Complete classification of compact four-manifolds with positive isotropic curvature

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Abstract

In this paper, we completely classify all compact 4-manifolds with positive isotropic curvature. We show that they are diffeomorphic to $S^4$, or $\mathbb{RP}^4$ or quotients of $S^3 \times \mathbb{R}$ by a cocompact fixed point free subgroup of the isometry group of the standard metric of $S^3 \times \mathbb{R}$, or a connected sum of them.

1 Introduction

Let $M$ be an $n$-dimensional Riemannian manifold. Recall that its curvature operator at $p \in M$ is the self adjoint linear endomorphism $\mathcal{R} : \wedge^2 T_p M \to \wedge^2 T_p M$ defined by

$$< \mathcal{R}(X \wedge Y), U \wedge V > = < Rm(X, Y)V, U >, \text{ for } X, Y, U, V \in T_p M.$$  

Here $<,>$ is the Riemannian metric and $Rm$ is the Riemann curvature tensor on $M$. The Riemannian metric $<,>$ can be extended either to a complex bilinear form $(,)$ or a Hermitian inner product $<<,>>$ on $T_p M \otimes \mathbb{C}$. We extend the curvature operator to a complex linear map on $\wedge^2 T_p M \otimes \mathbb{C}$, also denoted by $\mathcal{R}$. Then, to every two plane $\sigma \subset T_p M \otimes \mathbb{C}$, we can define the complex sectional curvature $K_C(\sigma)$ by

$$K_C(\sigma) = << \mathcal{R}(Z \wedge W), Z \wedge W >>$$

where $[Z, W]$ is a unitary basis of $\sigma$ with respect to $<<,>>$. We say that $M$ has positive isotropic curvature (PIC for short) if $K_C(\sigma) > 0$ whenever $\sigma \subset T_p M \otimes \mathbb{C}$ is a totally isotropic two plane for any $p \in M$. Here $\sigma$ is totally isotropic if $(Z, Z) = 0$ for any $Z \in \sigma$. To clarify the meaning of positive isotropic curvature, we have the following diagram for the relative strength of the positivity for various notions of curvatures.

$$\mathcal{R} > 0 \Rightarrow K_C > 0 \Rightarrow K > 0 \Rightarrow \text{Ric} > 0 \Rightarrow R > 0$$

$$\downarrow$$

pointwise 1/4 pinching $\Rightarrow$ PIC $\Rightarrow$ R > 0
Here, $K$ is the sectional curvature, i.e the restriction of $K_C$ on real 2 planes in $T_p M \otimes \mathbb{C}$, $Ric$ is the Ricci curvature and $R$ is the scalar curvature on $M$. The pointwise $1/4$ pinching condition means that for any $p \in M$, we have

$$1 < \frac{\max\{K(\sigma) : 2 \text{ plane } \sigma \subset T_p M\}}{\min\{K(\sigma) : 2 \text{ plane } \sigma \subset T_p M\}} \leq 4.$$ 

The notion of positive isotropic curvature was introduced in the paper of Micallef and Moore [16] in 1988 where they discovered that it can be used to control the stability of minimal surfaces just as the notion of positive sectional curvature can be used to control the stability of geodesics. Hence by using minimal surface theory, they proved

**Theorem (Micallef-Moore).** Let $M$ be a compact simply connected $n$-dimensional manifold with positive isotropic curvature where $n \geq 4$. Then $M$ is homeomorphic to a sphere.

In view of above diagram, for $n \geq 4$, if $M$ is a compact simply connected $n$-dimensional manifold with positive curvature operator or pointwise $1/4$ pinching, then $M$ is homeomorphic to a sphere. The latter generalizes the famous sphere theorem of Berger and Klingenberg. It is spectacular that, by using the Ricci flow, it was proved recently in Böhm-Wilking [2] and Brendle-Schoen [1] that a compact $n$-dimensional simply connected manifold with positive curvature operator or pointwise $1/4$ pinching is indeed diffeomorphic to the round sphere $S^n$.

In 1997, in a seminal paper [11], Hamilton initiated the study of positive isotropic curvature by Ricci flow. In dimension 4, he first proved that the condition of positive isotropic curvature is preserved under Ricci flow. Then, under the assumption that there is no essential incompressible space forms in the manifold, he developed a theory of Ricci flow with surgery to exploit the development of singularities in the Ricci flow to recover the topology of the manifold. Here an incompressible space form $N$ in a four manifold $M$ is a smooth submanifold diffeomorphic to a spherical space form $S^3/\Gamma$ such that the inclusion induces an injection from $\pi_1(N)$ to $\pi_1(M)$. It is essential unless $\Gamma = 1$ or $\Gamma = \mathbb{Z}_2$ and the normal bundle is unorientable. Hamilton’s paper contained some unjustified statements which were later supplemented by the paper of Chen and Zhu [5]. Their main result is

**Theorem (Hamilton).** Let $M$ be a compact four manifold with no essential incompressible space form. Then $M$ admits a metric with positive isotropic curvature if and only if it is diffeomorphic to $S^4$, $\mathbb{R}P^4$, $S^3 \times S^1$, $S^3 \tilde{\times} S^1$ (this is the quotient of $S^3 \times S^1$ by $\mathbb{Z}_2$ which acts by reflection and antipodal map on the first and second factor respectively), or a connected sum of them.

Clearly, each of the manifolds $S^4$, $\mathbb{R}P^4$, $S^3 \times S^1$, $S^3 \tilde{\times} S^1$ listed in the above theorem admits a metric with positive isotropic curvature. A theorem of Micallef and Wang [17] guarantees that the connected sum of compact manifolds with positive isotropic curvature also admits such a metric. Another useful observa-
tion is that the condition of no essential incompressible space form is automatically satisfied if \( \pi_1(M) \) is torsion free, i.e. contains no nontrivial element of finite order. Indeed, \( \Gamma \) in the above definition of essential incompressible space form must be trivial. So, if the fundamental group of a compact Riemannian four manifold \( M \) with positive isotropic curvature contains a normal torsion free subgroup of finite index, then a finite cover of \( M \) is diffeomorphic to \( S^4, S^3 \times S^1 \) or a connected sum of them. This shows the intimate connection between the topology and the fundamental group of a compact Riemannian manifold with positive isotropic curvature, at least in dimension 4.

For dimension greater than 4, it has been proved recently by Brendle and Schoen [1] that the condition of positive isotropic curvature is preserved under Ricci flow although there is yet no generalization of the curvature pinching estimates which is crucial in Hamilton’s analysis of [11]. Another interesting result for higher dimensional Riemannian manifold with positive isotropic curvature is the result of Fraser and Wolfson [7] [8] who proved that the fundamental group of any compact surface of genus \( g \geq 1 \) cannot occur as a subgroup of such manifold when its dimension is greater than 4.

Recently, Schoen [21] proposed the following

**Conjecture (Schoen).** For \( n \geq 4 \), let \( M \) be an \( n \)-dimensional compact Riemannian manifold with positive isotropic curvature. Then a finite cover of \( M \) is diffeomorphic to \( S^n, S^{n-1} \times S^1 \) or a connected sum of them. In particular, the fundamental group of \( M \) is virtually free.

The purpose of this paper is to prove the conjecture of Schoen when \( n = 4 \). Indeed, we obtain a more precise result. In particular, we know exactly what are the fundamental groups of such manifolds. Our main result is

**Main Theorem.** Let \( M \) be a compact 4-dimensional manifold. Then it admits a metric with positive isotropic curvature if and only if it is diffeomorphic to \( S^4, \mathbb{RP}^4, S^3 \times \mathbb{R} / G \) or a connected sum of them. Here \( G \) is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on \( S^3 \times \mathbb{R} \).

We give two immediate corollaries of our Main Theorem.

**Corollary 1.** The conjecture of Schoen is true for \( n = 4 \).

**Proof.** There is nothing to prove if \( M \) is diffeomorphic to \( S^4 \) or \( \mathbb{RP}^4 \). So, we may assume that \( M \) is diffeomorphic to \( m\mathbb{RP}^4 \# S^3 \times \mathbb{R} / G_1 \# \cdots \# S^3 \times \mathbb{R} / G_k \) for some nonnegative integer \( m \) and positive integer \( k \). The fundamental group of \( M \) is given by

\[
\underbrace{\mathbb{Z}_2 \ast \cdots \ast \mathbb{Z}_2}_{m \text{ times}} \ast G_1 \ast \cdots \ast G_k.
\]

Now a cocompact fixed point free discrete subgroup \( G \) of the isometry group of \( S^3 \times \mathbb{R} \) is always virtually infinite cyclic. This is because, by the cocompactness of the action of \( G \) on \( S^3 \times \mathbb{R} \), \( G \) always contains an element \( g \) which acts as translation on the second factor and the infinite cyclic subgroup generated by \( g \) must have finite index as it also acts cocompactly on \( S^3 \times \mathbb{R} \). Thus \( \pi_1(M) \) is the
free products of finite and virtually infinite cyclic groups. It is known that such group always contains a normal free subgroup of finite index. In particular, $\pi_1(M)$ contains a torsion free normal subgroup of finite index. By the remark after the statement of Hamilton’s Theorem, the conclusion in the conjecture of Schoen holds.

The second corollary concerns the classification of compact conformally flat Riemannian four manifolds with positive scalar curvature. We start with a digression of the geometry of Riemannian four manifold $M$. In this case, the bundle $\wedge^2 TM$ has a decomposition into the direct sum of its self-dual and anti-self-dual parts

$$\wedge^2 TM = \wedge^+_2 TM \oplus \wedge^-_2 TM.$$  

The curvature operator can then be decomposed as

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

where $A = W_+ + \frac{R}{12}$, $B = \text{Ric}$, $C = W_- + \frac{R}{12}$. Here $W_\pm$ are the self-dual and anti-self-dual Weyl curvature tensors respectively while $\text{Ric}$ is the trace free part of the Ricci curvature tensor. Denote the eigenvalues of the matrices $A$, $C$ and $\sqrt{BB^t}$ by $a_1 \leq a_2 \leq a_3$, $c_1 \leq c_2 \leq c_3$, $b_1 \leq b_2 \leq b_3$ respectively. It is known that the condition of positive isotropic curvature is equivalent to the conditions $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$. From this, it is clear that a compact conformally flat Riemannian four manifold with positive scalar curvature always has positive isotropic curvature.

Now it had been observed by Izeki [13] that a compact conformally flat Riemannian four manifold $M$ with positive scalar curvature always has a finite cover which is diffeomorphic to $S^4$, $\mathbb{R}P^4$, $S^3 \times S^1$ or a connected sum of them. The reason is this. Let $M$ be such a manifold, then by a result of Schoen and Yau [20], $\pi_1(M)$ is a Kleinian group. In particular, it is a finitely generated subgroup of a linear group, namely $SO(5,1)$. By Selberg’s Lemma, $\pi_1(M)$ contains a torsion free normal subgroup of finite index. Since such manifold always has positive isotropic curvature, we can again apply the above remark after the statement of Hamilton’s Theorem to conclude that $M$ has a finite cover which is diffeomorphic to $S^4$, $S^3 \times S^1$ or a connected sum of them.

Our Main Theorem gives a more precise classification of such manifolds.

**Corollary 2.** A compact four manifold admits a metric of positive isotropic curvature if and only if it admits a conformally flat metric of positive scalar curvature.

**Proof.** The manifolds $S^4$, $\mathbb{R}P^4$, $S^3 \times S^1 \mathbb{R}/G$ listed in the Main Theorem clearly admit conformally flat metrics of positive scalar curvature and we only have to invoke the fact that connected sum of conformally flat Riemannian manifolds with positive scalar curvature also admits such a metric. \qed
We remark that corollary 2 does not hold for dimension $n > 4$. The following example is taken from [17]. For any Riemann surface $\Sigma_g$ of genus $g \geq 2$ and $n > 4$, the manifold $M = \Sigma_g \times S^{n-2}$ admits a conformally flat metric of positive scalar curvature, however, because of the above mentioned result of Fraser and Wolfson [8], $M$ cannot admit a metric with positive isotropic curvature.

The proof of our Main Theorem naturally divides into two parts. The first part is analytical and the second part topological.

Our argument in the first part is based on the celebrated Hamilton-Perelman theory [11] [19] on the Ricci flow with surgery. To approach the topology of a compact four-manifold with positive isotropic curvature, we take it as initial data and evolve it by the Ricci flow. It is easy to see that the solution will blow up in finite time. By applying Hamilton’s curvature pinching estimates obtained in [11], we can get a complete understanding on the part around the singularities of the solution. Then we can perform Hamilton’s surgery procedure to cutoff the part around the singularities. After the surgery, due to the possible existence of essential incompressible space forms, we will get a closed (maybe not connected) orbifold with positive isotropic curvature. After studying Ricci flow on orbifold and obtaining a detailed singularity analysis for orbifold Ricci flow, we can use the orbifold as initial data to run the Ricci flow and to do surgeries again. By repeating this procedure and extending the arguments in the previous paper [5] of the first and the third authors to the orbifold case, we will be able to show that, after a finite number of surgeries and discarding a finite number of pieces which are diffeomorphic to spherical orbifolds $S^4/\Delta$ (here $\Delta$ denotes a finite subgroup of the orthogonal group $O(5)$) with at most isolated orbifold singularities, the solution becomes extinct. As a result, we prove that the initial manifold is diffeomorphic to an orbifold connected sum (see below or the precise definition given in section 2) of spherical orbifolds $S^4/\Delta$.

The second part concerns the recovery of the topology of the manifold from the orbifold connected sum. First of all, by an algebraic lemma, we know that a spherical orbifold $S^4/\Delta$ has either zero, one or two orbifold singularities. A spherical orbifold with no orbifold singularity is simply $S^4$ or $\mathbb{R}P^4$ while those with one or two orbifold singularities is, after removing an open neighborhood from each of its orbifold singularities, diffeomorphic to a smooth cap or a cylinder respectively. Here a cylinder $C(\Gamma)$ is given by $S^3/\Gamma \times [-1, 1]$ for some finite fixed point free subgroup $\Gamma$ of $SO(4)$ while a smooth cap $C_\sigma$ is given as the quotient of $S^3/\Gamma \times [-1, 1]$ by a group of order two generated by $\sigma(x, s) \mapsto (\sigma(x), -s)$ where $\sigma$ is a fixed point free isometric involution on $S^3/\Gamma$. Now, the orbifold connected sum of spherical orbifolds is formed in two steps. In the first step, to undo the surgeries in the Ricci flow which create orbifold singularities, we glue copies of the $C(\Gamma)$’s and $C_\sigma$’s along their diffeomorphic boundaries with suitable identifying maps to form a number of closed (compact) manifolds. It is not hard to see that, up to diffeomorphisms, they are essentially of two types: the self-gluing of the two ends of a cylinder $C(\Gamma)$ and the gluing of two smooth caps, $C_\sigma$ and $C_{\sigma'}$, with diffeomorphic boundaries by suitable diffeomorphisms.
on $S^3/\Gamma$. Since we know that any diffeomorphism on a three dimensional spherical space form is isotopic to an isometry. The resulting closed manifolds can be equipped with metrics which are locally isometric to $S^3 \times \mathbb{R}$. Now, the second step in the formation of the orbifold connected sum consists of two types of operations. The first is the usual connected sum of the above closed manifolds with $S^4$'s and $\mathbb{RP}^4$'s and the second is adding handles to them. Since the latter operation is in term equivalent to the connect sum of them with $S^3 \times S^1$ or $S^3 \tilde{\times} S^1$, our Main Theorem is proved.

