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Frontiers

Investigating a nonlinear dynamical model of COVID-19 disease under fuzzy Caputo, random and ABC fractional order derivative

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A R T I C L E   I N F O

Article history:
Received 15 July 2020
Revised 25 July 2020
Accepted 21 August 2020
Available online 25 August 2020

Keywords:
Qualitative theory
Fuzzy fractional dynamical system
Random fractional derivative
Mathematical model of COVID-19

A B S T R A C T

This paper is devoted to investigation of the fractional order fuzzy dynamical system, in our case, modeling the recent pandemic due to corona virus (COVID-19). The considered model is analyzed for exactness and uniqueness of solution by using fixed point theory approach. We have also provided the numerical solution of the nonlinear dynamical system with the help of some iterative method applying Caputo as well as Atangana-Baleanu and Caputo fractional type derivative. Also, random COVID-19 model described by a system of random differential equations was presented. At the end we have given some numerical approximation to illustrate the proposed method by applying different fractional values corresponding to uncertainty.

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1. Introduction

Here we discuss that in recent time a terrible outbreak, which has been originated from China is spreading throughout the world very rapidly. Thousands of people have been faced death due to this disease. The outbreak of a deadly and highly infected virus of the present era is a corona virus and it is identified in the Wuhan (Chinese city) on December 31, 2019 [1,2]. Since then it has been killed over 0.5 million of people, while the infected people are more than 13 millions in more than 180 countries. The history of this virus traced back to 1965, when Tyrrell and Bynoe have identified and they passaged a virus named B814 [3]. This virus is found in human embryonic tracheal organ cultures acquired from the respiratory tract of an adult [4].

Different researchers and policy makers are struggling to control the disease from further spreading. One big factor of spreading this disease is immigration of infected people from place to place which effect more people and hence cause spreading this disease. Therefore on international level, many countries of the world have banned air traffic for some time and also they have announced lock-down in cities so that some precautionary measure should be taken to reduce maximum loss of human lives. Also each country in the globe try to reduced unnecessary traveling of people and to reduce the cases of infection in their country [5]. Since scientists and researchers are trying to investigate cure or vaccine for the aforesaid outbreak so that in future such like pandemic may be controlled.

Understanding properly about the pandemic plays useful role in controlling of the disease in a society. Implementation of a suitable strategy against the disease transmission is another challenge. From medical engineering point of view, mathematical modeling approach is one of the key tool in order to handle these infectious diseases. Mathematical models have been established for different disease in history, for study, we refer [6–9]. Similarly the mentioned outbreak has been reported in large numbers of articles, reports, monographs, etc. (for detail see [1,2,10–17]).

The mathematical models are mostly differential and integral equations of integer order (IDEs). However, for the last few decades, the non-integer order differential equations (FDEs) can be used to formulate real phenomena with greater degree of precision and accuracy. Further, their applications can be found in different areas of physical and medical science, like engineering, economics, control theory, finance and in epidemiology. Modern calculus is the generalization of classical integer-order calculus. The increasing interest of using FDEs in modeling of real world problems is due to its various properties which are not found in IDEs. In contrast of IDEs which are local in nature, the FDEs are non-local and possesses the memory effects which make it more superior than IDEs. It is also because, in many situations the future state of the model depends not only upon the current state but also on the previous history[18–21]. These features enables FDEs to model the phenom-
Table 1
Description of the parameters given in model (1).

| Notation | Parameters description |
|----------|------------------------|
| $a$      | The population whose tests are negative |
| $d_0$    | Natural death |
| $b$      | The population whose tests are positive |
| $\mu$    | Death due to Corona |
| $K$      | Proportionality constant |
| $\kappa$ | Infected rate |
| $a'$     | Rate at which recovered individuals lose immunity |
| $\beta$  | Recovered rate |

ena having not only the non-Gaussian but also for non-Markovian behavior. Further, the classical IDEs are unable to provide the information in between two different integer values. Various type of fractional-order operators were introduced in existing literature to over come such limitations of integer-order derivatives. The applications of these fractional operators can be found in various fields.

The worm and interesting area of research in recent time is mostly devoted to investigate biological models of infectious diseases. Many investigation about the mathematical models are devoted to study stability theory, existence results and optimization, we refer few as [22–25].

Due to the current situations, many studies have been recorded on mathematical modeling of outbreak of COVID-19, we refer few as [26–29]. Currently this area of mathematical models for the coronavirus infectious diseases is warm area of research. Therefore in [26] authors considered the following model of four compartment including healthy or susceptible population $S(t)$, the exposed class $E(t)$, the infected population $I(t)$ and the removed class $R(t)$ (death due to corona or natural) at time $t$ as

\[
\begin{align*}
\frac{dS(t)}{dt} &= -K(t)S(t)(1 + \alpha I(t)) - d_0 S(t), \\
\frac{dE(t)}{dt} &= K(t)S(t)(1 + \alpha I(t)) - (d_0 + \kappa) E(t), \\
\frac{dI(t)}{dt} &= b + \alpha E(t) - (\mu + d_0 + \beta) I(t), \\
\frac{dR(t)}{dt} &= \beta I(t) - d_0 R(t).
\end{align*}
\]

(1)

\[
S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad R(0) = R_0.
\]

The details of the parameters written in the model (1), with complete descriptions are given in Table 1.

Differential equations and classical calculus have been extended to many field of applied and pure mathematics. Like classical calculus to modern and fuzzy calculus and differential equations to fractional order and fuzzy fractional order. In 1965, Zadeh [30] defined the fuzzy concepts about set theory. It is obvious that the idea of fuzzy has been discussed in variety of subjects like fuzziness for fixed point theory, fuzziness for topological, fuzziness for control system and fuzziness for automata and so on. In 1996 Chang and Zadeh [31] extend the notion of fuzzy set to fuzzy mapping and control. Many researcher generalized fuzzy mapping and control to define elementary fuzzy calculus. Dobrus and Prada [32] defined the concept of fuzzy integral equations. Therefore in data imprecision or vagueness is occurred, the fuzzy concept is better to described the parameters then instead of crisp model.

In eighteenth century Riemann and Liouville, Euler and Fourier provided useful results in basics of modern calculus. Because of their contribution the area of modern calculus was also established and some good research has been carried out later on. This is due to lots of applications of modern calculus in the filed of mathematical modeling, where several hereditary concepts and memory process have been explained clearly, while classical calculus failed to explain such type of properties. Fractional calculus is the globalization of classical calculus with greater degree of freedom in differential operator as compared to integer differential operator which is local in nature. The important applications of the said calculus may be traced out in [18–21,33–36]. Therefore, researchers and scientists have given very much interest in discussion of arbitrary order of derivatives and integrals. In fact fractional derivative is a definite integral which geometrically interpret the accumulation of the whole function or the whole spectrum which globalize it. On the other hand ordinary derivative is a special case of the fractional order. For analysis of differential equations for qualitative, numerical and optimization of solutions, we refer few as [37–42]. It is also remarkable that fractional differential operators have been defined by number of ways. It is well known fact that definit integral has no regular kernel, therefore both type of kernel have been involved in various definitions. One of the important definition which has very recently attracted the attention is the ABC derivative introduced by Atangana-Baleanu and Caputo [43] in 2016. The mentioned derivative exhibit the singular kernel by non-singular kernel and therefore were greatly studied [44–52].

2. Basic definitions

Here, we recall some definition from [18,19,53].

**Definition 1.** For a function say $y(t)$ we define fractional integral corresponding to $t$ as

\[
P^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha - 1} y(\zeta) d\zeta, \quad \alpha > 0,
\]

such that antiderivative converges to same value.

