Soliton Stars as Holographic Confined Fermi Liquids

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Fermi surfaces have been studied in several ads-cft context: extremal blackholes and electron stars.

Realistic condensed matter systems have a confined Fermi surface, in that the gapless excitations consists of excitations about the Fermi surface only.

Such confined Fermi surfaces have been discussed in the context of hard wall geometries (without backreaction).

We construct a full backreacted solution of fermions in ads soliton geometry in the Thomas-Fermi approximation.

Through a probe fermion analysis on our soliton star geometry we establish that we have large number of sharp Fermi surfaces.

We then go on to study the phase space of the known zero temperature solutions.
Outline

1. Fermionic fluid in the AdS soliton geometry
2. The electron star
3. Perturbative soliton star
4. Numerical soliton star
5. Comparison of various solutions
6. Probe fermions in the soliton star background
7. Conclusion
We consider the Maxwell-Einstein system coupled to a charged fermion which is treated in the Thomas-Fermi approximation.

The equation of motion takes the following form

\[ G_{\mu\nu} - 6g_{\mu\nu} = - \left( F_{\mu\chi} F_{\nu}^{\chi} - \frac{1}{4} F_{\chi\xi} F_{\xi\chi} g_{\mu\nu} \right) + \beta T_{\mu\nu}^{(f)} \right) \] \[ \nabla_{\mu} F_{\nu}^{\mu} = \beta j_{\nu}^{(f)}. \]

The fermion stress tensor and charge current are taken to be

\[ T_{\mu\nu}^{(f)} = (p(z) + \rho(z)) u_{\mu} u_{\nu} + p(z) g_{\mu\nu}, \quad j_{\nu}^{(f)} = \sigma(z) u_{\mu} \]

We assume that fermions sees only the local flat space-time

1. Compton wavelength is small compared to curvature scales.
2. Local fermionic density is high compared to curvature scales.

Also the interaction cross-section between two fermions are parametrically suppressed in our regime of approximation.
The local chemical potential in the bulk $\mu(z)$ is given by the time component of the gauge field in the local tangent space.

The energy and charge density is given by

$$
\rho(z) = \int_m^{\mu(z)} ES(E) dE,
$$

$$
\sigma(z) = \int_m^{\mu(z)} S(E) dE,
$$

The density of states for fermions in 5 dimensions is

$$
S(E) = E(E^2 - m^2),
$$

The pressure is given by the Gibbs-Duhem relation

$$
p(z) = -\rho(z) + \mu(z)\sigma(z).
$$
A Lifshitz solution

- The metric and gauge field ansatz

\[ ds^2 = -f(z)dt^2 + g(z)dz^2 + \frac{1}{z^2} \left( d\theta^2 + dx^2 + dy^2 \right), \quad A = h(z)dt. \]

- An exact solution of the Maxwell-Einstein-Fermion system with this symmetry ansatz has the form

\[ f(z) = \frac{1}{z^{2\alpha}}, \quad g(z) = \frac{g_L}{z^2}, \quad h(z) = \frac{h_L}{z^\alpha}. \]
The electron star is another solution of the same system with the same symmetry such that it reduces to this Lifshitz geometry in the IR.

For the star solution to exist the local chemical potential inside the star should be greater than the mass of the fermion

$$\mu(z) = \frac{h(z)}{\sqrt{f(z)}} \geq m.$$ 

This condition implies that there is a upper bound of mass ($m = 1$).

The inequality is saturated at the radius of the electron star.

The solution outside is a vacuum solution of the Einstein-Maxwell system which in this symmetry sector is simply the RN black hole solution.
Numerical solution and asymptotic data of Electron star

- We perturb away from the IR Lifshitz solution

\[ f(z) = \frac{1}{z^{2\alpha}} \left( 1 + \sum_i f_{es}^{(i)} z^i \right), \quad g(z) = \frac{g_L}{z^2} \left( 1 + \sum_i g_{es}^{(i)} z^i \right), \quad h(z) = \frac{h_L}{z^\alpha} \left( 1 + \sum_i h_{es}^{(i)} z^i \right). \]

- This perturbed solution is used to put the initial condition of the numerical solution.

- The radius of the star is obtained from the numerical solution by \( \mu(z_*) = m \) condition.

