The $q$-analogue of the alternating groups and its representations

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1 Introduction

Frobenius began the study of representation theory and character theory of the symmetric groups $\mathfrak{S}_n$ at the turn of the century [4]. In this study, he showed that there is one irreducible representation $\pi_\lambda$ of $\mathfrak{S}_n$ corresponding to each partition $\lambda$ of $n$ and this correspondence gives all of irreducible representations of $\mathfrak{S}_n$ up to equivalency. After this work, he also began the study of representation theory and character theory of the alternating groups $\mathfrak{A}_n$ at the opening of the 20th century [5]. In this study, he showed that there are branching rules when we restrict irreducible representation of the $\mathfrak{S}_n$ to $\mathfrak{A}_n$. We denote by $\tilde{\pi}_\lambda$ the restriction of $\pi_\lambda$ to $\mathfrak{A}_n$. If $\lambda$ is self-conjugate (that is symmetric figure as Young diagram along its diagonal), then $\tilde{\pi}_\lambda$ is decomposed into the direct sum of two irreducible representations whose degrees are coincide. But these two representations are mutually inequivalent. On the other hand, if $\lambda$ is non self-conjugate, then $\tilde{\pi}_\lambda$ is also the irreducible and $\tilde{\pi}_\lambda$ is equivalent to $\tilde{\pi}_{t\lambda}$ where $t\lambda$ is transpose of $\lambda$. These representations consist all of irreducible representations of $\mathfrak{A}_n$ up to equivalency (Theorem 3.4).

The concrete construction of irreducible representations of $\mathfrak{S}_n$ were given by Young. He gave irreducible representations of $\mathfrak{S}_n$ in two ways: ‘seminormal form’, and ‘orthogonal form’ (For detail, see [2]).

The Hecke algebra $\mathcal{H}_n(q)$ of type $A$ is considered as the $q$-analogue of the group ring $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group $\mathfrak{S}_n$, and its irreducible representations are constructed as well as in the case of $\mathfrak{S}_n$ under some assumptions about the value of the parameter $q$. The seminormal form for $\mathcal{H}_n(q)$ was given by Hoefsmit [9], and the orthogonal form was given by Wenzl [15]. It was not seem to be seen about the $q$-analogue of the group ring $\mathbb{C}[\mathfrak{A}_n]$ of the alternating group $\mathfrak{A}_n$ so far. In this context, we defined a subalgebra $\mathfrak{A}_n(q)$ of $\mathcal{H}_n(q)$ whose dimension as vector space over $\mathbb{C}$ is just a half of that of $\mathcal{H}_n(q)$. The group ring $\mathbb{C}[\mathfrak{A}_n]$ which is generated by following elements,

$$\{s_is_{i+1} \mid 1 \leq i \leq n-2\}$$

where $s_i$’s are simple permutations, has the dimension just a half of $\mathfrak{S}_n$ over $\mathbb{C}$. On the other hand, the
The subalgebra of $H_n(q)$ generated by following elements,

$$\{ g_ig_{i+1} \mid 1 \leq i \leq n-2 \}$$

where $g_i$’s are generators of $H_n(q)$, has the dimension more than a half of that of $H_n(q)$ because of quadratic relations,

$$g_i^2 = (q - 1)g_i + q \quad \text{for}\ i = 1, 2, \ldots, n-1$$

of $H_n(q)$. So instead of $g_i$, we take another generators $f_i$ of $H_n(q)$ as follows.

$$f_i = \frac{2g_i - (q - 1)}{q + 1} \quad \text{for}\ i = 1, 2, \ldots, n-1$$

Using these generators, we define the subalgebra $A_n(q)$ of $H_n(q)$ generated by following elements.

$$\{ f_if_{i+1} \mid 1 \leq i \leq n-2 \}$$

Just then, $A_n(q)$ has the dimension just a half of that of $H_n(q)$(Theorem 5.3). Moreover, when we take the limit $q \to 1$, $A_n(q)$ goes to $\mathbb{C}[A_n]$ and the generator $f_1f_{i+1}$ goes to $s_1s_{i+1}$. Therefore, we consider $A_n(q)$ as the $q$-analogue of the alternating group $A_n$. Furthermore, we show a presentation of $A_n(q)$(Theorem 5.5). In this presentation, quadratic relations are same as in the case of $A_n$, but cubic relations are slightly different. At the limit $q \to 1$, the defining relations of $A_n(q)$ coincide with those of $A_n$. Thus we can define the $q$-analogue of the alternating groups $A_n$ as follows.

**Definition 1.1.** Assume that $q \neq -1$ and $n > 2$. $A_n(q)$ is the algebra over $\mathbb{C}$ defined by the generators $y_1, y_2, \ldots, y_{n-2}$ and the following relations.

1. $y_i^3 = -\left(\frac{q - 1}{q + 1}\right)^2(y_i^2 - y_i) + 1$

2. $y_i^2 = 1 \quad \text{for}\ i > 1$

3. $(y_{i-1}y_i)^3 = -\left(\frac{q - 1}{q + 1}\right)^2\left\{ (y_{i-1}y_i)^2 - y_{i-1}y_i \right\} + 1 \quad \text{for}\ i = 2, 3, \ldots, n-2$

4. $(y_iy_j)^2 = 1 \quad \text{whenever}\ |i - j| > 1$

For $n = 2$, we define $A_2(q)$ the algebra over $\mathbb{C}$ generated by only the unit element.

We also analyze the representations of $A_n(q)$. Using the same notation as in the case of $\mathfrak{S}_n$(no confusion appears in this context), for the irreducible representation $\pi_\lambda$ of $H_n(q)$ corresponding to the partition $\lambda$ of $n$ (we identify $\lambda$ with a certain Young tableau), we denote by $\pi_\lambda$ the restriction of $\pi_\lambda$ to $A_n(q)$. Except of some ‘bad’ values of $q$, branching rules for $\mathfrak{S}_n \to A_n$ are valid to $H_n(q) \to A_n(q)$. More precisely, if $\lambda$ is non self-conjugate then $\pi_\lambda$ is irreducible, on the other hand, if $\lambda$ is self-conjugate then $\pi_\lambda$ is decomposed into two irreducible components $\pi_\lambda^+$ and $\pi_\lambda^-$. The following statement is the main result for representations of $A_n(q)$.
Theorem 6.5. Let $q$ be a complex number such that $q \neq 0$ and $q$ is not a $k$-th root of unity with $1 \leq k \leq n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be non self-conjugate Young diagrams and $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+q}$ be self-conjugate Young diagrams within $\Lambda_n$. Then representations $\tilde{\pi}_{\lambda_1}, \tilde{\pi}_{\lambda_2}, \ldots, \tilde{\pi}_{\lambda_p}, \tilde{\pi}_{\lambda_{p+1}}, \ldots, \tilde{\pi}_{\lambda_{p+q}}$, $\tilde{\pi}_{\lambda_{p+q}}$ are irreducible and not equivalent each other. These representations consist of all equivalent classes of irreducible representations of $\mathfrak{A}_n(q)$. Hence, $\mathfrak{A}_n(q)$ is semisimple.

We show this result by using relations of matrix elements of orthogonal representations of $\mathcal{H}_n(q)$. Thus, we get all of irreducible representations of $\mathfrak{A}_n(q)$ up to equivalence and show the semisimplicity of $\mathfrak{A}_n(q)$ except of some ‘bad’ values of $q$.

2 Orthogonal representations of the symmetric groups

In this section, we give a review of the concrete construction of an irreducible representation corresponding to each Young diagram, It is called ‘Orthogonal representation’, which was given by A.Young in 1932.

Let $\mathfrak{S}_n$ be the symmetric group, consisting of all permutations of $n$ things. Especially, $\mathfrak{S}_n$ permutes $n$ numbers $\{1, 2, \ldots, n\}$. $\mathfrak{S}_n$ is generated by elements $s_i (i = 1, 2, \ldots, n - 1)$ where $s_i = (i, i + 1)$ is a simple permutation, and have defining relations as follows.

1. $s_i^2 = 1$ for $i = 1, 2, \ldots, n - 1$
2. $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i = 1, 2, \ldots, n - 2$
3. $s_is_j = sjs_i$ whenever $|i - j| \geq 2$

We notice that $|\mathfrak{S}_n| = n!$. For $\sigma \in \mathfrak{S}_n$

$$\sigma = s_{i_1}s_{i_2}\ldots s_{i_r}$$

is an expression of $\sigma$. When $\sigma$ cannot be expressed by any form

$$\sigma = s_{j_1}s_{j_2}\ldots s_{j_t}$$

with $t < r$, we call $l(\sigma) = r$ the length of $\sigma$ and the expression

$$\sigma = s_{i_1}s_{i_2}\ldots s_{i_r}$$

is called the reduced expression of $\sigma$.

Consider a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{pmatrix}$$
and set the number of inversions of $\sigma$ as follows.

$$d_i = \sharp \{ j \mid j > k \text{ where } \sigma_k = i \text{ and } \sigma_j < i \}$$

Then, $\sigma$ has a reduced expression $\sigma = \ldots (s_{i-1} s_{i-2} \ldots s_{i-d_i})$

$$\ldots (s_{n-2} s_{n-3} \ldots s_{n-1-d_{n-1}})$$

$$(s_{n-1} s_{n-2} \ldots s_{n-d_n})$$

where the $i$-th contribution is only to be included if $d_i \geq 1$. The reduced expression described above is called the normal reduced expression. For the sake of convenience, we denote $s_{i-1} s_{i-2} \ldots s_{i-d_i}$ by $U_{i,d_i}$.