**Definition 2 ([54]).** Let we have a continuous fuzzy function $\tilde{y}$ on $[0, b] \subset \mathbb{R}$, we define fuzzy fractional integral in Riemann-Liouville sense corresponding to $t$ as

\[
P^\alpha \tilde{y}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha - 1} \tilde{y}(\zeta) d\zeta, \quad \alpha, \zeta \in (0, \infty).
\]

(3)

Further, if $\tilde{y} \in \mathcal{C}^\alpha[0, b] \cap L^1[0, b]$, where $\mathcal{C}^\alpha \subset [0, b]$ is the space of fuzzy continuous functions and $L^1 \subset [0, b]$ is a space of fuzzy Lebesgue integrable functions respectively, then fuzzy fractional integral is defined as

\[
\left[ P^\alpha \tilde{y}(t) \right]_p = [P^\alpha \tilde{y}_p(t), P^\alpha \tilde{y}_p(t)], \quad 0 \leq p \leq 1,
\]

such that

\[
P^\alpha \tilde{y}_p(t) = \int_0^t (t - \zeta)^{\alpha - 1} \tilde{y}_p(\zeta) \frac{1}{\Gamma(\alpha)} d\zeta, \quad \alpha, \zeta \in (0, \infty),
\]

\[
P^\alpha \tilde{y}_p(t) = \int_0^t (t - \zeta)^{\alpha - 1} \tilde{y}_p(\zeta) \frac{1}{\Gamma(\alpha)} d\zeta, \quad \alpha, \zeta \in (0, \infty).
\]

**Definition 3.** Consider a mapping, say $y(t)$, we define the Caputo arbitrary order derivative corresponding to $t$ as

\[
\frac{d^\alpha y(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \zeta)^{n - \alpha - 1} \frac{d^n y(\zeta)}{d\zeta^n} d\zeta, \quad \alpha > 0,
\]

where

\[
\frac{d^r y(t)}{dt^r} = (d^r y)(t) = (r - 1, 1 - r) \quad \text{and} \quad 0 \leq r \leq 1.
\]
Definition 4 ([54]). In some line for a function $\tilde{y} \in C^f[0, b] \cap L^f[0, b]$, such that $\tilde{y} = [\tilde{y}_p(t), \tilde{y}_p(t)]$, $p \in [0, 1]$ and $t_0 \in (0, b)$, then the fuzzy Caputo fractional derivative is given as
$$\left[\Gamma^D_p\tilde{y}_p(t)\right]_p = \left[\Gamma^D_p\tilde{y}_p(t), \Gamma^D_p\tilde{y}_p(t)\right],$$
where,
$$\Gamma^D_p\tilde{y}_p(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_{t_0}^{t} (t - \zeta)^{n-\alpha-1} \frac{d^n}{d\zeta^n} \tilde{y}_p(\zeta) d\zeta\right]_{\zeta = t_0},$$
$$\Gamma^D_p\tilde{y}_p(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_{t_0}^{t} (t - \zeta)^{n-\alpha-1} \frac{d^n}{d\zeta^n} \tilde{y}_p(\zeta) d\zeta\right]_{\zeta = t_0}.$$
such that the integration on the right sides converges and $n = [\alpha]$. Since $\alpha \in [0, 1]$ so $n = 1$.

Definition 5. The ABC fractional differentiation of a function $y(t)$ having the condition $y(t) \in H^1(0, \tau)$ is given by
$$\mathcal{ABC}\mathcal{D}_p^\alpha(y(t)) = \frac{\mathcal{ABC}(\alpha)}{1 - \alpha} \int_0^t \frac{d}{d\zeta} y(\zeta) [ \frac{-\alpha}{1 - \alpha} (t - \zeta) ] d\zeta.$$(4)

Definition 6. In some line for a function $\tilde{y} \in C^f[0, b] \cap L^f[0, b]$, such that $\tilde{y} = [\tilde{y}_p(t), \tilde{y}_p(t)]$, $p \in [0, 1]$ and $t_0 \in (0, b)$, then the fuzzy Atangana-Baleanu-Caputo (ABC) fractional derivative is given as
$$\left[\mathcal{ABC}\Gamma^D_p\tilde{y}_p(t)\right]_p = \left[\mathcal{ABC}\Gamma^D_p\tilde{y}_p(t), \mathcal{ABC}\Gamma^D_p\tilde{y}_p(t)\right],$$
where,
$$\mathcal{ABC}\Gamma^D_p\tilde{y}_p(t) = \frac{\mathcal{ABC}(\alpha)}{1 - \alpha} \left[\int_0^t \frac{d}{d\zeta} \tilde{y}_p(\zeta) [ \frac{-\alpha}{1 - \alpha} (t - \zeta) ] d\zeta\right]_{\zeta = t_0},$$
$$\mathcal{ABC}\Gamma^D_p\tilde{y}_p(t) = \frac{\mathcal{ABC}(\alpha)}{1 - \alpha} \left[\int_0^t \frac{d}{d\zeta} \tilde{y}_p(\zeta) [ \frac{-\alpha}{1 - \alpha} (t - \zeta) ] d\zeta\right]_{\zeta = t_0}.$$
such that the integration on the right sides converges and $n = [\alpha]$. Since $\alpha \in [0, 1]$ so $n = 1$.

Definition 7. Let $y(t) \in L[0, T]$, then the corresponding integral in ABC sense is given by
$$\mathcal{ABC}\mathcal{I}_p^\alpha(y(t)) = \frac{1 - \alpha}{\mathcal{ABC}(\alpha)} y(t) + \frac{\alpha}{\mathcal{ABC}(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} y(\zeta) d\zeta.$$(5)

Definition 8. Let we have a continuous fuzzy function $\tilde{y}$ on $[0, b] \subset R$, we define fuzzy fractional integral ABC sense corresponding to $t$ as
$$\mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}(t)) = \frac{1 - \alpha}{\mathcal{ABC}(\alpha)} \tilde{y}(t) + \frac{\alpha}{\mathcal{ABC}(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \tilde{y}(\zeta) d\zeta, \alpha, \zeta \in (0, \infty).$(6)

Further, if $\tilde{y} \in C^f[0, b] \cap L^f[0, b]$, where $C^f \in [0, b]$ is the space of fuzzy continuous functions and $L^f \in [0, b]$ is a space of fuzzy Lebesgue integrable functions respectively, then fuzzy fractional integral is defined as
$$\left[\mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}(t))\right]_p = \left[\mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}_p(t), \mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}_p(t)\right], \quad 0 \leq p \leq 1.$$
such that
$$\mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}_p(t)) = \frac{1 - \alpha}{\mathcal{ABC}(\alpha)} \tilde{y}_p(t) + \frac{\alpha}{\mathcal{ABC}(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \tilde{y}_p(\zeta) d\zeta, \quad \alpha, \zeta \in (0, \infty),$$
$$\mathcal{ABC}\mathcal{I}_p^\alpha(\tilde{y}_p(t)) = \frac{1 - \alpha}{\mathcal{ABC}(\alpha)} \tilde{y}_p(t) + \frac{\alpha}{\mathcal{ABC}(\alpha)\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha-1} \tilde{y}_p(\zeta) d\zeta, \quad \alpha, \zeta \in (0, \infty).$$

Lemma 1 ([54]). The Laplace transform (LT) of the Caputo derivative for $0 < \alpha \leq 1$ is given as
$$\mathcal{L}\left[\frac{d^\alpha \tilde{y}(t)}{dt^\alpha}\right] = s^\alpha \mathcal{L}[\tilde{y}(t)] - s^{\alpha-1}[\tilde{y}(0)].$(7)

Definition 9 ([54]). A fuzzy number of mapping $P: R \to [0, 1]$ if it satisfies the following conditions:
1. $P$ is upper semi continuous;
2. $P(\mu(y_1) + \mu(y_2)) \geq \min[P(y_1), P(y_2)];$
3. There exist $y_0 \in R$ such that $P(y_0) = 1$, e $P$ is normal;
4. $cI[y \in R, P(y) > 0]$ is compact.

