On the kinematics of large-scale peculiar motions

Eleni Tsaprazi\textsuperscript{1,2} and Christos G. Tsagas\textsuperscript{1,3}

\textsuperscript{1}Section of Astrophysics, Astronomy and Mechanics, Department of Physics
Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
\textsuperscript{2}Sorbonne Universités, UPMC Univ Paris 06, CNRS
Institut d’Astrophysique de Paris (IAP), 98bis Boulevard Arago, Paris 75014, France
\textsuperscript{3}DAMTP, Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK

Abstract

We consider the linear kinematics of large-scale peculiar motions in a perturbed Friedmann universe. In so doing, we introduce a 4-velocity “tilt”, relative to the smooth Hubble expansion, and take the viewpoint of the “real” observers that move along with the peculiar flow. Using relativistic cosmological perturbation theory we study the linear evolution of the peculiar velocity field, as well as the expansion/contraction-rate, the shear and the rotation of the bulk motion. Our solutions show that, depending on the initial conditions, all the aforementioned kinematic quantities could readily grow to cosmologically relevant values. These theoretical results seem to support reports from a number of peculiar-velocity surveys, claiming large-scale bulk flows which move considerably faster than previously anticipated.

1 Introduction

Large-scale peculiar motions, also referred to as “bulk flows”, are an established observational fact, confirmed by numerous surveys expending out to scales of several hundred Mpc (e.g. see \cite{ref1} for a representative though incomplete list). On theoretical grounds, peculiar velocities are predicted by all structure formation scenarios, as the direct and unavoidable outcome of the ever increasing inhomogeneity and anisotropy of the post-recombination universe. Currently, the magnitude and the scale-dependence of the measured peculiar velocity field are still under debate, with a number of studies reporting measurements in excess (maybe well in excess \cite{ref2}) of those theoretically anticipated. Most of the available studies, however, are either Newtonian or “quasi-Newtonian” and all are conducted in the rest-frame of the smooth Hubble expansion. The latter is defined as the coordinate system where the dipole of the Cosmic Microwave Background (CMB) spectrum vanishes. Nevertheless, even at the linear level, there are subtle differences between the Newtonian and the relativistic studies, given the very different way the two theories approach issues as fundamental as the time-vs-space relation and the nature of gravity itself. In addition, no real observer in the universe follows the CMB frame, but we all move relative to it. Our Local Group of galaxies, for example, “drifts” with respect to the smooth Hubble flow at approximately 600 km/sec. For these reasons, the present work looks into the question of large-scale peculiar velocities and of their evolution by employing relativistic cosmological
perturbation theory and by adopting the view point of a real observer, living in a typical galaxy (like our Milky Way) and moving relative to the CMB-frame. The linear nature of our analysis implies that our scales of interest are large enough to neglect all nonlinear effects, which in practice means scales in excess of 100 Mpc.

We begin with a brief introduction to the study of peculiar velocities within the framework of relativistic cosmology. In so doing we assume a perturbed Friedmann-Robertson-Walker (FRW) universe, filled with a pressureless fluid that can be baryonic, or low-energy cold dark matter (CDM), or both. We also allow for two families of observers. The “fictitious” ideal observers following the CMB-frame and their real counterparts moving relative to it. In addition to the peculiar velocity field, we also investigate the full spectrum of the peculiar kinematics, namely the volume expansion/contraction, the shear and the rotation of the bulk motion. Not surprisingly, we find that linear peculiar flows are triggered by the inhomogeneous environment of the post-recombination universe. Our analysis also provides the full set of differential equations monitoring the peculiar velocity field, its expansion/contraction-rate its shear distortion and its rotation after decoupling. The homogeneous parts of these formulae can be solved analytically and the solutions show increase, both in absolute terms and relative to the background Hubble expansion. More specifically, the velocity ($\tilde{v}$) of the bulk motion is found to grow as $\tilde{v} \propto a^2$, where $a = a(t)$ is the cosmological scale-factor. This implies that the strength of the peculiar motion relative to the universal expansion, described by the ratio $\tilde{v}/v_H$ – with $v_H$ representing the Hubble velocity on the corresponding length, grows as $\tilde{v}/v_H \propto a^{5/2}$. At the same time, the expansion/contraction-rate ($\tilde{\vartheta}$) of the bulk flow shows a scale-dependent growth, which reaches its maximum ($\tilde{\vartheta} \propto a$) on super-Hubble scales and tends to a constant (i.e. $\tilde{\vartheta} \rightarrow$ constant) deep inside the horizon. Finally, the peculiar shear and the peculiar verticity ($\tilde{\varsigma}$ and $\tilde{\varpi}$ respectively) are also found to grow as $\tilde{\varsigma}, \tilde{\varpi} \propto a$ independent of scale. To the best of our knowledge, the aforementioned growth-rates of the peculiar velocity field are considerably stronger than those previously reported in the theoretical literature. In this respect, our results seem to support recent surveys claiming large-scale bulk flows much larger and much faster than anticipated.

2 The peculiar velocity field

In relativity neither time nor space retain their absolute Newtonian notion. As a result, observers moving with respect to each other have their own measure of time and space and they generally experience different versions of what one might call “reality”.

