Abstract

We prove an $L^p(I, C^α(Ω))$ regularity result for a reaction-diffusion equation with mixed boundary conditions, symmetric $L^∞$ coefficients and an $L^∞$ initial condition. We provide explicit control of the $L^p(I, C^α(Ω))$ norm with respect to the data. To prove our result, we first establish $C^α(Ω)$ control of the stationary equation, extending a result by Haller-Dintelmann et al. (2009).

Keywords: Elliptic problems, Parabolic problems, Maximal regularity, Mixed boundary value problems

I Introduction

In this article we are interested in the $L^p(I, C^α(Ω))$ regularity of the solution $v$ to a reaction-diffusion equation of the form

$$d(tv - \text{div}(Dv)) + v = f, \quad \text{in } I \times Ω$$

$$v(0) = v_0, \quad \text{in } Ω$$

$$v(t, x) = 0, \quad t \in I, x \in Γ_D,$$

$$n \cdot D(x)v(t, x) = 0, \quad t \in I, x \in Γ_N.$$

where $Ω \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded Lipschitz domain, $\partial Ω = Γ_D \cup Γ_N$ is a partition of the boundary in Dirichlet and Neumann part, $I = [0, T]$ is a finite time interval, and $f \in L^p(I, L^2(Ω))$. In particular, we consider the case where the initial condition $v_0$ only has regularity $v_0 \in L^∞(Ω)$ and we make few assumptions on the remaining data. Our main result is the following.

Theorem 1. Let $Ω \subset \mathbb{R}^d$ with $d = 2, 3$ be a Lipschitz domain, $I = [0, T]$ a time interval, $\partial Ω = Γ_N \cup Γ_D$ a partition of the boundary into a Dirichlet and a Neumann part, where both $Γ_N$ and $Γ_D$ are allowed to have vanishing measure. Assume that $Ω \cup Γ_N$ is Gröger regular, let $f \in L^p(I, L^2(Ω))$ for $p \in [2, \infty)$, $D \in L^∞(Ω, \mathbb{R}^d)$ be symmetric and elliptic with ellipticity constant $ν > 0$. For $v_0 \in L^∞(Ω)$ denote by $v \in H^1(I, H^1_D(Ω), H^1_D(Ω)^*)$ the solution to

$$\int_I \langle \partial_t v, \partial_t \rangle_{H^1_D(Ω)} dt + \int_I \int_Ω Dv \cdot v dx dt = \int_I \int_Ω f \cdot v dx dt \quad \text{in } L^2(I, H^1_D(Ω)^*)$$

$$v(0) = v_0.$$

Then there is $β = β(p) \in (0, 1)$ such that $v \in L^p(I, C^β(Ω))$ and we may estimate

$$\|v\|_{L^p(I, C^β(Ω))} \leq C (Ω, T, ν, \|D\|_{L^∞(Ω, \mathbb{R}^d)}, p, β) \cdot \left[\|f\|_{L^p(I, L^2(Ω))} + \|v_0\|_{L^∞(Ω)}\right].$$

(1)
In the above estimate, if we fix $\Omega$ and $p$, only a lower bound for $v_0$ and upper bounds for $\|D\|$ and $T$ determine the value of the constant $C$. The regularity estimate is thus uniform for $v \in [c_\varepsilon, C_\varepsilon]$, $D \in L^\infty(\Omega, M_\varepsilon)$ with $\|D\| \leq C_B$ and time intervals $I^* = [0, T^*]$ with $T^* \leq T$.

The crucial detail in the above theorem is the fact that $v_0$ lies only in the space $L^\infty(\Omega)$ and not in the trace space for the initial conditions. Therefore, well known maximal regularity results, for example [Amann (1995)], cannot be applied directly. We thus split the problem into two equations, one with homogeneous right-hand side and one with homogeneous initial condition and analyze them separately. We remark that we are only concerned with spatial dimensions two and three and that our proof does not extend beyond this. The reason lies in the stationary counterpart of the result, Theorem 5.1 in [Haller-Dintelmann et al. (2009)], where this restriction on the dimension appears.

There are a number of reasons to study regularity properties of equations with non-smooth data. Often, mixed boundary conditions are dictated by concrete applications and this alone leads to a considerable loss of regularity, at least if regularity up to the boundary is needed, see [Savare (1997)]. [Kassmann and Madych (2004)]. Another reason to study problems with non-smooth data comes from multi-physics problems, i.e., coupled systems of equations. To prove existence results for coupled systems one usually employs an ansatz based on a fixed-point theorem and successively solves the equations. This leads to problems with low regularity data as it may be necessary to frame the fixed point problem in a low regularity setting.

Providing explicit norm control in the sense of Theorem 4 is useful, e.g., for (PDE constrained) optimization problems where the existence of a solution is established by the direct method of the calculus of variations and thus bounds on the solution independent of the data are required. In forthcoming work we discuss how our main result is crucial in establishing the existence of an optimal control function in a PDE constrained optimization problem stemming from tissue engineering.

The proof of Theorem 4 crucially relies on a counterpart for the stationary problem. The stationary result is available in the literature, see Theorem 5.1 in [Haller-Dintelmann et al. (2009)], albeit without the information on the norm control. We therefore revisit the proof of Theorem 5.1 in [Haller-Dintelmann et al. (2009)] and provide the missing estimates required for explicit norm control. For the reasons given above, this is of independent interest, so we provide Theorem 4 for a precise statement of our quantitative version of Theorem 5.1 in [Haller-Dintelmann et al. (2009)].

There are several results in the literature that treat the regularity of elliptic and parabolic equations subject to Dirichlet-Neumann conditions for non-smooth domains and rough coefficients. We briefly discuss the ones most closely related to our contribution. The mere Hölder regularity of elliptic equations for mixed boundary conditions was already established in [Haller-Dintelmann et al. (2009)], however without explicit control of the Hölder norm and the implications for parabolic problems where recognized in [Disser et al. (2017)]. There, a $C^\alpha(I \times \Omega)$ regularity result for a diffusion equation (without reaction term) was provided, however, the authors neither consider $L^\infty(\Omega)$ initial conditions (under which the $C^\alpha(I \times \Omega)$ regularity can not hold in general) nor do they provide explicit control of the Hölder norm in terms of the data. Other works focus on the maximal regularity of parabolic equations in distribution spaces, see for instance [Haller-Dintelmann and Rehberg (2011)] or maximal regularity questions for non-autonomous equations, see [Disser et al. (2017)].

1.1 Preliminaries and Notation

Let $\Omega \subset \mathbb{R}^d$, for $p \in [1, \infty]$ and $k \in \mathbb{N}$ we denote by $L^p(\Omega)$ the space of $p$-integrable functions, by $W^{k,p}(\Omega)$ the subset of $L^p(\Omega)$ of $k$-times weakly differentiable functions. When $\Gamma_D \subset \partial \Omega$, we denote by $W^{1,p}_0(\Omega)$ the subset of $W^{1,p}(\Omega)$ consisting of functions that vanish on $\Gamma_D$ in the trace sense. This space coincides with the closure of $C_\varepsilon^\infty(\Omega)$ in $W^{1,p}(\Omega)$ when $\Omega$ is a Lipschitz domain, see the definition below. If $p = 2$ we write $H^k(\Omega)$ and $H^1_0(\Omega)$ instead of $W^{k,2}(\Omega)$ and $W^{1,2}_0(\Omega)$. By $C^\alpha(\Omega)$ we denote the space of $\alpha$ Hölder continuous functions for $\alpha \in (0, 1)$. The topological dual space of a Banach space $X$ is denoted by $X'$.

