ON FRACTIONAL SCHRÖDINGER EQUATIONS
IN SOBOLEV SPACES

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(Communicated by Jose Canizo)

Abstract. Let $\sigma \in (0,1)$ with $\sigma \neq \frac{1}{2}$. We investigate the fractional nonlinear Schrödinger equation in $\mathbb{R}^d$:

$$i\partial_t u + (-\Delta)^{\sigma} u + \mu |u|^{p-1} u = 0, \quad u(0) = u_0 \in H^s,$$

where $(-\Delta)^{\sigma}$ is the Fourier multiplier of symbol $|\xi|^{2\sigma}$, and $\mu = \pm 1$. This model has been introduced by Laskin in quantum physics [23]. We establish local well-posedness and ill-posedness in Sobolev spaces for power-type nonlinearities.

1. Introduction. Let $\sigma \in (0,1)$ with $\sigma \neq \frac{1}{2}$. We consider the Cauchy problem for the fractional nonlinear Schrödinger equation in $\mathbb{R}^d$,

$$
\begin{cases}
  i\partial_t u + (-\Delta)^{\sigma} u + \mu |u|^{p-1} u = 0, \\
  u(0) = u_0 \in H^s,
\end{cases}
$$

(NLS$_{\sigma}$)

where $u = u(t,x) : I \times \mathbb{R}^d \to \mathbb{C}$ and $I \subset \mathbb{R}$ is a time interval containing 0. The equation is called focusing if $\mu = -1$, and defocusing if $\mu = 1$. The operator $(-\Delta)^{\sigma}$ is the so-called fractional Laplacian, that is, the Fourier multiplier with symbol $|\xi|^{2\sigma}$. The fractional laplacian is the infinitesimal generator of a certain Levy process [1]. A rather extensive study of the potential theoretic aspects of this operator can be found in [22].

The previous equation is a fundamental equation of fractional quantum mechanics, a generalization of the standard quantum mechanics extending the Feynman path integral to Levy processes [23].

The purpose of the present paper is to develop a general well-posedness and ill-posedness theory in Sobolev spaces. For the nonlinear Schrödinger equation ($\sigma = 1$), the equation is known to be locally well-posed in $H^s$, provided that $s$ is greater than or equal to the regularity invariant under scaling and that invariant

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2000 Mathematics Subject Classification. 35A01, 35B35, 35B45, 35B65.

Key words and phrases. Fractional Schrödinger, Sobolev spaces, Local and global well-posedness, ill-posedness.

Y.H. would like to thank IHÉS for their hospitality and support while he visited in the summer of 2014. Y.S. would like to thank the hospitality of the Department of Mathematics at University of Texas at Austin where part of the work was initiated. Y.S. acknowledges the support of ANR grants "HAB" and "NONLOCAL". We are thankful to Prof. Enno Lenzmann for a remark on the first theorem of this work which improved the result.

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under the Galilean transformation \[2\]. On the other hand, it is ill-posed in \(H^s\) otherwise \([20, 4, 3]\). Therefore, for the fractional case, it is natural to introduce two important exponents:

\[
s_c = \frac{d}{2} - \frac{2\sigma}{p-1}
\]

and

\[
s_g = \frac{1 - \sigma}{2}.
\]

Here, \(s_c\) is the scaling-critical regularity exponent in that for \(\lambda \geq 0\), the transformation

\[
u(t, x) \mapsto \frac{1}{\lambda^{2\sigma/(p-1)}} \nu \left( \frac{t}{\lambda^{2\sigma}}, \frac{x}{\lambda} \right), \quad u_0(x) \mapsto \frac{1}{\lambda^{2\sigma/(p-1)}} u_0 \left( \frac{x}{\lambda} \right)
\]

keeps the equation invariant, and \(s_g\) is the critical regularity in the "pseudo"-Galilean invariance (see the proof of ill-posedness below). One might conjecture that the fractional nonlinear Schrödinger equation is locally well-posed in \(H^s\) for 

\[
s \geq \max\{s_c, s_g\}
\]

and it is ill-posed in \(H^s\) if 

\[
s < \max\{s_c, s_g\}.
\]

An important feature of the equation under study is a loss of derivatives for the Strichartz estimates as proved in \([10]\). Unless additional assumptions are met such as radiality as in \([15]\), one has a loss of \(d(1 - \sigma)\) derivatives in the dispersion (see \((5)\)). When the nonlinearity is Hartree-type such as \((V * |u|^{p-1})u\), some amount of loss of derivatives can be absorbed by the regularizing property of the convoluted potential, depending on how nice the potential is (see \([6]\) and references there in). However, a loss of regularity happens to be an issue for the equation with power-type nonlinearity.

Several papers have been devoted to fractional Schrödinger equations with various nonlinearities. The above mentioned paper \([6]\) deals with a Hartree-type nonlinearity. Similarly, standing waves and their properties such as stability have been investigated in \([5, 7]\). In the appendix, we follow the Weinstein argument \([24]\) to prove global well-posedness below the ground state for the \(L^2\) critical fractional Schrödinger equation.

The one-dimensional case has been treated in \([8]\) for cubic nonlinearities, i.e. \(p = 3\), and \(\sigma \in \left(\frac{1}{2}, 1\right)\), where the authors employed bilinear estimates in Bourgain spaces. In this article, we consider a higher-dimensional version and general power nonlinearity, and we also include all \(\sigma \in (0, 1)\) except \(\sigma = \frac{1}{2}\). In this latter case, the Strichartz estimates do not hold and one has to use a different argument.

Our proof of local well-posedness is much simpler than that in \([8]\), and it relies only on the standard Strichartz estimates and simple \(L^p_t\) bounds. However, we emphasize that in \([8]\), the authors also derived well-posedness theory on the flat torus that is indeed the main goal of their paper, while our method cannot be applied to the periodic case due to weaker dispersion on \(\mathbb{T}^d\).

Under the flow of the equation (NLS\(_\sigma\), the following quantities are conserved:

\[
M[u] = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx \quad \text{(mass)},
\]

\[
E[u] = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{\mu}{p+1} |u(t, x)|^{p+1} \, dx \quad \text{(energy)}.
\]

It is interesting to investigate global aspects with large data using these conservation laws. However, in the present paper, we will not consider this interesting topic. We refer the reader to \([14]\) for a study of the energy-critical equation in the radial case,
following the seminal work of Kenig and Merle [19, 18]. We also do not consider blow-up phenomena, an aspect we will treat in a forthcoming work.

**Remark 1.** Throughout the paper, the standard notation \(|\nabla|^s\) stands for \((-\Delta)^{\frac{s}{2}}\).

### 1.1. Main results.

The goal of this paper is to show that \((\text{NLS}_\sigma)\) is locally well-posed in \(H^s\) for \(s \geq \max(s_c, s_g, 0)\), and it is ill-posed in \(H^s\) for \(s \in (s_c, 0)\). We start with well-posedness results.

**Theorem 1.1** (Local well-posedness in subcritical cases). Let

\[
\begin{cases}
  s > \max\{s_g, s_c\} & \text{when } d = 1 \text{ and } p > 1, \text{ or } d \geq 2 \text{ and } p \geq 3, \\
  s = s_g & \text{when } d = 1 \text{ and } 2 \leq p < 5.
\end{cases}
\]

Then, \((\text{NLS}_\sigma)\) is locally well-posed in \(H^s\).

**Theorem 1.2** (Local well-posedness in critical cases). Suppose that

\[
\begin{cases}
  p > 5 & \text{when } d = 1, \\
  p > 3 & \text{when } d \geq 2.
\end{cases}
\]

Then, \((\text{NLS}_\sigma)\) is locally well-posed in \(H^{s_c}\).

The proof of Theorem 1.2 is based on a new method, improving on estimates in [9]. This improvement, based on controlling the nonlinearity in a suitable space, is necessary due to the loss of derivatives in the Strichartz estimates.

As a by-product, we also prove small data scattering.

**Theorem 1.3** (Small data scattering). Suppose that

\[
\begin{cases}
  p > 5 & \text{when } d = 1, \\
  p > 3 & \text{when } d \geq 2.
\end{cases}
\]

Then, there exists \(\delta > 0\) such that if \(\|u_0\|_{H^{s_c}} < \delta\), then \(u(t)\) scatters in \(H^{s_c}\). Precisely, there exist \(u_\pm \in H^{s_c}\) such that

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it(-\Delta)^\sigma} u_\pm\|_{H^{s_c}} = 0.
\]

**Remark 2.** Contrary to the case \(\sigma \neq \frac{1}{2}\), when \(\sigma = \frac{1}{2}\), the fractional NLS does not have small data scattering. See [17].

