Anomalous magnetic interference of cross-type Josephson junctions exposed to oblique magnetic fields

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Abstract

Gauge-invariant phase difference and critical currents of cross-type Josephson junctions with thin and narrow superconducting strips exposed to three-dimensional magnetic fields are theoretically investigated. When a sandwich-type Josephson junction in the xy plane is exposed to parallel magnetic fields $H_x$ and $H_y$, the phase difference linearly depends on the spatial coordinates, $x$ and $y$, and the critical currents exhibit the standard Fraunhofer-type magnetic interference. The perpendicular field $H_z$, on the other hand, nonlinearly modulates the distribution of the phase difference and the critical currents as the functions of the oblique field exhibit anomalous magnetic interference. We obtain simple analytical expressions for critical currents of small cross-type junctions by neglecting the effects of self-field and trapped vortices. The resulting dc critical currents show anomalous and diverse interference patterns depending on the parallel and perpendicular magnetic fields.

Keywords: cross junction, critical current, magnetic interference

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the most fascinating behaviors of the static response of Josephson junctions is the interference patterns in the magnetic-field dependence of the dc critical current $I_c$. The $I_c$ of small junctions exhibit Fraunhofer-type interference patterns [1–3], and the interference behavior in response to a magnetic flux in steps of the flux quantum $\phi_0 = h/2e = 2.07 \times 10^{-15}$ Wb is a key phenomenon for application to ultrasensitive magnetometers, superconducting quantum interference devices (SQUIDs) [2, 4, 5].

Anomalous magnetic interference in $I_c$ of Josephson junctions has been extensively investigated by considering the effects of inhomogeneous current density in the junction [4, 6], trapped vortices [7–10], local current injection [11–13], ferromagnetic $\pi$ junctions [14–16], and semiconductor junctions [17]. Magnetic interference in $I_c$ are also analyzed to investigate the edge states in topological insulators [18–20] and in graphene [21].

Here we demonstrate that anomalous interference patterns appear even in the naive case of sandwich-type Josephson junctions in oblique magnetic fields. Although the response of junctions to parallel magnetic fields is well understood [4], only a limited number of works have been published on the response to perpendicular and/or oblique magnetic fields [7, 22–25]. The systematic response of dc critical current of Josephson junctions to oblique magnetic fields remains unclear.

In this paper, we theoretically investigate the dc critical current $I_c$ of cross-type junctions in oblique magnetic fields. We derive simple analytical expressions for $I_c$ as a function of the three-dimensional magnetic field $\mathbf{H} = (H_x, H_y, H_z)$ by assuming that the magnetic screening effect in small junctions is weak and that no vortices are trapped in the junctions. We
demonstrate a variety of anomalous interference patterns in the field dependence of $I_c$, although the analytical formulae of $I_c$ are quite simple.

This paper is organized as follows. The static two-dimensional distribution of the gauge-invariant phase difference in small sandwich-type Josephson junctions is considered in section 2. Analytical expressions for $I_c$ of cross-type Josephson junctions are derived and a variety of the interference patterns of $I_c$ as functions of the parallel and perpendicular (i.e. in-plane and out-of-plane) magnetic fields are demonstrated in section 3. Our results are summarized in section 4.

2. Gauge-invariant phase difference

In this section we consider the static two-dimensional distribution of the gauge-invariant phase difference in sandwich-type Josephson junctions by neglecting the effects of self-field and trapped vortices.

2.1. Basic equations for two-dimensional Josephson junctions

We consider a Josephson junction as shown in figure 1 in which a barrier layer of thickness $d_b$ is located at $|z| < d_b/2$ sandwiched between two superconducting layers.

The supercurrent density associated with the order parameter $\psi = |\psi| \exp(i\varphi)$ based on Ginzburg-Landau theory is given by [3, 4]:

$$ J = \frac{1}{\mu_0 \lambda^2} \left( \frac{\phi_0}{2\pi} \nabla \varphi - A \right), \quad (1) $$

where $\mu_0$ is the vacuum permeability, $\lambda$ is the London penetration depth, $\varphi$ is the phase, and $A$ is the vector potential related to the magnetic induction $B = \nabla \times A$. We assume that suppression of $|\psi|$ from its equilibrium value is negligible.

