A two-dimensional quantum walk with non-Hermitian skin effects

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We construct a two-dimensional, discrete-time quantum walk exhibiting non-Hermitian skin effects under open-boundary conditions. As a confirmation of the non-Hermitian bulk-boundary correspondence, we show that the emergence of topological edge states are consistent with Floquet winding numbers calculated using a non-Bloch band theory invoking time-dependent generalized Brillouin zones. Further, the non-Bloch topological invariants associated with the Floquet bands are captured by a non-Hermitian local Chern marker in real space, defined through local biorthogonal eigen wave functions of the non-unitary Floquet operator. Our work suggests the possibility of directly measuring non-Bloch topological invariants in quantum-walk dynamics, and would stimulate further studies of non-Hermitian Floquet topological phases where skin effects play a key role.

I. INTRODUCTION

Non-Hermitian topological phases arise in open systems with non-Hermitian effective Hamiltonians [1], and can exhibit remarkable properties with no counterparts in Hermitian settings. In this context, one of the most intensively discussed phenomena is the breakdown of conventional bulk-boundary correspondence [2–10], which can be restored through a non-Bloch band theory to account for the localization of nominal bulk eigenstates near boundaries [5–16], known as the non-Hermitian skin effects. So far, non-Hermitian skin effects and the corresponding non-Hermitian bulk-boundary correspondence have been experimentally observed in topoelectric circuits [17, 18], metamaterials [19], or photons [20, 21]. These experiments explore the sensitivity of eigen-energy spectra to boundary conditions, the localization of bulk wave functions near boundaries, or the correspondence between topological edge states with non-Bloch topological invariants, but a direct measurement of non-Bloch topological invariants is still elusive. Nor is there an experimental demonstration of non-Hermitian skin effects in quantum mechanical systems of higher dimensions, for want of readily accessible schemes.

In this work, we propose a two-dimensional, discrete-time quantum walk which features non-Hermitian skin effects and is amenable to existing control protocols on quantum simulation platforms such as photons and cold atoms. An exemplary Floquet system, discrete-time quantum walks under appropriate design acquire topological properties [22–34], characterized by a pair of Floquet winding numbers, which, in two dimensions, are intimately connected to Chern numbers of the corresponding Floquet bands [35]. For our proposed quantum walk, we find that the topological invariants capable of characterizing topological edge states should be calculated by a non-Bloch extension of the Floquet winding numbers, defined on a generalized Brillouin zone that is time-dependent within one driving period. More important, we show that non-Bloch topological invariants of the system can be revealed through a non-Hermitian local Chern marker in real space, which can be translated to the evaluation of single-particle matrices in real space and hence detectable through quasi-local measurements. Our study offers the interesting prospect of observing non-Hermitian skin effects and non-Bloch topological invariants in higher-dimensional non-Hermitian topological systems, and enriches the understanding of non-Hermitian Floquet topological phases.

The work is organized as follows. In Sec. II, we discuss the construction of the quantum walk and analyze
its quasienergy spectrum. In Sec. III, we show how bulk-boundary correspondence is preserved by introducing non-Bloch Floquet winding numbers and non-Bloch Chern numbers. A non-Hermitian local Chern marker is defined and discussed in Sec. IV, which agrees with the non-Bloch Chern number of the relevant Floquet band. Finally, we summarize in Sec. V.

II. MODEL AND QUASI-ENERGY SPECTRUM

We consider a discrete-time, non-unitary quantum walk on a bipartite square lattice, the Floquet operator of which being

$$U = T_4(\theta_1)T_3(-\theta_2)M(\gamma)T_2(-\theta_2)T_1(\theta_1 - \frac{\pi}{2}),$$  \hspace{1cm} (1)

with $T_j(\theta) = e^{i\theta \sigma_j}$ ($j = 1, 2, 3, 4$), imposing rotations in the basis of adjacent sublattice sites $\{|A\}, |B\}$ according to the sequence illustrated in Fig. 1. Here $\sigma_i$ ($i = x, y, z$) are the Pauli matrices. Non-unitarity is introduced through the on-site gain-loss operator $M = e^{\gamma \sigma_z}$, with $\gamma$ the gain-loss parameter.