Finally, our last theorem is the ill-posedness result. Note that our result is not optimal, since one should expect ill-posedness in \(H^s\) up to \(s_g = \frac{1-\sigma}{2}\), which is nonnegative. We hope to come back to this issue in a forthcoming work.

**Theorem 1.4** (Ill-posedness). Let \(d = 1, 2\) or \(3\) and \(\sigma \in (\frac{d}{4}, 1)\). If \(p\) is not an odd integer, we further assume that \(p \geq k + 1\), where \(k\) is an integer larger than \(\frac{d}{2}\). Then, \((\text{NLS}_\sigma)\) is ill-posed in \(H^s\) for \(s \in (s_c, 0)\).

An interesting feature of the previous ill-posedness result is the fact that, contrary to the standard NLS equation \((\sigma = 1)\) there is no exact Galilean invariance. However, one can introduce a new “pseudo-Galilean invariance” which is enough to our purposes. More precisely, for \(v \in \mathbb{R}^d\), we define the transformation

\[
G_v u(t, x) = e^{-iv.x} e^{it|v|^{2\sigma}} u(t, x - 2t\sigma|v|^{2(\sigma-1)}v).
\]

Note that when \(\sigma = 1\), \(G_v\) is simply a Galilean transformation, and that NLS is invariant under this transformation, that is, if \(u(t, x)\) solves NLS, so does \(G_v u(t, x)\).
However, when $\sigma \neq 1$, $(\text{NLS}_\sigma)$ is not exactly symmetric with respect to pseudo-Galilean transformations. This opens the construction of solitons for $(\text{NLS}_\sigma)$ which happen to be different from the ones constructed in the standard case $\sigma = 1$. Indeed, if we search for exact solutions of the type

$$u(t, x) = e^{i|t|^\sigma - \omega^2} e^{-i\omega \cdot x} Q_\omega (x - 2t\sigma|v|^{2(\sigma - 1)}v),$$

then the profile $Q_\omega$ solves the pseudo-differential equation

$$\mathcal{P}_\omega Q_\omega + \omega^{2\sigma} Q_\omega - |Q_\omega|^{p-1} Q_\omega = 0,$$

where

$$\mathcal{P}_\omega = e^{i\omega \cdot x} (-\Delta)^\sigma e^{-i\omega \cdot x} - |v|^{2\sigma} - 2i\sigma|v|^{2\sigma-2}v \cdot \nabla,$$

i.e., $\mathcal{P}_\omega$ is a Fourier multiplier $\mathcal{P}_{\omega} f(\xi) = p_{\omega}(\xi) \hat{f}(\xi)$, with symbol $p_{\omega}(\xi) = |\xi - v|^{2\sigma} - |v|^{2\sigma} + 2\sigma|v|^{2\sigma-2}v \cdot \xi$. (3)

We plan to come back to this issue in future works.

2. Strichartz estimates. In this section, we review Strichartz estimates for the linear fractional Schrödinger operators. We say that $(q, r)$ is admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad (q, r, d) \neq (2, \infty, 2).$$

We define the Strichartz norm by

$$\|u\|_{S_{q', r}(I)} := \|\nabla|^{-d(1-\sigma)(\frac{1}{q} - \frac{1}{2})} u\|_{L^{q}_{t} L^{r}_{x}},$$

where $I = [0, T]$. Let $\psi : \mathbb{R}^d \to [0, 1]$ be a compactly supported smooth function such that $\sum_{N \in 2\mathbb{Z}} \psi_N = 1$, where $\psi_N(\xi) = \psi(\frac{\xi}{N})$. For dyadic $N \in 2^\mathbb{Z}$, let $P_N$ be a Littlewood-Paley projection, that is, $P_N \hat{f}(\xi) = \psi(\frac{\xi}{N}) \hat{f}(\xi)$. Then, we define a slightly stronger Strichartz norm by

$$\|u\|_{\tilde{S}_{q', r}(I)} := \left( \sum_{N \in 2^\mathbb{Z}} \|P_N(\nabla|^{-d(1-\sigma)(\frac{1}{q} - \frac{1}{2})} u\|_{L^{q}_{t} L^{r}_{x}}^{2} \right)^{1/2}.$$

**Proposition 1** (Strichartz estimates [10]). For an admissible pair $(q, r)$, we have

$$\|e^{it(-\Delta)^\sigma} u_0\|_{S_{q', r}(I)}, \|e^{it(-\Delta)^\sigma} u_0\|_{\tilde{S}_{q', r}(I)} \lesssim \|u_0\|_{H^\sigma},$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{S_{q', r}(I)} \lesssim \|F\|_{L_{t}^{q} H_{\tilde{r}}}^\sigma,$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{\tilde{S}_{q', r}(I)} \lesssim \|F\|_{L_{t}^{q} H_{\tilde{r}}}^\sigma.$$

**Sketch of Proof.** By the standard stationary phase estimate, one can show that

$$\|e^{it(-\Delta)^\sigma} P_1\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{d}{2}},$$

and by scaling,

$$\|e^{it(-\Delta)^\sigma} P_N\|_{L^1 \to L^\infty} \lesssim N^{d(1-\sigma)} |t|^{-\frac{d}{2}}.$$

Then, it follows from the argument of Keel-Tao [21] that for any $I \subset \mathbb{R}$,

$$\|e^{it(-\Delta)^\sigma} P_N(\nabla|^{-d(1-\sigma)(\frac{1}{q} - \frac{1}{2})} u_0)\|_{L^{q}_{t} L^{r}_{x}} \lesssim \|P_N u_0\|_{H^\sigma},$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} P_N(\nabla|^{-d(1-\sigma)(\frac{1}{q} - \frac{1}{2})} F) ds \right\|_{L^{q}_{t} L^{r}_{x}} \lesssim \|P_N F\|_{L_{t}^{q} H_{\tilde{r}}}^\sigma.$$
Squaring the above inequalities and summing them over all dyadic numbers in $2^\mathbb{Z}$, we prove Strichartz estimates.

The loss of derivatives is due to the Knapp phenomenon (see [15]). However, in the radial case, one can overcome this loss as proved in [15], restricting then the admissible powers of the fractional laplacian. Indeed, in [15], it is proved that one has optimal Strichartz estimates if $\sigma$ is an admissible pair of the fractional laplacian. Indeed, in [15], it is proved that one has optimal Strichartz estimates if $\sigma \in (d/(2d-1), 1)$. In particular, the number $d/(2d-1)$ is larger than 1/2 and there is a gap between the Strichartz estimates for the wave operator $\sigma = 1/2$ and the one occurring for higher powers. This issue suggests that a new phenomenon might occur for this range of powers.

3. Local well-posedness. We establish local well-posedness of the fractional NLS by the standard contraction mapping argument based on Strichartz estimates. Due to loss of regularity in Strichartz estimates, our proof relies on the $L^2_x$ bounds (see Lemma 3.1 and 3.4).

3.1. Subcritical cases. First, we consider the case that $d = 1$, $p > 1$ and $s > \max\{s_g, s_c\}$, or $d \geq 2$, $p \geq 3$ and $s > \max\{s_g, s_c\}$. Since

$$
\begin{cases}
    s_g > s_c & \text{if } d = 1 \text{ and } 1 < p < 5, \\
    s_g = s_c & \text{if } d = 1 \text{ and } p = 5, \\
    s_g < s_c & \text{if } d = 1 \text{ and } p > 5, \text{ or if } d \geq 2 \text{ and } 3 \leq p < 5,
\end{cases}
$$

it can be divided into three sub-cases,

$$
\begin{cases}
    d = 1, \ 1 < p < 5 \text{ and } s > s_g, & (6) \\
    d = 1, \ p \geq 5 \text{ and } s > s_c, & (7) \\
    d \geq 2, \ p \geq 3 \text{ and } s > s_c. & (8)
\end{cases}
$$

For local well-posedness, we make use of the following space-time bound. Here, for notational convenience, we denote by $c^+$ a number strictly larger than $c$ but arbitrarily close to $c$, and similarly for $c^-$.

