Inequalities for the Ranks of Quantum States

Josh Cadney,1 Marcus Huber,2,1,3 Noah Linden,1 and Andreas Winter4,2,1

1School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom
2Física Teórica: Informació i Fenomens Quàntics,
Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain
3ICFO-Institut de Ciencies Fotòniques, 08860 Castelldefels, Barcelona, Spain
4ICREA – Institució Catalana de Recerca i Estudis Avançats,
Pg. Lluís Companys 23, ES-08010 Barcelona, Spain

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We investigate relations between the ranks of marginals of multipartite quantum states. These are the Schmidt ranks across all possible bipartitions and constitute a natural quantification of multipartite entanglement dimensionality. We show that there exist inequalities constraining the possible distribution of ranks. This is analogous to the case of von Neumann entropy (α-Rényi entropy for α = 1), where nontrivial inequalities constraining the distribution of entropies (such as e.g. strong subadditivity) are known. It was also recently discovered that all other α-Rényi entropies for α ∈ (0, 1) ∪ (1, ∞) satisfy only one trivial linear inequality (non-negativity) and the distribution of entropies for α ∈ (0, 1) is completely unconstrained beyond non-negativity. Our result resolves an important open question by showing that also the case of α = 0 (logarithm of the rank) is restricted by nontrivial linear relations and thus the cases of von Neumann entropy (i.e., α = 1) and 0-Rényi entropy are exceptionally interesting measures of entanglement in the multipartite setting.

I. INTRODUCTION

Entanglement is ubiquitous in multi-party quantum systems; today it is recognized to play a fundamental role as a resource in quantum information theory. Therefore there have been intensive investigations into quantifying this resource in an operational way (cf. [1] for a recent review). Apart from its applications in metrology and communication it is also necessary for exponential speedup in quantum computation [2]. Many continuous measures of entanglement however may be polynomially small in the size of the system while still admitting an exponential speedup [3]. In this scenario the Schmidt rank constitutes a natural measure of bipartite entanglement, but more importantly in terms of a resource theory it reveals the minimum entanglement dimensionality required to create a state via stochastic local operations and classical communication (SLOCC) [4]. The Schmidt rank is the number of non-zero terms in the Schmidt decomposition of $|\psi\rangle_{AB}$, or more simply, the minimum number of terms in a decomposition of $|\psi\rangle_{AB} = \sum_{i=1}^{r_A} |\alpha_i\rangle_A |\beta_i\rangle_B$ into a sum of product vectors. Equivalently, it is the rank of the reduced density matrix $\rho_A := \text{Tr}_B |\psi\rangle\langle\psi|$.

Recently this idea has been generalized to the multipartite setting by the introduction of the rank vector [2, 8]. The rank vector is a list of the Schmidt ranks across each possible bipartition of the state. Each element of this vector constitutes an SLOCC monotone and it can be used to reveal the dimensionality of genuine multipartite entanglement. For example, a tripartite state $|\psi\rangle_{ABC}$ has three bipartitions, $A : BC$, $B : AC$ and $AB : C$, giving three different ranks: $r_A$, $r_B$ and $r_{AB}$. It is then natural to ask what relations these ranks must satisfy. For example, the inequality

$$r_{AB} \leq r_A r_B$$

is a simple consequence of the relation

$$\text{supp}(\psi_{AB}) \subseteq \text{supp}(\psi_A) \otimes \text{supp}(\psi_B),$$

where supp(σ) denotes the support of the density matrix σ. In [2] it is shown that for tripartite pure states this is the only relevant constraint on the ranks, however, it is conjectured that for four and more parties there are further inequalities. Finding these would not only reveal a great amount of structure in multipartite Hilbert spaces, but also give a rich set of constraints for distributing entanglement in multipartite systems in the flavour of generalized monogamy relations. For example, the inequality above makes an interesting physical statement: The product of the entanglement dimensionality of $A$ with $BC$ and the entanglement dimensionality of $B$ with $AC$, is an upper bound to the entanglement dimensionality of $C$ with $AB$. 


References

[1] D. Petz, Rev. Math. Phys. 20, 1187 (2008).
[2] M. Huber and A. Winter, J. Math. Phys. 55, 032207 (2014).
[3] D. Aharonov and J. Anshu, Quantum Inf. Comput. 12, 1171 (2012).
[4] M. Christandl, R. König, R. Renner, and L. Vazirani, Comm. Math. Phys. 285, 317 (2009).
[5] J. Preskill, Quantum 4, 380 (2014).
[6] M. Dür, H. Briegel, J. Cirac, and P. Zoller, Phys. Rev. A 62, 062314 (2000).
[7] R. Renner and H. Winter, J. Math. Phys. 50, 042201 (2009).
[8] M. Halbmeier, M. Huber, and A. Winter, Phys. Rev. A 88, 052328 (2013).
An at first glance quite separate research topic is the study of entropy inequalities, which is important in both classical and quantum information theory. Given a multipartite system, how do the entropies of the different subsystems relate to each other? For the case of the von Neumann entropy, \( S(\rho) = -\text{Tr}\rho \log \rho \), they must satisfy the well-known strong subadditivity and weak monotonicity relations.

\[
S(\rho_A) + S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{AC}), \quad S(\rho_A) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC}).
\]

For four-party pure states these are the only constraints, but it is a major open question to determine whether or not further inequalities exist for five or more parties. Interestingly, the key ingredient in our proofs is the strong subadditivity of the von Neumann entropy. We then construct some states with interesting ranks, which violate some other inequalities, including one conjectured in \([5]\) (section IV). Finally, we present another inequality, which, if true, would complete the picture in the four-party case. We have so far been unable to find a general proof of this inequality, or a counterexample for that matter, but we present a proof for certain special cases (section VI).}

We conclude in section VI, referring several open problems to the attention of the reader.
Similarly, the 0-entropy vector of $|\psi\rangle_{ABCD}$ is

$$v_\psi = (\log r_A, \log r_B, \log r_C, \log r_D, \log r_{AB}, \log r_{AC}, \log r_{AD})$$  \hspace{1cm} (10)

We are interested in determining which vectors are rank/0-entropy vectors. Therefore, we define a set $\Sigma_n$ which is the set of all $n$-party rank vectors. More precisely,

$$\Sigma_n = \left\{ u \in \mathbb{N}^{2^n-1} : \exists d_1, \ldots, d_n \in \mathbb{N}, |\psi\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n} \text{ such that } u = r_\psi \right\}$$  \hspace{1cm} (11)

and in a completely analogous way we define $\Omega_n$ to be the set of all $n$-party 0-entropy vectors.

