Hopf Textures

Sun Hong Rhie\textsuperscript{1,2} and David P. Bennett\textsuperscript{2}

\textsuperscript{1}Center for Particle Astrophysics, University of California, Berkeley, CA 94720
\textsuperscript{2}Institute for Geophysics and Planetary Physics, Lawrence Livermore National Laboratory, Livermore, CA 94550

ABSTRACT

A Hopf texture is a vacuum field configuration of isovector fields which is an onto map from the space as a large three sphere to the vacuum manifold $S^2$. We construct a Hopf texture with spherically symmetric energy density and discuss the topological charge. A Hopf texture collapses, and we study the collapse process numerically. In our simulations, it is clear that a Hopf texture does not decay into a pair of monopoles. We also argue that the probability of forming Hopf textures in random processes is very small compared to that of global monopoles.

Submitted to Physical Review D
1. Introduction

Field theories with spontaneously broken symmetries have important roles in particle physics, early universe cosmology, and condensed matter physics. Many of the most interesting features of these theories involve topologically non-trivial field configurations. These can be classified by continuous maps between spheres: $\pi_a(S^b): a \geq b$. $\pi_1(S^1)$ describes cosmic strings\(^1\) (local or global), $\pi_2(S^2)$, describes magnetic monopoles (gauge) and global monopoles,\(^2,3\) $\pi_3(S^3)$, describes instantons, baryon number violating anomalies and global textures,\(^4\) and finally $\pi_3(S^2)$ is the relevant algebra for Hopf textures. The name Hopf texture refers to the name of the map $\pi: S^3 \rightarrow S^2(=CP^1)$, Hopf fibration. (generally the Hopf map is $\pi: S^{2n+1} \rightarrow CP^n$)\(^5\) The fibration induces the isomorphism: $\pi_3(S^3) \rightarrow \pi_3(S^2)$ which in a sense is a sole reason why $\pi_3(S^2)$ is nontrivial. $\pi_4(S^3) = Z/2$, ‘quantization’ of two-flavour skyrmions,\(^6\) $\pi_4(S^2) = Z_2$ and $\pi_4(S^4) = Z$ complete the nontrivial algebras up to $a = 4$, the dimension of conventional space-time.

In the standard big bang GUT cosmology the universe expands, cools and goes through symmetry breaking phase transitions. At a GUT scale phase transition, the formation of topological defects\(^7\) such as cosmic strings,\(^1\) global monopoles,\(^3\) or textures can serve as seeds for the formation of galaxies and large scale structure, and the dynamics of these objects is important for the study of the cosmic structure they may seed. In this paper we will consider the theories with symmetry breakings such as $O(3) \rightarrow O(2)$ which have two types of defects: global monopoles and Hopf textures.

When a global $O(3)$ is spontaneously broken to $O(2)$, the vacuum states of the theory have a one to one correspondence with the points on a two sphere. Higgs fields prefer to align themselves (like ferromagnets) within causally connected volumes in order to minimize the gradient energy pressure. Topological defects are believed to form in the early universe because the coherence length of a field undergoing a symmetry breaking phase transition must be small–smaller than the causal horizon length. On scales larger than the initial horizon length, the phases of the Higgs fields will be uncorrelated so that a finite density of topological defects will be inevitable.\(^7\) As the universe expands after the phase transition, the horizon length grows and larger volumes come into causal contact. This allows global monopoles to “find” antimonopoles and annihilate while Hopf textures can begin to feel the long range forces that cause them to collapse. As a consequence, the number of the defects per horizon volume reaches a limiting value shortly after the phase transition and remains constant. This is popularly referred to as scaling behaviour, and it has been confirmed by numerical simulations for cosmic strings,\(^1\) global monopoles,\(^3\) domain walls,\(^8\) and SO(4) textures.\(^9\)
Textures are different from other defects in that the mapping between \( S^3 \) and \( R^3 \) is not as simple as the mapping between \( S^1 \) or \( S^2 \) and \( R^3 \). The smaller spheres can be simply embedded in \( R^3 \), but to make the correspondence between \( S^3 \) and \( R^3 \), one has to identify a boundary surface in \( R^3 \) (typically at infinity) with a point in \( S^3 \). If we impose the boundary condition \( \vec{\phi} = c \) constant at infinity and consider space as a large three sphere, then a Hopf texture is a vacuum field configuration of isovector fields whose values wrap the vacuum manifold two sphere once and an \( SO(4) \) (or \( SU(2) \)) texture is a vacuum filed configuration of iso-four-vector fields whose values wrap around the vacuum manifold three sphere. However, the boundary condition is not reasonable in the context of the early universe, because the field will be uncorrelated on scales larger than the horizon. Therefore the probability to form integer charge textures is very small, and the dominant form of the structures will be ‘partial textures’ as was discussed by Borill et al.\(^{10}\) in the case of \( SO(4) \) textures. The notion of ‘partial texture’ makes sense because the non-integer ‘partial texture’ number is fairly well defined in the given patch of space which contains high density of ‘fibers’. This is due to that (as we shall see) the dynamical evolution tends to intensify the texture density until the moment of collapse when the texture number changes by 1.

Here we include a brief description of \( SO(4) \) textures which are simpler to understand than Hopf textures. A ‘partial \( SO(4) \) texture’ is a map from a patch of space onto a patch of three sphere, and the winding number is the fractional volume of the patch of three sphere wrapped by the patch of space. The dynamics tends to isolate the texture so that at later times one can define a boundary surrounding the texture that does have an approximately constant field value. For example, lets consider the evolution of a partial texture of total flux \( Q(> 1/2) \) inside a finite box with fixed boundary conditions. Such a configuration will evolve toward a configuration with a shrinking \( Q = 1 \) texture in the center and a partial texture of charge \( Q - 1 \) bound to the boundary surface. If there were a repulsive core preventing the texture from collapsing, the shrinking unit charge texture would be stabilized. An example of this would be the thermal (random) production of Skyrmions with integer topological charge (≡ baryon number).

