BOGOMOLOV’S CONJECTURE FOR HYPERELLIPTIC CURVES
OVER FUNCTION FIELDS

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1. Introduction

Let us fix a field $k$. Let $X$ be a smooth projective surface over $k$, $Y$ a smooth projective curve over $k$, and let $f : X \to Y$ be a generically smooth semistable curve of genus $g \geq 2$ over $Y$. Let $K$ be the function field of $Y$, $\overline{K}$ the algebraic closure of $K$, and let $C$ be the generic fiber of $f$. For $D \in \text{Pic}^1(C)(K)$, let

$$ j : C(\overline{K}) \to \text{Pic}^1(C)(\overline{K}) $$

be a morphism defined by $j(x) = x - D$, and $\| \cdot \|_{NT}$ the semi-norm arising from the Néron-Tate pairing on $\text{Pic}^1(C)(\overline{K})$. We set

$$ B_C(P; r) = \{x \in C(\overline{K}) \mid \|j(x) - P\|_{NT} \leq r\} $$

for $P \in \text{Pic}^0(C)(\overline{K})$ and $r \geq 0$, and set

$$ r_C(P) = \begin{cases} -\infty & \text{if } \#(B_C(P; 0)) = \infty, \\ \sup \{r \geq 0 \mid \#(B_C(P; r)) < \infty\} & \text{otherwise.} \end{cases} $$

Then, we have the following conjectures due to Bogomolov.

**Conjecture 1.1.** (Bogomolov’s conjecture). If $f$ is non-isotrivial, then $r_C(P) > 0$ for all $P$.

**Conjecture 1.2.** (Effective Bogomolov’s conjecture). If $f$ is non-isotrivial, then there exists an effectively calculated positive number $r_0$ such that

$$ \inf_{P \in \text{Pic}^0(C)(\overline{K})} r_C(P) \geq r_0. $$

In order to describe $r_0$ above, we introduce the types of nodes of a semistable curve. Let $C$ be a semistable curve of genus $g$ and $P$ a node of $C$. We can assign a number $i$ to the node $P$ in the following way. Let $\nu : C_P \to C$ be the partial normalization at $P$. If $C_P$ is connected, then $i = 0$. Otherwise, $i$ is the minimum of arithmetic genera of two connected components of $C_P$. We say the node $P$ of $C$ is of type $i$. We denote by $\delta_i(C)$ the number of nodes of type $i$. and by $\delta_i(X/Y)$ the number of nodes of type $i$ in all the fibers of $f : X \to Y$, i.e., $\delta_i(X/Y) = \sum_{y \in Y} \delta_i(X_y)$.

Moriwaki proved the following results.
(a) \((\text{char}(k) = 0)\). If \(f\) is not smooth and every singular fiber of \(f\) is a tree of stable components, then
\[
\inf_{P \in \text{Pic}^0(C) (\overline{K})} r_C(P) \geq \sqrt{\frac{(g-1)^2}{g(2g+1)} \left( \frac{g-1}{3} \delta_0(X/Y) + \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i(X/Y) \right)}.
\]

(b) \((\text{char}(k) \geq 0)\). If \(g = 2\), then \(f\) is not smooth and
\[
\inf_{P \in \text{Pic}^0(C) (\overline{K})} r_C(P) \geq \sqrt{\frac{2}{135} \delta_0(X/Y) + \frac{2}{5} \delta_1(X/Y)}.
\]

In this paper, we would like to prove the effective Bogomolov’s conjecture for generically smooth semistable hyperelliptic curves.

Let \(C\) be a semistable curve over \(k\). We say that \(C\) is a semistable hyperelliptic curve if there exist a valuation ring \(R\) with residue field \(k\) and a generically smooth semistable curve \(f : Z \to \text{Spec}(R)\) such that the generic fiber of \(f\) is a smooth hyperelliptic curve and the special fiber of \(f\) is \(C\). By the definition, \(C\) has an involution \(\iota\), and we can see that \(C/\langle \iota \rangle\) is a nodal curve which is a tree of \(\mathbb{P}^1\). For details, see [1].

Now let \(f : X \to Y\) be a generically smooth hyperelliptic semistable curve of genus \(g\) with the hyperelliptic involution \(\iota\). Let \(C\) be a fiber of \(f\), which is a semistable hyperelliptic curve over \(k\) with the involution \(\iota|_C\), and \(P\) a node of \(C\) of type 0. We can also assign a number \(j\) to the pair of nodes \((P, \iota(P))\) of type 0 in the following way. If \(P = \iota(P)\), we set \(j = 0\). If \(P \neq \iota(P)\), then the partial normalization at \(P\) and \(\iota(P)\) is a nodal curve with two connected components since \(C/\langle \iota \rangle\) is a tree of \(\mathbb{P}^1\). We set \(j\) to be the minimum of arithmetic genera of two connected components of \(C_{P,\iota(P)}\). We say that the node \(P\), or the pair of nodes \((P, \iota(P))\) is of type \((0, j)\), or of subtype \(j\). We denote by \(\xi_0(C)\) the number of nodes of type \((0,0)\), and by \(\xi_j(C)\) the number of such pairs of nodes of type \((0, j)\) for \(j \geq 1\). Moreover, we set
\[
\xi_j(X/Y) = \sum_{y \in Y} \xi_j(X_y).
\]

The following are the main results of this paper.

**Theorem 1.3.** \((\text{char}(k) = 0)\). We assume that \(f\) is hyperelliptic. Then, Bogomolov’s conjecture holds for \(f\). In addition, \(f\) is not smooth and
\[
\inf_{P \in \text{Pic}^0(C) (\overline{K})} r_C(P) \geq \sqrt{r_0},
\]
where \(r_0\) is a positive number given below.

(1) If \(g = 3, 4\), then
\[
\begin{align*}
r_0 &= \frac{(g-1)^2}{g(2g+1)} \left( \frac{2g-5}{12} \delta_0(X/Y) \\
&\quad + \sum_{j=1}^{[(g-1)/2]} (2j(g-1-j) - 1)\delta_j(X/Y) \right)
\end{align*}
\]
(2) If $g \geq 5$, then
\[
\begin{align*}
  r_0 & = \frac{(g - 1)^2}{g(2g + 1)} \left( \frac{(2g - 5)}{12} \xi_0(X/Y) \\
  & \quad + \sum_{j=1}^{[g-1]/2} \frac{2(3j(g - 1 - j) - g - 2)}{3} \xi_j(X/Y) + \sum_{i=1}^{[g/2]} 4i(g - i)\delta_i(X/Y) \right).
\end{align*}
\]

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2. Some remarks on the admissible constants

In this paper, we mean by a graph a topological graph in sense of [7] equipped with the set of edges and the set of vertices.

Let $G$ be a connected graph, and $\text{Vert}(G)$ (resp. $\text{Ed}(G)$) the set of vertices (resp. edges) of $G$. If $\sim$ is an equivalence relation in $\text{Ed}(G)$, we set $\text{Ed}(G)^\sim = \text{Ed}(G)/\sim$. Let $\bigoplus_{e \in \text{Ed}(G)} \mathbb{R} e$ (resp. $\bigoplus_{e \in \text{Ed}(\text{Ed}(G))} \mathbb{R} \bar{e}$) be the $\mathbb{R}$-vector space formally generated by $\text{Ed}(G)$ (resp. $\text{Ed}(G)^\sim$), and $\mathcal{M}(\text{Ed}(G))$ (resp. $\mathcal{M}(\text{Ed}(G)^\sim)$) the dual vector space of $\bigoplus_{e \in \text{Ed}(G)} \mathbb{R} e$ (resp. $\bigoplus_{e \in \text{Ed}^\sim} \mathbb{R} \bar{e}$). We express by $\{e^*\}_{e \in \text{Ed}(G)}$ (resp. $\{\bar{e}^*\}_{\bar{e} \in \text{Ed}(G)}$) the dual basis of $\mathcal{M}(\text{Ed}(G))$ (resp. $\mathcal{M}(\text{Ed}(G)^\sim)$) with respect to $\text{Ed}(G)$ (resp. $\text{Ed}(G)^\sim$). We have the natural projection $\bigoplus_{e \in \text{Ed}(G)} \mathbb{R} e \to \bigoplus_{\bar{e} \in \text{Ed}(G)^\sim} \mathbb{R} \bar{e}$ and the natural inclusion $\mathcal{M}(\text{Ed}(G)^\sim) \hookrightarrow \mathcal{M}(\text{Ed}(G))$. Set
\[
\mathcal{M}(\text{Ed}(G)^\sim)_{>0} = \left\{ \lambda : \bigoplus_{\bar{e} \in \text{Ed}(G)^\sim} \mathbb{R} \bar{e} \to \mathbb{R} \mid \lambda(\bar{e}) > 0 \text{ for any } e \in \text{Ed}(G) \right\}.
\]

Note that to give an element $\lambda \in \mathcal{M}(\text{Ed}(G)^\sim)_{>0}$ is nothing but to give length to each edge such that length of $e$ is $\lambda(\bar{e})$. In this sense, we sometimes call an element $\lambda \in \mathcal{M}(\text{Ed}(G)^\sim)_{>0}$ a Lebesgue measure on $G$, and call a graph equipped with a Lebesgue measure a metrized graph.

Now, we recall several facts on Green’s function on a metrized graph. For details on metrized graphs, see [7].

Let $(G; \lambda)$ be a connected metrized graph and $D$ an $\mathbb{R}$-divisor on $G$. If $\text{deg}(D) \neq -2$, then there is a unique measure $\mu_{(G; \lambda, D)}$ on $G$ and a unique function $g_{(G; \lambda, D)}$ on $G \times G$ with the following properties.

(a) $\int_G \mu_{(G; \lambda, D)} = 1$.

(b) $g_{(G; \lambda, D)}(x, y)$ is symmetric and continuous on $G \times G$.

(c) For a fixed $x \in G$, $\Delta_y(g_{(G; \lambda, D)}(x, y)) = \delta_x - \mu_{(G; \lambda, D)}$.

(d) For a fixed $x \in G$, $\int_G g_{(G; \lambda, D)}(x, y)\mu_{(G; \lambda, D)}(y) = 0$.

(e) $g_{(G; \lambda, D)}(D, y) + g_{(G; \lambda, D)}(y, y)$ is a constant function on $y \in G$.

The constant $g_{(G; \lambda, D)}(D, y) + g_{(G; \lambda, D)}(y, y)$ is denoted by $c(G; \lambda, D)$. Further, we set
\[
\epsilon(G; \lambda, D) = 2\text{deg}(D)c(G; \lambda, D) - g_{(G; \lambda, D)}(D, D),
\]
which we call the admissible constant of \((G; \lambda, D)\). In this paper, we consider polarizations on \(G\) supported in \(\text{Vert}(G)\) only.

Let \((G; \lambda, D)\) be a connected polarized metrized graph. We consider the following constants arising from \((G; \lambda, D)\). (In the following, \(e\) is an edge, \(P_e\) and \(Q_e\) are the terminal points of \(e\), \(e^o = e \setminus \{P_e, Q_e\}\), and \(P, Q \in \text{Vert}(G)\).)

\[
\begin{align*}
l_e &= \lambda(\bar{e}) : \text{the length of } e \\
r_{(G; \lambda)}(P, Q) &= : \text{the resistance between } P \text{ and } Q \\
r_e &= r_{G \setminus \{e^o\}}(P_e, Q_e) \\
g_{(G; \lambda, D)}(P, Q) \\
e(\lambda, D) \
\end{align*}
\]

If \(\gamma(G; \lambda, D)\) is one of the above constants, then it is easy to see that the function on \(\mathcal{M}(\text{Ed}(G)^\sim)_{>0}\) defined by

\[
\lambda \mapsto \gamma(G; \lambda, D)
\]

is a rational function. We denote these functions by the “similar” symbols as follows.

\[
\begin{align*}
X_{\bar{e}} : \lambda &\mapsto \lambda(\bar{e}) \\
r_G(P, Q) : \lambda &\mapsto r_{(G; \lambda)}(P, Q) \\
R_e : \lambda &\mapsto r_e \\
g_{(G, D)}(P, Q) : \lambda &\mapsto g_{(G, \lambda, D)}(P, Q) \\
\epsilon(G, \lambda, D) : \lambda &\mapsto \epsilon(G, \lambda, D)
\end{align*}
\]

When we do not have to emphasize \(\lambda\), we sometimes write \(\bar{\epsilon}(G, D)\) for \(\epsilon(G; \lambda, D)\), for example. Note that these rational functions can be viewed as elements of rational function field \(\mathbb{Q}(\{X_{\bar{e}}\}_{\bar{e} \in \text{Ed}(G)^\sim})\) generated by indeterminates \(\{X_{\bar{e}}\}_{\bar{e} \in \text{Ed}(G)^\sim}\).

