Linear combinations of graph eigenvalues

Vladimir Nikiforov
Department of Mathematical Sciences, University of Memphis,
Memphis TN 38152, USA, email: vnkifrv@memphis.edu

March 30, 2022

Abstract

Let \( \mu_1(G) \geq \ldots \geq \mu_n(G) \) be the eigenvalues of the adjacency matrix of a graph \( G \) of order \( n \), and \( G \) be the complement of \( G \).

Suppose \( F(G) \) is a fixed linear combination of \( \mu_1(G), \mu_{n-i+1}(G), \mu_i(G), \) and \( \mu_{n-i+1}(G), 1 \leq i \leq k \). We show that the limit

\[
\lim_{n \to \infty} \frac{1}{n} \max \{ F(G) : v(G) = n \}
\]

always exists. Moreover, the statement remains true if the maximum is taken over some restricted families like “\( K_r \)-free” or “\( r \)-partite” graphs.

We also show that

\[
\frac{29 + \sqrt{329}}{42} n - 25 \leq \max_{v(G)=n} \mu_1(G) + \mu_2(G) \leq \frac{2}{\sqrt{3}} n,
\]

answering in the negative a question of Gernert.

AMS classification: 15A42, 05C50

Keywords: extremal graph eigenvalues, linear combination of eigenvalues, multiplicative property

1 Introduction

Our notation is standard (e.g., see [11, 3, and 7]); in particular, all graphs are defined on the vertex set \([n] = \{1, \ldots, n\}\) and \( G \) stands for the complement of \( G \). We order the eigenvalues of the adjacency matrix of a graph \( G \) of order \( n \) as \( \mu_1(G) \geq \ldots \geq \mu_n(G) \).

Suppose \( k > 0 \) is a fixed integer and \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_k \), are fixed reals. For any graph \( G \) of order at least \( k \), let

\[
F(G) = \sum_{i=1}^{k} \alpha_i \mu_i(G) + \beta_i \mu_{n-i+1}(G) + \gamma_i \mu_i(G) + \delta_i \mu_{n-i+1}(G).
\]
For a given graph property \( \mathcal{F} \), i.e., a family of graphs closed under isomorphism, it is natural to look for \( \max \{ F(G) : G \in \mathcal{F}, v(G) = n \} \). Questions of this type have been studied, here is a partial list:

\[
\begin{align*}
&\max \{ \mu_1(G) + \mu_n(G) : G \text{ is } K_r\text{-free, } v(G) = n \} \quad \text{Brandt} \quad [2]; \\
&\max \{ \mu_1(G) - \mu_n(G) : v(G) = n \} \quad \text{Gregory, Hershkowitz, Kirkland} \quad [3]; \\
&\max \{ \mu_1(G) + \mu_2(G) : v(G) = n \} \quad \text{Gernert} \quad [4]; \\
&\max \{ \mu_i(G) + \mu_i(G) : v(G) = n \} \quad \text{Nosal} \quad [10], \text{ Nikiforov} \quad [8]; \\
&\max \{ \mu_i(G) + \mu_i(G) : v(G) = n \} \quad \text{Nikiforov} \quad [9].
\end{align*}
\]

One of the few sensible questions in such a general setup is the following one: does the limit

\[
\lim_{n \to \infty} \frac{1}{n} \max \{ F(G) : G \in \mathcal{F}, v(G) = n \}
\]

exist? We show that, under some mild conditions on \( \mathcal{F} \), this is always the case.

For any graph \( G = (V,E) \) and integer \( t \geq 1 \), write \( G^{(t)} \) for the graph obtained by replacing each vertex \( u \in V \) by a set \( V_u \) of \( t \) independent vertices and joining \( x \in V_u \) to \( y \in V_v \) if and only if \( uv \in E \).

Call a graph property \( \mathcal{F} \) multiplicative if:

(a) \( \mathcal{F} \) is closed under adding isolated vertices;
(b) \( G \in \mathcal{F} \) implies \( G^{(t)} \in \mathcal{F} \) for every \( t \geq 1 \). Note that “\( K_r\)-free”, “\( r\)-partite”, and “any graph” are multiplicative properties.

