On the indices of periodic points in $C^1$-generic wild homoclinic classes in dimension three

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Abstract

We study the dynamics of homoclinic classes on three dimensional manifolds under the robust absence of dominated splittings. We prove that if such a homoclinic class contains a volume-expanding periodic point, then, $C^1$-generically, it contains a hyperbolic periodic point whose index (dimension of the unstable manifold) is equal to two.

0 Notations

In this section, we collect some basic definitions which we frequently use throughout this article.

0.1 Some basic terminologies

We consider a closed (compact and boundaryless) smooth manifold $M$ with a Riemannian metric. In this article, we mainly treat the case where $\dim M = 3$. The space of $C^1$-diffeomorphisms of $M$ is denoted by $\text{Diff}^1(M)$. We fix a distance function on $\text{Diff}^1(M)$ derives from the Riemannian metric and furnish $\text{Diff}^1(M)$ with the $C^1$-topology. For $f \in \text{Diff}^1(M)$, we denote the set of periodic points of $f$ by $\text{Per}(f)$ and the set of hyperbolic periodic points of $f$ by $\text{Per}_h(f)$. For $P \in \text{Per}(f)$, by $\text{per}(P)$ we denote the period of $P$, i.e., the least positive integer $k$ that satisfies $f^k(P) = P$. For $x \in M$ we denote the orbit of $x$ by $O(x,f)$ or simply $O(x)$. We put $J(P) := \det(df_{\text{per}}(P))$ and call this value Jacobian of $P$. We say that a periodic point is volume-expanding (resp. volume-contracting, conservative) if $|J(P)| > 1$ (resp. $|J(P)| < 1$, $|J(P)| = 1$).

In the following, we assume $P$ is a hyperbolic periodic point. The index of a hyperbolic periodic point $P$ (denoted by $\text{ind}(P)$) is defined to be the dimension of the unstable manifold of $P$. By $W^s(P,f)$ (resp. $W^u(P,f)$) we denote the stable (resp. unstable) manifold of $P$. We also use the simplified notation $W^s(P)$ (resp. $W^u(P)$). By $H(P,f)$, or simply $H(P)$, we denote the homoclinic class of $P$, i.e., the closure of the set of points of transversal intersections of $W^u(P)$ and $W^s(P)$. Two hyperbolic periodic points $P$ and $Q$ are said to be homoclinically related if $W^u(P)$ and $W^u(Q)$, $W^u(Q)$ and $W^s(P)$ both have non-empty transversal intersections. If $g$ is a $C^1$-diffeomorphism sufficiently close to $f$, then one can define the continuation of $P$. We denote the continuation of $P$ for $g$ by $P(g)$. We say that $P$ has a homoclinic tangency if there exists a point of non-transversal intersection between stable manifold and unstable manifold of $P$, i.e., there exists a point $x \in W^s(P) \cap W^u(P)$ at which $T_x W^s(P)$ and $T_x W^u(P)$ do not span $T_x M$.

0.2 Linear cocycles and dominated splittings

Let $(\Sigma, f, E, A)$ be a linear cocycle, where $\Sigma$ is a topological space, $f$ is a homeomorphism of $\Sigma$, $E$ is a Riemannian vector bundle over $\Sigma$ and $A$ is a bundle map that is compatible with $f$, i.e., $A$ is a map $A : E \to E$, where for each $x \in \Sigma$, $A(x, \cdot)$ is the linear isomorphisms from $E(x)$ to $E(f(x))$. We also use the notation $A(x)$ in the sense of $A(x, \cdot)$ and denote the linear cocycle only with $E$ or $A$ when the meaning is clear from the context. In our application, we mainly treat linear cocycles where $\Sigma$ is some invariant set of $M$, $f$ is the restriction of a diffeomorphism to $\Sigma$, $E$ is the restriction of the tangent bundle to $\Sigma$, and $A$ is the differential of $f$ restricted to $E$. We can naturally define the $n$-times iteration of $A$, denoted by $A^n$, and inverse of $A$, denoted by $A^{-1}$. We say that a linear cocycle $(\Sigma, f, E, A)$ is periodic if each point $x \in \Sigma$ is periodic for $f$. A periodic linear cocycle is said to be diagonalizable if at each point, the return map (the composition of corresponding linear maps along the orbit) is diagonalizable.
On each fiber, there is a norm that derives from the Riemannian metric. We denote it by \(| \cdot |\). Then, a linear cocycle is said to be bounded by \(K > 0\) if the following inequality holds:

\[
\max \left\{ \sup_{x \in \Sigma} ||A(x)||, \sup_{x \in \Sigma} ||A^{-1}(x)|| \right\} < K.
\]

When \(A\) is the restriction of the differential of a diffeomorphism on a compact manifold, \(A\) is bounded by some constant. For a linear cocycle, we can canonically define invariant subcocycle, direct sum between some cocycles, and quotient of cocycles (for the details, see section 1 of [BDP]).

