Extreme value theory based confidence intervals for the parameters of a symmetric Lévy-stable distribution

Djamel Meraghni*, Louiza Soltane

Laboratory of Applied Mathematics, Mohamed Khider University, Biskra, Algeria

Abstract
We exploit the asymptotic normality of the extreme value theory (EVT) based estimators of the parameters of a symmetric Lévy-stable distribution, to construct confidence intervals. The accuracy of these intervals is evaluated through a simulation study.

Keywords: Asymptotic normality; Confidence bounds; Lévy-stable law; Extreme values; Hill Estimator.

MSC 2010 Subject Classification: 60E07; 62G20; 62G32; 62G05

*Corresponding author

E-mail addresses:
djmeraghni@yahoo.com (D. Meraghni)
louiza_stat@yahoo.com (L. Soltane)
1. Introduction

1.1. Lévy-stable Distributions.

The Lévy-stable distribution, also called stable, \( \alpha \)-stable or stable Paretian, represents a rich class of probability distributions. Introduced in 1920’s by Lévy Paul (1925), while investigating the behavior of normalized sums of independent identically distributed (iid) random variables (rv’s), it has got an increased attention in the last decades for at least two good reasons. First, it is theoretically supported by the generalized central limit theorem which states that the \( \alpha \)-stable law is the only possible limit distribution for properly normalized and centered sum of iid rv’s. Second, it allows skewness and fat tails meaning that it is suitable for data collected in areas as diverse as finance, hydrology, meteorology,... Indeed, a great deal of empirical evidence indicates that these data can be so heavy-tailed that they are poorly described by the largely used Gaussian distribution. In other words, the stable model provides a much better fit for heavy-tailed observations sets than the commonly adopted normal one does.

The extreme value theory (EVT), which proved to be an excellent tool in risk management, could be applied to estimate the parameters characterizing a stable distribution in order to determine the appropriate model for a given data set. In the sequel, let \( \overset{d}{=} \), \( \overset{p}{\rightarrow} \) and \( \overset{d}{\rightarrow} \) stand for equality in distribution, convergence in probability and convergence in distribution respectively and let \( \mathcal{N}(m, v^2) \) denote the normal distribution with mean \( m \in \mathbb{R} \) and variance \( v^2 > 0 \).

A rv \( X \) is said to be Lévy-stable if and only if, for \( n \geq 2 \), \( \exists a_n > 0, b_n \in \mathbb{R} \) such that

\[
\frac{(X_1 + \ldots + X_n) - b_n}{a_n} \overset{d}{=} X,
\]

where \( X_1, \ldots, X_n \) are independent copies of \( X \). It is shown that \( \exists 0 < \alpha \leq 2 \) such that \( a_n = n^{1/\alpha} \), (see, e.g., Feller, 1971).

Except from three special cases, a stable rv suffers from the lack of closed-form expressions for its distribution function (df) and probability density function (pdf). However, it is typically described by its characteristic function \( \varphi \) which has many representations. The most famous one is defined for \( t \in \mathbb{R} \) by

\[
\varphi(t) = \begin{cases} 
\exp \left\{ i \mu t - \sigma^\alpha |t|\alpha \left( 1 - i \beta \text{sign}(t) \tan \frac{\alpha \pi}{2} \right) \right\} & \text{for } \alpha \neq 1, \\
\exp \left\{ i \mu t - \sigma |t| \left( 1 + i \beta \text{sign}(t) \frac{\pi}{2} \log |t| \right) \right\} & \text{for } \alpha = 1,
\end{cases}
\]
where

\[ i^2 = -1 \quad \text{and} \quad \text{sign}(t) := \begin{cases} 
1 & \text{if} \ t > 0, \\
0 & \text{if} \ t = 0, \\
-1 & \text{if} \ t < 0.
\end{cases} \]

As we may see, this family of distributions is characterized by four parameters:

- \( 0 < \alpha \leq 2 \): stability index, tail exponent or shape parameter.
- \( \sigma > 0 \): scale parameter.
- \( -1 \leq \beta \leq 1 \): skewness parameter.
- \( \mu \in \mathbb{R} \): location parameter.

