THE DECAY CONSTANTS
OF PSEUDOSCALAR MESONS
IN A RELATIVISTIC QUARK MODEL

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Abstract

The decay constants of pseudoscalar mesons are calculated in a relativistic quark model which assumes that mesons are made of a valence quark antiquark pair and of an effective vacuum like component. The results are given in terms of quark masses and of some free parameters entering the expression of the internal wave functions of the mesons. By using the pion and kaon decay constants $F_{\pi^+} = 130.7$ MeV, $F_{K^+} = 159.8$ MeV to fix the parameters of the model one gets $60$ MeV $\leq F_{D^+} \leq 185$ MeV, $95$ MeV $\leq F_{D_s} \leq 230$ MeV, $80$ MeV $\leq F_{B^+} \leq 205$ MeV for the light quark masses $m_u = 5.1$ MeV, $m_d = 9.3$ MeV, $m_s = 175$ MeV and the heavy quark masses in the range: 1. GeV $\leq m_c \leq 1.6$ GeV, 4.1 GeV $\leq m_b \leq 4.5$ GeV. In the case of light neutral mesons one obtains with the same set of parameters $F_{\pi^0} \approx 138$ MeV, $F_\eta \approx 130$ MeV, $F_{\eta'} \approx 78$ MeV. The values are in agreement with the experimental data and other theoretical results.

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1 Introduction

The decay constants of pseudoscalar mesons have been treated by current algebra and PCAC like simple scale parameters relating the meson fields with the corresponding axial currents. In quark models they are expressed by means of the quark-antiquark annihilation amplitude [1], but, although simple in principle, the calculation of the decay constants and, in general, of the electro-weak form factors, is a difficult task due to the binding effects which escape a relativistic treatment.

A solution is to bypass the binding problem and work with free quarks. This is the way followed by QCD sum rules [2], which rely on the assumption of quark-hadron duality and relate the hadronic matrix elements with some quark and gluon transition amplitudes which can be evaluated within the perturbative QCD scheme. This is a fruitful method which produced most of the recent theoretical results.

Another solution is to find a suitable description of the quark annihilation in the bound states. The current assumption here is that the decay constants are proportional with the internal wave function of the meson at zero distance between the quarks [1]. Using various forms for the binding potential [3] one gets that $F$ behaves like $\psi(0) M^{-1/2}$, where $M$ is the meson mass. Potential models work in the case of heavy mesons only, but they are not Lorentz covariant and are improper in many cases involving heavy mesons, like, for instance, in heavy to light semileptonic decays, where the movement of the recoiling meson cannot be neglected. For this case and many others where the hadron movement is important one needs a relativistic treatment of the binding. Unfortunately, the best relativistic theory we have at hand, the field theory in the perturbative approach, is unable to give an easy answer to the binding problem. In our opinion its failure in describing the bound states is due to the lack of a relativistic equivalent of the binding energy. We thus suggest to renounce representing the binding by a series of some quantum exchanges, since binding is not a perturbative effect and look instead for a relativistic generalization of the binding energy which can be included in a suitable relativistic wave function for the free particles forming the composite system. In this way we hope to combine the valuable features of the potential models, which are suitable for describing the binding, but are improper for introducing the boost of the bound state, with those of the relativistic models which can boost the free states, but cannot describe the binding.

The model we propose has been recently applied to the weak radiative decays of pseudoscalar mesons [4] and to the decay $Z^0 \rightarrow \pi^0 \gamma$ [5]. In this paper we intend to
exploit further the properties of the model and to perform some predictions on the values of the decay constants of heavy mesons.

The specific assumption of the model is that mesons are made of a valence $q\bar{q}$ pair bound together by some collective oscillation modes of the quark gluonic field. The last ones are described by an effective vacuum-like component $\Phi$ carrying its own 4-momentum which is not subject of a mass-shell constraint since $\Phi$ is far from being elementary.

As it will be further shown, it is the presence of the field $\Phi$ which allows to keep the mesons and quarks on their mass shell while ensuring the energy-momentum conservation together with a continuous distribution of the quark relative momenta inside the mesons. One solves in this way the problem mentioned by Isgur, Scora, Grinstein and Wise [6] who noticed that a "mock meson" made of almost free quarks with a continuous distribution of the relative momentum has a false mass width reflecting the fact that the sum of the free quark momenta does not belong to a certain representation of the Lorentz group.

