Asymptotic behaviour of Dirichlet eigenvalues for homogeneous Hörmander operators and algebraic geometry approach

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Abstract

We study the Dirichlet eigenvalue problem of homogeneous Hörmander operators \( \triangle X = \sum_{j=1}^{m} X_j^2 \) on a bounded open domain containing the origin, where \( X_1, X_2, \ldots, X_m \) are linearly independent smooth vector fields in \( \mathbb{R}^n \) satisfying Hörmander’s condition and a suitable homogeneity property with respect to a family of non-isotropic dilations. Suppose that \( \Omega \) is a smooth open bounded domain in \( \mathbb{R}^n \) containing the origin. Combining the subelliptic heat kernel estimates, the resolution of singularities in algebraic geometry and some refined analysis involving convex geometry, we establish the explicit asymptotic behaviour \( \lambda_k \approx k^{\frac{2}{\alpha_0}} (\ln k)^{-\frac{2d_0}{\alpha_0}} \) as \( k \to +\infty \), where \( \lambda_k \) denotes the \( k \)-th Dirichlet eigenvalue of \( \triangle X \) on \( \Omega \), \( \alpha_0 \) is a positive rational number, and \( d_0 \) is a non-negative integer. Furthermore, we also give the optimal bounds of index \( \alpha_0 \), which depends on the homogeneous dimension associated with vector fields \( X_1, X_2, \ldots, X_m \).

Keywords: Hörmander’s condition, homogeneous Hörmander operators, Dirichlet eigenvalues, subelliptic heat kernel, resolution of singularities

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1. Introduction and main results

For \( n \geq 2 \), let \( X = (X_1, X_2, \ldots, X_m) \) be the real smooth vector fields defined on \( \mathbb{R}^n \) and satisfy the following assumptions:

(H.1) There exists a family of (non-isotropic) dilations \( \{\delta_t\}_{t>0} \) of the form

\[
\delta_t : \mathbb{R}^n \to \mathbb{R}^n, \quad \delta_t(x) = (t^{\alpha_1}x_1, t^{\alpha_2}x_2, \ldots, t^{\alpha_n}x_n),
\]

where \( 1 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \) are positive integers, such that \( X_1, X_2, \ldots, X_m \) are \( \delta_t \)-homogeneous of degree 1, i.e.

\[
X_j(f \circ \delta_t) = t(X_j f) \circ \delta_t, \quad \forall t > 0, \ f \in C^\infty(\mathbb{R}^n), \ j = 1, \ldots, m;
\]

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(H.2) The vector fields $X_1, X_2, \ldots, X_m$ are linearly independent in $\mathcal{X}(\mathbb{R}^n)$ (as linear differential operators) and satisfy the Hörmander’s condition at $0 \in \mathbb{R}^n$, i.e.

$$\dim\{Y(0) \mid Y \in \text{Lie}(X)\} = n.$$ 

Here $\text{Lie}(X)$ is the smallest Lie subalgebra in $\mathcal{X}(\mathbb{R}^n)$ containing $X = (X_1, X_2, \ldots, X_m)$, and $\mathcal{X}(\mathbb{R}^n)$ denotes the set of all smooth vector fields in $\mathbb{R}^n$, which is also a vector space over $\mathbb{R}$ equipped with the natural operations.

Then we denote the so-called $\delta_t$-homogeneous dimension of $(\mathbb{R}^n, \delta_t)$ by

$$Q := \sum_{j=1}^{n} \alpha_j. \quad (1.1)$$

Consider the following formally self-adjoint operator $\triangle_X$ generated by vector fields $X_1, X_2, \ldots, X_m$, i.e.

$$\triangle_X := -\sum_{i=1}^{m} X_i^* X_i,$$

where $X_i^* = -X_i - \text{div}X_i$ denotes the formal adjoint of $X_i$, and $\text{div}X_i$ is the divergence of $X_i$. Due to assumptions (H.1) and (H.2), we know the Lie algebra $\text{Lie}(X)$ is nilpotent of step $\alpha_n$, and the vector fields $X = (X_1, X_2, \ldots, X_m)$ satisfy the Hörmander’s condition in $\mathbb{R}^n$, i.e., there exists a smallest positive integer $r$ such that $X_1, X_2, \ldots, X_m$ together with their commutators of length at most $r$ span the tangent space $T_x(\mathbb{R}^n)$ at each point $x \in \mathbb{R}^n$ (see Proposition 2.9 below). The index $r$ is called the Hörmander index of $X$, which admits $r = \alpha_n$ under assumptions (H.1) and (H.2). Moreover, by assumption (H.1) and Proposition 2.6 below, we have $X_i^* = -X_i$ for $i = 1, \ldots, m$. Thus, $\triangle_X$ has the sum of square form

$$\triangle_X = \sum_{i=1}^{m} X_i^2. \quad (1.2)$$

The operator $\triangle_X$ in (1.2) under assumptions (H.1) and (H.2) is called the homogeneous Hörmander operator.

The class of the homogeneous Hörmander operators is quite large, and it contains meaningful degenerate operators widely studied in the literature. For instance, the sub-Laplacians on Carnot groups, the Grushin operators and the Martinet operators fall in this class. In recent years, the study of homogeneous Hörmander operators has received considerable attention. Employing the global lifting method constructed by Folland [37], Biagi-Bonfiglioli [10] proved the existence of global fundamental solution. Then, Biagi-Bonfiglioli-Bramanti [13] further established the global estimates for the fundamental solution. Meanwhile, the existence and Gaussian bounds of global heat kernels for homogeneous Hörmander operators have been investigated by Biagi-Bonfiglioli [11] and Biagi-Bramanti [14], respectively. In addition, we refer to [6, 9, 12, 15, 16] for more results of degenerate elliptic equations related to the homogeneous Hörmander operators.

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As it is well-known, the sub-Riemannian geometry is the natural geometric framework for subelliptic PDEs. The vector fields \( X = (X_1, X_2, \ldots, X_m) \) under Hörmander’s condition induce a canonical sub-Riemannian structure \((D, g)\) such that \( \mathbb{R}^n \) endowed with \((D, g)\) forms a sub-Riemannian manifold \((\mathbb{R}^n, D, g)\). Here \( D \) is the distribution, which is a family of linear subspaces \( D_x \subset T_x(\mathbb{R}^n) \) with \( D_x = \text{span}\{X_1(x), X_2(x), \ldots, X_m(x)\} \) depending smoothly on \( x \in \mathbb{R}^n \), and \( g \) denotes the sub-Riemannian metric. When \( D \) has a constant rank \( m \) on \( \mathbb{R}^n \) with \( m \leq n \) (i.e. the dimension \( \dim D_x = m \leq n \) for every \( x \in \mathbb{R}^n \)), \( D \) is the subbundle of \( \mathcal{T} \mathbb{R}^n \) and the vector fields \( X = (X_1, X_2, \ldots, X_m) \) are orthonormal concerning the sub-Riemannian metric \( g \). In this case, the Hörmander operator \( \triangle_X \) coincides with the sub-Laplacian on sub-Riemannian manifold \((\mathbb{R}^n, D, g)\) (see [39]). However, the rank of \( D \) may vary in general, and the setting above encompasses almost sub-Riemannian structures. More details on sub-Riemannian geometry can be found in [1, 7, 41, 50, 66, 70, 76].

We next introduce the sub-Riemannian flag to analyze the Lie algebra structure of distribution \( D \). Specifically, the sub-Riemannian flag at each \( x \in \mathbb{R}^n \) is the sequence of nested vector subspaces

\[
\{0\} = D^0_x \subset D_x = D^1_x \subset D^2_x \subset \cdots \subset D^{r(x)-1}_x \subset D^{r(x)}_x = T_x(\mathbb{R}^n)
\]

defined in terms of successive Lie brackets, and \( r(x) \leq r \) is the degree of nonholonomy at \( x \). Here for each \( 1 \leq j \leq r(x) \), \( D^j_x \) is a subspace of \( T_x(\mathbb{R}^n) \) spanned by all commutators of \( X_1, X_2, \ldots, X_m \) with length at most \( j \). Setting \( \nu_j(x) = \dim D^j_x \) for \( 1 \leq j \leq r(x) \) with \( \nu_0(x) := 0 \), we denote by

\[
\nu(x) := \sum_{j=1}^{r(x)} j(\nu_j(x) - \nu_{j-1}(x))
\]

the pointwise homogeneous dimension at \( x \) (see [67]). It follows from (1.3) that \( n \leq n + r(x) - 1 \leq \nu(x) \leq nr(x) \). Moreover, a point \( x \in \mathbb{R}^n \) is regular if, for every \( 1 \leq j \leq r(x) \), the dimension \( \nu_j(y) \) is a constant as \( y \) varies in an open neighbourhood of \( x \). Otherwise, \( x \) is said to be singular. Besides, a set \( S \subset \mathbb{R}^n \) is equiregular if every point of \( S \) is regular, and a set \( S \subset \mathbb{R}^n \) is said to be non-equiregular if it contains some singular points. The equiregular assumption is also known as the Métivier’s condition in PDEs (see [64]). For the equiregular connected set \( S \), the pointwise homogeneous dimension \( \nu(x) \) is a constant \( \nu \) which coincides with the Hausdorff dimension of \( S \) related to the vector fields \( X \), and this constant \( \nu \) is also called the Métivier’s index. Additionally, if the set \( S \subset \mathbb{R}^n \) is non-equiregular, we can introduce the so-called generalized Métivier’s index by

\[
\tilde{\nu}_S := \max_{x \in S} \nu(x).
\]

The generalized Métivier’s index is also known as the non-isotropic dimension (see [21, 22, 84]), which plays an important role in the geometry and functional settings associated with vector fields \( X \). Note that \( n + \max_{x \in S} r(x) - 1 \leq \tilde{\nu}_S < n\alpha_n \) for \( \alpha_n > 1 \), and \( \tilde{\nu}_S = \nu \) if the closure of \( S \) is equiregular and connected.
The present work is concerned with the Dirichlet eigenvalue problem of homogeneous Hörmander operator $\Delta_X$, i.e.

$$\begin{cases}
-\Delta_X u = \lambda u, & \text{on } \Omega; \\
u \in H^1_{X,0}(\Omega),
\end{cases} \quad (1.5)$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded open domain containing the origin, and $H^1_{X,0}(\Omega)$ is the weighted Sobolev space associated with vector fields $X$ (see Section 2 below). According to Hörmander’s condition, $-\Delta_X$ is a positive self-adjoint operator possessing discrete Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

When $X = (\partial_{x_1}, \ldots, \partial_{x_n})$, $\Delta_X$ is reduced to the standard Laplacian $\Delta$. The classical eigenvalue problems have been widely studied since Weyl [80] established the celebrated asymptotic formula:

$$\lambda_k \sim 4\pi \left( \frac{\vert \Omega \vert}{\Gamma \left( \frac{n}{2} + 1 \right)} \right)^{-\frac{2}{n}} \cdot k^\frac{2}{n} \quad \text{as } k \to +\infty,$$

where $\vert \Omega \vert$ is the $n$-dimensional Lebesgue measure of $\Omega$. Here we refer readers to [27–30, 46, 49, 55, 57, 61, 63, 69, 74, 80, 81] as well as the references therein.

For vector fields $X = (X_1, X_2, \ldots, X_m)$ under Hörmander’s condition, $\Delta_X$ is the self-adjoint Hörmander operator. The study of eigenvalue problems for Hörmander operator originated from Métivier [64] in 1976. Imposing the equiregular assumption on $\Omega$, he proved

$$\lambda_k \sim \left( \int_{\Omega} \gamma(x) dx \right)^{-\frac{2}{n}} \cdot k^\frac{2}{n} \quad \text{as } k \to +\infty,$$ \quad (1.6)

where $\gamma(x)$ is a positive continuous function on $\Omega$ and $\nu$ is the Métivier index of $\Omega$. However, the asymptotic behaviour of eigenvalues has not been fully understood when we drop the equiregular assumption. As far as we know, there are only a few asymptotic results in the general non-equiregular case. In 1981, Fefferman-Phong [36] proved that, for the closed eigenvalue problem of Hörmander operator on compact (closed) manifold $M$,

$$c_1 \int_M \frac{d\mu}{\mu(B_{d_X}(x, \lambda^{-\frac{1}{2}}))} \leq N(\lambda) \leq c_2 \int_M \frac{d\mu}{\mu(B_{d_X}(x, \lambda^{-\frac{1}{2}}))}$$ \quad (1.7)

holds for sufficiently large $\lambda$, where $N(\lambda) := \# \{ k \mid \lambda_k \leq \lambda \}$ is the spectral counting function, $c_2 > c_1 > 0$ are some constant depending on $X$ and $M$, $\mu$ is the smooth measure on $M$ and $B_{d_X}(x, r)$ is the subunit ball (defined in Section 2 below). However, for general Hörmander operators, the abstract integral in (1.7) cannot be calculated explicitly without further assumptions. In addition, the validity of estimation (1.7) is still an open problem for the Dirichlet eigenvalue of Hörmander operators.

Recently, Chen-Chen [22] dealt with the Dirichlet eigenvalue problem for Hörmander operator in non-equiregular case and established the explicit asymptotic formula under some
weak conditions. Specifically, denoting by $H := \{ x \in \Omega | \nu(x) = \tilde{\nu} \}$ the level set of pointwise homogeneous dimension attaining the maximum value, they proved that if $H$ possesses a positive measure, then

$$\lambda_k \sim \left( \frac{\Gamma \left( \frac{\tilde{\nu}_0}{2} + 1 \right)}{\int_H \gamma_0(x) dx} \right)^{\frac{2}{\tilde{\nu}_0}} \cdot k^{\frac{2}{\tilde{\nu}_0}} \quad \text{as} \quad k \to +\infty,$$

(1.8)

where $\tilde{\nu} := \max_{x \in \Omega} \nu(x)$ is the non-isotropic dimension (or generalized Métivier’s index) of $\Omega$ depending on vector fields $X$, and $\gamma_0$ is a positive measurable function on $\Omega$. We note that (1.8) generalizes Métivier’s asymptotic formula (1.6). Indeed, the equiregular assumption on $\Omega$ gives that $\tilde{\nu} = \nu$ and $H = \Omega$. Thus the condition $|H| = |\Omega| > 0$ is certainly satisfied. Furthermore, in the case of $|H| = 0$, [22] obtained that

$$\lim_{k \to +\infty} \frac{k^{\frac{2}{\tilde{\nu}_0}}}{\lambda_k} = 0.$$  

(1.9)

Observe that (1.9) only indicate $\lambda_k$ grows faster than $k^{\frac{2}{\tilde{\nu}_0}}$ as $k \to +\infty$. A natural question arises: In the case of $|H| = 0$, what is the exact growth rate of $\lambda_k$ as $k \to +\infty$?

It is worth pointing out that even if $|H| = 0$, the class of the homogeneous Hörmander operators remains to be quite large and contains a lot of crucial degenerate operators concerned in sub-Riemannian geometry. For example, the so-called Grushin operators, Bony operators and Martinet operators belong to this class (see Section 6 below).

In the present work, we will establish the explicit asymptotic estimates of Dirichlet eigenvalues for homogeneous Hörmander operators and give a quite satisfactory answer to the question above. Our investigations on the asymptotic behaviour of eigenvalues are also the follow-up studies of recent surveys [21, 22] on the eigenvalue problems of general Hörmander operators. When we consider the Hörmander operators in the case of $|H| = 0$, many new difficulties arise in the trace estimates of the subelliptic Dirichlet heat kernel. We shall invoke some new techniques involving the algebraic geometry and convex geometry to derive the explicit asymptotic estimates on Dirichlet eigenvalues. Our results in the case of $|H| = 0$ may shed light on general Hörmander operators where much less is known. For more related results on eigenvalue problems of degenerate elliptic operators, one can see [2, 3, 23, 25, 31, 32, 45, 56, 78] and the references therein.

Notations. Throughout this paper, we write $f(x) \approx g(x)$ if $C^{-1} g(x) \leq f(x) \leq C g(x)$, where $C > 0$ is some constant independent of the relevant variables in $f(x)$ and $g(x)$. Moreover, we say $f(x) \approx g(x)$ as $x \to x_0$ if there are some constants $C > 0$ and $\delta > 0$ such that $C^{-1} g(x) \leq f(x) \leq C g(x)$ holds for all $0 < |x - x_0| < \delta$.

We now state our main results. First, we obtain the following estimates of subelliptic Dirichlet heat kernel of homogeneous Hörmander operator.

**Theorem 1.1.** Let $X = (X_1, X_2, \ldots, X_m)$ be the real smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2). Suppose that $\Omega$ is a bounded open domain in $\mathbb{R}^n$, which contains the origin and has smooth boundary $\partial \Omega$. Then the Dirichlet heat kernel
$h_D(x,y,t)$ of $\triangle_X$ on $\Omega$ satisfies

$$
\int_\Omega h_D(x,x,t) dx \approx \int_\Omega \frac{dx}{|B_{d_X}(x, \sqrt{t})|} \quad \text{as} \quad t \to 0^+,
$$

(1.10)

where $|B_{d_X}(x,r)|$ denotes the $n$-dimensional Lebesgue measure of subunit ball $B_{d_X}(x,r)$.

Combining the theory of resolution of singularities and some delicate analysis associated with convex geometry, we then give the following explicit estimates of the integral in (1.10) related to the subunit ball.

**Theorem 1.2.** Let $X = (X_1, X_2, \ldots, X_m)$ be the real smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2), and $Q$ be the homogeneous dimension given by (1.1). Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded domain containing the origin in $\mathbb{R}^n$ and $w = \min_{x \in \mathbb{R}^n} \nu(x)$. Then we have

$$
\int_\Omega \frac{dx}{|B_{d_X}(x,r)|} \approx r^{-Q_0} |\ln r|^{d_0} \quad \text{as} \quad r \to 0^+,
$$

(1.11)

where $Q_0 \in \mathbb{Q}^+$ is a positive rational number and $d_0$ is a non-negative integer, which satisfy the following properties:

1. If $w = Q$, then $Q_0 = Q$ and $d_0 = 0$;
2. If $w \leq Q - 1$, then

$$
n \leq \max\{w, Q - \alpha(X)\} \leq Q_0 \leq Q - 1 \quad \text{and} \quad d_0 \in \{0, 1, \ldots, v\}.
$$

(1.12)

Here $v \leq n - 1$ is the number of degenerate components of $X$ and $\alpha(X)$ is the sum of degenerate indexes, which are defined in Definition 2.4 below.

**Remark 1.1.** Theorem 1.2 gives an explicit asymptotic behaviour of integral (1.11), which involves the term $r^{-Q_0} |\ln r|^{d_0}$ only. Moreover, the index $Q_0$ may not be an integer (see Example 6.4), and the bounds of $Q_0$ in (1.12) are optimal. In particular, the indexes $Q_0$ and $d_0$ can be calculated explicitly for specific homogeneous Hörmander vector fields by the blow-up technique in algebraic geometry and Proposition 4.7 below. We shall give some classical examples to show the optimality and present the calculation method in Section 6.

Finally, according to Theorem 1.1 and Theorem 1.2, we achieve the following explicit asymptotic estimate of Dirichlet eigenvalues.

**Theorem 1.3.** Let $X = (X_1, X_2, \ldots, X_m)$ and $\Omega$ satisfy the conditions in Theorem 1.1. Denote by $\lambda_k$ the $k$-th Dirichlet eigenvalue of problem (1.5). Then we have

$$
N(\lambda) \approx \int_\Omega \frac{dx}{|B_{d_X}(x, \lambda^{-\frac{1}{2}})|} \approx \lambda^{\frac{Q_0}{2}} (\ln \lambda)^{d_0} \quad \text{as} \quad \lambda \to +\infty,
$$

(1.13)

and

$$
\lambda_k \approx k^{\frac{Q_0}{v_0}} (\ln k)^{\frac{d_0}{v_0}} \quad \text{as} \quad k \to +\infty,
$$

(1.14)

where $N(\lambda) := \#\{k | \lambda_k \leq \lambda\}$ is the spectral counting function, $Q_0$ and $d_0$ are the indexes in Theorem 1.2.
Remark 1.2. Theorem 1.3 implies that the Fefferman-Phong’s estimate (1.7) also holds for Dirichlet eigenvalue problem (1.5). In addition, for the homogeneous vector fields $X$ defined on $\mathbb{R}^n$, Proposition 2.12 below shows an ingenious relationship between the homogeneous dimension $Q$ and the pointwise dimension $\nu(x)$, which says $Q = \nu(0) = \max_{x \in \Omega} \nu(x) = \tilde{\nu}$ if the domain $\Omega$ contains the origin in $\mathbb{R}^n$. Furthermore, according to Corollary 2.3 below, we only have two situations: $w = Q$ and $w \leq Q - 1$. If $w = Q$, we obtain $H = \{x \in \Omega|\nu(x) = \tilde{\nu}\} = \Omega$ and our asymptotic estimate (1.14) is compatible with the asymptotic formula (1.8); for $w \leq Q - 1$, it follows that $|H| = 0$ and (1.14) gives an explicit growth rate of $\lambda_k$ as $k \to +\infty$.

Remark 1.3. Theorem 1.3 also improves our previous estimates of Dirichlet eigenvalues for homogeneous Hörmander operators in [24].

The plan of the rest paper is as follows. In Section 2, we present some necessary preliminaries, including the weighted Sobolev space and related embedding results, the homogeneous functions and homogeneous vector fields, the subunit metric and volume estimates of subunit balls, and the classification of homogeneous Hörmander vector fields. In Section 3, we discuss the existence of the Dirichlet heat kernel for general self-adjoint Hörmander operators and establish some estimates of the subelliptic heat kernels. In Section 4, we study the explicit asymptotic behaviours of integral (1.11). The proofs of Theorem 1.1-Theorem 1.3 will be given in Section 5. Finally, as further applications of Theorem 1.2 and Theorem 1.3, we give some related examples in Section 6.

2. Preliminaries

We start with the weighted Sobolev spaces and some embedding results associated with vector fields.

2.1. Weighted Sobolev spaces and embedding results

In this part, we consider a system of real smooth vector fields $X = (X_1, X_2, \ldots, X_m)$ defined on an open subset $W \subset \mathbb{R}^n$ and satisfy the Hörmander’s condition on $W$.

We first introduce the weighted Sobolev spaces associated with $X$, which are the natural spaces when dealing with problems related to the Hörmander operators. Formally, the weighted Sobolev space, also known as the Folland-Stein space (cf. [83]), is a Hilbert space on $W$ defined by

$$H^1_X(W) = \{u \in L^2(W) \mid X_j u \in L^2(W), j = 1, \ldots, m\},$$

and endowed with the norm

$$\|u\|_{H^1_X(W)}^2 = \|u\|_{L^2(W)}^2 + \|X u\|_{L^2(W)}^2 = \|u\|_{L^2(W)}^2 + \sum_{j=1}^m \|X_j u\|_{L^2(W)}^2.$$

Let $\Omega \subset W \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary $\partial \Omega$. We denote by $H^1_{X,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1_X(W)$. It is well-known that $H^1_{X,0}(\Omega)$ is also a Hilbert space. Moreover, we have the following Poincaré inequality.
Proposition 2.1 (Poincaré Inequality). The first Dirichlet eigenvalue $\lambda_1$ of self-adjoint Hörmander operator $-\Delta_X = \sum_{j=1}^m X_j^* X_j$ is strictly positive. Moreover,

$$
\lambda_1 \int_\Omega |u|^2 \, dx \leq \int_\Omega |Xu|^2 \, dx \quad \forall u \in H^1_{X,0}(\Omega).
$$

(2.1)

It is worth pointing out that (2.1) is the Friedrichs-Poincaré type inequality, which is different from the Poincaré-Wirtinger type inequality extensively investigated in [38, 44, 51]. If there exists at least one vector field $X_j$ ($1 \leq j \leq m$) which can be globally straightened in $\Omega$, we can derive (2.1) in a direct way (see [82, Lemma 5]). Furthermore, (2.1) was also obtained in [54, Lemma 3.2] and [22, Proposition 2.1] under the extra non-characteristic condition on the boundary of $\Omega$, i.e., for any $x_0 \in \partial \Omega$, there exists at least one vector field $X_{j_0}$ ($1 \leq j_0 \leq m$) such that $X_{j_0}(x_0) \notin T_{x_0}(\partial \Omega)$.

When we deal with the general smooth domain without the non-characteristic assumption on $\partial \Omega$, there are many difficulties in studying the boundary regularity of weak solutions in the characteristic points. Thanks to Derridj [34], we know the characteristic set admits zero measure in $\partial \Omega$. Thus, we can use the trace theorem locally in the non-characteristic subset of the boundary to treat the Poincaré inequality (2.1) without the non-characteristic condition. We now give the proof of (2.1) in the interest of making the exposition reasonably self-contained.

Proof of Proposition 2.1. We set

$$
\lambda_1 = \inf_{\|\varphi\|_{L^2(\Omega)} = 1} \sup_{\varphi \in H^1_{X,0}(\Omega)} \|X\varphi\|_{L^2(\Omega)}^2.
$$

Suppose $\lambda_1 = 0$. Then there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ in $H^1_{X,0}(\Omega)$ such that $\|X\varphi_j\|_{L^2(\Omega)} \to 0$ with $\|\varphi_j\|_{L^2(\Omega)} = 1$. Observing that $H^1_{X,0}(\Omega)$ is compactly embedded into $L^2(\Omega)$ (see [33]), it follows from [77, Chapter I, Theorem 1.2] that there exists $\varphi_0 \in H^1_{X,0}(\Omega)$ satisfying $\|\varphi_0\|_{L^2(\Omega)} = 1$, $\Delta_X \varphi_0 = 0$ and $\|X\varphi_0\|_{L^2(\Omega)} = 0$. The hypoellipticity of $\Delta_X$ yields $\varphi_0 \in C^\infty(\Omega)$. Moreover, since $X_j \varphi_0 = 0$ on $\Omega$ for $1 \leq j \leq m$, we deduce from Hörmander’s condition that $\partial_{\epsilon_j} \varphi_0 = 0$ on $\Omega$ for $1 \leq j \leq n$, which means $\varphi_0$ must be a constant on $\Omega$.

For $x_0 \in \partial \Omega$, if $X_j(x_0) \notin T_{x_0}(\partial \Omega)$ for some $1 \leq j \leq m$, we say $x_0$ is a non-characteristic point of vector fields $X$. Otherwise, $x_0$ is called the characteristic point of vector fields $X$. Let $\partial \Omega = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0$ is the collection of all non-characteristic points in $\partial \Omega$, and $\Gamma_1 = \partial \Omega \setminus \Gamma_0$ denote the set of characteristic points in $\partial \Omega$. According to Derridj [33, 34], we have $\varphi_0 \in C^\infty(\Omega \cup \Gamma_0)$, and $\Gamma_1$ admits zero $(n - 1)$-dimensional measure in $\partial \Omega$.

We next show that $\varphi_0(x_0) = 0$ holds for some point $x_0 \in \Gamma_0$. For any small $\epsilon > 0$, we can choose a non-empty compact set $\Gamma_\epsilon$ such that $\Gamma_\epsilon \subset \Gamma_0 \subset \partial \Omega$ and $|\partial \Omega \setminus \Gamma_\epsilon|_{n-1} \leq \epsilon$. Owing to [34, Theorem 2], we can find a continuous trace operator $T_\epsilon : H^1_{X,0}(\Omega) \to L^2(\Gamma_\epsilon)$ such that $T_\epsilon(u) = u|_{\Gamma_\epsilon}$, for $u \in H^1_{X,0}(\Omega)$, $\cap C^\infty(\Omega \cup \Gamma_\epsilon)$. For any $u_0 \in H^1_{X,0}(\Omega)$, there exists a sequence $\{u_k\}_{k=1}^\infty \subset C^\infty(\Omega)$ such that $u_k \to u_0$ in $H^1_{X,0}(\Omega)$. Observing that $u_k \in C^\infty(\Omega) \subset H^1_{X,0}(\Omega)$, we have $T_\epsilon(u_k) \to T_\epsilon(u_0)$ and $T_\epsilon(u_0) = 0$.

As a result, $\varphi_0 \in H^1_{X,0}(\Omega)$ implies $T_\epsilon(\varphi_0) = 0$. Moreover, $\varphi_0 \in C^\infty(\Omega \cup \Gamma_\epsilon)$ due to $\Gamma_\epsilon \subset \Gamma_0$. Thus, we have $\varphi_0|_{\Gamma_\epsilon} = T_\epsilon(\varphi_0) = 0$, which indicates $\varphi_0(x_0) = 0$ for some $x_0 \in \Gamma_0$. 

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Combining this fact with \( \varphi_0 \in C^\infty(\Omega \cup \Gamma_0) \), it follows that \( \varphi_0(x) = 0 \) for all \( x \in \Omega \). This contradicts \( \|\varphi_0\|_{L^2(\Omega)} = 1 \).

In 2015, Yung [84] proved the following sharp subelliptic Sobolev embedding theorem.

**Proposition 2.2** (Sobolev Embedding Theorem). Denote by \( \hat{\nu} \) the generalized Métivier index of vector fields \( X \) on \( \Omega \). Then for \( 1 \leq p < \hat{\nu} \), there exists a constant \( C = C(\Omega, X) > 0 \), such that for all \( u \in C^\infty(\Omega) \), the inequality

\[
\|u\|_{L^q(\Omega)} \leq C \left( \| Xu \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)} \right)
\]

(2.2)

holds for \( q = \frac{\hat{\nu}p}{\hat{\nu} - p} \).

**Proof.** See [84, Corollary 1].

According to (2.1) and (2.2), we can derive the following weighted Sobolev inequality.

**Corollary 2.1** (Weighted Sobolev Inequality). There exists a constant \( C = C(\Omega, X) > 0 \), such that for any \( u \in H^1_{X,0}(\Omega) \),

\[
\left( \int_{\Omega} |u|^{\frac{\hat{\nu}p}{\hat{\nu} - p}} dx \right)^{\frac{\hat{\nu} - 2}{\hat{\nu}p}} \leq C \left( \int_{\Omega} |Xu|^2 dx \right)^{\frac{1}{2}}.
\]

(2.3)

On the other hand, we have the following sub-elliptic estimates.

**Proposition 2.3** (Subelliptic estimates I). For any open subset \( \Omega \subset W \), there exist constants \( \epsilon_0 > 0 \) and \( C > 0 \) such that

\[
\|u\|_{H^{s,0}(\mathbb{R}^n)} \leq C \left( \sum_{i=1}^m \|X_iu\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 \right) \quad \forall u \in H^1_{X,0}(\Omega),
\]

(2.4)

where \( \|u\|_{H^{s,0}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \) is the fractional Sobolev norm.

**Proof.** See [72, Theorem 17].

**Proposition 2.4** (Sub-elliptic estimates II). Assume \( \Omega \subset W \) is an open subset, and \( \phi, \phi_1 \in C^\infty_0(\Omega) \) are some functions such that \( \phi_1 \equiv 1 \) on the support of \( \phi \). Then there exists \( \epsilon > 0 \) so that for every \( s \geq 0 \), there is a constant \( C > 0 \) such that

\[
\|\phi u\|_{H^{s,0}(\mathbb{R}^n)} \leq C \left( \|\phi_1 \Delta_X u\|_{H^{s}(\mathbb{R}^n)} + \|\phi_1 u\|_{L^2(\mathbb{R}^n)} \right)
\]

(2.5)

holds for any \( u \in L^2(\Omega) \cap C^\infty(\Omega) \). Here \( \|u\|_{H^{s}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \) is the fractional Sobolev norm.

