A CHARACTERISTIC POLYNOMIAL FOR THE
TRANSITION PROBABILITY MATRIX OF A
CORRELATED RANDOM WALK ON A GRAPH

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Abstract

We define a correlated random walk (CRW) induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph $G$, and present a formula for the characteristic polynomial of the transition probability matrix of this CRW by using a determinant expression for the generalized weighted zeta function of $G$. As applications, we give the spectrum of the transition probability matrices for the CRWs induced from the Grover matrices of regular graphs and semi-regular bipartite graphs. Furthermore, we consider another type of the CRW on a graph.

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1 Introduction

Zeta functions of graphs started from the Ihara zeta functions of regular graphs by Ihara [6]. In [6], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada [15,16]. Hashimoto [4] generalized Ihara’s result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [1] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Morita [12] defined a generalized weighted zeta function of a digraph which contains various zeta functions of a graph or a digraph. Ide et al [5] presented a determinant expression for the above generalized weighted zeta function of a graph.

The time evolution matrix of a discrete-time quantum walk in a graph is closely related to the Ihara zeta function of a graph. A discrete-time quantum walk is a quantum analog of the classical random walk on a graph whose state vector is governed by a matrix called the time evolution matrix (see [8]). Ren et al. [13] gave a relationship between the discrete-time quantum walk and the Ihara zeta function of a graph. Konno and Sato [10] obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a graph.

In this paper, we define introduce a new correlated random walk induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph, and present a formula for the characteristic polynomial of its transition probability matrix.

In Section 2, we review for the Ihara zeta function and the generalized weighted zeta functions of a graph. In Section 3, we review for the Grover walk on a graph. In Section 4, we define a correlated random walk (CRW) induced from the time evolution matrix (the Grover matrix) of the Grover walk on a graph $G$, and present a formula for the characteristic polynomial of the transition probability matrix of this CRW. In Section 5, we give the spectrum of the transition probability matrix for this CRW of a regular graph. In Section 6, we present the spectrum for the transition probability matrix of this CRW of a semiregular bipartite graph. In Section 7, we present formulas for the characteristic polynomials of the transition probability matrices of another type of the CRW on a graph, and give the spectrum of its transition probability matrix.

2 Preliminaries

2.1 Zeta functions of graphs

Graphs and digraphs treated here are finite. Let $G$ be a connected graph and $D_G$ the symmetric digraph corresponding to $G$. Set $D(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$. For $e = (u,v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v,u)$ be the inverse of $e = (u,v)$. For $v \in V(G)$, the degree $\deg v = \deg v = d_v$ is the number of vertices adjacent to $v$ in $G$. A graph $G$ is called $k$-regular if $\deg v = k$ for each $v \in V(G)$.

A path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \cdots, e_n)$ of $n$ arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we write $P = (v_0, v_1, \cdots, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is called an $(o(P), t(P))$-path. We say that a path $P = (e_1, \cdots, e_n)$ has a backtracking if $e_{i+1}^{-1} = e_i$ for some $i$ $(1 \leq i \leq n - 1)$. A $(v, w)$-path is called a $v$-cycle (or $v$-closed path) if $v = w$. The inverse cycle of a cycle $C = (e_1, \cdots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \cdots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called equivalent if there exists a positive number $k$ such that $f_j = f_{j+k}$ for all $j$, where the subscripts are considered by modulo $m$. The inverse cycle of
2.2 The generalized weighted zeta functions of a graph

Let $G$ be a connected graph with $n$ vertices and $m$ edges, and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$, where $e_{m+i} = e_i^{-1}$ for $1 \leq i \leq m$. Furthermore, we consider two functions $\tau : D(G) \to \mathbb{C}$ and $\mu : D(G) \to \mathbb{C}$. Let $\theta : D(G) \times D(G) \to \mathbb{C}$ be a function such that

$$\theta(e, f) = \tau(f)\delta_{t(e)\alpha(f)} - \mu(f)\delta_{e^{-1}f}.$$ 

We introduce a $2m \times 2m$ matrix $M(\theta) = (M_{ef})_{e, f \in D(G)}$ as follows:

$$M_{ef} = \theta(e, f).$$

Then the generalized weighted zeta function $Z_G(u, \theta)$ of $G$ is defined as follows (see [12]):

$$Z_G(u, \theta) = \det(I_{2m} - uM(\theta))^{-1}.$$ 

We consider two $n \times n$ matrices $A_G(\theta) = (a_{uv})_{u, v \in V(G)}$ and $D_G(\theta) = (d_{uv})_{u, v \in V(G)}$ as follows:

$$a_{uv} = \begin{cases} \tau(e)/(1 - u^2\mu(e)\mu(e^{-1})) & \text{if } e(u, v) \in D(G), \\ 0 & \text{otherwise}, \end{cases}$$

$$d_{uv} = \begin{cases} \sum_{e(v) = u} \tau(e)\mu(e^{-1})(1 - u^2\mu(e)\mu(e^{-1})) & \text{if } u = v, \\ 0 & \text{otherwise}. \end{cases}$$

A determinant expression for the generalized weighted zeta function of a graph is given as follows (see [5]):
Theorem 2 (Ide, Ishikawa, Morita, Sato and Segawa) Let $G$ be a connected graph with $n$ vertices and $m$ edges, and let $\tau : D(G) \to \mathbb{C}$ and $\mu : D(G) \to \mathbb{C}$ be two functions. Then
\[
Z_G(u, \theta)^{-1} = \prod_{j=1}^{m} (1 - u^2 \mu(e_j)\mu(e_j^{-1})) \det(I_n - uA_G(\theta) + u^2D_G(\theta)),
\]
where $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ ($e_{m+j} = e_j^{-1}$ ($1 \leq j \leq m$)).