**Proposition 2.1.** The map

$$f : [0, 1] \times [0, 2] \times \ldots \times [0, n-1] \to S_n$$

defined by

$$f(d_2, d_3, \ldots, d_n) = U_{2,d_2} U_{3,d_3} \ldots U_{i,d_i} \ldots U_{n,d_n}$$

(where the $i$-th contribution is only to be included if $d_i \geq 1$, and $f(0, 0, \ldots, 0)$ is the unit element in $S_n$) is a bijection.

**Corollary 2.2.** Let $\mathbb{C}[S_n]$ be the group ring of $S_n$. Then $\mathbb{C}[S_n]$ has following basis as a vector space over $\mathbb{C}$.

$$\{U_{2,d_2} U_{3,d_3} \ldots U_{n,d_n} \mid (d_2, d_3, \ldots, d_n) \in [0, 1] \times [0, 2] \times \ldots \times [0, n-1]\}$$

Let $\Lambda_n$ be the set of all Young diagrams with $n$ boxes. We mean by a Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots ] \in \Lambda_n$ an array of $n$ boxes with $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row, and so on with $\lambda_1 \geq \lambda_2 \geq \ldots$. For $\lambda \in \Lambda_n$, the number of rows in which there are nonzero boxes is called the depth of $\lambda$ and we denote by $l(\lambda)$. We immediately observe that $\lambda_1 + \lambda_2 + \ldots + \lambda_{l(\lambda)} = n$. Formally, the Young diagram $\lambda \in \Lambda_n$ is defined by the subset of $\mathbb{N} \times \mathbb{N}$ as follows.

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i, 1 \leq i \leq l(\lambda)\}$$

The element $(i, j)$ is identified with the box at $i$-th row and $j$-th column of $\lambda$. We shall write $\mu < \lambda$ if $\mu$ is obtained from $\lambda$ by removing appropriate boxes.

As well known fact, for each Young diagram $\lambda \in \Lambda_n$, there is an irreducible representation of $S_n$ over the complex field $\mathbb{C}$ up to equivalence, and all irreducible representations of $S_n$ over $\mathbb{C}$ are parameterized by elements of $\Lambda_n$. 
Let $\text{Tab}(\lambda)$ be the set of all tableaux belonging to $\lambda$ and let $\text{STab}(\lambda)$ be the set of all standard tableaux belonging to $\lambda$. For $T \in \text{Tab}(\lambda)$, we mean by $T(i, j)$ the number on the box at $i$-th row and $j$-th column of $T$.

$\mathfrak{S}_n$ acts on $\text{Tab}(\lambda)$ as permutations of numbers on boxes. For $\sigma \in \mathfrak{S}_n$ and $T \in \text{Tab}(\lambda)$, $\sigma$ acts on $T$ as follows.

$$(\sigma \cdot T)(i, j) = \sigma(T(i, j))$$

We claim that this action is simply transitive.

**Definition 2.3.** Let $T$ be a tableau of shape $\lambda$, and $(i, j)$ the box at $i$-th row and $j$-th column of $T$. Then the class of $(i, j)$ is the value $j - i$. For $k = T(i, j)$, we mean by $\alpha_{k, T}$ the class of $(i, j)$.

From Definition 2.3, we have $\alpha_{1, T} = 0$ for each standard tableau $T$.

**Definition 2.4.** Let $k, l$ be numbers on boxes in a tableau $T$, then the axial distance $d_{T, k, l}$ from $k$ to $l$ in $T$ is defined as follows.

$$d_{T, k, l} = \alpha_{k, T} - \alpha_{l, T}$$

It follows immediately from Definition 2.4 that

$$d_{T, k, l} = -d_{T, l, k}$$

**Lemma 2.5.** If $T \in \text{STab}(\lambda)(\lambda \in \Lambda_n)$, then $d_{T, k, k+1}$ is nonzero for each $k$ with $1 \leq k \leq n - 1$.

**Proof.** If $(i, j)$ and $(i', j')$ have a same class, then there exists an integer $m$ such that $i = i' + m, j = j' + m$. Hence, boxes which have a same class are located on the line parallel to the diagonal of the Young diagram. For standard tableaux, numbers labeled on boxes strictly increase across each row and each column. Therefore, the box on which $k$ is labeled and the box on which $k + 1$ is labeled cannot be located on the same line parallel to diagonal.

Let $V_\lambda$ be the complex vector space with basis $\{v_T \mid T \in \text{STab}(\lambda)\}$. We define the linear map $\pi_\lambda(s_i)$ for elements $s_i = (i, i + 1) \in \mathfrak{S}_n$ on $V_\lambda$ in three ways according to the relation of location between the box on which $i$ is labeled and the box on which $i + 1$ is labeled in $T$.

1. If $i$ and $i + 1$ appear in the same row, $\pi_\lambda(s_i)v_T = v_T$.
2. If $i$ and $i + 1$ appear in the same column, $\pi_\lambda(s_i)v_T = -v_T$.
3. Otherwise, we observe that $s_i \cdot T$ is also in $\text{STab}(\lambda)$, and $\pi_\lambda(s_i)$ acts on the subspace $\mathbb{C}v_T \oplus \mathbb{C}v_{s_i \cdot T}$ of $V_\lambda$ defined by the matrix,

$$
\begin{bmatrix}
-\eta & \sqrt{1-\eta^2} \\
\sqrt{1-\eta^2} & \eta
\end{bmatrix}
$$

where $\eta = (d_{T,i,i+1})^{-1}$.

**Theorem 2.6.** For $\lambda, \mu \in \Lambda_n$ such that $\lambda \neq \mu$, we have $\pi_\lambda \not\equiv \pi_\mu$. Moreover, by the definition above, $\{\pi_\lambda \mid \lambda \in \Lambda_n\}$ is the set of all equivalent classes of irreducible representations of $S_n$.

*Proof.* See for example, [6] or [15].

## 3 Representations of the alternating groups

In this section, we review some facts about irreducible representations of the alternating groups. Every irreducible representation of the alternating groups is realized with the restriction of that of the symmetric groups. Furthermore, there is a beautiful branching rule for the restriction of irreducible representations of the symmetric groups to the alternating groups, which was discovered by Frobenius [5].

Let $\mathfrak{A}_n$ be the alternating group, consisting of all even permutations.

$$
\mathfrak{A}_n = \{\sigma \in S_n \mid l(\sigma) \text{ is even}\}
$$

As the fact, we can say that $\mathfrak{A}_n$ is generated by following elements.

$$
\mathfrak{A}_n = \langle s_1 s_{i+1} \mid 1 \leq i \leq n-2 \rangle
$$

We notice that $|\mathfrak{A}_n| = n!/2 = |S_n|/2$.

From Corollary 2.2 and the definition of $\mathfrak{A}_n$, the following statement holds.

**Proposition 3.1.** Let $\mathbb{C}[\mathfrak{A}_n]$ be the group ring of $\mathfrak{A}_n$. Then $\mathbb{C}[\mathfrak{A}_n]$ has following basis as a vector space over $\mathbb{C}$.

$$
\{\sigma = U_{2,d_2}U_{3,d_3} \ldots U_{n,d_n} \mid (d_1, d_2, \ldots, d_{n-1}) \\
\in [0,1] \times [0,2] \times \ldots \times [0, n-1] \quad l(\sigma) \text{ is even}\}
$$

For irreducible representations $\pi_\lambda$ of $S_n$, we denote by $\hat{\pi}_\lambda$ the restriction of $\pi_\lambda$ to $\mathfrak{A}_n$. For $\lambda \in \Lambda_n$ (resp. $T \in \text{STab}(\lambda)$), we mean by $'\lambda$ (resp. $'T$) its transpose. When $'\lambda = \lambda$, we say $\lambda$ self-conjugate.

**Proposition 3.2.** For every non self-conjugate $\lambda \in \Lambda_n$, $\hat{\pi}_\lambda$ is an irreducible representation of $\mathfrak{A}_n$ and $\hat{\pi}_\lambda \cong \hat{\pi}_{'\lambda}$.
Proposition 3.3. For every self-conjugate \( \lambda \in \Lambda_n \), \( \tilde{\pi}_\lambda \) is the direct sum of two irreducible representations \( \tilde{\pi}_\lambda^+ \) and \( \tilde{\pi}_\lambda^- \) with \( \deg(\tilde{\pi}_\lambda^+) = \deg(\tilde{\pi}_\lambda^-) \).

Theorem 3.4. Let \( \lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q} \) be non self-conjugate Young diagrams and \( \lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q} \) be self-conjugate Young diagrams within \( \Lambda_n \). Then irreducible representations \( \tilde{\pi}_{\lambda_1}, \tilde{\pi}_{\lambda_2}, \ldots, \tilde{\pi}_{\lambda_p}, \tilde{\pi}_{\lambda_{p+1}}, \ldots, \tilde{\pi}_{\lambda_{p+q}} \) are not equivalent each other and these consist of all equivalent classes of irreducible representations of \( A_n \).

Proof. See, for example [2] or [5].

It is well known that there is a presentation of \( A_n \) as follows.

Proposition 3.5. For \( n > 2 \), the generators \( x_1, x_2, \ldots, x_{n-2} \) and the following relations define a presentation of \( A_n \).

