The Fermion Mass Hierarchy and Neutrino Mixing Problem

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Abstract

The fermion mass problem is briefly reviewed. The observed hierarchy of quark and charged lepton masses strongly suggests the existence of an approximately conserved chiral flavour symmetry beyond the Standard Model. It is argued that in models of this type, the requirement of a natural explanation for both the atmospheric and solar neutrino problems leads to an essentially unique picture of neutrino masses and mixing angles. The anti-grand unification model is used as an explicit example to illustrate these ideas.

1 Introduction

The most striking feature of the charged fermion spectrum is the hierarchy of quark-lepton masses, ranging from the top quark with a mass of order the electroweak scale, $M_t = 175$ GeV, down to the electron of mass $1/2$ MeV. It therefore seems that the top quark mass may be understood in terms of physics already present at the electroweak scale, i.e. the Standard Model (SM) or possibly its minimal supersymmetric extension (MSSM); whereas the suppression of the other fermion masses requires flavour dynamics beyond the SM or MSSM.

One popular mechanism for generating the top quark mass is to assume that, at some high energy scale $M_X$, the running Yukawa coupling constant $g_t(M_X)$ for the top quark is of order unity or larger. It is attracted to its infra-red quasi-fixed
point value and, in the case of the MSSM, leads to a successful prediction:

\[ M_t \simeq (200 \text{ GeV}) \sin \beta \]  

(1)

for \( 1.5 \lesssim \tan \beta \lesssim 2.5 \). It is also possible to get a large \( \tan \beta \) solution, when all the third generation Yukawa couplings, \( g_t, g_b, g_{\tau} \), contribute significantly to the renormalisation group equations. I discussed this fixed point scenario at the sixth Lomonosov conference two years ago [1]. The top quark mass has also been calculated in the SM, using the so-called Multiple Point Principle (MPP) [2] according to which there should be another vacuum with essentially the same energy density as the usual SM vacuum. This principle requires the top quark and Higgs pole masses \( (M_t, M_H) \) to lie on the SM vacuum stability curve. Furthermore the vacuum expectation value (VEV) of the Higgs field in the second vacuum is expected to be of the same order of magnitude as the SM cut-off scale, which we take to be the Planck mass, giving the SM predictions:

\[ M_t = 173 \pm 5 \text{ GeV} \quad M_H = 135 \pm 9 \text{ GeV} \]  

(2)

as first reported at the previous Lomonosov conference [1].

It is natural [3] to try to explain the suppression of the other SM fermion masses in terms of selection rules due to approximate conservation laws. The mass \( m \) in the Dirac equation is essentially a transition amplitude between a left-handed fermion component \( \psi_L \) and its right-handed partner \( \psi_R \). If \( \psi_L \) and \( \psi_R \) have different quantum numbers under an approximate chiral symmetry group \( G \), the mass term is suppressed. So we are led to consider introducing a chiral flavour (gauge) symmetry beyond the SM group which, when unbroken, allows only the top quark Yukawa coupling to be non-zero. The other quark and lepton masses and mixing angles are then generated at some order in the VEV(s) responsible for breaking the symmetry, measured relative to the fundamental mass scale of the theory. Previously [4] I illustrated this mechanism for generating the fermion mass hierarchy using models which extend the MSSM by an Abelian \( U(1)_f \) flavour symmetry [4, 5] with Green-Schwarz anomaly cancellation. As pointed out in section 2, it is possible to generate a realistic mass hierarchy using an anomaly free \( SM \otimes U(1)^2 \) model. In this talk I will use the anti-grand unification model [2] (AGUT) to illustrate this approach to the charged fermion mass hierarchy and also to the neutrino mixing problem.

