(4+N)-Dimensional Elastic Manifolds in Random Media: 
A Renormalization-Group Analysis

H. Bucheli\textsuperscript{a}, O.S. Wagner\textsuperscript{a}, V.B. Geshkenbein\textsuperscript{a, b}, A.I. Larkin\textsuperscript{b, c}, and G. Blatter\textsuperscript{a}

\textsuperscript{a} Theoretische Physik, ETH-H"onggerberg, CH-8093 Z"urich, Switzerland
\textsuperscript{b} L. D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia
\textsuperscript{c} Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455

(August 1997)

Motivated by the problem of weak collective pinning of vortex lattices in high-temperature superconductors, we study the model system of a four-dimensional elastic manifold with \( N \) transverse degrees of freedom (4+N-model) in a quenched disorder environment. We assume the disorder to be weak and short-range correlated, and neglect thermal effects. Using a real-space functional renormalization group (FRG) approach, we derive a RG equation for the pinning-energy correlator up to two-loop correction. The solution of this equation allows us to calculate the size \( R_c \) of collectively pinned elastic domains as well as the critical force \( F_c \), i.e., the smallest external force needed to drive these domains. We find \( R_c \propto \delta_p^{-2\alpha_1} \exp(\alpha_1/\delta_p) \) and \( F_c \propto \delta_p^{-2\alpha_2} \exp(-2\alpha_1/\delta_p) \), where \( \delta_p \ll 1 \) parametrizes the disorder strength, \( \alpha_1 = (2/\pi)^{N/2} 8\pi^2/(N + 8) \), and \( \alpha_2 = 2(3N + 22)/(N + 8)^2 \). In contrast to lowest-order perturbation calculations which we briefly review, we thus arrive at determining both \( \alpha_1 \) (one-loop) and \( \alpha_2 \) (two-loop).

\section{I. INTRODUCTION}

The generic problem of a \( D \)-dimensional elastic manifold with \( N \) transverse degrees of freedom subject to quenched random impurities that can pin the manifold has attracted a great deal of attention for many years. It has applications in numerous areas of physics such as dislocations, spin systems, polymers, charge density waves, and vortex lattices in type-II superconductors. The case \( N = 1 \), for instance, corresponds to \( D \)-dimensional interfaces separating two coexisting phases in \((D + 1)\)-dimensional systems. On intermediate distances the flux lattice in superconductors is believed to behave like an elastic manifold with \( D = 3 \) and \( N = 2 \).

The crucial quantity for defining order in these systems is the disorder-averaged square of the relative displacement \( \langle u^2(x) \rangle := \langle \| u(x) - u(0) \| \rangle^2 \), which describes the roughening of the \( N \)-component displacement field \( u(x) \) of \( D \)-dimensional support. In general, \( \langle u^2(x) \rangle \) depends on the distance \( R := |x| \), the dimensionalities \( D \) and \( N \), and on the type of disorder. In the present paper, we choose the disorder to be weak, short-range correlated, and of Gaussian type. Under these assumptions it has been shown \cite{7} that in less than four dimensions \( (D \leq 4) \) quenched disorder always destroys the translational long-range order of the manifold, breaking it up into correlated domains of extent \( R_c \). Each of these domains behaves elastically independently and is pinned individually. The length scale \( R_c \) over which short-range order subsists is commonly referred to as the collective pinning radius and is usually defined by the criterion \( \langle u^2(R_c) \rangle \sim \xi^2 \), where \( \xi \) is the length scale describing the internal structure of the manifold \cite{hilling}. For distances \( R \) smaller than \( R_c \) (perturbative regime), the ground state of the system is unique and the mean square displacement \( \langle u^2(R) \rangle \) can be calculated perturbatively. On the other hand, for \( R > R_c \) (random manifold regime), there are many competing metastable minima inducing the so-called “wandering” of the manifold as described by the scaling law \( \langle u^2(R) \rangle \approx \xi^2 (R/R_c)^{2\alpha} (R \gg R_c) \). A lot of effort has gone into the determination of the wandering exponent \( \zeta \), using elaborate techniques such as the replica formalism combined with either variational approaches \cite{bunch,ferrenberg} or with the renormalization group (RG). Here, we make use of RG methods within the perturbative regime to determine the collective pinning radius \( R_c \). For that purpose, we will replace the estimate \( \langle u^2(R_c) \rangle \sim \xi^2 \) by an improved definition of \( R_c \).
based on the divergence of the fourth derivative of the pinning-energy correlator, signaling the appearance of competing ground states at this length scale. The collective pinning radius $R_c$ is not only a relevant quantity for characterizing order, but also determines dynamic properties such as the minimum force $F_c$ needed to move the pinned manifold (critical force) and the activation barrier $U_c$ for creep.

In this work we concentrate on the particularly interesting case $D = 4$, the upper critical dimension of this problem, where the mean-square displacement shows a logarithmic behavior in the perturbative regime \cite{14}, $(u^2(R)) \propto \ln R$, ($R < R_c$). This logarithmic situation motivates the use of renormalization group methods. The $(4+N)$-dimensional model system is strongly related to the problem of weak collective pinning of a $(3+2)$-dimensional manifold with dispersive elastic moduli, an issue which has received group methods. The $(4+N)$-regime \cite{10},

\[ ⟨ \] of this problem, where the mean-square displacement shows a logarithmic behavior in the perturbative regime. In a future paper \cite{20}, the RG technique developed below will be applied to that topic (see also Ref. \cite{19}).

The main objects of interest in our investigations are the collective pinning radius $R_c$ and the critical force density $F_c$. In section II, we introduce the non-dispersive $(4+N)$-model and define the static and dynamic Green’s functions as well as the relevant correlation functions. For simplicity, we neglect thermal effect, i.e., we set $T = 0$. Then, in section III, we derive $R_c$ and $F_c$ by means of lowest-order perturbation calculations combined with scaling techniques and the dynamic approach, and discuss the problem with these methods. Section IV is devoted to the renormalization group treatment of the pinning problem. We construct the RG transformation and derive a functional RG equation for the pinning energy correlator in one and two-loop approximation. We show how $R_c$ can be obtained by solving this equation and determine the critical force density $F_c$ using simple scaling relations. We discuss various aspects making this technique superior to those presented in section III. In a second step, we apply these results to the dynamical situation, where the system is driven with an external force. In particular, we determine the behavior of the friction coefficient $\eta$ under the action of the RG and give an alternative derivation of $F_c$. Finally, in section VI we summarize and present our conclusion.

II. MODEL

A. Non-dispersive $(4+N)$-model

We consider a four-dimensional elastic manifold in a $(4+N)$-dimensional space in the continuum limit: A point of the manifold is represented by a four-component vector $x$ (internal degrees of freedom), while the displacement relative to the equilibrium position at that point is characterized by the continuous $N$-component vector field $u(x)$ (transverse degrees of freedom). The elastic part of the free energy is given by the isotropic expression

\[ F_{el}[u] = \frac{1}{2} C \int d^4 x \, \nabla u^\alpha(x) \cdot \nabla u^\alpha(x), \]

where $\alpha = 1, 2, \ldots, N$; as usual, indices appearing twice are implicitly summed over. In the present work, the elastic modulus $C$ is taken to be a constant (non-dispersive model).

The elastic manifold is embedded in a random environment which we model as a Gaussian random pinning potential $U_{pin}(x, s)$ with zero mean and short-range correlations,

\[ \langle U_{pin}(x, s) U_{pin}(x', s') \rangle = \gamma_U \delta^4(x - x') \delta^N(s - s'). \]  

The parameter $\gamma_U$ is a measure of the disorder strength and is assumed to be small. Angular brackets $\langle \ldots \rangle$ denote the average over all possible realizations of disorder. Note that $U_{pin}$ depends on the internal as well as on the transverse degrees of freedom of the system. For a given configuration $u(x)$ and a random, but fixed disorder, the energy density arising from the interaction of the manifold with the disordered potential is the convolution of the potential $U_{pin}$ with the form factor $p(s)$:

\[ E_{pin}(x, u(x)) = \int d^N s \, U_{pin}(x, s) p(|s - u(x)|) . \]

The function $p(s)$ characterizes the internal structure of the manifold. In the following, we will adopt the simple expression
\[ p(s) = \exp \left( -\frac{s^2}{\xi^2} \right), \tag{3} \]

where \( \xi \) is chosen as the smallest resolvable length scale of the system. The effect of the form factor is to smear the random potential \( U_{\text{pin}} \) over a length \( \xi \), so that the typical effective distance between two ‘valleys’ in the energy landscape is of order \( \xi \).

