A Multiple Hypothesis Testing Approach to Low-Complexity Subspace Unmixing

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Abstract

Subspace-based signal processing has a rich history in the literature. Traditional focus in this direction has been on problems involving a few subspaces. But a number of problems in different application areas have emerged in recent years that involve significantly larger number of subspaces relative to the ambient dimension. It becomes imperative in such settings to first identify a smaller set of active subspaces that contribute to the observations before further information processing tasks can be carried out. We term this problem of identification of a small set of active subspaces among a huge collection of subspaces from a single (noisy) observation in the ambient space as subspace unmixing. In this paper, we formally pose the subspace unmixing problem, discuss its connections with problems in wireless communications, hyperspectral imaging, high-dimensional statistics and compressed sensing, and propose and analyze a low-complexity algorithm, termed marginal subspace detection (MSD), for subspace unmixing. The MSD algorithm turns the subspace unmixing problem into a multiple hypothesis testing (MHT) problem and our analysis helps control the family-wise error rate of this MHT problem at any level $\alpha \in [0, 1]$. Some other highlights of our analysis of the MSD algorithm include: (i) it is applicable to an arbitrary collection of subspaces on the Grassmann manifold; (ii) it relies on properties of the collection of subspaces that are computable in polynomial time; and (iii) it allows for linear scaling of the number of active subspaces as a function of the ambient dimension. Finally, we also present numerical results in the paper to better understand the performance of the MSD algorithm.

Index Terms

Average mixing coherence; family-wise error rate; Grassmann manifold; interference subspaces; local 2-subspace coherence; multiple hypothesis testing; subspace detection; subspace unmixing

I. INTRODUCTION

Subspace models, in which it is assumed that signals of interest lie on or near a low-dimensional subspace of a higher-dimensional Hilbert space $\mathcal{H}$, have a rich history in signal processing, machine learning, and statistics. While...
much of the classical literature in detection, estimation, classification, dimensionality reduction, etc., is based on
the subspace model, many of these results deal with a small number of subspaces, say, \( X := \{ S_1, S_2, \ldots, S_N \} \)
with each \( S_i \) a subspace of \( \mathcal{H} \), relative to the dimension of the Hilbert space: \( \dim(\mathcal{H}) := D \geq N \). Consider, for
instance, the classical subspace detection problem studied in [2]. In this problem, one deals with two subspaces—the
signal subspace and the interference subspace—and a solution to the detection problem involves a low-complexity
generalized likelihood ratio test [2]. However, proliferation of cheap sensors and low-cost semiconductor devices in
the modern world means we often find ourselves dealing with a significantly larger number of subspaces relative
to the extrinsic dimension, i.e., \( D \ll N \). But many of the classical subspace-based results do not generalize in
such “\( D \) smaller than \( N \)” settings either because of the breakdown of the stated assumptions or because of the
prohibitive complexity of the resulting solutions. In fact, without additional constraints, information processing in
such settings might well be a daunting, if not impossible, task.

One constraint that often comes to our rescue in this regard in many applications is the “principle of parsimony”:
while the total number of subspaces might be large, only a small number of them, say, \( n \propto D \), tend to be “active”
at any given instance. Mathematically, the \( D \)-dimensional observations \( y \in \mathcal{H} \) in this parsimonious setting can be
expressed as \( y \in F(X_A) + \text{noise} \), where \( X_A := \{ S_i \in X_N : S_i \text{ is active} \} \) denotes the set of active subspaces
with \( n := \# \{ i : S_i \text{ is active} \} \ll D \ll N \) and \( F(\cdot) \) denotes the rule that relates the set of active subspaces \( X_A \) to
the (noisy) observations. It is easy to convince oneself in this case that the classical subspace-based computational
machinery for information processing becomes available to us as soon as we have access to \( X_A \). One of the
fundamental challenges for information processing in the “\( D \) smaller than \( N \)” setting could then be described as
the recovery of the set of active subspaces, \( X_A \subset X_N \), from the \( D \)-dimensional observations \( y \in F(X_A) + \text{noise} \).

We term this problem the recovery of \( X_A \) from noisy observations as subspace unmixing. In this paper, we study
a special case of subspace unmixing that corresponds to the subspace sum model, i.e., \( F(X_A) := \sum_{i \in A} S_i \), where \( A := \{ i : S_i \text{ is active} \} \) denotes the indices of active subspaces. Before describing our main contributions in relation
to subspace unmixing from \( y \in \sum_{i \in A} S_i + \text{noise} \), we discuss some of its applications in different areas.

1) Multiuser Detection in Wireless Networks: Consider a wireless network comprising a large number of users
in which some of the users simultaneously transmit data to a base station. It is imperative for the base station in
this case to identify the users that are communicating with it at any given instance, which is termed as the problem
of multiuser detection. This problem of multiuser detection in wireless networks can also be posed as a subspace
unmixing problem. In this context, users in the network communicate with the base station using \( D \)-dimensional
codewords in \( \mathcal{H} \), each individual user is assigned a codebook that spans a low-dimensional subspace \( S_i \) of \( \mathcal{H} \), the
total number of users in the network is \( N \), the number of active users at any given instance is \( n \ll N \), and the base
station receives \( y \in \sum_{i \in A} S_i + \text{noise} \) due to the superposition property of the wireless medium, where \( A \) denotes
the indices of the users actively communicating with the base station.

2) Spectral Unmixing in Hyperspectral Remote Sensing: Hyperspectral remote sensing has a number of civilian
and defense applications, which typically involve identifying remote objects from their spectral signatures. Because
of the low spatial resolution of hyperspectral imaging systems in most of these applications, individual hyperspectral
pixels tend to comprise multiple objects (e.g., soil and vegetation). Spectral unmixing is the problem of decomposition of a “mixed” hyperspectral pixel into its constituent objects. In order to pose this spectral unmixing problem into the subspace unmixing problem studied in this paper, we need two assumptions that are often invoked in the literature. First, the spectral variability of each object in different scenes can be captured through a low-dimensional subspace. Second, the mixture of spectra of different objects into a hyperspectral pixel can be described by a linear model. The spectral unmixing problem under these assumptions is the subspace unmixing problem, with \( y \in \mathcal{H} \) denoting the \( D \)-dimensional hyperspectral pixel of an imaging system with \( D \) spectral bands, \( \{ S_i \subset \mathcal{H} \}_{i=1}^{N} \) denoting the low-dimensional subspaces of \( \mathcal{H} \) associated with the spectra of individual objects, \( N \) denoting the total number of objects of interest, and \( y \in \sum_{i \in A} S_i + \text{noise} \) with \( n := |A| \ll N \) since only a small number of objects are expected to contribute to a single hyperspectral pixel.

3) Group Model Selection in High-Dimensional Statistics: Model selection in statistical data analysis is the problem of learning the relationship between the samples of a dependent or response variable (e.g., the malignancy of a tumor, the health of a network) and the samples of independent or predictor variables (e.g., the expression data of genes, the traffic data in the network). There exist many applications in statistical model selection where the implication of a single predictor in the response variable implies presence of other related predictors in the true model. In such situations, the problem of model selection is often reformulated in a “group” setting. This problem of group model selection in high-dimensional settings, where the number of predictors tends to be much larger than the number of samples, can also be posed as the subspace unmixing problem. In this context, \( y \in \mathcal{H} \) denotes the \( D \)-dimensional response variable with \( D \) representing the total number of samples, \( N \) denotes the total number of groups of predictors that comprise the design matrix, \( \{ S_i \subset \mathcal{H} \}_{i=1}^{N} \) denotes the low-dimensional subspaces of \( \mathcal{H} \) spanned by each of the groups of predictors, and \( y \in \sum_{i \in A} S_i + \text{noise} \) with \( A \) denoting the indices of the groups of predictors that truly affect the response variable.

4) Sparsity Pattern Recovery in Block-Sparse Compressed Sensing: Compressed sensing is an alternative sampling paradigm for signals that have sparse representations in some orthonormal bases. In recent years, the canonical compressed sensing theory has been extended to the case of signals that have block-sparse representations in some orthonormal bases. Sparsity pattern recovery in block-sparse compressed sensing is the problem of identifying the nonzero “block coefficients” of the measured signal. The problem of sparsity pattern recovery in block-sparse compressed sensing, however, can also be posed as the subspace unmixing problem. In this context, \( y \in \mathcal{H} \) denotes the \( D \)-dimensional measurement vector with \( D \) being the total number of measurements, \( N \) denotes the total number of blocks of coefficients, \( \{ S_i \subset \mathcal{H} \}_{i=1}^{N} \) denotes the low-dimensional subspaces of \( \mathcal{H} \) spanned by the “blocks of columns” of the composite matrix \( \Phi \Psi \) with \( \Phi \) being the measurement matrix and \( \Psi \) being the sparsifying basis, and \( y \in \sum_{i \in A} S_i + \text{noise} \) with \( A \) denoting the indices of the nonzero blocks of coefficients of the signal in \( \Psi \).

A. Relationship to Prior Work and Our Contributions

Since the subspace unmixing problem has connections to a number of application areas, it is invariably related to prior works in some of those areas. In the context of multiuser detection, the work that is most closely related
to ours is [3]. However, the setup of [3] can be considered a restrictive version of the general subspace unmixing problem posed in here. Roughly speaking, the setup in [3] can be described as a *randomly modulated* subspace sum model, $y = \sum_{i \in A} \varepsilon_i S_i + \text{noise}$ with $\{\varepsilon_i\}_{i=1}^N$ being independent and identically distributed isotropic random variables. In addition, the results of [3] rely on parameters that cannot be easily translated into properties of the subspaces alone. Finally, [3] relies on a convex optimization procedure for multiuser detection that has superlinear (in $D$ and $N$) computational complexity.

In the context of group model selection and block-sparse compressed sensing, our work can be considered related to [4]–[17]. None of these works, however, help us understand the problem of subspace unmixing in its most general form. Some of these works, when translated into the general subspace unmixing problem, either consider only random subspaces [5]–[7] or study subspaces generated through a Kronecker operation [11]–[17]. Further, some of these works either focus on the randomly modulated subspace sum model [14], [15] or generate results that suggest that, fixing the dimensions of subspaces, the total number of active subspaces can at best scale as $O\left(\sqrt{D}\right)$ [7]–[10]—the so-called “square-root bottleneck.” Finally, many of these works either focus on computational approaches that have superlinear complexity [4]–[7], [10], [13], [16] or suggest that low-complexity approaches suffer from the “dynamic range of active subspaces” [9], [14].

In contrast to these and other related earlier works, the main contributions of this paper are as follows. First, it formally puts forth the problem of subspace unmixing that provides a mathematically unified view of many problems studied in other application areas. Second, it presents a low-complexity solution to the problem of subspace unmixing that has linear complexity in $D$, $N$, and the dimensions of the individual subspaces. Third, it presents a comprehensive analysis of the proposed solution, termed *marginal subspace detection* (MSD), that makes no assumptions about the structures of the individual subspaces. In particular, the resulting analysis relies on geometric measures of the subspaces that can be computed in polynomial time. Finally, the analysis neither suffers from the square-root bottleneck nor gets affected by the dynamic range of the active subspaces. We conclude by pointing out that a preliminary version of this work appeared in [13]. However, that work was focused primarily on group model selection, it did not account for noise in the observations, and the ensuing analysis lacked details in terms of the metrics of multiple hypothesis testing.

### B. Notation and Organization

The following notational convention is used throughout the rest of this paper. We use the standard notation $:=\text{t}$ to denote definitions of terms. The notation $|\cdot|$ is used for both the cardinality of a set and the absolute value of a real number. Similarly, $\|\cdot\|_2$ is used for both the $\ell_2$-norm of a vector and the operator 2-norm of a matrix. The notation $\setminus$ denotes the set difference operation. Finally, we make use of the following “Big–O” notation for scaling relations: $f(n) = O(g(n))$ if $\exists c_0 > 0, n_0 : \forall n \geq n_0, f(n) \leq c_0 g(n)$, $f(n) = \Omega(g(n))$ (alternatively, $f(n) \geq g(n)$) if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if $g(n) = O(f(n))$ and $f(n) = O(g(n))$.