Based on recent experimental progress of topological quantum walks with photons [28–34] or cold atoms [36], such a design is accessible under the flexible control of these systems. This is particularly so with single photons, where one-dimensional topological quantum walks with non-Hermitian skin effects have recently been implemented [20]. As we demonstrate below, the Floquet operator Eq. (1) drives a non-unitary quantum walk with non-Hermitian skin effects, but to characterize Floquet winding numbers that account for topological edge states, a generalized, time-dependent Brillouin zone would be needed, in sharp contrast to previous studies.

A prominent feature of non-Hermitian systems with skin effects is the sensitive dependence of the energy spectrum on boundary conditions. We show in Fig. 2 the quasienergy spectra $E$, associated with the Floquet Hamiltonian $H_F$ defined through $U = e^{-i H_F}$ (with a branch cut at $E = \pi$), where two distinct boundary conditions are considered: periodic boundary condition along both $x$ and $y$ directions [Fig. 2(a)(b)(c)(d)(e)] labeled as PBC; open boundary condition in the $x$ direction but periodic along $y$ [Fig. 2(f)(g)(h)(i)(j)], labeled as OBC. Typical of non-Hermitian Floquet topological phases, two quasienergy band gaps near $\text{Re} E = 0$ and $\text{Re} E = \pi$ are identified on the complex plane where edge states can appear under OBC. With changing parameters, these band gaps can close [Fig. 2(a)(g) at $\text{Re} E = 0$] and open up again [Fig. 2(e)(i)(j)], but gapless regimes generally exist [Fig. 2(b)(c)(d)(h)], which is a common feature for many two-dimensional non-Hermitian topological systems [6]. From the way the band gaps close and open, we identify these as line gaps according to the definition in Ref. [37]. Of particular importance, the gap closing point occurs at distinct parameters under different boundary conditions [Fig. 2(a)(h)], suggesting the breakdown of conventional bulk-boundary correspondence. This is more explicitly illustrated in Fig. 2(d)(i), where the system under PBC is gapless near $\text{Re} E = 0$, defying the definition of topological invariants, whereas the same gap is open and a pair of in-gap edge states appear under OBC.

To explore the mismatch in quasienergy spectrum under different boundary conditions, in Fig. 3(a), we show the gap-closing points and gapless regions under both

![Figure 2: Typical energy spectra of the system under PBC (a)(b)(c)(d)(e) and OBC (f)(g)(h)(i)(j), respectively, with the parameters: (a)(f) $\theta_1 = 0.78\pi$, (b)(g) $\theta_1 = 0.806\pi$, (c)(h) $\theta_1 = 0.83\pi$, (d)(i) $\theta_1 = 0.895\pi$, (e)(j) $\theta_1 = 0.92\pi$. For all subplots, we take $\theta_2 = 1.59\pi$ and $\gamma = 0.4$.](image-url)
boundary conditions in the parameter space of $\gamma$ and $\theta_1$. Apparently, the gapless region is larger under the OBC, and the mismatch of gap closing points under different boundary conditions increases with larger non-Hermiticity. Such a behavior is accompanied by the non-Hermitian skin effects [see Fig. 3(b)], with most eigenstates localized at the boundaries, which, according to the non-Bloch band theory, induce the breakdown of Hermitian bulk-boundary correspondence and give rise to the sensitivity of gap-closing parameters with respect to boundary conditions. A natural question then is whether the non-Bloch band theory should restore the bulk-boundary correspondence in our Floquet dynamics.

III. NON-BLOCH FLOQUET WINDING NUMBER

As discussed in Ref. [35], in a two-dimensional Floquet topological system, the bulk-boundary correspondence is governed by the Floquet winding numbers, which are related to Chern numbers of different Floquet bands in a straightforward manner. Here, we show that a non-Hermitian bulk-boundary correspondence can be established through the introduction of non-Bloch Floquet winding numbers, defined over the generalized Brillouin zone, conceptually similar to the static case.

