ANALYSIS OF A FREE BOUNDARY AT CONTACT POINTS
WITH LIPSCHITZ DATA

A. L. KARAKHANYAN AND H. SHAHGHOLIAN

Abstract. In this paper we consider a minimization problem for the functional
\[ J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u\leq 0\}} \]
in the upper half ball \( B_1^+ \subset \mathbb{R}^n, n \geq 2 \), subject to a Lipschitz continuous Dirichlet data on \( \partial B_1^+ \). More precisely we assume that \( 0 \in \partial \{u > 0\} \) and the derivative of the boundary data has a jump discontinuity. If \( 0 \in \partial (\{u > 0\} \cap B_1^+) \), then (for \( n = 2 \) or \( n \geq 3 \) and the one-phase case) we prove, among other things, that the free boundary \( \partial \{u > 0\} \) approaches the origin along one of the two possible planes given by
\[ \gamma x_1 = \pm x_2, \]
where \( \gamma \) is an explicit constant given by the boundary data and \( \lambda_{\pm} \) the constants seen in the definition of \( J(u) \). Moreover the speed of the approach to \( \gamma x_1 = x_2 \) is uniform.

Contents

1. Introduction 5142
2. Linear growth: A Heuristic discussion 5145
3. Main results 5146
4. Technicalities 5148
5. Proof of Theorem A 5153
6. Proof of Theorem B 5156
7. Largest and smallest global solutions 5158
8. Proof of Theorem C 5162
9. Proof of Theorem D 5167
10. Proof of Theorem E 5169
11. Appendix 1 5170
12. Appendix 2 5173
References 5175

Received by the editors May 22, 2012 and, in revised form, May 15, 2013.
2010 Mathematics Subject Classification. Primary 35R35.
Key words and phrases. Free boundary problem, regularity, contact points.
The second author was partially supported by the Swedish Research Council. The authors also thank Professor Carlos Kenig for several valuable comments. The first author thanks the Göran Gustafsson Foundation for visiting appointments to KTH.
1. Introduction

In this paper we consider the local minimizers of the functional

\[ J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+^2 \chi_{\{u > 0\}} + \lambda_-^2 \chi_{\{u \leq 0\}}, \]

where \( B_1^+ \subset \mathbb{R}^n, n \geq 2 \), is the upper half of the open unit ball, \( \chi_D \) is the characteristic function of \( D \subset \mathbb{R}^n \), \( \lambda_\pm \) are given positive constants and \( u = f \) on \( \partial B_1^+ \) with Lipschitz continuous \( f \). The local regularity of the minimizers \( u \) and the free boundary \( \partial \{u > 0\} \) were studied in [AC], [ACF], and [Gu]; notably it was shown that \( u \) is locally Lipschitz continuous.

The boundary regularity of \( u \) with smooth boundary data \( f \) such that \( |f(x)| \approx o(|x|) \) near the origin was considered in [KKS] where, assuming the origin is a contact point, the authors have proved that close to the origin, the free boundary approaches the plane \( \{x_1 = 0\} \) in a tangential fashion.

The objective of this paper is to consider boundary data that gives rise to non-tangential touch between the free and the fixed boundaries. Such problems appear naturally in the mathematical formulation of the so-called Dam problem for the water reservoirs (see [AG]). Other problems of this kind emerge in wake and cavity formations in stationary Eulerian flows moving through cylindrical domains (see [BZ], Chapters 1.9 and 9.5 for more applications).

Since the formulation of our main results requires some technical definitions, we refrain ourselves from giving an exact account of our main results here. However, in lay terms, one can say that our main result in this paper states that for a boundary data such as \( \alpha x_1 + x_2^+ - \alpha x_2^- \), the free boundary \( \Gamma(u) \) approaches the fixed one along one of the planes \( \gamma x_1 = \pm x_2 \), where

\[ \gamma = \sqrt{\frac{\lambda_+^2 - \lambda_-^2}{\alpha_+^2 - \alpha_-^2} - 1}. \]

We prove this when \( n = 2 \) for the two-phase problem and \( n \geq 3 \) for the one-phase problem; see Theorem C. The difficulty for a two-phase in higher dimensions comes from the classification of global homogeneous solutions that is not feasible by our technique.

1.1. Plan of the paper. The plan of this paper is as follows. In this introductory part we give the necessary notation and definitions to formulate the problem. Section 2 contains a heuristic discussion of the optimal regularity of solutions. The key point is the uniform linear growth of minimizers at the origin. We formulate the main results of this paper in Section 3. To deal with the boundary behavior of minimizers one needs to obtain up-to boundary uniform continuity near contact points. The proof of this result as well as a basic compactness theorem for blow-up sequences is contained in Section 4 and the Appendix. Section 5 takes care of the optimal regularity of minimizers to our functional. In Sections 6-8 we show that homogeneous global solutions in the one-phase case are two-dimensional, and hence independent of \( x_3, x_4, \ldots, x_n \). A stability result is given in Section 9. In fact Section 9 contains the proof of the main result of this paper, describing how the free boundary behaves close to the origin. Finally in Section 10 we give an example of a non-homogeneous global solution.
1.2. **Notation.** We will use the following notation throughout the paper:

- $C_0, C_n, \ldots$: generic constants,
- $\chi_D$: the characteristic function of the set $D \subset \mathbb{R}^n$, $n \geq 2$,
- $\mathcal{D}$: the closure of $D$,
- $\partial D$: the boundary of a set $D$,
- $x, x'$: $x = (x_1, \ldots, x_n)$, $x' = (0, x_2, \ldots, x_n)$,
- $\mathbb{R}^n_+, \mathbb{R}^n_-$: $\{x \in \mathbb{R}^n : x_1 > 0\}$, $\{x \in \mathbb{R}^n : x_1 < 0\}$,
- $\Pi$: $\{x \in \mathbb{R}^n : x_1 = 0\}$,
- $B_r(x), B^+_r(x)$: $\{y \in \mathbb{R}^n : |y - x| < r\}$, $B_r(x) \cap \mathbb{R}^n_+$,
- $B_r^+, B^+_r(0)$, $B^+_r(0)$,
- $B'_r$: $B_r \cap \Pi$,
- $S^+_r$: $\partial B_r \cap \mathbb{R}^n_+$,
- $\lambda_+, \Lambda$: are positive numbers and $\Lambda = \lambda^2_+ - \lambda^2_- \neq 0$,
- $\Gamma(u)$: $\partial\{u > 0\}$; the free boundary of $u$,
- $\Omega^+(u), \Omega^-(u)$: $\Omega^+(u) = \{x : u(x) > 0\}$, $\Omega^-(u) = \{x : u(x) < 0\}$,
- $K_\delta(x_0)$: the open cone $K_\delta = \{x \in \mathbb{R}^n_+ : |x - x'| > \delta|x - x_0|\}$,
- $K_\delta': \{x \in \mathbb{R}^n_+ : x_1 > \delta|x'|\}$,
- $\mathcal{P}_r, \mathcal{H}\mathcal{P}_\infty, \mathcal{P}^*_r$ see Definitions [1.3, 1.5, 1.6] and [1.9],
- $v^\pm$: $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$. Thus $v = v^+ - v^-$.

1.3. **Problem set-up.** Throughout this paper we assume

\[(1.1) \quad f(x) = \alpha_+ x^+_2 - \alpha_- x^-_2 + g(x),\]

where $\alpha_+, \alpha_-$ are non-negative constants such that $\alpha_+ + \alpha_- > 0$, and $g(x) \in C^{1,\alpha}(B^+_1)$, $g(x) = o(|x|)$. Typically $g(x) = C|x|^{1+\kappa}$ for positive constants $C$ and $\kappa$.

For a fixed domain $D \subset \mathbb{R}^n_+$ we put

$$J(u, D) = \int_D |\nabla u|^2 + \lambda^2_+ \chi_{\{u > 0\}} + \lambda^2_- \chi_{\{u < 0\}}.$$

When it is clear for which $D$ the functional $J$ is considered, we just write it as $J(u)$, omitting the explicit dependence on $D$. The case $D = B^+_R$ is of particular interest.

**Definition 1.1.** Let $\mathcal{K}_f(D) = \{w : w \in H^1(D), \ w - f \in H^1_0(D)\}$ be the class of admissible functions.

- A function $u$ is said to be a local minimizer of $J(u, D)$ if for any function $v \in \mathcal{K}_f(D)$ such that $u = v$ on $\partial D'$, for $D' \subset D$, it follows that
  $$J(u) \leq J(v).$$

- The class of local minimizers is denoted by $\mathcal{P}(D, n, \lambda_+, \alpha_+, g)$.

**Remark 1.2.** For $D = B^+_R$ we denote the corresponding class by $\mathcal{P}_r(n, \lambda_+, \alpha_+, g)$. We also set $\mathcal{P}_r(n, \lambda_+, \alpha_+) = \mathcal{P}_r(n, \lambda_+, \alpha_+, 0)$. It is worthwhile to point out that if $u \in \mathcal{P}_r(n, \lambda_+, \alpha_+, g)$, then $u_r(x) = \frac{u(rx)}{r}$ is $\mathcal{P}_1(n, \lambda_+, \alpha_+, g(rx)/r)$ by the scale invariance of $J(u, B^+_R)$.

If $D$ is a bounded domain, then, from the definition of $J(u, D)$, we have

\[(1.2) \quad J(u, D) = \lambda^2_-|D| + \int_D |\nabla u|^2 + \Lambda \chi_{\{u > 0\}},\]
where $\Lambda = \lambda_+^2 - \lambda_-^2 > 0$. In what follows we take
\begin{equation}
J(u, D) = \int_D |\nabla u|^2 + \Lambda \chi_{\{u>0\}}.
\end{equation}

Next we introduce a particular class of local minimizers $u$, such that the free boundary $\partial\{u > 0\}$ is $\delta$–**non-tangential** or $\delta$–**NT** for short.

**Definition 1.3.** Let $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$.

1° We say that the free boundary $\Gamma(u)$ is $\delta$–**non-tangential** (or $\delta$–**NT**) at $x_0 \in B'_r \cap \Gamma(u)$ if there exists a $\delta > 0$ such that
\begin{equation}
(B_{2r}^+(x_0) \setminus B_{r}^+(x_0)) \cap \partial\{u > 0\} \cap K_\delta(x_0) \neq \emptyset, \quad \forall \rho \in (0, r),
\end{equation}
where $K_\delta(x_0) = \{x \in \mathbb{R}_+^n : x_1 > \delta |x' - x'_0|\}$.

2° The class of all local minimizers in $B_R^+(x_0)$ with $\delta$–**NT** free boundary is denoted by $\mathcal{P}_r(x_0, n, \lambda_\pm, \alpha_\pm, g, \delta)$. When $x_0 = 0$ and $R = 1$ we often omit the dependence of $\mathcal{P}_r$ from $x_0$ and write $\mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g, \delta)$ for brevity.

One can interpret condition (1.4) geometrically as follows: There is a free boundary point at each intersection of the cone $K_\delta(x_0)$ with $B_{2r}^+(x_0) \setminus B_{r}^+(x_0)$, and hence the free boundary does not approach the plane $x_1 = 0$ rapidly as $r \to 0$. The next section contains more discussion on $\delta$–**NT** as a necessary condition for linear growth.

**Remark 1.4.** The $\delta$–**NT** assumption can be weakened as follows. Let $r > 0$ be small, $z \in \partial\{u > 0\}$ be a non-isolated point of the free boundary and assume that there is a point $x_r \in (B_{2r}(z) \setminus B_r(z)) \cap K_\delta$ such that
\begin{equation}
|u(x_r)| \leq C r, \quad \forall r > 0
\end{equation}
for some fixed constants $\delta, C$ independent from $r$. Then one can prove that $u$ grows linearly from the origin. It should be noted here that (1.5) is always true for the solutions to the one-phase problem provided that the origin is a non-isolated free boundary point; see (5.9).

1.4. **Blow-up limits and global solutions.** Let $u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g)$, $j = 1, 2, \ldots$, and let $x_0$ be a contact point, i.e. $x_0 \in \Gamma(u_j) \cap B'_1$. Typically $x_0 = 0$. For $r_j > 0$ we introduce the **blow-up sequence** of functions at $x_0$,
\begin{equation}
v_j(x) = \frac{u_j(x_0 + r_j x)}{r_j}, \quad r_j \downarrow 0 \text{ as } j \to \infty.
\end{equation}

If the sequence $v_j$ is bounded in a suitable space, then sending $r_j$ to $0$ we obtain a so called **blow-up limit** $u_0$. One of our main objectives in this paper is to classify the blow-up limits of the sequence $v_j$ in (1.6) as $j$ tends to infinity. It is noteworthy that, in general, the blow-up limit depends on the sequence $\{r_j\}_1^\infty$. Thus the blow-up limit $u_0$ is not unique. Hence it is natural to address the classification of blow-up limits. To do so we employ the monotonicity formula (1.10) and show that the blow-up at the contact points is only one of the functions (3.3) (see Sections 4.3 and 7.1).

The classification of all possible blow-up limits is based on geometric properties that these functions share, notably the linear growth and the homogeneity.
Definition 1.5. Let $u$ be a local minimizer in $\mathbb{R}^n_+$.

1° We say that $u$ is a **global solution** if $u \in \mathcal{P}_\infty$, where

$$
\mathcal{P}_\infty(C) = \bigcap_{r>0} \left\{ u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm) : u(0, x') = \alpha_+ x_2^+ - \alpha_- x_2^-, \ |u(x)| \leq C(x_1 + |x_2|) \right\}
$$

for some positive constant $C$ and $\mathcal{P}_r(n, \lambda_\pm, \alpha_\pm) = \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, 0)$.

2° The class of all **homogeneous global solutions** is denoted by

$$
\mathcal{H}\mathcal{P}_\infty(C) = \{ u \in \mathcal{P}_\infty : u(tx) = tu(x), \forall t > 0 \}.
$$

This definition requires some explanation. First we note that any blow-up limit of linearly growing $u$ is a global solution. Moreover it follows from the monotonicity theorem in Section 1.3 that the blow-up $u_0 \in \mathcal{H}\mathcal{P}_\infty$. The linear growth constant $C$ appearing in the definition must be consistent with the constants $\alpha_\pm$ that determine the boundary date. Clearly we must have $C \geq \max(\alpha_+, \alpha_-)$, otherwise at least one of $\alpha_\pm$ must be zero. A posteriori $\mathcal{H}\mathcal{P}_\infty$ contains only two functions, by Theorem C 3.3, linking $C$ with constants $\lambda_\pm$ too. In fact if $\frac{\lambda_+^2 - \lambda_-^2}{\alpha_+^2 - \alpha_-^2} - 1 < 0$, then $\mathcal{H}\mathcal{P}_\infty$ is empty. Therefore whenever constant $C$ is chosen large enough and $\frac{\lambda}{\alpha_+^2 - \alpha_-^2} - 1 \geq 0$, the resulted class of global homogeneous solutions is determined uniquely.

Finally we define the extreme global solutions and stability in order to classify the global solutions.

Definition 1.6.

1° $u \in \mathcal{P}_\infty(n, \lambda_\pm, \alpha_\pm)$ is said to be the smallest (resp. largest) global solution if for any $v \in \mathcal{P}_\infty(n, \lambda_\pm, \alpha_\pm)$ we have $u \leq v$ (resp. $u \geq v$).

2° The class of all local minimizers that after blow-up coincide with the smallest homogeneous global solution $v_S$:

$$
\mathcal{P}_r' = \left\{ u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g) : \lim_{r_j \to 0} \frac{u(r_j x)}{r_j} = v_S(x), \text{ for some sequence } r_j \right\}.
$$

If $u \in \mathcal{P}_r'$, then we say that $u$ is **stable**.

2. Linear growth: A Heuristic discussion

In analyzing the behavior of the free boundary one needs, in general, to start with the growth rate of the solution at free boundary points. Lipschitz regularity, up to the boundary, would be the most desirable property for minimizers of our functional. This property, or at least the linear growth property at the origin, is indispensable for the rest of the theory to follow.