The total free energy of the manifold in the presence of disorder is eventually

\[ \mathcal{F}[u] = \mathcal{F}_{\text{el}}[u] + \mathcal{F}_{\text{pin}}[u] = \int d^4x \left[ \frac{1}{2} C \nabla u^\alpha(x) \cdot \nabla u^\alpha(x) + E_{\text{pin}}(x, u(x)) \right], \tag{4} \]

with the pinning energy correlator

\[ (E_{\text{pin}}(x, u(x)) E_{\text{pin}}(x', u'(x'))) = \delta^4(x-x') K_\xi(u(x) - u'(x')), \tag{5} \]

where

\[ K_\xi(u) = \gamma U \xi^N \left( \frac{\pi}{2} \right)^{\frac{N}{2}} \exp \left( -\frac{u^2}{4\xi^2} \right), \quad u := |u|, \tag{6} \]

as obtained from Eqs. (1)–(3). The exponential in the correlator \( K_\xi(u) \) is directly related to the expression for the form factor, Eq. (3).

The equilibrium configuration \( u(x) \) of the system results from the competition between the elastic energy the system has to pay for a distortion and the pinning energy the system can win by accommodating to the impurity potential. Unfortunately, determining these configurations reveals an impossible undertaking because of the randomness of \( E_{\text{pin}} \). One must therefore content oneself with calculating disorder-averaged quantities, relying on the system’s assumed property of self-averaging.

### B. Green’s functions

Minimizing the free-energy functional (4) with respect to \( u(x) \),

\[ \frac{\delta}{\delta u^\alpha(x)} \mathcal{F}[u] = 0, \quad \alpha = 1, \ldots, N, \tag{7} \]

defines the equation of state,

\[ -C \nabla^2 u^\alpha(x) = F_{\text{pin}}^\alpha(x, u(x)), \tag{8} \]

where \( F_{\text{pin}} \) stands for the pinning force density,

\[ F_{\text{pin}}^\alpha(x, u(x)) := -\frac{\delta}{\delta u^\alpha(x)} \mathcal{F}_{\text{pin}}[u] = -\frac{\partial}{\partial u^\alpha} E_{\text{pin}}(x, u(x)). \tag{9} \]

The equation of state (8) can equivalently be written in the form

\[ u^\alpha(x) = \int d^4y \, G^{\alpha\beta}(x-y) F_{\text{pin}}^\beta(y, u(y)), \tag{10} \]

where

\[ G^{\alpha\beta}(x) = \delta^{\alpha\beta} G(x), \quad G(x) = \frac{1}{(2\pi)^2 C x^2}, \quad x := |x|, \tag{11} \]

is the static Green’s function of the system defined by the relation

\[ -C \nabla^2 G(x) = \delta^4(x). \]
For the dynamic approach in section III we need the time-dependent Green’s function \( G^{\alpha \beta}(x, t) \). We assume a dissipative dynamics for our elastic manifold which we characterize by the friction coefficient \( \eta \). With the equation of motion
\[
\left( -C \nabla^2 + \eta \frac{\partial}{\partial t} \right) u^{\alpha}(x, t) = F_{\text{pin}}^{\alpha}(x, u(x, t)),
\]
the dynamic Green’s function takes the form \( \hat{G}^{\alpha \beta}(k, \omega) = \frac{\delta^{\alpha \beta}}{(Ck^2 - i\eta_\omega)} \) in Fourier representation. Transforming back to real space, one readily obtains
\[
G^{\alpha \beta}(x, t) = \Theta(t) \frac{\eta}{16\pi^2 C_\omega^2} \exp \left( -\frac{\eta x^2}{4Ct} \right).
\]
The Heaviside-function \( \Theta(t) \) reflects the causality of the Green’s function.

C. Correlation functions

For later use we define the derivatives of \( F_{\text{pin}} \),
\[
F_{\text{pin}}^{\alpha_1 \cdots \alpha_l}(x, u(x)) := \frac{\partial}{\partial u^{\alpha_1}} \cdots \frac{\partial}{\partial u^{\alpha_l}} F_{\text{pin}}(x, u(x)), \quad \alpha_1, \ldots, \alpha_l = 1, \ldots N, \quad l = 1, 2, 3, \ldots,
\]
and the corresponding correlators,
\[
\langle F_{\text{pin}}^{\alpha_1 \cdots \alpha_l}(x, u(x)) F_{\text{pin}}^{\beta_1 \cdots \beta_n}(x', u'(x')) \rangle = (-1)^n \delta^l \left( x - x' \right) K_{\xi}^{\alpha_1 \cdots \alpha_l, \beta_1 \cdots \beta_n}(u(x) - u'(x')) ,
\]
where naturally \( K_{\xi}^{\alpha_1 \cdots \alpha_l, \beta_1 \cdots \beta_n}(u) := (\partial/\partial u^{\alpha_1}) (\partial/\partial u^{\alpha_2}) \cdots K_{\xi}(u) \). Each partial derivative \( \partial/\partial u^{\alpha} \) contributes a minus sign, thus giving rise to the factor \((-1)^n\). In particular, the force-force correlator is given by
\[
K_{\xi}^{\alpha \beta}(u) = -\gamma_U \xi^{N-2} \left( \frac{\pi}{2} \right)^{\frac{N}{2}} \left( \delta^{\beta \gamma} - \frac{u^\alpha u^\beta}{\xi^2} \right) \exp \left( -\frac{u^2}{2\xi^2} \right).
\]
Of special interest in section IV will be the curvature of the force at the origin,
\[
K_{\xi}^{\alpha \beta \gamma \delta}(0) = \Gamma_{\xi} \left( \delta^{\alpha \beta} \delta^{\gamma \delta} + \delta^{\alpha \gamma} \delta^{\beta \delta} + \delta^{\alpha \delta} \delta^{\beta \gamma} \right) =: \Gamma_{\xi} \Delta^{\alpha \beta \gamma \delta},
\]
with \( \Gamma_{\xi} := \gamma_U \xi^{N-4} \left( \frac{\pi}{2} \right)^{\frac{N}{2}} \).

III. PERTURBATION THEORY

In this section we present the usual ways to define and determine the collective pinning radius \( R_c \) and the critical force density \( F_c \). First, we calculate \( R_c \) by a lowest-order perturbation calculation, and assess \( F_c \) by dimensional estimates \[1\]. Second, we use the dynamic approach \[12,13\] to determine \( F_c \) in an alternative manner. We also point out the disadvantages of these simple methods.

A. Dimensional estimates

The collective pinning radius \( R_c \) is usually derived by computing the fluctuations of the displacement field \( u(x) \) induced by the random environment,
\[
\langle u^2(R) \rangle := \langle \left| u(x) - u(0) \right|^2 \rangle, \quad R := |x|,
\]
in the absence of an external force, \( F_{\text{ext}} = 0 \). The condition \( \langle u^2(R_c) \rangle \simeq \xi^2 \) defines the collective pinning radius \( R_c \) which represents the extension of a collectively pinned elastic domain. For small fluctuations,
\[ \langle u^2(R) \rangle \leq \xi^2, \text{ one can use a perturbative approach} \quad (11). \text{ The starting point is the integral equation} (10). \]

We expand the pinning force, \( F_{\text{pin}}(y, u(y)) = F_{\text{pin}}(y, 0) + O(u(y)) \), and only retain the lowest-order term, so that Eq. (10) reads \( (G^{\alpha \beta} = \delta^{\alpha \beta} G) \)

\[ u^\alpha(x) = \int d^4y \ G(x - y) F_{\text{pin}}^\alpha(y, 0). \quad (16) \]