To define the Floquet winding number, we first rewrite the Floquet operator in momentum space, in terms of a time-dependent effective Hamiltonian $H(k, t)$

$$U(k) = T e^{-i \int_0^T H(k,t')dt'},$$  \hspace{1cm} (2)

where $T$ is the time-ordering operator. The formally complicated $H(k, t)$ can be constructed in a stroboscopic fashion, by dividing each Floquet driving period (taken as unit time) into five steps (see Appendix). A time-period operator $U_\epsilon(k, t)$ ($\epsilon \in \{0, \pi\}$) is then introduced [35]

$$U_\epsilon(k, t) = \begin{cases} T e^{-i \int_0^T H(k, 2t')dt'}, & 0 \leq t < \frac{1}{2} \\ V_\epsilon(k, 2 - 2t), & \frac{1}{2} \leq t \leq 1 \end{cases}, \hspace{1cm} (3)$$

where $V_\epsilon(k, t) = e^{-i H_{\text{eff}}(k)t}$, with $H_{\text{eff}}(k) = i \ln U(k)$. The subscript $\epsilon$ indicates a branch cut at $\epsilon$ is taken when evaluating $\ln$. The Floquet winding number is then defined as [35]

$$W_\epsilon = \frac{1}{8\pi^2} \int dtdk_xdk_y Tr \left[ U_\epsilon^{-1} \partial_\epsilon U_\epsilon U_\epsilon^{-1} \partial_k U_\epsilon, U_\epsilon^{-1} \partial_{k_y} U_\epsilon \right]. \hspace{1cm} (4)$$

For completeness, we also define the Chern number of a given Floquet band

$$C = \frac{1}{2\pi i} \int dk_xdk_y Tr \left( \hat{P} \left[ \partial_{k_y}, \hat{P}, \partial_{k_y}, \hat{P} \right] \right), \hspace{1cm} (5)$$

where the operator $\hat{P} = \sum_n |\psi_{n,R}\rangle \langle \psi_{n,L}|$ is the projection onto a given Floquet band with $\psi_{n,L,R}$ the $n$th left (right) eigenstates of the band. Specifically, $U|\psi_{n,R}\rangle = \lambda_n|\psi_{n,R}\rangle$ and $U^\dagger|\psi_{n,L}\rangle = \lambda_n^*|\psi_{n,L}\rangle$. Here $\lambda_n$ is the $n$th eigenvalue of $U$.

For a unitary quantum walk with $\gamma = 0$, the winding number $W_0$ ($W_\pi$) dictates the number of edge states on the given edge and within the gap at $\epsilon$, according to the bulk-boundary correspondence of a two-dimensional Floquet system. The difference between these winding numbers corresponds to the Chern number of the Floquet band between the relevant gaps. For instance, $W_0 - W_\pi$ ($W_\pi - W_0$) corresponds to the Chern number of the left (right) Floquet band in our system [35], according to layout of Fig. 2.

In the non-unitary case with finite $\gamma$, topological edge states appearing on the boundaries not always have a correspondence in Floquet winding numbers calculated under the PBC. Specifically, as shown in Fig. 2(d)(i), when the system is gapless near $\epsilon (\epsilon = 0, \pi)$ under the PBC, imposing an OBC can open up the same gap, within which topological edge states emerge, indicating the breakdown of the conventional bulk-boundary correspondence.

To account for these boundary-dependent topological edge states and restore the bulk-boundary correspondence, we resort to the non-Bloch band theory, where non-Bloch topological invariants are evaluated over a generalized Brillouin zone, based on the non-Bloch nature of the bulk eigen wave functions under the OBC. For the strip-geometry considered here, the Bloch phase factor $e^{i\epsilon k_x x}$ of the bulk eigen wave functions along the $x$ direction is replaced by $\beta(p_x, k_y, t) := |\beta(p_x, k_y, t)| e^{ip_x}$, where $p_x$ is a phase parameter. The time dependence of $\beta(p_x, k_y, t)$ derives from the time-period operator $U_\epsilon(k, t)$, and is directly related to the micromotion of the Floquet dynamics. This is in sharp contrast to the one-dimensional quantum walk in Ref. [20], where the presence of chiral symmetry enables a simplified characterization of the Floquet winding number with time-independent $\beta$ [38].