The rest of this paper is organized as follows. In Sec. [14] we rigorously formulate the problem of subspace unmixing and define the relevant metrics used to measure the performance of subspace unmixing algorithms. In
Sec. [III] we describe our proposed algorithm for subspace unmixing and provide its analysis in terms of the requisite performance metrics. In Sec. [IV] we expand on our analysis of the proposed subspace unmixing algorithm and provide results that help understand its significance. In Sec. [V] we present some numerical results to support our analysis and we finally conclude in Sec. [VI].

II. PROBLEM FORMULATION

Consider a $D$-dimensional real Hilbert space $\mathcal{H}$ and the Grassmann manifold $\mathcal{G}(d, \mathcal{H})$ that denotes the collection of all $d$-dimensional subspaces of $\mathcal{H}$. Since finite-dimensional Hilbert spaces are isomorphic to Euclidean spaces, we will assume without loss of generality in the following that $\mathcal{H} = \mathbb{R}^D$ and hence $\mathcal{G}(d, \mathcal{H}) = \mathcal{G}(d, D)$. Next, consider a collection of $N \gg D/d \gg 1$ subspaces given by $\mathcal{X}_N = \{S_i \in \mathcal{G}(d, D), i = 1, \ldots, N\}$ such that $S_1, \ldots, S_N$ are pairwise disjoint: $S_i \cap S_j = \{0\}$ $\forall i, j = 1, \ldots, N, i \neq j$. Heuristically, this means each of the subspaces in $\mathcal{X}_N$ is low-dimensional and the subspaces collectively “fill” the ambient space $\mathbb{R}^D$.

The fundamental assumptions in the problem of subspace unmixing in this paper are that only a small number $n < D/d \ll N$ of the subspaces are active at any given instance and the observations $y \in \mathbb{R}^D$ correspond to a noisy version of an $x \in \mathbb{R}^D$ that lies in the sum of the active subspaces. Mathematically, we can formalize these assumptions by defining $\mathcal{A} = \{i : S_i \in \mathcal{X}_N \text{ is active}\}$, writing $x = \sum_{i \in \mathcal{A}} S_i$, and stating that the observations $y = x + \eta$, where $\eta \in \mathbb{R}^D$ denotes noise in the observations. For the sake of this exposition, we assume $\eta$ to be either bounded energy, deterministic error, i.e., $\|\eta\|_2 < \epsilon_n$, or independent and identically distributed (i.i.d.) Gaussian noise with variance $\sigma^2$, i.e., $\eta \sim \mathcal{N}(0, \sigma^2 I)$. The final detail we need in order to complete formulation of the problem of subspace unmixing is a mathematical model for generation of the “noiseless signal” $x \in \sum_{i \in \mathcal{A}} S_i$.

In this paper, we work with the following generative model:

- **Mixing Bases:** Each subspace $S_i$ in the collection $\mathcal{X}_N$ is associated with an orthonormal basis $\Phi_i \in \mathbb{R}^{D \times d}$, i.e., $\text{span}(\Phi_i) = S_i$ and $\Phi_i^T \Phi_i = I$.

- **Activity Pattern:** The set of indices of the active subspaces $\mathcal{A}$ is a random $n$-subset of $\{1, \ldots, N\}$ with $\Pr(\mathcal{A} = \{i_1, i_2, \ldots, i_n\}) = 1/\binom{N}{n}$.

- **Signal Generation:** There is a deterministic but unknown collection of “mixing coefficients” $\{\theta_j \in \mathbb{R}^d, j = 1, \ldots, n\}$ such that the noiseless signal $x$ is given by $x = \sum_{j=1}^n \Phi_i \theta_j$, where $\mathcal{A} = \{i_1, i_2, \ldots, i_n\}$.

Given this generative model, the goal of subspace unmixing in this paper is to identify the set of active subspaces $\mathcal{X}_A = \{S_i \in \mathcal{X}_N : i \in \mathcal{A}\}$ using knowledge of the collection of subspaces $\mathcal{X}_N$ and the noisy observations $y \in \mathbb{R}^D$. In particular, our focus is on unmixing solutions with linear (in $d$, $N$, and $D$) computational complexity.

A few remarks are in order now regarding the stated assumptions. First, the assumption of pairwise disjointness of the subspaces is much weaker than the assumption of linear independence of the subspaces, which is typically invoked in the literature on subspace-based information processing [2], [19]. In particular, while pairwise disjointness implies pairwise linear independence, it does not preclude the possibility of an element in one subspace being

1 Note that all results presented in this paper can be extended in a straightforward manner to the case of a complex Hilbert space.

2 The other commonly invoked assumption of orthogonal subspaces is of course impossible in the $D/d \ll N$ setting.
representable through a linear combination of elements in two or more subspaces. Second, while the generative model makes use of the mixing bases, these mixing bases might not be known to the unmixing algorithms in some applications; we will briefly discuss this further in the sequel. Third, the rather mild assumption on the randomness of the activity pattern can be interpreted as the lack of a priori information concerning the activity pattern of subspaces. Finally, those familiar with detection under the classical linear model [20, Sec. 7.7] will recognize the assumption \( x = \sum_{j=1}^{n} \Phi_{ij} \theta_j \) as a simple generalization of that setup for the problem of subspace unmixing.

A. Performance Metrics

In this paper, we address the problem of subspace unmixing by transforming it into a multiple hypothesis testing problem (cf. Sec. III). While several measures of error have been used over the years in multiple hypothesis testing problems, the two most widely accepted ones in the literature remain the family-wise error rate (FWER) and the false discovery rate (FDR) [21]. Mathematically, if we use \( \hat{A} \subset \{1, \ldots, N\} \) to denote an estimate of the indices of active subspaces returned by an unmixing algorithm then controlling the FWER at level \( \alpha \) in our setting means \( \text{FWER} := \Pr(\hat{A} \not\subset A) \leq \alpha \). In words, FWER \( \leq \alpha \) guarantees that the probability of declaring even one inactive subspace as active (i.e., a single false positive) is controlled at level \( \alpha \). On the other hand, controlling the FDR in our setting controls the expected proportion of inactive subspaces that are incorrectly declared as active by an unmixing algorithm [22].

While the FDR control is less stringent than the FWER control [22], our goal in this paper is control of the FWER. This is because control of the FDR in the case of dependent test statistics, which will be the case in our setting (cf. Sec. III), is a challenging research problem [23]. Finally, once we control the FWER at some level \( \alpha \), our goal is to have as large a fraction of active subspaces identified as active by the unmixing algorithm as possible. The results reported in the paper in this context will be given in terms of the non-discovery proportion (NDP), defined as \( \text{NDP} := \frac{|A \setminus \hat{A}|}{|A|} \).

B. Preliminaries

In this section, we introduce some definitions that will be used throughout the rest of this paper to characterize the performance of our proposed approach to subspace unmixing. It is not too difficult to convince oneself that the “hardness” of subspace unmixing problem should be a function of the “similarity” of the underlying subspaces: the more similar the subspaces in \( \mathcal{X}_N \), the more difficult it should be to tell them apart. In order to capture this intuition, we work with the similarity measure of subspace coherence in this paper, defined as:

\[
\gamma(S_i, S_j) := \max_{w \in S_i, z \in S_j} \frac{|\langle w, z \rangle|}{\|w\|_2 \|z\|_2},
\]

where \((S_i, S_j)\) denote two \( d \)-dimensional subspaces in \( \mathbb{R}^D \). Note that \( \gamma : \mathcal{G}(d, D) \times \mathcal{G}(d, D) \rightarrow [0, 1] \) simply measures cosine of the smallest principal angle between two subspaces and has appeared in earlier literature [10], [24]. In particular, given (any arbitrary) orthonormal bases \( U_i \) and \( U_j \) of \( S_i \) and \( S_j \), respectively, it follows that...
Clearly, the inactive subspaces of course contribute no energy to the observations, i.e., also follows from the definition of subspace coherence that these is termed noiseless signal assumption of pairwise disjointness, however, we have that too little energy to the final observations to the extent that they get buried in noise cannot be identified using any computational method. Finally, note that bases in relation to each of the subspaces in the collection.

**Definition 1** (Local 2-Subspace Coherence). Given a collection of subspaces \( \mathcal{X}_N = \{ \mathcal{S}_i \in \mathfrak{S}(d, D), i = 1, \ldots, N \} \), the local 2-subspace coherence of subspace \( \mathcal{S}_i \) is defined as \( \gamma_{2,i} := \max_{j \neq i, k \neq i : j \neq k} \left[ \gamma(\mathcal{S}_i, \mathcal{S}_j) + \gamma(\mathcal{S}_i, \mathcal{S}_k) \right] \).

In words, \( \gamma_{2,i} \) measures closeness of \( \mathcal{S}_i \) to the worst pair of subspaces in the collection \( \mathcal{X}_N^\perp := \mathcal{X}_N \setminus \{ \mathcal{S}_i \} \). It also follows from the definition of subspace coherence that \( \gamma_{2,i} \in [0, 2] \), with \( \gamma_{2,i} = 0 \) if and only if every subspace in \( \mathcal{X}_N^\perp \) is orthogonal to \( \mathcal{S}_i \), while \( \gamma_{2,i} = 2 \) if and only if two subspaces in \( \mathcal{X}_N^\perp \) are the same as \( \mathcal{S}_i \). Because of our assumption of pairwise disjointness, however, we have that \( \gamma_{2,i} \) is strictly less than 2 in this paper. We conclude our discussion of the local 2-subspace coherence by noting that it is trivially computable in polynomial time.

The next definition we need to characterize the performance of subspace unmixing is *active subspace energy*.

**Definition 2** (Active Subspace Energy). Given the set of indices of active subspaces \( \mathcal{A} = \{ i_1, i_2, \ldots, i_n \} \) and the noiseless signal \( x = \sum_{j=1}^{n} \Phi_{i_j} \theta_j \), the energy of the \( i_j \)-th active subspace is defined as \( E_{i_j} := \| \Phi_{i_j} \theta_j \|_2^2 \).

Inactive subspaces of course contribute no energy to the observations, i.e., \( E_i = 0 \) \( \forall i \notin \mathcal{A} \). But it is important for us to specify the energy of active subspaces for subspace unmixing. Indeed, active subspaces that contribute too little energy to the final observations to the extent that they get buried in noise cannot be identified using any computational method. Finally, note that \( E_{i_j} \equiv \| \theta_j \|_2^2 \) due to the orthonormal nature of the mixing bases.

Finally, the low-complexity algorithm proposed in this paper requires two additional definitions. The first one of these is termed *average mixing coherence* of individual subspaces, which measures the “niceness” of the mixing bases in relation to each of the subspaces in the collection \( \mathcal{X}_N \).

**Definition 3** (Average Mixing Coherence). Given a collection of subspaces \( \mathcal{X}_N = \{ \mathcal{S}_i \in \mathfrak{S}(d, D), i = 1, \ldots, N \} \) and the associated mixing bases \( \mathcal{B}_N := \{ \Phi_i : \text{span}(\Phi_i) = \mathcal{S}_i, \Phi_i^T \Phi_i = I, i = 1, \ldots, N \} \), the average mixing coherence of subspace \( \mathcal{S}_i \) is defined as \( \rho_i := \frac{1}{N-1} \left\| \sum_{j \neq i} \Phi_i^T \Phi_j \right\|_2 \).

Since we are introducing average mixing coherence for the first time in the literature\(^3\), it is worth understanding its behavior. First, unlike (local 2-)subspace coherence, it is not invariant to the choice of the mixing bases. Second, note that \( \rho_i \in [0, 1] \). To see this, observe that \( \rho_i = 0 \) if the subspaces in \( \mathcal{X}_N \) are orthogonal to each other. Further, we have from triangle inequality and the definition of subspace coherence that \( \rho_i \leq \sum_{j \neq i} \gamma(\mathcal{S}_i, \mathcal{S}_j)/(N-1) \leq 1 \).

Clearly, the *average subspace coherence* of the subspace \( \mathcal{S}_i \), defined as \( \tau_i := \sum_{j \neq i} \gamma(\mathcal{S}_i, \mathcal{S}_j)/(N-1) \), is a trivial upper bound on \( \rho_i \) and we will return to this point in the sequel. We once again conclude by noting that both the average mixing coherence, \( \rho_i \), and the average subspace coherence, \( \tau_i \), are trivially computable in polynomial time.

The final definition we need to characterize subspace unmixing is that of *cumulative active subspace energy*.