Let \(G\) be a connected graph and \(S\) a subset of \(\text{Ed}(G)^\sim\). We define \(G_S\) as the graph obtained by contracting all edges \(e\) with \(\bar{e} \in S\), and define \(G^S\) as \(G^S_{\text{Ed}(G)^\sim \setminus S}\). For a polarization \(D\) on \(G\), we also define \(D_S\) (resp. \(D^S\)) as the polarization on \(G_S\) (resp. \(G^S\)) induced by \(D\) in the following way. Let \(v\) be a vertex of \(G_S\) and \(\{v_1, \ldots, v_k\}\) the set of vertices of \(G\) which go to \(v\) when we contract the edges in \(S\). Then, we set the coefficient of \(v\) of \(D_S\) to be the sum of coefficients of all \(v_i\)'s of \(D\). Note that \(\deg(D_S) = \deg(D)\).

**Lemma 2.1.** In the same notation as above, we have

\[
\epsilon(G, D)(X_{\bar{e}_0} = 0) = \epsilon(G_{\bar{e}_0}, D_{\bar{e}_0})
\]

for any \(\bar{e}_0 \in \text{Ed}(G)^\sim\).

**Proof.** It is sufficient to show that

\[
\lim_{l_{\bar{e}_0} \to 0} \epsilon(G; \lambda, D) = \epsilon(G_{\bar{e}_0}; \lambda', D_{\bar{e}_0})
\]

for any \(\lambda = \sum_{\bar{e} \in \text{Ed}(G)^\sim} l_{\bar{e}} \bar{e}^\ast\), where \(\lambda' = \sum_{\bar{e} \in \text{Ed}(G_{\bar{e}_0})} l_{\bar{e}} \bar{e}^\ast\).
We may assume that all edges are connected closed interval. Let \( s_e : e \to [0,l_e] \) be a parameterization. Then, we can set
\[
g(x) := g(G;\lambda,D)(O,x) = \alpha_e s_e(x)^2 + \beta_e s_e(x) + \gamma_e 
\]
for some \( \alpha_e, \beta_e, \gamma_e \in \mathbb{R} \) and \( \alpha'_e, \beta'_e, \gamma'_e \in \mathbb{R} \).

The continuous condition on \( G, \Delta(g) = \delta_O - \mu(G;\lambda,D) \), the continuous condition on \( G' \) and \( \Delta(g') = \delta_O - \mu(G_D;\lambda) \) give a system of linear equations on \( \alpha_e, \beta_e, \gamma_e \) and \( \alpha'_e, \beta'_e, \gamma'_e \). It is easy to see that \( \alpha_e \to \alpha'_e, \beta_e \to \beta'_e \) when \( l_e \to 0 \). By the conditions \( \int_G g\mu(G;\lambda,D) \) and \( \int_{G_D} g\mu(G_D;\lambda) \), we also have \( \gamma_e \to \gamma'_e \), hence we obtain the lemma.

Let \( G_1 \) and \( G_2 \) be graphs. Fix vertices \( v_1 \in G_1 \) and \( v_2 \in G_2 \). The one-point-sum \( G_1 \vee G_2 \) with respect to \( v_1 \) and \( v_2 \) is defined as \( (G_1 \vee G_2)/v_1 \sim v_2 \). The set of edges is naturally defined by \( \text{Ed}(G_1 \vee G_2) = \text{Ed}(G_1) \amalg \text{Ed}(G_2) \) and the set of vertices is defined by \( \text{Vert}(G_1 \vee G_2) = \text{Vert}(G_1) \amalg \text{Vert}(G_2)/v_1 \sim v_2 \). If \( G_i \) has a Lebesgue measure \( \lambda_i \) for \( i = 1, 2 \), then \( G_1 \vee G_2 \) has the canonical Lebesgue measure given by \( \lambda = \lambda_1 + \lambda_2 \).

**Definition 2.2.** Let \( G \) be a connected graph which is not one point. \( G \) is said to be **reducible** if there exist two graphs \( G_1 \) and \( G_2 \) which are not one point such that \( G \) is a one-point-sum of \( G_1 \) and \( G_2 \). \( G \) is said to be **irreducible** if it is not reducible.

For any connected graph \( G \), we have the **irreducible decomposition** of \( G \). Set
\[
J = \{ v \in \text{Vert}(G) \mid G \setminus \{ v \} \text{ is not connected.}\}
\]
Let \( H^o \) be a connected component of \( G \setminus J \). Then, the closure of \( H^o \) is an irreducible subgraph of \( G \), and we call it an irreducible component of \( G \). Let \( \{ G_1, \ldots, G_n \} \) be the set of irreducible components. For a permutation \( \sigma : (1,2,\ldots,n) \mapsto (i_1, i_2, \ldots, i_n) \), we define a sequence of subgraphs of \( G \) inductively by
\[
H_{i_k}^o = G_{i_k}, \quad H_k^o = H_{k-1}^o \cup G_{i_k}.
\]
Then, we can easily see by the definition of \( \{ G_1, \ldots, G_n \} \) that there exists a permutation \( \sigma \) such that \( H_{k-1}^o \cup G_{i_k} \) is a one-point-sum of \( H_{k-1}^o \) and \( G_{i_k} \) for all \( k = 2, \ldots, n \). In this sense, we write \( G_1 \vee \cdots \vee G_n \) instead of \( G_1 \cup \cdots \cup G_n \) and call it the **irreducible decomposition** of \( G \). We denote by \( \text{Irr}(G) \) the set of all irreducible components of \( G \).

The next proposition implies that irreducible graphs are fundamental for calculating the admissible constants.

**Proposition 2.3.** Let \( G_1, G_2 \) and \( G = G_1 \vee G_2 \) be connected graphs, and \( D \) a polarization supported in \( \text{Vert}(G) \) with \( \deg D \neq -2 \). Let \( D_i \) be the polarization on \( G_i \) defined by \( D_i = D^{\text{Ed}(G_i)} \) for \( i = 1, 2 \). Then, we have
\[
\epsilon(G,D) = \epsilon(G_1,D_1) + \epsilon(G_2,D_2)
\]
as rational functions on \( \mathcal{M}(\text{Ed}(G))_{>0} \).

**Proof.** Let \( \lambda_1 \) and \( \lambda_2 \) be Lebesgue measures on \( G_1 \) and \( G_2 \) respectively. \( \lambda = \lambda_1 + \lambda_2 \) is a Lebesgue measure on \( G \). By [5, Lemma 3.7], we have
\[
\mu(G;\lambda,D) = \mu(G_1;\lambda_1,D_1) + \mu(G_2;\lambda_2,D_2) - \delta_O.
\]
where \( \{O\} = G_1 \cap G_2 \). Consider the following function on \( G \):

\[
g(x) = \begin{cases} 
g(G_1; \lambda_1, D_1)(O, x) + g(G_2; \lambda_2, D_2)(O, O) & \text{if } x \in G_1, \\
g(G_2; \lambda_2, D_2)(O, x) + g(G_1; \lambda_1, D_1)(O, O) & \text{if } x \in G_2.
\end{cases}
\]

Then, we can easily check that \( g \) is continuous on \( G \), \( \Delta(g) = \delta_O - \mu(G; \lambda, D) \), and \( \int_G g \mu(G; \lambda, D) = 0 \). Thus we have \( g(G; \lambda, D)(O, x) = g(x) \). Therefore, by \([6, \text{Lemma 4.1}]\), we obtain the formula.

\[\square\]

3. Calculations of the admissible constants for hyperelliptic graphs

3.1. Definitions and terminology. First of all, we give the definition of a particular class of graphs, called hyperelliptic graphs.

**Definition 3.1.** Let \( G \) be a connected graph, and \( \text{Vert}(G) \) (resp. \( \text{Ed}(G) \)) the set of vertices (resp. edges) of \( G \). Suppose that \( G \) has a homeomorphism \( \iota : G \to G \) such that \( \iota^2 \) is the identity on \( G \), called the involution on \( G \), which induces naturally an automorphism on \( \text{Vert}(G) \) and \( \text{Ed}(G) \) respectively. Then, \( (G, \text{Vert}(G), \text{Ed}(G), \iota) \), or simply \( G \), is called a hyperelliptic graph if it has the following properties.

1. Every edge is homeomorphic to the connected closed interval.
2. \( \iota(e) \neq e \) for any \( e \in \text{Ed}(G) \).
3. If \( v \) is a vertex with \( \iota(v) \neq v \), then there exist at least three edges which start from \( v \).
4. The topological space \( G/\langle \iota \rangle \) has no loops. (We call such a graph a tree.)

Note that \( G/\langle \iota \rangle \) is a connected graph whose vertices and edges are given by \( \text{Vert}(G)^\sim = \text{Vert}(G)/\langle \iota \rangle \) and \( \text{Ed}(G)^\sim = \text{Ed}(G)/\langle \iota \rangle \) respectively.

When we talk on a measure on a hyperelliptic graph, we always assume that it is invariant under the involution, i.e., an element of \( \mathcal{M}(\text{Ed}(G)^\sim)_{>0} \) with respect to the equivalence relation arising from \( \iota \).

**Example 3.2.** We shall give an example of hyperelliptic graphs, which is the main object in this paper. Let \( G_1 \) be the metrized graph by the configuration of a singular fiber \( C_1 \) of semistable hyperelliptic curve \( f : X \to Y \) as in the introduction. We assume that \( C_1 \) does not have nodes of positive type. \( \text{Vert}(G_1) \) and \( \text{Ed}(G_1) \) correspond to the set of irreducible components of \( C_1 \) and the set of nodes of \( C_1 \) respectively. The hyperelliptic involution \( \iota \) also acts on \( \text{Vert}(G_1) \) and \( \text{Ed}(G_1) \). Then, there may exist an edge \( e \) with \( \iota(e) = e \). Note that if such \( e \) is the connected closed interval, then the vertices which are the terminal points of \( e \) are moved to each other by \( \iota \). For \( e \) with \( \iota(e) = e \), let \( v_e \) be the point on \( e \) such that \( e \setminus (\{\text{vertices on } e\} \cup \{v_e\}) \) is a disjoint union of two open segments of same length. Now, let \( G_2 \) be the metrized graph which is same as \( G_1 \) as a metrized topological space, such that \( \text{Vert}(G_2) \) is the union of \( \text{Vert}(G_1) \) and the set of such \( v_e \)'s as above, and that \( \text{Ed}(G_2) \) is the segments in \( G_1 \) which connect two points in \( \text{Vert}(G_2) \). Then, we can make \( \iota \) act on \( G_2 \) such that \( \iota \) is a symmetric homeomorphism and \( \iota(e) \neq e \) for any \( e \in \text{Ed}(G_2) \). Let \( V \) be the subset of \( \text{Vert}(G_2) \) consisting of vertices \( v \) such that \( \iota(v) \neq v \) and there are only two edges starting from \( v \). Then, \( G_2 \setminus (\text{Vert}(G_2) \setminus V) \) is a disjoint union of open segments. Let \( G_3 \) be the metrized graph which is nothing but \( G_2 = G_1 \) as a metrized space, such that \( \text{Vert}(G_3) = \text{Vert}(G_2) \setminus V \) and \( \text{Ed}(G_3) \) is the set of segments in \( G_2 \) which connect two
points in $\text{Vert}(G_3)$. We can also make $\iota$ act on $G_3$ naturally such that $\iota$ is a symmetric homeomorphism. Noting, in addition, that $G_1/\langle \iota \rangle$ is a tree of $\mathbb{P}^1$, we can easily see by its construction that $G_3$ is a hyperelliptic graph with an $\iota$-invariant measure.

We fix the following terminology.

