DECOMPOSING WITH DIFFERENTIABLE FUNCTIONS

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Abstract. Let \( d_{m,n} \) denote the least number of sets into which \( m \)-dimensional Euclidean space can be decomposed so that each member of the partition has an \( n \)-dimensional tangent plane at each of its points. It will be shown to be consistent that \( d_{m,n} \) is different from \( d_{m,n} + 1 \). These cardinal will be shown to be closely related to the invariants associated with the problem of decomposing continuous functions into differentiable ones.

Keywords: cardinal invariant, Sacks real, tangent plane, covering number

1. Introduction

This paper is concerned with problems arising from partitioning sets into subsets with differentiability properties. If \( V \) is a vector space and \( v \in V \) then \( \langle v \rangle \) will be denoted the space spanned by \( v \).

Definition 1.1. Let \( \mathcal{P}_{k,n} \) denote the space of all subspaces of \( \mathbb{R}^n \) of dimension less than or equal to \( k \). Define a metric \( \rho \) on \( \mathcal{P}_{k,n} \) by letting

\[
\rho(V, W) = \max_{x \in V^+} \min_{y \in W^+} \arccos \left( \frac{x \cdot y}{2 \|x\| \|y\|} \right)
\]

where \( V^+ \) and \( W^+ \) are the non-zero elements of the corresponding subspaces. If \( v \in \mathbb{R}^n \) then \( \rho(\langle v \rangle, V) \) will be abbreviated to \( \rho(v, V) \) and if \( v' \in \mathbb{R}^n \) then \( \rho(\langle v \rangle, \langle v' \rangle) \) will be abbreviated to \( \rho(v, v') \).

The following definition plays a central role in this paper.

Definition 1.2. If \( A \subseteq \mathbb{R}^n \) and \( b \in \mathbb{R}^n \) then \( V \) will be said to be a TangentPlane \( k \)-dimensional tangent plane to \( A \) at \( b \) if and only if for every \( \epsilon > 0 \) there is some \( \delta > 0 \) such that \( \rho(V, a) < \epsilon \) for every \( a \in A \) such that \( 0 < \|a - b\| < \delta \). A subset \( A \subseteq \mathbb{R}^n \) will be said to be a \( k \)-smooth surface if for every \( b \in A \) there is \( V \in \mathcal{P}_{k,n} \) which is a \( k \)-dimensional tangent plane to \( A \) at \( b \).

Observe that if \( A \subseteq \mathbb{R}^n \) and \( V \) is a \( k \)-dimensional tangent plane to \( A \) at \( b \) and if there is no \( m \)-dimensional tangent plane to \( A \) at \( b \) then \( V \) is the unique \( k \)-dimensional tangent plane to \( A \) at \( b \). However it is possible that a \( k \)-dimensional smooth surface in \( \mathbb{R}^n \) may have \( m \)-dimensional tangent planes at some — or even all — points, where \( m < k \).

Lemma 1.1. If \( S \subseteq \mathbb{R}^m \) is an \( n \)-smooth surface then there is a Borel set \( S' \supseteq S \) which is also an \( n \)-smooth surface.

Proof: Choose \( D \), a countable dense subset \( S \) and let \( S' \) be the set of all \( x \in \mathbb{R}^m \) such that \( D \) has an \( n \)-dimensional tangent plane at \( x \).

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Definition 1.3. Let $D_{m,n}$ denote the $\sigma$-ideal generated by the $n$-smooth surfaces of $\mathbb{R}^m$.

The covering number of $D_{m,n}$ will be denoted by $\mathfrak{d}_{m,n}$. Unanswered questions about the additivity, cofinality and other invariants of $D_{m,n}$ naturally suggest themselves.

2. Lower Bounds

For the sake of the next theorem, let $D_{m,0}$ denote the $\sigma$-ideal generated by the discrete subsets of $\mathbb{R}^m$ and note that $\mathfrak{d}_{m,0} = 2^{\aleph_0}$.

Theorem 2.1. If $m > n \geq 1$ then $\mathfrak{d}_{m,n}^+ \geq \mathfrak{d}_{n,n-1}$.

Proof: Let $\mathfrak{d}_{m,n} = \kappa$ and suppose that $\kappa^+ < \mathfrak{d}_{n,n-1}$. Let $\{S_\alpha\}_{\alpha \in \kappa}$ be $n$-smooth surfaces such that $\bigcup_{\alpha \in \kappa} S_\alpha = \mathbb{R}^m$. Since $m > n$ it is possible to let $V \subseteq \mathbb{R}^m$ be an $n$-dimensional subspace and let $W$ be a 1-dimensional subspace orthogonal to $V$. Noting that $\kappa^+ < \mathfrak{d}_{n,n-1} \leq 2^{\aleph_0}$, let $\{w_\xi\}_{\xi \in \kappa^+}$ be distinct elements of $W$. For each $\xi \in \kappa^+$ let $S_{\xi, \alpha}$ consist of all those $s \in S_\alpha \cap (V + w_\xi)$ such that $V$ does not contain a tangent plane to $S_\alpha$ at $s$. (Recall that tangent planes have been defined as subspaces so it does not make sense to say that $V + w_\xi$ does not contain a tangent plane to $S_\alpha$ at $s$.) It follows that for each $s \in S_{\xi, \alpha}$ the plane tangent to $S_\alpha$ at $s$ intersects $V$ on a subspace of dimension smaller than $n$ and so $S_{\xi, \alpha}$ considered as subspace of $V$ under the obvious isomorphism, is a $(n-1)$-smooth surface. Since $\kappa^+ < \mathfrak{d}_{n,n-1}$ it follows that there is some $v \in V$ such that $v + w_\xi \notin S_{\xi, \alpha}$ for all $\xi \in \kappa^+$ and $\alpha \in \kappa$. There is an uncountable set $X \subseteq \kappa^+$ and $\alpha \in \kappa$ such that $v + w_\xi \in S_\alpha$ for each $\xi \in X$. Let $\xi \in X$ be such that $w_\xi$ is a limit of $\{v_\xi\}_{\xi \in X}$. Since $v + w_\xi \in S_\alpha \setminus S_{\xi, \alpha}$ it follows that $S_\alpha$ has a tangent plane at $v + w_\xi$ which is contained in $V$. This contradicts the fact that $W$ is orthogonal to $V$ because, since $v + w_\xi$ is a limit of $\{v + w_\xi\}_{\xi \in X}$, it must be that any tangent plane at $v + w_\xi$ includes $W$.

It is worth noting that if $k < n$ and $S \subseteq \mathbb{R}^n$ is $k$-smooth then $S$ is nowhere dense. Hence the covering number of $D_{m,n}$ is at least as great as that of the meagre ideal.