- The solution outside has the form

\[ f(z) = \chi^2 z^{-2} - m^{(es)} z^2 + \frac{2}{3} (Q^{(es)})^2 z^4, \]

\[ g(z) = \frac{\chi^2}{z^2 (1 - m^{(bh)} z^4 + \frac{2}{3} (Q^{(es)})^2 z^6)}, \]

\[ h(z) = \mu^{(es)} - Q^{(es)} z^2. \]

- The parameters \( m^{(es)}, Q^{(es)}, \) and \( \mu^{(es)} \) are determined by a first derivative patch with the numerical solution at the radius of the star.
Typical solution inside the electron star for $m = 1/3, \beta = 70.43$. 

Plot of the Numerical solution of Electron star
The metric and gauge field ansatz

\[ ds^2 = -f(z)dt^2 + g(z)dz^2 + k(z)d\theta^2 + \frac{1}{z^2} (dx^2 + dy^2), \quad A = h(z)dt. \]

The perturbative solution has the form

\[ f(z) = \frac{1}{z^2} + \beta f^{(1)}(z) + \mathcal{O}(\beta^2), \quad g(z) = \left( \frac{1}{z^2(1 - z^4)} \right) + \beta g^{(1)}(z) + \mathcal{O}(\beta^2), \]

\[ k(z) = \frac{1}{4} \left( \frac{1}{z^2} - z^2 \right) + \beta k^{(1)}(z) + \mathcal{O}(\beta^2), \quad h(z) = h_0 + \beta h^{(1)}(z) + \mathcal{O}(\beta^2). \]

The solution inside the star is obtained by imposing regular boundary conditions near the tip of the soliton.

The radius of the star is obtained by the condition

\[ \mu(z_r) \equiv \frac{h(z_r)}{\sqrt{f(z_r)}} = m. \]

The solution outside is then a solution of the vacuum Einstein-Maxwell system and is obtained by first derivative patching with the inner solution at the radius of the star.
Numerical solution of the soliton star

- For constructing the numerical solution the procedure followed is identical to the perturbative solution.
- We first solve the equations analytically near the tip of the soliton
  
  \[ f(z) = 1 + f_1(1 - z) + \mathcal{O}(1 - z)^2, \]
  \[ g(z) = \frac{g(-1)}{1 - z} + g_0 + \mathcal{O}(1 - z), \]
  \[ k(z) = 4g(-1)(1 - z) + k_2(1 - z)^2 + \mathcal{O}(1 - z)^2, \]
  \[ h(z) = h_0 + h_1(1 - z) + \mathcal{O}(1 - z)^2. \]

- This solution is used to set the IR initial value for the numerical solution.
- Soliton star has one parameter more than the electron star and we can take \( h_0 \) to be this parameter.
- We should of course have \( h_0 \geq m \), but \( h_0 \) also has an upper bound, where the solution becomes singular.
Comparison of Numerics with Perturbation theory

Solution for $\beta = 0.001$, $m = 1$, $h_0 = 4.1$. 

![Graph of f(z), g(z), k(z), and h(z)]
Asymptotic data

- Asymptotically the soliton star has the form
  \[ f(z) = f_\infty z^{-2} - m_\infty z^2 + \frac{2}{3} Qz^4 + \ldots \]
  \[ g(z) = \frac{1}{z^2} + \ldots \]
  \[ k(z) = k_\infty z^{-2} + k^{(1)}_\infty z^2 \ldots \]
  \[ h(z) = \mu - Qz^2 + \ldots \]

- These asymptotic parameter are read off from the numerical solution by an appropriate fitting analysis.
- We use a scale transformation to set the asymptotic $\theta$-circle radius to unity.
- The scale invariant parameters for our solution are then
  \[ T_{tt} k_\infty^2 \] and \[ Q = Qk_\infty^3 / \sqrt{f_\infty}. \]
- For technical convenience we consider the dimensionless ratio \[ \mathcal{R} = \frac{T_{tt}^3}{f_\infty Q^4}. \]
For both extremal RN blackhole and electron star $R$ is a constant number unlike the soliton star.

For the extremal RN blackhole $R = 81$

For the Electron star $R$ behaves in the following way
A condition for thermodynamic stability is

\[
\frac{d\tilde{\mu}}{dQ} = \frac{\partial^2 E}{\partial Q^2} \geq 0,
\]

If the bound gets violated then clumping of charge may be energetically favored compared to the homogeneous distribution of charge that we have in our case.

In such cases this thermodynamic instabilities may lead to a dynamical instability, the end point being a configuration that lie outside the symmetry sector analyses here.

