hat{\tau}_0^{-1}(\Delta_+)^{-1} \mathcal{P} \{-a_{11}^r(\xi + \hat{\varphi}) - a_{12}^r \psi + \lambda b_{11}^r(\xi + \hat{\varphi}) + \lambda b_{12}^r \psi\}.
\]

(3.19)

By (3.13), and (3.15)-(3.17), it is easy to check that there exist positive constant \(C_0\) and small positive constants \(\varepsilon_0\) and \(\delta_0\) such that \(G(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda)\) is continuous in \((\varepsilon, s, r)\) and \(C^1\)-continuous in \((\xi, \hat{\varphi}, \psi, \lambda)\) and satisfies

\[
||G(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda)||_{H^2(\Omega)} \leq C_0(\varepsilon + ||\varphi||_{L^2(\Omega)} + ||\psi||_{L^2(\Omega)}),
\]

and

\[\varepsilon_0 \delta_0 \leq \frac{1}{2C_0}, \quad (3.20)\]

for any \(0 \leq \varepsilon \leq \varepsilon_0, 0 \leq s, r \leq \delta_0, \forall (\xi, \hat{\varphi}, \psi) \in \mathbb{C} \times \hat{Y}_0 \times H^2_0\) and \(|\lambda| \leq 1\). By virtue of (3.20), we can apply the Implicit Function Theorem to solve \(\hat{\varphi}\) in (3.18), it follows that there exists a continuous function \(\Phi(\varepsilon, s, r, \xi, \psi, \lambda)\) defined on \(\Sigma_0 = \{\varepsilon \in [0, \varepsilon_0], 0 \leq s, r \leq \delta_0, |\lambda| \leq 1, \xi \in \mathbb{C}, \psi \in H^2_0\}\), which is linear in \((\xi, \psi)\) and smooth in \((\xi, \psi, \lambda)\) such that on \(\Sigma_0\) equation (3.18) can be represent as

\[
\hat{\varphi} = s \varepsilon \Phi(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda),
\]

(3.21)

and

\[
||\Phi(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda)||_{H^2(\Omega)} \leq 2C_0(||\xi||_{L^2(\Omega)} + ||\psi||_{L^2(\Omega)}) \text{ on } \Sigma_0.
\]

(3.22)

Putting (3.21) into the first and the third equations in (3.14), and using (3.15)-(3.17) for \(s, r > 0\) and \(\varepsilon > 0\), we can rewrite (3.14) as follows

\[
\mathcal{L}_{d_2}^{s, r}(X) = \lambda H_{d_2}^{s, r}(X) + \left(s \varepsilon \int_{\Omega} (-a_{11}^r + \lambda b_{11}^r) \Phi(\varepsilon, s, r, \xi, \hat{\varphi}, \psi, \lambda) dx \right).
\]

(3.23)

Obviously, system (3.23) is a regularly perturbed eigenvalue problem of the following limiting system

\[
\mathcal{L}_{d_2}(X) = \lambda H_{d_2}(X), \quad \mathcal{L}_{d_2} \text{ and } H_{d_2} \text{ defined by (2.25)},
\]

(3.24)

with small parameters \(s, r > 0\) for fixed \(\varepsilon > 0\). In the proof of Theorem 1.2, it is known that (3.24) has a simple positive eigenvalue \(\sigma_{d_2}\) for each fixed small \(\frac{a_2}{\lambda_1} - d_2 > 0\). Using (3.22) and by applying similar argument based on Lyapunov-Schmidt decomposition as in the proof of Theorem 1.2, we can similarly prove that for each fixed small \(\frac{a_2}{\lambda_1} - d_2 > 0\) and small enough \(s, r > 0\), system (3.23) has a
simple real eigenvalue near $\sigma_{d_2}$ denoted by $\lambda^+(d_2, s, r)$, with a real eigenfunction denoted by $(\xi_\lambda, \psi_\lambda)^\top$ satisfying

$$
\lambda^+(d_2, s, r) \to \sigma_{d_2} > 0, \text{ and } \left( \begin{array}{c} \xi_\lambda \\ \psi_\lambda \end{array} \right) \to \left( \begin{array}{c} \xi_{d_2} \\ \psi_{d_2} \end{array} \right) \text{ as } s, r \to 0^+,
$$

(3.25)

here we omit the details of the proof.

(3.25) also implies that $\lambda^+(d_2, s, r) > 0$ for small $\frac{d^2}{k_1} - d_2 > 0$ and sufficiently small $s, r > 0$, which proves Theorem 1.4.

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