STABILIZATION OF THE PETROVSKY-WAVE NONLINEAR COUPLED SYSTEM WITH STRONG DAMPING

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Abstract. This paper concerns the well-posedness and uniform stabilization of the Petrovsky-Wave Nonlinear coupled system with strong damping. Existence of global weak solutions for this problem is established by using the Galerkin method. Meanwhile, under a clever use of the multiplier method, we estimate the total energy decay rate.

1. Introduction and statement of main results

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of system of Petrovsky-wave of the type

\[
\begin{align*}
    u_1'' + \Delta^2 u_1 - a(x) \Delta u_2 - g_1(\Delta u_1') &= 0, & x \in \Omega, t \geq 0 \\
    u_2'' - \Delta u_2 - a(x) \Delta u_1 - g_2(\Delta u_2') &= 0, & x \in \Omega, t \geq 0 \\
    \Delta u_1 = u_1 = u_2 &= 0, & x \in \Gamma, t \geq 0 \\
    u_i(x,0) = u_0^i(x), & u'_i(x,0) = u_1^i(x), & x \in \Omega, \; i = 1, 2. 
\end{align*}
\]

(1.1)

Here \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) with regular boundary \( \Gamma \).

When \( a(x) = 0 \), the Petrovsky equation was treated by Komornik \[7\], where he used semigroup approach for setting the well posedness and he studied the strong stability by introducing a multiplier method combined with a nonlinear integral inequalities. Recently, Bahlil et al. \[4\], studied the system

\[
\begin{align*}
    u_1'' + a(x) u_2 + \Delta^2 u_1 - g_1(u_1'(x,t)) &= f_1(u_1, u_2), & \text{in} \; \Omega \times \mathbb{R}^+, \\
    u_2'' + a(x) u_1 - \Delta u_2 - g_2(u_2'(x,t)) &= f_2(u_1, u_2), & \text{in} \; \Omega \times \mathbb{R}^+, \\
    \partial_{\nu} u_1 = u_1 = v = u_2 &= 0 & \text{on} \; \Gamma \times \mathbb{R}^+, \tag{1.2}
\end{align*}
\]

for \( g_i \; (i = 1, 2) \) do not necessarily having a polynomial growth near the origin, by using Faedo-Galerkin method to prove the existence and uniqueness of solution and established energy decay results depending on \( g_i \). Guesmia \[6\] consider the problem \[(1.2)\] without Source Terms \( f_1 \) and \( f_2 \). He deal with global existence and uniform decay of solutions.

In this paper, we prove the global existence of weak solutions of the problem \[(1.1)\] by using the Galerkin method (see Lions \[11\]) we use some technique from \[4\] to establish an explicit and general decay result, depending on \( g_i \). The proof is based on a powerful tool which is the multiplier method \[12, 5\] and makes use of some properties of convex functions, and general Jensen and Young’s inequalities. These convexity arguments were

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by the following formula

\[ E(t) = \frac{1}{2} \int_{\Omega} |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla u_1|^2 + |\nabla u_2|^2 \, dx + \int_{\Omega} a(x) \Delta u_1 \Delta u_2 \, dx, \]

is a nonnegative.

**Proof.** Multiplying the first equation in (1.1) by \(-\Delta u_1\) and the second equation by \(-\Delta u_2\), integrating (by parts) over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u_1'|^2 + |\nabla u_2'|^2 + |\nabla u_1|^2 + |\nabla u_2|^2 \, dx + 2 \int_{\Omega} a(x) \Delta u_1 \Delta u_2 \, dx
\]

\[ = - \int_{\Omega} \Delta u_1' g_1(\Delta u_1') + \Delta u_2' g_2(\Delta u_2') \, dx. \]
Using H"older’s inequality, Sobolev embedding and the condition (1.3), we get
\[
\int_{\Omega} a(x) \Delta u_1 \Delta u_2 \, dx \geq -\frac{1}{2} \|a\|_{L^\infty(\Omega)} \left( \frac{\sqrt{\varepsilon}}{\sqrt{c}} \right) \int_{\Omega} |\Delta u_1| |\Delta u_2| \, dx \\
\geq -\frac{1}{2} \|a\|_{L^\infty(\Omega)} \int_{\Omega} \frac{1}{c} |\Delta u_1|^2 + c'|\Delta u_2|^2 \, dx \\
\geq -\frac{1}{2} \|a\|_{L^\infty(\Omega)} \int_{\Omega} \frac{c^2}{c} |\nabla \Delta u_1|^2 + c'|\Delta u_2|^2 \, dx \\
\geq -\frac{c}{2} \|a\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla \Delta u_1|^2 + |\Delta u_2|^2 \, dx
\]
then
\[
E(t) \geq \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 + |\nabla u_2|^2 + (1 - c' \|a\|_{L^\infty(\Omega)}) (|\nabla \Delta u_1|^2 + |\Delta u_2|^2) \, dx \\
\geq 0.
\]
Hence, \( E \) is a nonnegative function and its derivative is
\[
E'(t) = -\int_{\Omega} \Delta u_1' g_1(\Delta u_1') + \Delta u_2' g_2(\Delta u_2') \, dx. \tag{1.6}
\]

We are now in the position to state our two main results:

