EULER-LAGRANGE EQUATIONS FOR COMPOSITION FUNCTIONALS IN CALCULUS OF VARIATIONS ON TIME SCALES

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Abstract. In this paper we consider the problem of the calculus of variations for a functional which is the composition of a certain scalar function \( H \) with the delta integral of a vector valued field \( f \), i.e., of the form \( H \left( \int_a^b f(t, x(t), x(t))dt \right) \). Euler-Lagrange equations, natural boundary conditions for such problems as well as a necessary optimality condition for isoperimetric problems, on a general time scale, are given. A number of corollaries are obtained, and several examples illustrating the new results are discussed in detail.

1. Introduction. The calculus on time scales was introduced by Bernd Aulbach and Stefan Hilger in 1988 [6]. The new theory bridges the divide and extends the traditional areas of continuous and discrete analysis and the various dialects of \( q \)-calculus [14] into a single theory [11, 12, 20]. The calculus of variations on time scales was born with the works [2, 8, 18] and has interesting applications in Economics [3, 4, 5, 15, 26]. Currently, several researchers are getting interested in the new theory and contributing to its development (see, e.g., [7, 9, 10, 16, 21, 22, 23, 24, 25]).

The present work is dedicated to the study of general (non-classical) problems of calculus of variations on an arbitrary time scale \( T \). As a particular case, by choosing \( T = \mathbb{R} \), one gets the generalized calculus of variations [13] with functionals of the form

\[
H \left( \int_a^b f(t, x(t), x'(t))dt \right),
\]

where \( f \) has \( n \) components and \( H \) has \( n \) independent variables. Cases of calculus of variations as these appear in practical applications (see [13] and the references given therein) but cannot be solved using the classical theory. Therefore, an extension of this theory is needed.

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The paper is organized as follows. In Section 2, some preliminaries on time scales are presented. Our results are given in Section 3 and Section 4. We begin Section 3 by formulating the general (non-classical) problem of calculus of variations \((1)\) on an arbitrary time scale. We obtain a general formula for the Euler-Lagrange equations and natural boundary conditions for the general problem (Theorem 3.2), which are then applied to the product (Corollary 3.4) and the quotient (Corollary 3.7). In Section 4 we prove a necessary optimality condition for the general isoperimetric problem (Theorem 4.3 and Theorem 4.5). Throughout the paper several examples illustrating the new results are discussed in detail.

2. Preliminaries. The following definitions and theorems will serve as a short introduction to the calculus of time scales; they can be found in \([11, 12]\).

A nonempty closed subset of \(\mathbb{R}\) is called a time scale and it is denoted by \(\mathbb{T}\). The real numbers (\(\mathbb{R}\)), the integers (\(\mathbb{Z}\)), the natural numbers (\(\mathbb{N}\)), the \(h\)-numbers (\(h\mathbb{Z} := \{hz|z \in \mathbb{Z}\}\), where \(h > 0\) is a fixed real number), and the \(q\)-numbers (\(q^{\mathbb{N}_0} := \{q^k|k \in \mathbb{N}_0\}\), where \(q > 1\) is a fixed real number) are examples of time scales, as are \([0, \frac{1}{2}, 1]\), \([2, 3] \cup \mathbb{N}\), and \([-1, 1] \cup [2, 3]\), where \([-1, 1]\) and \([2, 3]\) are real number intervals. We assume that a time scale \(\mathbb{T}\) has the topology that it inherits from the real numbers with the standard topology.

Definition 2.1. For \(t \in \mathbb{T}\) we define he forward jump operator \(\sigma: \mathbb{T} \to \mathbb{T}\) by

\[
\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \quad \text{for all } t \in \mathbb{T},
\]

while the backward jump operator \(\rho: \mathbb{T} \to \mathbb{T}\) is defined by

\[
\rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \quad \text{for all } t \in \mathbb{T}.
\]

In this definition we consider \(\sigma(M) = M\) if \(\mathbb{T}\) has a maximum \(M\) and \(\rho(m) = m\) if \(\mathbb{T}\) has a minimum \(m\).

A point \(t \in \mathbb{T}\) is called right-dense, right-scattered, left-dense and left-scattered if \(\sigma(t) = t\), \(\sigma(t) > t\), \(\rho(t) = t\) and \(\rho(t) < t\), respectively. Points that are simultaneously right-scattered and left-scattered are called isolated. Points that are simultaneously right-dense and left-dense are called dense.

The graininess function \(\mu: \mathbb{T} \to [0, \infty)\) is defined by

\[
\mu(t) = \sigma(t) - t, \quad \text{for all } t \in \mathbb{T}.
\]

Example 2.2. If \(\mathbb{T} = \mathbb{R}\), then \(\sigma(t) = \rho(t) = t\) and \(\mu(t) = 0\). If \(\mathbb{T} = \mathbb{Z}\), then \(\sigma(t) = t + 1\), \(\rho(t) = t - 1\), and \(\mu(t) = 1\). On the other hand, if \(\mathbb{T} = q^{\mathbb{N}_0}\), where \(q > 1\) is a fixed real number, then we have \(\sigma(t) = qt\), \(\rho(t) = q^{-1}t\), and \(\mu(t) = (q - 1)t\).

Definition 2.3. A time scale \(\mathbb{T}\) is called regular if the following two conditions are satisfied:

(i) \(\sigma(\rho(t)) = t\), for all \(t \in \mathbb{T}\); and

(ii) \(\rho(\sigma(t)) = t\), for all \(t \in \mathbb{T}\).

