Primordial Magnetic Fields from Superconducting Cosmic Strings

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Abstract

Cosmic strings are stable topological defects that may have been created at a phase transition in the early universe. It is a growing belief that, for a wide range of theoretical models, such strings may be superconducting and carry substantial currents which have important astrophysical and cosmological effects. This paper explores the possibility of generation of a primordial magnetic field by a network of charged–current carrying cosmic strings. The field is created by vorticity, generated in the primordial plasma due to the strings’ motion and gravitational pull. In the case of superconducting strings formed at the breaking of grand unification, it is found that strong magnetic fields of high coherence can be generated in that way. Such fields could account for the observed galactic and intergalactic magnetic fields since they suffice to seed magnetic dynamos on galactic scales.

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1 Introduction

The origin of the observed galactic magnetic fields remains elusive. The magnetic field of the Milky Way and the nearby galaxies is of the order of a \( \mu \text{Gauss} \) and is coherent over \( k\text{pc} \) scales. In the Milky Way the magnetic field is orientated along the spiral arms alternating its direction from one arm to the other. This strongly suggests that the field is supported by a dynamo mechanism which arranges it along the spiral density waves \([1][2]\). Similar evidence for a dynamo mechanism comes from galaxies at a stage of rapid star formation (so called star-burst galaxies \([3]\)) where the magnetic flux needs to be amplified by a factor of \( \sim 5 \) to account for the observations \([4]\).

It is, therefore, a wide belief that the galactic magnetic fields are generated through a dynamo mechanism \([4][5]\). A number of mean field theory dynamo models exist in the literature, the most popular of which is the well known \( \alpha - \Omega \) dynamo. The basic idea of the dynamo mechanism is that a weak seed field could be amplified by the turbulent motion of ionised gas, which follows the differential rotation of the galaxy. The growth of the field is exponential and, thus, its strength can be increased several orders of magnitude in only a few e-foldings of amplification. When the field reaches the equipartition energy (\( \sim \mu \text{Gauss} \)) then its growth is suppressed by dynamical back-reaction.

If the time scale of growth of the field is no more than a galactic rotation period \( \sim 10^8 \text{yrs} \), then the field amplification factor since the collapse of the protogalaxy is of the order \( \sim 10^{13} \), given that the total number of galactic rotations is \( \sim 30 \). Thus, the seed field required has to be at least \( \sim 10^{-20} \text{Gauss} \) on the comoving scale of \( \sim 100 \text{kpc} \) at the time \( t_{gc} \sim 10^{15} \text{sec} \) of gravitational collapse. Since the collapse of the galaxies enhances their frozen-in magnetic field by a factor of \( (\rho_g/\rho_c)^{2/3} \sim 10^3 \) (where \( \rho_g \sim 10^{-24} \text{g cm}^{-3} \) is the typical mass density of a galaxy and \( \rho_c \simeq 2 \times 10^{-29}\Omega h^2 \text{g cm}^{-3} \) is the current cosmic mass density), the above seed field corresponds to a field of the order of \( \sim 10^{-23} \mu \text{Gauss} \) over the comoving scale of \( \sim 1 \text{Mpc} \). Assuming that the rms field scales as \( a^{-2} \) with the expansion of the universe (\( a \) is the scale factor of the universe, \( a \propto t^{2/3} \) in the matter era) we find that the magnitude of the seed field at the start of structure formation has to be at least \( \sim 10^{-22} \mu \text{Gauss} \times (t_{gc}/t_{eq})^{4/3} \sim 10^{-21} \mu \text{Gauss} \), where \( t_{eq} \sim 10^{11} \text{sec} \) is the time of equal matter and radiation energy densities.

The origin and nature of this seed field is still an open question. Many authors have argued that it could be produced by stellar winds and other explosions \([6]\) but such a field would be extremely incoherent over galactic scales. Significant incoherences of the field have been shown \([7]\) to destabilise and destroy the galactic dynamo and, therefore, it seems that coherency is a crucial factor for seeding the galactic magnetic fields. Also, the existing evidence for intergalactic fields \([8][9]\) led people to believe that the origin of galactic seed fields may be truly primordial.

Most of the attempts to create a primordial magnetic field in the early universe involve either inflation or phase transitions because the creation of a primordial field can occur only in out of thermal equilibrium conditions \([8]\).

At phase transitions primordial magnetic fields can be created on the surface of the bubble walls if the transition is a first order one \([10]\), or due to stochastic Higgs-field gradi-
ents [10]. Unfortunately though, phase transitions occur very early and the causal horizon is of much smaller comoving scale than the protogalactic one (e.g. for the electroweak transition the comoving scale of the horizon is $\sim 10^{-3} \text{pc}$). Thus, the generated magnetic field is too incoherent to successfully trigger the galactic dynamo.

Due to this fact, inflation has been considered as another option that could increase the coherency of the created magnetic field [11]. However, the conformal invariance of electromagnetism suggested that the field would scale as $a^{-2}$ during the inflationary period which, as a result, diminished the field strength to much lower values than the required seed field limit. In order to overcome this problem, additional terms have been introduced in the Lagrangian to break explicitly conformal invariance. Even so, most attempts produced too weak seed fields.

Other attempts to generate an adequate seed field involve string theory cosmology [12]. The problem has been pushed even further back in time, into a possible pre-big bang epoch of negative time, where dilaton inflation can take place. These models, although attractive, suffer from lack of understanding of the interface between the dilaton inflation era and the usual, post-big bang radiation era.

Perhaps the most realistic approach to the problem has been the cosmic string scenario, where the magnetic field is generated by vortical motions inside the wakes of cosmic strings [13]. Vortical generation of a magnetic field has been an early idea of Harrison [14], who considered the field to be created during the radiation era inside expanding spinning volumes of plasma (eddies). However, Rees has shown that Harrison’s eddies would be unstable and decay with cosmic expansion, whereas irrotational density perturbations (lumps) from curvature fluctuations would grow and dominate [7]. Rees suggested a different version of vortical magnetic field growth which is similar to Harrison’s idea but can be applied to a gravitationally bound spinning body.

In the cosmic string scenario vorticity is generated in the wakes of cosmic strings after structure formation begins. Therefore, the vortical eddies are gravitationally bound and do not suffer from instabilities. Matter in the trail of a string is substantially ionised and so the Harrison–Rees mechanism can still operate. The scale of coherency of the generated magnetic field is set by the scale of wiggles on the string and, for wakes created at $t_{eq}$ it can be up to $100 \, \text{kpc}$. The field strength is of the order of $\sim 10^{-18} \text{Gauss}$. Thus, cosmic strings are able to generate magnetic fields of enough strength and coherency to seed the galactic dynamo mechanism.

It is not very clear, though, whether stable vortical motions can be generated by the rapid, stochastic, motion of the string wiggles. An alternative mechanism by Avelino and Shellard [15] has overcome this problem by considering dynamical friction. In this model, vorticity is generated not by the wiggles but by the strings themselves, which drag matter behind them and introduce circular motions over inter-string scales. The magnetic field obtained though, is weak $\sim 10^{-23} \text{Gauss}$ and can only marginally seed the galactic dynamo. Also its coherency is very high $\sim 100 \, \text{Mpc}$, which may be incompatible with intergalactic field observations.

In this paper we employ the mechanism of Avelino and Shellard in the context of superconducting strings. It is an increasing belief that cosmic strings may be generically
superconducting and carry substantial currents \cite{16}. Charged current–carrying string networks may evolve in a much different way that the usual, non-supercconducting case. As shown by Dimopoulos and Davis \cite{17}, superconducting networks may be more tangled with slower moving strings. In this case we show that the magnetic field generated can be much stronger than in the non-supercconducting case and still be coherent over protogalactic scales $\sim 1 \text{ Mpc}$.

