Unified Bayesian Frameworks for Multi-criteria Decision-making Problems

Majid Mohammadi

*Department of Computer Science, Vrije Universiteit Amsterdam, The Netherlands

Abstract

This paper introduces Bayesian frameworks for tackling various aspects of multi-criteria decision-making (MCDM) problems, leveraging a probabilistic interpretation of MCDM methods and challenges. By harnessing the flexibility of Bayesian models, the proposed frameworks offer statistically elegant solutions to key challenges in MCDM, such as group decision-making problems and criteria correlation. Additionally, these models can accommodate diverse forms of uncertainty in decision makers’ (DMs) preferences, including normal and triangular distributions, as well as interval preferences. To address large-scale group MCDM scenarios, a probabilistic mixture model is developed, enabling the identification of homogeneous subgroups of DMs. Furthermore, a probabilistic ranking scheme is devised to assess the relative importance of criteria and alternatives based on DM(s) preferences. Through experimentation on various numerical examples, the proposed frameworks are validated, demonstrating their effectiveness and highlighting their distinguishing features in comparison to alternative methods.

Keywords: MCDM; Bayesian hierarchical model; mixture model; probabilistic ranking.

1. Introduction

Multi-criteria decision-making (MCDM) is a prominent field within Operations Research that focuses on evaluating alternatives based on multiple criteria, considering the preferences of one or more decision-makers (DMs). In MCDM, an essential step involves eliciting DMs’ preferences and translating them into priorities or criteria weights, which are used for ranking, sorting, or selecting the best alternatives.

Various MCDM methods exist, each with its own elicitation procedure and mathematical model for deriving priorities. However, these methods often lack flexibility in handling different types of preferences and uncertainties, requiring the development of new models for each specific scenario. For instance, if preferences are expressed using fuzzy numbers or intervals instead of a standard scale, a new mathematical model is typically needed. Additionally, group MCDM presents challenges in aggregating preferences from multiple DMs, especially when preferences differ or uncertainties are involved.

Bayesian statistics, a flexible statistical theory based on probability theory and Bayes’ theorem, has gained popularity across domains such as machine learning [1] and marketing science [2] due to its ability to handle various inputs/outputs and address complex challenges. However, the application of Bayesian models in the field of MCDM remains limited, with only a few studies exploring its potential.

This paper aims to bridge the gap between Bayesian statistics and MCDM by presenting unified Bayesian frameworks for addressing diverse MCDM problems. The proposed framework accommodates different MCDM methods (e.g., AHP, SWING, BWM) and considers both certain and uncertain preferences (e.g., 1-9 scale, intervals, triangular and normal distributions). Rather than comparing different methods or preference types, the focus is on deriving prioritization and other subsequent tasks (e.g., grouping the DMs) based on expressed preferences.

1.1. Related Works

Different MCDM methods. Numerous MCDM methods have been developed, including trade-off methods [3], AHP (Analytic Hierarchy Process) [4], ANP (Analytic Network Process) [5], BWM (Best-Worst Method) [6], SWING [7], SMART (Simple Multi-Attribute Rating Technique) [8], and ordinal regression [9][10], among others. Each method employs a distinct mechanism for eliciting DMs’ preferences and translating them into priorities. For example, the AHP involves comparing all pairs of criteria, creating a pairwise comparison matrix (PCM) that is used to calculate priorities using methods such as the eigenvalue method [4] or geometric mean method [11]. The BWM, on the other hand, requires selecting the best and worst criteria and conducting pairwise comparisons between them and the others [6], with priorities determined through a max-min optimization problem. Other methods, like point allocation, swing, and SMART, elicit preferences through a vector of pairwise comparisons between criteria, and priorities are derived by normalizing this vector.

Uncertain Preferences. Most existing MCDM methods are developed based on specific preference types, such as the 1-9 scale in AHP and BWM. However, these scales do not capture the uncertainty in DMs’ preferences. To address this gap, several extensions have been proposed to incorporate different types of uncertainties. For instance, preferences can be expressed as intervals or using distributions like triangular or normal.
normal (also known as triangular and Gaussian numbers in fuzzy MCDM). Changing the preference type from a standard scale to, for example, the triangular distribution or intervals requires developing new mathematical models to translate preferences into priorities. Consequently, the literature offers a plethora of methods that cater to various uncertain preferences [12, 13, 14, 15, 16, 17, 18, 19].

Aggregation in Group MCDM. There are two classes of methods for aggregation in group MCDM. The first approach is the aggregation of individual judgment (AIJ) [20, 21], which aggregates individual preferences before treating the aggregated preferences as a single DM problem to determine aggregated priorities. For example, in AHP, PCMs from multiple DMs are aggregated using the geometric mean, and the aggregated priorities are calculated using methods like the eigenvalue method applied to the aggregated PCM. However, this approach is effective for AHP but may not be applicable to other methods like BWM, where each DM may have different reference points (i.e., best and worst criteria) for pairwise comparisons. Additionally, if the preference type is changed to intervals, a different aggregation method is required, as computing the geometric mean of intervals is not straightforward. The second approach, aggregation of individual priorities (AIP) [22, 23], involves computing priorities for each DM separately and then aggregating them (e.g., by the geometric mean). However, the AIP approach may not be universally applicable, particularly when intervals represent priorities, as shown in [6] and [24].

Bayesian Prioritization. Bayesian models provide a means to translate preferences into priorities. However, there are few studies on using Bayesian statistics for MCDM problems. For example, Altuzarra et al. [25] transformed the calculation of priorities in AHP into a Bayesian regression problem and estimated the priorities accordingly. They employed the stochastic AHP [26, 27], which assumes PCM contamination by log-normal noise, and converted weight ratios into a regression problem by taking logarithms. They developed a Bayesian model with a closed-form solution for computing group priorities. While their Bayesian model handles group preferences, it is neither AIJ nor AIP, as their method takes preferences as input and simultaneously outputs individual and group priorities. Altuzarra et al. [28] used a similar Bayesian model to reach consensus in the AHP negotiation process for group MCDM, developing a consensus-reaching procedure based on disagreement measures between DMs. Moreno-Jiménez et al. [29] developed a Bayesian model for systemic decision-making, addressing complex multi-actor decision-making problems. Moreover, a Bayesian model for BWM was developed to consider preferences of multiple DMs within the BWM framework [30]. Unlike Bayesian models for group AHP, the Bayesian BWM approach avoids transforming the estimation problem into regression and can estimate individual priorities and aggregated priorities using a hierarchical model. Additionally, a Bayesian model was proposed for ordinal regression [31], identifying an additive utility function based on pairwise comparisons among alternatives.

Grouping Decision-maker. In some scenarios, there are homogeneous subgroups with distinct preferences among the DMs. In such cases, aggregating preferences may not provide useful information, and it may be more valuable to identify homogeneous subgroups and analyze their preferences separately. This is particularly relevant in large-scale group MCDM scenarios with numerous DMs, where subgroup identification aids in dimensionality reduction and finding representatives for each subgroup [32]. Classical clustering methods like K-means [33, 34], fuzzy c-means [35, 36], and hierarchical clustering [37] have been widely employed to group DMs. Additionally, Gargallo et al. [38] developed a Bayesian mixture model for grouping DMs based on their preferences, specifically for the AHP. They employed the stochastic AHP formulation and estimated a mixture of Gaussian distributions for priorities. This model was applied to an e-democracy use case in the City Council of Zaragoza for budget allocation among four options. A similar model was proposed for grouping DMs in large-scale group decision-making [39].

1.2. Motivations and Contributions

The field of multi-criteria decision-making (MCDM) faces several challenges, such as the development of new mathematical models for different MCDM problems based on specific methods and preference types. For instance, if preferences are expressed using uncertain formats like triangular distributions or intervals, existing models designed for conventional preferences become inapplicable. This limitation necessitates numerous mathematical models to cover all possible combinations of MCDM methods, preference types, and problem variations. Consequently, certain challenges are addressed only for specific methods and cannot be generalized to others. For example, the study of criteria correlation is primarily focused on the ANP and lacks generalization to other methods.

Although Bayesian models show promise for addressing MCDM problems, the current literature has limitations. Existing Bayesian models are limited to specific MCDM methods, such as the AHP, and often assume preferences expressed only on a 1-9 scale. Furthermore, these models treat group preferences as if they were expressed by a single decision-maker, and neglect the advantages offered by Bayesian statistics in handling criteria correlation and different types of preferences. Additionally, the creation of procedures for evaluating and ranking alternatives based on distributional priorities is often overlooked, despite the potential hindrance of using other MCDM methods that rely on point estimates of criteria weights.

To address these gaps, this paper aims to develop a comprehensive Bayesian framework that accommodates different MCDM problems, diverse MCDM methods (e.g., AHP, SWING, BWM), and various preference types (e.g., a 1-9 scale, intervals, normal and triangular distributions). The key contributions of this paper can be summarized as follows:

• Probabilistic Formulation of Group MCDM: We approach group MCDM from a probabilistic perspective, providing appropriate likelihood distributions for different MCDM methods with their standard preference types. By formulating group MCDM in a probabilistic framework,
we can leverage various Bayesian models within the proposed framework, addressing aggregation and criteria correlation (Section 2).

- **Handling Uncertain Preferences**: We study the impact of uncertain preferences on Bayesian modeling in MCDM. Specifically, we present a Bayesian formulation for interval preferences, as well as preferences expressed through normal and triangular distributions. By incorporating these uncertain preferences into the Bayesian framework, we enhance the flexibility and realism of MCDM models (Section 3).