**Theorem 1.2.** Let \((u_1^0, u_2^0) \in \overline{V}\) and \((u_1^1, u_2^1) \in V\) arbitrarily. Assume that (1.3) and (1.4) hold. Then, system (1.1) is a nonnegative function and its derivative is

\[
E(t) \leq \psi^{-1}(h(t) + \psi(E(0))), \quad \forall t \geq 0 \tag{1.7}
\]
where \( \psi(t) = \int_t^1 \frac{1}{\varphi(s)} \, ds \) for \( t > 0 \), \( h(t) = 0 \) for \( 0 \leq t \leq \frac{E(0)}{\varphi(E(0))} \) and

\[
h^{-1}(t) = t + \frac{\psi^{-1}(t + \psi(E(0)))}{\varphi^{-1}(t + \psi(E(0)))}, \quad \forall t \geq \frac{E(0)}{\varphi(E(0))}
\]

\[
\varphi(t) = \begin{cases} 
  \frac{t}{G'(\varepsilon_0 t)} & \text{if } G \text{ is linear on } [0, \varepsilon] \\
  t G'(\varepsilon_0 t) & \text{if } G'(0) = 0 \text{ and } G'' > 0 \text{ on } [0, \varepsilon]
\end{cases}
\]

where \( \omega \) and \( \varepsilon_0 \) are positive constants.

2. Some technical lemmas

**Lemma 2.1.** Let \( E : \mathbb{R}^+ \to \mathbb{R}^+ \) be a non-increasing differentiable function, \( \lambda \in \mathbb{R}^+ \) and \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) a convex and increasing function such that \( \varphi(0) = 0 \). Assume that

\[
\int_0^{+\infty} \varphi(E(t)) \, dt \leq E(s), \quad \forall s \geq 0 \]
\[
E'(t) \leq \lambda E(t), \quad \forall t \geq 0
\]

Then \( E \) satisfies the following estimate:

\[
E(t) \leq e^c \lambda \int_0^s \varphi^{-1}\left( e^{\lambda(s-t)} \right) \right) \varphi^{-1}\left( h(t) + \psi(E(0)) \right), \quad \forall t \geq 0 \tag{2.1}
\]

where \( \psi(t) = \int_t^1 \frac{1}{\varphi(s)} \, ds \), \( \forall t \geq 0 \).
where

\[ u(x,t) \in \mathbf{V} \]

\[ u(x,t) = 0 \quad \text{for} \quad t \in [0,T_0] \]

\[ K(t) = D(t) + \frac{\psi^{-1}(t + \psi(E(0)))}{\varphi(\psi^{-1}(t + \psi(E(0))))} e^{\lambda t}, \quad \forall t \geq 0 \]

\[ D(t) = \int_0^t e^{\lambda s} ds \quad \forall t \geq 0 \]

\[ T_0 = D^{-1}\left( \frac{E(0)}{\varphi(E(0))} \right), \quad \tau_0 = \begin{cases} 0 & \forall t > T_0 \\ T_0 & \forall t \in [0,T_0] \end{cases} \]

3. PROOF OF THEOREM 1.2

We will use the Faedo-Galerkin method [11] to prove the existence of a global solutions. Let \( T > 0 \) be fixed and denote by \( V^k \) the space generated by \( \{w_1^k, w_2^k, ..., w_i^k\} \), where the set \( \{w_i^k, k \in \mathbb{N}\} \) is a basis of \( V \).

We construct approximate solution \( u_i^k, \) \( k = 1, 2, 3, ..., \) in the form

\[ u_i^k(x,t) = \sum_{j=1}^{k} c_i^j(t)w_j^i(x), \]

where \( c_i^j (j = 1, 2, ..., k) \) are determined by the following ordinary differential equations

\[
\begin{aligned}
\begin{cases}
(u_i^k + \Delta^2 u_i^k - a(x)\Delta u_i^k - g_1(\Delta u_i^k), w_j^i) = 0 & \forall w_j^i \in V^k \\
(u_i^k - \Delta u_i^k - a(x)\Delta u_i^k - g_2(\Delta u_i^k), w_j^i) = 0 & \forall w_j^i \in V^k \\
u_i^k(0) = u_i^{0k}, \quad \dot{u}_i^k(0) = u_i^{1k}, & x \in \Omega, \quad i = 1, 2
\end{cases}
\end{aligned}

(3.1)
\]

with initial conditions

\[ u_i^k(0) = u_i^{0k} = \sum_{j=1}^{k} (u_j^0, w_j^i)w_j^i \rightarrow u_i^0, \quad \text{in} \quad H^4(\Omega) \cap H^2_0(\Omega) \quad \text{as} \quad k \rightarrow +\infty, \quad (3.2)\]

\[ u_{i}^k(0) = u_{i}^{0k} = \sum_{j=1}^{k} (u_j^0, w_j^i)w_j^i \rightarrow u_j^0, \quad \text{in} \quad H^4(\Omega) \cap H^2_0(\Omega) \quad \text{as} \quad k \rightarrow +\infty, \quad (3.3)\]

\[ \dot{u}_i^k(0) = u_i^{1k} = \sum_{j=1}^{k} (u_j^0, w_j^i)w_j^i \rightarrow u_j^1, \quad \text{in} \quad H^2(\Omega) \quad \text{as} \quad k \rightarrow +\infty. \quad (3.4)\]

\[ \dot{u}_i^k(0) = u_i^{1k} = \sum_{j=1}^{k} (u_j^0, w_j^i)w_j^i \rightarrow u_j^1, \quad \text{in} \quad H^2(\Omega) \quad \text{as} \quad k \rightarrow +\infty. \quad (3.5)\]

\[ -\Delta^2 u_i^0 + a(x)\Delta u_i^{0k} + g_1(\Delta u_i^{1k}) \rightarrow -\Delta^2 u_i^0 + a(x)\Delta u_i^0 + g_1(\Delta u_i^1), \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad k \rightarrow +\infty. \quad (3.6)\]

\[ \Delta u_i^{0k} + a(x)\Delta u_i^{0k} + g_2(\Delta u_i^{1k}) \rightarrow \Delta u_i^0 + a(x)\Delta u_i^0 + g_2(\Delta u_i^1), \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad k \rightarrow +\infty. \quad (3.7)\]

First, we are going to use some a priori estimates to show that \( t_k \rightarrow +\infty. \). Then, we will show that the sequence of solutions to (3.3) converges to a solution of (1.1) with the claimed smoothness.