Using a notation of Samorodnitsky and Taqqu (1994), a rv \( X \) with stable distribution will be written as \( X \sim S_{\alpha}(\sigma, \beta, \mu) \). The three cases where we have explicit formulas for the pdf are the very popular Gaussian distribution \( S_2(\sigma, 0, \mu) \) and the lesser known models of Cauchy \( S_1(\sigma, 0, \mu) \) and Lévy \( S_{1/2}(\sigma, 1, \mu) \). The tail exponent \( \alpha \), which is the most important among all four parameters, indicates the rate at which the tails of the distribution taper off. For \( 0 < \alpha < 2 \), the \( k \)th \( (k = 1, 2, \ldots) \) moment of a stable rv is finite if and only if \( k < \alpha \), whereas for \( \alpha = 2 \) all the moments exist. In particular, the distribution mean only exists when \( 1 < \alpha \leq 2 \) and is equal to the location parameter \( \mu \). For \( 0 < \alpha < 2 \), the variance is infinite and the distribution tails are asymptotically equivalent to those of a Pareto distribution, i.e., they exhibit a power-law behavior.

### 1.2. Heavy Tails Property of \( S_{\alpha}(\sigma, \beta, \mu) \)

In general, the upper and lower tails of a Lévy-stable distribution asymptotically exhibit a Pareto-like behavior, i.e., they fall off like a power function. The rate of decay is governed by the stability index: the smaller \( \alpha \), the slower the decay and hence the heavier the distribution tails, as shown in Figure 1.1.

More precisely, for a rv \( X \sim S_{\alpha}(\sigma, \beta, \mu) \), the following result holds (see e.g., Samorodnitsky and Taqqu, 1994, page 16).

\[
\lim_{x \to \infty} x^\alpha P(X > x) = C_{\alpha} \frac{1 + \beta}{2} \sigma^\alpha \quad \text{and} \quad \lim_{x \to \infty} x^\alpha P(X < -x) = C_{\alpha} \frac{1 - \beta}{2} \sigma^\alpha, \quad (1.1)
\]

where

\[
C_{\alpha} := \left( \int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \frac{2}{\pi} \Gamma(\alpha) \sin \left( \frac{\pi \alpha}{2} \right), \quad (1.2)
\]
with $\Gamma$ being the gamma function defined, for $u > 0$, by $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}dx$. From equations (1.1), we get what is specifically called tail balance conditions. That is, we have, as $x \to \infty$,

$$
\frac{P(X > x)}{P(|X| > x)} \to \frac{1 + \beta}{2} =: p \quad \text{and} \quad \frac{P(X < -x)}{P(|X| > x)} \to \frac{1 - \beta}{2} =: q = 1 - p, \quad (1.3)
$$

Let $F$ and $G$ denote the df’s of $X \sim S_\alpha(\sigma, \beta, \mu)$ and $Z = |X|$ respectively. It is obvious that $F$ and $G$ are related by

$$
G(x) = F(x) - F(-x), \quad x > 0.
$$

From relation (1.1), we get that the distribution tail of $Z$ satisfies

$$
1 - G(x) \sim C_\alpha \sigma^\alpha x^{-\alpha}, \quad \text{as} \quad x \to \infty, \quad (1.4)
$$

and

$$
\lim_{t \to \infty} \frac{1 - G(tx)}{1 - G(t)} = x^{-\alpha}, \quad x > 0. \quad (1.5)
$$

The latter means that $1 - G$ is regularly varying at infinity with index $-\alpha < 0$. For full details on regular variation, see, for instance, Appendix B in de Haan and Ferreira (2006). From Gnedenko (1943), relation (1.5) is equivalent to say that $G$ is in Fréchet maximum domain of attraction. More precisely, for a sample $Z_1, ..., Z_n$ $(n \geq 1)$ from the rv $Z$, we have

$$
\max (Z_1, ..., Z_n) \xrightarrow{d} \Phi_\alpha,
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.1}
\caption{Stable densities for different values of $\alpha$ with $\beta = \mu = 0$ and $\sigma = 1$. The solid line corresponds to the Gaussian model.}
\end{figure}
\( G^{-1}(u) := \inf \{ x \in \mathbb{R}, G(x) \geq u \}, \) \( 0 < u < 1, \) is the generalized inverse or quantile function of \( Z \) and
\[
\Phi_\alpha(x) := \begin{cases} 
\exp(-x^{-\alpha}), & x > 0 \\
0, & x \leq 0
\end{cases}
\]

For further details and a complete description of this class of distributions, we refer to the textbooks of Feller (1971), Zolotarev (1986), Samorodnitsky and Taqqu (1994) and Nolan (2001). On the other hand, there are available some very useful computer programs, such that "STABLE", "Xplore" and the package "stabledist" of the statistical software R (Ihaka and Gentleman, 1996), specially developed for numerical purposes (computing stable df’s and pdf’s, generating stable rv’s, estimating stable parameters,...).

