We consider the scaling of the mean square dipole moment in a plasma with logarithmic interactions in a two- and three-dimensional system. In both cases, we establish the existence of a low-temperature regime where the mean square dipole moment does not scale with system size and a high-temperature regime where it does scale with system size. Thus, there is a nonanalytic change in the polarizability of the system as a function of temperature, and hence a metal-insulator transition in both cases. The relevance of this transition in three dimensions to quantum phase transitions in 2 + 1-dimensional systems is briefly discussed.

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Gauge theories where matter fields are coupled to compact gauge-fields in 2+1 dimensions have recently been much studied as effective theories for Mott insulators with competing orders. Compact U(1) gauge theories in 2+1 dimensions are capable of sustaining topological defects in the gauge fields in the form of space-time instantons. Such theories generically offer the possibility of featuring confinement-deconfinement transitions associated with a proliferation of topological defects in the gauge sector. It is hoped that phase transitions in matter coupled such theories may be connected to difficult problems in condensed matter physics, such as the breakdown of Landau Fermi liquid theory and possibly also spin-charge separation in strongly correlated systems at zero temperature in two spatial dimensions. One such model considered recently is the compact abelian Higgs model where the gauge field is coupled to matter fields with the fundamental charge. By itself, the compact U(1) gauge sector may be mapped onto the 2+1-dimensional Coulomb gas with 1/R-interactions between “point charges”, i.e., the instantonic topological defects of the theory. As shown by Polyakov, the latter system is always in a metallic phase (when the instantons are regarded as electric charges). More recently, it was shown that when matter carrying the fundamental charge was coupled to the compact gauge sector, and critical matter field fluctuations were integrated out, the system could be mapped onto a gas of point charges interacting with a potential – ln(R) in 2+1 space-time dimensions, instead of 1/R. Using RG arguments, it was demonstrated that the system with the 2+1-dimensional logarithmic interaction may undergo a finite-temperature phase transition driven by the unbinding of dipole configurations, from a low-temperature dielectric regime to a high-temperature metallic regime (again when the instantons are regarded as electric charges). Here, we demonstrate this using Monte Carlo (MC) simulations.

Whether or not there exists a low-temperature dielectric regime separated from a high-temperature metallic regime in a three-dimensional system of point charges interacting with logarithmic interactions and overall charge-neutrality, is presently a matter of debate in both the condensed matter and lattice gauge theory literature. The theorem that dipoles cannot screen the Coulomb potential (in any dimension) does not apply to the case of a logarithmic pair potential between point charges in three dimensions. It is due to this inability of the dipoles to screen a Coulomb potential that the Kosterlitz-Thouless (KT) transition is possible: the low temperature dielectric phase is always critical below the KT transition temperature. In three dimensions, on the other hand, Debye-Hückel theory is essentially exact. Thus, starting from a system of charges, screening is such that the screening length cannot become infinite, and the system is always in the metallic phase. On the other hand, if one starts from a three-dimensional dipole system, the system always stays in the dielectric phase, since dipoles cannot screen. Once more, there is no phase transition. If some departure from the Coulomb potential occurs in three dimensions, dipoles could conceivably be able to screen. A 3D logarithmic interaction could in principle be screened to a potential that decays with separation between the charges. If so, no metal-insulator transition occurs at any finite temperature, analogous to the situation in a three-dimensional Coulomb plasma. If one thinks of the gas of point charges as a gas of instantonic defects in a compact gauge field, the lack of a metal-insulator transition due to non-standard screening by dipoles would correspond to permanent confinement in the compact gauge theory.

