THE SPACE CONSISTING OF UNIFORMLY CONTINUOUS FUNCTIONS ON
A METRIC MEASURE SPACE WITH THE $L^p$ NORM

KATSUHISA KOSHINO

ABSTRACT. Let $X = (X, d, \mathcal{M}, \mu)$ be a metric measure space, where $d$ is a metric on $X$, $\mathcal{M}$ is a
$\sigma$-algebra of $X$, and $\mu$ is a measure on $\mathcal{M}$. Suppose that $X$ is separable and locally compact, that
$\mathcal{M}$ contains the Borel sets of $X$, that for each $E \in \mathcal{M}$, there exists a Borel set $B \subset X$ such that
$E \subset B$ and $\mu(B \setminus E) = 0$, that for every non-empty open set $U \subset X$, $\mu(U) > 0$, that for all compact
sets $K \subset X$, $\mu(K) < \infty$, and that $X \setminus \{x \in X \mid \{x\} \in \mathcal{M} \text{ and } \mu(\{x\}) = 0\}$ is not dense in $X$. In
this paper, we shall show that the space of real-valued uniformly continuous functions on $X$ with
the $L^p$ norm, $1 \leq p < \infty$, is homeomorphic to the subspace consisting of sequences converging to
0 in the pseudo interior.

1. INTRODUCTION

Throughout this paper, we assume that spaces are Hausdorff, maps are continuous, but functions
are not necessarily continuous, and $1 \leq p < \infty$. Let $X = (X, d, \mathcal{M}, \mu)$ denote a metric measure
space, where $d$ is a metric on $X$, $\mathcal{M}$ is a $\sigma$-algebra of $X$, and $\mu$ is a measure on $\mathcal{M}$. Denote

$$X_0 = \{x \in X \mid \{x\} \in \mathcal{M} \text{ and } \mu(\{x\}) = 0\}.$$ 

A measure space $X$ is called to be Borel provided that $\mathcal{M}$ contains the Borel sets of $X$. We say
that a Borel measure space $X$ is regular if for each $E \in \mathcal{M}$, there is a Borel set $B \subset X$ such
that $E \subset B$ and $\mu(B \setminus E) = 0$. Notice that the Euclidean space $\mathbb{R}^n = (\mathbb{R}^n, \mathcal{M}, \mu)$, where $\mathcal{M}$ is
the Lebesgue measurable sets and $\mu$ is the Lebesgue measure, is regular Borel. A regular Borel
measure space $X$ satisfies the following stronger condition.

- For any $E \in \mathcal{M}$, there exist Borel sets $B_1$ and $B_2$ in $X$ such that $B_2 \subset E \subset B_1$ and
  $\mu(B_1 \setminus B_2) = 0$.

Indeed, if $X$ is regular Borel, then for every $E \in \mathcal{M}$, we can find Borel subsets $B_1$ and $C$ of $X$
such that $E \subset B_1$, $X \setminus E \subset C$, $\mu(B_1 \setminus E) = 0$ and $\mu(C \setminus (X \setminus E)) = 0$. Let $B_2 = X \setminus C$, so

$$\mu(B_1 \setminus B_2) = \mu(B_1 \setminus E) + \mu(E \setminus B_2) = \mu(B_1 \setminus E) + \mu(C \setminus (X \setminus E)) = 0.$$ 

We write the integral of a real-valued $\mathcal{M}$-measurable function $f(x)$ on $E \in \mathcal{M}$ with respect to
$\mu$ as $\int_E f(x) d\mu(x)$. Set

$$L^p(X) = \left\{ f : X \to \mathbb{R} \mid f \text{ is } \mathcal{M}\text{-measurable and } \int_X |f(x)|^p d\mu(x) < \infty \right\}$$

endowed with the following norm

$$\|f\|_p = \left( \int_X |f(x)|^p d\mu(x) \right)^{1/p},$$

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where two functions that are coincident almost everywhere are identified. Recall that a property holds almost everywhere if it holds on \( X \setminus E \) for some \( E \in \mathcal{M} \) with \( \mu(E) = 0 \). The function space \( L^p(X) \) is a Banach space, refer to [5, Theorem 4.8]. When \( X \) is a separable regular Borel metric measure space, \( L^p(X) \) is also separable, see [5, Theorem 4.13]. If for any \( n \in \mathbb{N} \), there is a pairwise disjoint family \( \{E_i\}_{1 \leq i \leq n} \subset \mathcal{M} \) such that each \( \mu(E_i) > 0 \), then \( L^p(X) \) is infinite-dimensional. Hence in the case that \( X \) is infinite, \( \mathcal{M} \) contains the open subsets of \( X \) and for each non-empty open set \( U \subset X, \mu(U) > 0 \), the space \( L^p(X) \) is infinite-dimensional. Denote the Hilbert cube by \( Q = [-1, 1]^\mathbb{N} \) and the pseudo interior by \( s = (-1, 1)^\mathbb{N} \). In the theory of infinite-dimensional topology, typical infinite-dimensional spaces, for example subspaces of \( Q \), have been detected among function spaces. Due to the efforts of R.D. Anderson [2] and M.I. Kadec [7], we have the following:

**Theorem 1.1.** Let \( X \) be an infinite separable regular Borel metric measure space. Suppose that any non-empty open subset of \( X \) is of positive measure. Then \( L^p(X) \) is an infinite-dimensional separable Banach space, so it is homeomorphic to \( s \).

In this paper, the topological type of the subspace
\[
C_u(X) = \{ f \in L^p(X) \mid f \text{ is uniformly continuous}\}
\]
will be studied. When \( X \) is compact, \( C_u(X) \) is coincident with the space
\[
C(X) = \{ f \in L^p(X) \mid f \text{ is continuous}\}.
\]

It is known that several function spaces are homeomorphic to the following subspace of \( s \),
\[
c_0 = \left\{ (x(n))_{n \in \mathbb{N}} \in s \mid \lim_{n \to \infty} x(n) = 0 \right\},
\]
refer to [8, 11, 10]. R. Cauty [6] proved the next theorem.

**Theorem 1.2.** Let \([0, 1] = ([0, 1], d, \mathcal{M}, \mu)\) be the closed unit interval, where \( d \) is the usual metric, \( \mathcal{M} \) is the Lebesgue measurable sets, and \( \mu \) is the Lebesgue measure. Then \( C([0, 1]) \) is homeomorphic to \( c_0 \).

More generally, we shall show the following:

**Main Theorem.** Let \( X \) be a separable locally compact regular Borel metric measure space. Suppose that for every non-empty open set \( U \subset X, \mu(U) > 0 \), that for each compact set \( K \subset X, \mu(K) < \infty \), and that \( X \setminus X_0 \) is not dense in \( X \). Then \( C_u(X) \) is homeomorphic to \( c_0 \).

## 2. Preliminaries

For each point \( x \in X \) and each positive number \( \delta > 0 \), put the open ball \( B(x, \delta) = \{ y \in X \mid d(x, y) < \delta \} \). Given subsets \( A, B \subset L^p(X) \), we denote their distance by \( \text{dist}(A, B) = \inf_{f \in A, g \in B} \|f - g\|_p \). For spaces \( A \subset Y \), the symbol \( \text{cl}_Y A \) stands for the closure of \( A \) in \( Y \). Recall that for functions \( f : Z \to Y \) and \( g : Z \to Y \), and for an open cover \( \mathcal{U} \) of \( Y \), \( f \) is \( \mathcal{U} \)-close to \( g \) provided that for each \( z \in Z \), there exists an open set \( U \in \mathcal{U} \) such that the doubleton \( \{f(z), g(z)\} \subset U \). We call a closed set \( A \) in a space \( Y \) a \( Z \)-set in \( Y \) if for each open cover \( \mathcal{U} \) of \( Y \), there exists a map \( f : Y \to Y \) such that \( f \) is \( \mathcal{U} \)-close to the identity map of \( Y \) and the image \( f(Y) \) misses \( A \). A \( Z_n \)-set is a countable union of \( Z \)-sets. A map \( f : Z \to Y \) is called to be a \( Z \)-embedding if \( f \) is an embedding and \( f(Z) \) is a \( Z \)-set in \( Y \). Given a class \( \mathcal{C} \) of spaces, we say that \( Y \) is strongly \( \mathcal{C} \)-universal if the following condition is satisfied.

