Trapping, anomalous transport and quasi-coherent structures in magnetically confined plasmas

Madalina Vlad and Florin Spineanu
National Institute for Laser, Plasma and Radiation Physics,
Association Euratom-MEdC, P.O.Box MG-36, Magurele, Bucharest, Romania

Abstract

Strong electrostatic turbulence in magnetically confined plasmas is characterized by trapping or eddying of particle trajectories produced by the $E \times B$ stochastic drift. Trapping is shown to produce strong effects on test particles and on test modes. It determines non-standard statistics of trajectories: non-Gaussian distribution, memory effects and coherence. Trapped trajectories form quasi-coherent structure. Trajectory trapping has strong nonlinear effects on the test modes on turbulent plasmas. We determine the growth rate of drift modes as function of the statistical characteristics of the background turbulence. We show that trapping provides the physical mechanism for the inverse cascade observed in drift turbulence and for the zonal flow generation.

Keywords: plasma turbulence, statistical approaches, test particle transport, Lagrangian methods

1 Introduction

A component of particle motion in magnetized plasmas is the stochastic electric drift produced by the electric field of the turbulence and by the confining magnetic field. This drift determines a trapping effect or eddy motion in turbulence with slow time variation [1]. Typical particle trajectories show sequences of trapping events (trajectory winding on almost closed paths) and long jumps. Numerical simulations have shown that the trapping process completely changes the statistical properties of the trajectories. Particle motion in a stochastic potential was extensively studied [2]-[4], but the process of trapping was not described until recently.

New statistical methods were developed [5], [6] that permitted to determine the effects of trapping. These are semi-analytical methods based on a set of deterministic trajectories obtained from the Eulerian correlation of the stochastic velocity. It was shown that trapping determines memory effects, quasi-coherent behavior and non-Gaussian distribution [6]. The trapped trajectories have quasi-coherent behavior and they form structures similar to fluid vortices. The diffusion coefficients decrease due to trapping and their scaling in the parameters of the stochastic field is modified. We have shown that anomalous diffusion appears due to collisions and average flows. A review of the effects of trapping on test particle statistics and on turbulent transport is presented in the first part of this paper.
The effects of trajectory trapping on the nonlinear dynamics of the test modes for the drift turbulence are presented in the second part of the paper. The semi-analytical methods developed for test particles are extended to test mode evolution in a turbulent magnetized plasmas. Test modes are usually studied for modelling wave-wave interaction in turbulent plasmas [7]. A different perspective is developed here by considering test modes on turbulent plasmas. They are described by nonlinear equations with the advection term containing the stochastic $\mathbf{E} \times \mathbf{B}$ drift described by a stochastic field with known statistical characteristics. The growth rate of the test modes is determined as function of these statistical parameters. We develop a Lagrangian approach of the type of that introduced by Dupree [8], [9]. The difference is that in Dupree’s method the stochastic trapping of trajectory was neglected and consequently the results can be applied to quasilinear turbulence. Our method takes into account the trapping and the non-standard statistics of trajectories that it yields and thus it is able to describe the nonlinear effects appearing in strong turbulence.

The paper is organized as follows. The test particle model is presented in Section 2. Section 3 contains a short description of the statistical methods. The nonlinear effects of trajectory trapping on test particle statistics and transport are presented in Section 4. The general physical explanation for the anomalous diffusion regimes appearing in the presence of trajectory trapping and the formation of trajectory structures are discussed in this section. The problem of test modes in turbulent plasmas for the case of drift turbulence is presented in Section 5 where the growth rate and the frequency are determined as function of the statistical characteristics of the turbulence. The complex effects of trajectory trapping on the drift modes are analyzed in Section 6. The conclusions are summarized in Section 7.

2 Test particle model

The test particle studies rely on known statistical characteristics of the stochastic field. They are determined from experimental studies or numerical simulations. The main aim of these studies is to determine the diffusion coefficients. The statistics of test particle trajectories provides the transport coefficients in turbulent plasmas without approaching the very complicated problem of self-consistent turbulence that explains the detailed mechanism of generation and saturation of the turbulent potential. The possible diffusion regimes can be obtained by considering various models for the statistics of the stochastic field.

