THE HADRONIC COUPLING CONSTANTS OF THE LOWEST HIDDEN-CHARM PENTAQUARK STATE WITH THE QCD SUM RULES IN SOLID QUARK-HADRON DUALITY

Zhi-Gang Wang\textsuperscript{1}, Hui-Juan Wang, Qi Xin
Department of Physics, North China Electric Power University, Baoding 071003, P. R. China

Abstract

In this article, we illustrate how to calculate the hadronic coupling constants of the pentaquark states with the QCD sum rules based on solid quark-hadron quality, then study the hadronic coupling constants of the lowest diquark-diquark-antiquark type hidden-charm pentaquark state with the spin-parity \( J^P = \frac{3}{2}^- \) in details, and calculate the partial decay widths. The total width \( \Gamma(P_{c}) = 14.32 \text{MeV} \) is compatible with the experimental value \( \Gamma(P_{c}(4410)) = 9.8 \pm 2.7 \pm 3.5 \text{MeV} \) from the LHCb collaboration, and favors assigning the \( P_{c}(4312) \) to be the \([ud][uc]\) pentaquark state with the \( J^P = \frac{1}{2}^+ \). The hadronic coupling constants have the relation \(|G_{P_D-\Sigma^{++}}|^2 = 2G_{P_D-\Sigma^+}^2 \gg |G_{P_D-\Lambda^+}|^2\), and favor the hadronic dressing mechanism. The \( P_{c}(4312) \) maybe have a diquark-diquark-antiquark type pentaquark core with the typical size of the \( qqg \) baryon states, the strong couplings to the meson-baryon pairs \( \bar{D}\Sigma \), lead to some pentaquark molecule components, and the \( P_{c}(4312) \) maybe spend a large time as the \( \bar{D}\Sigma_c \) molecular state.

PACS number: 12.39.Mk, 14.20.Lq, 12.38.Lg
Key words: Pentaquark states, QCD sum rules

1 Introduction

In 2015, the LHCb collaboration observed two pentaquark or pentaquark molecule candidates \( P_{c}(4380) \) and \( P_{c}(4450) \) in the \( J/\psi p \) invariant mass spectrum in the \( \Lambda_c^0 \rightarrow J/\psi K^- p \) decays \cite{1}. In 2019, also in the \( J/\psi p \) invariant mass spectrum, the LHCb collaboration observed a new narrow pentaquark or pentaquark molecule candidate \( P_{c}(4312) \) and confirmed the old structure \( P_{c}(4450) \), which consists of two narrow overlapping peaks \( P_{c}(4440) \) and \( P_{c}(4457) \) \cite{2}. There have been several possible interpretations for the nature of the \( P_{c}(4312) \), \( P_{c}(4440) \) and \( P_{c}(4457) \), such as the pentaquark molecular states \( \bar{D}\Sigma \), \( \bar{D}\Sigma_c \), compact diquark-diquark-antiquark type pentaquark states or diquark-triquark type pentaquark states \( \bar{D}\Sigma \), \( \bar{D}\Sigma_c \), color-octet-color-octet type pentaquark states \( \bar{D}\Sigma \), hadrocharmonium pentaquark states \( \bar{D}\Sigma_c \), etc.

In Refs.\cite{3, 4}, we perform comprehensive investigations of the spin-parity \( J^P = \frac{1}{2}^+, \frac{3}{2}^+ \) and \( \frac{5}{2}^+ \) diquark-diquark-antiquark type hidden-charm pentaquark states with the QCD sum rules by carrying out the operator product expansion up to the vacuum condensates of dimension 10 in a consistent way, and reproduce the experimental values of the masses of the \( P_{c}(4380) \) and \( P_{c}(4450) \) as the compact pentaquark states with the spin-parity \( J^P = \frac{1}{2}^+ \) and \( \frac{3}{2}^+ \), respectively. Furthermore, we obtain the lowest masses 4.29 \( \pm \) 0.13 GeV and 4.30 \( \pm \) 0.13 GeV for the scalar-diquark–scalar-diquark-antiquark type and scalar-diquark-axialvector-diquark-antiquark type hidden-charm pentaquark states with the spin-parity \( J^P = \frac{1}{2}^- \), respectively \cite{3}, which are consistent with the mass of the \( P_{c}(4312) \) observed later by the LHCb collaboration \cite{2}. Then we update the analysis by taking into account the vacuum condensates up to dimension 13 in a consistent way \cite{11}, and obtain more flatter platforms and better predictions. The new analysis indicates that the lowest scalar-diquark–scalar-diquark-antiquark type and axialvector-diquark-diquark-antiquark type compact hidden-charm pentaquark states with the spin-parity \( J^P = \frac{1}{2}^- \) have the masses 4.31 \( \pm \) 0.11 GeV and 4.34 \( \pm \) 0.14 GeV, respectively, which are all consistent with the mass of the \( P_{c}(4312) \).

On the other hand, in Ref.\cite{4}, we perform detailed investigations of the \( \bar{D}\Sigma_c, \bar{D}\Sigma_c^*, \bar{D}\Sigma_c, \bar{D}\Sigma_c^* \) pentaquark molecular states with the QCD sum rules by carrying out the operator product

\textsuperscript{1}E-mail: zgwang@aliyun.com.
expansion up to the vacuum condensates of dimension 13 in a consistent way. The theoretical predications favor assigning the $P_c(4312)$ to be the $D\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{1}{2}^-$, assigning the $P_c(4380)$ to be the $\bar{D}\Sigma^*_c$ pentaquark molecular state with the spin-parity $J^P = \frac{3}{2}^-$, and assigning the $P_c(4440/4457)$ to be the $\bar{D}^*\Sigma_c$ pentaquark molecular state with the spin-parity $J^P = \frac{3}{2}^-$ or the $\bar{D}^*\Sigma^*_c$ pentaquark molecular state with the spin-parity $J^P = \frac{5}{2}^-$. It is odd that the experimental values of the masses of the $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ can be reproduced both in the scenarios of the pentaquark states and pentaquark molecular states with the QCD sum rules. In fact, the diquark-diquark-antiquark type local pentaquark current with definite quantum numbers couples potentially to a definite compact pentaquark state, while this local current can be re-arranged into a special superposition of a series of color-singlet-color-singlet type currents, which couple potentially to the pentaquark molecular states or meson-baryon scattering states with the same quantum numbers \[11\]. We cannot exclude the possibility that a pentaquark state and a pentaquark molecular state with the same quantum numbers have about the same masses.

We can borrow some ideas from the nature of the light flavor scalar mesons, which provide a subject of an intense and continual controversy in establishing the meson spectrum, the more elusive things are the quark configurations of the $f_0(980)$ and $a_0(980)$, which have almost the degenerate masses. In the scenario of the hadronic dressing mechanism, the scalar mesons $f_0(980)$ and $a_0(980)$ have small or large $q\bar{q}$ cores of the typical $q\bar{q}$ meson size, or large $[qq][\bar{q}\bar{q}]_3$ cores in the relative S-wave with some $q\bar{q}$ components in the relative P-wave, the bare $q\bar{q}$ or $[qq][\bar{q}\bar{q}]_3$ cores are dressed by the hadronic interactions with the pseudoscalar mesons, the strong couplings to the hadronic channels enrich the pure $q\bar{q}$ or $[qq][\bar{q}\bar{q}]_3$ states with other components and spend part or most part of their lifetime as the virtual $K^+K^-$ or $K^0\bar{K}^0$ states \[13\] \[16\] \[17\] \[18\]. The QCD sum rules indicate that the nonet scalar mesons below 1 GeV are the two-quark-tetraquark mixing states with large or small two-quark components \[19\] \[20\]. Without introducing mixing effects in one way or the other, it is difficult to reproduce the experimental values of the masses of the nonet scalar mesons below 1 GeV \[21\] \[22\]. In summary, the QCD sum rules favor the hadronic dressing mechanism \[18\] \[19\] \[20\].

The hadronic dressing mechanism also works in interpreting the exotic $X$, $Y$ and $Z$ states. In Ref.\[23\], we choose the $[sc]_P[\bar{s}\bar{c}]_A - [sc]_A[\bar{s}\bar{c}]_P$ type tetraquark current to study the strong decays of the $Y(4660)$ with the QCD sum rules based on solid quark-hadron quality. The numerical values indicate that the hadronic coupling constants $|G_{Y\psi'}f_0| \gg |G_{YJ/\psi}f_0|$, which is consistent with the fact that the $Y(4660)$ is observed in the $\psi'\pi^+\pi^-$ invariant mass distribution, and favors the $\psi'f_0(980)$ molecule assignment \[23\]. Similar mechanism maybe exist for the pentaquark states and pentaquark molecular states.

In the article, we study the hadronic coupling constants of the lowest scalar-diquark-scalar-diquark-antiquark type hidden-charm pentaquark state with the QCD sum rules based on the solid quark-hadron duality, and study its two-body strong decays, and examine the hadronic dressing mechanism for the compact pentaquark states, and compromise the scenarios of the pentaquark states and pentaquark molecular states.

The article is arranged as follows: in Sect.2, we illustrate how to calculate the hadronic coupling constants of the hidden-charm pentaquark states with the QCD sum rules based on the solid quark-hadron quality; in Sect.3, we derive the QCD sum rules for the hadronic coupling constants of the lowest hidden-charm pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$; in Sect.4, we present the numerical results and discussions; and Sect.5 is reserved for our conclusion.
2 The hadronic coupling constants of the hidden-charm pentaquark states

In this section, we illustrate how to calculate the hadronic coupling constants of the hidden-charm pentaquark states with the QCD sum rules. Firstly, let us write down the three-point correlation functions $\Pi(p,q)$,

$$\Pi(p,q) = i^2 \int d^4x d^4y e^{iqy} \langle 0 | T \{ J_M(x) J_B(y) \bar{J}_P(0) \} | 0 \rangle,$$  \hspace{1cm} (1)

where $J_P(0) = J_P^0(0)\gamma^0$, the current $J_P(0)$ interpolates the hidden-charm pentaquark state $P_c$, the $J_M(x)$ and $J_B(y)$ interpolate the conventional meson $M$ and baryon $B$, respectively,

$$\langle 0 | J_P(0) | P_c(p') \rangle = \lambda_P U(p',s),$$

$$\langle 0 | J_M(0) | M(p) \rangle = \lambda_M,$$

$$\langle 0 | J_B(0) | B(q) \rangle = \lambda_B U(q,s),$$  \hspace{1cm} (2)

the $\lambda_A$, $\lambda_B$ and $\lambda_C$ are the pole residues or decay constants, the $U(p',s)$ and $U(q,s)$ are the Dirac spinors.

