Electric time in quantum cosmology

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Abstract
Effective quantum cosmology is formulated with a realistic global internal time given by the electric vector potential. New possibilities for the quantum behavior of space-time are found using a Wheeler–DeWitt setting, and the high-density regime is shown to be very sensitive to the specific form of state realized.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Even in simple isotropic models, the high-curvature regime of (loop) quantum cosmology remains poorly understood. At high curvature one expects strong quantum effects sensitive to what state the Universe is in, but the precise form of suitable states is unknown. (In this paper, we will show a new explicit example for this sensitivity.) The popular use of Gaussians or semiclassical states is hard to justify in this regime, and if one starts with a semiclassical state at low curvature and evolves toward larger curvatures, the quantum state depends on the dynamics in all cosmic phases passed through. Quantum ambiguities then prevent precise-enough knowledge of the state dynamics.

As a second important issue, the problem of time [1–3] remains unsolved, affecting the right choice of dynamics. The problem is usually evaded (but not solved) by using specific choices of global internal times which tend to be unrealistic near the big bang, such as a free massless scalar or dust. As part of the problem of time, it is not known how to transform quantum wave functions or entire Hilbert spaces between different internal times, and therefore
results found with one choice of global internal time do not necessarily hold for other choices. But if they depend on what time is used, they cannot be considered physical.

As elsewhere in physics, effective equations provide a better handle on reliable predictions. For a canonical setting such as canonical quantum cosmology, such equations do not refer to entire wave functions but rather to moments of a state [4, 5]. Only a small number of moments, chiefly fluctuations and the covariance, is needed in semiclassical regimes, and as one evolves toward stronger quantum regimes, one can self-consistently check when higher moments become relevant. Effective techniques for quantum constraints [6–8] also allow the use of realistic local internal times, and one can change between different times by mere gauge transformations [9–11]. In this paper, we will not make use of these latter techniques because they require rather involved discussions of constrained systems. Instead, we develop the effective framework of quantum cosmology for a new choice of internal time which is still global but more realistic at high density than a free massless scalar or dust: radiation. We demonstrate that a more realistic choice of internal time arises from electric fields. To be specific, we will assume a Wheeler–DeWitt representation, but our general discussions apply equally to different choices.

2. Radiation Hamiltonian and effective dynamics

The Hamiltonian constraint for spatially flat isotropic Friedmann–Lemaître–Robertson–Walker models with radiation is

\[ H = -\frac{3}{8\pi G}\gamma^{\alpha\beta}c^2\sqrt{|p^\gamma|} + \frac{E^2}{\sqrt{|p^\gamma|}} = 0, \]  

(1)

written in canonical gravitational variables \((c, p)\). These variables are related to the scale factor \(a\), the derivative being with respect to proper time, by \(|p| = a^2\) and \(c = -\gamma\text{sgn}(p)\) (with the Barbero–Immirzi parameter \(\gamma\) relevant in loop quantizations [16, 17]). We have Poisson brackets \([c, p] = 8\pi G/3\) while the momentum \(A\) of \(E\), \([A, E] = 1\), does not appear in the Hamiltonian. As usual, we assume a fixed normalization of the scale factor. If this choice is changed, \(E\) transforms like \(p\) and \(A\) like \(c\). All our following equations, including those of the quantum theory, will be invariant under such (non-canonical) transformations. (See [18] for more details.) The matter part is determined by the electric field \(E = |\vec{E}|\), assumed sufficiently small so as not to cause significant anisotropy. (One could extend the electric field to a triplet \(E^a\), akin to a triad, and preserve exact isotropy [13]. One of them would then provide internal time, while the others would appear as additional degrees of freedom.) By writing the constraint in the form of a Friedmann equation, dividing \(H\) by \(|p|^{3/2}\), one can easily confirm that the matter term provides the correct behavior for radiation: the \(E\)-term amounts to an energy density \(\rho = E^2/|p^\gamma|^2 = E^2/a^4\) with \(E\) constant because \(H\) does not depend on the momentum \(A\) conjugate to \(E\). (Alternatively, the Hamiltonian can be derived using the standard electric-field energy density: \(q_{ab}E^aE^b/\sqrt{\text{det}q}\) reduces to \(|p|E^2/|p|^{3/2}\) with an isotropic spatial metric \(q_{ab} = |p|\delta_{ab}\).)

The momentum of \(E\) is the electromagnetic vector potential and would contribute a non-zero term to \(H\) in the presence of a magnetic field. However, a magnetic field requires

\[\text{3 The use of electric fields in early universe cosmology has been implemented in inflationary theories [12–15]. In these articles, the electric fields considered have or acquire a strong intensity as they play the role of an inflaton and change by many orders of magnitude. Such models require the use of Bianchi universes [12], unless stochastic isotropy conditions [13] or special choices of the gauge [14, 15] are selected. Our clock field, by contrast, will be given by the vector potential, which can grow without affecting the intensity or causing much anisotropic back-reaction. (Just like some part of the vector potential, time is a gauge parameter.)}\n
\[\text{4 The absolute value is taken with the flat Euclidean metric, } |\vec{E}|^2 = \delta_{ab}E^aE^b, \text{ to keep the spatial metric } q_{ab} \text{ as a physical degree of freedom independent of } E.\]
deviations from homogeneity for the rotation of \( \mathbf{A} \) to be non-zero. In the symmetric context used here, the restriction to pure electric fields is therefore meaningful. Since the electric field is canonically conjugate to the vector potential \( A \), which does not appear in the constraint, \( E \) is constant and \( A \) can be used as a global internal time. We will call this choice electric time.

