ON SPHERICALLY SYMMETRIC STRUCTURES IN GR

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Abstract. We reconsider some subtle points concerning the relativistic treatment of the gravitational fields generated by spherically symmetric structures.

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Summary. – 1. On the Einsteinian fields generated by spherosymmetrical bodies. The continuity adjustments at the spherical boundary. – 2. On Birkhoff’s theorem. – 2bis. Physical spaces and Bildräume. – 3. Various forms of solution to the problem of the Einsteinian field generated by a mass point. Regular fields outside extended spherosymmetrical distributions of matter. – 4. Event horizons and physical reality. – 5. and 5bis. Geodesic motions of test particles and light-rays in the Einsteinian field of a collapsed spherically symmetric body $B$ with the minimal radius $(9/8)2m$. Gravitational actions of a repulsive kind. – 5ter. The event horizons are incapable of swallowing anything. – Appendices: Some observational consequences of sects. 5, 5bis, 5ter.

1. – As it is known $[\text{III}]$, the solution to the problem of the Einsteinian gravitational field outside a spherosymmetrical mass distribution at rest (extended or point-like) is given – if $r, \vartheta, \varphi$ are spherical polar coordinates – by the following spacetime interval:

$$ds^2 = \left[1 - \frac{2m}{f(r)}\right]c^2dt^2 - \left[1 - \frac{2m}{f(r)}\right]^{-1}\left[\frac{df(r)}{f(r)}\right]^2 - \left[\frac{df(r)}{f(r)}\right]^2d\vartheta^2 - \left[\frac{df(r)}{f(r)}\right]^2d\varphi^2;$$

where: $m \equiv GM/c^2$, $G$ is the gravitational constant; $M$ is the mass of the material distribution; $c$ is the speed of light in vacuo; $f(r)$ is any regular function of $r$, that gives the Newtonian potential $GM/r$ for large values of $r$. Eq.$[\text{III}]$ holds only for $r > 2m$: indeed, for $f(r) \leq 2m$ the $ds^2$ loses its physically essential pseudo-Riemannian character. On the other hand, it is not reasonable (as Marcel Brillouin and Nathan Rosen explicitly remarked) to invert, within $f(r) < 2m$, the roles of the radial and temporal coordinates, thus rendering time dependent a static metrical tensor: a quite unphysical result. Reality cannot be changed by decree.

If we choose $f(r) \equiv r$, we obtain the customary standard form of solution, which was discovered (independently) by Droste, Hilbert, and Weyl. According to a locus communis, this form would be dictated, as it were, by any intrinsically geometric approach. In reality, in any approach of this kind
one starts invariably with the choice $r^2d\omega^2$ for the angular term of $ds^2$, and this implies consequently the usual, standard expression for the other terms.

When one investigates the field of an extended spherically symmetric distribution of matter, one is also confronted with the problem of the continuity adjustment between the internal and the external values of the metric tensor $g_{jk}(x)$, $(j, k = 0, 1, 2, 3)$, $[(x) \equiv (x^0, x^1, x^2, x^3)]$, – and possibly of the continuity adjustment between the internal and the external values of the derivatives $\partial g_{jk}/\partial x^\alpha$, $(\alpha = 1, 2, 3)$, in perfect analogy with the Newtonian theory. In this more satisfactory treatment, the external value of $f(r)$ does not generally coincide with the value of $f(r)$ corresponding to the mass point endowed with the same mass $M$ of the extended distribution. This fact was clarified by Schwarzschild in his second fundamental memoir on GR [2], which solves rigorously the problem of the Einsteinian field generated by a homogeneous sphere of an incompressible fluid. The continuity adjustments of $g_{jk}(x)$ and of $\partial g_{jk}/\partial x^\alpha$ at the spherical boundary tell us that eq.(1) is externally valid not for the function $f(r) \equiv [r^3 + (2m)^3]^{1/3}$ characterizing the original Schwarzschildian form of solution for the gravitational field of a mass point [3], but for the function $f(r) \equiv (r^3 + \varrho)^{1/3}$, where $\varrho$ is a constant different from $2m$.