3. Partitioning the plane into a small number of differentiable sets

Sacks forcing will be denoted by $S$. The following definition generalizes Sacks forcing by introducing a forcing partial order which, it will be shown, is intermediate between the product an interation of a finite number of Sacks reals. Before giving the definition, define an indexed set $\{x_\xi\}_{\xi=0}^n \subseteq \mathbb{R}^m$ to be $\epsilon$-orthogonal if and only if $||\rho(x_i - x_0, x_j - x_0) - \pi/2|| < \epsilon$ whenever $1 \leq i < j \leq n$. Observe that elementary algebra implies that for each dimension $m$ there is $\epsilon(m) > 0$ such that if $\{x_\xi\}_{\xi=0}^n \subseteq \mathbb{R}^m$ is $\epsilon(m)$-orthogonal then for any subspace $V$ of dimension $k < n$ there is $i$ such that $1 \leq i \leq n$ and $\rho(x_i - x_0, V) > \epsilon(m)$.

Definition 3.1. The partial order $\mathcal{S}(m,n)$ is defined to consist of all Borel sets $B \subseteq \mathbb{R}^m$ such that there exists a family $\{C_\xi : \xi \in \mathbb{Z}^+(n+1)\}$ such that

1. $C_\xi$ is an open ball of radius less than $1/||\xi||$
2. $\bigcap_{\xi \in \omega} \bigcup_{\xi = m+1} C_\xi \subseteq B$
3. $\overline{C_\xi} \cap \overline{C_{\xi'}} = \emptyset$ if $||\xi|| = ||\xi'||$ and $\xi \neq \xi'$

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3. $\overline{C_\xi} \cap \overline{C_{\xi'}} = \emptyset$ if $||\xi|| = ||\xi'||$ and $\xi \neq \xi'$
4. \( \overline{C_\xi} \subseteq C_{\psi} \) if \( \xi' \subseteq \xi \)
5. If \( x_k \in C_{\xi \land k} \) then \( \{x_i\}_{i=0}^m \) is \( \epsilon \)-orthogonal.

The family \( \{C_\xi : \xi \in \mathcal{E}((n+1))\} \) will be called a witness to the fact that \( B \in S(m,n) \).

If \( B \in S(m,n) \) is such that there exist open balls \( C_\xi \) such that
\[
\bigcap_{i \in \omega} \bigcup_{\xi : i \rightarrow n+1} C_\xi = B
\]
and \( \{C_\xi : \xi \in \mathcal{E}((n+1))\} \) is a witness to the fact that \( B \in S(m,n) \) then \( B \) will be referred to as a natural member of \( S(m,n) \). It is obvious that the natural members of \( S(m,n) \) are dense in \( S(m,n) \).

**Lemma 3.1.** If \( k < n \), \( B \) is a natural member of \( S(m,n) \) and \( S \) is a \( k \)-dimensional smooth surface in \( \mathbb{R}^n \) then \( S \cap B \) is meagre relative to \( B \).

**Proof:** Since \( B \) is a natural member of \( S(m,n) \), there exist open balls \( \{C_\xi : \xi \in \mathcal{E}((n+1))\} \) such that \( \bigcap_{i \in \omega} \bigcup_{\xi : i \rightarrow n+1} C_\xi = B \). For each \( x \in B \) let \( \psi_x : \omega \rightarrow n+1 \) be the unique function satisfying that \( x \in C_{\psi_x(j)} \) for each \( j \in \omega \).

By Lemma 3.1 it may as well be assumed that \( S \) is Borel and, hence, satisfies the Property if Baire. If \( S \) is not meagre relative to \( B \) then let \( U \) be an open set in \( B \) such that \( B \cap U \cap S \) is comeagre in \( B \cap U \). Notice that the set of all \( x \in B \) such that \( \psi_x(j) = 0 \) for only finitely many \( j \) is a meagre set in \( B \). Let \( x \in B \cap U \cap S \) be such that \( \psi_x(j) = 0 \) for infinitely many \( j \in \omega \). Let \( V \) be a \( k \)-dimensional tangent plane to \( S \) at \( x \) and let \( \delta \) be such that \( \rho(V,a) < \epsilon(m) \) for every \( a \in A \) such that \( 0 < \|a - x\| < \delta \).

Let \( j \in \omega \) be such that \( j > 1/\epsilon(m), j > 1/\delta \), the open ball of radius \( 1/j \) around \( x \) is contained in \( U \) and \( \psi_x(j) = 0 \). It follows that \( C_{\psi_x(j)} \subseteq U \) and hence \( C_{\psi_x(j) \land t} \subseteq U \) for each \( i \in n \). It follows from Definition 3.3 that \( C_{(\psi_x(j)) \land t} \cap B \) is non-empty and open in \( B \). Hence it is possible to choose \( x_i \in C_{(\psi_x(j)) \land t} \cap B \) for each \( i \in n \) such that \( \{x_i\}_{i=0}^m \) is an \( \epsilon(m) \)-orthogonal family. Since the dimension of \( V \) is less than \( n \) it follows from the choice of \( \epsilon(m) \) such that \( \rho(V,x_i - x_0) > \epsilon(m) \) for some \( i \) between 1 and \( n \). This is a contradiction. ■

**Lemma 3.2.** If \( B \) is a natural member of \( S(m,n) \) and \( W \) is a dense \( G_\delta \) relative to \( B \) then \( W \in S(m,n) \).

**Proof:** The proof is standard. Let \( C_\xi \) be open balls witnessing that \( B \) is a natural member of \( S(m,n) \). Let \( W = \bigcap_{i \in \omega} U_i \) where each \( U_i \) is a dense open set.

Inductively choose \( C_\xi' \subseteq U_{|\xi|} \) such that \( C_\xi' = C_\eta \) for some \( \eta : k \rightarrow n + 1 \) where \( k \geq |\xi| \). Having chosen \( C_\xi' \), use the fact that \( C_\xi' = C_\eta \) implies that \( C_\xi' \) is open to choose some \( U \subseteq C_\xi' \cap U_{|\xi| + 1} \) which is open and nonempty relative to \( B \). This means that there is some \( \mu \geq \eta \) such that \( C_\mu \cap B \subseteq U \). Let \( C_{\xi(i)} = C_{\xi(i) \land t} \) for \( i \in n + 1 \). It follows that the family of sets \( C_\xi' \) witnesses that \( W \in S(m,n) \). ■

**Corollary 3.1.** If \( B \in S(m,n) \) and \( S \in D_{m,k} \) then \( B \setminus S \in S(m,n) \).

**Proof:** Without loss of generality, assume that \( B \) is natural. By Lemma 3.3 and the definition of \( D_{m,k} \), it follows that \( B \setminus S \) is comeagre relative to \( B \). Now apply Lemma 3.2. ■
Corollary 3.2. If $B \in \mathcal{S}(m,n)$ and $B \subseteq \bigcup_{k \in \omega} A_k$ where each $A_k$ is Borel then there is some $k \in \omega$ such that $B \cap A_k \in \mathcal{S}(m,n)$.

Proof: Without loss of generality, assume that $B$ is natural. For some $k$ it must be that $A_k$ is of second category relative to $B$. Let $U$ be a non-empty open set such that $A_k$ is comeagre in $B \cap U$. Then it is clear that $B \cap U \in \mathcal{S}(m,n)$ so applying Lemma 3.2 gives the desired result. ■

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Corollary 3.3. If $g \in \mathbb{R}^m$ is the generic real added by forcing with $\mathcal{S}(m,n)$ and $k < n$ then $g$ does not belong to any $k$-smooth surface with a Borel code in the ground model.