To realize \( A \)-evolution, we follow standard techniques of deparameterization and solve the constraint equation for the momentum

\[
p_A = E(c, p) = \pm \sqrt{\frac{3}{8\pi G^2}}|c|\sqrt{|p|}.
\]

As a function on the gravitational phase space \((c, p)\), \( E(c, p) \) provides Hamiltonian equations of motion for the classical \( c(A) \) and \( p(A) \),

\[
\frac{dc}{d\tau} = [c, E(c, p)] \quad \text{and} \quad \frac{dp}{d\tau} = \{p, E(c, p)\},
\]

as well as the basis for the quantum Hamiltonian of effective equations with respect to \( A \). To transform equations or solutions to proper time \( \tau \), we can multiply all \( d/d\tau \) by \( d\tau/dA \) by \( d\tau/dA = [A, H] = 2E/|p| \) using (1). We confirm the correct classical equations of motion

\[
\frac{d}{d\tau} \frac{a}{\gamma} \left( \frac{1}{2} \frac{a^2}{\gamma^2} \right) = -\frac{\text{sgn}(p)}{2\sqrt{|p|}} \frac{dc}{d\tau} = -\frac{2E}{\gamma p} \frac{dc}{d\tau} = -\frac{2E}{\gamma p} \{c, E\} = \mp \sqrt{\frac{8\pi G}{3}} \sqrt{|p|} \frac{|c|\text{sgn}(p)}{\gamma^2} = -\text{sgn}(p) \frac{c}{\gamma}.
\]

where we have substituted \( E \) for \( c \) in the second line and used the electromagnetic expressions for energy density \( \rho \) and pressure \( P = \frac{4}{3}\rho \) to compare with the standard acceleration equation.

(\text{In what follows, we will set } 8\pi G/3 = 1 \text{ and } \gamma = 1, \text{ so that } \{c, p\} = 1.)

2.1. Effective dynamics

The sign in (2) determines whether one considers solutions of positive or negative frequency with respect to time \( A \). Without loss of generality, we will use the negative choice, such that \( E(c, p) = -|c|\sqrt{|p|} \). Moreover, we can choose a definite sign of \( p \) (the orientation of space as measured by a triad) because we will consider only the approach to small \( p \), not a possible transition from positive to negative \( p \), or vice versa. For such a transition in dynamical terms to be described reliably, the Planck regime of quantum gravity would have to be much better understood than is possible at present. (To describe the transition non-singularly, we would have to refer to wave functions subject to a difference equation in loop quantum cosmology [19–21].) We will work with positive \( p > 0 \).

Finally, since \( E(c, p) \) is a conserved quantity, the sign of \( c\sqrt{p} \) never changes dynamically and we can drop the absolute value, the two sign options here merging with the explicit \( \pm \) in (2). Even for quantum states, the fact that the \( A \)-Hamiltonian \( E(c, p) \) and its quantization are conserved means that the absolute value can be dropped, provided that the expectation value \( \langle c\sqrt{p} \rangle \) is much larger than its quantum fluctuations. Truncating the whole state to a support of definite sign on the spectrum not just of \( |c\sqrt{p}| \) but also of \( c\sqrt{p} \) then ensures that no opposite-sign solutions mix. (These sign issues are the same as in harmonic cosmology obtained with a free massless scalar [22, 23]. For more details, see these papers or [18]. For the construction of corresponding Hilbert spaces in deparameterized quantum cosmology, see [24].)
Following the procedure of canonical effective equations [4, 5], the quantum Hamiltonian $E_Q$ is a function on the quantum phase space with coordinates given by expectation values and moments

$$\Delta(c^ap^b) = \langle(\hat{c} - \langle\hat{c}\rangle)^a(\hat{p} - \langle\hat{p}\rangle)^b\rangle_{\text{symm}}$$

(3)

of a state (using totally symmetric ordering). These variables allow a Poisson structure by extending

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} = \frac{\{\hat{A}, \hat{B}\}}{i\hbar},$$

(4)

defined for all operators $\hat{A}$ and $\hat{B}$, to products of expectation values by the Leibniz rule. There is a general formula for the Poisson bracket of two moments of arbitrary degree, which unfortunately is rather complicated. In most practical examples, especially for moments of low order, it is more convenient to compute their Poisson brackets directly from commutators. At second order, that is for fluctuations $(\Delta c)^2 = \Delta(c^2)$, $(\Delta p)^2 = \Delta(p^2)$ and the covariance $C_{cp} = \Delta(c^p)$, we have

$$\{(\Delta c)^2, (\Delta p)^2\} = 4C_{cp}, \quad \{(\Delta c)^2, C_{cp}\} = 2(\Delta c)^2, \quad \{(\Delta p)^2, C_{cp}\} = -2(\Delta p)^2.$$  

(5)

Semiclassical states are defined generally by the hierarchy $\Delta(c^ap^b) \sim O(h^{(a+b)/2})$ of the moments. This class of states is much more general than that of Gaussian wave functions, which would determine all moments in terms of at most two parameters. At the level of effective equations, sufficient generality of the states considered can thus be guaranteed, without giving rise to prejudices about the form of wave functions. With a semiclassical (or other) hierarchy, the set of infinitely many moments can be truncated to finitely many ones by approximation, allowing practical methods to study the approach to strong quantum regimes. High orders of the moments, if required, make the equations unwieldy, but the derivation as well as solutions of equations of motion for moments to rather high orders can be done with efficient computational codes [25]. Quantum corrections by higher moments are analogues of higher-curvature corrections for higher time derivatives in effective actions [26], amounting in quantum cosmology to important higher-curvature corrections.