1. \( x_1^3 = x_2^2 = \ldots = x_{n-2}^2 = 1 \)
2. \( (x_{i-1} x_i)^3 = 1 \) for \( i = 2, 3, \ldots, n-2 \)
3. \( (x_i x_j)^2 = 1 \) whenever \( |i - j| > 1 \)

4 Orthogonal representations of the Hecke algebra \( H_n(q) \)

Like the case of \( S_n \), all equivalent classes of irreducible representations of the Hecke algebra \( H_n(q) \) are parametrized by the elements in \( \Lambda_n \). In this section, we will review how to construct an irreducible representation of \( H_n(q) \) corresponding to \( \lambda \in \Lambda_n \), which was taken from Wenzl’s paper [15].

Since the way of construction is similar to the case of \( S_n \), and we do not discuss about representations of \( S_n \) any more, we will use the same notation as in the case of \( S_n \).

Let \( q \) be a complex number and we will mean by the Hecke algebra \( H_n(q) \) of type \( A_{n-1} \) the algebra over \( \mathbb{C} \) with generators \( 1, g_1, g_2, \ldots, g_{n-1} \) with the defining relations

1. \( g_i^2 = (q - 1) g_i + q \) for \( i = 1, 2, \ldots, n-1 \)
2. \( g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \) for \( i = 1, 2, \ldots, n-2 \)
3. \( g_i g_j = g_j g_i \) whenever \( |i - j| \geq 2 \)

When \( q \) is not a root of unity, it is known that \( H_n(q) \cong \mathbb{C}[S_n] \).

If \( g_{i_1} g_{i_2} \ldots g_{i_k} \) cannot be expressed by any linear combination whose terms are products of at most \( k - 1 \) generators, then \( g_{i_1} g_{i_2} \ldots g_{i_k} \) is called the reduced expression. Using the same notation as in the case of \( \mathbb{C}[S_n] \), we denote \( g_{i_1-1} g_{i_2-2} \ldots g_{i_d-d} \) by \( U_{i,d} \). As well known fact, we can show the following.
Proposition 4.1. \( H_n(q) \) has following basis as a vector space over \( \mathbb{C} \).

\[
\{ U_{2,d_2}U_{3,d_3} \ldots U_{i,d_i} \ldots U_{n,d_n} \mid (d_2, d_3, \ldots, d_n) \in [0,1] \times [0,2] \times \ldots \times [0,n-1] \}
\]

(where the \( i \)-th contribution is only to be included if \( d_i \geq 1 \), and \( U_{2,0}U_{3,0} \ldots U_{n,0} \) is the unit element in \( H_n(q) \))

Let \( V_\lambda \) be the complex vector space with basis \( \{ v_T \mid T \in \text{STab}(\lambda) \} \) for \( \lambda \in \Lambda_n \). We define the linear map \( \pi_\lambda(g_i) \) for elements \( g_i \in H_n(q) \) on \( V_\lambda \) in three ways according to the relation of location between the box on which \( i \) is labeled and the box on which \( i + 1 \) is labeled in \( T \).

1. If \( i \) and \( i + 1 \) appear in the same row, \( \pi_\lambda(g_i)v_T = qv_T \).
2. If \( i \) and \( i + 1 \) appear in the same column, \( \pi_\lambda(g_i)v_T = -v_T \).
3. Otherwise, we observe that \( s_i \cdot T \) is also in \( \text{STab}(\lambda) \), and \( \pi_\lambda(g_i) \) acts on the subspace \( \mathbb{C}v_T \oplus \mathbb{C}v_{s_i \cdot T} \) of \( V_\lambda \) defined by the matrix,

\[
\frac{1}{1 - q^d} \begin{pmatrix}
q^d(1 - q) & \sqrt{q(1 - q^{d-1})(1 - q^{d+1})} \\
\sqrt{q(1 - q^{d-1})(1 - q^{d+1})} & -(1 - q)
\end{pmatrix}
\]

where \( d = d_{T,i,i+1} \) is the axial distance from \( i \) to \( i + 1 \) in \( T \).

When \( q \neq 0 \) and \( q \) is not a \( k \)-th root of unity with \( 1 \leq k \leq n - 1 \), this matrix is always well-defined because of Lemma 2.5.

Theorem 4.2. For \( \lambda \in \Lambda_n \) and \( \mu \in \Lambda_n \) such that \( \lambda \neq \mu \), we have \( \pi_\lambda \not\cong \pi_\mu \). Moreover, by the definition above, \( \{ \pi_\lambda \mid \lambda \in \Lambda_n \} \) is all equivalent classes of irreducible representations of \( H_n(q) \).

To see the relation of representation matrix between \( \pi_\lambda \) and \( \pi_{t \lambda} \), we show the following lemma.

Lemma 4.3. For any \( T \in \text{STab}(\lambda) \) with \( \lambda \in \Lambda_n \), following holds.

\[
d_{T,i,j} = -d_{T,i,j}
\]

Therefore, we obtain the following immediately.

Proposition 4.4. The following relations between \( \pi_\lambda \) and \( \pi_{t \lambda} \) hold for every \( \lambda \in \Lambda_n \).

1. If \( \pi_\lambda(g_i)v_T = qv_T \), then \( \pi_{t \lambda}(g_i)v_{tT} = -v_{tT} \)
2. If \( \pi_\lambda(g_i)v_T = -v_T \), then \( \pi_{t \lambda}(g_i)v_{tT} = qv_{tT} \)
Proof. A straightforward computation.

Expanding in $H$ a presentation of $H$ Proposition 4.6.

From the definition of $U$ nonzero scalar multiplication of $i$ from the definition of $f$ we denote $f_i$ for $i = 1, 2, \ldots, n - 1$

$\pi$ is described by the matrix

$$
\frac{1}{1 - q^d} \begin{bmatrix}
-(1 - q) & \sqrt{q(1 - q^{d-1})(1 - q^{d+1})} \\
\sqrt{q(1 - q^{d-1})(1 - q^{d+1})} & q^d (1 - q)
\end{bmatrix}
$$

where $d = d_{T,i+1}$ is the axial distance from $i$ to $i + 1$ in $T$.

We introduce another presentation of $H_n(q)$. Assume $q \neq -1$ and set

$$
f_i = \frac{2g_i - (q - 1)}{q + 1} \quad \text{for} \ i = 1, 2, \ldots, n - 1
$$

**Proposition 4.5.** These generate $H_n(q)$ and constitute with the relations

1. $f_i^2 = 1$ for $i = 1, 2, \ldots, n - 1$
2. $f_if_{i+1}f_i = f_{i+1}f_if_{i+1} + \left(\frac{q - 1}{q + 1}\right)^2 (f_{i+1} - f_i) \quad \text{for} \ i = 1, 2, \ldots, n - 1$
3. $f_if_j = f_jf_i \quad \text{whenever} \ |i - j| \geq 2$

a presentation of $H_n(q)$.

**Proof.** A straightforward computation.

We denote $f_{i-1}f_{i-2}\ldots f_{i-d_i}$ with $i = 2, 3, \ldots, n$ and $d_i \in [0, i-1]$ by $U'_{i,d_i}$. In the same way as Proposition 4.1, we can show the following.

**Proposition 4.6.** $H_n(q)$ has following basis as a vector space over $\mathbb{C}$.

$$
\{U'_{2,d_2}U'_{3,d_3}\ldots U'_{i,d_i}\ldots U'_{n,d_n} \mid (d_2, d_3, \ldots, d_n) \\
\in [0, 1] \times [0, 2] \times \ldots \times [0, n - 1] \}
$$

(where the $i$-th contribution is only to be included if $d_i \geq 1$, and $U'_{2,0}U'_{3,0}\ldots U'_{n,0}$ is the unit element in $H_n(q)$)

**Proof.** Expanding $U'_{2,d_2}U'_{3,d_3}\ldots U'_{i,d_i}\ldots U'_{n,d_n}$ with $g_i$'s, the term which has the longest length is the nonzero scalar multiplication of $U_{2,d_2}U_{3,d_3}\ldots U_{i,d_i}\ldots U_{n,d_n}$ from the definition of $f_i$'s. Hence $U'_{2,d_2}U'_{3,d_3}\ldots U'_{i,d_i}\ldots U'_{n,d_n}$'s are linearly independent and constitute a basis of $H_n(q)$.

From the definition of $\pi_\lambda$ and Proposition 4.4, we observe the following.

**Proposition 4.7.** The following relations between $\pi_\lambda$ and $\pi_{\lambda_i}$ hold for every $\lambda \in \Lambda_n$.

1. $\pi_\lambda(f_i)v_T = v_T$ and $\pi_{\lambda_i}(f_i)v_T = -v_T$ if $i$ and $i + 1$ appear in the same row of $T$. 


2. \( \pi_\lambda(f_i)v_T = -v_T \) and \( \pi_\lambda(f_i)v_{sT} = v_{sT} \) if \( i \) and \( i+1 \) appear in the same column of \( T \).