**Definition 4** (Cumulative Active Subspace Energy). Given the set of indices of active subspaces \( \mathcal{A} \), the cumulative

\[ \gamma(\mathcal{S}_i, \mathcal{S}_j) := \| U_i^T U_j \|_2 \]. Since we are interested in unmixing any active collection of subspaces, we will be stating our main results in terms of the *local 2-subspace coherence* of individual subspaces, defined in the following.
active subspace energy is defined as $\mathcal{E}_A := \sum_{i \in A} \mathcal{E}_i$.

In words, cumulative active subspace energy can be considered a measure of “signal energy” and together with the noise energy/variance, it characterizes signal-to-noise ratio for the subspace unmixing problem.

III. MARGINAL SUBSPACE DETECTION FOR SUBSPACE UNMIXING

In this section, we present our low-complexity approach to subspace unmixing and characterize its performance in terms of the parameters introduced in Sec. II-B. Recall that the observations $y \in \mathbb{R}^D$ are given by $y = x + \eta$ with $x \in \sum_{i \in \mathcal{A}} S_i$. Assuming the cardinality of the set of indices of active subspaces, $n = |\mathcal{A}|$, is known, one can pose the subspace unmixing problem as an $M$-ary hypothesis testing problem with $M = \binom{N}{n}$. In this formulation, we have that the $k$-th hypothesis, $\mathcal{H}_k$, $k = 1, \ldots, M$, corresponds to one of the $M$ possible choices for the set $\mathcal{A}$. While an optimal theoretical strategy in this setting will be to derive the $M$-ary maximum likelihood decision rule, this will lead to superlinear computational complexity since one will have to evaluate $M = \binom{N}{n} \geq \left(\frac{N}{n}\right)^n$ test statistics, one for each of the $M$ hypotheses, in this formulation. Instead, since we are interested in low-complexity approaches in this paper, we approach the problem of subspace unmixing as $N$ individual binary hypothesis testing problems. An immediate benefit of this approach, which transforms the problem of subspace unmixing into a multiple hypothesis testing problem, is the computational complexity: we need only evaluate $N$ test statistics in this setting.

The challenges in this setting of course are specifying the decision rules for each of the $N$ binary hypotheses and understanding the performance of the corresponding low-complexity approach in terms of the FWER and the NDP. We address these challenges in the following by describing and analyzing this multiple hypothesis testing approach.

A. Marginal Subspace Detection

In order to solve the problem of subspace unmixing, we propose to work with $N$ binary hypothesis tests on the observations $y = x + \eta$, as defined below.

$$\mathcal{H}_0^k : x = \sum_{j=1}^{n} \Phi_{i_j}\theta_j \quad \text{s.t.} \quad k \notin \mathcal{A} = \{i_1, i_2, \ldots, i_n\}, \quad k = 1, \ldots, N, \tag{2}$$

$$\mathcal{H}_1^k : x = \sum_{j=1}^{n} \Phi_{i_j}\theta_j \quad \text{s.t.} \quad k \in \mathcal{A} = \{i_1, i_2, \ldots, i_n\}, \quad k = 1, \ldots, N. \tag{3}$$

In words, the null hypothesis $\mathcal{H}_0^k$ being true signifies that subspace $S_k$ is not active, while the alternative hypothesis $\mathcal{H}_1^k$ being true signifies that $S_k$ is active. Note that if we fix a $k \in \{1, \ldots, N\}$ then deciding between $\mathcal{H}_0^k$ and $\mathcal{H}_1^k$ is equivalent to detecting a subspace $S_k$ in the presence of an interference signal and additive noise. While this setup is reminiscent of the subspace detection problem studied in [2], the fundamental differences between the binary hypothesis test(s) in our problem and that in [2] are that: (i) the interference signal in [2] is assumed to come from a single, known subspace, while the interference signal in our problem setup is a function of the underlying activity pattern of the subspaces; and (ii) our problem setup involves multiple hypothesis tests (with dependent test statistics), which therefore requires control of the FWER. Nonetheless, since *matched subspace detectors* are
known to be (quasi-)optimal in subspace detection problems [2], we put forth the test statistics for our $N$ binary hypothesis tests that are based on matched subspace detectors.

Specifically, in order to decide between $\mathcal{H}_0^k$ and $\mathcal{H}_1^k$ for any given $k$, we compute the test statistic $T_k(y) := \|U_k^T y\|_2^2$, where $U_k$ denotes any orthonormal basis of the subspace $S_k$. Notice that $T_k(y)$ is invariant to the choice of the basis $U_k$ and therefore it can be computed without knowledge of the set of mixing bases $B_N$. In order to relate this test statistic to the classical subspace detection literature, note that $T_k(y) = \|U_k U_k^T y\|_2^2 = \|P_{S_k} y\|_2^2$. That is, the test statistic is equivalent to projecting the observations onto the subspace $S_k$ and computing the energy of the projected observations, which is the same operation that arises in the classical subspace detection literature [2], [20]. The final decision between $\mathcal{H}_0^k$ and $\mathcal{H}_1^k$ then involves comparing the test statistic against a threshold $\tau_k$:

$$T_k(y) \gtrless \tau_k, \quad k = 1, \ldots, N. \quad (4)$$

Once we obtain these marginal decisions, we can use them to obtain an estimate of the set of indices of the active subspaces by setting $\hat{A} = \{k : \mathcal{H}_1^k \text{ is accepted}\}$. We term this entire procedure, outlined in Algorithm 1, as marginal subspace detection (MSD) because of its reliance on detecting the presence of subspaces in the active set using marginal test statistics. The challenge then is understanding the behavior of the test statistics for each subspace under the two hypotheses and specifying values of the thresholds $\tau_k$ that lead to acceptable FWER and NDP figures. Further, a key aspect of any analysis of MSD involves understanding the number of active subspaces that can be tolerated by it as a function of the subspace collection $\mathcal{X}_N$, the ambient dimension $D$, the subspace dimension $d$, etc. In order to address these questions, one would ideally like to understand the distributions of the test statistics for each of the $N$ subspaces under the two different hypotheses. However, specifying these distributions under the generative model of Sec. III and ensuring that (i) the final results can be interpreted in terms of the geometry of the underlying subspaces, and (ii) the number of active subspaces can be allowed to be almost linear in $\frac{D}{d}$ appears to be an intractable analytical problem. We therefore instead focus on characterizing the (left and right) tail probabilities of the test statistics for each subspace under the two hypotheses.

**B. Tail Probabilities of the Test Statistics**

In this section, we evaluate $\Pr(T_k(y) \geq \tau | \mathcal{H}_0^k)$ and $\Pr(T_k(y) \leq \tau | \mathcal{H}_1^k)$. To this end, we assume an arbitrary (but fixed) $k \in \{1, \ldots, N\}$ in the following and first derive the right-tail probability under the null hypothesis, i.e.,

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**Algorithm 1** Marginal Subspace Detection (MSD) for Subspace Unmixing

**Input:** Collection $\mathcal{X}_N = \{S_i \in \Theta(d, D), i = 1, \ldots, N\}$, observations $y \in \mathbb{R}^D$, and thresholds $\{\tau_i > 0\}_{i=1}^N$

**Output:** An estimate $\hat{A} \subset \{1, \ldots, N\}$ of the set of indices of active subspaces

- $U_k \leftarrow$ An orthonormal basis of the subspace $S_k$, $k = 1, \ldots, N$ \{Initialization\}
- $T_k(y) \leftarrow \|U_k^T y\|_2^2$, $k = 1, \ldots, N$ \{Computation of test statistics\}
- $\hat{A} \leftarrow \{k \in \{1, \ldots, N\} : T_k(y) > \tau_k\}$ \{Decision rules for marginal detection\}
Let $y = \sum_{j=1}^{n} \Phi_j \theta_j + \eta$ and $k \notin A = \{i_1, i_2, \ldots, i_n\}$. In order to facilitate the forthcoming analysis, we note that since $T_k(y)$ is invariant to the choice of $U_k$, we have $T_k(y) = \left\| \sum_{j=1}^{n} U_k^T \Phi_i \theta_j + U_k^T \eta \right\|_2^2 = \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j + \Phi_k^T \eta \right\|_2^2$. We now state the result that characterizes the right-tail probability of $T_k(y)$ under the null hypothesis, $\mathcal{H}_0^k$.

**Lemma 1.** Under the null hypothesis $\mathcal{H}_0^k$ for any fixed $k \in \{1, \ldots, N\}$, the test statistic has the following right-tail probability:

1. In the case of bounded deterministic error $\eta$ and the assumption $\tau > (\epsilon_n + \rho_k \sqrt{n \epsilon A})^2$, we have
   \[
   \Pr\left( T_k(y) \geq \tau \big| \mathcal{H}_0^k \right) \leq c^2 \exp\left( -\frac{c_0(N-n)^2(\sqrt{\tau} - \epsilon_n - \rho_k \sqrt{n \epsilon A})}{N^2 \gamma_{2,k}^{\epsilon \mathcal{A}}} \right).
   \] (5)

2. In the case of i.i.d. Gaussian noise $\eta$, define $\epsilon := \sigma \sqrt{d + 2\delta + 2\sqrt{d \delta}}$ for any $\delta > 0$. Then, under the assumption $\tau > (\epsilon + \rho_k \sqrt{n \epsilon A})^2$, we have
   \[
   \Pr\left( T_k(y) \geq \tau \big| \mathcal{H}_0^k \right) \leq c^2 \exp\left( -\frac{c_0(N-n)^2(\sqrt{\tau} - \epsilon - \rho_k \sqrt{n \epsilon A})}{N^2 \gamma_{2,k}^{\epsilon \mathcal{A}}} \right) + \exp(-\delta).
   \] (6)

Here, the parameter $c_0 := \frac{1}{2 \sqrt{\epsilon_0}}$ is an absolute positive constant.

**Proof:** We begin by defining $\bar{T}_k(y) := \sqrt{T_k(y)}$ and noting $\bar{T}_k(y) \leq \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j \right\|_2 + \left\| \Phi_k^T \eta \right\|_2$. In order to characterize the right-tail probability of $T_k(y)$ under $\mathcal{H}_0^k$, it suffices to characterize the right-tail probabilities of $Z_k^1 := \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j \right\|_2$ and $Z_k^2 := \left\| \Phi_k^T \eta \right\|_2$. This is rather straightforward in the case of $Z_k^2$. In the case of deterministic error $\eta$, we have $Z_k^2 \geq \epsilon_n$ with zero probability. In the case of $\eta$ being distributed as $\mathcal{N}(0, \sigma^2 I)$, we have that $\eta_k := \Phi_k^T \eta \in \mathbb{R}^d \sim \mathcal{N}(0, \sigma^2 I)$. In that case, the right-tail probability of $Z_k^2$ can be obtained by relying on a concentration of measure result in [27] Sec. 4, Lem. 1 for the sum of squares of i.i.d. Gaussian random variables. Specifically, it follows from [27] that $\forall \delta_2 > 0$,

\[
\Pr \left( Z_k^2 \geq \sigma \sqrt{d + 2\delta_2 + 2\sqrt{d \delta_2}} \right) \leq \exp(-\delta_2).
\] (7)

We now focus on the right-tail probability of $Z_k^1$, conditioned on the null hypothesis. Recall that $\mathcal{A}$ is a random $n$-subset of $\{1, 2, \ldots, N\}$ with $\Pr(\mathcal{A} = \{i_1, i_2, \ldots, i_n\}) = 1/(\binom{N}{n})$. Therefore, defining $\bar{\Pi} := (\pi_1, \ldots, \pi_N)$ to be a random permutation of $\{1, \ldots, N\}$ and using $\Pi := (\pi_1, \ldots, \pi_n)$ to denote the first $n$-elements of $\bar{\Pi}$, the following equality holds in distribution:

\[
\left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j \right\|_2 : k \notin A \overset{\text{dist}}{=} \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j \right\|_2 : k \notin \Pi.
\] (8)

We now define a probability event $E_0^k := \{ \Pi = (\pi_1, \ldots, \pi_n) : k \notin \Pi \}$ and notice from (8) that

\[
\Pr( Z_k^1 \geq \delta_1 | \mathcal{H}_0^k ) = \Pr \left( \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_i \theta_j \right\|_2 \geq \delta_1 | E_0^k \right).
\] (9)

The rest of this proof relies heavily on a Banach-space-valued Azuma’s inequality (Proposition [1]) stated in Appendix [A]. In order to make use of Proposition [1] we construct an $\mathbb{R}^d$-valued Doob’s martingale $(M_0, M_1, \ldots, M_n)$.
on \( \sum_{j=1}^{n} \Phi_k \theta_j \) as follows:

\[
M_0 := \sum_{j=1}^{n} \Phi_k \mathbb{E}[\Phi_j | E_0] \theta_j, \quad \text{and}
\]

\[
M_\ell := \sum_{j=1}^{n} \Phi_k \mathbb{E}[\Phi_j | \pi_{\ell-1}, E_0] \theta_j, \quad \ell = 1, \ldots, n,
\]

where \( \pi_{\ell} := (\pi_1, \ldots, \pi_{\ell}) \) denotes the first \( \ell \) elements of \( \Pi \). The next step involves showing that the constructed martingale has bounded \( \ell_2 \) differences. In order for this, we define

\[
M_\ell(u) := \sum_{j=1}^{n} \Phi_k \mathbb{E}[\Phi_j | \pi_{\ell-1}, \pi_{\ell} = u, E_0] \theta_j
\]

for \( u \in \{1, \ldots, N\} \setminus \{k\} \) and \( \ell = 1, \ldots, n \). It can then be established using techniques very similar to the ones used in the method of bounded differences for scalar-valued martingales.