**Lemma 3.1 (L^{p-1}_t L^\infty_x$ bound).** Let

$$(q_0, r_0) = \begin{cases}
    (4^+, \infty^-) & \text{when (6) holds}, \\
    \left((p - 1)^+, \left(\frac{2d(p - 1)}{d(p - 1) - 4}\right)^-\right) & \text{when (7) or (8) holds},
\end{cases}$$

where $(q_0, r_0)$ is an admissible pair. Then,

$$\|u\|_{L^{p-1}_t L^\infty_x} \leq T^{0^+} \|u\|_{S^{q_0}_{r_0}(I)}. \tag{9}$$

**Proof.** If (6) holds, by Hölder inequality and Sobolev inequality (with $s > s_g$ and sufficiently large $\infty^-$ depending on $(s - s_g)$), we obtain

$$\|u\|_{L^{p-1}_t L^\infty_x} \leq T^{0^+} \|u\|_{L^{q_+}_t L^\infty_x} \leq T^{0^+} \|(|\nabla|^s|\nabla|^{-s})^ru\|_{L^{q_+}_t L^\infty_x} = T^{0^+} \|u\|_{S^{q_0}_{r_0}(I)}.$$  

Similarly, if (7) or (8) holds, then

$$\|u\|_{L^{p-1}_t L^\infty_x} \leq T^{0^+} \|u\|_{L^{q_0}_{r_0}} \leq T^{0^+} \|\nabla|^{-d(1-\sigma)(\frac{1}{2} - \frac{1}{r_0})u\|_{L^{q_0}_{r_0}} = T^{0^+} \|u\|_{S^{q_0}_{r_0}(I)},$$

since

$$\frac{1}{r_0} - s - d(1 - \sigma)(\frac{1}{2} - \frac{1}{r_0}) < 0. \tag{10}$$
Here, in the second inequality, we used the embedding $W^{s-d(1-\sigma)(\frac{1}{2}-\frac{1}{p_0})} \hookrightarrow L^\infty$ and $\|\langle \nabla \rangle^{-d(1-\sigma)(\frac{1}{2}-\frac{1}{p_0})} u\|_{L^\infty} \leq \|\langle \nabla \rangle^{-d(1-\sigma)(\frac{1}{2}-\frac{1}{p_0})} u\|_{L^{p_0}}$. 

We also employ the following useful lemma.

**Lemma 3.2.** Let $1 < q \leq \infty$, $1 < r < \infty$ and $s_1 \geq s_2$. Then, $B = \{ u : \| u \|_{L^q_{t,x} W^{s_2,r}_x} \leq R \}$, equipped with the norm $\| \cdot \|_{L^q_{t,x} W^{s_2,r}_x}$, is a complete metric space. 

**Proof.** We recall:

**Theorem 3.3** (Theorem 1.2.5 in [2]). Consider two Banach spaces $X \hookrightarrow Y$ and $1 < p, q \leq \infty$. Let $(f_n)_{n \geq 0}$ be a bounded sequence in $L^p(I, Y)$ and let $f : I \rightarrow Y$ be such that $f_n(t) \rightarrow f(t)$ in $Y$ as $n \rightarrow \infty$, for a.e. $t \in I$. If $(f_n)_{n \geq 0}$ is bounded in $L^p(I; X)$ and if $X$ is reflexive, then $f \in L^p(I; X)$ and $\| f \|_{L^p(I; X)} \leq \liminf_{n \rightarrow \infty} \| f_n \|_{L^p(I; X)}$.

Suppose that $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $B$. Then, $f_n$ converges to $f$ in $L^q_{t,x} W^{s_2,r}_x$. Moreover, it follows from Theorem 1.2.5 in [2] that

$$\| f \|_{L^q_{t,x} W^{s_2,r}_x} \leq \liminf_{n \rightarrow \infty} \| f_n \|_{L^q_{t,x} W^{s_2,r}_x} \leq R,$$

and thus $f \in B$. Therefore, we conclude that $B$ is complete. 

**Proof of Theorem 1.1 when $d = 1$ and $p > 1$, or $d \geq 2$ and $p \geq 3$.** We define the non-linear mapping $\Phi_{u_0}$ by

$$\Phi_{u_0}(u) := e^{it(-\Delta)^{\alpha}} u_0 + i \mu \int_0^t e^{i(t-s)(-\Delta)^{\alpha}} (|u|^{p-1} u)(s) ds.$$ 

Let

$$X_\alpha := L^\infty_{t,x} H^\alpha_x \cap S^\alpha_{q_0, r_0}(I),$$

where $(q_0, r_0)$ is an admissible pair in Lemma 3.1. Suppose that $s \geq \max\{s_g, s_c\}$. Then, by Strichartz estimates, the nonlinear estimate (34) and (9), we get

$$\| \Phi_{u_0}(u) \|_{X_\alpha} \leq \| u_0 \|_{H^\alpha} + \| |u|^{p-1} u \|_{L^1_{t,x} L^2_x}$$

$$\leq \| u_0 \|_{H^\alpha} + \| |u|^{p-1} u \|_{L^1_{t,x} L^2_x}^\frac{1}{2} \| u \|_{L^2_{t,x} L^\infty_x}^\frac{1}{2}$$

$$\leq \| u_0 \|_{H^\alpha} + T^{\alpha^+} \| u \|_{S^\alpha_{q_0, r_0}(I)}^\frac{p-1}{2} \| u \|_{L^2_{t,x} L^\infty_x}^\frac{1}{2}$$

and similarly

$$\| \Phi_{u_0}(u) - \Phi_{u_0}(v) \|_{X_\alpha} \leq \| |u|^{p-1} u - |v|^{p-1} v \|_{L^1_{t,x} L^2_x}$$

$$\leq \| (|u|^{p-1} + |v|^{p-1}) |u - v| \|_{L^1_{t,x} L^2_x}$$

$$\leq (\| |u|^{p-1} L^1_{t,x} L^\infty_x \| \| v \|^{p-1} L^1_{t,x} L^\infty_x \| |u - v| \|_{L^\infty_{t,x} L^2_x}$$

$$\leq T^{\alpha^+} (\| u \|_{S^\alpha_{q_0, r_0}(I)} + \| v \|_{S^\alpha_{q_0, r_0}(I)} \| u - v \|_{L^\infty_{t,x} L^2_x}$$

(11)

Thus, for sufficiently small $T > 0$, $\Phi_{u_0}$ is contractive on the ball

$$B := \{ u : \| u \|_{X_\alpha} \leq 2 \| u_0 \|_{H^\alpha} \},$$

equipped with the norm $\| \cdot \|_{X_\alpha}$, which is complete by Lemma 3.2. 

□
Remark 3. Lemma 3.2 allows us to avoid giving derivatives in (11). Indeed, if one takes the $X_s$ norm for the difference $\Phi_{u_0}(u) - \Phi_{u_0}(v)$, then one will obtain a bound with $\|\nabla^\alpha u\|_{L_x^p L_t^s}$, which is not necessarily finite for $u \in B$.

We consider the case $d = 1, 2 \leq p < 5$ and $s = s_\mu$. We deal with this case separately, because if one tries to obtain an $L_x^{p-1} L_s^\infty$ bound as in Lemma 3.1, one can see that the exponent $(q_0, r_0)$ should be chosen to be $(4, \infty)$ (see (10)). Then, one cannot make use of Lemma 3.2, since $L_x^\infty$ is not reflexive. For this reason, we control the $L_x^4 L_s^\infty$ norm by Strichartz estimates (12) instead of Lemma 3.1, and we restrict ourselves to the case $p \geq 2$ (since (13) requires $p \geq 2$).

Proof of Theorem 1.1 when $d = 1$ and $2 \leq p < 5$. Let

$$\|u\|_{X_s} := \|u\|_{L_t^\infty I} H_s^\alpha \cap L_t^4 I \infty,$$

where $I = [0, T)$, and define the nonlinear mapping $\Phi_{u_0}$ as above. Then, applying the one-dimensional Strichartz estimates

$$\|e^{it(-\Delta)^{\alpha}} u_0\|_{L_t^4 I_x^4 \cap L_t^4 I \infty} \lesssim \|u_0\|_{H^s},$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^{\alpha}} \Phi(s) \, ds \right\|_{L_t^4 I_x^4 \cap L_t^4 I \infty} \lesssim \|\Phi\|_{L_t^1 I_x^4 H_s^\alpha}, \quad (12)$$

the nonlinear estimate (34) and Hölder inequality, we get

$$\|\Phi_{u_0}(u)\|_{X_s} \lesssim \|u_0\|_{H^s} + \|u\|^{p-1} u\|_{L_t^4 I_x^4 H_s^\alpha} \lesssim \|u_0\|_{H^s} + T \frac{4}{s-\mu} \|u\|^{p-1} u\|_{L_t^4 I_x^4} \|u\|_{L_t^4 I_x^4 H_s^\alpha} \lesssim \|u_0\|_{H^s} + T \frac{4}{s-\mu} \|u\|_{X_s}.$$

Similarly, by Strichartz estimates,

$$\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_s} \lesssim \|u\|^{p-1} u - \|v\|^{p-1} v\|_{L_t^4 I_x^4 H_s^\alpha}.$$

By the fundamental theorem of calculus and the fractional Leibniz rule,

$$\|u\|^{p-1} u - \|v\|^{p-1} v\|_{X_s} \lesssim \left\| \int_0^1 p|v + \alpha(u - v)|^{p-1}|u - v| \, d\alpha \right\|_{H^s} \lesssim \int_0^1 \|v + \alpha(u - v)|^{p-1}(u - v)\|_{H^s} \, d\alpha \lesssim \int_0^1 \|v + \alpha(u - v)|^{p-1}|u - v| \, d\alpha + \|v + \alpha(u - v)|^{p-1}|H^s| \, u - v \, d\alpha.$$
Thus, it follows that
\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X^s}\nabla \leq T^{\frac{\delta}{2}} \left\{ \left( \|u\|_{L^{p_1}_{t,x}}^{p_1} + \|v\|_{L^{p_1}_{t,x}}^{p_1} \right) \|u - v\|_{L^{\infty}_{t,x}H_x^{s'}} + \left( \|u\|_{L^{p_2}_{t,x}}^{p_2} + \|v\|_{L^{p_2}_{t,x}}^{p_2} \right) \|u - v\|_{L^{\infty}_{t,x}H_x^{s'}} \right\}
\]
Choosing sufficiently small \( T > 0 \), we conclude that \( \Phi_{u_0} \) is a contraction on a ball \( B := \{ u : \|u\|_{X^s} \leq 2\|u_0\|_{H^s} \} \) equipped with the norm \( \| \cdot \|_{X^s} \).