The structure of this paper is as follows. In Section 2 we give a detailed overview of the field theory of string superconductivity in both the bosonic and fermionic case. In Section 3 we deal with the energy–momentum tensor of the superconducting string, which we use to describe the string spacetime and its consequences on particle deflection in Section 4. In Section 5 we calculate the primordial magnetic field generated. First we find the momentum transfer from the string to the plasma and the resulting rotational velocity of the plasma vortical motions. Then, we estimate the generated primordial magnetic field at the time when structure formation begins, taking also into account constraints coming from the observations of the microwave background anisotropy. Finally, in Section 6 we dicuss our results and give our conclusions. Throughout this paper, unless stated otherwise, we use natural units ($\hbar = c = 1$) for which the Planck mass is $m_P = 1.22 \times 10^{19} \text{GeV}$.

2 String superconductivity

String superconductivity was initially conceived by Witten \cite{18}. There are two types of models that give rise to superconducting strings depending on whether the current carriers are bosons or fermions. However, through a specific formalism, there is exact correspondence between the two cases in most aspects. In what follows, we give a description of bosonic and fermionic superconductivity and develop the general formalism to treat them both.

2.1 Bosonic superconductivity

We will describe bosonic superconducting strings in the context of a $U(1) \times U(1)_{em}$ theory as in \cite{18,19}. This theory involves two scalar fields: The $U(1)$ Higgs–field $\phi$, which is responsible for the formation of the vortex and the $U(1)_{em}$ field $\sigma$ which breaks electromagnetism inside the string and turns it superconducting. The Lagrangian density of the theory is,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + (D_\mu \sigma)^* D^\mu \sigma + (D_\mu \phi)^* D^\mu \phi - V(\phi, \sigma)$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$, and also $D_\mu \sigma = (\partial_\mu + ieA_\mu)\sigma$ and $D_\mu \phi = (\partial_\mu + igW_\mu)\phi$ with $W_\mu$ and $A_\mu$ being the gauge fields coupled to the vortex and the photon field respectively ($e$ and $g$ are the relevant gauge couplings). The potential $V(\phi, \sigma)$ is given by,
\[ V(\phi, \sigma) = \frac{1}{4} \lambda (\phi^2 - \eta^2)^2 + \frac{1}{4} \lambda' \sigma^4 + f \sigma^2 \phi^2 - m^2 \sigma^2 \]  \hspace{1cm} (2)

where \( \lambda, \lambda', f \leq 1 \) are coupling constants and \( \eta \) is the energy scale of the string.

Through the above potential it is possible to form a string by breaking \( U(1) \). For suitable values of parameters the coupling between the scalar fields may force the breaking of electromagnetism inside the string, making it superconducting.

One can rewrite the above potential as,

\[ V(\phi, \sigma) = \frac{1}{4} \lambda (\phi^2 - \eta^2)^2 + \frac{1}{4} \lambda' (\sigma^2 - \sigma_0^2)^2 + f \sigma^2 \phi^2 - m^2 \sigma^2 \]  \hspace{1cm} (3)

where \( \sigma_0 = \sqrt{2/\lambda'} m \) is the expectation value of \( \sigma \) inside the string core.

From the above it follows that, in order for \( U(1) \) to be broken and to form the string we require, \( V(\eta, 0) < V(0, \sigma_0) \Rightarrow \lambda \eta^4 > \lambda' \sigma_0^4 \) which gives the constraint,

\[ \left( \frac{m}{\eta} \right)^4 < \frac{\lambda \lambda'}{4} \]  \hspace{1cm} (4)

Furthermore, in order for electromagnetism not to be also broken outside the string we require that the unbroken state is a global minimum of \( V \), i.e. that \( \frac{\partial^2}{\partial \sigma^2} V(\eta, 0) > 0 \) which suggests that,

\[ \frac{m^2}{f \eta^2} < 1 \]  \hspace{1cm} (5)

In order for electromagnetism to be broken inside the string we require that the minimum of the potential in the core corresponds to a non-zero value of \( \sigma \), i.e. \( \frac{\partial^2}{\partial \sigma^2} V(0, \sigma \neq 0) > 0 \) which gives,

\[ m^2 > 0 \]  \hspace{1cm} (6)

Finally, in order for the \( \sigma \)-condensate to be contained inside the string we require,

\[ m^{-1}_\phi > (\sqrt{\lambda'} \sigma_0)^{-1} \Rightarrow \left( \frac{m}{\eta} \right)^2 > \frac{\lambda}{2} \]  \hspace{1cm} (7)

where \( m_\phi = \sqrt{\lambda} \eta \) is the mass of the Higgs particle. The mass of the \( \sigma \) particle is easily found as,

\[ m^2_\sigma = f \eta^2 - m^2 > 0 \]  \hspace{1cm} (8)

where we have also used (3).

By combining (3) and (5) we obtain,

\[ \lambda < \frac{4}{\lambda} \left( \frac{m}{\eta} \right)^4 < \lambda' \]  \hspace{1cm} (9)
Under the above conditions the theory admits a vortex solution with electromagnetism broken in its core. The $\sigma$ field may be parametrised as,

$$\sigma = \sigma(r) e^{i\psi(z,t)}$$

(10)

where we have assumed that the vortex core lies on the $z$-axis. The phase field $\psi$ may vary randomly along the string and wind up in a non-trivial way. This winding gives rise to the string current.

If one defines the current as, $J_\mu = \frac{\partial L}{\partial A_\mu} = -ie [\sigma^* D_\mu \sigma - \sigma (D_\mu \sigma)^*]$ it is straightforward to show that the string current is, $J_\mu = 2e\sigma(r)^2(\partial_\mu \psi + eA_\mu)$. Integrating over the string core gives,

$$J_a = 2Ke (\partial_a \psi + eA_a)$$

(11)

where the constant $K$ is given by,

$$K \equiv \int d^2 r \sigma^2 \simeq \frac{1}{\lambda'}$$

(12)

Varying (11) with respect to $\phi, \sigma, W_\mu, A_\mu$ and $\psi$ respectively, gives the following equations,

$$\square \phi - gW^\mu W_\mu \phi - \frac{1}{2}\lambda(\phi^2 - \eta^2)\phi - f\sigma^2 \phi = 0$$

(13)

$$\square \sigma - (f\phi^2 - m^2)\sigma - (\partial_\mu \psi + eA_\mu)(\partial^\mu \psi + eA^\mu)\sigma - \frac{1}{2}\lambda'\sigma^3 = 0$$

(14)

$$\partial_\mu G^{\mu \nu} = g^2 \phi^2 W^\nu$$

(15)

$$\partial_\mu F^{\mu \nu} = J^\nu$$

(16)

$$\partial_\mu J^\mu = 0$$

(17)

where $\square \equiv \partial_\mu \partial^\mu$.

The last equation of the above expresses the dynamical conservation of the string current.

### 2.2 Fermionic superconductivity

Following Witten [18] we introduce two fermion fields $\Psi_L$ and $\Psi_R$ which couple to the vortex field $\phi$ through a coupling $\lambda$. Then, the Lagrangian density may be written as [26],

$$\mathcal{L} = \overline{\Psi}_L i \slashed{D} \Psi_L + \overline{\Psi}_R i \slashed{D} \Psi_R - \lambda (\phi \overline{\Psi}_L \Psi_R + h.c.)$$

(18)
\[ \mathcal{D} \equiv \gamma^\mu D_\mu \quad \text{and} \quad D_\mu \Psi = (\partial_\mu + iqA_\mu)\Psi \] with \( \gamma^\mu \) being the Dirac matrices and \( q \) the gauge coupling.

As shown in [18], fermionic superconductivity resembles strongly the bosonic one apart from some minor aspects such as particle production. There is indeed a way to switch from one picture to the other. Let us introduce the scalar field \( y \) such that,

\[ \overline{\Psi} \gamma^a \Psi \equiv \frac{1}{\sqrt{\pi}} \varepsilon^{ab} \partial_a y \] (19)

where \( \varepsilon^{ab} \) is the 2-dimensional Levi-Civita tensor.

Then, with the equivalence, \( q \equiv -\sqrt{2\pi K}e \) the string current is,

\[ J^a \equiv -q \overline{\Psi} \gamma^a \Psi = \sqrt{2K}e \varepsilon^{ab} \partial_a y \] (20)

If we define \( y \) such that,

\[ \partial^a y = \sqrt{2K}e \varepsilon^{ab}(\partial_b \psi + eA_b) \] (21)

then the current in (20) is identified with the one given in (11). As shown in [18] the above definition of \( y \) is consistent. In terms of this formalism we can describe both the fermionic and the bosonic superconducting strings.

