Wavelet based regularization for Euclidean field theory*

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Abstract

It is shown that Euclidean field theory with polynomial interaction, can be regularized using the wavelet representation of the fields. The connections between wavelet based regularization and stochastic quantization are considered.

1 Introduction

The connections between quantum field theory and stochastic differential equations have been calling constant attention for quite a long time [1, 2]. We know that stochastic processes often posses self-similarity. The renormalization procedure used in quantum field theory is also based on the self-similarity. So, it is natural to use for the regularization of field theories the wavelet transform (WT), the decomposition with respect to the representation of the affine group. In this paper two ways of regularization are presented. First, the direct substitution of WT of the fields into the action functional leads to a field theory with scale-dependent coupling constants.

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Second, the WT, being substituted into the Parisi-Wu stochastic quantization scheme [3], provides a stochastic regularization with no extra vertexes introduced into the theory.

2 Scalar field theory on affine group

The Euclidean field theory is defined on $\mathbb{R}^d$ by the generating functional

$$W_E[J] = \mathcal{N} \int \mathcal{D}\phi \exp \left[ -S[\phi(x)] + \int d^d x J(x) \phi(x) \right], \quad (1)$$

where $S[\phi]$ is the Euclidean action. In the simplest case of a scalar field with the fourth power interaction

$$S[\phi] = \int d^d x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (2)$$

The $\phi^4$ theory is often referred to as the Ginsburg-Landau model for its ferromagnetic applications. The $\phi^3$ theory is also a useful model.

The perturbation expansion generated by the functional (1) is usually evaluated in $k$-space. The reformulation of the theory from the coordinate $(x)$ to momentum $(k)$ representation is a particular case of decomposition of a function with respect to the representation of a Lie group $G$. Let us consider the fourth power interaction model

$$\int V(x_1, x_2, x_3, x_4) \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) d^4x_1 d^4x_2 d^4x_3 d^4x_4.$$
Using the notation \( U(g)|\psi\rangle \equiv |g, \psi\rangle \), \( \langle \phi|g, \psi\rangle \equiv \phi(g) \), \( \langle g_1, \psi|D|g_2, \psi\rangle \equiv D(g_1, g_2) \), we can rewrite the generating functional (11) in the form

\[
W_G[J] = \int D\phi(g) \exp \left( -\frac{1}{2} \int_G \phi(g_1) D(g_1, g_2) \phi(g_2) d\mu(g_1) d\mu(g_2) - \frac{\lambda}{4!} \int_G V(g_1, g_2, g_3, g_4) \phi(g_1) \phi(g_2) \phi(g_3) \phi(g_4) d\mu(g_1) d\mu(g_2) d\mu(g_3) d\mu(g_4) + \int_G J(g) \phi(g) d\mu(g) \right),
\]

(4)

where \( V(g_1, g_2, g_3, g_4) \) is the result of the transform \( \phi(g) := \int \overline{U(g)} \psi(x) \phi(x) dx \) applied to \( V(x_1, x_2, x_3, x_4) \) in all arguments \( x_1, x_2, x_3, x_4 \).

Let us turn to the particular case of the affine group.

\[
x' = ax + b, \quad U(g)\psi(x) = a^{-d/2} \psi((x - b)/a), \quad x, x', b \in \mathbb{R}^d.
\]

(5)

The scalar field \( \phi(x) \) in the action \( S[\phi] \) can be written in the form of wavelet decomposition

\[
\phi(x) = C_\psi^{-1} \int \frac{1}{\sqrt{d\nu}} \psi \left( \frac{x-b}{a} \right) \phi_a(b) \frac{da db}{a^{d+1}}, \\
\phi_a(b) = \int \frac{1}{\sqrt{d\nu}} \psi \left( \frac{x-b}{a} \right) \phi(x) d^d x.
\]

(6)

In the scale-momentum \((a, k)\) representation the matrix element of the free field inverse propagator \( \langle a_1, b_1; \psi|D|a_2, b_2; \psi\rangle \) has the form

\[
D(a_1, a_2, k) = a_1^{d/2} \sqrt{\psi(a_1 k)} (k^2 + m^2) a_2^{d/2} \sqrt{\psi(a_2 k)}.
\]

(7)

The field theory (11) with the propagator \( D^{-1}(a_1, a_2, k) \) gives standard Feynman diagram technique, but with extra wavelet factor \( a^{d/2} \sqrt{\psi(ak)} \) on each line and the integrations over the measure \( d\mu(a, k) = \frac{d^d k}{(2\pi)^d} \frac{da}{a^{d+1}} \) instead of \( \frac{d^d k}{(2\pi)^d} \).