3.2. Scaling-critical cases. In the scaling-critical case, we use the following lemma, which plays the same role as (9). We note that the norms in the lemma are defined via the Littlewood-Paley projection in order to overcome the failure of the Sobolev embedding \( W^{s,p} \hookrightarrow L^q \), \( \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{d} \), when \( q = \infty \). Lemma 3.3 generalizes [9, Lemma 3.1].

**Lemma 3.4** (Scaling-critical \( L^{p-1}_{t;1}L^{\infty}_{x} \) bound).

\[
\|u\|_{L^{p-1}_{t;1}L^{\infty}_{x}} \leq \begin{cases} 
\|u\|_{S_{\sigma,2}(L^2_x)}^{\frac{\sigma}{p} - 5} \|u\|_{S_{\sigma,2}(L^2_x)}^{\sigma} & \text{when } d = 1 \text{ and } p > 3, \\
\|u\|_{S_{\sigma,2}(L^2_x)}^{\frac{\sigma}{p} - 3} \|u\|_{S_{\sigma,2}(L^2_x)}^{\sigma} & \text{when } d = 2 \text{ and } p > 3, \\
\|u\|_{S_{\sigma,2}(L^2_x)}^{\frac{\sigma}{p} - 3} \|u\|_{S_{\sigma,2}(L^2_x)}^{\sigma} & \text{when } d \geq 3 \text{ and } p > 3.
\end{cases}
\]

**Proof.** We will prove the lemma only when \( d \geq 3 \). By interpolation \( \|f\|_{L^{p_0}} \leq \|f\|_{L^{p_1}}^{1/\theta} \|f\|_{L^{p_2}}^{1-\theta} \), \( \frac{1}{p_0} = \frac{1}{p_1} + \frac{1-\theta}{p_2} \), \( 0 < \theta < 1 \), it suffices to show the lemma for rational \( (p-1) = \frac{m}{n} > 2 \) with \( \gcd(m,n) = 1 \). First, we estimate

\[
A(t) = \left[ \sum_{N} \|P_N u(t)\|_{L^2_x} \right]^{\frac{m}{n}} \sim \sum_{N_1 \geq \cdots \geq N_m} \prod_{i=1}^{m} \|P_N u(t)\|_{L^2_x}.
\]

Observe from Bernstein’s inequality that
\[
\|P_N u(t)\|_{L^2_x} \leq N^{-\frac{\sigma(p-3)}{p-1} d_N},
\]
\[
\|P_N u(t)\|_{L^2_x} \leq N^{\frac{2\sigma}{p-1} d_N},
\]
where
\[
d_N = \|P_N u(t)\|_{W^{s,-1(1-\sigma)}_{x},\frac{2\sigma}{p-1}},
\]
\[
d_N' = \|P_N u(t)\|_{H^{s'}}.
\]
As a consequence, we have
\[
\|P_N u(t)\|_{L^2_x} \leq \left( N^{\frac{\sigma(p-3)}{p-1} d_N} \right)^{\frac{1}{\theta} \left( N^{\frac{2\sigma}{p-1} d_N} \right)^{1-\theta}} = N^{\frac{\sigma(p-3)}{p-1} d_N \theta (d_N')^{1-\theta}},
\]
where \( \theta = \frac{1}{p-1} \). Hence, applying (15) for \( i = 1, \cdots, n \) and (17) for \( i = n+1, \cdots, m \), we bound \( A(t) \) by
\[
\leq \sum_{N_1 \geq \cdots \geq N_m} \left( \prod_{i=1}^{n} N_i^{-\frac{\sigma(p-3)}{p-1} d_N} \right) \left( \prod_{i=n+1}^{m} N_i^{\frac{\sigma(p-3)}{p-1} d_N} \right)^{\theta (d_N')^{1-\theta}}.
\]
For an arbitrarily small $\epsilon > 0$, we let
\[
\tilde{d}_N = \sum_{N' \in 2^{\mathbb{Z}}} \min \left( \frac{N}{N'}, \frac{N'}{N} \right)^\epsilon \tilde{d}_{N'}, \quad \tilde{d}'_N = \sum_{N' \in 2^{\mathbb{Z}}} \min \left( \frac{N}{N'}, \frac{N'}{N} \right)^\epsilon \tilde{d}'_{N'}.
\]
Then, since $d_N \leq d_N$ and $\tilde{d}_N \leq (\frac{N}{N'})^\epsilon \tilde{d}_{N'}$, and similarly for primes, $A(t)$ is bounded by
\[
\lesssim \sum_{N_1 \cdots N_m \in 2^{\mathbb{Z}}} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)}{p-1}} \left( \frac{N_i}{N} \right)^\epsilon \tilde{d}_{N_i} \right) \times \sum_{N_1 \cdots N_n \in 2^{\mathbb{Z}}} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)-(m-n)}{p-1}} \left( \frac{N_i}{N} \right)^{(m-n)\epsilon} (\tilde{d}_{N_i})^{(m-n)\theta} (\tilde{d}'_{N_i})^{(m-n)(1-\theta)} \right)
\]
Summing in $N_m, N_{m-1}, \ldots, N_{n+1}$ and using $m-n = (p-2)n$,
\[
A(t) \lesssim \sum_{N_1 \cdots N_n} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)}{p-1}} \left( \frac{N_i}{N} \right)^\epsilon \tilde{d}_{N_i} \right) \times \sum_{N_1 \cdots N_n} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)-(m-n)}{p-1}} \left( \frac{N_i}{N} \right)^{(m-n)\epsilon} (\tilde{d}_{N_i})^{(m-n)\theta} (\tilde{d}'_{N_i})^{(m-n)(1-\theta)} \right)
\]
and then summing in $N_n, N_{n-1}, \ldots, N_1$, we obtain that
\[
A(t) \lesssim \sum_{N_1} \tilde{d}_{N_1}^{n+(m-n)\theta} (\tilde{d}'_{N_1})^{(m-n)(1-\theta)} = \sum_{N_1} (\tilde{d}_{N_1})^{2n} (\tilde{d}'_{N_1})^{m-2n},
\]
which is, by Hölder inequality and Young’s inequality, bounded by
\[
\lesssim \|\tilde{d}_N\|^{2n}_{\ell_N^{(2n)}} (\tilde{d}'_N)^{m-2n}_{\ell_N^{(2n)}} = \|\tilde{d}_N\|^{2n}_{\ell_N^{(2n)}} \|\tilde{d}'_N\|^{m-2n}_{\ell_N^{(m-2n)}}
\]
Finally, by the estimate for $A(t)$, we prove that
\[
\|u\|^{p-1}_{L_{t,x}^{(p-1)}} \leq \int_I A(t)^{-1} dt = \int_I A(t)^{\frac{1}{2}} dt \leq \int_I \|d_N\|^{2n}_{\ell_N^{(2n)}} \|d_N\|^{m-2n}_{\ell_N^{(m-2n)}} dt \\
\leq \|d_N\|^{2n}_{\ell_N^{(2n)}} \|d'_N\|^{m-2n}_{\ell_N^{(m-2n)}} dt \leq \|d_N\|^{2n}_{L_{t,x}^{(2n)}} \|d'_N\|^{m-2n}_{L_{t,x}^{(m-2n)}} \\
\leq \|u\|^{2}_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)} \|u\|^{m-2n}_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)}
\]

