Rigidly rotating cylinders of charged dust

B.V. Ivanov*
Institute for Nuclear Research and Nuclear Energy,
Tzarigradsko Shausse 72, Sofia 1784, Bulgaria

The gravitational field of a rigidly rotating cylinder of charged dust is found analytically. The general and all regular solutions are divided into three classes. The acceleration and the vorticity of the dust are given as well as the conditions for the appearance of closed timelike curves.

04.20.Jb

I. INTRODUCTION

Uncharged dust with spherical symmetry exists only as a flat spacetime. The addition of charge allows the existence of non-trivial solutions representing the electrification of Minkowski spacetime [1-3]. On the contrary, when the symmetry is cylindrical, static charged dust configuration are still trivial. One needs at least rigid rotation to invoke charged analogs of the Lanczos-Van Stockum metric [4,5]. A solution with vanishing Lorentz force was found in Ref. [6]. A thorough study of this problem was performed in the works of Islam [7-14]. The emphasis, however, was on the axially symmetric rigid rotation. The charge-mass ratio was taken to be constant. Solutions with non-constant ratio were also found [15], one of them being the general solution with vanishing Lorentz force. Later the general solution of this type, when the charged dust is axisymmetric and differentially rotating, was derived [16-18]. Several cylindrically symmetric stationary solutions were obtained as particular cases.

In this paper we study from the beginning cylindrically symmetric rigidly rotating configurations of charged dust. The general and all regular solutions are found, they fall into three distinct cases. Connections are made with past results.

In Sec II the Einstein-Maxwell equations are written for the Papapetrou form of the metric. A master differential equation is given for $g_{tt}$ and all other dust characteristics are written in terms of this metric component. Expressions are given for the acceleration, vorticity and angular velocity. Criteria for elementary flatness and the appearance of closed timelike curves (CTC) are discussed. In Sec III a symmetry transformation is used to break the main equation into three different cases and their general solutions are given. They resemble the vacuum solution with its Weyl and Lewis classes. In Sec IV all regular solutions are derived and studied in more detail. They are appropriate for interior solutions. Connections with previous research are outlined. Sec V contains a short discussion.

II. FIELD EQUATIONS AND DUST CHARACTERISTICS

The metric of a stationary cylindrically symmetric spacetime has the Papapetrou form [19,20]

$$ds^2 = -f (dt + A d\varphi)^2 + f^{-1} \left[ e^{2k} (dr^2 + dz^2) + W^2 d\varphi^2 \right],$$

where $x^0 = t, x^1 = \varphi, x^2 = r, x^3 = z$ and the metric functions depend only on the radial coordinate. Dust has zero pressure so that $W = r$ and the Einstein equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} ,$$

$$T_{\mu\nu} = \rho u_\mu u_\nu + E_{\mu\nu} .$$

We use units with $G = c = 1$. In comoving coordinates the four-velocity of dust $u^\mu$ has just a time component $u^0 = f^{-1/2}$. The energy density is $\rho = mn$ where $m$ is the mass of the particles and $n$ is their number density. Furthermore

*E-mail: boyko@inrne.bas.bg
\[ 8\pi E_{\mu\nu} = 2F^\alpha_\mu F^\nu_\alpha - \frac{1}{2}g_{\mu\nu}L, \quad L = F_\alpha^\beta F^\alpha_\beta, \] (4)

\[ F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \] (5)

where \( A_\mu \) is the electromagnetic vector potential and derivatives are denoted by comma. In our case

\[ A_\mu = (\Phi, \Psi, 0, 0) \] (6)

and there is radial electric field and longitudinal magnetic field. The covariant Maxwell equations are

\[ (F^{\mu\nu})_{;\nu} = 4\pi J^\mu, \quad J^\mu = qnu^\mu = \alpha \rho u^\mu, \] (7)

where \( q \) is the charge of the particles and \( \alpha = q/m \). There are two non-trivial equations

\[ \Psi' - A\Phi' = \frac{br}{f}, \] (8)

\[ 4\pi \alpha \rho = -\frac{f^{3/2}e^{-2k}}{r} \left( \frac{r\Phi'}{f} + bA' \right)', \] (9)

