A new asymptotical flat and spherically symmetric solution in the generalized Einstein-Cartan-Kibble-Sciama gravity and gravitational lensing

Songbai Chen\textsuperscript{1,2,3,4}, Lu Zhang\textsuperscript{1,2}, Jiliang Jing\textsuperscript{1,2,3,4} \footnote{Corresponding author: csb3752@hunnu.edu.cn}, Jiliang Jing\textsuperscript{1,2,3,4} \footnote{jljing@hunnu.edu.cn}

\textsuperscript{1}Institute of Physics and Department of Physics, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{2}Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{3}Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha, Hunan 410081, People’s Republic of China
\textsuperscript{4}Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China

Abstract

In this paper we firstly present a new asymptotical flat and spherically symmetric solution in the generalized Einstein-Cartan-Kibble-Sciama (ECKS) theory of gravity and then investigate the propagation of photon in this background. This solution possesses three independent parameters which affect sharply photon sphere, deflection angle of light ray and gravitational lensing. Since the condition of existence of horizons is not inconsistent with that of photon sphere, there exists a special case where there is horizon but no photon sphere in this spacetime. Especially, we find that in this special case, the deflection angle of a light ray near the event horizon tends to a finite value rather than diverges, which is not explored in other spacetimes. We also study the strong gravitational lensing in this spacetime with the photon sphere and then probe how the spacetime parameters affects the coefficients in the strong field limit.

PACS numbers: 04.70.Dy, 95.30.Sf, 97.60.Lf
I. INTRODUCTION

General relativity is the most beautiful theory of gravity at present and it is a fundamental theoretical setting for the modern astrophysics and cosmology. However, the observed accelerating expansion of the current Universe [1–5] implies that some important ingredients could be missing in this theory. One of ingredients vanished Einstein theory is torsion, which is the antisymmetric part of the general affine connection. In the gravity theories with torsion, the gravitational field is described by both of spacetime metric and the torsion field, which means that the emergence of torsion will modify the feature of the gravitational interaction.

One of natural extensions of Einstein’s theory of gravity is the so-called ECKS theory of gravity [6, 7]. In this theory with torsion, the curvature and the torsion, respectively, are assumed to couple with the energy and momentum and the intrinsic angular momentum of matter. The gravitational repulsion effect arising from such a spinor-torsion coupling can avoid the formation of spacetime singularities in the region with extremely large densities, for example, in the interior of black holes and the very early stage of Universe [8–11]. In the low densities region, the ECSK theory and Einstein’s general relativity give indistinguishable predictions since the contribution from torsion to the Einstein equations is negligibly small. However, the torsion field is not dynamical in the ECSK theory since the torsion equation is an algebraic constraint rather than a partial differential equation, which means that the torsion field outside of matter distribution vanishes because it can not propagate as a wave in the spacetime.

In order to construct a dynamical torsion field, one can generalize ECSK theory by introducing higher order corrections in Lagrangian [12, 13]. These coupling terms between the spacetime torsion and curvature yield that both equations of motion for spacetime torsion and curvature are dynamical equations, and then ensure that the spacetime torsion can propagate in the spacetime even in the absence of spin of matter. Recently, some non-trivial static asymptotical flat vacuum solutions [14] are obtained in this generalized ECSK theory by solving Einstein field equations and torsion field equation. These non-trivial solutions describe the spacetimes with special structures, which is useful for detecting the effects originating from spacetime torsion within the gravitational interactions.

A natural question is that there are other asymptotical flat and spherically symmetric solutions in this generalize ECSK theory. In this paper, we will present a new asymptotical flat and spherically symmetric solution. Our new solution can recover to Schwarzschild solution and Reissner-Nordström one, which is
different from those obtained in Ref. [14]. Especially, as the parameters take certain special values, this solution can also reduce to the black hole solutions in the brane-world [15]. Gravitational lensing is a phenomenon of the deflection of light rays in the curve spacetime. It is well known that gravitational lensing can provide us a lot of important signatures about compact objects, which could help to identify black hole and verify alternative theories of gravity in their strong field regime [16–42]. Therefore, in this paper, we also further study the gravitational lensing in the spacetime described by our new solution.

The paper is organized as follows. In Sec. II, we will firstly present a new asymptotical flat and spherically symmetric solution in the generalized Einstein-Cartan-Kibble-Sciama (ECKS) theory of gravity. In Sec. III, we will investigate investigate the propagation of photon in this background and probe the effects of the spacetime parameters on the photon sphere, the deflection angles for light ray. We also analyze the coefficients in the strong field limit in the cases with photon sphere. Finally, we present a summary.

