GALOIS STRUCTURE OF THE HOLOMORPHIC POLY-DIFFERENTIALS OF CURVES

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Abstract. Suppose $X$ is a smooth projective geometrically irreducible curve over a perfect field $k$ of positive characteristic $p$. Let $G$ be a finite group acting faithfully on $X$ over $k$ such that $G$ has non-trivial, cyclic Sylow $p$-subgroups. In this paper we show that for $m > 1$, the decomposition of $H^0(X, \Omega^m_X)$ into a direct sum of indecomposable $kG$-modules is uniquely determined by the divisor class of a canonical divisor of $X/G$ together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$. This extends to arbitrary $m > 1$ the $m = 1$ case treated by the first author with T. Chinburg and A. Kontogeorgis. We discuss some applications to congruences between modular forms in characteristic 0, to the tangent space of the global deformation functor associated to $(X, G)$, and to the $kG$-module structure of Riemann-Roch spaces associated to divisors on $X$.

1. Introduction

Let $k$ be a perfect field, and let $X$ be a smooth projective geometrically irreducible curve over $k$. We denote by $\Omega_X$ the sheaf of relative differentials of $X$ over $k$. For a positive integer $m$, we denote by $\Omega^m_X$ the $m$-fold tensor product of $\Omega_X$ with itself over $\mathcal{O}_X$ and call it the sheaf of relative $m$-differentials. The space of holomorphic $m$-differentials of $X$ is then defined to be the space of global sections $H^0(X, \Omega^m_X)$. Suppose $G$ is a finite group acting faithfully on the right on $X$ over $k$. Then $G$ acts on the left on $\Omega^m_X$ and on $H^0(X, \Omega^m_X)$. In particular, $H^0(X, \Omega^m_X)$ is a left module for the group ring $kG$. Note that the $k$-dimension of $H^0(X, \Omega^m_X)$ is equal to the genus $g(X)$ of $X$ provided $m = 1$ or $g(X) \leq 1$, and its $k$-dimension equals $(2m - 1)(g(X) - 1)$ in all other cases. It is a classical question to ask how $H^0(X, \Omega^m_X)$ decomposes into a direct sum of indecomposable $kG$-modules and to give an explicit description of these indecomposables. This question was first raised by Hecke in [17], and it was answered by Chevalley and Weil in [9] (see also [19]) in the case when $k$ is algebraically closed and its characteristic does not divide $#G$.

For the remainder of the paper, we assume the characteristic of $k$ is a prime number $p$ that divides $#G$. In this case, $kG$ is not semisimple, and the determination of the $kG$-module structure of $H^0(X, \Omega^m_X)$ becomes more difficult. One reason is that there are indecomposable $kG$-modules that are not irreducible. Another reason is the appearance of wild ramification. Several authors have studied the $kG$-module structure of $H^0(X, \Omega^m_X)$ under various assumptions on the group $G$, the parameter $m$, and the ramification of the cover $X \to X/G$. Their research has often focused on cyclic groups or abelian $p$-groups or the case when $m = 1$.

A sample of previous results when $m = 1$ is as follows. The case when $G$ is cyclic was studied in [12] and [14], and it was completed in [29]. In [5], this was extended to the case when $G$ has cyclic Sylow $p$-subgroups. In [39], the case when $G$ is an elementary abelian $p$-group and $X/G \cong \mathbb{P}^1_k$ was studied. In [33], solvable groups $G$ were considered under some additional assumptions on the ramification of the cover $X \to X/G$. In [23] and, independently, in [37], the case when $G$ is arbitrary and the cover $X \to X/G$ is tamely ramified was analyzed. In [32], the subspace of the so-called semisimple holomorphic differentials was studied. Moreover,
in [28] the global sections of a $G$-equivariant locally free sheaf on $X$ were studied when $G$ is arbitrary and the cover $X \to X/G$ is weakly ramified. In [6], methods from modular $K$-theory were used to determine the decomposition of coherent cohomology groups into indecomposable direct summands.

Compared with the $m = 1$ case, less has been accomplished on the module structure of $H^0(X, \Omega^m_X)$ when $m > 1$. The results in [37] can be applied to the sheaf $\Omega^m_X$ for tamely ramified covers, and the results in [36] can be applied to $\Omega^m_X$ when $G$ is cyclic of order $p$. The methods in [6] can be used to extend the latter result to an arbitrary cyclic $p$-group. More recently, in [24], the $kG$-module structure of $H^0(X, \Omega^m_X)$ was determined for $m > 1$ when $G$ is a finite cyclic $p$-group or $G$ is an elementary abelian $p$-group and $X/G \cong \mathbb{P}^1_k$.

We would like to point out that the determination of the $kG$-module structure of $H^0(X, \Omega^m_X)$ for $m \geq 1$ can be divided into two basic approaches. The authors of [24, 25, 33, 44] have used so-called Boseck bases and Artin-Schreier extensions. In contrast, the authors of [5] have taken a more geometric approach, decomposing the cover $X \to X/G$ into a wildly ramified cover followed by a tamely ramified cover and treating the wildly ramified cover by working locally and applying the results of [37] to the tamely ramified cover. It is the latter approach we will use in this paper.

We now state our main result, which extends [5, Thm. 1.1] from $m = 1$ to arbitrary $m > 1$. Since $g(X) = 0$ implies $H^0(X, \Omega^m_X) = 0$ and since $g(X) = 1$ implies $H^0(X, \Omega^m_X) = k$ with trivial $G$-action, we focus on the case when $g(X) \geq 2$.

**Theorem 1.1.** Suppose $g(X) \geq 2$, $G$ has non-trivial, cyclic Sylow $p$-subgroups, and $m > 1$ is an integer. The $kG$-module structure of $H^0(X, \Omega^m_X)$ is uniquely determined by the divisor class of a canonical divisor of $X/G$, together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$.

The statement of this theorem differs from [5, Thm. 1.1] since if $m = 1$ then the $kG$-module structure of $H^0(X, \Omega_X)$ does not depend on the class of canonical divisors of $X/G$. We now highlight the similarities and main differences in the proofs when $m = 1$ versus when $m > 1$.

In both cases, we first reduce to the case when $G$ is $p$-hypo-elementary and $k$ is algebraically closed. We then let $I$ be a suitable normal $p$-subgroup of $G$ such that the cover $X \to X/I$ decomposes into a wildly ramified cover $X \to X/I$ followed by a tamely ramified cover $X/I \to X/G$. We analyze the wildly ramified cover by comparing filtrations of both the $kG$-module $H^0(X, \Omega^m_X)$ and the $O_X$-$G$-sheaf $\Omega^m_X$ that are given by the actions of the powers of the radical of $kI$. We analyze the tamely ramified cover using the results in [37]. The two main differences in the proofs when $m = 1$ versus when $m > 1$ are as follows:

1. Suppose $I$ is not trivial. When $m = 1$ then $H^1(X, -)$ applied to the top filtered piece of $\Omega^m_X$ does not vanish, and neither does $H^1(X, \Omega_X)$. On the other hand, when $m > 1$ then $H^1(X, -)$ applied to all filtered pieces of $\Omega^m_X$ vanishes. Since $H^1(X, \Omega^m_X) = 0$ when $g(X) \geq 2$ and $m > 1$, this is crucial for a dimension argument needed to compare the filtration of $H^0(X, \Omega^m_X)$ to the filtration of $\Omega^m_X$.

2. When $m = 1$ then one can use Serre duality to avoid having to consider the class of canonical divisors of $X/G$. When $m > 1$ then this canonical class is essential, together with the fundamental characters of the ramified points, to analyze the tamely ramified cover $X/I \to X/G$.

When $m = 2$, Theorem 1.1 leads to a formula for the dimension of the tangent space of the global deformation functor associated to the pair $(X, G)$ (see §3.4). We will also show how our method can be used to determine the $kG$-module structure of Riemann-Roch spaces associated to divisors on $X$ whose degrees are greater than $2g(X) - 2$ (see §4).

We next discuss implications of our main result Theorem 1.1 for the study of classical modular forms of even weight $2m$. Let $\ell \geq 3$ be a prime number with $\ell \neq p$, and let $X(\Gamma_\ell)$ be the (compactified) modular curve associated to the principal congruence subgroup $\Gamma_\ell$ of $SL(2, \mathbb{Z})$ of level $\ell$. Let $F$ be a number field that
is unramified over $p$ and that contains a primitive $\ell$th root of unity $\zeta_\ell$. Suppose $A \subset F$ is a Dedekind domain whose fraction field equals $F$ and that contains $\mathbb{Z}[(\frac{1}{\ell}, \zeta_\ell)]$. It follows from \cite{26, 27} that there is a proper smooth canonical model $\mathcal{X}_A(\ell)$ of $X(\Gamma_\ell)$. Moreover, $H^0(\mathcal{X}_A(\ell), \Omega_{\mathcal{X}_A(\ell)}^{\otimes m})$ is naturally identified with the $A$-lattice $\mathcal{S}_{2m}(A)$ of holomorphic cusp forms for $\Gamma_\ell$ of weight $2m$ that have $q$-expansion coefficients in $A$ at all the cusps, in the sense of \cite{26} §1.6. See \cite{33} for a brief summary of these results. Notice that in \cite{11} and other classical references, $\text{SL}(2, \mathbb{Z})$ acts on the right on $\mathcal{S}_{2m}(A)$. However, as usual, by letting the left action of a group element equal the right action of its inverse, we can convert right actions into left actions.

Let $\mathcal{V}(F, p)$ be the set of places $v$ of $F$ over $p$. For each such $v$, let $\mathcal{O}_{F,v}$ be the ring of integers of the completion $F_v$ of $F$ at $v$ and let $\mathfrak{m}_{F,v}$ be its maximal ideal. For the remainder of the introduction, we assume that $A \subset \mathcal{O}_{F,v}$ for all $v \in \mathcal{V}(F, p)$, and we let $k(v)$ be the residue field of $A$ modulo the maximal ideal $A \cap \mathfrak{m}_{F,v}$. Viewing all $k(v)$ inside a fixed algebraic closure $\overline{\mathbb{F}}_v$, let $k$ be an algebraically closed field containing $\overline{\mathbb{F}}_v$ and define $X_p(\ell) = k \otimes_A \mathcal{X}_A(\ell)$ to be the reduction of $\mathcal{X}_A(\ell)$ modulo $p$ over $k$. Letting $G = \text{PSL}(2, \mathbb{Z}/\ell) = \text{PSL}(2, \overline{\mathbb{F}}_\ell)$, we will prove the following result.

**Theorem 1.2.** Suppose $p \geq 3$ and $m > 1$, and assume that $A \subset \mathcal{O}_{F,v}$ for all $v \in \mathcal{V}(F, p)$. The $\mathcal{O}_{F,v}$-module

$$\mathcal{O}_{F,v} \otimes_A H^0(\mathcal{X}_A(\ell), \Omega^{\otimes m}_{\mathcal{X}_A(\ell)}) = \mathcal{O}_{F,v} \otimes_A \mathcal{S}_{2m}(A)$$

is a direct sum over blocks $\mathcal{B}$ of $\mathcal{O}_{F,v}G$ of modules of the form $P_{\mathcal{B}} \oplus U_{\mathcal{B}}$ in which $P_{\mathcal{B}}$ is a projective $B$-module and $U_{\mathcal{B}}$ is either the zero module or a single indecomposable non-projective $B$-module. Moreover, both $P_{\mathcal{B}}$ and the reduction $U_{\mathcal{B}}$ of $U_{\mathcal{B}}$ modulo $\mathfrak{m}_{F,v}$ are uniquely determined by the lower ramification groups and the fundamental characters of the closed points of $X_p(\ell)$ that are ramified in the cover $X_p(\ell) \to X_p(\ell)/G$.

Theorem [1, 2] extends \cite{5} Thm. 1.2] from $m = 1$ to arbitrary $m > 1$. Since $X_p(\ell)/G \cong \mathbb{P}^1_k$, we do not need to refer to the class of canonical divisors of $X_p(\ell)/G$. If $p > 3$ then the cover $X_p(\ell) \to X_p(\ell)/G \cong \mathbb{P}^1_k$ is tamely ramified, and we prove Theorem 1.2 using Nakajima’s work in \cite{37}. On the other hand, if $p = 3$ then the cover $X_3(\ell) \to X_3(\ell)/G \cong \mathbb{P}^1_k$ has wildly ramified points. In this case, we use Theorem 1.1 to determine the precise $kG$-module structure of $H^0(X_3(\ell), \Omega^{\otimes m}_{X_3(\ell)})$, see Theorem 1.3 below, which leads to Theorem 1.2 when $p = 3$. Since the Sylow $2$-subgroups of $G$ are not cyclic, our methods are not sufficient to treat the case when $p = 2$.

Similarly to \cite{5}, we use the general approach in \cite{28} to define congruences modulo $p$ between modular forms. Notice, however, that in contrast to \cite{28}, we consider cusp forms for the principal congruence subgroup $\Gamma_\ell$ (rather than for $\Gamma_0(\ell)$ or $\Gamma_1(\ell)$) and that we allow more general rings $T$ of Hecke operators to act (see below). Our goal is to use Theorem 1.2 to characterize when such congruences can arise from the decomposition of $F \otimes_A \mathcal{S}_{2m}(A)$ into $G$-isotypic pieces. We refer to \cite{11} Chap. 3 for a discussion of Hecke operators and their actions on modular forms (see also \cite{38}).

For $m > 1$, define $\mathcal{S}_{2m}(F) = F \otimes_A \mathcal{S}_{2m}(A)$ to be the space of cusp forms of weight $2m$ that have $q$-expansion coefficients in $F$ at all cusps, in the sense of \cite{26} §1.6. Let $T$ be a ring of Hecke operators acting on $\mathcal{S}_{2m}(F)$, and suppose there exists a decomposition

$$\mathcal{S}_{2m}(F) = E_1 \oplus E_2$$

into a direct sum of $F$-subspaces that are stable under the action of $T$. Let $\mathfrak{a}$ be an ideal of $A$. Following \cite{28}, we define a non-trivial congruence modulo $\mathfrak{a}$ linking $E_1$ and $E_2$ to be a pair $(f, g)$ of forms $f \in \mathcal{S}_{2m}(A) \cap E_1$ and $g \in \mathcal{S}_{2m}(A) \cap E_2$ such that

$$f \equiv g \mod \mathfrak{a} \cdot \mathcal{S}_{2m}(A) \quad \text{but} \quad f \notin \mathfrak{a} \cdot \mathcal{S}_{2m}(A).$$

Such congruences have played an important role in the development of the theory of modular forms, Galois representations and arithmetic geometry. For a further discussion, see for example \cite{13, 14}.
As above, let $G = \text{PSL}(2, F_\ell)$. We call a $T$-stable decomposition into $F$-subspaces of the form (1.1) $G$-isotypic if there exist two orthogonal central idempotents of $FG$ such that $1 = e_1 + e_2$ in $FG$ and

(1.2) \[ E_1 = e_1 S_{2m}(F) \quad \text{and} \quad E_2 = e_2 S_{2m}(F). \]

Similarly to [5 §7], we obtain that if the idempotents $e_1, e_2$ are fixed by the conjugation action of $\text{PGL}(2, F_\ell)$ on $G$ and if $T$ is the ring of Hecke operators of index prime to $\ell$, then substituting (1.2) in (1.1) leads to a $G$-isotypic $T$-stable decomposition; see Proposition 6.6.

We will show the following theorem which concerns non-trivial congruences arising from $G$-isotypic $T$-stable decompositions of $S_{2m}(F)$ and which extends [5, Thm. 1.3] from $m = 1$ to arbitrary $m > 1$.

**Theorem 1.3.** With the assumptions of Theorem 1.2 suppose further that $F$ contains a root of unity of order equal to the prime to $p$ part of the order of $G$. Let $a$ be the maximal ideal over $p$ in $A$ associated to $v \in \mathcal{V}(F, p)$. A $T$-stable decomposition (1.1) of $S_{2m}(F)$ that is $G$-isotypic, in the sense that it arises from orthogonal central idempotents as in (1.2), results in non-trivial congruences modulo $a$ between modular forms in $S_{2m}(F)$ if and only if the following is true. There is a block $B$ of $\mathcal{O}_{F,v}G$ such that when $P_B$ and $U_B$ are as in Theorem 1.2 $M_B = P_B \oplus U_B$ is not equal to the direct sum $(M_B \cap e_1 M_B) \oplus (M_B \cap e_2 M_B)$. For a given $B$, there will be orthogonal idempotents $e_1$ and $e_2$ for which this is true if and only if $B$ has non-trivial defect groups, and either $P_B \neq \{0\}$ or $F_v \otimes_{\mathcal{O}_{F,v}} U_B$ has two non-isomorphic irreducible constituents.

Let now $p = 3$ and let $X_3(\ell)$ be, as above, the reduction of $X_A(\ell)$ modulo 3 over an algebraically closed field $k$ of characteristic 3 containing all residue fields $k(v)$ of $A$ for $v \in \mathcal{V}(F, 3)$. If $\ell = 5$ then $X_3(\ell)$ has genus 0. For $\ell \geq 7$, Theorem 1.5 below gives a description of the isomorphism classes of all indecomposable $kG$-modules.

**Notation 1.4.** Let $\ell \geq 7$ be a prime number, let $G = \text{PSL}(2, F_\ell)$, and let $k$ be an algebraically closed field of characteristic 3.

(a) If $T$ is a simple $kG$-module, then $U^{(G)}_{T,b}$ denotes a uniserial $kG$-module with $b$ composition factors whose socle is isomorphic to $T$. The isomorphism class of $U^{(G)}_{T,b}$ is uniquely determined by $T$ and $b$ (see, for example, [2]).

(b) If $\ell \equiv -1 \mod 3$ then there exist indecomposable $kG$-modules that are not uniserial. These modules all belong to the principal block of $kG$. There are precisely two isomorphism classes of simple $kG$-modules belonging to the principal block, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $\tilde{T}_0$ of $k$-dimension $\ell - 1$. It follows, for example, from [2] that the isomorphism classes of the non-uniserial indecomposable $kG$-modules are all uniquely determined by their socles and their tops (i.e. radical quotients), together with the number of their composition factors that are isomorphic to $\tilde{T}_0$. There are three different types of such modules. Namely, $U^{(G)}_{T_0, T_0, b}$ (resp. $U^{(G)}_{\tilde{T}_0, T_0, b}$, resp. $U^{(G)}_{T_0, \tilde{T}_0, b}$) denotes an indecomposable $kG$-module that has $b$ composition factors isomorphic to $T_0$, whose socle is isomorphic to $T_0 \oplus \tilde{T}_0$ (resp. $\tilde{T}_0$, resp. $T_0 \oplus \tilde{T}_0$) and whose top is isomorphic to $\tilde{T}_0$ (resp. $T_0 \oplus \tilde{T}_0$, resp. $T_0 \oplus \tilde{T}_0$).

**Theorem 1.5.** Let $m > 1$ be an integer, let $\ell \geq 7$ be a prime number, and let $k$ be an algebraically closed field of characteristic 3 containing all residue fields $k(v)$ of $A$ for $v \in \mathcal{V}(F, 3)$. Define $X = X_3(\ell)$. 


Let $\epsilon = \pm 1$ be such that $\ell \equiv \epsilon \mod 3$. Write $\ell - \epsilon = 2 \cdot 3^n \cdot n'$ where $3$ does not divide $n'$. For $i \in \{0, 1\}$, define $\delta_i$ to be $1$ if $m \equiv i \mod 3$ and to be $0$ otherwise. Define $\delta_m \in \{0, 1\}$ by $m \equiv \delta_m \mod 2$.

(i) There exists a projective $kG$-module $Q_\ell$, depending on $\ell$, such that the following is true:

1. Suppose $\ell \equiv 1 \mod 4$ and $\ell \equiv -1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module. For $0 \leq t \leq (n' - 1)/2$, let $\overline{T}_t$ be representatives of simple $kG$-modules of $k$-dimension $\ell - 1$ such that $\overline{T}_0$ belongs to the principal block of $kG$. As a $kG$-module,

$$H^0(X, \Omega^\otimes_X) \cong \delta_1 \delta_m U^{(G)}_{T_0, T_0, (3^n, 1-1)/2} \oplus \delta_1 (1 - \delta_m) U^{(G)}_{T_0, T_0, (3^n, 1-1)/2}$$

$$\oplus \delta_0 (1 - \delta_m) U^{(G)}_{T_0, T_0, 3^n-1} \oplus \delta_0 \delta_m U^{(G)}_{T_0, 3^n-1} \oplus \sum_{t=1}^{(n' - 1)/2} \delta_1 U^{(G)}_{T_t, 3^n-1} \oplus \sum_{t=1}^{(n' - 1)/2} \delta_0 U^{(G)}_{T_t, 2, 3^n-1} \oplus Q_\ell.$$

2. Suppose $\ell \equiv -1 \mod 4$ and $\ell \equiv 1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module, and let $T_1$ be a simple $kG$-module of $k$-dimension $\ell$. For $1 \leq t \leq (n' - 1)/2$, let $\overline{T}_t$ be representatives of simple $kG$-modules of $k$-dimension $\ell + 1$. As a $kG$-module,

$$H^0(X, \Omega^\otimes_X) \cong \delta_1 \delta_m U^{(G)}_{T_0, 2, 3^n-1} \oplus \delta_1 (1 - \delta_m) U^{(G)}_{T_1, 2, 3^n-1}$$

$$\oplus \delta_0 (1 - \delta_m) U^{(G)}_{T_0, 3^n-1} \oplus \delta_0 \delta_m U^{(G)}_{T_1, 3^n-1} \oplus \sum_{t=1}^{(n' - 1)/2} \delta_1 U^{(G)}_{T_t, 2, 3^n-1} \oplus \sum_{t=1}^{(n' - 1)/2} \delta_0 U^{(G)}_{T_t, 3^n-1} \oplus Q_\ell.$$

3. Suppose $\ell \equiv 1 \mod 4$ and $\ell \equiv 1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module, and let $T_1$ be a simple $kG$-module of $k$-dimension $\ell$. For $1 \leq t \leq (n' - 2)/2$, let $\overline{T}_t$ be representatives of simple $kG$-modules of $k$-dimension $\ell + 1$. There exist simple $kG$-modules $T_{0, 1}$ and $T_{1, 0}$ of $k$-dimension $(\ell + 1)/2$ such that, as a $kG$-module,

$$H^0(X, \Omega^\otimes_X) \cong \delta_1 \delta_m \left( U^{(G)}_{T_0, 2, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right) \oplus \delta_1 (1 - \delta_m) \left( U^{(G)}_{T_1, 2, 3^n-1} \oplus U^{(G)}_{T_1, 1, 3^n-1} \right)$$

$$\oplus \delta_0 (1 - \delta_m) \left( U^{(G)}_{T_0, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right) \oplus \delta_0 \delta_m \left( U^{(G)}_{T_1, 1, 3^n-1} \oplus U^{(G)}_{T_1, 0, 3^n-1} \right) \oplus \sum_{t=1}^{(n' - 2)/2} \delta_1 \overline{U}^{(G)}_{T_t, 2, 3^n-1} \oplus \sum_{t=1}^{(n' - 2)/2} \delta_0 \overline{U}^{(G)}_{T_t, 3^n-1} \oplus Q_\ell.$$

4. Suppose $\ell \equiv -1 \mod 4$ and $\ell \equiv -1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module. For $0 \leq t \leq (n' - 2)/2$, let $\overline{T}_t$ be representatives of simple $kG$-modules of $k$-dimension $\ell - 1$ such that $\overline{T}_0$ belongs to the principal block of $kG$. There exist simple $kG$-modules $T_{0, 1}$ and $T_{1, 0}$ of $k$-dimension $(\ell - 1)/2$ such that, as a $kG$-module,

$$H^0(X, \Omega^\otimes_X) \cong \delta_1 \delta_m \left( U^{(G)}_{T_0, 2, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right) \oplus \delta_1 (1 - \delta_m) \left( U^{(G)}_{T_0, 2, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right)$$

$$\oplus \delta_0 (1 - \delta_m) \left( U^{(G)}_{T_0, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right) \oplus \delta_0 \delta_m \left( U^{(G)}_{T_0, 2, 3^n-1} \oplus U^{(G)}_{T_0, 1, 3^n-1} \right) \oplus \sum_{t=1}^{(n' - 2)/2} \delta_1 \overline{U}^{(G)}_{T_t, 3^n-1} \oplus \sum_{t=1}^{(n' - 2)/2} \delta_0 \overline{U}^{(G)}_{T_t, 2, 3^n-1} \oplus Q_\ell.$$

The multiplicities of the projective indecomposable $kG$-modules in $Q_\ell$ are known explicitly. In parts (3) and (4), there are two conjugacy classes of subgroups of $G$, represented by $H_1$ and $H_2$, that are isomorphic to the symmetric group $\Sigma_3$ such that the conjugates of $H_1$ (resp. $H_2$) occur (resp. do not occur) as inertia groups of closed points of $X$. This characterizes the simple $kG$-module $T_{0, 1}$ in
parts (3) and (4) as follows. The restriction of \( T_{0,1} \) to \( H_1 \) (resp. \( H_2 \)) is a direct sum of a projective module and a non-projective indecomposable module whose socle is the trivial simple module (resp. the simple module corresponding to the sign character).

(ii) Let \( k_1 \) be a perfect field of characteristic 3 containing all residue fields \( k(v) \) of \( A \) for \( v \in \mathcal{V}(F,3) \), and let \( k \) be an algebraic closure of \( k_1 \). Define \( X_1 = k_1 \otimes_2 X_1 \). Then

\[
  k \otimes_{k_1} H^0(X_1,\Omega_X^{\otimes m}) \cong H^0(X,\Omega_X^{\otimes m})
\]
as \( kG \)-modules, and the decomposition of \( H^0(X_1,\Omega_X^{\otimes m}) \) into indecomposable \( kG \)-modules is uniquely determined by the decomposition of \( H^0(X,\Omega_X^{\otimes m}) \) into indecomposable \( kG \)-modules. Moreover, the \( kG \)-module \( H^0(X,\Omega_X^{\otimes m}) \) is a direct sum over blocks \( B \) of \( kG \) of modules of the form \( P_{B} \otimes U_{B} \), in which \( P_{B} \) is a projective \( B \)-module and \( U_{B} \) is either the zero module or a single indecomposable non-projective \( B \)-module. In addition, one can determine \( P_{B} \) and \( U_{B} \) from the lower ramification groups and the fundamental characters of the closed points of \( X \) that are ramified in the cover \( X \to X/G \).

Theorem 1.5 extends [4, Thm. 1.4] from \( m = 1 \) to arbitrary \( m > 1 \). We prove Theorem 1.5 by applying Theorem 1.1 together with a description of the blocks of \( kG \) and their Brauer trees in [7].