Let \((\Sigma, f, E, A)\) be a linear cocycle and suppose \(E\) is a direct sum of two non-trivial invariant subbundles \((\Sigma, f, F, A)|_F\) and \((\Sigma, f, G, A)|_G\) (denoted by \(E = F \oplus G\), where \(A|_F\) is the restriction of \(A\) to \(F\). For a positive integer \(n\), we say that \(F \oplus G\) is an \(n\)-dominated splitting if following holds:

\[
\|A^n(x)|_F\|\|A^{-n}(f^n(x))|_G\| < 1/2, \text{ for all } x \in \Sigma.
\]

We say that a linear cocycle \((\Sigma, f, E, A)\) admits a dominated splitting if there exists two invariant subbundles \(F, G\) of \(E\) and an integer \(n\) such that \(E = F \oplus G\) is an \(n\)-dominated splitting. We say that an \(f\)-invariant set \(\Lambda \subset M\) admits a dominated splitting if the linear cocycle \((\Lambda, f, TM|_\Lambda, df)\) admits a dominated splitting.

### 0.3 Chain recurrence class and robust cycles

Let \(\varepsilon\) be a positive real number, \(x, y \in M\) and \(d(\cdot, \cdot)\) be a metric on \(M\). An \(\varepsilon\)-chain from \(x\) to \(y\) for \(f \in \text{Diff}^1(M)\) is a sequence \((x_i)_{i=1}^{n} (n \geq 2)\) in \(M\) satisfying \(d(f(x_i), x_{i+1}) < \varepsilon\) for \(1 \leq i < n\), \(x_1 = x\) and \(x_n = y\). Two points \(x, y\) are said to be chain equivalent if for any \(\varepsilon > 0\) there exist \(\varepsilon\)-chains from \(x\) to \(y\) and from \(y\) to \(x\). A point \(x \in M\) is said to be a chain recurrent point if \(x\) is chain equivalent to itself. For a chain recurrent point \(x\), its chain recurrence class of \(x\) is defined to be the set of the points that is chain equivalent to \(x\).

Let \(\Gamma\) and \(\Sigma\) be two transitive hyperbolic invariant sets for \(f\). We say that \(f\) \(\Gamma\) and \(\Sigma\) have a heterodimensional cycle if the following holds:

1. The indices (the dimension of the unstable manifolds) of the sets \(\Gamma\) and \(\Sigma\) are different.
2. The stable manifold of \(\Gamma\) meets the unstable manifold of \(\Sigma\) and the same holds for stable manifold of \(\Sigma\) and the unstable manifold of \(\Gamma\).

We say that the heterodimensional cycle associated to \(\Gamma\) and \(\Sigma\) is \(C^1\)-robust if there exists a \(C^1\)-neighborhood \(U\) of \(f\) such that for each \(g \in U\) there exists a heterodimensional cycle associated to the continuations \(\Gamma(g)\) of \(\Gamma\) and \(\Sigma(g)\) of \(\Sigma\).

### 1 Introduction

Let \(M\) be a compact smooth manifold without boundary. For \(f \in \text{Diff}^1(M)\) and \(P \in \text{Per}_h(f)\), its homoclinic class, denoted by \(H(P, f)\) (or \(H(P)\)), is defined to be the closure of the set of the points of the transversal intersection between the stable manifold and the unstable manifold of \(P\). The theory of Smale’s generalized horseshoe tells us that \(H(P)\) coincides with the closure of the set of hyperbolic periodic points that are homoclinically related to \(P\). In the study of uniformly hyperbolic systems, homoclinic classes play an important role and it is expected that they also play an important role in the research of non-hyperbolic dynamics (see chapter 10 of [BDV]).