3 The Grover walk on a graph

Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Set $d_j = d_{e_j} = \deg v_j$ for $i = 1, \ldots, n$. The Grover matrix $U = U(G) = (U_{ef})_{e,f \in R(G)}$ of $G$ is defined by
\[
U_{ef} = \begin{cases} 
2/d_{t(f)} (= 2/d_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise.}
\end{cases}
\]
The discrete-time quantum walk with the matrix $U$ as a time evolution matrix is called the Grover walk on $G$.

Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then the $n \times n$ matrix $T(G) = (T_{uv})_{u,v \in V(G)}$ is given as follows:
\[
T_{uv} = \begin{cases} 
1/(\deg GU) & \text{if } (u,v) \in D(G), \\
0 & \text{otherwise.}
\end{cases}
\]
Note that the matrix $T(G)$ is the transition matrix of the simple random walk on $G$(see [10]).

Theorem 3 (Konno and Sato) Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges. Then the characteristic polynomial for the Grover matrix $U$ of $G$ is given by
\[
\det(\lambda I_{2m} - U) = (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)I_n - 2\lambda T(G))
\]
\[
= (\lambda^2 + 1)^{m-n} \det((\lambda^2 + 1)I_n - 2\lambda A(G))
\]
\[
\frac{d_{v_1}, \ldots, d_{v_n}}{d_{v_1}, \ldots, d_{v_n}}.
\]

From this Theorem, the spectra of the Grover matrix on a graph is obtained by means of those of $T(G)$ (see [13]). Let $\text{Spec}(F)$ be the spectra of a square matrix $F$ .

Corollary 1 (Emms, Hancock, Severini and Wilson) Let $G$ be a connected graph with $n$ vertices and $m$ edges. The Grover matrix $U$ has $2n$ eigenvalues of the form
\[
\lambda = \lambda_T \pm i\sqrt{1 - \lambda_T^2},
\]
where $\lambda_T$ is an eigenvalue of the matrix $T(G)$. The remaining $2(m-n)$ eigenvalues of $U$ are $\pm 1$ with equal multiplicities.

4 A correlated random walk on a graph

Let $G$ be a connected graph with $n$ vertices and $m$ edges, and $U$ be the Grover matrix of $G$. Then we define a $2m \times 2m$ matrix $P = (P_{ef})_{e,f \in D(G)}$ as follows:
\[
P_{ef} = |U_{ef}|^2.
\]
Note that
\[
P_{ef} = \begin{cases} 
4/d_{o(f)}^2 (= 4/d_{o(e)}^2) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
(2/d_{t(f)} - 1)^2 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

The random walk with the matrix \( P \) as a transition probability matrix is called the correlated random walk (CRM) (with respect to the Grover matrix) on \( G \) (see [7,9]).

Let \( R = (R_{ef})_{e,f \in D(G)} \) be a \( 2m \times 2m \) matrix such that
\[
R_{ef} = \begin{cases} 
4/d_{o(f)}^2 (= 4/d_{o(e)}^2) & \text{if } o(e) = o(f) \text{ and } f \neq e, \\
(2/d_{o(f)} - 1)^2 & \text{if } f = e, \\
0 & \text{otherwise}.
\end{cases}
\]

Then we have
\[ P = J_0 R. \]

By Theorem 2, we obtain the following formula for \( P \).

**Theorem 4** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges, and let \( P \) be the transition probability matrix of the CRW with respect to the Grover matrix. Then
\[
\det(I_{2m} - uP) = \prod_{j=1}^{m} (1 - u^2(4/d_{o(e_j)}^2 - 1)(4/d_{t(e_j)}^2 - 1)) \det(I_n - uA_{CRW} + u^2D_{CRW}),
\]
where
\[
(A_{CRW})_{xy} = \begin{cases} 
4/d_{o(e)}^2(4/d_{t(e)}^{-1} - 1) & \text{if } (x,y) \in D(G), \\
0 & \text{otherwise}.
\end{cases}
\]
\[
(D_{CRW})_{xy} = \begin{cases} 
\sum_{o(e) = x} 4/d_{o(e)}^2(4/d_{t(e)}^{-1} - 1) & \text{if } x = y, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** For the matrix \( P \), we have
\[
P_{ef} = 4/d_{o(f)}^2 \delta_{t(f)o(e)} - (4/d_{o(e)}^2 - 1)\delta_{f^{-1}e}.
\]
The we let two functions \( \tau : D(G) \rightarrow \mathbb{C} \) and \( \mu : D(G) \rightarrow \mathbb{C} \) as follows:
\[
\tau(e) = \frac{4}{d_{o(e)}^2} \quad \text{and} \quad \mu(e) = \frac{4}{d_{o(e)}^2} - 1.
\]
Furthermore, let
\[
\theta(e,f) = \frac{4}{d_{o(f)}^2} \delta_{t(e)o(f)} - (4/d_{o(f)}^2 - 1)\delta_{e^{-1}f}.
\]
Then we have
\[ P = \iota M(\theta). \]
Thus, we obtain
\[
\det(I_{2m} - uP) = \det(I_{2m} - u \iota M(\theta)) = \det(I_{2m} - uM(\theta)) = Z_G(u, \theta)^{-1}.
\]
By Theorem 3, we have
\[
\det(I_{2m} - uP) = \prod_{j=1}^{m} (1 - u^2(4/d_{o(e_j)}^2 - 1)(4/d_{t(e_j)}^2 - 1)) \det(I_n - uA_{CRW} + u^2D_{CRW}),
\]