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1Recall that for a property \( P \) of functions, a function \( g \in \{ f \in L^p(X) \mid f \text{ satisfies the property } P \} \) if there exists \( f : X \to \mathbb{R} \) such that \( f \) satisfies the property \( P \) and \( g = f \) almost everywhere.
Let $A \in \mathcal{C}$ and $f : A \to Y$ be a map. Suppose that $B$ is a closed set in $A$ and the restriction $f|_B$ is a $Z$-embedding. Then for each open cover $\mathcal{U}$ of $Y$, there is a $Z$-embedding $g : A \to Y$ such that $g$ is $\mathcal{U}$-close to $f$ and $g|_B = f|_B$.

For spaces $Y \subset M$, $Y$ is homotopy dense in $M$ if $M$ admits a homotopy $h : M \times [0,1] \to M$ such that $h(M \times (0,1)) \subset Y$ and $h(y,0) = y$ for every $y \in M$. For a class $\mathcal{C}$, let $\mathcal{C}_\sigma$ be the class of spaces written as a countable union of closed subspaces that belong to $\mathcal{C}$. A space $Y$ is said to be a $\mathcal{C}$-absorbing set in $M$ provided that it satisfies the following conditions.

1. $Y \in \mathcal{C}_\sigma$ and is homotopy dense in $M$.
2. $Y$ is strongly $\mathcal{C}$-universal.
3. $Y$ is contained in a $Z_\sigma$-set in $M$.

Let $\mathcal{M}_2$ be the class of absolute $F_{\sigma\delta}$-spaces, that is, $Y \in \mathcal{M}_2$ if $Y$ is metrizable and is an $F_{\sigma\delta}$-set in any metrizable space $M$ containing $Y$ as a subspace. The space $c_0$ is an $\mathcal{M}_2$-absorbing set in $\mathfrak{s}$. According to Theorem 3.1 of [1], we can establish the following:

**Theorem 2.1.** Let $Y$ and $Z$ be an $\mathcal{M}_2$-absorbing set in $\mathfrak{s}$. Then $Y$ and $Z$ are homeomorphic.

3. The Borel complexity of $C_u(X)$ in $L^p(X)$

In this section, we will show that $C_u(X) \in \mathcal{M}_2$. The following proposition is of use, refer to Theorem 4.9 of [2].

**Proposition 3.1.** Let $f, f_k \in L^p(X)$, $k \in \mathbb{N}$. If $\|f - f_k\|_p \to 0$, then there exists a subsequence $\{f_{k(n)}\}$ such that $f_{k(n)} \to f$ almost everywhere.

For all positive numbers $\epsilon, \delta > 0$, let

$$A(\epsilon, \delta) = \{ f \in L^p(X) \mid \text{for almost every } x, y \in X, \text{ if } d(x,y) < \delta, \text{ then } |f(x) - f(y)| \leq \epsilon \}.$$

**Lemma 3.2.** For any $\epsilon, \delta > 0$, the subset $A(\epsilon, \delta)$ is closed in $L^p(X)$.

**Proof.** To prove that $A(\epsilon, \delta)$ is closed in $L^p(X)$, fix any sequence $\{f_k\}$ in $A(\epsilon, \delta)$ converging to $f \in L^p(X)$. We need only to show that $f \in A(\epsilon, \delta)$, that is, for almost every $x, y \in X$, if $d(x,y) < \delta$, then $|f(x) - f(y)| \leq \epsilon$. Since $\|f - f_k\|_p \to 0$, we can replace $\{f_k\}$ with a subsequence so that $f_k \to f$ almost everywhere by Proposition 3.1. Then there exists $E_0 \subset X$ such that $\mu(E_0) = 0$ and $f_k(x) \to f(x)$ for each $x \in X \setminus E_0$. On the other hand, because $f_k \in A(\epsilon, \delta)$ for each $k \in \mathbb{N}$, we can find $E_k \subset X$ with $\mu(E_k) = 0$ so that for any $x, y \in X \setminus E_k$, if $d(x,y) < \delta$, then $|f_k(x) - f_k(y)| \leq \epsilon$. Let $E = \bigcup_{k \in \mathbb{N}} E_k$ and take any $x, y \in X \setminus E$ with $d(x,y) < \delta$. Note that $\mu(E) = 0$. Then $|f_k(x) - f_k(y)| \leq \epsilon$ for every $k \in \mathbb{N}$, $f_k(x) \to f(x)$ and $f_k(y) \to f(y)$, which implies that $|f(x) - f(y)| \leq \epsilon$. We conclude that $f \in A(\epsilon, \delta)$. \qed

Note that a space $Y \in \mathcal{M}_2$ if and only if $Y$ can be embedded into a completely metrizable space as an $F_{\sigma\delta}$-set, see [1], Theorem 9.6. We prove the following:

**Proposition 3.3.** Suppose that for each $E \in \mathcal{M}$ with $\mu(E) = 0$, $X \setminus E$ is dense in $X$. Then $C_u(X)$ is an $F_{\sigma\delta}$-set in $L^p(X)$, and hence $C_u(X) \in \mathcal{M}_2$.

**Proof.** We shall show that $C_u(X) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/n, 1/m)$. Then it follows from Lemma 3.2 that $C_u(X)$ is $F_{\sigma\delta}$ in $L^p(X)$. Clearly, $C_u(X) \subset \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/n, 1/m)$. To prove that $C_u(X) \supset \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/n, 1/m)$, fix any function $f \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/n, 1/m)$. For each $n \in \mathbb{N}$, there exist $m_n \in \mathbb{N}$ and $E_n \subset X$ with $\mu(E_n) = 0$ such that if $x, y \in X \setminus E_n$ and $d(x,y) < 1/m_n$, then $|f(x) - f(y)| \leq 1/n$. Let $E = \bigcup_{n \in \mathbb{N}} E_n$, so the measure $\mu(E) = 0$. Then for any $n \in \mathbb{N}$ and $x, y \in X \setminus E$ with $d(x,y) < 1/m_n$, we have $|f(x) - f(y)| \leq 1/n$, which implies that $f|_{X \setminus E}$ is uniformly continuous. Since $X \setminus E$ is dense in $X$, the restriction $f|_{X \setminus E}$ can be extended over $X$ as
a uniformly continuous map, that is coincident with \( f \) almost everywhere. Therefore \( f \in C_u(X) \).
The proof is complete. □

4. THE \( Z_\sigma \)-SET PROPERTY OF \( C_u(X) \) IN \( L^p(X) \)

In this section, it is shown that \( C_u(X) \) is contained in some \( Z_\sigma \)-set in \( L^p(X) \). The next theorem is important on convergence of sequences in \( L^p(X) \), refer to Theorem 4.2 of [5].

**Theorem 4.1** (The Dominated Convergence Theorem). Let \( f, f_k \in L^1(X) \), \( k \in \mathbb{N} \). Suppose that \( f_k(x) \to f(x) \) for almost every \( x \in X \), and that there is a function \( g \in L^p(X) \) such that for any \( k \in \mathbb{N} \), \( |f_k(x)| \leq g(x) \) for almost every \( x \in X \). Then \( f, f_k \in L^p(X) \), \( k \in \mathbb{N} \), and \( \|f - f_k\|_p \to 0 \).

The following technical lemma will be very useful for detecting \( Z \)-sets in \( L^p(X) \).

**Lemma 4.2.** Let \( Y \) be a paracompact space, \( \phi : Y \to L^p(X) \) be a map, and \( a \in X_0 \) such that for all \( \lambda > 0 \), \( B(a, \lambda) \in \mathcal{M} \). Then for every map \( \epsilon : Y \to (0,1) \), there exists maps \( \psi : Y \to L^p(X) \) and \( \delta : Y \to (0,1) \) such that for each \( y \in Y \),

(i) \( \|\phi(y) - \psi(y)\|_p \leq \epsilon(y) \),

(ii) \( \psi(y)(B(a, \delta(y))) = \{0\} \).