### 2 AGUT Model and Fermion Mass Hierarchy

The AGUT model [2] is based on extending the SM gauge group, \( SMG = S(U(2) \otimes U(3)) \approx SU(3) \otimes SU(2) \otimes U(1) \), to the non-simple gauge group \( SMG^3 \otimes \)
near the Planck scale \( M_{Planck} \simeq 10^{19} \) GeV. This means there is a pure SM desert, without supersymmetry, up to within an order of magnitude or so below \( M_{Planck} \), and the SM gauge coupling constants are not unified but their values are predicted \[6\] using the MPP. This AGUT group \( SMG^3 \times U(1)_f \) at first seems rather complicated and arbitrary. In fact it can be rather simply characterised, as the maximal anomaly free subgroup \( G_{max} \) of the group \( U(45) \) of unitary transformations on the known quark and lepton Weyl fields, for which the SM irreducible representations remain irreducible under \( G_{max} \). However the main motivation for considering this group is provided by its successful phenomenological predictions/fits. The SM group is embedded in \( G_{max} \) as the diagonal subgroup of \( SMG_3 \) and, above the AGUT breaking energy scale, each of the three quark-lepton generations has its own set of SM-like gauge particles together with an Abelian \( U(1)_f \) gauge boson. The \( SMG^3 \) quantum numbers are assigned to the quarks and leptons in the obvious way and the \( U(1)_f \) charges \( Q_f \) are carried by just the right-handed fermions of the second and third proto-generations:

\[
Q_f(\tau_R) = Q_f(b_R) = Q_f(c_R) = 1 \quad Q_f(\mu_R) = Q_f(d_R) = Q_f(t_R) = -1
\]

We now choose the Higgs fields responsible for the breakdown of the \( SMG^3 \otimes U(1)_f \) group to the SM group and the various suppressions of the quark-lepton masses. Phenomenological arguments lead us to introduce a Higgs field \( S \) with a VEV of order unity in Planck units and three Higgs fields \( W, T \) and \( \xi \) with VEVs an order or magnitude smaller. Since the \( S \)-field does not suppress the fermion mass matrix elements, phenomenological arguments only determine the quantum numbers of the other Higgs fields modulo those of \( S \). With this choice of quantum numbers, tree diagrams of the type shown in Fig. \[\] generate the following order of magnitude effective SM Yukawa coupling matrices \[7\] for \( u \) and \( d \) type quarks:

\[
Y_U \simeq \begin{pmatrix}
WT^2\xi^2 & WT^2\xi & W^2T\xi \\
WT^2\xi & WT^2 & W^2T \\
\xi^3 & 1 & WT
\end{pmatrix}
\]

\[
Y_D \simeq \begin{pmatrix}
WT^2\xi^2 & WT^2\xi & T^3\xi \\
WT^2 & WT^2 & T^3 \\
W^2T^4\xi & W^2T^4 & WT
\end{pmatrix}
\]

for charged leptons. Here \( W, T \) and \( \xi \) denote the VEVs of the Higgs field in Planck units and we have assumed the presence of a rich spectrum of vector-like Dirac fermions with fundamental masses \( M_F \simeq M_{Planck} \) to mediate the symmetry breaking transitions. The corresponding set of Higgs field Abelian quantum numbers can be specified as charge vectors \( \vec{Q} \equiv (y_1/2, y_2/2, y_3/2, Q_f) \), where \( y_i/2 \) denotes the weak hypercharge for the \( i \)th proto-generation:

\[
\vec{Q}_W = (-1/6, -1/3, 1/2, -1/3) \quad \vec{Q}_T = (-1/6, 0, 1/6, 1/3) \quad \vec{Q}_\xi = (0, 0, 0, 1)
\]
The non-Abelian representations of the Higgs fields are, like the fermions, taken to be singlet or fundamental representations with their dualities and trialities determined by the natural generalisation of the SM charge quantisation rule. The quantum numbers $Q_{\Phi_{WS}}$ of the Weinberg-Salam Higgs field $\Phi_{WS}$ are chosen to ensure that the top quark Yukawa coupling is of order one and corresponds to an off-diagonal element of $Y_U$. We effectively only use the Abelian quantum numbers to determine the mass suppression factors and, since we took $<S>=1$, we could generate the same SM Yukawa matrix structure, eqs. (4) and (5), with an anomaly free $SMG \otimes U(1)_{f1} \otimes U(1)_{f2}$ model and a corresponding set of Higgs fields $W$, $T$, $\xi$ and $\Phi_{WS}$. The two flavour charges in such an Abelian extension of the SM could then be identified as $Q_{f1} = y_3/2$ and $Q_{f2} = 4y_1/2 - 2y_2/2 + Q_f$.