2.1 The 4-velocity “tilt”

Consider two families of observers with 4-velocities $u_a$ and $\tilde{u}_a$ respectively. Let us also assume that the latter family of observers has peculiar velocity $\tilde{v}_a$ relative to the first. The three aforementioned velocity fields are then related by the familiar Lorentz boost

$$\tilde{u}_a = \tilde{\gamma}(u_a + \tilde{v}_a),$$

where $u_a u^a = -1 = \tilde{u}_a \tilde{u}^a$, $u_a \tilde{v}^a = 0$, $\tilde{\gamma} = 1/\sqrt{1 - \tilde{v}^2}$ and $\tilde{v}^2 = \tilde{v}_a \tilde{v}^a < 1$ (see Fig. 1). The “tilt” between the two 4-velocity vectors is determined by the (hyperbolic) angle $\beta$, defined by $\cosh \beta = -\tilde{u}_a u^a = \tilde{\gamma}$. The latter ensures that $\beta = \ln(\tilde{\gamma} + \sqrt{\tilde{\gamma}^2 - 1})$, with $\beta > 0$ since $\tilde{\gamma} > 1$.[1]
2.2 The 1+3 threading

Introducing two 4-velocities means that there are two temporal directions (along $u_a$ and $\tilde{u}_a$) and two 3-dimensional spatial sections (orthogonal to $u_a$ and $\tilde{u}_a$ respectively). Projecting onto these 3-spaces is achieved by using the symmetric projection tensors

$$h_{ab} = g_{ab} + u_a u_b \quad \text{and} \quad \tilde{h}_{ab} = g_{ab} + \tilde{u}_a \tilde{u}_b,$$

(2)

with $h_{ab} u^a = 0 = \tilde{h}_{ab} \tilde{u}^b$ and $h_{a}^{a} = 3 = \tilde{h}_{a}^{a}$. Note that, when there is no vorticity, these two projectors also act as the metric tensors of their associated 3-dimensional hypersurfaces.

We may now proceed to define temporal and spatial differentiation relative to the two 4-velocity fields seen in Eq. (1). In particular, the time-derivatives are denoted by

$$\cdot = u^a \nabla_a \quad \text{and} \quad \tilde{\cdot} = \tilde{u}^a \nabla_a,$$

(3)

while the corresponding spatial gradients are

$$D_a = h_{a}^{b} \nabla_b \quad \text{and} \quad \tilde{D}_a = \tilde{h}_{a}^{b} \nabla_b.$$

(4)

Using the above, one can decompose any spacetime variable, operator and equation into their timelike and spacelike components (relative to the $u_a$ or the $\tilde{u}_a$ field), thus achieving an 1+3 threading of the host spacetime into time and space [4].

2.3 Kinematic decomposition

The irreducible kinematic variables of the two 4-velocity fields emerge after decomposing their gradients. More specifically, we have [4]

$$\nabla_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab} - A_a u_b.$$
In the above $\Theta = \mathcal{D}^a u_a$, $\sigma_{ab} = \mathcal{D}_{(b} u_{a)}$, $\omega_{ab} = \mathcal{D}_{[b} u_{a]}$ and $A_a = \ddot{u}_a$ are respectively the expansion/contraction scalar, the shear tensor, the vorticity tensor and the 4-acceleration vector of the $u_a$-field. Positive values for $\Theta$ mean expansion, while in the opposite case we have contraction. Nonzero shear implies changes in the shape (under constant volume) of the associated fluid element and non-vanishing vorticity ensures rotation. Finally, a nonzero 4-acceleration indicates the presence of non-gravitational forces. In an exactly analogous way we may write

$$\nabla_b \tilde{\tilde{\nu}}_a = \frac{1}{3} \tilde{\tilde{\vartheta}} \tilde{\tilde{\eta}}_{ab} + \tilde{\tilde{\varsigma}}_{ab} + \tilde{\tilde{\omega}}_{ab} - \frac{1}{3} \tilde{\tilde{\Theta}} \tilde{\tilde{\nu}}_b ,$$

(6)

for the gradient of the $\tilde{\tilde{\nu}}_a$-field. Note that the shear, the vorticity and the 4-acceleration are all spacelike, namely $\sigma_{ab} u^b = 0 = \omega_{ab} u^b = A_a u^a$ by construction, with exactly analogous constraints applying to their “tilded” counterparts (i.e. $\tilde{\tilde{\sigma}}_{ab} \tilde{\tilde{u}}^b = 0 = \tilde{\tilde{\omega}}_{ab} \tilde{\tilde{u}}^b = \tilde{\tilde{A}}_a \tilde{\tilde{u}}^a$).

No realistic fluid-flow is absolutely rigid, but instead it is expected to expand or contract, to change shape and to rotate, even by small amounts. It is therefore plausible to argue that large-scale bulk peculiar motions should behave in a similar manner. The expansion/contraction-rate, the shear distortion and the rotation of the $\tilde{\nu}_a$-field, technically speaking the irreducible variables of the peculiar kinematics, are obtained by splitting its spatial gradient. More specifically, taking the viewpoint of the tilded observer, we have

$$\tilde{\tilde{D}}_b \tilde{\tilde{\nu}}_a = \frac{1}{3} \tilde{\tilde{\vartheta}} \tilde{\tilde{\eta}}_{ab} + \tilde{\tilde{\varsigma}}_{ab} + \tilde{\tilde{\omega}}_{ab} ,$$