**Proof.** For each \( f \in L^p(X) \) and each \( A \in \mathcal{M} \), define a function \( f_A \in L^p(X) \) by

\[
f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}
\]

Remark that \( \|f_A\|_p = (\int_A |f(x)|^p d\mu(x))^{1/p} \). Given any \( y \in Y \), put

\[
\xi(y) = \sup \{0 < \eta \leq 1 \mid \|\phi(y)_{B(a,\eta)}\|_p < \epsilon(y) \}.
\]

According to Theorem 4.1, \( \xi(y) > 0 \) for all \( y \in Y \) because \( \mu(\{a\}) = 0 \). Then the function \( \xi : Y \to (0,1] \) is lower semi-continuous. Indeed, fix any \( y \in Y \) and \( \eta \in (0,\xi(y)) \). By the definition, there is \( \lambda \in (0,\epsilon(y)) \) such that \( \|\phi(y)_{B(a,\lambda(\eta) - \eta)}\|_p < \epsilon(y) - \lambda \). Due to the continuity of \( \phi \) and \( \epsilon \), we can find a neighborhood \( U \) of \( y \) such that for every \( y' \in U \), \( \|\phi(y) - \phi(y')\|_p < \lambda/2 \) and \( |\epsilon(y) - \epsilon(y')| < \lambda/2 \). Then

\[
\|\phi(y')_{B(a,\lambda(\eta) - \eta)}\|_p \leq \|\phi(y')_{B(a,\lambda(\eta) - \eta)} - \phi(y)_{B(a,\lambda(\eta) - \eta)}\|_p + \|\phi(y)_{B(a,\lambda(\eta) - \eta)}\|_p \\
\leq \|\phi(y') - \phi(y)\|_p + \|\phi(y)_{B(a,\lambda(\eta) - \eta)}\|_p \\
< \epsilon(y) - \lambda/2 < \epsilon(y').
\]

Therefore \( \xi(y') \geq \xi(y) - \eta \), which means that \( \xi \) is a lower semi-continuous function. Since \( Y \) is paracompact, there is a map \( \delta : Y \to (0,1) \) such that \( \delta(y) < \xi(y)/2 \) for each \( y \in Y \) by virtue of Theorem 2.7.6 of [9]. Then the desired map \( \psi : Y \to L^p(X) \) can be defined as follows:

\[
\psi(y)(x) = \begin{cases} 0 & \text{if } x \in B(a, \delta(y)), \\ ((d(a, x) - \delta(y))/\delta(y))\phi(y)(x) & \text{if } x \in B(a, 2\delta(y)) \setminus B(a, \delta(y)), \\ \phi(y)(x) & \text{if } x \in X \setminus B(a, 2\delta(y)). \end{cases}
\]

Condition (ii) follows from the definition immediately. Let us note that for each \( x \in B(a, 2\delta(y)) \setminus B(a, \delta(y)) \),

\[
|\phi(y)(x) - \psi(y)(x)| = |\phi(y)(x) - ((d(a, x) - \delta(y))/\delta(y))\phi(y)(x)| \\
= ((2\delta(y) - d(a, x))/\delta(y))|\phi(y)(x)| \leq |\phi(y)(x)|.
\]

\(^2\text{This is valid for any } p \in [1, \infty).\)
Since $2\delta(y) < \xi(y)$, we get

$$\|\phi(y) - \psi(y)\|_p \leq \|\phi(y)_{B(a,2\delta(y))}\|_p \leq \|\phi(y)_{B(a,\xi(y))}\|_p \leq \epsilon(y),$$

and hence condition (i) holds.

It remains to verify the continuity of $\psi$. Take any $y \in Y$ and $\lambda > 0$. Since $\phi$ and $\delta$ are continuous, we can choose a neighborhood $U$ of $y$ so that for each $y' \in U$,

(a) $\|\phi(y) - \phi(y')\|_p < \lambda/8$,
(b) $|\delta(y) - \delta(y')| < \delta(y)/2$,
(c) $|\delta(y) - \delta(y')|\|\phi(y)\|_p < \lambda\delta(y)/8$,
(d) $|1/\delta(y) - 1/\delta(y')|\|\phi(y)\|_p < \lambda/8$.

We shall prove that $\|\psi(y) - \psi(y')\|_p < \lambda$ only in the case that $\delta(y) \leq \delta(y')$ because it can be shown similarly in the other case. Note that $\delta(y') < 2\delta(y)$ by condition (b). Obviously, $\|\psi(y)_{B(a,\delta(y))} - \psi(y')_{B(a,\delta(y))}\|_p = 0$. Due to condition (c), we have

$$\|\psi(y)_{B(a,\delta(y'))\setminus B(a,\delta(y))} - \psi(y')_{B(a,\delta(y'))\setminus B(a,\delta(y))}\|_p$$

$$= \left(\int_{B(a,\delta(y'))\setminus B(a,\delta(y))} |\psi(y)(x) - \psi(y')(x)|^p d\mu(x)\right)^{1/p}$$

$$= \left(\int_{B(a,\delta(y'))\setminus B(a,\delta(y))} |(d(a,x) - \delta(y))/\delta(y)\phi(y)(x)|^p d\mu(x)\right)^{1/p}$$

$$\leq ((\delta(y') - \delta(y))/\delta(y))\left(\int_{B(a,\delta(y'))\setminus B(a,\delta(y))} \phi(y)(x)^p d\mu(x)\right)^{1/p}$$

$$\leq ((\delta(y') - \delta(y))/\delta(y))\left(\int_X |\phi(y)(x)|^p d\mu(x)\right)^{1/p} = ((\delta(y') - \delta(y))\|\phi(y)\|_p/\delta(y) < \lambda/8.$$
By conditions (a) and (d),

\[
\|\psi(y)_{B(a,2\delta(y))}\setminus B(a,\delta(y')) - \psi(y')_{B(a,2\delta(y))}\setminus B(a,\delta(y'))\|_p \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y')) |\psi(y)(x) - \psi(y')(x)|^p d\mu(x)\right)^{1/p} \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y')) \left|((d(a,x) - \delta(y))/\delta(y))\phi(y)(x) \right. \right. \\
\left. \left. - ((d(a,x) - \delta(y'))/\delta(y'))\phi(y')(x)\right|^p d\mu(x)\right)^{1/p} \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y')) \left|((d(a,x) - \delta(y))/\delta(y)) - (d(a,x) - \delta(y'))/\delta(y'))\phi(y)(x) \right. \right. \\
\left. \left. + (d(a,x) - \delta(y'))/\delta(y'))(\phi(y)(x) - \phi(y')(x))\right|^p d\mu(x)\right)^{1/p} \\
\leq (1/\delta(y) - 1/\delta(y')) \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y')) |\phi(y)(x)|^p d\mu(x)\right)^{1/p} \\
\leq 2(\delta'(y') - \delta(y))\left(\int_{B(a,2\delta(y'))}\setminus B(a,2\delta(y)) |\phi(y)(x)|^p d\mu(x)\right)^{1/p} \\
\leq 2(\delta'(y') - \delta(y))\left(\int_{B(a,2\delta(y'))}\setminus B(a,2\delta(y)) |\phi(y)(x)|^p d\mu(x)\right)^{1/p} \\
\leq 2(\delta'(y') - \delta(y))\left(\int_{X} |\phi(y)(x)|^p d\mu(x)\right)^{1/p} + \left(\int_{X} \phi(y)(x) - \phi(y')(x)|^p d\mu(x)\right)^{1/p} \\
= 2(\delta'(y') - \delta(y))\|\phi(y)\|_p + \|\phi(y) - \phi(y')\|_p < \lambda/4.
\]

Using conditions (a) and (c), we get

\[
\|\psi(y)_{B(a,2\delta(y'))}\setminus B(a,\delta(y)) - \psi(y')_{B(a,2\delta(y'))}\setminus B(a,\delta(y))\|_p \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y)) |\psi(y)(x) - \psi(y')(x)|^p d\mu(x)\right)^{1/p} \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y)) |\phi(y)(x) - ((d(a,x) - \delta(y))/\delta(y))\phi(y')(x)|^p d\mu(x)\right)^{1/p} \\
= \left(\int_{B(a,2\delta(y))}\setminus B(a,\delta(y)) \left|1 - (d(a,x) - \delta(y'))/\delta(y'))\phi(y)(x) \right. \right. \\
\left. \left. + ((d(a,x) - \delta(y'))/\delta(y'))(\phi(y)(x) - \phi(y')(x))\right|^p d\mu(x)\right)^{1/p} \\
\leq 2((\delta'(y') - \delta(y))/\delta(y))\left(\int_{B(a,2\delta(y'))}\setminus B(a,2\delta(y)) |\phi(y)(x)|^p d\mu(x)\right)^{1/p} \\
\leq 2((\delta'(y') - \delta(y))/\delta(y))\left(\int_{X} |\phi(y)(x)|^p d\mu(x)\right)^{1/p} + \left(\int_{X} \phi(y)(x) - \phi(y')(x)|^p d\mu(x)\right)^{1/p} \\
= 2((\delta'(y') - \delta(y))/\delta(y))\|\phi(y)\|_p + \|\phi(y) - \phi(y')\|_p < 3\lambda/8.
\]
It follows from condition (a) that
\[
\|\psi(y)x \setminus B(a,2\delta(y')) - \psi(y')x \setminus B(a,2\delta(y'))\|_p = \left( \int_{x \setminus B(a,2\delta(y'))} |\psi(y)(x) - \psi(y')(x)|^p \mu(x) \right)^{1/p}
\]
\[
= \left( \int_{x \setminus B(a,2\delta(y'))} |\phi(y)(x) - \phi(y')(x)|^p \mu(x) \right)^{1/p}
\]
\[
\leq \left( \int_x |\phi(y)(x) - \phi(y')(x)|^p \mu(x) \right)^{1/p}
\]
\[
= \|\phi(y) - \phi(y')\|_p < \lambda/8.
\]

