ISSUES AND RAMIFICATIONS IN QUANTIZED FRACTAL SPACE TIME: AN INTERFACE WITH QUANTUM SUPERSTRINGS

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Abstract
Recently a stochastic underpinning for space time has been considered, what may be called Quantized Fractal Space Time. This leads us to a number of very interesting consequences which are testable, and also provides a rationale for several otherwise inexplicable features in Particle Physics and Cosmology. These matters are investigated in the present paper.

1 The Background ZPF
We observe that the Bohm formulation discussed in detail in Chapter 3 converges to Nelson’s stochastic formulation in the context of the QMKNBH. Indeed Bohm’s non local potential as also Nelson’s three conditions merely describe the QMKNBH as a vortex, the mass being given by the self interaction, the radius of the vortex being the Compton wavelength. [1]. We can get a clue to the origin of Quantum Mechanical fluctuations: Following Smolin we observe that the non local stochastic theory becomes the classical local theory in the thermodynamic limit, in which $N$ the number of particles in the universe becomes infinitely large. However if $N$ is finite but large, these

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fluctuations are of the order $1/\sqrt{N}$ of the dimensions of the system, the universe in this case. Indeed as we will see in the next Section this provides a holistic rationale for the ”spooky” non-locality of Quantum Theory.

We now remark that in the above formulation elementary particles, typically electrons, can be thought of as ’twisted bits’ of the electromagnetic field. Indeed it was pointed out by Barut and co-workers that wave packet solutions of the mass less scalar fields appear as massive particles, while such solutions for the electromagnetic field would provide a formulation of the wave mechanics without assuming the Planck constant \cite{2}. For example this gives $E = l\omega$ rather than $E = \hbar\omega$, $l$ being the angular momentum and $E$ the energy and $\omega$ the frequency. Boudet \cite{3}, also questions the necessity of the Planck Constant. These theories do not give a value to Planck’s constant which merely appears as a proportionality factor, because all the equations considered are linear. All this as also the zitterbewegung formulation of Barut and Bracken and Hestenes described in Chapter 3, is superseded by the QMKNBH theory and electrons appear as twisted bits of the ZPF given by the relation, $E = \hbar\omega$, instead of Barut’s, $E = l\omega$, where $l$ is the angular momentum. The question, whether this characterizes the Planck constant, will be answered at the end of Section 5.

2 Stochastic Conservation Laws

Conservation Laws, as is universally known, play an important role in Physics, starting with the simplest such laws relating to momentum and energy. These laws provide rigid guidelines or constraints within which physical processes take place.

These laws are observational, though a theoretical facade can be given by relating them to underpinning symmetries \cite{4}.

Quantum Theory, including Quantum Field Theory is in conformity with the above picture. On the other hand the laws of Thermodynamics have a different connotation: They are not rigid in the sense that they are a statement about what is most likely to occur, or is an averaged out statement.

In our formulation, the Compton wavelength represents a statistical uncertainty (Cf. Chapter 6), given by

$$l \sim \frac{R}{\sqrt{N}} \quad (1)$$
By now (1) is a familiar relation. Given the above background we consider
the following simplified EPR experiment, discussed elsewhere[5].
Two structureless and spinless particles which are initially together, for ex-
ample in a bound state get separated and move in opposite directions along
the same straight line. A measurement of the momentum of one of the par-
ticles, say \( A \) gives us immediately the momentum of the other particle \( B \).
The latter is equal and opposite to the former owing to the conserva-
tion law of linear momentum. It is surprising that this statement should be true in
Quantum Theory also because the momentum of particle \( B \) does not have an
apriori value, but can only be determined by a separate acausal experiment
performed on it.
This is the well known non locality inherent in Quantum Theory. It ceases
to be mysterious if we recognize the fact that the conservation of momen-
tum is itself a non local statement because it is a direct consequence of the
homogeneity of space as we will see again in the next Chapter: Infact the
displacement operator \( \frac{\delta}{\delta x} \) is, given the homogeneity of space, independent
of \( x \) and this leads to the conservation of momentum in Quantum Theory
(cf.ref.[6]). The displacement \( \delta x \) which gives rise to the above displacement
operator is an instantaneous shift of origin corresponding to an infinite ve-
locity and is compatible with a closed system. It is valid if the instantaneous
displacement can also be considered to be an actual displacement in real
time \( \delta t \). This happens for stationary states, when the overall energy remains
constant.
It must be borne in mind that the space and time displacement operators
are on the same footing only in this case[7]. Indeed in relativistic Quantum
Mechanics, \( x \) and \( t \) are put on the same footing - but special relativity itself
deals with inertial, that is relatively unaccelerated frames.
Any field theory deals with different points at the same instant of time. But
if we are to have information about different points, then given the finite
velocity of light, we will get this information at different times. All this in-
formation can refer to the same instant of time only in a stationary situation.
We will return to this point. Further the field equations are obtained by a
suitable variational principle,
\[
\delta I = 0 \quad (2)
\]
In deducing these equations, the \( \delta \) operator which corresponds to an arbi-
trary variation, commutes with the space and time derivatives, that is the
momentum and energy operators which in our picture constitute a complete set of observables. As such the apparently arbitrary operator $\delta$ in (2) is constrained to be a function of these (stationary) variables\cite{8}.

All this underscores two facts: First we implicitly consider an apriori homogenous space, that is physical space. Secondly though we consider in the relativistic picture the space and time coordinates to be on the same footing, in fact they are not as pointed out by Wheeler\cite{9}. Our understanding or perception of the universe is based on ”all space (or as much of it as possible) at one instant of time”.