From this expression and by means of Eqs. (9) and (13), we obtain

\[ \langle u^\alpha(x)u^\alpha(x') \rangle = -K_\xi^{\alpha \alpha}(0) \int d^4y \ G(x - y) G(x' - y) + O(\gamma_U^2). \]

Next, we write the Green’s functions in Fourier representation; the mean square of the relative displacement then takes the form

\[ \langle u^2(R) \rangle = -2K_\xi^{\alpha \alpha}(0) \int \frac{dk}{(2\pi)^4} \left( 1 - \cos(k \cdot x) \right) \hat{G}(k)^2 + O(\gamma_U^2). \quad (17) \]

Surprisingly enough, this lowest-order perturbation expression is true to all orders in \( \gamma_U \) \quad (14) (this can be checked by keeping \( u(y) \) on the rhs. of Eq. (16) and going through the subsequent calculations). Two elements are responsible for this striking fact: the short-range correlation \( \propto \delta^4(x - y) \) in \( \langle F_{\text{pin}} E_{\text{pin}} \rangle \) and the implicit assumption made in this derivation that the system is in a unique state, which is only valid in the perturbative regime. The integral in Eq. (17) is invariant under space rotations; one can therefore choose \( x \) parallel to the \( x_4 \)-axis and introduce cylindrical coordinates \( (K^2 := k_1^2 + k_2^2 + k_3^2) \),

\[ \langle u^2(R) \rangle = - \frac{2K_\xi^{\alpha \alpha}(0)}{(2\pi)^4 C^2} \int_0^\infty dk_4 K^2 dK \int_{-\infty}^\infty dk_4 \frac{1 - \cos(k_4 R)}{(K^2 + k_4^2)^2}. \]

Performing the \( K \)-integration yields

\[ \langle u^2(R) \rangle = - \frac{K_\xi^{\alpha \alpha}(0)}{4\pi^2 C^2} \int_0^\infty dk_4 \frac{1 - \cos(k_4 R)}{k_4}. \quad (18) \]

The \( k_4 \)-integration produces a log-contribution which diverges for \( k_4 \to \infty \). For this reason, one has to introduce an upper cutoff given by the inverse of the smallest length scale in the problem which is \( \xi \).

With \( K_\xi^{\alpha \alpha}(0) = -\gamma_U \xi^{N-2} N(\pi/2)^{N/2} \), Eq. (14), one finally obtains

\[ \langle u^2(R) \rangle \approx \frac{N}{4\pi^2} \left( \frac{\pi}{2} \right)^{\frac{N}{2}} \delta_p \ln \left( \frac{R}{\xi} \right), \quad R \gg \xi, \quad \text{where} \quad \delta_p := \gamma_U \xi^{N-4}/C^2 \quad (19) \]

where \( \delta_p := \gamma_U \xi^{N-4}/C^2 \) is the dimensionless disorder parameter. The collective pinning radius defined through the relation \( \langle u^2(R_c) \rangle \approx \xi^2 \) then is

\[ R_c \approx \xi \exp \left( \frac{4\pi^2}{N} \left( \frac{\pi}{2} \right)^{\frac{N}{2}} \frac{1}{\delta_p} \right), \quad \text{where} \quad \delta_p := \gamma_U \xi^{N-4}/C^2 \quad (20) \]

which is exponentially large in the limit of weak pinning, \( \delta_p \ll 1 \). With the upper cutoff in Eq. (18) being given only up to a factor of order unity, the constant of proportionality in Eq. (20) is not unequivocally determined. Much more relevant, though, is the uncertainty in the numerical factor in the exponential function, which has its origin in the criterion \( \langle u^2(R_c) \rangle \approx \xi^2 \), where the constant of proportionality is assumed to be of order unity.

A rough estimate for the critical force density \( F_c \) can be gained by scaling. The typical elastic energy of a collectively pinned domain is

\[ U_c \sim C \left( \frac{\xi}{R_c} \right)^2 V_c, \]

where \( V_c := R_c^4 \) is the collective pinning volume. The energy gain arising from the action of an external force \( F_{\text{ext}} \) shifting the domain by \( \xi \) is

\[ U_{\text{ext}} \sim F_{\text{ext}} V_c \xi. \]

5
Comparison between both energy scales gives a scaling estimate for the critical force density,

\[ F_c \sim \frac{C}{\xi} \left( \frac{\xi}{R_c} \right)^2 \simeq \frac{C}{\xi} \exp \left( -\frac{8\pi^2}{N} \left( \frac{2}{N} \right)^{\frac{\xi}{\delta_p}} \right). \tag{21} \]

**B. Dynamic approach**

In the dynamic approach \[12,13\] the critical force \( F_c \) is determined directly without any reference to the pinning radius \( R_c \). To begin with, the system is driven by a large external force density \( F_{\text{ext}} \gg F_c \). In this regime, the elastic manifold does not noticeably feel the pinning potential and \( F_{\text{ext}} \) is only opposed by the friction force \( F_{\text{frict}} \) which we describe by the phenomenological expression \( F_{\text{frict}} = -\eta_0 v \), with the viscosity \( \eta_0 \) a material constant. The flow velocity \( v \) of the manifold is determined by the steady-flow condition

\[ F_{\text{frict}} + F_{\text{ext}} = 0 \quad \Rightarrow \quad v = v_o := \frac{1}{\eta_0} F_{\text{ext}}. \]

If \( F_{\text{ext}} \) is decreased, the system begins to sensibly interact with the randomly distributed impurities via the pinning force \( F_{\text{pin}} \). This becomes noticeable through fluctuations in the velocity field, leading to a reduction of the average velocity, \( v = v_o - \delta v \) with \( v_o \cdot \delta v > 0 \). The average of \( F_{\text{pin}} \) can be interpreted as an effective friction force with a velocity-dependent viscosity \( \delta \eta (v) \),

\[ \langle F_{\text{pin}} \rangle = -\delta \eta (v) v. \tag{22} \]

As before, the steady-flow velocity is given by the relation

\[ F_{\text{frict}} + \langle F_{\text{pin}} \rangle + F_{\text{ext}} = 0 \quad \Rightarrow \quad \eta(v) v := (\eta_0 + \delta \eta (v)) v = F_{\text{ext}}. \tag{23} \]

The applied force for which the relative fluctuations of the viscosity are of order unity then furnishes a useful criterion for the critical force density:

\[ \frac{\delta \eta (F_{\text{ext}})}{\eta_0} \bigg|_{F_{\text{ext}} = F_c} = 1. \tag{24} \]

Alternatively, a criterion can be formulated making use of the reduction in the flow velocity, \( \langle F_{\text{pin}} \rangle = -\eta_0 \delta v \), and the condition \( \delta v (v_c) = v_c \).

We now calculate the friction coefficient \( \delta \eta \) due to pinning within a perturbative approach up to first order in \( \delta p \). We start from Eq. (22),

\[ \nu^\alpha \delta \eta = -\langle F_{\text{pin}}^\alpha (x, vt + u(x, t)) \rangle, \]

and expand the pinning force into a Taylor series, keeping the first two terms,

\[ \nu^\alpha \delta \eta \approx -\langle F_{\text{pin}}^\alpha (x, vt) \rangle - \langle F_{\text{pin}}^\alpha \beta (x, vt) u^\beta (x, t) \rangle. \]

The first term vanishes; in the second one, we use the integral equation (14) generalized to the time-dependent case and expanded to lowest order,

\[ u^\alpha (x, t) = \int d^4 x' ds G(x - x', t - s) F_{\text{pin}}^\alpha (y, vs). \]

Remembering Eq. (13) and the definition \( F_{\text{pin}}^\alpha = -E_{\text{pin}}^\alpha \), one obtains

\[ \nu^\alpha \delta \eta \approx \int dt G(0, t) F_{\text{pin}}^\alpha (vt), \tag{25} \]
where we made use of the time-translation symmetry. Inserting the explicit expression for the correlator $K_{\xi}^{\alpha \beta \beta}$ and for the time-dependent Green’s function (12), we find:

$$\frac{\delta \eta(v)}{\eta_0} \simeq \frac{N + 2}{16\pi^2} \left( \frac{\pi^2}{2} \right)^{\frac{1}{2}} \delta_p \int_0^\infty \frac{dt}{t} \exp \left( -\frac{v^2 t^2}{2\xi^2} \right) + O(\delta_p^2).$$