In this work, we concentrate on the case where \( \beta = 0, \) that is when the distribution is symmetric about \( \mu. \) In this case, the characteristic function and the tail balance conditions respectively reduce to the simpler forms
\[
\varphi(t) = \exp\{ -\sigma^\alpha |t|^\alpha + i\mu t \}, \quad t \in \mathbb{R},
\]
and
\[
\lim_{x \to \infty} \frac{P(X > x)}{P(|X| > x)} = \lim_{x \to \infty} \frac{P(X < -x)}{P(|X| > x)} = \frac{1}{2}.
\]

The rest of the paper is organized as follows. Section 2, is devoted to a brief reminder on EVT-based estimators of the stable parameters. In Section 3, we use the asymptotic normality property of the estimators to build confidence intervals for parameters \( \alpha, \mu \) and \( \sigma. \) Finally, the accuracy of such intervals is investigated in a simulation study in Section 4.

2. EVT-BASED ESTIMATION

The lack of explicit forms for the df and pdf severely hampers the estimation of the distribution parameters. Nevertheless, several numerical procedures of estimation based on the sample quantiles, the sample characteristic function and maximum likelihood approaches, are proposed in the literature. In a comparative study Ojeda (2001) notices that maximum likelihood based methods are the most accurate but the slowest of all others. On the other hand, the nature of the Lévy-stable distribution tails suggests that EVT could play a major role in estimating its parameters. EVT is a classical topic in probability theory and mathematical
statistics, developed for the estimation of occurrence probability of rare events. It permits to extrapolate the behavior of distribution tails from the largest observed data. EVT techniques have proven to be very useful where estimation of tail-related quantities such as extreme value index, high quantiles, small exceedance probabilities and mean excess function, is needed. The domains of application of EVT include insurance (premium computation, large losses,...), finance (asset returns, exchange rate,...), hydrology (floods, drought,...), meteorology (extreme weather conditions,...), ecology (pollution peaks,...), telecommunications (network traffic,...), physics (nuclear reactions,...). EVT-based estimation approach has at least three advantages. It focuses only on tail behavior and does not assume a parametric form for the entire distribution. It provides estimators of explicit forms making estimate computation easier and more direct. Finally, it produces estimators which enjoy the asymptotic normality property leading to the construction of confidence bounds for the unknown parameters. A very good variety of textbooks may be consulted for a review of this topic and its multiple applications. We can cite, for instance, de Haan and Ferreira (2006), Embrechts et al. (1997), Reiss and Thomas (1997) and Beirlant et al. (2004).

2.1. Estimating the Stability Index.

The characteristic exponent $\alpha$ is the main parameter as it governs the behavior of the distribution tails. Many estimators are proposed for $\alpha$ via the EVT approach, among which the most popular is that introduced by Hill (Hill, 1975) as follows:

$$
\hat{\alpha}_n = \hat{\alpha}_n(k) := \left( \frac{1}{k} \sum_{i=1}^{k} \log Z_{n-i+1:n} - \log Z_{n-k:n} \right)^{-1},
$$

(2.6)

where $Z_{1:n} \leq ... \leq Z_{n:n}$ are the order statistics pertaining to a sample $(Z_1, ..., Z_n)$, $n \geq 1$, from the rv $Z$ and $k = k(n)$ is an integer sequence such that

$$
k \to \infty \text{ and } k/n \to 0 \text{ as } n \to \infty.
$$

(2.7)