Turning now to the potential models for the meson as a bound state, we recall that the distribution of the relative momentum is given by the Fourier transform of the internal wave function. The existence of a continuous distribution ensuring the $L^2$ integrability of the wave function appears to be a direct consequence of the binding potential.

In our model it is the effective field $\Phi$ which adjusts the continuous distribution of the relative momentum to the free particle behaviour required by relativistic covariance. This argument gives a substantial support to the assumption that $\Phi(Q)$ represents the excitations responsible for the confinement and allows us to consider $Q_\mu$ the relativistic generalization of the potential energy. In fact, $Q_0$ only is the analogue of the potential energy of the $qq$ system at rest. The spatial components $Q$ introduced for relativistic consistency could be rather related with the fluctuations of the confining oscillation modes of the quark gluonic field, not with the intensity of the binding forces.

An important ingredient of the model is the internal function of the compound system representing the hadron. In the lack of a dynamical equation for it, we shall use some trial functions allowing to ensure the integrability of the matrix elements of interest.

Finally we wish to stress once again that a real relativistic treatment of a system made of independent constituents requires the use of the momentum space. If the states were defined in the configuration space, it would be necessary to introduce an independent time coordinate for each component, which is nonsense.

In the next section we discuss shortly the dynamical assumptions of the model and give the expressions of the decay constants as functions of the quark and meson masses.

The numerical results obtained with exponential and gaussian internal functions are given in the third section. The fit of the pion and kaon decay constants with the experimental values is used to fix the parameters of the model.

We analyse the results in the fourth section and draw some general conclusions concerning the reliability of the model.
2 Calculation of the decay constants

The fundamental dynamical assumption of the model is that the interaction inside the quark system representing a hadron can be treated independently from the external interaction. The first one is a mean field effect and is taken into account by means of the internal wave function, while the external interaction is the effect of some specific quantum fluctuations. We recall that this is also the main assumption underlying the Furry representation in field theory [7].

The form we proposed for the meson state is [4]:

\[
|M_i(P)\rangle = \frac{i}{(2\pi)^3} \int \frac{d^3p}{e/m} \frac{d^3q}{e/m'} d^4Q \varphi(p, q; Q) \bar{u}(p) \Gamma_M \psi \delta(4)(p + q + Q - P) \\
\times \Phi^\dagger(Q) a^\dagger(p) b^\dagger(q) |0\rangle
\]

where \(a^\dagger, b^\dagger\) are the creation operators of the valence \(q\bar{q}\) pair; \(u, v\) are Dirac spinors and \(\Gamma_M\) is a Dirac matrix ensuring the relativistic coupling of the quark spins. The quarks are supposed to be free; their creation and annihilation operators satisfy canonical commutation relations and commute with \(\Phi^\dagger(Q)\), which describes the creation of a nonelementary excitation carrying the momentum \(Q_\mu\). The mass spectrum of the nonelementary excitations denoted by \(\Phi^\dagger\) and the internal distribution of momenta are described by the Lorentz invariant function \(\varphi(p, q; Q)\). A natural assumption is that \(\varphi\) is a time independent, equilibrium distribution since the hadrons are long living. This means that as long as a quark system like that described by (1) is the single one in the external state and as long as it does not emit and absorb any electroweak quanta, the distribution of momenta is given by \(\varphi(p, q; Q)\) and does not change. A straightforward consequence is that any time translation operator \(U_s(t, t')\) describing the evolution of a quark system under the action of strong forces only can be replaced by unity when acting on a state like (1). This fact will allow us to perform some simplifications in the calculation of the matrix elements of interest.

For a better understanding of the present model, a comparison of the expression (1) with the "mock meson" in Ref.[6] is most useful. A first remark is that the continuous distribution of the relative momentum in a meson made of a quark and of an antiquark only, introduced by hand more than 20 years ago [6,8], follows naturally in our model from the existence of a third component, the field \(\Phi\), which contributes to the meson momentum. A second remark is that, unlike the "mock meson", the expression (1) can be safely boosted due to the function \(\delta^4(p + q + Q - P)\) which guarantees that the sum of the internal momenta belongs to the representation of the Lorentz group having the meson mass as invariant.