For $0 < \alpha \leq 1$ it is known that, for any number of parties, the closure of the set of $\alpha$-entropy vectors is a convex cone (i.e. it is closed under addition and multiplication by positive real scalars) \cite{16}. This means that it can be characterized only in terms of the linear inequalities between the $\alpha$-entropy of different parts of system. Since $\Omega_n$ is a discrete set it is clearly not a cone. However, the results of \cite{2} imply that $\Omega_3$ is the intersection of a cone with the set of log-integer points. In particular, let $C$ be the closed cone defined by

$$C := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, x \leq y + z, y \leq x + z, z \leq x + y\}$$  \hspace{1cm} (12)

and let $\log \mathbb{N}^3$ denote the set of log-integer points, then $\Omega_3 = C \cap \log \mathbb{N}^3$. For example, if $a, b, c \in \mathbb{N}$ are such that $(\log a, \log b, \log c) \in C$ with $a \leq b \leq c$ then by taking the state

$$|\psi\rangle_{ABC} = \sum_{i=1}^a \sum_{j=1}^b |i\rangle_A |j\rangle_B |(i-1)b + j\rangle_C,$$  \hspace{1cm} (13)

and removing appropriate terms from the sum, we can arrive at a state with rank vector $(a, b, c)$.

It turns out that a similar statement does not hold for $\Omega_n$ with $n \geq 4$. However, it is true that $\Omega_n$ is still closed under addition, and hence also multiplication by positive integer scalars. For, if $|\psi\rangle$ and $|\phi\rangle$ have 0-entropy vectors $v_\psi$ and $v_\phi$ respectively, then the state $|\psi\rangle \otimes |\phi\rangle$ has 0-entropy vector $v_\psi + v_\phi$. This is one reason why we will be principally concerned with inequalities which are linear in the 0-entropies of a state, and therefore geometric in the ranks.

**Remark.** The theory outlined above has a classical counterpart, which has been studied in the past: Namely, to the joint distribution $P_{[n]}$ of $n$ discrete random variables $X_1, \ldots, X_n$, associate for each subset $I \subset [n]$ the cardinality $s_I$ of the support of $P_I$, which is the marginal distribution of $X_I = (X_i : i \in I)$, so that $\log s_I = H_0(X_I)$ is the classical 0-Rényi entropy of the variables $X_I$. Another way of looking at it is to observe that $s_I$ is precisely the size of the projection of the support of $P_{[n]}$ onto the coordinates $I$.

By introducing the $(n+1)$-party purification

$$|\psi\rangle_{012\ldots n} = \sum_{x_1\ldots x_n} \sqrt{P(x_1\ldots x_n)} |x_1\ldots x_n\rangle^0 |x_1\rangle^1 \cdots |x_n\rangle^n,$$  \hspace{1cm} (14)

we see that the $s_I, I \subset [n]$ are precisely the Schmidt ranks $r_I$ of this state, so that the classical case actually appears a special case of the quantum one, for states of the form \cite{14}.

What are the inequalities satisfied by the $s_I$ of a generic distribution? By the above observation, any constraint on the quantum ranks must necessarily hold for the classical supports, for instance submultiplicativity $s_{I\cup J} \leq s_I s_J$. But there is evidently more, such as monotonicity, i.e. $s_I \leq s_J$ for all $I \subset J$, which does not hold for the quantum ranks $r_I$. Further, for each $J \subset [n]$ and each $1 \leq k \leq |J|$ the following inequality holds \cite{18}

$$s_J^{(\frac{|J|-1}{k-1})} \leq \prod_{|J|=k} s_J,$$  \hspace{1cm} (15)

where the product is taken over all subsets $I \subset J$ of size $k$. Theorem 2 below shows that the case $|J| = 3$, $k = 2$ is true also for quantum ranks.
III. NEW RANK INEQUALITIES

In this section we prove two new inequalities for the ranks of a four-party state.

**Theorem 1.** Let $|\psi\rangle_{ABCD}$ be a four-party quantum state. Then $r_A \leq r_{AB} r_{AC}$.

*Proof.* By (2) we see that projecting $\mathcal{H}_A$ onto supp($\psi_A$) does not change $|\psi\rangle_{ABCD}$. Consequently we may assume, without loss of generality, that the single-party density matrix $\rho_A$ has full rank, i.e. $r_A = d_A$. The Schmidt decomposition across the $A:BCD$ partition then gives us $|\psi\rangle_{ABCD} = \sum_{i=1}^{d_A} \lambda_i |\alpha_i\rangle_A \otimes |\beta_i\rangle_{BCD}$, where $\{ |\alpha_i\rangle \}_{i=1}^{d_A}$ is an orthonormal basis of $\mathcal{H}_A$, the states $|\beta_i\rangle$ are also orthonormal, and $\lambda_i > 0$ for all $i$.

Consider the operator $\psi_A^{-\frac{1}{2}} = \sum_{i=1}^{d_A} \lambda_i^{-1} |\alpha_i\rangle \langle \alpha_i|$. It is an invertible linear map on $\mathcal{H}_A$. Therefore consider the operator $X := \psi_A^{-\frac{1}{2}} \otimes \mathbb{1}_B \otimes \mathbb{1}_C \otimes \mathbb{1}_D$, where $\mathbb{1}_B$ denotes the identity on $\mathcal{H}_B$, is a local invertible linear map on $\mathcal{H}_{ABCD}$, which will not change any of the ranks. Let $|\psi'\rangle_{ABCD} := \frac{1}{\sqrt{d_A}} X |\psi\rangle_{ABCD} = \sum_{i=1}^{d_A} \frac{1}{\sqrt{d_A}} |\alpha_i\rangle_A \otimes |\beta_i\rangle_{BCD}$.