However, in conventional theory this is at best an approximation. Moreover in our formulation, the particles are fluctuationally created out of a background ZPF, and, it is these $N$ particles that define physical space, which is no longer apriori as in the Newtonian formulation. It is only in the thermodynamic limit in which $N \to \infty$ and $l \to 0$, in (1), that we recover the above classical picture of a rigid homogenous space, with the conservation laws. In other words the above conservation laws are strictly valid in the thermodynamic limit, but are otherwise approximate, though very nearly correct because $N$ is so large.

Our formulation leads to a cosmology in which $\sqrt{N}$ particles are fluctuationally created from the background ZPF (Cf. Chapter 7), so that the violation of energy conservation is proportional to $\frac{1}{\sqrt{N}}$. From (1) also we could similarly infer that the violation of momentum conservation is proportional to $\frac{1}{\sqrt{N}}$ (per particle).

All this implies that there is a small but non-zero probability that the measurement of the particle $A$ in the above experiment will not give information about the particle $B$.

This last conclusion has also been drawn by Gaeta\cite{10} who considers a background Brownian or Nelson-Garbaczewski-Vigier noise (the ZPF referred to above) as sustaining Nelson’s Stochastic Mechanics (and the Schrodinger equation).

In conclusion, the conservation laws of Physics are conservation laws in the thermodynamic sense.
3 Quantized Space Time, Time’s Arrow and Parity Breakdown

The arrow of time has been a puzzle for a long time. As is well known, the laws of Newtonian Mechanics, Electromagnetism or Quantum Theory do not provide an arrow of time - they are equally valid under time reversal, with only one exception. This is in the well known problem of Kaon decay. On the other hand it is in Thermodynamics and Cosmology that we find an arrow of time [11]. Indeed it has been shown that stochastic processes are needed for irreversibility[12].

It is also true that there has been no theoretical rationale for the Kaon puzzle which we will touch upon shortly. We will try to find such a theoretical understanding in the context of our quantized space-time, $\sim \hbar/(\text{energy})$, that is the Compton time[13].

Let us start with one of the simplest quantum mechanical systems, one which can be in either of two states separated by a small energy[14]. The system flips from one state to another unpredictably and this ”life time” and the energy spread satisfy the Uncertainity Principle, so that the former is a Compton time. We have:

$$i\hbar \frac{d\psi_i}{dt} \approx i\hbar \left[ \frac{\psi_i(t + n\tau) - \psi_i(t)}{n\tau} \right] = \sum_{i=1}^{2} H_{ij} \psi_i$$

where, $H_{11} = H_{22}$ (which we set = 0 as only relative energies of the two levels are being considered) and $H_{12} = H_{21} = E$, by symmetry. Unlike in the usual theory where $\delta t = n\tau \rightarrow 0$, in the case of quantized space-time $n$ is a positive integer. So the second term of (3) reduces to

$$[E + i \frac{E^2\tau}{\hbar}] \psi_i = [E(1 + i)] \psi_i, \text{ as } \tau = \hbar/E$$

Interestingly, in the above analysis, in (4), the fact that the real and imaginary parts are of the same order is in fact borne out by experiment.

From (3) we see that the Hamiltonian is not Hermitian that is it admits complex Eigen values indicative of decay, if the life times of the states are $\sim \tau$. 

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In general this would imply the exotic fact that if a state starts out as $\psi_1$ and decays, then there would be a non zero probability of seeing in addition the decay products of the state $\psi_2$. In the process it is possible that some symmetries which are preserved in the decay of $\psi_1$ or $\psi_2$ separately, are violated. In this context we will now consider the Kaon puzzle. As is well known from the original work of Gellmann and Pais, the two state analysis above is applicable here\[15, 16\]. In the words of Penrose\[17\], ”the tiny fact of an almost completely hidden time-asymmetry seems genuinely to be present in the $K^0$-decay. It is hard to believe that nature is not, so to speak, trying to tell something through the results of this delicate and beautiful experiment.” On the other hand as Feynman put it\[18\], ”if there is any place where we have a chance to test the main principles of quantum mechanics in the purest way.....this is it.”

What happens in this well known problem is, that given $CP$ invariance, a beam of $K^0$ masons can be considered to be in a two state system as above, one being the short lived component $K^S$ which decays into two pions and the other being the long lived state $K^L$ which decays into three pions. In this case $E \sim 10^{10}h\[15\]$, so that $\tau \sim 10^{-10} sec$. After a lapse of time greater than the typical decay period, no two pion decays should be seen in a beam consisting initially of the $K^0$ particle. Otherwise there would be violation of $CP$ invariance and therefore also $T$ invariance. However exactly this violation was observed as early as 1964\[19\]. This violation of time reversal has now been confirmed directly by experiments at Fermilab and CERN\[20\].

We would like to point out that the Kaon puzzle has a natural explanation in the quantized time scenario discussed above. Further, we have shown that the discreteness leads to the non commutative geometry

$$[x, y] = 0(l^2), [x, p_x] = i\hbar[1 + l^2]$$  \(5\)

and similar equations. If terms $\sim l^2$ are neglected we get back the usual Quantum Theory. However retaining these terms, we deduced in Chapter 6 the Dirac equation. Moreover it can be seen that given (\(5\)) space reflection symmetry no longer holds. This violation is an $O(l^2)$ effect.