Proof: Suppose that $S$ is a $k$-smooth surface and $B \in \mathcal{S}(m,n)$ is such that $B \forces_{\mathcal{S}(m,n)} \forall \alpha \in \mathbb{R}^m$.

From Corollary 3.1 it follows that $B \setminus S \in \mathcal{S}(m,n)$. However $B \setminus S \forces_{\mathcal{S}(m,n)} \exists g \notin S$.

4. The iteration

The countable support iteration of length $\alpha$ of forcings $\mathcal{S}(m,n)$ will be denoted by $\mathbb{P}^\alpha_m$. As a convenience, by countable support will be meant that if $p \in \mathbb{P}_\alpha^m$ then $p(\gamma) = \mathbb{R}^m$ for all but countably many $\gamma \in \alpha$. From Corollary 3.3 of the previous section it follows that if $V$ is a model of the Generalized Continuum Hypothesis and $G$ is $\mathbb{P}_\alpha^m$ generic over $V$ then $\omega_k = \omega_2$ for each $k < n$ in $V[G]$. It will be shown in this section that $\omega_k = \omega_1$ in $V[G]$.

To this end, suppose that $V$ is a model of the Continuum Hypothesis, $p^* \in \mathbb{P}_\omega^m$ and that $p^* \forces_{\mathbb{P}_\omega^m} \exists x \in \mathbb{R}^m$. It follows that there must be some $\alpha \in \omega_2$ such that $p^* \forces_{\mathbb{P}_\alpha^m} \exists x \in V[G \cap \mathbb{P}_\alpha^m]$ and $p^* \forces_{\mathbb{P}_\beta^m} \forall x \notin V[G \cap \mathbb{P}_\beta^m]$ for all $\beta \in \alpha$. It must be shown that there is, in the ground model, an $n$-smooth surface $S \subseteq \mathbb{R}^m$ as well as a condition $q \leq p^*$ such that $q \forces_{\mathbb{P}_\omega^m} \exists x \in S$.

Fix $x$ and let $\mathcal{M} \prec H(\aleph_3)$ be a countable elementary submodel containing $p^*$, $x$ and $\omega$. Notice that if $p$ and $q$ belong to $\mathbb{P}_\alpha^m \cap \mathcal{M}$ then the $p \land q$ also belongs to $\mathcal{M}$. The exact definition of $p \land q$ is not important here, only that it is definable in $\mathcal{M}$. Observe that $p \land q$ exists in $\mathbb{P}_\alpha^m$ because each factor of the iteration has a naturally defined meet operation.

Let $\mathcal{P}(\mathbb{P}_\alpha^m \cap \mathcal{M})$ be given the Tychonoff product topology — in other words, it is homeomorphic to the Cantor set. Let $\mathcal{F}$ be the set of all $F \in \mathcal{P}(\mathbb{P}_\alpha^m \cap \mathcal{M})$ such that

- if $p \in F$ and $q \geq p$ then $q \in F$,
- if $p \in F$ and $q \in F$ then $p \land q \in F$ and $p \land q \neq \emptyset$,
- if $p \in F$, $\{q_i\}_{i \in \kappa} \subseteq \mathbb{P}_\alpha^m$ and there does not exist any $r \in \mathbb{P}_\alpha^m$ such that $r \leq u$ and $r \land q_i = \emptyset$ for each $i \in \kappa$ then $F \setminus \{q_i\}_{i \in \kappa} \neq \emptyset$.

Notice that $\mathcal{F}$ is a closed set in $\mathcal{P}(\mathbb{P}_\alpha^m \cap \mathcal{M})$. Next, let $\mathcal{G}$ be the set of all $F \in \mathcal{F}$ such that $F$ is $\mathbb{P}_\alpha^m$ generic over $\mathcal{M}$ and $p^* \in G$.

To see that $\mathcal{G}$ is a dense $G_\delta$ relative to $\mathcal{F}$, it suffices to show that if $D$ is a dense subset of $\mathbb{P}_\alpha^m$ then $\{F \in \mathcal{F} : F \cap D \neq \emptyset\}$
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is a dense open set in $F$. In order to verify this, let $V$ be an open set in $F$. It may be assumed that

$$V = \{ F \in F : p \in F \text{ and } \{ q_i \}_{i \in k} \cap F = \emptyset \}$$

for some $p \in P_{\alpha}^m$ and $\{ q_i \}_{i \in k} \subseteq P_{\alpha}^m$. Because $V$ is a non-empty open set in $F$, it must be that there is some $r \in P_{\alpha}^m$ such that $r \leq p$ and $r \wedge q_i = \emptyset$ for each $i \in k$. Let $r' \in D$ be such that $r' \leq r$ and observe that that $\{ F \in F : r' \in F \} \subseteq V$. Hence $G$ is a dense $G_{\delta}$ in a closed subspace of $P_{\alpha}^m$. It is worth noting that the third condition in the definition of $F$ implies that $G$ has a base consisting of sets of the form $U = \{ F \in G : p \in F \}$.

Let $\Psi : G \rightarrow R^n$ be the mapping defined by $\mathfrak{M}(G) \models x^G = \Psi(G)$ where $x^G$ is the interpretation of the name $x$ in $\mathfrak{M}(G)$. Notice that $\Psi$ is continuous.

Before continuing, some notation concerning fusion arguments will be established. Let $\{ f_i \}_{i \in \omega}$ be a sequence of finite functions satisfying the following properties:

- $f_i : \alpha \cap 2^\omega \rightarrow \omega$
- for each $i \in \omega$ there exists a unique ordinal $\beta(i) \in \alpha \cap 2^\omega$ such that $f_{i+1}(\gamma) = f_i(\gamma)$ unless $\gamma = \beta(i)$ and $f_{i+1}(\beta(i)) = f_i(\beta(i)) + 1$ where $f_i(\beta(i))$ is defined to be 0 if $\beta(i)$ is not in the domain of $f_i$.
- $\sup_{i \in \omega} f_i(\gamma) = \omega$ for each $\gamma \in \alpha \cap 2^\omega$.

For each condition $p \in P_{\alpha}^m$ the definition of $S(m, n)$ guarantees that, for each $\gamma \in \alpha \cap 2^\omega$, there is a name $\{ C_\xi^\gamma : \xi \in \check{\alpha} n + 1 \}$ such that $p \forces \gamma$ forces that $\{ C_\xi^\gamma : \xi \in \check{\alpha} n + 1 \}$ is a witness to the fact that $p(\gamma) \in S(m, n)$ and each $C_\xi^\gamma$ is an open ball whose radius and centre are rational. It will also be assumed that $C_\xi^\gamma = R^m$ for each $\gamma$.