Therefore we have
\[
\|\psi(y) - \psi(y')\|_p \leq \|\psi(y)B(a,\delta(y)) - \psi(y')B(a,\delta(y))\|_p
\]
\[
+ \|\psi(y)B(a,\delta(y)) \setminus B(a,\delta(y)) - \psi(y')B(a,\delta(y)) \setminus B(a,\delta(y))\|_p
\]
\[
+ \|\psi(y)B(a,2\delta(y)) \setminus B(a,\delta(y)) - \psi(y')B(a,2\delta(y)) \setminus B(a,\delta(y))\|_p
\]
\[
+ \|\psi(y)X \setminus B(a,2\delta(y')) - \psi(y')X \setminus B(a,2\delta(y'))\|_p
\]
\[
< 7\lambda/8 < \lambda.
\]
Consequently, \( \psi \) is continuous. Thus the proof is completed. \( \square \)

Remark 1. In the above lemma, for each \( y \in Y \), when \( \phi(y) \) is corresponding to a function almost everywhere, that is uniformly continuous and bounded on \( B(a, 2\delta(y)) \), we have \( \psi(y) \in C_\mu(X) \).

We show the following:

Lemma 4.3. Let \( a \in X_0 \) such that for each \( \lambda > 0 \), \( B(a, \lambda) \in \mathcal{M} \) and \( \mu(B(a, \lambda)) > 0 \), and for some \( \lambda' > 0 \), \( \mu(B(a, \lambda')) < \infty \). Suppose that \( A \subset L^p(X) \) and \( \xi : A \to (0, \infty) \) is a function such that for every \( f \in A \), \( f(x) = 0 \) for almost every \( x \in B(a, \xi(f)) \), and that \( B \) is a \( Z \)-set in \( L^p(X) \). If the union \( A \cup B \) is a closed set in \( L^p(X) \), then it is a \( Z \)-set.

Proof. Let \( \epsilon : L^p(X) \to (0,1) \) be a map. We shall construct a map \( \phi : L^p(X) \to L^p(X) \) so that \( \phi(L^p(X)) \cap (A \cup B) = \emptyset \) and \( \|\phi(f) - f\|_p < \epsilon(f) \) for every \( f \in L^p(X) \). Since \( B \) is a \( Z \)-set, there is a map \( \psi_1 : L^p(X) \to L^p(X) \setminus B \) such that \( \|\psi_1(f) - f\|_p < \epsilon(f)/3 \) for each \( f \in L^p(X) \). Using Lemma 4.2 we can obtain maps \( \psi_2 : L^p(X) \to L^p(X) \) and \( \delta : L^p(X) \to (0,1) \) such that for each \( f \in L^p(X) \),

(i) \( \|\psi_1(f) - \psi_2(f)\|_p \leq \min\{\epsilon(f), \text{dist}(\{\psi_1(f)\}, B)\}/3 \),
(ii) \( \psi_2(f)(B(a, \delta(f))) = \{0\} \).

Since \( \mu(B(a, \lambda')) < \infty \) for some \( \lambda' > 0 \),
\[
\lim_{k \to \infty} \mu(B(a, \lambda'/k)) = \mu\left( \bigcap_{k \in \mathbb{N}} B(a, \lambda'/k) \right) = \mu(\{a\}) = 0.
\]

So we can take \( \lambda > 0 \) so that \( \mu(B(a, \lambda)) \leq 1 \). Letting \( \psi_3 : L^p(X) \to L^p(X) \) be a map such that
\[
\psi_3(f)(x) = \begin{cases} 
\min\{\epsilon(f), \text{dist}(\{\psi_1(f)\}, B)\}/3 & \text{if } x \in B(a, \lambda), \\
0 & \text{if } x \in X \setminus B(a, \lambda),
\end{cases}
\]
we can defined the desired map \( \phi : L^p(X) \to L^p(X) \) by \( \phi(f) = \psi_2(f) + \psi_3(f) \). Since \( \psi_2 \) and \( \psi_3 \) are continuous, so is \( \phi \). It is easy to see that \( \phi(f) \notin A \) for any \( f \in L^p(X) \). Observe that by condition
(i),
\[\|\phi(f) - \psi_1(f)\|_p = \|\psi_2(f) + \psi_3(f) - \psi_1(f)\|_p \leq \|\psi_2(f) - \psi_1(f)\|_p + \|\psi_3(f)\|_p \leq 2 \min\{\epsilon(f), \text{dist}(\{\psi_1(f)\}, B)\}/3 < \text{dist}(\{\psi_1(f)\}, B),\]
which implies that \(\phi(f) \notin B\). Moreover,
\[\|\phi(f) - f\|_p \leq \|\phi(f) - \psi_1(f)\|_p + \|\psi_1(f) - f\|_p < 2 \min\{\epsilon(f), \text{dist}(\{\psi_1(f)\}, B)\}/3 + \epsilon(f)/3 \leq \epsilon(f).\]

The proof is completed. \(\square\)

Now we will prove that there exists a \(Z_\sigma\)-set in \(L^p(X)\) which contains \(C_a(X)\). Set \(C_a(X) = \{f \in L^p(X) \mid f|_{X\setminus E} \text{ is continuous for some } E \subset X \text{ with } \mu(E) = 0\}\). It is obvious that \(C_a(X) \subset C_u(X)\). We show the following proposition.

**Proposition 4.4.** Let \(X\) be separable. Suppose that for all points \(x \in X_0\), \(B(x, \lambda) \in M\) and \(\mu(B(x, \lambda)) > 0\) for each \(\lambda > 0\), and \(B(x, X(x)) < \infty\) for some \(\lambda(x) > 0\), and that \(X \setminus X_0\) is not dense in \(X\). Then \(C_a(X)\) is contained in some \(Z_\sigma\)-set in \(L^p(X)\), and hence so is \(C_u(X)\).

**Proof.** Notice that \(X \setminus \text{cl}_X (X \setminus X_0) \neq \emptyset\) and there is a countable open basis \(U\) of \(X \setminus \text{cl}_X (X \setminus X_0)\). We may assume that \(\emptyset \notin U\). For each \(n \in \mathbb{N}\) and each \(U \in U\), let
\[Z(n, U) = \{f \in L^p(X) \mid |f(x)| \geq 1/n \text{ for almost every } x \in U\}.
Then \(Z(n, U)\) is closed in \(L^p(X)\). Indeed, for every sequence \(\{f_k\} \subset Z(n, U)\) that converges to \(f \in L^p(X)\), by Proposition 3.1, replacing \(\{f_k\}\) with a subsequence, we have that \(f_k \to f\) almost everywhere. For almost every \(x \in U\), \(|f_k(x)| \geq 1/n\) and \(f_k(x) \to f(x)\), which implies that \(|f(x)| \geq 1/n\). Thus \(Z(n, U)\) is closed. Fix any \(a \in U\). According to Lemma 1.3, for each map \(\epsilon : L^p(X) \to (0, 1)\), we can choose maps \(\phi : L^p(X) \to L^p(X)\) and \(\delta : L^p(X) \to (0, 1)\) satisfying the following:

1. \(|\phi(f) - f\|_p < \epsilon(f)\), and
2. \(|\phi(f)(B(a, \delta(f)))| = \{0\}\) for any \(f \in L^p(X)\).
Recall that \(\mu(B(a, \delta(f))) > 0\). As is easily observed, \(\phi(L^p(X)) \cap Z(n, U) = \emptyset\). Hence \(Z(n, U)\) is a \(Z\)-set in \(L^p(X)\).