In the following, we employ a simple theory of self-interacting isovector scalar fields and construct a Hopf texture with spherically symmetric energy density as a transient configuration of the theory. Using the notion of fiber - equifield contour, we develop a way to understand the configuration particularly in terms of topological charge. We observe the linkage of fibers which results from the twist of transition functions (\( SO(2) \) gauge fields) of the three sphere as a fiber bundle (a Hopf bundle). To understand general configurations, we develop algebraic expressions and then discuss the notion of “partial winding” of Hopf textures and the
necessity of “gauge invariance.” We note that the total energy of a Hopf texture is proportional to the size of the texture and study the process of collapse of Hopf textures by computer simulations. We pay particular attention to collapse of a skewed Hopf texture because of the claim\textsuperscript{11} that a skewed Hopf texture in nematic liquid crystal (NLC) decays into a pair of a global monopoles and find that that is not the case in our simulation. Of course, our system of isovector scalar fields is somewhat different from that of the NLC because the ‘equilibrium manifold’ in NLC is \( RP^2 = S^2/\pm \), while the vacuum manifold in \( SO(3) \) field theory is \( S^2 \). The smallest integer charge Hopf texture in NLC wraps around the ‘equilibrium manifold’ twice. So we examined the evolution of \( Q = 2 \) Hopf texture of isovector scalar fields only to confirm the conclusion with \( Q = 1 \) case.

2. Construction of a Hopf Texture

Let’s consider a theory of a self-interacting isovector scalar field with a “negative mass squared” term. The Lagrangian is
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a - \frac{\lambda}{4} (\phi^a \phi^a - \eta^2)^2 ,
\]
and the equations of motion are
\[
\partial^2 \phi^a + \lambda (|\phi|^2 - \eta^2) \phi^a = 0 .
\]
The vacuum states are given by \(|\phi| = \eta\), the vacuum expectation value, and the set of the vacuum states is a two sphere. Here \( \eta \) is assumed to be \( \sim 10^{16} \) GeV, the generic GUT scale. A global monopole is a static solution
\[
\phi^a = \eta f(r) \frac{a^a}{r} ,
\]
where \( f(0) = 0, f(\infty) = 1 \). \( f(r) \) rapidly approaches unity as \( r \) becomes larger than the core size \( \delta = 1/\sqrt{\lambda \eta^2} \). When the scale of the physics of interest is much larger than the core size, we can assume that \( f(r) = 1 \) almost everywhere, and the \( SO(3) \) sigma model is a good approximation. The eq.(2.2) can be written as,
\[
\partial^2 \phi^a = \frac{\phi^b \phi^b \partial^2 \phi^b}{|\phi|^2} + \frac{\phi^a \partial^2 |\phi|^2}{2|\phi|^2} ,
\]
and in the limit \(|\phi| = \eta\), we obtain
\[
\partial^2 \phi^a - \frac{1}{\eta^2} \phi^a \phi^b \partial^2 \phi^b = 0 .
\]
\[
\phi^a \phi^a = \eta^2 .
\]
These are the defining equations of the system of isovector scalar fields we will
discuss in this paper.

A global monopole has a singularity at the center and the field values are the same along the radial lines stretching from the center to infinity. If the center is removed, the punctured space \( R^3 - \{\ast\} \) is homeomorphic to a ‘cylinder’ \( S^2 \times R \) and the trivial fibration \( \pi: S^2 \times R \rightarrow S^2 \) amounts to a ‘winding’. The number of ‘windings’ is called the topological charge. The triviality of the fibration makes the notion of fiber \( R \) rather redundant in the analyses of global monopoles, but when fields with two degrees of freedom are smoothly distributed almost everywhere in three dimensional space, there are always one dimensional curves along which the fields take the same values. A fiber is nothing but an equifield contour, and a monopole has a singularity because fibers converge. Of course, the region of the same field value can be a surface or a patch of space (or even the whole space), but not an isolated point. A fiber is either open or closed, and a finite open fiber ends on a monopole. A Hopf texture consists of closed fibers.

A Hopf texture is a mapping from space (a large “three sphere”) onto the vacuum manifold two sphere. The mapping is a nontrivial fibration \( \pi: S^3 \rightarrow S^2 \), where the fiber is \( S^1 \). In order to construct a Hopf texture, we have to find the projection map \( \pi \), and the first task is to find circles in \( S^3 \) (total space) and contract each of them to a point so that the points form \( S^2 \) (the base manifold). We note that \( S^3 \) is invariant under \( SO(4) \), and \( SO(4) \) has \( SO(2) \) subgroups which are circles and whose modules are also circles. Thus we parameterize \( S^3 \) such that the circles in the three sphere are manifest. For \( (x_1, x_2, x_3, x_4) \in S^3 \subset R^4 \),

\[
\begin{align*}
  x_1 &= \cos(\xi) \cos(\eta), & x_3 &= \sin(\xi) \cos(\zeta), \\
  x_2 &= \cos(\xi) \sin(\eta), & x_4 &= \sin(\xi) \sin(\zeta),
\end{align*}
\]

where \( 0 \leq \xi \leq \pi/2, \ 0 \leq \eta, \zeta \leq 2\pi \). The volume element in this parameterization is

\[
\sin(\xi) \cos(\xi) d\xi d\eta d\zeta .
\]

Consider an \( SO(2) \) subgroup which rotates the 1-2 plane and the 3-4 plane at the same time by \( \eta \mapsto \eta - \psi \) and \( \zeta \mapsto \zeta - \psi \). Now project out the \( SO(2) \) degree of freedom by taking \( \psi = \eta \), where \( \xi \neq 0 \). Then \( S^3 \) is mapped to a hemisphere \( HS^2 \), where the boundary \( S^1 \) corresponds to \( \xi = 0 \). Contracting the boundary to a point by a smooth map completes the Hopf fibration:

\[
\begin{align*}
  \phi_1 &= \sin(2\xi) \cos(\zeta - \eta) = 2(x_1x_3 + x_2x_4) , \\
  \phi_2 &= \sin(2\xi) \sin(\zeta - \eta) = 2(x_1x_4 - x_2x_3) , \\
  \phi_3 &= \cos(2\xi) = -1 + 2(x_1^2 + x_2^2) .
\end{align*}
\]

Finally we obtain a Hopf texture with winding number unity by mapping \( S^3 \) to
$R^3_\infty (\equiv R^3 \cup \{\infty\})$. We choose a stereographic projection which maps two spheres parallel to the equator in $S^3$ to concentric spherical shells in three space.

$$r(\chi) = D \tan\left(\frac{\chi}{2}\right), \quad (2.10)$$

where $\chi$ is the polar angle of $S^3$ in spherical coordinates ($\chi, \theta, \varphi$). $D$ is a scale parameter being the radius of the spherical shell to which the equator of three sphere is mapped. The field distribution, with $D = 1$, is

$$\phi_1 = \frac{4y(1 - r^2) + 8xz}{(1 + r^2)^2},$$

$$\phi_2 = \frac{4x(1 - r^2) - 8yz}{(1 + r^2)^2}, \quad (2.11)$$

$$\phi_3 = -1 + \frac{8(x^2 + y^2)}{(1 + r^2)^2},$$

and by replacing $\vec{x}$ by $\vec{x}/D$, one obtains the configuration for an arbitrary $D$.

