The minimal supersymmetric grand unified theory:  
I. symmetry breaking and the particle spectrum

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We discuss in detail the symmetry breaking and related issues in the minimal renormalizable supersymmetric grand unified theory. We compute the particle spectrum and study its impact on the physical scales of the theory. This provides a framework for the analysis of phenomenological implications of the theory, to be carried out in part II.

I. INTRODUCTION

It has been argued recently [1] that the minimal supersymmetric renormalizable grand unified theory is based on the $SO(10)$ gauge symmetry with the following minimal set of states [2, 3]

- three generations of 16-dimensional matter supermultiplets $16_F$
- $210_H, 126_H, \bar{126}_H$ and $10_H$ Higgs supermultiplets.

The theory is minimal in the sense of having a minimal set of parameters and most predictability. Its main features are:

1. exact R-parity conservation at all energies [4, 5, 6] and a stable LSP
2. natural smallness of neutrino masses through the see-saw mechanism [7, 8, 9, 10]
3. completely realistic fermion spectrum [11, 12, 13]
4. in the case of type II see-saw, it offers a natural connection between $b-\tau$ unification and a large atmospheric neutrino mixing angle [14]
5. both in the type I and type II see-saw cases, the 1-3 leptonic mixing angle turns out to be large, close to the upper experimental limit [15, 16, 17]
6. the loss of asymptotic freedom above $M_{GUT}$ and the existence of a new fundamental scale $M_F \simeq 10M_{GUT}$ where couplings become strong [18]

This theory thus should be confronted with experiment, which requires the detailed computation of the symmetry breaking. This is the aim of this paper. In the follow-up, we will discuss at length the phenomenological implications: proton decay, neutrino masses and mixings, R-parity, leptogenesis and related issues.

Some initial attempts in this direction have already been made [19, 20], but the complete analysis requires a more precise information about the particle states.

The paper is organized as follows. In the next section, we describe the construction of the theory and argue in favor of its minimality. In section III, we study the patterns of symmetry breaking allowed by the most general renormalizable superpotential. The spectrum is accordingly computed in section IV, and in section V we use it to determine the physical scales. Summary and outlook is left for the last section. Many technical details and useful tables are left for the Appendices.

II. THE THEORY: FIELDS AND INTERACTIONS

As in any $SO(10)$ theory, the matter superfields are 16-dimensional spinorial representations. The Higgs sector [2, 3, 10] contains as mentioned

$$\Phi(210); \Sigma(126); \Sigma(\bar{126}); H(10)$$ (1)
\( \Sigma \) is needed in order to give a large mass \( M_R \) to \( \nu^c \). \( \Sigma \) in order to preserve supersymmetry at \( M_R \) and \( \Phi(210) \) in order to complete the symmetry breaking down to MSSM. \( \Phi \) is the minimal choice that does the job and it also plays an important role in generating the correct fermion mass matrices (Section V). Having realistic fermionic spectrum necessitates also \( H \) on top of \( \Sigma \).

The most general renormalizable superpotential of the above fields is

\[
W_H = \frac{m_\Phi}{4!} \Phi^2 + \frac{m_\Sigma}{5!} \Sigma \Sigma + \lambda \Phi^3 + \eta \Phi \Sigma \Sigma + m_H H^2 + \frac{1}{4!} \Phi H (\alpha \Sigma + \bar{\alpha} \bar{\Sigma})
\]  

(2)

The simplicity of \( W_H \) is worth commenting on. It has only four different couplings and three mass terms, which facilitates the study of symmetry breaking. This is the advantage of large representations, which may have other defects.

The matter superfields are \( \Psi_a(16), a = 1, 2, 3 \). The most general Yukawa superpotential is given by

\[
W_Y = \Psi (Y_H H + Y_\Sigma \Sigma) \Psi.
\]  

(3)

with generation indices suppressed, \( Y_H \) and \( Y_\Sigma \) are symmetric matrices. This implies 15 real couplings in total. Namely, \( Y_H \) for example can be diagonalized and made real, which means 3 real couplings and \( Y_\Sigma \) has 6 complex or 12 real couplings.

The small number of couplings should be considered the main virtue of the theory, for it implies a large amount of predictivity. After rotating away the phases of Higgs superfields, the 7 (4 couplings and 3 masses) complex parameters of \( W_H \) become 10 real ones. Together with the single gauge coupling, we have thus 26 real parameters in total. Of course, we are not counting the supersymmetry breaking terms. This can be compared with the MSSM, which has the same number of couplings but describes far less phenomena. Similarly, one can show that the \( SU(5) \) supersymmetric theory has many more couplings [1]. As we have argued at length in [1], this theory should be considered the minimal supersymmetric GUT.

The study of symmetry breaking and fermion masses favor the Pati-Salam \( G_{422} = SU(4)_C \times SU(2)_L \times SU(2)_R \) language [20]. The decomposition of the above fields under \( G_{422} \) is given by

\[
H \equiv 10 = (6, 1, 1) + (1, 2, 2)
\]

\[
\Psi \equiv 16 = (4, 2, 1) + (4, 1, 2)
\]

\[
\Phi \equiv 210 = (15, 1, 1) + (1, 1, 3) + (15, 1, 3)
\]  

\[
+ (15, 3, 1) + (6, 2, 2) + (10, 2, 2) + (10, 2, 2)
\]

\[
\Sigma \equiv 126 = (100, 1, 3) + (10, 3, 1) + (6, 1, 1) + (15, 2, 2)
\]

\[
\bar{\Sigma} \equiv \bar{126} = (10, 1, 3) + (100, 3, 1) + (6, 1, 1) + (15, 2, 2)
\]

Of course, in order to study in detail the symmetry breaking and the particle spectrum, a complete decomposition under the Standard Model group of the above fields is required. We refer the reader to Appendix A for these details.

III. PATTERNS OF SYMMETRY BREAKING

The first step is the breaking of \( SO(10) \) down to the MSSM; and here \( H \) can be ignored. Only the MSSM singlets are allowed to take a vacuum expectation value (VEV). We shall call their VEVs

\[
p = \langle \Phi(1, 1, 1) \rangle; \quad a = \langle \Phi(1, 1, 15) \rangle; \quad \omega = \langle \Phi(1, 3, 15) \rangle
\]

\[
\sigma = \langle \Sigma(1, 3, 10) \rangle; \quad \bar{\sigma} = \langle \bar{\Sigma}(1, 3, 10) \rangle
\]  

(4)

The superpotential as a function of these VEVs is calculated to be

\[
W_H = m_\Phi (p^2 + 3a^2 + 6\omega^2) + 2\lambda (a^3 + 3p\omega^2 + 6a\omega^2)
\]

\[
+ m_\Sigma \sigma \bar{\sigma} + \eta \sigma \bar{\sigma} (p + 3a - 6\omega).
\]  

(5)
Vanishing of the D-terms implies $|\sigma| = |\bar{\sigma}|$, while from the F-terms we get

\begin{align}
2m_\Phi p + 6\lambda \omega^2 + \eta \sigma \bar{\sigma} &= 0 \\
2m_\Phi a + 2\lambda(a^2 + 2\omega^2) + \eta \sigma \bar{\sigma} &= 0 \\
2m_\Phi \omega + 2\lambda(p + 2a)\omega + \eta \sigma \bar{\sigma} &= 0 \\
\sigma [m_\Sigma + \eta(p + 3a - 6\omega)] &= 0
\end{align}

From these equations, we get the following set of degenerate SUSY-preserving vacua

1. $p = a = \omega = \sigma = 0$ – the SO(10)-preserving minimum.
2. $p = a = -\omega = -m_\Phi/3\lambda ; \sigma = 0$. As can be confirmed by calculating explicitly the gauge boson masses, this minimum has $SU(5) \times U(1)$ symmetry.
3. $p = a = -\omega = -m_\Sigma/10\eta ; \sigma \bar{\sigma} = m_\Sigma (10\eta m_\Phi - 3\lambda m_\Sigma)/(50\eta^3)$. This is the $SU(5)$ minimum, and includes the previous one for $\lambda m_\Sigma/\eta m_\Phi = 10/3$.
4. $p = \omega = \sigma = 0 ; a = -m_\Phi/\lambda$. This is obviously the left-right symmetric $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ minimum.
5. $p = a = \omega = -m_\Phi/3\lambda ; \sigma = 0$. This is again $SU(5) \times U(1)$ symmetric, but with the flipped $SU(5)$ assignments for the particle states.
6. $p = 3m_\Phi/\lambda ; a = -2m_\Phi/\lambda , \omega = \pm im_\Phi/\lambda ; \sigma = 0$. This minimum has symmetry $SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$.

7. 
\begin{align}
p = & -\frac{m_\Phi x(1 - 5x^2)}{\lambda (1 - x)^2} ; a = -\frac{m_\Phi (1 - 2x - x^2)}{\lambda (1 - x)} ; \omega = -\frac{m_\Phi x}{\lambda} ; \sigma \bar{\sigma} = \frac{2m_\Phi^2 x(1 - 3x)(1 + x^2)}{\eta \lambda (1 - x)^2} \\
-8x^3 + 15x^2 - 14x + 3 &= (x - 1)^2 \frac{\lambda m_\Sigma}{\eta m_\Phi}
\end{align}

For generic $x$, this is the Standard Model minimum. Of particular interest are the cases $x \sim 0$ and $x \sim 1$, which provide the chains with intermediate scales. The former case corresponds to the left-right symmetry, while the latter gives an intermediate $G_{122}$ (Pati-Salam) scale.

