A numerical approach toward the $p$-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$

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with Appendix B by François Brunault

Abstract

Restricting ourselves to elliptic curves over $\mathbb{Q}$, we reformulate the $p$-adic Beilinson conjecture due to Perrin-Riou, which is customized to our computational approach. We then develop a new algorithm for numerical verifications of the $p$-adic Beilinson conjecture, which is based on the theory of rigid cohomology and $F$-isocrystals.

1 Introduction

Since Dirichlet proved his famous class number formula in the 19th century, the regulator is one of the most important subjects in algebraic number theory. In the 20th century, several generalizations of Dirichlet’s formula were proposed in the context of higher $K$-theory by Lichtenbaum, Borel, etc. Especially, Beilinson’s conjecture on special values of motivic $L$-functions ([Bei], [Sch]) is an important milestone, and this mysterious conjecture is still attracting lots of researchers in arithmetic geometry.

The $p$-adic analogue of the Beilinson conjecture was formulated by Perrin-Riou [PR], which we call the $p$-adic Beilinson conjecture. Roughly speaking, it asserts a certain formula on the $p$-adic regulators and the special values of $p$-adic $L$-functions, which is conceptionally described as

$$\frac{p\text{-adic } L\text{-value}}{p\text{-adic regulator}} = \frac{L\text{-value}}{\text{Beilinson regulator}}.$$ 

However, the above involves a serious issue. In the present, there is no general definition or even candidate of the $p$-adic counterparts of motivic $L$-functions. She needed to formulate the statement together with the existence of $p$-adic $L$-functions simultaneously, and that causes the very complicated statement.

On the other hand, restricting ourselves to the case of elliptic curves over $\mathbb{Q}$, the $p$-adic $L$-functions are defined by the following people.

• Mazur and Swinnerton-Dyer [MS]

• Vishik [Vi], Amice and Vélu [AV]

• Pollack and Stevens [PS2], Bellaïche [B] (critical slope $p$-adic $L$-functions).
Thus one can re-formulate the $p$-adic Beilinson conjecture in terms of the above $p$-adic $L$-functions (cf. [PR 4.2]). However there still remains a problem. In general, the regulators are only determined up to multiplication by $\mathbb{Q}^\times$, in other words there is no canonical manner to remove the ambiguity of $\mathbb{Q}^\times$. The $p$-adic Beilinson conjecture is stated by fixing basis of several cohomology groups, though the choice does not matter by replacing the normalization of her $p$-adic $L$-functions if necessary. On the other hand, the normalization of the $p$-adic $L$-functions by Mazur and Swinnerton-Dyer et al is fixed and it is a natural one. Therefore we need to choose a suitable basis of cohomology groups according to this, and it is not so an obvious question.

In this paper we write down the $p$-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$ in terms of the $p$-adic $L$-functions by Mazur and Swinnerton-Dyer et al. To be precise, for an elliptic curve $E$ over $\mathbb{Q}$ and $\xi \in H^2_{\acute{e}t}(E, \mathbb{Q}(n+2))$, we introduce $R_{\infty}(h^1(E)(-n), \xi)$ the Beilinson regulator without ambiguity of $\mathbb{Q}^\times$, and $R_{p,\gamma}(h^1(E)(-n), \xi)$ the $p$-adic regulator without ambiguity of $\mathbb{Q}^\times$. Then our formulation of the $p$-adic Beilinson conjecture is described as follows (Conjecture 3.3):

$$\frac{L_{p,\gamma}(E, \omega^{-n-1}, -n)}{R_{p,\gamma}(h^1(E)(-n), \xi)} = \pm \frac{L'(E, -n)}{R_{\infty}(h^1(E)(-n), \xi)}, \quad n \geq 0. \quad (1.1)$$

We also give a formulation for elliptic modular forms of weight $k \geq 2$ with rational Fourier coefficients (Conjecture 3.1). Our statement features that it is more accessible than the original one, especially concerning numerical verifications. The latter half of this paper is devoted to give numerical verifications of our statement in case $n = 0$:

$$\frac{L_{p,\gamma}(E, \omega^{-1}, 0)}{R_{p,\gamma}(h^1(E), \xi)} = \pm \frac{L'(E, 0)}{R_{\infty}(h^1(E), \xi)}. \quad (1.2)$$

To do this, the central role is played by the theory of rigid cohomology and $F$-isocrystals. In particular, we use the deformation method by Lauder [L] and apply the result on effective bounds on overconvergence by Kedlaya-Tuitman [KT, Theorem 2.1]. We shall show,

**Theorem 1.1** Let $f : X \to S \subset \mathbb{P}^1_{\mathbb{Q}}$ be an elliptic fibration which has at least one split multiplicative fiber. Let $X_a = f^{-1}(a)$ be the fiber at a $\mathbb{Q}$-rational point $a \in S(\mathbb{Q})$. Let $\xi \in H^2_{\acute{e}t}(X, \mathbb{Q}(2))$. Then, under some mild assumption on $p$, there is an algorithm for computing the $p$-adic regulator

$$R_{p,\gamma}(h^1(X_a), \xi|_{X_a}) \quad (1.3)$$

modulo $p^n$ for each $n \geq 1$. See §4.7 for the down-to-earth algorithm.

For a particular example, there is a theoretical proof of (1.2).

**Theorem 1.2 (Theorem 5.2)** Let $E$ be the elliptic curve defined by an equation $y^2 = x(1-x)(1+3x)$ over $\mathbb{Q}$. Let

$$\xi = \left\{ \frac{y + 1 - x}{y - 1 + x}, \frac{4x^2}{(1-x)^2} \right\} \in K_2(E).$$

Let $p$ be a prime at which $E$ has a good reduction. Let $\gamma$ be a root of eigenpolynomial of $E$ modulo $p$ such that $\text{ord}_p(\gamma) < 1$. Then (1.2) holds.
The proof uses a formula of Rogers-Zudilin for computing the right hand side of (1.2), and Brunault’s formula for computing the left hand side, together with comparing the symbol $\xi$ with the Beilinson-Kato element (Appendix B).

There have been several papers concerning the $p$-adic Beilinson conjecture, Bannai-Kings [BK], Kings-Loeffler-Zerbes [KLZ], Niklas [N], Brunault [Br], Bertolini-Darmon [BD] and probably more. A noteworthy point is that our approach is entirely different from theirs. For example, many papers rely on Kato’s seminal paper [Ka3] where the Beilinson-Kato elements are introduced. Ours is less powerful than their methods while ours allows an approach to new examples such as the Beilinson-Kato elements vanish in the $p$-adic cohomology groups $H^1_f(G, H^1_{\text{et}}(E, Q_p(2)))$ (Remark 5.1). This also allows us to be convinced of our reformulation of the $p$-adic Beilinson conjecture.

The paper is organized as follows. §2 is a review of $p$-adic $L$-functions for elliptic modular forms, which includes the recent development of critical slope $L$-functions. We give a precise statement of the $p$-adic Beilinson conjecture for elliptic curves in §3 (Conjectures 3.1, 3.3). In §4, we provide the algorithm for computing the $p$-adic regulators (1.3), which is summed up in §4.7. With use of this algorithm, we give numerical verifications of the $p$-adic Beilinson conjecture for explicit examples of elliptic curves in §5 where we use SAGE or MAGMA for computing the special values of ($p$-adic) $L$-functions. The values of critical slope $p$-adic $L$-functions were kindly provided by Professor Robert Pollack. In §6 Appendix A, we sum up the standard results on the Gauss-Manin connection for elliptic fibrations, which are used especially in §4. Most results in Appendix A are probably known to experts. However since the authors do not find a suitable literature, we give a comprehensive exposition for the convenience of the reader. Finally we compare the symbol in Theorem 1.2 with the Beilinson-Kato element in §7 Appendix B.

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2 $p$-adic $L$-functions for elliptic modular forms

Here we recall some basic facts on the cyclotomic $p$-adic $L$-functions for elliptic modular forms. Let $N$ be a positive integer greater than 3. Fix a prime $p > 2$ not dividing $N$ and fix embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $K$ be a subfield of $\bar{\mathbb{Q}}$. Let $S_{k+2}(\Gamma_1(N), K)$ be the space of cuspforms of weight $k + 2$ for $\Gamma_1(N)$ whose Fourier coefficients belong to $K$. Let $f$ be a normalized eigenform of weight $k + 2$ for $\Gamma_1(N)$ with character $\chi_f$. We denote the $n$-th Fourier coefficient of $f$ by $a_n(f)$. Let $\alpha$ and $\beta$ be the two roots of $X^2 - a_p(f)X + \chi_f(p)p^{k+1}$ with $\text{ord}_p(\alpha) \leq \text{ord}_p(\beta)$. Then $f_{\alpha}(z) = f(z) - \beta f(pz)$ and $f_{\beta}(z) = f(z) - \alpha f(pz)$ are normalized cuspforms for $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$ of weight $k + 2$. Moreover they are also eigenforms for $U_p$-operator and satisfy $U_p f_{\alpha}(z) = \alpha f_{\alpha}(z)$ and $U_p f_{\beta}(z) = \beta f_{\beta}(z)$. $f_{\alpha}$ and $f_{\beta}$ are called $p$-refinements (or $p$-stabilizations) of $f$. If
ordin$
abla_p(\beta) = k + 1$, the $p$-refinement $f_\beta$ is called a cuspidal eigenvalue of critical slope. If $f$ is non-ordinary, then $\alpha$ and $\beta$ satisfy $\text{ord}_p(\alpha) \leq \text{ord}_p(\beta) < k + 1$. In this case, we have $p$-adic $L$-functions $L_{p,\alpha}(f, \chi, s)$ and $L_{p,\beta}(f, \chi, s)$ constructed by Vishik [VI] and Amice-Vélu [AV] for any Dirichlet characters $\chi$. If $f$ is ordinary, $\alpha$ satisfies $\text{ord}_p(\alpha) = 0$. Therefore we also have $p$-adic $L$-function $L_{p,\alpha}(f, \chi, s)$. For the critical slope case, Pollack-Stevens [PS1, PS2] and Bellaïche [B] constructed $p$-adic $L$-functions $L_{p,\beta}(f, \chi, s)$ under certain assumptions. In this section, we recall the constructions briefly.

### 2.1 Preliminaries

Let $\mathcal{H} = \mathbb{Z}[T_\ell \downarrow N], U_q(\ell[N])]$ be the Hecke algebra. For a commutative ring $R$, let $\mathcal{A}_k(R)$ denote the space of homogeneous polynomials of degree $k$ in $X$ and $Y$ with $R$-coefficients. Let $\Delta_0 = \text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. Then $\Delta_0$ is a left $GL_2(\mathbb{Q})$-module by the linear fractional transformation.

Let $Y_1(N)$ be the open modular curve over $\mathbb{Q}$ with level $\Gamma_1(N)$-structure and $\pi : \mathcal{E} \to Y_1(N)$ the universal elliptic curve. The line bundle $\omega = \pi_*\Omega^1_{\mathcal{E}/Y_1(N)}$ is naturally extended to the modular curve $X_1(N)$. Then we have an identification

$$S_{k+2}(\Gamma_1(N), K) = H^0(X_1(N)_K, \omega^k \otimes \Omega^1_{X_1(N)}).$$

For the local system $\mathcal{L}^B_k = \text{Sym}^k R^1\pi_*\mathbb{Z}$, we also have a canonical identification

$$H^1(Y_1(N)(\mathbb{C}), \mathcal{L}^B_k) \cong \text{Hom}_{\Gamma_1(N)}(\Delta_0, \mathcal{A}_k(\mathbb{Z})).$$

For simplicity, we denote $H^1_c(\Gamma_1(N), \mathcal{A}_k(R)) = H^1_c(Y_1(N)(\mathbb{C}), \mathcal{L}^B_k) \otimes R$ for a commutative ring $R$. The element $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ defines an involution. If $R$ is a $\mathbb{Z}[1/2]$-module, then we have a decomposition

$$H^1_c(\Gamma_1(N), \mathcal{A}_k(R)) = H^1_c(\Gamma_1(N), \mathcal{A}_k(R))^\oplus \oplus H^1_c(\Gamma_1(N), \mathcal{A}_k(R)^\ominus),$$

where $H^1_c(\Gamma_1(N), \mathcal{A}_k(R))^\oplus = \{ x \in H^1_c(\Gamma_1(N), \mathcal{A}_k(R)) | \epsilon(x) = \pm x \}$. It is known that the action of $\epsilon$ coincides with the action of the complex conjugation (see [Ka3, Proof of Lemma 7.19]). For a cuspidal form $g \in S_{k+2}(\Gamma_1(N), \mathbb{C})$, we define a cohomology class $\xi_g \in H^1_c(\Gamma_1(N), \mathcal{A}_k(\mathbb{C}))$ by

$$\xi_g(\{ x \} - \{ y \}) = 2\pi i \int_x^y g(z)(zX + Y)^k dz.$$

Let $K$ be a subfield of $\overline{\mathbb{Q}}_p$ and $\phi : \mathcal{H} \to K$ be an algebra homomorphism and $V$ a finite dimensional vector space over $K$ with an action of $\mathcal{H}$. For $\alpha \in K$ and $T \in \mathcal{H}$, let $V[\alpha, T]$ be the eigenspace for $T$ acting on $V$ with the eigenvalue $\alpha$ and we put

$$V[\phi] = \bigcap_{T \in \mathcal{H}} V[(\phi(T), T)].$$
Similarly, let \( V(\alpha,T) \) be the generalized eigenspace for \( T \) acting on \( V \) with the eigenvalue \( \alpha \) and we define the \( \phi \)-isotypic subspace \( V(\phi) \) of \( V \) by

\[
V(\phi) = \bigcap_{T \in \mathcal{H}} V(\phi(T),T).
\]

For a normalized eigenform \( f(z) \) in \( S_{k+2}(\Gamma_1(N), \mathbb{Q}) \), we put \( L = \mathbb{Q}(\{a_n(f)\}_{n=1}^{\infty}) \). It is well-known that \( L \) is a finite extension of \( \mathbb{Q} \). Then we can define the homomorphism \( \phi_f : \mathcal{H} \to L \) by \( \phi_f(T_\ell) = a_\ell(f) \) for \( \ell \nmid N \) and \( \phi_f(T_q) = a_q(f) \) for \( q|N \). For simplicity, we denote \( V[f] = V[\phi_f] \) and \( V(f) = V(\phi_f) \).

### 2.2 Result of Amice-Vélu and Vishik

Let \( f \) be a normalized newform in \( S_{k+2}(\Gamma_1(N), \mathbb{Q}) \) and we denote the complex conjugate cusform \( \sum_{n=1}^{\infty} a_n(f)q^n \) of \( f \) by \( f^\ast \). Let \( \nu \) be the valuation induced by the embedding \( L \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p \) and \( \mathcal{O}_{L,v} \), the discrete valuation ring defined by \( v \). We define

\[
H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_{L,v})) \pm[f] = H^1_c(\Gamma_1(N), \mathcal{P}_k(L)) \pm[f] \cap H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_{L,v})).
\]

Then it is known that \( H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_{L,v})) \pm[f] \) is a free \( \mathcal{O}_{L,v} \)-module of rank one. We fix a generator \( u^\pm \) in \( H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_{L,v})) \pm[f^\ast] \). For \( 0 \neq \eta \in S_{k+2}(\Gamma_1(N), L)[f^\ast] \) and \( u = (u^+, u^-) \), we define a complex number \( \Omega^\pm_j(\eta, u) \) by

\[
\xi^\pm_{\eta} = \Omega^\pm_j \cdot u^\pm.
\]

#### Remark 2.1

1. By the multiplicity one, \( \eta \) is a nonzero multiple of \( f \) by an element in \( L^\times \).

2. The generator \( u^\pm \) is determined only up to an element in \( \mathcal{O}_{L,v}^\times \). Since

\[
H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_L)) \pm[f^\ast] = H^1_c(\Gamma_1(N), \mathcal{P}_k(L)) \pm[f] \cap H^1_c(\Gamma_1(N), \mathcal{P}_k(\mathcal{O}_L))
\]

is not a free \( \mathcal{O}_L \)-module in general, we do not have a canonical choice of \( u^\pm \) to define the periods. Therefore, our \( p \)-adic \( L \)-functions will depend on the choice of \( u^\pm \).

Let \( f \) be a normalized eigenform of weight \( k + 2 \) for \( \Gamma_0(N) \) with character \( \chi_f \). Let \( \alpha \) be a root of \( X^2 - a_p(f)X + \chi_f(p)p^{k+1} \). For \( a, m \in \mathbb{Q} \) and \( P \in \mathcal{P}_k(\mathbb{C}) \), we put

\[
\chi^\pm_{f,\alpha}(a, m)(P) = \frac{\pi i}{|\Omega_f|} \left\{ \int_{-a/m}^{a/m} f_\alpha(z) P(-mz + a, 1) dz \pm \int_{-a/m}^{a/m} f_\alpha(z) P(mz + a, 1) dz \right\}.
\]

#### Theorem 2.2 (Vishik [V], Amice-Vélu [AV])

Let \( f \) be a normalized eigenform of weight \( k + 2 \) for \( \Gamma_0(N) \) with character \( \chi_f \). Let \( \alpha \) be a root of \( X^2 - a_p(f)X + \chi_f(p)p^{k+1} \) satisfying \( \text{ord}_p(\alpha) < k + 1 \). Then there exists a unique \( p \)-adic distribution \( \mu^\pm_{f,\alpha} \) on \( \mathbb{Z}_p^\times \) satisfying the following properties:
1. \[ \int_{a+p^jZ_p} x^j d\mu_{f,\alpha}^\pm = \chi_{f,\alpha}(a, p^\nu)(X^jY^{k-j}) \text{ for } \nu > 0 \text{ and } 0 \leq j \leq k. \]

2. For \(0 \leq i < \operatorname{ord}_p(\alpha)\), \(\sup_{a \in Z_p^\times} \left| \int_{a+p^iZ_p} (x-a)^i d\mu_{f,\alpha}^\pm \right|_p = o(p^{\nu(\operatorname{ord}_p(\alpha)-i)})\) when \(\nu\) tends to infinity (i.e. \(\mu_{f,\alpha}^\pm\) is ord\((\alpha)\)-admissible).

For \(x \in Z_p^\times\), we write \(x = \omega(x) \langle x \rangle\) with \(\omega(x) \in \mu_{p-1}(Z_p)\) and \(\langle x \rangle \in 1 + pZ_p\). Let \(\chi : Z_p^\times \to \overline{Q}^\times\) be a character of conductor \(p^n\). Now we set \(\mu_{f,\alpha} = \mu_{f,\alpha}^+ + \mu_{f,\alpha}^-\) and define the \(p\)-adic \(L\)-function \(L_{p,\alpha}(f, \chi, s) = L_{p,\alpha, \eta, u}(f, \chi, s)\) by

\[ L_{p,\alpha}(f, \chi, s) = \int_{Z_p^\times} \chi(x) \langle x \rangle^{s-1} d\mu_{f,\alpha} \]

for \(s \in Z_p\). Then we have the following interpolation property.

**Theorem 2.3** Assume that the root \(\alpha\) satisfies \(\operatorname{ord}_p(\alpha) < k + 1\). For a primitive character \(\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{Q}^\times\) and \(1 \leq j \leq k + 1\), we have

\[ L_{p,\alpha}(f, \omega^{j-1}\chi, j) = \begin{cases} (1 - \chi_f(p)p^{k+1-j}\alpha^{-1})(1 - p^{j-1-\alpha^{-1}}) \frac{L(f, j)}{\Omega_f^{(1,j)}} & \text{if } \chi = 1, \\ \left(\frac{p^{j\nu}}{\alpha^{a_\nu}}\right) \cdot \frac{(j-1)!}{(-2\pi i)^{j-1}\tau(\chi)} \cdot \frac{L(f, \chi, j)}{\Omega_f^{(\chi,j)}} & \text{if } \chi \neq 1, \end{cases} \]

where \(\epsilon(\chi, j) = (-1)^{(j-1)} \cdot \chi(-1)\) and \(\tau\) is the Gauss sum defined by

\[ \tau(\chi) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(a) e^{2\pi i a/p^n}. \]

### 2.3 Critical slope \(p\)-adic \(L\)-functions

If \(\operatorname{ord}_p(\beta) = k + 1\), it is said that \(\beta\) has a critical slope. Here we briefly recall the construction of critical slope \(p\)-adic \(L\)-functions following Pollack-Stevens [PS1, PS2]. Let \(p\) be a prime and \(N\) a positive integer satisfying \((p, N) = 1\). We denote \(\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)\) and \(S_0(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid p \nmid a, p \mid c, \det g \neq 0 \right\}\). For a right \(\mathbb{Z}_p[\mathbb{S}_0(p)]\)-module \(M\), we denote the locally constant sheaf associated to \(M\) on the open modular curve \(Y_{\Gamma_0}(\mathbb{C}) = \mathbb{H}/\Gamma_0\) by \(\tilde{M}\), where \(\mathbb{H}\) is the upper half plane. For simplicity, we assume that the order of the torsion elements of \(\Gamma_0\) are prime to \(p\) or \(M\) is a \(\mathbb{Z}_p\)-module with \(p > 3\). Recall that \(\Delta_0 = \text{Div}^0(\mathbb{P}^1(\mathbb{Q}))\) is the degree zero divisors on \(\mathbb{P}^1(\mathbb{Q})\). For \(\varphi \in \text{Hom}(\Delta_0, M)\) and \(g \in S_0(p)\), we put \(\langle \varphi | g(D) = \varphi(gD) | g \rangle\) for \(D \in \Delta_0\). Define Hecke operators \(T_\ell (\ell \nmid N)\) and \(U_q (q|N)\) by \(\varphi| T_\ell = \varphi \left[ \begin{array}{cc} \ell & 0 \\ 0 & 1 \end{array} \right] + \sum_{a=0}^{\ell-1} \varphi \left[ \begin{array}{cc} 1 & a \\ 0 & \ell \end{array} \right] \) and \(\varphi| U_q = \sum_{a=0}^{q-1} \varphi \left[ \begin{array}{cc} 1 & a \\ 0 & q \end{array} \right] \). We denote \(H^1_c(\Gamma_0, M) = H^1_c(Y_{\Gamma_0}(\mathbb{C}), M)\) as before. Then Ash-Stevens [AS, Proposition 4.6.] proved...
that there exists a canonical Hecke equivariant isomorphism $H_c^1(\Gamma_0, M) \cong \text{Hom}_{\Gamma_0}(\Delta_0, M)$, where $\text{Hom}_{\Gamma_0}(\Delta_0, M) = \{ \varphi \in \text{Hom}(\Delta_0, M) \mid \varphi|_g = \varphi \text{ for all } g \in \Gamma_0 \}$ is the $\Gamma_0$-invariant homomorphisms. From now on, we will identify these modules by the above isomorphism. If $M$ is a Banach module and $\Gamma_0$ acts on $M$ as unitary operators, then $H_c^1(\Gamma_0, M)$ is also a Banach space with the norm defined by $||\Psi|| = \sup_{D \in \Delta_0} ||\Psi(D)||$ for $\Psi \in H_c^1(\Gamma_0, M)$.

For $r \in |C_p^\times|$, we denote

$$B[Z_p, r] = \{ z \in C_p \mid \text{there exists } a \in Z_p \text{ such that } |z - a|_p \leq r \}.$$ 

Then $B[Z_p, r]$ is the $C_p$-valued points of a $Q_p$-affinoid variety. Let $A[r]$ be the $Q_p$-Banach algebra of $Q_p$-affinoid functions on $B[Z_p, r]$ with the norm $|| \cdot ||_r$ on $A[r]$ defined by $||f||_r = \sup_{z \in B[Z_p, r]} |f(z)|_p$. If $r > r'$, the restriction map $A[r] \to A[r']$ is injective, completely continuous and has dense image. Then we define the topological spaces $\mathcal{A}(Z_p) = \lim_{\rightarrow s > 0} A[s]$ and $\mathcal{A}^1(Z_p, r) = \lim_{r > s > 0} A[s]$ with the inductive limit topology. We have natural (continuous) inclusions

$$\mathcal{A}^1(Z_p, r) \hookrightarrow A[r] \hookrightarrow \mathcal{A}^1(Z_p, r). \quad (2.1)$$

Let $\mathcal{D}[r]$ be the space of continuous $Q_p$-linear functionals on $A[r]$. Similarly, denote the space of continuous $Q_p$-linear functionals on $\mathcal{A}(Z_p)$ (resp. $\mathcal{A}^1(Z_p, r)$) by $\mathcal{D}(Z_p)$ (resp. $\mathcal{D}^1(Z_p, r)$). The topologies are given by the strong topology. Then the space $\mathcal{D}[r]$ becomes a Banach space with the norm defined by

$$||\mu||_* = \sup_{0 \neq f \in A[r]} \frac{|\mu(f)|_p}{||f||_r}.$$ 

The family of norms $\{ || \cdot ||_s \}_{s > 0}$ (resp. $\{ || \cdot ||_s \}_{s > r}$) induces a topology on $\mathcal{D}(Z_p)$ (resp. $\mathcal{D}^1(Z_p, r)$). By taking the dual of the inclusions (2.1), we have continuous maps

$$\mathcal{D}(Z_p) \to \mathcal{D}[r] \to \mathcal{D}^1(Z_p, r).$$

Fix a positive integer $k$. For $g \in \Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z_p) \mid p \nmid a, p \nmid c, ad - bc \neq 0 \right\}$ and $f \in A[r]$, we set

$$(g \cdot_k f)(z) = (a + cz)^k \cdot f \left( \frac{b + dz}{a + cz} \right).$$

For $g \in \Sigma_0(p)$ and $\mu \in \mathcal{D}[r]$, we also set

$$(\mu|_{k}g)(f) = \mu(g \cdot_k f).$$

These actions induce the actions of $\Sigma_0(p)$ on $\mathcal{A}(Z_p)$, $\mathcal{A}^1(Z_p, r)$, $\mathcal{D}(Z_p)$ and $\mathcal{D}^1(Z_p, r)$. To emphasize the role of $k$ in these actions, we write these modules as $\mathcal{A}_k(Z_p)$, $\mathcal{A}^1_k(Z_p, r)$, $\mathcal{D}_k(Z_p)$, $\mathcal{D}_k[r]$ and $\mathcal{D}^1_k(Z_p, r)$. Moreover we denote $\mathcal{D}_k = \mathcal{D}_k(Z_p)$, $\mathcal{D}_k = \mathcal{D}_k[1]$ and $\mathcal{D}^1_k = \mathcal{D}^1_k(Z_p, 1)$ for simplicity.
Let \( \mathcal{D}_k = \mathcal{D}_k(\mathbb{Q}_p) \) be the space of homogeneous polynomials with \( \mathbb{Q}_p \)-coefficients of degree \( k \) in \( X \) and \( Y \). We endow the space \( \mathcal{D}_k \) with the structure of a right \( GL_2(\mathbb{Q}_p) \)-module by

\[
(P|g)(x) = P(dx - cY, -bX + aY),
\]

where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( P \in \mathcal{D}_k \). Then we have an \( \Sigma_0(p) \)-equivariant map

\[
\rho_k : \mathcal{D}_k^1 \to \mathcal{D}_k
\]
given by \( \mu \mapsto \int (Y - zX)^k d\mu(z) \) and it induces the specialization map

\[
\rho_k^c : H^1_c(\Gamma_0, \mathcal{D}_k^1) \to H^1_c(\Gamma_0, \mathcal{D}_k).
\]

Now we recall the slope decomposition of Banach spaces with a completely continuous operator. Let \( X \) be a Banach space over \( \mathbb{Q}_p \) with a completely continuous operator \( U \). Let \( Q \in \mathbb{Q}_p[T] \) be an irreducible polynomial with \( Q(0) \neq 0 \). By Riesz decomposition theorem [PS2, Theorem 4.1], we have the decomposition into a direct sum of two closed subspaces preserved by the operator \( U \): \( X \cong X(Q) \oplus X'(Q) \), where \( Q(U) \) is nilpotent on \( X(Q) \) and invertible on \( X'(Q) \). Moreover \( X(Q) \) is finite dimensional over \( \mathbb{Q}_p \). For an irreducible polynomial \( Q \in \mathbb{Q}_p[T] \), we set \( v_Q^i(Q) = sup_{\alpha} v_Q(\alpha) \), where \( \alpha \) runs over the all roots of \( Q \) in \( \mathbb{Q}_p \). For Banach space \( X \) over \( \mathbb{Q}_p \), with a completely continuous operator \( U \) and \( h \in \mathbb{R} \), we denote

\[
X^{(h)} = \bigoplus_{v_Q(\alpha) < h} X(Q),
\]

where \( Q \) runs over all monic irreducible polynomials over \( \mathbb{Q}_p \) satisfying \( v_Q(\alpha) < h \). Then it is known that the space \( X^{(h)} \) is finite dimensional.