As concerns the concrete form of the internal function \(\varphi(p, q; Q)\) some more comments are necessary. It must be said that we have no \textit{a priori} arguments for a particular form.
We expect, however, for $\varphi$ to be such as to ensure the convergence of the integrals over the internal momenta in the expressions of the physical amplitudes.

Related to this fact, we remark that if the fourth component of the momentum carried by the effective field $\Phi$ was positive in the rest frame of the meson, the function $\delta^{(3)}(p + q + Q - M)$ would provide some natural upper bound for the quark energies. The integrals in the expression (1) of a meson state would then extend over a finite range and their convergence would be easy to ensure.

This is however not the case since, as shown by the electromagnetic form factors, there is no upper bound for the quark energy in a hadron. One must then allow for negative values of $Q_0$ and introduce some definite cut-off functions to ensure the convergence of the integrals.

The trial functions we shall use in the next as internal functions cut-off the large values of $Q_0$ and $Q$ only, but, due to the presence of the function $\delta^4(p + q + Q - P)$ we expect for them to provide the necessary cut-off for the quark momenta too. A meson at rest is supposed to have an internal function of the following kind:

$$
\varphi(p, q; Q) = D_M \sigma(Q_0, Q)
$$

$\sigma(Q_0, Q) = \exp \left[ \frac{Q_0}{\alpha} - \frac{|Q|}{\beta} \right] \theta(Q^2)\theta(-Q_0)$ \hspace{1cm} (2a)

$\sigma(Q_0, Q) = \exp \left[ \frac{Q_0}{\alpha} - \frac{Q^2}{\beta^2} \right] \theta(Q^2)\theta(-Q_0)$ \hspace{1cm} (2b)

$\sigma(Q_0, Q) = \exp \left[ -\frac{Q_0^2}{\alpha^2} - \frac{Q^2}{\beta^2} \right] \theta(Q^2)\theta(-Q_0)$ \hspace{1cm} (2c)

where $M$ is the meson mass, $\alpha, \beta$ are the free parameters of the model ensuring the desired convergence of the integrals. Lorentz covariance of the internal function becomes obvious if one writes $Q_0$, and $|Q|$ as: $Q_0 = (P \cdot Q)/M, |Q| = \sqrt{(P \cdot Q)^2/M^2 - Q^2}$, where $P$ is the meson momentum. The functions $\theta$ in eqs. (2a), (2b), (2c) express the fact that $Q$ is time like and $Q_0$ negative, in agreement with our assumption that $Q_\mu$ is the relativistic generalization of the potential energy in the bound system.

Before proceeding to the evaluation of the decay constants, we have to do an explicit statement on the vacuum expectation value of the effective field. As mentioned above, the momentum carried by $\Phi^\dagger$ is not subject of a mass-shell constraint, since it represents the creation of a collective excitation, not of an elementary one. Accordingly, we assume that:

$$
\Phi(Q_1) \Phi(Q_2) \cdots \Phi^+(Q_n) = \Phi(Q_1 + Q_2 \cdots - Q_n)
$$

and that the vacuum expectation value of the effective field $\Phi(Q)$ is:

$$
\langle 0 | \Phi(Q) | 0 \rangle = \mu^4 \int d^4 X \exp(-iQ \cdot X) = (2\pi)^4 \mu^4 \delta^{(4)}(Q).
$$

(3)
We emphasize that the appearance of the function $\delta^{(4)}$ in (3) is essential for ensuring the overall energy-momentum conservation and for preserving the Lorentz covariance of the model.

The constant $\mu^4$ in eq. (3), introduced for dimensional reasons, is related with the volume of a large 4-dimensional box of interest for our problem by $\mu^4 = (VT)^{-1}$. A short comment on the box size will be given in the last section.