We call a bounded, open set $\Omega \subset \mathbb{R}^d$ a Lipschitz domain if $\overline{\Omega}$ is a Lipschitz manifold with boundary, this definition is adopted from [Grisvard (2011), Definition 1.2.1.2]. We denote the cube $[-1, 1]^d \subset \mathbb{R}^d$ by $Q$, its half $\{x \in Q \mid x_d \leq 0\}$ by $Q_-$, the hyperplane $\{x \in Q \mid x_d = 0\}$ by $\Sigma$ and $\{x \in \Sigma \mid x_{d-1} < 0\}$ by $\Sigma_0$. The next definition goes back to Gröger, see [Gröger (1989)].
Definition 2 (Gröger Regular Sets). Let $\Omega \subset \mathbb{R}^d$ be bounded and open and $\Gamma \subset \partial \Omega$ a relatively open set. We call $\Omega \cup \Gamma$ Gröger regular, if for every $x \in \partial \Omega$ there are open sets $U, V \subset \mathbb{R}^d$ with $x \in U$, and a bijective, bi-Lipschitz map $\phi : U \to V$, such that $\phi(x) = 0$ and $\phi(U \cap (\Omega \cup \Gamma))$ is either $Q_-, Q_- \cup \Sigma$ or $Q_- \cup \Sigma_0$.

It can easily be shown that Gröger regular sets $\Omega$ (no matter the choice $\Gamma \subset \partial \Omega$) are Lipschitz domains, we refer to ([Haller-Dintelmann et al., 2009], Theorem 5.1). The notion of Gröger regularity is very weak and many applications fall in this category. This claim is supported by the following characterization of Gröger regular sets in two and three dimensions that allow to check Gröger regularity almost “by appearance”.

Theorem 3 (Gröger Regular Sets in 2D, Theorem 5.2 in [Haller-Dintelmann et al. 2009]). Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $\Gamma \subset \partial \Omega$ be relatively open. Then $\Omega \cup \Gamma$ is Gröger regular if and only if $\bar{\Gamma} \cap (\partial \Omega \setminus \Gamma)$ is finite and no connected component of $\partial \Omega \setminus \Gamma$ consists of a single point.

Theorem 4 (Gröger Regular Sets in 3D, Theorem 5.4 in [Haller-Dintelmann et al. 2009]). Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and $\Gamma \subset \partial \Omega$ be relatively open. Then $\Omega \cup \Gamma$ is Gröger regular if and only if the following two conditions hold

(i) $\partial \Omega \setminus \Gamma$ is the closure of its interior.

(ii) For any $x \in \bar{\Gamma} \cap (\partial \Omega \setminus \Gamma)$ there is an open neighborhood $U_x$ of $x$ and a bi-Lipschitz map $\phi : U_x \cap \bar{\Gamma} \cap (\partial \Omega \setminus \Gamma) \to (-1, 1)$.

In the definition of Gröger regular sets, the local model $\{x \in Q \mid x_d < 0\}$ and $\{x \in Q \mid x_d = 0, x_{d-1} < 0\}$ is redundant. We also cite the following.

Lemma 5 (Lemma 4.10 in [Haller-Dintelmann et al., 2009]). There exists a bi-Lipschitz mapping $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ mapping $Q_- \cup \Sigma_0$ onto $Q_- \cup \Sigma$.

II An Elliptic Mixed Boundary Value Problem

In this section we prove a Hölder regularity result for linear elliptic equations with mixed boundary conditions and measurable, bounded coefficients with explicit control of the Hölder norm in terms of the data. It is the stationary counterpart of Theorem 1 and of independent interest. The theorem is in the spirit of Stampacchia (1960). However, we extend the results from Stampacchia (1960) to Lipschitz domains with a very weak compatibility condition on the Dirichlet-Neumann partition $\partial \Omega = \Gamma_D \cup \Gamma_N$ of the boundary. We follow closely the proof in Haller-Dintelmann et al. (2009) and extend it by explicitly controlling the appearing constants.

2.1 Elliptic Result

Theorem 6 (Quantitative Hölder Control for Mixed Boundary Value Problems). Let $\Omega \subset \mathbb{R}^d$ be bounded and open with $d \in \{2, 3, 4\}$, consider a partition $\partial \Omega = \Gamma_N \cup \Gamma_D$ into Neumann and Dirichlet boundary and assume that $\Omega \cup \Gamma_N$ is Gröger regular. Let $M \subset L^\infty(\Omega, \mathbb{R}^{d \times d})$ be a set of matrix-valued, measurable functions with a common lower bound $\nu > 0$ on the ellipticity constants and a common upper bound $M$ on the $L^\infty(\Omega, \mathbb{R}^{d \times d})$ norm. For $A \in M$ define the operator

$$- \operatorname{div}(A \nabla \cdot) + 1 : H^1_0(\Omega) \to H^1_0(\Omega)^*, \quad u \mapsto \int_{\Omega} A \nabla u \cdot \nabla \cdot u \cdot dx.$$  (2)

Then, for every $q > d$ and $A \in M$ there exists $\alpha > 0$ such that

$$(- \operatorname{div}(A \nabla \cdot) + 1)^{-1} : W^{1,q}_D(\Omega) \to C^{\alpha}(\Omega)$$

is continuous. Stronger, for all $A \in M$ we may choose the same $\alpha > 0$ and can estimate the operator norms

$$\sup_{A \in M} \|(- \operatorname{div}(A \nabla \cdot) + 1)^{-1}\|_{L(W^{1,q}_D(\Omega), C^{\alpha}(\Omega))} < \infty.$$  (3)

Proof. The idea of the proof is to localize the equation by a partition of unity, additionally employing the Lipschitz transformations from the definition of a Gröger regular set. Using a suitable reflection technique at the Neumann boundary, this allows to apply Hölder regularity results for pure, homogeneous Dirichlet problems either on a ball or a cuboid. In these cases quantitative regularity results exist. The details of the
proof are carried out throughout this section. As only the quantitative aspects of the transformations are missing, we pay special attention to these and keep the remaining aspects of the proof brief, referring to Haller-Dintelmann et al. (2009) when necessary.

2.2 Known Regularity Results

We review the known regularity results that we need in the proof of the main theorem. We begin with a classical Hölder regularity result for elliptic equations without mixed boundary conditions.

**Theorem 7** (Theorem C.2 in Kinderlehrer and Stampacchia (2000)). Let \( \Omega \subset \mathbb{R}^d \) be a ball or a cuboid, \( f \in L^q(\Omega, \mathbb{R}^d) \) with \( q > d \) and \( q > 2 \). Assume that \( A \in L^\infty(\Omega, \mathbb{R}^{d \times d}) \) is uniformly elliptic with ellipticity constant \( \nu > 0 \) and \( L^\infty(\Omega, \mathbb{R}^{d \times d}) \) bound \( M > 0 \). Then, there exist \( K = K(\nu, M, \Omega, d) \) and \( \alpha = \alpha(\nu, M, \Omega, d) \in (0, 1) \) such that for the solution \( u \in H^1_0(\Omega) \) of

\[
\int_{\Omega} A \nabla u \nabla (\cdot) \, dx = \int_{\Omega} f \cdot \nabla (\cdot) \, dx \quad \text{in } H^1_0(\Omega),
\]

it holds \( u \in C^0(\Omega) \) and

\[
\max_{\overline{B}(x)} u(x) - \min_{\overline{B}(x)} u(x) = \text{osc} \, u \leq K\|f\|_{L^q(\Omega, \mathbb{R}^d)} \cdot r^\alpha.
\]

**Proof.** In Kinderlehrer and Stampacchia (2000), the result is proven for domains of class \( s \) which trivially includes balls and cuboids. For us the result for balls and cuboids suffices. \( \square \)

The above result implies a control of the Hölder norm. We collect this fact in a Corollary.