3. Topological Charge

It should be obvious from the construction that the field configuration in eq. (2.11) wraps the vacuum manifold once and hence describes a texture of topological charge unity. Here we develop an algebraic understanding of the topology of the particular configuration in eq. (2.11) and extend it to arbitrary configurations.

The charge density of $\pi_n(S^n)$ can be constructed using the $\epsilon$-tensor which is antisymmetric and invariant under $SO(n)$. For strings ($n = 1$),

$$Q = \frac{1}{2\pi} \int_{S^1} \epsilon_{ab} \phi_a d\phi_b,$$

for monopoles ($n = 2$),

$$Q = \frac{1}{8\pi} \int_{S^2} \epsilon_{abc} \phi_a d\phi_b d\phi_c, \quad (3.1)$$

and for $SO(4)$ textures ($n = 3$),

$$Q = \frac{1}{12\pi^2} \int_{S^3} \epsilon_{abcd} \phi_a d\phi_b d\phi_c d\phi_d, \quad (3.2)$$

where three space is considered as a large three sphere $S^3$. The simplest field configuration of a $SO(4)$ texture is the homogeneous distribution of normal fields.
on $S^3$ ($\subset \mathbb{R}^4$) stereographically projected onto $\mathbb{R}^3$. This is a simple dimensional extension of the homogeneous normal field configuration on $S^2$ ($\subset \mathbb{R}^3$) which is just a cross section of the hedgehog monopole configuration on a two sphere surrounding the pole.

In order to understand Hopf textures, we need a different line of extension of the understanding of monopoles: fibration. We interpret the eq. (3.1) for a monopole as: one projects down the fibers, radial lines, and see how many times the field configuration on the cross section (here two sphere) covers the vacuum manifold two sphere. If we apply the same idea to the Hopf texture in eq. (2.11), we should project down the fibers, circles, and see how many times the field configuration on the cross section (say, ‘two sphere’) covers the vacuum manifold two sphere. Then the charge formula should be

$$Q = \frac{1}{8\pi} \int_{\text{twosphere}} \epsilon_{abc} \phi_a d\phi_b d\phi_c,$$

just as that of monopoles. One remaining task is the identification of the ‘two sphere’, the submanifold dual to Hopf texture charge density two form.

Let’s take a look at the field configuration in eq. (2.11). (see fig. 1.) The $z$-axis is a fiber corresponding to $\xi = \frac{\pi}{2}$ circle and the field configuration is axially symmetric around it. $\xi = 0$ fiber lies on the $x - y$ plane having the radius of $D$ (= 1) and other fibers foliate a tube, $S^1 \times S^1$, stringed by $\xi = 0$ fiber. For example, the fibers of $\phi_3 = 0$ ($\xi = \frac{\pi}{4}$) loop around the donut shell described by the equation $(\rho - \sqrt{2})^2 + z^2 = 1$, where $\rho^2 = x^2 + y^2$ so that each fiber winds once both of the $S^1$’s of $S^1 \times S^1$ (diagonal map from $S^1$ to $S^1 \times S^1$). Therefore a fiber on the torus links the other fibers on the same torus as well as both of the fibers $\xi = 0$ and $\xi = \frac{\pi}{2}$. (See fig. 2 for the linkage of two diagonal non-intersecting curves on a donut shell. Two non-diagonal non-intersecting closed curves on a torus have linking number zero.) In fact, space can be thought of as a family of tori parameterized by $\xi \in [0, \pi/2]$. In the limit of $\xi = 0$, the radius of the left $S^1$ converges to zero, $S^1 \times S^1 \rightarrow \{\ast\} \times S^1 \approx S^1$ (the $\xi = 0$ fiber), and in the limit of $\xi = \pi/2$, the right $S^1$ collapses to a point, $S^1 \times S^1 \rightarrow S^1 \times \{\ast\} \approx S^1$ (the $\xi = \pi/2$ fiber). Note that the left $S^1$ is parameterized by $\zeta$ which is ill-defined when $\xi = 0$, and right $S^1$ is parameterized by $\zeta$ which is ill-defined when $\xi = \frac{\pi}{2}$ (see eq. (2.7)). The linking of a fiber on an inner torus ($\xi$ smaller) and a fiber on an outer torus ($\xi$ larger) should be obvious because the fiber on the inner torus, as a curve in three space, can be deformed to the circle $\xi = 0$ without crossing the other fiber. Thus, any fiber is linked with all the other fibers in the Hopf texture with unit charge. In other words, if we choose any fiber, the total flux will go through the loop of
the fiber, and the ‘two sphere’ in eq. (3.3) is an arbitrary surface bounded by the fiber. For example, a half plane $\varphi = \text{constant}$, being bounded by the fiber $\xi = \frac{\pi}{2}$, is a ‘two sphere’. Indeed one obtains $Q = 1$ by integrating the charge density over the plane $\varphi = 0$. Actually, this was to be expected from the appearance of the hemisphere $HS^2$ bounded by $\xi = 0$ in the course of construction of the Hopf texture (below the eq. (2.8)). It should also be obvious that the fiber $\xi = 0$ is not any special than the others because of spherical symmetry.

(A simpler example of ‘twist and linking’ may be found in Möbius strip, $\pi : M\rightarrow S^1$, where the fiber is an interval $I = [0,1]$. $s(\in I) = \frac{1}{2}$ is a circle, the Möbius strip is a family of circles parameterized by $s \in [\frac{1}{2},1]$, and the circles link each other once.)

In summary, a surface subtending to the total flux of a Hopf texture with unit charge is not a closed surface in three space. It is an open surface bounded by a fiber, and we called it a ‘two sphere’ because its relative homotopy (the boundary is considered as one point, because the field value is constant on it) is equal to $\pi_2(S^2)$, whose algebra well known from widely discussed monopoles. Being derived from the same theory (the same fields and the same vacuum manifold), the charge density of Hopf texture has the same functional form as that of monopole, but Hopf textures are distinguishable from monopoles because of the distinction in the submanifolds to integrate the charge density over. For example, if we integrate Hopf charge density over a closed surface, which is the proper submanifold to count monopoles over, the integral vanishes. We can see this directly by integrating the charge density of the field configuration in eq. (2.11) over the $y = 0$ plane. This is related to the fact that $\pi_2(S^3) = 0$, namely the $y = 0$ plane can be deformed to infinity and contracted to a point.