The gauge-invariant phase difference $\theta(x,y)$ at the junction, which is associated with the phase $\varphi(x,y,z)$ and the $z$ component of the vector potential $A_z(x,y,z)$, is defined by [3, 4, 26]:

$$ \theta(x,y) = \varphi(x,y,+d_b/2) - \varphi(x,y,-d_b/2) $$

$$ - \frac{2\pi}{\phi_0} \int_{-d_b/2}^{+d_b/2} A_z(x,y,z) dz. \quad (2) $$

We calculate the contour integral of the vector potential along the rectangular contour $C_y$ which has vertices at $(x,y, \pm d_b/2)$ and $(x,y + \Delta y, \pm d_b/2)$ (figure 1). Using equations (1) and (2), we also obtain the following expression:

$$ \oint_{C_y} A \cdot ds = \int_0^{\Delta y} dy \int_{-d_b/2}^{+d_b/2} dz B_z = B_s d_y \Delta y, \quad (3) $$

for a thin junction layer $d_b \to 0$ and narrow contour width $\Delta y \to 0$. Using equations (1) and (2), we also obtain the following expression:

Figure 1. Schematic of the Josephson-junction structure in which a barrier layer of thickness $d_b$ located at $|z| < d_b/2$ between superconducting layers at $|z| > d_b/2$. $C_y$ and $C_z$ are closed rectangular contours of $\Delta y \times d_y$ in the $yz$ plane and of $\Delta x \times d_y$ in the $zx$ plane, respectively.

$$ \oint_{C_z} A \cdot ds = \int_{-\Delta x}^{\Delta x} dx \int_{-d_b/2}^{+d_b/2} dz B_y = B_s d_x \Delta y, $$

$$ + \int_{-d_b/2}^{+d_b/2} dz [A_z(x,y + \Delta y,z) - A_z(x,y,z)] $$

$$ = - \frac{\phi_0}{2\pi} \frac{\partial \theta(x,y)}{\partial y} - \Delta y $$

$$ + \mu_0 \lambda^2 \left[ J_s(x,y,+d_b/2) - J_s(x,y,-d_b/2) \right] \Delta y. \quad (4) $$

Equations (3) and (4) yield [3, 4, 7, 27, 28]:

$$ \frac{\phi_0}{2\pi} \frac{\partial \theta(x,y)}{\partial y} = \mu_0 \lambda^2 \left[ J_s(x,y,+d_b/2) - J_s(x,y,-d_b/2) \right] $$

$$ - B_s d_y \Delta y, \quad (5) $$

The contour integral along $C_y$ of $A$ also yields a similar equation:

$$ \frac{\phi_0}{2\pi} \frac{\partial \theta(x,y)}{\partial y} = \mu_0 \lambda^2 \left[ J_s(x,y,+d_b/2) - J_s(x,y,-d_b/2) \right] $$

$$ + B_s d_y \Delta y, \quad (6) $$

2.2. Cross-type junctions with thin superconducting nanostrips

We now consider cross-type junctions with two superconducting nanostrips consisting of a top strip of thickness $d_{s1}$ and width $w_1$ along the $y$ axis and a bottom strip of thickness $d_{s2}$ and width $w_2$ along the $x$ axis, as shown in figure 2.

We assume that the thicknesses of the strips are smaller than the London penetration depth, $d_{s1} < \lambda$ and $d_{s2} < \lambda$, and the widths are smaller than the Pearl length, $w_1 < \lambda^2/d_{s1}$ and $w_2 < \lambda^2/d_{s2}$, such that magnetic screening is weak in the superconducting nanostrips. The current density in thin and narrow superconducting strips is simply obtained by integrating the London equation, $\nabla \times J = -B/\mu_0 \lambda^2 \simeq -H/\lambda^2$, where $H = (H_x,H_y,H_z)$ is the applied uniform magnetic field.
The screening current density $J_1$ in the top strip and $J_2$ in the bottom strip are thus given by:

$$J_1 = \frac{1}{\lambda^2} \left[ -x H_z \dot{y} + \left( z - \frac{d_1 + d_1}{2} \right) (-H_z \dot{\hat{x}} + H_y \dot{\hat{y}}) \right],$$  \hspace{1cm} (7)$$

$$J_2 = \frac{1}{\lambda^2} \left[ y H_z \dot{x} + \left( z + \frac{d_2 + d_2}{2} \right) (-H_z \dot{\hat{x}} + H_y \dot{\hat{y}}) \right].$$  \hspace{1cm} (8)$$