In general, one cannot expect this property to hold, and one is forced to impose conditions to assure this. Indeed, a harmonic function in $B^+_1$ with merely Lipschitz data on $\{ x_1 = 0 \}$ is not Lipschitz. In such cases the extra logarithmic term enters into the game, and the solution will belong merely to the little-o Zygmund class

$$
|u(x)| \leq C|x| \log |x|^{-1}.
$$

In the one-phase case it is possible to obtain linear growth from the origin, provided the origin is a non-isolated free boundary point. In other words if there is a sequence of free boundary points in $\{ x_1 > 0 \}$ approaching the origin, then we expect linear
growth for the solutions. We will state and give a proof of this below. A similar result of this type was proven in [AC]. Observe that if, even in the one-phase case, we chose the boundary data large enough, e.g., \( \alpha_+^2 > \Lambda \), then one may show that the function \( u \) minimizing \( J \) is harmonic in the upper half ball; see Section 7.1. Thus, a harmonic function with Lipschitz data can impossibly be Lipschitz up to the boundary.

For the two-phase problem the analysis becomes much more complicated, and we could not find any complete theory. Since the Dirichlet data has two signs close to the origin

\[
f(x) \approx \alpha_+ x_2^+ - \alpha_- x_2^-,
\]

the free boundary \( \partial \{ u > 0 \} \) is always present in the upper half ball. The problem is that it might approach the fixed boundary \( \{ x_1 = 0 \} \) tangentially, and give rise to a non-Lipschitz behavior of the solution. (This argument does not apply to the one-phase case.) The reader may verify that if the free boundary (in two phase case) approaches tangentially to the fixed boundary and at the same time the solution is Lipschitz, then a blow-up limit would result in the fact that one of the phases vanishes but the boundary data is a two-phase data, and hence a contradiction would arise. This, in particular, suggests that for the two-phase problem, a natural condition to impose is that the free boundary does not touch the fixed one in a tangential fashion.

It is also not too hard to prove that there are certain Lipschitz boundary data, for which the solution is not Lipschitz and touches the fixed boundary tangentially. For the proof we would need a classification of the homogeneous global solution (as in Theorem C). Suppose \( n = 2 \); then the proof of Theorem C is more or less elementary in this case (see the proof). If we accept this result, for the moment, we see that for \( \alpha := \alpha_+ = \alpha_- \) and \( \Lambda > 0 \) one may conclude that the solution cannot be Lipschitz. Otherwise, if this was the case, then a blow-up of the solution would result in a global solution, with linear growth. Hence the classification theorem, Theorem C, would then suggest that the solution is \( u = \alpha x_2 \), but then the free boundary condition \( |\nabla u^+|^2 - |\nabla u^-|^2 = \Lambda > 0 \) fails.

From the representation \((3.3)\), we also see that if \( \alpha_+^2 - \alpha_-^2 > \Lambda \), then again an up to the boundary Lipschitz continuous solution cannot exist.

The question of finding optimal conditions, that assure linear growth for the minimizers from the origin, is still open. We have partially answered this question in Theorems A and B, below, under mild conditions.

3. MAIN RESULTS

In this section we state the main results of this paper. To begin our analysis we need the optimal growth estimate for a local minimizer \( u \) near the contact points. More precisely we have to show that \( u \in P_1(n, \lambda \pm, \alpha \pm, g) \) grows linearly away from \( z \in \partial \{ u > 0 \} \cap \Pi \). Clearly we can assume that \( z = 0 \).

**Theorem A.** Let \( u \in P_1(n, \lambda \pm, \alpha \pm, g) \) and let either of the following hold:

1° \( u \in P_1(n, \lambda \pm, \alpha \pm, g, \delta) \), i.e. the condition \((1.4)\) (or its weaker form \((1.5)\) ) is satisfied for some \( \delta > 0 \) and all \( r < 1 \).

2° \( \alpha_- = 0 \), \( g \geq 0 \) and the origin is a non-isolated free boundary point.
Then
\begin{equation}
|u(x)| \leq C|x|, \quad x \in B^+_\frac{1}{2},
\end{equation}
where \( C \) depends on \( n, \lambda_{\pm}, \alpha_{\pm}, \sup_{B_1} |u| \) and \( \delta, g. \)

As for part 2\(^a\) of Theorem A, let us note that the weak \( \delta-\)NT assumption \((1.5)\)
is always satisfied for one-phase problem; see \((6.9)\).

Our next result is an improvement of Theorem A in the following sense: Let \( u_0 \) be a blow-up of \( u \) at the origin; then \( |u_0(x)| \leq C|x| \) in \( \mathbb{R}^n_+ \) and \( u_0(x) = \alpha_+ x_2^+ - \alpha_- x_2^- \) on \( \Pi. \) However these are not enough to conclude that \( u_0 \in \mathcal{P}_\infty \), since the estimate
\( |u_0(x)| \leq C(x_1 + |x_2|) \) in the definition of \( \mathcal{P}_\infty \) does not follow immediately. Suppose \( T_{i,R}(x) = x + Re_i, i \neq 2, \) is the translation in \( e_i \) direction by \( R \in \mathbb{R}. \) Then \( u_0(T_{i,R}(x)) \) is also a minimizer, but possibly with a different constant \( C \) in the linear growth estimate. Does the boundary data \( \alpha_+ x_2^+ - \alpha_- x_2^- \), depending only on \( x_2 \), have any effect? Do we get the same growth for \( u_0(T_{i,R}(x))? \)

**Theorem B.** Let \( u \in \mathcal{P}_1(n, \lambda_{\pm}, \alpha_{\pm}, g) \) and suppose that there is \( C > 0 \) such that
\begin{equation}
|u(x)| \leq C|x - z|, \quad \forall z \in \partial \{ u > 0 \} \cap B_\frac{1}{2}.
\end{equation}

Then for any blow-up limit \( u_0 \) of \( u \) at the origin we have
\[ |u_0(x)| \leq C(x_1 + |x_2|). \]
In particular any blow-up limit of \( u \) belongs to \( \mathcal{P}_\infty(C) \).

Theorem B is used to classify homogeneous global solutions by employing a customary dimension reduction argument. Notably we show that if \( u \in \mathcal{H}\mathcal{P}_\infty \), then \( u \) depends only on \( x_1 \) and \( x_2 \) variables. Again we note that the growth estimate \( u(x) \leq C|x - z| \) is true for the one-phase case. As for the two-phase case, one can prove that the uniform \( \delta \) or weak \( \delta-\)NT condition (see \((1.5)\)) for each contact point \( z \in B_\frac{1}{2} \) will imply \( |u(x)| \leq C|x - z| \) in view of Theorem A.

To set forth the implications of Theorem B we return to the translated solution \( u_0(T_{i,R}(x)), i \geq 3. \) For arbitrary \( R_1 < R_2 \) one can show that \( \max(\{u_0(T_{i,R_1}(x))\}) \), \( u_0(T_{i,R_2}(x)) \) is a minimizer of \( J(u, B_1) \) with boundary values \( \max(\{u_0(T_{i,R_1}(x))\}) \), \( u_0(T_{i,R_2}(x)) \) on \( \partial B^+_1 \). Moreover by Theorem B the maximum of solutions has exactly the same linear growth as \( u_0 \). Thus we can construct a translation invariant maximal global solution. Repeating this argument for all \( i \geq 3 \) we obtain a maximal global solution depending on \( x_1 \) and \( x_2 \) only. The minimal solution is constructed analogously. Writing the Laplace operator in polar coordinates we obtain the classification of global homogeneous solutions.

**Theorem C.** In \( \mathbb{R}^2 \), there are only two homogeneous global solutions:
\begin{align}
v_L &= \alpha_+ (\gamma x_1 + x_2)^+ - \alpha_- (\gamma x_1 + x_2)^-, \\
v_S &= \alpha_+ (-\gamma x_1 + x_2)^+ - \alpha_- (-\gamma x_1 + x_2)^-,
\end{align}
where \( \gamma = \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2}} + 1. \) Thus \( \mathcal{H}\mathcal{P}_\infty = \{v_S, v_L\}. \)

This also holds in \( \mathbb{R}^n \), for \( n > 2, \) and for the one-phase case, with \( \alpha_- = \lambda_- = 0. \)
If \( \Lambda \leq \alpha_+^2 - \alpha_-^2, \) then there is no free boundary.

An obvious consequence of this theorem is that for any \( u \in \mathcal{P}_r, \) the angle of the touch between the free and fixed boundaries is dictated by the behavior of \( v_S \) or \( v_L. \)
From Theorem C one can deduce that the free boundary approaches the origin along the plane \( \{ x \in \mathbb{R}^n : \gamma x_1 = x_2 \} \). The approach is uniform for the small solution, but in general not for the large one. For the precise formulation we introduce some notation: Let \( \sigma \) be a modulus of continuity and consider

\[
K^+_\sigma := \left\{ x : x_1 > 0, x_2 > 0, \frac{x_2}{\gamma + \sigma(|x|)} < x_1 < \frac{x_2}{\gamma - \sigma(|x|)} \right\},
\]
\[
K^-_\sigma := \left\{ x : x_1 > 0, x_2 < 0, \frac{-x_2}{\gamma + \sigma(|x|)} < x_1 < \frac{-x_2}{\gamma - \sigma(|x|)} \right\}.
\]

**Theorem D.** Let \( u \in \mathcal{P}_r \) (see Section 1.2), and let \( v_S, v_L \) be defined by (3.3). We consider \( n = 2 \) for the two-phase problem and \( n \geq 3 \) for the one-phase problem. Then, close to the origin, \( \Gamma(u) \) touches tangentially one of the hyperplanes \( \Gamma(v_S) = \{ x \in \mathbb{R}^n : x_2 = \gamma x_1 \} \) or \( \Gamma(v_L) = \{ x \in \mathbb{R}^n : x_2 = -\gamma x_1 \} \). More precisely there exists a modulus of continuity \( \sigma(r) = \sigma(u, r) \) and \( r_0 \in (0, 1) \) such that for any \( r \in (0, r_0) \) either

\[
\Gamma(u) \subset B_r^+ \bigcap K^+_\sigma \quad \text{or} \quad \Gamma(u) \subset B_r^- \bigcap K^-_\sigma.
\]

If \( u \) touches the hyperplane \( \Gamma(v_S) \) (i.e. \( u \in \mathcal{P}_r^+ \)), then \( \sigma(r) \) and \( r_0 \) are independent of \( u \), and thus the neighborhood \( B_{r_0} \) is uniform.

It follows from the definition of \( \mathcal{P}_\infty \), and by Theorem B, that for \( u \in \mathcal{P}_r \), the limit \( u_j(x) = \frac{u(r_jx)}{r_j} \), \( r_j \downarrow 0 \), is a global solution. Furthermore, from Weiss’ formula [1], we have that the limit has to be a homogeneous function of degree one. Thus the blow-up limits belong to \( \mathcal{H}\mathcal{P}_\infty \). However the class of global solutions \( \mathcal{P}_\infty \) may contain non-homogeneous solutions, as our last theorem shows.

**Theorem E.** There exists a non-homogeneous global solution with boundary values \( \alpha_+ x_1^+ \).

A consequence of Theorem E is a kind of instability of the angle of touch, which amounts to the fact that if a free boundary is asymptotically close to \( v_S \), then by slight perturbation of the boundary data the free boundary may come close to \( v_L \), asymptotically. This constitutes the idea in the construction of global non-homogeneous solutions in Theorem E.

Theorem E exhibits the structure of the class of global solutions, namely the fact that there exist non-homogeneous functions in \( \mathcal{P}_\infty \). This is due to the following: If \( u_j \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm^j) \), then the blow-up sequence \( v_j = \frac{u_j(r_jx)}{r_j} \) converges to a global solution \( v_\infty \in \mathcal{P}_\infty(n, \lambda_\pm, \alpha_\infty^j) \) where \( \alpha_\infty^j = \lim_{j \to \infty} \alpha_\pm^j \). But it does not necessarily imply that \( v_\infty \) is homogeneous. If \( u = u_j \) and \( \alpha_\pm = \alpha_\infty^j \), then from Weiss’ monotonicity theorem it follows that \( v_\infty \) is homogeneous; see Section 4.3.

4. Technicalities

In this section we gather a number of useful properties that almost all local minimizers share. Some of these properties are of local nature and some hold true near the fixed boundary, e.g. Hölder continuity. Although the boundary extensions follow from standard techniques we have supplied the proofs for the reader’s convenience.

4.1. Uniform Hölder continuity for \( u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g) \). We begin with recalling some well-known facts, which can be found in [ACP].
Proposition 4.1. Let \( u \) be a local minimizer of \( J(u) \) in \( B_1^+ \) and \( \Lambda = \lambda_+^2 - \lambda_-^2 > 0 \). Then
1° \( u \) is a bounded subharmonic function in \( B_1^+ \) \[ACF\] Theorem 2.3],
2° \( u \) is harmonic in the interior of \( B_1^+ \setminus \{ u = 0 \} \) \[ACF\] Theorem 2.4],
3° \( u^+ \) is non-degenerate \[ACF\] Corollary 3.2],
4° if \( \text{meas} \{ u = 0 \} = 0 \), then \( |\nabla u^+|^2 - |\nabla u^-|^2 = \Lambda \) across the free boundary \( \Gamma(u) \) in some weak sense \[ACF\] Theorem 2.4].

The starting point in our study is the uniform Hölder continuity of local minimizers. It will allow us to translate some of the well-known local properties of \( u \) into the boundary case.

Lemma 4.2. Let \( u \in \mathcal{P}_1(n, \lambda_+, \alpha_\pm, g) \). Then \( u \) is bounded in \( B_\frac{1}{2}^+ \).

Proof. By Theorem 2.1 in \[ACF\] \( u \) is continuous in each subdomain \( D \subset B_1^+ \). Moreover by Proposition 4.1 \( u \) is harmonic in \( \{ u \neq 0 \} \), hence \( u^+ \) is subharmonic. Indeed, if \( x \in \Omega^+(u) \), then \( f_{B_r(x)} u^+ \geq u(x) \) for each \( r < r_0 \) such that \( B_{r_0}(x) \subset \Omega^+(u) \), otherwise \( f_{B_r(x)} u^+ \geq 0 = u^+(x) \) for \( x \not\in \Omega^+(u) \). Thus the mean value property is satisfied locally. Thus \( u^+ \) is subharmonic.

Let \( v \) be the harmonic lifting of \( u \), i.e. \( \Delta v = 0 \), \( v|_{\partial B_1^+} = u^+ \). From the maximum principle \( u^+ \leq v \) and \( \int_{B_1^+} |\nabla v|^2 \leq \int_{B_1^+} \nabla u^+)^2 \). In particular \( \|v\|_{H^1(B_1^+)} \leq C\|u\|_{H^1(B_1^+)} \) with some tame constant \( C \). This yields that \( v \in C^0(B_\frac{1}{2}^+) \). Hence \( u^+ \) is bounded in \( B_\frac{1}{2}^+ \).

By a similar argument one can show that \( u^- \) is bounded. \( \square \)

The next proposition is more general and can be applied to families of local minimizers.

Proposition 4.3. Let \( u \in \mathcal{P}_{R_0}(n, \lambda_\pm, \alpha_\pm, g) \) and
\[
\sup_{B_{2R}} |u| + \alpha_+ + \alpha_- + \Lambda + \|f\|_{C^{0,1}} \leq M, \quad 2R < R_0.
\]
Then there are positive constants \( \beta = \beta(n, R, M) \) and \( C = C(n, R, M) \) such that \( u \in C^\beta(B_R^+) \) and \( \|u\|_{C^\beta(B_R^+)} + \|u\|_{H^1(B_R^+)} \leq C \).

Proof. Let \( w \) be the harmonic lifting of \( u \) in \( B_{2R}^+ \). Because \( u - w \in H_0^1(B_{2R}^+) \), it follows that
\[
\int_{B_{2R}^+} |\nabla u|^2 - |\nabla w|^2 = \int_{B_{2R}^+} |\nabla (u - w)|^2 + \int_{B_{2R}^+} 2\nabla w \cdot \nabla (u - w) = \int_{B_{2R}^+} |\nabla (u - w)|^2.
\]
Then from \( J(u, B_{2R}^+) \leq J(w, B_{2R}^+) \) and the equality above we obtain
\[
\int_{B_{2R}^+} |\nabla (u - w)|^2 = \int_{B_{2R}^+} |\nabla u|^2 - |\nabla w|^2 \leq \int_{B_{2R}^+} \Lambda |\chi_{\{w > 0\}} - \chi_{\{u > 0\}}| \leq \Lambda |B_1| (2R)^n.
\]

Take \( \eta \in C^\infty_0(B_{2R}) \), \( \eta \equiv 1 \) in \( B_R \), \( 0 < \eta \leq 1 \) and \( |\nabla \eta| \leq \frac{C}{R} \) for some dimensional constant \( C \). Obviously \( (w - f)\eta^2 \in H_0^1(B_{2R}^+) \) can be used as a test function in the weak formulation of \( \Delta w = 0 \),
\[
\int_{B_{2R}^+} \eta^2 |\nabla w|^2 = \int_{B_{2R}^+} \nabla w \left[ \nabla f \eta^2 - 2\eta \nabla \eta (w - f) \right].
\]
Applying Cauchy-Schwarz inequality and the estimate $|\nabla \eta| \leq \frac{C}{R}$ we obtain Caccioppoli’s inequality

\begin{equation}
\int_{B_R^+} |\nabla w|^2 \leq 8 \int_{B_{2R}^+} \left[ |\nabla f|^2 + \frac{4C^2}{R^2} (w - f)^2 \right] \leq C_1,
\end{equation}

where $C_1 = 4|B_2|^2 M^2 \left[ (2R)^n + 16C^2 (2R)^{n-2} \right]$.