For \( r \in \mathbb{C}_p^\times \), \( \mathbb{D}_k[r] \) is a Banach space and hence \( H^1_c(\Gamma_0, \mathbb{D}_k[r]) \) is also a Banach space. Then the Hecke operator \( U = U_p \) acts on the space \( H^1_c(\Gamma_0, \mathbb{D}_k[r]) \) complete continuously. Therefore we have the space \( H^1_c(\Gamma_0, \mathbb{D}_k[r])^{(h)} \) for each \( h \in \mathbb{R} \). Since \( H^1_c(\Gamma_0, \mathcal{D}_k) \) and \( H^1_c(\Gamma_0, \mathcal{D}_k) \) are not Banach spaces, we define

\[
H^1_c(\Gamma_0, \mathcal{D}_k)^{(h)} = H^1_c(\Gamma_0, \mathcal{D}_k) \cap H^1_c(\Gamma_0, \mathbb{D}_k)^{(h)}
\]

and

\[
H^1_c(\Gamma_0, \mathcal{D}_k)^{(h)} = H^1_c(\Gamma_0, \mathcal{D}_k) \cap H^1_c(\Gamma_0, \mathbb{D}_k)^{(h)}
\]

for any \( r > 1 \). Here we take the intersections as subsets of \( \text{Hom}(\Delta_0, \mathbb{D}_k) \) or \( \text{Hom}(\Delta_0, \mathbb{D}_k[r]) \).

**Remark 2.4** By [PS2, Lemma 5.3], we have isomorphisms

\[
H^1_c(\Gamma_0, \mathcal{D}_k)^{(h)} \cong H^1_c(\Gamma_0, \mathbb{D}_k)^{(h)} \cong H^1_c(\Gamma_0, \mathcal{D}_k)^{(h)} \cong H^1_c(\Gamma_0, \mathbb{D}_k[r])^{(h)}
\]

for any \( h \in \mathbb{R} \) and \( r > 0 \) with \( r \in \mathbb{C}_p^\times \). Therefore the definition of \( H^1_c(\mathcal{D}_k)^{(h)} \) does not depend on the choice of \( r \).
For a normalized eigenform $f$, the order of torsion elements in $(\Gamma_1(N), \mathbb{Q}_p)$ is known that $f$ is a critical slope eigenform in $S_k(\Gamma_1(N), \mathbb{Q}_p)$. Assume that $f_\beta$ is a critical slope eigenform in $S_{k+2}(\Gamma_0, \mathbb{Q}_p)$, we put $K = \mathbb{Q}_p \{a_n f_\beta\}_{n=1}^\infty$. Then it is known that $(H^1_c(\Gamma_0, \mathcal{D}_k)(f_\beta))$ are 1 dimensional vector spaces over $K$. We choose $\varphi_{f_\beta} = \varphi_{f_\beta, \eta, u}$ in $(H^1_c(\Gamma_0, \mathcal{D}_k) \otimes K)(f_\beta)$ as

$$\varphi_{f_\beta}(\{x\} - \{y\}) = \frac{\pi i}{\Omega_f^{-1}} \left( \int_x^y f_\beta(z)(zX + Y)^k dz - \int_{-x}^{-y} f_\beta(z)(zX - Y)^k dz \right)$$

and put $\varphi_{f_\beta} = \varphi_{f_\beta}^+ + \varphi_{f_\beta}^-$. Recall that $\Omega_f = \Omega_f^-(\eta, u)$ is the period determined by $\eta$ and $u = (u^+, u^-)$.

Assume that $f_\beta$ is a critical slope eigenform in $S_{k+2}(\Gamma_0, \mathbb{Q}_p)$. Furthermore we suppose that $f_\beta$ is non-$\theta$-critical. Then by Theorem 2.6 there exists a unique element $\Phi_{f_\beta}$ such that the image of $\Phi_{f_\beta}$ under the specialization map $H^1_c(\Gamma_0, \mathcal{D}_k)(f_\beta) \rightarrow H^1_c(\Gamma_0, \mathcal{D}_k)(f_\beta)$ is equal to $\varphi_{f_\beta}$. Since we have the identification $H^1_c(\Gamma_0, \mathcal{D}_k) = \text{Hom}_{\Gamma_0}(\Delta_0, \mathcal{D}_k)$, $\mu_{f_\beta}$ defines an element in $\Phi(\mathbb{Z}_p^\times)$. Therefore we can define the critical slope $p$-adic $L$-function $L_{p,\beta}(f, \chi, s) = L_{p,\beta, \eta, u}(f, \chi, s)$ by

$$L_{p,\beta}(f, \chi, s) = \int_{\mathbb{Z}_p^\times} \chi(x)(x)^{s-1} d\mu_{f_\beta}$$

for each finite order character $\chi$ of $\mathbb{Z}_p^\times$. Then the critical slope $p$-adic $L$-function $L_{p,\beta}(f, \chi, s)$ satisfies the same interpolation property with Theorem 2.3. However the $p$-adic distribution $\mu_{f_\beta}$ is not characterized by the interpolation property in the critical slope case.

**Remark 2.7** If a root $\gamma$ of $X^2 - a_p(f) + \chi f(p)p^{k+1}$ satisfies $\text{ord}_p(\gamma) < k + 1$, then we also have a unique lift $\Phi_{f_\beta}$ of $\varphi_{f_\beta}$ by Theorem 2.5. In this case, we can show that $\Phi_{f_\beta}(\{\infty\} - \{0\}) = \mu_{f_\gamma}$. Proposition 6.3]. Therefore we have an alternative construction of the $p$-adic $L$-functions for non-critical slope cuspforms using overconvergent modular symbols.
\section{p-adic Beilinson conjecture for non-critical values of p-adic $L$-functions}

In this section, we give a reformulation of p-adic Beilinson conjecture for the special values of the cyclotomic p-adic $L$-functions in the case of cuspforms with rational Fourier coefficients. For the case of elliptic curves over $\mathbb{Q}$, the conjecture can be simplified. Since we will focus on the case of elliptic curves over $\mathbb{Q}$ later, we also explain the formulation of the conjecture for the special case. First, we begin with the review of Beilinson regulators and the syntomic regulators.

For a smooth scheme $X$ over a commutative ring $A$, we denote by $\Omega^\bullet_{X/A}$ the algebraic de Rham complex, and by $H^*_\text{dR}(X/A) := \mathbb{H}^*_\text{zar}(X, \Omega^\bullet_{X/A})$ the algebraic de Rham cohomology.

\subsection{Deligne-Beilinson cohomology and real regulators}

Let $X$ be a smooth quasi-projective variety over $\mathbb{C}$, and $\overline{X}$ a smooth completion such that $D := \overline{X} \setminus X$ is a normal crossing divisor (abbreviated NCD). Let $j : X \hookrightarrow \overline{X}$ be the open immersion. Let $\Omega^\bullet_{\text{an}}(\log D)$ denote the analytic de Rham complex with log poles along $D$, which is quasi-isomorphic to the de Rham complex $j_*\Omega^\bullet_{\text{an}}$. Let $\text{Fil}^r\Omega^\bullet_{\text{an}}(\log D) := \Omega^\bullet_{\text{an}}(\log D) / (\text{log}_r \Omega^\bullet_{\text{an}})$ be the stupid filtration.

For an integer $r \geq 0$, the Deligne-Beilinson complex is

$$Z(r)_{\partial, X, \overline{X}} := \text{Cone}[Rj_*Z_X(r) \oplus \text{Fil}^r\Omega^\bullet_{\text{an}}(\log D) \xrightarrow{\epsilon^{-1}} j_*\Omega^\bullet_{\text{an}}]$$

the cone of complexes of sheaves on the analytic site $\overline{X}$ where $Z_X(r) := (2\pi i)^rZ_X$ and $\epsilon : Z_X(r) \to \Omega^\bullet_{\text{an}}$ is the natural map, and $i : \text{Fil}^r\Omega^\bullet_{\text{an}}(\log D) \to \Omega^\bullet_{\text{an}}(\log D) \to j_*\Omega^\bullet_{\text{an}}$ is the composition. The Deligne-Beilinson cohomology is defined to be the cohomology

$$H^i_{\partial}(X, Z(r)) := \mathbb{H}^i(\overline{X}, Z(r)_{\partial, X, \overline{X}}).$$

This depends only on $X$, and functorial with respect to $X$ ([EV Lemma 2.8]). The Deligne-Beilinson cohomology $H^i_{\partial}(X, A(r))$ with coefficients in a ring $A \subset \mathbb{R}$ is defined by replacing $\mathbb{Z}$ with $A$ in the above. The distinguished triangle

$$\text{Cone}[\text{Fil}^r\Omega^\bullet_{\text{an}}(\log D) \to j_*\Omega^\bullet_{\text{an}}] \to Z(r)_{\partial, X, \overline{X}} \to Rj_*Z_X(r)$$

gives rise to an exact sequence

$$\cdots \to H^{i-1}_{\text{dR}}(X/\mathbb{C})/\text{Fil}^r \to H^i_{\partial}(X, Z(r)) \to H^i_{\text{dR}}(X(\mathbb{C}), Z(r)) \to H^i_{\text{dR}}(X/\mathbb{C})/\text{Fil}^r \to \cdots$$

where $\text{Fil}^r = \text{Fil}^rH^i_{\text{dR}}(X/\mathbb{C})$ denotes Deligne’s Hodge filtration. Here we note that the analytic de Rham cohomology $\mathbb{H}^i(X, \Omega^\bullet_{\text{an}})$ agrees with the algebraic de Rham cohomology $H^i_{\text{dR}}(X/\mathbb{C})$.

Let $X_{\mathbb{R}}$ be a smooth variety over $\mathbb{R}$, and $\overline{X}_{\mathbb{R}}$ a smooth compactification such that $D_{\mathbb{R}} := \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}$ is a NCD. Let $\overline{X} = \overline{X}_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ etc. and denote by $\overline{X}(\mathbb{C})$ the complex points. Then
the complex conjugation \( z \rightarrow \bar{z} \) induces the anti-holomorphic map \( F_\infty : \overline{X}(\mathbb{C}) \rightarrow \overline{X}(\mathbb{C}) \), which is called the infinite Frobenius. Let \( \omega \mapsto \overline{\omega} \) denote the complex conjugation on smooth differential forms on \( X(\mathbb{C}) \). The map

\[
\omega \mapsto F_\infty^*(\omega), \quad \omega \in \Omega^i_{X/\mathbb{C}}(\log D)
\]

induces maps

\[
F_\infty^{-1}Z_X(r) \rightarrow Z_X(r), \quad F_\infty^{-1}Z(r)_{\omega,X,\overline{X}} \rightarrow Z(r)_{\omega,X,\overline{X}}
\]

of sheaves on the smooth manifold \( \overline{X}(\mathbb{C}) \). We thus have an action on the cohomology \( H^i_p(X,\mathbb{Z}(r)), H^i_B(X(\mathbb{C}),\mathbb{Z}(r)) \) and so on, which we write by the same notation \( F_\infty \). By definition \( F_\infty^2 = \text{id} \). We write \( H^+ \) to be the \( F_\infty \)-fixed part, and \( H^- \) the anti-fixed part. We have

\[
H^i_B(X(\mathbb{C}),\mathbb{Z}(r))^+ = H^i_B(X(\mathbb{C}),\mathbb{Z}(s))^{(-1)^r-\ast} \otimes \mathbb{Z}(r-s).
\]

Define the real Deligne-Beilinson cohomology \( H^i_p(X_{\mathbb{R}}/\mathbb{R},\mathbb{R}(r)) \) to be the \( F_\infty \)-fixed part ([Sch §2])

\[
H^i_p(X_{\mathbb{R}}/\mathbb{R},\mathbb{R}(r)) := H^i_p(X,\mathbb{R}(r))^+.
\]

By the theory of the universal Chern classes, the Beilinson regulator map (the higher Chern class map)

\[
\text{reg}_{\mathcal{B},p}^{i,r} : K_{2r-i}(X) \longrightarrow H^i_p(X,\mathbb{Z}(r)) \tag{3.2}
\]

is defined ([Sch §4, p.29–30]). Let \( X_{\mathbb{R}} \) be a smooth variety over \( \mathbb{R} \), then it induces the real regulator map

\[
\text{reg}_{\mathcal{B},p}^{i,r} : K_{2r-i}(X_{\mathbb{R}}) \longrightarrow H^i_p(X_{\mathbb{R}}/\mathbb{R},\mathbb{R}(r)) \tag{3.3}
\]

to the real Deligne-Beilinson cohomology group.

### 3.2 Syntomic cohomology and Syntomic regulators

Let \( W \) be the Witt ring of a perfect field \( k \) of characteristic \( p > 0 \), and \( K := \text{Frac} W \). Let \( X \) be a smooth projective \( W \)-scheme. We write \( X_k := X \times_W k \) and \( X_K := X \times_W K \). Then the syntomic cohomology of Fontaine and Messing

\[
H^i_{\text{syn}}(X,\mathbb{Z}_p(r)), \quad r \geq 0
\]

is defined ([FM], [KaI]). There is an exact sequence

\[
\cdots \rightarrow \text{Fil}^r H^{i-1}_{\text{crys}}(X_k/W) \xrightarrow{1-\phi} H^{i-1}_{\text{crys}}(X_k/W) \rightarrow H^i_{\text{syn}}(X,\mathbb{Z}_p(r)) \rightarrow \text{Fil}^r H^i_{\text{crys}}(X_k/W) \rightarrow \cdots \tag{3.4}
\]

which is a counterpart of (3.1), where \( \phi \) is the \( p \)-th Frobenius on the crystalline cohomology \( H^i_{\text{crys}}(X_k/W) \), and \( \text{Fil}^r \) is the Hodge filtration via the comparison \( H^i_{\text{crys}}(X_k/W) \cong H^i_{\text{dR}}(X/W) \). If we drop the assumption that \( X \) is proper over \( W \), the construction of the syntomic cohomology in [FM] or [KaI] does not give a "correct" cohomology theory. For
such a variety, we need the rigid syntomic cohomology $H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r))$ by Besser [Be1] (see also [NN] 1B), which gives the correct cohomology theory.

Let $X$ be a smooth $W$-scheme. Then the syntomic regulator map

$$\text{reg}_{\text{rig-syn}}^{i,r}: K_{2r-i}(X) \longrightarrow H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r))$$

(3.5)
is defined ([Be1, Theorem 7.5], [NN, Theorem A]). This is compatible with the étale regulator map

$$\text{reg}_{\text{ét}}^{i,r}: K_{2r-i}(X_{K}) \longrightarrow H^i_{\text{ét}}(X_K, \mathbb{Q}_p(r))$$

(3.6)

via the syntomic period map (loc.cit. Theorem A (7))

$$c : H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r)) \rightarrow H^i_{\text{ét}}(X_K, \mathbb{Q}_p(r)).$$

If $X$ is projective over $W$, the rigid syntomic cohomology agrees with the syntomic cohomology of Fontaine-Messing. In this case we write the subscript “syn” instead of “rig-syn”.

Let $X$ be a smooth projective $W$-scheme. The composition $\rho$ in the following diagram

$$H^{i-1}_{\text{crys}}(X_k/W)/(1 - p^{-r}\phi)\text{Fil}^r \xrightarrow{\rho} H^i_{\text{syn}}(X, \mathbb{Q}_p(r)) \xrightarrow{c} H^i_{\text{ét}}(X, \mathbb{Q}_p(r))$$

induces a map

$$H^{i-1}_{\text{crys}}(X_k/W)/\text{Fil}^r \rightarrow H^1(G_K, H^{i-1}_{\text{ét}}(X_{K}, \mathbb{Q}_p(r))).$$

This agrees with the Bloch-Kato exponential map ([Be1 Proposition 9.11], [NN Proposition 1.1]).

### 3.3 $p$-adic Beilinson conjecture for modular forms

In this section, we assume that the Fourier coefficients of $f$ belong to $\mathbb{Q}$ for simplicity. For a projective smooth variety over $\mathbb{Q}$, let $H^j_{\text{mot}}(X, \mathbb{Q}(j)) = K_{2j-i}(X, \mathbb{Q}(j))$ be the motivic cohomology group and let $H^j_{\#}(X, \mathbb{Q}(j))_{\mathbb{Z}}$ denote the integral part defined by Scholl [Sc2]. Recall that $\pi : \mathcal{E} \rightarrow X_1(N)$ is the universal generalized elliptic curve over the modular curve. Let $\mathcal{E}^k = \mathcal{E} \times X_1(N) \cdots \times X_1(N)$ be the $k$-fold fiber product of $\mathcal{E}$ over the modular curve $X_1(N)$ and $W_k$ the canonical desingularization of $\mathcal{E}^k$ (See [Sc1] or an appendix in [BDP] by Brian Conrad for details). $W_k$ is a smooth projective variety over $\mathbb{Q}$ of dimension $k + 1$. Fix a prime $p$ not dividing $N$. Let $\mathcal{W}_k$ be the integral model over $\mathbb{Z}_p$. For $a \in (\mathbb{Z}/N\mathbb{Z})^k$, let $\sigma_a$ denote the automorphism on $\mathcal{E}^k$ obtained from the translation by the sections of order $N$, which naturally extends to $W_k$. Then we define the projector $\varepsilon_k^{(1)}$

$$\varepsilon_k^{(1)} = \frac{1}{N_k} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^k} \sigma_a.$$
Let $G_k$ be the symmetric group on $k$ letters. The multiplication by $-1$ on $G_k$ and the natural permutation action of $G_k$ on $E^k$ give rise to an action of the semi-direct product $\{\pm 1\}^k \times G_k$ on $E^k$. This action also extends to $W_k$. Let $\mu : \{\pm 1\}^k \times G_k \to \{\pm 1\}$ be the character which is the multiplication on $\{\pm 1\}^k$ and the sign character on $G_k$. Then

$$\varepsilon_k^{(2)} = \frac{1}{2^k \cdot k!} \sum_{\sigma \in \{\pm 1\}^k \times G_k} \mu(\sigma) \sigma \in \mathbb{Q}[\text{Aut}(W_k/X_1(N))]$$

defines a projector. The projectors $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$ commute, and therefore the composition $\varepsilon_k = \varepsilon_k^{(1)} \varepsilon_k^{(2)}$ defines a projector. Then the de Rham realization of the Chow motive $M = (W_k, \varepsilon_k)$ is canonically isomorphic to the parabolic de Rham cohomology:

$$H^*(\pi/\mathcal{O}_k) = H^k_{\text{par}}(X_1(N), \text{Sym}^k \mathcal{L}, \nabla),$$

where $\mathcal{L} = \pi_1^1 \pi_1(0 \to \mathcal{O}_k \to \Omega^1_{X_1(N)} \to 0)$ is the relative de Rham cohomology sheaf on $X_1(N)$. The space of cuspforms $S_{k+2}(\Gamma_1(N), \mathbb{Q}) = H^0(X_1(N), \mathcal{O}^k \otimes \Omega^1_{X_1(N)})$ is naturally identified with a subspace of the parabolic de Rham cohomology $H^1_{\text{par}}(X_1\mathcal{N}), \text{Sym}^k \mathcal{L}, \nabla)$. Moreover, we have $\text{Fil}^i \varepsilon_k H^*_{\text{par}}(W_k/\mathbb{Q}) = 0$, where $\text{Fil}^i$ denotes the $i$-th step in the Hodge filtration on $\varepsilon_k H^*_{\text{par}}(W_k/\mathbb{Q})$. For the Betti realization, we have a canonical identification

$$H^i_{\text{B}}(M, \mathbb{Q}) = \varepsilon_k H^i_{\text{B}}(W_k(\mathbb{Q}), \mathbb{Q}) = \varepsilon_k H^i_{\text{B}}(W_k(\mathbb{C}), \mathbb{Q}) = H^i_{\text{par}}(X_1(N)(\mathbb{C}), \mathcal{L}^B_{k, \mathbb{Q}}),$$

where $\mathcal{L}^B_{k, Q} = \mathcal{L}^B_k \otimes \mathbb{Q}$ and

$$H^1_{\text{par}}(X_1(N)(\mathbb{C}), \mathcal{L}^B_{k, \mathbb{Q}}) = \text{Im} \left[ H^1_{\text{B}}(X_1(N)(\mathbb{C}), \mathcal{L}^B_{k, \mathbb{Q}}) \to H^1(X_1(N)(\mathbb{C}), \mathcal{L}^B_{k, \mathbb{Q}}) \right]$$

is the parabolic Betti cohomology.

Let $f$ be a normalized newform in $S_{k+2}(\Gamma_1(N), \mathbb{Q})$. (Note that $f = f^*$ in this case). Then the element $u^\pm$ is determined only up to sign. Also we fix a non-zero element $\eta$ in $S_{k+2}(\Gamma_1(N), \mathbb{Q})[f] = H^0(X_1(N), \mathcal{O}^k \otimes \Omega^1_{X_1(N)})[f]$. As we mentioned above, $\eta$ can be identified with an element in $\varepsilon_k H^{k+1}_{\text{par}}(W_k/\mathbb{Q})[f]$ and $u^\pm$ can be viewed as an element in

$$H^0_{\text{B}}(M, \mathbb{Q})[f] = \varepsilon_k H^0_{\text{B}}(W_k(\mathbb{C}), \mathbb{Q})[f] = \varepsilon_k H^{k+1}_{\text{B}}(W_k(\mathbb{C}), \mathbb{Q})[f].$$

Let $\gamma$ be an eigenvalue of the Frobenius $\Phi$ on

$$(\varepsilon_k H^k_{\text{dr}}(W_k/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}_p})[f] \cong (\varepsilon_k H^k_{\text{cris}}(W_{k,F}_p/\mathbb{Z}_p) \otimes \overline{\mathbb{Q}_p})[f]$$

and choose a non-zero eigenvector $v_\gamma$ in $(\varepsilon_k H^{k+1}_{\text{dr}}(W_k/\mathbb{Q}_p) \otimes \mathbb{Q}_p)\Phi(\gamma))[f]$ with the eigenvalue $\gamma$. Let

$$\text{reg}_{\text{syn}} : \varepsilon_k H^{k+2}_{\text{dr}}(W_k, \mathbb{Q}(n+k+2)) \to \varepsilon_k H^{k+2}_{\text{syn}}(W_k, \mathbb{Q}(n+k+2)) \cong \varepsilon_k H^{k+1}_{\text{dr}}(W_k/\mathbb{Q}_p)$$



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be the syntomic regulator map. Let
\[ \text{pr}_{dR,f} : \varepsilon_k H^{k+1}_{dR}(W_k/Q_p) \to \varepsilon_k H^{k+1}_{dR}(W_k/Q_p)[f] \]
denote the projection to the \( f \)-isotypic subspace. Assume that \( \eta \) and \( v_\gamma \) are linearly independent over \( Q_p \). For an element \( \xi \in \varepsilon_k H^{k+2}_{dR}(W_k, Q(n + k + 2))_\mathbb{Z} \), we set
\[ R_{p,\gamma,\eta}(M(f)(-n), \xi) = \Gamma^*(-n)(1 - p^{1+n\gamma^{-1}}) \frac{\text{Tr}(\text{pr}_{dR,f} \circ \text{reg}_{\text{syn}}(\xi) \cup v_\gamma)}{\text{Tr}(\eta \cup v_\gamma)} \in Q_p(\gamma), \]
where
\[ \Gamma^*(-n) = \begin{cases} \Gamma(-n) & \text{if } n \leq -1, \\ \frac{(-1)^n}{n!} & \text{if } n \geq 0. \end{cases} \]

Note that \( R_{p,\gamma,\eta}(M(f)(-n), \xi) \) is independent of the choice of \( v_\gamma \) by the definition. On the other hand, we have the Beilinson regulator map
\[ \text{reg}_\phi : \varepsilon_k H^{k+2}_{dR}(W_k, Q(n + k + 2))_\mathbb{Z} \to \varepsilon_k H^{k+2}_{dR}(W_k, \mathbb{R}(n + k + 1))^+ \]
\[ \cong \varepsilon_k H^{k+1}_B(W_k(\mathbb{C}), \mathbb{R}(n + k + 1))^{-1} \cong \varepsilon_k H^{k+1}_B(W_k(\mathbb{C}), \mathbb{R})^{-1+n+k+1} \otimes \mathbb{R}(n + k + 1). \]

Choose a non-zero element \( \delta^\pm \in \varepsilon_k H^{k+1}_B(W_k(\mathbb{C}), \mathbb{R})^{-1}[f] \). We denote the projection to the \( f \)-isotypic subspace by
\[ \text{pr}_{B,f} : \varepsilon_k H^{k+1}_B(W_k(\mathbb{C}), \mathbb{R})^{-1} \otimes \mathbb{R}(m) \to \varepsilon_k H^{k+1}_B(W_k(\mathbb{C}), \mathbb{R})^{-1}[f] \otimes \mathbb{R}(m), \]
where \( m = n + k + 1 \). Now we define
\[ R_\infty(M(f)(-n), \xi) = \frac{1}{(2\pi i)^{n+k+1}} \frac{\text{Tr}(\text{pr}_{B,f} \circ \text{reg}_\phi(\xi) \cup \delta^{-1+n+k+1})}{\text{Tr}(\delta^{-1+n+k+1})} \in \mathbb{R}. \]

By the definition, it is clear that \( R_\infty(M(f)(-n), \xi) \) is independent of the choice of \( \delta^\pm \). By Beilinson conjecture, it is expected that the ratio \( \frac{L'(f,-n)}{R_\infty(M(f)(-n), \xi)} \) is a rational number.

**Conjecture 3.1** (\( p \)-adic Beilinson conjecture for modular forms \([PR]\)) Suppose that one of the following conditions holds:

1. \( \text{ord}_p(\gamma) < k + 1 \),
2. \( \text{ord}_p(\gamma) = k + 1 \) and the \( p \)-refinement \( f_\gamma \) is not \( \theta \)-critical.

Moreover we assume that \( \eta \) and \( v_\gamma \) are linearly independent over \( Q_p \). Then for \( n \geq 0 \), there exists a non-zero element
\[ \xi \in \varepsilon_k H^{k+2}_{dR}(W_k, Q(n + k + 2))_\mathbb{Z} \]
such that the ratio
\[ \frac{L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)} \]
is a non-zero rational number and we have
\[ \frac{L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)} = \pm \frac{L'(f, -n)}{R_\infty(M(f)(-n), \xi)}. \]
Remark 3.2 1. In the original formulation of $p$-adic Beilinson conjecture by Perrin-Riou [PR 4.2.2 Conjecture (CP($M, \tau$))], it is expected the existence of $p$-adic $L$-function satisfying the interpolation property in the critical range and $p$-adic Beilinson formula in the non-critical range. For the case of elliptic modular forms, we already have an appropriate $p$-adic $L$-function $L_{p,\gamma}(f, \chi, s)$ satisfying the interpolation property in the critical range. Therefore we can formulate the conjecture as more accessible form in this case. Moreover, Perrin-Riou used the étale regulator map to formulate the conjecture. Therefore the formula of the conjecture has a slightly different form. In general, the comparison between the étale regulator and the syntomic regulator is explained by Besser [Be1]. By [Be1, Remark 8.6.3], the difference is given by the factor $(1 - \Phi_p^N)$. Using this result, it is easy to see that our formulation is equivalent to Perrin-Riou’s formulation.

2. Our normalization of the $p$-adic $L$-functions is slightly different from Perrin-Riou’s normalization. In fact, Perrin-Riou’s $p$-adic $L$-functions should be the half of our $p$-adic $L$-function. For details, see [PR, 4.2.2 Conjecture (CP($M, \tau$)) and 4.3.4].

3. For $c \in \mathbb{Q}^\times$, it is easy to see

$$L_{p,\gamma,c\eta,u}(f, \omega^{-n-1}, -n) = c^{-1} \cdot L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)$$

and

$$R_{p,\gamma,c\eta}(M(f)(-n), \xi) = c^{-1} \cdot R_{p,\gamma,\eta}(M(f)(-n), \xi).$$

Therefore the ratio $\frac{L_{p,\gamma,c\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)}$ does not depend on the choice of $\eta$. Moreover, we can show that the conjecture [3.7] is also independent of the choice of $u = (u^+, u^-)$.

4. Bannai-Kings [BK] proved a version of $p$-adic Beilinson conjecture for Hecke motives over imaginary quadratic fields with class number one when $\text{ord}_p(\gamma) = 0$.

3.4 $p$-adic Beilinson conjecture for elliptic curves

Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$ and let $\omega_E$ denote the Néron differential. Let $u^\pm$ be the generator of $H^1(E(\mathbb{C}), \mathbb{Z})^\pm$ satisfying $\int_{u^+} \omega_E > 0$ and $i^{-1} \int_{u^-} \omega_E > 0$. Then we put $\Omega_E^+ = \int_{u^+} \omega_E$ and $\Omega_E^- = \int_{u^-} \omega_E$. It is known that

$$\frac{L(E, \chi, 1)}{\Omega_E^\chi(-1)} \in \mathbb{Q}(\chi)$$

for any Dirichlet character $\chi$. Let $f = f_E$ be the newform associated to $E$. For $a, m \in \mathbb{Q}$, we put

$$\lambda_E^\pm(a, m) = \frac{\pi i}{\Omega_E^\pm} \left\{ \int_{\infty}^{\infty} f(z)dz \pm \int_{-\infty}^{-\infty} f(z)dz \right\}.$$
Assume that $E$ has good reduction at $p$. Let $\gamma$ be a root of $X^2 - a_p(E)X + p$. Using the modular symbol $\lambda_{E, \gamma}$, we can construct the $p$-adic distribution $\mu_{E, \gamma}$ for any choice of the root $\gamma$ and we can define the $p$-adic $L$-function $L_{p, \gamma}(E, \chi, s) = \int_{\mathbb{Z}_p^\times} \chi(x) |x|^{s-1} d\mu_{E, \gamma}$ for $s \in \mathbb{Z}_p$.

