Local adiabatic quantum search with different paths

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Abstract

We report on a detailed analysis of generalization of the local adiabatic search algorithm. Instead of evolving directly from an initial ground state Hamiltonian to a solution Hamiltonian a different evolution path via a 'sure success' Hamiltonian is introduced and is shown that the time required to find an item in a database of size N can be made to be almost independent of the size of the database, asymptotically by modifying the Hamiltonian used to evolve the system. The complexity of the search algorithm in this and other similar approaches is shifted to construction of the Hamiltonian.

Key words: Quantum search algorithms; Local adiabatic search algorithm

1 Introduction

One of the promising approaches to quantum computing is provided by the adiabatic evolution method introduced by Farhi et. al.[1] and elaborated and developed by several groups [1,2,3]. While the standard paradigm of quantum computation uses quantum gates (i.e., unitary operators) applied to quantum registers, the main idea of the adiabatic computing algorithms is to encode the initial state and solution state of the problem at hand in properly chosen Hamiltonians. Generally, the meaning of the ”proper” is the requirement that the initial and the solution states are the ground states of an initial and final Hamiltonians. Then these two Hamiltonians are combined in a time-dependent way and system is let to evolve under the influence of this combined Hamiltonian. The speed of mixing is determined by the adiabatic theorem

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of quantum mechanics [4,5]. According to this theorem, if the evolution of a quantum system is governed by a Hamiltonian that varies slowly enough, the system will stay near its instantaneous ground state. The general time-complexity of adiabatic quantum computation is still an open problem [6]. The complexity of the algorithm is determined by the time needed for switching from initial to solution state "slowly enough". The switching speed depends inversely on the size of the spectral gap between the instantaneous ground and first excited states of the Hamiltonian.

One of the successful quantum algorithms is the database search algorithm of Grover [7], which has been examined in great details in many studies [8,9,10]. This algorithm shows a quadratic speedup in searching a database over classical algorithms. Adiabatic search algorithm in its local adiabatic version was shown to provide similar speedup in searching, namely \( O(\sqrt{N}) \) number of queries [3], where \( N \) is the number of items in the database. Also, some numerical evidence suggests that adiabatic algorithm might efficiently solve some instances of hard combinatorial problems, outperforming classical methods [11,12,13,14].

Farhi et al. [15] showed that one does not need to take a straight path from initial to final Hamiltonian. They showed that by applying a control Hamiltonain at the evolution stage an algorithmic failure can be turned to a success for 3-SAT problem. Recently, Boulatov and Smelyanskiy reported [16] a detailed theoretical study of using different paths for a version of 3-SAT problem.

In the present article, we show that one can get a better than square-root speedup by using different path approach of Ref. [15]. We use part of 'sure success' Hamiltonian of Bae and Kwon [17] as the control Hamiltonian. Added control Hamiltonian keeps the minimum spectral gap a constant plus a system size dependent term which goes to zero as the system size tends to get close to infinity. It is shown that complexity of the proposed algorithm approaches a constant almost independent of the size of the system.

2 Quantum Adiabatic Theorem

Consider a quantum system in state \( |\Psi(t)\rangle \) which evolves according to the Schrodinger equation

\[
\frac{i}{\hbar} \frac{d}{dt} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle
\]

where \( H(t) \) is the Hamiltonian governing the dynamics of the system (we
take $h = 1$). If the Hamiltonian is independent of time and the system is at its ground state, it will remain there forever. According to adiabatic theorem [4,5] if the Hamiltonian varies slowly enough, the state of the system will remain at the instantaneous ground state of the Hamiltonian for all $t$.

Let $|\Psi_k(t)\rangle$ be the eigenstate of $H(t)$, which satisfy

$$H(t) |\Psi_k(t)\rangle = E_k(t) |\Psi_k(t)\rangle$$

where $E_k(t)$ is the corresponding eigenvalue and $k$ labels the eigenvalues ($k = 0$ is the ground state).

The adiabacity conditions are related to the minimum energy gap of the spectrum of $H(t)$ and the transition to excited states of the system. These conditions can be made as follows: The minimum energy gap is defined as

$$g_{\text{min}} = \min_{0 \leq t \leq T} [E_1(t) - E_0(t)]$$

and the maximum of the matrix element of $dH/dt$ between the ground and the first excited state;

$$D_{\text{max}} = \max_{0 \leq t \leq T} \left| \langle \Psi_1(t) | \frac{dH}{dt} | \Psi_0(t) \rangle \right|$$

(1)

Adiabatic theorem states that, if we prepare the system at time $t = 0$ in its ground state $|\Psi_0(0)\rangle$ and let it evolve under the Hamiltonian $H(t)$ for a time $T$, then

$$|\langle \Psi_0(T) | \Psi(T) \rangle|^2 \geq 1 - \epsilon^2$$

provided that

$$\frac{D_{\text{max}}}{g_{\text{min}}^2} \leq \epsilon$$

(2)

where $\epsilon \ll 1$.