This solution includes:

- the 3rd (for $x = 1/2$), if $\lambda m_\Sigma/\eta m_\Phi = -5$
- the 4th (for $x = 0$), if $\lambda m_\Sigma/\eta m_\Phi = 3$
- the 5th (for $x = 1/3$), if $\lambda m_\Sigma/\eta m_\Phi = -2/3$
- the 6th (for $x = \pm i$), if $\lambda m_\Sigma/\eta m_\Phi = -3(1 \pm 2i)$

All this will become more transparent when we calculate the particle spectrum in the following section.

### IV. PARTICLE SPECTRUM

We refer the reader to Appendix A for the notation and decomposition in SM language, and to Appendix B for details of the calculation of the spectrum. The computations are performed using a nice method developed by He and Meljanac. We are interested in the physically realistic case 7, where the resulting symmetry is the MSSM. The unmixed states are given in Table II.

We give the mixing matrices for the rest of the states in Appendix C. In order to find the eigenvalues of a general matrix $M$ one needs to diagonalize the matrix $M^\dagger M$. Since our aim here is not the complete numerical and phenomenological analysis (left for the part II), we give for illustration the eigenvalues in the Hermitian case (Table
Notice that the $\Phi$ plays a central role in this through the as in the Georgi-Jarlskog scenario \[1, 22\]. This is why one ends up with a realistic matter spectrum in this theory. Hor $\bar{\Sigma}$ give masses to fermions which cannot be realistic for all three generations due to unknown fermion and sfermion mixing angles \[28, 29, 30\] (see however \[31\]), but to be on the safe side it would the order of the GUT scale, proton decay becomes typically too fast \[25, 26, 27\]. There is some uncertainty in this a main threat to any supersymmetric grand unified theory. It is well known that for generic values of their masses of all mix, and thus the light doublets are a mixture of the original ones. In other words, they all have nonvanishing fine-tuning one ends up with just one light pair, as is illustrated in Appendix C. What is crucial though is that they $(\Phi)$ is broken down to $SU(2)$ doublets. Without fine-tuning all of them have large masses, on the order of the GUT scale. With the minimal using a different method. We have confirmed their results, and the complete set of matrices is given in Appendix II). This case by definition means the CP conserving situation at the high scale of real couplings and real vevs and $\bar{\alpha} = \alpha$.

The mass matrices for the color triplets and the SM doublets that mix with $H(10)$ were first calculated in \[24\] using a different method. We have confirmed their results, and the complete set of matrices is given in Appendix 4 where we also discuss their eigenvalues. Let us briefly discuss the physical aspects of these systems. First the $SU(2)$ doublets. Without fine-tuning all of them have large masses, on the order of the GUT scale. With the minimal fine-tuning one ends up with just one light pair, as is illustrated in Appendix C. What is crucial though is that they all mix, and thus the light doublets are a mixture of the original ones. In other words, they all have nonvanishing VEVs. In particular, in the Pati-Salam language, both (2, 2, 1) and (2, 2, 15) fields will contribute to fermion masses as in the Georgi-Jarlskog scenario \[1, 22\]. This is why one ends up with a realistic matter spectrum in this theory. Notice that the $\Phi$ plays a central role in this through the $\alpha$ and $\bar{\alpha}$ couplings in \[24\]: without it one would have only $H$ or $\bar{\Sigma}$ give masses to fermions which cannot be realistic for all three generations.

Now, the color triplet superfields. They are responsible, either directly or indirectly for d=5 proton decay \[24, 24\], a main threat to any supersymmetric grand unified theory. It is well known that for generic values of their masses of the order of the GUT scale, proton decay becomes typically too fast \[24, 24, 24\]. There is some uncertainty in this due to unknown fermion and sfermion mixing angles \[28, 28, 30\] (see however \[31\]), but to be on the safe side it would be desirable to have these states weigh more \[32\].

A useful test of symmetry breaking is the identification of the would-have-been Goldstone bosons. When $SO(10)$ is broken down to $G_{321}$, 33 gauge bosons become massive (here and in what follows, we give values of $Y/2$ when specifying the SM group $G_{321}$ quantum numbers):

\begin{itemize}
  \item [i)] \( (X, Y) \), mediators of proton decay, with \( G_{321} = (3, 2, -5/6) \) and \( (3, 2, 5/6) \)
  \item [ii)] \( (X', Y') \), also mediators of proton decay, with \( G_{321} = (3, 2, 1/6) \) and \( (3, 2, -1/6) \)
  \item [iii)] \( (X_{PS}) \) the $G_{422}$ leptoquarks, responsible for rare decays, with \( G_{321} = (3, 1, 2/3) \) and \( (3, 1, -2/3) \)
  \item [iv)] \( (W_{R}) \), the mirror image of the $W$ bosons, with \( G_{321} = (1, 1, \pm 1) \) and \( (1, 1, 0) \)
\end{itemize}

A quick glance at Table III shows that the massless states have precisely the above quantum numbers.

### Table I: Masses of the unmixed states as functions of $x$ for the 7th symmetry breaking pattern.

| Field | SU(3)$_C \times$ SU(2)$_L \times$ U(1)$_Y$ | Mass/m$_\Phi$ |
|-------|---------------------------------|----------------|
| $\Phi$ | \((3, 1, +5/3, \bar{\Phi})\) | $8 x (2 x - 1) / (x - 1)^2$ |
| \((8, 1, +1)\) | | $4 (x^2 - 3 x + 1 + 3 x^3) / (x - 1)^2$ |
| \((1, 3, 0)\) | | $-2 (x^2 - 5 x + 1 + 7 x^3) / (x - 1)^2$ |
| \((3, 3, -2/3, \bar{\Phi})\) | | $-4 x (-1 + 3 x^2) / (x - 1)^2$ |
| \((8, 3, 0)\) | | $-4 (-x^2 + 2 x - 1 + 2 x^3) / (x - 1)^2$ |
| \((1, 2, +3/2)\) | | $-4 (-1 + x + 3 x^2) / (x - 1)$ |
| \((6, 2, -1/6, \bar{\Phi})\) | | $4 (-1 + x + x^2) / (x - 1)$ |
| \((6, 2, +5/6, \bar{\Phi})\) | | $4 (2 x - 1) / (x - 1)$ |
| $\Sigma, \bar{\Sigma}$ | \((1, 3, -1, (1, 3, +1)\) | $-4 (\eta/\lambda) x \left( 4 x^2 - 3 x + 1 \right) / (x - 1)^2$ |
| \((3, 3, -1/3, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( 7 x^3 - 7 x^2 + 5 x - 1 \right) / (x - 1)^2$ |
| \((6, 3, +1/3, \bar{\Sigma})\) | | $-4 (\eta/\lambda) \left( x^2 - x + 1 \right) / (x - 1)^2$ |
| \((1, 1, +2)\) | | $-12 x (\eta/\lambda)$ |
| \((3, 1, +4/3, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( 3 x^2 - 6 x + 1 \right) / (x - 1)$ |
| \((6, 1, +2/3, \bar{\Sigma})\) | | $-4 (\eta/\lambda) (1 - 3 x) / (x - 1)$ |
| \((6, 1, -1/3, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( x^2 - 7 x + 2 \right) / (x - 1)$ |
| \((6, 1, -4/3)\) | | $-4 (\eta/\lambda) \left( 2 x - 4 x + 1 \right) / (x - 1)$ |
| \((3, 2, +7/6, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( 6 x^3 - 10 x^2 + 7 x - 1 \right) / (x - 1)^2$ |
| \((3, 2, +7/6, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( x^2 - 6 x + 1 \right) / (x - 1)^2$ |
| \((3, 2, -7/6, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( 5 x^3 - 8 x^2 + 6 x - 1 \right) / (x - 1)^2$ |
| \((8, 2, +1/2)\) | | $-2 (\eta/\lambda) \left( 3 x^2 - 7 x + 2 \right) / (x - 1)^2$ |
| \((8, 2, -1/2, \bar{\Sigma})\) | | $-2 (\eta/\lambda) \left( 4 x^3 - 9 x^2 + 9 x - 2 \right) / (x - 1)^2$ |
two intermediate scales, since this gives us more freedom. Namely, after eliminating the unification coupling at the scale gets raised or lowered in this instance, and we will address the issue. A more interesting situation emerges with the intermediate scale. Let us discuss these two cases separately.

An important issue in supersymmetric unification is the possible existence of intermediate scales. Of course, the success of the MSSM couplings unification favors a single step breaking, and the intermediate scales cannot be too far from the GUT scale.

In the case of a single intermediate scale, the small uncertainty in the values of the couplings at $M_I$ tells us that $M_I$ can be at most an order or two of magnitude away from $M_X$. It is useful to know, though, whether the GUT scale gets raised or lowered in this instance, and we will address the issue. A more interesting situation emerges with two intermediate scales, since this gives us more freedom. Namely, after eliminating the unification coupling at the GUT scale, one has only two equations for three unknowns, one may in principle end up with interesting new physical scales. We first discuss the simpler case of single intermediate scale.

