Integrable evolution Hamiltonian equations of the third order with the Hamiltonian operator $D_x$

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ABSTRACT. All non-equivalent integrable evolution equations of third order of the form $u_t = D_x \frac{\delta H}{\delta u}$ are found.

1. Introduction.

We consider the third order integrable Hamiltonian evolution equations of the form

$$u_t = D_x \left( \frac{\delta H}{\delta u} \right) = D_x \left( \frac{\partial H}{\partial u} - D_x \frac{\partial H}{\partial u_x} \right).$$

(1.1)

Here $H(x, u, u_x)$ is the Hamiltonian and $D_x$ is the total $x$-derivative. The celebrated KdV equation with $H = -\frac{1}{4} u_x^2 + \frac{1}{8} u^3$ provides the simplest example of such an equation. The function $H$ is defined up to equivalence $H \to H + D_x f(x, u) + \lambda u$, where the function $f$ and the constant $\lambda$ are arbitrary.

Using the symmetry approach to integrability [1, 2], we obtain a complete list of canonical forms for integrable Hamiltonians $H$. Our proof of the classification statement contains an algorithm which allows to bring any integrable Hamiltonian to one of the canonical forms by canonical transformations.

1.1. Canonical transformations. Consider point transformations of the form

$$x = \varphi(y, v), \quad u = \psi(y, v).$$

(1.2)

The invertibility of the transformation is equivalent to the inequality $\Delta = \psi_v \varphi_y - \varphi_v \psi_y \neq 0$. Transformation (1.2) is called canonical if $\Delta = \psi_v \varphi_y - \varphi_v \psi_y = 1$. It is easy to verify that canonical transformations preserve the form of equation (1.1). The Hamiltonian of the resulting equation is given by

$$\tilde{H}(y, v, v_y) = H \left( \varphi(y, v), \psi(y, v), \frac{D_y(\psi)}{D_y(\varphi)} \right) D_y(\varphi).$$

(1.3)

Example. Linear transformations of the form

$$x = f(y), \quad u = \frac{v}{f} + g(y), \quad \tilde{H} = H f'.$$

are canonical for arbitrary functions $f$ and $g$.

Remark 1. If we consider only Hamiltonians that do not depend on $x$ explicitly, we still have non-trivial canonical transformations

$$x = f(v) + y, \quad u = v, \quad \tilde{H} = (f' v_y + 1) H.$$

$$x = v, \quad u = f(v) + y, \quad \tilde{H} = H v_y. \quad \square$$
Besides we use the following canonical transformations of a more general form:

1. Dilatations of the form
\[ t = \alpha \tilde{t}, \quad x = \beta y, \quad u = \gamma v, \quad \tilde{H} = \frac{\alpha}{\beta \gamma^2} H(\beta y, \gamma v) \]
are admissible for any \( H \).

2. If \( H \) does not depend on \( x \), then the Galilean transformation
\[ y = x + ct, \quad v = u, \quad \tilde{H} = H - \frac{1}{2} c v^2; \]
is admissible.

3. If \( H = c x u + h(u_x) \), where \( c \) is a constant, then the following transformation
\[ u \rightarrow u + ct, \quad H \rightarrow H - c xu \]
is admissible.

### 1.2. Integrability conditions.

Necessary integrability conditions for equations of the form
\[ u_t = F(x, u, u_x, u_{xx}, u_{xxx}) \]
are given by a series of conservation laws [1, 2]:
\[
\frac{d}{dt} \rho_n = \frac{d}{dx} \theta_n, \quad n = -1, 0, 1, \ldots
\]
where \( \rho_n \) are said to be the canonical densities. They can be defined by the following recursive formula presented here at the first time:
\[
\rho_{n+2} = \frac{1}{3} \rho_{-1} \left( \theta_n - F_0 \delta_{n,0} - F_1 \rho_n - F_2 D_x(\rho_n) - F_2 \sum_{-1}^{n} \rho_i \rho_j \right) - \frac{1}{3} (\rho_{-1})^{-2} D_x^2(\rho_n)
\]
\[ - (\rho_{-1})^{-2} \left( \frac{1}{2} D_x \sum_{-1}^{n} \rho_i \rho_j + \frac{1}{3} \sum_{0}^{n} \rho_i \rho_j \rho_k + \rho_{-1} \sum_{0}^{n+1} \rho_i \rho_j \right), \quad n = -2, -1, 0, \ldots,
\]
where \( \rho_n = \theta_n = 0 \) for \( n < -1 \), \( F_n = \partial F / \partial u_n \), \( \rho_{-1} = F_3^{-1/3} \), and \( \delta_{ik} \) is the Kronecker symbol.