**Definition 3.3.** Let $G$ be a hyperelliptic graph.

1. $v \in \text{Vert}(G)$ is said to be **fixed** if $\iota(v) = v$. We denote by $\text{Vert}_f(G)$ the set of fixed vertices.
2. $v \in \text{Vert}(G)$ is said to be **non-fixed** if $\iota(v) \neq v$. We denote by $\text{Vert}_{n,f}(G)$ the set of non-fixed vertices.
3. $e \in \text{Ed}(G)$ is said to be **disjoint** if $e \cap \iota(e) = \emptyset$. We denote by $\text{Ed}_0(G)$ the set of disjoint edges.
4. $e \in \text{Ed}(G)$ is said to be **one-jointed** if $e \cap \iota(e)$ is a set of one point. We denote by $\text{Ed}_1(G)$ the set of one-jointed edges.
5. $e \in \text{Ed}(G)$ is said to be **two-jointed** if $e \cap \iota(e)$ is a set of two points. We denote by $\text{Ed}_2(G)$ the set of two-jointed edges.

If $S$ is one of the above sets, we denote by $S^\sim$ the set $S/\langle \iota \rangle$, and we write $\overline{s}$ for the class of $s \in S$ in $S^\sim$.

Let us consider several lemmas concerning the above definitions.

**Lemma 3.4.** If $G_1$ is an irreducible component of a hyperelliptic graph $G$, then we have $\iota(G_1) = G_1$.

**Proof.** Let $\pi : G \to G/\langle \iota \rangle$ be the natural projection. Suppose $\iota(G_1) \neq G_1$. Let $P$ be a vertex of $G_1$ which joints $G_1$ with another component, and $H$ the subgraph containing $G_1$ such that $H \setminus \{P, \iota(P)\}$ is the connected component of $G \setminus \{P, \iota(P)\}$. We note the following claim.

**Claim 1.** Let $G_2$ be an irreducible component of $G$ different from $G_1$ having $P$ as a jointing point with $G_1$. Then, $G_2 \cap H = \{P\}$.

**Proof.** Since $P$ is a jointing point, $G_2 \setminus \{P\}$ is not contained in the connected component of $G \setminus \{P\}$ which $G_1 \setminus \{P\}$ belongs to.

Now, assume $P = \iota(P)$. Then, $\iota(G_1)$ is an irreducible component of $G$ with $P \in \iota(G_1)$. Since $\iota(G_1) \neq G_1$, we must have $\iota(G_1) \cap H = \{P\}$ by the claim, hence $\iota(H) \neq H$. Therefore, we see $\iota(H) \cap H = \{P\}$ by the definition of $H$, and $\pi|_H : H \to \pi(H)$ is an isomorphism, accordingly, $H$ is a tree. Take a terminal point $Q$ of $H$ different from $P$. Then, since $H$ is a tree, there exists only one edge starting from $Q$. This contradicts to the definition of hyperelliptic graphs. Therefore, we must have $\iota(P) \neq P$. Noting $\iota(H) \cap H \supset \{P, \iota(P)\}$ and the definition of $H$, we see again $H = \iota(H)$. Since $P$ and $\iota(P)$ are two distinct jointing points, $G \setminus (H \setminus \{P, \iota(P)\})$ has two connected components $G_3$ and $G_4$ with $\iota(G_3) = G_4$. Therefore, $\pi|_{G_3} : G_3 \to \pi(G_3)$ is an isomorphism, and we have a contradiction in the same way.

**Lemma 3.5.** If $G = G_1 \lor G_2$ is a hyperelliptic graph, then $\iota(O) = O$, where $O$ is the jointing point of $G_1$ and $G_2$ in $G$.

**Proof.** If $\iota(O) \neq O$, then $G_1 \cap G_2$ has two points $O$ and $\iota(O)$ by the above lemma, which contradicts to the assumption of this lemma.
In virtue of the above two lemmas, we see that $G$ is hyperelliptic if and only if every irreducible component is hyperelliptic.

The next lemma characterizes jointing points of a hyperelliptic graph.

**Lemma 3.6.** Let $G$ be a hyperelliptic graph, and $v$ a vertex of $G$. Then, $v$ is a jointing point of irreducible components if and only if $v$ is fixed and at least four edges start from $v$.

**Proof.** The “only if” part is obvious from the above two lemmas. We will show the “if” part.

Let $e_1$, $\iota(e_1)$, $e_2$ and $\iota(e_2)$ be four edges starting from $v$. If $v$ is not a jointing point, then $G \setminus \{v\}$ is connected, hence we can find a path connecting $e_1$ with $e_2$ in $G \setminus \{v\}$. This shows, however, that $G/\langle \iota \rangle$ has a loop, which contradicts to Definition 3.1 (4).

**Lemma 3.7.** If $e$ is a two-jointed edge, then $e \cup \iota(e)$ is an irreducible component of $G$.

**Proof.** By Lemma 3.6, it is sufficient to show that if $v$ is a vertex of $G$ with $\iota(v) \neq v$, then $\iota(v)$ and $v$ cannot be connected by one edge. Suppose that $\iota(v)$ and $v$ are connected by one edge $e$. If we suitably parameterize $e \cup \iota(e)$, then it is homeomorphic to the circle $S^1 = \{(\cos t, \sin t)\}$ and the action of $\iota$ on $e \cup \iota(e)$ is nothing but a map from $S^1$ to $S^1$ given by $t \mapsto t + \pi$. Therefore, the image of $e \cup \iota(e)$ in $G/\langle \iota \rangle$ is a circle, which is a contradiction. Thus, we can give the following definitions.

**Definition 3.8.** (1) An irreducible hyperelliptic graph $G$ is said to be simple if $G$ consists of two two-jointed edges.

(2) A hyperelliptic graph $G$ is said to be semisimple if every irreducible component of $G$ is simple.

We can easily see that the simple graph is uniquely determined. We denote by $SG$ the simple graph. (See Figure 1.)

\begin{figure}[h]
\centering
\includegraphics[width=0.1\textwidth]{SG}
\caption{SG}
\end{figure}

Let $\#\text{Irr}(G)$ be the number of irreducible components of $G$ and let $\#\text{Irr}_s(G)$ be that of irreducible components of $G$ which is simple.

**Remark 3.9.** The following are immediate from the definitions.

(1) We have $\#\text{Irr}(G) = \#\text{Irr}(G_\bar{e})$ for $\bar{e} \in \text{Ed}_0(G)\sim$.

(2) We have $\#\text{Irr}(G) < \#\text{Irr}(G_\bar{e})$ for $\bar{e} \in \text{Ed}_1(G)\sim$.

(3) We have $\#\text{Irr}(G) = \#\text{Irr}(G_\bar{e}) + 1$ for $\bar{e} \in \text{Ed}_2(G)\sim$.

**Definition 3.10.** Let $G$ be a hyperelliptic graph. We define the size of $G$, denoted by $sz(G)$, in the following way.
(1) If $G$ is irreducible, we set
\[ sz(G) = \begin{cases} 
1 & \text{if } G \text{ is simple,} \\
 #(\text{Ed}_1(G) \sim) - 1 & \text{otherwise.} 
\end{cases} \]

(2) If $G = G_1 \lor G_2 \lor \cdots \lor G_k$, where $G_i$ \((i = 1, 2, \cdots, k)\) is irreducible, we set
\[ sz(G) = \sum_{i=1}^{k} sz(G_i). \]

Note that
\[ sz(G) = #(\text{Ed}_1(G) \sim) - 1 + \#\text{Irr}_s(G) - \#\text{Irr}(G) \]
\[ = #(\text{Ed}_1(G) \sim) - \#\text{Irr}(G) + 2\#\text{Irr}_s(G). \]

**Definition 3.11.** An irreducible hyperelliptic graph $G$ of size $n$ is said to be \((n\text{-th})\) elementary if all edges of $G$ are one-jointed.

Note that $sz(G) > 1$ if $G$ is an elementary graph. The $n$-th elementary graph is uniquely determined, and we denote it by $G_n$. (See Figure 2.)

**Figure 2.** $G_n$

**Lemma 3.12.** Let $G$ be a hyperelliptic graph. Then, we have $sz(G) = sz(G_e)$ for $e \in \text{Ed}_0(G) \cup \text{Ed}_1(G)$.

**Proof.** We may assume that $G$ is irreducible.

If $e$ is disjoint, then it is obvious. Suppose $e$ is one-jointed. Let $v$ and $\iota(v)$ be the terminal points of $e \cup \iota(e)$, and suppose that $k$ disjoint edges and $(l + 1)$ one-jointed edges start from $v$. Then, $G_{\bar{e}}$ decomposes into $k$ non-simple components and $l$ simple components. Hence, we have
\[ sz(G_{\bar{e}}) = #\text{Ed}_1(G_{\bar{e}}) \sim - k + l \]
\[ = ( #\text{Ed}_1(G) \sim - 1 - l + k ) - k + l \]
\[ = sz(G). \]
\[ \square \]
We define functions $\nu^0_G$, $\nu^1_G$, and $\nu_G$ on $\text{Vert}_{n.f}(G)$ as follows.

\[
\nu^0_G(v) = \text{the number of disjoint edges which start from } v.
\]
\[
\nu^1_G(v) = \text{the number of one-jointed edges which start from } v.
\]
\[
\nu_G(v) = \nu^0_G(v) + \nu^1_G(v).
\]

Let $G$ be a hyperelliptic graph. Let $\{X_e\} e \in \text{Ed}(G)^\sim$ be the set of symbols as in section 2, and $V$ the $\mathbb{Q}$-vector space with basis $\{X_e\} e \in \text{Ed}(G)^\sim$. We denote the space of homogeneous polynomials of degree $d$ by $S^dV$, i.e., $S^dV$ is the $d$-th symmetric tensor product of $V$.

Put $n = \text{sz}(G)$. Now, we introduce important polynomials.

(1) Choose distinct $\bar{e}_1, \ldots, \bar{e}_n \in \text{Ed}(G)^\sim$. Set

\[
\delta_{\bar{e}_1, \ldots, \bar{e}_n} = \begin{cases} 
1 & \text{if } G^{\bar{e}_1, \ldots, \bar{e}_n} \text{ is a semisimple hyperelliptic graph of size } n, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
L_G = \sum_{\bar{e}_1, \ldots, \bar{e}_n} \delta_{\bar{e}_1, \ldots, \bar{e}_n} X_{\bar{e}_1} \cdots X_{\bar{e}_n} \in S^nV.
\]

(2) Choose distinct $\bar{e}_1, \ldots, \bar{e}_{n+1} \in \text{Ed}(G)^\sim$. If $\#(\text{Vert}_{n.f}(G^{\bar{e}_1, \ldots, \bar{e}_{n+1}})^\sim) = 1$, we denote by $\nu(\bar{v})$ the number of edges which start from a representative $v$ of the unique non-fixed vertex class $\bar{v}$ of $G^{\bar{e}_1, \ldots, \bar{e}_{n+1}}$. Then we set

\[
c_{\bar{e}_1, \ldots, \bar{e}_{n+1}} = \begin{cases} 
\nu(\bar{v}) - 2 & \text{if } \#(\text{Vert}_{n.f}(G^{\bar{e}_1, \ldots, \bar{e}_{n+1}})^\sim) = 1 \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
M_G = \sum_{\bar{e}_1, \ldots, \bar{e}_{n+1}} c_{\bar{e}_1, \ldots, \bar{e}_{n+1}} X_{\bar{e}_1} \cdots X_{\bar{e}_{n+1}} \in S^{n+1}V.
\]

Note that $M_G = 0$ if $G$ is semisimple.

In the case where $G$ is irreducible, $L_G$ and $M_G$ can be expressed in another way. Let $\bar{e}_1, \ldots, \bar{e}_k$ be distinct disjoint edges, and $v$ a non-fixed vertex of the graph $G_{\text{Ed}_0(G)^\sim \setminus \{\bar{e}_1, \ldots, \bar{e}_k\}}$. For simplicity, we set $G' := G_{\text{Ed}_0(G)^\sim \setminus \{\bar{e}_1, \ldots, \bar{e}_k\}}$. Set $\sigma_{(\bar{e}_1, \ldots, \bar{e}_k; \bar{v})}$ to be the $(\nu^1_{G'}(v) - 1)$-th elementary symmetric polynomial on $\{X_e\} e \in \text{Ed}_1(G')^\sim_{\bar{v}}$ and $\tau_{(\bar{e}_1, \ldots, \bar{e}_k; \bar{v})}$ to be the $\nu^1_{G'}(v)$-th elementary symmetric polynomial on $\{X_e\} e \in \text{Ed}_1(G')^\sim_{\bar{v}}$, where $\text{Ed}_1(G')^\sim_{\bar{v}}$ is the set of edge classes whose representatives start from $v$.

