Abstract

We extend to dimension \( n \geq 3 \) the concept of \( \rho \)-pair in a coloured graph and we prove the existence theorem for minimal rigid crystallizations of handle-free, closed \( n \)-manifolds.

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1 Introduction

The concept of \( \rho \)-pair in a 4-coloured graph was introduced for the first time by Sostenes Lins in [12]. Roughly speaking, it consists of two equally coloured edges, which belong to two or three bicoloured cycles. A graph with no \( \rho \)-pairs was then called rigid in the same paper, where the following basic result was proved:

Every handle-free, closed 3-manifold admits a rigid crystallization of minimal order.

The proof is based on the definition of a particular local move, called switching of a \( \rho \)-pair. Starting from any gem \( \Gamma \) of a closed, irreducible 3-manifold \( M \), a finite sequence of such moves, together with the cancelling
of a suitable number of 1-dipoles, produces a rigid crystallization $\Gamma'$ of the same manifold $M$, whose order is strictly less than the order of $\Gamma$.

The above existence theorem plays a fundamental rôle in the problem of generating automatically essential catalogues of 3-manifolds, with ”small” Heegaard genus and/or graph order (see, e.g., [12], [3], [5], [4], [14]).

In the present paper, we extend the concepts of $\rho$-pair, switching and rigidity to $(n+1)$-coloured graphs, for $n > 3$.

Our main result is the proof of the existence of a rigid crystallization of minimal order, for every handle-free $n$-dimensional, closed manifold. It will be used in a subsequent paper to generate the catalogue of all 4-dimensional, closed manifolds, represented by (rigid) crystallizations of ”small” order.

2 Notations

In the following all manifolds will be piecewise linear (PL), closed and, when not otherwise stated, connected. For the basic notions of PL topology, we refer to [17] and to [8]; ”$\cong$” will mean ”PL-homeomorphic”. For graph theory, see [9] and [18].

We will use the term ”graph” instead of ”multigraph”. Hence multiple edges are allowed, but loops are forbidden. As usual, $V(\Gamma)$ and $E(\Gamma)$ will denote the vertex-set and the edge-set of the graph $\Gamma$.

If $\Gamma$ is an oriented graph, then each edge $e$ is directed from its first endpoint $e(0)$ (also called tail) to its second endpoint $e(1)$ (called head).

An $(n+1)$-coloured graph is a pair $(\Gamma, \gamma)$, where $\Gamma$ is a graph, regular of degree $n+1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0, \ldots, n\}$ is a map with the property that, if $e$ and $f$ are adjacent edges of $E(\Gamma)$, then $\gamma(e) \neq \gamma(f)$. We shall often write $\Gamma$ instead of $(\Gamma, \gamma)$.

Let $B$ be a subset of $\Delta_n$. Then, the connected components of the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ are called $B$-residues of $(\Gamma, \gamma)$. Moreover, for each $c \in \Delta_n$, we set $\hat{c} = \Delta_n \setminus \{c\}$. If $B$ is a subset of $\Delta_n$, we define $g_B$ to be the number of $B$-residues of $\Gamma$; in particular, given any colour $c \in \Delta_n$, $g_{\hat{c}}$ denotes the number of components of the graph $\Gamma_{\hat{c}}$, obtained by deleting all edges coloured $c$ from $\Gamma$. If $i, j \in \Delta_n, i \neq j$, then $g_{ij}$ denotes the number of cycles of $\Gamma$, alternatively coloured $i$ and $j$, i.e. $g_{ij} = g_{\{i,j\}}$.

An isomorphism $\phi : \Gamma \to \Gamma'$ is called a coloured isomorphism between the $(n+1)$-coloured graphs $(\Gamma, \gamma)$ and $(\Gamma', \gamma')$ if there exists a permutation $\varphi$ of $\Delta_n$ such that $\varphi \circ \gamma = \gamma' \circ \phi$. 

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A pseudocomplex $K$ of dimension $n$ \cite{11} with a labelling on its vertices by $\Delta_n = \{0, \ldots, n\}$, which is injective on the vertex-set of each simplex of $K$ is called a \textit{coloured n-complex}.

It is easy to associate a coloured $n$-complex $K(\Gamma)$ to each $(n+1)$-coloured graph $\Gamma$, as follows:

- for each vertex $v$ of $\Gamma$, take an $n$-simplex $\sigma(v)$ and label its vertices by $\Delta_n$;
- if $v$ and $w$ are vertices of $\Gamma$ joined by a $c$-coloured edge ($c \in \Delta_n$), then identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertices labelled $c$.

