\[ Z^0 \rightarrow 2\gamma \] and the Twisted Coproduct of the Poincaré Group

A. P. Balachandran

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA.

S. G. Jo

Department of Physics, Kyungpook National University, Daegu, 702-701, Korea

and

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA.

Abstract

Yang’s theorem forbids the process \[ Z^0 \rightarrow 2\gamma \] in any Poincaré invariant theory if photons are bosons and their two-particle states transform under the Poincaré group in the standard way (under the standard coproduct of the Poincaré group). This is an important result as it does not depend on the assumptions of quantum field theory. Recent work on noncommutative geometry requires deforming the above coproduct by the Drinfel’d twist. We prove that \[ Z^0 \rightarrow 2\gamma \] is forbidden for the twisted coproduct as well. This result is also independent of the assumptions of quantum field theory. As an illustration of the use of our general formulae, we further show that \[ Z^0 \rightarrow \nu + \bar{\nu} \] is forbidden for the standard or twisted coproduct of the Poincaré group if the neutrino is massless, even if lepton number is violated. This is a special case of our general result that a massive particle of spin \( j \) cannot decay into two identical massless particles of the same helicity if \( j \) is odd, regardless of the coproduct used.

1 Introduction

Many years ago, Yang [1] proved the result that a massive spin 1 particle cannot decay into two photons. The proof required invariance under the Poincaré group \( \mathcal{P}_+ \) (without reflections), Bose statistics of photons and the assumption that the two photon states transformed in the standard way under \( \mathcal{P}_+ \). (Many books [2–6] treat the Poincaré group. See e.g. Balachandran and Trahern [7] and references therein.)

\[ \text{bal@phy.syr.edu} \]
\[ \text{sgjo@knu.ac.kr} \]
\[ \text{Permanent address} \]
Yang’s proof does not use quantum field theory (QFT). It forbids the decay \( Z^0 \rightarrow 2\gamma \). Limits on the branching ratio for such processes thus give tests on the standard assumptions about relativistic invariance and Bose symmetry which are insensitive to models of QFT. This result of Yang is thus of basic significance.

Charge conjugation invariance does forbid the decay \( Z^0 \rightarrow 2\gamma \). But the standard model does not have this invariance.

The structure of the Poincaré group \( P^\uparrow_+ \) does not uniquely dictate the two-particle Poincaré transformation law. If \( x \) denotes spacetime coordinate and the single particle wave functions \( \psi, \chi \) transform according to

\[
\psi \rightarrow \Lambda \psi, \quad \chi \rightarrow \Lambda \chi, \\
(\Lambda \psi)(x) := \psi(\Lambda^{-1}x), \quad (\Lambda \chi)(x) := \chi(\Lambda^{-1}x) 
\]  

(1.1)

under a Lorentz transformation \( \Lambda \), the two-particle wave function \( \psi \otimes \chi \) is customarily transformed according to

\[
\psi \otimes \chi \rightarrow (\Lambda \otimes \Lambda)(\psi \otimes \chi), \\
(\Lambda \otimes \Lambda)(\psi \otimes \chi)(x, y) = \psi(\Lambda^{-1}x) \chi(\Lambda^{-1}y). 
\]

(1.2)

But this rule involves the choice of a homomorphism \( \Delta_0 \) from the Lorentz group \( L^\uparrow_+ \) to \( L^\uparrow_+ \times L^\uparrow_+ \), namely,

\[
\Delta_0 (\Lambda) = \Lambda \times \Lambda. 
\]

(1.3)

More generally, for the Poincaré group \( P^\uparrow_+ \), we uncritically assume the homomorphism

\[
\Delta_0 (g) = g \times g, \quad g \in P^\uparrow_+. 
\]

(1.4)

The choice of \( \Delta_0 \) is not dictated by the Poincaré group and amounts to an additional assumption.

The Poincaré group in fact admits more general coproducts and hence more general transformation laws of multiparticle states. These coproducts are parametrised by an antisymmetric matrix \( \theta = (\theta^{\mu\nu}) \) with constant entries \( \theta^{\mu\nu} = -\theta^{\nu\mu} \) and are given by

\[
\Delta_\theta (g) = F_\theta^{-1}(g \otimes g)F_\theta, \\
F_\theta = e^{-iP_\nu \otimes \theta^{\mu\nu} P_\mu}, \\
P = (P_\mu) : \text{Four} - \text{momentum}. 
\]

(1.5)

\( F_\theta \) is known as the Drinfel’d twist [8]. This twisted coproduct has become central for the implementation of Poincaré invariance on the Moyal plane [9, 10].

The coproduct \( \Delta_0 \) defines the action of the Poincaré group on multiparticle states.

It is clear from \([1, 2]\) that its action on two-particle states commutes with the flip operator \( \tau \):

\[
\tau(\psi \otimes \chi) := \chi \otimes \psi. 
\]

(1.6)
Hence the subspaces with elements

\[ P_{\pm} (\psi \otimes \chi), \quad P_{\pm} = \frac{1}{2} (1 \pm \tau) \]  

(1.7)

are Poincaré invariant. Restriction to these subspaces is thus compatible with Poincaré invariance. In this way we are led to the concepts of bosons and fermions given by the projectors \( P_{\pm} \).

The transformation \( \tau \) generalizes to \( N \)-particle sectors where they generate the permutation group \( S_N \). The projectors \( P_{\pm} \) also generalize to \( N \)-particle sectors where they project to the two one-dimensional representations of \( S_N \).

But already at the two-particle level, the flip \( \tau \) fails to commute with \( \Delta_\theta(g) \). Instead, we must replace \( \tau \) by

\[ \tau_\theta = F_\theta^{-1} \tau F_\theta, \quad \tau_\theta^2 = 1 \otimes 1, \]
\[ \tau_0 = \tau, \]  

(1.8)

which does commute with \( \Delta_\theta(g) \) [11–14]. The twisted flip \( \tau_\theta \) is associated with the new projectors

\[ P_{\pm}^\theta = \frac{1}{2} (1 \pm \tau_\theta), \]
\[ P_{\pm}^0 = P_{\pm}. \]  

(1.9)

They define the twisted bosonic and fermionic subspaces with elements \( P_{\pm}^\theta (\psi \otimes \chi) \).

The transformation \( \tau_\theta \) as well generalizes to \( N \)-particle sectors [12].