In the Axiom A diffeomorphisms, every homoclinic class exhibits uniformly hyperbolic structure. That enables us to investigate the fine internal structure of dynamics of homoclinic classes. However, in the non-uniformly hyperbolic systems, homoclinic classes do not necessarily exhibit uniform hyperbolicities. This fact makes the study of non-hyperbolic dynamics difficult.

Even in the non-uniformly hyperbolic systems, a homoclinic class may exhibit weak form of hyperbolicity, such as partial hyperbolicity (see [BDV] for definition) or dominated splitting (see section 0.2 for definition). On the other hand, there do exist homoclinic classes that do not exhibit any kind of hyperbolicities in a robust fashion, and there are several indications that such kind of absence of hyperbolicities implies the
complexities of the dynamics. For example, Bonatti, Díaz, and Pujals [BDP] proved that the robust absence of the dominated splitting on a homoclinic class implies the $C^1$-Newhouse phenomenon, i.e., locally generic coexistence of infinitely many sinks or sources. Furthermore, in [BD1], Bonatti and Díaz showed, under the robust absence of dominated splitting and some conditions on the Jacobians, a homoclinic class exhibits very complicated dynamics named universal dynamics.

Thus, the following questions are interesting: What are the effects that the absence of dominated splitting on a homoclinic class gives rise to? Or, how the existence of the dominated splitting on a homoclinic class is disturbed? There are some results in this direction. For example, Gourmelon [Gou2] proved that under the absence of dominated splittings on a homoclinic class, one can create a homoclinic tangency inside the homoclinic class. Note that Wen [W] also proved similar result starting from preperiodic points. The result of Gan says that the existence of dominated splitting of index $i$ on preperiodic points is equivalent to the existence of $i$-eigenvalue gap (see [Gan] for precise definition). In this article, inspired by Abdenur, et al. ([ABCDW]), we investigate the index set of homoclinic classes that do not admit dominated splittings from the $C^1$-generic viewpoint.

To review the result of [ABCDW], we prepare one notation. Given $f \in \text{Diff}^1(M)$ and $P \in \text{Per}_h(f)$, the index set of $H(P, f)$ (denoted by $\text{ind}(H(P, f))$) is defined to be the set of integers that appear as a index of some periodic points in $H(P, f)$, i.e., we put

$$\text{ind}(H(P, f)) := \{k \in \mathbb{N} \mid \exists Q \in \text{Per}_h(f) \cap H(P, f), \ \text{ind}(Q) = k\}.$$ 

In [ABCDW], it was proved that for $C^1$-generic diffeomorphism, every homoclinic class has index set which is an interval in $\mathbb{N}$. The problem we pursue in this article is the following: Does the robust absence of dominated splittings give some restrictions on its index set?

Let us state our main result. For $f \in \text{Diff}^1(M)$ and $P \in \text{Per}_h(f)$, $H(P)$ is is said to be wild if it is robustly non-dominated, more precisely, there exists a neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ of $f$ such that for every $g \in \mathcal{U}$, the continuation $P(g)$ is defined and $H(P, g)$ does not admit any kind of dominated splittings.

Here is a partial answer to this question.

**Theorem 1.1.** For $C^1$-generic diffeomorphisms of a three-dimensional closed smooth manifold, if there exists a wild homoclinic class $H(P)$ that contains an index-one volume-expanding hyperbolic periodic point, then $2 \in \text{ind}(H(P))$.

Let us see an immediate corollary of this theorem.

**Corollary 1.** For $C^1$-generic diffeomorphisms of a three-dimensional closed smooth manifold, if there exists a wild homoclinic class that contains two hyperbolic periodic points such that one of them is volume-expanding and the other is volume-contracting. Then $\text{ind}(H(P)) = \{1, 2\}$.

Thus, under some assumptions on Jacobian, there does exist a restriction on the index of periodic points inside wild homoclinic classes.

This theorem can be interpreted as a qualification of homoclinic classes to be “basic pieces.” To explain this, let us review the intuitive idea of [BDP]. The wildness of a homoclinic class scatters its hyperbolicity to any direction. Thus by using their technique, it is not difficult to prove that, under the wildness, one can create an index bifurcation by an arbitrarily small perturbation. However, this argument does not tell us whether the bifurcation happens inside the homoclinic class or not. If homoclinic classes are to deserve as basic pieces, then it is desirable that a phenomenon which local (linear algebraic) information implies to happen can be observed inside the original homoclinic classes. Theorem [1.1] says that the bifurcation problem we discussed above is “well localized” in homoclinic classes.