The paper is organized as follows. In §2 we set up our notation and recall some results needed for the paper. In particular, we discuss equivariant sheaves in §2.1, we discuss canonical divisors, inverse differentials, and fundamental characters in §2.2 and we show in §2.3 how to reduce the proof of Theorem 1.1 to the case when \( G \) is a \( p \)-hypo-elementary group and \( k \) is algebraically closed. In §3 we prove Theorem 1.1 by decomposing the Galois cover \( X \to X/G \) into a wildly ramified cover \( X \to X/I \) followed by a tamely ramified cover \( X/I \to G \). We analyze the wildly ramified cover in §3.1 and the tamely ramified cover in §3.2. In §3.3 we summarize the key steps of the proof of Theorem 1.1 in Algorithm 3.7. In §3.4 we discuss an application of Theorem 1.1 to the deformation theory of curves; see Proposition 3.7. In §4 we illustrate Theorem 1.1 and Proposition 3.7 by considering a particular family of hyperelliptic curves. In §5 we discuss the holomorphic poly-differentials of the reductions \( X_p(\ell) \) of the modular curves modulo \( p \), and we prove Theorems 1.2 and 1.3 when \( p > 3 \). In §6 we fully determine the \( kPSL(2,\mathbb{F}_\ell) \)-module structure of \( H^0(X_3(\ell),\Omega_X^{\otimes m}(\ell)) \) when \( k \) is an algebraically closed field containing containing all residue fields \( k(v) \) of \( A \) for \( v \in \mathcal{V}(F,3) \); see Propositions 6.5.1, 6.5.2 for the precise statements. In particular, this proves Theorem 1.6, which we then use to prove Theorems 1.2 and 1.3 when \( p = 3 \). In §7 we show how our method can be used to determine the \( kG \)-module structure of Riemann-Roch spaces associated to divisors on \( X \) of degree greater than \( 2g(X) - 2 \); see Proposition 7.1.

Part of this paper is the Ph.D. thesis of the second author under the supervision of the first (see [45]).

2. Preliminaries

In this section, we set up our notation and recall some results needed for the paper. Throughout this section, \( X \) is a smooth projective irreducible curve over a perfect field \( k \) of positive characteristic \( p \), and \( G \) is a finite group acting faithfully on the right on \( X \) by automorphisms of \( X \) over \( k \). For \( x \in X \) and \( g \in G \), we write \( x.g = \bar{g}(x) \) where \( \bar{g} \) is the automorphism of \( X \) associated to \( g \).

2.1. Equivariant sheaves. Let \( B_G = \text{Maps}(G,k) \), and let \( \Gamma_G = \text{Spec}(B_G) \) be the constant group scheme over \( k \) associated to \( G \). We write \( \mu : \Gamma_G \times_k \Gamma_G \to \Gamma_G \) for the group law, and \( \varepsilon : \text{Spec}(k) \to \Gamma_G \) for the identity section. Note that \( \Gamma_G = \coprod_{g \in G} \Gamma_g \) for the closed subschemes \( \Gamma_g = \text{Spec}(B_G/J_g) \) where \( J_g \) is the kernel of the \( k \)-algebra homomorphism \( B_G \to k \) given by evaluation at \( g \in G \). The right action of \( G \) on \( X \) corresponds to a right action of \( \Gamma_G \) on \( X \), given by the following morphism \( \vartheta : X \times_k \Gamma_G \to X \). Writing
$X \times_k \Gamma_G = \coprod_{g \in G} X \times_k \Gamma_g$, $\vartheta$ is the unique morphism such that on each $X \times_k \Gamma_g$, $\vartheta$ is given as $g \circ p_{1,g}$ where $p_{1,g} : X \times_k \Gamma_g \to X$ is the projection onto the first component.

We recall from [43, §1.2] (see also [25, §1.3]) the notion of $\mathcal{O}_X$-$G$-modules, which are also called $G$-equivariant sheaves on $X$; see also [3 §2]. Let $p_1 : X \times_k \Gamma_G \to X$ (resp. $p_{12} : X \times_k \Gamma_G \times_k \Gamma_G \to X \times_k \Gamma_G$) denote the projection onto the first component (resp. the first and second components). A quasi-coherent (resp. coherent, resp. locally free coherent) $\mathcal{O}_X$-$G$-module is defined to be a quasi-coherent (resp. coherent, resp. locally free coherent) sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules together with an isomorphism of $\mathcal{O}_X \times_k \Gamma_G$-modules

$$\phi : \vartheta^* \mathcal{F} \to p_{1}^* \mathcal{F}$$

such that $\phi$ is associative, in the sense that it satisfies the cocycle condition

$$(p_{12}^* \phi) \circ ((\vartheta \times 1_{\Gamma_G})^* \phi) = (1_X \times \mu)^* \phi$$

on $X \times_k \Gamma_G \times_k \Gamma_G$. On the stalk level, the cocycle condition means that, for all $x \in X$ and all $g_1, g_2 \in G$, the isomorphism $\mathcal{F}_{x,(g_1,g_2)} \cong \mathcal{F}_x$ is the same as the composition $\mathcal{F}_{x,(g_1),g_2} \cong \mathcal{F}_{x,g_1} \cong \mathcal{F}_x$. In other words, the cocycle condition corresponds to the associativity of the group action of $G$ on $X$. The unitarity of the group action is also a consequence, by applying $(1_X \times 1 \times \varepsilon)^*$ to both sides of (2.1).

Using [10, §1.2.5]), we can equivalently define a quasi-coherent $\mathcal{O}_X$-$G$-module to be a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules with a compatible action of $G$ in the following sense. For $x \in X$ and $g \in H$, the actions of $g$ on $X$ and on $\mathcal{F}$ give isomorphisms of stalks $g : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ and $g : \mathcal{F}_{x,g} \to \mathcal{F}_x$ such that $g(\alpha f) = g(\alpha).g(f)$ for all $\alpha \in \mathcal{O}_{X,x}$ and $f \in \mathcal{F}_{x,g}$.

Note that $H^0(X, \mathcal{F})$ has a left $kG$-module structure as follows. Consider the composition of morphisms

$$H^0(X, \mathcal{F}) \xrightarrow{\vartheta^*} H^0(X \times_k \Gamma_G, \vartheta^* \mathcal{F}) \xrightarrow{\phi} H^0(X \times_k \Gamma_G, p_{1}^* \mathcal{F}) \hookrightarrow \text{Maps}(G, H^0(X, \mathcal{F}))$$

where the last arrow $\iota$ results from the identification

$$H^0(X \times_k \Gamma_G, p_{1}^* \mathcal{F}) = H^0(X, \mathcal{F}) \otimes_k H^0(\Gamma_G, \mathcal{O}_{\Gamma_G}) = H^0(X, \mathcal{F}) \otimes_k \text{Maps}(G, k) = \text{Maps}(G, H^0(X, \mathcal{F})).$$

Then, for $f \in H^0(X, \mathcal{F})$ and $g \in G$, we have $g.f = (\iota \circ \phi \circ \vartheta^*)(f)(g)$.

Let $k$ be a fixed algebraic closure of $k$, and let $\overline{X}$ and $\overline{\mathcal{F}}$ denote the base change of $X$ and $\mathcal{F}$ from $k$ to $\overline{k}$, respectively. By flat base change (see [10, Prop. III.9.3]), we obtain

$$\overline{k} \otimes_k H^0(X, \mathcal{F}) \cong H^0(\overline{X}, \overline{\mathcal{F}})$$

as $kG$-modules.

Let $I$ be a subgroup of $G$. Since $X$ is a smooth projective curve and $k$ is a perfect field, the quotient scheme $Y = X/I$ exists and is a smooth projective curve over $k$. Let $\pi : X \to Y$ denote the corresponding quotient morphism. The sheaf of rings $\pi_* \mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-$G$-module, and we identify $\mathcal{O}_Y$ with the subsheaf of $I$-invariants of $\pi_* \mathcal{O}_X$. Suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-$G$-module. Then $\pi_* \mathcal{F}$ is a quasi-coherent sheaf of $\pi_* \mathcal{O}_X$-modules with an action of $G$ that is compatible with the action of $G$ on $\pi_* \mathcal{O}_X$ over $\mathcal{O}_Y$. Hence we can view $\pi_* \mathcal{F}$ as a quasi-coherent $\mathcal{O}_Y$-$G$-module. Moreover, if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-$G$-module then $\pi_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$ is also a quasi-coherent $\mathcal{O}_Y$-$G$-module by letting $G$ act diagonally. Note that if $\mathcal{F}$ is coherent (resp. locally free coherent) as an $\mathcal{O}_X$-module and $\mathcal{G}$ is coherent (resp. locally free coherent) as an $\mathcal{O}_Y$-module, then so are $\pi_* \mathcal{F}$ and $\pi_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$ as $\mathcal{O}_Y$-$G$-modules.

Suppose additionally that $I$ is a normal subgroup of $G$, and that $J$ is an ideal of $kI$ that is taken to itself by the conjugation action of $G$ on $I$. Since $I$ acts trivially on $\mathcal{O}_Y$, we can regard $\pi_* \mathcal{F}$ as a module for the sheaf of group rings $\mathcal{O}_Y I$ on $Y$. We define the kernel $\mathcal{K} = K(\mathcal{F}, I, J)$ of $J$ acting on $\pi_* \mathcal{F}$ to be the sheaf of $\mathcal{O}_Y$-modules having sections over each open set $V$ of $Y$ equal to the kernel of $J$ acting on $\pi_* \mathcal{F}(V)$. Since we
assume that $J$ is taken to itself by the conjugation action of $G$ on $kI$, it follows that $K$ is a quasi-coherent $O_Y$-module.

2.2. Canonical divisors, inverse differentials, and fundamental characters. In this subsection, we suppose that $k$ is algebraically closed. Let $I$ be a normal subgroup of $G$, and let $Y = X/I$ with corresponding quotient morphism $\pi : X \to Y$. Define $\overline{G} = G/I$, and let $Z = X/G = Y/\overline{G}$ with corresponding quotient morphism $\lambda : Y \to Z$.

Let $K_Z$ be a canonical divisor on $Z$, and let $R_{Y/Z}$ be the ramification divisor of $Y$ over $Z$. By [16] Prop. IV.2.3,  

\[ K_Y = \lambda^* K_Z + R_{Y/Z} \]  

is a canonical divisor on $Y$. Moreover, $K_Y$ is $\overline{G}$-invariant, and, for all integers $m \geq 1$, there is an isomorphism $\Omega_Y^{\otimes m} \cong O_Y(mK_Y)$ of $O_Y$-$\overline{G}$-modules, which, by inflation from $\overline{G}$ to $G$, is also an $O_Y$-$G$-module isomorphism. Note that  

\[ K_X = \pi^* K_Y + R_{X/Y} = (\lambda \circ \pi)^* K_Z + R_{X/Z} \]  

is a canonical $G$-invariant divisor on $X$, and, for all integers $m \geq 1$, we have that $\Omega_X^{\otimes m} \cong O_X(mK_X)$ as $O_X$-$G$-modules.

Let $k(X)_X$ denote the constant sheaf on $X$ associated to the function field $k(X)$. By a fractional ideal sheaf on $X$ we will mean a subsheaf of $k(X)_X$ that is a coherent $O_X$-module. The inverse different $D_{X/Y}^{-1}$ is the largest fractional ideal sheaf on $X$ such that $\text{Tr}_{k(X)/k(Y)}(D_{X/Y}^{-1}) \subseteq O_Y$. In particular, $D_{X/Y}^{-1}$ is a locally free sheaf on $X$ of rank one, and, for all integers $m \geq 1$, we have that $\Omega_{X}^{\otimes m} \cong O_X(mK_X)$ as $O_X$-$G$-modules.

Hence we obtain, for all $m \geq 1$, an isomorphism  

\[ \Omega_{X}^{\otimes m} \cong \pi^* \Omega_Y^{\otimes m} \otimes_{O_Y} \pi_* D_{X/Y}^{-1,\otimes m} \]  

of $O_X$-$G$-modules. Applying $\pi_*$ to both sides of (2.5) and using the projection formula (see [16] Ex. II.5.1), we obtain, for all $m \geq 1$, an isomorphism  

\[ \pi_* \Omega_X^{\otimes m} \cong \Omega_Y^{\otimes m} \otimes_{O_Y} \pi_* D_{X/Y}^{-1,\otimes m} \]  

of $O_Y$-$G$-modules.

Let $x \in X$, and let $G_{x,0}$ be the inertia group of $x$ in $G$. The fundamental character of $G_{x,0}$ is the character $\theta_x : G_{x,0} \to k(x)^*$ given by the action of $G_{x,0}$ on the cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Note that $\theta_x$ factors through the maximal $p'$-quotient group $G_{x,1}/G_{x,0}$ of $G_{x,0}$. In particular, $\theta_x$ is the trivial character if $G_{x,0}$ is trivial or a $p$-group. Since we assume $k$ to be algebraically closed, we have $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \cong k$ and $k(x) = O_{X,x}/\mathfrak{m}_{X,x} \cong k$. If $\pi_x$ is a uniformizer for $\mathfrak{m}_{X,x}$ then $\theta_x$ is given by  

\[ \theta_x(g) = \frac{g \pi_x}{\pi_x} \mod \mathfrak{m}_{X,x} \quad \text{for all } g \in G_{x,0}. \]  

Similarly, for $y \in Y$, we can define the fundamental character $\theta_y : G_{y,0} \to k(y)^* = k^*$ using a uniformizer $\pi_y$ for $\mathfrak{m}_{Y,y}$. 

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Remark 2.1. Let $K_Y$ be as in \cite{2}, let $r \geq 0$ be an integer and let $D$ be a $G$-invariant divisor on $Y$. Fix a point $y \in Y$. Then the stalk of $\mathcal{O}_Y(rK_Y + D)$ at $y$ satisfies the equalities
\[
\mathcal{O}_Y(rK_Y + D)_y = m_{Y,y}^{-\text{ord}_y(rK_Y + D)} = \pi_y^{-\text{ord}_y(rK_Y + D)} \mathcal{O}_{Y,y}
\]
as fractional ideals of $\mathcal{O}_{Y,y}$. This implies that $\mathcal{O}_Y(rK_Y + D)_y \otimes_{\mathcal{O}_{Y,y}} k$ is a $k\overline{G}_{y,0}$-module of $k$-dimension one whose character equals $\theta_y \text{ord}_y(rK_Y + D)$. Suppose now that $r \geq 1$ and $K'_Z = K_Z + \text{div}(f)$ for some $f \in k(Z) = k(Y)^G$ and define $K'_Y = \lambda^* K'_Z + R_Y/Z$. Then the map
\[
\nu : \mathcal{O}_Y(rK'_Y + D)_y \to \mathcal{O}_Y(rK_Y + D)_y
\]
given by $\nu(h') = f^* h'$ for all $h' \in \mathcal{O}_Y(rK'_Y + D)_y$ is an $\mathcal{O}_{Y,y}\overline{G}_{y,0}$-module isomorphism. This implies that $\text{ord}_y(rK_Y + D) \equiv \text{ord}_y(rK'_Y + D) \mod \#\overline{G}_{y,0}$. In particular, the residue class $\text{ord}_y(rK_Y + D) \mod \#\overline{G}_{y,0}$ does not depend on the choice of canonical divisor $K_Z$ in \cite{2}.

Remark 2.2. Suppose $I$ is a normal $p$-subgroup of $G$ that contains the Sylow $p$-subgroups of the inertia groups of all closed points of $X$. Hence $\lambda : Y = X/I \to Z = X/G$ is tamely ramified. Let $y \in Y$ be a ramified point in this cover, and suppose $x \in X$ lies above $y$. Then $\overline{G}_{y,0}$ is a cyclic $p'$-group, and if $I_{x,0}$ is the inertia group of $x$ in $I$ then
\[
\overline{G}_{y,0} \cong G_{x,0}/I_{x,0}.
\]
In particular, $\theta_x$ factors through $\overline{G}_{y,0}$. By our assumptions on $I$, we have $\text{ord}_x(\pi_y) = \#I_{x,0}$, and we can choose $\pi_y := \pi_x^{\#I_{x,0}}$. Thus we can identify
\[
(2.8) \quad \theta_y = \theta_x^{\#I_{x,0}}
\]
on the maximal $p'$-quotient of $G_{x,0}$ which we identify with $\overline{G}_{y,0}$. In other words, $\theta_y$ is uniquely determined by $\theta_x$ and $\#I_{x,0}$. Since $\overline{G}_{y,0}$ is a group of order prime to $p$, we can identify $\theta_x$ and $\theta_y$ with their Brauer characters on $\overline{G}_{y,0}$ by fixing an isomorphism between the $(\#\overline{G}_{y,0})^{1/2}$ roots of unity both in $k$ and in the ring of infinite Witt vectors over $k$.

2.3. Reduction to $p$-hypo-elementary subgroups and algebraically closed base fields. In this subsection, we assume again that $k$ is perfect. We show how to reduce the proof of Theorem \cite{1} to the case when $k$ is algebraically closed and $G$ is a $p$-hypo-elementary group.

A $p$-hypo-elementary group is a group $H$ that is equal to a semidirect product $H = P \rtimes C$ for a normal $p$-subgroup $P$ and a cyclic $p'$-group $C$. The following result is a consequence of the Conlon Induction Theorem (see \cite{1} Thm. (80.51)); see also \cite{5} Lemma 3.2).

Lemma 2.3. Suppose $M$ is a finitely generated $kG$-module. Then the decomposition of $M$ into its indecomposable direct $kG$-module summands is uniquely determined by the decompositions of the restrictions $\text{Res}^G_H M$ of $M$ into a direct sum of indecomposable $kH$-modules as $H$ ranges over all $p$-hypo-elementary subgroups of $G$.

If $G$ has cyclic Sylow $p$-subgroups then so does every $p$-hypo-elementary subgroup $H$ of $G$. Let $\overline{k}$ be a fixed algebraic closure of $k$. The following result is straightforward; see \cite{5} Prop. 3.5).

Lemma 2.4. Let $H = P \rtimes C$ be a $p$-hypo-elementary group where $P$ is cyclic. If $M$ is a finitely generated $kH$-module, then its decomposition into a direct sum of indecomposable $kH$-modules is uniquely determined by the decomposition of $\overline{k} \otimes_k M$ into a direct sum of indecomposable $\overline{k}H$-modules.

In other words, if $G$ has cyclic Sylow $p$-subgroups and $F$ is a quasi-coherent $O_X$-$G$-module, then Lemmas \cite{2} and \cite{4} together with the isomorphism in \cite{2} show that the $kG$-module structure of $H^0(X, F)$ is uniquely
determined by the $\overline{k}H$-module structure of $H^0(X,\overline{\mathcal{F}})$ as $H$ ranges over all $p$-hypo-elementary subgroups of $G$.

The next remark describes the indecomposable $\overline{k}H$-modules when $H$ is a $p$-hypo-elementary group with a cyclic Sylow $p$-subgroup (see, for example, [3] pp. 35-37 & 42-43); see also [3] Remark 3.4.

Remark 2.5. Suppose $H = P \rtimes \chi C$ where $P = \langle \sigma \rangle$ is cyclic of order $p^n > 1$, $C = \langle \rho \rangle$ is cyclic of order $c$ relatively prime to $p$, and $\psi : C \to \text{Aut}(P)$. Then $\text{Aut}(P) \cong \mathbb{F}_p^* \times Q$ for an abelian $p$-group $Q$, and $\psi$ factors through a character $\chi : C \to \mathbb{F}_p^*$. To emphasize this character, we write $H = P \rtimes \chi C$. Note that the order of $\chi$ divides $p - 1$. For all $i \in \mathbb{Z}$, $\chi^i$ defines a simple $kC$-module of $k$-dimension one, which we denote by $T_{\chi^i}$. We also view $T_{\chi^i}$ as a $kH$-module by inflation.

Let $\zeta$ be a primitive $c^{th}$ root of unity in $\overline{k}$. For $0 \leq a \leq c - 1$, let $S_a$ be the simple $\overline{k}C$-module on which $\rho$ acts as $\zeta^a$. We also view $S_a$ as a $\overline{k}H$-module by inflation. For $i \in \mathbb{Z}$, define $S_{\chi^i} = \overline{k} \otimes_k T_{\chi^i}$ and, for $0 \leq a \leq c - 1$, define $\chi^i(a) \in \{0, 1, \ldots, c - 1\}$ to be such that $S_{\chi^i(a)} \cong S_a \otimes \overline{k}$. The projective cover of the trivial simple $\overline{k}H$-module $S_0$ is uniserial, in the sense that it has a unique composition series, with $p^n$ ascending composition factors of the form

$$S_0, S_{\chi^{-1}}, S_{\chi^{-2}}, \ldots, S_{\chi^{-(p - 2)}}, S_0, S_{\chi^{-1}}, \ldots, S_{\chi^{-(p - 2)}}, S_0.$$

More generally, the projective cover of the simple $\overline{k}H$-module $S_a$, for $0 \leq a \leq c - 1$, is uniserial with $p^n$ ascending composition factors of the form

$$S_a, S_{\chi^{-1}(a)}, S_{\chi^{-2}(a)}, \ldots, S_{\chi^{-(p - 2)(a)}}, S_a, S_{\chi^{-1}(a)}, \ldots, S_{\chi^{-(p - 2)(a)}}, S_a.$$

There are precisely $\#H$ isomorphism classes of indecomposable $\overline{k}H$-modules, and they are all uniserial. If $U$ is an indecomposable $\overline{k}H$-module, then it is uniquely determined by its socle, which is the kernel of the action of $(\sigma - 1)$ on $U$, and its $k$-dimension. For $0 \leq a \leq c - 1$ and $1 \leq b \leq p^m$, let $U_{a,b}$ be an indecomposable $\overline{k}H$-module with socle $S_a$ and $k$-dimension $b$. Then $U_{a,b}$ is uniserial and its $b$ ascending composition factors are equal to the first $b$ ascending composition factors in (2.10).

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We use the geometric method which was given for the case $m = 1$ in [3] §3-4. We adopt the conventions and notation from [2]. Throughout this section, we assume that $m > 1$ is a fixed integer.

By Lemmas 2.3 and 2.4 together with the isomorphism in (2.2), it is sufficient to prove Theorem 1.1 under the following assumptions, which we will make throughout this section.

Assumption 3.1. Let $k$ be an algebraically closed field of prime characteristic $p$. Assume $G = P \rtimes \chi C$ is a $p$-hypo-elementary group, where $P = \langle \sigma \rangle$ is a cyclic $p$-group of order $p^n > 1$, $C = \langle \rho \rangle$ is a cyclic $p'$-group of order $c$, and $\chi : C \to \mathbb{F}_p^*$ is a character. Let $X$ be a smooth projective curve over $k$ of genus $g(X) \geq 2$, and suppose there a faithful right action of $G$ on $X$ over $k$.

As in (2.3), we view $\chi$ as a character of $G$ by inflation, and we denote, for all $i \in \mathbb{Z}$, the one-dimensional $kG$-module with character $\chi^i$ by $S_{\chi^i}$.

Let $\Omega_X$ be the sheaf of relative differentials of $X$ over $k$, and let $\Omega_X^{\otimes m}$ be the $m$-fold tensor product of $\Omega_X$ with itself over $\mathcal{O}_X$, which we call the sheaf of relative $m$-differentials of $X$ over $k$. The sheaves $\Omega_X$ and $\Omega_X^{\otimes m}$ are coherent $\mathcal{O}_X$-$G$-modules that are locally free $\mathcal{O}_X$-modules of rank one. For any closed point $x$ of $X$ and integer $i \geq 0$, let $G_{x,i}$ denote the $i^{th}$ lower ramification subgroup of $G$. In other words, $G_{x,i}$ is the group of all elements in $G$ that fix $x$ and act trivially on $\mathcal{O}_{X,x}/m_{X,x}^{i+1}$. In particular, $G_{x,0}$ is the inertia group of $x$ in $G$.  

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Notation 3.2. Suppose Assumption 3.1 holds. Let $I = \langle \tau \rangle$ be the (cyclic) subgroup of $P$ generated by the Sylow $p$-subgroups of the inertia groups of all closed points of $X$. Suppose $#I = p^n$, where $0 \leq n_I \leq n$. Let $Y = X/I$, and let $\pi : X \to Y$ denote the quotient morphism. Define $G = G/I$, and let $Z = X/G = Y/G$ with corresponding quotient morphism $\lambda : Y \to Z$. Suppose $\mathcal{F}$ is a coherent $O_X$-module that is a locally free $O_X$-module of finite rank, and let $J = kI(\tau - 1)$ be the Jacobson radical of the group ring $kI$. For $0 \leq j \leq #I$, we denote by $H^0(X, \mathcal{F}(j))$ the kernel of the action of $J^j = kI(\tau - 1)^j$ on $H^0(X, \mathcal{F})$, and we denote by $\pi_* \mathcal{F}(j)$ the kernel of the action of $J^j$ on $\pi_* \mathcal{F}$ (see (3.2)). If $D$ is a divisor on $Y$ and $r$ is a positive integer, we denote by $\Omega^r_Y(D)$ the tensor product $\Omega^r_Y \otimes O_Y(D)$.

Note that $I$ is a normal subgroup of $G$, and that $\pi$ is wildly ramified, provided $#I > 1$, and that $\lambda$ is tamely ramified. Because $J^j$ is taken to itself by the conjugation action of $G$ on $I$, it follows that $H^0(Y, \Omega^m_X(j))$ is a $kG$-module and that $\pi_* \Omega^m_X(j)$ is a quasi-coherent $O_Y$-module. Moreover, since $\pi_* \Omega^m_X(j)$ is a subsheaf of a locally free coherent $O_Y$-module of finite rank, we obtain that $\pi_* \Omega^m_X(j)$ is a locally free coherent $O_Y$-module.

The steps to prove Theorem 1.1 are as follows. In §3.1, we consider the filtration $\{\pi_* \Omega^{m,(j)}_X\}_{j=0}^{#I}$ of the $G$-sheaf $\pi_* \Omega^m_X$ and compare it to the filtration $\{H^0(Y, \Omega^m_X(j))\}_{j=0}^{#I}$ of the $kG$-module $H^0(Y, \Omega^m_X)$. More precisely, we prove in Proposition 3.3 that, for $0 \leq j \leq #I - 1$, there exists a $G$-invariant divisor $D_j$ on $Y$ that is determined by the lower ramification groups of the closed points of $X$ that are ramified in the cover $\pi : X \to Y = X/I$ such that

\begin{equation}
\pi_* \Omega^{m,(j+1)}_X / \pi_* \Omega^{m,(j)}_X = S_{m,j} \otimes \Omega^m_Y(D_j)
\end{equation}

as $O_Y$-modules. Moreover, if $#I > 1$ then $\deg(D_j) > 2m - 2$ for all $0 \leq j \leq #I - 1$, which we show implies $H^1(Y, \pi_* \Omega^{m,(j+1)}_X / \pi_* \Omega^{m,(j)}_X) = 0$ for all $0 \leq j \leq #I - 1$. This is different from the case when $m = 1$ since there we have $H^1(Y, \pi_* \Omega^{1,(j+1)}_X / \pi_* \Omega^{1,(j)}_X) = k$. We then use a dimension argument to show in Lemma 3.4 that, for $0 \leq j \leq #I - 1$,

\begin{equation}
H^0(X, \Omega^m_X(j+1)) / H^0(X, \Omega^m_X(j)) \cong H^0(Y, \pi_* \Omega^{m,(j+1)}_X / \pi_* \Omega^{m,(j)}_X)
\end{equation}

as $kG$-modules. Hence it follows from (3.1) and (3.2) that, for $0 \leq j \leq #I - 1$, the $kG$-module structure of the quotient

\begin{equation}
H^0(X, \Omega^m_X(j+1)) / H^0(X, \Omega^m_X(j))
\end{equation}

is determined by $S_{m,j}$ and the $kG$-module structure of

\begin{equation}
H^0(Y, \Omega^m_Y(D_j)).
\end{equation}

In §3.2, we determine in Proposition 3.3 the $kG$-module structure of (3.4) using that the cover $\lambda : Y \to Z = Y/G$ is tamely ramified, which allows us to apply the results in §3.1. More precisely, we show that the $kG$-module structure of (3.4) is uniquely determined by the class $[K_Z]$ of a canonical divisor $K_Z$ of $Z$, together with the inertia groups and the fundamental characters of the closed points of $Y$ that are ramified in the cover $\lambda : Y \to Z$. This is different from the case when $m = 1$ since there one can use Serre duality to avoid the divisor class $[K_Z]$. At the end of §3.2, we prove Theorem 1.1 using Proposition 3.5, Lemma 3.4 and Proposition 3.3. In §3.3, we summarize the main steps in the proof of Theorem 1.1 in Algorithm 3.6. In §3.4, we discuss an application of Theorem 1.1 to the deformation theory of curves when $m = 2$ (see Proposition 3.7).
3.1. Analyzing the (wildly ramified) cover $\pi : X \to Y$. In this subsection, we prove the isomorphisms (3.1) and (3.2) under Assumption 3.1 and using Notation 3.2.