In this paper, we first analyze the space of two-photon state vectors for \( \theta^{\mu\nu} = 0 \). It consists of vectors of the form \( P_+ (\psi \otimes \chi) \). Using just group theory, we show that the reduction of the representation of the Poincaré group \( \mathcal{P}_+^1 \), acting by the coproduct \( \Delta_\theta \) on this space, does not contain its massive spin 1 representation. This proves Yang’s theorem.

Next, we repeat this analysis for the two-photon states given by the projector \( P_+^\theta \), the coproduct for \( \mathcal{P}_+^1 \) being \( \Delta_\theta \). We still find Yang’s result: This representation of the Poincaré group does not contain the massive spin 1 representation. The process \( Z^0 \to 2\gamma \) is still forbidden. We show also that this selection rule is a special case of a more general selection rule, valid for any \( \theta^{\mu\nu} \), forbidding the decay of a massive particle of spin \( j \) into two massless identical particles of the same helicity if \( j \) is odd.

Not all treatments of the standard model on the Moyal plane preserve Poincaré invariance. The first treatment of \( Z^0 \to 2\gamma \) in a model violating Lorentz invariance is due to [15]. More recent research on this subject can be found in [16]. Also in the approach advocated by [14], based on the twisted coproduct, for example, for reasons of locality, it breaks down when a process involves both gauge and matter fields. In this case, \( Z^0 \to 2\gamma \) need not be forbidden. Further analysis of this approach is needed for a precise statement.

In the next two sections, we summarize the construction of the unitary irreducible representations (UIRR’s) of the universal covering group \( \bar{\mathcal{P}}_+^1 \) of \( \mathcal{P}_+^1 \) for massive and massless
particles. (Not all zero mass UIRR’s are covered, only those of interest are described.) Yang’s theorem is then proved in section 4 and generalized to the twisted coproduct case in section 5. Section 6 contains brief concluding remarks.

2 Irreducible Representations of \( \mathcal{P}^+ \)

The Lie algebra of Poincaré group \( \mathcal{P}^+ \) is spanned by the 10 generators \( J_{\mu\nu} \) and \( P_\mu \) \((\mu, \nu \in \{0, 1, 2, 3\})\) which satisfy

\[
\begin{align*}
[J_{\alpha\beta}, J_{\mu\nu}] &= i(g_{\beta\mu}J_{\alpha\nu} + g_{\alpha\nu}J_{\beta\mu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\mu}), \\
[J_{\alpha\beta}, P_\mu] &= i(g_{\beta\mu}P_\alpha + g_{\alpha\mu}P_\beta), \\
[P_\mu, P_\nu] &= 0.
\end{align*}
\]

The Casimir operators of \( \mathcal{P}^+ \) are \( P^2 = P^\mu P_\mu \) and \( W^2 = W^\mu W_\mu \) where \( W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\nu\alpha} P^\beta \) is the Pauli-Lubanski operator. These are represented by constants in irreducible representations. We set \( P^2 = m^2 \) and consider only the cases \( m^2 \geq 0 \) and \( P_0 > 0 \).

2.1 Irreducible Representations for Massive Particles

The construction of the UIRR’s of \( \mathcal{P}^+ \) are described in many books, for example in [7]. Here we will briefly describe them.

For \( m^2 > 0 \), the UIRR’s of \( \mathcal{P}^+ \) are labeled by \( m \) and \( j \) with \( j = 0, \frac{1}{2}, 1, \cdots \). The representation space of each UIRR is spanned by \( \{| p \ j \lambda \rangle \} \) where \( p^\mu p_\mu = m^2 \) and \( \lambda = -j, -j+1, \cdots , j-1, j \). Here, \( p^\mu \) is a vector residing on the three-dimensional hyperboloid \( \{ p \in \mathbb{R}^4 \mid p^2 = m^2, \ p_0 > 0 \} \) and, consequently, the representation space is not compact. This is natural because the group itself is not compact. The basis states satisfy

\[
\begin{align*}
P^\mu | p \ j \lambda \rangle &= \ p^\mu | p \ j \lambda \rangle, \\
W^2 | p \ j \lambda \rangle &= -m^2 j(j+1) | p \ j \lambda \rangle, \\
\langle p' j' \lambda' | p j \lambda \rangle &= 2p_0 \delta_{j'j} \delta_{\lambda'\lambda} \delta^3(p' - p).
\end{align*}
\]

In order to understand the behavior of these states under the action of an arbitrary Lorentz transformation, we have to be more precise about the definition of the basis states.

For any given timelike 4-momentum \( p^\mu \) with positive \( p_0 \), there is a rest frame in which the momentum becomes \( \hat{k} = (m, 0, 0, 0) \). In this frame \( | \hat{k} \ j \lambda \rangle \) is defined as a state satisfying

\[
P^\mu | \hat{k} \ j \lambda \rangle, = \ \hat{k}^\mu | \hat{k} \ j \lambda \rangle,
\]
\[ L^2 | \hat{k} j \lambda \rangle = j(j + 1) | \hat{k} j \lambda \rangle \]
\[ L_3 | \hat{k} j \lambda \rangle = \lambda | \hat{k} j \lambda \rangle. \]  

(2.3)

Here, \( L_i = \frac{1}{2} \epsilon_{ijk} J_{jk} \) and \( L^2 = L_1^2 + L_2^2 + L_3^2 \). In the rest frame, \(| \hat{k} j \lambda \rangle\) transforms as usual under a spatial rotation \( R \):

\[ U(R) | \hat{k} j \lambda \rangle = D^j_{\lambda\lambda'}(R) | \hat{k} j \lambda' \rangle, \]  

(2.4)

\( D^j(R) \) being spin \( j \) rotation matrices. Also \( R \in SU(2) \) if \( j \in \{1/2, 3/2, \cdots\} \).