\[ J_0 R \]

\[ P = J_0 R. \]
Thus, we have
\[
(A_{CRW})_{xy} = \begin{cases} \
\frac{d + 4(4-d)}{d^2} & \text{if } (x, y) \in D(G), \\
0 & \text{otherwise},
\end{cases}
\]
\[
(D_{CRW})_{xy} = \begin{cases} \
\frac{4/d_x^2}{4/d_y^2} & \text{if } x = y, \\
0 & \text{otherwise}.
\end{cases}
\]

By Theorem 4, we obtain the spectrum of the transition probability matrices for the CRWs induced from the Grover matrices of regular graphs and semiregular bipartite graphs.

## 5 An application to the correlated random walk on a regular graph

We present spectra for the transition matrix of the correlated random walk on a regular graph with respect to the Grover matrix.

**Theorem 5** Let $G$ be a connected $d$-regular graph with $n$ vertices and $m$ edges, where $d \geq 2$. Furthermore, let $P$ be the transition probability matrix of the CRW with respect to the Grover matrix. Then

\[
\det(I_{2m} - uP) = \frac{(d^2 - u^2(4 - d)^2)^{m-n}}{d^{2m}} \det(d + (4 - d)u^2)I_n - 4uA(G).
\]

**Proof.** Let $G$ be a connected $d$-regular graph with $n$ vertices and $m$ edges, where $d \geq 2$. Then we have

\[
d_{v(e)} = d(e) = d \text{ for each } e \in D(G).
\]

Thus, we have

\[
1 - u^2(d_{v(e)} - 1) = \frac{d^2 - u^2(4 - d)^2}{d^2},
\]

\[
(A_{CRW})_{xy} = \frac{4/d_x^2}{1 - u^2(4/d_x - 1)(4/d_y - 1)} = \frac{4}{d^2} \text{ if } (x, y) \in D(G)
\]

and

\[
(D_{CRW})_{xy} = \sum_{o(e) = x} \frac{4/d_x^2(4/d_y - 1)}{1 - u^2(4/d_x - 1)(4/d_y - 1)} = \frac{4d(4 - d)}{d(d^2 - u^2(4 - d)^2)} = \frac{4(4 - d)}{d^2 - u^2(4 - d)^2}.
\]

Therefore, it follows that

\[
A_{CRW} = \frac{4}{d^2 - u^2(4 - d)^2} A(G) \text{ and } D_{CRW} = \frac{4(4 - d)}{d^2 - u^2(4 - d)^2} I_n.
\]

By Theorem 4, we have

\[
\det(I_{2m} - uP)
\]

\[
= \frac{(d^2 - u^2(4 - d)^2)^{m-n}}{d^{2m}} \det((d^2 - u^2(4 - d)^2)I_n - 4uA(G) + 4(4 - d)u^2 I_n)
\]

\[
= \frac{(d^2 - u^2(4 - d)^2)^{m-n}}{d^{2m}} \det((d + (4 - d)u^2)I_n - 4uA(G)).
\]

By substituting $u = 1/\lambda$, we obtain the following result.
Corollary 2 Let $G$ be a connected $d$-regular graph with $n$ vertices and $m$ edges, where $d \geq 2$. Furthermore, let $P$ be the transition probability matrix of the CRW with respect to the Grover matrix. Then
\[
\det(\lambda^2m - P) = \frac{(d^2\lambda^2 - (4 - d)^2)^{m-n}}{d^m} \det(d(d\lambda^2 + (4 - d))I_n - 4\lambda A(G))
\]
\[
= (\lambda^2 - (\frac{4}{d} - 1)^2)^{m-n} \lambda^n \det((\lambda + (\frac{4}{d} - 1))\frac{1}{\lambda}I_n - \frac{4}{d^2}A(G)).
\]

The second identity of Corollary 2 is considered as the spectral mapping theorem for $P$.

By Corollary 2, we obtain the spectra for the transition matrix $P$ of the CRW with respect to the Grover matrix on a regular graph.

Corollary 3 Let $G$ be a connected $d(\geq 2)$-regular graph with $n$ vertices and $m$ edges. Then the transition probability matrix $P$ has $2n$ eigenvalues of the form
\[
\lambda = \frac{2\lambda_A \pm \sqrt{4\lambda_A^2 - d^2(4 - d)}}{d^2},
\]
where $\lambda_A$ is an eigenvalue of the matrix $A(G)$. The remaining $2(m - n)$ eigenvalues of $P$ are $\pm(4 - d)/d$ with equal multiplicities $m - n$.