**Proof.** See [72, Theorem 18].
According to the classical Sobolev imbedding theorem we know that for \( s > \frac{n}{2} \), there exists a constant \( C > 0 \) such that
\[
\sup_{x \in \mathbb{R}^n} |u(x)| \leq C \|u\|_{H^s(\mathbb{R}^n)} \quad \forall u \in H^s(\mathbb{R}^n).
\] (2.6)

Therefore, (2.6) and Proposition 2.4 imply the following corollary.

**Corollary 2.2.** Let \( N \in \mathbb{N}^+ \) with \( N > \frac{n}{2} \) (where \( \epsilon \) was given in Proposition 2.4) and \( \xi(x) \in C_0^\infty(\Omega) \). If \( u \in L^2(\Omega) \cap C^\infty(\Omega) \) and \( (\Delta_X)^k u \in L^2(\Omega) \) for \( 1 \leq k \leq N \), then we have
\[
\sup_{x \in \Omega} |\xi(x)u(x)| \leq C N \sum_{k=0}^N \| (\Delta_X)^k u \|_{L^2(\Omega)}. \tag{2.7}
\]

### 2.2. \( \delta_t \)-homogeneous functions and vector fields

We then briefly recall some definitions and properties of \( \delta_t \)-homogeneous functions and \( \delta_t \)-homogeneous vector fields, and one can refer to [17] for more details.

**Definition 2.1 (\( \delta_t \)-homogeneous function).** A real function \( f \) defined on \( \mathbb{R}^n \) is called the \( \delta_t \)-homogeneous of degree \( \sigma \in \mathbb{R} \) if \( f \neq 0 \) and \( f \) satisfies
\[
f(\delta_t(x)) = t^\sigma f(x) \quad \forall x \in \mathbb{R}^n, \ t > 0.
\]

According to Definition 2.1, if \( f \) is a continuous function with \( \delta_t \)-homogeneous degree \( \sigma \) and \( f(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R}^n \), then \( \sigma \geq 0 \). Moreover, the continuous and \( \delta_t \)-homogeneous of degree 0 functions are precisely the non-zero constants (see [17, p.33]).

**Proposition 2.5 (Smooth \( \delta_t \)-homogeneous functions).** For any \( f \in C^\infty(\mathbb{R}^n) \), \( f \) is \( \delta_t \)-homogeneous of degree \( \sigma \in \mathbb{N} \) if and only if \( f \) is a polynomial function of the form
\[
f(x) = \sum_{\sum_{i=1}^n \alpha_i \beta_i = \sigma} c_{\beta_1, \ldots, \beta_n} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \tag{2.8}
\]
with some \( c_{\beta_1, \ldots, \beta_n} \neq 0 \), where \( \beta_1, \beta_2, \ldots, \beta_n \) are some non-negative integers.

Proof. See [17, Proposition 1.3.4]. \( \square \)

On the other hand, the \( \delta_t \)-homogeneous vector field is defined as follows.

**Definition 2.2.** Let \( Y \) be a non-identically-vanishing linear differential operator defined on \( \mathbb{R}^n \). We say \( Y \) is \( \delta_t \)-homogeneous of degree \( \sigma \in \mathbb{R} \) if
\[
Y(\varphi(\delta_t(x))) = t^\sigma (Y \varphi)(\delta_t(x)) \quad \forall \varphi \in C^\infty(\mathbb{R}^n), \ x \in \mathbb{R}^n, \ t > 0.
\]

The \( \delta_t \)-homogeneous smooth vector field admit the following properties.
Proposition 2.6 (Smooth $\delta$-homogeneous vector fields). Suppose $Y$ is a smooth non-vanishing vector field in $\mathbb{R}^n$ such that

$$Y = \sum_{j=1}^{n} \mu_j(x) \partial_{x_j}.$$ 

Then $Y$ is $\delta$-homogeneous of degree $\sigma \in \mathbb{N}$ if and only if $\mu_j$ is a polynomial function $\delta$-homogeneous of degree $\alpha_j - \sigma$ in the form of (2.8) (unless $\mu_j \equiv 0$). Moreover, for each $\mu_j$ with $\mu_j \neq 0$, we have $\alpha_j \geq \sigma$, and $Y$ satisfies

$$Y = \sum_{j \leq n, \alpha_j \geq \sigma} \mu_j(x) \partial_{x_j}.$$ 

In particular, if $\sigma \geq 1$, since $\mu_j$ is a $\delta$-homogeneous polynomial function of degree $\alpha_j - \sigma$, it follows from (2.8) that $\mu_j(x) = \mu_j(x_1, \ldots, x_{j-1})$ does not depend on the variables $x_j, \ldots, x_n$.

Proof. See [17, Proposition 1.3.5, Remark 1.3.7].

For $1 \leq j_i \leq m$, we let $J = (j_1, \ldots, j_k)$ be a multi-index with length $|J| = k$. Then there exists a commutator $X_J$ of length $k$ such that

$$X_J = [X_{j_1}, [X_{j_2}, \ldots [X_{j_{k-1}}, X_{j_k}]]].$$

Adopting the notations above, we have

Proposition 2.7. For $k \geq 1$, let $X^{(k)} = \{X_J|J = (j_1, \ldots, j_k), 1 \leq j_i \leq m, |J| = k\}$ be the set of all commutators of length $k$. Then for any $Y \in X^{(k)}$, $Y$ is the $\delta$-homogeneous of degree $k$ if and only if $Y \equiv 0$ (i.e. $Y$ is a zero vector field). In particular, $X^{(k)} = \{0\}$ for any $k > \alpha_n$.

Proof. See [17, Proposition 1.3.10].

Remark 2.1. Proposition 2.6 and Proposition 2.7 implies that $X^{(k_1)} \cap X^{(k_2)} = \{0\}$ for $k_1 \neq k_2$, and

$$\text{Lie}(X) = \text{span}X^{(1)} \oplus \cdots \oplus \text{span}X^{(\alpha_n)}.$$ 

2.3. Subunit metric and volume estimates of subunit balls

The essential geometric object we are concerned with is the subunit metric constructed on $\mathbb{R}^n$ via the vector fields $X = (X_1, X_2, \ldots, X_m)$ satisfying Hörmander’s condition, which plays a crucial role in estimating the heat kernel of homogeneous Hörmander operator.

Definition 2.3 (Subunit metric, see [67, 68]). For any $x, y \in \mathbb{R}^n$ and $\delta > 0$, let $C(x, y, \delta)$ be the collection of absolutely continuous mapping $\varphi : [0, 1] \to \mathbb{R}^n$, which satisfies $\varphi(0) = x, \varphi(1) = y$ and

$$\varphi'(t) = \sum_{i=1}^{m} a_i(t) X_i(\varphi(t))$$

with $\sum_{k=1}^{m} |a_k(t)|^2 \leq \delta^2$ for a.e. $t \in [0, 1]$. The subunit metric $d_X(x, y)$ is defined by

$$d_X(x, y) := \inf \{\delta > 0 | \exists \varphi \in C(x, y, \delta) \text{ with } \varphi(0) = x, \varphi(1) = y\}.$$
We mention that the sub-Riemannian manifold \((\mathbb{R}^n, D, g)\) has the natural structure of a metric space with Carnot-Carathéodory distances derived from the sub-Riemannian metric \(g\) (see [7, 50]). It follows from [52, Proposition 3.1] that the Carnot-Carathéodory distance on sub-Riemannian manifold \((\mathbb{R}^n, D, g)\) is equivalent to the subunit metric \(d_X\). Therefore, we only consider the subunit metric \(d_X\) throughout this paper.

Given any \(x \in \mathbb{R}^n\) and \(r > 0\), we denote by

\[
B_{d_X}(x, r) := \{ y \in \mathbb{R}^n \mid d_X(x, y) < r \}
\]

the subunit ball associated with subunit metric \(d_X(x, y)\). Owing to the assumption (H.1), the subunit metric \(d_X\) and subunit ball \(B_{d_X}(x, r)\) enjoy the homogeneity properties (see [13]):

1. For any \(x, y \in \mathbb{R}^n\) and \(t > 0\), \(d_X(\delta_t(x), \delta_t(y)) = td_X(x, y)\).
2. For any \(x, y \in \mathbb{R}^n\) and \(t, r > 0\), \(y \in B_{d_X}(x, r)\) if and only if \(\delta_t(y) \in B_{d_X}(\delta_t(x), r)\).
3. For any \(t, r > 0\) and \(x \in \mathbb{R}^n\), \(|B_{d_X}(\delta_t(x), tr)| = t^Q|B_{d_X}(x, r)|\), where \(|B_{d_X}(x, r)|\) denotes the \(n\)-dimensional Lebesgue measure of \(B_{d_X}(x, r)\).

To further estimate the volume of subunit ball, we recall some standard notations in [68].

Let \(Y_1, \ldots, Y_q\) be some enumeration of the components of \(X^{(1)}, \ldots, X^{(\alpha_n)}\), we say \(Y_i\) has formal degree \(d(Y_i) = k\), if \(Y_i\) is an element of \(X^{(k)}\). For each \(n\)-tuple of integers \(I = (i_1, \ldots, i_n)\) with \(1 \leq i_j \leq q\), we consider the function

\[
\lambda_I(x) := \det(Y_{i_1}, \ldots, Y_{i_n})(x),
\]

(2.9)

where \(\det(Y_{i_1}, \ldots, Y_{i_n})(x) = \det(b_{jk}(x))\) for \(Y_{i_j} = \sum_{k=1}^n b_{jk}(x) \partial_{x_k}\). We also define

\[
d(I) := d(Y_{i_1}) + \cdots + d(Y_{i_n}),
\]

and

\[
\Lambda(x, r) := \sum_I |\lambda_I(x)|r^{d(I)},
\]

(2.10)

where the sum is taken over all \(n\)-tuples. For the \(\delta_t\)-homogeneous vector fields \(X\), the functions \(\lambda_I\) have the following properties.

**Proposition 2.8.** Suppose \(X = (X_1, X_2, \ldots, X_m)\) are the smooth vector fields defined on \(\mathbb{R}^n\) and satisfy assumption (H.1). Then every \(\lambda_I\) given by (2.9) is a polynomial function. Furthermore, \(\lambda_I\) satisfies \(\lambda_I(\delta_t(x)) = t^Qd(I)\lambda_I(x)\) and

1. If \(d(I) < Q\), \(\lambda_I(0) = 0\);
2. If \(d(I) = Q\), \(\lambda_I(x) \equiv \lambda_I(0)\);
3. If \(d(I) > Q\), \(\lambda_I(x) \equiv 0\).

**Proof.** It follows from Proposition 2.6, Proposition 2.7 and (2.9) that \(\lambda_I\) is a polynomial function for every \(n\)-tuple \(I\). Then we show the homogeneity of \(\lambda_I\).
For each \( n \)-tuple of integers \( I = (i_1, \ldots, i_n) \) with \( 1 \leq i_j \leq q \), we let \( Y_{i_j} \) be the corresponding vector fields of \( I \) such that \( Y_{i_j} = \sum_{k=1}^n b_{j,k}(x) \partial_{x_k} \in X^{(d(Y_{i_j}))} \) for \( j = 1, \ldots, n \). Then, Proposition 2.7 implies that vector field \( Y_{i_j} \) is \( \delta_t \)-homogeneous of degree \( d(Y_{i_j}) \). By Proposition 2.6 we obtain that \( b_{j,k}(x) \) is either zero function or \( \delta_t \)-homogeneous polynomial function of degree \( \alpha_k - d(Y_{i_j}) \). Thus, it follows from (2.9) that for any \( t > 0 \),

\[
\lambda_I(\delta_t(x)) = \det(b_{j,k}(\delta_t(x)))) = \det(t^{\alpha_k-d(Y_{i_j})}b_{j,k}(x))
\]

\[
= \left( \prod_{k=1}^n t^{\alpha_k} \right) \left( \prod_{j=1}^n t^{-d(Y_{i_j})} \right) \lambda_I(x) = t^{Q-d(I)} \lambda_I(x). \tag{2.11}
\]

If \( d(I) < Q \), let \( t \to 0^+ \) in (2.11), then \( \lambda_I(0) = 0 \). For the case \( d(I) = Q \), let \( t \to 0^+ \) in (2.11) again, we have \( \lambda_I(0) = \lambda_I(0) \). Finally, if \( d(I) > Q \), observing that (2.11) gives \( t^{d(I)-Q} \lambda_I(\delta_t(x)) = \lambda_I(x) \), we get \( \lambda_I(x) \equiv 0 \) by taking \( t \to 0^+ \). The proof of Proposition 2.8 is completed. \( \square \)

Using Proposition 2.8, the validity of Hörmander condition can be derived from the assumptions (H.1) and (H.2).

**Proposition 2.9.** Let \( X = (X_1, X_2, \ldots, X_m) \) be the smooth vector fields defined on \( \mathbb{R}^n \) satisfying assumptions (H.1) and (H.2), then \( X \) satisfy Hörmander condition in \( \mathbb{R}^n \) with the Hörmander index \( r = \alpha_n \).

**Proof.** It derives from Proposition 2.8 that \( \lambda_I(0) = 0 \) holds for any \( n \)-tuple \( I \) with \( d(I) \neq Q \). If \( \lambda_I(0) = 0 \) for all \( n \)-tuples \( I \) such that \( d(I) = Q \), then any \( n \) vector fields \( Y_{i_1}, \ldots, Y_{i_n} \) belonging to \( \bigcup_{k=1}^n X^{(k)} \) cannot span \( \mathbb{R}^n \) at the origin, which implies

\[
\dim \{ Y(0) \mid Y \in \text{Lie}(X) \} < n.
\]

This is a contradiction to assumption (H.2). Hence, there exists an \( n \)-tuple \( I_0 \) such that \( d(I_0) = Q \) and \( \lambda_{I_0}(x) \equiv \lambda_{I_0}(0) \neq 0 \). That means

\[
\dim \{ Y(x) \mid Y \in \text{Lie}(X) \} = n \quad \forall x \in \mathbb{R}^n,
\]

which implies the validity of Hörmander condition in \( \mathbb{R}^n \). Furthermore, Proposition 2.7 indicates the Hörmander index \( r \leq \alpha_n \).

We then show that the Hörmander index \( r = \alpha_n \). By Remark 2.1,

\[
D_x^j = \text{span} \{ Y(x) \mid Y \in X^{(1)} \cup \cdots \cup X^{(j)} \} = \bigoplus_{i=1}^j \text{span} \{ Y(x) \mid Y \in X^{(i)} \}.
\]

If for all \( Y(x) = \sum_{j=1}^n \mu_j(x) \partial_{x_j} \in D_x^{\alpha_n-1} \), the last coefficient function \( \mu_n \equiv 0 \), then the assumption (H.2) is failed. Thus, there is a \( Y_0(x) \in D_x^{\alpha_n-1} \) such that \( \mu_n \neq 0 \). We next claim that the \( \delta_t \)-homogeneous degree of \( \mu_n(x) \) cannot be zero. Observe that \( Y_0(x) \in D_x^{\alpha_n-1} \) implies \( Y_0(x) \in \text{span} \{ Y(x) \mid Y \in X^{(i)} \} \) for some \( 1 \leq i \leq \alpha_n - 1 \). From Proposition 2.6 and Proposition 2.7, we conclude that \( \mu_n \) is a smooth \( \delta_t \)-homogeneous function of degree \( \alpha_n - i \) with \( \alpha_n - i \geq 1 \). This indicates \( \mu_n(0) = 0 \) and \( \dim D_x^{\alpha_n-1} < n \). Hence the Hörmander index \( r \geq \alpha_n \), and we have \( r = \alpha_n \). \( \square \)
We have the following the volume estimates of subunit ball.

**Proposition 2.10** (Global version Ball-Box theorem). For any $x \in \mathbb{R}^n$, there exist positive constants $C_1, C_2$ such that for any $x \in \mathbb{R}^n$ and any $r > 0$,

$$C_1 \Lambda(x, r) \leq |B_{dx}(x, r)| \leq C_2 \Lambda(x, r),$$

(2.12)

where $|B_{dx}(x, r)|$ is the $n$-dimensional Lebesgue measure of $B_{dx}(x, r)$.

**Proof.** The local version of Ball-Box theorem was obtained from the deep investigations on subelliptic metric and subunit metric carried out by Nagel-Stein-Wainger [68] and Morbidelli [67]. Precisely, for vector fields $X$ only satisfy Hörmander’s condition, (2.12) is restricted on some compact set $K \subset \mathbb{R}^n$ and $0 < r \leq r_0$, where $r_0$ is a positive constant depending on vector fields $X$ and compact set $K$. However, owing to homogeneous assumption (H.1), we can derive the global version of Ball-Box theorem by using a local-to-global homogeneity argument, starting from the local version of Ball-Box theorem. The proof of this global estimate can be found in [13, Theorem B].

Moreover, $\Lambda(x, r)$ has the following properties.

**Proposition 2.11.** For any $x \in \mathbb{R}^n$, the pointwise homogeneous dimension $\nu(x)$ satisfies

$$\nu(x) = \min\{d(I) | \lambda_I(x) \neq 0\} = \lim_{s \to 0^+} \frac{\ln \Lambda(x, s)}{\ln s}. \tag{2.13}$$

**Proof.** See [21, Proposition 2.2].

**Proposition 2.12.** Let $X = (X_1, X_2, \ldots, X_m)$ be the smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2). Then $n \leq w \leq \nu(x) \leq Q$ and $\nu(0) = Q$, where $w = \min_{x \in \mathbb{R}^n} \nu(x)$ and $Q$ is the homogeneous dimension given by (1.1). Additionally, the function $\Lambda(x, r)$ defined by (2.10) satisfies

$$\Lambda(x, r) = \sum_{k=\nu(x)}^{Q} f_k(x) r^k = \sum_{k=w}^{Q} f_k(x) r^k, \tag{2.14}$$

where $f_k(x) = \sum_{d(I)=k} |\lambda_I(x)|$ is a $\delta_t$-homogeneous non-negative continuous function of degree $Q - k$. Furthermore, $f_w(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^n$ and $f_Q(x) = f_Q(0) > 0$ for all $x \in \mathbb{R}^n$.

**Proof.** Observing that $\nu(x)$ is an integer value function with $\nu(x) \geq n$, we obtain $\nu(x_0) = \min_{x \in \mathbb{R}^n} \nu(x) = w$ for some $x_0 \in \mathbb{R}^n$. Recall that the proof of Proposition 2.9 implies $\lambda_{I_0}(x) \equiv \lambda_{I_0}(0) \neq 0$ holds for some $n$-tuple $I_0$ satisfying $d(I_0) = Q$. Thus, (2.13) gives $\nu(x) \leq Q$. Additionally, Proposition 2.8 yields that $\lambda_I(0) = 0$ for $d(I) \neq Q$, which indicates $\nu(0) = Q$. 

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Combining (2.10), Proposition 2.8 and Proposition 2.11,

\[ \Lambda(x, r) = \sum_I |\lambda_I(x)| r^{d(I)} = \sum_{k=\nu(x)}^Q f_k(x)r^k, \]  

(2.15)

where \( f_k(x) = \sum_{d(I)=k} |\lambda_I(x)| \). Moreover, using Proposition 2.11 and (2.15), we obtain

\[ \lim_{r \to 0^+} \frac{\ln \left( \sum_{k=\nu(x)}^Q f_k(x_0)r^k \right)}{\ln r} = \lim_{r \to 0^+} \frac{\ln \left( \sum_{k=\nu(x)}^Q f_k(x_0)r^k \right)}{\ln r}, \]

which gives \( f_w(x_0) \neq 0 \). That means

\[ \Lambda(x, r) = \sum_{k=\nu(x)}^Q f_k(x)r^k = \sum_{k=w}^Q f_k(x)r^k. \]

Finally, Proposition 2.8 indicates that \( f_Q(x) = f_Q(0) \geq |\lambda_0(0)| > 0 \) for all \( x \in \mathbb{R}^n \).

\[ \square \]

2.4. Classification of homogeneous Hörmander vector fields

In this part, we discuss the classification of homogeneous Hörmander vector fields. For this purpose, we give some useful propositions and definitions.

Proposition 2.13. Let \( X = (X_1, X_2, \ldots, X_m) \) be the smooth vector fields defined on \( \mathbb{R}^n \) satisfying assumptions (H.1) and (H.2). Denote by \( W := \{ x \in \mathbb{R}^n | \nu(x) = Q \} \). Then we have

(i) \( w = Q \) if and only if \( W = \mathbb{R}^n \);
(ii) \( w \leq Q - 1 \) if and only if \( |W| = 0 \), where \( |W| \) denotes the \( n \)-dimensional Lebesgue measure of \( W \).

Proof. If \( w = Q \), Proposition 2.12 implies \( \nu(x) = Q \) for any \( x \in \mathbb{R}^n \), which yields \( W = \mathbb{R}^n \). Clearly, \( W = \mathbb{R}^n \) gives \( w = \min_{x \in \mathbb{R}^n} \nu(x) = Q \). Thus, conclusion (i) is proved.

We then show conclusion (ii). Suppose that \( w \leq Q - 1 \), by Proposition 2.12 we have \( f_w(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R}^n \). Denoting by \( Z(f_w) := \{ x \in \mathbb{R}^n | f_w(x) = 0 \} \) the zeros of function \( f_w(x) \), we obtain from Proposition 2.8 that

\[ Z(f_w) = \bigcap_{d(I)=w} \{ x \in \mathbb{R}^n | \nu_I(x) = 0 \} \neq \emptyset. \]

Observing \( \lambda_I \) is the polynomial for each \( n \)-tuple \( I \), it follows that \( |\{ x \in \mathbb{R}^n | \lambda_I(x) = 0 \}| = 0 \) and \( |Z(f_w)| = 0 \) (see [65]). Moreover, for any \( x \in W \), we have \( w < \nu(x) = Q \). Combining (2.13) and (2.14),

\[ \nu(x) = Q = \lim_{r \to 0^+} \frac{\ln \left( \sum_{k=\nu(x)}^Q f_k(x)r^k \right)}{\ln r} = \lim_{r \to 0^+} \frac{\ln \left( \sum_{k=w}^Q f_k(x)r^k \right)}{\ln r}, \]
which implies $f_w(x) = 0$. This means $W \subset Z(f_w)$ and $|W| = 0$. On the other hand, assume that $|W| = 0$. If $w = Q$, by conclusion (i) we have $W = \mathbb{R}^n$, which contradicts the fact $|W| = 0$. □

Proposition 2.13 gives the following obvious corollary.

**Corollary 2.3.** Suppose that $X = (X_1, X_2, \ldots, X_m)$ and $\Omega$ satisfy the conditions in Theorem 1.1. Then $Q = \tilde{\nu}$, and the set $H = \{x \in \Omega | \nu(x) = \tilde{\nu}\} = W \cap \Omega$ and satisfies

1. $w = Q$ if and only if $H = \Omega$;
2. $w \leq Q - 1$ if and only if $|H| = 0$, where $|H|$ is the $n$-dimensional Lebesgue measure of $H$.

We next introduce the degenerate component and degenerate index, which will be useful in Section 4 below.

**Definition 2.4 (Degenerate component and degenerate index).** Let $X = (X_1, X_2, \ldots, X_m)$ be the smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2). We set $x = (x_1, x_2, \ldots, x_n)$. For each $1 \leq j \leq n$, if the function $\Lambda(x, r)$ depends on the variable $x_j$, then we say $x_j$ is the degenerate component of $X$. Otherwise, we call $x_j$ the non-degenerate component of $X$. Furthermore, for the degenerate component $x_j$, we say $\alpha_j$ is the degenerate index associated with $x_j$, and we denote by

$$\alpha(X) := \sum_{j \in A} \alpha_j \quad (2.17)$$

the sum of all degenerate indexes of the vector fields $X$, where

$$A := \{1 \leq j \leq n | x_j \text{ is the degenerate component of } X\}.$$

**Remark 2.2.** From Proposition 2.6, we know the last variable $x_n$ is the non-degenerate component of $X$.

**Remark 2.3.** The degenerate components of vector fields $X$ indicate the variable dependency of $\Lambda(x, r)$, which allow us to deal with $\Lambda(x, r)$ in lower-dimensional space. We mention that the degenerate components of vector fields $X$ differ from the dependent variables of vector fields $X$. For instance, consider the vector fields $X_1 = \partial_{x_1} + 2x_3\partial_{x_3}, X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ on $\mathbb{R}^3$. It follows that $X_1, X_2$ depend on variables $x_1, x_2$, but $\Lambda(x, r) = 4r^4$ is independent of the variables of $x$.

According to Proposition 2.12 and Definition 2.4, we have the following classification of homogeneous Hörmander vector fields.

**Proposition 2.14.** Assume that $X = (X_1, X_2, \ldots, X_m)$ and $\Omega$ satisfy the conditions in Theorem 1.1. Then

1. $w = Q$ if and only if all variables $x_1, \ldots, x_n$ are non-degenerate components of vector fields $X$;
2. $w \leq Q - 1$ if and only if the vector fields $X$ have at least one degenerate component.
3. Estimates of the subelliptic heat kernels

3.1. Subelliptic Dirichlet heat kernel

Let $X = (X_1, X_2, \ldots, X_m)$ be the real smooth vector fields defined on an open subset $W \subset \mathbb{R}^n$ and satisfy Hörmander’s condition on $W$. Assume that $\Omega \subset W$ is a bounded open domain with smooth boundary $\partial \Omega$. In this part, we shall construct the subelliptic Dirichlet heat kernel of general self-adjoint Hörmander operator $\triangle_X = -\sum_{j=1}^m X_j^* X_j$ on $\Omega$. We mention that the subelliptic Dirichlet heat kernel was studied in [22, Section 4] under the non-characteristic assumption on the boundary of $\Omega$. However, the non-characteristic assumption seems quite restrictive for general subelliptic Dirichlet eigenvalue problems. In view of this, we would like to give a modified construction of the subelliptic Dirichlet heat kernel on the general smooth domain without the non-characteristic condition.

The weak Dirichlet eigenvalue problem of self-adjoint Hörmander operator $\triangle_X$ is given by

$$\begin{cases} 
-\triangle_X u = \lambda u, & \text{on } \Omega; \\
u \in H^1_{X,0}(\Omega).
\end{cases}$$

It follows from Proposition 2.1 and Proposition 2.3 that problem (3.1) is well-defined. Let $\lambda_k$ and $\phi_k$ be the $k$-th Dirichlet eigenvalue and eigenfunction of problem (3.1), respectively. We know that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

Moreover, the corresponding eigenfunctions $\{\phi_k\}_{k=1}^\infty$ constitute an orthonormal basis of $L^2(\Omega)$ and also an orthogonal basis of $H^1_{X,0}(\Omega)$.

For each $\phi_i \in H^1_{X,0}(\Omega)$, we have $(\triangle_X + \lambda_i)\phi_i = 0$. According to the regularity result of Derridj in [34], it follows that $\phi_i \in C^\infty(\Omega \cup \Gamma_0)$ and $\phi_i|_{\Gamma_0} = 0$, where $\Gamma_0 \subset \partial \Omega$ is the non-characteristic set with $|\partial \Omega \setminus \Gamma_0|_{n-1} = 0$. Additionally, by the Moser iteration method, Proposition 2.1 and the chain rules in $H^1_{X,0}(\Omega)$ (see Proposition 7.3 in Appendix below), we can deduce the following $L^\infty$ estimate.

**Proposition 3.1.** There exists a positive constant depending on $X$ and $\Omega$ such that

$$\|\phi_i\|_{L^\infty(\Omega)} \leq C_1 \cdot \lambda_i^{\frac{s}{s-1}},$$

where $\nu$ is the generalized Métivier index on $\Omega$.

**Proof.** Since $-\triangle_X \phi_i = \lambda_i \phi_i$, then

$$\int_\Omega X \phi_i \cdot X u dx = \lambda_i \int_\Omega \phi_i u dx \quad \forall u \in H^1_{X,0}(\Omega).$$

For any $s \geq 2$ and any $M > 0$, we let $v_M = \min\{||\phi_i||, M\}$ and $u_M = (v_M)^{s-1} \cdot \text{sgn}(\phi_i)$. Observing that the functions $t \mapsto \min\{|t|, M\}$ and $t \mapsto (\min\{|t|, M\})^{s-1} \text{sgn}(t)$ are piecewise
$C^1$-smooth in $\mathbb{R}$ with the derivatives belonging to $L^\infty(\mathbb{R})$, it follows from Proposition 7.3 that $v_M, u_M \in H^1_{X,0}(\Omega)$ and

$$Xu_M = (s-1)v_M^{s-2}\text{sgn}(\phi_i)Xv_M.$$  

Using (3.3) and $\int_\Omega Xv_M \cdot X(|\phi_i| - v_M)dx = 0$, we have

$$\lambda_i \int_\Omega |\phi_i||v_M|^{s-1}dx = (s-1) \int_\Omega |v_M|^{s-2}X|\phi_i| \cdot Xv_Mdx = (s-1) \int_\Omega |v_M|^{s-2}|Xv_M|^2dx. \quad (3.4)$$

Similarly, we can deduce from Proposition 7.3 that $v_M^s \in H^1_{X,0}(\Omega)$, since the function $t \mapsto \min\{\min\{|t|, M\}\}$ is piecewise $C^1$-smooth in $\mathbb{R}$ and has derivative belonging to $L^\infty(\mathbb{R})$. Moreover, $v_M \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$. The weighted Sobolev inequality (2.3) yields

$$-(s-1) \int_\Omega v_M^{s-2}|Xv_M|^2dx = \frac{4(s-1)}{s^2} \int_\Omega |X(v_M^s)|^2dx \leq \frac{4C(s-1)}{s^2} \left( \int_\Omega |v_M|^\frac{s}{s-2}dx \right)^\frac{s-2}{s}, \quad (3.5)$$

where $C > 0$ is the Sobolev constant in (2.3) independent of $M$. Thus, we conclude from (3.4) and (3.5) that

$$(s-1) \int_\Omega v_M^{s-2}|Xv_M|^2dx \geq \frac{2C}{s} \left( \int_\Omega |v_M|^\frac{s}{s-2}dx \right)^\frac{s-2}{s}, \quad (3.6)$$

and

$$\lambda_i \int_\Omega |\phi_i||v_M|^{s-1}dx \geq \frac{2C}{s} \left( \int_\Omega |v_M|^\frac{s}{s-2}dx \right)^\frac{s-2}{s}. \quad (3.7)$$

If $\phi_i \in L^s(\Omega)$, (3.7) derives that

$$\int_\Omega |\phi_i|^sdx \geq \frac{2C}{s\lambda_i} \left( \int_\Omega |\phi_i|^\frac{s}{s-2}dx \right)^\frac{s-2}{s},$$

which means $\left( \frac{2C}{s\lambda_i} \right)^\frac{1}{s} \|\phi_i\|_{L^2(\Omega)} \leq \|\phi_i\|_{L^s(\Omega)}$ for $\beta = \frac{s}{s-2}$. Setting $s = 2\beta^j \geq 2$, respectively for $j = 0, 1, 2, \cdots$, then we have

$$\|\phi_i\|_{L^{2\beta^{j+1}}(\Omega)} \leq \left( \frac{\beta^j \lambda_i}{C} \right)^\frac{1}{2\beta^j} \|\phi_i\|_{L^{2\beta^j}(\Omega)}.$$
Proposition 3.2. For the $k$-th Dirichlet eigenvalue of $-\Delta_X$, we have
\[ \lambda_k \geq C \cdot k^{\frac{2\alpha}{\nu}} \quad \forall k \geq 1, \] 
where the positive constant $C$ depends on vector fields $X$ and $\Omega$, and $\epsilon_0$ is the positive constant given in Proposition 2.3.

