Contrôlabilité de systèmes multi-dimensionnels couplés en cascade par un nombre réduit de contrôles

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Résumé – Nous démontrons qu’il est possible de contrôler des systèmes de \( N \) équations d’évolution faiblement couplées en cascade par un nombre réduit de contrôles frontière ou localement distribués, le nombre de contrôle pouvant varier de 1 à \( N - 1 \). Nous donnons des applications aux systèmes couplés multi-dimensionnels en cascade hyperboliques, paraboliques et de Schrödinger.

Controllability of cascade coupled systems of multi-dimensional evolution PDE’s by a reduced number of controls

Abstract – We prove controllability results for abstract systems of weakly coupled \( N \) evolution equations in cascade by a reduced number of boundary or locally distributed controls ranging from a single up to \( N - 1 \) controls. We give applications to cascade coupled systems of \( N \) multi-dimensional-hyperbolic, parabolic and diffusive equations.

Version française abrégée.

1 Introduction

La contrôlabilité à zéro de systèmes couplés d’équations paraboliques ou diffusives par un nombre réduit de contrôles est une question ardue qui suscite beaucoup d’intérêt depuis plus d’une dizaine d’années, tout particulièrement dans les cas où les zones de couplage et de contrôle ne s’intersectent pas. Ces systèmes prennent la forme (1) avec \( \theta = 0 \) (resp. \( \theta = \pi/2 \)) dans le cas parabolique (resp. dans le cas de Schrödinger), et où \( \Omega \) est un ouvert à frontière suffisamment régulière dans \( \mathbb{R}^d \), \( Y = (y_1, \ldots, y_N) \) est l’état à contrôler, \( C \) est l’opérateur (borné) de couplage et \( B \) celui du contrôle (borné ou non borné) et \( v = (v_1, \ldots, v_m) \) le contrôle.

Ces systèmes ont été particulièrement étudiés dans le cas de systèmes paraboliques d’ordre 2 couplés en cascade, c’est-à-dire pour lesquels \( N = 2 \), \( m = 1 \) avec \( Bv = (0, v_1)_\omega \) et \( C \) est donné par (2). Ces systèmes apparaissent naturellement dans l’étude de l’existence de contrôles insensibilisants pour l’équation de la chaleur scalaire \([23, 10, 27, 11, 12, 28]\). Des résultats positifs de contrôlabilité à zéro \([27, 6, 7, 17, 18, 20]\) ont été obtenus dans les cas où \( O \cap \omega \neq \emptyset \). Kavian et de Teresa \([19]\) ont montré un résultat de continuation unique pour des systèmes paraboliques couplés en cascade d’ordre 2 dans les cas \( O \cap \omega = \emptyset \). Des résultats locaux de nulle contrôlabilité ont été obtenus dans le cas de deux équations paraboliques non linéairement couplées \([15]\).

L’article de synthèse \([8]\) donne l’état de l’art de ces dernières années sur les systèmes paraboliques couplés (en cascade ou sous des formes plus générales). Il présente notamment plusieurs résultats d’observabilité pour ces systèmes, basés sur des estimations de Carleman et des généralisations de la condition de Kalman en dimension infinie. Il souligne aussi les différences essentielles entre contrôle interne et frontière dans les cas de systèmes couplés.

Indépendamment, la question de la contrôlabilité exacte indirecte de systèmes hyperboliques symétriques d’ordre 2 de la forme (3) a été étudiée par l’auteur dans \([1, 2]\) en introduisant une méthode d’énergie à deux niveaux (cas de couplages coercifs). Ces résultats ont été récemment étendus par l’auteur et Léautaud \([3, 4]\) aux cas de couplings partiellement coercifs et ont permis de déduire des résultats de contrôlabilité à zéro de systèmes couplés paraboliques symétriques dans des cas où \( \omega \cap O \) (ou \( \omega \cap \Gamma_1 = \emptyset \) dans le cas de contrôle frontière). Dans un travail récent...
Rosier et de Teresa [26], ont obtenu des résultats positifs de contrôlabilité de systèmes couplés en cascade d’ordre 2 hyperboliques sous une hypothèse forte de périodicité du semi-groupe associé à une seule équation libre (sans couplage) avec applications au cas de systèmes couplés en cascade d’ordre 2 paraboliques ou de Schrödinger en dimension 1 d’espace (cas parabolique) ou dans des carrés (cas Schrödinger avec condition de Neumann) dans des cas où $\omega \cap O = \emptyset$. Dehman, Léautaud et Le Rousseau [21] ont montré un résultat de contrôlabilité pour des systèmes en cascade d’ordre 2 avec temps minimal de contrôle dans une variété riemannienne. L’approche repose sur une analyse micro-locale fine qui permet notamment de comprendre comment les géodésiques doivent rencontrer d’abord la zone d’observation puis la zone de couplage puis encore la zone d’observation pour un résultat positif d’observabilité.