A natural question is whether our Main Theorem and its proof can be extended to dimension greater than 4. We believe that the analytic part of our proof will go through once Hamilton's curvature pinching estimates in [11] can be extended to higher dimensions. Assuming that this has been done, most of the argument in the topological part of our proof will also go through. This will allow us to show that a compact Riemannian n-dimensional manifold $M$ with positive isotropic curvature is homeomorphic to $S^n$, $\mathbb{RP}^n$, $S^{n-1} \times \mathbb{R}/G$ or a connected sum of them. Here we only know that $G$ acts differentiably on $S^{n-1} \times \mathbb{R}$. The differences are due to the possible existence of (exotic) diffeomorphisms on a spherical space form $S^{n-1}/\Gamma$ which is not isotopic to an isometry. By the same argument as in our proof of Corollary 1, this result still implies a weaker form of the conjecture of Schoen, namely, $M$ has a finite cover which is homeomorphic to $S^n$, $S^{n-1} \times S^1$ or a connected sum of them.

Our paper is organized as follows. In section 2, we introduce some terminologies and state one of the main results of the paper, Theorem 2.1, which says that any 4-orbifold with positive isotropic curvature and with at most isolated singularities is diffeomorphic to an orbifold connected sum of spherical orbifolds $S^4/\Gamma$. In section 5, we identify these orbifold connected sums and prove the Main Theorem. The proof of Theorem 2.1 by Ricci flow, will occupy sections 3 and 4. Section 6 gives the proof of a geometric lemma which is used frequently in the paper.

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2 Orbifold connected sum

We generalize the construction of connected sum of manifolds to orbifolds with at most isolated singularities. For an orbifold $X, x \in X$, we use $\Gamma_x$ to denote the local uniformization group at $x$, namely, there is a open neighborhood $B_x \ni x$ with smooth boundary which is a quotient $\tilde{B}/\Gamma_x$, where $\tilde{B}$ is diffeomorphic to $\mathbb{R}^n$ and $\Gamma_x$ is a finite subgroup of linear transformations fixing the origin. Let $X_1, \cdots, X_p$ be $n$-dimensional orbifolds with at most isolated orbifold singularities. Let $x_1, x'_1, x_2, x'_2, \cdots, x_q, x'_q$ be $2q$ distinct points (not necessarily singular) on $X_1, \cdots, X_p$ such that for each pair $(x_j, x'_j), \Gamma_{x_j}$ is conjugate to $\Gamma_{x'_j}$ as linear subgroups. Assume $x_j \in X_i$ and $x'_j \in X'_i$ for $j = 1, \cdots, q$. Let $f_j$ be a diffeomorphism
from $\partial B_j$ and $\partial B_{j'}$ for $j = 1, 2, \cdots, q$. For each $j$, we remove $x_j$ and $x_j'$ from the orbifolds, and identify the boundary $\partial B_{j'}$ with $\partial B_j$ by using the diffeomorphism $f_j$. Let $f = (f_1, \cdots, f_q)$. We denote the resulting space by $\# f(X_1, \cdots, X_q)$. We call it an orbifold connected sum of $X_1, \cdots, X_q$. Here we emphasis that the diffeomorphism type of the resulting orbifold depends only on the isotopic class of $f$. Now we specify our construction to dimension 4.

One of the main efforts of this paper is to show the following:

**Theorem 2.1.** Let $(M^4, g)$ be a compact 4-dimensional manifold or orbifold with at most isolated singularities with positive isotropic curvature. Then $M^4$ is diffeomorphic to an orbifold connected sum of a finite number of spherical 4-orbifolds $X_1 = S^4/\Gamma_1, \cdots, X_j = S^4/\Gamma_j$, where each $\Gamma_i$ is a finite subgroup of the isometry group, $O(5)$, of the standard metric on $S^4$ so that the quotient orbifold $X_j$ has at most isolated singularities.

Now, we discuss some natural examples of compact four manifolds with positive isotropic curvature. We will describe their constructions from orbifold connected sums by spherical orbifolds.

In dimension 4, except for $S^4$ and $RP^4$, the best known examples of positive isotropic curvature are $S^3/\Gamma \times S^1$, where $\Gamma$ is a fixed point free finite subgroup of $SO(4)$. Clearly $\Gamma$ can also act isometrically on $S^4$ by fixing an axis. The orbifold $S^4/\Gamma$ has exactly two singularities $P$ and $P'$. Clearly, if one performs an orbifold connected sum on $S^4/\Gamma$ with itself by using the identity map as the identifying map, it gives $S^3/\Gamma \times S^1$. If we choose the identifying map $f$ (in $Diff(S^3/\Gamma)$) in a nontrivial isotopic class, then the connected sum may give some twisted product of $S^3/\Gamma$ and $S^1$. We denote the manifold by $S^3/\Gamma \times S^1$. By [15], the mapping class group of three dimensional spherical space form $S^3/\Gamma$ is a finite group. So for each $\Gamma$, there is only a finite number of diffeomorphism classes of $S^3/\Gamma \times S^1$. In particular, when $\Gamma = \{1\}$ and $f$ is an orientation reversing diffeomorphism, the resulting manifold is $S^3 \times S^1$, which is the only unoriented $S^3$ bundle over $S^1$.

If $S^3/\Gamma$ admits a fixed point free isometry $\sigma$ satisfying $\sigma^2 = 1$, then we can define a reflection $\delta$ on the 4-manifold $S^3/\Gamma \times \mathbb{R}$ by $\delta(x, s) = (\sigma(x), -s)$, where $x \in S^3/\Gamma, s \in \mathbb{R}$. The quotient $(S^3/\Gamma \times \mathbb{R})/\{1, \delta\}$ is a smooth four manifold with a neck like end $S^3/\Gamma \times \mathbb{R}$. We denote the manifold by $C^\nu_{\Gamma}$. If we think of the sphere $S^4$ as the compactification of $S^3/\Gamma \times \mathbb{R}$ by adding two points (north and south poles) at infinities of $S^3/\Gamma \times \mathbb{R}$, we can regard $\Gamma$ and $\delta$ as isometries of the standard $S^4$ in a natural manner. So $C^\nu_{\Gamma}$ is diffeomorphic to the smooth manifold obtained by removing the unique singularity from $S^4/\{\Gamma, \delta\}$. We call $C^\nu_{\Gamma}$ smooth cap.

Given two smooth caps $C^\nu_{\Gamma}$ and $C^\nu_{\Gamma'}$, if $\Gamma$ is conjugate to $\Gamma'$ (i.e. there is an isometry $\gamma$ of $S^3$ such that $\Gamma = \gamma \Gamma' \gamma^{-1}$), we can glue $C^\nu_{\Gamma}$ and $C^\nu_{\Gamma'}$ along their boundaries by a diffeomorphism $f : \partial C^\nu_{\Gamma} \to \partial C^\nu_{\Gamma'}$. Then we get a smooth manifold and we denote it by $C^\nu_{\Gamma} \cup_f C^\nu_{\Gamma'}$. Let $P, P'$ be the singularities of the orbifolds $S^4/\{\Gamma, \delta\}$ and $S^4/\{\Gamma', \delta'\}$ If we resolve these two singularities by orbifold connected sum with some diffeomorphism $f$ between the boundaries of a neighborhood of the singular points, we get $C^\nu_{\Gamma} \cup_f C^\nu_{\Gamma'}$. A simple example for $\Gamma = \Gamma' = \{1\}$ is $RP^4 \# RP^4,$
which is a quotient of $S^3 \times S^1$ by $\mathbb{Z}_2$ which acts by antipodal map and reflection on the first and second factor respectively.

The proof of theorem 2.1 will occupy sections 3, 4. The method is to use Ricci flow to deform the initial metric. By developing singularities, Ricci flow allows us to find the necks connecting these spherical orbifolds. We disconnect these spherical orbifolds by cutting off the necks between them. Let us start to consider Ricci flow.

Let $(M^4, g_0)$ be a compact 4-dimensional orbifold with at most isolated singularities with positive isotropic curvature. We deform the initial metric by the Ricci flow equation:

$$\frac{\partial g}{\partial t} = -2\text{Ric}, \quad g|_{t=0} = g_0.$$ (2.1)

Since the implicit function theorem or De Turck trick can also be applied on orbifolds, we have the short time solution $g(\cdot, t)$ of (2.1) (see [12], [9], [6]). Recall that as in the introduction, in dimension 4, the curvature operator has the following decomposition

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

and we denote the eigenvalues of matrices $A$, $C$ and $\sqrt{BB^t}$ by $a_1 \leq a_2 \leq a_3$, $c_1 \leq c_2 \leq c_3$, $b_1 \leq b_2 \leq b_3$ respectively. Since the maximum principle can also be applied on orbifolds, the positivity of isotropic curvature and improved pinching estimates of Hamilton are also preserved under the Ricci flow. We have

**Lemma 2.2.** (Theorem B1.1 and Theorem B2.3 of [11])

There exist positive constants $\rho, \Lambda, P < +\infty$ depending only on the initial metric, such that the solution to the Ricci flow (2.1) satisfies

$$a_1 + \rho > 0 \text{ and } c_1 + \rho > 0,$$

$$\max\{a_3, b_3, c_3\} \leq \Lambda (a_1 + \rho) \max\{a_3, b_3, c_3\} \leq \Lambda (c_1 + \rho),$$

$$\frac{\sqrt{(a_1 + \rho)(c_1 + \rho)}}{b_3} \leq 1 + \frac{\Lambda \rho}{\max\{\log (a_1 + \rho)(c_1 + \rho), 2\}}.$$ (2.2)

So any blowing up limit satisfies the following restricted isotropic curvature pinching condition

$$a_3 \leq \Lambda a_1, \quad c_3 \leq \Lambda c_1, \quad b_3^2 \leq a_1 c_1.$$ (2.3)

We can also define the same notion of $\kappa$ non-collapsed for a scale $r_0$ for solutions to the Ricci flow on orbifolds, namely, for any space time point $(x_0, t_0)$, the condition that $|\text{Rm}(x, t)| \leq r_0^{-2}$, for all $t \in [t_0 - r_0^2, t_0]$ and $x \in B_t(x_0, r_0)$, implies $\text{Vol}_{t_0}(B_t(x_0, r_0)) \geq \kappa r_0^4$. Since integration by parts and log-Sobolev inequality still hold on closed orbifolds, we can apply the same argument as in [18] (Theorem 4.1 of [18] or see Lemma 2.6.1 and Theorem 3.3.3 of [3] for the details) to show
Lemma 2.3. For any $T > 0$, there is a $\kappa$ depending on $T$ and the initial orbifold metric, such that the smooth solution to the Ricci flow which exists for $[0, T)$ is $\kappa$ non-collapsed for scales less than $\sqrt{T}$.

Since the scalar curvature is strictly positive, it follow from the standard maximum principle and the evolution equation of the scalar curvature that the solution must blow up at finite time. As in the smooth case, we will show that the geometric structure at any point with suitably large curvature is close to an ancient $\kappa$–solution. So it is important to investigate the structures of any ancient $\kappa$–solutions. This is done in section 3.

For the convenience of discussion, we need to fix some terminologies and notations.

In this paper, a (topological) neck is defined to be diffeomorphic to $S^3/\Gamma \times \mathbb{R}$. Here $\Gamma$ is a finite fixed point free subgroup of isometries of $S^3$. For caps, we define smooth caps consisting of $C^\Gamma_4$ and $B^4$ and we define two types of orbifold caps. The orbifold cap of Type I is obtained by crunching the boundary $S^3/\Gamma \times \{0\}$ of $S^3/\Gamma \times [0, 1)$ to a point. We denote it by $C^\Gamma_1$. By extending the action $\Gamma$ to isometric actions of $S^4$, it is clear $C^\Gamma_1$ is obtained by removing one singularity from the spherical orbifold $S^4/\Gamma$. To define the orbifold cap of type II, we first construct certain spherical orbifold in the following manner. We write the equation of $S^4$ as $x_1^2 + \cdots + x_5^2 = 1$, then the isometry $(x_1, x_2, \cdots, x_5) \rightarrow (x_1, -x_2, \cdots, -x_5)$ has exactly two fixed points $(1, 0, 0, 0, 0)$ and $(-1, 0, 0, 0, 0)$ with local uniformization group $\mathbb{Z}_2$. We denote this spherical orbifold by $S^4/(x, \pm x')$. The orbifold cap of Type II, denoted by $S^4/(x, \pm x') \setminus \bar{B}^4$, is obtained by removing a smooth point from spherical orbifold $S^4/(x, \pm x')$.

Roughly speaking, we will show in section 3 that either the ancient $\kappa$–solution is diffeomorphic to a global quotient $S^4/\Gamma$ or else it has local structures of necks, smooth caps, or orbifold caps of type I or II described in the above.

## 3 Ancient $\kappa$–solutions on orbifolds

Definition 3.1. We say a solution to the Ricci flow is an ancient $\kappa$–orbifold solution if it is a smooth complete nonflat solution to the Ricci flow on a four-orbifold with at most isolated singularities satisfying the following three conditions:

(i) the solution exists on the ancient time interval $t \in (-\infty, 0]$, and

(ii) it has positive isotropic curvature and bounded curvature, and satisfies the restricted isotropic curvature pinching condition,

$$a_3 \leq \Lambda a_1, \quad c_3 \leq \Lambda c_1, \quad b_3^2 \leq a_1 c_1,$$  \hspace{1cm} (3.1)

(iii) $\kappa$-noncollapsed on all scales for some $\kappa > 0$.

The purpose of this section is to describe the canonical neighborhood structure of ancient $\kappa$–orbifold solutions.
3.1 Curvature has null eigenvector

Theorem 3.2. Let $(X, g_t)$ be an ancient $\kappa$–orbifold solution defined in Definition 5.7 such that the curvature operator has nontrivial null eigenvector somewhere. Then we have

(i) if $X$ is smooth manifold, then either $X = (S^3/\Gamma) \times \mathbb{R}$ or $X = C^0_{\Gamma_z}$, for some fixed point free isometric subgroup $\Gamma$ or $\Gamma'$ of $S^3$ and $\sigma$ is an fixed point free isometry on $S^3/\Gamma'$ with $\sigma^2 = 1$;

(ii) if $X$ has singularities, then $X$ is diffeomorphic to $S^4/(x, \pm x') \setminus \mathbb{B}$. In particular, $X$ has exactly two singularities.

Proof. Suppose the curvature operator has nontrivial null eigenvector somewhere. Then the null eigenvectors exist everywhere in space time by Hamilton’s strong maximum principle [10].

Case 1: $X$ is a smooth manifold.

