We investigate the vacuum state of the lattice gauge theory with fermions in 2+1 dimensions. The vacuum in the Hermite form for the fermion part is obtained; the vacuum in the unitary form has been proposed by Luo and Chen. It is shown that the Hermite vacuum has a lower energy than the unitary one through the variational method.

1. Introduction

Quantum chromodynamics (QCD) is accepted as the model of the strong interaction. The lattice gauge theory is useful for understanding the low energy behavior of QCD. There are mainly two methods for formulating the lattice gauge theory. One method is the path integral formulation in the Euclidean space–time. In this formalism, the fermion part of the QCD action is integrated into the fermion determinant. Essentially, effects of the fermion appear through the fermion determinant. Its estimation is investigated in various ways. Another method is the Hamiltonian formulation in the Minkowski space. In this formalism, the main effort has been made for the pure gauge theory. The vacuum wave function is constructed in various ways. It is expressed in terms of the loops of the link variables. Moreover, the excited state orthogonal to the vacuum, i.e. the glueball
state, is constructed. One desires to understand the hadron dynamics analytically rather than numerically. However, there have been few attempts to construct the vacuum with fermions. Luo and Chen have investigated the vacuum assuming the unitary form for the fermion field with the variational method.

In this paper, we investigate the vacuum state of the lattice gauge theory in 2+1 dimensions. The lattice gauge theory in the (2+1)-dimensional space–time is often dealt with since the (2+1)-dimensional theory is simple. We expect that the results derived from the three-dimensional theory can be generalized to the (3+1)-dimensional theory for the most part.

In Sec. 2, the vacuum state of the lattice gauge theory with the fermion field is obtained in the strong coupling expansion. The essential idea is that the state $e^{R\mathcal{H}}|0\rangle$ is the vacuum, at least an eigenstate, if the Hamiltonian is expressed as $H = e^{-R\mathcal{H}_{\text{eff}}}e^{-R} + E_0$ in terms of the energy shift $E_0$ and certain operators $R$ and $\mathcal{H}_{\text{eff}}$, which annihilates the strong coupling vacuum $\mathcal{H}_{\text{eff}}|0\rangle = 0$. In Sec. 3, we justify the strong coupling vacuum through the $t$ expansion method, a useful tool for the Hamiltonian formulation. In Sec. 4, we show that this vacuum has a lower energy expectation value than that proposed by Luo and Chen. A discussion is given in Sec. 5.

2. Strong Coupling Vacuum

We start with the latticized Hamiltonian in 2+1 dimensions

$$H = H_G + H_F.$$  \hfill (1)

The gauge part of the Hamiltonian, $H_G = G + P$, is written as

$$G = \frac{g^2}{2} \sum_x E^a_i(x)E^a_i(x),$$  \hfill (2)

$$P = -\frac{1}{2g^2} \sum_x \sum_{ij} \text{tr} \left[ U_{ij}(x) + U_{ij}^\dagger(x) \right].$$  \hfill (3)
Vacuum State of Lattice Gauge Theory . . .

where \( g \) is the dimensionless coupling constant. In the Hamiltonian formulation, only spatial variables are discretized. The lattice spacing is assumed to be 1. The link variable \( U_i(x) \) is defined between the two adjacent sites \( x \) and \( x + \hat{i} \). The plaquette is defined by

\[
U_{ij}(x) = U_i(x)U_j(x + \hat{i})U_i^\dagger(x + \hat{j})U_j^\dagger(x). \tag{4}
\]