For our case, at any given time $t$ within one driving period, when the parameters $(p_x, k_y)$ vary, the allowed values of $\beta(p_x, k_y, t)$, dictated by the quasi-energy spectrum through the eigen equations of $U_\epsilon$, form a closed trajectory on the complex plane, representing the generalized Brillouin zone at time $t$ [see Fig. 3(c) and Appendix]. The non-Bloch Floquet winding numbers $\tilde{W}_\epsilon$ are then evaluated by making the substitution $(k_x, k_y) \rightarrow (p_x - i \ln(\beta(p_x, k_y, t))$, $k_y)$ in Eq. (4), where the time dependence is to be integrated over one period.

We show the calculated non-Bloch Floquet winding numbers in Fig. 3(d), where the light (dark) shaded region indicates gapless quasi-energy spectrum at Re$E = 0$ under PBC (both PBC and OBC). While $W_\pi = \tilde{W}_\pi = -1$ under all parameters shown in Fig. 3(d), the non-Bloch winding number $\tilde{W}_0$ takes quantized values only in gapped regions under the OBC. Most important, the non-Bloch winding number $\tilde{W}_0$ is quantized and correctly predicts the presence or absence of topological edge states in the gap near Re$E = 0$ [see Figs. 2 and 3(d)], thus
FIG. 3. (a) Gapless regions (for the gap near Re$E = 0$) in parameter space spanned by $\gamma$ and $\theta_1$. The light (dark) shaded region denote gapless regions under PBC (both PBC and OBC). The five black dots indicate the parameters used for different columns in Fig. 2, from left to right in the same sequential order. (b) Normalized spatial probability distributions, $p_l(m, k_y) = \sum_{s=A,B} |\langle m, s|\Psi_{n,R}\rangle k_y|^2$, for eigen states of $U$ under OBC, with $\theta_1 = 0.7\pi$, $k_y = 0.45\pi$ and varying $\gamma$. Here $|m, s\rangle_{k_y}$ indicates the sublattice state $s$ of the $m$th unit cell along the $x$ direction. $|\Psi_{n,R}\rangle k_y$ is the $n$th right eigenvector of $U$ under OBC for a given $k_y$. For our calculation, we take 76 unit cells (labeled by $m$) along the $x$ direction. (c) Generalized Brillouin zones, characterized by $\beta$ on the complex plane, at different times within one driving period, with $\theta_1 = 0.78\pi$, $k_y = 0.45\pi$ and $\gamma = 0.4$. (d) Non-Bloch Floquet winding number $W_0$ and Bloch winding number $W_0$ for $\gamma = 0.4$. The green solid (red dashed) line shows the calculated non-Bloch (Bloch) winding number $W_0$ ($W_0$), and the shaded regions denote gapless regions similar to (b). For all cases, we fix $\theta_2 = 1.59\pi$, where the band gap near Re$E = \pi$ remains open with $W_0 = -1$.

restoring the bulk-boundary correspondence. We find that edge states in both band gaps are chiral, propagating in a counterclockwise fashion along the boundary, similar to the Hermitian case.

Furthermore, by introducing the non-Bloch Chern number, defined by replacing $(k_x, k_y)$ in Eq. (5) with $(p_x - i n_c (|\beta(p_x, k_y, t)|), k_y)$, the relation between non-Bloch Floquet winding numbers and the non-Bloch Chern numbers of Floquet bands remains the same as that in the Hermitian case. For instance, in Fig. 2(i)(j), $\tilde{W}_0 = -1$ and $\tilde{W}_\pi = -1$, leading to vanishing non-Bloch Chern numbers for both Floquet bands. Nevertheless, topological edge states emerge in both quasienergy gaps, dictated by the non-Bloch Floquet winding number. We note that while the non-Bloch Floquet winding numbers $\tilde{W}_0$ is well-defined so long as the band gap at Re$E = \epsilon$ remains open, the non-Bloch Chern numbers are only well-defined when both gaps are open.