The integration produces a log-divergence for $t \to 0$. A lower cutoff is given by the time $t_\xi$ the elastic manifold needs to recover from a distortion on the smallest length scale $\xi$. Balancing elastic and dynamic terms in the Green’s function $G(x, t)$, Eq. (12), on the scale $\xi$, we obtain

$$t_\xi \simeq \frac{\eta_0 \xi^2}{4C}. \quad (26)$$

On the other hand, the exponential factor in the integrand, which originates from the correlator $K_{\xi}^{\alpha \beta \beta}$, provides an upper cutoff,

$$t_v \simeq \frac{\sqrt{2} \xi}{v}, \quad (27)$$

describing the time scale for the interaction between the system and the pinning centers at velocity $v$. With these two cutoffs, the ratio between $\delta \eta$ and $\eta_0$ becomes

$$\frac{\delta \eta(F_{\text{ext}})}{\eta_0} \simeq \frac{N + 2}{16\pi^2} \left( \frac{\pi^2}{2} \right)^{\frac{1}{2}} \delta_p \left[ \ln \left( \frac{4\sqrt{2}C}{\xi F_{\text{ext}}} \right) + \text{const.} \right] + O(\delta_p^2). \quad (28)$$

This expression is valid in the non-linear regime, i.e., for forces $F_{\text{ext}} < 4\sqrt{2}C/\xi$ (so that $t_\xi < t_v$), where the effective friction coefficient $\eta(F_{\text{ext}}) = F_{\text{ext}}/v(F_{\text{ext}})$ deviates from $\eta_0$. The critical force $F_c$ is found from the result (28), using the criterion $\delta \eta(F_c)/\eta_0 \simeq 1$ (for weak disorder $\delta_p \ll 1$, we expect $F_c$ to be much smaller than $C/\xi$, so that the constant in Eq. (28) can be neglected). We obtain

$$F_c \simeq 4\sqrt{2}C \xi \exp \left( -\frac{16\pi^2}{N + 2} \left( \frac{2}{\pi} \frac{N^2}{\delta_p} \right) \right), \quad (29)$$

which is indeed exponentially small. As in the previous method, both the constant of proportionality and the numerical factor in the exponential function can only be ascertained up to a number of order unity.

**IV. RENORMALIZATION GROUP ANALYSIS**

As we have seen in the previous section, the leading term in the fluctuations of the displacement field $u(x)$ shows a logarithmic behavior, $\langle u^2 \rangle \propto \ln R^2$, cf. Eq. (13). The logarithmic dependence weights all scales equally such that all length scales are relevant in the final result. This provides the motivation to apply the renormalization group (RG). Our guiding line for this analysis will be the work of Efetov and Larkin [14] dealing with the pinning problem of charge-density waves.

**A. Construction of the RG transformation**

The following real-space renormalization procedure was suggested by Khmel’nitskii and Larkin [13] for the problem of friction between two rough surfaces. In a recent publication [14], Balents, Bouchaud, and Mézard gave a precise formulation of this method within the framework of the momentum-shell RG and used it to investigate the random manifold regime (large scales $R > R_c$) of the $(4-\epsilon)+N$-model; here, we are interested in the perturbative regime (small scales $R < R_c$) with the aim to determine $R_c$.

Assume that we have renormalized the displacement up to a scale $R_1 > \xi$; let us denote the corresponding field by $u_{(1)}(x)$. The associated free-energy functional reads

$$\mathcal{F}[u_{(1)}] = \int d^4x \left[ \frac{1}{2} C \nabla u_{(1)}^\alpha(x) \cdot \nabla u_{(1)}^\alpha(x) + E_{\text{pin},(1)}(x, u_{(1)}(x)) \right].$$
Next, we go over to a larger scale \( R_2 > R_1 \) and separate \( u_{(1)}(x) \) into a far and a near-field contribution, \( u_{(1)}(x) = u_{(2)}(x) + w(x) \), with

\[
\begin{align*}
u_{(2)}(x) &= - \int_{|x-y| > R_2} d^4y \, G^{\alpha\beta}(x-y) \, E_{pin,(1)}^{\beta}(y, u_{(1)}(y)), \\
w^\alpha(x) &= - \int_{\Omega} d^4y \, G^{\alpha\beta}(x-y) \, E_{pin,(1)}^{\beta}(y, u_{(1)}(y)),
\end{align*}
\]

where \( \Omega := \{ x, y | R_1 < |x-y| < R_2 \} \). The free energy can then be written as

\[
F[u_{(2)}] = \int d^4x \left[ \frac{1}{2} C \nabla u_{(2)}^\alpha(x) \cdot \nabla u_{(2)}^\alpha(x) + E_{pin,(2)}(x, u_{(2)}(x)) \right]
\]

with the renormalized pinning energy density,

\[
E_{pin,(2)}(x, u_{(2)}(x)) = E_{pin,(1)}(x, u_{(1)}(x)) + \frac{1}{2} C \left[ \nabla w^\alpha(x) \cdot \nabla w^\alpha(x) + 2 \nabla u_{(2)}^\alpha(x) \cdot \nabla w^\alpha(x) \right].
\]

The mixed term \( \nabla u_{(2)}^\alpha \cdot \nabla w^\alpha \) is zero when integrated over and can be omitted. Integrating the quadratic term by parts, one finds

\[
E_{pin,(2)}(x, u_{(2)}(x)) = E_{pin,(1)}(x, u_{(1)}(x)) - \frac{1}{2} w^\alpha(x) C \nabla^2 w^\alpha(x).
\]

In the limit of zero temperature, out of all the possible near-field contributions \( w(x) \) only those minimizing the energy are relevant (in a statistical sense),

\[
\frac{\partial E_{pin,(2)}(x, u_{(2)}(x))}{\partial w^\alpha} = 0.
\]

Combining this condition with Eq. (31), we can relate \( w(x) \) to the pinning force \( E_{pin,(1)}^\alpha \),

\[
-C \nabla^2 w^\alpha(x) = -E_{pin,(1)}^\alpha(x, u_{(1)}^\alpha(x)).
\]

(32)

(It is interesting to note that by differentiating Eq. (31) with respect to \( u_{(2)}^\alpha \), one shows that for the pinning force

\[
E_{pin,(2)}^\alpha(x, u_{(2)}(x)) = E_{pin,(1)}^\alpha(x, u_{(1)}(x)) + O \left( \frac{\partial w}{\partial u_{(2)}(x)} \right),
\]

holds, which tells us that the force correlator \( K^{\alpha\beta}(u = 0) \) will not change under the RG.)

Eq. (31), together with Eqs. (30) and (32), is the starting point for our further considerations. It will allow us to derive a functional renormalization group (FRG) equation for the pinning energy correlator \( K_R(u) \), the subscript \( R \) denoting the scale of renormalization. We will then solve this equation by expanding the relevant correlators around \( u = 0 \). It turns out that the fourth derivative, \( K_R^{\alpha\beta\gamma\delta}(0) \), as well as higher-order even derivatives diverge at a finite scale which we identify with the collective pinning radius \( R_c \). Indeed, we know that on length scales smaller than \( R_c \) (perturbative regime), the system is in a unique state and the pinning energy as well as the related correlators are analytic functions in the field \( u \). At \( R_c \) the manifold starts to probe different energy `valleys’ and our calculation, which rests on the analyticity of the pinning energy, breaks down, reflected by the singular behavior of the curvature of \( K(u) \). In what follows we will take the divergence of \( K_R^{\alpha\beta\gamma\delta}(0) \) as an unambiguous definition of \( R_c \) which allows one to go beyond the simple estimate \( \langle u^2(R_c) \rangle \simeq \xi^2 \).