Proof. See [22, Proposition 3.2] and [26, Theorem 1.1].

Owing to Proposition 3.1 and Proposition 3.2, we can now construct the subelliptic Dirichlet heat kernel of general self-adjoint Hörmander operator on the bounded smooth open domain $\Omega$. Precisely, we have

Proposition 3.3. The self-adjoint Hörmander operator $\Delta_X$ admits a unique Dirichlet heat kernel $h_D(x,y,t)$ which is well defined on $(\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+$ by
\[ h_D(x,y,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y). \] 
Furthermore, $h_D(x,y,t)$ satisfies the following properties:

(1) For any fixed $y \in \Omega$, $h_D(x,y,t)$ is the solution of
\[ \left( \frac{\partial}{\partial t} - \Delta_X \right) h_D(x,y,t) = 0 \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+. \] 

(2) For any non-negative integer $k$,
\[ \partial^k_t h_D(x,y,t) \in C^\infty(\Omega \times \Omega \times \mathbb{R}^+) \cap C((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+) \] 
and
\[ \partial^k_t h_D(x,y,t) = \sum_{i=1}^{\infty} (-\lambda_i)^k e^{-\lambda_i t} \phi_i(x)\phi_i(y) \] 
for all $(x,y,t) \in (\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+$. Moreover, for any fixed $(y,t) \in \Omega \times \mathbb{R}^+$, $h_D(x,y,t) \in H^1_{X,0}(\Omega)$ with respect to $x$. 

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(3) For any $\varphi \in C_0^\infty(\Omega)$,
\[
\lim_{t \to 0^+} \int_\Omega h_D(x, y, t) \varphi(y) dy = \varphi(x).
\] (3.13)

(4) For any $t > 0$, $h_D(x, y, t) = 0$ if $x \in \Gamma_0$ or $y \in \Gamma_0$.

(5) For any $(x, y, t) \in \Omega \times \Omega \times \mathbb{R}^+$, $h_D(x, y, t) = h_D(y, x, t)$.

(6) For any $s, t > 0$,
\[
h_D(x, y, t + s) = \int_\Omega h_D(x, z, t) h_D(z, y, s) dz. \tag{3.14}
\]

(7) For any $(x, y, t) \in \Omega \times \Omega \times \mathbb{R}^+$,
\[
h_D(x, y, t) > 0. \tag{3.15}
\]

Additionally,
\[
\int_\Omega h_D(x, y, t) dy \leq 1 \text{ for all } (x, t) \in \Omega \times \mathbb{R}^+. \tag{3.16}
\]

(8) For $f_0 \in L^2(\Omega)$, the function
\[
f(x, t) = \int_\Omega h_D(x, y, t) f_0(y) dy \tag{3.17}
\]
solves the degenerate heat equation
\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) f(x, t) = 0, \quad \text{for } (x, t) \in \Omega \times \mathbb{R}^+, \tag{3.18}
\]
and satisfies
\[
\lim_{t \to 0^+} f(x, t) = f_0(x) \quad \text{in } L^2(\Omega),
\]
\[
f(x, t) \in C^\infty(\Omega \times \mathbb{R}^+) \cap C((\Omega \cup \Gamma_0) \times \mathbb{R}^+),
\]
\[
f(\cdot, t) \in H^1_{X,0}(\Omega) \quad \text{for all } t \in \mathbb{R}^+. \tag{3.19}
\]

Proof. By Proposition 3.1 and the fact $\phi_i \in C^\infty(\Omega \cup \Gamma_0)$, we have, for $t > 0$ and any non-negative integer $k$,
\[
|\partial_t^k e^{-\lambda_i t} \phi_i(x) \phi_i(y)| = \lambda_i^k e^{-\lambda_i t} |\phi_i(x)| \cdot |\phi_i(y)| \leq \lambda_i^k e^{-\lambda_i t} \|\phi_i\|^2_{L^\infty(\Omega)} \leq C_1^2 e^{-\lambda_i t} \lambda_i^\frac{k}{2}. \tag{3.20}
\]

Then, we introduce the following inequality
\[
e^{-\frac{z}{2} \alpha} \leq \alpha^\alpha e^{-\alpha} \quad \forall z > 0, \quad \alpha > 0. \tag{3.21}
\]
Putting $z = \frac{1}{2} \lambda_i t$ and $\alpha = \frac{\tilde{\nu}}{2} + k$ into (3.21), we get
\[
e^{-\lambda_i t \lambda_i^\frac{k}{2}} \leq (\tilde{\nu} + 2k)^{\tilde{\nu} + k} e^{-(\tilde{\nu} + k) t - (\tilde{\nu} + k)} e^{-\frac{1}{2} \lambda_i t}. \tag{3.22}
\]
Hence, (3.20), (3.22) and Proposition 3.2 together imply that

\[
\sum_{i=1}^{\infty} |\partial_i^k e^{-\lambda_i t} \phi_i(x)\phi_i(y)| \leq C_1^2 \cdot (\bar{\nu} + 2k)^{\frac{d}{2} + k} \bar{e}^{-\left(\frac{d}{2} + k\right)t - \left(\frac{d}{2} + k\right)k} \sum_{i=1}^{\infty} e^{-\frac{C_2}{2} \nu_i t},
\]

which implies the series

\[
\sum_{i=1}^{\infty} \partial_i^k e^{-\lambda_i t} \phi_i(x)\phi_i(y)
\]

is uniformly convergent on \((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times [a, \infty)\) for any \(a > 0\) and any non-negative integer \(k\). In particular, letting \(k = 0\) in (3.24), we obtain the sub-elliptic Dirichlet heat kernel \(h_D(x, y, t)\) given by \(h_D(x, y, t) := \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y)\) is well-defined on \((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+\). Moreover, it can be clearly seen that \(h_D(x, y, t)\) admits properties (4) and (5).

We denote by \(Sh_N(x, y, t)\) the sum of the first \(N\) terms of the series (3.9), i.e.

\[
Sh_N(x, y, t) := \sum_{i=1}^{N} e^{-\lambda_i t} \phi_i(x)\phi_i(y).
\]

Since

\[
\int_{\Omega} (X\phi_i(x)) \cdot (X\phi_j(x)) dx = \begin{cases} \lambda_i, & i = j; \\ 0, & i \neq j. \end{cases}
\]

For any fixed \((y, t) \in \Omega \times \mathbb{R}^+\), we also have

\[
\int_{\Omega} |X_x (Sh_{N+k}(x, y, t) - Sh_N(x, y, t))|^2 \, dx
\]

\[
= \int_{\Omega} \left| X_x \left( \sum_{i=N+1}^{N+k} e^{-\lambda_i t} \phi_i(x)\phi_i(y) \right) \right|^2 \, dx
\]

\[
= \sum_{i=N+1}^{N+k} e^{-2\lambda_i t} \lambda_i \phi_i^2(y) \to 0 \quad \text{(for any } k \in \mathbb{N}^+, \text{ as } N \to +\infty).\]

That means for any fixed \((y, t) \in \Omega \times \mathbb{R}^+\), \(Sh_N(x, y, t) \to h_D(x, y, t)\) in \(H^1_{X,0} (\Omega)\) as \(N \to +\infty\). Consequently, \(h_D(x, y, t) \in H^1_{X,0} (\Omega)\) with respect to \(x\) for any fixed \((y, t) \in \Omega \times \mathbb{R}^+\).

For any fixed \(y \in \Omega\) and \(N \in \mathbb{Z}^+\), \(u_{y,N}(x, t) = Sh_N(x, y, t)\) is a solution to (3.10). The uniform convergence of \(u_{y,N}(x, t)\) implies that \(h_D(x, y, t) \in \mathcal{D}'(\Omega \times \mathbb{R}^+)\) is a weak solution to (3.10) with respect to \((x, t)\). Analogously, \(h_D(x, y, t) \in \mathcal{D}'(\Omega \times \Omega \times \mathbb{R}^+)\) is also a weak solution of equation \(\partial_t - \frac{1}{2}(\Delta^{X,0}_X + \Delta^{y}_X)u(x, y, t) = 0\), since for each \(N\), \(Sh_N(x, y, t)\) is a solution of \(\partial_t - \frac{1}{2}(\Delta^{X,0}_X + \Delta^{y}_X)u(x, y, t) = 0\). Then the hypoellipticity of \(\partial_t - \frac{1}{2}(\Delta^{X,0}_X + \Delta^{y}_X)\)
indicates that \( h_D(x, y, t) \in C^\infty(\Omega \times \Omega \times \mathbb{R}^+) \). Moreover, the uniform convergence of series (3.24) on \((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times [a, +\infty)\) for any \( a > 0 \) and any non-negative integer \( k \) gives \( \partial_t^k h_D(x, y, t) \in C((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+) \) for \( k \in \mathbb{N} \). Thus, we obtain the properties (1) and (2).

Note that \( \{\phi_k\}_{k=1}^\infty \) constitutes an orthonormal basis of \( L^2(\Omega) \). Given a function \( f_0(x) \in L^2(\Omega) \), we have

\[
f_0(x) = \sum_{i=1}^\infty a_i \phi_i(x) \quad \text{in} \ L^2(\Omega),
\]

where \( a_i = \int_\Omega f_0(y) \phi_i(y) dy \). In terms of Parseval’s identity we know

\[
\sum_{i=1}^\infty a_i^2 = \| f_0 \|_{L^2(\Omega)}^2 < +\infty. \tag{3.26}
\]

Furthermore, for any \( t > 0 \), we have

\[
f(x, t) = \int_\Omega h_D(x, y, t) f_0(y) dy = \int_\Omega \left( \sum_{i=1}^\infty e^{-\lambda_i t} \phi_i(x) \phi_i(y) \right) \left( \sum_{j=1}^\infty a_j \phi_j(y) \right) dy
\]

\[
= \sum_{i=1}^\infty e^{-\lambda_i t} a_i \phi_i(x) \quad \text{in} \ L^2(\Omega).
\]

Since

\[
\sum_{i=1}^\infty e^{-\lambda_i t} |a_i| \cdot |\phi_i(x)| \leq \| f_0 \|_{L^2(\Omega)} \cdot \sum_{i=1}^\infty e^{-\lambda_i t} \cdot |\phi_i(x)| \leq C_1 \| f_0 \|_{L^2(\Omega)} \cdot \sum_{i=1}^\infty e^{-\lambda_i t} \lambda_i \phi_i,
\]

and

\[
\left\| \sum_{k=m}^{m+l} e^{-\lambda_k t} a_k \phi_k \right\|_{L^2(\Omega)} \leq \| f_0 \|_{L^2(\Omega)} \sum_{k=m}^{m+l} e^{-\lambda_k t} \lambda_k \to 0 \quad \text{for any} \ l \in \mathbb{N}^+ \quad \text{as} \ m \to +\infty,
\]

we know that the series \( \sum_{i=1}^\infty e^{-\lambda_i t} a_i \phi_i(x) \) converges uniformly on \((\Omega \cup \Gamma_0) \times [a, +\infty)\) for any \( a > 0 \) and also converges in \( H^1_{X,0}(\Omega) \) for any fixed \( t > 0 \). This implies \( f(x, t) \in C((\Omega \cup \Gamma_0) \times \mathbb{R}^+) \) is a weak solution to (3.18) and satisfies \( f(\cdot, t) \in H^1_{X,0}(\Omega) \) in (3.19). Additionally, the hypoellipticity of \( \partial_t - \Delta_X \) yields \( f(x, t) \in C^\infty(\Omega \times \mathbb{R}^+) \).

In order to verify that \( f(x, t) \) satisfies the initial condition in (3.19), it suffices to prove that \( f(x, t) = \int_\Omega h_D(x, y, t) f_0(y) dy \to f_0(x) \) as \( t \to 0^+ \) in \( L^2(\Omega) \). Indeed,

\[
\| f(\cdot, t) - f_0(\cdot) \|_{L^2(\Omega)}^2 = \left\| \sum_{i=1}^\infty (e^{-\lambda_i t} - 1) a_i \phi_i(\cdot) \right\|_{L^2(\Omega)}^2 = \sum_{i=1}^\infty a_i^2 (1 - e^{-\lambda_i t})^2.
\]
The identity (3.26) indicates that \( \sum_{i=1}^{\infty} a_i^2 (1-e^{-\lambda_i t})^2 \) converges uniformly on \([0, +\infty)\). Therefore

\[
\lim_{t \to 0^+} \| f(x, t) - f_0(x) \|_{L^2(\Omega)}^2 = \lim_{t \to 0^+} \sum_{i=1}^{\infty} a_i^2 (1-e^{-\lambda_i t})^2 = 0,
\]

which means that \( f(x, t) \) satisfies the initial condition in (3.19). Hence, we get the property (8).

Substituting \( u(x, t) = \int_{\Omega} h_D(x, y, t) \varphi(y) dy - \varphi(x) \) for any \( \varphi \in C_0^\infty(\Omega) \), it follows that \( u(x, t) \in L^2(\Omega) \cap C^\infty(\Omega) \) for \( t > 0 \). For any \( k \in \mathbb{N}^+ \), the properties (2) and (5) give

\[
(\Delta_X)^k u(x, t) = \int_{\Omega} (\Delta_X^k) h_D(x, y, t) \varphi(y) dy - (\Delta_X^k) \varphi(x) = \int_{\Omega} h_D(x, y, t) (\Delta_X^k) \varphi(y) dy - (\Delta_X^k) \varphi(x),
\]

which implies that \( (\Delta_X)^k u(x, t) \in L^2(\Omega) \) for any \( t > 0 \) and \( k \in \mathbb{N}^+ \). Using Corollary 2.2, we have for any \( \xi \in C_0^\infty(\Omega) \),

\[
\sup_{x \in \Omega} |\xi(x) u(x, t)| \leq C \sum_{k=0}^{N} \| (\Delta_X)^k u(x, t) \|_{L^2(\Omega)} = C \sum_{k=0}^{N} \left\| \int_{\Omega} h_D(x, y, t) (\Delta_X^k) \varphi(y) dy - (\Delta_X^k) \varphi(x) \right\|_{L^2(\Omega)}. \tag{3.27}
\]

As a result of property (8) and (3.27), for any cut-off function \( \xi \in C_0^\infty(\Omega) \) we have

\[
\lim_{t \to 0^+} \sup_{x \in \Omega} |\xi(x) u(x, t)| = \lim_{t \to 0^+} \sup_{x \in \Omega} \left| \xi(x) \cdot \left( \int_{\Omega} h_D(x, y, t) \varphi(y) dy - \varphi(x) \right) \right| = 0. \tag{3.28}
\]

Since the cut-off function \( \xi(x) \) is arbitrary, then for any given \( x \in \Omega \), (3.28) gives that

\[
\lim_{t \to 0^+} \int_{\Omega} h_D(x, y, t) \varphi(y) dy = \varphi(x).
\]

This completes the proof of property (3).

Besides, for \( t > 0 \) and \( s > 0 \), we obtain

\[
\int_{\Omega} h_D(x, z) h_D(z, y, s) dz = \int_{\Omega} \left( \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(z) \right) \left( \sum_{j=1}^{\infty} e^{-\lambda_j s} \phi_j(z) \phi_j(y) \right) dz = \sum_{i=1}^{\infty} e^{-\lambda_i (t+s)} \phi_i(x) \phi_i(y) = h_D(x, y, t + s),
\]

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which yields to property (6).

Finally, it remains to prove the property (7) and the uniqueness of $h_D(x, y, t)$.

We firstly show that $h_D(x, y, t) \geq 0$. If $h_D(x_0, y_0, t_0) < 0$ holds for some $(x_0, y_0, t_0) \in \Omega \times \Omega \times \mathbb{R}^+$, there exist $0 < \delta < t_0$ and $\alpha > 0$ such that $B(x_0, \delta) \subset \Omega$, $B(y_0, \delta) \subset \Omega$, and

$$h_D(x, y, t) < -\alpha < 0 \quad \text{for all} \quad (x, y, t) \in B(x_0, \delta) \times B(y_0, \delta) \times (t_0 - \delta, t_0 + \delta).$$

Then, we can find a function $f_0 \in C_0^\infty(B(y_0, \delta))$ with $0 \leq f_0 \leq 1$, such that

$$f_0(y) = \begin{cases} 
1, & y \in B(y_0, \frac{\delta}{2}) \\
0, & y \in \Omega \setminus B(y_0, \delta).
\end{cases}$$

Let

$$f(x, t) = \int_{\Omega} h_D(x, y, t)f_0(y)dy = \int_{B(y_0, \delta)} h_D(x, y, t)f_0(y)dy.$$ 

In particular, we have

$$f(x_0, t_0) = \int_{B(y_0, \delta)} h_D(x_0, y, t_0)f_0(y)dy < 0. \quad (3.29)$$

Given some $T > t_0 > 0$. Using property (8), we may regard $f(\cdot, t)$ as a path $f : (0, T) \to H^1_\lambda(\Omega)$. Moreover, for any $t \in (0, T)$, the uniformly convergence of $\sum_{i=1}^\infty e^{-\lambda t_0}a_i \phi_i(x)$ implies the strong derivative $\frac{df}{dt}$ exists in $L^2(\Omega)$ and satisfying

$$\frac{df}{dt} - \Delta_X f = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

According to Proposition 7.5, we deduce that $-f(\cdot, t) \leq 0$ mod $H^1_{\lambda, 0}(\Omega)$ due to $(-f(\cdot, t))_+ \in H^1_{\lambda, 0}(\Omega)$. Here $f_+ = \max\{f, 0\}$, and $f \leq 0$ mod $H^1_{\lambda, 0}(\Omega)$ means that $f \leq g$ for some $g \in H^1_{\lambda, 0}(\Omega)$. The inequality $|x_+ - y_+| \leq |x - y|$ gives that $(-f(\cdot, t))_+ \to (-f_0)_+ = 0$ in $L^2(\Omega)$ as $t \to 0^+$. By Proposition 7.6, we obtain $f(x, t) \geq 0$ for all $t \in (0, T)$ and $x \in \Omega$, which contradicts (3.29). Thus, $h_D(x, y, t) \geq 0$ on $\Omega \times \Omega \times \mathbb{R}^+$.

Secondly, we assume that $h_D(x', y', t') = 0$ for some $(x', y', t') \in \Omega \times \Omega \times \mathbb{R}^+$. Since $u(x, t) = h_D(x, y', t) \in C^\infty(\Omega \times \mathbb{R}^+)$ satisfies $(\partial_t - \Delta_X)u(x, t) = 0$ and $u(x, t) \geq 0$, the Bony’s parabolic type strong maximum principle (see [18, Theorem 3.2]) shows that $u(x, t) \equiv 0$ for all $0 < t \leq t'$ and all $x \in \Omega$. Now taking a function $f \in C_0^\infty(\Omega)$ such that $f(y') \neq 0$, then we have $\lim_{t \to 0^+} \int_{\Omega} h_D(x, y', t)f(x)dx = f(y')$, and yet it is contradictory due to $u(x, t) = h_D(x, y', t) \equiv 0$ for all $0 < t \leq t'$ and all $x \in \Omega$. Hence we conclude $h_D(x, y, t) > 0$ on $\Omega \times \Omega \times \mathbb{R}^+$.

Let $\Omega = \bigcup_{i=1}^\infty K_i$ with $K_i \subset K_{i+1}^\circ$, where $\{K_i\}_{i=1}^\infty$ are compact sets and $K_i^\circ$ is the interior of $K_i$. We then define a sequence of functions $f_i$ such that

$$f_i \in C_0^\infty(K_{i+1}^\circ) \subset C_0^\infty(\Omega), \quad 0 \leq f_i(x) \leq 1, \quad \text{with} \quad f_i(x) = \begin{cases} 
1, & x \in K_i, \\
0, & x \in (K_{i+1}^\circ)^c.
\end{cases}$$
Clearly, \( \lim_{i \to \infty} f_i(x) = 1_{\Omega}(x) \) and \( 0 \leq 1_{K_i}(x) \leq f_i(x) \leq f_{i+1}(x) \leq 1_{\Omega}(x) \). Here \( 1_E(x) \) denotes the indicator function of set \( E \) such that \( 1_E(x) = 1 \) for \( x \in E \) and \( 1_E(x) = 0 \) for \( x \notin E \). For any \( T > 0 \), we can also regard \( g(x,t) := \int_{\Omega} f_i(y)h_D(x,y,t)dy - 1 \) as a path \( g : (0,T) \to H^1_\mathcal{X}(\Omega) \). Besides, the strong derivative \( \frac{dg}{dt} \) exists in \( L^2(\Omega) \) and satisfying

\[
\frac{dg}{dt} - \Delta_X g = 0 \quad \text{in} \quad \mathcal{D}'(\Omega),
\]

and \( (g(\cdot,t))_+ \in H^1_{\mathcal{X},0}(\Omega) \) due to Remark 7.1. Similarly, \( g(\cdot,t) \to f_i - 1 \) in \( L^2(\Omega) \) implies \( (g(\cdot,t))_+ \to (f_i - 1)_+ = 0 \) in \( L^2(\Omega) \). Using Proposition 7.6 again, we obtain

\[
\int_{\Omega} f_i(y)h_D(x,y,t)dy \leq 1 \quad \text{for all} \quad i \in \mathbb{N}^+.
\]

By Lebesgue’s monotone convergence theorem, we have

\[
\int_{\Omega} h_D(x,y,t)dy = \lim_{i \to \infty} \int_{\Omega} f_i(y)h_D(x,y,t)dy \leq 1 \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+.
\]

Therefore, we complete the proof of property (7).

Finally, if \( \overline{f}(x,t) \) is another solution to (3.18) with the same initial condition \( f_0 \), the weak maximum principle indicates that the solution \( f(x,t) - \overline{f}(x,t) \) to (3.18) must be identically equal to 0. This yields the uniqueness of \( h_D(x,y,t) \).

The arguments of all above complete the proof of Proposition 3.3. \( \square \)

### 3.2. Comparison of subelliptic heat kernels

To further estimate the subelliptic Dirichlet heat kernel, we employ the comparison between Dirichlet heat kernel and global heat kernel. For the vector fields \( X = (X_1, X_2, \ldots, X_m) \) under the assumptions (H.1) and (H.2), Biagi-Bonfiglioli [11] proved that the homogeneous Hörmander operator \( \Delta_X \) possesses a global heat kernel \( h(x,y,t) \) satisfying the following properties:

1. For any fixed point \( y \in \mathbb{R}^n \), \( h(x,y,t) \) is the solution of

\[
\left( \frac{\partial}{\partial t} - \Delta_X \right) h(x,y,t) = 0 \quad \text{for all} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.
\]

2. For any \( x, y \in \mathbb{R}^n \),

\[
\lim_{t \to 0^+} h(x,y,t) = \delta_x(y),
\]

where \( \delta_x(y) \) is the Dirac distribution in \( \mathbb{R}^n \).

3. For any \( (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \),

\[
h(x, y, t) \geq 0.
\]

Moreover, for any \( (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \),

\[
\int_{\mathbb{R}^n} h(x, y, t)dy \leq 1.
\]
(4) For any $s, t > 0$,
\[ h(x, y, t + s) = \int_{\mathbb{R}^n} h(x, z, t)h(z, y, s)dz. \] (3.34)

(5) For any $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$,
\[ h(x, y, t) = h(y, x, t). \] (3.35)

(6)
\[ h(x, y, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+). \] (3.36)

The Gaussian bounds of subelliptic global heat kernel $h(x, y, t)$ for Hörmander operators on compact manifold were first obtained by Jerison-Sánchez-Calle [53] via the lifting-approximating approach proposed by Rothschild-Stein [72]. However, when we relax the compact manifold to $\mathbb{R}^n$, the Gaussian bounds require some extra assumptions on vector fields. Imposing the so-called uniform Hörmander’s condition on $C^\infty$ class vector fields, Kusuoka-Stroock [58, 59] proved the Gaussian bounds via the probabilistic approach. Furthermore, these results have been generalized by Bramanti-Brandolini-Lanconelli-Uguzzoni [19] to more general degenerate elliptic operators. For homogeneous Hörmander operators, Biagi-Bramanti [14] proved the following global Gaussian bounds of $h(x, y, t)$ by invoking Folland’s global lifting technique in [37].

Proposition 3.4. Let $X = (X_1, X_2, \ldots, X_m)$ be the smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2), and $h(x, y, t)$ be the global subelliptic heat kernel of homogeneous Hörmander operator $\triangle_X$. Then there exists a constant $A_1 > 1$ such that
\[ \frac{1}{A_1 |B_{\delta_1(x)}(x, \sqrt{t})|}e^{-\frac{A_1 \delta_1^2(x,y)}{A_1 t}} \leq h(x, y, t) \leq \frac{A_1}{|B_{\delta_1(x)}(x, \sqrt{t})|}e^{-\frac{\delta_1^2(x,y)}{A_1 t}} \] (3.37)
holds for all $x, y \in \mathbb{R}^n$ and all $t > 0$.

Proof. See [14, Theorem 2.4]. \qed

With the results above, we can establish the following comparison lemma of the subelliptic heat kernels, which is the key ingredient in the proof of Theorem 1.1.

Lemma 3.1. Assume that $X = (X_1, X_2, \ldots, X_m)$ and $\Omega$ satisfy the conditions in Theorem 1.1. Let $\delta_1(x) := \frac{\delta_X^2(x, \partial \Omega)}{A_1 Q}$ be the continuous function defined on $\Omega$, where $A_1$ is the positive constant in Proposition 3.4, $Q$ is the homogenous dimension, and $d_X(x, \partial \Omega) := \inf_{y \in \partial \Omega} d_X(x, y)$. Then for any $x \in \Omega$ and any $0 < t \leq \delta_1(x)$, we have
\[ 0 \leq h(x, x, t) - h_D(x, x, t) \leq \frac{A_1 C_3}{|B_{\delta_1(x)}(x, \sqrt{t})|}e^{-\frac{\delta_X^2(x, \partial \Omega)}{A_1 t}}, \] (3.38)
where $C_3$ is a positive constant depending on vector fields $X$. 

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Proof. According to the property (2) of subelliptic Dirichlet heat kernel \( h_D(x, y, t) \), we can extend \( h_D(x, y, t) \) to \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \) such that \( h_D(x, y, t) \) vanishes on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \setminus \overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^+ \).

Denote by

\[
E(x, y, t) := \begin{cases} 
  h(x, y, t) - h_D(x, y, t), & t > 0; \\
  0, & t \leq 0.
\end{cases}
\]

(3.39)

It follows from (3.10), (3.11), (3.30) and (3.36) that \( E(x, y, t) \in C^\infty(\Omega \times \Omega \times \mathbb{R}^+) \cap C((\Omega \cup \Gamma_0) \times (\Omega \cup \Gamma_0) \times \mathbb{R}^+) \), and for any fixed \( y \in \Omega \), \( E(x, y, t) \) is the solution of

\[
\left( \frac{\partial}{\partial t} - \Delta_X \right) E(x, y, t) = 0 \quad \text{for all} \quad (x, t) \in \Omega \times \mathbb{R}^+.
\]

(3.40)

Moreover,

\[
\lim_{t \to 0^+} \int_\Omega E(x, y, t) \varphi(x) dx = 0 \quad \text{for all} \quad \varphi \in C^\infty_0(\Omega).
\]

(3.41)

Then we prove that

\[
E(x, y, t) \geq 0 \quad \text{on} \quad \Omega \times \Omega \times \mathbb{R}^+.
\]

(3.42)

Suppose that \( E(x_0, y_0, t_0) < 0 \) for some \( (x_0, y_0, t_0) \in \Omega \times \Omega \times \mathbb{R}^+ \), there exists \( 0 < \delta < t_0 \) such that \( B(x_0, \delta) \subset \Omega \), \( B(y_0, \delta) \subset \Omega \) and

\[
E(x, y, t) < \frac{1}{2} E(x_0, y_0, t_0) < 0 \quad \text{on} \quad B(x_0, \delta) \times B(y_0, \delta) \times (t_0 - \delta, t_0 + \delta),
\]

where \( B(x, \delta) = \{ y \in \mathbb{R}^n \mid |x - y| < \delta \} \) denotes the classical Euclidean ball in \( \mathbb{R}^n \). Let \( f_0 \in C^\infty_0(B(y_0, \delta)) \subset C^\infty_0(\Omega) \) be a function satisfying \( 0 \leq f_0 \leq 1 \) and

\[
f_0(y) = \begin{cases} 
  1, & y \in B(y_0, \frac{\delta}{2}); \\
  0, & y \in \mathbb{R}^n \setminus B(y_0, \delta),
\end{cases}
\]

and

\[
f(x, t) = \int_\Omega E(x, y, t) f_0(y) dy = \int_{B(y_0, \delta)} E(x, y, t) f_0(y) dy.
\]

For any given \( T > 0 \) and any \( 0 < t < T \), we set

\[
f(x, t) = \int_\Omega h(x, y, t) f_0(y) dy - \int_\Omega h_D(x, y, t) f_0(y) dy := f_1(x, t) - f_2(x, t).
\]

The proof of Proposition 3.3 implies that \( f_2 \) admits the conditions (7.12) and (7.13) in Proposition 7.6. Thus, we only need to concerning with \( f_1 \). Clearly, \( \partial_t f_1 - \Delta_X f_1 = 0 \) on \( \mathbb{R}^n \times \mathbb{R}^+ \). The hypoellipticity of \( \partial_t - \Delta_X \) implies \( f_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \), which yields \( f_1(\cdot, t) \in H^1_{\text{loc}}(\Omega) \) for any \( 0 < t < T \). Moreover, since \( \partial_t h(x, y, t) \) is uniform continuity on \( \overline{\Omega} \times \overline{\Omega} \times [t - \varepsilon, t + \varepsilon] \) for small \( \varepsilon > 0 \), we can verify that the strong derivative \( \frac{df_1}{dt} = \partial_t f_1(\cdot, t) \in L^2(\Omega) \) and satisfies

\[
\frac{df_1}{dt} - \Delta_X f_1 = \frac{\partial f_1}{\partial t} - \Delta_X f_1 = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

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Besides, \( f_1(x, t) \geq 0 \) and Proposition 7.5 indicate that \(-f(x, t) \leq 0 \bmod H_{X, 0}^1(\Omega)\). By (3.41), \( \lim_{t \to 0^+} f(x, t) = 0 \) for all \( x \in \Omega \). Combining the fact \( 0 \leq f_0 \leq 1 \) with (3.16) and (3.33), we have

\[
|f(x, t)| \leq \int_{\Omega} |E(x, y, t)| dy \leq \int_{\Omega} h(x, y, t) dy + \int_{\Omega} h_D(x, y, t) dy \leq 2.
\]

Thus, the dominated convergence theorem gives \((-f(\cdot, t))_+ \to 0\) in \( L^2(\Omega) \) as \( t \to 0^+ \). According to Proposition 7.6, we have \( f(x, t) \geq 0 \) for all \( x \in \Omega \) and \( t \in (0, T) \). But the definition of \( f_0 \) shows that

\[
f(x_0, t_0) = \int_{B(y_0, \delta)} E(x_0, y, t_0) f_0(y) dy < 0.
\]

This yields a contradiction. As a result, \( E(x, y, t) \geq 0 \) on \( \Omega \times \Omega \times \mathbb{R}^+ \).