We use the four-spinor formulation for the fermion field in three dimensions. To avoid the doubling problem, we adopt the staggered fermion:

\[
H_F = T + m H_0:
\]

\[
T = \frac{i}{2} \sum_x \eta_i(x) \left[ \chi^\dagger(x)U_i(x)\chi(x + \hat{i}) - \chi^\dagger(x + \hat{i})U_i^\dagger(x)\chi(x) \right], \tag{5}
\]

\[
H_0 = \sum_x (-1)^x \chi^\dagger(x)\chi(x), \tag{6}
\]

where we have used the notations \( \eta_i(x) = \delta_{i1} + \delta_{i2}(-1)^x \) and \((-1)^x = (-1)^{x_1 + x_2}\). The (anti)commutators between the variables are defined by

\[
[E^a_l, U^{AB}_l] = -(T^aU)_l^{AB}\delta_{l,l'}, \tag{7}
\]

\[
\{\chi^A(x), \chi^B(y)\} = \delta_{AB}\delta_{xy}, \tag{8}
\]

where \( l \) denotes a link. We use the \( \gamma \) matrices in the representation \( \gamma^0 = \sigma_3 \otimes \sigma_3 \), \( \gamma^0\gamma^i = -1 \otimes \sigma_i \). The relation between the fundamental fermion and the Dirac spinor is given by

\[
\psi_{2\alpha + \beta - 2}(X) = \frac{1}{2\sqrt{2}} \sum_{\rho_1, \rho_2 = 0, 1} (\sigma_1^{\rho_1} \sigma_2^{\rho_2})_{\beta\alpha} \chi(2X + \rho), \tag{9}
\]

where the index of the Dirac spinor \( 2\alpha + \beta - 2 \) runs from 1 to 4 because \( \sigma_1^{\rho_1} \sigma_2^{\rho_2} \) is a \( 2 \times 2 \) matrix. We proceed to construct the vacuum state. We assume that \( g^2 \gg m \gg 1/g^2 \). The vacuum state of the pure gauge theory is developed in various ways. In one method, the vacuum state is given by \( |\tilde{0}\rangle = e^R|0\rangle \) if the Hamiltonian is expressed as

\[
H = e^{-R}H_{\text{eff}}e^{-R}, \tag{10}
\]

in the Hermitian operators \( R \) and \( H_{\text{eff}} \), which annihilates the strong coupling vacuum \( H_{\text{eff}}|0\rangle = 0 \). To obtain the vacuum state with the fermion field, we consider the strong
coupling $g \to \infty$ and heavy quark limit $m \to \infty$. The reason why we consider the heavy quark limit is that the vacuum expectation value of the link variable, contained by the fermion kinetic term, vanishes in the strong coupling limit or in the random phase. From the time evolution, $\chi(x,t)$ is the annihilation operator of the particle on the site $x$ when $(-1)^x = 1$, and the production operator of the antiparticle on the site $x$ when $(-1)^x = -1$. The vacuum state of the strong coupling and heavy quark Hamiltonian is defined by

$$E_a^a|0\rangle = 0, \quad (11a)$$
$$\chi(x)|0\rangle = 0 \text{ for } (-1)^x = 1, \quad (11b)$$
$$\chi^\dagger(x)|0\rangle = 0 \text{ for } (-1)^x = -1. \quad (11c)$$

Now, we express the Hamiltonian in the form (10). We adopt the expression

$$H = \frac{g^2}{2} \sum_x e^{-R} E_a^a(x) e^{R} E_a^a(x) e^{-R}$$
$$+ h_1 \sum_{(-1)^x = 1} e^{-R} \chi^\dagger(x) e^{R} \chi(x) e^{-R}$$
$$+ h_1 \sum_{(-1)^x = -1} e^{-R} \chi(x) e^{R} \chi^\dagger(x) e^{-R}$$
$$+ h_2 \sum_{(-1)^x = 1} e^{-R} \chi^\dagger_A(x) \chi_B(x + \hat{i}) e^{R} \chi_B^\dagger(x + \hat{i}) \chi_A(x) e^{-R}$$
$$+ h_2 \sum_{(-1)^x = -1} e^{-R} \chi_B^\dagger(x + \hat{i}) \chi_A(x) e^{R} \chi_A^\dagger(x + \hat{i}) \chi_B(x) e^{-R} + E_0, \quad (12)$$

where the parameters $h_1$, $h_2$, and $E_0$ are to be determined. For the pure gauge case, only the first term is needed out of the Hamiltonian. We have to add the four-fermion term to the expression (12) because the term without link variables $H_1$ appears from the electric part:

$$H_1 = \sum_{(-1)^x = 1} \chi^\dagger_A(x) \chi_B(x + \hat{i}) \chi_B^\dagger(x + \hat{i}) \chi_A(x) + \sum_{(-1)^x = -1} \chi^\dagger_B(x + \hat{i}) \chi_A(x) \chi_A^\dagger(x) \chi_B(x + \hat{i}). \quad (13)$$
We expect that \( R \) is expressed as

\[
R = r_1 R_1 + r_{2a} R_{2a} + r_{2b} R_{2b} + r_{2c} R_{2c} + r_{2d} R_{2d} + R_g + O\left(\frac{1}{g^6}\right),
\]

(14)

The subscript in \( R_n \) stands for the operator which contains \( n \) link variables. We have already known the gauge part \( R_g \) investigated in various papers. The lowest term of \( R_g \) is the plaquette in the strong coupling expansion:

\[
R_g = -\frac{1}{2g^2C(N_c)} P + O\left(\frac{1}{g^8}\right),
\]

(15)

where \( C(N_c) = \frac{N_c^2 - 1}{2N_c} \) is a Casimir invariant. We should note that we do not impose the plaquette ansatz investigated in Ref. [3] on the vacuum. The other operators in Eq. (14) are expressed as \( R_1 = T \) and

\[
\begin{align*}
R_{2a} & = \frac{N_c + 1}{2N_c} \sum_x \left[ \left( \chi^\dagger(x) U_i(x) \chi(x + \hat{i}) \right)^2 + \left( \chi^\dagger(x + \hat{i}) U_i^\dagger(x) \chi(x) \right)^2 \right], \\
R_{2b} & = \frac{1}{N_c} \sum_x : \chi^\dagger(x) U_i(x) \chi(x + \hat{i}) \cdot \chi^\dagger(x + \hat{i}) U_i^\dagger(x) \chi(x) :,
\end{align*}
\]

(16a, 16b)

\[
\begin{align*}
R_{2c} & = \sum_x \left[ E_i^a(x), - \sum_j \eta_j(x - \hat{j}) \eta_i(x) \left( \chi^\dagger(x - \hat{j}) U_j(x - \hat{j}) T^a U_i(x) \chi(x + \hat{i}) \\
& \quad + \chi^\dagger(x + \hat{i}) U_i^\dagger(x) T^a U_j^\dagger(x - \hat{j}) \chi(x - \hat{j}) \right) \\
& \quad + \sum_{j \neq i} \eta_j(x) \eta_i(x) \left( \chi^\dagger(x + \hat{j}) U_i^\dagger(x) T^a U_j(x) \chi(x + \hat{i}) \\
& \quad + \chi^\dagger(x + \hat{i}) U_i^\dagger(x) T^a U_j^\dagger(x) \chi(x + \hat{j}) \right) \\
& \quad + \sum_j \eta_j(x) \eta_i(x + \hat{i}) \left( \chi^\dagger(x) T^a U_i(x) U_j(x + \hat{i}) \chi(x + \hat{i} + \hat{j}) \\
& \quad + \chi^\dagger(x + \hat{i} + \hat{j}) U_i^\dagger(x) T^a U_j^\dagger(x) \chi(x) \right) \\
& \quad - \sum_{j \neq i} \eta_j(x) \eta_i(x + \hat{i} - \hat{j}) \left( \chi^\dagger(x) T^a U_i(x) U_j^\dagger(x + \hat{i} - \hat{j}) \chi(x + \hat{i} - \hat{j}) \\
& \quad + \chi^\dagger(x + \hat{i} - \hat{j}) U_i^\dagger(x + \hat{i} - \hat{j}) U_j^\dagger(x) T^a \chi(x) \right) \right],
\end{align*}
\]

(16c)

\[
R_{2d} = 2 \sum_x \left[ E_i^a(x), - \chi^\dagger(x) T^a \chi(x) + \chi^\dagger(x + \hat{i}) U_i^\dagger(x) T^a U_i(x) \chi(x + \hat{i}) \right],
\]