II. A BLACK HOLE SOLUTION IN THE GENERALIZED EINSTEIN-CARTAN-KIBBLE-SCIAMA GRAVITY

In this section, we will present a new black hole solution with torsion in the generalized ECKS theory of gravity. Let us start with the action [14]

$$S = \int d^4x \sqrt{-g} \left[ - \frac{1}{16\pi G} \left( \mathcal{L}_0 + a_1 \mathcal{L}_1 \right) \right],$$

(1)

where $a_1$ is a coupling constant, $\mathcal{L}_0$ is ECKS Lagrangian [12–14]

$$\mathcal{L}_0 = R + \frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q_{\delta\epsilon\eta} - \frac{1}{2} Q_{\alpha\delta\epsilon} Q_{\beta\gamma\eta} + 2 Q_{\alpha\beta} Q_{\alpha\beta},$$

(2)

and $\mathcal{L}_1$ is

$$\mathcal{L}_1 = R Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} \left[ Q^{\delta\epsilon\eta} \left( 2 Q_{\delta\epsilon\eta} + Q_{\delta\epsilon\eta} \right) + 8 Q_{\delta\epsilon\eta} \right] + Q_{\alpha\beta} Q_{\beta\gamma} Q^{\delta\epsilon\eta} Q_{\delta\epsilon\eta}.$$  

(3)

Here $R$ is Ricci Scalar. The tensor $Q_{\mu\nu}$ describes the spacetime torsion, which is defined by

$$Q_{\mu\nu} = \tilde{\Gamma}_\mu^\alpha - \tilde{\Gamma}_\nu^\alpha.$$  

(4)

It is the antisymmetric part of the general affine connection $\tilde{\Gamma}_\mu^\alpha$. The affine connection $\tilde{\Gamma}_\mu^\alpha$ is related to the Levi-Civita Christoffel connection $\Gamma_\mu^\alpha$ by

$$\tilde{\Gamma}_\mu^\alpha = \Gamma_\mu^\alpha + K_\mu^\alpha,$$

(5)
where $K_{\mu\nu}^\alpha$ is the contorsion tensor with a form

$$K_{\mu\nu}^\alpha = \frac{1}{2} \left[ Q_{\mu\nu}^\alpha - Q_{\mu\nu}^\alpha - Q_{\nu\mu}^\alpha \right].$$

As in Ref. [14], we focus on a static spherically symmetric vacuum solution with the metric

$$ds^2 = -Hdt^2 + \frac{dr^2}{F} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

and suppose that the non-vanishing components of the torsion tensor $Q_{\mu\nu}^\alpha$ for the solution (7) have the forms

$$Q^\alpha_{tr} = -Q^\alpha_{rt} = A, \quad Q^\alpha_{t\theta} = -Q^\alpha_{\theta t} = B, \quad Q^\alpha_{\theta\phi} = -Q^\alpha_{\phi\theta} = B.$$

Here $A$, $B$, $H$, and $F$ are only functions of the polar coordinate $r$. The choice (8) for the torsion tensor $Q_{\mu\nu}^\alpha$ satisfies the constraint $\mathcal{L}_H Q_{\mu\nu}^\alpha = 0$, which has been applied in cosmological setting [43]. Inserting Eqs. (7) and (8) into the action (11), we can find that the total Lagrangian has a form

$$\mathcal{L} = \sqrt{-g}(\mathcal{L}_0 + a_1 \mathcal{L}_1) = \frac{\sin \theta}{2H^2} \sqrt{\frac{H}{F}} \left[ H - 2a_1(A^2 + 2B^2) \right] \left[ 2r^2 FHH'' + HH'(r^2 F' + 4rF) 
+ 4H^2(rF' + F - 1) + 4r^2 HB(2A + B) \right].$$

Making use of Euler-Lagrange equation, we can obtain four second order differential equations

$$H^2 \left[ (rF' + F - 1)H - r^2 B(2A + B) \right] + a_1(A^2 + 2B^2) \left[ 6r^2 BH(2A + B) + 2H^2(rF' + F - 1) 
+ 2rHH'(rF' + 4F) + r^2 F(4H''H - 5H'^2) \right] - a_1 r H(A^2 + 2B^2)' [H(rF' + 4F) - 4rFH'] 
- 2a_1 r^2 FHH^2(A^2 + 2B^2)^2 = 0,$$

$$-H^2 \left[ (rF' + F - 1)H + r^2 B(2A + B) \right] + a_1(A^2 + 2B^2) \left[ 2r^2 BH(2A + B) + 2H^2(F - 1) + rFH'(2H 
+ rH') \right] + a_1 r FH(A^2 + 2B^2)' [H + rH'] = 0,$$

$$-r^2 BH^2 + 4a_1 r^2 HB(3A^2 + 2B^2) + a_1 A \left[ 4r^2 B^2 H + 4H^2(rF' + F - 1) - r^2 FH'^2 + rHH'(rF' + 4F) 
+ 2r^2 FHH'' \right] = 0,$$

$$-r^2 (A + B) H^2 + 2a_1 r^2 H \left[ B(A^2 + 4B^2) + A(A^2 + 6B^2) \right] + a_1 B \left[ 4H^2(rF' + F - 1) - r^2 FH'^2 
+ rHH'(rF' + 4F) + 2r^2 FHH'' \right] = 0.$$