Combining (3.15), (3.33), (3.39) and (3.42), it is easy to verify that \( u_y(x, t) = E(x, y, t) \) is locally integrable on \( \Omega \times \mathbb{R} \) for any fixed \( y \in \Omega \). Therefore, we can deduce from (3.40) and (3.41) that \( u_y(x, t) \) satisfies

\[
(\partial_t - \Delta_X) u_y(x, t) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega \times \mathbb{R}).
\]

The hypoellipticity of \( \partial_t - \Delta_X \) implies that for any fixed \( y \in \Omega \), \( u_y(x, t) = E(x, y, t) \in C^\infty(\Omega \times \mathbb{R}) \) satisfying

\[
\lim_{t \to 0^+} u_y(x, t) = \lim_{t \to 0^+} E(x, y, t) = E(x, y, 0) = 0 \quad \forall x \in \Omega.
\]

Denoting by \( \Omega_j := \{ x \in \Omega | d_X(x, \partial \Omega) > \frac{1}{j} \} \), we know that \( \Omega_j \) is an open subset in \( \Omega \) and satisfying

\[
\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \Omega_j \subset \Omega_{j+1} \subset \Omega. \tag{3.43}
\]

For any fixed \( y \in \Omega \), there exists \( j_0 \geq 1 \) such that \( y \in \Omega_j \) for all \( j \geq j_0 \). Since \( u_y(x, t) = E(x, y, t) \in C^\infty(\Omega_j \times [0, +\infty)) \) satisfies

\[
\begin{cases}
(\partial_t - \Delta_X) u_y(x, t) = 0, & \text{on } \Omega_j \times \mathbb{R}^+; \\
\text{lim}_{t \to 0^+} u_y(x, t) = 0, & \text{for all } x \in \overline{\Omega_j},
\end{cases}
\]

by weak parabolic maximum principle (see [19, Proposition 3.6] and [60, Proposition 2.2]), (3.15), (3.35) and Proposition 3.4, we have

\[
E(x, y, t) \leq \max_{z \in \partial \Omega_j, \ 0 \leq s \leq t} E(z, y, s) \leq \max_{z \in \partial \Omega_j, \ 0 \leq s \leq t} h(y, z, s) \leq \max_{z \in \partial \Omega_j, \ 0 \leq s \leq t} \frac{A_1}{A_1 s} e^{\frac{d_X^2(x,y)}{A_1 s}} \leq \max_{0 \leq s \leq t} \frac{A_1}{|B_{d_X}(y, \sqrt{s})|} e^{-\frac{d_X^2(y, \partial \Omega_j)}{A_1 s}} \quad \text{for any } (x, t) \in \Omega_j \times \mathbb{R}^+. \tag{3.44}
\]
where $A_1$ is the positive constant in Proposition 3.4. On the other hand, Proposition 2.10 implies that

$$|B_{d_X}(x, r_2)| \leq C_3 \left( \frac{r_2}{r_1} \right)^Q |B_{d_X}(x, r_1)|$$

(3.45)

holds for any $x \in \mathbb{R}^n$ and any $0 < r_1 < r_2$, where $Q$ is the homogeneous dimension and $C_3$ is a positive constant depending on vector fields $X$ (see [14, Remark 3.9]). Hence, by (3.44) and (3.45) we obtain for any fixed $y \in \Omega_j \subset \Omega$,

$$E(x, y, t) \leq \max_{0 \leq s \leq t} \frac{A_1}{|B_{d_X}(y, \sqrt{s})|} e^{-\frac{d^2(y, \partial \Omega_j)}{A_1 t}}$$

(3.46)

holds for any $(x, t) \in \Omega_j \times \mathbb{R}^+$. Note that the function $g(s) := s^{-\frac{2}{d_X}} e^{-\frac{d^2(y, \partial \Omega_j)}{A_1 t}}$ is increasing on $\left(0, \frac{2d^2(y, \partial \Omega_j)}{A_1 t}\right]$ and decreasing on $\left(\frac{2d^2(y, \partial \Omega_j)}{A_1 t}, +\infty\right)$ with $\lim_{s \to 0^+} g(s) = \lim_{s \to +\infty} g(s) = 0$. Therefore (3.46) gives that, for any fixed $y \in \Omega_j \subset \Omega$,

$$E(x, y, t) \leq \frac{A_1 C_3}{|B_{d_X}(y, \sqrt{t})|} e^{-\frac{d^2(y, \partial \Omega_j)}{A_1 t}}$$

(3.47)

for any $(x, t) \in \Omega_j \times \left(0, \frac{2d^2_X(y, \partial \Omega_j)}{A_1 t}\right]$. That means

$$E(x, y, t) \leq \frac{A_1 C_3}{|B_{d_X}(x, \sqrt{t})|} e^{-\frac{d^2(x, \partial \Omega)}{A_1 t}}$$

(3.48)

holds for any $x \in \Omega_j$ and any $0 < t \leq \frac{2d^2_X(x, \partial \Omega)}{A_1 t}$. We mention that the positive constants $A_1, C_3, Q$ in estimate (3.48) are independent of $\Omega_j$. Furthermore, for any $x \in \Omega$, $d_X(x, \partial \Omega_j) \to d_X(x, \partial \Omega)$ as $j \to +\infty$, and $x \in \Omega_j$ for sufficiently large $j$. Thus, taking $j \to +\infty$ in (3.48), we derive that

$$E(x, x, t) \leq \frac{A_1 C_3}{|B_{d_X}(x, \sqrt{t})|} e^{-\frac{d^2(x, \partial \Omega)}{A_1 t}}$$

(3.49)

holds for all $x \in \Omega$ and any $0 < t \leq \delta_1(x) := \frac{d^2(x, \partial \Omega)}{A_1 Q^\frac{1}{n}}$.

4. Asymptotic behaviour of integral of $\Lambda(x, r)^{-1}$

This section aims to study the explicit asymptotic behaviour of the following integral

$$J_\Omega(r) := \int_{\Omega} \frac{dx}{\Lambda(x, r)} \quad \text{as} \quad r \to 0^+, \quad (4.1)$$
where $\Lambda(x, r)$ is the function given by (2.10), and $\Omega$ is a bounded open domain in $\mathbb{R}^n$ containing the origin. Due to the global version of Ball-Box theorem (Proposition 2.10), the asymptotic behaviour of integral (4.1) will play an essential role in proving Theorem 1.2.

Inspired by Nagel-Stein-Wainger [68], one may divide the domain $\Omega$ into some subregions given by $\Omega_{I,r} := \{ x \in \Omega | \lambda_I(x) r^{d(I)} = \max_J |\lambda_j(x) r^{d(J)}| \}$ for any $0 < r < 1$. Then the estimate of $J_\Omega(r)$ amounts to estimating the integrals of $(\lambda_I(x) r^{d(I)})^{-1}$ on such subregions. However, these subregions $\Omega_{I,r}$ cannot be expressed explicitly for the vector fields only satisfying Hörmander’s condition, which causes many difficulties in the calculations of related integrals.

Fortunately, if we restrict ourselves to the homogeneous Hörmander vector fields, the corresponding smooth functions $\lambda_I$ are polynomials. This allows us to study the explicit asymptotic behaviour of integral $J_\Omega(r)$ by improving the above idea. Let us briefly describe our approach. According to the degenerate components and homogeneity property (H.1), we first show that $J_\Omega(r)$ is asymptotically equal to the integral of $\Lambda(x, r)^{-1}$ on any given $\nu$-dimensional bounded open domain $\Omega_0 \subset \mathbb{R}^\nu$ containing the origin in $\mathbb{R}^\nu$. Here $\nu$ is the number of degenerate components of vector fields $X$, and $\mathbb{R}^\nu$ denotes the projection of $\mathbb{R}^n$ in the directions of all degenerate components. Then, by employing the resolution of singularities in algebraic geometry, we can find a real analytic map $\rho$ defined on a real analytic manifold such that every $\lambda_I \circ \rho$ has the form $c(u)m(u)$ in some local coordinates, where $m(u)$ is a monomial and $c(u)$ is a non-vanishing analytic function. As a result, we have the following estimation chain:

\[
J_\Omega(r) = \int_{\Omega} \frac{dx}{\Lambda(x, r)} \approx \int_{\Omega_0} \frac{dx_{I_1} \cdots dx_{I_\nu}}{\sum_I |\lambda_I(x)| r^{d(I)}} \quad \text{(Lemma 4.1)}
\]

\[
\approx \sum_{j=1}^l \int_{(-1,1)^\nu} \frac{|u^{q_j}| du}{\sum_I |u^{p_{j,I}}| r^{d(I)}} \quad \text{(Resolution of singularities)}
\]

\[
\approx \sum_{j=1}^l \int_{(0,1)^\nu} \frac{u^{q_j} du}{\sum_I |u^{p_{j,I}}| r^{d(I)}} \approx \sum_{j=1}^l \int_{(0,\infty)^\nu} \frac{u^{q_j+1} du}{\sum_I e^{-(q_j+1,u)} r^{d(I)}} \quad \text{(Change coordinates)}
\]

where $l$ is a positive integer, $\mathbf{1} = (1, \ldots, 1)$ is the vector in $\mathbb{R}^\nu$, $q_j$ and $p_{j,I}$ are $\nu$-dimensional multi-indexes. The subregions in each integral can be characterized by some polyhedrons in the form of

\[
P_{I,r} = \{ u \in [0, \infty)^\nu | e^{-(p_{j,I},u)} r^{d(I)} = \max_j e^{-(p_{j,J},u)} r^{d(J)} \}
\]

\[
= \left\{ u \in [0, \infty)^\nu | (p_{j,I} - p_{j,J}, u) \leq (d(J) - d(I)) \left( \ln \frac{1}{r} \right) \right\}
\]

which implies

\[
J_\Omega(r) \approx \sum_{j=1}^l \int_{(0,\infty)^\nu} \frac{u^{q_j+1} du}{\sum_I e^{-(p_{j,I},u)} r^{d(I)}} \approx \sum_{j=1}^l \sum_I \frac{1}{r^{d(I)}} \int_{P_{I,r}} e^{(p_{j,I} - q_j - 1,u)} du \quad \text{for } 0 < r < 1.
\]
Owing to some refined analysis involving convex geometry, we finally obtain that

\[ \frac{1}{r^{d(1)}} \int_{P_{r,r}} e^{(p,1-q,1,1)} du \approx r^{-\alpha} | \ln r |^{\tilde{d}} \quad \text{as} \quad r \to 0^+ \]

with \( \alpha \in Q \) and \( \tilde{d} \in \{0, 1, \ldots, v\} \). Consequently,

\[ J_{\Omega}(r) \approx r^{-Q_0} | \ln r |^{d_0} \quad \text{as} \quad r \to 0^+ , \]

where \( Q_0 \in Q \) and \( d_0 \in \{0, 1, \ldots, v\} \). Moreover, we shall also give the optimal ranges of indexes \( Q_0 \) and \( d_0 \).

4.1. Simplification procedures of integral \( J_{\Omega}(r) \)

We start with the following useful lemma.

**Lemma 4.1.** For any given bounded open subsets \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{R}^n \) containing the origin, we have

\[ J_{\Omega_1}(r) \approx J_{\Omega_2}(r) \]

where the notation \( J_D(r) \) is defined as

\[ J_D(r) = \int_{D} \frac{dx}{\Lambda(x,r)} \quad \text{for set} \quad D \subset \mathbb{R}^n . \]

Moreover, let \( \{x_{i_1}, \ldots, x_{i_v}\} \) be the collection of all degenerate components of \( X \), and \( \mathbb{R}^{v} := \mathbb{R}_{x_{i_1}} \times \cdots \times \mathbb{R}_{x_{i_v}} \) be the projection of \( \mathbb{R}^n \) in directions \( \{x_{i_1}, \ldots, x_{i_v}\} \). Then we have

\[ J_{\Omega_1}(r) \approx J_{\Omega_3}(r) \approx J_{\Omega_3,v}(r) \approx J_{\Omega_4,v}(r) , \]

where \( \Omega_3 \) and \( \Omega_4 \) are any given bounded open sets in \( \mathbb{R}^v \) containing the origin, and the notation \( J_{D',v}(r) \) denotes

\[ J_{D',v}(r) = \int_{D'} \frac{dx_{i_1} \cdots dx_{i_v}}{\Lambda(x,r)} \quad \text{for set} \quad D' \subset \mathbb{R}^v . \]

**Proof.** Given any positive constant \( p > 0 \), we consider the corresponding \( n \)-dimensional \( \delta_r \)-box

\[ D(p) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | |x_j| < p^{\alpha_j}, j = 1, \ldots, n \} . \]

By Proposition 2.12, we have for any \( 0 < p < q \),

\[ J_{D(p)}(r) = \int_{D(p)} \frac{dx}{\Lambda(x,r)} = \int_{D(p)} \frac{dx}{\sum_{k=w}^{Q} f_k(x)r^{k}} \]

\[ = \left( \frac{p}{q} \right)^Q \int_{D(q)} \frac{dy}{\sum_{k=w}^{Q} f_k(\delta_{\frac{q}{p}}(y))r^{k}} \approx \left( \frac{p}{q} \right)^Q \int_{D(q)} \frac{dy}{\sum_{k=w}^{Q} \left( \frac{p}{q} \right)^{Q-k} f_k(y)r^{k}} \]

\[ \approx J_{D(q)}(r) . \]

In particular, for any fixed \( p > 0 \),

\[ J_{D(p)}(r) \approx J_{D(1)}(r) . \quad (4.2) \]
For any bounded open set \( \Omega_1 \subset \mathbb{R}^n \) containing the origin, there exist positive constants \( 0 < p < q \) such that \( D(p) \subset \Omega_1 \subset D(q) \). Applying (4.2) to \( J_{D(p)}(r) \leq J_{\Omega_1}(r) \leq J_{D(q)}(r) \), we derive \( J_{\Omega_1}(r) \approx J_{D(1)}(r) \). This indicates \( J_{\Omega_1}(r) \approx J_{\Omega_2}(r) \) for any bounded open sets \( \Omega_1, \Omega_2 \) in \( \mathbb{R}^n \) containing the origin.

Similarly, for any bounded open set \( \Omega_3 \subset \mathbb{R}^v \) containing the origin, there are \( 0 < p < q \) such that the projections of \( n \)-dimensional \( \delta_t \)-boxes \( D(p) \) and \( D(q) \) in directions \( \{x_{i_1}, \ldots, x_{i_v}\} \), denoted by \( D(p, v) \) and \( D(q, v) \), satisfy

\[
D(p, v) \subset \Omega_3 \subset D(q, v),
\]

where \( D(p, v) = \{ (x_{i_1}, \ldots, x_{i_v}) \in \mathbb{R}^v | |x_{i_j}| < p^{\alpha_{i_j}}, \; j = 1, \ldots, v \} \). From (4.3) we have

\[
J_{D(p,v),v}(r) \leq J_{\Omega_3,v}(r) \leq J_{D(q,v),v}(r).
\]

Moreover, a direct calculation yields that

\[
2^{n-v}p^{Q-\alpha(X)}J_{D(p,v),v}(r) = J_{D(p)}(r) \quad \text{and} \quad 2^{n-v}q^{Q-\alpha(X)}J_{D(q,v),v}(r) = J_{D(q)}(r),
\]

where \( \alpha(X) \) is sum of all degenerate indexes of vector fields \( X \) defined in (2.17). Hence

\[
J_{\Omega_1}(r) \approx J_{\Omega_2}(r) \approx J_{\Omega_3,v}(r) \approx J_{\Omega_4,v}(r)
\]

holds for any bounded open sets \( \Omega_1, \Omega_2 \subset \mathbb{R}^n \) and \( \Omega_3, \Omega_4 \subset \mathbb{R}^v \) containing the origin.

If the vector fields \( X \) admit a unique degenerate component \( x_j \), the non-constant functions \( \lambda_i(x) \) are monomials depending on variable \( x_j \). Note that there are vector fields \( X \) under assumptions (H.1) and (H.2) possessing a unique degenerate component. A well-known example is the Grushin type vector fields \( X = (\partial_{x_1}, x_1 \partial_{x_2}) \) in \( \mathbb{R}^2 \). We first deal with the asymptotic behaviour of \( J_{\Omega}(r) \) for this simple case.

**Proposition 4.1.** Let \( X = (X_1, X_2, \ldots, X_m) \) be the smooth vector fields defined on \( \mathbb{R}^n \) satisfying assumptions (H.1) and (H.2). Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded open domain containing the origin. If \( x_j \) is the unique degenerate component of \( X \) associated with the degenerate index \( \alpha_j \), then we have

\[
J_{\Omega}(r) \approx \left\{ \begin{array}{ll}
\frac{1}{r^{Q-\alpha_j}} |\ln r|, & \text{if } Q - w = \alpha_j; \\
\frac{1}{r^{Q-\alpha_j}}, & \text{if } Q - w > \alpha_j,
\end{array} \right. \quad \text{as } r \to 0^+,
\]

where \( Q \) is the homogeneous dimension and \( w = \min_{x \in \mathbb{R}^n} \nu(x) \).

**Proof.** Proposition 2.12 gives

\[
\Lambda(x_j, r) = \sum_{k=w}^{Q} f_k(x_j) r^k,
\]

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where \( f_k(x_j) = \sum_{d(I)=k} |\lambda_I(x_j)| \) is a \( \delta_r \)-homogeneous non-negative continuous function of degree \( Q - k \). For each \( n \)-tuple \( I \) with \( d(I) = k \), by Proposition 2.8 we obtain that \( \lambda_I(x_j) \) is the polynomial function in the form of (2.8) and satisfies \( \lambda_I(t^\alpha x_j) = t^{Q-k}\lambda_I(x_j) \). This means

\[
\lambda_I(x_j) = \begin{cases} 
\lambda_I(1)x_j^{\frac{Q-k}{\alpha_j}}, & \text{if } \frac{Q-k}{\alpha_j} \in \mathbb{N} \text{ and } \lambda_I(x_j) \neq 0; \\
0, & \text{if } \frac{Q-k}{\alpha_j} \notin \mathbb{N} \text{ or } \lambda_I(x_j) = 0.
\end{cases} \tag{4.5}
\]

If \( Q - w < \alpha_j \), by (4.5) and Proposition 2.14 we have \( f_w(x_j) = \sum_{d(I)=w} |\lambda_I(x_j)| = 0 \), which contradicts the fact \( f_w(x) \neq 0 \) in Proposition 2.12. Thus we only need to consider the following two cases:

- **Case 1**: \( Q - w = \alpha_j \). Combining (2.14) and (4.5), we have
  \[
  \Lambda(x_j, r) = c_w|x_j| r^{Q-\alpha_j} + f_Q r^Q,
  \]
  where \( c_w \) and \( f_Q \) are some positive constants. Using Lemma 4.1, we obtain
  \[
  J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_{(-1,1)} \frac{dx_j}{c_w|x_j| r^{Q-\alpha_j} + f_Q r^Q} = \frac{2}{r^{Q-\alpha_j}} \int_0^1 \frac{dz}{c_w z + f_Q r^{\alpha_j}} \approx \frac{1}{r^{Q-\alpha_j}} \ln r \quad \text{as } r \to 0^+.
  \]

- **Case 2**: \( Q - w > \alpha_j \). It follows from (2.14) and (4.5) that
  \[
  \Lambda(x_j, r) = \sum_{k=w}^{Q-1} c_k |x_j|^{\frac{Q-k}{\alpha_j}} r^k + f_Q r^Q,
  \]
  where \( c_w, f_Q > 0 \) are some positive constants, and \( c_j \geq 0 \) for \( w+1 \leq j \leq Q-1 \). Then, by Lemma 4.1 we have
  \[
  J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_{(-1,1)} \sum_{k=w}^{Q-1} \frac{dx_j}{c_k |x_j|^{\frac{Q-k}{\alpha_j}} r^k + f_Q r^Q} = 2 \int_0^1 \frac{dz}{\sum_{k=w}^{Q-1} c_k z^{\frac{Q-k}{\alpha_j}} r^k + f_Q r^Q} = \frac{2\alpha_j}{r^{Q-\alpha_j}} \int_0^\frac{1}{r^{Q-\alpha_j}} \frac{y^{\alpha_j-1} dy}{\sum_{k=w}^{Q-1} c_k y^{Q-k} + f_Q} \quad \text{as } r \to 0^+.
  \]
  Since \( Q - w - (\alpha_j - 1) > 1 \), there exist positive constant \( C > 0 \) such that
  \[
  C^{-1} \leq \int_0^\frac{1}{r^{Q-\alpha_j}} \frac{y^{\alpha_j-1} dy}{\sum_{k=w}^{Q-1} c_k y^{Q-k} + f_Q} \leq C
  \]
  holds for any \( r \in (0, 1) \).
Moreover, in local coordinates, the Jacobian \( J_\Omega(r) \approx \frac{1}{r^{2-\alpha_j}} \) as \( r \to 0^+ \).

We next pay attention to the general case that vector fields \( X \) admit more than one degenerate component. In this situation, the non-constant functions \( \lambda_i \) are polynomials possessing zeroes in \( \Omega \), and the estimates on asymptotic behaviour of \( J_\Omega(r) \) require more delicate analysis. We shall invoke the resolution of singularities theory to deal with the integral \( J_\Omega(r) \). As it is well-known, Hironaka’s celebrated theorem \cite{Hironaka} on the resolution of singularities is a deep result in algebraic geometry. It was first proved in 1964, and since then, many different variations of this theorem have been developed. The version we state below is due to Atiyah \cite{Atiyah}, which was also carried out by Watanabe \cite{Watanabe} and Lin \cite{Lin} in constructing singular learning theory.

**Proposition 4.2** (Resolution of singularities). Let \( f_1(x), \ldots, f_m(x) \) be a family of non-constant real analytic functions defined on an open set \( V \subset \mathbb{R}^N \) containing the origin. Suppose that \( f_i(0) = 0 \) for \( i = 1, \ldots, m \). Then there exists a triple \((M, W, \rho)\) where

(a) \( W \subset V \) is a neighbourhood of the origin in \( \mathbb{R}^N \);
(b) \( M \) is an \( N \)-dimensional real analytic manifold;
(c) \( \rho : M \to W \) is a real analytic map.

Furthermore, \((M, W, \rho)\) satisfies the following properties:

(A) \( \rho \) is proper, i.e. the inverse image of any compact set is compact;
(B) \( \rho \) is a real analytic isomorphism between \( M \setminus \bigcup_{i=1}^m Z(f_i \circ \rho) \) and \( W \setminus \bigcup_{i=1}^m Z(f_i) \), where
\[
Z(f_i \circ \rho) = \{ x \in M \mid f_i(\rho(x)) = 0 \} \quad \text{and} \quad Z(f_i) = \{ x \in W \mid f_i(x) = 0 \};
\]
(C) For any \( y \in \bigcup_{i=1}^m Z(f_i \circ \rho) \), there exists a local chart \( U_y \) with the coordinates \( u = (u_1, u_2, \ldots, u_N) \) such that \( y \) corresponds to the origin and
\[
(f_i \circ \rho)(u) = a_{y,i}(u)u^{p_{y,i}} = a_{y,i}(u_1, u_2, \ldots, u_N)u_1^{p_1}u_2^{p_2} \cdots u_N^{p_N} \quad \text{for} \quad i = 1, \ldots, m,
\]
where \( a_{y,i} \) is a non-vanishing analytic function on \( U_y \), and \( p_{y,i} = (p_1^y, p_2^y, \ldots, p_N^y) \) is an \( N \)-dimensional multi-index.\(^1\) Moreover, in local coordinates, the Jacobian \( J_\rho \) of map \( \rho \) has the form
\[
J_\rho(u) = b_y(u)u^{q_y} = b_y(u_1, u_2, \ldots, u_N)u_1^{q_1}u_2^{q_2} \cdots u_N^{q_N},
\]
where \( b_y \) is an analytic non-vanishing function on \( U_y \), and \( q_y = (q_1^y, q_2^y, \ldots, q_N^y) \) is an \( N \)-dimensional multi-index.

**Proof.** See \cite{Atiyah}, \cite[Theorem 2.8]{Watanabe} and \cite[Corollary 3.4]{Lin}.

Additionally, we have

\(^1\)Here we means that \( \psi_y(0) = 0 \) and \( (f_i \circ \rho)(\psi_y^{-1}(u)) = a_{y,i}(\psi_y^{-1}(u))u^{p_{y,i}} \), where \( \psi_y : U_y \to \mathbb{R}^N \) is the coordinate map on the chart \( U_y \). For convenience, we shall omit the coordinate map \( \psi_y \).
Lemma 4.2. The real analytic map $\rho : M \to W$ given in Proposition 4.2 is surjective. In particular, $\rho^{-1}(0) \neq \emptyset$.

Proof. Let

$$Z_W := \bigcup_{i=1}^{m} Z(f_i) = \left\{ x \in W \left| \prod_{i=1}^{m} f_i(x) = 0 \right\}$$

and

$$Z_M := \bigcup_{i=1}^{m} Z(f_i \circ \rho) = \left\{ x \in M \left| \prod_{i=1}^{m} f_i(\rho(x)) = 0 \right\}.$$  

The property (B) in Proposition 4.2 indicates that $\rho$ is an analytic isomorphism from $M \setminus Z_M$ to $W \setminus Z_W$. Since for each $1 \leq i \leq m$, $f_i$ and $f_i \circ \rho$ are both analytic and non-zero functions, we can deduce that $W \setminus Z_W$ and $M \setminus Z_M$ are dense in $W$ and $M$, respectively.

Then for any $y_0 \in Z_W$, we can choose a sequence $\{y_k\}_{k=1}^{\infty} \subset W \setminus Z_W$ such that $y_k \to y_0$ as $k \to +\infty$. According to the property (B) in Proposition 4.2, for $\{y_k\}_{k=1}^{\infty} \subset W \setminus Z_W$, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset M \setminus Z_M$ such that $\rho(x_k) = y_k$ holds for all $k \geq 1$. Set $K := \{y_k\}_{k=1}^{\infty} \cup \{y_0\} \subset W$. Clearly, $K$ is compact, and $\{x_k\}_{k=1}^{\infty} \subset \rho^{-1}(K)$ implies that $\rho^{-1}(K)$ is a non-empty compact subset in $M$. Therefore, there is a subsequence $\{x_{k_j}\}_{j=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $x_{k_j} \to x_0 \in \rho^{-1}(K) \subset M$ as $j \to +\infty$. Due to the continuity of $\rho$, we have $\rho(x_0) = y_0$. Hence, $\rho$ is a surjective map and $\rho^{-1}(0) \neq \emptyset$.

Combining Proposition 4.2 and Lemma 4.2, we can derive the following conclusion.

Proposition 4.3. Let $X = (X_1, X_2, \ldots, X_m)$ be the smooth vector fields defined on $\mathbb{R}^n$ satisfying assumptions (H.1) and (H.2). Suppose $X$ have $v$ degenerate components and $2 \leq v \leq n - 1$. Then for any bounded domain $\Omega$ in $\mathbb{R}^n$ containing the origin, we have

$$J_{\Omega}(r) = \int_{\Omega} \frac{dx}{A(x, r)} \approx \sum_{j=1}^{l} \int_{(-1,1)^v} \frac{|u^{q_j}|du}{\sum_{I} |u^{p_{I}}|d(I)},$$

where $l$ is a positive integer, $q_j$ and $p_{I}$ are $v$-dimensional multi-indexes.

Proof. Denote by $\{x_{i_1}, \ldots, x_{i_v}\}$ the collection of all degenerate components of vector fields $X$. Substituting by $x_{i_j} = z_j$ for $1 \leq j \leq v$, for any bounded subset $\Omega_0 \subset \mathbb{R}^v$ containing an open neighbourhood of the origin in $\mathbb{R}^v$, we obtain from Lemma 4.1 that

$$J_{\Omega}(r) \approx J_{\Omega_0}(r) := \int_{\Omega_0} \frac{dz}{A(z, r)},$$

where $z = (z_1, \ldots, z_v) \in \mathbb{R}^v$. We shall choose a suitable subset $\Omega_0$ to estimate $J_{\Omega}(r)$.

Note that $\lambda_I(z)$ is a polynomial in $\mathbb{R}^v$ for each $n$-tuple of integers $I = (i_1, \ldots, i_n)$ with $1 \leq i_j \leq q$. Let

$$\mathcal{B} := \{I = (i_1, \ldots, i_n) | 1 \leq i_j \leq q, 1 \leq j \leq n, \lambda_I(z) \text{ is a non-constant polynomial}\}.$$  

Clearly, $\mathcal{B} \neq \emptyset$. For any $I \in \mathcal{B}$, Proposition 2.8 yields $d(I) < Q$ and $\lambda_I(0) = 0$. Applying Proposition 4.2 to the family of polynomials $\{\lambda_I\}_{I \in \mathcal{B}}$, we obtain a triple $(M, W, \rho)$ where
(a) \( W \subset V \) is a neighbourhood of the origin in \( \mathbb{R}^n \);
(b) \( M \) is a \( v \)-dimensional real analytic manifold;
(c) \( \rho : M \to W \) is a real analytic map.

Furthermore, \( (M, W, \rho) \) satisfies the following properties:

(A) \( \rho \) is proper, i.e. the inverse image of any compact set is compact;
(B) \( \rho \) is a real analytic isomorphism between \( M \setminus \bigcup_{I \in \mathcal{B}} Z(\lambda_I \circ \rho) \) and \( W \setminus \bigcup_{I \in \mathcal{B}} Z(\lambda_I) \), where
\[
Z(\lambda_I \circ \rho) = \{ x \in M | \lambda_I(\rho(x)) = 0 \} \quad \text{and} \quad Z(\lambda_I) = \{ x \in W | \lambda_I(x) = 0 \};
\]
(C) For any \( y \in \bigcup_{I \in \mathcal{B}} Z(\lambda_I \circ \rho) \), there exists a local chart \( U_y \) with the coordinates \( u = (u_1, u_2, \ldots, u_v) \) such that \( y \) corresponds to the origin and
\[
(\lambda_I \circ \rho)(u) = a_{y, I}(u) u^{p_{y, I}} \quad \text{for all} \quad I \in \mathcal{B},
\]
where \( a_{y, I} \) is a non-vanishing analytic function on \( U_y \), and \( p_{y, I} \) is a \( v \)-dimensional multi-index. Furthermore, in local coordinates, the Jacobian \( J_\rho \) of map \( \rho \) has the form
\[
J_\rho(u) = b_y(u) u^{q_y},
\]
where \( b_y \) is a non-vanishing analytic function on \( U_y \), and \( q_y \) is a \( v \)-dimensional multi-index.

Using \( \lambda_I(0) = 0 \) and Lemma 4.2, we have \( \rho^{-1}(0) \neq \emptyset \) and \( \rho^{-1}(0) \subset \bigcup_{I \in \mathcal{B}} Z(\lambda_I \circ \rho) \). For any \( y \in \rho^{-1}(0) \), there exists a pre-compact open neighbourhood \( U_y \) such that the property (C) is satisfied on \( U_y \), and the analytic functions \( a_{y, I}(u), b_y(u) \) are non-vanishing on \( \overline{U_y} \). Moreover, we can find a bump function \( \Psi_y \) such that

- \( \Psi_y \in C^\infty(M) \) and \( 0 \leq \Psi_y \leq 1 \);
- \( \Psi_y(y) = 1 \);
- \( \text{supp } \Psi_y \subset \{ z \in M | \Psi_y(z) \neq 0 \} \subset U_y \).

Denoting by \( V_y := \{ z \in M | \Psi_y(z) > 0 \} \) the open neighborhood of \( y \), we have \( V_y \subset \text{supp } \Psi_y \subset U_y \). Observing that \( \{ V_y \}_{y \in \rho^{-1}(0)} \) is an open cover of compact set \( \rho^{-1}(0) \), there exists a finite subcover \( \{ V_{y_1}, V_{y_2}, \ldots, V_{y_l} \} \) associated with the bump functions \( \{ \Psi_{y_1}, \ldots, \Psi_{y_l} \} \) and pre-compact open neighborhoods \( \{ U_{y_1}, U_{y_2}, \ldots, U_{y_l} \} \) such that for each \( 1 \leq i \leq l \), we have \( V_{y_i} \subset \text{supp } \Psi_{y_i} \subset U_{y_i} \).