At the hadron side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $\bar{J}_P(0)$, $J_M(x)$, $J_B(y)$ into the three-point correlation functions $\Pi(p,q)$ and isolate the ground state contributions to obtain the result \cite{25,26},

$$\Pi(p,q) = -i \lambda_P \lambda_M \lambda_B \frac{(\not{p} + m_B) G_{PMB} \Gamma (\not{p'} + m_P)}{(m_M^2 - p^2)(m_B^2 - q^2)(m_P^2 - p'^2)} + \cdots,$$  \hspace{1cm} (3)

where $p' = p + q$, the $G_{PMB}$ are the hadronic coupling constants defined by

$$\langle M(p) B(q) | P_c(p') \rangle = G_{PMB} U(q) \Gamma U(p'),$$  \hspace{1cm} (4)

the $\Gamma$ are some Dirac $\gamma$-matrixes.

In the QCD sum rules, we take the quark-hadron duality to match the hadron representation with the QCD representation of the correlation functions,

$$\Pi_H(p,q) = \Pi_{QCD}(p,q),$$  \hspace{1cm} (5)

where we add the subscripts $H$ and $QCD$ to denote the hadron side and QCD side of the correlation functions, respectively. Let us write}

3
down the components $\Pi_H(p^2, p^2, q^2)$ explicitly,

$$
\Pi_H(p^2, p^2, q^2) = \frac{\lambda_P \lambda_M \lambda_B G_{PMB}}{(m^2_M - p^2)(m^2_B - q^2)(m^2_P - p^2)}
+ \frac{1}{(m^2_M - p^2)(m^2_P - p^2)} \int_{s^0_p}^{s^0_P} \int_{t^0}^{s^0_M} \int_{u^0}^{s^0_B} \frac{d^3 t \rho_{PB}(p^2, p^2, t)}{t - q^2}
+ \frac{1}{(m^2_B - q^2)(m^2_P - p^2)} \int_{s^0_p}^{s^0_P} \int_{t^0}^{s^0_M} \int_{u^0}^{s^0_B} \frac{d^3 t \rho_{PM}(p^2, t, q^2)}{t - p^2}
+ \frac{1}{(m^2_M - p^2)(m^2_B - q^2)} \int_{s^0_p}^{s^0_P} \int_{t^0}^{s^0_M} \int_{u^0}^{s^0_B} \frac{d^3 t \rho_{PM}(p^2, t, q^2) + \rho_{PB}(p^2, p^2, t)}{t - p^2}
+ \cdots , \tag{8}
$$

the four functions $\rho_{PB}(p^2, p^2, t), \rho_{PM}(p^2, t, q^2)$, $\rho_{PM}(t', p^2, q^2)$ and $\rho_{PB}(t', p^2, q^2)$ parameterize

the complex couplings or transitions between the ground states and the higher resonances or the continuum states.

We rewrite the correlation functions $\Pi_H(p^2, p^2, q^2)$ at the hadron side as

$$
\Pi_H(p^2, p^2, q^2) = \int_{(m_M + m_B)^2}^{s^0_P} \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} du \rho_H(s', s, u) \frac{\rho_H(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)}
+ \int_{s^0_p}^{s^0_P} \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} du \rho_H(s', s, u) \frac{\rho_H(s', s, u)}{(s - p^2)(u - q^2) + \cdots ,} \tag{9}
$$

through triple dispersion relation, where the $\rho_H(s', s, u)$ are the hadronic spectral densities,

$$
\rho_H(s', s, u) = \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{\text{Im} \rho_H(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3} , \tag{10}
$$

where the $\Delta_u^2$ and $\Delta_u^2$ are the thresholds, the $s^0_P, s^0_M, s^0_B$ are the continuum thresholds.

Now we carry out the operator product expansion at the QCD side in the deep Euclidean region $P^2 = -p^2 \gg \Lambda^2_{QCD}$ and $Q^2 = -q^2 \gg \Lambda^2_{QCD}$. However, we cannot write the correlation functions $\Pi_{QCD}(p^2, p^2, q^2)$ in the form,

$$
\Pi_{QCD}(p^2, p^2, q^2) = \int_{(m_M + m_B)^2}^{s^0_P} \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} du \rho_{QCD}(s', s, u) \frac{\rho_{QCD}(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)}
+ \int_{s^0_p}^{s^0_P} \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} du \rho_{QCD}(s', s, u) \frac{\rho_{QCD}(s', s, u)}{(s - p^2)(u - q^2) + \cdots ,} \tag{11}
$$

through triple dispersion relation analogously, because the QCD spectral densities $\rho_{QCD}(s', s, u)$ cannot exist,

$$
\rho_{QCD}(s', s, u) = \lim_{\epsilon_3 \to 0} \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{\text{Im} \rho_{QCD}(s' + i\epsilon_3, s + i\epsilon_2, u + i\epsilon_1)}{\pi^3} = 0 , \tag{12}
$$

and have to write the correlation functions $\Pi_{QCD}(p^2, p^2, q^2)$ in the form,

$$
\Pi_{QCD}(p^2, p^2, q^2) = \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} \int_{-\Delta_u^2}^{\Delta_u^2} du \rho_{QCD}(p^2, s, u) \frac{\rho_{QCD}(p^2, s, u)}{(s - p^2)(u - q^2) + \cdots ,} \tag{13}
$$

through double dispersion relation, where the $\rho_{QCD}(p^2, s, u)$ are the QCD spectral densities,

$$
\rho_{QCD}(p^2, s, u) = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \frac{\text{Im} \rho_{QCD}(p^2 + i\epsilon_2, s + i\epsilon_1)}{\pi^2} . \tag{14}
$$
Henceforth we will write the QCD spectral densities \( \rho_{QCD}(p^2, s, u) \) as \( \rho_{QCD}(s, u) \) for simplicity.

As the duality below the three continuum thresholds \( s_M^0, s_B^0 \) and \( s_B^0 \) cannot exist simultaneously,

\[
\int_{(m_M + m_B)^2}^{s_B^0} ds' \int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \frac{\rho_H(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)} \neq \int_{(m_M + m_B)^2}^{s_B^0} ds' \int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \frac{\rho_{QCD}(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)},
\]

we match the hadron side with the QCD side of the correlation functions \( \Pi(p^2, p^2, q^2) \) below two continuum thresholds \( s_M^0 \) and \( s_B^0 \) simultaneously, and carry out the integral over \( ds' \) firstly to obtain the solid duality \[23, 27, 28, 29\],

\[
\int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \frac{1}{(s - p^2)(u - q^2)} \left[ \int_{\Delta_2^0}^{s_B^0} ds' \frac{\rho_H(s', s, u)}{s' - p^2} \right] = \int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \frac{\rho_{QCD}(s, u)}{s - p^2}(u - q^2),
\]

where \( \Delta_2^0 = (m_M + m_B)^2 \). Now let us write down the quark-hadron duality explicitly,

\[
\int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} = \int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \int_{\Delta_2^0}^{s_M^0} ds' \frac{\rho_H(s', s, u)}{(s' - p^2)(s - p^2)(u - q^2)}
\]

\[
= \lambda_P \lambda_M \lambda_B G_{PMB} \frac{(m_M^2 - p^2)(m_B^2 - p^2)(m_B^2 - q^2)}{(m_B^2 - p^2)(m_B^2 - q^2)} + C_{P'M} + C_{P'B},
\]

where we introduce the parameters \( C_{P'M} \) and \( C_{P'B} \) to parameterize the net effects,

\[
C_{P'M} = \int_{p'_{\perp}}^{s_M^0} dt \frac{\rho_{P'M}(t, p'^2, q^2)}{t - p'^2},
\]

\[
C_{P'B} = \int_{p'_{\perp}}^{s_B^0} dt \frac{\rho_{P'B}(t, p'^2, q^2)}{t - p'^2}.
\]

From Eq.\[17\], we can see that the duality below the continuum thresholds \( s_M^0 \) and \( s_B^0 \) is solid.

In numerical calculations, we can take the unknown functions \( C_{P'M} \) and \( C_{P'B} \) as free parameters, and choose the suitable values to delete the contaminations of the higher resonances and continuum states to obtain the stable QCD sum rules. The parameters \( C_{P'M} \) and \( C_{P'B} \) are not necessary to be constants, they may depend on the Borel parameters, as there are complex interactions or transitions between the ground states and the higher resonances or the continuum states, after the double Borel transform, some net Borel parameter dependence may be induced.

If the \( M \) is a charmonium or bottomonium state and the \( B \) is a light flavor baryon state, we set \( p'^2 = p^2 \) and perform the double Borel transform with respect to the variables \( P^2 = -p^2 \) and \( Q^2 = -q^2 \) respectively to obtain the QCD sum rules,

\[
\frac{\lambda_P \lambda_M \lambda_B G_{PMB}}{m_P^2 - m_M^2} \left[ \exp \left( -\frac{m_M^2}{T_1^2} \right) - \exp \left( -\frac{m_P^2}{T_1^2} \right) \right] \exp \left( \frac{m_B^2}{T_2^2} \right) + (C_{P'M} + C_{P'B}) \exp \left( -\frac{m_M^2}{T_1^2} - \frac{m_B^2}{T_2^2} \right) = \int_{\Delta_2^0}^{s_M^0} ds \int_{\Delta_2^0}^{s_B^0} du \rho_{QCD}(s, u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

where the \( T_1^2 \) and \( T_2^2 \) are the Borel parameters. If the \( M \) is a heavy meson and the \( B \) is a heavy baryon state, we set \( p'^2 = 4q^2 \) and perform the double Borel transform with respect to the variables
\[ P^2 = -p^2 \text{ and } Q^2 = -q^2 \] respectively to obtain the QCD sum rules,
\[
\frac{\lambda_{\lambda M} \lambda_{\lambda B} G_{PMB}}{4 (m_{\bar{P}}^2 - m_B^2)} \left[ \exp \left( -\frac{m_B^2}{T_2^2} \right) - \exp \left( -\frac{\bar{m}_{\bar{P}}^2}{T_2^2} \right) \right] \exp \left( -\frac{m_{\bar{B}}^2}{T_1^2} \right) + \\
(C_{P' M} + C_{P' B}) \exp \left( -\frac{m_B^2}{T_2^2} - \frac{m_{\bar{M}}^2}{T_2^2} \right) = \int_{\Delta_s^2}^{\infty} ds \int_{\Delta_s^2}^{\infty} du \rho_{QCD}(s, u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right) ,
\]

where \( \bar{m}_{\bar{P}} = \frac{m_{\bar{P}}}{2} \).