In order to upper bound \( \|M_\ell(u) - M_\ell(v)\|_2 \), we define a \( D \times d \) matrix \( \bar{\Phi}_{\ell,j}^{u,v} \) as

\[
\bar{\Phi}_{\ell,j}^{u,v} := \mathbb{E}[\Phi_j | \pi_{\ell-1}, \pi_{\ell} = u, E_0] - \mathbb{E}[\Phi_j | \pi_{\ell-1}, \pi_{\ell} = v, E_0], \quad \ell = 1, \ldots, n,
\]

and note that \( \bar{\Phi}_{\ell,j}^{u,v} = 0 \) for \( j < \ell \) and \( \bar{\Phi}_{\ell,j}^{u,v} = \Phi_u - \Phi_v \) for \( j = \ell \). In addition, notice that the random variable \( \pi_j \) conditioned on \( \{\pi_{\ell-1}, \pi_{\ell} = u, E_0\} \) has a uniform distribution over \( \{1, \ldots, N\} \setminus \{\pi_{\ell-1}, u, k\} \), while \( \pi_j \) conditioned on \( \{\pi_{\ell-1}, \pi_{\ell} = v, E_0\} \) has a uniform distribution over \( \{1, \ldots, N\} \setminus \{\pi_{\ell-1}, v, k\} \). Therefore, we get \( \forall j > \ell \)

\[
\bar{\Phi}_{\ell,j}^{u,v} = \frac{1}{N - \ell - 1} (\Phi_u - \Phi_v).
\]

It now follows from the preceding discussion that

\[
\|M_\ell(u) - M_\ell(v)\|_2 = \left\| \sum_{j=1}^{n} \Phi_k \bar{\Phi}_{\ell,j}^{u,v} \theta_j \right\|_2 \leq \sum_{j=1}^{n} \left\| \Phi_k \bar{\Phi}_{\ell,j}^{u,v} \right\|_2 \|\theta_j\|_2 \leq \left\| \Phi_k (\Phi_u - \Phi_v) \right\|_2 \|\theta\|_2 + \frac{\sum_{j>\ell} \|\bar{\Phi}_{\ell,j}^{u,v}\|_2 \|\theta_j\|_2}{N - \ell - 1} \leq \gamma(S_k, S_u) + \gamma(S_k, S_v) \left( \|\theta\|_2 + \frac{\sum_{j>\ell} \|\theta_j\|_2}{N - \ell - 1} \right),
\]

where (a) is due to the triangle inequality and submultiplicativity of the operator norm. It then follows from (13), (10) and definition of the local 2-subspace coherence that

\[
\|M_\ell - M_{\ell-1}\|_2 \leq \gamma_{2,k} \left( \|\theta\|_2 + \frac{\sum_{j>\ell} \|\theta_j\|_2}{N - \ell - 1} \right).
\]

The final bound we need in order to utilize Proposition 1 is that on \( \|M_0\|_2 \). To this end, note that \( \pi_j \) conditioned on \( E_0^k \) has a uniform distribution over \( \{1, \ldots, N\} \setminus \{k\} \). It therefore follows that

\[
\|M_0\|_2 = \left\| \sum_{j=1}^{n} \Phi_k (\sum_{q=1}^{N} \Phi_q / (N - 1)) \theta_j \right\|_2 \leq \frac{1}{N - 1} \left\| \sum_{q=1}^{N} \Phi_k \Phi_q \right\|_2 \sum_{j=1}^{n} \|\theta_j\|_2 \leq \rho_k \sqrt{n} E_A.
\]
Here, (b) is again due to submultiplicativity of the operator norm, while (c) is due to definitions of the average mixing coherence and the cumulative active subspace energy as well as the triangle inequality and the Cauchy–Schwarz inequality. Next, we make use of [30, Lemma B.1] to note that $\zeta_{B}(\tau)$ defined in Proposition 1 satisfies $\zeta_{B}(\tau) \leq \tau^2 / 2$ for $(B, \| \cdot \|) \equiv (L_{2}(\mathbb{R}^{d}), \| \cdot \|_{2})$. Consequently, under the assumption $\delta_{1} > \rho_{k} \sqrt{N\mathcal{E}_{A}}$, it can be seen from our construction of the Doob martingale $(M_{0}, M_{1}, \ldots, M_{n})$ that

$$
\Pr\left( \left\| \sum_{j=1}^{n} \Phi_{k}^{j} \Phi_{\pi, j} \right\|_{2} \geq \delta_{1} \right) = \Pr\left( \| M_{n} \|_{2} \geq \delta_{1} \right) = \Pr\left( \| M_{n} \|_{2} \geq \delta_{1} - \| M_{0} \|_{2} \right)
$$

$$
\leq \Pr\left( \| M_{n} - M_{0} \|_{2} \geq \delta_{1} - \rho_{k} \sqrt{N\mathcal{E}_{A}} \right)
$$

$$
\leq e^{2} \exp \left( -c_{0}(\delta_{1} - \rho_{k} \sqrt{N\mathcal{E}_{A}})^{2} \right), \quad (19)
$$

where (d) is mainly due to the bound on $\| M_{0} \|_{2}$ in (18), while (e) follows from the Banach-space-valued Azuma inequality in Appendix A. In addition, we can establish using (17), the inequality $\sum_{j>\ell} \| \theta_{j} \|_{2} \leq \sqrt{N\mathcal{E}_{A}}$, and some tedious algebraic manipulations that

$$
\sum_{\ell=1}^{n} b_{\ell}^{2} = \gamma_{2,k}^{2} \sum_{\ell=1}^{n} \left( \| \theta_{\ell} \|_{2} + \sum_{j>\ell} \| \theta_{j} \|_{2} \right) \leq \gamma_{2,k}^{2} \mathcal{E}_{A} \left( \frac{N}{N - n} \right)^{2}. \quad (20)
$$

Combining (9), (19) and (20), we therefore obtain $\Pr(Z_{1}^{k} \geq \delta_{1}|H_{0}^{k}) \leq e^{2} \exp \left( -c_{0}(N-n)^{2}(\delta_{1} - \rho_{k} \sqrt{N\mathcal{E}_{A}})^{2} \right) / N^{2} \gamma_{2,k}^{2} \mathcal{E}_{A}^{2}$.

We now complete the proof by noting that

$$
\Pr\left( T_{1}(y) \geq \tau |H_{1}^{k} \right) = \Pr\left( \tilde{T}_{1}(y) \geq \sqrt{\tau} |H_{1}^{k} \right) \leq \Pr\left( Z_{1}^{k} + Z_{2}^{k} \geq \sqrt{\tau} |H_{0}^{k} \right)
$$

$$
\leq \Pr\left( Z_{1}^{k} + Z_{2}^{k} \geq \sqrt{\tau} |H_{0}^{k}, Z_{2}^{k} < \epsilon_{2} \right) + \Pr\left( Z_{2}^{k} \geq \epsilon_{2} |H_{0}^{k} \right)
$$

$$
\leq \Pr\left( Z_{1}^{k} \geq \sqrt{\tau} - \epsilon_{2} |H_{0}^{k} \right) + \Pr\left( Z_{2}^{k} \geq \epsilon_{2} \right). \quad (21)
$$

The two statements in the lemma now follow from the (probabilistic) bounds on $Z_{2}^{k}$ established at the start of the proof and the probabilistic bound on $Z_{1}^{k}$ obtained in the preceding paragraph.

Our next goal is evaluation of $\Pr\left( T_{1}(y) \leq \tau |H_{1}^{k} \right)$. In this regard, we once again fix an arbitrary $k \in \{1, \ldots, N\}$ and derive the left-tail probability under the alternative hypothesis, $H_{1}^{k}$, i.e., $y = \sum_{j=1}^{n} \Phi_{k} \theta_{j} + \eta$ such that the index $k \in A = \{i_{1}, i_{2}, \ldots, i_{n}\}$.

**Lemma 2.** Under the alternative hypothesis $H_{1}^{k}$ for any fixed $k \in \{1, \ldots, N\}$, the test statistic has the following left-tail probability:

1) In the case of bounded deterministic error $\eta$ and under the assumptions $\mathcal{E}_{k} > (\epsilon_{\eta} + \rho_{k} \sqrt{n(\mathcal{E}_{A} - \mathcal{E}_{k})})^{2}$ and $\tau < (\sqrt{\mathcal{E}_{k} - \epsilon_{\eta} - \rho_{k} \sqrt{n(\mathcal{E}_{A} - \mathcal{E}_{k})})^{2}$, we have

$$
\Pr\left( T_{k}(y) \leq \tau |H_{1}^{k} \right) \leq e^{2} \exp \left( -c_{0}(N-n)^{2}(\sqrt{\mathcal{E}_{k} - \sqrt{\tau} - \epsilon_{\eta} - \rho_{k} \sqrt{n(\mathcal{E}_{A} - \mathcal{E}_{k})})^{2} \right) / (2N-n)^{2} \gamma_{2,k}^{2} (\mathcal{E}_{A} - \mathcal{E}_{k})^{2}. \quad (22)
$$
2) In the case of i.i.d. Gaussian noise $\eta$, define $\epsilon := \sigma \sqrt{d + 2\delta + 2\sqrt{d\delta}}$ for any $\delta > 0$. Then, under the assumptions $E_k > (\epsilon + \rho_k \sqrt{n(E_A - E_k)})^2$ and $\tau < (\sqrt{E_k} - \epsilon - \rho_k \sqrt{n(E_A - E_k)})^2$, we have

$$\Pr(T_k(y) \leq \tau | \mathcal{H}_k^1) \leq e^{2\epsilon} \exp\left(\frac{c_0(N - n) - 2\epsilon \sqrt{2\rho_k(E_A - E_k)}}{(2N - n)^2 \rho_k (E_A - E_k)}\right) + \exp(-\delta). \quad (23)$$

Here, the parameter $c_0 := \frac{1}{2\rho_k}$ is an absolute positive constant.

**Proof:** We once again define $\tilde{T}_k(y) := \sqrt{T_k(y)}$ and note that $\tilde{T}_k(y) \geq \|\sum_{j=1}^n \Phi_k^T \Phi_j \theta_j\|_2 - \|\Phi_k^T \eta\|_2$. Therefore, characterization of the left-tail probability of $Z_k^1 := \|\sum_{j=1}^n \Phi_k^T \Phi_j \theta_j\|_2$ and the right-tail probability of $Z_k^2 := \|\Phi_k^T \eta\|_2$ under $\mathcal{H}_k^1$ helps us specify the left-tail probability of $T_k(y)$ under $\mathcal{H}_k^1$. Since the right-tail probability of $Z_k^2$ for both deterministic and stochastic errors has already been specified in the proof of Lemma 4, we need only focus on the left-tail probability of $Z_k^1$ under $\mathcal{H}_k^1$ in here.