The consistency of $\hat{\alpha}_n$ is proved in Mason (1982), while its almost sure convergence is established in Necir (2006a). For the asymptotic normality of $\hat{\alpha}_n$ (and other related estimators), it is required an additional assumption, known as the second-order condition of regular variation (see de Haan and Stadtmüller, 1996), which specifies the rate of convergence in (1.5). That is, we assume that there exist a constant $\rho < 0$, called second-order parameter, and a function $A$ tending to zero and not
changing sign near infinity, such that for any $x > 0$, we have
\[
\lim_{t \to \infty} \frac{(1 - G(tx)) / (1 - G(t)) - x^{-\alpha}}{A(t)} = x^{-\alpha} x^{\alpha \rho} - 1 / \rho / \alpha .
\] (2.8)

Note that when $1 < \alpha < 2$, the condition (2.8) is fulfilled. Indeed, using the expansion (to the second order) given in top of page 95 in Zolotarev (1986), yields that $G$ belongs to Hall’s class of heavy-tailed distributions (Hall, 1982), which in turn implies that (2.8) holds. A df $K$ is said to belong Hall’s class if
\[
1 - K(x) = cx^{-1 / \gamma} (1 + dx^{\rho / \gamma} + o(x^{\rho / \gamma})) , \text{ as } x \to \infty,
\] (2.9)

where $\gamma > 0$, $\rho \leq 0$, $c > 0$, and $d \neq 0$. Hall’s class, which is a subset of the more general family of models with second-order regularly varying tails, includes distributions (Burr, Fréchet,...) that are most commonly used in extreme event modelling. Among the works on the asymptotic normality of $\hat{\alpha}_n$, we can cite that of Peng (1998) who proved that, if (2.8) holds, then for an integer sequence $k$ satisfying (2.7) and $\lim_{n \to \infty} \sqrt{k} A(n/k) = \lambda$, with $\lambda$ finite, then
\[
\sqrt{k} (\hat{\alpha}_n - 1) \overset{d}{\to} N(\lambda / (1 - \rho), \alpha^{-2}) , \text{ as } n \to \infty.
\] (2.10)

Weron (2001) discussed the performance of Hill’s estimator $\hat{\alpha}_n$ and noted that for $\alpha \leq 1.5$ the estimation is quite reasonable but as $\alpha$ approaches 2, there is a significant overestimation when considering samples of typical size (for an illustration, see Figure 4.2 and Table 4.1). For such values of $\alpha$, a very large number of observations (a million or more) is needed in order to obtain acceptable estimates and avoid misleading inference on the stability index, because the true heavy tail nature of the distribution is visible only for extremely large datasets. Fortunately, this kind of datasets are available nowadays and their storage and treatment are made possible thanks to a very sophisticated technology.

The behavior of Hill’s estimator (and therefore that of EVT-based estimators) is affected by the number $k$ of upper order statistics to be used in estimate computations. One needs to locate where the distribution tails really begin because using too many data results in a big bias and too few observations lead to a substantial variance. Consequently, one has to make a trade-off between bias and variance in order to get an accurate estimate. To this end, it seems reasonable that minimizing the mean squared error allows for a compromise between the bias and variance components. On the other hand, there exist several algorithms and data-adaptive procedures for the selection of the optimal sample fraction of extreme values that guarantees the
2.2. Estimating the Location Parameter. The empirical mean \( \bar{X} := n^{-1} \sum_{i=1}^{n} X_i \), which is the natural estimator of the mean, is, in virtue of the central limit theorem, asymptotically normal provided that the second moment is finite. However, for \( X \sim S_\alpha(\sigma, \beta, \mu) \) with \( 1 < \alpha < 2 \), the latter theorem is not applicable because the variance of \( X \) is infinite. Therefore, the asymptotic normality of the sample mean \( \bar{X} \) is not established. To solve this problem, Peng (2001) proposed an asymptotically normal estimator \( \hat{\mu}_n \) for \( \mu \), based on the order statistics \( X_{1:n} \leq \ldots \leq X_{n:n} \) associated to a sample \((X_1, \ldots, X_n)\) from \( X \), as follows:

\[
\hat{\mu}_n = \hat{\mu}_n(k) := \hat{\mu}_n^{(1)} + \hat{\mu}_n^{(2)} + \hat{\mu}_n^{(3)},
\]

where

\[
\hat{\mu}_n^{(2)} = \hat{\mu}_n^{(2)}(k) := \frac{1}{n} \sum_{i=k+1}^{n-k} X_{i:n} \text{ (trimmed mean)},
\]