We consider in slab geometry an electrostatic turbulence represented by an electrostatic potential $\phi^e(x, t)$, where $x \equiv (x_1, x_2)$ are the Cartesian coordinates in the plane perpendicular to the confining magnetic field directed along $z$ axis, $\mathbf{B} = B\mathbf{e}_z$. The test particle motion in the guiding center approximation is determined by

$$\frac{dx(t)}{dt} = v(x, t) \equiv -\nabla \phi(x, t) \times \mathbf{e}_z,$$

where $x(t)$ represent the trajectory of the particle guiding center, $\nabla$ is the gradient in the $(x_1, x_2)$ plane and $\phi(x, t) = \phi^e(x, t)/B$. The electrostatic potential $\phi(x, t)$ is considered to be a stationary and homogeneous Gaussian stochastic field, with zero
average. It is completely determined by the two-point Eulerian correlation function (EC), $E(x, t)$, defined by

$$E(x, t) \equiv \langle \phi(x', t') \phi(x + x', t + t) \rangle. \quad (2)$$

The average $\langle ... \rangle$ is the statistical average over the realizations of $\phi(x, t)$, or the space and time average over $x$ and $t$. This function evidences three parameters that characterize the (isotropic) stochastic field: the amplitude $\Phi = \sqrt{E(0, 0)}$, the correlation time $\tau_c$, which is the decay time of the Eulerian correlation and the correlation length $\lambda_c$, which is the characteristic decay distance. These three parameters combine in a dimensionless Kubo number

$$K = \frac{\tau_c}{\tau_{fl}} \quad (3)$$

where $\tau_{fl} = \lambda_c / V$ is the time of flight of the particles over the correlation length and $V = \Phi / \lambda_c$ is the amplitude of the stochastic velocity.

The diffusion coefficient is determined as (see [10])

$$D_i(t) = \int_0^t d\tau L_{ii}(\tau) \quad (4)$$

where

$$L_{ij}(t; t_1) \equiv \langle v_i(0, 0) v_j(x(t), t) \rangle \quad (5)$$

is the correlation of the Lagrangian velocity (LVC). It is obtained using the decorrelation trajectory method, a semi-analytical approach presented below.

Equation (1) represents the nonlinear kernel of the test particle problem. The statistical methods will be presented for Eq. (1) for simplicity. They were developed to include complex models with other components of the motion (particle collisions, average flows, motion along the confining magnetic field, etc.). The effects of these components on the transport will be discussed in Section 4.

3 The nested subensemble approach

Test particle transport in magnetized plasmas in the nonlinear regime characterized by trajectory trapping was analytically studied only the last decade when the decorrelation trajectory method (DTM) [5] and the nested subensemble approach (NSA) [6] were developed. Trajectory trapping is essentially related to the invariance of the Lagrangian potential. Thus, a statistical method is adequate for the study of this process if it is compatible with the invariance of the potential. The NSA is the development of the DTM as a systematic expansion that obtains much more statistical information.

The main idea in NSA is to study the stochastic equation (1) in subensembles of realizations of the stochastic field. First the whole set of realizations $R$ is separated in subensembles $(S1)$, which contain all realizations with given values of the potential and of the velocity in the starting point of the trajectories $x = 0, t = 0$:

$$(S1) : \quad \phi(0, 0) = \phi^0, \quad v(0, 0) = v^0. \quad (6)$$
Then, each subensemble \((S1)\) is separated in subensembles \((S2)\) corresponding to fixed values of the second derivatives of the potential in \(x = 0, t = 0\)

\[
(S2) : \quad \phi_{ij}(0, 0) \equiv \frac{\partial^2 \phi(x, t)}{\partial x_i \partial x_j} \bigg|_{x=0,t=0} = \phi_{ij}^0 \tag{7}
\]

where \(ij = 11, 12, 22\). Continuing this procedure up to an order \(n\), a system of nested subensembles is constructed. The stochastic (Eulerian) potential and velocity in a subensemble are Gaussian fields but non-stationary and non-homogeneous, with space and time dependent averages and correlations. The correlations are zero in \(x = 0, t = 0\) and increase with the distance and time. The average potential and velocity performed in a subensemble depend on the parameters of that subensemble and of the subensembles that include it. They are determined by the Eulerian correlation of the potential (see [6] for details). The stochastic equation (??) is studied in each highest order subensemble \((Sn)\). The average Eulerian velocity determines an average motion in each \((Sn)\). Neglecting the fluctuations of the trajectories, the average trajectory in \((Sn)\), \(X(t; Sn)\), is obtained from

\[
\frac{dX(t; Sn)}{dt} = \varepsilon_{ij} \frac{\partial \Phi(X; Sn)}{\partial X_j}. \tag{8}
\]

This approximation consists in neglecting the fluctuations of the trajectories in the subensemble \((Sn)\). It is rather good because it is performed in the subensemble \((Sn)\) where the trajectories are similar due to the fact that they are super-determined. Besides the necessary and sufficient initial condition \(x(0) = 0\), they have supplementary initial conditions determined by the definition (6-7) of the subensembles. The strongest condition is the inial potential \(\phi(0, 0) = \phi^0\) that is a conserved quantity in the static case and determines comparable sizes of the trajectories in a subensemble. Moreover, the amplitude of the velocity fluctuations in \((Sn)\), the source of the trajectory fluctuations, is zero in the starting point of the trajectories and reaches the value corresponding to the whole set of realizations only asymptotically. This reduces the differences between the trajectories in \((Sn)\) and thus their fluctuations.