Substitution of equations (7) and (8) into equations (5) and (6) yields:

$$\frac{\phi_0}{2\pi \mu_0} \frac{\partial \theta(x,y)}{\partial x} = d_{\text{eff}} H_y - y H_z,$$  \hspace{1cm} (9)$$

$$\frac{\phi_0}{2\pi \mu_0} \frac{\partial \theta(x,y)}{\partial y} = -d_{\text{eff}} H_x - x H_z,$$  \hspace{1cm} (10)$$

where $d_{\text{eff}}$ is the effective junction thickness for thin superconducting films as follows\,[27]:

$$d_{\text{eff}} = d_1 + (d_1 + d_2)/2.$$  \hspace{1cm} (11)$$

Integrating equations (9) and (10), we obtain the resulting distribution of the gauge-invariant phase difference,

$$\theta(x,y) = \theta_0 + \frac{2\pi \mu_0}{\phi_0} \left[ d_{\text{eff}} (x H_y - y H_x) - x y H_z \right],$$  \hspace{1cm} (12)$$

where $\theta_0$ is the integral constant. Note that $\theta$ is given as the sum of the parallel-field contribution ($\propto x H_y - y H_x$) and the perpendicular-field contribution ($\propto x y H_z$)\,[7], and the simple formula of equation (12) is obtained for weak magnetic screening.

Responding to the perpendicular field $H_z$, the screening current flowing in superconducting strips produces an effective parallel magnetic field in the junctions\,[7], resulting in a modulation of $\theta$. Thus, the perpendicular field $H_z$ modulates $\theta$ through $J_3$ and $J_4$, as shown in the right-hand sides of equations (5) and (6).

3. dc critical currents of cross-type Josephson junctions

3.1. General expression for critical currents

The Josephson current density across the junction is $J_c = J_c \sin \theta$, where $J_c$ is the critical current density. The net current $I_c$ across the junction is given by:

$$I_c = \int_{-w_1/2}^{w_1/2} dx \int_{-w_2/2}^{w_2/2} dy J_c \sin \theta(x,y)$$

$$= J_c \int_{-w_1/2}^{w_1/2} dx \int_{-w_2/2}^{w_2/2} dy \exp \{i \theta_0 + \theta_1(x,y) \},$$  \hspace{1cm} (13)$$

where equation (12) is rewritten as $\theta(x,y) = \theta_0 + \theta_1(x,y)$ and,

$$\theta_1(x,y) = \frac{2x}{w_1} \beta - \frac{2y}{w_2} \alpha - \frac{4xy}{w_1 w_2} \gamma.$$  \hspace{1cm} (14)$$

The parameters $\alpha$, $\beta$, and $\gamma$ in equation (14) are defined by:

$$\alpha = \pi \Phi_z/\phi_0, \quad \Phi_z = \mu_0 H_w d_{\text{eff}},$$  \hspace{1cm} (15)$$

$$\beta = \pi \Phi_y/\phi_0, \quad \Phi_y = \mu_0 H_w d_{\text{eff}},$$  \hspace{1cm} (16)$$

$$\gamma = \pi \Phi_x/2\phi_0, \quad \Phi_x = \mu_0 H_w d_{\text{eff}},$$  \hspace{1cm} (17)$$

where $\Phi_x = \mu_0 H_w S_x$, $\Phi_y = \mu_0 H_w S_y$, and $\Phi_z = \mu_0 H_w S_z$ are the magnetic fluxes linked in the areas $S_x = w_2 d_{\text{eff}}$, $S_y = w_1 d_{\text{eff}}$, and $S_z = w_1 w_2$, respectively.

The dc critical current $I_c$ is obtained by maximizing $I_c$ with respect to $\theta_0$. Equation (13) is rewritten in the form $I_c = \text{Im} [\exp(\theta_0)]$, which has a maximum of $|\theta|$. We thus obtain $I_c = J_c \int dx \int dy |\exp(\theta_0)| \ [3]$. That is,

$$\frac{I_c(\Phi_x, \Phi_y, \Phi_z)}{I_{c0}} = \mathcal{I} \left( \frac{\Phi_x}{\phi_0}, \frac{\Phi_y}{\phi_0}, \frac{\Phi_z}{\phi_0} \right),$$  \hspace{1cm} (18)$$

where $I_{c0}$ is the critical current at zero magnetic field,

$$I_{c0} = I_c(0,0,0) = J_c w_1 w_2,$$  \hspace{1cm} (19)$$

and the complex-valued function $\mathcal{I}(\alpha, \beta, \gamma)$ is defined by:

$$\mathcal{I}(\alpha, \beta, \gamma) = \frac{1}{4} \int_{-1}^{1} dx' \int_{-1}^{1} dy' \exp \{i(\alpha x' + \beta x' + \gamma x'y') \}.$$  \hspace{1cm} (20)$$

The integral variables in equation (20) are $x' = 2x/w_1$ and $y' = 2y/w_2$. See appendix for details on $\mathcal{I}(\alpha, \beta, \gamma)$. The symmetry in $\mathcal{I}(\alpha, \beta, \gamma)$ shown in equation (A.1) leads to symmetry in the critical current $I_c \propto |\mathcal{I}|$,

$$I_c(-\Phi_x, \Phi_y, \Phi_z) = I_c(\Phi_x, -\Phi_y, \Phi_z) = I_c(\Phi_x, \Phi_y, -\Phi_z) = I_c(\Phi_x, \Phi_y, \Phi_z).$$  \hspace{1cm} (21)$$

We thus consider only the case when $\Phi_x \geq 0$, $\Phi_y \geq 0$, and $\Phi_z \geq 0$. 