We therefore consider a logarithmic plasma in two and three dimensions on the lattice using MC simulations, and study the polarizability of the system as a function of temperature. The screening properties of these systems determine whether they are insulators or metals and are governed by the dielectric constant ε, which in a low-density approximation is given by the polarizability p of the system, ε = 1+n_dΩ_d p. Here n_d is the dipole density, and Ω_d is the solid angle in a d-dimensional system. Since the polarizability of the system is proportional to...
the mean square separation $\langle s^2 \rangle \equiv \langle |\vec{r}_i - \vec{r}_j|^2 \rangle$ between the charges constituting the dipoles, it is natural to focus on $\langle s^2 \rangle$ in order to investigate whether the system is a dielectric or a metal. We find a low-temperature regime where dipoles are tightly bound, separated from a high-temperature regime where they are unbound. This implies the existence of a low-temperature insulating regime and a high-temperature metallic regime, separated by a genuine phase transition. For comparison and benchmark purposes, we compute the same quantities for the two-dimensional logarithmic plasma, where a metal-insulator transition in the form of a KT phase transition is known to exist \[12, 13\].

Let us first turn to the 2D system, it is well-known that dipoles begin to unbind at a critical temperature $T_{KT}$ \[13\] and at high temperatures the 2DCG is a fully ionized metallic plasma, separated from the low-temperature dielectric insulating phase by a genuine KT phase-transition. Thus, we expect no finite-size scaling of $\langle s^2 \rangle$ below the KT-transition, whereas we should expect $\langle s^2 \rangle \propto L^{\alpha(T)}$ with $\alpha(T) \leq 2$ at higher temperatures. Using an intuitive low density argument, neglecting screening effects \[13\], we can calculate the behaviour of $\langle s^2 \rangle$ to leading order in $L$,

$$\langle s^2 \rangle \propto \begin{cases} 
\text{Const.} & ; T < T_{KT} \\
L^{(T-T_{KT})/T} & ; T_{KT} < T < 2T_{KT} \\
L^2 & ; 2T_{KT} < T.
\end{cases} \quad (5)$$

Hence, $\alpha(T)$ is zero for low temperatures and a monotonically increasing function of temperature just above $T_{KT}$.

Including screening effects in 2D shows that this conclusion still holds, however the temperature at which it occurs is determined by screening. This is ultimately related to the fact that dipoles contribute only through a correction to the coefficient of the bare logarithmic pair-potential. Hence, in $2D$ the functional form of the renormalized potential is unaltered, only prefactors are changed. In three dimensions, it is far from obvious that dipoles do not have a much more disruptive effect on a bare logarithmic pair-potential.

To address the issue of scaling of $\langle s^2 \rangle$, we will use MC simulations and finite-size scaling to study $\langle s^2 \rangle$ both in the 2DCG as well as in the 3DLG. In our simulations, the particle number is not conserved. However, during the simulations the system is maintained electrically neutral. The MC moves involve creation and annihilation of charges, applying the Metropolis algorithm in this process. Starting at some randomly chosen lattice site, an attempt to insert a negative or positive charge at random at this site is made, with an opposite charge at a nearest-neighbour site. The move is accepted with probability $\exp(-\Delta E/T) = \exp(-\langle \mathcal{H}_{\text{new}} - \mathcal{H}_{\text{old}} \rangle/T)$, and this is done in all $d$ directions, before we move to the next site. It is clear that placing a charge on top of an opposite one corresponds to the annihilation of the existing one. In order to measure $s$ we have to keep track of which two charges belong to each other in a dipole, since there is no physical link defining the dipole. Two charges inserted into the lattice at the same MC move is chosen as a dipole, and if one of these charges subsequently is annihilated, the effect of this annihilation is to diffuse one of the charges in the dipole. Hence, such a diffusion may increase or decrease $s$ for this dipole \[15\]. One sweep is defined as going through all the sites in the lattice once, and at every tenth sweep we sample $s^2$ averaged over the system. The sample is, however, rejected if there are no changes in the system at this MC time, since there is no
information on \( \langle s^2 \rangle \) in such a configuration. The thermal average obtained at the end of the simulation is thus taken only over non-empty configurations.