Here we develop algebraic expressions to extend the intuitive understanding above. In particular, we note that the fiber is compact and the charge formula defined on a gauge slice can be integrated along the fiber. We will see the emergence of the connection, the gauge field. Let’s start with topological flux density

$$ \mathcal{B}_i = \frac{1}{8\pi} \epsilon_{ijk} \epsilon_{abc} \phi^a \partial_j \phi^b \partial_k \phi^c , $$

which we can derive from the eq. (3.3). The surface element corresponding to the flux density is

$$ dS^i = \frac{1}{2} \epsilon_{ijk} dx^{jk} ; \quad dx^{jk} \equiv dx^i dx^j . $$

We note that

$$ \nabla \times \mathcal{B} = 0 , \quad (3.4) $$

i.e., the flux is irrotational. If we compare it to magnetic fields, it is easy to un-
derstand that that is due to the absence of currents or line singularities. However, there are point singularities (monopoles), and the divergence does not vanish at the singularities.

\[ \nabla \cdot \mathbf{B} = Q \delta^3(x - x_0) \]

If there are no monopoles, the divergence vanishes everywhere, and this is the case we will discuss in the following. For Hopf textures,

\[ \nabla \cdot \mathbf{B} = 0 \]

and there exists a vector potential \( \mathbf{A} \) such that

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

The flux through an arbitrary surface \( S \)

\[ \int_S \mathbf{B} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{x} \]

and obviously the integral vanishes if the surface \( S \) is closed because \( \partial S = 0 \). In cartesian coordinates of the three sphere defined in eq. (2.7), it’s rather easy to find an \( \mathbf{A} \)

\[ B_i = \frac{1}{\pi} \varepsilon_{ijk}(\partial_j x_1 \partial_k x_2 + \partial_j x_3 \partial_k x_4) \]

\[ A_i = \frac{1}{2\pi}(x_1 \partial_i x_2 - x_2 \partial_i x_1) + \frac{1}{2\pi}(x_3 \partial_i x_4 - x_4 \partial_i x_3) \]  

(3.5)

One can see that \( \mathbf{A} \) is an \( SO(2) \) gauge field related to the simultaneous local rotation of the phases in the 1-2 plane and the 3-4 plane. Under the rotation, \( \mathbf{A} \rightarrow \mathbf{A} - \nabla \psi \), where \( \psi \) is the rotation angle. In spherical coordinates,

\[ B_i = \frac{1}{2\pi} \varepsilon_{ijk} \sin 2\xi \partial_j \xi + \sin^2 \xi \partial_k \zeta \]  

(3.6)

where the class \([\zeta - \eta]\) runs from 0 to \(2\pi\). In particular,

\[ \oint_{\text{fiber}} \mathbf{A} \cdot d\mathbf{x} = 1 \]  

(3.7)

because \(\zeta - \eta = \text{constant} \) along the fiber. This is consistent with the fact that a
‘two sphere’ is bounded by a fiber for a Hopf texture with charge unity.

\[
1 = \int_{\text{two sphere}} \mathbf{B} \cdot d\mathbf{S} = \oint_{\text{fiber}} \phi \cdot A \cdot dx .
\] (3.8)

(Of course, the eq. (3.7) is not true in general as we will demonstrate later with a field configuration of charge 2.) Now we can change the charge formula in eq. (3.3) into a volume integral by multiplying by 1.

\[
Q = \int_{\text{gaugeslice}} \mathbf{B} \cdot d\mathbf{S} \oint_{\text{fiber}} \phi \cdot A \cdot dx = \frac{1}{2} \int A_i B_j \epsilon_{jmn} \, dx^{imn} = \int A \cdot B \, d^3x .
\] (3.9)

The second equality holds because \( \mathbf{B} \) is the direction vector of the fiber.

\[
\nabla_B \tilde{\phi} = \mathbf{B} \cdot \nabla \tilde{\phi} = 0 ,
\]
i.e., \( \mathbf{B} \) is non-zero only when \( \nabla \phi = 0 \), which is the defining equation of fiber. The ‘differential charge’ is the fractional volume of the three sphere covered by \( d^3x \)

\[
\delta Q = A \cdot B \, d^3x = \frac{1}{2\pi^2} \sin \xi \cos \xi d\xi d\eta d\zeta .
\]

This displays the fact that the Hopf texture charge \( Q \) arises because three space \( R^3 \) is relatively a three sphere here. Of course, the fractional volume does not define a (partial) Hopf texture charge because it is not a well-defined number. It is gauge variant. If we let

\[
\Delta Q = \int_{\Delta} A \cdot B \, d^3x ,
\]
then

\[
\Delta Q \rightarrow \Delta Q - \oint \psi \mathbf{B} \cdot d\mathbf{S}
\]
under a gauge transformation. This is not invariant, because \( \psi \) has nontrivial (\( \neq \) constant) solutions. For example, in a gauge \( \nabla \cdot A = 0, \nabla^2 \psi = 0 \), and \( \psi = x \) is a solution. Well, that’s because \( \Delta Q \) measures the winding of the three sphere covered by the space patch \( \Delta \), whereas a meaningful measure will have to be the
winding of the vacuum manifold \( S^2 \). In other words, \( \Delta Q \) has to be associated with \( S^2 \) as well as with the total space \( S^3 \). The remedy is rather obvious: the integral must include integration over the fiber. If we rewrite eq. (3.9) as

\[
Q = \oint_{\text{fiber}} \mathbf{A} \cdot d\mathbf{x} \oint_{\partial S} \mathbf{A} \cdot d\mathbf{x} ,
\]

we see that a fiber and \( \partial S \) are the generators of a torus. This is one of the family of tori we discussed before. We also can write it as

\[
Q = \int_{S_f} \mathbf{B} \cdot dS \int_{S} \mathbf{B} \cdot dS ,
\]

where \( S_f \) is a surface bounded by a fiber, and we see that linking is essential because the first integral would vanish if there were no linking. For example, one can consider a vortex-like flux, say \( \mathbf{B} = \hat{\phi}/\rho^2 \), then the flux is orthogonal to the plane bounded by a fiber, and the integral vanishes. (or \( \mathbf{A} = \hat{z}/\rho \) and \( \mathbf{A} \cdot \mathbf{B} = 0 \).) A Hopf texture charge is a count of linked fibers.