The statistics of trajectories for the whole set of realizations (in particular the LVC) is obtained as weighted averages of these trajectories \(X(t; Sn)\). The weighting factor is the probability that a realization belongs to the subensemble \((Sn)\); it is analytically determined.

Essentially, this method reduces the problem of determining the statistical behavior of the stochastic trajectories to the calculation of weighted averages of some smooth, deterministic trajectories determined from the EC of the stochastic potential. This semi-analytical statistical approach (the nested subensemble method) is a systematic expansion that satisfies at each order \(n > 1\) all statistical conditions required by the invariance of the Lagrangian potential in the static case. The order \(n = 1\) corresponds to the decorrelation trajectory method introduced in [5]. In this case only the average potential is conserved.

The nested subensemble method is quickly convergent. This is a consequence of the fact that the mixing of periodic trajectories, which characterizes this nonlinear stochastic process, is directly described at each order of our approach. The results obtained in first order (the decorrelation trajectory method) for \(D(t)\) are practically
not modified in the second order [6]. Thus, the decorrelation trajectory method
is a good approximation for determining diffusion coefficients. The second order
nested subensemble method is important because it provides detailed statistical
information on trajectories: the probability of the displacements and of the distance
between neighboring trajectories in the whole ensemble of realizations and also in
the subensembles (S1). A high degree of coherence is so evidenced in the stochastic
motion of trapped trajectories.

4 Trapping effects on test particles

4.1 Trajectory structures

Detailed statistical information about particle trajectories was obtained using the
nested subensemble method [6]. This method determines the statistics of the tra-
jectories that start in points with given values of the potential. This permits to
evidence the high degree of coherence of the trapped trajectories.

The trapped trajectories correspond to large absolute values of the initial poten-
tial while the trajectories starting from points with the potential close to zero
perform long displacements before decorrelation. These two types of trajectories
have completely different statistical characteristics [6]. The trapped trajectories
have a quasi-coherent behavior. Their average displacement, dispersion and proba-
bility distribution function saturate in a time $\tau_s$. The time evolution of the square
distance between two trajectories is very slow showing that neighboring particles
have a coherent motion for a long time, much longer than $\tau_s$. They are character-
ized by a strong clump effect with the increase of the average square distance that
is slower than the Richardson law. These trajectories form structures, which are
similar with fluid vortices and represent eddying regions. The size and the built-
up time of the structures depend on the value of the initial potential. Trajectory
structures appear with all sizes, but their characteristic formation time increases
with the size. These structures or eddying regions are permanent in static stochas-
tic potentials. The saturation time $\tau_s$ represents the average time necessary for
the formation of the structure. In time dependent potentials the structures with
$\tau_s > \tau_c$ are destroyed and the corresponding trajectories contribute to the diffusion
process. These free trajectories have a continuously growing average displacement
and dispersion. They have incoherent behavior and the clump effect is absent. The
probability distribution functions for both types of trajectories are non-Gaussian.

The average size of the structures $S(K)$ in a time dependent potential is plotted
in Figure 1. One can see that for $K < 1$ the structures are absent ($S \equiv 0$) and that
they appear for $K > 1$ and continuously grow as $K$ increases. The dependence on
$K$ is a power low with the exponent dependent on the EC of the potential. The
exponent is 0.19 for the Gaussian EC and 0.35 for a large EC that decays as $1/r^2$. 
4.2 Anomalous diffusion regimes

Test particle studies connected with experimental measurements of the statistical properties of the turbulence provide the transport coefficients with the condition that there is space-time scale separation between the fluctuations and the average quantities. Particle density advected by the stochastic $\mathbf{E} \times \mathbf{B}$ drift in turbulent plasmas leads in these conditions to a diffusion equation for the average density with the diffusion coefficient given by the asymptotic value of Eq. (4). Recent numerical simulations [11] confirm a close agreement between the diffusion coefficient obtained from the density flux and the test particle diffusion coefficient. Experiment based studies of test particle transport permit to strongly simplify the complicated self-consistent problem of turbulence and to model the transport coefficients by means of test particle stochastic advection. The running diffusion coefficient $D(t)$ is defined as the time derivative of the mean square displacement of test particles and is determined according to Eq. (4) as the time integral of the Lagrangian velocity correlation (LVC). Thus, test particle approach is based on the evaluation of the LVC for given EC of the fluctuating potential.