Proof. By Corollary 2, we have
\[
\det(\lambda^2m - P) = \frac{(d^2\lambda^2 - (4 - d)^2)^{m-n}}{d^m} \prod_{\lambda_{A} \in \text{Spec}(A(G))} (d(d\lambda^2 + 4 - d) - 4\lambda_{A}\lambda)
\]
\[
= (\lambda^2 - (\frac{4}{d} - 1)^2)^{m-n} \prod_{\lambda_{A} \in \text{Spec}(A(G))} (d^2\lambda^2 - 4\lambda_{A}\lambda + d(4 - d)).
\]
Thus, solving
\[
d^2\lambda^2 - 4\lambda_{A}\lambda + d(4 - d) = 0,
\]
we obtain
\[
\lambda = \frac{2\lambda_A \pm \sqrt{4\lambda_A^2 - d^2(4 - d)}}{d^2}.
\]

In the case of $d = 4$, we consider $P = (P_{ef})_{e,f \in E(G)}$ be the transition probability matrix of the CRW with respect to the Grover matrix on a $d$-regular graph $G$. If $t(f) = o(e)$ and $f \neq e^{-1}$, then $P_{ef} = 4/d^2 = 4/4^2 = 1/4$. If $f = e^{-1}$, then $P_{ef} = 4/d^2 - (4/d - 1) = 4/4^2 - (4/4 - 1) = 1/4$. Thus, this CRW is considered to be a simple random walk on $G$ which the particle moves over each arc in terms of the same probability. Furthermore, an $n \times n$ Hadamard matrix is a unitary matrix whose elements have the absolute value $1/\sqrt{n}$ (see [2]). The Grover matrix of a $d$-regular graph is an Hadamard matrix if and only if $d = 4$.

6 An application to the correlated random walk on a semiregular bipartite graph

We present spectra for the transition probability matrix of the correlated random walk on a semiregular bipartite graph. Hashimoto [4] presented a determinant expression for the Ihara zeta function of a semiregular bipartite graph. We use an analogue of the method in the proof of Hashimoto’s result.

A bipartite graph $G = (V_1, V_2)$ is called $(q_1, q_2)$-semiregular if $\deg_G v = q_i$ for each $v \in V_i (i = 1, 2)$. For a $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph $G = (V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set $V_i$ and edge set $\{P : \text{reduced path } | | P | = 2; o(P), t(P) \in V_i\}$ for $i = 1, 2$. Then $G^{[1]}$ is $(q_1 + 1)q_2$-regular, and $G^{[2]}$ is $(q_2 + 1)q_1$-regular.
Theorem 6 Let $G = (V, W)$ be a connected $(r, s)$-semiregular bipartite graph with $n$ vertices and $e$ edges. Set $|V| = m$ and $|W| = n(m \leq n)$. Furthermore, let $P$ be the transition probability matrix of the CRW with respect to the Grover matrix of $G$, and

$$Spec(A(G)) = \{ \pm \lambda_1, \cdots, \pm \lambda_m, 0, \ldots, 0 \}.\]$$

Then

$$det(I_{2e} - uP) = (1 - u^2(4/r - 1)(4/s - 1))^{n-m} (1 - u^2(4/r - 1))^{n-m} \times \prod_{j=1}^{m} ((1 - u^2(4/s - 1))(1 - u^2(4/r - 1)) - 16 \frac{\lambda_j^2}{r^2 s^2} u^2).$$

Proof. Let $e \in D(G)$. If $o(e) \in V$, then

$$d_{o(e)} = r, \ d_{t(e)} = s.$$Thus, we have

$$1 - u^2(\frac{4}{d_{o(e)}} - 1)(\frac{4}{d_{t(e)}} - 1) = \frac{rs - u^2(4 - r)(4 - s)}{rs},$$

$$(A_{CRW})_{xy} = \frac{4/d_x^2}{1 - u^2(4/d_x - 1)(4/d_y - 1)}$$

$$= \begin{cases} \frac{4x}{rs - u^2(4-r)(4-s)} & \text{if } (x, y) \in D(G) \text{ and } x \in V, \\ \frac{1}{rs - u^2(4-r)(4-s)} & \text{if } (x, y) \in D(G) \text{ and } x \in W, \end{cases}$$

and

$$(D_{CRW})_{xx} = \sum_{o(e) = x} \frac{4/d_x^2(4/d_{t(e)}) - 1}{1 - u^2(4/d_x - 1)(4/d_{t(e)})}$$

$$= \begin{cases} r \cdot \frac{4(4-s)}{r(s - u^2(4-r)(4-s))} = \frac{4(4-s)}{rs - u^2(4-r)(4-s)} & \text{if } x \in V, \\ s \cdot \frac{4(4-r)}{s(r - u^2(4-r)(4-s))} = \frac{4(4-r)}{rs - u^2(4-r)(4-s)} & \text{if } x \in W. \end{cases}$$

Next, let $V = \{v_1, \cdots, v_m\}$ and $W = \{w_1, \cdots, w_n\}$. Arrange vertices of $G$ as follows: $v_1, \cdots, v_m; w_1, \cdots, w_n$. We consider the matrix $A = A(G)$ under this order. Then, let

$$A = \begin{bmatrix} 0 & E \\ t & 0 \end{bmatrix}.$$Since $A$ is symmetric, there exists an orthogonal matrix $F \in O(n)$ such that

$$EF = \begin{bmatrix} R & 0 \\ \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & 0 & \cdots & 0 \\ & \ddots & \vdots & \vdots & \vdots \\ & & \mu_m & 0 & \cdots \end{bmatrix}.$$Now, let

$$H = \begin{bmatrix} I_m & 0 \\ 0 & F \end{bmatrix}.$$Then we have

$$t^\dagger H A H = \begin{bmatrix} 0 & R & 0 \\ t^\dagger R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$Furthermore, let

