We construct new exact solutions of the focusing Nonlinear Schrödinger equation (NLSE). This is a soliton propagating on an unstable condensate. The Kuznetsov and Akhmediev solitons as well as the Peregrine instanton are particular cases of this new solution. We discuss applications of this new solution to the description of freak (rogue) waves in the ocean and in optical fibers.

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- NLSE via local matrix \( \bar{\partial} \) problem – We study solutions of the following NLSE

\[
i \varphi_t - \frac{1}{2} \varphi_{xx} - (|\varphi|^2 - A^2) \varphi = 0
\]  

(1)

with boundary conditions \(|\varphi|^2 \to A^2\) at \(x \to \pm \infty\). Here \(A = \overline{A}\) is a real constant. Equation (1) is the compatibility condition for the following overdetermined linear system for a matrix function \(\Psi\):

\[
\frac{\partial \Psi}{\partial x} = \hat{U} \Psi, \quad i \frac{\partial \Psi}{\partial t} = (\lambda \hat{U} + \hat{W}) \Psi
\]  

(2)

Here

\[
\hat{U} = I \lambda + u
\]  

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{pmatrix}
\]  

\[
\hat{W} = \frac{1}{2} \begin{pmatrix} |\varphi|^2 - |A|^2 & \varphi_x \\ \varphi_x^* & -|\varphi|^2 + |A|^2 \end{pmatrix}
\]  

(3)

If \(\varphi = 0\), system (2) has the following solution:

\[
\Psi_0 = \frac{1}{\sqrt{1 - q^2}} \begin{pmatrix} e^{\phi} & q \cdot e^{\phi} \\ q \cdot e^{\phi} & e^{-\phi} \end{pmatrix}
\]  

(4)

Here

\[
\phi = kx + \Omega t, \quad k^2 = \lambda^2 - A^2
\]

\[
\Omega = -i \lambda k, \quad q = -\frac{\lambda}{\lambda + k}
\]

In what follows we assume that \(k \to \lambda\) at \(|\lambda| \to \infty\). Then

\[
\Psi_0^{-1} \Psi_0 = \frac{1}{\sqrt{1 - q^2}} \begin{pmatrix} e^{-\phi} & -q \cdot e^{-\phi} \\ -q \cdot e^{\phi} & e^{\phi} \end{pmatrix}
\]  

(5)

Notice that

\[
\overline{k(-\lambda)} = -k(\lambda), \quad \overline{\Omega(-\lambda)} = -\Omega(\lambda), \quad \overline{\phi(-\lambda)} = \phi(\lambda)
\]

and

\[
\Psi_0^{-1}(\lambda) = \Psi_0^{-1}(\lambda)
\]  

(6)

We consider the \(\bar{\partial}\)-problem on the complex \(\lambda\)-plane. We are looking for a second order matrix \(\chi(\lambda, \overline{\lambda}, x, t)\) obeying the equation

\[
\frac{\partial \chi}{\partial \lambda} = \chi \cdot f(\lambda, \overline{\lambda}, x, t)
\]  

(7)

and normalized by condition \(\chi \to 1\) at \(|\lambda| \to \infty\). Here

\[
f = \psi_0 f_0(\lambda, \overline{\lambda}) \psi_0^{-1}
\]

\[
fo(\lambda, \overline{\lambda}) = f_0^+(\lambda, \overline{\lambda})
\]  

(8)

By virtue of (6) function \(f\) satisfies condition \(f(\lambda, \overline{\lambda}) = f^+(\lambda, \overline{\lambda})\). This proves that

\[
\chi^{-}(\lambda) = \chi^{+}(\overline{\lambda})
\]  

(9)

The function \(\chi\) has an asymptotic expansion at \(\lambda \to \infty\)

\[
\chi = 1 + \frac{R}{\lambda} + \cdots
\]

(10)

By virtue of (8) \(R^+ = R\). The function \(\chi\) satisfies the following system of equations

\[
\frac{\partial \chi}{\partial x} = \hat{U} \chi - \chi \hat{U}_0
\]

\[
i \frac{\partial \chi}{\partial t} = (\hat{U} + \hat{W}) - \lambda \chi \hat{U}_0
\]  

(11)

Here \(\hat{U}\) and \(\hat{W}\) are given by expressions (3). Because system (11) is overdetermined, we have the following expression for \(\varphi\):

\[
\varphi = A - 2R_{(12)}
\]  

(12)

For an arbitrary choice of matrix function \(fo(\lambda, \overline{\lambda})\) satisfying condition (8), function \(\varphi\) is the solution of equation (11).