The von Neumann entropy of the reduced state $\psi'_A$ is $\log d_A = \log r_A$. The strong subadditivity of von Neumann entropy (3) then gives

$$\log r_A = S(\psi'_A) \leq S(\psi'_{AB}) + S(\psi'_{AC}) - S(\psi'_{ABC}) \leq \log r_{AB} + \log r_{AC},$$

where we have used the fact that for any density matrix $\rho$ with rank $r$, $S(\rho) \leq \log r$. Taking the exponential of each side of this inequality gives the result. □

**Theorem 2.** Let $|\psi\rangle_{ABCD}$ be a four-party quantum state. Then $r_A^2 \leq r_{AB} r_{AC} r_{BC}$.

*Proof.* We follow the proof above, but this time we take the entropy inequality (4) and add to it the further inequality

$$S(\psi'_A) \leq S(\psi'_{BC}) + S(\psi'_{ABC}),$$

which is derived from (3) by taking the $B$ system to be trivial. This yields

$$2 \log r_A = 2S(\psi'_A) \leq S(\psi'_{AB}) + S(\psi'_{AC}) + S(\psi'_{BC}) \leq \log r_{AB} + \log r_{AC} + \log r_{BC}.$$

Again, taking exponentials gives the result. □

We have given a simple proof of each of the inequalities above. However, the crucial ingredient underpinning these proofs is the strong subadditivity of von Neumann entropy. This is a very famous, and highly non-trivial result. Although many proofs of strong subadditivity are known (e.g. [13, 19–21]), none provides any intuition regarding the ranks we consider. Furthermore, the argument we have employed is limited by the fact that by a local invertible filtering one can make only one reduced state equal to the maximally mixed state; indeed, if it were possible to find, for given $|\psi\rangle_{ABCD}$, another state $|\psi''\rangle_{ABCD}$ with the same Schmidt ranks for all bipartite partitions, but such that $\psi'_A$ and $\psi''_A$ are maximally mixed on their respective subsystems, then strong subadditivity (3) and weak monotonicity (4) would hold for $S_0$ – and we know that already to be false. It would, therefore, be desirable to give direct, self-contained proofs of Theorems 1 and 2. We are able to do so for the former.

**Direct proof of Theorem 1.** We begin with some preliminaries. Suppose that $|\psi\rangle_{AB}$ is a bipartite quantum state, and fix orthonormal bases $\{ |i\rangle_A \}_{i=1}^{d_A}$ and $\{ |j\rangle_B \}_{j=1}^{d_B}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$. Then, for some complex $\alpha_{ij}$, we can write

$$|\psi\rangle_{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} |ij\rangle_{AB} = \sum_{i=1}^{d_A} |i\rangle_A \otimes \left( \sum_{j=1}^{d_B} \alpha_{ij} |j\rangle_B \right) = \sum_{i=1}^{d_A} |i\rangle_A \otimes |\alpha_i\rangle_B.$$

(19)
where we define \(|\alpha_i\rangle_B := \sum_{j=1}^{d_B} \alpha_{ij} |j\rangle_B\). Now let \(M : \mathcal{H}_A \rightarrow \mathcal{H}_B\) be the linear map such that \(M |i\rangle_A = |\alpha_i\rangle_B\). Then we can write
\[
|\psi\rangle_{AB} = \sum_{i=1}^{d_A} |i\rangle_A \otimes M |i\rangle_A = (\mathbb{1}_A \otimes M) |\phi^+\rangle_{AA},
\]
(20)
where \(|\phi^+\rangle_{AA} := \sum_{i=1}^{d_A} |i\rangle_A |i\rangle_A\) is the (unnormalized) maximally entangled state between two copies of \(\mathcal{H}_A\).

We wish to calculate the reduced density matrix \(\psi_B := \Tr_A \psi\), which we can do as follows
\[
\psi_B = \sum_{j=1}^{d_A} (|j\rangle_A \otimes \mathbb{1}_B) |\psi\rangle_{AB} (|j\rangle_A \otimes \mathbb{1}_B)
= \sum_{j=1}^{d_A} (|j\rangle_A \otimes \mathbb{1}_B)(|i\rangle_A \otimes M |i\rangle_A)(|j\rangle_A \otimes M^\dagger)(|j\rangle_A \otimes \mathbb{1}_B)
= \sum_{j=1}^{d_A} \delta_{ij} \delta_{i'j} M |i\rangle_A M^\dagger
= \sum_{j=1}^{d_A} M |j\rangle_A M^\dagger = M \left( \sum_{j=1}^{d_A} |j\rangle_A \right) M^\dagger = MM^\dagger.
\]
(21)
From this we can conclude that the rank of \(|\psi\rangle_{AB}\) across the \(A : B\) partition is rank \(M\).

We now return to the problem at hand, in which we have a four-party state \(|\psi\rangle_{ABCD}\). Taking the Schmidt decomposition across the \(AB : CD\) partition gives
\[
|\psi\rangle_{ABCD} = \sum_{k=1}^{r_{AB}} |\eta_k\rangle_{AB} \otimes |\theta_k\rangle_{CD},
\]
(22)
where \(|\eta_k\rangle_{AB}\) and \(|\theta_k\rangle_{CD}\) are sets of mutually orthogonal (unnormalized) states. According to eq. (20), write \(|\eta_k\rangle_{AB} = (|\eta_k\rangle_A \otimes R_k |\phi^+\rangle_{AA}\) and \(|\theta_k\rangle_{CD} = (|\theta_k\rangle_C \otimes S_k |\phi^+\rangle_{CC}\). Then we have
\[
|\psi\rangle_{ABCD} = \sum_{k=1}^{r_{AB}} \sum_{i=1}^{d_A} \sum_{j=1}^{d_C} |i\rangle_A \otimes R_k |i\rangle_A \otimes |j\rangle_C \otimes S_k |j\rangle_C.
\]
(23)
By the argument above, this gives
\[
r_A = \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes |\theta_k\rangle_{CD} \right),
\]
(24)
r_{AC} = \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes S_k \right).
(25)
Now consider the following observation
\[
\sum_{k=1}^{r_{AB}} R_k \otimes |\theta_k\rangle_{CD} = \left( \sum_{k=1}^{r_{AB}} R_k \otimes \mathbb{1}_C \otimes S_k \right) \left( |\mathbb{1}_A \otimes |\phi^+\rangle\rangle_{CC} \right),
\]
(26)
which implies
\[
r_A = \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes |\theta_k\rangle_{CD} \right) \leq \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes \mathbb{1}_C \otimes S_k \right) = r_{AC} \text{rank}(\mathbb{1}_C).
\]
(27)
Notice that if we could replace rank(\(\mathbb{1}_C\)) by a term less than or equal to \(r_{AB}\) then we would be done. This is what the following lemma allows us to do.
Lemma 3. Suppose $S_1, \ldots, S_N$ are (non-zero) linear maps $S_i : \mathcal{H}_C \rightarrow \mathcal{H}_D$ such that $\text{Tr}(S_i^\dagger S_j) = 0$ whenever $i \neq j$. Then for some $K \leq N$ there is a rank $K$ projector $P$ on $\mathcal{H}_C$, and a state $|\phi\rangle_{CC}$, such that the vectors $\{(P \otimes S_k) |\phi\rangle_{CC}\}_{k=1}^N$ are linearly independent.