The quantum Hamiltonian is a power series in the moments of $c$ and $p$, obtained by Taylor expanding the quantized $(\hat{E}) = \langle E(\langle\hat{c}\rangle + (\hat{c} - \langle\hat{c}\rangle), \langle\hat{p}\rangle + (\hat{p} - \langle\hat{p}\rangle))\rangle$ in $\hat{c} - \langle\hat{c}\rangle$ and $\hat{p} - \langle\hat{p}\rangle$:

$$E_Q := \langle\hat{E}\rangle = E(\langle\hat{c}\rangle, \langle\hat{p}\rangle) + \sum_{a,b} \frac{1}{a!b!} \frac{\partial^{a+b}E(\langle\hat{c}\rangle, \langle\hat{p}\rangle)}{\partial (\hat{c})^a \partial (\hat{p})^b} \Delta(c^a p^b).$$

(6)

Since we assume an operator for $\hat{c}$ to exist, we will obtain the quantum Hamiltonian of a Wheeler–DeWitt quantization, as opposed to a loop quantization where only exponentials of $i\hbar c\hat{c}$, but no $\hat{c}$ exist [27, 18]. (A modification of the classical dynamics by holonomy corrections would be required in the latter case.) Choosing totally symmetric ordering for $c\sqrt{p}$ and expanding to quadratic terms with second-order moments, we have

$$E_Q = -c\sqrt{p} - \frac{C_{cp}}{2\sqrt{p}} + \frac{1}{8} \frac{(\Delta p)^2}{p^{3/2}} c + \cdots,$$

(7)

abbreviating $c = \langle\hat{c}\rangle$ and $p = \langle\hat{p}\rangle$ without risk of confusion. To this order, the Poisson structure provides effective equations

$$\frac{dp}{d\Lambda} = \sqrt{p} - \frac{1}{8} \frac{(\Delta p)^2}{p^{3/2}},$$

(8)

$$\frac{dc}{d\Lambda} = -\frac{c}{2\sqrt{p}} + \frac{C_{cp}}{4p^{3/2}} - \frac{3}{16} \frac{(\Delta p)^2}{p^{5/2}} c$$

(9)
\[ \frac{d(\Delta p)^2}{dA} = \frac{(\Delta p)^2}{\sqrt{p}} \]
\[ \frac{dC_{cp}}{dA} = \frac{1}{4} \frac{(\Delta p)^2}{p^{3/2}} c \]
\[ \frac{d(\Delta c)^2}{dA} = -\frac{(\Delta c)^2}{\sqrt{p}} + \frac{1}{2} \frac{C_{cp}}{p^{3/2}} \]

derived from Hamiltonian equations of motion \( df(c, p, \Delta(\cdot))/dA = \{ f, EQ(c, p, \Delta(\cdot)) \} \).

2.2. Solutions

The equations (8) and (10) for \( p \) and \( (\Delta p)^2 \) are not coupled to the other variables and can be solved separately. To do so, we introduce a new evolution parameter \( x \) by \( dx = p^{-1/2} dA \), so that (10) is completely decoupled: \( \frac{d(\Delta p)^2}{dx} = \frac{(\Delta p)^2}{\sqrt{p}} \) is solved by

\[ (\Delta p)^2(x) = (\Delta p)^2_0 e^x \]  \( (13) \)

with initial values at \( x = 0 \). Inserting this solution in (8) and rewriting it for \( p^2 \), we have the inhomogeneous differential equation \( dp^2/dx = 2p^2 - \frac{1}{4} (\Delta p)^2_0 e^x \) solved by

\[ p(x) = p_0 e^{\frac{1}{4} (\Delta p)^2_0} 1 - \frac{1}{4} (\Delta p)^2_0 (1 - e^{-x}) \].  \( (14) \)

The function is real and positive for all semiclassical values, and in fact in the whole range of \( (\Delta p)^2_0 \leq 2p_0 \). If \( (\Delta p)^2_0 > 2p_0 \), in which case we have to be much more careful trusting our effective equations but may still analyze (14) to suggest possible effects to be corroborated further, \( p(x) \) remains real only for \( x < -\ln \left( 1 - \frac{4p^2_0}{(\Delta p)^2_0} \right) \).

2.2.1. Potential effects at large fluctuations.

In this brief section we collect properties of our equations and solutions when fluctuations become large, keeping in mind that we would have to go to higher orders in effective equations to justify the implications. Nevertheless, it is interesting to see what the equations may indicate.

The first derivative of \( p \) by \( x \),

\[ \frac{dp}{dx} = e^{x/2} \frac{(p^2_0 - \frac{1}{4} (\Delta p)^2_0) e^x + \frac{1}{4} (\Delta p)^2_0}{\sqrt{(p^2_0 - \frac{1}{4} (\Delta p)^2_0) e^x + \frac{1}{4} (\Delta p)^2_0}} \]

becomes zero at

\[ \tilde{x} = \ln \left( \frac{\frac{1}{4} (\Delta p)^2_0}{-p^2_0 + \frac{1}{4} (\Delta p)^2_0} \right) \]

which is real and finite for \( (\Delta p)^2_0 > 4p^2_0 \). Turning points—bounces or recollapses—therefore require strong quantum effects and large relative fluctuations. (But they are certainly not implied by our equations. When relative fluctuations are large, we have a non-semiclassical regime and higher-order moments can no longer be assumed to be negligible. Nevertheless, it may be instructive to explore the possibilities.)