Here we denotes the set of all fuzzy numbers by $E$.

Definition 10. ([54]) The parametric form of a fuzzy number can be written in the form of order pair as
$$[\tilde{k}(r), \tilde{r}(r)],$$
such that $0 \leq r \leq 1$, which has the given properties:
1. \( k(r) \) is left continuous, bounded and increasing function over \([0,1]\) and right continuous at 0;
2. \( \overline{k}(r) \) is right continuous, bounded decreasing function over \([0,1]\) and right continuous at 0;
3. \( \overline{k}(r) \leq \overline{k}(r) \).

\( r \) is called crisp number if \( k(r) = \overline{k}(r) = r \).

Let \( E \) denote the set of upper semi-continuous, convex and normal fuzzy numbers with bounded \( \theta \)-level interval which yields that \( v \in E \), then \( \theta \)-level set

\[ [\nu]^{\theta} = \{ t : \nu(t) \geq \theta \}, \quad 0 \leq \theta \leq 1 \]

which is bounded and closed interval represented by

\[ [\nu]^{\theta} = [\underline{\nu}(\theta), \overline{\nu}(\theta)] \).

For arbitrary fuzzy number

\[ v = (\underline{\nu}(\theta), \overline{\nu}(\theta)), \quad w = (\underline{\nu}(\theta), \overline{\nu}(\theta)) \]

and for \( \kappa_1 \geq 0 \), various operations are defined as follow

1. Addition: \( (\nu(\theta) + w(\theta), \overline{\nu}(\theta) + \overline{\nu}(\theta)) = (\underline{\nu}(\theta) + \overline{\nu}(\theta), \overline{\nu}(\theta) + \overline{\nu}(\theta)) \).
2. Subtraction: \( (\nu(\theta) - w(\theta), \overline{\nu}(\theta) - \overline{\nu}(\theta)) = (\nu(\theta) - \nu(\theta), \nu(\theta) - \nu(\theta)) \).
3. Scaler multiplication: \( \kappa_1 v(\theta) = \begin{cases} \kappa_1 \nu(\theta), & \kappa_1 v(\theta) > 0 \\ \kappa_1 \nu(\theta), & \kappa_1 v(\theta) < 0 \end{cases} \)

**Definition 11** ([54]). Let \( D_1 : E \times E \to R \) be a mapping, \( v = (\underline{\nu}(\theta), \overline{\nu}(\theta)) \) and \( w = (\underline{\nu}(\theta), \overline{\nu}(\theta)) \) are any two fuzzy number in parametric form. Then the Hausdorff distance between \((v, w)\) is measured as

\[ D_1(v, w) = \sup_{\theta \in [0, 1]} \max[|\nu(\theta) - \nu(\theta)|, |\mu(\theta) - \mu(\theta)|] \]

In \( E \), a metric \( D_1 \) as defined above have following properties;

1. \( D_1(v + v, w + v) = D_1(v, w) \) for all \( v, v \in E \),
2. \( D_1(\kappa_1 v, \kappa_1 w) = |\kappa_1| D_1(v, w) \) for all \( \kappa_1 \in R, v, w \in E \)
3. \( D_1(v + \mu, w + v) \leq D_1(v, w) + D_1(v, w) \) for all \( v, w, \mu, v, w \in E \),
4. \( (E, D_1) \) is a complete metric space.

**Definition 12** ([54]). Suppose that \( y_1, y_2 \in E \), if there exist \( y_3 \in E \) such that

\[ y_1 = y_2 + y_3, \]

then \( y_3 \) is said to be \( H \)-difference of \( y_1 \) and \( y_2 \) and denoted as \( y_1 + y_2 \).

**Definition 13.** ([55]) Consider the fuzzy mapping \( h : R \to E \), then \( h \) is called a continuous for any fixed \( y_0 \in [\beta_1, \beta_2] \), if for every \( \epsilon > 0 \), there exist \( \delta > 0 \) such that if \( |y - y_0| \) which implies that

\[ D_1(h(y), h(y_0)) < \epsilon. \]

**Definition 14** ([54]). A levelwise continuous mapping \( h : [\beta_1, \beta_2] \subset R \to E \) is defined at \( a \in [\beta_1, \beta_2] \) if the set valued mapping \( h_0(y) = (h(y))^\theta \) is continuous at \( y = a \) with respect to the \( H \)-metric \( D_1 \) for all \( \theta \in [0, 1] \).

**Theorem 1.** ([57]) Consider

1. \( h(y) \) is a levelwise continuous function on \([a, a + y_0], y_0 > 0\);  
2. \( k(y, s) \) is a levelwise continuous function on \( \Delta : a \leq s \leq y \leq a + y_0 \) and \( D_1(v(y), h(y_0)) < y_1 \) where \( y_1 > 0 \);
3. For any \((y, s, v(s)), (y, s, w(s)) \in \Delta \), we have

\[ D_1(k(y, s, v(s))) \leq MD_1(v(s))^{\theta} \leq MD_1(v(s))^{\theta}. \]

where the constant \( M > 0 \) is given and for any \( \theta \in [0, 1] \). Then the level wise continuous solution \( v(y) \) exist and unique and defined for \( y \in (a, a + \theta) \), where \( \theta = \min[y_0, \frac{y_1}{w}] \) and \( N = D_1(k(y, s, v(s)), (y, s, w(s)) \in \Delta, \) where keep in mid that \( y = (u, v) \).

**Theorem 2** ([57]). Let \( \varphi_1 \) and \( \varphi_2 \) be fuzzy valued functions of corresponding order \( c > 0 \), which are piecewise continuous on \([0, \infty)\), then

\[ \mathcal{L}[\varphi_1 * \varphi_2](s) = \mathcal{L}[\varphi_1(s)] \cdot \mathcal{L}[\varphi_2(s)] \]

where \( \mathcal{L} \) represent the integral transform due to Laplace.

**Note:** For the qualitative analysis, we define Banach space

\[ \overline{Y} = \overline{Z} = C([0, T] \times R^4, R), \]

where \( \overline{Z} = C[0, T] \) under the fuzzy norm

\[ \|\overline{Y}\| = \|\Omega\| = \sup_{t \in [0, T]} |\overline{S}(t)| + |\overline{E}(t)| + |\overline{F}(t)| + |\overline{R}(t)|. \]
3. Positivity of solution

Lemma 2. The solution of the model (1) is bounded in the feasible region given by
\[ T = \{(S, E, I, R) \in \mathbb{R}^4_+ : 0 \leq N(t) \leq \frac{a+b}{d_0}\}. \]

Proof. As \( N(t) = S(t) + E(t) + I(t) + R(t) \), then By adding all equation of (1), we have
\[
\frac{dN}{dt} = a - KIS(1 + \alpha I) - d_0S + KIS(1 + \alpha I) - (d_0 + \kappa)E
   + b + \alpha E - (\mu + d_0 + \beta)I + \beta I - d_0R
   = a - d_0S - d_0E - d_0I - d_0R - \kappa E + \alpha E - \mu I
   \leq a - N(t)(d_0) + b
\]
\[
\frac{dN}{dt} + d_0N \leq a + b. \tag{9}
\]
Solving (9), we have
\[
N(t) \leq \frac{a + b}{d_0} + C \exp(-d_0 t). \tag{10}
\]
when \( t \to \infty \), \( N(t) \leq \frac{a + b}{d_0} \), hence the required result is received. \quad \Box