Since $u - f = 0$ in $B_{2R}^+$ we can apply Poincaré’s inequality to conclude that $\int_{B_{2R}^+} (u - f)^2 \leq \frac{c_0}{R^2} \int_{B_{2R}^+} |\nabla (u - f)|^2$ depends on the dimension $n$ and $\mathcal{H}^{n-1}(B_R')$, the $n-1$ dimensional Hausdorff measure of $B_R$.

Combining inequalities (4.1), (4.2) and Poincaré’s inequality we get

\begin{equation}
\int_{B_R^+} |\nabla u|^2 \leq 2 \left( \int_{B_R^+} |\nabla w|^2 + \int_{B_R^+} |\nabla (w - u)|^2 \right)
= 2(C_1 + \Lambda |B_1|(2R)^n) \equiv C_2;
\end{equation}

thereby

\begin{equation}
\int_{B_R^+} u^2 \leq 2 \int_{B_R^+} f^2 + 2 \int_{B_R^+} (u - f)^2
\leq 2 \left( M^2 \frac{|B_1|}{2} R^n + \frac{c_0}{R^2} \int_{B_R^+} |\nabla (u - f)|^2 \right)
\leq M^2 |B_1| R^n + \frac{4c_0}{R^2} \left( \int_{B_R^+} |\nabla u|^2 + \int_{B_R^+} |\nabla f|^2 \right)
\leq M^2 |B_1| R^n + \frac{4c_0}{R^2} (C_2 + M^2 \frac{|B_1|}{2} R^n) \equiv C_3,
\end{equation}

implying that $\|u\|_{H^1(B_R^+)} \leq \sqrt{C_2 + C_3} \equiv C_4$.

As for Hölder continuity let us note that in view of Theorem 7.19 of [GT] it is enough to show that for $B_r^+(z) \subset B_{2r}^+, z \in B_R', r < \frac{1}{2}$, we have

\begin{equation}
\int_{B_r^+(z)} |\nabla u| \leq C_5 r^{n-1+\beta},
\end{equation}

for some $\beta > 0$ and $C_5$ depending on $M, n$ and $R$. Indeed if $z \in B_R^+$ and $|z - z'| > \frac{1}{4}$ we get that $B_\frac{1}{2}(z) \subset B_r^+$, and by local continuity Theorem 2.1 of [ACF] $u$ is uniformly continuous with some $\beta > 0$ depending only on $\|u\|_{H^1(B_R^+)}$, $n$ and $M$.

Whilst for $r < \frac{1}{2}$ either $|z - z'| \leq r$ and $B_r^+(z) \subset B_{2r}(z')$ or $r < |z - z'| < \frac{1}{2}$.

First we deal with the case $z \in B_R^+$ and $B_{\frac{1}{2}r}(z) \subset B_R^+$. Let $v$ be the harmonic lifting of $u$ in $B_{4r}^+(z)$, i.e. $\Delta v = 0$ in $B_{4r}^+$ and $v - u \in H^1_0(B_{4r}^+(z))$. Since $J(u, B_{4r}^+) \leq J(v, B_{4r}^+)$ it follows that

$$
\int_{B_{4r}^+(z)} |\nabla u|^2 + \Lambda \chi_{\{u > 0\}} \leq \int_{B_{4r}^+(z)} |\nabla v|^2 + \Lambda \chi_{\{v > 0\}}.
$$
Thereby

\begin{equation}
\int_{B^+_r(z)} |\nabla u|^2 - |\nabla v|^2 = \int_{B^+_r} (\nabla u - \nabla v)(\nabla u + \nabla v) \\
= \int_{B^+_r(z)} |\nabla (u - v)|^2 \\
\leq \int_{B^+_r(z)} \Lambda \chi_{\{v > 0\}} - \Lambda \chi_{\{u > 0\}} \\
\leq M|B_4|r^n.
\end{equation}

From the triangle inequality we get

\begin{equation}
\int_{B^+_r(z)} |\nabla u| \leq \int_{B^+_r(z)} |\nabla (u - v)| + \int_{B^+_r(z)} |\nabla v| \\
\leq M|B_4|r^n + \int_{B^+_r(z)} |\nabla v|,
\end{equation}

where the last line follows from (4.6) and Cauchy-Schwarz inequality.

It remains to show that there are constants \( \beta \in (0,1), C_6 \) depending on \( M, R \) and \( n \) such that

\begin{equation}
\int_{B^+_r(z)} |\nabla v|^2 \leq C_6 r^{n-2+2\beta}.
\end{equation}

To see this take \( \eta \in C_0^\infty(B_4) \) such that \( \eta \equiv 1 \) in \( B_r, 0 \leq \eta \leq 1, \) and \( |\nabla \eta| \leq \frac{C}{r}, \)

\( C \) is a dimensional constant; then \( \eta^2(v - f) = 0 \) on \( \partial B^+_4 \) and we have from the weak formulation of harmonicity of \( v \)

\[ \int_{B^+_4} \nabla v[2\eta\nabla \eta(v - f) + \eta^2(\nabla v - \nabla f)] = 0. \]

Rearranging the terms and applying Hölder’s inequality we get

\[ \int_{B^+_4} \eta^2|\nabla v|^2 = -\int_{B^+_4} \nabla v\eta[2\nabla \eta(v - f) - \eta \nabla f] \\
\leq \varepsilon \int_{B^+_4} \eta^2|\nabla v|^2 + \frac{1}{\varepsilon} \int_{B^+_4} [2\nabla \eta(v - f) - \eta \nabla f]^2. \]

Choosing \( \varepsilon \) suitably small and recalling that \( \eta \equiv 1 \) in \( B_r \) we get the estimate

\begin{equation}
\int_{B^+_r} |\nabla v|^2 \leq \frac{C}{\varepsilon} \int_{B^+_4} [2\nabla \eta(v - f) - \eta \nabla f]^2.
\end{equation}

According to Lemma 1.2.4 in [K] \( v \) is Hölder continuous with some exponent \( \gamma = \gamma(n, M, R) \in (0,1), \) because \( |v| \leq M, ||v||_{H^1(B^+_4)} \leq M + ||u||_{H^1(B^+_4)} \) and we have the estimate

\begin{equation}
\int_{B^+_4} [2\nabla \eta(v - f) - \eta \nabla f]^2 \leq C_7 \sup_{B^+_4} |v - f| r^{n-1} + C_7 \sup_{B^+_4} |\nabla f| r^n \leq C_8 r^{n-1+\gamma},
\end{equation}

where \( C_8 \) depends only on \( n, M, R \) and to get the first inequality we used the estimate \( |\nabla \eta| \leq \frac{C}{r}. \) Thus choosing \( \beta = \frac{1+\gamma}{2} \) the result follows. Notice that \( \beta \) depends only on \( n, M, R. \)
Finally it remains to show (1.5) for $B_r^+ (z)$ with $z \in B_R^+$ and $r \leq |z - z'| \leq \frac{1}{2}$. Notice that (1.7) and (1.9) still hold for this case. As for the estimate (1.8), it follows from Poisson representation and the bound $|v| \leq M$. □

Remark 4.4. One can apply Proposition 1.3 to a countable family $\mathcal{P}_{R_j} (n, \lambda^j_{\pm}, \alpha^j_{\pm}, g_j)$, $j = 1, 2, \ldots$, as $R_j \to \infty$; see the proof of (5.3) and (5.4) below.

4.2. Implications of linear growth. The standard regularity result for free boundary problems states that the free boundary is smooth away from an ineluctable singular set of smaller co-dimension. The genus of regular points is characterized by flatness.

Mathematically the blow-up consists of scaling $u$ in small balls centered on the free boundary: for $u \in \mathcal{P}_1 (n, \lambda_{\pm}, \alpha_{\pm}, g)$ with linear growth at the origin, the scaled functions $v_j (x) = \frac{u(r_j x)}{r_j}$ are uniformly bounded as $r_j \searrow 0$. Since $f (0) = 0$, one readily verifies that $v_j \in \mathcal{P}_{1/r_j} (n, \alpha_{\pm}, \lambda_{\pm}, g_j)$, where $g_j (x) = \frac{g(r_j x)}{r_j}$. Clearly $v_j$ is defined in $B_{r_j}^+$ and provides a better picture of the free boundary at the origin.

Thus by scaling we obtain a sequence of function $v_j$ and a sequence of corresponding free boundaries $\Gamma_j = \Gamma (v_j)$. One expects that the convergence $v_j \to v_0$ implies $\Gamma_j \to \Gamma_0 = \Gamma (v_0)$ in Hausdorff distance, which will follow immediately from a compactness of $v_j$ in a suitable class of functions. For the reader’s convenience we recall Theorem 3.1 from [KKS].

Proposition 4.5 ([KKS]). Let $v_j$ be a blow-up sequence of $u_j$, as in (1.6), with $u_j \in \mathcal{P}_1 (n, \lambda_{\pm}, \alpha_{\pm}, g)$ and $x_0 = 0$. Further assume that $u_j$ have uniform linear growth. Then, after passing to a subsequence, there exists $v \in \mathcal{P}_\infty$ so that

1° $v_j \to v$ uniformly on compact subsets of $\mathbb{R}^n_+$ and in $C^\beta (E)$, $0 < \beta < 1$, for each $E \subset \mathbb{R}^n_+$,

2° for each $M$, $v_j \to v$ weakly in $H^1 (B_M^+)$,

3° for each $M$, $\chi \{ v_j > 0 \} \to \chi \{ v > 0 \}$ in $L^1 (B_M^+)$,

4° $\nabla v_j (x) \to \nabla v (x)$ for a.e. $x$,

5° for each $\delta > 0$, $E \subset B_M^+$, $\text{dist} (E, \Pi) \geq \delta$, $0 < r < \delta / 4$, for $j$ large

$$\partial \{ v_j > 0 \} \cap E \subset \bigcup_{x \in \{ v > 0 \} \cap E_{\delta/2}} B_r (x),$$

and

$$\partial \{ v > 0 \} \cap E \subset \bigcup_{x \in \{ v_j > 0 \} \cap E_{\delta/2}} B_r (x),$$

where $E_{\delta/2}$ is a $\delta/2$-neighborhood of $E$.

4.3. Weiss’ energy. It follows from [W1] (see also [W2]) that for any $u \in \mathcal{P}_\tau (n, \lambda_{\pm}, \alpha_{\pm}, 0)$

$$W (R, u, x_0) = W (R) = \frac{1}{R^n} \int_{B_R^+(x_0)} |\nabla u|^2 + \Lambda \chi \{ u > 0 \} - \frac{1}{R^{n+1}} \int_{S_R^+(x_0)} u^2$$

is a non-decreasing function of $R$, with $x_0 \in \Gamma (u), B_R (x_0) \subset B_r^+$, and

$$dW \frac{d}{dR} = \frac{1}{R^n} \int_{S_R^+(x_0)} (\nabla u \cdot v - \frac{u}{R})^2.$$
Proposition 4.6. Let $u \in \mathcal{P}_1(n, \lambda_{\pm}, \alpha_{\pm}, g)$ and $g(x) = C|x|^{1+\kappa}$. If $u$ has linear growth, then $W(R, u, 0)$ is a non-decreasing function of $R$ and
\[
\frac{dW}{dR} \geq \frac{1}{R^n} \int_{S_R^+} \left( \nabla u \cdot \nu - \frac{u}{R} \right)^2.
\]
In particular any blow-up limit of $u$ at the origin is a homogeneous function of degree one.

Remark 4.7. For clarity we take $g(x) = C|x|^{1+\kappa}$ with $C, \kappa > 0$. The case of more general $g(x) = o(|x|)$ can be dealt with similarly, namely one needs to add a corrective term to $W$ to maintain the monotonicity.

Proof. If $g \neq 0$ and $u \in \mathcal{P}_r(n, \alpha_{\pm}, \lambda_{\pm}, g)$, then some extra care is needed to prove the estimate from below for the derivative $W'$. See Lemma 11.1 in Appendix 1 for the proof.

It remains to show that $W(r, u, 0)$ is bounded when $r$ tends to zero. If $v$ is the harmonic lifting of $u$ in $B^+_4$ and $u$ has linear growth at 0, i.e. $\sup_{B^+_4} |u| \leq Cr$, then by maximum principle $\sup_{B^+_4} |v| \leq Cr$. From Caccioppoli’s inequality (4.9) we have
\[
\int_{B^+_r} |\nabla v|^2 \leq Cr^n.
\]
Hence
\[
\int_{B^+_r} |\nabla u|^2 \leq 2 \int_{B^+_r} |\nabla v|^2 + 2 \int_{B^+_r} |\nabla (u - v)|^2,
\]
which, in view of (4.6), implies that $W$ is bounded for small $r$, whenever $u \in \mathcal{P}_1(n, \lambda_{\pm}, \alpha_{\pm}, g)$ is a linearly growing solution. \qed

5. Proof of Theorem A

The proof of Theorem A consists of two parts. The first one deals with the two-phase problem. Our method is based on the dyadic scaling argument. If the statement of Theorem A fails, then it allows us to construct a linearly growing, non-degenerate harmonic function $v_0$ in $\mathbb{R}^n_+$ vanishing on $\partial \mathbb{R}^n_+$ and at some interior point of $\mathbb{R}^n_+$. The latter is due to $\delta$–NT condition; see (1.4). Thus, in view of the Liouville theorem, $v_0$ is zero, which contradicts the non-degeneracy of $v_0$.

5.1. Two-phase case. Set
\[
S(j, u) := \sup_{B^+_2} |u|.
\]
It suffices to show
\[
S(j + 1, u) \leq \max \left\{ \frac{c2^{-j}}{2}, \frac{S(j, u)}{2}, \ldots, \frac{S(0, u)}{2^{j+1}} \right\}
\]
for some positive constant $c$. Let us suppose that (5.1) is not true. Then there exists a sequence of minimizers $u_j \in \mathcal{P}_1(n, \lambda_{\pm}, \alpha_{\pm}, g)$ and a sequence of integers $k_j$ so that
\[
S(k_j + 1, u_j) > \max \left\{ \frac{j2^{-k_j}}{2}, \frac{S(k_j, u_j)}{2}, \ldots, \frac{S(0, u_j)}{2^{k_j+1}} \right\}.
\]
Observe that from Lemma 1.2 \(|u_j| \leq M\); hence \(k_j \to \infty\). Put
\[
v_j(x) = u_j(2^{-k_j}x) S(k_j + 1, u_j).
\]

We wish to show that (5.2) implies uniform up-to-boundary estimates for the sequence \(v_j\). In fact there are positive constants \(\alpha\) and \(C\) depending on \(R\) but independent of \(j\) such that the following estimates hold:
\[
\begin{align}
\|v_j\|_{C^\alpha(B_R^+)} & \leq C(R), \\
\|v_j\|_{H^1(B_R^+)} & \leq C(R).
\end{align}
\]

For brevity we denote
\[
\epsilon_j = \frac{2^{-k_j}}{S(k_j + 1, u_j)}, \quad f_j(x) = \epsilon_j(\alpha_+ x_2^+ - \alpha_- x_2^-) + g_j(x),
\]
where \(g_j(x) = \frac{\tilde{g}(2^{-k_j}x)}{S(k_j + 1, u_j)}\). Recall that by (5.2)
\[
\epsilon_j = \frac{2^{-k_j}}{S(k_j + 1, u_j)} \leq \frac{1}{j} \to 0;
\]
thereby \(f_j \to 0\) when \(j \to \infty\), for \(g(x) = o(|x|)\).