Let \(Z = C_a(X) \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{U \in U} Z(n, U)\). We shall show that \(\text{cl}_{L^p(X)} Z\) is a \(Z\)-set in \(L^p(X)\). Take any \(a \in X \setminus \text{cl}_X (X \setminus X_0)\) and \(\delta > 0\) such that \(B(a, \delta) \subset X \setminus \text{cl}_X (X \setminus X_0)\). For each \(f \in \text{cl}_{L^p(X)} Z\), we prove that \(f(x) = 0\) for almost every \(x \in B(a, \delta)\). There exists \(\{f_k\} \subset Z\) such that \(\|f_k - f\|_p \to 0\). By Proposition 3.1, replacing \(\{f_k\}\) with a subsequence, we can choose \(E_0 \subset X\) with \(\mu(E_0) = 0\) so that \(f_k(x) \to f(x)\) for any \(x \in B(a, \delta) \setminus E_0\). Since each \(f_k \in C_a(X)\), there is \(E_k \subset X\) such that \(\mu(E_k) = 0\) and \(f_k|_{X \setminus E_k}\) is continuous. Put \(E = \bigcup_{k \in \mathbb{N} \cup \{0\}} E_k\), so \(\mu(E) = 0\). Let any \(x \in B(a, \delta) \setminus E\). For all \(n \in \mathbb{N}\) and \(U \in U\), there is a point \(x(n, U) \in U \setminus E\) such that \(|f_k(x(n, U))| < 1/n\) because \(f_k \notin Z(n, U)\). Due to the continuity of \(f_k|_{X \setminus E}\), we have \(|f_k(x)| \leq 1/n\), which means that \(|f(x)| \leq 1/n\). Therefore \(f(x) = 0\) for almost every \(x \in B(a, \delta)\). Consequently, \(Z\) is a \(Z\)-set in \(L^p(X)\) by Lemma 1.3, so \(C_a(X)\) is contained in the \(Z_\sigma\)-set \(\text{cl}_{L^p(X)} Z \cup \bigcup_{n \in \mathbb{N}} \bigcup_{U \in U} Z(n, U)\). \(\square\)

### 5. The strong \(\mathfrak{M}_2\)-universality of \(C_u(X)\)

This section is devoted to proving that \(C_u(X)\) is strongly \(\mathfrak{M}_2\)-universal. Indeed, we will show the stronger result in Proposition 5.3. Given any pair of spaces \((M, Y)\), which means that \(Y \subset M\), and any pair of classes \((\mathfrak{A}, \mathfrak{C})\), we write \((M, Y) \in (\mathfrak{A}, \mathfrak{C})\) if \(M \in \mathfrak{A}\) and \(Y \in \mathfrak{C}\). A pair \((M, Y)\) is called to be strongly \((\mathfrak{A}, \mathfrak{C})\)-universal if the following condition holds.
Let $(A, D) \in (\mathfrak{A}, \mathfrak{C})$ and $B$ be a closed subset of $A$. Suppose that $f : A \to M$ is a map such that $f|_B$ is a $Z$-embedding and $(f|_B)^{-1}(Y) = B \cap D$. Then for each open cover $\mathcal{U}$ of $M$, there exists a $Z$-embedding $g : A \to M$ such that $g$ is $\mathcal{U}$-close to $f$, $g|_B = f|_B$ and $g^{-1}(Y) = D$.

Denote the class of compact metrizable spaces by $\mathfrak{M}_0$. Set

$$c_1 = \{(x(n))_{n \in \mathbb{N}} \in \mathfrak{S} \mid \lim_{n \to \infty} x(n) = 1\}.$$  

It is well known that the both pairs $(\mathfrak{S}, c_0)$ and $(\mathfrak{S}, c_1)$ are strongly $(\mathfrak{M}_0, \mathfrak{M}_2)$-universal, refer to [8]. A strong universality of a pair implies one of a space. By virtue of Theorems 1.7.9 and 1.3.2 of [3], we can establish the following:

**Proposition 5.1.** Let $(\mathfrak{A}, \mathfrak{C})$ be a pair of classes of metrizable spaces. Suppose that $M$ is a space homeomorphic to $\mathfrak{S}$ and $Y$ is a homotopy dense subspace in $M$. If $(M, Y)$ is strongly $(\mathfrak{A}, \mathfrak{C})$-universal, then $Y$ is strongly $\mathfrak{C}$-universal.

The next lemma will be used for proving Proposition 5.3.

**Lemma 5.2.** Let $Y$ be a space and $g : Y \to \mathfrak{S}$ be an injective map. Suppose that for every $E \in \mathcal{M}$ with $\mu(E) = 0$, $X \setminus E$ is dense in $X$, and that $x_m, x_\infty \in X$, $m \in \mathbb{N}$, are points such that $d(x_m, x_\infty) < 1$, $\{d(x_m, x_\infty)\}$ is a strictly decreasing sequence converging to 0 and $B(x_\infty, d(x_1, x_\infty)) \in \mathcal{M}$ with $\mu(B(x_\infty, d(x_1, x_\infty))) \leq 1$. Then for each map $\delta : Y \to (0, 1)$, there exists an injective map $\Phi : Y \to L^p(X)$ which satisfies the following conditions for every $y \in Y$.

1. $\|\Phi(y)\|_p \leq \delta(y)$.
2. $\Phi(y)(X \setminus B(x_\infty, d(x_{2k}, x_\infty))) = \{0\}$ if $2^{-k} \leq \delta(y) \leq 2^{-k+1}$, $k \in \mathbb{N}$.
3. $\Phi(y)(x_m) = \delta(y)$ for all $m \in \{2j + 1, \infty \mid j > k\}$ if $2^{-k} \leq \delta(y) \leq 2^{-k+1}$, $k \in \mathbb{N}$.
4. $\Phi(y)$ is continuous on $X \setminus \{x_\infty\}$.
5. $y \in g^{-1}(c_1)$ if and only if $\Phi(y) \in C(X)$.

**Proof.** For each $k \in \mathbb{N}$, setting

$$Y_k = \{y \in Y \mid 2^{-k} \leq \delta(y) \leq 2^{-k+1}\},$$

we have that $Y = \bigcup_{k \in \mathbb{N}} Y_k$. Define a map $f_k^i : Y_k \to [0, 1]$ as follows:

$$f_k^i(y) = \begin{cases} 0 & \text{if } i = 1, \\ \delta(y)(1 - \phi_k(y)) & \text{if } i = 2, \\ \delta(y)(1 - \phi_k(y))g(y)(1) & \text{if } i = 3, \\ \delta(y) & \text{if } i = 2j, j \geq 2, \\ \delta(y)((1 - \phi_k(y))g(y)((i - 1)/2) + \phi_k(y)g((i - 3)/2)) & \text{if } i = 2j + 1, j \geq 2, \end{cases}$$

where $\phi_k(y) = 2 - 2^k\delta(y)$. For each $m \in \mathbb{N}$, let

$$S_m = \{x \in X \mid r_m \leq d(x, x_\infty) \leq r_{m-1}\},$$

where $r_0 = 1$ and $r_m = d(x_m, x_\infty)$, and let $\psi_m : S_m \to [0, 1]$ be a map defined by

$$\psi_m(x) = (d(x, x_\infty) - r_m)/(r_{m-1} - r_m).$$

We define a map $\Phi_k : Y_k \to L^p(X)$, $k \in \mathbb{N}$, as follows:

$$\Phi_k(y)(x) = \begin{cases} \delta(y) & \text{if } x = x_\infty, \\ \psi_{2k+i}(x)f_k^i(y) + (1 - \psi_{2k+i}(x))f_k^{i+1}(y) & \text{if } x \in S_{2k+i}, \\ 0 & \text{if } d(x, x_\infty) \geq r_{2k}. \end{cases}$$
Verify that $\Phi_k(y) = \Phi_{k+1}(y)$ for all $y \in Y_k \cap Y_{k+1}$. Indeed, by the definition, $\Phi_k(y)(x_\infty) = \delta(y) = \Phi_{k+1}(y)(x_\infty)$, and $\Phi_k(y)(x) = 0 = \Phi_{k+1}(y)(x)$ for every $x \in X$ with $d(x, x_\infty) \geq r_{2k}$. We get $\phi_k(y) = 1$ and $\phi_{k+1}(y) = 0$ because $\delta(y) = 2^{-k}$, and hence $f_1^k(y) = f_2^k(y) = f_3^k(y) = 0$. Therefore for each $x \in S_{2k+1}$,

$$\Phi_k(y)(x) = \psi_{2k+1}(x)f_1^k(y) + (1 - \psi_{2k+1}(x))f_2^k(y) = 0 = \Phi_{k+1}(y)(x),$$

and for each $x \in S_{2k+2}$,

$$\Phi_k(y)(x) = \psi_{2k+2}(x)f_2^k(y) + (1 - \psi_{2k+2}(x))f_3^k(y) = 0 = \Phi_{k+1}(y)(x).$$

Moreover, $f_3^k(y) = 0 = f_1^{k+1}(y)$, $f_2^{k+1}(y) = \delta(y)g(y)(j) = f_2^{k+1}(y)$ and $f_2^{k+2}(y) = \delta(y) = f_2^{k+1}(y)$ for any $j \geq 1$, that is, $f_i^k(y) = f_i^{k+1}(y)$ for any $i \geq 1$. It follows that for each $x \in S_{2k+i+2}$, $i \geq 1$,

$$\Phi_k(y)(x) = \psi_{2k+i+2}(x)f_i^{k+1}(y) + (1 - \psi_{2k+i+2}(x))f_{i+1}^{k+1}(y) = \Phi_{k+1}(y)(x).$$

As a consequence, $\Phi_k(y) = \Phi_{k+1}(y)$.