**Theorem 1** For any multiplicative property \( \mathcal{F} \) the limit

\[
c = \lim_{n \to \infty} \frac{1}{n} \max \{ F(G) : G \in \mathcal{F}, v(G) = n \}
\]

exists. Moreover,

\[
c = \limsup \left\{ \frac{1}{|G|} F(G) : G \in \mathcal{F} \right\}.
\]

Note that, since the \( \alpha_i \)'s, \( \beta_i \)'s, \( \gamma_i \)'s, and \( \delta_i \)'s may have any sign, Theorem 1 implies that

\[
\lim_{n \to \infty} \frac{1}{n} \min \{ F(G) : G \in \mathcal{F}, v(G) = n \}
\]

exists as well.

Gernert [4] (see also Stevanovic [11]) has proved that the inequality

\[
\mu_1(G) + \mu_2(G) \leq v(G)
\]

holds if the graph \( G \) has fewer than 10 vertices or is one of the following types: regular, triangle-free, thoroidal, or planar; he consequently asked whether this inequality holds for any graph \( G \). We answer this question in the negative by showing that

\[
1.122n - 25 < \frac{29 + \sqrt{329}}{42} n - 25 \leq \max_{v(G)=n} \mu_1(G) + \mu_2(G) \leq \frac{2}{\sqrt{3}} n < 1.155n.
\]

(2)
2 Proofs

Given a graph $G$ and an integer $t > 0$, set $G^{[t]} = \overline{G(t)}$, i.e., $G^{[t]}$ is obtained from $G^{(t)}$ by joining all vertices within $V_u$ for every $u \in V$. The following two facts are derived by straightforward methods.

(i) The eigenvalues of $G^{(t)}$ are $t\mu_1(G), \ldots, t\mu_n(G)$ together with $n(t-1)$ additional 0’s.

(ii) The eigenvalues of $G^{[t]}$ are $t\mu_1(G)+t-1, \ldots, t\mu_n(G)+t-1$ together with $n(t-1)$ additional $(-1)$’s.

We shall prove (3) first. Fix some $t\mu_s(G)$, and inequality (3) follows from

\[ 0 \leq \mu_s(G^{[t]}) - t\mu_s(G) < \frac{tn}{\sqrt{n-k}}, \tag{3} \]

\[ 0 \geq \mu_{n-s+1}(G^{[t]}) - t\mu_{n-s+1}(G) > -\frac{tn}{\sqrt{n-k}}, \tag{4} \]

\[ 0 \leq \mu_s(G^{[t]}) - t\mu_s(G) + t - 1 < t + \frac{tn}{\sqrt{n-k}}, \tag{5} \]

\[ 0 \geq \mu_{n-s+1}(G^{[t]}) - t\mu_{n-s+1}(G) + t - 1 > -t - \frac{tn}{\sqrt{n-k}}. \tag{6} \]

**Proof** We shall prove (3) first. Fix some $s \in [k]$ and note that (i) implies that $G^{(t)}$ and $G$ have the same number of positive eigenvalues. In particular, $G^{(t)}$ has at most $n-1$ negative eigenvalues, and so $\mu_s(G^{(t)}) \geq 0$. If $\mu_s(G^{(t)}) > 0$, then $\mu_s(G) > 0$ and $\mu_s(G^{[t]}) = t\mu_s(G)$, so (3) holds. If $\mu_s(G^{(t)}) = 0$, then

\[ 0 \geq \mu_s(G) \geq \ldots \geq \mu_n(G), \]

and inequality (3) follows from

\[ (n-k)\mu_s^2(G) \leq (n-s+1)\mu_s^2(G) \leq \sum_{i=s}^n \mu_i^2(G) < n^2. \]

Next we shall prove (5). Note that (ii) implies that $G^{(t)}$ and $G$ have the same number of eigenvalues that are greater than $-1$. Since $G^{[t]}$ has at most $n-1$ eigenvalues that are less than $-1$, it follows that $\mu_s(G^{[t]}) \geq -1$. If $\mu_s(G^{[t]}) > -1$, then $\mu_s(G) > -1$ and $\mu_s(G^{[t]}) = t\mu_s(G) + t - 1$, so (5) holds. If $\mu_s(G^{[t]}) = -1$, then

\[ -1 \geq \mu_s(G) \geq \ldots \geq \mu_n(G), \]

and inequality (5) follows from

\[ (n-k)\mu_s^2(G) < (n-s+1)\mu_s^2(G) \leq \sum_{i=s}^n \mu_i^2(G) < n^2. \]
In view of (9), this is obvious if \( \mu \). Inequality (10) follows now from

\[
0 \leq \mu_s (G) - \mu_s (H) < 3 \sqrt{n}, \quad (7)
\]

\[
0 \geq \mu_{n-s+1} (G) - \mu_{n-s} (H) > -3 \sqrt{n}. \quad (8)
\]

We also need the following lemma.