This is not surprising. It has already been pointed out that the space time divide viz., $x + ict$ arises due to the zitterbewegung or double Weiner process in the Compton wavelength - and in this derivation terms $\sim (ct)^2 \sim l^2$ were neglected. However if these terms are retained, then we get a correction to the usual theory including special relativity. (We will come back to this point
To see this more clearly let us as in Chapter 3 as a first approximation treat the continuum as a series of discrete points separated by a distance \( l \), which then leads to

\[
Ea(x_n) = E_0a(x_n) - Aa(x_n + l) - Aa(x_n - l)
\]  

(6)

When \( l \) is made to tend to zero, it was shown that from (6) we recover the Schrodinger equation, and further, we have,

\[
E = E_0 - 2A\cos kl.
\]  

(7)

The zero of energy was chosen such that \( E = 2A = mc^2 \), the rest energy of the particle, in the limit \( l \to 0 \). However if we retain terms \( \sim l^2 \), then from (6) we will have instead

\[
\left| \frac{E}{mc^2} - 1 \right| \sim 0(l^2)
\]  

(8)

Equation (8) shows the correction to the energy mass formula, where again we recover the usual formula in the limit \( O(l^2) \approx 0 \).

It must be mentioned that all this would be true in principle for discrete space time, even if the minimum cut off was not at the Compton scale. Intuitively this should be obvious: Space time reflection symmetries are based on a space time continuum picture.

Let us now consider some further imprints of discrete space time\(^\[21\]\). First we consider the case of the neutral pion. As we saw in Chapter 4, this pion decays into an electron and a positron. Could we think of it as an electron-positron bound state also? In this case we have,

\[
\frac{mv^2}{r} = \frac{e^2}{r^2}
\]  

(9)

Consistently with the above formulation, if we take \( v = c \) from (8) we get the correct Compton wavelength \( l_\pi = r \) of the pion. However this appears to go against the fact that there would be pair annihilation with the release of two photons. Nevertheless if we consider discrete space time, the situation would be different. In this case the Schrodinger equation

\[
H\psi = E\psi
\]  

(10)
where $H$ contains the above Coulumb interaction could be written, in terms of the space and time separated wave function components as,

$$H\psi = E\phi T = \phi \hbar [\frac{T(t - \tau) - T}{\tau}]$$

(11)

where $\tau$ is the minimum time cut off which in the above work has been taken to be the Compton time. If, as usual we let $T = \exp(irt)$ we get

$$E = -\frac{2\hbar}{\tau} \sin \frac{\tau r}{2}$$

(12)

(12) shows that if,

$$|E| < \frac{2\hbar}{\tau}$$

(13) holds then there are stable bound states. Indeed inequality (13) holds good when $\tau$ is the Compton time and $E$ is the total energy $mc^2$. Even if inequality (13) is reversed, there are decaying states which are relatively stable around the cut off energy $\frac{2\hbar}{\tau}$.

This is the explanation for treating the pion as a bound state of an electron and a positron, as indeed is borne out by its decay mode. The situation is similar to the case of Bohr orbits– there also the electrons would according to classical ideas have collapsed into the nucleus and the atoms would have disappeared. In this case it is the discrete nature of space time which enables the pion to be a bound state as described by (8).

4 Magnetic Effects

If as discussed in Chapter 3 and subsequently, the electron is indeed a Kerr-Newman type charged black hole, it can be approximated by a solenoid and we could expect an Aharonov-Bohm type of effect, due to the vector potential $\vec{A}$ which would give rise to shift in the phase in a two slit experiment for example[22]. This shift is given by

$$\Delta\delta_B = \frac{e}{\hbar} \int \vec{A}.\vec{d}s$$

(14)

while the shift due to the electric charge would be

$$\Delta\delta_E = -\frac{e}{\hbar} \int A_0 dt$$

(15)
where $A_0$ is the electrostatic potential. In the above formulation we would have

$$\vec{A} \sim \frac{1}{c} A_0$$  \hspace{1cm} (16)

Substitution of (16) in (14) and (15) shows that the magnetic effect $\sim \frac{v}{c}$ times the electric effect.

Further, the magnetic component of a Kerr-Newman black hole, as we saw in Chapter 3 is given by

$$B_\hat{r} = \frac{2ea}{r^3} \cos \Theta + O\left(\frac{1}{r^4}\right), B_\hat{\Theta} = \frac{e \sin \Theta}{r^3} + O\left(\frac{1}{r^4}\right), B_\hat{\phi} = 0,$$  \hspace{1cm} (17)

while the electrical part is

$$E_\hat{r} = \frac{e}{r^2} + O\left(\frac{1}{r^3}\right), E_\hat{\Theta} = O\left(\frac{1}{r^4}\right), E_\hat{\phi} = 0,$$  \hspace{1cm} (18)

Equations (17) and (18) show that in addition to the usual dipole magnetic field, there is a shorter range magnetic field given by terms $\sim \frac{1}{r^4}$. In this context it is interesting to note that an extra so called $B^{(3)}$ magnetic field of shorter range and probably mediated by massive photons has indeed been observed and studied over the past few years[23].

5 Stochastic Holism and the Number of Arbitrary Parameters

The discrete space time or zitterbewegung has an underpinning that is stochastic. The picture leads to the goal of Wheeler’s ‘Law without Law’ as we saw in Chapter 6. Furthermore the picture that emerges is Machian. This is evident from equations like (2), (3) and (6) of Chapter 7– the micro depends on the macro. So the final picture that emerges is one of stochastic holism.