Choosing \( w_i^j = -2\Delta u_i^k \) in (3.1), we obtain

\[
\begin{aligned}
\frac{d}{dt} \int_\Omega |\nabla \dot{u}_i^k|^2 + |\nabla \dot{u}_i^k|^2 + |\nabla \Delta \dot{u}_i^k|^2 + |\Delta u_i^k|^2 \right| dx + 2a(x)\Delta u_i^k \Delta u_i^k dx \\
+ 2 \int_\Omega \Delta u_i^k g_1(\Delta u_i^k) dx + 2 \int_\Omega \Delta u_i^k g_2(\Delta u_i^k) dx = 0.
\end{aligned}
\]

(3.8)
and choosing \( w_i^k = \Delta^2 u_i^k \) in (3.1), implies
\[
\frac{d}{dt} \int_{\Omega} |\Delta u_i^k|^2 + |\Delta u_j^k|^2 + |\Delta^2 u_i^k|^2 + |\nabla \Delta u_2|^2 + 2a(x)\nabla \Delta u_1 \nabla u_2 \ dx \\
+ 2 \int_{\Omega} \nabla a(x) \Delta u_2 \nabla \Delta u_1^k \ dx + 2 \int_{\Omega} \nabla a(x) \Delta u_1^k \nabla \Delta u_2 \ dx \\
+ 2 \int_{\Omega} \nabla \Delta u_1^k \ g'_i(\Delta u_1^k) \ dx + 2 \int_{\Omega} |\nabla \Delta u_2^k|^2 \ g'_2(\Delta u_2^k) \ dx = 0 \tag{3.9}
\]

Summing (3.8) and (3.9), we obtain
\[
\frac{d}{dt} \int_{\Omega} (|\Delta u_1^k|^2 + |\Delta u_2^k|^2 + |\nabla u_1^k|^2 + |\nabla u_2^k|^2 + |\Delta^2 u_i^k|^2 + |\nabla \Delta u_1|^2 + |\Delta u_2|^2) \ dx \\
+ 2 \frac{d}{dt} \int_{\Omega} (a(x)\Delta u_1^k \Delta u_2^k + a(x)\nabla \Delta u_1 \nabla \Delta u_2^k) \ dx + 2 \int_{\Omega} \Delta u_1^k g_1(\Delta u_1^k) \ dx + 2 \int_{\Omega} \Delta u_2^k g_2(\Delta u_2^k) \ dx \\
+ 2 \int_{\Omega} |\nabla \Delta u_1^k|^2 g'_1(\Delta u_1^k) \ dx + 2 \int_{\Omega} |\nabla \Delta u_2^k|^2 g'_2(\Delta u_2^k) \ dx = 0 \tag{3.10}
\]

Using Hölder’s inequality and Sobolev embedding, we have
\[
2 |\int_{\Omega} a(x) \Delta u_1^k \Delta u_2^k \ dx| \leq \frac{2}{{\sqrt{c_0}}} \int_{\Omega} |a(x)||\Delta u_1^k||\Delta u_2^k| \ dx \\
\leq c' |a| \int_{\Omega} |\nabla \Delta u_1^k(\cdot, t)|^2 \ dx + c' |a| \int_{\Omega} |\Delta u_2^k(\cdot, t)|^2 \ dx \tag{3.11}
\]
and
\[
\left|2 \int_{\Omega} a(x) \nabla \Delta u_1^k \nabla \Delta u_2^k \ dx\right| \\
\leq 2|a| \int_{\Omega} |\nabla \Delta u_1^k||\nabla \Delta u_2^k| \ dx \\
\leq |a| \int_{\Omega} |\nabla \Delta u_1^k|^2 \ dx + |a| \int_{\Omega} |\nabla \Delta u_2^k|^2 \ dx \tag{3.12}
\]

By Hölder’s inequality, Sobolev embedding and the condition (1.4), we get
\[
2 |\int_{\Omega} \nabla a(x) \Delta u_1^k \nabla \Delta u_2^k \ dx| \leq 2 \int_{\Omega} |\nabla a(x)||\Delta u_1^k||\nabla \Delta u_2^k| \ dx \\
\leq 2 \int_{\Omega} |\nabla a(x)||\Delta u_2^k||\nabla \Delta u_1^k| |\nabla \Delta u_1^k| \sqrt{\frac{g_1'(\Delta u_1^k)}{\sqrt{\gamma_1}}} \ dx \\
\leq \int_{\Omega} |\nabla \Delta u_1^k|^2 g'_1(\Delta u_1^k) \ dx + \frac{c'}{\gamma_1} |\nabla a|^2 \int_{\Omega} |\Delta u_2^k|^2 \ dx \tag{3.13}
\]