Proof. Fix bases $\mathcal{B}_C = \{|i\rangle_C\}_{i=1}^d$ and $\mathcal{B}_D = \{|j\rangle_D\}_{j=1}^d$. Let $|\psi_k\rangle_{CD} := (P \otimes S_k) |\phi\rangle_{CC}$, where $P := \sum_{i=1}^N |i\rangle_C \langle i|_C$, and $|\phi\rangle_{CC} := \sum_{i=1}^N |ii\rangle_{CC}$. Then the $|\psi_k\rangle$ are linearly dependent if and only if there exist constants $\lambda_1, \ldots, \lambda_N$, not all zero, such that:

$$\sum_{k=1}^N \lambda_k |\psi_k\rangle_{CD} = 0 \iff \sum_{k=1}^N \sum_{i=1}^K |i\rangle_C \otimes \lambda_k S_k |i\rangle_C = 0$$

$$\iff \sum_{k=1}^N \lambda_k S_k |i\rangle_C = 0 \quad \forall 1 \leq i \leq K$$

$$\iff \text{The matrices formed by taking the first } K \text{ columns of each of } S_1, \ldots, S_N \text{ (when written with respect to } \mathcal{B}_C \text{ and } \mathcal{B}_D) \text{ are linearly dependent.}$$

We proceed by induction. The lemma clearly holds for $N = 1$. Suppose that it is true for $N = m$. Then there exists $\hat{K} \leq m$ such that the first $\hat{K}$ columns of the matrices $S_1, \ldots, S_m$ are linearly independent. Write $A^{(\hat{K})}$ for the first $\hat{K}$ columns of the matrix $A$. Consider matrix $S_{m+1}$. If $S_{m+1}^{(\hat{K})}$ is linearly independent of $S_1^{(\hat{K})}, \ldots, S_m^{(\hat{K})}$ then we are done. Otherwise we have unique constants $\mu_1, \ldots, \mu_m$ such that $S_{m+1}^{(\hat{K})} = \sum_{k=1}^m \mu_k S_k^{(\hat{K})}$. Because the $S_k$ are orthogonal under the Hilbert-Schmidt inner product, they must be linearly independent, and so we cannot have $S_{m+1} = \sum_{k=1}^m \mu_k S_k$. Therefore, there must be a column in which equality does not hold. By relabelling basis vectors, we may assume that this is the $(\hat{K} + 1)$-th column. Because of the uniqueness of the $\mu_k$ it follows that $S_1^{(\hat{K}+1)}, \ldots, S_{m+1}^{(\hat{K}+1)}$ are linearly independent.

In order to make use of this lemma, we first observe that $S_1, \ldots, S_{RA}$ are orthogonal under the Hilbert-Schmidt inner product. Indeed, since $\{\theta_k\}_{k=1}^{RA}$ are mutually orthogonal states, for $i \neq j$ we have

$$0 = \langle \theta_i | \theta_j \rangle = \sum_{k,l=1}^{dC} \langle (k)_C | (k)_C S_i^\dagger |l\rangle_C \langle l | S_j |l\rangle_C$$

$$= \sum_{k,l=1}^{dC} \delta_{kl} \langle (k)_C S_i^\dagger |S_j |l\rangle_C = \text{Tr}(S_i^\dagger S_j).$$

Let $K$, $P$ and $|\phi\rangle_{CC}$ be given by Lemma 3. Then we can easily construct a linear map $V_{CD} : \mathcal{H}_C \otimes \mathcal{H}_D \rightarrow \mathcal{H}_C \otimes \mathcal{H}_D$ which sends $(P \otimes S_k) |\phi\rangle_{CC}$ to $(1_C \otimes S_k) |\phi^+\rangle_{CC}$ for each $k$. Then we can write the following:

$$\sum_{k=1}^{RA} R_k \otimes |\theta_k\rangle_{CD} = (1_B \otimes V_{CD}) \left( \sum_{k=1}^{RA} R_k \otimes P \otimes S_k \right) (1_A \otimes |\phi\rangle_{CC}).$$

This implies

$$r_A \leq \text{rank} \left( \sum_{k=1}^{RA} R_k \otimes P \otimes S_k \right)$$

$$= K \text{ rank} \left( \sum_{k=1}^{RA} R_k \otimes S_k \right) \leq r_{AB} r_{AC},$$

and we are done. \[\square\]
IV. EXTREMAL STATES

Having established two further inequalities for the ranks of multiparty states, it is natural to ask whether there are any more. At this point it is useful to switch to the perspective of the 0-Rényi entropy, by taking the logarithm of the ranks. In this scenario, our inequalities become linear constraints on the set of possible entropy vectors.

Focusing on the four-party case, we have the following constraints:

\[
S_0(A) \geq 0, \\
S_0(A) + S_0(B) \geq S_0(AB), \\
S_0(AB) + S_0(AC) \geq S_0(A), \\
S_0(AB) + S_0(AC) + S_0(BC) \geq 2S_0(A),
\]

and all inequalities by permuting the names of the parties, and more generally by substituting pairwise disjoint subsets of parties for \( A, B, C \).