**Proposition 3.3.** Suppose $m > 1$, Assumption 3.1 and Notation 3.2 and fix $0 \leq j \leq p^n - 1$. The action of $O_Y$ and of $G$ on $\pi_*\Omega_X^m$ makes the quotient sheaf $L_j = \pi_*\Omega_X^{m(j+1)} / \pi_*\Omega_X^{m(j)}$ into a locally free coherent $O_Y$-module. There exists a $G$-invariant divisor $D_j$ on $Y$ with the following properties:

(i) There is an isomorphism of locally free coherent $O_Y$-modules between $L_j$ and $S_{\chi^{-1}} \otimes_k \Omega_Y^m(D_j)$.

(ii) The divisor $D_j$ is effective and determined by the lower ramification groups associated to the action of $I$ on $X$. More precisely, writing $D_j = \sum_{y \in Y} d_{y,j} y$, the coefficient $d_{y,j}$, for $y \in Y$, is given as follows. Let $x \in X$ lie above $y$, and let $I_x,0$ be the inertia group of $x$ in $I$. Write $\# I_x,0 = p^n$. Let $t \geq 0$ be the unique integer such that $p^{n_t-n_x} \leq t < p^{n_t-n_x}(t + 1)$. Then

\[
d_{y,j} = \left[ m \sum_{i \geq 0} (\# I_{x,i} - 1) - \sum_{i=1}^{n_x} d_{x,i} p^{n_x - t} i_t \right] / p^{n_x}
\]

where $a_1,t,\ldots,a_{n_x},t \in \{0,1,\ldots,p-1\}$ are given by the $p$-adic expansion of $t$,

\[
t = a_1,t + a_2,t p + \cdots a_{n_x},t p^{n_x-1},
\]

and $1 \leq i_1 < i_2 < \cdots < i_{n_x}$ are the $n_x$ jumps in the numbering of the lower ramification groups $I_{x,i}$.

In particular, $d_{y,j} = 0$ when $y$ is unramified in the cover $\pi : X \to Y$.

(iii) If $n_I \geq 1$, i.e. $\# I > 1$, then $\deg(D_j) > 2m - 2$ and $\deg(L_j) > 2g(Y) - 2$.

**Proof.** Suppose first that $n_I = 0$. Then $X = Y$, $\pi : X \to Y$ is the identity morphism, and $j = 0$. In this case, $L_j = \pi_*\Omega_X^m = \Omega_Y^m$. Hence $D_j$ is the zero divisor, and the formula (3.5) also gives $d_{y,j} = 0$ for all $y \in Y$.

For the remainder of the proof, we assume $n_I \geq 1$. The proofs of parts (i) and (ii) of Proposition 3.3 follow the same main steps as the proof of [1] Prop. 4.1 when $m = 1$. We will highlight the main points below. However, the proof of part (iii) is different from the case when $m = 1$.

To prove part (i), we note that by (2.1), we have an isomorphism

\[
\pi_*\Omega_X^m \cong \Omega_Y^m \otimes_{O_Y} \pi_*D_{X/Y}^{1,m}
\]

of $O_Y$-modules. Consider the short exact sequences of coherent $O_Y$-modules

\[
0 \to \pi_*\Omega_X^{m(j)} \to \pi_*\Omega_X^{m(j+1)} \to L_j \to 0
\]

and

\[
0 \to \pi_*D_{X/Y}^{1,m(j)} \to \pi_*D_{X/Y}^{1,m(j+1)} \to H_j \to 0
\]

where we use, as before, the notation $\pi_*D_{X/Y}^{1,m(j)}$ for the kernel of the action of $J^j = k(I - 1)^j$ on $\pi_*D_{X/Y}^{1,m}$. Since $I$ acts trivially on $O_Y$ and $\Omega_Y^m$ and since the functor $\Omega_Y^m \otimes_{O_Y} -$ is right exact, we can identify $L_j = \Omega_Y^m \otimes_{O_Y} H_j$ as coherent $O_Y$-modules.

We now show that $H_j$ is a line bundle for $O_Y$. Let $\eta_X$ (resp. $\eta_Y$) be the generic point on $X$ (resp. $Y$). Then for all $y \in Y$ and all $j \geq 0$, there is a canonical homomorphism $(\pi_*D_{X/Y}^{1,m(j)})_y \to (\pi_*D_{X/Y}^{1,m(j)})_{\eta_Y}$ between stalks. Since $(\pi_*D_{X/Y}^{1,m(j)})_{\eta_Y}$ is a vector space over $k(Y)$ and $\pi_*D_{X/Y}^{1,m(j)}$ is a locally free coherent $O_Y$-module, it follows that this homomorphism is injective. On the other hand, we can identify the stalk $(\pi_*D_{X/Y}^{1,m(j)})_{\eta_X} = (D_{X/Y}^{1,m(j)})_{\eta_X}$ and view the latter inside the function field $k(X)$. For all integers
Note that the stalk of \((3.8)\) \((\mathcal{H}_j)_y = (\pi_*\mathcal{D}_{X/Y}^{-1,\otimes m,(j)})_y \cap k(X)^{(j)}\). We obtain an injective homomorphism

\[
(\mathcal{H}_j)_y = (\pi_*\mathcal{D}_{X/Y}^{-1,\otimes m,(j+1)})_y \rightarrow (\pi_*\mathcal{D}_{X/Y}^{-1,\otimes m,(j)})_y \rightarrow k(X)^{(j+1)} / k(X)^{(j)}.
\]

Note that the module on the right is a one-dimensional vector space over \(k(Y) = k(X)^{Y}\), since, by the normal basis theorem, \(k(X)\) is a free rank one module over the group ring \(k(Y)I\). Since the image of \((\mathcal{H}_j)_y\) under the map \(\mathcal{D}_{X/Y}^\otimes\mathcal{O}_Y\) is a non-zero \(\mathcal{O}_Y\)-submodule of a free rank one \(k(Y)\)-module and since \(\mathcal{O}_Y\) is a principal ideal domain, it follows that \((\mathcal{H}_j)_y\) is a free \(\mathcal{O}_Y\)-module of rank one. This implies that \(\mathcal{H}_j\) is a line bundle for \(\mathcal{O}_Y\). Therefore, \(\mathcal{L}_j = \Omega_Y^\otimes \mathcal{O}_Y \mathcal{H}_j\) is also a line bundle for \(\mathcal{O}_Y\).

Using similar arguments as in the proof of [5, Prop. 4.1], it follows that there exists a \(G\)-invariant divisor \(D_j\) on \(Y\) such that there is an isomorphism

\[
(\tau - 1)^j : \mathcal{H}_j \rightarrow S_{X^{-j}} \otimes_k \mathcal{O}_Y(D_j)
\]

of \(\mathcal{O}_Y\)-modules. In particular, this shows that there is an isomorphism of \(\mathcal{O}_Y\)-modules between \(\mathcal{L}_j\) and \(S_{X^{-j}} \otimes_k \Omega_Y^\otimes (D_j)\), which proves part (i) of Proposition 3.3.

To prove part (ii) of the proposition, we write

\[
D_j = \sum_{y \in Y} d_{y,j} y.
\]

Fix \(y \in Y\), and suppose \(x \in X\) lies above \(y\). Let \(I_{x,0}\) be the inertia group of \(x\) in \(I = \langle \tau \rangle\). If \(\# I_{x,0} = p^{n_x} \leq p^n\) then \(I_{x,0} = \langle \tau_x \rangle\) for \(\tau_x = \tau^{p^n - n_x}\). Note that \(I_{x,0}\) is a normal subgroup of \(G\) and of \(I\). Let \(Y_x = X/I_{x,0}\) so that \(Y = Y_x/(I/I_{x,0})\), and let \(\alpha_x : X \rightarrow Y_x\) and \(\beta_x : Y_x \rightarrow Y\) denote the respective quotient morphisms, so that \(\beta_x = \text{étale over } y\) and \(\beta_x \circ \alpha_x = \tau\). Let \(y_x \in Y_x\) be a point above \(y\) and below \(x\). Then \(x\) is totally ramified over \(y_x\) for the action of \(I_{x,0}\), and \(y\) splits into \(p^{n_j - n_x}\) points in \(Y_x\) where \(y_x\) is one of them. By the transitivity of the inverse different and taking \(m\)-fold tensor products, we have

\[
\mathcal{D}_{X/Y}^{-1,\otimes m} = \mathcal{D}_{X/Y_x}^{-1,\otimes m} \otimes \mathcal{O}_X \alpha_x^\ast \mathcal{D}_{X/Y_x}^{-1,\otimes m}.
\]

Using similar arguments as in the proof of [5, Prop. 4.1], it follows that there exists a small open neighborhood \(V_y\) of \(y\) such that we have an equality

\[
\left(\mathcal{D}_{X/Y}^{-1,\otimes m}\right)|_{U_y} = \left(\mathcal{D}_{X/Y_x}^{-1,\otimes m}\right)|_{U_y}
\]

of the restrictions to the inverse image \(U_y = \pi^{-1}(V_y) \subset X\). Let \(J_x = \text{rad}(kI_{x,0}) = kI_{x,0}(\tau_x - 1)\). For \(0 \leq t \leq p^n - 1\), let \((\alpha_x)_s \mathcal{D}_{X/Y_x}^{-1,\otimes m,(t)}\) denote the kernel of the action of \(J_x^t = kI_{x,0}(\tau_x - 1)^t\) on \((\alpha_x)_s \mathcal{D}_{X/Y_x}^{-1,\otimes m}\). As above, it follows that, for \(0 \leq t \leq p^n - 1\), there exists a \(G\)-invariant divisor \(D'_{t,x}\) on \(Y_x\) such that

\[
(\alpha_x)_s \mathcal{D}_{X/Y_x}^{-1,\otimes m,(t+1)}/(\alpha_x)_s \mathcal{D}_{X/Y_x}^{-1,\otimes m,(t)} \cong \mathcal{O}_{Y_x}(D'_{t,x})
\]

as \(\mathcal{O}_{Y_x}\)-modules. Writing

\[
D'_{t,x} = \sum_{y' \in Y_x} d'_{y',x,t} y'
\]

and using similar arguments as in the proof of [5, Prop. 4.1], we obtain that

\[
d_{y,j} = d'_{y,x,t} \quad \text{when } t \text{ satisfies } p^{n_j - n_x} t \leq j < p^{n_j - n_x} (t + 1).
\]

Note that the stalk of \((\alpha_x)_s \mathcal{D}_{X/Y_x}^{-1,\otimes m}\) at \(y_x\) is naturally identified with the stalk of \(\mathcal{D}_{X/Y_x}^{-1,\otimes m}\) at \(x\) and that \(\mathcal{D}_{X/Y_x}^{-1,\otimes m} = \mathcal{O}_X(mR_{X/Y_x})\) as fractional ideal sheaves on \(X\). Therefore, using similar arguments as in the
it follows that 

\[ D \]

is as in formula \( \text{(3.12)} \), when \( t \) is the unique non-negative integer satisfying \( p^{n_i - n_x} t \leq j < p^{n_i - n_x} (t + 1) \). Because

\[ \sum_{i \geq 0} (#I_{x,t} - 1) = \sum_{t=1}^{n_x} (p - 1)p^{n_x - t}(i_t + 1) \]

for all \( 1 \leq t \leq n_x \), we obtain that \( D_j \) is an effective divisor, establishing part (ii) of Proposition 3.3.

It remains to prove part (iii) of the proposition. We first show that \( \deg(D_j) > 2m - 2 \). Using \( \text{(3.12)} \) and that \( 0 \leq a_{\ell,t} \leq p - 1 \) for all \( 1 \leq \ell \leq n_x \), we obtain

\[ d_{y,j} \geq \left\lfloor \sum_{t=1}^{n_x} ((m - 1)(p - 1) i_t + m(p - 1)) p^{n_x - t} \right\rfloor. \]

Suppose first that \( n_I \geq 3 \). This means that there exists at least one ramified point \( y \in Y \) such that \( #I_{x,0} = p^{n_x} = p^{n_i} \geq p^3 \) when \( x \in X \) lies above \( y \). Since \( i_{\ell,t} \geq 1 + (t - 1)p \) for all \( \ell \), \( \text{(3.13)} \) leads to the inequality

\[
d_{y,j} \geq \left\lfloor \sum_{t=1}^{3} ((m - 1)(p - 1)(1 + (t - 1)p) + m(p - 1)) p^{n_x - t} \right\rfloor.
\]

Because \( m \geq 1 \) and \( p \geq 2 \), we have \( p^3 + p^2 - 2p - \frac{2m - 1}{m - 1} \geq 0 \), which implies that \( d_{y,j} \geq 2m - 1 \) and hence \( \deg(D_j) \geq 2m - 2 > 2m - 2 \).

Suppose next that \( n_I = 2 \). Then there exists at least one ramified point \( y \in Y \) such that \( #I_{x,0} = p^{n_x} = p^{n_i} \geq p^2 \) when \( x \in X \) lies above \( y \). This implies by \( \text{(3.13)} \) that

\[
d_{y,j} \geq \left\lfloor (p - 1)p^{-2}((m - 1)(pi_1 + i_2) + m(p + 1)) \right\rfloor.
\]

If \( y \in Y \) is the unique ramified point in \( Y \) and \( i_1 + i_2 \leq 4 \) then \( i_1 \geq 1 \) and \( i_2 = 0 \) mod \( p \) implies that \( p = 2 \), \( i_1 = 1 \) and \( i_2 = 3 \). But then the Riemann-Hurwitz formula shows that \( g(X) = 1 \), which is a contradiction to Assumption 3.1. Therefore, we must have \( i_1 + i_2 > 4 \), which implies, since \( i_1 \geq 1 \), that \( pi_1 + i_2 \geq p + 4 \). Thus, \( \text{(3.14)} \) leads to

\[
d_{y,j} \geq \left\lfloor 2m - 1 + p^{n_x - t}(m - 1) \left( 3p - 5 - \frac{1}{m - 1} \right) \right\rfloor.
\]

Because \( m \geq 1 \) and \( p \geq 2 \), we then obtain \( 3p - 5 - \frac{1}{m - 1} \geq 0 \), which implies that \( d_{y,j} \geq 2m - 1 \) and hence \( \deg(D_j) > 2m - 2 \).

Suppose next that \( n_I = 1 \) and that there are \( r \geq 1 \) points \( y_1, \ldots, y_r \in Y \) that are ramified. Then \( \#I_{x,0} = p^i \) for \( s = 1, \ldots, r \) when \( x_s \in X \) lies above \( y_s \). Let \( i_{s,1} \) be the unique jump in the lower ramification groups \( I_{x,s,i} \), \( i \geq 0 \). This implies by \( \text{(3.13)} \) that

\[
d_{y,s,j} \geq \left\lfloor ((m - 1)i_{s,1} + m)(1 - p^{-1}) \right\rfloor
\]

for \( s = 1, \ldots, r \).

Assume first that \( r = 1 \). If the unique jump \( i_{1,1} = 1 \) then it follows by the Riemann-Hurwitz formula that \( g(X) = 0 \), which is a contradiction to Assumption 3.1. On the other hand, if \( p = 3 \) and the unique jump is \( i_{1,1} = 2 \) or if \( p = 2 \) and the unique jump is \( i_{1,1} = 3 \), then it follows by the Riemann-Hurwitz formula that
which implies $\deg(L) \geq 2m - 1 + \frac{m-2}{3}$ for $0 \leq j \leq p^ni - 1$, which is also a contradiction to Assumption 3.1. Therefore, if $r = 2$ and $p = 2$ then at least one of $i_1, i_1$ and $i_2, i_1$ must be bigger than one, say $i_1 > 1$. Hence, $d_{y_{i,j}} \geq |m + \frac{m-2}{3}| \geq m$ and $d_{y_{i,j}} \geq |m - \frac{1}{3}| = m - 1$, which implies $\deg(D_j) \geq m + m - 1 = 2m - 1 > 2m - 2$.

Lastly, assume that $r \geq 3$ and $p = 2$ or $r \geq 2$ and $p \geq 3$. If $p = 2$ then $d_{y_{i,j}} \geq |m - \frac{1}{2}| = m - 1$ for $1 \leq s \leq r$, and if $p \geq 3$ then $d_{y_{i,j}} \geq |m + \frac{m-2}{3}| \geq m$ for $1 \leq s \leq r$. Therefore, if $r \geq 3$ and $p = 2$ then $\deg(D_j) \geq 3m - 3 > 2m - 2$, and if $r \geq 2$ and $p \geq 3$ then $\deg(D_j) \geq 2m > 2m - 2$.

It remains to show $\deg(L_j) > 2g(Y) - 2$. Since $m > 1$ and $\deg(D_j) > 2m - 2$, we obtain

$$\deg(L_j) = \deg(\Omega_Y^m(D_j)) = m(2g(Y) - 2) + \deg(D_j) > m(2g(Y) - 2) + 2m - 2 > 2g(Y) - 2$$

completing the proof of Proposition 3.3.

Lemma 3.4. Suppose $m > 1$, Assumption 3.1 and Notation 3.2 For all $0 \leq j \leq p^ni - 1$, there are isomorphisms

$$H^0(X, \Omega_X^m(j+1)/H^0(X, \Omega_X^m(j)) \cong H^0(Y, \pi^*\Omega_X^m(j+1)/\pi^*\Omega_X^m(j)) \cong S_{X-j} \otimes_k H^0(Y, \Omega_Y^m(D_j))$$

of $kG$-modules, where $D_j$ is the divisor from Proposition 3.3.

Proof. As before, let $L_j = \pi^*\Omega_X^m(j+1)/\pi^*\Omega_X^m(j)$ for $0 \leq j \leq p^ni - 1$. It follows from Proposition 3.3 that there are $kG$-module isomorphisms

$$H^0(Y, L_j) \cong H^0(Y, S_{X-j} \otimes_k \Omega_Y^m(D_j)) \cong S_{X-j} \otimes_k H^0(Y, \Omega_Y^m(D_j))$$

for $0 \leq j \leq p^ni - 1$. Since $I$ acts trivially on all these modules, these are also $kG$-module isomorphisms.

Suppose first that $n_I = 0$. Then $X = Y$, $\pi : X \to Y$ is the identity morphism, and $j = 0$. Moreover, $L_j = \pi^*\Omega_X^m = \Omega_Y^m$ and $D_j$ is the zero divisor. Hence there is nothing more to prove.

For the remainder of the proof, we assume $n_I \geq 1$. Using similar arguments as in the proof of 4.2, we obtain an injective $kG$-module homomorphism

$$\gamma_j : H^0(X, \Omega_X^m(j+1)/H^0(X, \Omega_X^m(j)) \to H^0(Y, L_j) \quad \text{for } 0 \leq j \leq p^ni - 1.$$

To complete the proof of Lemma 3.4 we now show that

$$\dim_k H^0(X, \Omega_X^m(j+1)/H^0(X, \Omega_X^m(j)) = \dim_k H^0(Y, L_j) \quad \text{for } 0 \leq j \leq p^ni - 1.$$ 

To prove (3.16), we compute $\dim_k H^0(X, \Omega_X^m)$ in two ways. By part (iii) of Proposition 3.3, we have $H^1(Y, L_j) = 0$ for $0 \leq j \leq p^ni - 1$. Hence the Riemann-Roch theorem implies that

$$\dim_k H^0(Y, L_j) = \deg(L_j) + 1 - g(Y)$$
for $0 \leq j \leq p^{n_i} - 1$. Therefore, we obtain
\[
\dim_k H^0(X, \Omega_X^{\otimes m}) = \sum_{j=0}^{p^{n_i} - 1} \dim_k H^0(X, \Omega_X^{\otimes m})^{(j+1)} / H^0(X, \Omega_X^{\otimes m})^{(j)}
\]
\[
\leq \sum_{j=0}^{p^{n_i} - 1} \dim_k H^0(Y, L_j)
\]
\[
= \sum_{j=0}^{p^{n_i} - 1} \deg(L_j) + p^{m_i}(1 - g(Y)).
\]

Next we compute $\dim_k H^0(X, \Omega_X^{\otimes m})$ using the Riemann-Roch theorem for $\pi_*\Omega_X^{\otimes m} \cong \Omega_Y^{\otimes m} \otimes_{\mathcal{O}_Y} \pi_*D_{X/Y}^{-1, \otimes m}$ (see (2.6)). Since we assume $m > 1$ and $g(X) \geq 2$, we have
\[
\deg(\Omega_X^{\otimes m}) = m(2g(X) - 2) > 2g(X) - 2
\]
which implies that $H^1(X, \Omega_X^{\otimes m}) = 0$. Hence, we obtain
\[
\dim_k H^0(X, \Omega_X^{\otimes m}) = \deg(\pi_*\Omega_X^{\otimes m}) + \text{rank}_{\mathcal{O}_Y}(\pi_*\Omega_X^{\otimes m})(1 - g(Y))
\]
\[
= \sum_{j=0}^{p^{n_i} - 1} \deg(L_j) + p^{m_i}(1 - g(Y)).
\]

Therefore, the inequality in (3.18) must be an equality, which implies (3.16), completing the proof of Lemma 3.4.

\[\square\]

3.2. Analyzing the tamely ramified cover $\lambda : Y \to Z$. In this subsection, we determine the $kG$-module structure of (3.14) under Assumption 3.1 and using Notation 3.2.

Proposition 3.5. Suppose $m > 1$, Assumption 3.1 and Notation 3.2, and fix $0 \leq j \leq p^{n_i} - 1$. Let $D_j$ be the $G$-invariant divisor from Proposition 3.3. Let $K_Z$ be a canonical divisor on $Z = X/G = Y/G$ with divisor class $[K_Z]$. The $kG$-module structure of $H^0(Y, \Omega_Y^{\otimes m}(D_j))$ is uniquely determined by $[K_Z]$, together with the inertia groups and the fundamental characters of the closed points of $Y$ that are ramified in the cover $\lambda : Y \to Z$.

More precisely, $H^0(Y, \Omega_Y^{\otimes m}(D_j))$ is a projective $kG$-module whose Brauer character is given as follows. Let $Z_{\text{ram}}$ be the set of closed points in $Z$ that ramify in $Y$. For $z \in Z_{\text{ram}}$, let $y(z) \in Y$ be such that $\lambda(y(z)) = z$. Let
\[
K_Y = \lambda^*K_Z + R_{Y/Z},
\]
and define $\ell_{y(z), j} \in \{0, 1, \ldots, \#G_{y(z), 0} - 1\}$ by
\[
\ell_{y(z), j} = \text{ord}_{y(z)}(mK_Y + D_j) \mod \#G_{y(z), 0}.
\]
Let $\theta_{y(z)}$ be the fundamental character of $G_{y(z), 0}$ and view it as a Brauer character as in Remark 2.2. Then the Brauer character of $H^0(Y, \Omega_Y^{\otimes m}(D_j))$ is equal to
\[
\sum_{z \in Z_{\text{ram}}} \left( \sum_{d=1}^{\ell_{y(z), j}} \text{Ind}_{G_{y(z), 0}} \theta_{y(z)}^{-d} - \sum_{d=0}^{\#G_{y(z), 0} - 1} \text{Ind}_{G_{y(z), 0}} \theta_{y(z)}^{d} \right) + n_j \beta(kG)
\]
where $\beta(kG)$ is the Brauer character of $kG$ and
\[
n_j = \frac{(2m - 1)(g(Y) - 1) + \deg(D_j)}{\#G} + \frac{1}{\#G_{y(z), 0}} \left( \left\lfloor \frac{\#G_{y(z), 0} - 1}{2} \right\rfloor - \ell_{y(z), j} \right).
\]
Proof. As in [2.2] we have an isomorphism
\[ \Omega_Y^{\otimes m}(D_j) = \Omega_Y^{\otimes m} \otimes \mathcal{O}_Y \mathcal{O}_Y(D_j) \cong \mathcal{O}_Y(mK_Y + D_j) \]
of \( \mathcal{O}_Y \mathcal{G} \)-modules, where \( K_Y \) is as in (3.18). As in Remark 2.1 we see that \( \mathcal{O}_Y(mK_Y + D_j)_{y(z)} \otimes \mathcal{O}_Y,y(z) k \) is a \( k \mathcal{G}_{y(z),0} \)-module of \( k \)-dimension one whose character equals \( \theta_{y(z),i}^{\ell_{y(z),j}} \) when \( \ell_{y(z),j} \) is as in (3.19). Moreover, \( \ell_{y(z),j} \) does not depend on the choice of canonical divisor \( K_Z \) but only on its divisor class \([K_Z]\). We obtain that the Brauer character of \( \Omega_Y^{\otimes m}(D_j)_{y(z)} \otimes \mathcal{O}_Y,y(z) k \) must also equal \( \theta_{y(z),i}^{\ell_{y(z),j}} \).

By [37, Thm. 2] and using that \( H^1(\Omega_Y^{\otimes m}(D_j)) = 0 \) by part (iii) of Proposition 3.3 there exists a short exact sequence of \( k \mathcal{G} \)-modules
\[ 0 \to H^0(Y, \Omega_Y^{\otimes m}(D_j)) \to L^0_j \to L^1_j \to 0 \]
for certain finitely generated projective \( k \mathcal{G} \)-modules \( L^0_j \) and \( L^1_j \). This implies that \( H^0(\Omega_Y^{\otimes m}(D_j)) \) is a projective \( k \mathcal{G} \)-module and that its Brauer character \( \beta(H^0(\Omega_Y^{\otimes m}(D_j))) = \text{equal to the difference of the Brauer characters } \beta(L^0_j) - \beta(L^1_j) \). Using [37, Eq. (*) on p. 120], we obtain that the difference \( \beta(L^0_j) - \beta(L^1_j) \) is equal to the formula in (3.20) for some integer \( n_j \). Since the value of \( \beta(k \mathcal{G}) \) is zero at any non-trivial element of \( \mathcal{G} \) whose order is relatively prime to \( p \), it follows that \( n_j \) is determined by the values of all involved Brauer characters at the identity element \( e \mathcal{G} \) of \( \mathcal{G} \). Since \( H^1(\Omega_Y^{\otimes m}(D_j)) = 0 \), we have by the Riemann-Roch theorem that
\[ \beta(H^0(\Omega_Y^{\otimes m}(D_j))) = \dim_k H^0(\Omega_Y^{\otimes m}(D_j)) - \dim_k H^1(\Omega_Y^{\otimes m}(D_j)) = \deg(mK_Y + D_j) + 1 - g(Y) = (2m - 1)(g(Y) - 1) + \deg(D_j). \]

On the other hand, for any integer \( d \geq 0 \),
\[ \left( \text{Ind}_{\mathcal{G}_{y(z),0}}^{\mathcal{G}} \theta_{y(z),i}^{d} \right) \mathcal{G} = \frac{\# \mathcal{G}}{\# \mathcal{G}_{y(z),0}} \]
and \( \beta(k \mathcal{G}) \mathcal{G} = \# \mathcal{G} \), which leads to the formula (3.21) for \( n_j \). \( \square \)

Proof of Theorem 1.1. By Lemma 2.3 and Lemma 2.4 together with the isomorphism in (2.2), it is sufficient to prove Theorem 1.1 under Assumption 3.1. We use Notation 3.2, and let \( M = H^0(X, \Omega_X^{\otimes m}) \). By Proposition 3.3, Lemma 5.4 and Proposition 5.5, the \( k \mathcal{G} \)-module structure of the subquotient modules
\[ (3.22) \frac{M^{(j+1)}}{M^{(j)}}, \quad 0 \leq j \leq p^m - 1, \]
is uniquely determined by the lower ramification groups associated to the action of \( I \) on \( X \), the canonical divisor class on \( Z = X/G = Y/\mathcal{G} \), together with the inertia groups and fundamental characters of the closed points of \( Y \) that are ramified in the cover \( \lambda : Y \to Z \).