### 3 QCD sum rules for the hadronic coupling constants of the lowest hidden-charm pentaquark state with \( J^P = \frac{1}{2}^- \)

In the following, we write down the three-point correlation functions \( \Pi(p, q) \) and \( \Pi_\mu(p, q) \) in the QCD sum rules,
\[
\Pi(p, q) = i^2 \int d^4 x d^4 y e^{i p x - i q y} \langle 0 | \{ J_M(x) J_B(y) \bar{J}_P(0) \} | 0 \rangle , \tag{21}
\]
\[
\Pi_\mu(p, q) = i^2 \int d^4 x d^4 y e^{i p x - i q y} \langle 0 | \{ J_\mu(x) J_N(y) \bar{J}_P(0) \} | 0 \rangle , \tag{22}
\]

where \( J_M(x) = J_{\nu_e}(x), J_{D^0}(x), J_{D^-}(x), J_B(y) = J_{\Lambda^+}(y), J_{\Sigma^+_c}(y), J_{\Sigma^+_c}(y), J_N(y), \)
\[
J_{\nu_e}(x) = \bar{c}(x) \gamma_5 c(x) , \\
J_{D^0}(x) = \bar{c}(x) \gamma_5 u(x) , \\
J_{D^-}(x) = \bar{c}(x) i \gamma_5 d(x) , \\
J_\mu(x) = \bar{c}(x) \gamma_\mu c(x) , \tag{23}
\]
\[
J_{\Lambda^+}(y) = \epsilon^{ijk} u_1^T(y) C \gamma_5 d_j(y) c_k(y) , \\
J_{\Sigma^+_c}(y) = \epsilon^{ijk} u_1^T(y) C \gamma_\alpha u_j(y) \gamma^\alpha \gamma_5 c_k(y) , \\
J_{\Sigma^+_c}(y) = \epsilon^{ijk} u_1^T(y) C \gamma_\alpha c_j(y) \gamma^\alpha \gamma_5 d_k(y) , \\
J_N(y) = \epsilon^{ijk} u_1^T(y) C \gamma_\alpha u_j(y) \gamma^\alpha \gamma_5 d_k(y) , \tag{24}
\]
\[
J_P(0) = \epsilon^{i k} \epsilon^{j n m} u_j^T(0) C \gamma_5 d_k(0) u_m^T(0) C \gamma_5 c_n(0) C \bar{c}_a^T(0) , \tag{25}
\]

the \( a, i, j, \cdots \) are color indices. We choose the currents \( J_{\nu_e}(x), J_{D^0}(x), J_{D^-}(x), J_\mu(x), J_{\Lambda^+}(y), J_{\Sigma^+_c}(y), J_{\Sigma^+_c}(y), J_N(y) \) and \( J_P(0) \) to interpolate the \( \eta_c, \bar{D}^0, D^-, J/\psi, \Lambda^+_c, \Sigma^+_c, \Sigma^+_c, \) and \( P_c \), respectively. Henceforth we will write the proton as \( N \) instead of \( p \) to avoid confusing with the four momentum \( \mu_\rho \).

At the hadron side, we insert a complete set of intermediate hadron states with the same quantum numbers as the current operators \( J_{\nu_e}(x), J_{D^0}(x), J_{D^-}(x), J_\mu(x), J_{\Lambda^+}(y), J_{\Sigma^+_c}(y), J_{\Sigma^+_c}(y), J_N(y) \) and \( J_P(0) \) into the correlation functions \( \Pi(p, q) \) and \( \Pi_\mu(p, q) \) respectively to obtain the hadronic representation \( 23, 26 \), then we isolate the ground state contributions and write them explicitly,
\[
\Pi_{\nu_e, N}(p, q) = \frac{f_{\nu_e} m_{\bar{P}}^2 \lambda_{\nu_e} \lambda_N}{2 m_c} \frac{-i (q + m_N) (q' + m_P)}{(m_{\bar{P}}^2 - p^2) (m_{\bar{P}}^2 - q^2) (m_N^2 - q^2)} G_{\nu_e, N} + \cdots , \tag{26}
\]
\[ \Pi_{PD^0\Lambda^+}(p, q) = \frac{f_D m_D^2 \lambda_p \lambda_{\Lambda_c}}{m_c} \frac{-i (\not{q} + m_{\Lambda_c}) (\not{q'} + m_{\Lambda_c})}{(m_D^2 - p^2) (m_{\Lambda_c}^2 - q^2)} G_{PD^0\Lambda^+} + \cdots, \]  
\[ \Pi_{PD^0\Sigma_c^+}(p, q) = \frac{f_D m_D^2 \lambda_p \lambda_{\Sigma_c^+}}{m_c} \frac{-i (\not{q} + m_{\Sigma_c}) (\not{q'} + m_{\Sigma_c})}{(m_D^2 - p^2) (m_{\Sigma_c}^2 - q^2)} G_{PD^0\Sigma_c^+} + \cdots, \]  
\[ \Pi_{PD^-\Sigma_c^{++}}(p, q) = \frac{f_D m_D^2 \lambda_p \lambda_{\Sigma_c^{++}}}{m_c} \frac{-i (\not{q} + m_{\Sigma_c}) (\not{q'} + m_{\Sigma_c})}{(m_D^2 - p^2) (m_{\Sigma_c}^2 - q^2)} G_{PD^-\Sigma_c^{++}} + \cdots, \]  
\[ \Pi_{\mu}(p, q) = f_{J/\psi} m_{J/\psi} \lambda_p \lambda_N \langle - (\not{q} + m_N) \rangle \frac{G_V \gamma^\alpha - i \frac{G_T}{m_p + m_N} \sigma^\beta \not{p}}{(m_D^2 - p^2) (m_{J/\psi}^2 - p^2) (m_N^2 - q^2)} \left( -g_{\mu\alpha} + \frac{p_{\mu} p_{\alpha}}{p^2} \right) + \cdots, \]  
where we introduce the subscripts \( P_{q,N}, PD^0\Lambda^+, PD^0\Sigma_c^+ \) and \( PD^-\Sigma_c^{++} \) in the correlation functions \( \Pi(p, q) \) to distinguish the corresponding hadronic coupling constants, and have taken the standard definitions for the pole residues or decay constants \( \lambda_p, \lambda_N, \lambda_{\Lambda_c}, \lambda_{\Sigma_c}, f_{\eta_c}, f_D, f_{J/\psi}, \)  
\[ \langle 0 | J(0) | P_c(p') \rangle = \lambda_p U(p', s), \]
\[ \langle 0 | J_N(0) | N(q) \rangle = \lambda_N U(q, s), \]
\[ \langle 0 | J_{\Lambda_c}(0) | \Lambda_c(q) \rangle = \lambda_{\Lambda_c} U(q, s), \]
\[ \langle 0 | J_{\Sigma_c}(0) | \Sigma_c(q) \rangle = \lambda_{\Sigma_c} U(q, s), \]
\[ \langle 0 | J_{\eta_c}(0) | \eta_c(p) \rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2 m_c}, \]
\[ \langle 0 | J_D(0) | D(p) \rangle = \frac{f_D m_D^2}{m_c}, \]
\[ \langle 0 | J_{\mu}(0) | J/\psi(p) \rangle = f_{J/\psi} m_{J/\psi} \varepsilon_\mu(p, s), \]
and the hadronic coupling constants \( G_{P_{q,N}}, G_{PD^0\Lambda^+}, G_{PD^0\Sigma_c^+}, G_{PD^-\Sigma_c^{++}}, G_V \) and \( G_T, \)
\[ \langle \eta_c(p) | N(q) | P_c(p') \rangle = G_{P_{q,N}} \bar{U}(q) U(p'), \]
\[ \langle \bar{D}^0(p) | \Lambda_c^+(q) | P_c(p') \rangle = G_{PD^0\Lambda_c^+} \bar{U}(q) U(p'), \]
\[ \langle \bar{D}^0(p) | \Sigma_c^+(q) | P_c(p') \rangle = G_{PD^0\Sigma_c^+} \bar{U}(q) U(p'), \]
\[ \langle D^- (p) | \Sigma_c^{+++}(q) | P_c(p') \rangle = G_{PD^-\Sigma_c^{++}} \bar{U}(q) U(p'), \]
\[ \langle J/\psi(p) N(q) | P_c(p') \rangle = -i \bar{U}(q) \varepsilon_5^\alpha \left( G_V \gamma^\alpha - i \frac{G_T}{m_p + m_N} \sigma^\beta \not{p} \right) \gamma_5 U(p'), \]
the \( U(p', s), U(p, s) \) and \( U(q, s) \) are the Dirac spinors, and the \( \varepsilon_\mu \) is the polarization vector of the \( J/\psi. \)

In this article, we choose \( \Gamma' = \sigma_{\mu\nu}, \gamma_5 \not{f}, \gamma_5 \) in Eq. (31), and carry out the traces in the Dirac spinor space,
\[ \frac{1}{4} \text{Tr} \left[ \Pi_H (p, q) \sigma_{\mu\nu} \right] = \Pi_H (p^2, p^2, q^2) (p_\mu q_\nu - q_\mu p_\nu) + \cdots, \]
\[ \frac{1}{4} \text{Tr} \left[ \Pi_H^H (p, q) \gamma_5 \not{f} \right] = \Pi_H^H (p^2, p^2, q^2) q_\mu p \cdot z + \cdots, \]
\[ \frac{1}{4} \text{Tr} \left[ \Pi_H^H (p, q) \gamma_5 \right] = \Pi_H^H (p^2, p^2, q^2) q_\mu + \cdots, \]
and choose the tensor structures \( p_\mu q_\nu - q_\mu p_\nu \), \( q_\mu p \cdot z \) and \( q_\mu \) to study the hadronic coupling constants, where the \( z_\mu \) is a four vector. We neglect the explicit expressions of the correlation functions \( \Pi_H(p^2, p^2, q^2) \), \( \Pi_{1H}^1(p^2, p^2, q^2) \) and \( \Pi_{1H}^2(p^2, p^2, q^2) \) at the hadron side for simplicity.

At the QCD side of the correlation functions, we carry out the operator product expansion up to the vacuum condensates of dimension-10, the interval of the vacuum condensates is large enough to obtain stable QCD sum rules. Moreover, we assume vacuum saturation for the higher dimensional vacuum condensates. As the vacuum condensates are vacuum expectations of the quark-gluon operators, we take the truncations \( \mathcal{O}(a_s^n) \) with \( n \leq 10 \) and \( k \leq 1 \) in a consistent way, and write (components of) the correlation functions \( \Pi_{QCD}(p^2, p^2, q^2) \) as

\[
\Pi_{QCD}(p^2, p^2, q^2) = \int_{\Delta_2^2}^{s_D^{2}} ds \int_{\Delta_2^2}^{q_D^{2}} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} + \cdots, \tag{34}
\]

through double dispersion relation, where the \( \Pi_{QCD}(p^2, p^2, q^2) \) represent the corresponding correlation functions of the \( \Pi_H(p^2, p^2, q^2) \), \( \Pi_{1H}^1(p^2, p^2, q^2) \) and \( \Pi_{1H}^2(p^2, p^2, q^2) \) at the QCD side collectively.