Following \([11]\), let us define

\[
\mathbb{T}^\kappa = \begin{cases} 
\mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}) & \text{if } \sup \mathbb{T} < \infty \\
\mathbb{T} & \text{if } \sup \mathbb{T} = \infty.
\end{cases}
\]

Definition 2.4. We say that a function \(f: \mathbb{T} \to \mathbb{R}\) is delta differentiable at \(t \in \mathbb{T}^\kappa\) if there exists a number \(f^\Delta(t)\) such that for all \(\varepsilon > 0\) there is a neighborhood \(U\) of \(t\) (i.e., \(U = (t - \delta, t + \delta) \cap \mathbb{T}\) for some \(\delta > 0\)) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.
\]
We call \( f^\Delta(t) \) the delta derivative of \( f \) at \( t \) and \( f \) is said delta differentiable on \( T^\kappa \) provided \( f^\Delta(t) \) exists for all \( t \in T^\kappa \).

**Remark 2.5.** If \( t \in T \setminus T^\kappa \), then \( f^\Delta(t) \) is not uniquely defined, since for such a point \( t \), small neighborhoods \( U \) of \( t \) consist only of \( t \) and, besides, we have \( \sigma(t) = t \). For this reason, maximal left-scattered points are omitted in Definition 2.4.

Note that in right-dense points \( f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{\kappa(t)} \), provided \( f \) exists, and in right-scattered points \( f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \), provided \( f \) is continuous at \( t \).

**Example 2.6.** If \( T = \mathbb{R} \), then \( f^\Delta(t) = f'(t) \), i.e., the delta derivative coincides with the usual one. If \( T = \mathbb{Z} \), then \( f^\Delta(t) = \Delta f(t) = f(t+1) - f(t) \). If \( T = q^{10} \), \( q > 1 \), then \( f^\Delta(t) = \frac{(qt) - f(t)}{(q-1)t} \), i.e., we get the usual derivative of quantum calculus [19].

A function \( f : T \to \mathbb{R} \) is called rd-continuous if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by \( C_{rd} \) and the set of all delta differentiable functions with rd-continuous derivative by \( C_{rd}^1 \).

Now we introduce the concept of integral for a function \( f : T \to \mathbb{R} \).

Let \( a, b \in T \) with \( a \leq b \). We define the closed interval \([a, b]\) in \( T \) by

\[
[a, b] := \{ t \in T : a \leq t \leq b \}.
\]

Open intervals and half-open intervals in \( T \) are defined accordingly. In what follows all intervals will be time scale intervals.

It is known that rd-continuous function possess an antiderivative, i.e., there exists a function \( F \) with \( F^\Delta = f \), and in this case the delta integral is defined by

\[
\int_a^b f(t) \Delta t = F(b) - F(a)
\]

for all \( a, b \in T \).

The delta integral has the following properties:

(i) if \( f \in C_{rd} \) and \( t \in T^\kappa \), then
\[
\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t)(t') ;
\]

(ii) if \( a, b \in T \) and \( f, g \in C_{rd} \), then
\[
\int_a^b f(\sigma(t))g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t) \Delta t ,
\]
\[
\int_a^b f(t)g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t ;
\]

(iii) if \([a, b]\) consists of only isolated points, then
\[
\int_a^b f(t) \Delta t = \sum_{t \in [a, b]} \mu(t)f(t).
\]

**Example 2.7.** Let \( a, b \in T \) with \( a < b \). If \( T = \mathbb{R} \), then \( \int_a^b f(t) \Delta t = \int_a^b f(t) dt \), where the integral on the right-hand side is the classical Riemann integral. If \( T = \mathbb{Z} \), then \( \int_a^b f(t) \Delta t = \sum_{k=a}^{b-1} f(k) \). If \( T = q^{10} \), \( q > 1 \), then \( \int_a^b f(t) \Delta t = (1 - q) \sum_{t \in [a, b]} tf(t) \).
The Dubois-Reymond lemma of the calculus of variations on time scales will be useful for our purposes.

**Lemma 2.8.** (Lemma of Dubois-Reymond [8]) Let $T = [a, b]$ be a time scale with at least three points and let $g \in C_{rd}^1$, $g : T^\infty \to \mathbb{R}$. Then,

$$
\int_a^b g(t) \cdot \eta^\Delta(t) \Delta t = 0 \quad \text{for all } \eta \in C^1_{rd} \text{ with } \eta(a) = \eta(b) = 0
$$

if and only if $g(t) = c$ on $T^\infty$ for some $c \in \mathbb{R}$.

3. **Euler-Lagrange equations.** Let $T$ be a time scale. Throughout we let $A, B \in T$ with $A < B$. For an interval $[c, d] \cap T$ we simply write $[c, d]$. We also abbreviate $f \circ \sigma$ by $f^\sigma$. Now let $[a, b]$, with $a, b \in T$ and $b < A$, be a subinterval of $[A, B]$.

The general (non-classical) problem of the calculus of variations on time scales under our consideration consists of minimizing or maximizing a functional of the form

$$
\mathcal{L}[x] = H \left( \int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \ldots, \int_a^b f_n(t, x^\sigma(t), x^\Delta(t)) \Delta t \right),
$$

where $x \in C^1_{rd}$. Using parentheses around the end-point conditions means that these conditions may or may not be present. We assume that:

(i) the function $H : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives with respect to its arguments and we denote them by $H'_i$, $i = 1, \ldots, n$;

(ii) functions $(t, y, v) \to f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, $i = 1, \ldots, n$, have partial continuous derivatives with respect to $y, v$ for all $t \in [a, b]$ and we denote them by $f_{iy}, f_{iv}$;

(iii) $f_i, f_{iy}, f_{iv}, i = 1, \ldots, n$, are rd-continuous in $t$ for all $x \in C^1_{rd}$.

A function $x \in C^1_{rd}$ is said to be an admissible function provided that it satisfies the end-points conditions (if any is given).

Let us consider the following norm in $C^1_{rd}$:

$$
||x|| = \sup_{t \in [a, b]} |x^\sigma(t)| + \sup_{t \in [a, b]} |x^\Delta(t)|.
$$

**Definition 3.1.** An admissible function $\tilde{x}$ is said to be a weak local minimizer (respectively weak local maximizer) for (1) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \leq \mathcal{L}[x]$ (respectively $\mathcal{L}[\tilde{x}] \geq \mathcal{L}[x]$) for all admissible $x$ with $||x - \tilde{x}|| < \delta$.

