Nonlinear dynamics of topological helicity wave in a frustrated skyrmion string

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A skyrmion in frustrated magnetic system has the helicity degree of freedom. A skyrmion string is formed in a frustrated layered system, which is well described by the XY model owing to the exchange coupling between adjacent layers. We consider a system where the interlayer exchange couplings are alternating, where the dimerized XY model is materialized, whose linear limit is the Su-Schrieffer-Heeger model. We argue that it is a nonlinear topological system. We study the quench dynamics of the helicity wave under the initial condition that the helicity of the skyrmion in the bottommost layer is rotated. It yields a good signal to detect whether the system is topological or trivial. Our results show that the helicity dynamics of the skyrmion string have a rich physics in the modulated exchange-coupled system.

I. INTRODUCTION

Topological solitons are particlelike excitations in continuum field theory [1], which can also be found in magnetic systems [2–14]. For example, the magnetic skyrmion is a representative topological soliton in magnetic materials with chiral or frustrated exchange interactions, which has been extensively studied in the past decade [2–16]. Magnetic skyrmions are versatile objects that can be used for different types of spintronic applications [2–14], mainly including information storage and logic computing [17–22]. A recent research further suggests the possibility of using skyrmions for quantum computing, where information is stored in the quantum degree of helicity [23]. However, the helicity of a skyrmion stabilized by the Dzyaloshinskii-Moriya interaction (DMI) is usually fixed [24]. Namely, the skyrmion stabilized by the bulk-type DMI shows a Bloch-type helicity structure [25, 26], while the one stabilized by the interface-induced DMI shows a Néel-type helicity [27–33]. On the other hand, the skyrmion in a frustrated magnet without DMI do have the helicity degree of freedom provided that the magnetic dipole-dipole interaction (DDI) is weak [24, 34–37]. The presence of DDI may lead to the Bloch-type helicity of a frustrated skyrmion [24, 38]. Most recently, three-dimensional (3D) skyrmion strings attract great attention from the field [39–55], which are materialized in layered magnets and thick magnetic bulks. Therefore, it is an interesting problem to study the helicity dynamics of a 3D skyrmion string in frustrated magnetic system without DMI.

Meanwhile, topological insulators are also among the hottest topics in condensed-matter physics. They are mainly studied in the linear theory. Recently, they are generalized to nonlinear systems including photonic [56–65], mechanical [66–68], and electric circuit [69–71] systems. The quench dynamics under the initial condition with only edge site being excited is a powerful method to differentiate the topological and trivial phases [63, 68, 71]. A finite standing wave is excised in the topological phase but not in the trivial phase.

There are some reports on a topological phase in magnetic systems with low-energy magnon excitations [72–76]. Spin-wave dynamics has been studied in dimerized spin-torque oscillator arrays by using the Holstein-Primakoff transformation [77]. It simulates the Su-Schrieffer-Heeger (SSH) model in magnetic systems, where the bonding is dimerized. The SSH model is a typical model of a topological insulator. Nonlinear dynamics of the non-Hermitian SSH model has also been studied in the same system [78].

In this work, we study the dynamics of the helicity wave along a 3D skyrmion string in a layered frustrated magnet, where the interlayer couplings are alternating. An effective model is described by the dynamical XY model with dimerization, which is a nonlinear model. We study the quench dynamics, where only the helicity at the bottommost magnetic layer is rotated at the initial time. The nonlinearity is controlled by the rotation angle. When the rotation angle is small, the system is approximated by a linear model, and the dynamical XY model is reduced to a kind of the dynamical SSH model. We find that the distinction between the topological and trivial phases remains even in the nonlinear regime.