- **Bayesian Mixture Model for Grouping Decision-Makers**: We introduce a Bayesian mixture model to identify homogeneous subgroups of decision-makers with distinct preferences. This model is applicable to MCDM methods with both certain and uncertain preferences, providing a powerful tool for large-scale group MCDM scenarios (Section 4).

- **Evaluation, Ranking, and Sorting of Alternatives**: We address the crucial task of evaluating, ranking, and sorting alternatives based on distributional priorities obtained from the Bayesian models. By incorporating distributional priorities into the decision-making process, we overcome the limitations of other MCDM methods that rely on point estimates of criteria weights, enabling a more comprehensive and robust analysis of alternatives (Section 5).

The contributions of the proposed unified Bayesian framework to the field of MCDM are summarized in Figure 1. Through numerical examples, we validate the effectiveness of the proposed frameworks and highlight their advantages compared to existing methods (Section 6). The paper concludes with a discussion of the findings, implications, and directions for future research in the field of Bayesian modeling for MCDM (Section 7).

2. Bayesian Group Decision-making

In this section, we present a Bayesian framework for modeling group decision-making and discuss the likelihood functions for various well-known MCDM methods. We also consider the correlation between criteria within the Bayesian modeling.

Let us first specify the notation used in this article. The set of n criteria is shown by $C = \{c_1, c_2, ..., c_n\}$, and we assume that there are m alternatives $A_1, A_2, ..., A_m$ to be evaluated based on A, each row of which is denoted by $a_i$, containing the performance of $A_i$ for the given set of criteria. The number of DMs is denoted by $R$, and $w_r$, $r = 1, ..., R$, represents the criteria weights for the $r^{th}$ DM. For group MCDM, the aggregated weight is shown by $w^*$, and $w^k$, $k = 1, ..., K$, represents the weights of $K$ subgroups of DMs. The preferences of DMs given by any MCDM method are also shown by $g_r$, $r = 1, ..., R$. We also use the colon in the sub- or superscript to show a set of variables, so $w^r:R$, for example, denotes the set $\{w^r\}_{r=1}^R$. Whenever the superscripts like $r$ for a DM are not used, the variable is assumed to be from an arbitrary DM, e.g., $g$ without a superscript is the preference of an arbitrary DM.
2.1. Bayesian Modeling of Group MCDM

To develop a Bayesian model for group MCDM, we need to specify the inputs to the model and the expected output. The inputs are the preferences expressed by all decision-makers using any of the MCDM methods, and the outputs are the criteria weights for each individual decision-maker and the aggregated weights representing the overall priorities of the group. Therefore, we are interested in the joint probability distribution over $w^{1,R}$ and $w^*$ given the preferences $g^{1,R}$, i.e.,

$$P(w^{1,R}, w^*|g^{1,R}). \tag{1}$$

If we compute or estimate the above joint probability distribution, we can obtain the probability distribution of each variable (i.e., $w^*$ or $w^*$) by marginalizing over all the other variables. The above joint distribution can be written using Bayes’ theorem:

$$P(w^{1,R}, w^*|g^{1,R}) \propto P(g^{1,R}|w^{1,R}, w^*)P(w^{1,R}, w^*)$$

$$= P(g^{1,R}|w^{1,R}, w^*)P(w^{1,R}|w^*)P(w^*), \tag{2}$$

where the second equality is acquired by using the probability chain rule on the second term on the right-hand side of the equation, and we use the proportion sign $\propto$ since the denominator in the Bayes’ rule is ignored as it only serves for normalization purposes for all the estimated parameters.

Equation (2) can be simplified based on two critical observations. First, we consider the preferences of all decision-makers to be independent of each other. This means that the priorities of decision-maker $r$ (i.e., $w^r$) depend only on its own preferences (i.e., $g^r$) and not on the preferences of other decision-makers. This independence leads to the following equation:

$$P(g^{1,R}|w^{1,R}, w^*) = \prod_{r=1}^{R} P(g^r|w^r, w^*).$$

In addition, the relation between $g^r$, $w^r$, and $w^*$ is Markovian: We expect to compute $w^*$s based on $g^r$‘s and then compute $w^*$ based on $w^*$‘s. As a result of this observation, $g^r$ is conditionally independent of $w^r$ given $w^*$, i.e.,

$$P(g^r|w^*, w^r) = P(g^r|w^*), \quad r = 1, ..., R.$$

Considering these two observations, we can rewrite the first term on the right-hand side of equation (2) as follows:

$$P(g^{1,R}|w^{1,R}, w^*) = \prod_{r=1}^{R} P(g^r|w^r, w^*) = \prod_{r=1}^{R} P(g^r|w^r). \tag{3}$$

where the first equality is based on the independence between the decision-makers’ preferences, and the second equality is obtained by assuming the conditional independence of $g^r$ and $w^r$ when $w^*$ is known. Plugging equation (3) into equation (2), it follows:

$$P(w^{1,R}, w^*|g^{1,R}) \propto P(g^{1,R}|w^{1,R}, w^*)P(w^{1,R}|w^*)P(w^*)$$

$$= P(w^*) \prod_{r=1}^{R} P(g^r|w^r)P(w^r|w^*). \tag{4}$$

Equation (4) reveals that, for group MCDM, we need to specify two distributions: $P(g^r|w^r)$, which represents the translation of a decision-maker’s preferences into a distribution over priorities, and $P(w^r|w^*)$, which describes how individual priorities are combined to form the aggregated priorities.

The Bayesian modeling approach offers several advantages as described by equation (4):

- The distribution $P(w^r|w^*)$ is independent of $g^r$, which means that once this distribution is specified, it can be used for all MCDM methods and preference types.
- Since $P(g^r|w^*)$ is independent of other priorities $w^r$, $r' \neq r$, we only need to identify the relationship between preferences and priorities for a single decision-maker, and the specification of the remaining parameter remains untapped. This flexibility allows us to use the Bayesian framework with various (and possibly new) MCDM methods.
- The preferences $g^r$ can vary among decision-makers, coming from different MCDM methods or expressed using different preference types. However, the Bayesian model developed based on equation (4) can still compute individual and aggregated weights by providing a distribution for $g^r$ given $w^r$. This property allows for flexibility in preference expression and enables data pooling from different studies that used other MCDM methods for preference elicitation.

Next, we will specify the appropriate likelihood functions for $P(g^r|w^r)$ in equation (4) for well-known MCDM methods. We will also discuss the proper distributions of priorities in cases where the criteria are considered dependent or independent.

2.2. From Preferences to Priorities: Likelihood Distributions for Several MCDM Methods

Different MCDM methods elicit preferences from decision-makers in distinct ways, requiring an intermediate step to translate them into priorities. However, for the proposed Bayesian framework in equation (4), we only need to specify the corresponding likelihood function $P(g^r|w^r)$, which represents the translation of the given preferences $g^r$ to priorities $w^r$. In the following, we discuss some distributions that can adequately serve as likelihood functions for different MCDM methods.

**Multinomial Distribution**

Several MCDM methods model preferences as a vector of integers, with each element representing the importance of the associated criterion compared to others. We discuss three such methods below:

- **Point Allocation Method:** In this method, a decision-maker (DM) allocates a certain number of points to each criterion, with more critical criteria receiving higher points. Typically, the DM is asked to allocate 100 points across all the considered criteria.
- **SMART (Simple Multi-attribute Rating Technique):** This method involves two processes: rating the alternatives and weighting the criteria [3]. For criteria weighting, the DM is asked to rank the criteria from best to worst in terms of their importance. The least and most important criteria are assigned values of 10 and 100, respectively, while the criteria in between are assigned values between 10 and 100.

- **Swing Method:** In this method, the DM is first asked to select the worst alternative for each criterion [4]. Then, they are allowed to change the outcome of only one criterion to the best alternative. The first criterion selected by the DM is considered the most important and is assigned a high value (typically 100). The remaining criteria are evaluated relative to the most important criterion, and their assigned values are determined accordingly.

The result of the elicitation process in each of the above methods is a vector of numbers, with each element's magnitude being commensurate to the importance of the corresponding criterion. For example, let \( A'_r \) be the vector of numbers from the above methods for the \( r \)th DM, then the weight of each criterion is computed by dividing each element by the sum of all the elements as:

\[
w'_j = \frac{A'_j}{\sum_{i=1}^{n} A'_i}, \quad j = 1, \ldots, n. \tag{5}
\]

We now discuss that the likelihood function can be modeled by the multinomial distribution, i.e.,

\[
A'_r | w' \sim \text{Multinomial}(w'), \tag{6}
\]

where \( \sim \) indicates that the term on the left follows the distribution on the right. If we apply the statistical inference given the multinomial distribution to estimate the weights \( w' \), we get:

\[
w'_j \approx \frac{A'_j}{\sum_{i=1}^{n} A'_i}, \quad j = 1, \ldots, n. \tag{7}
\]

The approximation sign \( \approx \) is used in the above equation because different statistical inference methods (e.g., frequentist or Bayesian) may yield slightly different results. However, the outcome is still in the neighborhood of the fraction of each element in the given vector to the sum of all the numbers. The proximity of equation (7) and equation (5) indicates that the multinomial distribution can properly model the preferences expressed in the methods mentioned above.

---

### Two Multinomial Distributions

In the Best-Worst Method (BWM), the DM is asked to select the best and worst criteria and perform pairwise comparisons between the best criterion and all others, as well as between all others and the worst criterion. The elicitation process involves the following three main steps for each DM:

- The \( r \)th DM selects the best and worst criteria, denoted as \( c_B \) and \( c_W \) from the criteria set \( C \).