Next theorem gives necessary optimality conditions for problem (1).

**Theorem 3.2.** If $\tilde{x}$ is a weak local solution of the problem (1), then the Euler-Lagrange equation

$$
\sum_{i=1}^n H'_i(F_1[\tilde{x}], \ldots, F_n[\tilde{x}]) \left( f_{iy}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{iv}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right) = 0
$$

holds for all $t \in [a, b]^\infty$, where $F_i[\tilde{x}] = \int_a^b f_i(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t$, $i = 1, \ldots, n$. Moreover, if $\tilde{x}(a)$ is not specified, then

$$
\sum_{i=1}^n H'_i(F_1[\tilde{x}], \ldots, F_n[\tilde{x}]) f_{iv}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) = 0.
$$
and if \( x(b) \) is not specified, then

\[
\sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\rho(b), \tilde{x}^{\sigma}(\rho(b)), \tilde{x}^{\Delta}(\rho(b))) + \int_{\rho(b)}^{b} f_{iv}(t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) \Delta t \right) = 0. \tag{4}
\]

**Proof.** Suppose that \( \mathcal{L}[x] \) has a weak local extremum at \( \tilde{x} \). For an admissible variation \( h \in C_{rd}^1 \) we define a function \( \phi : \mathbb{R} \to \mathbb{R} \) by \( \phi(\varepsilon) = \mathcal{L}[\tilde{x} + \varepsilon h] \). We do not require \( h(a) = 0 \) or \( h(b) = 0 \) in case \( x(a) \) or \( x(b) \), respectively, is free (it is possible that both are free). A necessary condition for \( \tilde{x} \) to be an extremizer for \( \mathcal{L}[x] \) is given by \( \phi'(\varepsilon)|_{\varepsilon=0} = 0 \). Using the chain rule for obtaining the derivative of a composed function we get

\[
\phi'(\varepsilon)|_{\varepsilon=0} = \sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \int_{a}^{b} \left[ f_{iv}(\bullet) h^{\sigma}(t) + f_{iv}(\bullet) h^{\Delta}(t) \right] \Delta t,
\]

where \( (\bullet) = (t, \tilde{x}^{\sigma}(t), \tilde{x}^{\Delta}(t)) \). Integration by parts of the first term of the integrand gives

\[
\int_{a}^{b} f_{iv}(\bullet) h^{\sigma}(t) \Delta t = \int_{a}^{b} f_{iv}(\circ) \Delta \tau h(t)|_{l=a}^{t} - \int_{a}^{b} \left( \int_{a}^{t} f_{iv}(\circ) \Delta \tau h^{\Delta}(t) \right) \Delta t,
\]

where \( (\circ) = (\tau, \tilde{x}^{\sigma}(\tau), \tilde{x}^{\Delta}(\tau)) \). The necessary condition \( \phi'(\varepsilon)|_{\varepsilon=0} = 0 \) can be written as

\[
0 = \int_{a}^{b} h^{\Delta}(t) \sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_{a}^{t} f_{iv}(\circ) \Delta \tau \right) \Delta t + \sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( \int_{a}^{t} f_{iv}(\circ) \Delta \tau h(t) \right)|_{l=a}^{t}. \tag{5}
\]

In particular, equation (5) holds for all variations which are zero at both ends. For all such \( h \)'s the second term in (5) is zero and by the Dubois-Reymond Lemma 2.8, we have

\[
\sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_{a}^{t} f_{iv}(\circ) \Delta \tau \right) \Delta t = c, \tag{6}
\]

for some \( c \in \mathbb{R} \) and all \( t \in [a, b] \). Hence, equation (2) holds for all \( t \in [a, b]^{c} \). Equation (5) must be satisfied for all admissible values of \( h(a) \) and \( h(b) \). Consequently, equations (5) and (6) imply that

\[
0 = \left( c + \sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \int_{a}^{b} f_{iv}(\bullet) \Delta t \right) h(b) - \left( c + \sum_{i=1}^{n} H'_i(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \int_{a}^{a} f_{iv}(\bullet) \Delta t \right) h(a).
\]
From the properties of the delta integral and from (6), it follows that
\[
0 = h(b) \left\{ \sum_{i=1}^{n} H'_i(F_1[\tilde{x}], \ldots, F_n[\tilde{x}]) \left( f_{iv}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right) \right\} - h(a) \left\{ \sum_{i=1}^{n} H'_i(F_1[\tilde{x}], \ldots, F_n[\tilde{x}]) f_{iv}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) \right\}
\]

(7)

If \(x(t)\) is not preassigned at either end-point, then \(h(a)\) and \(h(b)\) are both completely arbitrary and we conclude that their coefficients in (7) must each vanish. It follows that condition (3) holds when \(x(a)\) is not given, and condition (4) holds when \(x(b)\) is not given. \(\square\)

\textbf{Remark 3.3.} Let \(T\) be a regular time scale. Then from the properties of the delta integral we have
\[
\int_{\rho(b)}^{b} f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t = \mu(\rho(t)) f_{iy}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))).
\]

Therefore (4) can be written in the form
\[
\sum_{i=1}^{n} H'_i(F_1[\tilde{x}], \ldots, F_n[\tilde{x}]) \left\{ f_{iv}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \mu(\rho(t)) f_{iy}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) \right\}.
\]

Choosing \(T = \mathbb{R}\) in Theorem 3.2 we immediately obtain Theorem 3.1 and Equation (4.1) in [13]. The Euler-Lagrange Equation for the product functional can be deduced from Theorem 3.2.

