TRAVELING WAVES FOR NONLOCAL LOTKA-VOLTERRA COMPETITION SYSTEMS

BANG-SHENG HAN†.§, ZHI-CHENG WANG†,* and ZENGJI DU‡

† School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu, 730000, China
§ School of Mathematics, Southwest Jiaotong University
Chengdu, Sichuan, 611756, China
‡ School of Mathematics and Statistics, Jiangsu Normal University
Xuzhou, Jiangsu, 221116, China

(Communicated by Shigui Ruan)

ABSTRACT. In this paper, we study the traveling wave solutions of a Lotka-Volterra diffusion competition system with nonlocal terms. We prove that there exists traveling wave solutions of the system connecting equilibrium \((0, 0)\) to some unknown positive steady state for wave speed \(c > c^* = \max\{2, 2\sqrt{d_1r}\}\) and there is no such traveling wave solutions for \(c < c^*\), where \(d\) and \(r\) respectively corresponds to the diffusion coefficients and intrinsic rate of an competition species. Furthermore, we also demonstrate the unknown steady state just is the positive equilibrium of the system when the nonlocal delays only appears in the interspecific competition term, which implies that the nonlocal delay appearing in the interspecific competition terms does not affect the existence of traveling wave solutions. Finally, for a specific kernel function, some numerical simulations are given to show that the traveling wave solutions may connect the zero equilibrium to a periodic steady state.

1. Introduction.

This paper is concerned with the following Lotka-Volterra diffusion-competition system with nonlocal terms:

\[
\begin{align*}
    u_t - d_1 u_{xx} &= r_1 u \left( 1 - b_1 (\phi_1 \ast u) - a_1 (\phi_2 \ast v) \right), \\
    v_t - d_2 v_{xx} &= r_2 v \left( 1 - b_2 (\phi_3 \ast v) - a_2 (\phi_4 \ast u) \right),
\end{align*}
\]

where \(d_i, r_i, a_i, b_i > 0 \quad (i = 1, 2)\) are constants and

\[(\phi_i \ast u)(x,t) := \int_{\mathbb{R}} \phi_i(x-y)u(y,t)dy \quad \text{for} \quad x \in \mathbb{R}, \quad i = 1, 2, 3, 4.\]

Here the unknown functions \(u(x,t), v(x,t)\) denote the densities of two competing species, the positive constants \(d_1, d_2\) are the diffusion coefficients of species \(u\) and \(v\), the positive constants \(a_1, a_2\) are the inter-specific competition coefficients of species \(u\) and \(v\), the positive constants \(b_1, b_2\) are the intra-specific competition coefficients of species \(u\) and \(v\), and the positive constants \(r_1, r_2\) are the growth rates of species.

2010 Mathematics Subject Classification. 35C07, 35K40, 35K57, 35R10.

Key words and phrases. Lotka-Volterra diffusion-competition system, nonlocal, traveling wave solution, existence, numerical simulation.

* Corresponding author: Zhi-Cheng Wang.
Here, the kernel \( \phi_i(x) \) (\( i = 1, 2, 3, 4 \)) are bounded functions and satisfy

(K1) \( \phi_i(x) \geq 0 \) and \( \int_{\mathbb{R}} \phi_i(x) dx = 1 \);

(K2) \( \int_{\mathbb{R}} \phi_i(y)e^{\lambda y} dy < \infty \) for any \( \lambda \in (0, \max\{1, \sqrt{7}\}) \).

It is well known that system (2) (resp. (1)) can further be simplified into the classical Lotka-Volterra competition system

\[
\begin{aligned}
&u_t - u_{xx} = u(1 - u(a_1 v), \\
v_t - dv_{xx} = rv(1 - v(a_2 u),
\end{aligned}
\]

if \( \phi_i \) (\( i = 1, 2, 3, 4 \)) are replaced by the Dirac delta functions. Obviously, system (3) always has three equilibria \((0, 0), (1, 0), (0, 1)\). Especially, if \( a_1, a_2 < 1 \) or \( a_1, a_2 > 1 \), there is the fourth equilibrium (co-existence state)

\[
(u^*, v^*) = \left( \frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2} \right).
\]

For the traveling wave solutions and the spreading speed of system (3), there were many studies, see [5, 9, 22, 38, 42, 29, 13, 23]. For more results about traveling wave solutions of system (3), we refer to [15, 43, 28] and the references therein (especially the review paper [15]).

Due to the realistic models in applications, time delays and nonlocal delays have been widely incorporated into kinds of reaction diffusion systems. The earlier studies about traveling wave solutions of the delayed reaction-diffusion systems mostly considered the cases that the reaction term satisfies quasimonotonicity or non-quasimonotonicity conditions, see [25, 26, 37, 41, 48, 49]. By using Schauder’s fixed point theorem, the cross iteration scheme and the upper and lower solutions method, Li et al. [32] established the existence of traveling wave solutions of a delayed Lotka-Volterra competition systems when the nonlinearity satisfies weak quasimonotonicity or weak non-quasimonotonicity conditions and the delays (occurring in intraspecific competition terms) are sufficiently small. Pan [40] further proved the existence of traveling wave solutions in a Gilpin-Ayala types competition diffusion system under weak quasimonotonicity or weak non-quasimonotonicity conditions. By applying the upper-lower technique and the monotone iteration, Lv and Wang [35] showed that the delayed Lotka-Volterra competitive system admits traveling wave solutions connecting the two semi-trivial equilibriums when the delay is sufficiently small. For more results about traveling wave solutions about the delayed Lotka-Volterra competition system with small delay, we can refer to [31, 46, 27, 24] and the references therein. Recently, Fang and Wu [6] considered the following time-delayed model

\[
\begin{aligned}
&u_t - u_{xx} = u(1 - u(t - \tau, x) - a_1 v), \\
v_t - dv_{xx} = rv(1 - v(t - \tau, x) - a_2 u).
\end{aligned}
\]
They proved that system (5) admits monotone traveling wave solutions connecting 
(0, 0) to the coexistence state if and only if \( c \geq \max \{2, 2\sqrt{\mu} \} \) and \( \tau \leq \tau(c) \) for 
one \( \tau(c) > 0 \). Lin and Ruan [34] proved the existence and the asymptotic behavior of 
traveling wave solutions about a Lotka-Volterra competition systems with distributed delays by using Schauder’s fixed point theorem, contracting rectangles. Note that in [6, 34], the delay does not need to be sufficiently small.

In contrast to the studies on the traveling wave solutions of the delayed Lotka-Volterra system, the study about the nonlocal Lotka-Volterra system is relatively fewer. Gourley and Ruan [12] studied the following system

\[
\begin{align*}
\frac{du_1}{dt}(x,t) &= d_1 \frac{\partial^2 u_1}{\partial x^2}(x,t) + r_1 u_1(x,t) \left[1 - a_1 u_1(x,t) - b_1 (g_1 \ast u_2)(x,t)\right], \\
\frac{du_2}{dt}(x,t) &= d_2 \frac{\partial^2 u_2}{\partial x^2}(x,t) + r_2 u_2(x,t) \left[1 - a_2 u_2(x,t) - b_2 (g_2 \ast u_1)(x,t)\right],
\end{align*}
\]

where \( r_i, a_i, b_i, \ i = 1, 2 \) are all positive constants and

\[
\begin{align*}
(g_1 \ast u_2)(x,t) &= \int_{-\infty}^{0} \int_{\mathbb{R}} \frac{e^{\tau t}}{\sqrt{4\pi \tau^2}} e^{\frac{x^2}{4\tau}} u_2(x - y, t + s) dy ds, \\
(g_2 \ast u_1)(x,t) &= \int_{-\infty}^{0} \int_{\mathbb{R}} \frac{e^{\tau t}}{\sqrt{4\pi \tau^2}} e^{\frac{x^2}{4\tau}} u_1(x - y, t + s) dy ds.
\end{align*}
\]

By using linear chain techniques and geometric singular perturbation theory, they established the existence of traveling wave solutions connecting two semitrivial equilibria of system (6) when \( b_2 < a_1, \ a_2 < b_1 \) and \( \tau_1, \tau_2 \) are sufficiently small. Later, through transforming system (6) into reaction-diffusion system coupled by four equations without delay, Lin and Li [33] proved that system (6) admits traveling wave solutions connecting \((0, 0)\) to the positive steady state when \( a_1 < b_2 \) and \( a_2 < b_1 \). Some other results about the traveling wave solutions of Lotka-Volterra system or the similar equations with nonlocal term can be referred to [14, 36, 44, 45, 47].

It is worth noting that the most studies above considered the case that the delays (or nonlocal delays) appearing in intra-specific competition terms are sufficiently small or there are no delays in intra-specific competition terms. In fact, even for the Fisher-KPP equation with (nonlocal) delay, the studies of its traveling wave solutions mainly considered the case that the delay is sufficiently small before. More recently, there have been some great progress on traveling wave solutions of the Fisher-KPP equation with delay or nonlocal delay. Particularly, Berestycki et al. [3] considered the following nonlocal Fisher-KPP equation

\[
\begin{align*}
u_t - \nu_{xx} &= \mu \nu (1 - \phi \ast \nu), \quad x \in \mathbb{R},
\end{align*}
\]

and proved that (7) admits traveling wave solutions connecting 0 to an unknown positive steady state for \( c \geq c^* = 2\sqrt{\mu} \) and there is no such traveling wave solutions for \( c < c^* \). It should be emphasized that the work of Berestycki et al. [3] did not require that the nonlocality is weak (that is, the nonlocal delay is small). After that, many important results about traveling wave solution and the spreading speed of equation (7) have been obtained, see [1, 2, 11, 8, 7, 20, 16, 17, 18, 19, 39] and the references therein. In addition, traveling wave solutions of delayed Fisher-KPP equation was also studied deeply, see [4, 11, 21, 30] and the references therein.

Inspired by [3, 17, 18], in this paper we study traveling wave solutions of system (2) when the nonlocality without any restriction. In addition, throughout this paper, we will assume that \( a_1 \) and \( a_2 \) satisfies the following hypotheses.

(H) \( 0 < a_1, a_2 < 1 \).
Now, we state our results as follows.

**Theorem 1.1.** Assume that (H) holds and the kernel \( \phi_i(x) (i = 1, 2, 3, 4) \) satisfy (K1) and (K2). Then, for any \( c > c^* = \max \left\{ 2, 2\sqrt{d} \right\} \), there exists a traveling wave solution \((u(x - ct), v(x - ct))\) satisfying
\[
\begin{align*}
-u''(x) - cu'(x) &= u(x) \left(1 - (\phi_1 \ast u)(x) - a_1(\phi_2 \ast v)(x)\right), \ x \in \mathbb{R}, \\
-dv''(x) - cv'(x) &= rv(x) \left(1 - (\phi_3 \ast v)(x) - a_2(\phi_4 \ast u)(x)\right), \ x \in \mathbb{R},
\end{align*}
\]
and the boundary conditions
\[
\liminf_{x \to -\infty} (u(x) + v(x)) > 0, \quad \lim_{x \to +\infty} u(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} v(x) = 0,
\]
where both \(u(x)\) and \(v(x)\) are positive functions. In particular, the wave profiles \(u\) and \(v\) decrease on \([0, +\infty)\) for some \(Z_0 > 0\) (which may depend on \(c\)). Further, if \(a_1 < \frac{1}{\sqrt{d}}\) and \(a_2 < \frac{1}{\sqrt{d}}\), where
\[
M = \max \left\{ \frac{4}{3} \left(\int_{-\sqrt{d} \frac{1}{2}}^{0} \phi_1(y) dy\right)^{-1}, \frac{4}{3} \left(\int_{-\sqrt{d} \frac{1}{2}}^{0} \phi_3(y) dy\right)^{-1} \right\},
\]
then the traveling wave solution \((u(x - ct), v(x - ct))\) satisfies
\[
\liminf_{x \to -\infty} u(x) > 0 \quad \text{and} \quad \liminf_{x \to -\infty} v(x) > 0.
\]
Besides, there is no such traveling wave solution \((u(x - ct), v(x - ct))\) satisfying (8) and (9) for \(c < c^*\).