\[
\hat{\mu}_n^{(1)} = \hat{\mu}_n^{(1)}(k) := \frac{k}{n} X_{k:n} \frac{\hat{\alpha}_n^{(1)}}{\hat{\alpha}_n^{(1)} - 1} \quad \text{and} \quad \hat{\mu}_n^{(3)} = \hat{\mu}_n^{(3)}(k) := \frac{k}{n} X_{n-k+1:n} \frac{\hat{\alpha}_n^{(3)}}{\hat{\alpha}_n^{(3)} - 1},
\]

with

\[
\hat{\alpha}_n^{(1)} = \hat{\alpha}_n^{(1)}(k) := \left( \frac{1}{k} \sum_{i=1}^{k} \log (-X_{i:n}) - \log (-X_{k:n}) \right)^{-1},
\]

and

\[
\hat{\alpha}_n^{(3)} = \hat{\alpha}_n^{(3)}(k) := \left( \frac{1}{k} \sum_{i=1}^{k} \log X_{n-i+1:n} - \log X_{n-k:n} \right)^{-1},
\]

being consistent estimators of \( \alpha \) as well. The strong limiting behavior of \( \hat{\mu}_n \) is studied in Necir (2006b) when constructing a nonparametric sequential test with power 1 for \( \mu \). For the asymptotic normality of \( \hat{\mu}_n \), we notice that, by the expansion (to the second order) and the relationship between the tails of \( X \) respectively given in pages 95 and 65 of Zolotarev (1986), both tails of \( F \) satisfy the definition of Hall’s model (2.9). Peng (2001) proved that, with a suitable choice of \( k \),

\[
\left( \frac{v}{\tau(k/n)} (\hat{\mu}_n - \mu) \right)^{d} \rightarrow \mathcal{N}(0, \delta^2), \quad \text{as} \ n \rightarrow \infty,
\]

where

\[
\delta^2 := 1 + \left( \frac{(2 - \alpha)(2\alpha^2 - 2\alpha + 1)}{2(\alpha - 1)^4} + \frac{(2 - \alpha)}{(\alpha - 1)} \right),
\]

and

\[
\tau^2(s) := \int_{s}^{1-s} \int_{s}^{1-s} (u \wedge v - uv) dF^{-1}(u)dF^{-1}(v), \quad 0 < s < 1.
\]
It is shown in Peng (2001) that, as \( n \to \infty \),
\[
\sqrt{k/n} F^{-1}(k/n) \tau(k/n) \xrightarrow{p} - \left( \frac{2 - \alpha}{2 (p^{2/\alpha} + (1 - p)^{2/\alpha})} \right)^{1/2} (1 - p)^{1/\alpha},
\]
which in our case (\( \beta = 0 \)), may be rewritten into
\[
\tau(k/n) \sim -\frac{2\sqrt{k/n} F^{-1}(k/n)}{\sqrt{2 - \alpha}}, \text{ as } n \to \infty.
\] (2.12)

2.3. Estimating the Scale Parameter. By combining relations (1.4) and (1.2), with some approximations, Meraghni and Necir (2007) provided a consistent estimator \( \hat{\sigma}_n \) to the scale parameter \( \sigma \) as follows:

\[
\hat{\sigma}_n := Z_{n-k:n} \left( \frac{k\pi}{2n \Gamma(\hat{\alpha}_n) \sin \frac{\pi\hat{\alpha}_n}{2}} \right)^{1/\hat{\alpha}_n},
\]
and proved that, with an adequate sequence \( k \),
\[
\frac{\sqrt{k}}{\log(k/n)} (\log \hat{\sigma}_n - \log \sigma) \xrightarrow{d} N\left( \lambda/(1 - \rho), \alpha^{-2} \right), \text{ as } n \to \infty.
\] (2.13)

3. Confidence bounds

Let us fix the confidence level of estimation to be 0 < 1 - \( a < 1 \) and let \( z_a \) denote the \((1 - a)\)-quantile of the standard Gaussian distribution. The first step, in the process of confidence interval construction, is to determine the optimal sample fraction, that we denote by \( k^* \), of extreme observations involved in estimate computation. To this end, we adopt the methodology of Neves and Fraga Alves (2004) who discussed and evaluated the performance of the procedure proposed by Reiss and Thomas (1997). The latter consists in taking as optimal the value of \( k \) that minimizes
\[
RT(k) := \frac{1}{k} \sum_{i=1}^{k} i^\theta |\hat{\alpha}_n(i) - med(\hat{\alpha}_n(1),...,\hat{\alpha}_n(k))|,
\] (3.14)
where \( med \) stands for the median and 0 \( \leq \theta \leq 1/2 \). In other words, we have
\[
k^* := \arg\min_k RT(k).
\]

Since we will be interested in the range 0 < 1/2 < 1/\( \alpha < 1 \), then we choose \( \theta = 0.3 \) as indicated in Neves and Fraga Alves (2004).

