Perturbative Spectrum of a Yukawa-Higgs Model with
Ginsparg-Wilson Fermions

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(Dated: February 1, 2008)

Abstract

A Yukawa-Higgs model with Ginsparg-Wilson (GW) fermions, proposed recently by Bhattacharya, Martin and Poppitz as a possible lattice formulation of chiral gauge theories, is studied. A simple argument shows that the gauge boson always acquires mass by the Stückelberg (or, in a broad sense, Higgs) mechanism, regardless of strength of interactions. The gauge symmetry is spontaneously broken. When the gauge coupling constant is small, the physical spectrum of the model consists of massless fermions, massive fermions and massive vector bosons.

PACS numbers: 11.15.Ha, 11.30.Rd, 11.15.Ex

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Recently, Bhattacharya, Martin and Poppitz [1] proposed a Yukawa-Higgs model with GW fermions as a possible lattice formulation of chiral gauge theories. (For reviews on various approaches on this problem, see Refs. [2, 3, 4, 5].) This approach was subsequently studied by analytical and numerical methods [6, 7]. The idea [1] is that half the fermion sector (“mirror fermions”) in a vector-like theory decouples, forming heavy composite fermions by strong Yukawa interactions and, at the same time, the gauge symmetry is not spontaneously broken by keeping the Higgs sector in a symmetric phase (by choosing a coupling \( \kappa \) small; see below). They argued that, in this way, a desired pattern of spectrum as chiral gauge theory, that is, massless Weyl fermions interacting via massless gauge bosons, can be realized.

If this scenario comes true, it implies a great simplification because the lattice chiral gauge theory formulated in Refs. [8, 9] on the basis of the GW relation requires ingenious construction of the fermion integration measure. An “ideal” measure must be consistent with the locality, gauge invariance and smoothness and its construction is far from being trivial. Although an explicit way of construction is known for (anomaly-free) U(1) gauge theories [8, 10, 11, 12] (and for the electroweak SU(2)_L \times U(1)_Y theory [13, 14]), for general non-abelian theories the way of construction has been known only to all orders of perturbation theory [15]. (The existence of an ideal measure in perturbation theory was shown in Refs. [16, 17].) Construction of the fermion integration measure in a non-perturbative level is a mathematically complex problem requiring, first of all, non-abelian generalization of a local cohomology argument on the lattice [10, 11, 12, 18, 19, 20, 21, 22] that is so far available only for the gauge group U(1). On the other hand, as we will review below, the fermion integration measure in the proposal of Ref. [1] is quite simple. Therefore, there is hope such that the mirror fermions and the Higgs field “dynamically” provide an ideal integration measure of massless Weyl fermions while evading the above complexity.

In this brief report, we show that the model unfortunately fails to meet above expectations. The physical vector boson always acquires mass by the Stückelberg (or Higgs) mechanism, regardless of strength of interactions. In this sense, the gauge symmetry is always spontaneously broken. Our argumentation to show this is very simple and kinematical. That is, it relies only on a symmetrical structure of the model. Because of the simplicity of this argument, we believe that some workers in this field have already arrived at the conclusion identical to ours. In fact, it has been known that a compact Higgs field (see below)
can be interpreted as a Stückelberg field; see, for example, Ref. [23]. On the other hand, it appears that the point we want to emphasize below is not so well-appreciated.

As an example, we take the so-called “345” model studied in Ref. [1]. The target theory is a two-dimensional U(1) chiral gauge theory that contains two left-handed Weyl fermions (their U(1) charges are 3 and 4, respectively) and one right-handed Weyl fermion (its U(1) charge is 5). Since $3^2 + 4^2 = 5^2$, this system is free from the gauge anomaly (the issue of the gauge anomaly plays no central role in what follows, however). The partition function of the model, according to Refs. [1, 6, 7], is defined by

$$
Z = \int \prod_x \left( \prod_\mu dU(x, \mu) \right) d\phi(x) \left( \prod_{q=0,3,4,5} d\psi_q(x) d\bar{\psi}_q(x) \right) e^{-S},
$$