Figure 2. Schematic of a cross-type Josephson junction with two superconducting nanostrips. The barrier layer is located at $|x| < w_1/2$, $|y| < w_2/2$, and $|z| < d_2/2$. The top superconducting strip is located at $|x| < w_1/2$, $|y| < w_2/2$, and $d_1/2 < z < -d_1/2$. The bottom superconducting strip is located at $|x| < \infty$, $|y| < w_2/2$, and $-d_2/2 < z < -d_2/2$, where $d_2 < \lambda$ and $w_2 < \lambda^2/d_2$. The parameters $\alpha$, $\beta$, and $\gamma$ are defined by:

\[ \alpha = \frac{2\pi \mu_0}{\phi_0} \Phi_z, \quad \beta = \frac{2\pi \mu_0}{\phi_0} \Phi_y, \quad \gamma = \frac{2\pi \mu_0}{\phi_0} \Phi_x. \]
3.2. Critical currents in parallel or perpendicular fields

The \( I_c \) for parallel magnetic fields (i.e. \( \Phi_z = 0 \)) is obtained from equation (A.2), and is given by

\[
\frac{I_c(\Phi_x, \Phi_y, 0)}{I_{c0}} = \frac{\sin(\pi \Phi_y / \phi_0) \sin(\pi \Phi_y / \phi_0)}{\pi \Phi_y / \phi_0},
\]

which shows the standard Fraunhofer-like interference [4].

The \( I_c \) for perpendicular magnetic fields (i.e. \( \Phi_x = \Phi_y = 0 \)) is obtained from equation (A.3), and is given by [7]:

\[
\frac{I_c(0, 0, \Phi_z)}{I_{c0}} = \frac{\sin(2\pi \Phi_z / 2\phi_0)}{2\pi \phi_0},
\]

where \( \text{Si}(z) = \int_0^z \sin(t) / t \, dt \) is the sine integral [29]. Note that \( I_c(0, 0, \Phi_z) \) monotonically decreases with increasing \( \Phi_z \) and shows no interference patterns [7]. The contribution of the phase difference \( \theta \) from the perpendicular field \( H_z \) is proportional to \( H_z \), as shown in equation (12), and the modulation of \( \theta \) due to \( H_z \) results in the decrease of the critical currents as \( I_c \sim 1 / H_z \) for large \( H_z \) [7, 24].

3.3. Critical currents in two-dimensional oblique fields

The \( I_c \) for \( \Phi_z = 0 \) is obtained from equation (A.4), and is given by:

\[
\frac{I_c(\Phi_x, 0, \Phi_y)}{I_{c0}} = \frac{\text{Si}[\pi(\Phi_x + \Phi_y/2) / \phi_0] - \text{Si}[\pi(\Phi_x - \Phi_y/2) / \phi_0]}{\pi \Phi_y / \phi_0}.
\]

For \( \Phi_y = \Phi_z / 2 \) and \( \Phi_z = 0 \) we have \( I_c(\Phi_z/2, 0, \Phi_z) = I_c(0, 0, 2\Phi_z) \).

Figures 3 and 4 show the plots of \( I_c \) vs \( \Phi_y \) and \( I_c \) vs \( \Phi_z \), respectively, where \( I_c \) is obtained from equation (24). The plot of \( I_c \) vs \( \Phi_y \) for \( \Phi_z = 0 \) in figure 3(a) corresponds to the Fraunhofer-like interference pattern given by equation (22) for \( \Phi_z = 0 \). The plot of \( I_c \) vs \( \Phi_z \) for \( \Phi_z = 0 \) in figure 4(a) shows the monotonic dependence given by equation (23). Interference patterns appear when \( |\Phi_z| < 2|\Phi_y| \). As shown in figures 3(a) and 4(a), the \( I_c \) for \( \Phi_z < 2|\Phi_y| \) is much smaller than \( I_{c0} \) when \( \Phi_y / \phi_0 \approx 1, 2, 3, 4, \ldots \) and \( \Phi_z / \phi_0 \approx 0, 1, 3, 5, \ldots \). As shown in figures 3(b) and 4(b), the \( I_c \) for \( \Phi_z < 2|\Phi_y| \) is much smaller than \( I_{c0} \) also when \( \Phi_z / \phi_0 \approx 3/2, 5/2, 7/2, 9/2, \ldots \) and \( \Phi_z / \phi_0 \approx 2, 4, 6, 8, \ldots \). The \( I_c \) with the magnetic interference is generally suppressed for \( |\Phi_z| < 2|\Phi_y| \), while the \( I_c \) without the magnetic interference is relatively large for \( |\Phi_z| > 2|\Phi_y| \). Therefore, the \( I_c \) temporarily increases with \( \Phi_z \) at \( |\Phi_z| \sim 2|\Phi_y| \), as seen in figure 4.