### 2.3 The effective action

Non-superconducting cosmic strings are described by the well-known Goto–Nambu action [26],

\[ S = -\mu \int d^2 \xi \sqrt{-\gamma} \] (22)

where \( \mu \) is the energy per unit length of the string, \( \xi^a \) are the string world-sheet coordinates and \( \gamma \) is the determinant of the world-sheet metric \( \gamma_{ab} = \partial_a x^\mu \partial_b x^\mu \). The relation between the spacetime and the world-sheet coordinates may be expressed as,

\[ x^\mu = x^\mu(\xi^a) + n^A_\mu r^A \] (23)

where \( n^A_\mu \) are vectors perpendicular to the string, that form a basis for the 2-dimensional perpendicular space spanned by the \( r^A \) coordinates for which, \( r^2 = r^A r_A \). The relation between the string metric \( \gamma_{ab} \) and the spacetime metric \( g_{\mu\nu} \) to first order is [13],

\[ g_{\mu\nu} \approx \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & \delta_{AB} \end{pmatrix} \] (24)

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1We will use \( a, b, c \) e.t.c. indeces to denote the components on the 2-dimensional world-sheet of the string, whereas \( \lambda, \mu, \nu \), e.t.c. indeces will be used to denote the components in 4-dimensions. The projection of a vector \( V^\mu \) on the string world-sheet is \( V^a = (\partial^a x^\mu)V^\mu \) where \( x^\mu \) are the 4-dimensional space coordinates. In a similar way we can project higher rank tensors. Finally, the \( A, B, C \) e.t.c. indeces will be used for the 2-dimensional space perpendicular to the string.
In the case of superconducting string the action has to account for the existence of a current. In the bosonic picture this, in fact, is equivalent with considering the addition of the $\sigma$ field kinetic–energy term:\[2\]

$$\Delta S_J = \int d^2 \xi d^2 r \sqrt{-\gamma} |D_a \sigma|^2 = \int d^2 r \sigma(r)^2 \int d^2 \xi \sqrt{-\gamma} |\partial_a \psi + e A_a|^2 =$$

$$= -\frac{1}{2} \int d^2 \xi \sqrt{-\gamma} (\partial y)^2$$

(25)

where $(\partial y)^2 \equiv (\partial_a y)(\partial^a y)$ and we have used (10), (12) and (21).

Thus, with the addition of the current term the Goto–Nambu action becomes [19],

$$\Delta S = -\int d^2 \xi \sqrt{-\gamma} [\mu + \frac{1}{2} (\partial y)^2] = -\int d^2 \xi \sqrt{-\gamma} (\mu - \frac{J^2}{4Ke^2})$$

(26)

where we have also used (20).

Including the usual Maxwell terms, the total action for a superconducting string is,

$$S = -\frac{1}{4} \int d^4 x \sqrt{-g} F_{\mu \nu} F_{\mu \nu} - \int d^2 \xi [\sqrt{-\gamma} (\mu - \frac{J^2}{4Ke^2}) + J^a A_a]$$

(27)

In terms of $y$ the above action is [18] [27] \ldots [31],

$$S = -\frac{1}{4} \int d^4 x \sqrt{-g} F_{\mu \nu} F_{\mu \nu} - \int d^2 \xi \{\sqrt{-\gamma} [\mu + \frac{1}{2} (\partial y)^2] - \sqrt{2Ke} \varepsilon^{ab}(\partial_a y)A_b\}$$

(28)

where $g$ is the determinant of the spacetime metric and we have used (20).

The above action can be recovered equivalently using the fermionic action [26],

$$\Delta S_J = -\int d^2 \xi \sqrt{-\gamma} (\bar{\Psi} i \gamma \partial \Psi) = -\int d^2 \xi \sqrt{-\gamma} [\frac{1}{2} (\partial y)^2 - \sqrt{2Ke} \varepsilon^{ab}(\partial_a y)A_b]$$

(29)

where we have used (13).

Varying the action with respect to $x^\mu, A_\mu$ and $y$ yields the following equations of motion respectively,

$$\left[\mu + \frac{1}{2} (\partial y)^2\right] \Box_2 x^\mu - (\partial^\mu y)(\partial_b x^\mu) \Box_2 y - (\partial^a y)(\partial^b y)(\partial_a \partial_b x^\mu) + J_\nu F^{\mu \nu} = 0$$

(30)

$$\Box_2 y = \sqrt{2Ke} \varepsilon^{ab} \partial_a A_b$$

(31)

$$\partial_\mu F^{\mu \nu} = \Box A^\nu = J^\nu$$

(32)

\[2\]The potential energy due to the $\sigma$ field is included into $\mu$. 

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where $\Box_2 \equiv \partial_a \partial^a$ and we have used the Coulomb gauge, $\partial_\mu A^\mu = 0$ for the Maxwell’s equations. Note also that, $2\varepsilon^{ab} \partial_a A_b = \varepsilon^{ab} F_{ab}$.

Using (11) and (20) the field equation (31) reduces to the trivial expression, $\varepsilon^{ab} \partial_a \partial_b \psi = 0$, which in fact led to the introduction of $y$ in [18]. Also, from (21) one can easily show that,

$$\Box_2 \psi = -e \partial_a A^a$$  \hspace{1cm} (33)

Using the above and (11) one can obtain the current conservation equation (17). Employing the Coulomb gauge we obtain, $\partial_a \psi = \text{const.}$, i.e. the gradient of the phase field $\psi$ remains constant along the string [22].

Finally, from (20) and (31) we obtain, $\partial_a J_b = 2Ke^2 F_{ab}$. Taking the time component of this, we find,

$$\frac{\partial J}{\partial t} = 2Ke^2 E$$  \hspace{1cm} (34)

where $E \equiv \varepsilon^{ab} \partial_a A_b$ is the electric field, externally applied on the string.

The above justifies the fact that the strings are considered to be superconducting. However, as shown in [18], (34) breaks down when the current grows very large. Indeed, there is a maximum current over which the string loses its superconducting properties. This occurs when the energy of the current either permits unwindings of the phase field (bosonic case) or allows current carriers to escape from the string (fermionic case). In both cases the maximum current cannot exceed, $J_{\text{max}} \sim e\sqrt{\mu}$ [18].

### 2.4 Back-reaction and external fields

The self-inductance of the string may be defined as follows [20],

$$L \equiv \frac{1 + 4Ke^2 \ln(\Lambda R)}{2Ke^2} \approx 2 \ln(\Lambda R)$$  \hspace{1cm} (35)

where $\Lambda^{-1} \sim (\sqrt{\lambda} \eta)^{-1}$ is the string width and $R$ is a suitable cut-off radius usually taken as the curvature radius of the string or the inter-string distance of the string network. In the above $\ln(\Lambda R) \approx 100 \gg 1$.

Considering self-inductance effects (34) is modified as [26],

$$\frac{dJ}{dt} = 2Ke^2 (E - L \frac{dJ}{dt}) = 2Ke^2 E$$  \hspace{1cm} (36)

where the renormalised charge is,

$$\tilde{e}^2 \equiv \frac{e^2}{1 + 2Ke^2 L} \approx \frac{1}{2KL}$$  \hspace{1cm} (37)

Considering back-reaction effects the photon field may be expressed as [18] [30],

$$\hat{A}^\mu = 2 \int d^2 \xi \sqrt{-\gamma} J^a(\xi) \partial_\alpha x^\mu(\xi) \Theta[x^0 - x^0(\xi)] \delta([x - x(\xi)]^2)$$  \hspace{1cm} (38)
where the step function $\Theta$ signifies the initiation of the current.