\textbf{Corollary 3.4.} If \(\tilde{x}\) is a solution of the problem
\[
\mathcal{L}[x] = \left( \int_{a}^{b} f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t \right) \left( \int_{a}^{b} f_2(t, x^\sigma(t), x^\Delta(t)) \Delta t \right),
\]
\[
(x(a) = x_a), \quad (x(b) = x_b),
\]
then the Euler-Lagrange equation
\[
F_2[\tilde{x}] \left( f_{1v}(\rho(t), \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right)
+ F_1[\tilde{x}] \left( f_{2v}(\rho(t), \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right) = 0
\]
holds for all \(t \in [a, b]^{\infty}\). Moreover, if \(x(a)\) is not specified, then
\[
F_2[\tilde{x}] f_{1v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) + F_1[\tilde{x}] f_{2v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) = 0;
\]
if \(x(b)\) is not specified, then
\[
F_2[\tilde{x}] \left( f_{1v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right)
+ F_1[\tilde{x}] \left( f_{2v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^{b} f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right) = 0.
\]

\textbf{Remark 3.5.} In the particular case \(T = \mathbb{R}\), Corollary 3.4 gives Equation (3.17) in [13].
Example 3.6. Consider the problem

\[
\text{minimize } \mathcal{L}[x] = \left( \int_{0}^{1} (x^\Delta(t))^2 \Delta t \right) \left( \int_{0}^{1} t x^\Delta(t) \Delta t \right) \quad (8)
\]

\[x(0) = 0, \quad x(1) = 1.\]

If \( \bar{x} \) is a local minimum of (8), then the Euler-Lagrange equation must hold, i.e.,

\[2 \bar{x}^\Delta(t) Q_2 + Q_1 = 0, \quad (9)\]

where

\[Q_1 = \mathcal{F}_1[\bar{x}] = \int_{0}^{1} (\bar{x}^\Delta(t))^2 \Delta t, \quad Q_2 = \mathcal{F}_2[\bar{x}] = \int_{0}^{1} t \bar{x}^\Delta(t) \Delta t.\]

If \( Q_2 = 0 \), then also \( Q_1 = 0 \). This contradicts the fact that on any time scale a global minimizer for the problem

\[
\text{minimize } \mathcal{F}_1[x] = \int_{0}^{1} (x^\Delta(t))^2 \Delta t \\
x(0) = 0, \quad x(1) = 1
\]

is \( \bar{x}(t) = t \) and \( \mathcal{F}_1[\bar{x}] = 1 \). Hence, \( Q_2 \neq 0 \) and equation (9) implies that candidate solutions for problem (8) are those satisfying the delta differential equation

\[\bar{x}^\Delta(t) = -\frac{Q_1}{2Q_2}, \quad (10)\]

subject to boundary conditions \( x(0) = 0 \) and \( x(1) = 1 \). Solving equation (10) we obtain

\[x(t) = -\frac{Q_1}{2Q_2} \int_{0}^{t} \tau \Delta \tau + 1 + \frac{Q_1}{2Q_2} \int_{0}^{1} \tau \Delta \tau. \]

Therefore, a solution of (10) depends on the time scale. Let us consider, for example, \( \mathbb{T} = \mathbb{R} \) and \( \mathbb{T} = \{0, \frac{1}{2}, 1\} \). On \( \mathbb{T} = \mathbb{R} \) we obtain

\[x(t) = -\frac{Q_1}{4Q_2} \frac{t^2}{2} + \frac{4Q_2 + Q_1}{4Q_2} t. \quad (11)\]

Substituting (11) into functionals \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) gives

\[
\begin{cases}
48Q_2^2 + Q_1^2 = Q_1 \\
12Q_2^2 - Q_1 = Q_2.
\end{cases} \quad (12)
\]

Solving the system of equations (12) we obtain

\[
\begin{cases}
Q_1 = 0 \\
Q_2 = 0
\end{cases} \quad \begin{cases}
Q_1 = \frac{1}{3} \\
Q_2 = \frac{1}{3}
\end{cases}
\]

Therefore,

\[\bar{x}(t) = -t^2 + 2t \]

is a candidate extremizer for problem (8) on \( \mathbb{T} = \mathbb{R} \). Note that nothing can be concluded as to whether \( \bar{x} \) gives a minimum, a maximum, or neither of these for \( \mathcal{L} \).

The solution of (10) on \( \mathbb{T} = \{0, \frac{1}{2}, 1\} \) is

\[x(t) = \begin{cases}
0 & \text{if } t = 0 \\
\frac{1}{2} + \frac{Q_1}{16Q_2} & \text{if } t = \frac{1}{2} \\
1 & \text{if } t = 1.
\end{cases} \quad (13)\]
Constants $Q_1$ and $Q_2$ are determined by substituting (13) into functionals $\mathcal{F}_1$ and $\mathcal{F}_2$. The resulting system of equations is
\[
\begin{align*}
1 + \frac{Q_1^2}{64Q_2^2} &= Q_1 \\
\frac{1}{Q_2} - \frac{Q_1}{64Q_2^2} &= Q_2.
\end{align*}
\] (14)
Since system of equations (14) has no real solutions, we conclude that there exists no extremizer for problem (8) on $\mathbb{T} = \{0, \frac{1}{2}, 1\}$ among the set of functions that we consider to be admissible.

Assuming that the denominator does not vanish, the Euler-Lagrange equation for the quotient problem can be deduced from Theorem 3.2.

**Corollary 3.7.** If $\tilde{x}$ is a solution of the problem
\[
\mathcal{L}[x] = \int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t - \int_a^b f_2(t, x^\sigma(t), x^\Delta(t)) \Delta t,
\] where $x(a) = x_a$, $(x(b) = x_b)$, then the Euler-Lagrange equation
\[
f_{1v}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - Q \left( f_{2v}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right) = 0
\] holds for all $t \in [a, b]^\kappa$, where $Q = \frac{\mathcal{F}_1[\tilde{x}]}{\mathcal{F}_2[\tilde{x}]}$. Moreover, if $x(a)$ is not specified, then
\[
f_{1v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) - Q f_{2v}(a, \tilde{x}^\sigma(a), \tilde{x}^\Delta(a)) = 0;
\] if $x(b)$ is not specified, then
\[
f_{1v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^b f_{1y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t - Q \left( f_{2v}(\rho(b), \tilde{x}^\sigma(\rho(b)), \tilde{x}^\Delta(\rho(b))) + \int_{\rho(b)}^b f_{2y}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \Delta t \right) = 0.
\]