Consider the scaled functional
\[
\tilde{J}_j(v, B_R^+) = \int_{B_R^+} |\nabla v|^2 + \epsilon_j^2 \lambda \chi_{\{v > 0\}}.
\]

If \(u_j \in P_1(n, \lambda_+, \alpha_+, g)\), then \(v_j \in P_{2\epsilon_j}(n, \epsilon_j \alpha_+, \epsilon_j \lambda_+ + g_j)\) provided \(R < 2k_j\). Indeed by a simple calculation we have
\[
\tilde{J}_j(v_j, B_R^+) = \int_{B_R^+} |\nabla v_j|^2 + \epsilon_j^2 \lambda \chi_{\{v_j > 0\}}
\]
\[
= \epsilon_j^2 2^{k_jn} \int_{B_{R/2^{k_j}}^+} |\nabla u_j|^2 + \lambda \chi_{\{u_j > 0\}}
\]
\[
= \epsilon_j^2 2^{k_jn} J(u_j, B_{R/2^{k_j}}^+).
\]

Furthermore for fixed \(R = 2^m\) we infer from (5.2) that
\[
\begin{align}
& \sup_{B_{R/2}^+} |v_j| = 1, \\
& \sup_{B_{2^m}^+} |v_j| \leq C 2^m, R = 2^m < 2^k, m \text{ is fixed.}
\end{align}
\]

Now we can apply Proposition 1.3 with \(\sup |v_j| + \epsilon_j(\alpha_+ + \alpha_- + \lambda_+ + \lambda_-) + \|f_j\|_{C^{0,1}} \leq M\) with \(M = 2^{m+1}\) and the estimates (5.3) and (5.4) follow.

Thereby we can extract a subsequence \(v_{j_k}\) which converges to some function \(v_0\) such that the following holds: for any fixed \(R > 0\)
\[
\begin{align}
(i) & \quad v_{j_k} \to v_0 \text{ in } C^\beta(B_R^+), \quad v_{j_k} \to v_0 \text{ weakly in } H^1_{loc}(\mathbb{R}_+^n), \\
(ii) & \quad \sup_{B_{R/2}^+} |v_0| = 1, \quad v_0(x) = 0, x \in \Pi \quad \text{by } C^\beta \text{ regularity,} \\
(iii) & \quad \Delta v_0 = 0 \text{ in } x_1 > 0, \\
(iv) & \quad v_0 \text{ has linear growth,} \\
(v) & \quad v_0(y_0) = 0 \text{ for some interior point } y_0 \text{ (by (1.4)).}
\end{align}
\]
Once all claims in (5.8) are proven we may use Liouville’s theorem for harmonic functions in \( \mathbb{R}^n_+ \) (utilizing (iii) and (iv)) to conclude \( v_0(x) = ax_1 \) for some constant \( a \neq 0 \). But then (ii), (v) and (vi) are in direct contradiction, and hence our supposition (5.2) is false.

Now we proceed by proving (5.8). The first claim follows from standard compactness arguments. The second one follows from (5.5) and the convergence of the traces of \( v_j \) in view of Hölder continuity.

Let us prove the third claim. Let \( D \subset \overline{B}^+_R \) be a domain and let \( R > 0 \) be fixed. Then \( v_j \in \mathcal{P}_{2^+}(u, \epsilon_j \alpha_{\pm}, \epsilon_j \lambda_{\pm}, g_j) \) for the scaled functional \( \tilde{J}_j \), defined by (5.7). Observe that for each \( \psi \in C_0^\infty(D) \)

\[
\tilde{J}_j(\psi, D) \to \int_D |\nabla \psi|^2 \quad \text{as} \quad j \to \infty.
\]

By (5.3) and (5.4), \( v_0 \) exists and \( \int_D |\nabla v_0|^2 \leq \liminf_{k \to \infty} \int_D |\nabla v_{j_k}|^2 \). According to (5.5) \( f_0 = v_0 = 0 \) on \( \Pi = \{ x : x_1 = 0 \} \), where \( f_0 = \lim_{j \to \infty} f_j \) uniformly.

Now let us take \( \psi \in H^1_0(D) \); then

\[
\tilde{J}_j(v_j, D) \leq \tilde{J}_j(v_j + \psi, D)
\]

or equivalently

\[
\int_D \epsilon^2_j \Lambda \chi_{\{v_j > 0\}} \leq \int_D -2\nabla v_j \nabla \psi + |\nabla \psi|^2 + \epsilon^2_j \Lambda \chi_{\{v_j + \psi > 0\}}.
\]

Thus by sending \( j_k \) to \( \infty \) and utilizing the weak convergence of gradients \( \nabla v_{j_k} \rightharpoonup \nabla v_0 \) in \( L^2(B^+_R) \), we conclude

\[
0 \leq \int_D -2\nabla v_0 \nabla \psi + |\nabla \psi|^2
\]

and upon adding \( \int_D |\nabla v_0|^2 \) to both sides we infer

\[
\int_D |\nabla v_0|^2 \leq \int_D |\nabla (v_0 - \psi)|^2.
\]

Since \( C_0^\infty(D) \) is dense in \( H^1_0(D) \) we conclude the proof of the third claim in (5.8).

The fourth claim follows from (5.2) as indicated above. Hence it remains to prove the fifth claim. By our assumption (1.4) (resp. (1.5)) there exists \( x_j \in B^+_{2^{-\epsilon_j}} \cap K_\delta \) such that \( u_j(x_j) = 0 \) (resp. \( |u(x_j)| \leq C |x_j| \)). Thereby

\[
\frac{1}{2} \leq \frac{|x_j|}{2^{k_j}} \leq 1.
\]

If we set \( y_j = \frac{x_j}{2^{k_j}} \), then one can easily verify that \( y_j \in (B_1 \setminus B_{1/2}) \cap K_\delta \) and \( y_j \to y_0 \), for some \( y_0 \in (B^+_1 \setminus B^+_{1/2}) \cap K_\delta \). Clearly \( v_0(y_0) = 0 \) by (5.5) and Hölder continuity.

Now the proof of (3.1), for the two-phase case is complete.

5.2. One-phase case. To prove (5.1) in Theorem A 2\(^c\), we need to work out condition (v) in (5.8), because the others follow as above. For the two-phase case, (v) was justified by assumption (1.4) whilst for one-phase case, (1.4) was replaced by the condition that the origin is a non-isolated free boundary point. Indeed, this would be enough to force through a similar condition as that in (v) of (5.8). However, the analysis is slightly more delicate and needs care.
Suppose for a sequence \( k_j \to \infty \) we have \( \{u = 0\} \cap (B_{\rho_{k_j}} \setminus B_{\rho_{k_j+1}}) \cap K_\delta \neq \emptyset \), where \( \rho_{k_j} = \frac{1}{2^{k_j}} \). We consider the family of balls \( B_{d_j}(\zeta) \) with \( d_j = \frac{1}{4}\rho_{k_j} \) such that \( \zeta \in S_{\frac{3}{2}\rho_{k_j}}^+ \) and \( B_{d_j}(\zeta) \) is above the free boundary \( \Gamma(u) \). Then in this family of balls there is one that touches the free boundary \( \Gamma(u) \). Let \( B_{d_j}(z) \) be such a ball touching the free boundary at \( z_0 \). Clearly \( u \) is positive and harmonic inside \( B_{d_j}(z) \) and attains its minimum at \( z_0 \), therefore we can apply Lemma 11.19 from [CS] to get the estimate

\[
(5.9) \quad u(z) \leq Cd_j \frac{\partial u}{\partial \nu}(z_0),
\]

where \( \nu \) is the inner normal to \( B_{d_j}(z) \) at \( z_0 \). Then by Theorem 6.3 in [AC] we have \( |\nabla u(z_0)| \leq \lambda_+ \), which in conjunction with Harnack’s inequality implies

\[
\sup_{B_{1/4}(z)} u \leq C_0 u(z) \leq C_0 C\lambda_+ d_j = \frac{C_0 C\lambda_+}{4} \rho_{k_j}.
\]

Hence

\[
(5.10) \quad \sup_{B_{1/4}(z)} u \leq \frac{C_0 C\lambda_+}{4} \rho_{k_j}.
\]

For scaled functions \( v_j(x) = \frac{u(\rho_{k_j} x)}{S(k_j+1, u_j)} \) it follows from (5.10) that there exists a ball \( B_{1/4}(y_0) \subset B_{1}^+ \setminus B_{1/2}^+ \) such that

\[
\sup_{B_{1/4}(y_0)} v_j \leq C \frac{\rho_{k_j}}{S(k_j+1, u_j)} = C\epsilon_j \to 0
\]

by (5.6), which gives (v) in (5.8) for the one-phase case.

The proofs of the remaining claims of (5.8) are the same as for the two-phase case and one will have the final contradictory conclusion.

6. Proof of Theorem B

It follows from the proof of Theorem A (2) that \( u \geq 0 \) grows linearly away from the origin, provided the origin is a non-isolated free boundary point. We can replace
the origin by any non-isolated free boundary point \( z \) near the origin and apply the same argument to show that for \( u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g) \) there exists a tame constant \( C \) such that the growth estimate

\[
0 \leq |u(x)| \leq C|x - z|
\]

holds for any \( z \in B_r' \cap \partial\Omega^+(u) \) for some \( r > 0 \).

In order to conclude (6.1) for the two-phase solutions we further require the \( \delta - \text{NT} \) condition to be satisfied in some neighborhood of the origin. Notice that in the two-phase case, by the Hölder continuity of \( u \), the origin is automatically a non-isolated free boundary point.

Our goal is to prove that the free boundary \( \Gamma(u) \) remains within a cone \( C_{\delta_0} = \{ x : x_1 \geq \delta_0 |x_2| \} \) in some neighborhood of the origin. This will be enough to prove Theorem B, because for the free boundary of the blow-up it implies \( \Gamma(u_0) \subset C_{\delta_0} \). Thus the uniform \( \delta - \text{NT} \) condition will be satisfied for \( u_0 \), with \( \delta_0 = \delta \), and the result will follow from Theorem A via a standard scaling argument.

**Lemma 6.1.** Let \( u \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g) \). If the \( \delta - \text{NT} \) assumption (1.4) is satisfied for any free boundary point \( x_0 \in B_{1/2} ' \), then there exists a tame constant \( \delta_0 \) such that

\[
|z_2| \leq \delta_0, \quad \forall z \in \Gamma(u) \cap B_{1/2}^+.
\]

In particular for any blow-up limit \( u_0 \) the inclusion \( \Gamma(u_0) \subset C_{\delta_0} \) is true.

**Proof.** It follows from the uniform \( \delta - \text{NT} \) condition and the discussion above that (6.1) is true. Suppose (6.2) fails. Then there exists a sequence \( z^k \in B_{1/2}^+ \cap \Gamma(u_k) \) of free boundary points of \( u_k \in \mathcal{P}_1(n, \lambda_\pm, \alpha_\pm, g) \) such that

\[
|z_2^k| \geq k z_1^k, \quad k > k_0,
\]

for sufficiently large \( k_0 \in \mathbb{N} \). Setting \( d_k = z_1^k, f_0(x) = \alpha_+ x_2^+ - \alpha_- x_2^- \) we have

\[
|z_2^k| \geq k d_k, \quad |f_0(p^k)| = \begin{cases} \alpha_+ |z_2^k| & \text{if } z_2^k > 0, \\ \alpha_- |z_2^k| & \text{if } z_2^k < 0, \end{cases}
\]

where \( p^k \) is the projection of \( z^k \) onto \( \Pi \). In any case we get that \( |f(p^k)| \geq k \min(\alpha_+, \alpha_-) d_k \). Put \( r_k = |z^k - \xi_k| \), where \( \xi_k = (0, 0, z_3^k, \ldots, z_n^k) \) is the projection of \( z^k \) onto \( \Pi \cap \{x_2 = 0\} \). Then from triangle inequality \( r_k = |z^k - \xi_k| \geq |z_2^k| - z_1^k > (k - 1) d_k \) implying \( d_k / r_k \leq 1 / k - 1 \). In particular

\[
1 \geq |y_2^k| = \frac{|z_2^k|}{r_k} \geq 1 - \frac{d_k}{r_k} \geq 1 - \frac{1}{k - 1}.
\]

Now introduce the scaled functions

\[
v_k(y) = \frac{u_k(\xi^k + r_k y)}{r_k}, \quad y \in B_2^+.
\]

The points \( y^k = \frac{z^k - \xi^k}{r_k} \) are on the half sphere \( S_1^+ \), and from (6.3) we get

\[
y_1^k = \frac{d_k}{r_k} < \frac{1}{k - 1}.
\]
From (6.1) we have $|u_k(x)| \leq C|x - \xi^k|$, since $\xi^k \in B'_k \cap \Gamma(u_k)$. Therefore it follows that $|v_k(y)| \leq C|y|$, with constant $C$ independent of $k$. Furthermore $v_k$ is a local minimizer of $J(\cdot, B^+_2)$ because

$$\int_{B^+_1} |\nabla v_k|^2 + \Lambda \chi_{\{v_k>0\}} = \frac{1}{r_k^n} \int_{B_k} |\nabla u|^2 + \Lambda \chi_{\{u>0\}}.$$  

Thus $v_k \in \mathcal{P}_2(n, \lambda, \alpha, g_k)$, where $g_k(x) = \frac{g(\xi + r_ky)}{r_k}$ and $\lim_{k \to \infty} g_k = 0$ uniformly.

By Proposition 4.3 it follows that $v_k$ is bounded in $C^{\beta}(B^+_3) \cap H^1(B^+_3)$ for some positive $\beta \in (0, 1)$. Then for a subsequence $k_j$, $v_{k_j} \to v_0$ in $C^\beta(B^+_3)$, $\nabla v_{k_j} \rightharpoonup \nabla v_0$ weakly in $L^2(B^+_3)$ and $y^{k_j} \to y^0$, where $y^0$ is a free boundary point. From (6.3) $|y^{k_j}_2| = \frac{|z^{\frac{1}{2}}_2|}{r_k} \to 1$ and $y \in S^+_1$. But then $y^0_1 = 0, |y^0_2| = 1$, and this contradicts $f_0(y^0) = \alpha \neq 0$.

Let $v(x) = \frac{u_0(\xi + Rx)}{R}$; then (6.2) translates to the free boundary of the blow-up function $u_0$ implying that $\Gamma(u_0) \subset C_{\delta_0}$. Hence we have the uniform $\delta$-NT condition for each $z \in \Gamma(u_0) \cap \Pi$. From Theorem A we have $|u(x)| \leq C|x|$. Returning to $u$ we conclude $|u(x)| \leq C|z - \xi| \leq C(x_1 + |x_2|)$, and this finishes the proof of Theorem B.

7. LARGEST AND SMALLEST GLOBAL SOLUTIONS

Before embarking into the details we briefly go over the main steps of the proof. First we notice that the global solutions enjoy ordering. This implies that there are smallest and largest global homogeneous solutions which we denote respectively by $v_S$ and $v_L$. It follows from the scale and translation invariance that $v_S$ and $v_L$ depend only on $x_1$ and $x_2$. Hence we can explicitly compute them. Moreover $v_S$ has larger $W$-energy implying that the free boundary of any global homogeneous solution, distinct from $v_L$ and $v_S$, cannot touch $\Gamma(v_S)$ or $\Gamma(v_L)$ tangentially.

Thus if there is a third global homogeneous solution $u$, then we can construct a new one which is symmetric in $x_3, x_4, \ldots, x_n$ variables and neither of the functions $v_S, v_L$ coincides with $u$. Thus without loss of generality we may assume that $u$ is
symmetric in $x_3, x_4, \ldots, x_n$ variables. Then a dimension reduction argument will
finish the proof since the only 2D solutions are $v_S$ and $v_L$.

7.1. Largest and smallest solutions in $\mathcal{P}_\infty$. We recall \textbf{1.3}

$$J(u, B_R^+) = \int_{B_R^+} |\nabla u|^2 + \lambda \chi_{\{u > 0\}}.$$ 

Let $v_1, v_2$ be two minimizers of $J(u, B_R^+)$ and $v_1 \leq v_2$ (resp. $v_1 \geq v_2$) on $\partial B_R^+$. Then it is easy to see that $\max(v_1, v_2)$ (resp. $(\min(v_1, v_2))$ is a minimizer of $J(u, B_R^+)$ with boundary values $v_2$ (resp. $v_1$).