In this case, it is known from Lemma 3.2 in [5] that the universal cover of $X$ is $S^3 \times \mathbb{R}$. Let $\Gamma$ be the group of deck transformations. We claim that the second components (acting on $\mathbb{R}$ isometrically) of $\Gamma$ must contain no translations. Otherwise $X$ is compact. Note that the flat $\mathbb{R}$ factor does not move during the Ricci flow, and the spherical factor becomes very large when time goes to $-\infty$. This contradicts with the $\kappa$–noncollapsing assumption. Let $\Gamma = \Gamma^0 \cup \Gamma^1$ where the second components of $\Gamma^0$ and $\Gamma^1$ act on $\mathbb{R}$ as an identity or reflection respectively. If $\Gamma^1$ is empty, $X = (S^3/\Gamma) \times \mathbb{R}$, where $\Gamma$ acts on $S^3$ isometrically and has no fixed point. If $\Gamma^1$ is not empty, by picking $\sigma \in \Gamma_z^1$, then it satisfies $\sigma^2 \in \Gamma^0$ and $\sigma \Gamma^0 = \Gamma^1$. It is clear that $X$ is obtained by taking quotient of $(S^3/\Gamma^0) \times \mathbb{R}$ by $\sigma$. Hence $X = C^0_{\Gamma_z^0}$ by using our notation in section 2.

We remark that the $(S^3/\Gamma) \times \mathbb{R}$ has two ends, but $C^0_{\Gamma_z^0}$ has only one end.

Case 2: $X$ is an orbifold with nonempty isolated singularities.

Since $X$ has local geometry of model $S^3 \times \mathbb{R}$, $X$ must be a global quotient of $S^3 \times \mathbb{R}$ by [22], namely, $X = S^3 \times \mathbb{R}/\Gamma$, where $\Gamma$ is a subgroup of standard isometries of $S^3 \times \mathbb{R}$.

Note that the fixed points of $\Gamma$ are isolated. For fixed point $z \in S^3 \times \mathbb{R}$, denote $\Gamma_z = \{ \gamma(z) = z, \gamma \in \Gamma \}$. Let $\Gamma^0_z$ be the minimal subgroup of $\Gamma$ containing all $\Gamma_z$. Then $\Gamma^0_z$ is a normal subgroup of $\Gamma$. We claim the action of $G = \Gamma/\Gamma^0_z$ on $S^3 \times \mathbb{R}/\Gamma^0_z$ has no fixed point. Indeed, if there are some $g \in G$ and $x \in S^3 \times \mathbb{R}$ such that $g \Gamma^0_z(x) = \Gamma^0_z(x)$, this will imply $gx = \gamma x$ for some $\gamma \in \Gamma^0_z$. Hence $\gamma^{-1}g \in \Gamma_x \subset \Gamma^0_z$ and $g \in \Gamma^0_z$.

Pick $\Gamma_z \neq \{1\}$. Let $\Gamma_z = \Gamma_z^0 \cup \Gamma_z^1$ where the $\mathbb{R}$ components of $\Gamma_z^0$ and $\Gamma_z^1$ act on $\mathbb{R}$ as an identity or reflection separately. We assume $z = (0, o)$ where $0 \in \mathbb{R}$ and $o \in S^3$. Since $(0, o)$ is the unique fixed point of each $\gamma \in \Gamma_z$, this implies $\Gamma_z^0 = \{1\}$ (otherwise a nontrivial element of $\Gamma_z^0$ will fix the whole $\{o\} \times \mathbb{R}$), and $\Gamma_z^1 = \{o_z\}$, where the $S^3$ component of $o_z$ acts antipodally on the geodesic spheres (isometric to scalings $S^3$) of $S^3$ at $o$, since $o_z$ has no fixed point on the geodesic spheres.

Note the $\mathbb{R}$ components of $\Gamma$ must contain no translations. If we would have two elements in $\Gamma$ whose $\mathbb{R}$ components reflecting around points with
different $\mathbb{R}$ coordinates, then this will produce an element in $\Gamma$ with nontrivial translation on $\mathbb{R}$ factor. This particularly implies that the all fixed points of $\Gamma$ have same $\mathbb{R}$ coordinates. We may assume these fixed points lie in $S^3 \times \{0\}$. We denote their reflections by $\sigma_x, \sigma_{y}, \cdots$. We can associate an equator (which is the unique invariant equator) to a $\sigma_z$ in an obvious way. Note the action $\sigma_x \sigma_y$ on $\mathbb{R}$ is trivial. If $\sigma_x \neq \sigma_y$, then $\sigma_x \sigma_y = id$ on the great circle C defined by the intersection of their equators, this implies $\sigma_x \sigma_y$ fixes every point of $\mathbb{R} \times C$, the contradiction shows that there are exactly two fixed points $A, B$ of $\Gamma$ lying antipodally on $S^3$.

Let $G = \Gamma/\Gamma_0$. We claim $G = 1$. Indeed, if $G \neq 1$, we pick $1 \neq g \in G$, then $g$ is a fixed point free isometry of $(\mathbb{R} \times S^3)/\Gamma_0$. We must have $g(A) = B$ and $g(B) = A$ and then $g^2 = 1$. Since $g$ sends geodesics connecting $A$ to $B$ to geodesics connecting $B$ to $A$, this implies $g$ sends the $\mathbb{RP}^2$ (as quotient of equator by $\sigma_z$ in the above) to itself without fix points. This is impossible. So we have showed $G$ is trivial. That means $(S^3 \times \mathbb{R})/\Gamma_0 = X$. By using our notation in section 2, $X$ is diffeomorphic to $S^1/(\mathbb{Z}, \pm \epsilon') \times \mathbb{B}^4$. We note that in this case $X$ has only one end, which is diffeomorphic to $S^3 \times \mathbb{R}$.

### 3.2 Positive curvature operator case

In this section, we investigate the canonical neighborhood structure for all cases of the ancient $\kappa$–orbifold solution. If the orbifold admits no singularity, this has been done in Theorem 3.8 in [5]. We recall

**Theorem 3.3.** (Theorem 3.8 in [5]) For every $\epsilon > 0$ one can find positive constants $C_1 = C_1(\epsilon), C_2 = C_2(\epsilon)$ such that for each point $(x, t)$ in every four-dimensional ancient $\kappa$–manifold solution (for some $\kappa > 0$) with restricted isotropic curvature pinching and with positive curvature operator, there is a radius $r$, $0 < r < C_1(R(x, t))^{1/2}$, so that some open neighborhood $B_i(x, r) \subset B \subset B_i(x, 2r)$ falls into one of the following three categories:

(a) $B$ is an **evolving $\epsilon$-neck** (in the sense that it is the time slice at time $t$ of the parabolic region $\{(x', t')| x' \in B, t' \in [t - \epsilon^{-2}R(x, t)^{-1}, t]\}$ which is, after scaling with factor $R(x, t)$ and shifting the time $t$ to 0, $\epsilon$-close (in $C^{1,\epsilon}$ topology) to the subset $(\mathbb{I} \times S^3) \times [-\epsilon^{-2}, 0]$ of the evolving round cylinder $\mathbb{R} \times S^3$, having scalar curvature one and length $2\epsilon^{-1}$ to $\mathbb{I}$ at time zero, or

(b) $B$ is an **evolving $\epsilon$-cap** (in the sense that it is the time slice at the time $t$ of an evolving metric on open $B^4$ or $\mathbb{RP}^4 \setminus \overline{B^4}$ such that the region outside some suitable compact subset of $B^4$ or $\mathbb{RP}^4 \setminus \overline{B^4}$ is an evolving $\epsilon$-neck), or

(c) $B$ is a compact manifold (without boundary) with positive curvature operator (thus it is diffeomorphic to $S^4$ or $\mathbb{RP}^4$); furthermore, the scalar curvature of the ancient $\kappa$-solution in $B$ at time $t$ is between $C_2^{-1}R(x, t)$ and $C_2R(x, t)$.

The key difficulty in analyzing the local structure of ancient $\kappa$–solution is the collapsing of the solution in the presence of orbifold singularities with big local uniformization groups. First of all, we need to generalize the concept of
\( \varepsilon \)-neck or \( \varepsilon \)-cap to orbifold solutions with at most isolated singularities, the point is that we allow a suitable isometric group to act on the usual necks and caps. Recall in this paper, we define (topologically) that neck is (diffeomorphic to) \( S^3/\Gamma \times \mathbb{R} \); and smooth cap is \( C_\Gamma \) and orbifold cap contains two types: type I: \( C_\Gamma \), and type II: \( S^4/(x, \pm x') \setminus \bar{B} \). The motivation to define the orbifold caps to contain only the above two types is from the consideration of canonical neighborhoods in this paper.

**Definition 3.4.** Fix \( \varepsilon > 0 \) and a space time point \((x, t)\). Let \( B \subset X \) be a space open subset containing \( x \),

(i) we call \( B \) an **evolving \( \varepsilon \)-neck** around \((x, t)\) if it is the time slice at time \( t \) of the parabolic region \((x', t')|x' \in B, t' \in [t - \varepsilon^{-2}R(x, t)^{-1}, t]\) which satisfies that there is a diffeomorphism \( \varphi : \mathbb{I} \times (S^3/\Gamma) \rightarrow B \) such that, after pulling back the solution \((\varphi^* g(\cdot, \cdot))\) to \( \mathbb{I} \times S^3 \), scaling with factor \( R(x, t) \) and shifting the time \( t \) to 0, the solution is \( \varepsilon \)-close (in \( C^{1/2} \) topology) to the subset \( (\mathbb{I} \times S^3) \times [-\varepsilon^{-2}, 0] \) of the evolving round cylinder \( \mathbb{R} \times S^3 \), having scalar curvature one and length \( 2\varepsilon^{-1} \) to \( \mathbb{I} \) at time zero,

(ii) we call \( B \) an **evolving \( \varepsilon \)-cap** if it is the time slice at the time \( t \) of an evolving metric on open smooth caps \( C_\Gamma \) and orbifold caps of the above two types \( C_\Gamma \) and \( S^4/(x, \pm x') \setminus \bar{B} \) such that the region outside some suitable compact subset is an evolving \( \varepsilon \)-neck around some point in the sense of (i).

Let us start with the following elliptic type curvature estimate for our orbifold solution. The idea of proof is to find out a global uniformization space which is not collapsed and investigate the isometric group action on it.

**Proposition 3.5.** There is a universal positive function \( \omega : [0, \infty) \rightarrow [0, \infty) \) such that for any an ancient \( \kappa \)-orbifold solution on 4-orbifold \( X \), we have

\[
R(x, t) \leq R(y, t) \omega(R(y, t) d(x, y)^2)
\]

for any \( x, y \in X, t \in (-\infty, 0] \).

**Proof.** This proposition for the case that \( X \) is a smooth manifold has been established in [5] (see Theorem 3.5 and Proposition 3.3 in [5]). Thus we may always assume that \( X \) has at least one (orbifold) singularity.

Case 1: Curvature operator has zero (eigenvalue) somewhere. Then by section [3,1] the scalar curvature is constant. So the proposition holds trivially in this case.

Case 2: \( X \) is compact with positive curvature operator. By the work of Hamilton, if we continue to evolve the metric, the metric will become rounder and rounder. On the other hand, by our \( \kappa \)-noncollasing assumption, and the compactness theorem of [14], we can extract a convergent subsequence to get a limit which is compact and round. From this, we know the orbifold \( X \) is diffeomorphic to a compact orbifold with positive constant sectional curvature and with at most isolated singularities. By [22], there is a finite subgroup \( G \subset ISO(S^3) \) of isometries of \( S^3 \) such that \( S^3/G \) is diffeomorphic to \( X \). Let \( \pi : S^3 \rightarrow X \) be the naturally defined smooth map, and \( \bar{g}(\cdot, t) = \pi^* g(\cdot, t) \) be
the induced $G$ invariant solution of Ricci flow on smooth manifold $S^4$. Now we check the $\kappa$–noncollapsing of $\tilde{g}$. Suppose $\tilde{R}(\cdot, t) \leq r^{-2}$ on $\tilde{B}_t(\tilde{x}, r)$ for all $t \in [t_0 - r^2, t_0]$. Let $x = \pi(\tilde{x}) \in X$, $\gamma$ be a geodesic in $X$ of length $\leq r$ with $x = \gamma(0)$. Then $\gamma$ has a lift of geodesic $\tilde{\gamma}$ (which may not be unique) in $S^4$ with $\lambda(0) = \tilde{x}$, and $L(\tilde{\gamma}) = L(\gamma)$. This fact implies $\pi : \tilde{B}_t(\tilde{x}, r) \to B_t(x, r)$ is surjective. This implies the curvature of $X$ is still bounded by $r^{-2}$ on $B_t(x, r) \times [t_0 - r^2, t_0]$, and hence $\text{vol}_{\tilde{B}_t}(\tilde{B}(\tilde{x}, r)) \geq \text{vol}_{B_t}(B(x, r)) \geq kr^4$ by the $\kappa$–noncollapsing assumption. So we have showed that the solution $\tilde{g}$ is an ancient $\kappa$–solution on smooth manifold. By [5] (Theorem 3.5 and Proposition 3.3 in [5]), $\tilde{g}(\cdot, t)$ is $k_0$–noncollapsed for some universal constant $k_0$. Furthermore, there is a universal positive function $\omega$ such that

$$\tilde{R}(\tilde{x}, t) \leq \tilde{R}(\tilde{y}, t)\omega(\tilde{R}(\tilde{y}, t)d(\tilde{x}, \tilde{y})^2) \quad (3.2)$$

for the curvature of induced Ricci flow $\tilde{g}(\cdot, t)$ at any two points $\tilde{x}, \tilde{y} \in R^4, t \in (0, \infty]$. For any pair of points $x, y \in X$, we draw minimal geodesic $\gamma$ connecting $x, y$ in $X$, $\gamma$ can be lifted to a geodesic $\tilde{\gamma} \subset R^4$ connecting two points $\tilde{x} \in \Phi^{-1}(x), \tilde{y} = \Phi^{-1}(y)$. Since $\tilde{d}(\tilde{x}, \tilde{y}) \leq L(\tilde{\gamma}) = d(x, y)$ and $R(x, t) = \tilde{R}(\tilde{x}, t)$ and $R(y, t) = \tilde{R}(\tilde{y}, t)$, by (3.2), we get

$$R(x, t) \leq R(y, t)\omega(R(y, t)d(x, y)^2).$$

Case 3: We assume $X$ is noncompact and has positive curvature operator. Let $P$ be a fixed singularity of $X$. We define a Busemann function $\varphi$ at time $-1$ in the following way:

$$\varphi(x) = \sup_{\gamma} \lim_{s \to +\infty} (s - d_{-1}(x, \gamma(s)))$$

where the sup is taken over all normal geodesic ray $\gamma$ originating from $P$. It is well-known that $\varphi$ is convex (with respect to the metric at time $-1$) and of Lipschitz constant $\leq 1$ and proper. Deforming $\varphi$ by the heat equation

$$\frac{\partial u}{\partial t} = \Delta_iu$$

with $u|_{t=1} = \varphi$. By a straightforward computation, we have

$$\frac{\partial}{\partial t}u_{ij} = \Delta u_{ij} + s\gamma_{km}g^{ln}R_{klij}u_{mn} - \frac{1}{2}(s^2R_{ik}u_{ij} + s^2R_{jk}u_{ij})$$

where $u_{ij} = \nabla^2_i u$ are the Hessian of $u$. Noting the curvature operator is positive, by maximum principle, we have $\nabla^2 u \geq 0$ is preserved. Moreover we have $\nabla^2 u > 0$ at $t = 0$ by the following reasons. The kernel of $\nabla^2 u$ is a parallel distribution by strong maximum principle of Hamilton [10]. If the kernel is nontrivial, then either the space splits product $\mathbb{R} \times \Sigma$ locally or the space admits a linear function ($\nabla^2 u = 0$). Both cases have contradiction with the strict positive curvature operator.