After some algebra the above reduces to [18] [19] [26],

$$\hat{A}^a = -2 \ln(\Lambda R) J^a \simeq -LJ^a$$  (39)

Using the above, (24) and (31) we obtain [26],

$$\Box_2 \tilde{y} = \frac{1}{2} \sqrt{2K} \tilde{e} \varepsilon_{ab} \hat{F}^{ab}$$  (40)

which is the renormalised version of (31). In the above $\tilde{y} \equiv (e/\tilde{e})y$ and $\hat{F}^{ab}$ is the external field strength. The right-hand side of (40) is the Lorentz force on the string since $\varepsilon_{ab} \hat{F}^{ab} = \varepsilon^{0a}[E_a + \varepsilon_{aij}(\partial_0 x^i)B^j]$, where $E_a$ and $B_i$ are the external electric and magnetic fields respectively [26].

Similarly, for the string current we have [28] ... [31],

$$J^\mu = \int d^2 \xi \sqrt{-\gamma} J^a \partial_a x^\mu(\xi) \delta^{(4)}(x - x(\xi)) =$$

$$= \int d^2 \xi \sqrt{-\gamma} \varepsilon_{ab} \partial_b \tilde{y} \partial_a x^\mu(\xi) \delta^{(4)}(x - x(\xi))$$  (41)

From the above, since the quantity $ey$ is unaltered by renormalisation, we conclude that the string current is unaffected by charge renormalisation. Also, (40) suggests that, in the absence of external fields $\Box_2 \tilde{y} = 0$ [28]. Using the renormalised version of (24) we can immediately obtain current conservation.

Equivalently, using (31) and (39) one can show that,

$$J_a = \frac{1}{\tilde{e}L} \partial_a \psi$$  (42)

which again suggests that the current is conserved since $\partial_a \psi = \text{const.}$.

Finally, we can obtain the renormalised version of the action by considering the inductance energy of the string [20], $\mathcal{E} = \frac{1}{2}LJ^2$. Thus, the addition to the Goto–Nambu action due to the string current is,

$$\Delta S = \frac{1}{2} \int d^2 \xi \sqrt{-\gamma} LJ^2$$  (43)

Adding the above to (22) and using also (37) we find,

$$\Delta S = \int d^2 \xi \sqrt{-\gamma} \left(\mu - \frac{J^2}{4K\tilde{e}^2}\right)$$  (44)

which is the renormalised analogue of (20).

In a similar way we can regard the total action in (27) and (28) by considering the external fields and the renormalised values of the physical quantities. From now on, unless stated otherwise, we will refer to the renormalised values of the latter. For economy, we will drop the tildes and hats.
2.5 The string current

Suppose that the string is lying along the $z$-axis. Then, $J^\mu = \delta^2(r)J^\mu(z, t)$. With the use of (20) we find [29],

$$J^\mu(z, t) = \sqrt{2Ke} \left( \frac{\partial y}{\partial z}, 0, 0, -\frac{\partial y}{\partial t} \right)$$  \hspace{1cm} (45)

If $\rho_\epsilon$ is the charge density inside the string then, $\frac{\partial}{\partial t}\rho_\epsilon \equiv J_z$ and $\frac{\partial}{\partial z}\rho_\epsilon \equiv J_0$. In view of (15) we can identify,

$$\rho_\epsilon = -\sqrt{2Ke}\frac{q}{\sqrt{\pi}}y$$  \hspace{1cm} (46)

that is, the field $y$ is a measure of the charge density along the string.

Now, the Maxwell’s equations give,

$$\frac{1}{r} \frac{\partial}{\partial r}(r\partial_r A^\mu) = 4\pi J^\mu\delta^2(r) \Rightarrow \partial_r A^\mu = \frac{2J^\mu}{r}$$  \hspace{1cm} (47)

Using the above we can find the electric and magnetic field around the string,

$$E_r = \frac{2J^0}{r}$$  \hspace{1cm} (48)

$$B_\theta = \frac{2J^z}{r}$$

The last of the above is the well-known Biot–Savart law for a line current.

Assuming now that, the charge density along the string is uniformly distributed, $J_0 = 0$ and $J^\mu = (0, 0, 0, J)$. Then, the electric field vanishes and the only non-zero components of the electromagnetic field strength are [25] [31],

$$F_{Az} = \frac{2J}{r}(\frac{r^A}{r})$$  \hspace{1cm} (49)

3 The energy–momentum tensor

3.1 The string current component

Using (20) the action (25) may be written as,

$$\Delta S_J = \frac{1}{4Ke^2} \int d^2\xi \sqrt{-\gamma} J^2$$  \hspace{1cm} (50)

Varying the above action with respect to the world-sheet metric we obtain the energy–momentum tensor of the string current [19] [20].
\[ \Theta^{ab} \equiv \frac{1}{\sqrt{-\gamma}} \frac{\delta \Delta S_f}{\delta \gamma_{ab}} = \frac{1}{4Ke^2} [2J^a J^b - \gamma^{ab} J^2] = \]
\[ = \frac{1}{2} (\partial y)^2 \gamma_{ab} + (\partial_a y)(\partial_b y) \quad (51) \]

It can be easily checked that the above tensor is traceless. Using (40) we find,

\[ \partial_b \Theta^b_a = \partial_a y \Box_2 y = \frac{1}{2} \gamma^a_{\mu} F_{\mu \nu} \partial_a x^\nu \quad (52) \]

In the absence of external fields the above becomes, \( \partial_b \Theta^b_a = 0 \). In view of (51) this gives,

\[ J \cdot E = 0 \quad \text{and} \quad J \times B = 0 \quad (53) \]

which are obviously satisfied by the self-fields (48) of the string.

### 3.2 With no electromagnetic back-reaction

From (28) the energy–momentum tensor of the string is given by [26],

\[ T^{ab} = \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = -\gamma^{ab} [\mu + \frac{1}{2} (\partial y)^2] + \partial_a y \partial_b y \Rightarrow \]

\[ T^\mu_{\nu} = -(\mu \gamma^{ab} - \Theta^{ab}) \partial_a x^\mu \partial_b x^\nu \quad (54) \]

Using (71) with \( J^\mu \equiv (0, 0, 0, J) \) it is easy to see that [31],

\[ \Theta^0_0 = -\Theta^z_z = -\frac{J^2}{4Ke^2} \quad (55) \]

Thus, we can write (54) as [19],

\[ T^\mu_{\nu} = -\delta^{(2)}(r) \text{diag}(\mu + \frac{J^2}{4Ke^2}; 0, 0, \mu - \frac{J^2}{4Ke^2}) \quad (56) \]

Note that the above is a core solution which does not include electromagnetic back-reaction effects.
3.3 With electromagnetic back-reaction

The electromagnetic energy–momentum tensor is,

\[ T_{\mu\nu}^{em} = F_{\lambda}^{\mu} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \]  

(57)

where \( F^{\mu\nu} \) refers to the self-fields of the string.

Using (49) one easily finds,

\[ T_{00}^{em} = T_{zz}^{em} = \frac{2 J^2}{r^2} \]  

(58)

\[ T_{AB}^{em} = \frac{2 J^2}{r^2} \left( \frac{2 r^A r^B}{r^2} - \delta^{AB} \right) \]

With a suitable rotation of the \( r^A \) coordinates (58) can be written as \([23][24],\)

\[ [T_{em}(r)]^\mu_\nu = \frac{2 J^2}{r^2} \text{diag}(-1, 1, -1, 1) \]  

(59)

The above tensor is traceless but not well-defined at \( r \to 0 \). In order to overcome this problem we employ a method introduced by Linet \([31]\) and expand the components of \( T_{\mu\nu}^{em} \) as distributions around the string core (see also \([25]\)). We use the fact that

\[ 2/r^2 = \Delta_2 [\ln(r/r_0)]^2 \]  

where \( \Delta_2 \equiv 1/r \partial_r (r \partial_r) \) is the 2-dimensional Laplacian and \( r_0 \) is the radius of the string core.\(^3\) The energy–momentum tensor (58) is, then, written as,

\[ T_{00}^{em} = T_{zz}^{em} = J^2 \Delta_2 [\ln(r/r_0)]^2 \]  

\[ T_{AB}^{em} = -2 J^2 \partial_A \partial_B [\ln(r/r_0)] \]  

(60)