Another way of expressing the above point is by observing that the interactions are relational. For example, in the equation leading to (7), of Chapter 7, if the number of particles in the universe tends to 1, then as we saw in Chapter 4, the gravitational and electromagnetic interactions would be equal, this happening at the Planck scale, where the Compton wavelength equals the Schwarzchild radius[24].

Infact as was shown in Chapter 6, when $N$ the number of particles in the
universe is 1 we have a Planck particle with a short life time $\sim 10^{-42}$ secs due to the Hawking radiation but with $N \sim 10^{80}$ particles as in the present universe we have the pion as the typical particle with a stable life time $\sim$ of the age of the universe due to the Hagedorn on radiation.

Let us now consider the following aspect [25]: It is well known that there are 18 arbitrary parameters in contemporary physics. We on the other hand have been working with the micro physical constants referred to earlier viz., the electron (or pion) mass or Compton wavelength, the Planck constant, the fundamental unit of charge and the velocity of light. These along with the number of particles $N$ as the only free parameter can generate the mass, radius and age of the universe as also the Hubble constant.

If we closely look at the equations (11) of Chapter 4 or (7) of Chapter 7, giving the gravitational and electromagnetic strength ratios, we can actually deduce the relation,

$$l = \frac{e^2}{mc^2} \quad (19)$$

In other words we have deduced the pion mass in terms of the electron mass, or, given the pion mass and the electron mass, we have deduced the fine structure constant. From the point of view of the order of magnitude theory in which the distinction between the electron, pion and proton gets blurred, what equation (13) means is, that the Planck constant itself depends on $e$ and $c$ (and $m$). Further in the Kerr-Newman type characterisation of the electron, in Chapter 3 the charge $e$ is really equivalent to the spinorial tensor density ($n = 1$). In this sense $e$ also is pre determined and we are left with a minimum length viz. the Compton length and a minimum time viz. the Compton time (or a maximal velocity $c$) as the only fundamental microphysical constants.

Let us try to further refine this line of thought. We observe that a discrete space time picture leads to the non commutative geometry alluded to earlier (5).

Infact we would have in this case, more fully,

$$[x, y] = 0(l^2), [x, p_x] = i\hbar[1 + l^2], [t, E] = i\hbar[1 + \tau^2] \quad (20)$$

What (20) means is that there is a higher order correction to the Heisenberg Uncertainty Principle. Infact from (20) we can easily conclude that there is
an extra energy $E'$ given by

$$\frac{E'}{mc^2} \sim \tau^2 \sim t^4 \sim \frac{1}{\sqrt{N}}$$

(21)

In (21), the appearance of $\frac{1}{\sqrt{N}}$ where $N$ is the number of particles in the universe appears at first sight to be purely accidental: We have not deduced it. However this is not so. Indeed from the picture of the fluctuational creation of particles alluded to in section 2, we get

$$\frac{E'}{mc^2} \sim \frac{1}{\sqrt{N}}$$

(22)

It can be seen that (22) and (21), deduced from two totally different stand-
points, are in fact the same. A consequence is the following fact: We have just seen that the micro physical constants namely an elementary particle mass, for example the electron mass $m$ (or Compton wavelength), a universal maximal velocity $c$ together with $N$ the number of particles in the universe were the only free parameters or arbitrary constants. From (21) we can see that there is a further narrowing down to just two arbitrary parameters, for example the maximal velocity $c$ and $N$. Given these two, the microphysical constants, including the Planck constant can be characterized, thus answering the question at the end of section 1. It must be emphasized that what is required is a universal maximum velocity in principle - its exact value is not important. Then, $N$ becomes the only parameter! All this is very much in the spirit of Feynman’s quotation in Chapter 1 as also the ancient Upanishadic tradition of seeing nature as different aspects of one phenomenon.

6 The Origin of a Metric

We first make a few preliminary remarks. When we talk of a metric or the distance between two ”points” or ”particles”, a concept that is implicit is that of topological ”nearness” - we require an underpinning of a suitably large number of ”open” sets [26]. Let us now abandon the absolute or background space time and consider, for simplicity, a universe (or set) that consists solely of two particles. The question of the distance between these particles (quite apart from the question of the observer) becomes meaningless. Indeed, this is
so for a universe consisting of a finite number of particles. For, we could isolate any two of them, and the distance between them would have no meaning. We can intuitively appreciate that we would in fact need distances of intermediate points. So for a meaningful distance, the concepts of open sets, connectedness and the like reenter in which case such an isolation would not be possible.

More formally let us define a neighbourhood of a particle (or point) A of a set of particles as a subset which contains A and at least one other distinct element. Now, given two particles (or points) A and B, let us consider a neighbourhood containing both of them, \( n(A, B) \) say. We require a non-empty set containing at least one of A and B and at least one other particle C, such that \( n(A, B) \supset n(A, C) \), and so on. Strictly, this "nested" sequence should not terminate. For, if it does, then we end up with a set \( n(A, P) \) consisting of two isolated "particles" or points, and the "distance" \( d(A, P) \) is meaningless. For practical purposes, in the spirit of Wheeler’s approximation, this sequence has to be very large.

Such an approximation has an immediate application. Our universe consists of some \( N \sim 10^{80} \) particles (or points), each point being "defined" within the Compton wavelength \( l \). Inside \( l \), space time in the usual sense breaks down - we have the unphysical zitterbewegung effects. Indeed \( l \) for a Planck particle of mass \( \sim 10^{-5} \text{gm} \) is precisely the Planck scale.