II. HELICITY DYNAMICS IN FRUSTRATED SKYRMION STRING

A rigid nanoscale skyrmion is a centrosymmetric swirling spin texture, whose collective coordinates are the skyrmion center and the helicity \( \eta (0 \leq \eta < 2\pi) \). The spin texture located at the coordinate center is parametrized as

\[
\mathbf{m}(r) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\]

with

\[
\phi = Q\phi + \eta + \pi/2,
\]

where \( \phi \) is the azimuthal angle \( (0 \leq \phi < 2\pi) \) satisfying

\[
x = r \cos \varphi, \quad y = r \sin \varphi,
\]

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V is the skyrmion number counting how many times \( m \) magnetic DDI energy is neglected in the present work. However, we analyze only the dynamics of the helicity by accounting for the presence of the factor \( \eta \), which is different from the conventional definition by the Néel-type structure [25, 26] (Néel-type [27–33] structure is realized by the DMI is absent. The skyrmion energy depends on the helicity between adjacent layers is described by the XY model, 

\[
H_{\text{inter}} = - \sum_i J_i \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right),
\]

(7)

because \( S_i^z = 0 \). By inserting Eq. (1) into this equation with \( \theta = \pi/2 \), we obtain 

\[
H_{\text{inter}} = - \sum_i J_i \cos (\eta_i - \eta_{i+1}).
\]

(8)

The kinetic term is given by [79]

\[
H_{\text{kine}} = m \sum_i \frac{\eta_i^2}{2},
\]

(9)

from which the equations of motion are derived,

\[
\begin{align*}
\dot{\eta}_i &= - \sum_i \left[ J_i \sin (\eta_i - \eta_{i+1}) + J_{i-1} \sin (\eta_i - \eta_{i-1}) \right] \\
& \quad - 2V \sum_i \sin 2\eta_i.
\end{align*}
\]

(11)

We may choose the alternating interlayer coupling,

\[
J_i = J \left( 1 + \lambda (-1)^i \right),
\]

(12)

or \( J_i = J_A \) for even \( i \) and \( J_i = J_B \) for odd \( i \) with

\[
J_A = J (1 + \lambda), \quad J_B = J (1 - \lambda).
\]

(13)

The exchange coupling can be controlled by modulating the thickness of the spacer between adjacent magnetic layers. In this way, the skyrmion string is made a dimerized system.

The equations of motion read

\[
\begin{align*}
m \frac{d^2 \eta_{2n-1}}{dt^2} &= J_A \sin (\eta_{2n} - \eta_{2n-1}) \\
& \quad + J_B \sin (\eta_{2n-2} - \eta_{2n-1}) - 2V \sin 2\eta_{2n-1},
\end{align*}
\]

(14)

\[
\begin{align*}
m \frac{d^2 \eta_{2n}}{dt^2} &= J_B \sin (\eta_{2n+1} - \eta_{2n}) \\
& \quad + J_A \sin (\eta_{2n-1} - \eta_{2n}) - 2V \sin 2\eta_{2n}.
\end{align*}
\]

(15)

We analyze the system under the initial condition,

\[
\eta_i(t) = \xi \pi \delta_{1,1}, \quad \eta_i(t) = 0 \quad \text{at} \quad t = 0.
\]

(16)

Namely, we rotate the helicity of the bottommost layer initially, and investigate how a helicity wave propagates along the skyrmion string as time evolves.
We note that the helicity of skyrmion can be controlled by applying a spin current [24]. As shown in Fig. 1, we assume that the layered frustrated magnet is placed upon a heavy-metal substrate, which may be fabricated experimentally in a bottom-up fashion. If we apply a charge current in the heavy-metal substrate, which may be fabricated experimentally in a bottom-up fashion. If we apply a charge current in the heavy-metal substrate, a vertical spin current will be generated and injected into the bottommost magnetic layer due to the spin Hall effect, which only drives the rotation of the skyrmion helicity in the bottommost magnetic layer. In addition to the rotation of the helicity, the center of the skyrmion also rotates [24].