**Lemma 3** Let \( G \) be a graph of order \( n \) and \( H \) be an induced subgraph of \( G \) of order \( n-1 \). Then for every \( 1 \leq s \leq 3n/4 \),

\[
0 \leq \mu_s (G) - \mu_s (H) < 3 \sqrt{n}, \quad (7)
\]

\[
0 \geq \mu_{n-s+1} (G) - \mu_{n-s} (H) > -3 \sqrt{n}. \quad (8)
\]

**Proof** We shall assume that \( V (G) = \{1, \ldots, n\} \) and \( V (H) = \{1, \ldots, n-1\} \). Let \( A \) be the adjacency matrix of \( G \) and let \( A_1 \) be the \( n \times n \) symmetric matrix obtained from \( A \) by zeroing its \( n \)th row and column. Since the adjacency matrix of \( H \) is the principal submatrix of \( A \) in the first \( n-1 \) columns and rows, the eigenvalues of \( A_1 \) are \( \mu_1 (H), \ldots, \mu_{n-1} (H) \) together with an additional 0. This implies that, for every \( s \in [n-1] \),

\[
\mu_s (A_1) = \left\{ \begin{array}{ll}
\mu_s (H), & \text{if } \mu_s (A_1) > 0 \\
\mu_{s-1} (H), & \text{if } \mu_s (A_1) \leq 0
\end{array} \right. \quad (9)
\]

We first show that, for every \( s \in [n-1] \),

\[
\mu_s (A_1) - \mu_s (H) \leq \frac{n}{\sqrt{n-s}}. \quad (10)
\]

In view of (9), this is obvious if \( \mu_s (A_1) > 0 \). If \( \mu_s (A_1) \leq 0 \), again in view of (9), we have

\[
\mu_s (A_1) - \mu_s (H) = \mu_{s-1} (H) - \mu_s (H) \leq |\mu_s (H)|
\]

Inequality (10) follows now from

\[
(n-s) \mu_s^2 (G) \leq (n-s+1) \mu_s^2 (G) \leq \sum_{i=s}^{n} \mu_i^2 (G) < n^2.
\]

Likewise, with proper changes of signs, we can show that, for every \( s \in [n-1] \),

\[
\mu_{n-s+1} (A_1) - \mu_{n-s} (H) \geq -\frac{n}{\sqrt{n-s}}.
\]

Having proved (10) we turn to the proof of (7) and (8). Note that the first inequalities in both (7) and (8) follow by Cauchy interlacing theorem. On the other hand, Weyl’s inequalities imply that

\[
\mu_n (A - A_1) \leq \mu_s (A) - \mu_s (A_1) \leq \mu_1 (A - A_1).
\]

Obviously, \( \mu_1 (A - A_1) \) is maximal when the off-diagonal entries of the \( n \)th row and column of \( A \) are 1’s. Thus, \( \mu_1 (A - A_1) \leq \sqrt{n-1} \) and \( \mu_n (A - A_1) = -\mu_1 (A - A_1) \geq -\sqrt{n-1} \). Hence,

\[
\mu_s (G) - \mu_s (H) = \mu_s (A) - \mu_s (A_1) + \mu_s (A_1) - \mu_s (H) \leq \sqrt{n-1} + \frac{n}{\sqrt{n-s}} < 3 \sqrt{n}.
\]
Likewise,
\[
\mu_{n-s+1}(G) - \mu_{n-s}(H) = \mu_{n-s+1}(A) - \mu_{n-s+1}(A_1) + \mu_{n-s+1}(A_1) - \mu_{n-s}(H) \\
\geq -\sqrt{n-1} - \frac{n}{\sqrt{n-s}} > -3\sqrt{n},
\]
completing the proof of Lemma 3. \qed

Corollary 4 Let \(G_1\) be a graph of order \(n\) and \(G_2\) be an induced subgraph of \(G_1\) of order \(n-l\). Then, for every \(1 \leq s \leq 3(n-l)/4\),
\[
|\mu_s(G_1) - \mu_s(G_2)| < 3l\sqrt{n}, \\
|\mu_{n-s+1}(G_1) - \mu_{n-l-s+1}(G_2)| < 3l\sqrt{n}.
\]

