Electronic properties of the Dirac and Weyl systems with first- and higher-order dispersion in non-Fermi-liquid picture

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We investigate the non-Fermi-liquid behaviors of the 2D and 3D Dirac/Weyl systems with low-order and higher order dispersion. The self-energy correction, symmetry, free energy, optical conductivity, density of states, and spectral function are studied. We found that, for Dirac/Weyl systems with higher order dispersion, the non-Fermi-liquid features remain even at finite chemical potential, and they are distinct from the ones in Fermi-liquid picture and the conventional non-Fermi-liquid picture. The Landau damping of the longitudinal excitations within random-phase-approximation (RPA) for the non-Fermi-liquid case are also discussed.

Keywords: non-Fermi-liquid behavior; Dirac semimetal; multi-Weyl semimetal; self-energy correction; longitudinal excitations.

1 Introduction

Different to the metals where the Fermi-liquid theory is valid, for the topological insulator or when near the quantum critical in modern condensed matter physics, the non-Fermi-liquid theory is required. In Fermi-liquid theory, the excitations near the Fermi surface (usually within the order parameter fluctuation gap) are Fermionic, that results in the uniform spin susceptibility\cite{1} in contrast to the one in topological insulator\cite{2}, and also leads to the linear-temperature-dependence of the electronic specific heat rather than the logarithmic divergent one as found in heavy-fermion system as well as the superconductors. In this letter, we investigate the non-Fermi-liquid behaviors of the 2D Dirac system with first-order dispersion in continuum model. The exchange-induced Fermionic and Bosonic self-energy correction as well as other observable quantities are calculated and also for these of the Dirac/Weyl systems with higher dispersion. Considering the disorder effect origin from the impurities, the polaron as a excited quasiparticle are important when consider the many-body effect (many-electron effect), and the disorder-induced self-energy\cite{3} describes the impurity-Fermions (for Fermionic polaron) or impurity-Bosons (for Bosonic polaron) interaction and with the impurity dressed by the corresponding particle-hole excitations. Besides, since the spin rotation is missing in the Dirac $\delta$-type impurity field, the spin structure is fixed, and the spin of impurity and that of majority particles are usually opposite, that provies the opportunity to form the Cooper pair and the strongly bound dimer.

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2 Self-energy correction in 2D Dirac system

The isotropic (2+1)D Dirac Fermions coupled to long-range Coulomb interaction can be described by the effective action

\[ S = \int d\tau d^2 r \{ \psi^\dagger [\partial_\tau - ig\phi + H_0(k)]\psi + \frac{1}{2}[(\partial_x \phi)^2 + (\partial_y \phi)^2] \}, \] (1)

where \( \psi \) is the Fermion field and \( \phi \) is the Bosonic field which describes the long-range instantaneous Coulomb interaction and related to the order parameter, and the Fermions couples \( H \) constant. The \( \psi \) where \( \phi \) is the Fermion field and \( \psi \) is the Fermion field.

\[ H_0(k) = \hbar v_F(\eta k_x \tau_x + k_y \tau_y) + Ds_z \tau_z - \mu, \] (2)

whose eigenvalues can be obtained by solving \( \det(H - E) = 0 \) as

\[ E_{k\pm} = \pm \sqrt{\hbar^2 v_F^2 k^2 + D^2 + \mu^2 - 2\eta \hbar v_F k \mu \cos \theta - \mu} \approx \pm \sqrt{\hbar^2 v_F^2 k^2 + D^2 - \mu}, \] (3)

where \( \theta = \arctan \frac{k_y}{k_x} \). By defining the scattered momentum as \( k' = k + q \), the exchange-induced Fermion self-energy is given by

\[ \Sigma(q) = -g^2 \int \frac{d\Omega}{2\pi} \frac{d^2 k}{(2\pi)^2} g_0(\Omega + i0, k') D_0(k), \] (4)

and its independent of the Boson Matsubara frequency \( \omega = 2m\pi T \) due to the instantaneous approximation of the Coulomb interaction in one-loop order. The instantaneous Coulomb interaction is given by the scalar potential which has the propagator

\[ \langle T\phi(t)\phi(t',r') \rangle = -i\delta(t-t') \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} e^{ik(r-r')} \] (5)

and that leads to the nonrelativistic features with the broken Lorentz invariance, as widely seen in the non-perturbative RG analysis \[4\] (while in perturbative RG analysis the instantaneous approximation is sometimes unadopted due to the effect of vector potential \[5\]).

\[ G_0(\Omega + i0, k') = \frac{1}{\Omega + i0 - H_0} \] is the bare Green’s function (Fermion propagator). The infinitesimal quantity \( i0 \) (corresponds to the scattering rate or the Fermionic damping rate) is important for the convergence of the integral and its sign is the same as that of the frequency (here we assuming the positive frequency). The Pauli exclusion principle also enforces \( i0 \rightarrow 0 \) in the static limit. \( D_0(k) = 1/k^2 \) is the bare Boson propagator. Thus for static case, we have for a single Fermion species

\[ \Sigma(q) = -\frac{g^2}{(2\pi)^2} \left[ \text{Li}_2\left(\frac{-D + \hbar v_F \Lambda + \mu + \Omega + i0}{\hbar v_F \Lambda}\right) - \text{Li}_2\left(\frac{-D - \hbar v_F \Lambda + \mu + \Omega + i0}{\hbar v_F \Lambda}\right) \right. \]

\[ + \ln(D - \hbar v_F \Lambda - \mu - \Omega - i0) \ln(\frac{-D + \mu + \Omega + i0}{\hbar v_F \Lambda}) \]

\[ - \ln(D + \hbar v_F \Lambda - \mu - \Omega - i0) \ln(\frac{-D + \mu + \Omega + i0}{\hbar v_F \Lambda}) - (\ln(-\Lambda) - \ln\Lambda) \ln(-D + \mu + \Omega + i0) \] (6)

|\( \Omega \)}
Unlike the momentum shell integration, we only applied the ultraviolet cutoff here to deal the non-Fermi liquid case with nearly zero gap (and chemical potential). Ultraviolet cutoff during the calculation is important to prevent the divergence of integral. At higher temperature, the repulsive Coulomb interaction competes with the attractive electron-phonon coupling. The unscreened on-site Coulomb repulsion averts the double occupation of the lattice sites and thus closing the gap, while the electron-phonon coupling is opposite. The lowest-order contribution to the exchange-induced self-energy reads

$$\Sigma(k, \omega) = T \sum_\Omega \int \frac{d^2 q}{(2\pi)^2} \frac{U V(q, \Omega)}{G(k - q, \omega - \Omega)} G(k, \Omega) G(k', \Omega'),$$

(7)

where $U$ is the Coulomb repulsion potential and

$$V(q, \omega) = - \int \frac{d^2 q}{(2\pi)^2} \frac{G(k, \Omega) G(k', \Omega')}{G(k, \Omega) G(k', \Omega')} \frac{1}{2 \pi},$$