where $\mu$ runs from 0 to 1. In this expression, $U(x, \mu)$ denotes the U(1) link variables and $\phi(x) \in U(1)$ is a compact Higgs field. $dU(x, \mu)$ and $d\phi(x)$ are the Haar measures. There are four fermion fields, $\psi_0(x)$, $\psi_3(x)$, $\psi_4(x)$ and $\psi_5(x)$. The first one $\psi_0$ is a spectator having no U(1) charge and it is introduced to form appropriate Yukawa interactions below. Note that the integration measure of the fermions is trivial in a sense that it is a simple product of Grassmann integrals (like that in lattice QCD). This point is quite different from construction of the fermion integration measure in the framework of Refs. [8, 9] that requires a careful choice of basis vectors in which the Weyl fermion fields are expanded. The total action is given by

$$
S = S_G + S_\kappa + S_{\text{light}} + S_{\text{mirror}}.
$$

We do not need to specify an explicit form of the gauge action $S_G$, although we assume that it belongs to a same universality class as the plaquette action. What is important to us is its invariance under the lattice gauge transformation ($\hat{\mu}$ denotes a unit vector in the $\mu$-direction and the lattice spacing $a$ is set to 1 in most part of this paper)

$$
U(x, \mu) \rightarrow \Lambda(x)U(x, \mu)\Lambda(x + \hat{\mu})^{-1},
$$

where $\Lambda(x) \in U(1)$. The kinetic term of the Higgs field $S_\kappa$ is

$$
S_\kappa = \kappa \sum_x \sum_\mu \text{Re} \left\{ 1 - \phi(x)^{-1}U(x, \mu)\phi(x + \hat{\mu}) \right\},
$$

where we have assumed that the field $\phi(x)$ has the U(1) charge +1. The gauge transformation
of $\phi$ is thus given by

$$\phi(x) \to \Lambda(x)\phi(x).$$

(5)

Of course, $S_\kappa$ is invariant under the gauge transformations (3) and (5). The actions of "light" fermions, which correspond to massless Weyl fermions in the target theory, are given by

$$S_{\text{light}} = \sum_x \left\{ \bar{\psi}_{0,+} D_0 \psi_{0,+} + \bar{\psi}_{3,+} D_3 \psi_{3,+} + \bar{\psi}_{4,-} D_4 \psi_{4,-} + \bar{\psi}_{5,+} D_5 \psi_{5,+} \right\}$$

(6)

and, for "mirror" ones

$$S_{\text{mirror}} = \sum_x \left\{ \bar{\psi}_{0,-} D_0 \psi_{0,-} + \bar{\psi}_{3,+} D_3 \psi_{3,+} + \bar{\psi}_{4,+} D_4 \psi_{4,+} + \bar{\psi}_{5,-} D_5 \psi_{5,-} \right\}
+ y \sum_x \left\{ \bar{\psi}_{0,-} (\phi^{-1})^3 \psi_{3,+} + \bar{\psi}_{3,+} (\phi) \psi_{0,-} + \bar{\psi}_{0,-} (\phi^{-1})^3 \psi_{4,+} + \bar{\psi}_{4,+} (\phi) \psi_{0,-} + \bar{\psi}_{3,+} (\phi^{-1})^2 \psi_{5,-} + \bar{\psi}_{5,-} (\phi)^2 \psi_{3,+} + \bar{\psi}_{4,+} (\phi^{-1}) \psi_{5,-} + \bar{\psi}_{5,-} (\phi) \psi_{4,+} \right\}
+ h \sum_x \left\{ \psi_{0,+}^T B(\phi^{-1})^3 \psi_{3,+}^T - \bar{\psi}_{3,+} B(\phi) \bar{\psi}_{0,-}^T + \psi_{0,-}^T B(\phi^{-1}) \psi_{4,+}^T - \bar{\psi}_{4,+} B(\phi) \bar{\psi}_{0,-}^T + \psi_{3,+}^T B(\phi^{-1}) \psi_{5,-}^T - \bar{\psi}_{5,-} B(\phi) \bar{\psi}_{3,+}^T + \psi_{4,+}^T B(\phi^{-1}) \psi_{5,-}^T - \bar{\psi}_{5,-} B(\phi) \bar{\psi}_{4,+}^T \right\},$$

(7)

where $B$ denotes the charge conjugation matrix in two dimensions.