If $M$ is a manifold of dimension $n$ and $\Gamma$ an $(n+1)$-coloured graph such that $|K(\Gamma)| \cong M$, then, following Lins \cite{12}, we say that $\Gamma$ is a \textit{gem (graph-encoded-manifold) representing} $M$.

Note that $\Gamma$ is a gem of an $n$-manifold $M$ iff, for every colour $c \in \Delta_n$, each $\hat{c}$-residue represents $S^{n-1}$. Moreover, $M$ is orientable iff $\Gamma$ is bipartite.

If, for each $c \in \Delta_n$, $\Gamma_{\hat{c}}$ is connected, then the corresponding coloured complex $K(\Gamma)$ has exactly $(n+1)$ vertices (one for each colour $c \in \Delta_n$); in this case both $\Gamma$ and $K(\Gamma)$ are called \textit{contracted}. A contracted gem $\Gamma$, representing an $n$-manifold $M$, is called a \textit{crystallization} of $M$.

The \textit{existence theorem} of crystallizations for every $n$-manifold $M$ was proved by Pezzana \cite{15, 16}. Surveys on crystallizations theory can be found in \cite{7, 2}.

Let $x, y$ be two vertices of an $(n+1)$-coloured graph $\Gamma$ joined by $k$ edges \{e$_1, \ldots, e_k$\} with $\gamma(e_h) = i_h$, for $h = 1, \ldots, k$. We call $\Theta = \{x, y\}$ a \textit{dipole of type} $k$, \textit{involving colours} $i_1, \ldots, i_k$, iff $x$ and $y$ belong to different $(\Delta_n \setminus \{i_1, \ldots, i_k\})$-residues of $\Gamma$.

In this case a new $(n+1)$-coloured graph $\Gamma'$ can be obtained by deleting $x, y$ and all their incident edges from $\Gamma$ and then joining, for each $i \in \Delta_n \setminus \{i_1, \ldots, i_k\}$, the vertex $i$-adjacent to $x$ to the vertex $i$-adjacent to $y$. $\Gamma'$ is said to be obtained from $\Gamma$ by \textit{cancelling (or deleting)} the $k$-dipole $\Theta$. Conversely $\Gamma$ is said to be obtained from $\Gamma'$ by \textit{adding the} $k$-dipole $\Theta$.

By a \textit{dipole move}, we mean either the \textit{adding} or the \textit{cancelling} of a dipole from a gem $\Gamma$.

As proved in \cite{6}, \textit{two gems $\Gamma$ and $\Gamma'$ represent PL-homeomorphic manifolds iff they can be obtained from each other by a finite sequence of dipole moves}.

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An $n$-dipole $\Theta = (x, y)$ is often called a blob (see [13], where a different calculus for gems is introduced). If $c$ is the (only) colour not involved in the blob $\Theta$, and $x', y'$ are the vertices $c$-adjacent to $x$ and $y$ respectively, then the cancelling of $\Theta$ from $\Gamma$ produces (in $\Gamma'$) a new $c$-coloured edge $e'$, joining $x'$ with $y'$. Following Lins, we call the inverse procedure the adding of a blob on the edge $e'$.

Two vertices $x, y$ of an $(n + 1)$-coloured graph $\Gamma$ are called completely separated if, for each colour $c \in \Delta_n$, $x$ and $y$ belong to two different $c$-residues. The fusion graph $\Gamma_{\text{fus}}(x, y)$ is obtained simply by deleting $x$ and $y$ from $\Gamma$ and then by gluing together the "hanging edges" with the same colours.

The following result was first proved, for case (a), in [12] and, for case (b), in [12] $(n = 3)$ and in [10].

**Lemma 1** Let $x, y$ be two completely separated vertices of a (possibly disconnected) graph $\Gamma$.

(a) If $x$ and $y$ belong to the (only) two different components $\Gamma'$ and $\Gamma''$ of $\Gamma$, representing two $n$-dimensional manifolds $M'$ and $M''$ respectively, then $\Gamma_{\text{fus}}(x, y)$ is a gem of a connected sum $M' \# M''$.