Similarly, we have
\[
2 |\int_{\Omega} \nabla a(x) \Delta u_1^k \nabla \Delta u_2^k \ dx| \leq \int_{\Omega} |\nabla \Delta u_2^k|^2 g'_2(\Delta u_2^k) \ dx + \frac{c'}{\gamma_1} |\nabla a|^2 \int_{\Omega} |\Delta u_2^k|^2 \ dx \\
\leq \int_{\Omega} |\nabla \Delta u_2^k|^2 g'_2(\Delta u_2^k) \ dx + \frac{c'}{\gamma_1} |\nabla a|^2 \int_{\Omega} |\Delta u_2^k|^2 \ dx \tag{3.14}
\]

Reporting (3.11) - (3.14), into (3.10) and integrating over \((0, t)\), we find
\[
\begin{align*}
F^k(t) + 2 \int_0^t \int_{\Omega} \Delta u_1^k (s) g_1(\Delta u_1^k(s)) \ dx \ dt + 2 \int_0^t \int_{\Omega} \Delta u_2^k (s) g_2(\Delta u_2^k(s)) \ dx \ dt \\
+ \int_0^t \int_{\Omega} |\nabla \Delta u_1^k(s)|^2 g'_1(\Delta u_1^k(s)) \ dx \ dt + \int_0^t \int_{\Omega} |\nabla \Delta u_2^k(s)|^2 g'_2(\Delta u_2^k(s)) \ dx \ dt \\
\leq F^k(0) + C_1 \int_0^t F^k(s) \ dx \ ds, \quad \forall t \in [0, t_k]
\end{align*}
\]

and
\[
\begin{align*}
\int_0^t \int_{\Omega} \Delta u_1^k (s) g_1(\Delta u_1^k(s)) \ dx \ dt + \int_0^t \int_{\Omega} \Delta u_2^k (s) g_2(\Delta u_2^k(s)) \ dx \ dt \\
+ \int_0^t \int_{\Omega} |\nabla \Delta u_1^k(s)|^2 g'_1(\Delta u_1^k(s)) \ dx \ dt + \int_0^t \int_{\Omega} |\nabla \Delta u_2^k(s)|^2 g'_2(\Delta u_2^k(s)) \ dx \ dt \\
\leq F^k(0) + C_1 \int_0^t F^k(s) \ dx \ ds, \quad \forall t \in [0, t_k]
\end{align*}
\]
where
\[ F^k(t) = \int_\Omega |\Delta u_1^k(t)|^2 + |\Delta u_2^k(t)|^2 + |\nabla u_1^k(t)|^2 + |\nabla u_2^k(t)|^2 + |\Delta^2 u_1^k(t)|^2 \, dx \]
\[ + (1 - c'|a| - |a|) \int_\Omega |\nabla u_1^k(t)|^2 \, dx + (1 - c'|a|) \int_\Omega |\nabla u_2^k(t)|^2 \, dx + (1 - |a|) \int_\Omega |\nabla \Delta u_2^k(t)|^2 \, dx \]
and \( C_1 \) is a positive constant depending only on \( |a| \), \( |\nabla a| \) and \( \tau_1 \).
So that, thanks to the monotonicity condition on the function \( g_i \) and using Gronwall’s lemma, we conclude that
\[ u_1^k \text{ is bounded in } L^\infty(0, T; H^4(\Omega) \cap H_0^2(\Omega)) \] (3.15)
\[ u_2^k \text{ is bounded in } L^\infty(0, T; H^2(\Omega) \cap H_0^2(\Omega)) \] (3.16)
\[ \Delta u_1^k g_i(\Delta u_1^k) \text{ is bounded in } L^1(A). \] (3.19)

where \( A = \Omega \times (0, T) \).

We assume first \( t < T \) and let \( 0 < \xi < T - t \). Set
\[ u_i^{k, \xi}(x, t) = u_i^k(x, t + \xi), \]
\[ U^{k, \xi} = u_1^{k, \xi}(x, t) - u_2^{k, \xi}(x, t), \]
and
\[ D^{k, \xi} = u_2^{k, \xi}(x, t). \]

Then, \( U^{k, \xi} \) solves the differential equation
\[ (\ddot{U}^{k, \xi} + \Delta^2 U^{k, \xi} - a(x)\Delta U^{k, \xi} - g_1(\Delta u_1^k) - g_1(\Delta u_1^k), u_1^k) = 0, \quad \forall u_1^k \in V^k. \] (3.20)
and \( D^{k, \xi} \) solves
\[ (\ddot{D}^{k, \xi} - a(x)\Delta U^{k, \xi} - g_2(\Delta u_2^k) - g_2(\Delta u_2^k), u_2^k) = 0, \quad \forall u_2^k \in V^k. \] (3.21)

Choosing \( u_1^{\xi} = -\Delta U^{k, \xi} \) in (3.20) and \( u_2^{\xi} = \Delta D^{k, \xi} \) in (3.21), and using the fact that \( g_i \) is nondecreasing, we find
\[
\frac{d}{dt} \int_\Omega (|\nabla U^{k, \xi}(x, t)|^2 + |\nabla D^{k, \xi}(x, t)|^2 + |\nabla \Delta U^{k, \xi}(x, t)|^2 + |\Delta D^{k, \xi}(x, t)|^2) \, dx \\
+ 2\frac{d}{dt} \int_\Omega a(x)\Delta U^{k, \xi}(x, t)\Delta D^{k, \xi}(x, t) \, dx \leq 0, \quad \forall t \geq 0.
\]