### V. Intermediate Scales?

#### A. One intermediate scale

The seventh (general) pattern of symmetry breaking allows a discussion of intermediate scales as a particular choice of $x$. For example, $x \sim 0$ means only a nonvanishing, or a L-R symmetric step, while $x \sim 1$ implies a Pati-Salam intermediate scale. Let us discuss these two cases separately.

1) $x \sim 0$. In this limit, $(p, \omega, \sigma) \ll a$ leaves only $G_{3221} = SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, the well known parity conserving extension of the SM, well fit for the understanding of neutrino masses. Most of the states get a mass $M_X = a$, except for those that get and intermediate scale mass $M_I \sim (p, \omega, \sigma)$. The measure of the intermediate scale is precisely $x \propto M_I/M_X$. Because of left-right symmetry, the following $G_{3221}$ states in $\Sigma$ must have mass $M_I$:

$$\Sigma : (1, 3, 1; \pm 1), (1, 1, 3; \pm 1).$$
There are however other states, belonging to the \( \Phi \) multiplet, that have an intermediate scale mass:

\[
\Phi : (3, 3, 1; -2/3), (\bar{3}, 3, 1; 2/3), (3, 1, 3; -2/3), (\bar{3}, 1, 3, 2/3).
\]

It is straightforward in this case to show that the lower the intermediate scale is, the lower the GUT scale becomes. This is very bad for the already existing problem of \( D = 5 \) proton decay, and should be discarded.

\( i) \ x \sim 1. \) In this case, \( p = M_X \) is the larger scale, while \( a \sim \sigma \sim M_I \), and \( \omega \) is of order \( M_I^2/M_X \). This is the case of the \( G_{422} = SU(4)_C \times SU(2)_L \times SU(2)_R \) Pati-Salam intermediate scale, of great phenomenological interest. States of \( G_{422} \) with an intermediate scale mass \( M_I \) are

\[
\Phi : (15, 1, 1), (10, 2, 2), (\bar{10}, 2, 2)
\]

\[
\Sigma, \bar{\Sigma} : (\bar{10}, 1, 3), (10, 1, 3)
\]

However, some states in \( \Sigma(\bar{10}, 1, 3) \bar{\Sigma}(10, 1, 3) \) are lighter, with a mass \( \sim M_I^2/M_X \). These are doubly charged color singlets

\[
\delta^{++}, \delta^{--},
\]

often appearing in supersymmetric left-right unification [5, 41].

As in the left-right case it can be easily shown that the GUT scale gets lowered and we discard this case too.

### B. Two intermediate scales

We have seen that \( x \sim 1 \) allows for the \( G_{422} \) intermediate scale. From eq. (10) it follows that \( \sigma \) can be made arbitrarily small by taking \( \eta \gg \lambda \). In such case, one ends up with the hierarchy \( p \gg a \gg \sigma, \omega \). This means a three-step breaking

\[
SO(10) \rightarrow SU(4)_C \times SU(2)_L \times SU(2)_R \rightarrow SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \rightarrow MSSM
\]  

(12)

Two possibilities arise:

1. \( \sigma > \omega \), with one-step breaking of \( SU(2)_R \times U(1)_{B-L} \).

2. \( \omega > \sigma \), the case with an intermediate (the third) \( B - L \) symmetry. \( SU(2)_R \times U(1)_{B-L} \) gets first broken down to \( U(1)_R \times U(1)_{B-L} \), and \( \sigma \) later completes the breaking.

Let us examine the first case. Calling \( p = M_X, a = M_{PS}, \sigma = M_R \), the particles that run and their masses are given in Table I

The scales in Table I are in correct order provided \( M_{PS}^3 > M_X^3, M_X M_R > 10M_{PS}^2 \). It can immediately be seen that the enormous \( b_i \) coefficients do not allow for intermediate scales, there are simply too many states running, and this is confirmed by a straightforward calculation. Figure 1. shows the scales \( M_X, M_{PS} \) and \( M_R \) as functions of the unification constant, allowing for the errors in the value of \( \alpha_3 = 0.117 \pm 0.002 \).

If \( M_{PS}^3 < M_X^3, M_X M_R \), or if we consider case 2, the same conclusion is reached: only the usual single-step breaking is possible.

Strictly speaking, the complete spectrum that we present allows for a more profound analysis of the unification of the gauge couplings. One need not assume the order of magnitude equality of the masses in a fixed multiplet, but incorporate precisely the impact of every individual eigenstate mass. This may be worth doing but is beyond the scope of the present paper.

### VI. SUMMARY AND OUTLOOK

In this work we addressed a computational (technical) investigation of the minimal renormalizable supersymmetric \( SO(10) \) theory, such as a detailed study of the symmetry breaking and the calculation of the mass spectrum. These calculations are necessary for a detailed study of the phenomenological manifestations of this theory, namely unification of the couplings, fermion masses, proton decay, leptogenesis, flavor violating processes, that will be performed in the sequel of this work (part 2).
TABLE III: Particles that run and their approximate masses for case B-1

| $G_{422}$ State                                      | Mass      | $b_1$  | $b_2$  | $b_3$  |
|------------------------------------------------------|-----------|--------|--------|--------|
| $H(6, 1, 1)$                                          |           |        |        |        |
| $\Phi(15, 1, 1)$, except for color octets            | $M_{PS}$  | -421/5 | -51    | -65    |
| $\Sigma(10, 1, 3)$, $\Sigma(10, 1, 3)$, except for color singlets |           |        |        |        |
| $\Phi(15, 3, 1)$, $\Phi(15, 1, 3)$                   | $M_R^2/M_X/M_{PS}^2$ | -291/5 | -51    | -48    |
| color singlets from $\Sigma(10, 1, 3)$, $\Sigma(10, 1, 3)$, except for doubly – charged, $\delta^{++}, \delta^{--}$ | $M_R$    | -153/5 | -21    | -24    |
| doubly – charged $\delta^{++}, \delta^{--}$         | $10M_{PS}^2/M_X$ | -33    | -21    | -24    |
| $\Phi(10, 2, 2), \Phi(10, 2, 2)$                     | $M_R^2/M_{PS}$ | -141/5 | -21    | -24    |
| color octets from $\Phi(15, 1, 1)$                   |           |        |        |        |

FIG. 1: $M_X$, $M_{PS}$ and $M_R$ as functions of the unification constant, for case B-1. Dotted lines are results for lowest and highest values of $\alpha_3$. 
In order to illustrate the calculations, we considered the question of whether intermediate (gauge) scales are permitted in this theory, reaching a negative conclusion. (Note however that it is possible that some particles of the theory can accidentally turn out to be light). A more detailed study, where the size of the threshold correction will be quantified, will be performed in part 2. Also, we studied the conditions imposed by the minimal fine tuning, that is needed to obtain the correct MSSM spectrum and that further reduces the number of free parameters. We demonstrated that the composition of the light higgs particles $H_u$ and $H_d$ is fixed in terms of the fundamental parameters of the theory.

A possible use of our results is for example the issue of the nature of the see-saw mechanism, i.e. whether it is of type I or type II [37, 38]. A simple and natural way to have a type I see-saw in the theory is to break the left-right symmetry at the GUT scale with the SU(2)$_R$ symmetry breaking scale much smaller [39]. This amounts to $p \gg \sigma, \omega$, but it enters into the conflict with the unification constraints as discussed before, and even more important, creates more danger for the $d = 5$ proton decay by lowering the masses of some colour triplets.

On the other hand, a type II see-saw can be achieved by having a small mass of the lefthanded triplet through a judicious choice (fine-tuning) of the parameter $x$. Again, this can create conflict with the gauge coupling unification and ought to be checked. It is probably most natural to have both type I and type II compete on equal grounds.

In summary, the minimal $SO(10)$ theory is a very constrained theory, with few free parameters. In this work, we prepared the necessary tools to explore this theory in detail. We are not yet in position to assess which are its detailed predictions and as a matter of fact we cannot exclude that this theory will eventually fail. In view of several experimental and theoretical considerations (e.g., on neutrino and fermion masses) we believe that such an exploration is largely worth the effort.

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VIII. NOTE ADDED

When these calculations were completed, a new paper appeared, also addressing the study of the minimal $SO(10)$ model [40]. There are some points of disagreement, and here we would like to argue in favor of our result.

1. In their table 1, there are two $2 \times 2$ matrices, whereas we have no such matrices. However, their existence would imply that $\Sigma$ mixes with itself (and similarly for $\bar{\Sigma}$), while there is not such a coupling in the theory.

2. Their eq.(4.2) describes a $4 \times 4$ mass matrix of the particles with SM numbers $(3,2,1/6)$, whereas we have a $3 \times 3$ mass matrix, and a decoupled state, precisely, the component of $\Sigma$ with PS numbers $(15,2,2)$. Our explanation of the existence of a decoupled state is the following one: the absence of a self coupling of $\Sigma$ (and $\bar{\Sigma}$) sets to zero one of their entries; the absence of the other couplings is due to the selection rule on $T_R$.

3. After their eq.(5.7), it is argued that one should set $\alpha \neq \bar{\alpha}$, if one wants to avoid the equality $Y_u = Y_d$. Indeed, they note that when $\alpha = \bar{\alpha}$, the components of the vector that represent $H_u$ are the same of the one of $H_d$.

