Characteristics of QCD Phase Transitions  
in an Extended Skyrme Model on $S^3$

Joon Ha Kim, Sooman Yee$^1$ and Hyun Kyu Lee

Dept. of Physics, Hanyang University, Seoul 133-791 Korea

e-mail adress: kjh@hepth.hanyang.ac.kr

We study the characteristics of the QCD phase transitions in dense hadronic matter using the Skyrme model constructed on $S^3$. We find numerically the localized solutions on $S^3$ using the extended Skyrme model which implements correctly the scale symmetry of QCD. The transition from the localized phase to the delocalized phase is found to be of first order at the critical radius of the hypersphere, $L_c$. The chiral restoration and the gluon decondensation also take place at the same critical size.

keywords: extended Skyrme model, hypersphere, phase transition, scale symmetry, chiral symmetry restoration, gluon decondensation.

$^1$Permanent address: Dept. of Physics, Soonchunhyang Univ., Onyang P.O.B.97, Chungnam 336-600 Korea
The Skyrme model\(^1\) describing baryons as topological solitons (skyrmion) arising from the effective meson theory of QCD in the large \(N_c\)-limit has been proved to be quite successful in calculating the various physical properties of baryons which are in fair quantitative agreements with experimental data. Given the success of the Skyrme model, there have been also interesting developments to investigate the behavior of hadronic matter at higher density. One of the interesting methods, pioneered by Manton and Ruback\(^2, 3\) is to put one skyrmion on a hypersphere \(S^3(\rho)\) with three dimensional volume \(2\pi^2\rho^3\). Although the obvious disadvantage of this method is that the physical space is not a hypersphere with curvature, this approach has been adopted to investigate the problems of dense matter in the Skyrme model motivated by the observation that it gives, in a simple framework, similar results as in the lattice skyrmion approach\(^4\) and also by its mathematical simplicity. The hypersphere approach has been studied not only in the Skyrme model\(^5, 6\) but also recently in the Nambu-Jona-Lasinio model\(^7\).

The skyrmion on the hypersphere of infinitely large radius is well localized so that it has the same properties as an isolated skyrmion in the flat space. The finite size solution which is well localized is possible due to the presence of the interaction term (Skyrme term or higher derivative terms), which prevents it from being collapsed. However, by shrinking the radius of the hypersphere, the effect of interaction term becomes much stronger than the kinetic term because of the curvature of the hypersphere. Therefore, it is possible that for the radius smaller than a certain critical radius \(\rho_c\), the localization mechanism due to the kinetic term becomes inactive and the skyrmion will spread out over the whole hypersphere uniformly. Manton \(^3\) interpreted it as a signature for the quark-hadron phase transition. Moreover, in relation to the chiral symmetry, the observation\(^8\) of vanishing order parameter \(\langle \sigma \rangle_0\) at the critical density leads to the interpretation that it would be a signature for chiral symmetry restoration.

In QCD, one of the important order parameter, is the gluon condensate \(\langle 0|F_{\mu\nu}F_{\mu\nu}|0\rangle\), which breaks the scale invariance in the hadron phase of QCD. The simplest way to implement this scale symmetry of QCD is to introduce a scalar field, \(\chi\), a dilaton or a glueball field in the effective Lagrangian. The scale anomaly of QCD in low energy physics is provided by the effective potential of \(\chi\) of the following generic form,

\[
\frac{B}{4} \chi^4 (\ln \chi^4 - 1),
\]

(1)
where $B$ is related to the bag constant. Eq. (3) has its minimum at nonzero $\chi_0 = \langle \chi \rangle_0$ which correctly reproduces the scale anomaly \cite{9}, and it is related to the gluon condensate,

$$\chi_0^4 = -\frac{\beta(g)}{2g} \langle 0 | F^{\mu\nu} F_{\mu\nu} | 0 \rangle$$

(2)

Recently the scale symmetry property of QCD in the effective Lagrangian has been proved to be very useful in discussing the QCD phase transition \cite{10} at finite temperature and finite density and also in investigating the medium effect on the parameters in the effective Lagrangian \cite{11}. Gomm, et al \cite{12} studied the bag formation in the Skyrme model using the effective Lagrangian in flat space, $R^3$, defined as

$$L = \frac{f^2}{4} \chi^2 Tr L_\mu L^\mu + \frac{1}{32\epsilon_s} Tr [L_\mu L_\nu]^2 - \frac{C}{2} \partial_\mu \chi \partial^\mu \chi - \frac{B}{4} \chi^4 (\ln \chi^4 - 1),$$