2. – Birkhoff’s theorem – i.e., the assertion that the $ds^2$ outside of any extended spherically symmetrical distribution of matter does not depend on a possible material motion which keeps the spherical symmetry (for instance, a rhythmical pulsation of the sphere), and satisfies accordingly eq.(II) – is usually demonstrated for the standard form $[f(r) \equiv r]$. It is however intuitive that the theorem is true for any choice of the function $f(r)$. A trivial formal proof runs as follows. To be determinate, let us consider the treatment given by Landau and Lifchitz in sect.97 of their book [4]. As it is well known, it is always possible to start from the following expression of the $ds^2$ – cf. eq.(97,2) of [4]:

$$(2) \quad ds^2 = \exp[\nu(r,t)]c^2dt^2 - \exp[\lambda(r,t)]dr^2 - r^2d\omega^2,$$

which holds within and without the material medium. The functions $\lambda(r,t)$ and $\nu(r,t)$ will be determined by Einstein equations. (Of course, with the above choice for the angular part, Landau and Lifchitz will obtain the standard form of $ds^2$ for the external region).

Now, put in eq.(2), in lieu of the usual polar coordinate $r$, a generic function $f(r)$ of it, and call $u$ this new radial coordinate. We remark immediately that all the computations of sect.97 of [4] remain valid also in this case. Thus, one arrives at these results: i) the function $\lambda$ does not depend on time, and ii):

$$(3) \quad \lambda(u) + \nu(u,t) = \text{a function } F(t) \quad ;$$

it is easy to infer from eq.(3), by means of a suitable change of time variable: $t \rightarrow t' = \psi(t)$, that the external field is always time independent and satisfies eq.(II). Q.e.d. –
2bis. – One remarks usually that the above choice (see eq.(2)) \( r^2 \omega^2 \) for the angular term implies that the surface \( r = \text{constant} \) has the area \( A = 4\pi(\text{constant})^2 \): a Euclidean formula! The explanation is simple: this formula does not give the “natural” expression of the area of the surface \( r = \text{constant} \), but its expression as measured in a suitable three-dimensional Bildraum, an auxiliary (and physically fictitious) flat space. The difference between a physical space and a “picture space” is conceptually essential. However, few authors (e.g., Weyl [2] and Fock [5]) point out explicitly this diversity.

Let us observe that in many instances the intervention of a Bildraum cannot be avoided – and for a plain reason. For clarity’s sake, let us consider again Schwarzschild’s problem. We do not know a priori the precise structure of the spacetime manifold generated by our gravitating mass. Consequently, we are not able to introduce a curvilinear coordinate system that is “really adapted” to the manifold geometry. In practice, we are obliged to start with a coordinate system suggested by simplicity’s considerations, in primis by the symmetry properties of the problem.

3. – The standard form of solution \([f(r) \equiv r]\), when considered for the field of a mass point, has a “hard” singularity at \( r = 0 \) (i.e., a singularity for which Kretschmann’s scalar is infinite) and a “soft” singularity at \( r = 2m \) (Kretschmann’s scalar of a finite value). (Of course, this form is physically and mathematically valid only for \( r > 2m \), contrary to a diffuse belief.)

It is instructive to compare the above form with other forms of solution, in particular with Fock’s form, for which \( f(r) \equiv r + m \) [5], and with Schwarzschild’s [3] and Brillouin’s [3] forms for which \( f(r) \equiv [r^3 + (2m)^3]^{1/3} \) and \( f(r) = r + 2m \), respectively. For a moment, and only for clarity’s sake, let us call \( r' \) the radial coordinate of standard form, and with \( r'' \) and \( r''' \), respectively, the radial coordinates of Fock’s form and of Schwarzschild’s and Brillouin’s forms. Fock’s \( r'' \) has its origin \( (r'' = 0) \) at \( r' = m \); Schwarzschild’s and Brillouin’s \( r''' \) has its origin \( (r''' = 0) \) at \( r' = 2m \). Thus the “hard” singularity at \( r' = 0 \) of standards \( ds^2 \), which belongs to the unphysical region \( 0 \leq r' \leq 2m \) that impairs the pseudo-Riemannian character of the interval, has been removed. Fock’s \( ds^2 \) holds only for \( r'' > m \). Schwarzschild’s and Brillouin’s \( ds^2 \)’s hold only for \( r''' > 0 \): they are maximally extended. It is evident that the above forms, considered for \( r' > 2m, r'' > m, r''' > 0 \) respectively, are diffeomorphic, and therefore mathematically and physically equivalent.