In particular, we have the following interpolation property: For a primitive character $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathbb{Q}^\times$,

$$L_{p, \gamma}(E, \chi, 1) = \begin{cases} (1 - \gamma^{-1})^2 \frac{L(E, 1)}{\Omega_E} & \text{if } \chi = 1, \\ \left(\frac{p}{\gamma}\right)^\nu \left(\frac{1}{\tau(\chi)} \right) \frac{L(E, \chi, 1)}{\Omega_E^{\chi(-1)}} & \text{if } \chi \neq 1. \end{cases}$$

For elliptic curves over $\mathbb{Q}$, the statement of $p$-adic Beilinson conjecture can be slightly simplified as follows. Note that $\gamma$ is also an eigenvalue of the Frobenius $\Phi$ on $H^1_{\text{dR}}(E/\mathbb{Q}_p) \cong H^1_{\text{crys}}(E_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p$. Choose a non-zero eigenvector $v_{\gamma}$ in $H^1_{\text{dR}}(E/\mathbb{Q}_p) \otimes \mathbb{Q}_p$ with the eigenvalue $\gamma$. Let

$$\text{reg}_{\text{syn}} : H^2_{\text{dR}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}} \to H^2_{\text{syn}}(E, \mathbb{Q}_p(n+2)) \cong H^1_{\text{dR}}(E/\mathbb{Q}_p)$$

be the syntomic regulator map. Suppose $\Phi(\omega_E) \neq \gamma \omega_E$. Note that this is equivalent to the condition that $\omega_E$ and $v_{\gamma}$ are linearly independent. Then we set

$$R_{p, \gamma}(h^1(E)(-n), \xi) = \Gamma^*(-n)(1 - p^{n+1}\gamma^{-1}) \frac{\text{Tr}(\text{reg}_{\text{syn}}(\xi) \cup v_{\gamma})}{\text{Tr}(\omega_E \cup v_{\gamma})} \in \mathbb{Q}_p(\gamma).$$

Let

$$\text{reg}_{\partial} : H^2_{\text{dR}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}} \to H^2_{\partial}(E, \mathbb{R}(n+2))$$

$$\cong H^1_B(E(\mathbb{C}), \mathbb{R}(n+1))^+$$

$$\cong \text{Hom}(H_1(E(\mathbb{C}), \mathbb{Q})^{(-1)^{n+1}}, \mathbb{R}(n+1)).$$

be the Beilinson regulator map and we denote

$$R_{\infty}(h^1(E)(-n), \xi) = \frac{1}{(2\pi i)^{n+1} \text{reg}_{\partial}(\xi)(u^{-1})^{n+1}} \in \mathbb{R}.$$ 

By Beilinson conjecture, it is expected that the ratio $\frac{L'(E, -n)}{R_{\infty}(h^1(E)(-n), \xi)}$ is a rational number.

**Conjecture 3.3 (p-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$)** Assume that one of the following conditions holds:

1. $\text{ord}_p(\gamma) < 1$,
2. $\text{ord}_p(\gamma) = 1$ and $f_{\gamma}$ is $\theta$-critical.
Moreover we suppose that $\Phi(\omega_E) \neq \gamma \omega_E$. Then for $n \geq 0$, there exists a non-zero element $\xi \in H^2_{dR}(E, \mathbb{Q}(n+2))_\mathbb{Z}$ such that

$$\frac{L_{p, \gamma}(E, \omega^{-n-1}, -n)}{R_{p, \gamma}(h^1(E)(-n), \xi)} = \pm \frac{L'(E, -n)}{R_\infty(h^1(E)(-n), \xi)}.$$

**Remark 3.4**

1. The assumption $\Phi(\omega_E) \neq \gamma \omega_E$ excludes only the case that $E$ has good reduction at $p$, $E$ has complex multiplication and $\gamma$ has critical slope.

2. By the rank part of the Beilinson conjecture, it is expected that

$$\dim \mathbb{Q} H^2_{dR}(E, \mathbb{Q}(n+2))_\mathbb{Z} = 1.$$

3. Brunault [Br] gave an explicit formula for étale regulator of Beilinson-Kato elements.

4. Bertolini-Darmon [BD] gave an explicit formula for syntomic regulator and Beilinson regulator of Beilinson-Kato elements in the case of weight two cuspforms under some technical assumptions. However, their result does not cover Conjecture 3.3.

5. In [A2], we proposed conjectures on the relation between the special values of $p$-adic $L$-functions $L_{p, \alpha}(E, \omega^{-1}, 0)$ and the special values of $p$-adic hypergeometric functions ([A2 Conjecture 4.30 – 4.35]). Using the results in [A2], it is easy to see that these conjectures follow from Conjecture 3.3 and the rank part of the Beilinson conjecture.

For $n = 0$, we will denote $R_{p, \gamma}(E, \xi) = R_{p, \gamma}(h^1(E), \xi)$ and $R_\infty(E, \xi) = R_\infty(h^1(E), \xi)$ for simplicity.

### 4 Syntomic regulators and overconvergent functions

In this section we work over the Witt ring $W = W(\mathbb{F})$ of a perfect field $\mathbb{F}$ of characteristic $p > 0$ endowed with the Frobenius $F$. Put $K := \text{Frac}(W)$ the fractional field. Let $A$ be a faithfully flat $W$-algebra. We mean by a $p^n$-th Frobenius on $A$ an endomorphism $\sigma$ such that $\sigma(x) \equiv x^{p^n} \mod pA$ for all $x \in A$ and that $\sigma$ is compatible with the $p^n$-th Frobenius on $W$. We also write $x^\sigma$ instead of $\sigma(x)$. For a $W$-algebra $A$ of finite type, we denote by $A^\dagger$ the weak completion. Namely if $A = W[T_1, \ldots, T_n]$, then $A^\dagger = W[T_1, \ldots, T_n]^\dagger$ is the ring of power series $\sum a_\alpha T^\alpha$ such that for some $r > 1$, $|a_\alpha| r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$, and if $A = W[T_1, \ldots, T_n]/I$, then $A^\dagger = W[T_1, \ldots, T_n]^\dagger/IW[T_1, \ldots, T_n]^\dagger$.

#### 4.1 Setting and Notation

Let $C$ be a smooth projective scheme over $W$ of relative dimension 1. Let

$$f : Y \longrightarrow C$$

\[\text{Shinichi Kobayashi pointed out this fact.}\]
be an elliptic fibration over $W$, which means that $Y$ is a smooth $W$-scheme and $f$ is a projective flat morphism with a section $e : C \to Y$ whose general fiber is an elliptic curve.

Let $Z \subset C$ be a closed subscheme such that $f$ is smooth over $S := C \setminus Z$. Put $X := f^{-1}(S)$ and $A := \mathcal{O}(S)$. Write $A_K := A \otimes_W K$, $Y_K := Y \times_W K$, $A_F := A/pA$, $Y_F := Y \times_W W/pW$ etc.

We suppose that the following conditions $A_1$, \ldots, $A_3$ are satisfied.

$A_1$ $Z$ is a disjoint union of $W$-rational points $\{ P_i \in S(W) \}$, and each $D_i := f^{-1}(P_i)$ is a relative NCD (possibly a smooth divisor) over $W$.

$A_2$ $H^1_{\text{zar}}(X, \Omega^1_{X/A})$ is a free $A$-module of rank one.

The condition $A_2$ is equivalent to that $H^1_{\text{zar}}(O_X)$ is a free $A$-module of rank one by the Serre duality. Then $H^1_{\text{dR}}(X/A)$ is a free $A$-module of rank 2 by Lemma \ref{lem_dR_rank2}.

$A_3$ There is a point $P_0$ such that $f$ has a split multiplicative reduction at a point $P_0$. Let $\lambda$ be the uniformizer of the local ring at the point $P_0$ and let $E_W[[\lambda]] := Y \times_C \text{Spec} W[[\lambda]]$ be the Tate curve over the formal neighborhood of $P_0$. Let

$$q = q(\lambda) = \kappa \lambda^n(1 + a_1 \lambda + \cdots) \in \lambda W[[\lambda]] \quad (4.1)$$

be the multiplicative period. Then $\kappa \in W^\times$ and $n$ is prime to $p$.

The multiplicative period $q$ is characterized by the functional $j$-invariant $j(\lambda)$, namely it is the unique power series satisfying

$$j(\lambda) = \frac{1}{q} + 744 + 196884q + \cdots.$$

Let $\tau(\lambda) \in \lambda K[[\lambda]]$ be defined by $q = \kappa \lambda^n \exp(\tau(\lambda))$,

$$\tau(\lambda) = \log(1 + a_1 \lambda + \cdots) \in \lambda K[[\lambda]]. \quad (4.2)$$

See Proposition \ref{prop_jInvariant} \ref{prop_jInvariant} for differential equations of $q$ and $\tau(\lambda)$.

\section{F-isocrystal and category Fil-$F$-MIC($A_K$)}

Let $A^\dagger$ be the weak completion of $A$. We write $A^\dagger_K = A^\dagger \otimes_W K$. Fix a $p$-th Frobenius $\sigma$ on $A^\dagger$. In [AM, 2.1] we introduce a category Fil-$F$-MIC($S$) = Fil-$F$-MIC($S, \sigma$), which consists of collections of datum $(H_{dR}, H_{\text{rig}}, c, \Phi, \nabla, \text{Fil}^\bullet)$ such that

- $H_{dR}$ is a finitely generated $A_K$-module,
- $H_{\text{rig}}$ is a finitely generated $A^\dagger_K$-module,
- $c : H_{\text{rig}} \cong H_{dR} \otimes_{A_K} A^\dagger_K$, the comparison,
- $\Phi : \sigma^* H_{\text{rig}} \cong H_{\text{rig}}$ is an isomorphism of $A^\dagger_K$-module,
\[ \nabla : H_{dR} \to \Omega^1_{S/K} \otimes H_{dR} \text{ is an integrable connection that satisfies } \Phi \nabla = \nabla \Phi. \]

- \( \text{Fil}^* \) is a finite descending filtration on \( H_{dR} \) of locally free \( A_K \)-module (i.e. each graded piece is locally free), that satisfies \( \nabla (\text{Fil}^i) \subset \Omega^1 \otimes \text{Fil}^{i-1} \).

Let \( \text{Fil}^* \) denote the Hodge filtration on the de Rham cohomology, and \( \nabla \) the Gauss-Manin connection. Then one has an object

\[ H^i(X/S) := (H_{dR}^i(X_K/S_K), H_{\text{rig}}^i(X_{\mathbb{Z}/p}) \otimes c, \Phi, \nabla, \text{Fil}^*) \quad (4.3) \]

in \( \text{Fil}^* \)-\( F \)-\( \text{MIC}(S) \). For an integer \( r \), the Tate object \( A(r) \in \text{Fil}^* \)-\( F \)-\( \text{MIC}(S) \) is defined in a customary way (loc.cit.), and we define the Tate twist \( M(r) \) by

\[ M(r) = M \otimes A_S(r) = (M_{dR}, M_{\text{rig}}, \nabla, p^{-r} \Phi, \text{Fil}^{*+r}) \in \text{Fil}^* \)-\( F \)-\( \text{MIC}(S) \).

### 4.3 Tate curve

Let \( \mathcal{E} = \mathcal{E}_{W[[\lambda]]} \) be the Tate curve as in A3. As is well-known, there is a uniformization

\[ \rho : \hat{\mathbb{G}}_m := \text{Spec} \lim_n \left( W/p^n[u, u^{-1}] \right) \to \mathcal{E} \]

over \( W[[\lambda]] \). Let \( \hat{R} := W((\lambda))^\wedge \) be the \( p \)-adic completion of the ring of Laurent power series \( W((\lambda)) = W[[\lambda]][\lambda^{-1}] \). Write \( \hat{R}_K := \hat{R} \otimes_W K \). The de Rham cohomology \( H_{dR}^1(\mathcal{E}/\hat{R}) \) is a free \( \hat{R} \)-module of rank 2. Following [AM, §4.1], we define a basis (which we call the de Rham symplectic basis)

\[ \hat{\omega}, \hat{\eta} \in H^1_{dR}(\mathcal{E}/\hat{R}) = H^1_{dR}(\mathcal{E} \times W[[\lambda]] \hat{R}/\hat{R}) \]

in the following way. Firstly \( \hat{\omega} \in \Gamma(\mathcal{E}, \Omega^1_{\mathcal{E}/\hat{R}}) \) is a differential form of first kind such that \( \rho^* \hat{\omega} = du/u \). Note that this depends on the uniformization \( \rho \). Let \( \nabla : H_{dR}^1(\mathcal{E}/\hat{R}_K) \to \Omega^1_{\hat{R}_K/K} \otimes H_{dR}^1(\mathcal{E}/\hat{R}_K) \) be the Gauss-Manin connection, where \( \Omega^1_{\hat{R}_K/K} \) denotes the module of continuous 1-forms. Since \( \text{Ker}(\nabla) \cong K \), one can define the unique element \( \hat{\eta} \) such that \( \nabla(\hat{\eta}) = 0 \) and \( \text{Tr}(\hat{\omega} \cup \hat{\eta}) = 1 \), where \( \text{Tr} : H_{dR}^1(\mathcal{E}/\hat{R}_K) \to \hat{R}_K \) is the trace map.

We shall give an explicit description of \( \{ \hat{\omega}, \hat{\eta} \} \) in case that \( p \geq 5 \) and \( \mathcal{E} \) is defined by \( y^2 = 4x^3 - g_2 x - g_3 \) (Proposition 6.6, (6.15)).

**Proposition 4.1**

\[ (\nabla(\hat{\omega}), \nabla(\hat{\eta})) = (\hat{\omega}, \hat{\eta}) \begin{pmatrix} 0 & 0 \\ \frac{\partial \eta}{\partial x} & 0 \end{pmatrix} \]

**Proof.** [AM, Proposition 4.1]. \( \square \)

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Let $H^1(X/A) \in \text{Fil-F-MIC}(A_K)$ be the object in (4.3). Suppose that $\sigma$ extends to a $p$-th Frobenius on $\hat{R} = W((\lambda))^\wedge$ (e.g. $\sigma(\lambda) = c\lambda^p$ with $c \in 1 + pW$). Then, according to [AM, 5.1], it gives rise to an object

$$H^1(\mathcal{E}/\hat{R}) = (H^1_{\text{dR}}(\mathcal{E}/\hat{R}) \otimes_W K, \nabla, \Phi)$$

in the category $F\text{-MIC}(\hat{R}_K) = F\text{-MIC}(\hat{R}_K, \sigma)$.

**Proposition 4.2** Define

$$\log(\sigma)(x) := p^{-1} \log\left(\frac{x^p}{x^\sigma}\right) = -p^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \frac{x^p}{x^\sigma}\right)^n$$

for $x \in \hat{R}^\times$. If $x \in W^\times$, then we also write $\log(F)(x) = \log(\sigma)(x)$. Put

$$\tau(\sigma)(\lambda) := \log(\sigma)(q) = \log(F)(\kappa) + n \log(\sigma)(\lambda) + \tau(\lambda) - p^{-1} \tau(\lambda)^\sigma \in W[[\lambda]].$$

Then we have

$$\left(\Phi(\omega) \quad \Phi(\eta)\right) = \left(\omega \quad \eta\right) \left(\begin{array}{cc} p & 0 \\ -p\tau(\sigma)(\lambda) & 1 \end{array}\right).$$

**Proof.** [AM, Theorem 5.4].

Let $\omega, \eta \in H^1_{\text{dR}}(X/A)$ be a free $A$-basis such that $\omega \in \Gamma(X, \Omega^1_{X/A}) \cong A$. Let $\hat{\omega}, \hat{\eta} \in H^1_{\text{dR}}(\mathcal{E}/\hat{R})$ be the de Rham symplectic basis as before. Define $F(\lambda) \in \hat{R}$ and $f_i(\lambda) \in \hat{R}$ by

$$\omega = F(\lambda)\hat{\omega}, \quad \eta = f_1(\lambda)\hat{\omega} + f_2(\lambda)\hat{\eta}. \quad (4.5)$$

In case that $p \geq 5$ and $\mathcal{E}$ is defined by $y^2 = 4x^3 - g_2x - g_3$, we shall give an explicit description of $F(\lambda)$ and $f_i(\lambda)$ (Proposition 6.6 (6.15) together with (6.14)).

**Proposition 4.3** Let $\sigma$ be a $p$-th Frobenius on $A^\dagger$ such that it extends on $\hat{R}$. Let

$$F_{11} = \frac{F(\lambda)^\sigma}{F(\lambda)} \left(1 + \tau(\sigma)(\lambda)\frac{f_1(\lambda)}{f_2(\lambda)}\right)$$

$$F_{12} = \frac{1}{F(\lambda)} \left(pf_1(\lambda)^\sigma - f_1(\lambda)f_2(\lambda)^\sigma - p\tau(\sigma)(\lambda)f_1(\lambda)^\sigma\right)$$

$$F_{21} = -\tau(\sigma)(\lambda)\frac{F(\lambda)^\sigma}{f_2(\lambda)}$$

$$F_{22} = \frac{f_2(\lambda)^\sigma - p\tau(\sigma)(\lambda)f_1(\lambda)^\sigma}{f_2(\lambda)}.$$

Then we have

$$\left(\Phi(\omega) \quad \Phi(\eta)\right) = \left(\omega \quad \eta\right) \left(\begin{array}{cc} p F_{11} & F_{12} \\ p F_{21} & F_{22} \end{array}\right).$$

Moreover $F_{ij} \in A^\dagger$.  

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Proof. The representation matrix of $\Phi$ is straightforward from Proposition 4.2 together with (4.5). Then this implies that $F_{ij} \in A_K^\dag = A^\dag \otimes W K$. We further show $F_{ij} \in A^\dag$. By a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A^\dag & \overset{p}{\longrightarrow} & A^\dag & \longrightarrow & A/pA & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \cap & & \downarrow & & \\
0 & \longrightarrow & \hat{R} & \overset{p}{\longrightarrow} & \hat{R} & \longrightarrow & \mathbb{F}((\lambda)) & \longrightarrow & 0
\end{array}
\]

the $W$-module $\hat{R}/A^\dag$ is $p$-torsion free, and this implies that $A^\dag = A_K^\dag \cap \hat{R}$. Therefore it is enough to show $F_{ij} \in \hat{R}$. To do this, it is enough to show that $F(\lambda), f_2(\lambda) \in \hat{R}$. However, since $\{\omega, \eta\}$ is a free $A$-basis of $H^1_{\mathrm{dR}}(X/A)$, one has that $\text{Tr}(\omega \cup \eta) = F(\lambda)f_2(\lambda) \in A \times A$ as required. □

Corollary 4.4 Let $a = \text{Spec} A/m$ be a $W$-rational point of $S$, and $X_a$ the fiber at $a$, which is an elliptic curve over $W$. Put $X_{a,p} := X_a \times_W \mathbb{F}_p$. Suppose $\sigma^{-1}(mA^\dag) \supset mA^\dag$. Then the representation matrix of the $p$-th Frobenius $\Phi$ on $H^1_{\mathrm{rig}}(X_a, F_p/F_p)$ is

\[
\begin{pmatrix}
pF_{11}(a) & F_{12}(a) \\
pF_{21}(a) & F_{22}(a)
\end{pmatrix},
\]

where for $f \in A_K^\dag$ we write by $f(a) \in A_K^\dag/mA_K^\dag \cong K$ the reduction modulo $mA_K^\dag$.

4.4 Regulator formula for $K_2$ of elliptic curves

Let

$$\xi \in K_2(X)$$

be an element of Quillen’s $K_2$. Let $G_\xi(\lambda) \in \hat{R}$ be defined by

$$d\log(\xi) = G_\xi(\lambda)\frac{d\lambda}{\lambda} \wedge \hat{\omega} \in \Gamma(\mathcal{E}, \Omega^2_{\mathcal{E}/W}).$$

Since this has at worst log pole and $\rho^*\hat{\omega} = du/u$, we have

$$G_\xi(\lambda) = l + a_1\lambda + a_2\lambda^2 + \cdots \in W[[\lambda]]. \quad (4.6)$$

The constant term $l$ is an integer (possibly zero). Indeed, let $\delta$ be the composition of the maps

$$K_2(\mathcal{E}) \overset{\text{tame}}{\longrightarrow} K'_1(D) \overset{j^*}{\longrightarrow} K_1(\mathbb{G}_{m,W}) \overset{K_1(\mathbb{G}_{m,K}) \cong K[u, u^{-1}] \times \text{ord}_u \mathbb{Z}}{\longrightarrow} \mathbb{Z},$$

where $D \subset \mathcal{E}$ is the central fiber at $\lambda = 0$ which is the Néron $n$-gon over $W$, and $j : \mathbb{G}_{m,W} \hookrightarrow D$ is the open immersion. Then $l = \pm \delta(\xi)$. According to [AM, §2.5], one can associate to $\xi$ the 1-extension

$$0 \longrightarrow H^1(X/A)(2) \longrightarrow M_\xi(X/A) \longrightarrow A \longrightarrow 0$$

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in Fil-$F$-MIC($A_K, \sigma$), where $A = A(0)$ is the trivial object of rank 1. Let $e_\xi \in Fil^0 M_\xi(X_K/A_K)_{\text{dR}}$ be the unique lifting of $1 \in A_K$. Define $\varepsilon_i^{(\sigma)}(\lambda) \in A^1_K$ by

$$e_\xi - \Phi(e_\xi) = \varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta. \quad (4.7)$$

**Theorem 4.5** Let $a = \text{Spec}A/m$ be a $W$-rational point of $S$, and $X_a$ the fiber at $a$, which is an elliptic curve over $W$. Suppose $\sigma^{-1}(mA^1) \supset mA^1$. Let

$$\text{reg}_{\text{syn}} : K_2(X_a) \longrightarrow H^2_{\text{syn}}(X_a, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(X_a/K)$$

be the syntomic regulator map. Then

$$\text{reg}_{\text{syn}}(\xi|_{X_a}) = \varepsilon_1^{(\sigma)}(a)\omega + \varepsilon_2^{(\sigma)}(a)\eta,$$

where for $f \in A^1_K$ we write by $f(a) \in A^1_K/mA^1_K \cong K$ the reduction modulo $mA^1_K$.

**Proof.** This is immediate from [AM, Proposition 6.4].

Define $E_i^{(\sigma)}(\lambda) \in \hat{R}_K$ by

$$\varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta = E_1^{(\sigma)}(\lambda)\hat{\omega} + E_2^{(\sigma)}(\lambda)\hat{\eta} \quad (4.8)$$

or explicitly

$$E_1^{(\sigma)}(\lambda) = F(\lambda)\varepsilon_1^{(\sigma)}(\lambda) + f_1(\lambda)\varepsilon_2^{(\sigma)}(\lambda) \quad (4.9)$$

$$E_2^{(\sigma)}(\lambda) = f_2(\lambda)\varepsilon_2^{(\sigma)}(\lambda), \quad (4.10)$$

where $F(\lambda)$ and $f_1(\lambda)$ are as in (4.5).

**Theorem 4.6** Let $\sigma$ be a $p$-th Frobenius on $A^1$ such that it extends on $\hat{R}$. Define $f'(\lambda)$ by $f'(\lambda)d\lambda = df(\lambda)$. Then we have

$$\left( E_1^{(\sigma)}(\lambda) \right)' = \frac{G_\xi(\lambda)}{\lambda} - p^{-1}G_\xi(\lambda)^{\sigma}(\lambda^{\sigma})' \quad (4.11)$$

$$\left( E_2^{(\sigma)}(\lambda) \right)' = -E_1^{(\sigma)}(\lambda)\frac{q'q}{q} + p^{-1}G_\xi(\lambda)^{\sigma}(\lambda^{\sigma})' \log^{(\sigma)}(q). \quad (4.12)$$

If $\lambda^\sigma = c\lambda^p$ for some $c \in 1 + pW$, then $E_1^{(\sigma)}(\lambda), E_2^{(\sigma)}(\lambda) \in W[[\lambda]] \otimes W K^2$ and

$$E_1^{(\sigma)}(0) = \frac{l}{n} \log^{(p)}(\kappa) - lp^{-1} \log(c). \quad (4.13)$$

²More precisely, we shall see that $E_1^{(\sigma)}(\lambda) \in W[[\lambda]]$ by Lemma 4.12 below.
Conjecture 4.7 Suppose \( \lambda^\sigma = c \lambda^p \) for some \( c \in 1 + pW \). Let \( D \subset \mathcal{E} \) be the central fiber at \( \lambda = 0 \) which is the \( \text{Néron} \) \( n \)-gon over \( W \), and \( D_\bullet \rightarrow D \) the simplicial scheme. Then there exists a map (specialization map)

\[
\text{Sp} : H^2_{\mathcal{E}}(\mathcal{E} \times_{W[[\lambda]]} W((\lambda)), \mathbb{Q}(2)) \rightarrow H^2_{\mathcal{E}}(D_\bullet, \mathbb{Q}(2))
\]

which is induced by the “nearby cycle functor with respect to a parameter \( \kappa^\frac{1}{n} \lambda \)”, and we have

\[
E_2^{(\sigma)}(0) = -\frac{1}{2n} (\log^F(\kappa) - np^{-1} \log(c))^2 + R_{\text{syn}}(\text{Sp}(\xi)),
\]

where \( R_{\text{syn}} \) is the composition of the following arrows

\[
H^2_{\mathcal{E}}(D_\bullet, \mathbb{Q}(2)) \xrightarrow{\text{Rep}_{\text{syn}}} H^2_{\text{syn}}(D_\bullet, \mathbb{Q}_p(2)) \xrightarrow{\cong} H^1_{\text{dR}}(D_\bullet/W) \xrightarrow{\delta_{n+1}} W.
\]

In particular, \( R_{\text{syn}}(\text{Sp}(\xi)) \) is a \( \mathbb{Q} \)-linear combination of \( \ln^{(p)}(\epsilon) \)'s with \( \epsilon \in W \cap \overline{\mathbb{Q}} \), where \( \ln^{(p)}(z) = \sum_{p \mid n} z^n/n^r \).

### 4.5 Generalization of \( p \)-adic Hypergeometric functions of logarithmic type

**Lemma 4.8** ([VdP] Proposition 7.12) Let \( \sigma, \Phi \) and \( F_{ij} \) be as in Proposition 4.3. Put

\[
\widehat{B} := \lim_n (A/p^n A_1[F_{22}]) = \lim_n (A/p^n A[F_{22}])
\]
and \( \hat{B}_K := \hat{B} \otimes_W K \). Note that \( \hat{B} \neq \{0\} \) as \( F_{22} \equiv f_2(\lambda)^{p-1} \neq 0 \mod p\hat{R} \).

Then
\[
\frac{F(\lambda)}{F(\lambda)^\sigma}, \quad \frac{E_1(\sigma)(\lambda)}{F(\lambda)}, \quad \frac{f_1(\lambda)}{F(\lambda)} \in \hat{B}_K.
\]

**Proof.** For \( f(\lambda) \in \hat{B} \), one has
\[
\Phi(f(\lambda)\omega + \eta) = (F_{11} + pF(\lambda)^\sigma F_{11})\omega + (F_{22} + pF(\lambda)^\sigma F_{21})\eta
\]
by Proposition \[4.3\]. Let \( V := \{ f(\lambda)\omega + \eta \mid f(\lambda) \in \hat{B} \} \). Consider a map
\[
S : V \to V, \quad S(f(\lambda)\omega + \eta) = \frac{F_{11}}{F_{22} + pF(\lambda)^\sigma F_{21}}\omega + \eta.
\]
This is well-defined as \( F_{22} + pF(\lambda)^\sigma F_{21} \in \hat{B}^\times \) and we have
\[
S(f_1(\lambda)\omega + \eta) - S(f_2(\lambda)\omega + \eta) = \frac{p(f_1(\lambda) - f_2(\lambda))D}{(F_{22} + pF_1(\lambda)^\sigma F_{21})(F_{22} + pF_2(\lambda)^\sigma F_{21})},
\]
where \( D := F_{11}F_{22} - F_{12}F_{12} \in (A^\dagger)^\times \). This shows that \( S \) is a contraction map with respect to the norm \( \| \cdot \| \) on \( \hat{R} = W((\lambda))^\wedge \) which is defined by \( \| \sum a_i \lambda^i \| := \sup \{|a_i| \mid i \in \mathbb{Z} \} \). Hence there is a unique fixed point \( u_0 = f_0(\lambda)\omega + \eta \in \hat{V} \), namely
\[
S(u_0) = u_0 \iff \Phi(u_0) = (F_{22} + pF_0(\lambda)^\sigma F_{21})u_0.
\]
Replace \( \hat{B} \) with \( \hat{R} \) and consider the similar map \( \hat{S} \) on \( \hat{V} := \{ f(\lambda)\omega + \eta \mid f(\lambda) \in \hat{R} \} \). Then one also has a unique fixed point \( \hat{u}_0 = f_0(\lambda)\omega + \eta \in \hat{V} \) of \( \hat{S} \), and then it turns out that \( u_0 = \hat{u}_0 \) by the uniqueness. Recall from \( (4.5) \)
\[
f_2(\lambda)\hat{\eta} = -\frac{f_1(\lambda)}{F(\lambda)}\omega + \eta.
\]
By Proposition \( 4.2 \), one has that \( \Phi(f_2(\lambda)\hat{\eta}) = f_2(\lambda)^\sigma\hat{\eta} \). This means that \( -f_1(\lambda)/F(\lambda)\omega + \eta \in \hat{V} \) is the unique fixed point of \( \hat{S} \),
\[
\hat{u}_0 = -\frac{f_1(\lambda)}{F(\lambda)}\omega + \eta.
\]
Hence
\[
-\frac{f_1(\lambda)}{F(\lambda)}\omega + \eta = f_0(\lambda)\omega + \eta, \quad \frac{f_2(\lambda)}{f_2(\lambda)} = F_{22} + pF_0(\lambda)^\sigma F_{21},
\]
and this immediately implies \( f_1(\lambda)/F(\lambda), f_2(\lambda)^\sigma/f_2(\lambda) \in \hat{B} \). Moreover since \( \text{Tr}(\omega \cup \eta) = F(\lambda)f_2(\lambda) \in A^\times \) (see the proof of Proposition \( 4.3 \)), one also has \( F(\lambda)/F(\lambda)^\sigma \in \hat{B} \). Finally, we have
\[
\text{Tr}(E_1(\sigma)(\lambda)\hat{\omega} + E_2(\sigma)(\lambda)\hat{\eta} \cup u_0) = \text{Tr}(E_1(\sigma)(\lambda)\hat{\omega} \cup u_0) = E_1(\sigma)(\lambda)/F(\lambda) \cdot \text{Tr}(\omega \cup \eta) \in \hat{B}_K
\]
from \( (4.8) \). This implies \( E_1(\sigma)(\lambda)/F(\lambda) \in \hat{B}_K \).

\[3\] Indeed \( F(\lambda)f_2(\lambda) \in A^\times \) as is in the proof of Proposition \( 4.3 \). Note that \( f_2(\lambda)^{p-1} \in \hat{R}/p\hat{R} = k((\lambda)) \) belongs to \( A/pA \) as so does \( (F_{22} \mod pA) \). However this does not imply \( f_2(\lambda)^{p-1} \in (A/pA)^\times \) because \( F(\lambda)^{p-1} \notin A/pA \).
**Corollary 4.9 (Unit root formula)** Suppose $W = \mathbb{Z}_p$ for simplicity. Let $a = \text{Spec} A/m$ be a $\mathbb{Z}_p$-rational point of $S$, and $X_a$ the fiber at $a$, which is an elliptic curve over $\mathbb{Z}_p$. Suppose $\sigma^{-1}(mA^1) \supset mA^1$. Suppose further that $X_{a,F_p} := X_a \times_{\mathbb{Z}_p} \mathbb{F}_p$ is an ordinary elliptic curve. Then the unit root of $X_{a,F_p}$ is

$$
\frac{F(\lambda)}{F(\lambda)^\sigma} \bigg|_{\lambda=a},
$$

where $f(\lambda)|_{\lambda=a} := (f(\lambda) \mod m\hat{B}_K) \in \hat{B}_K/m\hat{B}_K = K$.