Now we fix the time $t = 0$. Notice that $u$ is still a proper function, so by strict convexity of $u$, we know $u$ has a unique critical point, which is the minimal
point. We claim the minimal point is just the singular point \( P \) we specified in the beginning. Hence there are no other singularities. The argument is in the following. Let \( \pi: \bar{U} \to U, U = \bar{U}/\Gamma \) be the local uniformization near \( P \). Then \( \bar{u} = u \circ \pi \in \Gamma \) invariant, and we have \( d\gamma(Vu)(P) = Vu(P) \) for any \( \gamma \in \Gamma \). Since \( \Gamma \) has isolated fixed point, we have \( \sum_{\gamma \in \Gamma} d\gamma(Vu)(P) = 0 \) and \( Vu(P) = 0 \) consequently.

Let \( \xi = \frac{Vu}{|Vu|} \) be a vector field which is singular at \( P \). Now we consider the map \( \Phi: C_\gamma X = \text{Cone}(S^3/\Gamma) \to X \) defined by

\[
\Phi(v, s) = \alpha_v(s)
\]

where \( \alpha_v(s) \) is the integral curve of \( \xi \) with \( \alpha_v(0) = P \) and \( \alpha_v'(0) = v \). By using \( \nabla^2 u(P) > 0 \), the \( \frac{Vu}{|Vu|} \) can take any value, so the above map is defined. Clearly, \( \Phi \) is a global orbifold diffeomorphim. We define \( \tilde{\Phi}: \mathbb{R}^4 = \text{Cone}(S^3) \to X \) by

\[
\tilde{\Phi} = \Phi \cdot \pi
\]

where \( \pi: \text{Cone}(S^3) \to \text{Cone}(S^3/\Gamma) \) is the natural projection. Define

\[
\tilde{g}(\cdot, t) = \tilde{\Phi}^* g(\cdot, t).
\]

Then \( \tilde{g}(\cdot, t) \) is a smooth complete ancient \( \kappa \) solution on smooth manifolds \( \mathbb{R}^4 \) with positive curvature operator and restricted isotropic pinching condition. Moreover by \([5]\) again, \( \tilde{g}(\cdot, t) \) is \( k_0 \) noncollapsed for some universal constant \( k_0 \), and same argument as in Case 2 completes the proof. 

\[ \square \]

**Corollary 3.6.** Let \( g \) be an ancient \( \kappa \)-orbifold solution on complete noncompact 4-orbifold \( X \) with positive curvature operator and nonempty isolated singularities, then there is at most one singularity and there is a finite group of isometries \( \Gamma \subset \text{ISO}(\mathbb{R}^4) \) of standard \( \mathbb{R}^4 \), such that \( O \) is the only fixed point for any element of \( \Gamma \), and \( X \) is diffeomorphic to \( \mathbb{R}^4/\Gamma \) as orbifolds.

**Theorem 3.7.** For every \( \varepsilon > 0 \) one can find positive constants \( C_1 = C_1(\varepsilon), C_2 = C_2(\varepsilon) \) such that for each point \( (x, t) \) in every complete noncompact four-dimensional ancient \( \kappa \)-orbifold solution with positive curvature operator, there is a radius \( r \), \( \frac{1}{C_1(R(x, t))^{1/2}} < r < C_1(R(x, t))^{1/2} \), so that some open neighborhood \( B(x, r) \subset B(x, 2r) \) falls into one of the following two categories:

(a) \( B \) is an *evolving \( \varepsilon \)-neck* around \( (x, t) \),

(b) \( B \) is an *evolving \( \varepsilon \)-cap* of Type I.

Moreover, the scalar curvature in \( B \) at time \( t \) is between \( C_2^{-1} R(x, t) \) and \( C_2 R(x, t) \).

**Proof.** We denote the unique singularity by \( O \). By corollary 3.6, \( X \) is diffeomorphic to \( \bar{X}/\Gamma \), where \( \bar{X} \) is diffeomorphic to \( \mathbb{R}^4 \), and \( \Gamma \subset \text{ISO}(\mathbb{R}^4) \) fixes the origin, denoted also by \( O \). Let \( \tilde{g} \) be the pulled back solution on \( \bar{X} \), which is a \( \Gamma \)-invariant solution on \( \bar{X} \). Note that the solution \( \tilde{g} \) is also \( \kappa \)-noncollapsed and therefore \( k_0 \)-noncollapsed for some universal \( k_0 \) by theorem 3.5 in \([5]\). Fix time \( t = 0 \). Now by the proof of Theorem 3.8 in \([5]\), there is a point \( x_0 \in \bar{X} \), such that for any given small \( \varepsilon > 0 \), there is a constant \( D(\varepsilon) > 0 \) depending only on \( \varepsilon \) such that any \( (x, 0) \) satisfying \( R(x_0, 0)d_0(x, x_0)^2 \geq D(\varepsilon) \) admits an evolving
above parametrization descents to a parametrization $\phi$ they align themselves to a global isometry of the standard $S^3$. So we conclude that $\hat{\Gamma}$. We claim the $R$ factor of $\hat{\Gamma}$ acts isometrically on $S^3 \times \mathbb{I}$ with the standard metric. We claim the $R$ factors of $\hat{\Gamma}$ have no translations. Indeed, suppose there is one $\gamma \in \Gamma$ such that the $R$ factor of $\gamma$ is a translation $s \mapsto s + L$ with $L > 0$. Otherwise, we consider $\gamma^{-1}$. So any point in finite region will be mapped to very far by $\gamma^m$ as $m \to \infty$. Since the $\gamma^m$ are isometries, and the manifold at infinity splits off a line, we conclude the curvature operator is not strictly positive in finite region. This is a contradiction with our assumption. The $R$ factors of $\hat{\Gamma}$ also contain no reflections, otherwise the manifold will contain two ends and splits off a line globally. So we conclude that $\hat{\Gamma}$ only acts on the factor $S^3$. This implies that the above parametrization descents to a parametrization $\phi : S^3/\Gamma \times (A, B) \to X$. 

$\epsilon$-neck around it. We scale the solution so that $R(x_0, 0) = 1$. In the following, we describe the canonical parametrization of necks which was given by Hamilton in the section C of [11]. We will use Hamilton's canonical parametrization to parametrize all the points outside a ball of radius $D(\epsilon) + 1$ centered at $x_0$ by a canonical diffeomorphism $\Phi$ from $S^3 \times \mathbb{I}$, where $\mathbb{I} \in \mathbb{R}$ is an interval.

For any $z \in \bar{X}$ with $d_0(z, x_0)^2 \geq D(\epsilon)$, there is a unique constant mean curvature hypersurface $S_z \subset \bar{X}$ passing through $z$. Each $(S_z, \bar{g})$ can be parametrized by a harmonic diffeomorphism from standard sphere $(S^3, \bar{g})$ to it, since the (induced) metrics $\bar{g}$ and $\bar{g}$ is very close. The coordinate function of factor $R \ni s$ can also be uniquely chosen in the following way: let

$$area(S^3 \times \{s\}, \bar{g}) = vol(S^3)r(s)^3,$$

we require function $s$ satisfies

$$Vol(S^3 \times \{s_1, s_2\}, \bar{g}) = vol(S^3) \int_{s_1}^{s_2} r(s)^3 ds.$$ 

Notice that the above harmonic diffeomorphisms are unique up to a rotations of $(S^3, \bar{g})$, since the induced metrics are close to the standard one. We require if $V$ is an infinitesimal rotation on $(S^3 \times \{z\}, \bar{g})$, and $W$ is the unit vector field which is $\bar{g}$ orthonormal to the sphere $S^3 \times \{z\}$, then

$$\int_{S^3 \times \{z\}} \bar{g}(V, W) = 0. \tag{3.3}$$

The above parameterization $\Phi : S^3 \times (A, B) \to \bar{X}$ can be extended on one end so that it covers all points outside a ball of radius $D(\epsilon) + 1$ centered at $x_0$. Without loss of generality, we assume as $z \to B$, the points on the manifold $\bar{X}$ divergent to infinity.

Let $\tilde{g} = \Phi^* \bar{g}$. Let $\gamma \in \Gamma$, and $\tilde{\gamma} = \Phi^{-1} \gamma \Phi$. Since $\gamma$ is an isometry of $\bar{g}$, it sends constant mean curvature spheres to constant mean curvature spheres. So $\tilde{\gamma}$ preserves the foliation of the horizontal spheres. So the uniqueness of harmonic maps in this case implies $\tilde{\gamma}$ is isometry in $S^3$ factor. The specific choice of coordinate $s \in \mathbb{R}$ implies the $R$ component of $\tilde{\gamma}$ is an isometry of $R$, and independent of the factor $S^3$. The (3.3) straighten out the rotations so that they align themselves to a global isometry of the standard $S^3 \times \mathbb{I}$, $\mathbb{I} = (A, B)$. So the group $\hat{\Gamma} = \Phi^{-1} \Gamma \Phi$ acts isometrically on $S^3 \times \mathbb{I}$ with the standard metric.

We claim the $R$ factors of $\hat{\Gamma}$ have no translations. Indeed, suppose there is one $\gamma \in \Gamma$ such that the $R$ factor of $\gamma$ is a translation $s \mapsto s + L$ with $L > 0$. Otherwise, we consider $\gamma^{-1}$. So any point in finite region will be mapped to very far by $\gamma^m$ as $m \to \infty$. Since the $\gamma^m$ are isometries, and the manifold at infinity splits off a line, we conclude the curvature operator is not strictly positive in finite region. This is a contradiction with our assumption. The $R$ factors of $\hat{\Gamma}$ also contain no reflections, otherwise the manifold will contain two ends and splits off a line globally. So we conclude that $\hat{\Gamma}$ only acts on the factor $S^3$. This implies that the above parametrization descents to a parametrization $\phi : S^3/\Gamma \times (A, B) \to X$. 

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Since as $\varepsilon \to 0$, after normalizations, metric $\hat{g}$ will converge in $C^\infty_{\text{loc}}$ topology to the standard one. This implies the following fact: for any given $\varepsilon > 0$, there is $\tilde{\varepsilon} > 0$ such that if $\varepsilon < \tilde{\varepsilon}$, then for any point $P \in S^3 \times (A, B)$, the metric $\hat{g}$ on $S^3 \times (A, B)$ around $P$ is $\varepsilon$− close to the standard one after scaling with the factor $\hat{R}(P)$.

We can also show the point $O$ has distance $\leq \sqrt{D(\varepsilon)} + 1$ with $x_0$. Indeed, if $d_0(x_0, O) \geq \sqrt{D(\varepsilon)} + 1$, then $O$ is covered by the parameterization $\Phi : S^3 \times (A, B) \to \tilde{X}$. Let $O = \Phi(\tilde{x}, \tilde{\delta}), \tilde{x} \in S^3, \tilde{\delta} \in (A, B)$. Since the group $\Gamma$ only acts on the factor $S^3$, we conclude that $\hat{\Gamma}$ fixes every point on $\{\tilde{\delta}\} \times (A, B)$. This is a contradiction.

Now we are ready to prove the theorem. For the given $\varepsilon > 0$, there is a $\tilde{\varepsilon} > 0$ defined in the above. For any point $x \in X$ with $d_0(O, x) \geq 2D(\tilde{\varepsilon})$, a suitable portion $S^3/\Gamma \times (\Lambda' \Lambda')$ of $S^3/\Gamma \times (A, B)$ in the above parametrization will give a $\varepsilon$− neighborhood of $x$. Let $\tilde{x} \in \tilde{X}$ satisfy $d_0(\tilde{x}, O) = 10D(\tilde{\varepsilon})$, denote the constant mean curvature hypersurface passing through $\tilde{x}$ by $\tilde{\Sigma}$. By Theorem G1.1 in [11], $\Sigma$ bounds a open set $\Omega$ which is differentiable ball $B^4$ in $\tilde{X}$. $\Omega$ is $\Gamma$− invariant, and $\Omega/\Gamma$ contains an $\varepsilon$− neck with its end. The curvature estimate on $\Omega/\Gamma$ follows from the above Proposition 3.5. Thus we only need to show $\Omega/\Gamma$ is diffeomorphic to the orbifold cap $C_1$ of type I.

Let $\phi : X \to \mathbb{R}$ be the Busemann function at time $t = 0$ on $X$ constructed around the singular point $O$. Let $u_0$ be a family of strictly convex smooth perturbation of $\phi$ as in Proposition 3.5 such that $u_0 = \phi$. By considering the integral curves of $u_0$ as in Proposition 3.5, one can show the level sets $u_0^{-1}(c)$ of $u_0$ diffeomorphic to $C_1$.

Let $f$ be the function of coordinate $\mathbb{R}$ on the parametrization $\phi : S^3/\Gamma \times (A, B) \to \tilde{X}$. By a geometric argument, one can show $Vf$ is almost parallel (with error controlled by $\varepsilon$) to the $Vf$, and so does $\nabla u_0$ for small $\delta$. By blending the function $u_0$ and a multiple of $f$ by a bump function, we get a function $\psi$, whose gradient curves gave a diffeomorphism from $f^{-1}(-\infty, c)$ and $u_0^{-1}(-\infty, c)$ by Morse theory. This particularly shows that $\Omega/\Gamma$ is diffeomorphic to $C_1$.

The proof of the theorem is completed. $\square$

We summarize the results obtained in this section:

**Theorem 3.8.** For every $\varepsilon > 0$ one can find positive constants $C_1 = C_1(\varepsilon), C_2 = C_2(\varepsilon)$, such that for every four-dimensional ancient $\kappa$-orbifold solution $(X, g_1)$, for each point $(x, t)$, there is a radius $r, \frac{1}{2}(R(x, t))^{-\frac{1}{2}} < r < C_1(R(x, t))^{-\frac{1}{2}}$, so that some open neighborhood $B_0(x, r) \subset B \subset B_t(x, 2r)$ falls into one of the following two categories:

a) $B$ is an evolving $\varepsilon$-neck around $(x, t)$,

b) $B$ is an evolving $\varepsilon$-cap,

c) $X$ is diffeomorphic to a closed spherical orbifold $S^4/\Gamma$ with at most isolated singularities.

Moreover, the scalar curvature in $B$ at case (a) and (b) at time $t$ is between $C_2^{-1} R(x, t)$ and $C_2 R(x, t)$.

**Proof.** By Theorem 3.2 and Theorem 3.7, we only need to consider the case when $X$ is compact with positive curvature operator. In this case, we continue
to evolve the metric by Ricci flow. Since the scalar curvature is strictly positive, the solution will blow up in finite time. By using the $\kappa$–noncollapsing in [18] and the compactness theorem in [14], we can scale the solution in space time around a sequence of points and extract a convergent subsequence. Moreover, the limit is still an orbifold with at most isolated singularities by [14]. By the pinching estimate of Hamilton [10], the Riemannian metric in the limit orbifold has constant sectional curvature. So it is a global quotient of sphere. □

4 Surgerical solutions

4.1 Surgery at first singular time

Since the scalar curvature at initial time is strictly positive, it follows from the maximum principle and the evolution equation of the scalar curvature that the curvature must blow up at some finite time $0 < T < \infty$. Note that the canonical structures of ancient $\kappa$–orbifold solutions have been completely described in the last section. Combining with a technical geometric lemma (Proposition 6.1 in the appendix), we have the similar singularity structure theorem before time $T$ as in the manifold case (see Theorem 4.1 in [5]).