The turbulent transport in magnetized plasmas is a strongly nonlinear process. It is characterized by the trapping of the trajectories, which determines a strong influence on the transport coefficient and on the statistical characteristics of the trajectories. The transport induced by the $\mathbf{E} \times \mathbf{B}$ stochastic drift in electrostatic turbulence [12] (including effects of collisions [13], average flows [14], motion along magnetic field [15], effect of magnetic shear [16]) and the transport in magnetic turbulence [17], [18] were studied in a series of papers using the decorrelation trajectory method. It was also shown that a direct transport (an average velocity) appears in turbulent magnetized plasmas due to the inhomogeneity of the magnetic field [19]-[21]. This statistical method was developed for the study of complex processes as the zonal flow generation [22], [23].

The results of all these studies are rather unexpected when the nonlinear effects
are strong. The diffusion coefficients are completely different of those obtained in quasilinear conditions. A rich class of anomalous diffusion regimes is obtained for which the dependence on the parameters is completely different compared to the scaling obtained in quasilinear turbulence. All the components of particle motion (parallel motion, collisions, average flows, etc.) have strong influence on the diffusion coefficients in the non-linear regimes characterized by the presence of trajectory trapping.

The reason for these anomalous transport regimes can be understood by analyzing the shape of the correlation of the Lagrangian velocity for particles moving by the $\mathbf{E} \times \mathbf{B}$ drift in a static potential [24]. In the absence of trapping, the typical LVC for a static field is a function that decay to zero in a time of the order $\tau_{fl} = \lambda_c/V$. This leads to Bohm type asymptotic diffusion coefficients $D_B = V^2 \tau_{fl} = V \lambda_c$. Only a constant $c$ is influenced by the EC of the stochastic field and the diffusion coefficient is $D = cD_B$ for all EC’s. In the case of the $\mathbf{E} \times \mathbf{B}$ drift, a completely different shape of the LVC is obtained for static potentials due to trajectory trapping. A typical example of the LVC is presented in Figure 2. This function decays to zero in a time of the order $\tau_{fl}$ but at later times it becomes negative, it reaches a minimum and then it decays to zero having a long, negative tail. The tail has power law decay with an exponent that depends on the EC of the potential [12]. The positive and negative parts compensate such that the integral of $L(t)$, the running diffusion coefficient $D(t)$, decays to zero. The transport in static potential is thus subdiffusive. The long time tail of the LVC shows that the stochastic trajectories in static potential have a long time memory.

This stochastic process is unstable in the sense that any weak perturbation produces a strong influence on the transport. A perturbation represents a decorrelation mechanism and its strength is characterized by a decorrelation time $\tau_d$. The weak perturbations correspond to long decorrelation times, $\tau_d > \tau_{fl}$. In the absence of trapping, such a weak perturbation does not produce a modification of the diffusion coefficient because the LVC is zero at $t > \tau_{fl}$. In the presence of trapping, which is characterized by long time LVC as in Figure 2, such perturbation influences the tail of the LVC and destroys the equilibrium between the positive and the negative parts. Consequently, the diffusion coefficient is a decreasing function of $\tau_d$. It means that when the decorrelation mechanism becomes stronger ($\tau_d$ decreases) the transport increases. This is a consequence of the fact that the long time LVC is negative. This behavior is completely different of that obtained in stochastic fields that do not produce trapping. In this case, the transport is stable to the weak perturbations. An influence of the decorrelation can appear only when the later is strong such that $\tau_d < \tau_{fl}$ and it determines the increase of the diffusion coefficient with the increase of $\tau_d$. This inverse behavior appearing in the presence of trapping is determined by the fact that a stronger perturbation (with smaller $\tau_d$) liberates a larger number of trajectories, which contribute to the diffusion.
Figure 2: Typical Lagrangian velocity correlation in static potential.

The decorrelation can be produced for instance by the time variation of the stochastic potential, which produces the decay of both Eulerian and Lagrangian correlations after the correlation time $\tau_c$. The decorrelation time in this case is $\tau_c$ and it is usually represented by a dimensionless parameter, the Kubo number defined by Eq. (3). The transport becomes diffusive with an asymptotic diffusion coefficient that scales as $D_{tr} = cV \lambda_{tr} K^\gamma$, with $\gamma$ in the interval $[-1, 0]$ (trapping scaling [12]). The diffusion coefficient is a decreasing function of $\tau_c$ in the nonlinear regime $K > 1$.

