Intrinsic Equations For a Relaxed Elastic Line of Second Kind on an Oriented Surface

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Abstract

Let \( \alpha (s) \) be an arc on a connected oriented surface \( S \) in \( E^3 \), parameterized by arc length \( s \), with torsion \( \tau \) and length \( l \). The total square torsion \( F \) of \( \alpha \) is defined by \( F = \int_0^l \tau^2 ds \). The arc \( \alpha \) is called a relaxed elastic line of second kind if it is an extremal for the variational problem of minimizing the value of \( F \) within the family of all arcs of length \( l \) on \( S \) having the same initial point and initial direction as \( \alpha \). In this study, we obtain differential equation and boundary conditions for a relaxed elastic line of second kind on an oriented surface.

1 Introduction

Let \( \alpha (s) \) denote an arc on a connected oriented surface \( S \) in \( E^3 \), parameterized by arc length \( s \), \( 0 \leq s \leq l \), with curvature \( \kappa (s) \). The total square curvature \( K \) of \( \alpha \) is defined by

\[
K = \int_0^l \kappa^2 ds.
\]  

(1)

An arc is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of \( K \) within the family of all arcs of length \( l \) on \( S \) having the same initial point and initial direction as \( \alpha \) [5]. In [5] they derive the intrinsic equations for a relaxed elastic line on an oriented surface. Hilbert and Cohn-Vossen [3] incorrectly suggested a flexible knitting needle, constrained to conform to a surface, as one model for a geodesic on a surface. This model actually gives a relaxed elastic line on the surface, and is not generally a geodesic unless the surface lies in a plane or on a sphere [5]. Physical motivation for study of the problem of elastic lines on surfaces may be found in the nucleosome core particle [4], [6].

There are several papers about this kind of minimization problems [9], [2], [10], [7].
In [8] authors handled the problem of minimizing the total square torsion on an oriented surface and defined the relaxed elastic line of second kind. However, they only gave Euler-Lagrange equations for this problem.

In this paper, we obtain intrinsic equations for a relaxed elastic line of second kind. We give a differential equation and three boundary conditions.

\section{Derivation of equations}

Let $\alpha (s)$ denote an arc on a connected oriented surface $S$ in $E^3$, parameterized by arc length $s$, $0 \leq s \leq l$, with torsion $\tau (s)$. The total square torsion $F$ of $\alpha$ is defined by

\[ F = \int_0^l \tau^2 ds. \]  

\textbf{Definition 1} The arc $\alpha$ is called a relaxed elastic line of second kind if it is an extremal for the variational problem of minimizing the value of $F$ within the family of all arcs of length $l$ on $S$ having the same initial point and initial direction as $\alpha$ [8].

We assume that the arc $\alpha$ is smooth enough to have derivatives up to required order and parameterized with arc length. Also, for technical reasons we assume that $\kappa \neq 0$, $\forall s$ on $\alpha$. On $\alpha$, let $T (s) = \alpha' (s)$ denote the unit tangent vector field, $n (s)$ denote the unit surface normal vector field to $S$ and $Q (s) = nxT$. Then, $\{ T, Q, n \}$ gives an orthonormal basis on $\alpha$ and $\{ T, Q \}$ gives a basis for the vectors tangent to $S$ at $\alpha (s)$. The frame $\{ T, Q, n \}$ is called Darboux frame. Derivative equations for the Darboux frame is

\[
\begin{bmatrix}
T' \\
Q' \\
n'
\end{bmatrix}
= \begin{bmatrix}
0 & \kappa_g & \kappa_n \\
-\kappa_g & 0 & \tau_g \\
-\kappa_n & -\tau_g & 0
\end{bmatrix}
\begin{bmatrix}
T \\
Q \\
n
\end{bmatrix},
\]

(3)

where $\kappa_g$, $\kappa_n$ and $\tau_g$ are geodesic curvature, normal curvature and geodesic torsion, respectively [1]. The square curvature $\kappa^2$ and the torsion $\tau$ of $\alpha$ on $S$ are given by

\[
\begin{aligned}
\kappa^2 &= \kappa_g^2 + \kappa_n^2, \\
\tau &= \tau_g + \frac{\kappa_g \kappa_n' - \kappa_n \kappa_g'}{\kappa_g^2 + \kappa_n^2}.
\end{aligned}
\]