**Proof.** We first note that $X_{a,F_p}$ is ordinary if and only if the absolute Frobenius on $H^1(\mathcal{O}_{X_{a,F_p}})$ does not vanish, and this is equivalent to that $F_{\lambda}^{-1}(\lambda)|_{\lambda=a} \neq 0 \mod p$ in Proposition 4.3. Hence $a$ is a point of $\text{Spec} \hat{B}_K$ and the special value $F(\lambda)/F(\lambda)^\sigma|_{\lambda=a}$ makes sense. As is shown in the proof of Lemma 4.8, the unit root vector is $u_0|_{\lambda=a}$ and the unit root is

$$
\frac{f_2(\lambda)^\sigma}{f_2(\lambda)} \bigg|_{\lambda=a} = \frac{F(\lambda)}{F(\lambda)^\sigma} \bigg|_{\lambda=a} \times \frac{d(\lambda)^\sigma}{d(\lambda)} \bigg|_{\lambda=a} = \frac{F(\lambda)}{F(\lambda)^\sigma} \bigg|_{\lambda=a},
$$

where $d(\lambda) := F(\lambda)f_2(\lambda) \in A^\times$. \hfill \Box

**Definition 4.10**

$$
\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda) := E^{(\sigma)}_{Y}(\lambda)/F(\lambda) = \frac{\text{Tr}((e_{\xi} - \Phi(e_{\xi})) \cup \eta)}{\text{Tr}(\omega \cup \eta)} \in \hat{B}_K,
$$

where $\text{Tr} : H^2_{\text{dR}}(\mathcal{O}/\hat{R}_K) \to \hat{R}_K$ is the trace map.

By definition, $\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda) = a^{-1}\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda)$ for $a \in A^\times$ and

$$
\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda) = r_1\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda) + r_2\mathcal{F}^{(\sigma)}_{Y/C,\xi,\omega}(\lambda).
$$

**Example 4.11** Let $p \geq 3$ and $W = \mathbb{Z}_p$. Let $f : Y \to \mathbb{P}^1$ be the Legendre family

$$
f^{-1}(\lambda) : y^2 = x(1-x)(1-(1-\lambda)x)
$$

of elliptic curves. This is smooth over $S = \text{Spec} A$, where $A := \mathbb{Z}_p[\lambda, (\lambda-\lambda^2)^{-1}]$, and this has a split multiplicative reduction at $\lambda = 0$. Let $\omega = dx/y, \eta = xdx/y$ be a free $A$-basis of $H^1_{\text{dR}}(X/A)$. Then

$$
F(\lambda) = 2F_1\left(\frac{1}{2}, \frac{1}{2} ; \lambda\right)
$$

is the hypergeometric power series. Let

$$
\xi := \left\{ \frac{y-x+1}{y+x-1}, -\frac{(x-1)^2}{\lambda x^2} \right\} \in K_2(X).
$$
Let $\sigma$ be the $p$-th Frobenius on $\mathbb{Z}_p[\lambda, (\lambda - \lambda^2)^{-1}]$ defined by $\sigma(\lambda) = c\lambda^p$ with $c \in 1 + p\mathbb{Z}_p$. Then
\[
\mathcal{F}_{Y/\mathbb{P}^1, \xi, \omega}(\sigma) = \frac{1}{F(\lambda)} \left[ -\log(F)(16) - p^{-1}\log(c) + \int_0^\lambda (F(\lambda) - F(\lambda^p)) \frac{d\lambda}{\lambda} \right]
\]
is the hypergeometric functions $\mathcal{F}_2^{(\sigma)}(\lambda)$ of logarithmic type introduced in [A2].

### 4.6 Bounds on $p$-adic expansions of $\varepsilon_k^{(\sigma)}(\lambda)$

In this section we give effective bounds of rational functions $\varepsilon_k^{(\sigma)}(\lambda)$ modulo $p^n$ according to [KT]. The main results are Theorems 4.15 and 4.16, which play key roles in later sections. We note that Kedlaya and Tuitman give a theorem in more general situation ([KT, Theorem 2.1]). However, as we need a little more precise statement, we give a self-contained proof for the sake of completeness.

We assume $p \geq 3$. Let $f : Y \to C$ be as in §4.1 which satisfies A1, A2 and A3. For a $W$-rational point $P_i \in Z$, let $\lambda_i$ be a uniformizer of the local ring at $P_i$. Put $Y_i := Y \times_C \text{Spec} W[[\lambda_i]]$ and $D_i := f^{-1}(P_i)$ the central fiber which is a relative NCD over $W$ by A1. Put $X_i = Y_i \setminus D_i$. We consider the following condition.

**A4** The multiplicity of each component of $D_i$ is prime to $p$. Moreover put
\[
H_i := \text{Im}[H^1_{\text{zar}}(Y_i, \Omega^*_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{\text{dR}}(X_i/W((\lambda_i)))]
\]
and
\[
\text{Fil}^1 H_i := \text{Im}[H^0_{\text{zar}}(Y_i, \Omega^1_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{\text{dR}}(X_i/W((\lambda_i)))].
\]

Then, both of $\text{Fil}^1 H_i$ and $H_i/\text{Fil}^1 H_i$ are free $W[[\lambda_i]]$-modules of rank 1.

We endow the log structures on $Y_i$ and $\text{Spec} W[[\lambda_i]]$ arising from the divisors $D_i$ and $\lambda_i = 0$ respectively. Since the multiplicity of each component of $D_i$ is prime to $p$, the morphism $(Y_i, D_i) \to (\text{Spec} W[[\lambda_i]], (\lambda_i))$ of log schemes is log-smooth in the sense of [Ka2] §6. One can define the crystalline cohomology $H^*_{\text{crys}}(Y_i(D_i)) := H^*_{\text{crys}}((Y, D_i)/(W[[\lambda_i]], (\lambda_i)))$ (loc.cit.) and also the syntomic cohomology $H^*_{\text{syn}}(Y_i(D_i), \mathbb{Z}_p(r))$ with log pole along $D_i$ (cf. [AM] §8.1). By [Ka2] Theorem (6.4)), there is the natural isomorphism
\[
H^k_{\text{crys}}(Y_i(D_i)) \cong H^k_{\text{zar}}(Y_i, \Omega^*_{Y_i/W[[\lambda_i]]}(\log D_i))
\]
so that the Frobenius $\Phi$ acts on the right hand side, if $\sigma(\lambda_i) = c\lambda_i^p$ with $c \in 1 + pW$.

**Lemma 4.12** Assume A4 is satisfied. Let $\{\omega_i, \eta_i\}$ be a $W[[\lambda_i]]$-basis of $H_i$. Suppose that $\sigma$ is defined by $\lambda_i^c = c\lambda_i^p$ for some $c \in 1 + pW$. Define $\varepsilon^{(\sigma)}(\lambda_i) \in A_K^1$ by
\[
\varepsilon(\Phi(\varepsilon)) = \varepsilon^{(\sigma)}(\lambda_i) \omega_i + \varepsilon^{(\sigma)}(\lambda_i) \eta_i \in H^1_{\text{dR}}(X_i/W((\lambda_i))) \otimes W((\lambda_i)) W((\lambda_i))^\wedge, \quad (4.14)
\]
where $W((\lambda_i))^\wedge$ denotes the $p$-adic completion. Then $\varepsilon^{(\sigma)}_1(\lambda_i)$ and $\varepsilon^{(\sigma)}_2(\lambda_i)$ belong to the subspace $W[[\lambda_i]] \cap A_K^1 = W[[\lambda_i]] \cap A^1$. 

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Proof. Let $H^\ast_{\text{syn}}(Y_i(D_i), \mathbb{Z}_p(r))$ be the syntomic cohomology group with log pole along the divisor $D_i$ ([AM, §8.1]). Let $\text{reg}_\ast : K_2(X) \to H^2_{\text{syn}}(Y_i(D_i), \mathbb{Z}_p(2))$ be the log syntomic regulator map, and $\text{reg}_{\text{rig-syn}} : K_2(X) \to H^2_{\text{rig-syn}}(X, \mathbb{Q}_p(2))$ the rigid-syntomic regulator map. Let $X_i := Y_i \setminus D_i$. Thanks to the compatibility of the two regulator maps, there is a commutative diagram

$$
\begin{array}{ccc}
K_2(X) & \xrightarrow{\text{reg}_{\text{rig-syn}}} & H^2_{\text{rig-syn}}(X, \mathbb{Q}_p(2)) \\
\downarrow{\text{reg}_\ast} & & \downarrow{\phi_{X_i}} \\
H^2_{\text{syn}}(Y_i(D_i), \mathbb{Z}_p(2)) & \xrightarrow{\phi_{Y_i}} & H^1_{\text{zar}}(\Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)) \\
& & \downarrow{\text{H}_1^\text{dR}(X_i/W((\lambda_i)))}
\end{array}
$$

where $\phi_{Y_i} = \phi_{Y_i/W[[\lambda_i]]}$ is the map constructed in [AM (8.16)]. Then one can show in the same way as the proof of [AM, Proposition 6.4] that

$$
\phi_{X_i} \circ \text{reg}_{\text{rig-syn}}(\xi) = \pm(e_\xi - \Phi(e_\xi)).
$$

Now Lemma 4.12 is immediate because

$$
\text{Im}[H^1_{\text{zar}}(\Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)) \to \text{H}_1^\text{dR}(X_i/W((\lambda_i)))] = W[[\lambda_i]]\omega_i + W[[\lambda_i]]\eta_i.
$$

\[\Box\]

Lemma 4.13 (Transformation of Frobenius) Let $\sigma$ be a $p$-th Frobenius on $A^1$. Let $F\text{-MIC}(A^\dagger_K, \sigma)$ be the category of objects $(H, \nabla, \Phi)$, where $(H, \nabla)$ is an integrable connection of free $A^\dagger_K$-module, and $\Phi$ is a $p$-th Frobenius on $H$ compatible with $\sigma$. Let $\partial = \nabla_{d/d\lambda}$. Let $\sigma'$ be another Frobenius on $A^1$. Thanks to the natural equivalence $F\text{-MIC}(A^\dagger_K, \sigma) \cong F\text{-MIC}(A^\dagger_K, \sigma')$ one can associate $(H, \nabla, \Phi') \in F\text{-MIC}(A^\dagger_K, \sigma)$ to an object $(H, \nabla, \Phi) \in F\text{-MIC}(A^\dagger_K, \sigma)$. Then, for $x \in H$ we have

$$
\Phi'(x) - \Phi(x) = \sum_{k=1}^{\infty} \frac{(\lambda_i^{\sigma'} - \lambda_i^\sigma)^k}{k!} \Phi \partial^k x,
$$

where $\partial := \nabla_{d/d\lambda_i}$.

\[\Box\]

Lemma 4.14 Assume that A4 is satisfied. Let $\{\omega_i, \eta_i\}$ be a $W[[\lambda_i]]$-basis of $H_i$ such that $\omega_i \in \text{Fil}^1 H_i$. Let $\sigma$ be a $p$-th Frobenius on $A^1$. Assume that $\lambda_i^\sigma \in W[[\lambda_i]]$ so that it naturally induces a $p$-th Frobenius on $W[[\lambda_i]]$. Let $\varepsilon_1^{(\sigma)}(\lambda_i)$ and $\varepsilon_2^{(\sigma)}(\lambda_i)$ be defined by (4.14). Put

$$
\varepsilon_n := \max\{i \in \mathbb{Z}_{\geq 0} \mid \text{ord}_p(p^i/i!) < n\}.
$$

Put $l_i := \min\{\text{ord}_{\lambda_i}(\lambda_i^\sigma - \lambda_i^\sigma)\}, p \in \mathbb{Z}_{\geq 0}$, where $\text{ord}_{\lambda_i} : K((\lambda_i)) \to \mathbb{Z}$ is the order function with respect to $\lambda_i$. Then

$$
\varepsilon_1^{(\sigma)}(\lambda_i), \varepsilon_2^{(\sigma)}(\lambda_i) \in \lambda_i^{l_i-p}\varepsilon_n^2W[[\lambda_i]] + p^n W[[\lambda_i, \lambda_i^{-1}]]
$$

for any $n \geq 1$.

Notice that $\varepsilon_n < \infty$ as we assumed $p > 2$.  

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Proof. Let $\sigma_0$ be the $p$-th Frobenius on $W[[\lambda_i]]$ such that $\sigma_0(\lambda_i) = \lambda_i^p$, and $\Phi_0$ the Frobenius with respect to $\sigma_0$. By Lemma 4.13 we have

$$\Phi(e_\xi) - \Phi_0(e_\xi) = \sum_{k=1}^{\infty} \frac{(\lambda_i^\sigma - \lambda_i^p)^k}{k!} \Phi_0 \partial^k e_\xi.$$  \hspace{1cm} (4.16)

Write the right hand side by $G_1(\lambda_i) \omega_i + G_2(\lambda_i) \eta_i$, so that we have

$$\varepsilon_1^{(\sigma)}(\lambda) - \varepsilon_1^{(\sigma)}(\lambda) = G_1(\lambda_i), \quad \varepsilon_2^{(\sigma)}(\lambda) - \varepsilon_2^{(\sigma)}(\lambda) = G_2(\lambda_i).$$

It follows from Lemma 4.12 that $\varepsilon_1^{(\sigma)}(\lambda_i)$ and $\varepsilon_2^{(\sigma)}(\lambda_i)$ belong to $W[[\lambda_i]]$. Hence it is enough to show

$$G_1(\lambda_i), G_2(\lambda_i) \in \lambda_i^{[(1-p)\epsilon_n]} W[[\lambda_i]] + p^n W[[\lambda_i, \lambda_i^{-1}]].$$  \hspace{1cm} (4.17)

The differential operator $\partial$ is induced from the connection

$$\nabla : H^1(Y_i, \Omega^*_i/W[[\lambda_i]](\log D_i)) \rightarrow \frac{d\lambda_i}{\lambda_i} \otimes H^1(Y_i, \Omega^*_i/W[[\lambda_i]](\log D_i)).$$

Moreover since $\nabla e_\xi = d\log(\xi) \in \frac{d\lambda_i}{\lambda_i} \otimes \Gamma(Y_i, \Omega^1_i/W[[\lambda_i]](\log D_i))$, we have

$$\partial e_\xi \in \lambda_i^{-1} W[[\lambda_i]] \omega_i, \quad \partial^k e_\xi \in \lambda_i^{-k} W[[\lambda_i]] \omega_i + W[[\lambda_i]] \eta_i$$

for all $k \geq 2$. Since $\Phi_0$ acts on the cohomology group $H^1(Y_i, \Omega^*_i/W[[\lambda_i]](\log D_i))$, there are $u_k(\lambda_i), v_k(\lambda_i) \in W[[\lambda_i]]$ such that

$$\Phi_0 \partial^k e_\xi = \lambda_i^{-kp} p^{-2} (u_k(\lambda_i) \omega_i + v_k(\lambda_i) \eta_i)$$  \hspace{1cm} (4.18)

for each $k \in \mathbb{Z}_{\geq 1}$, where “$p^{-2}$” comes from the Tate twist $H^1(X/A) \otimes A(2)$. Notice that since $\Phi_0 \omega_i \equiv 0$ modulo $p$, one has

$$\Phi_0 \partial e_\xi \in \lambda_i^{-p} p^{-1} W[[\lambda_i]] \Phi_0 \omega_i \subset \lambda_i^{-p} p^{-1} (W[[\lambda_i]] \omega_i + W[[\lambda_i]] \eta_i)$$

and hence

$$u_1(\lambda_i), v_1(\lambda_i) \in p W[[\lambda_i]].$$

Applying (4.18) to (4.16), we have

$$G_1(\lambda_i) \omega_i + G_2(\lambda_i) \eta_i = \sum_{k=1}^{\infty} \frac{(\lambda_i^\sigma - \lambda_i^p)^k}{k!} \lambda_i^{-kp} p^{-2} (u_k(\lambda_i) \omega_i + v_k(\lambda_i) \eta_i)$$

$$= \sum_{k=1}^{\infty} \frac{p^{k-2}}{k!} \left( \frac{p^{-1}(\lambda_i^\sigma - \lambda_i^p)}{\lambda_i^p} \right)^k (u_k(\lambda_i) \omega_i + v_k(\lambda_i) \eta_i).$$

Now (4.17) is immediate from this on noticing $p^{-1}(\lambda_i^\sigma - \lambda_i^p)/\lambda_i^p \in \lambda_i^{1-p} W[[\lambda_i]].$
Theorem 4.15 Suppose that \( C = \mathbb{P}^1_W \) and \( f : Y \to \mathbb{P}^1 \) satisfies A1, . . . , A3 and A4 for all \( i \). Let \( \lambda \) be an inhomogeneous coordinates of \( \mathbb{P}^1 \). We assume that \( f \) has a split multiplicative reduction at \( \lambda = 0 \) as in A3. Let \( \{a_0, a_1, \ldots, a_m\} \subset W \) be the subset such that \( D_i := f^{-1}(a_i) \) is a singular fiber, and set \( a_0 = 0 \). Put \( D_\infty := f^{-1}(\infty) \) (possibly a smooth fiber). Put \( \lambda_i := \lambda - a_i \) for \( i \in \{0, 1, \ldots, m\} \) and \( \lambda_\infty := \lambda^{-1} \). Suppose further that the following condition is satisfied.

A5 There are \( \omega \in \Gamma(\Omega^1_{X/A}) \) and \( \eta \in H^1_{dR}(X/A) \) such that for each \( i \in \{0, 1, \ldots, m, \infty\} \), \( \{\lambda^a_i \omega, \lambda^{b_i} \eta\} \) is a \( W[[\lambda_i]] \)-basis of

\[
H_i = \text{Im} \left[ H^1(Y_i, \Omega^1_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{dR}(X_i/W((\lambda_i))) \right]
\]

for some \( a_i, b_i \in \mathbb{Z} \).

Let \( \sigma \) be a \( p \)-th Frobenius on \( A^\dagger \) such that \( \lambda^\sigma = c \lambda^p \) with \( c \in 1 + pW \). Define \( \varepsilon_k^{(\sigma)}(\lambda) \in A^\dagger_K \) by

\[
e_k - \Phi(e_k) = \varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta.
\]

Put \( l_i := \min\{\text{ord}_{\lambda_i}(\lambda_i^\sigma - \lambda_i^\xi), p\} \). Let \( e_n \) be as in (4.15), and

\[
d_{1,n} := \sum_{i=0}^m (-a_i + (p - l_i)e_n + 2) - a_\infty, \quad d_{2,n} := \sum_{i=0}^m (-b_i + (p - l_i)e_n + 2) - b_\infty.
\]

Then \( \varepsilon_k^{(\sigma)}(\lambda) \in A^\dagger \) and there are polynomials \( f_{k,n}(\lambda) \) of degree \( \leq d_{k,n} \) such that

\[
\varepsilon_1^{(\sigma)}(\lambda) \equiv \frac{f_{1,n}(\lambda)}{\prod_{i=0}^m \lambda_i^{-a_i + (p - l_i)e_n + 2}} \mod p^n A^\dagger, \quad (4.20)
\]

\[
\varepsilon_2^{(\sigma)}(\lambda) \equiv \frac{f_{2,n}(\lambda)}{\prod_{i=0}^m \lambda_i^{-b_i + (p - l_i)e_n + 2}} \mod p^n A^\dagger. \quad (4.21)
\]

Proof. Put \( \omega_i := \lambda_i^a \omega \) and \( \eta_i := \lambda_i^b \eta \). Then

\[
\varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta = \lambda_i^{-a_i}\varepsilon_1^{(\sigma)}(\lambda)\omega_i + \lambda_i^{-b_i}\varepsilon_2^{(\sigma)}(\lambda)\eta_i
\]

for \( i \in \{0, 1, \ldots, m, \infty\} \). Now the assertion is immediate from Lemma 4.14 (estimate lemma).

In a similar way to the above, we can obtain an estimate of the \( p \)-adic expansion of the Frobenius matrix on \( H^1_{\text{rig}}(X_F/S_F) \).

Theorem 4.16 Let the notation and assumption be as in Theorem 4.15. Let

\[
(\Phi(\omega) \quad \Phi(\eta)) = \begin{pmatrix} pF_{11} & pF_{12} \\ pF_{21} & pF_{22} \end{pmatrix}.
\]

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Let $e_n$ be as in (4.15), and put

$$d_{11,n} := \sum_{i=0}^{m} \left( pa_i - a_i + (p - l_i)e_n \right) + pa_{\infty} - a_{\infty},$$

$$d_{21,n} := \sum_{i=0}^{m} \left( pa_i - b_i + (p - l_i)e_n \right) + pb_{\infty} - b_{\infty},$$

$$d_{12,n} := \sum_{i=0}^{m} \left( pb_i - a_i + (p - l_i)e_n \right) + pb_{\infty} - a_{\infty},$$

$$d_{22,n} := \sum_{i=0}^{m} \left( pb_i - b_i + (p - l_i)e_n \right) + pb_{\infty} - b_{\infty}.$$

Then there are polynomials $f_{ij,n}(\lambda) \in W[\lambda]$ of degree $\leq d_{ij,n}$ such that

$$pF_{11} \equiv \left( \prod_{i=0}^{m} (\lambda_i^p)^{a_i} \lambda_i^{-a_i + (p - l_i)e_n} \right)^{-1} f_{11,n}(\lambda) \mod p^nA^l, \quad (4.22)$$

$$pF_{21} \equiv \left( \prod_{i=0}^{m} (\lambda_i^p)^{a_i} \lambda_i^{-b_i + (p - l_i)e_n} \right)^{-1} f_{21,n}(\lambda) \mod p^nA^l, \quad (4.23)$$

$$F_{12} \equiv \left( \prod_{i=0}^{m} (\lambda_i^p)^{b_i} \lambda_i^{-a_i + (p - l_i)e_n} \right)^{-1} f_{12,n}(\lambda) \mod p^nA^l, \quad (4.24)$$

$$F_{22} \equiv \left( \prod_{i=0}^{m} (\lambda_i^p)^{b_i} \lambda_i^{-b_i + (p - l_i)e_n} \right)^{-1} f_{22,n}(\lambda) \mod p^nA^l. \quad (4.25)$$

**Proof.** Put $\omega_i := \lambda_i^p \omega$ and $\eta_i := \lambda_i^b \eta$ for $i \in \{0, 1, \ldots, m, \infty \}$, and

$$pF_{11}^{(i)} \omega_i + pF_{21}^{(i)} \eta_i := \Phi(\omega_i) = (\lambda_i^p)^{a_i} \lambda_i^{-a_i} pF_{11} \cdot \omega_i + (\lambda_i^p)^{a_i} \lambda_i^{-b_i} pF_{21} \cdot \eta_i,$$

$$F_{12}^{(i)} \omega_i + F_{22}^{(i)} \eta_i := \Phi(\eta_i) = (\lambda_i^p)^{b_i} \lambda_i^{-a_i} F_{12} \cdot \omega_i + (\lambda_i^p)^{b_i} \lambda_i^{-b_i} F_{22} \cdot \eta_i.$$

Let $\sigma_0(\lambda_i) = \lambda_i^p$, and $\Phi_0$ the associated Frobenius on $H^1_{\text{rig}}(X_0/S_0)$. Since $\Phi_0$ acts on $H^1(Y, \Omega^1_{Y/W[[\lambda]]} \log D_i)$, we have the “integrality”

$$\Phi_0(\omega_i), \Phi_0(\eta_i) \in W[[\lambda_i]] \omega_i + W[[\lambda_i]] \eta_i. \quad (4.26)$$

It follows from Lemma 4.13 (transformation of Frobenius) that there are $u_k(\lambda_i), v_k(\lambda_i) \in W[[\lambda_i]]$ such that

$$\Phi(\omega_i) - \Phi_0(\omega_i) = \sum_{k=1}^{\infty} \frac{(\lambda_i^p - \lambda_i^p)^k}{k!} \lambda_i^{-k} p^k (u_k(\lambda_i) \omega_i + v_k(\lambda_i) \eta_i) \quad (4.27)$$

$$= \sum_{k=1}^{\infty} \frac{p^k}{k!} \left( \frac{l(\lambda_i)}{\lambda_i^p} \right)^k (u_k(\lambda_i) \omega_i + v_k(\lambda_i) \eta_i) \quad (4.28)$$
In this section, we demonstrate how to compute the syntomic regulators of $\lambda_i^{-k}W[[\lambda_i]]\omega_i + \lambda_i^{-k}W[[\lambda_i]]\eta_i$. By (4.26) and (4.28), we have “estimates”

$$pF_{11}^{(i)}, pF_{21}^{(i)} \in \lambda_i^{(\ell_i-p)e_n}W[[\lambda_i]] + p^aW[[\lambda_i, \lambda_i^{-1}]]$$

or equivalently

$$pF_{11} \in \frac{1}{(\lambda_i^\sigma)^{a_i} \lambda_i^{-a_i+p-l_i)e_n}W[[\lambda_i]] + p^aW[[\lambda_i, \lambda_i^{-1}]]$$

$$pF_{21} \in \frac{1}{(\lambda_i^\sigma)^{a_i} \lambda_i^{-b_i+p-l_i)e_n}W[[\lambda_i]] + p^aW[[\lambda_i, \lambda_i^{-1}]]$$

for all $i \in \{0, 1, \ldots, m, \infty\}$. Now (4.22) and (4.23) follow from this. In the same way, we have

$$F_{12} \in \frac{1}{(\lambda_i^\sigma)^{b_i} \lambda_i^{-a_i+p-l_i)e_n}W[[\lambda_i]] + p^aW[[\lambda_i, \lambda_i^{-1}]]$$

$$F_{22} \in \frac{1}{(\lambda_i^\sigma)^{b_i} \lambda_i^{-b_i+p-l_i)e_n}W[[\lambda_i]] + p^aW[[\lambda_i, \lambda_i^{-1}]]$$

for all $i \in \{0, 1, \ldots, m, \infty\}$, and hence (4.24) and (4.25) follow. This completes the proof. □

4.7 Computing Syntomic Regulators

In this section, we demonstrate how to compute the syntomic regulators of $K_2$ of elliptic curves. The key is Theorem 4.15. In what follows, $p \geq 3$ and $W = W(F_{p'})$ is the Witt ring of the finite field $F_{p'}$ with $p'$-elements. Put $K := \text{Frac}(W)$.