Figure 5 shows a density plot of \( I_c \) as a function of \( (\Phi_x, \Phi_y) \). For \( |\Phi_z| < 2|\Phi_y| \) [i.e. the right-lower region below the dashed line in the \( (\Phi_x, \Phi_y) \) plane], the stepwise dark lines corresponding to \( I_c \ll I_{c0} \) clearly demonstrate the interference patterns of \( I_c(\Phi_x, 0, \Phi_z) \), which are consistent with the behavior shown in figures 3 and 4.

Substitution of \( \alpha = (n + 1/2)\pi \) or \( \gamma = (m + 1/2)\pi \) (where \( n \) and \( m \) are integers) in equation (A.5) yields:

\[
\frac{I_c(\Phi_x, 0, \Phi_z)}{I_{c0}} \sim \frac{\sin(\pi \Phi_x / \phi_0) \sin(\pi \Phi_z / 2\phi_0)}{\pi \Phi_x / \phi_0 \pi \Phi_z / 2\phi_0}.
\]
for \( \Phi_x/\phi_0 = n + 1/2 \) or \( \Phi_x/\phi_0 = 2m + 1 \). Equation (25) roughly explains the interference patterns shown in figures 3(a) and 4(b). Substitution of \( n = m \pi \) or \( \gamma = m \pi \) (where \( n \) and \( m \) are integers) into equation (A.5) yields:

\[
\frac{I_c(\Phi_x, 0, \Phi_z)}{I_{c0}} \sim \frac{\cos(\pi \Phi_x/\phi_0)}{(\pi \Phi_x/\phi_0)^2} \cos(\pi \Phi_z/2\phi_0),
\]

for \( \Phi_x/\phi_0 = n \) or \( \Phi_x/\phi_0 = 2m \). Equation (26) roughly reproduces the interference patterns shown in figures 3(b) and 4(a). Equation (A.6) yields:

\[
\frac{I_c(\Phi_x, 0, \Phi_z)}{I_{c0}} \sim \frac{\phi_0}{\Phi_z} \cos(\pi \Phi_x/\phi_0) \cos(\pi \Phi_z/2\phi_0),
\]

which roughly explains the behavior of \( I_c \) without interference shown in figures 3 and 4.

### 3.4. Critical currents in three-dimensional oblique fields

Figure 6 shows density plots of \( I_c \) as functions of \( (\Phi_x, \Phi_y) \). The interference patterns of \( I_c \) disappear in the bright regions of \( |\Phi_x| > 2 \max(|\Phi_x|, |\Phi_y|) \). Dark lines or spots corresponding to the case where \( I_c \ll I_{c0} \) exhibit peculiar interference patterns with horizontal and vertical checkerboard patterns seen for integer \( \Phi_x/\phi_0 \) (upper panels of figure 6) and diagonal checkerboard patterns seen for half-integer \( \Phi_x/\phi_0 \) (lower panels of figure 6). These patterns are roughly reproduced by the simplified expression in equation (A.13).

Figure 7 shows the density plots of \( I_c \) as functions of \( (\Phi_x, \Phi_y, \Phi_z) \), where \( \Phi_{j0} \equiv (\Phi_{jx}^0 + \Phi_{jy}^0)^{1/2} \). The parameter \( \vartheta \equiv \arctan(\Phi_x/\Phi_y) = \arctan([w_1/w_2](H_x/H_y)) \) corresponds to the field angle in the \( xy \) plane when \( w_1 = w_2 \). If we simply denote the \( \vartheta \) dependence of the critical current as \( I_c(\vartheta) \), the \( I_c(\vartheta) \) has a periodicity expressed as \( I_c(\pi/2 + \vartheta) = I_c(\pi/2 - \vartheta) = I_c(\vartheta) \), as seen in figure 6. The \( I_c(\vartheta) \) for \( 0 < \vartheta \leq \pi/4 \) is plotted in figure 7. Noticeable \( I_c \)-interference texture like weaving patterns, which can also be reproduced by equation (A.13), are seen in the region of \( \Phi_z < 2H_{f1} \min(\cos \vartheta, \sin \vartheta) \).