B. One-loop FRG equation

We first construct a formal expansion in \( w \) of \( E_{pin,(2)} \), Eq. (31), which we will then use to derive a perturbation series in \( \delta_p \) of \( \langle E_{pin,(2)} \rangle \). Thereby, we will relate the energy correlators at renormalization scales \( R_2 \) and \( R_1 \), \( K_{R_2} \) and \( K_{R_1} \), respectively; this will directly lead to the FRG equation for...
Let us furthermore define the operator $K_R(u)$. To simplify the notation, we will omit the subscript ‘pin’ and use the abbreviations

$$E_j := E_{\text{pin},(2)}(x_j, u_2(x_j)), \quad E^\beta_j := E_{\text{pin},(1)}(x_j, u_2(x_j)),$$

$$K_{ij} := K_{R_2}(u_2(x_i) - u'_2(x_j)).$$

Let us furthermore define the operator

$$G^\alpha_{ij} (\cdots) := \int_\Omega d^4 x_j G^\alpha(x_i - x_j) (\cdots).$$

We begin by expanding the first term on the rhs. of Eq. (31) with respect to $w$.

$$w(x_0) := \frac{1}{\Omega} E_0^\alpha u^{(1)}(x_0) w^{\alpha_1}(x_0) \cdots w^{\alpha_n}(x_0).$$

Next, we replace everywhere $w$ by its implicit definition, Eq. (31). The force function $E_{(1)}^\beta$ under the integral is in turn written as a Taylor expansion as above. We proceed iteratively, thereby eliminating $w$ from our equation. Grouping the different terms with increasing number of Green’s functions, we arrive at

$$-E_{(1)}(x_0, u_1(x_0)) = -E_0 + E_0^\mu G_{01}^\mu E_1^\nu - E_0^\mu G_{01}^\mu E_1^{\nu\rho} G_{12}^\rho E_2^\sigma - \frac{1}{2!} E_0^\mu G_{01}^\mu E_1^{\nu\rho} G_{12}^{\rho\sigma} E_2^\tau + \frac{2}{2!} E_0^\mu G_{01}^\mu E_1^{\nu\rho} G_{12}^{\rho\sigma} G_{23}^{\lambda} E_3^\tau + \frac{1}{3!} E_0^\mu G_{01}^\mu E_1^{\nu\rho} G_{12}^{\rho\sigma} G_{23}^{\lambda} G_{34}^{\mu} E_4^\tau + \cdots.$$

In a diagrammatic language this can be written as

A bullet with subscript ‘$j$’ and $l$ legs $\alpha_1, \ldots, \alpha_l$ stands for the expression $(-E_{(1)}^{\alpha_1 \cdots \alpha_l})/l!$, whereas a solid line with the indices ‘$\mu$’ and ‘$\nu$’ joining two bullets ‘$i$’ and ‘$j$’ represents $G^\mu_{ij}$, i.e., the Green’s function $G^\mu(x_i - x_j)$, plus an integration over $x_j \in \Omega$. The factor 2 in front of the penultimate term is the symmetry factor which accounts for the fact that the expansion gives the two equivalent diagrams

In a similar way, we expand the second term on the rhs. of Eq. (31), making use of Eq. (32):

$$-\frac{1}{2} w^\alpha(x_0) C \nabla^2 w^\alpha(x_0) = -\frac{1}{2} w^\alpha(x_0) E_{(1)}^\alpha(x_0, u_2(x_0) + w(x_0)) = \ldots.$$

Adding the series for $E_{(1)}(x_0, u_1(x_0))$ and $-\frac{1}{2} w^\alpha(x_0) C \nabla^2 w^\alpha(x_0)$ we finally obtain
Let us now turn to the energy-energy correlation function, \( \langle E_0 E_0' \rangle \), where \( x_j \equiv x'_j \). We write both energy functions in an expansion as done above. Using the linearity of the averaging procedure, we obtain a series of \( n \)-point correlators of the form \( \langle E_0^{(i_1)\alpha_1\beta_1} \cdots E_0^{(i_n)\alpha_n\beta_n} \rangle \). Assuming Gaussian disorder, we can make use of Wick’s theorem: \( n \)-point functions with odd \( n \) vanish and those with even \( n \) can be decomposed into a sum of products of 2-point correlators. For instance, application of Wick’s theorem transforms the second-order term

\[
G_{01}^{\mu\nu} G_{0'1'}^{\rho\sigma} \langle E_0^\mu E_0'^\rho E_1^\nu E_1'^\sigma \rangle
\]

into the expression

\[
G_{01}^{\mu\nu} G_{0'1'}^{\rho\sigma} \left[ \langle E_0^\mu E_0'^\rho \rangle \langle E_1^\nu E_1'^\sigma \rangle + \langle E_0^\mu E_1'^\sigma \rangle \langle E_0'^\rho E_1^\nu \rangle + \langle E_0'^\rho E_0^\mu \rangle \langle E_1'^\sigma E_1^\nu \rangle \right],
\]

diagrammatically represented by

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{diagram1.png}
\end{array}
\end{array}
\]

Dotted lines stand for 2-point correlation functions; the number of these lines indicates the order in \( \delta_p \) of the corresponding diagram.

The number of diagrams is considerably reduced by the special properties of the correlations. First, the correlator \( K_{R_1}(u) \), being an even function in \( u \), has all its odd derivatives vanishing at \( u = 0 \), \( K_{R_1}^{\alpha_1 \cdots \alpha_l}(0) = 0 \) \( (l = \text{odd}) \). Therefore, each term containing an odd correlator connecting two points ‘\( i \)’ and ‘\( j \)’ on the same ‘tree’ (i.e., already linked by a solid line) vanishes,

\[
\langle E_i^{\alpha_1 \cdots \alpha_l} E_j^{\beta_1 \cdots \beta_n} \rangle = (-1)^n \delta^4(x_i - x_j) K_{R_1}^{\alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_n}(u(2)(x_i) - u(2)(x_j))
\]

\[
= (-1)^n K_{R_1}^{\alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_n}(0) = 0 \quad (l + n = \text{odd}).
\]

Furthermore, terms with a correlation between two neighboring points ‘\( i \)’ and ‘\( j \)’ joined by a solid line, \( \langle E_i^{\alpha_1 \cdots \alpha_l} E_j^{\beta_1 \cdots \beta_n} \rangle \propto \delta^4(x_i - x_j) \), give no contribution, because the delta function has no overlap with the domain of integration \( \Omega \). An example is given by the last diagram above. The remaining non-vanishing diagrams up to second order in \( \delta_p \) (zero and one-loop diagrams) are
\[ \langle E_0 E'_0 \rangle = 1 + \frac{1}{4} \begin{array}{c} \text{square} \\ \end{array} + \frac{1}{4} \begin{array}{c} \text{X} \\ \end{array} + 2 \cdot \frac{1}{2} \begin{array}{c} \text{dashed} \\ \end{array} . \tag{34} \]

The factor 2 in front of the last term counts the number of equivalent contributions that appear when multiplying the two power series for the energies. Note that so far no special assumption about \( G^{\mu \nu} \) has been made. If the matrix of Green’s functions is symmetric, \( G^{\mu \nu} = G^{\nu \mu} \), the second and third diagrams in Eq. (34) are equal.

The energy correlator \( K_{R_2}(\mathbf{u}(x_0) - \mathbf{u}'(x_0)) = \tilde{K}_{00} \) at scale \( R_2 \) is obtained by integrating Eq. (34) over \( x_0 \):

\[ \tilde{K}_{00} = \int d^4 x_0' \langle E_0 E'_{0} \rangle . \]

By way of illustration we discuss the calculation for the second diagram in Eq. (34),

\[ \int d^4 x_0' \{ D2 \} = -K^{\mu \rho}_{00} \int d^4 x_1 \ G^{\mu \nu}(x_0 - x_1) G^{\rho \sigma}(x_0 - x_1) K^{\nu \sigma}_{11} . \]