Here \( b \) is an integration constant, specific for the charged case, and \( ' \) means derivative with respect to \( r \). The electromagnetic invariant \( L \) reads

\[ L = \frac{1}{2}e^{-2k} (b^2 - \Phi'^2). \] (10)

We use the same combinations of Ricci tensor components as in the neutral case \[13\]. The Einstein-Maxwell equations become

\[ \frac{f^2 A'}{r} = a_0 + 4b\Phi, \] (11)

\[ f \left( f'' + \frac{f'}{r} \right) - f'^2 = \frac{f^4 A'^2}{r^2} + 8\pi \rho e^{2k} + 2f (b^2 + \Phi'^2), \] (12)

\[ \frac{2k'}{r} - \frac{f'^2}{2f^2} + \frac{f^2 A'^2}{2r^2} = \frac{2}{f} (b^2 - \Phi'^2), \] (13)

\[ 2k'' + \frac{f'^2}{2f^2} + \frac{f^2 A'^2}{2r^2} = \frac{2}{f} (b^2 + \Phi'^2), \] (14)

where \( a_0 \) is a constant. The Lorentz force is given by

\[ m \left( \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha \right) = qF^{\mu\nu}u_\nu \] (15)

and vanishes when \( \Phi' = 0 \). Its \( r \) component supplies a relation between \( f \) and \( \Phi \)

\[ f = (b_0 - \alpha \Phi)^2, \] (16)

where \( b_0 \) is an arbitrary constant. A system of 6 independent equations for the 6 variables \( f, A, k, \rho, \Phi, \Psi \) is given by Eqs (8,9,11-13,16). The charge \( q \) and the mass \( m \) are free parameters.

One can write the r.h.s. of Eq (11) as

\[ a_0 + 4b\Phi = b_1 - \frac{4b}{\alpha} \sqrt{f}, \quad b_1 = a_0 + \frac{4bb_0}{\alpha}, \] (17)
where $b_1$ is a redefinition of $b_0$. Inserting Eqs (9,11,16,17) into Eq (12) we obtain an ordinary differential equation for $f$
\[
(q^2 - m^2) \left( f'' + \frac{f'}{r} - \frac{f'^2}{f} \right) + 2 \left( 4m^2 - q^2 \right) b^2 - \frac{6mqbb_1}{\sqrt{f}} + \frac{q^2b_1^2}{f} = 0. \tag{18}
\]

The other characteristics of the charged dust are expressed through $f$. Thus, $\Phi$ is given by Eq (17), $A$ is obtained by integrating Eq (11)
\[
A = b_1 \int \frac{rdr}{f^2} - \frac{4b}{\alpha} \int \frac{rdr}{f^{3/2}}. \tag{19}
\]
Then $\Psi$ and $\rho$ are obtained from Eqs (8,9) which become
\[
\Psi' = \frac{br}{f} - \frac{Af'}{2\alpha f^{1/2}}, \tag{20}
\]
\[
4\pi\alpha\rho e^{2k} = \frac{f''}{2\alpha} + \frac{f'}{2\alpha r} - \frac{3f^2}{4\alpha f} - \frac{bb_1}{f^{3/2}} + \frac{4b^2}{\alpha}. \tag{21}
\]
Finally, $k$ follows from Eq (13)
\[
\frac{2k'}{r} = \frac{\alpha^2 - 1}{2\alpha^2} \frac{f'^2}{f^2} - \frac{b_1^2}{2f^2} + \frac{4bb_1}{\alpha f^{3/2}} + \frac{2(\alpha^2 - 4)b^2}{\alpha^2 f}. \tag{22}
\]

The characteristics of the four-velocity in the metric (1) are of special interest \[21\]. The expansion and shear vanish. The acceleration has only a radial component $v^r$ and its magnitude is
\[
v = \left( \sqrt{f} \right)' e^{-k} = -\alpha \Phi e^{-k}. \tag{23}
\]
The vorticity vector has only a longitudinal component $w^z$ and magnitude
\[
w = \frac{A'}{2r} f^{3/2} e^{-k} = \left( \frac{b_1}{2f^{1/2}} - \frac{2b}{\alpha} \right) e^{-k}. \tag{24}
\]