4. Global and local stability analysis

For stability, we have to find the equilibrium points for (1) as
\[ D_t(S(t)) = 0, \]
\[ D_t(E(t)) = 0, \]
\[ D_t(I(t)) = 0, \]
\[ D_t(R(t)) = 0. \]

We have \( E_0 = \left( \frac{a + b}{d_0}, 0, 0, 0 \right) \) is the disease free equilibrium point of (1). Next, we have to find the basic reproduction number as

Theorem 3. The basic reproduction number for (1) is computed as
\[ R_0 = \frac{\alpha Ka}{d_0(d_0 + \kappa)(\mu + d_0 + \beta)}. \]

Proof. Let we take 2nd and 3rd equation of (1) for finding reproduction number as
\[ X = \begin{bmatrix} E \\ I \end{bmatrix}, \]
or
\[ \frac{dX}{dt} = \begin{bmatrix} \frac{dS}{dt} = KIS(1 + \alpha I) - (d_0 + \kappa)E \\ \frac{dE}{dt} = b + \alpha E - (\mu + d_0 + \beta)I \end{bmatrix}. \]

take \( \frac{dS}{dt} = F - V \), \( F \) is the non-linear term and \( V \) is the linear term given as under
\[ F = \begin{bmatrix} KIS(1 + \alpha I) \\ 0 \end{bmatrix}, \]
and
\[ V = \begin{bmatrix} (d_0 + \kappa)E \\ -(b + \alpha E) + (\mu + d_0 + \beta)I \end{bmatrix}. \]

taking jacobian of \( F \) and \( V \) to obtain the next generation matrix as follows
\[ F = \mathcal{F} = \begin{bmatrix} 0 & KS + 2\alpha KS \\ 0 & 0 \end{bmatrix}, \]
and
\[ V = \mathcal{V} = \begin{bmatrix} (d_0 + \kappa) & 0 \\ -\alpha & \mu + d_0 + \beta \end{bmatrix}. \]

so the next generation matrix is
\[ \mathcal{FV}^{-1} = \begin{bmatrix} \frac{\alpha(KS + 2\alpha KS)}{(d_0 + \kappa)(\mu + d_0 + \beta)} & 0 \\ 0 & 0 \end{bmatrix}. \]
Now $R_0$ is equal to leading eigen value of the next generation matrix $FV^{-1}$ at disease free equilibrium point $E_0 = (\frac{\Phi_1}{\alpha}, 0, 0, 0)$
\[
\rho(FV^{-1})|_{E_0} = \begin{bmatrix}
\frac{\alpha K_a}{d_0(d_0 + \kappa)} & 0 \\
0 & 0
\end{bmatrix}.
\] (11)

So the reproduction number is given by
\[
R_0 = \frac{\alpha K_a}{d_0(d_0 + \kappa)(\mu + d_0 + \beta)}.
\]
Hence the required result is proved. □

**Theorem 4. Statement** The pandemic free equilibrium point of (1) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

**Proof.** Let we denote the right hand side of (1) by $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, respectively then the “Jacobian matrix” for (1) can be computed as
\[
J = \begin{bmatrix}
\frac{\partial}{\partial t}(\Phi_1) & \frac{\partial}{\partial t}(\Phi_1) & \frac{\partial}{\partial t}(\Phi_1) & \frac{\partial}{\partial t}(\Phi_1) \\
\frac{\partial}{\partial t}(\Phi_2) & \frac{\partial}{\partial t}(\Phi_2) & \frac{\partial}{\partial t}(\Phi_2) & \frac{\partial}{\partial t}(\Phi_2) \\
\frac{\partial}{\partial t}(\Phi_3) & \frac{\partial}{\partial t}(\Phi_3) & \frac{\partial}{\partial t}(\Phi_3) & \frac{\partial}{\partial t}(\Phi_3) \\
\frac{\partial}{\partial t}(\Phi_4) & \frac{\partial}{\partial t}(\Phi_4) & \frac{\partial}{\partial t}(\Phi_4) & \frac{\partial}{\partial t}(\Phi_4)
\end{bmatrix}
\]
or
\[
J_{E_0} = \begin{bmatrix}
-d_0 & 0 & -\frac{K_a}{d_0} & 0 \\
0 & -(d_0 + \kappa) & 0 & 0 \\
0 & 0 & -(\mu + d_0 + \beta) & 0 \\
0 & 0 & 0 & -d_0
\end{bmatrix}
\]
Now the characteristics equation can be find as
\[
\text{Det}(J_{E_0} - \lambda) = \begin{vmatrix}
-d_0 - \lambda & 0 & -\frac{K_a}{d_0} & 0 \\
0 & -(d_0 + \kappa) - \lambda & 0 & 0 \\
0 & 0 & -(\mu + d_0 + \beta) - \lambda & 0 \\
0 & 0 & 0 & -d_0 - \lambda
\end{vmatrix} = 0,
\]
so the real parts of all the eigen values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are negative, $\lambda_4 = R_0 - 1$. imply that $\lambda_4$ must be negative if $R_0 < 1$, hence proved. □

5. Iterative solution of (1)

5.1. Numerical solution of (1) by euler method in sense of caputo derivative

Now here we have to evaluate approximate solution of the model (1) and the numerical simulations will be achieved by the “Euler iterative method”. For this we apply the arbitrary order Caputo derivative to establish a numerical procedure for the simulation of our considered model (1). To develop a numerical scheme, we go ahead with the model (1) as
\[
\begin{align*}
\text{D}^\theta_0 S(t) &= \Phi_1(t, S(t), E(t), I(t), R(t)) \\
\text{D}^\theta_0 E(t) &= \Phi_2(t, S(t), E(t), I(t), R(t)) \\
\text{D}^\theta_0 I(t) &= \Phi_3(t, S(t), E(t), I(t), R(t)) \\
\text{D}^\theta_0 R(t) &= \Phi_4(t, S(t), E(t), I(t), R(t)),
\end{align*}
\] (12)
\[
S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad 0 < \theta \leq 1.
\]
Let $[0, \theta]$ be set of points, on which we must have to evaluate the series solution of the model (12). Actually, we cannot not evaluate the functions $S(t), E(t), I(t), R(t)$ which will be the solution the IVP (12). Instead of this, an interval $(t_p, S(t_p))$ is produced from which the points are used for our iterative procedure. For this, we further divide the set of points $[0, \theta]$ into $j$ sub-intervals $[t_p, t_{p+1}]$ of equal difference $h = \theta / n$ only using the nodes $t_p = ph$ for $p = 0, 1, \ldots, n$. Consider that
\[
S(t), I(t), E(t), S(t), E(t), I(t), R(t)
\]
are continues on $[0, T]$. Applying the generalized Euler’s or Taylor’s method about $t = t_0 = 0$ to the considered model expressed in (12) and for each value $t$ take value $a$, the expression for $t_1$, we have
\[
\begin{align*}
S(t_1) = S(t_0) + \Phi_1(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t_0^\theta}{\Gamma(\theta + 1)} + \text{D}^\theta_0 S(t)|_{t_0 = a} \frac{t^\theta}{\Gamma(2\theta + 1)} \\
E(t_1) = E(t_0) + \Phi_2(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t_0^\theta}{\Gamma(\theta + 1)} + \text{D}^\theta_0 E(t)|_{t_0 = a} \frac{t^\theta}{\Gamma(2\theta + 1)} \\
I(t_1) = I(t_0) + \Phi_3(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t_0^\theta}{\Gamma(\theta + 1)} + \text{D}^\theta_0 I(t)|_{t_0 = a} \frac{t^\theta}{\Gamma(2\theta + 1)} \\
R(t_1) = R(t_0) + \Phi_4(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t_0^\theta}{\Gamma(\theta + 1)} + \text{D}^\theta_0 R(t)|_{t_0 = a} \frac{t^\theta}{\Gamma(2\theta + 1)}
\end{align*}
\] (13)
Let the difference between two successive point is $h$ will be chosen small enough, then we may ignore the higher-order term from involving $h^{20}$ and get the results from (13) as