However, the fields remain different, since the role of $\Sigma$ and $\bar{\Sigma}$ gets exchanged when going from $H_u$ to $H_d$.

Similar computations have been performed by C.S. Aulakh and A. Girdhar, to appear soon.

APPENDIX A: DECOMPOSITION OF THE $SO(10)$ REPRESENTATIONS

1. Conventions

Duality in $SO(2n)$ is defined as:

$$\Sigma_{a_1...a_n}^{d} = \frac{-i^n}{n!} \epsilon_{a_1...a_n,b_1...b_n} \Sigma_{b_1...b_n}$$  \hspace{1cm} (A1)
SO(10) indices are labeled by latin subscripts, and is decomposed so that \( i = 1..4 \) is reserved for SO(4) and \( i = 5..10 \) for SO(6).

The color states in the fundamental 6 of SO(6) are given by indices:

\[
\begin{align*}
    r & : 5 + i6 \\
    3_{-2/3} & : 7 + i8 \\
    g & : 9 + i0
\end{align*}
\]

with \( B - L = -2/3 \). Then the 2-index 15 of SU(4) is represented by:

\[
\begin{align*}
    1_0 & \quad r\bar{r} + b\bar{b} + g\bar{g} = [56 + 78 + 90] \\
    3_{4/3} & \quad r\bar{g} = [59 - 60 - i50 - i69] \\
    3_{-4/3} & \quad r\bar{g} = [59 - 60 + i50 + i69] \\
    8_0 & \quad r\bar{r} - b\bar{b} = [56 - 78]
\end{align*}
\]

The 3-index 10:

\[
\begin{align*}
    1_{-2} & \quad rbg = [579 - 689 - 670 - 580 + i(679 + 589 + 570 - 680)] \\
    3_{-2/3} & \quad (r\bar{r} + g\bar{g})b = -2i[567 + 790 + i568 + i890] \\
    6_{2/3} & \quad (-r\bar{r} + g\bar{g})\bar{b} = [568 - 890 - i790 + i567]
\end{align*}
\]

The 10 is obtained by conjugation of above. The 4-index 15 and the 5-index 6 are obtained using dualization, from the 2- and 1-index. SU(2)_R doublets are given by:

\[
\begin{align*}
    T_{3R} = +1/2, \ T_{3L} = +1/2 : \ [-1 + i2] \\
    T_{3R} = -1/2, \ T_{3L} = +1/2 : \ [3 + i4] \\
    T_{3R} = +1/2, \ T_{3L} = -1/2 : \ [3 - i4] \\
    T_{3R} = -1/2, \ T_{3L} = -1/2 : \ [1 + i2]
\end{align*}
\]

And 2-index 3 + 1:

\[
\begin{align*}
    T_{3R} = +1 : \ [14 + 23 + i(13 - 24)] \\
    T_{3R} = 0 : \ [12 + 34] \\
    T_{3R} = -1 : \ [14 + 23 - i(13 - 24)] \\
    T_{3R} = 0 : \ [12 - 34]
\end{align*}
\]

The 3-index doublets are dual to the 1-index.

2. States

Standard Model states \( \Phi_J, \Sigma_J, \Sigma^*_J, H_J \) are combinations (from here on capital indices label states)

\[
\Phi_J = c^a_{\alpha\beta\gamma\delta} \Phi_{\alpha\beta\gamma\delta}
\]

and so on. One color representative \((7 + i8)\) is chosen for each color multiplet and the \( T_{3L} = 0 \) for left-handed triplets.

Using indices for the states is not very practical, so for identification purposes SM states are labeled with a subindex:
with a shorthand notation for $T^3_R$. So, e.g. the fields that get a vev would be labeled:

\[
\Omega = \Phi_{113}^0 \quad \Sigma = \Phi_{113}^- \quad \bar{\Sigma} = \Phi_{113}^+
\]

Ambiguities: The singlet in $\Phi(1,1,1)$ is called $\Phi_P$, while that in $\Phi(1,1,15)$ is called $\Phi_A$. There are two $\Phi_{322}^{\pm}$, one in $(2,2,6)$, called $\Phi_{T \pm}$ and one in $(2,2,10)$ called $\Phi_{J \pm}$. Color singlets like $\Phi_{122}$ in $(2,2,10)$ are distinguished from those in $(2,2,6)$ by calling them $\Phi_{122}$. All states are normalized canonically in the kinetic term.

3. Standard Model decomposition

We give the Standard Model states in terms of $SO(10)$ indices, labeled by $SU(3), SU(2)_L, SU(2)_T^R$ in Tables IV-VII. For $SU(3)$ states, only one color combination is given, and for $SU(2)_L$ triplets only the $T_{3L} = 0$.

States of $\Sigma$ ($\bar{\Sigma}$) are defined already self-dual (anti-self dual), but the dual part is not written in the table, for shortness, nor is the normalization factor included. States formed by the (antisymmetrized) linear combination of $N$ components of $\Phi_{ijkl}$ will be normalized by $1/\sqrt{4!N}$. Similarly , states from $\Sigma_{ijklm}$ ($\bar{\Sigma}_{ijklm}$) will be normalized by $1/\sqrt{5!N}$, where $N$ counts the field components and their dual (anti-dual) parts. States from $H_i$ have a $1/\sqrt{N}$ factor.

The above states $\Phi$ are normalized canonically in a sense that the Kahler potential for them is $K = \Phi^\dagger \Phi$. It is easy to show that this corresponds to the original $SO(10)$ Kahler

\[
K = \frac{1}{4!} \Phi^\dagger_{ijkl} \Phi_{ijkl} + \frac{1}{5!} \Sigma^\dagger_{ijklm} \Sigma_{ijklm} + \frac{1}{6!} \bar{\Sigma}^\dagger_{ijklm} \bar{\Sigma}_{ijklm} + H^\dagger_i H_i
\]  

APPENDIX B: CALCULATION OF THE SPECTRUM

1. Symmetry breaking

The Higgs superpotential is:

\[
W_H = \frac{m_{\Phi}}{4!} \Phi^\dagger_{ijkl} \Phi_{ijkl} + \frac{\lambda}{4!} \Phi^\dagger_{ijkl} \Phi_{klmn} \Phi_{mnij}
+ \frac{m_{\Sigma}}{5!} \Sigma^\dagger_{ijklm} \Sigma_{ijklm} + \frac{\eta}{4!} \Phi^\dagger_{ijkl} \Sigma_{ijmn} \Sigma_{klmn}
+ m_H H_i H_i + \frac{1}{4!} \Phi^\dagger_{ijkl} H_m (\alpha \Sigma_{ijklm} + \bar{\alpha} \bar{\Sigma}_{ijklm})
\]

Using the above conventions, the Standard Model singlet fields that get a VEV are

\[
\langle \Phi_{1234} \rangle = p : \quad \langle \Phi_{5678} \rangle = \langle \Phi_{5690} \rangle = \langle \Phi_{7890} \rangle = a
\]
\[
\langle \Phi_{1256} \rangle = \langle \Phi_{1278} \rangle = \langle \Phi_{1290} \rangle = \langle \Phi_{3456} \rangle = \langle \Phi_{3478} \rangle = \langle \Phi_{3490} \rangle = \omega
\]
\[
\langle \Sigma_{a+1,b+3,c+5,d+7,e+9} \rangle = \frac{1}{25/2} (i)^{(-a-b+c+d+e)} \sigma
\]
\[
\langle \bar{\Sigma}_{a+1,b+3,c+5,d+7,e+9} \rangle = \frac{1}{25/2} (-i)^{(-a-b+c+d+e)} \sigma
\]

with $a, b, c, d, e$, running for 0 to 1. The superpotential for this fields is then calculated to be

\[
W_H = m_{\Phi} \left( p^2 + 3a^2 + 6\omega^2 \right) + 2\lambda \left( a^3 + 3p\omega^2 + 6a\omega^2 \right)
+ m_{\Sigma} \sigma \bar{\sigma} + \eta \sigma \bar{\sigma} \left( p + 3a - 6\omega \right).
\]
TABLE IV: Decomposition of states in the $210$ representation.

Patterns of symmetry breaking are given in the text. We give a brief summary in Table XX, specifying the Pseudo-Goldstone Bosons (PGB) when they exist. The states that get their masses only after supersymmetry breaking can be found in our mass matrices below.