Then, we can easily see that

\[
L_G = \sum_{\bar{e}_1, \ldots, \bar{e}_k \in \text{Ed}_0(G)^\sim} \left( \prod_{\bar{v}} \sigma_{(\bar{e}_1, \ldots, \bar{e}_k; \bar{v})} \right) X_{\bar{e}_1} \cdots X_{\bar{e}_k}
\]

\[
M_G = \sum_{\bar{e}_1, \ldots, \bar{e}_k \in \text{Ed}_0(G)^\sim} \left( \sum_{\bar{v}} \left( (\nu(\bar{v}) - 2) \tau_{(\bar{e}_1, \ldots, \bar{e}_k; \bar{v})} \prod_{\bar{v}' \neq \bar{v}} \sigma_{(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}')} \right) \right) X_{\bar{e}_1} \cdots X_{\bar{e}_k},
\]
where \( \bar{v} \) runs over \( \text{Vert}_{n,1}(G_{\text{Ed}_0(G)^\sim \setminus \set{\bar{e}_1, \ldots, \bar{e}_k}})^\sim \). Note that \( L_G \) is an irreducible polynomial if \( G \) is an irreducible hyperelliptic graph.

**Remark 3.13.** Let \( G = G_1 \vee G_2 \) be a hyperelliptic graph of size \( n \).

1. For distinct \( \bar{e}_1, \ldots, \bar{e}_k \in \text{Ed}(G_1)^\sim \) and distinct \( \bar{e}_{k+1}, \ldots, \bar{e}_n \in \text{Ed}(G_2)^\sim \), \( G^{\bar{e}_1, \ldots, \bar{e}_k}_{\bar{e}_{k+1}, \ldots, \bar{e}_n} \) is a semisimple hyperelliptic graph of size \( n \) if and only if \( G^{\bar{e}_1, \ldots, \bar{e}_k}_{\bar{e}_{k+1}, \ldots, \bar{e}_n} \) is a semisimple hyperelliptic graph of size \( \text{sz}(G_1) \) (resp. \( \text{sz}(G_2) \)).
2. For distinct \( \bar{e}_1, \ldots, \bar{e}_k \in \text{Ed}(G_1)^\sim \) and distinct \( \bar{e}_{k+1}, \ldots, \bar{e}_{n+1} \in \text{Ed}(G_2)^\sim \), \( G^{\bar{e}_1, \ldots, \bar{e}_k}_{\bar{e}_{k+1}, \ldots, \bar{e}_{n+1}} \) is a one-point-sum of semisimple hyperelliptic graphs and the \( l \)-th elementary hyperelliptic graph if and only if one of \( \{G^{\bar{e}_1, \ldots, \bar{e}_k}_{\bar{e}_1, \bar{e}_{k+1}, \ldots, \bar{e}_{n+1}}, G^{\bar{e}_1, \bar{e}_{k+1}, \ldots, \bar{e}_{n+1}}_{\bar{e}_1, \ldots, \bar{e}_{k+1}}\} \), say \( G^{\bar{e}_1, \ldots, \bar{e}_k}_{\bar{e}_1, \bar{e}_{k+1}, \ldots, \bar{e}_{n+1}} \), is a semisimple hyperelliptic graph of size \( \text{sz}(G_1) \), and the other, say \( G^{\bar{e}_1, \bar{e}_{k+1}, \ldots, \bar{e}_{n+1}}_{\bar{e}_1, \ldots, \bar{e}_{k+1}} \), is a one-point-sum of \( (\text{sz}(G_2) - l) \) simple hyperelliptic graphs and the \( l \)-th elementary hyperelliptic graph.

The next lemma is simple, but important for our latter purpose.

**Lemma 3.14.** Let \( G \) be a hyperelliptic graph.

1. \( L_{G_{\bar{e}}} = L_G(X_{\bar{e}} = 0), M_{G_{\bar{e}}} = M_G(X_{\bar{e}} = 0) \).
2. If \( G = G_1 \vee G_2 \), then we have

\[
\frac{M_G}{L_G} = \frac{M_{G_1}}{L_{G_1}} + \frac{M_{G_2}}{L_{G_2}}.
\]

**Proof.** (1) is obvious from the definitions. For (2), it is sufficient to show that \( L_{G_1}L_{G_2} = L_G \) and \( M_{G_1}L_{G_2} + L_{G_1}M_{G_2} = M_G \), but they are also obvious from the definitions and the above remarks.

Let \( G \) be a hyperelliptic graph, and \( D \) a polarization on \( G \) with \( \iota(D) = D \). For any \( \bar{e} \in \text{Ed}(G)^\sim \), \( G^\bar{e} \) is simple and \( D^\bar{e} = aP + bQ \), where \( P \) and \( Q \) are the vertices and \( a, b \in \mathbb{R} \).

Set \( w(\bar{e}) = \min\{a, b\} \).

The following theorem is a key result for our main theorem.

**Theorem 3.15.** Let \( G \) be a hyperelliptic graph, and \( D \) a polarization given by

\[
D = \sum_{v \in \text{Vert}_{n,1}(G)} (\nu_G(v) - 2)v + \sum_{v' \in \text{Vert}_{1}(G)} a_{v'}v',
\]

where \( a_{v'} \in \mathbb{R} \). Then, if \( \deg(D) + 2 \neq 0 \), we have

\[
\epsilon(G, D) = \sum_{\bar{e} \in \text{Ed}(G)^\sim} \left( \frac{2}{3\deg(D) + 2} + \frac{w(\bar{e})(\deg(D) - w(\bar{e}))}{\deg(D) + 2} \right) X_{\bar{e}} + 2 \frac{\deg(D) M_G}{3\deg(D) + 2 L_G},
\]

as rational functions on the length of each edge, i.e., on \( \mathcal{M}(\text{Ed}(G)^\sim)_{> 0} \).

In the rest of this section, we will give the proof of Theorem 3.15.

3.2. **Preliminaries to the proof of Theorem 3.15.** First of all, let us begin with direct calculations of the admissible constants for \( SG \) and \( G_{n-1} \) \((n > 2)\).
Proposition 3.16. (1) Let \( P \) and \( Q \) be the two vertices of \( SG \), and \( D = aP + bQ \) a polarization on \( SG \). Then, we have
\[
\epsilon(SG, D) = \left( \frac{2 \deg(D)}{3 \deg(D) + 2} + \frac{ab}{\deg(D) + 2} \right) X_\epsilon.
\]

(2) Let \( Q \) be a non-fixed vertex of \( G_n \), \( \{e_1, \ldots, e_n\} \) the set of edges which start from \( Q \), \( P_i \) the other vertex of \( e_i \) for \( i = 1, \ldots, n \), and
\[
D = (n-2)Q + (n-2)\iota(Q) + \sum_{i=1}^{n} a_i P_i
\]
a hyperelliptic polarization on \( G_{n-1} \) with \( \deg(D) + 2 \neq 0 \). Then, we have
\[
\epsilon(G_{n-1}, D) = \sum_{i=1}^{n} \left( \frac{2 \deg(D)}{3 \deg(D) + 2} + \frac{a_i(\deg(D) - a_i)}{\deg(D) + 2} \right) X_{\epsilon_i} + \frac{2 \deg(D)}{3 \deg(D) + 2} \frac{(n-2)\sigma_n}{\sigma_{n-1}},
\]
where \( \sigma_k \) is the \( k \)-th elementary symmetric polynomial on \( \{X_{\epsilon_i}\} \).

Note that \( L_{G_{n-1}} = \sigma_{n-1} \) and \( M_{G_{n-1}} = (n-2)\sigma_n \).

Proof. For (1), it is easy to see by [Lemma 3.7, Proposition 4.2, Corollary 4.3]. We will prove (2).

Let \( \lambda = m_1\overline{e}_1 + \cdots + m_n\overline{e}_n \) be a Lebesgue measure on \( G_n \). Set
\[
\overline{\sigma}_k = \sigma_k(m_1, \ldots, m_n)
\]
\[
\overline{\sigma}_k^{(i)} = \text{the } k \text{-th elementary symmetric polynomial on } \{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n\}.
\]

By [Lemma 3.7,], we have
\[
\mu(G_n; \lambda, D) = \frac{1}{\deg(D) + 2} \left( \sum_{i=1}^{n} a_i \delta_i + \sum_{i=1}^{n} \frac{\overline{\sigma}_n}{\delta_{n-1}} (de_i + d\iota(e_i)) \right).
\]

Let
\[
s_i : e_i \to [0, m_i]
\]
be an arc-length parameter such that \( s_i(P_i) = 0 \) and \( s_i(Q) = m_i \). We denote by the same symbol \( s_i \) the parameter \( s_i \circ \iota \) on \( \iota(e_i) \). Consider the following function on \( G_{n-1} \):
\[
g(x) = \begin{cases} 
\frac{\overline{\sigma}_n}{2(\deg(D) + 2)\delta_{n-2}} s_1(x)^2 & \text{on } e_1 \text{ or } \iota(e_1), \\
\frac{a_1}{2(\deg(D) + 2) - 1} s_i(x) + \gamma_1 & \text{on } e_i \text{ or } \iota(e_i) \text{ for } i \neq 1,
\end{cases}
\]
\[
\frac{1}{2(\deg(D) + 2)\delta_{n-2}} s_i(x)^2 + \frac{a_i}{2(\deg(D) + 2)} s_i(x) + \gamma_i.
\]
where
\[
\gamma_1 = \frac{2}{3(\deg(D) + 2)^2} \left( \bar{\sigma}_1 + (n - 2) \frac{\bar{\sigma}_n}{\bar{\sigma}_{n-1}} \right) + \frac{(\deg(D) - a_1)(\deg(D) - a_1 + 2)}{2(\deg(D) + 2)^2} m_1 + \sum_{j=1} a_j(a_j + 2) \frac{m_j}{2(\deg(D) + 2)^2},
\]
and
\[
\gamma_i = \frac{2}{3(\deg(D) + 2)^2} \left( \bar{\sigma}_1 + (n - 2) \frac{\bar{\sigma}_n}{\bar{\sigma}_{n-1}} \right) + \sum_{j\neq i} a_j(a_j + 2) \frac{m_j}{2(\deg(D) + 2)^2} + \frac{a_1 - (a_1 + 1)(\deg(D) - a_1 + 2)}{(\deg(D) + 2)^2} m_1 + \frac{a_i - (a_i + 1)(\deg(D) - a_i + 2)}{(\deg(D) + 2)^2} m_i
\]
if \(i \neq 1\). Then, we can check by direct calculations that \(g\) is continuous, \(\Delta(g) = \delta_{P_1} - \mu_{(G;\lambda,D)}\), and \(\int g_{(G;\lambda,D)} = 0\). Thus, \(g_{(G;\lambda,D)}(P_1, x) = g(x)\), and by [3, Lemma 4.1], we obtain the formula.

Following two lemmas are fundamental for hyperelliptic graphs.

**Lemma 3.17.** Let \((G, D)\) be a polarized hyperelliptic graph with a Lebesgue measure \(\lambda = \sum e \lambda e^*\). Then, the admissible metric is given by
\[
\mu_{(G;\lambda,D)} = \frac{1}{\deg(D) + 2} \left( \delta_D - \delta_K + \sum_{e \in \text{Ed}(G)} \frac{P_G^e(\lambda)}{L_G(\lambda)} de \right),
\]
where \(P_G^e\) is the coefficient of \(X_e\) of \(L_G\) when \(L_G\) is regarded as a polynomial on \(X_e\).

**Proof.** We will prove the lemma by induction on the size of \(G\).

If \(\text{sz}(G) \leq 2\), we obtain the formula by direct calculations.

Now, suppose that \(\text{sz}(G) > 2\). We may assume that \(G\) is irreducible. For any \(e_0 \in \text{Ed}(G)\), there exists a non-fixed vertex \(v\) such that at least two one-jointed edges \(e_1\) and \(e_2\) start from \(v\) and that \(\bar{e}_1\) and \(\bar{e}_2\) are different from \(\bar{e}_0\). Since this lemma is true for a hyperelliptic graph \(G'_e\), if it is true for \(G'\), we may assume that \(\nu(v) = 3\), i.e., exactly three edges start from \(v\). Note that the third edge \(\bar{e}_3\) which starts from \(v\) is a disjoint edge since \(\text{sz}(G) > 2\). Let \(H_1\) be the subgraph generated by \(\{e_1, e_2, e_3\} \cup \iota(\{e_1, e_2, e_3\})\) and \(G_1\) a metrized graph characterized by the following conditions.