Now we match the hadron side with the QCD side of the correlation functions \( \Pi(p^2, p^2, q^2) \), and carry out the integral over \( ds' \) firstly to obtain the solid duality, then write down the quark-hadron duality explicitly,

\[
\int_{4m_c^2}^{s_D^{2}} ds \int_{0}^{m_c^2} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} = \frac{f_{J/\psi} m_{J/\psi}^{2} \lambda_{P} \lambda_{N}}{2 m_c} \frac{G_{P\psi N}}{(m_p - p^2)(m_{J/\psi} - p^2)(m_{J/\psi} - q^2)} + \cdots,
\]

\[
\int_{m_c^2}^{s_D^{2}} ds \int_{m_c^2}^{q_D^{2}} du \frac{\partial^0 \lambda_{P} \lambda_{N}}{(s - p^2)(u - q^2)} = \frac{f_{D} m_{D}^{2} \lambda_{P} \lambda_{N}}{m_c} \frac{G_{P\psi N}}{(m_p - p^2)(m_{D} - p^2)(m_{D} - q^2)} + \cdots,
\]

\[
\int_{m_c^2}^{s_D^{2}} ds \int_{m_c^2}^{q_D^{2}} du \frac{\partial^0 \Sigma^{+}}{(s - p^2)(u - q^2)} = \frac{f_{D} m_{D}^{2} \lambda_{P} \lambda_{N}}{m_c} \frac{G_{P\psi N}}{(m_p - p^2)(m_{D} - p^2)(m_{D} - q^2)} + \cdots,
\]

\[
\int_{m_c^2}^{s_D^{2}} ds \int_{m_c^2}^{q_D^{2}} du \frac{\partial^{-} \Sigma^{++}}{(s - p^2)(u - q^2)} = \frac{f_{D} m_{D}^{2} \lambda_{P} \lambda_{N}}{m_c} \frac{G_{P\psi N}}{(m_p - p^2)(m_{D} - p^2)(m_{D} - q^2)} + \cdots,
\]

\[
\int_{4m_c^2}^{s_D^{2}} ds \int_{0}^{q_D^{2}} du \frac{\rho_{QCD}(s, u)}{(s - p^2)(u - q^2)} = \frac{f_{J/\psi} m_{J/\psi}^{2} \lambda_{P} \lambda_{N}}{m_p - p^2} \frac{G_{T} - G_{V}}{m_{J/\psi}^2 - p^2(m_{J/\psi}^2 - q^2)} + \cdots.
\]
\[
\int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \frac{\rho_{QCD}(s,u)}{(s-p^2)(u-q^2)} = f_{J/\psi} m_{J/\psi} \lambda_p \lambda_N \frac{(m_P - m_N) G_V - G_T \frac{m_{J/\psi}^2}{m_P + m_N}}{(m_P^2 - p^2)(m_{J/\psi}^2 - p^2)(m_N^2 - q^2)}
\]

\[
+ \frac{C_{P^*J/\psi,2} + C_{P^*N,2}}{(m_{J/\psi}^2 - p^2)(m_N^2 - q^2)},
\]

where the parameters \(C_{P^*n_c} + C_{P^*N}, C_{P^*\bar{D}0} + C_{P^*\Lambda^+_c}, C_{P^*D^0} + C_{P^*\Sigma^+_c}, C_{P^*D^*} + C_{P^*\Pi^*_c}, C_{P^*J/\psi,1} + C_{P^*J/\psi,2} + C_{P^*N,2}\) are defined according to Eq. (18).

We perform double Borel transform with respect to the variables \(P^2 = -p^2\) and \(Q^2 = -q^2\) respectively according to Eqs. (19)- (20) to obtain the QCD sum rules,

\[
\frac{f_{n_c} m_{n_c} \lambda_p \lambda_N}{2m_c} \frac{G_{Pn_cN}}{m_P^2 - m_{n_c}^2} \left[ \exp \left( -\frac{m_{n_c}^2}{T_1^2} - \frac{m_{n_c}^2}{T_2^2} \right) - \exp \left( -\frac{m_P^2}{T_1^2} - \frac{m_P^2}{T_2^2} \right) \right] \exp \left( -\frac{m_N^2}{T_2^2} \right) + \]

\[
(C_{P^*\nu} + C_{P^*\nu}) \exp \left( -\frac{m_{n_c}^2}{T_2^2} - \frac{m_{n_c}^2}{T_1^2} \right) = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \rho_{QCD}(s,u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

(41)

\[
\frac{f_{Dn_c} m_{Dn_c} \lambda_p \lambda_{\Lambda^+_c}}{4m_c} \frac{G_{PD\Lambda^+_c}}{m_P^2 - m_{n_c}^2} \left[ \exp \left( -\frac{m_{n_c}^2}{T_2^2} - \frac{m_{n_c}^2}{T_1^2} \right) - \exp \left( -\frac{m_P^2}{T_2^2} - \frac{m_P^2}{T_1^2} \right) \right] \exp \left( -\frac{m_D^2}{T_2^2} \right) + \]

\[
(C_{P^*\bar{D}0} + C_{P^*\Sigma^+_c}) \exp \left( -\frac{m_{n_c}^2}{T_2^2} - \frac{m_{n_c}^2}{T_1^2} \right) = \int_{m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \rho_{QCD}(s,u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

(42)

\[
\frac{f_{Dn_c} m_{Dn_c} \lambda_p \lambda_{\Sigma^+_c}}{4m_c} \frac{G_{PD\Sigma^+_c}}{m_P^2 - m_{n_c}^2} \left[ \exp \left( -\frac{m_{n_c}^2}{T_2^2} - \frac{m_{n_c}^2}{T_1^2} \right) - \exp \left( -\frac{m_P^2}{T_2^2} - \frac{m_P^2}{T_1^2} \right) \right] \exp \left( -\frac{m_D^2}{T_2^2} \right) + \]

\[
(C_{P^*\bar{D}0} + C_{P^*\Sigma^+_c}) \exp \left( -\frac{m_{n_c}^2}{T_1^2} - \frac{m_{n_c}^2}{T_2^2} \right) = \int_{m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \rho_{QCD}(s,u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

(43)

\[
\frac{f_{Dn_c} m_{Dn_c} \lambda_p \lambda_{\Sigma^{++}_c}}{4m_c} \frac{G_{PD\Sigma^{++}_c}}{m_P^2 - m_{n_c}^2} \left[ \exp \left( -\frac{m_{n_c}^2}{T_2^2} - \frac{m_{n_c}^2}{T_1^2} \right) - \exp \left( -\frac{m_P^2}{T_2^2} - \frac{m_P^2}{T_1^2} \right) \right] \exp \left( -\frac{m_D^2}{T_2^2} \right) + \]

\[
(C_{P^*\bar{D}0} + C_{P^*\Sigma^{++}_c}) \exp \left( -\frac{m_{n_c}^2}{T_1^2} - \frac{m_{n_c}^2}{T_2^2} \right) = \int_{m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \rho_{QCD}(s,u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

(44)

\[
f_{J/\psi} m_{J/\psi} \lambda_p \lambda_N \frac{G_{V/T}}{m_P^2 - m_{J/\psi}^2} \left[ \exp \left( -\frac{m_{J/\psi}^2}{T_1^2} - \frac{m_{J/\psi}^2}{T_2^2} \right) - \exp \left( -\frac{m_P^2}{T_1^2} - \frac{m_P^2}{T_2^2} \right) \right] \exp \left( -\frac{m_N^2}{T_2^2} \right) + \]

\[
C_{V/T} \exp \left( -\frac{m_{J/\psi}^2}{T_1^2} - \frac{m_{J/\psi}^2}{T_2^2} \right) = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{s_N} du \rho_{QCD}(s,u) \exp \left( -\frac{s}{T_1^2} - \frac{u}{T_2^2} \right),
\]

(45)
Data Group [28]. Moreover, we take into account the energy-scale dependence of the parameters, where $t$ to the acceptable energy scale $\mu$.

At the hadron side, we take the hadronic parameters as $m_{J/\psi} = 3.0969$ GeV, $m_N = 0.93827$ GeV, $m_{P^0} = 1.86484$ GeV, $m_{L^+} = 2.28646$ GeV, $m_{\Sigma^+} = 2.4529$ GeV from the Particle Data Group [28], and $\sqrt{s_{J/\psi}} = 3.6$ GeV, $\sqrt{s_{P}^0} = 3.5$ GeV, $\sqrt{s_N} = 1.3$ GeV, $f_{J/\psi} = 0.418$ GeV, $f_{P^0} = 0.387$ GeV [29], $\sqrt{s_{DD}^0} = 2.5$ GeV, $f_D = 0.208$ GeV [30], $\lambda_N = 0.032$ GeV$^3$ [31], $\sqrt{s_{\Lambda_c}} = 3.1$ GeV, $\lambda_{\Lambda_c} = 0.022$ GeV$^3$ [32], $\sqrt{s_{\Sigma_c}} = 3.2$ GeV, $\lambda_{\Sigma_c^+} = 0.045$ GeV$^3$ [33], $m_P = 4.31$ GeV, $\lambda_P = 1.40 \times 10^{-3}$ GeV$^6$ [34] from the QCD sum rules.

At the QCD side, we take the standard values of the vacuum condensates $\langle \bar{q}q \rangle = -(0.24 \pm 0.01)$ GeV$^3$, $\langle \bar{q}g_s Gq \rangle = m_q^2 \langle \bar{q}q \rangle$, $m_\sigma = (0.8 \pm 0.1)$ GeV$^2$, $\langle \sigma_G G \rangle = (0.33$ GeV$)^4$ at the energy scale $\mu = 1$ GeV [26, 25, 34], and choose the $\overline{MS}$ mass $m_c(m_c) = (1.275 \pm 0.025)$ GeV from the Particle Data Group [28]. Moreover, we take into account the energy-scale dependence of the parameters,

$$\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(1\text{GeV}) \left[ \frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{\mu}{2}};$$

$$\langle \bar{q}g_s \sigma Gq \rangle(\mu) = \langle \bar{q}g_s \sigma Gq \rangle(1\text{GeV}) \left[ \frac{\alpha_s(1\text{GeV})}{\alpha_s(\mu)} \right]^{\frac{\mu}{2}};$$

$$m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{\mu}{2}};$$

$$\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1 \log t}{b_0^2} + \frac{b_2^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t} \right],$$

where $t = \log \frac{\mu^2}{b_0^2}$, $b_0 = \frac{33 - 2n_f}{12\pi}$, $b_1 = \frac{153 - 19n_f}{24\pi^2}$, $b_2 = \frac{2857 - 663n_f + 225n_f^2}{192\pi^4}$, $\Lambda = 210$ MeV, 292 MeV and 332 MeV for the flavors $n_f = 5, 4$ and 3, respectively [28, 35], and evolve all the parameters to the acceptable energy scale $\mu$ with $n_f = 4$ to extract the hadronic coupling constants $G_{P_0N}, G_{P_0D\Sigma^+}, G_{P_0D\Sigma^+}, G_V$ and $G_T$.

The best energy scale of the QCD spectral density in the QCD sum rules for the lowest diquark-diquark-antiquark type hidden-charm pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$ is $\mu = \ldots$
2.3 GeV [6], which is fixed by the energy scale formula \( \mu = \sqrt{M_{J/\psi N}^2/\mu - (2M_c)^2} \) with the effective c-quark mass \( M_c = 1.82 \text{ GeV} \) in the case of the constituents are the charmed diquark states in the color antitriplet. The energy scale \( \mu = 2.3 \text{ GeV} \) is too large in the QCD sum rules for the mesons \( \eta_c, D^0, D^+, J/\psi \) and baryons \( \Lambda_c^+, \Sigma_c^+, \Sigma_c^+, N \). In this article, we take the energy scales of the QCD spectral densities to be \( \mu = \frac{m_{J/\psi N}}{2} = 1.5 \text{ GeV} \), which is acceptable for the charmed mesons and charmonium states at least based on our previous studies [33].