Integrating in \([0, t]\), we get
\[
\int_\Omega (|\nabla U^{k, \xi}(t)|^2 + |\nabla D^{k, \xi}(t)|^2) \, dx + (1 - c'|a|) \int_\Omega |\nabla \Delta U^{k, \xi}(t)|^2 + |\Delta D^{k, \xi}(t)|^2 \, dx \\
\leq C_2 \int_\Omega (|\nabla U^{k, \xi}(0)|^2 + |\nabla D^{k, \xi}(0)|^2) \int_\Omega |\nabla \Delta U^{k, \xi}(0)|^2 + |\Delta D^{k, \xi}(0)|^2 \, dx
\]
and \( C_2 \) is a positive constant depending only on \( |a| \) and \( c' \).
Dividing by \( \xi^2 \), and letting \( \xi \to 0 \), we find
\[
\int_\Omega (|\nabla u_1^k(t)|^2 + |\nabla u_2^k(t)|^2 + |\nabla \Delta u_1^k(t)|^2 + |\Delta u_2^k(t)|^2) \, dx \\
\leq C_2' \int_\Omega (|\nabla u_1^k(0)|^2 + |\nabla u_2^k(0)|^2 + |\nabla \Delta u_1^k|^2 + |\Delta u_2^k|^2) \, dx
\]

We estimate \( \|\nabla u_1^k(0)\| \). Choosing \( v = -\Delta u_1^k \) and \( t = 0 \) in (3.1), we obtain that
\[ \|\nabla u_1^k(0)\|^2 = \int_\Omega \nabla u_1^k(0) \nabla (-\Delta u_1^k - a(x)u_2^k + g_1(\Delta u_1^k)) \, dx. \]
and
\[ \|\nabla u_2^k(0)\|^2 = \int_\Omega \nabla u_2^k(0) \nabla (\Delta u_2^k - a(x)u_k^k + g_2(\Delta u_2^k)) \, dx. \]
By (3.4), (3.5) and (3.22), we deduce that $C$ and $\int (\Delta u_h^k (0))$. Using (3.27), (3.29) and (3.32), we have $u_1^h$ is bounded in $L^\infty (0, T; H^1_0 (\Omega))$. For each $m$ and $u_2^k$ is bounded in $L^\infty (0, T; H^1_0 (\Omega))$. By (3.21), (3.25) and (3.26), we deduce that $g_t (\Delta u_h^m) \rightarrow \chi$, where $H^2 \to \Omega$. Applying Dunford-Pettis and Banach-Alaglo-Bourbaki theorems, we conclude from (3.15-3.19) and (3.20) that there exists a subsequence $\{u_h^n\}$ of $\{u_h\}$ such that $g_t (\Delta u_h^m) \rightarrow \chi$, weak-star in $L^2 (A)$.

As $(u_1^m, u_2^m)$ is bounded in $L^\infty \to u_1^m, u_2^m)$, we have $(u_1^m, u_2^m) \rightarrow (u_1, u_2)$, strong in $L^2 (0, T; H)$. (3.32)

In the other hand, using (3.27), (3.29) and (3.30), we have

\[
\int_0^T \int_\Omega \left( \frac{\partial u_1^m}{\partial t} + \Delta u_1^m - a(x) \Delta u_2^m \right) w \frac{dx}{dt} \rightarrow \\
\int_0^T \int_\Omega \left( \frac{\partial u_1^m}{\partial t} + \Delta u_1^m - a(x) \Delta u_2^m \right) w \frac{dx}{dt},
\]

\[
\int_0^T \int_\Omega \left( \frac{\partial u_2^m}{\partial t} + \Delta u_2^m - a(x) \Delta u_1^m \right) w \frac{dx}{dt} \rightarrow \\
\int_0^T \int_\Omega \left( \frac{\partial u_2^m}{\partial t} + \Delta u_2^m - a(x) \Delta u_1^m \right) w \frac{dx}{dt},
\]

\[
\int_0^T \int_\Omega g_t (\Delta u_1^m) w \frac{dx}{dt} \rightarrow \int_0^T \int_\Omega g_t (\Delta u_1^m) w \frac{dx}{dt},
\]

when $m \rightarrow +\infty$.

**Lemma 3.1.** For each $T > 0$, $g_t (\Delta u_t) \in L^1 (A)$, $\|g_t (\Delta u_t)\|_{L^1 (A)} \leq K$, where $K$ is a constant independent of $t$ and $g_t (\Delta u_t) \rightarrow g_t (\Delta u_t)$ in $L^1 (A)$. 