### 4. Discussion and summary

We considered the response of sandwich-type Josephson junctions in three-dimensional magnetic fields \( \mathbf{H} = (H_x, H_y, H_z) \). It should be noted that the response to the perpendicular field \( H_z \) is much more sensitive than that to the parallel magnetic field \( H_{\parallel} \) if the junction dimensions in the \( xy \) plane are much larger than the effective junction thickness \( w_1 \gg d_{eff} \), where \( H_j \equiv (H_{x}^2 + H_{y}^2)^{1/2} \) and \( w_j \equiv \min(w_1, w_2) \). Even when \( |H_x| \sim |H_y| \), we have \( |\Phi_x/\Phi_0| \geq |H_x/H_{\parallel}|(w_j/d_{eff}) \gg 1 \) from equations (15)–(17). Therefore, when the dependence of \( I_c \) on the field angle from the \( xy \) plane \( \vartheta_0 = \arctan(H_x/H_{\parallel}) \) is investigated, the theoretical results presented in this paper may be observed in the narrow angle range of \( |\vartheta| < d_{eff}/w_j \ll 1 \). The present theory considers the case where the applied magnetic field is smaller than the lower critical field and no vortices are present in the junctions. In perpendicular geometry, vortices are easily to penetrate into the junctions, and the perpendicular field \( H_z \) should be small enough.

The junction geometry and configuration of the superconducting films affect the interference behavior of \( I_c \). The \( I_c \) for overlap-type junctions in perpendicular fields \( H \) show Fraunhofer-type interference patterns [22, 23], whereas \( I_c \) for cross-type junctions monotonically decrease with \( H \), without interference [7]. Monaco et al [24] numerically showed that interference patterns in \( I_c \) vs \( H_z \) strongly depend on the configuration of the superconducting strips. Although we consider only cross-type junctions in this paper, the \( I_c \) interference in oblique fields for other geometries (e.g. overlap-type junctions) is expected to show different patterns from our results.

In this paper we assume that the critical current density \( J_c \) is homogeneous in the Josephson junctions, although inhomogeneous \( J_c \) affects the interference patterns of \( I_c \) [4, 6]. The inhomogeneous current distributions arising from the edge states in topological insulators [18–20] and in graphene [21] are investigated by analyzing the interference patterns of \( I_c \). It would be interesting to consider the effects of the inhomogeneous \( I_c(x, y) \) due to the edge states in topological materials upon the interference patterns of the sandwich-type junctions in three-dimensional magnetic fields.

To summarize, we theoretically investigate the dc critical current \( I_c \) of cross-type Josephson junctions with thin and narrow superconducting strips exposed to three-dimensional magnetic fields \( (\Phi_x, \Phi_y, \Phi_z) \) for weak magnetic screening in the junctions and in the strips. The standard Fraunhofer-type magnetic interference for parallel magnetic fields is strongly modulated by the perpendicular field, and the magnetic interference disappear when \( |\Phi_z| > 2 \max(|\Phi_x|, |\Phi_y|) \). Phase manipulation by the perpendicular field results in a variety of the magnetic interference of \( I_c \), and might be useful for developing novel Josephson-junction devices.
Figure 6. Logarithmic density plots of the critical current $I_c(\Phi_x, \Phi_y, \Phi_z)/I_{c0}$ as a function of $\alpha/\pi = \Phi_x/\phi_0$ and $\beta/\pi = \Phi_y/\phi_0$ when $2\gamma/\pi = \Phi_z/\phi_0$ is integer (upper panels) and half integer (lower panels). The color scale in these plots is the same as in figure 5. Vertical and horizontal dashed lines correspond to $\Phi_z = 2\Phi_x$ and $\Phi_z = 2\Phi_y$, respectively.

Figure 7. Logarithmic density plots of the critical current $I_c(\Phi_x \cos \vartheta, \Phi_y \sin \vartheta, \Phi_z)/I_{c0}$ as a function of $\Phi_z \equiv (\Phi_x^2 + \Phi_y^2)^{1/2}$ and $\Phi_z$ for $\vartheta \equiv \arctan(\Phi_y/\Phi_x) = 0.10\pi, 0.15\pi, 0.19\pi, 0.25\pi$. The color scale in these plots is the same as in figure 5. Upper and lower dashed lines correspond to $\Phi_z = 2\Phi_x = 2\Phi_y \cos \vartheta$ and $\Phi_z = 2\Phi_y = 2\Phi_y \sin \vartheta$, respectively.