In order to characterize $\Pr(Z_k^1 \leq \delta_1 | \mathcal{H}_k^1)$, we once again define $\Pi := (\pi_1, \ldots, \pi_N)$ to be a random permutation of $\{1, \ldots, N\}$ and use $\Pi := (\pi_1, \ldots, \pi_n)$ to denote the first $n$-elements of $\Pi$. We then have the following equality in distribution:

$$\left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 : k \in A \overset{dist}{=} \left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 : k \in \Pi. \quad (24)$$

We now define a probability event $E_k^i := \{\Pi = (\pi_1, \ldots, \pi_n) : k \in \Pi\}$ and notice from (24) that

$$\Pr(Z_k^1 \leq \delta_1 | \mathcal{H}_k^1) = \Pr\left(\left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \leq \delta_1 | E_k^i\right). \quad (25)$$

Next, we fix an arbitrary $i \in \{1, \ldots, n\}$ and define another probability event $E_2^i := \{\pi_i = k\}$. It then follows that

$$\Pr\left(\left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \leq \delta_1 | E_k^i\right) = \sum_{i=1}^n \Pr\left(\left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \leq \delta_1 | E_k^i, E_2^i\right) \Pr(E_2^i | E_k^i)$$

$$= \sum_{i=1}^n \Pr\left(\left\| \theta_i + \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \leq \delta_1 | E_k^i, E_2^i\right) \Pr(E_2^i | E_k^i)$$

$$\leq \sum_{i=1}^n \Pr\left(\left\| \sum_{j=1}^n \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \geq \sqrt{E_k^i} - \delta_1 | E_2^i\right) \Pr(E_2^i | E_k^i), \quad (26)$$

where (a) follows for the facts that (i) $\|\theta_i + \sum_{j \neq i} \Phi_k^T \Phi_{\pi_j} \theta_j\|_2 \geq \|\theta_i\|_2 - \|\sum_{j \neq i} \Phi_k^T \Phi_{\pi_j} \theta_j\|_2$, (ii) $\|\theta_i\|_2$ conditioned on $E_2^i$ is $\sqrt{E_k^i}$, and (iii) $E_2^i \subset E_k^i$. It can be seen from (25) and (26) that our main challenge now becomes specifying the right-tail probability of $\|\sum_{j \neq i} \Phi_k^T \Phi_{\pi_j} \theta_j\|_2$ conditioned on $E_2^i$. To this end, we once again rely on Proposition 4 in Appendix A.

Specifically, we construct an $\mathbb{R}^d$-valued Doob martingale $(M_0, M_1, \ldots, M_{n-1})$ on $\sum_{j \neq i} \Phi_k^T \Phi_{\pi_j} \theta_j$ as follows.
We first define $\Pi^{-i} := (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n)$ and then define
\begin{align}
M_0 := \sum_{j=1}^{n} \Phi_k^T \mathbb{E}[\Phi_{\pi_j} \mid E_2] \theta_j, \quad \text{and} \\
M_\ell := \sum_{j=1}^{n} \Phi_k^T \mathbb{E}[\Phi_{\pi_j} \mid \pi_1^{i-1}, \ell, E_2] \theta_j, \quad \ell = 1, \ldots, n-1,
\end{align}
\tag{27}
\tag{28}
where $\pi_1^{i,\ell}$ denotes the first $\ell$ elements of $\Pi^{-i}$. The next step in the proof involves showing $\|M_\ell - M_{\ell-1}\|_2$ is bounded for all $\ell \in \{1, \ldots, n-1\}$. To do this, we use $\pi_\ell^{-i}$ to denote the $\ell$-th element of $\Pi^{-i}$ and define
\begin{align}
M_\ell(u) := \sum_{j=1}^{n} \Phi_k^T \mathbb{E}[\Phi_{\pi_j} \mid \pi_1^{i-1,\ell-1}, \pi_\ell^{-i} = u, E_2] \theta_j
\end{align}
\tag{29}
for $u \in \{1, \ldots, N\} \setminus \{k\}$ and $\ell = 1, \ldots, n-1$. It then follows from the argument in Lemma $\text{[1]}$ that $\|M_\ell - M_{\ell-1}\|_2 \leq \sup_{u,v} \|M_\ell(u) - M_\ell(v)\|_2$. We now define a $D \times d$ matrix $\tilde{\Phi}_{\ell,j}^{u,v}$ as
\begin{align}
\tilde{\Phi}_{\ell,j}^{u,v} := \mathbb{E}[\Phi_{\pi_j} \mid \pi_1^{i-1,\ell-1}, \pi_\ell^{-i} = u, E_2] - \mathbb{E}[\Phi_{\pi_j} \mid \pi_1^{i-1,\ell-1}, \pi_\ell^{-i} = v, E_2], \quad \ell = 1, \ldots, n.
\end{align}
\tag{30}
It is then easy to see that $\forall j > \ell + 1, j \neq i$, the random variable $\pi_j$ conditioned on the events $\{\pi_1^{i-1,\ell-1}, \pi_\ell^{-i} = u, E_2\}$ and $\{\pi_1^{i-1,\ell-1}, \pi_\ell^{-i} = v, E_2\}$ has a uniform distribution over the sets $\{1, \ldots, N\} \setminus \{\pi_1^{i-1,\ell-1}, u, k\}$ and $\{1, \ldots, N\} \setminus \{\pi_1^{i-1,\ell-1}, v, k\}$, respectively. It therefore follows $\forall j > \ell + 1, j \neq i$ that $\tilde{\Phi}_{\ell,j}^{u,v} = \frac{1}{N-1} (\Phi_u - \Phi_v)$.

In order to evaluate $\tilde{\Phi}_{\ell,j}^{u,v}$ for $\ell \geq \ell + 1, j \neq i$, we need to consider three cases for the index $\ell$. In the first case of $\ell \geq i$, it can be seen that $\tilde{\Phi}_{\ell,j}^{u,v} = 0 \forall j \leq \ell$ and $\tilde{\Phi}_{\ell,j}^{u,v} = \Phi_u - \Phi_v$ for $j = \ell + 1$. In the second case of $\ell = i - 1$, it can similarly be seen that $\tilde{\Phi}_{\ell,j}^{u,v} = 0 \forall j < \ell$ and $j = \ell + 1$, while $\tilde{\Phi}_{\ell,j}^{u,v} = \Phi_u - \Phi_v$ for $j = \ell$. In the final case of $\ell < i - 1$, it can be further argued that $\tilde{\Phi}_{\ell,j}^{u,v} = 0 \forall j < \ell$, $\tilde{\Phi}_{\ell,j}^{u,v} = \Phi_u - \Phi_v$ for $j = \ell$, and $\tilde{\Phi}_{\ell,j}^{u,v} = \frac{1}{N-1} (\Phi_u - \Phi_v)$ for $j = \ell + 1$. Combining all these facts together, we have the following upper bound:
\begin{align}
\|M_\ell(u) - M_\ell(v)\|_2 &= \| \sum_{j=1}^{n} \Phi_k^T \tilde{\Phi}_{\ell,j}^{u,v} \theta_j \|_2 \\
&\overset{(b)}{\leq} \sum_{j=1}^{n} \| \Phi_k^T \tilde{\Phi}_{\ell,j}^{u,v} \|_2 \| \theta_j \|_2 \\
&\overset{(c)}{\leq} ||\Phi_k^T (\Phi_u - \Phi_v) ||_2 \left( \|\theta_\ell\|_2 1_{(\ell \neq i)} + \|\theta_{\ell+1}\|_2 1_{(\ell \neq i-1)} + \sum_{j > \ell + 1} \|\theta_j\|_2 \frac{1}{N - \ell - 1} \right) \\
&\leq (\gamma(S_k, S_u) + \gamma(S_k, S_v)) \left( \|\theta_\ell\|_2 1_{(\ell \neq i)} + \|\theta_{\ell+1}\|_2 1_{(\ell \neq i-1)} + \sum_{j > \ell + 1} \|\theta_j\|_2 \frac{1}{N - \ell - 1} \right).
\end{align}
\tag{31}
Here, (b) and (c) follow from the preceding facts that $\tilde{\Phi}_{\ell,j}^{u,v} = 0 \forall j < \ell$ and $\|\Phi_k^T \tilde{\Phi}_{\ell,j}^{u,v} \|_2 \leq \|\Phi_k^T (\Phi_u - \Phi_v) \|_2$ for $j = \ell$ and $j = \ell + 1$. Consequently, it follows from (31) and definition of the local 2-subspace coherence that
\begin{align}
\|M_\ell - M_{\ell-1}\|_2 \leq \gamma_{2,k} \left( \|\theta_\ell\|_2 1_{(\ell \neq i)} + \|\theta_{\ell+1}\|_2 1_{(\ell \neq i-1)} + \sum_{j > \ell + 1} \|\theta_j\|_2 \frac{1}{N - \ell - 1} \right) \leq \frac{b_\ell}{b_{\ell}} \|\theta_{\ell+1}\|_2 1_{(\ell \neq i-1)} + \sum_{j > \ell + 1} \|\theta_j\|_2 \frac{1}{N - \ell - 1}.
\end{align}
\tag{32}
The next step needed to utilize Proposition 1 involves an upper bound on \( \| M_0 \|_2 \), which is given as follows:

\[
\| M_0 \|_2 = \left\| \sum_{j \neq i} \Phi_k^T \left( \sum_{q=1}^{N} \Phi_q \right) \theta_j \right\|_2 \leq \frac{1}{N-1} \left\| \sum_{q=1}^{N} \Phi_k^T \Phi_q \right\|_2 \left\| \sum_{j \neq i} \theta_j \right\|_2 \leq \rho_k \sqrt{(n-1)(E_A - E_k)}. \quad (33)
\]

Here, \((d)\) primarily follows from the fact that, conditioned on \( E_2 \), \( \sum_{j \neq i} \| \theta_j \|_2^2 = E_A - E_k \).

Our construction of the Doob martingale, Proposition 1 in Appendix A, \([30, \text{Lemma B.1}]\) and the assumption \( \sqrt{E_k} - \delta_1 > \rho_k \sqrt{n(E_A - E_k)} \) now provides us the following upper bound:

\[
\Pr \left( \left\| \sum_{j=1}^{n} \Phi_k^T \Phi_{\pi_j} \theta_j \right\|_2 \geq \sqrt{E_k} - \delta_1 \mid E_2 \right) = \Pr \left( \| M_{n-1} \|_2 \geq \sqrt{E_k} - \delta_1 \mid E_2 \right)
\]

\[= \Pr \left( \| M_{n-1} \|_2 - \| M_0 \|_2 \geq \sqrt{E_k} - \delta_1 - \| M_0 \|_2 \mid E_2 \right) \]

\[\leq \Pr \left( \| M_{n-1} - M_0 \|_2 \geq \sqrt{E_k} - \delta_1 - \rho_k \sqrt{n(E_A - E_k)} \mid E_2 \right) \]

\[\leq e^2 \exp \left( -\frac{c_0 \left( \sqrt{E_k} - \delta_1 - \rho_k \sqrt{n(E_A - E_k)} \right)^2}{\sum_{\ell=1}^{n-1} b_{\ell}^2 \gamma_{2,k} (E_A - E_k)} \right), \quad (34)\]

where \((e)\) is primarily due to the bound on \( \| M_0 \|_2 \) in \((33)\). Further, it can be shown using \((32)\), the inequality \( \sum_{\ell=1}^{n-1} \| \theta_{\ell} \|_2 \| \theta_{\ell+1} \|_2 \| \theta_{\ell-i} \|_2 \leq (E_A - E_k) \), and some tedious manipulations that the following holds:

\[\sum_{\ell=1}^{n-1} b_{\ell}^2 \leq \gamma_{2,k}^2 (E_A - E_k) \left( \frac{2N-n}{N-n} \right)^2. \quad (35)\]

Combining \((25)\), \((26)\), \((34)\) and \((35)\), we obtain \( \Pr (Z_1^k \leq \delta_1 | \mathcal{H}_1^k) \leq e^2 \exp \left( -\frac{c_0 (N-n)^2 \left( \sqrt{E_k} - \delta_1 - \rho_k \sqrt{n(E_A - E_k)} \right)^2}{(2N-n)^2 \gamma_{2,k}^2 (E_A - E_k)} \right). \)