The expressions (6) and (7) need some explanation. The subscript $q$ of the lattice Dirac operators $D_q$ ($q = 0, 3, 4$ or 5) indicates the U(1) charge of the fermion it acts. In the lattice Dirac operator $D_q$, the link variables are contained with the representation $(U(x, \mu))^q$. The Dirac operator $D_q$ must be gauge covariant. That is, under the gauge transformation (3), it transforms as $D_q \to (\Lambda)^q D_q (\Lambda^{-1})^q$. It is also assumed that $D_q$ satisfies the GW relation (24)

$$\gamma_5 D_q + D_q \gamma_5 = D_q \gamma_5 D_q.$$

(8)

Neuberger’s operator (25, 26) is simplest among such lattice Dirac operators. Defining the combination $\hat{\gamma}_{q,5} = \gamma_5 (1 - D_q)$, one has from the GW relation

$$(\hat{\gamma}_{q,5})^2 = 1, \quad D_q \hat{\gamma}_{q,5} = -\gamma_5 D_q$$

(9)

and hence $\hat{\gamma}_{q,5}$ is a lattice analogue of the $\gamma_5$ (27, 28, 29). We also introduce projection operators

$$\hat{P}_{q,\pm} = \frac{1}{2} (1 \pm \hat{\gamma}_{q,5}), \quad P_\pm = \frac{1}{2} (1 \pm \gamma_5)$$

(10)
and define chiral components of lattice fermions by
\[ \psi_{q, \pm}(x) \equiv \hat{P}_{q, \pm}\psi_q(x), \quad \bar{\psi}_{q, \pm}(x) \equiv \bar{\psi}_q(x)P_{\pm} \] (11)
for each \( q \). Note that, because of the property (9), the action of a lattice Dirac fermion completely decomposes into the right- and the left-handed parts
\[ \bar{\psi}_q(x)D_q\psi_q(x) = \bar{\psi}_{q,+}(x)D_q\psi_{q,+}(x) + \bar{\psi}_{q,-}(x)D_q\psi_{q,-}(x). \] (12)
As emphasized in Refs. [1, 6, 7], this complete chiral separation of a lattice action is peculiar to formulation based on the lattice Dirac operator satisfying the GW relation. Since the Dirac operator is gauge covariant, so are the projection operators \( \hat{P}_{q, \pm} \to (\Lambda^\pm q) \hat{P}_q \) and of course \( P_{\pm} \to (\Lambda^\pm q) P_q \). Then the actions (6) and (7) are clearly invariant under the simultaneous gauge transformations (3), (5) and
\[ \psi_q(x) \to (\Lambda(x))^q \psi_q(x), \quad \bar{\psi}_q(x) \to \bar{\psi}_q(x)(\Lambda(x)^{-1})^q. \] (13)
The action for light fermions \( S_{\text{light}} \) is identical to the action of the Weyl fermions that would be taken in the formulation of Ref. [8]. See also Ref. [29].

The Yukawa interactions in Eq. (7) are chosen [1] so that they break all global (vector as well as chiral) \( U(1) \) transformations of mirror fermions, \( \psi_{0,-}, \psi_{3,+}, \psi_{4,+}, \) and \( \psi_{5,-} \), except the global \( U(1) \) part of the gauge transformations (13) and (5).