When \(\phi_1(x) = \phi_3(x) = \delta(x)\), we have the following theorem.

**Theorem 1.2.** Suppose that (H) holds and the kernel \(\phi_i(x) (i = 2, 4)\) satisfy (K1) and (K2). Suppose that \(\phi_1(x) = \phi_3(x) = \delta(x)\), where \(\delta(x)\) is the Dirac function. Then, for any \(c > c^* = \max \left\{ 2, 2\sqrt{d} \right\} \), there exists a traveling wave solution \((u(x - ct), v(x - ct))\) satisfying
\[
\begin{align*}
-u''(x) - cu'(x) &= u(x) \left(1 - u(x) - a_1(\phi_2 \ast v)(x)\right), \ x \in \mathbb{R}, \\
-dv''(x) - cv'(x) &= rv(x) \left(1 - v(x) - a_2(\phi_4 \ast u)(x)\right), \ x \in \mathbb{R},
\end{align*}
\]
and the boundary conditions
\[
\lim_{x \to +\infty} u(x) = \lim_{x \to +\infty} v(x) = 0, \quad \lim_{x \to +\infty} u(x) = u^* \quad \text{and} \quad \lim_{x \to +\infty} v(x) = v^*,
\]
where \((u^*, v^*)\) is defined by (4). In addition, there is no such traveling wave solution \((u(x - ct), v(x - ct))\) satisfying (12) and (13) for \(c < c^*\).

This paper is organized as follows. In Section 2, we establish the existence of traveling wave solutions of system (1) connecting the equilibrium \((0, 0)\) and an unknown positive steady state, that is Theorem 1.1. Furthermore, we consider a special case of system (1) (that is, \(\phi_1(x) = \phi_3(x) = \delta(x)\)) and prove Theorem 1.2 in Section 3. In Section 4, by numerical simulations and the stability analysis, we show that the unknown steady state can be a periodic steady state.
2. Existence of traveling wave solutions of system (2). In this section, we show that system (2) admits traveling wave solutions connecting \((0, 0)\) to an unknown positive steady state. Let \(c > \max \{2, 2\sqrt{d}\}\). Setting \(\xi = x - ct\) and looking for solutions of system (2) with the form of \((u(x, t), v(x, t)) = (U(\xi), V(\xi))\), we can get

\[
\begin{aligned}
-cU'(\xi) - U''(\xi) &= U(\xi)(1 - (\phi_1 \ast U)(\xi) - a_1 (\phi_2 \ast V)(\xi)), \\
-cV'(\xi) - dV''(\xi) &= rV(\xi)(1 - (\phi_3 \ast V)(\xi) - a_2 (\phi_4 \ast U)(\xi)),
\end{aligned}
\]

where

\[(\phi_1 \ast w)(\xi) = \int_R \phi_1(\eta)w(\xi - \eta)d\eta, \quad \xi \in \mathbb{R}.
\]

Our method is to first focus on a two-point boundary value problem on a finite interval and then take the limit of solutions of the problem as the interval passes to the whole line. Specifically, the solutions of two-point boundary value problem are obtained by constructing super- and subsolutions and using Schauder’s fixed point theorem.

**Supersolution.** Take

\[
\bar{p}_c(x) = e^{-\lambda_c x} \quad \text{and} \quad \bar{q}_c(x) = e^{-\zeta_c x}, \quad \forall x \in \mathbb{R},
\]

where \(\lambda_c > 0\) is the smaller root of the equation

\[
\lambda_c^2 - c\lambda_c + 1 = 0,
\]

and \(\zeta_c > 0\) is the smaller root of the equation

\[
d\zeta_c^2 - c\zeta_c + r = 0.
\]

Then one has

\[
\begin{aligned}
-\bar{p}_c'' - c\bar{p}_c' &= \bar{p}_c \quad \text{and} \quad -d\bar{q}_c'' - c\bar{q}_c' &= r\bar{q}_c \quad \text{in} \, \mathbb{R}.
\end{aligned}
\]

**Subsolution.** Take

\[
\underline{p}_c(x) = e^{-\lambda_c x} - A e^{-(\lambda_c + \varepsilon)x} \quad \text{and} \quad \underline{q}_c(x) = e^{-\zeta_c x} - B e^{-(\zeta_c + \varepsilon)x}
\]

where \(\varepsilon \in (0, \min\{\lambda_c, \zeta_c\})\) is small enough such that

\[
r_c = -(\lambda_c + \varepsilon)^2 + c(\lambda_c + \varepsilon) - 1 > 0,
\]

and

\[
t_c = -d(\zeta_c + \varepsilon)^2 + c(\zeta_c + \varepsilon) - r > 0.
\]

In addition, \(A > 1\) is large enough such that

\[
\frac{\ln A}{\varepsilon} > \max \left\{ \frac{1}{\lambda_c - \varepsilon} \ln \frac{2Z_1^c}{\Delta_c}, \frac{1}{\zeta_c - \varepsilon} \ln \frac{2a_1 Z_2^c}{\Delta_c} \right\},
\]

and \(B > 1\) is large enough such that

\[
\frac{\ln B}{\varepsilon} > \max \left\{ \frac{1}{\lambda_c - \varepsilon} \ln \frac{2r a_2 Z_3^c}{\Delta_c}, \frac{1}{\zeta_c - \varepsilon} \ln \frac{2r Z_3^c}{\Delta_c} \right\},
\]

where

\[
Z_i^c = \int_R \phi_i(y) e^{\lambda_i y} dy, \quad i = 1, 4,
\]

and

\[
Z_i^c = \int_R \phi_i(y) e^{\zeta_i y} dy, \quad i = 2, 3.
\]
Following from the condition (K2), we know that $Z_i^c$ is well defined. Then for $x$ satisfying $\bar{p}_c(x) > 0$ and $\bar{q}_c(x) > 0$, that is, $x > \max \left\{ \frac{\ln A}{\epsilon}, \frac{\ln B}{\epsilon} \right\}$, we have

$$-c'p''_c - p''_c + p'_c (\phi_1 * \bar{p}_c) + a_1 p'_c (\phi_2 * \bar{q}_c)$$

$$= -c \left[ -\lambda_c e^{-\lambda_c x} + A(\lambda_c + \epsilon) e^{-(\lambda_c + \epsilon)x} \right] - \left[ \lambda_c^2 e^{-\lambda_c x} - A(\lambda_c + \epsilon)^2 e^{-(\lambda_c + \epsilon)x} \right]$$

$$- \left[ e^{-\lambda_c x} - A e^{-(\lambda_c + \epsilon)x} \right] + \left[ e^{-\lambda_c x} - A e^{-(\lambda_c + \epsilon)x} \right] (Z_1^e e^{-\lambda_c x} + a_1 Z_2^e e^{-\lambda_c x})$$

$$= e^{-\lambda_c x} (c\lambda_c - \lambda_c^2 - 1) + A e^{-(\lambda_c + \epsilon)x} \left[ -c(\lambda_c + \epsilon) + (\lambda_c + \epsilon)^2 + 1 \right]$$

$$+ e^{-\lambda_c x} \left[ -A e^{-(\lambda_c + \epsilon)x} + a_1 Z_2^e e^{-\lambda_c x} \right]$$

$$\leq 0$$

and

$$-c'q''_c - d q''_c - r q'_c + r q'_c (\phi_3 * \bar{q}_c) + a_2 q'_c (\phi_4 * \bar{p}_c)$$

$$= -c \left[ -\zeta_c e^{-\zeta_c x} + B(\zeta_c + \epsilon) e^{-(\zeta_c + \epsilon)x} \right] - d \left[ \zeta_c^2 e^{-\zeta_c x} - B(\zeta_c + \epsilon)^2 e^{-(\zeta_c + \epsilon)x} \right]$$

$$- r \left[ e^{-\zeta_c x} - B e^{-(\zeta_c + \epsilon)x} \right] + r \left[ e^{-\zeta_c x} - B e^{-(\zeta_c + \epsilon)x} \right] \left( Z_3^e e^{-\zeta_c x} + a_2 Z_4^e e^{-\lambda_c x} \right)$$

$$= e^{-\zeta_c x} \left( c\zeta_c - d\zeta_c^2 - r \right) + B e^{-(\zeta_c + \epsilon)x} \left[ -c(\zeta_c + \epsilon) + d(\zeta_c + \epsilon)^2 + r \right]$$

$$+ r \left[ e^{-\zeta_c x} - B e^{-(\zeta_c + \epsilon)x} \right] \left( Z_3^e e^{-\zeta_c x} + a_2 Z_4^e e^{-\lambda_c x} \right)$$

$$\leq e^{-(\zeta_c + \epsilon)x} \left[ -B \zeta_c + r Z_3^e e^{-(\zeta_c + \epsilon)x} + a_2 r Z_4^e e^{-(\zeta_c + \epsilon)x} \right]$$

$$< 0.$$
where \( a > \max \left\{ \frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon} \right\} \) and

\[
\Psi(x) = \begin{cases} u(a), & x > a, \\ u(x), & x \in [-a, a], \\ u(-a), & x < -a, \end{cases} \quad \Psi(x) = \begin{cases} v(a), & x > a, \\ v(x), & x \in [-a, a], \\ v(-a), & x < -a. \end{cases}
\]

In order to obtain the existence of the problem (18), we consider the following two-point boundary value problem

\[
\begin{align*}
-u'' - (\phi_1 \ast \overline{\Psi} + a_1(\phi_2 \ast \overline{\Psi}))u &= u_0, \\
-cv'' + r(\phi_3 \ast \overline{\Psi} + a_2(\phi_4 \ast \overline{\Psi}))v &= r v_0,
\end{align*}
\]

(19)

where \((u_0, v_0) \in \mathcal{M}_a\) and

\[
\overline{\Psi}_0(x) = \begin{cases} u_0(a), & x > a, \\ u_0(x), & x \in [-a, a], \\ u_0(-a), & x < -a, \end{cases} \quad \Psi_0(x) = \begin{cases} v_0(a), & x > a, \\ v_0(x), & x \in [-a, a], \\ v_0(-a), & x < -a. \end{cases}
\]

In addition, the convex set \( \mathcal{M}_a \) is defined as

\[
\mathcal{M}_a = \left\{ (u, v) \in C([-a, a], \mathbb{R}^2) : \begin{array}{l}
\overline{\Psi}_c(x) \leq u(x) \leq \overline{\Psi}_c(x), \quad x \in (-a, a), \\
\overline{\Psi}_c(x) \leq v(x) \leq \overline{\Psi}_c(x), \quad x \in (-a, a), \\
u(\pm a) = \overline{\Psi}_c(\pm a), \quad v(\pm a) = \overline{\Psi}_c(\pm a),
\end{array} \right\}
\]

Let \( \Psi_a \) be the solution mapping of the problem (19). Namely, \( \Psi_a(u_0, v_0) = (u, v) \). It is clear that a solution of the problem (18) is a fixed point of the problem (19). It is easy to know \( \Psi_a \) is compact and continuous. It suffices to prove that the set \( \mathcal{M}_a \) is invariant for the mapping \( \Psi_a \). Given \((u_0, v_0) \in \mathcal{M}_a\), since \((u, v) \equiv (0,0)\) is a subsolution of the problem (19), we have \( u(x) > 0, v(x) > 0 \) for any \( x \in (-a, a) \).