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Appendix. The complex-valued function $I(\alpha, \beta, \gamma)$

We now present the mathematical formulae for the complex-valued critical current $I(\alpha, \beta, \gamma)$ defined by equation (20). Because of the following symmetries, we need only consider the case when $\alpha \geq 0$, $\beta > 0$, and $\gamma > 0$:

$$I(-\alpha, \beta, \gamma) = I(\alpha, -\beta, \gamma) = I(\alpha, \beta, -\gamma) = I^*(\alpha, \beta, \gamma),$$

where $I^*$ is the complex conjugate of $I$.

The $I$ for $\gamma = 0$ and for $\alpha = \beta = 0$ are respectively given by:

$$I(\alpha, \beta, 0) = (\sin \alpha \sin \beta)/\alpha \beta,$$  \hspace{1cm} (A.2)

$$I(0, 0, \gamma) = \text{Si}(\gamma)/\gamma.$$  \hspace{1cm} (A.3)

The $I$ for $\beta = 0$ is:

$$I(\alpha, 0, \gamma) = \frac{1}{2\gamma} [\text{Si}(\alpha + \gamma) - \text{Si}(\alpha - \gamma)].$$  \hspace{1cm} (A.4)

The asymptotic behavior of equation (A.4) for $|\alpha| \gg |\gamma|$ is:

$$I(\alpha, 0, \gamma) \sim \frac{\sin \alpha \sin \gamma}{\alpha \gamma} + \frac{\cos \alpha}{\alpha^2} (\cos \gamma - \frac{\sin \gamma}{\gamma}),$$  \hspace{1cm} (A.5)
and that for $|\alpha| \ll |\gamma|$ is:
\[
\mathcal{I}(\alpha, 0, \gamma) \sim \frac{\pi}{2\gamma} - \cos \alpha \frac{\cos \gamma}{\gamma^2}.
\]  
(A.6)

Equation (20) is generally rewritten as:
\[
\mathcal{I}(\alpha, \beta, \gamma) = \frac{ie^{-i\alpha\beta/\gamma}}{4\gamma} \left[ F(\eta_1) - F(\eta_2) - F(\eta_3) + F(\eta_4) \right].
\]  
(A.7)

where
\[
\eta_1 = (\alpha + \gamma)(\beta + \gamma)/\gamma,
\]
\[
\eta_2 = (\alpha - \gamma)(\beta + \gamma)/\gamma,
\]
\[
\eta_3 = (\alpha + \gamma)(\beta - \gamma)/\gamma,
\]
\[
\eta_4 = (\alpha - \gamma)(\beta - \gamma)/\gamma.
\]
(A.8, A.9, A.10, A.11)

The complex-valued function $F(z)$ in equation (A.7) is defined by:
\[
F(z) = \int_0^z \frac{1 - \sin t}{t} dt = -i \text{Si}(z) - C(z) + C + \ln z,
\]  
(A.12)

where $C(z) = -\int_z^\infty \frac{dt}{t} \cos t$ is the cosine integral and $C = 0.577 \ldots$ is Euler’s constant [29]. The general asymptotic behavior of equation (A.7) for small $\gamma$ (i.e. for $\max(|\alpha|, |\beta|) \gg |\gamma|$) is:
\[
\mathcal{I}(\alpha, \beta, \gamma) \sim \frac{1}{\alpha \beta} \left( \sin \alpha \sin \beta \cos \gamma - i \cos \alpha \cos \beta \sin \gamma \right),
\]  
(A.13)

and that for large $\gamma$ (i.e. for $\max(|\alpha|, |\beta|) \ll |\gamma|$) is:
\[
\mathcal{I}(\alpha, \beta, \gamma) \sim \frac{\pi}{2\gamma} - \cos \alpha \frac{\cos \gamma}{\gamma^2}
\]
\[
- \frac{e^{i\alpha\beta/\gamma}}{\gamma^2} (\cos \alpha \cos \beta \cos \gamma - i \sin \alpha \sin \beta \sin \gamma).
\]  
(A.14)