Going back from \( \hat{k} = (m, 0, 0, 0) \) to the given \( p^\mu \) is achieved by a Lorentz transformation. However, there are many Lorentz transformations which fulfill this job. The ambiguity comes from the existence of a non-trivial stability group of \( \hat{k} \), which, in this case, is the rotation subgroup. We fix the ambiguity by choosing the Lorentz transformation \( L(p) \) which transforms \( \hat{k} \) to \( p \), i.e. \( p = L(p)\hat{k} \), as follows:

\[ L(p) = e^{-i\alpha J_{12}} e^{-i\beta J_{31}} e^{i\alpha J_{12}} e^{-i\delta J_{03}}. \]  

(2.5)

The values of \( \alpha, \beta \) are fixed by the spatial part of \( p^\mu \) and that of \( \delta \) is fixed by the time component of \( p^\mu \). With this \( L(p) \), we define our general basis state \( | p j \lambda \rangle \) by

\[ | p j \lambda \rangle = U(L(p)) | \hat{k} j \lambda \rangle. \]  

(2.6)

In order to see how \( | p j \lambda \rangle \) transforms under an arbitrary Lorentz transformation \( \Lambda \), we consider

\[ U(\Lambda) | p j \lambda \rangle = U(L(\Lambda p)) U(L^{-1}(\Lambda p)) U(\Lambda) U(L(p)) | \hat{k} j \lambda \rangle \]
\[ = U(L(\Lambda p)) U(L^{-1}(\Lambda p) \Lambda L(p)) | \hat{k} j \lambda \rangle. \]  

(2.7)

Here, \( L(\Lambda p) \) is the Lorentz transformation of the form given in (2.5), which maps \( \hat{k} \) to \( \Lambda p \). Notice that \( L^{-1}(\Lambda p) \Lambda L(p) \) leaves \( \hat{k} \) invariant. Therefore, it must be a pure spatial rotation. We denote it by \( R(\Lambda, p) \). Using (2.4), we get

\[ U(\Lambda) | p j \lambda \rangle = D^j_{\lambda\lambda'}(R(\Lambda, p)) | \Lambda p j \lambda' \rangle. \]  

(2.8)

We see that the first two equations in (2.2) can be derived using (2.3) and (2.6).

This representation of the Poincaré group is unitary for the scalar product given by (2.2).

We denote the vector space spanned by \( \{ | p j \lambda \rangle \} \) as \( V(\lambda) \).

### 2.2 Irreducible Representations for Massless Particles

Now we consider the case \( m = 0 \). In this case, the UIRR’s of \( \mathcal{P}_+ \) are characterized by a continuous parameter \( \rho \) with \( 0 \leq \rho < \infty \) and the sign of energy \( \text{sign} (p_0) \).

For a given \( \rho \) with \( \rho > 0 \) and a given sign \( p_0 \), there are two irreducible representations. The representation space is spanned by \( \{ | p \lambda \rho \ (\text{sign} \ p_0) \rangle \} \) with \( p^\mu p_\mu = 0 \). For the first
irreducible representation, \( \lambda = \cdots, -1, 0, 1, \cdots \), while for the second irreducible representation, \( \lambda = \cdots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots \). Under \( 2\pi \) rotation, the first set of states are invariant while the second states change sign. The basis states satisfy

\[
P^\mu | p \, \lambda \, \rho \, (\text{sign } p_0) \rangle = p^\mu | p \, \lambda \, \rho \, (\text{sign } p_0) \rangle
\]

\[
W^2 | p \, \lambda \, \rho \, (\text{sign } p_0) \rangle = -\rho^2 | p \, \lambda \, \rho \, (\text{sign } p_0) \rangle.
\]

We skip the analysis of the behavior of these states under an arbitrary Lorentz transformation.

For \( \rho = 0 \), there are an infinite number of inequivalent UIRR’s. They are labelled by helicity \( \lambda \) with \( \lambda \in \{ \cdots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \cdots \} \) and by sign \( p_0 \). We fix sign \( p_0 \) to be positive as that is the case of interest. Each representation space is then spanned by \( \{ | p \, \lambda \rangle \mid p^2 = 0, p_0 > 0 \} \) for a fixed \( \lambda \). Note that distinct \( \lambda \) define inequivalent irreducible representations of \( \mathcal{P}^\perp_+ \).

Photons are described by the UIRR’s with \( \rho = 0 \) and \( \lambda = \pm 1 \). Integral values of \( \lambda \) give UIRR’s of \( \mathcal{P}^\perp_+ \).

Let us discuss the behavior of \( | p \, \lambda \rangle \) under the action of an arbitrary Lorentz transformation. For any light-like four-momentum \( p^\mu \) with positive \( p_0 \), there is a frame in which the momentum becomes \( \hat{k} = (\omega, 0, 0, \omega) \). The stability group of \( \hat{k} \) is the group generated by \( \{ \Pi_1, \Pi_2, L_3 \} \) where \( \Pi_1 = J_{10} - J_{13} \) and \( \Pi_2 = J_{20} - J_{23} \). Their commutation relations are

\[
[L_3, \Pi_1] = i \Pi_2,
\]

\[
[L_3, \Pi_2] = -i \Pi_1,
\]

\[
[\Pi_1, \Pi_2] = 0.
\]

This group is isomorphic to the Euclidean group in two dimensions. In the frame where the four-momentum is \( \hat{k}^\mu \), \( | \hat{k} \, \lambda \rangle \) is defined as a state satisfying

\[
P^\mu | \hat{k} \, \lambda \rangle = \hat{k}^\mu | \hat{k} \, \lambda \rangle,
\]

\[
L_3 | \hat{k} \, \lambda \rangle = \lambda | \hat{k} \, \lambda \rangle
\]

\[
\Pi_i | \hat{k} \, \lambda \rangle = 0.
\]

As in the massive case, we introduce a Lorenz transformation \( L(p) \) of the form \((2.5)\), which maps \( \hat{k} \) to a given light-like 4 momentum \( p^\mu \). With this \( L(p) \), \( | p \, \lambda \rangle \) is defined as

\[
| p \, \lambda \rangle = U(L(p)) | \hat{k} \, \lambda \rangle.
\]

Under an arbitrary Lorentz transformation \( \Lambda \), we have
\[ U(\Lambda) \mid p \lambda \rangle = U(L(\Lambda p))U(L^{-1}(\Lambda p)\Delta L(p)) \mid \hat{k} \lambda \rangle, \tag{2.13} \]

where \( L^{-1}(\Lambda p)\Delta L(p) \) is an element of the stability group of \( \hat{k} = (\omega, 0, 0, \omega) \). The action of the stability group on \( \mid \hat{k} \lambda \rangle \) is given in Eq. (2.11). Therefore, the above equation is equal to \( \mid \Lambda p \lambda \rangle \) times a phase factor.

We normalize the states by
\[ \langle p' \lambda' \mid p \lambda \rangle = 2p_0 \delta_{\lambda\lambda'} \delta^3(p' - p). \tag{2.14} \]

Using (2.9) and (2.14), we can show that the above representations for \( m = 0 \) are unitary.