However, since \( \Delta_2 \ln(r/r_0) = \frac{1}{2} \delta^{(2)}(r) \), one finds, \( \partial_A T_{em}^{AB} = -J^2 \delta^{AB} \delta^{(2)}(r) \) and \( T_{\mu\nu}^{em} \) is not conserved. In order to have conservation we need to add the term \( J^2 \delta^{AB} \delta^{(2)}(r) \) on \( T_{em}^{AB} \). The addition of this term suggests that the core solution (56) for the string energy–momentum tensor becomes \([32][31]\).

\[ T_{\mu}^{\nu} = -\delta^{(2)}(r) \text{diag}(\mu + \frac{J^2}{4 K e^2}, -\frac{J^2}{2}, \frac{J^2}{2}, \mu - \frac{J^2}{4 K e^2}) \]  

(61)

The above is a core solution that takes into account electromagnetic back-reaction. As can be seen in \([31]\), the string current increases the linear mass density \( \mu \) of the string by a factor \( \frac{1}{2} L J^2 \) which corresponds to the self-inductance energy density of the current.\(^3\)

\(^3\)The core radius is defined as the distance where the string gauge fields assume their long-range logarithmic behaviour whereas the rest of the vortex fields become negligible. The above results are also in agreement with \([23]\) if one takes into account that \( A(r_0) = -L J = -J/2 K e^2 \).
Also, it generates pressure due to the inertia of the charge carriers \(^{[24]}\), both towards the perpendicular direction (\(T_1^1\) and \(T_2^2\) terms) and along the string. The latter also reduces the total string tension \(T_z^z\).

In fact, \((61)\) suggests that, for high enough currents, the string tension may be diminished to zero. In that case the strings would not have a driving force to untangle. Also, string loops would not collapse, but remain stable and form the so called “springs” \([19]\)\(^{20}\)\(^{33}\). However, it has been argued \([22]\) that the back-reaction of the current would shrink the \(\sigma\)-condensate in the string core (in the bosonic case) thus reducing the pressure from the charge carriers that counteracts the string tension. In overall, the string tension may not decrease as rapidly due to the current as implied by \((61)\). Indeed, numerical simulations \([34]\) have shown that the string tension is little affected by changes of the string current. It would be more accurate, then, to express the core solution as,

\[
T^\mu_\nu = -\delta^{(2)}(r) \text{diag}(U, W, W, T) \quad (62)
\]

where \(W = -J^2/2\) \(^{[25]}\). Then the overall solution for the string energy–momentum tensor may be written as,

\[
\begin{align*}
T^{00}_\text{em} &= U \delta^{(2)}(r) + J^2 \Delta_2 [\ln(r/r_0)]^2 \\
T^{zz}_\text{em} &= -T \delta^{(2)}(r) + J^2 \Delta_2 [\ln(r/r_0)]^2 \\
T^{AB}_\text{em} &= J^2 \delta^{AB} \delta^{(2)}(r) - 2J^2 \partial_A \partial_B [\ln(r/r_0)]
\end{align*}
\quad (63)
\]

Thus, the following should be regarded as extreme estimates,

\[
U \simeq \mu + \frac{J^2}{4Ke^2} \quad (64)
\]

\[
T \simeq \mu - \frac{J^2}{4Ke^2} \quad (65)
\]

4 The string spacetime

4.1 The string metric

Using \((63)\) one can solve the Einstein equations,

\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_\lambda^\lambda = 8\pi G T^{\mu\nu} \quad (66)
\]

where \(G = m_p^2\) is Newton’s gravitational constant.

In first order in \(G\) the metric is found to be \(^{[23]}\)\(^{24}\),
\[ ds^2 = \{1 + 4G[J^2 + (U - T)] \ln(r/r_0) + 4GJ^2[\ln(r/r_0)]^2\}(-dt^2 + dr^2) \]
\[ + \{1 - 8G(U + \frac{J^2}{2}) - 4G[J^2 - (U - T)] \ln(r/r_0) + 4GJ^2[\ln(r/r_0)]^2\}r^2d\theta^2 \]
\[ + \{1 + 4G[J^2 - (U - T)] \ln(r/r_0) - 4GJ^2[\ln(r/r_0)]^2\}dz^2 \] (67)

where \( J = 2W \).

The metric of the spacetime perpendicular to the string may be found by setting, \( dz = 0 \). Then the above gives,
\[ ds^2_\perp = (1 - h_{00})[-dt^2 + dr^2 + (1 - b)r^2d\theta^2] \] (68)
where,
\[ h_{00} = -4G[J^2 + (U - T)] \ln(r/r_0) - 4GJ^2[\ln(r/r_0)]^2 \] (69)

and to first order in \( G \),
\[ 1 - b = \frac{1 - 8G(U + J^2/2) - 4G[J^2 - (U - T)] \ln(r/r_0) + 4GJ^2[\ln(r/r_0)]^2}{1 + 4G[J^2 + (U - T)] \ln(r/r_0) + 4GJ^2[\ln(r/r_0)]^2} \]
\[ \simeq 1 - 8G(U + \frac{J^2}{2}) - 8GJ^2 \ln(r/r_0) \] (70)

From (68) we see that the spacetime around a superconducting string resembles the conical spacetime of a non-superconducting string. Indeed, if we set \( J = 0 \) and \( U = T = \mu \), then \( h_{00} = 0 \) and \( (1 - b) = (1 - 8G\mu) \), and (68) reduces to the non-superconducting string, conical–spacetime solution with deficit angle \( \delta = 8\pi G\mu \). In the case of non-vanishing current, though, the deficit angle is,
\[ \delta = b\pi = 8\pi G\{U + J^2[\frac{1}{2} + \ln(r/r_0)]\} \] (71)

Thus, the spacetime is not exactly conical since \( \delta \) is dependent on \( r \). However, this logarithmic dependence is very weak and, if one is interested in astrophysical effects (such as primordial magnetic field generation) then the logarithmic dependence may well be approximated as,
\[ \ln(r/r_0) \simeq \ln(\Lambda R) \] (72)

Under this approximation (71) is in good agreement with the more rigorous calculations of [23] [27]. Now, using (64) and (37) we find,
\[ \delta \simeq 8\pi G[\mu + \frac{1}{2}(QJ)^2] \]  

(73)

where,

\[ Q \equiv \sqrt{1 + 4\ln(\Lambda R)} \simeq 20 \]  

(74)

### 4.2 The gravitational field

From (67) it can be seen that the string metric deviations from Minkowski spacetime are of the order of \( G\mu \leq 10^{-6} \). Thus, we can use linear theory to approximate the gravitational field. The metric, then, can be written as,

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]  

(75)

where \( \eta_{\mu\nu} = (-1, 1, 1, 1) \) is the Minkowski metric.

The geodesic equation is,

\[ \frac{d^2 x^i}{d\tau^2} + \Gamma^i_{00} = 0 \]  

(76)

where \( i \) denotes the spacial coordinates and \( \tau \simeq t \) is the proper time. Since, \( \Gamma^i_{00} = -\frac{1}{2}\partial_i h_{00} \) we find,

\[ f = \frac{1}{2} \nabla h_{00} \]  

(77)

Inserting (69) into the above we obtain,

\[ f(r) = -\frac{2GJ^2}{r} \left[ 1 + \frac{U - T}{J^2} + 2\ln(r/r_0) \right] \]  

(78)

Using (37), (64), (65) and the approximation (72) we find,

\[ f(r) \simeq -\frac{2G(QJ)^2}{r} \]  

(79)

The above is an attractive gravitational force similar to the one of a massive rod of linear mass density \( \sim (QJ)^2 \).

In (79) we have used the extreme values (64) and (65) of the string linear mass density and tension. As mentioned, though, the string tension may be larger than suggested by (65). In that case \( Q \) in the expression (79) would decrease and the gravitational pull of the string will be weakened since it depends on the difference \( (U - T) \) as shown in (78). However, (71) suggests that, even in the extreme case, when \( U = T \), the attractive force is decreased only by a factor of 2.
4.3 Particle deflection

Writing the metric (68) in cartesian coordinates we have,

$$\text{ds}_⊥^2 = (1 - h_{00})(-dt^2 + dx_kdx^k)$$  \hfill (80)

where the $k = 1, 2$ and we have assumed that the string lies on the $z$-axis. Note that we need to extract from the above a wedge of angle width $\delta$ \[35\]. Then,

$$d\tau^2 = -d^2s_⊥ = (1 - h_{00})dt^2(1 - \dot{x}_k\dot{x}^k)$$  \hfill (81)

where the dots signify derivation with respect to time.