Since for all closed points \( x \in X \) that are ramified in the cover \( \lambda \circ \pi : X \to Z = X/G \), the inertia group \( I_{x,0} \) is the (unique) Sylow \( p \)-subgroup of the inertia group \( G_{x,0} \), we obtain \( I_{x,0} = G_{x,1} \) and \( I_{x,i} = G_{x,i} \), for all \( i \geq 1 \). In other words, the lower ramification groups associated to the action of \( I \) on \( X \) are uniquely determined by the lower ramification groups associated to the action of \( G \) on \( X \).

Suppose \( y \in Y \) is ramified in the cover \( \lambda : Y \to Z \). Let \( x \in X \) be a point in \( X \) with \( \pi(x) = y \). Then \( x \) is ramified in the cover \( \lambda \circ \pi : X \to Z = X/G \). Let \( \theta_y \) be the fundamental character of the inertia group \( G_{x,0} \) and let \( \theta_x \) be the fundamental character of the inertia group \( G_{x,i} \). It follows as in Remark 2.2 that \( G_{y,0} \) can be identified with the quotient group \( G_{x,0}/I_{x,0} \) and that we can identify \( \theta_y = \theta_x^{\# I_{x,0}} \) on \( G_{y,0} = G_{x,0}/I_{x,0} \). In other words, the inertia groups and fundamental characters of the closed points of \( Y \) that are ramified in
the cover $\lambda : Y \to Z$ are uniquely determined by the inertia groups and fundamental characters of the closed points of $X$ that are ramified in the cover $\lambda \circ \pi : X \to Z = X/G$.

It remains to show that the $k\overline{G}$-module structures of the subquotients in $\Omega_X^{\otimes m}$ uniquely determine the $kG$-module structure of $M$. This follows, as in the proof of [5, Thm. 1.1], by using the description of the indecomposable $kG$-modules in Remark 2.3.

\section{Algorithm resulting from the proof of Theorem 1.1}

The following algorithm summarizes the main steps in the proof of Theorem 1.1 as outlined in 3.1 and 3.2.

\textbf{Algorithm 3.6.} Suppose $m > 1$, Assumption 3.1 and Notation 3.2. Let $M = H^0(X, \Omega_X^{\otimes m})$.

1. Fix $0 \leq i \leq p^{n_1} - 1$. For $y \in Y$, let $x \in X$ be a point above $y$, and let $I_{x,0}$ be the inertia group of $x$ inside $I$ of order $p^{n_x}$. Let $1 \leq i_1 < i_2 < \cdots < i_{n_x}$ be the $n_x$ jumps in the numbering of the lower ramification groups $I_{x,i}$. Let $0 \leq t \leq p^{n_x} - 1$ be the unique integer such that $p^{n_x - n_x}t \leq j < p^{n_x - n_x}(t + 1)$ and write $t = a_{1,t} + a_{2,t}p + \cdots + a_{n_x,t}p^{n_x-1}$ with $0 \leq a_{n_x,t} \leq p - 1$. Let

$$d_{y,j} = \left[ m \sum_{i \geq 0} (\#I_{x,i} - 1) - \sum_{\ell = 1}^{n_x} a_{\ell,t} p^{n_x - \ell} \right]$$

and define

$$D_j = \sum_{y \in Y} d_{y,j}y.$$ 

By Proposition 3.3 and Lemma 3.4, there is a $k\overline{G}$-module isomorphism

$$M^{(j+1)}/M^{(j)} \cong S_{X-j} \otimes_k H^0(Y, \Omega_Y^{\otimes m}(D_j)).$$

2. Fix $0 \leq j \leq p^{n_1} - 1$. Let $Z_{\text{ram}}$ be the set of closed points in $Z = X/G = Y/\overline{G}$ that ramify in the cover $\lambda : Y \to Z$. For $z \in Z_{\text{ram}}$, let $y(z) \in Y$ be a point above $z$. Let $K_Y = \lambda^*K_Z + R_{Y/Z}$ and define $\ell_{y(z),j} \in \{ 0, 1, \ldots, \#\overline{G}_{y(z),0} - 1 \}$ by

$$\ell_{y(z),j} = \text{ord}_{y(z)}(mK_Y + D_j) \mod \#\overline{G}_{y(z),0}.$$ 

Let $\theta_{y(z)}$ be the fundamental character of $\overline{G}_{y(z),0}$ and view it as a Brauer character as in Remark 2.2. By Lemma 3.4 and Proposition 3.5, the $k\overline{G}$-module $S_{X^\lambda} \otimes_k (M^{(j+1)}/M^{(j)})$ is projective and its Brauer character $\beta(j)$ is equal to

$$\beta(j) = \sum_{z \in Z_{\text{ram}}} \left( \sum_{d=1}^{\ell_{y(z),j}} \text{Ind}_{\overline{G}_{y(z),0}}^{\overline{G}_{y(z),0}} \theta_{y(z)}^{-d} - \sum_{d=0}^{\#\overline{G}_{y(z),0} - 1} d \text{Ind}_{\overline{G}_{y(z),0}}^{\overline{G}_{y(z),0}} \theta_{y(z)}^d \right) + n_j \beta(k\overline{G})$$

where

$$n_j = \frac{(2m - 1)(g(Y) - 1)}{\#G} + \sum_{z \in Z_{\text{ram}}} \frac{1}{\#\overline{G}_{y(z),0}} \left( \frac{\#\overline{G}_{y(z),0} - 1}{2} - \ell_{y(z),j} \right).$$

3. Use the notation from Remark 2.3. Fix integers $a, b$ with $0 \leq a \leq c - 1$ and $1 \leq b \leq p^n$. Write $b = b' + b'' p^{n-n_1}$ with $0 \leq b' < p^{n-n_1}$ and $0 \leq b'' \leq p$. Then the number $n_{a,b}$ of indecomposable direct $kG$-module summands of $M$ that are isomorphic to $U_{a,b}$ is given as follows (see, for example, the proof of [5, Thm. 1.1]):

(a) If $b' \geq 1$ then $n_{a,b}$ equals the number of indecomposable direct $k\overline{G}$-module summands of $M^{(b''+1)}/M^{(b''+1)}$ with socle $S_{X^{-b''}(a)}$ and $k$-dimension $b'$. 

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(b) If $b' = 0$ then $b'' \geq 1$. In this case, define $n_1(a, b)$ to be the number of indecomposable direct $kG$-module summands of $M(b'')/M(b''-1)$ with socle $S_{X-(b''-1)}(a)$ and $k$-dimension $p^{n-n_1}$. Also, define $n_2(a, b)$ to be the number of indecomposable direct $kG$-module summands of $M(b''+1)/M(b'')$ with socle $S_{X-b''}(a)$, where we set $n_2(a, b) = 0$ if $b'' = p^n$. Then $n_{a,b} = n_1(a, b) - n_2(a, b)$.

### 3.4. An application to deformation theory.

In this subsection, we show that Theorem 1.1 and its proof have applications to deformation theory and the equivariant deformation problem, as introduced in [4].

Let $X$ be a smooth projective curve of genus $g(X) \geq 2$ defined over an algebraically closed field $k$ of prime characteristic $p$, and suppose $G$ is a finite group acting faithfully on the right on $X$ over $k$. Let $\mathcal{C}$ be the category of commutative local Artinian $k$-algebras with residue field $k$. A lift of the pair $(X, G)$ over an object $A$ in $\mathcal{C}$ is a proper smooth curve $X' \to \text{Spec}(A)$, together with a group homomorphism $G \to \text{Aut}_A(X')$ such that there is a $G$-equivariant isomorphism $X' \otimes_A k \cong X$. A deformation of $(X, G)$ over $A$ is a $G$-equivariant isomorphism class of such lifts inducing the identity on $X$. The global deformation functor

$$D_{gl} : \mathcal{C} \to \text{Sets}$$

sends an object $A$ of $\mathcal{C}$ to the set of deformations of $(X, G)$ over $A$; see [4] for details. It follows from [4] §2 that $D_{gl}$ has a versal deformation ring in the sense of [31] and that the tangent space $t_{D_{gl}}$ of $D_{gl}$ is isomorphic to the equivariant cohomology $H^1(G, T_X)$ of $(X, G)$ where $T_X$ is the tangent sheaf on $X$. Using the arguments in [31] §3 and in [30] §7, we obtain, as in the beginning of the proof of [30] Thm. 7.1, that

$$\dim_k t_{D_{gl}} = \dim_k H^1(G, T_X) = \dim_k H^0(X, \Omega_X^\otimes 2)_G$$

where the subscript $G$ denotes the $G$-coinvariants of a $kG$-module.

**Proposition 3.7.** Suppose $G$ has non-trivial cyclic Sylow $p$-subgroups. If $g(X) \geq 2$ then the $k$-dimension of the tangent space of the global deformation functor $D_{gl}$ is uniquely determined by the class of canonical divisors of $X/G$, together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$.

If we suppose Assumption 3.1 and Notation 3.2, then we obtain

$$\dim_k t_{D_{gl}} = \sum \left\{ n_{a,b} \right\}_{\{a,b\} : S_a \cong S_{a-1} \otimes 1}$$

where $n_{a,b}$, for $0 \leq a \leq c - 1$ and $1 \leq b \leq p^n$, is as defined in step (3) of Algorithm 3.6 when applied to $m = 2$.

**Proof.** The first paragraph of the statement follows from Theorem 1.1 using (3.23).

Suppose now Assumption 3.1 and Notation 3.2. By step (3) of Algorithm 3.6, the number of indecomposable direct $kG$-module summands of $H^0(X, \Omega_X^\otimes 2)_G$ that are isomorphic to $U_{a,b}$ (for $0 \leq a \leq c - 1$ and $1 \leq b \leq p^n$) is given by $n_{a,b}$. Since taking $G$-coinvariants defines an additive functor $(-)_G$ from the category of finitely generated $kG$-modules to the category of $k$-vector spaces, it follows from (3.23) that

$$\dim_k t_{D_{gl}} = \sum_{a=0}^{c-1} \sum_{b=1}^{p^n} n_{a,b} \dim_k (U_{a,b})_G.$$

Since each $U_{a,b}$ is uniserial, it follows that $(U_{a,b})_G \neq 0$ if and only if $U_{a,b}/\text{rad}(U_{a,b}) \cong S_0$ is the trivial simple $kG$-module. It follows from Remark 2.5 that

$$U_{a,b}/\text{rad}(U_{a,b}) \cong S_{X^{a+1}(a)}.$$

Since the latter module is isomorphic to $S_0$ if and only if $S_a \cong S_{X^{a-1}(a)}$, the formula in (3.24) follows. □
Remark 3.8. Suppose Assumption 4.1 holds and use Notation 3.2. Recall that the order of the character \( \chi : C \to \mathbb{F}_p^* \) divides \( p - 1 \). Hence it follows that for \( \ell \in \{0,1,\ldots,n\} \), the only non-zero value of \( n_{a,\ell'} \) is \( n_{0,\ell'} \).

Remark 3.9. Proposition 4.7 generalizes [29, Cor. 2.3] from cyclic \( p \)-groups to \( p \)-hypo-elementary groups. In [30, Thm. 7.1], the \( k \)-dimension of \( t_{D_q} \) has been determined for any finite group \( G \) satisfying the additional assumption that \( \dim_k M^G = \dim_k M_G \) for all finitely generated \( kG \)-modules \( M \). However, for an arbitrary \( p \)-hypo-elementary group \( G \), this additional assumption does not hold.

4. A family of hyperelliptic curves

In this section, we illustrate Theorem 1.1 and in particular Algorithm 3.6 and Proposition 3.7 by considering a particular family of hyperelliptic curves. We make the following assumptions throughout this section.

Assumption 4.1. Let \( m > 1 \) be an integer, let \( p > 3m \) be a prime number, and let \( k \) be an algebraically closed field of characteristic \( p \). Write the function field of \( \mathbb{P}^1_k \) as \( k(\mathbb{P}^1_k) = k(t) \), and consider the quadratic cover \( X \) of \( \mathbb{P}^1_k \) with function field \( k(X) = k(t)(\sqrt{f}) \) where

\[
\begin{align*}
f(t) &= t^{p^2} - t = \prod_{\alpha \in k^1(\mathbb{F}_{p^2})} (t - \alpha).
\end{align*}
\]

Let \( P = \langle \sigma \rangle \) be a cyclic group of order \( p \), and let \( C = \langle \rho \rangle \) be a cyclic group of order \( 2(p - 1) \). Fix an isomorphism \( \xi : C \to \mathbb{Z}^2 \) and define \( \chi : C \to \mu_{p-1} = \mathbb{F}_p^* \subset k^* \) to be \( \chi = \xi^2 \). Define \( G \) to be the semidirect product \( G = P \rtimes \chi C \), and let \( G \) act on \( k(X) \) as follows:

\[
\begin{align*}
\sigma(t) &= t + 1, & \sigma(\sqrt{f}) &= \sqrt{f}, \\
\rho(t) &= \chi^{-1}(\rho)t, & \rho(\sqrt{f}) &= \xi^{-1}(\rho)\sqrt{f}.
\end{align*}
\]

In [4.3], we first determine the data needed to apply Algorithm 3.6. In [4.2], we apply Algorithm 3.6 to determine the precise \( kG \)-module structure for \( H^0(X,\Omega_X^m) \). In [4.3], we apply Proposition 3.7 to determine the \( k \)-dimension of the tangent space \( t_{D_{q1}} \).

4.1. Lower ramification groups, fundamental characters, and canonical divisors. We first determine the lower ramification groups of the action of \( G \) on \( X \). Define \( \nu = \rho^{p-1} \in C \). Then \( \nu \) has order 2 and acts on \( k(X) \) as \( \nu(t) = t \) and \( \nu(\sqrt{f}) = -\sqrt{f} \). The cover \( X \to X/G \cong \mathbb{P}^1_k \) has 3 branch points. Identifying \( Z = X/G \) with \( \mathbb{P}^1_k \), these are given by \( z_0 = 0, z_1 = 1, \) and \( z_\infty = \infty \). For \( u \in \{0,1,\infty\} \), let \( x_u \in X \) be above \( z_u \). Then the orbit \( G.x_0 \) contains \( p \) points of \( X \), the orbit \( G.x_1 \) contains \( p^2 - p \) points of \( X \), and the orbit \( G.x_\infty = \{x_\infty\} \). The lower ramification groups are as follows:

\[
\begin{align*}
G_{x_0,0} &= \langle \rho \rangle, & G_{x_0,1} &= \{e_G\}, \\
G_{x_1,0} &= \langle \nu \rangle, & G_{x_1,1} &= \{e_G\}, \\
G_{x_\infty,0} &= G, & G_{x_\infty,1} &= \langle \sigma \rangle = G_{x_\infty,2}, & G_{x_\infty,3} &= \{e_G\}.
\end{align*}
\]

Let \( I \) be the subgroup of \( G \) generated by the Sylow \( p \)-subgroups of the inertia groups of all closed points in \( X \) and let \( Y = X/I \). Since \( x_\infty \in X \) is the only point whose inertia group has order divisible by \( p \), it follows that \( I = I_{x_\infty,0} = P = \langle \sigma \rangle \), so that \( \tau = \sigma \). Moreover, \( x_\infty \) is the unique point that is ramified in the cover \( \pi : X \to Y \), and it is totally ramified.

Let \( Z = X/G \) and consider the cover \( \lambda : Y \to Z \). For \( u \in \{0,1,\infty\} \), let \( y_u = \pi(x_u) \). If \( \bar{G} = G/I \), the orbits \( \bar{G}.y_0 \) and \( \bar{G}.y_\infty \) are each singletons, and the orbit \( \bar{G}.y_1 \) contains \( p - 1 \) points of \( Y \). The points in these
orbits are the only points in $Y$ that are ramified in the cover $\lambda : Y \to Z$. Identifying $G = \langle \rho \rangle$, we obtain the following inertia groups:

$$\overline{G}_{y_0,0} = \langle \rho \rangle = \overline{G}_{y_\infty,0} \quad \text{and} \quad \overline{G}_{y_1,0} = \langle \nu \rangle.$$  

By the Riemann-Hurwitz formula, it follows that $g(Y) = (p - 1)/2$. For $u \in \{0, 1, \infty \}$, the uniformizer $\pi_u$ of $\mathcal{O}_{X,x_u}$ can be chosen as follows:

$$\pi_0 = \sqrt{f} = \pi_1 \quad \text{and} \quad \pi_{\infty} = \sqrt{f} t^{-(p^2 + 1)/2}.$$  

The corresponding fundamental characters are given as:

$$\begin{align*}
\theta_{y_0} &= \xi^{-1}, \\
\theta_{y_1} &= \text{Res}^{(\rho)}_{(\nu)}(\xi^{-1}), \\
\theta_{y_\infty} &= (\xi^{-1}, \chi(p^2 + 1)/2)^p = \xi^p \in \mathbb{C}.
\end{align*}$$

where we use that the order of $\xi$ is $2(p - 1)$ and that $p^3 \equiv p \mod 2(p - 1)$. Note that since $\xi^{-1}(\nu) = -1$, we have

$$\text{Ind}^{(\rho)}_{(\nu)}(\theta_{y_1}) = \text{Ind}^{(\rho)}_{(\nu)} \left( \text{Res}^{(\rho)}_{(\nu)}(\xi^{-1}) \right) = \xi^{-1} \sum_{i=0}^{p-2} \chi^i = \sum_{i=0}^{p-2} \xi^{2i+1}.$$  

Since $Z = X/G \cong \mathbb{P}^1_k$ and $z_\infty$ corresponds to $\infty$ in $\mathbb{P}^1_k$, we have that $K_Z = -2 z_\infty$ is a canonical divisor for $Z$. Therefore, $K_Y = \lambda^* K_Z + R_Y/Z$ is a canonical divisor on $Y$. Using (4.1) to determine $R_Y/Z$, we obtain

$$K_Y = (2p - 3) y_0 + \sum_{y \subset G, y_1} y - (2p - 1) y_\infty.$$  

4.2. The $kG$-module structure of the holomorphic poly-differentials. We apply Algorithm 3.6 to determine the $kG$-module structure for $M = H^0(X, \Omega^m_X)$ for all $m > 1$. As seen in (3.6), $I = P = \langle \sigma \rangle$, and hence $n_I = 1$. Moreover, $y_\infty \in Y$ is the only point in $Y$ that ramifies in $X$, and $I_{y_\infty,i} = I$ for $i \in \{0, 1, 2\}$ and $I_{y_\infty, i}$ is trivial for $i \geq 3$. By step (1) of the algorithm, we have $p$ divisors $D_j$ on $Y$, for $0 \leq j \leq p - 1$, which are given as $D_j = d_{y_\infty,j} y_\infty$ where

$$d_{y_\infty,j} = \left\lceil \frac{3m(p - 1) - 2j}{p} \right\rceil = \left\lceil \frac{3m - 3m + 2j}{p} \right\rceil.$$  

Since we assume $p > 3m$, we can rewrite this as follows. Define

$$\delta_m = \begin{cases} 
0, & m \text{ even}, \\
1, & m \text{ odd},
\end{cases}$$

and define

$$\begin{align*}
A_1 &= \left\{ 0, 1, \ldots, \frac{p - 3m - 1 + \delta_m}{2} \right\}, \\
A_2 &= \left\{ \frac{p - 3m - 1 + \delta_m}{2}, \ldots, \frac{2p - 3m - \delta_m}{2} \right\}, \\
A_3 &= \left\{ \frac{2p - 3m - \delta_m}{2} + 1, \ldots, p - 1 \right\}.
\end{align*}$$

Then $D_j = (3m - \ell) y_\infty$ if $j \in A_\ell$ for $1 \leq \ell \leq 3$.  

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For step (2) of Algorithm 3.6 we note that $Z_{\text{ram}} = \{z_0, z_1, z_{\infty}\}$. Recall that $y_u \in Y$ lies above $z_u$ for $u \in \{0, 1, \infty\}$. Using (4.11) and $K_Y$ from (4.4), we obtain, for $0 \leq j \leq p - 1$,

$$\ell_{y_u,j} = \begin{cases} 2p - 2 - m, & u = 0, \\ \delta_m, & u = 1, \\ 2m - \ell, & u = \infty, j \in A_\ell \text{ for } 1 \leq \ell \leq 3. \end{cases}$$

We now use the fundamental characters from (4.2) to determine the Brauer character of $S_{\chi_j} \otimes_k (M^{(j+1)}/M^{(j)})$, for $0 \leq j \leq p - 1$, as in step (2) of Algorithm 3.6. Suppose $j \in \mathcal{A}_\ell$ for $1 \leq \ell \leq 3$. Then

$$n_j = \frac{(2m - 1)(g(Y) - 1) + \text{deg}(D_j)}{2(p - 1)} + \sum_{u \in \{0, 1, \infty\}} \frac{1}{\#G_{y_u,0}} \left( \frac{\#G_{y_u,0} - 1}{2} - \ell_{y_u,j} \right) = \frac{m - \delta_m}{2}.$$ 

We therefore obtain from step (2) of Algorithm 3.6 using in particular (4.3), that the Brauer character $\beta(j)$ of $S_{\chi_j} \otimes_k (M^{(j+1)}/M^{(j)})$ is equal to

$$\beta(j) = \sum_{u \in \{0, 1, \infty\}} \left( \sum_{d=1}^{2p-2-m} \text{Ind}_{G_{y_u,0}}^{G_{y_u,0}} \theta_{y_u}^{-d} - \sum_{d=0}^{p-2} \text{Ind}_{G_{y_u,0}}^{G_{y_u,0}} \theta_{y_u}^{d} \right) + n_j \beta(kG)$$

$$= \left( \sum_{d=1}^{2p-2-m} \xi^d - \sum_{d=1}^{2p-3} \frac{d}{2(p-1)} \xi^{-d} \right) + \left( \delta_m \sum_{i=0}^{p-2} \xi^{2i+1} - \frac{1}{2} \sum_{i=0}^{p-2} \xi^{2i+1} \right)$$

$$+ \left( \sum_{d=1}^{2m-\ell} \xi^{-pd} - \sum_{d=1}^{2p-3} \frac{d}{2(p-1)} \xi^{pd} \right) + \frac{m - \delta_m}{2} \sum_{i=0}^{p-2} \left( \xi^{2i} + \xi^{2i+1} \right).$$

Rewriting $\beta(j)$ as a sum of $\xi^{2i}$ and $\xi^{2i+1}$, for $0 \leq i \leq p - 2$, we obtain

$$\beta(j) = \sum_{i=0}^{p-1-(m+\delta_m)/2} \left( \frac{m - \delta_m}{2} \right) \xi^{2i} + \left( \frac{m - \delta_m}{2} + 1 - \delta_{\ell,3} \right) \xi^{2p-2m}$$

$$+ \sum_{i=p-(m+\delta_m)/2}^{(p-3)/2-m} \left( \frac{m + \delta_m}{2} - 1 \right) \xi^{2i+1} + \left( \frac{m + \delta_m}{2} - 1 + \delta_{\ell,1} \right) \xi^{p-2m}$$

$$+ \sum_{i=p-1-(m-\delta_m)/2}^{p-2-(m-\delta_m)/2} \left( \frac{m + \delta_m}{2} \right) \xi^{2i+1} + \left( \frac{m + \delta_m}{2} - 1 \right) \xi^{2i+1}.$$ 

Finally, we use step (3) of Algorithm 3.6 to determine the $kG$-module structure of $M = H^0(X, \Omega_X^m)$. We fix integers $a, b$ with $0 \leq a \leq 2p - 3$ and $1 \leq b \leq p$, and we want to determine the the number $n_{a,b}$ of indecomposable direct $kG$-module summands of $M$ that are isomorphic to $U_{a,b}$.

Suppose first that $b = p$, leading to the projective $kG$-modules that are direct summands of $M$. By step (3) of Algorithm 3.6 we have

$$n_{a,p} = \#\{\text{summands of } \beta(p-1) \text{ equal to } \xi^a\}.$$ 

Since $\ell = 3$ for $j = p - 1$, we obtain from (4.6):

$$n_{2i,p} = \begin{cases} \frac{m - \delta_m}{2} + 1, & i \in E_1, \\ \frac{m - \delta_m}{2}, & i \in E_2, \end{cases} \quad \text{and} \quad n_{2i+1,p} = \begin{cases} \frac{m + \delta_m}{2}, & i \in O_1, \\ \frac{m + \delta_m}{2} - 1, & i \in O_2, \end{cases}$$

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that the only non-zero coefficients of non-projective direct summands of $H_a$ is

$$
E_1 = \{p - m + 1, p - m + 2, \ldots, p - 1 - m + \delta_m \}, \quad E_2 = \{0, 1, \ldots, p - 2\} - E_1,
$$

$$
O_1 = \{\frac{p - 1}{2} - m + 1, \frac{p - 1}{2} - m + 2, \ldots, p - 2 - m + \delta_m \}, \quad O_2 = \{0, 1, \ldots, p - 2\} - O_1.
$$

Next suppose that $1 \leq b \leq p - 1$. By step (3) of Algorithm 3.6 we have

$$
n_{a,b} = \#\{\text{summands of } \beta(b - 1) \text{ equal to } \xi^a\} - \#\{\text{summands of } \beta(b) \text{ equal to } \xi^a\}.
$$

In other words, the non-zero $n_{a,b}$ occur precisely for those values of $a$ for which there is a “jump” in the coefficient of $\xi^a$ when comparing the formula (4.8) for $j = b - 1$ and $j = b$. It follows from (4.6) and the definition of the sets $A_1, A_2,$ and $A_3$ that these jumps occur at $b = \frac{p - 3m + 1 + \delta_m}{2}$ and at $b = \frac{2p - 3m + 2 - \delta_m}{2}$. The jump in the coefficient of $\xi^a$ for $b = \frac{p - 3m + 1 + \delta_m}{2}$ occurs at $a = p - 2m$, whereas the jump in the coefficient of $\xi^a$ for $b = \frac{2p - 3m + 2 - \delta_m}{2}$ occurs at $a = 2p - 2m$. Therefore, $n_{a,b} = 1$ for these two pairs $(a,b)$ and $n_{a,b} = 0$ for all other pairs $(a,b)$ with $1 \leq b \leq p - 1$. We obtain the following result:

**Proposition 4.2.** Under Assumption 4.1 there is an isomorphism of $kG$-modules

$$
H^0(X, \Omega_X^{\otimes m}) \cong U_{p-2m, 2p-3m+2-\delta_m} \oplus \bigoplus_{i \in E_1} \left( \frac{m - \delta_m}{2} + 1 \right) U_{2i_1 + 1, p} \oplus \bigoplus_{i \in E_2} \left( \frac{m - \delta_m}{2} \right) U_{2i_1, p} \oplus \bigoplus_{i \in O_1} \left( \frac{m + \delta_m}{2} \right) U_{2i_1, p} \oplus \bigoplus_{i \in O_2} \left( \frac{m + \delta_m}{2} - 1 \right) U_{2i_1 + 1, p}
$$

where $\delta_m$ is as in 4.3 and $E_1, E_2, O_1, O_2$ are as in 4.7.