Nous généralisons les résultats de contrôle aux cas de systèmes couplés en cascade hyperboliques, paraboliques ou de Schrödinger d’ordre 2 sans hypothèse de périodicité dans les Théorèmes 2.1, 2.2 et 2.3 donnés dans la partie anglaise pour des domaines avec bord. Nous indiquons par ailleurs que ces différents résultats se généralisent aux cas de systèmes couplés en cascade hyperboliques, paraboliques ou de Schrödinger d’ordre $N$ avec $N-p$ contrôles, avec $N \geq 2$ et $p$ variant de $N-1$ à 1 et des régions de couplage qui n’intersectent pas les zones de contrôle (frontière ou localement distribué). En particulier, nous montrons qu’il est possible de contrôler un système couplé multi-dimensionnel en cascade d’ordre $N$, hyperbolique parabolique ou de type diffusif, par un seul contrôle frontière ou localement distribué, la zone de contrôle n’intersectant aucune des zones de couplages localisés. Par contre, notre approche ne donne pas le temps minimal de contrôle, contrairement à [21].

Ces résultats sont basés sur une généralisation de la méthode d’énergie à deux niveaux [2] et de son extension récente [4] introduite pour des systèmes couplés symétriques hyperboliques, à des systèmes couplés en cascade hyperboliques d’ordre $N$, $N \geq 2$.

Cette Note est dédiée à la mémoire de mon père Abdallah Boussouira.

1 Introduction

The question of null controllability results for coupled parabolic or diffuse equations is a challenging issue since more than a decade, especially in the cases of localized coupling and control regions with empty intersection and in case of boundary control and localized couplings as well. Such $N$-coupled parabolic or diffusive control systems are given as

$$\begin{cases}
e^{i\theta} y_t - \Delta y + \mathcal{C} y = B v \text{, in } Q_T = \Omega \times (0, T), \\
y = 0 \text{, on } \Sigma_T = \partial \Omega \times (0, T), \\
y(0, .) = y_0(., \text{ in } \Omega),
\end{cases} \tag{1}$$

with $\theta = 0$ (resp. $\theta = \pi/2$) in the parabolic case (resp. for Schrödinger case) and where $\Omega$ is an open non-empty subset in $\mathbb{R}^d$ with a smooth boundary $\Gamma$, $Y = (y_1, \ldots, y_N)$ is the state to be controlled, $\mathcal{C}$ is a coupling bounded operator on $(L^2(\Omega))^N$, $B$ is either a bounded control operator from $(L^2(\Omega))^m$ to $(L^2(\Omega))^N$ or may act only on a part of the boundary of $\Omega$ for some components of the above system, and $v \in L^2((0, T); (L^2(\Omega))^m)$ is the control.

The above systems have received a lot of attention in the case of cascade 2-coupled parabolic systems, that is when $N = 2$ and $\mathcal{C}$ has the form

$$\mathcal{C} = \begin{pmatrix} 0 & 1_0 \\ 0 & 0 \end{pmatrix} \tag{2}$$
where \( m = 1 \) and \( Bv = (0, v1v) \). Here \( O \) and \( \omega \) are open non-empty subsets of \( \Omega \) standing respectively for the coupling and control regions and \( 1O \) stands for the characteristic function of the set \( O \). Cascade systems appear naturally when studying insensitizing controls for the heat equation\([23, 10, 27, 11, 12, 28]\).

De Teresa\([27]\) has studied null controllability results for 2-coupled cascade parabolic systems, motivated by the determination of insensitizing controls for the heat equation in the case \( \omega \cap O \neq \emptyset \). We also refer to\([27, 6, 7, 17, 18, 20, 15]\) for results on null controllability results on coupled parabolic systems by a single control force for either constant coupling operators and locally distributed control, or localized coupling operators and locally distributed control regions with a non-empty intersection between control and coupling regions. These results are based on Carleman estimates for the observability of the adjoint system. In the case \( \omega \cap O = \emptyset \), Kavian et de Teresa\([19]\) proved a unique continuation result for a 2-coupled cascade systems of parabolic equations. Local null controllability results have been obtained for nonlinearly coupled 2-systems of parabolic equations\([15]\). The survey paper\([8]\) presents the state-of-the-art on coupled parabolic systems. In particular, it focuses on observability results for the adjoint system based on Carleman estimates and generalizations of the Kalman rank condition in infinite dimensions. It also stresses fundamental differences between localized and boundary controllability in this context.