3. Otherwise, the representation matrix of \( \pi_\lambda(f_i) \) acting on subspace of \( V_\lambda \) spanned by \( (v_T, v_{sT}) \), and 
\( \pi_\lambda(f_i) \) acting on subspace of \( V_\lambda \) spanned by \( (v_T, v_{sT}) \) is given by

\[
\pi_\lambda(f_i) \sim \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \quad \pi_\lambda(f_i) \sim \begin{bmatrix} -A & -B \\ -B & A \end{bmatrix}
\]

where

\[
A = \frac{(1 - q)(1 + q^d)}{(1 + q)(1 - q^d)}
\]

\[
B = \frac{2 \sqrt{q^d(q^{d-1}) - (1 - q^d)}}{(1 + q)(1 - q^d)}
\]

\[d = d_{T,i,i+1}\]

5 The subalgebra \( \mathfrak{A}_n(q) \) of \( \mathcal{H}_n(q) \)

From Proposition 4.5, we observe that parity of length of each expression by \( f_i \)'s is preserved even if we choose another expression. To be exact, if an expression of even (resp. odd) length is expressed in a linear combination of other expressions, then the length of each term is even (resp. odd). So we can consider the subalgebra \( \mathfrak{A}_n(q) \) generated by all elements of even length like the case of the alternating groups. We will show this subalgebra has dimension just a half of that of \( \mathcal{H}_n(q) \). Moreover, we will show a presentation of \( \mathfrak{A}_n(q) \), which is a \( q \)-analogue of that of \( \mathfrak{A}_n \). Comparing with basic relations of \( \mathfrak{A}_n \) shown in Proposition 3.5, quadratic relations are same as in the case of \( \mathfrak{A}_n \), but cubic relations are slightly different.

Now we define the subalgebra \( \mathfrak{A}_n(q) \) of \( \mathcal{H}_n(q) \).

**Definition 5.1.** For \( n \geq 3 \) and \( q \neq -1 \), we define \( \mathfrak{A}_n(q) \) the subalgebra over \( \mathbb{C} \) of \( \mathcal{H}_n(q) \) generated by \( f_if_{i+1} \) with \( 1 \leq i \leq n-1 \).

To be consistent with the alternating group \( \mathfrak{A}_2 \), we set \( \mathfrak{A}_2(q) \) the subalgebra of \( \mathcal{H}_2(q) \) generated by the unit element.

**Proposition 5.2.** \( \mathfrak{A}_n(q) \) consists of all elements whose terms have even length in the expression by \( f_i \) with \( 1 \leq i \leq n-1 \).

**Proof.** From Proposition 4.5, the multiplication of two monomial elements of even length is expressed by elements whose terms have even length. Hence it is enough to see that \( f_jf_k \) (for \( 1 \leq j, k \leq n-1 \) \( j \neq k \))
are generated by \( f_1f_i \)'s. If \( j = 1 \) and \( k > 1 \) or \( j > 2 \) and \( k = 1 \), then \( f_jf_k \) is the generator itself. For \( j > 2 \) and \( k > 2 \), \( f_1f_jf_1f_k = f_jf_k \). For \( j = 2 \), we obtain from Proposition 4.5 that

\[
f_2f_k = (f_1f_2)^2f_1f_k - \left(\frac{q-1}{q+1}\right)^2(f_1f_k - f_1f_2f_1f_k)
\]

Thus, we complete the proof. \( \square \)

**Theorem 5.3.** As a vector space over the field \( \mathbb{C} \), \( \mathfrak{A}_n(q) \) has dimension \( n!/2 \).

\[
\dim_{\mathbb{C}} \mathfrak{A}_n(q) = \frac{n!}{2}
\]

**Proof.** Consider all of elements which have even length in the set,

\[
\{ U'_{2,d_2}U'_{3,d_3} \ldots U'_{i,d_i} \ldots U'_{n,d_n} \mid (d_2,d_3,\ldots,d_n) \\
\in [0,1] \times [0,2] \times \ldots \times [0,n-1] \}
\]

Then they are linearly independent over \( \mathbb{C} \) in \( \mathfrak{A}_n(q) \). Thus it is sufficient to prove that any element in \( \mathfrak{A}_n(q) \) is expressed as a linear combination of the form,

\[
\{ U'_{2,d_2}U'_{3,d_3} \ldots U'_{i,d_i} \ldots U'_{n,d_n} \mid \text{length is even} \}
\]

We will prove this by induction on the length of elements in \( \mathfrak{A}_n(q) \). Consider the element \( f_{i_1}f_{i_2} \ldots f_{i_{2k}} \) in \( \mathfrak{A}_n(q) \). In the exchange and deletion process of expressions of words using relations of Proposition 4.5, leading terms of right hand sides of Proposition 4.5 bring us same calculations as in the case of \( \mathfrak{S}_n \), and remaining terms calculations for the products of at most \( 2(k-1) \) \( f_i \)'s. By the induction assumption, remaining terms can be expressed with a linear combination of desired form, hence \( f_{i_1}f_{i_2} \ldots f_{i_{2k}} \) can be also. \( \square \)

Next we will give a presentation of \( \mathfrak{A}_n(q) \).

**Proposition 5.4.** Let \( y_i = f_1f_{i+1} \) for \( i = 1,2,\ldots,n-2 \). Then \( y_i \)'s satisfy the following relations.

1. \( y_1^3 = -\left(\frac{q-1}{q+1}\right)^2(y_1^2 - y_1) + 1 \)
2. \( y_i^2 = 1 \quad \text{for } i > 1 \)
3. \( (y_{i-1}y_i)^3 = -\left(\frac{q-1}{q+1}\right)^2 \left( (y_{i-1}y_i)^2 - y_{i-1}y_i \right) + 1 \quad \text{for } i = 2,3,\ldots,n-2 \)
4. \( (y_iy_j)^2 = 1 \quad \text{whenever } |i-j| > 1 \)
Proof. Relations 2 and 4 are obvious. For relation 1,

\[(y_1)^3 = f_1 f_2 f_1 f_2 f_1 f_2 \]
\[= \left\{ f_2 f_1 + \left( \frac{q-1}{q+1} \right)^2 (1 - f_1 f_2) \right\} f_1 f_2 \]
\[= 1 + \left( \frac{q-1}{q+1} \right)^2 \left\{ f_1 f_2 - (f_1 f_2)^2 \right\} \]
\[= - \left( \frac{q-1}{q+1} \right)^2 (y_1^3 - y_1) + 1 \]

For relation 3,

\[(y_{i-1} y_i)^3 = f_1 f_i f_1 f_{i+1} f_1 f_{i+1} f_1 f_{i+1} f_1 f_i f_{i+1} \]
\[= f_1 f_i f_1 f_{i+1} f_1 f_{i+1} f_1 f_{i+1} f_1 f_i f_{i+1} \]
\[= \left\{ f_1 f_{i+1} f_1 f_{i+1} + \left( \frac{q-1}{q+1} \right)^2 (f_1 f_{i+1} - f_1 f_i) \right\} f_{i+1} f_i f_{i+1} \]
\[= 1 + \left( \frac{q-1}{q+1} \right)^2 \left( f_1 f_i f_{i+1} - f_1 f_{i+1} f_i f_{i+1} \right) \]
\[= 1 + \left( \frac{q-1}{q+1} \right)^2 \left\{ y_{i-1} y_i - (y_{i-1} y_i)^2 \right\} \]

Thus we complete the proof. \qed

**Theorem 5.5.** The generators \( y_i \)'s and relations given in Proposition 5.4 define a presentation of \( \mathfrak{A}_n(q) \).

We will prove this theorem at the end of this section.

Assume that \( q \neq -1 \). Let \( \mathfrak{A}_n(q)(n \geq 3) \) be an algebra over \( \mathbb{C} \) generated by elements \( a_1, a_2, \ldots, a_{n-2} \) with defining relations,

1. \( a_1^2 = - \left( \frac{q-1}{q+1} \right)^2 (a_1^2 - a_1) + 1 \)
2. \( a_i^2 = 1 \) for \( i > 1 \)
3. \( (a_{i-1} a_i)^3 = - \left( \frac{q-1}{q+1} \right)^2 \left\{ (a_{i-1} a_i)^2 - a_{i-1} a_i \right\} + 1 \) for \( i = 2, 3, \ldots, n-2 \)
4. \( (a_i a_j)^2 = 1 \) whenever \( |i-j| > 1 \)

For \( n = 2 \), we set \( \mathfrak{A}_2(q) \) an algebra over \( \mathbb{C} \) generated by only the unit element.

To prove the Theorem 5.5, we shall begin with the following Lemma.

**Lemma 5.6.** The following relations hold for \( \mathfrak{A}_n(q) \).

1. \( a_1^{-1} = a_1^2 + \left( \frac{q-1}{q+1} \right)^2 (a_1 - 1) \)
2. \( a_i^{-1} = a_i \) for \( i = 2, 3, \ldots, n-2 \)
3. \( a_{i-1} a_i a_{i-1} = a_i a_{i-1}^{-1} a_i - \left( \frac{q-1}{q+1} \right)^2 (a_{i-1} - a_i) \) for \( i = 2, 3, \ldots, n-2 \)
4. \( a_i a_{i-1} = a_{i-1}^{-1} a_i a_{i-1} - \left( \frac{q - 1}{q + 1} \right)^2 (a_i - a_{i-1}) \) for \( i = 2, 3, \ldots, n - 2 \)

5. \( a_i a_1 = a_1^{-1} a_i \) \( a_1 a_i = a_i a_1^{-1} \) for \( i > 2 \)

6. \( a_i a_j = a_j a_i \) whenever \( |i - j| > 1 \) and \( i, j > 1 \)

Proof. A straightforward computation. \(\Box\)

**Proposition 5.7.** For \( n > 3 \), \( \tilde{A}_n(q) \) is generated as a vector space over \( \mathbb{C} \) by the monomials with at most one occurrence of \( a_{n-2} \).