$$\alpha = \frac{4}{rs - u^2(4-r)(4-s)}.$$
Then we have
\[ \mathbf{A}_{CRW} = \begin{bmatrix} 0 & \alpha s/r \mathbf{E} \\ \alpha r/s \mathbf{E} & 0 \end{bmatrix}, \]
and
\[ \mathbf{D}_{CRW} = \begin{bmatrix} \alpha(4-s)\mathbf{I}_m & 0 \\ 0 & \alpha(4-r)\mathbf{I}_n \end{bmatrix}. \]
Thus, we have
\[ \mathbf{t} \mathbf{HA}_{CRW} \mathbf{H} = \begin{bmatrix} 0 & \alpha s/r \mathbf{R} \\ \alpha r/s \mathbf{R} & 0 \end{bmatrix} \]
and
\[ \mathbf{t} \mathbf{HD}_{CRW} \mathbf{H} = \begin{bmatrix} \alpha(4-s)\mathbf{I}_m & 0 \\ 0 & \alpha(4-r)\mathbf{I}_n \end{bmatrix}. \]

By Theorem 4,
\[
\begin{aligned}
\det(\mathbf{I}_2 - u\mathbf{P}) &= \frac{(rs-u^2(4-r)(4-s))^{s}}{r^s s^r} \det(\mathbf{I}_{\nu} - u\mathbf{A}_{CRW} + u^2\mathbf{D}_{CRW}) \\
&= \frac{(rs-u^2(4-r)(4-s))^{s}}{r^s s^r} \det \left( \begin{bmatrix} \mathbf{I}_m + \alpha(4-s)u^2\mathbf{I}_m & -\alpha su/r \mathbf{R} \\ -\alpha ru/s \mathbf{R} & \mathbf{I}_m + \alpha(4-r)u^2\mathbf{I}_m \end{bmatrix} \right) \cdot \det \left( \begin{bmatrix} \mathbf{I}_m & 0 \\ 0 & \frac{1}{1+\alpha(4-s)u^2} \alpha su/r \mathbf{R} \end{bmatrix} \right) \\
&= \frac{(rs-u^2(4-r)(4-s))^{s+m-2}}{r^s s^r} (rs + u^2(4-r)s)^{n-m} \cdot \det \left( \begin{bmatrix} (1+\alpha(4-s)u^2)\mathbf{I}_m & 0 \\ -\alpha ru/s \mathbf{R} & (1+\alpha(4-r)u^2)\mathbf{I}_m - \frac{\alpha^2 u^2}{1+\alpha(4-s)u^2} \mathbf{RR} \end{bmatrix} \right) \\
&= \frac{(rs-u^2(4-r)(4-s))^{s+m-2}}{r^s s^r} (rs + u^2(4-r)s)^{n-m} \cdot \det((1+\alpha(4-s)u^2)(1+\alpha(4-r)u^2)\mathbf{I}_m - \alpha^2 u^2 \mathbf{RR}).
\end{aligned}
\]

Since \( \mathbf{A} \) is symmetric, \( \mathbf{t} \mathbf{RR} \) is symmetric and positive semi-definite, i.e., the eigenvalues of \( \mathbf{t} \mathbf{RR} \) are of form:
\[ \lambda_1^2, \ldots, \lambda_m^2 (\lambda_1, \ldots, \lambda_m \geq 0). \]
Furthermore, we have
\[ \det(\lambda \mathbf{I}_{\nu} - \mathbf{A}(G)) = \lambda^{n-m} \det(\lambda^2 - \mathbf{t} \mathbf{RR}), \]
and so,
\[ \text{Spec}(\mathbf{A}(G)) = \{ \pm \lambda_1, \ldots, \pm \lambda_m, 0, \ldots, 0 \}. \]
Then the transition matrix has the form
\[ \text{det}(I_{2r} - uP) \]
\[ = \frac{(rs-u^2(4-r)(4-s))^{n-m}}{rs} (rs + u^2(4-r)s)^{n-m} \]
\[ \times \prod_{j=1}^{m} ((1 + \alpha(4-s)u^2)(1 + \alpha(4-r)u^2)I_m - \alpha^2 \lambda_j^2 u^2) \]
\[ = \frac{(rs-u^2(4-r)(4-s))^{n-m}}{rs} (rs + u^2(4-r)s)^{n-m} \]
\[ \times \prod_{j=1}^{m} \frac{rs+u^2(4-r)s}{rs-u^2(4-r)(4-s)} \]
\[ \times \frac{16u^2}{\lambda_j^2 (rs-u^2(4-r)(4-s))^2} \]
\[ = \frac{(rs-u^2(4-r)(4-s))^{n-m}}{rs} (rs + u^2(4-r)s)^{n-m} \]
\[ \times \prod_{j=1}^{m} (rs(s+u^2(4-s))(r+u^2(4-r)) - 16\lambda_j^2 u^2) \]
\[ = (1 - u^2(4/r - 1)(4/s - 1))^n - \nu(1 + u^2(4/r - 1))^n - m \]
\[ \times \prod_{j=1}^{m} ((1 + u^2(4/s - 1))(1 + u^2(4/r - 1)) - 16\lambda_j^2 u^2). \]

\[ \square \]

Now, let \( u = 1/\lambda \). Then we obtain the following result.