Then, using the relations (1) and (3) we get for the norm of a single meson state the following expression:

$$
\langle M_i(P') | M_j(P) \rangle = (2\pi)^3 \delta_{ij} \delta^{(3)}(P - P') \delta(E - E') (2\pi)^4 \mu^4
$$

$$
\times \int \frac{d^3 p}{e/m} \frac{d^3 q}{e/m'} \frac{d^4 Q}{\delta^{(4)}(p + q + Q - P)} \varphi(p, q; Q)^2 \text{Tr} \left( \frac{\hat{p} + m}{2m} \Gamma_M \frac{\hat{q} - m'}{2m'} \Gamma_{M'} \right).
$$

The function $\delta^{(3)}(P - P') \delta(E - E')$ in eq. (4) originating from $\delta^{(4)}(p + q + Q - P)$ in the definition of a single meson state can also be written as $\frac{2}{M} \delta^{(3)}(P - P') \delta(M - M')$. It reflects the existence of a continuous mass spectrum for the complex system representing the meson and cannot be modified without renouncing the real Lorentz invariance of the model. Its appearance in the expression of the norm forces us to treat the physical meson as a mixed state, defined with the aid of the diagonal density matrix $\rho(M)$ which satisfies the normalization condition

$$
\int \rho(M) \, dM = M_0.
$$

where $M_0$ is the central value of the mass distribution. Consequently, the density of states in the phase space must be modified by replacing $\frac{1}{(2\pi)^3} \frac{d^4 P}{2E}$ with $\frac{1}{(2\pi)^3} \frac{d^4 P}{2E} \rho(M, M_0) \frac{dM}{M_0}$. If the dependence of the matrix elements on the meson mass $M$ is rather smooth and the width of the mass distribution function is small, one can replace $M$ by $M_0$ in the expressions of the matrix elements and perform the integral over meson masses in the new expression of the density of states by using the normalization condition (5). The calculation can then proceed like in the old case.

The matrix element of interest for the leptonic decay of a meson, written in the lowest order of perturbation with respect to the weak interaction is:

$$
\langle 0 | U_s(+\infty, 0) A_\mu(0) U_s(0, -\infty | M(P) \rangle = i \frac{F_M}{P_\mu},
$$

where the operator $U_s(t, t')$ describes the evolution of a system under the action of strong interaction among the constituents, and $A_\mu$ is the free-field weak current of interest in the process. It is important to notice that in soft processes, like, for instance the present one, the perturbative expansion of $U_s$ is improper. For this reason we shall not consider the virtual states generated by the evolution operator in the perturbative approach, but merely look at the real modifications which could appear in the distribution of flavours and
momenta during the time translation. In the above case no such changes could appear, since the real vacuum and the single meson state are stable states whose content does not change under the action of strong interaction and consequently both time translation operators in eq. (6) can be replaced by unity.

By using the relation (3), the canonical anticommutation relations of the fermionic operators and integrating over the internal momenta, we obtain from the matrix element (6) the following expression for the decay constants:

\[ F_M = (2\pi)^4 \mu^4 D_M \frac{2\pi \sqrt{3} p (m + m')}{M} \left[ 1 - \frac{(m - m')^2}{M^2} \right] \]  

where \( p = \frac{M}{2} \sqrt{\left[ 1 - \frac{(m + m')^2}{M^2} \right] \left[ 1 - \frac{(m - m')^2}{M^2} \right]} \) and the factor \( \sqrt{3} \) comes from the colours.

It is worthwhile noticing that the leptonic decay constant in eq. (7) is proportional with the internal wave function at \( Q_\mu = 0 \), which means the absence of any other excitations beside the valence quarks. Expressing this result in more general terms, one may say that the leptonic decay constants are proportional with the value of the internal function at vanishing contribution from the binding effects. This is in remarkable agreement with the old assumption that \( F_P \) is proportional with the internal wave function at vanishing distance between the quarks [1], since, according to the asymptotic freedom, this is the point in the configuration space where the confining forces vanish. It is a strong argument for considering the present model as a real relativistic generalization of the potential models.

By integrating now over the quark momenta in eq.(4) and introducing the expression of the normalization constant \( D_M \) as given by (7), we get the expression of the decay constants in terms of the model parameters \( \alpha, \beta, \mu \) and of the quark masses:

\[ F_M = (2\pi)^2 \mu^2 (12\pi^3)^{1/2} M (m + m') \left( 1 - \frac{(m + m')^2}{M^2} \right)^{1/2} \left( 1 - \frac{(m - m')^2}{M^2} \right)^{3/2} \]

\[ \times \int_{Q^2 \leq Q_0^2} dQ_0 Q^2 d|Q| \sigma^2(Q_0, Q) \left[ \frac{(M - Q_0)^2 - Q^2 - (m - m')^2}{M^2 [(M - Q_0)^2 - Q^2]} \right] \]

\[ \times \sqrt{[(M - Q_0)^2 - Q^2]^2 - 2[(M - Q_0)^2 - Q^2](m^2 + m'^2) + (m^2 - m'^2)^2}}^{-1/2}. \]  