We choose the values of the free parameters as \( C_{P\eta_c} + C_{P\eta_c} = -1.167 \times 10^{-5} \text{ GeV}^3 \), \( C_{P\eta_c} + C_{P\eta_c} = 1.24 \times 10^{-6} \text{GeV}^3/\mu^2 \), \( C_{P\eta_c} + C_{P\eta_c} = -5.14 \times 10^{-6} \text{GeV}^7/\mu^2 \), \( C_V = 2.406 \times 10^{-5} \text{GeV}^9 \), \( C_T = 4.12 \times 10^{-6} \text{GeV}^8 \sqrt{T^2} \) to obtain flat platforms in the Borel windows \( T^2 = (4.7 - 5.7) \text{ GeV}^2 \), \((2.3 - 3.1) \text{ GeV}^2 \), \((1.9 - 2.7) \text{ GeV}^2 \), \((3.5 - 4.5) \text{ GeV}^2 \), \((3.1 - 4.1) \text{ GeV}^2 \) for the hadronic coupling constants \( G_{P\eta_c}, G_{P\eta_c}, G_{P\eta_c}, G_{P\eta_c}, G_V, G_T \), respectively. We fit those values to obtain the same intervals of the flat platforms \( T_{\text{max}}^2 - T_{\text{min}}^2 = 1.0 \text{ GeV}^2 \) and \( 0.8 \text{ GeV}^2 \) for the hadronic coupling constants \( G_{PMB} \) with \( M = \text{charmonium states} \) and \( D \) mesons, respectively [23, 27], where the \( T_{\text{max}}^2 \) and \( T_{\text{min}}^2 \) are the maximum and minimum of the Borel parameters, respectively.

Finally, we take into account the uncertainties of the input parameters, and obtain the values of the hadronic coupling constants \( G_{P\eta_c}, G_{P\eta_c}, G_{P\eta_c}, G_{P\eta_c}, G_V, G_T \), which are shown in Fig[1].

\[
G_{P\eta_c,N} = -0.40 \pm 0.50, \\
G_{P\eta_c} = 0.24 \pm 0.18, \\
G_{P\eta_c} = -1.15 \pm 0.36, \\
G_V = 0.35 \pm 0.94, \\
G_T = 0.11 \pm 0.30. 
\]  

As the uncertainties of the hadronic coupling constants are very large, we choose the central values to calculate the partial decay widths as a crude estimation,

\[
\Gamma(P_c \to \eta_c N) = 31.3972 G_{P\eta_c,N}^2 \text{MeV} = 5.02 \text{MeV}, \\
\Gamma(P_c \to \bar{D}^0 \Lambda_{c}^+) = 49.4472 G_{P\bar{D}^0 \Lambda_{c}^+}^2 \text{MeV} = 2.85 \text{MeV}, \\
\Gamma(P_c \to J/\psi N) = 29.5806 G_{V}^2 - 97.1516 G_V G_T + 80.2825 G_V^2 = 6.45 \text{MeV}. 
\]  

If we saturate the decay width of the \( P_c \) with the two-body strong decays to the \( \eta_c N, \bar{D}^0 \Lambda_{c}^+ \) and \( J/\psi N \), we can obtain the total width \( \Gamma(P_c) = 14.32 \text{MeV} \), which is compatible with the experimental value \( \Gamma(P_c,4312) = 9.8 \pm 2.7 \pm 3.7 \text{MeV} \) from the LHCb collaboration [2]. The present calculations support assigning the \( P_c(4312) \) to be the diquark-diquark-antiquark type hidden-charm pentaquark state with the spin-parity \( J^P = \frac{1}{2}^- \). The \( P_c(4312) \) maybe have a diquark-diquark-antiquark type pentaquark core with the typical size of the \( qqqq \) type baryon states, the strong couplings to the meson-baryon pairs \( \bar{D}^0 \Sigma_{c}^+ \) and \( \bar{D}^0 \Sigma_{c}^+ \) lead to some pentaquark molecule components according to the large hadronic coupling constants \( |G_{\bar{D}^0 \Sigma_{c}^+}| = \sqrt{2}|G_{\bar{D}^0 \Sigma_{c}^+}| \gg |G_{\bar{D}^0 \Sigma_{c}^+}| \), and the \( P_c(4312) \) maybe spend a large time as the \( \bar{D}^0 \Sigma_{c}^+ \) and \( \bar{D}^0 \Sigma_{c}^+ \) molecular states, just in the case of the \( f_0(980), a_0(980) \) and \( Y(4660) \). In Ref.[6], we assign the \( P_c(4312) \) to be the \( D \Sigma_{c} \) pentaquark molecular state with the spin-parity \( J^P = \frac{1}{2}^- \) tentatively, and explore its two-body strong decays with the QCD sum rules, and obtain the partial decay widths \( \Gamma(P_c(4312) \to \eta_c N) = 0.255 \text{MeV} \) and \( \Gamma(P_c(4312) \to J/\psi N) = 9.296 \text{MeV} \). The \( P_c(4312) \) has quite different branching fractions in the scenarios of the pentaquark state and pentaquark molecular state. We can search for the \( P_c(4312) \) in the \( \eta_c N, \bar{D}^0 \Lambda_{c}^+ \) and \( J/\psi N \) invariant mass spectrum, and measure the branching fractions \( B_r(P_c(4312) \to \eta_c N, \bar{D}^0 \Lambda_{c}^+, J/\psi N) \) precisely, which maybe shed light on the nature of the \( P_c(4312) \) unambiguously and test the predictions of the QCD sum rules.
Figure 1: The hadronic coupling constants with variations of the Borel parameters $T^2$, where the (I), (II), (III), (IV) and (V) correspond to the $G_{P_{NI}}, G_{P_{D_{0}}^{0}}$, $G_{P_{D_{0}^{+}}^{0}}$, $G_{V}$ and $G_{T}$, respectively.
5 Conclusion

In this article, we illustrate how to calculate the hadronic coupling constants of the hidden-charm pentaquark states with the QCD sum rules based on solid quark-hadron quality, then study the hadronic coupling constants of the lowest diquark-diquark-antiquark type pentaquark state with the spin-parity $J^P = \frac{1}{2}^-$ in a consistent way. The predicted width $\Gamma(P_c) = 14.32$ MeV is compatible with the experimental data $\Gamma_{P_c}(3122) = 9.8 \pm 2.7\pm^{+3.7}_{-3.0}$ MeV from the LHCb collaboration, and favors assigning the $P_c(3122)$ to be the $[ud][uc]\bar{c}$ type compact pentaquark state with the spin-parity $J^P = \frac{1}{2}^−$. The $P_c(3122)$ maybe have a diquark-diquark-antiquark type pentaquark core with the typical size of the $qqq$ type baryon states, the strong couplings to the meson-baryon pairs $D^0\Sigma_c^+$ and $D^-\Sigma_c^{++}$ lead to some pentaquark molecule components according to the large hadronic coupling constants $|G_{PD^0\Sigma_c^+}| = \sqrt{2}|G_{PD^0\Sigma_c^{++}}| \gg |G_{PD^0\Lambda^+_c}|$, just in the case of the $f_0(980)$, $a_0(980)$ and $Y(4660)$. The $P_c(3122)$ has quite different branching fractions in the scenarios of the pentaquark state and pentaquark molecular state, we can distinguish the two scenarios unambiguously by measuring the branching fractions $Br (P_c(3122) \to \eta_c N, D^0\Lambda^+_c, J/\psi N)$ precisely.

Appendix

The explicit expressions of the QCD spectral densities $\rho_{QCD}^{n_c N}(s, u)$, $\rho_{QCD}^{D^0\Lambda^+_c}(s, u)$, $\rho_{QCD}^{D^0\Sigma_c^+}(s, u)$, $\rho_{QCD}^{J/\psi N}(s, u)$ and $\rho_{QCD}^{J/\psi N, 2}(s, u)$, 