### 3.1. Multiply Charged HopfTextures

Here we would like to discuss a couple of multiple integer charge configurations. They are very minimal extensions of the case of \( Q = 1 \), but they should provide a taste of general configurations. Let’s first consider a non-trivial field configuration of \( Q = 0 \) given by a map \( \phi' \). (It is understood that the spherical coordinates of \( S^3 \) are functions on \( R^3 \) as given by the inverse of the stereographic projection in eq. (2.10).)

\[
\begin{align*}
\phi'_1 & = \sin(4\xi) \cos(\zeta - \eta) = 2\phi_3\phi_1 \\
\phi'_2 & = \sin(4\xi) \sin(\zeta - \eta) = 2\phi_3\phi_2 \\
\phi'_3 & = \cos(4\xi) = 2\phi_3^2 - 1
\end{align*}
\]

\[
B_i = \frac{1}{\pi} \epsilon_{ijk} \sin 4\xi \partial_j \xi \partial_k [\zeta - \eta] ,
\]

\[
A_k = \frac{1}{2\pi} (\cos^2 2\xi \partial_k \eta + \sin^2 2\xi \partial_k \zeta) .
\]

The total charge is zero, because the vacuum manifold is wrapped twice in the
opposite senses. For example, if we integrate over $y = 0$ half-plane,

$$\int B = 0 .$$

However,

$$\oint_{fiber} A = 1 ,$$

and according to Stokes' theorem, the fiber can not be the boundary of the cross section of the total flux. As before, space is considered as a family of tori parameterized by $\xi$, and the whole torus of $(\rho - \sqrt{2})^2 + z^2 = 1$ is a ‘fiber’ with $\phi_3 = 1$. The total flux inside the torus is 1, and that outside of it is $-1$. The fibers of the same field value inside and outside the torus are distinct loops, and they link each other. There is no oriented surface bounded by a pair of linked loops. Now we consider a field configuration of $Q = 2$.

$$\phi''_1 = \sin(2\xi) \cos 2(\zeta - \eta) = \frac{\phi^2_1 - \phi^2_2}{\sqrt{\phi^2_1 + \phi^2_2}}$$

$$\phi''_2 = \sin(2\xi) \sin 2(\zeta - \eta) = \frac{2\phi_1 \phi_2}{\sqrt{\phi^2_1 + \phi^2_2}}$$

$$\phi''_3 = \cos(2\xi) = \phi_3 ,$$

(3.13)

Here each (non-degenerate) torus has two fibers with the same field values: for example, $\zeta = \eta$ and $\zeta = \eta + \pi$. A gauge slice, e.g., $y = 0$ half plane, cut through both of them in the same sense, and the flux density shows the double strength.

$$B_i = \frac{1}{2\pi} \epsilon_{ijk} \sin 2\xi \partial_j \xi \partial_k (\zeta - \eta) ,$$

and

$$A_k = \frac{1}{\pi} (\cos^2 \xi \partial_k \eta + \sin^2 \xi \partial_k \zeta) .$$

If we integrate over a fiber,

$$\oint_{fiber} A = 2 ,$$

and the construction of the volume integral formula for the charge can not be given by (3.9). Let’s define the “linking number” of a Hopf texture as: the “linking
“linking number” is one if a fiber with one field value links with another fiber with another field value, the “linking number” is two if a fiber with one field value links with other two fibers with another field value, the “linking number” is four if two fibers with the same field value link with other two fibers with another field value, etc.

Then one can see that the field configuration in eq. (3.13) has a “linking number” of four, and the configuration (3.11) has a “linking number” zero because any fiber is linked with two fibers with the same value and the opposite sense. One also can see that the eq. (3.9) delivers the correct “linking number” for integer charge Hopf textures.

3.2. The Scarcity of Partial Hopf Textures

Now let’s consider partial Hopf textures. We can construct one in the following manner: take a $Q = 1$ configuration, and consider a cube centered on the center of the Hopf texture. Imagine that this cube “cuts” the fibers that it crosses, so that there is a maximum radius torus $T_{\text{max}}$ which is intact. Next, extend the field values on the walls of the cube radially to infinity. Then the total flux is given by the flux within $T_{\text{max}}$, but the “linking number” of the partial torus is one according to the definition above because the “linking number” does not care about the total number of the fibers involved in the linking. The “linking number” cares only about how many “identical fibers” are involved. In fact, if we take additional account of the linking of the loops on the tori with infinite fibers, the total flux of fibers involved is the same as that of $Q = 1$. In other words, for any partial Hopf texture, there is a counterpart integer Hopf texture. Now recall that

$$1 = \oint_{\text{fiber}} A = \int_{S_f} B.$$ 

For example, the topological flux through the fiber $\xi = 0$ (the last ‘torus’ to be cut) has to be one. Generally

$$\oint_{\text{fiber}} A = n \quad ; \quad n \in \text{integer}.$$ (3.14)

In other words, a partial Hopf texture cannot be an isolated donut of linked fibers of the total flux of an arbitrary real number. The donut has to be linked by topological flux of another partial Hopf texture or infinite fibers such that the condition (3.14) is met. The implication is that the probability to form a partial Hopf texture is comparable to the probability to form a full charge Hopf texture, and so the number density of any Hopf texture in the early universe is expected to be very small. This
is very different from the case of $SO(4)$ textures. (Simplistically speaking, Hopf textures are more global than $SO(4)$ textures.) We can make a crude estimate of the density of Hopf textures per horizon volume by doing this calculation in a toy universe in which the field takes only certain discrete values at points on a cubic grid in space. (The field is assumed to vary smoothly between the grid points.) This yields an estimate of about $10^{-4}$ Hopf textures per horizon volume.

In general, the flux of Hopf textures, if any, coexist with that of global monopoles, and the total flux of the fibers through an arbitrary surface cannot be expressed as a loop integral over the boundary of the surface. That is because a gauge fields corresponding to monopole flux is singular, or the integral of the flux over a surface bounded by a given loop jumps as the surface passes the poles.

$$\int_{S} \mathbf{B} \cdot dS \neq \oint_{\partial S} \mathbf{A} \cdot dx .$$

For example, we can consider a configuration where the flux of a pair of monopole and antimonopole is wedged along the $z$-axis through the unit charge Hopf texture. The vector potential $\mathbf{A}$ along a closed fiber must be the same as that of the unit charge Hopf texture, and its loop integration must be equal to 1. However, the total flux going through a surface bounded by a fiber is two if the monopole flux goes through the surface. Thus, counting Hopf textures, e.g., in a simulation, is rather hopeless because of its global nature. The volume integral (3.9) doesn’t really help for identifying the flux either, because $\mathbf{A}$ field is not a functional of the primary fields $\vec{\phi}$.