IV. LOCAL CHERN MARKER

While topological invariants are considered global characters of the system, local topological markers have been identified recently, both for Hermitian [39–42] and non-Hermitian [43] topological systems, which can distinguish different topological phases through quasi-local probes in real space. Based on the Hermitian construction in Ref. [40], we adopt the following local Chern marker for our two-dimensional quantum-walk dynamics

\[
c(m) = -\frac{4\pi}{A_c} \text{Im} \sum_{s=A,B} \langle r_{m,s}| \hat{P} \hat{x} \hat{y} \hat{g} \hat{P}| r_{m,s} \rangle,
\]

where $A_c$ is the area of a unit cell in real space [see Fig. 1], $|r_{m,s}\rangle$ labels the sublattice state $s$ in the $m$th unit cell, $Q = 1 - \hat{P}$, and $\hat{x}$ and $\hat{y}$ are the position operators. Eq. 6 extends the previous definition in Ref. [40]
to non-Hermitian settings by considering a biorthogonal construction, but is distinct from the local marker in Ref. [43], which is also defined for a non-Hermitian topological system with skin effects.

Under OBC along the $x$ direction, the local Chern marker should be a function of position in the $x$ direction, which is shown in Fig. 4 under fixed parameters with $W_0 = 0$ and $W_\pi = -1$. The calculated local Chern marker is $\sim 1$ sufficiently away from the boundaries, consistent with the non-Bloch Chern number of the corresponding Floquet band. Deviations are observed close to the boundary, similar to the behavior of local Chern marker in a Hermitian topological system [40]. We then show the variation of the non-Hermitian local Chern number across the topological phase transition [see Fig. 4]. Here, the Chern marker is quantized to the non-Bloch Chern number calculated with Eq. (5), provided the quasienergy gap remains open.

Similar to the Hermitian case, we find that the projection operator $\hat{P}$ is exponentially localized in real space when both band gaps are open and the Chern number of the corresponding Floquet band is well-defined. This makes the local Chern marker in Eq. (6) a quasi-local probe in real space, which can in principle be detected by measuring the matrix elements of single-particle density matrix [40].

V. CONCLUSION

We show that the non-Bloch band theory is crucial in establishing non-Hermitian bulk-boundary correspondence for two-dimensional, discrete-time quantum walks. The resulting non-Bloch Floquet winding numbers, defined over the generalize Brillouin zone, correctly predict the emergence of topological edge states in the quasienergy gaps, and are related to non-Bloch Chern numbers of the Floquet bands. A non-Hermitian local Chern marker is introduced to characterize non-Bloch Chern numbers in real space, whose quasi-local nature offers the possibility of the direct detection of non-Bloch topological invariants in future experiments. Our results should be applicable to general non-Hermitian Floquet topological systems.

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Appendix A: The effective Hamiltonian in momentum space

The explicit form of the time-dependent, effective Hamiltonian $H(k, t)$ in Eq. (2) is constructed as the following

$$H(k, t) = \left( \begin{array}{cc} 0 & b(t)e^{-ik_x} + d(t)e^{-ik_y} \\ a(t)e^{-ik_x} + c(t)e^{-ik_y} & 0 \end{array} \right) + H.c. + \gamma(t) \begin{bmatrix} 0 & 0 \\ 0 & -\gamma(t) \end{bmatrix},$$

with the step-wise, time-dependent parameters

$$a(t) = i(\frac{\pi}{2} - \theta_2)G(t - \frac{1}{10}),$$
$$b(t) = -i\theta_2G(t - \frac{3}{10}),$$
$$\gamma(t) = i\gammaG(t - \frac{5}{10}),$$
$$c(t) = i\theta_2G(t - \frac{7}{10}),$$
$$d(t) = i\theta_1G(t - \frac{9}{10}),$$
$$G(t) = 5\Theta(t + \frac{1}{10})\Theta(\frac{1}{10} - t).$$

Here $\Theta(t)$ is the Heaviside step function. The time-dependent coefficients divide one Floquet period into five segments, each implementing a gate operation.
Appendix B: Generalized Brillouin zone

We now outline the recipe for calculating the generalized Brillouin zone, which is encoded in $\beta(p_x, k_y, t)$ under open boundary conditions. Following Eq. (3) in the main text, we have

$$U_x(k, t) = \begin{cases} e^{-iH_1(k)10t}, & 0 \leq t \leq \frac{1}{10} \\ e^{-iH_2(k)(10-\frac{1}{10})}e^{-iH_1(k)}, & \frac{1}{10} < t \leq \frac{2}{10} \\ e^{-im(k)(10-\frac{2}{10})}e^{-iH_2(k)e^{-iH_1(k)}}, & \frac{2}{10} < t \leq \frac{3}{10} \\ e^{-iH_3(k)(10-\frac{3}{10})}e^{-im(k)}e^{-iH_2(k)}e^{-iH_1(k)}, & \frac{3}{10} < t \leq \frac{4}{10} \\ e^{-iH_4(k)(10-\frac{4}{10})}e^{-iH_3(k)}e^{-im}e^{-iH_2(k)}e^{-iH_1(k)}, & \frac{4}{10} < t \leq \frac{5}{10} \\ e^{-iH_x^m(k)(2-2t)}, & \frac{1}{2} < t \leq 1 \end{cases}$$