The proof of the lemma can now be completed by noting that

\[
\Pr \left( T_k(y) \leq \tau \mid \mathcal{H}_1^k \right) = \Pr \left( T_k(y) \leq \sqrt{\tau} \mid \mathcal{H}_1^k \right) \leq \Pr \left( Z_1^k - Z_2^k \leq \sqrt{\tau} \mid \mathcal{H}_1^k \right)
\]

\[\leq \Pr \left( Z_1^k - Z_2^k \leq \sqrt{\tau} \mid \mathcal{H}_1^k, Z_2^k < \epsilon_2 \right) + \Pr \left( Z_2^k \geq \epsilon_2 \mid \mathcal{H}_1^k \right)
\]

\[\leq \Pr \left( Z_1^k \leq \sqrt{\tau} + \epsilon_2 \mid \mathcal{H}_1^k \right) + \Pr \left( Z_2^k \geq \epsilon_2 \right). \quad (36)\]

The two statements in the lemma now follow from the (probabilistic) bounds on \( Z_2^k \) established at the start of the proof of Lemma 1 and the probabilistic bound on \( Z_1^k \) obtained in the preceding paragraph.

\[\bull\]

C. Performance of Marginal Subspace Detection

In this section, we will leverage Lemma 1 to control the FWER of MSD at a prescribed level \( \alpha \). In addition, we will make use of Lemma 2 to understand the NDP performance of MSD when its FWER is controlled at level \( \alpha \). Before proceeding with these goals, however, it is instructive to provide an intuitive interpretation of Lemmas 1 and 2 for individual subspaces (i.e., in the absence of a formal correction for multiple hypothesis testing \([21], [22]\)). We provide such an interpretation in the following for the case of bounded deterministic error \( \eta \), with the understanding that extensions of our arguments to the case of i.i.d. Gaussian noise \( \eta \) are straightforward.
Lemma 1 characterizes the probability of individually rejecting the null hypothesis $H_{0k}$ when it is true (i.e., declaring the subspace $S_k$ to be active when it is inactive). Suppose for the sake of argument that $H_{0k}$ is true and $S_k$ is orthogonal to every subspace in $X_N \setminus \{S_k\}$, in which case the $k$-th test statistic reduces to $T_k(y) \equiv \|\eta\|^2_2$. It is then easy to see in this hypothetical setting that the decision threshold $\tau_k$ must be above the noise floor, $\tau_k > \epsilon_n^2$, to ensure one does not reject $H_{0k}$ when it is true. Lemma 1 effectively generalizes this straightforward observation to the case when the $S_k$ cannot be orthogonal to every subspace in $X_N \setminus \{S_k\}$. First, the lemma states in this case that an effective noise floor, defined as $\epsilon_{\text{eff}}^2 := (\epsilon_n + \rho_k \sqrt{m^2_A})^2$, appears in the problem and the decision threshold must now be above this effective noise floor, $\tau_k > \epsilon_{\text{eff}}^2$, to ensure one does not reject $H_{0k}$ when it is true. It can be seen from the definition of the effective noise floor that $\epsilon_{\text{eff}}$ has an intuitive additive form, with the first term $\epsilon_n$ being due to the additive error $\eta$ and the second term $\rho_k \sqrt{m^2_A}$ being due to the mixing with non-orthogonal subspaces. In particular, $\epsilon_{\text{eff}} \setminus \epsilon_n$ as the average mixing coherence $\rho_k \setminus 0$ (recall that $\rho_k \equiv 0$ for the case of $S_k$ being orthogonal to the subspaces in $X_N \setminus \{S_k\}$). Second, once a threshold above the effective noise floor is chosen, the lemma states that the probability of rejecting the true $H_{0k}$ decreases exponentially as the gap between the threshold and the effective noise floor increases and/or the local 2-subspace coherence $\gamma_{2,k}$ of $S_k$ decreases.

In particular, the probability of rejecting the true $H_{0k}$ in this case has the intuitively pleasing characteristic that it approaches zero exponentially fast as $\gamma_{2,k} \setminus 0$ (recall that $\gamma_{2,k} \equiv 0$ for the case of $S_k$ being orthogonal to the subspaces in $X_N \setminus \{S_k\}$).

We now shift our focus to Lemma 2 which specifies the probability of individually rejecting the alternative hypothesis $H_{1k}$ when it is true (i.e., declaring the subspace $S_k$ to be inactive when it is indeed active). It is once again instructive to first understand the hypothetical scenario of $S_k$ being orthogonal to every subspace in $X_N \setminus \{S_k\}$.

In this case, the $k$-th test statistic under $H_{1k}$ being true reduces to $T_k(y) \equiv \|x_{S_k} + U_k^T \eta\|^2_2$, where $x_{S_k}$ denotes the component of the noiseless signal $x$ that is contributed by the subspace $S_k$. Notice in this hypothetical setting that the rotated additive error $U_k^T \eta$ can in principle be antipodally aligned with the signal component $x_{S_k}$, thereby reducing the value of $T_k(y)$. It is therefore easy to argue in this idealistic setup that ensuring one does accept $H_{1k}$ when it is true requires: (i) the energy of the subspace $S_k$ to be above the noise floor, $E_k > \epsilon_n^2$, so that the test statistic remains strictly positive; and (ii) the decision threshold $\tau_k$ to be below the subspace-to-noise gap, $\tau_k < (\sqrt{E_k} - \epsilon_n)^2$, so that the antipodal alignment of $U_k^T \eta$ with $x_{S_k}$ does not result in a false negative. We now return to the statement of Lemma 2 and note that it also effectively generalizes these straightforward observations to the case when the $S_k$ cannot be orthogonal to every subspace in $X_N \setminus \{S_k\}$. First, similar to the case of Lemma 1 this lemma states in this case that an effective noise floor, defined as $\epsilon_{\text{eff}}^2 := (\epsilon_n + \rho_k \sqrt{m^2_A - E_k})^2$, appears in the problem and the energy of the subspace $S_k$ must now be above this effective noise floor, $E_k > \epsilon_{\text{eff}}^2$, to ensure that the test statistic remains strictly positive. In addition, we once again have an intuitive additive form of $\epsilon_{\text{eff}}$, with its first term being due to the additive error $\eta$, its second term being due to the mixing with non-orthogonal subspaces, and $\epsilon_{\text{eff}} \setminus \epsilon_n$ as the average mixing coherence $\rho_k \setminus 0$. Second, the lemma states that the decision threshold must now be below the subspace-to-effective-noise gap, $\tau_k < (\sqrt{E_k} - \epsilon_{\text{eff}})^2$. Third, once a threshold below the subspace-to-effective-noise gap is chosen, the lemma states that the probability of rejecting the true $H_{1k}$
decreases exponentially as the gap between \((\sqrt{\.\gamma_{2,k}} - \epsilon_{\text{eff}})^2\) and the threshold increases and/or the local 2-subspace coherence \(\gamma_{2,k}\) of \(S_k\) decreases. In particular, Lemma 2 once again has the intuitively pleasing characteristic that the probability of rejecting the true \(\mathcal{H}_1^k\) approaches zero exponentially fast as \(\gamma_{2,k} \rightarrow 0\).

Roughly speaking, it can be seen from the preceding discussion that increasing the values of the decision thresholds \(\{\tau_k\}\) in MSD should decrease the FWER. Such a decrease in the FWER of course will come at the expense of an increase in the NDP. We will specify this relationship between the \(\tau_k\)'s and the NDP in the following. But we first characterize one possible choice of the \(\tau_k\)'s that helps control the FWER of MSD at a predetermined level \(\alpha\). The following theorem makes use of Lemma 1 and the Bonferroni correction for multiple hypothesis testing [21].

**Theorem 1.** The family-wise error rate of the marginal subspace detection (Algorithm 1) can be controlled at any level \(\alpha \in [0, 1]\) by selecting the decision thresholds \(\{\tau_k\}_{k=1}^N\) as follows:

1) **In the case of bounded deterministic error \(\eta\), select**

\[
\tau_k = \left( \epsilon_{\eta} + \rho_k \sqrt{nE_A} + \frac{\gamma_{2,k}N}{N-n} \sqrt{c_0^{-1}E_A \log \left( \frac{e^{2N}}{\alpha} \right)} \right)^2, \quad k = 1, \ldots, N.
\]

2) **In the case of i.i.d. Gaussian noise \(\eta\), select**

\[
\tau_k = \left( \sigma \sqrt{d + 2 \log \left( \frac{2N}{\alpha} \right)} + 2 \sqrt{d \log \left( \frac{2N}{\alpha} \right)} + \rho_k \sqrt{nE_A} + \frac{\gamma_{2,k}N}{N-n} \sqrt{c_0^{-1}E_A \log \left( \frac{e^{2N}}{\alpha} \right)} \right)^2, \quad k = 1, \ldots, N.
\]

**Proof:** The Bonferroni correction for multiple hypothesis testing dictates that the FWER of the MSD is guaranteed to be controlled at a level \(\alpha \in [0, 1]\) as long as the probability of false positive of each individual hypothesis is controlled at level \(\frac{\alpha}{N}\) [21], i.e., \(Pr(T_k(y) \geq \tau_k | \mathcal{H}_0) \leq \frac{\alpha}{N}\). The statement for the bounded deterministic error \(\eta\) can now be shown to hold by plugging the prescribed decision thresholds into Lemma 1. Similarly, the statement for the i.i.d. Gaussian noise \(\eta\) can be shown to hold by plugging \(\delta := \log \left( \frac{2N}{\alpha} \right)\) and the prescribed decision thresholds into Lemma 1.

A few remarks are in order now regarding Theorem 1. We once again limit our discussion to the case of bounded deterministic error, since its extension to the case of i.i.d. Gaussian noise is straightforward. In the case of deterministic error \(\eta\), Theorem 1 requires the decision thresholds to be of the form \(\tau_k = (\epsilon_{\eta} + \epsilon_{m,1} + \epsilon_{m,2})^2\), where \(\epsilon_{\eta}\) captures the effects of the additive error, \(\epsilon_{m,1}\) is due to the mixing with non-orthogonal subspaces, and \(\epsilon_{m,2}\) captures the effects of both the mixing with non-orthogonal subspaces and the FWER \(\alpha\). Other factors that affect the chosen thresholds include the total number of subspaces, the number of active subspaces, and the cumulative active subspace energy. But perhaps the most interesting aspect of Theorem 1 is the fact that as the mixing subspaces become “closer” to being orthogonal, the chosen thresholds start approaching the noise floor \(\epsilon_{\eta}^2\): \(\tau_k \rightarrow \epsilon_{\eta}^2\) as \(\rho_k, \gamma_{2,k} \rightarrow 0\).

While Theorem 1 helps control the FWER of MSD, it does not shed light on the corresponding NDP figure for MSD. In order to completely characterize the performance of MSD, therefore, we also need the following theorem.

---

4In here, we are suppressing the dependence of \(\epsilon_{m,1}\) and \(\epsilon_{m,2}\) on the subspace index \(k\) for ease of notation.
Theorem 2. Suppose the family-wise error rate of the marginal subspace detection (Algorithm 1) is controlled at level $\alpha \in [0, 1]$ by selecting the decision thresholds $\{\tau_k\}_{k=1}^N$ specified in Theorem 1. Then the estimate of the indices of active subspaces returned by MSD satisfies $\hat{A} \supset A_*$ with probability exceeding $1 - \varepsilon$, where:

1) In the case of bounded deterministic error $\eta$, we have $\varepsilon := N^{-1} + \alpha$ and

$$A_* := \left\{ i \in A : \mathcal{E}_i > \left( 2\eta + \rho_i \sqrt{n \mathcal{E}_{i,i}} + \frac{\gamma_{2,i} N}{N - n} \sqrt{c_0^{-1} \mathcal{E}_{2,i}} \right)^2 \right\}$$

with parameters $\mathcal{E}_{1,i} := \left( \sqrt{\mathcal{E}_A} + \sqrt{\mathcal{E}_A - \mathcal{E}_i} \right)^2$ and $\mathcal{E}_{2,i} := \left( \sqrt{\mathcal{E}_A \log(\frac{e^2 N}{\alpha})} + (2 - \frac{n}{N}) \sqrt{2(\mathcal{E}_A - \mathcal{E}_i) \log(eN)} \right)^2$.