(4)

Suppose that $\alpha$ lies in a coordinate patch $(u, v) \rightarrow x(u, v)$ of $S$, and let $x_u = \partial x / \partial u$, $x_v = \partial x / \partial v$. Then, $\alpha$ is expressed as

\[ \alpha (s) = x (u(s), v(s)), \quad 0 \leq s \leq l, \]

with

\[ T (s) = \alpha' (s) = \frac{du}{ds} x_u + \frac{dv}{ds} x_v \]
and

\[ Q(s) = p(s)x_u + q(s)x_v \]

for suitable scalar functions \( p(s) \) and \( q(s) \).

Now, we will define variational fields for our problem. In order to obtain variational arcs of length \( l \), we need to extend \( \alpha \) to an arc \( \alpha^*(s) \) defined for \( 0 \leq s \leq l^* \), with \( l^* > l \) but sufficiently close to \( l \) so that \( \alpha^* \) lies in the coordinate patch. Let \( \mu(s), \ 0 \leq s \leq l^* \), be a scalar function of class \( C^2 \), not vanishing identically. Define

\[ \eta(s) = \mu(s)p^*(s), \quad \zeta(s) = \mu(s)q^*(s). \]

Then,

\[ \eta(s)x_u + \zeta(s)x_v = \mu(s)Q(s) \quad (5) \]

along \( \alpha \). Also assume that

\[ \mu(0) = 0, \quad \mu'(0) = 0, \quad \mu''(0) = 0. \quad (6) \]

Now, define

\[ \beta(\sigma;t) = x(u(\sigma) + t\eta(\sigma), v(\sigma) + t\zeta(\sigma)), \quad (7) \]

for \( 0 \leq \sigma \leq l^* \). For \( |t| < \varepsilon_1 \) (where \( \varepsilon_1 > 0 \) depends upon the choice of \( \alpha^* \) and of \( \mu \)), the point \( \beta(\sigma;t) \) lies in the coordinate patch. For fixed \( t \), \( \beta(\sigma;t) \) gives an arc with the same initial point and initial direction as \( \alpha \), because of (6). For \( t = 0 \), \( \beta(\sigma;0) \) is the same as \( \alpha^* \) and \( \sigma \) is arc length. For \( t \neq 0 \), the parameter \( \sigma \) is not arc length in general.

For fixed \( t \), \( |t| < \varepsilon_1 \), let \( L^*(t) \) denote the length of the arc \( \beta(\sigma;t), 0 \leq \sigma \leq l^* \). Then,

\[ L^*(t) = \int_0^l \sqrt{\left( \frac{\partial \beta}{\partial \sigma} \cdot \frac{\partial \beta}{\partial \sigma} \right)} \, d\sigma \quad (8) \]

with

\[ L^*(0) = l^* > l. \quad (9) \]

By (4) and (5), \( L^*(t) \) is continuous and differentiable in \( t \). Particularly, it follows from (4) that

\[ L^*(t) > \frac{l + l^*}{2} > l \quad \text{for} \quad |t| < \varepsilon \quad (10) \]

for a suitable \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_1 \). Because of (10) one can restrict \( \beta(\sigma;t), 0 \leq |t| < \varepsilon \), to an arc of length \( l \) by restricting the parameter \( \sigma \) to an interval \( 0 \leq \sigma \leq \lambda(t) \leq l^* \) by requiring

3
\[ \int_0^{\lambda(t)} \sqrt{\left( \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right)} \, d\sigma = l. \]  

(11)

Note that \( \lambda(0) = l \). The function \( \lambda(t) \) need not be determined explicitly, but we shall need

\[ \frac{d\lambda}{dt} \bigg|_{t=0} = \int_0^l \mu \kappa_g ds. \]  

(12)

The proof of (12) and of other results will depend on calculations from (7) such as

\[ \frac{\partial \beta}{\partial \sigma} \bigg|_{t=0} = T, \quad 0 \leq s \leq l, \]  