3 Reduction of the Direct Product of Two Massless States: No Twist

The direct product of two UIRR’s of the Poincaré group can be reduced into a direct sum of UIRR’s. We consider the product of two massless representations. Here, we exclude \( p_0 \neq 0 \) and sign \( p_0 < 0 \) massless representations. The product states are then massive except when two massless states have parallel momenta. In this exceptional case, the product representation is also irreducible:
\[ \mid p_1 \lambda_1 \rangle \mid p_2 \lambda_2 \rangle \sim \mid p_1 + p_2 \lambda_1 + \lambda_2 \rangle. \tag{3.1} \]

Note that this relation is defined up to a normalization factor. We do not consider this case further. It does not affect the process \( Z^0 \to 2\gamma \).

We consider a two massless-particle state with fixed helicities \( \lambda_i \) \((i = 1, 2)\). A general state can be expressed as a linear sum of the basis states \( \{ \mid p_1 \lambda_1 \rangle \mid p_2 \lambda_2 \rangle \} \). The representation space \( V(\lambda_1) \otimes V(\lambda_2) \) spanned by the basis is irreducible with respect to the direct product of the two Poincaré groups. However, under the diagonal subgroup, this space is reducible.

The reduction of the direct product of two massless representations can be summarized by the following formula:
\[ \mid \lambda_1 \lambda_2 \hat{p} j \mu \rangle = \int_{SU(2)} d\mu(R)D_{\mu \lambda_1 \lambda_2}^{j*}(R)\Delta_0(R) \mid q_1 \lambda_1 \rangle \mid q_2 \lambda_2 \rangle. \tag{3.2} \]

Here, \( d\mu(R) \) is the invariant Haar measure on the \( SU(2) \) group manifold. It is normalized by \( \int_{SU(2)} d\mu(R) = 1 \). The momenta of the two particles are fixed by \( q_1 = (q, 0, 0, q) \) and \( q_2 = (q, 0, 0, -q) \) with positive \( q \). Therefore, the state is described in the center of momentum frame and \( \hat{p} = (M, 0, 0, 0) \) with \( M = 2q \) as the mass of the two particle system.

We can understand this crucial formula as follows. We have to verify that the left-hand side transforms under \( SU(2) \) like a vector with angular momentum \( j \) and its third component \( \mu \). Now under \( S \in SU(2) \), \( \mid \lambda_1 \lambda_2 \hat{p} j \mu \rangle \) transforms to \( \int_{SU(2)} d\mu(R)D_{\mu \lambda_1 \lambda_2}^{j*}(R)\Delta_0(S)\Delta_0(R) \mid q_1 \lambda_1 \rangle \mid q_2 \lambda_2 \rangle \). Using \( \Delta_0(S)\Delta_0(R) = \Delta_0(SR) \) and the invariance of the measure,
the transformed state can be shown to be \( D_j^{\alpha\mu}(S) \mid \lambda_1\lambda_2 \hat{p} j \mu \rangle \), which verifies the validity of (3.2).

The state in an arbitrary frame can be obtained by the corresponding Lorentz transformation as in the single particle case:

\[
| \lambda_1\lambda_2 p j \mu \rangle = \Delta_0(L(p)) | \lambda_1\lambda_2 \hat{p} j \mu \rangle. \tag{3.3}
\]

It can be shown that the states \( | \lambda_1\lambda_2 \hat{p} j \mu \rangle \) with \( \mu = -j, -j + 1, \cdots, j - 1, j \) and their Lorentz transforms form a basis for a UIRR labelled by \( \{\lambda_1, \lambda_2, M, j\} \). We denote the space as \( \tilde{V}(\lambda_1, \lambda_2, M, j) \). It can also be shown that any state in \( V(\lambda_1) \otimes V(\lambda_2) \) can be expressed as a superposition of \( | \lambda_1\lambda_2 p j \mu \rangle \) with different \( \{M, j\} \). It shows that

\[
V(\lambda_1) \otimes V(\lambda_2) = \bigoplus_{M,j} \tilde{V}(\lambda_1, \lambda_2, M, j). \tag{3.4}
\]

On the right hand side of this expression, the value of \( M \) runs over all positive values and the value of \( j \) is lower-bounded by \( | \lambda_1 - \lambda_2 | \).

Note that we have considered only the cases \( M > 0 \) in the above discussion.

In order to obtain Clebsch-Gordan coefficients, we write, for \( R \in SU(2) \),

\[
R = e^{-i\alpha J_{12}} e^{-i\beta J_{31}} e^{-i\gamma J_{12}},
\]

\[
d\mu(R) = \frac{1}{16\pi^2} \, d\alpha \, d\cos\beta \, d\gamma, \ \alpha \in [0, 2\pi], \ \beta \in [0, \pi], \ \gamma \in [0, 4\pi]. \tag{3.5}
\]

Then, (3.2) becomes

\[
| \lambda_1\lambda_2 \hat{p} j \mu \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\alpha \int_{-1}^1 d\cos\beta \, d^j_{\mu,\lambda_1-\lambda_2}(\beta(\vec{p}_1)) \, e^{i(\mu - \lambda_1 - \lambda_2)\alpha(\vec{p}_1)} | p_1 \lambda_1 \rangle | p_2 \lambda_2 \rangle_{CM}. \tag{3.6}
\]

Here, \( p_1 = (p_{10}, \vec{p}_1) \) and \( d^j_{\mu,\lambda_1-\lambda_2}(\beta) = D^j_{\mu,\lambda_1-\lambda_2}(e^{-i\beta J_{31}}) \). Coordinates \( (\alpha(\vec{p}_1), \beta(\vec{p}_1)) \) are the azimuthal and polar angles of \( \vec{p}_1 \). The subscript ‘CM’ denotes the ‘center-of-momentum’ frame where \( p_2 = (p_{20}, \vec{p}_2) \) with \( \vec{p}_1 + \vec{p}_2 = 0 \). Therefore, the corresponding angles of \( \vec{p}_2 \) are \( (\alpha + \pi, \pi - \beta) \).