Let \( \Psi := \sum_{j=1}^l \Psi_{y_j} \). It follows that \( \Psi \in C^\infty(M) \) and
\[
\rho^{-1}(0) \subset V_0 \subset U_0 \subset M,
\]
where \( U_0 := \bigcup_{j=1}^l U_{y_j} \) and \( V_0 := \bigcup_{j=1}^l V_{y_j} = \{ x \in M | \Psi(x) > 0 \} \) are pre-compact open sets in \( M \). For each \( 1 \leq i \leq l \), denoting by \( \varphi_i := \frac{\Psi_{y_i}}{\Psi} \), we have \( \varphi_i \in C^\infty(V_0) \), \( 0 \leq \varphi_i(y) \leq 1 \), \( \text{supp } \varphi_i \subset \text{supp } \Psi_{y_i} \subset U_{y_i} \), and \( \sum_{j=1}^l \varphi_j(y) = 1 \) for all \( y \in V_0 \).
Clearly, \( \rho(V_0) \) is a bounded set in \( \mathbb{R}^v \) containing the origin. We further show that \( \rho(V_0) \) contains an open neighbourhood of the origin in \( \mathbb{R}^v \). Suppose the origin in \( \mathbb{R}^v \) is not in the interior of \( \rho(V_0) \). Then there exists a sequence \( \{y_k\}_{k=1}^{\infty} \subset W \setminus \rho(V_0) \) such that \( y_k \to 0 \in \mathbb{R}^v \) as \( k \to +\infty \). It follows from Lemma 4.2 that \( \rho : M \to W \) is a surjective map. Therefore, we can find a corresponding sequence \( \{x_k\}_{k=1}^{\infty} \subset M \setminus V_0 \) such that \( \rho(x_k) = y_k \) for all \( k \geq 1 \). Setting \( K_1 := \{y_k\}_{k=1}^{\infty} \cup \{0\} \), it turns out that \( K_1 \) is a compact subset in \( \mathbb{R}^v \). This indicates \( \rho^{-1}(K_1) \) is a non-empty compact subset in \( M \) since \( \{x_k\}_{k=1}^{\infty} \subset \rho^{-1}(K_1) \). Thus, there exists a subsequence \( \{x_{k_j}\}_{j=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty} \) which converges to some point \( x_0 \in \rho^{-1}(K_1) \subset M \). The continuity of \( \rho \) gives \( \rho(x_0) = 0 \) and \( x_0 \in \rho^{-1}(0) \subset V_0 \). Recalling that \( V_0 \) is an open subset in \( M \), it follows \( \{x_{k_j}\}_{j=0}^{\infty} \subset V_0 \) for some \( j_0 \in \mathbb{N}^+ \), which contradicts \( \{x_k\}_{k=1}^{\infty} \subset M \setminus V_0 \). Consequently, \( \rho(V_0) \) contains an open neighbourhood of the origin in \( \mathbb{R}^v \).

For ease of notations, we denote by \( Z_W = \bigcup_{I \in \mathcal{B}} Z(\lambda_I) \) and \( Z_M = \bigcup_{I \in \mathcal{B}} Z(\lambda_I \circ \rho) \). Since for each \( I \in \mathcal{B} \), \( \lambda_I \) and \( \lambda_I \circ \rho \) are non-identically-vanishing analytic functions, we can deduce from [65] that \( Z_W \) and \( Z_M \) have \( v \)-dimensional zero measure. Thus the property (B) yields that

\[
J_{\rho(V_0),v}(r) = \int_{\rho(V_0)} \frac{dz}{\Lambda(z,r)} = \int_{\rho(V_0)} \frac{dz}{\sum_I |\lambda_I(z)||r^{d(I)}}
\]

\[
= \int_{V_0} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}} \leq \int_{V_0} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}}.
\]

where \( dy \) is the volume element in \( M \), and \( J_\rho \) denotes the Jacobian of the transition from \( dz \) to \( dy \).

Moreover, in local coordinates, by property (C) we have

\[
\int_{V_0} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}} = \int_{\text{supp} \varphi_j} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}}
\]

\[
\leq \int_{U_{y_j}} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}} = \int_{U_{y_j}} \frac{\int_{U_{y_j}} b_{y_j}(u)|u^{q_j}||du}{\sum_I |a_{y_j,I}(u)|r^{d(I)}}
\]

where \( q_j \) and \( p_{j,I} \) are \( v \)-dimensional multi-indexes, \( b_{y_j}(u) \) and \( a_{y_j,I}(u) \) are non-vanishing analytic functions on the compact set \( \overline{U_{y_j}} \). Hence, we obtain

\[
\int_{U_{y_j}} \frac{\int_{U_{y_j}} b_{y_j}(u)|u^{q_j}||du}{\sum_I |a_{y_j,I}(u)|r^{d(I)}} \approx \int_{(-1,1)^v} \frac{|u^{q_j}||du}{\sum_I |u^{p_{j,I}}|r^{d(I)}}.
\]

On the other hand, since \( \varphi_j(y_j) > 0 \), there exists an open neighbourhood \( D_j \) of \( y_j \) such that \( \varphi_j(y_j) > \frac{1}{2} \varphi_j(y_j) > 0 \) for all \( y \in D_j \). This means \( D_j \subset \text{supp} \varphi_j \subset U_{y_j} \). Therefore,

\[
\int_{V_0} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}} \geq \int_{D_j} \frac{\int_M |J_\rho(y)||\varphi_j(y)|dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}} \geq \frac{1}{2} \varphi_j(y_j) \int_{D_j} \frac{\int_M |J_\rho(y)||dy}{\sum_I |\lambda_I(\rho(y))||r^{d(I)}}
\]

\[
= \frac{1}{2} \varphi_j(y_j) \int_{D_j} \frac{\int_{U_{y_j}} b_{y_j}(u)|u^{q_j}||du}{\sum_I |a_{y_j,I}(u)|r^{d(I)}} \approx \int_{(-1,1)^v} \frac{|u^{q_j}||du}{\sum_I |u^{p_{j,I}}|r^{d(I)}}.
\]
Since \( \rho(V_0) \) is a bounded set containing an open neighbourhood of the origin in \( \mathbb{R}^v \), by taking \( \Omega_0 = \rho(V_0) \) we conclude from (4.7)-(4.11) that

\[
J_{\Omega}(r) \approx \sum_{j=1}^i \int_{(-1,1)^v} \frac{|u^a|}{|u^{p_j,t}|^{rd(t)}} du.
\] (4.12)

\[\square\]

4.2. Estimates of integrals of rational functions

This part aims to construct the explicit asymptotic behaviour of the integral

\[
\int_{(-1,1)^v} \frac{|u^a|}{\sum_I |u^{p_j,t}|^{rd(t)}} du \quad \text{as} \quad r \to 0^+.
\] (4.13)

Let us consider a class of general integrals including (4.13). Suppose \( \mathcal{G} \) is the collection of finitely many index pairs \( (a,s) \), wherein each index pair \( (a,s) \), \( a = (a_1, \ldots, a_N) \) denotes the \( N \)-dimensional multi-index and \( s \in [0, +\infty) \) is a non-negative constant. Given a fixed \( N \)-dimensional multi-index \( b \), we set

\[
I(r) := \int_{[0,1]^N} \frac{x^b dx}{\sum_{(a,s) \in \mathcal{G}} x^{a_s}} \frac{dx}{x^a} \text{ for } 0 < r < 1.
\] (4.14)

Furthermore, we assume that \( I(r) < +\infty \) for any \( 0 < r < 1 \). Obviously, \( I(r) \) reduces to the integral in (4.13) if we choose suitable \( v \)-dimensional multi-index \( b \) and index pairs \( (a,s) \). With some refined analysis involving the convex geometry, we shall give the explicit asymptotic behaviour of \( I(r) \) as \( r \to 0^+ \). In the interest of making the exposition reasonably self-contained, we recall some preliminary definitions, notations and results in convex geometry, and one can refer to [20, 48, 71] for more details.

Suppose that \( N \geq 2 \) is a positive integer, and \( \mathbb{R}^N \) is the \( N \)-dimensional Euclidean space equipped with the Euclidean distance

\[
|x - y|_N := \sqrt{\langle x - y, x - y \rangle},
\]

where \( \langle x, y \rangle := \sum_{i=1}^N x_i y_i \) for the points \( x = (x_1, \ldots, x_N) \) and \( y = (y_1, \ldots, y_N) \) in \( \mathbb{R}^N \). For a subset set \( A \) in \( \mathbb{R}^N \), we denote by \( \text{conv}(A) \) the convex hull of \( A \), which is the intersection of all convex sets containing set \( A \). Obviously, if \( A \subset B \subset \mathbb{R}^N \), then \( \text{conv}(A) \subset \text{conv}(B) \). Moreover, we denote by

\[
A + B := \{x + y | x \in A, y \in B\}
\]

the Minkowski sum of two sets of position vectors \( A \) and \( B \) in \( \mathbb{R}^N \).

We then introduce the affine set and the dimension of the convex set. A set \( M \subset \mathbb{R}^N \) is called the affine set if \( tx + (1-t)y \in M \) for any \( x, y \in M \) and any \( t \in \mathbb{R} \). If an affine set \( M \neq \emptyset \), then there is a unique linear subspace \( L \subset \mathbb{R}^N \) such that \( M = \{y\} + L \) for any \( y \in M \). The dimension of the affine set \( M \) is defined by \( \dim M := \dim L \). Moreover, the intersection of affine sets is also an affine set. For a convex set \( A \subset \mathbb{R}^N \), the intersection of
all affine sets containing $A$ is called the affine hull of $A$, denoted by $\text{aff } A$. The dimension of convex $A$ is defined as the dimension of affine hull of $A$, i.e. $\dim A := \dim(\text{aff } A)$.

For any convex set $A \subset \mathbb{R}^N$, we denote by 
\[
\text{relint}(A) := \{ x \in \text{aff } A \mid \exists \varepsilon > 0, B^N_N(x) \cap \text{aff } A \subset A \}
\]
the relative interior of $A$, where $B^N_N(x) = \{ y \in \mathbb{R}^N \mid \| y - x \|_N < \varepsilon \}$ denotes the Euclidean ball in $\mathbb{R}^N$. It is known that for any non-empty convex set $A$, relint$(A)$ is a non-empty convex set. Furthermore, $\text{relint}(A) = \overline{A}$ and $\text{relint}(\overline{A}) = \text{relint}(A)$. If $\dim A = N$, we have $\text{aff } A = \mathbb{R}^N$ and $\text{relint}(A) = A^o$, where $A^o$ denotes the interior of $A$ in $\mathbb{R}^N$.

As an important class of convex set, the polyhedron in $\mathbb{R}^N$ is defined by the intersection of finitely many closed half-spaces. Here the closed half-space in $\mathbb{R}^N$ is the set $\{ x \in \mathbb{R}^N \mid f(x) \leq 0 \}$ associated with the given affine function $f$. In addition, a polytope is the convex hull of finitely many points. According to [20, Corollary 8.7] and [48, Theorem 1.20], we know that a non-empty bounded polyhedron is a polytope and vice versa.

For a given polyhedron $P \subset \mathbb{R}^N$, if there is an affine function $f$ such that $P \subset \{ x \in \mathbb{R}^N \mid f(x) \leq 0 \}$ and $F := \{ x \in \mathbb{R}^N \mid f(x) = 0 \} \cap P \neq \emptyset$, then we call $F$ the face of $P$. We say $F$ is a $d$-face if $\dim F = d$. In particular, the 0-face is called the vertex, and we will not distinguish between 0-faces and vertices. From the definitions above, we can see that the face of a polyhedron is also a polyhedron. Moreover, it is worth pointing out that, for an unbounded polyhedron $P \subset \mathbb{R}^N$, there exists at least one non-zero vector $q \in \mathbb{R}^N$ such that $p + \lambda q \in P$ for all $\lambda \geq 0$ and $p \in P$, where the non-zero vector $q$ is known as the direction of polyhedron $P$ (see [71, Theorem 8.4]).

In this part, we shall deal with the volumes of convex sets with different dimensions. For a clearer presentation, it is desirable to write $V_m(A)$ as the $m$-dimensional volume of a convex set $A$. Especially, if $A$ is a $d$-dimensional convex set with $d \geq 1$, we have $V_d(A) > 0$ while $V_m(A) = 0$ provided $m > d$. Furthermore, we will agree with $V_0(x) = 0$ for any point $x \in \mathbb{R}^N$.

Let us start our estimates on $I(r)$. We first show that $I(r)$ is asymptotically equal to the sum of some integrals on polyhedrons. Precisely, we have

**Proposition 4.4.** For each index pair $(a, s) \in \mathcal{G}$, let 
\[
P_{a,s} := \{ y \in [0, +\infty)^N \mid f_{(a,s),(a',s')}(y) := (a-a', y) - (s'-s) \leq 0, \forall (a',s') \in \mathcal{G} \} \quad (4.15)
\]
be a polyhedron in $[0, +\infty)^N$, and 
\[
\phi_a(y) := \langle a - b - 1, y \rangle \quad (4.16)
\]
be a linear function with $1 := e_1 + e_2 + \cdots + e_N$, where $e_j := (0, \ldots, 1, \ldots, 0)$ denotes the unit vector in $\mathbb{R}^N$. If $I(r) < +\infty$ for any $r \in (0, 1)$, then there exists a positive constant $C < 1$ only depending on $\mathcal{G}$, such that
\[
C \sum_{(a,s) \in \mathcal{G}^o} J_{a,s}(r) \leq I(r) \leq \sum_{(a,s) \in \mathcal{G}^o} J_{a,s}(r) \quad \text{for all } r \in (0, 1), \quad (4.17)
\]
where

\[ J_{a,s}(r) := \left( \ln \frac{1}{r} \right)^N \frac{1}{r^s} \int_{P_{a,s}} e^{(\ln \frac{1}{r})\phi_a(y)} dy, \]  

(4.18)

and \( \mathfrak{G}^o := \{(a, s) \in \mathfrak{G} | V_N(P_{a,s}) > 0\} \) denotes the collection of index pairs \((a, s)\) such that the polyhedron \(P_{a,s}\) possesses positive \(N\)-dimensional volume.

**Proof.** Substituting \( x_i = e^{-y_i} \) for \( i = 1, \ldots, N \), we have

\[ I(r) = \int_{[0,1]^N} \frac{x^b dx}{\sum_{(a,s) \in \mathfrak{G}} x^a r^s} = \int_{[0,\infty]^N} \frac{e^{-(b+1)y} dy}{\sum_{(a,s) \in \mathfrak{G}} e^{-(a,y)r^s}}. \]  

(4.19)

Then, for each index pair \((a, s)\) \(\in\) \(\mathfrak{G}\) and \(0 < r < 1\), we define

\[ A_{a,s}(r) := \left\{ y \in [0, +\infty)^N \mid (a - a', y) \leq (s' - s) \left( \ln \frac{1}{r} \right), \ \forall (a', s') \in \mathfrak{G} \right\}. \]  

(4.20)

Clearly, \( A_{a,s}(r) \) is a polyhedron in \([0, +\infty)^N\) depending on the index pair \((a, s)\) and the parameter \(r\). Note that \( A_{a,s}(r) \) can be rewritten as

\[ A_{a,s}(r) = \left\{ y \in [0, +\infty)^N \mid e^{-(a,y)r^s} = \max_{(a', s') \in \mathfrak{G}} e^{-(a', y)r^s} \right\}. \]

We next claim that the polyhedrons \( A_{a,s}(r) \) and \( P_{a,s} \) satisfy the following properties:

(i) For \(0 < r < 1\), let \( T_r(x) := (\ln \frac{1}{r}) x \) be the automorphism in \(\mathbb{R}^N\), then

\[ T_r(P_{a,s}) = A_{a,s}(r). \]

(ii) For \(0 < r < 1\), we have

\[ \bigcup_{(a,s) \in \mathfrak{G}} A_{a,s}(r) = \bigcup_{(a,s) \in \mathfrak{G}} P_{a,s} = [0, +\infty)^N. \]

(iii) For \(0 < r < 1\), \( V_N(A_{a_1,s_1}(r) \cap A_{a_2,s_2}(r)) = 0 \) and \( V_N(P_{a_1,s_1} \cap P_{a_2,s_2}) = 0 \) if \((a_1, s_1) \neq (a_2, s_2)\).

Clearly, (4.15) and (4.20) yield property (i). Since for any \(0 < r < 1\), \( A_{a,s}(r) \) is isomorphic to \( P_{a,s} \), we only need to verify properties (ii) and (iii) for either of \( A_{a,s}(r) \) and \( P_{a,s} \). Observe that \( \mathfrak{G} \) is the collection of finitely many index pairs. Hence, for any \( y_0 \in [0, +\infty)^N \) and \(0 < r < 1\), there exists an index pair \((a, s) \in \mathfrak{G}\) such that

\[ e^{-(a,y_0)r^s} = \max_{(a', s') \in \mathfrak{G}} e^{-(a', y_0)r^s'}, \]

which yields \( y_0 \in A_{a,s}(r) \) and \( \bigcup_{(a,s) \in \mathfrak{G}} A_{a,s}(r) = [0, +\infty)^N \). Furthermore, (4.15) gives that \( P_{a_1,s_1} \cap P_{a_2,s_2} \subset \{ y \in [0, +\infty)^N | f_{(a_1,s_1)(a_2,s_2)}(y) = 0 \} \) for \((a_1, s_1) \neq (a_2, s_2)\). Since \( y \in \)
\[0, +\infty)^N f_{(a_1, s_1), (a_2, s_2)}(y) = 0\] is a hyperplane with dimension at most \(N - 1\), it follows that \(V_N(P_{a_1, s_1} \cap P_{a_2, s_2}) = 0\).

Owing to the properties (i)-(iii) above, we have

\[
I(r) = \int_{(0, +\infty)^N} e^{-b(1, y)} dy = \sum_{(a, s) \in \mathfrak{G}} \int_{A_{a,s}(r)} \frac{e^{-b(1, y)}}{r^{a,s}} dy,
\]

(4.21)

where \(\mathfrak{G} := \{(a, s) \in \mathfrak{G} | V_N(P_{a,s}) > 0\}\). Furthermore, for each \((a, s) \in \mathfrak{G}^\circ\),

\[
\frac{C}{r^a} \int_{A_{a,s}(r)} e^{-b(1, y)} dy \leq \int_{A_{a,s}(r)} \frac{e^{-b(1, y)}}{r^{a,s}} dy \leq \frac{1}{r^a} \int_{A_{a,s}(r)} e^{-b(1, y)} dy,
\]

(4.22)

where \(0 < C < 1\) is a positive constant only depending on \(\mathfrak{G}\). Denote by \(\phi_a(y) := \langle a - b - 1, y \rangle\). It follows from (4.22) and property (i) that, for any \(0 < r < 1\),

\[
\frac{C}{r^a} \int_{T_r(P_{a,s})} e^{\phi_a(y)} dy \leq \int_{A_{a,s}(r)} \frac{e^{-b(1, y)}}{r^{a,s}} dy \leq \frac{1}{r^a} \int_{T_r(P_{a,s})} e^{\phi_a(y)} dy.
\]

(4.23)

Consequently, by (4.21) and (4.23) we obtain

\[
C \sum_{(a, s) \in \mathfrak{G}^\circ} \frac{1}{r^a} \left( \ln \frac{1}{r} \right)^N \int_{P_{a,s}} e^{\left( \ln \frac{1}{r} \right) \phi_a(y)} dy \leq I(r) \leq \sum_{(a, s) \in \mathfrak{G}^\circ} \frac{1}{r^a} \left( \ln \frac{1}{r} \right)^N \int_{P_{a,s}} e^{\left( \ln \frac{1}{r} \right) \phi_a(y)} dy
\]

(4.24)

holds for all \(r \in (0, 1)\).

We now turn to the asymptotic behaviour of \(J_{a,s}(r)\) as \(r \to 0^+\) under the assumptions in Proposition 4.4. It follows from (4.17) that for any \(r \in (0, 1)\), \(I(r) < +\infty\) if and only if

\[
J_{a,s}(r) < +\infty \quad \text{for all } (a, s) \in \mathfrak{G}^\circ.
\]

(4.25)

For the index pair \((a, s) \in \mathfrak{G}^\circ\), if \(a - b - 1 = 0\), then \(\phi_a(y) \equiv 0\). In this case, \(V_N(P_{a,s}) < +\infty\) and \(P_{a,s}\) is bounded. As a result,

\[
J_{a,s}(r) = \left( \ln \frac{1}{r} \right)^N \frac{1}{r^a} V_N(P_{a,s}) \quad \text{for } a - b - 1 = 0, 0 < r < 1.
\]

(4.26)

When \(a - b - 1 \neq 0\), the asymptotic estimates of integral \(J_{a,s}(r)\) will be carried out by following results.

**Lemma 4.3.** For any fixed multi-index \(b\) and the index pair \((a, s) \in \mathfrak{G}^\circ\) with \(a - b - 1 \neq 0\), \(\phi_a(x) = \langle a - b - 1, x \rangle\) attains its maximum on the polyhedron \(P_{a,s}\). Furthermore, denoting by \(m_{a,s} := \max_{x \in P_{a,s}} \phi_a(x)\), the set \(M_{a,s} := \{x \in P_{a,s} | \phi_a(x) = m_{a,s}\}\) is a bounded face of polyhedron \(P_{a,s}\) with dimension \(0 \leq d_{a,s} \leq N - 1\).
Proof. Due to the theory of linear programming (see [42, p.15]), it follows that if $\phi_a$ attains its maximum on the polyhedron $P_{a,s}$, then $M_{a,s}$ is a $d_{a,s}$-face of $P_{a,s}$ with $0 \leq d_{a,s} \leq N - 1$. Thus, it remains to prove that the maximum of $\phi_a$ is achieved, and $M_{a,s}$ is bounded.

Since the polyhedron $P_{a,s} \subset [0, +\infty)^N$ cannot contain any line in $\mathbb{R}^N$, we can deduce from [8, Theorem 2.6] that $P_{a,s}$ has at least one extreme point (i.e. vertex, see [8, Theorem 2.3]). If $\phi_a$ cannot attain its maximum on $P_{a,s}$, then [8, Theorem 2.8] gives sup$_{x \in P_{a,s}} \phi_a(x) = +\infty$, which implies $P_{a,s}$ is an unbounded closed set in $\mathbb{R}^N$. Additionally, [8, Corollary 2.5] indicates that $\phi_a(P_{a,s})$ is also a polyhedron in $\mathbb{R}^N$, and thus $[c, +\infty) \subset \phi_a(P_{a,s})$ for some constant $c > 0$.

Let us consider the affine function $g_\varphi(y) := \phi_a(y) - \varphi$ for some $\varphi > 0$. For any $\varphi > c$, there is a point $y_0 \in P_{a,s}^\circ$ such that $g_\varphi(y_0) > 0$. Observing that $P_{a,s}$ is an $N$-dimensional polyhedron with $V_N(P_{a,s}) > 0$, we have relint$(P_{a,s}) = P_{a,s}^\circ \neq \emptyset$, which allows us to choose a $\bar{y}_0$ in some neighbourhood of $y_0$ such that $\bar{y}_0 \in P_{a,s}^\circ$ and $g_\varphi(\bar{y}_0) > 0$. Hence for any $\varphi > c$, the open set $D_\varphi$ given by $D_\varphi := \{y \in \mathbb{R}^N | g_\varphi(y) > 0\} \cap P_{a,s}^\circ$ is non-empty and satisfies $V_N(D_\varphi) > 0$. Then we further show that

$$
\lim_{\varphi \to +\infty} V_N(D_\varphi) = +\infty. \tag{4.27}
$$

Assume there exists a sequence $\{\varphi_k\}_{k \geq 1}$ such that $c < \varphi_k < \varphi_{k+1}$, $\varphi_k \to +\infty$ as $k \to +\infty$, and $0 < V_N(D_{\varphi_k}) \leq c_1$ for some $c_1 > 0$ and all $k \geq 1$. Since $D_{\varphi_k}$ is a polyhedron possessing non-empty interior $D_{\varphi_k}^{\interior}$ and satisfying $0 < V_N(D_{\varphi_k}^{\interior}) \leq c_1$, we can deduce that $D_{\varphi_k}$ is the polytope for each $k \geq 1$. Thus, $\phi_a$ attains its maximum $m_k := \sup_{x \in D_{\varphi_k}} \phi_a(x)$ on the compact set $D_{\varphi_k}$. Clearly, $D_{\varphi_{k+1}} \subseteq D_{\varphi_k}$ and $m_{k+1} \leq m_k$ for $k \geq 1$. Because $\varphi_k \to +\infty$ as $k \to +\infty$, we obtain $\varphi_k \to m_1$ and $D_{\varphi_k} \neq \emptyset$ for some integer $k_0 \geq 1$. For any $y \in D_{\varphi_{k_0}}$, we have $\phi_a(y) > \varphi_{k_0}$, which contradicts $\phi_a(y) \leq m_{k_0} \leq m_1 < \varphi_{k_0}$. As a result, $\lim_{\varphi \to +\infty} V_N(D_\varphi) = +\infty$.

According to the arguments above, if $\phi_a$ cannot attain its maximum on the polyhedron $P_{a,s}$, then for any $r \in (0, 1)$,

$$
\int_{P_{a,s}} e^{(\ln \frac{1}{r})\phi_a(y)} dy \geq \int_{D_\varphi} e^{(\ln \frac{1}{r})\phi_a(y)} dy \geq e^{(\ln \frac{1}{r})\varphi} V_N(D_\varphi) \to +\infty
$$

as $\varphi \to +\infty$. This contradicts (4.25). Hence, we conclude the linear function $\phi_a$ must attain its maximum $m_{a,s} := \max_{x \in P_{a,s}} \phi_a(x)$ on the polyhedron $P_{a,s}$.

The set $M_{a,s} := \{x \in P_{a,s} | \phi_a(x) = m_{a,s}\}$ is a face of polyhedron $P_{a,s}$, which is also a polyhedron in $\mathbb{R}^N$. If $M_{a,s}$ is an unbounded polyhedron, there exists a direction $q \in \mathbb{R}^N$ such that $\{p + tq | t \geq 0\} \subset M_{a,s} \subset P_{a,s}$ for any $t \geq 0$ and any $p \in M_{a,s}$. It follows from [71, Theorem 8.3] that $q$ is also the direction of polyhedron $P_{a,s}$. For any $x_0 \in M_{a,s}$, we have $\phi_a(x_0 + tq) = \phi_a(x_0) = m_{a,s}$ for all $t \geq 0$, which implies $\phi_a(q) = 0$. Since $a - b - 1 \neq 0$, we let $B_R^N(x_0 + tq) = \{y \in \mathbb{R}^N | |y - x_0 - tq|_N < R\}$ be the $N$-dimensional ball whose centre is $x_0 + tq$ with radius $R = \frac{m_{a,s} + 1}{\sum_{j=1}^{N} |a_j - b_j|}$ $> 0$. It follows that $B_R^N(x_0) \cap P_{a,s}^\circ \neq \emptyset$ and

$$
\phi_a(y) \geq m_{a,s} - |m_{a,s}| - 1 \quad \text{for all} \quad y \in B_R^N(x_0) \cap P_{a,s}. \tag{4.28}
$$
Meanwhile, for any $y \in B_N^R(x_0) \cap P_{a,s}$, we have $y + tq \in P_{a,s}$ and $y + tq \in B_N^R(x_0 + tq)$ for any $t \geq 0$. That means

$$B_N^R(x_0) \cap P_{a,s} + \{tq\} \subset B_N^R(x_0 + tq) \cap P_{a,s}$$

and

$$V_N(B_N^R(x_0 + tq) \cap P_{a,s}) \geq V_N(B_N^R(x_0) \cap P_{a,s}) > 0$$

holds for any $t \geq 0$. Denoting by $\Gamma_R := \bigcup_{t \geq 0} B_N^R(x_0 + tq)$, we can verify that $\Gamma_R$ is a convex set and $V_N(\Gamma_R \cap P_{a,s}) = +\infty$. Furthermore, by using $\phi_a(q) = 0$ and (4.28), we have

$$\phi_a(y) \geq m_{a,s} - |m_{a,s}| - 1 \quad \text{for all} \quad y \in \Gamma_R \cap P_{a,s}.$$

Thus, we get

$$\int_{P_{a,s}} e^{(\ln \frac{1}{r})\phi_a(y)} dy \geq \int_{\Gamma_R \cap P_{a,s}} e^{(\ln \frac{1}{r})\phi_a(y)} dy \geq e^{(\ln \frac{1}{r})(m_{a,s} - |m_{a,s}| - 1)} V_N(\Gamma_R \cap P_{a,s}) = +\infty$$

holds for any $0 < r < 1$, which also contradicts (4.25). Consequently, $M_{a,s}$ is a bounded $d_{a,s}$-face of $P_{a,s}$ with $0 \leq d_{a,s} \leq N - 1$.

Lemma 4.43 derives the following result of $J_{a,s}(r)$ for $a - b - 1 \neq 0$.

**Proposition 4.5.** Suppose that $(a, s) \in \mathcal{G}^0$ with $a - b - 1 \neq 0$. For the integral $J_{a,s}(r)$ defined by (4.18) with $0 < r < 1$, we have

$$J_{a,s}(r) = \left(\ln \frac{1}{r}\right)^N \frac{C_0}{r^s + m_{a,s}} \int_0^{+\infty} e^{-(\ln \frac{1}{r})x_N} \left(\int_{\mathbb{R}^{N-1}} 1_{\mathcal{H}(P_{a,s})}(x)dx_1 \ldots dx_{N-1}\right) dx_N, \quad (4.29)$$

where $C_0 > 0$ is a positive constant only depending on the multi-indexes $a$ and $b$, $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ is a non-degenerate affine transform such that $\mathcal{H}(M_{a,s}) = \mathcal{H}(P_{a,s}) \cap \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N = 0\}$ and $\mathcal{H}(P_{a,s}) \subset \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N \geq 0\}$. Furthermore, we can see the dimensions of $M_{a,s}$ and $P_{a,s}$ are invariant under the non-degenerate affine transform $\mathcal{H}$.