Indeed testing $\max(v_1, v_2)$ against $v_2$ in $B_R^+$ and $\min(v_1, v_2)$ against $v_1$ we get

(7.1) \[J(v_2, B_R^+) \leq J(\max(v_1, v_2), B_R^+),\]

$$J(v_1, B_R^+) \leq J(\min(v_1, v_2), B_R^+).$$

Clearly

$$J(\max(v_1, v_2), B_R^+) = \int_{B_R^+ \cap \{v_1 > v_2\}} |\nabla v_1|^2 + \lambda \chi_{\{v_1 > 0\}}$$

$$+ \int_{B_R^+ \cap \{v_1 \leq v_2\}} |\nabla v_2|^2 + \lambda \chi_{\{v_2 > 0\}},$$

$$J(\min(v_1, v_2), B_R^+) = \int_{B_R^+ \cap \{v_1 > v_2\}} |\nabla v_2|^2 + \lambda \chi_{\{v_2 > 0\}}$$

$$+ \int_{B_R^+ \cap \{v_1 \leq v_2\}} |\nabla v_1|^2 + \lambda \chi_{\{v_1 > 0\}},$$

which gives

(7.2) \[J(\max(v_1, v_2), B_R^+) + J(\min(v_1, v_2), B_R^+) = J(v_1, B_R^+) + J(v_2, B_R^+).\]

Hence (7.1) in conjunction with (7.2) implies

$$J(v_1, B_R^+) = J(\min(v_1, v_2), B_R^+),$$

$$J(v_2, B_R^+) = J(\max(v_1, v_2), B_R^+).$$

Upon applying this observation to a finite number of minimizers we obtain

\textbf{Lemma 7.1.} If $v_1, \ldots, v_N$ are minimizers on $B_R^+$ and $v_1 \leq v_2 \leq \cdots \leq v_N$ on $\partial B_R^+$ (resp. $v_1 \geq v_2 \geq \cdots \geq v_N$), then $v_R^L = \max(v_1, \ldots, v_N)$ (resp. $v_S^R = \min(v_1, \ldots, v_N)$) is a minimizer of $J^R$ with boundary values $v_N$ on $\partial B_R^+$.

Employing a compactness argument it follows that there exists a largest and a
smallest minimizer denoted respectively by $v_R^L$ and $v_S^R$.

By definition, for any $u \in \mathcal{P}_R(n, \lambda_\pm, \alpha_\pm) \cap \mathcal{P}_\infty$ we have

$$v_S^R(x) \leq u(x) \leq v_R^L(x), \quad x \in B_R^+.$$ 

Moreover by Definition \textbf{1.5} $v_S^R$ and $v_R^L$ have uniform linear growth, i.e. $|v_S^R|, |v_R^L| \leq C(x_1 + |x_2|)$ for some tame constant $C$ independent of $R$. Sending $R \to \infty$ and utilizing the linear growth Proposition \textbf{4.3} we infer that $v_L^R \to v_L$ uniformly and weakly in $H^1_{\text{loc}}$. Furthermore $v_L \in \mathcal{P}_\infty$.

Indeed let $\varphi \in C_0^\infty(B_\rho^+)$, let $\rho$ be fixed and let $\rho < R$; then $v_L^R$ is a minimizer and we have

$$J(v_L^R, B_\rho^+) \leq J(v_L^R + \varphi, B_\rho^+), \quad \forall B_\rho^+ \subset \mathbb{R}_+^n.$$
More explicitly it can be rewritten as \( \int_{B^\rho_+} \Lambda \chi_{\{v_R > 0\}} \leq \int_{B^\rho_+} 2\nabla v_L^R \cdot \nabla \varphi + |\nabla \varphi|^2 + \Lambda \chi_{\{v_R + \varphi > 0\}} \).

By a customary compactness argument and weak convergence of gradients we get
\[
J(v_L, B^\rho_+) \leq J(v_L + \varphi, B^\rho_+), \quad \forall \varphi \in C_0^\infty(B^\rho_+).
\]
The same argument leads to the existence of \( v_S \) — the smallest global homogeneous solution. Thus
\[
v_S \leq u \leq v_L, \quad \forall u \in \mathcal{P}_\infty.
\]

Since the class \( \mathcal{P}_\infty \) is scale and \( e_3, \ldots, e_n \) translation invariant, it follows that \( v_S, v_L \) are homogeneous and depend only on \( x_1 \) and \( x_2 \) variables.

Now let us explicitly compute \( v_L \) and \( v_S \). For this we write the Laplacian in polar coordinates:
\[
\Delta w = 1/r \left[ \frac{\partial(rw)}{\partial r} + \frac{\partial}{\partial \varphi} \left( \frac{w}{r} \varphi \right) \right] = 1/r [g(\varphi) + g''(\varphi)],
\]
where \( w = rg(\varphi) \). Recall that \( v_S, v_L \) are harmonic outside of the zero set by Proposition 4.1. This implies that \( g \) is a linear combination of \( \sin \varphi \) and \( \cos \varphi \).

Therefore the largest and smallest solutions are linear combinations of \( x_1 \) and \( x_2 \).

Assume that
\[
v^+ = ax_1 + bx_2, \quad \text{in } \Omega^+(v), \quad v^- = Ax_1 + Bx_2, \quad \text{in } \Omega^-(v),
\]
where \( v^+ \) and \( v^- \) are respectively the positive and negative parts of \( v \) and \( a, b, A, B \) are constants to be determined. The boundary condition \( v = \alpha_+ x_2^+ - \alpha_- x_2^- \) on \( \Pi \) implies \( b = \alpha_+, B = \alpha_- \).

Let us assume that the free boundary \( \Gamma(v) \) is given by
\[
x_1 = x_2 \tan \theta.
\]
Both \( v^+ \) and \( v^- \) must vanish on \( \Gamma(v) \). Hence
\[
0 = ax_1 + bx_2 = ax_2 \tan \theta + \alpha_+ x_2 = x_2(a \tan \theta + \alpha_+)
\]
and we easily find that
\[
a = -\frac{\alpha_+}{\tan \theta} = -\alpha_+ \cot \theta.
\]
Similarly
\[
A = -\frac{\alpha_-}{\tan \theta} = -\alpha_- \cot \theta.
\]

Summarizing we have
\[
v^+ = \alpha_+(-x_1 \cot \theta + x_2), \quad v^- = \alpha_-(-x_1 \cot \theta + x_2).
\]
Note that \( \cot \theta \) takes only two values, positive and negative, corresponding respectively to large and small solutions. To evaluate \( \cot \theta \) we need to use the gradient jump condition \( |\nabla v^+|^2 - |\nabla v^-|^2 = \Lambda \), which is now satisfied in the classical sense; see Proposition 4.1. Substitution of \( v \) into this identity gives
\[
\alpha_+^2(1 + \cot^2 \theta) - \alpha_-^2(1 + \cot^2 \theta) = \Lambda
\]
or equivalently
\[
\cot \theta = \pm \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1}.
\]
Note that if \( \Lambda \leq \alpha^2_+ - \alpha^2_- \), then there is no free boundary. Summarizing we get that
\[
\begin{align*}
    v_L &= \alpha_+ (\gamma x_1 + x_2)^+ - \alpha_- (\gamma x_1 + x_2)^-, \\
    v_S &= \alpha_+ (-\gamma x_1 + x_2)^+ - \alpha_- (-\gamma x_1 + x_2)^-, \\
    \gamma &= \sqrt{\frac{\Lambda}{\alpha^2_+ - \alpha^2_-}} - 1.
\end{align*}
\]

The above discussion is summarized in the following proposition.

**Proposition 7.2.** The largest and smallest solutions \( v_L, v_S \) are given by (7.3), and these are the only two-dimensional homogeneous global solutions.

7.2. Comparison of \( W \)-energy. The aim of this section is to show that \( v_S \) has bigger \( W \)-energy than \( v_L \). For all values of \( \alpha_\pm \) for which \( v_S \neq v_L \) we have
\[
W(1, v_S, 0) > W(1, v_L, 0).
\]

As a consequence we get that the largest solution is stable in the following sense:

**Proposition 7.3.** Let \( u \in P_1(n, \lambda_\pm, \alpha_\pm, g) \) and suppose there is \( R_0 \in (0, 1) \) such that \( W(R_0, u, 0) < W(1, v_S, 0) \). Then any blow-up limit \( u_0 \) coincides with \( v_L \).

**Proof.** To check this we recall the monotonicity of \( W \), and infer that \( W(0^+, u, 0) = W(1, u_0, 0) < W(1, v_S, 0) \), which in view of Theorem C and \( W(1, v_S, 0) \geq W(1, v_L, 0) \) implies that \( u_0 = v_L \).

Now it remains to show (7.4). If \( v \) is a homogeneous solution, then \( W \) is constant; hence it suffices to compute \( W(1, \cdot, 0) \). By Green’s formula
\[
\int_{B^+_1} |\nabla v|^2 = \int_{\partial B^+_1} v \frac{\partial v}{\partial \nu}.
\]

We can easily compute
\[
W(1, v) = \int_{\partial B^+_1} v \frac{\partial v}{\partial \nu} + \int_{B^+_1} \Lambda \chi\{v > 0\} - \int_{S^+_1} v^2
\]
\[
= \int_{B^+_1} v \frac{\partial v}{\partial \nu} + \int_{B^+_1} \Lambda \chi\{v > 0\},
\]

where the last equality follows from \( v(x) = x \cdot \nabla v(x) \) on \( S^+_1 = \partial B^+_1 \cap \mathbb{R}^n_+ \). In particular one can take \( v \) to be \( v_L \) or \( v_S \).

Now let \( \theta \in (0, \pi/2) \) be determined from
\[
\cot \theta = \sqrt{\frac{\Lambda}{\alpha^2_+ - \alpha^2_-}} - 1.
\]

Utilizing the explicit form of \( v_S \) one can readily verify that
\[
\int_{B^+_1} \Lambda \chi\{v_S > 0\} = \int_0^1 \int_{S^+_1} \Lambda \chi\{v_S > 0\} = \frac{1}{n} \int_{S^+_1} \Lambda \chi\{v_S > 0\} = \frac{\Lambda \theta}{2\pi} \omega_n,
\]

where \( \omega_n \) is the volume of the \( n \)-dimensional unit ball. Similarly
\[
\int_{B^+_1} \Lambda \chi\{v_L > 0\} = \frac{\Lambda (\pi - \theta)}{2\pi} \omega_n.
\]
Next we notice that \( \frac{\partial v}{\partial \nu} = -\frac{\partial v}{\partial x_1} \), on \( B'_1 \). Therefore we have

\[
- \int_{B'_1} v_S \frac{\partial v_S}{\partial x_1} = - \int_{B'_1 \cap \{x_2 > 0\}} \alpha_+ x_2 (-\gamma \alpha_+) - \int_{B'_1 \cap \{x_2 < 0\}} \alpha_- x_2 (-\gamma \alpha_-)
\]

\[
= \gamma \alpha_+^2 \int_{B'_1 \cap \{x_2 > 0\}} x_2 + \gamma \alpha_-^2 \int_{B'_1 \cap \{x_2 < 0\}} x_2
\]

\[
= \gamma (\alpha_+^2 - \alpha_-^2) \int_{B'_1 \cap \{x_2 > 0\}} x_2
\]

\[
= \frac{\omega_{n-2}}{n} \gamma (\alpha_+^2 - \alpha_-^2),
\]

where \( \gamma = \cot \theta = \sqrt{\frac{\Lambda}{\alpha_+^2 - \alpha_-^2} - 1} \). Hence

\[
W(1, v_S, 0) = \gamma (\alpha_+^2 - \alpha_-^2) \frac{\omega_{n-2}}{n} + \Lambda \frac{\theta \omega_n}{2\pi},
\]

and similarly one can see that

\[
W(1, v_L, 0) = -\gamma (\alpha_+^2 - \alpha_-^2) \frac{\omega_{n-2}}{n} + \Lambda \frac{(\pi - \theta) \omega_n}{2\pi}.
\]

Summarizing we have that

\[
W(1, v_S, 0) - W(1, v_L, 0) = \Lambda \frac{\omega_{n-2}}{n} \left[ 2\gamma \frac{\alpha_+^2 - \alpha_-^2}{\Lambda} + \frac{(\pi - \theta) \omega_n}{2\pi} \right].
\]

Using the explicit computation for \( \omega_n \) we obtain

\[
n \frac{\omega_n}{\omega_{n-2}} = 2\pi.
\]

Finally we observe that \( \sin^2 \theta = \frac{\alpha_+^2 - \alpha_-^2}{\Lambda} \); hence

\[
2\gamma \frac{\alpha_+^2 - \alpha_-^2}{\Lambda} + \frac{(\pi - \theta) \omega_n}{2\pi} \omega_{n-2} = 2\gamma \frac{\alpha_+^2 - \alpha_-^2}{\Lambda} + (\pi - \theta) \omega_{n-2}
\]

\[
= 2 \left[ \cot \theta \sin^2 \theta - \frac{(\pi - \theta)}{2} \right]
\]

\[
= \sin(2\theta) - 2\theta + \pi 
\]

\[\geq 0.\]

Therefore

\[
W(1, v_S, 0) \geq W(1, v_L, 0)
\]

and equality holds if and only if \( \theta = \pi/2 \). \( \square \)

8. Proof of Theorem C

8.1. Free boundary as a generalized minimal surface. The aim of this section is to classify homogeneous global solutions. For \( n = 2 \) this was done in Proposition 7.2. Therefore from now on we shall assume \( n > 2, \alpha_- = \lambda_- = 0 \) (i.e. the one-phase case). Notice that the condition \( \lambda_- = 0 \) can be dropped due to the formulas (1.2) and (1.3). We recall that if \( u \) is a global solution, and hence a local minimizer, of \( J \) for one-phase problem, then

\[
(8.1) \quad \sup_{B_r(x_0)} |\nabla u|^2 \leq \Lambda + C(x_0) r^\alpha
\]
for any \( x_0 \in \mathbb{R}^n_+ \), \( B_r(x_0) \subset \mathbb{R}^n_+ \) and \( C(x_0) \) depends on dist\((x_0, \Pi)\); see Theorem 6.3 of [AC]. As a result we obtain that for any free boundary point \( x_0 \) the following estimate holds:

\[
(8.2) \quad \limsup_{x \to x_0} |\nabla u(x)|^2 \leq \Lambda.
\]

Our first task is to show that the estimate (8.2) holds in supp\( u \).

**Lemma 8.1.** Let \( u \) be a global homogeneous solution. Then for \( z^0 \in \Gamma \cup \{ x_1 = 0, x_2 > 0 \} \)

\[
\limsup_{z \to z^0} \frac{|u(z) - u(z^0)|}{|z - z^0|} \leq \sqrt{\Lambda}.
\]

**Proof.** To see this let \( z_0 \in \Pi \) and \( u(z^0) > 0 \). Then there is \( r > 0 \) such that \( u \in C^1(B_r^+(z^0)) \). Thus the tangential derivatives are controlled by \( \alpha_+ \leq \Lambda \). As for the normal derivative we notice that from the definition of \( v_S \) and \( v_L \) we have that (since \( \alpha_- = 0 \))

\[
|\nabla v_S| = |\nabla v_L| = \sqrt{\Lambda}.
\]

But \( v_S \leq u \leq v_L \) and \( v_S = u = v_L \) on \( \Pi \), hence it is enough to estimate the \( x_1 \)-derivative. Indeed, from the estimate \( v_S \leq u \leq v_L \) and \( v_S(z^0) = v_L(z^0) = u(z^0) \) we get

\[
\frac{\partial v_S(z^0)}{\partial x_1} \leq \frac{\partial u(z^0)}{\partial x_1} \leq \frac{\partial v_L(z^0)}{\partial x_1}.
\]

Therefore \( |\nabla u(z^0)|^2 \leq \Lambda \).

It is also apparent by the free boundary condition (8.1) that \( |\nabla u|^2 \leq \Lambda \) on the free boundary. \( \square \)

**Lemma 8.2.** Let \( u \) be a global homogeneous solution. Then

1° The following estimate is true:

\[
(8.3) \quad \sup_{x \in \mathbb{R}^n_+ \cap \{ u > 0 \}} |\nabla u(x)|^2 \leq \Lambda.
\]

2° In particular \( \Gamma(u) \) is a generalized surface of non-positive outward mean curvature.

It should be remarked that the estimate (8.3) does not hold for non-homogeneous global solutions; see Remark 10.4.