III. LINEAR THEORY

A. SSH model

We first investigate the system where the initial helicity rotation $\xi \pi$ is tiny, $|\xi| \ll 1$, which we call the linear regime. Indeed, the equations of motion are linearized and given by

$$m \eta_i = - [J_i (\eta_i - \eta_{i+1}) + J_{i-1} (\eta_i - \eta_{i-1})] - 4V \eta_i,$$

(17)

since we may assume $|\eta_i - \eta_{i+1}| \ll 1$ and $|\eta_i| \ll 1$. They are summarized as

$$m \eta_i = \sum_j H^\text{Linear}_{ij} \eta_j,$$

(18)

where

$$H^\text{Linear}_{ij} = H^\text{SSH}_{ij} - (J_A + J_B + 4V) \delta_{ij},$$

(19)

with

$$H^\text{SSH}_{ij} = J_A (\delta_{2i,2j-1} + \delta_{2j,2i-1}) + J_B (\delta_{2i,2j+1} + \delta_{2j,2i+1}).$$

(20)

This is the SSH model. It is expressed as

$$H^\text{SSH} (k) = \begin{pmatrix} 0 & J_A + J_B e^{-ik} \\ J_A + J_B e^{ik} & 0 \end{pmatrix},$$

(21)

in the momentum space.

The SSH model (20) is illustrated in Fig. 2, where blue disks stand for skyrmions located in the layer $i$, while red and blue lines indicate the couplings $J_A$ and $J_B$, respectively. There are two special limits, that is, the system has two isolated edges at $i = 1$ and $L$ for $\lambda = -1$ as in Fig. 2(a), while all of the states are dimerized for $\lambda = 1$ as in Fig. 2(c).

![FIG. 2. Illustration of a skyrmion string, where blue disks represent skyrmions, while red and green bonds represent the couplings $J_A$ and $J_B$. (a) Isolated limit $\lambda = 1$, (b) intermediate state $|\lambda| < 1$ and (c) dimer limit $\lambda = 1$.](image)

We show $\psi^\text{b}_n$ localized at the bottom edge $n = 1$ and the top edge $n = L$ of the skyrmion string. They correspond to the two isolated disks in Fig. 2(a).

We consider the homogeneous system with $\lambda = 0$, where the equations of motion are simply given by

$$m \eta_i = J \sum_i [\eta_{i+1} + \eta_{i-1} - 2\eta_i] - 4V \eta_i.$$
The helicity wave along the skyrmion string.

The effective mass into Eq. (24), we obtain the velocity of the helicity wave as

\[ m \eta_1 = -J \sin (\eta_1 - \eta_2), \]
\[ m \eta_2 = -J \sin (\eta_2 - \eta_1), \]

which are equivalent to

\[ m (\eta_1 + \eta_2) = 0, \]
\[ m (\eta_1 - \eta_2) = -2J \sin (\eta_1 - \eta_2). \]

The solution is given by

\[ \eta_1(t) = -\eta_2 + c_3 = am \left[ \frac{\sqrt{(4J + c_1)(t + c_2)^2}}{2}, \frac{8J}{4J + c_1} \right], \]

where \( c_1, c_2, \) and \( c_3 \) are determined from the initial condition, or by solving \( \eta_1(0) = \xi \pi \). This solution indicates that there is an oscillation in the two layers at the bottom edge at \( \lambda = 1 \), and hence we call it a dimer state. Numerical analysis shows the emergence of the dimer state also at \( \lambda \neq 1 \), forming a dimer phase in the nonlinear regime, as we will discuss.

IV. NONLINEAR THEORY

We proceed to analyze the system where the initial helicity rotation \( \xi \pi \) is not tiny, which we call the nonlinear regime.

A. Dimer system

We have noticed the emergence of an isolated point at \( \lambda = 1 \) in the energy spectrum of the linear theory as in Fig. 5(c). We explore the physics of this point at \( \lambda = 1 \), where the system is perfectly dimerized as in Fig. 2(c). We set \( V = 0 \) in order to obtain analytic solution. In this case, the equations of motion are simply given by

\[ m \eta_1 = -J \sin (\eta_1 - \eta_2), \]
\[ m \eta_2 = -J \sin (\eta_2 - \eta_1), \]

We have set \( m = 1, J = 1 \) and \( V = 0.1 \). We have used a finite chain with length \( L = 50 \).

The continuum limit reads

\[ m \eta (x) = J \frac{\partial^2 \eta}{\partial x^2} - 4V \eta, \]

which is the wave equation. By inserting the linear wave ansatz

\[ \eta = \exp [i (kx - \omega t)] \]

into Eq. (24), we obtain the velocity of the helicity wave as

\[ v = \frac{\omega}{k} = \sqrt{\frac{J - 4V/k^2}{m}}. \]

The effective mass \( m \) is determined in terms of the velocity of the helicity wave along the skyrmion string.