With \( R_1 < |x_0 - x_1| < R_2 \) and \( u_{(2)} \) smooth over \( R_1 \), we can replace \( K^{\mu \rho}_{11} \) by \( K^{\nu \sigma}_{00} \) and extract it from under the integral. Inserting the explicit expression for the Green’s function, Eq. (11), and going over to spherical coordinates, one easily carries out the remaining integral:

\[ I_1 := \int d^4 x_1 G(x_0 - x_1)^2 = -\frac{S_3}{(2\pi)^4 C^2} \int_{R_1^2}^R \frac{dR^2}{R_1^2} = I \ln \left( \frac{R_2^2}{R_1^2} \right) , \tag{35} \]

where \( S_3 = 2\pi^2 \) is the surface of the unit sphere in 4 dimensions and \( I := S_3/(2(2\pi)^4 C^2) \). The other diagrams are calculated in a similar way. Collecting all the terms, one finds

\[ K_{R_2}(\mathbf{u}) = K_{R_1}(\mathbf{u}) + I \left( \frac{1}{2} K_{R_1}(\mathbf{u}) K_{R_1}(\mathbf{u}) - K_{R_1}^{\mu \rho}(\mathbf{u}) K_{R_1}^{\mu \rho}(\mathbf{u}) \right) \ln \left( \frac{R_2^2}{R_1^2} \right) , \tag{36} \]

which yields the functional RG equation for the pinning energy correlator,

\[ \frac{\partial K_R(\mathbf{u})}{\partial \ln R^2} = I \left( \frac{1}{2} K_{R}^{\mu \rho}(\mathbf{u}) K_{R}^{\mu \rho}(\mathbf{u}) - K_{R}^{\mu \rho}(\mathbf{u}) K_{R}^{\mu \rho}(\mathbf{u}) \right) , \tag{37} \]

the initial condition being given by the function \( K_{\xi}(\mathbf{u}) \), Eq. (11). This result was first obtained by Fisher \[ \text{[16]} \] for the \((4+1)\)-model by means of the replica formalism.

Eq. (37) is equivalent to a system of coupled differential equations for the expansion coefficients \( K_{R}^{\alpha_1 \cdots \alpha_{2n+1}}(0) \) (recall that all odd correlators \( K_{R}^{\alpha_1 \cdots \alpha_{2n+1}}(0) = 0 \)):

\[
\begin{align*}
\frac{dK_R(0)}{d\ln R^2} & = -\frac{1}{2} I K_0^{\alpha \beta}(0) K_0^{\alpha \beta}(0), \\
\frac{dK_{R}^{\alpha \beta}(0)}{d\ln R^2} & = 0 , \\
\frac{dK_{R}^{\alpha \beta \gamma \delta}(0)}{d\ln R^2} & = I \left( K_{R}^{\alpha \beta \rho \sigma}(0) K_{R}^{\gamma \delta \rho \sigma}(0) + K_{R}^{\alpha \gamma \rho \sigma}(0) K_{R}^{\beta \delta \rho \sigma}(0) + K_{R}^{\alpha \delta \rho \sigma}(0) K_{R}^{\beta \gamma \rho \sigma}(0) \right) , \\
\end{align*}
\tag{38}
\]

The first two coefficients are readily given by

\[ K_R(0) = K_{\xi}(0) \left( 1 - \frac{1}{2} N_\delta \ln \frac{R^2}{\xi^2} \right) , \]

\[ K_{R}^{\alpha \beta}(0) = K_{\xi}^{\alpha \beta}(0) = K_{\xi}(0) \frac{1}{\xi^2} \delta^{\alpha \beta} , \]
where $\delta^{\alpha\beta}\delta^{\alpha\beta} = N$ and Eqs. (13) and (14) have been used. Eq. (38) can be simplified, if one assumes that the tensorial structure of $K_R^{\alpha\beta\gamma\delta}(0)$ is conserved under a RG transformation. In analogy with Eq. (13), we define $\Gamma_R$ by $K_R^{\alpha\beta\gamma\delta}(0) =: \Gamma_R \Delta^{\alpha\beta\gamma\delta}$. Replacing this expression into Eq. (38), setting $\gamma = \alpha$ and $\delta = \beta$, and summing over all indices, one finds after some algebra

$$\frac{d\Gamma_R}{d\ln R^2} = (N + 8) I \Gamma_R^2 =: A_1 \Gamma_R^2,$$

(39)

with the initial condition $\Gamma_\xi = \gamma u_\xi^{N-4} \left( \frac{\xi}{\Delta} \right)^{\frac{2}{\delta_p}}$. This equation is easily integrated,

$$\frac{\Gamma_R}{\Gamma_\xi} = \frac{1}{1 - A_1 \Gamma_\xi \ln(R^2/\xi^2)},$$

(40)

the solution being valid for $\xi \leq R < R_c$. $\Gamma_R$ diverges at the collective pinning radius

$$R_c = \xi \exp \left( \frac{1}{2A_1 \Gamma_\xi} \right) = \xi \exp \left( \frac{8\pi^2}{N + 8} \left( \frac{2}{\pi} \right)^{\frac{2}{\delta_p}} \frac{1}{\delta_p} \right).$$

(41)

At that scale $K_R(u)$ stops being an analytic function of $u$ and for larger scales the system is no longer in a unique ground state. One may show by induction that all the higher derivatives, $K_R^{\alpha_1\ldots\alpha_{2n}}(0)$, $n = 3, 4, \ldots$, – if nonzero – exhibit a singularity at the same scale $R_c$. The one-loop RG result for $R_c$, Eq. (41), confirms the lowest-order perturbation calculation, Eq. (20), except for the argument of the exponential, which is smaller (for $N \leq 8$) by a factor $2N/(N + 8)$. Note that because the renormalization is arbitrarily started at the length scale $\xi$, the present RG method cannot fix the constant of proportionality in Eq. (41) just as the lowest-order perturbation calculation in section III.A. However, with this approach there is no unknown constant as in the condition $\langle u^2(R_c) \rangle \approx \xi^2$; the numerical factor inside the exponential function depends only on $\Gamma_\xi$, i.e., the precise choice for the form factor, Eq. (3). Another advantage of this definition is that it provides a possibility for computing higher-order corrections to $R_c$ as will be done in the next paragraph.

C. Two-loop FRG equation

The number of diagrams inflates with increasing order of $\delta_p$. There were only 3 non-vanishing second-order (one-loop) terms, there are already 20 non-vanishing third-order (two-loop) diagrams. With the assumption of symmetric Green’s functions, $G^{\mu\nu} = G^{\nu\mu}$, only 14 topologically different diagrams are left. Two different integral expressions occur; they are of the form

$$I_2 = \int_{\Omega_1} dx_1 G(x_0 - x_1)^2 \int_{\Omega_2} dx_2 G(x_1 - x_2)^2,$$

$$I_3 = \int_{\Omega_1} dx_1 G(x_0 - x_1)^2 \int_{\Omega_3} dx_2 G(x_0 - x_2) G(x_1 - x_2),$$

where the integration domains are such that $|x_0 - x_1|$, $|x_0 - x_2|$, and $|x_1 - x_2|$ are between $R_1$ and $R_2$. Accordingly, the two-loop diagrams can be regrouped into 2 classes; they are shown in Figs. 1 and 2. The factor in front of the diagrams enumerates the occurrence of this term in the series expansion.
Recalling that a $l$-leg bullet carries a factor $1/l!$, we calculate these diagrams as demonstrated in the previous paragraph. The third-order contributions to $K_{R_2}$, Eq. (36), are found to be

$$K_{R_2}^{2-loop}(u) = \left[ \frac{1}{2} \left( K_{R_1}^{\mu \rho}(u) - K_{R_1}^{\mu \rho}(0) \right) \left( K_{R_1}^{\kappa \tau}(u) - K_{R_1}^{\kappa \tau}(0) \right) K_{R_1}^{\mu \rho \kappa \tau}(u) \right] I_2 + \left[ \left( K_{R_1}^{\mu \rho}(u) - K_{R_1}^{\mu \rho}(0) \right) K_{R_1}^{\kappa \tau \mu}(u) K_{R_1}^{\kappa \tau \rho}(u) \right] I_3.$$  

(42)

Here, we have already taken into account the diagonal structure of the Green’s matrix, $G^{\mu \nu} = \delta^{\mu \nu} G$. In section III.B we have seen that the relevant quantity for determining $R_R$ is $\Gamma_R$, which is essentially the fourth derivative of $K(u)$. We therefore differentiate Eqs. (30) and (12) twice with respect to $u^\alpha$ and twice with respect to $u^\beta$, set $u = 0$ and use the definition of $\Gamma_R$, $K_R^{\alpha \alpha \beta \beta}(0) = \Gamma_R \Delta^{\alpha \beta} = \Gamma_R N(N + 2)$. After summing over all indices, we eventually obtain
\[ \Gamma_{R_2} = \Gamma_{R_1} + \Gamma_{R_1}(N+8)I_1 + \Gamma_{R_1}^2(N^2 + 6N + 20)I_2 + \Gamma_{R_1}^3(20N + 88)I_3. \]  