3 Local Adiabatic Evolution Algorithm

The problem we consider here is to find a marked item in a set of $N$ items. Items in the system are labeled by $n$ qubits, so the dimension of the Hilbert
space of the system is \(N = 2^n\). The initial state of the system is the uniform superposition of all basis states, which also includes the unknown marked state \(|m\rangle\)

\[
|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle
\]  

(3)

where \(|i\rangle\) are the basis states with \(i = 0, \ldots, N - 1\). \(|\psi_0\rangle\) is the zero energy ground state of

\[H_0 = I - |\psi_0\rangle\langle\psi_0|\]

whose ground state is \(|\psi_0\rangle\) with energy 0. Let the marked state Hamiltonian be

\[H_m = I - |m\rangle\langle m|\]

whose ground state, the marked state \(|m\rangle\), is unknown. The time-dependent Hamiltonian that interpolates between these two Hamiltonians is

\[H(s) = (1 - s)H_0 + sH_m\]  

(4)

where the mixing parameter \(s = s(t)\) is a monotonic function with \(s(0) = 0\) and \(s(T) = 1\), \(T\) being the required running time of the algorithm. The algorithm is to prepare the system in the state \(|\psi(0)\rangle = |\psi_0\rangle\) and then apply \(H(s)\) for a time interval \(T\) to end up with the state \(|m\rangle\). Roland and Cerf [3] shown that complexity of this algorithm when applied in local adiabatic scheme scales with \(\sqrt{N}\), similar to Grover’s algorithm and this result is optimal for any chose of the time evolution of \(s(t)\).

One can generalize the time-dependent Hamiltonian of Eq. 4 by adding to it a Hamiltonian of the form [18,17]

\[H_D = a(s)|\psi_0\rangle\langle m| + b(s)|m\rangle\langle\psi_0|\]

where we have chosen \(a(s) = b(s) = s(1 - s)\) so that the Hamiltonian at \(s = 0\) is \(H_0\) and at \(s = 1\) it is \(H_m\). \(H_D\) acts only during the adiabatic evolution of the system. So its effect is to change the path taken by the system for going from the initial state \(|\psi_0\rangle\) to final state \(|m\rangle\). A similar extra Hamiltonian was shown to convert an algorithmic failure to a success by Farhi et.al. [15]. Now the time-dependent Hamiltonian of the system is

\[H(s) = (1 - s)H_0 + sH_m - H_D\]  

(5)
which in matrix representation can be written as

\[
H = E \begin{pmatrix}
(s-1)(x+x-1) & (s-1)(s+x)\sqrt{1-x^2} \\
(s-1)(s+x)\sqrt{1-x^2} & s(1-s)x^2
\end{pmatrix}
\]

where \( x = \langle m|\psi_0 \rangle = 1/\sqrt{N} \). Two lowest eigenvalues of this Hamiltonian are plotted as function of \( s \) in Fig. 1. These eigenvalues are separated by a gap

\[
g(s) = \sqrt{1-4s(1-s)(1-x(x+1)-s(1-s))}
\]

note that the minimum of \( g(s) \) is at \( s = 1/2 \) with value \( g_{\text{min}} = 1/2 + 1/\sqrt{N} \).

The eigenvectors corresponding to two lowest eigenvalues are

\[
|E_0; s\rangle = \frac{(1-g(s)-2s)|m\rangle + (s-1)(s+x)(1+\sqrt{1-x^2})|\psi_0\rangle}{\sqrt{4(s-1)^2(s+x)^2(1-x^2) + (1-g(s)-2s+2(s-1)x(s+x))^2}}
\]

\[
|E_1; s\rangle = \frac{(1+g(s)-2s)|m\rangle + (s-1)(s+x)(1+\sqrt{1-x^2})|\psi_0\rangle}{\sqrt{4(s-1)^2(s+x)^2(1-x^2) + (1+g(s)-2s+2(s-1)x(s+x))^2}}
\]

Fig. 1. Eigenvalues of the two lowest states of the time-dependent Hamiltonian \( \tilde{H}(s) \) as a function of reduced time \( s \) for \( N = 64 \).