Setting $H = 2a_1(A^2 + 2B^2)$ and $A = -\frac{1}{2} B$ as in Ref. [14], one can find that the equation (12) is satisfied naturally and other three equations can be reduced to a single equation

$$2r^2 F B'' + r(rF' + 4F)B' + 2(rF' + F - 1)B = 0.$$
which is consistent with that in Ref. [14]. This means that once B is selected as a proper form, we can obtain the function F by solving the second order differential equation (14). Letting $B = \sqrt{1 - \frac{2m}{r}}$ and $a_1 = \frac{\gamma}{4}$, we can obtain the solution

$$H = 1 - \frac{2m}{r}, \quad F = \frac{(r - 2m)(2r + \alpha)}{r(2r - 3m)},$$

(15)

where $\alpha$ is an integral constant. If taking $F = 1 - \frac{2\gamma m}{r}$ and $a_1 = \frac{9}{4}$, we can obtain the solution

$$H = \left(1 - \frac{1}{\gamma} + \frac{1}{\gamma} \sqrt{1 - \frac{2\gamma m}{r}}\right)^2, \quad F = 1 - \frac{2\gamma m}{r}.$$

(16)

Interestingly, these two solutions (15) and (16), respectively, have the same forms as the black hole solutions I and II in the brane-world [15]. This implies that there exist certain unknown connection between the generalized ECKS theory of gravity [1] and the braneworld gravity [15]. Here, we set the function $F = 1 - \frac{2\gamma m}{r} + \frac{q^2}{r^2}$ and the coupling parameter $a_1 = \frac{9}{4}$, and then obtain a new black hole solution in the generalized ECKS theory of gravity [1]

$$F = 1 - \frac{2\gamma m}{r} + \frac{q^2}{r^2},$$

$$H = \frac{1}{(\gamma^2 m^2 - q^2)^2} \left[\gamma(\gamma - 1)m^2 + \frac{(1 - \gamma)q^2 m + (\gamma m^2 - q^2)\sqrt{r^2 - 2\gamma mr + q^2}}{r}\right]^2,$$

(17)

where $m$, $q$, and $\gamma$ are constants. This solution is asymptotically flat since both of functions $F$ and $H$ tend to 1 as $r$ approaches to spatial infinity. As $\gamma = 1$, we find that the solution (17) reduces to the Reissner-Nordström type. As $q = 0$, it becomes the black hole solution (16). As in the case of the black hole solutions I and II in the brane-world [15], the solution (17) will be expressed in terms of the ADM mass $m$ and the parameterized post-newtonian (PPN) parameter $\beta_0 = \frac{\gamma + 1}{2}$. The parameter $q$ is similar to a “tidal-like charge”. The position of outer event horizon lies at

$$r_H = \gamma m + \sqrt{\gamma^2 m^2 - q^2},$$

(18)

which is defined by equation $F = 0$. Especially, the surface gravity constant of event horizon for the black hole (17) is

$$\kappa = \frac{1}{2} \left| \frac{\sqrt{\gamma^2 r^2 - \gamma \gamma g_{tt}}}{-g_{tt}} \right|_{r = r_H} = \frac{1}{2} \left| \frac{F(r) \, dH(r)}{H(r) \, dr} \right|_{r = r_H} = \frac{\gamma m^2 - q^2}{\sqrt{\gamma^2 m^2 - q^2} (\gamma m + \sqrt{\gamma^2 m^2 - q^2})^2}.$$

(19)

Thus, the presence of parameter $\gamma$ will bring some particular spacetime properties differed from those in the Einstein’s general relativity, which could make a great deal influence on the propagation of photon in the spacetime with metric functions (17).
III. THE DEFLECTION ANGLE FOR LIGHT RAY AND STRONG GRAVITATIONAL LENSING IN THE SPACETIME WITH TORSION

Let us to study the deflection angle for light ray and the corresponding gravitational lensing in the background of a black hole spacetime with torsion (17). Due to the nature of the spherically symmetric spacetime, we here consider only the case where both the source and the observer lie in the equatorial plane so that the orbit of the photon is limited on the same plane in the background spacetime. With the condition $\theta = \pi/2$, the metric (17) can be expressed as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\phi^2,$$

with

$$A(r) = H, \quad B(r) = 1/F, \quad C(r) = r^2.$$ (21)

