Predicting Critical Speeds in Rotordynamics: A New Method

J.D. Knight and L.N. Virgin
Dept. Mechanical Engineering, Duke University, Durham, NC 27708-0300, U.S.A.
E-mail: l.virgin@duke.edu

R.H. Plaut
Dept. Civil and Environmental Engineering, Virginia Tech, Blacksburg, VA 24061, USA.

Abstract.
In rotordynamics, it is often important to be able to predict critical speeds. The passage through resonance is generally difficult to model. Rotating shafts with a disk are analyzed in this study, and experiments are conducted with one and two disks on a shaft. The approach presented here involves the use of a relatively simple prediction technique, and since it is a black-box data-based approach, it is suitable for in-situ applications.

1. Introduction
The new method to predict critical speeds of rotors is quite simple. Measurements of translational steady-state amplitudes are recorded at a number of relatively low rotational speeds. These data are manipulated in three alternative formats, based on elementary theoretical analysis of a Jeffcott rotor model. Using extrapolation or linear interpolation, critical speeds are predicted. Data from an experimental rig are used to verify the approach.

1.1. The Southwell Plot
The inspiration for the new method is the Southwell plot [1], which was developed to predict critical buckling loads of structures. For example, consider the pin-ended column shown in figure 1(a). Suppose it has a half-sine imperfection with central deflection $\delta_0$ and is subject to a compressive axial load $P$. The overall lateral central deflection measured from the straight configuration is given by $Q = \delta_0 + \delta$, and it can be shown that $Q = \delta_0/(1 - P/P_E)$, i.e., the load effectively magnifies the initial imperfection as the critical value $P_E$ is approached (figure 1(b)). Southwell realized a useful opportunity, in a practical testing situation, by re-arranging to obtain $\delta/P = \delta/P_E + \delta_0/P_E$. This represents the form of a straight line $y = mx + c$, in which $y \equiv \delta/P$ and $x \equiv \delta$, and importantly the slope is $m \equiv 1/P_E$. Thus, measuring several different axial loads $P$ and corresponding additional transverse midpoint deflections $\delta$, these data can be plotted in the plane of $\delta/P$ versus $\delta$, and a straight line is fit to those points. This is shown in figure 1(c). The inverse of the slope of the line furnishes an approximation of the critical value $P = P_E$ [2]. Other applications have included buckling of plates and shells, and lateral buckling of beams, and have involved various modifications of the Southwell plot (e.g., [3, 4]).
Figure 1. The Southwell plot. (a) a pin-ended column, (b) the force-deflection relation, (c) the alternative axes furnishing a straight line.

2. Basic Rotordynamic Modeling

Since the growth of motion as a rotor’s critical speed is approached is of a similar form to the growth of deflection as a column’s critical load is approached, we shall adapt the Southwell procedure for a rotordynamics context. That is, we will use measurements of amplitudes \( A \) of a rotor at various low rotational speeds (angular velocities) \( \omega \) to predict the first critical speed (where \( A \) has its first local maximum when plotted as a function of \( \omega \)). The new method is motivated by the theoretical behavior of a simple undamped Jeffcott rotor, for which the shaft is flexible, massless, and represented by equivalent translational springs, the disk with mass \( M \) is unbalanced and located at midspan, and the supports are rigid [5, 6, 7]. The disk has an eccentricity, \( e \), associated with an unbalance mass \( m \), such that \( e = mu/M \). The maximum distance from the original shaft center to the deflected shaft center during steady-state motion is the measured amplitude.

Figure 2(a) shows a schematic Jeffcott rotor. The geometry and bearings result in the rotational analog of a simply-supported beam’s frequencies and mode shapes. Part (b) of this figure shows a schematic of a sweep-up in rotational rate, with the maximum resonant response associated with the critical speed. The goal of this work is to use information about the low-amplitude response (and its rate of increase) to predict the critical speed, without actually reaching it.

Figure 2. (a) the Jeffcott rotor, (b) a typical sweep up in rotational speed passing through resonance.
The translational steady-state synchronous motion of the disk is \( x(t) = A \sin(\omega t - \phi) \) with amplitude

\[
A = \frac{e\omega^2}{|\omega_0^2 - \omega^2|}
\]

(1)

where \( \omega_0 \) is the natural vibration frequency and \( A \) is the maximum radial amplitude of the geometric center of the disk at rotational speed \( \omega \). The critical speed \( \omega_{cr} \) is equal to \( \omega_0 \). Thus we see that the amplitude grows as \( \omega^2 \to \omega_0^2 \).