The condition for elementary flatness is \[20,21\]
\[
\lim_{r\to 0} e^{-k} \left( r^2 - f^2 A^2 \right)^{1/2} = 1. \tag{25}
\]
We shall show that the second term in the bracket decouples for regular solutions so that $k(0) = 1$ follows. Then the angular velocity $\Omega = w(0)$ of the rigid body rotation becomes
\[
\Omega = \frac{b_1}{2f(0)^{1/2}} - \frac{2b}{\alpha}. \tag{26}
\]
The appearance of CTC is governed by $g_{11}$
\[
g_{11} = f^{-1} (r + fA) (r - fA). \tag{27}
\]
The sign of $A$ determines the direction of rotation. For definiteness we accept that $A > 0$. The condition for CTC is $g_{11} < 0$ which means
\[
A > \frac{r}{f}. \tag{28}
\]

The key equation (18) was obtained by Islam \[14\], Eq (6.102) in the Lewis formulation of the problem. We have rederived it in the Papapetrou picture because it is simpler and draws parallels with uncharged rotating perfect fluids with $\gamma$-law equation of state \[21\]. Eq (18) is quite complicated and seems non-integrable. Different particular cases were studied by Islam. In the next section we simplify considerably this equation and find its general solution.
III. GENERAL SOLUTION

The derivation of Eq (18) makes it possible to trace the appearance of the constants $b, b_1$. The potential $\Phi$ of the electric field is defined up to a constant, $\Phi \to \Phi + \Phi_0$. Under this transformation Eqs (16,17) give

$$b_0 \to b_0 + \alpha \Phi_0, \quad b_1 \to b_1 + 4b \Phi_0.$$  \hfill (29)

In Ref. [14] $\Phi_0$ was used to nullify $b_0$. However, we can nullify $b_1$ instead, as long as $b \neq 0$. Hence, either $b = 0$ or $b_1 = 0$. Eq (18) simplifies drastically. This mechanism does not work when $\Phi$ is constant and only magnetic field is present. Then the Lorentz force vanishes and $f$ is also a constant. In conclusion, there are three cases

1) $\Phi = 0$, $f = f_0$. Eq (18) turns into a quadratic equation for $f_0^{1/2}$

$$2 \left(4m^2 - q^2\right) b^2 f_0 - 6mqbb_1 \sqrt{f_0} + q^2 b_1^2 = 0.$$  \hfill (30)

The case $q = \pm m$ eliminates the derivatives of $f$ in Eq (18) and again leads to Eq (30).

2) $b = 0$. Eq (18) becomes

$$\left(\frac{rf'}{f}\right)' = \lambda_2 \frac{r}{f^2}, \quad \lambda_2 = \frac{q^2 b_1^2}{2(m^2 - q^2)}.$$  \hfill (31)

3) $b_1 = 0$. Eq (18) now reads

$$\left(\frac{rf'}{f}\right)' = 2\lambda_3 \frac{r}{f}, \quad \lambda_3 = \frac{(4m^2 - q^2) b^2}{m^2 - q^2}.$$  \hfill (32)

The last two equations are written together in a simple manner if we introduce $f = e^{2u}$

$$(ru)' = \lambda_i r e^{-n_i u}.$$  \hfill (33)

Here $i = 2, 3$ and $n_2 = 4$, $n_3 = 2$. One can easily see that in the neutral vacuum case ($\rho = \Phi = a = b = 0$) Eqs (11,12) give exactly Eq (31) or (33) with $i = 2$ and negative $\lambda_2$. This is in fact Eq (A4) from Ref. [22] which leads to the three classes of Lewis solutions [23–25]. Thus the charged dust solutions are similar to the uncharged vacuum solutions, but as we shall see, for realistic situations $q^2 < m^2$ and $\lambda_2$ is positive.