We can use symbolic computation software, for instance LRS \cite{22} to find the extremal rays of the cone determined by these inequalities. We obtain eight families of rays (up to permuting the parties) spanned by the vectors listed below.

| Family | \( A \) | \( B \) | \( C \) | \( D \) | \( AB \) | \( AC \) | \( AD \) |
|--------|--------|--------|--------|--------|--------|--------|--------|
| 1      | 1      | 1      | 0      | 0      | 0      | 0      | 1      |
| 2      | 1      | 1      | 1      | 1      | 2      | 2      | 2      |
| 3      | 1      | 1      | 1      | 1      | 1      | 1      | 1      |
| 4      | 2      | 2      | 1      | 1      | 2      | 1      | 1      |
| 5      | 2      | 2      | 2      | 1      | 2      | 1      | 1      |
| 6      | 3      | 3      | 3      | 1      | 2      | 2      | 2      |
| 7      | 2      | 2      | 2      | 1      | 3      | 1      | 1      |
| 8      | 1      | 1      | 1      | 1      | 2      | 1      | 0      |

If we can find states with 0-entropy vectors on, or arbitrarily close to these rays, then we can conclude that there are no more linear inequalities. Unfortunately, we were only able to do so for the first six families. Consider the following states:

\[
|\psi_1\rangle_{ABCD} = (|00\rangle_{AB} + |11\rangle_{AB})|00\rangle_{CD},
\]

\[
|\psi_2\rangle_{ABCD} = \sum_{i,j=0}^{2} |i\rangle_A |j\rangle_B |i+j\rangle_C |i+2j\rangle_D,
\]

where, in the second state, addition takes place modulo 3. These states have rank vectors:

\[
r_{\psi_1} = (2, 2, 1, 1, 1, 2, 2),
\]

\[
r_{\psi_2} = (3, 3, 3, 9, 9, 9),
\]

from which it follows that their 0-entropy vectors lie on rays 1 and 2. Now consider the states:

\[
|\psi_3\rangle_{ABC} = |\phi_d^+\rangle_{AB} \oplus |\phi_d^+\rangle_{CD},
\]

\[
|\psi_4\rangle_{A_1A_2B_1B_2C_1C_2D} = |\phi_d^+\rangle_{A_1C_1} |\phi_d^+\rangle_{A_2D} \oplus |\phi_d^+\rangle_{B_1C_1} |\phi_d^+\rangle_{B_2D} ;
\]

\[
|\psi_5\rangle_{A_1A_2B_1B_2C_1C_2D} = |\phi_d^+\rangle_{A_1C_1} |\phi_d^+\rangle_{A_2D} \oplus |\phi_d^+\rangle_{B_1C_1} |\phi_d^+\rangle_{B_2D} \oplus |\phi_d^+\rangle_{A_1C_1} |\phi_d^+\rangle_{B_1C_2} |\phi_d^+\rangle_{B_2D} ;
\]

\[
|\psi_6\rangle_{A_1A_2A_3B_1B_2B_3C_1C_2C_3D} = |\phi_d^+\rangle_{A_1B_1} |\phi_d^+\rangle_{A_2C_1} |\phi_d^+\rangle_{A_3D} \oplus |\phi_d^+\rangle_{A_1B_2} |\phi_d^+\rangle_{A_2C_2} |\phi_d^+\rangle_{A_3D} \oplus |\phi_d^+\rangle_{A_1B_3} |\phi_d^+\rangle_{A_2C_3} |\phi_d^+\rangle_{A_3D} \\
+ |\phi_d^+\rangle_{A_1C_1} |\phi_d^+\rangle_{B_1C_2} |\phi_d^+\rangle_{B_2C_3} |\phi_d^+\rangle_{B_3D} .
\]

Here, we have used some notation that requires explanation. As before, the state \( |\phi_d^+\rangle_{AB} \) is \( \sum_{i=1}^d |i\rangle_A |i\rangle_B \).

Where a system is missing from a state, it is assumed to be present, but in an unentangled pure state.
Finally, the orthogonal sum of two states, denoted $|\psi\rangle \oplus |\eta\rangle$, is the superposition of $|\psi\rangle$ and $|\eta\rangle$ embedded in orthogonal parts of the local Hilbert spaces. For example,

$$|\phi^+_d\rangle_{AB} = |\phi^+_d\rangle_{AB} |11\rangle_{CD} = \sum_{i=1}^{d} |ii11\rangle_{ABCD},$$

$$|\phi^+_d\rangle_{CD} = |11\rangle_{AB} |\phi^+_d\rangle_{CD} = \sum_{i=1}^{d} |11ii\rangle_{ABCD},$$

and we could have

$$|\psi_3\rangle_{ABCD} = \sum_{i=1}^{d} |ii11\rangle_{ABCD} + \sum_{i=1}^{d} (d+1)_{A}|d+i\rangle_{B}|d+i\rangle_{C}|d+i\rangle_{D}. \quad (37)$$

Notice that if states $|\psi\rangle$ and $|\eta\rangle$ have rank vectors $r_\psi$ and $r_\eta$, then $|\psi\rangle \oplus |\eta\rangle$ has rank vector $r_\psi + r_\eta$. Using this fact, we can see that states $|\psi_3\rangle, \ldots, |\psi_6\rangle$ have rank vectors:

$$r_{\psi_3} = (d+1, d+1, d+1, d+1, 2d, 2d),$$

$$r_{\psi_4} = (d^2+1, d^2+1, 2d, 2d, 2d^2, 2d, 2d),$$

$$r_{\psi_5} = (d^2+d+1, d^2+d+1, d^2+2d, 2d+1, 3d^2, 3d, 3d),$$

$$r_{\psi_6} = (d^3+2d, d^3+2d, d^3+2d, 3d^3, 3d^3, 3d^3, 3d^3). \quad (38)$$

It is clear that for large $d$, the leading term in each component will dominate, and for the 0-entropy (once we have taken the log) only the leading exponent matters. More precisely, let $v_\psi$ be the 0-entropy vector of state $|\psi\rangle$. Then, $\lim_{d \to \infty} \frac{1}{\log d} v_\psi$ is the $i$th vector in the table above, for each $i = 3, 4, 5, 6$.