**Remark 3.8.** In the particular situation $\mathbb{T} = \mathbb{R}$, Corollary 3.7 gives Equation (3.21) in [13].

**Example 3.9.** Consider the problem
\[
\text{minimize} \quad \mathcal{L}[x] = \frac{\int_0^2 (x^\Delta(t))^2 \Delta t}{\int_0^2 (x(t))^2 \Delta t}, \quad \text{subject to} \quad x(0) = 0, \quad x(2) = 4.
\] (15)
If $\tilde{x}$ is a local minimizer for (15), then the Euler-Lagrange equation must hold, i.e.,
\[
0 = [2\tilde{x}^\Delta(t) - Q(1 + 2\tilde{x}^\Delta(t))]^\Delta, \quad t \in [0, 2]^\kappa,
\]
where
\[
Q = \frac{\int_0^2 (\tilde{x}^\Delta(t))^2 \Delta t}{\int_0^2 (\tilde{x}^\Delta(t) + (\tilde{x}^\Delta(t))^2) \Delta t}.
\]
Therefore,
\[
0 = 2\tilde{x}^\Delta(t) - Q2\tilde{x}^\Delta(t), \quad t \in [0, 2]^\kappa.
\]
Thus \( \dddot{x}(t) = 0 \) or \( Q = 1 \). The solution of the delta differential equation
\[
\dddot{x}(t) = 0,
\]
\[
x(0) = 0, \quad x(2) = 4
\]
does not depend on the time scale and it is \( \dot{x}(t) = 2t \). Observe that \( \mathcal{L}[\dddot{x}] = \frac{2}{3} < 1 \). Therefore, \( \dddot{x} \) is a candidate local minimizer for problem (15).

**Example 3.10.** Consider the problem
\[
\begin{align*}
\text{extremize} \quad & \mathcal{L}[x] = \int_0^1 t x'(t) \Delta t \bigg/ \int_0^1 x'(t)^2 \Delta t, \\
x(0) = 0, \quad x(1) = 1
\end{align*}
\]
(16)
The Euler-Lagrange equation for this problem is
\[
0 = 1 - 2Q x'(t),
\]
where \( Q \) is the value of functional \( \mathcal{L} \) in a solution of (16). Since \( Q \neq 0 \), it follows that
\[
\dddot{x}(t) = \frac{1}{2Q}.
\]
(17)
Solving equation (17) we obtain
\[
x(t) = \frac{1}{2Q} \int_0^t \tau \Delta \tau + 1 - \frac{1}{2Q} \int_0^1 \tau \Delta \tau.
\]
Therefore, a solution of (17) depends on the time scale. Let us consider, for example, \( T = \mathbb{R} \) and \( T = \{0, \frac{1}{2}, 1\} \). On \( T = \mathbb{R} \) we obtain
\[
x(t) = \frac{1}{4Q} t^2 + \frac{4Q - 1}{4Q} t.
\]
(18)
Substituting (18) into functional \( \mathcal{L} \) yields
\[
\frac{24Q^2 + 2Q}{48Q^2 + 1} = Q.
\]
(19)
Solving equation (19) we obtain \( Q \in \left\{ \frac{1}{4} - \frac{\sqrt{3}}{6}, 0, \frac{1}{4} + \frac{\sqrt{3}}{6} \right\} \). Therefore,
\[
x_1(t) = \frac{3}{3 - 2\sqrt{3}} t^2 + \frac{2\sqrt{3}}{2\sqrt{3} - 3} t
\]
is a candidate local minimizer while
\[
x_2(t) = \frac{3}{3 + 2\sqrt{3}} t^2 + \frac{2\sqrt{3}}{2\sqrt{3} + 3} t
\]
is a candidate local maximizer for problem (16) on \( T = \mathbb{R} \). The solution of (17) on \( T = \{0, \frac{1}{2}, 1\} \) is
\[
x(t) = \begin{cases}
0 & \text{if } t = 0 \\
\frac{1}{2} - \frac{1}{16Q} & \text{if } t = \frac{1}{2} \\
1 & \text{if } t = 1.
\end{cases}
\]
(20)
The constant \( Q \) is determined by substituting (20) into \( \mathcal{L} \). The resulting equation is
\[
\frac{1}{4} + \frac{1}{32Q} = Q + \frac{1}{64Q}.
\]
(21)
Solving (21) we obtain $Q \in \left\{ \frac{1}{8} - \frac{\sqrt{2}}{8}, \frac{1}{8} + \frac{\sqrt{2}}{8} \right\}$ and stationary functions are

\[ x_1(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{\sqrt{2}}{2\sqrt{2} - 2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1, \end{cases} \quad (22) \]

and

\[ x_2(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{\sqrt{2}}{2\sqrt{2} + 2} & \text{if } t = \frac{1}{2} \\ 1 & \text{if } t = 1. \end{cases} \quad (23) \]

Figure 1. The extremal minimizer of Example 3.10 for $T = \mathbb{R}$ and $T = \{0, \frac{1}{2}, 1\}$.

Figure 2. The extremal maximizer of Example 3.10 for $T = \mathbb{R}$ and $T = \{0, \frac{1}{2}, 1\}$.

Therefore (22) is a candidate local minimizer while (23) is a candidate local maximizer for problem (16) on $T = \{0, \frac{1}{2}, 1\}$.