(13)

which gives

\[ \frac{\partial^2 \beta}{\partial \sigma^2} \bigg|_{t=0} = T' = \kappa_g Q + \kappa_n n. \]  

(14)

Also

\[ \frac{\partial \beta}{\partial t} \bigg|_{t=0} = \mu Q \]  

(15)

because of (5). Further differentiation of (15) gives

\[ \frac{\partial^2 \beta}{\partial t \partial \sigma} \bigg|_{t=0} = \frac{\partial^2 \beta}{\partial \sigma \partial t} \bigg|_{t=0} = \mu' Q + \mu Q' = -\mu \kappa_g T + \mu' Q + \mu \tau_g n \]  

(16)

and using (3),

\[ \frac{\partial^3 \beta}{\partial t \partial \sigma^2} \bigg|_{t=0} = \left( -2 \mu' \kappa_g - \mu \kappa_g' \right) T + \left( \mu'' - \mu \kappa_g^2 - \mu \tau_g^2 \right) Q \]  

(17)

\[ + \left( 2 \mu' \tau_g + \mu \tau_g' - \mu \kappa_g \kappa_n \right) n. \]

Also using (14) we have

\[ \frac{\partial^3 \beta}{\partial \sigma^3} \bigg|_{t=0} = -\left( \kappa_g^2 + \kappa_n^2 \right) T + \left( \kappa_g' + \kappa_n \tau_g \right) Q + \left( \kappa_g'' + \kappa_g \tau_g \right) n \]  

(18)

and by (15)

\[ \frac{\partial^4 \beta}{\partial t \partial \sigma^3} \bigg|_{t=0} = \left( -3 \mu'' \kappa_g + 3 \mu' \kappa_g' + 2 \mu \kappa_g'' + \mu \kappa_g' \tau_g + 2 \mu \kappa_n \tau_g' \right) + \left( 3 \mu' \kappa_n \tau_g - \mu \kappa_g \kappa_n^2 - \mu \kappa_g^3 - \mu \tau_g^2 \right) T \]  

(19)

\[ - \left( 3 \mu' \kappa_g^2 + 3 \mu' \tau_g^2 + 3 \mu \kappa_g \kappa_g' + 3 \mu \tau_g \tau_g' - \mu'' \right) Q \]  

\[ - \left( 3 \mu' \kappa_n \kappa_g + 2 \mu \kappa_g' \kappa_n + \mu \kappa_g^2 \tau_g - 3 \mu' \tau_g \right) \]  

\[ - 3 \mu' \tau_g' - \mu \tau_g'' + \mu \kappa_g \kappa_n' + \mu \tau_g^3 + \mu \kappa_n^2 \tau_g \right) n. \]
Now, let $F(t)$ denote the functional of a relaxed elastic line of second kind for the arc $\beta(\sigma; t)$, $0 \leq \sigma \leq \lambda(t)$, $|t| < \varepsilon$. Since, in general, $\sigma$ is not the arc length for $t \neq 0$ functional (2) can be calculated as follows:

$$F(t) = \int_0^{\lambda(t)} \left( \frac{\partial \beta}{\partial \sigma} \times \frac{\partial^2 \beta}{\partial \sigma^2} + \frac{\partial^3 \beta}{\partial \sigma^3} \right) \left( \frac{\partial \beta}{\partial \sigma} \times \frac{\partial^2 \beta}{\partial \sigma^2} \right)^2 d\sigma.$$

A necessary condition for $\alpha$ to be an extremal is that

$$\left. \frac{dF}{dt} \right|_{t=0} = 0$$

for arbitrary $\mu$ satisfying (6). In calculating $dF/dt$; we give explicitly only terms that do not vanish for $t = 0$. The omitted terms are those with the factor

$$\left( \frac{\partial^2 \beta}{\partial \sigma^2} \frac{\partial \beta}{\partial \sigma} \right)$$

which vanishes at $t = 0$ because $\langle T, T' \rangle = 0$. Thus, using (4), (12−14) and (16−19) we get