(16d)

where the normal ordering is taken in the meaning of Eqs. (14). The graph of the operator
$R_1$, which implies the creation (or annihilation) of the quark pair, is shown in Fig. 1(a). The operators $R_{2a}$, $R_{2b}$, $R_{2p}$ imply the creation (or annihilation) of the two quark pairs, the creation after the annihilation of the quark pair, and the creation (or annihilation) of the plaquette respectively, shown in Figs. 1(b)–1(d). By requiring the equality of Eqs. (1) and (12) order by order, we find the parameters to be

\begin{align}
h_1 &= m + \frac{4N_c}{N_c^2 - 1} \frac{1}{g^2} + O\left(\frac{1}{g^4}\right), \quad (17a) \\
h_2 &= -\frac{2N_c}{(N_c - 1)(N_c^2 - 1)} \frac{1}{g^2} + O\left(\frac{1}{g^4}\right), \quad (17b) \\
r_1 &= \frac{2}{g^2 C(N_c)} \left[ 1 - \frac{4m}{g^2 C(N_c)} \right] + O\left(\frac{1}{g^6}\right), \quad (17c) \\
r_{2a} &= -\frac{N_c}{N_c^2 - N_c - 2} \left[ \frac{2}{g^2 C(N_c)} \right]^2 + O\left(\frac{1}{g^6}\right), \quad (17d) \\
r_{2b} &= -\frac{1}{N_c} \left[ \frac{2}{g^2 C(N_c)} \right]^2 + O\left(\frac{1}{g^6}\right), \quad (17e)
\end{align}
Vacuum State of Lattice Gauge Theory

\[ r_{2c} = \frac{1}{2C(N_c)} \left[ \frac{2}{g^2 C(N_c)} \right]^2 + O\left( \frac{1}{g^6} \right), \tag{17f} \]

\[ r_{2d} = \frac{1}{2N_c} \left[ \frac{2}{g^2 C(N_c)} \right]^2 + O\left( \frac{1}{g^6} \right), \tag{17g} \]

\[ E_0 = -\frac{4N_c^2}{N_c^2 - 1/2g^2} L^2 + O\left( \frac{1}{g^4} \right), \tag{17h} \]

for sufficiently large \( N_c \). The number of the lattice sites is assumed to be \( L^2 = \sum x \). The deviation of the parameter \( h_1 \) from the quark mass \( m \) shows the mass generation in the effective Hamiltonian. The Hamiltonian (12) reproduces the normal-ordered form of the expression (1) up to order \( 1/g^2 \). Consequently, the vacuum state is \( \langle \tilde{0} \rangle = e^R\langle 0 \rangle \). It is an eigenstate of the Hamiltonian

\[ H|\tilde{0}\rangle = E_0|\tilde{0}\rangle. \tag{18} \]

We find that the pseudoscalar meson \( \psi^\dagger \gamma^0 \gamma^5 \psi |\tilde{0}\rangle \) and the vector meson \( \psi^\dagger \gamma^0 \gamma^1 \psi |\tilde{0}\rangle \) with zero momentum take the same energy expectation value,

\[ E = \frac{g^2}{2} C(N_c) + 2m + E_0 + \frac{1}{g^2 C(N_c)}, \tag{19} \]

where we have used the additional \( \gamma \) matrix defined by \( \gamma^5 = -\sigma_2 \otimes 1 \).

3. Comparison with \( t \) Expansion

In the previous section, we have obtained the vacuum in the strong coupling expansion. It is instructive to compare our vacuum with that obtained by the \( t \) expansion. The \( t \) expansion vacuum is given as a trial state with the damping factor \( e^{-\bar{H}t} \) in the limit \( t \to \infty \). The real vacuum is the lowest energy state that survives in this limit:

\[ \lim_{t \to \infty} \frac{e^{-\bar{H}t}|0\rangle}{\langle 0|e^{-2\bar{H}t}|0\rangle^{1/2}} = \frac{|\tilde{0}\rangle}{\langle \tilde{0}|\tilde{0}\rangle^{1/2}}, \tag{20} \]