We can see the option for turning points directly from the equations of motion, especially (8). When fluctuations become large, the moment terms in the quantum Hamiltonian provide new possibilities for potential turning points of \( p(A) \). We have \( dp/dA = 0 \) for \( (\Delta p)^2 = 8p^2 \),
requiring large relative volume fluctuations. To test whether this point can be a minimum, potentially corresponding to a bounce, we compute the second derivative of \( p \) by \( A \),

\[
\frac{d^2 p}{dA^2} = \frac{1}{2} - \frac{3}{128} \frac{(\Delta p)^4}{p^3} \tag{15}
\]

using the equations of motion. (All terms linear in \((\Delta p)^2\) cancel.) For \((\Delta p)^2 = 8p_0^2\), \(d^2 p/dA^2 = -1/4\), indicating a maximum of \( p \) and therefore a recollapse. However, if \((\Delta p)^2\) in (15) is significant, higher moments may easily contribute and change the behavior. The precise form of the turning point found here is therefore an explicit example for an effect that is highly sensitive to the precise form of quantum state. For other examples in terms of wave functions, see the solutions given in [28, 29].

Quantum fluctuations could therefore trigger a recollapse, reminiscent of the effect pointed out in [30, 31]. At the turning point, using our solution (14), we have

\[
p(x) = \frac{\Delta p_0}{4 \sqrt{1 - 4p_0^2/(\Delta p_0)^2}} > \frac{1}{2} p_0.
\]

The recollapse value \( p(x) \) may be large if \((\Delta p_0)^2\) is close to \(4p_0^2\), and it is equal to \( p_0\) for \((\Delta p_0)^2 = 8p_0^2\). (If \((\Delta p_0)^2 = 8p_0^2, x < \ln 2\) in order for \( p(x) \) to be real.)

We can now transform back to \( A \) by integrating

\[
dA = \sqrt{p} \, dx = \sqrt{p_0} e^{x^2/2} \sqrt{1 - \frac{1}{4} \frac{(\Delta p_0)^2}{p_0^2} (1 - e^{-x})} \, dx.
\]

The result can be expressed in terms of hypergeometric functions. For instance, using for illustrative purposes the value \((\Delta p_0)^2 = 8p_0^2\), we find

\[
A(x) = A_0 - 2 \sqrt{p_0} \, e^{x^2/2} \left( 1 - \frac{1}{4} x^2 \right) + \mathcal{O}(x^3). \tag{16}
\]

in which \( \Gamma \) represents the Euler function.

Unfortunately, this function and especially its inversion for \( x(A) \), to be inserted in \( p(x) \), are complicated. We can proceed further with additional approximations. First, as one example to explore the strong quantum regime, we may assume that the initial value \((\Delta p_0)^2\) is close to \(8p_0^2\). Thus \( x \to 0 \), and if we are interested in what happens close to \( x \to 0 \), we can expand

\[
\frac{dA}{dx} \sim \sqrt{p_0} \left( 1 - \frac{1}{4} x^2 \right) + \mathcal{O}(x^3).
\]

Integrating from 0 to \( x \) with \( A_0 = 0 \), we find

\[
A(x) \sim \sqrt{p_0} x \left( 1 - \frac{1}{12} x^2 \right) + \mathcal{O}(x^4). \tag{17}
\]

inverted by

\[
x = A \sqrt{p_0} \left( 1 + \frac{1}{12} \left( \frac{A}{\sqrt{p_0}} \right)^2 \right) + \mathcal{O}(A^4).
\]

We can then go back to (14), expand it as \( p(x) = p_0 \left( 1 - \frac{1}{2} x^2 \right) + \mathcal{O}(x^3) \), and find

\[
p(A) = p_0 \left( 1 - \frac{1}{2} \left( \frac{A}{\sqrt{p_0}} \right)^2 \right) + \mathcal{O}(A^3).
\]

To test the approximation, we show a graph of the exact \( A(x) \) as well as its comparison with the function (17) in figure 1. The approximation is good within 10% even for rather large values of \( x \).
Figure 1. The plot of $A(x)$, as recovered exactly (top). We may confront this plot with values found in (17), by plotting the ratio between the graph on the left and the graph of the function in (17). It is then immediate to check (bottom) that the approximation works quite well (the difference is less than 10%) everywhere, except toward the upper boundary of the admissible range of values for $x$, where the difference rises up to 10%.

2.2.2. Small fluctuations. For small relative initial fluctuations $(\Delta p_0^2/p_0^2) \ll 1$, for which our effective equations are reliable, and sufficiently small $x$, we can expand the square root and integrate $dA \sim \sqrt{p_0} e^{x/2} \left(1 - \frac{1}{16} \left(\frac{(\Delta p_0^2/p_0^2)}{1 - e^{-x}}\right) \right) dx$ to

$$A(x) \sim 2\sqrt{p_0} \left(1 - \frac{1}{16} \left(\frac{(\Delta p_0^2/p_0^2)}{1 - e^{-x}}\right) \right) e^{x/2} - \frac{1}{16} \left(\frac{(\Delta p_0^2/p_0^2)}{1 - e^{-x}}\right) + A_0$$

with $b = 1 - \frac{1}{8} (\Delta p_0^2/p_0^2)$. (We will consider only positive $x$, so that the contribution from fluctuations does not increase for larger values. Integrating backwards to negative $x$ would have to be done more carefully.) To express $x$ in terms of $A$, we write

$$x = 2\text{arsinh}\left(\frac{1}{2} \left(1 - \left(1 - \frac{1}{8} (\Delta p_0^2/p_0^2)^2\right)^{-1/2}\right) \frac{(A - A_0)/\sqrt{p_0}}{\sqrt{p_0}} \right)$$

$$- \text{arcosh}\left(\left(1 - \left(1 - \frac{1}{8} (\Delta p_0^2/p_0^2)^2\right)^{-1/2}\right)\right).$$
We emphasize that we had to assume relative fluctuations to be small only at one time, which could be in a semiclassical regime. Our solutions are then valid even in stronger quantum regimes (until, of course, higher moments grow large).