The most characteristic feature of the AGUT Yukawa matrices $Y_U$, $Y_D$ and $Y_E$ is that their diagonals are equal order of magnitudewise. Apart from the $t$ and $c$ quarks, the fermion mass eigenvalues are given by the diagonal elements and hence the AGUT model simulates the GUT SU(5) mass predictions, namely the degeneracy of the $d$-quarks with the charged leptons in the corresponding generations. However, we only predict these degeneracies at the Planck scale as far as order of magnitude is concerned, and not exactly! This gives much better agreement with experiment than exact SU(5) predictions, which are rather bad unless more Weinberg-Salam Higgs fields are included a la Georgi-Jarlskog’s factor 3 mechanism. Also note that we predict the $u$ quark to be degenerate with the $d$ quark and the electron. In addition we have the following order of magnitude Planck scale relations:

$$m_0^3 \simeq m_t m_c m_s \quad V_{ub} \simeq V_{td} \simeq V_{us} V_{cb}$$
Table 1: Best fit to experimental data. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|   | $m_u$  | $m_d$  | $m_e$  | $m_c$  | $m_s$  | $m_\mu$ |
|---|--------|--------|--------|--------|--------|---------|
| Fit| 3.6 MeV| 7.0 MeV| 0.87 MeV| 1.02 GeV| 400 MeV| 88 MeV |
| Data| 4 MeV | 9 MeV | 0.5 MeV | 1.4 GeV | 200 MeV | 105 MeV |

\[
V_{us} \simeq V_{cd} \simeq \sqrt{\frac{m_d}{m_s}} \quad V_{cb} \simeq V_{ts} \simeq \frac{m_s^2}{m_c m_b}
\]  \hspace{1cm} (8)

and predict the CP-violating area of the “unitarity triangle” to be given order of magnitudewise by $J \simeq V_{us} V_{cb} V_{ub}$. The results of such a three parameter order of magnitude fit to the data are given in Table 1.

3 Neutrino Mixing Problem

In this section we consider the generic textures of the neutrino mass matrix, which arise in models with a natural fermion mass hierarchy due to an approximately conserved chiral flavour symmetry. We then require that the atmospheric and solar neutrino problems be explained by the eigenvalues and mixing angles generated by diagonalising the neutrino mass matrix. In this way we are led to a picture in which the electron and muon neutrinos are quasi-degenerate in mass with maximal mixing between them, and both atmospheric and solar neutrino data result from $\nu_\mu \leftrightarrow \nu_e$ vacuum oscillations.

The effective three generation light neutrino mass matrix $M_\nu$, generated by interactions beyond the SM, couples the left-handed neutrinos with the right-handed anti-neutrinos:

\[
\mathcal{L}_m = (M_\nu)_{ij} \nu_{Li} C\nu_{Lj} + \text{h.c.} \quad (9)
\]

By its very definition $M_\nu$ is symmetric. The overall neutrino mass scale is not really understand and is usually set by hand, by an appropriate choice of $M_F$ in the “see-saw” mass scale $\frac{(\Phi_W^2)}{M_F}$ or of the VEV of a weak isortriplet Higgs field $\Delta$. In models with approximately conserved chiral charges, its matrix elements are generally of different orders of magnitude, except for the equalities enforced by the symmetry $M_\nu = M_\nu^T$. The largest neutrino mass eigenvalue is then given
by the largest matrix element of $M_\nu$. If it happens to be one of a pair of equal off-diagonal elements, we get two very closely degenerate states as the heaviest neutrinos and the third neutrino will be much lighter and, in first approximation, will not mix with the other two [10]. If the largest element happens to be a diagonal element, it will mean that the heaviest neutrino is a Majorana neutrino, the mass of which is given by this matrix element, and it will not be even order of magnitude-wise degenerate with the other, lighter neutrinos. These lighter neutrinos may or may not get their masses from off-diagonal elements and thus, in first approximation, be degenerate.