By definition,
\[
\sum_{a}^{b} \rho_{I_1} \cdots \rho_{I_k} = \sum_{I_1 \geq a, 1 \leq s \leq k, \sum_{1}^{k} I_s = b} \rho_{I_1} \cdots \rho_{I_k}.
\]
In particular,
\[
\sum_{-1}^{-2} \rho_i \rho_j = \rho_{-1}^2, \quad \sum_{-1}^{1} \rho_i \rho_j = 2 \rho_{-1} \rho_0, \quad \sum_{0}^{1} \rho_i \rho_j \rho_k = 0,
\]
\[ \sum_{0}^{1} \rho_i \rho_j \rho_k = \rho_0^3, \quad \sum_{0}^{1} \rho_i \rho_j \rho_k = 3 \rho_0^2 \rho_1, \ldots \]
The first three canonical densities are given by

\[ \rho_{-1} = F_3^{-1/3}, \quad \rho_0 = -D_x(\ln \rho_{-1}) - \frac{1}{3} F_2 \rho_{-1}, \]

\[ \rho_1 = \frac{1}{3} \theta_{-1} \rho_{-1} - \frac{1}{3} F_1 \rho_{-1}^2 + F_2 \rho_{-1} D_x(\rho_{-1}) + \frac{1}{9} F_2^2 \rho_{-1}^5 + \frac{1}{3} \rho_{-1}^2 F_2 \]

\[ + \frac{2}{3} D_x(\rho_{-1}^2 D_x(\rho_{-1})) + \frac{1}{3} \rho_{-1}^{-3} (D_x(\rho_{-1}))^2. \]

Notice that the flux \( \theta_{-1} \) of the first canonical conservation law \( (1.5) \) is involved in the formula for \( \rho_1 \).

The integrability conditions lead to some partial differential equations for the right hand side \( F \) of \( (1.4) \). We don’t explain here how to derive these PDEs (see for example \[3\], where this technique is described in details).

For equations \( (1.1) \) any canonical density \( \rho_n \) can be expressed in terms of the Hamiltonian \( H \) and \( \theta_{-1}, \theta_0, \ldots, \theta_{n-3}, \theta_{n-2} \). In particular,

\[ \rho_{-1} = -\left( \frac{\partial^2 H}{\partial u_1^2} \right)^{-1/3}. \]

Let us denote \( \rho_{-1} = a \). Then

\[ \frac{\partial^2 H}{\partial u_1^2} = -a^{-3}, \quad a = a(x, u, u_x). \]

The integrability conditions provide PDEs for \( H \), which allow us to obtain a complete list of integrable Hamiltonians. It is known \[1\] that for Hamiltonian equations all even integrability conditions are trivial. Almost all the information of integrable Hamiltonians will be derived for the fist and third integrability conditions.

2. Classification statement.

Theorem 1. Any non-linear equation of the form \( (1.1) \) that has infinite hierarchy of higher symmetries

\[ u_{x_k} = F_k(x, u, u_x, \ldots), \quad k = 1, 2, \ldots \]

is canonically equivalent to one of the following equation:

\[ u_t = D_x \left( \frac{u_{xx}}{a^3} - \frac{3 a'}{2 a^4} u_x^2 + \frac{\partial}{\partial u} \frac{P(u)}{a} \right), \quad H = -\frac{u_x^2}{2 a^3} + \frac{P(u)}{a}, \]  \hspace{1cm} (2.1)

where \( a = c_1 u^2 + c_2 u + c_3 \),

\[ u_t = D_x \left( \frac{u_{xx}}{u^3} - \frac{3 a u_x^2}{2 u^4} + P(x) u^2 \right), \quad H = -\frac{u_x^2}{2 u^3} + \frac{1}{3} P(x) u^3, \]  \hspace{1cm} (2.2)

\[ u_t = D_x \left( \frac{u_{xx}}{\sqrt{u_x + P(u)}}^3 + 3 \frac{P'(u)}{\sqrt{u_x + P(u)}} - \frac{P(u) P'(u)}{(u_x + P(u))^{3/2}} \right), \quad H = 4 \sqrt{u_x + P(u)}. \]  \hspace{1cm} (2.3)