(3)

which has the correct QCD scaling law. The glueball field $\chi$ is normalized as $\chi/\chi_0 = 1$, $C$ and $B$ are related to $\chi_0^2$ and the bag constant $(B_B = B\chi_0^4/4)$ respectively. Some years ago, Reinhardt and Dang \cite{13} investigated the behavior of the skyrmion in the dense baryonic matter using the same Lagrangian on $S^3$. They found that there is a phase transition into delocalized Skyrmion triggered by the vanishing gluon condensate. But, it is not easy to read out the order of the phase transition and the critical radius from their calculations. In this work, we carry out the numerical analysis in more detail to investigate the characteristics of phase transition, particularly to see whether it is first or second order phase transition.

We use the hedgehog ansatz for the skyrmion field configuration on $S^3$,

$$U = \exp[i \vec{r} \cdot \hat{r}(\theta, \phi) f(\mu)],$$

(4)

where $(\mu, \theta, \phi)$ are polar coordinates on $S^3(\rho)$ with $0 \leq \mu, \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ and the metric is given by

$$ds^2 = \rho^2 (d \mu^2 + \sin^2 \mu d \theta^2 + \sin^2 \mu \sin^2 \theta d \phi^2).$$

(5)

The static energy of the system can be obtained from eq.(3) using hedgehog ansatz for chiral field and classical field for $\chi(\mu)$,

$$E = \int \sqrt{g} dV L,$$
\[ \begin{align*}
E_2 &= 4\pi L \int_0^\pi d\mu \sin^2 \mu \left( f'^2 + \frac{2 \sin^2 f}{\sin^2 \mu} \right), \\
E_4 &= \frac{4\pi}{L} \int_0^\pi d\mu \sin^2 \mu \left( 2f'^2 + \frac{\sin^2 f}{\sin^2 \mu} \right), \\
E_\chi &= 8\pi e_2^2 L \int_0^\pi d\mu \sin^2 \mu \left\{ \frac{C}{2} \chi'^2 + L^2 \left[ \frac{B}{4} \chi^4 (\ln \chi^4 - 1) + \frac{B}{4} \right] \right\},
\end{align*} \]

where \( C = C/(e_4 f_\pi)^2, \ B = B/(e_4 f_\pi)^2 \) and \( L \) is the dimensionless radius defined as \( L = e_4 f_\pi \rho \).

Extremizing the static energy, we get the Euler equations for \( \chi(\mu) \) and \( f(\mu) \),

\[ \begin{align*}
f''(L\chi^2 + \frac{2 \sin^2 f}{\sin^2 \mu}) + 2Lf'(\chi' + \chi^2 \cot \mu) \\
- \frac{\sin 2f}{\sin^2 \mu} \left[ L\chi^2 + \frac{1}{L} \left( \frac{\sin^2 f}{\sin^2 \mu} - f'^2 \right) \right] &= 0, \\
\chi'' + 2 \cot \mu \chi' - \frac{1}{e_4} \left[ 4L^2 B \chi^3 \ln \chi + \frac{1}{e_4^2} \chi(f'^2 + \frac{2 \sin^2 f}{\sin^2 \mu}) \right] &= 0.
\end{align*} \]

where the chiral profile function and the glueball field satisfy the boundary conditions, for the \( B = 1 \) baryon,

\[ f(0) = 0, \ f(\pi) = \pi, \ \chi'(0) = 0, \ \chi'(\pi) = 0. \]

One can easily see that the solutions of the identity map \( f(\mu) = \mu \) with \( \chi(\mu) = 0 \) are the solutions of the above Euler equations for any radius, \( L \). They are the delocalized solutions with no bag formation\[12\].