### 4.3. The tangent space of the global deformation functor

Under Assumption 4.1 we now use Proposition 4.2 to determine the dimension of the tangent space $t_{D_{gl}}$ of the global deformation functor $D_{gl}$ associated to the pair $(X,G)$. We use Proposition 4.2 for $m = 2$.

Let $b \in \{1, 2, \ldots, p\}$. By the equality in (3.24), we must find all $a \in \{0, 1, \ldots, 2p - 3\}$ such that $S_a \cong S_{\chi^{b-1}}$. Since $\chi = \xi^2$, it follows that $S_{\chi^{b-1}} = S_{2(b-1)}$. Hence, we need to find all $a \in \{0, 1, \ldots, 2p - 3\}$ such that

$$
2(b - 1) \equiv a \mod 2(p - 1).
$$

If $b = p$ then it follows by Remark 3.8 that the only value of $a$ that satisfies (4.8) is $a = 0$. By Proposition 3.7 applied to $m = 2$, the coefficient $n_{0,p}$ of $U_{0,p}$ equals $(m - \delta_m)/2 = 1$. We next consider $b \in \{0, 1, \ldots, p - 1\}$. The only value of $a$ that satisfies (4.8) is $a = 2(b - 1)$. However, Proposition 3.7 applied to $m = 2$, shows that the only non-zero coefficients of non-projective direct summands of $H^0(X, \Omega_X^{\otimes 2})$ are

$$
n_{p-4,2-p}, \quad n_{2p-4,p-2}.
$$

Therefore, there are no pairs $(a,b)$ satisfying (4.8) when $b \in \{0, 1, \ldots, p - 1\}$.

We obtain the following result:

**Lemma 4.3.** Suppose Assumption 4.1 holds, and let $P_0$ be the projective $kG$-module cover of the trivial simple $kG$-module $S_0$. The $k$-dimension of the tangent space $t_{D_{gl}}$ of the global deformation functor $D_{gl}$ associated to the pair $(X,G)$ is equal to the multiplicity of $P_0$ as a direct summand of $H^0(X, \Omega_X^{\otimes 2})$, which equals one.
5. Holomorphic poly-differentials of the modular curves $X(\Gamma_\ell)$ modulo $p$

In this section, we discuss Theorem 1.1 for modular curves in positive characteristic. In particular, this has applications to congruences between modular forms in positive characteristic. The case of holomorphic differentials was considered in [5 §5-7].

The geometric theory of modular forms and the associated arithmetic theory of moduli spaces of elliptic curves were initiated by Deligne-Rapoport [12], Katz [26] and Katz-Mazur [27] (see also [20]). In what follows, we will use the set up and notation from [5 §5].

Let $\ell \geq 3$ be a prime number, and let $\Gamma_\ell$ be the principal congruence subgroup of $\text{SL}(2,\mathbb{Z})$ of level $\ell$. The compactification of the quotient of the complex upper half plane by the action of $\Gamma_\ell$ defines a compact Riemann surface $X(\Gamma_\ell)$, called the modular curve associated to $\Gamma_\ell$. If $A$ is a Dedekind domain containing $\mathbb{Z}[\frac{1}{\ell}, \zeta_\ell]$, then there exists a proper smooth canonical model $X_A(\ell)$ of $X(\Gamma_\ell)$ over $A$ (see [20, 27]). Note that $X_A(\ell)$ has geometrically connected fibers that all have the same genus. Moreover, for all $m \geq 1$, the global sections $H^0(X_A(\ell), \Omega_{X_A(\ell)}^m)$ are naturally identified with the $A$-lattice $S_{2m}(A)$ of holomorphic weight $2m$ cusp forms for $\Gamma_\ell$ that have $q$-expansion coefficients in $A$ at all the cusps, in the sense of [20 §1.6].

Suppose now $p \neq \ell$ is a prime number. Let $F$ be a number field that is unramified over $p$ and that contains a primitive $\ell^\text{th}$ root of unity $\zeta_\ell$. Suppose $A$ is a Dedekind subring of $F$ that has fraction field $F$ and that contains $\mathbb{Z}[\frac{1}{\ell}, \zeta_\ell]$. Let $\mathcal{V}(F,p)$ be the set of places $v$ of $F$ over $p$, and let $\mathcal{O}_{F,v}$ be the ring of integers of the completion $F_v$ of $F$ at $v$. We assume $A$ is contained in $\mathcal{O}_{F,v}$ for all $v \in \mathcal{V}(F,p)$. Let $X_A(\ell)$ be the smooth projective canonical model of $X(\Gamma_\ell)$ over $A$ as above.

For $v \in \mathcal{V}(F,p)$, let $\mathfrak{m}_{F,v}$ be the maximal ideal of $\mathcal{O}_{F,v}$. Define $\mathcal{P}_v = A \cap \mathfrak{m}_{F,v}$ which is a maximal ideal over $p$ in $A$, and define $k(v) = A/\mathcal{P}_v$ to be the corresponding residue field. Then

$$X_v(\ell) = k(v) \otimes_A X_A(\ell)$$

is a smooth projective curve over $k(v)$, and

$$(A/pA) \otimes_A X_A(\ell) = \prod_{v \in \mathcal{V}(F,p)} X_v(\ell).$$

Since $k(v)$ is a finite field, we can identify its algebraic closure $\overline{k(v)}$ with a fixed algebraic closure $\overline{F}_p$ of $F_p$ for all $v \in \mathcal{V}(F,p)$. Let $k$ be an algebraically closed field containing $\overline{F}_p$, and hence containing $k(v)$ for all $v \in \mathcal{V}(F,p)$. Then the reduction of $X_A(\ell)$ modulo $p$ over $k$, which is denoted by $X_p(\ell)$ in [3], is defined as

$$X_p(\ell) = k \otimes_{k(v)} X_v(\ell)$$

for all $v \in \mathcal{V}(F,p)$. Similarly to [3 §5], it follows that when $k = \overline{F}_p$ in (5.2), then, for all $m \geq 1$, there is an isomorphism

$$\overline{F}_p \otimes_{\mathbb{Z}} H^0(X_A(\ell), \Omega_{X_A(\ell)}^m) \cong H^0(X_p(\ell), \Omega_{X_p(\ell)}^m)^{[F:Q]}$$

which is equivariant with respect to the action of $\text{SL}(2, \mathbb{Z}/\ell)$ on $X(\ell)$.

Let $G = \text{PSL}(2, \mathbb{Z}/\ell) = \text{PSL}(2, \mathbb{F}_\ell)$, let $k$ be an algebraically closed field containing $\overline{F}_p$, and let $X_p(\ell)$ be as in [32]. By [3 Thm. 1.1], if $\ell \geq 7$ then $\text{Aut}(X_p(\ell)) = G$ unless $p = 3$ and $\ell \in \{7, 11\}$. Moreover, $\text{Aut}(X_3(7)) \cong \text{PGU}(3, \mathbb{F}_3)$ and $\text{Aut}(X_3(11)) \cong M_{11}$. If $\ell < 7$ then $X_p(\ell)$ has genus 0, whereas if $\ell \geq 7$ then the genus $g(X_p(\ell))$ is given as (see, for example, [3 Cor. 3.2])

$$g(X_p(\ell)) = \frac{(\ell - 1)(\ell + 1)(\ell - 6)}{24}.$$ 

Suppose now that $\ell \geq 7$, and define $X = X_p(\ell)$. In [34 Prop. 5.5], it is shown that the genus of $X/G$ is zero, and the lower ramification groups associated to the cover $X \to X/G$ are determined. It follows that if $p > 3$ then the ramification of $X \to X/G$ is tame and the $kG$-module structure of the holomorphic
poly-differentials $H^0(X, \Omega_X^{\otimes m})$ can be determined using [27] Thm. 2 or [23] Thm. 3. If $p = 3$ then Theorem 1.1 can be used to determine the $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ (see §6.5). On the other hand, if $p = 2$ then the Sylow 2-subgroups of $G$ are not cyclic and the methods of this article are not sufficient to treat this case.

When the ramification of $X \to X/G$ is tame, we obtain the following result.

**Lemma 5.1.** Suppose $m > 1$, $p > 3$ and $p \neq \ell \geq 7$. Let $X = X_p(\ell)$, and let $k$ be an algebraically closed field containing $\mathbb{F}_p$.

(i) The $kG$-module $H^0(X, \Omega_X^{\otimes m})$ is projective.

(ii) Let $v \in V(F, p)$, let $k_1$ be a perfect field containing $k(v)$, and let $k$ be an algebraic closure of $k_1$. Define $X_1 = k_1 \otimes_{k(v)} \mathcal{X}_v(\ell)$ where $\mathcal{X}_v(\ell)$ is as in §6.1. The $k_1G$-module $H^0(X_1, \Omega_{X_1}^{\otimes m})$ is projective.

The $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ as in (i) and the $k_1G$-module structure of $H^0(X_1, \Omega_{X_1}^{\otimes m})$ as in (ii) are both determined by the lower ramification groups and the fundamental characters associated to the cover $X \to X/G$.

**Proof.** Since we assume that $m > 1$ and since $g(X) \geq 3$ by §5.3, it follows that $\deg(\Omega_{X_1}^{\otimes m}) > 2g(X) - 2$, which implies $H^1(X, \Omega_X^{\otimes m}) = 0$. By [27] Thm. 2, this means that there exist finitely generated projective $kG$-modules $L^0$ and $L^1$ together with an exact sequence of $kG$-modules

$$0 \to H^0(X, \Omega_X^{\otimes m}) \to L^0 \to L^1 \to 0.$$

Therefore, the sequence (5.4) splits and $H^0(X, \Omega_X^{\otimes m})$ is a projective $kG$-module. The remainder of Lemma 5.1 is proved using similar arguments as in the proof of [5] Lemma 5.2]. □

**Proof of Theorems 1.2 and 1.3 when $p > 3$.** Using Lemma 5.1 instead of [5] Lemma 5.2, Theorems 1.2 and 1.3 when $p > 3$ are proved by using similar arguments as in the proof of [5] Thms. 2.1 and 2.3]. □

6. **Holomorphic poly-differentials of the modular curves $X_3(\ell)$**

In this section, we determine the $kG$-module structure of the holomorphic poly-differentials of the reduction $X_3(\ell)$ of $\mathcal{X}_4(\ell)$ modulo 3. In particular, we will prove Theorem 1.5 and Theorems 1.2 and 1.3 when $p = 3$. We make the following assumptions throughout this section.

**Assumption 6.1.** Let $m > 1$ be an integer, let $\ell \geq 7$ be a prime number, let $p = 3$, and let $X = X_3(\ell)$ be as in §5.2. Let $k$ be an algebraically closed field containing $\mathbb{F}_3$, and let $G = PSL(2, \mathbb{F}_\ell)$.

The following summarizes the most relevant notation we will use throughout this section; see [5] §6. Note that the subgroup structure of $G = PSL(2, \mathbb{F}_\ell)$ is well known (see, for example, [18] §II.8).

**Notation 6.2.** Suppose Assumption 6.1 holds. Write

$$m = 3 \cdot m' + i_m \quad \text{where } i_m \in \{0, 1, 2\}.$$

For $i \in \{0, 1\}$, define

$$\delta_i = \begin{cases} 1, & i_m = i, \\ 0, & \text{otherwise}, \end{cases}$$

and define

$$\delta_m = \begin{cases} 0, & m \equiv 0 \mod 2 \\ 1, & m \equiv 1 \mod 2. \end{cases}$$

Let $\epsilon, \epsilon' \in \{\pm 1\}$ be such that

$$\ell \equiv \epsilon \mod 3 \quad \text{and} \quad \ell \equiv \epsilon' \mod 4.$$

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Write

\[ \ell - \epsilon = 3^n \cdot 2 \cdot n' \]  

where 3 does not divide \( n' \).

We fix subgroups of \( G \) as follows:

(a) a cyclic subgroup \( V = \langle v \rangle \) of order \((\ell - \epsilon)/2 = 3^n \cdot n'\);

(b) two dihedral groups \( \Delta_1 = \langle v', s \rangle \) and \( \Delta_2 = \langle v', vs \rangle \) of order \( 2 \cdot 3^n \), where \( v' = v^n \in V \) is an element of order \( 3^n \) and \( s \in N_G(V) - V \) is an element of order 2;

(c) the unique subgroup \( I = \langle \tau \rangle \) of order 3 in each of \( V, \Delta_1, \Delta_2 \) where \( \tau = (v')^{3^{n-1}} \);

(d) a cyclic subgroup \( W = \langle w \rangle \) of order \((\ell + \epsilon)/2\);

(e) a cyclic subgroup \( R \) of order \( \ell \).

We define \( P = \langle v' \rangle \), we define \( P_1 = I \), and we define \( N_1 = N_G(P_1) \). If \( \epsilon = -\epsilon' \), i.e. if \( \ell \equiv -\epsilon \mod 4 \), we also define \( \Delta = \Delta_1 \).

**Remark 6.3.** Using Notation 6.2 \( P \) is a Sylow 3-subgroup of \( G \) of order \( 3^n \), \( P_1 \) is the unique subgroup of \( P \) of order 3 and its normalizer \( N_1 = N_G(P_1) \) equals \( N_G(V) = \langle v, s \rangle \) which is a dihedral group of order \( \ell - \epsilon \). Moreover, \( N_G(W) \) is a dihedral group of order \( \ell + \epsilon \), and \( N_G(R) \) is a semidirect product with normal subgroup \( R \) and cyclic quotient group of order \((\ell - 1)/2\). Additionally,

(i) if \( \epsilon = -\epsilon' \), then \( n' \) is odd, and \( \Delta = \Delta_1 \) is a representative of the unique conjugacy class of dihedral groups of order \( 2 \cdot 3^n \) in \( G \);

(ii) if \( \epsilon = \epsilon' \), then \( n' \) is even, and \( \Delta_1 \) and \( \Delta_2 \) are representatives of the two distinct conjugacy classes of dihedral groups of order \( 2 \cdot 3^n \) in \( G \).

We follow the strategy used in [5, §6] for \( m = 1 \) and adapt it to our situation when \( m > 1 \). In §6.1 we list, respectively determine, the data needed to apply Algorithm 3.6 to determine the \( \ell - \epsilon \)-module structure of \( \text{Res}^G_H \mathcal{H}^0(X, \Omega^m_X) \) for \( \Gamma \in \{ V, \Delta_1, \Delta_2 \} \). In §6.2 we first apply Algorithm 3.6 to determine the precise \( k\Gamma \)-module structure of \( \text{Res}^G_H \mathcal{H}^0(X, \Omega^m_X) \) for \( \Gamma \in \{ V, \Delta_1, \Delta_2 \} \). We then use this to determine the **stable** \( kN_1 \)-module structure of \( \text{Res}^{G/K} \mathcal{H}^0(X, \Omega^m_X) \), which means that we determine the non-projective indecomposable direct \( kN_1 \)-module summands \( \text{Res}^{G/K} \mathcal{H}^0(X, \Omega^m_X) \). In §6.3 we use that the Green correspondence induces an equivalence between the **stable** module categories for \( kN_1 \) and \( kG \) to determine the **stable** \( kG \)-module structure of \( \mathcal{H}^0(X, \Omega^m_X) \). In §§6.3 and 6.4 we determine the Brauer character of the \( kG \)-module \( \mathcal{H}^0(X, \Omega^m_X) \). In §6.5 we use §6.3 and §6.4 to determine the precise \( kG \)-module structure of \( \mathcal{H}^0(X, \Omega^m_X) \). We have to consider 4 cases, depending on the values of \( \epsilon, \epsilon' \in \{ \pm 1 \} \). In §6.6 we prove Theorem 1.5 and in §6.7 we prove Theorems 1.2 and 1.3. Finally, in §6.8 we prove Proposition 6.6 which shows how to construct non-trivial \( G \)-isotypic \( T \)-stable decompositions of the space of even weight cusp forms when \( T \) is the ring of Hecke operators that have index prime to the level.

**6.1. Lower ramification groups, fundamental characters, and canonical divisors.** Suppose

\[ \Gamma \in \{ V, \Delta_1, \Delta_2 \} \].

The lower ramification groups of the action of \( \Gamma \) on \( X \) were determined in [5, §6] using [34, Prop. 5.5]. We summarize the relevant information about the non-trivial inertia groups \( \Gamma_{x,0} \) and their first lower ramification groups \( \Gamma_{x,1} \) in Tables 6.1 and 6.2, listing only non-conjugate subgroups \( \Gamma \). Note that for all \( x \in X \), \( \Gamma_{x,2} \) is trivial. Moreover, if \( \epsilon = \epsilon' \) then all inertia groups \( W_{x,0} \) are trivial.

In particular, for \( \Gamma \in \{ V, \Delta_1, \Delta_2 \} \), the subgroup of \( \Gamma \) generated by the Sylow 3-subgroups of the inertia groups in \( \Gamma \) of all closed points in \( X \) is equal to the unique subgroup \( I = \langle \tau \rangle \) of \( \Gamma \) of order 3. Let \( Y = X/I \),
and let \( \pi : X \to Y \) be the corresponding Galois cover with Galois group \( I \). Since there are \( 3^{n-1} \cdot n' \) points in \( Y \) that ramify in \( X \), the Riemann-Hurwitz theorem shows

\[
g(Y) - 1 = 3^{n-1}n' \cdot \frac{(\ell + \epsilon)(\ell - 6) - 8}{12}.
\]

The following remark summarizes the fundamental characters for the cases when \( I \leq \Gamma \); see [5, §6].

**Remark 6.4.** Suppose \( I \leq \Gamma \), i.e., \( \Gamma \in \{V, \Delta_1, \Delta_2\} \). Let \( Y = X/I \), let \( Z = X/\Gamma \), and let \( \lambda : Y \to Z \) be the corresponding cover which is Galois with Galois group \( \Gamma = \Gamma/I \). Let \( Z_{\text{ram}} \) be the set of closed points in \( Z \) that ramify in the cover \( \lambda \).

(a) Suppose \( \Gamma = V \). Then \( \#Z_{\text{ram}} = 0 \) and there are no non-trivial fundamental characters.

(b) Suppose \( \Gamma \in \{\Delta_1, \Delta_2\} \). Then \( \#Z_{\text{ram}} = (\ell - \epsilon)/2 \). For each \( z \in Z_{\text{ram}} \), let \( y(z) \) be a point in \( Y \) lying above \( z \). Then \( \Gamma_{y(z), 0} \) is a subgroup of order 2 in \( \Gamma \) and the fundamental character \( \theta_{y(z)} \) is the unique non-trivial character of \( \Gamma_{y(z), 0} \). In particular, for \( z \in Z_{\text{ram}} \), the Brauer characters \( \text{Ind}_{\Gamma_{y(z), 0}}^{\Gamma} \theta_{y(z)} \) are all equal to the Brauer character of the projective indecomposable \( k\Gamma \)-module whose socle is non-trivial.

In the next remark we provide information, as needed for step (2) of Algorithm 3.6, on the canonical divisors for \( Z = X/\Gamma \) and for \( Y = X/I \) when \( \Gamma \in \{\Delta_1, \Delta_2\} \). Note that we do not need to consider \( \Gamma = V \) since the cover \( X/I \to X/V \) is unramified.

**Remark 6.5.** Suppose \( \Gamma \in \{\Delta_1, \Delta_2\} \). Let \( Y = X/I \), \( Z = X/\Gamma \), and let \( \pi : X \to Y \) and \( \lambda : Y \to Z \) be the corresponding Galois covers with Galois groups \( I \) and \( \Gamma = \Gamma/I \), respectively. The cover \( X \to X/G \cong \mathbb{P}^1_k \) factors into the Galois cover \( \lambda \circ \pi : X \to Z \) followed by a separable morphism \( f : Z = X/\Gamma \to X/G \) which

| Subgroup \( \Gamma \leq G \) | Isomorphism type of \( \Gamma_{x, 0} \) | Isomorphism type of \( \Gamma_{x, 1} \) | Number of points \( x \in X \) with inertia group \( \Gamma_{x, 0} \) |
|------------------------|-----------------|-----------------|-----------------|
| \( V \) | \( \mathbb{Z}/3 \) | \( \mathbb{Z}/3 \) | \( 3^{n-1} \cdot n' \) |
| \( \Delta_1 = \Delta_1 \) | \( \Sigma_3 \) | \( \mathbb{Z}/3 \) | \( 3^{n-1} \) |
| \( W \) | \( \mathbb{Z}/2 \) | \( 1 \) | \( \frac{\ell - \epsilon}{2} \) |
| \( R \) | \( \mathbb{Z}/\ell \) | \( 1 \) | \( \frac{\ell - 1}{2} \) |

Table 6.1. Non-trivial inertia groups for certain subgroups \( \Gamma \) of \( G = \text{PSL}(2, \ell) \) if \( \epsilon = -\epsilon' \)

| Subgroup \( \Gamma \leq G \) | Isomorphism type of \( \Gamma_{x, 0} \) | Isomorphism type of \( \Gamma_{x, 1} \) | Number of points \( x \in X \) with inertia group \( \Gamma_{x, 0} \) |
|------------------------|-----------------|-----------------|-----------------|
| \( V \) | \( \mathbb{Z}/3 \) | \( \mathbb{Z}/3 \) | \( 3^{n-1} \cdot n' \) |
| \( \Delta_1 \) | \( \Sigma_3 \) | \( \mathbb{Z}/3 \) | \( 2 \cdot 3^{n-1} \) |
| \( \Delta_2 \) | \( \mathbb{Z}/3 \) | \( \mathbb{Z}/3 \) | \( 3^{n-1} \cdot (n' - 2) \) |
| \( R \) | \( \mathbb{Z}/\ell \) | \( 1 \) | \( \frac{\ell - 1}{2} \) |

Table 6.2. Non-trivial inertia groups for certain subgroups \( \Gamma \) of \( G = \text{PSL}(2, \ell) \) if \( \epsilon = \epsilon' \)
is not Galois. A canonical divisor on $X/G \cong \mathbb{P}^1_k$ is given by $K_{X/G} = -2\infty$, and a canonical divisor on $Z$ is given by

$$K_Z = f^*(-2\infty) + R_f$$

where $R_f$ is the ramification divisor of $f$ (see [10] Prop. IV.2.3]). Therefore,

$$K_Y = \lambda^*K_Z + R_{Y/Z} = (f \circ \lambda)^*(-2\infty) + \lambda^*R_f + R_{Y/Z}.$$ 

Let now $z \in Z_{\text{ram}}$ and let $y(z) \in Y$ be above it. By [34] Table on p. 193], $z$ and $y(z)$ lie above $0 \in X/G \cong \mathbb{P}^1_k$, which means that the coefficient of $y(z)$ in $(f \circ \lambda)^*(-2\infty)$ is zero. Since $\overline{\Gamma}_{y(z),0} \cong \mathbb{Z}/2$, we obtain that the coefficient of $y(z)$ in $R_{Y/Z}$ is 1. One the other hand, the coefficient of $y(z)$ in $\lambda^*R_f$ equals $e_{y(z)/z} \cdot d_{z/0}$, where $e_{y(z)/z} = 2$ is the ramification index of $y(z)$ over $z$ and $d_{z/0}$ is the different exponent of $z$ over $0 \in X/G \cong \mathbb{P}^1_k$. Using the transitivity of the different, we obtain for a point $x(z) \in X$ above $z$ that

$$d_{x(z)/0} = e_{x(z)/z} \cdot d_{z/0} + d_{x(z)/z}.$$ 

By [34] Table on p. 193], $d_{x(z)/0} = (6 - 1) + (3 - 1) = 7$. We obtain the following values for $e_{x(z)/z}$ and $d_{z/0}$, and hence for $d_{z/0}$.

(i) Suppose $\epsilon = -\epsilon'$. In this case, we only need to consider $\Gamma = \Delta = \Delta_1$.

It follows from Table 6.1 and Remark 6.4 that there is precisely one point $z_1 \in Z_{\text{ram}}$ with $e_{x(z_1)/z_1} = 6$ and $d_{x(z_1)/z_1} = 7$, and there are precisely $(\ell - \epsilon')/2 - 1$ points $z_2, \ldots, z_{(\ell - \epsilon')/2} \in Z_{\text{ram}}$ with $e_{x(z_i)/z_i} = 2$ and $d_{x(z_i)/z_i} = 1$ for $2 \leq i \leq (\ell - \epsilon')/2$. Therefore, we obtain $d_{z_1/0} = 0$ and $d_{z_i/0} = 3$ for $2 \leq i \leq (\ell - \epsilon')/2$, which means

$$\text{ord}_{y(z_1)}(K_Y) = \begin{cases} 1, & i = 1, \\ 7, & 2 \leq i \leq (\ell - \epsilon')/2, \end{cases}$$

if $\Gamma = \Delta$.

(ii) Suppose $\epsilon = \epsilon'$. In this case, we need to consider $\Gamma \in \{\Delta_1, \Delta_2\}$.

If $\Gamma = \Delta_1$, it follows from Table 6.2 and Remark 6.4 that there are precisely two points $z_1, z_2 \in Z_{\text{ram}}$ with $e_{x(z_1)/z_1} = 6$ and $d_{x(z_1)/z_1} = 7$ for $i = 1, 2$, and there are precisely $(\ell - \epsilon')/2 - 2$ points $z_3, \ldots, z_{(\ell - \epsilon')/2} \in Z_{\text{ram}}$ with $e_{x(z_i)/z_i} = 2$ and $d_{x(z_i)/z_i} = 1$ for $3 \leq i \leq (\ell - \epsilon')/2$. Therefore, we obtain $d_{z_1/0} = 0$ for $i = 1, 2$, and $d_{z_i/0} = 3$ for $3 \leq i \leq (\ell - \epsilon')/2$, which means

$$\text{ord}_{y(z_1)}(K_Y) = \begin{cases} 1, & i = 1, 2, \\ 7, & 3 \leq i \leq (\ell - \epsilon')/2, \end{cases}$$

if $\Gamma = \Delta_1$.

If $\Gamma = \Delta_2$, it follows from Table 6.2 and Remark 6.4 that there are precisely $(\ell - \epsilon')/2$ points $z_1, \ldots, z_{(\ell - \epsilon')/2} \in Z_{\text{ram}}$ with $e_{x(z_i)/z_i} = 2$ and $d_{x(z_i)/z_i} = 1$ for $1 \leq i \leq (\ell - \epsilon')/2$. Therefore, we obtain $d_{z_1/0} = 3$ for $1 \leq i \leq (\ell - \epsilon')/2$, which means

$$\text{ord}_{y(z_1)}(K_Y) = 7, \quad 1 \leq i \leq (\ell - \epsilon')/2,$$

if $\Gamma = \Delta_2$.