Recalling the power law dependence of the coupling constants on the cutoff momentum resulting from the Wilson expansion, we can define a scalar field model on the affine group, with the coupling constant dependent on scale. Say, the fourth power interaction can be written as

\[
V[\phi] = \int \frac{\lambda(a)}{4!} \phi_a^4(b) d\mu(a, b), \quad \lambda(a) \sim a^{\nu}.
\]

(8)
The one-loop order contribution to the Green function $G_2$ in the theory with interaction (8) can be evaluated [11] by integration over $z = a k$:

$$
\int a^\nu a^d |\hat{\psi}(ak)|^2 \frac{d^d k}{k^2 + m^2} (2\pi)^d a^{d+1} = \int \frac{d^d k}{(2\pi)^d} \frac{C^{(\nu)}_{\psi} k^{-\nu}}{k^2 + m^2}, C^{(\nu)}_{\psi} = \int |\hat{\psi}(z)|^2 \frac{dz}{z^{1-\nu}}. \quad (9)
$$

Therefore, there are no UV divergences for $\nu > d - 2$. This is a kind of asymptotically free theory which is hardly appropriate, say, to spin systems. What is required to get a finite theory is an interaction vanishing outside a given domain of scales. Such model is presented in the next section by means of the stochastic quantization framework.

## 3 Stochastic quantization with wavelets

Let us remind the basic ideas of the stochastic quantization [3, 14, 12, 13]. Let $S[\phi]$ be an action of the field $\phi(x)$. Instead of calculation of the physical Green functions, it is possible to introduce the “extra-time” variable $\tau$: $\phi(x) \rightarrow \phi(x, \tau)$ and evaluate the moments $\langle \phi(x_1, \tau_1) \ldots \phi(x_m, \tau_m) \rangle_{\eta}$ by averaging over the random process $\phi(x, \tau, \cdot)$ governed by the Langevin equation with the Gaussian random force

$$
\dot{\phi}(x, \tau) + \frac{\sigma^2}{2} \frac{\delta S}{\delta \phi(x, \tau)} = \eta(x, \tau), \langle \eta(x, \tau)\eta(x', \tau') \rangle = \sigma^2 \delta(x - x')\delta(\tau - \tau'). \quad (10)
$$

The physical Green functions are then obtained by taking the limit $\tau_1 = \ldots = \tau_m = T \rightarrow \infty$.

The stochastic quantization procedure has been considered as a perspective candidate for the regularization of gauge theories, for it respects local gauge symmetries in a natural way. However a $\delta$-correlated Gaussian random noise in the Langevin equation still yields sharp singularities in the perturbation theory. For this reason a number of modifications based on the noise regularization $\eta(x, \tau) \rightarrow \int dy R_{xy}(\partial^2) \eta(y, \tau)$ have been proposed [11, 17, 15].

In this paper, following [13], we start with the random processes defined directly in wavelet space. The use of the wavelet coefficients instead of the original stochastic processes provides an extra analytical flexibility of the method: there exist more than one set of random functions $W(a, b, \cdot)$ the images of which have coinciding correlation functions. It is easy to check that the random process generated by wavelet coefficients with the correlation
From this the one-loop contribution the stochastic Green function follows: the same correlation function as the white noise has \([17]\).

As an example, let us consider the Kardar-Parisi-Zhang equation \([16]\):

$$\dot{Z} - \nu \Delta Z = \frac{\lambda}{2} (\nabla Z)^2 + \eta. \quad (11)$$

Substitution of wavelet transform

$$Z(x) = C^{-1} \int \exp(i(kx - k_0 t)) \partial^4 \hat{\psi}(a k) \hat{Z}(a,k) \frac{d^{d+1} k}{(2\pi)^{d+1} a^{d+1}}$$

into (11), with the random force of the form

$$\langle \hat{\eta}(a_1,k_1) \hat{\eta}(a_2,k_2) \rangle = C_{\psi}(2\pi)^{d+1} \delta^{d+1}(k_1 + k_2) a_1^{d+1} \delta(a_1 - a_2) D(a_2,k_2), \quad (12)$$