F-term contribution to the masses are calculated by pieces, defining:

\[(\mathcal{H})_{a_{1},a_{2},a_{3},a_{4}} = \frac{1}{6} [(\Phi)_{a_{1},a_{2},i,j} \Phi_{i,j,a_{3},a_{4}} + \{a.s\}] \]  \hspace{1cm} \text{(B4)}

\[(\mathcal{H})_{a_{1},a_{2},a_{3},a_{4}} = \frac{1}{10} [(\Sigma)_{a_{1},a_{2},i,j,k} \Sigma_{i,j,k,a_{3},a_{4}} + \{a.s\}] \]  \hspace{1cm} \text{(B5)}

\[(\mathcal{J})_{a_{1},a_{2},a_{3},a_{4},a_{5}} = \frac{1}{2} \left[ \frac{1}{10} (\Phi_{a_{1},a_{2},i,j} \Sigma_{i,j,a_{3},a_{4},a_{5}} + \{a.s\}) + \{\text{dual}\} \right] \]  \hspace{1cm} \text{(B6)}

\[(\mathcal{J})_{a_{1},a_{2},a_{3},a_{4},a_{5}} = \frac{1}{2} \left[ \frac{1}{10} (\Sigma_{i,j,a_{3},a_{4},a_{5}} \Phi_{i,j,a_{1},a_{2}} + \{a.s\}) + \{\text{dual}\} \right] \]  \hspace{1cm} \text{(B7)}
TABLE V: Decomposition of states in the $210$ representation.

| $G_{422}$ | State in $\Phi$ | SO(10) indices |
|-----------|-----------------|----------------|
| 1,1,2     | $122^+$         | $[-1,5,7,9] + [1,6,8,9] + [1,5,8,0] + [1,6,7,0]$ |
|           | $122^-$         | $[-i[1,5,7,9] - i[1,5,8,9] - i[1,6,7,9] + i[1,6,8,0]] - i[1 \to 2]$ |
| 1,1,1     | $311^+$         | $[3,5,7,9] + [3,6,8,9] + [3,5,8,0] - [3,6,7,0]$ |
|           | $311^-$         | $[3,5,7,9] + i[3,5,8,9] + i[3,6,7,9] - i[3,6,8,0]] + i[3 \to 4]$ |

TABLE VI: Decomposition of states in the $10$ representation.

| $G_{422}$ | State in $H$ | SO(10) indices |
|-----------|-------------|----------------|
| 1,1,1     | $122^+$     | $[-1] + i[2]$ |
|           | $122^-$     | $[3] + i[4]$  |
| 3,1,1     | $311$       | $[7] + i[8]$  |
|           | $311$       | $[7] - i[8]$  |

\[(\mathcal{K}_\sigma)_{a_1,a_2,a_3,a_4} = (\Sigma)_{a_1,a_2,a_3,a_4,i}H_i\]  
\[(\mathcal{K})_{a_1,a_2,a_3,a_4,a_5} = \frac{1}{2} \left( \frac{1}{5} (\Phi)_{a_1,a_2,a_3,a_4}H_{a_5} + \{a.s\} \right) + \{\text{dual}\} \]  

(B8)  
(B9)

(and similar for $\Sigma$). Then, omitting $SO(10)$ indices

\[F_\Phi = 2m_\Phi\Phi + 6\lambda H + \eta(H_\sigma + H_\bar{\sigma}) + \alpha K_\sigma + \bar{\alpha} K_\bar{\sigma}\]  
\[F_\Sigma = m_\Sigma\Sigma + 5\eta J + 5\eta J_\sigma + 5\alpha k\]  

(B10)

To get the F-terms for the states, the combinations

\[F_{\Phi J} = \bar{c}_{J}^{abcd} F_{\Phi abcd}\]

(B11)

are found, where $\bar{c}_{J}$ are the coefficients of $\bar{\Phi} J = \bar{c}_{J}^{abcd} \Phi_{abdc}$. Note that the mass terms are of the form $\bar{\Phi} \Phi$ and of course in the case of SM singlets $\Phi = \bar{\Phi}$. Results are given in Tables VIII and IX.

2. Masses

We give masses for eigenstates, or the mixing matrices, for the most general pattern of symmetry breaking. States/matrices are identified by the hypercharge $Y/2$. 
For mixed states, these are the fermion mass matrices:

\[
\frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \left( \begin{array}{cccc}
2m_\Phi & 0 & \lambda 2 \sqrt{6}\omega & \eta \sigma \\
0 & 2(m_\Phi + 2\lambda a) & \lambda \sqrt{2}\omega & -i\eta \sigma \\
\lambda \sqrt{6}\omega & \lambda \sqrt{2}\omega & 2(m_\Phi + \lambda (p + 2a)) & -i\eta \sqrt{3}\sigma \\
-i\eta \sigma & -i\eta \sqrt{3}\sigma & i\eta \sqrt{6}\sigma & m_\Sigma + \eta (p + 3a - 6\omega)
\end{array} \right) \quad (B12)
\]

\[
\frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \left( \begin{array}{cccc}
2(m_\Phi + \lambda a) & -\lambda \sqrt{2}\omega & -\eta \sqrt{2}\sigma \\
-\lambda \sqrt{2}\omega & 2(m_\Phi + \lambda (p + a)) & -\eta \sqrt{2}\sigma \\
-\eta \sqrt{2}\sigma & -\eta \sqrt{2}\sigma & m_\Sigma + \eta (p + a - 2\omega)
\end{array} \right) \quad (B13)
\]
\[Y/2 = (1,-1)\]
\[\varphi = (\Phi_{113^+}, \Sigma_{113^0})\]
\[\overline{\varphi} = (\Phi_{113^-}, \Sigma_{113^0})\]
\[
\begin{pmatrix}
2(m_\Phi + \lambda(p + 2a)) & i\eta\sqrt{6}\sigma \\
-i\eta\sqrt{6}\sigma & m_\Sigma + \eta(p + 3a)
\end{pmatrix}
\] (B14)

\[Y/2 = (0,0)\]
\[\varphi = (\Phi_{811}, \Phi_{813^0})\]
\[\overline{\varphi} = (\Phi_{811}, \Phi_{813^0})\]
\[
\begin{pmatrix}
2(m_\Phi - \lambda a) & -\lambda 2\sqrt{2}\omega \\
-\lambda 2\sqrt{2}\omega & 2(m_\Phi + \lambda(p - a))
\end{pmatrix}
\] (B15)

\[Y/2 = (1/6,-1/6)\]
\[\varphi = (\Phi_{I^+, \Phi_{II^+}, \Sigma_{322^-}})\]
\[\overline{\varphi} = (\Phi_{I^-}, \Phi_{II^-}, \Sigma_{322^+})\]
\[
\begin{pmatrix}
2(m_\Phi - \lambda \omega) & -\lambda 2\sqrt{2}\omega & -\eta\sqrt{2}\sigma \\
-\lambda 2\sqrt{2}\omega & 2(m_\Phi + \lambda(a - \omega)) & -\eta\sqrt{2}\sigma \\
-\eta\sqrt{2}\sigma & -\eta\sqrt{2}\sigma & m_\Sigma + \eta(a - 3\omega)
\end{pmatrix}
\] (B16)

\[Y/2 = (-5/6,5/6)\]
\[\varphi = (\Phi_{I^-}, \Phi_{II^-})\]
\[\overline{\varphi} = (\Phi_{I^+}, \Phi_{II^+})\]
\[
\begin{pmatrix}
2(m_\Phi + \lambda \omega) & \lambda 2\sqrt{2}\omega \\
\lambda 2\sqrt{2}\omega & 2(m_\Phi + \lambda(a + \omega))
\end{pmatrix}
\] (B17)

\[Y/2 = (-1/2,1/2)\]
\[\varphi = (H_{122^-}, \Sigma_{122^-}, \Sigma_{122^+}, \Phi_{122^+})\]
\[\overline{\varphi} = (H_{122^+}, \Sigma_{-122^+}, \Sigma_{122^+}, \Phi_{122^-})\]
\[
\begin{pmatrix}
2m_H & \alpha\sqrt{\frac{3}{2}}(a - \omega) & \bar{\alpha}\sqrt{\frac{3}{2}}(a + \omega) & \alpha\sigma \\
\bar{\alpha}\sqrt{\frac{3}{2}}(a - \omega) & m_\Sigma + 2\eta(a - \omega) & 0 & \eta\sqrt{6}\sigma \\
\alpha\sqrt{\frac{3}{2}}(a + \omega) & 0 & m_\Sigma + 2\eta(a + \omega) & 0 \\
-\bar{\alpha}\sigma & -\eta\sqrt{6}\sigma & 0 & -2(m_\Phi + 3\lambda(a - \omega))
\end{pmatrix}
\] (B18)

\[Y/2 = (1/3,-1/3)\]
\[\varphi = (H_{311}, \Sigma_{311}, \Sigma_{311}, \Sigma_{313^0}, \Phi_{313^+})\]
\[\overline{\varphi} = (H_{311}, \Sigma_{311}, \Sigma_{311}, \Sigma_{313^0}, \Phi_{313^-})\]
\[
\begin{pmatrix}
2m_H & \frac{\alpha}{\sqrt{2}}(p + a) & \frac{\bar{\alpha}}{\sqrt{2}}(p - a) & -\alpha 2i\omega & \alpha i\sigma \\
\frac{\alpha}{\sqrt{2}}(p + a) & m_\Sigma & 0 & -\eta 2i\sqrt{2}\omega & \eta i\sqrt{2}\sigma \\
\frac{\bar{\alpha}}{\sqrt{2}}(p - a) & 0 & m_\Sigma & 0 & 0 \\
\bar{\alpha}2i\omega & \eta 2i\sqrt{2}\omega & 0 & m_\Sigma + \eta(p + a) & \eta 2\sigma \\
-\bar{\alpha}i\sigma & -\eta i\sqrt{2}\sigma & 0 & \eta 2\sigma & 2(m_\Phi + \lambda(p + a - 4\omega))
\end{pmatrix}
\] (B19)
APPENDIX C: PROPERTIES OF THE MIXING MATRICES