$$S(t_1) = S(t_0) + \Phi_1(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$E(t_1) = E(t_0) + \Phi_2(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$I(t_1) = I(t_0) + \Phi_2(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$R(t_1) = R(t_0) + \Phi_2(t_0, S(t_0), E(t_0), I(t_0), R(t_0)) \frac{t^0}{\Gamma(\theta + 1)}.$$  \hspace{1cm} (14)

On repeating the same fashion, a sequence of points that approximates the solution $(S(t), E(t), I(t), R(t))$ is formed. A general formula about $t_{p+1} = t_p + h$ is

$$S(t_{p+1}) = S(t_p) + \Phi_1(t_p, S(t_p), E(t_p), I(t_p), R(t_p)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$E(t_{p+1}) = E(t_p) + \Phi_2(t_p, S(t_p), E(t_p), I(t_p), R(t_p)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$I(t_{p+1}) = I(t_p) + \Phi_2(t_p, S(t_p), E(t_p), I(t_p), R(t_p)) \frac{t^0}{\Gamma(\theta + 1)}$$

$$R(t_{p+1}) = R(t_p) + \Phi_2(t_p, S(t_p), E(t_p), I(t_p), R(t_p)) \frac{t^0}{\Gamma(\theta + 1)}.$$  \hspace{1cm} (15)

where $p = 0, 1, 2, \ldots, n - 1.$

5.2. Numerical solution of (1) by Adams-Bashforth method in sense of ABC derivative

The model (1) has been solved approximately by Adams-Bashforth method in zeeshan et al[58] as

$$S(t_{n+1}) = S(0) + \frac{(1 - \theta)}{\Delta ABC(\theta)} \left[ \Phi_1(S(t_n), E(t_n), I(t_n), R(t_n), t_n) \right]$$

$$+ \frac{\theta}{\Delta ABC(\theta)} \sum_{p=0}^{n} \left( \Phi_1(S(t_p), E(t_p), I(t_p), R(t_p), t_p) \frac{t^0}{\Gamma(\theta + 2)} \right)$$

$$\Delta^0 \left[ (n + 1 - p)^0 (n - p + 2 + \theta) \right]$$

$$- \Phi_1(S(t_{p-1}), E(t_{p-1}), I(t_{p-1}), R(t_{p-1}), t_{p-1}) \frac{t^0}{\Gamma(\theta + 2)} \Delta^0 \left[ (n + 1 - p)^0 (n - p + 1 + \theta) \right].$$

$$E(t_{n+1}) = E(0) + \frac{(1 - \theta)}{\Delta ABC(\theta)} \left[ \Phi_2(S(t_n), E(t_n), I(t_n), R(t_n), t_n) \right]$$

$$+ \frac{\theta}{\Delta ABC(\theta)} \sum_{p=0}^{n} \left( \Phi_2(S(t_p), E(t_p), I(t_p), R(t_p), t_p) \frac{t^0}{\Gamma(\theta + 2)} \right)$$

$$\Delta^0 \left[ (n + 1 - p)^0 (n - p + 2 + \theta) \right]$$

$$- \Phi_2(S(t_{p-1}), E(t_{p-1}), I(t_{p-1}), R(t_{p-1}), t_{p-1}) \frac{t^0}{\Gamma(\theta + 2)} \Delta^0 \left[ (n + 1 - p)^0 (n - p + 1 + \theta) \right].$$

$$I(t_{n+1}) = I(0) + \frac{(1 - \theta)}{\Delta ABC(\theta)} \left[ \Phi_3(S(t_n), E(t_n), I(t_n), R(t_n), t_n) \right]$$

$$+ \frac{\theta}{\Delta ABC(\theta)} \sum_{p=0}^{n} \left( \Phi_3(S(t_p), E(t_p), I(t_p), R(t_p), t_p) \frac{t^0}{\Gamma(\theta + 2)} \right)$$

$$\Delta^0 \left[ (n + 1 - p)^0 (n - p + 2 + \theta) \right]$$

$$- \Phi_3(S(t_{p-1}), E(t_{p-1}), I(t_{p-1}), R(t_{p-1}), t_{p-1}) \frac{t^0}{\Gamma(\theta + 2)} \Delta^0 \left[ (n + 1 - p)^0 (n - p + 1 + \theta) \right].$$
So we can compare Caputo Euler method and ABC Adams-bashforth method given as in Figs. 1 and 2 respectively. We have simulated the results corresponding to the given data in Table 2 and the parameters values are given in the Table 2 below as:

We take the following initial values $S(0) = 0.323$, $E(0) = 0.21$, $I(0) = 0.22$, $R(0) = 0.21$ in million.
Table 2

Description of the parameters given in model (1).

| Notation | Parameters description                                      | values                |
|----------|------------------------------------------------------------|-----------------------|
| $a$      | The population whose test is negative                      | 0.00250281 millions   |
| $d_0$    | Natural death rate                                          | 0.00000004/million    |
| $b$      | The population whose test is positive                      | 0.006656 millions     |
| $\mu$    | Death due to Corona                                         | 0.0109                |
| $\kappa$ | The rate constant characterizing the infection              | 0.000024              |
| $\alpha$ | Rate at which recovered individuals lose immunity           | 0.000005/million      |
| $\beta$  | Recovered rate                                              | 0.75                  |

6. Random COVID-19 disease model

In the nature, the coefficients of the model given by Eq. (1) are random numbers, in this section, we present the random COVID-19 disease model transforming the coefficients of system given by Eq. (1) to random variables [59]. Using normal distribution, we will obtain the new set of random coefficients

$$ A \sim \Delta(\sigma_1, \vartheta_1^2); \quad B \sim \Delta(\sigma_2, \vartheta_2^2); \quad C \sim \Delta(\sigma_3, \vartheta_3^2); \quad D \sim \Delta(\sigma_4, \vartheta_4^2); \quad E \sim \Delta(\sigma_5, \vartheta_5^2); $$

$$ F \sim \Delta(\sigma_6, \vartheta_6^2); \quad G \sim \Delta(\sigma_7, \vartheta_7^2). $$

(16)

where $\sigma_m, \vartheta_m, m \in (1, 8)$, these parameters corresponds to the means and standard deviations of the normal distributions, respectively. These random variables are expected to be a sum of independent quantities often have a normal distribution, also the mean values of these distributions will be chosen according to the numerical values of the coefficients.