(8)

is the fluctuation exchange potential. Here we approximate the irreducible vertex function to the on-site Hubbard interaction, and the resulting exchange self-energy is obviously beyonds the instantaneous approximation

while for the attractive phonon-mediated interaction, similarly, the self-energy reads

$$\Sigma(k, \omega) = T \sum_\Omega \int \frac{d^2 q}{(2\pi)^2} U_{e-ph} P(q, \Omega) G(k - q, \omega - \Omega),$$

(9)

with the phonon propagator (in lowest-order)

$$P(q, \Omega) = \frac{\Omega_{ph}^2}{\Omega_{ph}^2 + \Omega^2},$$

(10)

where $U_{e-ph}$ is the electron-phonon coupling parameter and $\Omega_{ph}$ is the phonon frequency. For strong enough on-site attractive Hubbard interaction, the charge-density-wave (CDW) phase or the gapless semimetal phase will unstable to the s-wave superconducting phase, and thus the symmetry described by $\langle \psi^+ | \sigma_{x/y} | \psi^- \rangle = 0$ is broken (± refers to the up and down spin respectively) which don’t consider the orbital degree of freedom. For 2D Dirac semimetal, due to the absence of the impurity scattering in the Dirac point with zero density of states, the short-range interaction is weak and insufficient to destabilize the Dirac Fermions. For superconducting phase without the Coulomb repulsion (instantaneous Coulomb interaction) and the disorder, the Lorentz invariance is possible with isotropic Fermion and Boson velocities (i.e., in case of the supersymmetry which interchanges bosons and fermions), which can be realized at low-energy by a metallic (polarizable) superstrate. Through the minimum model in Eq.(2), the time-reversal symmetry can be shown as

$$\Theta H_0(k) \Theta^{-1} = \tau_y H_0^*(k) \tau_y = h v_F (-\eta k_x \tau_x - k_y \tau_y) + D \tau_z \sigma_z - \mu = H_0(-k),$$

(11)

where $\Theta = i \tau_y \hat{K}$ is the time-reversal operator. While for the 2D lattice model

$$H_0^l(k) = \sum_k c_k^\dagger [t \sin k_x \tau_y + d_z(k) \tau_z] c_k - \sum_k c_k^\dagger \mu c_k,$$

(12)

where $d_z(k)$ is the momentum-dependent gap function. For this lattice model, the particle-hole symmetry at half-filling can be revealed by

$$\Xi H_0^l(k) \Xi^{-1} = \tau_y H_0^{l*}(k) \tau_y = \sum_k c_k^\dagger [-t \sin k_x \tau_y - d_z(k) \tau_z] c_k = H^l_0(-k),$$

(13)

3
where $\Xi = \tau_y K$ is the particle-hole operator. Topologically, $\Theta^2 = 0, \pm 1$ corresponds to the time-reversal symmetry and $\Xi^2 = 0, \pm 1$ corresponds to the particle-hole symmetry. Although the time-reversal symmetry and the inversion symmetry are broken in the presence of gap function or by the charge-density-wave (CDW) order formed by polarized electrons, the symmetry described by the product of $\Theta$ and the in-plane mirror reflection operator $M_x$ could be preserved, i.e., $\Theta M_x$ which protect the semimetal nature against the weak short-range interaction. The weak short-range interaction can’t be taken into account the RG analysis, while the frequency-dependent self-energy in a non-Fermi liquid system is proportional to the anomalous dimension and the RG parameter (the logarithmic term). The anomalous dimension also implies the missing of the pole structure of the Green’s function, which correspond to the electron addition and removal energies in the noninteracting case.

In one-loop order, the bare Boson propagator (phonon or the photon) is modified as $D(\Omega, k) = (k^2 - \Sigma_b(\Omega, k))^{-1}$ where $\Sigma_b(\omega, k)$ is the Boson self-energy in density-density correlation form:

$$\Sigma_b(\omega, q) = g_s g_v T^2 g^2 \int \frac{d^2 k}{(2\pi)^2} \text{Tr}[\sigma_0 G_0(\Omega', k')\sigma_0 G_0(\Omega, k)]$$

$$= g_s g_v T^2 g^2 \int \frac{d^2 k}{(2\pi)^2} 4 \sum s s' \frac{k k' + D^2}{\varepsilon_\varepsilon' + \varepsilon_\varepsilon' + \omega + i0}$$

where we use the discrete values of the frequency since otherwise the above integral becomes zero, and the formula

$$T \sum_{\Omega} \frac{1}{-\Omega - i0 - \mu + \varepsilon_\varepsilon' - \Omega - i0 + \varepsilon_\varepsilon'} = \frac{N_F(\varepsilon_\varepsilon_\varepsilon' + \varepsilon_\varepsilon_\varepsilon' + \omega + i0)}{\varepsilon_\varepsilon_\varepsilon' - \varepsilon_\varepsilon_\varepsilon' + \omega + i0}$$

is used. Here $\Omega' = \Omega + \omega$ and $g_s g_v = 2$ denotes the Fermion species (spin and valley degrees of freedom). $N_F(x) = (1 + e^{x/T})^{-1}$ is the Fermi-distribution function. The above expression also implies that the Boson self-energy is related to the equation of motion for the bare Green’s function. Through the Ward identity $\gamma = \partial_\xi \Sigma$ which is independent of both the external frequency and the scattering wavevector, the vertex function can be obtained as

$$\gamma = -\frac{g^2}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 - \Sigma_b(\Omega, q)} \left[ \frac{1}{q^2 - \Sigma_b(\Omega, q)} \right]$$

To simplify the calculation, we restrict us to the gapless case, then the Boson self-energy becomes

$$\Sigma_b(\Omega, q) = g_s g_v g^2 \int \frac{dk}{(2\pi)^2} \frac{1}{4} \left[ \frac{k k'}{\varepsilon_\varepsilon' + \varepsilon_\varepsilon' - \Omega - i0} \right]$$

$$= g_s g_v g^2 \int \frac{kd k}{(2\pi)^2} \frac{1}{16 \hbar^2 v^2} \{ \text{ln}(\varepsilon_\varepsilon_\varepsilon' + \Omega + i0) - \text{ln}(\varepsilon_\varepsilon_\varepsilon_\varepsilon' + 2\hbar v_F \Lambda) \}$$

where $s s' = -1$ here due to the dominating interband transition. Then the vertex function at half-filling is

$$\gamma = -\frac{g^2}{2(2\pi)^2} \int dq \left[ \frac{q}{q^2 - 2a \hbar v_F \Lambda + (\text{ln}(\varepsilon_\varepsilon_\varepsilon' + \Omega + i0) - \text{ln}(\varepsilon_\varepsilon_\varepsilon_\varepsilon' + \Omega + i0)) \right]$$

$$- \text{ln}(\varepsilon_\varepsilon_\varepsilon_\varepsilon' + \Omega + i0)]^{-1}.$$
3 Disorder effect and free-energy in grand-canonical ensemble

For the disordered system, the Fermion self-energy in lowest-order approximation reads
\[ \Sigma_{LO} = n_i V_{k,k'}/\hbar \]
where \( n_i \) is the impurity concentration and \( V_{k,k'} \) is the scattering potential. For localized potential, the disorder-induced self-energy could be momentum-independent due to the rotational invariance. In low-order perturbation theory, the self-energy matrix contains a gap function in the diagonal element, while the non-diagonal elements are missing.