For other types of perturbations, their interaction with the trapping process produces more complicated nonlinear effects. For instance, particle collisions lead to the generation of a positive bump on the tail of the LVC [13] due to the property of the 2-dimensional Brownian motion of returning in the already visited places. Other decorrelation mechanisms appearing in plasmas are average component of the velocity like poloidal rotation [14] or the parallel motion that determines decorrelation when the potential has a finite correlation length along the confining magnetic field. The effects of an average component of the velocity are discussed in Section 5.1. in connection with drift turbulence.

5 Test modes on drift turbulence

Test particle trajectories are strongly related to plasma turbulence. The dynamics of the plasma basically results from the Vlasov-Maxwell system of equations, which represents the conservation laws for the distribution functions along particle trajectories. Studies of plasma turbulence based on trajectories were initiated by Dupree [8], [9] and developed especially in the years seventies (see the review paper [7] and references there in). These methods do not account for trajectory trapping and thus they apply to the quasilinear regime or to unmagnetized plasmas. A very important problem that has to be understood is the effect of the non-standard statistical characteristics of the test particle trajectories on the evolution of the instabilities and of turbulence in magnetized plasmas.
We extend the Lagrangian methods of the type of [9], [25], [26] to the nonlinear regime characterized by trapping. We study linear modes on turbulent plasma with the statistical characteristics of the turbulence considered known. The dispersion relation for such test modes is determined as function of the characteristics of the turbulence. We consider the drift instability in slab geometry with constant magnetic field. The combined effect of the parallel motion of electrons (non-adiabatic response) and finite Larmor radius of the ions destabilizes the drift waves.

The gyrokinetic equations are not linearized around the unperturbed state as in the linear theory but around a turbulent state with known spectrum. The perturbations of the electron and ion distribution functions are obtained from the gyrokinetic equation using the characteristics method as integrals along test particle trajectories of the source terms determined by the average density gradient.

The background turbulence produces two modifications of the equation for the linear modes. One consists in the stochastic $E \times B$ drift that appears in the trajectories and the other is the fluctuation of the diamagnetic velocity. Both effects are important for ions while the response of the electrons is approximately the same as in quiescent plasma. They depend on the parameters of the turbulence.

### 5.1 The statistics of the characteristics

The solution for the potential in the zero Larmor radius limit is

$$\phi(x, z, t) = \phi_0(x - V_* t, z),$$  

where $\phi_0$ is the initial condition and $V_*$ is the diamagnetic velocity. This shows that the potential is not changed but displaced with the diamagnetic velocity. The finite Larmor radius effects consist in the modification of the amplitude and of the shape of the potential, but this appears on a much slower time scale.

The ordering of the characteristic times for the drift turbulence is

$$\tau^e || \ll \tau_* \ll \tau_c \ll \tau^i ||,$$

where $\tau^e ||, \tau^i ||$ are the parallel decorrelation times for electrons and ions ($\tau^{e,i}_|| = \lambda_||/v_{th}^{e,i}$ with $\lambda_||$ the parallel correlation length and $v_{th}^{e,i}$ the thermal velocity), $\tau_* = \lambda_e/V_*$ is the characteristic time for the potential drift and $\tau_c$ is the correlation time of the potential. The linear and nonlinear regimes are determined by the position of the time of flight in this ordering. The latter is much smaller than $\tau_c$ and much larger than $\tau^e ||$. The statistical characteristics of the trajectories essentially depend on the ratio $\tau_*/\tau_{fl}$.

The quasilinear case corresponds to $\tau_*/\tau_{fl} \ll 1 (V/V_* \ll 1)$, which means turbulence with the amplitude of the $E \times B$ drift smaller than the diamagnetic velocity. The motion of the potential produces in this case a fast decorrelation and trapping does not appear. The probability of displacements is Gaussian and the diffusion coefficient is $D_{ql} = V^2 \tau_*$.

The nonlinear case corresponds to $\tau_*/\tau_{fl} > 1 (V/V_* > 1)$. The motion of the potential is slow and trajectory structures produced by trapping exist in this case.