The geodesics for the photon in the spacetime (20) obey

$$\frac{dt}{d\lambda} = \frac{1}{A(r)},$$

$$\frac{d\phi}{d\lambda} = \frac{J}{C(r)},$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{B(r)}\left[\frac{1}{A(r)} - \frac{J^2}{C(r)}\right].$$ (24)

where $J$ is the angular momentum of the photon and $\lambda$ is an affine parameter along the null geodesics. Here, the energy of photon is set to $E = 1$. For a black hole spacetime (17), the impact parameter $u(r_0)$ for a photon can be expressed as

$$u(r_0) = J(r_0) = \frac{r_0}{\sqrt{H(r_0)}},$$ (25)

where $r_0$ is the closest distance between the photon and black hole. It is well known that the photon sphere plays an important role in the propagation of photon in a curve spacetime. In the background of a black hole spacetime (17), the radius of the photon sphere $r_{ps}$ is the largest real root of the equation

$$A(r)C'(r) - A'(r)C(r) = 2\left[m(\gamma - 1)q^2 - \gamma mr + (q^2 - \gamma m^2)\sqrt{r^2 - 2\gamma mr + q^2}\right]$$

$$\times \left[\gamma m^2 - q^2(r^2 - 3\gamma mr + 2q^2) + m(\gamma - 1)(\gamma mr - 2q^2)\sqrt{r^2 - 2\gamma mr + q^2}\right] = 0.$$ (26)

As expected, the radius of the photon sphere $r_{ps}$ depends on both the parameters $q$ and $\gamma$ of the black hole. However, the appearance of $\gamma$ yields that the form of the photon sphere radius become very complicated in
FIG. 1: Change of the photon sphere radius $r_{ps}$ with parameters $q$ and $\gamma$ in the background of a black hole spacetime with torsion (17). Here, we set $m = 1$.

FIG. 2: The boundary of the existence of horizon (red dashed line) and of the photon sphere (blue line) in the background of a black hole spacetime with torsion (17). Here, we set $m = 1$.

In this case. In Fig.1, we present the variety of the photon sphere $r_{ps}$ with the parameter $q$ and $\gamma$ of by solving eq. (26) numerically. It is show that the photon sphere radius increases with the parameter $\gamma$ and decreases with $q$. The change of $r_{ps}$ with $q$ is similar to that in the Reissner-Nordström spacetime. Moreover, we find that the photon sphere $r_{ps}$ exists only in the regime $q < q_c$. The value of the upper limit $q_c$ depends on the parameter $\gamma$, and its form can be expressed as

$$q_c = \begin{cases} 
  \frac{m}{2\sqrt{6}} \left[ \frac{W_{2/3} + 2(9\gamma^2 + 4)W_{1/3} + (9\gamma^2 - 8)^2}{W_{2/3}} \right]^{1/2}, & 0 < \gamma \leq 1, \\
  \frac{m}{4\sqrt{3}} \left[ \frac{(-1 - \sqrt{3})W_{2/3} + 4(9\gamma^2 + 4)W_{1/3} + (-1 + \sqrt{3})(81\gamma^4 + 64)^2}{W_{2/3}} \right]^{1/2}, & 1 \leq \gamma \leq \frac{4}{3}, \\
  \sqrt{\gamma}, & \gamma \geq \frac{4}{3},
\end{cases}$$

(27)
where

\[ W = 54 \gamma^2 \sqrt{2 (27 \gamma^3 - 18 \gamma - 8)(3 \gamma - 4)^3 |\gamma - 1| + 2187 \gamma^6 - 5832 \gamma^5 + 4860 \gamma^4 - 1728 \gamma^2 + 512}. \] (28)

Comparing Eq. (18) with Eq. (27), it is obvious that the condition of existence of horizons is not inconsistent with that of the photon sphere. This means that in the parameter panel \((\gamma, q)\), the whole region is split into four regions I-IV by those critical curves. It is shown in Fig.(2) where the red dashed line and the blue solid line, respectively, denote the boundary of the existence of horizon and of the photon sphere. When the parameters \((\gamma, q)\) lie in the region I, there exist both horizon and photon sphere radius \(r_{ps}\), which is similar to that of in the static black hole spacetime in the Einstein’s general relativity. When \((\gamma, q)\) lie in the region IV, there is no horizon and no photon sphere, which corresponds to case of strong naked singularity where the singularity is completely naked [20, 25]. When \((\gamma, q)\) is located in region II, there exists only photon sphere but no horizon, which corresponds to case of weak naked singularity where the singularity is covered by the photon sphere [20, 25]. When \((\gamma, q)\) lies in the region III, there is horizon but no photon sphere. These four situations are similar to those in the black hole spacetime with a torsion [44]. Thus, the presence of torsion changes the spacetime structure which will affect the propagation of photon in the background spacetime.