(7)

with $\tilde{\vartheta} = \tilde{\tilde{D}}^a \tilde{\tilde{v}}_a$, $\tilde{\varsigma}_{ab} = \tilde{\tilde{D}}_{(b} \tilde{\tilde{v}}_{a)}$ and $\tilde{\tilde{\omega}}_{ab} = \tilde{\tilde{D}}_{[b} \tilde{\tilde{v}}_{a]}$ representing the peculiar expansion/contraction, the peculiar shear and the peculiar vorticity respectively.\(^1\)

### 2.4 Linear relations between the two frames

So far we have not imposed any constraints on the peculiar velocity, which means that our definitions and our formulae hold for arbitrarily fast relative motions in a general spacetime. Hereafter, we will only consider non-relativistic peculiar velocities with $\tilde{\nu}^2 \ll 1$ and

$$\tilde{\nu}_a \simeq u_a + \tilde{\nu}_a ,$$

(8)

since $\tilde{\gamma} \simeq 1$. We will also assume that the host spacetime is a perturbed, almost-FRW universe. Finally, the $u_a$-field will be identified with the coordinate system of the Hubble flow, while the tilded observers will be located in a typical galaxy like our Milky Way.\(^2\) Then, the kinematic variables defined in § 2.3 are related by the linear expressions

$$\tilde{\tilde{\Theta}} = \Theta + \tilde{\vartheta} , \quad \tilde{\tilde{\sigma}}_{ab} = \sigma_{ab} + \tilde{\varsigma}_{ab} ,$$

(9)

$$\tilde{\tilde{\omega}}_{ab} = \omega_{ab} + \tilde{\tilde{\omega}}_{ab} \quad \text{and} \quad \tilde{\tilde{A}}_a = A_a + \tilde{\nu}_a' + \frac{1}{3} \Theta \tilde{\tilde{\nu}}_a ,$$

(10)

\(^1\)The scalar $\tilde{\vartheta}$ monitors the volume expansion/contraction of the bulk motion. When $\tilde{\vartheta} > 0$ the peculiar flow expands and in the opposite case it contracts. The peculiar shear tensor follows changes in the shape of the bulk flow, say from spherical to spheroidal, while the vorticity tensor contains information about its rotation. At the linear level we have $|\tilde{\vartheta}|/\Theta \ll 1$, $\tilde{\varsigma}/\Theta \ll 1$ and $\tilde{\tilde{\omega}}/\Theta \ll 1$ by default, with $2\tilde{\varsigma}^2 = \tilde{\varsigma}_{ab} \tilde{\varsigma}^{ab}$ and $2\tilde{\tilde{\omega}}^2 = \tilde{\tilde{\omega}}_{ab} \tilde{\tilde{\omega}}^{ab}$.

\(^2\)Although relativity postulates the absence of preferred coordinate systems, the universal expansion naturally selects the CMB frame as the reference system relative to which peculiar velocities should be defined and measured.
where $\tilde{v}'_a = \tilde{u}^b \nabla_b \tilde{v}_a$ is the time derivative of the peculiar velocity in the tilded frame (not to be confused with the “peculiar gravitational acceleration” of the Newtonian treatments). Similarly, the dynamical variables measured in the two frames are related by

$$
\tilde{\rho} = \rho, \quad \tilde{\dot{p}} = p, \quad \tilde{q}_a = q_a - (\rho + p)\tilde{v}_a \quad \text{and} \quad \tilde{\pi}_{ab} = \pi_{ab}, \quad (11)
$$

to first approximation [6]. Here $\rho$ is the energy density, $p$ is the isotropic pressure, $q_a$ is the energy flux and $\pi_{ab}$ is the viscosity of the matter as measured in the $u_a$-frame, while their tilded counterparts are associated with the $\tilde{u}_a$-field.

In what follows we will use the above linear relations to study the kinematic evolution of large-scale peculiar motions in a perturbed Friedmannian universe. Of particular importance for our purposes are relations (10b) and (11c). The former implies that, even when $A_a$ vanishes in the CMB frame, the tilded observers measure an (effective) nonzero 4-acceleration solely because of their peculiar motion. In an analogous way, Eq. (11c) ensures that there is an effective nonzero energy-flux vector in the tilded frame, as a result of its relative motion alone (i.e. $\tilde{q}_a \neq 0$, although $q_a = 0$ in the coordinate system of the smooth Hubble flow).

3 Linear sources of peculiar flows

Large-scale peculiar motions are treated as a result of the increasing inhomogeneity and anisotropy of our universe, triggered by the ongoing structure-formation process. The latter starts in earnest after recombination, once the baryons have decoupled from the background radiation field.