- The DM expresses their preferences for \( c_B \) relative to the other criteria using a number on a scale from 1 to 9, where 1 indicates equal importance and 9 indicates significantly higher importance. This step results in the creation of the “Best-to-Others” vector, denoted as \( A'_B \) for DM \( r \).

- The DM is then asked to express their preferences for all other criteria relative to \( c_W \) using the 1-9 scale. This step leads to the creation of the “Others-to-Worst” vector, denoted as \( A'_W \) for DM \( r \).

The BWM computes the criteria weights by solving an optimization problem to find a set of priorities that minimizes the maximum distance to the elicited pairwise comparisons. Thus, the weight of the best criterion is approximately proportional to the elicited pairwise comparisons:

\[
\frac{w_B}{w_j} \approx A_{Bj}, \quad j = 1, \ldots, n, \tag{8}
\]

where \( w_B \) is the weight of the best criterion. By the same token, the weight ratio of the worst criterion to the other criteria should be commensurate to \( A_W \):

\[
\frac{w_j}{w_W} \approx A_{Wj}, \quad j = 1, \ldots, n, \tag{9}
\]

where \( w_W \) is the weight of the worst criterion.

Since we have two types of pairwise comparisons, i.e., \( g' = [A'_B, A'_W] \), the likelihood function in equation (4) can be written as:

\[
P(g'|w') = P(A'_B | A'_W | w') = P(A'_B | w')P(A'_W | w'), \tag{10}
\]

where the last equality is obtained since the vectors \( A'_B \) and \( A'_W \) are elicited independently. In \( A_W \), as suggested by equation (4), the magnitude of each element is commensurate to the importance of the corresponding criterion, meaning that the multinomial distribution can be used to model \( A_W \):

\[
A'_W | w' \sim \text{multinomial}(w'). \tag{11}
\]

Applying the statistical inference to this equation, one gets:

\[
w_j \approx \frac{A_{Wj}}{\sum_{i=1}^{n} A_{Wi}},
\quad w_W \approx \frac{A_{WW}}{\sum_{i=1}^{n} A_{Wi}}. \tag{12}
\]

By dividing the above two equations, it follows:

\[
\frac{w_j}{w_W} \approx \frac{A_{Wj}}{A_{WW}} = A_{Wj}, \tag{13}
\]

where the last equality is obtained since \( A_{WW} = 1 \). Therefore, the result of statistical inference techniques when \( A_W \) is modeled by a multinomial distribution (as in equation (13)) yields a similar result to the original BWM (as in equation (9)). Similarly, the “Best-to-Others” can be modeled using a multinomial distribution, but the criteria weights are commensurate with the inverse of elements in \( A_B \) since it contains the comparison of
the best criterion to others, meaning that a higher value in $A_B$ denotes a lesser important criterion. Thus, one can write:

$$A_B' \sim \text{multinomial}(1/w').$$  \hfill (14)

Writing the ratios between the elements of $A_B'$ and the corresponding weight, equation (13) readily follows from equation (14). Thus, the preferences expressed in the BWM form can be adequately modeled by using two multinomial distributions.

**Dirichlet distributions**

The multinomial distribution can be used if the preferences of DMs are expressed by some positive integers. If the real numbers are instead being used, the multinomial distribution cannot be utilized. In the AHP, we have a complete set of pairwise comparisons between all the criteria under study, where the pairwise comparison can be stated by the 1-9 scale as well as its inverse, i.e., $\{\frac{1}{9}, \frac{1}{6}, \ldots, 1.2, \ldots, 9\}$. Therefore, the multinomial distribution cannot be used for such preferences.

Let $M'$ be the PCM elicited from the $r^{th}$ DM and $\hat{M}'$ be its normalized version obtained by dividing each column by its sum. Each column of $\hat{M}'$ is potential criteria weights, as the criteria weights are computed by the geometric mean of all columns in the AHP geometric mean method. As a result, the likelihood function in equation (4) can be written as:

$$P(g'|w') = P(M'|w') = \prod_{j=1}^{n} P(\hat{M}'|w'),$$  \hfill (15)

where $\hat{M}_j$ is the $j^{th}$ column of matrix $\hat{M}$, and the last equality is derived based on the assumption that columns in a PCM are conditionally independent of each other given the associated weights. This assumption is mild since it is typically assumed that the pairwise comparisons are entirely independent of each other. Still, here we assume that the columns are only conditionally independent (and not each element of them) given the associated weights of the DM.

The columns of $\hat{M}'$ are normalized, satisfying non-negativity and unit-sum constraints. To model them, we use the Dirichlet distribution, which adheres to these constraints. We parameterize the distribution based on its mean as follows:

$$\hat{M}_j' \sim \text{Dir}(\gamma'w'),$$  \hfill (16)

where $\text{Dir}$ is the Dirichlet distribution, $\gamma'$ is a positive number, also known as concentration parameter, and $w'$ acts as the mean of the distribution. The standard Dirichlet distribution takes a vector of numbers and returns a probability distribution, while equation (16) is based on a positive number and its mean. In order words, equation (16) assumes that the columns of $\hat{M}'$ are in the neighborhood of the weights $w'$, with the closeness determined by the parameter $\gamma'$. So, if all the columns represent the same weights (i.e., the fully-consistent case), $\gamma'$ would take a higher value, while its value decreases otherwise. Since $\gamma'$ is a random variable and needs to be estimated, we need to specify a prior for it. We use the gamma distribution as the prior, as it ensures non-negativity:

$$\gamma' \sim \text{gamma}(a, b),$$  \hfill (17)

where $a$ and $b$ are the shape and rate parameters, respectively. Setting $a = b = 0.01$ ensures that the mean of the gamma distribution is 1 and its standard deviation is 100, following the maximum entropy principle. The large standard deviation indicates a lesser impact of the prior, and the values of $\gamma'$ are determined based on the DMs’ input preferences.

### 2.3. Priorities Aggregation: Likelihood Distributions of Weights

This section discusses the appropriate distributions for the weight terms in equation (4). The weights in MCDM must satisfy the non-negativity and unit-sum (or constant-sum) constraints. Such data is known as compositional data in statistics.

**Definition 2.1** (41). A composition $w \in \mathbb{R}^n$ is a vector with positive components $w_1, \ldots, w_n$ whose sum is one (or any other fixed number).

Compositions lie on multi-dimensional simplices, which means that typical distributions such as univariate and multivariate normal distributions cannot be used to model this type of data. However, there are proper distributions for data on simplices. In the following, we discuss two such distributions.

**Dirichlet Distribution**

As was used to model the preferences of DMs in the AHP, the Dirichlet is arguably the most well-known distribution for modeling compositions. To use this distribution in equation (4), we can write:

$$w' \sim \text{Dir}(\gamma'w')$$  \hfill (18)

where $\gamma' > 0$ is the concentration parameter, and the aggregated weight $w^*$ is the mean of the Dirichlet distribution. equation (18) is the probabilistic way of aggregation: It assumes that the criteria weights of all DMs is around the aggregated weights $w^*$, and $\gamma'$ governs the proximity of $w'$ to $w^*$. During the inference process, $w^*$ is estimated as it represents the center of the weights from different DMs, providing a proper aggregation from a probabilistic perspective. Since $\gamma'$ is another parameter that needs to be estimated, we again use the gamma distribution as a prior for it:

$$\gamma' \sim \text{gamma}(a^*, b^*),$$  \hfill (19)

where $a^*$ and $b^*$ are the shape and rate parameters, respectively, and are set to be 0.01. For the aggregated priorities $w^*$, we can again use the Dirichlet distribution:

$$w^* \sim \text{Dir}(\alpha),$$  \hfill (20)

where $\alpha \in \mathbb{R}^n$ is a positive vector and a hyperparameter to the model. Following the maximum entropy principle, we set the values of the elements in $\alpha$ to $1/n$.

Figure 2 shows the graphical model of group decision-making without assuming any correlation between criteria. In this figure, the unshaded circles represent the random variables, while the shaded circle represents the observed variable, i.e., the preferences $g$. The box represents the replicates, indicating that $w$ and $g$ are replicated $R$ times, where $R$ is the number of DMs.
Remark 2.2. The Bayesian model depicted in Figure 2 does not have a closed-form solution. However, various Markov chain Monte Carlo (MCMC) sampling methods are available that can adequately estimate the posterior distribution of the model. The result of such estimation is a set of samples from the posterior distributions, which can be used to reconstruct the posterior distribution or for subsequent analyses. We employ MCMC sampling for all the Bayesian models discussed throughout the remainder of this article.

Logistic-Normal Distribution

The Dirichlet distribution assumes a strong independence assumption among the criteria under study. If the correlation between the criteria needs to be taken into account, a more flexible distribution is required. The logistic-normal distribution is one of the well-known distributions on the simplex that can handle criteria correlation. In the logistic-normal distribution, we first transform the compositions from the simplex to the real space by using the log-ratio transformation. Then, the transformed data in the real space can be modeled using the normal distribution, as it lies in the real space and not on the simplex. To achieve this, we can use the centered log-ratio transformation (CLR) to transform compositions into the real space.