The known forms of solution by Lemaître (1933), Synge (1950), Finkelstein (1958), Kruskal and Szekeres (1960) are superfluous exertions; moreover, they make an essential use of coordinate transformations the derivatives of which are singular at \( r' = 2m \) “in just the appropriate way for providing a transformed metric that is regular there” (Antoci and Liebscher, 2001). We observe that the singularity \( r' = 2m \) (or \( r'' = m \), or \( r''' = 0 \)) corresponds to the existence of a gravitating mass point – and therefore it does not represent a defect of the coordinate chart.
It is not difficult to see that eq. (1) admits of infinite functions \( f(r) \) such that the corresponding \( ds^2 \) is \textit{everywhere regular} for \( r \geq 0 \) \[6\]; e.g., \( f(r) \equiv r + 3m \) (i.e., \( r' \equiv r^{IV} + 3m \geq 3m \)).

These regular solutions can be interpreted as representing the \textit{external} values of the \( g_{jk} \)'s of various spherically symmetric distributions with different structures and different radii. We shall see that a particularly significant solution of this kind is that for which \( f(r) \equiv r + (9/8)2m \) – i.e., \( r' \equiv r^{IV} + (9/8)2m \geq (9/8)2m \).

4. – The so-called \textit{event-horizons} corresponding to the singular geometric \textit{loci} of the various forms of solution for a mass point – in particular, \( r = 2m \) for the standard form, \( r = m \) for Fock’s form, \( r = 0 \) for Schwarzschild’s and Brillouin’s forms – have a very dubious physical meaning, if considered with unprejudiced mind (see sect. \textbf{5ter}).

We remark that: \( i) \) as it can be rigorously demonstrated \[7\], the spherosymmetrical gravitational collapse of a massive (or supermassive) celestial body with a time-dependent pressure ends in a small structure endowed with a \textbf{finite} volume; \( ii) \) the minimal radius of spherically symmetric bodies (of a given mass \( M \)) consisting of perfect fluids at rest (in particular, of incompressible fluids \[2\]) is equal to \((9/8)(2m)\), see \[2\], \[8\].

Last, but not least, we wish to emphasize that it is not legitimate to hypothesize the existence of an event horizon in a celestial body which is a member of a binary system – an assumption that many authors do. Indeed, as it was pointed out by McVittie many years ago (in 1972), an existence theorem would be needed to show that Einstein equations contain solutions which represent a binary system of stars having as a member a gravitating mass point. A simple analogy with Newton gravitational theory is \textbf{not} sufficient.

5. – The precise meaning of mentioned (see sect. \textbf{4}) minimal radius \((9/8)/2m\) is the following: it is “\textit{der außen gemessene Radius}” \( P_{a,\text{min}} \) (see \[2\]) by means of which the spherical volume \( V_{a,\text{min}} \) of the considered material distribution is measured in a given \textit{Bildraum} (not in the real spacetime manifold!) by \((4/3)\pi P_{a,\text{min}}^3\).

In the \textit{standard} coordinate system we have \( P_a \equiv r_a \), where \( r_a \) is the radial coordinate of the points of the spherical body.

The final volume \( V_{\text{fin}} \) of a collapsed massive star with time-dependent pressure and mass density, \( p(t) \) and \( \varrho(t) \), is \[7\]:

\[
V_{\text{fin}} = \frac{4}{3} \pi \left[ \varepsilon_0^2 \left( 1 + \varepsilon_0 \right)^{-2} \right] r_b^3,
\]

where \( \varepsilon_0 \equiv p(0)/(c^2 \varrho(0)) \), and \( r_b \) is the radial coordinate – in a Friedmann’s coordinate system – of the spherical boundary at \( t = t_0 = 0 \). Both \( V_{a,\text{min}} \) and \( V_{\text{fin}} \) are defined in flat \textit{Bildräume}, which are \textit{in abstracto} the same three-dimensional “picture space”. Accordingly, we can put \( [P_{a,\text{min}} = r_{a,\text{min}} = (9/8)2m] \):
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\[
\frac{4}{3} \pi [(9/8)2m]^3 = \frac{4}{3} \pi \left[ \varepsilon_0^2 (1 + \varepsilon_0)^{-2} \right] r_b^3 ,
\]
from which:

\[
\frac{9}{4} m = \left[ \varepsilon_0^{2/3} (1 + \varepsilon_0)^{-2/3} \right] r_b ;
\]
for a given mass \( M \), suitable choices of \( \varepsilon_0 \) and \( r_b \) allow us to satisfy this equation.