**Corollary 8.** Assume we are in the situation of Theorem 7. Then

\((- \text{div } AV)^{-1} : W^{1,q}_0(\Omega)^* \rightarrow C^0(\Omega)\)

is well defined and continuous with its operatornorm bounded by

\[
\|(- \text{div } AV)^{-1}\|_{L(W^{1,q}_0(\Omega)^*, C^0(\Omega))} \leq K,
\]

with \( K = K(\nu, M, \Omega, d) \), however, possibly different from the constant \( K \) in Theorem 7.

**Proof.** We begin by showing that (4) yields a bound on the \( C^0(\Omega) \) norm of a solution \( u \) to \(- \text{div } (AVu) = f\).

To this end, take \( x, y \in \overline{\Omega} \) and consider the closed ball around \( x \) with radius \( r = |x - y| \). Then, \( y \in B_r(x) \) and (4) yields

\[
|u(x) - u(y)| \leq \text{osc} \, u \leq K\|f\|_{L^q(\Omega, \mathbb{R}^d)}|x - y|^\alpha,
\]

hence

\[
|u|_{C^0(\Omega)} \leq K\|f\|_{L^q(\Omega, \mathbb{R}^d)}.
\]

To bound the \( C^0(\Omega) \) norm of \( u \), note that \( u \) vanishes on the boundary of \( \Omega \). Let \( x \in \overline{\Omega} \) and \( x_0 \in \partial \Omega \) and use again (4) to estimate

\[
|u(x)| \leq |u(x) - u(x_0)| \leq K\|f\|_{L^q(\Omega, \mathbb{R}^d)}|x - x_0|^\alpha \leq K\|f\|_{L^q(\Omega, \mathbb{R}^d)} \text{diam}(\Omega)^\alpha.
\]

Hence,

\[
|u|_{C^0(\Omega)} \leq K \max(1, \text{diam}(\Omega)^\alpha)|f|_{L^q(\Omega, \mathbb{R}^d)}.
\]

To conclude the proof, note that any abstract functional \( \phi \in W^{1,q}_0(\Omega)^* \) can be written in the form

\[
\phi = \int_{\Omega} f \cdot \nabla (\cdot) \, dx
\]

for some \( f \in L^q(\Omega, \mathbb{R}^d) \) and clearly it holds for a constant \( c = c(\Omega, d) \)

\[
\|\phi\|_{W^{1,q}_0(\Omega)^*} \leq \|f\|_{L^q(\Omega, \mathbb{R}^d)} \leq c \cdot \|\phi\|_{W^{1,q}_0(\Omega)^*}.
\]

Thus, we can estimate the operatornorm

\[
\|(- \text{div } AV)^{-1}\|_{L(W^{1,q}_0(\Omega)^*, C^0(\Omega))} \leq c \cdot K \max(1, \text{diam}(\Omega)^\alpha)
\]

as asserted. \( \square \)
The next result concerns higher integrability of the gradient of the solution of an elliptic equation subjected to mixed boundary conditions. It is essentially to Gröger, see for example Gröger (1989); Gröger and Rehbberg (1989) for the original work and Haller-Dintelmann et al. (2016) for a more recent proof that weakens the assumptions on the domain even further. However, we stay in the realm of Gröger regular sets as this seems general enough for the applications we have in mind.

**Theorem 9** (Higher Gradient Integrability, Theorem 5.6 in Haller-Dintelmann et al. (2016)). Let $\mathcal{M} \subset L^\infty(\Omega, \mathbb{R}^{d \times d})$ be a set of matrix valued functions with a common lower bound $\nu > 0$ on the ellipticity constants and a common upper bound $M > 0$ on the $L^\infty(\Omega, \mathbb{R}^{d \times d})$ norm. Furthermore, assume that $\Omega \cup \Gamma_N$ is Gröger regular. Then, there is an open interval $I_M$ around 2 such that for all $A \in \mathcal{M}$ and $p \in I_M$

$$-
abla(\nabla \psi) + 1 : W^{1,p}_D(\Omega) \rightarrow W^{1,p}_D(\Omega)^*$$

is a linear homeomorphism and we have

$$\sup_{p \in I_M} \sup_{A \in \mathcal{M}} \left\| \left( -\nabla(\nabla \psi) + 1 \right)^{-1} \right\|_{L^p_0(\Omega), L^p(\Omega)} < \infty.$$  

*Proof.* This is a specialized version of Theorem 5.6 in Haller-Dintelmann et al. (2016). We need to guarantee that our assumptions imply the Assumptions 2.3, 3.1 and 5.4 in the notation of that paper (which they a fortiori do). In fact, Gröger regular sets are Lipschitz domains and this ensures Assumption 2.3 in Haller-Dintelmann et al. (2016) and also Assumption 4.11 there. Then, Assumption 4.11 implies Assumption 3.1 as shown in Theorem 4.15 in Haller-Dintelmann et al. (2016). Finally, Assumption 5.4 only requires ellipticity and measurability of the functions $A \in \mathcal{M}$, a fact that we also assumed. \hfill \bbox

### 2.3 Technical Lemmas

As the strategy to prove Theorem 6 consists of localization techniques we investigate in the following technical lemmas how this affects the Hölder control we are interested in. The localization goes through three possible stages: i) a localization by a partition of unity. This involves analyzing how the equation is changed when the solution is multiplied by a smooth cut-off function, ii) in the vicinity of $\partial \Omega$, the Lipschitz transformations to cuboids from the definition of Gröger regular sets need to be employed. This yields a pure Dirichlet problem for the Dirichlet boundary, iii) at the Neumann boundary a reflection technique is used to also produce a pure Dirichlet problem.

The following is a quantitative version of Lemma 4.6 in Haller-Dintelmann et al. (2009).

**Lemma 10.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded with a partition $\partial \Omega = \Gamma_D \cup \Gamma_N$ in Dirichlet and Neumann boundary parts. Furthermore, let $\Omega \cup \Gamma_N$ be regular and $U \subset \mathbb{R}^d$ open, such that $\Omega_* := \Omega \cap U$ is also a Lipschitz domain. Furthermore, set $\Gamma_* := \Gamma_D \cap U$ and let $\eta \in C^0(\mathbb{R}^d)$ with support in $U$. For arbitrary but fixed $q \in [1, \infty)$ define the maps

(i) The multiplication-restriction operator

$$R_\eta : W^{1,q}_D(\Omega) \rightarrow W^{1,q}_*(\Omega_*), \quad v \mapsto \eta v |_{\Omega_*}.$$  

(ii) The multiplication-extension operator

$$E_\eta : W^{1,q}_*(\Omega_*) \rightarrow W^{1,q}_D(\Omega), \quad v \mapsto \tilde{\eta} v.$$  

Here, the map $v \mapsto \tilde{\eta}$ denotes the extension by zero outside of $\Omega_*$. Then, both maps are well defined, linear and continuous and we may estimate

$$\|\eta v |_{\Omega_*}\|_{W^{1,q}_*(\Omega_*)} \leq 2\|\eta\|_{C(\Omega_*)} \|v\|_{W^{1,q}_D(\Omega)} \quad \& \quad \|\tilde{\eta} v\|_{W^{1,q}_D(\Omega)} \leq 2\|\eta\|_{C(\Omega_*)} \|v\|_{W^{1,q}_*(\Omega_*)}.$$  

*Proof.* The well definedness of $R_\eta$ and $E_\eta$ was established in Lemma 4.6 in Haller-Dintelmann et al. (2009). The estimates can be computed in the following way

$$\|\eta v |_{\Omega_*}\|_{W^{1,q}_*(\Omega_*)} = \|\eta v\|_{L^q(\Omega)} + \|\nabla(\eta v)|_{L^q(\Omega)} \leq \|\eta \|_{L^\infty(\Omega_*)} \|v\|_{L^q(\Omega)} + \|\nabla v\|_{L^q(\Omega_*)} \leq 2\|\eta\|_{C(\Omega_*)} \|v\|_{W^{1,q}_D(\Omega)},$$

\hfill \bbox
We also need a quantitative version of Lemma 4.7 in Haller-Dintelmann et al. (2009).