(43)

Let us compute the two integrals \( I_2 \) and \( I_3 \). \( I_2 \) is found at once, cf. Eq. (35),

\[ I_2 = (I_1)^2 = I^2 \ln^2 \left( \frac{R_2^2}{R_1^2} \right). \]

(44)

The calculation of \( I_3 \) is less trivial. It can be written in the form

\[ I_3 = \frac{1}{(2\pi)^8 C^2} \int_{\Omega_1} d^4x \int_{\Omega_2} y^2(x + y)^2, \]

where the integration domains \( \Omega_1 \) and \( \Omega_2 \) are such that \(|x|, |y|, \) and \(|x + y| \) are taken between \( R_1 \) and \( R_2 \). This integral is well known from \( \phi^4 \)-field theory [18]. In the limit \( R_1 \to 0, R_2 \to \infty \), it is logarithmically divergent, the divergent contributions being of the form \( \ln(R_2/R_1) \) and \( \ln^2(R_2/R_1) \). To extract these terms, we take \( R_2 \gg R_1 \), so that the condition \( R_1 < |x+y| < R_2 \) can be omitted with good approximation. We introduce spherical coordinates as usual, \( R := |x|, r := |y| \), and choose the \( y_4 \)-axis parallel to \( x \):

\[ I_3 \approx \frac{4\pi S_3}{(2\pi)^8 C^2} \int_{R_1^2} \frac{R_2^2}{2R} \int_{R_1^2} \frac{R_2^2}{2R} dr \int_0^\pi \frac{\sin^2 \vartheta d\vartheta}{R^2 + r^2 + 2Rr \cos \vartheta}, \quad R_2 \gg R_1. \]

The \( dr^2 \)-integration is split into two parts, \( R_1^2 < r^2 < R^2 \) and \( R^2 < r^2 < R_2^2 \); for each part the \( d\vartheta \)-integration is carried out with help of the method of residua by transforming to coordinates \( z := \exp(i\vartheta) \).

The remaining integrals pose no problem and one obtains

\[ I_3 = I^2 \left[ \ln \left( \frac{R_2^2}{R_1^2} \right) + \frac{1}{2} \ln^2 \left( \frac{R_2^2}{R_1^2} \right) + \text{const.} + O \left( \frac{R_1^2}{R_2^2} \right) \right]. \]

(45)

Inserting the expressions for \( I_2 \) and \( I_3 \), Eqs. (44) and (45) into Eq. (43), we obtain the logarithmic corrections up to two-loop approximation:

\[ \frac{\Gamma_{R_2}}{\Gamma_{R_1}} = 1 + A_1 \Gamma_{R_1} \ln \left( \frac{R_2^2}{R_1^2} \right) + A_2 \Gamma_{R_1}^2 \ln^2 \left( \frac{R_2^2}{R_1^2} \right) + A_2 \Gamma_{R_1} \ln \left( \frac{R_2^2}{R_1^2} \right), \]

(46)

with \( A_1 = (N+8)I \) and \( A_2 = (20N + 88)I^2 \). The fact that the two-loop term \( \propto \ln^2 \) is the exact square of the one-loop contribution \((A_1 \Gamma_{R_1} \ln)\) provides a check for the correctness of the one-loop RG equation, Eq. (39). Indeed, from Eq. (40) one deduces

\[ \frac{\Gamma_{R_2}}{\Gamma_{R_1}} = 1 + A_1 \Gamma_{R_1} \ln \left( \frac{R_2^2}{R_1^2} \right) + A_2 \Gamma_{R_1}^2 \ln^2 \left( \frac{R_2^2}{R_1^2} \right) + O(\delta_p^3); \]

this shows that the \( \ln^2 \)-contribution in Eq. (46) is already taken into account in the one-loop equation and need therefore not be considered explicitly. The two-loop RG equation for \( \Gamma_R \) then reads:

\[ \frac{d\Gamma_R}{d\ln R^2} = A_1 \Gamma_R^2 + A_2 \Gamma_R^3. \]

(47)

Similarly, the two-loop FRG equation for \( K_R(u) \) is inferred from Eqs. (36) and (42), by omitting all the \( \ln^2 \)-terms in \( I_2 \) and \( I_3 \):
\[
\frac{\partial K_R(u)}{\partial \ln R^2} = I \left( \frac{1}{2} K_R^{\mu \rho}(u) K_R^{\mu \rho}(u) - K_R^{\mu \rho}(u) K_R^{\mu \rho}(0) \right) + \frac{I^2}{2} \left( K_R^{\mu \rho}(u) - K_R^{\mu \rho}(0) \right) K_R^{\kappa \tau \rho}(u) K_R^{\kappa \tau \rho}(u). \tag{48}
\]

The integration of the differential equation (47),
\[
\int_{\ln \xi^2}^{\ln R^2} d \ln R^2 = \int_{\Gamma_\xi}^{\Gamma_{R}} \frac{d \Gamma}{A_1 \Gamma^2 + A_2 \Gamma^2},
\]

yields the solution
\[
\ln \left( \frac{R^2}{\xi^2} \right) = -\frac{1}{A_1 \Gamma_R} + \frac{1}{A_1 \Gamma_\xi} + \frac{A_2}{A_1^2} \ln \left( 1 + \frac{A_1}{A_2 \Gamma_R} \right) - \frac{A_2}{A_1^2} \ln \left( 1 + \frac{A_1}{A_2 \Gamma_\xi} \right).
\]

To extract \( R_c \), we take the limit \( \Gamma_R \to \infty \) and use the assumption of weak pinning, so that \( A_2 \Gamma_\xi/A_1 \propto \delta_p \ll 1 \):
\[
\ln \left( \frac{R^2}{\xi^2} \right) = \frac{1}{A_1 \Gamma_\xi} + \frac{A_2}{A_1^2} \ln \left( \frac{A_2 \Gamma_\xi}{A_1} \right).
\]

Solving with respect to \( R_c \) and inserting the explicit expressions for \( A_1, A_2, I, \) and \( \Gamma_\xi \), leads to the final result for the collective pinning radius,
\[
R_c \simeq \xi \delta_p^{\alpha_2} \exp \left( \frac{\alpha_1}{\delta_p} \right), \tag{49}
\]

where
\[
\alpha_1 = \frac{8\pi^2}{N + 8} \left( \frac{2}{\pi} \right) \frac{\delta}{\kappa}, \quad \text{and} \quad \alpha_2 = \frac{2(5N + 22)}{(N + 8)^2}.
\]

Comparison with the one-loop result, Eq. (11), shows that the two-loop calculation produces an algebraic correction with exponent \( \alpha_2 \) (the constant \( \alpha_1 \) in the exponential function remains unchanged). Finally, scaling arguments provide us with an improved result for the critical force density,
\[
F_c \simeq \frac{C}{\kappa} \left( \frac{\xi}{R_c} \right)^2 = \frac{C}{\xi} \delta_p^{-2\alpha_2} \exp \left( -\frac{2\alpha_1}{\delta_p} \right).
\]

### D. Dynamic approach and RG

In this paragraph we apply the above results to the dynamic approach introduced in Sec. III.B. We investigate the behavior of the friction coefficient \( \eta(\nu) = \eta_0 + \delta \eta(\nu) \) of Eq. (23) under the renormalization group and arrive at an estimate for the critical force density \( F_c \) in an alternative way.