1. Standard Model vacuum

In the Standard Model vacuum, we have

\[ p = \frac{m_\Phi }{\lambda } \frac{x(1-5x^2)}{(1-x)^2}; \quad a = \frac{m_\Phi }{\lambda } \frac{1-2x-x^2}{(1-x)}; \quad \omega = \frac{m_\Phi }{\lambda } \frac{x}{(1-x)}; \quad \sigma^2 = \frac{2m_\Phi^2}{\eta \lambda } \frac{x(1-3x)(1+x^2)}{(1-x)^2}; \]  

(C1)

with

\[ -8x^3 + 15x^2 - 14x + 3 = (x - 1)^2 \frac{\lambda m_\Sigma}{\eta m_\Phi } \]  

(C2)

Eigenstates are listed in Tables I and II. We give here the mixing matrices for the SM singlets, color triplets and \( SU(2)_L \) doublets for the SM vacua parametrized by \( x \). Matrices are in units of \( m_\Phi \).

\[ \mathbf{Y}/2 = 0 \]

\[ \varphi = (\Phi_+, \Phi_A, \Phi_\Omega, \Sigma_{113}^+, \Sigma_{113}^-) \]

\[ \varphi = (\Phi_+, \Phi_A, \Phi_\Omega, \Sigma_{113}^+, \Sigma_{113}^-) \]

\[ \begin{pmatrix}
2 & 0 & -2\sqrt{6}x & i\sqrt{\frac{\lambda}{x}}s(x) & -i\sqrt{\frac{\lambda}{x}}s(x) \\
0 & -2\frac{2x^2 + 3x - 1}{x-1} & -4\sqrt{2x} & i\sqrt{\frac{\lambda}{2x}}s(x) & -i\sqrt{\frac{\lambda}{2x}}s(x) \\
-2\sqrt{6}x & -4\sqrt{2x} & 2(1+x^2)(3x-1) & -i\sqrt{\frac{\lambda}{2}}\sqrt{6} s(x) & i\sqrt{\frac{\lambda}{2}}\sqrt{6} s(x) \\
-i\sqrt{\frac{\lambda}{x}}s(x) & -i\sqrt{\frac{\lambda}{2x}}s(x) & i\sqrt{\frac{\lambda}{2}}\sqrt{6} s(x) & 0 & 0 \\
i\sqrt{\frac{\lambda}{x}}s(x) & i\sqrt{\frac{\lambda}{2x}}s(x) & -i\sqrt{\frac{\lambda}{2}}\sqrt{6} s(x) & 0 & 0
\end{pmatrix} \]  

(C3)

where

\[ s(x) = \pm \frac{\sqrt{2x(1-3x)(1+x^2)}}{(x-1)} \]

\[ \mathbf{Y}/2 = (-1/2, 1/2) \]

\[ \varphi = (H_{122}, \Sigma_{112}^-, \Sigma_{122}^-, \Phi_{122}^+) \]

\[ \varphi = (H_{122}^+, \Sigma_{112}^+, \Sigma_{122}^+, \Phi_{122}^-) \]

\[ \begin{pmatrix}
2m_H/m_\Phi & \frac{2m_H}{\sqrt{2x}} & \frac{2m_H}{\sqrt{2x}} & \frac{2m_H}{\sqrt{2x}} & \frac{2m_H}{\sqrt{2x}} \\
\frac{a}{\sqrt{2x}} \frac{1-3x}{x-1} & -\frac{a}{\sqrt{2x}} \frac{8x^3 - 9x^2 + 6x - 1}{(x-1)^2} & 0 & \frac{a}{\sqrt{2x}} \frac{\sqrt{6}}{\sqrt{\lambda}} s(x) \\
\frac{a}{\sqrt{2x}} \frac{2x^2 - x + 1}{x-1} & 0 & -\frac{a}{\sqrt{2x}} \frac{12x^3 - 17x^2 + 10x - 1}{(x-1)^2} & 0 & -4\frac{a}{\sqrt{2x}} \frac{\sqrt{6}}{\sqrt{\lambda}} s(x)
\end{pmatrix} \]  

(C4)

\[ \mathbf{Y}/2 = (1/3, -1/3) \]

\[ \varphi = (H_{311}, \Sigma_{311}, \Sigma_{311}^+, \Phi_{313}^-) \]

\[ \varphi = (H_{311}, \Sigma_{311}^+, \Sigma_{311}^-, \Phi_{313}^+) \]

\[ \begin{pmatrix}
2m_H/m_\Phi & \frac{a}{\sqrt{2x}} \frac{1}{x-1} & \frac{a}{\sqrt{2x}} \frac{1}{x-1} & \frac{a}{\sqrt{2x}} \frac{1}{x-1} & \frac{a}{\sqrt{2x}} \frac{1}{x-1} \\
\frac{a}{\sqrt{2x}} \frac{4x^3 - 2x^2 + 2x - 1}{(x-1)^2} & -\frac{a}{\sqrt{2x}} \frac{8x^3 - 15x^2 + 14x - 3}{x(x-1)^2} & 0 & \frac{a}{\sqrt{2x}} \frac{\sqrt{2}}{\sqrt{\lambda}} s(x) \\
\frac{a}{\sqrt{2x}} \frac{1}{x-1} & 0 & -\frac{a}{\sqrt{2x}} \frac{8x^3 - 15x^2 + 14x - 3}{x(x-1)^2} & 0 & 0 \\
-2\frac{a}{\sqrt{2x}} s(x) & -2\frac{a}{\sqrt{2x}} s(x) & 0 & -2\frac{a}{\sqrt{2x}} \frac{2x^2 - x + 1}{x-1} & 2\frac{a}{\sqrt{2x}} \frac{\sqrt{2}}{\sqrt{\lambda}} s(x)
\end{pmatrix} \]  

(C5)
2. Explicit expression of the determinants

There is some interest in having an expression of the determinant of the matrices. One reason is that one wants that some determinants are not small. Another reason is the following. Let us consider the 1 loop running of \( n \) particles with masses \( m_1, m_2 \ldots m_n \), that mix among them through the \( n \times n \) matrix \( \mathcal{M} \). Since they have same beta function coefficient \( b \), their contribution is \( b \log(m_1/M) + b \log(m_2/M) + \ldots b \log(m_n/M) = b \log(\prod_{i=1}^{n} m_i/M^n) \). Of course, the masses \( m_i \) have to be positive numbers here and the zero modes have to be excluded. Thus one does not need exactly the determinant of the matrix, but the square root of \( \mathcal{M}^\dagger \mathcal{M} \), with the zero modes removed, that we call ‘reduced determinant’. (In the case of a matrix \( 1 \times 1 \), one has simply to take the absolute value.) For matrices \( n \times n \) with a single zero mode, namely \( m_1 = 0 \), the reduced determinant can be calculated either explicitly or by a simpler formula:

\[
\prod_{i=2}^{n} m_i = \left[ \text{Det}'(\mathcal{M}\mathcal{M}^\dagger) \right]^{1/2} = \text{Abs} \left[ \text{Det}'(\mathcal{M}) / \langle e|f \rangle \right]
\]  

(C6)

where the symbol \( \text{Det}' \) means \( \text{Det}'(\mathcal{M}) = \lim_{\epsilon \rightarrow 0} \text{Det}(\mathcal{M} + \epsilon \mathbf{1})/\epsilon \). At the denominator of the last expression we have \( \langle e|f \rangle = \sum_{i=1}^{n} e_i^* f_i \), namely the scalar product of the unit vectors of the left and right zero modes defined by \( \mathcal{M} f = 0 \) and \( \mathcal{M}^\dagger e = 0 \).

In the following, we show the determinants “\( \text{Det} \)” of the mass matrices and the reduced determinants “\( \text{Det}' \)” identified by their \( G_{321} \) quantum numbers.

\[
\text{Det}(8, 1, 0) = 8 m_\Phi^2 \frac{(x + 1) (2 x - 1) (x^3 + 6 x^2 - 7 x + 2)}{(x - 1)^3} 
\]

(C7)

\[
\text{Det}(3, 1, -1/3) = -32 m_\Phi^5 \frac{\alpha \bar{\eta}^2 x (x + 1) (2 x - 1)^2 (3 x - 1)^2 p_{16}}{(x - 1)^9 p_3 p_5} 
\]

(C8)

\[
\text{Det}'(1, 1, 1) = 2 m_\Phi \frac{(3 x - 1) (x^2 + 1) - 3 \frac{\bar{\eta}}{\lambda} (x - 1)^2 x}{(x - 1)^2}
\]

(C9)

\[
\text{Det}'(3, 2, -5/6) = -2 m_\Phi \frac{3 x^2 - 2 x + 1}{x - 1} 
\]

(C10)

\[
\text{Det}'(3, 1, 2/3) = -8 m_\Phi^2 x \frac{(3 x - 1) x (x^2 + 1) - \frac{x}{\lambda} (3 x^4 + 4 x^2 - 4 x + 1)}{(x - 1)^3}
\]