4. Numerical Study of Collapse

The energy density of the Hopf texture configuration given in eq. (2.9) is

$$\mathcal{E} = \frac{16D^2}{(D^2 + r^2)^2} .$$

The total energy $E = 16\pi^2D$ is proportional to the scale parameter $D$ which implies that the texture is unstable to collapse. In the non-linear sigma model ($\lambda \to \infty$ limit), the Hopf texture will shrink to a point. With a more realistic potential, when $D$ gets sufficiently small, the energy density will be large enough so that the field can climb over the potential barrier and the Hopf texture can unwind. The initial gradient energy will be released mainly as goldstone boson radiation with a small amount of massive Higgs radiation coming from the last stages of the collapse.
Even though the energy density, (4.1) is spherically symmetric, the field configuration, (2.9) is only axially symmetric. Thus the process of collapse cannot be spherically symmetric, and the family of configurations parameterized by the size $D$ in eq. (2.11) are not dynamically related.\footnote{13}

Because of the lack of symmetry, it is probably fruitless to look for an analytic solution to the collapse of a Hopf Texture. Here we study the collapse numerically by evolving the eq. (2.2) with the constraints that $\vec{\phi}^2 \equiv 1$ and $\vec{\phi} \cdot \vec{\phi} = 0$. Unlike its continuum counterpart, the numerical “nonlinear sigma-model” does not force $\vec{\phi}^2 = 1$ except at the locations of the grid points. Thus, objects such as global monopoles or unwinding textures can be present in the numerical “nonlinear sigma-model” simulations. In fact, it is not difficult to show that the numerical sigma model is indeed the limit of the numerical version of (2.2) when $\lambda \to \infty$. The advantage of the numerical “nonlinear sigma-model” is that one does not need to use extremely small timesteps to prevent the instabilities in the massive degree of freedom that occur when $\lambda$ is large. (The timesteps do have to be much smaller than the spacial steps in order to evolve short wavelength modes properly, however.) The results displayed below have used timesteps of $\Delta t = 0.032 \Delta x$ for the “nonlinear sigma-model” and $\Delta t = 0.08 \Delta x$ for the “$\lambda \phi^4$” simulation. These parameters always resulted in global energy conservation of better than 0.1\% (typically $\sim 0.01\%$).

In our numerical simulations, we chose an initial configuration of unit charge Hopf texture whose total flux is contained in a finite volume. We chose the following map from $S^3 \rightarrow R^3$, in place of the eq. (2.10),

$$r(\chi) = \frac{R}{\pi} \chi,$$

and ran the following initial field configuration.

$$\phi_1 = 2 \sin^2 \left( \frac{\pi r}{R} \right) \frac{x z}{r^2} + \sin \left( \frac{2 \pi r}{R} \right) \frac{y}{r},$$

$$\phi_2 = 2 \sin^2 \left( \frac{\pi r}{R} \right) \frac{y z}{r^2} - \sin \left( \frac{2 \pi r}{R} \right) \frac{x}{r},$$

$$\phi_3 = 1 - 2 \sin^2 \left( \frac{\pi r}{R} \right) \frac{x^2 + y^2}{r^2},$$

(4.2)

where $R$ is the radius of the Hopf texture. A time series of the collapse of this configuration is shown in Fig. 3. This figure shows the $y = 0$ cross section of eq. (4.2). Note that the sign of $\phi_3$ cannot be determined from the information in this figure, but that these configurations have a $\phi_3(-x) = \phi_3(x)$ reflection symmetry (in the $y = 0$ plane). Fig. 3(a) shows the $y = 0$ cross section of the initial
configuration with only 1 out of every 4 points shown for clarity. Figs. 3(b) and (c) show the same cross section (magnified by a factor of 2) just before and just after the decay of the Hopf texture. Fig. 3(d) shows this the field configuration about $8\Delta x$ after the collapse. Note the similarity of the configurations long before (Fig. 3(a)) and very shortly before the collapse (Fig. 3(b)).

We have examined our simulations of Hopf texture collapse with a monopole detection algorithm in order to test the claim by Chuang, et al.\textsuperscript{11} that Hopf textures in uniaxial nematic liquid crystals decay into global monopoles. This algorithm was designed to work with the “nonlinear sigma-model” simulations, and it detects monopoles in the grid cells by smoothing interpolating the field from the vertices to the sides of the cells. For the simulations with axially symmetric initial conditions, we found that the monopole detection algorithm never detected any monopoles.

Fig. 4 shows the flow of a selected set of fibers during the collapse of the symmetric Hopf texture. After an initial transient, each fiber tends to move at a nearly constant velocity until collapse, but these velocities are not radial. So as expected, there seems to be no self-similar collapse mode as there is for $SO(4)$ textures.

In order to understand how a more generic Hopf texture might collapse we have also studied the collapse of an asymmetric Hopf texture with an initial configuration given by,

$$
\begin{align*}
\phi_1 &= 2\sin^2\left(\frac{\pi r}{R_\phi}\right) \frac{xz}{r^2} + \sin\left(\frac{2\pi r}{R_\phi}\right) \frac{y}{r}, \\
\phi_2 &= 2\sin^2\left(\frac{\pi r}{R_\phi}\right) \frac{yz}{r^2} - \sin\left(\frac{2\pi r}{R_\phi}\right) \frac{x}{r}, \\
\phi_3 &= 1 - 2\sin^2\left(\frac{\pi r}{R_\phi}\right) \frac{x^2 + y^2}{r^2},
\end{align*}
$$

(4.3)

where

$$
R_\phi \equiv \frac{R}{3} \left(2 + \frac{x}{\sqrt{x^2 + y^2}}\right).
$$

(4.4)

Figure 5 shows a cross section of a time series of a collapsing asymmetric Hopf texture. Note that the cross section shown is orthogonal to the cross section shown in Fig. 3. For the symmetric collapse, this cross section (the $z = 0$) plane would display the axial symmetry.