Let us now to discuss the behavior of the deflection angle of light ray in the spacetime described by a metric with torsion (17). For the photon coming from infinite, the deflection angle in a curve spacetime can be expressed as

\[ \alpha(r_0) = I(r_0) - \pi, \] (29)

where \(r_0\) is the closest approach distance and \(I(r_0)\) is

\[ I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)} dr}{\sqrt{C(r)} \sqrt{\frac{C(r) A(r_0)}{C(r_0) A(r)}} - 1}. \] (30)

In Figs.(3)-(5), we plot the change curves of the deflection angle \(\alpha(r_0)\) with the distance of approach \(r_0\) for different parameters \(\gamma\) and \(q\) in the spacetime with torsion (17). For the cases with photon sphere, i.e., the parameters \((\gamma, q)\) lie in the region I or II in Fig.(2), the deflection angle for different \(\gamma\) and \(q\) strictly increases with the decreases of the closest distance of approach \(r_0\) and finally becomes infinite as \(r_0\) tends to the respective photon sphere radius \(r_{ps}\), i.e., \(\lim_{r_0 \to r_{ps}} \alpha(r_0) = \infty\), which is shown in Fig.(3). In Fig.(4), we present the deflection angle in the case in which the parameters \((\gamma, q)\) lie in the region IV in Fig.(2), there is no horizon and no photon sphere and singularity is naked completely. It is shown that the deflection angle of the light ray closing to the singularity tends to a finite value \(-\pi\) for different \(\gamma\) and \(q\), which means that
FIG. 3: Deflection angle $\alpha(r_0)$ as a function of the closest distance of approach $r_0$ for the cases with photon sphere. The panels in the upper and bottom rows correspond to the cases in which the parameters $(\gamma, q)$ are located in the regions I and II in Fig.(2), respectively. Here, we set $m = 1$.

FIG. 4: Deflection angle $\alpha(r_0)$ as a function of the closest distance of approach $r_0$ for the case without photon sphere and horizon in which the parameters $(\gamma, q)$ lie in the region IV in Fig.(2). Here, we set $m = 1$.

the photon could not be captured by the compact object so that the photon goes back along the original direction in this situation. This behavior can be regarded as a common feature of gravitational lensing by strong naked singularity. As the parameters $(\gamma, q)$ lie in the region III in Fig.(2), there exists horizon but no photon sphere, we find that the deflection angle of the light finally becomes a finite value as $r_0$ tends to the
FIG. 5: Deflection angle $\alpha(r_0)$ as a function of the closest distance of approach $r_0$ for the case with horizon and no photon sphere in which the parameters $(\gamma, q)$ lie in the region $III$ in Fig.(2). Here, we set $m = 1$.

respective event horizon radius $r_H$, i.e., $\lim_{r_0 \to r_H} \alpha(r_0) = \alpha_{r_H}$. This behavior differs from those in the black hole with a torsion considered in Ref. [44] in which the deflection angle of the light finally becomes unlimited large as $r_0$ tends to the event horizon radius. It could be understood by a fact that due to nonexistence of photon sphere the photon is captured directly by black hole before it make infinite complete loops around the central object in this case. Moreover, we find that the deflection angle $\alpha_{r_H}$ increases with the parameter $\gamma$ and decreases with $q$. In the far-field limit, the deflection angle can be approximated as

$$\alpha|_{r_0 \to \infty} \approx \frac{2(\gamma + 1)m}{r_0} + \frac{(6\pi - 8)(\gamma + 1)m^2 + 3\pi(\gamma^2m^2 - q^2)}{4r_0^2},$$

which means that $\lim_{r_0 \to \infty} \alpha(r_0) = 0$ for all values of parameters $\gamma$ and $q$, which is a common feature in all asymptotical flat spacetimes.

Finally, we will study the strong gravitational lensing by a compact object [17] with the photon sphere and then probe how the parameters $\gamma$ and $q$ affects the coefficients in the strong field limit. Using of the method developed by Bozza [21], one can define a variable

$$z = 1 - \frac{r_0}{r},$$

and then rewrote the integral (30) as

$$I(r_0) = \int_0^1 R(z, r_0)f(z, r_0)dz,$$

with

$$R(z, r_0) = 2\sqrt{A(r)B(r)C(r)},$$

$$f(z, r_0) = \frac{1}{\sqrt{A(r_0)C(r) - A(r)C(r_0)}},$$

(35)
The function $R(z, r_0)$ is regular for all values of $z$ and $r_0$, but $f(z, r_0)$ diverges as $z$ tends to zero. Thus, one can split the integral into the divergent part $I_D(r_0)$ and the regular part $I_R(r_0)$,

$$I_D(r_0) = \int_0^1 R(0, r_{ps}) f_0(z, r_0) dz,$$
$$I_R(r_0) = \int_0^1 [R(z, r_0) f(z, r_0) - R(0, r_{ps}) f_0(z, r_0)] dz. \tag{36}$$