Finally, we can integrate all equations (8)–(12) perturbatively if we assume moments to be small throughout the whole evolution. At zeroth order, we first solve the classical equations, ignoring all moments. We obtain

\[ p_{\text{classical}}(A) = \left( \sqrt{p_0} + A/2 \right)^2 \quad \text{and} \quad c_{\text{classical}}(A) = \frac{c_0\sqrt{p_0}}{\sqrt{p_0} + A/2} \propto \frac{1}{\sqrt{p_{\text{classical}}(A)}} \]  

(18)

with initial values \( p_0 \) and \( c_0 \) when \( A = 0 \). These solutions can then be assumed in the equations of motion for moments to find approximate solutions for the latter. We obtain

\[(\Delta p)^2(A) \propto p(A), \quad C_{cp}(A) \propto -c(A) + \text{const} \quad \text{and} \quad (\Delta c)^2(A) \propto c^4 + \text{const}c^3 + \text{const}''c^2.\]

(19)

(The first of these equations is, of course, consistent with our full solutions (13) and (14) for small \((\Delta p)^2/p_0^2\).) In particular, relative fluctuations \((\Delta p)^2/p^2 \propto p^{-1}, C_{cp}/(cp) \propto p^{-1}\) and \((\Delta c)^2/c^2 \propto \text{const}''\) remain small at small \(c\). Moreover, the uncertainty product \((\Delta c)^2(\Delta p)^2\) is bounded from below by \(\text{const}''\), and the uncertainty relation will never be violated if we choose appropriate values for the constants.

These solutions indicate that effective equations get better and better when one evolves toward larger \(p\), but give rise to strong quantum effects with growing relative fluctuations at small \(p\).

3. Effective equations in terms of proper time

The transformation from internal times to proper time after quantization may not be obvious because it requires a careful look at quantum corrections of different but related expressions—the internal-time Hamiltonian and the Hamiltonian constraint. Nevertheless, once an internal time has been chosen, there is a unique procedure to transform equations or solutions back to proper time. We will first go through the general procedure to highlight ambiguities and difficulties of deparameterized quantizations, as part of the problem of time.

To obtain equations of motion in proper time in our classical deparameterized model, we transformed \(d/dA\) to \(d/dr = 2E/\sqrt{p}d/dA\) with constant \(E\), computing \(dA/dr = [A, H[N]]\) with a Hamiltonian constraint \(H\) of lapse function \(N = 1\). In the deparameterized quantum model, we quantize \(E(c, p)\) after having solved the Hamiltonian constraint equation for \(E\); we do not quantize \(H\) itself. After deparameterized quantization, we can go back to a Hamiltonian constraint

\[ H_Q = \frac{E^2}{\sqrt{p}} - \frac{E_Q(c, p, \Delta(\cdot))^2}{\sqrt{p}} = \frac{E^2}{\sqrt{p}} - \frac{(c\sqrt{p} + \frac{1}{2}C_{cp}p^{-1/2} - \frac{1}{4}(\Delta p)^2cp^{-3/2} + \cdots)^2}{\sqrt{p}} \]

(20)

with quantum corrections in \(E_Q(c, p, \Delta(\cdot))\). By construction, this corrected Hamiltonian constraint gives rise to the deparameterized model we started with if the momentum \(A\) of \(E\) is chosen as time. We therefore transform \(A\)-derivatives to proper-time derivatives by using \(dA/dr = [A, H_Q] = 2E/\sqrt{p} \approx 2E_Q(c, p, \Delta(\cdot))/\sqrt{p}\), the latter equation holding on shell when \(H_Q = 0\).

With this procedure, the relationship between internal-time intervals \(dA\) and proper-time intervals \(d\tau\) in terms of \(E\) does not differ from the classical one, except that a new function \(E_Q\) is used. The identification of \(E\) with \(E_Q\), without further quantum corrections, comes about because \(A\) has been chosen as internal time. As a time variable, it is not quantized and retains
its classical form in quantum evolution equations. By construction, the momentum $p_A$ of $A$ at the quantum level is then $E_Q$, which is used in the electromagnetic Hamiltonian to compute the relation between $A$ and proper time.