Similar expressions can be written for the decay constants of neutral mesons. Defining them like in Ref. [8, p.1444], one has:

\[ F_{M^0} = (2\pi)^2 \mu^2 (24\pi^3)^{1/2} \sum_{i=u,d,s} \kappa_i^2 m_i \left( 1 - \frac{4m_i^2}{M^2} \right)^{1/2} \]
\[
\times \left\{ \sum_{i=u,d,s} \kappa_i^2 \int dQ_0 \, |Q| d|Q| \, \sigma^2(Q_0, Q) \right. \\
\left. \times \sqrt{\left[ \left(1 + \frac{Q_0^2}{M^2} \right)^2 - \frac{Q^2}{M^2} \right] \left[ \left(1 + \frac{Q_0^2}{M^2} \right)^2 - \frac{Q^2}{M^2} - \frac{4m_i^2}{M^2} \right]} \right\}^{-1/2}
\]

where \( m_i \) are the quark masses, \( \kappa_i = a(\lambda_3)_{ii} + b(\lambda_8)_{ii} + c(\lambda_0)_{ii} \) with \( \lambda_j \) the Gell-Mann matrices [8, p.1288], \( a = 1; \ b = c = 0 \) for \( \pi^0 \), \( a = 0; \ b = \cos\theta_P; \ c = -\sin\theta_P \) for \( \eta \), \( a = 0; \ b = \sin\theta_P; \ c = \cos\theta_P \) for \( \eta' \) and \( \theta_P = -10^0 \) or \( \theta_P = -23^0 \) [8, p.1320].

### 3 Numerical results

Before proceeding to the numerical calculations we have to analyse the relation of the model parameters \( \alpha, \beta, \mu \), with the general features of the bound \( q\bar{q} \) system.

First of all we remind that \( Q_0 \) is the analogue of the potential energy in the nonrelativistic models. In the present approach it is the energy of the oscillation modes of the quark-gluonic field confining the valence quarks inside the meson. Its cut-off parameter, \( \alpha \), must be chosen in such a way as to ensure a relative stability of \( F_P \) with the increase of \( M_P \). (See eq. (8)). Our tests with \( \alpha = \rho \sqrt{m \, m'/M/(m+m')} \), \( \alpha = \rho \sqrt{m \, m'/M} \), \( \alpha = \rho \, (m + m') \) and \( \alpha = \rho \, M \) where \( M \) is the meson mass and \( \rho \) a universal parameter, proved that the last choice is the best. All the others either lead to very small values for the decay constants of the heavy mesons or do not allow to fit pion and kaon decay constants with the same set of parameters, as required from the beginning.

The same stability argument forces us to introduce an additional cut-off for \( |Q| \), the momentum carried by the effective component, since the simple requirement for \( Q^2 \) to be positive would lead to a too strong increase of \( F_P \) with the meson mass. We recall that \( |Q| \) has been introduced for relativistic consistency; we did not relate it with the potential energy, but rather with some fluctuations in the momentum carried by the collective excitations denoted by \( \Phi \). The parameter \( \beta \) in eqs.(2a,b,c) is hence a measure of the fluctuation amplitude and we shall assume that it does not depend on the quark or meson masses because the vacuum-like excitations are not sensitive to the flavours. However, we expect for \( \beta \) to be smaller than the cut-off parameter of \( Q_0 \) in the case of heavy mesons, because the fluctuation effect must be negligible in their cases.

The parameter \( (2\pi)^4 \mu^4 \) is assumed to be an universal constant, related with the volume of the 4-dimensional box relevant for the process. Its independence on the masses will be used to fix the parameters \( \alpha \) and \( \beta \) of the model. Our procedure is to introduce the well known values of the pion and kaon decay constants \( F_\pi = 130.7 \, MeV, \ F_K = 159.8 \, MeV \) [9] in the equation (8) and search for the values of the parameters \( \alpha \) and \( \beta \) yielding the same
value for $\mu$. The calculations have been done using the values of the light quark masses $m_u = 5.1$ MeV, $m_d = 9.3$ MeV, $m_s = 175$ MeV resulting from the chiral perturbation theory [10] and the heavy quark masses in the range quoted by Particle Data [9]. The results obtained using the trial functions (2a), (2b), (2c) are listed in the following tables. The quark masses and the decay constants are given in MeV.