Now, our argument is based on a simple change of integration variables in Eq. (11). Instead of gauge variant original variables \( U(x, \mu), \psi_q(x) \) and \( \bar{\psi}_q(x) \), one may use gauge invariant ones
\[ U'(x, \mu) = \phi(x)^{-1}U(x, \mu)\phi(x + \hat{\mu}), \quad \psi'_q(x) = (\phi(x)^{-1})^q\psi_q(x), \quad \bar{\psi}'_q(x) = \bar{\psi}_q(x)(\phi(x))^q. \] (14)
For any fixed configuration of \( \phi(x) \), the Jacobian from \( \{U(x, \mu), \psi_q(x), \bar{\psi}_q(x)\} \) to \( \{U'(x, \mu), \psi'_q(x), \bar{\psi}'_q(x)\} \) is unity because \( \phi(x) \in U(1) \) and the numbers of integration variables \( \psi_q(x) \) and \( \bar{\psi}_q(x) \) are same. It is obvious that the action \( S \), when expressed in terms of these primed variables, does not contain the \( \phi \)-field anymore. This is simply a reflection of the gauge invariance of the action and the fact that the compact field \( \phi(x) \in U(1) \) can be regarded as a parameter of the lattice gauge transformation. Then, since \( \phi \) is compact, we can integrate it out from the partition function.
After this change of variables, the kinetic term of the $\phi$-field becomes the mass term of the (gauge invariant) vector boson

$$S_\kappa = \kappa \sum_x \sum_\mu \text{Re} \{1 - U'(x, \mu)\}.$$  \hfill (15)

Thus we see that the vector boson acquires mass by the Stückelberg (or, in a broad sense, Higgs) mechanism \[37\]. (For a review on the Stückelberg mechanism, see Ref. \[30\].) Our choice of the primed variables (14) corresponds to the so-called unitary gauge and one can say that the gauge symmetry is spontaneously broken. In terms of the primed variables, the Yukawa interactions in $S_{\text{mirror}}$ become mass terms of mirror fermions. Note that the above argument holds regardless of strength of interactions. In the present two-dimensional theory, the dimensionless gauge coupling constant $ag$ goes to zero in the continuum limit $a \rightarrow 0$. For $ag \ll 1$, the situation relevant in the continuum limit, the spectrum of the model consists of massless fermions, massive fermions and *massive* vector bosons, interacting through chiral couplings. The mass of the massive fermions is $O(y/a)$ or $O(h/a)$. The mass of the vector boson is, on the other hand, $O(\kappa g)$. Since the variables (14) are gauge invariant, this is a physical spectrum. This perturbative physical spectrum differs from the one, that might be expected in chiral gauge theories in the perturbative regime.

In the above example, the Higgs field has the U(1)-charge +1 and this charge is, according to the terminology of Ref. \[31\], the “fundamental representation”. In fact, our argument above is nothing but the argument used in Ref. \[31\] to show that lattice gauge models with a compact Higgs field in the fundamental representation are in the Higgs phase. The presence of fermions is not relevant in this argument. Here, one cannot repeat an argument of Ref. \[31\] which shows the existence of the Coulomb phase (in which the gauge symmetry is not spontaneously broken) for $\kappa \ll 1$, because that argument is based on the presence of a phase transition in pure gauge models. In two-dimensional gauge models, such a phase transition does not occur.

A similar argument can be repeated for the two-dimensional “1-0” model \[6, 7\] that contains two fermions with the U(1) charges +1 and 0, respectively. The target chiral gauge theory of this model is anomalous because $1^2 \neq 0$ but nevertheless our argument proceeds without any essential change. We again have massless fermions, massive fermions and massive vector bosons. This is very natural because two-dimensional anomalous U(1) chiral gauge theory would be consistent, if the vector boson is allowed to be massive \[32, 33\].
In Refs. [1, 6, 7], the authors are considering the limit $a_g = 0$, where $a_g$ is the dimensionless gauge coupling constant, as a first approximation. Then they completely neglect the gauge fields including the gauge degrees of freedom. What we wanted to emphasize in this note is that this kind of approximation which neglects the underlying gauge symmetry can sometimes be misleading. In other words, the nature of the spontaneous breaking of a continuous symmetry crucially depends on whether the symmetry is global or local (i.e., gauged). For example, global symmetries cannot be spontaneously broken in two dimensions [34], while the Higgs mechanism in two dimensions itself is not prohibited.