Proof. We claim that
\[ g(\Delta u') \in L^1(A). \]
Indeed, since \( g \) is continuous, we deduce from \( 3.30 \)
\[ g_i(\Delta u_i^k) \rightarrow g_i(\Delta u_i') \quad \text{almost everywhere in } A. \] (3.35)
\[ \Delta u_i^k g_i(\Delta u_i^k) \rightarrow \Delta u_i' g_i(\Delta u_i') \quad \text{almost everywhere in } A. \]
Hence, by \( 3.10 \) and Fatou’s Lemma, we have
\[ \int_0^T \int_\Omega \Delta u_i'(x,t) g_i(\Delta u_i'(x,t)) \, dx \, dt \leq K_1, \quad \text{for } T > 0 \] (3.36)
Now, we can estimate \( \int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))| \, dx \, dt \). By Cauchy-Schwarz inequality, we have
\[ \int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))| \, dx \, dt \leq c |A|^{1/2} \left( \int_0^T \int_\Omega \left| g_i(\Delta u_i'(x,t)) \right|^2 \, dx \, dt \right)^{1/2}. \]
Using \( 1.11 \) and \( 3.36 \), we obtain
\[ \int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))|^2 \, dx \, dt \leq \int_0^T \int_{\{\Delta u_i'|e\leq r\}} \Delta u_i' g_i(\Delta u_i') \, dx \, dt + \int_0^T \int_{\{\Delta u_i'|e>0\}} c G^{-1}(\Delta u_i' g_i(\Delta u_i')) \, dx \, dt \]
\[ \leq c \int_0^T \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt + c G^{-1} \left( \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt \right) \]
\[ \leq c \int_0^T \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt + c' G^*(1) + c'' \int_\Omega \Delta u_i' g_i(\Delta u_i') \, dx \, dt \leq cK_1 + c' G^*(1), \quad \text{for } T > 0. \]
Then
\[ \int_0^T \int_\Omega |g_i(\Delta u_i'(x,t))| \, dx \, dt \leq K_1, \quad \text{for } T > 0. \]
Let \( E \subset \Omega \times [0,T] \) and set
\[ E_1 = \{(x,t) \in E : |\Delta u_i^m(x,t)| \leq \frac{1}{\sqrt{|E|}}\}, \quad E_2 = E \setminus E_1, \]
where \( |E| \) is the measure of \( E \). If \( M(r) = \inf \{|s| : s \in \mathbb{R} \text{ and } |g_i(s)| \geq r\} \)
\[ \int_E |g_i(\Delta u_i^m)| \, dx \, dt \leq c\sqrt{|E|} + \left( M\left( \frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\Delta u_i^m g_i(\Delta u_i^m)| \, dx \, dt. \]
By applying \( 3.10 \) we deduce that
\[ \sup_m \int_E g_i(\Delta u_i^m) \, dx \, dt \rightarrow 0, \quad \text{when } |E| \rightarrow 0. \]
From Vitali’s convergence theorem we deduce that
\[ g_i(\Delta u_i^m) \rightarrow g_i(\Delta u_i') \quad \text{in } L^1(A). \]
This completes the proof. \( \square \)

Then \( 3.31 \) implies that
\[ g_i(\Delta u_i^m) \rightarrow g_i(\Delta u_i'), \quad \text{weak-star in } L^2([0,T] \times \Omega). \]
We deduce, for all \( v \in L^2([0,T] \times L^2(\Omega)), \) that
\[ \int_0^T \int_\Omega g_i(\Delta u_i^m) w \, dx \, dt \rightarrow \int_0^T \int_\Omega g_i(\Delta u_i') w \, dx \, dt. \]
Finally we have shown that, for all \( w \in L^2([0,T] \times L^2(\Omega)) \):
\[ \int_0^T \int_\Omega \left( u_1''(x,t) + \Delta^2 u_1(x,t) - a(x) \Delta u_1(x,t) - g_1(\Delta u_i'(x,t)) \right) w \, dx \, dt = 0. \]
and
\[ \int_0^T \int_\Omega \left( u_2''(x,t) - \Delta u_2(x,t) - a(x) \Delta u_1(x,t) - g_2(\Delta u_i'(x,t)) \right) w \, dx \, dt = 0. \]
Therefore, \((u_1, u_2)\) are solutions for the problem (1.1).

4. PROOF OF THEOREM 1.3

From now on, we denote by \(c\) various positive constants which may be different on different occurrences. Multiplying the first equation of (1.1) by \(-\frac{2}{E}\Delta u_1\), we obtain

\[
0 = \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_1 (u''_1 + \Delta^2 u_1 - a(x) \Delta u_2 + g_1(\Delta u'_1)) \, dx \, dt
- \left[ \frac{\varphi(E)}{E} \int_\Omega u'_1 \Delta u_1 \, dx \right]_S^T + \int_S^T \left( \frac{\varphi(E)}{E} \right)' \int_\Omega \Delta u_1 u'_1 \, dx \, dt
- 2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u'_1|^2 \, dx \, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega (|\nabla u'_1|^2 + |\nabla u_2|^2) \, dx \, dt
+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega a(x) \Delta u_1 \Delta u_2 \, dx \, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_1 g_1(\Delta u'_1) \, dx \, dt.
\]

Similarly, we have

\[
0 = \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_2 (u''_2 + \Delta^2 u_2 - a(x) \Delta u_1 + g_2(\Delta u'_2)) \, dx \, dt
- \left[ \frac{\varphi(E)}{E} \int_\Omega u'_2 \Delta u_2 \, dx \right]_S^T + \int_S^T \left( \frac{\varphi(E)}{E} \right)' \int_\Omega \Delta u_2 u'_2 \, dx \, dt
- 2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u'_2|^2 \, dx \, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega (|\nabla u'_2|^2 + |\nabla u_1|^2) \, dx \, dt
+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega a(x) \Delta u_2 \Delta u_1 \, dx \, dt + \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_2 g_2(\Delta u'_2) \, dx \, dt.
\]

Taking their sum, we obtain

\[
\int_S^T \varphi(E) \, dt \leq \left[ \frac{\varphi(E)}{E} \int_\Omega u'_1 \Delta u_1 + u'_2 \Delta u_2 \, dx \right]_S^T \leq \varphi(E(S))
- \int_S^T \left( \frac{\varphi(E)}{E} \right)' \int_\Omega \Delta u_1 u'_1 + \Delta u_2 u'_2 \, dx \, dt
+ 2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u'_1|^2 + |\nabla u'_2|^2 \, dx \, dt
- \int_S^T \frac{\varphi(E)}{E} \int_\Omega \Delta u_1, g_1(\Delta u'_1) + \Delta u_2, g_2(\Delta u'_2) \, dx \, dt.
\]