\hfill \square

**Proof of Theorem 1.2.** For simplicity, we assume that $d \geq 3$. Indeed, with little modifications, we can prove the theorem when $d = 1, 2$. We define $\Phi_{u_0}(u)$ as in the proof of Theorem 1.1. Then, by Strichartz estimates, the nonlinear estimate (34) and (14), we have
\[
\|\Phi_{u_0}(u)\|_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)} \leq \|e^{it(-\Delta)^{\sigma}} u_0\|_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)} + c_0 \|u\|^{p-1}_{L_{t,x}^{(2n)}} + c_1 \|u\|^{p-1}_{L_{t,x}^{(2n)}} + c_2 \|u\|^{p-2}_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)}
\]
\[
\leq \|e^{it(-\Delta)^{\sigma}} u_0\|_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)} + c_1 \|u\|^{p-1}_{L_{t,x}^{(2n)}} + c_2 \|u\|^{p-2}_{S_{(\epsilon)}^{\frac{p}{2},\frac{d}{2}}(I)}.
\]
Similarly, one can show that
\[ \| \Phi_{u_0}(u) \|_{\mathcal{S}_{\varepsilon,2}^p(I)} \leq c \| u_0 \|_{H^\varepsilon} + c \| u \|_{\mathcal{S}_{\varepsilon,2}^p(I)} \| u \|_{\mathcal{S}_{\varepsilon,2}^{p-2}(I)} \]
and
\[ \| \Phi_{u_0}(u) - \Phi_{u_0}(v) \|_{\mathcal{S}_0^{1/2} \mathcal{D}^{-1/2} + \mathcal{S}_0^{1/2} \mathcal{D}^{-1/2}(I)} \leq c(\| u \|_{\mathcal{S}_{\varepsilon,2}^{p-1}(I)}^{p-1} + \| v \|_{\mathcal{S}_{\varepsilon,2}^{p-1}(I)}^{p-1}) \| u - v \|_{L_{t,x}^\varepsilon L_x^2} \]
\[ \leq c(\| u \|_{\mathcal{S}_{\varepsilon,2}^{p}(I)}^{p-2} + \| v \|_{\mathcal{S}_{\varepsilon,2}^{p}(I)}^{p-2}) \| u - v \|_{L_{t,x}^\varepsilon L_x^2}^2. \]

Now we let \( \delta = \delta(c, \| u_0 \|_{H^\varepsilon}) > 0 \) be a sufficiently small number to be chosen later, and then we pick \( T = T(u_0, \delta) > 0 \) such that
\[ \| e^{it(-\Delta)\varepsilon} u_0 \|_{\mathcal{S}_{\varepsilon,2}^{1/2} \mathcal{D}^{-1/2}(I)} \leq \delta, \]
Define
\[ B = \left\{ u : \| u \|_{\mathcal{S}_{\varepsilon,2}^{1/2} \mathcal{D}^{-1/2}(I)} \leq 2\delta \text{ and } \| u \|_{\mathcal{S}_{\varepsilon,2}^{p-2}(I)} \leq 2c \| u_0 \|_{H^\varepsilon} \right\} \]
equipped with the norm
\[ \| u \|_X := \| u \|_{\mathcal{S}_0^{1/2} \mathcal{D}^{-1/2} + \mathcal{S}_0^{1/2} \mathcal{D}^{-1/2}(I)} + \| u \|_{\mathcal{S}_0^{p}(I)}. \]

Then, for \( u \in B \), we have
\[ \| \Phi_{u_0}(u) \|_{\mathcal{S}_{\varepsilon,2}^{1/2} \mathcal{D}^{-1/2}(I)} \leq \delta + c(2\delta)^2(2c \| u_0 \|_{H^\varepsilon})^{p-2} \leq 2\delta, \]
\[ \| \Phi_{u_0}(u) \|_{\mathcal{S}_{\varepsilon,2}^{p-2}(I)} \leq c\| u_0 \|_{H^\varepsilon} + c(2\delta)^2(2c \| u_0 \|_{H^\varepsilon})^{p-2} \leq 2c \| u_0 \|_{H^\varepsilon}. \]

Choosing sufficiently small \( \delta > 0 \), we prove that \( \Phi_{u_0} \) maps \( B \) to itself. Similarly, one can show
\[ \| \Phi_{u_0}(u) - \Phi_{u_0}(v) \|_X \leq \frac{1}{2} \| u - v \|_X. \]
Therefore, it follows that \( \Phi_{u_0} \) is a contraction mapping in \( B \). \( \square \)

**Remark 4.** (i) In the proofs, the \( L_{t,x}^p \) norm bounds are crucial for the following reason. In Proposition 1, there is a loss of regularity in the estimates except the following trivial ones below
\[ \| e^{it(-\Delta)\varepsilon} u_0 \|_{L_{t,x}^\varepsilon L_x^2} = \| u_0 \|_{L^2} \]
and
\[ \left\| \int_0^t e^{i(s)\varepsilon} F(s) ds \right\|_{L_{t,x}^\varepsilon L_x^2} \leq \| F \|_{L_{t,x}^\varepsilon L_x^2}. \]

Hence, when we estimate the \( L_{t,x}^\varepsilon H_x^s \) norm of the integral term in \( \Phi_{u_0}(u) \), we are forced to use the trivial estimate
\[ \left\| \int_0^t e^{i(s)\varepsilon} |u|^{p-1} u(s) ds \right\|_{L_{t,x}^\varepsilon H_x^s} \leq \| u \|_{L_{t,x}^\varepsilon H_x^s}^p \leq \| u \|_{L_{t,x}^\varepsilon H_x^s}. \]

Indeed, otherwise, we have a higher regularity norm on the right hand side. Then, we cannot close the contraction mapping argument. Moreover, if \( u_0 \in H^s \), there is no good bound for \( \| e^{it(-\Delta)\varepsilon} u_0 \|_{L_{t,x}^\varepsilon W_x^{s,r}} \) except the trivial one \((q, r) = (\infty, 2)\). Thus, we are forced to bound the right hand side of (3.10) by
\[ \| u \|_{L_{t,x}^\varepsilon H_x^s}. \]
Therefore, we should have a good control on \( \|u\|_{L_{t,x}^{p-1}L_x^\infty} \).

(ii) When \( p \leq 3 \), the \( L_{t,x}^{p-1}L_x^\infty \) norm is scaling-supercritical. Thus, based on our method, the assumptions on \( p \) in Theorem 1.1 and 1.2 are optimal except \( p = 3 \) in the critical case.

4. Small data scattering.

Proof of Theorem 1.3. For simplicity, we consider the case \( d \geq 3 \) only. It follows from the estimates in the proof of Theorem 1.2 that if \( \|u_0\|_{H^s} \) is small enough, then

\[
\|u(t)\|_{L_{t,x}^{p-1}L_x^\infty} + \|u(t)\|_{L_{t,x}^{p+1}H_x^\infty} \lesssim \|u(t)\|_{S^{sc}^\infty}^{(\infty)}(\mathbb{R}) + \|u(t)\|_{S^{sc}^{2d}}^{(\infty)}(\mathbb{R}) \lesssim \|u_0\|_{H^s} < \infty.
\]

By Strichartz estimates, the fractional chain rule and (14), we prove that

\[
e^{-iT_1(-\Delta)^\nu}u(T_1) - e^{-iT_2(-\Delta)^\nu}u(T_2)\|_{H^s} = \left\| \int_{T_1}^{T_2} e^{-i(s-T_2)(-\Delta)^\nu}(|u|^{p-1}u)(s)\,ds \right\|_{H^s}
\]

\[
\lesssim \|u(t)\|_{L_{t\in[T_1,T_2]}^{p-1}L_x^\infty} \|u(t)\|_{L_{t\in[T_1,T_2]}^{p+1}H_x^\infty} \to 0
\]

as \( T_1, T_2 \to \pm \infty \). Thus, the limits

\[
u(t) = \lim_{t \to \pm \infty} e^{-i(t-(\Delta)^\nu}\|u\|_{H^s} = \|e^{-i(t-(\Delta)^\nu}u(t) - u(t)\|_{H^s} \to 0
\]

as \( t \to \pm \infty \). \( \square \)

5. Ill-posedness. We will prove Theorem 1.4 following the strategy in [3]. Throughout this section, we assume that \( d = 1, 2 \) or \( 3 \) and \( \frac{d}{2} \leq \sigma < 1 \). If \( p \) is not an odd integer, we further assume that \( p \geq k + 1 \), where \( k \) is the smallest integer greater than \( \frac{d}{2} \).

First, we construct an almost non-dispersive solution by small dispersion analysis.