Suppose that $r \in P_{\alpha}^m$, $\Gamma \in [\alpha \cap 2^\omega]^{<\omega}$ and $A$ is a function with domain $\Gamma$ such that $A(\gamma)$ is a $P_{\gamma}^m$, $\Gamma$-name for a member of $S(m, n)$ for each $\gamma$ in $\Gamma$. Then $r[A]$ denotes the condition defined by

$$r[A](\gamma) = \begin{cases} r(\gamma) & \text{if } \gamma \notin \Gamma \\ r(\gamma) \cap A(\gamma) & \text{if } \gamma \in \Gamma \end{cases}$$

For $i \in \omega$, $p \in P_{\alpha}^m$ and $h \in \prod_{\gamma \in \alpha \cap 2^\omega} f_{i}(\gamma)(n + 1)$ the notion of what it means for $p$ to be determined is the same as in $[2]$. In particular, if $\Gamma \in [\alpha \cap 2^\omega]^{<\omega}$ and $f : \Gamma \rightarrow \omega$ then let

$$T_f = \bigcup_{\beta} \prod_{\gamma \in \Gamma \cap \beta} f_{i}(\gamma)(n + 1)$$

and note that $T_f$ is a tree under inclusion. Define a condition $p \in P_{\alpha}^m$ to be $f$-determined with respect to $\{ C_\xi^\gamma : \xi \in \check{\alpha} n + 1 \}$ if there is a function $A$ with domain $T_f$ such that if

$$h \in \prod_{\gamma \in \Gamma \cap \beta(n + 1)} f_{i}(\gamma)(n + 1)$$

then, for each $\gamma \in \Gamma \cap (\beta + 1)$, $A(h)(\gamma)$ is an $m$-ball with rational centre and radius such that $p[A(h) \upharpoonright \beta] \forces_{P_{\beta}} ^m \text{“} c_{h(\beta)}^\beta = A(h)(\beta) \text{”}$ and, furthermore, $A(h) \upharpoonright \beta = A(h) \upharpoonright (\beta)$ for every $\beta$.

By a fusion sequence will be meant a sequence of conditions $\{ p_i \}_{i \in \omega}$ such that
• for each $i \in \omega$ and $\gamma$ in the support of $f_i$ there is a sequence
  \[ \{ C_{\xi,i}^\gamma : \xi \in \mathbf{z}(n + 1) \} \]
  such that $p_i \upharpoonright \gamma$ forces that \( \{ C_{\xi,i}^\gamma : \xi \in \mathbf{z}(n + 1) \} \) is a witness to the fact that
  $p_i(\gamma) \in \mathcal{S}(m,n)$
  • each $p_i$ is $f_i$-determined with respect to \( \{ C_{\xi,i}^\gamma : \xi \in \mathbf{z}(n + 1) \} \) by a function
    $A_i$ defined on $T_i = T_{f_i}$
  • $p_{i+1} \upharpoonright \gamma$ forces that \( C_{\xi,i}^\gamma = C_{\xi,i+1}^\gamma \) for each $i \in \omega$, $\gamma \in \alpha \cap \mathfrak{M}$ and $\xi$ such that
    $|\xi| \leq f_i(\gamma)$

Notice that, since the sets $T_i$ are all disjoint, there is no ambiguity in letting $A = \bigcup_{i \in \omega} A_i$ and saying that $A$ witnesses that $p_i$ is $f_i$-determined with respect to

\[ \{ C_{\xi,i}^\gamma : \xi \in \mathbf{z}(n + 1) \} \]

rather than that $\{ A_i \}_{i \in \omega}$ witnesses this. This situation will be referred to by saying that $A$ witnesses that $\{ p_i \}$ is a fusion sequence. If $G \in \mathcal{G}$ then define $H_i(G)$ to be the unique, maximal member of $T_i$ such that $p_i[A(H_i(G))] \in G$ provided that such a maximal member exists at all. Finally, if $h$ is a maximal element of $T_i$ then let $\mathcal{U}(h)$ be the open set in $\mathcal{G}$ defined by $\mathcal{U}(h) = \mathcal{U}(p_i[A(h)])$ and note that $i$ is the uniquely determined as the only integer such that $h$ is a maximal member of $T_i$. Of course, $\mathcal{U}(h)$ and $H_i(G)$ are only defined in the context of a given fusion sequence but, since this will always be clear, it will not be added to the notation.

Two facts are worth noting. First, it is easy to verify that the fusion of such a sequence is in $\mathbb{P}_\alpha^{m,n}$. Second, if $G$ and $G'$ are in $\mathcal{G}$ and $H_i(G)$ and $H_i(G')$ are defined and equal for all $i \in \omega$ then $G = G'$.

Two cases will now be considered depending on whether or not $\alpha$ is a limit ordinal.

### Lemma 4.1

If $\alpha$ is limit ordinal then there is a $1$-smooth curve in $\mathbb{R}^m$, $L$, and a condition $r \leq p^*$ such that $q \upharpoonright \mathbb{P}_\alpha^{m,n} \forces \text{"}x \in L\text{"}$.

**Proof:** For each $G \in \mathcal{G}$ let $\tau(G)$ be the set of all $V \in P_{1,m}$ such that for all $\epsilon > 0$, $q \in G$ and $\beta \in \alpha$ there is $G' \in \mathcal{G}$ such that $G' \cap \mathbb{P}_\beta^{m,n} = G \cap \mathbb{P}_\beta^{m,n}$, $q \in G'$ and $\rho(V, \Psi(G) - \Psi(G')) \leq \epsilon$. The first thing to observe is that if $G \in \mathcal{G}$ then $\tau(G) \neq \emptyset$.

To see this, first observe that $\tau(G) = \bigcap_{\beta \in \alpha \cap \mathfrak{M}} \bigcap_{q \in G} \overline{\tau(G,\beta,q)}$ where

$\tau(G,\beta,q) = \{ (\Psi(G) - \Psi(G')) : G' \cap \mathbb{P}_\beta^{m,n} = G \cap \mathbb{P}_\beta^{m,n}, q \in G' \text{ and } \Psi(G) \neq \Psi(G') \}$

and, since $P_{1,m}$ is compact, it suffices to show that if $\beta \in \alpha \cap \mathfrak{M}$ and $q \in G$ then $\tau(G,\beta,q) \neq \emptyset$. Since $p^* \forces \mathbb{P}_\alpha^{m,n} \forces \text{"}x \notin V[G \cap \mathbb{P}_\beta^{m,n}]\text{"}$ for every $\beta \in \alpha$ it follows that there must be $G'$ containing both $p^*$ and $q$ which is $\mathbb{P}_\alpha^{m,n} / (G \cap \mathbb{P}_\beta^{m,n})$ generic over $\mathfrak{M}[G \cap \mathbb{P}_\beta^{m,n}]$ and such that the interpretation of $x$ in $\mathfrak{M}[G \cap \mathbb{P}_\beta^{m,n} * G']$ is different from the interpretation of $x$ in $\mathfrak{M}[G]$.