Hence, we have

\[
\begin{align*}
-c \overline{\Psi}_c' - \overline{\Psi}_c' + (\phi_1 \ast \overline{\Psi}_0 + a_1(\phi_2 \ast \overline{\Psi}_0))\overline{\Psi}_c &= \overline{\Psi}_c' - \overline{\Psi}_c' + (\phi_1 \ast \overline{\Psi}_0 + a_1(\phi_2 \ast \overline{\Psi}_0))\overline{\Psi}_c \\
&\geq -c \overline{\Psi}_c' + \overline{\Psi}_c' \\
&= \overline{\Psi}_c' \\
&\geq u_0
\end{align*}
\]

and

\[
\begin{align*}
-c \overline{\Psi}_c' - \overline{\Psi}_c' + r(\phi_3 \ast \overline{\Psi}_0 + a_2(\phi_4 \ast \overline{\Psi}_0))\overline{\Psi}_c &= r \overline{\Psi}_c' - r \overline{\Psi}_c' \\
&\geq r v_0
\end{align*}
\]

where \( u(\pm a) = \overline{\Psi}_c(\pm a) \leq \overline{\Psi}_c(\pm a) \) and \( v(\pm a) = \overline{\Psi}_c(\pm a) \leq \overline{\Psi}_c(\pm a) \). The maximum principle implies that \( u(x) \leq \overline{\Psi}_c(x) \) and \( v(x) \leq \overline{\Psi}_c(x) \) for all \( x \in (-a, a) \).

On the other hand, for any \( x \in \left( \frac{\ln A}{\varepsilon}, a \right) \), we have

\[
\begin{align*}
-c \overline{\Psi}_c' - \overline{\Psi}_c' + (\phi_1 \ast \overline{\Psi}_0 + a_1(\phi_2 \ast \overline{\Psi}_0))\overline{\Psi}_c &= \overline{\Psi}_c' - \overline{\Psi}_c' + (\phi_1 \ast \overline{\Psi}_0 + a_1(\phi_2 \ast \overline{\Psi}_0))\overline{\Psi}_c \\
&\leq -c \overline{\Psi}_c' + \overline{\Psi}_c' + (\phi_1 \ast \overline{\Psi}_c + a_1(\phi_2 \ast \overline{\Psi}_c))\overline{\Psi}_c
\end{align*}
\]
Let
\[ \text{Proof.} \]
Lemma 2.1.
\[ \begin{align*}
1 \quad \text{Lemma.} \\
\text{Next, we prove the lemma. By evaluating} \\
\text{which does not depend on} \quad a \quad \text{and} \quad c > c^* \\
\text{(where} \quad c^* = \max \left\{ 2, 2 \sqrt{dr} \right\} \text{)} \quad \text{such that each solution of the problem} \quad (18) \\
\text{satisfies} \quad 0 \leq u_a(x) \leq M \quad \text{and} \quad 0 \leq v_a(x) \leq M \\
\text{for all} \quad a > \max \left\{ \frac{1}{\varepsilon} \ln \frac{A(\lambda_a + \varepsilon)}{A_e}, \frac{1}{\varepsilon} \ln \frac{B(\zeta_a + \varepsilon)}{\zeta_e} \right\} \text{ and all} \quad x \in [-a, a].
\end{align*} \]

\textbf{Proof.} Let \( x_M, x_N \in [-a, a] \) satisfy
\[ M_u = \max_{x \in [-a, a]} u_a(x) = u_a(x_M), \quad M_v = \max_{x \in [-a, a]} v_a(x) = v_a(x_N). \]

First, we show that \( x_M, x_N \in [-a, a] \). Since \( u_a(a) = \tilde{p}_c(a), \quad v_a(a) = \tilde{q}_c(a) \), and \( \tilde{p}_c(x), \tilde{q}_c(x) \) are decreasing for \( x > \max \left\{ \frac{1}{\varepsilon} \ln \frac{A(\lambda_a + \varepsilon)}{A_e}, \frac{1}{\varepsilon} \ln \frac{B(\zeta_a + \varepsilon)}{\zeta_e} \right\} \), then we know that \( x_M, x_N \in [-a, a] \).

Next, we prove the lemma. By evaluating
\[ -cu_a' - u''_a = u_a(1 - \phi_1 * \pi_a - a_1(\phi_2 * \pi_a)) \]
at \( x_M \) and
\[ -cu_a' - dv_a'' = rv_a(1 - \phi_3 * \pi_a - a_2(\phi_4 * \pi_a)) \]
at \( x_N \), we can get
\[ \begin{align*}
1 - (\phi_1 * \pi_a)(x_M) - a_1(\phi_2 * \pi_a)(x_M) & \geq 0, \\
1 - (\phi_3 * \pi_a)(x_N) - a_2(\phi_4 * \pi_a)(x_N) & \geq 0,
\end{align*} \]
which implies
\[ (\phi_1 * \pi_a)(x_M) \leq 1 \quad \text{and} \quad (\phi_3 * \pi_a)(x_N) \leq 1. \]

In addition, we also have
\[ -cu_a' - u''_a \leq u_a \leq M_u \]
and
\[ -cu_a' - dv_a'' \leq rv_a \leq rM_v. \]
Then
\[(u' e^{cx})' \geq -Ma e^{cx} \text{ and } (dv' e^{\bar{\alpha}x})' \geq -rM_v e^{\bar{\alpha}x}\.\]
Integrating the previous inequality from $x_M$ to $x > x_M$ and integrating the last inequality from $x_N$ to $x > x_N$, we get
\[u'_a(x) \geq -\frac{M_u}{c} \frac{1 - e^{-c(x-x_M)}}{x}, \quad x \in [x_M, a),\]
and
\[u'_a(x) \geq -\frac{rM_v}{c} \frac{1 - e^{-\bar{\alpha}(x-x_N)}}{x}, \quad x \in [x_N, a)\]
due to the fact $u'_a(x_M) = 0$ and $v'_a(x_N) = 0$. Furthermore, integrating again, for $x \geq x_M$, we have
\[
u_a(x) \geq M_v \frac{2}{c} \int_{x_M}^x e^{-\bar{\alpha}(x-x_N)} dx = M_v \frac{2}{c} \int_{x_N}^x e^{-\bar{\alpha}(x-x_N)} dx\]
where $h(y) = \frac{e^{-\lambda y} + y - 1}{y^2}$ and for $x \geq x_N$, we have
\[
u_a(x) \geq M_u - \frac{rM_v}{c} (x - x_M) + \frac{rM_v}{c} e^{\bar{\alpha}x} \int_{x_N}^x e^{-\bar{\alpha}x} dx = M_u - \frac{rM_v}{c} (x - x_M) + \frac{rM_v}{c} e^{\bar{\alpha}x} \int_{x_N}^x e^{-\bar{\alpha}x} dx\]
and integrating the last
\[
1 \geq M_u \left[ 1 - \frac{1}{2}(a - x_M)^2 \right], \quad 1 \geq M_v \left[ 1 - \frac{r}{2d}(a - x_N)^2 \right].
\]
By $\int_{-\sqrt{\frac{d}{2}}}^{0} \phi_1(y)dy < 1$, let $x_0 := \sqrt{\frac{d}{2}}$ and $1 - \frac{y^2}{2} \geq 1 - \frac{\frac{d}{2}}{2} := \frac{3}{4}$ for any $y \in [0, x_0]$, we get

$$M_u \leq \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_1(y)dy \right)^{-1}.$$  

Similarly, take $\tilde{a}_0 = \sqrt{\frac{d}{2}}$ and let $y_0 := \sqrt{\frac{d}{2}}$, then if $x_N \in (a - y_0, a)$, it follows from (23) that

$$M_v \leq \left[ 1 - \frac{r}{2d}(a-x_N)^2 \right]^{-1} \leq \left( 1 - \frac{r}{2d}y_0^2 \right)^{-1} = \frac{4}{3}.$$  

If $x_N \in (-a, a - y_0)$, it follows from (22) that

$$1 \geq (\phi_3 * v)(x_N) = \int_{\mathbb{R}} \phi_3(y)\uppi_a(x_N - y)dy \geq \int_{-y_0}^{0} \phi_3(y)\uppi_a(x_N - y)dy \geq M_v \int_{-y_0}^{0} \phi_3(y) \left( 1 - \frac{ry^2}{2d} \right) dy.$$  

Furthermore, it follows from $y_0 := \sqrt{\frac{d}{2}}$ and $1 - \frac{y^2}{2d} \geq 1 - \frac{\frac{d}{2}}{2d} := \frac{3}{4}$ for any $y \in [0, y_0]$ that

$$M_v \leq \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_3(y)dy \right)^{-1}.$$  

Thus, we always have

$$M_u \leq \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_1(y)dy \right)^{-1} \quad \text{and} \quad M_v \leq \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_3(y)dy \right)^{-1}.$$  

Choose

$$M = \max \left\{ \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_1(y)dy \right)^{-1}, \frac{4}{3} \left( \int_{-\sqrt{\frac{d}{2}}}^{0} \phi_3(y)dy \right)^{-1} \right\},$$

then we can get the inequality (20). This completes the proof. \(\square\)

*Take the limit of $(u_a, v_a)$ as $a \to +\infty$. It follows from the standard elliptic estimates and Lemma 2.1, we see that there exists $M_0 > 0$ such that

$$\|u_a\|_{C^{2,\alpha}(-\frac{1}{2}, \frac{1}{2})} \leq M_0 \quad \text{for any} \quad a > \max \left\{ \frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon} \right\},$$

and

$$\|v_a\|_{C^{2,\alpha}(-\frac{1}{2}, \frac{1}{2})} \leq M_0 \quad \text{for any} \quad a > \max \left\{ \frac{\ln A}{\varepsilon}, \frac{\ln B}{\varepsilon} \right\},$$

where $\alpha \in (0, 1)$ is some constant. Letting $a \to +\infty$ (possibly along a subsequence), then we know $u_a \to u$ and $v_a \to v$ in $C^{2,\alpha}_{loc}(\mathbb{R})$, and $(u(x), v(x))$ satisfies

$$-cu' - u'' = u(1 - \phi_1 * u - a_1(\phi_2 * v)), \quad x \in \mathbb{R},$$

and

$$-cu' - dv'' = rv(1 - \phi_3 * v - a_2(\phi_4 * u)), \quad x \in \mathbb{R}.$$  

Moreover, we know that $\tilde{p}_c(x) \leq u(x) \leq \min \{M, \tilde{p}_c(x)\}$ and $\tilde{q}_c(x) \leq v(x) \leq \min \{M, \tilde{q}_c(x)\}$, which implies

$$\lim_{x \to +\infty} u(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} v(x) = 0.$$
In the following, we divide four steps to complete the proof of Theorem 1.1.

**Step 1.** We show that there exists a $Z_0 > 0$ such that $u(x)$ and $v(x)$ are monotonically decreasing for $x > Z_0$.

We use a contradiction argument. Assume that $u(x)$ is not eventually monotonic as $x \to +\infty$, then there exists a sequence $z_n \to +\infty$ such that $u(x)$ achieves a local minimum at $z_n$ and $u(z_n) \to 0$, $v(z_n) \to 0$. Since

$$-cu'(z_n) - u''(z_n) = u(z_n)(1 - (\phi_1 * u)(z_n) - a_1(\phi_2 * v)(z_n)),$$

then, for any $n \in \mathbb{N}$, we have

$$\phi_1 * u)(z_n) + a_1(\phi_2 * v)(z_n) \geq 1. \quad (24)$$

On the other hand, since $u(x)$ and $v(x)$ are bounded in $C^2(\mathbb{R})$, and $\lim_{x \to +\infty} u(x) = \lim_{x \to +\infty} v(x) = 0$, it is easy to get that

$$(\phi_1 * u)(z_n) \to 0 \quad \text{and} \quad (\phi_2 * v)(z_n) \to 0 \quad \text{as} \quad n \to \infty,$$

which contradicts (24). Therefore, $u(x)$ is eventually monotonic. By using the same method, we can also get that $v(x)$ is eventually monotonic.