The test particle motion in a drifting potential is obtained by a Galilean transformation from the motion produced by a stochastic $E \times B$ drift and an average
velocity $V_d$. This process was studied in [17]. It was shown that strips of opened contour lines of the effective potential $\phi + xV_d$ appear due to an average velocity $V_d$ and that the width of these strips increases with $V_d$ until they completely eliminate the closed contour lines (for $V_d > V$). The Lagrangian correlation of the velocity in the presence of an average velocity $V_d < V$ does not decay to zero as in Figure 2, but it has a positive asymptotic values at $t \to \infty$. Consequently the transport along the average velocity is superdiffusive in the static potential and diffusive with large diffusion coefficient (proportional with the average velocity) in the time dependent case. A part of the particles are trapped and the other move on the strips of opened contour lines of the effective potential. The invariance of the distribution of the Lagrangian velocity shows that the average velocity of the free particles $V'_f$ fulfills the condition

$$n_{fr}V'_f = V_d, \quad (11)$$

and thus it is larger than the average velocity ($V'_f > V_d$). $n_{tr}$ is the fraction of trapped trajectories and $n_{fr}$ is the fraction of free trajectories at a moment ($n_{tr} + n_{fr} = 1$).

This physical image leads, by changing the reference frame, to the following paradigm of the statistics of trajectories produced by the $E \times B$ drift in a moving potential. The trapped particles (structures) are advected by the moving potential while the other particles have an average motion in the opposite direction with a velocity $V_{fr}$ such that

$$n_{fr}V_{fr} + n_{tr}V_s = 0, \quad (12)$$

which is the equivalent of Eq. (11). This shows that there are particle flows in opposite directions induced by the drifting potential if the amplitude of the stochastic $E \times B$ velocity is larger than the velocity of the potential. This determines a splitting of the probability of displacements in two parts: the probability of trapped and the probability of free particles. The first is a picked function that has constant width and moves with the velocity $V_s$. The second, is a Gaussian like function with an average displacement $\langle x_2 \rangle_{fr} = V_{fr}t = -V_s t n_{tr}/n_{fr}$. The probability of displacements at $t < \tau_c$ is modeled by

$$P(x, y, t) = n_{tr}G(x, y - V_s t; S_x, S_y) + n_{fr}G(x, y - V_{fr} t; S_x + 2D_x t, S_y + 2D_y t) \quad (13)$$

where $G(x, y; S_x, S_y)$ is the 2-dimensional Gaussian distribution with dispersion $S_x, S_y$. We have considered for simplicity the distribution of trapped particles as a Gaussian function but with small (fixed) dispersion. The shape of this function does not change much these estimations. The free trajectories have dispersion that grows linearly in time.

5.2 The growth rate of drift modes in turbulent plasma

The average propagator of for a mode with frequency $\omega$ and wave number $k = (k_1, k_2)$ is evaluated using the above results on trajectory statistics. It depends on
the size $S(K)$ of the structures and on the fractions of trapped and free particles:

$$
\int_{-\infty}^{t} d\tau \langle \exp (-i k \cdot x(\tau)) \rangle \exp (i \omega (t - \tau))
= i \exp (-k_i^2 S_i^2) \left[ \frac{n_{tr}}{\omega + k_y V_x} + \frac{n_{fr}}{\omega + k_y V_{fr} + i k_i^2 D_i} \right]
$$

(14)

where $x(\tau)$ is the trajectory in the moving potential integrated backward in time with the condition $x$ at time $t$.

The solution of the dispersion relation is obtained as

$$
\omega = k_y V_*^{\text{eff}}
$$

(15)

$$
V_*^{\text{eff}} = \frac{V_0 \mathcal{F} (n_{fr} - n_{tr}) + 2 n_{tr}}{2 - \Gamma_0 \mathcal{F}}
$$

(16)

$$
\mathcal{F} = \exp \left( -\frac{1}{2} k_i^2 S_i^2 \right)
$$

(17)

$$
\gamma = \frac{\sqrt{\pi}}{|k_z| v_{Te}} \frac{k_y^2 (V_* - V_*^{\text{eff}})}{2 - \Gamma_0 \mathcal{F}} \left( V_*^{\text{eff}} - \frac{n_{tr}}{n_{fr}} V_* \right) - k_i^2 D_i \frac{2 - \Gamma_0 \mathcal{F} n_{tr}}{2 - \Gamma_0 \mathcal{F}} + k_i k_j R_{ij} V_*^{\text{eff}}
$$

(18)

where $\Gamma_0 = \exp (-b) I_0 (b)$, $b = k_2^2 \rho_L^2 / 2$ and $\rho_L$ is the ion Larmor radius. The tensor $R_{ij}$ has the dimension of a length and is defined by

$$
R_{ji}(\tau, t) \equiv \int_{\tau}^{t} d\theta' \int_{-\infty}^{\tau - \theta'} d\theta M_{ji}(|\theta|)
$$

(19)

where $M_{ij}$ is the Lagrangian correlation

$$
M_{ji}(|\theta' - \theta|) \equiv \langle v_j \left( x^i(\theta'), z, \theta' \right) \partial_2 v_i \left( x^i(\theta), z, \theta \right) \rangle,
$$

(20)

and $v_j$ is the $E \times B$ drift velocity.