Definition 2.3 (H1). The centered log-ratios (CLR) of a composition like \( w \) is a vector like \( \hat{w} \in \mathbb{R}^n \) whose elements are calculated as:

\[
\hat{w}_j = \log\left(\frac{w_j}{\prod_i w_i^{1/m}}\right) \tag{21}
\]

The vector \( \hat{w} \) in the above equation lies on the n-dimensional real space. To use the logistic-normal distribution, we first transform the weights from all the DMs, as well as the aggregated weights, into the real space:

\[
\hat{w}' = \text{CLR}(w'), \quad r = 1, \ldots, R, \\
\hat{w}^* = \text{CLR}(w^*). \tag{22}
\]

Then, the transformed weights \( \hat{w}'s \) can be modeled using the multivariate normal distribution, i.e.,

\[
\hat{w}' \sim \text{MVN}(\hat{w}^*, \Sigma), \tag{23}
\]

where \( \text{MVN} \) is the multivariate normal distribution, \( \hat{w}' \) is the mean, and \( \Sigma \) is the covariance matrix of the normal distribution containing the correlation between the criteria. Equation (23) adheres to the probabilistic aggregation as in equation (18), since \( \hat{w}' \) is assumed to be at the center of \( \hat{w}^* \)'s. Similarly, we need to specify the prior on \( \hat{w}' \). To that end, we use another multivariate normal distribution:

\[
\hat{w}^* \sim \text{MVN}(\mu, \Sigma), \tag{24}
\]

where \( \mu \) is a hyperparameter for the prior distribution, and we set all of its elements to \( 1/n \) to ensure an uninformative prior. The covariance matrix for the multivariate normal distribution needs to be available prior to conducting the inference. It can be elicited from the DM or computed from the performance matrix by calculating the correlation between the performance of different alternatives on pairs of criteria. The covariance matrix is then used to estimate the weights, taking into account the correlation between the criteria.

Figure 2 shows the graphical model for the group decision-making with the correlation between the criteria being considered.

Remark 2.4. The proposed Bayesian group decision-making model, as in equation (2), can also be applied to the preferences of a single DM. In this case, the estimation of \( \hat{w}' \) is not required, and terms \( P(w'|w) \) and \( P(w^*) \) are removed from equation (2). In this case, the weight distributions encompass the range of possible priorities based on the expressed preferences. If a DM is fully consistent, the standard deviation of the weight distribution is infinitesimal, indicating minimal uncertainty in the priorities. As inconsistency increases, the standard deviation of the weight distribution grows, reflecting higher uncertainty in the model.

2.4. Credal Ranking: A Probabilistic Ranking Scheme

When priorities are computed deterministically as a compositional vector, a criterion is considered more important than another if its corresponding weight is higher. However, this statement may not hold true if the weights are not deterministic, as in the case of interval estimates [24] with overlaps between criteria weights or the distributional estimates obtained from the
proposed Bayesian model. In the case of distributional priorities, it is possible to provide more information about the level of certainty when comparing the importance of criteria. The following two definitions lay the foundation for developing such a ranking scheme.

Definition 2.5 (30). For two criteria $c_i$ and $c_j$, the credal ordering $O$ is a quadruple $O(c_i, c_j, r, d)$, where:

- $r$ represents the relation between the two criteria, i.e., $\geq, \leq, =$;
- $d \in [0, 1]$ is the confidence level associated with the relation $r$ between the criteria.

The confidence level in Definition 2.5 needs to be computed based on the distributional priorities. These confidence levels provide decision-makers with more information about the level of certainty in the preferences of the DMs and how one can model them in the preferences elicited from the DMs.

To model uncertain inputs, we seek to estimate the following joint probability:

$$P(w^{1\cdot R}, w^*, g^{1\cdot R}|\hat{g}^{1\cdot R}),$$

(27)

where $\hat{g}^{1\cdot R}$ is the preferences of DMs expressed in an uncertain way (e.g., intervals), $g^{1\cdot R}$ are random variables that need to be estimated based on the uncertain preferences $\hat{g}^{1\cdot R}$. In contrast to equation (2) where $g^{1\cdot R}$ is given, in equation (27), $g^{1\cdot R}$ are random variables that need to be estimated based on the uncertain preferences $\hat{g}^{1\cdot R}$.

Applying Bayes’ rule to the joint probability in equation (27), we have:

$$P(w^{1\cdot R}, w^*, g^{1\cdot R}|\hat{g}^{1\cdot R}) \propto P(g^{1\cdot R}|\hat{g}^{1\cdot R})P(g^{1\cdot R}|w^{1\cdot R}, w^*)P(w^{1\cdot R}, w^*)$$

$$= P(w^*) \prod_{r=1}^{R} P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})P(g_{r}^{1\cdot R}|w^r)P(w^r|w^*).$$

(28)

In equation (28), there is an additional term $P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})$ in the likelihood function compared to equations (2) and (6). The remaining parts of the equation are the same. From a Bayesian perspective, this means that we can use the model specifications provided in the previous section for $P(g_{r}^{1\cdot R}|w^r)$ and $P(w^r|w^*)$, and add another level to the hierarchical model for $P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})$ to handle the uncertainty. Figure 4 shows the probabilistic graphical model presented by equation (28).

In the following, we discuss different types of uncertainty in the preferences of the DMs and how one can model them using equation (28), as well as their effects on other terms in

Figure 3: The graphical model of the group decision-making model with criteria correlation being considered.

**Definition 2.6.** For a set of criteria $C = \{c_1, ..., c_n\}$, the credal ranking contains the credal ordering between all pairs of criteria in $C$.

### 3. Preferences with Uncertainty

The preferences elicited from the DMs could carry a level of uncertainty. For example, the preferences could be expressed as intervals or by some distributions like triangular. The Bayesian models put forward in the previous section cannot directly address uncertain preferences. However, many uncertain preferences can be modeled using probability density functions (PDFs). For example, interval preferences can be adequately modeled by the uniform distribution, where the values within the interval are considered equally likely.

To model uncertain inputs, we seek to estimate the following joint probability:

$$P(w^{1\cdot R}, w^*, g^{1\cdot R}|\hat{g}^{1\cdot R}),$$

(27)

where $\hat{g}^{1\cdot R}$ is the preferences of DMs expressed in an uncertain way (e.g., intervals), $g^{1\cdot R}$ are random variables that need to be estimated based on the uncertain preferences $\hat{g}^{1\cdot R}$.

Applying Bayes’ rule to the joint probability in equation (27), we have:

$$P(w^{1\cdot R}, w^*, g^{1\cdot R}|\hat{g}^{1\cdot R}) \propto P(g^{1\cdot R}|\hat{g}^{1\cdot R})P(g^{1\cdot R}|w^{1\cdot R}, w^*)P(w^{1\cdot R}, w^*)$$

$$= P(w^*) \prod_{r=1}^{R} P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})P(g_{r}^{1\cdot R}|w^r)P(w^r|w^*).$$

(28)

In equation (28), there is an additional term $P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})$ in the likelihood function compared to equations (2) and (6). The remaining parts of the equation are the same. From a Bayesian perspective, this means that we can use the model specifications provided in the previous section for $P(g_{r}^{1\cdot R}|w^r)$ and $P(w^r|w^*)$, and add another level to the hierarchical model for $P(\hat{g}_{r}^{1\cdot R}|g_{r}^{1\cdot R})$ to handle the uncertainty. Figure 4 shows the probabilistic graphical model presented by equation (28).
we focus only on modeling di-
equation (28), particularly the likelihood function for individ-
ual weight, i.e., \( P(q' | w') \). For brevity and to avoid repetition, we focus only on modeling different uncertain preferences for the BWM. Nonetheless, the results can be generalized to other methods discussed in the previous section.

3.1. Uncertainty as Normal Distribution

Uncertainty in the preferences of DMs can be modeled by using a normal distribution, which is also used as a number in fuzzy logic/theory [42, 43]. To model the normal uncertain inputs in the BWM, we assign a mean and a standard deviation for each of the “Best-to-Others” and “Others-to-Worst” vectors. Let \( \mu_{A_B} \) and \( \mu_{A_W} \) be the mean vectors, and \( \sigma_{A_B} \) and \( \sigma_{A_W} \) be their corresponding standard deviation. The assumption is that the \( A_B \) is in the neighborhood of \( \mu_{A_B} \), and this closeness is governed by the given standard deviation. This can be mathe-
matically expressed as:

\[
\begin{align*}
\mu_{A_B} & \sim N(A_B^r, \sigma_{A_B}^r), \quad r = 1, \ldots, R, \\
\mu_{A_W} & \sim N(A_W^r, \sigma_{A_W}^r), \quad r = 1, \ldots, R.
\end{align*}
\tag{29}
\]

where \( N \) is a univariate normal distribution. Therefore, the \( A_B^r \)'s could be estimated based on the given mean and standard deviation. This indeed affects the use of the multinomial distribution for modeling the term \( P(A_B^r | w') \), because the values of \( A_B^r \) and \( A_W^r \) are sampled from a normal distribution and are no longer integers. Instead, we need to use the Dirichlet distribution as:

\[
\begin{align*}
A_B^r & \sim \text{Dir}(\gamma^r | w'), \quad r = 1, \ldots, R, \\
A_W^r & \sim \text{Dir}(\gamma^r | w'), \quad r = 1, \ldots, R, \\
\gamma^r & \sim \text{gamma}(a, b), \quad r = 1, \ldots, R.
\end{align*}
\tag{30}
\]

where the parameters of the gamma distribution are set as \( a = b = 0.01 \) to follow the maximum entropy principle. If we use the normal inputs (or other uncertain types of inputs) for the preferences, the multinomial distribution must also be replaced with the Dirichlet for other methods, such as swing, SMART, and point allocation.

Remark 3.1. The normal distribution is defined on the real space, so it can take any values, including negative ones, which might not be acceptable for many MCDM methods. However, the likelihood of such an occurrence is not very high given a proper standard deviation. Nonetheless, one can use the truncated normal distribution, where the distribution is bounded only by the acceptable values of the parameter. Such a feature is supported by almost any probabilistic language for estimating the posterior distribution.