(Situation). Let $f : Y \to \mathbb{P}^1$ be an elliptic fibration whose generic fiber is defined by an affine equation

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in K(\lambda),$$

where $\lambda$ is an inhomogeneous coordinate of $\mathbb{P}^1$. Put

$$\omega = \frac{dx}{y}, \quad \eta = \frac{x\,dx}{y}.$$  

We assume that $f$ satisfies the conditions A1, \ldots, A3 in §4.1, and the multiplicity of each component of an arbitrary singular fiber is prime to $p$. Then the conditions A4 and A5 in §4.6 are satisfied by Proposition 6.10. Let $\{a_0, a_1, \ldots, a_m\} \subset W$ be the subset such that $D_i := f^{-1}(a_i)$ is a singular fiber ($f^{-1}(\infty)$ may be a singular fiber). We suppose that $a_0 = 0$ and $f$ has a split multiplicative reduction at $\lambda = 0$ as in A3. Put $S := \text{Spec}W[\lambda, \prod_i (\lambda - a_i)^{-1}]$ and $X := f^{-1}(S)$.

Let $\xi \in K_2(X)$. Let $\sigma$ be a $p$-th Frobenius on $W[\lambda, \prod_i (\lambda - a_i)^{-1}]$ given by

$$\sigma(\lambda) = c\lambda^p, \quad c \in 1 + pW.$$  

(4.29)
Let
\[ e_\xi - \Phi(e_\xi) = e_1^{(\sigma)}(\lambda) \omega + e_2^{(\sigma)}(\lambda) \eta. \]

Let \( s \geq 1 \) be an integer, and \( a \in W \) an element such that \( a \not\equiv a_i \mod p \) for all \( i \). Our goal is to compute
\[ e_1^{(\sigma)}(a), e_2^{(\sigma)}(a) \mod p^s W \] (4.30) explicitly. In case \( \sigma(\lambda) = a^{p^s} \), this gives the syntomic regulator
\[ \text{reg}_{\text{syn}}(\xi|_{X_a}) = e_1^{(\sigma)}(a) \omega + e_2^{(\sigma)}(a) \eta \in H^2_{\text{syn}}(X_a; \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(X_a/K) \]
for the fiber \( X_a := f^{-1}(a) \) (see Theorem 4.5).

(Notation). Replacing the variables \( x, y \) with \( \lambda x, \lambda y \) if necessary, we take an affine equation
\[ y^2 = 4x^3 - g_2x - g_3 \]
with \( \text{ord}_\lambda(g_2) = \text{ord}_\lambda(g_3) = 0 \) and \( n := \text{ord}_\lambda(\Delta) > 0 \), where \( \Delta := g_3^2 - 27g_2^3 \). For \( f(\lambda) \in K((\lambda)) \), we denote by \( f'(\lambda) = df(\lambda)/d\lambda \) the derivative with respect to \( \lambda \).

(i) Put \( \lambda_i := \lambda - a_i \) for \( i \in \{0, 1, \ldots, m\} \) and \( \lambda_\infty := \lambda^{-1} \). For \( i \in \{0, 1, \ldots, m\} \), put \( l_i := \min\{\text{ord}_\lambda(\lambda_i^n - \lambda_i^p), p\} \). It follows from Proposition 6.10 that for each \( i \in \{0, 1, \ldots, m, \infty\} \), there are integers \( a_i, b_i \) such that
\[ \lambda_i^{a_i} \omega, \lambda_i^{b_i} \eta \]
is a free \( W[[\lambda_i]] \)-basis of \( H_i \), where \( H_i \) is as in Theorem 4.15, A5.

(ii) (cf. (4.1)). Put \( J := g_2^3 / \Delta \). The \( j \)-invariant is \( 1728J \). Put
\[ \kappa := \frac{\lambda^{-n}}{1728J} \bigg|_{\lambda=0}, \quad \text{where } n := -\text{ord}_\lambda(J), \]
which belongs to \( W^\times \) by A3.

(iii) (cf. Proposition 6.5). We fix a square root
\[ c_0 := \sqrt{-\frac{g_2(0)}{18g_3(0)}}. \]

By A3, \( f \) has a split multiplicative reduction at \( \lambda = 0 \) and the fiber \( f^{-1}(0) \) is a relative NCD over \( W \), so that we have \( c_0 \in W^\times \).

(iv) (cf. (4.5) and Proposition 6.5, 6.14). Let \( (12g_2)^{-1/4} \in W[[\lambda]] \) be the power series with the constant term \( c_0 \). Put
\[ F(\lambda) := (12g_2)^{-1/4} F_1 \left( \frac{1}{12}, \frac{5}{12}; J^{-1} \right). \]
(v) (cf. (4.2) and Proposition 6.7). Let \( \tau(\lambda) \in \lambda K[[\lambda]] \) be the power series determined by
\[
\frac{d\tau(\lambda)}{d\lambda} = 3\frac{2 g_2 g_3 - 3g'_2 g_3}{2\Delta F(\lambda)^2} - \frac{n}{\lambda}.
\]
By Proposition 6.7, \( q := \kappa^\lambda \exp(\tau(\lambda)) \) is the multiplicative period of the Tate curve \( \mathcal{E} = X \times_A \text{Spec} W((\lambda)) \).

(vi) (cf. (4.4)). Put
\[
\tau^{(\sigma)}(\lambda) := \log^{(\sigma)}(q) = \log(F) - np^{-1} \log(c) + \tau(\lambda) - p^{-1} \tau(\lambda)^\sigma.
\]

(vii) (cf. §4.4). For \( \xi \in K_2(X) \), let \( g_\xi(\lambda) \in A \) be defined by \( d\log(\xi) = g_\xi(\lambda) \frac{d\lambda}{\lambda} \wedge dx \wedge dy \). Put
\[
G_\xi(\lambda) := g_\xi(\lambda)F(\lambda) = l + a_1 \lambda + a_2 \lambda^2 + \cdots \in W[[\lambda]].
\]
Note that \( l \) is an integer.

(viii) (cf. Proposition 6.6). Put
\[
H(\lambda) := \frac{2\Delta}{3(2g_2 g_3 - 3g'_2 g_3)} \left( F'(\lambda) + \frac{\Delta'}{12\Delta} F(\lambda) \right).
\]

(ix) (cf. Theorem 4.6 (4.11)). Let \( E_1^{(\sigma)}(\lambda) \in W[[\lambda]] \) be defined by
\[
(E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^\sigma}{\lambda}, \quad E_1^{(\sigma)}(0) := \frac{l}{n} \log(F) - \frac{l}{p} \log(c).
\]

**Step 1.** Compute the power series
\[
F(\lambda) = f_0 + f_1 \lambda + f_2 \lambda^2 + \cdots + f_N \lambda^N + O(\lambda^{N+1}),
\]
\[
H(\lambda) = h_0 + h_1 \lambda + h_2 \lambda^2 + \cdots + h_N \lambda^N + O(\lambda^{N+1}),
\]
\[
E_1^{(\sigma)}(\lambda) = A_0 + A_1 \lambda + A_2 \lambda^2 + \cdots + A_N \lambda^N + O(\lambda^{N+1}).
\]
Moreover, let
\[
E_2^{(\sigma)}(\lambda) = B_0 + B_1 \lambda + B_2 \lambda^2 + \cdots + B_N \lambda^N + O(\lambda^{N+1})
\]
be the power series satisfying
\[
(E_2^{(\sigma)}(\lambda))' = -E_1^{(\sigma)}(\lambda) \frac{q'}{q} + \frac{G_\xi(\lambda)^\sigma}{\lambda} \tau^{(\sigma)}(\lambda)
\]
\[
= -E_1^{(\sigma)}(\lambda) \left( \frac{n}{\lambda} + \tau'(\lambda) \right) + \frac{G_\xi(\lambda)^\sigma}{\lambda} \tau^{(\sigma)}(\lambda)
\]
Each \( B_i \) is determined except the constant term \( B_0 \) which shall be computed in the next step. Here we take \( N \) to be a large integer according to the Step 2 and 3. See Remark 4.17 (4.32) for an explicit lower bound of \( N \).

**Step 2 (Computing \( E_2^{(\sigma)}(0) \)).** Let
\[
\varepsilon_1^{(\sigma)}(\lambda) := \frac{E_1^{(\sigma)}(\lambda)}{F(\lambda)} - H(\lambda)E_2^{(\sigma)}(\lambda), \quad \varepsilon_2^{(\sigma)}(\lambda) := F(\lambda)E_2^{(\sigma)}(\lambda)
\]
(cf. (4.9), (4.10) and (6.15)). We compute the constant \( E_2^{(\sigma)}(0) = B_0 \) modulo \( p^s \) (without assuming Conjecture 4.7). We first note that \( B \) is uniquely determined by the fact that \( \varepsilon_2^{(\sigma)}(\lambda) = F(\lambda)E_2^{(\sigma)}(\lambda) \) is an overconvergent function (i.e. \( F(\lambda)E_2^{(\sigma)}(\lambda) \in \mathcal{A}_K^1 \)). Indeed, if there is another constant \( B'_0 \)\((\neq B_0)\) such that \( F(\lambda)(B'_0 + B_1 \lambda + \cdots) \) is an overconvergent function, then \( F(\lambda)(B'_0)(\lambda) - F(\lambda)(B_0 + B_1 \lambda + \cdots) = (B_0 - B'_0)F(\lambda) \) is also overconvergent. This is impossible. Indeed, let \( \zeta \in W^\times \) be a root of unity such that \( X_\zeta \subset \mathcal{F}_{p^r} \) is an ordinary elliptic curve. If \( F(\lambda) \) were overconvergent, then the special value of \( F(\lambda) \) at \( \lambda = \zeta \) makes sense. Then one has
\[
\gamma_\zeta := \left. \frac{F(\lambda)}{F(\lambda^{p^s})} \right|_{\lambda = \zeta} = \frac{F(\zeta)}{F(\zeta^{p^s})} = \frac{F(\zeta)}{F(\zeta)} = 1.
\]
This contradicts with the fact that \( \gamma_\zeta \) is the unit root of the \( p^r \)-th Frobenius of \( X_\zeta \subset \mathcal{F}_{p^r} \) (Corollary 4.9). In a practical way, one can use Theorem 4.15 to compute \( B_0 \). Namely letting \( e_s := \max\{ i \in \mathbb{Z}_{\geq 0} \mid \text{ord}_p(p^r/i!) < s \} \) be as in (4.15) and \( d_{2,s} := \sum_{i=0}^m (-b_i + (p - l_i)e_{s+2}) - b_\infty \) as in (4.19), one finds the unique
\[
B_0 \mod p^s W
\]
satisfying that for a large \( s' \geq s \),
\[
\varepsilon_2^{(\sigma)}(\lambda) \equiv \text{polynomial of degree } \leq d_{2,s'} \mod p^{s'} A^\dagger
\]
or equivalently, the power series
\[
F(\lambda)(B_0 + B_1 \lambda + \cdots) \prod_{i=0}^m (\lambda - a_i)^{-b_i + (p - l_i)e_{s+2}} \in (W/p^{s'} W)[[\lambda]]
\]
modulo \( p^{s'} \) terminates at least at the degree \( d_{2,s'} \). See Remark 4.17 (4.32) for an explicit lower bound of \( s' \).

**Step 3 (Computing \( \varepsilon_k^{(\sigma)}(a) \)).** Compute the power series \( \varepsilon_k^{(\sigma)}(\lambda) \). We again apply Theorem 4.15 Compute the power series
\[
f_{1,s}(\lambda) := \varepsilon_1^{(\sigma)}(\lambda) \prod_{i=0}^m (\lambda - a_i)^{-a_i + (p - l_i)e_{s+2}}, \quad f_{2,s}(\lambda) := \varepsilon_2^{(\sigma)}(\lambda) \prod_{i=0}^m (\lambda - a_i)^{-b_i + (p - l_i)e_{s+2}}
\]
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modulo $p^sW[[\lambda]]$ which terminates at least at the degree $d_{1,s}$ and $d_{2,s}$ respectively. Then one has
\[
\varepsilon_1^{(\sigma)}(a) \equiv \prod_{i=0}^{m}(a-a_i)^{-a_i+(p-l_i)e_{i+s+2}} \mod p^sW,
\]
\[
\varepsilon_2^{(\sigma)}(a) \equiv \prod_{i=0}^{m}(a-a_i)^{-b_i+(p-l_i)e_{i+s+2}} \mod p^sW.
\]

**Remark 4.17** Suppose that there are two constants $B_0, B_0^*$ such that $B_0 \neq B_0^* \mod p^s$ and
\[
F(\lambda)(B_0 + B_1\lambda + \cdots) \prod_{i=0}^{m}(\lambda - a_i)^{-b_i+(p-l_i)e_{i+s+2}} \in (W/p^sW)[[\lambda]]
\]

and
\[
F(\lambda)(B_0^* + B_1\lambda + \cdots) \prod_{i=0}^{m}(\lambda - a_i)^{-b_i+(p-l_i)e_{i+s+2}} \in (W/p^sW)[[\lambda]]
\]
terminate at the degree $d_{2,s'}$. Since \(\text{ord}_p(B_0 - B_0^*) < s\),
\[
F(\lambda)\prod_{i=0}^{m}(\lambda - a_i)^{-b_i+(p-l_i)e_{i+s+2}} \in (W/p^{s'+s+1}W)[[\lambda]] \quad (4.31)
\]
terminates at the degree $d_{2,s'}$. This implies that the unit root $\gamma_\zeta$ of the $p^s$-th Frobenius on $X_\zeta \times W \mathbb{F}_{p^s}$ satisfies
\[
\gamma_\zeta = \left. \frac{F(\lambda)}{F(\lambda^{p^s})} \right|_{\lambda=\zeta} \equiv 1 \mod p^{s'+s+1}W.
\]

Conversely letting
\[
N_\zeta := \min\{i \in \mathbb{Z}_{\geq 1} \mid \gamma_\zeta \equiv 1 \mod p^iW\},
\]
for an arbitrary $s' \geq N_\zeta + s - 1$, the power series $(4.31)$ cannot terminate at the degree $d_{2,s'}$. Therefore, the constant $B_0 \mod p^s$ is characterized by the condition that
\[
F(\lambda)(B_0 + B_1\lambda + \cdots) \prod_{i=0}^{m}(\lambda - a_i)^{-b_i+(p-l_i)e_{i+s+2}} \in (W/p^sW)[[\lambda]]
\]
terminates at least at the degree $d_{2,s'}$ for all $s' \geq N_\zeta + s - 1$.

Summing up the above, we get a bound of $N$ in **Step 1** and a bound of $s'$ in **Step 2** as follows. Choose $\zeta \in W$ such that $X_\zeta \times \mathbb{F}_{p^s}$ is ordinary. Then take $s'$ such that
\[
s' \geq N_\zeta + s - 1. \quad (4.32)
\]

Let $F(\lambda)\prod_{i=0}^{m}(\lambda - a_i)^{-b_i+(p-l_i)e_{i+s+2}} = \sum_{i=0}^{\infty} C_i\lambda^i$. Find an integer $M > d_{2,s'}$ such that
\[
C_M \not\equiv 0 \mod p^{s'-s+1}W.
\]

Then take $N$ such that
\[
N > \max\{d_{1,s}, d_{2,s}, M\}. \quad (4.33)
\]

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4.8 Computing the Frobenius matrix on $H^1_{\text{rig}}(X_F/S_F)$

We keep the situation and notation in §4.7. Let

$$(\Phi(\omega) \quad \Phi(\eta)) = (\omega \quad \eta) \begin{pmatrix} pF_{11}(\lambda) & F_{12}(\lambda) \\ pF_{21}(\lambda) & F_{22}(\lambda) \end{pmatrix}$$

be the matrix of the $p$-th Frobenius on $H^1_{\text{rig}}(X_F/S_F)$. It follows from Proposition 4.3 together with Proposition 6.6 (6.15) that one has

$$F_{11}(\lambda) = F(\lambda) \sigma^{-1} F(\lambda)^{\sigma},$$
$$F_{12}(\lambda) = pH(\lambda) - H(\lambda) F(\lambda)\sigma + p\tau(\sigma)(\lambda) H(\lambda) F(\lambda)^\sigma, \quad \lambda \sigma = c\lambda^p - a_i F_i$$
$$F_{21}(\lambda) = -\tau(\sigma)(\lambda) F(\lambda) F(\lambda)^\sigma, \quad \lambda \sigma = c\lambda F_i$$
$$F_{22}(\lambda) = F(\lambda) - p\tau(\sigma)(\lambda) H(\lambda) F(\lambda).$$

One can apply Theorem 4.16 to compute

$$\begin{pmatrix} pF_{11}(a) & F_{12}(a) \\ pF_{21}(a) & F_{22}(a) \end{pmatrix} \mod p^s.$$

Namely compute the power series

$$f_{11,s}(\lambda) := pF_{11}(\lambda) \prod_{i=0}^m (\lambda_i^p)^{a_i} \lambda_i^{-(a_i-(p-l_i)e_s)} \in (W/p^sW)[[\lambda]]$$

with coefficients in $W/p^sW$ which terminates at least at the degree $d_{11,s}$, where we note $\lambda_i^p = (\lambda - a_i)^\sigma = c\lambda^p - a_i F_i$ (see (4.29)). Then one computes $F_{11}(a)$ by

$$pF_{11}(a) \equiv f_{11,s}(a) \cdot \left( \prod_{i=0}^m (ca^p - a_i F_i)^{a_i} (a - a_i)^{-a_i+(p-l_i)e_s} \right)^{-1} \mod p^s W.$$

One can also compute $pF_{12}$, $F_{21}$ and $F_{22}$ in the same way.

5 Numerical Verifications of the $p$-adic Beilinson conjecture for $K_2$

We apply the algorithm in §4.7 to compute the syntomic regulators, and give numerical verifications of the $p$-adic Beilinson conjecture for $K_2$ (Conjecture 3.3, $n = 0$) of the following two examples

- $y^2 = x^3 - 2x^2 + (1-a)x$ with $a \in \mathbb{Q} \setminus \{0, 1\}$,
- $y^2 = x(1-x)(1-(1-a)x)$ with $a \in \mathbb{Q} \setminus \{0, 1\}$.
5.1 \[ y^2 = x^3 - 2x^2 + (1 - \lambda)x \]

We first discuss the elliptic fibration in [A2, Theorem 4.29]. By a simple replacement of variables, it is a fibration \( f : Y \to \mathbb{P}^1 \) defined by a Weierstrass equation

\[
y^2 = 4x^3 - g_2x - g_3, \quad g_2 := 4\lambda + \frac{4}{3}, \quad g_3 := \frac{8}{3}\lambda - \frac{8}{27}
\]

over the base ring \( \mathbb{Z}(p) = \mathbb{Q} \cap \mathbb{Z}_p \) with \( p \geq 5 \). Put

\[
\omega = \frac{dx}{y}, \quad \eta = \frac{xdx}{y}.
\]

The functional \( j \)-invariant is \[ \frac{64(3\lambda + 1)^3}{\lambda(1 - \lambda)^2} \]. \( f \) has a multiplicative reduction of Kodaira type I\(_1\) at \( \lambda = 0 \) which is split over \( \mathbb{Z}(p) \), a multiplicative reduction of type I\(_2\) at \( \lambda = 1 \), and an additive reduction of type III\(^*\) at \( \lambda = \infty \). One can check that \( f \) satisfies all the conditions A1, \ldots, A3 in §4.1 and A4, A5 in §4.6 (detail is left to the reader). Put \( S = \text{Spec} \mathbb{Z}(p)[\lambda, (\lambda - \lambda^2)^{-1}] \) and \( X = f^{-1}(S) \). Write \( X_{\mathbb{Z}_p} = X \times_{\mathbb{Z}(p)} \mathbb{Z}_p \), \( X_{\mathbb{F}_p} = X \times_{\mathbb{Z}(p)} \mathbb{F}_p \) etc. Let \( \sigma : \mathbb{Z}_p[[\lambda]] \to \mathbb{Z}_p[[\lambda]] \) be the \( p \)-th Frobenius given by \( \sigma(\lambda) = c\lambda^p \) with \( c \in 1 + p\mathbb{Z}_p \).

Let

\[
\xi := \left\{ \begin{array}{l}
3y + 6x - 2 \\
3y - 6x + 2
\end{array} \right. \quad \frac{-9\lambda(3x + 2)}{(3x - 1)^3} \in K_2(X)
\]

be a symbol (cf. [A2, Theorem 4.29]). A direct calculation yields

\[
d\log(\xi) = 2d\lambda \frac{dx}{y}.
\]

Put

\[
F_{\frac{1}{2}, \frac{3}{4}}(\lambda) = 2F_1 \left( \frac{1}{4}, \frac{3}{4}; \lambda \right).
\]

The notation in §4.7 in our case is explicitly given as follows.

(i) \( (a_0, b_0) = (a_1, b_1) = (0, 0) \) and \( (a_\infty, b_\infty) = (0, 1) \).

(ii) Since \( 1728J = 64(3\lambda + 1)^3/(\lambda(1 - \lambda)^2) \), one has \( n = \text{ord}_\lambda(J) = 1 \) and \( \kappa = 1/64 \).

(iii) One has \( c_0^2 = -g_2(0)/(18g_3(0)) = 1/4 \). We take \( c_0 = 1/2 \).

(iv) One can show

\[
F(\lambda) := (12g_2)^{-\frac{1}{2}}2F_1 \left( \frac{1}{12}, \frac{5}{12}; J^{-1} \right) = \frac{1}{2}F_{\frac{1}{2}, \frac{3}{4}}(\lambda)
\]

in the following way. Recall from the proof of Proposition 6.5 that \( F(\lambda) \) is characterized by

\[
F(\lambda) = \frac{1}{2\pi i} \int_{\delta_\lambda} \omega.
\]
By theorem [6.4] we have
\[
(\partial_\lambda (\omega) \quad \partial_\lambda (\eta)) = (\omega \quad \eta) \begin{pmatrix} \frac{3\lambda-1}{12(\lambda-\lambda^2)} & -\frac{3\lambda+1}{36(\lambda-\lambda^2)} \\ \frac{1}{4(\lambda-\lambda^2)} & \frac{3\lambda+1}{12(\lambda-\lambda^2)} \end{pmatrix},
\]
where \(\partial_\lambda := \nabla_{d/d\lambda}\). Hence
\[
((\lambda - \lambda^2)\partial^2_\lambda + (1 - 2\lambda)\partial_\lambda - \frac{3}{4})\omega = 0.
\]
This implies that \(F(\lambda)\) satisfies
\[
(\lambda - \lambda^2) \frac{d^2}{d\lambda^2} F(\lambda) + (1 - 2\lambda) \frac{d}{d\lambda} F(\lambda) - \frac{3}{4} F(\lambda) = 0. \tag{5.3}
\]
This is the hypergeometric differential equation (e.g. [NIST, 15.10]). Hence \(F(\lambda) = c F_{\frac{1}{4}, \frac{3}{4}}(\lambda)\) for some constant \(c\). Since the constant term of \(F(\lambda)\) is \(c_0 = 1/2\) by definition, one has \(c = 1/2\).

(v) The multiplicative period is \(q = 64^{-1}\lambda \exp(\tau(\lambda))\) with
\[
\lambda \frac{d}{d\lambda} \tau(\lambda) = -1 + (1 - \lambda)^{-1} F_{\frac{1}{4}, \frac{3}{4}}(\lambda)^{-2}, \quad \tau(0) = 0.
\]
Explicitly,
\[
\tau(\lambda) = -\frac{5}{8} \lambda - \frac{269}{1024} \lambda^2 - \frac{1939}{12288} \lambda^3 - \frac{922253}{8388608} \lambda^4 + \cdots.
\]

(vi) \(\tau^{(\sigma)}(\lambda) = -p^{-1} \log(64^{p-1}c) + \tau(\lambda) - p^{-1}\tau(\lambda)^\sigma\).

(vii) By (5.2), one has \(g_\xi(\lambda) = 2\) and \(G_\xi(\lambda) = 2 F(\lambda) = F_{\frac{1}{4}, \frac{3}{4}}(\lambda)\) (hence \(l = 1\)).

(viii)
\[
H(\lambda) = 2(\lambda - \lambda^2) F_{\frac{1}{4}, \frac{3}{4}}'(\lambda) + \frac{1}{6} (1 - 3\lambda) F_{\frac{1}{4}, \frac{3}{4}}(\lambda)
\]
\[
= \frac{1}{6} - \frac{3}{32} \lambda - \frac{85}{2048} \lambda^2 - \frac{875}{32768} \lambda^3 - \frac{165165}{8388608} \lambda^4 + \cdots.
\]

(ix) Let \(E_1^{(\sigma)}(\lambda) \in \mathbb{Z}_p[[\lambda]]\) be defined by
\[
(E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^\sigma}{\lambda}, \quad E_1^{(\sigma)}(0) := -p^{-1} \log(64^{p-1}c).
\]
Let \( E_2^{(\sigma)}(\lambda) = C + a_1 \lambda + \cdots \) be the power series satisfying
\[
\frac{d}{d\lambda} E_2^{(\sigma)}(\lambda) = -E_1^{(\sigma)}(\lambda) \left( \frac{1}{\lambda} + \tau'(\lambda) \right) + \frac{G_\xi(\lambda)^\sigma}{\lambda} \tau(\lambda) \\
= -E_1^{(\sigma)}(\lambda) \left( (1-\lambda)^{-1} F_{\frac{1}{4},\frac{3}{4}}(\lambda)^{-2} \right) + \frac{G_\xi(\lambda)^\sigma}{\lambda} \tau(\lambda).
\]
This determines \( E_2^{(\sigma)}(\lambda) \) except the constant term \( C \). We expect
\[
C = -\frac{1}{2} \left( p^{-1} \log(64^p - 1) \right)^2
\]
according to Conjecture 4.7. The authors do not know a proof of this, while it does not matter toward numerical computation thanks to the method in Step 2 in §4.7.