(b) If $\Gamma$ is a gem of a (connected) $n$-manifold $M$, then $\Gamma_{\text{fus}}(x, y)$ is a gem of $M \# \mathbb{H}$, where $\mathbb{H}$ is either $S^{n-1} \times S^1$ or $S^{n-1} \tilde{\times} S^1$ (i.e. the orientable or non-orientable $(n - 1)$-sphere bundle over $S^1$).

Note that such a manifold $\mathbb{H}$ is often called a handle (of dimension $n$). A manifold $M$ is called handle-free if it admits no handles as connected summands (i.e. if $M$ is not homeomorphic to $M' \# \mathbb{H}$, $M'$ being any $n$-manifold).

### 3 Switching of $\rho$-pairs.

Let $(\Gamma, \gamma)$ be an $(n + 1)$-coloured graph. Let further $(e, f)$ be any pair of edges, both coloured $c$, of $\Gamma$.

If we delete $e, f$ from $\Gamma$, we obtain an edge-coloured graph $\Gamma'$, with exactly four vertices of degree $n$ (namely, the endpoints $u, v$ of $e$ and the endpoints $w, z$ of $f$).

Now, there are exactly two $(n + 1)$-coloured graphs $(\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)$ (different from $(\Gamma, \gamma)$) which can be obtained by adding two new edges (both
coloured c) to $\Gamma$: such edges are either $e_1, f_1$, joining $u$ with $w$ and $v$ with $z$ respectively, or $e_2, f_2$, joining $u$ with $z$ and $v$ with $w$ respectively. (See Figure 1, Figure 1a and Figure 1b, where, w.l.o.g., we consider $c = 0$)

![Figure 1](image)

We will say that $(\Gamma_1, \gamma_1)$ and $(\Gamma_2, \gamma_2)$ are obtained from $(\Gamma, \gamma)$ by a switching on the pair $(e, f)$.
Actually, we are interested in particular pair of equally coloured edges of $\Gamma$. More precisely:

**Definition 2** A pair $R = (e, f)$ of edges of $\Gamma$, with $\gamma(e) = \gamma(f) = c$, will be called:

(a) a $\rho_n$-pair involving colour $c$ if, for each colour $i \in \Delta_n \setminus \{c\}$, we have $\Gamma_{\{c,i\}}(e) = \Gamma_{\{c,i\}}(f)$;

(b) a $\rho_{n-1}$-pair, involving colour $c$, if there exists a colour $d \neq c$, such that:

\[(b_1) \quad \Gamma_{\{c,d\}}(e) \neq \Gamma_{\{c,d\}}(f), \text{ and} \]
\[(b_2) \quad \text{for each colour } j \in \Delta_n \setminus \{c, d\}, \Gamma_{\{c,j\}}(e) = \Gamma_{\{c,j\}}(f). \]

The colour $d$ of above will be said to be *not involved* in the $\rho_{n-1}$-pair $R$. By a $\rho$-pair, we will mean for short either a $\rho_n$-pair or a $\rho_{n-1}$-pair.
Theorem 3 Let \((\Gamma, \gamma)\) be an \((n + 1)\)-coloured graph, \(R = (e, f)\) be a \(\rho\)-pair of \(\Gamma\) and let \((\Gamma_1, \gamma_1)\) be obtained from \((\Gamma, \gamma)\) by any switching of \(R\). Then:

(a) if \(R\) is a \(\rho_{n-1}\)-pair, then \(\Gamma\) and \(\Gamma_1\) have the same number of components;

(b) if \(R\) is a \(\rho_n\)-pair, then \(\Gamma_1\) has at most one more component than \(\Gamma\).

Proof. As before, let us call \(u, v\) the endpoints of \(e\) and \(w, z\) the endpoints of \(f\). Let further \(\overline{\Gamma}\) be the graph obtained by deleting \(e\) and \(f\) from \(\Gamma\).

As it is easy to check, \(u, v, w\) and \(z\) lie in the same component of \(\overline{\Gamma}\) (and therefore of \(\Gamma_1\)).

(a) If \(R\) is a \(\rho_{n-1}\)-pair, then \(u, v, w\) and \(z\) also lie in the same component of \(\overline{\Gamma}\) (and therefore of \(\Gamma_1\)).

For, let \(d\) be the colour not involved in \(R\). By definition of \(\rho_{n-1}\)-pair, \(\Gamma\{c, d\}(e)\) and \(\Gamma\{c, d\}(f)\) are two different bicoloured cycles of \(\Gamma\), the first one containing \(e\) and the second one containing \(f\).