The second step is to compute the estimate values which correspond to the optimal number \( k^* \). Note that, for parameters \( \alpha \) and \( \sigma \), we only use the top observations in the \( Z \)-sample, whereas for \( \mu \), we use the whole \( X \)-sample. Finally, we exploit the asymptotic normality results (2.10), (2.11) and (2.13) to get asymptotic confidence
bounds for $\alpha$, $\mu$ and $\sigma$ respectively. If we set $\hat{\alpha}^* := \hat{\alpha}_n(k^*)$, $\hat{\mu}^* := \hat{\mu}_n(k^*)$ and $\hat{\sigma}^* := \hat{\sigma}_n(k^*)$, then the respective $(1 - a) \times 100\%$-confidence intervals for parameters $\alpha$, $\mu$ and $\sigma$ are
\[
\left( 1 + z_{a/2}/\sqrt{k^*} \right) \hat{\alpha}^*, \left( 1 - z_{a/2}/\sqrt{k^*} \right) \hat{\alpha}^*,
\]
\[
\left( \hat{\mu}^* - z_{a/2}/\sqrt{n} \right) \hat{\delta}^* \hat{\tau}^*, \left( \hat{\mu}^* + z_{a/2}/\sqrt{n} \right) \hat{\delta}^* \hat{\tau}^*,
\]
and
\[
\exp \left\{ \frac{\log \hat{\alpha}^* + z_{a/2} \log(k^*/n)}{\hat{\alpha}^* \sqrt{k^*}} \right\}, \exp \left\{ \frac{\log \hat{\sigma}^* - z_{a/2} \log(k^*/n)}{\hat{\sigma}^* \sqrt{k^*}} \right\},
\]
where
\[
\hat{\delta}^* := \left( 1 + \frac{(2 - \hat{\alpha}^*) (2 \hat{\alpha}^2 - 2 \hat{\alpha}^* + 1)}{2 (\hat{\alpha}^* - 1)^2} + \frac{(2 - \hat{\alpha}^*)}{(\hat{\alpha}^* - 1)} \right)^{1/2}
\]
and $\hat{\tau}^* := \frac{-2 \sqrt{k^*/nX_{k^*/n}}}{\sqrt{2 - \hat{\alpha}^*}}$.

4. SIMULATION STUDY

We carry out a simulation study, by means of the statistical software R (Ihaka and Gentleman, 1996), to illustrate the finite sample behaviors of the three estimators $\hat{\alpha}_n$, $\hat{\mu}_n$ and $\hat{\sigma}_n$ by computing their absolute biases (abs bias) and mean squared errors (mse). We also evaluate the accuracy of the confidence intervals (conf int) through their lengths and coverage probabilities (cov prob). But first, we start by graphically checking Weron’s note (Weron, 2001) on Hill’s estimator of the stability index. It is noteworthy that, for each experiment, we make 1000 replications then we take our overall results by averaging over all the individual results obtained at the end of each repetition.

We see on the right graph of Figure 4.2, which is based on samples of size 5000 from stable distributions $S_{1.1}(1,0,0)$ and $S_{1.8}(1,0,0)$, that there is no intermediate number $k$ which gives a good estimate for $\alpha$ and that estimates can even be above the Lévy-stable regime. On the other side, the left panel shows that for $\alpha = 1.1$, accurate estimates could be obtained for $k \in \{150, ..., 500\}$.