The random-phase approximation (RPA) results is valid only in the long-wavelength limit as well as the low-energy limit for large flavor number analysis, in which case the Eliashberg theory as well as the Migdal’s theorem are valid. In this case the Boson propagator (as well as the Boson-frequency-related spin susceptibility) is overdamped due to the small Boson velocity and small external Boson momentum (compared to the Fermionic ones) with the Landau damping. The above results are correct for the low-energy Fermions excitations (within the band gap) for RPA which with chemical potential much larger than \( k_B T \). Inversely, the non-Fermi-liquid feature emerges for the case of \( \mu < k_B T \). The strong screening effect by polarized Fermions to the disorder also provides the possibility to recover the Fermi liquid within the spectrum gap of the order parameter fluctuations (of the order of \( D^2/W \) where \( W \) is the Fermion bandwidth) which with coherent Bosonic excitations except when the disorder-induced linewidth\[^{10}\] is larger than the excitation energy. In the mean time, the excitation gap gives rise to the dissipation effect which is related to the free energy and the conductivity. Further, the response function is nonzero even at \( q = 0 \) for the Bosonic frequency in the range \( \omega < \nu_F q < 2D < 2\mu \). While for the Bosonic frequency larger than \( \nu_F q \), the transverse spin excitation (still within the band gap) is the Goldstone spin wave and thus it’s gapless in the long-wavelength limit, in contrast to the longitudinal excitations which is gapped even at \( q = 0 \).

Unlike the weak short-range interaction, the electron-electron interaction mediated by the gapless Bosonic mode is long ranged at the quantum critical point and with the gapless critical fluctuation of the order parameter (about the Bosonic excitations) which can be described by the Ginzburg-Landau function. The Ginzburg-Landau function (the free energy) here describing the order parameter fluctuation does not contains the term \( \phi^* \partial_t \phi \) due to the particle-hole symmetry as state above, where \( \phi \) is the two component complex amplitude. For bipartite system, the particle-hole symmetry suggests the exitence of the zero-energy modes which satisfy \( \phi_A = \pm i\phi_B \). In the presence of particle-hole symmetry the ac Hall conductivity vanishes while the dc conductivity is preserved\[^{11}\]. At finite temperature, the free energy can be obtained by the following partition function base on the Fermion propagator as
\[
Z = \det \left[ \frac{1}{G_0(k,\Omega)T} \right] = \prod_{\epsilon} \left( \frac{\Omega + i0 - H_0}{T} \right)^2, \quad (19)
\]
then the free energy density reads
\[
F = -T \lim_{n \to \infty} \int \frac{d^2 k}{(2\pi)^2} \ln Z = -T \lim_{n \to \infty} \int \frac{d^2 k}{(2\pi)^2} \ln \left[ \frac{H_0^2}{T^{2n+2}} \frac{\Gamma(1-H_0+n)}{\Gamma(1-H_0)^2} \right]. \quad (20)
\]
The above integral can be analytically solved as

\[ F = -T \lim_{n \to \infty} \frac{1}{2\pi i} \int \frac{1}{2} 2\pi \hbar^2 v_F^2 k^2 \ln \left( -\frac{2D^2}{\hbar^2 v_F^2} \right) \left[ -2D^2 \ln(H_0) - 2\hbar^2 v_F^2 k^2 \ln\left( -H_0 + n + 1 \right) + 2\hbar^2 v_F^2 k^2 \ln\left( -H_0 + 1 \right) \right. \\
+ \left\{ \hbar^2 v_F^2 k^2 \ln\left( \frac{H_0}{T^{2n+2}} - \psi(-3) \left( -H_0 + n + 1 \right) + 4\psi(-3) \left( -H_0 + 1 \right) \right) \\
- 4\hbar v_F k \psi(-2) (-H_0 + n + 1) + 4\hbar v_F k \psi(-2) (-H_0 + 1) - 2\mu^2 \ln(H_0) + 4D \mu \ln(H_0) \right\} \right|_k, \]

and for semimetal at half-filling it reduces to

\[ F = -T \lim_{n \to \infty} \frac{1}{2\pi i} \int \frac{1}{2} 2\pi \hbar^2 v_F^2 k^2 \ln \left( -\hbar v_F k + n + 1 \right) + \hbar^2 v_F^2 k^2 \ln\left( \frac{H_0}{T^{2n+2}} - \psi(-3) \left( -\hbar v_F k + n + 1 \right) \right) \\
+ 4\hbar v_F k \psi(-2) (-\hbar v_F k + 1) + 4\psi(-3) (-\hbar v_F k + 1) + k^2 \ln(1 - \hbar v_F k) \right\} \right|_k, \]

where \( \ln \Gamma \) denotes the logarithm of the Gamma function and \( \psi^{(n)} \) is the \( n \)th derivative of the digamma function. By using the approximational relation [12]

\[ \ln \left( \frac{\omega^2 + H_0^2}{T^2} \right) \approx \frac{H_0}{T} + 2\ln(1 + e^{-H_0/T}), \]

then the free energy density can be rewritten as

\[ F = -T \left[ 1 \right] \frac{2T^2 \text{Li}_2(-e^{\hbar v_F k / T})}{\hbar^2 v_F} + \frac{1}{3} k^2 \left( \frac{2\hbar v_F k}{T} + 3\ln N_F(-\hbar v_F k) - 3\ln N_F(\hbar v_F k) \right) \\
- \frac{2kT \text{Li}_2(-e^{\hbar v_F k / T})}{\hbar v_F} \right|_k, \]

where \( \text{Li}_n(x) \) is the polylogarithm function. Consider the many-body effect to the 2D Dirac system, the perturbations can be taken into account in the grand-canonical ensemble, where we rewrite the tight-binding model Hamiltonian as

\[ H = H_0 + H_E + H_D \]

\[ = \sum_{ij} (tc_i c_j) + \sum_{ij} \frac{1}{2} U_{ij} n_i n_j \\
+ V_0 \sum_i c_i^\dagger c_i \delta(r - r_i), \]

where \( ij \) is the nearest neighbor sites and \( t \) is the nearest neighbor hopping. \( n = c_i^\dagger c_j \). \( U_{ij} \) is the Coulomb interaction strength in second term which is the Coulomb exchange interaction-related term (bilinear). \( V_0 \) is the impurity scattering potential (magnetic or nonmagnetic) in the third term which is the disorder-related term. The creation and annihilation operators are all the particle one here. Then the free energy density (grand potential) is still \( F = -T\ln Z \) but with the partition function in interacting case as

\[ Z(i\Omega, U_{ij}, V_0) = \text{Tr} \exp\left[ -\frac{H + N\mu}{T} \right] \]

\[ = \int D\psi D\psi^* e^S, \]