**Lemma 11.** Let $\Omega, \Gamma_N, \Gamma_D, \mathcal{U}, \eta, \Omega_*, \Gamma_*, R_0$ and $E_\eta$ be as in Lemma 10 and denote by $A_\eta$ the restriction of a function $A \in L^{\infty}(\Omega, \mathbb{R}^{d \times d})$ to the set $\Omega_*$. For $f \in H^1_D(\Omega)^*$ denote by $v_f \in H^1_D(\Omega)$ the function that satisfies

$$- \text{div}(A v_f) + v_f = f, \quad \text{in } H^1_D(\Omega)^*.$$ 

Define the maps

(i) The adjoint map of $E_\eta$ for $q \in (1, \infty)$

$$E^*_\eta : W^{1,q'}_\Gamma(\Omega)^* \to W^{1,q'}_\Gamma(\Omega), \quad f \mapsto f(\overline{\eta(\cdot)}) =: f_*.$$ 

(ii) The functional $T_{v_f}$

$$T_{v_f} : H^1_D(\Omega) \to \mathbb{R}, \quad w \mapsto \iint_{\Omega_*} v_A \nabla \eta \nabla w \, dx.$$ 

Then, the localization of $v_f$ by $\eta$, i.e., $u_f := (\eta v)_\Omega$, satisfies the equation

$$- \text{div}(A \nabla u_f) = -(\eta v)_{\Omega} - (A \nabla v_f)_{\Omega}, (\nabla \eta)_{\Omega} + T_{v_f} + f_* =: f^* \quad \text{in } H^1_D(\Omega)^*. \quad (5)$$

Furthermore, if $2 \leq d \leq 4$ and $f \in W^{1,q'}_\Gamma(\Omega)^*$ with $q > d$, then there exists $p > d$ such that $f^* \in W^{1,p'}_\Gamma(\Omega)^*$ and the map

$$\text{Loc} : W^{1,q'}_\Gamma(\Omega)^* \to W^{1,p'}_\Gamma(\Omega)^*, \quad f \mapsto f^*$$

possesses an estimate on its operator norm only depending on $\nu$ and $\Omega$, i.e.,

$$||f^*||_{W^{1,p'}_\Gamma(\Omega)^*} \leq C(\Omega, \nu, M)||f||_{W^{1,q'}_\Gamma(\Omega)^*}. \quad (6)$$

**Proof.** Our extension of Lemma 4.7 in Haller-Dintelmann et al. (2009) is the explicit norm control in (6). To this end, we treat the terms in (5) separately. First, note that there is $\epsilon > 0$ such that

$$W^{1,4/3-\epsilon}_\Gamma(\Omega_*) \hookrightarrow L^{4/3}(\Omega_*)$$

and we set $p_1' = 4/3 - \epsilon$ which implies $p_1 > 4$. We then compute for $w \in W^{1,q'}_\Gamma(\Omega_*)$

$$\iint_{\Omega_*} \eta v_f \, dx \leq ||\eta||_{L^{\infty}(\Omega_*)} ||v||_{L^{q'}(\Omega_*)} ||w||_{L^{p_1'}(\Omega_*)} \leq C(\Omega) ||\eta||_{L^{\infty}(\Omega_*)} ||v||_{H^1_D(\Omega)} ||w||_{W^{1,p_1'}_\Gamma(\Omega_*)} \leq C(\Omega, \nu) ||\eta||_{L^{\infty}(\Omega_*)} ||f||_{H^{1,q'}_\Gamma(\Omega)} ||w||_{W^{1,p_1'}_\Gamma(\Omega_*)} \leq C(\Omega, \nu) ||\eta||_{L^{\infty}(\Omega_*)} ||f||_{W^{1,q'}_\Gamma(\Omega)} ||w||_{W^{1,p_1'}_\Gamma(\Omega_*)}$$

Taking suprema over unit balls in $W^{1,q'}_\Gamma(\Omega_*)$ and $W^{1,q'}_\Gamma(\Omega)^*$ we get that the map

$$W^{1,q'}_\Gamma(\Omega)^* \to W^{1,p_1'}_\Gamma(\Omega_*)^*, \quad f \mapsto - \iint_{\Omega_*} \eta v_f \, dx$$

has its operator norm bounded by $C(\Omega, \nu)||\eta||_{L^{\infty}(\Omega_*)}$.

For the second term, note that we may factorize for all small enough $\epsilon > 0$ using Theorem 9

$$W^{1,q'}_\Gamma(\Omega)^* \hookrightarrow W^{1,2+\epsilon}(\Omega)^* \to W^{1,2+\epsilon}_\Gamma(\Omega) \to L^{2+\epsilon}(\Omega_*) \hookrightarrow W^{1,2+\epsilon}_\Gamma(\Omega_*)^*$$

given by

$$f \mapsto f \mapsto v_f \mapsto A \nabla v_f \nabla \eta \Omega_* \mapsto \iint_{\Omega_*} A \nabla v_f \nabla \eta \Omega_* \, dx,$$
The latter also implies the continuity of the embedding
\[ L^{2+\epsilon}(\Omega) \hookrightarrow W^{1,p'}_{1,*}(\Omega)^*. \]

The operator norm of the composition then essentially relies on the operator norm of
\[ W^{1,2+\epsilon'}(\Omega)^* \hookrightarrow W^{1,2+\epsilon}(\Omega), \quad f \mapsto v_f. \]

However, Theorem 9 shows that this is uniform with respect to the ellipticity constant \( \nu \) of \( A \), its \( L^\infty(\Omega, \mathbb{R}^{d\times d}) \) bound for \( A \) and all small \( \epsilon > 0 \).

The third term works similarly. Following [Haller-Dintelmann et al. (2009)] there is \( \epsilon > 0 \) such that
\[ W^{1,2+\epsilon}(\Omega) \hookrightarrow L^{4+\delta}(\Omega) \]
for a \( \delta = \delta(d) > 0 \). We estimate for \( w \in W^{1,4+\delta'}(\Omega) \)
\[ \langle T f, w \rangle_{W^{1,4+\delta'}(\Omega)} \leq \|v\|_{L^{4+\delta}(\Omega)} \|\|L_{\nu}(\Omega, \mathbb{R}^{d\times d})\|\|v\|_{L^{4+\delta'}(\Omega)} \]
\[ \leq C(\nu, M, \Omega) \|f\|_{L^{4+\delta'}(\Omega)} \|\|L_{\nu}(\Omega, \mathbb{R}^{d\times d})\|\|v\|_{W^{1,4+\delta'}(\Omega)}. \]

The constant \( C(\nu, M, \Omega) \) is again determined through Theorem 9. We set \( p_2 = 4 + \delta \).