Lemma 5.1 (Small dispersion analysis). Given a Schwartz function \( \phi_0 \), let \( \phi^{(\nu)}(t, x) \) be the solution to the fractional NLS

\[
i\partial_t u + \nu^{2\sigma}(-\Delta)^\sigma u + \mu|u|^{p-1}u = 0, \quad u(0) = \phi_0,
\]

and \( \phi^{(0)}(t, x) \) be the solution to the ODE with no dispersion

\[
i\partial_t u + \mu|u|^{p-1}u = 0, \quad u(0) = \phi_0,
\]

that is,

\[
\phi^{(0)}(t, x) = \phi_0(x)e^{it\omega|\phi_0(x)|^{p-1}}.
\]

Then there exist \( C, c > 0 \) such that if \( 0 < \nu \leq c \) is sufficiently small, then

\[
\|\phi^{(\nu)}(t) - \phi^{(0)}(t)\|_{H^k} \leq C\nu^{2\sigma}
\]

for all \( |t| \leq c|\log \nu|^c \).

Proof. The proof closely follows the proof of Lemma 2.1 in [3]. \( \square \)

Obviously, \( \phi^{(\nu)}(t, \nu x) \) is a solution to \( (\text{NLS}_\nu) \). Moreover, \( \phi^{(\nu)}(t, \nu x) \) is bounded and almost flat in the following sense.
Corollary 1. Let $\phi^{(v)}$, $\nu$ and $c$ be in Lemma 5.1. Let $s \geq 0$. Then,
\[ \| \phi^{(v)}(t, \nu x) \|_{L^2_x} \leq 1 \] (22)
and
\[ \| \phi^{(v)}(t, \nu x) \|_{H^s_x} \lesssim \nu^{s-\frac{d}{2}} (c|\log \nu|^c)^s \] (23)
for all $|t| \leq c|\log \nu|^c$.

Proof. Since $k > \frac{d}{2}$, by the Sobolev inequality, we have
\[ \| \phi^{(v)}(t, \nu x) - \phi^{(0)}(t, \nu x) \|_{L^2_x} = \| \phi^{(v)}(t, x) - \phi^{(0)}(t, x) \|_{L^2_x} \] \[ \lesssim \| \phi^{(v)}(t) - \phi^{(0)}(t) \|_{H^k} \lesssim \nu^{2\sigma}. \]

Then, (22) follows from the explicit formula (20) for $\phi^{(0)}(t, x)$. It follows from (21) and (20) that
\[ \| \phi^{(v)}(t, \nu x) \|_{H^s_x} \lesssim \nu^{s-\frac{d}{2}} (\| \phi^{(0)}(t) \|_{H^s} + \| \phi^{(v)}(t) - \phi^{(0)}(t) \|_{H^s}) \sim \nu^{s-\frac{d}{2}} (c|\log \nu|^c)^s. \]

For $v \in \mathbb{R}^d$, we define the pseudo-Galilean transformation by
\[ G_v u(t, x) = e^{-i v \cdot x} e^{i t |v|^{2\sigma}} u(t, x - 2t\sigma |v|^{2(\sigma-1)} v). \]

Note that when $\sigma = 1$, $G_v$ is simply a Galilean transformation, and that NLS is invariant under this transformation, that is, if $u(t, x)$ solves NLS, so does $G_v u(t, x)$. However, when $\sigma \neq 1$, (NLS$_\sigma$) is not exactly symmetric with respect to pseudo-Galilean transformations. Indeed, if $u(t)$ solves (NLS$_\sigma$), then $\tilde{u}(t) = G_v u(t)$ obeys (NLS$_\sigma$) with an error term
\[ i\tilde{\partial}_t \tilde{u} + (-\Delta)^{\sigma} \tilde{u} + \omega |\tilde{u}|^{p-1} \tilde{u} = e^{i t |v|^{2\sigma}} e^{-i v \cdot x} (\hat{E} u)(t, x - 2t\sigma |v|^{2(\sigma-1)} v), \] (24)
where
\[ \hat{E} u(\xi) = E(\xi) \hat{u}(\xi) \]
with
\[ E(\xi) = |\xi - v|^{2\sigma} - |\xi|^{2\sigma} - |v|^{2\sigma} + 2\sigma |v|^{2(\sigma-1)} v \cdot \xi. \]

However, we note that
\[ |E(\xi)| \lesssim |\xi|^{2\sigma}. \] (25)

Indeed, if $|\xi| \leq \frac{|v|}{100}$, then
\[ E(\xi) = \|v\|^{2\sigma} \left( \left| \frac{v}{|v|} - \frac{\xi}{|\xi|} \right|^2 + 2\sigma \frac{v}{|v|} \cdot \frac{\xi}{|\xi|} \right) - |\xi|^{2\sigma} \lesssim |v|^{2\sigma} \frac{\|\xi\|^2}{|v|^2} + |\xi|^{2\sigma} \lesssim |\xi|^{2\sigma}. \]

Otherwise, $E(\xi) \lesssim |\xi|^{2\sigma} + |v|^{2\sigma} + |\xi|^{2\sigma} + |v|^{2\sigma} + 2\sigma |v|^{2(\sigma-1)} |\xi| \lesssim |\xi|^{2\sigma}$.

Therefore, one would expect an almost symmetry for an almost flat solution $u(t)$, such as $\phi^{(v)}(t, \nu x)$ in Lemma 5.1. Precisely, we have the following lemma.

Lemma 5.2 (Pseudo-Galilean transformation). Let $\phi^{(v)}$, $\nu$ and $c$ be in Lemma 5.1. For $v \in \mathbb{R}^d$, we define
\[ \tilde{u}(t, x) = (G_v \phi^{(v)}(\cdot, \nu))(t, x) = e^{-i v \cdot x} e^{i t |v|^{2\sigma}} \phi^{(v)}(t, \nu (x - 2t\sigma |v|^{2(\sigma-1)} v)), \]
and let $u(t, x)$ be the solution to (NLS$_\sigma$) with the same initial data
\[ e^{-i v \cdot x} \phi^{(v)}(0, \nu x) = e^{-i v \cdot x} \phi_0(0, \nu x). \] (26)
\[ \| e^{iv\cdot x}(u(t) - \tilde{u}(t)) \|_{L^2_v} \lesssim \nu^\delta \]  
for all \( |t| \leq c |\log \nu|^c \).

**Remark 5.** When \( p = 3 \), in [8] the authors were able to use the counterexample in [4]. This counterexample is constructed by pseudo-conformal symmetry and Galilean transformation. A good thing is that this solution is very small in high Sobolev norms, too. Somehow, this smallness allows [8] to show that the error in pseudo-Galilean transformation is also small. However, when \( p > 3 \), the counterexample in [4] does not work. Later, Christ, Colliander and Tao [3] constructed a different counterexample which works for more general \( p \). Unfortunately, this counterexample is not small in high Sobolev norms. It is very large instead. In particular, for our purposes, it is hard to control the error from pseudo-Galilean transformation. But, our new counterexample still has small high Sobolev norms. It is very large instead. Somehow, this smallness allows [8] to show that the error in pseudo-Galilean transformation is almost invariant. We also remark that the condition \( \sigma > \frac{3}{4} \) is to guarantee smallness of the error (see (28)).

**Proof of Lemma 5.2.** Let \( R(t) = (u - \tilde{u})(t) \). Then, \( R(t) \) satisfies
\[
i \partial_t R + (-\Delta)^\sigma R = \mu(|\tilde{u}|^{p-1} \tilde{u} - |u|^{p-1} u) - e^{it|v|^{2\sigma}} (E\phi^{(\nu)}(t, \nu(x - 2\sigma t|v|^{2(\sigma-1)}v)),
\]
or equivalently
\[
R(t) = i \int_0^t e^{i(t-s)(-\Delta)^\sigma} \left\{ \mu(|\tilde{u}|^{p-1} \tilde{u} - |u|^{p-1} u)(s)
+ e^{i|v|^{2\sigma}} (E\phi^{(\nu)}(s, \nu(x - 2\sigma s|v|^{2(\sigma-1)}v)) \right\} ds.
\]
Hence, by a trivial estimate, we get
\[
\| e^{iv\cdot x} R(t) \|_{H^k} \leq \int_0^t \| e^{iv\cdot x} (|u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u})(s) \|_{H^k} + \| E\phi^{(\nu)}(s, \nu) \|_{H^k} ds
= \int_0^t I(s) + II(s) ds.
\]
First, by (25) and (23), we show that
\[
\int_0^t II(s) ds \lesssim \int_0^t \sum_{j=0}^k \| \phi^{(\nu)}(s, \nu) \|_{H^{j+2}} ds \sim (c |\log \nu|^c)^{1+2\sigma - \frac{q}{2}} t^{2\sigma - \frac{q}{2}}.
\]
For \( I(s) \), expanding \( u = \tilde{u} + R \) and then applying Hölder inequality and Sobolev inequalities, we bound \( I(s) \) by
\[
\lesssim \sum_{j=1}^p \| e^{iv\cdot x} R \|_{H^k}^j.
\]
For example, when \( p = 3 \),
\[
I(s) \leq 2 \| e^{iv\cdot x} \tilde{u}^2 e^{iv\cdot x} R \|_{H^k} + \| (e^{iv\cdot x} \tilde{u})^2 e^{iv\cdot x} R \|_{H^k} + 2 \| e^{iv\cdot x} \tilde{u} e^{iv\cdot x} R \|_{H^k}
+ \int e^{iv\cdot x} \tilde{u} (e^{iv\cdot x} R)^2 e^{iv\cdot x} R |_{H^k}
=: I_1(s) + I_2(s) + I_3(s) + I_4(s) + I_5(s).
\]
Consider
\[ I_1(s) = \sum_{|\alpha| \leq k} \| \nabla_2^{\alpha_1} \cdots \nabla_2^{\alpha_d} (e^{iv \cdot x} \tilde{u}_x^2 e^{iv \cdot x} R)(s) \|_{L^2} =: \sum_{|\alpha| \leq k} I_{1,\alpha}(s), \]
where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \) is a multi-index with \( |\alpha| = \sum_{i=1}^d \alpha_i \). Observe that whenever a derivative hits
\[ e^{iv \cdot x} \tilde{u}(s) = e^{i s |v|^{2\sigma}} \phi^{(\nu)}(s, \nu(x - 2s|v|^{2(\sigma - 1)})) \]
we get a small factor \( \nu \). Hence, after distributing derivatives by the Leibniz rule, the worst term we have in \( I_{1,\alpha}(s) \) is
\[ \| e^{iv \cdot x} \tilde{u}(s)^2 \nabla^\alpha e^{iv \cdot x} R(s) \|_{L^2}, \]
which is, by (22), bounded by
\[ \| e^{iv \cdot x} \tilde{u}(s) \|_{L_x^2} \| \nabla^\alpha e^{iv \cdot x} R(s) \|_{L^2} \sim \| \nabla^\alpha e^{iv \cdot x} R(s) \|_{L^2}. \]
Likewise, we estimate other terms.

