An integrable localized approximation for interaction of two nearly anti-parallel sheets of the generalized vorticity in 2D ideal electron-magnetohydrodynamic flows

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The formalism of frozen-in vortex lines for two-dimensional (2D) flows in ideal incompressible electron magnetohydrodynamics (EMHD) is formulated. A localized approximation for nonlinear dynamics of two close sheets of the generalized vorticity is suggested and its integrability by the hodograph method is demonstrated.

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I. GENERAL REMARKS

This work is devoted to analytical study of ideal incompressible EMHD flows (see, e.g., and references therein about EMHD and its applications). Our primary goal here is to consider a simplified 1D problem that has many similar qualitative properties with the problem about attractive interaction of two distributed currents in 2D ideal EMHD, that was numerically simulated recently with a high resolution [1]. More precisely, we introduce and partially analyse local approximations for particular class of the 2D ideal EMHD flows, that are reduced in mathematical sense to dynamics of a single or few 1D objects, the vortex lines. The most interesting result of present work is the demonstration of exact solvability by the known hodograph method of long-scale dynamics in the unstable vortex structure constituted by two nearly anti-parallel sheets of the generalized vorticity in 2D ideal EMHD.

As known, the EMHD model approximately describes dynamics of the low-inertial electron component of plasma in situations when the heavy ion component is almost motionless and serves just to provide a neutralizing background for electrically charged electron fluid and to keep a constant concentration of the electrons. The (divergence-free in this case) electric current \(-en\mathbf{v}(r, t)\) creates the quasi-stationary magnetic field,

\[ \mathbf{B}(r, t) = \frac{4\pi en}{c} \text{curl}^{-1} \mathbf{v}, \]

which contributes to the generalized electron vorticity,

\[ \Omega \equiv \text{curl} \mathbf{v} - \frac{e}{mc} \mathbf{B}. \]

The most simple way how to derive the ideal EMHD equation of motion is just to use the well known fact that the generalized vorticity in an ideal homogeneous fluid is frozen-in,

\[ \Omega_t = \text{curl}[\mathbf{v} \times \Omega]. \]

As the result, the corresponding equation of motion can be represented in the remarkable form

\[ \Omega_t = \text{curl} \left[ \text{curl} \left( \frac{\delta \mathcal{H}(\Omega)}{\delta \Omega} \right) \times \Omega \right], \]

where the Hamiltonian functional of ideal incompressible EMHD in the Fourier representation is given by the expression

\[ \mathcal{H}(\Omega) = \frac{d^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\Omega_k^2}{1 + d_e^2 k^2}. \]

Here \(D = 2\) or \(D = 3\) depending on dimensionality of the problem and the electron inertial length

\[ d_e = (mc^2/4\pi e^2 n)^{1/2} \]

is introduced. Below we normalize all length scales to this quantity.

II. VORTEX LINE REPRESENTATION OF 2D IDEAL EMHD

Our analytical approach is based on the representation of ideal EMHD in terms of frozen-in lines of the generalized vorticity \(\Omega(r, t)\), as described, for instance, in [2, 3]. The general form (4) of equation of motion allows one to represent the field \(\Omega(r, t)\) through the shapes of frozen-in vortex lines (the so called formalism of vortex lines),

\[ \Omega(r, t) = \int_{\mathcal{N}} d^2\nu \int \delta(r - R(\nu, \xi, t)) R_\xi(\nu, \xi, t) d\xi, \]

where \(\delta(\ldots)\) is the 3D delta-function, \(\mathcal{N}\) is some fixed 2D manifold depending on the particular problem, \(\nu \in \mathcal{N}\) is a label of vortex line, \(\xi\) is an arbitrary longitudinal parameter along the line. Dynamics of the line shape \(R(\nu, \xi, t) = (X(\nu, \xi, t), Y(\nu, \xi, t), Z(\nu, \xi, t))\) is determined by the variational principle \(\delta(\mathcal{L})/\delta R(\nu, \xi, t) = 0\), with the Lagrangian of the form

\[ \mathcal{L} = \int_{\mathcal{N}} d^2\nu \int \left[ (R_\xi \times \mathbf{R}_\nu) \cdot \mathbf{D}(R) \right] d\xi - \mathcal{H}\{\Omega\{R\}\}. \]
where the vector function $\mathbf{D}(\mathbf{R})$ must satisfy the only relation
\begin{equation}
(\nabla \mathbf{R} \cdot \mathbf{D}(\mathbf{R})) = 1. \tag{9}
\end{equation}

Below we take $\mathbf{D}(\mathbf{R}) = (0, Y, 0)$.