3.1 Peculiar velocities

Let us assume an almost-FRW cosmology filled with pressure-free matter. This can be baryons, or low-energy CDM, or a mixture of both. The absence of pressure means that we can set $A_a = 0$ in the Hubble frame. Then relation (10d), which also serves as the linear propagation equation of the peculiar velocity in the bulk-flow frame, recasts as

$$
\tilde{v}'_a = -H \tilde{v}_a + \tilde{A}_a, \quad (12)
$$

with $H = \Theta/3$ being the Hubble parameter measured in the CMB frame. The above relation makes the 4-acceleration the sole source of peculiar velocities (at the linear perturbative level) and also the key to the subsequent evolution of the $\tilde{v}_a$-field. In the quasi-Newtonian approach, the effects of the 4-acceleration were accounted for by writing $\tilde{A}_a$ as the spatial gradient of an effective gravitational potential [6]. This, however, pre-assumes that the perturbed spacetime is both irrotational and shear-free. Moreover, the necessary propagation formula of $\tilde{A}_a$ was obtained after introducing an ansatz for the time evolution of the aforementioned potential [6]. Here we will not make these assumptions. Instead, by turning to relativistic cosmological perturbation theory, we will obtain analytical expressions for both the 4-acceleration vector and its time-derivative. More specifically, written in the rest-frame of the peculiar motion, the linear evolution formula of the density inhomogeneities ($\tilde{\Delta}_a$) reads [7]

$$
\tilde{\Delta}'_a = -\tilde{Z}_a - 3aH \tilde{A}_a - \frac{a}{\rho} \tilde{D}_a \tilde{D}^b \tilde{q}_b, \quad (13)
$$
given that the bulk-flow observers see nonzero 4-acceleration and an effective energy flux (see §2.4 earlier).\footnote{By recasting Eq. (13) relative to the Hubble frame, one recovers the familiar standard expression $\Delta_a = -Z_a$, with $\Delta_a = (a/\rho)D_a\rho$ and $Z_a = aD_a\Theta$ [4], which reflects the absence of peculiar motions in the CMB frame.}

Note that $\Delta_a = (a/\rho)D_a\rho$ and $Z_a = aD_a\Theta$ to first approximation, where $a = a(t)$ is the cosmological scale-factor – defined by $\dot{a}/a = H$. Solving the above for $\dot{A}_a$, substituting the resulting expression into the right-hand side of (12) and keeping in mind that $\ddot{q}_a = -\rho\ddot{v}_a$ (recall that $q_a = 0$ – see Eq. (11c) earlier), we arrive at

$$\ddot{v}_a' = -H\dot{v}_a + \frac{1}{3H}\dot{D}_a\dot{\vartheta} - \frac{1}{3aH} \left( \dot{\Delta}_a' + \dot{Z}_a \right),$$

(14)
since $\dot{\vartheta} = \tilde{D}^a\tilde{v}_a$. The latter vanishes in the absence of drift motions, leaving $\dot{\Delta}_a'$ and $\dot{Z}_a$ as the sole sources of peculiar velocities at the linear perturbative level. Therefore, the $\dot{v}_a$-field is induced by the increasing inhomogeneity of the post-recombination universe and more specifically by temporal variations in the distribution of the density-gradients and by spatial variations in the universal expansion (described by $\dot{\Delta}_a'$ and $\dot{Z}_a$ respectively).

### 3.2 Peculiar expansion/contraction, shear and rotation

All the information regarding the irreducible kinematics of the peculiar motion, namely the generation and evolution of the peculiar expansion/contraction, shear and rotation (see decomposition (7) in §2.3), is encoded in the $\tilde{D}_a\tilde{v}_a$-gradient and its time-derivatives. The linear sources of these kinematic quantities, in particular, follow from the first time-derivative of $\tilde{D}_a\tilde{v}_a$. This is obtained by taking the spatial gradient of Eq. (14) and then applying the linear commutation law $\tilde{D}_a\tilde{v}_a' = (\tilde{D}_b\tilde{v}_a)' + HD_b\tilde{\vartheta}_a$ between the temporal and the spatial derivatives of first-order vectors (e.g. see [6] for a list of such formulae). The resulting linear expression reads

$$\left( \tilde{D}_b\tilde{v}_a \right)' = -2H\tilde{D}_b\tilde{v}_a + \frac{1}{3H}\tilde{D}_b\tilde{D}_a\dot{\vartheta} - \frac{1}{3a^2H} \left( \dot{\Delta}_{ab}' + \dot{\Theta}_{ab} \right),$$

(15)
where $\dot{\Delta}_{ab} = a\tilde{D}_b\tilde{\Delta}_a$ and $\dot{Z}_{ab} = a\tilde{D}_b\tilde{Z}_a$ by construction.\footnote{The spacelike vector $\Delta_a$ contains collective information on the density inhomogeneity, which can be decoded by taking the spatial gradient $\Delta_{ab} = a\tilde{D}_a\Delta_b$ and then splitting it into its irreducible parts as $\Delta_{ab} = (\Delta/3)\tilde{h}_{ab} + \tilde{\Delta}_{(ab)} + \tilde{\Delta}_{(ab)}$, where $\Delta = \Delta_a^a$. This scalar describes overdensities or underdensities in the distribution of the matter. The symmetric and trace-free tensor $\Delta_{(ab)}$ monitors changes in the shape of the inhomogeneity (under constant volume), while its antisymmetric counterpart $\Delta_{(ab)}$ is associated with vortices in the density distribution. It goes without saying that an exactly analogous decomposition applies to the 3-gradient $Z_{ab}$ [4].}

The trace of the above leads to the linear evolution formulae of the peculiar expansion/contraction

$$\dot{\vartheta}' = -2H\dot{\vartheta} + \frac{1}{3H}\tilde{D}^2\dot{\vartheta} - \frac{1}{3a^2H} \left( \dot{\Delta}' + \dot{\Theta} \right),$$

(16)
with $\dot{\Delta} = \Delta^{ab}_a$ and $\dot{\Theta} = \Theta^{ab}_a$. Therefore, time-varying overdensities/underdensities in the matter distribution, together with scalar inhomogeneities in the universal expansion (described by $\dot{\Delta}'$ and $\dot{\Theta}$ respectively), can force a bulk flow to expand or contract locally.