Now define the desired map $\Phi : Y \to L^p(X)$ by $\Phi(y) = \Phi_k(y)$ if $y \in Y_k$. Evidently, conditions (1), (2), (3) and (4) follows from the definition of $\Phi$. We will check condition (5). Firstly, let us show the only if part. Take any $y \in g^{-1}(c_1)$, where $y \in Y_k$ for some $k \in \mathbb{N}$, and let $\epsilon \in (0, \delta(y))$. Since $g(y) \in c_1$, there exists $i_0 \in \mathbb{N}$ such that if $i \geq i_0$, then $g(y)(i) > 1 - \epsilon/\delta(y)$. Let any $i \geq 2i_0+3$ and any point $x \in S_{2k+i}$. In the case that $i$ is even, $f_i^k(y) = \delta(y)$. In the case that $i$ is odd,

$$f_i^k(y) = \delta(y)((1 - \phi_k(y))g(y)((i - 1)/2) + \phi_k(y)g(y)((i - 3)/2))$$

$$> \delta(y)((1 - \phi_k(y))(1 - \epsilon/\delta(y)) + \phi_k(y)(1 - \epsilon/\delta(y))) = \delta(y) - \epsilon.$$

Therefore we get that

$$\psi_{2k+i}(x)f_i^k(y) + (1 - \psi_{2k+i}(x))f_{i+1}^k(y) > \psi_{2k+i}(x)(\delta(y) - \epsilon) + (1 - \psi_{2k+i}(x))(\delta(y) - \epsilon)$$

$$= \delta(y) - \epsilon.$$

It follows that

$$|\Phi(y)(x_\infty) - \Phi(y)(x)| = |\delta(y) - (\psi_{2k+i}(x)f_i^k(y) + (1 - \psi_{2k+i}(x))f_{i+1}^k(y))|$$

$$< \delta(y) - (\delta(y) - \epsilon) = \epsilon,$$

which means that the function $\Phi(y)$ is continuous at $x_\infty$. Moreover, $\Phi(y)$ is continuous on $X \setminus \{x_\infty\}$ due to (4), so $\Phi(y) \in C(X)$.

Next, to prove the if part, fix any $y \in Y$ such that $\Phi(y) \in C(X)$. Then $y \in Y_k$ and $\phi_k(y) > 0$ for some $k \in \mathbb{N}$. For each $\epsilon \in (0, 1)$, let $\epsilon' = \epsilon\phi_k(y)\delta(y)$. Since $\Phi(y)$ is coincident with a function continuous at the point $x_\infty$, we can find a subset $E \subset X$ with $\mu(E) = 0$ and an even number $i_0 \geq 4$ such that for every $z, z' \in B(x_\infty, r_{2k+i_0-2}) \setminus E$, $|\Phi(y)(z) - \Phi(y)(z')| < \epsilon'$. By the combination of condition (4) with the density of $X \setminus E \subset X$, for every $i \geq i_0$, there is a point $z_i \in B(x_\infty, r_{2k+i_0-2}) \setminus E$, which is sufficiently close to $x_{2k+i-1}$, such that $|\Phi(y)(x_{2k+i-2}) - \Phi(y)(z_i)| < \epsilon'$. Therefore, $\Phi(y) \in C(X)$.
\( \epsilon' / 3 \). Hence for any odd number \( i \geq i_0 \),
\[
|f^k_{i}(y) - \phi(y)| = |f^k_{i}(y) - f^k_{i_0}(y)|
\]
\[
= |(\psi_{2k+i}(x_{2k+i-1})f^k_{i}(y) + (1 - \psi_{2k+i}(x_{2k+i-1}))f^k_{i+1}(y))
- (\psi_{2k+i}(x_{2k+i-1})f^k_{i_0}(y) + (1 - \psi_{2k+i}(x_{2k+i-1}))f^k_{i_0+1}(y))|
\]
\[
= |\Phi(y)(x_{2k+i-1}) - \Phi(y)(x_{2k+i-1})| + |\Phi(y)(z_i) - \Phi(y)(z_{i_0})|
\]
\[
< \epsilon'.
\]
It follows that for each \( j \geq (i_0 - 2)/2 \),
\[
g(y)(j) = (f^k_{j-3}(y)/2^j + (1 - \phi_k(y))g(y)(j + 1))/\phi_k(y)
\]
\[
> (f^k_{j-3}(y)/2^j + (1 - \phi_k(y)))/\phi_k(y)
\]
\[
> ((\delta(y) - \epsilon')/\delta(y) - (1 - \phi_k(y)))/\phi_k(y)
\]
\[
= ((\delta(y) - \epsilon \phi_k(y))\delta(y))/\delta(y) - (1 - \phi_k(y)))/\phi_k(y) = 1 - \epsilon,
\]
which implies that \( g(y) \in \mathcal{C}_1 \).

Finally, we shall verify that \( \Phi \) is injective. Let any \( y_1, y_2 \in Y \) with \( \Phi(y_1) = \Phi(y_2) \). Remark that there is \( E \subset X \) with \( \mu(E) = 0 \) such that for each point \( x \in X \setminus E \), \( \Phi(y_1)(x) = \Phi(y_2)(x) \).

By condition (4) and the density of \( X \setminus \{x_{i_0}\} \) in \( X \setminus \{x_{i_0}\} \), we can see that \( \Phi(y_1)|_{X \setminus \{x_{i_0}\}} = \Phi(y_2)|_{X \setminus \{x_{i_0}\}} \). For some \( k_i \in \mathbb{N} \), \( i = 1, 2 \), the point \( y_i \in Y_{k_i} \). Letting \( k = \max\{k_i \mid i = 1, 2\} \), we have
\[
\Phi(y_i)(x_{2k+i}) = \Phi_k(y_i)(x_{2k+i}) = \psi_{2k+i}(x_{2k+i})f^k_{i}(y_i) + (1 - \psi_{2k+i}(x_{2k+i}))f^k_{i+1}(y_i)
\]
\[
= f^k_{i}(y_i) = \delta(y_i),
\]
so \( \delta(y_1) = \delta(y_2) \). Thus the both points \( y_1 \) and \( y_2 \) are contained in \( Y_k \) and
\[
\phi_k(y_1) = 2 - 2^k \delta(y_1) = 2 - 2^k \delta(y_2) = \phi_k(y_2).
\]
Furthermore, for every \( i \in \mathbb{N} \), we get
\[
f^k_{i+1}(y_i) = \Phi_k(y_i)(x_{2k+i}) = \Phi(y_i)(x_{2k+i}) = \Phi_k(y_2)(x_{2k+i}) = f^k_{i+1}(y_2),
\]
which means that \( f^k_j(y_1) = f^k_j(y_2) \) for each \( j \geq 2 \). When \( \phi_k(y_1) = 1 \), for all \( j \in \mathbb{N} \),
\[
g(y_1)(j) = f^k_{j+3}(y_1)/\delta(y_1) = f^k_{j+3}(y_2)/\delta(y_2) = g(y_2)(j).
\]
When \( \phi_k(y_1) \neq 1 \), we see that
\[
g(y_1)(1) = f^k_3(y_1)/(1 - \phi_k(y_1))\delta(y_1) = f^k_3(y_2)/(1 - \phi_k(y_2))\delta(y_2) = g(y_2)(1).
\]
Supposing that \( g(y_1)(j) = g(y_2)(j) \) for some \( j \in \mathbb{N} \), we can obtain
\[
g(y_1)(j + 1) = (f^k_{j+3}(y_1)/\delta(y_1) - \phi_k(y_1)g(y_1)(j))/(1 - \phi_k(y_1))
\]
\[
= (f^k_{j+3}(y_2)/\delta(y_2) - \phi_k(y_2)g(y_2)(j))/(1 - \phi_k(y_2)) = g(y_2)(j + 1).
\]
By induction, it follows that \( g(y_1)(j) = g(y_2)(j) \) for any \( j \in \mathbb{N} \), that is, \( g(y_1) = g(y_2) \). By virtue of the injectivity of \( g \), we have \( y_1 = y_2 \), so \( \Phi \) is an injection. The proof is completed. \( \blacktriangleleft \)

Remark 2. In the above lemma, if there is a compact set \( K \subset X \) that contains \( B(x_{\infty}, d(x_1, x_{\infty})) \), the function \( \Phi(y) \) has a compact support for each \( y \in Y \). Hence when \( \Phi(y) \) is continuous, it is uniformly continuous, so condition (5) can be rewritten as follows:
(5) $y \in g^{-1}(c_1)$ if and only if $\Phi(y) \in C_u(X)$.