The conventions (2.5) and (2.6) for defining the basis state have to be carefully followed to obtain (3.6). We illustrate how the calculation is done for the factors involving \( \lambda_2 \) in (3.6). First note that the \( \gamma \) dependent terms in (3.2) cancel out. So we focus on the relevant term coming from \( \Delta_0(R) \) and \( | q_2 \lambda_2 \rangle \). It is

\[
e^{-i\alpha(\vec{p}_1)J_{12}} e^{-i\beta(\vec{p}_1)J_{31}} | q_2 \lambda_2 \rangle = e^{-i\alpha(\vec{p}_1)J_{12}} e^{-i\beta(\vec{p}_1)J_{31}} e^{-i\pi J_{31}} | q_1 \lambda_2 \rangle
\]

\[
= e^{-i(\alpha(\vec{p}_1)+\pi)J_{12}} e^{i\beta(\vec{p}_1)J_{31}} e^{i\pi J_{12}} e^{-i\pi J_{31}} | q_1 \lambda_2 \rangle
\]

\[
= e^{-i(\alpha(\vec{p}_1)+\pi)J_{12}} e^{-i(\pi - \beta(\vec{p}_1))J_{31}} e^{i(\alpha(\vec{p}_1)+\pi)J_{12}} e^{-i(\alpha(\vec{p}_1)+2\pi)J_{12}} | q_1 \lambda_2 \rangle
\]

8
\[ e^{-i(\alpha(\vec{p}_1) + 2\pi)\lambda_2} \langle p_2 \lambda_2 | p_2 \lambda_2 \rangle = (-1)^{2\lambda_2} e^{-i\alpha(\vec{p}_1)\lambda_2} \langle p_2 \lambda_2 | p_2 \lambda_2 \rangle. \quad (3.7) \]

The factor \((-1)^{2\lambda_2}\) is an overall factor and will be absorbed into a new definition of the state \( | \lambda_1 \lambda_2 \vec{p} j j\mu \rangle \). The \(\lambda_2\)-dependence of the second index in \(d^{ij}_{\mu \lambda_1 - \lambda_2}(\beta(\vec{p}_1))\) comes directly from \(D^{ij}_{\mu \lambda_1 - \lambda_2}(R)\) in (3.2). We thus account for the \(\lambda_2\)-terms in (3.6).

Inverting (3.2) we get
\[
\Delta_0(R) | q_1 \lambda_1 \rangle | q_2 \lambda_2 \rangle = \sum_{j,\mu} (2j + 1) D^{ij}_{\mu \lambda_1 - \lambda_2}(R) | \lambda_1 \lambda_2 \vec{p} j j\mu \rangle. \quad (3.8)
\]

From this and using (3.5) we have

\[
| p_1 \lambda_1 \rangle | p_2 \lambda_2 \rangle_{CM} = \sum_{j,\mu} (2j + 1) e^{-i(\mu - \lambda_1 - \lambda_2)\alpha(\vec{p}_1)} D^{ij}_{\mu \lambda_1 - \lambda_2}(\beta(\vec{p}_1)) | \lambda_1 \lambda_2 \vec{p} j j\mu \rangle. \quad (3.9)
\]

The Clebsch-Gordan coefficients in the center-of-momentum frame are determined by (3.6) and (3.9). Relations in the general frame can be obtained by Lorentz transforming these two equations. We thus get

\[
\langle k_1 \lambda_1 | \langle k_2 \lambda_2 | \lambda_1 \lambda_2 \vec{p} j j\mu \rangle = \frac{1}{\pi} d^{ij}_{\mu \lambda_1 - \lambda_2} [\beta(\vec{k}_1)] e^{i(\mu - \lambda_1 - \lambda_2)\alpha(\vec{k}_1)} \delta(\vec{k}_1^\ast - q) \delta(q + k_1).
\]

and
\[
\langle \lambda_1 \lambda_2 p' j' \mu' | \lambda_1 \lambda_2 p j \mu \rangle = \frac{2}{\pi(2j + 1)} \delta_{j'j} \delta_{\mu'\mu} \delta(\vec{q} - \vec{p}). \quad (3.10)
\]

We can get (3.11) quickly as follows. All but the overall normalization factor \(2/\pi(2j + 1)\) in (3.11) is fixed by general considerations. To get the overall factor, we put \(p = \vec{p}\) and use (3.10). Then (3.10) vanishes unless \(\langle k_1 \lambda_1 | \langle k_2 \lambda_2 | \lambda_1 \lambda_2 \vec{p} j j\mu \rangle = \text{C.M.} \langle p_1 \lambda_1 | \langle p_2 \lambda_2 |. \]

Substituting for the former in (3.10) by the latter from (3.9), we get the factor \(2/\pi(2j + 1)\) in (3.11). The factor 2 comes because the total center-of-momentum energy is twice the energy of either particle and \(\delta(x) = 2\delta(2x)\).

### 4 The Case of Two Identical Particles

When we consider two identical particles, the product state must be either symmetrized or anti-symmetrized depending on the spin of the particle. The reduction formula should be modified accordingly. For the case of massless particles, we get

\[
| \lambda_1 \lambda_2 \vec{p} j j\mu \rangle_{S,A} = \int_{SO(3)} d\mu(R) D^x(R)^{ij}_{\mu \lambda_1 - \lambda_2} \Delta_0(R) \frac{1 \pm \tau}{2} | q_1 \lambda_1 \rangle | q_2 \lambda_2 \rangle. \quad (4.1)
\]

Here, \(\tau\) is the flip operator,
\[
\tau | q_1 \lambda_1 \rangle | q_2 \lambda_2 \rangle = | q_2 \lambda_2 \rangle | q_1 \lambda_1 \rangle, \quad (4.2)
\]
and \( S(A) \) denotes the symmetric (anti-symmetric) state. We take \( + \) if the particles are tensorial (their helicities are integral) and we take \( - \) if they are spinorial (their helicities are \( \pm 1/2, \pm 3/2, \cdots \)). Note here that the two helicities \( \lambda_1 \) and \( \lambda_2 \) may be different. Massless particle states with different helicities never mix under the Poincaré group \( P_+^1 \). However, the disconnected component of the Poincaré group will mix different helicity states. For example, under parity, helicity changes sign so that the helicity of the photon can be \( \pm 1 \).