Let us reintroduce our theorem from a different viewpoint. Aiming at the global understanding of $C^1$-dynamical systems, Palis suggested the famous Palis conjecture (see [P]), that is, every $(C^1)$-diffeomorphism away from Axiom A and no-cycle diffeomorphisms can be approximated by a diffeomorphism with heterodimensional cycle or homoclinic tangency. Recently, Bonatti and Díaz asked a stronger version of this conjecture:

**Problem** (Question 1.2 in [BD2]). Let $M$ be a smooth closed manifold. Does there exist a $C^1$-open and dense subset $\mathcal{O} \subset \text{Diff}^1(M)$ such that every $f \in \mathcal{O}$ either verifies the Axiom A and the no-cycle condition or has a $C^1$-robust heterodimensional cycle.
Our study gives a partial answer to this question. In fact, we can prove the following:

**Theorem 1.2.** Let $f$ be a $C^1$-diffeomorphism of a three-dimensional closed smooth manifold. If $f$ has a wild homoclinic class that contains an index-1 volume-expanding hyperbolic periodic point, then $f$ can be approximated by a diffeomorphism with a robust heterodimensional cycle.

Note that Wen [W] and Gourmelon [Gou2] already gave positive answers to the Palis conjecture under similar hypothesis. The novelty of our results is that we can create a connection between two saddles. Roughly speaking, outside Axiom A diffeomorphisms with no-cycles, by linear algebraic arguments and Franks’ lemma, it is not difficult to create an index bifurcation with an arbitrarily small perturbation. On the other hand, in general it is difficult to create a cycle between two saddles, since we need the information about the recurrence between two saddles. Our proof suggests one scenario to create a connection between two saddles.

In this article, we confined our attention to three-dimensional cases. Let us see what happens the other dimensions. In dimension two, it is easy to determine $\text{ind}(H(P))$. It is $\{0\}$ (sink), $\{2\}$ (source), or $\{1\}$. Note that there is no example of wild homoclinic classes in dimension two (in $C^1$-topology). If one can construct such a homoclinic class, it serves as a counter-example of the conjecture of Smale about the density of the Axiom A and no-cycle condition diffeomorphisms in $C^1$-topology (see [S]). The study of higher dimensional cases are of natural interest. So far, the complete solution of the higher dimensional cases are not obtained.

In [Sh], the author gave an example of generically wild homoclinic classes on four dimensional manifold whose index set exhibits an index deficiency in a robust fashion (for the precise definition, see [Sh]). The example of the homoclinic class that satisfies the assumption of the theorem can be found in [BD1].

As far as I know, this is the only mechanism that assures the wildness of some homoclinic classes. Thus it is interesting to study whether there is another mechanism that leads the creation of a wild homoclinic class. For example, the following question is interesting to study:

**Problem.** Can one find $f \in \text{Diff}^1(M)$ ($\dim(M) = 3$) such that for some hyperbolic periodic point $P$, $H(P)$ is wild and robustly $\text{ind}(H(P)) = \{1\}$?

Finally, let us see the organization of this article. In section 2 we introduce our strategy for the proofs of Theorem 1.1 and 1.2. We also furnish some part of their proof. The rest of the proofs are given in section 3 and section 4. More explanation on the contents of section 2 and section 4 can be found at the end of the section 2.

## 2 Outline of the proof

In this section, we see the strategy for the proof of Theorem 1.1 and 1.2. First, we explain how we will prove Theorem 1.2. The proof is divided into two propositions.

We say a hyperbolic periodic point $P$ has a homothetic tangency if $P$ has a homoclinic tangency and the restrictions of $df^{\text{per}}(P)$ to the stable spaces and unstable spaces are both homotheties (a linear endomorphism of a linear space is said to be a homothety if it is equal to $r\text{Id}$, where $r$ is a positive real number and $\text{Id}$ is the identity map).

Roughly speaking, the first proposition states that, under the robust absence of dominated splittings, one can create a homothetic tangency inside the homoclinic class by an arbitrarily small perturbation.

**Proposition 1.** Let $f \in \text{Diff}^1(M)$ with $\dim M = 3$ and $P$ be a volume-expanding index-1 hyperbolic periodic point of $f$. If $H(P)$ is wild then one can find a $C^1$-diffeomorphism $g$ arbitrarily $C^1$-close to $f$ such that the following properties hold.