By exploiting now some beautiful computations by Hilbert \([9]\) of the geodesic lines in the spacetime manifold of a gravitating mass point, we shall exhibit a significant diagram (see Fig. 1) which shows the squared velocity \((dr/cdt)^2\) of a test particle in radial motion through the external region of a spherically symmetric body of the minimal radius \((9/8)/2m\).

Hilbert begins with a fundamental remark: it is always possible to find a suitable coordinate system for which both the gravitating centre and the test particle are at rest \([10]\): a consequence of “plasticity” of the reference frames of GR. (Accordingly, the notion of affine geodesic completeness of a manifold is not of primary importance in GR).

Then, Hilbert finds the most general expression of geodesic lines, and investigates in detail the motions on the orbits \( r = \text{constant} \), and the radial ones.

The circular motions are restricted by the following relations (Hilbert’s \( \alpha \) is equal to our \( 2m \)):

\[
(7) \quad r > \frac{3}{2}(2m) > \frac{9}{8}(2m) ,
\]

\[
(8) \quad \frac{v}{c} < \frac{1}{\sqrt{3}} ,
\]

where \( v = c(m/r)^{1/2} \) is the ordinary velocity (Hilbert puts \( c = 1 \)). They are a striking consequence of spatio-temporal curvature, which acts repulsively for small values of the radial coordinate \( r \). Inequality \((7)\) represents a reinforcement of \( r > 2m \), the validity condition of standard \( ds^2 \). The meaning of inequality \((8)\) is the following: when the coordinate \( r \) of the circular orbit decreases, the particle velocity tends to the maximal value \( c/\sqrt{3} \) – in contrast with Newton theory, for which this velocity increases illimitably, because there is no restriction as inequality \((7)\).

Hilbert remarks that Schwarzschild’s result \([3]\) \( v < c/\sqrt{2} \) has been obtained as a consequence of inequality \( r > 2m \), which is weaker than relation \((7)\). (Of course, Hilbert has made the formal translation from Schwarzschild’s coordinate frame to the standard one.)

Finally, Hilbert points out that the general equation of motion of the test particle \([11]\) admits as solutions infinite curves that approach indefinitely by spiraling every allowed circular orbit – as it is required by Poincarè’s general theory of orbits \([12]\). For the circular motions of light we have a coordinate radius \( r = (3/2)(2m) \) and a velocity \( v = c/\sqrt{3} \). There are infinite Poincarè’s curves that approach indefinitely by spiraling this circular trajectory.
For the radial motions of a test particle the action of the spacetime curvature is quite peculiar, as we shall see. The differential equation of these motions is \((2m < r < \infty)\) \[13\]:

\[
\frac{1}{c^2} \frac{d^2 r}{dt^2} - \frac{3}{2} \frac{2m}{r(r-2m)} \left( \frac{dr}{c dt} \right)^2 + \frac{m(r-2m)}{r^3} = 0 ,
\]

with the following integral \((A\) is an integration constant with negative values; we have \((2/3) \leq |A| \leq 1)\):

\[
\left( \frac{dr}{c dt} \right)^2 = \left( \frac{r-2m}{r} \right)^2 + A \left( \frac{r-2m}{r} \right)^3 .
\]

According to eq.\((9)\), the acceleration is negative (gravitation acts attractively), or positive (gravitation acts repulsively) when, respectively:

\[
\left| \frac{dr}{c dt} \right| < \frac{1}{\sqrt{3}} \frac{r-2m}{r} ,
\]

or

\[
\left| \frac{dr}{c dt} \right| > \frac{1}{\sqrt{3}} \frac{r-2m}{r} .
\]

It is instructive to consider the case \(A = -1\); the test particle starts from \(r = \infty\) with zero velocity: \(dr/dt = 0\).