Here \( P \) is an arbitrary polynomial of degree not greater then 4, \( c_i \) are arbitrary constants.
Remark 2. Using translations \( u \to u + c \), dilatations \( u \to \lambda u \), \( t \to \alpha t \), \( x \to \beta x \), and the Galilean transformation, one can reduce (2.1) to one of the following canonical forms:

\[
\begin{align*}
    u_t &= D_x (u_{xx} + u^3), \quad H = -\frac{1}{2} u_x^2 + \frac{1}{4} u^4, \\
    u_t &= D_x (u_{xx} + u^2), \quad H = -\frac{1}{2} u_x^2 + \frac{1}{3} u^3, \\
    u_t &= D_x \left( \frac{u_{xx}}{a^3} - \frac{3 u^2 x}{2 a^4} + c_1 u^2 + \frac{c_2}{u^2} \right), \quad H = -\frac{u_x^2}{2 a^3} + \frac{1}{3} c_1 u^3 - c_2 u^{-1}, \\
    u_t &= D_x \left( \frac{u_{xx}}{a^3} - 3 \frac{u_x u^2}{a^4} + c_1 \frac{c - u^2}{a^2} - 2 c_2 \frac{u}{a^2} \right), \quad H = -\frac{u_x^2}{2 a^3} + \frac{c_1 u + c_2}{a},
\end{align*}
\]

where \( a = u^2 + c \). □

Proof of Theorem 1. It follows from the first integrability condition \((n = -1 \text{ in (1.5)})\) that

\[
\frac{d}{dx} \left( a^3 \frac{\partial^2 a}{\partial u_x^2} \right) = 0. \tag{2.4}
\]

The solution of (2.4) is given by

\[
a = \sqrt{a_1 u_x^2 + a_2 u_x + a_3}, \tag{2.5}
\]

where \( a_i = a_i(x, u) \) and

\[
a_2^2 - 4 a_1 a_3 = \text{const.} \tag{2.6}
\]

Under canonical transformations the function \( a \) transforms as follows

\[
\tilde{a} = \sqrt{\tilde{a}_1 v_y^2 + \tilde{a}_2 v_y + \tilde{a}_3}, \quad \text{where} \quad \tilde{a}_1 = a_1 \varphi_x^2 - a_2 \varphi_x \varphi_u + a_3 \varphi_u^2.
\]

Hence we can reduce \( a_1 \) to zero by an appropriate canonical transformation. Taking into account (2.6), we see that it suffices to consider the following two cases: (A) \( a = a(x, u) \) and (B) \( a = \sqrt{u_x + q(x, u)} \) (if \( \tilde{a}_1 = 0 \) then \( \tilde{a}_2 \) is a constant, which can be brought to 1 by a dilatation).

2.1. Case A. In this case the Hamiltonian is given by

\[
H = h(x, u) - \frac{u_x^2}{2 a^3}, \quad a = a(x, u). \tag{2.7}
\]

The first integrability condition implies the following differential equations

\[
\frac{\partial^4 a}{\partial u^3} = 0, \quad \frac{\partial}{\partial x} \left( \frac{\partial^2 a^2}{\partial u_x^2} - 3 \left( \frac{\partial a}{\partial u} \right)^2 \right) = 0,
\]

therefore

\[
a = s_1 u_x^2 + s_2 u + s_3, \quad s_i = s_i(x), \tag{2.8}
\]

and

\[
\frac{d}{dx} (s_2^2 - 4 s_1 s_3) = 0. \tag{2.9}
\]
It is easy to verify that we can reduce the functions \(s_1(x), s_2(x), s_3(x)\) to constants by a canonical tranformation of the form
\[
x = f(y), \quad u = \frac{v}{f'} + g(y).
\]
So we obtain
\[
H = h(x, u) - \frac{u_x^2}{2 a^3}, \quad a = c_1 u^2 + c_2 u + c_3. \tag{2.10}
\]

The third integrability condition implies
\[
a \frac{\partial^5 h}{\partial u^5} + 5 a' \frac{\partial^4 h}{\partial u^4} + 10 a'' \frac{\partial^3 h}{\partial u^3} = 0.
\]
Substituting \(g/a\) for \(h\), we get \(g^{(5)} = 0\) and therefore
\[
H = \frac{r_1 u^4 + r_2 u^3 + r_3 u^2 + r_4 u + r_5}{c_1 u^2 + c_2 u + c_3} - \frac{u_x^2}{2 (c_1 u^2 + c_2 u + c_3)^2}, \quad r_i = r_i(x). \tag{2.11}
\]

To determine the \(x\)-dependence of the functions \(r_i(x)\) we consider several subcases.