If we had quantized the Hamiltonian constraint and then solved it or deparameterized it after quantization, additional quantum corrections with moment terms from $E^2/\sqrt{p}$ would have resulted. These corrections are not included in a deparameterized quantization or in a quantization of the corresponding reduced phase space. A single deparameterized quantization is consistent as long as one does not ask how it may be related to other possible deparameterizations with other choices of internal time. However, if one allows for different internal times, the resulting quantum theories are not likely to be equivalent. In each case, one first solves the classical Hamiltonian constraint equation $H = 0$ for a different variable, $E$ in the choice made here or some other degree of freedom in a different model. The resulting internal-time Hamiltonians then take into account quantum corrections which differ from each other and from the corrections that a quantization of the original Hamiltonian constraint would imply. Deparameterized quantizations not only ignore some quantum corrections in the Hamiltonian constraint, the terms ignored even differ depending on which internal time one chooses. Predictions of different models are then unlikely to be equivalent, and a single deparameterized model cannot be considered physical unless one can show how its results are related to those of different time choices. Our considerations in this paper, as well as most results in quantum cosmology that rely on quantization after deparameterization, therefore cannot be considered complete. Our aim here is not to derive complete effects but rather to give examples for the different features that various choices of internal time can give rise to.

After these cautionary remarks, we now continue with our discussion of electric time. Multiplying our previous equations with $2E_Q/\sqrt{p}$, we obtain

$$\frac{dp}{d\tau} = E_Q \left(2 - \frac{1}{4} \frac{(\Delta p)^2}{p^2}\right) = -2c\sqrt{p} - \frac{C_{ep}}{\sqrt{p}} + \frac{1}{2}(\Delta p)^2 \frac{c}{p^{3/2}}$$

$$\frac{dc}{d\tau} = E_Q \left(-\frac{c}{p} + \frac{C_{ep}^2}{2p^2} - \frac{3}{8} \frac{(\Delta p)^2}{p^3} c\right) = \frac{c^2}{\sqrt{p}} + \frac{1}{4}(\Delta p)^2 \frac{c^2}{p^{3/2}}$$

$$\frac{d(\Delta p)^2}{d\tau} = 2E_Q \frac{(\Delta p)^2}{p} = -2(\Delta p)^2 \frac{c}{\sqrt{p}}$$

$$\frac{dC_{ep}}{d\tau} = \frac{1}{2}E_Q \frac{(\Delta p)^2}{p^2} c = -\frac{1}{2}(\Delta p)^2 \frac{c^2}{p^{3/2}}$$

$$\frac{d(\Delta c)^2}{d\tau} = E_Q \left(-\frac{2}{2} \frac{(\Delta c)^2}{p} + \frac{C_{ep}^2}{p^2}\right) = 2(\Delta c)^2 \frac{c}{\sqrt{p}} - C_{ep}^2 \frac{c^2}{p^{3/2}}$$

where, consistent with our approximation, we have ignored quadratic terms in the small second-order moments because they would compete with higher-order moments which are ignored here. As one can check explicitly, the moments satisfy

$$\frac{d}{d\tau} ((\Delta p)^2(\Delta c)^2 - C_{ep}^2) = 0$$

so that the uncertainty product is preserved. Any departure of a state from saturating the uncertainty relation

$$(\Delta p)^2(\Delta c)^2 - C_{ep}^2 \geq \frac{\hbar^2}{4}$$

(27)
remains constant. If an initial state saturates the uncertainty relation, it will keep saturating it in this regime, and we obtain a dynamical coherent state.

From the first of these equations it follows that the classical relation $c = -\frac{1}{2} \dot{p}/\sqrt{p} = -\dot{a}$, which descends from the isotropic reduction of the Ashtekar–Barbero connection $A^i_a = \Gamma^i_a - \gamma K^i_a$ ($\gamma = 1$) for a spatially flat cosmological model, receives quantum corrections: writing $p$ in terms of the scale factor, we have

$$c = -\frac{\dot{a} + \frac{1}{2} C_{cp}/p}{1 - \frac{1}{2} (\Delta p)^2/p^2} = -\dot{a} \left( 1 + \frac{1}{4} (\Delta p)^2/p^2 \right) - \frac{1}{2} C_{cp}/a^2 + \cdots .$$

(We cannot easily transform the moments to those of $a$ and $\dot{a}$ because these variables are nonlinearly related to $p$ and $c$.) With this relation, we can use the equation for $dc/dr$ to compute the acceleration equation

$$\frac{\dot{a}}{a} = -\left( 1 - \frac{1}{4} (\Delta p)^2/p^2 \right) \left( \frac{\dot{a}}{a} \right)^2,$$

ignoring products of moments. For small relative fluctuations, the equation is consistent with the classical one for radiation. Only the $p$-fluctuation enters, while all covariance terms cancel. Our equations do not provide strong indications for acceleration from large fluctuations, for when fluctuations are large, higher-order moments must be included as well. Such terms could have the right sign for acceleration, but would require further analysis beyond the orders used here.

It is interesting to analyze these equations near the time when $\dot{a} = 0 = \dot{p}$ (which, as we recall, requires large fluctuations), where we have small $c = -\frac{1}{2} C_{cp}/a^2$ and can ignore $c^2$-terms in our equations. In addition to $a$, $c$ is then nearly constant and we can integrate the moment equations

$$\frac{d(\Delta p)^2}{d\tau} = -2(\Delta p)^2 c a,$$

$$\frac{dC_{cp}}{d\tau} = 0,$$

$$\frac{d(\Delta c)^2}{d\tau} = 2(\Delta c)^2 c a$$

(30)

$$(\Delta p)^2(\tau) \sim (\Delta p)^2_0 \exp(-2(\tau - \tau_1)c/a), \quad (\Delta c)^2(\tau) \sim (\Delta c)^2_0 \exp(2(\tau - \tau_1)c/a)$$

(31)

where $\tau_1$ is the time when $\dot{a} = 0$. Toward larger $\tau$, $\Delta p$ decreases exponentially from its initial value $\Delta p_1$, assumed large to make a turning point possible (yet not guaranteed). On the other hand, $p$ itself should change by a power law given the present assumptions, since equation (29) makes acceleration impossible. At least one side of a potential turning point could therefore become semiclassical, but again we emphasize that our equations themselves have to be amended by higher-moment terms near any possible turning point where fluctuations would have to be large. Our solutions indicate that consistent cosmological scenarios could be achieved, but they have to be pushed to higher orders before they can be determined conclusively.