Since \(E\) is non-increasing, we find that

\[
\left[ \frac{\varphi(E)}{E} \int_\Omega u'_1 \Delta u_1 + u'_2 \Delta u_2 \, dx \right]_S^T \leq c \varphi(E(S))
- \left[ \int_S^T \left( \frac{\varphi(E)}{E} \right)' \int_\Omega \Delta u_1 u'_1 + \Delta u_2 u'_2 \, dx \, dt \right] \leq c \varphi(E(S))
\]

Using these estimates, we conclude from (4.1) that

\[
\int_S^T \varphi(E) \, dt \leq C \varphi(E(S)) + 2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u'_1|^2 + |\nabla u'_2|^2 \, dx \, dt
+ \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| g_1(\Delta u'_1) + |\Delta u_2| g_2(\Delta u'_2) \, dx \, dt.
\]

Now, we estimate the terms of the right-hand side of (4.2) in order to apply the results of Lemma 2.1.

As in Komornik [7], we consider the following partition of \(\Omega\),

\[
\Omega^\epsilon = \{ x \in \Omega : |\Delta u'_1| > \epsilon \}, \quad \Omega^\epsilon = \{ x \in \Omega : |\Delta u'_1| \leq \epsilon \}
\]
We distinguish two cases:

**Case 1.** $G$ is linear on $[0, \varepsilon]$. By using Sobolev embedding and Young's inequality, we obtain

\[
\int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| |g_1(\Delta u_1')| \, dx \, dt + \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2 \, dx \, dt \\
\leq \varepsilon \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| |g_2(\Delta u_2')| \, dx \, dt + \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_2'|^2 \, dx \, dt \\
\leq \varepsilon C \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt.
\]

Summing (4.3) and noting that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

\[
\int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| |g_1(\Delta u_1')| + |\Delta u_2| |g_2(\Delta u_2')| \, dx \, dt \\
+ \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 \, dx \, dt \\
\leq \varepsilon C \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt + C(\varepsilon) \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt
\]

and

\[
\int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| |g(\Delta u_1')| + |\Delta u_2| |g(\Delta u_2')| \, dx \, dt \\
+ \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2 + |\nabla u_2'|^2 \, dx \, dt \\
\leq \varepsilon C \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt + C(\varepsilon) \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt
\]

Inserting these two inequalities into (4.2) and choosing $\varepsilon > 0$ small enough, we deduce that

\[
\int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| |g_1(\Delta u_1')| \, dx \, dt \\
+ \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2 \, dx \, dt \\
\leq \varepsilon C \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_2| g_2(\Delta u_2') \, dx \, dt + c \varphi(E(S))
\]

Since, choosing $\varphi(s) = s$, we deduce from (4.3) that

\[
\int_\Omega T \varphi(E(t)) \, dx \, dt \leq c \varphi(E(S))
\]

**Case 2.** $G'(0) = 0$, $G'' > 0$ on $[0, \varepsilon]$.

Using (4.3) and the fact that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

\[
\int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\nabla u_1'|^2 + |\Delta u_1| |g(\Delta u_1')| \, dx \, dt \\
\leq \varepsilon C \int_\Omega T \frac{\varphi(E)}{E} \int_\Omega |\Delta u_1| g_1(\Delta u_1') \, dx \, dt
\]
and
\[
\int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\nabla u_1|^2 + |\Delta u_1| \cdot |g_1(\Delta u_1')| \, dx \, dt \\
\leq \varepsilon C \int_S^T \varphi(E) \, dt + \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} \Delta u_2 g_2(\Delta u_2') \, dx \, dt
\]

Summing these two inequalities, we have
\[
\int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\nabla u_1|^2 + |\Delta u_1| \cdot |g_1(\Delta u_1')| \, dx \, dt \\
+ \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\nabla u_2|^2 + |\Delta u_2| \cdot |g_2(\Delta u_2')| \, dx \, dt \\
\leq \varepsilon C \int_S^T \varphi(E) \, dt + C \varphi(E(S))
\]

and exploit Jensen’s inequality and the concavity of $G^{-1}$ to obtain
\[
\int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\Delta u_1| \cdot |g_1(\Delta u_1')| + |\nabla u_1|^2 \, dx \, dt \\
\leq \varepsilon C \int_s^T \varphi(E) \int_{\Omega^*} |\nabla u_1|^2 + |g_1(\Delta u_1')|^2 \, dx \, dt \\
+ \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\nabla u_1|^2 + |g_1(\Delta u_1')|^2 \, dx \, dt \\
\leq \varepsilon C \int_S^T \varphi(E) \, dt + C(\varepsilon) \int_S^T \frac{\varphi(E)}{E} |\Omega| G^{-1} \left( \frac{1}{|\Omega|} \int_{\Omega} \Delta u_1' \Delta u_1' \right) \, dt
\]

Similarly, we have
\[
\int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\Delta u_2| \cdot |g_2(\Delta u_2')| + |\nabla u_2|^2 \, dx \, dt \\
\leq \varepsilon \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\Delta u_2|^2 + |g_2(\Delta u_2')|^2 \, dx \, dt \\
+ \int_S^T \frac{\varphi(E)}{E} \int_{\Omega^*} |\Delta u_2|^2 + |g_2(\Delta u_2')|^2 \, dx \, dt
\]