**Example 3.11.** Consider the problem

\[
\begin{align*}
\text{extremize} & \quad L[x] = \frac{\int_{a}^{b} \left( (x^\Delta(t))^2 - q(t)(x^\sigma(t))^2 \right) \Delta t}{\int_{a}^{b} (x^\sigma(t))^2 \Delta t}, \\
x(a) = 0, \quad x(b) = 0,
\end{align*}
\]

(24)

where $q : [a, b] \to \mathbb{R}$ is a continuous function. The Euler-Lagrange equation for this problem is

\[ x^\Delta(t) + q(t)x^\sigma(t) + Qx^\sigma(t) = 0, \quad (25) \]

subject to

\[ x(a) = 0, \quad x(b) = 0, \quad (26) \]

where $Q$ is the value of functional $L$ in a solution of (24). It is easily seen that (25)–(26) is a case of the Sturm-Liouville eigenvalue problem on time scales (see [1] and [17]). It follows that the problem of determining eigenfunctions of (25) subject to (26) is equivalent to the problem of determining functions satisfying (26) which render $L$ stationary.
4. Isoperimetric problems. Let us consider now the general (non-classical) isoperimetric problem on time scales. The problem consists of minimizing or maximizing

\[ \mathcal{L}[x] = H \left( \int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \ldots, \int_a^b f_m(t, x^\sigma(t), x^\Delta(t)) \Delta t \right), \]

in the class of functions \( x \in C^1_{rd} \) satisfying the boundary conditions

\[ x(a) = x_a, \quad x(b) = x_b \tag{28} \]

and the constraint

\[ \mathcal{K}[x] = \mathcal{P} \left( \int_a^b g_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \ldots, \int_a^b g_m(t, x^\sigma(t), x^\Delta(t)) \Delta t \right) = k, \tag{29} \]

where \( x_a, x_b, k \) are given real numbers. We assume that:

(i) functions \( H : \mathbb{R}^n \to \mathbb{R} \) and \( P : \mathbb{R}^m \to \mathbb{R} \) have continuous partial derivatives with respect to their arguments and we denote them by \( H_i', i = 1, \ldots, n \), and \( P_i', i = 1, \ldots, m; \)
(ii) functions \( (t, y, v) \to f_i(t, y, v), i = 1, \ldots, n \), and \( (t, y, v) \to g_j(t, y, v), j = 1, \ldots, m \), from \([a, b] \times \mathbb{R}^2 \) to \( \mathbb{R} \) have partial continuous derivatives with respect to \( y, v \) for all \( t \in [a, b] \) and we denote them by \( f_{iy}, f_{iv} \) and \( g_{jy}, g_{jv} \);
(iii) \( f_i, f_{iy}, f_{iv}, i = 1, \ldots, n \), and \( g_j, g_{jy}, g_{jv}, j = 1, \ldots, m \), are rd-continuous in \( t \) for all \( x \in C^1_{rd} \).

**Definition 4.1.** An admissible function \( \tilde{x} \) is said to be a weak local minimizer (respectively weak local maximizer) for the isoperimetric problem \((27)–(29)\) if there exists \( \delta > 0 \) such that \( \mathcal{L}[	ilde{x}] \leq \mathcal{L}[x] \) (respectively \( \mathcal{L}[	ilde{x}] \geq \mathcal{L}[x] \)) for all admissible \( x \) satisfying the boundary conditions \( (28) \), the isoperimetric constraint \( (29) \), and \( \|x - \tilde{x}\|_1 < \delta \).

**Definition 4.2.** We say that \( \tilde{x} \) is an extremal for \( \mathcal{K} \) if

\[
\sum_{i=1}^m P_i' \left( G_1[\tilde{x}], \ldots, G_m[\tilde{x}] \right) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\circ) \Delta \tau \right) = c,
\]

where \( (\bullet) = (t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \) and \( (\circ) = (\tau, \tilde{x}^\sigma(\tau), \tilde{x}^\Delta(\tau)) \), for some constant \( c \) and for all \( t \in [a, b] \). An extremizer (i.e., a weak local minimizer or a weak local maximizer) for the problem \((27)–(29)\) that is not an extremal for \( \mathcal{K} \) is said to be a normal extremizer; otherwise (i.e., if it is an extremal for \( \mathcal{K} \)), the extremizer is said to be abnormal.

**Theorem 4.3.** If \( \tilde{x} \) is a normal extremizer for the isoperimetric problem \((27)–(29)\), then there exists a real \( \lambda \) such that

\[
\sum_{i=1}^n H_i' \left( F_1[\tilde{x}], \ldots, F_n[\tilde{x}] \right) \left( f_{iv}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - f_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right)
- \lambda \sum_{i=1}^m P_i' \left( G_1[\tilde{x}], \ldots, G_m[\tilde{x}] \right) \left( g_{iv}^\Delta(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) - g_{iy}(t, \tilde{x}^\sigma(t), \tilde{x}^\Delta(t)) \right) = 0 \tag{30}
\]

for all \( t \in [a, b]^c \).

**Proof.** Consider a variation of \( \tilde{x} \), say \( \tilde{x} = \tilde{x} + \varepsilon_1 h_1 + \varepsilon_2 h_2 \), where \( h_i \in C^1_{rd} \) and \( h_i(a) = h_i(b) = 0, i = 1, 2 \), and \( \varepsilon_i \) is a sufficiently small parameter \( \varepsilon_1 \) and \( \varepsilon_2 \) must
be such that \(\|\bar{x} - \tilde{x}\|_1 < \delta\) for some \(\delta > 0\). Here, \(h_1\) is an arbitrary fixed function and \(h_2\) is a fixed function that will be chosen later. Define the real function

\[
K(\varepsilon_1, \varepsilon_2) = K[\bar{x}] = P \left( \int_a^b g_1(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \ldots, \int_a^b g_m(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t \right) - k.
\]

We have

\[
\frac{\partial K}{\partial \varepsilon_2} \bigg|_{(0,0)} = \sum_{i=1}^m P'_i(G_1[\bar{x}], \ldots, G_m[\bar{x}]) \int_a^b \left[ g_{iv}(\bullet) h_{2i}^2(t) + g_{iv}(\bullet) h_{2i}^\Delta(t) \right] \Delta t,
\]

where \((\bullet) = (t, \bar{x}^\sigma(t), \bar{x}^\Delta(t))\). Since \(h_2(a) = h_2(b) = 0\), integration by parts gives

\[
\int_a^b h_{2i}^\Delta(t) \sum_{i=1}^m P'_i(G_1[\bar{x}], \ldots, G_m[\bar{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\omega) \Delta \tau \right) \Delta t,
\]

where \((\omega) = (t, \bar{x}^\sigma(t), \bar{x}^\Delta(t))\). By Lemma 2.8, there exists \(h_2\) such that \(\frac{\partial K}{\partial \varepsilon_2} \bigg|_{(0,0)} \neq 0\). Since \(K(0,0) = 0\), by the implicit function theorem we conclude that there exists a function \(\varepsilon_2\) defined in the neighborhood of zero, such that \(K(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0\), i.e., we may choose a subset of variations \(\bar{x}\) satisfying the isoperimetric constraint.