4. Numerical solutions

We return to the Hamiltonian system generated by (7), to seek self-consistent solutions. Here we only illustrate differences between classical and quantum solutions and test the validity of our approximations, which turn out to be very good. A detailed analysis especially near potential turning points could, as motivated by [32], reveal a wave function of the Universe that entails parity violation and non-trivial chiral effects. However, not just the numerics but also higher orders of our effective equations would then have to be developed.

We attempt to solve the system with certain initial conditions for $c(A)|_{A=A_0}$ and the other variables. We may assume the normalization of $p(A)$ initially, such that $p(A)|_{A=A_0} = 1$. We then
Figure 2. Solutions of effective equations for expectation values, plotted as their ratios to the classical solutions $c_{\text{classical}}(A)$ (top) and $p_{\text{classical}}(A)$ (bottom). Initial fluctuations have been set to rather large values—$\Delta p_0 = \Delta c_0 = p_0 = |c_0| = 1$ with $C_{cp} = 0$—to show the implications of quantum corrections more clearly. Nevertheless, the ratios to the classical solutions (with the same initial values $p_0 = |c_0| = 1$) stay close to one.

We therefore parameterize solutions in terms of two positive real numbers, namely $(\Delta p)^2_{A=0}, (\Delta c)^2_{A=0} \in \mathbb{R}^+$, and a real number $C_{cp} \in \mathbb{R}$, which together with $p(A)|_{A=A_0}$ and $c(A)|_{A=A_0}$ represent our initial conditions for the system. The moments must obey the uncertainty relation (27). With these initial values, we solve (8)–(12).

Figure 2 shows good agreement with the classical solutions even for rather large fluctuations, with $\Delta p_0 = p_0$ and $\Delta c_0 = |c_0|$ ($C_{cp} = 0$). In the initial regime of large fluctuations, the ratios quickly depart from the value one, assumed by choosing as classical
initial conditions the initial expectation values of the effective equations. After this initial dip, as \( p \) grows the ratios asymptote to a constant over a rather large range of electric time \( A \). Note that the constant approached by the ratio of solutions for \( c(A) \) is visibly different from one, while the ratio for solutions of \( p(A) \) is very close to one. Since these two constants are different from each other, the deviations from the value one cannot be explained by a different normalization of the scale factor in effective and classical solutions, which would affect both ratios, the one of \( |p| = a^2 \) and the one of \( c \) related to \( \dot{a} \). Instead, the reason for the ratios deviating from one can be seen from equation (7). If fluctuation terms are included, especially large ones amounting to our sample initial values, the effective solution refers to an \( E_0 \) different from the one of the classical solution. The rate of proper time will then change relative to the rate of internal time, implying a rescaling of \( c \) by a constant. (Recall that \( E_0 \) is a constant of motion.) This constant rescaling of \( c \) is seen in our plots, which therefore is not a failure of the classical limit. The agreement with (7) can be confirmed quantitatively. For \( C_{ep}^0 = 0 \), as chosen here, we have \( E_0 = -c_0\sqrt{p_0}(1 - \frac{1}{2}(\Delta p)^2/p_0^2) + \cdots \). Our choice of initial fluctuations reduces \( E_0 \) by 12.5\% compared to the classical \( E \). Accordingly, the effective solution for \( c(A) \) is rescaled by \( E_0/E = 0.875 \) (with the classical \( E = -c_0\sqrt{p_0} \) as per our initial expectation values). This is indeed the constant approached by the numerical solution.

This observation raises an interesting conceptual question. For our Hamiltonian system, we use \( E_0 \) as the momentum of time, with implications only for the rate of proper time. However, it has a physical meaning as the electric field and would therefore be measurable if our system were realistic. The choice of an initial quantum state in a regime of strong fluctuations (or correlations) could have sizeable implications in semiclassical regimes. Or put differently, if one requires effective solutions to reproduce the values of all observables once a semiclassical regime is reached, possible initial states are restricted. With \( E_0 \) in (7) being an observable, one cannot just use the equations of motion generated by the Hamiltonian \( E_0 = E_0(c, p, \ldots) \) but must also keep it as a constraint on the allowed initial values even if the system has been deparameterized. In our example, if one would like to use the same initial values for \( c \) and \( p \) as in classical solutions as well as the same classical limit at large volume (some kind of final condition) it is not possible to have just fluctuation terms, but correlations would have to be chosen so as to produce the classical value for \( E_0 = E \).

We proceed with further illustrations of the effective Hamiltonian system far from the classical limit. Numerical solutions assuming \( c_0 = 0 \) (near a potential turning point) together with \( (\Delta p)^2 = (\Delta c)^2 = 1 \) and \( C_{ep}^0 = 3/4 \) are shown in figure 3. The plot for \( p(A) \) is increasing in \( A \), with a certain rate amounting to the expansion of the universe. This plot, as well as the one for the initially growing \( c(A) \), may seem to indicate acceleration in the increasing branch, but this is realized only in terms of internal time \( A \) not in terms of proper time; see (29). Note also that the range of \( A \) in figure 3 is much smaller than the range used in figure 2 to show the approach of the ratios to one. The plots shown in figure 3 remain in the regime in which the ratios do not yet asymptote to one, and correspondingly this latter asymptotic behavior is reached for much larger values of \( p \) and smaller ones of \( c \).