Collecting all,
\[ \| e^{iv \cdot x} R(t) \|_{H^k} \lesssim (c |\log \nu|^{-1} + 2s - \frac{2}{2} \nu^{2\sigma} - \frac{d}{2}) + \int_0^t \sum_{j=1}^p \| e^{iv \cdot x} R(s) \|_{H^k} ds \]
for \( |t| \leq c |\log \nu|^{-1} \). Then, by the standard nonlinear iteration argument, we prove the lemma.

Since we have solutions almost symmetric with respect to the pseudo-Galilean transformations, we can make use of the following decoherence lemma to construct counterexamples for local well-posedness.

**Lemma 5.3 (Decoherence).** Let \( s < 0 \). Fix a nonzero Schwartz function \( \psi \). For \( a, a' \in [\frac{1}{2}, 1], 0 < \nu \leq \lambda < 1 \) and \( \nu \in \mathbb{R}^d \) with \( |\nu| \geq 1 \), we define
\[ \tilde{u}_{(a,\nu)}^{(a',\nu',\lambda)}(t,x) := G_\nu \left( \lambda^{-\frac{2\sigma}{d}} \phi^{(a,\nu)}(\lambda^{-(2\sigma)}, \lambda^{-1} \nu) \right)(t,x), \]
where \( \phi^{(a,\nu)} \) is the solution to (19) with initial data \( \psi \). Then, we have
\[ \| \tilde{u}_{(a,\nu)}^{(a',\nu',\lambda)}(0) \|_{H^s}, \| \tilde{u}_{(a,\nu)}^{(a',\nu',\lambda)}(0) \|_{H^s} \leq C|\psi|^s \lambda^{\frac{2\sigma}{d}} \langle \frac{1}{\nu} \rangle^{d/2}, \]
\[ \| \tilde{u}_{(a,\nu)}^{(a',\nu',\lambda)}(0) - \tilde{u}_{(a',\nu',\lambda)}^{(a',\nu',\lambda)}(0) \|_{H^s} \leq C|\psi|^s \lambda^{\frac{2\sigma}{d}} \langle \frac{1}{\nu} \rangle^{d/2} |a - a'| \]
and
\[ \| \tilde{u}_{(a,\nu)}^{(a,\nu',\lambda)}(t) - \tilde{u}_{(a',\nu',\lambda)}^{(a',\nu',\lambda)}(t) \|_{H^s} \]
\[ \geq c \| \psi \|^{s} \lambda^{\frac{2\sigma}{d}} \langle \frac{1}{\nu} \rangle^{d/2} \left\{ \| (\phi^{(a,\nu)}(\frac{1}{\lambda\nu}) - \phi^{(a',\nu)}(\frac{1}{\lambda\nu})) \|_{L^2} - C|\log \nu| C(\frac{1}{\nu})^{-k} |v|^{-s-k} \right\} \]
for all \( |t| \leq c |\log \nu|^{-1} \).

**Proof.** The proof closely follows the proof of Lemma 3.1 in [3].

**Proof of Theorem 1.4.** The proof is very similar to that of Theorem 1 in [3] except that in the last step, we need to use Lemma 5.2 due to lack of exact symmetry. We give a proof for the readers’ convenience.

Let \( \epsilon > 0 \) be a given but arbitrarily small number. Let \( \nu = \lambda^{\alpha} \), where \( \alpha > 0 \) is a small number to be chosen later. Then, we pick \( \nu \in \mathbb{R}^d \) such that
\[ \lambda^{-\frac{2\sigma}{d}} |\psi|^s (\lambda/\nu)^{d/2} = \epsilon \iff |\psi| = \nu^{\frac{1}{2}} \left( \frac{2\sigma}{d} \right) + \frac{2\sigma}{d} \right) \epsilon^{1/s}. \]
Note that since $s < 0$, $\frac{1}{2} \left( \frac{d(1-\alpha)}{r} + \frac{2\alpha}{p-1} \right) = \frac{1}{2} \left( \frac{d}{2} - \alpha \epsilon \right) < 0$ for sufficiently small $\alpha$, and thus $|v| \geq 1$. Hence, it follows from Lemma 5.3 that

$$\|\tilde{u}(a,v,\lambda,v)(0)\|_{H^s} \leq C \varepsilon,$$  \hspace{1em} (30)

$$\|\tilde{u}(a,v,\lambda,v)(0) - \tilde{u}(a',v,\lambda,v)(0)\|_{H^s} \leq C \varepsilon |a - a'|,$$  \hspace{1em} (31)

and

$$\|\tilde{u}(a,v,\lambda,v)(t) - \tilde{u}(a',v,\lambda,v)(t)\|_{H^s} \geq C \varepsilon \left\{ \|\phi(a,\nu)\left(\frac{x}{t^{\frac{1}{2}}}\right) - \phi(a',\nu)\left(\frac{x}{t^{\frac{1}{2}}\sigma}\right)\|_{L^2} - C |\log \nu| C \left(\frac{\lambda}{\nu}\right)^{\kappa} |v|^{-s-k} \right\}$$

for all $|t| \leq c \log \nu^{\kappa \lambda}$. Now we observe from the explicit formula (20) for $\phi(a,\nu)$ and (19) that there exists $T > 0$ such that $\|\phi(a,\nu)(T) - \phi(a',\nu)(T)\|_{L^2} \geq c$. Moreover, if $\alpha > 0$ is sufficiently small, $C |\log \nu| C \left(\frac{\lambda}{\nu}\right)^{\kappa} |v|^{-s-k} \rightarrow 0$ as $\nu \rightarrow 0$. Therefore, for $\nu$ small enough, we have

$$\|\tilde{u}(a,v,\lambda,v)(\lambda^{2\sigma} T) - \tilde{u}(a',v,\lambda,v)(\lambda^{2\sigma} T)\|_{H^s} \geq C \varepsilon.$$  \hspace{1em} (32)

Next, we replace $\tilde{u}(a,v,\lambda,v)$ and $\tilde{u}(a',v,\lambda,v)$ in (6.11), (6.12) and (6.13) by $u(a,v,\lambda,v)$ and $u(a',v,\lambda,v)$ by Lemma 5.2 with $O(\nu^{\kappa \lambda})$ error. Then, making $|a - a'|$ arbitrarily small and then sending $\nu \rightarrow 0$ (so, $\lambda^{2\sigma} T \rightarrow 0$), we complete the proof. \qed

Appendix A. Nonlinear estimate. In general, the fractional chain rule

$$\|\nabla|^{s} F(u)\|_{L^r} \leq \|F'(u)\|_{L^{r_1}} \|\nabla|^{s} u\|_{L^{r_2}},$$  \hspace{1em} (33)

where $0 < s < 1$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $1 < r, r_1, r_2 < \infty$ (see [11], for instance), is employed to prove local well-posedness of a dispersive PDE having a power-type nonlinearity $|u|^{p-1} u$ in $H^s$ when $0 < s < 1$ and $p$ is not an odd integer. In this paper, due to a loss of regularity in Strichartz estimates (Proposition 1), we make use of the endpoint fractional chain rule ($r_1 = \infty$). Although we expect the endpoint case can be proved by modifying the proof in the non-endpoint case, we could not find a reference in the literature. Thus, for completeness, we give the proof of what we need.