where we have used the Hamiltonian divided by \( g^2 \), \( \bar{H} = H/g^2 \), for the convenience of the order counting. We also use the notations \( \bar{G}, \bar{P}, \bar{T}, \) and \( \bar{H}_0 \) for the electric part,
the plaquette term, the fermion kinetic term, and the fermion mass term divided by $g^2$ respectively. Since our vacuum is given in the strong coupling expansion, we use a spoiled version of the $t$ expansion, i.e. we do not apply the continuation to extract the physical value, Padé approximation, etc. The terms multiplied by $(\bar{G} + m\bar{H}_0)^n$ for all $n$ should be considered together because its eigenvalue is usually of order $g^0$. Out of the $t$ expansion vacuum, the terms proportional to $|0\rangle$ are

$$|0'\rangle = |0\rangle + \left(\frac{1}{2g^2}\right)^2 \sum_l \sum_{n=2}^{\infty} \frac{(-t)^n}{n!} N_c \left(\bar{G} + m\bar{H}_0\right)^{n-2} |0\rangle. \quad (21)$$

These terms are almost from the expansion of the exponential function

$$|0'\rangle = \left[1 + \frac{1}{4} N_c \frac{1}{g^2} C^{-2} \left(e^{-Ct} - 1 + Ct \right) 2L^2 \right] |0\rangle, \quad (22)$$

where $C = C(N_c)/2 + 2m/g^2$ is the eigenvalue of $\bar{G} + m\bar{H}_0$ for the quark pair state $\bar{T}|0\rangle$.

The quark pair states, shown in Fig. 2, are assembled to

$$|l\rangle = \frac{1}{C} \left(e^{-Ct} - 1 \right) \bar{T}|0\rangle. \quad (23)$$

The double quark pair states are summed to

$$|ll'\rangle = -\frac{1}{4g^4 2C^2} \left(1 + e^{-2Ct} - 2e^{-Ct}\right) \sum_{l \neq l'} \eta_l \eta_{l'} \chi^\dagger U_l \chi \cdot \chi^\dagger U_{l'} \chi |0\rangle, \quad (24)$$
for $l \neq l'$ and
\[
|ll\rangle = -\frac{1}{4g^4} \frac{1}{1 - r} \left( 1 - r + r e^{-Ct} - e^{-Cr} \right) \frac{1}{rC^2} \sum_l \left( \chi^\dagger U_l \chi \right)^2 |0\rangle,
\]
(25)
for the same $l$ where $Cr = (N_c^2 - N_c - 2)/2N_c + 4m/g^2$ is the eigenvalue of $\tilde{G} + m\tilde{H}_0$ for the state $(\chi^\dagger U_l \chi)^2 |0\rangle$. The plaquette state is
\[
|P\rangle = \left\{ \exp \left[ -2C(N_c)t \right] - 1 \right\} \frac{1}{2C(N_c)} \tilde{P}|0\rangle.
\]
(26)
The $t$ vacuum state in the limit $t \to \infty$ is consistent with the vacuum $e^R|0\rangle$,
\[
\lim_{t \to \infty} \left[ |0'\rangle + |l\rangle + |ll'\rangle + |ll\rangle + |P\rangle \right]/\langle 0|e^{-2Ht}|0\rangle^{1/2}
= \frac{1 + r_1 R_1 + r_2a R_2a + R_{2g} + \frac{1}{2}r_2^2 R_1^2}{\langle 0|0\rangle^{1/2}} |0\rangle.
\]
(27)
up to order $1/g^4$.

4. Variational Vacuum

So far, we have investigated the vacuum in the strong coupling expansion. Luo and Chen have proposed the vacuum in the unitary form for the fermion part. The unitary vacuum contradicts the one we have obtained, which we call the Hermite vacuum. In this section, we compare the two vacua through the variational method. We show that a lower energy minimum is realized for the hermite vacuum than for the unitary one. To compare the two vacua on the equal condition, we adopt the same form for the gauge part in both cases. We use the one-plaquette formulation in which the gauge part $R_g$ is truncated up to the plaquette term, because we can calculate the vacuum expectation value without variation. For the fermion part, we use the one-link approximation $R_f = \theta_f T$ as a trial state with a variational parameter $\theta_f$ and the kinetic term of the fermion $T$ defined in Eq. 6.