2) In the case of i.i.d. Gaussian noise $\eta$, we have $\varepsilon := N^{-1} + \frac{3}{2} \alpha$ and

$$A_* := \left\{ i \in A : \mathcal{E}_i > \left( 2\varepsilon + \rho_i \sqrt{n \mathcal{E}_{i,i}} + \frac{\gamma_{2,i} N}{N - n} \sqrt{c_0^{-1} \mathcal{E}_{2,i}} \right)^2 \right\}$$

with the three parameters $\varepsilon := \sigma \sqrt{d + 2 \log(\frac{2N}{\alpha})} + 2 \sqrt{d \log(\frac{2N}{\alpha})}$, $\mathcal{E}_{1,i} := \left( \sqrt{\mathcal{E}_A} + \sqrt{\mathcal{E}_A - \mathcal{E}_i} \right)^2$ and $\mathcal{E}_{2,i} := \left( \sqrt{\mathcal{E}_A \log(\frac{e^2 N}{\alpha})} + (2 - \frac{n}{N}) \sqrt{2(\mathcal{E}_A - \mathcal{E}_i) \log(eN)} \right)^2$.

Proof: In order to prove the statement for the bounded deterministic error $\eta$, pick an arbitrary $i \in A_*$ and notice that the assumptions within Lemma 2 for the subspace $\mathcal{S}_i \in \mathcal{A}_N$ are satisfied by virtue of the definition of $A_*$ and the choice of the decision thresholds in Theorem 1. It therefore follows from (22) in Lemma 2 that $i \not\in \hat{A}$ with probability at most $N^{-2}$. We can therefore conclude by a simple union bound argument that $A_* \not\subset \hat{A}$ with probability at most $N^{-1}$. The statement now follows from a final union bound over the events $A_* \not\subset \hat{A}$ and $\hat{A} \not\subset A_*$, where the second event is needed since we are simultaneously controlling the FWER at level $\alpha$. Likewise, the statement for the i.i.d. Gaussian noise $\eta$ can be shown to hold by first plugging $\delta := \log(\frac{2N}{\alpha})$ into (23) in Lemma 2 and then making use of similar union bound arguments.

Remark 1. An astute reader will notice that we are being loose in our union bounds for the case of i.i.d. Gaussian noise. Indeed, we are double counting the event that the sum of squares of $d$ i.i.d. Gaussian random variables exceeds $\varepsilon^2$, once during Lemma 1 (which is used for FWER calculations) and once during Lemma 2 (which is used for this theorem). In fact, it can be shown through a better bookkeeping of probability events that $\varepsilon = N^{-1} + \alpha$ for i.i.d. Gaussian noise also. Nonetheless, we prefer the stated theorem because of the simplicity of its proof.

It can be seen from Theorem 2 that if one controls the FWER of the MSD using Theorem 1 then its NDP figure satisfies $\text{NDP} \leq \frac{|A_\ast| |A_*|}{n}$ with probability exceeding $1 - N^{-1} - \Theta(\alpha)$. Since $A_* \subset A_*$, it then follows that the NDP figure is the smallest when the cardinality of $A_*$ is the largest. It is therefore instructive to understand the nature of $A_*$, which is the set of indices of active subspaces that are guaranteed to be identified as active by the MSD algorithm. Theorem 2 tells us that any active subspace whose energy is not “too small” is a member of $A_*$. Specifically, in the case of bounded deterministic error, the threshold that determines whether the energy of an active subspace is large or small for the purposes of identification by MSD takes the form $(2\eta + \hat{\varepsilon}_{m,1} + \hat{\varepsilon}_{m,2})^2$. Here, similar to the case of Theorem 1 we observe that $\hat{\varepsilon}_{m,1}$ and $\hat{\varepsilon}_{m,2}$ are pseudo-noise terms that appear only due to the mixing with non-orthogonal subspaces and that depend upon additional factors such as the total number of
subspaces, the number of active subspaces, the cumulative active subspace energy, and the FWER. In particular, we once again have the intuitive result that \( \tilde{\epsilon}_{m,1}, \tilde{\epsilon}_{m,2} \to 0 \) as \( \rho_i, \gamma_{2,i} \to 0 \), implying that any active subspace whose energy is on the order of the noise floor will be declared as active by the MSD algorithm in this setting. Since this is the best that any subspace unmixing algorithm can be expected to accomplish, one can argue that the MSD algorithm performs near-optimal subspace unmixing for the case when the average mixing coherences and the local 2-subspace coherences of individual subspaces in the collection \( \mathcal{X}_N \) are significantly small. Finally, note that this intuitive understanding of MSD can be easily extended to the case of i.i.d. Gaussian noise, with the major difference being that \( \epsilon_\eta \) in that case gets replaced by

\[
\epsilon = \sigma \sqrt{d + 2 \log \left( \frac{2Na}{\alpha} \right) + 2 \sqrt{d \log \left( \frac{2Na}{\alpha} \right)}}.
\]

IV. MARGINAL SUBSPACE DETECTION AND SUBSPACE COHERENCE CONDITIONS

We have established in Sec. [III] that the FWER of MSD can be controlled at any level \( \alpha \in [0, 1] \) through appropriate selection of the decision thresholds (cf. Theorem [1]). Further, we have shown that the selected thresholds enable the MSD algorithm to identify all active subspaces whose energies exceed effective noise floors characterized by additive error/noise, average mixing coherences, local 2-subspace coherences, etc. (cf. Theorem [2]). Most importantly, these effective noise floors approach the “true” noise floor as the average mixing coherences and the local 2-subspace coherences of individual subspaces approach zero, suggesting near-optimal nature of MSD for such collections of mixing subspaces in the “D smaller than N” setting. But we have presented no mathematical evidence to suggest the average mixing coherences and the local 2-subspace coherences of individual subspaces can indeed be small enough for the effective noise floors of Theorem [2] to be on the order of \( \text{(true noise floor + o(1))} \). Our goal in this section, therefore, is to provide evidence to this effect by arguing for the existence of collection of subspaces whose average mixing coherences and local 2-subspace coherences approach zero at significantly fast rates.

Recall from the statement of Theorem [2] and the subsequent discussion that the effective noise floor for the \( i \)-th subspace involves additive pseudo-noise terms of the form

\[
\epsilon'_i := \rho_i \sqrt{n \mathcal{E}_{1,i}} + \frac{\gamma_{2,i}N}{N-n} \sqrt{\mathcal{E}_{2,i}},
\]

where \( \sqrt{\mathcal{E}_{1,i}} = \Theta \left( \sqrt{\mathcal{E}_A} \right) \) and \( \sqrt{\mathcal{E}_{2,i}} = \Theta \left( \sqrt{\mathcal{E}_A \log(N/\alpha)} \right) \). Since we are assuming that the number of active subspaces \( n = O(N) \), it follows that \( \epsilon'_i = \Theta \left( \rho_i \sqrt{n \mathcal{E}_A} \right) + \Theta \left( \gamma_{2,i} \sqrt{\mathcal{E}_A \log(N/\alpha)} \right) \). In order to ensure \( \epsilon'_i = o(1) \), therefore, we need the following two conditions to hold:

\[
\rho_i = O \left( \frac{1}{\sqrt{n \mathcal{E}_A}} \right), \quad \text{and}
\]

\[
\gamma_{2,i} = O \left( \frac{1}{\sqrt{\mathcal{E}_A \log(N/\alpha)}} \right).
\]

Together, we term the conditions (38) and (39) as subspace coherence conditions. Both these conditions are effectively statements about the geometry of the mixing subspaces and the corresponding mixing bases. In order to understand the implications of these two conditions, we parameterize the cumulative active subspace energy as

\[5\text{We are once again suppressing the dependence of } \tilde{\epsilon}_{m,1} \text{ and } \tilde{\epsilon}_{m,2} \text{ on the subspace index for ease of notation.} \]
$E_A = \Theta(n^4)$ for $\delta \in [0, 1]$. Here, $\delta = 0$ corresponds to one extreme of the cumulative active subspace energy staying constant as the number of active subspaces increases, while $\delta = 1$ corresponds to other extreme of the cumulative active subspace energy increasing linearly with the number of active subspaces.

We now turn our attention to the extreme of $\delta = 1$, in which case the subspace coherence conditions reduce to $\rho_i = O(n^{-1})$ and $\gamma_{2,i} = O(n^{-1/2} \log^{-1/2}(N/\alpha))$. We are interested in this setting in understanding whether there indeed exist subspaces and mixing bases that satisfy these conditions. We have the following theorem in this regard, which also sheds light on the maximum number of active subspaces that can be tolerated by the MSD algorithm.

**Theorem 3.** Suppose the number of active subspaces satisfies $n \leq \min \left\{ \sqrt{N} - 1, \frac{c_1^2 D(N-1)}{(Nd-D) \log(N/(\alpha))} \right\}$ for some constant $c_1 \in (0, 1)$. Then there exist collections of subspaces $X_N = \{S_i \in \Theta(d, D), i = 1, \ldots, N\}$ and corresponding mixing bases $B_N = \{\Phi_i : \text{span}(\Phi_i) = S_i, \Phi_i\Phi_i^T = I, i = 1, \ldots, N\}$ such that $\rho_i \leq n^{-1}$ and $\gamma_{2,i} \leq c_2 n^{-1/2} \log^{-1/2}(N/\alpha)$ for $i = 1, \ldots, N$, where $c_2 \geq \max\{2c_1, 1\}$ is a positive numerical constant.

**Proof:** The proof of this theorem follows from a combination of results reported in [25]. To begin, note from the definition of local 2-subspace coherence that $\frac{\rho_{i, j}}{\log}\leq \mu(X_N) := \max_{i \neq j} \gamma(S_i, S_j)$. We now argue there exist $X_N$’s such that $\mu(X_N) = 0.5c_2 n^{-1/2} \log^{-1/2}(N/\alpha)$, which in turn implies $\gamma_{2,i} \leq c_2 n^{-1/2} \log^{-1/2}(N/\alpha)$ for such collections of subspaces. The quantity $\mu(X_N)$, termed worst-case subspace coherence, has been investigated extensively in the literature [25], [31]. The first thing we need to be careful about is the fact from [31] Th. 3.6 [25] Th. 1 that $\mu(X_N) \geq \sqrt{\frac{Nd-D}{D(N-1)}}$, which is ensured by the conditions $n \leq \frac{c_1^2 D(N-1)}{(Nd-D) \log(N/\alpha)}$ and $c_2 \geq 2c_1$. The existence of such collections of subspaces now follows from [25], which establishes that the worst-case subspace coherences of many collections of subspaces (including subspaces drawn uniformly at random from $\Theta(d, D)$) come very close to meeting the lower bound $\sqrt{\frac{Nd-D}{D(N-1)}}$.

In order to complete the proof, we next need to establish that if a collection of subspaces has $\mu(X_N) = 0.5c_2 n^{-1/2} \log^{-1/2}(N/\alpha)$ then there exists at least one corresponding mixing bases for that collection such that $\rho_i \leq n^{-1}$. In this regard, note that $\rho_i \leq \nu(B_N) := \max_i \rho_i$. The quantity $\nu(B_N)$, termed average group/block coherence, was introduced in [11] and investigated further in [25]. In particular, it follows from [25] Lemma 7] that every collection of subspaces $X_N$ has at least one mixing bases with $\nu(B_N) \leq \frac{\sqrt{N}+1}{N}$, which can in turn be upper bounded by $n^{-1}$ for $n \leq \sqrt{N} - 1$.

Recall that our problem formulation calls for $n < D/d \ll N$. Theorem helps quantify these inequalities for the case of linear scaling of cumulative active subspace energy. Specifically, note that $\frac{D(N-1)}{(Nd-D) \log(N/(\alpha))} = O\left(\frac{D}{d \log(N/\alpha)}\right)$ for large $N$. We therefore have that Theorem allows the number of active subspaces to scale linearly with the extrinsic dimension $D$ modulo a logarithmic factor. Stated differently, Theorem establishes that the total number of active dimensions, $nd$, can be proportional to the extrinsic dimension $D$, while the total number of subspaces in the collection, $N$, affect the number of active dimensions only through a logarithmic factor. Combining Theorem with the earlier discussion, therefore, one can conclude that the MSD algorithm does not suffer from the “square-root bottleneck” of $nd = O(\sqrt{D})$ despite the fact that its performance is being characterized in terms of polynomial-time computable measures. Finally, we note that the constraint $n = O(\sqrt{N})$ in Theorem appears due to our use of [25].
Lemma 7], which not only guarantees existence of appropriate mixing bases but also provides a polynomial-time algorithm for obtaining those mixing bases. If one were interested in merely proving existence of mixing bases then this condition can be relaxed to \( n = O(N) \) by making use of [25] Th. 8 instead in the proof.