Since $\mathcal{G}$ and $P_{1,m}$ are both Polish spaces and $\tau$ is a Borel subset of $\mathcal{G} \times P_{1,m}$, it is possible to appeal to the von Neumann Selection Theorem to find a Baire measurable function $\Delta : \mathcal{G} \to P_{1,m}$ such that $\Delta(G) \in \tau(G)$ for each $G \in \mathcal{G}$. Let $W \subseteq \mathcal{G}$ be a dense $G_\delta$ such that $\Delta \upharpoonright W$ is continuous. Let $W_n$ be dense open sets such that $W = \bigcap_{n \in \omega} W_n$.

The next step will be to construct a fusion sequence $\{ p_i \}$ and $A$ which witnesses this so that the following conditions are satisfied
To see that this induction can be carried out, suppose that
\[
\{\text{defined. From the last induction hypothesis it follows that there is some} \ q \ \text{such that} \\
\{\text{show that there is} \ i > \ \text{constructed satisfying the induction requirements. Let} \ G \\
\text{maximal elements of} \ T \ \text{members of} \ p \\
\text{proof is complete. In particular, let} \ L \ \text{be the image under} \ \Psi \ \text{of the set of all} \ G \in \mathcal{G} \\
\text{first, it will be shown that if such a fusion sequence can be constructed, then the} \\
\text{next claim will be used in showing that the desired fusion sequence can be} \\
\text{constructed.} \\
\text{Claim 1. Suppose that} \ \epsilon > 0, \ q \in \mathbb{P}^{m,n}_\alpha \ \text{and} \ \beta \in \alpha. \ \text{Then there exist conditions} \\
\text{such that} \ q_i \leq q \ \text{for} \ i \in n + 1 \\
\text{if} \ \beta = q_{i'} | \ \beta \ \text{for} \ i \ \text{and} \ i' \ \text{in} \ n + 1 \\
\text{if} \ \{G_i\}_{i=0}^n \subseteq W \ \text{and} \ q_i \in G_i \ \text{for each} \ i \leq n \\
\rho(\Psi(G_i) - \Psi(G_j), \Delta(G_i)) < \epsilon \\
\text{so long as} \ i \neq j \\
\text{4.} \ q_i \models_{\mathbb{P}^{m,n}} \ "x \in E_i" \ \text{for closed balls} \ E_i \ \text{such that} \ E_i \cap E_{i'} = \emptyset \ \text{if} \ i \neq i' \\
\text{Assuming the claim, suppose that} \ p_i \ \text{and} \ A_i \ \text{of the fusion sequence have been} \\
\text{constructed satisfying the induction requirements. Let} \ \{h_s\}_{s \in k} \ \text{enumerate all the} \\
\text{maximal elements of} \ T_i \ \text{and let} \ \mu \in \alpha \cap \mathfrak{M} \ \text{contain the domain of} \ f_{i+1}. \ \text{Let} \ q_y^{y,j} = p_i[A(h_s)] \ \text{for each} \ s \in k. \ \text{Proceed by induction on} \ y \ \text{to construct} \ q_y^{y,j} \ \text{such that} \\
\text{q}^{y,j}_s \leq q^{y-1,j}_s \ \text{for each} \ y \leq k \\
\text{if} \ \mu' \leq \mu \ \text{and} \ h_x | \mu' = h_x | \mu' \ \text{then} \ q^{y,j}_s \ | \mu' = q^{y,j'}_s | \mu' \\
\text{u}(q^{y+1,j}_y) \subseteq W_i \ \text{for} \ y \leq k \ \text{and} \ j \in n + 1 \\
\text{if} q^{y+1,j}_y \in G \ \in W, \ q^{y+1,j}_y \in G' \ \in W \ \text{and} \ j \neq j' \ \text{then} \ \rho(\Psi(G) - \Psi(G'), \Delta(G)) < 1/i \\
\text{for all} \ y \leq k \ \text{and} \ j \leq n \\
q^{y+1,j}_y \models_{\mathbb{P}^{m,n}} \ \beta(i) |_{\mathbb{P}^{m,n}_{\beta(i)}} \ "C^{y(j),i}_{h_y(\beta(i))} = A^{j,h_y}" \\
q_y^{y+1,j} \models_{\mathbb{P}^{m,n}} \ "x \in E'_y" \ \text{where} \ \{E'_y\}_{y \in n + 1} \ \text{is a pairwise disjoint collection of} \\
\text{closed balls} \\
\text{if} \ y \leq s \ \text{the} \ q^{y,j}_s = q^{y,j'}_y \ \text{for all} \ j \ \text{and} \ j' \ \text{in} \ n + 1 \ \\
\text{To see that this induction can be carried out, suppose that} \ \{q^{y,j}_y\}_{y \in k} \ \text{have been} \\
\text{defined. From the last induction hypothesis it follows that there is some} \ q'' \ \text{such that} \\
q'' = q^{y,j}_y \ \text{for all} \ j \in n + 1. \ \text{Since} \ W_i \ \text{is dense open it is possible to find} \ q' \leq q''
such that $U(q') \subseteq W_i$. Extend $q'$ to $q$ such that
\[ q \upharpoonright \beta(i) \upharpoonright_{\Delta_{\beta(i)}} \text{for each } j \leq n. \]
Use the claim to find conditions $\{q_y^{y+1,j}\}_{j \in n+1}$ such that
\begin{itemize}
  \item $q_y^{y+1,j} \leq q$ for $i \in n + 1$
  \item $q_y^{y+1,j} \upharpoonright \mu = q_y^{y+1,j'} \upharpoonright \mu$ for $j$ and $j'$ in $n + 1$
  \item if $\{G_j\}_{j \in n+1} \subseteq W$ and $q_y^{y+1,j} \in G_j$ for each $j \in n + 1$ then
    \[ \rho(\Psi(G_j) - \Psi(G_{j'}), \Delta(G_j)) < 1/(i + 1) \]
\end{itemize}
so long as $j \neq j'$
\begin{itemize}
  \item $q_y^{y+1,j} \upharpoonright_{\Delta_{\beta(i)}} \text{for closed balls } E_j$ such that $E_j \cap E'_{j'} = \emptyset$ if $j \neq j'$
\end{itemize}
If $s \neq y$ let $b(s)$ be the least member of the domain of $f_i$ such that $h_y(b(s)) \neq h_y(b(s))$ and define $q_y^{y+1,j}$ to be the least upper bound of $q_y^{y+1,j} \upharpoonright b(s)$ and $q_y^{y,j}$. It is easily verified that all of the induction requirements are satisfied.