**Step 2.** We show that there exists no traveling wave solutions for speed $c < c^*$. We use a contradiction argument. Assume that for $c < c^*$, there exist a traveling wave solution satisfying (8) and (9). Take a sequence $z_n$ satisfying $z_n \to +\infty$ as $n \to \infty$. Let $u_n(x) = u(x + z_n)/u(z_n)$, $v_n(x) = v(x + z_n)/v(z_n)$, then we have

$$-u''_n(x) - cu'_n(x) = u_n(x) \left(1 - \phi_1 \ast \tilde{u}_n(x) - a_1(\phi_2 \ast \tilde{v}_n)(x)\right) \quad \text{in} \quad x \in \mathbb{R},$$

and

$$-dv''_n(x) - cu'_n(x) = rv_n(x) \left(1 - \phi_3 \ast \tilde{v}_n(x) - a_2(\phi_4 \ast \tilde{u}_n)(x)\right) \quad \text{in} \quad x \in \mathbb{R},$$

where $u_n(x) = u(x + z_n)$ and $v_n(x) = v(x + z_n)$. It is noting that $u_n(0) = v_n(0) = 1$ and $u_n(x), v_n(x)$ are decreasing in $(Z_0 - z_n, +\infty)$ for $n \in \mathbb{N}$, where $Z_0$ is defined by Step 1. Since $u(x) \to 0(x \to +\infty)$ and $v(x) \to 0(x \to +\infty)$, we know that $(u_n, v_n) \to (0, 0)$ locally uniformly in $x$ as $n \to +\infty$. Let $(u_n, v_n) \to (\tilde{u}(x), \tilde{v}(x))$ in $C^2_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. Then we have

$$-\tilde{u}'' - c\tilde{u}' = \tilde{u} \quad \text{in} \quad \mathbb{R} \quad (25)$$

and

$$-d\tilde{v}'' - c\tilde{v}' = r\tilde{v} \quad \text{in} \quad \mathbb{R}. \quad (26)$$

Clearly, $\tilde{u}()$ and $\tilde{v}()$ are non-increasing on $\mathbb{R}$ and satisfy $\tilde{u}(0) = \tilde{v}(0) = 1$. In addition, we can show that $\tilde{u}()$ and $\tilde{v}()$ are positive on $\mathbb{R}$. In fact, if there exists some point $x_0 \in \mathbb{R}$ such that $\tilde{u}(x_0) = 0$, then $\tilde{u}(x) = 0$ for any $x > x_0$. Now by the uniqueness of solutions of ordinary differential equations, we have $\tilde{u}(\cdot) \equiv 0$ on $\mathbb{R}$, which contradicts the fact $\tilde{u}(0) = 1$. Therefore, $\tilde{u}(x) > 0$ for $x \in \mathbb{R}$. Similarly, $\tilde{v}(x) > 0$ for $x \in \mathbb{R}$.

By the positivity of $\tilde{u}$ and $\tilde{v}$, we further have $\tilde{u}'(x) > 0$ and $\tilde{v}'(x) > 0$ for $x \in \mathbb{R}$. We now have $c \geq \max\{2, 2\sqrt{d}\}$ because equation (25) admits such a solution $\tilde{u}$ if and only if $c \geq 2$, and equation (26) admits such a solution $\tilde{v}$ if and only if $c \geq 2\sqrt{d}$. Thus, there exists no traveling wave solutions for speed $c < \min\{2, 2\sqrt{d}\}$.

**Step 3.** We show that

$$\liminf_{x \to +\infty}(u(x) + v(x)) > 0.$$

By a contradiction, we assume that there exists a sequence $y_n$ satisfying $y_n \to -\infty$ as $n \to +\infty$ such that $u(y_n) \to 0$ and $v(y_n) \to 0$ as $n \to +\infty$. Take $\tilde{u}(x) = u(x)$ for any $x \geq y_n$ and $\tilde{v}(x) = v(x)$ for any $x \geq y_n$. Since $\tilde{u}(y_n) \to 0$ and $\tilde{v}(y_n) \to 0$ as $n \to +\infty$, we have

$$\lim_{x \to +\infty}(u(x) + v(x)) = \lim_{n \to +\infty}(u(y_n) + v(y_n)) = 0,$$

which is a contradiction. Therefore, $\liminf_{x \to +\infty}(u(x) + v(x)) > 0$.

**Step 4.** We show that $u(x)$ and $v(x)$ are unbounded. We use a contradiction argument. Assume that $u(x)$ and $v(x)$ are bounded as $x \to +\infty$. Then, for any $n \in \mathbb{N}$, we have

$$u_n(x) = u(x + z_n)/u(z_n) \to u(x) \quad \text{as} \quad n \to \infty.$$
$u(-x), \tilde{v}(x) = v(-x)$ and $\tilde{c} = -c$, then $(\tilde{u}(y_n), \tilde{v}(y_n)) \to (0,0)$ and $(\tilde{u}(x), \tilde{v}(x))$ satisfies

$$\begin{cases}
-\tilde{c}u'' - \tilde{u}' = \tilde{u} (1 - \phi_1 \otimes \tilde{u} - a_1 (\phi_2 \otimes \tilde{v})), \\
-\tilde{c}v'' - \tilde{v}' = \tilde{v} (1 - \phi_3 \otimes \tilde{v} - a_2 (\phi_4 \otimes \tilde{u})),
\end{cases}$$

where $(\phi_i \otimes u)(x) = \int_{\mathbb{R}} \phi_i(y) w(x+y)dy$, $i = 1, 2, 3, 4$. Similar to Step 2, one has $\tilde{c} \geq \max \{2, 2\sqrt{\frac{C}{R}}\}$, which implies $c \leq -\max \{2, 2\sqrt{\frac{C}{R}}\}$. This is a contradiction.

**Step 4.** Finally, we show that there hold

$$\liminf_{x \to -\infty} u(x) > 0 \quad \text{and} \quad \liminf_{x \to -\infty} v(x) > 0,$$

if $a_1 < \frac{1}{M}$ and $a_2 < \frac{1}{M}$, where $M$ is defined by (10).

By a contradiction, without loss of generality we assume that

$$\liminf_{x \to -\infty} u(x) = 0,$$

then there must hold one of the following two cases:

**Case 1.** There exists a sequence $x_n \to -\infty$ as $n \to +\infty$, such that $u(x)$ attains local minimum at $x_n$ and $u(x_n) \to 0$ as $n \to +\infty$.

Recall that $0 < u(x) \leq M$ and $0 < v(x) \leq M$ for any $x \in \mathbb{R}$. In addition, $u(x)$ satisfies

$$-u''(x) - cu'(x) - (1 - (\phi_1 \ast u)(x)) (x) - a_1 (\phi_2 \ast v)(x) u(x) = 0 \quad \forall x \in \mathbb{R}.$$

Let $\tilde{u}_n(x) := u(x_n + x)$ and $\tilde{v}_n(x) := v(x_n + x)$ for any $x \in \mathbb{R}$. Then $\tilde{u}_n(x) := u(x_n + x)$ satisfies

$$-\tilde{u}_n''(x) - c\tilde{u}_n'(x) - (1 - (\phi_1 \ast \tilde{u}_n)(x)) (x) - a_1 (\phi_2 \ast \tilde{v}_n)(x) \tilde{u}_n(x) = 0 \quad \forall x \in \mathbb{R}.$$

Since $|1 - (\phi_1 \ast \tilde{u}_n)(x) - a_1 (\phi_2 \ast \tilde{v}_n)(x)| \leq 1 + M + a_1 M$ for any $x \in \mathbb{R}$, then by the Harnack inequality (see Gilbarg and Trudinger [10, Theorem 8.20]), for any $Z > 0$ there exists a constant $C > 0$, which only depends on $Z > 0$, such that

$$\sup_{x \in [-Z, Z]} \tilde{u}_n(x) \leq C\tilde{u}_n(0), \quad \forall n \in \mathbb{N}. \quad (27)$$

It follows from $\lim_{n \to \infty} \tilde{u}_n(0) = \lim_{n \to \infty} u(x_n) = 0$ that for any $\delta \in \left(0, \frac{1-a_1 M}{2}\right)$, there exists $N \in \mathbb{N}$ such that $\tilde{u}_n(x) \leq \delta$ for any $x \in (-Z, Z)$ and $n > N$, namely, $u(x) \leq \delta$ for any $x \in (x_n - Z, x_n + Z)$ and $n > N$. Now we show that

$$\lim_{n \to \infty} (\phi_1 \ast u)(x_n) = 0.$$ 

Fix any $\varepsilon > 0$. Then there exists $Z > 0$ such that

$$\int_{(-\infty, -Z] \cup [Z, \infty)} \phi_1(y)dy < \frac{\varepsilon}{2M}.$$ 

Take $\delta < \frac{\varepsilon}{2}$. Then there exists $N \in \mathbb{N}$ such that $u(x_n + y) = \tilde{u}_n(y) < \delta$ for any $y \in [-Z, Z]$ and $n > N$. Thus, we have

$$(\phi_1 \ast u)(x_n) = \int_{(-\infty, -Z] \cup [Z, \infty)} \phi_1(y)u(x_n - y)dy + \int_{-Z}^{Z} \phi_1(y)u(x_n - y)dy < \varepsilon$$

for any $n > N$. This implies that $\lim_{n \to \infty} (\phi_1 \ast u)(x_n) = 0$. Hence,

$$-cu'(x_n) - u''(x_n) = u'(x_n)(1 - (\phi_1 \ast u)(x_n) - a_1 (\phi_2 \ast v)(x_n)) \geq u(x_n)(1 - a_1 M - (\phi_1 \ast u)(x_n)) > 0$$

for sufficiently large $n$. On the other hand, since $u(x)$ attains local minimum at $x_n$, then

$$-cu'(x_n) - u''(x_n) \leq 0.$$ 

Obviously, there is a contradiction.
Case 2. \( \lim_{x \to -\infty} u(x) = 0 \) and there exists a large \( Z > 0 \) such that \( u'(x) \geq 0, \forall x < -Z \).