The distributed random variable $Z = \Delta(\sigma_m, \vartheta_m^2), m \in (1, 8)$ can be written as $\Psi = \sigma_m + \vartheta_m Y_m, m \in (1, 8)$, where $Y - \Delta(0, 1)$ is the standard normally distributed random variable. These random variables can be rewritten involving the standard normally distributed random variable, following this idea, we have

$$ A = \sigma_1 + \vartheta_1 Y_1; \quad B = \sigma_2 + \vartheta_2 Y_2; \quad C = \sigma_3 + \vartheta_3 Y_3; \quad D = \sigma_4 + \vartheta_4 Y_4; \quad E = \sigma_5 + \vartheta_5 Y_5; $$

$$ F = \sigma_6 + \vartheta_6 Y_6; \quad G = \sigma_7 + \vartheta_7 Y_7. $$

(17)

The initial conditions are $S(0) = 4, E(0) = 0, I(0) = 0$ and $R(0) = 0$. For the random variables $Y_m, m \in (1, 8)$, the distribution is $\Delta(0, 1)$. Setting the appropriate values of $a_m, s_m$, yields

$$ A = 0.73 \times 10^6 + 0.02 Y_1; \quad B = 0.546 \times 10^6 + 0.028 Y_2; \quad C = 0.68 \times 10^6 + 0.019 Y_3; $$

$$ D = 2.17 + 0.0347 Y_4; \quad E = 2.07 + 0.011 Y_5; \quad F = 2.2 \times 10^6 + 0.011 Y_6; \quad G = 3.1 + 0.0317 Y_7. $$

(18)

Substituting the parameters described in Eq. (18) into the model given by Eq. (1), we get

$$ \begin{align*}
\frac{dS(t)}{dt} &= A - E(t)S(t)(1 + F(t)) - BS(t), \\
\frac{dE(t)}{dt} &= E(t)S(t)(1 + F(t)) - (B + E)E(t), \\
\frac{dl(t)}{dt} &= C + FE(t) - (D + B + G)I(t), \\
\frac{dR(t)}{dt} &= GI(t) - BR(t).
\end{align*} $$

(19)

The solution of the model (19) can be obtained applying the Adams-Bashforth method [60]. The Numerical scheme is given by

$$ S_{n+1} = S_0 + \frac{3}{2} h \left\{ A - E_n(t_n)S_n(t_n)(1 + F_n(t_n)) - BS_n(t_n) \right\} - $$

$$ - \frac{1}{2} h \left\{ A - E_{n-1}(t_{n-1})S_{n-1}(t_{n-1})(1 + F_{n-1}(t_{n-1})) - BS_{n-1}(t_{n-1}) \right\}, $$

$$ E_{n+1} = E_0 + \frac{3}{2} h \left\{ E_n(t_n)S_n(t_n)(1 + F_n(t_n)) - (B + E)E_n(t_n) \right\} - $$

$$ - \frac{1}{2} h \left\{ E_{n-1}(t_{n-1})S_{n-1}(t_{n-1})(1 + F_{n-1}(t_{n-1})) - (B + E)E_{n-1}(t_{n-1}) \right\}, $$

$$ I_{n+1} = I_0 + \frac{3}{2} h \left\{ C + FE_n(t_n) - (D + B + G)I_n(t_n) \right\} - $$

$$ - \frac{1}{2} h \left\{ C + FE_{n-1}(t_{n-1}) - (D + B + G)I_{n-1}(t_{n-1}) \right\}. $$

(20)
M.U. Rahman, M. Arfan and K. Shah et al. / Chaos, Solitons and Fractals 140 (2020) 110232

Fig. 3. Random COVID-19 Disease model for $\Delta = 0.3$.

Fig. 4. Random COVID-19 Disease model for $\Delta = 0.4$.

$$R_{n+1} = R_0 + \frac{3}{2} h \left\{ G + I_n(t_n) - BR_n(t_n) \right\}$$

$$- \frac{1}{2} h \left\{ G + I_{n-1}(t_{n-1}) - BR_{n-1}(t_{n-1}) \right\}.$$

The numerical results given in Figs. 3(a)-3(d), 4(a)-4(d), 5(a)-5(d) and 6(a)-6(d) shows numerical simulations of the random solution of our model.

From our figures obtained, we show that exposed and infected individuals show a sharp increase for all values of $\Delta$ due to the high transmissibility of COVID-19 virus [61]. In some countries the number of labs for sample testing is limited which leads to delay in confirmation of infected individual present a huge risk for susceptible population. If in these countries the response on lab confirmation of COVID-19 positive cases is more fast, we can isolate infected individual and break the chain of transmission on time. Several countries taking this pandemic seriously and implementing severe and extreme control measures like educating and following World Health Organization preventive measures. For this reason is very important implement control measures like social distancing, mitigation, containment and restrict gathering of people.

7. Iterative solution of (2)

7.1. Numerical solution of (2) by Adams-Bashforth method in caputo sense

In this section, we are giving approximate solutions of fractional order model (2) under the Caputo derivative by fractional Adams-Bashforth method. Then the numerical simulations are got via the suggested scheme. To this aim, we employ the fractional AB method to establish a numerical procedure for the simulation of our considered model (2). To produce a numerical scheme, we go ahead with the
model (2) can be written for simplicity as
\[
\begin{align*}
\mathcal{C}D^\alpha_t(S(t)) &= \Phi_1(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R}, t) = a - K(t)\tilde{S}(t)(1 + \alpha(t)) - d_0\tilde{S}(t), \\
\mathcal{C}D^\alpha_t(E(t)) &= \Phi_2(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R}, t) = K(t)\tilde{S}(t)(\infty + \alpha(t)) - (d_0 + \kappa)\tilde{E}(t), \\
\mathcal{C}D^\alpha_t(I(t)) &= \Phi_3(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R}, t) = b + \alpha\tilde{E}(t) - (\mu + d_0 + \beta)\tilde{I}(t), \\
\mathcal{C}D^\alpha_t(R(t)) &= \Phi_4(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R}, t) = \beta\tilde{I}(t) - d_0\tilde{R}(t), \\
\tilde{S}(0) &= k_0(t)S_0, \quad \tilde{E}(0) = k_0(E_0), \quad \tilde{I}(0) = k_0(I_0), \quad \tilde{R}(0) = k_0(R_0).
\end{align*}
\]
Taking integration of the first equation of (21) in Caputo sense, we get
\[
\tilde{S}(t) - S(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha-1} \Phi_1(\tilde{S}(y), \tilde{E}(y), \tilde{I}(y), \tilde{R}(y), t) dy.
\]
Set \( t = t_{n+1} \) for \( n = 0, 1, 2, \ldots \), it follows that
\[
\tilde{S}(t_{n+1}) - S(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - y)^{\alpha-1} \Phi_1(\tilde{S}(y), \tilde{E}(y), \tilde{I}(y), \tilde{R}(y), t) dy
\]
\[
= \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{n} \left( t_{n+1} - y \right)^{\alpha-1} \Phi_1(\tilde{S}(y), \tilde{E}(y), \tilde{I}(y), \tilde{R}(y), t) dy.
\]
Now, we approximate the function \( \Phi_1(\tilde{S}(t), \tilde{E}(t), \tilde{I}(t), \tilde{R}(t), t) \) on the interval \([t_p, t_{p+1}]\) through the interpolation polynomial as follows
\[
\Phi_1(\tilde{S}(t), \tilde{E}(t), \tilde{I}(t), \tilde{R}(t), t) \approx \frac{\Phi_1(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t_p) - \Phi_1(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t_{p-1})}{(t - t_{p-1}) - (t - t_p)}
\]
which implies that

\[
\tilde{S}(t_{n+1}) = S(0) + \frac{1}{\Gamma(\theta)} \sum_{p=0}^{n} \left( \Phi_1 \left( \tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t \right) \right) \Delta \\
\times \int_{p}^{t_{p+1}} (y - t_{n+1})(t_{n+1} - y)^{\theta-1} dy \\
- \Phi_1 \left( \tilde{S}(t_{p-1}), \tilde{E}(t_{p}), \tilde{I}(t_{p}), \tilde{R}(t_{p}), t \right) \int_{p}^{t_{p+1}} (y - t_{n+1})(t_{n+1} - y)^{\theta-1} dy \\
= S(0) + \frac{1}{\Gamma(\theta)} \sum_{p=0}^{n} \left( \Phi_1 \left( \tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t \right) \right) \Delta \\
- \Phi_1 \left( \tilde{S}(t_{p-1}), \tilde{E}(t_{p}), \tilde{I}(t_{p}), \tilde{R}(t_{p}), t \right) \Delta \\
\int_{p}^{t_{p+1}} (y - t_{n+1})(t_{n+1} - y)^{\theta-1} dy.
\]