Finally, the mapping \( f \mapsto f_* \) is nothing but \( E_n \) and thus \( \|E_n\| = \|E_n\| \), the latter already being computed in Lemma 10. To conclude the proof we take \( p = \min(p_1, p_2, p_3) \). \( \square \)

We reproduce the following proposition from [Haller-Dintelmann et al. (2009)] as the notation it introduces and its content are essential for the remainder of the section.

**Proposition 12** (Proposition 4.9 in [Haller-Dintelmann et al. (2009)]). Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain, let \( \Gamma_N \) be an open subset of its boundary and denote by \( \Gamma_D \) its complement in \( \partial \Omega \). Let \( \phi \) be bi-Lipschitz mapping defined on a neighborhood of \( \Omega \) into \( \mathbb{R}^d \) and denote \( \phi(\Omega) = \overline{\Omega} \) and \( \phi(\Gamma_D) = \overline{\Gamma_D} \). Then the following holds:

(i) For any \( p \in (1, \infty) \), the mapping \( \phi \) induces a linear homeomorphism
\[ \Phi_p : W^{1,p}_D(\overline{\Omega}) \rightarrow W^{1,p}_D(\Omega), \quad u \mapsto u \circ \phi. \]

(ii) If \( A \) is a member of \( L^\infty(\Omega, \mathbb{R}^{d\times d}) \), then
\[ -\Phi_p^* \circ \text{div}(A\nabla \Phi_p(\cdot)) = -\text{div}(\widetilde{A}(\cdot)) \]
with
\[ \widetilde{A}(y) = \frac{D\phi(\phi^{-1}(y))}{\det(D\phi(\phi^{-1}(y)))} A(\phi^{-1}(y))(D\phi(\phi^{-1}(y))^T \phi^{-1}(y)) \]
for almost all \( y \in \overline{\Omega} \).

(iii) If \( A \) is uniformly elliptic and essentially bounded, then so is \( \widetilde{A} \).

The last result we need is a reflection procedure that allows to transform a mixed Neumann-Dirichlet problem on the model domain \( Q_- \cup \Sigma \) to a pure Dirichlet problem on \( Q_- \) and thus makes Corollary 8 applicable. It is based on Proposition 4.11 in [Haller-Dintelmann et al. (2009)].

**Lemma 13** (Reflection Principle). For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we set \( x_- = (x_1, \ldots, x_{d-1}, x_d) \) and for a matrix \( A \in \mathbb{R}^{d\times d} \) we define
\[ A_{jk}^- = \begin{cases} A_{jk} & \text{if } j, k < d, \\ -A_{jk} & \text{if } j = d, k \neq d \text{ or } k = d \text{ and } j \neq d, \\ A_{jk} & \text{if } j = k = d. \end{cases} \]

Now let \( A \) denote a member of \( L^\infty(Q_-, \mathbb{R}^{d\times d}) \) and define a member of \( L^\infty(Q, \mathbb{R}^{d\times d}) \) via
\[ \hat{A}(x) = \begin{cases} A(x) & \text{if } x \in Q, \\ (A(x_-))^+ & \text{if } x_- \in Q_. \end{cases} \]

Let us set \( \Gamma_D = \partial Q_- \setminus \Sigma \). Then for any fixed \( p \in (1, \infty) \) it holds:

7
(i) If \( v \in W_{1,0}^{1,p}(Q_-) \) satisfies \( \text{div}(AVv) = f \in W_{1,0}^{1,p}(Q_-)' \), then \( \text{div}(A\hat{\nabla}\hat{v}) = \hat{f} \in W_{0,1}^{1,p}(Q_-)' \) holds for

\[
\hat{v}(x) = \begin{cases} 
v(x) & \text{if } x \in Q, \\
v(x) & \text{if } x \not\in Q.
\end{cases}
\]

and \( \langle \hat{f}, u \rangle_{W_{0,1}^{1,p}(Q_-)} = \langle f, u|_{Q_-} + u|_{Q_+}, u \rangle_{W_{1,0}^{1,p}(Q_-)} \), where \( u_-(x) = u(x-) \).

(ii) The map

\[ W_{1,0}^{1,p}(Q_-)' \to W_{0,1}^{1,p}(Q_-)', \quad f \mapsto \hat{f} \]

is continuous.

(iii) Furthermore, if \( A \in L^{\infty}(Q_-, \mathbb{R}^{d \times d}) \) has ellipticity constant \( v \) and \( L^\infty \) bound \( M \), then so does \( \hat{A} \).

\[ A^{-1} \xi \cdot \xi = A\hat{\xi} \cdot \hat{\xi}, \]

where \( \hat{\xi} = (-\xi_1, \ldots, -\xi_{d-1}, \xi_d) \). This implies

\[ \inf_{\xi \neq 0} A^{-1} \xi \cdot \xi = \inf_{\xi \neq 0} A\hat{\xi} \cdot \hat{\xi} \geq v|\xi|^2. \]

Furthermore, it holds ||A|| = ||A|| in the Frobenius norm, hence \( \hat{A} \) and \( A \) share its bound as members of \( L^{\infty}(Q_-, \mathbb{R}^{d \times d}) \). Finally, we provide the computations for the above equality

\[
A^{-1} \xi \cdot \xi = \sum_{i,j=1}^{d-1} A_{ij} \xi_i \xi_j + \sum_{i=1}^{d-1} (-A_{ii}) \xi_i \xi_i + \sum_{j=1}^{d-1} (-A_{ij}) \xi_j \xi_d + A_{dd} \xi_d^2 \\
= \sum_{i=1}^{d-1} A_{ii} (-\xi_i)^2 + \sum_{i=1}^{d-1} A_{ii} \xi_i \xi_i + \sum_{j=1}^{d-1} A_{ij} (-\xi_j) \xi_d + A_{dd} \xi_d^2 \\
= A\hat{\xi} \cdot \hat{\xi}.
\]

\[ \square \]

### 2.4 Proof of the Main Result

**Proof of Theorem 6** We follow the steps in [Haller-Dintelmann et al. (2009)]. For every \( x \in \Omega \) choose a ball \( B_x \subset \Omega \) centered at \( x \) and contained in \( \Omega \). For every \( x \in \partial \Omega \), by the definition of Gröger regularity, there exists an open neighborhood \( U_x \) of \( x \) and an open set \( W_x \) together with a bi-Lipschitz map \( \Psi_x : U_x \to W_x \) such that

\[ \Psi_x((\Omega \cup \Gamma_N) \cap U_x) = Q_- \] or \[ \Psi_x((\Omega \cup \Gamma_N) \cap U_x) = Q_- \cup \Sigma \]

depending on \( x \in \partial \Omega \). The system \( \{ U_x \}_{x \in \partial \Omega} \cup \{ B_x \}_{x \in \Omega} \) forms an open covering of \( \overline{\Omega} \). We choose a finite subcovering \( U_{x_1}, \ldots, U_{x_k}, B_{y_1}, \ldots, B_{y_l} \) and a subordinated smooth partition of unity \( \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_l \). Let \( A \in M \), \( q > d \) and \( f \in W_{1,0}^{1,p}(\Omega)' \) and denote by \( v \) the solution of

\[ -\text{div}(AVv) + v = f, \quad \text{in } H_{1,0}^{1}(\Omega)'. \]