3.2. Interval Preferences

A DM can express their preferences via intervals, indicating that all values within the interval are valid for the corresponding pairwise comparison. From a probabilistic perspective, the values within the interval are considered equally likely, allowing us to use the uniform distribution to model the intervals.

We can model interval preferences using the truncated normal distribution, where the mean is the midpoint of the interval and the standard deviation is very broad. Alternatively, the beta distribution with parameters \( \alpha = \beta = 1 \) is equivalent to the uniform distribution in the \([0, 1]\) interval, and it can be fur-
ther shifted or stretched to accommodate any other intervals expressed by the DMs.

We can also treat interval preferences as a prior uniform distribution and use the capabilities of Bayesian samplers to im-
pose such a prior. Typically, we only specify the limits of the variables, and the samplers draw values uniformly from within those limits unless other distributions are imposed. In our implementation, we take advantage of this approach to define a uniform distribution for interval values.

Remark 3.2. There is a distinction between specifying the bounds of a variable for a prior uniform distribution and using the truncated normal distribution with a standard deviation. In the former, the variable is assumed to follow a uniform distribution, and the posterior of the variable will reflect this assumption. In the latter, the posterior may be more focused around specific values within the intervals, incorporating the values of a pairwise comparison that align more closely with other elicited preferences.

3.3. Triangular-valued Preferences

Another way of expressing uncertainty is through triangular numbers, which are commonly used in fuzzy MCDM [44]. The triangular distribution can effectively model triangular numbers. However, the triangular distribution is not differentiable, making it difficult for Bayesian samplers to handle.

To address this issue, one approach is to rewrite the trian-
gular distribution as the subtraction of two uniform distributions. By doing so, we can avoid the non-smoothness of the triangular distribution, enabling sampling based on the uniform distributions.

Alternatively, we can use the built-in functions provided by probabilistic languages to augment the log density of the posterior distribution. For example, for a symmetric triangular distri-
bution in the interval \([\alpha, \beta]\), we can write:

\[
\text{triangular}(x | \alpha, \beta) = \frac{2}{\beta - \alpha} \left( 1 - \left| x - \frac{\alpha + \beta}{2} \right| \right). \tag{31}
\]

When dealing with the Bayesian model for triangular num-
bers based on the above formula, we need to specify the limits of \( x \). It is worth noting that the aforementioned procedure for the triangular distribution is not differentiable, but there are typ-
ically built-in methods for handling absolute values that can be utilized instead.

Let \( \hat{g} \) represent a pairwise comparison expressed by a sym-
metric triangular number containing \( \alpha \) and \( \beta \). We can then use the absolute value function (available in many Bayesian sampling languages) and augment the log density of the posterior distribution to impose the prior triangular distribution on the corresponding pairwise comparisons.
Remark 3.3. Dealing with the symmetric triangular distribution is simpler from a sampling perspective. However, this does not mean that asymmetric triangular distributions cannot be handled, but different strategies are required to enable sampling. Nonetheless, symmetric triangular distributions are widely used and can address many MCDM problems.

Remark 3.4. From a sampling perspective, the triangular distribution is not ideal due to its non-smoothness. A suitable differentiable surrogate is the beta distribution, which can take on different shapes similar to other triangular distributions. The differentiability of the beta distribution leads to more stable Bayesian sampling and better posterior estimation. However, the beta distribution has not been extensively studied within the MCDM community, and further research is needed to explore its pros and cons for MCDM problems, which is beyond the scope of this article.

4. Grouping Decision-makers: A Bayesian Mixture Model

In this section, we present a Bayesian model for grouping DMs on their preferences, in contrast to the Bayesian models discussed thus far, which were developed for aggregation. This model can be used in various problems, such as identifying the coalition of actors in decision-making problems and finding subgroups of users with similar interests for better-customized recommendations. Furthermore, developing such a model can also be helpful in the negotiation process for group MCDM problems [45], where identifying subgroups can facilitate better negotiation and ultimately lead to consensus.

In the group decision-making model discussed in the previous section, one of the parameters to be estimated is the aggregated criteria weights, representing the overall preferences of the group. Now, we aim to identify multiple homogeneous subgroups of DMs with similar preferences.

Let $Z$ be the number of homogeneous subgroups of the DMs, and $\Omega_r$, $z = 1, ..., Z$, represent the center of each group. Furthermore, let $\theta^r \in R^Z$ denote the membership of each DM in different subgroups, where $\theta^r_j \geq 0$ for $j = 1, ..., Z$, and $\sum_j \theta^r_j = 1$. Also, let $y^r \in \{1, ..., Z\}$ be the group label of DM $r$. Therefore, for a Bayesian mixture model, we need to estimate the following joint distribution:

$$P(\theta^1:R, y^1:R, w^1:R, \Omega^1:Z | g^1:R),$$

where $g^r$ is the preferences of DM $r$ expressed by any of the MCDM methods. If the Bayes’ rule is applied, the joint distribution can be written as:

$$P(g^1:R, y^1:R, w^1:R, \Omega^1:Z | g^1:R) = \prod_{r=1}^R P(\theta^r)P(y^r | \theta^r)P(w^r | \theta^r, \Omega^r)P(g^r | w^r)$$

In equation (33), the term $P(g^r | w^r)$ has been extensively discussed in Section 2 allowing the preferences $g^r$ to be expressed using any MCDM method, and the appropriate likelihood function will also be used here. Additionally, the uncertain inputs discussed in Section 3 can be incorporated into the mixture, similar to equation (29). However, for brevity, we will focus only on certain preferences in this section.

Now, we specify the proper distribution for each term in equation (33). Since $\theta^r$ is a discrete probability distribution representing the membership of the $r^{th}$ DM in all subgroups, we can model it using a Dirichlet distribution:

$$\theta^r \sim \text{Dir}(\lambda), \quad \theta^r \in R^Z, \quad r = 1, ..., R, \quad (34)$$

where $\lambda \in R^Z$ is a hyperparameter. The values in $\theta^r$ denote the probabilities of the $r^{th}$ DM belonging to each of the subgroups. To identify the specific subgroup, we can write:

$$y^r \sim \text{cat}(\theta^r), \quad r = 1, ..., R, \quad (35)$$

where $\text{cat}$ is the categorical distribution. The categorical distribution selects an index based on the input $\theta^r$, which is indeed the group label of the associated DM. Given the subgroup of each DM, we can identify the relation between the DMs’ criteria weights and the corresponding subgroup center as:

$$w^r \sim \text{Dir}(y^r \Omega^r), \quad r = 1, ..., R, \quad y^r \in \{1, ..., Z\}, \quad (36)$$

where $y^r$ is a positive constant. Similar to equation (18), equation (36) indicates that the weight of DM $r$, given its subgroup $y^r$, is centered around the center of its subgroup, and the closeness to the center of the subgroup is governed by $y^r$. The center of each subgroup should also represent a set of priorities,
similar to \( w^* \) in the group decision-making model, so it can be modeled as:

\[
\Omega^z \sim \text{Dir}(\delta), \quad z = 1, ..., Z,
\]

where \( \delta \in \mathbb{R}^Z \) is a hyperparameter. Also, \( y^z \) in equation (36) can be modeled by the gamma distribution with \( a \) and \( b \) parameters, i.e.,

\[
y^z \sim \text{gamma}(a, b), \quad z = 1, ..., Z.
\]

As a result of the mixture model, we can identify the subgroups of DMs, represented by \( w^{1:R} \). The hyperparameters \( \lambda \) and \( \delta \) are set to uniformly 0.01 for all components, and the parameters of the gamma distribution are set to \( a = b = 0.01 \). Figure 5 plots the graphical model for grouping DMs in group MCDM.

The only limitation of the proposed model is the discrete value \( y^z \) obtained by sampling from a categorical distribution. This discrete variable often leads to instability of the sampling procedure or might not even be supported by the standard Bayesian samplers. To overcome this issue, we can compute the marginal distribution over the continuous variables by summing out the discrete variable \( y^z \). To that end, we can write:

\[
P(\theta^{1:R}, w^{1:R}, \Omega^{1:Z}, y^{1:Z} | g^{1:R}, \lambda, \delta, a, b) \\
\propto P(g^{1:R} | w^{1:R})P(\theta^{1:R} | \Omega^{1:Z} | \delta)P(y^{1:Z} | a, b) \\
P(w^{1:R} | \theta^{1:R}, \Omega^{1:Z}, y^{1:Z}) \\
= \prod_{r=1}^{R} P(g^r | w^r)P(\theta^r | \lambda) \prod_{z=1}^{Z} P(\Omega^z | \delta)P(y^z | a, b) \\
\prod_{r=1}^{R} \prod_{z=1}^{Z} P(w^r | \theta^r, \Omega^z, y^z).
\]

All the terms in equation (39) have specific distributions, except for the last term on the right. The last term can be written as:

\[
P(w^r | \theta^r, \Omega^z, y^z) = \sum_{y^{r'}=1}^{Z} P(w^r | y^{r'}, \Omega^z, y^{r'}) \\
= \sum_{y^{r'}=1}^{Z} P(y^{r'} | \theta^r)P(w^r | \Omega^z, y^{r'}) \\
= \theta^r_y P(w^r | \Omega^z, y^z),
\]

where the last equality is obtained since \( P(y^r | \theta^r) = \theta^r_y \). As a result, the discrete variable \( y^z \) is marginalized out. Consequently, the need for sampling a discrete variable is eliminated, leading to a more stable implementation of the model.