For these four rays, we were unable to find states with 0-entropy vectors actually on the ray. Interestingly, in the case of ray 3, it is easy to see that such a state cannot be found. This is because, for any state $|\psi\rangle$ on the ray, $S_0(AD) = 0$, which implies that the rank across the $AD : BC$ partition is 1, so we can write $|\psi\rangle_{ABCD} = |\eta\rangle_{AD} |\theta\rangle_{BC}$. It follows that $r_{AB} = r_{A|r_B}$ and so $S_0(AB) = S_0(A) + S_0(B)$, which does not hold for a state on ray 3. Similarly, one can show that it is not possible to find states on ray 6, but we do not know this for rays 4 and 5.

**Remark.** Notice that for large enough $d$, $|\psi_3\rangle, |\psi_4\rangle, |\psi_5\rangle$ and $|\psi_6\rangle$ all violate the inequality

$$r_{A|r_B r_C} \leq r_{A|r_B r_C r_D},$$

which had been conjectured in [3]. Furthermore, all four provide further counterexamples to strong subadditivity.

**V. ANOTHER INEQUALITY?**

In the previous section we took our known inequalities for the 0-entropy, and computed the corresponding set of extremal rays. We then demonstrated that 6 of the 8 families of these extremal rays can be approximated by 0-entropy vectors. However, we were unable to find such a construction for rays 7 and 8.

Now we apply the process in reverse. We have found 6 families of 0-entropy vectors which are extremal rays. We can again use the LRS software to compute the set of inequalities to which they correspond. It turns out that this set is just the known inequalities, with one further inequality added, which we present as an hypothesis.

**Hypothesis 1.** Let $|\psi\rangle_{ABCD}$ be a four-party quantum state. Then $r_{BC} \leq r_{AB r_A r_C} r_{AD}$.

If we could prove this hypothesis, then we would have a complete picture of the four-party linear inequalities for the 0-entropy.

Before presenting some partial results towards this hypothesis, we first write it in a different form.
Hypothesis 2. Let $R_1, \ldots, R_K$ be $m_1 \times n_1$ complex matrices, and let $S_1, \ldots, S_K$ be $m_2 \times n_2$ complex matrices. Then

$$\text{rank} \left( \sum_{i=1}^{K} R_i \otimes S^T_i \right) \leq K \text{rank} \left( \sum_{i=1}^{K} R_i \otimes S_i \right). \quad (40)$$

Lemma 4. Hypothesis [1] is equivalent to Hypothesis [2].

Proof. $\implies$ Let $|\psi\rangle_{ABCD}$ be a four-party quantum state, and let us write it in the same way as (23):

$$|\psi\rangle_{ABCD} = \sum_{k=1}^{r_{AB}} \sum_{i=1}^{d_A} \sum_{j=1}^{d_C} |i\rangle_A \otimes R_k |i\rangle_A \otimes |j\rangle_C \otimes S_k |j\rangle_C. \quad (41)$$

Now, instead of considering $R_k, S_k$ as linear maps, we consider them as matrices written in the standard bases. From eq. (25) we have

$$r_{AC} = \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes S_k \right) \quad (42)$$

Further, by using the identity $(\mathbb{1}_C \otimes S_k) |\phi^+\rangle_{CC} = (S^T_k \otimes \mathbb{1}_C) |\phi^+\rangle_{DD}$, we can write

$$|\psi\rangle_{ABCD} = \sum_{k=1}^{r_{AB}} \sum_{i=1}^{d_A} \sum_{j=1}^{d_C} |i\rangle_A \otimes R_k |i\rangle_A \otimes S^T_k |j\rangle_D \otimes |j\rangle_D, \quad (43)$$

from which it follows that

$$r_{AD} = \text{rank} \left( \sum_{k=1}^{r_{AB}} R_k \otimes S^T_k \right). \quad (44)$$

Hypothesis [2] then implies $r_{BC} \leq r_{AB} r_{AC}$ for $|\psi\rangle$.

$\implies$ Suppose that Hypothesis [2] is false. Then there exist matrices $R_k, S_k$ for $1 \leq k \leq K$ such that

$$\text{rank} \left( \sum_{i=1}^{K} R_i \otimes S^T_i \right) > K \text{rank} \left( \sum_{i=1}^{K} R_i \otimes S_i \right). \quad (45)$$

Consider the quantum state

$$|\psi\rangle_{ABCD} = \sum_{k=1}^{K} \sum_{i=1}^{d_A} \sum_{j=1}^{d_C} |i\rangle_A \otimes R_k |i\rangle_A \otimes |j\rangle_C \otimes S_k |j\rangle_C. \quad (46)$$

For this state we have

$$r_{AC} = \text{rank} \left( \sum_{k=1}^{K} R_k \otimes S_k \right), \quad (47)$$

$$r_{AD} = \text{rank} \left( \sum_{k=1}^{K} R_k \otimes S^T_k \right), \quad (48)$$

and, since $r_{AB} = \dim \text{span} \{ (\mathbb{1}_C \otimes S_k) |\phi^+\rangle_{CC} : 1 \leq k \leq K \}$, we have $r_{AB} \leq K$. Putting this together we have a state $|\psi\rangle$ for which

$$r_{BC} = \text{rank} \left( \sum_{k=1}^{K} R_k \otimes S^T_k \right) > K \text{rank} \left( \sum_{i=1}^{K} R_i \otimes S_i \right) \geq r_{AB} r_{AC}, \quad (49)$$

so Hypothesis [1] must also be false. \qed
Theorem 5. Let $|\Psi\rangle_{ABCD}$ be a four-party quantum state, with $r_{AB} \leq 2$. Then $r_{BC} \leq r_{AB} r_{AC}$.

Proof. By the argument above, it suffices to prove Hypothesis 2 for the cases $K = 1$ and $K = 2$.

$K = 1$: rank $R_1 \otimes S_1 = (\text{rank } R_1)(\text{rank } S_1) = (\text{rank } R_1)(\text{rank } S_1^T)$, using the fact that the row rank and the column rank of a matrix are the same, and equal to its rank.