Expanding the argument of the square root in $f(z, r_0)$ to the second order in $z$, one can obtain

$$f_0(z, r_0) = \frac{1}{\sqrt{\alpha(r_0) z + \beta(r_0) z^2}} \tag{37}$$

with

$$\alpha(r_0) = \frac{2}{(\gamma^2 - q^2)^2 \sqrt{r_0^2 - 2\gamma r_0 + q^2}} \left[ (\gamma - 1)(q^2 - \gamma r_0) + (q^2 - \gamma) \sqrt{r_0^2 - 2\gamma r_0 + q^2} \right] \times \left[ (\gamma - q^2)(r_0^2 - 3\gamma r_0 + 2q^2) + (\gamma - 1)(\gamma r_0 - 2q^2) \sqrt{r_0^2 - 2\gamma r_0 + q^2} \right],$$

$$\beta(r_0) = \frac{1}{(\gamma^2 - q^2)^2} \left\{ (\gamma - 1)(q^2 - \gamma) \frac{r_0(\gamma r_0 - q^2)[2r_0^3 - 6\gamma r_0^2 - 2\gamma q^2 + 3(\gamma^2 + q^2)r_0]}{(r_0^2 - 2\gamma r_0 + q^2)^{3/2}} \right. + \frac{2r_0(\gamma - r_0)(\gamma r_0 - 2q^2)}{\sqrt{r_0^2 - 2\gamma r_0 + q^2}} \left. - 2(3\gamma r_0 - 2q^2) \sqrt{r_0^2 - 2\gamma r_0 + q^2} \right\} + 2q^6 - 6\gamma^3 r_0
+ \left[ 3r_0^2 - 6\gamma r_0 + 2(\gamma^2 - 4\gamma + 1) \right] q^4 - 2\gamma [3r_0^2 + 3(\gamma^2 - 4\gamma + 1)r_0 - \gamma] q^2 + 3\gamma^2 r_0^2 (\gamma^2 - 2\gamma + 2) \right\}. \tag{38}$$

It is obvious that the coefficient $\alpha(r_0)$ vanishes as $r_0$ tends to the radius of photon sphere $r_{ps}$, and then the leading term of the divergence in $f_0(z, r_0)$ is $z^{-1}$. This means that near the photon sphere the deflection angle of light ray can be expressed as

$$\alpha(\theta) = -\bar{a} \ln \left( \frac{\theta D_{OL}}{u_{ps}} - 1 \right) + \bar{b} + O(u - u_{ps}), \tag{39}$$

with

$$\bar{a} = \frac{R(0, r_{ps})}{2 \sqrt{\beta(r_{ps})}},$$
$$\bar{b} = -\pi + b_R + \bar{a} \ln \frac{r_{ps}^2 C''(r_{ps}) A(r_{ps})}{u_{ps} \sqrt{A'(r_{ps}) A(r_{ps})}} - C(r_{ps}) A'(r_{ps}) C(r_{ps}) \right],$$
$$b_R = I_R(r_{ps}), \quad u_{ps} = \frac{r_{ps}}{\sqrt{A(r_{ps})}}. \tag{40}$$

which indicates clearly that the deflection angle diverges logarithmically where the light is close to the photon sphere. The quantity $D_{OL}$ is the distance between gravitational lens object and observer, $\bar{a}$ and $\bar{b}$ are the strong field limit coefficients which depend on the spacetime functions at the photon sphere. The changes of the coefficients ($\bar{a}$ and $\bar{b}$) with the parameters $\gamma$ and $q$ is shown in Fig.(6). For the fixed $\gamma$, the coefficient
FIG. 6: Change of the strong deflection limit coefficients $\bar{a}$ and $\bar{b}$ with the parameters $\gamma$ and $q$ in the spacetime with torsion (17).

$\bar{a}$ increases monotonously with $q$. With increase of $q$, the coefficient $\bar{b}$ first increases and then decreases for the smaller $\gamma$, but first decreases and then increases for the larger $\gamma$. With increase of the parameter $\gamma$, $\bar{a}$ increases monotonously as $q < 1$ and decreases as $q > 1$, $\bar{b}$ first decreases and then increases as $q < 1$. For the case with $q > 1$, $\bar{b}$ increases with $\gamma$ as the quantity $q - 1$ is small. With the further increase of $q$, the change of $\bar{b}$ shows gradually a tendency of first decrease and then increase.