(a) \(\text{Ed}(G_1) = \{e'_1, e''_1\}\), \(e'_1\) and \(e''_1\) are just the closed connected intervals, and \(e'_1\) intersects with \(e''_1\) in one point \(v'_0\).

(b) \(l_{e'_1} = l_{e''_1} = l_{e_3} + \frac{l_{e_1} l_{e_2}}{l_{e_1} + l_{e_2}}\).

Let \(v'_1\) (resp. \(v''_1\)) be the terminal point of \(e'_1\) (resp. \(e''_1\)) which is not \(v'_0\), and \(v_1\) the terminal point of \(e_3\) which is not \(v\). We would like to consider another graph \(G''\) which we obtain from \(G\) by replacing the subgraph \(H_1\) by \(G_1\): the graph constructed in the following way.

1. Remove \(H_1 \setminus \{v_1, \iota(v_1)\}\) from \(G\).
2. Connect \(G_1\) with \(G \setminus (H_1 \setminus \{v_1, \iota(v_1)\})\) by identifying \(v'_1\) with \(v_1\), and \(v''_1\) with \(\iota(v_1)\). (See Figure 3)
$G'$ is again a hyperelliptic graph of size $(s_z(G) - 1)$. Note that we can naturally see each $e \in \text{Ed}(G) \setminus (\{e_1, e_2, e_3\} \cup \iota(\{e_1, e_2, e_3\}))$ as an edge of $G'$. When we regard $e$ as an edge of $G'$, we denote it by $e'$. Let $\lambda'$ be a Lebesgue measure on $G'$ such that

$$l_{\bar{e}'} = \frac{l_{\bar{e}} + l_{\bar{e}_1} l_{\bar{e}_2}}{l_{\bar{e}_1} + l_{\bar{e}_2}},$$

$$l_{e'} = l_{\bar{e}} \quad \text{for } e' \neq \bar{e}'_1.$$

By the definition of $(G'; \lambda')$, we have

$$\frac{1}{l_{\bar{e}} + r_{\bar{e}}} = \begin{cases} 
\frac{1}{l_{e'} + r_{e'}} & \text{if } \bar{e} \neq \bar{e}_1, \bar{e}_2, \bar{e}_3, \\
\frac{1}{l_{\bar{e}'_1} + r_{\bar{e}'_1}} & \text{if } \bar{e} = \bar{e}_3.
\end{cases}$$

On the other hand, we have

$$\frac{2}{l_{\bar{e}_0} + r_{\bar{e}_0}} = \frac{P_{G'}^{\bar{e}_0}(\lambda')}{L_{G'}(\lambda')},$$

by the induction hypothesis. Therefore, the following suffices for our lemma since $(X_{\bar{e}_1} + X_{\bar{e}_2})P_{G}^{\bar{e}_0}(\{Y_{e'}\}) = P_{G}^{\bar{e}_0}$ is automatic if the following is shown:

$$(X_{\bar{e}_1} + X_{\bar{e}_2})L_{G'}(\{Y_{e'}\}) = L_G,$$

where

$$Y_{\bar{e}'} = \begin{cases} 
X_{\bar{e}} & \text{if } \bar{e} \neq \bar{e}_1, \bar{e}_2, \bar{e}_3 \\
X_{\bar{e}_3} + \frac{X_{\bar{e}_1} X_{\bar{e}_2}}{X_{\bar{e}_1} + X_{\bar{e}_2}} & \text{if } \bar{e}' = \bar{e}'_1.
\end{cases}$$

This can be checked by direct calculations if we use the second expression of $L_G$.

Lemma 3.18. Let $G$ be a hyperelliptic graph and $D$ a polarization on $G$ such that the coefficient of every non-fixed vertex $v$ in $D$ is $\nu_G(v) - 2$. Then, there is a homogeneous polynomial $F$ of degree $(s_z(G) + 1)$ such that

$$\epsilon(G, D) = \frac{F}{L_G}.$$  

Proof. We may assume that $G$ is irreducible. The following is a key claim.
Claim 2. In the same situation, let $O$ be a fixed vertex, $e$ a one-jointed edge starting from $O$, and $P_1$ the other terminal vertex of $e$. Assume that there exists another one-jointed edge starting from $P_1$. Then, for any vertex $P$, there exists a homogeneous polynomial $F_P$ of degree $(sz(G) + 1)$ with

$$g_{(G,D)}(O, P) = \frac{F_P}{L_G}.$$  

Proof. We will show the claim by induction on $sz(G)$. For $sz(G) = 2$, we have already obtained the claim in Proposition 3.16. Assume that we have the claim for $sz(G) \leq n$. To simplify the notations, we only prove the claim for the graph like Figure 4. We can prove the claim for general hyperelliptic graphs in the same method.

![Figure 4. $G$](image_url)

In virtue of [3, Proposition 4.2], it is sufficient to show the case that $D = P_1 + \cdots + P_n + i(P_1) + \cdots + i(P_n)$. Let us fix an arbitrary Lebesgue measure $\lambda$ on $G$ invariant under the involution, and fix arc-length parameters $s_i$ and $t_j$ on $e_i$ and $f_j$ such that $s_i(P_i) = 0$, $s_i(P_{i+1}) = l_i$, $t_j(Q_j) = 0$, and $t_j(P_j) = m_j$, where $l_i$ is the length of $e_i$ and $m_j$ is the length of $f_j$. Let $\mu$ be the admissible metric of $(G; \lambda, D)$. Set

$$g(x) = g_{(G; \lambda, D)}(O, x) = \begin{cases} \alpha_is_i(x)^2 + \beta_is_i(x) + \gamma_i & \text{on } e_i, \\ A_jt_j(x)^2 + C_j & \text{on } f_j. \end{cases}$$

We know by Lemma 3.17 that $\alpha_i$ and $A_j$ is of form $(\text{poly.})/\bar{L}_G$, (i.e., for example, there exists a homogeneous polynomial $H_i$ of degree $(sz(G) - 1)$ determined by $(G, D)$ and $i$ such that $\alpha_i = H_i(\lambda)/L_G(\lambda)$.) We can determine $\beta_j$’s inductively in the following way. The first order differential equation at $P_n$ which comes from $\Delta g(x) = \delta_O - \mu$ gives

$$-(2\alpha_{n-1}l_{n-1} + \beta_{n-1}) + (-2A_n m_n) + (-2A_{n+1}m_{n+1}) = 0,$$

hence we see $\beta_{n-1} = (\text{poly.})/\bar{L}_G$. Suppose that we know $\beta_k = (\text{poly.})/\bar{L}_G$ for $k \geq k_0$. The first order differential equation at $P_{k_0}$ gives

$$(-2\alpha_{k_0-1}l_{k_0-1} - \beta_{k_0-1}) + (-2A_{k_0} m_{k_0}) + \beta_{k_0} = 0,$$

hence we see $\beta_{k_0-1} = (\text{poly.})/\bar{L}_G$. Thus, we have shown that all $\beta_j$’s are of form $(\text{poly.})/\bar{L}_G$. 


By the continuity of $g$, we have

\[
\begin{cases}
\alpha_{i-1}l_i^2 + \beta_{i-1}l_{i-1} + \gamma_{i-1} = \gamma_i & \text{for } i = 1, \ldots, n-1, \\
A_j m_j^2 + C_j = \gamma_j & \text{for } j = 1, \ldots, n-1, \\
A_n m_n^2 + C_n = \alpha_{n-1}l_{n-1}^2 + \beta_{n-1}l_{n-1} + \gamma_{n-1} \\
A_{n+1} m_{n+1}^2 + C_{n+1} = \alpha_{n-1}l_{n-1}^2 + \beta_{n-1}l_{n-1} + \gamma_{n-1},
\end{cases}
\]

hence, in order to obtain the claim, it is sufficient to show that $\gamma_0 = (\text{poly.})/\bar{L}_G$.

By the condition $\int_G g \mu = 0$, we have

\[
0 = 2 \sum_{i=0}^{n-1}\left(\frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \gamma_i\right)(2\alpha_i l_i) + 2 \sum_{j=1}^{n+1}\left(\frac{1}{3} A_j m_j^2 + C_j\right)(2A_j m_j)
\]

\[
= 2 \sum_{i=1}^{n-1}\left(\frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \sum_{k=1}^{i-1}(\alpha_k l_k^2 + \beta_k l_k) + \alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0\right)(2\alpha_i l_i)
\]

\[
+ 2 \sum_{j=2}^{n}\left(\frac{1}{3} A_j m_j^2 + \sum_{k=1}^{j-1}(\alpha_k l_k^2 + \beta_k l_k) - A_j m_j^2 + \alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0\right)(2A_j m_j)
\]

\[
+ 2 \left(\frac{1}{3} A_{n+1} m_{n+1}^2 + \sum_{k=1}^{n}(\alpha_k l_k^2 + \beta_k l_k) - A_{n+1} m_{n+1}^2 + \alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0\right)(2A_{n+1} m_{n+1})
\]

\[
+ 2 \left(\frac{1}{3} \alpha_0 l_0^2 + \frac{1}{2} \beta_0 l_0 - (\alpha_0 l_0^2 + \beta_0 l_0) + \alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0\right)(2\alpha_0 l_0)
\]

\[
+ 2 \left(\frac{1}{3} A_1 m_1^2 - A_1 m_1^2 + \alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0\right)(2A_1 m_1)
\]

\[
= (\alpha_0 l_0^2 + \beta_0 l_0 + \gamma_0)\left(2 \sum_{i=0}^{n-1} 2\alpha_i l_i + 2 \sum_{j=1}^{n+1} 2A_j m_j\right)
\]

\[
+ 2 \sum_{i=1}^{n-1}\left(\frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \sum_{k=1}^{i-1}(\alpha_k l_k^2 + \beta_k l_k)\right)(2\alpha_i l_i)
\]

\[
+ 2 \sum_{j=2}^{n}\left(-\frac{2}{3} A_j m_j^2 + \sum_{k=1}^{j-1}(\alpha_k l_k^2 + \beta_k l_k)\right)(2A_j m_j)
\]

\[
+ 2 \left(-\frac{2}{3} A_{n+1} m_{n+1}^2 + \sum_{k=1}^{n}(\alpha_k l_k^2 + \beta_k l_k)\right)(2A_{n+1} m_{n+1})
\]

\[
+ 2 \left(-\frac{2}{3} \alpha_0 l_0^2 - \frac{1}{2} \beta_0 l_0\right)(2\alpha_0 l_0) + 2 \left(-\frac{2}{3} A_1 m_1^2\right)(2A_1 m_1).
\]
By $f_G \mu = 1$, we have
\[ 2 \left( \sum_{i=0}^{n} 2\alpha_i l_i + \sum_{j=1}^{n+1} 2A_j m_j \right) = 1 \]
and thus,
\[ -\gamma_0 = 2 \left( -\frac{2}{3} \alpha_0 l_0^2 - \frac{1}{2} \beta_0 l_0 \right) (2\alpha_0 l_0) + 2 \left( -\frac{3}{2} A_1 m_1^2 \right) (2A_1 m_1) \]
\[ + 2 \sum_{i=1}^{n-1} \left( \frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \sum_{k=1}^{i-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2\alpha_i l_i) \]
\[ + 2 \sum_{j=2}^{n} \left( -\frac{2}{3} A_j m_j^2 + \sum_{k=1}^{j-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2A_j m_j) \]
\[ + 2 \left( -\frac{2}{3} A_{n+1} m_{n+1}^2 + \sum_{k=1}^{n-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2A_{n+1} m_{n+1}) + (\alpha_0 l_0^2 + \beta_0 l_0). \]

From the argument so far, we see that $g_{(G,D)}(O, O) = (\text{homog.poly. of deg } (2sz(G) + 1))/L_G^2$ as rational expressions on $\{X_e\}_{e \in \mathbb{N}}$. Since $L_G$ is irreducible, it is enough for our claim to show that $g_{(G,D)}(O, O) = (\text{poly.})/(X_{e_0} + X_{f_1})^a L_G$ for some nonnegative integer $a$.

Let $G'$ be the graph like Figure 5.