$K = 2$: Let $M = R_1 \otimes S_1 + R_2 \otimes S_2$, where $R_1, R_2$ are $m_1 \times n_1$ complex matrices, and $S_1, S_2$ are $m_2 \times n_2$ complex matrices. This means we can write

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n_1} \\ M_{21} & M_{22} & \cdots & M_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m_1,1} & M_{m_1,2} & \cdots & M_{m_1,n_1} \end{pmatrix},$$

where $M_{ij} = (R_1)_{ij} S_1 + (R_2)_{ij} S_2$. We refer to the matrices $M_{ij}$ as the ‘blocks’ of $M$. Let $M^T$ denote the partial transpose of $M$, which is the matrix obtained by taking the transpose of each block of $M$. We aim to show that rank $M^T \leq 2$ rank $M$.

Notice the following:

(i) Let $U := \text{span}\{S_1, S_2\}$. Then $U$ is a 2-dimensional vector space of matrices, and $M_{ij} \in U$ for all $i, j$.

(ii) Let $V := \mathcal{C}(S_1|S_2)$, where $\mathcal{C}(A)$ denotes the span of the columns of matrix $A$, and $S_1|S_2$ is a shorthand for the block matrix $(S_1|S_2)$. Then every column of each $M_{ij}$ is in $V$.

(iii) Suppose that $T_1, T_2$ are linearly independent blocks of $M$. Then clearly we have span$\{T_1, T_2\} = U$. Further, by applying elementary column operations we obtain

$$\mathcal{C}(T_1|T_2) = \mathcal{C}(T_1|T_2|0|0)$$
$$= \mathcal{C}(T_1|T_2|S_1|S_2)$$
$$= \mathcal{C}(0|S_1|S_2)$$
$$= \mathcal{C}(S_1|S_2)$$
$$= V.$$  

In particular, we must have dim $V \leq \text{rank } T_1 + \text{rank } T_2$, and hence rank $T_i \geq \frac{1}{2} \text{dim } V$ for at least one value of $i$.

(iv) Each row of $M^T$ is in the space $V \oplus V \oplus \cdots \oplus V$ (where there are $n_1$ copies of $V$) and so, for example, rank $M^T \leq n_1 \text{ dim } V$.

Suppose that $E$ is an $n_1 \times n_1$ invertible matrix, and let $I$ be the $n_2 \times n_2$ identity matrix. Then, rank $M(E \otimes I) = \text{rank } M$ and rank $[M(E \otimes I)]^T = \text{rank } M^T (E \otimes I) = \text{rank } M^T$, so, if we choose to, we can replace $M$ with $M(E \otimes I)$ for any invertible $E$. In the case where $E$ performs elementary column operations, the effect of $(E \otimes I)$ acting on $M$ on the right is to perform block-wise column operations on $M$. (By an analogous argument, we may also perform block-wise row operations on $M$.)

Suppose that $M_{11}$ and $M_{12}$ are linearly independent matrices. Then, for all $j \geq 3$, $M_{ij} \in \text{span}\{M_{11}, M_{12}\} = U$, and by applying block-wise column operations we may assume (as far as the ranks are concerned) that $M$ has the form

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{m_1,1} & M_{m_1,2} & M_{m_1,3} & \cdots & M_{m_1,n_1} \end{pmatrix}.$$  

Let

$$M' := \begin{pmatrix} M_{23} & \cdots & M_{2n_1} \\ \vdots & \ddots & \vdots \\ M_{m_1,3} & \cdots & M_{m_1,n_1} \end{pmatrix},$$  

then we can replace $M$ by the block matrix $M'$ and the original Hypothesis 2 for $M$ will be equivalent to Hypothesis 2 for $M'$.
and consider the following inequalities, which hold for arbitrary block matrices:

$$\text{rank } A + \text{rank } C \leq \text{rank } \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \leq \text{rank } \begin{pmatrix} A \\ B \end{pmatrix} + \text{rank } C.$$  (54)

Applying the second inequality to $M^\Gamma$, and using the same argument as comment (iv) above, we obtain

$$\text{rank } M^\Gamma \leq 2 \dim V + \text{rank } M'^\Gamma.$$  (55)

Applying the first inequality to $M$ we obtain

$$\text{rank } M \geq \text{rank}(M_{11}|M_{12}) + \text{rank } M'$$

$$= \dim V + \text{rank } M'. $$  (56)

Therefore, if $M$ is a counterexample to Hypothesis 2, i.e. $\text{rank } M^\Gamma > 2 \text{rank } M$, then $M'$ is also a counterexample.

Suppose, for contradiction, that a counterexample to Hypothesis 2 exists, with $K = 2$. Let $M$ be a minimal such counterexample, in the sense that $m_1 + n_1$ takes the smallest possible value. By the argument above we may assume that $M_{11}$ and $M_{12}$ are not linearly independent. Further, by applying block-wise row and column operations, we may assume that no two blocks of $M$ in the same row are linearly independent.

Now, $M$ must contain 2 linearly independent blocks, or else it would be possible to write $M = R \otimes S$, which implies $\text{rank } M^\Gamma = \text{rank } M$. Suppose $T_1, T_2$ are linearly independent blocks. By comment (iii) above, we may assume that $\text{rank } T_1 \geq \frac{1}{2} \dim V$. By swapping block-wise rows and columns, we may also assume that $T_1$ is $M_{11}$. Since each row has only one linearly independent block, using further block-wise column operations, we can assume the matrix has the form

$$M = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ M_{21} & M_{22} & \cdots & M_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m_11} & M_{m_12} & \cdots & M_{m_1n_1} \end{pmatrix}. $$  (57)

Now,

$$\text{rank } M^\Gamma \leq \dim V + \text{rank } M'^\Gamma, $$  (58)

where $M'$ is the part of the matrix below the blocks of zeroes. Also,

$$\text{rank } M \geq \text{rank } T_1 + \text{rank } M' \geq \frac{1}{2} \dim V + \text{rank } M'. $$  (59)

Since $\text{rank } M^\Gamma > 2 \text{rank } M$, this gives

$$\dim V + \text{rank } M'^\Gamma > 2 \left( \frac{1}{2} \dim V + \text{rank } M' \right), $$  (60)

and so

$$\text{rank } M'^\Gamma > 2 \text{rank } M', $$  (61)

which is a contradiction to the assumption that $M$ was a minimal counterexample. Hence there can be no counterexample and the theorem is proved.