Since \( \lim_{x \to -\infty} u(x) = 0 \) and \( \lim \inf_{x \to -\infty}(u(x) + v(x)) > 0 \), then we know that

\[
\lim_{x \to -\infty} v(x) > 0.
\]

It follows that there exists a sequence \( x_n \to -\infty \) \((n \to +\infty)\) such that

\[
\lim_{n \to +\infty} v(x_n) = \lim \inf_{x \to -\infty} v(x) = A > 0,
\]

and

\[
\lim_{n \to +\infty} u(x_n) = 0,
\]

where \( A \) is a constant. Let \( u_n(x) = u(x + x_n)/u(x_n), v_n(x) = v(x + x_n)/v(x_n), \) \( \tilde{u}_n(x) = u(x + x_n) \) and \( \tilde{v}_n(x) = v(x + x_n) \), then we have

\[
-cu'_n(x) - u''_n(x) = u_n(x)(1 - (\phi_1 * \tilde{u}_n)(x) - a_1(\phi_2 * \tilde{v}_n)(x)) \quad \forall x \in \mathbb{R}.
\]

Assume that \( u_n(x) \) converges to \( \tilde{u}(x) \) and \( v_n(x) \) converges to \( \tilde{v}(x) \) in \( C^2_{loc}(\mathbb{R}) \) as \( n \to \infty \). Since \( \lim_{n \to \infty} \tilde{u}_n(0) = 0 \), it follows from the Harnack inequality (see (27)) that \( \tilde{u}_n(x) \to 0 \) in \( C^2_{loc}(\mathbb{R}) \) as \( n \to \infty \). Then applying the \( L^p \) interior estimates and the imbedding theorem (see Gilbarg and Trudinger [10]), we have \( u_n(x) \to 0 \) in \( C^2_{loc}(\mathbb{R}) \) as \( n \to \infty \). In addition, assume that \( \tilde{v}_n(x) \to \tilde{v}(x) \) in \( C^2_{loc}(\mathbb{R}) \) when \( n \to +\infty \). Since \( u'(x) \geq 0 \) for \( x < -Z \), then we have \( \tilde{u}(x) \geq 0 \) for any \( x \in \mathbb{R} \). Since \( \tilde{u}(x) \) satisfies

\[
-c\tilde{u}'(x) - \tilde{u}''(x) = \tilde{u}(x)(1 - a_1(\phi_2 * \tilde{v}))(x), \quad \forall x \in \mathbb{R},
\]

we obtain

\[
-c\tilde{u}(x) + c\tilde{u}'(0) - \tilde{u}'(0) = \int_0^x \tilde{u}(y)(1 - a_1(\phi_2 * \tilde{v}))(y)dy > (1 - a_1M)\tilde{u}(0)x \quad (28)
\]

for any \( x > 0 \). Since \( \tilde{u}(x) > 0, \tilde{u}'(x) \geq 0 \) and \( \tilde{u}(0) = 1 \), then for sufficiently large \( x \), the inequality (28) is not true, which is a contradiction. Thus, one has \( \lim \inf_{x \to -\infty} u(x) > 0 \). Consequently, the proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2. In this section we prove Theorem 1.2, namely consider the case of \( \phi_1(x) = \phi_3(x) = \delta(x) \) in system (2). Then system (2) reduces to

\[
\begin{cases}
 u_t = u_{xx} + u(1 - u - a_1(\phi_2 * v)), \\
 v_t = d v_{xx} + rv(1 - v - a_2(\phi_4 * u)),
\end{cases} \quad (29)
\]

and the corresponding traveling wave solutions satisfies

\[
\begin{align*}
-cu' - u'' &= u(1 - u - a_1(\phi_2 * v)), \\
-cd v' - dv'' &= rv(1 - v - a_2(\phi_4 * u)).
\end{align*}
\]

In order to prove Theorem 1.2, we first establish the existence of solutions of system (30) for any \( c > c^* = \max\{2, 2 \sqrt{d}r\} \). The method is similar to that in Section 2. In the following we fix \( c > c^* = \max\{2, 2 \sqrt{d}r\} \).

Supersolution. Take

\[
\bar{\rho}_c(x) = \min\{e^{-\lambda_c x}, 1\} \quad \text{and} \quad \bar{\eta}_c(x) = \min\{e^{-\zeta_c x}, 1\}, \quad \forall x \in \mathbb{R},
\]

where \( \lambda_c \) and \( \zeta_c \) is defined in (14) and (15). Then one has

\[
-c\bar{\rho}_c' - c\bar{\rho}_c'' \geq \bar{\rho}_c(1 - \bar{\rho}_c) \quad \text{and} \quad -d\bar{\eta}_c' - d\bar{\eta}_c'' \geq r\bar{\eta}_c(1 - \bar{\eta}_c) \quad \text{in} \quad \mathbb{R}.
\]
Subsolution. Take
\[ p^c(x) = e^{-\lambda_c x} - \tilde{A}e^{-(\lambda_c + \varepsilon)x} \quad \text{and} \quad q^c(x) = e^{-\zeta_c x} - \tilde{B}e^{-(\zeta_c + \varepsilon)x} \]
where \( \varepsilon \in (0, \min\{\lambda_c, \zeta_c\}) \) is small enough such that
\[ \kappa_c = - (\lambda_c + \varepsilon)^2 + c(\lambda_c + \varepsilon) - 1 > 0, \]
and
\[ \iota_c = - d(\zeta_c + \varepsilon)^2 + c(\zeta_c + \varepsilon) - r > 0. \]
In addition, \( \tilde{A} > 1 \) is large enough such that
\[ \frac{\ln \tilde{A}}{\varepsilon} > \max \left\{ \frac{1}{\lambda_c - \varepsilon} \ln \frac{2}{A\kappa_c}, \frac{1}{\zeta_c - \varepsilon} \ln \frac{2a_1 Z_0^c}{A\kappa_c} \right\}, \]
and \( \tilde{B} > 1 \) is large enough such that
\[ \frac{\ln \tilde{B}}{\varepsilon} > \max \left\{ \frac{1}{\lambda_c - \varepsilon} \ln \frac{2r a_2 Z_0^c}{B\iota_c}, \frac{1}{\zeta_c - \varepsilon} \ln \frac{2r}{B\iota_c} \right\}, \]
where \( Z_0^c \) and \( Z_0^c \) is defined in (16) and (17) respectively. Furthermore, let
\[ \tilde{p}_c(x) = \max \left\{ 0, p^c(x) \right\} \quad \text{and} \quad \tilde{q}_c(x) = \max \left\{ 0, q^c(x) \right\}, \quad x \in \mathbb{R}, \]
then similar to those done in Section 2 for \( \tilde{p}_c(x) \) and \( \tilde{q}_c(x) \), one has
\[ -c\tilde{p}_c'' - \tilde{p}_c' \leq \tilde{p}_c'' - (\tilde{p}_c)^2 - a_1 \tilde{p}_c(\phi_2 * \overline{q}_c) \]
for any \( x \neq \frac{\ln \tilde{A}}{\varepsilon} \) and
\[ -c\tilde{q}_c'' - \tilde{q}_c' \leq \tilde{q}_c'' - r(\tilde{q}_c)^2 - r a_2 \tilde{q}_c(\phi_4 * \overline{p}_c) \]
for any \( x \neq \frac{\ln \tilde{B}}{\varepsilon} \).

A two-point boundary value problem. For \( c > \max \left\{ 2, 2\sqrt{dr} \right\} \), we consider the following problem in a finite domain \((-a, a)\):
\[
\begin{align*}
- cu' - u'' &= u \left(1 - u - a_1 (\phi_2 * \overline{u})\right), \\
- cu' - dv'' &= v \left(1 - v - a_2 (\phi_4 * \overline{v})\right), \\
\quad u(\pm a) &= \tilde{p}_c(\pm a), \quad v(\pm a) = \tilde{q}_c(\pm a),
\end{align*}
\]
(31)
where \( a > \max \left\{ \frac{\ln \tilde{A}}{\varepsilon}, \frac{\ln \tilde{B}}{\varepsilon} \right\} \). In order to obtain the existence of the problem (31), we consider the following two-point boundary value problem
\[
\begin{align*}
- cu' - u'' + u_0 u + a_1 (\phi_2 * \overline{u}_0) u &= u_0, \\
- cu' - dv'' + v_0 v + a_2 (\phi_4 * \overline{v}_0) v &= v_0, \\
\quad u(\pm a) &= \tilde{p}_c(\pm a), \quad v(\pm a) = \tilde{q}_c(\pm a),
\end{align*}
\]
(32)
where \( (u_0, v_0) \in \mathcal{M}_a \) and
\[
\overline{u}_0(x) = \begin{cases} 
  u_0(a), & x > a, \\
  \overline{u}_0(x), & x \in [-a, a], \\
  u_0(-a), & x < -a,
\end{cases} \quad \overline{v}_0(x) = \begin{cases} 
  v_0(a), & x > a, \\
  \overline{v}_0(x), & x \in [-a, a], \\
  v_0(-a), & x < -a.
\end{cases}
\]
Here the convex set $\mathcal{M}_a$ is defined as
\[ \mathcal{M}_a = \left\{ (u, v) \in C([-a, a], \mathbb{R}^2) \mid \begin{cases} \tilde{p}_c(x) \leq u(x) \leq \tilde{p}_a(x), & x \in (-a, a), \\ \tilde{q}_c(x) \leq v(x) \leq \tilde{q}_a(x), & x \in (-a, a), \\ u(\pm a) = \tilde{p}_c(\pm a), & v(\pm a) = \tilde{q}_c(\pm a) \end{cases} \right\}. \]

Let $\Phi_a$ be the solution mapping of the problem (32). Namely, $\Phi_a(u_0, v_0) = (u, v)$. Similar the arguments in Section 2, we have that $\Phi_a$ has a fixed point $(u_0, v_0)$ in $\mathcal{M}_a$, which is just the solution of (31). In addition, it is easy to see that $u_0(x) \leq 1$ and $v_0(x) \leq 1$. Let $a \to +\infty$ (possibly along a subsequence), then we know $u_0 \to u$ and $v_0 \to v$ in $C^2_{\text{loc}}(\mathbb{R})$ and $(u, v)$ satisfies
\[ \begin{cases} -cu' - u'' = u(1 - u - a_1\phi_2 * v), & x \in \mathbb{R}, \\ -cv' - dv'' = rv(1 - v - a_2\phi_4 * u), & x \in \mathbb{R}. \end{cases} \]

Moreover, we have that
\[ \tilde{p}_a(x) \leq u(x) \leq \tilde{p}_c(x) \quad \text{and} \quad \tilde{q}_a(x) \leq u(x) \leq \tilde{q}_c(x), \]
which combining the strong maximum principle implies that $0 < u(x) < 1$, $0 < v(x) < 1$, and
\[ \lim_{x \to +\infty} u(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} v(x) = 0. \]

In particular, we have the following observation.

**Lemma 3.1.** The solution $(u, v)$ of system (30) satisfies
\[ 0 < \liminf_{x \to -\infty} u(x) \leq \limsup_{x \to -\infty} u(x) < 1 \quad \text{and} \quad 0 < \liminf_{x \to -\infty} v(x) \leq \limsup_{x \to -\infty} v(x) < 1. \] (33)

**Proof.** (i) Firstly, we show $0 < \liminf_{x \to -\infty} u(x)$ and $0 < \liminf_{x \to -\infty} v(x)$. We use a contradiction argument. Assume that (33) is not true, then there holds one of
\[ \liminf_{x \to -\infty} u(x) = 0 \quad \text{and} \quad \liminf_{x \to -\infty} v(x) = 0. \]

Without loss of generality, we assume that $\liminf_{x \to -\infty} u(x) = 0$, which includes two cases: (i) there exist a sequence $x_n \to -\infty$ ($n \to \infty$), such that $u(x_n) \to 0$ as $n \to \infty$ and $u(x)$ achieves a local minimum at $x_n$; (ii) $\lim_{x \to -\infty} u(x) = 0$ and there exist a $Z > 0$, such that $u'(x) \geq 0$ for any $x < -Z$.

If the case (i) is true, since $u(x)$ attains local minimum at $x_n$, then
\[ u'(x_n) = 0, \quad u''(x_n) \geq 0. \] (34)

Following from (34) and
\[ -cu'(x_n) - u''(x_n) = u(x_n)(1 - u(x_n) - a_1(\phi_2 * v)(x_n)), \]
we have
\[ 1 - u(x_n) - a_1(\phi_2 * v)(x_n) \leq 0, \quad \forall n \in \mathbb{N}. \]

On the other hand, since $u(x_n) \to 0$ and $v \leq 1$, then
\[ 1 - u(x_n) - a_1(\phi_2 * v)(x_n) \geq 1 - u(x_n) - a_1 \to 1 - a_1 \]
when $n \to +\infty$. There is a contradiction.