**Theorem 4.1.** Given small $\epsilon > 0$, there is $r = r(T) > 0$ depending on $\epsilon, T$ and the initial metric such that for any point $(x_0, t_0)$ with $Q = R(x_0, t_0) \geq r^{-2}$, the solution in the parabolic region $\{(x, t) \in X \times [0, T) | d_p^2(x, x_0) < \epsilon^{-2}Q^{-1}, t_0 - \epsilon^{-2}Q^{-1} < t \leq t_0\}$ is, after scaling by the factor $Q$, $\epsilon$-close (in $C^{1,1}$-topology) to the corresponding subset of some ancient $\kappa$-orbifold solution with restricted isotropic curvature pinching (2.3) and with at most isolated orbifold singularities.

**Proof.** First of all, we may assume the orbifold is not diffeomorphic to a spherical orbifold $S^4/\Gamma$, otherwise we are in case c) in Theorem 3.8. We argue by contradiction as in manifold case [5].

We choose a point $(x_0, t_0)$ almost critically violating the conclusion of the theorem. We scale the solution around $(x_0, t_0)$ with factor $R(x_0, t_0)$ and shift the time $t_0$ to 0. The key point of the proof is to bound the curvature. Note that we still have $\kappa$–noncollapsing condition (Lemma 2.3), and compactness theorem [14] for $\kappa$–noncollapsed Ricci flow solutions on orbifolds with isolated singularities. By the canonical neighborhood decomposition theorem for ancient $\kappa$– solutions, we can show the curvature is bounded in bounded normalized distance with $x_0$. The boundedness of curvature on the limit space follows from Proposition 6.1. We have all the ingredients we need to mimic the same proof in the manifold case [5] to show that we can extract a convergent subsequence which converges to an ancient $\kappa$-orbifold solution. This is a contradiction. □

We denote by $\Omega$ the open set of points where curvature become bounded as $t \to \infty$. Denote by $\tilde{g}_t$ the limit of $g_t$ on $\Omega$ as $t \to T$.

Fix $0 < \delta << \epsilon$, and let $\rho = \rho(T) = \delta r(T)$, and $\Omega_{\rho} = \{x \in X | R \leq \rho^{-2}\}$. If $\Omega_{\rho}$ is empty, then by Theorem 4.1 and Theorem 3.8 $X$ is either diffeomorphic to a
spherical orbifold $S^4/\Gamma$ with at most isolated singularities, or $X$ is covered by $\varepsilon$–necks and $\varepsilon$–caps. For the latter case, if there occurs no caps, $X$ is covered by $\varepsilon$–necks, hence diffeomorphic to $S^3/\Gamma \times S^1$ or $S^3/\Gamma \times S^1$; if there are caps, we have four types of caps: $C'_I, C_I, S^4/(x, \pm x') \setminus \bar{B}_4, B^4$ and hence $X$ is diffeomorphic to either smooth manifolds $S^4, \mathbb{R}P^4, C'_I \cup J, C_I$, or one of the orbifolds $C'_I \cup J, C'_I, C_I \cup J, C_I, S^4/(x, \pm x'), S^4/(x, \pm x') \# \mathbb{R}P^4, S^4/(x, \pm x') \# S^4/(x, \pm x')$. So we conclude that if $\Omega_\rho$ is empty, the $X$ is diffeomorphic to a spherical orbifold $S^4/\Gamma$ with at most isolated singularities or a connected sum of two spherical orbifolds $S^4/\Gamma_1$ and $S^4/\Gamma_2$ with at most isolated singularities. While if the solution, near the time $T$, has positive curvature operator, it follows from the proof of Theorem 3.8 that $X$ is diffeomorphic to a spherical orbifold with at most isolated singularities. Thus, when $\Omega_\rho$ is empty or the solution becomes to have positive curvature operator everywhere, we stop the the procedure here and say that the solution becomes extinct.

We then may assume that $\Omega_\rho \neq \emptyset$ and any point outside $\Omega_\rho$ has a $\varepsilon$–neck or $\varepsilon$–cap neighborhood. We are interested in those $\varepsilon$–horns $H$ (consisting of $\varepsilon$–necks) whose one end is in $\Omega_\rho$ and the curvature becomes unbounded on other end. We will perform surgeries on these horns. First of all, we need the existence of finer necks (than $\varepsilon$) in the $\varepsilon$–horn $H$. The reason to find a finer neck to perform surgeries is to quantitatively control the accumulations of the errors caused by surgeries.

**Proposition 4.2.** For the arbitrarily given small $0 < \delta << \varepsilon$, there is an $0 < h < \delta \rho$ depending only on $\delta$ and $\varepsilon$, and independent of non-collapsing parameter $\kappa$ such that if a point $x$ on the $\varepsilon$–horn $H$ whose finite end is in $\Omega_\rho$ has curvature $\geq h^{-2}$, then there is a $\delta$–neck around it.

The argument is a bit different from Lemma 5.2 in [5]. The reason is that, the canonical neighborhoods in [5] are universally non-collapsed, but in the present situation we do not know it a priori.

**Proof.** There is a fixed point free finite group of isometries $\Gamma \in ISO(S^3)$ so that we can apply Hamilton’s parametrization to parametrize the whole $H, \Phi_\Gamma : (S^3/\Gamma) \times (A, B) \rightarrow H$, where $\Phi_\Gamma$ is a diffeomorphism. Denote by $\Phi : S^3 \times (A, B) \rightarrow H$ the natural projection. Without loss of generality, we assume $\Phi(S^3 \times \{s\})$ has nonempty intersection with $\Omega_\rho$, as $s \rightarrow A$, and curvature becomes unbounded as $s \rightarrow B$. To prove the claim, we argue by contradiction. Suppose $x_j \in H$ is a sequence of points with $\mathring{R}(x_j) \geq h^{-2} \rightarrow \infty$ but $x_j$ has no $\delta$–neck neighborhood. We pull back the solution to $S^3 \times (A, B)$, scale with factor $\mathring{R}(x_j)$ around $x_j$, shift the time $T$ to 0. Note the rescaled solution on $S^3 \times (A, B)$ is smooth (without orbifold singularities) and uniformly non-collapsed. We apply the same argument of step 2 in Theorem 4.1 in [5] to show that the curvature is bounded in any fixed finite ball around point $x_j$ for the rescaled solution, otherwise we get a piece of non-flat nonnegatively curved metric cone as a blow up limit, which contradicts with Hamilton’s strong maximum principle (see [10]). This implies the two ends of $S^3 \times (A, B)$ are very far from point $x_j$ (in the normalized distance). We then
extract (around \((\bar{x}, T)\)) a convergent subsequence so that the limit splits off a line by the Toponogov splitting theorem. By (3.1), the limit is the standard \(S^3 \times \mathbb{R}\). Since the solution is \(\Gamma\) invariant, it descends to \(H\) and gives a \(\delta\)-neck around \(x_j\) as \(j\) large enough. This is a contradiction. □

Let us describe the Hamilton’s surgery along the \(\delta\)-neck \(N\) with scalar curvature \(h\) in the center \(\bar{x}\). We assume the normalization \((\text{of } S^3 \times (A, B)\) with some factor) is so that the metric \(h^{-2}\Phi^*g_{0}\) on \((\bar{x}, s_0) \in S^3 \times (A, B)\) is \(\delta'\) close to standard neck metric \(ds^2 = dz^2 + ds^2_{S^3}\) on \(S^3 \times \mathbb{R}\) of scalar curvature 1, where \(\delta' = \delta'(\delta)\) satisfies \(\lim_{\delta \to 0} \delta' = 0\). We assume the center of the \(\delta\)-neck has \(R\) coordinate \(z = 0\).

The surgery is to cut open the neck (in Hamilton’s parametrization) and glue back caps \((\mathbb{B}^4, \tilde{g})\) by conformal pinching the metric \(\bar{g}\) and bending it with the standard cap metric \((\pi)\). We describe the construction on the left hand \((\text{of coordinate } R)\) \((\text{corresponding to the finite part connecting to } \Omega_{\rho})\)

\[
\tilde{g} = \begin{cases} 
\bar{g}, & z = 0, \\
e^{-2f}\bar{g}, & z \in [0, 2], \\
\phi e^{-2f}\bar{g} + (1 - \phi)e^{-2/h^2g_0}, & z \in [2, 3], \\
h^2e^{-2f}g_0, & z \in [3, c'],
\end{cases}
\]

where \(f\) is some fixed function and \(g_0\) is the standard metric. We also perform the same surgery procedure on the right hand with parameters \(\bar{z} \in [0, 4] (\bar{z} = 8 - z)\).

Since the group \(\Gamma\) acts isometrically on the factor \(S^3\) of \(S^3 \times \mathbb{R}\), the above surgery procedure on Hamilton’s parametrization descends to a surgery on the space \(X\) by cutting off a \(\delta\)-neck and gluing back two orbifold caps \(C\) separately. We call the above procedure as a \(\delta\)-cutoff surgery.

Now at least the proof of justification of pinching estimates of Hamilton can be carried through without changing a word.

**Lemma 4.3.** (Hamilton [11] D3.1, Justification of the pinching assumption)

There are universal positive constants \(\delta_0\), such that for any \(\bar{T}\) there is a constant \(h_0 > 0\) depending on the initial metric and \(\bar{T}\) such that if we perform above \(\delta\)-cutoff surgery at a \(\delta\)-neck of radius \(h\) at time \(T \leq \bar{T}\) with \(\delta < \delta_0\) and \(h^{-2} \geq h_0^{-2}\), such that after the surgery, the pinching condition (2.2) still holds at all points at time \(T\).

### 4.2 A priori assumptions

We can define the notion of Ricci flow with surgeries in the same way as in [5] by replacing manifolds with orbifolds with at most isolated singularities. As in [5], the solutions to Ricci flow with surgery in this paper are obtained by performing concrete surgeries. We cut open a neck in a horn and glue back two caps. This makes the all connected components after surgeries are also closed orbifolds with at most isolated singularities. Notice that each neck in the horn
is diffeomorphic to $S^3/\Gamma$. If $\Gamma$ is trivial, we glue the usual caps $B^4$; if $\Gamma$ is no
trivial, we glue back orbifold caps $C_\Gamma$, this produces new orbifold singularities
(tips of the caps).

To understand the topology, we are interested in the solutions with good
properties. Namely, we would like to construct a long time solution satisfying
the a priori assumptions consisting of the pinching assumption and the
canonical neighborhood assumption.

**Pinching assumption:** There exist positive constants $\rho, \Lambda, \rho' < +\infty$ such that
there hold
\[
a_1 + \rho > 0 \text{ and } c_1 + \rho > 0, \\
\max\{a_3, b_3, c_3\} \leq \Lambda(a_1 + \rho) \text{ and } \max\{a_3, b_3, c_3\} \leq \Lambda(c_1 + \rho),
\]
and
\[
\frac{b_3}{\sqrt{(a_1 + \rho)(c_1 + \rho)}} \leq 1 + \frac{\Lambda e^{\rho t}}{\max[\log \sqrt{(a_1 + \rho)(c_1 + \rho)}, 2]}.
\]
everywhere.

**Canonical neighborhood assumption (with accuracy $\varepsilon$):** Let $g_t$ be a solution
to the Ricci flow with surgery staring with $(\Omega, \varepsilon)$. For the given $\varepsilon > 0$, there exist
two constants $C_1(\varepsilon), C_2(\varepsilon)$ and a non-increasing positive function $r$ on $[0, +\infty)$ with
the following properties. For every point $(x, t)$ where the scalar curvature $R(x, t)$
is at least $r^{-\frac{3}{2}}(t)$, there is an open neighborhood $B, B_t(x, \varepsilon) \subset B \subset B_t(x, 2\varepsilon)$ with
$0 < \sigma < C_1(\varepsilon)R(x, t)^{-\frac{1}{2}}$, which falls into one of the following three categories:
(a) $B$ is a strong $\varepsilon$-neck,
(b) $B$ is an $\varepsilon$-cap,
(c) at time $t$, $X$ is diffeomorphic to a closed spherical orbifold $S^4/\Gamma$ with at most
isolated singularities.

Moreover, for (a) and (b), the scalar curvature in $B$ at time $t$ is between $C_2^{-1}R(x, t)$ and
$C_2R(x, t)$, and satisfies the gradient estimate
\[
|\nabla R| < \eta R^2 \text{ and } |\frac{\partial R}{\partial t}| < \eta R^2,
\]
where $\eta$ is a universal constant and the definitions of $\varepsilon$-cap and strong $\varepsilon$-neck will be
given in the next paragraph.

We give the precise definitions of $\varepsilon$-cap, and strong $\varepsilon$-neck in the following.
First, we say an open set $B$ on an orbifold is an $\varepsilon$-neck if there is a diffeomorphism
$\varphi : \Omega \times (S^3/\Gamma) \to B$ such that the pulled back metric $(\varphi)^*g_t$, scaling with
some factor, is $\varepsilon$-close (in $C^{[-1]}$ topology) to the standard metric $\Omega \times (S^3/\Gamma)$ with
scalar curvature 1 and $\Omega = (-\varepsilon^{-1}, \varepsilon^{-1})$. An open set $B$ is $\varepsilon$-cap if $B$ is diffeomorphic
to smooth cap $B^4$, $C_\Gamma$ orbifold cap of Type I, II, $C_\Gamma$ or $S^4/(x, \pm x')\backslash \mathbb{B}^4$, and
the region around the end is an $\varepsilon$-neck. A strong $\varepsilon$-neck $B$ at $(x, t)$ is the time slice at time $t$ of the parabolic region $\{(x', t') | x' \in B, t' \in [t - R(x, t)^{-1}, t]\}$ where
the solution is well-defined and has the property that there is a diffeomorphism $\varphi : \Omega \times (S^3/\Gamma) \to B$ such that , the pulling back solution $(\varphi)^*g(\cdot, \cdot)$ scaling with
factor $R(x, t)$ and shifting the time $t$ to 0, is $\varepsilon$-close(in $C^{[-1]}$ topology) to the
subset $(\mathcal{I} \times S^3/\Gamma) \times [-1, 0]$ of the evolving round cylinder $\mathbb{R} \times (S^3/\Gamma)$, having scalar curvature one and length $2\varepsilon^{-1}$ to $\mathcal{I}$ at time zero.

In order to take limits for surgerical orbifold solutions, we need the non-collapsed condition. Let $\kappa$ be a positive constant. We say the solution is $\kappa$–noncollapsed on the scales less than $\rho$ if it satisfies the following property: if

$$|Rm(\cdot, \cdot)| \leq r^{-2}$$

on $P(x_0, t_0, r, -r^2) = \{(x', t') \mid x' \in B_r(x_0, r), t' \in [t_0 - r^2, t_0]\}$ and $r < \rho$, then we have

$$\text{Vol}_6(B_0(x_0, r)) \geq \kappa r^4.$$ 

Since we are dealing with solutions with surgeries, the parabolic neighborhood $P(x_0, t_0, r, -r^2)$ is a little bizarre, the condition $|Rm(x, t)| \leq r^{-2}$ is imposed on the place where the solution is defined.

We will inductively construct a long time solution $g(t)$ satisfying the a priori assumptions. In section 4.1, we actually have constructed a solution satisfying a priori assumptions for a period of time. In order to extend our solution for a longer time inductively, we need to do surgery repeatedly. In particular, we need that there exist sufficient fine necks in horns of surgical solutions and the estimate has to be quantitative. The following statement is similar to Proposition 4.2, but the situation is a bit different, since we are dealing with solutions with surgery.