VI. CONCLUSIONS

We have introduced the problem of determining the universal inequalities between the ranks of partial traces of a general $n$-party pure state. Specifically, taking logarithms, the linear inequalities between the
0-Rényi entropies became the object of our study, and we showed that by using strong subadditivity for the von Neumann entropy we can derive two new inequalities for the ranks. The search for further inequalities lead, in the case of $n = 4$ parties, to several interesting families of states, and an hypothetical third inequality, which if true, would result in a complete description of the linear inequalities for the 0-Rényi entropy. However, a proof or refutation of this hypothesis remains the major open problem of our work.

Note that we were able to give purely algebraic proofs for all the inequalities we found, except Theorem 2; to go beyond statements derived from strong subadditivity, it seems fruitful to develop an algebraic proof of Theorem 2, too, but this has eluded us so far. Returning to the basic setup of section II, we also note that it is far from clear in which sense $\Omega_n$, the set of 0-entropy vectors, is best described by linear inequalities. Obviously, $\Omega_n$ is not a cone since it is a discrete set (a certain subset of $\log N^{2^{n-1}-1}$), so this is a valid and important question. Possible answers are suggested by the case $n = 3$, where we observed that the closed cone $C_3$ generated by $\Omega_3$ has the property that $\Omega_3 = C_3 \cap \log N^3$. The analogue of this is not true for general $n$. However, could it be at least the case that every log-integer point in the interior of $C_n$ is in $\Omega_n$? Or if not that, every interior log-integer point of sufficiently large norm? Any such statement would provide at least partial justification for focusing on the linear inequalities, and we refer them to the careful attention of the reader.

Finally, we close this discussion with a suggestion for further extension of our theory: While here we considered the Schmidt rank only for all bipartitions, tensor rank is a natural multi-party generalization (see [23] and [24] for the general concept, and [24] for its appearance in quantum information), so we might be tempted to associate to each state vector $|\psi\rangle$ a vector of tensor ranks, one for each partition of the ground set $[n]$ into arbitrarily many parts. Their number is known as the Bell number. It is a very interesting, yet wide-open problem, to find the universal inequalities between the tensor ranks of the various partitions of the $n$ parties.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K.l Horodecki. Quantum entanglement. Rev. Mod. Phys., 81:865–942, 2009.
[2] G. Vidal. Efficient Classical Simulation of Slightly Entangled Quantum Computations. Phys. Rev. Lett., 91:147902, 2003.
[3] M. van den Nest. Universal quantum computation with little entanglement. Phys. Rev. Lett., 110:060504, 2013.
[4] H.-K. Lo and S. Popescu. Concentrating entanglement by local actions: Beyond mean values. Phys. Rev. A, 63:022301, 2001.
[5] M. Huber and J.I. de Vicente. Structure of Multidimensional Entanglement in Multipartite Systems. Phys. Rev. Lett., 110:030501, 2013.
[6] M. Huber, M. Perarnau-Llobet, and J. de Vicente. The entropy vector formalism and the structure of multidimensional entanglement in multipartite systems. arXiv:1307.5541v1, 2013.
[7] Z. Zhang and R.W. Yeung. On the Characterization of Entropy Function via Information Inequalities. IEEE Trans. Inf. Theory, 44(4):1440–1452, 1998.
[8] F. Matúš. Infinitely many information inequalities. In Proc. IEEE Int. Symp. Inf. Theory, pages 41–44, 2007.
[9] R. Dougherty, C. Freiling, and K. Zeger. Non-Shannon Information Inequalities in Four Random Variables. arXiv:1104.3602, 2011.
[10] N. Pippenger. The Inequalities of Quantum Information Theory. IEEE Trans. Inf. Theory, 49(4):773–789, 2003.
[11] N. Linden and A. Winter. A New Inequality for the von Neumann Entropy. *Commun. Math. Phys.*, 259:129–138, 2005.

[12] J. Cadney, N. Linden, and A. Winter. Infinitely Many Constrained Inequalities for the von Neumann Entropy. *IEEE Trans. Inf. Theory*, 58(6):3657–3663, 2012.

[13] E.H. Lieb and M.-B. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.*, 14:1938–1941, 1973.

[14] N. Linden, E. Maneva, S. Massar, S. Popescu, D. Roberts, B. Schumacher, J.A. Smolin, and A.V. Thapliyal. Unpublished, 2005.

[15] B. Ibinson. *Quantum Information and Entropy*. PhD thesis, University of Bristol, 2007.

[16] N. Linden, M. Mosonyi, and A. Winter. The structure of Rényi entropic inequalities. *arXiv:1212.0248v2*, 2012.

[17] Note that this is slightly different from [2], where the rank vector is formed by listing the ranks in decreasing order.

[18] A.J. Schwenk and J.I. Munro. How small can the mean shadow of a set be? *Amer. Math. Monthly*, 90(5):325–329, 1983.

[19] B. Groisman, S. Popescu, and A. Winter. Quantum, classical, and total amount of correlations in a quantum state. *Phys. Rev. A*, 72(3):032317, 2005.

[20] M.-B. Ruskai. Another short and elementary proof of strong subadditivity of quantum entropy. *Rep. Math. Phys.*, 60:1–12, 2007.

[21] N. Beaudry and R. Renner. An intuitive proof of the data processing inequality. *Quantum Inf. Comput.*, 12:432–441, 2012.

[22] Available as a free download at [http://cgm.cs.mcgill.ca/~avis/C/lrs.html](http://cgm.cs.mcgill.ca/~avis/C/lrs.html).

[23] V. Strassen. Rank and Optimal Computation of Generic Tensors. *Lin. Alg. Appl.*, 52+53:645–685, 1983.

[24] P. Bürgisser, M. Clausen, and M.A. Shokrollahi. *Algebraic Complexity Theory*. Springer Verlag, 1997.

[25] J. Eisert and H.-J. Briegel. Schmidt measure as a tool for quantifying multiparticle entanglement. *Phys. Rev. A*, 64:022306, 2001.