$$
\rho_{QCD}^{n_c N}(s, u) = -\frac{m_c}{2048\pi^6} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \left[ -\frac{m_c}{12\pi^2} \int_{x_1}^{x_f} dx \delta(u) + \frac{1}{4608\pi^4} \frac{m_c}{72\pi^2} \int_{x_1}^{x_f} dx \left( 1 + x \right) \delta(s - \bar{m}_c^2) + \frac{1}{192\pi^2 T_2^2} \int_{x_1}^{x_f} dx \delta(u) \right] \\
+ \frac{m_c}{18432\pi^4 T_4^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{1024\pi^4 T_2^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{6144\pi^4 T_4^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{1087\pi^4 T_4^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{36T_4^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{432T_2^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{11m_c}{9216\pi^2 T_2^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right] \\
+ \frac{m_c}{1536\pi^2 T_2^2} \frac{1}{\pi} \left[ \frac{f_{\pi}^2}{\alpha_s \Lambda^2} \int_{x_1}^{x_f} dx \int_{x_1}^{x_f} dx \delta(s - \bar{m}_c^2) \right], \quad (50)
$$
\begin{align*}
\mathcal{D}^{\rho_{QCD}}_{\rho_{QCD}} (s, u) &= \frac{9m_c}{4096\pi^6} \int_{x_i}^1 dx \int_{y_i}^1 dy (1-x)y(1-y)^2 \left( u - \bar{m}_y^2 \right)^2 \\
- \frac{3\langle \bar{q}q \rangle}{1024\pi^4} &\int_{y_i}^1 dy y(1-y)^2 \delta \left( s - m_x^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
+ \frac{m_c^2\langle \bar{q}q \rangle}{128\pi^3} &\int_{x_i}^1 dx \int_{y_i}^1 dy (1-x)(1-y) \\
+ \frac{\langle \bar{q}q, \sigma Gq \rangle}{4096\pi^2 T_1} &\int_{y_i}^1 dy y(1-y)^2 \left( 4 + \frac{s}{T_1^2} \right) \delta \left( s - m_x^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
+ \frac{m_c^2\langle \bar{q}q, \sigma Gq \rangle}{1024\pi^4} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(1-x)(1-2y)}{y} \delta \left( u - \bar{m}_y^2 \right) \\
+ \frac{\langle \bar{q}q, \sigma Gq \rangle}{8192\pi^4} &\int_{y_i}^1 dy (5y - 3) \delta \left( s - m_x^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
+ \frac{m_c^2\langle \bar{q}q, \sigma Gq \rangle}{6144\pi^4} &\int_{x_i}^1 dx \int_{y_i}^1 dy y(1-y)(3u - s) \delta \left( s - \bar{m}_y^2 \right) \\
- \frac{m_c^3\langle \bar{q}q, \sigma Gq \rangle}{12288\pi^4} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(2 - 7x)(1-y)}{x} \delta \left( s - \bar{m}_y^2 \right) \\
+ \frac{m_c^3\langle \bar{q}q, \sigma Gq \rangle}{8192\pi^4 T_1 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(1-x)y(1-y)^2}{x^3} \delta \left( s - \bar{m}_y^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
+ \frac{m_c^3\langle \bar{q}q, \sigma Gq \rangle}{8192\pi^4 T_1 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(3 - 4x)y(1-y)^2}{x^2} \delta \left( s - \bar{m}_y^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
+ \frac{m_c^3\langle \bar{q}q, \sigma Gq \rangle}{4096\pi^4 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(1-x)(1-y)^2}{y^2} \delta \left( u - \bar{m}_y^2 \right) \\
- \frac{m_c \langle \bar{q}q, \sigma Gq \rangle}{16384\pi^4 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy (1-x) \delta (u - 1) \\
- \frac{m_c \langle \bar{q}q, \sigma Gq \rangle}{16384\pi^4 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy (1-x+y)(1-y)^2(3u - s) \delta \left( s - \bar{m}_y^2 \right) \\
+ \frac{m_c \langle \bar{q}q, \sigma Gq \rangle}{3072\pi^2 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(2x - 1)(1-y)(5y - 3)}{x} \delta \left( s - \bar{m}_y^2 \right) \left( u - \bar{m}_y^2 \right) \\
+ \frac{m_c^2\langle \bar{q}q \rangle}{6144\pi^4 T_1} &\int_{y_i}^1 dy \frac{(1-y)^2}{y^2} \delta \left( s - m_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
+ \frac{m_c^4\langle \bar{q}q \rangle}{2304\pi^2 T_1^2} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(1-x)(1-y)}{x^3} \delta \left( s - m_x^2 \right) \\
+ \frac{m_c^2\langle \bar{q}q \rangle}{768\pi^2 T_2^2} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{(1-x)(1-y)}{y^2} \left( 1 - \frac{u}{3T_2} \right) \delta \left( u - \bar{m}_y^2 \right) \\
+ \frac{m_c^2\langle \bar{q}q \rangle}{4608\pi^2 T_2^2} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{8xy - 2x - 7y + 4}{xy} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
+ \frac{m_c^2\langle \bar{q}q \rangle}{18432\pi^2 \pi} &\int_{x_i}^1 dx \int_{y_i}^1 dy \frac{8xy - 2x - 7y + 4}{xy} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - \bar{m}_y^2 \right)
\end{align*}
\[
\frac{\langle \bar{q}q \rangle}{18432 \pi^2} \left( \frac{\alpha_s G}{\pi} \right) \int_{y_i}^1 dy \int_{y_i}^1 dy_y (5y - 2)(3u - s) \delta (s - \bar{m}_z^2) \delta (u - \bar{m}_y^2)
\]
\[
- \frac{\langle \bar{q}q \rangle}{1024 \pi^2} \left( \frac{\alpha_s G}{\pi} \right) \int_{y_i}^1 dy_y \delta (s - m_c^2)
\]
\[
+ \frac{\langle \bar{q}q \rangle}{12288 \pi^2 T_1} \left( \frac{\alpha_s G}{\pi} \right) \int_{y_i}^1 dy (5y - 3)(1 - y) \delta (s - m_z^2) (u - \bar{m}_y^2)
\]
\[
+ \frac{m_c^2 \langle \bar{q}q \rangle}{768 \pi^2 T_1} \left( \frac{\alpha_s G}{\pi} \right) \int_{x_i}^1 dx \int_{y_i}^1 dy \frac{1 - y}{x^2} \delta (s - \bar{m}_z^2)
\]
\[
- \frac{m_c \langle \bar{q}q \rangle}{96 \pi^2} \int_{y_i}^1 dy (1 - y) \delta (s - m_z^2) + \frac{m_c \langle \bar{q}q \rangle}{64 \pi^2} \int_{y_i}^1 dx (1 - x) \delta (u - m_c^2)
\]
\[
- \frac{m_c \langle \bar{q}q \rangle}{3072 \pi^2} \int_{y_i}^1 dy \left( \frac{4 - 5y}{y} \right) \delta (s - m_c^2) \delta (u - \bar{m}_y^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle}{9216 \pi^2 T_1^2} \int_{y_i}^1 dy (1 - y) \left( -5 + \frac{24s}{T_1^2} \right) \delta (s - m_c^2)
\]
\[
- \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle}{128 \pi^2 T_2^2} \int_{x_i}^1 dx (1 - x) \left( 1 + \frac{u}{T_2^2} \right) \delta (u - m_c^2)
\]
\[
- \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle}{1024 \pi^2} \int_{x_i}^1 dx \delta (s - \bar{m}_z^2) \delta (u - m_c^2)
\]
\[
- \frac{\langle \bar{q}q \rangle^3}{48} \delta (s - m_c^2) \delta (u - m_c^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{3072 \pi^2 T_1^2} \int_{y_i}^1 dy \left( 1 - 2y \right) \delta (s - m_c^2) \delta (u - \bar{m}_y^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{3072 \pi^2 T_2^2} \int_{x_i}^1 dx (1 - x) \delta (u - m_c^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{4096 \pi^2 T_2^2} \int_{x_i}^1 dx \left( 1 + \frac{u}{T_2^2} \right) \delta (s - \bar{m}_z^2) \delta (u - m_c^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{4096 \pi^2 T_2^2} \delta (s - m_c^2) \delta (u - \bar{m}_z^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{294912 \pi^2 T_1^2} \int_{y_i}^1 dy \left( 96 - 191y \right) \delta (s - \bar{m}_y^2) \delta (u - m_c^2)
\]
\[
+ \frac{5m_c^3 \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{36864 \pi^2 T_1^2} \int_{y_i}^1 dy (1 - y) \delta (s - m_c^2)
\]
\[
- \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{6144 \pi^2 T_2^2} \int_{y_i}^1 dy \frac{1}{y} \delta (s - m_c^2) \delta (u - \bar{m}_y^2)
\]
\[
+ \frac{7m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{36864 \pi^2 T_2^2} \int_{x_i}^1 dx \frac{1}{x} \delta (s - \bar{m}_z^2) \delta (u - m_c^2)
\]
\[
+ \frac{m_c \langle \bar{q}q \rangle \langle qg, \sigma Gq \rangle^2}{221184 \pi^2 T_2^2} \int_{x_i}^1 dx \frac{8 - 37x}{x} \delta (s - \bar{m}_z^2) \delta (u - m_c^2)
\]
\[
+ \frac{m_c^3 \langle \bar{q}q \rangle^2}{1728 T_1^2} \left( 1 - \frac{m_c^2}{2T_1^2} \right) \left( \frac{\alpha_s G}{\pi} \right) \int_{y_i}^1 dy (1 - y) \delta (s - m_c^2)
\]
\[
\frac{m_c^3 \langle \bar{q} q \rangle^2}{1152 T_1^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x) \delta (s - m_c^2)}{x^3} \delta (u - m_c^2) \\
- \frac{m_c^3 \langle \bar{q} q \rangle^2}{1152 T_2^4} \left( \frac{1 - m_c^2}{T_2^2} \right) \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx (1-x) \delta (u - m_c^2) \\
+ \frac{m_c^3 \langle \bar{q} q \rangle^2}{1152 T_1^4} \left( \frac{m_c^2}{T_2^2} \right) \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx (1-x) \delta (u - m_c^2) \\
+ m_c \langle \bar{q} q \rangle \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \left( \frac{1 - m_c^2}{x^2} \right) \frac{1}{T_2^2} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx (1-x) \delta (u - m_c^2) \\
+ m_c \langle \bar{q} q \rangle \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \left( \frac{1 - m_c^2}{x^2} \right) \frac{1}{T_2^2} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx (1-x) \delta (u - m_c^2) \\
- \frac{m_c \langle \bar{q} q \rangle}{1536 T_2^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{y_i}^{1} dy \frac{1 - 2x}{x} \delta (s - m_c^2) \delta (u - m_c^2) \\
+ \frac{m_c \langle \bar{q} q \rangle}{1152 T_2^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \frac{3 - 4x}{x^2} \delta (s - m_c^2) \delta (u - m_c^2) \\
- \frac{m_c \langle \bar{q} q \rangle}{13824 T_2^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{y_i}^{1} dy \frac{y + 2}{y} \delta (s - m_c^2) \delta (u - m_c^2) \\
- \frac{m_c \langle \bar{q} q \rangle}{3456 T_2^4} \left( \frac{\alpha_s G G}{\pi} \right) \delta (s - m_c^2) \delta (u - m_c^2) \\
\]

\[
\rho_{QCD}^{\psi \Sigma^+} (s, u) = - \frac{3 m_c}{1024 \pi^6} \int_{x_i}^{1} dx \int_{y_i}^{1} dy (1-x)(1-y)^2 (u - m_u^2)^2 \\
+ \frac{\langle \bar{q} q \rangle}{256 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy (1-y)^2 \delta (s - m_u^2) (u - m_u^2)^2 \\
+ \frac{3 m_c^2 \langle \bar{q} q \rangle}{128 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy (1-x)(1-y) \\
+ \frac{\langle \bar{q} q \rangle \sigma_{Gq}}{512 \pi^4} \int_{y_i}^{1} dy (1-y) \delta (s - m_u^2) (u - m_u^2) \\
+ \frac{m_c^2 \langle \bar{q} q \rangle \sigma_{Gq}}{1024 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy (1-y) \delta (s - m_u^2) \\
+ \frac{m_c^2 \langle \bar{q} q \rangle \sigma_{Gq}}{1024 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)}{y} \delta (s - m_u^2) (u - m_u^2)^2 \\
- \frac{m_c^3 \langle \bar{q} q \rangle}{1024 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{y^2} \delta (u - m_u^2) \\
+ \frac{m_c^3 \langle \bar{q} q \rangle}{3072 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{y^2} \delta (u - m_u^2) \\
- \frac{m_c}{2048 \pi T_1^4} \left( \frac{1 - m_c^2}{3T_1^2} \right) \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{x^2} \delta (s - m_c^2) (u - m_u^2)^2 \\
+ \frac{m_c^3 \langle \bar{q} q \rangle}{3072 \pi^4} \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{y^2} \delta (u - m_u^2) \\
- \frac{m_c}{2048 \pi T_1^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy (1-x)(1-y) \\
- \frac{m_c}{2048 \pi T_1^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{x} \delta (s - m_c^2) (u - m_u^2)^2 \\
+ \frac{m_c}{3072 \pi^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(1-x)(1-y)^2}{x} \delta (s - m_c^2) (u - m_u^2)^2 \\
- \frac{m_c}{73728 \pi^4} \left( \frac{\alpha_s G G}{\pi} \right) \int_{x_i}^{1} dx \int_{y_i}^{1} dy \frac{(y + 2)(1-y)^2}{y} (3u - s) \delta (s - m_c^2) \\
\]