**Subcase A.1.** Let \(c_1 = c_2 = 0\). Then the Hamiltonian is equivalent to
\[
H = -\frac{1}{2} u_x^2 + \frac{1}{4} q_1 u^4 + \frac{1}{3} q_2 u^3 + \frac{1}{2} q_3 u^2 + q_4 u,
\]
where \(q_i = q_i(x)\). The third integrability condition is equivalent to relations
\[
q_1 = k_1, \quad 2q_2 q'_2 = 3 k_1 q'_3, \quad 2q''_2 + 2 q_2 q_3 = 4 k_1 q'_4, \tag{2.12}
\]
where \(k_1\) is a constant.

If \(k_1 \neq 0\), then we normalize it to 1 by \(u \to u k_1^{-1/2}\) and reduce \(q_2\) to zero by \(u \to u - q_2(x)/3\). It follows from \(2.12\) that now we have \(q'_2 = q_4 = 0\). Thus the Hamiltonian is equivalent to \(2.14\) and we arrive at the modified KdV equation.

If \(k_1 = 0\), then \(2.12\) implies \(q_2 = k_2\), where \(k_2\) is a constant. Suppose that \(k_2 \neq 0\). Then we normalize \(k_2\) to 1 and reduce \(q_3\) to zero by \(u \to u - q_3(x)/2\). It follows from the fifth integrability condition that \(q'_4 = 0\) and we obtain the KdV equation \(2.14\).

At last, if \(k_1 = k_2 = 0\), then corresponding equation becomes linear:
\[
 u_t = D_x(u_{xx} + f(x) u + g(x)). \tag{2.13}
\]
where \(f\) and \(g\) are arbitrary functions.

**Subcase A.2.** Suppose \(c_1 = 0, c_2 \neq 0\). Then the Hamiltonian is equivalent to
\[
H = -\frac{u_x^2}{2 u^3} + \frac{1}{3} q_1 u^3 + \frac{1}{2} q_2 u^2 - q_3 u^{-1} + q_4 u,
\]
where \(q_1, q_2, q_3\) and \(q_4\) are functions of \(x\). It follows from the third integrability condition that
\[
 q'_1 q_2 - 2 q_1 q'_2 = 0, \quad q''_2 - q_2 q'_3 - 2 q_2 q_3 = 0, \quad q'_4 = 0. \tag{2.14}
\]
Since \(q_4\) is a constant, without loss of generality we put \(q_4 = 0\).

Under canonical transformation
\[
y = f(x), \quad v = u/f',
\]
the Hamiltonian transforms as follows:
\[
\tilde{H} = -\frac{v_y^2}{2v^3} + \frac{1}{3} q_1 f'^2 v^3 + \frac{1}{2} q_2 f' v^2 - (v f')^{-1} \left( f' f''' - \frac{3}{2} f''^2 + q_3 f'^2 \right).
\]
Consider the following cases: (a) \( q_2 \neq 0 \) and (b) \( q_2 = 0 \).

In the case (a) taking \( \int 1/q_2 \, dx \) for \( f \), we bring \( q_2 \) to 1. Then it follows from (2.14) that \( q'_1 = q'_3 = 0 \) and we get an equation equivalent to (2.11).

In the case (b) taking for \( f \) any nonconstant solution of equation \( f' f''' - \frac{3}{2} f''^2 + q_3 f'^2 \), we reduce \( q_3 \) to zero. Then the fifth integrability condition implies \( q^{(5)}_1 = 0 \) and we arrive at equation (2.2).

Remark 3. If we consider the normalization \( q_3 = 0 \) instead of \( q_2 = 1 \) in the case (a), then we get
\[
u_t = D_x \left( \frac{u_{xx}}{u^2} - \frac{3}{2} \frac{u_x^2}{u^4} + c \frac{q_2^2}{u^2} + q_2(x) u \right), \quad H = -\frac{u_x^2}{2} + \frac{1}{3} c q_2^2(x) u^3 + \frac{1}{2} q_2(x) u^2, \quad q''_2 = 0.
\]
We have chosen the canonical form (2.11) since the corresponding Hamiltonian does not depend on \( x \) explicitly. □