- Solitonic solution – Let us choose

\[
fo(\lambda, \overline{\lambda}) = \begin{pmatrix} 0 & \alpha(\lambda, \overline{\lambda}) \\
\overline{\alpha(\lambda, \overline{\lambda})} & 0 \end{pmatrix}
\]

The function \(f\) now takes the form:

\[
f(\lambda, \overline{\lambda}, x, t) = ae^{2\phi} A + \alpha(\lambda, \overline{\lambda}) B e^{-2\phi}
\]

The matrices \(A, B\) are degenerate

\[
A_{\alpha\beta} = a_{\alpha} b_{\beta}, \quad B_{\alpha\beta} = c_{\alpha} d_{\beta}
\]

\[
a = (1, q), \quad b = (-q, 1); \quad c = (q, 1), \quad d = (1, -q)
\]

and

\[
(\vec{a} \cdot \vec{b}) = a_{\alpha} b_{\alpha} = 0, \quad (\vec{c} \cdot \vec{d}) = c_{\alpha} d_{\alpha} = 0
\]

We now choose

\[
\alpha(\lambda, \overline{\lambda}) = C \delta(\lambda - \eta)
\]

Here \(\delta(\lambda - \eta)\) is the two-dimensional \(\delta\)-function. Later on, the existence of \(\delta\)-function allows us to work with two values of function \(\phi\):

\[
\phi(\eta) = k(\eta)(x - int), \quad \phi(-\eta) = -\bar{\phi}
\]

We now assume that \(\eta(\eta) = q, \quad \eta(-\eta) = -\overline{q}\). In a neighborhood of \(\lambda = \eta, \lambda = -\overline{\eta}\) we expand \(A\) and \(B\):

\[
A = A_0 + A_1(\lambda - \eta), \quad B = B_0 + B_1(\lambda + \overline{\eta})
\]

\[
A_0 = \begin{pmatrix} q & 1 \\ -q^2 & q \end{pmatrix}, \quad B_0 = \begin{pmatrix} q & -q^2 \\ 1 & q \end{pmatrix}
\]

\[
A_1 = -\frac{q}{k} \begin{pmatrix} 1 & 0 \\ -2q & 0 \end{pmatrix}, \quad B_1 = -\frac{q}{k} \begin{pmatrix} 0 & 1 \\ 1 & -2q \end{pmatrix}
\]
Note that now \( \vec{\sigma} = \vec{b} \) and \( \vec{d} = \vec{a} \). We will find a solution of the \( \vec{\mathcal{D}} \) problem (7) in the form of rational functions with two poles:

\[
\chi = 1 + \frac{U}{\lambda - \eta} + \frac{V}{\lambda + \eta} \tag{13}
\]

where \( U, V \) are constant degenerate matrices:

\[
U_{\alpha\beta} = u_{\alpha}b_{\beta}, \quad V_{\alpha\beta} = v_{\alpha}a_{\beta}
\]

To avoid a singularity we require

\[
UA_0 = 0, \quad VB_0 = 0
\]

Substituting (13) into (7) we end up with a simple linear system of equations for \( u_{\alpha} \) and \( v_{\alpha} \):

\[
u_{\alpha} \left( 1 + \frac{q|\eta|^2}{k} Ce^{2\phi} \right) - \frac{1 + |\eta|^2}{\eta + \eta} Ce^{2\phi} v_{\alpha} = a_{\alpha} C e^{2\phi}
\]

\[
\frac{1 + |\eta|^2}{\eta + \eta} Ce^{2\phi} u_{\alpha} + (1 + \frac{q|\eta|^2}{k} Ce^{2\phi}) v_{\alpha} = b_{\alpha} C e^{2\phi}
\]

By virtue of (15) the solution of equation (11) is given as follows

\[
\phi = A - 2(u_1 + v_1|\eta|)
\]

It is convenient to present final answer in terms of a uniforming variable \( \xi \):

\[
\lambda = A \left( \xi + \frac{1}{\xi} \right), \quad k = A \left( \xi - \frac{1}{\xi} \right), \quad \xi + \xi \neq 0
\]

Then \( \xi = Re^{-i\alpha} \). Now:

\[
\phi = \frac{1}{2}(ax - \gamma t) + i \frac{1}{2}(kx - \omega t)
\]