TABLE I. Decay constants of heavy mesons. The indices (a), (b), (c) correspond to the trial functions (2a), (2b), (2c).

| $\alpha_{(i)}$, $\beta_{(i)}$ | $D(c\bar{d})(1869)$ | $D_s(c\bar{s})(1969)$ | $B(u\bar{b})(5279)$ |
|-----------------------------|---------------------|-----------------------|----------------------|
| $(2\pi)^4 \mu_{(i)}^4$     | $m_c$ | $F_D$ | $m_c$ | $F_{D_s}$ | $m_b$ | $F_B$ |
| $\alpha_{(a)}$ = 0.075 $M$ | 1000. | 126. | 1000. | 152. | 4100. | 107. |
| $\beta_{(a)}$ = 0.096 MeV | 1300. | 109. | 1300. | 141. | 4300. | 92.  |
| $(2\pi)^4 \mu_{(a)}^4$   | 1600. | 56.  | 1600. | 91.  | 4500. | 64.  |
| $\alpha_{(a)}$ = 0.05 $M$ | 1000. | 125. | 1000. | 151. | 4100. | 111. |
| $\beta_{(a)}$ = 0.065 MeV | 1300. | 111. | 1300. | 142. | 4300. | 95.  |
| $(2\pi)^4 \mu_{(a)}^4$   | 1600. | 60.  | 1600. | 95.  | 4500. | 77.  |
| $\alpha_{(a)}$ = 0.025 $M$ | 1000. | 121. | 1000. | 144. | 4100. | 98.  |
| $\beta_{(a)}$ = 0.032 MeV | 1300. | 111. | 1300. | 139. | 4300. | 87.  |
| $(2\pi)^4 \mu_{(a)}^4$   | 1600. | 65.  | 1600. | 98.  | 4500. | 72.  |
| $\alpha_{(b)}$ = 0.075 $M$ | 1000. | 155. | 1000. | 188. | 4100. | 148. |
| $\beta_{(b)}$ = 0.082 MeV | 1300. | 137. | 1300. | 177. | 4300. | 126. |
| $(2\pi)^4 \mu_{(a)}^4$   | 1600. | 73.  | 1600. | 118. | 4500. | 102. |
| $\alpha_{(b)}$ = 0.05 $M$ | 1000. | 156. | 1000. | 190. | 4100. | 153. |
| $\beta_{(b)}$ = 0.054 MeV | 1300. | 140. | 1300. | 180. | 4300. | 132. |
| $(2\pi)^4 \mu_{(b)}^4$   | 1600. | 79.  | 1600. | 124. | 4500. | 108. |
| $\alpha_{(b)}$ = 0.025 $M$ | 1000. | 155. | 1000. | 188. | 4100. | 156. |
| $\beta_{(b)}$ = 0.027 MeV | 1300. | 143. | 1300. | 181. | 4300. | 136. |
| $(2\pi)^4 \mu_{(b)}^4$   | 1600. | 85.  | 1600. | 129. | 4500. | 114. |
| $\alpha_{(c)}$ = 0.075 $M$ | 1000. | 185. | 1000. | 227. | 4100. | 203. |
| $\beta_{(c)}$ = 0.04 MeV  | 1300. | 168. | 1300. | 216. | 4300. | 177. |
| $(2\pi)^4 \mu_{(c)}^4$   | 1600. | 95.  | 1600. | 149. | 4500. | 144. |
| $\alpha_{(c)}$ = 0.05 $M$ | 1000. | 183. | 1000. | 223. | 4100. | 204. |
| $\beta_{(c)}$ = 0.027 MeV | 1300. | 168. | 1300. | 214. | 4300. | 178. |
| $(2\pi)^4 \mu_{(c)}^4$   | 1600. | 99.  | 1600. | 152. | 4500. | 148. |
| $\alpha_{(c)}$ = 0.025 $M$ | 1000. | 176. | 1000. | 213. | 4100. | 198. |
| $\beta_{(c)}$ = 0.014 MeV | 1300. | 163. | 1300. | 207. | 4300. | 175. |
| $(2\pi)^4 \mu_{(c)}^4$   | 1600. | 99.  | 1600. | 151. | 4500. | 147. |
Using the equation (9) and the same sets of parameters as above we calculated also the decay constants of the lightest pseudoscalar mesons. The results are quoted in the next table.