$$\left. \frac{dF}{dt} \right|_{t=0} = \int_0^l \mu \left\{ \kappa_g \tau^2 \left[ -(\kappa_g \kappa^2 + \kappa_n \tau_g \kappa^2 + (\kappa_n - \tau_g')) \left( \kappa_n' - \kappa_n \tau_g \right) \right] 
- \left( \kappa_n'^2 + \tau_g'^2 \right) \left( \kappa_n' + \kappa_n \tau_g \right) - \kappa_g \left( 2 \kappa_n \kappa_n + \kappa_n^2 \tau_g - \tau_g'' + \kappa_n \kappa_n' + \tau_g'^3 + \kappa_n^2 \tau_g \right) 
+ 3 \kappa_n \left( \kappa_n \kappa_n' + \tau_g \tau_g' \right) - 2 \tau \left( -\kappa_g \kappa^2 + \kappa_n \left( \tau_g' - \kappa_n \tau_n \right) - \kappa_n \left( \kappa_n^2 + \tau_g^2 \right) \right) \right\} ds$$

$$+ 2 \int_0^l \mu \frac{\tau}{\kappa^2} \left[ -\kappa_n \kappa^2 - 2 \tau_g \left( \kappa_n' - \kappa_n \tau_g \right) - 3 \kappa_n \left( \kappa_n \kappa_n' - \tau_g' \right) + 3 \kappa_n \left( \kappa_n^2 + \tau_g^2 \right) 
- 4 \kappa_n \tau_g \tau_g \right] ds + 2 \int_0^l \mu'' \frac{\tau}{\kappa^2} \left( \kappa_n' + 4 \kappa_n \tau_g - \kappa_n \tau_n \right) ds - 2 \int_0^l \mu'' \frac{\kappa_n \tau}{\kappa^2} ds.$$

However using integration by parts and (16) we have
\[
\frac{dF}{dt} \bigg|_{t=0} = \int_0^l \mu \left\{ \kappa_g \tau^2 (l) + 2 \frac{\tau}{\kappa^2} 
- \kappa_g \kappa^2 \left( \kappa_g'^2 + \kappa_g'^2 \tau_g + \kappa_g'^2 \tau_g^2 - \kappa_g'^2 \kappa_g'^2 \right) 
+ 3 \kappa_g \left( \kappa_g' \kappa_g' \kappa_g' \tau_g + 2 \tau \left( \kappa_g^{(2)} - \kappa_g \kappa_g' \kappa_g' \kappa_g' \tau_g + \kappa_g^2 \left( \kappa_g'^2 + \kappa_g'^2 \right) \right) \right) 
- \kappa_g \kappa_g'^2 \tau_g - 3 \kappa_g \left( \kappa_g \kappa_g' \tau_g - \kappa_g' \tau_g + 3 \kappa_g \right) \kappa_g'^2 + \kappa_g'^2 \right) \right\} d\tau 
\]

In order to have
\[
\frac{dF}{dt} \bigg|_{t=0} = 0
\]
for any choice of the function \( \mu (s) \) satisfying (13) with arbitrary values \( \mu (l) \), \( \mu' (l) \) and \( \mu'' (l) \) the given arc must satisfy three boundary conditions

\[
\left[ \frac{\tau}{\kappa^2} \left( \kappa_g'^2 + \kappa_g'^2 \tau_g + \kappa_g'^2 \tau_g^2 - \kappa_g'^2 \kappa_g'^2 \right) \right]_{s=l} = 0
\]

\[
\left[ \frac{\tau}{\kappa^2} \left( \kappa_g'^2 + \kappa_g'^2 \tau_g + \kappa_g'^2 \tau_g^2 - \kappa_g'^2 \kappa_g'^2 \right) \right]_{s=l} = 0,
\]

\[
\kappa_g (l) \tau (l) = 0
\]