Hence there are two paths of \(\overline{\Gamma}\), which join \(u\) with \(v\) and \(w\) with \(z\), respectively.

On the other hand, if \(j\) is any colour, \(j \neq c, d\), then \(\Gamma\{c, j\}(e) = \Gamma\{c, j\}(f)\) is a single bicoloured cycle, containing both \(e\) and \(f\).

This proves that there is a path of \(\overline{\Gamma}\), which joins \(u\) with either \(w\) or \(z\). This completes the proof of (a).

(b) If \(i \in \Delta_n \setminus \{c\}\), then by definition of \(\rho_n\)-pair, \(\Gamma\{c, i\}(e) = \Gamma\{c, i\}(f)\). This proves that there are two paths of \(\overline{\Gamma}\), the first one joining \(u\) with either endpoint of \(f\), \(w\) say, and the second one joining \(z\) with \(v\).

This shows that \(\overline{\Gamma}\) (hence also \(\Gamma_1\)) has at most one more component than \(\Gamma\). \(\blacksquare\)

In the following, we will show that in some particular, but geometrically relevant, cases, it is possible to choose a "preferred" way to switch a pair of equally coloured edges of \((\Gamma, \gamma)\).

CASE (A): \(\Gamma\) bipartite.

If \(R = (e, f)\) is any pair of edges, both coloured \(c\) (in particular, if \(R\) is a \(\rho\)-pair) of a bipartite \((n + 1)\)-coloured graph \((\Gamma, \gamma)\), then it is easy to see that only one of the two possible switching of \(R\) is again bipartite.

If, further, \(V_0, V_1\) are the two bipartition classes of \(\Gamma\) and we orient \(e, f\) from \(V_0\) to \(V_1\), so that their tails \(x_0 = e(0), y_0 = f(0)\) belong to \(V_0\), and their heads \(x_1 = e(1), y_1 = f(1)\) belong to \(V_1\), the (bipartite) switching \((\Gamma', \gamma')\) of \(R\) is obtained as follows:

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(I) delete $e$ and $f$ from $\Gamma$ (thus obtaining $\Gamma$);

(II) join $x_0$ with $y_1$ and $x_1$ with $y_0$ by two new edges $e', f'$, both coloured $c$.

**CASE (B): $\Gamma$ non bipartite, with bipartite residues.**

If $\Gamma$ is a non bipartite graph, but for each colour $i$, $\Gamma_i$ has bipartite components (residues), then we shall consider two subcases

Subcase (B1): $R = (e, f)$ is a $\rho_{n-1}$-pair of $\Gamma$, involving colour $c$ and not involving colour $d$.

Let $\Xi$ be the residue of $\Gamma_d$ containing both $e$ and $f$ (note that $e$ and $f$ belong to the same $i$-residue, because for every colour $i \neq c, d$, $\Gamma_{\{c,i\}}(e) = \Gamma_{\{c,i\}}(f)$.)

Let $V_0, V_1$ be the two bipartition classes of $\Xi$ (recall that $\Xi$ is bipartite), As in Case (A), let us orient $e$ from $V_0$ to $V_1$. Now, the switching of $R = (e, f)$ is the $(n + 1)$-coloured graph $(\Gamma', \gamma')$, obtained as before (Case (A)):

(I) delete $e$ and $f$ from $\Gamma$;

(II) join $x_0 = e(0)$ with $y_1 = f(1)$ and $x_1 = e(1)$ with $y_0 = f(0)$ by two new edges $e', f'$, both coloured $c$.

Subcase (B2): $R = (e, f)$ is a $\rho_n$-pair (involving colour $c$) of $\Gamma$ and $n \geq 3$.

Let us orient arbitrarily the edge $e$, as before, let us call $x_0 = e(0)$ and $x_1 = e(1)$. Let now $i$ be any colour different from $c$. The orientation on $e$ induces a coherent orientation on all edges of the cycle $\Gamma_{\{c,i\}}(e)$ and, in particular, on the edge $f$ (with the induced orientation).

Now, we shall prove that the orientation on $f$ (and hence its tail and its head) is independent from the choice of colour $i$ ($i \neq c$).

For, let $h$ be any colour of $\Delta_n$, $h \neq c$, and let $y_0^h, y_1^h$ be the tail and the head of the edge $f$, with the orientations induced by the cycle $\Gamma_{\{c,h\}}(e)$ ($e$ being oriented as before).