3. Prediction Based on Alternative Formats

Along the lines of the Southwell plot, if \( \omega < \omega_0 \), Eq. 1 can be written as

\[
A = \omega_0^2 (A/\omega^2) - e
\]

(2)

This represents the equation of a straight line in which the slope gives the square of the critical speed, \( \omega_0^2 \), if the axes \((x, y) = (A/\omega^2, A)\) are chosen (regardless of the value of \( e \)). We shall make use of this format in “Plot 1” a little later. We can also creatively choose other axes in order to isolate information about the critical speed. For example, Eq. 1 can also be written as

\[
A\omega^2 = A\omega_0^2 - e\omega^2
\]

(3)

and now this can be exploited since it is also a straight line with a slope corresponding to the square of the critical speed if \((x, y) = (A, A\omega^2)\). We shall make use of this format in “Plot 2”. Finally, we can also re-arrange Eq. 1 into the form

\[
\omega^2 / A = \omega_0^2 / e - \omega^2 / e
\]

(4)

so that now the intercept with the \( x \)-axis provides an estimate of the critical speed (squared) if \((x, y) = (\omega^2, \omega^2 / A)\), and we shall use this format in “Plot 3”.

These three formats (Eqs. 2-4) generate the plots shown in figure 3. For Plot 1, \( A \) is the vertical axis and \( A/\omega^2 \) is the horizontal axis, resulting in a straight line with slope \( \omega_0^2 \), which is equal to \( \omega_{cr}^2 \). It is proposed that for other rotor systems, the slope of a straight line fit to measured data points taken at low rotational speeds and plotted with these axes \((A/\omega^2, A)\) will furnish an estimate for the square of the first critical speed. (Higher critical speeds also could be estimated using measured amplitudes at rotational speeds approaching those critical speeds, and indeed, critical speeds might be estimated under reducing rotational speeds starting from high rates of rotation.)

For Plot 2, the vertical axis is \( A\omega^2 \) and the horizontal axis is \( A \). The slope yields an approximate value of \( \omega_{cr}^2 \) (in this case there is a minor effect due to \( e \)). Finally, in Plot 3 the vertical axis is \( \omega^2 / A \) and the horizontal axis is \( \omega^2 \), and an approximate value of \( \omega_{cr}^2 \) is obtained by extrapolating data points to the horizontal axis. The extrapolation need not be linear, depending on damping and other effects to be discussed.

The accuracy of the new method depends on the number of amplitude measurements and the rotational speeds at which they are recorded (relative to the critical speed).

3.1. Effect of Damping

Steady-state motion of a Jeffcott rotor with viscous damping is considered. The damping ratio is \( \zeta \) and the amplitude of the rotor is [6]

\[
A = \frac{e\omega^2}{[(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2]^{1/2}}
\]

(5)
The maximum value of $A$ occurs at the critical speed

$$\omega_{cr} = \frac{\omega_0}{(1 - 2\zeta^2)^{1/2}}$$

(6)

As an example, assume that $\omega_0 = 1$ and $\varepsilon = 1$, so that one can interpret $A$ as the dimensional amplitude divided by the dimensional eccentricity, and $\omega$ as the dimensional rotational speed divided by the natural frequency. Figure 4(a) shows the steady-state responses of the damped system in terms of amplitude vs. frequency, for some typical damping ratios. For relatively low rotational speeds the damping does not have much effect on the response.

We can recast Eq. 5 into the format referred to as Plot 1 ($A$ versus $A/\omega^2$), and the result is shown in figure 4(b). We see a close-to-linear relation for relatively low amplitude, regardless of damping, and this slope gives the critical speed ($\omega_0 = 1$). The approach is indicated by the red arrows. There is also a degree of linearity in this relation under decreasing speed (the green arrows). The alternative plots (2 and 3) give similar results [8], but we will defer direct comparison till the next section, in which experimental data are used to assess the utility of the approach in a more practical context.

4. Experiments
A series of tests was performed using the Bently Nevada Rotor Kit, a test-bed specifically designed to illustrate various rotordynamic behaviors. The system is shown in figure 5. Two sets of proximity
Figure 5. The Bently Nevada experimental rig, with two disks near the shaft center, and proximity probes.

probes were used to measure the X and Y coordinates of the response for different rotation rates. The average amplitude \( A = \sqrt{X^2 + Y^2} \) was then computed.

A single disk was placed centrally on the shaft, and preliminary testing suggested a critical speed in the vicinity of 1700 - 1900 rpm. The amplitude (\( A \)) was measured at discrete rotational speeds (\( \omega \)). Data generated from using three eccentric (unbalance) masses are superimposed in figure 6. From part (a) we see that the magnitude of the unbalance masses changes (scales) the amplitude but not the critical speed: this is a linear system.