6.2. The stable $kN_1$-module structure of the holomorphic poly-differentials. We first calculate the restrictions of $H^0(X, \Omega_X^{\text{log}})$ to the subgroups $\Gamma \in \{V, \Delta_1, \Delta_2\}$ of $N_1$. We then use that every $kN_1$-module is uniquely determined by its restrictions to these subgroups to find the stable $kN_1$-module structure of $\text{Res}_{N_1}^G H^0(X, \Omega_X^{\text{log}})$, which means that we find the non-projective indecomposable direct $kN_1$-module summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^{\text{log}})$, together with their multiplicities.

As noted in [33], for each $\Gamma \in \{V, \Delta_1, \Delta_2\}$, the subgroup of $\Gamma$ that is generated by the Sylow 3-subgroups of the inertia groups of all closed points in $X$ is equal to the unique subgroup $I = \langle \tau \rangle$ of $\Gamma$ of order 3. Let $Y = X/I$, and let $\pi : X \to Y$ be the corresponding Galois cover with Galois group $I$. There are precisely $3^{n-1} \cdot n'$ closed points $x$ in $X$ with $\Gamma_{x,0} \supseteq I$. For $1 \leq t \leq n'$, let $y_{t,1}, \ldots, y_{t,3^{n-1}} \in Y$ be points that ramify in $X$. For any ramification point $x \in X$ of the cover $\pi : X \to Y$, we have $\# I_{x,0} = \# I_{x,1} = 3$ and $\# I_{x,2} = 1$.}
If \( y \in Y \) ramifies in \( X \) and \( x \in X \) is a point above it, we obtain from step (1) of Algorithm \textbf{3.6} that, for \( 0 \leq j \leq 2 \),

\[
d_{y,j} = \left\lfloor \frac{4m-j}{3} \right\rfloor.
\]

Hence, for \( 0 \leq j \leq 2 \), the divisor \( D_j \) is given as

\[
D_j = \sum_{\ell=1}^{\ell} \sum_{i=1}^{m} \left\lfloor \frac{4m-j}{3} \right\rfloor y_{\ell,i}.
\]

Writing \( m = 3 \cdot m' + i_m \) as in Notation \textbf{6.2} where \( i_m \in \{0, 1, 2\} \), we obtain

\[
\left\lfloor \frac{4m-j}{3} \right\rfloor = 4m' + \left\lfloor \frac{4i_m-j}{3} \right\rfloor = \begin{cases} 
4m' - 1, & i_m = 0, j \in \{2, 1\}, \\
4m', & (i_m = 0, j = 0) \text{ or } (i_m = 1, j = 2), \\
4m' + 1, & i_m = 1, j \in \{1, 0\}, \\
4m' + 2, & i_m = 2, j \in \{2, 1, 0\},
\end{cases}
\]

and hence the degree of \( D_j \) is given as

\[
\deg(D_j) = 3^{n-1} \cdot n' \cdot \left\lfloor \frac{4m-j}{3} \right\rfloor.
\]

\textbf{6.2.1. The \( kv \)-module structure.} By Notation \textbf{6.2}, we have \( V \cong (\mathbb{Z}/3^n) \times (\mathbb{Z}/n') \) where 3 does not divide \( n' \). If \( Z = X/V \), then \( \lambda : Y \to Z \) is unramified with Galois group \( \overline{\Gamma} = V/I \), which means that \( Z_{\text{ram}} = \emptyset \). Hence by step (2) of Algorithm \textbf{3.6} applied to \( M_V = \text{Res}^G_V H^0(X, \Omega_X^{\otimes m}) \), we have, for \( 0 \leq j \leq 2 \), that the Brauer character \( \beta_V(j) \) of \( M_{V'}^{(j+1)}/M_{V'}^{(j)} \) is equal to

\[
\beta_V(j) = n_j(V) \beta(kV)
\]

where

\[
n_j(V) = \frac{(2m-1)(g(Y) - 1) + \deg(D_j)}{\# \overline{\Gamma}}.
\]

Since \( \# \overline{\Gamma} = 3^{n-1} \cdot n' \), it follows from \textbf{6.6} and \textbf{6.7} that

\[
n_2(V) = \begin{cases} 
n_1(V) = n_0(V) - 1, & i_m = 0, \\
n_1(V) - 1 = n_0(V) - 1, & i_m = 1, \\
n_1(V) = n_0(V), & i_m = 2.
\end{cases}
\]

Using the notation of Remark \textbf{2.7}, there are \( n' \) isomorphism classes of simple \( kv \)-modules, represented by \( S^{(V)}_0, S^{(V)}_1, \ldots, S^{(V)}_{n'-1} \), where we use the superscript \( (V) \) to indicate these are simple \( kv \)-modules. Writing \( U^{(V)}_{a,b} \) for an indecomposable \( kv \)-module of \( k \)-dimension \( b \) with socle isomorphic to \( S^{(V)}_a \), it follows from step (3) of Algorithm \textbf{3.6} that

\[
\text{Res}^G_V H^0(X, \Omega_X^{\otimes m}) \cong n_2(V) kV \oplus \delta_1 \bigoplus_{t=0}^{n'-1} U^{(V)}_{t,2^{3n-1}} \oplus \delta_0 \bigoplus_{t=0}^{n'-1} U^{(V)}_{t,3^{n-1}}
\]

where \( \delta_0, \delta_1 \) are as in Notation \textbf{6.2} and, using \textbf{6.1} and \textbf{6.7},

\[
n_2(V) = (2m - 1) \left( \frac{(\ell - 6)(\ell + e) - 8}{12} + \left\lfloor \frac{4m-2}{3} \right\rfloor \right).
\]
6.2.2. The $k\Delta_1$- and $k\Delta_2$-module structures. Let $\Gamma \in \{\Delta_1, \Delta_2\}$, where we only consider $\Gamma = \Delta = \Delta_1$ if $\epsilon = -\epsilon'$. By Notation 6.2 we have $\Gamma \cong (\mathbb{Z}/3^n) \times \chi^3(\mathbb{Z}/2)$. In particular, there are precisely two isomorphism classes of simple $k\Gamma$-modules, represented by $S_0(\Gamma)$ and $S_1(\Gamma)$, and $S_\chi \cong S_1(\Gamma)$. If $Z = X/\Gamma$, then $\lambda : Y \rightarrow Z$ is tamely ramified with Galois group $\Gamma = \Gamma/I$. By Remark 6.4, $\#Z_{\text{ram}} = (\ell - \epsilon')/2$. Moreover, if $z \in Z_{\text{ram}}$ and $y(z) \in Y$ lies above $z$, then $\Gamma_{y(z),0}$ is a subgroup of order 2 in $\Gamma$ and the fundamental character $\theta_{y(z)}$ is the unique non-trivial character of $\Gamma_{y(z),0}$. Hence by step (2) of Algorithm 3.6 applied to $M_\Gamma = \text{Res}_\Gamma^G H^0(X, \Omega_X^m)$, we have, for $0 \leq j \leq 2$, that the Brauer character $\beta_\Gamma(j)$ of $S_{\chi j,\delta}(\text{Ind}_\Gamma^Y (M_\Gamma^{(j+1)}/M_\Gamma^{(j)})$ is equal to (using that $\ell_{y(z),j} \in \{0, 1\}$ for $z \in Z_{\text{ram}}$)

$$\beta_\Gamma(j) = \sum_{z \in Z_{\text{ram}}} \left( \ell_{y(z),j} - \frac{1}{2} \right) \text{Ind}_{\Gamma_{y(z),0}}^{\Gamma} \theta_{y(z)} + n_j(\Gamma) \beta(k\Gamma)$$

where

$$n_j(\Gamma) = \frac{(2m - 1)(g(Y) - 1) + \deg(D_j)}{\#\Gamma} + \sum_{z \in Z_{\text{ram}}} \frac{1}{2} \left( \frac{1}{2} - \ell_{y(z),j} \right),$$

and, for $z \in Z_{\text{ram}}$, $\ell_{y(z),j} \in \{0, 1\}$ is defined by

$$\ell_{y(z),j} \equiv \text{ord}_{\text{y}(z)}(mK_Y + D_j) \mod 2.$$

Let $Z_{\text{ram}} = \{z_1, \ldots, z_{(\ell - \epsilon')/2}\}$. Define

$$(6.8)\quad i_0 = \begin{cases} 1, & \epsilon = -\epsilon', \Gamma = \Delta = \Delta_1, \\ 2, & \epsilon = \epsilon', \Gamma = \Delta_1, \\ 0, & \epsilon = \epsilon', \Gamma = \Delta_2. \end{cases}$$

By Remark 6.5 we have that precisely the first $i_0$ points in $Z_{\text{ram}}$ are among the points $y_{t,1}, \ldots, y_{t,3n-1}$ ($1 \leq t \leq n'$) in $Y$ occurring in $D_j$ in (6.8). Hence, we obtain by (6.2), (6.3) and (6.4) that

$$\text{ord}_{\text{y}(z_i)}(mK_Y + D_j) = \begin{cases} m + \left\lfloor \frac{4m - j}{3m} \right\rfloor, & 1 \leq i \leq i_0, \\ 7m, & i_0 + 1 \leq i \leq (\ell - \epsilon')/2. \end{cases}$$

It follows that

$$(6.9)\quad \ell_{y(z_i),j} = \begin{cases} 0, & m - j \equiv 0, 1, 2 \mod 6, \\ 1, & m - j \equiv 3, 4, 5 \mod 6, \end{cases} \quad \text{for } 1 \leq i \leq i_0,$$

and

$$\ell_{y(z_i),j} - \delta_m \quad \text{for } i_0 + 1 \leq i \leq (\ell - \epsilon')/2,$$

where $\delta_m$ is as in Notation 6.2. Moreover, we have, for $1 \leq i \leq i_0$,

$$\ell_{y(z_i),j} - \delta_m = \ell_{j}(1 - 2\delta_m),$$

where

$$(6.10)\quad \ell_{j} = \begin{cases} \delta_1, & j = 0, \\ \delta_0 + \delta_1, & j = 1, \\ \delta_0, & j = 2, \end{cases}$$

and $\delta_0, \delta_1$ are as in Notation 6.2.

Let $P(\Gamma, 0)$ (resp. $P(\Gamma, 1)$) be a projective indecomposable $k\Gamma$-module with trivial (resp. non-trivial) socle. Then $\text{Ind}_{\Gamma_{y(z),0}}^{\Gamma} \theta_{y(z)} = \beta(P(\Gamma, 1))$, for all $z \in Z_{\text{ram}}$, and $\beta(k\Gamma) = \beta(P(\Gamma, 0)) + \beta(P(\Gamma, 1))$. Hence it
follows that
\[
\beta_\Gamma(j) = n_j(\Gamma) \beta(P(\Gamma,0)) + \left(n_j(\Gamma) + i_0 \left(\ell_{j(z_1,i)} - \delta_m\right) + \delta_m \frac{\ell - \ell'}{2} - \frac{\ell - \ell'}{4}\right) \beta(P(\Gamma,1))
\]
\[
= n_j(\Gamma) \beta(P(\Gamma,0)) + \left(n_j(\Gamma) + (1 - 2\delta_m) \left(i_0 \left(\ell_j - \frac{\ell - \ell'}{4}\right)\right)\right) \beta(P(\Gamma,1))
\]
and
\[
n_j(\Gamma) = \frac{(2m-1)(\delta(Y) - 1) + \deg(D_j)}{\#_\Gamma} - \frac{1 - 2\delta_m}{2} \left(i_0 \left(\ell_j - \frac{\ell - \ell'}{4}\right)\right)
\]
where \(\delta_m\) is as in Notation 6.2, \(i_0\) is as in 6.8 and \(\ell_j\) is as in 6.10. Since \(\#_\Gamma = 2 \cdot 3^n-1\), it follows from 6.7, 6.8 and 6.10 that
\[
n_2(\Gamma) = \begin{cases} 
 n_1(\Gamma) = n_0(\Gamma) - \frac{n' + i_0(1 - 2\delta_m)}{2}, & i_m = 0, \\
n_1(\Gamma) - \frac{n' - i_0(1 - 2\delta_m)}{2} = n_0(\Gamma) - \frac{n' - i_0(1 - 2\delta_m)}{2}, & i_m = 1, \\
n_1(\Gamma) = n_0(\Gamma), & i_m = 2.
\end{cases}
\]
Writing \(U^{(\Gamma)}_{a,b}\) for an indecomposable \(k\Gamma\)-module of \(k\)-dimension \(b\) with socle isomorphic to \(S^{(\Gamma)}_a\), it follows from step (3) of Algorithms 8.6 that, for \(\Gamma \in \{\Delta_1, \Delta_2\},
\]
\[
\text{Res}_{\Gamma}^G H^0(\mathbf{X}, \Omega_X^{\otimes m}) \cong \ n_2(\Gamma) U_{0,3^n}^{(\Gamma)} \oplus \left(n_2(\Gamma) + (1 - 2\delta_m) \left(i_0 \delta_0 - \frac{\ell - \ell'}{4}\right)\right) U_{1,3^n}^{(\Gamma)}
\]
\[
\oplus \delta_1 \left(\frac{n' - i_0(1 - 2\delta_m)}{2} U_{0,2,3^n-1}^{(\Gamma)} \oplus \frac{n' + i_0(1 - 2\delta_m)}{2} U_{1,2,3^n-1}^{(\Gamma)}\right)
\]
\[
\oplus \delta_0 \left(\frac{n' + i_0(1 - 2\delta_m)}{2} U_{0,3^n-1}^{(\Gamma)} \oplus \frac{n' - i_0(1 - 2\delta_m)}{2} U_{1,3^n-1}^{(\Gamma)}\right)
\]
where \(\delta_0, \delta_1, \delta_m\) are as in Notation 6.2, \(i_0\) is as in 6.8, and, using 6.1 and 6.7,
\[
n_2(\Gamma) = \frac{n'}{2} \left(\frac{(2m-1)(\ell + \ell)(\ell - 6) - 8}{12} + \left[\frac{4m}{3} - 2\right]\right) - \frac{1 - 2\delta_m}{2} \left(i_0 \delta_0 - \frac{\ell - \ell'}{4}\right).
\]

6.2.3. The stable \(kN_1\)-module structure when \(\epsilon = -\ell'\). We use 6.2.1 and 6.2.2 to determine the stable \(kN_1\)-module structure of \(H^0(\mathbf{X}, \Omega_X^{\otimes m})\) when \(\epsilon = -\ell'\). Using Notation 6.2 \(P = \langle v' \rangle\) is a Sylow 3-subgroup of \(G\) and \(P_1 = I\) is the unique subgroup of \(P\) of order 3. Hence \(N_1 = N_G(P) = \langle v, s \rangle\) is a dihedral group of order \(\ell = 2 \cdot 3^n \cdot n'\). There are \(2 + (n' - 1)/2\) isomorphism classes of simple \(kN_1\)-modules. These are represented by 2 one-dimensional \(kN_1\)-modules \(S_0^{(N_1)}\) and \(S_1^{(N_1)}\) such that \(S_i^{(N_1)}(\Delta)\) restricts to \(S_i^{(\Delta)}\) for \(i \in \{0, 1\}\), together with \((n' - 1)/2\) two-dimensional simple \(kN_1\)-modules \(S_1^{(N_1)}, \ldots, \sim S_{(n' - 1)/2}^{(N_1)}\), where \(\sim S_i^{(N_1)} = \text{Ind}_{N_1}^G S_i^{(V)}\) for \(1 \leq t \leq (n' - 1)/2\). The indecomposable \(kN_1\)-modules are uniserial, where the projective modules all have length \(3^n\). For \(i \in \{0, 1\}\), the projective cover of \(S_i^{(N_1)}\) has ascending composition factors
\[
S_i^{(N_1)}, S_{n-i}^{(N_1)}, S_i^{(N_1)}(\Delta) \cdots, S_{n-i}^{(N_1)}(\Delta), S_i^{(N_1)}(\Delta).
\]
For \(t \in \{1, \ldots, (n' - 1)/2\}\), the composition factors of the projective cover of \(S_t^{(N_1)}\) are all isomorphic to \(S_t^{(N_1)}(\Delta)\). For \(i \in \{0, 1\}\), we write \(U_{a,b}^{(N_1)}\) for an indecomposable \(kN_1\)-module of \(k\)-dimension \(b\) whose socle is isomorphic to \(S_i^{(N_1)}\). For \(t \in \{1, \ldots, (n' - 1)/2\}\), we write \(\tilde{U}_{a,b}^{(N_1)}\) for an indecomposable \(kN_1\)-module of \(k\)-dimension \(2b\) whose socle is isomorphic to \(S_t^{(N_1)}(\Delta)\). By 6.2.1 and by 6.2.2 for \(\Gamma = \Delta\), we obtain that the non-projective indecomposable direct summands of \(\text{Res}_{N_1}^G H^0(\mathbf{X}, \Omega_X^{\otimes m})\), with their multiplicities, are given
by the direct sum

\[(6.11) \quad \delta_1 \left( \delta_m U_{0,2}^{(N_i)} \oplus (1 - \delta_m) U_{1,2}^{(N_i)} \oplus \bigoplus_{t=1}^{(n'-1)/2} \tilde{U}_{t,2}^{(N_i)} \right) \]

\[\oplus \delta_0 \left( (1 - \delta_m) U_{0,3}^{(N_i)} \oplus \delta_m U_{1,3}^{(N_i)} \oplus \bigoplus_{t=1}^{(n'-1)/2} \tilde{U}_{t,3}^{(N_i)} \right)\]

where \(\delta_0, \delta_1, \delta_m\) are as in Notation 6.2.

6.2.4. The stable \(kN_1\)-module structure when \(\epsilon = \epsilon'\). We use again 6.2.1 and 6.2.2 to determine the stable \(kN_1\)-module structure of \(H^0(X, \Omega_X^{\otimes m})\) when \(\epsilon = \epsilon'\). Using Notation 6.2, \(P = \langle \nu' \rangle\) is a Sylow 3-subgroup of \(G\) and \(P_1 = I\) is the unique subgroup of \(P\) of order 3. Hence \(N_1 = N_G(P) = \langle \nu, s \rangle\) is a dihedral group of order \(\ell - \epsilon = 2 \cdot 3^n \cdot n'\). There are \(4 + (n'/2 - 1)\) isomorphism classes of simple \(kN_1\)-modules. These are represented by 4 one-dimensional \(kN_1\)-modules \(S_{0,0}^{(N_i)}, S_{0,1}^{(N_i)}, S_{1,0}^{(N_i)}\) and \(S_{1,1}^{(N_i)}\) such that \(S_{i_1, i_2}^{(N_i)}\) restricts to \(S_{i_1}^{(\Delta_1)}\) and to \(S_{i_2}^{(\Delta_2)}\) for \(i_1, i_2 \in \{0, 1\}\), together with \((n'/2 - 1)\) two-dimensional simple \(kN_1\)-modules \(\tilde{S}_1^{(N_i)}, \ldots, \tilde{S}_{(n'/2 - 1)}^{(N_i)}\), where \(\tilde{S}_t^{(N_i)} = \text{Ind}_V^N S_t^{(V)}\) for \(1 \leq t \leq (n'/2 - 1)\). The indecomposable \(kN_1\)-modules are uniserial, where the projective modules all have length \(3^n\). For \(i \in \{0, 1\}\), the projective cover of \(S_{i,i}^{(N_i)}\) has ascending composition factors

\[S_{i,1}^{(N_i)}, S_{1,i}^{(N_i)}, S_{1,i,1}^{(N_i)}, \ldots, S_{1,i,1,1}^{(N_i)}, S_{1,i}^{(N_i)}\]

and the projective cover of \(S_{i,1}^{(N_i)}\) has ascending composition factors

\[S_{i,1}^{(N_i)}, S_{1,i}^{(N_i)}, S_{1,i,1}^{(N_i)}, \ldots, S_{1,i,1,i}^{(N_i)}, S_{1,i}^{(N_i)}\]

For \(t \in \{1, \ldots, (n'/2 - 1)\}\), the composition factors of the projective cover of \(\tilde{S}_t^{(N_i)}\) are all isomorphic to \(\tilde{S}_t^{(N_i)}\). For \(i_1, i_2 \in \{0, 1\}\), we write \(U_{i_1,i_2}^{(N_i)}\) for an indecomposable \(kN_1\)-module of \(k\)-dimension \(b\) whose socle is isomorphic to \(S_{i_1,i_2}^{(N_i)}\). For \(t \in \{1, \ldots, (n'/2 - 1)\}\), we write \(\tilde{U}_{t,b}^{(N_i)}\) for an indecomposable \(kN_1\)-module of \(k\)-dimension \(2b\) whose socle is isomorphic to \(\tilde{S}_t^{(N_i)}\). By 6.2.1 and by 6.2.2, for \(\Gamma \in \{\Delta_1, \Delta_2\}\), we obtain that the non-projective indecomposable direct summands of \(\text{Res}_{N_1}^G H^0(X, \Omega_X^{\otimes m})\), with their multiplicities, are given by the direct sum

\[(6.12) \quad \delta_1 \left( \delta_m \left( U_{0,2}^{(N_i)} \oplus U_{1,2}^{(N_i)} \right) \oplus (1 - \delta_m) \left( U_{1,1,2}^{(N_i)} \oplus U_{1,0,2}^{(N_i)} \right) \oplus \bigoplus_{t=1}^{(n'/2-1)} \tilde{U}_{t,2}^{(N_i)} \right) \]

\[\oplus \delta_0 \left( (1 - \delta_m) \left( U_{0,3}^{(N_i)} \oplus U_{1,0,3}^{(N_i)} \right) \oplus \delta_m \left( U_{1,1,0}^{(N_i)} \oplus U_{1,0,3}^{(N_i)} \right) \oplus \bigoplus_{t=1}^{(n'/2-1)} \tilde{U}_{t,3}^{(N_i)} \right)\]

where \(\delta_0, \delta_1, \delta_m\) are as in Notation 6.2.

6.3. The stable \(kG\)-module structure of the holomorphic poly-differentials. In this section, we determine the non-projective indecomposable \(kG\)-modules that are direct summands of \(H^0(X, \Omega_X^{\otimes m})\), together with their multiplicities. These \(kG\)-modules are precisely the Green correspondents of the non-projective indecomposable direct \(kN_1\)-module summands of \(\text{Res}_{N_1}^G H^0(X, \Omega_X^{\otimes m})\). We use 6.2.3 and 6.2.4, together with \([7, \S III-VI]\), to determine these Green correspondents; see also \([5, \S 6.4]\). We have to consider four cases.
6.3.1. **The stable kG-module structure when \( \ell' = -\varepsilon \) and \( \ell' = 1 \).** This is the case when \( \ell \equiv 1 \mod 4 \) and \( \ell \equiv -1 \mod 3 \). By \([6.2.3]\) the non-projective indecomposable direct summands of \( \text{Res}^G_{N_1} H^0(X, \Omega^\otimes_{X}^m) \), with their multiplicities, are those in the direct sum \([6.11]\). We determine the Green correspondents of these summands, using the information in \([7] \S IV\); see also \([8] \S 6.4.1\]. There are \( 1 + (n' - 1)/2 \) blocks of \( kG \) of maximal defect \( n \), consisting of the principal block \( B_0 \) and \( (n' - 1)/2 \) blocks \( B_1, \ldots, B_{(n'-1)/2} \), and there are \( 1 + (\ell - 1)/4 \) blocks of \( kG \) of defect 0. There are precisely two isomorphism classes of simple \( kG \)-modules that belong to \( B_0 \), represented by the trivial simple \( kG \)-module \( T_0 \) and a simple \( kG \)-module \( T_0 \) of \( k \)-dimension \( \ell - 1 \). For each \( t \in \{1, \ldots, (n' - 1)/2\} \), there is precisely one isomorphism class of simple \( kG \)-modules belonging to \( B_t \), represented by a simple \( kG \)-module \( \overline{T}_t \) of \( k \)-dimension \( \ell - 1 \).

To determine the Green correspondents of the non-projective indecomposable direct \( kN_1 \)-module summands of \( \text{Res}^G_{N_1} H^0(X, \Omega^\otimes_{X}^m) \), we use that there is a stable equivalence between the module categories of \( kG \) and \( kN_1 \). This allows us to use the results from \([2] \S X.1\] on almost split sequences to be able to detect the Green correspondents. As in \([5] \S 6.4.1\], the Green correspondent of \( S_0^{(N_1)} \) is \( T_0 \), the Green correspondent of \( S_0^{(N_1)} \) is a uniserial \( kG \)-module of length \( 3^n - 1 \) whose composition factors are all isomorphic to \( \overline{T}_0 \), and the Green correspondent of \( S_0^{(N_1)} \), for \( 1 \leq t \leq (n' - 1)/2 \), is a uniserial \( kG \)-module of length \( 3^n - 1 \) whose composition factors are all isomorphic to \( \overline{T}_t \). Following the irreducible homomorphisms in the stable Auslander-Reiten quiver of \( B_0 \) (resp. \( B_t \), for \( 1 \leq t \leq (n' - 1)/2 \)), we obtain the indecomposable \( kG \)-modules that are the Green correspondents of the indecomposable \( kN_1 \)-modules occurring in \([6.11]\).

Using Notation \([1.4]\) we obtain that the non-projective indecomposable direct \( kG \)-module summands of \( H^0(X, \Omega^\otimes_{X}^m) \), with their multiplicities, are given by

\[
(6.13) \quad \delta_1 \left( \delta_m U^{(G)}_{T_0, T_0,(3^n-1)/2} \oplus (1 - \delta_m) U^{(G)}_{T_0, T_0,(3^{n'-1}+1)/2} \oplus \bigoplus_{t=1}^{(n' - 1)/2} U^{(G)}_{T_t, 3^n - 1} \right) \\
\quad \quad \quad \quad \quad \oplus \delta_0 \left( (1 - \delta_m) U^{(G)}_{T_0, T_0,3^{n-1}+1} \oplus \delta_m U^{(G)}_{T_0, T_0,3^n - 1} \oplus \bigoplus_{t=1}^{(n' - 1)/2} U^{(G)}_{T_t, 2 \cdot 3^n - 1} \right).
\]

In particular, for every possible value of \( \delta_0, \delta_1, \delta_m \), the module in \( (6.13) \) contains, for each block \( B \) of \( kG \), at most one non-projective indecomposable \( kG \)-module belonging to \( B \) as direct summand.

6.3.2. **The stable kG-module structure when \( \ell' = -\varepsilon \) and \( \ell' = -1 \).** This is the case when \( \ell \equiv -1 \mod 4 \) and \( \ell \equiv 1 \mod 3 \). By \([6.2.3]\) the non-projective indecomposable direct summands of \( \text{Res}^G_{N_1} H^0(X, \Omega^\otimes_{X}^m) \), with their multiplicities, are again those in the direct sum \([6.11]\). We determine the Green correspondents of these summands, using the information in \([7] \S IV\); see also \([8] \S 6.4.2\]. There are \( 1 + (n' - 1)/2 \) blocks of \( kG \) of maximal defect \( n \), consisting of the principal block \( B_0 \) and \( (n' - 1)/2 \) blocks \( B_1, \ldots, B_{(n'-1)/2} \), and there are \( 1 + (\ell + 1)/4 \) blocks of \( kG \) of defect 0. There are precisely two isomorphism classes of simple \( kG \)-modules that belong to \( B_0 \), represented by the trivial simple \( kG \)-module \( T_0 \) and a simple \( kG \)-module \( T_1 \) of \( k \)-dimension \( \ell \). For each \( t \in \{1, \ldots, (n' - 1)/2\} \), there is precisely one isomorphism class of simple \( kG \)-modules belonging to \( B_t \), represented by a simple \( kG \)-module \( \overline{T}_t \) of \( k \)-dimension \( \ell + 1 \).