We analyze the dependence of boundary chemical potential on charge density $Q$, and find that in certain cases this kind of instability is indeed indicated.
Various values of $\beta$ are 0.1 (red), 1 (blue), 5 (green), 15 (yellow), 40.7 (red), 70.4 (blue).
Schematic Plot of various possible cases

Case 1 (Green)

Case 2 (White)

Case 3 (Blue)

Soliton stars as Fermi liquids

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Probe Fermions in the soliton star background

- We analyze the Dirac equation in the soliton star background in WKB approximation

\[
\left( \Gamma^a e_a^M \mathcal{D}_M - m \right) \psi = 0.
\]

- We now make the following ansatz for the spinor field

\[
\psi = \frac{z}{(f(z)k(z))^{1/4}} e^{i(-\omega t + p\theta + k_1 x + k_2 y)} \Psi.
\]

- With this the Dirac equation reduces to

\[
\left( \frac{1}{\sqrt{g(z)}} \Gamma^\mu \partial_\mu - m + izK_\mu \Gamma^\mu \right) \psi = 0,
\]

with \( K_\mu = \{- \frac{1}{z \sqrt{f(z)}} \left( \omega + q h(z) \right), p, k_1, k_2 \} \).

- Setting \( k_2 = 0 \) and \( k_1 = k \), we now take the scaling limit

\[
m \to \gamma m, \quad q \to \gamma, \quad \omega \to \gamma \omega, \quad k \to \gamma k.
\]

- Effective Schrödinger equation has the form

\[
\partial_z^2 \Phi_1 - \gamma^2 V(z) \Phi_1 = 0, \quad V(z) = g(z) \left( m^2 + z^2 k^2 - \frac{(\omega + h(z))^2}{f(z)} \right).
\]
Plots of the effective potential at zero frequency

\begin{align*}
    V(z) &= \left[1 - \frac{m}{z}\right]^{\beta} \left[1 - \frac{h_0}{z}\right] \\
    m &= 0.33, \quad \beta = 70.43, \quad h_0 = 0.4 \\
    m &= 0.33, \quad \beta = 70.43, \quad h_0 = 0.65 \\
    m &= 0.33, \quad \beta = 70.43, \quad h_0 = 0.7 \\
    m &= 0.33, \quad \beta = 1, \quad h_0 = 0.6 \\
    m &= 0.33, \quad \beta = 1, \quad h_0 = 1.5 \\
    m &= 0.33, \quad \beta = 1, \quad h_0 = 2.3 \\
    m &= 1, \quad \beta = 1, \quad h_0 = 1.3 \\
    m &= 1, \quad \beta = 1, \quad h_0 = 2.0 \\
    m &= 1, \quad \beta = 1, \quad h_0 = 2.4
\end{align*}
Let us define the following quantities

\[ X(z) = \gamma \int_{z}^{z^*} dz \sqrt{V(z)}; \quad Y(z) = \gamma \int_{z^*}^{z} dz \sqrt{-V(z)}; \quad \hat{X} = X(0); \quad \hat{Y} = Y(1). \]

We have to require that the fermionic wavefunction vanishes at the IR tip.

Solving the Schrödinger equation in the large \( \gamma \) limit, the poles of the boundary Greens functions occur at

\[ \hat{Y} + \frac{\pi}{4} = n\pi - 2e^{-2\hat{X}} \Rightarrow \hat{Y} \approx n\pi. \]

The zero frequency limit of this equation provides the Fermi-momentum.

Note that these are sharp Fermi surfaces without any imaginary part of frequency, unlike electron stars.

Also we have large number of Fermi surfaces just like a theory of free fermions in the adjoint representation in the large \( N \) limit (which has \( \mathcal{O}(N^2) \) Fermi surfaces).
We studied a holographic model of confined Fermi liquid by putting fermions in the AdS soliton geometry.

All the charge in our solution is carried by the fermionic fluid that constitutes the star and hence Luttinger theorem holds true for our solution.

Our solution is characterized by many sharp Fermi surfaces as revealed by the probe fermion analysis.

We have revealed an interesting phase structure for the zero temperature solutions and understanding finite temperature deformation of this would be interesting.

We would also like to learn more about the inhomogeneous solutions whose existence was hinted by the thermodynamic instability.

It would be interesting to understand other properties of these Fermi surface (from the CFT point of view). In particular it would be physically interesting to know how the effect of anomalies in the microscopic theory are captured by the Fermi surface or excitation about it.