Let \( a \in \mathbb{Z}_p \) such that \( a \not\equiv 0, 1 \mod p \), and \( X_a \) the fiber at \( \lambda = a \). Let \( \sigma_a(\lambda) = a^{1-p} \lambda^p \), and
\[
\text{reg}_{\text{syn}}(\xi|_{\lambda=a}) = \varepsilon_1^{(\sigma_a)}(a) \frac{dx}{y} + \varepsilon_2^{(\sigma_a)}(a) \frac{xdx}{y}.
\]
If \( a = 2, 4, 8 \), then the symbol \( \xi \) is integral. Running the steps in §4.7, we obtain the following table:

| \( p \) | \( \varepsilon_1^{(\sigma_a)}(a) \) | \( \varepsilon_2^{(\sigma_a)}(a) \) |
|---------|-----------------|-----------------|
| 5       | 19674787784 \( (p^{15}) \) | 14084682764 \( (p^{15}) \) |
| 7       | 2590802242826 \( (p^{15}) \) | 1117519637850 \( (p^{15}) \) |
| 11      | 7965734565 \( (p^{10}) \) | 13678445088 \( (p^{10}) \) |
| 13      | 11926785562 \( (p^{10}) \) | 7281360585 \( (p^{10}) \) |
| 17      | 278698244 \( (p^8) \) | 3681687094 \( (p^8) \) |

\( a = 2 \)

| \( p \) | \( \varepsilon_1^{(\sigma_a)}(a) \) | \( \varepsilon_2^{(\sigma_a)}(a) \) |
|---------|-----------------|-----------------|
| 5       | 10899377190 \( (p^{15}) \) | 18907289609 \( (p^{15}) \) |
| 7       | 1079749670299 \( (p^{15}) \) | 380030532560 \( (p^{15}) \) |
| 11      | 1981716464 \( (p^{10}) \) | 23499057595 \( (p^{10}) \) |
| 13      | 99992661864 \( (p^{10}) \) | 5427943535 \( (p^{10}) \) |
| 17      | 5624041300 \( (p^8) \) | 6557787442 \( (p^8) \) |

\( a = 4 \)

| \( p \) | \( \varepsilon_1^{(\sigma_a)}(a) \) | \( \varepsilon_2^{(\sigma_a)}(a) \) |
|---------|-----------------|-----------------|
| 5       | 25874535623 \( (p^{15}) \) | 28248957885 \( (p^{15}) \) |
| 7       | bad reduction | |
| 11      | 11113392970 \( (p^{10}) \) | 10168663913 \( (p^{10}) \) |
| 13      | 50210889322 \( (p^{10}) \) | 1878713357 \( (p^{10}) \) |
| 17      | 4803397460 \( (p^8) \) | 657810945 \( (p^8) \) |

\( a = 8 \)

Next we compute the \( p \)-adic regulators
\[
R_{p,\gamma}(X_a, \xi) = R_{p,\gamma}(h^1(X_a), \xi) = (1 - p^{-1}) \frac{\text{Tr}(\text{reg}_{\text{syn}}(\xi|_{\lambda=a}) \cup v_\gamma)}{\text{Tr}(\omega_{X_a} \cup v_\gamma)}
\]
introduced in §3.4, where \( \gamma \) is a unit or non-unit root of \( X_a \), \( v_\gamma \in H^1_{\text{dR}}(X_a/\mathbb{Q}_p) \otimes \mathbb{Q}_p \) is the eigenvector of \( \Phi \otimes 1_{\mathbb{Q}_p} \) with eigenvalue \( \gamma \) and \( \omega_{X_a} \) is the Néron differential. To do this we first compute the representation matrix

\[
(\Phi(\omega) \quad \Phi(\eta)) = (\omega \quad \eta) \begin{pmatrix} pF_{11}(\lambda) & F_{12}(\lambda) \\ pF_{21}(\lambda) & F_{22}(\lambda) \end{pmatrix} \bigg|_{\lambda = a}
\]

of the \( p \)-th Frobenius on \( H_{\text{crys}}(X_a, \mathbb{F}_p/\mathbb{Z}_p) \). Let us see the case \( a = 4 \) and \( p = 5 \) (the other cases are done in the same way). Let \( \alpha \) be the unit root, and \( \beta = p\alpha^{-1} \) the non-unit root. By counting the \( \mathbb{F}_5 \)-rational points of \( X_{a,\mathbb{F}_5} \), one has that the trace of the Frobenius is \(-2\), and hence \( \alpha = -1 + 2\sqrt{-1} \equiv 22219310363 \mod 5^5 \). Thanks to the method in §4.8 one has

\[
\begin{pmatrix} pF_{11}(a) & F_{12}(a) \\ pF_{21}(a) & F_{22}(a) \end{pmatrix} \equiv \begin{pmatrix} 30465757535 & 3584355845 \\ 11560944585 & 518205885 \end{pmatrix} \mod 5^{15}.
\] (5.4)

Hence the eigenvector \( v_\alpha \) satisfies

\[
v_\alpha \equiv 710786365\omega + \eta \mod 5^{14},
\]

and the eigenvector \( v_\beta \) for the non-unit root \( \beta \) satisfies

\[
v_\beta \equiv \omega + 28450379180\eta \mod 5^{15}.
\]

In case \( a = 2, 4, 8 \), it follows by computing the minimal Weierstrass equation (e.g. Magma [Magma]) that one has \( \omega_{X_a} = \omega \). We thus have

\[
\frac{\text{Tr}(\text{reg}_{\text{sym}}(\xi_{|\lambda = a}) \cup v_\alpha)}{\text{Tr}(\omega_{X_a} \cup v_\alpha)} \equiv (\varepsilon_{1}^{(\sigma)}(a) - 710786365\varepsilon_{2}^{(\sigma)}(a)) \frac{\text{Tr}(\omega \cup \eta)}{\text{Tr}(\omega_{X_a} \cup \eta)} \equiv 3956230280 \mod 5^{14},
\]

\[
\frac{\text{Tr}(\text{reg}_{\text{sym}}(\xi_{|\lambda = a}) \cup v_\beta)}{\text{Tr}(\omega_{X_a} \cup v_\beta)} \equiv \frac{28450379180\varepsilon_{1}^{(\sigma)}(a) - \varepsilon_{2}^{(\sigma)}(a)}{28450379180} \frac{\text{Tr}(\omega \cup \eta)}{\text{Tr}(\omega_{X_a} \cup \eta)} \equiv 5^{-1} \cdot 5576309542 \mod 5^{13},
\]

where \( a \equiv b \mod p^a \) for \( a, b \in \mathbb{Q}_p \) means that \( a - b \in p^a \mathbb{Z}_p \). Hence

\[
R_{5,a}(X_a, \xi) \equiv 5110910605 \mod 5^{14},
\]

\[
R_{5,5a^{-1}}(X_a, \xi) \equiv 5^{-1} \cdot 2495229428 \mod 5^{13}.
\]

If \( a = 4 \) and \( p = 7 \), then \( X_a \) has a supersingular reduction. In this case the Frobenius is

\[
\begin{pmatrix} pF_{11}(a) & F_{12}(a) \\ pF_{21}(a) & F_{22}(a) \end{pmatrix} \equiv \begin{pmatrix} 3603314253994 & 263103376372 \\ 899507565369 & 1144247255949 \end{pmatrix} \mod 7^{15},
\] (5.5)

and the eigenvector \( v_{\sqrt{-7}} \) for \( \sqrt{-7} \) satisfies

\[
v_{\sqrt{-7}} \equiv \omega + (2066673311059 + 1689081159560\sqrt{-7})\eta \mod 7^{15}\mathbb{Z}_p[\sqrt{-7}].
\]
Hence

\[ R_{r,\sqrt{-7}}(X_a, \xi) = (1 + \sqrt{-7}) \frac{\text{Tr}([\text{reg}_{\text{syn}}(\xi|t=a) \cup v_{\sqrt{-7}}])}{\text{Tr}(\omega_{X_a} \cup v_{\sqrt{-7}})} \]

\[ \equiv (1 + \sqrt{-7}) \frac{C\varepsilon(\sigma(a)) - \varepsilon(\sigma(a))}{C} \]

\[ \equiv 224156767500 + 7^{-1} \cdot 577165355498\sqrt{-7} \]

modulo \(7^{13}\mathbb{Z}_p[\sqrt{-7}]\). Other results are given in the following table, where \(\alpha\) denotes the unit root or \(\sqrt{-p}\):

### Table of \(p\)-adic regulators

| \(p\) | \(R_{p,\alpha}(X_a, \xi)\) | \(R_{p,\beta}(X_a, \xi)\) |
|-------|-----------------|------------------|
| 5     | 5162207171(\(p^{15}\)) | 5^{-1} \cdot 4698651004(\(p^{13}\)) |
| 7     | 1898050197813(\(p^{15}\)) | 47494881888(\(p^{13}\)) |
| 11    | 4688873516(\(p^{10}\)) | 4077822(\(p^{8}\)) |
| 13    | 109756622389(\(p^{10}\)) | 13^{-1} \cdot 1014980225(\(p^{8}\)) |
| 17    | 1874620233(\(p^{8}\)) | 17^{-1} \cdot 239203868(\(p^{6}\)) |

| \(p\) | \(R_{p,\alpha}(X_a, \xi)\) | \(R_{p,\beta}(X_a, \xi)\) |
|-------|-----------------|------------------|
| 5     | 5110910605(\(p^{14}\)) | 5^{-1} \cdot 2495229428(\(p^{13}\)) |
| 7     | 224156767500 + 7^{-1} \cdot 577165355498\sqrt{-7}(\(p^{13}\)) | (supersingular) |
| 11    | 23123035879(\(p^{10}\)) | 11^{-1} \cdot 1673260489(\(p^{8}\)) |
| 13    | 131267814396(\(p^{10}\)) | 13^{-1} \cdot 8897873775(\(p^{8}\)) |
| 17    | 4659831644(\(p^{8}\)) | 17^{-1} \cdot 401683799(\(p^{6}\)) |

We compare the above regulators with the \(p\)-adic \(L\)-values. One can compute the special values \(L_{p,\alpha}(X_a, \omega^{-1}, 0)\) by SAGE. The values \(L_{p,\beta}(X_a, \omega^{-1}, 0)\) of the critical slope \(L\)-function are kindly provided by Professor Robert Pollack:

| \(p\) | \(L_{p,\beta}(X_a, \omega^{-1}, 0)\) |
|-------|----------------------------------|
| 5     | 2208371/5(\(p^{10}\)) |
| 7     | 1125190415157500314434008856467880(\(p^{30}\)) |
| 11    | 88350501278121136930215849171019600321077(\(p^{30}\)) |
The above indicate the following:

| $a$ | $R_{p,\alpha}/L_{p,\alpha}(X_a, \omega^{-1}, 0)$ | $R_{p,\beta}/L_{p,\beta}(X_a, \omega^{-1}, 0)$ |
|-----|---------------------------------|---------------------------------|
| 2   | $-1$                            | $-1$                            |
| 4   | $-2$                            | $-2$                            |
| 8   | $1/2$                           |                                  |

Next we compute the Beilinson regulators. Let $X_\lambda := f^{-1}(\lambda)$ be the fiber at $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$, an elliptic curve over $\mathbb{R}$. Let $F_\infty$ denote the infinite Frobenius on $X_\lambda(\mathbb{C})$. Fix generators $u^\pm \in H_1(X_\lambda(\mathbb{C}), \mathbb{Z})^{F_\infty=\pm1} \cong \mathbb{Z}$. Since the elliptic curve $X_\lambda$ is the rectangle case (i.e. $4x^3 - g_2x - g_3$ has 3 distinct real roots), one has $H_1(X_\lambda(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}u^+ + \mathbb{Z}u^-$. Let

$$\text{reg}_{\mathbb{R}}: K_2(X_\lambda) \rightarrow H^2_{\mathbb{R}}(X_\lambda, \mathbb{R}(2)) \cong \text{Hom}(H_1(X_\lambda(\mathbb{C}), \mathbb{Z}), \mathbb{R}(1))$$

be the Beilinson regulator map. Then

$$R_\infty(X_\lambda, \xi) = \frac{1}{2\pi i} \langle \text{reg}_{\mathbb{R}}(\xi), u^- \rangle.$$

Note that in general

$$\langle \text{reg}_{\mathbb{R}}(\xi), u^+ \rangle \in \mathbb{R}, \quad \langle \text{reg}_{\mathbb{R}}(\xi), u^- \rangle \in \mathbb{R}(1). \quad (5.6)$$

We claim

$$\frac{1}{2\pi i} \langle \text{reg}_{\mathbb{R}}(\xi), u^- \rangle = \pm \text{Re} \left[ -\log 64 + \log \lambda + \frac{3\lambda}{16} \text{F}_3 \left( \frac{5}{4}, \frac{7}{4}, 1, 1; 2, 2, 2; \lambda \right) \right]. \quad (5.7)$$

If $0 < \lambda < 1$, the equation $4x^3 - g_2x - g_3$ has 3 distinct real roots

$$-\frac{2}{3} < \frac{1}{3} - \sqrt{\lambda} < \frac{1}{3} + \sqrt{\lambda}.$$

Hence $u^\pm$ are the cycles in the $x$-plane displayed as follows:

![Cycle Diagram](image)

$Q_1 = \frac{1}{3} - \sqrt{\lambda}, Q_2 = \frac{1}{3} + \sqrt{\lambda}, 0 < \lambda < 1$

If $\lambda > 1$, the equation $4x^3 - g_2x - g_3$ has 3 distinct real roots

$$\frac{1}{3} - \sqrt{\lambda} < -\frac{2}{3} < \frac{1}{3} + \sqrt{\lambda},$$

the generator $u^\pm$ are as follows:
If $0 < \lambda < 1$ the cycle $u^-$ agrees with the vanishing cycle $\delta_\lambda$ at $\lambda = 0$. We move $\lambda$ to a point in the interval $(1, \infty)$. Then $\delta_\lambda = u^- + mu^+$ with some $m \in \mathbb{Z}$. Hence it follows from (5.6) that

$$\frac{1}{2\pi i} \langle \text{reg}_R(\xi), u^- \rangle = \text{Re} \left( \frac{1}{2\pi i} \langle \text{reg}_R(\xi), \delta_\lambda \rangle \right)$$

in both cases of $0 < \lambda < 1$ and $\lambda > 1$. By (5.2) one has

$$f(\lambda) := \lambda \frac{d}{d\lambda} \langle \text{reg}_R(\xi), \delta_\lambda \rangle = 2 \int_{\delta_\lambda} \frac{dx}{y} = 2 \int_{\delta_\lambda} \frac{d\lambda}{y} = 4 \int_{\delta_\lambda} \frac{dx}{y}$$

which is a single valued analytic function on $0 \leq \arg(z-1) < 2\pi$. Since $\delta_\lambda$ is the vanishing cycle, $f(\lambda)$ is holomorphic at $\lambda = 0$. Moreover it satisfies the hypergeometric differential equation (5.3). Therefore $f(\lambda) = c F_{\frac{1}{4}, \frac{3}{4}}(\lambda)$ with some constant $c$. Since

$$f(0) = \lim_{\lambda \to +0} 4 \int_{\frac{1}{4}+\sqrt{\lambda}}^{\frac{1}{4}+\sqrt{\lambda}} \frac{dx}{y} = \pm 2\pi i,$$

one concludes $f(\lambda) = \pm 2\pi i F_{\frac{1}{4}, \frac{3}{4}}(\lambda)$. Hence we have

$$\frac{1}{2\pi i} \langle \text{reg}_R(\xi), u^- \rangle = \text{Re} \left( \frac{1}{2\pi i} \langle \text{reg}_R(\xi), \delta_\lambda \rangle \right) = \pm \text{Re} \left[ C + \log \lambda + 3\lambda \frac{1}{16} F_3 \left( \frac{5}{4}, \frac{7}{4}, 1, 1; 2, 2, \lambda \right) \right]$$

with some constant $C$. One can determine $C$ by a similar argument to the proof of [A1, Theorem 3.2]. We thus have (5.7).

We may take

$$R_\infty(X_\lambda, \xi) = \text{Re} \left[ -\log 64 + \log \lambda + \frac{3\lambda}{16} F_3 \left( \frac{5}{4}, \frac{7}{4}, 1, 1; 2, 2, \lambda \right) \right]$$

by replacing $u^-$ with $-u^-$ if necessary. Here is the table of numerical computations (of precision 20) by MATHEMATICA for $F_3$ and MAGMA for $L$-values:

| $a$ | $R_\infty(X_\lambda, \xi)$ | $L'(X_\lambda, 0)$ | $R_\infty/L'(X_\lambda, 0)$ |
|-----|---------------------------|---------------------|----------------------------|
| 2   | -3.01582754200021         | 3.01582754200021    | -1                         |
| 4   | -2.47055987085139         | 1.23527993542569    | -2                         |
| 8   | -2.04611026378501         | -4.09222052757003   | 0.5                        |

We thus have numerical verifications on the $p$-adic Beilinson conjecture for $K_2$ of the above elliptic curves.

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Remark 5.1 Let \( z_{BK} \in \lim_{\longleftarrow n} K_2(Y(Np^n))_\mathbb{Q} \) be the Beilinson-Kato element. We denote by \( z_{BK}(2) \in K_2(E)_\mathbb{Q} \) its image for \( E \) an elliptic curve of conductor \( N \). Brunault\(^3\) showed that if \( E \) does not have CM, then

\[
R_{p,\alpha}(E, z_{BK}(2)) = \prod_{l \mid N} (1 - a_l(E)) \frac{L(E, 1)\Omega^2}{\iota(f, f)} L_{p,\alpha}(E, \omega^{-1}, 0). \tag{5.9}
\]

In particular the \( p \)-adic regulator vanishes when \( a_l(E) = 1 \) for some \( l \mid N \) or \( L(E, 1) = 0 \) (he further expects \( z_{BK}(2) = 0 \), loc.cit. Question 12). Concerning \( y^2 = x^3 - 2x^2 + (1 - a)x \), if \( a = 4 \) then \( a_3(E) = 1 \) and if \( a = 8 \) then \( L(E, 1) = 0 \). In these cases one cannot use \( z_{BK} \) to discuss the \( p \)-adic Beilinson conjecture for \( K_2 \). Instead we use the symbol \( \xi \) in (5.1), and obtain the numerical verifications.

5.2 Legendre family \( y^2 = x(1 - x)(1 - (1 - \lambda)x) \)

Next example is the Legendre family given by a Weierstrass equation \( y_0^2 = x_0(1 - x_0)(1 - (1 - \lambda)x_0) \). Let \( p \geq 5 \) be a prime. Replace the variables \( (x, y) = (x_0^{-1} + (\lambda - 2)/3, 2yx_0^{-2}) \).

We have a family

\[
f : Y \longrightarrow \mathbb{P}^1
\]

of elliptic curves over \( \mathbb{Z}_p = \mathbb{Z}_p \cap \mathbb{Q} \) whose generic fiber is given by a Weierstrass equation

\[
y^2 = 4x^3 - g_2x - g_3, \quad g_2 := \frac{4}{3}(\lambda^2 - \lambda + 1), \quad g_3 := -\frac{4}{27}(\lambda + 1)(\lambda - 2)(2\lambda - 1).
\]

Put

\[
\omega := \frac{dx}{y} = -\frac{dx_0}{2y}; \quad \eta := \frac{xdx}{y} = -\left(\frac{x_0^{-1} + \frac{\lambda - 2}{3}}{2y} \right) \frac{dx_0}{2y_0} = -\left(1 - \lambda\right)x_0 + \frac{\lambda - 2}{3} \right) \frac{dx_0}{2y_0}.
\]

One can check that \( f \) satisfies all the conditions \( A_1, \ldots, A_3 \) in §4.1 and \( A_4, A_5 \) in §4.6 (detail is left to the reader). The functional \( j \)-invariant is \( 256(\lambda^2 - \lambda + 1)^3/(\lambda^2(1 - \lambda)^2) \). The singular fibers appear at \( \lambda = 0, 1, \infty \), and the Kodaira symbols are \( I_2, I_2 \) and \( I_3 \) respectively. Moreover the fiber at \( \lambda = 0 \) is split multiplicative over \( \mathbb{Z}_p \), while so is not the fiber at \( \lambda = 1 \).

Put \( S := \text{Spec}\mathbb{Z}_p[\lambda, (\lambda - \lambda^2)^{-1}] \) and \( X := f^{-1}(S) \). Let

\[
\xi = \left\{ \frac{y_0 + 1 - x_0}{y_0 - 1 + x_0}, \frac{\lambda x_0^2}{(1 - x_0^2)} \right\} \in K_2(X) \tag{5.10}
\]

\(^3\)Brunault [Br] Théorème 38 (115)] obtains a formula in terms of the composition of the étale regulator map and the Bloch-Kato exponential map

\[
\log \text{reg}_f : K_2(E) \longrightarrow H^1_{\text{dR}}(E/\mathbb{Q}_p).
\]

By [Bel] Proposition 9.11, this agrees with the syntomic regulator map, more precisely \((1 - p^{-2}\phi)\log \text{reg}_f = \text{reg}_{\text{syn}}\), where \( \phi \) is the \( p \)-th Frobenius on \( H^1_{\text{dR}}(E/\mathbb{Q}_p) \cong H^1_{\text{cryst}}(E_{p^\flat}/\mathbb{Z}_p) \otimes \mathbb{Q} \) (see also §4.2). Hence one has \( \text{Tr}(\text{reg}_{\text{syn}}(z) \cup v_0) = (1 - p^{-1}\alpha^{-1})\text{Tr}(\log \text{reg}_f(z) \cup v_0) \) and then [Br Théorème 38 (115)] reads (5.9).
be a symbol (cf. [A2, (4.52)]). One immediately has
\[ d\log(\xi) = -\frac{d\lambda}{\lambda} \frac{dx_0}{y_0} = 2 \lambda \frac{d\lambda}{dx} y_0. \]

Put \( \Delta := g_2^3 - 27 g_3^2 \), and
\[ F_{\frac{1}{2}, \frac{1}{2}}(\lambda) = 2 F_1 \left( \frac{1}{2}, \frac{1}{2}; \lambda \right). \]

The notation in §4.7 in our case is explicitly given as follows.

(i) \((a_0, b_0) = (a_1, b_1) = (0, 0) \) and \((a_\infty, b_\infty) = (0, 1) \).

(ii) \( n = -\text{ord}_\lambda(J) = 2 \), \( \kappa = 1/256 \).

(iii) One has \( c_0^2 = -g_2(0)/(18g_3(0)) = 1/4 \). We take \( c_0 = 1/2 \).

(iv) One can show
\[ F(\lambda) = \frac{1}{2} F_{\frac{1}{2}, \frac{1}{2}}(\lambda) \]
in the same way as in §5.1(iv).

(v) The multiplicative period is \( q = 256^{-1}l^2 \exp(\tau(\lambda)) \) with
\[ \lambda \frac{d}{d\lambda} \tau(\lambda) = -2 + 2(1 - \lambda)^{-1} F_{\frac{1}{2}, \frac{1}{2}}(\lambda)^{-2}, \quad \tau(0) = 0. \]

Explicitly,
\[ \tau(\lambda) = -\lambda - \frac{13}{32} \lambda^2 - \frac{23}{96} \lambda^3 - \frac{2701}{16384} \lambda^4 + \ldots. \]

(vi) \( \tau^{(\sigma)}(\lambda) = -p^{-1} \log(256^{p-1}c^2) + \tau(\lambda) - p^{-1} \tau(\lambda)^{\sigma}. \)

(vii) By (5.2), one has \( g_\xi(\lambda) = 2 \) and \( G_\xi(\lambda) = 2F(\lambda) = F_{\frac{1}{2}, \frac{1}{2}}(\lambda) \) (hence \( l = 1 \)).

(viii)
\[ H(\lambda) = (\lambda - \lambda^2) F'_{\frac{1}{2}, \frac{1}{2}}(\lambda) + \frac{1}{6} (1 - 2\lambda) F_{\frac{1}{2}, \frac{1}{2}}(\lambda) \]
\[ = \frac{1}{6} - \frac{1}{24} \lambda - \frac{11}{384} \lambda^2 - \frac{29}{1536} \lambda^3 - \frac{1375}{98304} \lambda^4 + \ldots. \]

(ix) Let \( E_1^{(\sigma)}(\lambda) \in \mathbb{Z}_p[[\lambda]] \) be defined by
\[ (E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^{\sigma}}{\lambda}, \quad E_1^{(\sigma)}(0) := -p^{-1} \log(16^{p-1}c). \]
Let $E_2^{(\sigma)}(\lambda) = C + a_1 \lambda + \cdots$ be the power series satisfying
\[
\frac{d}{d\lambda} E_2^{(\sigma)}(\lambda) = -E_1^{(\sigma)}(\lambda) \left( \frac{2}{\lambda} + \tau'(\lambda) \right) + \frac{G\xi(\lambda)^\sigma}{\lambda} \tau^{(\sigma)}(\lambda)
\]
\[
= -2 \frac{E_1^{(\sigma)}(\lambda)}{\lambda} \left( (1 - \lambda)^{-1} F_{\frac{1}{2}, \frac{1}{2}}(\lambda)^{-2} \right) + \frac{G\xi(\lambda)^\sigma}{\lambda} \tau^{(\sigma)}(\lambda).
\]
This determines $E_2^{(\sigma)}(\lambda)$ except the constant term $C$. We expect
\[
C = \frac{1}{4} (p^{-1} \log(256p^{-1}c^2))^2 = - (p^{-1} \log(16p^{-1}c))^2
\]
according to Conjecture 4.7 while it does not matter toward numerical computation thanks to the method in Step 2 in §4.7. Let $X_a$ be the fiber at $\lambda = a$. Let $\sigma_a(\lambda) = a^{-1-\lambda}$, and
\[
\text{reg}_{\text{syn}}(\xi|_{\lambda=a}) = \varepsilon_1^{(\sigma_a)}(a) \frac{dx}{y} + \varepsilon_2^{(\sigma_a)}(a) \frac{dx}{y}.
\]
We now discuss a particular case $a = 4$. In this case, $X_4$ is isogenous to $X_0(24)$ and the Néron differential $\omega_{X_4}$ is $dx/y$.

**Theorem 5.2** Let $p$ be a prime at which $X_4$ has a good reduction (including a supersingular reduction). Then Conjecture 3.3 for $X_4$ is true in case $n = 0$ and $\text{ord}_p(\gamma) < 1$.

**Proof.** Noticing
\[
\lim_{\lambda \to +0} \lambda \frac{d}{d\lambda} \langle \text{reg}_{\text{syn}}(\xi), u^- \rangle = 2 \lim_{\lambda \to +0} \int_1^{(1-\lambda)^{-1}} \frac{dx_0}{y_0} = 2\pi i
\]
on one can show that the Beilinson regulator is
\[
R_{\infty}(X_a, \xi) = \frac{1}{2\pi i} \langle \text{reg}_{\text{syn}}(\xi), u^- \rangle = \text{Re} \left[ -\log 16 + \log a + \frac{a}{4} F_3 \left( \frac{3}{2}, \frac{3}{2}, 1, 1, 1 \right) \right]
\]
in the same way as in §5.1 (the orientation of $u^-$ is chosen suitably). By [A1] Lemma 3.5, this is replaced with
\[
-2 \text{Re} \left[ a^{-\frac{1}{2}} F_2 \left( \frac{1}{2}, \frac{1}{2} ; 1, \frac{1}{2} ; \frac{1}{a} \right) \right]
\]
when $a \in \mathbb{R}_{>0}$. A formula of Rogers and Zudilin ([RZ] Theorems 2, p.399 and (6), p.386) yields
\[
\frac{\pi^2}{12} F_2 \left( \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{1}{4} \right) = L(E_{24}, 2) = \frac{\pi^2}{6} L'(E_{24}, 0).
\]
We thus have
\[
R_{\infty}(X_4, \xi)/L'(X_4, 0) = -2.
\]
On the other hand, Theorem 7.6 in Appendix B yields that $\xi$ agrees with the Beilinson-Kato element $4z_{\text{BK}, X_4}$. Since $X_4$ does not have CM, one can apply Brunault’s formula (5.9) ([Br] Théorème 38 (115)). We thus have
\[
R_{p, \alpha}(X_4, \xi)/L_{p, \alpha}(X_4, \omega^{-1}, 0) = -2.
\]
Hence Conjecture 3.3 for $n = 0$ and $\gamma = \alpha$ follows. □
Remark 5.3 \((5.12)\) also gives an example of \([A2, \text{Conjecture 4.30}]\),

\[
(1 - p\alpha^{-1})\mathcal{F}_{\frac{1}{2}, \frac{3}{2}}(t)|_{t=4} = -L_{p, \alpha}(X_4, \omega^{-1}, 0)
\]

for \(p\) such that \(X_4\) has a good ordinary reduction.

We also discuss the case \(\gamma = \beta\), namely \(p\)-adic regulators and the critical slope \(p\)-adic \(L\)-values. In the same way as in \(\S 5.1\), one has the table:

| \(p\) | \(\varepsilon_1^{(\sigma_\alpha)}(a)|_{a=4}\) | \(\varepsilon_2^{(\sigma_\alpha)}(a)|_{a=4}\) |
|------|-----------------|-----------------|
| 5    | 14203707162(p^{13}) | 2689502325(p^{15}) |
| 7    | 3834481697564(p^{15}) | 1564968189963(p^{15}) |
| 11   | 19820018884(p^{10}) | 8395655104(p^{10}) |
| 13   | 11468932902(p^{10}) | 10787680898(p^{10}) |
| 17   | 159074202(p^{8}) | 5132326047(p^{8}) |

The \(p\)-adic regulators

\[
R_{p, \beta}(X_4, \xi) = (1 - p\beta^{-1})\frac{\text{Tr}(\text{reg}_{\text{sym}}(\xi|_{\lambda=4}) \cup v_\beta)}{\text{Tr}(\omega_{X_4} \cup v_\beta)}
\]

are as follows:

| \(p\) | \(R_{p, \beta}(X_4, \xi)\) |
|------|-----------------|
| 5    | 2763422021(p^{13}) |
| 7    | (supersingular) |
| 11   | 11^{-1} \cdot 114484571(p^{8}) |
| 13   | 13^{-1} \cdot 12923433(p^{8}) |
| 17   | 17^{-1} \cdot 23698690(p^{8}) |

On the other hand, \(L_{5, \beta}(X_a, \omega^{-1}, 0) = 305462 \mod 5^8\) and \(L_{11, \beta}(E, \omega, 0) = 1453013467/11 \mod 11^8\) (computed by R. Pollack). This indicates

\[
R_{p, \beta}(X_4, \xi)/L_{p, \beta}(X_4, \omega^{-1}, 0) = -2
\]

as desired.

6 Appendix A: Gauss-Manin connection for elliptic fibrations

For a smooth scheme \(Y\) over \(T\), we denote by \(\Omega^q_{Y/T} = \bigwedge^q_\mathcal{O}_Y \Omega^1_{Y/T}\) the sheaf of relative differential \(q\)-forms on \(Y\) over \(T\).