The coproduct \( \Delta_0(R) \) and \( \tau \) commute and we can write

\[
| \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} = \frac{1 \pm \tau}{2} | \lambda_1 \lambda_2 \hat{p} j \mu \rangle. \tag{4.3}
\]

The action of \( \tau \) on \( | \lambda_1 \lambda_2 \hat{p} j \mu \rangle \) changes the order of the two one-particle states and we get

\[
\tau | \lambda_1 \lambda_2 \hat{p} j \mu \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\alpha \int_{-1}^1 d\cos \beta \ d^j \mu_{\lambda_1 - \lambda_2} (\beta) \ e^{i(\mu - \lambda_1 - \lambda_2) \alpha} | p_2 \lambda_2 \rangle \ p_1 \lambda_1 \rangle_{\text{CM}}. \tag{4.4}
\]

Here, the momenta of two particles are given by \( p_1 = (q, \vec{p}_1) \) and \( p_2 = (q, -\vec{p}_1) \) with the direction of \( \vec{p}_1 \) denoted by \( (\alpha, \beta) \). Identifying \( | p_1 \lambda_1 \rangle \) by \( | (\alpha, \beta) \lambda_1 \rangle \), we have

\[
| p_2 \lambda_2 \rangle \ p_1 \lambda_1 \rangle_{\text{CM}} = | -\overrightarrow{(\alpha, \beta)} \lambda_2 \rangle \ | \overrightarrow{(\alpha, \beta)} \lambda_1 \rangle. \tag{4.5}
\]

Using \( -\overrightarrow{(\alpha, \beta)} = (\alpha + \pi, \pi - \beta) \), the above state can be written as

\[
| p_2 \lambda_2 \rangle \ p_1 \lambda_1 \rangle_{\text{CM}} = | (\alpha + \pi, \pi - \beta) \lambda_2 \rangle \ | (\alpha + \pi, \pi - \beta) \lambda_1 \rangle. \tag{4.6}
\]

We now change the integration variables from \( \alpha \) and \( \beta \) to \( \tilde{\alpha} = \alpha + \pi \) and \( \tilde{\beta} = \pi - \beta \) and get

\[
\tau | \lambda_1 \lambda_2 \hat{p} j \mu \rangle = (-1)^{j+\lambda_1+\lambda_2} \frac{1}{4\pi} \int_0^{2\pi} d\tilde{\alpha} \int_{-1}^1 d\cos \tilde{\beta} \ d^j \mu_{\lambda_2 - \lambda_1} (\tilde{\beta}) \ e^{i(\mu - \lambda_1 - \lambda_2)\tilde{\alpha}} | \overrightarrow{(\tilde{\alpha}, \tilde{\beta})} \lambda_2 \rangle \ | -\overrightarrow{(\tilde{\alpha}, \tilde{\beta})} \lambda_1 \rangle. \tag{4.7}
\]

Here, we have used the identity

\[
d^j \mu_{(\pi - \beta)} = (-1)^{(j+\mu)\rho} e^{i(\rho - \beta)^2} \ d^j \mu_{(-\mu)(\beta)}. \tag{4.8}
\]

This identity is well-known in angular momentum theory \[17\]. Comparing this with (3.6), we have,

\[
\tau | \lambda_1 \lambda_2 \hat{p} j \mu \rangle = (-1)^{j+\lambda_1+\lambda_2} | \lambda_2 \lambda_1 \hat{p} j \mu \rangle, \tag{4.9}
\]

and therefore,

\[
| \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} = \frac{1}{2} \left( | \lambda_1 \lambda_2 \hat{p} j \mu \rangle \pm (-1)^{(j+\lambda_1+\lambda_2)} | \lambda_2 \lambda_1 \hat{p} j \mu \rangle \right). \tag{4.10}
\]

This equation determines the selection rules. For example, Yang’s argument about the forbidden decay of \( Z^0 \rightarrow 2\gamma \) can be easily explained using this equation as follows.
The particle $Z^0$ has spin $j = 1$. Therefore, the two photons after the $Z^0$ decay at rest cannot have opposite helicities by angular momentum conservation. For if the two photons have opposite helicities, then $|\lambda_1 - \lambda_2| = 2$ and the minimum value for $j$ is 2. This is bigger than the spin of $Z^0$ which is 1.

Now we assume that the two photons after decay have the same helicity, that is, $\lambda_1 = \lambda_2 = \lambda$. In this case, (4.10) becomes

$$|\lambda \lambda \hat{p} j \mu\rangle_s = \frac{1}{2} \left( 1 + (-1)^{(j+\lambda)} \right) |\lambda \lambda \hat{p} j \mu\rangle.$$ (4.11)

We choose + because photon is a boson. Now substituting $j = 1$ and $\lambda = \pm 1$, we find that the right hand side vanishes. This means that two photon states cannot have any $j = 1$ component. Consequently, the decay $Z^0 \rightarrow 2\gamma$ is forbidden.

So far, we have considered the standard coproduct of the Poincaré group acting on the tensor product states. In the next section, we introduce a new coproduct and investigate how to reduce the direct product of two irreducible representations with this new coproduct.

## 5 Twisted Coproduct

We now replace the coproduct $\Delta_0(R)$ by the twisted coproduct $\Delta_\theta(R)$ to define a new action of Poincaré transformation on the direct product states as was discussed in the introduction. The direct product of two irreducible representations of the Poincaré group is also reducible under the action of this twisted coproduct. The way to reduce the direct product space is the same as in the untwisted coproduct case except that the untwisted coproduct $\Delta_0(R)$ should be replaced by the twisted coproduct $\Delta_\theta(R)$. For the case of two massless particle systems, we have

$$|\lambda_1 \lambda_2 \hat{p} j \mu\rangle_\theta = \int_{SU(2)} d\mu(R) D^*_{\mu \lambda_1 - \lambda_2}(R) \Delta_\theta(R) |q_1 \lambda_1\rangle |q_2 \lambda_2\rangle.$$ (5.1)

It can be shown that the subspace generated by the above states forms an irreducible subspace under the twisted coproduct action of the Poincaré group. That is, the state $|\lambda_1 \lambda_2 \hat{p} j \mu\rangle_\theta$ transforms under the action of the twisted coproduct of the Poincaré group as if it is a single particle state with mass $2q$ and spin $j$ just like the way that $|\lambda_1 \lambda_2 \hat{p} j \mu\rangle$ transforms under the action of the untwisted coproduct. It can also be shown that the collection of $\{|\lambda_1 \lambda_2 \hat{p} j \mu\rangle_\theta\}$ and their Lorentz transformations with different $\lambda_1, \lambda_2, j$ form a complete set for the direct product space. Note here that the two particle state on the right hand side of (5.1) is taken to be the ordinary tensor product state. If we use the star(or twisted) tensor product state instead defined by [13, 18]

$$|\Psi\rangle \otimes_{\theta} |\Phi\rangle = F^{-1}_{\theta} |\Psi\rangle \otimes |\Phi\rangle,$$ (5.2)

there will be an extra overall phase factor on the right hand side of (5.1), which is quite irrelevant in the following arguments.