**Proof.** Without loss of generality, we may suppose that $a_N - b_N - 1 \neq 0$. By Lemma 4.43, $\phi_a$ attains its maximum $m_{a,s}$ on the bounded face $M_{a,s}$ of polyhedron $P_{a,s}$. Then for any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we denote by

$$\mathcal{H}(x) := (x_1, \ldots, x_{N-1}, m_{a,s} - \phi_a(x)).$$

Clearly, $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ is a non-degenerate affine transform such that $\mathcal{H}(M_{a,s}) = \mathcal{H}(P_{a,s}) \cap \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N = 0\}$ and $\mathcal{H}(P_{a,s}) \subset \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N \geq 0\}$. Moreover, $\dim M_{a,s} = \dim \mathcal{H}(M_{a,s})$ and $\dim P_{a,s} = \dim \mathcal{H}(P_{a,s})$. Thus, the Fubini’s theorem derives

$$J_{a,s}(r) = \left(\ln \frac{1}{r}\right)^N \frac{C_0}{r^s + m_{a,s}} \int_{P_{a,s}} e^{(\ln \frac{1}{r})\phi_a(y)} dy = \frac{(\ln \frac{1}{r})^N}{r^s [1 + b_N - a_N]} \int_{\mathcal{H}(P_{a,s})} e^{-(\ln \frac{1}{r})(x_N - m_{a,s})} dx$$

$$= \frac{(\ln \frac{1}{r})^N}{r^s [1 + b_N - a_N]} \int_{\mathbb{R}^N} 1_{\mathcal{H}(P_{a,s})}(x)e^{-(\ln \frac{1}{r})(x_N - m_{a,s})} dx$$

$$= \frac{(\ln \frac{1}{r})^N}{r^s + m_{a,s} [1 + b_N - a_N]} \int_0^{+\infty} e^{-(\ln \frac{1}{r})x_N} \left(\int_{\mathbb{R}^{N-1}} 1_{\mathcal{H}(P_{a,s})}(x)dx_1 \ldots dx_{N-1}\right) dx_N.$$
As a consequence of Proposition 4.5, in the case of \( a - b = 1 \neq 0 \), the asymptotics of \( J_{a,b}(r) \) can be achieved by estimating

\[
S(x_N) := \int_{\mathbb{R}^{N-1}} 1_{H(P_{a,b})}(x) dx_1 \cdots dx_{N-1}.
\]

To proceed with our estimation, we decompose \( \mathbb{R}^N \) into \( \mathbb{R}^N = E \oplus E^\perp \), where

\[
E := \text{span}\{e_j | 1 \leq j \leq N - 1\}.
\]

We also let \( E_+ := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N > 0\} \). Let \( E_d \subset E \) be a \( d \)-dimensional linear subspace such that \( 1 \leq d \leq N - 1 \). For any \( x \in E_d \) and \( r > 0 \), we denote by

\[
B^d_\varepsilon(x) := \{y \in E_d | |y - x|_N < r\}
\]

the \( d \)-dimensional ball in \( E_d \). Especially, if \( d = 0 \), we agree with \( E_0 = \{0\} \) and in this case the 0-dimensional ball is \( \{0\} \) which is still denoted by \( B^0_\varepsilon(0) \). Additionally, \( B^N_\varepsilon(x) := \{y \in \mathbb{R}^N | |y - x|_N < r\} \) denotes the \( N \)-dimensional Euclidean ball in \( \mathbb{R}^N \).

Consider the quadruple \((E_d,x_0,u,\varepsilon)\), where \( E_d \subset E \) is a \( d \)-dimensional linear subspace, \( \varepsilon > 0 \) is some positive constant, and \( x_0 \in E, u \in S^{N-1} \cap E_+ \) are some points. Then for any \( r > 0 \), we define the corresponding set which varies along \( t \geq 0 \), i.e.

\[
F(t) := B^d_\varepsilon(0) + B^{N-1}_{\varepsilon t}(0) + \{x_0 + t\eta u\},
\]

where \( \eta = \langle u, e_N \rangle^{-1} \). Clearly, (4.31) gives

\[
F(t) \subset \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N = t\}
\]

for all \( t \geq 0 \), which means \( F(t_1) \cap F(t_2) = \emptyset \) if \( t_1 \neq t_2 \). In particular, \( F(0) = \{x_0\} + B^d_\varepsilon(0) \). Meanwhile, (4.31) also indicates that \( F(t) \) is convex and \( V_{N-1}(F(t)) > 0 \) for all \( t > 0 \). Furthermore, we have

**Lemma 4.4.** For any \( t \geq 0 \), the set \( F(t) \) given by (4.31) satisfies

\[
V_{N-1}(F(t)) = c_d(\varepsilon t)^{N-1-d} + c_{d-1}(\varepsilon t)^{N-d} + \cdots + c_0(\varepsilon t)^{N-1},
\]

where \( c_j > 0 \) for \( j = 0, \ldots, d \).

**Proof.** The transform invariance of Lebesgue measure gives that

\[
V_{N-1}(F(t)) = V_{N-1}\left(\overline{B^d_\varepsilon(0) + B^{N-1}_{\varepsilon t}(0)}\right).
\]

By Steiner’s formula (see [48, Theorem 3.10]), we get

\[
V_{N-1}\left(\overline{B^d_\varepsilon(0) + B^{N-1}_{\varepsilon t}(0)}\right) = \sum_{i=0}^{N-1} \frac{(N-1)!(\varepsilon t)^i}{(N-1-i)!i!} V\left(B^d_\varepsilon(0), \ldots, B^d_\varepsilon(0), B^{N-1}_{\varepsilon t}(0), \ldots, B^{N-1}_{\varepsilon t}(0)\right),
\]

(4.34)
where

\[ V \left( B_{t}^d(0), \ldots, B_{t}^d(0) \right) \] \[ V \left( B_{1}^{N-1}(0), \ldots, B_{1}^{N-1}(0) \right), \]

abbreviated as \( V^i \), is called the quermassintegral or \((N - 1)\)-dimensional mixed volume. According to [48, Remark 3.15], we deduce that if \( N - 1 - i > d \), then \( V^i = 0 \) while \( V^i > 0 \) if \( N - 1 - i \leq d \), since there are at most \( d \) segments in \( B_{t}^d(0) \) with linearly independent directions. Hence, we conclude from (4.33) and (4.34) that

\[ V_{N-1}(F(t)) = c_d(\varepsilon t)^{N-1-d} + c_{d-1}(\varepsilon t)^{N-d} + \cdots + c_0(\varepsilon t)^{N-1}, \]

where \( c_j > 0 \) for \( j = 0, \ldots, d \).

Then for any given \( \delta > 0 \), we define a tube by

\[ T(\delta) := \bigcup_{0 \leq \varepsilon \leq \delta} F(t). \tag{4.35} \]

The tube \( T(\delta) \) in \( \mathbb{R}^N \) admits the following property:

**Lemma 4.5.** For any \( \delta > 0 \), \( T(\delta) = \text{conv}(F(\delta) \cup F(0)) \).

**Proof.** For any \( x \in T(\delta) \), by (4.31) and (4.35) we know there exists a \( t \in [0, \delta] \) such that

\[ x = a + t\varepsilon b + x_0 + t\eta u \in F(t), \]

where \( a \in B_{t}^d(0) \) and \( b \in B_{1}^{N-1}(0) \). Since \( 0 \leq t \leq \delta \), we have

\[ x = \left( 1 - \frac{t}{\delta} \right) (a + x_0) + \frac{t}{\delta}(a + \delta \varepsilon b + x_0 + \delta \eta u) \in \text{conv}(F(\delta) \cup F(0)), \]

which gives \( T(\delta) \subset \text{conv}(F(\delta) \cup F(0)) \).

We next prove that \( T(\delta) \) is convex. For \( x_1, x_2 \in T(\delta) \), there exist \( t_1, t_2 \in [0, \delta] \), \( a_1, a_2 \in B_{t}^d(0) \) and \( b_1, b_2 \in B_{1}^{N-1}(0) \) such that

\[ x_i = a_i + t_i \varepsilon b_i + x_0 + t_i \eta u \in F(t_i) \quad \text{for} \quad i = 1, 2. \]

Thus for any \( \lambda \in [0, 1] \),

\[ \lambda x_1 + (1 - \lambda)x_2 = [\lambda a_1 + (1 - \lambda)a_2] + [\lambda t_1 \varepsilon b_1 + (1 - \lambda)t_2 \varepsilon b_2] + x_0 + [\lambda t_1 + (1 - \lambda)t_2] \eta u. \]

It follows that

\[ \lambda a_1 + (1 - \lambda)a_2 \in B_{t}^d(0) \quad \text{and} \quad \lambda t_1 \varepsilon b_1 + (1 - \lambda)t_2 \varepsilon b_2 \in B_{1}^{N-1+(1-\lambda)t_2}(0), \]

which yields \( \lambda x_1 + (1 - \lambda)x_2 \in F(\lambda t_1 + (1 - \lambda)t_2) \subset T(\delta) \), and thus \( T(\delta) \) is convex. Consequently,

\[ \text{conv}(F(\delta) \cup F(0)) \subset \text{conv}(T(\delta)) = T(\delta) \subset \text{conv}(F(\delta) \cup F(0)). \]

\[ \square \]
On the other hand, the plane section of polyhedron in $\mathbb{R}^N$ satisfies the following property.

**Lemma 4.6.** Suppose $P \subset \mathbb{R}^N$ is a polyhedron with $V_N(P) > 0$, and $f(x)$ is an affine function. If the plane section $Q := \{x \in \mathbb{R}^N | f(x) = 0\} \cap P \neq \emptyset$ is not a face of $P$, then $V_{N-1}(Q) > 0$.

**Proof.** By $V_N(P) > 0$ we obtain that $P$ has non-empty $N$-dimensional relative interior $\text{relint}(P)$ satisfying $\text{relint}(P) = P^\circ$. Hence, $V_{N-1}(Q) > 0$ amounts to proving that $\{x \in \mathbb{R}^N | f(x) = 0\} \cap \text{relint}(P) \neq \emptyset$.

Suppose that $\{x \in \mathbb{R}^N | f(x) = 0\} \cap \text{relint}(P) = \emptyset$. Since $f$ is continuous and $\text{relint}(P)$ is connected, we obtain either $P \subset \{x \in \mathbb{R}^N | f(x) \leq 0\}$ or $P \subset \{x \in \mathbb{R}^N | f(x) \geq 0\}$. In both cases, $Q = \{x \in \mathbb{R}^N | f(x) = 0\} \cap P$ is the face of $P$, which leads a contradiction. Consequently, we have $V_{N-1}(Q) > 0$. \[\square\]

Let us return to the estimates of integral $S(x_N)$. For a given set $M \subset \mathbb{R}^N$, we introduce the $(N-1)$-dimensional volume of the section of $M$, i.e.

$$X_M(t) := \int_{\mathbb{R}^{N-1}} 1_M(y_1, \ldots, y_{N-1}, t)dy_1 \cdots dy_{N-1} \quad \text{for } t \geq 0. \quad (4.36)$$

Combining (4.30) and (4.36), we have

$$S(t) = \int_{\mathbb{R}^{N-1}} 1_{H(a,s)}(x_1, \ldots, x_{N-1}, t)dx_1 \cdots dx_{N-1} = X_{H(a,s)}(t). \quad (4.37)$$

Recalling $F(t_1) \cap F(t_2) = \emptyset$ if $t_1 \neq t_2$, (4.35) and (4.36) give that

$$X_{T(\delta)}(t) = \int_{\mathbb{R}^{N-1}} 1_{T(\delta)}(y_1, \ldots, y_{N-1}, t)dy_1 \cdots dy_{N-1}$$

$$= \int_{\mathbb{R}^{N-1}} 1_{F(t)}(y_1, \ldots, y_{N-1}, t)dy_1 \cdots dy_{N-1} = V_{N-1}(F(t)). \quad (4.38)$$

Due to Lemma 4.4-Lemma 4.6, we have the following estimates of $S(t)$.

**Lemma 4.7.** For $(a, s) \in \mathcal{G}^o$ with $a - b - 1 \neq 0$, let $\mathcal{H}(P_{a,s})$ and $\mathcal{H}(M_{a,s})$ be the sets given in Proposition 4.5. Denote by $d_{a,s} = \dim M_{a,s}$. Then there exists a $\delta > 0$ such that

$$p_2(t) \leq S(t) \leq p_1(t) \quad \text{for all } 0 \leq t \leq \delta,$$

where $p_i(t) = v_{i,d_{a,s}} t^{N-1-d_{a,s}} + \cdots + v_{i,0} t^{N-1}$ is the polynomial with $v_{i,j} > 0$ for $i = 1, 2$ and $j = 0, \ldots, d_{a,s}$.

**Proof.** It follows from Lemma 4.3 and Proposition 4.5 that $\mathcal{H}(M_{a,s})$ is a bounded $d_{a,s}$-face of the polyhedron $\mathcal{H}(P_{a,s})$, which implies $\mathcal{H}(M_{a,s})$ is a polytope. Let us choose a $\delta > 0$ such that the plane section $\mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | x_N = \delta\}$ possesses positive $(N-1)$-dimensional volume. Observe that the polyhedron $\mathcal{H}(P_{a,s}) \subset \mathcal{E}_+$ has finite vertices in $\mathcal{E}_+$ (see [8, Corollary 2.1]). If $\mathcal{H}(P_{a,s})$ has $m$ vertices in $\mathcal{E}_+$, we denote by $0 < z_1 \leq z_2 \leq \cdots \leq z_m$
the orthogonal projections on $E^\perp$ of these vertexes. Then, we take a $\delta \in (0, z_1)$ and let $g(x) := x_N - \delta$. Obviously, the non-empty set $\mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | g(x) = 0\}$ is not the face of $\mathcal{H}(P_{a,s})$ in this case, since $g(x) < 0$ for $x \in \mathcal{H}(M_{a,s}) \subset \mathcal{H}(P_{a,s})$ and $g(y) > 0$ for $y := (y_1, \ldots, y_{N-1}, z_1)$ being the vertex of $\mathcal{H}(P_{a,s})$. On the other hand, if all the vertexes of $\mathcal{H}(P_{a,s})$ lie in $E$, we can find a point $w = (w_1, \ldots, w_{N-1}, w_N) \in \text{relint}(\mathcal{H}(P_{a,s})) \subset E_+$ with $w_N > 0$ due to $V_N(\mathcal{H}(P_{a,s})) > 0$. By taking $\delta \in (0, w_N)$ and using the similar arguments as above, we deduce the non-empty set $\mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | x_N = \delta\}$ is not the face of $\mathcal{H}(P_{a,s})$. Hence in both cases, Lemma 4.6 indicates that the plane section $\mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | x_N = \delta\}$ possesses positive $(N - 1)$-dimensional volume.

According to the boundedness of $\mathcal{H}(M_{a,s})$ and [71, Corollary 8.4.1], the plane section $Q(t) := \mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | x_N = t\}$ is bounded for any $t \in [0, \delta]$. Therefore

$$P(\delta) := \mathcal{H}(P_{a,s}) \cap \{x \in \mathbb{R}^N | x_N \leq \delta\} = \bigcup_{0 \leq t \leq \delta} Q(t)$$

is a bounded polyhedron (i.e. polytope) with its vertexes lying in $Q(0) \cup Q(\delta)$, because $\mathcal{H}(P_{a,s})$ has no vertexes in $\{x \in \mathbb{R}^N | 0 < x_N < \delta\}$ by construction. From [48, Theorem 1.21] and [8, Theorem 2.3] we have $P(\delta) = \text{conv}(Q(0) \cup Q(\delta))$. Moreover, $Q(\delta)$ is an $(N - 1)$-face of $P(\delta)$ and $V_{N-1}(Q(\delta)) > 0$.

Since $\mathcal{H}(M_{a,s})$ has non-empty $d_{a,s}$-dimensional relative interior $\text{relint}(\mathcal{H}(M_{a,s}))$, for any $x_0 \in \text{relint}(\mathcal{H}(M_{a,s}))$, there is a unique $d_{a,s}$-dimensional linear subspace $E_{d_{a,s}} \subset E$ and a $d_{a,s}$-dimensional ball $B_{d_{a,s}}(0) \subset E_{d_{a,s}}$ such that

$$\mathcal{H}(M_{a,s}) \subset \{x_0\} + \overline{B_{d_{a,s}}(0)}.$$ 

In particular, in the case of $d_{a,s} = 0$, $\mathcal{H}(M_{a,s}) = \{x_0\} \subset E$ and $B_{d_{a,s}}^1(0) = \{0\}$.

By the definition of $F(t)$ in (4.31), we can find a quadruple $(E_{d_{a,s}}, x_0, u_1, \varepsilon_1)$ with $u_1 \in S^{N-1} \cap E_+$ and sufficiently large $\varepsilon_1 > 0$ such that the corresponding set $F_1(t)$ satisfies $Q(\delta) \subset F_1(\delta)$ and $Q(0) \subset F_1(0)$. Here $F_1(t)$ is given by

$$F_1(t) := \overline{B_{d_{a,s}}(0)} + B_{N-1}^{u_1}(0) + \{x_0 + t\eta_1 u_1\}$$

for $0 \leq t \leq \delta$ with $\eta_1 = \langle u_1, e_N \rangle^{-1}$. According to Lemma 4.5, we obtain

$$P(\delta) = \text{conv}(Q(0) \cup Q(\delta)) \subset \text{conv}(F_1(0) \cup F_1(\delta)) = \bigcup_{0 \leq t \leq \delta} F_1(t) := T_1(\delta),$$

and

$$Q(t) \subset F_1(t) \quad \text{for all} \quad 0 \leq t \leq \delta. \quad (4.39)$$

Furthermore, there exists a $u_2 \in S^{N-1} \cap E_+$ such that $x_0 + \delta \eta_2 u_2 \in \text{relint}(Q(\delta))$ with $\eta_2 = \langle u_2, e_N \rangle^{-1}$, since $Q(\delta)$ is an $(N - 1)$-dimensional non-empty polytope. It follows that $Q(\delta) - \{x_0 + \delta \eta_2 u_2\}$ is a bounded closed subset in $E$, and $0 \in \text{relint}(Q(\delta) - \{x_0 + \delta \eta_2 u_2\})$. Thus, there is an $(N - 1)$-dimensional ball $B_{N-1}^{r_3}(0) \subset E$ such that

$$\{x_0 + \delta \eta_2 u_2\} + B_{N-1}^{r_3}(0) \subset Q(\delta).$$
Similarly, we can also find a $d_{a,s}$-dimensional ball $B_{r_2}^{d_{a,s}}(0) \subset E_{d_{a,s}}$ such that $0 < r_2 < r_3$ and

$$
\{x_0\} + B_{r_2}^{d_{a,s}}(0) \subset \mathcal{H}(M_{a,s}).
$$

Hence, for the quadruple $(E_{d_{a,s}}, x_0, u_2, \varepsilon_2)$ with $\varepsilon_2 = \frac{r_3 - r_2}{\delta} > 0$, the corresponding set

$$
F_2(t) := B_{r_2}^{d_{a,s}}(0) + B_{r_2}^{-1}(0) + \{x_0 + t\eta_2u_2\}
$$

satisfies

$$
F_2(\delta) = \{x_0 + \delta\eta_2u_2\} + B_{r_2}^{d_{a,s}}(0) + B_{r_2}^{-1}(0)
\subset \{x_0 + \delta\eta_2u_2\} + B_{r_2+\delta}^{-1}(0) = \{x_0 + \delta\eta_2u_2\} + B_{r_3}^{-1}(0) \subset Q(\delta)
$$

and

$$
F_2(0) = \{x_0\} + B_{r_2}^{d_{a,s}}(0) \subset \mathcal{H}(M_{a,s}).
$$

Therefore, we obtain

$$
P(\delta) = \text{conv}(Q(0) \cup Q(\delta)) \supset \text{conv}(F_2(0) \cup F_2(\delta)) = \bigcup_{0 \leq t \leq \delta} F_2(t) := T_2(\delta)
$$

and

$$
F_2(t) \subset Q(t) \quad \text{for all } 0 \leq t \leq \delta.
$$

As a result of (4.39) and (4.40), we have

$$
X_{T_2(\delta)}(t) \leq S(t) \leq X_{T_1(\delta)}(t) \quad \text{for all } 0 \leq t \leq \delta.
$$

Consequently, it follows from Lemma 4.4, (4.38) and (4.41) that

$$
p_2(t) \leq S(t) \leq p_1(t) \quad \text{for all } 0 \leq t \leq \delta,
$$

where $p_i(t) = v_{i,d_{a,s}}t^{N-1-d_{a,s}} + \cdots + v_{i,0}t^{N-1}$ is the polynomial with $v_{i,j} > 0$ for $i = 1, 2$ and $j = 0, \ldots, d_{a,s}$.

**Lemma 4.8.** For $(a, s) \in \mathcal{S}^0$ with $a - b - 1 \neq 0$, let $\mathcal{H}(P_{a,s})$ and $\mathcal{H}(M_{a,s})$ be the sets given in Proposition 4.5. Suppose that $\mathcal{H}(P_{a,s})$ is unbounded and $\delta > 0$ is the positive constant given in Lemma 4.7, then there exists a positive constant $C_1$ such that

$$
S(t) \leq C_1 t^{N-1} \quad \text{for all } t \geq \delta.
$$

**Proof.** Denoted by $C_{\mathcal{H}(P_{a,s})} := \{y \in \mathbb{R}^N | x + \gamma y \in \mathcal{H}(P_{a,s}), \forall x \in \mathcal{H}(P_{a,s}), \forall \gamma \geq 0\}$ the recession cone consisting of the origin in $\mathbb{R}^N$ and all directions of $\mathcal{H}(P_{a,s})$. The unboundedness of $\mathcal{H}(P_{a,s})$ indicates that $C_{\mathcal{H}(P_{a,s})}$ contains at least a non-zero vector. In addition, it follows from [71, Theorem 8.2] that $C_{\mathcal{H}(P_{a,s})}$ is closed. Owing to the Minkowski-Weyl’s decomposition theorem of polyhedron (see [73, Section 8]), we have

$$
\mathcal{H}(P_{a,s}) = P_1 + C_{\mathcal{H}(P_{a,s})},
$$

(4.42)
where $P_1$ is a polytope.

Since the plane section $Q(t) = \mathcal{H}(P_{a,s}) \cap \{ x \in \mathbb{R}^N | x_N = t \}$ is bounded for any $t \in [0, \delta]$ and $\mathcal{H}(P_{a,s}) \subset B_{+}$, we can deduce that for any non-zero vector $z = (z_1, \ldots, z_N) \in C_{\mathcal{H}(P_{a,s})}$, the last coordinate $z_N > 0$. Then, let $f(z) := \langle z, e_N \rangle$ be the continuous function defined on the compact set $C_{\mathcal{H}(P_{a,s})} \cap S^{N-1}$. It follows that $f(z) \geq c_0 > 0$ holds for all $z \in C_{\mathcal{H}(P_{a,s})} \cap S^{N-1}$ and some $0 < c_0 < 1$. Thus, $\frac{r_N}{|z|} \geq c_0 > 0$ holds for any non-zero vector $z \in C_{\mathcal{H}(P_{a,s})}$.

Observing that $P_1$ is a polytope, the boundedness of $P_1$ allows us to find an $N$-dimensional ball $B_{r_4}^N(0)$ such that $P_1 \subset B_{r_4}^N(0)$. Therefore, (4.42) gives

$$\mathcal{H}(P_{a,s}) = P_1 + C_{\mathcal{H}(P_{a,s})} \subset B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}. \tag{4.43}$$

Combining (4.36), (4.37) and (4.43), we obtain

$$S(t) = X_{\mathcal{H}(P_{a,s})}(t) = V_{N-1}(\mathcal{H}(P_{a,s}) \cap \{ x \in \mathbb{R}^N | x_N = t \})$$

$$\leq V_{N-1}(B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}) \cap \{ x \in \mathbb{R}^N | x_N = t \}). \tag{4.44}$$

We next estimate the upper bound of $V_{N-1}(B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}) \cap \{ x \in \mathbb{R}^N | x_N = t \}$. For any $x \in B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}$, we have $x = y + z$, where $y \in B_{r_4}^N(0)$ and $z \in C_{\mathcal{H}(P_{a,s})}$. Additionally, denoting by $x = (x_1, \ldots, x_N) = (y_1 + z_1, \ldots, y_N + z_N)$, we know $|y| \leq r_4$ and $\frac{r_N}{|z|} \geq c_0 > 0$ if $|z| \neq 0$. Setting $x_N = y_N + z_N = t > 0$, the upper bound will be estimated in following two cases:

- **Case 1:** $|z|_N \neq 0$. It follows that

$$0 < c_0 |z|_N \leq z_N = t - y_N \leq t + r_4, \tag{4.45}$$

which means

$$z_1^2 + \cdots + z_{N-1}^2 \leq \frac{1 - c_0^2}{c_0^2} z_N^2. \tag{4.46}$$

Combining (4.45) and (4.46), we obtain $|z_j| \leq \frac{1}{c_0^2} z_N \leq \frac{t + r_4}{c_0}$ and $|x_j| \leq |y_j| + |z_j| \leq \frac{t + r_4}{c_0} + r_4$ for $1 \leq j \leq N - 1$. Thus, for $t > 0$ we have

$$V_{N-1}(B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}) \cap \{ x \in \mathbb{R}^N | x_N = t \}) \leq 2^{N-1} \left( \frac{t + r_4}{c_0} + r_4 \right)^{N-1}. \tag{4.47}$$

- **Case 2:** $|z|_N = 0$. In this case, $y_N = t > 0$ and $|x_j| = |y_j| \leq r_4$ for $1 \leq j \leq N - 1$. Hence, we also have

$$V_{N-1}(B_{r_4}^N(0) + C_{\mathcal{H}(P_{a,s})}) \cap \{ x \in \mathbb{R}^N | x_N = t \}) \leq (2r_4)^{N-1} \leq 2^{N-1} \left( \frac{t + r_4}{c_0} + r_4 \right)^{N-1}. \tag{4.48}$$

As a consequence of (4.44), (4.47) and (4.48), there is a positive constant $C_1 > 0$ such that

$$S(t) \leq C_1 t^{N-1} \quad \text{for all } t \geq \delta. \quad \Box$$

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According to Proposition 4.5, Lemma 4.7 and Lemma 4.8, we can now give the asymptotic behaviour of integral \( J_{a,s}(r) \) for \( a - b - 1 \neq 0 \).

**Proposition 4.6.** For \( (a,s) \in \mathbb{G}^o \) with \( a - b - 1 \neq 0 \), let \( \mathcal{H}(P_{a,s}) \) and \( \mathcal{H}(M_{a,s}) \) be the sets given in Proposition 4.5. Then the integral \( J_{a,s}(r) \) given by (4.29) admits the following asymptotic behaviour:

\[
J_{a,s}(r) \approx \frac{|\ln r|^{d_{a,s}}}{r^{s+m_{a,s}}} \quad \text{as} \quad r \to 0^+,
\]

(4.49)

where \( m_{a,s} \) and \( d_{a,s} \) are the constants defined in Lemma 4.3 above.

*Proof.* Lemma 4.7 gives

\[
p_2(t) \leq S(t) \leq p_1(t) \quad \text{for all} \quad 0 \leq t \leq \delta,
\]

where \( \delta > 0 \) is a positive constant, and \( p_i(t) = v_i,d_{a,s}t^{N-1-d_{a,s}} + \cdots + v_i,0t^{N-1} \) with \( v_i,j > 0 \) for \( i = 1,2 \) and \( j = 0, \ldots, d_{a,s} \). Then, by (4.29) and (4.30) we have for \( 0 < r < 1 \),

\[
J_{a,s}(r) = \left( \ln \frac{1}{r} \right)^N \frac{C_0}{r^{s+m_{a,s}}} \int_0^{\delta} e^{-\left(\ln \frac{1}{r}\right)t} S(t) dt + \left( \ln \frac{1}{r} \right)^N \frac{C_0}{r^{s+m_{a,s}}} \int_{\delta}^{+\infty} e^{-\left(\ln \frac{1}{r}\right)t} S(t) dt
\]

\[
:= J_{a,s,1}(r) + J_{a,s,2}(r),
\]

(4.50)

where \( \delta > 0 \) is the positive constant given in Lemma 4.7.

A direct calculation yields that for \( i = 1,2 \) and \( 0 < r < 1 \),

\[
\int_0^{\delta} e^{-\left(\ln \frac{1}{r}\right)t} p_i(t) dt = \sum_{k=N-1-d_{a,s}}^{N-1} v_{i,N-1-k} \int_0^{\delta} e^{-\left(\ln \frac{1}{r}\right)t} t \cdot \cdot \cdot t_{j=0} dt
\]

\[
= \sum_{k=N-1-d_{a,s}}^{N-1} \frac{v_{i,N-1-k}}{(\ln \frac{1}{r})^{k+1}} \int_0^{\delta} \frac{1}{u} e^{-u} u^k du
\]

\[
\approx \frac{1}{(\ln \frac{1}{r})^{N-d_{a,s}}} \quad \text{as} \quad r \to 0^+,
\]

(4.51)

which gives

\[
J_{a,s,1}(r) = \left( \ln \frac{1}{r} \right)^N \frac{C_0}{r^{s+m_{a,s}}} \int_0^{\delta} e^{-\left(\ln \frac{1}{r}\right)t} S(t) dt \approx \frac{|\ln r|^{d_{a,s}}}{r^{s+m_{a,s}}} \quad \text{as} \quad r \to 0^+.
\]

(4.52)

Furthermore, by Lemma 4.8 we have \( S(t) \leq C_1 t^{N-1} \) for all \( t \geq \delta \). Hence, for \( 0 < r < 1 \)

\[
0 \leq \left( \ln \frac{1}{r} \right)^{N-d_{a,s}} \int_{\delta}^{+\infty} e^{-\left(\ln \frac{1}{r}\right)t} S(t) dt \leq C_1 \left( \ln \frac{1}{r} \right)^{N-d_{a,s}} \int_{\delta}^{+\infty} e^{-\left(\ln \frac{1}{r}\right)t} t^{N-1} dt
\]

\[
= C_1 \left( \ln \frac{1}{r} \right)^{-d_{a,s}} \int_{\delta \ln \frac{1}{r}}^{+\infty} e^{-u} u^{N-1} du \to 0 \quad \text{as} \quad r \to 0^+.
\]

(4.53)
It follows from (4.53) that
\[
J_{a,s,2}(r) = \left( \ln \frac{1}{r} \right)^N \frac{C_0}{r^{s+m_{a,s}}} \int_{\delta}^{+\infty} e^{-(\ln \frac{1}{r})^t} S(t) dt = o \left( \frac{|\ln r|^{d_{a,s}}}{r^{s+m_{a,s}}} \right) \quad \text{as } r \to 0^+.
\] (4.54)

Combining (4.50), (4.52) and (4.54), we have
\[
J_{a,s}(r) = \left( \ln \frac{1}{r} \right)^N \frac{C_0}{r^{s+m_{a,s}}} \int_{0}^{+\infty} e^{-(\ln \frac{1}{r})^t} S(t) dt \approx \frac{|\ln r|^{d_{a,s}}}{r^{s+m_{a,s}}} \quad \text{as } r \to 0^+.
\]

\[\square\]

Finally, let us summarize the asymptotic results in this part.