The results from the simulations on the 2DCG are presented in Fig. 1. A total of \( 10^5 \) MC sweeps at each temperature are used to produce the first plot. From the \( \langle s^2 \rangle \)-data of system sizes \( L = 8 \), 12, 16, 20, 24, 28, 32, 36, 40 and 48, we extract \( \alpha \) for temperatures in the interval (1.0, 1.52), which is shown in the second plot. It is evident that there are two distinct regimes of temperatures, one in which the charges of almost all dipoles are bound as tightly as possible, the separation of the charges correspond to the lattice constant. In the high-temperature regime the dipoles have started to separate, reflected by a scaling of \( \langle s^2 \rangle \sim L^{\alpha(T)} \) with the system size. The two regimes are necessarily separated by a phase transition, since in the low-temperature regime \( \alpha(T) = 0 \) while in the high-temperature regime \( \alpha(T) \neq 0 \). This necessarily implies a non-analytic behavior of \( \alpha(T) \). An attempt to determine the transition temperature from these plots yields approximately 1.32. This is slightly less than the early results of Saito and Müller-Krumbhaar of 1.35 in our units [10], but is in excellent agreement with much more recent simulations by Olsson [17]. It provides confidence in the method of locating the critical point by monitoring the quantity \( \langle s^2 \rangle \), even when the system is subjected to periodic boundary conditions [12].

Exactly the same simulation technique is applied to the 3DLG and the results are shown in Fig. 1. The finite-size analysis of \( \langle s^2 \rangle \) in order to extract \( \alpha \) is here done on the basis of system sizes \( L = 8 \), 12, 16, 20, 24, 28, 32, 36 and 40 and up to \( 2 \cdot 10^5 \) MC sweeps are used. As in the 2DCG we see that the system exhibits two distinct regimes, one insulating regime consisting of tightly bound dipoles and one metallic. There is no scaling of \( \langle s^2 \rangle \) with \( L \) below \( T \simeq 0.32 \) with the system size. Note that the change in scaling of the mean square dipole moment occurs at a significantly lower temperature than in the 2DCG. This is to be expected, since there is more configurational entropy available in 3D than in 2D. It is worthwhile comparing this result with the one obtained in Ref. [8] where the coupling \( 4\pi^2/t \) corresponds to the temperature \( T \) here. There the critical value \( t_c = 12\pi^2 \) was obtained, corresponding to \( T_c = 1/3 \) in our case and agreeing well with our numerical result.

The main result is that in the 3DLG and the 2DCG, a low-temperature regime exists where positive and negative charges are bound in tight dipole pairs. This regime is separated from a high-temperature regime where at least a finite fraction of charges are free. The results obtained in three dimensions have the same features as those found in 2D. The scaling exponent \( \alpha(T) \) for \( \langle s^2 \rangle \) is zero in the low-temperature phase and positive in the high-temperature phase. Such a change in \( \alpha(T) \) cannot be analytical, and therefore the two scaling regimes must be separated from each other via a phase transition. Since \( \langle s^2 \rangle \) is a measure of the polarizability of the system, we conclude that the above demonstrates a non-analytic change in the polarizability of the system, i.e., a non-analytic change in the dielectric function of the system as a function of temperature. Hence, a system of point charges with overall charge neutrality interacting with a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Results from MC simulations of the 2DCG, where 1.0 \cdot 10^5 sweeps were used at each temperature. a) \( \langle s^2 \rangle \) versus \( T \) for a selection of the simulated system sizes \( L = 8 \), 12, 16, 20, 24, 28, 32, 36, 40 and 48. Errorbars are smaller than the symbols used. b) \( \alpha \) versus \( T \) found from fitting the data of \( \langle s^2 \rangle \) at different \( T \) to \( AL^\alpha \) where \( A \) is a constant. A selection of such fits are shown in c). We note that \( \langle s^2 \rangle \) is practically independent of \( L \) up to a certain \( T \).}
\end{figure}
bare logarithmic interaction undergoes a metal-insulator transition in both 2D and 3D.

The 3DLG can be shown to be equivalent to an anomalous sine-Gordon theory in \( d = 3 \) where the usual \( k^2 \) dispersion is replaced by a \(|k|^3\) one [8], which is non-analytic. This leads to technical difficulties in a standard renormalization group (RG) calculation, where in a perturbative treatment only analytic singularities are generated. Therefore, in a standard RG analysis an analytic correction \( \sim k^2 \) to the dispersion can be generated. On the other hand, since \(|k|^3\) is non-analytic, it cannot be renormalized within a standard RG analysis. It is conceivable that this is one of the reasons why a recent RG analysis of this theory did not find evidence for a dielectric phase [9].