Proof. For \( n = 4 \), Let \( M \) be a monomial in \( \tilde{A}_4(q) \) with at least two occurrences of \( a_2 \). Displaying two consecutive occurrences of \( a_2 \) in \( M \), we write \( M = M_1 a_2 a_2 M_3 \), where we can assume that \( M_2 \) is a monomial in \( a_1 \), that may be 1 or \( a_1 \) or \( a_1^2 \). For the first case, \( a_2 a_2 = 1 \). For the second case, from the relation 4 of Lemma 5.6,

\[
a_2 a_1 a_2 = a_1^{-1} a_2 a_1^{-1} - \left( \frac{q - 1}{q + 1} \right)^2 (a_2 - a_1^{-1})
\]

Hence we get the desired form. For the third case, from the relation 1, 3 of Lemma 5.6,

\[
a_2 a_1^2 a_2 = a_2 \left( a_1^{-1} - \left( \frac{q - 1}{q + 1} \right)^2 (a_1 - 1) \right) a_2
\]

\[
= a_1 a_2 a_1 + \left( \frac{q - 1}{q + 1} \right)^2 (a_1 - a_2)
\]

\[
- \left( \frac{q - 1}{q + 1} \right)^2 (a_2 a_1 a_2 - 1)
\]

Applying the second case to the term \( a_1 a_2 a_1 \), we get the desired form for the third case. Thus we complete the proof for \( n = 4 \).

Next, let \( M \) be a monomial in \( \tilde{A}_n(q) \) with at least two occurrences of \( a_{n-2} \). Displaying two consecutive occurrences of \( a_2 \) in \( M \), we write \( M = M_1 a_{n-2} a_2 a_{n-2} M_3 \), where we can assume that \( M_2 \) is a monomial in \( a_1, a_2 \ldots a_{n-3} \). We can assume by induction that \( M_2 \) contains \( a_{n-3} \) at most once. If \( M_2 \) does not contain \( a_{n-3} \) at all, then by the relation 2, 5, 6 of Lemma 5.6,

\[
M = M_1 M_2^2 a_{n-3} M_3 = M_1 M_2 M_3^2
\]

for some \( M_2' \in \tilde{A}_{n-1}(q) \). In this case, \( a_{n-3} \) is vanished. If \( M_2 \) contains \( a_{n-3} \) exactly once, we can write \( M_2 = M_4 a_{n-3} M_5 \) with \( M_4, M_5 \) in \( a_1, a_2 \ldots a_{n-4} \) and then by the relations 5, 6 of Lemma 5.6,

\[
M = M_1 a_{n-2} a_2 a_{n-3} M_5 a_{n-2} M_3 = M_1 M_4' a_{n-2} a_{n-3} a_{n-2} M_5' M_3
\]

for some \( M_4', M_5' \in \tilde{A}_{n-1}(q) \). By the relation 4 of Lemma 5.6,

\[
M = M_1 M_4' \left( a_{n-3}^{-2} a_{n-2}^{-1} - \left( \frac{q - 1}{q + 1} \right)^2 (a_{n-2} - a_{n-3}^{-1}) \right) M_5' M_3
\]

reducing the number of occurrence of \( a_{n-2} \). \(\Box\)
Consider the following sets of monomials.

\[ S_1 = \{1, a_1, a_1^2\} \]
\[ S_2 = \{1, a_2, a_2a_1, a_2^2\} \]
\vdots
\[ S_i = \{1, a_i, a_ia_{i-1}, \ldots, a_2a_1, a_ia_{i-1} \ldots a_2a_1^2\} \]
\vdots
\[ S_{n-2} = \{1, a_{n-2}, a_{n-2}a_{n-3}, \ldots, a_{n-2}a_{n-3} \ldots a_2a_1, a_{n-2}a_{n-3} \ldots a_2a_1^2\} \]

We shall say that \( M_0 = U_1U_2 \ldots U_{n-2} \) is a monomial in normal form in \( \mathfrak{A}_n(q) \), if \( U_i \in S_i \) for \( i = 1, 2, \ldots, n-2 \). There are \( n!/2 \) of monomials in normal form in \( \mathfrak{A}_n(q) \).

**Proposition 5.8.** The monomials in normal form in \( \mathfrak{A}_n(q) \) generate \( \mathfrak{A}_n(q) \) as a vector space over \( \mathbb{C} \). In particular, \( \dim_{\mathbb{C}} \mathfrak{A}_n(q) \leq n!/2 \)

**Proof.** The proof will be by induction. For \( n = 3 \), it is obvious. Let \( M \) be an element in \( \mathfrak{A}_n(q) \). If \( M \) contains no \( a_{n-2} \), then \( M \in \mathfrak{A}_{n-1}(q) \). Hence by the induction assumption, \( M \) is expressed by a linear combination of monomials in normal form in \( \mathfrak{A}_n(q) \). If \( M \) contains \( a_{n-2} \), then by Proposition 5.7, we can write \( M = M_1a_{n-2}M_2 \) where \( M_1, M_2 \in \mathfrak{A}_{n-1}(q) \). By induction, \( M_2 \) is a linear combination of the form \( U_1U_2 \ldots U_{n-3} \) with \( U_i \in S_i \) for \( i = 1, 2, \ldots, n-3 \). By the relations 5, 6 of Lemma 5.6, we have

\[ M_1a_{n-2}U_1U_2 \ldots U_{n-3} = M'_1a_{n-2}U_{n-3} \]

for some \( M'_1 \) in \( \mathfrak{A}_{n-1}(q) \). By induction again, \( M'_1 \) is a linear combination of monomials of the form \( U_1U_2 \ldots U_{n-3} \) with \( U_i \in S_i \). Thus \( M \) is a linear combination of monomials \( U_1U_2 \ldots U_{n-2} \) as desired. \( \square \)

**Proof of Theorem 5.5.** Consider the map \( \varphi : \mathfrak{A}_n(q) \rightarrow \mathfrak{A}_n(q) \) such that \( \varphi(a_i) = y_i \). \( \varphi \) defines a homomorphism of algebras. We immediately observe that this map is surjective. Hence it is enough to see that the dimension of \( \mathfrak{A}_n(q) \) is no more than the dimension of \( \mathfrak{A}_n(q) \), but this was already shown in Proposition 5.8. \( \square \)

### 6 Representations of \( \mathfrak{A}_n(q) \)

In this section, we analyze representations of \( \mathfrak{A}_n(q) \) by restricting the representations we know from \( \mathcal{H}_n(q) \). There is a close relationship between representations of \( \mathfrak{S}_n \) and those of \( \mathfrak{A}_n \) already shown in section 2. We will see this phenomenon in the case of \( \mathcal{H}_n(q) \) and \( \mathfrak{A}_n(q) \). For irreducible representations \( \pi_\lambda \) of \( \mathcal{H}_n(q) \), we denote by \( \hat{\pi}_\lambda \) the restriction of \( \pi_\lambda \) to \( \mathfrak{A}_n(q) \). For \( g \in \mathfrak{A}_n(q) \),

\[
\hat{\pi}_\lambda(g)v_T = \sum_{T' \in \text{Stab}(\lambda)} g_{T'\leftarrow T}^\lambda v_{T'}
\]
where $g_{T,T'}^\lambda$’s are complex numbers. For $T, T' \in \text{STab}(\lambda)$, there exists $\sigma \in S_n$ such that $\sigma \cdot T = T'$. We define the distance $d(T, T')$ from $T$ to $T'$ by $l(\sigma)$.

Lemma 6.1. For $g \in \mathfrak{A}_n(q)$,

$$g_{T,T'}^\lambda = g_{T',T}^\lambda$$

Proof. It is enough to prove for the case $g \in \{y_i \mid 1 \leq i \leq n - 2\}$. The element $f_i$ acts on $v_T$ in three ways according to where '$i$' and '$i + 1$' appear in the standard tableau $T$. Hence, for the action of $y_i = f_1 f_{i+1}$, we must consider places of $1, 2, i + 1$ and $i + 2$. We claim that only two cases are possible for arrangements of 1 and 2: they appear in the same row, or appear in the same column. If $i > 1$, then there is no interaction between the action of $f_1$ and $f_{i+1}$. But if $i = 1$, then actions of $f_1$ and $f_2$ interact. Hence the action of $y_1$ is more complicated. So, we show equalities of matrix elements as the case may be.

The case $i > 1$.

1. If 1 and 2 appear in the same row(resp. column) of $T$, and $i + 1$ and $i + 2$ also appear in the same row(resp. column) of $T$, then 1 and 2 appear in the same column(resp. row) of $^tT$ and $i + 1$ and $i + 2$ also appear in the same column(resp. row) of $^tT$. In this case, we easily get $\tilde{\pi}_\lambda(y_i) v_T = v_T$ and $\tilde{\pi}_t^\lambda(y_i) v_{^tT} = v_{^tT}$.

2. If 1 and 2 appear in the same row(resp. column) of $T$, and $i + 1$ and $i + 2$ appear in the same column(resp. row) of $T$, then 1 and 2 appear in the same row(resp. column) of $^tT$ and $i + 1$ and $i + 2$ appear in the same row(resp. column) of $^tT$. In this case, we easily get $\tilde{\pi}_\lambda(y_i) v_T = -v_T$, and $\tilde{\pi}_t^\lambda(y_i) v_{^tT} = -v_{^tT}$.

3. If 1 and 2 appear in the same row(resp. column) of $T$, and $i + 1$ and $i + 2$ appear in neither the same row nor the same column of $T$, then 1 and 2 appear in the same column(resp. row) of $^tT$.