For the estimation of the shape parameter $\alpha$, we generate 3000 observations of symmetric $\alpha$-stable distributions $S_{\alpha}(\sigma,0,0)$ with several values for parameters $\alpha$ and $\sigma$. The results are summarized in Table 4.1, where we note that, as expected, the smaller the parameter values, the better the estimation. The bottom of the table (corresponding to $\alpha = 1.8$) shows that the estimation is very poor for large $\alpha$-values and confirms the graphical conclusion we made about the irrelevance of Hill’s estimator, for large stability indices, when built on the basis of datasets of
Figure 4.2. Plots of Hill’s estimator (based on samples of size 5000) of the stability index $\alpha$ vs. the number $k$ of upper order statistics for $\alpha = 1.1$ (left) and $\alpha = 1.8$ (right). The horizontal line represents the true value of $\alpha$.

typical sizes. For this reason, only the values of $\alpha$ that are less than or equal to 1.5 will be considered thereafter. We gather the simulation results in Table 4.2 for the location parameter $\mu$ and in Table 4.3 for the scale parameter $\sigma$. The former shows that the more $\alpha$ gets away from 1, the estimation of $\mu$ gets better and better while the latter indicates that the estimation of $\sigma$ is not good when $\alpha$ is around 1.5 but for smaller values, it might be considered as acceptable. It is to be noted that, in regards to the estimation of $\mu$, the results are extremely poor when the stability index is very close to 1. This may be explained by the fact that, in this work, we only consider $\alpha$-values lying between 1 and 2 and in this case the location parameter is equal to the distribution mean and is estimated as such. When $\alpha$ is less than or equal to 1, the mean does not exist and therefore the EVT-based estimation of $\mu$ is very bad when $\alpha$ is near 1.

References

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J., 2004. Statistics of extremes: Theory and applications. Wiley.

Cheng, S. and Peng, L., 2001. Confidence intervals for the tail index. Bernoulli 7, 751-760.
| $\alpha$ | $\sigma$ | $\hat{\alpha}_n$ | abs bias | mse  | conf int        | length | cov prob |
|---------|---------|------------------|----------|------|----------------|--------|----------|
| 0.1     | 1.114   | 0.014            | 0.011    | 0.978, 1.301 | 0.323 | 0.91     |
| 1.1     | 0.5     | 1.110            | 0.010    | 0.975, 1.294 | 0.319 | 0.91     |
|         | 1.0     | 1.114            | 0.014    | 0.979, 1.298 | 0.319 | 0.90     |
| 0.1     | 1.227   | 0.027            | 0.013    | 1.075, 1.435 | 0.359 | 0.91     |
| 1.2     | 0.5     | 1.217            | 0.017    | 1.066, 1.422 | 0.356 | 0.91     |
|         | 1.0     | 1.231            | 0.031    | 1.081, 1.436 | 0.354 | 0.89     |
| 0.1     | 1.633   | 0.133            | 0.045    | 1.421, 1.931 | 0.510 | 0.63     |
| 1.5     | 0.5     | 1.630            | 0.130    | 1.419, 1.923 | 0.504 | 0.65     |
|         | 1.0     | 1.627            | 0.127    | 1.414, 1.927 | 0.513 | 0.65     |
| 0.1     | 2.403   | 0.603            | 0.492    | 2.013, 3.004 | 0.991 | 0.36     |
| 1.8     | 0.5     | 2.418            | 0.618    | 2.032, 3.009 | 0.978 | 0.34     |
|         | 1.0     | 2.405            | 0.605    | 2.020, 2.994 | 0.974 | 0.36     |

Table 4.1. Simulation results of the estimation of the shape parameter of a symmetric Lvy-stable distribution based on 1000 samples of 3000 observations.

| $\alpha$ | $\sigma$ | $\hat{\mu}_n$ | abs bias | mse  | conf int | length | cov prob |
|---------|---------|----------------|----------|------|----------|--------|----------|
| 0.1     | 0.025   | 0.025          | 0.035    | -0.432, 0.482 | 0.915 | 0.89    |
| 1.2     | -0.058  | 0.058          | 0.213    | -2.158, 2.042 | 4.200 | 0.88    |
|         | 0.1     | 0.171          | 0.171    | -2.878, 3.220 | 6.098 | 0.85    |
| 0.1     | 0.001   | 0.001          | 0.002    | -0.094, 0.096 | 0.190 | 0.92    |
| 1.3     | 0.004   | 0.004          | 0.04     | -0.471, 0.479 | 0.950 | 0.92    |
|         | 0.1     | 0.008          | 0.008    | -0.942, 0.958 | 1.900 | 0.92    |
| 0.1     | 0.003   | 0.003          | 0.000    | -0.038, 0.044 | 0.083 | 0.94    |
| 1.5     | -0.010  | 0.010          | 0.005    | -0.207, 0.186 | 0.393 | 0.94    |
|         | 0.1     | 0.013          | 0.013    | -0.394, 0.421 | 0.814 | 0.94    |

Table 4.2. Simulation results of the estimation of the location parameter of a symmetric Lvy-stable distribution based on 1000 samples of 3000 observations.