Isolating the symmetric and trace-free component of (15), we obtain the linear propagation equation of the peculiar shear, namely

$$\zeta_{ab}' = -2H\zeta_{ab} + \frac{1}{3H}\tilde{D}_b\tilde{D}_a\dot{\vartheta} - \frac{1}{3a^2H} \left( \dot{\Delta}_{(ab)}' + \dot{\Theta}_{(ab)} \right).$$

(17)
The above implies that $\tilde{\Delta}'_{(ab)}$, $\tilde{Z}_{(ab)}$, and $\tilde{D}_{(b}\tilde{D}_a)\tilde{\vartheta}$ are the linear sources of shear-like perturbations in the $\tilde{v}_a$-field. The former represents (non-static) shape-distortions in the density inhomogeneity, while the last two monitor anisotropic gradients in the universal expansion and in the (local) peculiar expansion/contraction respectively.

Finally, the evolution of the peculiar vorticity follows from the skew part of Eq. (15). To first approximation, the latter reads

$$
\tilde{\omega}'_{ab} = -2H\tilde{\omega}_{ab} - \frac{1}{3a^2H} \left( \tilde{\Delta}'_{[ab]} + \tilde{Z}_{[ab]} \right),
$$

(18)
given that $\tilde{D}_{[b}\tilde{D}_a)\tilde{\vartheta} = \tilde{\vartheta}'\tilde{\omega}_{ab} - \text{see [6]}$ – is a second-order quantity. Accordingly, time-varying vortices in the matter density and rotational inhomogeneities in the background expansion (described by $\tilde{\Delta}'_{[ab]}$ and $\tilde{Z}_{[ab]}$ respectively) serve as linear sources of local peculiar rotation.

4 Linear evolution of peculiar flows

A key novelty in our approach is that it addresses the 4-acceleration issue analytically, without introducing any assumptions about its nature and evolution (see §3.1 previously). So, in what follows, we will provide the linear propagation formula of the 4-acceleration vector.

4.1 The key differential equations

Taking the time derivative of Eq. (13), recalling that $\dot{H} = -H^2[1 - \frac{\Omega}{2}]$ in the FRW background (with $\Omega = \kappa \rho / 3H^2$ being the associated density parameter) and using the linear commutation law $(\tilde{D}_a\tilde{\vartheta})' = \tilde{D}_a\tilde{\vartheta}' - H\tilde{D}_a\tilde{\vartheta}$ between the temporal and the spatial derivatives of first-order scalars [6], provides the propagation equation

$$
\tilde{A}'_a = \frac{1}{2} H\Omega (\tilde{v}_a' + H\tilde{v}_a) + \frac{1}{3H} \tilde{D}_a\tilde{\vartheta}' - \frac{1}{3aH} \left( \tilde{\Delta}'_a + \tilde{Z}'_a \right),
$$

(19)

which monitors the linear evolution of the 4-acceleration relative to the bulk-flow frame. We have therefore obtained analytical linear expressions for both the 4-acceleration and its time derivative (given by Eqs. (13) and (19) respectively), while avoiding the restrictions of the quasi-Newtonian analysis (see §3.1 previously).

The system of (12) and (19) determines the linear evolution of the peculiar velocity field. In particular, taking the time-derivative of Eq. (12), and then using (19), leads to

$$
\tilde{v}''_a = -H \left( 1 - \frac{1}{2} \Omega \right) \tilde{v}'_a + H^2(1 + \Omega) \tilde{v}_a + \frac{1}{3H} \tilde{D}_a\tilde{\vartheta}' - \frac{1}{3aH} \left( \tilde{\Delta}''_a + \tilde{Z}''_a \right).
$$

(20)

In addition, employing the linear commutation laws $	ilde{D}_b\tilde{v}'_a = (\tilde{D}_b\tilde{v}_a)' + H\tilde{D}_b\tilde{v}_a$ and $	ilde{D}_b\tilde{v}''_a = (\tilde{D}_b\tilde{v}_a)'' + 2H(\tilde{D}_b\tilde{v}_a)' - (H^2\Omega/2)\tilde{D}_b\tilde{v}_a$, the linearised spatial gradient of the above gives

$$
\left( \tilde{D}_b\tilde{v}_a \right)'' = -3H \left( 1 - \frac{1}{6} \Omega \right) \left( \tilde{D}_b\tilde{v}_a \right)' + 2H^2\Omega \tilde{D}_b\tilde{v}_a + \frac{1}{3H} \tilde{D}_b\tilde{D}_b\tilde{\vartheta}'
\quad - \frac{1}{3a^2H} \left( \tilde{\Delta}''_{ab} + \tilde{Z}''_{ab} \right).
$$

(21)
These two differential equations govern the linear kinematics of large-scale peculiar motions in an almost-FRW universe filled with pressureless (baryonic or/and CDM) matter, as seen by observers “living” inside the aforementioned bulk peculiar flows.