Set
\[ D' = P_1 + \cdots + P_n + \nu(P_1) + \cdots + \nu(P_n) + 2O'. \]

Let $\lambda'$ be the invariant measure by which the length of $e_i$ is $l_i$ for $i = 1, \ldots, n - 1$, the length of $f_j$ is $m_j$ for $j = 2, \ldots, n + 1$, and the length of $e'_0$ is $l_0 m_1/(l_0 + m_1)$. Set as before, for $i = 0, \ldots, n - 1$ and $j = 2, \ldots, n + 1$,
\[ g'(x) := g_{(G', \lambda', D')}(O', x) = \begin{cases} 
\alpha_0' s_0'(x)^2 + \beta_0' s_0'(x) + \gamma_0' & \text{on } e'_0, \\
\alpha_i' s_i'(x)^2 + \beta_i' s_i'(x) + \gamma_i' & \text{on } e_i, \\
A_j' t_j'(x)^2 + C_j' & \text{on } f_j.
\end{cases} \]

Of course, we have $\alpha_i' = \alpha_i$ for $i \neq 0$ and $A_j' = A_j$ for $j \neq 1$, and by the procedure in determining the $\beta_i$'s of $G$, we also have $\beta_i' = \beta_i$ for $i \neq 0$. Here, by the condition $\int_{G'} g' \mu' = 0$, \[ \int_{G'} g' \mu' = 0, \]
we have
\[
0 = 2 \sum_{i=1}^{n-1} \left( \frac{1}{3} \alpha_i' l_i'^2 + \frac{1}{2} \beta_i' l_i' + \gamma_i' \right) (2 \alpha_i' l_i') + 2 \sum_{j=1}^{n+1} \left( \frac{1}{3} A_j' m_j'^2 + C_j' \right) (2 A_j' m_j') + \frac{2}{\deg D + 2} \gamma_0
\]
\[
= (\alpha_0' l_0'^2 + \beta_0' l_0' + \gamma_0) \left( 2 \sum_{i=0}^{n-1} 2 \alpha_i' l_i' + 2 \sum_{j=1}^{n+1} 2 A_j' m_j' + \frac{2}{\deg D + 2} \right)
\]
\[
+ 2 \sum_{i=1}^{n-1} \left( \frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \sum_{k=1}^{i-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 \alpha_i l_i)
\]
\[
+ 2 \sum_{j=2}^{n} \left( -\frac{2}{3} A_j m_j^2 + \sum_{k=1}^{j-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 A_j m_j)
\]
\[
+ 2 \left( -\frac{2}{3} A_{n+1} m_{n+1}^2 + \sum_{k=1}^{n-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 A_{n+1} m_{n+1})
\]
\[
- 2 \left( \frac{2}{3} \alpha_0' l_0'^2 + \frac{1}{2} \beta_0' l_0' \right) (2 \alpha_0' l_0') - \frac{2}{\deg D + 2} (\alpha_0' l_0'^2 + \beta_0' l_0')
\]

hence, noting \( \int C' \mu' = 1 \) as before, we have
\[
2 \sum_{i=1}^{n-1} \left( \frac{1}{3} \alpha_i l_i^2 + \frac{1}{2} \beta_i l_i + \sum_{k=1}^{i-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 \alpha_i l_i)
\]
\[
+ 2 \sum_{j=2}^{n} \left( -\frac{2}{3} A_j m_j^2 + \sum_{k=1}^{j-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 A_j m_j)
\]
\[
+ 2 \left( -\frac{2}{3} A_{n+1} m_{n+1}^2 + \sum_{k=1}^{n-1} (\alpha_k l_k^2 + \beta_k l_k) \right) (2 A_{n+1} m_{n+1})
\]
\[
= 2 \left( \frac{2}{3} \alpha_0' l_0'^2 + \frac{1}{2} \beta_0' l_0' \right) (2 \alpha_0' l_0') + \frac{2}{\deg D + 2} (\alpha_0' l_0'^2 + \beta_0' l_0') - (\alpha_0' l_0'^2 + \beta_0' l_0' + \gamma_0).
\]

Therefore,
\[
-\gamma_0 = 2 \left( -\frac{2}{3} \alpha_0' l_0'^2 - \frac{1}{2} \beta_0 l_0 \right) (2 \alpha_0 l_0) + 2 \left( -\frac{2}{3} A_1 m_1^2 \right) (2 A_1 m_1) + 2 \left( \frac{2}{3} \alpha_0' l_0'^2 + \frac{1}{2} \beta_0' l_0' \right) (2 \alpha_0' l_0')
\]
\[
+ \frac{2}{\deg D + 2} (\alpha_0' l_0'^2 + \beta_0' l_0') - (\alpha_0' l_0'^2 + \beta_0' l_0' + \gamma_0) + (\alpha_0 l_0^2 + \beta_0 l_0)
\]
\[
= \frac{8}{3} (\alpha_0' l_0' + A_1 m_1^2 - \alpha_0' l_0'^2)
\]
\[
- \beta_0 l_0 (2 \alpha_0 l_0) + \beta_0' l_0' (2 \alpha_0' l_0') + \frac{2}{\deg D + 2} (\alpha_0' l_0'^2 + \beta_0' l_0')
\]
\[
- (\alpha_0' l_0'^2 + \beta_0' l_0') + (\alpha_0 l_0^2 + \beta_0 l_0) - \gamma_0.
\]
\(\beta_0\) (resp. \(\beta_0'\)) can be calculated with the first order differential equation at \(O\) (resp. \(O'\)), and this calculation shows that \(\beta_0\) and \(\beta_0'\) are just rational numbers independent of the measure \(\lambda\). Noting that

\[
\alpha_0' = \frac{1}{(\deg(D) + 2) l_{e_0}' + r_{e_0}'} = \frac{1}{(\deg(D) + 2) l_1 + r_1} = \alpha_1,
\]

we see that

\[
\alpha_0' = \frac{p_{\tilde{G}}^f(\lambda)}{2(\deg(D) + 2)L_G(\lambda)}.
\]

Now, look at the graph \(G'\). If we regard \(e_0'' = e_0' \cup e_1\) and \(\iota'(e_0'') = \iota'(e_0') \cup \iota(e_1)\) as one edge, \(G'\) is a hyperelliptic graph of size \(sz(G) - 1\). Hence by the induction hypothesis, there exists a homogeneous polynomial \(F_1\) on \(\{X_{e}\}_{e \in Ed(G')}\) independent of \(\lambda\) such that \(\gamma_0' = F_1(\lambda')/L_{G'}(\lambda')\). As we saw in the proof of Lemma 3.17,

\[
X_{e_0'} = X_{e_0}X_{\tilde{f}_1} + X_{\tilde{e}_1}
\]

and

\[(X_{e_0} + X_{\tilde{f}_1})L_{G'} = L_G,
\]

hence we have a homogeneous polynomial \(F_2\) on \(\{X_e\}_{e \in Ed(G')}\) independent of \(\lambda\) and a nonnegative integer \(a\) independent of \(\lambda\) such that \(\gamma_0' = F_2(\lambda)/(l_0 + m_1)^aL_G(\lambda)\). Consequently, we see that all terms in

\[-\frac{8}{3}(\alpha_0^2l_0^3 + A_1^2m_1^3 - \alpha_0'^2l_0'^3)
- \beta_0l_0(2\alpha_0l_0) + \beta_0'l_0(2\alpha_0'l_0) + \frac{2}{\deg D + 2}(\alpha_0'l_0^2 + \beta_0'l_0')
- (\alpha_0'^2 + \beta_0'l_0) + (\alpha_0l_0^2 + \beta_0l_0) - \gamma'_0
\]

but the first line are of form \((\text{poly.})/(l_0 + m_1)^a\tilde{L}_G\), hence it suffices to show that the first line is also of that form, i.e., there exist a homogeneous polynomial \(F_3\) and a nonnegative integer \(a\) such that

\[
\alpha_0^2l_0^3 + A_1^2m_1^3 - \alpha_0'^2l_0'^3 = \frac{F_3(\lambda)}{(l_0 + m_1)^a\tilde{L}_G(\lambda)}.
\]

Now for simplicity, set \(X_0 = X_{e_0}, X_1 = X_{\tilde{f}_1}, P_0 = P_{\tilde{G}}^{e_0}, P_1 = P_{\tilde{G}}^{\tilde{f}_1}, \) and \(P_2 = P_{\tilde{G}}^{\tilde{e}_1}\).

\[
\sum_{i=0}^{n-1} 2\alpha_i l_i + \sum_{j=1}^{n+1} 2A_j m_j = \frac{1}{2}
\]

\[
\frac{1}{\deg D + 2} + \sum_{i=0}^{n-1} 2\alpha_i l_i' + \sum_{j=2}^{n+1} 2A_j m_j' = \frac{1}{2},
\]
which come from $\int_G \mu = 1$ and $\int_G \mu' = 1$, give us

$$2\alpha' l_0' = -\frac{1}{\deg D + 2} + 2A_1 m_1 + 2\alpha_0 l_0.$$  

Noting

$$\alpha_0 = \frac{P_0(\lambda)}{2(\deg(D) + 2)L_G(\lambda)},$$

$$A_1 = \frac{P_1(\lambda)}{2(\deg(D) + 2)L_G(\lambda)},$$

$$\alpha_0' = \alpha_1 = \frac{P_2(\lambda)}{2(\deg(D) + 2)L_G(\lambda)},$$

we have

$$P_2 X_0 X_1 \equiv (X_0 + X_1)(P_0 X_0 + P_1 X_1) \quad \text{mod } L_G.$$  

Thus, we are reduced to show

$$(X_0 + X_1)P_0^2 X_0^3 + (X_0 + X_1)P_1^2 X_1^3 - (P_0 X_0 + P_1 X_1)^2 X_0 X_1 \equiv 0 \quad \text{mod } L_G$$  

Let $B$ be a polynomial such that $X_0 X_1 B$ is the sum of all the monomials of $L_G$ which are divisible by $X_0 X_1$. Then, we have

$$L_G = P_0 X_0 + P_1 X_1 - X_0 X_1 B$$

since any monomial in $L_G$ is divisible by $X_0$ or $X_1$. Moreover, since $(P_0 - X_1 B)X_0 = L_G - P_1 X_1$, no monomials in $P_0 - X_1 B$ are divisible by $X_0$ or $X_1$. Hence by symmetricity on $X_0$ and $X_1$, $P_1 - X_0 B = P_0 - X_1 B$ and denote this polynomial by $C$, which is free from $X_0$ and $X_1$. Note that

$$P_0 X_0 + P_1 X_1 \equiv X_0 X_1 B \quad \text{mod } L_G$$

and

$$L_G = P_0 X_0 + P_1 X_1 - X_0 X_1 B$$

$$= X_0 X_1 B + X_0 C + X_0 X_1 B + X_1 C - X_0 X_1 B$$

$$= X_0 X_1 B + (X_0 + X_1)C.$$  

Then, we see

$$(X_0 + X_1)P_0^2 X_0^3 + (X_0 + X_1)P_1^2 X_1^3 - (P_0 X_0 + P_1 X_1)^2 X_0 X_1$$

$$= (X_0 + X_1)P_0 X_0^2 (P_0 X_0 + P_1 X_1) + (X_0 + X_1)P_1 X_1 (P_1 X_1^2 - P_0 X_0^2) - (P_0 X_0 + P_1 X_1)^2 X_0 X_1$$

$$\equiv (X_0 + X_1)P_0 X_0^2 (X_0 X_1 B)$$

$$+ (X_0 + X_1)P_1 X_1 (X_1^2 X_0 B + X_1^2 C - X_2^2 X_0 B - X_2^2 C) - (X_0 X_1 B)^2 X_0 X_1 \quad \text{mod } L_G$$

$$= (X_0 + X_1)P_0 X_0^2 (X_0 X_1 B) + (X_1^2 - X_0^2)P_1 X_1 (X_0 X_1 B)$$

$$+ (X_0 + X_1)(X_1^2 - X_0^2)P_1 X_1 C - (X_0 X_1 B)^2 X_0 X_1.$$
Now
\[(X_0 + X_1)P_0 X_0^2 + (X_1^2 - X_0^2) P_1 X_1 - (X_0X_1 B) X_0 X_1\]
\[= (X_0 + X_1)(X_0X_1 B + X_0^2 C) - (X_0X_1 B) X_0 X_1 + (X_1^2 - X_0^2) P_1 X_1\]
\[= X_0^2(X_0X_1 B + (X_0 + X_1) C) + (X_1^2 - X_0^2) P_1 X_1\]
\[\equiv (X_1^2 - X_0^2) P_1 X_1 \mod L_G.\]