**Corollary 4** Let \( G = (V, W) \) be a connected \((r, s)\)-semiregular bipartite graph with \( \nu \) vertices and \( \epsilon \) edges. Set \( |V| = m \) and \( |W| = n(m \leq n) \). Furthermore, let \( P \) be the transition probability matrix of the CRW with respect to the Grover matrix and
\[ \text{Spec}(A(G)) = \{ \pm \lambda_1, \ldots, \pm \lambda_m, 0, \ldots, 0 \}. \]

Then
\[ \text{det}(\lambda I_{2r} - P) = (\lambda^2 - (4/r - 1)(4/s - 1))^\nu (\lambda^2 + (4/r - 1))^n - m \]
\[ \times \prod_{j=1}^{m} ((\lambda^2 + (4/s - 1))(\lambda^2 + (4/r - 1)) - 16\lambda_j^2 u^2). \]

By Corollary 4, we obtain the spectra for the transition probability matrix \( P \) of the CRW with respect to the Grover matrix of a semiregular bipartite graph.

**Corollary 5** Let \( G = (V, W) \) be a connected \((r, s)\)-semiregular bipartite graph with \( \nu \) vertices and \( \epsilon \) edges. Set \( |V| = m \) and \( |W| = n(m \leq n) \). Furthermore, let \( P \) be the transition probability matrix of the CRW with respect to the Grover matrix and
\[ \text{Spec}(A(G)) = \{ \pm \lambda_1, \ldots, \pm \lambda_m, 0, \ldots, 0 \}. \]

Then the transition matrix \( P \) has 2\( \epsilon \) eigenvalues of the form

1. 4\( r \)m eigenvalues:
\[ \lambda = \pm \sqrt{\frac{2r^2 s^2 - 4rs^2 - 4r^2 s + 16\lambda_j^2 \pm \sqrt{(2r^2 s^2 - 4rs^2 - 4r^2 s + 16\lambda_j^2)^2 - 4r^2 s^3(4-r)(4-s)}}{2r^2 s^2}}; \]
2. \(2n - 2m\) eigenvalues:

\[
\lambda = \pm i \sqrt{\frac{4}{r} - 1};
\]

3. \(2(\epsilon - \nu)\) eigenvalues:

\[
\lambda = \pm \sqrt{\left(\frac{4}{r} - 1\right)\left(\frac{4}{s} - 1\right)}.
\]

Proof. Solving

\[
(\lambda^2 + (4/s - 1))(\lambda^2 + (4/r - 1)) - 16\frac{\lambda^2}{r^2s^2}\lambda^2 = 0,
\]

i.e.,

\[
\lambda^4 + \left(\frac{4}{r} + \frac{4}{s} - 2 - \frac{16\lambda^2}{r^2s^2}\right)\lambda^2 + \left(\frac{4}{r} - 1\right)\left(\frac{4}{s} - 1\right) = 0,
\]

we obtain

\[
\lambda = \pm \sqrt{\frac{1}{2}(2 - \frac{4}{r} - \frac{4}{s} + \frac{16\lambda^2}{r^2s^2}) \pm \sqrt{(2 - \frac{4}{r} - \frac{4}{s} + \frac{16\lambda^2}{r^2s^2})^2 - 4\left(\frac{4}{r} - 1\right)\left(\frac{4}{s} - 1\right),}
\]

i.e.,

\[
\lambda = \pm \sqrt{\frac{2r^2s^2 - 4rs^2 - 4r^2s + 16\lambda^2}{2r^2s^2}} \pm \sqrt{(2r^2s^2 - 4rs^2 - 4r^2s + 16\lambda^2)^2 - 4r^3s^3(4-r)(4-s)};
\]

\(\Box\)

7 Another type of the correlated random walk on a cycle graph

The CRW is defined by the following transition probability matrix \(P\) on the one-dimensional lattice:

\[
P = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix},
\]

where

\(a + c = b + d = 1, \ a, b, c, d \in [0, 1]\).

As for the CRW, see [7,9], for example.

We formulate a CRW on the arc set of a graph with respect to the above matrix \(P\). The cycle graph is a connected 2-regular graph. Let \(C_n\) be the cycle graph with \(n\) vertices and \(n\) edges. Furthermore, let \(V(C_n) = \{v_1, \ldots, v_n\}\) and \(e_j = (v_j, v_{j+1})(1 \leq j \leq n)\), where the subscripts are considered by modulo \(m\). Then we introduce a \(2n \times 2n\) matrix \(U = (U_{ef})_{e,f \in D(C_n)}\) as follows:

\[
U_{ef} = \begin{cases}
d & \text{if } t(f) = o(e), f \neq e^{-1} \text{ and } f = e_j, \\
b & \text{if } f = e^{-1} \text{ and } f = e_j, \\
a & \text{if } t(f) = o(e), f \neq e^{-1} \text{ and } f = e_j^{-1}, \\
c & \text{if } f = e^{-1} \text{ and } f = e_j^{-1}, \\
0 & \text{otherwise.}
\end{cases}
\]
Note that $U$ is be able to write as follows:

$$U = \begin{bmatrix} dQ^{-1} & dI_n \\ bI_n & aQ \end{bmatrix},$$

where $Q = P_\sigma$ is the permutation matrix of $\sigma = (12\ldots n)$. The CRW with $U$ with a transition probability matrix is called the second type of CRW on $C_n$ with respect to the above matrix $P$.