Eq (33) is solved by introducing $s = \ln r$ and $y = n_i u - 2 \ln r$. It becomes

$$y_{ss} = n_i \lambda_i e^{-y}$$  \hfill (34)

and is integrated after multiplication with $y_{ss}$. The general solution is

$$f_i = \left(\alpha r \sqrt[4]{e^1} + n_i \lambda_i r^{-\sqrt[4]{e^1}}\right)^{4/n_i},$$  \hfill (35)

where $a$ and $c$ are integration constants. The function $f$ depends on four parameters: $a, c, \alpha$ and either $b$ or $b_1$. The constant $c$ is real and when $c > 0$ Eq (35) is analogous to the vacuum Weyl class. When $c$ is negative, $c = -\beta^2$, $f$ can be kept real if $a$ becomes complex and satisfies

$$|a|^2 = -\frac{n_i \lambda_i}{8 \beta^2}.$$  \hfill (36)

A necessary condition is $\lambda_i < 0$. This requires $\alpha^2 > 1$ for $i = 2$ and $1 < \alpha^2 < 4$ for $i = 3$. Then we can write $a = |a| e^{i\sigma}$ and

$$f_i = \left[2 |a| r \cos (\sigma + \beta \ln r)\right]^{4/n_i}.$$  \hfill (37)

This is an analog of the vacuum Lewis class. The free parameters are $\sigma, \beta, \alpha$ and $b_1 (b)$.

The other characteristics follow from $f$ and are real too. The integrations in Eq (22) are not explicit in general. Curiously, in the case $b = 0$ the function $A$ may be written as in the vacuum

$$A = \frac{b_1 r \sqrt[4]{e^1}}{2a \sqrt[4]{e^1}} + A_0,$$  \hfill (38)

where $A_0$ becomes a complex constant in the Lewis class.

The vacuum solutions are exterior ones and the singularities on the axis do not matter. The rotating charged dust is supposed to serve as an interior solution, so the study of regular metrics is really important.
IV. REGULAR SOLUTIONS

Case 1. Solutions with vanishing Lorentz force. They have $\Phi = 0$ and constant $f$ given by Eq (30)

$$\sqrt{f_0} = \frac{b_1 \alpha (3 \pm \sqrt{1+2\alpha^2})}{2b (4-\alpha^2)},$$

(39)

unless $q^2 = 4m^2$. The special case $q^2 = m^2$ gives the simple result

$$f_0 = \frac{2 \pm \sqrt{3}}{6} \left(\frac{b_1}{b}\right)^2.$$  (40)

In the case $q^2 = 4m^2$ Eq (30) is linear and

$$f_0 = \frac{1}{9} \left(\frac{b_1}{b}\right)^2.$$  (41)

if $b_1 \neq 0$. When $b_1$ vanishes, $f_0$ remains unfixed. The other functions read

$$A = \frac{a_0 r^2}{2f_0^2}, \quad \Psi = \frac{b r^2}{2f_0}, \quad a_0 = b_1 - \frac{4b}{\alpha} \sqrt{f_0},$$

(42)

$$k = \frac{k_0 r^2}{4}, \quad k_0 = \frac{2b^2}{f_0} - \frac{a_0^2}{2f_0^2},$$

(43)

$$\rho = -\frac{ba_0}{4\pi \alpha \sqrt{f_0} e^{-2k}}, \quad L = \frac{b^2}{2} e^{-2k}.$$  (44)

Obviously $A$ decouples from the elementary flatness condition Eq (25). The solution has three parameters; $\alpha, b, b_1$. The constant $b$ sets the strength of the magnetic field and $ba_0/\alpha$ must be negative to ensure the positivity of the energy density. The latter decreases monotonically when $k_0$ is positive, but never vanishes.

One can recover the Lanczos solution by setting $b = -a_0 q/2 \to 0$ and $f_0 = 1$, since Eq (30) becomes trivial. Then $k_0$ is always negative and $\rho$ increases with the distance. The presence of magnetic field invokes $b$ in $k_0$ and makes it positive when the rotationally induced $a_0$ is compensated.

Eq (23) shows that there is no acceleration of the dust particles. The angular velocity and the vorticity are

$$\Omega = \frac{a_0}{2\sqrt{f_0}}, \quad w = \Omega e^{-k}.$$  (45)

CTC appear when $r$ is bigger than

$$r_0 = \frac{\sqrt{f_0}}{\Omega}.$$  (46)

The solution is regular and realistic. It was found in Ref. [3], see also Eq (6.96) from Ref. [14], where $f_0 = 1$ was chosen.