Now, if $h$ is a maximal member of $T_{i+1}$ then let $j(h) \leq n$ and $h^*$ be a maximal member of $T_i$ such that $h(\beta(i)) = h^*(\beta(i)) \land j(h)$ and, if $\gamma \neq \beta(i)$ then $h(\gamma) = h^*(\gamma)$. Let $q_h = q_y^{h^*,j(h)}$. Let
\[ A_{i+1}(h^*,\gamma) = \begin{cases} A_i(h^*,\gamma) & \text{if } \gamma \neq \beta(i) \\ A_i(h,\gamma) & \text{if } \gamma = \beta(i) \end{cases} \]
Define $p_{i+1}$ to be the join of all the conditions $q_h$ as $h$ ranges over all maximal members of $T_{i+1}$. Note that if $y = y'$ then by the induction hypothesis on $p_i$, $\Psi(\mathcal{U}(h_y))$ and $\Psi(\mathcal{U}(h_{y'}))$ have disjoint closures so it may be assumed that $E_y \cap E'_{y'} = \emptyset$. Thus the three requirements of the desired fusion sequence are satisfied.

All that remains to be done is to prove the claim. To this end, let $q$, $\epsilon$ and $\beta$ be given. Using the fact that $W$ is dense, let $G^* \in W$ be arbitrary such that $q \in G$. Using the continuity of $\Delta$ on $W$, find $q' \in G$ such that $G' \in W$ and $q' \in G'$ then $\rho(\Delta(G), \Delta(G')) < \epsilon/2$ and let $p^{-1} = q'$. First notice that it suffices to construct by induction on $i$ a sequence $\{\langle E_i, E_i, G_i, p_i, p_i \rangle\}_{i \in n+1}$ such that for each $i \in n + 1$
\begin{enumerate}
  \item $p_i \leq p^{-1}$
  \item $p_i \leq p^{-1}$
  \item $G_i \cap \mathbb{P}^m = G^* \cap \mathbb{P}^m$
  \item $p_i \in G^*$ for each $i$
  \item $p_i \in G_i$ for each $i$
  \item $E_i$ and $E_i$ are disjoint closed subsets of $\mathbb{R}^m$
  \item $E_i' \cup E_{i+1} \subseteq E_i'$
  \item $\Psi(\mathcal{U}(p_i))$ is contained in the interior of $E_i$
  \item $\Psi(\mathcal{U}(p_i))$ is contained in the interior of $E_i$
  \item $\rho(x - x', \Delta(G^*)) < \epsilon/2$ if $x \in E_i$ and $x' \in E_i$
\end{enumerate}
The reason this suffices is that, having done so, using conditions (3) and (5) it is possible to find a single $p \in G^* \cap \mathbb{P}^m$ extending each $p_i$. Let $q_i$ be the greatest lower bound of both $p_i$ and $p$. It follows that $q_i \leq p_i \leq p_i^{-1} \leq p^{-1} = q$. Moreover, $q_i \upharpoonright \beta = p$ for each $i \in n + 1$. Also, it follows from conditions (6) and (7) that the sets $\{E_i\}_{i \in n+1}$ are pairwise disjoint closed sets and $p_i \upharpoonright \mathcal{U} = \{x \in E_i\}$ by condition (9). Finally, suppose that $i \neq j$, $\{G_i, G_j\} \subseteq W$, $q_i \in G_i$ and $q_i \in G_j$. Then $\rho(\Delta(G_i), \Delta(G_j)) < \epsilon/2$ because $q_i \leq q'$. Hence
\[ \rho(\Psi(G_i) - \Psi(G_j), \Delta(G_i)) < \epsilon \]
by condition (10).

To carry out the induction, suppose that \((E_i, E^i, G_i, p_i, p^i))_{i \in J}\) have been constructed. From the definition of \(\Delta(G^*)\) it follows that there is \(G_J\) such that

- \(G_J \cap \mathbb{P}^{m,n}_\beta = G^* \cap \mathbb{P}^{m,n}_\beta\)
- \(p^{j-1} \in G_J\)
- \(\rho(\Psi(G^*) - \Psi(G_J), \Delta(G^*)) < \epsilon/4\)
- \(\Psi(G^*) \neq \Psi(G_J)\)

Let \(E^j\) and \(E_J\) be disjoint closed neighbourhoods of \(\Psi(G^*)\) and \(\Psi(G_J)\) respectively such that \(\rho(x' - x, \Delta(G^*)) < \epsilon/2\) for any \(x \in E^j\) and \(x' \in E_J\). Since induction hypothesis (8) implies that \(\Psi(U(p^{j-1}))\) is contained in the interior of \(E^{j-1}\) and \(p^{j-1} \in G_J\) it follows that it may be assumed that \(E^j \cup E_J \subseteq E^{j-1}\). From the continuity of \(\Psi\) it is possible to find \(p_j \in G_J\) and \(p^j \in G^*\) extending \(p^{j-1}\) such that \(\Psi(U(p_j))\) is contained in the interior of \(E_J\) and \(\Psi(U(p^j))\) is contained in the interior of \(E^j\). All of the induction hypotheses are now satisfied.

The possibility that \(\alpha\) is a successor must now be considered. The proof has the same structure as the limit case but the details are different. Only the successor case requires the use of higher dimensional tangent planes.

**Lemma 4.2.** If \(\alpha\) is a successor ordinal then there is an \(n\)-smooth surface in \(\mathbb{R}^m\), successor \(L\), and a condition \(\rho \leq p^*\) such that \(q \models \text{“}x \in L\text{”}\).

**Proof:** Let \(\alpha = \beta + 1\). For each \(G \in G\) let \(\tau(G)\) be the set of all \(V \in P_{n,m}\) such that for all \(\epsilon > 0\) and \(q \in G\) there is a family \(a \in [G]^{\tau < \infty}\) such that

1. \(G \in a\)
2. \(a\) is \(\epsilon\)-orthogonal under some indexing
3. \(G' \cap \mathbb{P}^{m,n}_\beta = G \cap \mathbb{P}^{m,n}_\beta\) for each \(G' \in a\)
4. \(q \in \cap a\)
5. \(\Psi \upharpoonright a\) is one-to-one
6. \(\rho(V, \Psi(G') - \Psi(G'')) < \epsilon\) for \(\{G', G''\} \in [a]^2\)

As in Lemma 4.1, it must be noted that if \(G \in G\) then \(\tau(G) \neq \emptyset\). Notice that

\[
\tau(G) \supseteq \bigcap_{q \in G} \bigcap_{\epsilon > 0} \tau(G, q, \epsilon)
\]

where \(\tau(G, q, \epsilon)\) is defined to be the set of all spaces generated by

\[\{\Psi(G') - \Psi(G') : \{G', G''\} \in [a]^2\}\]

where \(a\) satisfies condition (1) to (5) with respect to \(q\) and \(\epsilon\). Since \(P_{n,m}\) is compact, it suffices to show that if \(q \in G\) then \(\tau(G, q, \epsilon) \neq \emptyset\). In \(M[G \cap \mathbb{P}^{m,n}_\beta]\) the name \(q(\beta)\) is interpreted as a condition in \(S(m, n)\). It follows that in \(M[G \cap \mathbb{P}^{m,n}_\beta]\) there must be open sets \(\{C_i\}_{i=0}^n\) such that \(C_i \cap q(\beta) \in S(m, n)\) and for any selection \(x_i \in C_i\) the family \(a = \{x_i\}_{i=0}^n\) is \(\epsilon\)-orthogonal. Since

\[p^* \models \text{“}x \notin M[G \cap \mathbb{P}^{m,n}_\beta]\text{”}\]

it is an easy matter to choose \(G_i'\) which is \(S(m, n)\) generic over \(M[G \cap \mathbb{P}^{m,n}_\beta]\), \(q_i \in G_i'\) and such that the interpretation of \(x\) in \(M[G \cap \mathbb{P}^{m,n}_\beta * G_i']\) is different from the interpretation of \(x\) in \(M[G \cap \mathbb{P}^{m,n}_\beta * G_i']\) unless \(i = j\). It follows that, letting \(G_i = G \cap \mathbb{P}^{m,n}_\beta * G_i'\), the space generated by

\[\{\Psi(G) - \Psi(G_i)\}_{i=1}^n\]
belongs to $\tau(G, q, \epsilon)$.