Danielsson, J., de Haan, L., Peng, L. and de Vries, C.G., 2001. Using a bootstrap method to choose the sample fraction in tail index estimation. J. Multivariate
Analysis 76, 226-248.

Embrechts, P., Klüppelberg, C. and Mikosch, T., 1997. Modelling extremal events for insurance and finance. Springer-Verlag, New York.

Feller, W., 1971. An introduction to probability theory and its applications, 2nd ed. Wiley, New York.

Fereira, A. and de Vries, C. G., 2004. Optimal confidence intervals for the tail index and high quantiles. Tinbergen Institute Discussion Paper 090/2.

Gnedenko, B.V., 1943. Sur la distribution limite du terme maximum d’une série aléatoire. Annales de Mathématiques 44, 423-453.

de Haan, L. and Stadtmüller, U., 1996. Generalized regular variation of second order. J. Australian Math. Soc. (Series A) 61, 381-395.

de Haan, L. and Ferreira, A., 2006. Extreme Value Theory: An Introduction. Springer.

Hall, P., 1982. On some simple estimates of an exponent of regular variation. Journal of the Royal Statistical Society 44, 37-42.

Hill, B.M., 1975. A simple general approach to inference about the tail of a distribution. Ann. Statist. 3, 1163-1174.

Ihaka, R. and Gentleman, R., 1996. R: A language for data analysis and graphics. J. Computational Graphics and Statistics 5, 299-314.

Lévy, P., 1925. Calcul des probabilités. Paris, Gauthier-Villars.
Mason, D.M. 1982. Laws of large numbers for sums of extreme values. Ann. Probab. \textbf{10}, 756-764.

Meraghni, D. and Necir, A., 2007. Estimating the scale parameter of a Lévy-stable distribution via the extreme value approach. Methodol Comput Appl Probab \textbf{9}, 557-572.

Necir, A., 2006a. A functional law of the iterated logarithm for kernel-type estimators of the tail index. J. Statist. Plann. Inference \textbf{136}, 780–802.

Necir, A., 2006b. A nonparametric sequential test with power 1 for the mean of Lévy-stable distributions with infinite variance. Methodol Comput Appl Probab \textbf{8}, 321-343.

Neves, C. and Fraga Alves, M.I., 2004. Reiss and Thomas’ automatic aelection of the number of extremes. Computational Statistics and Data Analysis \textbf{47}, 689-704.

Nolan, J.P. 2001. Maximum likelihood estimation and diagnostics for stable distributions. In O.E. Barndorff-Nielsen, T. Mikosch, S. Resnick (eds.), Lévy Processes, Birkhäuser, Boston.

Ojeda, D., 2001. Comparison of stable estimators. Ph.D. Thesis, Department of Mathematics and Statistics, American University.

Peng, L., 1998. Asymptotically unbiased estimators for the extreme value index. Statist. Probab Lett \textbf{38}, 107-115.

Peng, L., 2001. Estimating the mean of a heavy-tailed distribution. Statist. Probab Lett \textbf{52}, 255-264.

Reiss, R.D. and Thomas, M., 1997. Statistical analysis of extreme values with applications to insurance, finance, hydrology and other fields. Birkhäuser.

Samorodnitsky, G. and Taqqu, M.S., 1994. Stable non-Gaussian random processes: Stochastic models with infinite variance. Chapman & Hall, New York.

Weron, R. 2001. Levy-stable distributions revisited : tail index > 2 does not exclude the Levy-stable regime. Internat. J. Modern Phys. C, \textbf{12}, 209-223.

Zolotarev, V.M., 1986. One-dimensional stable distributions. American Mathematical Society. Providence, Rhode Island.