Now, we define a function $w : D(C_n) \rightarrow \mathbb{R}$ as follows:

$$w(e) = \begin{cases} d & \text{if } e = e_j (1 \leq j \leq n), \\ a & \text{if } e = e_j^{-1} (1 \leq j \leq n). \end{cases}$$

Furthermore, let an $n \times n$ matrix $W(C_n) = (w_{uv})_{u,v \in V(C_n)}$ as follows:

$$w_{uv} = \begin{cases} w(u,v) & \text{if } (u,v) \in D(C_n), \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $U$ is given as follows.

**Theorem 7** Let $C_n$ be the cycle graph with $n$ vertices, and $U$ the transition probability matrix of the second type of CRW on $C_n$. Then

$$\det(\lambda I_{2n} - U) = \det((\lambda^2 + (ad - bc))I_n - \lambda W(C_n)).$$

**Proof.** At first, we consider two $2n \times 2n$ matrices $2n \times 2n$ matrices $B = (B_{ef})_{e,f \in D(C_n)}$ and $J = (J_{ef})_{e,f \in D(C_n)}$ as follows:

$$B_{ef} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \quad J_{ef} = \begin{cases} b - a & \text{if } f = e^{-1} \text{ and } e = e_j, \\ c - d & \text{if } f = e^{-1} \text{ and } e = e_j^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$U = \imath B + \imath J.$$ 

Now, we define two $2n \times n$ matrices $K = (K_{ev})_{e \in D(C_n) : v \in V(C_n)}$ and $L = (L_{ev})_{e \in D(C_n), v \in V(C_n)}$ as follows:

$$K_{ev} = \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise,} \end{cases} \quad L_{ev} = \begin{cases} w(e) & \text{if } o(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$K \imath L = B, \quad LK = W(C_n).$$

If $A$ and $B$ are an $m \times n$ matrix and an $n \times m$ matrix, respectively, then we have

$$\det(I_m - AB) = \det(I_n - BA).$$

Thus,

$$\det(I_{2n} - uU) = \det(I_{2n} - u(\imath B + \imath J))$$

$$= \det(I_{2n} - u(B + J))$$

$$= \det(I_{2n} - uJ - uB)$$

$$= \det(I_{2n} - uJ - uK \imath L)$$

$$= \det(I_{2n} - uK \imath L(I_{2n} - uJ)^{-1}) \det(I_{2n} - uJ)$$

$$= \det(I_n - u \imath L(I_{2n} - uJ)^{-1}K) \det(I_{2n} - uJ).$$
But, we have
\[
\det(I_{2n} - uJ) = \begin{bmatrix} I_n & -(b - a)uI_n \\ -(c - d)uI_n & I_n \end{bmatrix} \cdot \begin{bmatrix} I_n & (b - a)uI_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_n \quad 0 \\ -(c - d)uI_n \quad I_n - u^2(b - a)(c - d)I_n \end{bmatrix} = (1 - (a - c)(d - b)u^2)^n.
\]

Furthermore, we have
\[
(I_{2n} - uJ)^{-1} = \frac{1}{1 - (a - b)(d - c)u^2} (I_{2n} + uJ).
\]

Therefore, it follows that
\[
\det(I_{2n} - uU) = (1 - (a - b)(d - c)u^2)^n \det(I_n - u/(1 - (a - b)(d - c)u^2) \mathcal{L}(I_{2n} + uJ)K)
\]
\[
= \det((1 - (a - b)(d - c)u^2)I_n - u \mathcal{L}K - u \mathcal{L}JK)
\]
\[
= \det((1 - (a - b)(d - c)u^2)I_n - uW(C_n) - u^2 \mathcal{L}JK).
\]

The matrix \( \mathcal{L}JK \) is a diagonal, and its \((v_i, v_i)\) entry is equal to
\[
(c - d)w(e_{i-1}^-) + (b - a)w(e_i) = (c - d)a + (b - a)d = c + bd - 2ad.
\]

That is,
\[
\mathcal{L}JK = (ab + cd - 2ad)I_n.
\]

Thus,
\[
\det(I_{2n} - uU) = \det((1 - (a - b)(d - c)u^2)I_n - uW(C_n) - u^2(ac + bd - 2ad)I_n)
\]
\[
= \det(((1 + (ad - bc)u^2)I_n - uW(C_n)).
\]

Substituting \( u = 1/\lambda \), the result follows. \( \square \)

By Theorem 7, we obtain the spectra for the transition probability matrix \( U \) of the second type of the CRW on \( C_n \). The matrix \( W(C_n) \) is given as follows:

\[
W(C_n) = \begin{bmatrix}
0 & d & 0 & \ldots & a \\
0 & a & 0 & d & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & d \\
0 & 0 & 0 & \ldots & a & 0
\end{bmatrix}
\]

**Corollary 6** Let \( C_n \) be the cycle graph with \( n \) vertices, and \( U \) the transition probability matrix of the second type of CRW on \( C_n \). Then the transition probability matrix \( U \) has \( 2n \) eigenvalues of the form

\[
\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4(ad - bc)}}{2}, \quad \mu \in \text{Spec}(W(C_n)).
\]
Proof. At first, we have

$$\det(I_{2n} - uU) = \prod_{\mu \in \text{Spec}(W(C_n))} (\lambda^2 - \mu \lambda + (ad - bc)),$$

Solving

$$\lambda^2 - \mu \lambda + (ad - bc) = 0,$$

we obtain

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4(ad - bc)}}{2}.$$

Now, we consider the case of $$a = b = c = d = 1/2$$. Then the matrix $$W(C_n)$$ is equal to $$W(C_n) = \frac{1}{2}A(C_n)$$.