Then we use the partition of unity to write

\[ v = \sum_{i=1}^{k} \eta_i v + \sum_{j=1}^{l} \xi_j v \]

and we need to estimate \( ||\eta_i v||_{C^0(\Omega)} \) and \( ||\xi_j v||_{C^0(\Omega)} \). This leads to three cases that need to be treated differently: First, the \( \xi_j v \) on the balls \( B_{y_j} \), then \( \eta_i v \) when \( (\Omega \cup \Gamma_N) \cap U_x \) equals \( Q_- \) and finally the case when \( (\Omega \cup \Gamma_N) \cap U_x = Q_- \cup \Sigma \).
**First Case.** We show that the Hölder norm of the \( \zeta_j \) can be controlled in terms of \( C(B_{x_j}, W M) \| f \|_{W^1_0}(\Omega) \). To this end, we employ Lemma \([11]\) with \( \mathcal{U} = B_{x_j} \) hence \( \Omega = B_{x_j} \) and \( \Gamma = \emptyset \). Then \( \zeta_j \) satisfies an equation of the form
\[
- \nabla(A_j \nabla(\zeta_j)) = g_j \quad \text{in } W^1_0(B_{x_j})
\]
with \( p_j > d \) and it holds
\[
\| g_j \|_{W^1_0(B_{x_j})} \leq C(B_{x_j}, W M) \cdot \| f \|_{W^1_0(\Omega)}.
\]
Hence, by Corollary \([8]\) there is \( \alpha_j \in (0, 1) \) such that
\[
\| \zeta_j \|_{L^\infty(\Omega)} = \| \zeta_j \|_{L^\infty(B_{x_j})} \leq C(B_{x_j}, W M) \cdot \| f \|_{W^1_0(\Omega)}.
\]

**Second Case.** Here we assume that we use \( \eta_j \) subordinated to \( U_j \) with
\[
\Psi_x((\Omega \cup \Gamma_N) \cap U_x) = Q_-. \tag{7}
\]
Setting \( \Omega_j = \Omega \cap U_j \), Lemma \([10]\) shows that \( \eta_j \) is a member of \( H^1_0(\Omega_j) \) and Lemma \([11]\) implies that \( \eta_j \) solves
\[
- \nabla(A_j \nabla(\eta_j)) = f_j \quad \text{in } H^1_0(\Omega_j)^\ast
\]
with \( f_j \in W^1_0(\Omega_j)^\ast \) and \( p_j > d \) and again
\[
\| f \|_{W^1_0(\Omega_j)} \leq C(\Omega_j, W M) \cdot \| f \|_{W^1_0(\Omega)}.
\]
Now, transform the function to \( Q_- \) using Proposition \([12]\) with \( \phi = \Psi^{-1}_x \) setting
\[
\psi_j := \Phi_j(\eta_j) = (\eta_j) \circ \Psi^{-1}_x.
\]
As we assumed \([7]\), \( \eta_j \) is a member of \( H^1_0(\Omega_j) \) and \( \psi_j \) is a member of \( H^1_0(\Omega_j) \). Furthermore, \( \psi_j \) satisfies and equation of the form
\[
- \nabla(\tilde{A} \nabla(\psi_j)) = h_j := (\Phi_j)^{-1} f_j \quad \text{in } W^1_0(\Omega_j)^\ast
\]
and by Corollary \([8]\) there is \( \alpha_j \in (0, 1) \) such that \( \psi_j \in C^\alpha(\Omega_j) \) with
\[
\| \psi_j \|_{C^\alpha(\Omega_j)} \leq C(\nu, M, \Omega_j) \cdot \| h_j \|_{W^1_0(\Omega_j)^\ast}
\]
where we used that \( \tilde{A} \) is still a bounded, measurable, elliptic matrix with possibly different boundedness and ellipticity constants, however controlled through the geometry of \( \Omega_j \). As Lipschitz maps preserve Hölder continuity in a controlled way we also have
\[
\| \eta_j \|_{C^\alpha(\Omega_j)} \leq C(\Omega_j) \cdot \| \psi_j \|_{C^\alpha(\Omega_j)}.
\]
Finally, we may estimate
\[
\| \eta_j \|_{C^\alpha(\Omega)} = \| \eta_j \|_{C^\alpha(\Omega)} \leq C(\Omega_j) \cdot \| \psi_j \|_{C^\alpha(\Omega_j)} \leq C(\nu, M, \Omega_j) \cdot \| (\Phi_j)^{-1} f_j \|_{W^1_0(\Omega_j)^\ast}
\]
\[
\leq C(\nu, M, \Omega_j) \cdot \| f_j \|_{W^1_0(\Omega_j)^\ast}
\]
\[
\leq C(\nu, M, \Omega_j) \cdot \| f \|_{W^1_0(\Omega)^\ast}.
\]

**Third Case.** We use the same notation as in the second case but now it holds
\[
\Psi_x((\Omega \cup \Gamma_N) \cap U_x) = Q_- \cup \Sigma.
\]
Setting \( \Gamma_j = \partial \Omega_j \setminus \Gamma_N \), it holds again \(- \nabla(A_j \nabla(\eta_j)) = f_j \) in \( H^1_0(\Omega_j)^\ast \) with \( f_j \in W^1_0(\Omega_j)^\ast \) and \( p_j > d \) and an estimate of the form
\[
\| f_j \|_{W^1_0(\Omega_j)^\ast} \leq C(\Omega_j, \nu, M) \cdot \| f \|_{W^1_0(\Omega)^\ast}.
\]
Now we transform to $Q_-$ as in the second case and then use the reflection principle, see Lemma 13 to transform to $Q$. This yields $\psi_j$ and $\hat{\psi}_j$, the latter solving a homogeneous problem on $Q$, the former as above, however, with a Neumann condition on $\Sigma$. We may estimate for a suitable $\alpha_j \in (0,1)$

$$\|\eta_j\|_{C^0(\Omega)} = \|\eta_j\|_{C^0(\Omega_j)} \leq C(\Omega_j) \cdot \|\psi_j\|_{C^0(Q_j)} \leq C(\Omega) \cdot \|\hat{\psi}_j\|_{C^0(Q_j)} \leq C(\nu, M, \Omega_j) \cdot \|\hat{\psi}_j\|_{W_{\text{loc}}^{1,\nu}(\Omega_j)} \leq C(\nu, M, \Omega_j) \cdot \|\hat{\psi}_j\|_{W_{\text{loc}}^{1,\nu}(Q_j)} \leq C(\nu, M, \Omega_j) \cdot \|\hat{\psi}_j\|_{W_{\text{loc}}^{1,\nu}(Q_j^c)}.$$  

Taking the minimal $\alpha_j$ concludes the proof. 

\[ \square \]

### III Parabolic Hölder Regularity

In this section, we prove Theorem[1] which we restate here for the readers convenience.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a Lipschitz domain, $I = [0, T]$ a time interval, $\partial \Omega = \Gamma_N \cup \Gamma_D$ a partition of the boundary into a Dirichlet and a Neumann part, where both $\Gamma_N$ and $\Gamma_D$ are allowed to have vanishing measure. Assume that $\Omega \cup \Gamma_N$ is Gröger regular, let $f \in L^p(I, L^2(\Omega))$ for $p \in [2, \infty)$, $D \in L^\infty(\Omega, \mathcal{M})$ with ellipticity constant $\nu > 0$ and let $k > 0$ be a constant. For $v_0 \in L^\infty(\Omega)$ denote by $v \in H^1(I, H^1_D(\Omega))$ the solution to

$$\int_I \langle d_i v, \nu \rangle_{H^1(\Omega)} dt + \int \int_D D\nabla v \cdot k \nabla (\cdot) dx dt = \int I \int \nu f(\cdot) dx dt \quad \text{in} \quad L^2(I, H^1_D(\Omega))'$$

$$v(0) = v_0.$$  

Then there is $\beta = \beta(p) \in (0,1)$ such that $v \in L^p(I, C^0(\Omega))$ and we may estimate

$$\|v\|_{L^p(I, C^0(\Omega))} \leq C\left( \Omega, T, \nu, ||D||_{L^\infty(\Omega, \mathbb{R}^{d \times d})}, p, \beta \right) \cdot \left[ \|f\|_{L^p(I, L^2(\Omega))} + \|v_0\|_{L^\infty(\Omega)} \right].$$

In the above estimate, if we fix $\Omega$ and $p$, only a lower bound for $\nu$ and upper bounds for $||D||$ and $T$ determine the value of the constant $C$. This provides uniformity for $v \in [c_\nu, C_\nu]$, $D \in L^\infty(\Omega, \mathcal{M})$ with $||D|| \leq C_\beta$ and time intervals $I' = [0, T']$ with $T' \leq T$.