(B1)

where

$$H_1(k) = \left(\frac{\pi}{2} - \theta_1\right)\begin{pmatrix} 0 & -ie^{ik_y} \\ ie^{-ik_x} & 0 \end{pmatrix},$$

$$H_2(k) = -\theta_2\begin{pmatrix} 0 & ie^{-ik_x} \\ -ie^{ik_x} & 0 \end{pmatrix},$$

$$m(k) = i\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$H_3(k) = \theta_2\begin{pmatrix} 0 & -ie^{ik_y} \\ ie^{-ik_y} & 0 \end{pmatrix},$$

$$H_4(k) = \theta_1\begin{pmatrix} 0 & ie^{-ik_y} \\ -ie^{ik_y} & 0 \end{pmatrix}. $$

(B2)  

(B3)  

(B4)  

(B5)  

(B6)

The step-wise $H(k, t)$ and $U_x(k, t)$ lead to a time-dependent $\beta(p_x, k_y, t)$ that is also step-wise in its functional form. As an example, we show the calculation $\beta$ for $t \in (\frac{2}{5}, \frac{1}{2})$.

With open boundaries in the $x$ direction, $k_y$ is still a good quantum number, and system is reduced to a one-dimensional quantum walk with an additional parameter $k_y$. Fourier transforming Eq. (B1) over $k_x$, we find the time-period operator for the reduced one-dimensional quantum walk within the time interval $t \in (\frac{2}{5}, \frac{1}{2})$

$$U_x(k_y, t) = \sum_m |m, A\rangle \langle m, A| \otimes \bar{\bar{A}} + |m, B\rangle \langle m, B| \otimes \bar{\bar{B}} +$$

$$|m, A\rangle \langle m, B| \otimes \bar{\bar{C}} + |m, B\rangle \langle m, A| \otimes \bar{\bar{D}} +$$

$$|m, B\rangle \langle m + 1, A| \otimes \bar{\bar{E}} + |m + 1, A\rangle \langle m, B| \otimes \bar{\bar{F}} +$$

$$|m, A\rangle \langle m + 1, A| \otimes \bar{\bar{G}} + |m + 1, A\rangle \langle m, A| \otimes \bar{\bar{H}} +$$

$$|m - 1, B\rangle \langle m, B| \otimes \bar{\bar{J}} + |m, B\rangle \langle m - 1, B| \otimes \bar{\bar{K}},$$

(B7)  

(B8)  

(B9)  

(B10)  

(B11)

where $|m, A\rangle$ labels sublattice site $A$ of the $m$ the unit cell along $x$, and

$$\bar{\bar{A}} = \sin \theta_1 \cos \theta_2 A_1 A_2 M_1,$$

$$\bar{\bar{B}} = \sin \theta_1 \cos \theta_2 B_1 B_2 M_2,$$

$$\bar{\bar{C}} = A_1 A_2 M_1 (-\cos \theta_1 \cos \theta_2 P_1 + \sin \theta_1 \sin \theta_2 P_0),$$

$$\bar{\bar{D}} = B_1 B_2 M_2 (\cos \theta_1 \cos \theta_2 P_1 - \sin \theta_1 \sin \theta_2 P_0),$$

$$\bar{\bar{E}} = B_1 B_2 M_2 (-\cos \theta_1 \cos \theta_2 P_0 + \sin \theta_1 \sin \theta_2 P_1),$$

$$\bar{\bar{F}} = A_1 A_2 M_1 (\cos \theta_1 \cos \theta_2 P_0 - \sin \theta_1 \sin \theta_2 P_1),$$

$$\bar{\bar{G}} = A_1 A_2 M_1 (-\cos \theta_1 \sin \theta_2 P_0),$$

$$\bar{\bar{H}} = A_1 A_2 M_1 (-\cos \theta_1 \sin \theta_2 P_1),$$

$$\bar{\bar{J}} = B_1 B_2 M_2 (-\cos \theta_1 \sin \theta_2 P_1),$$

$$\bar{\bar{K}} = B_1 B_2 M_2 (-\cos \theta_1 \sin \theta_2 P_0),$$

(B12)  