Let \(K\) be a field of characteristic zero. Let \(U/S\) be a smooth projective family of curves over the base field \(K\). Let

\[
\nabla : R^1f_*\Omega^*_{U/S} \longrightarrow \Omega^1_S \otimes R^1f_*\Omega^*_{U/S}
\]
be the Gauss-Manin connection, where we simply write \( \Omega^q_S = \Omega^q_{S/K} \). This is defined to be the connecting homomorphism

\[
R^1 f_* \Omega^*_{U/S} \to R^2 f_* (f^* \Omega^1_S \otimes \Omega^*_{U/S}) \cong \Omega^1_S \otimes R^2 f_* \Omega^*_{U/S} \cong \Omega^1_S \otimes R^1 f_* \Omega^*_{U/S}
\]

which arises from an exact sequence

\[
0 \to f^* \Omega^1_S \otimes \Omega^*_{U/S} \to \Omega^*_U \to \Omega^*_{U/S} \to 0, \quad \Omega^*_U := \Omega^*_U / \operatorname{Im}(f^* \Omega^2_S \otimes \Omega^*_{U/S})
\]

(cf. [Ha3] Ch.III, §4). Here the first isomorphism in (6.1) is the projection formula, and the second one is due to the identification \( R^q f_* \Omega^*_{U/S} \cong R^{q-1} f_* \Omega^*_{U/S} \) with which we should be careful about “sign”. Indeed the differential of the complex \( \Omega^*_{U/S} \) is “\(-d\)"

\[
\Omega^*_{U/S} : \mathcal{O}_U \to \Omega^1_{U/S}
\]

so that we need to arrange the sign to make an isomorphism between \( R^q f_* \Omega^*_{U/S} \) and \( R^{q+1} f_* \Omega^*_{U/S} \). We make it by a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_U & \xrightarrow{d} & \Omega^1_{U/S} \\
-\text{id} & \downarrow & \text{id} \\
\mathcal{O}_U & \xrightarrow{d} & \Omega^1_{U/S}.
\end{array}
\]

Then \( \nabla \) satisfies the usual Leibniz rule

\[
\nabla(gx) = dg \otimes x + g \nabla(x), \quad x \in \Gamma(S, R^1 f_* \Omega^*_{U/S}), \ g \in \mathcal{O}(S).
\]

### 6.1 Family of hyperelliptic curves

Let \( S \) be an irreducible affine smooth variety over \( K \). Let \( f(x) \in \mathcal{O}_S(S)[x] \) be a polynomial of degree \( 2g + 1 \) or \( 2g + 2 \) which has no multiple roots over any geometric points \( \bar{x} \in S \). Then it defines a smooth family of hyperelliptic curves \( f : U \to S \) defined by the Weierstrass equation \( y^2 = f(x) \). To be more precise, let \( z = 1/x, \ u = y/x^{g+1} \) and put \( g(z) = z^{2g+2} f(1/z) \). Let

\[
U_0 = \operatorname{Spec} \mathcal{O}_S(S)[x, y]/(y^2 - f(x)), \quad U_{\infty} = \operatorname{Spec} \mathcal{O}_S(S)[z, u]/(u^2 - g(z)).
\]

Then \( U \) is obtained by gluing \( U_0 \) and \( U_{\infty} \) via identification \( z = 1/x, \ u = y/x^{g+1} \). We assume that there is a section \( e : S \to U \). Then

\[
x^{i-1} \frac{dx}{y} = -z^{g-i} \frac{dz}{u}, \quad \frac{y}{x^i} = \frac{u}{z^{g+1-i}}, \quad 1 \leq i \leq g.
\]

We shall compute the Gauss-Manin connection

\[
\nabla : H^1_{\text{dR}}(U/S) \to \Omega^1_S \otimes H^1_{\text{dR}}(U/S), \quad H^q_{\text{dR}}(U/S) := H^q_{\text{zar}}(U, \Omega^*_{U/S})
\]

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(we use the same symbol “$\Omega^1_S$” for $\Gamma(S, \Omega^1_S)$, since it will be clear from the context which is meant). To do this, we describe the de Rham cohomology in terms of the Čech complex. Write
\[
\check{C}^0(\mathcal{F}) := \Gamma(U_0, \mathcal{F}) \oplus \Gamma(U_\infty, \mathcal{F}), \quad \check{C}^1(\mathcal{F}) := \Gamma(U_0 \cap U_\infty, \mathcal{F})
\]
for a (Zariski) sheaf $\mathcal{F}$. Then the double complex
\[
\begin{array}{ccc}
\check{C}^0(\mathcal{O}_U) & \longrightarrow & \check{C}^0(\Omega^1_{U/S}) \\
\delta & & \downarrow \\
\check{C}^1(\mathcal{O}_U) & \longrightarrow & \check{C}^1(\Omega^1_{U/S})
\end{array}
\]
gives rise to the total complex
\[
\check{C}^\bullet(U/S) : \check{C}^0(\mathcal{O}_U) \xrightarrow{\partial} \check{C}^0(\Omega^1_{U/S}) \xrightarrow{\delta} \check{C}^1(\mathcal{O}_U) \xrightarrow{\partial} \check{C}^1(\Omega^1_{U/S})
\]
of $R$-modules starting from degree 0, and the cohomology of it is the de Rham cohomology $H^\bullet_{\text{dR}}(U/S)$:
\[
H^q_{\text{dR}}(U/S) = H^q(\check{C}^\bullet(U/S)), \quad q \geq 0.
\]
Elements of $H^1_{\text{dR}}(U/S)$ are represented by cocycles
\[
(f) \times (x_0, x_\infty) \text{ with } df = x_0 - x_\infty.
\]

**Lemma 6.1** Suppose
\[
f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathcal{O}(S)
\]
with $n = 2g + 1$ or $2g + 2$. Put
\[
\omega_i := (0) \times \left( \frac{x^{i-1}dx}{y} - \frac{z^{g-i}dz}{u} \right), \quad (6.5)
\]
\[
\omega^*_i := \left( \frac{y}{x^i} \right) \times \left( \sum_{m > i} (m/2 - i)a_m x^{m-i-1} \right) \frac{dx}{y} - \left( \sum_{m \leq i} (m/2 - i)a_m z^{g-m+i} \right) \frac{dz}{u}, \quad (6.6)
\]
for $1 \leq i \leq g$. Then they give a basis of $H^1_{\text{dR}}(U/S)$. Moreover $6.5$ span the image of $\Gamma(U, \Omega^1_{U/S}) \hookrightarrow H^1_{\text{dR}}(U/S)$.

**Proof.** Exercise.

**Lemma 6.2** There are the following equivalence relations:
\[
\begin{align*}
(x^iy^i) \times (0, 0) & \equiv (0) \times (-d(x^iy^i), 0) \mod \text{Im} \check{C}^0(\mathcal{O}_U), \\
(z^iu^i) \times (0, 0) & \equiv (0) \times (0, d(z^iu^i)) \mod \text{Im} \check{C}^0(\mathcal{O}_U).
\end{align*}
\]

**Proof.** Straightforward from the definition.
6.2 Computation of Gauss-Manin connection

Let us compute $\nabla(\omega_i)$ and $\nabla(\omega^*_i)$. Recall that there is the exact sequence

$$0 \rightarrow \hat{C}^*(f^*\Omega^1_S \otimes \Omega^{* -1}_{U/S}) \rightarrow \hat{C}^*(\Omega^1_U) \rightarrow \hat{C}^*(\Omega^{*}_{U/S}) \rightarrow 0 \quad (6.7)$$

and it gives rise to the connecting homomorphism

$$\delta : H^1_{\text{dR}}(U/S) = H^1(\hat{C}^*(\Omega^{*}_{U/S})) \rightarrow H^2(U, f^*\Omega^1_S \otimes \Omega^{* -1}_{U/S}) = H^2(\hat{C}^*(f^*\Omega^1_S \otimes \Omega^{*}_{U/S})).$$

Recall the isomorphism

$$\hat{C}^*(f^*\Omega^1_S \otimes \Omega^{* -1}_{U/S}) \xrightarrow{\cong} \Omega^1_S \otimes \hat{C}^*(\Omega^{*}_{U/S})$$

induced from (6.2). It induces the isomorphism

$$\iota : H^2(U, f^*\Omega^1_S \otimes \Omega^{* -1}_{U/S}) \xrightarrow{\cong} \Omega^1_S \otimes H^1_{\text{dR}}(U/S).$$

By definition we have $\nabla = \iota \delta$ the Gauss-Manin connection (6.4). Let us write down the maps $\delta$ and $\iota$ in terms of Cech cocycles. The differential operator $\mathcal{D}$ on the total complex of the middle term of (6.7) is given as follows

$$\mathcal{D} : \hat{C}^1(\mathcal{O}_U) \times \hat{C}^0(\Omega^1_U) \rightarrow \hat{C}^1(\Omega^1_U) \times \hat{C}^0(\Omega^2_U),$$

$$(\alpha) \times (\beta_0, \beta_\infty) \longmapsto (-d\alpha + \beta_0 - \beta_\infty) \times (d\beta_0, d\beta_\infty).$$

We denote a lifting of $(z_0, \hat{z}_\infty) \in \hat{C}^0(\Omega^1_U)$ by $(\hat{z}_0, \hat{z}_\infty) \in \hat{C}^0(\Omega^1_U)$. Then for $(\alpha) \times (z_0, \hat{z}_\infty) \in H^1_{\text{dR}}(U/S)$ one has

$$(\alpha) \times (z_0, \hat{z}_\infty) \xrightarrow{\delta} \mathcal{D}((\alpha) \times (\hat{z}_0, \hat{z}_\infty)) = (-d\alpha + \hat{z}_0 - \hat{z}_\infty) \times (d\hat{z}_0, d\hat{z}_\infty) \quad (6.8)$$

$$\in \hat{C}^1(f^*\Omega^1_S) \times \hat{C}^0(f^*\Omega^1_S \otimes \Omega^{*}_{U/S}) \quad (6.9)$$

The isomorphism $\iota$ is given by

$$(g dt) \times (dt \wedge z_0, dt \wedge \hat{z}_\infty) \longmapsto dt \otimes [(-g) \times (z_0, \hat{z}_\infty)] \quad (6.10)$$

(the “sign” appears in the above due to (6.2)).

To compute $\nabla(\omega_i)$ and $\nabla(\omega^*_i)$ for the basis in Lemma 6.1, there remains to compute lifting of $dx/y$ and $dz/u$.

**Lemma 6.3** Let $A, B \in \mathcal{O}(S)[x]$ and $C, D \in \mathcal{O}(S)[z]$ satisfying

$$Af + B \frac{\partial f}{\partial x} = 1, \quad Cg + D \frac{\partial g}{\partial z} = 1.$$

Put differential 1-forms

$$\frac{\widehat{dx}}{y} := \frac{Af dx + Bdf}{y} = Aydx + 2Bdy \in \Gamma(U_0, \Omega^1_U),$$

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\[
\frac{dz}{u} := Cgdz + Ddg = Cudz + 2Ddu \in \Gamma(U_\infty, \Omega_U^1).
\]

Then
\[
x^i \frac{dx}{y} \in \Gamma(U_0, \Omega_U^1), \quad z^i \frac{dz}{u} \in \Gamma(U_\infty, \Omega_U^1)
\]
are liftings of \(x^i dx/y \in \Gamma(U_0, \Omega_{U/S}^1)\) and \(z^i dz/u \in \Gamma(U_\infty, \Omega_{U/S}^1)\) respectively.

**Proof.** Straightforward. \(\square\)

By using the liftings in Lemma 6.3, one can compute the map \(\delta\). With use of Lemma 6.2, one finally obtains the connection matrix of \(\nabla\).

Here is an explicit formula in case of elliptic fibration.

**Theorem 6.4** Let \(S\) be a smooth affine curve and \(f : U \to S\) a projective smooth family of elliptic curves whose affine form is given by a Weierstrass equation \(y^2 = 4x^3 - g_2x - g_3\) with \(\Delta := g_3^2 - 27g_2^3 \in \mathcal{O}_S(S)^\times\). Suppose that \(\Omega_S^1\) is a free \(\mathcal{O}_S\)-module with a base \(d\lambda \in \Gamma(S, \Omega_S^1)\).

For \(f \in \mathcal{O}_S(S)\), we define \(f' \in \mathcal{O}_S(S)\) by
\[
df = f' d\lambda.
\]

Let
\[
\omega := \left(0\right) \times \left(\frac{dx}{y}, -\frac{dz}{u}\right), \quad \eta := \left(\frac{y}{2x}\right) \times \left(\frac{g_2z + 2g_3z^2}{4u}dz\right).
\]

be elements in \(H^1_{dR}(U/S)\). Then we have
\[
\nabla \omega = -\frac{\Delta'}{12\Delta} d\lambda \otimes \omega + \frac{3}{2\Delta} (2g_2g_3' - 3g_2'g_3) d\lambda \otimes \eta,
\]
\[
\nabla \eta = -\frac{g_2(2g_2g_3' - 3g_2'g_3)}{8\Delta} d\lambda \otimes \omega + \frac{\Delta'}{12\Delta} d\lambda \otimes \eta.
\]

**Proposition 6.5** Let \(g_2, g_3 \in K[[\lambda]]\) be power series such that \(g_2(0) \neq 0, g_3(0) \neq 0\) and \(\Delta = g_3^2 - 27g_2^3 \in \lambda K[[\lambda]]\). Let \(f : U \to S = \text{Spec} K[[\lambda]]\) be the family of elliptic curves given by an affine equation
\[
y^2 = 4x^3 - g_2x - g_3
\]
with semistable reduction at \(\lambda = 0\). Let \((12g_2)^{-1/4} = c_0 + c_1\lambda + \cdots \in K[[\lambda]]\) denote the power series such that the initial term satisfies \(c_0^2 = -18^{-1}g_2(0)g_3(0)^{-1}\). Let \(F(\lambda)\) be defined by
\[
\rho^* \omega = F(\lambda) \frac{du}{u}.
\]

Then
\[
F(\lambda) = (12g_2)^{-\frac{1}{4}}F_1\left(\frac{1}{12}, \frac{5}{12}; J^{-1}\right), \quad \text{where} \ J := g_2^3/(g_2^3 - 27g_3^2). \tag{6.14}
\]

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Proof. We may assume $K = \mathbb{C}$. Denote by $U^{an}_\lambda = f^{-1}(\lambda)$ the analytic torus for $|\lambda| \ll 1$. Let $\delta_\lambda \in H_1(U^{an}_\lambda, \mathbb{Z})$ be the vanishing cycle. Then

$$F(\lambda) = \frac{1}{2\pi i} \int_{\delta_\lambda} \omega.$$ 

This is the unique solution of the Picard-Fuchs equation \([SB, (1.3)]\), and then \([6.14]\) is proven in \([SB, (1.5)]\). \qed

**Proposition 6.6** Let the notation be as in Proposition 6.5. Put

$$H(\lambda) := \frac{2\Delta}{3(2g_2g_3' - 3g'_2g_3)} \left( F'(\lambda) + \frac{\Delta'}{12\Delta} F(\lambda) \right)$$

and

$$\widehat{\omega} := \frac{1}{F(\lambda)} \omega, \quad \widehat{\eta} := -H(\lambda) \omega + F(\lambda) \eta$$

(\(\Longleftrightarrow \omega = F(\lambda) \widehat{\omega}, \ \eta = H(\lambda) \widehat{\omega} + \frac{1}{F(\lambda)} \widehat{\eta}\)).

Then \(\{\widehat{\omega}, \widehat{\eta}\}\) forms a de Rham symplectic basis which satisfies

$$\nabla(\widehat{\omega}) = \frac{dq}{q} \otimes \widehat{\eta},$$

where \(q = \kappa \lambda^n + \cdots \in \lambda K[[\lambda]]\) is the multiplicative period.

Proof. By (6.12) and (6.13), it is straightforward to see that

$$\nabla(\widehat{\eta}) = 0, \quad \nabla(\widehat{\omega}) = \frac{3(2g_2g_3' - 3g'_2g_3)}{2\Delta F(\lambda)^2} d\lambda \otimes \widehat{\eta}.$$ 

Since

$$\nabla(\omega \cup \eta) = \nabla(\omega) \cup \eta + \omega \cup \nabla(\eta) = 0,$$

\(\widehat{\omega} \cup \widehat{\eta} = C \neq 0\) is a nonzero constant. Hence \(\{\widehat{\omega}, C^{-1} \widehat{\eta}\}\) forms a de Rham symplectic basis. It follows from Proposition 4.1 that one has

$$\frac{dq}{q} = C \frac{1}{F(\lambda)^2} \frac{3(2g_2g_3' - 3g'_2g_3)}{2\Delta} d\lambda = -C \frac{3g_2g_3}{2F(\lambda)^2\Delta} \left( \frac{g'_2}{g_2} - 2 \frac{g'_3}{g_3} \right) d\lambda.$$ 

(6.16)

Recall the formula (6.14)

$$F(\lambda) = (12g_2)^{-\frac{1}{4}} F_1 \left( \frac{1}{12}, \frac{5}{12}; J^{-1} \right) = c_0 + c_1 \lambda + \cdots, \quad J := \frac{g_3^2}{g_2} (g_2 - 27 g_3^2)$$

with \(c_0 = -18^{-1} g_2(0) g_3(0)^{-1}\). Take the residue of (6.16) at \(\lambda = 0\). Since

$$d \log(g_3^{-2} \Delta + 27) = d \log \frac{g_3^2}{g_3^3} = \left( \frac{g'_2}{g_2} - 2 \frac{g'_3}{g_3} \right) d\lambda = \left( \frac{cng_3(0)^{-2}}{27} \lambda^{n-1} + \text{(higher terms)} \right) d\lambda$$

where \(\Delta = c\lambda^n + \cdots\) with \(c \neq 0\), one has

$$-C \frac{3g_2(0)g_3(0)n g_3(0)^{-2}}{2c_0^2} = n \quad \Longleftrightarrow \quad C = 1.$$ 

\[\Box\]
In the above proof, we showed the following.

**Proposition 6.7** Let \( \tau(\lambda) \in \lambda K[\lambda] \) be defined by \( q = \kappa \lambda^n \exp(\tau(\lambda)) \). Then
\[
\frac{dq}{q} = \frac{3(2g_2g_3 - 3g'_2g_3)}{2\Delta F(\lambda)^2} d\lambda, \quad \frac{d\tau}{d\lambda} = \frac{3(2g_2g_3 - 3g'_2g_3)}{2\Delta F(\lambda)^2} - \frac{n}{\lambda}.
\] (6.17)

### 6.3 Deligne’s canonical extension

Let \( \overline{S} \) be a smooth curve over \( \mathbb{C} \), and \( j : S := \overline{S} - \{ P \} \hookrightarrow \overline{S} \) an inclusion. Let \((\mathcal{H}, \nabla)\) be a vector bundle with integrable connection over \( S \). Then there is unique subbundle \( \mathcal{H}_e \subset j^* \mathcal{H} \) satisfying the following conditions (cf. [Z] (17)).

- The connection extends to have log pole, \( \nabla : \mathcal{H}_e \to \Omega^1_S(\log P) \otimes \mathcal{H}_e \),
- each eigenvalue \( \alpha \) of \( \text{Res}_P(\nabla) \) satisfies \( 0 \leq \text{Re}(\alpha) < 1 \).

The extended bundle \((\mathcal{H}_e, \nabla)\) is called Deligne’s canonical extension. The inclusion map
\[
[\mathcal{H}_e \to \Omega^1_S(\log P) \otimes \mathcal{H}_e] \longrightarrow [j_* \mathcal{H} \to \Omega^1_S \otimes j_* \mathcal{H}]
\]
is a quasi-isomorphism of complexes of sheaves. Besides \( \exp(-2\pi i \text{Res}_P(\nabla)) \) coincides with the monodromy operator on \( H_C = \text{Ker}(\nabla^{en}) \) around \( P \) (cf. [S], (2.21)).

Let \( f : U \to \overline{S} \) be a projective flat morphism of nonsingular algebraic varieties such that \( f \) is smooth over \( S \) and \( D = f^{-1}(P) \) is a NCD in \( \overline{U} \). Let \( U = f^{-1}(S) \). Let \((\mathcal{H}, \nabla)\) be the higher direct image \((R^i f_* \Omega^i_U, \nabla)\). Then all eigenvalues of \( \text{Res}_P(\nabla) \) are rational numbers. There is the natural isomorphism
\[
R^i f_* \Omega^i_U |_{\overline{S}}(\log D) \cong \mathcal{H}_e.
\] (6.18)

In particular, the left hand side is a locally free \( \mathcal{O}_{\overline{S}} \)-module. If \( f \) is a morphism of smooth \( K \)-schemes, then Deligne’s extension \((R^i f_* \Omega^i_U, \nabla)\) can be obviously defined over \( K \).

We have seen how to compute a connection matrix of the Gauss-Manin connection for a family of hyperelliptic curves. In case of an elliptic fibration, they are simply given as follows.

**Proposition 6.8** Let \( f : \mathcal{E} \to \text{Spec}K[[\lambda]] \) be an elliptic fibration defined by a minimal Weierstrass equation \( y^2 = 4x^3 - g_2x - g_3 \) with \( g_2, g_3 \in K[[\lambda]] \), \( \Delta := g_3^3 - 27g_2^2 \neq 0 \). Then \( \mathcal{H}_e \) has a basis \( \{ \omega, \eta \} \) (resp. \( \{ \lambda \omega, \eta \} \)) if \( f \) has a semistable or smooth reduction (resp. additive reduction).

**Proof.** Since we have Theorem [6.4], we can show the above by case-by-case analysis based on \((\text{ord}(g_2), \text{ord}(g_3))\). The detail is left to the reader. \( \square \)
6.4 The de Rham cohomology of elliptic curves over a ring

Let $U$ be a smooth scheme over a commutative ring $A$. For a point $s$ of Spec$A$, we denote the residue field by $k(s)$, and write $U_s := U \times_A k(s)$. We define the de Rham cohomology $H^i_{\text{dR}}(U/A)$ in the same way as before,

$$H^i_{\text{dR}}(U/A) := \mathbb{H}^i_{\text{zar}}(U, \Omega^\bullet_{U/A}), \quad i \in \mathbb{Z}_{\geq 0}.$$ 

**Lemma 6.9** Let $A$ be a noetherian integral domain. Let $U \to \text{Spec}A$ be a smooth projective morphism of relative dimension one with geometrically connected fibers. Then, for any $i, j, l \geq 0$, $H^i(U, \Omega^j_{U/A})$ and $H^i_{\text{dR}}(U/A)$ are locally free $A$-modules and there are the canonical isomorphisms

$$H^i(U, \Omega^j_{U/A}) \otimes_A k(s) \xrightarrow{\cong} H^i(U_s, \Omega^j_{U_s/k(s)}), \quad (6.19)$$

$$H^i_{\text{dR}}(U/A) \otimes_A k(s) \xrightarrow{\cong} H^i_{\text{dR}}(U_s/k(s)) \quad (6.20)$$

for all points $s \in \text{Spec}A$. Moreover there is a (non-canonical) isomorphism

$$H^i_{\text{dR}}(U/A) \cong H^0(\Omega^1_{U/A}) \oplus H^1(\mathcal{O}_U). \quad (6.21)$$

**Proof.** Since $\dim_{k(s)} H^i(U_s, \Omega^1_{U_s/k(s)})$ is constant with respect to $s$, one can apply [Ha1, III,12.9], so that $H^i(U, \Omega^1_{U/A})$ is a locally free $A$-module and the isomorphism $(6.19)$ follows. The natural map $H^1(\Omega^1_{U/A}) \to H^2_{\text{dR}}(U/A)$ is surjective as $H^2(\mathcal{O}_U) = 0$. There are the trace maps $\text{Tr} : H^1(\Omega^1_{U_s/k(s)}) \to k(s)$ (cf. [Ha2, VII,4.1]) and $\text{Tr} : H^2_{\text{dR}}(U_s/k(s)) \to k(s)$ (cf. [Ha3, Proposition (2.2)]) which are compatible. This implies that the map $H^1(\Omega^1_{U_s/k(s)}) \to H^2_{\text{dR}}(U_s/k(s))$ is a non-zero surjective map, and hence bijective as $\dim_{k(s)} H^1(\Omega^1_{U_s/k(s)}) = 1$. Let

$$\begin{array}{c}
H^1(\Omega^1_{U/A}) \otimes_A k(s) \xrightarrow{\text{surj.}} H^2_{\text{dR}}(U/A) \otimes_A k(s) \\
\cong \\
H^1(\Omega^1_{U_s/k(s)}) \xrightarrow{\cong} H^2_{\text{dR}}(U_s/k(s))
\end{array}$$

be a commutative diagram. Hence all arrows are bijective and all terms are one-dimensional for all points $s \in \text{Spec}A$. We thus have $(6.20)$ in case $l = 2$ and that $H^2_{\text{dR}}(U/A)$ is locally free of rank 1, since $A$ is a noetherian integral domain ([Ha1, II, 8.9]). Moreover we have an isomorphism $H^1(\Omega^1_{U/A}) \xrightarrow{\cong} H^2_{\text{dR}}(U/A)$ by Nakayama's lemma, so that we have an exact sequence

$$0 \to H^0(\Omega^1_{U/A}) \to H^1_{\text{dR}}(U/A) \to H^1(\mathcal{O}_U) \to 0. \quad (6.22)$$

Since the right term is a projective $A$-module, the sequence splits. Hence one has a (non-canonical) isomorphism $(6.21)$, and also that $H^i_{\text{dR}}(U/A)$ is a locally free $A$-module. Finally,
the isomorphism (6.20) in case $l = 1$ follows from a commutative diagram

\[
0 \longrightarrow H^0(U^1_{\lambda/A}) \otimes k(s) \longrightarrow H^1_{\text{dr}}(U/A) \otimes k(s) \longrightarrow H^1(\mathcal{O}_U) \otimes k(s) \longrightarrow 0
\]

\[
\cong \quad \cong
\]

\[
0 \longrightarrow H^0(U^1_{\lambda/k(s)}) \longrightarrow H^1_{\text{dr}}(U_s/k(s)) \longrightarrow H^1(\mathcal{O}_{U_s}) \longrightarrow 0,
\]

where the right and left isomorphisms follow from (6.19).

Let $W$ be a commutative ring of characteristic zero, and let $f : U \to S$ be a projective smooth morphism of smooth $W$-schemes. Then one can define the Gauss-Manin connection

\[
\nabla : H^1_{\text{dr}}(U/S) \longrightarrow \Omega^*_S \otimes H^1_{\text{dr}}(U/S)
\]

in the same way as before, where $\Omega^*_S = \Omega^*_{S/W}$. Moreover if $f$ extends to a projective flat family $\overline{f} : \overline{U} \to \overline{S} = S \cup \{P\}$ such that $P$ is a $W$-rational point and $D = f^{-1}(P)$ is a relative NCD, then one can also define Deligne’s canonical extension

\[(R^i\overline{f}_*\Omega^*_{U/S}(\log D), \nabla).\]

The sheaf $R^i\overline{f}_*\Omega^*_{U/S}(\log D)$ is a coherent $\mathcal{O}_{\overline{S}}$-module, but not necessarily a locally free $\mathcal{O}_{\overline{S}}$-module. It seems difficult to ask whether it is locally free or not in a general situation. The following gives a partial answer in case of elliptic fibrations.

**Proposition 6.10** Suppose that $W$ is a noetherian integral domain of characteristic zero such that 2 is invertible in $W$. Put $K := \text{Frac}W$. Let $f : \mathcal{E} \to \text{Spec}W[[\lambda]]$ be an elliptic fibration given by a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ with $g_2, g_3 \in W[[\lambda]]$, $\Delta := g_2^3 - 27g_3^2 \in W((\lambda))^\times$, such that the central fiber $D = f^{-1}(0)$ is a relative NCD. Suppose that $y^2 = 4x^3 - g_2x - g_3$ is minimal as a Weierstrass equation over $K[[\lambda]]$. Let $E := \mathcal{E} \times_{W[[\lambda]]} W((\lambda))$ and put $\omega := dx/y$ and $\eta := xdx/y$. Then

\[
H^1_{\text{dr}}(E/W((\lambda))) = W((\lambda))\omega + W((\lambda))\eta
\] (6.23)

is a free $W((\lambda))$-module of rank 2. Put

\[H := \text{Im}[H^1(\mathcal{E}, \Omega^*_{\mathcal{E}/W[[\lambda]]}(\log D)) \to H^1_{\text{dr}}(E/W((\lambda)))].\]

Let $\omega_0 = \omega = dx/y$ (resp. $\omega_0 = \lambda\omega$) if $f$ has a smooth or multiplicative reduction (resp. an additive reduction). Then

\[H \subset W[[\lambda]]\omega_0 + W[[\lambda]]\eta.\] (6.24)

The equality holds if the multiplicity of any component of $D$ is invertible in $W$.

**Proof.** Write $E_s := E \times_{W((\lambda))} k(s)$ for $s \in \text{Spec}W((\lambda))$. Then $\{dx/y|_{E_s}, xdx/y|_{E_s}\}$ is a $k(s)$-basis of $H^1_{\text{dr}}(E_s/k(s))$. Therefore it follows from Lemma 6.9 and Nakayama’s
lemma that $H^1_{dR}(E/W((\lambda)))$ is a free $W((\lambda))$-module with basis $\{\omega, \eta\}$. We show (6.24).

Write $E_K := \mathcal{E} \times_{W[[\lambda]]} K[[\lambda]]$, $E_K := E \times_{W[[\lambda]]} K[[\lambda]]$ etc. Put

$$H_K := \text{Im} [H^1(\mathcal{E}_K, \Omega^{\lambda}_{\mathcal{E}_K/K[[\lambda]]}(\log D)) \to H^1_{dR}(E_K/K((\lambda)))].$$

By Proposition 6.8, $H_K$ is a free $K[[\lambda]]$-module with basis $\{\omega_0, \eta\}$. Hence

$$H \subset \left(W((\lambda))\omega_0 + W((\lambda))\eta\right) \cap H_K = W[[\lambda]]\omega_0 + W[[\lambda]]\eta$$

as required. The last statement is more delicate. Recall the descriptions (6.12) and (6.13)

$$\omega = (0) \times \left(\lambda^e \frac{dx}{y}, -\lambda^e \frac{dz}{u}\right), \quad \eta = \left(\frac{y}{2x}\right) \times \left(\frac{xdx}{y}, \frac{(g_2 z + 2 g_3 z^2)dz}{4u}\right) \in H^1_{dR}(\mathcal{E}/W((\lambda)))$$

in terms of Cech cocycles, where we put $e = 0$ if $D$ is multiplicative and $e = 1$ if additive. Let $U_0^\circ := \text{Spec} W[[\lambda]][x, y]/(y^2 - 4x^3 + g_2 x + g_3)$ and $U_\infty^\circ := \text{Spec} W[[\lambda]][z, u]/(u^2 - 4z + g_2 z^3 + g_3 z^4)$, and $\rho : \mathcal{E} \to \mathcal{E}^*$ the blow-ups. Let $U_0 := \rho^{-1}(U_0^*)$ and $U_\infty := \rho^{-1}(U_\infty^*)$. To prove $\omega_0, \eta \in H$, we show that the above cocycles belong to

$$\Gamma(U_0 \cap U_\infty, \mathcal{E}^*) \times \Gamma(U_0, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D) \oplus \Gamma(U_\infty, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)).$$

Since $y/2x \in \Gamma(U_0 \cap U_\infty, \mathcal{E}^*)$, it is enough to show that

$$\lambda^e \frac{dx}{y} \in \Gamma(U_0, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)), \quad \lambda^e \frac{dz}{u} \in \Gamma(U_\infty, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)),$$

and

$$\frac{xdx}{y} \in \Gamma(U_0, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)), \quad \frac{(g_2 z + 2 g_3 z^2)dz}{u} \in \Gamma(U_\infty, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)).$$

We first show (6.25). We know that

$$\lambda^e \frac{dx}{y} \in \Gamma(U_0, \Omega^1_{\mathcal{E}/K[[\lambda]]}(\log D)), \quad \lambda^e \frac{dz}{u} \in \Gamma(U_\infty, \Omega^1_{\mathcal{E}/K[[\lambda]]}(\log D))$$

by Proposition 6.8. On the other hand, since $\Gamma(\mathcal{E}, \Omega^1_{\mathcal{E}/W((\lambda))})$ is a free $W((\lambda))$-module with basis $dx/y = -dz/u$, one has

$$\lambda^{e+m} \frac{dx}{y} \in \Gamma(U_0, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)), \quad \lambda^{e+m} \frac{dz}{u} \in \Gamma(U_\infty, \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D))$$

for some large integer $m$. By the assumption, the multiplicity of any component of $D$ is invertible in $W$. One sees that $\Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)$ is locally free as $f$ is locally given by $\lambda \mapsto x_1^r x_2^s$, where $r_i$ denotes the multiplicity. One can check that the map $a$ in the following diagram is injective:

$$\begin{array}{ccccc}
0 & \to & \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D) & \xrightarrow{\lambda^m} & \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D) & \to & \Omega^1_{\mathcal{E}/W[[\lambda]]}(\log D)/\lambda^m & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_{\mathcal{E}/K[[\lambda]]}(\log D) & \xrightarrow{\lambda^m} & \Omega^1_{\mathcal{E}/K[[\lambda]]}(\log D) & \to & \Omega^1_{\mathcal{E}/K[[\lambda]]}(\log D)/\lambda^m & \to & 0.
\end{array}$$
Now the diagram chase with use of (6.27) and (6.28) implies (6.25).