Several effects appear in the test modes characteristics due to the background turbulence. The spreading of ion trajectories produces the diffusion $D_i$ that influences the growth rate (18) both in linear and nonlinear conditions. This term is similar to the result of Dupree, but the values of $D_i$ is influenced by trapping. Beside this, there are several influences that appear only in the nonlinear regime. The first is the factor $\mathcal{F}$ given by Eq. (17), which is produced by the trajectory structures. It determines essentially the modification of the mode frequency. The flows of the ions induced by the drifting potential are represented by the fractions $n_{tr}$ and $n_{fr}$. The tensor $R_{ij}$ is determined by the fluctuations of the diamagnetic velocity due to the background turbulence. We will analyze each of these processes in the next section.

6 Trapping effects on the test modes

The trajectory trapping has a complex influence on the mode. This can be understood by considering the evolution of the drift turbulence starting from a stochastic
potential with very small amplitude as it can be deduced from the growth rates of the test modes.

The trajectories are Gaussian, there is no trapping in such potential and the only effect of the background turbulence is the diffusion of ion trajectories that produce resonance broadening. The well known results of drift modes in quasilinear turbulence are obtained

\[ \omega = k_y V_s \frac{\Gamma_0}{2 - \Gamma_0}, \quad \gamma = \frac{\sqrt{\pi}}{|k_z| v_{Te}} \frac{(k_y V_s - \omega) (\omega - k_y V_s)}{2 - \Gamma_0} - k_i^2 D_{ql}, \]

(21)

where \( D_x = D_y = D_{ql} = V^2 \lambda_c / V_s \). This shows that the modes with large \( k \) are damped due to ion trajectory diffusion as the amplitude potential increases. The maximum of the spectrum is for \( \omega = k_y V_s / 2 \) and corresponds to \( k_{\perp} \rho_L \sim 1 \).

When the nonlinear stage is attained for \( V > V_* \), the first effect is produced when the fraction of trapped trajectories is still small by the quasi-coherent component of ion motion. The structures of ion trajectories determine the \( F \) factor (17), which modifies the effective diamagnetic frequency (16) and the frequency \( \omega \). At this stage the flows can be neglected \( (n_{tr} \approx 0, n_{fr} \approx 1) \) and \( R_{ji} \approx 0 \) in Eqs. (15)-(18), so that only the factor \( F \) is important. It is interesting to note that this factors appears in Eqs. (15)-(18) as a multiple of \( \Gamma_0 \), although they come from different sources (\( F \) from the propagator and \( \Gamma_0 \) from gyro-average of the mode potential). This shows that the trapping or eddying motion has the same attenuation effect as the gyro-average. The maximum of the spectrum that appears for \( \omega = k_y V_s / 2 \) is obtained for smaller \( k_{\perp} \), of the order of the size of the trajectory structures \( k_{\perp} S \sim 1 \). This means that the unstable range of the wave numbers is displaced toward small values. The maximum growth rate is not changed but displaced at values of the order \( 1 / S \). Consequently, in this stage both the amplitude of the turbulence and its correlation length increase.

At larger amplitude of the background potential, when the fraction or trapped ions becomes comparable with the fraction of free ions, the ion flows induced by the moving potential become important. These flows determine the increase of the effective diamagnetic velocity (16) toward the diamagnetic velocity and the modification of the growth rate of the drift modes. The latter decreases and for \( n_{tr} = n_{fr} \) it is negative. The evolution of the amplitude becomes slower and eventually the growth rates vanishes and changes the sign. Thus, the flows of the ions induced by the moving potential produce the damping of the drift modes.

The fluctuations of the diamagnetic velocity due to background turbulence determine a direct contribution to the growth rate (the tensor \( R_{ij} \)). This term is zero for homogeneous and isotropic turbulence and strongly depends on the parameters of the anisotropy. The \( i = j = 1 \) component corresponds to zonal flows. Preliminary results show that it appears for trapped particles due to the anisotropy induced by ion flows with the moving potential.

7 Summary and conclusions

We have discussed the problem of stochastic advection of test particles by the \( E \times B \) drift in turbulent plasmas. We have shown that trajectory trapping or eddying have
complex nonlinear effects on the statistical characteristics of the trajectories and on the transport. The nonlinear effects are very strong in the case of static potentials. The trajectories are non-Gaussian, there is statistical memory, coherence and they form structures. These properties persist if the system is weakly perturbed by time variation of the potential or by other components of the motion (collisions, poloidal rotation, parallel motion). The memory effect (long tail of the LVC) determines anomalous diffusion regimes.