\[ = \int D\psi D\psi^* e^S, \]
where the path integral runs over the Grassman variables, $N$ is the particle number operator, and with the action $S$ reads

$$S(i\Omega, U_{ij}, V_0) = \sum_{\delta_{ij}} \psi(i\Omega) [(i\Omega + \mu) \delta_{ij} - H_0 - H_D] \psi^*(i\Omega) - \int_0^{1/T} d\tau H_E(\tau), \quad (27)$$

where $\psi(i\Omega)$ is the real Grassmann variable, and the first term is always positive summing over the Fermionic frequency. The perturbaed Green’s function which satisfies the Dyson relation $G = G_0 + G\Sigma G_0$ (or $\Sigma = G_0^{-1} - G^{-1}$), can be obtained by the ratio of the partition functions as

$$G(i\Omega, U_{ij}, V_0) = -\frac{Z'(i\Omega, U_{ij}, V_0)}{Z(i\Omega, U_{ij}, V_0)} = -\frac{\int D\psi D\psi^* e^{S}\psi(i\Omega)\psi^*(i\Omega)}{\int D\psi D\psi^* e^{S}}, \quad (28)$$

The case of $G(i\Omega, U_{ij}, V_0) \neq 1$ clearly indicates broken of the supersymmetry[13]. We consider the $\delta$-type impurity potential in above disorder-term, which indicates the Born approximation. In such case the spin structure is fixed as also been observed in the surface state of the 3D topological insulators, thus the spin rotation is missing which can be observed in the nodal-line semimetals, and the spin current operator becomes zero in the helicity basis. The Born approximation guarantees the sign-invariance of the momentum before and after the scattering, and the reversed scattering amplitude has the same value with the origin one[14]. In grand canonical ensemble, the spin current vanishes in the thermodynamical limit (and thus with infinite $\mu$) due to the vanishing spin density even when beyong the Born approximation. Beyong the $\delta$-type impurity field, the extrinsic spin current emergents and the scattering of both the impurity and the majority particle (with opposite spins) create the Fermionic polaron, and the optical Hall conductivity (in fact this is the only case where the transverse conductivity equals to the Hall conductivity) of the polaron determines the current in direction orthogonal to the external force[13], is related to the current operator by $\sigma_{xy} = J_x/(-\nabla_y V)$ in linear response theory where $V$ is the external potential. The current operator here is much smaller than the one in QED just like the group velocity operator which is much smaller than the speed of light. While for the Bosonic polaron for the system immersed in the Bose-Einstein condensates, the interaction is stronger than the Fermionic one due to the higher compresibility of the BEC compared to the Fermionic media. The Bosonic polaron is formed by the Fermionic impurity which dressed by the majority Bosonic excitations.

### 4 Optical conductivity

The destoryed Fermi-liquid behavior can be observed by the singular Bosonic susceptibility at the nesting wavevector, and it can’t be found even at low-energy limit (far away from the quantum critical point) when the ultraviolet cutoff applied is infinite during the calculation, as could be found in the many-electron system, for example, when the Fermions coupled to the 1D Ising variable[16] or the fluctuation transverse gauge field[17] or the longitudinal Bosonic excitation[1]. The non-Fermi-liquid phenomenons are widely observed in the heavy-Fermi sys-

**7**mtem and the cuprate materials[18, 19], including the logarithmic divergent specific heat which related to the free energy by $C_V = -T \frac{\partial^2 E}{\partial T^2}$. In the presence of monochromatic light, the
nondiagonal part of the optical conductivity can be obtained by summing over the eigenvalues:

\[
\sigma_{xy}(\omega) = i\hbar e^2 \int \frac{d^2k}{(2\pi)^2} \sum_{\varepsilon, \varepsilon'} \frac{[N_F(\varepsilon) - N_F(\varepsilon')] \langle \varepsilon | v_x | \varepsilon' \rangle \langle \varepsilon' | v_y | \varepsilon \rangle}{(\varepsilon - \varepsilon') (\varepsilon - \varepsilon' + \omega + i0)}
\]

\[
= i\hbar e^2 \int \frac{d^2k}{(2\pi)^2} \sum_{\varepsilon, \varepsilon'} \frac{[N_F(\varepsilon) - N_F(\varepsilon')] \frac{\omega}{\varepsilon} [(1 - \frac{\delta}{\varepsilon})(1 + \frac{\delta}{\varepsilon}) - (1 - \frac{\delta}{\varepsilon})(1 + \frac{\delta}{\varepsilon})]}{(\varepsilon - \varepsilon') (\varepsilon - \varepsilon' + \omega + i0)},
\]

where only the retarded Green’s function is used in contrast to the Streda one\[^{20}\]. Here the identify

\[
N_F(\varepsilon)(1 - N_F(\varepsilon'))(1 - \varepsilon' / T) = N_F(\varepsilon) - N_F(\varepsilon').
\]

is used. The velocity matrix elements are

\[
\langle \varepsilon | v_x | \varepsilon' \rangle = \frac{v_F}{2} [s \sqrt{1 + \frac{D}{\varepsilon}} \sqrt{1 - \frac{D}{\varepsilon'}} + s' \sqrt{1 - \frac{D}{\varepsilon}} \sqrt{1 + \frac{D}{\varepsilon'}} (1 - \delta_{ss'}),
\]

\[
\langle \varepsilon' | v_y | \varepsilon \rangle = \frac{iv_F}{2} [s' \sqrt{1 + \frac{D}{\varepsilon}} \sqrt{1 - \frac{D}{\varepsilon'}} - s \sqrt{1 - \frac{D}{\varepsilon}} \sqrt{1 + \frac{D}{\varepsilon'}} (1 - \delta_{ss'}),
\]

where the electron/hole indices have \( ss' = -1 \) during the optical transition due to the Pauli exclusion principle, while the spin indices \( \sigma \) before and after transition are invariant when (both the intrinsic and extrinsic) Rashba-coupling are negligible, otherwise the spin index changes since it’s no more the good quantum number. In case that the Fermi level lies within the band gap, the diagonal elements of the conductivity are zero (and thus implies the \( C_4 \) symmetry of the system since \( \sigma_{xx} = \sigma_{yy} \)), while the non-diagonal elements becomes independent of the Dirac-mass due to the vanishing classical term\[^{21}\]. Note that the result of \( \sigma_{xx} = \sigma_{yy} \) as well as the \( \sigma_{xy} = -\sigma_{yx} \) also appear in the optical limit with \( q \rightarrow 0 \) (also called the local limit) Taking into account the effect of chemical potential, then the Hall conductivity is composed of two parts:

\[
\sigma_{xy}(\omega) = i\hbar e^2 \int \frac{d^2k}{(2\pi)^2} \Theta(\Lambda - \varepsilon) \left[ \frac{\langle \varepsilon | v_x | \varepsilon' \rangle \langle \varepsilon' | v_y | \varepsilon \rangle}{(\varepsilon - \varepsilon') (\varepsilon - \varepsilon' + \omega + i0)} + \frac{\langle \varepsilon' | v_x | \varepsilon \rangle \langle \varepsilon | v_y | \varepsilon' \rangle}{(\varepsilon - \varepsilon') (\varepsilon + \varepsilon' + \omega + i0)} \right],
\]

