Super solutions of the model RB

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1 Introduction

Constraint satisfaction problems (CSPs) have come to play a prominent role in the area of computer science, statistical physics and information theory. A CSP instance involves a set of variables and a set of constraints, and the task is to find a solution (an assignment of variables that satisfies all the constraints) or to prove the unsolvability. Model RB [1] is a CSP model proposed to overcome the trivial unsatisfiability of standard Model B [2]. The satisfiability phase transition and exact threshold points have been established [1], and the benchmarks generated from model RB have been widely used in algorithm competitions [3]. Studies from the statistical physics perspective are also fruitful [4–6], where more structures of solution space such as clustering phase are revealed.

An instance I of Model RB contains a finite set V = {x₁, ..., xₙ} of variables and m = nlnn constraints (r > 0 is a constant). The domain of each variable xᵢ is D = {1, 2, ..., d}, where d = nⁿ (α > 0 is a constant). Each constraint Cᵢ contains k randomly chosen distinct variables Xᵢ = {xᵢ₁, xᵢ₂, ..., xᵢₖ}, and a set of satisfying tuples of values Rᵢ ⊂ Dᵈ with |Rᵢ| = pd⁻⁴ (0 < p < 1 is a constant and q = 1 − p). Constraint Cᵢ is satisfied if the tuple of values assigned to Xᵢ is in the corresponding relation Rᵢ.

Super solution is a concept introduced to formalize a solution with a certain degree of robustness or stability in many combinatorial optimization and decision problems. To quantify the robustness, (a,b)-super solution was introduced to constraint programming in [7]. Specifically, an (a,b)-super solution is one in which if the values assigned to a variables are no longer available, the solution can be repaired by assigning these variables with a new values and at most b other variables. (1,0)-super solutions have been studied for model RB with fixed k = 2 and d = √n in [8] and for random (3+p)-SAT in [9]. In this paper, we consider the (1,1)-super solution of model RB with no restrictions on k and d, and by asymptotic first-moment arguments we prove the following results.

Theorem 1 Let y be the number of (1,1)-super solutions of the model RB. Then

\[
\lim_{r \to 0} \mathbb{E} [Y] = \begin{cases} 
0, & \text{if } r > -\frac{\alpha}{\ln p} \\
+\infty, & \text{if } r < -\frac{\alpha}{\ln p} 
\end{cases}
\]

2 Proof of Theorem 1

We will use same notations as [8] up to Eq. (1). Let \( \Delta(\sigma, \tau) = \{|x_i|: \sigma(x_i) \neq \tau(x_i), \sigma, \tau \in D^i\} \), and \( \sigma \models C \) be that \( \sigma \) satisfies \( C \); let \( S(\sigma) \) be the event that \( \sigma \) is a solution for I; \( R(\sigma) \) be that there exists another solution \( \tau \) for I such that \( \Delta(\sigma, \tau) = \{x_i\} \), or \( \Delta(\sigma, \tau) = \{x_i, x_j\} \), \( j \neq i \); \( W(\sigma) \) be that \( \sigma \) is a (1,1)-super solution for I. Let \( M \subset X^n \) be all possible multi-sets of m unordered constraints \(|M| = \binom{n}{k}^m\), and \( E(e) \) be the event that \( e \in M \) is selected as the set of constraints of the random instance. Then for any \( e \in M \), \( P(E(e)) = \binom{n}{k}^{-m} \) and \( P(S(\sigma)|E(e)) = p^m. \) Now we have

\[
P(W(\sigma)) = P(S(\sigma) \cap \bigcap_{i < \infty} R_i(\sigma))
= \binom{n}{k}^{-m} \sum_{e \in M} P(R_i(\sigma) \cap S(\sigma) \cap E(e)).
\]

By inclusion/exclusion, we give the following bounds which are tight enough to estimate \( \mathbb{E} [Y] \).

\[
1 - \sum_{i = 1}^{n} P(R_i(\sigma) \cap S(\sigma) \cap E(e)) \leq P((\cap_{i < \infty} R_i(\sigma) \cap S(\sigma) \cap E(e)) + \\
1 - \sum_{i = 1}^{n} P(R_i(\sigma) \cap S(\sigma) \cap E(e)) + \\
\sum_{1 < j \leq n} P(R_j(\sigma) \cap S(\sigma) \cap E(e)).
\]

In the following, assume that \( \sigma \) is a solution of I, and \( C(x_i) \) is the set of constraints containing the variable \( x_i \), \( 1 \leq i \leq n \). Let \( \rho = 1 - q^{1+(d-1)(k-1)} \).

Lemma 1 Suppose \( y \in D_1(\sigma(x_i)) \), and \( \tau \) is an assignment such that \( \tau(x_i) = y, \ |\Delta(\sigma, \tau)| = 1 \text{ or } 2 \), then for any \( C \in C(x_i) \), \( P(\tau | \models C) = \rho \).

Proof. Note that there are \( 1 + q^{d-1}(k-1) \) possible ways for \( \tau \)
to assign variables in $C$, thus $P_\omega \notin C = q_1^{1+(d-1)(k-1)}$. Hence Lemma 1 follows.