**Proposition 4.4.** Suppose we have a solution to the Ricci flow with surgery on $(0, T)$ satisfying the a priori assumptions in the above, and the solution becomes singular as $t \to T$. For the arbitrarily given small $0 < \delta < \varepsilon$, there is an $0 < h < \delta \rho(T) = \delta^2 r(T)$ depending only on $\delta$, $\varepsilon$, and $r(T)$ such that if at the time $T$, a point $x$ on an $\varepsilon$–horn $H$ whose finite end is in $\Omega_{\rho(T)}$ has curvature $\geq h^{-2}$, then there is a $\delta$–neck around it.

**Proof.** We observe that the canonical neighborhoods of the points in the $\varepsilon$–horn $H$ (far from the end) are all strong $\varepsilon$–necks. The solution around any point $\hat{x}$ on $H$ with $R(\hat{x}, T) \geq h^{-2}$ has existed for a previous time interval $(T - R(\hat{x}, T)^{-1}, T)$. Suppose the proposition is not true. We use Hamilton’s parametrization $\Phi : S^3 \times (A, B) \to H$ to pull back the solution on $S^3 \times (A, B)$. By the same argument of Proposition 4.2, we extract a convergent subsequence from the parabolic scalings around suitable points $\hat{x}$ with $R(\hat{x}, T) \geq h^{-2} \to \infty$. The limit solution is just the standard solution on $S^3 \times \mathbb{R}$ which exists at least on the time interval $(-1, 0]$ after shifting the origin. Moreover, the solution on all points (on the original space) at normalized time $-1 + \frac{1}{10}h$ still has strong $\varepsilon$–neck neighborhoods and the scalar curvature is $\leq 1$ as $h^{-1} \to \infty$. So we can actually extract a subsequence so that the limit solution is defined at least on $[-2, 0]$. Since the solution is $\Gamma$–invariant, this gives a $\delta$–neck as $h^{-1}$ is very large. This is a contradiction. \hfill $\square$

Now we justify the uniform $\kappa$–noncollapsing under the assumption of canonical neighborhoods with accuracy $\varepsilon$ for some parameter $\tilde{r}$ which may
be very small. The key point is that even if we perform \( \delta \)-cutoff surgeries with sufficient fine \( \delta \) which depends on \( \bar{r} \), the noncollapsing constant \( \kappa \) we obtained is uniform and independent of \( \bar{r} \). In Lemma 5.5 in [5], the same estimate was deduced when the space is smooth. The fact that the canonical neighborhoods in [5] are not collapsed played a crucial role in the proof there. In the current context, at a priori, the canonical neighborhoods may be sufficiently collapsed. We need a different argument. Our idea is the following. When the scale is not too small comparing with the canonical neighborhood parameter \( \bar{r} \), we observe that the surgery is performed far away and the argument of Perelman’s Jacobian comparison theorem can be modified to apply as in the smooth case [5].

When the scale is small, we first show the space has a canonical geometric neck near the point and then extend the canonical geometric neck to form a long geometric tube so that the other end of the tube has a neck of big scale. After showing the neck with big scale is noncollapsing, we will get a control on the order of the fundamental group of the neck which in turn gives the control on the noncollapsing of the original neck with small scale.

**Lemma 4.5.** Given a compact four-orbifold with positive isotropic curvature and given small \( \epsilon > 0 \) and a positive integer \( l \). Suppose we have constructed the sequences \( 0, \delta_j > 0, r_j > 0, \kappa_j > 0, 0 \leq j \leq l - 1 \), such that any solution to the Ricci flow with surgery on \([0, T]\), with \( T \in [l\epsilon^2, (l + 1)\epsilon^2] \) and with the four-orbifold as the initial data, obtained by \( \delta(t) \)-cutoff surgeries with \( \delta(t) \leq \delta_0 \), satisfies the following three properties:

(i) the pinching assumption holds on \([0, T]\),

(ii) if \( \delta(t) \leq \delta_j \) on \([j\epsilon^2, (j + 1)\epsilon^2]\), for all \( 0 \leq j \leq l - 1 \), then the canonical neighborhood assumption (with accuracy \( \epsilon \)) holds with parameter \( r_j > 0 \) on each \([j\epsilon^2, (j + 1)\epsilon^2]\) for all \( 0 \leq j \leq l - 1 \);

(iii) if \( \delta(t) \leq \delta_j \) on \([j\epsilon^2, (j + 1)\epsilon^2]\), for all \( 0 \leq j \leq l - 1 \), then it is \( \kappa_j > 0 \) noncollapsed on \([j\epsilon^2, (j + 1)\epsilon^2]\) for all scales less than \( \epsilon \), for all \( 0 \leq j \leq l - 1 \).

Then there exists a \( \kappa_j = \kappa_j(\kappa_{j-1}, r_{j-1}, \epsilon) > 0 \) and for any \( \bar{r} > 0 \), there exists \( \tilde{\delta}_j = \tilde{\delta}_j(\kappa_{j-1}, \bar{r}, \epsilon) > 0 \) such that any solution to Ricci flow with \( \delta(t) \)-cutoff surgeries on \([0, T')\) for some \( T' \in [l\epsilon^2, (l + 1)\epsilon^2] \) is \( \kappa_j \)-noncollapsed on \([(l - 1)\epsilon^2, T')\) for all scales less than \( \epsilon \), if

(a) it satisfies the canonical neighborhood assumption (with accuracy \( \epsilon \)) with parameter \( \bar{r} \) on \([l\epsilon^2, T')\);

(b) for each \( t \in [l\epsilon^2, T') \), on each connected components of the solution, there is a point \( x \) on it such that \( R(x, t) \leq \bar{r}^{-2} \);

(c) \( \delta(t) \leq \delta_j \) on \([j\epsilon^2, (j + 1)\epsilon^2]\), for all \( 0 \leq j \leq l - 1 \), and \( \delta(t) \leq \bar{r} \) on \([(l - 1)\epsilon^2, T')\).

**Proof.** Suppose \( R(\cdot, \cdot) \leq r_0^{-2} \) on \( P(x_0, t_0, r_0, -r_0^2) = \{(x', t') \mid x' \in B_{r_0}(x_0, r), t' \in [t_0 - r_0^2, t_0]\} \), we will estimate \( vol(B_{r_0}(x_0, r))/r_0^4 \) from below.

Step 1: In this step, we deal with the estimates on scales not too small comparing with \( \bar{r} \). We assume \( r_0 \geq \epsilon^{(l-1)} \), where \( C(\epsilon) \) is some fixed constant (to be determined later) depending only on \( \epsilon \). In this case, we adapt the proof of Lemma 5.5 in [5] as follows.

Since the surgeries occur in place where the curvature is bigger than \( \delta^{-2}\bar{r}^{-2} \), which is much larger than \( \bar{r}^{-2} \), we first modify the argument of Lemma 5.5 in
to show any $L$ geodesic $γ(τ), τ ∈ [0, τ]$ $(τ ≤ t₀ − (l − 1)ε²)$, starting from $(x₀, t₀)$ with reduced length $≤ ε^{-1}$, stays far away from the place where surgeries occur. More precisely, we claim that if some $γ(τ₀)$ is not far from some cap which is glued by surgery procedure at time $t = t₀ − τ₀$, then the reduced length of $γ$ defined by

$$\frac{1}{2} \sqrt{τ} \int_0^τ \sqrt{R(γ(τ), τ)} + |γ(τ)|^2 dτ$$

is $≥ 25ε^{-1}$.

This estimate for manifold case was established in (5.8) on page 238 of [5]. Let us recall the proof of this estimate for the manifold case given in [5]. Note that the place performed $δ-$cutoff surgery is deeply inside the horn under normalization and the parabolic region $P(x₀, t₀, r₀, r₀)$ is far from it by curvature estimates for canonical neighborhoods. Thus at the time $t = t₀ − τ₀$, the point $γ(τ₀)$ lies deeply inside a very long tube and the segment $γ(τ), τ ∈ [0, τ₀]$, tends to escape from the tube. If $γ(τ)$ escapes from the very long tube within short time $≤ CR(x₁, t₀ − τ₀)^{-1}$ from $τ₀$, where $C$ is some universal constant and $x₁$ is a point in the neck where surgery takes place, then $∫_0^τ |γ(τ)|^2 dτ$ contributes a big quantity to the above integral since the tube is quite long. However if $γ(τ)$ stays a while $≥ CR(x₁, t₀ − τ₀)^{-1}$ on the long tube, then $∫_0^τ Rdτ$ contributes a large quantity to the above integral, since for any $1 > ζ > 0$, we have the estimate

$$R(x, t) ≥ R(x₁, t₀ − τ₀) \frac{\text{Const.}}{τ − τ₀} − R(x₁, t₀ − τ₀)(t − t₀ + τ₀)$$

on $γ|_{[t₀ − τ₀, t₀]}$, when $δ$ is small enough and $γ(τ)$ stays not far from the cap.

All the above arguments of [5] still work in our present orbifold case except the verification of the last statement on the estimate of the scalar curvature on the tube. In [5], the proof of the above estimate on the scalar curvature on the tube was given as follows. Recale the solution with factor $R(x₁, t₀ − τ₀)$ around $(x₀, t₀ − τ₀)$. Since the necks in the manifold case of [5] are not collapsed, we can extract a convergent limit as $δ → ∞$. The limit, called standard solution, is rotationally symmetric, exists exactly on the time interval $[0, \frac{1}{r³}]$ and has curvature estimates $\frac{\text{Const.}}{τ − τ₀}$ at time $s$. But in the current orbifold case, at a priori, we do not know whether the necks in the canonical neighborhoods are collapsed or not. Our new argument is to use Hamilton’s canonical parametrization for (the part of) horn: $Φ : S³ × (−L, L)$ such that the surgery is taken place on $[0, 4]$, and there is finite group $Γ$ of global isometric actions of $S³$, such that $Φ$ is $Γ$ invariant, and $Φ : S³/Γ × (−L, L)$ is diffeomorphic to its image and each $S³/Γ$ is mapped to a constant mean curvature hypersurface. Moreover the pull back metric on $S³ × (−L, L)$ (after scaling) is very close to the standard cylinder. We perform a standard surgery on $S³ × (−L, L)$ by cutting open the neck and glue back a cap, denote the resulting space by $Y$. Clearly, we can require $Φ$ to be extended and defined on $Y$ to the space after surgery, and the pull back metric is close to the standard capped infinite cylinder. We pull back the solution also
to Y. Note that the gradient estimate in the canonical neighborhood assumption implies a curvature bound for the solutions. Then as $\delta \to 0$, we can apply the uniqueness theorem [4] to show that the solutions on $Y$ around point near the cap converge to a standard solution. So the above estimate on the scalar curvature also holds in our present case.

After proving that any $L$ geodesic of reduced length $< 25\varepsilon^{-1}$ does not touch the surgery region, one can apply the same argument of Lemma 5.5 in [5] of using Perelman's Jacobian comparison to bound $\text{vol}_{10}(B_{0}(x_{0}, r_{0}))/r_{0}^{4}$ from below by constant depending only on $\varepsilon, \kappa_{k-1}, \eta_{k-1}$ (see [5], pages 238-241, for the details).

Step 2: In this step, we deal with the estimates on scales less than $\varepsilon_{0}$. This case is easier in [5] because the space has no singularities and the canonical neighborhoods are not collapsed there. In our present orbifold case, at a priori, the canonical neighborhoods in our definitions may be sufficiently collapsed.

So we need a new argument.

Clearly, we may assume $R(x', t') = r_{0}$ for some point on $P(x_{0}, t_{0}, r_{0}, -r_{0}^{2}) = \{(x', t') \mid x' \in B_{0}((x_{0}, r), t') \in [0 - r_{0}^{2}, t_{0}]\}$, otherwise we enlarge $r_{0}$. Since $r_{0} \leq \frac{r_{0}}{\varepsilon}$, by the definition of canonical neighborhoods, we can choose $C(\varepsilon)$ large enough so that every point in $B_{t_{0}}(x_{0}, r_{0})$ has curvature $\geq \tilde{r}^{-2}$. In particular, the point $x_{0}$ at the time $t_{0}$ has a canonical neighborhood, which is a strong $\varepsilon$-neck or $\varepsilon$-cap. For both cases, the canonical neighborhood contain an $\varepsilon$-neck $N$ which is close to $(-\varepsilon^{-1}, \varepsilon^{-1}) \times (S^{3}/\Gamma)$. Clearly, in order to get the $\kappa$-noncollapsing, we only need to bound the order $|\Gamma|$ of the group $\Gamma$ from above.

Now we consider one of the boundaries $\partial N$ of $N$. Since the curvature is $\geq \tilde{r}^{-2}$ there, there is an $\varepsilon$-neck or $\varepsilon$-cap adjacent to $N$. If it is the $\varepsilon$-cap adjacent to $N$, we stop for this end and consider the other boundary of $N$. If it is a $\varepsilon$-neck adjacent (denoted by $N'$) to $N$, and $N'$ contains a point having curvature $\leq C(\varepsilon)^{2}\varepsilon^{-2}$, then we also stop. Otherwise, $N \cup N'$ form a longer (topological) neck, we consider the boundary of $N'$ and continue the argument. We do the same argument for the another boundary of $N$. Since there is a point $\bar{x}$ on space such that $R(\bar{x}, t_{0}) \leq \tilde{r}^{-2}$ by assumption (b), there must be an extension of one boundary of $N$ such that the final adjacent neck or cap having a point with curvature $\leq C(\varepsilon)^{2}\varepsilon^{-2}$. By canonical neighborhood assumption, the curvature at the final neck or cap are $\leq C(\varepsilon)^{2}\varepsilon^{-2}$. We conclude that there is a tube $T$ consisting of $\varepsilon$-necks such that $T$ contains the initial neck $N$ and another $\varepsilon$-neck $N_{1}$ where the curvatures are $\leq C(\varepsilon)^{2}\varepsilon^{-2}$. By step 1, we can bound

$$\frac{\text{vol}_{10}(N_{1})}{\varepsilon^{3}\text{diam}(N_{1})^{4}} \geq \frac{1}{C(\varepsilon, \kappa_{k-1}, \eta_{k-1})}$$

(4.1)

from below uniformly. By using Hamilton’s canonical parametrization $\Phi : S^{3} \times (A, B)$ to parametrize $T$, $\Gamma$ acts isometrically on the factor $S^{3}$ on the whole $S^{3} \times (A, B)$. This gives $|\Gamma|\text{vol}_{10}(N_{1}) \leq C(\varepsilon)\text{diam}(N_{1})^{4}$. By combining with (4.1), we get a uniform upper bound of $|\Gamma|$.

The proof of the theorem is completed.

$\square$
Theorem 4.6. Given a compact four-dimensional orbifold \((X, g)\) with positive isotropic curvature and with at most isolated singularities. Given any fixed small constant \(\varepsilon > 0\), one can find three non-increasing positive and continuous functions \(\delta(t), \tilde{r}(t)\) and \(\tilde{\kappa}(t)\) defined on whole \([0, +\infty)\) with the following properties. For arbitrarily given positive continuous function \(\delta(t) \leq \tilde{\delta}(t)\) on \([0, +\infty)\), the Ricci flow with \(\delta(t)\)-cutoff surgery, starting with \(g\), admits a solution satisfying the a priori assumption (with accuracy \(\varepsilon\) with \(r = \tilde{r}(t)\)) and \(\kappa\)-noncollapsing (with \(\kappa = \tilde{\kappa}(t)\)) on a maximal time interval \([0, T)\) with \(T < +\infty\) and becoming extinct at \(T\). Moreover, the solution is obtained by performing at most finite number of \(\delta\)-cutoff surgeries on \([0, T)\).