The above construction of U(1) models can be generalized to four dimensions. Our conclusion on the massive vector boson is similar, except the point that now the models should be used with finite lattice spacings, because the models are not renormalizable (in the first place, due to Yukawa couplings with a compact Higgs field).

Finally, we comment on generalization to a non-abelian compact gauge group $G$. Natural generalization of the Higgs action is

$$S_\kappa = \kappa \sum_x \sum_\mu \text{Re} \text{tr} \left\{ 1 - \phi(x)^{-1} U(x, \mu) \phi(x + \hat{\mu}) \right\}, \quad (16)$$

where the compact Higgs field $\phi(x)$ is $G$-valued and the Higgs field transforms as $\phi(x) \to \Lambda(x) \phi(x)$ under the lattice gauge transformation. The fermion actions would be replaced by

$$S_{\text{light}} = \sum_x \{ \overline{\chi}_+ D_0 \chi_+ + \overline{\psi}_- D \psi_- \},$$

$$S_{\text{mirror}} = \sum_x \{ \overline{\chi}_- D_0 \chi_- + \overline{\psi}_+ D \psi_+ \}$$

$$+ y \sum_x \{ \overline{\chi}_- R(\phi^{-1}) \psi_+ + \overline{\psi}_+ R(\phi) \chi_- \} + h \sum_x \{ \chi^T BR(\phi^{-1}) \psi_+ - \overline{\psi}_+ BR(\phi) \chi_- \}, \quad (17)$$

where $B$ denotes the charge conjugation matrix. We assumed that the fermion $\psi$ belongs to a unitary (generally reducible) representation $R$ of $G$ and, $R(\phi)$, for example, denotes the Higgs field in that representation. $\chi$ is a spectator (gauge singlet) and we have to introduce $\text{dim} R$ spectators. The lattice Dirac operators and the chirality projections are defined according to the gauge representations of the fermions. We do not write down an explicit form of gauge transformations, etc, because generalization from the abelian case is
obvious. Now, we may make change of variables (that corresponds to the unitary gauge)

\[ U'(x, \mu) = \phi(x)^{-1} U(x, \mu) \phi(x + \hat{\mu}), \]
\[ \psi'(x) = R(\phi(x)^{-1}) \psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x) R(\phi(x)). \] (18)

Then the total action becomes independent of the Higgs field \( \phi \) and we have the physical spectrum similar to that of the above U(1) case. Note that, in this model, all vector bosons become massive and the \( G \) gauge symmetry is completely broken. The unitary gauge is equivalent to take \( \phi(x) \equiv 1 \) and this configuration is not invariant under any non-trivial gauge transformation. Thus, with the above construction, it is impossible to leave some subgroup \( H \), such as the U(1)\(_{\text{EM}}\) within the standard model SU(3) \( \times \) SU(2)\(_L\) \( \times \) U(1)\(_Y\), unbroken.

The \( G \)-valued compact Higgs field precisely corresponds to the “fundamental representation” case considered in Ref. [31] and our conclusion is consistent with that of Ref. [31]; the model is in the Higgs phase. In two dimensions, because of the absence of a phase transition in the pure gauge sector, an argument of Ref. [31] for the existence of the Coulomb phase does not apply. In four dimensions, non-abelian models with massive vector bosons in which the mass is provided by the Stückelberg (not Higgs in a limited sense) mechanism is not renormalizable and the model should be used with finite lattice spacings. On a related issue, see Ref. [35].

In conclusion, the Yukawa-Higgs model with GW fermions proposed in Ref. [1] regrettably cannot be a starting point for lattice formulation of chiral gauge theories, because the gauge symmetry is always spontaneously broken.

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The importance of this phenomenon in a somewhat different context was stressed to me by Yoshio Kikukawa.

Here, by the Stückelberg mechanism, we mean the situation in which the gauge boson acquires mass by completely absorbing all scalar fields and all the scalar fields correspond to gauge degrees of freedom.