We now assume the following property\(^2\): Given two distinct elements (or even subsets) A and B, there is a neighbourhood \( N_{A_1} \) such that A belongs to \( N_{A_1} \), B does not belong to \( N_{A_1} \), and also given any \( N_{A_1} \), there exists a neighbourhood \( N_{A_2} \) such that \( A \subset N_{A_2} \subset N_{A_1} \), that is there exists an infinite sequence of neighbourhoods between A and B. In other words we introduce topological closeness.

From here, as in the derivation of Urysohn’s lemma\(^2\), we could define a mapping \( f \) such that \( f(A) = 0 \) and \( f(B) = 1 \) and which takes on all intermediate values. We could now define a metric, \( d(A, B) = |f(A) - f(B)| \). We could easily verify that this satisfies the properties of a metric.

It must be remarked that the metric turns out to be again, a result of a global or a series of larger sets, unlike the usual local picture in which it is the other way round.
7 Kaluza-Klein Theories and Quantized Super Strings

In Chapter 1, we briefly alluded to string theory. Though our subsequent considerations were in a different class, there is a surprising interface, as we will now see. Our starting point is the fact encountered in Chapter 6 that the fractal dimension of a Brownian quantum path is 2. This was further analysed and it was explained that this is symptomatic of Quantized Fractal space time and it was shown that in fact the coordinate $x$ becomes $x + \nu ct$. The complex coordinates or equivalently non-Hermitian position operators are symptomatic of the unphysical zitterbewegung which is eliminated after an averaging over the Compton scale. In this picture the fluctuationnal creation of particles was taken into account in a consistent cosmological scheme in Chapter 7.

It is well known that the generalization of the complex $x$ coordinate to three dimensions leads to quarternions $\mathbb{H}$, and the Pauli spin matrices.

We next return to the model of an electron as a Quantum Mechanical Kerr-Newman Black Hole. Infact in Chapter 3, we deduced electromagnetism in two ways. The first was by considering an imaginary shift,

$$x^\mu \rightarrow x^\mu + \imath a^\mu, \quad (a^\mu \sim \text{Compton scale}) \quad (23)$$

in a Quantum Mechanical context. This lead to

$$\imath \hbar \frac{\partial}{\partial x^\mu} \rightarrow \imath \hbar \frac{\partial}{\partial x^\mu} + \frac{\hbar}{a^\mu} \quad (24)$$

and the second term on the right side of (24) was shown to be the electromagnetic vector potential $A^\mu$,

$$A^\mu = \frac{\hbar}{a^\mu} \quad (25)$$

The second was by taking into account the fact that at the Compton scale, it is the so called negative energy two spinors $\chi$ of the Dirac bispinor that dominate where,

$$\chi \rightarrow -\chi$$

under reflections. This lead to the tensor density property,

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu} \quad (26)$$
the second term on the right side of (26) being identified with $A^\mu$,

$$A^\mu = \hbar \Gamma^\mu_\nu$$  \hspace{1cm} (27)

It was pointed out that (27) is formally and mathematically identical to Weyl’s original formulation, except that here it arises due to the purely Quantum Mechanical spinorial behaviour whereas Weyl had put it by hand. Another early scheme for the unification of gravitation and electromagnetism as referred to earlier was that put forward by Kaluza and Klein\[29, 30, 31\] in which an extra dimension was introduced and taken to be curled up. This idea has resurfaced in recent years in String Theory.

We will first show that the characterization of $A^\mu$ in (25) is identical to a Kaluza Klein formulation. Then we will show that equations (26) and (27) really denote the fact that the geometry around an electron is non-integrable. Finally we will show that in fact both (24) or (25) and (26) or (27) are the same formulations (as can be guessed heuristically by comparing (24) and (26)).

We first observe that the transformation (23) can be written as,

$$x^i \rightarrow x^i + \alpha^5 x_5$$  \hspace{1cm} (28)

where $\alpha_{i5}$ in (28) will represent a small shift from the Minkowski metric $g_{ij}$, and $i, j = 1, 2, 3, 4, 5, x^5$ being a fifth coordinate introduced for purely mathematical conversion.

Owing to (28), we will have,

$$g_{ij}dx^i dx^j \rightarrow g_{ij}dx^i dx^j + (g_{ij} \alpha^5)dx^i dx_5$$  \hspace{1cm} (29)

In Kaluza’s formulation,

$$A_\mu \propto g_{\mu 5}$$  \hspace{1cm} (30)

Comparison of (30), (28) and (29) with (23) and (24) shows that indeed this is the case. That is, the formulation given in (23) and (24) could be thought of as introducing a fifth curled up dimension, as in the Kaluza-Klein theory. To see why the Quantum Mechanical formulation (26) and (27) corresponds to Weyl’s theory, we start with the effect of an infinitesimal parallel displacement of a vector\[32\].

$$\delta a^\sigma = -\Gamma^\sigma_\mu a^\mu dx^\nu$$  \hspace{1cm} (31)
As is well known, (31) represents the extra effect in displacements, due to the curvature of space - in a flat space, the right side would vanish. Considering partial derivatives with respect to the \( \mu \)th coordinate, this would mean that, due to (31)

\[
\frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu
\] (32)

The second term on the right side of (32) can be written as:

\[-\Gamma^\lambda_{\mu\nu} g^\nu_{\lambda} a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma\]

where we have utilized the property that in the above formulation as seen in Chapter 3,

\[g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\]

\(\eta_{\mu\nu}\) being the Minkowski metric and \(h_{\mu\nu}\) a small correction whose square is neglected.