The process of trajectory trapping also influences the evolution of the turbulence. Recent results on test modes on turbulent plasmas are presented. They are based on a Lagrangian method that takes into account the trapping or eddying of the ions. The growth rate and the frequency of the drift modes on turbulent plasmas are estimated as function of the characteristics of the turbulence. The effects of the background turbulence appear in particle trajectories (characteristics of Vlasov equations) and in the fluctuations of the diamagnetic velocity produced by the density fluctuations. We show that the nonlinear process of trapping, which determines non-standard statistical properties of trajectories, has a very strong and complex influence on the evolution of the turbulence. It appears when the amplitude of the $E \times B$ drift becomes larger than the diamagnetic velocity.

A different physical perspective on the nonlinear evolution of drift waves is obtained. The main role is played by the trapping of the ions in the stochastic potential that moves with the diamagnetic velocity. We show that the moving potential determines flows of the ions when the amplitude of the $E \times B$ velocity is larger than the diamagnetic velocity. A part of the ions are trapped and move with the potential while the other ions drift in the opposite direction. These opposite (zonal) flows compensate such that the average velocity is zero. The evolution of the turbulence toward large wave lengths (the inverse cascade) is determined by ion trapping, which averages the potential and determines a smaller effective diamagnetic velocity. The ion flows produced by the moving potential determine the decay of the growth rate and eventually the damping of the drift modes and generate zonal flows due to their nonlinear interaction with the fluctuations of the diamagnetic velocity.

**References**

[1] R. H. Kraichnan, *Phys. Fluids* **19**, 22 (1970).

[2] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).

[3] G. Falkovich, K. Gawedzki and M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001).

[4] Balescu R., *Aspects of Anomalous Transport in Plasmas*, Institute of Physics Publishing (IoP), Bristol and Philadelphia, 2005.

[5] Vlad M., Spineanu F., Misguich J.H., Balescu R., *Phys.Rev.E* **58** (1998) 7359.

[6] Vlad M. and Spineanu F., *Phys. Rev. E* **70** (2004) 056304.

[7] Krommes J. A., *Phys. Reports* **360** (2002) 1.
[8] Dupree T. H., *Phys. Fluids* 9 (1966) 1773.

[9] Dupree T. H., *Phys. Fluids* 15 (1972) 334.

[10] Taylor G. I., *Proc. London Math. Soc.* 20 (1921) 196.

[11] Basu R., Jessen T., Naulin V., Rasmussen J.J., *Physics of Plasmas* 10 (2003) 2696.

[12] Vlad M., Spineanu F., Misguich J. H., Reusse J.-D., Balescu R., Itoh K., Itoh S.-I., *Plasma Phys. Control. Fusion* 46 (2004) 1051.

[13] Vlad M., Spineanu F., Misguich J. H. and Balescu R., *Phys. Rev. E* 61 (2000) 3023.

[14] Vlad M., Spineanu F., Misguich J. H. and Balescu R., *Phys. Rev. E* 63 (2001) 066304.

[15] Vlad M., Spineanu F., Misguich J. H. and Balescu R., *Nuclear Fusion* 42 (2002) 157.

[16] Petrisor I., Negrea M., Weyssow B., *Physica Scripta* 75 (2007) 1.

[17] Vlad M., Spineanu F., Misguich J.H., Balescu R., *Physical Review E* 67 (2003) 026406.

[18] Negrea M., Petrisor I., Balescu R., *Physical Review E* 70 (2004) 046409.

[19] Vlad M., Spineanu F., Benkadda S., *Phys. Rev. Letters* 96 (2006) 085001.

[20] Vlad M., Spineanu F., Benkadda S., *Phys. Plasmas* 15 (2008) 032306.

[21] Vlad M., Spineanu F., Benkadda S., *Plasma Phys. Control. Fusion* 50 (2008) 065007.

[22] Balescu R., *Physical Review E* 68 (2003) 046409.

[23] Balescu R., Petrisor I., Negrea M., *Plasma Phys. Control. Fusion* 47 (2005) 2145.

[24] Vlad M. and Spineanu F., *Physica Scripta* T107 (2004) 204.

[25] Spineanu F., Vlad M., *Phys. Letters A* 133 (1988) 319.

[26] Vlad M., Spineanu F., Misguich J.H., *Plasma Phys. Control. Fusion* 36 (1994) 95.