Now we show the following:

**Proposition 5.3.** Suppose that for every $E \in \mathcal{M}$ with $\mu(E) = 0$, the complement $X \setminus E$ is dense in $X$, and that there are distinct points $x_0, x_\infty \in X_0$ such that for any $\lambda > 0$, $B(x_\infty, \lambda) \in \mathcal{M}$, and $x_\infty$ has a compact neighborhood $K \subset X$ with $\mu(K) < \infty$. If $C_u(X)$ is homotopy dense in $L^p(X)$, then the pair $(L^p(X), C_u(X))$ is strongly $\mathcal{M}_0\mathcal{M}_2$-universal, and hence $C_u(X)$ is strongly $\mathcal{M}_2$-universal.

**Proof.** The latter half follows from the strong $(\mathcal{M}_0, \mathcal{M}_2)$-universality of $(L^p(X), C_u(X))$ and Proposition 5.1. We shall show the first half. Suppose that $(A, D) \in (\mathcal{M}_0, \mathcal{M}_2)$, $B$ is a closed set in $A$, and $\Phi : A \to L^p(X)$ is a map such that $\Phi|_B$ is a $Z$-embedding and $(\Phi|_B)^{-1}(C_u(X)) = B \cap D$. For each $\varepsilon > 0$, let us construct a $Z$-embedding $\Psi : A \to L^p(X)$ such that $\|\Psi(a) - \Phi(a)\|_p < \varepsilon$ for every $a \in A$, $\Psi|_B = \Phi|_B$ and $\Psi^{-1}(C_u(X)) = D$. We can assume that $\Phi(B) \cap \Phi(A \setminus B) = \emptyset$ because $\Phi(B)$ is a $Z$-set in $L^p(X)$. Let $\delta : A \to [0,1)$ be a map defined by

$$\delta(a) = \min\{\varepsilon, \text{dist}(\{\Phi(a)\}, \Phi(B))\} / 4.$$ 

As is easily observed, $\delta(a) = 0$ if and only if $a \in B$. Since $C_u(X)$ is homotopy dense in $L^p(X)$, there is a homotopy $h : L^p(X) \times [0,1] \to L^p(X)$ such that $h(f,0) = f$, $h(f,t) \in C_u(X)$ for all $f \in L^p(X)$ and $t \in (0,1]$, and moreover, $\|h(f,t) - f\|_p \leq t$ for all $f \in L^p(X)$ and $t \in [0,1]$. Define a map $\phi : A \to L^p(X)$ by setting $\phi(a) = h(\Phi(a), \delta(a))$. Notice that

$$\|\phi(a) - \Phi(a)\|_p = \|\Phi(a), \delta(a)\|_p - \Phi(a)\|_p \leq \delta(a)$$

for every $a \in A$, and that $\phi(A \setminus B) \subset C_u(X)$.

Take $0 < \lambda \leq d(x_0, x_\infty) / 2$ such that $B(x_\infty, \lambda) \subset K$ and $\mu(B(x_\infty, \lambda)) \leq 1$. According to Lemma 4.2, we can find maps $\psi : A \setminus B \to L^p(X)$, $\xi : A \setminus B \to (0, \lambda)$ and $\eta : A \setminus B \to (0, \lambda)$ so that for any $a \in A \setminus B$, $\xi(a) \leq \delta(a)$ and

(i) $\|\phi(a) - \psi(a)\|_p \leq \delta(a)$,

(ii) $\psi(a)(B(x_\infty, \xi(a))) = \{0\}$,

(iii) $\psi(a)(B(x_0, \eta(a))) = \{0\}$.

Since $\phi(A \setminus B) \subset C_u(X)$, we may assume that $\psi(B \setminus A) \subset C_u(X)$, see Remark 1. Put

$$A_k = \{a \in A \mid 2^{-k} \leq \xi(a) \leq 2^{-k+1}\}$$

for each $k \in \mathbb{N}$, so every $A_k$ is compact and $A \setminus B = \bigcup_{k \in \mathbb{N}} A_k$. It follows from the assumption that $X \setminus \{x_\infty\}$ is dense in $X$. Choose a point $x_1 \in X \setminus \{x_\infty\}$ such that $d(x_1, x_\infty) < \min\{\xi(a) \mid a \in A_1\}$. Moreover, we can inductively find $x_m \in X \setminus \{x_\infty\}$ for any $m \geq 2$ so that

$$d(x_m, x_\infty) \leq \min\{1 / m, d(x_{m-1}, x_\infty), \xi(a) \mid a \in A_m\}.$$ 

For simplicity, let $r_m = d(x_m, x_\infty)$ for each $m \in \mathbb{N}$. Then $\{r_m\}$ is strictly decreasing to 0 and for any $a \in A_k$ and $k \in \mathbb{N}$, $\psi(a)(B(x_\infty, r_{2k})) = \{0\}$. By virtue of the strong $(\mathcal{M}_0, \mathcal{M}_2)$-universality of $(s,c_1)$, we can obtain an embedding $g : A \to s$ so that $g^{-1}(c_1) = D$. Applying Lemma 5.2 and Remark 2 take an injective map $\psi' : A \setminus B \to L^p(X)$ so that the following conditions hold for every $a \in A \setminus B$.

1. $\|\psi'(a)\|_p \leq \xi(a)$.
2. $\psi'(a)(X \setminus B(x_\infty, r_{2k})) = \{0\}$ if $a \in A_k$, $k \in \mathbb{N}$.
3. $\psi'(a)(x_m) = \xi(a)$ for all $m \in \{2j + 1, \infty \mid j > k\}$ if $a \in A_k$, $k \in \mathbb{N}$.

Recall that every compact set in $s$ is a $Z$-set, refer to Theorem 1.1.14 and Proposition 1.4.9 of [3]. Under our assumption in Main Theorem, the space $L^p(X)$ is homeomorphic to $s$ by Theorem 1.14 and hence the image of any map from $A \in \mathcal{M}_0$ is a $Z$-set in $L^p(X)$. 


(4) \( \psi'(a) \) is continuous on \( X \setminus \{ x_\infty \} \).

(5) \( a \in D \setminus B \) if and only if \( \psi'(a) \in C_u(X) \).

Let \( \psi' : A \setminus B \to L^p(X) \) be a map defined by \( \psi'(a) = \psi(a) + \psi'(a) \). Since \( \psi \) and \( \psi' \) are continuous, so is \( \psi'' \). Due to conditions (i) and (1), for every \( a \in A \setminus B \),

\[
\| \phi(a) - \psi''(a) \|_p = \| \phi(a) - (\psi(a) + \psi'(a)) \|_p \leq \| \phi(a) - \psi(a) \|_p + \| \psi'(a) \|_p \\
\leq \delta(a) + \xi(a) \leq 2\delta(a).
\]

By virtue of condition (5), we have that \( a \in D \setminus B \) if and only if \( \psi''(a) \in C_u(X) \). To verify that \( \psi'' \) is an injection, fix any \( a_1, a_2 \in A \setminus B \) with \( \psi''(a_1) = \psi''(a_2) \), where we get some \( k_1, k_2 \in \mathbb{N} \) such that \( a_1 \in A_{k_1} \) and \( a_2 \in A_{k_2} \) respectively. Let \( k = \max\{k_i \mid i = 1, 2\} \). According to (ii), almost everywhere on \( B(x_\infty, r_{2k}) \),

\[
\psi'(a_1)(x) = \psi''(a_1)(x) = \psi''(a_2)(x) = \psi'(a_2)(x).
\]

Since \( \psi'(a_i), i = 1, 2, \) is continuous on \( B(x_\infty, r_{2k}) \setminus \{ x_\infty \} \) by (4), and for any \( E \in \mathcal{M} \) with \( \mu(E) = 0, X \setminus E \) is dense in \( X \) by the assumption, we can see that \( \psi'(a_1)(x) = \psi'(a_2)(x) \) for every \( x \in B(x_\infty, r_{2k}) \setminus \{ x_\infty \} \), and therefore especially,

\[
\xi(a_1) = \psi'(a_1)(x_{2k+3}) = \psi'(a_2)(x_{2k+3}) = \xi(a_2).
\]

Thus \( a_1, a_2 \in A_k \). On the other hand, since \( \psi'(a_i)(X \setminus B(x_\infty, r_{2k})) = \{0\}, i = 1, 2, \) due to condition (2), we have that \( \psi'(a_1) \) and \( \psi'(a_2) \) are coincident almost everywhere on \( X \). It follows from the injectivity of \( \psi' \) that \( a_1 = a_2 \). Consequently, the map \( \psi'' \) is an injection.