As in Lemma 4, it is possible to find a Baire measurable function $\Delta : \mathcal{G} \to P_{n,m}$ such that $\Delta(G) \in \tau(G)$ for each $G \in \mathcal{G}$. Let $W \subseteq \mathcal{G}$ be a dense $\mathcal{G}$ such that $\Delta \upharpoonright W$ is continuous. Let $W_n$ be dense open sets such that $W = \bigcup_{n \in \omega} W_n$.

The next step will be to construct a fusion sequence $\{p_i\}$ and $A$, which witnesses this, so that the following conditions are satisfied

- if $h$ is a maximal member of $T_i$ then $U(h) \subseteq W_i$
- if $h_0$ and $h_1$ are distinct maximal members of $T_i$ then the images $\Psi(U(h_0))$ and $\Psi(U(h_1))$ have disjoint closures
- if $G$ belongs to $W$, $H_i(G) = H_i(G')$ and $H_{i+1}(G) \neq H_{i+1}(G')$ then $\rho(\Psi(G) - \Psi(G'), \Delta(G)) < 1/i$

First, it will be shown that if such a fusion sequence can be constructed, then the proof is complete. In particular, let $L$ be the image under $\Psi$ of the set of all $G \in \mathcal{G}$ such that $H_i(G)$ is defined for all $i \in \omega$. Let $r$ be the fusion of the sequence $\{p_i\}_{i \in \omega}$. Note that $\Psi^{-1} \upharpoonright L$ is defined.

The proof that $L$ is $n$-smooth is the same as in Lemma 4. Suppose that $x = \Psi(G) \in L$ and let $\epsilon > 0$. It suffices to show that there is $\delta > 0$ such that if $x' \in L$ and $\|x - x'\| < \delta$ then $\rho(\Delta(G), x - x') < \epsilon$. To this end, let $i > 1/\epsilon$ and let $\delta$ be so small that if $h$ and $h'$ are distinct maximal members of $T_i$ then the distance between $\Psi(U(h))$ and $\Psi(U(h'))$ is greater than $\delta$. Therefore, if $\|x - x'\| < \delta$ it follows that $H_i(G') = H_i(G)$ where $G'$ is such that $\Psi(G') = x'$. Let $j \in \omega$ be the greatest integer such that $H_j(G) = H_j(G')$. The first property guarantees that $G$ belongs to $W$ and, since $H_{j+1}(G) \neq H_{j+1}(G')$, it follows that $\rho(\Delta(G), x - x') = \rho(\Delta(G), \Psi(G) - \Psi(G')) < 1/j < 1/i < \epsilon$.

To see that the fusion sequence can be found, suppose that $p_i$ and $A_i$ of the fusion sequence have been constructed satisfying the induction requirements. Let $\{h_s\}_{s \in k}$ enumerate all the maximal elements of $T_i$. Let $q_s^{y,j} = p_i[A(h_s)]$ for each $s \in k$. Proceed by induction on $y$ to construct $q_s^{y,j}$ such that

- $q_s^{y,j} \leq q_s^{y-1,j}$ for each $y \leq k$
- if $\mu \leq \alpha$ and $h_s \upharpoonright \mu = h_s' \upharpoonright \mu$ then $q_s^{y,j} \upharpoonright \mu = q_s^{y,j'} \upharpoonright \mu$
- $U(q_s^{y+1,j}) \subseteq W_i$ for $y \leq k$ and $j \in n + 1$
- if $q_s^{y+1,j} \in G \subseteq W$, $q_s^{y+1,j'} \in G' \subseteq W$ and $j \neq j'$ then $\rho(\Psi(G) - \Psi(G'), \Delta(G)) < 1/i$
- for all $y \leq k$ and $j \leq n$ $q_s^{y+1,j} \upharpoonright \beta(i) \upharpoonright \rho_{p_m,n} \upharpoonright \beta(i) \upharpoonright \beta(i) \upharpoonright \beta(i) \cup \lambda_j = A^j h_y$
- $q_s^{y+1,j} \upharpoonright \rho_{p_m,n} \upharpoonright \beta(i) \upharpoonright \beta(i) \cup \lambda_j = A^j h_y$

To see that this induction can be carried out, suppose that $\{q_s^{y,j}\}_{s \in k}$ have been defined. From the last induction hypothesis it follows that there is some $q''$ such that $q'' = q_s^{y,j}$ for all $j \in n + 1$. Since $W_i$ is dense open it is possible to find $q' \leq q''$ such that $U(q') \subseteq W_i$. Extend $q'$ to $q$ such that

$$q \upharpoonright \rho_{p_m,n} \upharpoonright \beta(i) \cup \lambda_j = A^j h_y$$

for each $j \leq n$.

Using the fact that $W$ is dense in $\mathcal{G}$, let $G \in W$ be such that $q \in G$. Using the definition of $\tau(G)$, let $\tilde{\epsilon} < 1/\tilde{i} + 1$ and let $a \in [G]^{n+1}$ be $\tilde{\epsilon}$-orthogonal such that

- $G \in a$
• $G' \cap \mathbb{P}_\beta^{m,n} = G \cap \mathbb{P}_\beta^{m,n}$ for each $G' \in a$

• $q \in \cap a$

• $\Psi \restriction a$ is one-to-one

• $\rho(\Delta(G), \Psi(G')) < \epsilon$ for $\{G', G''\} \in [a]^2$

It is therefore possible to choose $\hat{q}_i \in G_i$ as well as $A_{i, h_y}$ such that $\hat{q}_i \upharpoonright \beta = \hat{q}_0 \upharpoonright \beta$ and such that $\hat{q}_0 \restriction \mathbb{P}_\beta^{m,n} \ " A_{i, h_y} = \hat{q}_i(\beta) ^ \wedge$ and, moreover, if $z_i \in A_{i, h_y}$ for each $i \leq n$ then $\{z_i\}_{i=0}^n$ is fraction $1$-orthogonal. Using the continuity of $\Delta$ and $\Psi$ as well as the fact that $p^* \restriction \mathbb{P}_\beta^{m,n} \ " x \notin \mathfrak{M}[G \cap \mathbb{P}_\beta^{m,n}]$ it is possible to extend each $\hat{q}_i$ to $q^{y+1,j}_i$ such that

• $q^{y+1,j}_i \leq q$ for $i \in n + 1$

• $q^{y+1,j}_i \upharpoonright \beta = q^{y+1,j}'_i \upharpoonright \beta$ for $j$ and $j'$ in $n + 1$

• if $\{G_j\}_{j \in \omega} \subseteq W$ and $q^{y+1,j}_i \in G_j$ for each $j \in n + 1$ then

$$
\rho(\Psi(G_j) - \Psi(G_{j'}), \Delta(G_j)) < 1/(i + 1)
$$

so long as $j \neq j'$

If $s \neq y$ let $b(s)$ be the least member of the domain of $f_i$ such that $h_x(b(s)) \neq h_y(b(s))$ and define $q^{y+1,j}_i$ to be the least upper bound of $q^{y+1,j}_i \upharpoonright b(s)$ and $q^{y,j}_i$. It is easily verified that all of the induction requirements are satisfied.