IV. SUMMARY

In this paper we firstly present a new black hole solution in the generalized ECKS gravity with three independent parameters $m$, $q$ and $\gamma$ and then investigate the propagation of photon in this background. We find that these spacetime parameters affect sharply photon sphere, deflection angle of light ray and strong gravitational lensing. The photon sphere exists only in the regime $q < q_c$ and the value of $q_c$ depends on the parameter $\gamma$. In the regime where photon sphere exists, the radius of photon sphere increases with the parameter $\gamma$, but decreases with $q$. Moreover, the condition of existence of horizons is not inconsistent with that of photon sphere, which yields that the whole region in the panel $(\gamma, q)$ can be split into four regions by
the boundaries of the existence of horizon and of the photon sphere. In the cases with photon sphere, the deflection angle of the light ray near the photon sphere diverges logarithmically, which is similar to those in the usual spacetime of a black hole or a weak naked singularity in the strong-field limit. In the case without photon sphere and horizon, the deflection angle of the light ray closing very to the singularity approaches a finite value $-\pi$, which does not depend on spacetime parameters $\gamma$ and $q$. It should be a common feature of the deflection angle of light ray near the static strong naked singularity. Furthermore, we also find that there exists a special case in which there is horizon but no photon sphere for the spacetime (17) as in the spacetime with a torsion [44]. However, we find that the deflection angle of the light ray near the event horizon tends to a finite value in this case, which differs from those in the black hole with a torsion considered in Ref. [44] where the deflection angle of the light finally becomes diverges logarithmically. It could be attribute to that the photon is captured directly by black hole before it make infinite complete loops around the central object in this case. Finally, we study the strong gravitational lensing by a compact object (17) with the photon sphere and then probe how the parameters $\gamma$ and $q$ affects the coefficients in the strong field limit.

V. ACKNOWLEDGMENTS

This work was partially supported by the Scientific Research Fund of Hunan Provincial Education Department Grant No. 17A124. J. Jing’s work was partially supported by the National Natural Science Foundation of China under Grant No. 11475061.

[1] A. G. Riess et al., *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, Astron. J. 116 1009 (1998).
[2] A. G. Riess et al., *New Hubble Space Telescope Discoveries of Type Ia Supernovae at z > 1: Narrowing Constraints on the Early Behavior of Dark Energy*, Astrophys. J. 659 98 (2007).
[3] P. de Bernardis et al., *A Flat Universe from High-Resolution Maps of the Cosmic Microwave Background Radiation*, Nature (London) 404 955 (2000).
[4] S. Perlmutter et al., *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*, Astrophys. J. 517 565 (1999).
[5] R. A. Knop et al., *New Constraints on $\Omega_M$, $\Omega_\Lambda$, and $w$ from an Independent Set of Eleven High-Redshift Supernovae Observed with HST*, Astrophys. J. 598 102 (2003).
[6] E. Cartan, C. R. Acad. Sci. (Paris) 174 593 (1922); E. Cartan, Ann. Ec. Norm. Sup. 40 325 (1923); E. Cartan, Ann. Ec. Norm. Sup. 41 1 (1924); E. Cartan, Ann. Ec. Norm. Sup. 42 17 (1925).
[7] T. W. B. Kibble, J. Math. Phys. 2 212 (1961); D. W. Sciama, Rev. Mod. Phys. 36 463 (1964); F. Hehl and E.
Kroner, Z. Phys. 187 478 (1965); F. Hehl, Abb. Braunschweig. Wiss. Ges. 18 98 (1966); A. Trautman, Bull. Polon. Acad. Sci. 20 185 (1972); A. Trautman, Bull. Polon. Acad. Sci. 20 503 (1972).

[8] W. Kopczyński, Phys. Lett. A 39, 219 (1972); Phys. Lett. A 43, 63 (1973).

[9] A. Trautman, Nature (Phys. Sci.) 242, 7 (1973); J. Tafel, Phys. Lett. A 45, 341 (1973); F. Hehl, P. von der Heyde, and G. Kerlick, Phys. Rev. D 10, 1066 (1974); B. Kuchowicz, Gen. Relativ. Gravit. 9, 511 (1978); M. Gasperini, Phys. Rev. Lett. 56, 2873 (1986); M. Gasperini, Gen. Relativ. Gravit. 30, 1703 (1998).

[10] N. Poplawski, Phys. Rev. D, 85 107502 (2012); Gen. Relativ. and Gravit., 44 1007 (2012); Phys. Lett. B, 690 73 (2010).

[11] S. D. Brechet, M. P. Hobson, A. N. Lasenby, Class. Quantum Grav. 25 245016 (2008).

[12] Y. N. Obukhov, V. N. Ponomarev and V. V. Zhytnikov, Gen. Rel. Grav. 21 1107 (1989); Y. N. Obukhov, Int. J. Geom. Meth. Mod. Phys. 3 95 (2006).

[13] S. M. Christensen, J. Phys. A: Math. Gen. 13 3001 (1980).

[14] H. Shabani, and A. Hadi Ziaie, arXiv:1709.06512.

[15] R. Casadio, A. Fabbri, L. Mazzacurati, New black holes in the braneworld?, Phys. Rev. D 65, 084040 (2002).

[16] A. Einstein, Lens-Like Action of a Star by the Deviation of Light in the Gravitational Field, Science 84 506 (1936).

[17] C. Darwin, The gravity field of a particle, Proc. R. Soc. London 249 180 (1959).

[18] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Role of the scalar field in gravitational lensing, Astron. Astrophys. 337 1 (1998).

[19] K. S. Virbhadra and G. F. R. Ellis, Schwarzschild black hole lensing, Phys. Rev. D 62 084003 (2000); C. M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, The geometry of photon surfaces, J. Math. Phys. (N.Y.) 42 818 (2001).