Fig. 5(a) shows the initial configuration while Fig. 5(b) shows the configuration immediately before the texture has Hopf texture has decayed. Figs. 5(c) at $t = 15.0\Delta x$ shows the configuration just after the instant of collapse, and Figs. 5(c)–(f) show the field evolution after the Hopf texture collapse.
In their observational study of the collapse of Hopf textures in Nematic liquid crystals, Chuang et al.\textsuperscript{11} claimed that Hopf textures generally decay by producing a global monopole-antimonopole pair. This monopole-antimonopole pair is supposed to be produced at the point where the Hopf texture first unwinds (on the negative $x$-axis in Fig. 5). The “poles” appear to be pulled in a roughly semicircular path (along the circle, the generator of tori parameterized by $\eta$, in terms of our convention) until they annihilate on the “other side” of the Hopf texture (on the positive $x$-axis in Fig. 5). NLC consists of rod-like molecules, and the order parameter is the orientation of the rod. The rods do not carry senses ($\pm$), and hence flipping the rods is a degeneracy operation. For example, would-be monopoles and antimonopoles in $SO(3)$ field theory are degenerate in NLC. (Imagine the hedgehog configuration with each vector drawn without an arrowhead.) The ‘equilibrium manifold’ is $RP^2 = S^2/\pm$, and the configuration in eq. (2.11) wraps around the ‘equilibrium manifold’ twice. Since $RP^2$ can not be embedded in three space, there is no half Hopf texture covering the ‘equilibrium manifold’ once. That being understood, we were curious about how the flow of open ‘two spheres’ can change into a flow of closed two spheres converging into two parting monopoles. Our simulations do show some resemblance to this scenario in the gradient energy density, but we do not observe the production of any monopole-antimonopole pairs except on scale below our spacial resolution. During the unwinding, the neighbouring fields in a cell rotate in the opposite directions in our discrete sigma model, and this amounts to fields’ climbing over the potential barrier in the core in continuum $\phi^4$ theory. Because of the rotation in the opposite directions, there are cells which seem to contain monopoles during that brief transition time, but they are not really monopoles with consistent extension of the fields in the neighbouring cells outside the core.

This last point can also be seen by examining the output of two different monopole detection algorithms: our standard monopole detection algorithm (described in Ref. 3), and another algorithm which simply searches for points with $\vec{\phi}^2 \ll 1$. Clearly this later algorithm cannot be used with the “non-linear sigma model” simulations. For these simulations, the monopole detection algorithm does “detect” some monopoles, but these have been determined to be numerical artifacts because their location depends on the grid spacing but is nearly independent of all other parameters such as the position in physical units. This diagnosis is confirmed by the $\vec{\phi}^2 \ll 1$ detection algorithm for some runs with finite $\lambda$. The maximum distance that these “phantom monopoles-antimonopole pairs” get from each other in any of these simulations is only about $2\Delta x$ independent of the size of the initial Hopf texture, so we are confident that they are not physical. In our simulations, a pair (or pairs) of monopoles were not materialized nor traveled along a ‘loop’ over many cells.
Fig. 6 shows the flow of a selected set of fibers during the collapse of the asymmetric Hopf texture. The evolution of each side individually resembles the collapse of the symmetric Hopf texture (Fig. 4), but there is a difference just after the collapse. For the symmetric Hopf texture, many of the fibers disappear at the time of the collapse and do not ever reappear (others reappear much later). All of the fibers shown in Fig. 6 crossing the \( y = 0 \) plane on the “small” \( (x < 0) \) side of the Hopf texture disappear at the time of the Hopf texture collapse \( (t = 14.4\Delta x) \), but then they reappear at \( t = 16.0\Delta x \) with the opposite orientation. This means that the fibers have all converged at a point and jumped over the potential barrier to “untwist” the Hopf texture. In the configuration at \( t = 16.0\Delta x \) the fibers are arranged in a toroidal configuration as before, but now the twist has been removed. This means that the same fibers are always on the inside of the torus, so the inner fibers can shrink to a point and disappear. This process proceeds with the outer fibers moving toward the inside and then disappearing.

Finally, we have made some rather simple comparisons of the evolution of the source term \( (8\pi \dot{\phi}^2) \) for the growth of density fluctuations in dark matter. We have compared the evolution of \( 8\pi \dot{\phi}^2 \) for the collapsing texture simulations described above and for simulations of the annihilation of a single monopole-antimonopole pair. We found that although the two configuration had similar initial energies and coherence scales, the collapsing Hopf texture collapsed more rapidly and developed a higher kinetic energy by a factor of 2 or so. In addition, collapsing Hopf textures also concentrated much of their energy into a small central region during and after the collapse. These factors suggest that the density fluctuations due to collapsing Hopf textures might be more prominent than the density fluctuations due to monopole-antimonopole annihilation. On the other hand, monopoles and antimonopoles are much more numerous, so the most prominent annihilation induced fluctuations might still be more prominent than the Hopf texture collapse induced fluctuations. These questions can probably only be convincingly resolved through by coupling gravity to the numerical simulations.
5. Discussions

This work started as the first step to find a way to measure the relative population of Hopf textures in comparison to that of monopoles in the simulations of global monopole and texture seeded cosmic structure formation. Along the way, we became convinced that the Hopf textures are very rare and the density fluctuation pattern due to Hopf textures are not so dramatically different from that of monopoles. Perivolaropoulos has claimed that a $\pi_2$ texture of $SO(3)$ field theory propagates with a planar structure, but the author used cylindrically symmetric equations of motion ignoring the non-cylindrical modes. In practice, the non-cylindrical modes prevent the ‘planar structure’ from being relevant.

Perhaps it is a different story if the system is antiferromagnetic as in chiral spin liquid (CSL). Laughlin, Zou, and Libby in Ref. 16 claimed to have identified the gauge field of a non-abelian magnetic monopole which is responsible for non-vanishing Berry phases (for transport of spinons around a loop) in CSL. P. B. Wiegmann recently argued that the ‘topological solitons’ of CSL are $SO(3)$-magnetic monopoles (Laughlin and his colleagues found to their distress that their gauge fields do not take values in $so(3)$ but in a subspace of $su(4)$), and they must correspond to ‘Hopf magnetic textures’ (global ‘Hopf textures’ in a magnetic system). The implication of Wiegmann’s claim that global monopoles can be identified with Hopf textures is a bit puzzling. Nonetheless, it will be interesting to know if $SO(3)$ topological objects play roles in the dynamics of the antiferromagnetic system. If we recall the work by Anderson that the antiferromagnetic has a long-range order in the ground state, where the ground state is a singlet state, we can imagine a situation that as the system cools, domains of Néel state begin to form such that the direction of the total spin of a sublattice in each domain is random. Then the difference of the total spins of the alternate sublattices resemble the isovector Higgs fields as an order parameter. If they are relevant at all, one would like to know what provides the stability of them (e.g., textures collapse; what forms the core of monopoles), what the effects of doping would be, how they will translate into the real situation where the electromagnetic interaction is turned on, and so on.
6. Conclusions

We have explicitly constructed singly and multiply charged Hopf texture configurations in an isovector scalar field theory which is spontaneously broken from $SO(3)$ to $SO(2)$, and derived general formulas for the Hopf texture charge. We have emphasized the importance of linking and gauge invariance. We argued that partial Hopf textures are as rare as full Hopf textures in the early universe, and made a crude estimation that the number density of horizon volumes should be smaller than $10^{-4}$.