If the case (ii) holds, let $w(x, t) = u(x - ct)$, since $u \leq 1$ and $v \leq 1$, then
\[ \partial_tw \geq \partial_{xx}w + u(1 - a_1 - w). \]
Consider the equation
\[ \partial_t \hat{u} = \partial_{xx} \hat{u} + \hat{u}(1 - a_1 - \hat{u}). \]

From the asymptotic propagation theory of Fisher-KPP equation, we know that, for any \( \varphi_0(x) \) with compact support and \( 0 \leq \varphi_0(x) \leq 1 - a_1 \), one has
\[ \lim_{t \to +\infty, |x| \leq c_1 t} \hat{u}(x, t; \varphi_0) = 1 - a_1, \quad 0 < c_1 < c_1', \]
where \( c_1' = 2\sqrt{1 - a_1} \). Furthermore, let \( \varphi_0(x) < w(x, 0) \) for any \( x \in \mathbb{R} \), one has \( w(x, t) \geq \hat{u}(x, t; \varphi_0) \) and hence
\[ w(0, t) \geq \hat{u}(0, t; \varphi_0), \]
which implies
\[ 0 = \lim_{t \to +\infty} w(0, t) \geq \lim_{t \to +\infty} \hat{u}(0, t; \varphi_0) = 1 - a_1 > 0. \]
This is a contradiction.

Combining the above two cases, we conclude \( \liminf_{x \to -\infty} u(x) > 0 \). Similarly, we can also get \( \liminf_{x \to -\infty} v(x) > 0 \).

(ii) Secondly, we show \( \limsup_{x \to -\infty} u(x) < 1 \) and \( \limsup_{x \to -\infty} v(x) < 1 \). We also use a contradiction argument. Without loss of generality, we assume that \( \limsup_{x \to -\infty} u(x) = 1 \). Then there exists \( x_n \to -\infty \) \( (n \to \infty) \) such that \( u(x_n) \to 1 \) as \( n \to \infty \). Let \( u_n(x) = u(x + x_n) \) and \( v_n(x) = v(x + x_n) \) for \( x \in \mathbb{R} \). Then \( (u_n, v_n) \to (u_*, v_*) \) in \( C^2_{loc}(\mathbb{R}) \) as \( n \to \infty \). In particular, \( (u_*, v_*) \) satisfies \( u_*(0) = 1, 0 < u_*(x) \leq 1, (\phi_2 \ast v_*)(0) \leq \liminf_{x \to -\infty} v(x) > 0 \), and
\[ -c_1 u'_*(0) - u''_*(0) = u_*(0)(1 - u(0) - a_1(\phi_2 \ast v_*)(0)). \]
Due to \( u'_*(0) = 0 \) and \( u''_*(0) \leq 0 \), there is a contradiction. This completes the proof of the lemma.

To complete the proof of Theorem 1.2, we need only to show that
\[ \lim_{x \to -\infty} u(x) = u^* \quad \text{and} \quad \lim_{x \to -\infty} v(x) = v^*. \]
For \( s \in [0, 1] \), define
\[
\begin{align*}
    a_1(s) &= su^*, \\
    b_1(s) &= su^* + (1 - s), \\
    a_2(s) &= su^*, \\
    b_2(s) &= su^* + (1 - s), \\
    a(s) &= (a_1(s), a_2(s)), \\
    b(s) &= (b_1(s), b_2(s)).
\end{align*}
\]

By a direct computation, we have that for any \( s \in (0, 1) \) and \( u = (u_1, u_2) \in [a(s), b(s)] \), we have
\[
\begin{align*}
    f_1(u_1, u_2) &= u_1(a_1(s), b_1(s), u_1, 0) > 0 & \text{if} & \quad u_1 = a_1(s), \\
    f_1(u_1, u_2) &< 0 & \text{if} & \quad u_1 = b_1(s), \\
    f_2(u_1, u_2) &= u_2(a_2(s), b_2(s), u_2, 0) > 0 & \text{if} & \quad u_2 = a_2(s), \\
    f_2(u_1, u_2) &> 0 & \text{if} & \quad u_2 = a_2(s),
\end{align*}
\]
where
\[ f_1(u_1, u_2) = u_1(1 - u_1 - a_1 u_2), \quad f_2(u_1, u_2) = u_2(1 - u_2 - a_2 u_1). \]

Denote
\[
\liminf_{x \to -\infty} u(x) = u^*, \quad \limsup_{x \to -\infty} u(x) = u^*.
\]
and
\[ \liminf_{x \to -\infty} v(x) = \underline{v}^*, \quad \limsup_{x \to -\infty} v(x) = \overline{v}^*. \]

Then there exists \( s_0 \in (0, 1] \) such that
\[ a_1(s) < \underline{v}^* \leq \overline{v}^* < b_1(s) \quad \text{and} \quad a_2(s) < \underline{v}^* \leq \overline{v}^* < b_2(s) \]
for any \( s \in [0, s_0) \). To complete the proof, it is sufficient to show
\[ a_1(s) < \underline{v}^* \leq \overline{w}^* < b_1(s), \quad a_2(s) < \underline{v}^* \leq \overline{w}^* < b_2(s), \quad s \in [0, 1) \]
and
\[ a_1(1) = u^* = \overline{v}^* = b_1(1) = u^*, \quad a_2(1) = u^* = \overline{v}^* = b_2(1) = v^*. \]

Otherwise, we assume, without loss of generality, that there exists a \( s_1 \in (0, 1) \) such that \( \underline{v}^* = a_1(s_1) \) and
\[ a_1(s) < \underline{v}^* \leq \overline{w}^* < b_1(s), \quad a_2(s) < \underline{v}^* \leq \overline{w}^* < b_2(s) \]
for \( s \in (0, s_1) \). Note that \( 0 < u \leq 1 \) and \( 0 < v \leq 1 \). Let \( w_k(x) := u(k + x) \) and \( \tilde{w}_k(x) := v(k + x) \) for any \( x \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Then there holds
\[ -w''_k(x) - cw'_k(x) - (1 - w_k(x) - a_1(\phi_2 * \tilde{w}_k)(x))w_k(x) = 0 \quad \forall x \in \mathbb{R}. \]

Since \( 0 < w_k(x) \leq 1 \) and \( 0 < \tilde{w}_k(x) \leq 1 \) for any \( x \in \mathbb{R} \) and \( k \in \mathbb{Z} \), it follows from the \( L^p \) interior estimates (see [10, Theorem 9.11]) that there exists some \( C_1 > 0 \) such that
\[ \|w_k\|_{L^p; (-2, 2)} \leq C_1 \|w_k\|_{L^p; (-3, 3)} \leq C_1 6^\frac{p}{2} \]
for any \( k \in \mathbb{Z} \), where \( p > 1 \), \( \| \cdot \|_{L^p; (-2, 2)} \) and \( \| \cdot \|_{L^p; (-3, 3)} \) denote the norms of the space \( W^{2,p}((-2, 2)) \) and \( L^p((-3, 3)) \), respectively. Furthermore, by the imbedding theorem, there exists some \( C_2 > 0 \) such that \( |w'_k(x)| \leq C_2 \) for any \( x \in [-2, 2] \) and \( k \in \mathbb{Z} \). By the arbitrariness of \( k \in \mathbb{Z} \), we get
\[ |u'(x)| \leq C_2 \quad \forall x \in \mathbb{R}. \]

Similarly, we can get
\[ |v'(x)| \leq C_2 \quad \forall x \in \mathbb{R} \]
for some constant \( C_2 > 0 \). By the equations satisfied by \( u \) and \( v \), we further have that there exists a constant \( C_3 > 0 \) such that
\[ |u''(x)|, |v''(x)| \leq C_3 \quad \forall x \in \mathbb{R}, \]
which implies that \( u', v', u'', v'' \) are bounded on \( \mathbb{R} \). In the following we consider two cases: (i) \( \underline{v}^* < \overline{v}^* \); (ii) \( \underline{v}^* = \overline{v}^* \).

If \( \underline{v}^* < \overline{v}^* \), then there exists a sequence \( x_n \) satisfying \( \lim_{n \to +\infty} x_n = -\infty \) such that
\[ \liminf_{x \to -\infty} u(x) = \lim_{n \to +\infty} u(x_n) = \underline{v}^* = a_1(s_1) = \limsup_{x \to -\infty} u(x) \leq b_1(s_1) \]
and
\[ \liminf_{n \to +\infty} (-cu'(x_n) - u''(x_n)) \leq 0. \]

In addition, it is clear that
\[ a_2(s_1) \leq \liminf_{n \to +\infty} (\phi_2 * v)(x_n) \leq \limsup_{n \to +\infty} (\phi_2 * v)(x_n) \leq b_2(s_1). \]

Using (35), we have
\[ 0 \geq \liminf_{n \to +\infty} (-cu'(x_n) - u''(x_n)) \]
\[ = \liminf_{n \to +\infty} f_1(u(x_n), (\phi_2 * v)(x_n)) \]
and we have
\[ f_1 \left( a_1(s_1), \limsup_{n \to +\infty} (\phi_2 * v)(x_n) \right) \geq f_1 \left( a_1(s_1), b_2(s_1) \right) > 0, \]
which is a contradiction.

If \( u^* = \pi^* = a_1(s_1) \), then we have
\[
\lim_{x \to -\infty} u(x) = a_1(s_1), \quad \lim_{x \to -\infty} u'(x) = \lim_{x \to -\infty} u''(x) = 0.
\]
and
\[
a_2(s_1) \leq \liminf_{n \to +\infty} (\phi_2 * v)(x) \leq \limsup_{n \to +\infty} (\phi_2 * v)(x) \leq b_2(s_1).
\]
Using (35) again, we have
\[
0 = \liminf_{x \to -\infty} (-cu'(x) - u''(x)) = \liminf_{x \to -\infty} f_1(u(x), (\phi_2 * v)(x)) \geq f_1 \left( a_1(s_1), \limsup_{x \to -\infty} (\phi_2 * v)(x) \right) \geq f_1 (a_1(s_1), b_2(s_1)) > 0
\]
which is also a contradiction.

To complete the proof of Theorem 1.2, the remainder is to show the nonexistence of traveling wave solutions for \( c < c^* \), which can be done by an argument similar to those in Theorem 1.1. Thus, we have completed the proof of Theorem 1.2.

4. Numerical simulations. In Section 2, the existence of traveling wave solutions connecting \((0, 0)\) to an unknown positive steady state for all speeds \( c > c^* \) is proved. Furthermore, we consider the special case that \( \phi_1(x) = \phi_3(x) = \delta(x) \) (\( \delta(x) \) is Dirac function) in Section 3. However, we do not know what the shape of the wave profiles will be. In this section, we will investigate these features of traveling wave solutions by numerical simulations and give some explanations about those results. For convenience of calculations, let \( \phi_2(x) = \phi_4(x) = \delta(x) \) and \( \phi_1(x) = \phi_3(x) = \phi(x) \), then (2) can simplify into
\[
\begin{cases}
  u_t - u_{xx} = u(1 - \phi * u - a_1v), \\
  v_t - dv_{xx} = rv(1 - \phi * v - a_2u).
\end{cases}
\]
(36)

In the following, we always take a specific kernel function \( \phi(x) = \phi_\sigma(x) = \frac{\sigma}{2} e^{-\frac{|x|}{\sigma^2}} \), where \( A = \frac{\sigma^2}{2} > 0 \), \( a \in \left( \frac{2}{3}, \sqrt{\frac{2}{3}} \right) \).