**Strategy of the Proof.** Here we discuss only the main ideas and provide the details in the course of the section. The first ingredient in the proof is the $C^r(\Omega)$ regularity result for the stationary operator, see Theorem[5]. This opens the door for maximal parabolic regularity results, however, the initial value as a member of $L^\infty(\Omega)$ does not suffice for a direct application of the theory, which would require $v_0$ to be a member of $H^1_{0,\text{cir}}(\Omega)$, the trace space in this situation, compare to Arendt et al. (2017). Therefore, we propose to use the superposition principle for linear operators to split the equation into

$$d_1 v_1 + D v_1 = f,$$

$$v_1(0) = 0$$

and

$$d_2 v_2 + D v_2 = 0,$$

$$v_2(0) = v_0.$$  

The linearity of the equation implies that $v = v_1 + v_2$. This gives us the advantage to analyze $v_1$ and $v_2$ separately. Now, $v_1$ can be treated by a combination of the maximal regularity results in Amann (1995) and Theorem[6]. For $v_2$ we can quantify the norm blow-up at the initial time-point using standard results from Brezis (2010). More precisely, it holds

$$\|v_2(t)\|_{C^0(\Omega)} \leq C \cdot \left( \frac{1}{t} \|v_0\|_{L^2(\Omega)} + 1 \right)$$

and using an interpolation result we are able to mitigate the singularity of $t \mapsto t^{-1}$ by reducing the Hölder exponent.  

\[ \square \]
3.1 Proof of the Main Result

We need some basic facts from semi-group theory for linear, unbounded operators in a Hilbert space $H$, that is operators of the form $M : \text{dom}(M) \subseteq H \rightarrow H$. However, we started with a linear, bounded and coercive operator defined on a full space $X$ taking values in its dual, i.e., $M \in \mathcal{L}(X, X^*)$. If we are given a Gelfand triple structure $(i, X, H)$, that is $X$ and $H$ are Hilbert spaces and $i : X \rightarrow H$ is an embedding with dense image, i.e., linear, continuous and bounded, we see that the two concepts are closely related.

**Definition 14.** Let $(i, X, H)$ be a Gelfand triple and $M \in \mathcal{L}(X, X^*)$ a coercive bounded linear operator. We define its part in $H$ as follows

$$\text{dom}(M) := \{v \in X \mid \text{there is } f \in H \text{ with } (f, \cdot)_H = Mv\}$$

and

$$M : \text{dom}(M) \subset H \rightarrow H, \quad Mv = R^{-1}(Mv)$$

where $R$ denotes the Riesz isometry of $H$.

**Remark 15.** Note that the above definition suppresses the embedding $i$ in various places, treating it like a set-theoretic inclusion. Furthermore, we stress that $M$ is well defined as a map since for every $Mv$ there is at most one $f \in H$ satisfying $(f, \cdot)_H = Mv$ as $i(X)$ is dense in $H$ by assumption.

**Lemma 16.** Let $(i, X, H)$ be a Gelfand triple and $M \in \mathcal{L}(X, X^*)$ a coercive, bounded linear operator. Then, its part $M$ in $H$ is maximal monotone, thus densely defined. If $M$ is self-adjoint\(^1\) as a member of $\mathcal{L}(X, X^*)$, then $M$ is self-adjoint as a densely defined operator in $H$.

**Proof.** Let $u, v \in \text{dom}(M)$ and note that by the definition of $M$ it holds

$$(Mu, v)_H = (R^{-1}(M(u)), v)_H = (M(v), u)_X. \quad (8)$$

This identity makes clear that the coercivity of $M$ implies the monotonicity of $M$. Additionally,

$$\text{Id}_H + M : \text{dom}(M) \rightarrow H$$

is bijective and hence $M$ is maximal monotone. If $M$ is self-adjoint, then (8) shows that $M$ is symmetric. However, linear symmetric maximal monotone operators are self-adjoint, see Brezis (2010). $\square$

The following Proposition is tailored to allow the application of Hille-Yosida’s celebrated theorem on solutions to the Cauchy problem.

**Proposition 17.** Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a bounded domain with a partition of the boundary into Dirichlet and Neumann part $\partial \Omega = \Gamma_D \cup \Gamma_N$. Both $\Gamma_D$ and $\Gamma_N$ are allowed to have vanishing measure. We assume that $\Omega \cup \Gamma_N$ is Gröger regular. Further, let $D \in L^\infty(\Omega, \mathcal{M}_\Omega)$ be given and assume it is elliptic with ellipticity constant $\nu > 0$. Let $k > 0$, we define the operator

$$M : H^1_0(\Omega) \rightarrow H^1_0(\Omega)^*, \quad Mv = \int_\Omega Dv \nabla \cdot \nu \, dx.$$

Then its part in $L^2(\Omega)$ is maximal monotone and self-adjoint. Further, there exists $\alpha > 0$ such that we have the embedding

$$\left(\text{dom}(M), ||\cdot||_{L^2(\Omega)} + ||\cdot||_{L^2(\Omega)}\right) \hookrightarrow C^\alpha(\Omega)$$

together with the estimate

$$||u||_{C^\alpha(\Omega)} \leq C(\Omega, \nu, ||D||_{L^\infty(\Omega, \mathcal{M}_\Omega)}) \cdot ||u||_{\text{dom}(M)}.$$

Here, the constant $C$ is precisely $||M^{-1}||_{L^2(\Omega), C^\alpha(\Omega)}$ and depends only on a lower bound for the ellipticity constant and an upper bound on $||D||_{L^\infty(\Omega, \mathcal{M}_\Omega)}$.

**Proof.** Using the Gelfand triple $(\text{Id}_{L^2(\Omega)}, H^1_0(\Omega), L^2(\Omega))$, we can apply Lemma 16 and deduce the maximal monotonicity of $M$. Further, the symmetry assumption on $D$ implies that $M$ is self-adjoint, again through Lemma 16. It remains to show the embedding into Hölder spaces – essentially due to Theorem 6– which yields the existence of $\alpha > 0$ such that

$$M^{-1} : L^2(\Omega) \rightarrow C^\alpha(\Omega)$$

\(^1\)We call a map $T \in \mathcal{L}(X, X')$ self-adjoint if $T^* \circ J = T$, where $J : X \rightarrow X''$ is the natural isometric embedding of a Banach space into its bi-dual and $T^*$ denotes the usual adjoint map.
is well defined and continuous. This requires the assumption $d = 1, 2, 3$. To see that the graph norm on $\text{dom}(M)$ controls the $\alpha$-Hölder norm, we let $u \in \text{dom}(M) \subset C^\alpha(\Omega)$. Then there exists a unique $f \in L^2(\Omega)$ such that $u = M^{-1}f$ and we compute

$$
\|u\|_{C^\alpha(\Omega)} = \|M^{-1}f\|_{C^\alpha(\Omega)} \leq C\|f\|_{L^2(\Omega)} = C\|Mu\|_{L^2(\Omega)} \leq C\|u\|_{\text{dom}(M)}.
$$