16
\begin{align}
&\quad - \frac{m_c^2 \langle \bar{q}q \rangle}{2304 \pi^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_1}^{1} dy \frac{(1-y)^2}{y^2} \delta \left( s - m_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
&\quad + \frac{\langle \bar{q}q \rangle}{768 \pi^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_1}^{1} dy (1-y) \delta \left( s - m_x^2 \right) \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{1536 \pi^2 T_2^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{(1-x)}{y} \delta \left( u - \bar{m}_y^2 \right) \\
&\quad + \frac{\langle \bar{q}q \rangle}{2304 \pi^2 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_1}^{1} dy y (1-y) \delta \left( s - m_x^2 \right) \left( u - \bar{m}_y^2 \right) \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{256 \pi^2 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{(1-y)}{x} \delta \left( s - \bar{m}_x^2 \right) \\
&\quad - \frac{m_c^2 \langle \bar{q}q \rangle}{4608 \pi^2 T_1^2} \left( 1 - \frac{m_x^2}{2T_1^2} \right) \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_1}^{1} dy y (1-y)^2 \delta \left( s - m_x^2 \right) \left( u - \bar{m}_y^2 \right)^2 \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{1536 \pi^2 T_2^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx (1-x) \delta \left( u - m_x^2 \right) \\
&\quad + \frac{\langle \bar{q}q \rangle}{9216 \pi^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{(1+y)(3u-s)}{y} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{3072 \pi^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{x}{y} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{256 \pi^2 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{(1-x)(1-y)}{x^2} \left( 1 - \frac{s}{3T_1^2} \right) \delta \left( s - \bar{m}_x^2 \right) \\
&\quad + \frac{m_c^2 \langle \bar{q}q \rangle}{256 \pi^2 T_2^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \int_{y_1}^{1} dy \frac{(1-x)(1-y)}{y^2} \left( 1 - \frac{u}{3T_2^2} \right) \delta \left( u - \bar{m}_y^2 \right) \\
&\quad - \frac{m_c^2 \langle \bar{q}q \rangle^2}{32 \pi^2} \int_{y_1}^{1} dy (1-y) \delta \left( s - m_x^2 \right) - \frac{m_c^2 \langle \bar{q}q \rangle^2}{96 \pi^2} \int_{x_1}^{1} dx (1-x) \delta \left( u - m_x^2 \right) \\
&\quad + \frac{m_c \langle \bar{q}q \rangle \langle \bar{q}q \sigma Gq \rangle}{384 \pi^2 T_2^2} \int_{x_1}^{1} dx (1-x) \left( 1 + \frac{2u}{T_2^2} \right) \delta \left( u - m_x^2 \right) \\
&\quad + \frac{m_c \langle \bar{q}q \rangle \langle \bar{q}q \sigma Gq \rangle}{1536 \pi^2} \int_{y_1}^{1} dy \frac{25y - 14}{y} \delta \left( s - m_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
&\quad - \frac{m_c \langle \bar{q}q \rangle \langle \bar{q}q \sigma Gq \rangle}{768 \pi^2 T_1^2} \int_{y_1}^{1} dy \frac{1}{y} \left( 1 - \frac{6s}{T_1^2} \right) \delta \left( s - m_x^2 \right) \\
&\quad - \frac{m_c \langle \bar{q}q \rangle \langle \bar{q}q \sigma Gq \rangle}{4608 \pi^2} \int_{x_1}^{1} dx \frac{9 - 2x}{x} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - m_x^2 \right) \\
&\quad - \frac{m_c \langle \bar{q}q \rangle^2}{576 \pi^2 T_1^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \frac{1}{x} \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - m_x^2 \right) \\
&- \frac{m_c \langle \bar{q}q \rangle^2}{1152 \pi^2 T_2^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_1}^{1} dy \frac{1}{y} \delta \left( s - m_x^2 \right) \delta \left( u - \bar{m}_y^2 \right) \\
&- \frac{m_c \langle \bar{q}q \rangle^2}{1152 \pi^2 T_2^2} \langle \frac{\alpha_s GG}{\pi} \rangle \delta \left( s - m_x^2 \right) \delta \left( u - m_y^2 \right) \\
&- \frac{m_c \langle \bar{q}q \rangle^2}{576 \pi^2 T_1^2} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{x_1}^{1} dx \frac{1 - x}{x^2} \left( 1 - \frac{s}{3T_1^2} \right) \delta \left( s - \bar{m}_x^2 \right) \delta \left( u - m_x^2 \right)
\end{align}
\[ - \frac{m_c \langle \bar{q}q \rangle^2 \alpha_s GG}{192 T_f^2 \pi} \int_{y_i}^1 dy \frac{1 - y}{y^2} \left(1 - \frac{u}{3 T_f^2}\right) \delta(s - m_c^2) \delta(u - \bar{m}_y^2) \\
+ \frac{m_c \langle \bar{q}q \rangle^2 \alpha_s GG}{6912 T_f^2 \pi} \int_{x_i}^1 dx \frac{1 - 2x}{x} \delta(s - \bar{m}_x^2) \delta(u - m_c^2) \\
- \frac{m_c \langle \bar{q}q \rangle^2 \alpha_s GG}{2304 T_f^2 \pi} \int_{y_i}^1 dy \frac{1 - 2y}{y} \delta(s - m_c^2) \delta(u - \bar{m}_y^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle^2}{576 T_f^1} \left(1 - \frac{m_c^2}{2 T_f^1}\right) \left(\frac{\alpha_s GG}{\pi}\right) \int_{y_i}^1 dy (1 - y) \delta(s - m_c^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle^2}{1728 T_f^6} \left(1 - \frac{m_c^2}{T_f^2}\right) \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^1 dx (1 - x) \delta(u - m_c^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle^2}{2048 T_f^2} \left(1 - \frac{m_c^2}{T_f^1}\right) \delta(s - m_c^2) \delta(u - \bar{m}_y^2) + \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{3072 \pi^2 T_f^6} \delta(s - m_c^2) \delta(u - m_c^2) \\
+ m_c \langle \bar{q}q \rangle \sigma Gq \frac{21}{512 \pi^2 T_f^1} \int_{y_i}^1 dy (s - m_c^2) + \frac{m_c^3 \langle \bar{q}q \rangle \sigma Gq}{3072 \pi^2 T_f^6} \int_{y_i}^1 dy (1 - y) \delta(s - m_c^2) \\
- \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{18432 \pi^2 T_f^1} \int_{x_i}^1 dx \frac{1}{x} \delta(s - \bar{m}_x^2) \delta(u - m_c^2) \\
+ \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{36864 \pi^2 T_f^2} \int_{x_i}^1 dx \frac{9 - 2x}{x} \left(1 + \frac{2u}{T_f^2}\right) \delta(s - \bar{m}_x^2) \delta(u - m_c^2) \\
+ \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{3072 \pi^2 T_f^1} \int_{y_i}^1 dy \frac{6 - 13y}{y} \left(1 + \frac{s}{T_f^2}\right) \delta(s - m_c^2) \delta(u - \bar{m}_y^2) \\
- \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{9216 \pi^2 T_f^2} \int_{y_i}^1 dy \frac{1}{y} \delta(s - m_c^2) \delta(u - \bar{m}_y^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle \sigma Gq}{147456 \pi^2 T_f^1} \int_{y_i}^1 dy \frac{379y - 326}{y} \delta(s - m_c^2) \delta(u - \bar{m}_y^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle \sigma Gq}{1536 \pi^2 T_f^2} \left(1 - \frac{m_c^2}{T_f^2}\right) \int_{x_i}^1 dx (1 - x) \delta(u - m_c^2), \quad (52) \]

\[ \rho^{J/\psi N.1}_{QCD}(s, u) = - \frac{m_c}{2048 \pi^0} \int_{x_i}^{x_f} dx u^2 - \frac{m_c \langle \bar{q}q \rangle^2}{12 \pi^2} \int_{x_i}^{x_f} dx \delta(u) \]

\[ + \frac{\langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma Gq}{9216 \pi^4} \int_{x_i}^{x_f} dx \left(1 + x\right) u \delta(s - \bar{m}_x^2) + \frac{7m_c \langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma Gq}{192 \pi^2 T_f^2} \int_{x_i}^{x_f} dx \delta(u) \]

\[ + \frac{m_c \langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma Gq}{576 \pi^2} \int_{x_i}^{x_f} dx \frac{3x - 5}{x} \delta(s - \bar{m}_x^2) \delta(u) - \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{1024 \pi^4} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^{x_f} dx \]

\[ - \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{9216 \pi^4} \int_{x_i}^{x_f} dx \frac{1}{x(1-x)} u \left(1 - \frac{u}{T_f^2}\right) \delta(s - \bar{m}_x^2) \\
- \frac{m_c \langle \bar{q}q \rangle \sigma Gq}{6144 \pi^2 T_f^1} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^{x_f} dx \frac{1}{x^2} \left(1 - \frac{m_c^2}{3x T_f^1}\right) u^2 \delta(s - \bar{m}_x^2) \\
+ \frac{m_c^3 \langle \bar{q}q \rangle^2}{1087144 \pi^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^{x_f} dx \frac{1}{x^3} \delta(s - \bar{m}_x^2) \delta(u) \\
- \frac{m_c \langle \bar{q}q \rangle^2}{432 \pi^2 T_f^2} \left(\frac{\alpha_s GG}{\pi}\right) \int_{x_i}^{x_f} dx \frac{1}{x} \delta(s - \bar{m}_x^2) \delta(u) \]
\[ + \frac{m_c(q\bar{q})^2 \alpha_s \sigma Gq}{216 T_1^2} \int_{x_i}^{x_f} dx \left( 6 x^2 - 13 x - 6 \right) \alpha (s - \bar{m}_c^2) \delta (u) \]
\[ + \frac{m_c(q\bar{q}, \sigma Gq)^2}{4608 \pi^2 T_2^2} \int_{x_i}^{x_f} dx \frac{7 - 4x}{x} \delta (s - \bar{m}_c^2) \delta (u) \]
\[ + \frac{m_c(q\bar{q}, \sigma Gq)^2}{221184 \pi^2 T_2^2} \int_{x_i}^{x_f} dx \frac{61 x + 40}{x(1 - x)} \delta (s - \bar{m}_c^2) \delta (u) , \tag{53} \]