On the other hand and independently, the question of controllability of symmetric weakly 2-coupled hyperbolic systems by a single control has been first addressed by the author in\([1, 2]\) by means of a two-level energy method. These systems have the form

\[
\begin{cases}
y_{1,tt} - \Delta y_1 + Cy_2 = Bv, & \text{in } Q_T = \Omega \times (0, T), \\
y_{2,tt} - \Delta y_2 + C^*y_1 = 0, & \text{in } Q_T = \Omega \times (0, T), \\
y_i = 0 & \text{in } \Sigma_T = \partial \Omega \times (0, T), \\
y_i(0, \cdot) = y^0_i(\cdot) & \text{in } \Omega, \end{cases}
\]

This method has been introduced in\([1, 2]\) in a general abstract setting to prove positive controllability results for coercive bounded coupling operators \( C \) (case of globally distributed couplings) and unbounded control operators (case of boundary control). These results have been recently extended by the author and Léautaud in\([3, 4]\) to the case of symmetric weakly 2-coupled hyperbolic systems with localized couplings and localized as well as boundary control. Moreover, using the transmutation method\([24, 25]\), applications to symmetric 2-coupled systems of parabolic and diffusive equations have also been deduced. These results are valid for multi-dimensional wave-like equations under the condition that both the coupling and control regions satisfy the Geometric Control Condition of Bardos Lebeau Ranch\([9]\) (see also\([13, 14]\) for weaker smoothness assumptions on \( \Omega \) and the coefficients of the elliptic operator \( A \)), in particular for cases \( Cz = pz \) and \( Bv = bv \) where \( p \) and \( b \) are nonnegative functions with supports containing respectively \( \overline{\Omega} \) and \( \overline{\omega} \) with \( \omega \cap O = \emptyset \). In a recent work, Rosier and De Teresa\([26]\) considered a 2-coupled system of cascade hyperbolic equations under a strong hypothesis, that is a periodicity assumption of the semigroup associated to a single uncoupled equation. They give applications to 2-coupled systems of cascade one-dimensional heat equations and to 2-coupled systems of cascade Schrödinger equations in a \( n \)-dimensional interval with empty intersection between the control and coupling regions. The method is linked to Däger’s\([16]\) approach and strongly relies on the periodicity assumption of the semigroup for the single free equation. There is a recent very interesting result for 2 coupled cascade systems with localized control by Dehman Léautaud Le Rousseau\([21]\) in a \( C^\infty \) compact connected riemannian manifold without boundary with characterization of the minimal control time using micro-local analysis. Besides the obtention of the minimal control time, another interesting feature of this result is that it allows to understand
how geodesics have to meet first the observation region then the coupling one and then again the observation one for a positive observability result.

This Note concerns the exact controllability of coupled $N$ systems of second order hyperbolic abstract equations in cascade by a reduced number of either boundary or locally distributed controls. We give sufficient conditions on the control and coupling operators for exact controllability to hold in case of $N - p$ controls, $p$ varying from 1 to $N - 1$. We then give applications to cascade systems of wave equations, parabolic equations and Schrödinger equations. Our result is valid for locally distributed as well as boundary controls but it does not give the minimal control time.

We first introduce the abstract setting. Let $H$ and $G$ denote Hilbert spaces with respective norm $| \cdot |, | \cdot |_{G}$ and scalar product $\langle \cdot , \cdot \rangle , \langle \cdot , \cdot \rangle_{G}$. We consider the following control cascade system

$$
\begin{align*}
\begin{cases}
y''_{1} + Ay_{1} + C^{*}y_{2} = 0 , \\
y''_{2} + Ay_{2} = Bu , \\
(y_{i}, y'_{i})(0) = (y_{0}^{i}, y_{1}^{i}) & \text{for } i = 1, 2 ,
\end{cases}
\end{align*}
$$

where $A$ satisfies

$$
(A1) \quad \begin{cases}
A : D(A) \subset H \mapsto H , A^{*} = A , \\
\exists \omega > 0 , |Au| \geq \omega|u| & \forall u \in D(A) ,
\end{cases}
$$

and where $C$ is a bounded operator in $H$, $B \in \mathcal{L}(G, H)$ (resp. $B \in \mathcal{L}(G, (D(A))')$) is the control operator in the case of bounded (resp. unbounded) control, and $v$ is the control. We set $B^{*}(w, w') = B^{*}w'$ (resp. $B^{*}(w, w') = B^{*}w$) when $B \in \mathcal{L}(G, H)$ (resp. $B \in \mathcal{L}(G, (D(A))')$).