and the differential equation

\[
\kappa_g \tau^2 (l) + 2 \frac{\tau}{\kappa^2} \left( \kappa_g \kappa^2 \left( \tau_g + \tau \right) + 2 \kappa_g \kappa \kappa_g' - 2 \kappa_g \kappa^2 \tau_g - \kappa_g' \tau_g + 4 \kappa_g \tau_g \right) 
- 2 \kappa_g^2 \kappa_g' - 2 \kappa_g^2 \tau_g - \kappa_g^2 \tau_g + 2 \kappa_g \tau_g \kappa_g' \kappa_g' \tau_g + 2 \kappa_g \tau_g \kappa_g'^2 + \kappa_g \tau_g \kappa_g'^2 + \kappa_g \tau_g \kappa_g'^2 
- 2 \left[ \frac{\tau}{\kappa^2} \left( \kappa_g'^2 + \kappa_g'^2 \tau_g - \kappa_g'^2 \kappa_g'^2 \right) \right] 
+ 2 \left[ \frac{\tau}{\kappa^2} \left( \kappa_g'^2 + \kappa_g'^2 \tau_g + 3 \kappa_g \tau_g \right. \right. \left. - \kappa_g' \tau_g \right) 
+ 2 \left( \frac{\kappa_g \tau}{\kappa^2} \right)'' \right] = 0.
\]
Observe that, any planar curve, namely a curve with zero torsion, satisfies the above differential equation and the boundary conditions and it is a relaxed elastic line of second kind. Thus, we have the following theorem:

**Theorem 2** The intrinsic equations for a relaxed elastic line of second kind on a connected oriented surface in Euclidean 3-space are given by the differential equation \( (23) \) with the boundary conditions \( (20) - (22) \) at the free end, where \( \kappa_g, \kappa_n \) and \( \tau_g \) are the geodesic curvature, the normal curvature and the geodesic torsion as functions of the arc length along the curve.

**Corollary 3** A geodesic on an oriented surface is a relaxed elastic line of second kind if it satisfies the differential equation

\[
\frac{\tau_g}{\kappa_n} \left( 4\kappa_n \tau_g' - 2\kappa_n \tau'_g \right) - \left( \frac{\tau_g}{\kappa_n} \left( \kappa^2_g - \kappa_n^2 \right) \right)' + \left( \frac{\tau_g}{\kappa_n} \left( \kappa_n' + \kappa_n \tau_g \right) \right)'' + \left( \frac{\tau_g}{\kappa_n} \right)''' = 0,
\]

and the boundary conditions

\[ \tau_g (l) = \tau'_g (l) = \tau''_g (l) = 0. \]

**Corollary 4** A line of curvature on an oriented surface is a relaxed elastic line of second kind if it satisfies the differential equation

\[
\kappa_g \tau^2 \left( l \right) + \frac{2 \kappa_g \tau}{\kappa^2} \left( 3\kappa_g \tau + 2\kappa_n \kappa'_g - 2\kappa_g \kappa'_n \right) + \left( \kappa'_n \tau + \kappa_n \tau' \right) + 2 \left( \frac{\kappa_n \tau}{\kappa^2} \right)'' + 2 \left( \frac{\tau \kappa_n}{\kappa^2} \right)''' = 0
\]

and the boundary conditions

\[
\left. \left( \frac{\tau}{\kappa^2} \left( \kappa_n' - \kappa_g \tau \right) \right) \right|_{s=l} + \left. \left( \frac{\kappa_n \tau}{\kappa^2} \right)'' \right|_{s=l} = 0,
\]

\[
\left. \left( \frac{\tau}{\kappa^2} \left( \kappa_n' - \kappa_g \tau \right) \right) \right|_{s=l} + \left. \left( \frac{\kappa_n \tau}{\kappa^2} \right)' \right|_{s=l} = 0,
\]

\[ \kappa_n (l) \tau (l) = 0. \]

**Corollary 5** An asymptotic curve on an oriented surface is a relaxed elastic line of second kind if it satisfies the differential equation

\[
\frac{\tau_g}{\kappa^2_g} \left( 2\kappa^3_g \tau_g - \kappa'_g \tau'_g + \kappa_g \tau''_g \right) - \left( \frac{\tau_g}{\kappa^2_g} \left( 3\kappa_g \tau'_g - 2\tau_g \kappa'_g \right) \right)' + 2 \left( \frac{\tau^2_g}{\kappa^2_g} \right)'' = 0
\]

and the boundary condition

\[ \tau_g (l) = 0. \]
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