Let $m_i = |C(x_i)|$, from Lemma 1, we have

$$P(R_\sigma \in D) = (1 - \rho^{m_i})^{d-1}. \quad (3)$$

**Lemma 2** Suppose $y \in D\setminus \{\sigma(x_i)\}$, $z \in D\setminus \{\sigma(x_j)\}$. If $\tau, \omega$ be assignments that $\tau(x_i) = y$ and $|\Delta(\tau, \sigma)| = 1 \text{ or } 2$; $\omega(x_j) = z$ and $|\Delta(\omega, \sigma)| = 1 \text{ or } 2$. Then for any $C \subseteq C(x_i) \cap C(x_j)$.

$$P(\tau, \omega \models C) = 1 - 2q^{1+(d-1)(k-1)} + q^{1+2(d-1)(k-1)}. \quad (4)$$

**Proof.** Note that there is exactly one common assignment of $\tau(C)$ and $\omega(C)$: $\tau(x_i) = \omega(x_i) = y$ and $\tau(x_j) = \omega(x_j) = z$. Therefore

$$P(\tau, \omega \models C) = 1 - P(\tau \notin C) \cdot P(\omega \notin C) + P(\tau, \omega \notin C)$$

$$= 1 - 2q^{1+(d-1)(k-1)} + q^{1+2(d-1)(k-1)}. \quad \square$$

Let $\tau, \omega$ be in Lemma 2, $l_{ij} = |C(x_i) \cap C(x_j)|$, then

$$P(S(\tau) \cup S(\omega) \models C(\sigma) \cap E(e))$$

$$= P(\tau \models C(x_i)) + P(\omega \models C(x_j)) - P(\tau \models C(x_i), \omega \models C(x_j))$$

$$= \rho^{m_i} + \rho^{m_j} - \rho^{m_i+m_j} \cdot 2l_{ij} \left(1 - 2q^{1+(d-1)(k-1)} + q^{1+2(d-1)(k-1)}\right)^{l_{ij}}$$

$$\geq \rho^{\min(m_i,m_j)}.$$ 

From Lemma 2, we can deduce that

$$P(R_\sigma \in D) \cap R_\sigma \models C = (1 - \rho^{\min(m_i,m_j)})^{d-1}. \quad (5)$$

Combining Eqs. (2), (3) and (4) we have

$$1 - \sum_{i=1}^{n} (1 - \rho^{m_i})^{d-1}$$

$$\leq P(R_\sigma \models C) = (1 - \rho^{\min(m_i,m_j)})^{d-1}. \quad (5)$$

- **Subcritical area**: $r > \frac{\alpha}{-\ln p}$. By Eqs. (1) and (5),

$$E[Y] \leq d^n \rho^m \sum_{e \in M} \left(1 - \sum_{i=1}^{n} (1 - \rho^{m_i})^{d-1}\right).$$

Note that $\rho^{m_i} \geq 1 - m_i q^{1+(d-1)(k-1)}$ and $q < 1$, thus

$$\sum_{i=1}^{n} (1 - \rho^{m_i})^{d-1} \leq q^{1+2(d-1)(k-1)} n (r n \ln n)^{d-1}$$

$$= \exp(1 + o(1)) (d-1)^2 (k-1) \ln q = o(1).$$

Therefore if $\alpha + r \ln p > 0$, then $E[Y]$ tends to infinity since $d^n \rho^m = (\alpha + r \ln p) n \ln n$.

- **Supercritical area**:

If $\alpha + r \ln p < 0$, by Eqs. (1) and (5),

$$E[Y] \leq d^n \rho^m \sum_{i=1}^{n} R(\sigma) \cap E(e))$$

$$\leq d^n \rho^m \left(1 - \sum_{i=1}^{n} \left(1 - \rho^{m_i}\right)^{d-1}\right)$$

$$+ d^n \rho^m \sum_{1 \leq i < j \leq n} \left(1 - \rho^{\min(m_i,m_j)}\right)^{(d-1)^2}.$$ 

Note that $m_i, m_j \leq r n \ln n$, $d = n^\alpha$ and $q < 1$, then

$$(1 + o(1)) (k-1)^2 \ln q = o(1).$$

Therefore if $\alpha + r \ln p > 0$, then

$$E[Y] \leq 2 d^n \rho^m = 2 \exp((\alpha + r \ln p) n \ln n) = o(1).$$

### 3 Conclusion

In this paper, we showed that the expected number of $(1,1)$-super solutions of model RB undergo phase transitions from 0 to infinity. Interestingly, the threshold is exactly the same with standard satisfiability threshold. Moreover, the restrictions of $r$ in Theorem 1 is independent of the constraint size $k$. Super solutions can be viewed as a certain type of “locally maximal” subset of the standard solutions, thus we believe our results reflect from another aspect the structure of solution space and stability of solutions. For future work, whether $r = \frac{\alpha}{-\ln p}$ is a critical point of being almost surely $(1,1)$-satisfiable to unsatisfiable is still open.

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