### TABLE II. The decay constants of the lightest neutral mesons.

| The trial function | $\pi^0(135)$ | $\eta(547)$ | $\eta(547)$ | $\eta'(958)$ | $\eta'(958)$ |
|--------------------|--------------|-------------|-------------|--------------|--------------|
| $\theta_P = -10^\circ$ | 137.-139.   | 128.-139.   | 77.-78.     | 67.-69.      | 94.-97.      |
| $\theta_P = -23^\circ$ | 131.        | 131.        | 77.-78.     | 75.-76.      | 105.-107.    |
| $\lambda = 10^\circ$ | 129.-132.   | 76.-78.     | 77.-81.     | 108.-114.    |
| $\lambda = -23^\circ$ | 129.-132.   | 76.-78.     | 77.-81.     | 108.-114.    |

### 4 Comments and conclusions

Analysing the numerical results in Table I, one notices that, for each of the tested internal function, the decay constants do not change significantly when passing from one set of parameters $\alpha$, $\beta$, $\mu$ to another set which fits the values of $F_\pi$ and $F_K$. Indeed, for a change with 200% of $\alpha$ and $\beta$, $(4\pi)^4\mu^4$ changes with two orders of magnitude, while the theoretical values of the decay constants change with less than 10%.

The decay constants are more sensitive at the variation of the heavy quark masses and could be used in principle for a more precise determination of the last ones. The comparison with the experimental values $F_D \leq 300$, $F_{D_s} = 232 \pm 45 \pm 20 \pm 48$ MeV, or $F_{D_s} = 344 \pm 37 \pm 52 \pm 42$ MeV [9, p.1443] and with the values yielded by QCD sum rules $F_D \approx (1.35 \pm 0.04 \pm 0.06)F_\pi$, $F_{D_s} \approx (1.55 \pm 0.10)F_\pi$, $F_B \approx (1.49 \pm 0.06 \pm 0.05)F_\pi$ [11] $F_B = 185 \pm 40$ MeV [12] shows that the agreement is better at the lowest values of the heavy quark masses. Things look mainly the same for any of the trial functions, but the best fit of the data seems to be done with the internal function (2c). Of course, this is just a qualitative estimate. A more reliable test could be provided by the fit of the weak or electromagnetic form factors, which are very sensitive at the form of the internal function.

In the case of neutral mesons, we found $(F_\eta)_{th}$ in the range $128$ MeV $- 139$ MeV for $\theta_P = -10^\circ$, which is in agreement with $F_\eta = 133 \pm 10$ MeV quoted in Ref.[9], but $(F_{\pi^0})_{th} \approx 138$ MeV, slightly larger than $F_{\pi^0} = 119 \pm 4$ MeV in Ref.[9]. One sees also that the calculated values of $F_{\eta'}$, both for $\theta_P = 10^\circ$ and $\theta_P = -23^\circ$ are smaller than $F_{\eta'} = 126 \pm 7$ MeV, quoted in Ref.[9, p.1444] The differences noticed above must not be taken too seriously because of the large uncertainties entering the values quoted in Ref.[9]. They come from the extrapolation on the meson mass shell when deriving $F_{P^0}$ with the aid of the axial anomaly but, for $\eta$, $\eta'$ they come also from the uncertainties in the mixing angle.
Resuming, one may say that the present model yields reasonable values for the decay constants of light and heavy mesons using the same set of parameters, which is quite remarkable.

A last comment concerns the parameter \( \mu \). As it can be seen from Table I, its values resulting from the fit of the pion and kaon decay constants are in the range 0.2 – 0.7 MeV. Recalling that \( \mu^{-4} \) is equal to the 4-dimensional volume \( VT \) of a very large box containing the meson, we get a box size of about 300 – 1000 fm, quite large in comparison with the meson size which is less than 1 fm. The large value found for the box size, as well as the relative independence of the results in Tables I and II on the value of \( \mu \), if \( \mu \) is sufficiently small, are strong arguments for the consistency of the present model.

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