[20] K. S. Virbhadra and G. F. R. Ellis, Gravitational lensing by naked singularities, Phys. Rev. D 65 103004 (2002).

[21] V. Bozza, Gravitational lensing in the strong field limit, Phys. Rev. D 66 103001 (2002).

[22] V. Bozza, Quasiequatorial gravitational lensing by spinning black holes in the strong field limit, Phys. Rev. D 67 103006 (2003); V. Bozza, F. De Luca, G. Scarpetta, and M. Sereno, Analytic Kerr black hole lensing for equatorial observers in the strong deflection limit, Phys. Rev. D 72 083003 (2005).

[23] V. Bozza, F. De Luca, and G. Scarpetta, Kerr black hole lensing for generic observers in the strong deflection limit, Phys. Rev. D 74 063001(2006).

[24] G. N. Gyulchev and S. S. Yazadjiev, Kerr-Sen dilaton-axion black hole lensing in the strong deflection limit, Phys. Rev. D 75 023006 (2007).

[25] G. N. Gyulchev and S. S. Yazadjiev, Gravitational Lensing by Rotating Naked Singularities, Phys. Rev. D 78 083004 (2008).

[26] S. Frittelly, T. P. Kling, E. T. Newman, Spacetime perspective of Schwarzschild lensing, Phys. Rev. D 61 064021 (2000).

[27] V. Bozza, S. Capozziello, G. Lovane, G. Scarpetta, Strong field limit of black hole gravitational lensing, Gen. Rel. and Grav. 33 1535 (2001).

[28] E. F. Eiroa, G. E. Romero, D. F. Torres, Reissner-Nordstrom black hole lensing, Phys. Rev. D 66 024010 (2002).

[29] R. Whisker, Strong gravitational lensing by braneworld black holes, Phys. Rev. D 71 064004 (2005).

[30] A. Bhadra, Gravitational lensing by a charged black hole of string theory, Phys. Rev. D 67 103009 (2003).

[31] S. Chen and J. Jing, Phys. Rev. D 80 024036 (2009); Y. Liu, S. Chen and J. Jing, Phys. Rev. D 81 124017 (2010);
S. Chen, Y. Liu and J. Jing, Phys. Rev. D 83, 124019 (2011); S. Chen and J. Jing, Phys. Rev. D 85, 124029 (2012); S. Chen, S. Wang, Y. Huang, J. Jing, S. Wang, Phys. Rev. D 95, 104017 (2017).

[32] S. Wang, S. Chen, J. Jing, J. Cosmol. Astropart. Phys. 11, 020 (2016);
[33] T. Ghosh, S. Sengupta, Strong gravitational lensing across dilaton anti-de Sitter black hole, Phys. Rev. D 81, 044013 (2010).
[34] A. N. Aliev, P. Talazan, Gravitational Effects of Rotating Braneworld Black Holes, Phys. Rev. D 80, 044023 (2009).
[35] H. Sotani and U. Miyamoto, Strong gravitational lensing by an electrically charged black hole in Eddington-inspired Born-Infeld gravity, Phys. Rev. D 92, 044052 (2015).
[36] S. Sahu, K. Lochan and D. Narasimha, Gravitational Lensing by Self-Dual Black Holes in Loop Quantum Gravity, Phys. Rev. D 91, 063001 (2015).
[37] C. Ding, C. Liu, Y. Xiao, L. Jiang and R. Cai, Strong gravitational lensing in a black-hole spacetime dominated by dark energy, Phys. Rev. D 88, 104007 (2013); E. F. Eiroa and C. M. Sendra, Regular phantom black hole gravitational lensing, Phys. Rev. D 88, 103007 (2013).
[38] S. Wei, Y. Liu, C. Fu and K. Yang, J. Cosmol. Astropart. Phys. 1210, 053 (2012).
[39] S. Wei, Y. Liu, Equatorial and quasi-equatorial gravitational lensing by Kerr black hole pierced by a cosmic string, Phys. Rev. D 85, 064044 (2012).
[40] G. V. Kraniotis, Precise analytic treatment of Kerr and Kerr-(anti) de Sitter black holes as gravitational lenses, Class. Quant. Grav. 28, 085021 (2011).
[41] J. Sadeghi, H. Vaez, Strong gravitational lensing in a charged squashed Kaluza-Klein Gödel black hole, Phys. Lett. B 728, 170 (2014).
[42] J. Sadeghi, A. Banijamali and H. Vaez, Strong Gravitational Lensing in a Charged Squashed Kaluza-Klein Black hole, Astrophys. Space Sci. 343, 559 (2013).
[43] M. Tsamparlis, Phys. Lett. A 75, 27 (1979); Phys. Rev. D 24, 1451 (1981).
[44] L. Zhang, S. Chen, J. Jing, Gravitational lensing by a black hole with torsion, arXiv:1712.00160.