**Proposition 4.7.** Let \( \mathcal{G} \) be the collection of finitely many index pairs \((a, s)\) and \( b \) be an \( N \)-dimensional multi-index. Assume that for each index pair \((a, s) \in \mathcal{G}, a = (a_1, \ldots, a_N)\) is an \( N \)-dimensional multi-index and \( s \in [0, +\infty) \) is a non-negative constant. Suppose further that
\[
I(r) = \int_{(0,1)^N} \frac{x^b dx}{\sum_{(a,s) \in \mathcal{G}} x^a r^s} < +\infty \quad \text{for all } 0 < r < 1,
\]
then we have
\[
I(r) \approx \sum_{(a,s) \in \mathcal{G}^0} r^{-s-m_{a,s}} |\ln r|^{d_{a,s}} \approx r^{-\alpha_0} |\ln r|^{d_0} \quad \text{as } r \to 0^+.
\] (4.55)

Here \( \mathcal{G}^0 = \{(a, s) \in \mathcal{G}|V_N(P_{a,s}) > 0\}, P_{a,s} \) is the polyhedron defined by (4.15), \( m_{a,s} = \max_{x \in P_{a,s}} \phi_a(x) \) and \( d_{a,s} = \dim M_{a,s}, \) where \( \phi_a(y) = \langle a - b - 1, y \rangle \) and \( M_{a,s} = \{ x \in P_{a,s}|\phi_a(x) = m_{a,s}\}. \) Additionally, the indexes \( \alpha_0, d_0 \) in (4.55) are given by
\[
\alpha_0 := \max \{ s + m_{a,s}|(a, s) \in \mathcal{G}^0 \} \quad \text{and} \quad d_0 := \max \{ d_{a,s}|(a, s) \in \mathcal{G}^0, s + m_{a,s} = \alpha_0\}. 
\] (4.56)

Furthermore, if for every index pair \((a, s) \in \mathcal{G} \) the corresponding index \( s \) is a non-negative integer, then \( \alpha_0 \in \mathbb{Q}. \)

**Proof.** Clearly, (4.55) is a direct consequence of (4.17), (4.26) and (4.49). Moreover, if for every index pair \((a, s) \in \mathcal{G} \) the corresponding index \( s \) is a non-negative integer, then all \( P_{a,s} \) given by (4.15) are rational polyhedrons (see [73]). For any index pair \((a, s) \in \mathcal{G}, \) since \( \phi_a(y) = \langle a - b - 1, y \rangle \) is a rational transformation from \( \mathbb{R}^N \) to \( \mathbb{R}, \) it follows from [5] that \( \phi_a(P_{a,s}) \) is also a rational polyhedron in \( \mathbb{R}. \) Using Lemma 4.3, we obtain \( \phi_a(P_{a,s}) = \{x \in \mathbb{R}|x \leq m_{a,s}\} \) or \( \phi_a(P_{a,s}) = \{x \in \mathbb{R}|c \leq x \leq m_{a,s}\} \) for some constant \( c \leq m_{a,s}, \) which are both rational polyhedrons. This means \( m_{a,s} \in \mathbb{Q} \) for all \((a, s) \in \mathcal{G} \) and \( \alpha_0 \in \mathbb{Q}. \) \[\square\]

Consequently, we have the following explicit asymptotic estimate of \( J_{\Omega}(r). \)
Proposition 4.8. Let $X = (X_1, X_2, \ldots, X_m)$ be the smooth vector fields defined on $\mathbb{R}^n$ satisfying the assumptions (H.1) and (H.2). Suppose that $\{x_i, \ldots, x_n\}$ is the collection of all degenerate components of $X$, and $\Omega \subset \mathbb{R}^n$ is a bounded open domain containing the origin. Then

$$J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx r^{-Q_0} |\ln r|^{d_0} \quad \text{as} \quad r \to 0^+,$$

(4.57)

where $Q_0 \in \mathbb{Q}$ and $d_0 \in \{0, 1, \ldots, v\}$ with $0 \leq v \leq n - 1$.

Proof. Remark 2.2 indicates $0 \leq v \leq n - 1$. For $v = 0$, it follows from Proposition 2.12 and Proposition 2.14 that $\Lambda(x, r) = f_Q(r)$ with $f_Q > 0$, which gives (4.57). Clearly, Proposition 4.1 yields (4.57) in the case of $v = 1$. For $2 \leq v \leq n - 1$, by Proposition 4.3 we have

$$J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \sum_{j=1}^{l} \int_{(-1,1)^v} \frac{|u^{q_j}|du}{\|w^{p_j,t}r^{d(l)}\}} \approx \sum_{j=1}^{l} \int_{(0,1)^v} \frac{u^{q_j}du}{\|w^{p_j,t}r^{d(l)}\}},

(4.58)

where $l \in \mathbb{N}^+$ is some positive integer, $q_j$ and $p_{j,t}$ are $v$-dimensional multi-indexes. Owing to Proposition 4.7, we obtain for $1 \leq j \leq l$,

$$\int_{(0,1)^v} \frac{u^{q_j}du}{\|w^{p_j,t}r^{d(l)}\}} \approx r^{-\tilde{\alpha}_j} |\ln r|^{\tilde{d}_j} \quad \text{as} \quad r \to 0^+,$$

(4.59)

where $\tilde{\alpha}_j \in \mathbb{Q}$ and $\tilde{d}_j \in \{0, 1, \ldots, v\}$. Hence, combining (4.58) and (4.59) we get

$$J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx r^{-Q_0} |\ln r|^{d_0} \quad \text{as} \quad r \to 0^+,$$

where $Q_0 \in \mathbb{Q}$ and $d_0 \in \{0, 1, \ldots, v\}$ with $0 \leq v \leq n - 1$. \hfill \Box

5. Proofs of main results

5.1. Proof of Theorem 1.1

Proof of Theorem 1.1. By Proposition 3.4, (3.39) and (3.42), we have

$$\int_\Omega h_D(x, x, t)dx \leq A_1 \int_\Omega \frac{dx}{|B_{d_X}(x, \sqrt{t})|} \quad \text{for all} \quad t > 0.$$

(5.1)

According to Lemma 3.1 and Proposition 3.4, for any compact subset $K \subset \Omega$, there are positive constants $\delta_1(K) := \frac{\delta_{\mathbb{R}^n}(K, \partial K)}{A_1 Q}$ and $\delta_2(K)$ with $0 < \delta_2(K) \leq \delta_1(K)$, such that for any $0 < t \leq \delta_2(K)$, we have

$$\int_\Omega h_D(x, x, t)dx \geq \int_K h_D(x, x, t)dx = \int_K h(x, x, t)dx - \int_K E(x, x, t)dx

\geq \int_K \frac{1}{A_1 |B_{d_X}(x, \sqrt{t})|} dx - \int_K \frac{A_1 C_3}{|B_{d_X}(x, \sqrt{t})|} e^{-\frac{\delta_{\mathbb{R}^n}(x, \partial K)}{A_1 t}} dx

\geq \left( \frac{1}{A_1} - A_1 C_3 e^{-\frac{\delta_{\mathbb{R}^n}(x, \partial K)}{A_1 t}} \right) \int_K \frac{dx}{|B_{d_X}(x, \sqrt{t})|} \geq \frac{1}{2A_1} \int_K \frac{dx}{|B_{d_X}(x, \sqrt{t})|}.$$

(5.2)
On the other hand, for $0 < \varepsilon < 1$, we let $K_0 := \delta_0(\Omega)$ be the compact subset in $\mathbb{R}^n$, where $\delta_0(\Omega) := \{\delta_0(x)|x \in \Omega\}$, and $\delta_0(x)$ is the dilation given in assumption (H.1). Then, we show that $K_0 \subset \Omega$ holds for some $\varepsilon_0 \in (0,1)$. Since $\Omega$ is a bounded open set containing the origin, there exist $0 < R_1 < R_2$ such that $B(0,R_1) \subset \Omega \subset B(0,R_2)$, where $B(0,R) := \{x \in \mathbb{R}^n||x| < R\}$ denotes the classical Euclidean ball in $\mathbb{R}^n$. Observing that for any $x = (x_1,\ldots,x_n) \in \delta_0(B(0,R_2))$ with $0 < \varepsilon < 1$, there is a unique $y = (y_1,\ldots,y_n) \in B(0,R_2)$ such that $x = \delta_0(y)$. Thus $|x|^2 = \varepsilon^2 y_1^2 + \cdots + \varepsilon^2 y_n^2 \leq \varepsilon^2(\varepsilon_1^2 + \cdots + \varepsilon_n^2) < \varepsilon^2 R_2^2$, which gives $\delta_0(B(0,R_2)) \subset B(0,\varepsilon R_2)$. In particular, taking $\varepsilon_0 = \frac{R}{\varepsilon R_2} \in (0,1)$, we get $\delta_0(\Omega) \subset \delta_0(B(0,R_2)) \subset B\left(0,\frac{R}{\varepsilon}\right)$. This means $K_0 = \delta_0^2(\Omega) \subset B\left(0,\frac{R}{\varepsilon}\right) \subset B(0,R_1) \subset \Omega$. Next, by (3.45) and the homogeneity property (3) of $B_{dx}(x,r)$ we have

$$\int_{K_{\varepsilon_0}} \frac{dx}{|B_{dx}(x,\sqrt{t})|} = \int_{\delta_0(\Omega)} \frac{dx}{|B_{dx}(x,\sqrt{t})|} = \int_\Omega \frac{\varepsilon_0^2 dy}{|B_{dx}(\delta_0(y),\sqrt{t})|} = \int_\Omega \frac{dy}{|B_{dx}(y,\frac{\sqrt{t}}{\varepsilon_0})|} \tag{5.3}$$

Hence, it follows from (5.1)-(5.3) that

$$\int_\Omega h_D(x,x,t)dx \approx \int_\Omega \frac{dx}{|B_{dx}(x,\sqrt{t})|} \text{ as } t \to 0^+.$$

The proof of Theorem 1.1 is complete. \hfill \Box

5.2. Proof of Theorem 1.2

Before we prove Theorem 1.2, we would like to give a useful lemma.

Lemma 5.1. Let $f$ be a function such that $f(x) \approx x^{-\mu_0}|\ln x|^b$ as $x \to 0^+$ with some $\mu_0 \in \mathbb{R}$ and $b > 0$. Denote by $g_a(x) := x^a f(x)$ for $a > 0$. If there are some positive constants $\mu_2 > \mu_1 > 0$ such that

1. $\liminf_{x \to 0^+} g_{\mu_1}(x) > 0$;
2. $\lim_{x \to 0^+} g_{\mu_2 + \varepsilon}(x) = 0$ holds for any $\varepsilon \in (0,1)$.

Then $\mu_0 \in [\mu_1,\mu_2]$.

Proof. Since $f(x) \approx x^{-\mu_0}|\ln x|^b$ as $x \to 0^+$, there are some constants $0 < c_1 \leq c_2 < +\infty$ and $\delta > 0$ such that

$$c_1 x^{-\mu_0}|\ln x|^b \leq f(x) \leq c_2 x^{-\mu_0}|\ln x|^b \quad \text{for all } x \in (0,\delta).$$

Suppose $\mu_1 > \mu_0$, it follows

$$\liminf_{x \to 0^+} g_{\mu_1}(x) \leq c_2 \limsup_{x \to 0^+} x^{\mu_1 - \mu_0}|\ln x|^b = 0,$$
which contradicts condition (1). This means $\mu_0 \geq \mu_1$. Additionally, if we assume $\mu_2 < \mu_0$, there exists a positive constant $l > 1$ such that $\varepsilon = \frac{1}{l}(\mu_0 - \mu_2) \in (0, 1)$. Then

$$g_{\mu_2 + \varepsilon}(x) = x^{1/\mu_0 + (1 - l)/(\mu_2 - \mu_0)}f(x) \geq c_1x^{(1 - l)/(\mu_2 - \mu_0)}|\ln x|^b$$

holds for all $x \in (0, \delta)$. This implies $\lim \inf_{x \to 0^+} g_{\mu_2 + \varepsilon}(x) = +\infty$, which contradicts condition (2). Hence, we have $\mu_0 \in [\mu_1, \mu_2]$.

**Proof of Theorem 1.2.** Proposition 2.10 yields

$$\int_{\Omega} \frac{dx}{|B_{dx}(x, r)|} \approx J_{\Omega}(r) = \int_{\Omega} \frac{dx}{\Lambda(x, r)}. \quad (5.4)$$

Thus, we only need to concern the explicit asymptotic behaviour of $J_{\Omega}(r)$ as $r \to 0^+$. Our estimates will be derived in the following two cases:

- **Case 1:** $w = Q$. By Proposition 2.12 and (5.4) we have
  $$J_{\Omega}(r) = \int_{\Omega} \frac{dx}{\Lambda(x, r)} \approx \frac{|\Omega|}{\Omega(0)} \cdot \frac{1}{r^Q}.$$  

- **Case 2:** $w \leq Q - 1$. Suppose $\{x_{i_1}, \ldots, x_{i_v}\}$ is the collection of all degenerate components of vector fields $X$ associated with the degenerate indexes $\{\alpha_{i_1}, \ldots, \alpha_{i_v}\}$, and $\alpha(X) = \alpha_{i_1} + \cdots + \alpha_{i_v}$ is the sum of all degenerate indexes. Proposition 4.8 yields that
  $$J_{\Omega}(r) \approx \int_{\Omega} \frac{dx}{\Lambda(x, r)} \approx r^{-Q_0}|\ln r|^{d_0} \quad \text{as} \quad r \to 0^+,$$  

where $Q_0 \in \mathbb{Q}$ and $d_0 \in \{0, 1, \ldots, v\}$ with $0 \leq v \leq n - 1$. Hence, it remains to establish the bounds of index $Q_0$. Using Proposition 2.12 and Lemma 4.1, we obtain

$$J_{\Omega}(r) \approx J_{(-1, 1)^v, w}(r) = \int_{(-1, 1)^v} \frac{dx_{i_1} \cdots dx_{i_v}}{\Lambda(x, r)} = \frac{1}{r^Q} \int_{(-1, 1)^v} \frac{dx_{i_1} \cdots dx_{i_w}}{\sum_{k=w}^{Q} f_k(x)^{r-k-Q}} \quad (5.7)$$

Here we change the variables $y_j = r^{-\alpha_{i_j}}x_{i_j}$ for $1 \leq j \leq v$ in the last step in (5.7). Then, we consider the function

$$g_a(r) := r^a J_{\Omega}(r) \quad \text{for} \quad r > 0.$$
It follows from Proposition 2.12 that \( \Lambda(y,1) = \sum_{k=w}^{Q} f_k(y) \geq f_Q(0) > 0 \). Therefore, (5.7) gives

\[
\liminf_{r \to 0^+} g_{Q-\alpha}(x)(r) = \liminf_{r \to 0^+} r^{Q-\alpha(X)} J_\Omega(r) \\
\geq C \cdot \liminf_{r \to 0^+} \int_{\Pi_{j=1}^{n} (-r^{-\alpha_j}, r^{-\alpha_j})} dy_1 \cdots dy_n \Lambda(y,1)^{-1} > 0,
\]

(5.8)

where \( C > 0 \) is some positive constant and the last term in (5.8) is finite or positive infinity. Furthermore, we have

\[
g_{w}(r) = r^{w} J_\Omega(r) = \int_{\Omega} \frac{dx}{f_{w}(x) + f_{w+1}(x)r + \cdots + f_{Q}(x)r^{Q-w}} \geq \int_{\Omega} \frac{dx}{\sum_{k=w}^{Q} f_k(x)} > 0 \quad \text{for all} \quad 0 < r < 1,
\]

(5.9)

which implies \( \liminf_{r \to 0^+} g_{w}(r) > 0 \). Hence, \( \mu_1 := \max\{Q - \alpha(X), w\} > 0 \) satisfies the condition (1) in Lemma 5.1. Next, we show that for any \( \varepsilon \in (0,1) \),

\[
\lim_{r \to 0^+} g_{Q-1+\varepsilon}(r) = \lim_{r \to 0^+} r^{Q-1+\varepsilon} J_\Omega(r) = 0,
\]

(5.10)

which indicates that \( \mu_2 := Q - 1 > 0 \) verifies the condition (2) in Lemma 5.1. Using Proposition 2.12 again, we obtain \( f_w(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R}^n \) and \( f_Q(x) = f_Q(0) > 0 \) for all \( x \in \mathbb{R}^n \), provided \( w \leq Q - 1 \). That means \( \lambda_\Omega(x) \neq 0 \) holds for some \( n \)-tuple \( \tilde{I} \) with \( d(\tilde{I}) = w \). For any \( \varepsilon \in (0,1) \) and \( r > 0 \), we have

\[
0 \leq g_{Q-1+\varepsilon}(r) = r^{Q-1+\varepsilon} J_\Omega(r) \leq r^{Q-1} \int_{\Omega} \frac{dx}{\Lambda(x,r)} \leq \int_{\Omega} \frac{dx}{\lambda_{\tilde{I}}(x)r^{w-Q+1} + f_Q(0)r}.
\]

(5.11)

On the other hand, we mention that the set \( Z(\lambda_{\tilde{I}}) := \{ x \in \mathbb{R}^n | \lambda_{\tilde{I}}(x) = 0 \} \) has zero \( n \)-dimensional measure since \( \lambda_{\tilde{I}} \) is a polynomial. Hence,

\[
h_{\varepsilon}(x,r) := \frac{r^\varepsilon}{\lambda_{\tilde{I}}(x)r^{w-Q+1} + f_Q(0)r} = \frac{r^{Q-1-w+\varepsilon}}{\lambda_{\tilde{I}}(x)}
\]

satisfies that

\[
\int_{\Omega} \frac{r^\varepsilon dx}{\lambda_{\tilde{I}}(x)r^{w-Q+1} + f_Q(0)r} = \int_{\Omega \setminus Z(\lambda_{\tilde{I}})} h_{\varepsilon}(x,r) dx,
\]

(5.12)

and

\[
\lim_{r \to 0^+} h_{\varepsilon}(x,r) = 0 \quad \text{for all} \quad x \in \Omega \setminus Z(\lambda_{\tilde{I}}).
\]

(5.13)
Moreover, for any $x \in \Omega \setminus Z(\lambda \tilde{f}),$

$$\frac{1}{h_\varepsilon(x, r)} = \frac{|\lambda \tilde{f}(x)|}{r^{Q+\varepsilon-w-1}} + f_Q(0)r^{1-\varepsilon} \geq \frac{1}{C}|\lambda \tilde{f}(x)|^{\frac{1}{Q-w}}$$

holds for some positive constant $C > 0$, which means

$$h_\varepsilon(x, r) \leq \frac{C}{|\lambda \tilde{f}(x)|^{\frac{1}{Q-w}}} \quad \text{for all } x \in \Omega \setminus Z(\lambda \tilde{f}). \quad (5.14)$$

It follows from (2.8) and Proposition 2.8 that $\lambda \tilde{f}$ is a $\delta_\varepsilon$-homogeneous polynomial of degree $Q - w$ which has the form

$$\lambda \tilde{f}(x) = \sum_{\sum_{i=1}^n \alpha_i \beta_i = Q-w} c_{\beta_1, \ldots, \beta_n} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are some non-negative integers. Observing that for each monomial $c_{\beta_1, \ldots, \beta_n} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, we have

$$1 - \varepsilon \frac{Q-w}{Q-w} (\beta_1 + \cdots + \beta_n) \leq 1 - \varepsilon \frac{Q-w}{Q-w} (\alpha_1 \beta_1 + \cdots + \alpha_n \beta_n) = 1 - \varepsilon < 1.$$

Thus, by means of [35, Proposition 4.1] we get

$$\int_\Omega \frac{dx}{|\lambda \tilde{f}(x)|^{\frac{1}{Q-w}}} < +\infty. \quad (5.15)$$

Combining (5.11)-(5.15) and Lebesgue’s dominated convergence theorem, we conclude

$$\lim_{r \to 0^+} g_{Q-1+\varepsilon}(r) = 0 \quad \text{for any } \varepsilon \in (0, 1).$$

Consequently, Lemma 5.1 yields

$$n \leq \max\{Q - \alpha(X), w\} \leq Q_0 \leq Q - 1.$$

The proof of Theorem 1.2 is complete. \( \square \)

5.3. Proof of Theorem 1.3

Proof of Theorem 1.3. According to Theorem 1.1, Theorem 1.2 and Proposition 3.3, we have

$$\sum_{k=1}^\infty e^{-t \lambda_k} \approx \int_\Omega \frac{dx}{|B_{d_k}(x, \sqrt{t})|} \approx t^{-\frac{Q_0}{2}} |\ln t|^{d_0} \quad \text{as } t \to 0^+, \quad (5.16)$$

where $Q_0$ and $d_0$ are the indexes given in Theorem 1.2. Then, applying the Tauberian theorem (see [75, Proposition B.0.13]) to (5.16), we get

$$N(\lambda) \approx \int_\Omega \frac{dx}{|B_{d_k}(x, \lambda^{-\frac{1}{2}})|} \approx \lambda^{\frac{Q_0}{2}} |\ln \lambda|^{d_0} \quad \text{as } \lambda \to +\infty, \quad (5.17)$$
where \( N(\lambda) := \{k|\lambda_k \leq \lambda\} \) is the counting function.

We next derive the explicit asymptotic behaviour of \( \lambda_k \). By (5.17), there exist positive constants \( M > e^2 \) and \( C > 1 \), such that for \( \lambda > M \),

\[
0 < \frac{1}{C} \lambda^{Q_0} \lambda^{d_0} \leq N(\lambda) \leq C \lambda^{Q_0} \lambda^{d_0}.
\] (5.18)

Because \( \lambda_k \to +\infty \) as \( k \to +\infty \), (5.18) yields

\[
k \leq N(\lambda_k) \leq C \lambda_k^{Q_0} \lambda_k^{d_0} \text{ for all } k \geq k_0,
\] (5.19)

where \( k_0 \) is a positive integer such that \( \lambda_k > M \) for any \( k \geq k_0 \). For \( \lambda > \lambda_1 \) we define

\[
M(\lambda) := \lim_{p \to \lambda^-} N(p).
\] (5.20)

Clearly, \( M(\lambda) \) is a left continuous function with \( M(\lambda_k) < k \) for all \( k \geq 1 \). For any \( \lambda_0 > M \), (5.18) and (5.20) indicates

\[
M(\lambda_0) = \lim_{p \to \lambda_0^-} N(p) \geq \lim_{p \to \lambda_0^-} \left( \frac{p^{Q_0}}{C} \lambda_k^{d_0} \right) = \frac{1}{C} \lambda_0^{Q_0} \lambda_0^{d_0}.
\] (5.21)

Combining (5.19)-(5.21) we get

\[
\frac{1}{C} \lambda_k^{Q_0} \lambda_k^{d_0} \leq k \leq C \lambda_k^{Q_0} \lambda_k^{d_0} \text{ for all } k \geq k_0.
\] (5.22)

As a consequence of (5.22), we obtain

\[
\limsup_{k \to +\infty} \frac{\lambda_k (\ln k)^{Q_0}}{k^{Q_0}} \leq \limsup_{k \to +\infty} \frac{\ln C \lambda_k^{d_0} \ln \lambda_k + d_0 \ln \ln \lambda_k^{Q_0}}{C \lambda_k^{Q_0} \ln \lambda_k^{Q_0}} = C^{\frac{Q_0}{Q_0}} \left( \frac{Q_0}{2} \right)^{\frac{Q_0}{Q_0}},
\] (5.23)

and

\[
\liminf_{k \to +\infty} \frac{\lambda_k (\ln k)^{Q_0}}{k^{Q_0}} \geq \liminf_{k \to +\infty} \frac{\ln C \lambda_k^{d_0} \ln \lambda_k + d_0 \ln \ln \lambda_k^{Q_0}}{C \lambda_k^{Q_0} \ln \lambda_k^{Q_0}} = C^{\frac{Q_0}{Q_0}} \left( \frac{Q_0}{2} \right)^{\frac{Q_0}{Q_0}}.
\] (5.24)

That means \( \lambda_k \approx k^{\frac{Q_0}{Q_0}} (\ln k)^{\frac{Q_0}{Q_0}} \) as \( k \to +\infty \). The proof of Theorem 1.3 is complete. \( \square \)

6. Some examples

In this section, as further applications of Theorem 1.2 and Theorem 1.3, we give some examples.
Example 6.1. For \( t \in \mathbb{N}^+ \), we let \( X = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}, x_1^l \partial_{x_n}) \) be the Grushin type vector fields defined on \( \mathbb{R}^n \). The Grushin operator (see [43]) generated by \( X \) is given by

\[
\triangle_G := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + x_1^2 \frac{\partial^2}{\partial x_n^2}.
\]

Obviously, \( X \) satisfy the assumption (H.1) with the dilation \( \delta_t(x) = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \ldots, t^{\alpha_{n-1}} x_{n-1}, t x_n) \), and homogeneous dimension \( Q = n + l \). Also, \( X \) satisfy Hörmander’s condition in \( \mathbb{R}^n \) with the Hörmander index \( r = l + 1 \). Moreover, \( x_1 \) is the unique degenerate component of \( X \) with corresponding degenerate index \( \alpha_1 = 1 \), and \( w = \min_{x \in \mathbb{R}^n} \nu(x) = n \).

Assume \( \Omega \subset \mathbb{R}^n \) is a bounded smooth open domain containing the origin. Hence, we can conclude from Proposition 4.1 that

\[
J_\Omega(r) = \int_{\Omega} \frac{dx}{\Lambda(x, r)} \approx \begin{cases} \frac{1}{r^{Q-1}} |\ln r|, & \text{if } l = 1; \\ \frac{1}{r^{Q-1}}, & \text{if } l > 1, \end{cases} \quad \text{as } r \to 0^+,
\]

which gives \( Q_0 = Q - 1 = \nu - 1 \) and

\[
d_0 = \begin{cases} 1, & \text{if } l = 1; \\ 0, & \text{if } l > 1. \end{cases}
\]

Denote by \( \lambda_k \) the \( k \)-th Dirichlet eigenvalue of the Grushin operator \( \triangle_G \) on \( \Omega \). It follows from Theorem 1.3 that

\[
\lambda_k \approx k^{\frac{2}{Q_0}} (\ln k)^{-\frac{2}{Q_0}} \approx \begin{cases} \left( \frac{k}{\ln k} \right)^{\frac{2}{Q-1}}, & \text{if } l = 1; \\ k^{\frac{2}{2^l-Q}}, & \text{if } l > 1. \end{cases} \quad \text{as } k \to +\infty. \tag{6.1}
\]

Remark 6.1. The estimates (6.1) of Dirichlet eigenvalues for Grushin operator \( \triangle_G \) improves Chen-Luo’s estimates in [26, Theorem 1.2].

Example 6.2. Let \( X = (X_1, X_2) \) with

\[
X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2} + x_1^2 \partial_{x_3} + \cdots + x_1^{n-1} \partial_{x_n}
\]

be the Bony type vector fields defined on \( \mathbb{R}^n \) (see [18]). The Bony operator generated by \( X \) is given by

\[
\triangle_B = \partial_{x_1}^2 + (x_1 \partial_{x_2} + x_1^2 \partial_{x_3} + \cdots + x_1^{n-1} \partial_{x_n})^2,
\]

which satisfies Hörmander’s condition but nevertheless with a very degenerate characteristic form. A direct calculation shows that \( X \) satisfy the assumption (H.1) with the dilation \( \delta_t(x) = (t x_1, t^2 x_2, \ldots, t^n x_n) \).
and the homogeneous dimension \( Q = \frac{n(n+1)}{2} \). In addition, \( x_1 \) is the unique degenerate component of \( X \) with corresponding degenerate index \( \alpha_1 = 1 \), and \( w = \min_{x \in \mathbb{R}^n} \nu(x) = Q - n + 1 \).

Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded smooth open domain containing the origin. According to Proposition 4.1,

\[
J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \begin{cases} 
\frac{1}{r^{Q-n}} |\ln r|, & \text{if } n = 2; \\
\frac{1}{r^{Q-n}}, & \text{if } n \geq 3,
\end{cases} \quad \text{as } r \to 0^+,
\]

which gives that

\[
\lambda_k \approx \begin{cases} 
\left( \frac{k}{\ln k} \right)^{\frac{2}{Q-1}}, & \text{if } n = 2; \\
k^{\frac{2}{Q-1}}, & \text{if } n \geq 3,
\end{cases} \quad \text{as } k \to +\infty, \tag{6.2}
\]

where \( \lambda_k \) is the \( k \)-th Dirichlet eigenvalue of the Bony operator on \( \Omega \).

**Example 6.3.** Let \( X = (X_1, X_2) \) be the Martinet type vector fields defined on \( \mathbb{R}^3 \), where \( X_1 = \partial x_1 \) and \( X_2 = \partial x_2 + x_1^2 \partial x_3 \). The Martinet operator generated by \( X \) is given by

\[
\Delta_M := \partial^2_{x_1} + (\partial x_2 + x_1^2 \partial x_3)^2.
\]

We can deduce that \( X \) satisfy the assumption (H.1) with the dilation \( \delta_t(x) = (tx_1, tx_2, t^3 x_3) \), and homogeneous dimension \( Q = 5 \). Meanwhile, \( X \) satisfy Hörmander’s condition in \( \mathbb{R}^3 \) with the Hörmander index \( r = 3 \). Moreover, \( x_1 \) is the unique degenerate component of \( X \) with corresponding degenerate index \( \alpha_1 = 1 \), and \( w = \min_{x \in \mathbb{R}^3} \nu(x) = 4 \).

Suppose that \( \Omega \subset \mathbb{R}^3 \) is a bounded smooth open domain containing the origin. Owing to Proposition 4.1 and Theorem 1.3, we can deduce that

\[
\lambda_k \approx \left( \frac{k}{\ln k} \right)^{\frac{2}{Q-1}}, \quad \text{as } k \to +\infty, \tag{6.3}
\]

where \( \lambda_k \) denotes the \( k \)-th Dirichlet eigenvalue of \( \Delta_M \) on \( \Omega \).

We mention that the three examples above only possess the unique degenerate component \( x_1 \). In this case, Proposition 4.1 together with Theorem 1.3 give an explicit asymptotic behaviour of Dirichlet eigenvalue with exact growth rate. Furthermore, Example 6.1 and Example 6.2 indicates the upper bound \( Q_0 \leq Q - 1 \) in (1.12) for index \( Q_0 \) is optimal. The following example will illustrate that the index \( Q_0 \) in Theorem 1.3 could be fractional and present a calculation method for indexes \( Q_0 \) and \( d_0 \).

**Example 6.4.** Let us consider the vector fields \( X = (X_1, X_2, X_3) \) defined on \( \mathbb{R}^3 \) such that

\[
X_1 = \partial x_1, \quad X_2 = x_1 \partial x_2 + x_2 \partial x_3, \quad \text{and} \quad X_3 = x_1^2 \partial x_3.
\]

The dilation of \( X \) is given by \( \delta_t(x) = (tx_1, t^2 x_2, t^3 x_3) \), which implies the homogeneous dimension \( Q = 6 \). Clearly, \( X \) satisfy Hörmander’s condition in \( \mathbb{R}^3 \) with the Hörmander index \( r = 3 \).
Assume $\Omega \subset \mathbb{R}^3$ is a bounded smooth open domain containing the origin. It follows that

$$\Lambda(x, r) \approx |x_1|^3 r^3 + (|x_1|^2 + |x_2|) r^4 + |x_1|^5 r + r^6.$$  

Therefore, Lemma 4.1 gives

$$J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_{(0,1)^2} \frac{dx_1 dx_2}{x_1^4 r^3 + (x_1^2 + x_2) r^4 + x_1 r^5 + r^6}. \quad (6.4)$$

Let us estimate (6.4) by the method carried out in Proposition 4.7. The set of index pairs is given by

$$\mathcal{G} = \{(a_1, a_2, s) | (3, 0, 3), (2, 0, 4), (0, 1, 4), (1, 0, 5), (0, 0, 6)\}.$$  

For each index pair $(a_1, a_2, s) \in \mathcal{G}$, we let

$$P_{a_1,a_2,s} = \{(y_1, y_2) \in [0, +\infty)^2 | (a_1 - a'_1) y_1 + (a_2 - a'_2) y_2 \leq s' - s, \forall (a'_1, a'_2, s') \in \mathcal{G}\}$$

be the polyhedron in $[0, +\infty)^2$, and $\phi_{a_1,a_2}(y) = (a_1 - 1) y_1 + (a_2 - 1) y_2$ be the corresponding linear function. It follows from Proposition 4.4 that

$$J_\Omega(r) \approx J_{3,0,3}(r) + J_{2,0,4}(r) + J_{0,1,4}(r) + J_{1,0,5}(r) + J_{0,0,6}(r), \quad (6.5)$$

where

$$J_{a_1,a_2,s}(r) := \left(\ln\frac{1}{r}\right)^2 \frac{1}{r^3} \int_{P_{a_1,a_2,s}} e^{\left(\ln\frac{1}{r}\right) \phi_{a_1,a_2}(y)} dy. \quad (6.6)$$

According to Lemma 4.3, we can find the maximum $m_{a_1,a_2,s}$ of $\phi_{a_1,a_2}$ in polyhedron $P_{a_1,a_2,s}$, if $V_2(P_{a_1,a_2,s}) \neq 0$, and the dimension $d_{a_1,a_2,s} = \dim\{x \in P_{a_1,a_2,s} | \phi_{a_1,a_2}(x) = m_{a_1,a_2,s}\}$.