As in \([5] \S 6.4.2\], the Green correspondent of \( S_0^{(N_1)} \) (resp. \( S_1^{(N_1)} \)) is \( T_0 \) (resp. \( T_1 \)), and the Green correspondent of \( S_t^{(N_1)} \), for \( 1 \leq t \leq (n' - 1)/2 \), is \( \overline{T}_t \). Using similar arguments as in \([6.8.1]\) and Notation \([1.4]\) we obtain that the non-projective indecomposable direct \( kG \)-module summands of \( H^0(X, \Omega^\otimes_{X}^m) \), with their
In particular, for every possible value of $\delta_0, \delta_1, \delta_m$, the module in (6.14) contains, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as direct summand.

6.3.3. The stable $kG$-module structure when $\epsilon' = \epsilon$ and $\epsilon' = 1$. This is the case when $\ell \equiv 1 \mod 4$ and $\ell \equiv 1 \mod 3$. By (6.2.3), the non-projective indecomposable direct summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are those in the direct sum (6.12). We determine the Green correspondents of these summands, using the information in [3] §III; see also [5] §6.4.3. There are $1 + (n'/2)$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_{00}$, another block $B_{01}$, and $(n'/2 - 1)$ blocks $B_1, \ldots, B_{(n'/2 - 1)}$, and there are $(\ell - 1)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_{00}$ (resp. $B_{01}$), represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $T_1$ of $k$-dimension $\ell$ (resp. by two simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of $k$-dimension $(\ell + 1)/2$). For each $t \in \{1, \ldots, (n'/2 - 1)\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $T_t$ of $k$-dimension $\ell - 1$.

In the case of $\ell \equiv 1 \mod 3$, the Green correspondent of $S_{0,1}^{(N_1)}$ (resp. $S_{1,1}^{(N_1)}$) is $T_0$ (resp. $T_1$), and the Green correspondent of $S_t^{(N_1)}$ for $1 \leq t \leq (n'/2 - 1)$, is $T_t$. On the other hand, the Green correspondent of $S_{0,1}^{(N_1)}$ is one of $T_{0,1}$ or $T_{1,0}$. We relabel the simple $kG$-modules, if necessary, to be able to assume that the Green correspondent of $S_{0,1}^{(N_1)}$ (resp. $S_{1,1}^{(N_1)}$) is $T_{0,1}$ (resp. $T_{1,0}$). Using similar arguments as in §6.3.1 and Notation 1.4, we obtain that the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are given by

\begin{equation}
(6.15) \quad \delta_1 \left( \delta_m \left( U_{T_0,2}^{(G)} \oplus U_{T_1,2}^{(G)} \right) \oplus (1 - \delta_m) \left( U_{T_0,1}^{(G)} \oplus U_{T_1,1}^{(G)} \right) \right) \oplus \left( \frac{n'/2 - 1}{2} \right) \bigoplus_{t=1} U_{T_t,2}^{(G)}
\end{equation}

\begin{equation}
\oplus \delta_0 \left( (1 - \delta_m) \left( U_{T_0,3}^{(G)} \oplus U_{T_1,3}^{(G)} \right) \oplus \delta_m \left( U_{T_0,1}^{(G)} \oplus U_{T_1,1}^{(G)} \right) \right) \oplus \left( \frac{n'-2}{2} \right) \bigoplus_{t=1} U_{T_t,3}^{(G)}
\end{equation}

In particular, for every possible value of $\delta_0, \delta_1, \delta_m$, the module in (6.15) contains, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as direct summand.

6.3.4. The stable $kG$-module structure when $\epsilon' = \epsilon$ and $\epsilon' = -1$. This is the case when $\ell \equiv -1 \mod 4$ and $\ell \equiv -1 \mod 3$. By (6.2.4), the non-projective indecomposable direct summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are again those in the direct sum (6.12). We determine the Green correspondents of these summands, using the information in [3] §VI; see also [5] §6.4.4. There are $1 + (n'/2)$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_{00}$, another block $B_{01}$, and $(n'/2 - 1)$ blocks $B_1, \ldots, B_{(n'/2 - 1)}$, and there are $(\ell - 3)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_{00}$ (resp. $B_{01}$), represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $T_0$ of $k$-dimension $\ell - 1$ (resp. by two simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of $k$-dimension $(\ell - 1)/2$). For each $t \in \{1, \ldots, (n'/2 - 1)\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $T_t$ of $k$-dimension $\ell - 1$.
As in [§6.4.4], the Green correspondent of $S^{(N_1)}_{0,0}$ (resp. $S^{(N_1)}_{1,1}$) is $T_0$ (resp. a uniserial $kG$-module of length $(3^n - 1)/2$ whose composition factors are all isomorphic to $\tilde{T}_0$), and the Green correspondent of $S^{(N_1)}_1$, for $1 \leq t \leq (n'/2-1)$, is a uniserial $kG$-module of length $3^n - 1$ whose composition factors are all isomorphic to $\tilde{T}_t$. On the other hand, the Green correspondent of $S^{(N_1)}_{0,1}$ is a uniserial $kG$-module of length $3^n - 1$ whose socle is isomorphic to either $T_{0,1}$ or $T_{1,0}$. By relabeling the simple $kG$-modules, if necessary, we are able to assume that the Green correspondent of $S^{(N_1)}_{0,1}$ (resp. $S^{(N_1)}_{1,0}$) is a uniserial $kG$-module of length $3^n - 1$ whose socle is isomorphic to $T_{0,1}$ (resp. $T_{1,0}$). Using similar arguments as in [§6.3.4 and Notation I.14], we obtain that the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega^m_X)$, with their multiplicities, are given by

$$\delta_1 \left( \delta_m \left( U'(G)_{T_0, T_0, 3^{n-1}+1} \oplus U'(G)_{T_0, T_0, 3^n} \right) \oplus \left( 1 - \delta_m \right) \left( U'(G)_{T_0, T_0, 3^n} \right) \right) \oplus \delta_0 \left( \left( 1 - \delta_m \right) \left( U'(G)_{T_0, T_0, 3^n} \right) \oplus \delta_m \left( U'(G)_{T_0, T_0, 3^{n-1}} \right) \right) \oplus \left( \frac{n'-2}{2} - 1 \right) \left( U'(G)_{T_0, T_0, 3^n} \right).$$

In particular, for every possible value of $\delta_0, \delta_1, \delta_m$, the module in (6.14) contains, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as direct summand.

6.4. The Brauer character of $H^0(X, \Omega^m_X)$ as a $kG$-module. The goal of this section is to compute the Brauer character of the $kG$-module $H^0(X, \Omega^m_X)$. In other words, we determine the values of the Brauer character $\beta(H^0(X, \Omega^m_X))$ at all elements $g \in G$ that are 3-regular, i.e. whose order is not divisible by 3. By [§I.8], the elements of order $\ell$ fall into 2 conjugacy classes. Let $r_1$ and $r_2$ be representatives of these conjugacy classes. Since all subgroups of $G$ of order $\ell$ are conjugate, we can assume, without loss of generality, that $R = \langle r_1 \rangle = \langle r_2 \rangle$. In fact, if $1 \leq \mu \leq \ell - 1$ is such that $\mathbb{F}_q^\ast = \langle \mu \rangle$ then we can choose $r_2 = r_1^\mu$. Moreover, for $i \in \{1, 2\}$ and $1 \leq a \leq (\ell - 1)/2$, we have that $(r_i)^{a^2}$ is conjugate to $r_1$. All elements $g \in G$ of a given order $\neq \ell$ lie in a single conjugacy class. We first determine the value of the Brauer character $\beta(H^0(X, \Omega^m_X))$ at $r_1$ and $r_2$, and then consider representatives for the conjugacy classes of the other 3-regular elements in $G$. We use the notation and some of the calculations from [§6.3].

6.4.1. The Brauer character of $H^0(X, \Omega^m_X)$ at elements of order $\ell$. By Tables I.31 and I.32 we have $R_{x,0} \in \{1, R\}$ for all closed points $x \in X$, and there are precisely $(\ell - 1)/2$ closed points $x$ in $X$ with $R_{x,0} = R$. Let $Y = X$ and $Z = X/R$, and let $\lambda : Y = X \to Z$ be the corresponding Galois cover with Galois group $R$, which is ramified. Moreover, let $Z_{\text{ram}}$ be the set of closed points in $Z$ that ramify in the cover $\lambda$, and let $f : Z = X/R \to X/G$ be the separable morphism such that $f \circ \lambda$ is the cover $X \to X/G \cong \mathbb{P}^1_k$.

As in Remark I.5, a canonical divisor on $X/G \cong \mathbb{P}^1_k$ is given by $K_{X/G} = -2\varepsilon$, and a canonical divisor on $Z$ is given by $K_Z = f^*(-2\varepsilon) + R_f$. Therefore,

$$K_X = K_Y = \lambda^*K_Z + R_{X/Z} = (f \circ \lambda)^*(-2\varepsilon) + \lambda^*R_f + R_{X/Z}.$$ 

Let $z \in Z_{\text{ram}}$ and let $x(z) \in X = Y$ be above it. By [I.34] Table on p. 193, $z$ and $x(z)$ lie above $\infty \in X/G \cong \mathbb{P}^1_k$, which means that the coefficient of $x(z)$ in $(f \circ \lambda)^*(-2\varepsilon)$ equals $-2\varepsilon$, since the ramification index of $x(z)$ over $\infty$ is $e_{x(z)/\infty} = \ell$. On the other hand, the coefficient of $x(z)$ in $R_{X/Z}$ is $\ell - 1$. Finally, the coefficient of $x(z)$ in $\lambda^*R_f$ equals $e_{x(z)/z} \cdot d_{z/\infty}$, where $e_{x(z)/z} = \ell$ is the ramification index of $x(z)$ over $z$ and $d_{z/\infty}$ is the different exponent of $z$ over $\infty \in X/G \cong \mathbb{P}^1_k$. Using the transitivity of the different and that $d_{z/\infty} = \ell - 1$ by [I.34] Table on p. 193, we obtain $d_{z/\infty} = 0$, which implies that the coefficient of $x(z)$ in $\lambda^*R_f$ is 0. Therefore,

$$\text{ord}_{x(z)}(K_X) = -2\varepsilon + \ell - 1 = -\ell - 1.$$
By step (2) of Algorithm 3.6 it follows that the Brauer character of $\text{Res}_{G}^{R} H^0(X, \Omega_{X}^{\otimes m})$ is equal to

$$
\sum_{z \in \mathbb{Z}_{\text{ram}}} \left( \sum_{d=1}^{\ell-1} \theta_{x(z)}^{-d} - \sum_{d=0}^{\ell-1} \frac{d}{\ell} \theta_{x(z)}^{d} \right) + n_0(R) \beta(kR)
$$

where

$$
n_0(R) = \frac{1}{\ell} (2m - 1) (g(X) - 1) + \sum_{z \in \mathbb{Z}_{\text{ram}}} \frac{1}{\ell} \left( \frac{\ell - 1}{2} - \ell_{x(z),0} \right),
$$

and, for $z \in \mathbb{Z}_{\text{ram}}$, $\ell_{x(z),0} \in \{0, 1, \ldots, \ell - 1\}$ is defined by $\ell_{x(z),0} = \text{ord}_{x(z)}(mK_X) \mod \ell$. By (6.17), we have

$$
\ell_{x(z),0} = \ell - 1 - m_\ell, \quad \text{where} \ m_\ell \in \{0, 1, \ldots, \ell - 1\} \ \text{satisfies} \ m_\ell \equiv m - 1 \mod \ell.
$$

Fix $z_0 \in \mathbb{Z}_{\text{ram}}$ and let $\xi_\ell$ be a primitive $\ell$th root of unity such that $\theta_{x(z_0)}(r_1) = \xi_\ell$. Then it follows that

$$
\{ \theta_{x(z)}(r_1) ; z \in \mathbb{Z}_{\text{ram}} \} = \{ (\xi_\ell)^a^2 ; 1 \leq a \leq (\ell - 1)/2 \}.
$$

Using Gauss sums, we see, similarly to [3] §6.3.1, that there exists a choice of square root of $\epsilon' \ell$, say $\sqrt{\epsilon' \ell}$, such that, for all positive integers $d$,

$$
\sum_{a=1}^{(\ell - 1)/2} (\epsilon_\ell)^{a^2d} = -1 + \left( \frac{d}{\ell} \right) \sqrt{\epsilon' \ell} \quad \text{and} \quad \sum_{a=1}^{(\ell - 1)/2} (\epsilon_\ell)^{a^2d} = -1 - \left( \frac{d}{\ell} \right) \sqrt{\epsilon' \ell}
$$

where $\left( \frac{d}{\ell} \right)$ denotes the Legendre symbol. Using (6.13), (6.19) and (6.20), we get

$$
\sum_{z \in \mathbb{Z}_{\text{ram}}} \ell_{x(z),0} \theta_{x(z)}^{-d}(r_1) = -\frac{\ell - 1}{2} + m_\ell - \sum_{d=1}^{m_\ell} \left( \frac{d}{\ell} \right) \sqrt{\epsilon' \ell}.
$$

Letting $h_\ell = h_{\mathbb{Q}(\sqrt{-\ell})}$ be the class number of $\mathbb{Q}(\sqrt{-\ell})$, it follows by [3] §6.3.1 that

$$
\sum_{z \in \mathbb{Z}_{\text{ram}}} \left( -\sum_{d=0}^{\ell-1} \frac{d}{\ell} \theta_{x(z)}^{d} \right)(r_1) = \begin{cases} \\
-\frac{\ell - 1}{4} + h_\ell \sqrt{-\ell}, & \epsilon' = 1, \\
-\frac{\ell - 1}{4} + \frac{h_\ell}{2} \sqrt{-\ell}, & \epsilon' = -1.
\end{cases}
$$

Since $r_2 = r_1^{\mu}$ and $\beta(kR)(r_b) = 0$ for $b = 1, 2$, we obtain

$$
\beta(H^0(X, \Omega_{X}^{\otimes m}))(r_b) = \begin{cases} \\
-\frac{\ell - 1}{4} + \frac{m_\ell + (-1)^b \sum_{d=1}^{m_\ell} \left( \frac{d}{\ell} \right) \sqrt{\ell}}{2}, & \epsilon' = 1, \\
-\frac{\ell - 1}{4} + \frac{m_\ell - (-1)^b \left( h_\ell - \sum_{d=1}^{m_\ell} \left( \frac{d}{\ell} \right) \right) \sqrt{-\ell}}{2}, & \epsilon' = -1,
\end{cases}
$$

for $b = 1, 2$, where $m_\ell \in \{0, 1, \ldots, \ell - 1\}$ satisfies $m_\ell \equiv m - 1 \mod \ell$.

6.4.2. The Brauer character of $H^0(X, \Omega_{X}^{\otimes m})$. By Notation 6.22 and Remark 6.3, $v$ is an element of order $(\ell - \epsilon)/2 = 3^n \cdot n'$, where $n'$ is not divisible by 3, $s$ is an element of order 2, and $w$ is an element of order $(\ell + \epsilon)/2$. Let $v'' = v^{3^n}$ be of order $n'$. Then a full set of representatives for the conjugacy classes of 3-regular elements of $G$ is given by

$$\{ e_G, r_b, s, (v'')^i, w^j \}$$

where $b = 1, 2, 1 \leq i < \frac{n'}{3}$ and $1 \leq j < \frac{\ell + \epsilon}{3}$. 36
From (6.22), we know the values of $\beta(H^0(X, \Omega_X^{\oplus m}))$ at $r_b$, for $b = 1, 2$. The other values of $\beta(H^0(X, \Omega_X^{\oplus m}))$ are as follows:

\begin{align}
(6.23) & \quad \beta(H^0(X, \Omega_X^{\oplus m}))(e_G) = (2m - 1) \frac{(\ell^2 - 1)(\ell - 6)}{24}, \\
(6.24) & \quad \beta(H^0(X, \Omega_X^{\oplus m}))(s) = (1 - 2\delta_m) \frac{\ell - \epsilon'}{4}, \\
(6.25) & \quad \beta(H^0(X, \Omega_X^{\oplus m}))((v''i)^j) = 0, \\
(6.26) & \quad \beta(H^0(X, \Omega_X^{\oplus m}))(w^j) = 0.
\end{align}

where $1 \leq i < n_i'$ and $1 \leq j < \frac{\ell + \epsilon}{4}$. In particular, $(v''i), w^j \notin \{e_G, s\}$. Note that we obtain (6.23), (6.24) and (6.25) from (6.21) and (6.22). Similarly, we obtain (6.26) by applying Algorithm 3.6 to $M_W = \text{Res}_W^G H^0(X, \Omega_X^{\oplus m})$.

6.5. The $kG$-module structure of $H^0(X, \Omega_X^{\oplus m})$. In this section, we determine the $kG$-module structure of $H^0(X, \Omega_X^{\oplus m})$ using (6.3) and (6.4). Let $\xi_n'$ be a fixed primitive $(n')^{th}$ root of unity, and let $\xi_w$ be a fixed primitive $((\ell + \epsilon)/2)^{th}$ root of unity. We use the Brauer characters of $G$ given in Table 6.3 which result from restricting certain ordinary characters to the 3-regular conjugacy classes of $G$ (see [7, III-VI] for the ordinary character tables of $G$). Note that these Brauer characters are all irreducible, except for $\delta_0^\epsilon$ when $\epsilon = 1$.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
& $(v''i)^j$ & & & \\
& $(1 \leq i < n_i')$ & & & \\
\hline
$\tilde{\delta}_1^\epsilon$ & $\ell + \epsilon$ & $\epsilon$ & $\epsilon |\epsilon' + \epsilon| ( -1)^j$ & $\epsilon (\xi_{w'})^{ti} + (\xi_w)^{ -ti}$ & $0$

$(0 \leq t < \frac{n_i'}{2})$ & & & & \\
\hline
$\gamma_a$ & $\frac{\ell' + \epsilon'}{2}$ & $\frac{\epsilon' + \epsilon + ( -1)^{a+1} \sqrt{\epsilon + \epsilon'} }{2}$ & $\epsilon' ( -1)^{(\ell' - \epsilon')/4}$ & $\epsilon' \frac{\epsilon' + \epsilon|}{2} ( -1)^i$ & $\epsilon' \frac{\epsilon' - \epsilon|}{2} ( -1)^j$

$(a = 1, 2)$ & & & & \\
\hline
$\eta_{w'}^G$ & $\ell - \epsilon$ & $-\epsilon$ & $-\epsilon |\epsilon' - \epsilon| ( -1)^u$ & $0$ & $-\epsilon (\xi_{w'})^{uj} + (\xi_w)^{ -uj}$

$(1 \leq u < \frac{\ell + \epsilon}{4})$ & & & & \\
\hline
\end{tabular}
\caption{Important Brauer characters of $G = \text{PSL}(2, \mathbb{F}_q)$.}
\end{table}

Let $\tilde{\beta}$ be the Brauer character of the largest projective direct summand $\tilde{M}$ of $H^0(X, \Omega_X^{\oplus m})$. Let $\text{IBr}(kG)$ denote the set of Brauer characters of simple $kG$-modules, and for each $\phi \in \text{IBr}(kG)$, let $E(\phi)$ be a simple $kG$-module with Brauer character $\phi$.

We need to consider 4 cases. In each case, we first describe the Brauer characters of the simple $kG$-modules occurring as composition factors in the non-projective indecomposable direct summands of $H^0(X, \Omega_X^{\oplus m})$, as given by (6.13) - (6.16). Using (6.22) - (6.26) and Table 6.3, this leads to the Brauer character $\tilde{\beta}$. For all $\phi \in \text{IBr}(kG)$, we then determine

$$\langle \tilde{\beta}, \phi \rangle = \frac{1}{\#G} \sum_{x \in G_3} \tilde{\beta}(x^{-1}) \phi(x)$$

where $G_3$ denotes the 3-regular elements of $G$. Since $\langle \tilde{\beta}, \phi \rangle$ equals the multiplicity of the projective $kG$-module cover $P(G, E(\phi))$ as a direct summand of the largest projective direct summand $\tilde{M}$ of $H^0(X, \Omega_X^{\oplus m})$, this then leads in each case to the precise $kG$-module structure of $H^0(X, \Omega_X^{\oplus m})$. 

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6.5.1. The $kG$-module structure of $H^0(\mathcal{X}, \Omega^m_\mathcal{X})$ when $\epsilon = -\epsilon'$. By [6.3.1] and [6.3.2], the non-projective indecomposable direct $kG$-module summands of $H^0(\mathcal{X}, \Omega^m_\mathcal{X})$ are given by [6.13] when $\epsilon' = 1$ and $\epsilon = -1$ and by [6.14] when $\epsilon' = -1$ and $\epsilon = 1$. In both cases, let $\tilde{\psi}$ denote the Brauer character of the trivial simple $kG$-module $T_0$. If $\epsilon = -1$ then $\tilde{\delta}^*_0$ is the Brauer character of the simple $kG$-module $\tilde{T}_0$. If $\epsilon = 1$ then the Brauer character of the simple $kG$-module $T_1$ is given by $\psi_1 = \tilde{\delta}^*_0 - \tilde{\psi}_0$. In both cases, for $1 \leq t \leq (n' - 1)/2$, the Brauer character of the simple $kG$-module $\tilde{T}_t$ is equal to $\tilde{\delta}^*_t$.

There are $1 + (\ell + \epsilon)/4$ additional Brauer characters of simple $kG$-modules that are also projective, given by $\gamma_\alpha$, $a = 1, 2$, and $\eta^G_\alpha$ where $1 \leq u \leq \ell + \epsilon - 1$. Therefore,

$$
\text{IBr}(kG) = \left\{ \psi_0, \psi'_0, \tilde{\psi}_1, \gamma_\alpha, \eta^G_\alpha : 1 \leq t \leq (n' - 1)/2, 1 \leq a \leq 2, 1 \leq u \leq \frac{\ell + \epsilon - 1}{4} \right\}
$$

where $\psi_0 = \tilde{\delta}^*_0$ if $\epsilon = -1$ and $\psi'_0 = \psi_1$ if $\epsilon = 1$, i.e. $\psi'_0 = \tilde{\delta}^*_0 - \frac{1 + \epsilon}{2} \tilde{\psi}_0$.

If $\tilde{\beta}$ is the Brauer character of the largest projective direct summand $M$ of $H^0(\mathcal{X}, \Omega^m_\mathcal{X})$ then we obtain:

\[
\beta_{\psi_0} = \frac{m - 2 - 2(\delta_1 + 2\delta_0) + 3\delta_0(\delta_0 - 1)}{\ell}, \quad \beta_{\psi'_0} = \frac{m - 1 - m_\ell}{\ell}, \quad \beta_{\tilde{\delta}^*_1} = \frac{2(2m - 1)(\ell - 6 + \epsilon) - 4(1 + \epsilon \delta_1 + \frac{3 + \epsilon}{2} \delta_0) - 6\epsilon}{\ell}, \quad \beta_{\tilde{\delta}^*_0} = \frac{2(2m - 1)(\ell - 6 + \epsilon) - 4(1 + \epsilon \delta_1 + \frac{3 - \epsilon}{2} \delta_0) - 6\epsilon}{\ell}.
\]

\[
\beta_{\tilde{\delta}^*_1} + \frac{(-1)^{n - 1}}{2} \left( 1 + \epsilon h_0 \sum_{d = 1}^{m_\ell} \left( \frac{d}{\ell} \right) \right),
\]

\[
\beta_{\gamma_\alpha} = \frac{2(2m - 1)(\ell - 6 + \epsilon) + 6\epsilon (1 - (1 - 2\delta_0)(-1)^{(\ell - \epsilon')/4}) + \epsilon m_1 - m_\ell}{\ell},
\]

\[
\beta_{\eta^G_\alpha} = \frac{(2m - 1)(\ell - 6 + \epsilon) + 6\epsilon (1 - (1 - 2\delta_0)(-1)^n) + \epsilon m_1 - m_\ell}{\ell},
\]

for $1 \leq t \leq (n' - 1)/2$, $a = 1, 2$, and $1 \leq u \leq \frac{\ell + \epsilon - 1}{4}$. We obtain the following two results.

**Proposition 6.5.1.** Let $\ell \equiv 1 \mod 4$ and $\ell \equiv -1 \mod 3$. Using Notation 1.4 we obtain that

$$
H^0(\mathcal{X}, \Omega^m_\mathcal{X}) \cong \delta_1 \delta_0 U^{(G)}_{T_0, T_0, (3^n - 1)/2} \oplus \delta_0 (1 - \delta_0) U^{(G)}_{T_0, T_0, (3^n - 1)/2} \oplus \delta_1 (1 - \delta_m) U^{(G)}_{T_0, T_0, (3^n - 1)/2} \oplus \delta_0 (1 - \delta_m) U^{(G)}_{T_0, T_0, 3^{n - 1}} \oplus \delta_0 \delta_0 U^{(G)}_{T_0, 3^{n - 1}} \oplus \bigoplus_{t = 1}^{(n' - 1)/2} \delta_1 U^{(G)}_{T_1, 3^n - 1} \oplus \bigoplus_{t = 1}^{(n' - 1)/2} \delta_0 U^{(G)}_{T_1, 3^n - 1} \oplus \bigoplus_{\phi \in \text{IBr}(kG)} \langle \tilde{\beta}, \phi \rangle P(G, E(\phi))
$$

as $kG$-modules, where $\langle \tilde{\beta}, \phi \rangle$ is as in [6.27] - [6.31] for $\phi \in \text{IBr}(kG)$ and $\epsilon = -1$.