Turning to the other two cases, Eq (37) shows that $f$ from the Lewis class is always singular at the axis, while Eq (35) indicates that the Weyl class is regular only for $c = 1$. For simplicity, let us take $a = n_i \lambda_i/8$, so that $f(0) = 1$. Eq (35) becomes

$$f_i = \left(1 + \frac{n_i \lambda_i}{8} r^2\right)^{4/n_i},$$

(47)

which is the general regular solution for non-constant $f$.

Case 2: $b = 0$. This case is close to the vacuum solution and we have

$$f_2 = 1 + \frac{\lambda_2}{2} r^2,$$  (48)
The function $A$ again decouples from Eq (25). Eq (50) shows that $\rho(0) > 0$ when $\lambda_2 > 0 (q^2 < m^2)$. Then, however, $e^{-2k}$ monotonically increases and so does $L$. The acceleration also increases, while the vorticity increases when $q^2 < m^2/2$ and decreases when $m^2/2 < q^2 < m^2$. The energy density has a zero at $r_0 = 2/\sqrt{\lambda_2}$ and turns negative, so the junction to an exterior should be made before this point. According to Eq (28) CTC exist when $\Omega r > 1$. They will appear in the interior if $q^2 < \frac{3}{2} m^2$. Some of the above relations were found in the Lewis form of the metric by Islam [14].

Case 3; $b_1 = 0$. Then $\lambda_3$ is positive except for $m^2 < q^2 < 4 m^2$ and we have

$$f_3 = \left(1 + \frac{\lambda_3}{4} \right)^2,$$

$$A = \frac{4b(1 - f_3)}{\alpha \lambda_3 f_3}, \quad \Psi' = \left(b - \frac{\lambda_3}{2\alpha} \right) \frac{r}{f_3} + \frac{\lambda_3 r}{2\alpha},$$

$$4\pi \rho = \left[ \frac{b^2}{\alpha^2} \left( \frac{8m^2 - 5q^2}{m^2 - q^2} - \frac{\lambda_3}{4\alpha^2} r^2 \right) \right] e^{-2k}, \quad e^{-2k} = f_3^{\lambda_3/\alpha^2},$$

$$\Phi' = -\frac{\lambda_3}{2\alpha} r, \quad L = \frac{1}{2} \left( b^2 - \frac{\lambda_3^2}{4\alpha^2} r^2 \right) e^{-2k},$$

$$v = -\frac{\alpha \lambda_3}{4b} r w, \quad w = \Omega f_3^{\frac{\lambda_3}{2\alpha}}, \quad \Omega = -\frac{2b}{\alpha}.\quad (57)$$

We accept $\Omega > 0$ and $q > 0$, so that $b < 0$. We have $A > 0$ and again $A \sim r^2$ for small $r$, decoupling in Eq (25). Now there are two intervals where $\rho(0)$ is positive, $q^2 < m^2$ and $q^2 > \frac{3}{2} m^2$. The function $e^{-2k}$ increases with the distance when $q^2 < 4m^2$ and decreases when $q^2 > 4m^2$. In the second case $\rho$ also decreases monotonically. This shows that realistic solutions are possible even when $q^2 > m^2$ contrary to the assertions in Ref. [14]. In any case the density has a zero and changes sign at

$$r_0^2 = \frac{4(\alpha^2 - 1)(5\alpha^2 - 8)}{\alpha^2 - 4 b^2}.$$

Finally, CTC exist when the inequality

$$\Omega r \left(1 + \frac{\lambda_3}{8} r^2 \right) > 1$$

is satisfied. It is slightly more complicated than in the previous two cases. CTC appear in the interior solution (i.e. before $r_0$) if
Here $\alpha^2 = 4 + \gamma$. This is true for small $\gamma$ but for big positive $\gamma$ the r.h.s. dominates. The situation is more intricate for negative $\gamma$ where the inequality does not hold around the zeroes of the polynomial in the l.h.s.