**Proof of Lemma 8.2** Suppose the statement of the lemma fails; then there is a maximizing sequence \( x^j \) with the property that \( |\nabla u(x^j)|^2 \to \Lambda + \epsilon_0 > \Lambda \). By zero-degree harmonicity of \( |\nabla u|^2 \) we may assume \( x^j \) are on the unit sphere. Also by subharmonicity of \( |\nabla u|^2 \) we assume that \( x^j \) tend to the boundary of \( \{ u > 0 \} \cap \{ x_1 > 0 \} \). By Lemma 8.1 the sequence \( x^j \) cannot converge to either of the boundaries (free or fixed). Hence it converges to the “corner”-points \( \{ x_1 = x_2 = 0, |x| = 1 \} \).

Let \( r_j = \text{dist}(x^j, \Gamma \cup \Pi) \); then we have three different possibilities:

- **Case 1**: \( \text{dist}(x^j, \Gamma) \approx x^j_1 \Rightarrow r_j \approx \text{dist}(x^j, \Gamma) \approx x^j_1 \),
- **Case 2**: \( \text{dist}(x^j, \Gamma) = o(\text{dist}(x^j, \Pi)) \Rightarrow r_j = o(x^j_1) \),
- **Case 3**: \( \text{dist}(x^j, \Pi) = o(\text{dist}(x^j, \Gamma)) \Rightarrow r_j = o(\text{dist}(x^j, \Gamma)) \).
Notice that $x^j_1 = \text{dist}(x^j, \Pi)$. We shall see that all these cases will lead to a contradiction.

Case 1. Let $\tilde{x}^j$ be the closest corner point on the $(n - 2)$-dimensional unit sphere, i.e. $\tilde{x}^j \in \{x_1 = x_2 = 0, \ |x| = 1\} = S^{n-2}$, in the first case, and in the other two cases the closest point on the boundary to $x^j$ (we again assume this close point is on the unit sphere).

Now let $d_j = |x^j - \tilde{x}^j|$ and scale $u$ at $\tilde{x}^j$ with $d_j$: $$u_j(x) = \frac{u(\tilde{x}^j + d_j x) - u(\tilde{x}^j)}{d_j}.$$ Note that $d_j \approx r_j \approx x^j_1$ translates to $u_j$ as follows: there is $y^j \in S^n, y^j_3 = \cdots = y^j_n = 0$ such that $y^j_1 = \text{dist}(y^j, \Gamma(u_j)) \approx 1$ and

\begin{equation}
\lim_{j \to \infty} |\nabla u_j(y^j)|^2 = \Lambda + \varepsilon_0. \tag{8.4}
\end{equation}

Clearly $u_j$ should be considered in a new domain, which is a scaled version of the support of $u$ at $\tilde{x}^j$ and which contains supp $v_S$. In the two other cases below the support of $u_j$ converges to a half space.

Next we see that in all cases $u_j$ converges to a limit function $u_0$ (at least for a subsequence) with the further property that $|\nabla u_0(y^0)|^2 = \Lambda + \varepsilon_0$ (here $y^0 = \lim_{j \to \infty} y^j$, again for a subsequence). In particular, and by construction, $|\nabla u_0(x)|^2$ takes maximum at $y^0$, an interior point to the support of $u_0$. Hence by the strong maximum principle it must be constant, and therefore $|\nabla u_0(x)|^2 = \Lambda + \varepsilon_0$ in the support of $u_0$. This in turn implies $u_0$ is linear. But $u_0$ is a global minimizer, hence $|\nabla u_0|^2 = \Lambda$ in supp $u_0$, which is in contradiction with (8.4).

Case 2. Let $u_j(x) = \frac{u(\tilde{x}^j + r_j x)}{r_j}$. We proceed as in Case 1 and extract a subsequence for which $u_j \to u_0$ and $u_0$ is a global minimizer. Furthermore (8.4) holds with $y^0 = \lim y^j$, but in this case $y^j_1 \in \Gamma(u_j), y^j_3 = \cdots = y^j_n = 0$. This implies that $|\nabla u_0(y^0)|^2 = \Lambda + \varepsilon_0$, which is in contradiction with (8.1).

Now, in the first two cases, the free boundary is present (due to the length of scale $r_j$). In first case, we obtain a global minimizer in $\mathbb{R}^n_+$ with boundary data as before. At the same time we have $u_0$ is linear, which results into the fact that $u_0$ is one of the functions $v_L, v_S$. But then this contradicts the fact that $|\nabla u_0|^2 = \Lambda + \varepsilon_0$.

Case 3. Now the last case gives us scaling with center at the fixed boundary. Here we use both the small and the large solutions to bound the scaled function. Indeed, for $\tilde{x}^j$ being the projection of $x^j$ onto $\Pi$, we have

$$u_j(x) = \frac{u(r_j x + \tilde{x}^j) - u(\tilde{x}^j)}{r_j} = \frac{u(r_j x + \tilde{x}^j) - u(\tilde{x}^j)}{r_j} - \frac{u(r_j x + \tilde{x}^j) - \alpha \tilde{x}^j_2}{r_j},$$

and hence the scaled versions of $v_S$ and $v_L$ at $\tilde{x}^j$ satisfy

$$(v_S)_j \leq u_j \leq (v_L)_j.$$

Hence the blow-up limits keep the order

\begin{equation}
v_S = (v_S)_0 \leq u_0 \leq (v_L)_0 = v_L. \tag{8.5}\end{equation}

Now as before we have $|\nabla u_0|^2 = \Lambda + \varepsilon_0$, and this is impossible due to (8.5) and the fact that $|\nabla v_L|^2 = |\nabla v_S|^2 = \Lambda$. 


Now we turn to the proof of the second statement of Lemma 8.2, namely that \( \Gamma(u_0) \) is a generalized surface of non-positive outward mean curvature. Let \( S \subset \partial_{\text{red}}\{u > 0\} \) be a portion of the free boundary of \( u \) and \( S' \) a small perturbation of \( S \) such that \( S' \subset \{u > 0\} \) and \( \partial S = \partial S' \). Then
\[
\mathcal{H}^{n-1}(S) \leq \mathcal{H}^{n-1}(S'),
\]
i.e. \( \partial_{\text{red}}\{u > 0\} \) is a generalized surface of non-positive outer mean curvature. Notice that by Lemma 12.3 \( \partial\{u > 0\} \) has finite perimeter in \( B_1 \). Thus \( \mathcal{H}^{n-1}(S) < \infty \).

To prove this we take the domains \( G, G_0 \) such that \( \partial G = S \cup S' \) and \( \overline{G} \subset G_0 \subset \mathbb{R}^n_+ \). Then we have
\[
0 = \int_G \Delta u = \int_S \partial_{\nu} u + \int_{S'} \partial_{\nu} u.
\]
On \( S \) we have that \( \partial_{\nu} u(x) = |\nabla u(x)| = \sqrt{\Lambda} \), for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \Gamma(u) \cap G_0 \), whereas on \( S' \), \( |\nabla u| \leq \sqrt{\Lambda} \) by \( \text{(S.2)} \). Comparing the integrals over \( S \) and \( S' \) we get that
\[
\sqrt{\Lambda} \mathcal{H}^{n-1}(S) = \int_S \partial_{\nu} u = -\int_{S'} \partial_{\nu} u \leq \sqrt{\Lambda} \mathcal{H}^{n-1}(S').
\]
After canceling \( \sqrt{\Lambda} \) the result follows. \( \square \)

### 8.2. Preliminary lemmas

Suppose that \( u \) is a third global homogeneous solution, which by Section 7.1 satisfies \( v_S \leq u \leq v_L \). In particular the free boundary \( \Gamma(u) \) lies in between the planes \( \Gamma_S \), and \( \Gamma_L \).

We first need a lemma that shows that the free boundary is locally a graph.

**Proposition 8.3.** Let \( u \) be a global homogeneous minimizer and let \( \Gamma(u) \) touch tangentially the free boundary of \( v_L \), at some point \( x^0 = (0,0,x^0_3,\ldots,x^0_n) \) with \( |x^0| = 1 \). Then in a small neighborhood of \( x^0 \) the free boundary \( \Gamma(u) \) is a \( C^1 \) graph in the direction normal to \( \Gamma_L \) in the upper half plane. Moreover the normal vector to \( \Gamma(u) \) is continuous up to the point \( x^0 \), and hence by homogeneity this holds on the axis \( tx^0 \) (\( t > 0 \)).

Let \( \Pi_0 = \{x \in \mathbb{R}^n : x_1 = x_2 = 0\} \) and \( x^0 \in \Pi_0 \setminus \{0\} \) be any given free boundary point close enough to \( \Pi_0 \). Let further \( \tilde{x}^0 \) be the projection of \( x^0 \) onto \( \Gamma_L \). Then by tangential touch between the free boundary \( \Gamma(u) \) and \( \Gamma_L \) (which is a flat plane) one has that \( |x^0 - \tilde{x}^0| = o(x^0_1) \). In particular for \( r_0 = x^0_1 \), sufficiently small, we have that, in the ball \( B_{r_0}(\tilde{x}_0) \), the free boundary \( \Gamma(u) \) is flat enough to satisfy the hypothesis of Theorem 8.1 in [AC]. In particular, in the direction of the plane \( \Gamma_L \) the free boundary is a \( C^1 \) graph locally in \( B_{\frac{r_0}{2}}(\tilde{x}_0) \). From here it follows that \( \Gamma(u) \), seen from the plane \( \Gamma_L \), is a \( C^1 \) graph over \( \Gamma_L \cap B_{\frac{r_0}{2}}(\tilde{x}_0) \).

It is now elementary to show that the normal of \( \Gamma(u) \) is continuous up to the point \( x^0 \). Indeed, if this fails, then there is a sequence \( x^j \) on the free boundary with normal \( \nu^j \) staying uniformly away from the normal \( \nu^L \) of \( \Gamma_L \), \( |\nu^j - \nu^L| > \varepsilon_0 > 0 \). Scaling \( u \) at \( x^j \) with \( r_j = \text{dist}(x^j,\Gamma_L) \) we have a limit global minimizer in \( \mathbb{R}^n \) (observe that this is due to tangential touch). On the other hand the free boundary will then become a plane, on one side of a scaled version of the plane \( \Gamma_L \), but with the normal at the origin being \( \nu^0 \), with \( |\nu^0 - \nu^L| > \varepsilon_0 > 0 \). This is impossible. \( \square \)

**Lemma 8.4.** Let \( u \in \mathcal{H}^\infty, \Gamma(u) = \partial\{u > 0\} \). If \( u \geq 0, \alpha_- = 0 \), then \( \Gamma(u) \) does not touch \( \Gamma_L \) tangentially.
Case 1. Let us suppose that $\Gamma(u)$ touches $\Gamma_L$ at $x^0$ and $x_1^0 > 0$. To conclude that this is a contradiction we use the free boundary condition and Hopf lemma. Notice that in order to use Hopf’s lemma, we need (at least $C^1,Dini$) regularity of $\Gamma(u)$ near $x^0$.

It follows from the one side flatness and the classical regularity result of Theorem 8.1 in [AC]. Then we can apply Hopf’s maximum principle to infer

$$\frac{\partial(u-v_L)}{\partial \nu}(x^0) > 0,$$

which is a contradiction in view of the tangential touch condition.

Case 2. We first choose a new coordinates system such that in new coordinates $y = (y_1, y_2, \ldots, y_n)$ we have $\Gamma_L = \{y \in \mathbb{R}^n : y_n = 0, y_{n-1} > 0\}$ and $\{u > 0\} \subset \{y \in \mathbb{R}^n : y_n < 0\}$. Now let us assume that $\Gamma(u)$ touches the free boundary of the larger solution $v_L$ at $y^0 \neq 0$ and $y_1^0 = 0$. Then by Proposition 8.3 the free boundary is locally a smooth graph, seen from the plane $\Gamma_L$. In particular, near $y^0$ the free boundary can be represented as $y_n = h(y')$, $y' = (y_1, y_2, \ldots, y_{n-1})$ and $h$ is a subsolution to the minimal surface equation in the weak sense.

Indeed, let $\mathbb{R}_+^{n-1} = \{y \in \mathbb{R}^n : y_n = 0, y_{n-1} > 0\}$ and $\bar{B} \subset \mathbb{R}_+^{n-1}$ be a ball touching $\partial \mathbb{R}_+^{n-1}$ at $y^0$. Then by Lemma 8.2 the surface area functional will increase, if we replace $h$ by $h_\epsilon = h + \epsilon \varphi$ for any $\varphi \in C_0^\infty(\bar{B}), \varphi \leq 0$ and $\epsilon > 0$ is small. This comparison yields

$$\mathcal{M}h(y') = \text{div} \left( \frac{Dh(y')}{\sqrt{1 + |\nabla h(y')|^2}} \right) \geq 0 \quad \text{weakly in } \bar{B}.$$}

Thus we have

$$\begin{cases} 
\mathcal{M}h(y') \geq 0 & \text{in } \bar{B}, \\
h(y') \leq 0 & \text{in } \bar{B}, \\
h(y^0) = 0 & y^0 \in \partial \bar{B}.
\end{cases}$$

By Hopf’s principle

$$\frac{\partial h(y^0)}{\partial y_{n-1}} > 0,$$

which is in contradiction with the tangential touch of $\Gamma(u)$ and $\mathbb{R}_+^{n-1}$.}

From Lemma 8.4 we know that $\Gamma(u)$ cannot touch $\Gamma_L$. Using this observation we can construct yet another global minimizer $\tilde{u}$ such that it is two dimensional and distinct from $v_L$ and $v_S$. This, however, will contradict Proposition 8.2 and the proof of Theorem C will finish.

Thus to complete the proof of Theorem C we need to construct $\tilde{u}$. This is done by the next lemma.

**Lemma 8.5.** Let $\mathbb{V}_\epsilon = \{x \in \mathbb{R}_+^2 : x_2 < -((\gamma - \epsilon)x_1)\}$ for small $\epsilon > 0$. If $\mathbb{V}_\epsilon \subset \{u = 0\}$, then there is a two-dimensional global solution $\tilde{u}$ which is distinct from $v_L$ and $v_S$. 


Proof. Suppose $V_\epsilon \subset \{u = 0\}$ for some $\epsilon > 0$. Then we can construct a global solution $\tilde{u}$ such that $\tilde{u} \geq u$, $\tilde{u}$ is two dimensional and $\Gamma(\tilde{u}) \subset \mathbb{R}^n_+ \setminus V_\epsilon$.

For $r > 0$ fixed and $x \in B^+_r$, we put $g_r(x) = \sup\{u(x + \ell T), \ell \in \mathbb{R}, T \in \mathbb{R}\}$ where

$$S^{n-2} = \{\ell = (0, 0, \ell_3, \ldots, \ell_n), \ell_3^2 + \ell_4^2 + \cdots + \ell_n^2 = 1\}.$$

Let $w \in P_r(n, \lambda_\pm, \alpha_+, 0, g_r)$, i.e. $w$ is a local minimizer of $J(\cdot, \partial B^+_r)$ with $w = g_r$ on $\partial B^+_r$; see Remark 7.1. From Lemma 7.1 we infer that $\tilde{u}_r = \sup w$ is a local minimizer and $\tilde{u}_r \geq u$ for any $w \in P_r(n, \lambda_\pm, \alpha_+, 0, g_r)$. In particular $\tilde{u}_r \geq u$ in $B^+_r$.

Taking $r_j \to \infty$, we have from Proposition 4.3 that there is a subsequence $r_{jk}$ such that $\tilde{u}_{r_{jk}} \to \tilde{u}_0$ locally in $H^1$ and $C^0$ and $\tilde{u}_0 \in P_\infty$. Because $P_r(n, \lambda_\pm, \alpha_+, 0, g_r)$ is translation invariant for each $\ell \in S^{n-1}$, it follows that $\tilde{u}_0$ is a two-dimensional solution. The condition $V_\epsilon \subset \{u = 0\}$ translates to $\tilde{u}_0$ and we get that $\Gamma(\tilde{u}_0) \subset \mathbb{R}^n_+ \setminus V_\epsilon$. Furthermore $\tilde{u}_0 \geq u$.

Since $v_L$ and $v_S$ are the only two-dimensional homogeneous global solutions, we conclude that $\epsilon = 0$; see Proposition 7.2. $\square$

Remark 8.6. It is noteworthy that the classification of global homogeneous solutions for the two-phase case would have been available if one already knew that the free boundary is regular. Indeed, if we a priori know that the free boundary is regular, then one can apply the maximum principle to $|\nabla u|^2$ in the set $\{u > 0\}$, and find out that the maximum must be on the boundary (either free or fixed). Actually, an argument similar to that of the proof of Lemma 8.2 would then result in the fact that the maximum is exactly on the boundary.