Proof. The pinching assumption is justified in Lemma 4.3. To justify the canonical neighborhood assumption, we can apply the same argument as in manifold case, because we have all ingredients we need to mimic the proof of Proposition 5.4 in [5]. We note the surgery does not occur on the place where the scalar curvature achieves its minimum. Then by applying the maximum principle to the scalar curvature equation \((\frac{\partial}{\partial t} - \triangle)R = 2|\text{Ric}|^2\), we conclude that the surgical solution must be extinct in finite time. To prove the finiteness of the number of surgeries, we need to check the \(\kappa\)-noncollapsing for the solution. In fact, the \(\kappa\)-noncollapsing follows from Lemma 4.3. Therefore, the proof of the theorem is completed.

\[
\square
\]

4.3 Recovering the topology

Proof. of Theorem 2.1

Consider a surgical solution, obtained by the previous theorem, to the Ricci flow with surgery on a maximal time interval \([0, T)\) with \(T < +\infty\). Now we can recover the topology of the initial orbifold as follows.

Suppose our surgeries times are \(0 < t_1 < t_2 < \cdots, t_k < T\). For a surgery time \(t_p\), after surgeries, denote by \(M_1^p, M_2^p, \cdots, M_{i_p}^p\) the all connected components either containing no points of \(\Omega_{p(0)}\) or having positive curvature operator. The rest connected components are denoted by \(N_1^p, \cdots, N_{i_p}^p\). Recall that our construction for the surgical solution is to stop the Ricci flow on those \(M_l^p\) for \(l = 1, \cdots, i_p\) and to continue the Ricci flow on \(N_l^p\) for \(l = 1, \cdots, i_p\). Note that all connected components at time \(T^-\) either contain no points of \(\Omega_{p(0)}\) or have positive curvature operator. We denote them by \(M_{i_p+1}, M_{i_p+2}, \cdots, M_{i_p+i_p}^p\), they are actually \(N_1^p, \cdots, N_{i_p}^p\). We collect all these \(M_l^p\)’s in a set \(S = \{M_1^p, \cdots, M_{i_p+i_p}^p\}\). For each \(M_{i_l}^p\), we will mark a finite number of points \(P_{i_l}^p\) \(l = 1, 2, \cdots, i_p\), in the following inductive way.

At first surgery time \(t_1\), we perform a cut-off surgery along a \(\delta\)-horn \(H\), i.e. we cut open \(\delta\)-horn along a neck \(N\) and glue back a cap or orbifold cap to the finite part of the horn connected to \(\Omega_p\). Remember we also glue back a cap or orbifold cap to the infinite part of the horn (so called horn-shape end). We denote the tips of these two caps by \(P\) and \(\bar{P}\), denote these two caps...
by $C^p$ and $C^b$ respectively. Through the neck $N$ we cut open, the surgery procedure establishes a diffeomorphism $\varphi_{p}$ from $\partial C^p$ to $\partial C^b$. Let $S_p$ and $S_b$ be the unit tangent spheres at $P$ and $\bar{P}$, then $\varphi_{p}$ induces an isotopic class $\varphi_{p}$ of diffeomorphism from $S_p$ to $S_b$. We assign the pairs $(P, S_p)$ and $(\bar{P}, S_b)$ to the manifolds or orbifolds where they are located. Inductively, at surgery time $t_p$, for each $N^{p-1}$ with some points already been marked by the previous steps, we leave these marked points alone, and add new points produced by performing surgeries at $t_p$ on $N^{p-1}$. Note that the previous marked points may be separated to lie in different connected components after surgeries. Once a component $M_j$ is terminated at a surgery time $t_j$, then there is no more points assigned to it in any later surgery times. We collect all these marked points $(P, S_p), (\bar{P}, S_b)$ and isotopic classes $\varphi_{p}$ together.

Now we investigate the topology of each $M_j \in \mathcal{S}$. We know at time $t_j$, $M_j$ is either diffeomorphic to a spherical orbifold $S^4/\Gamma$ with at most isolated singularities, or it is covered by $\varepsilon$ – necks and $\varepsilon$ – caps. Now we consider the latter case.

If $M_j$ contains no caps, then $M_j$ is diffeomorphic to smooth manifold $S^3/\Gamma \times S^1$ or $S^3/\Gamma \times S^1$.

If $M_j$ contains caps, then $M_j$ is diffeomorphic to either smooth manifold $S^4, \mathbb{R}P^4, C^j \cup f C^j$, or one of the orbifolds $C^j \cup f C^j, C^j \cup f C^j, S^4/(x, \pm x^i)$, $S^4/(x, \pm x^i)\#\mathbb{R}P^4, S^4/(x, \pm x^i)\#S^4/(x, \pm x^i)$.

So we conclude that each $M_j$ is diffeomorphic to a connected sum of at most two spherical orbifolds $S^4/\Gamma_i^1$ and $S^4/\Gamma_i^2$. For each $(P, S_p), (\bar{P}, S_b)$ and $\varphi_{p}$, we know reversing the surgery procedure is to do connected sum by removing the pair of points $P, \bar{P}$ and using $\varphi_{p}$ to identify the boundaries. Therefore, our original orbifold is diffeomorphic to the connected sum of spherical orbifolds $S^4/\Gamma$ with at most isolated singularities. This completes the proof of Theorem 2.1.

5 Proof of Main Theorem

The main purpose of this section is to deduce the Main Theorem from Theorem 2.1 We need several lemmas on the group actions on the sphere $S^n$.

**Lemma 5.1.** Let $G \subset SO(2n+1)(n \geq 2)$ be a finite subgroup such that each nontrivial element in $G$ has exactly one eigenvalue equal to 1. Then there is a common nonzero vector $0 \neq v \in \mathbb{R}^{2n+1}$ such that for all $g \in G$ we have $g(v) = v$.

**Proof.** The idea of the proof is similar to the classification of fixed point free finite subgroups of the isometry group of $S^{2n+1}$ in [23]. We divide our argument into two cases.

Case i): $|G|$ is even. In this case, there is an element of order 2 by Cauchy theorem. We denote this element by $\sigma$. We claim that $\sigma$ is the unique element
of order 2 in $G$. Indeed, suppose $\sigma'$ is another distinct order 2 element. Note that, by our assumption, $\sigma$ and $\sigma'$ must have one eigenvalue equal to 1 and $2n$ eigenvalues equal to $-1$. Let $E_1$ and $E_2$ be the eigenspaces with eigenvalue $-1$ of $\sigma$ and $\sigma'$ respectively. Clearly $\sigma \sigma'^{-1} = 1$ on $E_1 \cap E_2$. Since $n \geq 2$, the intersection $E_1 \cap E_2$ has dimension $\geq 2n - 1 \geq 3$, this implies that $\sigma = \sigma'$ on the whole space. This is a contradiction.

By the uniqueness of $\sigma$, we know that $g^{-1} \sigma g = \sigma$ for any $g \in G$. Suppose $\sigma(v) = v$ for $|v| = 1$, then $\sigma g(v) = g(v)$. Hence $g(v) = v$ or $g(v) = -v$. We claim that $g(v) = -v$ cannot happen. The reason is as follows. Let $g(u) = u$ for $|u| = 1$, then $g^2(u) = u$. By combining with $g^2(v) = v$, we know that either $g^2 = 1$ or $v = \pm u$. If $g(v) = -v$, then $v$ cannot be $\pm u$, so $g$ has order 2, and equal to $\sigma$ by the uniqueness of order 2 element, this contradicts with $\sigma(v) = v$. So we have showed that $g(v) = v$ for any $g \in G$.

Case ii): $|G|$ is odd. First, we show that every subgroup of order $p^2$ ($p$ is a prime number) of $G$ is cyclic. Namely, we will show that $G$ satisfies the $p^2$ condition.

Indeed, suppose $H$ is a noncyclic subgroup of order $p^2$ for some prime number $p$. Since a group of order $p^2$ with $p$ prime must be abelian, we can apply the same argument as case (i) to conclude that there is a unit vector $v$ fixed by the whole group. Let $W \equiv \mathbb{R}^{2n}$ be the orthogonal complement of $v$ in $\mathbb{R}^{2n+1}$. Then $H$ induces a fixed point free action on the unit sphere $S^{2n-1}$ of $W$. So for any $v' \in S^{2n-1}$, we have $0 = \sum_{g \in H} g(v')$. On the other hand, since $G$ is abelian and noncyclic, we conclude that each nontrivial element has order exactly $p$, the intersection of any two distinct order $p$ groups contains only the identity. Let $H_i, i = 1, \cdots, m, (m \geq 2)$ be the subgroups in $H$ of order $p$, then for any $v' \in S^{2n-1}$,

$$0 = \sum_{g \in G} g(v') = \sum_{i=1}^{m} \sum_{g \in H_i} g(v') = (m - 1)v' = -(m - 1)v',$$

where we have used the fact $\sum_{g \in H_i} g v' = 0$ since $H_i$ also acts freely on $S^{2n-1}$. The contradiction shows that $H$ is cyclic.

The fact that $G$ satisfies $p^2$ condition implies that every Sylow subgroup of $G$ is cyclic (see Theorem 5.3.2 in [23], note that since $|G|$ is odd, so must be $p$). By Burnside theorem (see Theorem 5.4.1 in [23]), once we know that every Sylow subgroup of $G$ is cyclic, then $G$ is generated by two elements $A$ and $B$ with defining relations

$$A^m = B^n = 1, \quad BAB^{-1} = A', \quad |G| = mn,$$

$$((r - 1)n, m) = 1, \quad r^n \equiv 1 (mod \ m).$$

Let $A(v) = v$ for $|v| = 1$. We will show that $B(v) = v$. Indeed, by the relation $BAB^{-1} = A'$, we have $AB^{-1}(v) = B^{-1}(v)$. This implies $B^{-1}(v) = v$ or $B^{-1}(v) = -v$. $B^{-1}(v) = -v$ will not happen, because it implies $B^{-2} = 1$ by the argument in case i). This will imply the order of the group is even, which is a contradiction with our assumption. So $v$ is fixed by the whole group $G$.

\[\square\]
Lemma 5.2. Let $G \subset O(2n + 1)(n \geq 2)$ be a finite group of orthogonal matrices such that each nontrivial element in $G$ has at most one eigenvalue equal to 1. Then there is a finite group $G' \subset SO(2n)$ acting freely on the sphere $S^{2n-1}$ and a character $\chi : G' \to \{\pm 1\}$ such that after conjugation, the group $G = \left\{ \begin{pmatrix} \chi(g) & 0 \\ 0 & g \end{pmatrix} : g \in G' \right\}$.

Proof. Let $G_0 = G \cap SO(2n + 1)$. If $G_0 = G$, then from Lemma 5.1 we are done by choosing $G' = \text{the restriction of } G \text{ on the orthogonal complement of } v \text{ (the common unit vector fixed by } G)$ and $\chi \equiv 1$. If $G_0 \neq G$, then $G_0$ is an index 2 normal subgroup of $G$. Since an element of $G_0$ must has 1 as its eigenvalue, it has exactly one eigenvalue 1 by our assumption. By Lemma 5.1 there is a common unit vector $v$ fixed by $G_0$. For any $g \in G \setminus G_0$, we claim $g(v) = -v$. The argument is as follows.

Since $g^2 \in G_0$, we have $g^2(v) = v$. Let $E = \text{span}[v, g(v)]$. We will show $\dim E = 1$. Indeed, suppose $\dim E = 2$. Since $g(v+g(v)) = v+g(v)$ and $g(v-g(v)) = -(v-g(v))$, $E$ is an invariant subspace of $g$, $\dim E^\perp$ is odd and $\det(g|_{E^\perp}) = 1$. So $g$ has another fixed nonzero vector in $E^\perp$. This contradiction shows that $g(v) = v$ or $g(v) = -v$. If $g(v) = v$, then $\det(g|_{v^\perp}) = -1$, and hence $g$ must have another fixed vector in $\{v\}^\perp$ since $\dim\{v\}^\perp$ is even, this again contradicts with the assumption that $g$ has at most one eigenvalue 1. This proves our claim.

Next, we show that $G$ acts freely on the unit sphere of $\{v\}^\perp$. For this, we only need to check for any $g \in G \setminus G_0$, $g$ has no nonzero fixed vector in $\{v\}^\perp$. But if this is not true, we have $g^2 = 1$, this implies that $g$ has one eigenvalue 1 (by assumption) and $2n$ eigenvalues $-1$, which contradicts with $\det(g) = -1$. To finish the proof, we only have to take $G' = \text{the restriction of } G \text{ on } \{v\}^\perp$ and $\chi$ is the character which takes value 1 on $G_0$ and -1 otherwise. □

In the following, we prove Main Theorem by using Theorem 2.1 and Lemmas 5.1 and 5.2.

Proof. of Main Theorem. With the help of the above lemmas, we can now describe the structure of the spherical orbifolds $S^4/\Gamma$ appearing in Theorem 2.1. Since the resulting quotient space $S^4/\Gamma$ has at most a pair of antipodal fixed points, so the group $\Gamma$ satisfies the assumptions in Lemma 5.2. There are three cases for $S^4/\Gamma$. The first case is $\Gamma$ acts on $S^4$ freely, the resulting space is a smooth manifold diffeomorphic to $S^4$ or $\mathbb{R}P^4$. The second case is $\Gamma \neq \{1\}$ and $\Gamma \subset SO(5)$. Assume $S^4 \subset \mathbb{R}^5$ has equation $x_1^2 + x_2^2 + \cdots + x_5^2 = 1$. By Lemma 5.2, we may assume the north pole $P = (0, 0, 0, 0, 1)$ and south pole $-P = (0, 0, 0, 0, -1)$ of $S^4$ are the fixed points of $\Gamma$. Let $S^3 = S^4 \cap \{x_5 = 0\}$, the group action fixes $x_5$ coordinate, so $S^4/\Gamma \setminus \{P, -P\}$ is diffeomorphic to $S^3/\Gamma \times (0, 1)$. The third case is $\Gamma_0 = \Gamma \cap SO(5) \subseteq \Gamma$. By Lemma 5.2, we may also assume the north pole $P$ and south pole $-P$ of $S^4$ are the fixed points of $\Gamma_0$. The group $\Gamma$ can act on $S^3 \times \mathbb{R}$ in a natural way, and we can equip a metric on $S^4/\Gamma \setminus \{P\}$, which is locally isometric to $S^3 \times \mathbb{R}$ and the manifold has only one end which is isometric to $S^3/\Gamma_0 \times [0, \infty)$.