(C11)

\[
\text{Det}'(3, 2, 1/6) = -4 m_\Phi^2 \frac{2 x (x - 1)^2 (x^2 + 1) + \frac{\bar{\eta}}{\lambda} (3 x - 1) (2 x^4 + x^2 - 2 x + 1)}{(x - 1)^3}
\]

(C12)

\[
\text{Det}'(1, 1, 0) = 128 m_\Phi^4 \frac{\eta}{\lambda} \frac{x (x + 1) (2 x - 1) (3 x - 1) (x^2 + 1) (x^3 - 3 x^2 + 2 x - 1)}{(x - 1)^5}
\]

(C13)

\[
\text{Det}'(1, 2, 1/2) = 2 m_\Phi^3 \frac{\alpha \bar{\eta} \lambda^2 (6 (x - 1)^2 p_{14} + \frac{\bar{\eta}}{\lambda} x p_{15}) + 2 (\frac{\bar{\eta}}{\lambda} p_3 p_5)^2}{(x - 1)^5 p_3 p_5}
\]

(C14)

The relevant polynomials are:
Finally, we give the expressions of the (unnormalized) left and right zero-modes, useful to calculate the reduced determinants:

\[ \begin{align*}
&f(1, 1, 1) = (-i\sqrt{6}\omega, \sigma) \\
&e^*(1, 1, 1) = (i\sqrt{6}\omega, \sigma)
\end{align*} \]

\[ \begin{align*}
&f(3, 2, -5/6) = (-\sqrt{2}x, x - 1) \\
&e^*(3, 2, -5/6) = f(3, 2, -5/6)
\end{align*} \]

\[ \begin{align*}
&f(3, 1, -2/3) = (-\sqrt{2}a, 2\omega, \sigma) \\
&e^*(3, 1, -2/3) = f(3, 1, -2/3)
\end{align*} \]  

(C15)

\[ \begin{align*}
&f(3, 2, -1/6) = ((\omega - p)/\sqrt{2}, \omega - a, \sigma) \\
&e^*(3, 2, -1/6) = f(3, 2, -1/6)
\end{align*} \]

\[ \begin{align*}
&f(1, 1, 0) = (0, 0, 0, 1, 1) \\
&e^*(1, 1, 0) = f(1, 1, 0)
\end{align*} \]  

(C16)

The expressions for the zero modes of the doublet matrix (=the light higgses) are given and discussed in the next Section.

3. Arranging 2 light doublets

In order to arrange the right spectrum in low energy theory, one has to impose that the determinant of the mass matrix of the doublets vanishes in first approximation. Since no coupling or mass of the theory is zero, this condition (referred as “minimal fine-tuning”) reads as follows:

\[ m_H = m_\Phi \frac{\alpha \bar{\alpha}}{2\eta \lambda (x - 1)p_3p_5} \]  

(C17)

where the three polynomials of \( x \), namely \( p_3, p_5 \) and \( p_{10} \) are defined above. From this condition, one finds the expressions of the zero-modes:

\[ H_d \propto \frac{2p_5}{x - 1} H_{122} - \sqrt{6} \frac{\bar{\alpha}}{\eta} (3x - 1)(x^3 + 5x - 1) \Sigma_{122} - \sqrt{6} \frac{\alpha}{\eta} (2x - 1)(x + 1) \frac{p_5}{p_3} \Sigma_{122} + \bar{\alpha} \frac{\sigma}{m_\Phi} p'_3 \Phi_{122}, \]  

(C18)

\[ H_u \propto \frac{2p_5}{x - 1} H_{122} + \sqrt{6} \frac{\alpha}{\eta} (3x - 1)(x^3 + 5x - 1) \Sigma_{122} + \sqrt{6} \frac{\bar{\alpha}}{\eta} (2x - 1)(x + 1) \frac{p_5}{p_3} \Sigma_{122} + \alpha \frac{\sigma}{m_\Phi} p'_3 \Phi_{122} \]  

(C19)

(the proper normalization of \( H_d \) is obtained in an obvious manner). The components that matter for the coupling of \( H_d \) to light particles are the first and the third one, while those for the coupling of \( H_u \) are the first two.

It should be noted that the mixing between the 10- and 126-light component depends just on two parameters, namely \( x \) and \( \alpha/\eta \). Furthermore, the parameter \( \alpha/\eta \) always multiplies the coupling \( Y_S \). Finally, we illustrate the dependence on \( x \) by giving a pair of examples (from here on, we have in mind the case \( |\alpha/\eta| \sim 1 \)). If we ask that \( H_u \) has a reduced 126 component, we are either at \( x \sim 1/3 \) or close at one of the roots of \( x^3 + 5x - 1 \); the real one is close to 0.2. Conversely, it is interesting to note that there is a root of \( p_3 \) in the vicinity of \( x = 0.12 \), and there 126 component of \( H_d \) happens to be large.
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| $G_{422}$ | State in $\Sigma$ | SO(10) indices |
|---------|------------------|----------------|
| 6,1,1   | 311              | $[1,2,3,4,7] + i[1,2,3,4,8]$ |
|         | 311              | $[1,2,3,4,7] - i[1,2,3,4,8]$ |
| 10,3,1  | 131              | $(1,2,5,7,9) - (1,2,6,8,9) - (1,2,5,8,0) - (1,2,6,7,0)$ |
|         |                  | $+ i[1,2,5,7,9] + i[1,2,5,8,9] + i[1,2,6,7,9] - i[1,2,6,8,0]$ |
|         | 331              | $- (1,2 \rightarrow 3,4)$ |
|         | 631              | $(1,2,5,6,8) - (1,2,8,9,0) - i[1,2,5,6,7] - i[1,2,7,9,0]$ |
|         |                  | $- (1,2 \rightarrow 3,4)$ |
| 10,1,3  | 113              | $- i[(1,3,5,7,9) - (1,3,6,8,9) - (1,3,5,8,0) - (1,3,6,7,0)$ |
|         |                  | $- i[1,3,5,7,0] - i[1,3,5,8,9] - i[1,3,6,7,9] + i[1,3,6,8,0]$ |
|         |                  | $- (1,3 \rightarrow 2,4) + i[1,3 \rightarrow 1,4] + i[1,3 \rightarrow 2,3]$ |
|         | 113              | $[(1,2,5,7,9) - (1,2,6,8,9) - (1,2,5,8,0) - (1,2,6,7,0)$ |
|         |                  | $- i[1,2,5,7,0] - i[1,2,5,8,9] - i[1,2,6,7,9] + i[1,2,6,8,0]$ |
|         |                  | $+ (1,2 \rightarrow 3,4)$ |
|         | 113              | $(1,3,5,7,9) - i[1,3,6,8,9] - i[1,3,5,8,0] - i[1,3,6,7,0]$ |
|         |                  | $+ i[1,3,5,7,0] + i[1,3,5,8,9] + i[1,3,6,7,9] - i[1,3,6,8,0]$ |
|         |                  | $- (1,3 \rightarrow 2,4) - i[1,3 \rightarrow 1,4] - i[1,3 \rightarrow 2,3]$ |
|         | 312              | $[(1,2,5,6,8) - (1,2,8,9,0) + i[1,2,5,6,7] + i[1,2,7,9,0]$ |
|         |                  | $+ (1,2 \rightarrow 3,4)$ |
|         | 313              | $[(1,3,5,6,8) + i[1,3,8,9,0] - (1,3,5,6,7) - (1,3,7,9,0)$ |
|         |                  | $- (1,3 \rightarrow 2,4) - i[1,3 \rightarrow 1,4] - i[1,3 \rightarrow 2,3]$ |
|         | 613              | $(1,2,5,6,8) - i[1,2,8,9,0] - i[1,2,5,6,7] + i[1,2,7,9,0]$ |
|         |                  | $+ (1,2 \rightarrow 3,4)$ |
|         | 613              | $(1,3,5,6,8) - i[1,3,8,9,0] + i[1,3,5,6,7] - i[1,3,7,9,0]$ |
|         |                  | $- (1,3 \rightarrow 2,4) - i[1,3 \rightarrow 1,4] - i[1,3 \rightarrow 2,3]$ |
| 15,2,2  | 122              | $- (1,5,6,7,8) - (1,5,6,9,0) - (1,7,8,9,0)$ |
|         |                  | $- i(1 \rightarrow 2)$ |
|         | 122              | $(+3,5,6,7,8) + [3,5,6,9,0] + [3,7,8,9,0]$ |
|         |                  | $+ i[3 \rightarrow 4]$ |
|         | 322              | $- i[1,6,7,8,9] - (1,5,7,8,0) - i[1,5,7,8,9] + i[1,6,7,8,0]$ |
|         |                  | $- i(1 \rightarrow 2)$ |
|         | 322              | $(+3,6,7,8,9) + [3,5,7,8,9] + i[3,6,7,8,0]$ |
|         |                  | $+ i(3 \rightarrow 4)$ |
|         | 322              | $(+1,6,7,8,9) + [1,5,7,8,0] - i[1,5,7,8,9] + i[1,6,7,8,0]$ |
|         |                  | $- i(1 \rightarrow 2)$ |
|         | 322              | $(+3,6,7,8,9) + [3,5,7,8,9] - i[3,6,7,8,0]$ |
|         |                  | $- i(3 \rightarrow 4)$ |
|         | 822              | $- (1,7,8,9,0) + [1,5,6,9,0] - i(1 \rightarrow 2)$ |
|         | 822              | $(+3,7,8,9,0) - [3,5,6,9,0] + i(3 \rightarrow 4)$ |