**Case 3a:** $b_1 = 0$, $q^2 = 4m^2$. Then $\lambda_3 = 0$ and the previous solution belongs also to Case 1 with $f_0 = 1$, $k_0 = 0$. The energy density and $L$ are constant. However, the general formula in Eq (35) gives also another solution, singular at the origin

$$f = c_1 r^{c_2},$$

where $c_i$ are some constants. It leads to

$$A = c_3 r^{2-3c_2/2}, \quad k = \frac{3}{16} c_2^2 \ln r,$$

$$\Phi' = -\frac{\sqrt{c_1 c_2}}{4} r^{c_2/2-1}, \quad \Psi' = \frac{b}{c_1} \left( 1 + \frac{c_2}{2} \right) r^{1-c_2}.$$ (62)

This solution was studied in Refs. [8,11,12] where $\rho$ can be found too. The importance of a general solution, encompassing both cases was addressed there. We would like to mention also that $\Psi \sim r^{2-c_2}$ except for $c_2 = 2$ when $\Psi \sim \ln r$. This particular case was studied in Ref. [12] where erroneously $\Psi$ was taken as a constant.

**V. DISCUSSION**

The master equation (18) of the problem studied in this paper has been known for a long time. It looks formidable, non-integrable and even more complicated than the Painlevé equations and transcendent, that appear in the electrovac case [26]. However, one of the two main constants in it can be always set to zero by changing the electric potential. This allows to obtain the general solution, which breaks into three distinct cases. One is the peculiar case with vanishing Lorentz force. The neutral dust solution of Lanczos can be obtained from it by taking a specific limit. The other two cases resemble the vacuum solution and possess both Weyl and Lewis classes of solutions. CTC appear already in the Weyl class.

The general solutions are singular at the origin. We have derived in addition all regular metrics which serve as interior charged dust solutions. They have been found in the past by Islam but as particular cases of a supposedly non-integrable equation. Here they have been systematized in an exhausting classification scheme, some corrections were made and their properties were further studied. We have stressed the advantages of the Papapetrou form of the metric for this and related problems [11,21,22]. As usual, the Weyl-Papapetrou-Majumdar type of connection given by Eq (16) makes the problem analytically solvable in elementary functions, unlike the electrovacuum case. Matching these interior charged dust solutions to third Painlevé transcendent seems too complicated, although some concrete cases are known [14].

[1] Ivanov B V 2002 J. Math. Phys. 43 1029
[2] Ivanov B V 2002 Phys. Rev. D 65 104001
[3] Bonnor W B 1960 Z. Phys. 160 59
[4] Lanczos K 1924 Z. Phys. 21, 73, 1997 Gen. Rel. Grav. 29 363
[5] van Stockum W J 1937 Proc. R. Soc. Edinburgh A 57 135
[6] Som M M and Raychaudhuri A K 1968 Proc. R. Soc. A 304 81
[7] Islam J N 1977 Proc. R. Soc. A 353 523
[8] Islam J N 1978 Proc. R. Soc. A 362 329
[9] Islam J N 1979 Proc. R. Soc. A 367 271
[10] Islam J N 1980 Proc. R. Soc. A 372 111
[11] Islam J N 1983 Proc. R. Soc. A 385 189
[12] Boachie L A and Islam J N 1983 Phys. Lett. A 93 321
[13] Islam J N 1983 Proc. R. Soc. A 389 291
[14] Islam J N 1985 *Rotating Fields in General Relativity* (Cambridge: Cambridge University Press)
[15] Bonnor W B 1980 J. Phys. A 13 3465
[16] Van den Bergh N and Wils P 1984 Class. Quantum Grav. 1 199
[17] Wils P and Van den Bergh N 1984 Class. Quantum Grav. 1 399
[18] Islam J N, Van den Bergh N and Wils P 1984 Class. Quantum Grav. 1 705
[19] Sklavenites D 1999 Class. Quantum Grav. 16 2753
[20] Kramer D, Stephani H, Herlt E and MacCallum M A H 1980 *Exact Solutions of Einstein’s Field Equations* (Cambridge: Cambridge University Press)
[21] Ivanov B V 2002 gr-qc/0205023 to appear in Class. Quantum Grav.
[22] Tipler F J 1974 Phys. Rev. D 9 2203
[23] Lewis T 1932 Proc. R. Soc. A 136 176
[24] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 Gen. Rel. Grav. 27 859
[25] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 Class. Quantum Grav. 12 111
[26] MacCallum M A H 1983 J. Phys. A 16 3853