Next we show (6.26). It follows from (6.25) that

$$\chi \frac{dx}{y} \in \Gamma(U_0, \Omega^{1}_{\mathcal{E}/W[[\lambda]]}(\log D)), \quad \lambda^e \frac{(g_2 z + 2g_3 z^2)dz}{u} \in \Gamma(U_{\infty}, \Omega^{1}_{\mathcal{E}/W[[\lambda]]}(\log D)).$$

If we show

$$\frac{dx}{y} \in \Gamma(U_{0,K}, \Omega^{1}_{\mathcal{E}_K/K[[\lambda]]}(\log D_K)), \quad \frac{(g_2 z + 2g_3 z^2)dz}{u} \in \Gamma(U_{\infty,K}, \Omega^{1}_{\mathcal{E}_K/K[[\lambda]]}(\log D_K)),$$

then, in the same way as above, one can show (6.26) by the diagram chase, which completes the proof of Proposition 6.10. Since $U_{\infty} \to \text{Spec} W[[\lambda]]$ is smooth, one has $dz/u \in \Gamma(U_{\infty}, \Omega^{1}_{\mathcal{E}/W[[\lambda]]})$, and hence the latter follows. The rest is to show

$$\frac{dx}{y} \in \Gamma(U_{0,K}, \Omega^{1}_{\mathcal{E}_K/K[[\lambda]]}(\log D_K)). \tag{6.29}$$

If $e = 0$, there is nothing to prove. Suppose $e = 1$, namely $D_K$ is additive. Let $T_K$ be the singular points of $D_K$ and put $\mathcal{E}^0_K := \mathcal{E}_K \setminus T_K$. It is enough to show

$$\frac{dx}{y} \in \Gamma(U_0 \setminus T_K, \Omega^{1}_{\mathcal{E}^0_K/K[[\lambda]]}(\log D_K)) \tag{6.30}$$

as $\text{codim}(T_K) = 2$. Let $\text{Spec} K[[\lambda_0]] \to \text{Spec} K[[\lambda]]$ be given by $\lambda^n = \lambda$ and consider a cartesian diagram

$$\begin{array}{cccc}
\mathcal{E}^0_{0,K} & \xrightarrow{j} & \mathcal{E}^0_{0,K} & \xrightarrow{\tilde{f}} K[[\lambda_0]] \\
\downarrow & & \downarrow & \downarrow \\
\mathcal{E}^0_K & & K[[\lambda]] & \\
\end{array}$$

with $j$ the normalization. The morphism $\tilde{f}$ is locally given by $(u, v) \mapsto \lambda = u^{n_i}$ around a component of the central fiber, where $n_i$ is the multiplicity. Take $n = \text{lcm}(n_i)$. Then $\mathcal{E}^0_{0,K}$ is locally defined by $u^{n_i} = \lambda^n$ in $(u, v, \lambda)$-space, and hence $\mathcal{E}^0_{0,K}$ is regular and $\tilde{f}$ is smooth. Let $U_{0,K} \subset \mathcal{E}^0_{0,K}$ be the inverse image of $U_{0,K}$, and $\tilde{D}_K$ the central fiber of $\tilde{f}$. Then, to show (6.30), it is enough to show

$$\frac{dx}{y} \in \Gamma(U_{0,K}, \Omega^{1}_{\mathcal{E}^0_{0,K}/K[[\lambda_0]]}(\log \tilde{D}_K)).$$

Let $x_0 = x/\lambda_0^m$, $y_0 = y/\lambda_0^l$, $g_{2,0} = g_2/\lambda_0^{2m}$ and $g_{3,0} = g_3/\lambda_0^{3m}$ with $3m = 2l$ such that $y_0^2 = 4x_0^3 - g_{2,0}x_0 - g_{3,0}$ is minimal over $K[[\lambda_0]]$. Since the Kodaira type of $\tilde{f}$ is I$_s$ with $s \geq 0$, one has

$$\frac{dx}{y} = \lambda_0^{2m-1} \frac{x_0 dx_0}{y_0} \in \Gamma(U_{0,K}, \Omega^{1}_{\mathcal{E}^0_{0,K}/K[[\lambda_0]]}) \subset \Gamma(U_{0,K}, \Omega^{1}_{\mathcal{E}^0_{0,K}/K[[\lambda_0]]}(\log \tilde{D}_K))$$

as $2m - l > 0$. This completes the proof. \hfill \Box

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Appendix B: Comparing elements in $K_2$ of elliptic curves

by François Brunault

In this appendix, we use results of Goncharov and Levin [GL] to compare the element $\xi$ in (5.10) and the Beilinson–Kato element in $K_2$ of the elliptic curve $X_0(24)$.

7.1 Describing $K_2$ of elliptic curves

Goncharov and Levin gave an explicit description of $K_2$ of an elliptic curve using the so-called elliptic Bloch group. In this section we recall this construction and state the result needed later to compare the two elements in $K_2$ (see Theorem 7.5).

Let $E$ be an elliptic curve defined over a number field $k$. To describe Quillen’s $K$-group $K_2(E)$, our starting point is the localisation map $K_2(E) \to K_2(k(E))$, where $k(E)$ is the function field of $E$. The group $K_2(k(E))$ can be described using Matsumoto’s theorem: for any field $F$, we have an isomorphism

$$K_2(F) \cong \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle x \otimes (1 - x) : x \in F \setminus \{0, 1\} \rangle}.$$

The class of $x \otimes y$ in $K_2(F)$ is denoted by $\{x, y\}$ and is called a Milnor symbol. The relations $\{x, 1 - x\} = 0$ are called the Steinberg relations.

Let $\mathbb{Z}[E(k)]$ be the group algebra of $E(k)$. Consider the Bloch map

$$\beta : \overline{k}(E)^\times \times \overline{k}(E)^\times \to \mathbb{Z}[E(k)]$$

$$(f, g) \mapsto \sum_{i, j} m_i n_j (p_i - q_j),$$

where $\text{div}(f) = \sum_i m_i (p_i)$ and $\text{div}(g) = \sum_j n_j (q_j)$ are the divisors of $f$ and $g$. The map $\beta$ is bilinear, so it induces a map

$$\overline{k}(E)^\times \otimes_{\mathbb{Z}} \overline{k}(E)^\times \to \mathbb{Z}[E(k)],$$

which we still denote by $\beta$.

Let $I$ be the augmentation ideal of $\mathbb{Z}[E(k)]$. The group $P$ of principal divisors on $E/k$ is generated by the divisors

$$(p + q) - (p) - (q) + (0) = ((p) - (0)) \cdot ((q) - (0))$$

with $p, q \in E(k)$,

so we have $P = I^2$ and $I/I^2 \cong E(k)$. It follows that $\beta$ takes values in $I^4$, and the image of $\beta$ generates $I^4$. Following Goncharov and Levin, we now define the elliptic Bloch group of $E$.

**Definition 7.1** Let $R^*_3(E/k)$ be the subgroup of $\mathbb{Z}[E(k)]$ generated by the divisors $\beta(f, 1 - f)$ with $f \in E(k)$, $f \neq 0, 1$. The elliptic Bloch group of $E/k$ is

$$B^*_3(E/k) = \frac{I^4}{R^*_3(E/k)}.$$
The elliptic Bloch group of $E/k$ is

$$B_3^*(E) = B_3^*(E/\overline{k})^{\text{Gal}(\overline{k}/k)}.$$ 

By Matsumoto’s theorem and the definition of the elliptic Bloch group, the map $\beta$ gives rise to a commutative diagram

$$\begin{array}{ccc}
K_2(\overline{k}(E)) & \longrightarrow & B_3^*(E/\overline{k}) \\
\downarrow & & \downarrow \\
K_2(k(E)) & \longrightarrow & B_3^*(E).
\end{array}$$

Goncharov and Levin \cite{GL} proved the following fundamental result.

**Theorem 7.2** The composite map

$$K_2(E) \otimes \mathbb{Q} \longrightarrow K_2(k(E)) \otimes \mathbb{Q} \longrightarrow B_3^*(E) \otimes \mathbb{Q}$$

is injective.

In fact, Goncharov and Levin showed that modulo torsion, $K_2(E)$ is isomorphic to the kernel of an explicit map

$$\delta_3 : B_3^*(E) \longrightarrow (\overline{k}^\times \otimes E(\overline{k}))^{\text{Gal}(\overline{k}/k)}.$$ 

**Proof of Theorem 7.2** By Quillen’s localisation theorem, there is a long exact sequence

$$\cdots \longrightarrow \bigoplus_{p \in E} K_2(k(p)) \longrightarrow K_2(E) \longrightarrow K_2(k(E)) \overset{\partial}{\longrightarrow} \bigoplus_{p \in E} k(p)^\times \longrightarrow \cdots,$$

where $p$ runs over the closed points of $E$; see the exact sequence in the proof of \cite{Q} Theorem 5.4, with $i = 2$ and $p = 0$. Tensoring with $\mathbb{Q}$ and using the fact that $K_2$ of a number field is a torsion group \cite{Ga}, we get an isomorphism $K_2(E) \otimes \mathbb{Q} \cong \ker(\partial) \otimes \mathbb{Q}$.

By \cite{GL} Theorem 3.8, the natural map $K_2(k(E)) \rightarrow B_3^*(E)$ induces an isomorphism

$$\left(\frac{H^0(E, K_2)}{\text{Tor}(k^\times, E(k)) + K_2(k)}\right) \otimes \mathbb{Q} \cong \ker(\delta_3) \otimes \mathbb{Q},$$

where $H^0(E, K_2) = \ker(\partial)$. Since $\text{Tor}(k^\times, E(k))$ is a torsion group \cite{W} Proposition 3.1.2(a) and $K_2(k)$ is also torsion, we get the result. \qed

We will need to work with the full group of divisors $\mathbb{Z}[E(\overline{k})]$, using (a modification of) the group $B_3(E)$ introduced in \cite{GL} Definition 3.1. The difference is that we use the $m$-distribution relations only for $m = -1$. 

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Definition 7.3 Let $R_3(E/k)$ be the subgroup of $\mathbb{Z}[E(k)]$ generated by the divisors $\beta(f, 1 - f)$ with $f \notin k(E)$, $f \neq 0, 1$ as well as the divisors $(p) + (-p)$ with $p \in E(k)$. We define

$$B_3(E/k) = \frac{\mathbb{Z}[E(k)]}{R_3(E/k)}$$

and

$$B_3(E) = B_3(E/k)^{\text{Gal}(k/k)}.$$

Goncharov and Levin proved the following result (compare [GL, Proposition 3.19(a)]).

Proposition 7.4 The canonical map $B_3^*(E) \otimes \mathbb{Q} \to B_3(E) \otimes \mathbb{Q}$ is injective.

Proof. It suffices to establish the result for $E/\overline{k}$. Let $D = \sum n_i[p_i] \in I^4$ be a divisor belonging to $R_3(E/\overline{k})$. Write $D = D' + D''$ with $D' \in R_3^*(E/\overline{k})$ and $D'' = \sum m_j((q_j) + (-q_j))$. The divisor $D''$ belongs to $I^4$ and is invariant under the elliptic involution $\sigma : p \mapsto -p$ on $E$. Thus we can write

$$2D'' = D'' + \sigma(D'') = \beta\left(\sum_{\ell} (f_\ell \otimes g_\ell) + \sigma^*(f_\ell \otimes g_\ell)\right)$$

for some rational functions $f_\ell, g_\ell$. By [GL] Lemma 3.21, for any rational functions $f, g$ on $E/\overline{k}$, we have $\sigma^*\{f, g\} = -\{f, g\}$ in $K_2(\overline{k}(E))/\{\overline{k}, \overline{k}(E)^x\}$. It follows that $(f \otimes g) + \sigma^*(f \otimes g)$ is a linear combination of Steinberg relations and of elements $\lambda \otimes h$ with $\lambda \in \overline{k}^x$ and $h \in \overline{k}(E)^x$. Applying the map $\beta$ and noting that $\beta(\lambda \otimes h) = 0$, we get $2D'' \in R_3^*(E/\overline{k})$ as desired. \qed

Putting together Theorem 7.2 and Proposition 7.4, we get the following result.

Theorem 7.5 The composite map

$$\beta : K_2(E) \otimes \mathbb{Q} \longrightarrow K_2(k(E)) \otimes \mathbb{Q} \longrightarrow B_3(E) \otimes \mathbb{Q},$$

sending an element $\sum_i\{f_i, g_i\}$ to the class of the divisor $\sum_i \beta(f_i, g_i)$, is injective.

Thanks to Theorem 7.5, any equality in $K_2(E) \otimes \mathbb{Q}$ can be proved by applying the map $\beta$ and comparing the divisors. Of course, the difficult part is to find the necessary Steinberg relations. In the following sections, we use this strategy to compare an “elliptic” and a “modular” element in $K_2$ of the elliptic curve $X_0(24)$.

7.2 Special elements in $K_2$ of $X_0(24)$

7.2.1 The minimal model

The curve $E_4$ in the Legendre family is given by the equation $y^2 = x(1 - x)(1 + 3x)$. A minimal Weierstrass equation is

$$E : Y^2 = X^3 - X^2 - 4X + 4,$$
obtained with the change of variables \((X, Y) = (1 - 3x, -3y)\). This is the curve 24a1 in the Cremona database \([Cr]\). The Néron differential is (up to sign)
\[
\omega_E = \frac{dX}{2Y} = \frac{dx}{2y}
\]
The torsion subgroup of \(E\) is isomorphic to \(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\), generated by the points \(p_1 = (0, 2)\) of order 4, and \(p_2 = (1, 0)\) of order 2.

### 7.2.2 The modular parametrisation

The curve \(E\) is in fact isomorphic to the modular curve \(X_0(24)\). We denote by \(\varphi : X_0(24) \rightarrow E\) the modular parametrisation, normalised so that \(\varphi(\infty) = 0\) and \(\varphi^*(\omega_E) = \omega_f = 2\pi if(z)dz\), where \(f\) is the unique newform of weight 2 and level \(\Gamma_0(24)\).

The modular curve \(X_0(24)\) has 8 cusps: \(\infty, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}\). They are all rational, and therefore correspond via \(\varphi\) to the 8 rationals points on \(E\). We now make explicit this bijection. Let \(\Lambda \subset \mathbb{C}\) be the lattice of periods of \(\omega_E = dX/(2Y)\). We have a canonical isomorphism
\[
\gamma : \mathbb{C}/\Lambda \xrightarrow{\cong} E(\mathbb{C})
\]
such that \(\gamma^*(\omega_E) = dz\) (so that \(\gamma^{-1}(p) = \int_0^p \omega_E\)). The idea is the following: given a point \(\tau \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})\), we have
\[
\varphi(\tau) = \gamma \left( \int_0^{\varphi(\tau)} \omega_E \right) = \gamma \left( \int_{\infty}^{\tau} \omega_f \right).
\]
The last integral, as well as the map \(\gamma\), can be computed using PARI/GP \([PARI]\). Note that although the computation is numerical, we know that if \(\tau\) is a cusp, then \(\int_{\infty}^{\tau} \omega_f\) belongs to the lattice \(\frac{1}{4}\Lambda\), hence its value can be ascertained.

We used the following PARI/GP code to compute the images of the cusps of \(X_0(24)\) by \(\varphi\).

```pari
e = ellinit("24a1");
mf = mfinit([24, 2]);
f = mfeigenbasis(mf)[1];
symb = mfsymbol(mf, f);
phiE(c) = ellztopoint(e, polcoef(mfsymboleval(symb, [oo, c])*2*Pi*I, 0));
apply(phiE, [oo, 0, 1/2, 1/3, 1/4, 1/6, 1/8, 1/12])
```

The results are shown in the following table.

| \(c\) | \(\infty\) | 0 | \(\frac{1}{2}\) | \(\frac{1}{3}\) | \(\frac{1}{4}\) | \(\frac{1}{6}\) | \(\frac{1}{8}\) | \(\frac{1}{12}\) |
|-------|---------|-----|---------|---------|---------|---------|---------|---------|
| \(\varphi(c)\) | 0 | 3p_1 + p_2 | p_1 + p_2 | 3p_1 | 2p_1 + p_2 | p_1 | p_2 | 2p_1 |
| \((X, Y)\) | \((0 : 1 : 0)\) | \((4, -6)\) | \((4, 6)\) | \((0, -2)\) | \((-2, 0)\) | \((0, 2)\) | \((1, 0)\) | \((2, 0)\) |
The sign of the functional equation of the \( L \)-function \( L(E, s) \) is \(+1\), hence the Atkin-Lehner involution \( W_{24} : \tau \mapsto -1/(24\tau) \) satisfies \( W_{24}(f) = -f \). This implies that the map \( W_{24} : E \to E \) has the form \( p \mapsto p_0 - p \) for some rational point \( p_0 \) on \( E \), and the table above gives \( p_0 = 3p_1 + p_2 \).

7.2.3 The Beilinson–Kato element

Recall the definition of the Beilinson–Kato element \( z_E \) in \( K_2(E) \otimes \mathbb{Q} \) (see [Br, Définition 9.3]):

\[
    z_E = \varphi^* \left( \frac{1}{2} (u_{24}, W_{24}(u_{24}))' \right),
\]

where \( u_N \) is the following product of Siegel units

\[
    u_N = \prod_{a \in \mathbb{Z}/N\mathbb{Z}^\times \atop b \in \mathbb{Z}/N\mathbb{Z}} g_{a,b},
\]

and the superscript ‘ means addition of Milnor symbols \( \{\lambda, v\} \) with \( \lambda \in \mathbb{Q}^\times \) and \( v \in \mathcal{O}(Y_0(24))^\times \) in order to obtain an element of \( K_2(X_0(24)) \otimes \mathbb{Q} \). Since the symbols \( \{\lambda, v\} \) are killed by \( \beta \), we can safely ignore them in the computation.

We wish to compute the divisor of the modular unit \( u_{24} \). Let us work more generally with \( u_N \). From the definition of Siegel units as infinite products, we know that the order of vanishing of \( g_{a,b} \) at the cusp \( \infty \) of \( X(N) \) is equal to \( NB_2(\{a/N\})/2 \), where \( B_2(x) = x^2 - x + 1/6 \) is the second Bernoulli polynomial and \( \{t\} = t - \lfloor t \rfloor \) is the fractional part of \( t \). Moreover, we have the transformation formula \( g_{a,b} \circ \alpha = g_{(a,b)\alpha} \) in \( \mathcal{O}(Y(N))^\times \otimes \mathbb{Q} \) for any \( \alpha \in \text{SL}_2(\mathbb{Z}) \). Using this, we can compute the order of vanishing of \( g_{a,b} \) at any cusp.

Since we are working with \( X_0(N) \) instead of \( X(N) \), we need to take into account the widths of the cusps of \( X_0(N) \). The width of the cusp \( 1/d \in X_0(N) \) is

\[
    w(1/d) = \frac{N}{d \cdot \text{gcd}(d, N/d)}.\]

Since \( 1/d = (1 \ 0 \ 0 \ 1)\infty \), we have

\[
    \text{ord}_{1/d}(u_N) = \sum_{a \in \mathbb{Z}/N\mathbb{Z}^\times \atop b \in \mathbb{Z}/N\mathbb{Z}} \text{ord}_{1/d}(g_{a,b})
    = \sum_{a \in \mathbb{Z}/N\mathbb{Z}^\times \atop b \in \mathbb{Z}/N\mathbb{Z}} w(1/d) \text{ord}_\infty(g_{a+db,b})
    = \frac{w(1/d)}{2} \sum_{a \in \mathbb{Z}/N\mathbb{Z}^\times \atop b \in \mathbb{Z}/N\mathbb{Z}} B_2(\{a/db\})
    = \frac{d \varphi(N)}{2 \text{gcd}(d, N/d) \varphi(d)} \sum_{a \in \mathbb{Z}/d\mathbb{Z}^\times} B_2(\{a/d\}).
\]
Here we used the distribution relation for the periodic Bernoulli polynomials,

\[ B_n\{mt\} = m^{n-1} \sum_{k=0}^{m-1} B_n\left\{ t + \frac{k}{m} \right\} \quad (m \geq 1). \]

We deduce the order of vanishing of \( u_{24} \) at each cusp of \( X_0(24) \), and therefore its divisor:

\[ \text{div}(u_{24}) = \frac{1}{6}(\infty) + \frac{2}{3}(0) - \frac{1}{3}(1/2) - \frac{2}{3}(1/3) - \frac{1}{6}(1/4) + \frac{1}{3}(1/6) - \frac{1}{6}(1/8) + \frac{1}{6}(1/12). \]

The fractions appearing here mean that \( u_{24} \) is only an element of \( \mathcal{O}(Y_0(24))^\times \otimes \mathbb{Q} \), in other words some power of \( u_{24} \) is a modular unit.

Applying the modular parametrisation \( \varphi \), we get

\[ \text{div}(u_{24}) = \frac{1}{6}(0) + 4(3p_1 + p_2) - 2(p_1 + p_2) - 4(3p_1) + 2(p_1) - (2p_1 + p_2) - (p_2) + (2p_1). \] (7.1)

Applying the Atkin-Lehner involution, we also have

\[ \text{div}(W_{24}(u_{24})) = \frac{1}{6}(3p_1 + p_2) + 4(0) - 2(2p_1) - 4(p_2) + 2(2p_1 + p_2) - (p_1) - (3p_1) + (p_1 + p_2). \] (7.2)

### 7.2.4 The symbol \( \xi \) in (5.10)

Recall

\[ \xi = \{ f, g \} \in K_2(E_4) \otimes \mathbb{Q}, \quad f = \frac{y - x + 1}{y + x - 1}, \quad g = -\frac{(x - 1)^2}{4x^2}. \] (7.3)

Using MAGMA, we can find the divisors of \( f \) and \( g \). Here is the code:

```magma
A2<x,y> := AffinePlane(Rationals());
C := Curve(A2, y^2-x*(1-x)*(1+3*x));
Cbar := ProjectiveClosure(C);
E, phi := EllipticCurve(Cbar);
Emin, psi := MinimalModel(E);
F<x,y> := FunctionField(Cbar);
f := (y-x+1)/(y+x-1);
g := -(x-1)^2/(4*x^2);
div_f := Decomposition(Divisor(f));
div_g := Decomposition(Divisor(g));
print "div(f) =", [p[2], psi(phi(RepresentativePoint(p[1]))) : p in div_f];
print "div(g) =", [p[2], psi(phi(RepresentativePoint(p[1]))) : p in div_g];
```

We obtain

\[ \text{div}(f) = -(3p_1) + (p_1) - (3p_1 + p_2) + (p_1 + p_2) \] (7.4)

\[ \text{div}(g) = 4(2p_1 + p_2) - 4(p_2). \]
7.3 Comparing the divisors

We are now going to apply $\beta$ to the two elements in $K_2(E) \otimes \mathbb{Q}$, and compare the results. For the Beilinson–Kato element, we find using (7.1) and (7.2) that

$$\beta(u_{24}, W_{24}(u_{24})) = \frac{1}{36} (8(0) - 8(p_2) + 28(p_1) - 28(p_1 + p_2) + 8(2p_1) - 8(2p_1 + p_2) - 44(3p_1) + 44(3p_1 + p_2)).$$

In the group $B_3(E) \otimes \mathbb{Q}$, we have the relation $(p) + (-p) = 0$ for any point $p$, hence $(p) = 0$ if $p$ is 2-torsion. So we can remove the 2-torsion points from the above divisor. In fact, we can express everything in terms of $p_1$ and $p_1 + p_2$ alone. We find

$$\beta(z_E) = 2(p_1) - 2(p_1 + p_2),$$

and thus

$$\beta(z_E) = (p_1) - (p_1 + p_2). \quad (7.5)$$

We proceed similarly for the element $\xi$ in (5.10). Using (7.4), we compute

$$\beta(f, g) = -8(p_1) - 8(p_1 + p_2) + 8(3p_1) + 8(3p_1 + p_2),$$

which gives

$$\beta(\xi) = -16(p_1) - 16(p_1 + p_2). \quad (7.6)$$

The divisors $\beta(z_E)$ and $\beta(\xi)$ are not proportional, which suggests that there should be a non-trivial Steinberg relation involving $p_1$ and $p_1 + p_2$. We can determine it experimentally by computing the elliptic dilogarithm of these points. Let us denote by $D_E : E(\mathbb{C}) \to \mathbb{R}$ the Bloch elliptic dilogarithm. Using PARI/GP, we find numerically

$$5D_E(p_1) + 3D_E(p_1 + p_2) \approx 0. \quad (7.7)$$

This means that we should have $5(p_1) + 3(p_1 + p_2) = 0$ in the group $B_3(E) \otimes \mathbb{Q}$. We will prove that this is indeed the case, by exhibiting a Steinberg relation.

We search for a rational function $h$ on $E$ such that the zeros and poles of both $h$ and $1 - h$ are among the 8 torsion points of $E$. To do this, we use Mellit’s technique of incident lines [Mc]; see also [GL] Proof of Lemma 3.29.

We view $E$ as a non-singular plane cubic. We generate all the lines passing only through the 8 torsion points of $E$. Say we have found three such lines $\ell_1, \ell_2, \ell_3$ which, moreover, meet at a point $p_0$ of $\mathbb{P}^2$. If the lines are pairwise distinct, we may choose equations $f_1, f_2, f_3$ for these lines satisfying $f_1 + f_2 = f_3$. Then $h = f_1/f_3$ has the property that the divisors of $h$ and $1 - h$ are supported at the torsion points. In particular $\beta(h, 1 - h)$ is also supported at the torsion points, which gives a relation in $B_3(E) \otimes \mathbb{Q}$.

If the intersection point $p_0$ lies on the curve, then the above relation is trivial: it is contained in the subgroup generated by the divisors $(p) + (-p)$. If, however, $p_0$ does not lie on the curve, then we usually get something interesting. It turns out that this method of incident lines works remarkably well in practice.
Using a computer, it is possible to search for all incident lines, and determine the associated Steinberg relations. In the present situation, we find the lines $\ell_1, \ell_2$ defined by the equations

$$f_1 = -\frac{1}{4}(X + Y - 2), \quad f_2 = \frac{1}{4}(X + Y + 2).$$

We have $f_1 + f_2 = 1$, so that the lines are parallel (taking $\ell_3$ to be the line at infinity, the lines $\ell_1, \ell_2, \ell_3$ are incident, so this is a particular case of the situation above). The divisors of these functions are given by

$$\text{div}(f_1) = 2(p_1) + (2p_1) - 3(0)$$
$$\text{div}(f_2) = 3(p_1) + (2p_1 + p_2) + (3p_1 + p_2) - 3(0)$$

and the associated Steinberg relation is

$$\beta(f_1, f_2) = 9(0) + (p_2) - 9(p_1) - 3(p_1 + p_2) - (2p_1) - (2p_1 + p_2) + (3p_1) + 3(3p_1 + p_2) \equiv -10(p_1) - 6(p_1 + p_2) \in B_3(E) \otimes \mathbb{Q}.$$

This shows that indeed $5(p_1) + 3(p_1 + p_2) = 0$ in $B_3(E) \otimes \mathbb{Q}$. Thus (7.5) and (7.6) simplify:

$$\bar{\beta}(z_E) = (p_1) + \frac{5}{3}(p_1) = \frac{8}{3}(p_1)$$

and

$$\bar{\beta}(\xi) = -16(p_1) - 16 \times -\frac{5}{3}(p_1) = \frac{32}{3}(p_1).$$

Using Theorem 7.5, we deduce

**Theorem 7.6** $\xi = 4z_E$ in $K_2(E) \otimes \mathbb{Q}$.

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