Now we apply this formalism to the 2D case, when the three-component field $\Omega$ does not depend on the $z$-coordinate. The field $\Omega(x, y, t)$ can be parameterized by two scalar functions, $\Psi(x, y, t)$ and $\Phi(x, y, t)$,
\begin{equation}
\Omega = (\partial_y \Psi, -\partial_x \Psi, \Phi). \tag{10}
\end{equation}

Because of the freezing-in property, the $\Psi$-function is just transported by the $xy$-component of the velocity field, that results in conservation of the integrals
\begin{equation}
I_F = \int F(\Psi)d^2\mathbf{r} = \text{const} \tag{11}
\end{equation}
with arbitrary function $F(\Psi)$. If initially $\Psi(x, y, 0)$ was piecewise constant, then at any time we have a flow with cylindrical sheets of frozen-in generalized vorticity. Each such cylinder is numbered by a number $a = 1..N$, has a constant in time value $C_a$ of the jump of $\Psi(x, y)$, and consists of a family of closed (if $\Phi(x, y, 0) = 0$) vortex lines with identical shape but with different shift along $z$-axis, $(X_a(\xi, t), Y_a(\xi, t), Z_a(\xi, t) + \eta)$, where $\xi$ is a longitudinal parameter along a line, $\eta$ is the shift. Obviously, the number $a$ together with the sift $\eta$ serve in this case as the 2D label $\nu$.

For 2D ideal EMHD in the physical space we have from Eq.(1) the double integral
\begin{equation}
\mathcal{H}\{\Omega\} \propto \frac{1}{2} \int \int K_0(|r_1 - r_2|)(\Omega(r_1) \cdot \Omega(r_2))d^2r_1d^2r_2, \tag{12}
\end{equation}
where $K_0(\cdot)$ is the modified Bessel function of the second kind. We do not write the exact coefficient in front of this expression since it only influences on a time scale and thus is not very interesting for us.

As follows from equations written above, dynamics of this set of contours in 2D ideal incompressible EMHD is determined by the Lagrangian
\begin{align*}
\mathcal{L} &= \sum_a C_a \int Y_a(Z'_a X'_a - X'_a Z'_a)d\xi \\
- &\frac{1}{2} \sum_{a,b} C_a C_b \int K_0 \left( \sqrt{(X_{a1} - X_{b2})^2 + (Y_{a1} - Y_{b2})^2} \right) \\
&\times \left( Z'_{a1} Z'_{b2} + X'_{a1} X'_{b2} + Y'_{a1} Y'_{b2} \right)d\xi_1d\xi_2, \tag{13}
\end{align*}
where the new constants $C_a$ are proportional to the corresponding jumps of $\Psi$ function, $X'_{a1} = \partial_{\xi_1} X_a(\xi_1, t)$ and so on.

For a given contour number, locally, a Cartesian coordinate can be used as the longitudinal parameter, for instance, the $x$-coordinate. In this case the function $Y_a(x, t)$ plays the role of the canonical coordinate, while $Z_a(x, t)$ plays the role of the canonical momentum. Thus, we have a “natural” system with the Hamiltonian being the sum of a quadratic on the generalized momentum “kinetic energy” and a “potential energy” $\mathcal{H}(\Psi)$ depending on the shape of the contours in $xy$-plane, or, in other words, on the $\Psi$ function. In EMHD the “potential energy” describes the interaction between parallel electric currents.

At this point it is interesting to compare the 2D EMHD with the usual Eulerian 2D hydrodynamics, which differs from (1) by the log-function instead of the $K_0$-function. In that case $\Psi$ function is just the $z$-component of the velocity field, and the “potential energy” $\mathcal{H}(\Psi)$ is an integral of motion for Eulerian 2D flows, as follows from the expression
\begin{equation}
\mathcal{H}_{\text{Euler2D}} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{|\Omega_k(\Psi)|^2}{k^2} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} |\Psi_k|^2 \tag{14}
\end{equation}
and from Eq.(1) with $F(\Psi) = \Psi^2$. Equations of motion, that follow from the variational principle with the Lagrangian like (13), but with “$-\log$” instead of “$K_0$”, are such that this term does not have influence on the contour dynamics in $xy$-plane, only it adds a linear function of the time to $Z$-coordinate of a vortex line. This property corresponds to conservation of the $z$-component of the velocity in 2D Eulerian flows for each moving element of the fluid. Obviously, in 2D EMHD such conservation does not take place.