That is, (32) becomes,

\[
\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}
\] (33)

The relation (33) is the same as the relation (26).

We will next show the correspondence between (33) or (27) or (26) and (25) or (24). To see this simply we note that the geodesic equation is,

\[
\dot{u}^\mu \equiv \frac{du^\mu}{ds} = \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma
\] (34)

We also use the fact that in the Quantum Mechanical Kerr-Newman Black Hole model referred to, we have as in Chapter 3

\[u^\mu = c \quad \text{for} \quad \mu = 1, 2 \quad \text{and} 3,\]

while,

\[|\dot{u}^\mu| = |u^\mu| \frac{mc^2}{\hbar}\]

So, from (34) we get,

\[\Gamma^\mu_{\nu\mu} = \frac{1}{\dot{a}^\nu}, |\dot{a}^\nu| = \frac{\hbar}{mc}\]

This establishes the required identity.

We now come to the interface with Quantum Super Strings. We have already seen that the Quantized Fractal space time referred to really leads to
a non-commutative geometry, given by (20). It was also seen that these relations directly lead to the Dirac equation. Quantized Fractal space time is the underpinning for Quantum Mechanical spin or the Quantum Mechanical Kerr-Newman Black Hole, that is ultimately equations like (24) or (25) and (26) or (27).

It is also true that both the Kaluza Klein formulation and the non commutative geometry (20) hold in the theory of Quantum Superstrings (QSS). Infact we get from here a clue to the mysterious six extra curled up dimensions of Quantum Superstring Theory. For this we observe that (20) gives an additional contribution to the Heisenberg Uncertainity Principle and we can easily deduce

$$\Delta p \Delta x \sim \hbar l^2$$

Remembering that at this Compton scale

$$\Delta p \sim mc$$

It follows that

$$\Delta x \sim l^3$$

(35)

as $l \sim 10^{-11} \text{cms}$ for the electron we recover from (35) the Planck Scale, as well as a rationale for the peculiar fact that the Planck Scale is the cube of the electron Compton scale.

More importantly, what (35) shows is, that at this level, the single dimension along the $x$ axis shows up as being three dimensional. That is there are two extra dimensions, in the unphysical region below the Compton scale. As this is true for the $y$ and $z$ coordinates also, there are a total of six curled up or unphysical or inaccessible dimensions in the context of the preceding section.

If we start with equations (23) to (25) which are related to QFST (Quantized Fractal space time) and the non-commutative relation (20) we obtain a unification of electromagnetism and gravitation. On the other hand if we consider the spinorial behaviour of the Dirac wave function, we get (26) or (27). The former has been seen to be the same as the Kaluza formulation while the latter is formally similar to the Weyl formulation - but in this case (27) is not put in by hand. Rather it is a Quantum Mechanical consequence. We have thus shown that these two approaches are the same. The extra dimensions are thus seen to be confined to the unphysical Compton scale - classically speaking they are curled up or inaccessible.
In a sense this is not surprising. The bridge between the two approaches was the Kerr-Newman metric which uses, though without a clear physical meaning in classical theory, the transformation (23). The reason why an imaginary shift is associated with spin is to be found in the Quantum Mechanical zitterbewegung and the consequent QFST.

Wheeler remarked as quoted in Chapter 4 [9], ’’the most evident shortcoming of the geometrodynamic model as it stands is this, that it fails to supply any completely natural place for spin 1/2 in general and for the neutrino, in particular”, while ”it is impossible to accept any description of elementary particles that does not have a place for spin half.” In fact the bridge between the two is the transformation (23). It introduces spin half into general relativity and curvature to the electron theory, via the equation (27) or (32).

In this context it is interesting to note that El Naschie has given the fractal formulation of gravitation[33].

Thus apparently disparate concepts like the Kaluza Klein and Weyl formulations, Quantum Mechanical Black Holes, Quantized Fractal space-time and QSS are seen to have a harmonious overlap, in the context of QFST with its roots in the fluctuational creation of particles[34, 35].

It is worth pointing out some of the similarities between String theories and our formulation. The former started off, by considering one dimensional extended objects or strings, the extension being of the order of the proton Compton wavelength, vibrating and rotating with the speed of light (Cf. refs. given in Chapter 1). Not only could space-time points and singularities be fudged, but further the angular momenta were proportional to the squares of the masses, defining the well known Regge trajectories, as also in our formulation (Cf. Chapter 12, Equation (14)). All this is not surprising.

In particular, QSS deals with Planck length phenomena, the Kaluza-Klein curled up extra dimensions and leads to the non-commutative geometry (24). QFST on the other hand, deals with phenomena at the Compton scale, space-time being unphysical below this scale. Yet it leads us back to the Planck scale, the same number of extra, curled up, Kaluza-Klein dimensions and the same non-commutative geometry (24), once the meaning of (35) (or the modification of the Heisenberg Uncertainty Principle) is recognized. In this interpretation, the situation is similar to the fractal one dimensional Brownian path becoming two (or three) dimensional. The key is the transformation (23), which we first encountered right at the beginning, in Chapter 3 itself. It conceals zitterbewegung, leads to the Kerr-Newman metric, QFST and
what not!
Finally it is worth emphasizing that both in Strings and in our formulation
the Compton wavelength extension provides a rationale for the dual reso-
nance model, which originated from the Regge trajectories and then gave
the initial motivation for String theory.