Let $X_1, \cdots, X_i$ be the orbifolds appearing in Theorem 2.1. The orbifold connected sum procedure can be described in two steps, the first step is to resolve all
singularities of $X_1, \cdots, X_l$ which appear pairwise in the surgery procedures of the Ricci flow by orbifold connected sums, the resulting spaces consists of finite number of smooth closed manifolds, denoted by $Y_1, \cdots, Y_l$. The next step is to perform the orbifold connected sums among these manifolds $Y_1, \cdots, Y_l$ and a finite number of $S^4$, $\mathbb{RP}^4$. Now we investigate the topology of each components $Y_i$ in the first step mentioned above. We remove all singularities from all $X_j$, the orbifolds falling in the second case give us necks $S^3/\Gamma \times (0, 1)$, the orbifolds in the third case give us caps of the form $C^\sigma_{1,0}$. Since in the first step, each end of one such neck has to be joined with a cap, or another neck, producing a longer cap or neck; the end of each cap has to be joined with a neck or another cap. So each $Y_i$ is diffeomorphic to either $S^4$, $\mathbb{RP}^4$ or the manifold (denoted by $S^3/\Gamma \times [0, 1]/f$ temporarily) obtained by gluing the boundaries of $S^3/\Gamma \times [0, 1]$ by utilizing a diffeomorphism $f : S^3/\Gamma \times [0, 1] \to S^3/\Gamma \times [1]$, or the manifold $C^\sigma_{1,0} \cup f C^\sigma_{1,0}$ obtained by gluing the boundaries $\partial C^\sigma_{1,0}$ and $\partial C^\sigma_{1,0}$ by a diffeomorphism $f''$. It is known that two spherical three space forms are isometric if they are diffeomorphic. So in the manifold $C^\sigma_{1,0} \cup f C^\sigma_{1,0}$, the group $\Gamma$ and $\Gamma'$ are conjugate (in $O(4)$). After a conjugation, we have $\Gamma = \Gamma'$, and assume $C^\sigma_{1,0} \cup f C^\sigma_{1,0} = C^\sigma_{1,0} \cup f C^\sigma_{1,0}$. Since the diffeomorphism types of $S^3/\Gamma \times [0, 1]/f$ and $C^\sigma_{1,0} \cup f C^\sigma_{1,0}$ remain unchanged if we deform $f$ and $f'$ isotopically. Moreover, by [15], the diffeomorphisms $f, f' \in Diff(S^3/\Gamma)$ are isotopic to isometries $I_f, I_f' \in ISO(S^3/\Gamma)$. So if we equip $S^3/\Gamma \times [0, 1]$ with the product standard metric, the induced metric on $S^3/\Gamma \times [0, 1]/I_f$ is locally isometric to $S^3 \times \mathbb{R}$. Similarly, if we equip $C^\sigma_{1,0}$ and $C^\sigma_{1,0}$ the standard metric which is locally isometric to $S^3 \times \mathbb{R}$ and the end is isometric to product $S^3/\Gamma \times [0, 1)$, then the induced metric on $C^\sigma_{1,0} \cup f I_f C^\sigma_{1,0}$ is locally isometric to $S^3/\Gamma \times \mathbb{R}$. In both cases, the universal covers are $S^3 \times \mathbb{R}$. Now we have shown that each $Y_i$ is diffeomorphic to either $S^4$, $\mathbb{RP}^4$ or $S^3 \times \mathbb{R}/G$, where $G$ is a fixed point free cocompact discrete subgroup of the isometries of standard metric on $S^3 \times \mathbb{R}$. Therefore, the manifold $M$ is diffeomorphic to an orbifold connected sum of $S^4$, or $\mathbb{RP}^4$ or $S^3 \times \mathbb{R}/G$. Note that doing orbifold connected sum through two embedded 3-spheres on a connected smooth manifold is equivalent to doing the usual connected sum of this manifold with $S^3 \times S^1$ or $S^1 \times S^1$, also, doing orbifold connected sum between two connected smooth manifolds is just doing the usual connected sum by suitably choosing the orientations of the embedded 3-spheres. Therefore, we have shown the manifold $M$ is diffeomorphic to the usual connected sum of $S^4$, or $\mathbb{RP}^4$ or $S^3 \times \mathbb{R}/G$.

Corollary 5.3. A compact 4-orbifold with at most isolated singularities with positive isotropic curvature is diffeomorphic to the connected sum $\# (S^3 \times \mathbb{R}/G_i) \# (S^4/\Gamma_j)$, where $G_i$ and $\Gamma_j$ are standard group actions, the connected sum is in the usual sense.

**Proof.** By using the same proof of the Main Theorem, we only need to consider those components which are diffeomorphic to either $C^\sigma_{1,0} \cup I_f C^\sigma_{1,0}$, or $C^\sigma_{1,0} \cup I_f C^\sigma_{1,0}$. Note that by [15], $f$ is isotopic to an isometry $f'$ of $S^3/\Gamma$, which can be naturally extended to a diffeomorphism of $C^\sigma_{1,0}$ to itself. This gives a diffeomorphism from
Appendix

Let \( \varepsilon \) be a positive constant. We call an open subset \( N \subset X \) in an metric space \( \text{GH }\varepsilon\text{-neck of radius } r \) if \( r^{-1}N \) is homeomorphic and Gromov-Hausdorff \( \varepsilon \)-close to a neck \( S \times I \) where \( S \) is some Alexandrov space with nonnegative curvature and without boundary and \( \text{diam}(S) \leq 1/\varepsilon \) and \( I = (-\varepsilon^{-1}, \varepsilon^{-1}) \).

**Proposition 6.1.** There exists a constant \( \varepsilon_0 = \varepsilon_0(n) > 0 \) such that for any complete noncompact \( n \)-dimensional intrinsic Alexandrov space \( X \) with nonnegative curvature, there is a positive constant \( r_0 > 0 \) and a compact set \( K \subset X \) such that any \( \text{GH }\varepsilon\text{-neck of radius } r \leq r_0 \) on \( X \) with \( \varepsilon \leq \varepsilon_0 \) must be contained in \( K \) entirely.

**Proof.** When the space is smooth manifold, and the topology defining the \( \varepsilon \)-neck is in \( C^{1,1} \), the proof is given by [5]. Now we modify the arguments to the present situation, the key observation is that we essentially used only triangle comparison in [5]. Here we include the proof for completeness.

We argue by contradiction. Suppose there exists a sequence of positive constants \( \varepsilon^a \to 0 \) and a sequence of \( n \)-dimensional complete noncompact pointed Alexandrov space \( (X^a, P^a) \) with nonnegative curvature such that for each fixed \( \alpha \), there exists a sequence of \( \text{GH }\varepsilon^a\text{-necks } N_k \) of radius \( r_k \leq 1/k \) on \( X^a \), and \( N_k \subset X \setminus B(P^a, k) \). Recall that by the definition of Gromov-Hausdorff distance, there is metric space \( Z_k \) containing isometric embeddings \( s \) of \( r_k^{-1}N_k \) and \( S \times I \) such that \( S \times I \subset B(s(r_kN_k)) \) and \( r_k^{-1}N_k \subset B(s(S \times I)) \). Let \( P_k \in r_k^{-1}N_k \) be a point having distance \( \leq \varepsilon^a \) with \( S \times \{0\} \) (in \( Z_k \)). Then we have \( d(P^a, P_k) \to \infty \) as \( k \to \infty \).

Let \( \alpha \) be fixed and sufficiently large. Connecting each \( P_k \) to \( P^a \) by a minimizing geodesic \( \gamma_k \), passing to subsequence, we may assume the angle \( \theta_{kl} \) between geodesic \( \gamma_k \) and \( \gamma_l \) at \( P^a \) is very small and tends to zero as \( k, l \to +\infty \), and the length of \( \gamma_{k+1} \) is much bigger than the length of \( \gamma_k \). Let us connect \( P_k \) to \( P_{l} \) by a minimizing geodesic \( \eta_{kl} \).

For any three points \( A, B, C \subset X \), we use \( \bar{\Delta}ABC \) to denote corresponding triangle in plane \( \mathbb{P} \) with \( d(A, B) = |\bar{A}\bar{B}|, d(A, C) = |\bar{A}\bar{C}|, d(B, C) = |\bar{B}\bar{C}| \), and we also use \( \bar{\Delta}ABC \) to denote the angle of \( \bar{\Delta}ABC \) at \( \bar{B} \).

Clearly, \( \bar{Z}P_k\bar{P}_k\bar{P}_l \) is close to \( \pi \) by comparison. Let \( P'_k \in \gamma_k \cap \partial N_k \) and \( P'' \in \eta_{kl} \cap \partial N_k \) then it is clear for any point \( x \in \partial N_k \), we have either \( \bar{Z}P_k\bar{P}_k\bar{x} \) is small and \( \bar{Z}P_k\bar{P}_k\bar{x} \) is close to \( \pi \), or \( \bar{Z}P_k\bar{P}_l\bar{x} \) is close to \( \pi \) and \( \bar{Z}P_k\bar{P}_l\bar{x} \) is small. This depends on \( \bar{x} \) lies which connected component of \( \partial N_k \).

By using the above facts and triangle comparison (see [5]), we can show that as \( k \) large enough, each minimizing geodesic \( \gamma \) with \( l > k \), connecting \( P^a \) to \( P_l \), must go through the whole \( N_k \).

Hence by taking a limit, we get a geodesic ray \( \gamma \) emanating from \( P \) which passes through all the necks \( N_k \), \( k = 1, 2, \cdots \), except a finite number of them. Throwing these finite number of necks, we may assume \( \gamma \) passes through all

\[ S^1/\Gamma, \partial \} \text{ or } S^4/\Gamma \text{ to } C_T \cup f \cdot C_T, \text{ or } C_T \cup f \cdot C_T. \]
necks $N_k, k = 1, 2, \cdots$. Denote the center sphere of $N_k$ by $S_k$, and their intersection points with $\gamma$ by $p_k \in S_k \cap \gamma$, for $k = 1, 2, \cdots$.

Take a sequence points $\gamma(m)$ with $m = 1, 2, \cdots$. For each fixed neck $N_k$, arbitrarily choose a point $q_k \in N_k$ near the center sphere $S_k$, draw a geodesic segment $\gamma^{km}$ from $q_k$ to $\gamma(m)$. Now we can show by triangle comparison that for any fixed neck $N_l$ with $l > k$, $\gamma^{km}$ will pass through $N_l$ for all sufficiently large $m$.

For any $s > 0$, choose two points $\hat{p}_k$ on $\overline{p_k \gamma(m)} \subset \gamma$ and $\hat{q}_k$ on $\overline{q_k \gamma(m)} \subset \gamma^{km}$ with $d(p_k, \hat{p}_k) = d(q_k, \hat{q}_k) = s$. By Toponogov comparison theorem, we have

$$\lim_{m \to \infty} \frac{d(\hat{p}_k, \hat{q}_k)}{d(p_k, q_k)} \geq 1.$$ 

Letting $m \to \infty$, we see that $\gamma^{km}$ has a convergent subsequence whose limit $\gamma^k$ is a geodesic ray passing through all $N_l$ with $l > k$. Denote by $p_l = \gamma(t_l), j = 1, 2, \cdots$. From the above computation, we deduce that

$$d(p_k, q_k) \leq d(\gamma(t_k + s), \gamma^k(s)).$$

for all $s > 0$.

Let $\varphi(x) = \lim_{t \to \infty}(t - d(x, \gamma(t)))$ be the Busemann function constructed from the ray $\gamma$. By the definition of Busemann function $\varphi$ associated to the ray $\gamma$, we see that $\varphi(\gamma^k(s_1)) - \varphi(\gamma^k(s_2)) = s_1 - s_2$ for any $s_1, s_2 \geq 0$. Consequently, by investigating the value of $\varphi$ on $\partial N_l$ and linearity of $\varphi \mid_{\gamma}$, we know for each $l > k$, we have $\gamma^k(t_l - t_k) \in \varphi^{-1}(\varphi(p_l)) \cap N_l$. This implies that the diameter of $\varphi^{-1}(\varphi(p_l)) \cap N_l$ is not greater than the diameter of $\varphi^{-1}(\varphi(p_l)) \cap N_l$ for any $l > k$, which is a contradiction as $l$ much larger than $k$. The proposition is proved.

\[\square\]

**Remark 6.2.** Without introducing a compact set $K$, the conclusion of Proposition 6.1 may not be true. The counter examples can be given by cones with small aperture.

**References**

[1] Brendle, S., and Schoen, R., *Classification of manifolds with weakly 1/4-pinched curvature*, Acta Math. 200 (2008), no. 1, 1-13.

[2] Böhm, C., and Wilking, B. *Manifolds with positive curvature operator are space forms*, Ann. of Math. (2) 167(2008), no.3, 1079-1097.

[3] Cao, H. D. and Zhu, X. P., *A complete proof of Poincare and geometrization conjectures–application of Hamilton-Perelman theory of Ricci flow*, Asian J. math. 102 (2006), 165-492.

[4] Chen, B. L. and Zhu, X. P., *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Diff. Geom. 74 (2006), 119-154.
[5] Chen, B. L. and Zhu, X. P., Ricci flow with surgery on four-manifolds with positive isotropic curvature, J. Diff. Geom. 74 (2006), 177-264.

[6] De Turck, D., Deforming metrics in the direction of their Ricci tensors J.Diff. Geom. 18 (1983), 157-162.

[7] Fraser, Ailana M., Fundamental groups of manifolds with positive isotropic curvature, Ann. of Math. (2) 158 (2003), no. 1, 345-354.

[8] Fraser, Ailana and Wolfson Jon, The fundamental group of manifolds of positive isotropic curvature and surface groups, Duke Math. J. 133 (2006), no. 2, 325-334.

[9] Hamilton, R. S., Three manifolds with positive Ricci curvature , J. Diff. Geom. 17 (1982), 255-306.

[10] Hamilton, R. S., Four-manifolds with positive curvature operator, J. Differential Geom. 24 (1986), 153-179.

[11] Hamilton, R. S., Four manifolds with positive isotropic curvature, Commu. in Analysis and Geometry,5(1997),1-92. (or see, Collected Papers on Ricci Flow, Edited by H.D.Cao, B.Chow, S.C.Chu and S.T.Yau, International Press 2002).

[12] Hamilton,R.S., Three-orbifolds with positive Ricci curvature, 521-524 Collected Papers on Ricci Flow, Edited by H.D.Cao, B.Chow, S.C.Chu and S.T.Yau, International Press 2002).

[13] Izeki, H., Limit sets of Kleinian groups and conformally flat Riemannian manifolds, Invent. Math. 122 (1995), 603-625.

[14] Lu, P., A compactness property for solutions of the Ricci flow on orbifolds American Journal of Mathematics, 123(2001),1103-1134.

[15] Mccullough, Darryl, Isometries of elliptic 3-manifolds, J. London Math. Soc. (2) 65(2002), no 1, 167-182.

[16] Micallef, M. and Moore, J. D., Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, Ann. of Math.(2) 127(1988)199-227.

[17] Micallef, M. and Wang, M., Metrics with nonnegative isotropic curvatures, Duke Math. J. 72 (1993), no. 3, 649-672.

[18] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.

[19] Perelman, G., Ricci flow with surgery on three manifolds, arXiv: math. DG/0303109.
[20] Schoen, R. and Yau, S.T., *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. 92 (1988), 47-71.

[21] Schoen, R., *Open problems proposed in Pacific Northwest Geometry Seminar, 2007-fall* (at Univ. of Oregon).

[22] Thurston, W. *Geometry and topology of three manifolds*, Lecture notes, Princeton University, 1979.

[23] Wolf, J., *Spaces of constant curvature*, Wilmington: Publish or Perish, 1984.