(B13)  

(B14)  

(B15)  

(B16)  

(B17)  

(B18)  

(B19)  

(B20)  

(B21)  

(B22)
with
\[ M_1 = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, \quad M_2 = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ A_1 = \begin{pmatrix} \cos \theta'_1 & \sin \theta'_1 e^{-iky} \\ -\sin \theta'_1 & \cos \theta'_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \cos \theta'_1 & -\sin \theta'_1 e^{iky} \\ \sin \theta'_1 & \cos \theta'_1 \end{pmatrix}, \]
\[ A_2 = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 e^{-iky} \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \cos \theta_2 & \sin \theta_2 e^{iky} \\ -\sin \theta_2 & \cos \theta_2 \end{pmatrix}. \] (B23)

Here $\theta'_1 = (10t - 4)\theta_1$, for $t \in \left(\frac{2}{5}, \frac{1}{2}\right]$.

Assuming a bulk state ansatz
\[ |\psi\rangle = \sum_m \beta^{-2m}(|2m\rangle \otimes |\phi\rangle + \beta^{-1}|2m + 1\rangle \otimes \sigma_x|\phi\rangle), \] (B24)
we have
\[ U_x(k_y, t)|\psi\rangle = \sum_m \beta^{-2m}(|2m\rangle \otimes (\tilde{A} + \beta^{-1}\tilde{C}\sigma_x) + |2m + 1\rangle \otimes (\tilde{D} + \beta^{-1}\tilde{B}\sigma_x) + |2m + 2\rangle \otimes (\tilde{H} + \beta^{-1}\tilde{F}\sigma_x)) \] (B25)
\[ + |2m + 3\rangle \otimes (\beta^{-1}\tilde{K}\sigma_x) + |2m - 1\rangle \otimes (\tilde{E} + \beta^{-1}\tilde{J}\sigma_x) + |2m - 2\rangle \otimes \tilde{G})|\phi\rangle \] (B26)
\[ = \lambda \sum_m \beta^{-2m}(|2m\rangle \otimes |\phi\rangle + \beta^{-1}|2m + 1\rangle \otimes \sigma_x|\phi\rangle). \] (B27)

Non-trivial solution of the eigen equation above exists only if
\[ \det(\tilde{A} - \lambda + \beta^{-1}\tilde{C}\sigma_x + \beta^{-2}\tilde{G} + \beta\tilde{F}\sigma_x + \beta^2\tilde{H}) = 0. \] (B28)

At any given instant within the time range $\left(\frac{2}{5}, \frac{1}{2}\right]$, we numerically solve the eigen spectrum $\lambda$ of an open chain, from which we get four non-zero solutions $\beta$ using Eq. (B28). We keep two solutions that satisfy $|\beta_1| = |\beta_2|$, and use the solution to fix the generalized Brillouin zone.

The calculation of $\beta$ in other time intervals are similar. In particular, for $t \in \left(\frac{1}{2}, \frac{3}{2}\right]$, $\beta$ is independent of time.