8 Resolution, Unification and the Core of the
Electron

El Naschie\cite{30} has referred to the fact that there is no apriori fixed length
scale (the Biedenharn conjecture). Indeed it has been argued in the above
context that depending on our scale of resolution, we encounter electro-
magnetism well outside the Compton wavelength, strong interactions at the
Compton wavelength or slightly below it and only gravitation at the Planck
scale. The differences between the various interactions are a manif estation
of the resolution.
In this connection it may be noted that we can refer to the core of the elec-
tron $\sim 10^{-20} cms$, as indeed has been experimentally noticed by Dehmelt and
co-workers\cite{37}. It is interesting that this can be deduced in the context of
the electron as a Quantum Mechanical Kerr-Newman Black Hole.
It was shown in Chapter 3 that for distances of the order of the Compton
wavelength the potential is given in its QCD form

$$V \approx -\frac{\beta M}{r} + 8\beta M\left(\frac{Mc^2}{h}\right)^2 r$$

(36)

For small values of $r$ the potential (36) can be written as

$$V \approx A e^{-\mu^2 r^2}, \quad \mu = \frac{Mc^2}{h}$$

(37)

It follows from (37) that

$$r \sim \frac{1}{\mu} \sim 10^{-21} cm.$$  

(38)

Curiously enough in (37), $r$ appears as a time, which is to be expected be-
cause at the horizon of a black hole $r$ and $t$ interchange roles.
One could reach the same conclusion, as given in equation (38) from a different angle. In the Schrodinger equation which is used in QCD, with the potential given by (36), one could verify that the wave function is of the type \( f(r) e^{-\mu r^2} \), where the same \( \mu \) appears in (37). Thus, once again we have a wave packet which is negligible outside the distance given by (38).

It may be noted that Brodsky and Drell\[38\] had suggested from a very different viewpoint viz., the anomalous magnetic moment of the electron, that its size would be limited by \( 10^{-20} cm \). The result (38) as pointed out, was experimentally confirmed by Dehmelt and co-workers.

9 Levels of Physics

We now return to the relation (3) or (20) which expresses the underlying non-commutative geometry of space-time. What we would like to point out is that we are seeing here different levels of physics. Indeed, rewriting (3) or (20) as,

\[
[x, u_x] = i[l + l^3],
\]

we can see that if \( l = 0 \), we have classical physics, while if \( 0(l^3) = 0 \), we have Quantum Mechanics and finally if \( 0(l^3) \neq 0 \) we have the above discussed fractal picture, and from another point of view, the superstring picture.

Interestingly, in our case the electron Compton wavelength \( l \sim 10^{-11} cm \), so that \( 0(l^3) \sim 10^{-33} \) as in string theory.

The expansion in terms of \( l \) given above can be continued\[39\], and thus one could in principle go into deeper levels as well.

10 Gravitation and Black Holes

In our formulation we have not invoked the full non linear Theory of General Relativity. General Relativity itself comes up as an approximation, in its linear version and also through the fact that while \( G \) the gravitational constant, varies with time, over intervals small compared to the age of the universe, it is approximately constant. (Dirac however reconciles the variation of \( G \) with General Relativity by invoking the so called gravitational units of measurement\[40\], the units of our common usage being the atomic units). The question arises, is it possible to accommodate Black Holes within
such a non General Relativistic formulation? We will now show that Black Holes could also be understood without invoking General Relativity at all. We start by defining a Black Hole as an object at the surface of which, the escape velocity equals the maximum possible velocity in the universe viz., the velocity of light. We next use the well known equation of Keplerian orbits[41],

\[
\frac{1}{r} = \frac{GM}{L^2}(1 + e\cos\theta)
\]  

(39)

where \(L\), the so called impact parameter is given by, \(R_c\), where \(R\) is the point of closest approach, in our case a point on the surface of the object and \(c\) is the velocity of approach, in our case the velocity of light.

Choosing \(\theta = 0\) and \(e \approx 1\), we can deduce from (39)

\[R = \frac{2GM}{c^2} \]  

(40)

Equation (40) gives the Schwarzschild radius for a Black Hole and can be deduced from the full General Relativistic theory.

We will now use (40) to exhibit Black Holes at three different scales, the micro, the macro and the cosmic scales.

Our starting point is the observation that a Planck mass, \(10^{-5}gms\) at the Planck length, \(10^{-33}cms\) satisfies (40) and, as such is a Schwarzschild Black Hole. As pointed out Rosen has used non-relativistic Quantum Theory to show that such a particle is a mini universe.

We next come to stellar scales. It is well known that for an electron gas in a highly dense mass we have[42]

\[K\left(\frac{M^{4/3}}{R^4} - \frac{M^{2/3}}{R^2}\right) = K'\frac{M^2}{R^4}\]  

(41)

where

\[\left(\frac{K}{K'}\right) = \left(\frac{27\pi}{64\alpha}\right)\left(\frac{hc}{\gamma m_p^2}\right) \approx 10^{40}\]  

(42)

and

\[\bar{M} = \frac{9\pi}{8} \frac{M}{m_p} \quad \bar{R} = \frac{R}{(\hbar/m_e c)},\]

\(M\) is the mass, \(R\) the radius of the body, \(m_p\) and \(m_e\) are the proton and electron masses and \(\hbar\) is the reduced Planck Constant. From (41) and (42)
it is easy to see that for $\bar{M} < 10^{60}$, there are highly condensed planet sized stars. (Infact these considerations lead to the Chandrasekhar limit in stellar theory). We can also verify that for $\bar{M}$ approaching $10^{60}$ corresponding to a mass $\sim 10^{36} gms$, or roughly a hundred to a thousand times the solar mass, the radius $R$ gets smaller and smaller and would be $\sim 10^8 cms$, so as to satisfy (40) and give a Black Hole in broad agreement with theory. (On 13th September, 2000, NASA announced the discovery of exactly such Black Holes.)