If we set for brevity \(x := r/(2m)\) and \(y := (dr/c dt)^2\), we see that the function

\[
y(x) = \left( \frac{x-1}{x} \right)^2 \left[ 1 - \frac{x-1}{x} \right] ; \quad 1 < x < \infty ,
\]

reaches its maximum value \(2^2/3^3\) at \(x = 3\): \(y(3) = 2^2/3^3\). At \(x = 9/8\), we have \(y(9/8) = 2^4/3^6\); \((9/8) = 1, 125)\); \(y(9/8)/y(3) = 2/3^3\); \(\sqrt{y(9/8)} = (dr/c dt)_{r=(9/8)2m} = 2\sqrt{2}/3^3 \approx 0, 104757\). In Fig. 1 we give a diagram of function \(13\) for some significant values. We see the impressive fall of particle velocity for \(x < 3\): a quite “anti-Newtonian” result!

For a radial motion of light, we have immediately from \(ds^2 = 0\) that

\[
\left| \frac{dr}{c dt} \right| = \frac{r-2m}{r} ; \quad (A = 0) ;
\]

by virtue of inequality \(12\), eq.\((14)\) tells us that light is repulsed by our extended body. If a light-ray starts with velocity \(c\) at \(r = \infty\), it arrives at \(r = (9/8)2m\) with velocity \((1/9)c\).

Finally, as Schwarzschild demonstrated \(2\), the light arrives at the centre of the body of radius \((9/8)2m\) with a velocity equal to zero \(14\): an important result for explaining the observational data concerning some X-ray novae \(15\).

Newtonian limit of eq.\((9)\): if \(2m\) and the particle velocity \(dr/dt\) are small, eq.\((9)\) is approximately equal to
\[
\frac{d^2 r}{dt^2} = -\frac{c^2 m}{r^2} = -\frac{GM}{r^2}.
\]

5bis. – If \( A := -|A|, x := r/(2m), y := (dr/c dt)^2 \), eq. (10) can be written as follows:

\[
y(x) = \left(\frac{x - 1}{x}\right)^2 \left[1 - |A|\frac{x - 1}{x}\right], \quad (1 < x < \infty).
\]

The value \( x = x_M \) for which \( y(x_M) \) gives the maximal value of \( y(x) \) is:

\[
x_M = \frac{3|A|}{3|A| - 2} \quad ; \quad \left(\frac{2}{3} \leq |A| \leq 1\right);
\]

if \( |A| = 2/3 \), we have \( x_M = \infty \), \( y(x_M) = 1/3 \), i.e. \( dr/dt = c/\sqrt{3} \). For any value of the mass of gravitating centre (extended or point-like), a test particle which starts from \( r = r_M = \infty \) with velocity \( v = c/\sqrt{3} \) must travel against a repulsive gravitational action. It will arrive at \( r = (9/8)2m \) with the velocity \( v = (5/27)(c/\sqrt{3}) \).

For \( |A| = 1 \), we know (see sect.5) that the velocity at \( r = (9/8)2m \) is \( v = (2\sqrt{2}/27)c \), if it had started from infinite with a zero velocity; in this case, the gravitational actions of Sun, planets, white dwarfs, and neutron stars are only attractive.

For \( |A| = 0.8 \), \( x_M = 6.0 \); for \( |A| = 0.83 \), \( x_M = 5.75 \); for \( |A| = 0.89 \), \( x_M = 4.0 \).

For \( x_M = 6.0 \), \( r_M = 6 \cdot 2m \); for a neutron star \( 2m \approx 5.3 \) km, \( r_M \approx 31.8 \) km; the star radius \( P_a \approx 10 \) km. Thus along \( 21.8 \) km gravitation acts repulsively.

5ter. – The belief in the physical significance of the event horizons encounters a great difficulty. As it follows from previous sects.5, 5bis, when the gravitating body is a mass point, the test particles and the light rays do not reach generally in their motion the surface \( r = 2m \); only in the radial motions (see eqs. (9), (10), (14), in particular) they can approach this surface, but they arrive at it with velocities \( dr/dt \) and accelerations \( d^2 r/dt^2 \) which are equal to zero. (Of course, we have an identical conclusion if we describe the Einsteinian field of the gravitating mass point with the forms of solution, e.g., by Schwarzschild and Brillouin [3] or by Fock [5]: test particles and light rays arrive at \( r = 0 \), resp. at \( r = m \), with velocities and accelerations equal to zero.)