By means of numerical simulations, we have followed the process of Hopf texture collapse with both symmetric and asymmetric initial conditions. We have shown that in neither of these cases does the Hopf texture produce a monopole-antimonopole pair when it collapses. This is in contrast to the observation by Chuang, et al. who claimed that the Hopf textures in nematic liquid crystals decayed through the production of global monopole-antimonopole pairs. It is possible that this discrepancy is due to the differences in the topology or the dynamics between the nematic liquid crystals and our $SO(3)$ scalar field theory. If this is the reason, then it suggests that the nematic liquid crystals might be of only limited use for the study of the evolution of topological defects in a cosmological context.

Finally, we briefly discussed the role of Hopf textures in seeding density fluctuations and found that although Hopf textures are much rarer than annihilating monopole-antimonopole pairs, their density fluctuations maybe somewhat more prominent.

Acknowledgements: This work was supported in part the U.S. Department of Energy at the Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48 and by the NSF grant No. PHY-9109414.
REFERENCES

1. G. W. Gibbons, S. W. Hawking, and T. Vachaspati, eds., *The Formation and Evolution of Cosmic Strings*, Cambridge University Press, Cambridge, 1990.

2. M. Barriola, and A. Vilenkin, *Phys. Rev. Lett.*, 63, 341, (1989).

3. D. P. Bennett and S. H. Rhie, *Phys. Rev. Lett.*, 67, 1709 (1990).

4. F. Wilczek and A. Zee, *Phys. Rev. Lett.*, 51, (1983) 2250; N. Turok, *Phys. Rev. Lett.*, 63, 2625, (1989).

5. E. H. Spanier. *Algebraic Topology*. Springer-Verlag, 1966, pages 91 and 377.

6. E. Witten, *Nucl. Phys.*, B223, 433 (1983).

7. T. W. Kibble, *J. Phys. A* 9 (1976) 1387.

8. W. H. Press, B. S. Ryden, and D. N. Spergel *Astrophys. J.* 357, 293 (1990).

9. D. N. Spergel, N. Turok, W. H. Press, and B. S. Ryden, *Phys. Rev. D43*, 1038 (1991).

10. J. Borill, E. J. Copeland, and A. R. Liddle, *Phys. Lett.* B258, (1991) 310.

11. I. Chuang, R. Durrer, N. Turok, and B. Yurke, *Science*, 251, 1336 (1991).

12. In NLC, there are strings, \( \pi_1(RP^2) = \pi_1(S^2/\pm) = \pm \), and \( \nabla \times B \neq 0 \).

13. It is interesting to note that a sphaleron has axially symmetric fields and spherically symmetric energy density. However, a sphaleron is a static solution even though unstable, while there is no static Hopf texture. Axial symmetry of a sphaleron is in the vector fields differently from that of Hopf textures which are scalar fields. Higgs field of the standard sphaleron is spherically symmetric. See N. S. Manton, *Phys. Rev. D28*, 2019 (1983).

14. D. P. Bennett, S. H. Rhie, and D. Weinberg, in preparation, 1992.

15. L. Perivolaropoulos, Brown University preprint BROWN-HET-775 (1990); R. Lees and T. Prokopec, Brown University preprint BROWN-HET-778 (1990).

16. X. G. Wen *et al.*, *Phys Rev. B39* (1989) 11413; L. D. Laughlin and Z. Zou, *Phys. Rev. B41* (1989) 664; Stephen B. Libby, Z. Zou, and L. D. Laughlin *Nucl. Phys. B348* (1991) 693.

17. P. B. Wiegmann, *Nonabelian gauge theory of quantum antiferromagnetism in three dimensions and fractional quantum numbers of magnetic solitons*, IAS preprint (1991).

18. P. W. Anderson, *Phys. Rev.* 86 (1952) 694.
FIGURE CAPTIONS

1) The explicit field configuration on the right show all the information but the sign of $\phi_2$. The schematic diagram on the left presents the extra information. The triplet of signs in each region represents the sign of fields $(\phi_1, \phi_2, \phi_3)$ in the order.

2) (a) Two diagonal non-intersecting curves on a donut shell can be made by drawing parallel diagonal lines on a rectangle, identifying the opposite edges and stretching the flat torus to embed it in space. (b) After one pair of edges are identified, the curves are parallel spirals on a cylinder. If we fix the end points and consider the curves as situated in three space, they are equivalent to two open strings braided once. (c) Identifying the other edges closes the strings to loops which are linked.

3) This figure shows a time series of the $y = 0$ cross section of a collapsing axially symmetric Hopf texture as computed in a $96^3$ “nonlinear sigma-model” simulation. Only $\phi_1$ and $\phi_3$ are shown which means that only the sign of $\phi_2$ is not determined by the direction and length of the vector shown. (a) shows the full cross section of the initial configuration given by eq. (4.2) with only every fourth grid point plotted. (b)–(d) show the central quarter of the $y = 0$ cross section at times $t = 25.6$, $28.8$, and $35.2\Delta x$ respectively. In (b)–(d), every point in the central region is shown.

4) The time evolution of various fibers intersecting the $y = 0$ plane is shown at time intervals of $3.2\Delta x$ from $t = 0$ to Hopf texture collapse at $t \simeq 27\Delta x$. The solid lines and filled circles represent the evolution of the $\phi_3 = -1$ fiber, while the other curves represent fibers in the $\vec{\phi} = (\cos \alpha, \sin \alpha, 0)$ family where $\alpha$ is an odd multiple of $\pi/8$. Each fiber is represented by a different symbol. (Since each fiber is a loop on a torus, they each intersect the $y = 0$ plane twice.)

5) This figure shows a time series of the $z = 0$ cross section of a collapsing asymmetric Hopf texture as computed in a $96^3$ “nonlinear sigma-model” simulation. $\phi_1$ and $\phi_2$ are shown, so the sign of $\phi_3$ is not determined by the direction and length of the vector shown. (a) shows the cross section of the initial configuration given by eq. (4.3) with only every fourth grid point plotted. (b)–(f) show the central quarter of the $y = 0$ cross section at times $t = 12.8$, $14.4$, $15.0$, $16.6$, and $19.8\Delta x$ respectively. Note that the figures (b)–(f) are magnified by a factor of $\sim 1.36$ with respect to figure (a).

6) The time evolution of various fibers intersecting the $y = 0$ plane is shown at time intervals of $1.6\Delta x$ from $t = 0$ to just after the collapse of the asymmetric Hopf texture at $t \simeq 14\Delta x$. The solid lines and filled circles represent the
evolution of the $\phi_3 = -1$ fiber, while the other curves represent fibers in the $\vec{\phi} = (\cos \alpha, \sin \alpha, 0)$ family where $\alpha$ is an odd multiple of $\pi/8$. 