Suppose now the maximum is on the free boundary. Then at such a maximum point $x^0$ (which is a maximum point for both $|\nabla u|^2$ due to Bernoulli boundary condition $|\nabla u|^2 = \Lambda + |\nabla u^{-}|^2$) one gets that $\partial_\nu |\nabla u^+(x^0)|^2 < 0$, where $\nu$ is the unit normal pointing inwards support of $u^+$. From here along with a possible regularity of free boundary it follows that $2u^+_{\nu\nu}u^+_\nu(x^0) < 0$, which along with $u^+_\nu(x^0) > 0$ gives that $u^+_{\nu\nu}(x^0) < 0$. By representation of Laplacian on the free boundary we get $0 = \Delta u^+ = \Delta_S u^+ + H u^+_\nu + u^+_{\nu\nu}$, and since $\Delta_S u^+ = 0$, $u^+_{\nu\nu}(x^0) > 0$, and $u^+_{\nu\nu}(x^0) < 0$, we arrive at $H(x^0) > 0$. A similar argument applied to $u^-$ gives us the converse $H(x^0) < 0$, and we shall have a contradiction, unless $|\nabla u|$ is constant.

Next suppose the maximum for $|\nabla u|^2$ is on the fixed boundary $x^0 \in \{x_1 = 0\}$. Then we have by a similar argument $u_1(x^0)u_{11}(x^0) < 0$. Now with a representation of the Laplacian on $\{x_1 = 0\}$ along with linearity of the boundary data we have $0 = Hu_1 + u_{11}$. Since the fixed boundary is a flat surface we have $H = 0$, and hence $u_{11} = 0$ on the fixed boundary. This contradicts $u_1(x^0)u_{11}(x^0) < 0$.

9. Proof of Theorem D

Now we are ready to produce the proof of Theorem D, exhibiting the non-tangential behavior of the free boundary.

Non-uniform approach. Take $u \in P_1(n, \lambda_\pm, \alpha_+, 0, g)$ and let $u_0$ be a blow-up of $u$ at the origin. Then by Propositions 4.5 and 4.6 $u_0 \in H_P(\infty)$. From Theorem C, $u_0$ is either $v_S$ or $v_L$. Suppose that $u_0 = v_S$. Let us consider the cone

$$K^+ := \left\{ x \in \mathbb{R}^n_+, x_2 > 0, \frac{x_2}{\gamma + \sigma} < x_1 < \frac{x_2}{\gamma - \sigma} \right\}.$$
for small \( \sigma > 0 \) (cf. (3.4)). Then we claim that for each \( \sigma > 0 \) there exists a \( r_\sigma > 0 \) such that for any \( r \in (0, r_\sigma) \), the following holds:

\[
(9.1) \quad \Gamma(u) \subset B_r^+ \cap K_\sigma^+.
\]

This would suffice to conclude the tangential touch, since the modulus of continuity can be constructed by inverting the relation \( \sigma \to r_\sigma \).

Suppose (9.1) fails. Then there is a sequence of free boundary points \( x^j \in \Gamma(u), |x^j| \to 0, u \in P_1(n, \lambda_\pm, \alpha_+, 0, g) \) such that \( x^j \notin K_\sigma^+ \) for some fixed \( \sigma > 0 \).

Set \( r_j = |x^j| \) and consider the limit of the sequence \( u_j(x) = u(r_j x) / r_j \). In view of Theorem A, \( u_j \)'s are bounded, and therefore by Proposition 4.6 and Theorem B for a subsequence \( u_{j_m} \to u_0 \in \mathcal{H}P_\infty \). Moreover the sequence of points \( y^j = x^j / |x^j| \in \partial B_1^+ \) is such that \( y^j \notin K_\sigma^+ \), and again by compactness this leads to the existence of \( y^0 \in \partial B_1^+ \setminus K_\sigma^+ \) such that \( u_0(y^0) = 0 \).

From the monotonicity formula of Weiss, Proposition 4.6, one can also show that \( u_0 \in \mathcal{H}P_\infty \) (see Section 4.3) and hence we can invoke Theorem C to conclude that \( u_0 = v_L \). This contradicts the fact that \( y^0 \in \partial B_1^+ \setminus K_\sigma^+ \), and the proof of the first part is completed. The case when \( u_0 = v_L \) is treated analogously.

**Uniform approach.** To show the uniformity in the second statement of Theorem D, we shall argue in the same way as above, but let \( u \) change during the scaling. In other words we define \( v_j(x) = u_j(x) / r_j \) with \( u_j \in P_1(n, \lambda_\pm, \alpha_+, 0, g) \), i.e. \( \lim_{r \to 0} W(r, v_j, 0) = W(1, v_S, 0) \). As above the scaled functions will converge to a global solution \( v_0 \), but \( v_0 \) is not necessarily homogeneous, and this is the only difference between the two cases.

Nevertheless, the assumption that \( \lim_{r \to 0} W(r, u_j, 0) = W(0^+, u_j, 0) = W(1, v_S, 0) \) for fixed \( j \) implies that \( W(tr_j, u_j, 0) = W(1, u_j(tr_j x) / tr_j, 0) = W(t, v_j, 0) \geq W(1, v_S, 0) \) by monotonicity of \( W \) (see Proposition 4.6) and after having sent \( t \to 0 \). This yields

\[
W(t, v_0, 0) = \lim_{r_j \to 0} W(tr_j, u_j, 0) = \lim_{r_j \to 0} W(t, v_j, 0) \geq W(1, v_S, 0),
\]

where \( v_0 \) is the global limit of a subsequence of \( v_j \). The first inequality follows from strong convergence of \( \nabla v_j \) in \( L^2 \), since \( \nabla v_j \) is a bounded sequence in \( L^\infty \), and hence we can apply Theorem 1 from [Z] and Proposition 4.5 to a suitable subsequence \( \{r_j^j\} \subset \{r_j\} \).

Next, the blow-down of \( v_0 \) at infinity, i.e. consider the scaling \( v_0(r x) / r \) with \( r \to \infty \) which results in a new homogeneous global solutions \( v_{00} \). From the monotonicity formula, Proposition 4.6 we have

\[
W(1, v_S, 0) \leq W(t, v_0, 0) \leq W(\infty, v_0, 0) = W(1, v_{00}, 0).
\]

Since \( v_{00} \) is homogeneous, we can apply Theorem C to conclude \( v_{00} \) is either \( v_L \) or \( v_S \). By the energy comparison [7.4] we should then have \( v_{00} = v_S \). Therefore \( W(t, v_0, 0) = W(1, v_S, 0) \) for any \( t > 0 \), hence \( v_0 \) is homogeneous by Proposition 4.6 Now, as in the previous case, contradiction comes from the fact that \( y^0 \in \partial B_1^+ \setminus K_\sigma^+ \). \(\square\)
10. Proof of Theorem E

10.1. Instability. The problem studied in this paper is highly unstable in the sense that changing the boundary data, no matter how small, may result in a different behavior of the touch between the free and the fixed boundary. This behavior was already alluded to in Theorem D, where we could not prove uniform behavior for solutions that touch tangentially $\Gamma(v_L)$, at the same time that the uniformity worked well for the class $\mathcal{P}_\nu(n, \lambda, \alpha, g)$.

To illustrate this phenomenon, take $\alpha_-=0$ and consider the largest homogeneous global solution $v_L$ as in Theorem C. Consider now the minimization problem in the upper half ball using the restriction of suitably scaled $v_L$ on the boundary of $B_1^+$ as boundary data. Now we know that the functional itself is a minimizer. Next let us decrease the data on the plane $\Pi$ to $(\alpha_+ - \varepsilon)x_2^+$. A minimizer $u^\varepsilon$ of the functional with this boundary value on $\Pi$ will exist, say take the smallest minimizer with boundary values $u^\varepsilon \leq v_L$ on $S^+_1$, so that $u^\varepsilon \leq v_L$. In particular this means that the free boundary for this minimizer will not touch the origin. Indeed, if it touches the origin, then we can blow up $u^\varepsilon$ at the origin, since by Theorem A $u^\varepsilon$ has linear growth at the origin, and obtain a global minimizer $u^0_\varepsilon$, with data $(\alpha_+ - \varepsilon)x_2$ on $\Pi$.

Now from the classification of the homogeneous global solutions, Theorem C, we must have that $u^0_\varepsilon < v_L$, and thus $u^0_\varepsilon = v_S^\varepsilon$ in $B_1^+$.

This means that the free boundary cannot touch the origin, for any $\varepsilon > 0$. In particular, by Theorem 5.1 in [KKS], we must have that it touches the fixed boundary tangentially at some point $x^0$ with $x_2^0 < 0$.

10.2. Non-homogeneous global solutions. In this section we show the existence of a global solution which is non-homogeneous. We follow a perturbation method used in [AS].

Let $\alpha_- = 0,0 < \alpha_+ < 1, \Lambda = 1$ and set $f^\varepsilon = (\alpha_+ - \varepsilon)x_2^+$. Now consider a minimizer of our functional in $B_1^+$, with admissible functions having boundary data $f^\varepsilon$ on $\Pi$ and $(\alpha_+ - \varepsilon)(-\gamma x_1 + x_2)^+$ on $S^+_1$, where $\gamma = \sqrt{\frac{1}{\alpha_+}} - 1$.

Let $\gamma_\varepsilon = \sqrt{\frac{1}{(\alpha_+ - \varepsilon)^2}} - 1$; then from Theorem C $v_L^\varepsilon = (\alpha_+ - \varepsilon)(-\gamma_\varepsilon x_1 + x_2)^+$ is the largest global homogeneous solution with boundary values $f^\varepsilon$ on $\Pi$. Notice that $\gamma_\varepsilon > \gamma, f^\varepsilon \leq \alpha_+ x_2^+$, implying that $v_L \geq (\alpha_+ - \varepsilon)(-\gamma_\varepsilon x_2 + x_1)^+$ on $S^+_1$. Consider the class of local minimizers

$$K_\varepsilon = \{u \in H^1(B_1^+), u = (\alpha_+ - \varepsilon)(-\gamma x_1 + x_2)^+ \text{ on } \partial S^+_1, u \text{ is a local minimizer of } J\}.$$

Then from the results of Section 7 $u^\varepsilon = \inf_{K_\varepsilon} u$ is a minimizer. Furthermore $u^\varepsilon \leq \min(u_\varepsilon, v_L) \leq v_L$.

For $\varepsilon$ fixed, any blow-up of $u^\varepsilon$ at the origin is a homogeneous global solution $u^0_\varepsilon$, which in view of the inequality $u^\varepsilon \leq v_L$, implies $u^0_\varepsilon \leq v_L$. Now $u^0_\varepsilon$ is a global homogeneous solution with boundary data $f^\varepsilon$, and hence it must equal one of the functions $(\alpha_+ - \varepsilon)(\pm \gamma_\varepsilon x_2 + x_1)^+$. The only way for $u^0_\varepsilon$ to be as above and satisfy

$$u^\varepsilon_0 \leq v_L = \alpha_+(-\gamma x_2 + x_1)^+$$

is that $u^\varepsilon_0 = (\alpha_+ - \varepsilon)(\gamma_\varepsilon x_2 - x_1)^+ = v_S^\varepsilon$. This in turn suggests that the free boundary $\Gamma(u^\varepsilon)$ starts at the origin with a tangential touch to $\Gamma(v_S^\varepsilon)$, the smallest global solution with boundary data $f^\varepsilon$. Since the free boundary divides $B_1^+$ into
two parts, it has to end on $S^+_1$; see Figure 3. In particular $\Gamma(u^\varepsilon)$ cuts the $x_1$-axis at some point $x^\varepsilon = (r_\varepsilon, 0)$. Now we consider the blow up of $u^\varepsilon$ with respect to $r_\varepsilon$. Observe that $r_\varepsilon \to 0$, and thereby, utilizing Proposition 4.5 and choosing a suitable subsequence, we obtain a global solution $u_0$ with boundary data $\alpha+\lambda x_1$, and with $\Gamma(u_0) \ni (1, 0) = \lim_{\varepsilon \to 0} x^\varepsilon/r_\varepsilon$. It follows from Theorem C that this solution cannot be homogeneous.

Remark 10.1. It should be remarked that in the above example of non-homogeneous global solutions, we have $|\nabla u|^2 \not\leq \Lambda$. Indeed, if this was true, then one may apply the maximum principle to $|\nabla u|^2$ in $\{u > 0\}$ and obtain a maximum on the free boundary (the free boundary is regular in 2-space dimension). Hence, by Hopf’s lemma one obtains $\partial_\nu |\nabla u|^2 > 0$, where $\nu$ is the unit normal on the free boundary pointing outside the support of $u$. In particular $u_\nu u_{\nu\nu} > 0$. Since $u_\nu = |\nabla u| = \sqrt{\Lambda}$ we will have $u_{\nu\nu} > 0$ on the free boundary. Using the representation of Laplacian on the free boundary $\Delta u = \Delta_S u + H u_\nu + u_{\nu\nu}$, where $H$ is the mean curvature, we conclude the convexity of the free boundary. This contradicts the geometry of the example above.

11. Appendix 1

In this section we prove that any blow-up limit of $u \in \mathcal{P}_r$ is a homogeneous function of degree one. The case when $g = 0$ immediately follows from [Work1], Section 2. When $g \neq 0$ some extra care is needed, because the comparison of $u$ with its homogeneous extension $u_l(x) = \frac{|x|}{t} u(t \frac{x}{|x|})$ in $B^+_1$ fails on the flat portion of the boundary, i.e. $u(x) \neq u_l(x)$ when $x \in \Pi \cap B_t$.

To fix the ideas we consider the model case $g(x) = C|x|^{1+\kappa}$ with $\kappa > 0$ and $C = \text{const}$. Since $\rho^{-1} g(\rho x) \to 0$ as $\rho \downarrow 0$, it follows that $u$ and $v = u - g$ have the same blow-ups at the origin.

Lemma 11.1. Let $u \in \mathcal{P}_r(n, \lambda_\pm, \alpha_\pm, g)$. Set $v = u - g$ where $g(x) = C|x|^{1+\kappa}, \kappa > 0$. Then

$$\tilde{W}(t) = \frac{1}{t^n} \int_{B^+_1} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} - \frac{1}{t^{n+1}} \int_{\partial B^+_1} v^2 + \frac{C_1}{\kappa} t^\kappa$$
is a nondecreasing function of \( t \). Furthermore

\[
(11.1) \quad \frac{d}{dt} \left\{ \frac{1}{t^n} \int_{B^+_1} |\nabla v|^2 + \Lambda \chi_{\{v>-g\}} - \frac{1}{t^{n+1}} \int_{\partial B^+_1} v^2 + \frac{C_1}{\kappa} t^\kappa \right\} \geq \frac{1}{t^n} \int_{\partial B^+_1} \left( \nabla v \cdot \nu - \frac{v}{t} \right)^2.
\]

Proof. Let \( \varphi \in H^1_0(B^+_r), r \in (0,1) \), and let us define \( v = u - g, v_\varphi = u + \varphi - g \) in \( B^+_i \). Then \( J(u) \leq J(u + \varphi) \) transforms into \( J(v + g) \leq J(v_\varphi + g) \). Employing Green’s identity we obtain

\[
\int_{B^+_r} |\nabla u|^2 = \int_{B^+_r} |\nabla v|^2 + 2 \int_{B^+_r} \nabla v \cdot \nabla g + \int_{B^+_r} |\nabla g|^2
= \int_{B^+_r} |\nabla v|^2 - v(2\Delta g) + 2 \int_{\partial B^+_r} v(\nabla g \cdot \nu) + \int_{B^+_r} |\nabla g|^2.
\]

Utilizing this computation and the fact \( v_\varphi - v = H^1_0(B^+_r) \), we see that if \( u \) is a minimizer of \( J(u, B^+_r) \), subject to its own boundary values on \( \partial B^+_r \), then \( v \) is a minimizer of

\[
(11.2) \quad \bar{J}(v) = \int_{B^+_r} |\nabla v|^2 - v(2\Delta g) + \Lambda \chi_{\{v>-g\}},
\]

because \( 2 \int_{\partial B^+_r} w(\nabla g \cdot \nu) + \int_{B^+_r} |\nabla g|^2 \) is constant for any \( w \in H^1(B^+_r) \), \( w|_{\partial B^+_r} = v|_{\partial B^+_r} \).