Proof Let \(\{v_1, \ldots, v_l\} = V(G_1) \setminus V(G_2)\). Set \(H_0 = G_1\); for every \(i \in [l]\), let \(H_i\) be the subgraph of \(G_1\) induced by the set \(V(G_1) \setminus \{v_1, \ldots, v_l\}\); clearly, \(H_l = G_2\). Since \(H_{i+1}\) is an induced subgraph of \(H_i\) with \(|H_{i+1}| = |H_i| - 1\), Lemma 3 implies that for every \(1 \leq s \leq 3(n-l)/4\),
\[
|\mu_s(G_1) - \mu_s(G_2)| \leq \sum_{i=0}^{l-1} |\mu_s(H_i) - \mu_s(H_{i+1})| \leq \sum_{i=0}^{l-1} 3\sqrt{n-i} < 3l\sqrt{n},
\]
\[
|\mu_{n-s+1}(G_1) - \mu_{n-l-s+1}(G_2)| \leq \sum_{i=0}^{l-1} |\mu_{n-i+s+1}(H_i) - \mu_{n-i-1-s+1}(H_{i+1})| \leq \sum_{i=0}^{l-1} 3\sqrt{n-i} < 3l\sqrt{n},
\]
completing the proof of the corollary. \qed

Proof of Theorem Set \(\varphi(n) = \frac{1}{n} \max \{F(G) : G \in \mathcal{F}, v(G) = n\}\)

Let \(M = \sum_{i=1}^{k} |\alpha_i| + |\beta_i| + |\gamma_i| + |\delta_i|\) and set
\[
c = \lim_{n \to \infty} \sup \varphi(n).
\]

Since \(|F(G)| \leq Mn\), the value \(c\) is defined. We shall prove that, in fact, \(c\) satisfies (11).

Note first if \(t \geq 2\), and \(n > 4k/3\), then, for any \(i \in [k]\), Lemma 2 implies that
\[
F(G^{(i)}) - tF(G) \geq -M \left(t + \frac{tn}{\sqrt{n-k}}\right) \geq -M \left(t + 2t\sqrt{n}\right) \geq -3Mt\sqrt{n}. \quad (11)
\]
Select \( \varepsilon > 0 \) and let \( G \in \mathcal{F} \) be a graph of order \( n > (3M/\varepsilon)^2 \) such that
\[
c + \varepsilon \geq \varphi(n) = \frac{F(G)}{n} \geq c - \varepsilon.
\]

Suppose \( N \geq n \left[ n \max \{2, (|c|/\varepsilon + 1), (3M/\varepsilon)^2\} \right] \); therefore the value \( t = \lfloor N/n \rfloor \) satisfies
\[
t \geq n \max \{2, (|c|/\varepsilon + 1), (3M/\varepsilon)^2\}.
\]
We shall show that \( \varphi(N) \geq c - 4\varepsilon \), which implies the assertion.

Let \( G_1 \) be the union of \( G(t) \) and \( N - tn \) isolated vertices. Clearly \( v(G_1) = N \) and, since \( \mathcal{F} \) is multiplicative, \( G_1 \in \mathcal{F} \). In view of \( N - tn < n \), Corollary 4 implies that
\[
F(G_1) \geq F(G(t)) - 3Mn\sqrt{N}.
\]
Even of \( \varphi(N) \geq F(G_1)/N \) and (11),
\[
\varphi(N) \geq \frac{F(G(t)) - 3Mn\sqrt{N}}{N} \geq \frac{iF(G) - 3Mt\sqrt{n} - 3Mn\sqrt{N}}{N}.
\]

We find that
\[
\varphi(N) \geq \frac{F(G)}{n} - \frac{F(G)(N - tn)}{nnN} - \frac{3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
\geq \varphi(n) - \frac{|\varphi(n)|}{N} - \frac{3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
\geq \varphi(n) - \frac{n^2 (|c| + |\varepsilon|) + 3Mt\sqrt{n} + 3Mn\sqrt{N}}{N} \\
\geq \varphi(n) - \frac{n^2 (|c| + |\varepsilon|)}{nt} - 3M\sqrt{nt} - 3M\sqrt{n} \geq c - 4\varepsilon,
\]
completing the proof of Theorem 1.

We turn now to the proof of inequality (2); we present it in two propositions.