**Proposition 6.5.2.** Let $\ell \equiv -1 \mod 4$ and $\ell \equiv 1 \mod 3$. Using Notation 1.4 we obtain that

$$
H^0(\mathcal{X}, \Omega^m_\mathcal{X}) \cong \delta_1 \delta_0 U^{(G)}_{T_0, 2 \cdot 3^{n - 1}} \oplus \delta_0 (1 - \delta_0) U^{(G)}_{T_1, 2 \cdot 3^{n - 1}} \oplus \delta_1 (1 - \delta_m) U^{(G)}_{T_0, 3^{n - 1}} \oplus \delta_0 \delta_0 U^{(G)}_{T_0, 3^{n - 1}} \oplus \delta_0 (1 - \delta_m) U^{(G)}_{T_0, 3^{n - 1}} \oplus \bigoplus_{t = 1}^{(n' - 1)/2} \delta_1 U^{(G)}_{T_1, 2 \cdot 3^n - 1} \oplus \bigoplus_{t = 1}^{(n' - 1)/2} \delta_0 U^{(G)}_{T_1, 3^n - 1} \oplus \bigoplus_{\phi \in \text{IBr}(kG)} \langle \tilde{\beta}, \phi \rangle P(G, E(\phi))
$$

as $kG$-modules, where $\langle \tilde{\beta}, \phi \rangle$ is as in [6.27] - [6.31] for $\phi \in \text{IBr}(kG)$ and $\epsilon = 1$.
6.5.2. The $kG$-module structure of $H^0(X, \Omega^m_X)$ when $\epsilon = \epsilon'$. By \[6.3.3\] and \[6.3.4\] the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega^m_X)$ are given by \[6.15\] when $\epsilon' = 1 = \epsilon$ and by \[6.16\] when $\epsilon' = -1 = \epsilon$. In both cases, let $\psi_0$ denote the Brauer character of the trivial simple $kG$-module $T_0$. If $\epsilon = 1$ then the Brauer character of the simple $kG$-module $T_1$ is given by $\psi_1 = \tilde{\delta}_0^* - \psi_0$. If $\epsilon = -1$ then $\tilde{\delta}_0^*$ is the Brauer character of the simple $kG$-module $\bar{T}_1$. In both cases, for $1 \leq t \leq (n'/2 - 1)$, the Brauer character of the simple $kG$-module $\bar{T}_t$ is equal to $\tilde{\delta}_0^*$. Moreover, the Brauer characters of the simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ are given by $\gamma_1$ and $\gamma_2$. Note that these characters only differ with respect to their values on the elements of order $\ell$ in $G$. Since we have already chosen a square root of $\epsilon' \ell$ to obtain \[6.20\] and \[6.22\], we define $s_0 \in \{\pm 1\}$ such that the Brauer character $\psi_{0,1}$ of $T_{0,1}$ satisfies

$$\psi_{0,1}(r_1) = \frac{\epsilon' + s_0 \sqrt{\epsilon' \ell}}{2}.$$ 

There are $(\ell + \epsilon - 2)/4$ additional Brauer characters of simple $kG$-modules that are also projective, given by $u^G_\ell$, $1 \leq u \leq \frac{\ell + \epsilon - 2}{4}$. Therefore,

$$\text{IBr}(kG) = \left\{ \psi_0, \psi'_0, \psi_{0,1}, \tilde{\delta}_0^*, u^G_\ell ; 1 \leq t \leq (n'/2 - 1), 1 \leq u \leq \frac{\ell + \epsilon - 2}{4} \right\},$$

where $\psi'_0 = \psi_1$ if $\epsilon = 1$ and $\psi'_0 = \tilde{\delta}_0^*$ if $\epsilon = -1$, i.e. $\psi'_0 = \tilde{\delta}_0^* - \frac{1 + \epsilon}{2} \psi_0$.

If $\tilde{\beta}$ is the Brauer character of the largest projective direct summand $\tilde{M}$ of $H^0(X, \Omega^m_X)$ then we obtain:

\begin{align*}
6.3.3 & \langle \tilde{\beta}, \psi_0 \rangle = \frac{m - 2(\delta_1 + 2\delta_0) + 3\delta_m(2\delta_0 - 1)}{6} = \frac{m - 1 - m_\ell}{\ell}, \\
6.3.4 & \langle \tilde{\beta}, \psi'_0 \rangle = \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{1 + \epsilon}{2} \delta_1 + \frac{3 - \epsilon}{2} \delta_0) - 12\epsilon \delta_m}{12} = \frac{m - 1 - m_\ell}{\ell} - \frac{1 + \epsilon}{2} \langle \tilde{\beta}, \psi_0 \rangle, \\
6.3.5 & \langle \tilde{\beta}, \psi_{0,1} \rangle = \frac{2(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{1 + \epsilon}{2} \delta_1 + \frac{3 - \epsilon}{2} \delta_0)}{24} \\
& \quad - 6\epsilon \left(1 - (1 - 2\delta_m) \left((-1)^{\ell-\epsilon} / 4 - (-1)^{\ell} \right) \left((1 - \epsilon) \delta_1 + (1 + \epsilon) \delta_0 \right) \right) - \frac{m - 1 - m_\ell}{2\ell} \\
& \quad + \frac{(-1)^s_0 \ell}{2} \left(1 - \epsilon \right)^2 \left(- \frac{m_\ell}{\ell} - \sum_{d=1}^m \left(\frac{d^2}{\ell} \right) \right), \\
6.3.6 & \langle \tilde{\beta}, \tilde{\delta}_1 \rangle = \frac{2(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{1 + \epsilon}{2} \delta_1 + \frac{3 - \epsilon}{2} \delta_0) - 6\epsilon \left(1 - (1 - 2\delta_m) (-1)^\ell \right)}{12} \left(\frac{m - 1 - m_\ell}{\ell} \right), \\
6.3.7 & \langle \tilde{\beta}, u^G_\ell \rangle = \frac{2(2m - 1)(\ell - 6 + \epsilon) + 6\epsilon}{12} \left(\frac{m - 1 - m_\ell}{\ell} \right),
\end{align*}

for $i \in \{0, 1\}$, $1 \leq t \leq (n'/2 - 1)$, and $1 \leq u \leq (\ell + \epsilon - 2)/4$. We obtain the following two results.

**Proposition 6.5.3.** Let $\ell \equiv 1 \pmod{4}$ and $\ell \equiv 1 \pmod{3}$. Using Notation \[1.31\] we obtain that

$$H^0(X, \Omega^m_X) \cong \delta_1 \delta_m \left( U^{(G)}_{T_0, 3^{n-1}} \oplus U^{(G)}_{T_0, 2^{n-1}, 3^{n-1}} \right) \oplus \delta_1 (1 - \delta_m) \left( U^{(G)}_{T_1, 2^{n-1}} \oplus U^{(G)}_{T_0, 2^{n-1}, 3^{n-1}} \right) \oplus \delta_0 (1 - \delta_m) \left( U^{(G)}_{T_0, 3^{n-1}} \oplus U^{(G)}_{T_1, 3^{n-1}} \right) \oplus \delta_0 \delta_m \left( U^{(G)}_{T_1, 3^{n-1}} \oplus U^{(G)}_{T_0, 3^{n-1}} \right) \oplus \bigoplus_{t=1}^{(n'/2 - 1)} \delta_1 U^{(G)}_{T_t, 3^{n-1}} \oplus \bigoplus_{t=1}^{(n'/2 - 1)} \delta_0 U^{(G)}_{T_t, 3^{n-1}} \oplus \bigoplus_{\phi \in \text{IBr}(kG)} \langle \tilde{\beta}, \phi \rangle P(G, E(\phi))$$

as $kG$-modules, where $\langle \tilde{\beta}, \phi \rangle$ is as in \[6.33\] - \[6.37\] for $\phi \in \text{IBr}(kG)$ and $\epsilon = 1$. 39
Proposition 6.5.4. Let $\ell \equiv -1 \mod 4$ and $\ell \equiv -1 \mod 3$. Using Notation 1.4 we obtain that

$$H^0(X, \Omega^m_X) \cong \delta_1 \delta_m \left( U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \right) \oplus \delta_1 (1 - \delta_m) \left( U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \right) \oplus \delta_0 (1 - \delta_m) \left( U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \right) \oplus \delta_0 \delta_m \left( U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \oplus U_{T_0, T_0}^{(G)} \right)$$

as $kG$-modules, where $\langle \beta, \phi \rangle$ is as in (6.33) - (6.34) for $\phi \in IBr(kG)$ and $\epsilon = -1$.

6.6. Proof of Theorem 1.5. Part (i) of Theorem 1.5 follows directly from Propositions 6.5.1 - 6.5.4. Notice that the sign $s_{\mathcal{F}}$ from (6.32) depends on the relationship between the socle of the Green correspondent of $T_{0,1}$ and the values of the Brauer character of $T_{0,1}$ on elements of order $\ell$. As in Theorem 1.3 let $H_1$ and $H_2$ be representatives of the two conjugacy classes of subgroups of $G$ that are isomorphic to $\Sigma_3$. By our definition of $\Delta_1$ and $\Delta_2$ in Notation 6.2 we can choose $H_1 \leq \Delta_1$ and $H_2 \leq \Delta_2$. Recalling our definition of $S_{0,1}^{(N)}$, we see that the restriction of $T_{0,1}$ to $H_1$ (resp. $H_2$) is the direct sum of a 2-dimensional uniserial module whose socle is the trivial simple module (resp. the simple module corresponding to the sign character) and a projective module.

Using Propositions 6.5.1 - 6.5.4 instead of [5] Props. 6.4.1 - 6.4.4 and Theorem 1.1 instead of [5] Thm. 1.1, part (ii) of Theorem 1.5 is proved by using similar arguments as in the proof of [5] Thm. 1.4. Note that a canonical divisor on $X/G \cong \mathbb{P}^1_k$ is given by $K_{X/G} = -2\omega$. Moreover, we already observed at the end of each of the sections 6.3.3 - 6.3.4 that for every possible value of $\delta_0, \delta_1, \delta_m$ and every block $B$ of $kG$, there is at most one non-projective indecomposable $kG$-module belonging to $B$ that occurs as direct summand of $H^0(X, \Omega^m_X)$.

6.7. Proof of Theorems 1.2 and 1.3 when $p = 3$. Using Propositions 6.5.1 - 6.5.4 instead of [5] Props. 6.4.1 - 6.4.4 and Theorem 1.5 instead of [5] Thm. 1.4, Theorems 1.2 and 1.3 are proved by using similar arguments as in the proof of [5] Thms. 1.2 and 1.3.

6.8. Isotypic Hecke stable decompositions of even weight cusp forms. Following [5] §7, we briefly describe how to construct non-trivial $G$-isotypic $\mathbb{T}$-stable decompositions of the space of even weight cusp forms when $\mathbb{T}$ is the ring of Hecke operators that have index prime to the level.

Suppose $N \geq 3$ and $m \geq 1$ are integers and $F$ is a number field. Let $\Gamma = \text{SL}(2, \mathbb{Z})$ and let $\Gamma_N$ denote its principal congruence subgroup of level $N$. Let $\mathcal{S}_{2m}(F)$ be the space of weight $2m$ cusp forms for $\Gamma_N$ that have $q$-expansion coefficients in $F$ at all cusp, in the sense of [29] §1.6. By [41] §6.1-6.2, together with flat base change, it follows that $\mathcal{S}_{2m}(F)$ coincides with the space of all weight $2m$ cusp forms for $\Gamma_N$ whose Fourier expansions with respect to $e^{2\pi i z/N}$ have coefficients in $F$.

Let $\Gamma = \text{SL}(2, \mathbb{Z}/N) = \Gamma / \Gamma_N$ and let $G = \text{PSL}(2, \mathbb{Z}/N) = \Gamma / (\Gamma_N, \pm I)$ where $I$ denotes the $2 \times 2$ identity matrix. When

$$\Delta_N = \{ \alpha \in \text{Mat}(2, \mathbb{Z}) : \det(\alpha) > 0 \text{ and } \text{gcd}(\det(\alpha), N) = 1 \} \quad \text{and} \quad \Delta'_N = \left\{ \alpha \in \Delta_N : \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \mod N \text{ for some } x \in (\mathbb{Z}/N)^* \right\},$$

define $R(\Gamma, \Delta_N)$ (resp. $R(\Gamma_N, \Delta'_N)$) to be the free $\mathbb{Z}$-module generated by the double cosets $\Gamma \alpha \Gamma$ for $\alpha \in \Delta_N$ (resp. $\Gamma_N \alpha \Gamma_N$ for $\alpha \in \Delta'_N$).
By [11 §3.1], $R(\Gamma, \Delta_N)$ and $R(\Gamma_N, \Delta_N')$ are rings, and by [11 Prop. 3.31], the map $\Gamma_N \alpha \Gamma_N \mapsto \Gamma \alpha \Gamma$, for $\alpha \in \Delta_N'$, induces an isomorphism between $R(\Gamma, \Delta_N')$ and $R(\Gamma, \Delta_N)$. Define

$$\mathcal{T} = R(\Gamma, \Delta_N') \otimes \mathbb{Q}$$

which is called the ring of Hecke operators of index prime to $N$. Namely, for each positive integer $n$ that is relatively prime to $N$, let $\rho_n(n)$ be a set of representatives $\alpha \in \Delta_N'$ of all distinct double cosets in $\Gamma_N \backslash \Delta_N' / \Gamma_N$ such that $\det(\alpha) = n$, and let

$$T'(n) = \sum_{\alpha \in \rho'_n(n)} \Gamma_N \alpha \Gamma_N.$$

Then, by [11 Thm. 3.34], $\mathcal{T}$ is the $\mathbb{Q}$-algebra generated by all $T'(n)$ when $n$ ranges over all positive integers that are relatively prime to $N$.

We have the following right action of $R(\Gamma, \Delta_N')$, and hence of $\mathcal{T}$, on $f \in S_{2m}(F)$. For $\alpha \in \Delta_N'$, write $\Gamma_N \alpha \Gamma_N = \bigcup_i \Gamma \alpha_i$ as a finite disjoint union of right cosets, and define

$$f|_{\Gamma_N \alpha \Gamma_N} = \det(\alpha)^{m-1} \cdot \sum_i f|_{\alpha_i}$$

where for a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Q})$ and $z$ in the complex upper half plane we let

$$(f|\gamma)(z) = \det(\gamma)^m \left( cz + d \right)^{-2m} f \left( \frac{az + b}{cz + d} \right).$$

In particular, for all $r \in \mathbb{Q}$, we have $(f| r \mathbb{1})(z) = f(z)$.

Note that, for $\alpha \in \Delta_N'$, the right action on $S_{2m}(F)$ by the double coset $\Gamma_N \alpha \Gamma_N$ defines an $F$-linear transformation on $S_{2m}(F)$, which we denote by $[\Gamma_N \alpha \Gamma_N]$. By [11 Thm. 3.41], the $F$-linear transformations $[\Gamma_N \alpha \Gamma_N]$ on $S_{2m}(F)$, with $\alpha \in \Delta_N'$, are mutually commutative, and normal with respect to the Petersson inner product on $S_{2m}(F)$. In particular, there exists an $F$-basis of $S_{2m}(F)$ consisting of common eigenfunctions of the linear transformations $[\Gamma_N \alpha \Gamma_N]$ for all $\alpha \in \Delta_N'$.

We obtain well-defined right actions by $\Gamma = \text{SL}(2, \mathbb{Z}/N)$ (resp. $G = \text{PSL}(2, \mathbb{Z}/N)$) on $S_{2m}(F)$ by defining $f \ast \varphi = f|\gamma$ (resp. $f \ast \overline{\varphi} = f|\gamma$) if $\gamma \in \Gamma$ has image $\varphi \in \Gamma$ (resp. $\overline{\varphi} \in G$). As usual, these right actions can be made into left actions by defining the left action of a group element to be the right action of its inverse.

Using similar arguments as in [5 §7], we obtain the following result.

**Proposition 6.6.** Suppose $e_1, e_2$ are orthogonal central idempotents of $FG$ such that $1 = e_1 + e_2$ and each $e_i$ is fixed by the conjugation action of $\text{PGL}(2, \mathbb{Z}/N)$ on $G$. Then setting $E_i = S_{2m}(F)e_i$ for $i \in \{1, 2\}$ gives a $G$-isotypic $\mathcal{T}$-stable decomposition of $S_{2m}(F)$, as defined in [11 §1] and [11 §2].

7. **Galois structure of Riemann-Roch spaces associated to divisors of large degree**

In this section, we discuss a generalization of our main result Theorem 1.1 to the Riemann-Roch space associated to a $G$-invariant divisor on $X$ of large degree. Information about the module structure of such Riemann-Roch spaces is useful in the context of algebraic geometry codes. For some previous work on this subject, mostly in the case when $kG$ is semisimple, see, for example, [21] [22], [15], [18].

As in the introduction, we assume that $k$ is a perfect field of prime characteristic $p$, $X$ is a smooth projective geometrically irreducible curve over $k$, and $G$ is a finite group acting faithfully on the right on $X$ over $k$. We prove the following result.

**Proposition 7.1.** Suppose $G$ has non-trivial cyclic Sylow $p$-subgroups. Let $E$ be a $G$-invariant divisor on $X$ with $\deg(E) > 2g(X) - 2$ and write $E = \sum_{x \in X} e_x x$. Then the $kG$-module structure of the Riemann-Roch space $H^0(X, \mathcal{O}_X(E))$ is uniquely determined by the coefficients $e_x$ of $E$, together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$. 

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The basic idea for the proof of Proposition 7.1 is to take the arguments in §3 that pertain to the \( m \)-fold tensor product \( D_{X/Y}^{-1,\otimes m} \) of the inverse different and to adjust them to the more general fractional ideal sheaf \( \mathcal{O}_X(E) \) on \( X \).

**Remark 7.2.** Suppose \( m > 1 \) is an integer. Letting \( Y, Z, \pi \) and \( \lambda \) be as in Notation 3.2 we have, as in §2.2 that \( \Omega_{\mathcal{O}_X(mK_X)}^m \cong \mathcal{O}_X(mK_X) \) as \( \mathcal{O}_X \)-modules, where \( K_X \) is a canonical divisor on \( X \) given by \( K_X = (\lambda \circ \pi)^* K_Z + R_{X/Z} \) for a canonical divisor \( K_Z \) on \( Z \). This implies in particular that the coefficients of the divisor \( mK_X \) are determined by the canonical divisor \( K_Z \) together with the lower ramification groups of the closed points of \( X \) that are ramified in the cover \( \lambda \circ \pi : X \to Z \). Since \( \deg(mK_X) = m(2g(X) - 2) > 2g(X) - 2 \) provided \( g(X) \geq 2 \), Proposition 7.1 provides a generalization of Theorem 1.1.

We now prove Proposition 7.1 and then state in Remark 7.3 how Algorithm 3.6 changes in this situation.

**Proof of Proposition 7.1.** As at the beginning of §3 it is sufficient to prove Proposition 7.1 under Assumption 3.1 and using Notation 3.2 In particular, for \( 0 \leq j \leq p^m - 1 \), we denote by \( H^0(X, \mathcal{O}_X(E))^{(j)} \) the kernel of the action of \( J^j = kI - (\tau - 1)^j \) on \( H^0(X, \mathcal{O}_X(E)) \), and we denote by \( \pi_* \mathcal{O}_X(E)^{(j)} \) the kernel of the action of \( J^j \) on \( \pi_* \mathcal{O}_X(E) \) (see 2.1).

Fix \( 0 \leq j \leq p^m - 1 \), and define

\[
\mathcal{E}_j = \pi_* \mathcal{O}_X(E)^{(j+1)}/\pi_* \mathcal{O}_X(E)^{(j)}.
\]

Suppose first that \( n_I = 0 \). Then \( X = Y, \pi : X \to Y \) is the identity morphism, and \( j = 0 \). In this case, \( \mathcal{E}_0 = \pi_* \mathcal{O}_X(E) = \mathcal{O}_X(E) \). Define \( D_0 = E \), which is a \( G \)-invariant divisor on \( Y = X \). Hence \( \mathcal{E}_0 = S_{X^{-1}} \otimes_k \mathcal{O}_X(D_0) \). Moreover, \( \deg(D_0) > 2g(Y) - 2 \).

Suppose next that \( n_I \geq 1 \). Using a similar argument as for \( \mathcal{H}_j \) in the proof of Proposition 3.3 we see that \( \mathcal{E}_j \) is a line bundle for \( \mathcal{O}_Y \) and that there exists a \( G \)-invariant divisor \( D_j \) on \( Y \) such that there is an isomorphism

\[
(\tau - 1)^j : \mathcal{E}_j \to S_{X^{-1}} \otimes_k \mathcal{O}_Y(D_j)
\]

of \( \mathcal{O}_Y \)-modules. As in the proof of part (ii) of Proposition 3.3 we write

\[
D_j = \sum_{y \in Y} d_{y,j} y.
\]

Fix \( y \in Y \), and suppose \( x = x(y) \in X \) lies above \( y \). Let \( I_{x,0} \) be the inertia group of \( x \) in \( I = \langle \tau \rangle \) and write \( I_{x,0} = \langle \tau_x \rangle \) for \( \tau_x = \tau^{p^{n_x - n_x}} \) when \( \# I_{x,0} = p^{n_x} \leq p^n \). Let \( Y_x = X/I_{x,0} \) and let \( \alpha_x : X \to Y_x \) and \( \beta_x : Y_x \to Y \) denote the respective quotient morphisms, so that \( \beta_x \) is étale over \( y \) and \( \beta_x \circ \alpha_x = \pi \). Let \( y_x \in Y_x \) be a point above \( y \) and below \( x \). Then \( x \) is totally ramified over \( y_x \) for the action of \( I_{x,0} \), and \( y_x \) splits into \( p^{n_x} \) points in \( Y_x \) where \( y_x \) is one of them. Consider the divisor \( E_x = e_x \cdot \sum_{x' \in \pi^{-1}(y)} x' \) on \( X \). Then at all points \( x' \in X \) over \( y \), the stalks of \( \mathcal{O}_X(E)_{x'} \) and \( \mathcal{O}_X(E_x)_{x'} \) are the same. Hence there exists a small open neighborhood \( V_y \) of \( y \) such that we have an equality

\[
(\mathcal{O}_X(E)) \mid_{U_y} = (\mathcal{O}_X(E_x)) \mid_{U_y}
\]

of the restrictions of \( \mathcal{O}_X(E) \) and \( \mathcal{O}_X(E_x) \) to the inverse image \( U_y = \pi^{-1}(V_y) \subset X \). Using the same arguments as in the proof of Proposition 3.3 where we replace \( D_{X/Y}^{-1,\otimes m} \) by \( \mathcal{O}_X(E_x) \), we obtain that

\[
d_{y,j} = \left| e_x - \sum_{\ell=1}^{n_x} a_{\ell,t} p^{n_x-\ell} \tau_x^{\ell t} \right|
\]

when \( t \) is the unique non-negative integer satisfying \( p^{n_x} \cdot t \leq j < p^{n_x} \cdot (t + 1) \) and \( a_{\ell,t} \) and \( \tau_x^{\ell t} \) are as in the statement of Proposition 3.3. Notice that the formula (7.2) is also valid when \( I_{x,0} \) is the trivial subgroup of \( I \), since then \( n_x = 0 \), \( X = Y_x \), \( \alpha_x : X \to Y_x \) is the identity morphism, and \( t = 0 \). In this
case, \((\alpha_x), \mathcal{O}_X(E_x)^{(t+1)}/(\alpha_x), \mathcal{O}_X(E_x)^{(t)} = \mathcal{O}_X(E_x), \) resulting in \(d_{y,j} = e_x\) for \(t = 0\) and \(0 \leq j < p^{nt},\) which coincides with the formula (7.2) when \(n_x = 0.\)

We next show that \(\deg(D_j) > 2g(Y) - 2\) by using our assumption that \(2g(X) - 2 < \deg(E) = \sum_{x' \in X} e_{x'}\). By the Riemann-Hurwitz formula and using (3.2), together with the fact that \(E\) is \(G\)-invariant and that \(a_{\ell,t} \leq p - 1\) for all \(\ell, t,\) we obtain

\[
2g(Y) - 2 = \frac{1}{p^{n_{y,j}}}
\left(2g(X) - 2 - \sum_{x' \in X} (p - 1)\sum_{\ell=1}^{n_{x'}} (p^{n_{x'}-\ell}(i_{\ell} + 1)) \right) < \sum_{x' \in X} \frac{e_{x'} - \sum_{\ell=1}^{n_{x'}} (p - 1)p^{n_{x'}-\ell}(i_{\ell} + 1)}{p^{n_{y,j}}}
\]

where, for each \(y \in Y, x(y)\) stands for one particular point in \(X\) above \(y.\)

It follows that there are \(kG\)-module isomorphisms \(H^0(Y, E_j) \cong S_{\chi - j} \otimes_k H^0(Y, \mathcal{O}_Y(D_j))\) for \(0 \leq j < p^{nt} - 1.\) Since \(\deg(E) > 2g(X) - 2\) and \(\deg(D_j) > 2g(Y) - 2\) for all \(j,\) we can use the same arguments as in the proof of Proposition 3.4 where we replace \(\Omega^{\otimes m}_Y\) by \(\mathcal{O}_X(E)\) and \(\Omega^{\otimes m}_Y(D_j)\) by \(\mathcal{O}_Y(D_j),\) to obtain isomorphisms

\[
H^0(X, \mathcal{O}_X(E))^{(j+1)}/H^0(X, \mathcal{O}_X(E))^{(j)} \cong H^0(Y, E_j) \cong S_{\chi - j} \otimes_k H^0(Y, \mathcal{O}_Y(D_j))
\]

of \(kG\)-modules for all \(0 \leq j < p^{nt} - 1.\)

Let \(Z_{\text{ram}}\) be the set of closed points in \(Z\) that ramify in \(Y.\) For \(z \in Z_{\text{ram}}, \) let \(y(z) \in Y\) be such that \(\lambda(y(z)) = z.\) Define \(\ell_{y(z),j} \in \{0, 1, \ldots, \#G_{y(z),0} - 1\}\) by

\[
(7.3) \quad \ell_{y(z),j} \equiv \text{ord}_{y(z)}(D_j) \mod \#G_{y(z),0}.
\]

Using the same arguments as in the proof of Proposition 3.5 where we replace \(\Omega^{\otimes m}_Y(D_j)\) by \(\mathcal{O}_Y(D_j)\) and \(mK_Y + D_j\) by \(D_j,\) it follows that the Brauer character of \(H^0(Y, \mathcal{O}_Y(D_j))\) is equal to

\[
\sum_{z \in Z_{\text{ram}}} \left( \sum_{d=1}^{\#G_{y(z),0}} \frac{\text{Ind}_{G_{y(z),0}}^{G_y} \theta_{-d} y(z)}{\#G_{y(z),0}} \right) + n_j \beta(kG)
\]

where \(\beta(kG)\) is the Brauer character of \(kG\) and

\[
n_j = \frac{\deg(D_j)}{\#G} + 1 - g(Y) + \sum_{z \in Z_{\text{ram}}} \frac{1}{\#G_{y(z),0}} \left( \frac{\#G_{y(z),0} - 1}{2} - \ell_{y(z),j} \right).
\]

We can now use the same arguments as at the end of 3.2 to complete the proof of Proposition 7.1.

\[\square\]

Remark 7.3. Suppose \(E\) is as in Proposition 7.1. Going through the proof of this proposition, we see that we obtain a very similar algorithm to Algorithm 3.6 to determine the \(kG\)-module structure of \(M = H^0(X, \mathcal{O}_X(E))\) under Assumption 3.1 and using Notation 3.2. More precisely, we just need to make the following adjustments in steps (1) and (2):

1. In step (1), we need to replace
   - \(\sum_{x} (\#I_{x,i} - 1)\) by \(e_x\) in the formula for \(d_{y,j},\) and
   - \(H^0(\mathcal{O}_{Y}^{\otimes m}(D_j))\) by \(H^0(Y, \mathcal{O}_Y(D_j))\) in the isomorphism for \(M^{(j+1)}/M^{(j)}.\)

2. In step (2), we need to drop \(K_Y = \lambda^* K_Z + R_{Y/Z},\) and we need to replace
   - \(mK_Y + D_j\) by \(D_j\) in the definition of \(\ell_{y(z),j},\) and
   - \(\frac{(2m - 1)(g(Y) - 1) + \deg(D_j)}{\#G}\) by \(\frac{\deg(D_j) + 1 - g(Y)}{\#G}\) in the definition of \(n_j.\)
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