Thus it remains to prove that any blow-up limit of \( v \) at the origin is a homogeneous function of degree one.

Let \( t > 0 \) be small and take \( v_t(x) = \frac{|x|}{t} v(t \frac{x}{|x|}) \); then on \( \partial B_t \) \( v_t \) agrees with \( v \) and it follows that \( \bar{J}(v) \leq \bar{J}(v_t) \). Using the homogeneity of \( v_t \) and the identities

\[
\nabla v_t(x) = \frac{x}{t |x|} v(t \frac{x}{|x|}) + \nabla v(t \frac{x}{|x|}) - \nabla v(t \frac{x}{|x|}) \cdot \frac{x}{|x|},
\]

\[
|\nabla v_t|^2 = t^{-2} v^2(t \frac{x}{|x|}) + \left| \nabla v(t \frac{x}{|x|}) \right|^2 - \left( \nabla v(t \frac{x}{|x|}) \cdot \frac{x}{|x|} \right)^2,
\]

one can easily compute

\[
\int_{B^+_r} |\nabla v_t|^2 + \Lambda \chi_{\{v_t>-g\}}
= \int_{B^+_r} \left[ \frac{x}{t |x|} v(r \frac{x}{|x|}) + \nabla v(t \frac{x}{|x|}) - \nabla v(t \frac{x}{|x|}) \cdot \frac{x}{|x|} \right]^2 + \Lambda \chi_{\{v_t>-g\}}
= \int_0^t \int_{\partial B^+_r} \left[ t^{-2} v^2(t \frac{x}{|x|}) + \left| \nabla v(t \frac{x}{|x|}) \right|^2 - \left( \nabla v(t \frac{x}{|x|}) \cdot \frac{x}{|x|} \right)^2 \right] + \Lambda \chi_{\{v_t>-g\}}
= \frac{t}{n} \int_{\partial B^+_r} |\nabla v|^2 + \frac{t}{n} \int_{\partial B^+_r} \left[ \frac{v^2}{t^2} - (\nabla v \cdot \nu)^2 \right] + \int_{B^+_r} \Lambda \chi_{\{v_t>-g\}}.
\]
To deal with the last integral, we first notice that \( \{v_t(x) > -g(x)\} \subset \{v(t \frac{x}{|x|}) > -Ct^{1+\kappa}\} \). Indeed if \( x \in \{v_t(x) > -g(x)\} \), then \( \frac{|x|}{t}v(t \frac{x}{|x|}) > -C|x|^{1+\kappa} \), or equivalently \( v(t \frac{x}{|x|}) > -Ct|x|^{\kappa} \). But \( |x| \leq t \) since \( x \in B^+_t \). Thus \( -Ct|x|^\kappa \leq -Ct^{1+\kappa} = -g(t) \).

In particular we get that \( \int_{B^+_t} \Lambda \chi_{\{v_t > -g\}} \leq \int_{B^+_t} \Lambda \chi_{\{v(t \frac{x}{|x|}) > -g(t)\}} \) which, after applying Fubini’s theorem, yields

\[
\int_{B^+_t} \Lambda \chi_{\{v(t \frac{x}{|x|}) > -g(t)\}} = \frac{1}{n} \int_{\partial B^+_t} \Lambda \chi_{\{v > -g\}}.
\]

Next we notice that if \( w \in H^1(B^+_t) \) and \( |w(x)| \leq C|x|, x \in B^+_t \), then \( \left| \int_{B^+_t} w(2\Delta g) \right| \leq C_1 t^{n+\kappa} \) with some tame constant \( C_1 \). Therefore comparing the \( \tilde{J} \) energies in \( B^+_t \), we get

\[
(11.3) \quad 0 \leq \tilde{J}(v_t) - \tilde{J}(v)
\]

\[
\leq \frac{t}{n} \int_{\partial B^+_t} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} + \frac{t}{n} \int_{\partial B^+_t} \left[ \frac{v^2}{t^2} - (\nabla v \cdot v)^2 \right] + \int_{B^+_t} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}}
\]

\[
+ \frac{t}{n} \int_{\partial B^+_t} \left[ \frac{v^2}{t^2} - (\nabla v \cdot v)^2 \right] + C_1 t^{n+\kappa}
\]

\[
= \frac{t^{n+1} d}{n} \int_{\partial B^+_t} \left[ \frac{1}{t^n} \int_{\partial B^+_t} \frac{1}{t^n} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} \right] - \frac{t}{n} \int_{\partial B^+_t} \left( \nabla v \cdot \nu - \frac{v}{t} \right)^2 - \frac{2t}{n} \int_{\partial B^+_t} \frac{v}{t^2} \left[ \nabla v \cdot \nu - \frac{v}{t} \right] + C_1 t^{n+\kappa}.
\]

Multiplying both sides by \( nt^{-n-1} \) we conclude that

\[
\frac{d}{dt} \left\{ \frac{1}{t^n} \int_{B^+_t} |\nabla v|^2 + \Lambda \chi_{\{v > -g\}} - \frac{1}{t^{n+1}} \int_{\partial B^+_t} v^2 + \frac{C_1}{k} t^{\kappa} \right\} \geq \frac{1}{t^n} \int_{\partial B^+_t} \left( \nabla v \cdot \nu - \frac{v}{t} \right)^2.
\]

\( \square \)

**Remark 11.2.** This argument shows that \( g \) can be replaced by any homogeneous polynomial or function of degree \( m > 1 \).

**Corollary 11.3.** Let \( v \) be as in Lemma [11.1]. Then any blow-up limit of \( v \) at the origin is homogeneous of degree one. In particular any blow-up of \( u \) is homogeneous of degree one.

**Proof.** The first statement follows exactly as in [W1], Section 2. To show that the blow-up of \( u \) is homogeneous we need to notice that \( g(rx)r^{-1} \to 0 \) uniformly as \( r \to 0 \). Hence the blow-ups of \( u \) and \( v \) coincide. \( \square \)
12. Appendix 2

We shall discuss the rectifiability of the free boundary in $B_1$.

**Lemma 12.1.** Let $u$ be a global homogeneous minimizer and let $\Gamma(u)$ touch tangentially the free boundary of $v_L$. Then $u$ is non-degenerate, i.e. there is a tame constant $c > 0$ such that for any $x \in \Gamma(u)$ the following estimate is true:

\[
\sup_{B_r^+(x)} u \geq c r, \quad \forall B_r(x) \subset \mathbb{R}^n. \tag{12.1}
\]

**Remark 12.2.** In [AC], a different form of non-degeneracy is proven (see Lemma 3.4 in [AC]), namely

\[
\int_{\partial B_r(x)} u \geq cr, \quad x \in \partial\{u > 0\}.
\]

This integral inequality implies that there is $y \in \partial B_r(x)$ such that $u(y) \geq cr$. But $u$ is subharmonic, therefore $\sup_{B_r(x)} u \geq cr$.

**Proof.** Let $x \in \partial\{u > 0\}$ and set $\delta(x) = \text{dist}(x, \Pi)$. If $\delta(x) \geq r$, then $B_r(x) \subset \mathbb{R}^n_+$. Taking $u_r(y) = \frac{u(x + ry)}{r}$, $y \in B_1$ and employing Lemma 8.2 in [AC], we see that $u_r$ is a local minimizer. Hence from Remark 12.2 we obtain $\sup_{B_{\frac{r}{2}}} u_r \geq c$, which after scaling back implies the desired result.

Now assume that $\delta(x) < r$. We consider two possible scenarios:

**Case a) $\frac{r}{1000} \leq \delta(x)$.** Then using Remark 12.2 in $B_{\delta(x)}(x)$ we get

\[
\sup_{B_{\delta(x)}^+(x)} u \geq \sup_{B_{\delta(x)}} u \geq \frac{c\delta(x)}{2} \geq \frac{cr}{2000}.
\]

**Case b) $\delta(x) < \frac{r}{1000}$.** Let $R(x) = \text{dist}(x, \Pi_0)$, where $\Pi_0 = \{x \in \mathbb{R}^n : x_1 = x_2 = 0\}$, and take $x_0 \in \Pi_0$ such that $R(x) = |x - x_0|$. Notice that $R(x) \sim \delta(x)$, because $\Gamma(u)$ touches $\Gamma_L$ tangentially. This means that there are two positive constants $a, b$ such that $aR(x) \leq \delta(x) \leq bR(x)$ if $x$ is close to $\Pi$ (see the definitions of the cones $K_\sigma$). We have $r > 1000\delta(x) \geq a1000R(x)$, yielding $R(x) \leq \frac{r}{a1000}$. In particular

\[
\rho = r - R(x) \geq r - \frac{r}{a1000} \geq \frac{r}{1000}.
\]

Observing that $B^+_{\rho}(x_0) \subset B^+_{\rho}(x)$ we get

\[
\sup_{B^+_{\rho}(x)} u \geq \sup_{B^+_{\rho}(x_0)} u \geq \sup_{B^+_{\rho}(x_0)} u_S \geq c\rho \geq \frac{cr}{1000}.
\]

**Lemma 12.3.** Let $u$ be as in Lemma 12.1. Then

\[
\mathcal{H}^{n-1}(B_1 \cap \partial\{u > 0\}) < \infty.
\]

**Proof.** For each open ball $B_r(x) \subset \mathbb{R}^n$ let $B^+_r(x) = B_r(x) \cap \mathbb{R}^n_+$. Introduce the measure $\mu = \Delta u$. Clearly $\mu$ is a non-negative Radon measure, because $\int_{B^+_r(x)} \mu = \int_{\partial B^+_r(x)} \nabla u \cdot \nu \leq C r^{n-1}$. Hence for any compact $D \subset \mathbb{R}^n$ we can cover $\overline{D \cap \mathbb{R}^n_+}$ by a finite number of balls, which yields $\mu(D \cap \mathbb{R}^n_+) < \infty$. 

\[\square\]
Next we want to show that there is a positive constant $c_0$ such that for each $x \in B_1^+ \cap \Gamma(u)$ we have
\begin{equation}
(12.2) \quad \int_{B_1^+(x)} \mu \geq c_0 r^{n-1} \quad \text{if } r > 0 \text{ is small.}
\end{equation}
From (12.2) one can conclude the proof of the lemma by employing a standard covering argument.

First we note that by Lemma 12.1 $u$ is nondegenerate, that is, there is a constant $c > 0$ such that
\begin{equation}
(12.3) \quad \sup_{B_1^+(x)} u \geq cr, \quad \forall x \in B_1^+ \cap \Gamma(u)
\end{equation}
for small $r > 0$. Now suppose that (12.2) fails. Then there is a sequence of free boundary points $x_j \in \Gamma(u)$ and a sequence of positive numbers $r_j > 0$ such that
\begin{equation}
(12.4) \quad \int_{B_{r_j}(x_j)} \mu \leq \frac{r_j^{n-1}}{j}.
\end{equation}

First, let us suppose that there is a subsequence $r_{j(m)}$ such that $B_{r_{j(m)}} \cap \Pi_0 \neq \emptyset$. Let $x_j^0 \in \Pi_0$ and dist$(x_j, \Pi_0) = |x_j - x_j^0|$. Then consider $v_m(x) = \frac{u(x_j^0 + r_{j(m)}x)}{r_{j(m)}}, x \in B_2^+$. From Proposition 4.5 and Lemma 8.2 we get $v_m \to v_0, \mu_m \rightharpoonup \mu_0$, where $\mu_m = \Delta v_m$ and $\Delta v_0 = \mu_0$, at least for a subsequence $m_k$, and $v_0$ is a local minimizer. Moreover (12.4) translates to
\[ \int_{B_1^+(y_0)} \mu_0 = 0 \]
for some $y_0 \in \Gamma_L$, i.e. $v_0$ is harmonic in $B_1^+(y_0)$. From the strong maximum principle we conclude $v_0 = 0$, which is in contradiction with the non-degeneracy of $v_{m_k}$ and $v_0$.

Finally let us assume that $B_{r_j}(x_j) \cap \Pi_0 = \emptyset$ for any $j$. Denote $\delta_j = \operatorname{dist}(x_j, \Pi)$. From the tangential touch of $\Gamma(u)$ and $\Gamma_L$ it follows that $aR_j \leq \delta_j \leq bR_j$, where $R_j = \operatorname{dist}(x_j, \Pi_0)$. Thus we have $r_j < R_j$. If, moreover, $r_j \geq \delta_j$, then applying Theorem 4.3 of [AC] to $\frac{u(x_j + r_j x)}{r_j}$ we will conclude a contradiction if $j$ is large enough.

Thus without loss of generality we may assume that $\delta_j < r_j < R_j$. Introduce $w_j(y) = \frac{u(x_j + \delta_j)}{\delta_j}, y \in B_1$; then
\[ \int_{B_{\delta_j}(x_j)} \mu \leq \int_{B_{r_j}(x_j)} \mu \leq \frac{r_j^{n-1}}{j} \leq \frac{\delta_j^{n-1}}{ja^{n-1}}. \]
Hence for $\Delta w_j = \mu_j$ we have $\int_{B_1} \mu_j \leq \frac{1}{ja^{n-1}}$. On the other hand $\sup_{B_{\frac{1}{2}}} w_j \geq c$. Extracting a subsequence for which $w_j \to w_0, \Delta w_j \to \Delta w_0$ in $B_1$ at least for a subsequence, where $w_0$ is a local minimizer in $B_1$; see Proposition 4.5. But $\int_{B_1} \Delta w_j \leq \frac{1}{ja^{n-1}} \to 0$. Thus $w_0 \geq 0$ is harmonic and non-degenerate in $B_1$ and $w_0(0) = 0$. Hence by the strong maximum principle $w_0 = 0$, which is in contradiction with $\sup_{B_{\frac{1}{2}}} w_0 \geq c$. \qed
References

[AC] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. 325 (1981), 105–144. MR618549 (83a:49011)

[ACF] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. 282 (1984), no. 2, 431–461, DOI 10.2307/1999245. MR732100 (85h:49014)

[AG] Hans Wilhelm Alt and Gianni Gilardi, *The behavior of the free boundary for the dam problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 9 (1982), no. 4, 571–626. MR693780 (85c:35089a)

[AS] John Andersson and Henrik Shahgholian, *Global solutions of the obstacle problem in half-spaces, and their impact on local stability*, Calc. Var. Partial Differential Equations 23 (2005), no. 3, 271–279, DOI 10.1007/s00526-004-0299-0. MR2142064 (2006b:35352)

[BZ] Garrett Birkhoff and E. H. Zarantonello, *Jets, wakes, and cavities*, Academic Press Inc., Publishers, New York, 1957. MR0088230 (19,486f)

[CKS] Luis A. Caffarelli, Lavi Karp, and Henrik Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. of Math. (2) 151 (2000), no. 1, 269–292, DOI 10.2307/121117. MR1745013 (2001a:35188)

[CS] Luis Caffarelli and Sandro Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005. MR2145284 (2006k:35310)

[GT] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)

[Gu] Alex Gurevich, *Boundary regularity for free boundary problems*, Comm. Pure Appl. Math. 52 (1999), no. 3, 363–403, DOI 10.1002/(SICI)1097-0312(199903)52:3<363::AID-CPA3>3.3.CO;2-L. MR1656068 (99m:35272)

[KKS] A. L. Karakhanyan, C. E. Kenig, and H. Shahgholian, *The behavior of the free boundary near the fixed boundary for a minimization problem*, Calc. Var. Partial Differential Equations 28 (2007), no. 1, 15–31, DOI 10.1007/s00526-006-0029-x. MR2267752 (2008d:35240)

[K] Carlos E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. 83, published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994. MR1282720 (96a:35040)

[W1] Georg Sebastian Weiss, *Partial regularity for a minimum problem with free boundary*, J. Geom. Anal. 9 (1999), no. 2, 317–326, DOI 10.1007/BF02921941. MR1759450 (2001b:49053)

[W2] Georg S. Weiss, *Boundary monotonicity formulae and applications to free boundary problems. I. The elliptic case*, Electron. J. Differential Equations (2004), No. 44, 12 pp. (electronic). MR2047400 (2004m:35286)

[Z] Tullio Zolezzi, *On weak convergence in $L^\infty$*, Indiana Univ. Math. J. 23 (1973/74), 765–766. MR0328576 (48 #6918)

Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, King’s Buildings, Mayfield Road, EH9 3JZ, Edinburgh, Scotland

E-mail address: aram.karakhanyan@ed.ac.uk

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden

E-mail address: henriksh@math.kth.se