5. Alternative Evaluation and Sorting: A Probabilistic Approach

In MCDM, multiple alternatives are typically involved, and the objective is to evaluate and rank them based on a set of criteria. The evaluation of alternatives usually involves computing their utilities, rankings, or sorting them based on the given preferences. In the Bayesian models presented in this article, the criteria weights are modeled as probability distributions, which means that conventional methods for processing alternatives based on deterministic priorities are not directly applicable. This section aims to address this gap and explore different approaches for evaluating and sorting alternatives in the context of distributional priorities.

5.1. Alternative Utility Computation: Aggregation with Expectation

The utility of an alternative is typically calculated based on their performance on the criteria under study and the corresponding criteria weights. The most straightforward method to calculate the utility of an alternative is arguably the weighted sum model (WSM), defined as:

\[
U_{WSM}(A_i) = \sum_{j=1}^{n} a_i w_j = a_i^T w,
\]
where $U_{WSM}(A_i)$ represents the utility of alternative $A_i$ based on the WSM, $a_{ij}$ is the performance of $A_i$ on criterion $j$, and $w_j$ is the weight of criterion $j$. In the following, we define the utility of an alternative with respect to the weight distribution from the Bayesian models.

**Definition 5.1.** Given the weight distribution $w$ over $n$ criteria $C = \{c_1, c_2, ..., c_n\}$, the utility of alternative $A_i$ is defined as the mathematical expectation of $a_i$ with respect to the weight distribution $w$:

$$U_{WSM}^B(A_i) = \mathbb{E}_w(a_i^T w),$$

(42)

where $U_{WSM}^B$ is the Bayesian utility function concerning the WSM, and $\mathbb{E}_w$ is the mathematical expectation with respect to $w$.

The computation in equation (42) requires the expectation of the weight distribution. Since we have $Q$ samples from the weight distribution through the MCMC sampling, this expectation can be estimated from the available $Q$ samples as:

$$U(A_i) = \mathbb{E}_w(a_i^T w) = a_i^T \mathbb{E}_w(w) = a_i^T \left( \frac{1}{Q} \sum_{q=1}^{Q} w^q \right),$$

(43)

where the first equality is written based on the properties of the mathematical expectation, and $w^q$ is the $q^{th}$ sample of the weight distribution from the MCMC sampling.

**Remark 5.2.** Equation (42) is similar to the WSM in equation (41), but the main difference is that in equation (42), the weighted sum is replaced by the average weighted sum over the samples of the weight distribution. These equations are conceptually different, with one important distinction being that the criteria weights in the WSM are assumed to be independent and cannot have correlation, whereas the weight distribution in equation (42) can capture the correlation between criteria weights.

**Remark 5.3.** Definition 5.1 can be generalized, as equation (42) represents the expectation of the WSM over the weight distribution. Let $G(a_i, w)$ be an arbitrary aggregation operator that computes the utility of alternative $a_i$ with respect to the priorities $w$. Then, the utility of $a_i$ based on the weight distribution can be written as:

$$U_{G}^B(A_i) = \mathbb{E}_w (G(a_i, w)) = \frac{1}{Q} \sum_{q=1}^{Q} G(a_i, w^q),$$

(44)

where $U_{G}^B(A_i)$ represents the Bayesian utility function of $A_i$ based on the aggregation operator $G$.

5.2. Comparing Alternatives: Credal Ranking for Alternatives

In typical MCDM problems with deterministic criteria weights, alternatives can be ranked based on computed utilities, as shown in equation (41). The decision to compare alternatives is relatively straightforward: if the utility of one alternative is higher than that of another, it is ranked higher. We can employ a similar approach for distributional weights by computing the utility based on Definition 5.1. However, a comparison based solely on the expected value with respect to the weight distribution ignores the dispersion in the weight distribution, which is one of the advantages of using probabilistic models. Therefore, it is possible to take the dispersion of the weights into account and provide a probabilistic ranking scheme for the alternatives.

**Definition 5.4.** For a pair of alternatives $A_i$ and $A_j$, the confidence of $A_i$ being superior to $A_j$ with respect to the weight distribution $w$ can be defined as:

$$P(A_i > A_j) = \mathbb{E}_w \left( G(a_i, w) > G(a_j, w) \right).$$

(45)

where $P(A_i > A_j)$ is the confidence of $A_i$ being superior to $A_j$, and $G$ is an aggregation operator.

By using the definition of the mathematical expectation, equation (45) could be further written as:

$$P(A_i > A_j) = \int_w I(G(a_i, w) > G(a_j, w)) P(w) dw,$$

(46)

where $I$ is an indicator function returning one if the condition in the parenthesis is satisfied and zero otherwise. We can now estimate the integral in equation (46) by using $Q$ available samples from the MCMC samples and write:

$$P(A_i > A_j) = \frac{1}{Q} \sum_{q=1}^{Q} I(G(a_i, w^q) > G(a_j, w^q)).$$

(47)

With equation (47), we can establish a credal ranking for the alternatives, similar to the credal ranking of criteria. Consequently, the confidence for each pair of alternatives (i.e., credal ordering of two alternatives) can be computed, and a credal ranking of all alternatives can be established accordingly.

5.3. Sorting Alternatives: A Bayesian Mixture Model

Sorting alternatives in MCDM refers to grouping the alternatives according to the preferences of decision-makers. Since the weights are distributional in the Bayesian models, sorting alternatives also requires careful consideration. One approach would be to use the aggregated utility of each alternative based on equation (42) or (44) and then apply conventional clustering methods, such as K-means. However, this approach would overlook the dispersion of alternative utilities arising from the probabilistic nature of the weights. In this section, we develop a Bayesian sorting algorithm that groups alternatives according to the preferences of a group. This approach is similar to the mixture model for grouping decision-makers but includes different specifications to account for the utilities of alternatives.

Let $u_{i,j} = 1, ..., m$, represent the utility of alternative $i$, computed using an aggregation function such as $G$ based on the alternative’s performance $a_i$ and the priorities $w$. Since $w$ is distributional, the values of $u_{i,j}$ are also uncertain and include the
dispersion of the utility as well. Let also $\hat{\Omega}, \hat{\varphi}$ be the center of clusters for $\hat{Z}$ groups of alternatives, $\hat{y}_{1:m}$ be the clusters of the alternatives, and $\hat{\theta}_{1:m}$ include the cluster probability of the alternatives, $\hat{y}_i \in R^2$, $i = 1, ..., m$. From a Bayesian perspective, we need to estimate the following joint distribution:

$$P(\hat{y}_{1:m}, \hat{\theta}_{1:m}, \hat{\Omega}, \hat{\varphi}| u_{1:m}) \propto \prod_{i=1}^{m} P(\hat{y}_i, \hat{\theta}_i) P(\hat{\theta}_i) \prod_{i=1}^{m} P(\hat{\Omega}, \hat{\varphi}) \prod_{i=1}^{m} P(u_i | \hat{y}_i, \hat{\Omega}, \hat{\varphi}). \quad (48)$$

We now specify the distributions of the terms in equation (48). First, the prior distributions can be modeled as:

$$\hat{\theta}_i \sim \text{Dir}(\lambda), \quad \hat{\theta}_i \in R^2, i = 1, ..., m,$$

$$\hat{\Omega} \sim \mathcal{N}(\mu, \sigma), \quad \hat{\varphi} = 1, ..., \hat{Z}, \quad (49)$$

where $\lambda \in R^2$ is the prior vector of the Dirichlet distribution specified as $[0, 1]$ and $\sigma$ is set to one since $u_i \in [0, 1]$. Given $\hat{\theta}_i$, the cluster $\hat{y}_i$ of alternative $i$ can be modeled as:

$$\hat{y}_i \sim \text{cat}(\hat{\theta}_i), \quad i = 1, ..., m. \quad (50)$$

Given the cluster $\hat{y}_i$ of alternative $i$, the relation between the utility of an alternative and the center of clusters $\hat{\Omega}$ can be written as:

$$\hat{\Omega}_{\hat{y}_i} \sim \mathcal{N}(u_i, \sigma), \quad (51)$$

where $\sigma$ is a hyperparameter and is set to one here because $u_i \in [0, 1]$. This sorting method also involves estimating a discrete parameter $\hat{\gamma}$, which can make the overall sampling process unstable or sometimes impossible for certain Bayesian sampling methods. Similar to the mixture model presented in Section 4, we can marginalize out $\hat{\gamma}$ and compute the marginal distribution over the continuous variables. To avoid repetition, we omit the computations for summing the discrete parameters. By solving this model, we obtain the centers of different clusters for alternatives and the membership of each alternative in the identified clusters.

6. Numerical Examples

In this section, we present a series of numerical examples to validate and showcase the effectiveness of the proposed Bayesian models. The experiments aim to highlight the advantages of the models and compare their results with other established methods. Since the true criteria weights for the DMs are not available in MCDM a priori, the proposed Bayesian models are validated by juxtaposing their results with other well-known and widely accepted methods and taking the proximity of the results as an indicator for the validity of results of the proposed models. In addition, we demonstrate the advantages of the proposed model, e.g., the credal ranking or criteria correlation, by studying the outcome of the models in the examples.

As discussed, the Bayesian models developed in this article do not have closed-form solutions. Therefore, Markov Chain Monte Carlo (MCMC) sampling is used to estimate the posterior distributions. Various programming languages and libraries can be used for this purpose. In this article, the models were implemented using Stan [46], a well-known platform for conducting Bayesian statistical inference. The Python interface of Stan was used to analyze and process the preferences.