**TABLE VII:** Decomposition of states in the $126$ representation
| $F_\Phi$  | $6(H_p + H_s)$ | $6\mathcal{H}_\omega$ | $\mathcal{H}_s, \mathcal{H}_\omega$ | $K_s, K_\omega$ |
|---------|----------------|---------------------|------------------|------------------|
| $F_{\Phi P}$ | $4a \Phi_A$ | $2\sqrt{3}\omega \Phi_{11}$ | $i(\sigma \Sigma_{113^-} - \sigma \Sigma_{113^+})$ | |
| $F_{\Phi A}$ | $2a \Phi_{311}$ | $-2\sqrt{2}\omega \Phi_{313^0}$ | $-\sqrt{2}\sigma \Sigma_{313^-}$ | |
| $F_{\Phi_{311}}$ | $2a \Phi_{311}$ | $-2\sqrt{2}\omega \Phi_{313^0}$ | $-\sqrt{2}\sigma \Sigma_{313^-}$ | |
| $F_{\Phi_{813}}$ | $-2a \Phi_{811}$ | $-2\sqrt{2}\omega \Phi_{813^0}$ | | |
| $F_{\Phi_{113}^+}$ | $(p + 2a)\Phi_{113^+}$ | $2\omega (\sqrt{6}\Phi_p + 2\sqrt{2}\Phi_A)$ | $\sqrt{6}\sigma \Sigma_{113^0}$ | $i\sigma H_{113}$ |
| $F_{\Phi_{113}^-}$ | $(p - 2a)\Phi_{113^-}$ | $-\sqrt{6}\sigma \Sigma_{113^0}$ | $-i\sigma H_{113}$ | |
| $F_{\Phi_{313}^+}$ | $(p + a)\Phi_{313^+}$ | $8\omega \Phi_{313^0}$ | $-2\sigma \Sigma_{313^-}$ | |
| $F_{\Phi_{313}^-}$ | $(p + a)\Phi_{313^-}$ | $-8\omega \Phi_{313^0}$ | $-i\sqrt{2}\sigma (\Sigma_{311} - i\sqrt{2}\Sigma_{313})$ | $i\sigma H_{313}$ |
| $F_{\Phi_{313}^0}$ | $(p + a)\Phi_{313}$ | $-2\sqrt{2}\omega \Phi_{311}$ | $-2\sigma \Sigma_{313^-}$ | |
| $F_{\Phi_{813}^+}$ | $(p + a)\Phi_{813^+}$ | $8\omega \Phi_{313^-}$ | | |
| $F_{\Phi_{813}^-}$ | $(p - a)\Phi_{813^-}$ | $-8\omega \Phi_{313^-}$ | | |
| $F_{\Phi_{813}^0}$ | $(p - a)\Phi_{813}$ | $-2\sqrt{2}\omega \Phi_{811}$ | | |
| $F_{\Phi_{131}}$ | $(p + 2a)\Phi_{131}^+$ | $2\omega (\Phi_f^+ + \sqrt{2}\Phi_{f f}^+)$ | $-\sqrt{2}\sigma \Sigma_{322^-}$ | |
| $F_{\Phi_{131}^-}$ | $(p - 2a)\Phi_{131^-}$ | $2\omega (\Phi_f^- + \sqrt{2}\Phi_{f f}^-)$ | $-\sqrt{2}\sigma \Sigma_{322^-}$ | |
| $F_{\Phi_{122}^+}$ | $6a \Phi_{122^+}$ | $-6\omega \Phi_{122^+}$ | $\sqrt{6}\sigma \Sigma_{122^-}$ | $\sigma H_{122}$ |
| $F_{\Phi_{122}^-}$ | $6a \Phi_{122^-}$ | $6\omega \Phi_{122^-}$ | | |
| $F_{\Phi_{122}^0}$ | $2a \Phi_{122}$ | $-2\omega (\Phi_{f f}^+ + \sqrt{2}\Phi_{f f}^+)$ | $-2\sigma \Sigma_{322^-}$ | |
| $F_{\Phi_{122}^0}$ | $2a \Phi_{122}$ | $2\omega (\Phi_{f f}^- + \sqrt{2}\Phi_{f f}^-)$ | | |
| $F_{\Phi_{622}^+}$ | $-2a \Phi_{622^+}$ | $2\omega \Phi_{622^+}$ | | |
| $F_{\Phi_{622}^-}$ | $-2a \Phi_{622^-}$ | $-2\omega \Phi_{622^-}$ | | |
| $F_{\Phi_{622}^0}$ | $-2a \Phi_{622}$ | | | |

**TABLE VIII:** F-term contribution for states in $\Phi$. 
\[
\begin{array}{|c|c|c|c|}
\hline
F_\Sigma & 5(F_p + F_a) & 5J_a & 5J_\sigma & 5(K_p + K_a + K_\omega) \\
\hline
F_{S,311} & -2i\sqrt{2} \omega \Sigma_{313} & \sqrt{2}i\sigma \Phi_{333}^+ & \sqrt{2}(p - a)H_{311} & \sqrt{2}(p + a)H_{311} \\
F_{S,331} & (p + a)\Sigma_{331} & 2i\omega \Sigma_{331} & \sqrt{2}i\sigma \Phi_{333}^+ & \sqrt{2}(p - a)H_{311} \\
F_{S,331} & (-p + a)\Sigma_{331} & 2i\omega \Sigma_{331} & \sqrt{2}i\sigma \Phi_{333}^+ & \sqrt{2}(p + a)H_{311} \\
F_{S,113} & (p + 3a)\Sigma_{113}^+ & -6\omega \Sigma_{113}^+ & -i\sqrt{6} \omega \Phi_{113}^+ & -i\sqrt{6} \omega \Phi_{113}^+ \\
F_{S,113} & (p + 3a)\Sigma_{113}^- & -6\omega \Sigma_{113}^- & -i\sqrt{6} \omega \Phi_{113}^+ & -i\sqrt{6} \omega \Phi_{113}^+ \\
F_{S,113} & (p + a)\Sigma_{113}^+ & 2\omega \Sigma_{113}^+ & 2\omega \Sigma_{113}^+ & 2\omega \Sigma_{113}^+ \\
F_{S,113} & (p + a)\Sigma_{113}^- & 2\omega \Sigma_{113}^- & 2\omega \Sigma_{113}^- & 2\omega \Sigma_{113}^- \\
F_{S,013} & (p + a)\Sigma_{013}^+ & -2\omega \Sigma_{013}^+ & -2\omega \Sigma_{013}^+ & -2\omega \Sigma_{013}^+ \\
F_{S,013} & (p + a)\Sigma_{013}^- & -2\omega \Sigma_{013}^- & -2\omega \Sigma_{013}^- & -2\omega \Sigma_{013}^- \\
F_{S,922} & (p + a)\Sigma_{922}^+ & 2\omega \Sigma_{922}^+ & 2\omega \Sigma_{922}^+ & 2\omega \Sigma_{922}^+ \\
F_{S,922} & (p + a)\Sigma_{922}^- & 2\omega \Sigma_{922}^- & 2\omega \Sigma_{922}^- & 2\omega \Sigma_{922}^- \\
F_{S,622} & (p + a)\Sigma_{622}^+ & -2\omega \Sigma_{622}^+ & -2\omega \Sigma_{622}^+ & -2\omega \Sigma_{622}^+ \\
F_{S,622} & (p + a)\Sigma_{622}^- & -2\omega \Sigma_{622}^- & -2\omega \Sigma_{622}^- & -2\omega \Sigma_{622}^- \\
F_{S,822} & (p + a)\Sigma_{822}^+ & -2\omega \Sigma_{822}^+ & -2\omega \Sigma_{822}^+ & -2\omega \Sigma_{822}^+ \\
F_{S,822} & (p + a)\Sigma_{822}^- & -2\omega \Sigma_{822}^- & -2\omega \Sigma_{822}^- & -2\omega \Sigma_{822}^- \\
\hline
\end{array}
\]

**TABLE IX:** F-term contribution for states in \( \Sigma \).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Vevs} & \text{SO}(10) \rightarrow & \text{P.G.B.} \\
\hline
1 & p = a = w = \sigma = 0 & & \text{---} \\
2 & p = a = w = -m/3\lambda; \sigma = 0 & SU(5) \times U(1) & \text{---} \\
3 & p = a = w = -M/10\eta; \sigma^2 = M(10m - 3\lambda M)/50\eta^3 & SU(5) & \text{---} \\
4 & p = w = \sigma = 0; a = -m/\lambda & SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_{B-L} & (3 + 3, 1, 3, 3 + 3, 3, 1) \\
5 & p = a = -w = -m/3\lambda; \sigma = 0 & SU(5) \times U(1) & \text{---} \\
6 & p = -3m/\lambda; a = -2m/\lambda; w = \pm im/\lambda; \sigma = 0 & SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L} & (8, 3, 1) \\
7 & \text{eqns. 10-11} & SU(3)_c \times SU(2)_L \times U(1) & \text{depend on } x \\
\hline
\end{array}
\]

**TABLE X:** Patterns of symmetry breaking