Let now $j \in \Delta_n, j \neq i, c$. In order to prove that $y_0^1 = y_0^1$ (and, as a consequence $y_1^1 = y_1^1$), let us consider a further colour $k$, with $k \neq i, j, c$.

Note that such a colour $k$ must exist, since $n \geq 3$ and therefore $\Delta_n$ contains at least four colours.

Let now $\Xi$ be the $k$-residue of $\Gamma$, which contains $e$. $\Xi$ is bipartite and contains both the cycles $\Gamma_{\{c,i\}}(e)$ and $\Gamma_{\{c,j\}}(e)$. As a consequence, $y_0^1 = y_0^1$. 8
In fact, supposing on the contrary, \( y_0^1 = y_1^1 \), we could construct an odd cycle of \( \Xi \).

The construction of the switching \((\Gamma', \gamma')\) of the \( \rho \)-pair \( R = (e, f) \) can now be performed as in the above cases:

(I) delete \( e \) and \( f \) from \((\Gamma, \gamma)\);

(II) join \( x_0 \) with \( y_1^1 \) and \( x_1 \) with \( y_0^1 \) by two new edges \( e', f' \), both coloured \( c \).

Remark 1 The above cases include all \( \rho \)-pairs of gems representing orientable \( n \)-manifolds (Case (A)), all \( \rho_{n-1} \)-pairs of gems representing non orientable \( n \)-manifolds (Case (B1)) and all \( \rho_n \)-pairs of gems representing non orientable \( n \)-manifolds, with \( n \geq 3 \) (Case (B2)).

The only remaining case is that of a \( \rho_2 \)-pair of a gem \( \Gamma \) representing a non orientable surface, for which it is not always possible the choice of a standard switching.

In fact, for \( n = 2 \), the procedure described in Case (B2) doesn’t work, as it depends on the choice of the colour \( i \).  

4 Main results

The present section is devoted to prove the following Theorems 4 and 8, which concern the geometrical meaning of switching \( \rho \)-pairs in gems of \( n \)-dimensional manifolds.

As in Section 2, let \( \mathbb{H} \) be a handle, i.e. either \((S^{n-1} \times S^1)\) or \((S^{n-1} \tilde{\times} S^1)\).

**Theorem 4** Let \((\Gamma, \gamma)\) be a gem of a (connected) \( n \)-manifold \( M \), \( n \geq 3 \), \( R = (e, f) \) be a \( \rho_n \)-pair in \( \Gamma \) and let \((\Gamma', \gamma')\) be the \((n+1)\)-coloured graph, obtained by switching \( R \). Then:

(a) if \((\Gamma', \gamma')\) splits into two connected components, \((\Gamma'_1, \gamma'_1)\) and \((\Gamma'_2, \gamma'_2)\) say, then they are gems of two \( n \)-manifolds \( M'_1 \) and \( M'_2 \) respectively, and \( M \cong M'_1 \# M'_2 \).

\(^1\) The case \( n = 2 \) is completely analyzed in [4], also for graphs representing surfaces with non-empty boundary.
(b) if $(\Gamma', \gamma')$ is connected, then it is a gem of an $n$-manifold $M'$ such that $M \cong M' \# \mathbb{H}$.

Moreover, if $(\Gamma, \gamma)$ is a crystallization of $M$, then $(\Gamma', \gamma')$ must be connected, and only case (b) may occur.

In order to prove Theorem 4, we need some further construction and a double sequence of Lemmas, which will be proved by induction on $n$.

**Lemma 5 – step n**  Let $(\Sigma, \sigma)$ be a gem of the $n$-sphere $S^n$, $n \geq 2$, $R = (e, f)$ be a $\rho_n$-pair of $\Sigma$ and let $(\Sigma', \sigma')$ be obtained by switching $R$. Then $\Sigma'$ splits into two connected components, both representing $S^n$.