6.1. Certain Preferences: Validation by AHP

The first example aims to validate the proposed Bayesian AHP model by using an example with certain preferences. The original AHP method, which uses a 1-9 scale, is employed. The example is adopted from [28], where five criteria are evaluated by six DMs. Each DM provides their PCMs representing the preference intensities among the criteria. The PCMs provided by the DMs are as follows:

Using the provided PCMs, the proposed Bayesian AHP model is applied to derive the criteria weights for each DM, as well as the aggregate criteria weights summarizing the overall preference of the group. The results of the proposed Bayesian model are then compared with those of the geometric mean method [11] and the Bayesian model in [28]. Table 1 presents the criteria weights for each individual DM, as well as the aggregate weights obtained from the three methods. The first row for each DM represents the weights computed using the geometric mean method, the second row represents the mean of weights distribution calculated using the Bayesian model in [25], and the third row represents the mean of weights distribution obtained from the proposed Bayesian model for AHP. We expect that the mean of the distribution from the proposed method will be in the neighborhood of the geometric mean method, which is a natural central tendency for weight distribution. The results in Table 1 demonstrate that the proposed Bayesian model yields similar means compared to the geometric mean method and the Bayesian model in [25], indicating the validity of the obtained results. Given that the proposed Bayesian model’s results are valid, we now study the credal ranking of criteria based on the aggre-
gated weights. The credal ranking includes a set of credal orderings, each identifying the extent to which a criterion is more important than another. To simplify the credal ranking, we use a weighted, directed graph to visualize the rankings and corresponding confidence levels. Figure 6a shows a directed graph visualizing the credal ranking for the example discussed above. The nodes in the graph are the criteria, their labels are a mixture of their names and weights’ distribution mean, and each directed edge like $A \xrightarrow{p} B$ indicates that criterion $A$ is more important than $B$ with a confidence level of $p$. If the nodes are positioned higher, they are more important based on the computed priorities. We also removed the trivial edges with $p = 1$ unless they are immediately ranked after another node, so if $A$ is positioned higher than $B$ with no edge in between, it implies that $A$ is more important than $B$ with $p = 1$. According to this graph, $C_1$ is more important than $C_2$ with a confidence level of one, as the corresponding edge suggests. Also, $C_1$ is more important than $C_4$ (as well as other criteria) with a confidence level of one, while the corresponding edge is removed for simplicity, and the superiority is acquired based on the position of $C_1$ (that is higher than $C_4$). Also, the associated edge suggests that $C_4$ is more important than $C_3$ with a confidence level of 0.86. So, in sum, the credal ranking could be written as $C_1 \succ C_2 \succ C_4 \succ C_3 \succ C_5$.

We can also visualize the weight distribution of each criterion, either with respect to the preferences of a DM or the aggregated weights. Figure 6b shows the weight distributions of the criteria with respect to the aggregated weights for the above example. Note that the distributions are centered at the mean shown in Table 1, and the standard deviation is affected by the possible inconsistency of the DM as well as the disagreement among the group of DMs. Nevertheless, estimating such a standard deviation is why we can develop a probabilistic ranking scheme, i.e., credal ranking, and provides the DMs with more information on the superiority of criteria. Figure 6b indicates that only the weight distributions of $C_3$ and $C_4$ overlap with $C_4$ having higher weight values, resulting in a credal ordering with a confidence level of 0.86. The other weight distributions have no overlap and the corresponding confidence level is thus one.

![Figure 6: The credal ranking as well as aggregated weight distributions based on the proposed Bayesian model for the AHP. (a) The credal ranking of criteria based on the aggregated weight of six DMs; (b) The aggregated weight distribution of the five criteria under study.](image)
6.2. Uncertain Preferences

For the cases where the preferences of the DMs entail a level of uncertainty, the Bayesian models put forward concerning the types of uncertainty could be used. For validating such models, we use the fact that certain preferences could also be modeled by the uncertain preferences as well. For example, the intervals shrunken to a point is a certain representation of preferences in an uncertain format, so is a normal distribution with an infinitesimal standard deviation. Since the Bayesian models with certain preferences are evaluated and validated in the previous example, we take advantage of the uncertain expression of certain preferences to validate the Bayesian models for the uncertain preferences. The expectation is that the results of the uncertain Bayesian models with certain preferences are the same as those of the Bayesian models with certain preferences.

To experiment, we use an example of the BWM with eight alternatives based on the distributional criteria weights. To that end, we do a case study evaluating different algorithms based on multiple performance metrics. More specifically, we study the evaluation and ranking of ontology alignment algorithms on a matching task based on five performance metrics: execution time, precision, recall, recall+, and consistency. We express these preferences with different uncertain (but equivalent) formats, and the corresponding uncertain preferences are subjected to the associated Bayesian models. Table 2 shows the mean of the distribution of the weight from Bayesian models with different types of preferences. Since the uncertain preferences are an expression of certain preferences in equation (52), we use the BWM input data presented in Section 6.2 and apply the Bayesian models with correlated and independent criteria. Table 3 shows the mean of the aggregated weight distributions for the two models. Based on this table, the aggregated weight in the correlated model is very similar to the independent model, as expected. As a result, this experiment validates the legitimacy of the correlated Bayesian model based on the logistic-normal distribution. The difference could also be caused because the logistic-normal and the Dirichlet distributions are not identical, but one can approximate the Dirichlet distribution with the logistic-normal distribution. This fact justifies the small difference between the weight estimation of the two models in Table 3. However, the results of the correlated model are still valid since the difference is negligible.

6.3. Weight Estimation with Criteria Correlation

When the criteria are correlated, it can influence the calculation of the criteria weights if the Bayesian model for the correlated criteria is used. To validate such Bayesian models, we use the BWM input data presented in Section 6.2 and apply the Bayesian models with correlated and independent criteria. Table 3 shows the mean of the aggregated weight distributions for the two models. Based on this table, the aggregated weight in the correlated model is very similar to the independent model, as expected. As a result, this experiment validates the legitimacy of the correlated Bayesian model based on the logistic-normal distribution. The difference could also be caused because the logistic-normal and the Dirichlet distributions are not identical, but one can approximate the Dirichlet distribution with the logistic-normal distribution. This fact justifies the small difference between the weight estimation of the two models in Table 3. However, the results of the correlated model are still valid since the difference is negligible.

6.4. Evaluating and Ranking of Alternatives

In this section, we study the evaluation and ranking of alternatives based on the distributional criteria weights. To that end, we do a case study evaluating different algorithms based on multiple performance metrics. More specifically, we study the evaluation and ranking of ontology alignment algorithms on a matching task based on five performance metrics: execution time, precision, recall, recall+, and consistency. We use the BWM input data presented in Section 6.2 and apply the Bayesian models with correlated and independent criteria. Table 3 shows the mean of the aggregated weight distributions for the two models. Based on this table, the aggregated weight in the correlated model is very similar to the independent model, as expected. As a result, this experiment validates the legitimacy of the correlated Bayesian model based on the logistic-normal distribution. The difference could also be caused because the logistic-normal and the Dirichlet distributions are not identical, but one can approximate the Dirichlet distribution with the logistic-normal distribution. This fact justifies the small difference between the weight estimation of the two models in Table 3. However, the results of the correlated model are still valid since the difference is negligible.

|         | C1  | C2  | C3  | C4  | C5  |
|---------|-----|-----|-----|-----|-----|
| DM1     | 0.491 | 0.232 | 0.092 | 0.138 | 0.046 |
|         | 0.491 | 0.232 | 0.092 | 0.138 | 0.046 |
|         | 0.465 | 0.241 | 0.103 | 0.136 | 0.055 |
| DM2     | 0.481 | 0.230 | 0.117 | 0.120 | 0.032 |
|         | 0.480 | 0.249 | 0.117 | 0.120 | 0.032 |
|         | 0.448 | 0.249 | 0.116 | 0.131 | 0.056 |
| DM3     | 0.302 | 0.320 | 0.102 | 0.184 | 0.092 |
|         | 0.301 | 0.318 | 0.102 | 0.183 | 0.092 |
|         | 0.411 | 0.260 | 0.114 | 0.143 | 0.072 |
| DM4     | 0.451 | 0.183 | 0.211 | 0.074 | 0.081 |
|         | 0.451 | 0.183 | 0.210 | 0.073 | 0.081 |
|         | 0.438 | 0.237 | 0.132 | 0.122 | 0.070 |
| DM5     | 0.407 | 0.291 | 0.085 | 0.147 | 0.070 |
|         | 0.407 | 0.290 | 0.084 | 0.147 | 0.071 |
|         | 0.429 | 0.259 | 0.107 | 0.138 | 0.068 |
| DM6     | 0.475 | 0.261 | 0.098 | 0.120 | 0.046 |
|         | 0.475 | 0.261 | 0.098 | 0.120 | 0.046 |
|         | 0.469 | 0.259 | 0.100 | 0.123 | 0.049 |
| Aggregated | 0.439 | 0.258 | 0.114 | 0.129 | 0.059 |
|         | 0.437 | 0.258 | 0.114 | 0.128 | 0.058 |
|         | 0.443 | 0.251 | 0.112 | 0.132 | 0.062 |