leads to the integral equation

$$(-\omega + \nu k^2) \hat{Z}(a,k) = \eta(a,k) - \frac{1}{2} a^2 \hat{\psi}(a k) C_{\psi}^{-1} \int \langle a_1 a_2 \rangle \frac{\partial^4 \hat{\psi}(a_1 k_1) \hat{\psi}(a_2 (k - k_1))}{k_1(k - k_1) \hat{Z}(a_1,k_1) \hat{Z}(a_2,k - k_1) \frac{d^{d+1} k_1}{(2\pi)^{d+1} a_1^{d+1} a_2^{d+1}}}.$$  

From this the one-loop contribution the stochastic Green function follows:

$$G(k) = G_0(k) - \lambda^2 G_0^2(k) \int \frac{d^{d+1} k_1}{(2\pi)^{d+1}} \Delta(k_1) \frac{k_1(k - k_1) |G_0(k_1)|^2 k k_1 G_0(k - k_1) + O(\lambda^4)}, \quad (13)$$

where \(G_0^{-1}(k) = -\omega + \nu k^2\). The difference from the standard approach \([16]\) is in the appearance of the effective force correlator

$$\Delta(k) \equiv C_{\psi}^{-1} \int \frac{d a}{a} |\hat{\psi}(a k)|^2 D(a,k), \quad (14)$$

which has the meaning of the effective force averaged over all scales.

Let us consider a single-band forcing \([17]\) \(D(a,k) = \delta(a - a_0) D(k)\) and the “Mexican hat” wavelet \(\hat{\psi}(k) = (2\pi)^{d/2} (-i k_1)^2 \exp(-k_1^2/2)\). In the leading order in small parameter \(x = |k|/|k_1| \ll 1\), the contribution to the stochastic Green function is:

$$G(k) = G_0(k) + \lambda^2 G_0^2(k) \frac{S_d}{(2\pi)^{d}} \frac{2 a_0^3 k^2 d - 2}{\nu^2} \frac{d}{8d} \int_0^\infty D(q) e^{-a_0^2 q^2} q^{d+1} dq + O(\lambda^4).$$
4 Langevin equation for the $\phi^3$ theory with scale-dependent noise

Let us apply the same scale-dependent noise (12) to the Langevin equation for $\phi^3$ theory. The standard procedure of the stochastic quantization then comes from the Langevin equation [6]

\[
\dot{\phi}(x, \tau) + \left[ -\Delta \phi + m^2 \phi + \frac{\lambda}{2!} \phi^2 \right] = \eta(x, \tau).
\] (15)

Applying the wavelet transform to this equation, we get:

\[
(-i\omega + k^2 + m^2) \hat{\phi}(a, k) = \hat{\eta}(a, k) - \frac{\lambda}{2} a^2 \hat{\psi}(a k) C^{-2}_\phi \int (a_1 a_2) \frac{d^d q}{(2\pi)^d} \Delta(q)|G_0(q)|^2 G_0(k - q) + O(\lambda^4).
\] (16)

Iterating the integral equation (16), we yield the correction to the stochastic Green function

\[
G(k) = G_0(k) + \lambda^2 G_0^2(k) \int \frac{d^{d+1} q}{(2\pi)^{d+1}} \Delta(q)|G_0(q)|^2 G_0(k - q) + O(\lambda^4).
\] (17)

The analytical expressions for the stochastic Green functions, can be obtained in the $\omega \to 0$ limit. As an example, we take the $D(a, q) = \delta(a - a_0) D(q)$ for $\phi^3$ theory and the Mexican hat wavelet. The one loop contribution to the stochastic Green function $G(k) = G_0(k) + G_0^2 \lambda^2 I_3 + O(\lambda^4)$ is

\[
\lim_{\omega \to 0} I_3^2 = \int \frac{d^d q}{(2\pi)^d} \Delta(q) \frac{1}{2(q^2 + m^2)} \cdot \frac{1}{q^2 + (k - q)^2 + 2m^2}.
\]

The same procedure can be applied in higher loops. As it can be seen, for constant or compactly supported $D(q)$ all integrals are finite due to the exponential factor coming from wavelet $\psi$.

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