Therefore, we have
\[(X_0 + X_1)P_0 X_0^3 + (X_0 + X_1)P_1^2 X_1^3 - (P_0X_0 + P_1X_1) X_0 X_1\]
\[\equiv (X_1^2 - X_0^2) P_1 X_1 (X_0X_1 B) + (X_0 + X_1)(X_1^2 - X_0^2) P_1 X_1 C \mod L_G\]
\[= (X_1^2 - X_0^2) P_1 X_1 (X_0X_1 B + (X_0 + X_1) C)\]
\[\equiv 0 \mod L_G,\]

thus, we complete the proof of Claim 2.

\[\square\]

**Claim 3.** Let \( O \) be a fixed vertex, \( e \) an edge starting from \( O \), and \( P \) the other terminal vertex of \( e \). Then, we have
\[ r_G(O, P) = \frac{\text{homog. poly. of deg } sz(G) + 1}{L_G}. \]

**Proof.** Since \( 2/(X_\bar{e} + R_\bar{e}) = P_G^\bar{e}/L_G \), we have
\[ R_\bar{e} = \frac{2L_G}{P_G^\bar{e}} - X_\bar{e}. \]

Therefore,
\[ r_G(O, P) = \frac{X_\bar{e} R_\bar{e}}{X_\bar{e} + R_\bar{e}} = \frac{(2L_G/P_G^\bar{e})X_\bar{e} - X_\bar{e}^2}{2L_G/P_G^\bar{e}} = \frac{2L_G X_\bar{e} - X_\bar{e}^2 P_G^\bar{e}}{2L_G},\]
and thus, we obtain the claim.

\[\square\]

**Claim 4.** Let \( O \) be a fixed vertex as in Claim 3. Then, for any vertex \( P \), we have
\[ r_G(O, P) = \frac{\text{homog. poly. of deg } sz(G) + 1}{L_G}. \]

**Proof.** If \( P \) is a fixed vertex, we can easily see that \( r_G(O, P) \) itself is a homogeneous polynomial of degree 1. In the argument below, hence, we assume that \( P \) is a non-fixed vertex.

We will prove the claim by induction on \( sz(G) \).

If \( sz(G) = 2 \), then we see that the claim is true by Claim 4 or by direct calculations.
Suppose $sz(G) > 2$. First we show the claim in the case where there are at least two one-jointed edges starting from $P$. If $O'$ is the other terminal point of a one-jointed edge starting from $P$, then

$$r_G(O', P) = \frac{\text{homog. poly. of deg } sz(G) + 1}{L_G}$$

by Claim $3$. On the other hand, we know

$$r_G(O, P) = g_{(G,D)}(O, O) - 2g_{(G,D)}(O, P) + g_{(G,D)}(P, P)$$
$$r_G(O', P) = g_{(G,D)}(O', O') - 2g_{(G,D)}(O', P) + g_{(G,D)}(P, P),$$

hence we have

$$r_G(O, P) - r_G(O', P) = \frac{\text{homog. poly. of deg } sz(G) + 1}{L_G}$$

by Claim $2$. Therefore, we obtain Claim $4$. Next, we assume that there exists at most one one-jointed edge starting from $P$. Then, $G$ is not elementary and we can find two distinct non-fixed vertices $P_1$ and $P_2$ whose classes in $\text{Vert}(G)$ are different from $P$, such that there exist at least two one-jointed edges $e_{i,1}$ and $e_{i,2}$ starting from $P_i$ for $i = 1, 2$. Let $O_i$ be the other terminal point of $e_{i,1}$ for $i = 1, 2$, which is a non-fixed vertex. Then, by the induction hypothesis and the same argument in the proof of Lemma 3.17 or Claim $2$, we can see that there is a nonnegative integer $a'$ with

$$r_G(O_{1}, P_{1}) = \text{homog. poly. of deg } (sz(G) + 1 + a') \frac{F_{1}}{(X_{\bar{e}_{2,1}} + X_{\bar{e}_{2,2}})^{a'}L_G}.$$

Hence, again by Claim $2$ and

$$r_G(O_{1}, P_{1}) = g_{(G,D)}(O_{1}, O_{1}) - 2g_{(G,D)}(O_{1}, P_{1}) + g_{(G,D)}(P_{1}, P_{1}),$$

we see that there exist a nonnegative integer $a$ and a homogeneous polynomial $F_1$ of degree $(sz(G) + 1 + a)$ which is coprime to $(X_{\bar{e}_{2,1}} + X_{\bar{e}_{2,2}})$, such that

$$r_G(O, P) = \frac{F_1}{(X_{\bar{e}_{2,1}} + X_{\bar{e}_{2,2}})^{a}L_G}.$$

In the same way, we see that there exist a nonnegative integer $b$ and a homogeneous polynomial $F_2$ of degree $(sz(G) + 1 + b)$ which is not divisible by $(X_{\bar{e}_{1,1}} + X_{\bar{e}_{1,2}})$, such that

$$r_G(O, P) = \frac{F_2}{(X_{\bar{e}_{1,1}} + X_{\bar{e}_{1,2}})^{b}L_G}.$$

Therefore, we have

$$(X_{\bar{e}_{1,1}} + X_{\bar{e}_{1,2}})^b F_1 = (X_{\bar{e}_{2,1}} + X_{\bar{e}_{2,2}})^a F_2,$$

and both $a$ and $b$ are equal to 0 by the choice of $F_1$ and $F_2$. \hfill \Box

In virtue of Claim $2$, Claim $4$ and [3, Lemma 4.1], we obtain the Lemma 3.18. \hfill \Box
3.3. **Proof of Theorem 3.15.** Now, we are ready to prove Theorem 3.15. We will prove the theorem by induction on the size of $G$.

If $sz(G) = 1$ or 2, it is Proposition 3.16. We assume $sz(G) > 2$.

**Step 1.** Suppose that $G$ is a one-point-sum of two graphs $G_1$ and $G_2$. Let $D_i$ be the polarization $D_{Ed(G_i)}^{\sim}$ on $G_i$ for $i = 1, 2$. Then, $sz(G_i) < sz(G)$ for $i = 1, 2$, we obtain the formula by the induction hypothesis and Lemma 3.14.

**Step 2.** Suppose that $G$ is irreducible and $n = sz(G)$. Let us consider the following claim.

**Claim 5.** For $P \in S^{n+1}V$, we assume that (1) $P(X_{\bar{e}} = 0) = 0$ for any $\bar{e} \in Ed_1(G)^{\sim}$, and (2) $P(\{X_{\bar{e}} = 0\}_{\bar{e} \in Ed_0(G)^{\sim}}) = 0$. Then we have $P = 0$.

**Proof.** Let $\lambda X_{\bar{e}_1} \cdots X_{\bar{e}_n} X_{\bar{e}_{n+1}}$ be a monomial in $P$.

If there is a disjoint edge class in $\{\bar{e}_1, \ldots, \bar{e}_{n+1}\}$, then we have a one-jointed one which dose not appear in $\{\bar{e}_1, \ldots, \bar{e}_{n+1}\}$. Therefore, we have $\lambda = 0$ by (1).

If there is no disjoint edge class in $\{\bar{e}_1, \ldots, \bar{e}_{n+1}\}$, then all of them are one-jointed. Hence, we have also $\lambda = 0$ by (2). \hfill $\Box$

Let us go back to the proof of Theorem 3.15. Set

$$F = \sum_{\bar{e} \in Ed(G)^{\sim}} \left( \frac{2}{3} \frac{\deg(D)}{\deg(D) + 2} + \frac{w(\bar{e}) (\deg(D) - w(\bar{e}))}{\deg(D) + 2} \right) X_{\bar{e}} + \frac{2}{3} \frac{\deg(D)}{\deg(D) + 2} M_G.$$

Since we know by Lemma 3.18

$$\epsilon(G, D) = \frac{P}{L_G}$$

for some homogeneous polynomial $P$ of degree $n + 1$, in virtue of the claim, it is sufficient to prove the following (1) and (2).

(1)

$$F(X_{\bar{e}_0} = 0) = \epsilon(G_1 \vee G_2, D')$$

for any $\bar{e}_0 \in Ed_1(G)^{\sim}$, where $G_1 \vee G_2 = G_{\bar{e}_0}$ and $D' = D_{\bar{e}_0}$.

(2)

$$F(\{X_{\bar{e}} = 0\}_{\bar{e} \in Ed_0(G)^{\sim}}) = \epsilon(G_n, D''),$$

where $D'' = (n - 1)Q + (n - 1)\iota(Q) + \sum_{i=1}^{n+1} a_i P_i$ in the same notation as Proposition 3.16.

(1) is equivalent to

$$\frac{M_G}{L_G} (X_{\bar{e}} = 0) = \frac{M_{G_1}}{L_{G_1}} + \frac{M_{G_2}}{L_{G_2}}$$

by Step 1, which is nothing but Lemma 3.14. (2) is obvious from Proposition 3.16. Thus, we have achieved the conclusion.
4. Proof of the main theorem

In this section, we consider metrized graphs only, hence we denote the admissible constants by \( \epsilon(G, D) \) instead of \( \epsilon(G; \lambda, D) \).

We need one more lemma:

**Lemma 4.1.** Let \( G \) be an irreducible hyperelliptic metrized graph with the Lebesgue measure \( \lambda = \sum_{\bar{e} \in \text{Ed}(G)} l_\bar{e} \bar{e}^* \). Then, we have

\[
\frac{M_G}{L_G} \leq \sum_{\bar{e} \in \text{Ed}_0(G)} l_\bar{e} + \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_\bar{e},
\]

where \( L_G = L_G(\lambda) \) and \( M_G = M_G(\lambda) \). Moreover, if \( \text{sz}(G) \leq 4 \), then we have

\[
\frac{M_G}{L_G} \leq \frac{1}{2} \sum_{\bar{e} \in \text{Ed}_0(G)} l_\bar{e} + \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_\bar{e}.
\]

**Proof.** Let us consider the following:

\[
\bar{L}_G \left( \sum_{\bar{e} \in \text{Ed}_0(G)} l_\bar{e} + \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_\bar{e} \right) = \left( \sum_{\bar{e}_1, \ldots, \bar{e}_k} \prod_{\bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) l_{\bar{e}_1} \cdots l_{\bar{e}_k} \right) \left( \sum_{\bar{e} \in \text{Ed}_0(G)} l_\bar{e} \right) \\
+ \left( \sum_{\bar{e}_1, \ldots, \bar{e}_k} \prod_{\bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) l_{\bar{e}_1} \cdots l_{\bar{e}_k} \right) \left( \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_\bar{e} \right),
\]

where \( \bar{e}_1, \ldots, \bar{e}_k \in \text{Ed}_0(G) \), and \( \bar{v} \) runs over all non-fixed vertices of \( G_{\text{Ed}_0(G)} \setminus \{\bar{e}_1, \ldots, \bar{e}_k\} \).

Firstly, we would like to estimate the second line of the right-hand-side of the above equality. We know that an inequality

\[
(a_1 + \cdots + a_k)(a_1a_2 \cdots a_{k-1} + a_2a_3 \cdots a_k + \cdots + a_ka_1 \cdots a_{k-2}) \geq k^2a_1a_2 \cdots a_k
\]

holds for positive numbers. Hence, noting that \( \text{Ed}_1(G) = \text{Ed}_1(G_{\text{Ed}_0(G)} \setminus \{\bar{e}_1, \ldots, \bar{e}_k\}) \), we have

\[
\left( \prod_{\bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) \right) \left( \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_0(G)} l_\bar{e} \right) = \frac{1}{4} \sum_{\bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) \left( \sum_{\bar{e} \in \text{Ed}_1(G)} l_\bar{e} \right) \prod_{\bar{v} \neq \bar{v}'} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}') \\
\geq \sum_{\bar{v}} \left( \frac{(\nu^1(\bar{v}))^2}{4} \tau(\bar{v}_1, \ldots, \bar{v}_k; \bar{v}) \prod_{\bar{v} \neq \bar{v}'} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}') \right),
\]

where \( \nu^1(\bar{v}) = \nu^1(G)(\bar{v}) \) is the number of one-jointed edges of \( G' = G_{\text{Ed}_0(G)} \setminus \{\bar{e}_1, \ldots, \bar{e}_k\} \) starting from \( v \). We also write \( \nu^0(\bar{v}) \) and \( \nu(\bar{v}) \) omitting \( G_{\text{Ed}_0(G)} \setminus \{\bar{e}_1, \ldots, \bar{e}_k\} \).