Since Theorem 3 guarantees existence of subspaces and mixing bases that satisfy the subspace coherence conditions for \( \delta = 1 \), it also guarantees the same for any other sublinear scaling \((0 \leq \delta < 1)\) of cumulative active subspace energy. Indeed, as \( \delta \searrow 0 \), the subspace coherence conditions (cf. 38 and 39) only become more relaxed. In fact, it turns out that the order-wise performance of the MSD algorithm no longer remains a function of the mixing bases for certain collections of subspaces when cumulative active subspace energy reaches the other extreme of \( \delta = 0 \). This assertion follows from the following theorem and the fact that \( \delta = 0 \) reduces the subspace coherence conditions to \( \rho_i = O(n^{-1/2}) \) and \( \gamma_{2,i} = O(\log^{-1/2}(N/\alpha)) \).

**Theorem 4.** Suppose the number of active subspaces satisfies \( n \leq \frac{c_3 D(N-1)}{Nd-D} \) for some constant \( c_3 \in (0,1) \) and the total number of subspaces in the collection \( \mathcal{X}_N \) satisfies \( N \leq \alpha \exp(n/4) \). In such cases, there exist collections of subspaces that satisfy \( \mu(\mathcal{X}_N) := \max_{i \neq j} \gamma(S_i, S_j) \leq n^{-1/2} \). Further, all such collections satisfy \( \rho_i \leq n^{-1/2} \) and \( \gamma_{2,i} \leq \log^{-1/2}(N/\alpha) \) for \( i = 1, \ldots, N \).

**Proof:** The proof of this theorem also mainly follows from [25], which establishes that there exist many collections of subspaces for which \( \mu(\mathcal{X}_N) = \sqrt{\frac{N d - D}{c_3 D (N - 1)}} \) for appropriate constants \( c_3 \in (0,1) \). Under the condition \( n \leq \frac{c_3 D (N-1)}{Nd-D} \), therefore, it follows that \( \mu(\mathcal{X}_N) \leq n^{-1/2} \). Since \( \gamma_{2,i} \leq 2 \mu(\mathcal{X}_N) \), in turn obtain \( \gamma_{2,i} \leq \log^{-1/2}(N/\alpha) \) under the condition \( N \leq \alpha \exp(n/4) \). Finally, we have from the definition of the average mixing coherence that \( \rho_i \leq \mu(\mathcal{X}_N) \), which in turn implies \( \rho_i \leq n^{-1/2} \) and this completes the proof of the theorem. \( \blacksquare \)

Once again, notice that Theorem 4 allows linear scaling of the number of active dimensions as a function of the extrinsic dimension. In words, Theorem 4 tells us that MSD performs well for unmixing of collections of subspaces that are approximately equi-isoclinic [31], defined as ones with same principal angles between any two subspaces, regardless of the underlying mixing bases as long as the cumulative active subspace energy does not scale with the number of active subspaces.

**V. Numerical Results**

In this section, we report results of numerical experiments that further shed light on the relationships between the local 2-subspace coherences, the average mixing coherences, and the MSD algorithm for the problem of subspace unmixing. The subspaces used in all these experiments are independently drawn at random from \( \mathfrak{G}(d, D) \) according to the natural uniform measure induced by the Haar measure on the Stiefel manifold \( \mathbb{S}(d, D) \), which is defined as \( \mathbb{S}(d, D) := \{ U \in \mathbb{R}^{D \times d} : U^T U = I \} \). Computationally, we accomplish this by resorting to the numerical algorithm proposed in [32] for random drawing of elements from \( \mathbb{S}(d, D) \) according to the Haar measure. In doing so, we not only generate subspaces \( \mathcal{X}_N = \{ S_i \}_{i=1}^N \) from \( \mathfrak{G}(d, D) \), but we also generate the associated mixing bases \( \mathcal{B}_N = \{ \Phi_i \}_{i=1}^N \) from \( \mathbb{S}(d, D) \). Mathematically, given a subspace \( S_i \in \mathfrak{G}(d, D) \) and its equivalence class in the Stiefel manifold \( [S_i] \subset \mathbb{S}(d, D) \), its associated mixing basis \( \Phi_i \in \mathbb{S}(d, D) \) is effectively drawn at random from \([S_i]\)
Fig. 1. Plots of local 2-subspace coherences and average mixing coherences for different values of $d$, $D$, and $N$. (a) and (b) correspond to local 2-subspace coherences for $N = 1500$ and $N = 2000$, respectively. (c) and (d) correspond to average mixing coherences for $N = 1500$ and $N = 2000$, respectively. The error bars in the plots depict the range of the two coherences for the different subspaces.

according to the Haar measure on $[S_i]$. It is important to note here that once we generate the $S_i$’s and the $\Phi_i$’s, they remain fixed throughout our experiments. In other words, our results are not averaged over different realizations of the subspaces and the mixing bases; rather, they correspond to a fixed set of subspaces and mixing bases.

Our first set of experiments evaluates the local 2-subspace coherences of the $S_i$’s and the average mixing coherences of the corresponding $\Phi_i$’s for different values of $d$, $D$, and $N$. The results of these experiments are reported in Figs. 1 and 2. Specifically, Fig. 1(a) and Fig. 1(b) plot $\sum_{i=1}^{N} \gamma_{2,i}/N$ as well as the range of the $\gamma_{2,i}$’s using error bars for $N = 1500$ and $N = 2000$, respectively. Similarly, Fig. 1(c) and Fig. 1(d) plot $\sum_{i=1}^{N} \rho_i/N$ as well as the range of the $\rho_i$’s using error bars for $N = 1500$ and $N = 2000$, respectively. It can be seen from these figures that both the local 2-subspace coherence and the average mixing coherence decrease with an increase...
in $D$, while they increase with an increase in $d$. In addition, it appears from these figures that the $\gamma_{2,i}$'s and the $\rho_i$'s start concentrating around their average values for large values of $D$. Careful examination of Fig. 1, however, suggests a contrasting behavior of the two coherences for increasing $N$. While increasing $N$ from 1500 to 2000 seems to increase the $\gamma_{2,i}$'s slightly, this increase seems to have an opposite effect on the $\rho_i$'s. We attribute this behavior of the average mixing coherence to the “random walk nature” of its definition, although a comprehensive understanding of this phenomenon is beyond the scope of this paper. One of the most important things to notice from Fig. 1 is that the average mixing coherences tend to be about two orders of magnitude smaller than the local 2-subspace coherences, which is indeed desired according to the discussion in Sec. IV. Finally, since the error bars in Fig. 1 do not necessarily give an insight into the distribution of the $\gamma_{2,i}$'s and the $\rho_i$'s, we also plot histograms of the two coherences in Fig. 2 for $N = 2000$ corresponding to $D = 600$ (Figs. 2a and 2c) and $D = 1400$.
Our second set of experiments evaluates the performance of the MSD algorithm for subspace unmixing. We run these experiments for fixed subspaces and mixing bases for the following four sets of choices for the \((d, D, N)\) parameters: \((3, 600, 2000)\), \((3, 1400, 2000)\), \((15, 600, 2000)\), and \((15, 1400, 2000)\). The results reported for these experiments are averaged over 5000 realizations of subspace activity patterns, mixing coefficients, and additive Gaussian noise. In all these experiments, we use \(\sigma = 0.01\) and \(E_A = n\), divided equally among all active subspaces, which means that all active subspaces lie above the additive noise floor. In terms of the selection of thresholds for Algorithm\(^1\) we rely on Theorem\(^1\) with a small caveat. Since our analysis uses a number of bounds, it invariably results in conservative thresholds. In order to remedy this, we use the thresholds \(\tilde{\tau}_k := c_1^2 \tau_k\) with \(\tau_k\) as in Theorem\(^1\) but using \(c_0 = 1\) and \(c_1 \in (0, 1)\). We learn this new parameter \(c_1\) using cross validation and set \(c_1 = 0.136\) and \(c_1 = 0.107\) for \(d = 3\) and \(d = 15\), respectively. Finally, we set the final thresholds to control the FWER in all these experiments at level \(\alpha = 0.1\).

The results of these experiments for our choices of the parameters are reported in Fig.\(^3\) (a) and Fig.\(^3\) (b) for \(d = 3\) and \(d = 15\), respectively. We not only plot the FWER and the NDP in these figures, but we also plot another metric of false-discovery proportion (FDP), defined as \(\text{FDP} := \frac{|\hat{A} \setminus A|}{|A|}\), as a measure of the FDR. Indeed, the expectation of the FDP is the FDR \([21]\). It is instructive to compare the FWER plots for \(D = 600\) and \(D = 1400\) in these figures. We can see from Fig.\(^2\) that the \(\gamma_{2,i}\)'s and the \(p_i\)'s are smaller for \(D = 1400\), which means that the thresholds \(\tilde{\tau}_k\)'s are also smaller for \(D = 1400\) (cf. Theorem\(^1\)). But Fig.\(^3\) shows that the FWER for \(D = 1400\) mostly remains below \(D = 600\), which suggests that Theorem\(^1\) is indeed capturing the correct relationship between the FWER of MSD and the properties of the underlying subspaces. In addition, the NDP plots in these figures for \(D = 600\) and \(D = 1400\) also help validate Theorem\(^2\). Specifically, Theorem\(^2\) suggests that the NDP of MSD should remain...
small for larger values of $n$ as long as the $\gamma_{2,i}$’s and the $\rho_i$’s remain small. Stated differently, since the $\gamma_{2,i}$’s and the $\rho_i$’s are smaller for $D = 1400$ than for $D = 600$ (cf. Fig 2), Theorem 2 translates into a smaller NDP figure for larger values of $n$ for $D = 1400$. It can be seen from the NDP plots in Fig. 3 that this is indeed the case.

VI. CONCLUSION

In this paper, we motivated and posed the problem of subspace unmixing as well as discussed its connections with problems in wireless communications, hyperspectral imaging, high-dimensional statistics and compressed sensing. We proposed and analyzed a low-complexity algorithm, termed \textit{marginal subspace detection} (MSD), that solves the subspace unmixing problem under the subspace sum model by turning it into a multiple hypothesis testing problem. We showed that the MSD algorithm can be used to control the family-wise error rate at any level $\alpha \in [0, 1]$ for an arbitrary collection of subspaces on the Grassmann manifold. We also established that the MSD algorithm allows for linear scaling of the number of active subspaces as a function of the ambient dimension. Numerical results presented in the paper further validated the usefulness of the MSD algorithm and the accompanying analysis. Future work in this direction includes design and analysis of algorithms that perform better than the MSD algorithm as well as study of the subspace unmixing problem under mixing models other than the subspace sum model.

APPENDIX A

BANACH-SPACE-VALUED AZUMA’S INEQUALITY

In this appendix, we state a Banach-space-valued concentration inequality from [33] that is central to some of the proofs in this paper.

\textbf{Proposition 1} (Banach-Space-Valued Azuma’s Inequality). Fix $s > 0$ and assume that a Banach space $(B, \| \cdot \|)$ satisfies

$$\zeta_B(\tau) := \sup_{u, v \in B, \|u\| = \|v\| = 1} \left\{ \frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 \right\} \leq s \tau^2$$

for all $\tau > 0$. Let $\{M_k\}_{k=0}^\infty$ be a $B$-valued martingale satisfying the pointwise bound $\|M_k - M_{k-1}\| \leq b_k$ for all $k \in \mathbb{N}$, where $\{b_k\}_{k=1}^\infty$ is a sequence of positive numbers. Then for every $\delta > 0$ and $k \in \mathbb{N}$, we have

$$\Pr(\|M_k - M_0\| \geq \delta) \leq \exp\left( - \frac{c_0 \delta^2}{\sum_{\ell=1}^k b_\ell^2} \right),$$

where $c_0 := \frac{e^{-1}}{256}$ is an absolute constant.

\textbf{Remark 2}. Theorem 1.5 in [33] does not explicitly specify $c_0$ and also states the constant in front of $\exp(\cdot)$ to be $e^{s+2}$. Proposition 1 stated in its current form, however, can be obtained from the proof of Theorem 1.5 in [33].

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