4.2 Peculiar velocity

To this point, we have considered a perturbed Friedmann universe without imposing any constraints on its spatial curvature. Hereafter, we will confine to an Einstein-de Sitter background by setting \( \Omega = 1 \) and \( H = 2/3t \). Then, isolating the homogeneous component of (20), that is ignoring the effects of the higher-order derivatives seen in the last two terms, we have

\[
9t^2 \ddot{v}_a'' + 3t \dddot{v}_a - 8 \dot{v}_a = 0.
\]

(22)

The latter accepts the power-law solution

\[
\ddot{v} = C_1 t^{4/3} + C_2 t^{-2/3} = C_3 a^2 + C_4 a^{-1},
\]

(23)
on all scales where the inhomogeneous part of Eq. (20) is subdominant. Hence, linear peculiar-velocity perturbations increase as \( \ddot{v} \propto t^{4/3} \propto a^2 \) (since \( a \propto t^{2/3} \) after equipartition). Practically speaking, this means that peculiar velocities can grow by up to six orders of magnitude between recombination and today (recall that \( a_0/a_{RC} \sim 10^3 \)). To the best of our knowledge, the aforementioned growth-rate is considerably larger than those reported in earlier theoretical studies. The quasi-Newtonian treatment, in particular, arrived at \( v \propto t^{1/3} \propto a^{1/2} \) [6].

Following solution (23) we deduce that \( \ddot{v}/v_H \propto t^{5/3} \propto a^{5/2} \) after decoupling, with \( v_H \propto a^{-1/2} \) representing the Hubble velocity at the time. Therefore, the ratio \( \ddot{v}/v_H \), which monitors the relative strength of the \( \ddot{v}_a \)-field, can increase by 15/2 orders of magnitude between recombination and the present time. Clearly, the residual magnitudes also depend on the initial conditions. Nevertheless, our study suggests that, although our universe may have entered the dust epoch with negligibly small peculiar velocities, the latter could have readily reached cosmologically relevant values by today. Taken at face value, this result seems to support peculiar-velocity surveys reporting bulk flows considerably larger and much faster than generally expected [1, 2].

4.3 Peculiar expansion/contraction

Let us go back to Eq. (21), which governs the linear evolution of the peculiar-velocity gradients. Assuming again a spatially flat FRW background, the trace of (21) gives

\[
\ddot{\vartheta} = -\frac{5H}{2} \dot{\vartheta}' + 2H^2 \ddot{\vartheta} + \frac{1}{3H} \dddot{\vartheta} + \frac{1}{3a^2 H} \left( \dddot{\Lambda} + \dddot{Z} \right).
\]

(24)

Keeping only the homogeneous part of the above and applying a simple Fourier decomposition to the perturbation, leads to the following differential equation for the \( n \)-th harmonic mode

\[
\ddot{\vartheta}_n'' = -\frac{5}{3} \left[ 1 + \frac{2}{15} \left( \frac{\lambda_H}{\lambda_n} \right)^2 \right] \ddot{\vartheta}_n' + \frac{8}{9t^2} \ddot{\vartheta}_n.'
\]

(25)

\footnote{The fact that the inhomogeneous component of (20) is comprised of the first and the second time-derivatives of spatial gradients suggests that the associated physics should be relevant on relatively small scales only.}

\footnote{We introduce the familiar harmonic splitting \( \ddot{\vartheta} = \sum_n \ddot{\vartheta}_n Q^{(n)} \), where \( D_n \ddot{\vartheta}_n = 0 \) and \( Q^{(n)} \) are standard scalar harmonic functions with \( \dot{Q}^{(n)} = 0 \) and \( \ddot{Q}^{(n)} = -(n/a)^2 Q^{(n)} \) (e.g. see [4]).}
Here, $\lambda_H = 1/H$ is the Hubble radius and $\lambda_n = a/n$ is the physical scale of the perturbation (i.e. of the bulk flow), with $n$ being the associated (comoving) wavenumber. On super-Hubble lengths, where $\lambda_H/\lambda_n \ll 1$, Eq. (25) reduces to

$$9t^2 \ddot{\vartheta} + 15t \dot{\vartheta}' - 8\vartheta = 0,$$

with a solution of the form

$$\ddot{\vartheta} = C_1 t^{2/3} + C_2 t^{-4/3} = C_3 a + C_4 a^{-2}.$$  \hspace{1cm} (27)

Therefore, throughout the dust epoch, $\ddot{\vartheta}$ grows proportionally to the dimensions of the expanding universe, as long as the perturbation remains outside the Hubble scale. At horizon crossing, namely when $\lambda_H/\lambda_n = 1$, the solution of Eq. (25),

$$\ddot{\vartheta} = C_1 t^{2(\sqrt{2/2} - 2)/9} + C_2 t^{-2(\sqrt{2/2} + 2)/9} = C_3 a^{(\sqrt{2/2} - 2)/3} + C_4 a^{-(\sqrt{2/2} + 2)/3},$$

shows a slight decrease in the growth-rate of $\ddot{\vartheta}$. The slowing-down effect enhances as we move down to progressively smaller lengths, with $\ddot{\vartheta}$ tending to a constant well inside the horizon.