B. Quench dynamics of helicity wave

We analyze the quench dynamics of the system under the initial condition (16) numerically for \(-1 \leq \lambda \leq 1\) and for \( 0 \leq \xi \leq 1 \). The time evolution of \( \sin \eta_i \) is shown in Fig. 4. There is a finite stationary oscillation at the site \( \xi = 1 \) in the topological phase as shown in Fig. 4(a2), but this is not the case in the trivial phase as shown in Fig. 4(b2). This feature holds also for strong nonlinear regime as shown in Fig. 4(c2) and (d2).

Figure 4 indicates that the amplitude after long enough time is a good signal to detect whether the system is topological or trivial. To detect it quantitatively, we define an indicator with the use of the maximum value of \( \sin |\eta_1| \) as

\[ \Phi(\lambda, \xi) = \max_{0.9T < t < T} [\sin |\eta_1|], \]

taking large enough \( T \) so that the time evolution of \( \eta_1 \) becomes stationary. We show \( \Phi(\lambda, \xi) \) for \(-1 \leq \lambda \leq 1\) by taking typical values of \( \xi \) in Fig. 5. In the weak nonlinear regime, \( \Phi \) is finite in the topological phase while it is almost zero in the trivial phase, as shown in Fig. 5(a). As the increase of the nonlinearity \( (\xi \to 1) \), the finite region of \( \Phi \) is expanded in the vicinity of \( \lambda = 1 \), forming the dimer phase, as shown in Fig. 5(b)~(d).

There is only slight difference in the indicator \( \Phi \) for various \( V \) in the topological phase as shown in Fig. 5. On the other hand, the peak value of \( \Phi \) is identical between \( \lambda = 1 \) and \( \lambda = -1 \) due to the energy conservation.
FIG. 5. Phase indicator $\Phi$ as a function of the dimerization $\lambda$. (a) $\xi = 0.1$, (b) $\xi = 0.5$, $\xi = 0.75$ and $\xi = 1$. Black curves indicate $V = 0$, magenta curves indicate $V = 0.1$ and cyan curves indicate $V = 0.2$. We have set $V = 0$, $m = 1$, $J = 1$, $L = 50$ and $T = 100$.

FIG. 6. Phase indicator $\Phi$ in the $\lambda$-$\xi$ plane. (a) Bird’s eye’s view and (b) the top view. (c) The color palette indicates the amplitude in (a) and (b). We have set $V = 0$, $m = 1$, $J = 1$, $L = 50$ and $T = 100$. (d) Schematic illustration of the phase diagram.

C. Phase diagram

The indicator $\Phi$ is shown in the $\lambda$-$\xi$ plane in Fig. 6. We find three phases: the topological, trivial and dimer phases. In the isolated limit $\lambda = -1$, $\Phi = \sin \xi \pi$ for $\xi \leq 1/2$ and $\Phi = 1$ for $1/2 \leq \xi \leq 1$. The topological distinction is hard to see for $\xi \simeq 0$ because $\sin \xi \pi$ is very small. However, there is a clear distinction between the topological and trivial phase as shown in Fig. 5(a). The phase boundary between the topological and trivial phase is always $\lambda = 0$ even in the nonlinear regime. On the other hand, the dimer phase emerges in the trivial phase when the nonlinearity exists. The region of the dimer phase consists of a point at $\xi = 0$ but occupies a quite large area for $\xi \gtrsim 1/2$.

V. DISCUSSION AND CONCLUSION

In conclusion, we have explored the helicity dynamics of a skyrmion string in a layered frustrated magnet, where the interlayer coupling is alternating. The topological physics of the SSH model well survives although the governing equation is nonlinear model. Our results show that an introduction of the interlayer degree of freedom may give us a rich physics in the dynamics of a skyrmion string. It is an interesting problem to use a skyrmion string as an information transmission channel.

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