III. LOCALIZED APPROXIMATIONS

A. The case of a single contour

For practical analytical calculations the system (13) is not very convenient because of the non-locality. However, since the $K_0$-function is exponentially small at large values of its argument, it is possible to introduce local approximations for long-scale dynamics. Let us first have a single contour of a large size $\Lambda \gg 1$. Then for smooth configurations approximate local equations of motion (with the time appropriately rescaled) can be obtained by varying the expression
\begin{align*}
\mathcal{L}_{\text{single}} &\approx \int Y_a(Z'_a X'_a - X'_a Z'_a)d\xi \\
- &\frac{1}{2} \int \sqrt{X'^2 + Y'^2} \left( 1 + \frac{Z'^2}{X'^2 + Y'^2} \right) d\xi, \tag{15}
\end{align*}
which naturally arises after we perform one integration in the double integral in Eq.(13) with (almost) straight shape of the adjacent piece (a few units of $d_0$) of the contour. Although for us this system seems to be very interesting and deserving much attention, now we concentrate on another case and consider unstable vortex structure constituted by two close contours.
While From (18) and (20) we see that the method; see, e.g., [11] for a particular case). Indeed, since \( \rho \) is equal to 2 for convenience. Then, after introducing new quantities \( \rho = 2Y \) and \( \mu = Z' \), as well as the function \( H(\rho, \mu) \),

\[
H(\rho, \mu) = (1 + \mu^2)(1 - e^{-\rho}),
\]

it is possible to write down the corresponding equations of motion in the following remarkable general form,

\[
\mu = \frac{\partial Z}{\partial x}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} H_\rho(\rho, \mu) = 0, \quad \frac{\partial Z}{\partial t} + H_\mu(\rho, \mu) = 0.
\]

As known, any nonlinear system of such form can be locally reduced to a linear equation after taking as the new independent variables \( \rho \) and \( \mu \) (the so called hodograph method; see, e.g., [11] for a particular case). Indeed, since from (18) and (20) we see

\[
dZ = \mu dx - H_\rho dt,
\]

it is useful to introduce an auxiliary function \( \chi(\rho, \mu) \) as

\[
\chi = Z - x\mu + tH_\rho
\]

in order to obtain

\[
d\chi = -xd\mu + tH_\rho d\rho + tH_\mu d\mu.
\]

From here we easily derive

\[
t = \frac{\chi_\rho}{H_\rho}, \quad x = H_\mu - \chi_\mu.
\]

After that we rewrite Eq. (19) as

\[
\frac{\partial(\rho, x)}{\partial(\rho, t)} - H_{\mu\rho} \frac{\partial(\rho, t)}{\partial(t, x)} - H_{\mu\mu} \frac{\partial(\mu, t)}{\partial(t, x)} = 0
\]

and multiply it by the Jacobian \( \partial(t, x)/\partial(\rho, \mu) \):

\[
\frac{\partial(\rho, x)}{\partial(\rho, \mu)} - H_{\mu\rho} \frac{\partial(\rho, t)}{\partial(\rho, \mu)} - H_{\mu\mu} \frac{\partial(\mu, t)}{\partial(\rho, \mu)} = 0.
\]

Thus, now we have

\[
x - H_{\mu\rho} t_\mu + H_{\mu\mu} t_\rho = 0.
\]

Differentiating this equation over \( \rho \) with taking into account Eqs. (21, 22) and subsequent simplifying give us the linear partial differential equation for the function \( t(\rho, \mu) \):

\[
(H_{\mu\rho} t_\rho - H_{\mu\rho} t_\mu) = 0.
\]

It is also useful to write down here the general equation for the function \( \chi(\rho, \mu) \):

\[
(H_{\rho\mu} \chi_\rho/H_{\rho\rho})_\rho - \chi_{\mu\mu} = 0.
\]