We calculate the energy expectation value for the Hermite vacuum $\exp(R_f) \exp(R_g)|0\rangle$ and the unitary one, $\exp(iR_f) \exp(R_g)|0\rangle$. The Hamiltonian can effectively be written as
\[ \langle H \rangle = \langle H_{\text{eff}} \rangle \] under the trial state, where

\[ H_{\text{eff}} = \left[ 1 + \frac{g^2}{2} C(N_c) \theta_f^2 \right] T + \left[ m + \frac{g^2}{2} C(N_c) \theta_f^2 \right] H_0 + \frac{g^2}{8} \theta_f^2 (R_{2a} + R_{2b} - H_1) \]
\[ - \frac{g^2}{4} N_e C(N_c) L^2 \theta_f^2, \quad (28) \]

for the Hermite and unitary vacuum respectively. Once the effective Hamiltonian is expressed in the link variable, we can estimate the energy expectation value using the one-plaquette formulation. The link graph with crossings can be factorized into bubbles and a diagram without crossings. Now, the energy density \( \mathcal{E} = \langle H \rangle / (N_e L^2) \) is expressed as a function of \( \theta_f \),

\[ \mathcal{E}(\theta_f) = -\frac{g^2}{4} C(N_c) \theta_f^2 + \left[ m + \frac{g^2}{2} C(N_c) \theta_f^2 \right] (A \theta_f + B) \]
\[ + \left[ 1 + \frac{g^2}{2} C(N_c) \theta_f - \frac{g^2}{8} C(N_c) \theta_f^3 \right] A, \quad (30) \]

\[ \mathcal{E}(\theta_f) = \frac{g^2}{4} C(N_c) \theta_f^2 + \left[ m - \frac{g^2}{2} C(N_c) \theta_f^2 \right] B, \quad (31) \]

for the Hermite and the unitary vacuum respectively, where \( A \) and \( B \) are polynomials of \( \theta_f \):

\[ A = \sum_{m,n} (-1)^{m+n-1} Y_{mn} (m+n)! (m+n)! (2m+2n-1)! 2^{2m+2n-1}, \quad (32a) \]
\[ B = -4 \sum_{m,n} (-1)^{m+n-1} Y_{mn} (m+n) \left( \frac{\theta_f}{2} \right)^{2m+2n}. \quad (32b) \]

The summation is taken over \( m, n \), the extension of the loop graph in the direction 1,2. We show the result for the SU(3) case below. The Hermite vacuum has a lower energy minimum than the unitary one, as in Fig. 3. The variational parameter \( \theta_f \) of the Hermite vacuum takes a nontrivial value, and that of the unitary vacuum zero in the
FIG. 3 The average energy $\mathcal{E} = \langle H \rangle / (N_c L^2)$ as a function of the variational parameter $\theta_f$ at $1/g^2 = 0.7$. The solid curve for the Hermite vacuum and the dashed curve for the unitary one.

one-plaquette approximation. From the relation (32), the chiral condensate is expressed by the polynomials (33):

$$\langle \bar{\psi} \psi \rangle = \frac{N_c}{4} \left( -\frac{1}{2} + A\theta_f + B \right),$$

(33)

$$\langle \bar{\psi} \psi \rangle = \frac{N_c}{4} \left( -\frac{1}{2} + B \right),$$

(34)

for the Hermite and the unitary vacuum respectively. The chiral condensate shows the scaling behavior in the vicinity of $1/g^2 = 0.66$, as in Fig. 4. The scaled chiral condensate $\langle \bar{\psi} \psi \rangle / g^4$ is estimated at $-0.074 \times 3$. To reach this value, a higher order calculation is required for the unitary vacuum, as in Ref. [3].