5. Results

Using the new prediction approach we re-plot the data in the suggested ways and obtain the results in figure 6(b-d). For ‘Plot 1’ (part (b)), a linear least-squares fit to the initial unbalance mass eccentricity data (the small crosses) gives a slope of \( 2.747 \times 10^{-5} \) and thus \( \omega_{cr}^2 = 36,403 \) and \( \omega_{cr} = 190.8 \text{ rad/s} \) and a critical speed of 1822 rpm. The response from other ranges of excitation can be used. The higher eccentricity was achieved by adding a second mass unbalance to the disk (signified by the closed circles) and were also fit to give a slope of \( 2.775 \times 10^{-5} \) and thus \( \omega_{cr}^2 = 36,032 \) and \( \omega_{cr} = 189.82 \text{ rad/s} \) and a critical speed of 1813 rpm. Some data were also taken with no unbalance masses attached (the open circles), i.e., some unavoidable eccentricity associated with the shaft itself, and these data resulted in a prediction of the critical speed of 1824 rpm.

Predictions based on Plots 2 and 3 give similarly accurate estimates of the critical speed: within a couple of percentages points depending on the range over which the data are fit.

We see that the linearity of the plot is questionable for low excitation frequencies (and hence low response amplitudes) for Plot 1, and in Plot 3 it turns out that a quadratic fit is the appropriate basis for extrapolation. It was determined that these effects are primarily associated with some shaft bow in the experimental rotor [8]. However, the general conclusion drawn from these data is that the new plot axes can provide useful (accurate) predictive information regarding the approach of a critical speed.

As a final confirmation, consider the case in which a second disk is added near the center of the
Figure 6. (a) the amplitude response as a function of rpm, (b-d) conversion into the new axes (Plots 1, 2, and 3) for prediction.

In this case we might expect the critical speed to be lowered by a factor of approximately $1/\sqrt{2}$. This is the case shown in figure 5. Some sample time series are shown in figures 7(a) and (b), together with the corresponding orbit in part (c) from which we extract the amplitude. At this rate of rotation (84 rad/s) there is a modest signal-to-noise ratio. Since the noise level remains fairly constant, the higher rotation rates and hence responses typically result in a high signal-to-noise ratio. For example, for a rotation rate of 400 rpm the average amplitude ($A$) is $0.1724 \text{ mm}$ with a standard deviation ($\sigma$) of $0.0120 \text{ mm}$; for 800 rpm, $A = 0.2714 \text{ mm}, \sigma = 0.0174 \text{ mm}$ (shown in figure 7), and for 1100 rpm, $A = 0.5950 \text{ mm}, \sigma = 0.0520 \text{ mm}$. However, as we shall see, this effect does not seem to adversely influence the prediction since the noise is averaged out in the amplitude measure. In effect, this means that the approach appears to be well-suited to practical in-situ applications and by no means limited to the desk-top experiments used here.

We next plot the various predictions for this case in figure 8. In general we focus attention on the data points leading up to the first critical speed. These are indicated in red in part (a). We also see from this diagram that the critical speed appears to be close to the response reaching $A \approx 0.65 \text{ mm}$, taken at $\omega = 1300 \text{ rpm}$ ($\equiv 136 \text{ rad/s}$). Of course, in practice we might be reluctant to reach the critical speed. In Plot 1 we only make use of the final three red data points in order to fit the data. This is due to a shaft bow effect that seems to have a profound (magnifying) effect for very low amplitudes/speeds when using the Plot 1 axes [8]. However, this slope gives $\omega_{cr}^2 = 18,785$ and thus $\omega_{cr} = 137.1 \text{ rad/s}$. Using all the red data points for Plot 2 leads to a prediction of $\omega_{cr}^2 = 17,788$ and thus $\omega_{cr} = 133.4 \text{ rad/s}$. Finally, and again using all the red data points and fitting with a quadratic and the Plot 3 format leads to $\omega_{cr}^2 = 18,055$ and thus $\omega_{cr} = 134.4 \text{ rad/s}$. Note the axes
Figure 7. A typical response (800 rpm). (a) time series from the X-direction sensor, (b) time series from the Y-direction sensor, (c) orbit.

Figure 8. Two disks at the shaft center, (a) the amplitude response as a function of rpm, (b-d) conversion into the new axes (Plots 1, 2, and 3) for prediction.
in part (d) do not extend to zero in the plot, but the fit does. All these predictions appear to be quite accurate.

It is interesting to note that the approximate critical speed for the two-disk case is indeed very close to a factor of $1/\sqrt{2}$ in comparison with the single-disk case, thus placing a good deal of confidence in the lumped-mass assumption on which the initial concept was based.

6. Concluding Remarks
This paper has shown that it is possible to exploit simple relationships involving the rotational speed and response amplitude of a rotating shaft as it approaches a critical speed, in order to predict that critical speed. This approach has been shown to work well in other circumstances [8]. Here it has been extended to predict the first critical speed of a rotor with two disks. Since there is no need for a theoretical model (the method is entirely data-driven), this approach is ideally suited to in-situ measurements in the field, in which the environmental changes might invalidate a model even under relatively high-fidelity modeling.

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