Using $h_{00} \ll 1$ and $\Gamma^i_{00} = -\frac{1}{2}\partial_i h_{00}$, we insert the above to the geodesic equation (76) and obtain \[35\],

$$2\ddot{x}^i = (1 - \dot{x}_k\dot{x}^k)\partial^i h_{00}$$  \hfill (82)

where $i = 1, 2$.

The above gives the acceleration felt by the particles due to the gravitational pull of the string field, in the frame of the string.

Suppose that the string moves in the $x$-direction with constant velocity $-v = -\sqrt{\dot{x}_k\dot{x}^k}$. Then for a particle in the position $(x, y)$ we have initially, $x = vt$ and $y = \text{const}$. The velocity boost felt by the particle towards the $y$-direction after its encounter with the string is \[26\],

$$v_y = \int_{-\infty}^{\infty} \dot{y}dt = \frac{1}{2v\gamma^2} \int_{-\infty}^{\infty} \partial_y h_{00}dx$$  \hfill (83)

where $\gamma^{-1} \equiv \sqrt{1 - v^2}$ is the Lorentz factor and we have also used (82).

Using (69) and (72) the above gives,

$$v_y = -\frac{2\pi G(QJ)^2}{v\gamma^2}$$  \hfill (84)

Taking into account the deficit angle, the relative boost between two particles on the opposite sides of the string is, $\Delta v_y = v\delta + 2|v_y|$. Switching to the particle frame gives,

$$u = \gamma \Delta v_y = 8\pi G\mu v\gamma + 4\pi G(QJ)^2(v\gamma + \frac{1}{v\gamma})$$  \hfill (85)

where we have also used (73).

4.4 Deceleration of the string

Avelino and Shellard \[15\] have realised that the deflection of particles in the string space-time results into a net drag of the plasma behind the string. This is due to the fact that the magnitude of the particle velocity is unaltered after its interaction with the string. The back-reaction of this effect is a decelerating force on the string. The velocity of plasma
dragging may be estimated by Taylor expansion of the particle velocity. In the lowest order in $G$ we find,

$$\delta v_x \simeq -\frac{1}{2} \frac{u^2}{v} = -32\pi^2 G^2 \mu v \gamma^2 [\mu + J^2 (1 + \frac{1}{v^2 \gamma^2})] - 8\pi^2 G^2 (QJ)^4 v \gamma^2 (1 + \frac{1}{v^2 \gamma^2})$$ (86)

The momentum change of the string is, $dp = (\rho \delta v_x) dx dy dz$, where $\rho = 3/32\pi G t^2$ is the energy density of the universe with $\Omega = 1$. Thus, the drag force per unit length on the string is,

$$f_x = \int \frac{dp}{dt dy} dx dy = 2 R \rho v \delta v_x$$ (87)

where $R \sim vt$ is the inter-string distance of the network [17].

Inserting (86) into the above we find,

$$f_x = -6\pi G H v \{2\mu^2 v^2 \gamma^2 + \frac{1}{2} (QJ)^4 (v \gamma + \frac{1}{v \gamma})^2 + 2\mu J^2 (v^2 \gamma^2 + 1)\}$$ (88)

where $H^{-1} = 2t$ is the Hubble radius.

The above force should be compared with the plasma friction force on the strings, $f_s = \max \{\rho v/m, Jv\sqrt{\rho}\}$, [17] where $m$ is the particle mass. It is easy to show that throughout the range of $J$ the above drag force is always subdominant and, thus, it does not affect the network evolution.

### 5 Primordial magnetic field generation

Although the motion of the strings is not significantly affected by the dragging of the plasma, the latter may gain substantial momentum during this process. Such momentum may introduce turbulence which could generate a primordial magnetic field.

#### 5.1 Momentum transfer

In the one scale model the scale of the string network inter-string distance is comparable to the string curvature radius, $R \sim vt$. A string segment of length $\sim R$ may transfer momentum to the plasma contained in an inter-string volume $\sim R^3$. In that way the network may introduce vortical motions to the plasma on inter-string scales. An estimate of the plasma rotational velocity $v_{rot}$ may be obtained as follows,

4For example, in the case of non-superconducting strings ($J = 0$) the drag force dominates string friction at the temperature, $T_f \sim (G\mu)^2 m_P$. Comparing this with the temperature $T_* \sim (G\mu) m_P$ when the network reaches the horizon scaling solution [17], we see that $f_x$ has no effect during the friction era of the network. During horizon scaling we can compare $f_x$ with the string tension $\sim (\mu/t)$ and verify that the latter dominates at all times.

17
\[ FR \simeq \frac{1}{2} M v_{\text{rot}}^2 \]  

where \( M \sim \rho R^3 \) is the mass of an inter-string volume and \( F \) is the total force on the plasma by a string segment of length \( R \). From the above the rotational velocity is estimated as,

\[ v_{\text{rot}} \sim \frac{\sqrt{GF}}{v} \]

The total force may be obtained from (88) as follows,

\[ F = \int_0^R f_x dz = v tf_x \Rightarrow \]

\[ F = 3\pi Gv^2[2\mu^2v^2\gamma^2 + \frac{1}{2}(QJ)^4(v\gamma + \frac{1}{v\gamma})^2 + 2\mu J^2(v^2\gamma^2 + 1)] \]

As shown in [17], for currents greater than a critical value \( J_c \) the network can never exit the friction domination era of its evolution. In [17] \( J_c \) is estimated as,

\[ J_c \sim \sqrt{G\mu} \]

However, even for such strong currents the network does reach a scaling solution during which the strings move with a terminal velocity given by [17],

\[ v_T^2 \sim \frac{G\mu}{\sqrt{GJ^2}} \]

As implied in [17] the string current is influential on the behaviour of the network only if \( J \geq J_c \). Thus, we will consider this case only. Using (93) the total force (91) becomes,

\[ F = 3\pi[2(G\mu)^2(\frac{\mu}{J^2})\mu + \frac{1}{2}Q^4(G\mu)^2J^2 + \frac{1}{2}Q^4(GJ^2)^2 + Q^4(GJ^2)^2 + 2\mu J^2(\mu^2 + 2\mu^2 \gamma^2 + 2\mu^2)] \]

In the above the 2nd, the 4th and the 5th terms of the right hand side may be disregarded by means of the condition \( J > J_c \). By comparing the remaining terms with each other it can be shown that the last term remains also always subdominant. Therefore, the above can be written as,

\[ F = 6\pi J^2[(G\mu)^2(\frac{\mu}{J^2})^2 + \frac{1}{2}Q^4(GJ^2)] \]

5Note that \( J < J_c \Rightarrow v = 1 \).
\[
F \simeq \begin{cases} 
3\pi G(QJ)^4 & J_{c1} < J \leq J_{\text{max}} \\
6\pi (G\mu)^2 (\mu/J^2)\mu & J_{c} < J \leq J_{c1} \\
6\pi G\mu^2 & J \leq J_{c}
\end{cases}
\]

where the critical current \( J_{c1} \) is,

\[
J_{c1} \equiv Q^{-2/3}(G\mu)^{1/6} \sqrt{\mu}
\]  

For currents stronger that the above value the current dependent term in (85) becomes dominant, i.e. the gravitational pull of the string is actually felt by the particles.

Inserting (96) into (90) and using also (93) we find,

\[
v_{\text{rot}} \simeq \begin{cases} 
\sqrt{3\pi}Q^2(G\mu)^{-1/4}(GJ^2)(\mu/J^2)^{-1/4} & J_{c1} < J \leq J_{\text{max}} \\
\sqrt{6\pi}(G\mu)^{5/4}(\mu/J^2)^{1/4} & J_{c} < J \leq J_{c1} \\
\sqrt{6\pi}(G\mu) & J \leq J_{c}
\end{cases}
\]

5.2 The Harrison–Rees magnetic field

Harrison [14] suggested first that turbulence in an expanding universe may generate a magnetic field since the turbulent velocity of the plasma would be different for the ions and the electrons.