Using the notation of the end of section 4, set

$$\tilde{\pi}_\lambda(f_{i+1}) = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

for the basis $v_T, v_{s_{i+1}}, T$.

and we get

$$\tilde{\pi}_\lambda(y_i) v_T = Av_T + Bv_{s_{i+1}} (\text{resp. } -Av_T - Bv_{s_{i+1}})$$

$$\tilde{\pi}_t^\lambda(y_i) v_{^tT} = Av_{^tT} + Bv_{s_{i+1}, ^tT} (\text{resp. } -Av_{^tT} - Bv_{s_{i+1}, ^tT})$$

The case $i = 1$.

By exchanging $T$ and $^tT$, we observe that it is enough to check only in two cases: 1, 2 and 3 appear in the same row, 1 and 2 appear in the same row and 1 and 3 appear in the same column.
1. If 1, 2 and 3 appear in the same row of $T$, then 1, 2 and 3 appear in the same column of $^tT$. In this case, we easily get $\tilde{\pi}_\lambda(y_1)v_T = v_T$ and $\tilde{\pi}_t\lambda(y_1)v_T = v_T$.

2. 1 and 2 appear in the same row and 1 and 3 appear in the same column of $T$, then 1 and 2 appear in the same column and 1 and 3 appear in the same row of $^tT$. We observe that $f_1$ acts on $v_T, v_{s_2.T}, v_{s_2.T}$ as scalar multiplication. Using the notation of the end of section 4, set

$$\tilde{\pi}_\lambda(f_2) = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$ for the basis $v_T, v_{s_2.T}$

and we get

$$\tilde{\pi}_\lambda(y_1)v_T = Av_T - Bv_{s_2.T}$$
$$\tilde{\pi}_t\lambda(y_1)v_T = Av_T - Bv_{s_2.T}$$

\[\square\]

**Proposition 6.2.** For $\lambda \in \Lambda_n$,

$$\tilde{\pi}_\lambda \cong \tilde{\pi}_t\lambda$$

**Proof.** From Lemma 6.1, representation matrices of $\tilde{\pi}_\lambda$ and $\tilde{\pi}_t\lambda$ for generators of $\mathfrak{A}_n(q)$ are coincide. Therefore, the above statement holds. \[\square\]

For $\lambda \in \Lambda_n$ with $n > 1$, we set

$$STab(\lambda)^+ = \{ T \in STab(\lambda) \mid T(1, 1) = 1, T(1, 2) = 2 \}$$
$$STab(\lambda)^- = \{ T \in STab(\lambda) \mid T(1, 1) = 1, T(2, 1) = 2 \}$$

Then, $STab(\lambda)$ is a disjoint union of $STab(\lambda)^+$ and $STab(\lambda)^-$. If $\lambda$ is self-conjugate, then $T \in STab(\lambda)^+$ if and only if $^tT \in STab(\lambda)^-$. Therefore, $STab(\lambda)^+$ corresponds to $STab(\lambda)^-$ one to one and onto by mapping, $T \rightarrow ^tT$.

Let $\lambda$ be self-conjugate. Let $\tilde{\mathcal{V}}_\lambda$ be the representation space of $\tilde{\pi}_\lambda$. Now, we will introduce two subspaces $\tilde{\mathcal{V}}_\lambda^+$ and $\tilde{\mathcal{V}}_\lambda^-$ of $\tilde{\mathcal{V}}_\lambda$ as follows.

$$\tilde{\mathcal{V}}_\lambda^+ = \bigoplus_{T \in STab(\lambda)^+} \mathbb{C}(v_T + v_{^tT})$$
$$\tilde{\mathcal{V}}_\lambda^- = \bigoplus_{T \in STab(\lambda)^+} \mathbb{C}(v_T - v_{^tT})$$

Then, $\tilde{\mathcal{V}}_\lambda$ is a direct sum of $\tilde{\mathcal{V}}_\lambda^+$ and $\tilde{\mathcal{V}}_\lambda^-$ as vector space over $\mathbb{C}$. 

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Proposition 6.3. Let $\lambda \in \Lambda_n$ be self-conjugate. Then $\tilde{V}_\lambda^+$ and $\tilde{V}_\lambda^-$ are $\mathfrak{A}_n(q)$-submodules of $\tilde{V}_\lambda$, and $\tilde{V}_\lambda$ has a $\mathfrak{A}_n(q)$-submodule decomposition as follows.

$$\tilde{V}_\lambda = \tilde{V}_\lambda^+ \oplus \tilde{V}_\lambda^-$$

as $\mathfrak{A}_n(q)$-module

Proof. Let $g \in \mathfrak{A}_n(q)$. We set $T \in \text{STab}(\lambda)^+$. From Lemma 6.1, we get following calculation for the element $v_T + v_T$ of $\tilde{V}_\lambda^+$.

$$\hat{\pi}_\lambda(g)(v_T + v_T) = \sum_{T' \in \text{STab}(\lambda)} g^\lambda_{TT'} (v_{T'} + v_T)$$

$$= \sum_{T' \in \text{STab}(\lambda)^+} g^\lambda_{TT'} (v_{T'} + v_T)$$

$$+ \sum_{T' \in \text{STab}(\lambda)^-} g^\lambda_{TT'} (v_{T'} + v_T)$$

$$= \sum_{T' \in \text{STab}(\lambda)^+} g^\lambda_{TT'} (v_{T'} + v_T)$$

$$+ \sum_{T' \in \text{STab}(\lambda)^+} (g^\lambda_{TT'} + g^\lambda_{T'T'}) (v_{T'} + v_T)$$

Thus, $\hat{\pi}_\lambda(g)(v_T + v_T)$ is in $\tilde{V}_\lambda^+$.

Next, we get following calculation for the element $v_T - v_T$ of $\tilde{V}_\lambda^-$. 

$$\hat{\pi}_\lambda(g)(v_T - v_T) = \sum_{T' \in \text{STab}(\lambda)} g^\lambda_{TT'} (v_{T'} - v_T)$$

$$= \sum_{T' \in \text{STab}(\lambda)^+} g^\lambda_{TT'} (v_{T'} - v_T)$$

$$+ \sum_{T' \in \text{STab}(\lambda)^-} g^\lambda_{TT'} (v_{T'} - v_T)$$

$$= \sum_{T' \in \text{STab}(\lambda)^+} g^\lambda_{TT'} (v_{T'} - v_T)$$

$$+ \sum_{T' \in \text{STab}(\lambda)^+} (g^\lambda_{TT'} - g^\lambda_{T'T'}) (v_{T'} - v_T)$$

Thus, $\hat{\pi}_\lambda(g)(v_T - v_T)$ is in $\tilde{V}_\lambda^-$. 

We denote by $\hat{\pi}_\lambda^+$ the representation of $\mathfrak{A}_n(q)$ corresponding to $\tilde{V}_\lambda^+$ and $\hat{\pi}_\lambda^-$ corresponding to $\tilde{V}_\lambda^-$. 

Next, we will show the irreducibilities of these representations and semisimplicity of $\mathfrak{A}_n(q)$. Let $\lambda \in \Lambda_n$ be non self-conjugate. Then $\tilde{V}_\lambda$ and $\tilde{V}_\lambda$ are isomorphic as $\mathfrak{A}_n(q)$-modules and the direct sum $\tilde{V}_\lambda \oplus \tilde{V}_\lambda$
has another decomposition as \( \mathbb{C} \) vector spaces,

\[
\tilde{V}_\lambda \oplus \tilde{V}_\lambda = \tilde{V}_\lambda^+ \oplus \tilde{V}_\lambda^{-}
\]

where \( \tilde{V}_\lambda^+ \) and \( \tilde{V}_\lambda^- \) defined as follows.

\[
\tilde{V}_\lambda^+ = \bigoplus_{T \in \text{STab}(\lambda)} \mathbb{C}(v_T + v_{T'})
\]

\[
\tilde{V}_\lambda^- = \bigoplus_{T \in \text{STab}(\lambda)} \mathbb{C}(v_T - v_{T'})
\]

**Proposition 6.4.** If \( \lambda \) is non self-conjugate, then the above decomposition of \( \tilde{V}_\lambda \oplus \tilde{V}_\lambda \),

\[
\tilde{V}_\lambda \oplus \tilde{V}_\lambda = \tilde{V}_\lambda^+ \oplus \tilde{V}_\lambda^-
\]

is a submodule decomposition of \( \mathfrak{A}_n(q) \).

**Proof.** For \( v_T + v_{T'} \in \tilde{V}_\lambda^+ \subset \tilde{V}_\lambda \oplus \tilde{V}_\lambda \), we consider the direct sum of irreducible representations \( \tilde{\pi}_\lambda \) and \( \tilde{\pi}_{\lambda'} \). For \( g \in \mathfrak{A}_n(q) \), we have from Lemma 6.1 the following.

\[
(\tilde{\pi}_\lambda \oplus \tilde{\pi}_{\lambda'})(v_T + v_{T'}) = \sum_{T' \in \text{STab}(\lambda)} g_{T'\lambda'}^\lambda(v_{T'} + v_{T'})
\]

Thus, \( (\tilde{\pi}_\lambda \oplus \tilde{\pi}_{\lambda'})(v_T + v_{T'}) \) is in \( \tilde{V}_\lambda^+ \).

The Similar calculation is valid for \( \tilde{V}_\lambda^- \), hence we omit the calculation for \( \tilde{V}_\lambda^- \).