\[
\sigma_{xy}(\omega, \mu) = i\hbar e^2 \int \frac{d^2k}{(2\pi)^2} \Theta(\mu - \varepsilon) \left\{ \frac{\langle \varepsilon' | v_x | \varepsilon \rangle \langle \varepsilon | v_y | \varepsilon \rangle}{(\varepsilon + \varepsilon') (\varepsilon - \varepsilon' + \omega + i0)} + \frac{\langle \varepsilon' | v_x | \varepsilon \rangle \langle \varepsilon | v_y | \varepsilon' \rangle}{(\varepsilon - \varepsilon') (\varepsilon + \varepsilon' + \omega + i0)} \right\}
\]

\[
- \left[ \frac{\langle \varepsilon | v_x | \varepsilon' \rangle \langle \varepsilon' | v_y | \varepsilon \rangle}{(\varepsilon - \varepsilon') (\varepsilon - \varepsilon' + \omega + i0)} + \frac{\langle \varepsilon' | v_x | \varepsilon \rangle \langle \varepsilon | v_y | \varepsilon' \rangle}{(\varepsilon - \varepsilon') (\varepsilon + \varepsilon' + \omega + i0)} \right],
\]

where the first part corresponds to the case that the chemical potential is small than the Dirac-mass while the second part is opposites. While the longitudinal optical conductivity reads

\[
\sigma_{xx}(\omega) = \sigma_{xx}^{\text{inter}}(\omega) = -e^2 \pi \hbar \int \frac{d^2k}{(2\pi)^2} \frac{|\langle \varepsilon | v_x | \varepsilon' \rangle|^2}{\varepsilon - \varepsilon'} \delta(\omega + i0 + \varepsilon - \varepsilon'),
\]

where the intraband part vanishes unless at finite temperature and at infrared limit (nearly zero photon energy).

Different to the hopping-current-related conductivity, the dissipation-current-related conductivity remains finite in static limit and proportional to \( i / \omega + \pi \delta(\omega) \)\[^{11}\] where \( \delta \) is the Kronecker \( \delta \) function here, but this part of the conductivity is negligible when under a magnetic field or at low-temperature. For the case of large band gap, the frequency about the optical transition during the intraband process is much larger than the interband one as shown in the WSe\(_2\)\[^{21}\].
5 Observable quantities in Dirac/Weyl systems with higher-order dispersion

Next we discuss the 2D topological insulator (TI) with higher-order dispersion (similar to the multi-Weyl semimetal) and small (but momentum-dependent) gap, whose Hamiltonian reads

\[
H_0^m = \frac{k_0^2-m^2 a}{2} \hbar v_F k^m (\hat{k} \cdot \sigma) + (c_1 + c_2 k^m) \sigma_z \tau_z - \mu, \tag{34}
\]

where we assume the Dirac-mass is momentum-dependent and controlled by the material-related constant \(c_1\) and \(c_2\), the \(k_0\) is another material-related constant in unit of momentum \([22]\) and \(a\) is the lattice spacing \([23]\), e.g., it equals to \(\sqrt{3}/2\) times of the lattice constant in the graphene-like hexagonal lattice system. The term \((\hat{k} \cdot \sigma)\) here only appears in the chiral systems with spin-momentum locking, while for the non-chiral systems, it’s usually replaced by the spin operator \(\sigma_z\) and the interband transition also vanishes in such case. The momentum-dependent mass term \((c_1 + c_2 k^m)\) here is similar to the effect of next-nearest-neighbor (intrinsic) Rashba coupling. Here we note that, we try to present a discussion for in the 2D Dirac system extend to the generic order \(m\) which also related to the Chern number in gapless case, and it’s not just applicable to the 2D TI, but also to the multilayer TI which with a single 2D Dirac node per surface Brillouin zone \([24, 25]\). The order \(m\) controls the in-plane band dispersion, for example, \(m = 1\) for the (topologically protected; which not exists currently in 3D real space \([26]\)) linear Dirac dispersion, \(m = 2\) for the quadratic dispersion, \(m = 3, 4\) for the trigonal warping system as found in the monolayer MoS\(_2\) \([27]\).

In such case the velocity operators can be obtained by using the relation \(v_\alpha = \frac{\partial H_0^m}{\partial \vec{k}_\alpha}(\alpha = x, y, z)\):

\[
v_x = \begin{pmatrix}
(m\frac{k_0^2-m^2 a}{2} \hbar v_F k^m)^{1-\frac{1}{m}} e^{-i(m-1)\theta} & 0 \\
0 & (m\frac{k_0^2-m^2 a}{2} \hbar v_F k^m)^{1-\frac{1}{m}} e^{i(m-1)\theta}
\end{pmatrix} \sigma_x + m c_2 k^{m-1} \cos \theta \sigma_3,
\]

\[
v_y = \begin{pmatrix}
0 & (m\frac{k_0^2-m^2 a}{2} \hbar v_F k^m)^{1-\frac{1}{m}} e^{-i(m-1)\theta} \\
(m\frac{k_0^2-m^2 a}{2} \hbar v_F k^m)^{1-\frac{1}{m}} e^{i(m-1)\theta} & 0
\end{pmatrix} \sigma_y + m c_2 k^{m-1} \sin \theta \sigma_3,
\]

where the relation \(\frac{\partial k_\alpha}{\partial k_\alpha} = \frac{k_\alpha}{k}\ (\alpha = x, y, z)\) is used. Here we write the third Pauli matrix as \(\sigma_3\) to distinguish from the spin operator in \(z\) direction. For the above multi-node dispersion, the eigenvectors are

\[
|\Psi_+\rangle = \frac{e^{ik_x x}}{\sqrt{2L_x}} \begin{pmatrix}
\sqrt{1+2\epsilon} \Phi_\epsilon(y) \\
\sqrt{1-2\epsilon} e^{im\theta} \Phi_\epsilon'(y)
\end{pmatrix},
\]

\[
|\Psi_-\rangle = \frac{e^{ik_x x}}{\sqrt{2L_x}} \begin{pmatrix}
-\sqrt{1-2\epsilon} \Phi_\epsilon(y) \\
\sqrt{1+2\epsilon} e^{im\theta} \Phi_\epsilon'(y)
\end{pmatrix},
\]

where we pick the momentum component \(k_x = (k^m)^{1/2} \cos \theta\) as a good quantum number, \(\Phi_\epsilon(y)\) is the harmonic oscillation wave function. Then the velocity matrix elements can be obtained base on the above velocity operators,

\[
\langle \epsilon | v_x | \epsilon' \rangle = \mp \frac{m \xi^{-1/m} \sqrt{2k^{-2} \xi^2 \epsilon^2 (1 + \cos(2\theta))}}{\sqrt{2k}} \left(1 - \delta_{s,s'}\right),
\]