Now, if $h$ is a maximal member of $T_{i+1}$ then let $j(h) \in n + 1$ and $h^*$ be a maximal member of $T_i$ such that $h(\beta(i)) = h^*(\beta(i)) \cup j(h)$ and, if $\gamma \neq \beta(i)$ then $h(\gamma) = h^*(\gamma)$. Let $q_h = q^{k(h)}$. Let

$$
A_{i+1}(h)(\gamma) = \begin{cases} 
A_i(h^*)(\gamma) & \text{if } \gamma \neq \beta(i) \\
A_i(\gamma, h^*) & \text{if } \gamma = \beta(i) \neq \beta \\
A_{j, h^*} & \text{if } \gamma = \beta(i) = \beta
\end{cases}
$$

Define $p_{i+1}$ to be the join of all the conditions $q_h$ as $h$ ranges over all maximal members of $T_{i+1}$. This satisfies the three requirements of the desired fusion sequence.

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Lemmas 4.2 and 4.3 together show that the ground model smooth sets are sufficient to cover all real added by iteratively adding reals with $\mathcal{S}(m, n)$. Combined with Corollary 4.3, this immediately gives the following theorem.

**Theorem 4.1.** If $1 \leq n < m \in \omega$, then it is consistent, relative to the consistency of set theory itself, that $\text{cov}(\mathcal{D}_{m,n}) < \text{cov}(\mathcal{D}_{m,n+1})$.

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5. DECOMPOSING CONTINUOUS FUNCTIONS

In [1] the authors consider the following question: If $\mathcal{A}$ and $\mathcal{B}$ are two families of functions between Polish spaces, what is the least cardinal $\kappa$ such that every member of $\mathcal{A}$ can be decomposed into $\kappa$ members of $\mathcal{B}$ — this cardinal $\kappa$ the authors call $\text{dec}(\mathcal{A}, \mathcal{B})$. The most natural class to consider for $\mathcal{B}$ is the class of continuous functions and the problem to which it gives rise had been posed by Lusin who wondered whether every Borel function could be decomposed in countably many continuous functions. In more recent times, it has been shown by Abraham and Shelah that every function of size less than $\epsilon$ can be decomposed into countably many continuous — and even monotonic — functions [2]. Various results concerning $\text{dec}(\mathcal{B}_\infty, \mathcal{C})$ where $\mathcal{B}_\infty$ is the class of pointwise limits of continuous function and $\mathcal{C}$ is the class of continuous functions can be found in [2.1, 2.2] and [3].
The question of determining $\text{dec}(C, D)$ where $D$ is the class of functions which are differentiable on their domain was raised by M. Morayne and J. Cichoń. It was not known whether it is consistent that $\text{dec}(C, D) < \mathfrak{b}$ and the best lower bound for $\text{dec}(C, D)$ was noted by Morayne to be the additivity of the null ideal. The following result improves the lower bound and shows that $\text{dec}(C, D) < \mathfrak{b}$ is indeed consistent.

**Theorem 5.1.** $\text{dec}(C, D) = \text{cov}(D_{\mathcal{E}, \infty})$.

**Proof:** To show that $\text{dec}(C, D) \leq \text{cov}(D_{\mathcal{E}, \infty})$ let $\kappa = \text{dec}(C, D)$ and let $\{S_\eta\}_{\eta \in \kappa}$ be a decomposition of $\mathbb{R}^2$ into 1-smooth curves. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, let $f_\eta f \cap S_\eta$.

To see that $f_\eta$ is differentiable on its domain, let $x$ be in the domain of $f_\eta$. Let $m$ be the slope of the tangent line to $S_\eta$ at the point $(x, f_\eta(x))$ allowing the possibility that $m = \infty$. Then, using the fact that $f$ is continuous, it is easily verified that $f_\eta'(x) = m$.

To show that $\text{dec}(C, D) \geq \text{cov}(D_{\mathcal{E}, \infty})$ let $A \subseteq \mathbb{R}$ be a perfect set and $\theta \in (0, \pi/4)$ be such that if $a$ and $a'$ are distinct points in $A \times A$ then angle formed by the horizontal axis and the line connecting $a$ and $a'$ is different from $\theta$. Let $H_\theta$ be the function which projects $\mathbb{R}^2$ to the vertical axis along the line at angle $\theta$ with respect to the horizontal axis. Let $H_{\pi/2} : \mathbb{R} \to \mathbb{R}$ be the orthogonal projection onto the horizontal axis. Since $H_\theta \upharpoonright (A \times A)$ is one-to-one and $A \times A$ is compact, it follows that $H_\theta^{-1}$ is continuous and, hence, so is $H = H_{\pi/2} \circ H_\theta^{-1}$. Since the domain of $H$ is compact, it can be extended to a continuous function on the entire real line.

Now suppose that $\{X_\eta\}_{\eta \in \kappa}$ are subsets of $\mathbb{R}$ such that $H \upharpoonright X_\eta$ is differentiable for each $\eta \in \kappa$. To see that $H_\theta^{-1}(X_\eta)$ is a 1-smooth curve, suppose that $(x, y) \in H_\theta^{-1}(X_\eta)$ is a point at which $H_\theta^{-1}(X_\eta)$ does not have a 1-dimensional tangent. In this case it is possible to find distinct slopes $m_0$ and $m_1$ and sequences $\{(x_n^i, y_n^i)\}_{n \in \omega}$ for each $i \in 2$ such that

$$\lim_{n \to \infty} \frac{y_n^i - y_n^{i'}}{x_n^i - x_n^{i'}} = m_i$$

allowing the possibility of an infinite limit. Let $w_n^i = H_\theta^{-1}(x_n^i, y_n^i)$ and $w = H_\theta^{-1}(x, y)$. It follows from the linearity of $H_\theta$ that there are distinct $M_0$ and $M_1$ such that

$$\lim_{n \to \infty} \frac{x_n^i - x_n^{i'}}{w_n^i - w_n^{i'}} = M_i$$

and this contradicts the differentiability of $H$ at $w$ since $H(w_n^i) = x_n^i$ and $H(w) = x$.

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