The map \( \psi'' \) can be extended to the desired map \( \Psi : A \to L^p(X) \) by \( \Psi|_B = \Phi|_B \) because for each \( a \in A \setminus B \),

\[
\| \Phi(a) - \psi''(a) \|_p \leq \| \Phi(a) - \phi(a) \|_p + \| \phi(a) - \psi''(a) \|_p \leq 3\delta(a) \\
= 3 \min\{\epsilon, \text{dist}(\{\Phi(a)\}, \Phi(B))\}/4 < \text{dist}(\{\Phi(a)\}, \Phi(B)).
\]

Observe that \( \| \Phi(a) - \Psi(a) \|_p < \epsilon \) for any \( a \in A \) and that

\[
\Psi(A \setminus B) = \psi''(A \setminus B) \subset L^p(X) \setminus \Phi(B) = L^p(X) \setminus \Psi(B).
\]

We see \( \Psi^{-1}(C_u(X)) = D \) due to that \( a \in D \setminus B \) if and only if \( \psi''(a) \in C_u(X) \) and the assumption that \( (\Phi|_B)^{-1}(C_u(X)) = B \cap D \). It remains to prove that \( \Psi \) is a \( Z \)-embedding. Since \( \Psi|_B = \Phi|_B \) is a \( Z \)-embedding and \( \Psi|_{A \setminus B} = \psi'' \) is an injective map, \( \Psi \) is an embedding. For every \( a \in A \setminus B \), \( \psi'(a)(X \setminus B(x_\infty, \lambda)) = \{0\} \) by (2), and \( B(x_0, \eta(a)) \subset X \setminus B(x_\infty, \lambda) \) by the definition. It follows from (iii) that for any \( x \in B(x_0, \eta(a)) \),

\[
\Psi(a)(x) = \psi''(a)(x) = \psi(a)(x) + \psi'(a)(x) = 0.
\]

According to Lemma 1, the image \( \Psi(A) = \Psi(A \setminus B) \cup \Psi(B) \) is a \( Z \)-set in \( L^p(X) \). We conclude that \( \Psi \) is a \( Z \)-embedding. \( \square \)

### 6. Proof of Main Theorem

Now we shall prove Main Theorem. In the case that a measure space \( X \) is regular Borel, and that there exists a countable family \( \{ U_n \} \) of open sets in \( X \) such that \( X = \bigcup_{n \in \mathbb{N}} U_n, U_n \subset U_{n+1} \) and \( \mu(U_n) < \infty \) for all \( n \in \mathbb{N} \), each function of \( L^p(X) \) can be approximated by a bounded map that vanishes outside \( U_n \) for some \( n \in \mathbb{N} \). Let

\[
C_c(X) = \{ f \in L^p(X) \mid f \text{ is a continuous function with a compact support} \}.
\]

The space \( C_c(X) \) is a convex subset of \( L^p(X) \) and \( C_c(X) \subset C_u(X) \). We can show the following proposition.
Proposition 6.1. Let $X$ a separable locally compact regular Borel metric measure space. Suppose that for each compact subset $K \subset X$, $\mu(K) < \infty$. Then the space $C_c(X)$ is homotopy dense in $L^p(X)$, and hence so is $C_u(X)$.

Proof. Since $X$ is separable and locally compact, we can find a countable family $\{U_n\}$ of open subsets so that $X = \bigcup_{n \in \mathbb{N}} U_n$, $\text{cl}_X U_n \subset U_{n+1}$ and $\text{cl}_X U_n$ is compact for any $n \in \mathbb{N}$. By the assumption, each $\mu(U_n) \leq \mu(\text{cl}_X U_n) < \infty$. Therefore $C_c(X)$ is a dense convex subset in the normed linear space $L^p(X)$. It follows from the combination of Theorem 6.8.9 with Corollary 6.8.5 of [9] that $C_c(X)$ is homotopy dense in $L^p(X)$. □

As is easily observed, if for every $E \in \mathcal{M}$ with $\mu(E) = 0$, the complement $X \setminus E$ is dense in $X$, then for any non-empty open subset $U \in \mathcal{M}$, $\mu(U) > 0$. Moreover, when $\mathcal{M}$ contains the open sets in $X$, the converse is valid. Indeed, suppose that there is a subset $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $X \setminus E$ is not dense in $X$, we can obtain an non-empty open subset $U \subset X$, that is contained in $E$. Then

$$0 < \mu(U) \leq \mu(E) = 0,$$

which is a contradiction.

Proof of Main Theorem. Remark that for each $E \subset X$ with $\mu(E) = 0$, $X \setminus E$ is dense in $X$. By Proposition 3.3, we have $C_u(X) \in \mathcal{M}_2 \subset (\mathcal{M}_2)_\sigma$. Due to Propositions 6.1 $C_u(X)$ is homotopy dense in $L^p(X)$. By Proposition 4.4 there exists a $Z_\sigma$-set in $L^p(X)$ that contains $C_u(X)$. Since $X \setminus X_0$ is not dense in $X$, $X_0$ is uncountable, so we can choose distinct points $x_0, x_\infty \in X_0$. Because $X$ is locally compact and any compact subset is of finite measure, the point $x_\infty$ has a compact neighborhood with a finite measure. According to Proposition 5.3 $C_u(X)$ is strongly $\mathcal{M}_2$-universal. Hence the space $C_u(X)$ is an $\mathcal{M}_2$-absorbing set in $L^p(X)$. Combining this with Theorems 11.1 and 2.1 we conclude that $C_u(X)$ is homeomorphic to $c_0$. □

7. Appendix

In the theory of infinite-dimensional topology, it is important to consider pairs of spaces. A pair $(M, Y) \in (\mathfrak{A}, \mathfrak{C})_\sigma$ if $M$ can be expressed as a countable union of closed subsets $M_n$, $n \in \mathbb{N}$, and $(M_n, M_n \cap Y) \in (\mathfrak{A}, \mathfrak{C})$. We say that $(M, Y)$ is an $(\mathfrak{A}, \mathfrak{C})$-absorbing pair if the following conditions are satisfied.

1. $(M, Y)$ is strongly $(\mathfrak{A}, \mathfrak{C})$-universal.
2. $Y$ is contained in some $Z_\sigma$-set $Z$ of $M$ such that $(Z, Y) \in (\mathfrak{A}, \mathfrak{C})_\sigma$.

The pairs $(s, c_0)$ and $(Q, c_0)$ are $(\mathcal{M}_0, \mathcal{M}_2)$-absorbing. As a consequence of Theorem 1.7.6 of [3], we have the following:

Theorem 7.1. Let $\mathfrak{A}$ and $\mathfrak{C}$ be classes of metrizable spaces. Suppose that both $(M, Y)$ and $(M', Y')$ are $(\mathfrak{A}, \mathfrak{C})$-absorbing pairs, and both $M$ and $M'$ are topological copies of $s$ or $Q$. Then $(M, Y)$ is homeomorphic to $(M', Y')$.

The following question naturally arises.

Problem 1. Is the pair $(L^p(X), C_u(X))$ homeomorphic to $(s, c_0)$?

In the paper [11], some continuous function space $C$ endowed with the hypograph topology admits a compactification $\overline{C}$ consisting of upper semi-continuous set valued functions such that the pair $(\overline{C}, C)$ is homeomorphic to $(Q, c_0)$. Let us ask the following:

Problem 2. Does the space $C_u(X)$ have a “natural” compactification $\overline{C_u(X)}$ such that $(\overline{C_u(X)}, C_u(X))$ is homeomorphic to $(Q, c_0)$?
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(Katsuhisa Koshino) Faculty of Engineering, Kanagawa University, Yokohama, 221-8686, Japan

E-mail address: ft160229no@kanagawa-u.ac.jp