\[
\langle \epsilon | v_y | \epsilon' \rangle = \mp \frac{idk^{-2} \xi^{-1/m} \sqrt{-2k^{-2} \xi^2 \epsilon^2 (1 - \cos(2\theta))}}{\sqrt{2k}} \left(1 - \delta_{s,s'}\right). \tag{37}
\]
In noninteracting case, the Bosonic propagator is overdamped with the gapless longitudinal excitations (or order parameter fluctuation) by the Landau damping, it’s also found that the Landau damping of the multi-Weyl semimetal is weaker than that of the marginal Fermi liquid which is distinct from the normal non-Fermi-liquid states. The dispersion of the multi-node Landau damping of the multi-Weyl semimetal is weaker than that of the marginal Fermi liquid excitations (or order parameter fluctuation) by the Landau damping, it’s also found that the

\[ \varepsilon^m = \pm \sqrt{c_1^2 + 2c_1c_2k^m + c_2^2k^{2m} + \mu^2 + k^{2m}\xi^2 - 2\eta k^m\mu\xi\cos\theta} \]

where we rewrite the term \( \frac{k^{2-m}}{2} \hbar v_F \) as the scale-dependent parameter \( \xi \). At half-filling, the Fermi level crosses the multi-band touching point and the Coulomb interaction remains long-ranged as described by the Bosonic field, due to the poor screening to the electron-electron Coulomb interaction. The imaginary part of the exchange-induced self-energy is related to the quasiparticle relaxation time (lifetime), while the real part is related to the interaction strength and the quasiparticle weight. Beyond the instantaneous approximation induced by the scalar potential, the exchange-induced self-energy containing Bosonic frequency reads

\[ \Sigma^m(\omega, k) = -g^2 \int \frac{d\Omega}{2\pi} \frac{d^2k}{(2\pi)^2} G_0(\omega + \Omega, k') \frac{1}{q^2 - \Sigma_b(\Omega, q)}, \]  

where the last term of the above expression is the dressed Coulomb potential. \( \Sigma^m_0(\Omega, q) \) is the Bosonic self-energy (i.e., the dynamical polarization here) which reads

\[ \Sigma_b^m(\Omega, q) = g_s g_v g^2 \int \frac{d\Omega}{2\pi} \frac{d^2k}{(2\pi)^2} \text{Tr}[\sigma_0 G_0^m(\Omega' k') \sigma_0 G_0^m(\Omega, k)]. \]

then the multi-Dirac-node bare Green’s function reads

\[ G_0^m(\Omega, k) = \frac{\Omega + i0 + H_0^m}{(\Omega + i0)^2 - H_0^{m2}} = \frac{\Omega + i0 + \xi k^m(\hat{k} \cdot \sigma) + (c_1 + c_2 k^m)\sigma_z \tau_z - \mu}{(\Omega + i0)^2 - (\xi k^m(\hat{k} \cdot \sigma) + (c_1 + c_2 k^m)\sigma_z \tau_z - \mu)^2}. \]

Base on this Green’s function, the dynamical polarization is available by the above expression but it’s too verbose to express which contains a hypergeometric function whose parameters are all related to the order \( m \). We then turn to a more concise expression at zero-temperature limit which reads

\[ \Sigma_b^m(\Omega, q) = \int \frac{d^2k}{(2\pi)^2} \sum_{ss'} \frac{g_s g_v}{2} \sum_{ss'} \frac{1 + ss'\cos b}{\Omega + i0 + s\varepsilon^m - s'\varepsilon^{m'}} \cos b = \frac{k + q\cos a}{\sqrt{k^2 + q^2 + 2kq\cos a}} - 1 - \frac{q^2\sin^2 a}{2k^2} + O(q^3), \]

\[ \varepsilon^m - \varepsilon^{m'} = -\frac{mqk^{m-1}(c_1 c_2 + c_2^2 k^m + \xi^2 k^m)}{\sqrt{c_1^2 + 2c_1c_2 k^m + c_2^2 k^{2m} + \xi^2 k^{2m}}} + O(q^2), \]

where \( b \) is the angle between \( k \) and \( k' \) and \( a \) is the angle between \( k \) and \( q \). The dynamical
polarization can be devided into the intraband and interband parts:

\[
\Sigma_b^m (\Omega, q) = \Sigma_b^{m,\text{intra}} (\Omega, q) + \Sigma_b^{m,\text{inter}} (\Omega, q),
\]

\[
\Sigma_b^{m,\text{intra}} (\Omega, q) = \frac{g_s g_v}{2} \int_0^{2\pi} d\theta \int_{-k_F}^{k_F} dk d\varepsilon \frac{1}{\varepsilon^\prime + i0 + \varepsilon_m - \varepsilon_m'} \left( \frac{1}{\varepsilon + i0 - \varepsilon_m + \varepsilon_m'}(1 + \cos \theta) - \frac{1}{\varepsilon + i0 - \varepsilon_m - \varepsilon_m'}(1 - \cos \theta) \right),
\]

\[
\Sigma_b^{m,\text{inter}} (\Omega, q) = \frac{g_s g_v}{2} \int_0^{2\pi} d\theta \int_{-k_F}^{k_F} dk d\varepsilon \frac{1}{\varepsilon^\prime + i0 + \varepsilon_m + \varepsilon_m'} \left( \frac{1}{\varepsilon + i0 - \varepsilon_m + \varepsilon_m'}(1 + \cos \theta) - \frac{1}{\varepsilon + i0 - \varepsilon_m - \varepsilon_m'}(1 - \cos \theta) \right),
\]

where the first term of the intraband part can be obtained after some algebra as

\[
\Sigma_b^{m,\text{intra}} (\Omega, q) = k^{1-m} \sqrt{c_1^2 + 2c_1c_2k_m + (c_2^2 + \xi^2)} k^{2m}
\]

\[
[4k^2(m-1)F_1(3-m, -1; -1/2, 1/2, 1/2; m; c_1 c_2 + \sqrt{-c_1^2\xi^2}, -c_1 c_2 + \sqrt{-c_1^2\xi^2})] - (m-3)q^2 \sin^2 \alpha F_1(1/m - 1; -1/2, 1/2, 1/2, m; c_1 c_2 + \sqrt{-c_1^2\xi^2}, -c_1 c_2 + \sqrt{-c_1^2\xi^2})],
\]

\[
2c_1 c_2 (m-3)mq \sqrt{-\sqrt{-c_1^2\xi^2} + c_1 c_2 + c_2 k_m + \xi^2 k_m} \frac{1}{c_1 c_2 - \sqrt{-c_1^2\xi^2} \sqrt{-c_1^2\xi^2} + c_1 c_2 + c_2 k_m + \xi^2 k_m} \bigg|_{k_F}^{\infty},
\]

where \( F_1(a; b, b'; c; d, d') = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)m!(b')_n}{m!n!(c)_{m+n}} d^m q^n \) is the Appell hypergeometric function. The other three terms can be obtained through the same way, and then the exchange-induced self-energy can also be obtained. Note that for the non-chiral Fermions, like the ones in 2D electron gas, \( \Sigma_b^{m,\text{inter}} (\Omega, q) \) vanishes since \( \cos \theta = 1 \).