Numerical evolutions of the second-order moments are perhaps more instructive tests of the equations and solutions we found earlier because they amount to new independent degrees of freedom. We therefore provide several plots comparing various forms of moment solutions. For numerical evolutions of the second-order moments we use the same \( A \)-range as in figure 3. We should therefore remain in a rather strong quantum regime for the whole plot. Nevertheless, we find good agreement with our approximate analytical solutions such as (19) and (31). Note that some of these previous solutions required additional approximations beyond the original moment expansion, such as using classical solutions (without quantum back-reaction even from the second-order moments) before solving the moment equations. This weak effect of
quantum back-reaction may perhaps be taken as an indication that our ignorance of moment terms of third and higher orders does not invalidate our qualitative picture even in stronger quantum regimes, but further analysis certainly remains necessary.

More specifically, figure 4 shows the three second-order moments. The covariance of position and momentum is almost stable in the range studied. The $c$-variance decreases with $A$ and quite soon reaches a vanishing value, while $p$-variance keeps increasing. Qualitatively, both fluctuations evolve in good agreement with (31). Figure 5 provides a more detailed comparison of the squared $p$-variance with the expectation value $p(A)$. Within this range, $(\Delta p)^2$ increases with the same rate as $p(A)$, in accordance with (19). Even though the $p$-variance increases, relative fluctuations $\Delta p/p$ decrease and indicate an approach to semiclassical behavior. The good agreement with our analytical solutions and approximations is further illustrated in.
Figure 4. The variance of $\hat{p}$ (thin dashed), of $c$ (solid), and the covariance between the two operators (thick dashed) for a quantum state of the universe that minimizes the uncertainty relation (27). Good agreement for instance with (31) is realized. (Initial values as in figure 3.)

Figure 5. The $p$-variance $(\Delta p)^2(A)$ (dashed) compared with the expectation value $p(A)$ (solid). Both functions increase in nearly the same way even for large initial fluctuations as in figure 2, confirming our analytical solutions. We have $(\Delta p)^2 > p$, but relative fluctuations $(\Delta p)^2/p^2$ become very small at large $A$.

5. Conclusions

We have laid out the basic description of deparameterized quantum cosmology with time provided by the electric field. The only matter source required to formulate time evolution is
radiation, expected to be significant in any early-universe model. No artificial matter sources such as dust or free massless scalars are required.

Our discussion in this paper does not include modifications suggested by loop quantum cosmology, such as holonomy and inverse-triad corrections. The former would prevent us from using $c$ as an operator and in moments, making the Hamiltonian more nonlinear by replacing $c$ with functions such as $\sin(\delta c)/\delta$. Such modifications certainly alter the high-curvature behavior and are expected to compete with fluctuation terms. The magnitude of holonomy corrections depends on the parameter $\delta$ used in this modification (a quantization ambiguity). However, if $\delta$ is related to the Planck length, for instance by $\delta \sim \ell_P/\sqrt{\rho}$ as often assumed, holonomy corrections are of the size $\ell_P^2/\ell_\chi^2$ with the Hubble scale $\ell_\chi$, the same size expected for higher-curvature corrections. Quantum back-reaction by moments, on the other hand, contributes higher-derivative terms, also related to higher-curvature corrections. If a loop quantization is used, it is therefore impossible to study either holonomy corrections or quantum back-reaction by fluctuations in isolation. In this paper, to provide a manageable first discussion of quantum cosmology with electric time, we have therefore decided to ignore loop effects so that our solutions refer to quantum back-reaction in Wheeler–DeWitt models. (The alternative, ignoring quantum back-reaction but keeping holonomy corrections, is not consistent unless one restricts oneself to models in which quantum back-reaction is absent or weak. These are only the models of harmonic cosmology [22, 33] or kinetic-dominated ones [34–36].) Moreover, recent results in off-shell loop quantum gravity [37–43] have shown that holonomy corrections in loop quantum cosmology must be treated with care because they trigger signature change at high density [44]. Their evolution equations must therefore stop before interesting high-density effects can be realized.

Our discussion of the quantum dynamics was done mainly at the effective level of Wheeler–DeWitt quantum cosmology, allowing us to analyze large classes of states without tying us down to specific wave functions (such as Gaussians). Considering quantum corrections by fluctuations, we have found several new possibilities for early-universe dynamics, showing its high sensitivity to specific forms of states. The small-volume behavior in electric time
is rather different from that with a free massless scalar as time. In the latter case, the singularity is approached exponentially with respect to the scalar (or by a power law in proper time) while electric time, which does not give rise to a harmonic model, leads to a more complicated behavior. These differences do not fully pertain to the different source types, which make already the classical dynamics depend on the model used; they rather affect more general properties of quantum back-reaction. Although we have pointed out the possibility of inequivalent quantum corrections obtained in different deparameterizations (section 3), it remains to be seen whether these differences are solely due to the different matter sources or due to effects of choices of internal times. Before this question is answered, it is not clear how generic current results of (loop) quantum cosmology are, since they have been obtained mainly with a scalar internal time. To decide this question, effective-constraint methods [6, 7] would be suitable but are rather complicated to perform. We therefore end this paper by concluding that much remains to be achieved before the high-curvature regime of quantum cosmology can be controlled, including a better understanding of the role of internal times.

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