The only appearing constant is the operator norm of $M^{-1}$ and Theorem 6 guarantees a suitable bound of this norm. \hfill \Box

**Theorem 18.** Assume we are in the situation of Proposition 17. Then for every $v_0 \in L^2(\Omega)$ there exists $\alpha > 0$ and $v \in C^1((0, T], L^2(\Omega)) \cap C^0((0, T], C^\alpha(\Omega))$

solving

$$
v'(t) + Mv(t) = 0 \quad \text{on } (0, T]
\quad \quad \quad \quad \quad \quad v(0) = v_0 \tag{9}
$$

Furthermore, it holds

$$
\|v(t)\|_{C^\alpha(\Omega)} \leq C(\Omega, \nu, |D|_{L^\infty}) \left(1 + \frac{1}{T}\right)\|v_0\|_{L^2(\Omega)}.
$$

More precisely, the constant $C(\Omega, |D|, |D|_{L^\infty})$ is the operator norm of the embedding $\text{dom}(M) \hookrightarrow C^\alpha(\Omega)$.

**Proof.** From Theorem 7.7 in [Brezis 2010] it follows that

$$
\|Mv(t)\|_{L^2(\Omega)} \leq \frac{1}{T}\|v_0\|_{L^2(\Omega)} \quad \text{and} \quad \|v(t)\|_{L^2(\Omega)} \leq \|v_0\|_{L^2(\Omega)}.
$$

Using this and the embedding $\text{dom}(M) \hookrightarrow C^\alpha(\Omega)$, we get

$$
\|v(t)\|_{C^\alpha(\Omega)} \leq C\|v(t)\|_{\text{dom}(M)} = C\|v(t)\|_{L^2(\Omega)} + C\|Mv(t)\|_{L^2(\Omega)} \leq C\|v_0\|_{L^2(\Omega)} + \frac{C}{T}\|v_0\|_{L^2(\Omega)}.
$$

\hfill \Box

**Theorem 19.** Assume we are in the situation of Proposition 17 and assume that $v_0 \in L^\infty(\Omega)$ and denote by $v \in C^1((0, T], L^2(\Omega))$ the solution to (9). Then for every $q \in (1, \infty)$ there exists $\beta = \beta(q)$ such that $v$ is a member of $L^q(I, C^\beta(\Omega)) \cap L^\infty(I, C^\alpha(\Omega))$. Furthermore, we can bound the $L^q(I, C^\beta(\Omega))$ norm depending on the data of the problem in the following way

$$
\|v\|_{L^q(I, C^\beta(\Omega))} \leq C(\Omega, \nu, |D|_{L^\infty}, |\nu|_{L^\infty}, I, \alpha, q). \tag{10}
$$

**Proof.** Let $p > q$ be fixed. Choose $\beta > 0$ such that $\alpha/p > \beta$. Then we can estimate for every $u \in C^\alpha(\Omega)$

$$
\|u\|_{C^\beta(\Omega)} \leq C \cdot \|u\|_{C^\alpha(\Omega)}^{1/p} |u|_{C^\alpha(\Omega)}^{1/p} + \|u\|_{C^\alpha(\Omega)}.
$$

To see this compute

$$
[u]_\beta = \sup_{x \neq y} \frac{|u(x) - u(y)|^{1-p}|u(x) - u(y)|^{1/p}}{|x - y|^{\alpha/p + \beta - \alpha/p}} = \sup_{x \neq y} |u(x) - u(y)|^{1-1/p} |x - y|^{\alpha/p - \beta} \cdot \left[\frac{|u(x) - u(y)|}{|x - y|^\alpha}\right]^{1/p} \leq (2\|u\|_{C^\alpha(\Omega)})^{1-1/p} \text{diam}(\Omega)^{\alpha/p - \beta} |u|_{\alpha}^{1/p}.
$$

Using the following estimate

$$
\|v(t)\|_{C^\beta(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)}
$$
and the above estimates of the $C^0$ norm and Theorem\cite{Amann1995} we obtain
\[
\|v(t)\|_{C^0(\Omega)} \leq (2\|v(t)\|_{C^0(\Omega)})^{1-1/p} \text{diam}(\Omega)^{p-1/p}[v(t)] + \|v(t)\|_{C^0(\Omega)}
\]
\[
\leq \max\{1, 2\|v_0\|\} \cdot \max\{1, \text{diam}(\Omega)\} \cdot [v(t)] + \|v_0\|_{C^0(\Omega)}
\]
\[
\leq C\left(\|v_0\|_{C^0(\Omega)}, \Omega\right) \cdot [v(t)] + \|v_0\|_{C^0(\Omega)}
\]
\[
\leq C\left(\|v_0\|_{C^0(\Omega)}, \nu, |D|_{L^\infty(\Omega)}\right) \cdot \left(1 + \frac{1}{T}\right)^{1/2}
\]
Inferring $q/p < 1$ then shows the integrability of $\|v(t)\|_{C^0(\Omega)}$ and the asserted bound. \hfill \Box

**Remark 20.** The constant in \cite{Amann1995} only depends on the length of the interval $I$, a lower bound for $v$ and an upper bound for $\|D\|_{L^\infty(\Omega)}$, hence is uniform for suitable families of operators and time intervals.

Finally we cite a known result from \cite{Amann1995} to treat the case with the vanishing initial condition.

**Theorem 21.** Assume we are in the situation of Proposition\cite{Amann1995} Let $f \in L^r(I, L^2(\Omega))$ with $p \in [2, \infty)$ and denote by $u$ the solution to
\[
u'(t) + Mu(t) = f \quad \text{on } (0, T)
\]
\[
u(0) = 0.
\]
Then it holds $u \in W^{1,r}(I, L^2(\Omega)) \cap L^r(I, \text{dom}(M))$ with the estimate
\[
\|u\|_{W^{1,r}(I, L^2(\Omega))} \leq C(v, |D|_{L^\infty(\Omega)}, \nu, p, I) \cdot \|f\|_{L^r(I, L^2(\Omega))}
\]
where $C(v, |D|_{L^\infty(\Omega)}, \nu, p, I)$ does depend on a lower bound for $v$, on an upper bound for $\|D\|_{L^\infty}$ and the upper bound $T$ of the time interval $I = [0, T]$.

**Proof.** We apply Theorem 4.10.8 in \cite{Amann1995}, using $E_0 = L^2(\Omega)$, $E_1 = \text{dom}(M)$. The requirement of $E_0$ being an UMD space holds as it is a Hilbert space, the other requirements can be shown using the fact that $M$ is self-adjoint and coercive, i.e., a member of $BIF(L^2(\Omega); 1, 0)$ in the terminology of \cite{Amann1995}. As we consider a problem with homogeneous initial conditions we don’t need to concern ourselves with the trace space for the initial conditions. \hfill \Box

**Completion of the proof of Theorem\cite{Amann1995}** Employing Theorem\cite{Amann1995} for $v_2$ and Theorem\cite{Amann1995} for $v_1$ we conclude that
\[
\|v\|_{L^r(I, C^0(\Omega))} \leq \|v_1\|_{L^r(I, C^0(\Omega))} + \|v_2\|_{L^r(I, C^0(\Omega))} \leq C(v, |D|_{L^\infty}, p, I, \beta) \cdot \left(\|f\|_{L^r(I, L^2(\Omega))} + \|v_0\|_{L^\infty(\Omega)}\right).
\]

\hfill \Box

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