We also set $H_{k} = D(A^{k/2})$ for $k \in \mathbb{N}$, with the convention $H_{0} = H$. The set $H_{k}$ is equipped with the norm $| \cdot |_{k}$ defined by $|A^{k/2} \cdot |$ and the associated scalar product. It is a Hilbert space.

We denote by $H_{-k}$ the dual space of $H_{k}$ with the pivot space $H$. We equip $H_{-k}$ with the norm $| \cdot |_{-k} = |A^{-k/2} \cdot |$. We also define the local natural energies as

$$
e_{1}(W)(t) = \frac{1}{2}\left(|A^{1/2}w|^{2} + |w'|^{2}\right) , \quad k \in \mathbb{Z} , i = 1, \ldots , n ,
$$

where $W = (w, w')$.

We are interested in the indirect exact controllability by $L^{2}$ controls for the above system. That is, we are concerned with identifying if: for a sufficiently large time $T$, for all initial data $(y_{1}^{0}, y_{2}^{0}, y_{1}^{1}, y_{2}^{1})$ in a suitable space, it is possible to find a control $v \in L^{2}((0, T); G)$ such that the solution $Y = (y_{1}, y_{2}, y'_{1}, y'_{2})$ of (1) satisfies $Y(T) = 0$. Here the control appears only in the equation for the second component, thus if exact controllability holds it means that the first component is indirectly controlled, indeed through the coupling with a directly controlled equation.

We shall assume that, the adjoint of $B$ is an admissible observation for one equation, that is

$$
(A2) \quad \forall \, T > 0 \exists \, C > 0 , \text{ such that all the solutions } w \text{ of } w'' + Aw = f \text{ satisfy}
\int_{0}^{T} \|B^{*}(w, w')\|_{G}^{2} dt \leq C \left( e_{1}(W(0)) + e_{1}(W(T)) + \int_{0}^{T} e_{1}(W(t)) dt + \int_{0}^{T} \|f\|_{H}^{2} dt \right) .
$$

where $W = (w, w')$. Thanks to this hypothesis, the solution of (1) can be defined by the method of transposition [22]. More precisely, for any $Y_{0} = (y_{0}^{0}, y_{0}^{1}, y_{1}^{0}, y_{1}^{1}) \in H_{1} \times H_{2} \times H_{0} \times H_{1}$ (resp. any $Y_{0} \in H_{0} \times H_{1} \times H_{-1} \times H_{0}$) and any $v \in L^{2}((0, T); G)$, (1) admits a unique solution $Y \in \mathcal{C}([0, T]; H_{1} \times H_{2} \times H_{0} \times H_{1})$ (resp. $\mathcal{C}([0, T]; H_{0} \times H_{1} \times H_{-1} \times H_{0})$ when $B \in \mathcal{L}(G, H)$ (resp. $B \in \mathcal{L}(G, (D(A))'$). We refer to [2, 4] for more details.
2 Main results for 2 coupled cascade systems

We assume the following observability inequalities for a single equation

\[
\begin{align*}
\exists T_1 > 0, T_2 > 0, \text{ such that all the solutions } w \text{ of } w'' + Aw = 0 \text{ satisfy} \\
\int_0^T \| B^*(w, w') \|_G^2 dt \geq C_1(T) e_1(W(0), \forall T > T_1, \\
\int_0^T \| \Pi_p w' \|_{H}^2 dt \geq C_2(T) e_1(W(0), \forall T > T_2,
\end{align*}
\]

and that \( C \) satisfies

\[
\begin{align*}
C \in \mathcal{L}(H_k) \text{ for } k \in \{0, 1, 2\}, ||C|| = \beta, |Cw|^2 \leq \beta(Cw, w) \forall w \in H, \\
\exists \alpha > 0 \text{ and } \Pi_p \in \mathcal{L}(H) \text{ such that, } \alpha |\Pi_p w|^2 \leq \langle Cw, w \rangle \forall w \in H.
\end{align*}
\]

The main results of this Note are the following.