Now we have to calculate the integrals \(I_{p+1,0}\) and \(I_{p,0}\) as

\[
I_{p+1,0} = \int_{p}^{t_{p+1}} (t - t_{p-1})(t_{n+1} - t)^{\theta-1} dt \\
= -\frac{\Delta^{\theta+1}}{\theta} \left( p + 1 - (p - 1) \right) (n + 1 - (p + 1))^{\theta} - (p - (p - 1)) (n + 1 - p)^{\theta} \\
- \frac{\Delta^{\theta+1}}{\theta(\theta - 1)} \left[ (n + 1 - (p + 1))^{\theta+1} - (n + 1 - p)^{\theta+1} \right].
\]

and

\[
I_{p,0} = \int_{p}^{t_{p+1}} (t - t_{p})(t_{n+1} - t)^{\theta-1} dt \\
= -\frac{\Delta^{\theta+1}}{\theta} \left( (p + 1 - (p - 1)) (n + 1 - (p + 1))^{\theta} - (p - (p - 1)) (n + 1 - p)^{\theta} \right) \\
- \frac{\Delta^{\theta+1}}{\theta(\theta - 1)} \left[ (n + 1 - (p + 1))^{\theta+1} - (n + 1 - p)^{\theta+1} \right].
\]

substituting (23) and (24) in (22), we have as follows

\[
\tilde{S}(t_{n+1}) = S(0) + \frac{1}{\Gamma(\theta)} \sum_{p=0}^{n} \left( \Phi_1 \left( \tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t \right) \right) \Delta \\
\times \Delta^{\theta} \left[ (n + 1 - p)^{\theta} (n - q + 2 + \theta) - (n - p)^{\theta} (n - p + 2 + 2\theta) \right].
\]
\[-\Phi_1(\bar{S}(t_{p-1}), \bar{E}(t_p), \bar{I}(t_p), \bar{R}(t_p), t_{p-1}) \Delta^\theta \]
\[\times \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right].\]

Similarly for the remaining three equations of (21) we can write the iterative method as

\[\bar{E}(t_{n+1}) = E(0) + \sum_{p=0}^{n} \Phi_1(\bar{S}(t_{p-1}), \bar{E}(t_{p-1}), \bar{I}(t_{p-1}), \bar{R}(t_{p-1}), t) \]
\[\Delta^\theta \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right]
\[\times \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right].\]

\[\bar{I}(t_{n+1}) = I(0) + \sum_{p=0}^{n} \Phi_1(\bar{S}(t_{p-1}), \bar{E}(t_{p-1}), \bar{I}(t_{p-1}), \bar{R}(t_{p-1}), t) \]
\[\Delta^\theta \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right]
\[\times \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right].\]

\[\bar{R}(t_{n+1}) = R(0) + \sum_{p=0}^{n} \Phi_1(\bar{S}(t_{p-1}), \bar{E}(t_{p-1}), \bar{I}(t_{p-1}), \bar{R}(t_{p-1}), t) \]
\[\Delta^\theta \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right]
\[\times \left[ (n + 1 - p)^\theta (n - p + 1 + \theta) \right].\]

7.2. Numerical solution of (2) by fractional Adams-Bashforth method in sense of ABC derivative

In this section, we are giving approximate solutions of fractional order model (2) under the ABC derivative by fractional Adams-Bashforth method. Then the numerical simulations are got via the suggested scheme. To this aim, we employ the fractional AB method to establish a numerical procedure for the simulation of our considered model (2). To produce a numerical scheme, we go ahead with the model (21) can be written for simplicity as

\[
\begin{align*}
\text{ABC-D}^\alpha_\text{AB} (S(t)) &= H_1(\bar{S}, \bar{E}, \bar{I}, \bar{R}, t) = a - K\bar{I}(t)\bar{S}(t)(1 + \alpha\bar{I}(t)) - d_0\bar{S}(t), \\
\text{ABC-D}^\alpha_\text{AB} (E(t)) &= H_2(\bar{S}, \bar{E}, \bar{I}, \bar{R}, t) = K\bar{I}(t)\bar{S}(t)(1 + \alpha\bar{I}(t)) - (d_0 + \kappa)\bar{E}(t), \\
\text{ABC-D}^\alpha_\text{AB} (I(t)) &= H_3(\bar{S}, \bar{E}, \bar{I}, \bar{R}, t) = b + a\bar{E}(t) - (\mu + d_0 + \beta)\bar{I}(t), \\
\text{ABC-D}^\alpha_\text{AB} (R(t)) &= H_4(\bar{S}, \bar{E}, \bar{I}, \bar{R}, t) = \beta\bar{I}(t) - d_0\bar{R}(t). \\
\end{align*}
\]

Taking integration of the first equation of (25) in ABC sense, we get

\[
\frac{(1 - \alpha)}{\text{ABC}(\alpha)} \int_0^t (t - y)^{\alpha - 1} H_1(\bar{S}(y), \bar{E}(y), \bar{I}(y), \bar{R}(y), t) dy.
\]

Set \( t = t_{n+1} \) for \( n = 0, 1, 2 \ldots \), it follows that

\[
\bar{S}(t_{n+1}) - S(0) = \frac{(1 - \alpha)}{\text{ABC}(\alpha)} \int_0^{t_{n+1}} (t - y)^{\alpha - 1} H_1(\bar{S}(y), \bar{E}(y), \bar{I}(y), \bar{R}(y), t) dy.
\]
\[
\begin{align*}
&+ \frac{\alpha}{\text{ABC}(\alpha) \Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - y)^{\alpha-1} H_1 \left( \tilde{S}(y), \tilde{E}(y), \tilde{I}(y), \tilde{R}(y), t \right) dy, \\
&= \left( 1 - \alpha \right) \frac{\alpha}{\text{ABC}(\alpha)} \left[ H_1 \left( \tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t \right) \right] \\
&+ \frac{\alpha}{\text{ABC}(\alpha) \Gamma(\alpha)} \sum_{p=0}^{n} \int_{t_p}^{t_{p+1}} (t_{p+1} - y)^{\alpha-1} H_1 \left( \tilde{S}(y), \tilde{E}(y), \tilde{I}(y), \tilde{R}(y), t \right) dy \\
\end{align*}
\]