_Ag = \begin{bmatrix} 3 & 4 & 6 & 1 & 5 & 2 & 9 & 7 \\ 1 & 2 & 8 & 4 & 5 & 3 & 9 & 6 \\ 2 & 2 & 3 & 1 & 5 & 5 & 9 & 8 \\ 2 & 1 & 8 & 2 & 9 & 3 & 8 & 8 \\ 2 & 4 & 9 & 1 & 4 & 3 & 5 & 5 \\ 1 & 2 & 9 & 1 & 3 & 5 & 5 & 4 \end{bmatrix}, \quad _{A_W} = \begin{bmatrix} 7 & 6 & 4 & 9 & 5 & 8 & 1 & 3 \\ 9 & 8 & 2 & 5 & 4 & 5 & 1 & 3 \\ 8 & 8 & 5 & 9 & 5 & 5 & 1 & 2 \\ 8 & 9 & 2 & 8 & 1 & 8 & 2 & 2 \\ 8 & 6 & 1 & 9 & 6 & 7 & 4 & 4 \\ 9 & 8 & 1 & 9 & 7 & 5 & 5 & 6 \end{bmatrix}_. (52)
Table 2: The mean of the aggregated weights for the BWM with Bayesian models with different inputs.

|                  | C1    | C2    | C3    | C4    | C5    | C6    | C7    | C8    |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| Certain Preferences | 0.203 | 0.171 | 0.060 | 0.218 | 0.090 | 0.130 | 0.054 | 0.071 |
| Normal Preferences  | 0.204 | 0.172 | 0.059 | 0.215 | 0.090 | 0.131 | 0.053 | 0.071 |
| Interval Preferences | 0.203 | 0.174 | 0.060 | 0.208 | 0.091 | 0.135 | 0.054 | 0.072 |
| Triangular Preferences | 0.203 | 0.171 | 0.059 | 0.219 | 0.090 | 0.129 | 0.053 | 0.071 |

Table 3: The mean of the aggregated weight distribution for Bayesian models of the correlated and independent criteria.

|                  | C1    | C2    | C3    | C4    | C5    | C6    | C7    | C8    |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| Independent Model | 0.203 | 0.171 | 0.060 | 0.218 | 0.090 | 0.130 | 0.054 | 0.071 |
| Correlated Model  | 0.203 | 0.175 | 0.059 | 0.220 | 0.090 | 0.128 | 0.052 | 0.069 |

Table 4: Evaluating the ontology matching algorithms based on five performance metrics

| Algorithms   | Time (0.083) | Precision (0.247) | Recall (0.24) | Recall+ (0.323) | Consistency (0.107) | Agg. utility |
|--------------|--------------|-------------------|---------------|-----------------|---------------------|--------------|
| LogMapBio    | 0            | 0.89              | 1             | 1               | 1                   | 0.889        |
| SANOM        | 0.962        | 0.89              | 0.923         | 0.829           | 0                   | 0.789        |
| LogMapLite   | 1            | 0.96              | 0.802         | 0.382           | 0                   | 0.636        |
| KEPLER       | 0.714        | 0.96              | 0.813         | 0.421           | 0                   | 0.627        |
| Lily         | 0.671        | 0.87              | 0.879         | 0.684           | 0                   | 0.702        |
| ALIN         | 0.68         | 1                 | 0.67          | 0               | 1                   | 0.571        |

7. Conclusion, Discussion, and Future Works

This paper presented several Bayesian models for various MCDM tasks and addressed several challenges in a statistically sound manner. In the following, the summarize of the contributions of this article and discuss the critical points and venues for future research.

7.1. Conclusions

In particular, the following points can summarize the contributions of the models put forward in this article:

- **Group MCDM**: The proposed Bayesian models handled the preferences of a group of DMs and provided criteria weights for each individual and the entire group. A probabilistic ranking scheme for the criteria was also introduced, allowing for the assessment of the extent to which one criterion is superior to another.

- **Uncertain Preferences**: The models accounted for uncertainty in preferences by incorporating an additional level in the Bayesian hierarchical framework. Different types of uncertainties, such as normal and triangular distributions, as well as intervals, were shown to be modeled within the proposed Bayesian frameworks.

- **Criteria Correlation**: The models considered the correlation between criteria by employing appropriate distributions over the criteria weights. The logistic-normal distribution was used to model the priorities, enabling the incorporation of criteria correlation by tuning the covariance matrix in the distribution.

Aside from the aggregated utility of each alternative, we further calculate the extent to which one algorithm is better than another, given their performance and the preferences of experts. This enables us to probabilistically rank the alternatives discussed in equation (47). Similar to the criteria ranking discussed in Section 6.1, the algorithms can be ranked using the credal ranking and visualized by a weighted, directed graph. Figure 7a shows the credal ranking of the six algorithms based on the Bayesian WSM. The nodes in this graph are the algorithms (or, generally speaking, alternatives), and the interpretation of the credal ranking in Figure 7a is identical to the one presented in Section 6.1. If an algorithm is positioned higher, it is better, and the weight of the corresponding edge shows the confidence of such a relationship.

Since the weight of performance metrics is distributional, the consequent computed utility for each alternative is also a distribution. We use the WSM and compute a distribution for each algorithm discussed in Table 4. Figure 7b shows the utility distribution of the algorithms, where each distribution is centered around the aggregated utility shown in the last column of Table 4.
Figure 7: (a) The credal ranking of algorithms based on five performance metrics and their importance according to the Bayesian WSM; (b) The utility distributions of algorithms based on the Bayesian WSM.
7.2. Discussion

The proposed Bayesian models for different MCDM tasks can be viewed as frameworks that can accommodate other MCDM methods and leverage the features discussed in this article. Embedding other methods into the Bayesian models requires finding the likelihood function of the preferences to the priorities, i.e., identifying the distribution for the term $P(g'|w')$ in equation (27). By doing so, one can incorporate their methods into the Bayesian models and take advantage of features such as handling group decision-making or considering criteria correlation for estimating criteria weights. Similarly, different types of preferences can be accommodated by identifying a suitable likelihood function in equation (27).

Considering criteria correlation in estimating criteria weights is another significant aspect of the proposed Bayesian models. The ANP is one of the few MCDM methods that can handle criteria correlation and estimate weights accordingly. The Bayesian models in this article can address criteria correlation for any MCDM method by using the covariance matrix in the logistic-normal distribution. Future research could explore different ways of constructing the covariance matrix for the logistic-normal distribution, such as leveraging Choquet integral-based distributions. These approaches would enable the consideration of possible interactions among criteria in estimating criteria weights.

The ability of the proposed models to handle preferences expressed by different methods or different types of preferences is another valuable feature. This property allows for flexibility in expressing preferences, enabling the analysis of preferences expressed in different formats and providing further insights into the MCDM problem at hand.

The proposed Bayesian models can also contribute to building consensus in group decision-making. For example, the Bayesian model for group decision-making can identify the criteria weights of each individual, which can be used to assess the differences or similarities among decision-makers. This information can facilitate negotiation and consensus-building among the involved decision-makers. The Bayesian mixture model for grouping decision-makers can also aid in the consensus-building process by identifying different subgroups of decision-makers. The existence of multiple subgroups with distinct weights indicates a group disagreement, which can inform the negotiation procedure in group MCDM.

7.3. Future Research

There are several avenues for future research. One critical requirement for the proposed Bayesian models is to assess the consistency of decision-makers. Instead of the traditional approach in MCDM, which categorizes decision-makers as either consistent or not, a probabilistic consistency measure could be developed. This measure would allow for quantifying the extent to which a decision-maker is consistent based on their preferences. One possible approach is to study the parameter $\gamma$ in equation (16), as a higher value of $\gamma$ indicates a higher level of decision-maker consistency. However, further research is required to quantify the level of consistency of the DMs.

Ordinal regression models are another essential class of MCDM methods that quantify the importance and marginal utility of each criterion. However, unlike the MCDM methods studied in this article, ordinal regression models are based on comparisons among alternatives (e.g., ranking or pairwise comparisons). Therefore, future research could focus on finding a likelihood function to adapt the proposed Bayesian models for ordinal regression.

Another intriguing development would be the application of Bayesian nonparametrics for grouping decision-makers in group MCDM and sorting problems. In the Bayesian models proposed in this article for grouping decision-makers and sorting alternatives, the number of groups/classes needs to be specified in advance. By employing Bayesian nonparametrics, the models themselves can identify the number of groups/classes without requiring prior knowledge.

Other types of preferences, particularly different fuzzy numbers, could also be studied to be incorporated into a Bayesian model. The Bayesian model presented in this article, as discussed in equation (27), may not be directly applicable in cases where a generative distribution for $P(g|g')$ cannot be identified. In such cases, more complex Bayesian models must be developed to accommodate these types of preferences.

References

[1] S. Theodoridis, Machine learning: a Bayesian and optimization perspective. Academic press, 2015.
[2] P. E. Rossi and G. M. Allenby, “Bayesian statistics and marketing,” Marketing Science, vol. 22, no. 3, pp. 304–328, 2003.
[3] R. L. Keeney, H. Raiffa et al., Decisions with multiple objectives: preferences and value trade-offs. John Wiely & Sons, Inc., 1976.
[4] T. L. Saaty, “A scaling method for priorities in hierarchical structures,” Journal of mathematical psychology, vol. 15, no. 3, pp. 234–281, 1977.
[5] ———, Decision making for leaders: the analytic hierarchy process for decisions in a complex world. RWS publications, 1990.
[6] J. Rezaei, “Best-worst multi-criteria decision-making method,” Omega, vol. 53, pp. 49–57, 2015.
[7] J. Mustajoki, R. P. Hämäläinen, and A. Salo, “Decision support by interval smart/swing—incorporating imprecision in the smart and swing methods,” Decision Sciences, vol. 36, no. 2, pp. 317–339, 2005.