Thus, the localized approximation (16) appears to be integrable in the sense that it is reduced to solution of a linear equation. However, the functions \( t(\rho, \mu) \) and \( x(\rho, \mu) \) are multi-valued in general case. Therefore statement of the Cauchy problem for the time evolution of the system (originally the Cauchy problem was formulated in \( (t, x) \)-representation in terms of initial functions \( \rho_0(x) \) and \( \mu_0(x) \) at \( t = 0 \)) now becomes much more complicated, since in \( (\rho, \mu) \)-plane initial data are placed on the parametrically given curve \( \rho = \rho_0(x), \mu = \mu_0(x) \) which can have self-intersections. It should be noted here that for \( \chi(\rho, \mu) \) initial data are determined directly by Eqs. (21, 22), while for \( t(\rho, \mu) \) their determination needs additional differentiation of Eq. (23) over \( \rho \).

Besides this, the particular function \( H(\rho, \mu) \) given by Eq. (17) results in the elliptic partial differential equation for the function \( t(\rho, \mu) \),

\[
2[(1 - e^{-\rho})t_\rho + e^{-\rho}((1 + \mu^2)t_{\mu\mu} = 0,
\]

in contrast to the usual 1D gas-dynamic case described in [11], where the corresponding equation is hyperbolic. Generally speaking, the ellipticity makes the Cauchy problem ill-posed in the mathematical sense, if the initial data are not very smooth. However, for sufficiently
smooth initial data the problem remains correctly formulated though still difficult for complete solution. Nevertheless, the linear equation seems to have an advantage, and we hope with its help to investigate more easily the problem of classification of possible singularities in this system. In the future work we will discuss how the quantity $\rho$ can tend to zero at some point $x$.

IV. CONCLUDING REMARK

It should be also noted that an analogous approach is useful in studying another unstable vortex structure, the pair of anti-parallel vortex filaments in the usual hydrodynamics and in other hydrodynamic-type models [12] (the corresponding instability in Eulerian hydrodynamics is known as the Crow instability). For instance, if we consider nonlinear development of the Crow instability in long-scale limit, then the localized approximation for symmetric (respectively to the plane $y = 0$) dynamics of the vortex pair gives us the Hamiltonian

$$H_{\text{Crow}} \propto \int \sqrt{X'^2 + Z'^2} \ln \left( \frac{Y}{\epsilon} \right) d\xi,$$

where $\epsilon$ is the (small) width of the filaments. Taking the $x$-coordinate as a longitudinal parameter $\xi$, we have the system like [11-20], but with $H(\rho, \mu) = H_{\text{Crow}}(\rho, \mu)$,

$$H_{\text{Crow}} = \sqrt{1 + \mu^2} \ln \rho.$$

Investigation of the corresponding linear equation for the function $t(\rho, \mu)$ is now in progress.

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[1] A.S. Kingsep, K.V. Chukbar, and V.V. Yan’kov, in Reviews of Plasma Physics edited by B. Kadomtsev (Consultants Bureau, New York, 1990), Vol. 16, p.243.
[2] K. Avinash, S. V. Bulanov, T. Esirkepov, P. Kaw, F. Pegoraro, P. V. Sasorov, and A. Sen, Phys. Plasmas 5, 2849 (1998).
[3] D. Biskamp, E. Schwarz, A. Zeiler, A. Celani, and J. F. Drake, Phys. Plasmas 6, 751 (1999).
[4] N. Attico, F. Califano, and F. Pegoraro, Phys. Plasmas 7, 2381 (2000).
[5] A. Fruchtman, Phys. Fluids B 3, 1908 (1991).
[6] S. B. Swanekamp, J. M. Grossmann, A. Fruchtman, B. V. Oliver, and P. F. Ottinger, Phys. Plasmas 3, 3556 (1996).
[7] V.P. Ruban, Phys. Rev. E 65, 047401 (2002).
[8] V. P. Ruban and S. L. Senchenko, LANL E-print physics/0204087.
[9] R. Grauer, private communication (2002).
[10] V.P. Ruban, Phys. Rev. E 64, 036305 (2001).
[11] L.D. Landau and E.M. Lifshitz, Hydrodynamics, (Nauka, Moscow, 1988), in chapter “One-dimensional motion of a compressible gas”.[12] V.P. Ruban, D.I. Podolsky, and J.J. Rasmussen, Phys. Rev. E 63, 056306 (2001);