Let us now consider the real function

\[
\bar{L}(\varepsilon_1, \varepsilon_2) = \bar{L}[\bar{x}] = H \left( \int_a^b f_1(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \ldots, \int_a^b f_n(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t \right).
\]

By hypothesis, \((0,0)\) is an extremal of \(\bar{L}\) subject to the constraint \(K = 0\) and \(\nabla K(0,0) \neq 0\). By the Lagrange multiplier rule, there exists some real \(\lambda\) such that \(\nabla(\bar{L}(0,0) - \lambda K(0,0)) = 0\). Having in mind that \(h_1(a) = h_1(b) = 0\), we can write

\[
\frac{\partial \bar{L}}{\partial \varepsilon_1} \bigg|_{(0,0)} = \int_a^b h_{1i}^\Delta(t) \sum_{i=1}^n H'_i(F_1[\bar{x}], \ldots, F_n[\bar{x}]) \left( f_{iv}(\bullet) - \int_a^t f_{iy}(\omega) \Delta \tau \right) \Delta t
\]

and

\[
\frac{\partial \bar{K}}{\partial \varepsilon_1} \bigg|_{(0,0)} = \int_a^b h_{1i}^\Delta(t) \sum_{i=1}^m P'_i(G_1[\bar{x}], \ldots, G_m[\bar{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\omega) \Delta \tau \right) \Delta t.
\]

Therefore,

\[
\int_a^b h_{1i}^\Delta(t) \left\{ \sum_{i=1}^n H'_i(F_1[\bar{x}], \ldots, F_n[\bar{x}]) \left( f_{iv}(\bullet) - \int_a^t f_{iy}(\omega) \Delta \tau \right) \\
- \lambda \sum_{i=1}^m P'_i(G_1[\bar{x}], \ldots, G_m[\bar{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\omega) \Delta \tau \right) \right\} \Delta t = 0. \tag{31}
\]

As (31) holds for any \(h_1\), by Lemma 2.8, we have

\[
\sum_{i=1}^n H'_i(F_1[\bar{x}], \ldots, F_n[\bar{x}]) \left( f_{iv}(\bullet) - \int_a^t f_{iy}(\omega) \Delta \tau \right) \\
- \lambda \sum_{i=1}^m P'_i(G_1[\bar{x}], \ldots, G_m[\bar{x}]) \left( g_{iv}(\bullet) - \int_a^t g_{iy}(\omega) \Delta \tau \right) = c, \tag{32}
\]

for some \(c \in \mathbb{R}\). Applying the delta derivative to both sides of equation (32), we get (30). \qed
Remark 4.4. Choosing $H, P : \mathbb{R} \to \mathbb{R}$ and $H = P = id$ in Theorem 4.3 we immediately obtain Theorem 3.4 in [17] and a particular case of Theorem 3.4 in [21].

One can easily cover abnormal extremizers within our result by introducing an extra multiplier $\lambda_0$.

Theorem 4.5. If $\tilde{x}$ is an extremizer for the isoperimetric problem \((27)−(29)\), then there exist two constants $\lambda_0$ and $\lambda$, not both zero, such that

\[
\lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(t, \tilde{x}(t), \tilde{x}'(t)) - f_{iy}(t, \tilde{x}(t), \tilde{x}'(t)) \right)
- \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left( g_{iv}(t, \tilde{x}(t), \tilde{x}'(t)) - g_{iy}(t, \tilde{x}(t), \tilde{x}'(t)) \right) = 0 \tag{33}
\]

for all $t \in [a, b]^n$.

Proof. Following the proof of Theorem 4.3, since $(0, 0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$, the extended Lagrange multiplier rule (see, for instance, [27, Theorem 4.1.3]) asserts the existence of reals $\lambda_0$ and $\lambda$, not both zero, such that $\nabla(\lambda_0 L(0, 0) - \lambda K(0, 0)) = 0$. Therefore,

\[
\lambda_0 \frac{\partial L}{\partial \varepsilon_1} \bigg|_{(0, 0)} - \lambda \frac{\partial K}{\partial \varepsilon_1} \bigg|_{(0, 0)} = 0
\]

\[
\Leftrightarrow \int_{a}^{b} h(t) \left\{ \lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_{a}^{t} f_{iy}(\circ) d\tau \right)
- \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left( g_{iv}(\bullet) - \int_{a}^{t} g_{iy}(\circ) d\tau \right) \right\} \Delta t = 0. \tag{34}
\]

Since (34) holds for any $h_1$, it follows by Lemma 2.8 that

\[
\lambda_0 \sum_{i=1}^{n} H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( f_{iv}(\bullet) - \int_{a}^{t} f_{iy}(\circ) d\tau \right)
- \lambda \sum_{i=1}^{m} P_i'(\mathcal{G}_1[\tilde{x}], \ldots, \mathcal{G}_m[\tilde{x}]) \left( g_{iv}(\bullet) - \int_{a}^{t} g_{iy}(\circ) d\tau \right) = c. \tag{35}
\]

for some $c \in \mathbb{R}$. The desired condition (33) follows by delta differentiation of (35). \]

Remark 4.6. If $\tilde{x}$ is a normal extremizer for the isoperimetric problem \((27)−(29)\), then we can choose $\lambda_0 = 1$ in Theorem 4.5 and obtain Theorem 4.3. For abnormal extremizers, Theorem 4.5 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq 0$ guarantees that Theorem 4.5 is a useful necessary condition.