We proceed to find numerical solutions with a typical parameter set\[12\]:

\[ e_4 = 4.5, \ f_\pi = 93 MeV, \ C = 3.29 \times 10^2 (MeV)^2, \ B = (220 MeV)^4 / e. \]

We can find two numerical solutions which are localized in addition to the identity map solution above mentioned. In Fig. 1, we plot the static energies of the solutions with respect to the radius of the hypersphere, \( L \). It is found that the two set of the localized solutions (curves (a) and (c) in Fig. 1) are possible for \( L > L_t (\approx 1.9) \). One set of the solutions (curve (c) in Fig. 1) is
not of our interest. Since we are interested in the lower energy configurations. At \( L = L_c (\approx 1.9) \) the energy of the identity map solutions (curve (b)) is equal to the energy of the localized solutions (curve (a)). For a smaller size of the hypersphere, \( L < L_c \), the identity map solutions (delocalized phase) are the lower energy solutions while for the larger hypersphere, \( L > L_c \), the localized solutions are the minimum energy solutions. Hence there is a phase transition from the localized to the delocalized phase at \( L_c \), which is of first order. The characteristic of the phase transition shown in Fig.1 is quite different from the results without \( \chi \) field in the effective Lagrangian, where the phase transition is smooth and found to be of second order \( (L_c = \sqrt{2}) \). Also the changes of the configurations of \( f(\mu) \) and \( \chi(\mu) \) with \( L \) are found out to be abrupt at \( L_c \) as shown in Fig. 2(a) and 2(b) respectively. Since the delocalized solutions characterized by \( \chi(\mu) = 0 \) everywhere correspond to vanishing gluon condensate, the phase transition at \( L_c \) from the localized phase with nonvanishing \( \chi(\mu) \) at \( \mu = \pi \) to delocalized phase implies a discontinuous change of the gluon condensate at \( L_c \). These results can be considered as evidences of the first order phase transition[14] triggered by the glueball field.

The order parameter, \( \langle \sigma \rangle_0 \), which is defined by

\[
\langle \sigma \rangle_0 = -\frac{2}{\pi} \int_0^\pi d\mu \sin^2 \mu \cos f(\mu).
\]

(11)

has been introduced as a useful quantity which effectively measures the chiral symmetry breaking. As shown in Fig.3, one can see the discontinuous change of the order parameter \( \langle \sigma \rangle_0 \) at \( L_c \). This suggests that the chiral symmetry is restored at the same radius \( L_c \) and the phase transition is of first order.

Some remarks are in order for the localized solutions. It is found that the localized solutions (the curves (a) and (c) in Fig.1) merge at \( L_t \) and the localized solutions are no more possible for \( L < L_t \) in numerical calculation. In order to see whether it is an numerical artifact, we look into the gradient of the profile function at \( \mu = 0 \), \( f'(0) \) as a function of \( L \). As shown in Fig. 4, \( f'(0)'s \) of the two localized solutions are smoothly joined at \( L_t \) and continuously reverse their slopes respect to \( L \), \( \frac{df'(0)}{dL} \). This observation provides at least an partial explanation of the merging behavior of the two set of the localized solutions and the absence of localized solutions for \( L < L_t \) other than the identity map solutions.

It is also interesting to note that our analyses indicate that the metastable baryonic state, (a) in Fig. 1, may possibly exist between \( L_t < L < L_c \), since
the energy of this state is not much bigger than the identity map solution (b) in Fig. 1.

In conclusion, in the frame work of the Skyrme model on \( S^3 \) with scale symmetry implemented, we find the evidence for the first order phase transition at the critical radius of the hypersphere, \( L_c \); from the localized phase to the delocalized phase, the chiral restoration and the gluon decondensation.

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Figure Captions

Fig. 1 The static energies with respect to the dimensionless radius, $L$ for two set of the localized numerical solutions, (a) and (c), and for the identity map solutions, (b).

Fig. 2 The numerical solutions of the chiral profile function $f(\mu)$, (a) and the glueball field $\chi(\mu)$, (b).

Fig. 3 The order parameter $\langle \sigma \rangle_0$. The dahed line is for a model without glue ball field and the solid line for the extended model which is implemented by the glueball field.

Fig. 4 The slopes of the chiral profile function at $\mu = 0$, $f'(0)$: The curve (a) corresponds to the lower energy localized solutions, the curve (a) in Fig. 1, and the curve (c) corresponds to the higher energy localized solutions, the curve (c) in Fig. 1.
This figure "fig1-1.png" is available in "png" format from:

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Fig. 2 (a)

\[ f(\mu) \]

\[ \mu \]

\[ L_c = 1.98 \]
\[ L_t = 1.9 \]

L_i L_t
Fig. 2 (b)
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Fig. 4

The graph shows a plot of $f'(0)$ against $L$. The key points are labeled as $L_t$ and $L_c$. The identity map is also indicated on the graph.
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