**Theorem 6.5.** Let \( q \) be a complex number such that \( q \neq 0 \) and \( q \) is not a \( k \)-th root of unity with \( 1 \leq k \leq n \). Let \( \lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_p, \lambda_p \) be non self-conjugate Young diagrams and \( \lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+q} \) be self-conjugate Young diagrams within \( \Lambda_n \). Then representations \( \tilde{\pi}_{\lambda_1}, \tilde{\pi}_{\lambda_2}, \ldots, \tilde{\pi}_{\lambda_p}, \tilde{\pi}_{\lambda_{p+1}}, \tilde{\pi}_{\lambda_{p+2}}, \ldots, \tilde{\pi}_{\lambda_{p+q}} \) are irreducible and not equivalent each other. These representations consist of all equivalent classes of irreducible representations of \( \mathfrak{A}_n(q) \). Hence, \( \mathfrak{A}_n(q) \) is semisimple.

**Proof.** At first, we will show the semisimplicity of \( \mathfrak{A}_n(q) \) under the assumptions of irreducibilities and mutual inequalities of these representations. We consider the map

\[
\tilde{\pi}_n : x \in \mathfrak{A}_n(q) \longrightarrow \tilde{\pi}_{\lambda_1}(x) \oplus \ldots \oplus \tilde{\pi}_{\lambda_p}(x) \oplus \tilde{\pi}_{\lambda_{p+1}}^+(x) \oplus \tilde{\pi}_{\lambda_{p+1}}^-(x) \oplus \ldots \oplus \tilde{\pi}_{\lambda_{p+q}}^+(x) \oplus \tilde{\pi}_{\lambda_{p+q}}^-(x)
\]

Then, by theorems of Burnside and Frobenius-Schur, \( \mathfrak{A}_n(q) \) has a quotient \( \tilde{\pi}_n(\mathfrak{A}_n(q)) \) isomorphic to the semisimple algebra \( \oplus \text{End}_\mathbb{C} V \) where \( V \) runs over irreducible representation spaces listed in the statement of theorem.

This semisimple algebra has dimension \( n!/2 \) and we already show that \( \mathfrak{A}_n(q) \) has dimension \( n!/2 \), thus \( \mathfrak{A}_n(q) \) is isomorphic to \( \oplus \text{End}_\mathbb{C} V \) and semisimple.
Next, we will show the irreducibilities and mutual inequalities of these representations by induction.

For $n = 2$, it is obvious. Indeed, $\mathfrak{A}_2(q)$ is generated by the unit element, and both $\tilde{\pi}_{(2)}$ and $\tilde{\pi}_{(1)}$ are identity maps.

For $n = 3$, $\tilde{\pi}_{(3)}$ and $\tilde{\pi}_{(13)}$ are identity map and

$$\tilde{V}_{(2,1)} = \tilde{V}^+_{(2,1)} \oplus \tilde{V}^-_{(2,1)}$$

where,

$$\tilde{V}^+_{(2,1)} = \mathbb{C}(v_T + v_T')$$

$$\tilde{V}^-_{(2,1)} = \mathbb{C}(v_T - v_T')$$

with standard tableau $T$ as follows.

$$T(1,1) = 1 \quad T(1,2) = 2 \quad T(2,1) = 3$$

We get the following from the proof of Lemma 6.1, Proposition 6.3,

$$\tilde{\pi}^\pm_{(1)}(y_1)(v_T \pm v_T') = \frac{1 + q^2 \pm 2\sqrt{q(1 + q + q^2)}}{(1 + q)^2} (v_T \pm v_T')$$

$$\tilde{\pi}^\pm_{(2)}(y_1^2)(v_T \pm v_T') = \left\{ \frac{1 + q^2 \pm 2\sqrt{q(1 + q + q^2)}}{(1 + q)^2} \right\}^2 (v_T \pm v_T')$$

Since every representation has degree 1, these representations are irreducible. Moreover, since $q$ is neither 0 nor $k$-th root of unity with $1 \leq k \leq 3$, we immediately have $\sqrt{q(1 + q + q^2)}$ is nonzero. Hence they are mutually inequivalent.

Let $n > 3$. By induction assumption, $\mathfrak{A}_{n-1}(q)$ is a semisimple algebra with central primitive idempotents $z_{\lambda_1}, z_{\lambda_2}, \ldots, z_{\lambda_{\nu+1}}, z_{\lambda_{\nu+2}}, \ldots, z_{\lambda'_{\nu+1}}, z_{\lambda'_{\nu+2}}, \ldots$ with $\lambda_1, \ldots, \lambda_{\nu+1}, \lambda'_{\nu+1}, \ldots, \lambda'_{\nu+2}, \ldots$ are non-self-conjugate and $\lambda', \lambda'_{\nu+1}, \ldots, \lambda'_{\nu+2}, \ldots$ are self-conjugate.

Let $\lambda \in \Lambda_n$ be non-self-conjugate. Then there is no diagram $\lambda' \in \Lambda_{n-1}$ such that $\lambda'$ is non-self-conjugate and $\lambda' < \lambda$ and $\lambda' < \lambda$. Indeed, for such diagram, we immediately obtain, $\lambda' < \lambda'$ and $\lambda' < \lambda$. Hence, $\lambda = \lambda$ but this is contradiction. When we restrict $\tilde{V}_\lambda$ to $\mathfrak{A}_{n-1}(q)$-module, we can write(for the reason, see [15]),

$$\tilde{V}_\lambda = \tilde{V}_{\lambda_{i_1}} \oplus \tilde{V}_{\lambda_{i_2}} \oplus \ldots \oplus \tilde{V}_{\lambda_{i_r}} \oplus \tilde{V}_{\lambda'_{i_{r+1}}} \oplus \ldots \oplus \tilde{V}_{\lambda'_{i_{s}}}$$

where $\lambda_{i_j}$'s are all of elements in $\Lambda_n$ such that $\lambda_{i_j} < \lambda$. We suppose that $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}$ are non self-conjugate and $\lambda'_{i_{r+1}}, \ldots, \lambda'_{i_{s}}$ are self-conjugate. We observe that $s$ is at most 1 because it is impossible to remove one box and add another box with keeping the self-conjugacy. We can write by induction,

$$\tilde{V}_\lambda = \tilde{V}_{\lambda'_{i_1}} \oplus \tilde{V}_{\lambda'_{i_2}} \oplus \ldots \oplus \tilde{V}_{\lambda'_{i_r}}$$
or

\[ \tilde{V}_\lambda = \tilde{V}_{\lambda'_{1}} \oplus \tilde{V}_{\lambda'_{2}} \oplus \ldots \oplus \tilde{V}_{\lambda'_{r}} \oplus \tilde{V}_{\lambda'_{r+1}}^+ \oplus \tilde{V}_{\lambda'_{r+1}}^- \]

with each subspace is irreducible \( A_{n-1}(q) \)-module and inequivalent each other. For \( \lambda', \tilde{\lambda}' \in \Lambda_{n-1} \) such that \( \lambda' < \lambda \) and \( \tilde{\lambda}' < \lambda \), there is exactly one \( \lambda'' \in \Lambda_{n-2} \) such that \( \lambda'' < \lambda' \) and \( \lambda'' < \tilde{\lambda}' \). Let \( T \in \mathrm{STab}(\lambda) \) be such that the tableau obtained from \( T \) by removing \( n \)-th box is shape \( \lambda' \) and the tableau obtained from \( T \) by removing \( n \)-th box and \( n - 1 \)-th box is shape \( \lambda'' \). Similarly, let \( \tilde{T} \in \mathrm{STab}(\lambda) \) be such that the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box is shape \( \tilde{\lambda}' \) and the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box and \( n - 1 \)-th box is shape \( \lambda'' \).

If \( \lambda' \) is non self-conjugate, then \( v_T \in \tilde{V}_{\lambda'_{r}} \), and we get,

\[ \tilde{\pi}_\lambda(y_{n-2})v_T = Av_T + Bv_{\tilde{T}} \]

Since \( q \neq 0 \) and \( q \) is not a \( k \)-th root of unity with \( 1 \leq k \leq n \), \( B \) is well-defined and nonzero. If \( \tilde{\lambda}' \) is non self-conjugate, then

\[ \tilde{\pi}_\lambda(z_{\lambda'})\tilde{\pi}_\lambda(y_{n-2})v_T = Bv_{\tilde{T}} \in \tilde{V}_{\lambda'} \]

If \( \tilde{\lambda}' \) is self-conjugate, then as the submodule decomposition

\[ \tilde{V}_{\lambda'} = \tilde{V}_{\lambda'}^+ \oplus \tilde{V}_{\lambda'}^- \]

we can write \( v_{\tilde{T}} \) as follows

\[ v_{\tilde{T}} = \frac{1}{2}(v_{\tilde{T}}^+ + v_{\tilde{T}}^-) + \frac{1}{2}(v_{\tilde{T}} - v_{\tilde{T}}^-) \]

where \( \tilde{T} \in \mathrm{STab}(\lambda) \) be such that the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box is shape \( \tilde{\lambda}' \) and the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box and \( n - 1 \)-th box is shape \( \tilde{\lambda}' \). \( v_{\tilde{T}}^+ + v_{\tilde{T}}^- \in \tilde{V}_{\lambda'}^+ \) and \( v_{\tilde{T}} - v_{\tilde{T}}^- \in \tilde{V}_{\lambda'}^- \). Hence

\[ \tilde{\pi}_\lambda(z_{\lambda'})\tilde{\pi}_\lambda(y_{n-2})v_T = B\left(\frac{1}{2}(v_{\tilde{T}}^+ + v_{\tilde{T}}^-) + \frac{1}{2}(v_{\tilde{T}} - v_{\tilde{T}}^-)\right) \]

If \( \lambda' \) is self-conjugate, then other Young diagrams \( \tilde{\lambda}' \in \Lambda_{n-1} \) with \( \tilde{\lambda}' < \lambda \) are non self-conjugate. We set \( v_T + v_{\tilde{T}} \in \tilde{V}_{\lambda'}^+ \) and \( v_T - v_{\tilde{T}} \in \tilde{V}_{\lambda'}^- \) where \( \tilde{T} \in \mathrm{STab}(\lambda) \) be such that the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box is shape \( \lambda' \) and the tableau obtained from \( \tilde{T} \) by removing \( n \)-th box and \( n - 1 \)-th box is shape \( \tilde{\lambda}' \). Then we get,

\[ \tilde{\pi}_\lambda(y_{n-2})(v_T + v_{\tilde{T}}) = Av_T + Bv_{\tilde{T}} + \tilde{\pi}_\lambda(y_{n-2})(v_{\tilde{T}}) \]
and elements in $\tilde{V}_\lambda$, do not appear in $\tilde{\pi}_\lambda(y_{n-2})(v_T)$. Hence,

$$\tilde{\pi}_\lambda(z_{\tilde{\lambda}})\tilde{\pi}_\lambda(y_{n-2})(v_T + v_T) = Bv_T \in \tilde{V}_\lambda,$$

is nonzero.