**Theorem 2.1** Assume the hypotheses \((A1) - (A4)\).

- (i) Let \( B^*(w, w') = B^{*} w' \) with \( B \in \mathcal{L}(G, H) \). Then, there exists a time \( T^* \geq \max(T_1, T_2) \) such that for all \( T > T^* \), and all \( Y_0 \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_0 \times \mathcal{H}_1 \), there exists a control function \( v \in L^2((0, T); G) \) such that the solution \( Y = (y_1, y_2, y'_1, y'_2) \) of \((4)\) satisfies \( Y(T) = 0 \).

- (ii) Let \( B^*(w, w') = B^{*} w \) with \( B \in \mathcal{L}(G, H') \). Then, there exists a time \( T^* \geq \max(T_1, T_2) \) such that for all \( T > T^* \), and all \( Y_0 \in \mathcal{H}_0 \times \mathcal{H}_1 \times \mathcal{H}_0 \times \mathcal{H}_1 \), there exists a control function \( v \in L^2((0, T); G) \) such that the solution \( Y = (y_1, y_2, y'_1, y'_2) \) of \((4)\) satisfies \( Y(T) = 0 \).

Let us now give applications to 2-coupled cascade parabolic and Schrödinger systems. We consider the locally distributed control system

\[
\begin{align*}
e^{i\theta} y_{1,t} - \Delta y_1 + cy_2 = 0, & \text{ in } Q_T = \Omega \times (0, T), \\
e^{i\theta} y_{2,t} - \Delta y_2 = bv, & \text{ in } Q_T = \Omega \times (0, T), \\
y_1 = y_2 = 0, & \text{ on } \Sigma_T = \partial \Omega \times (0, T), \\
y_{i}(0, .) = y_{i}^0(\cdot), & \text{ in } \Omega, i = 1, 2,
\end{align*}
\]

and the boundary control system

\[
\begin{align*}
e^{i\theta} y_{1,t} - \Delta y_1 + cy_2 = 0, & \text{ in } Q_T = \Omega \times (0, T), \\
e^{i\theta} y_{2,t} - \Delta y_2 = 0, & \text{ in } Q_T = \Omega \times (0, T), \\
y_1 = 0, y_2 = bv, & \text{ on } \Sigma_T = \partial \Omega \times (0, T), \\
y_{i}(0, .) = y_{i}^0(\cdot), & \text{ in } \Omega, i = 1, 2,
\end{align*}
\]

where we assume for the sequel \( \theta \in [-\pi/2, \pi/2] \), \( c \geq 0 \) on \( \Omega \), \{\{c > 0\} \supset \overline{O} \) and \( b \geq 0 \) on \( \Omega \), \{\{b > 0\} \supset \overline{\omega} \) (resp. \( b \geq 0 \) on \( \Gamma \), \{\{b > 0\} \supset \overline{\Gamma_1} \) in the case of system \((10)\) (resp. \((11)\)), where \( O \) and \( \omega \) are open subsets of \( \Omega \), and where \( \Gamma_1 \subset \Gamma \).

**Theorem 2.2** Assume that the subsets \( O \) and \( \omega \) (resp. \( O \) and \( \Gamma_1 \)) satisfy the Geometric Control Condition and that \( \theta = 0 \). Then, for all \( T > 0 \), for all initial data \((y_1^0, y_2^0) \in (L^2(\Omega))^2 \) (resp. \((y_1^0, y_2^0) \in (H^{-1}(\Omega))^2 \)), there exists a control \( v \in L^2((0, T) \times \Omega) \) (resp. \( v \in L^2((0, T) \times \Gamma_1) \)) such that the solution of \((10)\) (resp. \((11)\)) satisfies \((y_1, y_2)(T, .) = 0 \) in \( \Omega \).

**Theorem 2.3** Assume that the subsets \( O \) and \( \omega \) (resp. \( O \) and \( \Gamma_1 \)) satisfy the Geometric Control Condition and that \( \theta = (-\pi/2, \pi/2) \). Then, for all \( T > 0 \), for all initial data \((y_1^0, y_2^0) \in \)
$L^2(\Omega) \times H^1_0(\Omega)$ (resp. $(y_1^0, y_2^0) \in H^{-1}(\Omega) \times L^2(\Omega)$), there exists a control $v \in L^2((0,T) \times \Omega)$ (resp. $v \in L^2((0,T) \times \Gamma_1)$) such that the solution of (10) (resp. (11)) satisfies $(y_1, y_2)(T,.) = 0$ in $\Omega$.