Let \( \phi_\sigma^+ (x) = \frac{\sigma}{2} e^{-\frac{|x|}{\sigma^2}} \) and \( \phi_\sigma^- (x) = -\frac{1}{\sigma} e^{-\frac{|x|}{\sigma^2}} \). Define
\[
\tilde{u}(t, x) = (\phi_\sigma^+ * u) (t, x), \quad \tilde{u}(t, x) = (\phi_\sigma^- * u) (t, x),
\]
and
\[
\tilde{v}(t, x) = (\phi_\sigma^+ * v) (t, x), \quad \tilde{v}(t, x) = (\phi_\sigma^- * v) (t, x),
\]
then we have
\[
\tilde{u}_{xx} = -\frac{a^2}{\sigma^2} (3u - \tilde{u}), \quad \tilde{u}_{xx} = -\frac{1}{\sigma^2} (-2u - \tilde{u}),
\]
and
\[ \ddot{v}_{xx} = -\frac{a^2}{\sigma^2} (3v - \ddot{v}), \quad \ddot{v}_{xx} = -\frac{1}{\sigma^2} (-2v - \ddot{v}). \]

Consequently system (36) reduces to
\[
\begin{align*}
\begin{cases}
  u_t - u_{xx} = u(1 - \ddot{u} - \ddot{a} - a_1 v), \\
  v_t - dv_{xx} = rv(1 - \ddot{v} - \ddot{a} - a_2 u), \\
  0 = \ddot{u}_{xx} + \frac{a^2}{\sigma^2} (3u - \ddot{u}), \\
  0 = \ddot{u}_{xx} + \frac{1}{\sigma^2} (-2u - \ddot{u}), \\
  0 = \ddot{v}_{xx} + \frac{3}{\sigma^2} (3v - \ddot{v}), \\
  0 = \ddot{v}_{xx} + \frac{1}{\sigma^2} (-2v - \ddot{v}).
\end{cases}
\end{align*}
\]

(37)

Obviously, system (37) have positive equilibrium
\[
(u^*, v^*, \ddot{u}^*, \ddot{v}^*) = \left(\frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2}, \frac{3(1 - a_1)}{1 - a_1 a_2} - \frac{2(1 - a_1)}{1 - a_1 a_2}, \frac{3(1 - a_2)}{1 - a_1 a_2} - \frac{2(1 - a_2)}{1 - a_1 a_2}\right).
\]

(38)

Next, we give initial value conditions. Take
\[
u(x, 0) = \begin{cases} 
\frac{1-a_1}{1-a_1 a_2}, & \text{for } x \leq L_0, \\
0, & \text{for } x > L_0,
\end{cases}
\]

(39)

and
\[
v(x, 0) = \begin{cases} 
\frac{1-a_2}{1-a_1 a_2}, & \text{for } x \leq L_0, \\
0, & \text{for } x > L_0.
\end{cases}
\]

(40)

By the definition of \( \ddot{u}(t, x) \) and \( \ddot{u}(t, x) \), we know that
\[
\ddot{u}(x, 0) = \int_{\mathbb{R}} \frac{A}{\sigma} e^{-\frac{1}{\sigma}|x-y|} u(y, 0) dy
\]

\[
= \begin{cases} 
3 \frac{1-a_1}{1-a_1 a_2} \left(1 - \frac{1}{2} e^{\frac{a(x-L_0)}{\sigma}}\right), & \text{for } x \leq L_0, \\
3 \frac{1-a_2}{1-a_1 a_2} e^{-\frac{a(x-L_0)}{\sigma}}, & \text{for } x > L_0,
\end{cases}
\]

(41)

and
\[
\ddot{u}(x, 0) = \int_{\mathbb{R}} -\frac{1}{\sigma} e^{-\frac{1}{\sigma}|x-y|} v(y, 0) dy
\]

\[
= \begin{cases} 
-2 \frac{1-a_1}{1-a_1 a_2} \left(1 - \frac{1}{2} e^{\frac{a(x-L_0)}{\sigma}}\right), & \text{for } x \leq L_0, \\
- \frac{1-a_2}{1-a_1 a_2} e^{\frac{a(x-L_0)}{\sigma}}, & \text{for } x > L_0.
\end{cases}
\]

(42)

Similarly, we can obtain
\[
\ddot{v}(x, 0) = \int_{\mathbb{R}} \frac{A}{\sigma} e^{-\frac{1}{\sigma}|x-y|} v(y, 0) dy
\]

\[
= \begin{cases} 
3 \frac{1-a_2}{1-a_1 a_2} \left(1 - \frac{1}{2} e^{\frac{a(x-L_0)}{\sigma}}\right), & \text{for } x \leq L_0, \\
3 \frac{1-a_2}{1-a_1 a_2} e^{-\frac{a(x-L_0)}{\sigma}}, & \text{for } x > L_0,
\end{cases}
\]

(43)

and
\[
\ddot{v}(x, 0) = \int_{\mathbb{R}} -\frac{1}{\sigma} e^{-\frac{1}{\sigma}|x-y|} v(y, 0) dy
\]
Before giving numerical simulation, we also need to know the boundary condition. Here, the zero-flux boundary condition is considered. Along with (39)-(44), system (37) can be simulated through the pedpe package in Matlab (see Fig 1 and Fig 2).

\[
\begin{aligned}
\frac{2(1-u_2)}{1-a_1a_2} \left( 1 - \frac{1}{2} e^{-\frac{x-x_0}{\sigma}} \right), & \quad \text{for } x \leq L_0, \\
-\frac{1-a_2}{1-a_1a_2} e^{-\frac{x-x_0}{\sigma}}, & \quad \text{for } x > L_0.
\end{aligned}
\]

(44)

From Figs 1 and 2, it can be seen that the solution \((u(x,t), v(x,t))\) of system (36) firstly occurs a 'hump' as \(\sigma\) increases. If further increase \(\sigma\), then the stability of \(u = u^* = \frac{1-a_1}{1-a_1a_2}\) and \(v = v^* = \frac{1-a_2}{1-a_1a_2}\) will change and a periodic steady state will arise around \(u = u^*\) and \(v = v^*\) respectively.

Now we give a simple explanation of this phenomenon. Linearizing system (37) near the equilibrium \((u^*, v^*, \tilde{u}^*, \tilde{v}^*, \hat{v}^*)\) defined in (38), we can get

\[
\begin{aligned}
&u_t - u_{xx} = -a_1 \cdot \frac{1-a_2}{1-a_1a_2} u - \frac{1-a_1}{1-a_1a_2} \tilde{u} - \frac{1-a_1}{1-a_1a_2} \hat{v}, \\
v_t - dv_{xx} = -a_2 \frac{1-a_1}{1-a_1a_2} u - r \frac{1-a_2}{1-a_1a_2} \tilde{v} - r \frac{1-a_2}{1-a_1a_2} \hat{v}, \\
0 = \tilde{u}_{xx} + \frac{1}{\sigma^2} (3a^2 u - a^2 \tilde{u}), \\
0 = \tilde{u}_{xx} + \frac{1}{\sigma^2} (-2u - \tilde{u}), \\
0 = \tilde{v}_{xx} + \frac{1}{\sigma^2} (3a^2 u - a^2 \tilde{v}), \\
0 = \tilde{v}_{xx} + \frac{1}{\sigma^2} (-2u - \tilde{v}).
\end{aligned}
\]

(45)
Figure 2. The time and space evolution of \( v(x, t) \) in nonlocal equation (36) with kernel \( \phi_{\sigma}(x) = \frac{3a}{2\sigma} e^{-\frac{a}{\sigma}|x|} - \frac{1}{2} e^{-\frac{|x|}{\sigma}} \). Our computational domain is \( x \in [0, 85], \ t \in [0, 30] \). The corresponding parameter values are: \( L_0 = 15, \ a = 0.7, \ a_1 = 0.4, \ a_2 = 0.5, \ d = 0.1, \ r = 2 \) and \( \sigma \) follows by 0.3, 0.6, 1.2, 1.5.

Take the test function with the form of

\[
\begin{pmatrix}
  u \\
  v \\
  \tilde{u} \\
  \tilde{v}
\end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix}
  C_1^k \\
  C_2^k \\
  C_3^k \\
  C_4^k \\
  C_5^k \\
  C_6^k
\end{pmatrix} e^{\lambda + ikx},
\]

where \( \lambda \) is the growth rate of perturbations in time \( t \), \( i \) is the imaginary unit and \( k \) is the wave number. Substituting (46) into (45) gives

\[
\begin{vmatrix}
  -k^2 - \lambda & -a_1 \frac{1-a_1}{1-a_1 a_2} & -a_1 \frac{1-a_1}{1-a_1 a_2} & -a_1 \frac{1-a_1}{1-a_1 a_2} & -a_1 \frac{1-a_1}{1-a_1 a_2} & 0 \\
  -a_1 \frac{1-a_1}{1-a_1 a_2} & 0 & 0 & 0 & 0 & 0 \\
  -a_2 \frac{1-a_2}{1-a_1 a_2} \frac{3a_1}{\sigma^2} & 0 & -a_1 \frac{1-a_1}{1-a_1 a_2} & 0 & 0 & 0 \\
  -a_2 \frac{1-a_2}{1-a_1 a_2} \frac{3a_1}{\sigma^2} & 0 & 0 & 0 & 0 & 0 \\
  -a_2 \frac{1-a_2}{1-a_1 a_2} \frac{3a_1}{\sigma^2} & 0 & -a_1 \frac{1-a_1}{1-a_1 a_2} \frac{3a_1}{\sigma^2} & -k^2 & 0 & 0 \\
  0 & 0 & 0 & 0 & -a_1 \frac{1-a_1}{1-a_1 a_2} \frac{3a_1}{\sigma^2} & -k^2
\end{vmatrix} = 0,
\]

which is equivalent to

\[ B \lambda^2 + D \lambda + E = 0, \]
where
\[ B = \left( \frac{1}{\sigma^2} + k^2 \right)^2 \left( \frac{a^2}{\sigma^2} + k^2 \right)^2, \]
\[ D = (d + 1)k^2 \left( \frac{1}{\sigma^2} + k^2 \right)^2 \left( \frac{a^2}{\sigma^2} + k^2 \right)^2 + \frac{3a^2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \left( \frac{a^2}{\sigma^2} + k^2 \right) \left( \frac{1}{\sigma^2} + k^2 \right)^2 \]
\[ - \frac{2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \left( \frac{1}{\sigma^2} + k^2 \right) \left( \frac{a^2}{\sigma^2} + k^2 \right)^2 + \frac{3a^2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \left( \frac{a^2}{\sigma^2} + k^2 \right)^2, \]
\[ E = \left( \frac{1}{\sigma^2} + k^2 \right)^2 \left( \frac{a^2}{\sigma^2} + k^2 \right)^2 \left( dk^4 - ra_1a_2 \frac{(1 - a_1)(1 - a_2)}{(1 - a_1a_2)^2} \right) \]
\[ + dk^2 \left[ \frac{3a^2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \left( \frac{a^2}{\sigma^2} + k^2 \right) \left( \frac{1}{\sigma^2} + k^2 \right)^2 \right. \]
\[ - \frac{2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \left( \frac{1}{\sigma^2} + k^2 \right) \left( \frac{a^2}{\sigma^2} + k^2 \right)^2 \left[ \frac{2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \left( \frac{a^2}{\sigma^2} + k^2 \right) - \frac{3a^2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \left( \frac{1}{\sigma^2} + k^2 \right) \right] \]
\[ - \left. k^2 \left( \frac{1}{\sigma^2} + k^2 \right) \left( \frac{a^2}{\sigma^2} + k^2 \right) \right]. \]