Let $G^*$ denote the dual function of the convex function $G$ in the sense of Young (see Arnold [8] p. 64). Then $G^*$ is the Legendre transform of $G$, which is given by (see Arnold [4] p. 61-62) i.e.,
\[
G^*(s) = \sup_{t \in \mathbb{R}^*} (st - G(t)).
\]

Then $G^*$ is given by
\[
G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \forall s \geq 0
\]

and satisfies the following inequality
\[
st \leq G^*(s) + G(t) \quad \forall s, t \geq 0
\]

Choosing $\varphi(s) = sG'(\varepsilon s)$, we obtain
\[
G^* \left( \frac{\varphi(s)}{s} \right) = sG'(\varepsilon s) = \varepsilon sG'(\varepsilon s) - G(\varepsilon s) \leq \varepsilon \varphi(s)
\]
Making use of (1.11) and (1.10), we have
\[
\int_S^T \varphi(E) E^{-1} \Omega^{-1} \int_\Omega \Delta u'_1 g_1(\Delta u'_1) \, dx \, dt \\
\leq c \int_S^T G'(\varphi(E)) \, dt + c \int_S^T \int_\Omega \Delta u'_1 g_1(\Delta u'_1) \, dx \, dt \\
\leq c \int_S^T \varphi(E) \, dt + c \int_S^T \int_\Omega \Delta u'_1 g_1(\Delta u'_1) \, dx \, dt
\] (4.11)

Summing (4.9) and (4.10) and using (4.11), we obtain
\[
\int_S^T \varphi(E) E^{-1} \Omega^{-1} \int_\Omega |\Delta u_1| |g_1(\Delta u'_1)| + |\nabla u'_1|^2 \, dx \, dt + \int_S^T \varphi(E) E^{-1} \Omega^{-1} \int_\Omega |\Delta u_2| |g_2(\Delta u'_2)| + |\nabla u'_2|^2 \, dx \, dt \\
\leq \varepsilon C - \int_S^T \varphi(E) \, dt + C(\varepsilon) E(S)
\] (4.12)

Then, choosing \( \varepsilon > 0 \) small enough and substitution of (4.10) and (4.12) into (4.2) gives
\[
\int_S^T \varphi(E(t)) \, dt \leq c(E(S) + \varphi(E(S))) \\
\leq c\left(1 + \frac{\varphi(E(S))}{E(S)}\right)E(S) \leq cE(S), \quad \forall S \geq 0
\]

Using Lemma 2.1 in the particular case where \( \Psi(s) = \omega \varphi(s) \) we deduce from (1.7) our estimate (4.2). The proof of Theorem 3.5 is now complete.

**Example 4.1.** Let \( g_1 \) be given by \( g_1(s) = s^p (-\ln s)^q \) where \( p \geq 1 \) and \( q \in \mathbb{R} \) on \( [0, \varepsilon] \). Then \( g_1'(s) = s^{p-1} (-\ln s)^q - q \) which is an increasing function in the right neighborhood of 0 (if \( q = 0 \) we can take \( \varepsilon = 1 \)). The function \( G \) is defined in the neighborhood of 0 by

\[
G(s) = cs^{\frac{p+1}{q}} (-\ln \sqrt{s})^q
\]

and we have

\[
G'(s) = cs^{\frac{p+1}{q}} (-\ln \sqrt{s})^{q-1} \left( -\frac{p+1}{2} - \frac{q}{2} \right), \quad \text{when } s \text{ is near 0}
\]

Thus

\[
\varphi(s) = cs^{\frac{p+1}{q}} (-\ln \sqrt{s})^{q-1} \left( -\frac{p+1}{2} - \frac{q}{2} \right), \quad \text{when } s \text{ is near 0}
\]

and

\[
\psi(t) = c \int_1^t s^{\frac{p+1}{q}} (-\ln \sqrt{s})^{q-1} \left( -\frac{p+1}{2} - \frac{q}{2} \right) \, ds \\
= c \int_1^t (\ln z)^{q-1} \left( -\frac{p+1}{2} \ln z - \frac{q}{2} \right) \, dz, \quad \text{when } t \text{ is near 0}
\]

We obtain in the neighborhood of 0

\[
\psi(t) = \begin{cases} 
\frac{1}{ct \frac{p+1}{q} (\ln t)^q} & \text{if } q > 1 \\
c(-\ln t)^{q-1} & \text{if } p = 1, q < 1 \\
c(-\ln(-\ln t)) & \text{if } p = 1, q = 1
\end{cases}
\]

and then in the neighborhood of \( +\infty \)

\[
\psi^{-1}(t) = \begin{cases} 
t^{\frac{p+1}{q}} (\ln t)^{-\frac{q}{2}} & \text{if } q > 1 \\
c^q e^{-\frac{q}{2} t} & \text{if } p = 1, q < 1 \\
c e^{-t} & \text{if } p = 1, q = 1
\end{cases}
\]
Since \( h(t) = t \) as \( t \) tends to infinity, we obtain

\[
E(t) \leq \begin{cases} 
ct^{\frac{2}{p-1}} (\ln t)^{\frac{2}{p-1}} & \text{if } q > 1 \\
ct^{-\frac{1}{q-1}} & \text{if } p = 1, q < 1 \\
ct^{-e^t} & \text{if } p = 1, q = 1
\end{cases}
\]

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