**Proposition 5** If \( G \) is a graph of order \( n \), then \( \mu_1(G) + \mu_2(G) \leq (2/\sqrt{3}) n \).

**Proof** Setting \( m = e(G) \), we see that
\[
\mu_1^2(G) + \mu_2^2(G) \leq \mu_1^2(G) + \ldots + \mu_n^2(G) = 2m.
\]
(12)
If \( m \leq n^2/4 \), the result follows from
\[
\mu_1(G) + \mu_2(G) \leq \sqrt{2 (\mu_1^2(G) + \mu_2^2(G))} \leq 2\sqrt{m} \leq n,
\]
so we shall assume that \( m > n^2/4 \). From [12], we clearly have

\[
\mu_1 (G) + \mu_2 (G) \leq \sqrt{2m - \mu_2^2 (G) + \mu_2 (G)}.
\]

The value \( \sqrt{2m - x^2 + x} \) is increasing in \( x \) for \( x \leq m \). On the other hand, Weyl’s inequalities imply that

\[
\mu_2 (G) + \mu_n (\overline{G}) \leq \mu_2 (K_n) = -1.
\]

Hence, if \( G \neq K_n \), we have \( \mu_2 (G) \geq 0 \) and so, \( \mu_2^2 (G) < \mu_n^2 (\overline{G}) \); if \( G = K_n \), then \( \mu_2^2 (G) = \mu_n^2 (\overline{G}) + 1 \); thus we always have

\[
\mu_2^2 (G) \leq \mu_n^2 (\overline{G}) + 1.
\]

From

\[
\mu_2^2 (G) \leq \mu_n^2 (\overline{G}) + 1 \leq e (\overline{G}) + 1 \leq \frac{n (n-1)}{2} + 1 - m \leq \frac{n^2}{2} - m < m,
\]

we see that

\[
\mu_1 (G) + \mu_2 (G) \leq \sqrt{3m - n^2/2} + \sqrt{n^2/2 - m}.
\]

The right-hand side of this inequality is maximal for \( m = n^2/3 \) and the result follows. \( \Box \)

**Proposition 6** For every \( n \geq 21 \) there exists a graph of order \( n \) with

\[
\mu_1 (G) + \mu_2 (G) > \frac{29 + \sqrt{329}}{42} n - 25.
\]

**Proof** Suppose \( n \geq 21 \), set \( k = \lfloor n/21 \rfloor \), let \( G_1 \) be the union of two copies of \( K_{8k} \) and \( G_2 \) be the join of \( K_{5k} \) and \( G_1 \); clearly \( v (G_2) = 21k \). Add \( n - 21k \) isolated vertices to \( G_2 \) and write \( G \) for the resulting graph. By Cauchy interlacing theorem, we have

\[
\mu_1 (G) \geq \mu_1 (G_2), \quad \mu_2 (G) \geq \mu_2 (G_2) \geq \mu_2 (G_1) = 8k - 1.
\]

Since the graphs \( K_{5k} \) and \( G_1 \) are regular, a theorem of Finck and Grohmann [5] (see also [1], Theorem 2.8) implies that \( \mu_1 (G_2) \) is the positive root of the equation

\[
(x - 5k + 1) (x - 8k + 1) - 80k^2 = 0.
\]

Hence,

\[
\mu_1 (G) + \mu_2 (G) \geq \frac{(29k - 4) + k\sqrt{329}}{2} > \frac{(29 (n-20) - 84) + (n-20) \sqrt{329}}{42}
\]

\[
> \frac{29 + \sqrt{329}}{42} n - 25,
\]

completing the proof. \( \Box \)

**Acknowledgments** Part of this research was completed while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2006. The author is also indebted to Béla Bollobás for his kind support.

Finally, the referees’ criticisms helped to correct the first version of the paper.
References

[1] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[2] S. Brandt, The local density of triangle-free graphs, *Discrete Math.* 183 (1998), 17–25.

[3] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.

[4] D. Gernert, personal communication.

[5] H.J. Finck, G. Grohmann, Vollständiges Produkt, chromatische Zahl und charakteristisches Polynom regulärer Graphen. I. (German) *Wiss. Z. Techn. Hochsch. Ilmenau* 11 (1965) 1–3.

[6] D. Gregory, D. Hershkowitz, S. Kirkland, The spread of the spectrum of a graph, *Linear Algebra Appl.* 332/334 (2001), 23–35.

[7] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. xiii+561 pp.

[8] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combin. Probab. Comput.* 11 (2002), 179–189.

[9] V. Nikiforov, Eigenvalue problems of Nordhaus-Gaddum type, to appear in *Discrete Math*.

[10] E. Nosal, Eigenvalues of Graphs, Master’s thesis, University of Calgary, 1970.

[11] D. Stevanovic, www.sgt.pep.ufrj.br/home_arquivos/prob_abertos.html.