\[ \rho_{QCD}^{J/\psi N^2}(s, u) = \frac{1}{2048 \pi^6} \int_{x_i}^{x_f} dx \left[ x s + x(1 - x) \left( s - \bar{m}_c^2 \right) \right] u^2 \]
\[ + \frac{(q\bar{q})^2}{12 \pi^2} \int_{x_i}^{x_f} dx \left[ x s + x(1 - x) \left( s - \bar{m}_c^2 \right) \right] \delta (u) \]
\[ + \frac{7(q\bar{q})(q\bar{q}, \sigma Gq)}{192 \pi^2 T_2^2} \int_{x_i}^{x_f} dx \left[ x s + x(1 - x) \left( s - \bar{m}_c^2 \right) \right] \delta (u) \]
\[ + \frac{(q\bar{q})(q\bar{q}, \sigma Gq)}{144 \pi^2} \int_{x_i}^{x_f} dx \left[ s \delta (s - \bar{m}_c^2) + 1 \right] \delta (u) \]
\[ + \frac{m_c^2}{18432 \pi^4 T_1^4} \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{\pi} \int_{x_i}^{x_f} dx \left[ \frac{1}{x^2} \left( 1 - \frac{s}{T_1^2} \right) + \frac{1}{x} \right] u^2 \delta (s - \bar{m}_c^2) \]
\[ + \frac{1}{1024 \pi^4} \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{x_i} \int_{x_i}^{x_f} dx \left[ x s + x(1 - x) \left( s - \bar{m}_c^2 \right) \right] \delta (u) \]
\[ + \frac{1}{2304 \pi^4} \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{x_i} \int_{x_i}^{x_f} dx \left[ s \delta (s - \bar{m}_c^2) + 1 \right] u \]
\[ + \frac{3}{3686 \pi^4} \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{x_i} \int_{x_i}^{x_f} dx \left[ 1 + \frac{s}{T_1^2} - \frac{s}{2x(1-x)T_1^2} \right] u^2 \delta (s - \bar{m}_c^2) \]
\[ + \frac{m_c^2(q\bar{q})^2}{108 T_1^2} \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{\pi} \int_{x_i}^{x_f} dx \left[ \frac{1}{x^2} \left( 1 - \frac{s}{T_2^2} \right) + \frac{1}{x} \right] \delta (s - \bar{m}_c^2) \delta (u) \]
\[ + \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{216 T_1^2} \int_{x_i}^{x_f} dx \left[ 1 + \frac{s}{T_1^2} - \frac{s}{2x(1-x)T_1^2} \right] \delta (s - \bar{m}_c^2) \delta (u) \]
\[ + \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{216 T_1^2} \int_{x_i}^{x_f} dx \left[ \frac{1}{x^2} \left( 1 - \frac{s}{T_2^2} \right) + \frac{1}{x} \right] \delta (s - \bar{m}_c^2) \delta (u) \]
\[ + \frac{(q\bar{q})^2 \alpha_s \sigma Gq}{2304 \pi^2 T_2^2} \int_{x_i}^{x_f} dx \left[ 7 - \frac{44 s}{T_1^2} - \frac{14 s}{x(1-x)T_1^2} \right] \delta (s - \bar{m}_c^2) \delta (u) , \tag{54} \]

where \( x_f = \frac{1 + \sqrt{1 - 4 m^2/s}}{2} \), \( x_i = \frac{1 - \sqrt{1 - 4 m^2/s}}{2} \), \( m^2 = \frac{m^2}{x(1-x)} \) in Eq. (50) and Eqs. (53)-(54), \( x_i = \frac{m^2}{s} \), \( y_i = \frac{m^2}{u} \), \( \bar{m}_x = \frac{m^2}{x} \), \( \bar{m}_y = \frac{m^2}{y} \) in Eqs. (51)-(52), \( \int_{x_i}^{x_f} dx \rightarrow \int_0^1 dx \), \( \int_{x_i}^{x_f} dx \rightarrow \int_0^1 dx \) and \( \int_{y_i}^{y_f} dy \rightarrow \int_0^1 dy \), when the \( \delta \) functions \( \delta(s - \bar{m}_c^2) \), \( \delta(s - \bar{m}_c^2) \) and \( \delta(u - \bar{m}_y^2) \) appear.

**Acknowledgements**

This work is supported by National Natural Science Foundation, Grant Number 11775079.
References

[1] R. Aaij et al, Phys. Rev. Lett. 115 (2015) 072001.

[2] R. Aaij et al, Phys. Rev. Lett. 122 (2019) 222001.

[3] R. Chen, X. Liu, X. Q. Li and S. L. Zhu, Phys. Rev. Lett. 115 (2015) 132002; H. X. Chen, W. Chen, X. Liu, T. G. Steele and S. L. Zhu, Phys. Rev. Lett. 115 (2015) 172001; L. Roca, J. Nieves and E. Oset, Phys. Rev. D92 (2015) 094003; J. He, Phys. Lett. B753 (2016) 547; H. Huang, C. Deng, J. Ping and F. Wang, Eur. Phys. J. C76 (2016) 624; F. K. Guo, U. G. Meissner, W. Wang and Z. Yang, Phys. Rev. D92 (2015) 071502; U. G. Meissner and J. A. Oller, Phys. Lett. B751 (2015) 59; T. J. Burns, Eur. Phys. J. A51 (2015) 152; K. Azizi, Y. Sarac and H. Sundu, Phys. Rev. D95 (2017) 094016; K. Azizi, Y. Sarac and H. Sundu, Phys. Lett. B782 (2018) 694.

[4] Z. G. Wang, Int. J. Mod. Phys. A34 (2019) 1950097.

[5] R. Chen, Z. F. Sun, X. Liu and S. L. Zhu, Phys. Rev. D100 (2019) 011502; H. X. Chen, W. Chen and S. L. Zhu, Phys. Rev. D100 (2019) 051501; M. Z. Liu, Y. W. Pan, F. Z. Peng, M. S. Sanchez, L. S. Geng, A. Hosaka and M. P. Valderrama, Phys. Rev. Lett. 122 (2019) 242001; F. K. Guo, H. J. Jing, U. G. Meissner and S. Sakai, Phys. Rev. D99 (2019) 091501; J. He, Eur. Phys. J. C79 (2019) 393; C. J. Xiao, Y. Huang, Y. B. Dong, L. S. Geng and D. Y. Chen, Phys. Rev. D100 (2019) 014022; Y. Shimizu, Y. Yamaguchi and M. Harada, arXiv:1904.00587; H. Huang, J. He and J. Ping, arXiv:1904.00221; Z. H. Guo and J. A. Oller, Phys. Lett. B793 (2019) 144; J. R. Zhang, Eur. Phys. J. C79 (2019) 1001; Q. Wu and D. Y. Chen, Phys. Rev. D100 (2019) 114002; X. Y. Wang, X. R. Chen and J. He, Phys. Rev. D99 (2019) 114007; S. Sakai, H. J. Jing and F. K. Guo, Phys. Rev. D100 (2019) 074007; T. J. Burns and E. S. Swanson, Phys. Rev. D100 (2019) 114033; H. W. Ke, M. Li, X. H. Liu and X. Q. Li, Phys. Rev. D101 (2020) 014024.

[6] Z. G. Wang and X. Wang, arXiv:1907.04582.

[7] L. Maiani, A. D. Polosa and V. Riquer, Phys. Lett. B749 (2015) 289; V. V. Anisovich, M. A. Matveev, J. Nyiri, A. V. Sarantsev and A. N. Semenova, arXiv:1507.07652; G. N. Li, M. He and X. G. He, JHEP 1512 (2015) 128; R. Ghosh, A. Bhattacharya and B. Chakrabarti, Phys. Part. Nucl. Lett. 14 (2017) 550; V. V. Anisovich, M. A. Matveev, J. Nyiri, A. V. Sarantsev and A. N. Semenova, Int. J. Mod. Phys. A30 (2015) 1550190.

[8] Z. G. Wang, Eur. Phys. J. C76 (2016) 70; Z. G. Wang, Eur. Phys. J. C76 (2016) 142; Z. G. Wang, Nucl. Phys. B913 (2016) 163; J. X. Zhang, Z. G. Wang and Z. Y. Di, Acta Phys. Polon. B48 (2017) 2013.

[9] Z. G. Wang and T. Huang, Eur. Phys. J. C76 (2016) 43.

[10] A. Ali and A. Y. Parkhomenko, Phys. Lett. B793 (2019) 365; R. Zhu, X. Liu, H. Huang and C. F. Qiao, Phys. Lett. B797 (2019) 134869; J. B. Cheng and Y. R. Liu, Phys. Rev. D100 (2019) 054002.

[11] Z. G. Wang, Int. J. Mod. Phys. A35 (2020) 2050003.

[12] R. F. Lebed, Phys. Rev. D92 (2015) 114030; R. F. Lebed, Phys. Lett. B749 (2015) 454; R. Zhu and C. F. Qiao, Phys. Lett. B756 (2016) 259.

[13] A. Pimikov, H. J. Lee and P. M. Zhang, Phys. Rev. D101 (2020) 014002.

[14] M. I. Eides, V. Y. Petrov and M. V. Polyakov, arXiv:1904.11616.
[15] N. A. Tornqvist, Z. Phys. C68 (1995) 647; M. Boglione and M. R. Pennington, Phys. Rev. Lett. 79 (1997) 1998.
[16] F. E. Close and N. A. Tornqvist, J. Phys. G28 (2002) R249.
[17] C. Amsler and N. A. Tornqvist, Phys. Rept. 389 (2004) 61.
[18] P. Colangelo and F. De Fazio, Phys. Lett. B559 (2003) 49; Z. G. Wang, W. M. Yang and S. L. Wan, Eur. Phys. J. C37 (2004) 223.
[19] Z. G. Wang, Eur. Phys. J. C76 (2016) 427.
[20] J. Sugiyama, T. Nakamura, N. Ishii, T. Nishikawa and M. Oka, Phys. Rev. D76 (2007) 114010.
[21] H. J. Lee, Eur. Phys. J. A30 (2006) 423.
[22] H. J. Lee, K. S. Kim and H. Kim, Phys. Rev. D100 (2019) 034021.
[23] Z. G. Wang, Eur. Phys. J. C79 (2019) 184.
[24] Z. G. Wang and X. H. Zhang, Commun. Theor. Phys. 54 (2010) 323; R. M. Albuquerque, M. Nielsen and R. Rodrigues da Silva, Phys. Rev. D84 (2011) 116004.
[25] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.
[26] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1.
[27] Z. G. Wang and J. X. Zhang, Eur. Phys. J. C78 (2018) 14; Z. G. Wang, Int. J. Mod. Phys. A34 (2019) 1950110; Z. G. Wang and Z. Y. Di, Eur. Phys. J. C79 (2019) 72; Z. G. Wang, Acta Phys. Polon. B51 (2020) 435.
[28] M. Tanabashi et al, Phys. Rev. D98 (2018) 030001.
[29] D. Becirevic, G. Duplancic, B. Klajn, B. Melic and F. Sanfilippo, Nucl. Phys. B883 (2014) 306.
[30] Z. G. Wang, Eur. Phys. J. C75 (2015) 427.
[31] B. L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006) 232.
[32] Z. G. Wang, Eur. Phys. J. C68 (2010) 479.
[33] Z. G. Wang, Phys. Lett. B685 (2010) 59.
[34] P. Colangelo and A. Khodjamirian, hep-ph/0010175.
[35] S. Narison and R. Tarrach, Phys. Lett. 125 B (1983) 217.
[36] Z. G. Wang, Eur. Phys. J. C74 (2014) 2874; Z. G. Wang, Commun. Theor. Phys. 63 (2015) 466; Z. G. Wang and Y. F. Tian, Int. J. Mod. Phys. A30 (2015) 1550004.
[37] Z. G. Wang, Eur. Phys. J. C76 (2016) 387.
[38] Z. G. Wang and T. Huang, Phys. Rev. D89 (2014) 054019.