For index pairs $(2, 0, 4)$ and $(1, 0, 5)$, we have $V_2(P_{2,0,4}) = V_2(P_{1,0,5}) = 0$, which implies $J_{2,0,4}(r) = J_{1,0,5}(r) = 0$. For other index pairs, employing the linear programming and Proposition 4.6 we obtain

1. $(3, 0, 3)$: $m_{3,0,3} = \frac{2}{3}, d_{3,0,3} = 0$ and $J_{3,0,3}(r) \approx r^{-\frac{11}{3}}$ as $r \to 0^+$.
2. $(0, 1, 4)$: $m_{0,1,4} = -\frac{1}{3}, d_{0,1,4} = 0$ and $J_{0,1,4}(r) \approx r^{-1}$ as $r \to 0^+$.
3. $(0, 0, 6)$: $m_{0,0,6} = -3, d_{0,0,6} = 0$ and $J_{0,0,6}(r) \approx r^{-3}$ as $r \to 0^+$.

Thus, $J_\Omega(r) \approx r^{-\frac{11}{3}}$ as $r \to 0^+$. That means $\lambda_k \approx k^{\frac{11}{3}}$ as $k \to +\infty$, where $\lambda_k$ is the $k$-th Dirichlet eigenvalue of the operator $\Delta_X = X_1^2 + X_2^2 + X_3^2$ on $\Omega$.

Finally, we give two examples satisfying $Q_0 = Q - \alpha(X) > w$ and $Q - \alpha(X) < w = Q_0$ respectively, which also indicate the lower bound $Q_0 \geq \max\{Q - \alpha(X), w\}$ in (1.12) is optimal.

**Example 6.5.** $(Q_0 = Q - \alpha(X) > w)$ Let us consider the vector fields $X = (X_1, X_2, X_3)$ in $\mathbb{R}^3$, where

$$X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2} + x_1^3 \partial_{x_3}, \quad \text{and} \quad X_3 = x_1 x_2 \partial_{x_3}.$$

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The corresponding dilation is given by \( \delta_r(x) = (tx_1, t^2x_2, t^4x_3) \), and the homogeneous dimension \( Q = 7 \). Moreover, \( X \) satisfy Hörmander’s condition in \( \mathbb{R}^3 \) with the Hörmander index \( r = 4 \).

Suppose \( \Omega \subset \mathbb{R}^3 \) is a bounded smooth open domain containing the origin. Then a direct calculation gives that

\[
\Lambda(x, r) \approx |x_1|^2|x_2|^3 + (|x_1x_2| + |x_1|^3)r^4 + (|x_2| + |x_1|^2)r^5 + |x_1|r^6 + r^7, \tag{6.7}
\]

which implies \( Q - \alpha(X) = 4 > w = 3 \). Therefore, by Lemma 4.1 and Proposition 4.7 we have

\[
J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_{(0,1)^2} \frac{dx_1dx_2}{x_1^2x_2r^3 + (x_1x_2 + x_1^3)r^4 + (x_2 + x_1^2)r^5 + x_1r^6 + r^7} \approx \frac{1}{r^4} \ln r \quad \text{as} \quad r \to 0^+. \tag{6.8}
\]

That means \( Q_0 = Q - \alpha(X) = 4 > w = 3 \) and \( d_0 = 1 \), which yields \( \lambda_k \approx \frac{k^4}{2}(\ln k)^{-\frac{1}{2}} \) as \( k \to +\infty \).

**Example 6.6.** \((Q_0 = w > Q - \alpha(X))\) Consider the vector fields \( X = (X_1, X_2) \) in \( \mathbb{R}^3 \), where

\[
X_1 = \partial_{x_1} - x_2^2\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1^2\partial_{x_3}.
\]

The corresponding dilation is given by \( \delta_t(x) = (tx_1, tx_2, t^3x_3) \), and the homogeneous dimension \( Q = 5 \). Obviously, \( X \) satisfy Hörmander’s condition in \( \mathbb{R}^3 \) with the Hörmander index \( r = 3 \).

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded smooth open domain containing the origin. Then

\[
\Lambda(x, r) \approx |x_1 + x_2|^{r^4} + r^5, \tag{6.9}
\]

which gives \( w = 4 > Q - \alpha(X) = 3 \). Thus, it follows from Lemma 4.1 and (6.9) that

\[
J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_{(-1,1)^2} \frac{dx_1dx_2}{x_1 + x_2}|r^4 + r^5|. \tag{6.10}
\]

A general method to transform the integrand \(|x_1 + x_2|^{r^4} + r^5|^{-1}\) into the situation described in the property \( (C) \) of Proposition 4.2 is the blow-up technique in algebraic geometry (see [79, Section 3]), but it comprises many tedious calculations. Instead, we would like to consider another variable transformation to deal with (6.10). Changing \( x_1 + x_2 = u_1 \) and \( x_2 = u_2 \) in (6.10), we obtain

\[
\int_{(-1,1)^2} \frac{dx_1dx_2}{x_1 + x_2}|r^4 + r^5| = \int_M \frac{du_1du_2}{u_1|r^4 + r^5|}, \tag{6.11}
\]

where \( M = \{(u_1, u_2) \in \mathbb{R}^2; 0 < u_1 < u_2 + 1, -1 < u_2 < 1\} \) is an open domain containing the origin. Hence, using Lemma 4.1 and Proposition 4.7 again, we have

\[
J_\Omega(r) = \int_\Omega \frac{dx}{\Lambda(x, r)} \approx \int_M \frac{du_1du_2}{u_1|r^4 + r^5|} \approx \int_0^1 \frac{du_1}{u_1|r^4 + r^5|} \approx \frac{1}{r^4} \ln r \quad \text{as} \quad r \to 0+. \tag{6.12}
\]
That means \( Q_0 = w = 4 > Q - \alpha(X) = 3 \) and \( d_0 = 1 \), which yields \( \lambda_k \approx k^{\frac{1}{2}}(\ln k)^{-\frac{1}{2}} \) as \( k \to +\infty \).

**Remark 6.2.** The change of variables in (6.11) is a simple application of resolution of singularities. In this case, the \((M, W, \rho)\) in Proposition 4.2 can be given by

\[
M = \{(u_1, u_2) \in \mathbb{R}^2 | u_2 - 1 < u_1 < u_2 + 1, -1 < u_2 < 1\}, \quad W = (-1, 1)^2,
\]

and \( \rho : M \to W \) is the real analytic map such that \( \rho(u_1, u_2) = (u_1 - u_2, u_2) \).

7. Appendix

7.1. Further results in weighted Sobolev spaces

In this part, we introduce the chain rules in the weighted Sobolev spaces, which helps us characterize the function space \( H^1_{X,0}(\Omega) \). We suppose that \( X = (X_1, X_2, \ldots, X_m) \) defined on an open subset \( W \subset \mathbb{R}^n \) satisfying the Hörmander’s condition, and \( \Omega \subset W \) is a bounded smooth open domain.

**Definition 7.1.** A real-valued function \( f \) defined on a non-empty open set \( O \subset \mathbb{R} \) is piecewise \( C^1 \)-smooth on \( O \) if \( f \in C(O) \) and there exists a finite collection of functions \( f_i \in C^1(O) \), \( i = 1, \ldots, l \), such that \( f(x) \in \{f_i(x) | i \in \{1, \ldots, l\}\} \) for all \( x \in O \).

**Proposition 7.1.** Let \( u \in H^1_X(\Omega) \) and \( \varphi \) be a piecewise \( C^1 \)-smooth function on \( \mathbb{R} \) with \( \varphi' \in L^\infty(\mathbb{R}) \). Then we have

1. The functions \( u_+, u_- \), \( |u| \in H^1_X(\Omega) \). Moreover, for any \( c \in \mathbb{R} \),

\[
X(u - c) = \begin{cases} 
Xu, & \text{on } \{x \in \Omega | u(x) > c\}; \\
0, & \text{on } \{x \in \Omega | u(x) \leq c\};
\end{cases}
\]

and

\[
X(u - c) = \begin{cases} 
0, & \text{on } \{x \in \Omega | u(x) \geq c\}; \\
-Xu, & \text{on } \{x \in \Omega | u(x) < c\}.
\end{cases}
\]

2. \( \varphi(u) \in H^1_X(\Omega) \) and

\[
X(\varphi(u)) = \varphi'(u)Xu. \tag{7.1}
\]

**Proof.** The conclusion (1) is derived in [38, Lemma 3.5], which gives

\[
Xu = 0 \quad \text{on } \{x \in \Omega | u(x) = c\} \quad \text{for any } c \in \mathbb{R}. \tag{7.2}
\]

By an induction argument, the proof of conclusion (2) is reduced to the case that \( \varphi(u) \) has one corner at \( u = c \) for some constant \( c \in \mathbb{R} \). Let \( \varphi_1, \varphi_2 \in C^1(\mathbb{R}) \) satisfy \( \varphi_1, \varphi'_2 \in L^\infty(\mathbb{R}) \), and

\[
\varphi(u) = \begin{cases} 
\varphi_1(u), & u \geq c; \\
\varphi_2(u), & u < c.
\end{cases}
\]
Owing to (7.2), we can define \( \varphi'(u)Xu = 0 \) on \( \{ x \in \Omega | u(x) = c \} \), irrespective of whether \( \varphi'(u) \) is defined. Then, since \( \varphi(u) = \varphi_1(c+(u-c)_+) + \varphi_2(c-(u-c)_-) - \varphi_1(c) \), the conclusion (2) follows from [38, Lemma 3.5] and conclusion (1).

**Proposition 7.2.** For any \( u \in H^1_X(\Omega) \), if \( \text{supp} \ u \) is a compact subset in \( \Omega \), then \( u \in H^1_{X,0}(\Omega) \).

**Proof.** Since \( \text{supp} \ u \) is a compact subset in \( \Omega \), we can find a function \( f \in C^\infty_0(\Omega) \) such that \( f \equiv 1 \) on \( \text{supp} \ u \). By using the Meyers-Serrin theorem (see [38, Theorem 1.13]) for vector fields, there exists a sequence \( \{ \psi_k \} \subset C^\infty(\Omega) \cap H^1_X(\Omega) \) such that \( \psi_k \to u \) in \( H^1_X(\Omega) \). Observing that \( f \psi_k \in C^\infty_0(\Omega) \) and

\[
\| f \psi_k - u \|^2_{H^1_{X,0}(\Omega)} = \| f \psi_k - f u \|^2_{H^1_{X,0}(\Omega)}
\leq \| f(\psi_k - u) \|^2_{L^2(\Omega)} + \| X(f \psi_k) - X(f u) \|^2_{L^2(\Omega)}
\leq C(\| \psi_k - u \|^2_{L^2(\Omega)} + \| Xu - Xu \|^2_{L^2(\Omega)})
\]

holds for some positive constant \( C > 0 \), we conclude that \( u \in H^1_{X,0}(\Omega) \).

**Proposition 7.3.** Let \( G \) be a piecewise \( C^1 \)-smooth function on \( \mathbb{R} \) with \( G' \in L^\infty(\mathbb{R}) \) and \( G(0) = 0 \). Then for any \( u \in H^1_{X,0}(\Omega) \), we have \( G(u) \in H^1_{X,0}(\Omega) \) and

\[
XG(u) = G'(u)Xu.
\]  
(7.3)

**Proof.** Since \( G \) is a piecewise \( C^1 \)-smooth function on \( \mathbb{R} \) with \( G' \in L^\infty(\mathbb{R}) \), the identity (7.3) follows from (7.1). Thus, it remains to prove \( G(u) \in H^1_{X,0}(\Omega) \) for all \( u \in H^1_{X,0}(\Omega) \).

For any \( u \in H^1_{X,0}(\Omega) \), we can choose a sequence \( \{ u_k \} \subset C^\infty_0(\Omega) \) such that \( u_k \to u \) in \( H^1_{X,0}(\Omega) \) and \( \| u_k \|_{H^1_{X,0}(\Omega)} \leq C \) for all \( k \geq 1 \) and some positive \( C > 0 \) independent of \( k \). Denoted by \( K := \| G' \|_{L^\infty(\mathbb{R})} \). It follows that \( G \) is uniformly Lipschitz continuous in \( \mathbb{R} \) with

\[
|G(x) - G(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}.
\]  
(7.4)

From \( G(0) = 0 \), we have \( |G(u_k)| \leq K|u_k| \), which implies \( G(u_k) \in L^2(\Omega) \) and \( \| G(u_k) \|_{L^2(\Omega)} \leq CK \) for all \( k \geq 1 \). Then (7.4) indicates that

\[
|G(u_k(x)) - G(u(x))| \leq K|u_k(x) - u(x)| \quad \text{a.e. on } \Omega,
\]

which gives \( \| G(u_k) - G(u) \|_{L^2(\Omega)} \leq K \| u_k - u \|_{L^2(\Omega)} \), and \( G(u_k) \to G(u) \) in \( L^2(\Omega) \).

On the other hand, by (7.3) we obtain \( \| XG(u_k) \|_{L^2(\Omega)} = \| G'(u_k)Xu_k \|_{L^2(\Omega)} \leq CK \) for all \( k \geq 1 \). That means \( G(u_k) \in H^1_X(\Omega) \) and \( \| G(u_k) \|_{H^1_X(\Omega)} \leq C_1 \) for all \( k \geq 1 \) and some constant \( C_1 > 0 \). Besides, \( G(0) = 0 \) implies \( \text{supp} \ G(u_k) \subset \text{supp} \ u_k \). According to Proposition 7.2, \( G(u_k) \in H^1_{X,0}(\Omega) \) for all \( k \geq 1 \). Hence, we can find a subsequence \( \{ G(u_{k_l}) \} \subset \{ G(u_k) \} \) such that \( G(u_{k_l}) \to G_0 \) weakly in \( H^1_{X,0}(\Omega) \) for some \( G_0 \in H^1_{X,0}(\Omega) \). Recalling that the embedding \( H^1_{X,0}(\Omega) \to L^2(\Omega) \) is compact, we obtain \( G(u_{k_l}) \to G_0 \) in \( L^2(\Omega) \), which implies \( G_0 = G(u) \) and \( G(u) \in H^1_{X,0}(\Omega) \).

**Remark 7.1.** From Proposition 7.3, we see that for any \( u \in H^1_{X,0}(\Omega) \), \( u_+, u_- \) \( |u| \) all belongs to \( H^1_{X,0}(\Omega) \). Moreover, \( (u - c)_+ \in H^1_{X,0}(\Omega) \) holds for any constant \( c \geq 0 \).
**Proposition 7.4.** Let \( u \in H^1_{X,0}(\Omega) \) be a non-negative function. Then there exists a sequence \( \{u_k\} \subset C_0^\infty(\Omega) \) such that \( u_k \geq 0 \) and \( u_k \to u \) in \( H^1_{X,0}(\Omega) \).

**Proof.** Consider the functions \( \psi(t) = t_+ \) and \( \varphi(t) = 1_{\mathbb{R}^+}(t) \). By \([40, \text{Example 5.3}]\), there exists a non-negative function sequence \( \{\psi_k\} \subset C^\infty(\mathbb{R}) \) satisfying

\[ \psi_k(0) = 0, \quad \sup_k \sup_{t \in \mathbb{R}} |\psi'_k(t)| < \infty, \quad (7.5) \]

\[ \psi_k(t) \to \psi(t), \quad \text{and} \quad \psi'_k(t) \to \varphi(t), \quad \text{for all} \ t \in \mathbb{R}. \quad (7.6) \]

From Proposition 7.3, we have \( \psi_k(u) \in H^1_{X,0}(\Omega) \) and \( X\psi_k(u) = \psi'_k(u)Xu \). Besides, for any \( u \in H^1_{X,0}(\Omega) \), \((7.6)\) gives \( \psi_k(u) \to \psi(u) \) and \( \psi'_k(u) \to \varphi(u) \) a.e. on \( \Omega \). Hence, \( \psi(u) \) and \( \varphi(u) \) are measurable. By \((7.5)\) there is a constant \( C > 0 \) such that \( |\psi_k(t)| \leq C|t| \), for all \( k \geq 1 \) and \( t \in \mathbb{R} \), which implies \( |\psi(t)| \leq C|t| \) and \( \psi(u) \in L^2(\Omega) \). Using \((7.6)\), we also have \( \varphi \in L^\infty(\mathbb{R}) \), whence \( \varphi(u)X_j u \in L^2(\Omega) \).

Then, we show that

\[ \psi_k(u) \to \psi(u) = u \text{ in } L^2(\Omega), \quad \text{and} \quad X\psi_k(u) \to \varphi(u)Xu = Xu \text{ in } L^2(\Omega), \quad (7.7) \]

which implies \( \psi_k(u) \to u \) in \( H^1_{X,0}(\Omega) \). Since \( |\psi_k(t) - \psi(t)| \leq 2C|t| \), the dominated convergence theorem gives that \( \psi_k(u) \to \psi(u) \) in \( L^2(\Omega) \). Similarly, we have

\[ \int_\Omega |X\psi_k(u) - \varphi(u)Xu|^2 dx = \int_\Omega |\psi'_k(u) - \varphi(u)|^2 |Xu|^2 dx \to 0. \]

Thus, \((7.7)\) is achieved.

On the other hand, the definition of \( H^1_{X,0}(\Omega) \) implies there is a sequence \( \{v_j\} \subset C_0^\infty(\Omega) \) such that \( v_j \to u \) in \( H^1_{X,0}(\Omega) \). By selecting a subsequence, we may assume that \( v_{j_1}(x) \to u(x) \) for almost all \( x \in \Omega \). In addition, for each function \( \psi_k \), we have \( \psi_k(v_j) \in C_0^\infty(\Omega) \) and \( \psi_k(v_j) \geq 0 \) for all \( j \geq 1 \).

We next show that for \( l \to +\infty \),

\[ \psi_k(v_{j_l}) \to \psi_k(u) \text{ in } L^2(\Omega), \quad \text{and} \quad X\psi_k(v_{j_l}) \to \psi'_k(u)Xu \text{ in } L^2(\Omega). \quad (7.8) \]

That means \( \psi_k(v_{j_l}) \to \psi_k(u) \) in \( H^1_{X,0}(\Omega) \). It follows from \((7.5)\) that

\[ \int_\Omega |\psi_k(v_{j_l}) - \psi_k(u)|^2 dx \leq C \int_\Omega |v_{j_l} - u|^2 dx, \]

which yields \( \psi(u_{j_l}) \to \psi(u) \) in \( L^2(\Omega) \). Meanwhile,

\[ |X\psi_k(v_{j_l}) - \psi'_k(u)Xu| = |\psi'_k(v_{j_l})Xv_{j_l} - \psi'_k(u)Xu| \]

\[ \leq |\psi'_k(v_{j_l})(Xv_{j_l} - Xu)| + |(\psi'_k(v_{j_l}) - \psi'_k(u))Xu|. \]

That means

\[ \|X\psi_k(v_{j_l}) - \psi'_k(u)Xu\|_{L^2(\Omega)} \leq C\|Xv_{j_l} - Xu\|_{L^2(\Omega)} + \|\psi'_k(v_{j_l}) - \psi'_k(u)Xu\|_{L^2(\Omega)}. \]
By construction, we have \( \psi'_k(v_j(x)) \to \psi'_l(u(x)) \) for almost all \( x \in \Omega \) as \( l \to +\infty \). Note that \( \|\psi'_k(v_j) - \psi'_l(u)\|X \|u\|^2 \) is dominated by \( 4C^2 \|X u\|^2 \in L^1(\Omega) \). Hence, \( X \psi_k(v_j) \to \psi'_l(u)X u \) in \( L^2(\Omega) \) and (7.8) is proved.

As a result of (7.7) and (7.8), the non-negative function \( u \in H^1_{X,0}(\Omega) \) can be approximated by a sequence of non-negative functions from \( C^\infty_0(\Omega) \).

Suppose that \( u, v \) are two measurable functions on \( \Omega \), we write \( u \leq w \mod H^1_{X,0}(\Omega) \) if \( u \leq w + w_0 \) for some \( w_0 \in H^1_{X,0}(\Omega) \). Then we have

**Proposition 7.5.** If \( u \in H^1_X(\Omega) \), then

\[
    u \leq 0 \mod H^1_{X,0}(\Omega)
\]

holds if and only if \( u_+ \in H^1_{X,0}(\Omega) \).

**Proof.** If \( u_+ \in H^1_{X,0}(\Omega) \) then we have \( u \leq 0 \mod H^1_{X,0}(\Omega) \) because \( u \leq u_+ \). Thus, it remains to prove that if \( u \leq v \) for some \( v \in H^1_{X,0}(\Omega) \) then \( u_+ \in H^1_{X,0}(\Omega) \).

Assume first that \( v \in C^\infty_0(\Omega) \), and let \( \varphi \in C^\infty_0(\Omega) \) be a cutoff function such that \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) on \( \text{supp} v \). Then we have the following identity:

\[
    u_+ = ((1 - \varphi)v + \varphi u)_+.
\]

Indeed, (7.9) is trivial for \( \varphi = 1 \). If \( \varphi < 1 \), then \( u \leq v = 0 \), so that the both sides of (7.9) vanish.

We next show that \( \varphi u \in H^1_{X,0}(\Omega) \). Recalling that \( u \in H^1_X(\Omega) \), by using the Meyers-Serrin theorem (see [38, Theorem 1.13]) we can find a sequence \( \{u_k\} \subset C^\infty(\Omega) \cap H^1_X(\Omega) \) such that \( u_k \to u \) in \( H^1_X(\Omega) \). Clearly, \( \{u_k\varphi\} \subset C^\infty(\Omega) \). Using

\[
    \|X(u_k\varphi) - X(u\varphi)\|_{L^2(\Omega)} \leq \|\varphi(Xu_k - Xu)\|_{L^2(\Omega)} + \|(u_k - u)X\varphi\|_{L^2(\Omega)}
\]

\[
    \leq C(\|Xu_k - Xu\|_{L^2(\Omega)} + \|u_k - u\|_{L^2(\Omega)}),
\]

and

\[
    \|u_k\varphi - u\varphi\|_{L^2(\Omega)} \leq C\|u_k - u\|_{L^2(\Omega)},
\]

we can deduce that \( \varphi u_k \to \varphi u \) in \( H^1_{X,0}(\Omega) \), and therefore \( \varphi u \in H^1_{X,0}(\Omega) \).

Since \( (1 - \varphi)v \in C^\infty_0(\Omega) \), it follows that \( (1 - \varphi)v + \varphi u \in H^1_{X,0}(\Omega) \). By Remark 7.1 and (7.9), we conclude that \( u_+ \in H^1_{X,0}(\Omega) \) for \( v \in C^\infty_0(\Omega) \).

We next consider the general case that \( v \in H^1_{X,0}(\Omega) \). Let \( \{v_k\} \subset C^\infty(\Omega) \) such that \( v_k \to v \) in \( H^1_{X,0}(\Omega) \). Then by \( u \leq v \) we can define a sequence \( u_k \) such that

\[
    u_k := u + (v_k - v) \leq v_k.
\]

The arguments above implies that \( (u_k)_+ \in H^1_{X,0}(\Omega) \) and

\[
    \|u_k - u\|_{H^1_{X,0}(\Omega)} = \|v_k - v\|_{H^1_{X,0}(\Omega)} \to 0.
\]

Thus, the conclusion amounts to proving that \( \|(u_k)_+ - u_+\|_{H^1_{X,0}(\Omega)} \to 0 \) for some subsequence \( \{u_{k_j}\} \subset \{u_k\} \).
By choosing the subsequence, we can find \( \{u_{k_j}\} \subset \{u_k\} \) such that \( u_{k_j}(x) \to u(x) \) for almost all \( x \in \Omega \). Clearly, the estimate \(|(u_{k_j})_+ - u_+| \leq |u_{k_j} - u| \) implies \((u_{k_j})_+ \to u_+\) in \( L^2(\Omega) \). Then, we have

\[
\|X(\mathbb{1}_{R}+ u_+)- Xu_+\|_{L^2(\Omega)} \\
\leq \|\mathbb{1}_{R}+(u_{k_j})(Xu_{k_j} - Xu)\|_{L^2(\Omega)} + \|(\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u))Xu\|_{L^2(\Omega)} \\
\leq \|Xu_{k_j} - Xu\|_{L^2(\Omega)} + \|((\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u))Xu\|_{L^2(\Omega)}.
\]

(7.10)

The second term in last step of (7.10) is equal to

\[
\left( \int_{\Omega} |\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u)|^2 |Xu|^2 dx \right)^{\frac{1}{2}}.
\]

(7.11)

Consider the following two sets:

\[
S_1 := \{ x \in \Omega \mid u_{k_j}(x) \to u(x) \text{ as } j \to \infty \} \quad \text{and} \quad S_2 := \{ x \in \Omega \mid u(x) = 0 \}.
\]

We mention that \( t = 0 \) is the unique discontinuous point for the function \( t \mapsto \mathbb{1}_{R}+(t) \). By construction, we have \(|S_1| = 0\). Besides, we obtain from (7.2) that \( Xu = 0 \) on \( S_2 \). Hence,

\[
\left( \int_{\Omega} |\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u)|^2 |Xu|^2 dx \right)^{\frac{1}{2}} = \left( \int_{\Omega \setminus (S_1 \cup S_2)} |\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u)|^2 |Xu|^2 dx \right)^{\frac{1}{2}}.
\]

In the set \( \Omega \setminus (S_1 \cup S_2) \), we have \( u_{k_j}(x) \to u(x) \neq 0 \), which implies by the continuity of \( \mathbb{1}_{R}+ \) in \( \mathbb{R} \setminus \{0\} \) that \( \mathbb{1}_{R}+(u_{k_j}) \to \mathbb{1}_{R}+(u) \). Moreover, by \( |\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u)|^2 |Xu|^2 \leq 4|Xu|^2 \) and the dominated convergence theorem, we finally obtain that

\[
\left( \int_{\Omega \setminus (S_1 \cup S_2)} |\mathbb{1}_{R}+(u_{k_j}) - \mathbb{1}_{R}+(u)|^2 |Xu|^2 dx \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad j \to +\infty.
\]

Consequently, \( u_+ \in H_{X,0}^{1,0}(\Omega) \). \( \Box \)

### 7.2. Degenerate weak parabolic maximum principle

In this part, we construct the weak parabolic maximum principle for the Hörmander operators.

**Proposition 7.6** (Weak parabolic maximum principle). For any given \( T > 0 \), let \( u : (0,T) \to H_{X}^{1}(\Omega) \) be a path that satisfies the following conditions:

1. For any \( t \in (0,T) \), the strong derivative \( \frac{du}{dt} \) exists in \( L^2(\Omega) \) and satisfies

\[
\frac{du}{dt} - \triangle_X u \leq 0,
\]

(7.12)

where \( \triangle_X \) is understood as an operator in \( \mathcal{D}'(\Omega) \).
2. For any $t \in (0, T)$,
   \[ u(\cdot, t) \leq 0 \quad \text{mod} \ H^1_{X,0}(\Omega). \]  
   \tag{7.13}

3. $u_+(\cdot, t) \to 0$ in $L^2(\Omega)$ as $t \to 0$.

Then $u(\cdot, t) \leq 0$ for all $t \in (0, T)$.

Proof. The inequality (7.12) implies that for any fixed $t \in (0, T)$ and any $v \in C_0^\infty(\Omega)$, we have
   \[ \int_\Omega \frac{du}{dt}vdx \leq - \int_\Omega Xu \cdot Xvdx. \]  
   \tag{7.14}

Owing to Proposition 7.4, we can extend (7.14) to all $v \in H^1_{X,0}(\Omega)$ with $v \geq 0$.

Consider the following smooth bump function
   \[ h_0(s) := \begin{cases} 
   e^{-\frac{1}{(s-1)^2}}, & 0 < s < 2; \\
   0, & s \geq 2 \text{ or } s \leq 0.
   \end{cases} \]

Let $\varphi$ be the solution of following ODE in $\mathbb{R}$:
   \[
   \begin{cases}
   \varphi''(s) = h_0(s), & \text{for } s \in \mathbb{R}; \\
   \varphi'(0) = 0, \\
   \varphi(0) = 0.
   \end{cases}
   \tag{7.15}
   
It follows that $\varphi \in C^\infty(\mathbb{R})$ satisfying
   \[
   \begin{cases}
   \varphi(s) = 0, & s \leq 0; \\
   \varphi(s) > 0, & s > 0; \\
   0 \leq \varphi'(s) \leq C, & s \in \mathbb{R}; \\
   \varphi'(s) > 0, & s > 0.
   \end{cases}
   \tag{7.16}
   
Here $C > 0$ is some positive constant.

Combining (7.13) and Proposition 7.5, $u_+(\cdot, t) \in H^1_{X,0}(\Omega)$ for any $t \in (0, T)$. Therefore, by Proposition 7.3 and (7.16) we have $\varphi(u(\cdot, t)) = \varphi(u_+(\cdot, t)) \in H^1_{X,0}(\Omega)$ and $X\varphi(u) = \varphi'(u_+)Xu_+ = \varphi'(u)Xu$. Setting $v = \varphi(u(\cdot, t))$ as the test function in (7.14), we obtain
   \[ \int_\Omega \frac{du}{dt} \varphi(u(x,t))dx \leq - \int_\Omega Xu \cdot X\varphi(u(x,t))dx = - \int_\Omega \varphi'(u)|Xu|^2dx. \]  
   \tag{7.17}

On the other hand, we have
   \[ \frac{d}{dt} \int_\Omega u(x,t)\varphi(u(x,t))dx = \int_\Omega \frac{du}{dt}(\varphi(u) + \psi(u))dx, \]  
   \tag{7.18}

where $\psi(s) := \varphi'(s)s$. Clearly, $\psi(s) = 0$ for $s \leq 0$, and $\psi(s) > 0$ for $s > 0$. Observing that
   \[ \psi'(s) = \varphi''(s)s + \varphi'(s), \]  

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we can deduce from (7.15) and (7.16) that \(0 \leq \psi'(s) \leq C\) for \(s \in \mathbb{R}\), and \(\psi'(s) > 0\) for \(s > 0\). That means \(\psi \in C^\infty(\mathbb{R})\) satisfying (7.16) for suitable constant \(C > 0\). Using the similar arguments, we can deduce that

\[
\int_\Omega \frac{du}{dt} \psi(u) dx = - \int_\Omega \psi'(u)| Xu |^2 dx \leq 0. \tag{7.19}
\]

It derives from (7.17)-(7.19) that the function \(t \mapsto \int_\Omega u(x,t)\varphi(u(x,t))dx\) is decreasing on \((0,T)\). Since \(\varphi(s) \leq Cs\) for all \(s \geq 0\), we obtain that

\[
0 \leq \int_\Omega u(x,t)\varphi(u(x,t))dx = \int_\Omega u_+(x,t)\varphi(u_+(x,t))dx \leq C \int_\Omega |u_+(x,t)|^2 dx.
\]

By \(u_+(\cdot, t) \to 0\) in \(L^2(\Omega)\) as \(t \to 0^+\), we have \(\lim_{t \to 0^+} \int_\Omega u(x,t)\varphi(u(x,t))dx = 0\), and thus

\[
\int_\Omega u(x,t)\varphi(u(x,t))dx = \int_\Omega u_+(x,t)\varphi(u_+(x,t))dx = 0 \quad \forall t \in (0,T).
\]

Consequently, \(u_+(x,t) = 0\) for all \(t \in (0,T)\). \(\square\)

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