The lepton mixing matrix $U$ is defined analogously to the usual CKM quark mixing matrix, in terms of the unitary transformations $U_\nu$ and $U_E$, on the left-handed lepton fields, which diagonalise the squared neutrino mass matrix $M_\nu M_\nu^\dagger$ and the squared charged lepton mass matrix $M_E M_E^\dagger$ respectively:

$$U = U_\nu^\dagger U_E$$

(10)

The charged lepton unitary transformation $U_E$ is expected to be quasi-diagonal, with small off-diagonal elements due to the charged lepton mass hierarchy. On the other hand when there is a quasi-degenerate pair of neutrinos, because off-diagonal elements dominate their masses, the mixing angle contribution from $U_\nu$ will be very close to $\pi/4$.

We are thereby led to consider the four possible textures given in Table 2. With texture 1 the neutrino spectrum is hierarchical and has small mixing angles like the charged fermion families. Textures 2–4 correspond to having a pair of quasi-degenerate neutrinos with essentially maximal mixing and a third essentially unmixed Majorana neutrino which may be $\nu_\tau$ or $\nu_\mu$ or $\nu_e$.

The atmospheric neutrino problem corresponds to a deficit of muon neutrinos [11], which could be explained by $\nu_\mu \leftrightarrow \nu_\tau$ or $\nu_\mu \leftrightarrow \nu_e$ oscillations with $\Delta m_{\text{atmos}}^2 \approx 10^{-2} \text{ eV}^2$ and strong mixing $\sin^2 2\theta > \sim 0.7$. Textures 1 and 3 leave $\nu_\mu$ weakly mixed and are thereby ruled out by our requirement of explaining the atmospheric neutrino problem. We also want to explain the solar neutrino problem and this requires mixing with the electron neutrino. The only small mixing solution is the MSW solution which has $\Delta m_{\text{MSW}}^2 \approx 10^{-5} \text{ eV}^2 \ll \Delta m_{\text{atmos}}^2$. Thus the MSW solution would require texture 4, but with a much greater degree of degeneracy between $\nu_e$ and one of the other eigenstates than the degeneracy between $\nu_\mu$ and $\nu_\tau$ naturally generated by the symmetry of the mass matrix. This would require an extreme fine-tuning of parameters, which we rule out as unnatural.

We are therefore left with a unique structure—texture 2—for the neutrino mass matrix in our approach. This structure corresponds to strong mixing of quasi-degenerate electron and muon neutrinos, with an essentially isolated Majorana tau neutrino. Both the atmospheric and solar neutrino problems are
Table 2: Neutrino mass matrix textures. The parameters $A$, $B$ and $C$ are of different orders of magnitude, giving a hierarchy of eigenvalues. The symbol $\bullet$ is used to denote relatively small elements responsible for small mixings and small mass splittings between otherwise degenerate eigenvalues. The mixing angles are estimated assuming the contributions from the charged lepton matrix are small.

|   | $A \bullet \bullet$ | Diagonal | No strong mixings $\theta$’s small |
|---|---------------------|----------|-----------------------------------|
| 1 | $\bullet A \bullet$ | $\nu_e \leftrightarrow \nu_\mu$ | $m_{\nu_e} \simeq m_{\nu_\mu}$ |
|   | $B \bullet \bullet$ | mix strongly, $\nu_\tau$ isolated | $\sin^2 2\theta_{e\mu} \simeq 1$ |
|   | $\bullet \bullet C$ | $m_{\nu_\tau} \simeq m_{\nu_\tau}$ | other $\theta$’s small |

| 2 | $\bullet A A$ | $\nu_e \leftrightarrow \nu_\tau$ | $m_{\nu_\tau} \simeq m_{\nu_\tau}$ |
|   | $A \bullet \bullet$ | mix strongly, $\nu_\mu$ isolated | $\sin^2 2\theta_{e\tau} \simeq 1$ |
|   | $\bullet \bullet C$ | $\nu_\mu$ isolated | other $\theta$’s small |

| 3 | $\bullet \bullet A$ | $\nu_\mu \leftrightarrow \nu_\tau$ | $m_{\nu_\mu} \simeq m_{\nu_\tau}$ |
|   | $B \bullet \bullet$ | mix strongly, $\nu_\mu$ isolated | $\sin^2 2\theta_{\mu\tau} \simeq 1$ |
|   | $\bullet \bullet A$ | $\nu_\mu$ isolated | other $\theta$’s small |