Let us go on to the estimation of the first term. For an arbitrary \( i, \bar{e}_i \in \{\bar{e}_1, \ldots, \bar{e}_k\} \), let \( v_{i,1} \) and \( v_{i,2} \) be the terminal points of \( \bar{e}_i \), and \( v_i \) the vertex of \( G_{\text{Ed}_0(G)} \setminus \{\bar{e}_1, \ldots, \bar{e}_k\} \) such that \( v_{i,1} \) and \( v_{i,2} \) go to \( v_i \) when we contract \( \bar{e}_i \). Then, we can easily check that

\[
\sigma(\bar{e}_1, \ldots, \bar{e}_i, \bar{e}_{i+1}, \ldots, \bar{e}_k; \bar{v}_i) = \tau(\bar{e}_1, \ldots, \bar{e}_k; v_{i,1}) \sigma(\bar{e}_1, \ldots, \bar{e}_k; v_{i,2}) + \sigma(\bar{e}_1, \ldots, \bar{e}_k; v_{i,1}) \tau(\bar{e}_1, \ldots, \bar{e}_k; v_{i,2}).
\]
If we use this formula, we see that

\[
\sum_{i=1}^{k} \left( \prod_{\emptyset \neq \emptyset_1} \sigma(\bar{e}_1, \ldots, \bar{e}_{i-1}, \bar{e}_{i+1}, \ldots, \bar{e}_k; \emptyset) \right) = \sum_{i=1}^{k} \left( \tau(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_1) \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_2) \prod_{\emptyset \neq \emptyset_1} \sigma(\bar{e}_1, \ldots, \bar{e}_{i-1}, \bar{e}_{i+1}, \ldots, \bar{e}_k; \emptyset) \right) + \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_1) \tau(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_2) \prod_{\emptyset \neq \emptyset_1, \emptyset_2} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) \]

\[
= \sum_{i=1}^{k} \left( \tau(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_1) \prod_{\emptyset \neq \emptyset_1} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) + \tau(\bar{e}_1, \ldots, \bar{e}_k; \emptyset_2) \prod_{\emptyset \neq \emptyset_1, \emptyset_2} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) \right)
\]

\[
= \sum_{\emptyset} \left( \nu^0(\bar{v}) \tau(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) \prod_{\emptyset' \neq \emptyset} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset') \right).
\]

We have, therefore,

\[
\left( \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \prod_{\emptyset} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) l_{\bar{e}_1} \ldots l_{\bar{e}_k} \right) \left( \sum_{\bar{e} \in E\bar{d}_0(G)} l_{\bar{e}} \right) = \sum_{\bar{e} \in E\bar{d}_0(G)} \left( \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \prod_{\emptyset} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \emptyset) l_{\bar{e}_1} \ldots l_{\bar{e}_k} l_{\bar{e}} \right)
\]

\[
\geq \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \sum_{i=1}^{k} \left( \prod_{\emptyset} \sigma(\bar{e}_1, \ldots, \bar{e}_{i-1}, \bar{e}_{i+1}, \ldots, \bar{e}_k; \emptyset) \right) l_{\bar{e}_1} \ldots l_{\bar{e}_k} \quad (k \geq 1)
\]

\[
= \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \left( \sum_{j=1}^{k} \left( \nu^0(\bar{v}_j) \tau(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}_j) \prod_{\emptyset \neq \bar{v}_j} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) \right) l_{\bar{e}_1} \ldots l_{\bar{e}_k} \right).
\]
Thus, we have
\[ \bar{L}_G \left( \sum_{\bar{e} \in \text{Ed}_0(G)} l_{\bar{e}} + \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_{\bar{e}} \right) \geq \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \left( \sum_{\bar{v}} \left( \frac{\left( \nu^1(\bar{v}) \right)^2}{4} + \nu^0(\bar{v}) \right) \tau(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) \prod_{\bar{v}' \neq \bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}') \right) l_{\bar{e}_1} \cdots l_{\bar{e}_k}. \]

Since
\[ \frac{\left( \nu^1(\bar{v}) \right)^2}{4} + \nu^0(\bar{v}) \geq \nu^1(\bar{v}) + \nu^0(\bar{v}) - 2 = \nu(\bar{v}) - 2 \]
for any \( v \), we obtain the first inequality.

We have also
\[ \bar{L}_G \left( \frac{1}{2} \sum_{\bar{e} \in \text{Ed}_0(G)} l_{\bar{e}} + \frac{1}{4} \sum_{\bar{e} \in \text{Ed}_1(G)} l_{\bar{e}} \right) \geq \sum_{\bar{e}_1, \ldots, \bar{e}_k \text{ all distinct}} \left( \sum_{\bar{v}} \left( \frac{\left( \nu^1(\bar{v}) \right)^2}{4} + \frac{\nu^0(\bar{v})}{2} \right) \tau(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}) \prod_{\bar{v}' \neq \bar{v}} \sigma(\bar{e}_1, \ldots, \bar{e}_k; \bar{v}') \right) l_{\bar{e}_1} \cdots l_{\bar{e}_k}. \]

If \( \text{sz}(G) \leq 4 \), then we see \( \#\text{Ed}_0(G) \leq 2 \). Therefore, we have
\[ \frac{\left( \nu^1(\bar{v}) \right)^2}{4} + \frac{\nu^0(\bar{v})}{2} \geq \nu^1(\bar{v}) + \nu^0(\bar{v}) - 2 = \nu(\bar{v}) - 2, \]
and we obtain the second inequality.

**Corollary 4.2.** In the same notation as that of Theorem 3.15, we have the following inequalities.

(1) \[ \epsilon(G, D) \leq \sum_{w(\bar{e}) \neq 0} \left( \frac{4}{3} \frac{\deg(D)}{\deg(D) + 2} + \frac{w(\bar{e})(\deg(D) - w(\bar{e}))}{\deg(D) + 2} \right) l_{\bar{e}} + \sum_{w(\bar{e}) = 0} \frac{5}{6} \frac{\deg(D)}{\deg(D) + 2} l_{\bar{e}} \]

for any \( G \) with a measure \( \lambda = \sum l_{\bar{e}} \bar{e}^* \).

(2) \[ \epsilon(G, D) \leq \sum_{w(\bar{e}) \neq 0} \left( \frac{\deg(D)}{\deg(D) + 2} + \frac{w(\bar{e})(\deg(D) - w(\bar{e}))}{\deg(D) + 2} \right) l_{\bar{e}} + \sum_{w(\bar{e}) = 0} \frac{5}{6} \frac{\deg(D)}{\deg(D) + 2} l_{\bar{e}} \]

if every irreducible component of \( G \) is of size less than 5.

**Proof.** Since we know that \( e \) is one-jointed if \( w(e) = 0 \), this is immediate from Theorem 3.15 and the above lemma. \( \square \)
Let us start the proof of the main theorem. First of all, note the following fact (cf. [7, Theorem 5.6] [3, Corollary 2.3] [4, Theorem 2.1]). If \((\omega^a_{X/Y}, \omega^a_{X/Y}) > 0\), then we have

\[
\inf_{P \in \text{Pic}^0(C(K)}) r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y}, \omega^a_{X/Y})},
\]

where \((, \rangle_a\) is the admissible pairing.

Let \((G_y, \omega_y)\) be the polarized metrized graph by the configuration of \(X_y\). By the definition of the admissible pairing, we can see

\[
(\omega^a_{X/Y}, \omega^a_{X/Y}) = (\omega_{X/Y}, \omega_{X/Y}) - \sum_{y \in Y} \epsilon(G_y, \omega_y).
\]

In virtue of [1, Proposition 4.7] and Noether’s formula, we have

\[
(\omega_{X/Y}, \omega_{X/Y}) = \frac{g-1}{2g+1} \xi_0(X/Y) + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} \frac{6j(g-1-j) + 2(g-1)}{2g+1} \xi_j(X/Y)
\]

\[
+ \sum_{i=1}^{\lfloor g/2 \rfloor} \left( \frac{12i(g-i)}{2g+1} - 1 \right) \delta_i(X/Y).
\]

Let \((G_1, D_1)\) (resp. \((G_2, D_2)\)) be the polarized metrized graph obtained from \((G_y, \omega_y)\) by contracting all edges which correspond to nodes of positive type (resp. of type 0). Then, \(G_1\) is a hyperelliptic graph as we saw in Example 3.2 if we suitably redefine the set of vertices and the set of edges, equipped with the involution invariant measure. Moreover, the divisor \(D_1\) is supported in the “new” set of vertices since the coefficient of each vertex corresponding to a \((-2)\)-rational component is 0. On the other hand, \((G_2, D_2)\) is a tree.

Firstly, we will talk on the case of \(g \geq 5\). By the definition of the polarized metrized dual graph, Proposition 2.3 and Corollary 4.2 (1), we have

\[
\epsilon(G_1, D_1) \leq \frac{5(g-1)}{12g} \xi_0(X_y) + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} \left( \frac{4(g-1)}{3g} + \frac{2j(g-1-j)}{g} \right) \xi_j(X_y),
\]

\[
\epsilon(G_2, D_2) \leq \sum_{i=1}^{\lfloor g/2 \rfloor} \left( \frac{4i(g-1)}{g} - 1 \right) \delta_i(X_y),
\]

and again by Proposition 2.3, we have

\[
\epsilon(G_y, \omega_y) \leq \frac{5(g-1)}{12g} \xi_0(X_y) + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} \left( \frac{4(g-1)}{3g} + \frac{2j(g-1-j)}{g} \right) \xi_j(X_y)
\]

\[
+ \sum_{i=1}^{\lfloor g/2 \rfloor} \left( \frac{4i(g-1)}{g} - 1 \right) \delta_i(X_y).
\]
Therefore,
\[
(\omega_{X/Y}^a, \omega_{X/Y}^a)_a \geq \frac{(g - 1)(2g - 5)}{12g(2g + 1)} \xi_0(X/Y) \\
+ \sum_{j=1}^{[(g-1)/2]} \frac{2(g - 1)(3j(g - 1 - j) - g - 2)}{3g(2g + 1)} \xi_j(X/Y) \\
+ \sum_{i=1}^{[g/2]} \frac{4(g - 1)i(g - i)}{g(2g + 1)} \delta_i(X/Y).
\]

Now since \( g \geq 5 \), we have
\[
3j(g - 1 - j) - g - 2 \geq 3(g - 2) - g - 2 = 2(g - 4) > 0,
\]
which shows \((\omega_{X/Y}^a, \omega_{X/Y}^a)_a > 0\). Thus, we obtain our theorem for \( g \geq 5 \).

Secondly, suppose that \( g \leq 4 \). If \( sz(G) > 4 \), then we can easily see that the degree of the polarization is larger than six. On the other hand, the degree of the polarization is necessarily equal to \( 2g - 2 \leq 6 \), which is a contradiction. Therefore, we can use the inequality of Corollary 4.2 (2). In the same way as above, we obtain an inequality
\[
\epsilon(G_y, \omega_y) \leq \frac{5(g - 1)}{12g} \xi_0(X_y) + \sum_{j=1}^{[(g-1)/2]} \left( \frac{g - 1}{g} + \frac{2j(g - 1 - j)}{g} \right) \xi_j(X_y) \\
+ \sum_{i=1}^{[g/2]} \left( \frac{4i(g - 1)}{g} - 1 \right) \delta_i(X_y),
\]
and
\[
(\omega_{X/Y}^a, \omega_{X/Y}^a)_a \geq \frac{(g - 1)(2g - 5)}{12g(2g + 1)} \xi_0(X/Y) \\
+ \sum_{j=1}^{[(g-1)/2]} \frac{(g - 1)(2j(g - 1 - j) - 1)}{g(2g + 1)} \xi_j(X/Y) \\
+ \sum_{i=1}^{[g/2]} \frac{4(g - 1)i(g - i)}{g(2g + 1)} \delta_i(X/Y).
\]

Since all the coefficients of \( \xi_j(X/Y) \) and \( \delta_i(X/Y) \) are positive if \( g \geq 3 \), we obtained the theorem for \( g = 3, 4 \).

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