It is important to show the mass of the pseudoscalar meson as a Goldstone boson caused by the chiral symmetry breaking. The pseudoscalar meson state is expressed by $\exp(R_f + R_g) a^\dagger |0\rangle$ in the same $R_f$ but with the different parameter $\theta_f$, where the production operator in the strong coupling limit is

$$a^\dagger = \sum_x \left[ \chi^\dagger (2x) U_2 (2x) \chi (2x + 2) + \chi^\dagger (2x + 1 + 2) U_2^f (2x + 1) \chi (2x + 1) \right].$$

(35)
FIG. 4 The scaled chiral condensate $\langle \bar{\psi}\psi \rangle/(g^4N_c)$ as a function of $1/g^2$. The scaling behavior is observed in the vicinity of $1/g^2 = 0.66$.

The mass of the pseudo-scalar meson, orthogonalized to the vacuum, is obtained as the energy difference from the vacuum energy

$$M_{PS} = \frac{1}{N} \sum_{m'n} (-1)^{n-1} Y^{2m'n} \left[ \frac{n^2 + 1}{2} \right] \theta_{j'}^{m' + 2n - 3} \left\{ \left[ m + \frac{g^2}{2} C(N_c) \theta_{j'}^2 \right] \theta_j C_{2m'n}^{(1)} + \left[ 1 + \frac{g^2}{4} C(N_c) \theta_{j'}^2 \right] C_{2m'n}^{(2)} - \frac{g^2}{4} C(N_c) \theta_{j'}^2 C_{2m'n}^{(3)} \right\} + \frac{g^2}{2} C(N_c),$$

where the coefficients are defined by

$$C_{mn}^{(1)} = 2(m + n)D_{mn} - (m + n - 1) \left( \frac{1}{2} \right)^{2m + 2n - 1},$$

$$C_{mn}^{(2)} = 2D_{mn} - \left( \frac{1}{2} \right)^{2m + 2n - 3},$$

$$C_{mn}^{(3)} = (m + n - 1)D_{mn},$$

$$D_{mn} = \frac{(m + n - 1)!(m + n - 1)!}{(2m + 2n - 2)!}.$$
5. Discussion

We discuss our results in this section. The vacuum state of the lattice gauge theory with the fermion field is investigated. The vacuum in the unitary form for the fermion part proposed in Ref. 6 is different from the Hermite vacuum obtained in the present paper. There is no reason to adopt the vacuum in the unitary form. A principle is required for the determination of the vacuum form. We have proposed the vacuum form inspired from the strong coupling expansion. It is shown that the Hermite vacuum has a lower energy than the unitary one through the variational method.

The chiral condensate of the Hermite vacuum in the one-link approximation is consistent with that of the unitary one in the two-link approximation in Ref. 6. This implies that the Hermite vacuum has a better convergence than the unitary one, in which only a trivial value is obtained in the one-link approximation.

The mass of the pseudoscalar meson is estimated. Although its scaled value does not vanish completely, it is consistent because the chiral symmetry is broken explicitly to solve the doubling problem.
We should note that our formulation of the vacuum structure breaks down in the strong coupling expansion for the SU(2) case. The factor $1/m$ appears in the coefficient of the term $R_{2a}$. Such a factor in the vacuum suggests that another formalism is required in the SU(2) case.
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**Figure Captions**

Fig. 1. Graphs of the operators: (a) the creation of the quark pair $R_1$, (b) the creation of the two quark pairs $R_{2a}$, (c) the reproduction of the quark pair $R_{2b}$, (d) the creation of the plaquette $R_{2p}$.

Fig. 2. Graph of the quark pair state in $t$ expansion. Dashed lines denote the operator $\bar{G} + m\bar{H}_0$.

Fig. 3. The average energy $E = \langle H \rangle / (N_c L^2)$ as a function of the variational parameter $\theta_f$ at $1/g^2 = 0.7$; the solid curve for the Hermite vacuum and the dashed curve for the unitary one.

Fig. 4. The scaled chiral condensate $\langle \bar{\psi}\psi \rangle / (g^4 N_c)$ as a function of $1/g^2$. The scaling behavior is observed in the vicinity of $1/g^2 = 0.66$.

Fig. 5. The scaled mass of the pseudoscalar meson $M_{PS}/g^2$ as a function of $1/g^2$. 