The action of the twisted coproduct on the tensor product state is
\[ \Delta_\theta(g) \left| q_1 \lambda_1 \right. \left| q_2 \lambda_2 \right) = e^{-\frac{i}{2} q_1 \wedge q_2 F_\theta^{-1} \Delta_\theta(g)} \left| q_1 \lambda_1 \left| q_2 \lambda_2 \right) , \]  

and therefore

\[ \left| \lambda_1 \lambda_2 \hat{p} j \mu \right) = e^{-\frac{i}{2} q_1 \wedge q_2 F_\theta^{-1}} \left| \lambda_1 \lambda_2 \hat{p} j \mu \right). \]  

(5.4)

Here, \( p \wedge q = p \mu \mu' q \nu \). Substituting (3.6) in this equation, we get

\[ \left| \lambda_1 \lambda_2 \hat{p} j \lambda \right) = \frac{1}{4\pi} \int_0^{2\pi} d\alpha \int_{-1}^{1} d\cos \beta d^j \mu, \lambda_1 - \lambda_2(\beta) e^{i(\mu - \lambda_1 - \lambda_2)\alpha} e^{\frac{i}{2}(p_1 \wedge p_2 - q_1 \wedge q_2)} \left| p_1 \lambda_1 \right| p_2 \lambda_2\right)_{CM} . \]  

(5.5)

If \( \theta^{0i} = 0 \), then since \( \hat{p}_1, \hat{p}_2(q_1, q_2) \) are antiparallel in the center-of-momentum frame, \( p_1 \wedge p_2 = q_1 \wedge q_2 = 0 \) and \( |\lambda_1 \lambda_2 \hat{p} j \mu \rangle \) and \( |\lambda_1 \lambda_2 \hat{p} j \mu \rangle \) are identical. However, using (3.3) the twisted state in an arbitrary frame is seen to be

\[ \left| \lambda_1 \lambda_2 p j \mu \right) = e^{-\frac{i}{2} q_1 \wedge q_2 F_\theta^{-1}} |\lambda_1 \lambda_2 p j \mu \rangle , \]  

(5.6)

so that \( |\lambda_1 \lambda_2 p j \mu \rangle \) and \( |\lambda_1 \lambda_2 p j \mu \rangle \) will in general be different if \( \theta^{ij} \neq 0 \) even if \( \theta^{0i} = 0 \).

The Clebsch-Gordan coefficients are modified:

\[ \left| p_1 \lambda_1 \right| p_2 \lambda_2\right)_{CM} = e^{\frac{i}{2}(q_1 \wedge q_2 - p_1 \wedge p_2)} \sum_{j, \mu} (2j + 1) e^{-i(\mu - \lambda_1 - \lambda_2)\alpha} d^j \mu, \lambda_1 - \lambda_2(\beta) \left| \lambda_1 \lambda_2 \hat{p} j \mu \right)_{\theta} , \]  

(5.7)

\[ \langle k_1 \mu_1 | k_2 \mu_2 | \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{\theta} = e^{\frac{i}{2}(k_1 \wedge k_2 - q_1 \wedge q_2)} \langle k_1 \mu_1 | k_2 \mu_2 | \lambda_1 \lambda_2 \hat{p} j \mu \rangle , \]  

(5.8)

\[ \theta \langle \lambda_1' \lambda_2' \mu' j' \mu' | \lambda_1 \lambda_2 p j \mu \rangle_{\theta} = \langle \lambda_1' \lambda_2' \mu' j' \mu' | \lambda_1 \lambda_2 p j \mu \rangle . \]  

(5.9)

Finally, we discuss the tensor product of two identical particle states. With the twisted coproduct, symmetrization or antisymmetrization should be done not with \( \tau \) but with \( \tau_\theta \) defined in the introduction. With this twisted flip operator, we get

\[ \left| \lambda_1 \lambda_2 \hat{p} j \mu \right)^{S,A}_{\theta} = \frac{1 \pm \tau_\theta}{2} \left| \lambda_1 \lambda_2 \hat{p} j \mu \right)_{\theta} . \]  

(5.10)

Substituting (5.11) into above equation, we obtain

\[ \left| \lambda_1 \lambda_2 \hat{p} j \mu \right)^{S,A}_{\theta} = \int_{SU(2)} d\mu(R) D^j_{\mu} \lambda_1 - \lambda_2(R) \frac{1 \pm \tau_\theta}{2} \Delta_\theta(R) \left| q_1 \lambda_1 \right| q_2 \lambda_2) . \]  

(5.11)

Using the relations \( 1 \pm \tau_\theta = F_\theta^{-1}(1 \pm \tau) F_\theta \) and \( \Delta_\theta(R) = F_\theta^{-1} \Delta_\theta(R) F_\theta \), we get

\[ \left| \lambda_1 \lambda_2 \hat{p} j \mu \right)^{S,A}_{\theta} = e^{-\frac{i}{2} q_1 \wedge q_2 F_\theta^{-1}} \int_{SU(2)} d\mu(R) D^j_{\mu} \lambda_1 - \lambda_2(R) \frac{1 \pm \tau}{2} \Delta(R) \left| q_1 \lambda_1 \right| q_2 \lambda_2) . \]  

(5.12)
Comparing this result with (4.1), we obtain
\[ | \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} = e^{-\frac{1}{2} q_1 \wedge q_2 F_{\theta}^{-1}} | \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} \]  
(5.13)
and
\[ | \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} = \frac{1}{2} \left( | \lambda_1 \lambda_2 \hat{p} j \mu \rangle_{S,A} \mp (-1)^{(j+\lambda_1+\lambda_2)} | \lambda_2 \lambda_1 \hat{p} j \mu \rangle_{S,A} \right). \]  
(5.14)
Here we used (5.6). In case \( \lambda_1 = \lambda_2 = \lambda \), we thus have
\[ | \lambda \lambda \hat{p} j \mu \rangle_{S,A} = \frac{1}{2} \left( 1 \pm (-1)^{(j+2\lambda)} \right) | \lambda \lambda \hat{p} j \mu \rangle_{S,A}. \]  
(5.15)
and consequently, the selection rules are not altered by twisting the coproduct. The decay \( Z_0 \to 2\gamma \) is forbidden even with the twisted coproduct. Note that this result is somehow expected because the twist operator carries only momentum (and no spin) degrees of freedom. The relative phase in (5.14), and consequently in (5.15), is not altered by the introduction of the twist.