Consider a rotating volume \( V \) of plasma. Suppose that the angular velocities \( \omega_m \) and \( \omega_r \) of the ion and the electron fluid respectively are uniform inside \( V \). Then, since \( V \propto a^{-3} \), we find that,

\[
\rho_m V = \text{const.} \quad \text{and} \quad \rho_r V^{4/3} = \text{const.}
\]

where \( \rho_m \propto a^{-3} \) is the ion density, which scales as pressureless matter, and \( \rho_r \propto a^{-4} \) is the electron density, which scales as radiation due to the strong coupling between the electrons and the photons through Thompson scattering. The angular momentum \( I = \rho \omega V^{5/3} \) of each plasma component has to be conserved. This suggests that,

\[
\omega_m \propto V^{-2/3} \propto a^{-2} \quad \text{and} \quad \omega_r \propto V^{-1/3} \propto a^{-1}
\]

Thus, the ion fluid spins down faster than the electron-photon gas. Consequently, a current is generated which creates a magnetic field in the volume \( V \).

Rees, however, has shown that expanding volumes of spinning plasma are unstable and decay with cosmic expansion compared to irrotational density perturbations [1]. He
suggested instead a different version of vortical magnetic field generation involving the scattering of the electrons on the microwave background radiation. This would tend to damp the vortical motions of the electrons in contrast to the ions which would stay unaffected. The result is again the generation of currents but, this time, it is the electron fluid that slows down. The mechanism applies to gravitationally bound spinning bodies such as those formed when structure formation begins.

In both cases, the Maxwell’s equations suggest \[ B \simeq -\frac{m}{e}w \] (101)

where \( m \sim 1 \text{GeV} \) is the nucleon mass and \( w \) is the vorticity given by,

\[ w = \nabla \times v_{\text{rot}} \] (102)

From the above the magnetic field generated over inter-string distances is,

\[ B \simeq \frac{m}{e} \left( \frac{v_{\text{rot}}}{R} \right) \] (103)

The turbulence inside the wake of cosmic strings is expected to ionise the plasma even after decoupling [26]. Also for superconducting cosmic strings the existence of a magnetocylinder around the core [17] is expected to induce further charge separation due to the difference of the inertia of the scattered particles. Since \( R \propto t \), from (103) it is evident that the stronger field will be generated at early times. Thus, we will calculate the magnetic field generated at \( t_{\text{eq}} \sim 10^{11} \text{sec} \), the time of equal matter and radiation densities, since this is the earliest that large scale streaming of the plasma is possible. From (98) and (5) we obtain,

\[ B_{\text{eq}} \simeq \frac{\sqrt{6\pi} m}{et_{\text{eq}}} \left\{ \begin{array}{c} \frac{1}{\sqrt{2}} Q^2 (G\mu)^{-1/2} (GJ^2)(\mu/J^2)^{-1/2} \quad J_{\text{cl}} < J \leq J_{\text{max}} \\ \frac{G\mu}{J} \quad J \leq J_{\text{cl}} \end{array} \right. \] (104)

For \( J = J_{\text{max}} \simeq \sqrt{\mu} \) the above gives,

\[ B_{\text{eq}}^{\text{max}} \sim 10^{-13} \sqrt{G\mu} \text{ Gauss} \] (105)

which is coherent over comoving scales,

\[ l \sim (v_T t_{\text{eq}}) \left( \frac{t_{\text{pr}}}{t_{\text{eq}}} \right)^{2/3} \sim 10^2 (G\mu)^{1/4} \text{Mpc} \] (106)

where \( t_{\text{pr}} \sim 10^{18} \text{sec} \) is the present time.

For energy scales corresponding to a grand unified theory (GUT) phase transition, \( G\mu \sim 10^{-6} \) and the above suggest that \( B_{\text{eq}}^{\text{max}} \sim 10^{-16} \text{Gauss} \) with a coherency of the order of \( l \sim 1 \text{Mpc} \) which is quite sufficient to seed the galactic magnetic fields. However, as we discuss below, this estimate may be over-optimistic.
5.3 Temperature anisotropy constraint

The deficit angle of the string spacetime apart from deflecting the trajectories of particles affects light propagation as well [26]. From (68) by setting $ds = 0$ we see that photons are boosted by the deficit angle even though the prefactor $(1 - h_{00})$ is irrelevant. As a result a string moving in front of radiation will blueshift light due to the Doppler effect. Thus, a string network is expected to generate temperature anisotropies on the microwave background radiation. These cannot exceed the observed values,

$$\frac{(\Delta T)}{T}_{\text{rms}} \leq 10^{-6} \quad (107)$$

The anisotropy generated by a single string is [20],

$$\frac{(\Delta T)}{T}_s \sim \delta v \gamma \quad (108)$$

The rms anisotropy due to a network of cosmic strings is estimated as [17] [30],

$$\frac{(\Delta T)}{T}_{\text{rms}} \sim \frac{H}{R} \left(\frac{\Delta T}{T}_s\right) \quad (109)$$

From the above and (73) we find that for superconducting cosmic strings,

$$\frac{(\Delta T)}{T}_{\text{rms}} \sim \delta \sim G[\mu + \frac{1}{2}(QJ)^2] \quad (110)$$

Thus, for $G\mu \leq 10^{-8}$ even the maximum current could not challenge the observations. However, for GUT strings one cannot allow the current to reach its maximum value. Indeed, there is some doubt whether superconducting strings may attain the maximum current [24] since the energy density of the current $\frac{1}{2}LJ^2 \simeq \ln(\Lambda R)J^2$ should not exceed the string linear mass density $\mu$ [29]. This implies that the current term in (110) should always remain subdominant. Therefore,

$$J \leq J_{c2} \equiv \frac{\sqrt{\mu}}{Q} \quad (111)$$

By evaluating the (104) with $J = J_{c2}$ we obtain the maximum permissible magnetic field for GUT energy scales,

$$B_{eq}^{\text{max}} \sim 10^{-15}Q^{-1}\sqrt{G\mu} \text{Gauss} \sim 10^{-19} \text{Gauss} \quad (112)$$

with coherence length,

$$l \sim 10^2 \sqrt{Q} (G\mu)^{1/4} \text{Mpc} \sim 1 \text{Mpc} \quad (113)$$

The above field is of sufficient strength and coherency to seed the galactic dynamo process and generate the observed galactic magnetic fields.
6 Discussion and conclusions

We have shown that GUT superconducting cosmic strings are able to generate turbulence on inter-string scales, which gives rise to a primordial magnetic field of enough strength to seed the dynamo process in galaxies and account for the observed galactic magnetic fields. Moreover, the generated field is coherent over very large scales, comparable with the protogalactic ones before gravitational collapse commenced. Turbulence and coherent rotation on these scales may also be related to the fragmentation process of galaxy formation. Furthermore, the existence of a primordial field, coherent over an entire protogalaxy, may have played a crucial role in removing angular momentum in a similar way as in the case of the collapse of protostellar clouds [5][37].

Due to excessive microwave temperature anisotropies, a maximum string current may not be acceptable. However, even under this constraint, the magnetic field generated is still adequate to seed the galactic dynamo. A possibly stronger constraint may be implemented by considering the density inhomogeneities due to the string wakes, since their growth is enhanced not only due to the deficit angle but also, due to the gravitational pull of the strings [26]. However, the magnitude of the overdensities is strongly dependent on the dark matter model used. Indeed, string wake overdensities may be substantially suppressed in the case of hot dark matter, due to free streaming [26].

In our estimates of the generated magnetic field we have used the extreme values (64) and (65) for the string linear mass density and tension. As we mentioned though, the string tension may be larger than suggested by (65). This however, would relax the temperature anisotropy constraint and the density inhomogeneity constraint even more.

We have implicitly assumed that the string current remains constant during the network evolution. Indeed, the current is dynamically conserved as shown by (17). Also, in the bosonic case (42) suggests that current conservation is ensured on topological grounds. However, most of the field theory of Section 2, concerns a straight and infinite superconducting string. A realistic string may be much more complicated. Still, it can be shown [17] that even in this case the string current remains constant when the network reaches a scaling solution.