Here

\[
ax = A\left( R - \frac{1}{R} \right) \cos(\alpha), \quad \gamma = -\frac{A^2}{2} \left( R^2 + \frac{1}{R^2} \right) \sin(2\alpha)
\]

\[
k = A\left( R + \frac{1}{R} \right) \sin(\alpha), \quad \omega = \frac{A^2}{2} \left( R^2 - \frac{1}{R^2} \right) \cos(2\alpha)
\]

After the proper choice of \( C \) we finish with

\[
\phi = \frac{A e^{2i\alpha}}{2} \left( \frac{2 \cos(2\alpha) \cosh(u + w) - \frac{1}{R^2} \cos(v) \cosh(w)}{\cosh(u + w) - \frac{1}{R^2} \cos(v)} + \frac{2 \sin(2\alpha) \sinh(u + w) + (R^2 - \frac{1}{R^2}) \sin(v)}{\cosh(u + w) - \frac{1}{R^2} \cos(v)} \right)
\]

Here

\[
u = \phi - \phi^* = ax - \gamma t, \quad v = \phi - \phi^* = i(kx - \omega t)
\]

\[
a = \frac{1}{2R} \left( R^2 \cos(\alpha) \right), \quad w = \ln(a)
\]

Note that

\[
\phi \to A \quad \text{at} \quad x \to -\infty
\]

\[
\phi \to A e^{4i\alpha} \quad \text{at} \quad x \to +\infty
\]

\[
|\phi|^2 = A^2 \quad \text{at} \quad x \to \pm \infty
\]

Fig. 1 shows a typical solitonic solution at the moment of maximum and minimum amplitude.

![FIG. 1: Typical solitonic solution at the moment of maximum (blue) and minimum (red) of amplitude. \((R = 2, \alpha = \frac{\pi}{16})\) (16)](image)

In our notation, we obtain Kuznetssov’s solution in the case of real \( \xi \). When the pole is near the real axis, our solution looks like a Kuznetsov soliton moving with constant speed. This speed tends to zero in the limit \( Im(\xi) \to 0 \). Thus, there is a broad area of Kuznetsov-like solutions near the real \( \xi \) axis. Fig. 2 shows a typical solution corresponding to this case. The blue curve corresponds to the moment of maximum value of \( |\phi|^2 \), while the red one corresponds to its minimal value.

![FIG. 2: Kuznetsov-like solution at the moment of maximum (blue) and minimum (red) of amplitude \((R = 2, \alpha = \frac{\pi}{16})\). (17)](image)

The Akhmediev case appears for \( |\xi|^2 = 1 \). Near this area the solution is a wave train moving with constant
speed and without changing its shape. We obtain the approximate expression for the envelope \( s(x,t) \) of this wave train in the case of \( \text{Arg}(\xi) = \frac{\pi}{4} \):

\[
|s|^2 = A \left( 1 + Q \frac{\cosh(u + w)}{(a^2 \cosh^2(u + w) - 4)} \right)
\]

(15)

Here

\[
Q = a(6 + R^4 + \frac{1}{R^4} - 4a^2)
\]

(16)

For an arbitrary value of \( \text{Arg}(\xi) \) the envelope \( s \) behaves similarly but the expression for it is more complicated. Fig. 3 shows a typical solitonic solution near \( |\xi|^2 = 1 \) and its envelope as calculated by (15).

\[|s|^2, A=1\]

FIG. 3: Soliton and its envelope. \((R = 1.01, \alpha = \frac{\pi}{4})\)

The speed and size of the wave train increases without bound in the limit case when \( \xi \) tends to the circle \( |\xi|^2 = 1 \). Thus the Akhmediev solution can be understood as a special case when the size and speed of the wave train tends to infinity.

Using the presented method one can construct multisoliton solutions with different values \( \xi_1, \ldots, \xi_n \) of the uniformizing parameter \( \xi \). By taking all \( \xi_i \rightarrow 1 \) one can obtain "multi-instanton" solutions found recently in articles [19, 20].

Conclusion — We have discovered new solitonic solutions of the NLSE. This discovery essentially increases the number of candidate solutions for the analytic description of rogue waves. We expect that similar solutions exist in more exact models than NLSE. Since the found solutions change the condensate phase they can hardly be used for this purpose directly. However, the two-soliton solutions with complex conjugated values of poles \( (\eta, \tilde{\eta}) \) are relevant candidates which could compete with homoclinic instanton-like solutions.

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* Electronic address: zakharov@math.arizona.edu
† Electronic address: agelash@gmail.com

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