Equation (5.15) shows that a massive particle of spin \( j \) cannot decay into a pair of identical massless particles of helicity \( \lambda \) if \( 1 + (-1)^{2\lambda}(-1)^{j+2\lambda} = 1 + (-1)^j = 0 \). This is so for any value of the twist \( \theta^{\mu\nu} \). Thus \( Z_0 \) cannot decay into two massless neutrinos of helicity \( \lambda \) for any value of \( \theta^{\mu\nu} \) even if lepton number is violated.

6 Concluding Remarks

We note that a relation of the form (5.11) is correct even for two identical massive particles. In that case, (5.11) is replaced by
\[ | j\lambda_1 \lambda_2 \hat{p} J \mu \rangle_{S,A} = \int_{SU(2)} d\mu(R) D^J_{\mu \lambda_1-\lambda_2}(R) \frac{1 \pm \tau_\theta}{2} \Delta_\theta(R) | q_1 j \lambda_1 \rangle \langle q_2 j \lambda_2|. \]  
(6.1)
This reduces as before to
\[ | j\lambda_1 \lambda_2 \hat{p} J \mu \rangle_{S,A} = e^{-\frac{1}{2} q_1 \wedge q_2 F_{\theta}^{-1}} | j\lambda_1 \lambda_2 \hat{p} J \mu \rangle_{S,A}. \]  
(6.2)
It follows that if \( \mathcal{P}^I_+ \)-invariance for \( \theta^{\mu\nu} = 0 \) forbids the decay of a spin \( J \) particle into two identical spin \( j \) particles, then \( \mathcal{P}^I_+ \)-invariance for \( \theta^{\mu\nu} \neq 0 \) also forbids it.

It is easy to show in a similar manner that if a decay into two non-identical particles is forbidden by \( \mathcal{P}^I_+ \)-invariance for \( \theta^{\mu\nu} = 0 \), it remains forbidden by \( \mathcal{P}^I_+ \)-invariance for \( \theta^{\mu\nu} \neq 0 \).

Yang’s result and those of this paper require two basic assumptions: (a) the \( S \)-operator \( S \) is invariant under \( \mathcal{P}^I_+ \), and (b) if \( \psi_B(\psi_F) \) has a possibly twisted Bose(Fermi) symmetry, \( S \psi_B(S \psi_F) \) has the same symmetry.

But not all QFT’s on the Moyal plane share these properties. There is in particular an approach to gauge theories with matter [14] which for non-abelian gauge groups gives Lorentz non-invariant \( S \)-operators violating the Pauli principle. This violation of Lorentz invariance by \( S \) comes from the non-locality of QFT’s on the Moyal plane.
The standard model can be deformed along the lines of this approach. The fate of the process \(Z_0 \rightarrow 2\gamma\) in this deformed model is yet to be studied.

Acknowledgments

A.P.B. thanks H. S. Mani for bringing Yang’s paper to his attention and for emphasizing the significance of the process \(Z_0 \rightarrow 2\gamma\). The work of A.P.B. is supported in part by US Department of Energy under grant number DE-FG02-85ER40231. S.G.J. is supported by the International Cooperation Research Program of the Ministry of Science and Technology of Korea.

References

[1] C. N. Yang, Phys. Rev. 77, 242 (1950).

[2] I. M. Gel’fand, R. A. Minlos and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Representations*, (Pergamon Press Limited, Oxford, 1963).

[3] T. Kahan, *Theory of Groups in Classical and Quantum Physics*, (American Elsevier Publishing Co., Inc., New York, 1966).

[4] M. A. Naimark, *Linear Representations of the Lorentz Group*, (Pergamon Press Limited, Oxford, 1964).

[5] E. P. Wigner, *Theoretical Physics*, (International Atomic Energy Agency, Vienna, 1963).

[6] A. P. Balachandran, *The Poincaré Group*, (Syracuse University preprint 1206-212, 1969); J. Strathdee, J. F. Boyce, R. Delbourgo and A. Salam, *Partial Wave Analysis*, (International Center for Theoretical Physics, Trieste, 1967).

[7] A. P. Balachandran and C. G. Trahern, *Lectures on Group Theory for Physicists*, (Bibliopolis, Napoli, 1984).

[8] V. G. Drin’feld, Leningrad Math. J. 1, 1419 (1990).

[9] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B 604, 98 (2004) [arXiv:hep-th/0408069].

[10] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. 22, 3511 (2005) [arXiv:hep-th/0504183].

[11] G. Fiore and P. Schupp, Nucl. Phys. B 470, 211 (1996) [arXiv:hep-th/9508047]; R. Oeckl, Nucl. Phys. B 581, 559 (2000) [arXiv:hep-th/0003018].
[12] A. P. Balachandran, G. Mangano, A. Pinzul and S. Vaidya, Int. J. Mod. Phys. A 21, 3111 (2006) \[arXiv:hep-th/0508002\]; A. P. Balachandran, A. Pinzul and B. Qureshi, Phys. Lett. B 634, 434 (2006) \[arXiv:hep-th/0508151\]; A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi and S. Vaidya, Phys. Rev. D 75, 045009 (2007) \[arXiv:hep-th/0608179\].

[13] P. Matlock, Phys. Rev. D 71, 126007 (2005) \[arXiv:hep-th/0504084\]; J. M. Gracia-Bondia, F. Lizzi, F. R. Ruiz and P. Vitale, Phys. Rev. D 74, 025014 (2006) \[arXiv:hep-th/0604206\]; F. Lizzi, S. Vaidya and P. Vitale, Phys. Rev. D 73, 125020 (2006) \[arXiv:hep-th/0601056\]; S. Vaidya, \[arXiv:0707.3858\]; P. Aschieri, F. Lizzi and P. Vitale, \[arXiv:0708.3002\].

[14] A. P. Balachandran, B. A. Qureshi, A. Pinzul and S. Vaidya, \[arXiv:hep-th/0608138\].

[15] W. Behr, N. G. Deshpande, G. Duplanćić, P. Schupp, J. Trampetić and J. Wess, Eur. Phys. J. C 29, 441 (2003) \[arXiv:hep-th/0202121\].

[16] M. Burić, D. Latas, V. Radovanović and J. Trampetić, \[arXiv:hep-th/0611299\]; J. Trampetić, \[arXiv:hep-th/0704.0559\] and references therein.

[17] L. C. Biedenharn and H. Van Dam, *Quantum Theory of Angular Momentum*, (Academic Press, New York, 1965).

[18] G. Fiore and J. Wess, \[arXiv:hep-th/0701078\].