Now, we approximate the function \( H_1(\tilde{S}(t), \tilde{E}(t), \tilde{I}(t), \tilde{R}(t), t) \) on the interval \([t_p, t_{p+1}]\) through the interpolation polynomial as follows

\[
H_1(\tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t) \approx \frac{H_1(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t)}{\Delta} (t - t_{p-1}) \\
+ \frac{H_1(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t)}{\Delta} (t - t_{p-1}),
\]

which implies that

\[
\begin{align*}
\tilde{S}(t_{n+1}) &= S(0) + \left( 1 - \alpha \right) \frac{\alpha}{\text{ABC}(\alpha)} \left[ H_1(\tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t_n) \right] \\
&+ \frac{\alpha}{\text{ABC}(\alpha) \Gamma(\alpha)} \sum_{p=0}^{n} \left( \frac{H_1(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t_p)}{\Gamma(\alpha + 2)} \right) \frac{1}{\Delta} \left[ (n + 1 - p)^{\alpha} (n - p + 2 + \alpha) \right. \\
&\left. - (n - p)^{\alpha} (n - p + 2 + 2\alpha) \right] \\
&- \frac{H_1(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t_{p-1})}{\Gamma(\alpha + 2)} \left[ (n + 1 - p)^{\alpha+1} - (n - p)^{\alpha} (n - p + 1 + \alpha) \right].
\end{align*}
\]

Similarly, for the remaining three equations of the model, we can write the iterative method as

\[
\tilde{E}(t_{n+1}) = \tilde{E}_0(t) + \left( 1 - \alpha \right) \frac{\alpha}{\text{ABC}(\alpha)} \left[ H_2(\tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t_n) \right] \\
+ \frac{\alpha}{\text{ABC}(\alpha) \Gamma(\alpha)} \sum_{p=0}^{n} \left( \frac{H_2(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t_p)}{\Gamma(\alpha + 2)} \right) \frac{1}{\Delta} \left[ (n + 1 - p)^{\alpha} (n - p + 2 + \alpha) \right. \\
&\left. - (n - p)^{\alpha} (n - p + 2 + 2\alpha) \right] \\
&- \frac{H_2(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t_{p-1})}{\Gamma(\alpha + 2)} \left[ (n + 1 - p)^{\alpha+1} - (n - p)^{\alpha} (n - p + 1 + \alpha) \right].
\]

\[
\tilde{I}(t_{n+1}) = \tilde{I}_0(t) + \left( 1 - \alpha \right) \frac{\alpha}{\text{ABC}(\alpha)} \left[ H_3(\tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t_n) \right] \\
+ \frac{\alpha}{\text{ABC}(\alpha) \Gamma(\alpha)} \sum_{p=0}^{n} \left( \frac{H_3(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t_p)}{\Gamma(\alpha + 2)} \right) \frac{1}{\Delta} \left[ (n + 1 - p)^{\alpha} (n - p + 2 + \alpha) \right. \\
&\left. - (n - p)^{\alpha} (n - p + 2 + 2\alpha) \right] \\
&- \frac{H_3(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t_{p-1})}{\Gamma(\alpha + 2)} \left[ (n + 1 - p)^{\alpha+1} - (n - p)^{\alpha} (n - p + 1 + \alpha) \right].
\]
\[ \tilde{R}(t_{n+1}) = \tilde{k}(t) R(t) + \frac{(1-\alpha)}{\text{ABC}(\alpha)} \left[ H_4(\tilde{S}(t_n), \tilde{E}(t_n), \tilde{I}(t_n), \tilde{R}(t_n), t_n) \right] \\
+ \frac{\alpha}{\text{ABC}(\alpha)} \sum_{p=0}^{n} \left[ \frac{H_4(\tilde{S}(t_p), \tilde{E}(t_p), \tilde{I}(t_p), \tilde{R}(t_p), t_p)}{\Gamma(\alpha + 2)} \right] \Delta^\alpha \left[ (n+1-p)^\alpha (n-p+2+\alpha) \right] \\
- (n-p)^\alpha (n-p+2+2\alpha) \\
- \frac{H_4(\tilde{S}(t_{p-1}), \tilde{E}(t_{p-1}), \tilde{I}(t_{p-1}), \tilde{R}(t_{p-1}), t_{p-1})}{\Gamma(\alpha + 2)} \Delta^\alpha \left[ (n+1-p)^{\alpha+1} - (n-p)^{\alpha} (n-p+1+\alpha) \right]. \]

So we can compare fuzzy Caputo iteration with fuzzy ABC iterative method as in figur 1 and 2. We take the following initial values \( S(0) = 4 \), \( E(0) = 0 \), \( I(0) = 0 \), \( R(0) = 0 \). Susceptible about whom it was fear and they were tested. Since fuzzy logic is used as a vigorous technique for modeling in medical practice where uncertainty lies in data. Here corresponding to the fuzzy concept we have presented the fuzzy upper and lower approximate solutions for various compartments corresponding to different values of fraction order in \( tr \) space. We see under uncertainty the decrease in susceptibility means that more people are to be exposed to the infection. As a results infected class may be increased for which cause in increase in fatality rate and also more people are infected so more recovery may be possible hence this class also raise with different. The concerned fuzzy dynamics for the novel coronaries is presented in Figs. 7-10 by using the parametric values of the Table 2.

Here in Figs. 11 - 14 we plot the numerical results corresponding to different fractional orders.

From Figs. 11-14, we see that as susceptibility is decreasing as a results the exposed class increases. Consequently the infection class also increase. If there is no precautionary measure then more fattily cases will occurs so recovered class also raise up. The concerned dynamics are more faster on small fractional order and as order enlarging the process become slightly slow.
Fig. 8. Graphical presentation of first three terms fuzzy solutions for different fractional order of exposed class.
Fig. 9. Graphical presentation of first three terms fuzzy solutions for different fractional order of infected class.
Fig. 10. Graphical presentation of first three terms fuzzy solutions for different fractional order of recovered class.
Fig. 11. Dynamical behavior of COVID-19 considered model under ABC fractional order derivative for order 0.75.
Fig. 12. Dynamical behavior of COVID-19 considered model under ABC fractional order derivative for order 0.85.
Fig. 13. Dynamical behavior of COVID-19 considered model under ABC fractional order derivative for order 0.95.
Fig. 14. Dynamical behavior of COVID-19 considered model under ABC fractional order derivative for order 1.0.

8. Conclusion

A comprehensive study has been carried out in this article about a COVID-19 mathematical model of four compartments. The considered model has been investigated from qualitative aspect and proved that such model has at least one solution by using fixed point results. Further since uncertainty always lies in such of data so in this work a fuzzy and random approach have used to deal the considered novel COVID-19 model. Existence of such models has been proved via using fuzzy fixed point results. Considering a novel Laplace Adomian method we have presented some approximation results. The concerned results have presented through surfaces plots against uncertain and taking different fractional order. Also numerical solutions for the random model were obtained applying the Adams-Bashforth method. The graphical results demonstrate that random approach provides superior and more suitable results than the ordinary COVID-19 model. These results are consistent with real-time data trends of COVID-19 transmission to humans in many countries with limited action on disease control. The fuzzy and random concepts can also be used in practice problems where data are uncertain. Since recently the nonlocal derivatives of fractional order have got much attention. So we have also investigated the proposed model under ABC derivative by using fractional Adam Bashforth method for numerical simulation. In further research, we study the applicability of the proposed schemes in another epidemic models and some related problem such as stability and optimal control problems.

Declaration of Competing Interest

The authors declare no conflict of interest.
CRediT authorship contribution statement

Mati ur Rahman: Formal analysis, Methodology, Writing - review & editing.
Muhammad Arfan: Formal analysis, Investigation, Writing - review & editing.
Kamal Shah: Formal analysis, Methodology, Conceptualization, Validation, Writing - review & editing.
J.F. Gómez-Aguilar: Software, Validation, Conceptualization, Methodology, Writing - review & editing.

Acknowledgments

José Francisco Gómez Aguilar acknowledges the support provided by CONACYT: cátedras CONACYT para jóvenes investigadores 2014 and SNI-CONACyT.

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