Accordingly, the event horizons are incapable of swallowing anything! –

Unfortunately, in the current literature the classic memoirs by Schwarzschild, Hilbert and Levi-Civita are ignored or disfigured [16].
APPENDIX A

The title of the first paper quoted in [15] is “X-ray novae and the evidence from black hole event horizons”. – The abstract runs as follows: “We discuss new observations of X-ray novae which provide strong evidence that black holes have event horizons. Optical observations of 13 X-ray novae indicate that these binary stars contain collapsed objects too heavy to be stable neutron stars. The objects have been identified as black hole candidates. X-ray observations of several of these X-ray novae in quiescence with the Chandra X-ray Observatory show that the systems are approximately 100 times fainter than nearly identical X-ray novae containing neutron stars. The advection-dominated accretion flow [ADAF] model provides a natural explanation for the difference. In this model, the accreting gas reaches the accretor at the center with a large amount of thermal energy. If the accretor is a black hole, the thermal energy will disappear through the event horizon, and the object will be very dim. If the accretor is a neutron star or any other object with a surface, the energy will be radiated from the surface, and the object will be bright. […]”. –

By virtue of the results of sects. 5, 5bis, 5ter, the above explanation of the observational data by means of the notion of event horizon is not reasonable.

A simple explanation is obtained if we consider, in lieu of a gravitating mass point, a gravitating small body $B$ of radius $(9/8)2m$. As we have seen in sect. 5, when a light-ray in radial motion reaches $B$, it goes through $B$ and arrives at its center with a velocity equal to zero. The accretor is now the body $B$. The accreting gas in radial motion reaches $B$ with a large amount of thermal energy, and with a remarkable speed. This very hot material heats up star $B$. A considerable amount of the energy will go through our accretor as e.m. radiation, and will arrive at its centre with a velocity equal to zero. In other terms, a significant part of the radiation will not escape from the stellar surface, but will be “absorbed” by object $B$, which will be dim.

We remark that, since $(9/8)2m < (3/2)2m$, no Poincaré’s spiraling orbit of particle, or light-ray, can reach the surface of $B$.

APPENDIX B

The title of the second paper quoted in [15] is “X-ray QPOs in Black-Hole Binary Systems”, where QPOs means “quasiperiodic oscillations”. In the second paragraph of the “Introduction” we read: “Observations with the Rossi X-ray Timing Explorer (RXTE) have pioneered efforts to further study black holes and their occasional relativistic jets via broad-band X-ray observations during active states of accretion. The X-ray timing and
spectral properties convey information about physical processes that occur near the black hole event horizon, and one of the primary research goals is to obtain constraints on the black hole mass and spire using predictions of general relativity (GR) in the strong field regime."

This paper and the second review article quoted in [15] concern the event horizons of BH’s with spin, i.e. the event horizons of Kerr’s corpuscles [17]. We shall give in a next Note a reasonable explanation of the observational data reported in these articles.

Figure 1. Diagram of $y(x) = [(x - 1)/x]^2[1 - (x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; $\max(3,0,4/27)$; $[y(9/8)]^{1/2} = 2\sqrt{2}/27$.

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[10] For two different generalizations of this result, see: i) L. Landau et E. Lifchitz [4], sect.82, p.291, footnote 2; ii) Weyl [2], p.268.

[11] See Hilbert [9], formula (46).

[12] See e.g. E.T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies* etc., Fourth Edition, 1937 (Cambridge University Press, Cambridge) 1979, Chapt. XV.

[13] See Hilbert [9], formula (53) and foll. –

[14] For a trivial oversight, formula (44) of paper [2] concerns in reality 1/v, and not v (light speed).

[15] See R. Narayan et al., arXiv:astro-ph/0107387 v2 (October 8th, 2001); R. Remillard et al., arXiv:astro-ph/0208402 v1 (August 21st, 2002). A review article by J.E. McClintock and R.A. Remillard, entitled “Black Hole Binaries”, in arXiv:astro-ph/0306213 v4 (June 23rd, 2004). A review article by J.E. McClintock entitled “The Spin of the Near-Extreme Kerr Black Hole GRS 1915+105”, in *Astropys. J.*, 652518 (2006-06-04).

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