Remark 4. If we consider the normalization \( q_1 = 1 \) instead of \( q_3 = 0 \) in the case (b), then we obtain
\[
u_t = D_x \left( \frac{u_{xx}}{u^2} - \frac{3}{2} \frac{u_x^2}{u^4} + u^2 + \frac{3}{2} \frac{\varphi(x)}{u^2} \right),
\]
where \( \varphi'^2 = 4 \varphi^3 - 2 \varphi - q_3, \varphi' \neq 0 \). It is another canonical form for equation (2.2). □

Subcase A.3. Suppose \( c_1 \neq 0 \) in (2.11). Using the dilatation \( u \to u c_1^{-1/2} \), we normalize \( c_1 \) to 1. Then the translation \( u \to u - c_2/2 \) reduces \( a \) to the form \( a = u^2 + c \). In this case the first integrability condition yields \( \partial H/\partial x = 0 \). Therefore all the functions \( r_i \) in (2.11) are constants and the equation is equivalent to (2.11).

2.2. Case B. Consider Hamiltonians of the form
\[
H = h(x, u) + 4 a, \quad a = \sqrt{u_x + q(x, u)}.
\]
It follows from the first integrability condition that
\[
\frac{\partial^3 h}{\partial u^3} = 0, \quad \frac{\partial^3 h}{\partial x^2 \partial u} - 2 q \frac{\partial^2 h}{\partial x \partial u^2} + \frac{\partial q}{\partial u} \frac{\partial^2 h}{\partial x \partial u} - \frac{\partial q}{\partial x} \frac{\partial^2 h}{\partial u^2} = 0.
\]
The third integrability condition implies one more simple PDE: \( \partial^5 q/\partial u^5 = 0 \). Solving this equation and (2.15) we find that
\[
q = q_1 u^4 + q_2 u^3 + q_3 u^2 + q_4 u + q_5, \quad h = \frac{1}{2} h_1 u^2 + h_2 u + h_3,
\]
where \( q_i = q_i(x), h_i = h_i(x) \). Substituting \( q \) and \( h \) in (2.16) we obtain the following system:
\[
2 q_1 h'_1 - q'_1 h_1 = 0, \quad q_2 h'_1 - q'_2 h_1 + 4 q_1 h_2 = 0, \quad q'_1 h_1 - 3 q_2 h'_2 = 0, \\
h''_1 + 2 q_3 h'_2 - q_4 h'_1 - q'_4 h_1 = 0, \quad h''_2 - 2 q_5 h'_1 + q_3 h'_2 - q'_5 h_1 = 0.
\]
The canonical transformation

\[ y = \varphi(x), \quad v = \frac{u}{\varphi'} + \psi(x), \]

changes the Hamiltonian as follows:

\[ \tilde{H} = \frac{1}{2} \tilde{h}_1 v^2 + \tilde{h}_2 v + \tilde{h}_3 + 4 \sqrt{v_\psi + \tilde{q}}, \]

where

\[ \tilde{h}_1 = \varphi'h_1, \quad \tilde{h}_2 = h_2 - \varphi'\psi h_1, \quad \tilde{q} = Q + (\varphi'^-2(v-\psi))', \]

\[ Q = q_1(\varphi'^2(v-\psi)^4 + q_2\varphi'(v-\psi)^3 + q_3(v-\psi)^2 + q_4(\varphi'^{-1}(v-\psi) + q_5(\varphi'^{-2}. \tag{2.19}) \]

If \( h_1 \neq 0 \) then we put \( \varphi' = 1/h_1 \) and \( \psi = h_2 \) to get \( h_1 = 1 \) and \( h_2 = 0 \). Now it follows from (2.18) that \( \partial q_i/\partial x = 0 \). Since \( H \) does not depend on \( x \) we remove the term \( \frac{1}{2} u^2 \) in \( H \) by the Galilean transformation and obtain equation (2.3).

If \( h_1 = 0 \) but \( h_2' \neq 0 \) then equations (2.18) lead to \( q_1 = q_2 = q_3 = 0, h_2'^{-1} + q_4 h_2 = 0 \). It follows from the formula

\[ \tilde{q} = \frac{v}{\varphi'^2}(\varphi'' + q_4 \varphi') + \frac{1}{\varphi'^2}(q_5 - (\psi \varphi')) \]

that there exist \( \varphi \) and \( \psi \) such that \( \tilde{q} = 0 \). In this case \( h_2 = cx \), where \( c \) is a constant. So, we obtain

\[ H = cx u + 4 \sqrt{u_x}, \quad u_t = D_x \left( \frac{u_{xx}}{u_x^{3/2}} + c x \right). \]

Using the transformation \( u \to u + ct \), we bring \( c \) to zero and arrive at a particular case of equation (2.3).