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References

[1] Rowell J M 1963 Magnetic field dependence of the Josephson tunnel current Phys. Rev. Lett. 11 200
[2] Jaklevic R C, Lambre J, Silver A H and Mercereau J E 1964 Quantum interference effects in Josephson tunneling Phys. Rev. Lett. 12 159
[3] Josephson B D 1965 Supercurrents through barriers Adv. Phys. 14 419
[4] Barone A and Paterno G 1982 Physics and Applications of the Josephson Effect (New York: Wiley)
[5] Clarke J and Braginski A I (eds) 2004 The SQUID Handbook (New York: Wiley)
[6] Dynes R C and Fulton T A 1971 Supercurrent density distribution in Josephson junctions Phys. Rev. B 3 3015
[7] Miller S, Biagi K, Clem J and Finne more D 1985 Critical currents of cross-type superconducting-normal-superconducting junctions in perpendic ular magnetic fields Phys. Rev. B 31 2684
[8] Gubankov V N, Lisitsi kski M P, Serpuchenko I L and Fistul M V 1991 Effect of Abrikosov vortices on the critical current of a Josephson junction Sov. Phys. JETP 73 734 (http://jetp.ras.ru/cgi-bin/e/index/e/73/4/p734?a=list)
[9] Golod T, Rydh A and Krasnov V M 2010 Detection of the Phase Shift from a Single Abrikosov Vortex Phys. Rev. Lett. 104 227003
[10] Clem J R 2011 Effect of nearby Pearl vortices upon the $I_c$ versus B characteristics of planar Josephson junctions in thin and narrow superconducting strips Phys. Rev. B 84 134502
[11] Nappi C, Lisitsinski M P and Cristiano R 2002 Fraunhofer critical-current diffraction pattern in annular Josephson junctions with injected current Phys. Rev. B 65 132516
[12] Goldobin E, Sterck A, Gaber T, Koele D and Kleiner R 2004 Dynamics of semifluxons in Nb Long Josephson $0\pi$ junctions Phys. Rev. Lett. 92 057005
[13] Kogan V G and Mints R G 2014 Effect of current injection into thin-film Josephson junctions Phys. Rev. B 90 184504
[14] Kemmler M et al 2010 Magnetic interference patterns in 0–$\pi$ superconductor/insulator/ferromagnet/superconductor Josephson junctions: effects of asymmetry between 0 and $\pi$ regions Phys. Rev. B 81 054522
[15] Alidoust M, Sewell G and Linder J 2012 Non-fraunhofer interference pattern in inhomogeneous ferromagnetic Josephson junctions Phys. Rev. Lett. 108 037001
[16] Börschö K, Komori S, Buzdin A I and Robinson J W A 2019 Fraunhofer patterns in magnetic Josephson junctions with non-uniform magnetic susceptibility Sci. Rep. 9 5616
[17] Suominen H J, Danon J, Kjaergaard M, Flensberg K., Shabani J, Palmström C J, Nichelle F and Marcus C M 2017 Anomalous Fraunhofer interference in epitaxial superconductor-semiconductor Josephson junctions Phys. Rev. B 95 035307
[18] Lee J H, Lee G-H, Park J, Lee J, Nam S-G, Shin Y-S, Kim J S and Lee H-J 2014 Local and nonlocal fraunhofer-like pattern from an edge-stepped topological surface Josephson current distribution Nano Lett. 14 5029
[19] Hart S, Ren H, Wagner T, Leubner P, Mühlbauer M, Brüne C, Buhmann H, Molenkamp L W and Yacoby A 2014 Induced superconductivity in the quantum spin Hall edge Nat. Phys. 10 638
[20] Prithvi V, Beukman A J A, Qu F, Cassidy M C, Charpentier C, Wegscheider W and Kouwenhoven L P 2015 Edge-mode superconductivity in a two-dimensional topological insulator Nat. Nanotechnol. 10 593
[21] Allen M T, Shantoko O, Fulga I C, Akhmerov A R, Watanabe K, Taniguchi T, Jarillo-Herrero P, Levitov L S and Yacoby A 2016 Spatially resolved edge currents and guided-wave electronic states in graphene Nature Phys. 12 128
[22] Rosenberg I and Chen J T 1975 Effect of transverse magnetic fields on dc josephson current Phys. Rev. Lett. 35 303
[23] Hebard A F and Fulton T A 1975 Josephson junctions in transverse magnetic fields Phys. Rev. Lett. 35 1310
[24] Monaco R, Aaroe M, Mygind J and Kochelet S V P 2008 Static properties of small Josephson tunnel junctions in a transverse magnetic field J. Appl. Phys. 104 023906
[25] Watanabe N, Nakayama A, Abe S and Aizawa K 2005 Magnetic field dependence of Josephson current by applying the external magnetic field perpendicular to the Josephson junction J. Appl. Phys. 97 10B116

[26] Anderson P W and Rowell J M 1963 Probable observation of the Josephson superconducting tunneling effect Phys. Rev. Lett. 10 230

[27] Weihnacht M 1969 Influence of film thickness on D. C. Josephson current Phys. Status Solidi b 32 K169

[28] Gurevich A 1992 Nonlocal Josephson electrodynamics and pinning in superconductors Phys. Rev. B 46 3187

[29] Gradshtein I S, Ryshik I M and Jeffrey A 1994 Table of Integrals, Series and Products 5th edn (London: Academic)