Let now $\Gamma, R = (e, f), \Gamma'$ be as in the statement of Theorem 4. Recall that (n being $\geq 3$) any orientation of $e$ induces a coherent orientation on $f$. As in Section 2, let $e(0), f(0), e(1)$ and $f(1)$, be the ends of $e$ and $f$, so that $e$ is directed from $e(0)$ to $e(1)$ and $f$ is directed from $f(0)$ to $f(1)$. Furthermore, after the switching, the new edges $e', f'$ of $\Gamma'$ join $e(0)$ with $f(1)$ and $e(1)$ with $f(0)$ respectively. Denote by $\tilde{\Gamma}$ the $(n + 1)$-coloured graph obtained by adding a blob (i.e. an $n$-dipole), with vertices $A$ and $B$ on the edge $f'$ of $\Gamma'$ (see Figure 2)

**Lemma 6 – step n**  With the above notations, if $\Gamma$ is a gem of a (connected) $n$-manifold $M$, $n \geq 3$, then:

(i) $\Gamma'$ (hence also $\tilde{\Gamma}$) is a gem of a (possibly disconnected) $n$-manifold $M'$;

(ii) $e(0)$ and $B$ are two completely separated vertices of $\tilde{\Gamma}$; moreover $\Gamma$ coincides with $\Gamma_{\text{fus}}(e(0), B)$.
Proof. First of all, we repeat here the proof of Lemma 5, step 2, which is exactly Corollary 13 of [1].

Let \((\Sigma, \sigma)\) a 3-coloured, bipartite graph representing \(S^2\). Let \(R\) be a \(\rho_2\)-pair in \(\Sigma\) involving colour \(c \in \Delta_2\). Then, by switching \(R\) in the only possible way, we obtain a new graph \((\Sigma', \sigma')\), either connected or with two connected components. Moreover, if we denote by \(d, k\) the further two colours of \(\Delta_2\), then \(\Sigma'\) has the same number of \((d, k)\)-coloured cycles (\(\hat{c}\)-residues) and one more \((c, h)\)-coloured cycle (\(\hat{h}\)-residue), for \(h = d, k\).
Hence \( \chi(\Sigma') = \chi(\Sigma) + 2 = 4 \). This implies that \( \Sigma' \) must have two connected components, both representing \( S^2 \).

Now, assuming Lemma \[ \text{5} \] step \( n - 1 \), we prove Lemma \[ \text{6} \] step \( n \).

For, let us suppose \( \Gamma \) to be a gem of the \( n \)-manifold \( M \). As a consequence, for each colour \( i \in \Delta_n \), all \( \hat{i} \)-residues are gems of \( S^{n-1} \). Now, suppose \( R = (e, f) \) to be a \( \rho_n \)-pair of \( \Gamma \), involving color \( c \), whose switching produces the graph \( \Gamma' \).

Of course, the switching of \( R \) has no effects on the \( \hat{c} \)-residues of \( \Gamma \). Hence, each \( \hat{c} \)-residue of \( \Gamma' \) is colour-isomorphic to the corresponding one of \( \Gamma \), and therefore represents \( S^{n-1} \). Let now \( i \) be any colour different from \( c \) and let \( \Xi \) be the \( i \)-residue containing \( R \). Of course, \( R \) is a \( \rho_{n-1} \)-pair of \( \Xi \) (where \( \Xi \) is a gem of \( S^{n-1} \)). Hence, by Lemma \[ \text{5} \] step \( n - 1 \), the switching of \( R \) splits \( \Xi \) into two new \( i \)-residues of \( \Gamma' \), both representing \( S^{n-1} \).

Now, assuming Lemma \[ \text{6} \] step \( n \), we prove Lemma \[ \text{5} \] step \( n \). Let \( \Sigma, R = (e, f), \Sigma' \) be as in the statement of Lemma \[ \text{5} \] Let further \( \hat{\Sigma} \) be obtained by adding a blob \( \Theta = (A, B) \) on the edge \( f' \) of \( \Sigma' \). Hence, by Lemma \[ \text{5} \] step \( n \), \( \Sigma' \) and \( \hat{\Sigma} \) are both gems of an \( n \)-manifold \( M' \); moreover \( e(0) \) and \( B \) are completely separated vertices of \( \hat{\Sigma} \), and \( \Sigma \) is isomorphic to \( \hat{\Sigma} \text{fus}(e(0), B) \). If \( \Sigma' \) (hence also \( \hat{\Sigma} \)) is connected, then, by Lemma \[ \text{4} \] the manifold represented by \( \hat{\Sigma} \) must have a handle \( \mathbb{H} \) as a direct summand, but this is impossible, since \( \Sigma \) represents \( S^n \), by hypothesis. Hence \( \Sigma' \) (and \( \hat{\Sigma} \)) must split into two components \( \Sigma'_1, \Sigma'_2 \) say, representing two connected \( n \)-manifolds \( M'_1, M'_2 \) respectively, so that \( S^n \cong M'_1 \# M'_2 \). But this implies that both \( M'_1, M'_2 \) are gems of \( S^n \), too.