| 4 | $B \bullet \bullet$ | $\nu_\mu \leftrightarrow \nu_\tau$ | $m_{\nu_\mu} \simeq m_{\nu_\tau}$ |
|   | $\bullet \bullet A$ | mix strongly, $\nu_\mu$ isolated | $\sin^2 2\theta_{\mu\tau} \simeq 1$ |
|   | $\bullet \bullet A$ | $\nu_\mu$ isolated | other $\theta$’s small |

then solved by $\nu_\mu \leftrightarrow \nu_e$ vacuum oscillations with close to maximal mixing and $\Delta m^2_{\mu\mu} \simeq 10^{-2}$ eV$^2$. This structure leads to an energy independent electron neutrino flux suppression factor of 1/2 in all solar neutrino experiments. Also it requires the LSND evidence for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations should prove to be unfounded. Since we have a hierarchical structure, we expect the mass splitting $\Delta m_{e\mu}$ to be at least one order of magnitude smaller than $m_{\nu_e}$ and $m_{\nu_\mu}$. We can also use the experimental limit $m_{\nu_e} \lesssim 10$ eV. So we obtain the order of magnitude estimate $m_{\nu_e} \simeq m_{\nu_\mu} \sim 1$ eV, which makes $\nu_e$ and $\nu_\mu$ candidates for hot dark matter. Due to the hierarchical structure of $M_\nu$, the mass of the tau neutrino should deviate from the other two quasi-degenerate mass eigenvalues by orders of magnitude. So, using the cosmological upper bound of 40 eV for stable neutrinos, we expect $\nu_\tau$ to be much lighter than $\nu_e$ and to be only slightly mixed.

It is possible to construct an explicit example of such a neutrino mixing structure, using the AGUT model with a triplet (under $SU(2)$ of the SM) Higgs field $\Delta$ having the Abelian charge vector:

$$\vec{Q}_W = (-1/2, -1/2, 0, 0)$$  \hspace{1cm} (11)

The corresponding charged lepton and neutrino mass matrices are:

$$M_E \sim \phi_{WS} \begin{pmatrix} W^T T^2 \xi^2 & W^T T^2 \xi^3 & W T^4 \xi^2 \\ W^T T^2 \xi^5 & W^T T^2 & W T^4 \xi^2 \\ W T^5 \xi^3 & W^2 T^4 & W T \end{pmatrix}, \quad M_\nu \sim \Delta \begin{pmatrix} \xi^3 & 1 & T^3 \xi^2 \\ 1 & \xi^3 & T^3 \xi \\ (T^3 \xi)^2 & T^3 \xi & T^3 W^3 \xi \end{pmatrix}$$  \hspace{1cm} (12)
which give, using the VEVs from the fit of Table 1, \( m_{\nu_e} \approx m_{\nu_\mu} \approx 2 \text{ eV}, \) \( m_{\nu_\tau} \approx 2 \times 10^{-7} \text{ eV}, \) \( \Delta m^2_{e\mu} \approx 8 \times 10^{-3} \text{ eV}^2 \) and \( \sin^2 2\theta_{e\mu} \approx 1. \)

Our scenario of neutrino mixing is readily testable by long baseline reactor neutrino oscillation experiments and, since the Lomono sov conference, initial results from the CHOOZ reactor experiment have become available \[13\]. This experiment finds (at 90% confidence level) no evidence for neutrino oscillations in the \( \bar{\nu}_e \) disappearance mode for \( \Delta m^2 \gtrsim 10^{-3} \text{ eV}^2 \) and maximum mixing, in conflict with the \( \nu_\mu \leftrightarrow \nu_e \) oscillation solution to the atmospheric neutrino problem and, hence, with our scenario. If both the atmospheric and solar neutrino problems and the CHOOZ results are upheld, \( M_\nu \) cannot\[1\] have any of the textures given in Table 2 and we must conclude that the dynamics underlying the structure of the neutrino mass matrix is not understood.

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\[1\] In principle the charged leptons could contribute significantly to the \( \mu - \tau \) mixing angle, via \( U_{e\gamma} \), without having the \( \nu_\mu \) and \( \nu_\tau \) masses quasi-degenerate. However it is difficult to make the contribution from \( U_{e\gamma} \) sufficiently large that \( \sin^2 2\theta_{\mu\tau} \gtrsim 0.7 \) arises naturally in this way.