While for the 3D Dirac semimetal [28, 29, 30] like the Na\textsubscript{3} Bi or Cd\textsubscript{3} As\textsubscript{2}, PtTe\textsubscript{2}, the the chiral anomaly emerges since in odd space dimensions the anti-commute relation about the \( \gamma_5 \) matrix is allowed, and each Dirac node resolved into two Weyl nodes [29] arrange along the \( z \)-direction of the momentum space and with opposite chirality. The Hamiltonian of the simplest 3D Dirac and Weyl semimetal are \( H = \sum_i h v_i k_i \sigma_i (i = x, y, z) \) and \( H = \sum_i h v_i k_i \sigma_i + \chi h v_z k_z - \chi \delta k_z (i = x, y) \), respectively. The chiral effect gives the sign \( \pm \) to \( \xi \). In 3D Dirac/Weyl semimetal, the perturbations can remove the nodal line and leave the nodes [31], while the nodes can not to be removed but can only to be shifted [24]. As we mentioned above, since the spin rotation is missing due to the Dirac \( \delta \)-type impurity field, the rotational invariance is presented, which is also partly due to the disorder averaging [32, 33], and thus the disorder-induced self-energy is independent of the external momentum, which reads

\[
\Sigma^D (\omega) = \frac{\eta_i v^2}{h^2} \int \frac{d^3 k}{(2\pi)^3} \Gamma_0 G_0^m T(\omega, k) \Gamma_0,
\]

where \( \Gamma_0 \) is the irreducible vertex function which doesn’t contains the Levi-Civita symbol here unlike the one in Ref. [24]. The vertex correction vanishes when it contains only the exchange-induced self-energy correction in instantaneous approximation, which can be obtained by the Ward identity \( \frac{\partial^2 \delta^2 \hat{G}(\Omega, k)}{\partial \omega \partial q} = \Gamma(\Omega, k) \), besides, the vertex correction also vanishes in the large species-case (large \( g \)) or when the integration momentum shell vanishes (the RG flow parameter \( \ell = 1 \). \( \eta_i \) here is the impurity concentration, \( V_0 \) is the impurity scattering potential (a scalar potential when only with the nonmagnetic impurity and without the magnetic impurity). For the 3D
Dirac-system, the Dirac nodes can be divided into the Weyl nodes along the \( z \)-direction by using the projection operator\(^{34} \), and the Hamiltonian reads

\[
H_0^{\text{mt}} = \frac{k_0^{2-m}a}{2} \hbar v_F k^m (\mathbf{k} \cdot \mathbf{\sigma}) + \chi \hbar v_z (k_z - \chi \delta k_z) \sigma_z + (c_3 + c_4 (k_z - \chi \delta k_z)^n) \sigma_z \tau_z - \mu_x, \quad (46)
\]

where \( \mu_x \) is the chemical potential with chirality \( \chi = \pm 1 \) and \( \mu_+ = \mu_- = \mu \) in undoped case. \( \delta k_z \) is the distance in momentum space removed from the previous Dirac node which explicitly breaks the time-reversal symmetry. \( v_z = \frac{\alpha}{\hbar} \sin(\delta k_z) \) is the \( z \)-direction velocity. Here we define \( k_x = k \cos \theta, k_y = k \sin \theta, k = k \sin \varphi, k_z = k \cos \varphi \), and still use the definition \( \frac{k_0^{2-m}a}{2} \hbar v_F = \xi \). The first term of the above Hamiltonian contains no out-of-plane components, which indicates the untitled type-I Weyl semimetal when the mass term is missing. The mass term is dominated by the momentum \( k_z \) here rather than the in-plane momentum as shown in the previous model and it explicitly breaks the inversion symmetry. Then eigenvalues can be obtained as

\[
\varepsilon^{\text{mt}} = -\mu - \hbar v_z \left( \delta k_z - \chi k_z \right) + \sqrt{[c_3 + c_4 (k_z - \chi \delta k_z)^n]^2 + k_0^{2m} \xi^2 + (\delta k_z - k_z \chi)^2 - 2 \eta \hbar v_z (\delta k_z - k_z \chi) \cos \theta} \approx -\mu - \hbar v_z \left( \delta k_z - \chi k_z \right) \mp \sqrt{[c_3 + c_4 (k_z - \chi \delta k_z)^n]^2 + k_0^{2m} \xi^2}. \quad (47)
\]

The imaginary part and real part of the bare Green’s function (Fermion propagator) \( G_0^{\text{mt}}(\omega, k) \) are

\[
\text{Im} G_0^{\text{mt}}(\omega, k) = -\pi \delta(\omega - \varepsilon^{\text{mt}}),
\]

\[
\text{Re} G_0^{\text{mt}}(i\omega, k) = \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega}{\omega^2 - (i\omega)^2} \text{Im} G_0^{\text{mt}}(\omega, k) = -\frac{2\varepsilon^{\text{mt}} \theta(\varepsilon^{\text{mt}})}{(\varepsilon^{\text{mt}})^2 + \omega^2}, \quad (48)
\]

where the Sokhotski-Plemelj theorem and Kramers-Kronig relation are used. \( \theta(x) \) here is the step function.

Using the eigenvalue

\[
\varepsilon^{\text{mt}} = -\mu - \hbar v_z \left( \delta k_z - \chi (k \cos \varphi) \right) \mp \sqrt{c_3^2 + (k \sin \varphi)^2 m \xi^2}, \quad (49)
\]

where we further set \( c_4 = 0 \), then after some algebra, the above disorder-induced self-energy can be obtained as

\[
\Sigma^D(\omega) = \frac{n_i V_0^2}{\hbar^2} \int_0^{2\pi} d\theta \int_0^\pi \frac{1}{m} (\sin \varphi)^{-2m} d\varphi \int_0^\Lambda k^2 \Gamma_0 G_0^{\text{mt}}(\omega, k) \Gamma_0
\]

\[
= \frac{2\pi n_i V_0^2}{m \hbar^2} \int_0^\pi (\sin \varphi)^{-2m} d\varphi
\]

\[
= \frac{1}{12} \Lambda^3 \left\{ -4 \delta k_z \hbar v_z \left[ 4 c_3^2 m \sqrt{\xi^2 (\Lambda \sin \varphi)^{2m}} + \frac{1}{2} F_1 \left( \frac{3}{2}, \frac{3}{2}; \frac{3}{2}; \frac{\xi^2 (\Lambda \sin \varphi)^{2m}}{c_3^2} \right) \right] \right\} \right\}
\]

\[
= \frac{12 \sqrt{c_3^2 + \xi^2 (\Lambda \sin \varphi)^{2m}}}{m + 3} + 3 \hbar v_z \Lambda \cos \varphi - 4 \mu
\]

\[
, \quad (50)
\]