We emphasize that although it is known that a metal-insulator transition occurs in 2D via a KT transition [13], our numerics by themselves do not demonstrate this. To establish the KT nature of the transition on purely numerical grounds requires convincing numerical evidence that there is a universal jump in the inverse dielectric constant at the transition. The main point of the present simulations is that they settle the difficult question of whether a low-temperature dielectric regime exists at all in a 3DLG, which is a system not subject to standard electrostatics, and for which the usual theorems on screening of charges by dipoles do not apply. The answer is in the affirmative, and we have applied our method also to the 2D case for comparison and as a benchmark on the correctness of method of monitoring \( \langle s^2 \rangle \) to establish the existence of two scaling regimes separated by a phase transition.

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References:

[1] S. Sachdev and K. Park, Ann. Phys. (N.Y.) 298, 58 (2002); S. Sachdev, Rev. Mod. Phys. 75, 913 (2003).
[2] R. Moessner, S.L. Sondhi, and E. Fradkin, Phys. Rev. B 65, 024504 (2002).
[3] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M.P.A. Fisher, cond-mat/0311326.
[4] A. M. Polyakov, Nucl. Phys. B 120, 429 (1977).
[5] E. Fradkin and S. H. Shenker, Phys. Rev. D 19, 3682 (1979); see also N. Nagaosa and P. A. Lee, Phys. Rev. B 61, 9166 (2000).
[6] M. Einhorn and R. Savit, Phys. Rev. D 19, 1198 (1979).
[7] T. Senthil and M. P. A. Fisher, Phys. Rev. B 62, 7850 (2000).
[8] H. Kleinert, F. S. Nogueira, and A. Sudbø, Phys. Rev. Lett., 88, 232001 (2002); Nucl. Phys. B 666, 361 (2003).

FIG. 2: Results from simulations of the 3DLG. Up to \( 2 \cdot 10^5 \) MC sweeps were used at each temperature. a) \( \langle s^2 \rangle \) vs \( T \) for some of the simulated system sizes \( L = 8, 12, 16, 20, 24, 28, 32, 36 \) and 40. Error bars are smaller than the symbols used. b) \( \alpha(T) \) vs \( T \) found from fitting the data of \( \langle s^2 \rangle \) at different \( T \) to \( AL^\alpha \) where \( A \) is a constant. A selection of such fits are shown in c). It is evident that a low-temperature regime where \( \langle s^2 \rangle \) is independent of \( L \) exists in the 3D log-gas in the same that that it exists in the 2D log-gas.
[9] I. F. Herbut and B.H. Seradjeh, Phys. Rev. Lett. 91, 171601 (2003); I. F. Herbut, B.H. Seradjeh, S. Sachdev, and G. Murthy, Phys. Rev. B 68, 195110 (2003); M. J. Case, B.H. Seradjeh, and I.F. Herbut, Nucl. Phys. B 676, 572 (2004).

[10] M.N. Chernodub, E.-M. Ilgenfritz, and A. Schiller, Phys. Lett. B 547, 269 (2002); ibid. 555, 206 (2003).

[11] J. Fröhlich and T. Spencer, J. Stat. Phys. 24, 617 (1981).

[12] P. Minnhagen, Rev. Mod. Phys., 59, 1001 (1987).

[13] J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).

[14] J.-R. Lee and S. Teitel, Phys. Rev. B 46, 3247 (1992). See Eqs. (8) and (10).

[15] Periodic boundary conditions are enforced by decomposing the dipole-vectors in cartesian components $s_\alpha, \alpha \in (x, y, z)$ and letting $s_\alpha \to L - s_\alpha$ if $s_\alpha > L/2$.

[16] Y. Saito and H. Müller-Krumbhaar, Phys. Rev. B 23, 308 (1980).

[17] P. Olsson (private communication).