In order to pave the way for our real-space RG, we rewrite Eq. (23) through the Fourier representation of the Green’s function, cf. Eq. (12),
\[
\nu^\alpha \delta \eta = \int_0^\infty dt K_\xi^{\alpha \beta}(vt) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{\eta} \exp \left( -\frac{Ck^2 t}{\eta} \right),
\]

where we have consistently replaced \( \eta_0 \) by \( \eta \). Since disorder is weak, the critical velocity \( \nu_c \) is expected to be small. Hence, we expand the correlator \( K_\xi^{\alpha \beta}(vt) \) into a power series around \( vt = 0 \) and retain the first non-vanishing term, proportional to \( \Gamma_\xi \):
\[
\delta \eta \simeq (N + 2) \frac{\Gamma_\xi}{\eta} \int \frac{d^4 k}{(2\pi)^4} \int_0^\infty dt \exp \left( -\frac{Ck^2 t}{\eta} \right).
\]
The time and angular integrations are easily carried out,
\[ \delta \eta = (N + 2) \Gamma_\xi \eta I \int \frac{dk^2}{k^2}. \]
This form is appropriate to make contact to our results. Let us derive a RG equation for the viscosity: Assume we have renormalized \( \eta \) up to a scale \( R_1 > \xi \). Going over to the larger scale \( R_2 \), the viscosity is increased by
\[ \eta_{R_2} - \eta_{R_1} = (N + 2) I \Gamma_{R_1} \eta_{R_1} \int_{1/R_2^2}^{1/R_1^2} \frac{dk^2}{k^2} = (N + 2) I \Gamma_{R_1} \eta_{R_1} \ln \left( \frac{R_2^2}{R_1^2} \right). \]
As a result, the desired one-loop RG equation for the viscosity reads
\[ \frac{d \eta_R}{d \ln R^2} = (N + 2) I \Gamma_R \eta_R. \]  
(50)
The corresponding formula for the \((4+1)\)-model has recently been derived by Leschhorn et al. \cite{17,18}, using essentially the same method. Eq. (50) is solved by separation of variables,
\[ \int_{\eta_0}^{\eta_R} \frac{d \eta}{\eta} = (N + 2) I \int_{\Gamma_\xi}^{\Gamma_R} \frac{d \Gamma}{\Gamma} \frac{d \ln R^2}{d \Gamma}. \]  
(51)
Use of the one-loop renormalization flow of \( \Gamma_R \), Eqs. (29) and (30), yields
\[ 1 + \frac{\delta \eta_R}{\eta_0} = \frac{\eta_R}{\eta_0} = \left( \frac{\Gamma_R}{\Gamma_\xi} \right)^\sigma = \left( A_1 \Gamma_\xi \ln \left( \frac{R_2^2}{R_1^2} \right) \right)^{-\sigma}, \]  
(52)
with \( \sigma = (N + 2)/(N + 8) \).
In section III.B we defined the critical force \( F_c \) as the applied force \( F_{ext} \) for which \( \delta \eta(F_{ext}) \simeq \eta_0 \), reflecting the onset of nonlinearity in the velocity-force characteristics as \( F_{ext} \) is lowered. Here, we employ an alternative criterion, which we believe is better adapted to the problem. If a (finite) critical force \( F_c > 0 \) exists, then \( \lim_{F_{ext} \to F_c^+} v(F_{ext}) = 0 \). As a consequence,
\[ \lim_{F_{ext} \to F_c^+} \eta(F_{ext}) = \lim_{v(F_{ext}) \to \infty} F_{ext} \]
and we can define \( F_c \) via the singularity of the effective viscosity \( \eta = \eta_0 + \delta \eta \). Obviously, \( \eta_R \) diverges for \( R \to R_c \), the critical force \( F_c \) is thus related to the collective pinning radius \( R_c \). From scaling we know that
\[ F_{ext} \simeq \frac{C \xi^2}{R_c^2}. \]  
(53)
Recalling Eq. (31), we thus find
\[ F_c \simeq \frac{C \xi^2}{R_c^2} = \frac{C}{\xi} \exp \left( \frac{16 \pi^2}{N + 8} \left( \frac{2}{\pi} \right) \frac{N}{\delta p} \frac{1}{\delta p} \right), \]  
(54)
which essentially agrees with the lowest-order perturbation result Eq. (29), the argument of the exponential function being smaller by a factor \( \sigma = (N + 2)/(N + 8) \). As usual, the number in front of the exponential remains unspecified. With the help of the scaling relation (53), the viscosity, see Eq. (52), can be written as a function of \( F_{ext} \),
\[ \frac{\eta(F_{ext})}{\eta_0} = \left( A_1 \Gamma_\xi \ln \frac{F_{ext}}{F_c} \right)^{-\sigma} = \left[ 1 + \frac{N + 8}{16 \pi^2} \left( \frac{\pi}{2} \right) \frac{N}{\delta p} \ln \left( \frac{C}{\xi F_{ext}} \right) \right]^{-\sigma}, \]  
(55)
providing us with an explicit form of the velocity-force characteristics \( v = F_{ext}/\eta(F_{ext}) \) versus \( F_{ext} \). Whereas the lowest-order perturbation result, Eq. (28), is only valid for applied forces \( F_c \ll F_{ext} < C/\xi \), this expression holds down to much smaller forces and thus better describes the non-linear regime. Expanding Eq. (28) for small \( \delta p \) and comparing the resulting series with Eq. (28), one can check that both expressions coincide (for \( F_{ext} \ll C/\xi \)) up to first order in \( \delta p \) as they ought to.
V. SUMMARY AND CONCLUSION

Using a RG approach, we have calculated the collective pinning radius \( R_c \) as well as the critical force density \( F_c \) for a \((4+N)\)-dimensional elastic manifold subject to point defects. We have confined ourselves to the case of zero temperature and have assumed weak and short-range disorder of Gaussian type.

By means of a real-space renormalization procedure a functional RG equation for the pinning energy correlator \( K_R(u) \) was found in one and two-loop approximation. It has been shown that the fourth derivative of that correlator taken at zero \( u \), \( K^{\alpha\beta\gamma\delta}_R(0) = \Gamma_R^{\alpha\beta\gamma\delta} \), possesses a singularity at some finite scale of renormalization which we have identified with the collective pinning radius \( R_c \). To one-loop correction we have derived the expression

\[
R_c = \alpha_0 \xi \exp(\alpha_1/\delta_p), \quad \alpha_1 = \left(\frac{2}{\pi}\right)^{N/2} \frac{8\pi^2}{(N+8)}, \quad \text{Eq. (41)}
\]

that confirms the result obtained by a simple lowest-order perturbation calculation, Eq. (20), the argument of the exponential function differing by a factor \( \frac{2N}{(N+8)} \). In two-loop approximation we have found an additional algebraic factor \( \propto \delta_p^{\alpha_2} \) with an exponent \( \alpha_2 = \frac{2(5N+22)}{(N+8)^2} \), see Eq. (49). Contrary to the perturbation calculation, the RG approach succeeds in fixing both \( \alpha_1 \) and \( \alpha_2 \). The constant of proportionality \( \alpha_0 \), however, remains undetermined in either of the two methods.

Next, we have derived a RG equation for the viscosity \( \eta \) that appears when driving the system by an external force, cf. Eq. (50). The previously obtained result for \( \Gamma_R \) has allowed us to solve this equation, see Eq. (55). In comparison with the lowest-order perturbation result, Eq. (28), the RG solution provides us with an improved description of the velocity-force characteristics in the non-linear regime where the disorder potential considerably influences the vortex motion. We have defined the critical force \( F_c \) via the singularity of the viscosity \( \eta \) and indicated how \( F_c \) and \( R_c \) are both related. As was the case for \( R_c \), the one-loop expression we have found for \( F_c \) agrees with the result obtained by means of dimensional estimates and the dynamical approach, Eqs. (21) and (24) respectively, except that the number in the exponential function is now fixed.

The improved result for \( R_c \) provides a firm starting point for the scaling analysis in the random manifold regime (large scales \( R > R_c \)). In a forthcoming publication [20], we apply the same method to the physical case of a three-dimensional system of vortices in type-II superconductors with dispersive elastic moduli and gain quantitative refinements of the weak collective pinning theory.

ACKNOWLEDGMENTS

The authors are grateful to G.T. Zimanyi, L. Balents, M. Mégard, T. Giamarchi, and P. le Doussal for valuable discussion on the subject and gratefully acknowledge the support by the Swiss National Foundation.
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