This concludes the proof of Lemmas \[ \text{5} \] and \[ \text{6} \].

**Proof.** (of Theorem \[ \text{4} \]) The proof of Theorem \[ \text{4} \] (a) and (b), is now a direct consequence of Lemma \[ \text{6} \] Step \( n \), and Lemma \[ \text{4} \].

If, further, \( \Gamma \hat{c} \) is connected, \( c \) being the colour involved in \( R \) (in particular, if \( \Gamma \) is a crystallization of \( M \)), then \( \Gamma' \) must be connected, too, and therefore \( M \cong M' \# \mathbb{H} \).
As a consequence of Theorem 4 and of Corollary 13 of [1], we have the following

**Corollary 7** If \((\Sigma, \sigma)\) is a crystallization of the \(n\)-sphere \(S^n\), \(n \geq 2\) then it can’t contain any \(\rho_n\)-pair.

**Theorem 8** Let \((\Gamma, \gamma)\) be a gem of a (connected) \(n\)-manifold \(M\), \(R = (e, f)\) be a \(\rho_{n-1}\)-pair of \(\Gamma\) and let \((\Gamma', \gamma')\) be obtained by switching \(R\). Then \(\Gamma'\) is a gem of the same manifold \(M\).

**Proof.** W.l.o.g., let us suppose \(c = 0\) to be the colour involved and \(d = n\) the one not involved in \(R\). By Theorem 2, \(\Gamma'\) has the same number of connected components as \(\Gamma\) and, by performing the switching, it is bipartite (resp. non-bipartite) iff \(\Gamma\) is.

Consider the graph \(\tilde{\Gamma}\), obtained by replacing the neighborhood of \(R\) in \(\Gamma\) (Figure 3a), with the graph of Figure 3b. The switching of \(R\) can be thought
as the replacing of the neighborhood of $R = (e, f)$ by the neighborhood of $R' = (e', f')$ (see Figure 1a). Consider now the graph $\tilde{\Gamma}$ obtained by replacing the above neighborhood by the graph of Figure 5, where $\Theta_1$ ($\Theta_2$ resp.) is formed by two vertices $A', e(0)$ ($B', f(1)$ resp.) joined by $n - 1$ edges coloured $1, \ldots, n - 1$.

Figure 3a
We will describe two sequences of dipole moves, joining $\tilde{\Gamma}$ with $\Gamma$ and $\Gamma'$ respectively, thus proving that $\Gamma$, $\Gamma'$ are gems of PL-homeomorphic manifolds.

The first sequence starts by considering $\delta_1 = (A, A')$, which is a 1-dipole. In fact, $\tilde{\Gamma}_n(A') = \delta_1$, whose further end is $e(1)$; hence the $\hat{n}$-residue $\tilde{\Gamma}_n(A)$ is different from $\delta_1$. By deleting, the 1-dipole $\delta_1$ from $\tilde{\Gamma}$, yields a 2-dipole $\delta_2$ with ends $B, B'$, in fact $\tilde{\Gamma}_n(B')$ is a multiple edge whose further end is $f(1)$ and which differs from the $\hat{n}$-residue $\tilde{\Gamma}_n(B)$. By cancelling $\delta_2$, too, we obtain $\Gamma$ (Figg. 3c and 3d).

$\Theta_1$ and $\Theta_2$ are $(n-1)$-dipoles, since the $(0, n)$-residue containing $A', B'$ is a quadrilateral cycle whose vertices are $A, B, A', B'$ only. By deleting them from $\Gamma$ (Figg. 3e and 3f), we obtain $\Gamma'$.
Figure 3c  Figure 3d
5 Rigid gems

Definition 9 An \((n + 1)\)-coloured graph \((\Gamma, \gamma)\), \(n \geq 3\), is called rigid iff it has no \(\rho\)-pairs.