The matrix element in Eq. 1 can be written as

\[
\left\langle \frac{dH}{dt} \right\rangle_{1,0} = \frac{ds}{dt} \left\langle \frac{d\tilde{H}}{ds} \right\rangle_{1,0}
= \frac{ds}{dt}(H_m - H_0 + (1-2s)H_D)_{1,0}
\]
\[ \langle H_m - H_0 + (1 - 2s)H_D \rangle_{1,0} \leq 1 \] [3].

In the local adiabatic approximation one tries to adjust the evolution rate of the \( s \) for infinitesimal time intervals \( dt \) and apply the adiabacity condition locally to each of these intervals. The adiabacity condition for the local adiabatic evolution would be

\[ \left| \frac{ds}{dt} \right| \leq \epsilon \frac{g^2(s)}{\langle \frac{dH}{ds} \rangle_{1,0}} \]

for all times \( t \). Using Eq. 6, the evolution of the Hamiltonian is chosen to evolve at a rate that is solution of

\[ \frac{ds}{dt} = \epsilon g^2(s) = \epsilon \left( 1 - 4s(1 - s) \left( 1 - \frac{1}{N} - \frac{1}{\sqrt{N}} + s(s - 1) \right) \right) \tag{8} \]

Integral of this equation involves the logs of the roots of \( g^2(s) \) and is displayed in Eq. 9. \( s(t) \) obtained from this integral is shown in Fig. 2 along with the \( s(t) \) of Roland and Cerf’s algorithm.

\[ t = \frac{1}{\sqrt{2h(N)}} \left( \frac{1}{k_2(N)} \tanh^{-1} \left[ \frac{2\sqrt{2}k_2(N)s}{2 - k_2(N)^2 - 4s} \right] - \frac{1}{k_1(N)} \tanh^{-1} \left[ \frac{2\sqrt{2}k_1(N)s}{2 - k_1(N)^2 - 4s} \right] \right) \tag{9} \]

where

\[ h(N) = \frac{1}{N} \sqrt{(1 - N)(2\sqrt{N} + 1)} \]
\[ k_1(N) = \sqrt{-2 - 4h(N) + \frac{4}{N} + \frac{4}{\sqrt{N}}} \]
\[ k_2(N) = \sqrt{-2 + 4h(N) + \frac{4}{N} + \frac{4}{\sqrt{N}}} \]

As can be seen from the figure, added term \( H_D \) changes \( H(s) \) much faster compared to the original Hamiltonian of Ref. [3]. The computation time of the algorithm is evaluated by taking \( s = 1 \) in the solution of Eq. 8, and is plotted against \( N \) in Fig. 3 along with the same quantity from Ref. [3]. The asymptotic form of Eq. 8 for \( s = 1 \) is given by

\[ T = \left( 1 + \frac{\pi}{4} \right) + O \left( 1/\sqrt{N} \right) \tag{10} \]
Fig. 2. Dynamic evolution of the Hamiltonian that drives the initial ground state to the solution state: plot of the evolution function $s(t)$ for $N = 64$. The dotted line is for the same quantity calculated by using the Hamiltonian of Ref. [3].

which show that as the system size $N$ approaches infinity, the required running time approaches the constant $(1 + \frac{\pi}{4})$.

Fig. 3. Computation time vs the size of the problem. The dashed and dotted line is for the $\sqrt{N}$ and Hamiltonian of Ref. [3], respectively.

The main factor that determines the required evolution time $T$ is the behavior of minimum gap $g_{\text{min}}$. This energy difference usually reaches its minimum at an avoided level-crossing. If this approach is exponentially fast with a growing particle number then the algorithm is not efficient. The effect of extra term $H_D$ is to keep these two levels far apart. Our result seem to be similar to one obtained by Das et al. [2] who use Eq. 4 with different coefficients for $H_0$ and $H_m$ terms. One can, actually have a minimum gap that is larger than Eq. 6 by choosing $a(s) = b(s) = \sqrt{s(s + 1)}$ in the definition of the driving Hamiltonian $H_D$. In that case, the running time approaches 1 for the large $N$ limit.
4 Conclusion

In summary, we have presented the results of an analysis of a generalized adiabatic quantum search algorithm. Our analysis shows that the optimal speed associated with Grover’s search algorithm can be improved by a careful choice of the time-dependent Hamiltonian. This analysis does not deal with the complexity of implementing the driving Hamiltonian. Hamiltonian of adiabatic algorithms includes searched state $|m\rangle$ terms. The construction of these terms are thought to be analogous to having access to an oracle in original Grover algorithm. It seems that the time-complexity of searching in present approach is shifted from searching process to construction of the Hamiltonian.

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