The same discussion is valid for $v_T - v_T \in \tilde{V}_\lambda^-$ and we get,

$$\tilde{\pi}_\lambda(z_{\tilde{\lambda}})\tilde{\pi}_\lambda(y_{n-2})(v_T - v_T) = Bv_T \in \tilde{V}_\lambda^-,$$

Because $\tilde{\lambda} \in \Lambda_{n-1}$ is arbitrary with $\tilde{\lambda} < \lambda$, every $\mathfrak{A}_n(q)$-submodule $W$ of $\tilde{V}_\lambda$ includes whole $\tilde{V}_\lambda$, therefore $\tilde{V}_\lambda$ is irreducible.

Let $\lambda \in \Lambda_n$ be self-conjugate. When we restrict $\tilde{V}_\lambda$ to $\mathfrak{A}_{n-1}(q)$-module, we can write (see [15] again),

$$\tilde{V}_\lambda = \tilde{V}_{\lambda''} \oplus \tilde{V}_{\lambda'1} \oplus \ldots \oplus \tilde{V}_{\lambda'r} \oplus \tilde{V}_{\lambda'1'r} \oplus \ldots \oplus \tilde{V}_{\lambda'1'r+s},$$

where $\lambda', \tilde{\lambda}' \in \Lambda_{n-1}$ such that $\lambda' < \lambda$ and $\tilde{\lambda}' < \lambda$. There is exactly one $\lambda'' \in \Lambda_{n-2}$ such that $\lambda'' < \lambda'$ and $\lambda'' < \tilde{\lambda}'$. Let $T \in \text{Stab}(\lambda)$ be such that the tableau obtained from $T$ by removing $n$-th box is shape $\lambda'$ and the tableau obtained from $T$ by removing $n$-th box and $n - 1$-th box is shape $\lambda''$. Similarly, let $\tilde{T} \in \text{Stab}(\lambda)$ be such that the tableau obtained from $\tilde{T}$ by removing $n$-th box is shape $\tilde{\lambda}'$ and the tableau obtained from $\tilde{T}$ by removing $n$-th box and $n - 1$-th box is shape $\lambda''$.

For $v_T + v_T \in \tilde{V}_{\lambda'}^+$,

$$\tilde{\pi}_\lambda(y_{n-2})(v_T + v_T) = Av_T + Bv_T + A\pi v_T + B\pi v_T$$

$$= A(v_T + v_T) + B(v_T + v_T)$$

where $B(v_T + v_T)$ is a nonzero element in $\tilde{V}_{\lambda'}^+$. Hence,

$$\tilde{\pi}_\lambda(z_{\tilde{\lambda}})\tilde{\pi}_\lambda(y_{n-2})(v_T + v_T) = B(v_T + v_T) \in \tilde{V}_{\lambda'}^+$$

Because $\tilde{\lambda}' \in \Lambda_{n-1}$ is arbitrary with $\tilde{\lambda}' < \lambda$, every $\mathfrak{A}_n(q)$-submodule $W$ of $\tilde{V}_{\lambda'}^+$ includes whole $\tilde{V}_{\lambda'}^+$, hence $\tilde{V}_{\lambda'}^+$ is irreducible. The same argument is valid for the case in $v_T - v_T \in \tilde{V}_{\lambda}^-$. Thus the irreducibilities of $\tilde{V}_{\lambda'}^+$ and $\tilde{V}_{\lambda}^-$ were proved.
Finally, we show inequivalencies of $\tilde{V}_\lambda$’s as $\mathfrak{A}_n(q)$-module. If $\lambda$ is self-conjugate, then there is at most one self-conjugate $\lambda'$ in $\Lambda_{n-1}$ or $\Lambda_{n-2}$ such that $\lambda' < \lambda$. In this case, $\tilde{V}_\lambda^+$ has a direct summand $\tilde{V}_{\lambda'}^+$ and does not have $\tilde{V}_\lambda^-$ as $\mathfrak{A}_{n-1}(q)$ nor $\mathfrak{A}_{n-2}(q)$-module, and $\tilde{V}_\lambda^-$ has a direct summand $\tilde{V}_{\lambda'}^-$ and does not have $\tilde{V}_\lambda^+$ as $\mathfrak{A}_{n-1}(q)$ nor $\mathfrak{A}_{n-2}(q)$-module. On the other hand, if $\mu$ is non self-conjugate, then $\tilde{V}_\mu$ has both $\tilde{V}_\lambda^+$ and $\tilde{V}_\lambda^-$ or has neither $\tilde{V}_\lambda^+$ nor $\tilde{V}_\lambda^-$. Hence, $\tilde{V}_\lambda^+$ and $\tilde{V}_\lambda^-$ and $\tilde{V}_\mu$ are mutually inequivalent.

Next, we consider the case that different Young diagrams $\lambda$ and $\mu$ are both self-conjugate or the case that different Young diagrams $\lambda$ and $\mu$ are both non self-conjugate with $\mathcal{N}\lambda \neq \mu$. In the former case, the set of all $\lambda' \in \Lambda_{n-1}$ such that $\lambda' < \lambda$ do not coincide with the set of all $\mu' \in \Lambda_{n-1}$ such that $\mu' < \mu$. Hence, $\tilde{V}_\lambda$ and $\tilde{V}_\mu$ are inequivalent already as $\mathfrak{A}_{n-1}(q)$-module. Since $\tilde{V}_\lambda^+ \cong \tilde{V}_\lambda^-$ as $\mathfrak{A}_{n-1}(q)$-module and $\tilde{V}_\lambda = \tilde{V}_\lambda^+ \oplus \tilde{V}_\lambda^-$ as $\mathfrak{A}_{n-1}(q)$-module, $\tilde{V}_\lambda^+$ and $\tilde{V}_\lambda^-$ are mutually inequivalent already as $\mathfrak{A}_{n-1}(q)$-module. In the latter case, the set of all $\lambda', \mathcal{N}\lambda' \in \Lambda_{n-1}$ such that $\lambda' < \lambda$ do not coincide with the set of all $\mu', \mathcal{N}\mu' \in \Lambda_{n-1}$ such that $\mu' < \mu$. Hence, $\tilde{V}_\lambda$ and $\tilde{V}_\mu$ are mutually inequivalent already as $\mathfrak{A}_{n-1}(q)$-module.

Finally, we shall show the multiplicity of the irreducible $\mathcal{H}_n(q)$-module $\tilde{V}_\mu$ in the induced module $\text{Ind}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(\tilde{V})$ where $\tilde{V}$ is an irreducible $\mathfrak{A}_n(q)$-module. We use the following Proposition known as Nakayama relation.

**Proposition 6.6.** If $S$ is a subring of a ring $R$, $M$ is a $R$-module and $N$ is a $S$-module then there is a natural homomorphism.

$$\text{Hom}_S(N, \text{Res}^R_S(M)) \cong \text{Hom}_R(\text{Ind}^R_S(N), M)$$

**Proof.** See for example, [1].

Using the Proposition 6.6 and Schur’s Lemma, We have the following.

**Proposition 6.7.** If $\lambda \in \Lambda_n$ is non self-conjugate then the following holds.

$$\text{Ind}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(\tilde{V}_\lambda) \cong V_\lambda \oplus V_\lambda$$

If $\lambda \in \Lambda_n$ is self-conjugate then the following holds.

$$\text{Ind}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(\tilde{V}_\lambda^+) \cong \text{Ind}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(\tilde{V}_\lambda^-) \cong V_\lambda$$

**Proof.** Directly from Proposition 6.6 we have

$$\text{Hom}_{\mathcal{H}_n(q)}(\text{Ind}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(\tilde{V}_\lambda), V_\mu) \cong \text{Hom}_{\mathfrak{A}_n(q)}(\tilde{V}_\lambda, \text{Res}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(V_\mu))$$

and the Schur’s Lemma means that its dimension is the multiplicity of $\tilde{V}_\lambda$ in the restriction $\text{Res}_{\mathfrak{A}_n(q)}^{\mathcal{H}_n(q)}(V_\mu)$. Multiplicities of irreducible $\mathfrak{A}_n(q)$-modules in the restrictions of irreducible $\mathcal{H}_n(q)$-modules are already shown in Proposition 6.2 and Theorem 6.5 for the non self-conjugate case and in Proposition 6.3 and Theorem 6.5 for the self-conjugate case. 

□
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