Our treatment is similar to this of Avelino and Shellard [15]. In their work, however, they consider the case of wiggly strings, which also have an attractive gravitational field due to the difference between their renormalised linear mass density and tension. The latter is a result of the tangled shape of wiggly strings, which, in effect, increases the string length (and thus, the linear mass density) between two fixed points on the string while also decreasing the tension due to the random orientation of the string microstructure [38]. For superconducting strings, microstructure is suppressed by electromagnetic radiation emission [39] and the difference between the linear mass density and tension is due to the existence of a current, which increases the energy content of the string while generating pressure and reducing, thus, the string tension [24]. Therefore, the gravitational field of superconducting strings arises in a qualitatively different way that the one of wiggly strings. Moreover, there are also quantitative differences between the two cases, since, for strong enough currents, a superconducting string network reaches a different scaling solution, due
to excessive plasma friction [7]. As a result, the inter-string distance is much smaller than
the horizon and the strings are slow moving. The above enable superconducting strings
to generate much stronger magnetic fields than wiggly strings. In fact, since the string
microstructure is suppressed by the existence of a current, wiggly strings may be regarded
as the limit of superconducting strings when the current is very small.

The superconducting string model provides a realistic mechanism for primordial mag-
netic field generation. The field generated by natural (GUT-scale) values of the parameters
is coherent over protogalactic scale and strong enough to seed the dynamo process in galax-
ies. Thus, our mechanism may be considered as a prime candidate to explain the observed
galactic magnetic fields.

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References

[1] E.N. Parker, “Galactic Dynamos and other questions on the origin of Magnetic
Fields” in Proceedings of Critical Dialogues in Cosmology. ed. N. Turok, World
Scientific, in press.

[2] L. Mestel and K. Subramanian, Mon. Not. R. Astr. Soc. 248 (1991) 677;
G. Rüdiger, D. Elstner and M. Shultz, “Galactic dynamo: Modes and models” in
The Cosmic Dynamo, ed. F. Krause, K.H. Rädler and G. Rüdiger, Kluwer Aca-
demic Publishers, Dordrecht, Holland, p. 321.

[3] P.P. Kronberg, Rep. Prog. Phys. (1994) 325.

[4] S.I. Vainstein and A.A. Rutzmaikin, A. Zh. 48 (1971) 902; H.K. Moffat,
Magnetic Field Generation in Electrically Conducting Fluids, CUP 1978;
E.N. Parker, Cosmological Magnetic Fields, OUP, Oxford 1979; A.A. Ruzmaikin,
A.A. Shukurov and D.D. Sokoloff, Magnetic Fields of Galaxies, Kluwer, Dordrecht
1988; S.I. Vainstein and A.A. Rutzmaikin, Sov. Astr. 15 (1992) 714; R. Beck,
A. Brandenburg, D. Moss, A. Shukurov and D. Sokoloff, An. Rev. Astron. Ast-
rophys. 34 (1996) 155.

[5] Y.B. Zel’dovich, A.A. Ruzmaikin and D.D. Sokoloff, Magnetic Fields in Astro-
physics, McGraw-Hill, New York 1983.

[6] R.M. Kulsrud and S.W. Anderson, Ap. J. 396 (1992) 606.
[7] M.J. Rees, Q. Jl. R. Astr. Soc. 28 (1987) 197.

[8] S. Davidson, Phys. Lett. B 380 (1996) 253.

[9] C.J. Hogan, Phys. Rev. Lett. 51 (1983) 1488; J. Quashnock, A. Loeb and D.N. Spergel, Ap. J. 344 (1989) L49; B. Cheng and A.V. Olinto, Phys. Rev. D 50 (1994) 2421; G. Baym, D. Bödeker and L. McLerran, Phys. Rev. D 53 (1996) 662; G. Sigl, A. Olinto and K. Jedamzik, Phys. Rev. D 55 (1997) 4582.

[10] T. Vachaspati, Phys. Lett. B 265 (1991) 258; K. Enqvist and P. Olesen, Phys. Lett. B 319 (1993) 178; A.P. Martin and A.C. Davis, Phys. Lett. B 360 (1995) 71; A.C. Davis and K. Dimopoulos, Phys. Rev. D 55 (1997) 7398.

[11] M.S. Turner, L.M. Widrow, Phys. Rev. D 37 (1988) 2743; B. Ratra, Ap. J. 391 (1992) L1; W.D. Garretson, G.B. Field and S.M. Caroll, Phys. Rev. D 46 (1992) 5346; A.D. Dolgov, Phys. Rev. D 48 (1993) 2499; F.D. Mazzitelli and F.M. Spedalieri, Phys. Rev. D 52 (1995) 6694.

[12] D. Lemoine and M. Lemoine, Phys. Rev. D 52 (1995) 1955; M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. 75 (1995) 3796.

[13] T. Vachaspati and A. Vilenkin, Phys. Rev. Lett. 67 (1991) 1057; T. Vachaspati, Phys. Rev. D 45 (1992) 3487; D.N. Vollick, Phys. Rev. D 48 (1993) 3585.

[14] E.R. Harrison, Nature 224 (1969) 1089; Phys. Rev. Lett. 30 (1973) 188.

[15] P.P. Avelino and E.P.S. Shellard, Phys. Rev. D51 (1995) 5946.

[16] A.C. Davis and W.B. Perkins, Phys. Lett. B 390 (1997) 107.

[17] K. Dimopoulos and A.C. Davis, preprint DAMTP-97-8 [hep-ph/9705302].

[18] E. Witten, Nucl. Phys. B 249 (1985) 557.

[19] E. Copeland, M. Hindmarsh and N. Turok, Phys. Rev. Lett. 58 (1987) 1910; Nucl. Phys. B 306 (1988) 908.

[20] C.T. Hill, H.M. Hodges and M.S. Turner, Phys. Rev. Lett. 59 (1987) 2493; Phys. Rev. D 37 (1988) 263.

[21] M. Hindmarsh, Phys. Lett. B 225 (1989) 127.

[22] R.L. Davis and E.P.S. Shellard, Phys. Lett. B 207 (1988) 404; ibid. 209 (1988) 485.

[23] T.M. Helliwell and D.A. Konkowski, Phys. Lett. A 143 (1990) 438.

[24] A. Babul, T. Piran and D.N. Spergel, Phys. Lett. B 202 (1988) 307; ibid. 209 (1988) 477.
[25] P. Peter and D. Puy, Phys. Rev. D 48 (1993) 5546.

[26] A. Vilenkin and E.P.S. Shellard, Cosmic Strings and other Topological Defects, Cambridge Monographs on Mathematical Physics, CUP, Cambridge 1994.

[27] D.N. Spergel, T. Piran and J. Goodman, Nucl. Phys. B 291 (1987) 847.

[28] A. Vilenkin and T. Vachaspati, Phys. Rev. Lett. 58 (1987) 1041.

[29] M. Aryal, A. Vilenkin and T. Vachaspati, Phys. Lett. B 194 (1987) 25.

[30] P. Amsterdamski, Phys. Rev. D 39 (1989) 1524.

[31] B. Linet, Cl. Quant. Grav. 6 (1989) 435; Phys. Lett. A 146 (1990) 159.

[32] I. Moss and S. Poletti, Phys. Lett. B 199 (1987) 34.

[33] D. Haws, M. Hindmarsh and N. Turok, Phys. Lett. B 209 (1988) 255.

[34] P. Peter, Phys. Rev. D 46 (1992) 3335.

[35] T. Vachaspati, Phys. Rev. D 45 (1992) 3487.

[36] L. Perivolaropoulos, Phys. Lett. B 298 (1993) 305.

[37] L. Mestel, Physica Scripta T11 (1986) 53.

[38] B. Carter, Phys. Rev. D 41 (1990) 3886; A. Vilenkin, ibid. 41 (1990) 3038.

[39] B. Pacyński, Ap. J. 335 (1988) 525; R. Plaga, ibid. 424 (1994) L9.
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