Since
\[ D = \frac{a^4}{\sigma^8} \cdot \frac{1 - a_1}{1 - a_1a_2} \cdot \frac{1}{\sigma^4} \cdot \frac{a^2}{\sigma^2} - \frac{2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \cdot \frac{1}{\sigma^2} \cdot \frac{a^4}{\sigma^4} \]
\[ + \frac{3a^2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \cdot \frac{1}{\sigma^4} \cdot \frac{a^2}{\sigma^2} - \frac{2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \cdot \frac{a^4}{\sigma^4} \cdot \frac{1}{\sigma^2} \]
\[ = \frac{a^4}{\sigma^8} \cdot \frac{1 - a_1}{1 - a_1a_2} + \frac{ra^4}{\sigma^8} \cdot \frac{1 - a_2}{1 - a_1a_2} > 0 \] (47)
when \( k = 0 \), the Hopf bifurcation can not arise. However,
\[ E = - \frac{a^4}{\sigma^8} ra_1a_2 \cdot \frac{(1 - a_1)(1 - a_2)}{(1 - a_1a_2)^2} + \left( \frac{2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \cdot \frac{a^2}{\sigma^2} - \frac{3a^2r}{\sigma^2} \cdot \frac{1 - a_2}{1 - a_1a_2} \cdot \frac{1}{\sigma^2} \right) \]
\[ \times \left( \frac{2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \cdot \frac{a^2}{\sigma^2} - \frac{3a^2}{\sigma^2} \cdot \frac{1 - a_1}{1 - a_1a_2} \cdot \frac{1}{\sigma^2} \right) \]
\[ = \frac{ra^4(1 - a_1a_2)}{\sigma^8} \cdot \frac{(1 - a_1)(1 - a_2)}{(1 - a_1a_2)^2} > 0 \] (48)
when \( k = 0 \). In addition, there exists \( k > 0 \) satisfying \( E < 0 \) when \( \sigma \to +\infty \). Then it is easy to know that there exist \( k_T > 0 \) and \( \sigma_T > 0 \) such that
\[ \begin{align*}
E'(k_T) &= 0, \\
E(k_T, \sigma_T) &= 0.
\end{align*} \]
Combining (47) and (48), we know that the Turing bifurcation will occur in system (36) at the critical wave number $k_T$ when $\sigma = \sigma_T$. Therefore, it can be seen that the traveling wave solution connects $(0,0)$ and a periodic steady state in Fig 1 and Fig 2.

**Acknowledgments.** The authors are grateful to the anonymous referees for their insightful comments and suggestions helping to the improvement of the manuscript. The first author was supported by NNSF of China (11801470). The second author were supported by NNSF of China (11371179, 11731005) and the Fundamental Research Funds for the Central Universities (lzujbky-2017-ot09). The third author was supported by NNSF of China (11871251, 11771185).

**REFERENCES**

[1] M. Alfaro and J. Coville, Rapid traveling waves in the nonlocal Fisher equation connect two unstable states, *Appl. Math. Lett.*, 25 (2012), 2095–2099.

[2] M. Alfaro, J. Coville and G. Raoul, Traveling waves in a nonlocal reaction-diffusion equation as a model for a population structured by a space variable and a phenotypic trait, *Comm. Partial Differential Equations*, 38 (2013), 2126–2154.

[3] H. Berestycki, G. Nadim, B. Perthame and L. Ryzhik, The non-local Fisher-KPP equation: Travelling waves and steady states, *Nonlinearity*, 22 (2009), 2813–2844.

[4] O. Bonnefon, J. Garnier, F. Hamel and L. Roques, Inside dyeanics of delayed traveling waves, *Math. Model. Nat. Phenom.*, 8 (2013), 42–59.

[5] C. Conley and R. Gardner, An application of the generalized Mores index to traveling wave solutions of a competition reaction-diffusion model, *Indiana Univ. Math. J.*, 33 (1984), 319–343.

[6] J. Fang and J. H. Wu, Monotone traveling waves for delayed Lotka-Volterra competition systems, *Discrete Contin. Dyn. Syst.*, 32 (2012), 3043–3058.

[7] J. Fang and X.-Q. Zhao, Monotone wave fronts of the nonlocal Fisher-KPP equation, *Nonlinearity*, 24 (2011), 3043–3054.

[8] G. Faye and M. Holzer, Modulated traveling fronts for a nonlocal Fisher-KPP equation: A dynamical systems approach, *J. Differential Equations*, 258 (2015), 2257–2289.

[9] R. A. Gardner, Existence and stability of traveling wave solutions of competition models: A degree theoretic approach, *J. Differential Equations*, 44 (1982), 343–364.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] A. Gomez and S. Trofimchuk, Monotone traveling wavefronts of the KPP-Fisher delayed equation, *J. Differential Equations*, 250 (2011), 1767–1787.

[12] S. A. Gourley and S. G. Ruan, Convergence and traveling fronts in functional differential equations with nonlocal terms: A competition model, *SIAM J. Math. Anal.*, 35 (2003), 806–822.

[13] J.-S. Guo and X. Liang, The minimal speed of traveling fronts for the Lotka-Volterra competition system, *J. Dynam. Differential Equations*, 23 (2011), 353–363.

[14] S.-J. Guo and J. Zimmer, Stability of travelling wavefronts in discrete reaction-diffusion equations with nonlocal delay effects, *Nonlinearity*, 28 (2015), 463–492.

[15] J.-S. Guo and C.-H. Wu, Recent developments on wave propagation in 2-species competition systems, *Discrete Contin. Dyn. Syst. Ser. B*, 17 (2012), 2713–2724.

[16] F. Hamel and L. Ryzhik, On the nonlocal Fisher-KPP equation: Steady states, spreading speed and global bounds, *Nonlinearity*, 27 (2014), 2735–2753.

[17] B.-S. Han and Z.-C. Wang, Traveling wave solutions in a nonlocal reaction-diffusion population model, *Commun. Pure Appl. Anal.*, 15 (2016), 1057–1076.

[18] B.-S. Han and Z.-C. Wang, Traveling waves for the nonlocal diffusive single species model with allee effect, *J. Math. Anal. Appl.*, 443 (2016), 243–264.

[19] B.-S. Han and Y. H. Yang, An integro-PDE model with variable motility, *Nonlinear Anal. Real World Appl.*, 45 (2019), 186–199.

[20] K. Hasik, J. Kopfová, P. Nábělková and S. Trofimchuk, Traveling waves in the nonlocal KPP-Fisher equation: Different roles of the right and the left interactions, *J. Differential Equations*, 260 (2016), 6130–6175.
A. Huang and P. X. Weng, Slowly oscillating wavefronts of the Fisher-KPP delayed equation, *Discrete Contin. Dyn. Syst.*, 34 (2014), 3511–3533.

Y. Hosono, Singular perturbation analysis of traveling waves for diffusive Lotka-Volterra competitive models, *Numerical and Applied Mathematics, IMACS Ann. Comput. Appl. Math.*, IMACS Trans. Sci. Comput. ’88, Baltzer, Basel, 1 (1988), 687–692.

Y. Hosono, The minimal spread of traveling fronts for a diffusive Lotka-Volterra competition model, *Bull. Math. Biol.*, 66 (1998), 435–448.

A. Huang and P. X. Weng, Traveling wavefronts for a Lotka-Volterra system of type-K with delays, *Nonlinear Anal. Real World Appl.*, 14 (2013), 1114–1129.

J. H. Huang and X. F. Zou, Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays, *J. Math. Anal. Appl.*, 271 (2002), 455–466.

J. H. Huang and X. F. Zou, Existence of traveling wavefronts of delayed reaction-diffusion systems without monotonicity, *Discrete Cont. Dyn. Syst.*, 9 (2003), 925–936.

J. H. Huang and X. F. Zou, Traveling wave solutions in delayed reaction diffusion systems with partial monotonicity, *Acta Math. Appl. Sin. Engl. Ser.*, 22 (2006), 243–256.

J. I. Kanel and L. Zhou, Existence of wave front solutions and estimates of wave speed for a competition-diffusion system, *Nonlinear Anal.*, 27 (1996), 579–587.

Y. Kan-on, Parameter dependence of propagation speed of traveling waves for competition-diffusion equations, *SIAM J. Math. Anal.*, 26 (1995), 340–363.

M. K. Kwong and C. H. Ou, Existence and nonexistence of monotone traveling waves for the delayed Fisher equation, *J. Differential Equations*, 249 (2010), 728–745.

K. Li and X. Li, Traveling wave solutions in a delayed diffusive competition system, *Nonlinear Anal.*, 75 (2012), 3705–3722.

W.-T. Li, G. Lin and S. G. Ruan, Existence of travelling wave solutions in delayed reaction-diffusion systems with applications to diffusion-competition systems, *Nonlinearity*, 19 (2006), 1253–1273.

G. Lin and W.-T. Li, Bistable wavefronts in a diffusive and competitive Lotka-Volterra type system with nonlocal delays, *J. Differential Equations*, 244 (2008), 487–513.

G. Lin and S. G. Ruan, Traveling wave solutions for delayed reaction-diffusion systems and applications to diffusive Lotka-Volterra competition models with distributed delays, *J. Dynam. Differential Equations*, 26 (2014), 583–605.

G.-Y. Lv and M. X. Wang, Traveling wave front in diffusive and competitive Lotka-Volterra system with delays, *Nonlinear Anal. Real World Appl.*, 11 (2010), 1323–1329.

G.-Y. Lv and M. Wang, Traveling wave front and stability as planar wave of reaction diffusion equations with nonlocal delays, *Z. Angew. Math. Phys.*, 64 (2013), 1005–1023.

S. W. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differential Equations*, 171 (2001), 294–314.

A. Okubo, P. K. Maini, M. H. Williamson and J. D. Murray, On the spatial spread of grey squirrel in Britatin, *Proc. R. Soc. Lond. B.*, 238 (1989), 113–125.

C. H. Ou and J. H. Wu, Traveling wavefronts in a delayed food-limited population model, *SIAM J. Math. Anal.*, 39 (2007), 103–125.

S. X. Pan, Traveling wave solutions in delayed diffusion systems via a cross iteration scheme, *Nonlinear Anal. Real World Appl.*, 10 (2009), 2807–2818.

K. W. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, *Trans. Amer. Math. Soc.*, 302 (1987), 587–615.

M.-M. Tang and P. C. Fife, Propagating fronts for competing species equations with diffusion, *Arch. Rational Mech. Anal.*, 73 (1980), 69–77.

J. H. van Vuuren, The existence of travelling plane waves in a general class of competition-diffusion systems, *IMA J. Appl. Math.*, 55 (1995), 135–148.

X. P. Yang and Y. F. Wang, Travelling wave and global attractivity in a competition-diffusion system with nonlocal delays, *Comput. Math. Appl.*, 59 (2010), 3338–3350.

L.-H. Yao, Z.-X. Yu and R. Yuan, Spreading speed and traveling waves for a nonmonotone reaction-diffusion model with distributed delay and nonlocal effect, *Appl. Math. Model.*, 35 (2011), 2916–2929.

Z.-X. Yu and R. Yuan, Traveling waves of delayed reaction-diffusion systems with applications, *Nonlinear Anal. Real World Appl.*, 12 (2011), 2475–2488.

Z.-C. Wang, W.-T. Li and S. G. Ruan, Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays, *J. Differential Equations*, 222 (2006), 185–232.
[48] J. H. Wu and X. F. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *J. Dynam. Differential Equations*, 13 (2001), 651–687.

[49] X. F. Zou and J. H. Wu, Existence of traveling wave fronts in delayed reaction-diffusion systems via the monotone iteration method, *Proc. Amer. Math. Soc.*, 125 (1997), 2589–2598.

Received February 2019; revised May 2019.

E-mail address: hanbangsheng@swjtu.edu.cn
E-mail address: wangzhch@lzu.edu.cn
E-mail address: duzengji1863.com