The Hubble parameter of the background Friedmann universe drops like $H \propto 1/t$ on all scales. We therefore deduce that, after matter-radiation equality, the relative-strength ratio $\ddot{\vartheta}/H$ grows as $\ddot{\vartheta}/H \propto t^{5/3} \propto a^{5/2}$ on super-Hubble lengths. Well inside the horizon, on the other hand, we find that $\ddot{\vartheta}/H \propto t \propto a^{3/2}$. Given that $a_0/a_{RC} \sim 10^9$, with $a_0$ and $a_{RC}$ corresponding to the present and to the recombination time respectively, peculiar velocity perturbations that cross inside the horizon at decoupling, with (say) $\ddot{\vartheta}/H \simeq 10^{-7}$ at the time, could readily reach values of $\ddot{\vartheta}/H \gtrsim 10^{-2}$ on scales spanning few hundred Mpc today.

### 4.4 Peculiar shear and vorticity

Isolating the symmetric and trace-free component of (21), we obtain the evolution formula of the peculiar shear. The skew part of the same expression, on the other hand, leads to the propagation equation of the peculiar vorticity. More specifically, assuming a flat FRW background with $\Omega = 1$, we arrive at

$$\ddot{\zeta}''_{ab} = -\frac{5}{2} H \dot{\zeta}'_{ab} + 2H^2 \dot{\zeta}_{ab} + \frac{1}{3H} D_{(a} \dot{D}_{b)} \ddot{\vartheta}' = -\frac{1}{3a^2 H} \left( \dddot{\Delta}_{(ab)} + \ddot{Z}'_{(ab)} \right)$$  \hspace{1cm} (30)

and

$$\ddot{\zeta}''_{ab} = -\frac{5}{2} H \ddot{\zeta}_{ab} + 2H^2 \ddot{\zeta}_{ab} - \frac{1}{3a^2 H} \left( \dddot{\Delta}_{[ab]} + \ddot{Z}'_{[ab]} \right),$$  \hspace{1cm} (31)

One can demonstrate the scale-dependence in the evolution of the peculiar expansion/contraction scalar by setting $\lambda_H/\lambda_n = \alpha$ and then expressing the solution of Eq. (25) in terms of the $\alpha$-parameter as

$$\ddot{\vartheta} = C_1 t^{\beta_1} + C_2 t^{\beta_2} = C_3 a^{3\beta_1/2} + C_4 a^{3\beta_2/2},$$

with $\beta_{1,2} = -[a^2 + 3 \pm \sqrt{a^4 + 6a^2 + 8}] / 9$. Then, well outside the horizon, where $\alpha \to 0$, we find that $\beta_1 \simeq 2/3$ and $\beta_2 \simeq -4/3$, thus recovering solution (27). Scales deep inside the Hubble radius, on the other hand, have $1/\alpha \to 0$. There, a simple (linear) Taylor expansion of the exponents gives $\beta_{1,2} \simeq -[a^2 + 3 \pm \alpha^2(1 + 3/\alpha^2)] / 9$, namely $\beta_1 \simeq 0$ and $\beta_2 \simeq -2a^2/9$. In other words, on sufficiently small scales the value $\ddot{\vartheta}$ tends to a constant.
respectively. When pressureless (baryonic or/and CDM) matter dominates the energy density of the universe, we have $a \propto t^{2/3}$ and $H = 2/3t$. In such a case, the homogeneous component of Eq. (30) recasts as

$$9t^2 \tilde{\varsigma}_{ab}^{\nabla t} + 15t \tilde{\varsigma}_{ab}^{t} - 8\tilde{\varsigma}_{ab} = 0$$

(32)

and accepts the power-law solution

$$\tilde{\varsigma} = C_1 t^{2/3} + C_2 t^{-4/3} = C_3 a + C_4 a^{-2},$$

(33)

on scales where the inhomogeneous part of (30) is subdominant (see also footnote 5 in § 4.2). There, the peculiar shear grows proportionally to the dimensions of the post-recombination host universe. The same is also true for the peculiar vorticity, since (after equipartition) the homogeneous component of expression (31) is formally identical to Eq. (32). Therefore, on lengths where one can ignore the higher-order derivatives seen in the last term of (31), the peculiar vorticity also grows as $\tilde{\varpi} \propto t^{2/3} \propto a$.

These results imply that, after decoupling, the relative strength of the peculiar shear and that of the peculiar vorticity grow as $\tilde{\varsigma}/H, \tilde{\varpi}/H \propto t^{5/3} \propto a^{5/2}$. Given that $a_0/a_{RC} \sim 10^3$, the aforementioned ratios can increase by 15/2 orders of magnitude between recombination and the present time. Therefore, at least on wavelengths where the inhomogeneous parts of Eqs. (30) and (31) are negligible, large-scale peculiar motions could start with negligibly small amounts of kinematic shear and vorticity (say with $\tilde{\varsigma}/H, \tilde{\varpi}/H \sim 10^{-10}$) at decoupling and still reach cosmologically relevant local ratios (close to $\tilde{\varsigma}/H, \tilde{\varpi}/H \sim 10^{-2}$) today. Although the residual values and the associated scales depend on the initial conditions as well, it is probably unsafe to a priori treat the observed large-scale bulk flows as shear-free or irrotational.

5 Discussion

The size and the speed of the large-scale peculiar velocity fields reported in a number of recent bulk-flow surveys appear to exceed the standard expectations. Taken at face value, these reports might imply gaps in our theoretical understanding of the way bulk peculiar motions evolve. After all, the available studies are essentially Newtonian in nature, despite the fact that the scales involved are good fraction of the Hubble horizon and the subtlety of the relative-motion effects. With these in mind, we have attempted a relativistic analysis of peculiar velocities in a perturbed FRW universe, containing pressureless matter. The latter can be baryonic or CDM in nature, or a mixture of both. Also, given that no real observer follows the Hubble flow, we have adopted a “tilted” cosmological model and conducted our analysis in the rest-frame of a typical galaxy (like our Milky Way), which moves relative to the smooth universal expansion.