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[1] H. J. Carmichael, Phys. Rev. Lett. 70, 2273 (1993).
[2] T. E. Lee, Phys. Rev. Lett. 16, 133903 (2016).
[3] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018).
[4] A. McDonald, T. Peregrine, and A. C. Clerk, Phys. Rev. X 8, 041031 (2018).
[5] S. Yao and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
[6] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018).
[7] K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019).
[8] C. H. Lee and R. Thomale, Phys. Rev. B 99, 201103(R) (2019).
[9] K. Zhang, Z. Yang, and C. Fang, arXiv:1910.01131 (2019).
[10] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. 124, 086810 (2020).
[11] X.-Z. Zhang and J.-B. Gong, Phys. Rev. B 101, 045415 (2020).
[12] Q.-B. Zeng, Y.-B. Yang and Y. Xu, Phys. Rev. B 101, 020201(R) (2020).
[13] N. Okuma, K. Kawabata, K. Shiozaki and M. Sato, Phys. Rev. Lett. 124, 086801 (2020).
[14] X.-R. Wang, C.-X. Guo and S.-P. Kou, Phys. Rev. B 101, 121116(R) (2020).
[15] C.-H. Lee, and R. Thomale, Phys. Rev. B 99, 201103(R) (2019).
[16] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. 123, 170401 (2019).
[17] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, Klessling, T., L. W. Molenkamp, L. C. H., A. Szameit, M. Greiter, and R. Thomale, arXiv:1907.11562 (2019).
[18] T. Hofmann, T. Helbig, F. Schindler, N. Salgo, M. Brzezińska, M. Greiter, T. Klessling, D. Wolf, A. Vollhardt, A. Kabaši, C. H. Lee, A. Bilušić, R. Thomale, T. Neupert, arXiv:1908.02759 (2019).
[19] A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, arXiv:1907.16119 (2019).
[20] L. Xiao, T.-S. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, Nat. Phys. (2020).
[21] S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, and A. Szameit, Science 368, 311 (2020).
[22] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler, Phys. Rev. A 82, 033429 (2010).
[23] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys.
Rev. B 82, 235114 (2010).
[24] J. Cayssol, B. Dora, F. Simon, and R. Moessner, Phys. Status Solidi RRL 7, 101 (2013).
[25] S. Barkhofen, T. Nitsche, F. Elster, L. Lorz, A. Gbris, I. Jex, and C. Silberhorn Phys. Rev. A 96, 033846 (2017).
[26] J. K. Ashth and H. Obuse, Phy. Rev. B 88, 121406(R) (2013).
[27] M. A. Broome, A. Fedrizzi, B. P. Lanyon, I. Kassal, A. Aspuru-Guzik, and A. G. White, Phys. Rev. Lett. 104, 153602 (2010).
[28] T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, Nat. Commun. 3, 882 (2012).
[29] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. 115, 040402 (2015).
[30] X. Zhan, L. Xiao, Z. Bian, K. Wang, X. Qiu, B. C. Sanders, W. Yi, and P. Xue, Phys. Rev. Lett. 119, 130501 (2017).
[31] L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nat. Phys. 13, 1117 (2017).
[32] C. Chen, X. Ding, J. Qin, Y. He, Y.-H. Luo, M.-C. Chen, C. Liu, X.-L. Wang, W.-J. Zhang, H. Li, L.-X. You, Z. Wang, D.-W. Wang, B. C. Sanders, C.-Y. Lu and J.-W. Pan, Phys. Rev. Lett. 121, 100502 (2018).
[33] F. Cardano, M. Maffei, F. Massa, B. Piccirillo, C. de Lisio, G. De Filippis, V. Cataudella, E. Santamato, and L. Marrucci, Nat. Commun. 7, 11439 (2016).
[34] F. Cardano, A. D’Errico, A. Dauphin, M. Maffei, B. Piccirillo, C. de Lisio, G. De Filippis, V. Cataudella, E. Santamato, L. Marrucci, M. Lewenstein, and P. Massignan, Nat. Commun. 8, 15516 (2017).
[35] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X 3, 031005 (2013).
[36] D. Xie, T.-S. Deng, T. Xiao, W. Gou, T. Chen, W. Yi, and B. Yan, Phys. Rev. Lett. 124, 050502 (2020).
[37] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X 9, 041015 (2019).
[38] S. Yao, Z. Yan, and Z. Wang, Phys. Rev. B 96, 195303 (2017).
[39] R. Bianco and R. Resta, Phys. Rev. B 84, 241106 (2011).
[40] M. D. Caio, G. Möller, N. R. Cooper, M. J. Bhaseen, Nat. Phys. 15, 257 (2019).
[41] L. Privitera and G. E. Santoro, Phy. Rev. B 93, 241406(R) (2016).
[42] O. Pozo, C. Repellin, and A. G. Grushin, Phys. Rev. Lett. 123, 247401 (2019).
[43] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. 123, 246801 (2019).