If \( h_1 = h_2' = 0 \) then without loss of generality we put \( h_2 = h_3 = 0 \). Thus, we have shown that in all cases the functions \( h_1, h_2, h_3 \) can be reduced to zeros.

Now we normalize the polynomial \( q \) prove that all coefficients of \( q \) can be reduced to constants. If \( q_1 \neq 0 \) we use the normalization \( q_1 = 1, q_2 = 0 \). If \( q_1 = 0, q_2 \neq 0 \) then we normalize \( q_2 \) and \( q_3 \) by 1 and 0 correspondingly. In the case \( q_1 = q_2 = 0 \) we may put \( q_4 = q_5 = 0 \). In each of these cases the third integrability condition gives rise to \( q_1 = 0 \) for all remaining coefficients of \( q \). The corresponding equations can be obtained from (2.3) by translations \( u \to u + c \) and dilatations \( u \to \lambda u, t \to \alpha t, x \to \beta x \). \( \square \)

### 2.3. Integrability of equations (2.1)–(2.3)

Equations (2.1a) and (2.1b) are known to be integrable by the inverse scattering method. Equations (2.1c), (2.1d) and (2.3) can be reduced to known integrable equations of the form [4]

\[ v_t = v_y y + G(x, v, v_y), \tag{2.20} \]

by the standard reciprocal transformation (see [2], section 1.4)

\[ dy = \rho_{-1} dx + \theta_{-1} dt, \quad v(t, y) = u(t, x), \tag{2.21} \]

where \( \rho_{-1} \) is the first canonical density and \( \theta_{-1} \) is the correspondent flux. Notice that this transformation is always applicable if \( \rho_{-1} \) depends on \( u \) only and the r.h.s. of the equation does
not depend on $x$. Sometimes (2.21) can be applied in the case when $\rho - 1$ depends on $u$ and $u_x$. That is the case for equation (2.3).

To reduce the equation for $v$ to an usual form some additional point transformation $v = f(w)$ can be needed. For equation (2.1b) we have $\rho - 1 = u$. Taking $v = e^w$, we obtain

$$w_t = w_{yyy} - \frac{1}{2} w_y^3 + w_y \left( c_1 e^{2w} - 3 c_2 e^{-2w} \right).$$

This equation was found by F. Calogero and A. Degasperis and independently by A. Fokas.

In the case of equation (2.1c) we have $\rho - 1 = u$. If $c \neq 0$ we put $c = -k^2/4$, $v = \frac{k}{2} \tanh(w/2)$. As a result we get the same (up to the Galilean transformation) Calogero–Degasperis equation

$$w_t = w_{yyy} - \frac{1}{2} w_y^3 + w_y \left( \tilde{c}_1 e^{2w} + \tilde{c}_2 e^{-2w} \right) - c_3 w_y,$$

where $\tilde{c}_1 = 3/2 k^{-2}(2 c_2 + k c_1)$, $\tilde{c}_2 = 3/2 k^{-2}(2 c_2 - k c_1)$, $c_3 = 6 c_2 k^{-2}$. In the case $c = 0$ we put $v = 1/w$ to obtain the mKdV equation:

$$w_t = w_{yyy} + 12 c_2 w^2w_y + 6 c_1 w w_y.$$

Equation (2.3) is related to one more equation found by Calogero and Degasperis:

$$v_t = v_{yyy} - \frac{3}{8} \frac{(D_y(Q + v_y^2))^2}{v_y(v_y + Q)} + \frac{1}{2} Q'' v_y, \quad Q = 4P,$$

by the transformation (2.21).

Since the right hand side of equation (2.2) depends on $x$ we can not apply transformation (2.21) straightforwardly. Instead we perform the substitution $u \rightarrow u_x$ to get the potential form

$$u_t = \frac{u_{xxx}}{u_x^2} - \frac{3}{2} \frac{u_x^2}{u_x^4} + P(x)u_x^2.$$

The hodograph transformation $y = u(t, x)$, $v(t, y) = x$ brings the latter equation to the Krichever–Novikov equation

$$v_t = v_{yyy} - \frac{3v_{yy}^2}{2v_y} - \frac{P(v)}{v_y}.$$

**Acknowledgments.** The authors would like to thank B. Dubrovin and M. Pavlov for useful discussions. The research was partially supported by the RFBR grant 14-01-00751. VS is thankful to IHES for its support and hospitality.

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