Theorem 10 The \((n + 1)\)-coloured graph \((\Gamma, \gamma)\), \(n \geq 3\), is rigid iff for each \(i \in \Delta_n\), the graph \(\Gamma_i\) has no \(\rho_{n-1}\)-pairs.\(^\text{2}\)

\(^{2}\)Note that, for \(n = 2\), the concept of rigidity has no interest at all. In fact, if \(\Gamma\) is a 3-coloured graph representing a closed surface, then it contains \(\rho\)-pairs: \(\rho_2\)-pairs, if \(\Gamma\) is a crystallization, either \(\rho_1\)-pairs or \(\rho_2\)-pairs, otherwise. Hence, given any closed surface \(M^2\), it can’t exist any rigid crystallization of \(M^2\).
Proof. Suppose that \((\Gamma, \gamma)\) is rigid and that there is a colour \(i \in \Delta_n\) such that \(\Gamma_i\) has a \(\rho_n\)-pair \(R = (e, f)\) of colour \(c \in \Delta_n - \{i\}\). Then \(R\) is a \(\rho\)-pair in \(\Gamma\) too, and \(\Gamma\) can’t be rigid.

Conversely, If for each \(i \in \Delta_n\), \((\Gamma)_i\) contains no \(\rho_{n-1}\)-pairs, but \((\Gamma, \gamma)\) isn’t rigid, then \(\Gamma\) contains at least a \(\rho\)-pair \(R = (e, f)\).

If \(R\) is a \(\rho_n\)-pair, then \(R\) is a \(\rho_{n-1}\)-pair in \((\Gamma)_i\), for each \(i \in \Delta_n\).

If \(R\) is a \(\rho_{n-1}\)-pair, and \(d\) is the non-involved colour, then \(R\) is a \(\rho\)-pair in \(\Gamma_i\).

Theorem 11 Every closed, connected, handle-free \(n\)-manifold \(M^n\), \(n \geq 3\), admits a rigid crystallization.

Moreover, if \((\Gamma, \gamma)\) is a crystallization of a closed, connected, handle-free \(n\)-manifold \(M^n\) of order \(p\), then there exists a rigid crystallization of \(M^n\) of order \(\leq p\).

Proof. Starting from any gem of \(M^n\) by cancelling a suitable number of 1-dipoles, we always can obtain a crystallization of \(M^n\) (see [7]). Suppose now that \(\Gamma\) is a crystallization of \(M^n\); if \(\Gamma\) is rigid, then it is the request crystallization.

If \(\Gamma\) has some \(\rho_{n-1}\)-pair \(R = (e, f)\), of colour \(c \in \Delta_n\) and non involving colour \(d \in \Delta_n \setminus \{c\}\), then consider the connected component \(\Xi\) of \(\Gamma_{d}\) containing both \(e\) and \(f\). Since \(M^n\) is a manifold, \(\Xi\) represents \(S^{n-1}\) and \(R\) is a \(\rho_{n-1}\)-pair in \(\Gamma_{d}\), again. For Lemma 5 by switching \(R\) in \(\Gamma_{d}\), we obtain two connected components, both representing \(S^{n-1}\); since \(\Gamma_{d}\) is connected (Theorem 3), then there is at least a 1-dipole in \(\Gamma_{d}\), whose cancellation reduces the vertex-number.

If \(\Gamma\) has some \(\rho_{n}\)-pair \(R = (e, f)\), of colour \(c \in \Delta_n\), then, for each colour \(i \in \Delta_n \setminus \{c\}\), the connected component of \(\Gamma_i\) containing \(e\) and \(f\), represents \(S^{n-1}\) and \(R\) is a \(\rho_{n-1}\)-pair in \(\Gamma_i\), as before, by switching \(R\) in \(\Gamma_i\), we obtain two connected components, both representing \(S^{n-1}\); since \(\Gamma_{i}\) is connected (Theorem 3), then there is at least a 1-dipole in \(\Gamma_{i}\), whose cancellation reduces the vertex-number, for each \(i \in \Delta_n \setminus \{c\}\).

Note that the minimal crystallizations of \(S^{n-1} \times S^1\) and \(S^{n-1} \times \hat{S}^1\) are not rigid (see, e.g., [10]). Hence the second statement of Theorem 10 is false for handles.

In dimension 3, there exist rigid crystallizations for \(S^2 \times S^1\) and \(S^2 \times S^1\). The minimal ones have order 20 for \(S^2 \times S^1\) and order 14 for \(S^2 \times \hat{S}^1\).
For $n > 3$, it is easy to construct a rigid crystallization of $S^{n-1} \times S^1$, if $n$ is even, and of $S^{n-1} \# S^1$, if $n$ is odd, both of order $2(2^n - 1)$.

The remaining cases are still open.

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