**Lemma A.1** (Nonlinear estimate). Suppose that $p > 1$ and $0 < s < 1$. Then, we have

$$\|\nabla|^{s} (|u|^{p-1} u)\|_{L^{r_2}} \leq \|u\|_{L^{r_1}}^{p-1} \|\nabla|^{s} u\|_{L^{r_2}}.$$  \hspace{1em} (34)

**Proof.** We recall Proposition 4 in [Stein, Singular integrals and differentiability properties ..., page 139]:

$$\|\nabla|^{s} f\|_{L^2} \sim \left( \int_{\mathbb{R}^d} \frac{\|f(x+y) - f(x)\|_{L^2}^2}{|y|^{d+2s}} dy \right)^{1/2}, \quad s \in (0, 1).$$  \hspace{1em} (35)

Therefore, by the point-wise bound for $(|u|^{p-1} u)(x+y) - (|u|^{p-1} u)(x)$, we obtain

$$\|(|u|^{p-1} u)(x+y) - (|u|^{p-1} u)(x)\|_{L^2} \leq \|(u(x+y)|^{p-1} + |u(x)|^{p-1})|u(x+y) - u(x)|\|_{L^2} \leq \|u\|_{L^{r_1}}^{p-1} \|u(x+y) - u(x)\|_{L^2}.$$  \hspace{1em} (36)

Then, (33) follows from (35). \qed
Appendix B. Ground states and $L^2$ critical equations. In the present Appendix, we investigate global existence and blow-up for (NLS$\sigma$) in two regimes: when $p = 1 + \frac{4\sigma}{d}$, i.e. the $L^2$-critical regime and when $1 + \frac{4\sigma}{d} < p < 1 + \frac{4\sigma}{d-2\sigma}$, i.e. the $L^2$-supercritical and $H^\sigma$-subcritical regime.

An important object, as noticed by Weinstein [24] is the so-called ground state. It is the unique (up to symmetries) solution of the elliptic equation

$$(-\Delta)^\sigma Q + Q - Q^p = 0. \tag{36}$$

Such a solution has been proved to exist in [12] (see also [13]).

We prove the following results.

**Theorem B.1.** Let $p = 1 + \frac{4\sigma}{d}$. If $u_0 \in H^\sigma$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then $u(t)$ exists globally in $H^\sigma$.

**Theorem B.2.** Let $1 + \frac{4\sigma}{d} < p < 1 + \frac{4\sigma}{d-2\sigma}$. Suppose that

$$M[u_0]^\theta E[u_0]^{1-\theta} < M[Q]^\theta E[Q]^{1-\theta}, \quad (37)$$

where $\theta = \frac{2\sigma(p-1)-d(p-1)}{2\sigma(p-1)}$ and $(1-\theta) = \frac{d(p-1)-4\sigma}{2\sigma(p-1)}$. If $\|u_0\|_{L^2}^{\theta} \|u_0\|_{H^\sigma}^{1-\theta} < \|Q\|_{L^2}^{\theta} \|Q\|_{H^\sigma}^{1-\theta}$, then $u(t)$ exists globally in $H^\sigma$.

We consider the ground state $Q$

$$(-\Delta)^\sigma Q + Q - Q^p = 0. \tag{38}$$

Multiplying the equation by $Q$ (and $x \cdot \nabla Q$), integrating over $\mathbb{R}^d$ and then applying integration by parts, we obtain

$$\|Q\|_{H^\sigma}^2 + \|Q\|_{L^2}^2 - \|Q\|_{L^{p+1}}^{p+1} = 0,$$

$$-\left(\frac{d}{2} - \sigma\right)\|Q\|_{H^\sigma}^2 - \frac{d}{2} \|Q\|_{L^2}^2 + \frac{d}{p+1} \|Q\|_{L^{p+1}}^{p+1} = 0. \tag{39}$$

Solving the equation for $\|Q\|_{H^\sigma}^2$ and $\|Q\|_{L^{p+1}}^{p+1}$, we get

$$\|Q\|_{H^\sigma}^2 = \frac{d(p-1)}{2\sigma(p+1) - d(p-1)} \|Q\|_{L^2}^2, \tag{40}$$

$$\|Q\|_{L^{p+1}}^{p+1} = \frac{2\sigma(p+1)}{2\sigma(p+1) - d(p-1)} \|Q\|_{L^2}^2.$$ 

Let $c_{GN}$ be the sharp constant for the Gagliardo-Nirenberg inequality, that is,

$$\|u\|_{L^{p+1}}^{p+1} \leq c_{GN} \|u\|_{L^2}^{p+1 - \frac{d}{2\sigma(p-1)}} \|u\|_{H^\sigma}^{\frac{d}{2\sigma(p-1)}}. \tag{41}$$

It is known that such a constant is attained at the ground state $Q$ (see [12]). Therefore, by (40), we have

$$c_{GN} = \frac{\|Q\|_{L^{p+1}}^{p+1}}{\|Q\|_{L^2}^{p+1 - \frac{d}{2\sigma(p-1)} - \frac{d}{2\sigma(p-1)}}} \|Q\|_{L^2}^{\frac{d}{2\sigma(p-1)}}$$

$$= \frac{2\sigma(p+1)}{\|Q\|_{L^2}^{p-1} [d(p-1)]^{\frac{d}{2\sigma(p-1)} - \frac{d}{2\sigma(p-1)}}} \left[2\sigma(p+1) - d(p-1)\right]^{\frac{d}{2\sigma(p-1)}}. \tag{42}$$

**Proof of Theorem B.1.** Then, by (42), we have

$$c_{GN} = \frac{p+1}{2\|Q\|_{L^2}^{p-1}}. \tag{43}$$
Hence, by Gagliardo-Nirenberg inequality and the conservation laws, if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), we have
\[
E[u_0] = E[u] = \frac{1}{2} \|u\|_{H^s}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}}^{p+1} \\
\geq \frac{1}{2} \|u\|_{H^s}^2 \left( 1 - \frac{\|u\|_{L^2}^{p-1}}{\|Q\|_{L^2}^{p-1}} \right) = \frac{1}{2} \|u\|_{H^s}^2 \left( 1 - \frac{\|u_0\|_{L^2}^{p-1}}{\|Q\|_{L^2}^{p-1}} \right).
\]
\( (44) \)

Therefore, \( u(t) \) exists globally in time.

\( \square \)

**Proof of Theorem B.2.** By the conservation laws and Gagliardo-Nirenberg inequality, we have
\[
M[u_0]^{\frac{2}{p+1}} E[u_0] = M[u]^{\frac{2}{p+1}} E[u] \\
\geq \frac{1}{2} \|u\|_{L^2}^2 \|u\|_{H^s}^2 - \frac{c_{GN}}{p+1} \|u\|_{L^2}^{\frac{2}{p+1}} + (p+1) - \frac{\|u\|_{H^s}}{\|u\|_{L^2}} \|u\|_{L^2}^{(p-1)} \\
= f\left( \|u\|_{L^2}^\theta \|u\|_{H^s}^{1-\theta} \right),
\]

where
\[
f(x) = \frac{1}{2} x^{\frac{2}{p+1}} - \frac{c_{GN}}{p+1} x^{\frac{2}{p}} + (p-1).
\]
\( (46) \)

Then, by differentiating \( f(x) \), one can show that
\[
x_0 = \left[ \frac{2\theta(p+1)}{c_{GN}d(p-1)} \right]^{\frac{1}{p-1}} = \|Q\|_{L^2}^\theta \|Q\|_{H^s}^{1-\theta}
\]
\( (47) \)

is a local maximum, \( f(x) \) is increasing in \([0, x_0]\), \( f(0) = 0 \) and
\[
f(x_0) = f(\|Q\|_{L^2}^\theta \|Q\|_{H^s}^{1-\theta}) = M[Q]^{\frac{\theta}{p+1}} E[Q].
\]
\( (48) \)

Therefore, by continuity of \( u(t) \) in \( H^s \), we conclude that if \( \|u_0\|_{L^2}^\theta \|u_0\|_{H^s}^{1-\theta} < x_0 \) and \( f(\|u_0\|_{L^2}^\theta \|u_0\|_{H^s}^{1-\theta}) < f(x_0) \), then \( f(\|u(t)\|_{L^2}^\theta \|u(t)\|_{H^s}^{1-\theta}) < f(x_0) \) for all \( t \).

\( \square \)

**Remark 6.** By adapting the arguments in [16], one can prove blow-up.

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Received December 2014; revised June 2015.

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