By Corollary 6, we obtain the spectra for the transition probability matrix $$U$$ of the second type of CRW on $$C_n$$.

Corollary 7 Let $$C_n$$ be the cycle graph with $$n$$ vertices, and $$U$$ the transition probability matrix of the second type of the CRW on $$C_n$$. Assume that $$a = b = c = d = 1/2$$. Then the transition probability matrix $$U$$ has $$n$$ eigenvalues of the form

$$\lambda = \cos \theta_j, \quad \theta_j = \frac{2\pi j}{n} (j = 0, 1, \ldots, n - 1) \quad (\star).$$

The remaining $$n$$ eigenvalues of $$U$$ are 0 with multiplicities $$n$$.

Proof. It is known that the spectrum of $$A(C_n)$$ are

$$2 \cos \theta_j, \quad \theta_j = \frac{2\pi j}{n} (j = 0, 1, \ldots, n - 1).$$

Note that the spectrum of (\star) are those of the transition probability matrix of the simple random walk on a cycle graph $$C_n$$.

We can generalize the result for $$a = b = c = d = 1/2$$ on $$C_n$$ to a $$d$$-regular graph($$d \geq 2$$). Let $$G$$ be a connected $$d$$-regular graph with $$n$$ vertices and $$m$$ edges. Furthermore, let $$P$$ be the $$d \times d$$ matrix as follows:

$$P = \frac{1}{d}J_d,$$

where $$J_d$$ is the matrix whose elements are all one. Let $$U = (U_{ef})_{e,f \in D(G)}$$ be the transition probability matrix of a CRW on $$G$$ with respect to $$P$$. Then we have

$$U_{ef} = \begin{cases} 
1/d & \text{if } t(e) = o(f), \\
0 & \text{otherwise},
\end{cases}$$

and so,

$$U = \frac{1}{d}B.$$

Similarly to The proof of Theorem 7, we obtain the following result.

Theorem 8 Let $$G$$ be a connected $$d$$-regular graph with $$n$$ vertices and $$m$$ edges. Furthermore, let $$U$$ the transition probability matrix of the CRW on $$G$$ with respect to $$P = 1/dJ_d$$. Then

$$\det(\lambda I_{2m} - U) = \lambda^{2m-n} \det(\lambda I_n - \frac{1}{d}A(G)).$$
Thus,

**Corollary 8** Let $G$ be a connected $d$-regular graph with $n$ vertices and $m$ edges. Furthermore, let $U$ the transition probability matrix of the CRW on $G$ with respect to $P = 1/dJ_d$. Then the transition probability matrix $U$ has $n$ eigenvalues of the form

$$
\lambda = \frac{1}{d} \lambda_A, \ \lambda_A \in \text{Spec}(A(G)).
$$

The remaining $2(m - n)$ eigenvalues of $U$ are 0 with multiplicities $2m - n$.

**References**

[1] H. Bass, The Ihara-Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992) 717-797.

[2] I. Bengtsson, W. Bruzda, A. Ericsson, J-A. Larsson, W. Tadej and K. Zyczkowski, Mutually unbiased bases and Hadamard matrices of order six, Journal of Mathematical Physics, 48, 052106 (2007).

[3] D. Foata and D. Zeilberger, A combinatorial proof of Bass’s evaluations of the Ihara-Selberg zeta function for graphs, Trans. Amer. Math. Soc. 351 (1999), 2257-2274.

[4] K. Hashimoto, Zeta Functions of Finite Graphs and Representations of $p$-Adic Groups, Adv. Stud. Pure Math. Vol. 15, Academic Press, New York, 1989, pp. 211-280.

[5] Y. Ide, A. Ishikawa, H. Morita, I. Sato and E. Segawa, The Ihara expression for the generalized weighted zeta function of a simple graph, preprint.

[6] Y. Ihara, On discrete subgroups of the two by two projective linear group over $p$-adic fields, J. Math. Soc. Japan 18 (1966) 219-235.

[7] N. Konno, Quantum Walks (in Japanese), Sangyou Tosho, Tokyo (2008).

[8] N. Konno, Quantum Walks, In: Lecture Notes in Mathematics: Vol.1954, pp.309-452, Springer-Verlag, Heidelberg (2008)

[9] N. Konno, Limit theorems and absorption problems for one-dimensional correlated random walks, Stochastic Models 25 (2009), 28-49.

[10] N. Konno and I. Sato, On the relation between quantum walks and zeta functions, Quantum Inf. Process. 11 (2012), no. 2, 341-349.

[11] M. Kotani and T. Sunada, Zeta functions of finite graphs, J. Math. Sci. U. Tokyo 7 (2000), 7-25.

[12] H. Morita, Ruelle zeta functions for finite digraphs, Linear Algebra and its Applications 603 (2020), 329-358.

[13] P. Ren, T. Aleksic, D. Emms, R. C. Wilson and E. R. Hancock, Quantum walks, Ihara zeta functions and cospectrality in regular graphs, Quantum Inf. Process. 10 (2011), 405-417.

[14] H. M. Stark and A. A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996), 124-165.

[15] T. Sunada, L-Functions in Geometry and Some Applications, in Lecture Notes in Math., Vol. 1201, Springer-Verlag, New York, 1986, pp. 266-284.

[16] T. Sunada, Fundamental Groups and Laplacians (in Japanese), Kinokuniya, Tokyo, 1988.

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