Corollary 4.7. (i) If $\tilde{x}$ is an extremizer for the isoperimetric problem

\[
\text{extremize } \mathcal{L}[x] = \left( \int_{a}^{b} f_1(t, \tilde{x}(t), \tilde{x}'(t)) \Delta t \right) \left( \int_{a}^{b} f_2(t, \tilde{x}(t), \tilde{x}'(t)) \Delta t \right),
\]

\[
x(a) = x_a, \quad x(b) = x_b,
\]
subject to the constraint
\[ K[x] = \left( \int_a^b g_1(t, x'(t), x^\Delta(t))\Delta t \right) \left( \int_a^b g_2(t, x'(t), x^\Delta(t))\Delta t \right) = k, \]
then there exist two constants \( \lambda_0 \) and \( \lambda \), not both zero, such that
\[ \lambda_0 \left\{ F_2[x] \left( f^\Delta_1(t, \dot{x}(t), \ddot{x}(t)) - f_{1y}(t, \ddot{x}(t), \ddot{x}(t)) \right) + F_1[x] \left( f^\Delta_2(t, \dot{x}(t), \ddot{x}(t)) - f_{2y}(t, \ddot{x}(t), \ddot{x}(t)) \right) \right\} \]
\[ - \lambda \left\{ G_2[x] \left( g^\Delta_1(t, \dot{x}(t), \ddot{x}(t)) - g_{1y}(t, \ddot{x}(t), \ddot{x}(t)) \right) + G_1[x] \left( g^\Delta_2(t, \dot{x}(t), \ddot{x}(t)) - g_{2y}(t, \ddot{x}(t), \ddot{x}(t)) \right) \right\} = 0 \]
for all \( t \in [a, b] \).

(ii) Assume that denominators of \( \mathcal{L} \) and \( \mathcal{K} \) do not vanish. If \( \ddot{x} \) is an extremizer for the isoperimetric problem
\[ \text{extremize } \mathcal{L}[x] = \frac{\int_a^b f_1(t, x'(t), x^\Delta(t))\Delta t}{\int_a^b f_2(t, x'(t), x^\Delta(t))\Delta t}, \quad x(a) = x_a, \quad x(b) = x_b, \]
subject to the constraint
\[ K[x] = \frac{\int_a^b g_1(t, x'(t), x^\Delta(t))\Delta t}{\int_a^b g_2(t, x'(t), x^\Delta(t))\Delta t} = k, \]
then there exist two constants \( \lambda_0 \) and \( \lambda \), not both zero, such that
\[ \lambda_0 \left\{ \mathcal{G}_2[\ddot{x}] \left( f^\Delta_1(t, \dot{x}(t), \ddot{x}(t)) - f_{1y}(t, \ddot{x}(t), \ddot{x}(t)) \right) - \mathcal{G}_2[\ddot{x}] \mathcal{Q}_L \left( f^\Delta_2(t, \dot{x}(t), \ddot{x}(t)) - f_{2y}(t, \ddot{x}(t), \ddot{x}(t)) \right) \right\} \]
\[ - \lambda \left\{ \mathcal{G}_2[\ddot{x}] \left( g^\Delta_1(t, \dot{x}(t), \ddot{x}(t)) - g_{1y}(t, \ddot{x}(t), \ddot{x}(t)) \right) + \mathcal{G}_2[\ddot{x}] \mathcal{Q}_K \left( g^\Delta_2(t, \dot{x}(t), \ddot{x}(t)) - g_{2y}(t, \ddot{x}(t), \ddot{x}(t)) \right) \right\} = 0 \]
holds for all \( t \in [a, b] \), where \( \mathcal{Q}_L = \frac{F_1[\ddot{x}]}{F_2[\ddot{x}]} \) and \( \mathcal{Q}_K = \frac{G_1[\ddot{x}]}{G_2[\ddot{x}]} \).

**Example 4.8.** Consider the problem
\[ \text{extremize } \mathcal{L}[x] = \frac{\int_0^1 (x^\Delta(t))^2\Delta t}{\int_0^1 tx^\Delta(t)\Delta t}, \quad x(0) = 0, \quad x(1) = 1, \]
subject to the constraint
\[ K[x] = \int_0^1 tx^\Delta(t)\Delta t = 1. \]
Since
\[ g_\sigma(t, x'(t), x^\Delta(t)) - \int_0^t g_\sigma(\tau, x_\sigma(\tau), x^\Delta(\tau))\Delta \tau = t \]
there are no abnormal extremals for the problem (36)–(37). Applying Theorem 4.3, we get the delta differential equation
\[ 2x^\Delta - Q - \lambda = 0, \]
where \( Q \) is the value of functional \( \mathcal{L} \) in a solution of (36)–(37). Solving equation (38) we obtain
\[ x(t) = \frac{Q + \lambda}{2} \int_0^t \tau \Delta \tau + 1 - \frac{Q + \lambda}{2} \int_0^1 \tau \Delta \tau. \]
Therefore, a solution of (38) depends on the time scale. Let us consider, for example, $T = \mathbb{R}$ and $T = \{0, \frac{1}{2}, 1\}$. On $T = \mathbb{R}$ we obtain

$$x(t) = 3t^2 - 2t$$

as a candidate local minimizer while on $T = \{0, \frac{1}{2}, 1\}$

$$x(t) = \begin{cases} 
0 & \text{if } t = 0 \\
-1 & \text{if } t = \frac{1}{2} \\
1 & \text{if } t = 1.
\end{cases}$$

is a candidate local minimizer for the problem (36)–(37).

**Figure 3.** The extremal minimizer of Example 4.8 for $T = \mathbb{R}$ and $T = \{0, \frac{1}{2}, 1\}$.

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