where \( _2F_1(a, b; c; d) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} \frac{d^n}{n!} \) is the hypergeometric function. In such a 3D Dirac system, the Fermion spectral function in the absence of the quasiparticle scattering (noninteracting case) reads

\[
A_0^{\text{mt}} = \delta(\omega - \varepsilon^{\text{mt}}), \quad (51)
\]
where the locations of the sharp peaks are surely related to the above band dispersion. The longitudinal optical conductivity \( \sigma_{zz} \) is finite unless when a pair of Weyl nodes tilted in parallel direction in momentum space. The density of states for the 3D Dirac semimetal at half-filling can then be obtained as

\[
D^{mT} = \int \frac{d^3k}{(2\pi)^3} A_0^{mT} \\
= \int \frac{d^3k}{(2\pi)^3} \delta(\omega - (\hbar v_z k_z + k^m \xi)) \\
\approx \frac{1}{\pi} \int dk_z \int dk \frac{c}{c^2 + (\hbar v_z k_z + k^m \xi)^2} \\
= \text{const.} + \text{const.}k \\
- \frac{ikm}{2\hbar v_z \xi k_m - i\hbar v_z k_z} - \frac{\xi k_m}{-\hbar v_z k_z} - \frac{1}{m-1} F_1 \left( -1, m; m - 1; i\xi k_m + c - i\hbar v_z k_z \right) \\
- \frac{k \tan^{-1}(\frac{\xi k_m - i\hbar v_z k_z}{\hbar v_z})}{\hbar v_z},
\]

(52)

where \( c \) is the small quantity used in the Lorentzian representation.

Base on the spectral function in the absence of the self-energy correction, the optical conductivity per Weyl node in the presence of nonzero Fermionic Matsubara frequency can be obtained by the Kubo formula

\[
\sigma_{zz}(\omega) = \frac{\text{Im}\Pi_{zz}(\omega + i0)}{\omega},
\]

(53)

then we have

\[
\text{Re} \sigma_{zz}(\omega) = \epsilon^2 \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{N_F(\Omega - \mu) - N_F(\Omega' - \mu)}{\Omega' - \Omega + \omega + i0} \text{Tr}[\hat{v}_z A_0^{mT}(\Omega', k) \hat{v}_z A_0^{mT}(\Omega, k)] \\
= \frac{\epsilon^2}{2\omega + 2i0} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left( \frac{A_1^{mT}(\Omega') - A_3^{mT}(\Omega')}{A_2^{mT}(\Omega') - A_4^{mT}(\Omega')} \right),
\]

(54)

where we define \( \Omega' = \Omega + \omega \), \( A_4^{mT} \) is the spectral function after optical transition. The velocity operator is \( \hat{v}_z = v_z \sigma_z \). The components of the spectral function read

\[
A_1^{mT}(\Omega) = \pi[\delta(\Omega + \varepsilon^{mT}) + \delta(\Omega - \varepsilon^{mT})], \\
A_2^{mT}(\Omega) = \pi[\delta(\Omega + \varepsilon^{mT}) - \delta(\Omega - \varepsilon^{mT})], \\
A_3^{mT}(\Omega) = \pi[-\delta(\Omega + \varepsilon^{mT}) - \delta(\Omega - \varepsilon^{mT})], \\
A_4^{mT}(\Omega) = \pi[-\delta(\Omega + \varepsilon^{mT}) + \delta(\Omega - \varepsilon^{mT})],
\]

(55)
then the above optical conductivity can be rewritten as
\[
\text{Re}\sigma_{zz}(\omega) = \frac{e^2}{2\omega + 2i0} \int_{\mu - \omega}^{\mu + \omega} \frac{d\Omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} 4\nu_z^2 \left( -\pi\delta(\Omega' - \varepsilon^{mT}) - \pi\delta(\varepsilon^{mT} - \varepsilon^{mT'}) \text{sgn}(\varepsilon^{mT'}) \right)
\]
\[
= \frac{e^2}{2\omega + 2i0} \int_{\mu - \omega}^{\mu + \omega} \frac{d\Omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} 4\nu_z^2 \pi^2 \delta(\Omega - \varepsilon^{mT}) (\delta(\Omega' - \varepsilon^{mT} - \delta(\varepsilon^{mT} - \varepsilon^{mT'}) \text{sgn}(\varepsilon^{mT}) \text{sgn}(\varepsilon^{mT'}))
\]
\[= \frac{2\nu_z^2 \pi^2 e^2}{\omega + i0} \int_{\mu - \omega}^{\mu + \omega} \frac{d\Omega}{2\pi m} \int_0^{2\pi} d\theta \int_0^\pi d\varphi (\sin\varphi)^{2-m} \int_0^\Lambda dk k^2 \delta(\Omega - \varepsilon^{mT}) (\delta(\Omega' - \varepsilon^{mT} - \delta(\varepsilon^{mT} - \varepsilon^{mT'}) \text{sgn}(\varepsilon^{mT}) \text{sgn}(\varepsilon^{mT'})).
\]
\]
\[\varepsilon^{mT'}\] is the eigenenergy after the optical transition, and we note that, since we assume the hopping strength is isotropic and real (i.e., does not consider the chiral kind of hopping), the eigenenergy (dispersion) is real. Then the above integral can be solved by using the Lorentzian representation \(\delta(x) = \frac{c}{\pi(c^2 + x^2)}\) again where \(c\) is a small quantity related to the quasiparticle scattering.

The spectral function including the disorder-induced self-energy effect reads\[5, 36\]
\[
A^{mT}(\Omega) = \frac{|\text{Im}\Sigma^D(\Omega)|}{(\Omega + \text{Re}\Sigma^D(\Omega) - \varepsilon^{mT})^2 + (\text{Im}\Sigma^D(\Omega))^2},
\]
which contains the informations about not only the dispersion but also the quasiparticle residue and the Fermion relaxation. In the presence of the screened long-range Coulomb interaction by the collective excitations in non-Fermi-liquid state (but with finite chemical potential), the above perturbed spectral function also related to the excitation damping, like the plasmon mode which damped into the particle-hole excitations due to the non-zero imaginary part of the polarization function (Bosonic self-energy) as we studied\[37, 38, 39, 40\].

6 Conclusion

In conclusion, we investigate the self-energy correction, symmetry, free energy, transverse optical conductivity of the 2D Dirac system in non-Fermi-liquid state. The non-Fermi-liquid behaviors of the 2D and 3D Dirac/Weyl systems with higher order dispersion are also studied and we found that the non-Fermi-liquid features remain even at finite chemical potential, and they are distinct from the Fermi-liquid picture and the conventional non-Fermi-liquid picture. In the presence of the impurity scattering, the Fermionic/Bosonic polaron formed by dressing the Fermion/Boson majority particles as widely found in ultracold Fermi gases\[41, 42\] and BEC\[43\], respectively, and they are also important in studying the perturbation effect as done in this paper within the contact potential (Dirac \(\delta\)-type impurity field) context. In the presence of the gapless order parameter fluctuation, the Landau damping of the longitudinal excitations within RPA for the non-Fermi-liquid case are also discussed.

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