Even at the linear level, the kinematics and the dynamics of the universe appear different in the coordinate system of the bulk peculiar flow than in the frame of the Hubble expansion, simply because of relative motion effects. For instance, although the cosmic medium may look like a perfect fluid in the CMB frame, it will appear imperfect to the real observers solely due to their motion with respect to the mean universal expansion. Taking these fairly well known relativistic effects into account and employing linear (relativistic) cosmological perturbation theory, enabled us to obtain analytical expressions for the 4-acceleration vector and for its time
derivative. We have therefore gone a step further than the quasi-Newtonian approach, where an effective potential and an evolution ansatz were introduced to address the 4-acceleration issue.

As expected, the linear analysis confirmed that peculiar motions are the result of the ongoing structure formation process, and more specifically of the increasing inhomogeneity of the post-recombination universe. Our study also provided the first (to the best of our knowledge) relativistic insight to the evolution of the peculiar kinematics. Technically speaking, this was achieved through a set of four differential equations, monitoring the linear propagation of the peculiar velocity itself, as well as that of the associated irreducible kinematic quantities. The latter are the (local) expansion/contraction, the shear and the rotation of the bulk flow. Solving the aforementioned differential formulae analytically, we found substantial growth for all aspects of the peculiar motion. Moreover, the strength of the peculiar-velocity field, relative to the background Hubble expansion of the post-recombination universe, was found to increase in time as well. The residual magnitudes, of course, also depend on the initial conditions at decoupling. In order to obtain analytical solutions for the bulk-flow kinematics, we had to confine to the homogeneous part of the associated differential equations. Typically, this means that our results may not apply on scales where the inhomogeneous components dominate. We expect these scales to be relatively small, though additional work is still necessary to establish the associated lower threshold.\footnote{In order to include the inhomogeneous part of differential equations \ref{eq:20}, \ref{eq:24} and \ref{eq:30} into their solutions, one would most likely have to go beyond the boundaries of the analytical study and into the numerical regime.} Having said that, our results so far do suggest that, although peculiar motions may start quite negligible at the onset of the structure-formation epoch, they can achieve cosmologically relevant status by today and on scales large enough to encompass bulk flows of several hundred Mpc. On these grounds, one should not be surprised to measure fast peculiar velocity fields affecting domains considerably larger than it is generally expected.

**Acknowledgements:** ET acknowledges support from an internship fellowship by the Centre National de la Recherche Scientifique (CNRS) at the Institut d’Astrophysique de Paris (IAP), during the later stages of this work. CGT wishes to acknowledge support from a visiting fellowship by Clare Hall College and visitor support by DAMTP at the University of Cambridge, where part of this work was conducted.

**References**

[1] R. Watkins, H.A. Feldman and M.J. Hudson, Mon. Not. R. Astron. Soc. **392**, 743 (2009); G. Lavaux, R.B. Tully, R. Mohayaee, S. Colombi, Astrophys. J. **709**, 483 (2010); A. Nusser and M. Davis, Astrophys. J. **736**, 93 (2011); J. Colin, R. Mohayaee, S. Sarkar and A. Shafieloo, Mon. Not. R. Astron. Soc. **414**, 264 (2011); Macaulay E., et al, Mon. Not. R. Astron. Soc. **425**, 1709 (2012); Ma Y.-Z. and Pan J., Mon. Not. R. Astron. Soc. **437**, 1996 (2014); C. Magoulas et al, *IAU Symposium* **308**, 336 (2016); M. Rameez, R. Mohayaee, S. Sarkar and J. Colin, Mon. Not. R. Astron. Soc. **477**, 1772 (2018).

[2] A. Kashlinsky, F. Atrio-Barandela, D. Kocevski and H. Ebeling, Astrophys. J. **686**, L49 (2009); A. Kashlinsky, F. Atrio-Barandela and H. Ebeling, Astrophys. J. **732**, 1 (2011); A.
Abate, H.A. Feldman, Mon. Not. R. Astron. Soc. 419, 3482 (2012); J. Colin, R. Mohayaee, S. Sarkar and A. Shafieloo, Mon. Not. R. Astron. Soc. 471, 1045 (2017).

[3] A.R. King and G.F.R. Ellis, Commun. Math. Phys. 31, 209, (1973).

[4] C.G. Tsagas, A. Challinor and R. Maartens, Phys. Rep. 465, 61 (2008); G.F.R. Ellis, R. Maartens and M.A.H. MacCallum, Relativistic Cosmology (Cambridge University Press, Cambridge, 2012).

[5] G.F.R. Ellis and C.G. Tsagas, Phys. Rev. D 66, 124015 (2002); C.G. Tsagas and M.I. Kadiltzoglou, Phys. Rev. D 88, 083501 (2013).

[6] R. Maartens, Phys. Rev. D 58, 124006 (1998); G.F.R. Ellis, H. van Elst and R. Maartens, Class. Quantum Grav. 18, 5115 (2001).

[7] C.G. Tsagas and M.I. Kadiltzoglou, Phys. Rev. D 92, 043515 (2015).