**Remarks** The above geometric conditions on the coupling region $O$ and the control region $\omega$ (resp. $\Gamma_1$) hold for various examples of subsets $O$ and $\omega$ (resp. $O$ and $\Gamma_1$) such that $O \cap \omega = \emptyset$ (resp. $O \cap \Gamma_1 = \emptyset$) for one-dimensional as well as multi-dimensional sets $\Omega$. In particular it holds for arbitrary open non-empty subsets $O$ and $\omega$ in the one-dimensional case.

The above theorems strongly extends Rosier and de Teresa’s results. This result shows that null controllability of 2-coupled cascade parabolic systems holds in a multi-dimensional setting with empty intersection between the coupling and control regions in a general situation including boundary control. It also extends the results for 2-coupled cascade Schrödinger systems in a multi-dimensional setting without any further periodicity assumption. Nevertheless it does not provide the minimal time control as in Dehman, Léautaud and Le Rousseau result for domains in a multi-dimensional setting without boundary.

One can note also that the results of this Note hold without smallness conditions on the coupling operators.

### 3 Further generalizations to $N$ coupled cascade systems

We more generally consider $N$-coupled control hyperbolic systems driven by $N - p$ controls, where $p \in \{1, \ldots, N - 1\}$ as follows

\[
\begin{cases}
    y_1'' + Ay_1 + C_{21}^* y_2 + \ldots + C_{N1}^* y_N = 0, \\
y_2'' + Ay_2 + C_{32}^* y_3 + \ldots + C_{N2}^* y_N = 0, \\
\vdots \\
y_p'' + Ay_p + C_{p+1}^* y_{p+1} + \ldots + C_{Np}^* y_N = 0, \\
\vdots \\
y_p'' + Ay_p + C_{p+1}^* y_{p+1} + \ldots + C_{Np}^* y_N = B_{p+1}v_{p+1}, \\
\vdots \\
y_{N-1}' + Ay_{N-1} + C_{N,N-1}^* y_N = B_{N-1}v_{N-1}, \\
\vdots \\
y_N'' + Ay_N = B_N v_N, \\
(y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, \ldots, N,
\end{cases}
\]

where $A$ satisfies (A1), the coupling operators $C_{ij}$ are bounded in $H$ for all $i \in \{2, \ldots, N\}$ and all $j \in \{1, \ldots, i-1\}$. We recover system (12) when $N = 2$, setting $C_{21} = C$, $B_2 = B$ and $v_2 = v$. For each $k \in \{p+1, \ldots, N\}$, the control operators $B_k$ can either satisfy $B_k \in \mathcal{L}(G_k, H)$ (bounded case) or $B_k \in \mathcal{L}(G_k, (D(A))^\ell)$ (unbounded case) where $G_k$ are given Hilbert spaces. Moreover we consider the case of $L^2$ controls, that is, we assume that the controls $v_k \in L^2((0,T); G_k)$ for $k \in \{p+1, \ldots, N\}$.

In [5], we give sufficient conditions on the coupling operators $C_{ij}$ for $i \in \{2, \ldots, N\}$, $j \in \{1, \ldots, i-1\}$, on the control operators $B_k$ for $k \in \{p+1, \ldots, N\}$, so that for sufficiently large time $T$, there exist controls $v_k \in L^2((0,T); G_k)$ for $k \in \{p+1, \ldots, N\}$, so that the solution $Y = (y_1, \ldots, y_N)^T$ of (12) satisfies $Y(T) = 0$. Thanks to the transmutation method, we give further applications to the null controllability of $N$-coupled parabolic or diffusive control cascade
systems by either 1, 2, up to \( N - 1 \) controls, each of them possibly chosen either locally distributed or localized on a part of the boundary. Moreover these results hold for localized couplings and localized or boundary controls such that the none of the coupling regions meet the control regions. In particular, we give non trivial examples of \( N \)-coupled systems with \( N \) an arbitrary integer greater than 2 which can be driven to equilibrium at time \( T \) by a single either locally distributed or boundary control. The control spaces depend on the number of requested controls.

These results are based on a generalization of the two-level energy method [2] and its recent extension [4] for 2-coupled symmetric hyperbolic control systems under a smallness condition on the coupling operator, to \( N \)-coupled cascade hyperbolic control systems without smallness conditions. They also rely on the obtaining of suitable observability estimates for the adjoint system.

This Note is dedicated to the memory of my father Abdallah Boussouira.

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