Finally for the universe as a whole, using only the theory of Newtonian gravitation, it is well known that we can deduce, as we saw in Chapter 7,

$$R \sim \frac{GM}{c^2}$$  \hspace{1cm} (43)

where this time $R \sim 10^{28} cms$ is the radius of the universe and $M \sim 10^{55} gms$ is the mass of the universe.

Equation (43) is the same as (40) and suggests that the universe itself is a Black Hole. It is remarkable that if we consider the universe to be a Schwarzschild Black Hole as suggested by (43), the time taken by a ray of light to traverse the universe equals the age of the universe $\sim 10^{17} secs$ as shown elsewhere [43].

11 Dimensionality and the Field and Particle Approach

In a recent paper, Castro, Granik and El Naschie have given a rationale for the three dimensionality of our physical space within the framework of a Cantorian fractal space time and El Naschie’s earlier work thereon[44]. An ensemble is used and the value for the average dimension involving the golden mean is deduced close to the value of our $3 + 1$ dimensions. We now make a few remarks based on an approach which is in the spirit of the above considerations.

Our starting point is the fact that the fractal dimension of a quantum path is two, which, it has been argued in Chapter 6 is described by the coordinates $(x, ict)$. Infact this lead to the Dirac equation of the spin half electron. Given the spin half, it was pointed out that it is then possible to deduce the dimensionality of an ensemble of such particles, which turns out to be three.
There is another way of looking at this. If we generalise from the one space dimensional case and the complex \((x, it)\) plane to three dimensions, we in fact obtain the four dimensional case and the Theory of Quarternions, which are based on the Pauli Spin Matrices\(^{28}\). As has been noted by Sachs, had Hamilton identified the fourth coordinate in the above generalisation with time, then he would have anticipated Special Relativity itself. It must be observed that the Pauli Spin Matrices which denote the Quantum Mechanical spin half form, again, a non commutative structure.

Curiously enough the above consideration in the complex plane can have an interesting connection with an unproven nearly hundred year old conjecture of Poincare.

Poincare had conjectured that the fact that closed loops could be shrunk to points on a two dimensional surface topologically equivalent to the surface of a sphere can be generalised to three dimensions also\(^{45}\). After all these years the conjecture has remained unproven. We will now see why the three dimensional generalisation is not possible.

We firstly observe that a two dimensional surface on which closed smooth loops can be shrunk continuously to arbitrarily small sizes is simply connected. On such a surface we can define complex coordinates following the hydrodynamical route exploiting the well known connection between the two. If we consider laminar motion of an incompressible fluid we will have\(^{46}\)

\[ \nabla \cdot \vec{V} = 0 \quad (44) \]

Equation \((44)\) defines, as is well known, the stream function \(\psi\) such that

\[ \vec{V} = \nabla \times \psi \vec{e}_z \quad (45) \]

where \(\vec{e}_z\) is the unit vector in the \(z\) direction.

Further, as the flow is irrotational, as well, we have

\[ \nabla \times \vec{V} = 0 \quad (46) \]

Equation \((46)\) implies that there is a velocity potential \(\phi\) such that,

\[ \vec{V} = \nabla \phi \quad (47) \]

The equations \((45)\) and \((47)\) show that the functions \(\psi\) and \(\phi\) satisfy the Cauchy-Reimann equations of complex analysis\(^{14}\).
So it is possible to characterise the fluid elements by a complex variable

\[ z = x + iy \]  

The question is can we generalise equation (48) to three dimensions? Infact as we saw a generalisation leads not to three but to four dimensions, with the three Pauli spin matrices \( \vec{\sigma} \) replacing \( i \). Further these Pauli spin matrices do not commute, and characterise spin or vorticity. This close connection can be established by other arguments as well ([48]).

This is not surprising - the reason lies in equation (45) or equivalently in the multiplication law of complex numbers. (Infact, there is a general tendency toloverlook this fact and this leads to the mistaken impression that complex numbers are just an ordered pair of numbers, which latter are usually associated with vectors.)

The above considerations give an explanation for the 3 + 1 dimensionality of space time ([48]). Moreover equations like (45) and (48) re-emphasize the hydrodynamical model discussed earlier. Incidentally as Barrow ([50]) puts it, "Interestingly, the number of dimensions of space which we experience in the large plays an important role.... It also ensures that wave phenomena behave in a coherent fashion. Were there four dimensions of space, then simple waves would not travel at one speed in free space, and hence we would simultaneously receive waves that were emitted at different times. Moreover, in any world but one having three large dimensions of space, waves would become distorted as they travelled. Such reverberation and distortion would render any high-fidelity signalling impossible. Since so much of the physical universe, from brain waves to quantum waves, relies upon travelling waves we appreciate the key role played by the dimensionality of our space in rendering its contents intelligible to us."

We make a final remark. We saw in Chapters 1 and 2 that while the contemporary Field approach is based on guage interactions and spin 1 Bosons, these Bosons, as seen in Chapter 9 are not the Quantized Vortices, but rather their bound states - they can be thought of as, approximately steamlines. On the other hand, our approach has been based on Fermions, spin half particles, which are like the Quantized Vortices encountered in Chapter 3.

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