1. There exists a volume-expanding hyperbolic periodic point $Q$ of index 1.
2. Let $P(g)$ be the continuation of $P$. Then two periodic points $P(g)$ and $Q$ are homoclinically related.
3. $Q$ has a homothetic tangency.

The second proposition says that from a homothetic tangency one can create a heterodimensional cycle by an arbitrarily $C^1$-small perturbation.
Proof of Theorem 1.1. Then, \( P \) are less than \( \Gamma \) Lemma 2.2.

Lemma 2.1 (Theorem 1 in BDKS). Let \( f \) be a \( C^1 \)-diffeomorphism with a heterodimensional cycle associated to saddles \( Q \) and \( R \) with \( \text{ind}(Q) - \text{ind}(R) = \pm 1 \). Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then there are diffeomorphisms \( g \) arbitrarily \( C^1 \)-close to \( f \) with robust heterodimensional cycles associated to two transitive hyperbolic sets containing the continuations \( Q(g) \) and \( R(g) \).

We can summarize the results of Proposition 1, Proposition 2, and Lemma 2.1 as follows:

Proposition 2. Let \( f \in \text{Diff}^1(M) \) with \( \dim M = 3 \) and \( Q \) be a volume-expanding hyperbolic periodic point of \( f \) with \( \text{ind}(Q) = 1 \). If \( Q \) has a homothetic tangency, then one can find a \( C^1 \)-diffeomorphism \( g \) arbitrarily \( C^1 \)-close to \( f \) such that the following properties hold.

1. There exists a hyperbolic periodic point \( R \) of \( g \) with \( \text{ind}(R) = 2 \).
2. \( g \) has a heterodimensional cycle associated to two periodic points \( Q(g) \) and \( R(g) \).

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Lemma 2.1 (Theorem 1 in BDKS). Let \( f \) be a \( C^1 \)-diffeomorphism with a heterodimensional cycle associated to saddles \( Q \) and \( R \) with \( \text{ind}(Q) - \text{ind}(R) = \pm 1 \). Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then there are diffeomorphisms \( g \) arbitrarily \( C^1 \)-close to \( f \) with robust heterodimensional cycles associated to two transitive hyperbolic sets containing the continuations \( Q(g) \) and \( R(g) \).

We can summarize the results of Proposition 1, Proposition 2, and Lemma 2.1 as follows:

Proposition 3. Let \( f \in \text{Diff}^1(M) \) with \( \dim M = 3 \) and \( P \) be a volume-expanding index-1 hyperbolic periodic point of \( f \). If \( H(P) \) is wild then one can find a \( C^1 \)-diffeomorphism \( g \) arbitrarily \( C^1 \)-close to \( f \) and a hyperbolic periodic point \( R \) of \( g \) with \( \text{ind}(R) = 2 \) such that \( g \) has a robust heterodimensional cycle associated to two transitive hyperbolic sets containing \( P(g) \) and \( R(g) \).

It is clear that Proposition 3 implies Theorem 1.2.

Now, let us see how we prove Theorem 1.1 using Proposition 3. For the proof, we need some generic properties about \( C^1 \)-diffeomorphisms listed below.

(R1) We denote the set of Kupka-Smale diffeomorphisms by \( \mathcal{R}_1 \) and the set of Kupka-Smale diffeomorphisms such that none of the periodic points are conservative by \( \mathcal{R}'_1 \). These are residual sets in \( \text{Diff}^1(M) \). The genericity of \( \mathcal{R}_1 \) is well known. One can prove the genericity of \( \mathcal{R}'_1 \) by modifying the usual proof of Kupka-Smale theorem (see R, for example) with paying attention to the fact that the volume-conservativeness of a hyperbolic periodic point is a fragile property (i.e., can be destroyed by an arbitrarily \( C^1 \)-small perturbation).

(R2) By \( \mathcal{R}_2 \) we denote the set of diffeomorphisms such that following holds: Any chain recurrence class \( C \) containing a hyperbolic periodic point \( P \) satisfies \( C = H(P) \). See BC.

(R3) By \( \mathcal{R}_3 \) we denote the set of the diffeomorphisms \( f \) that satisfying the following: Let \( P \) and \( Q \) be hyperbolic periodic points with \( \text{ind}(P) = \text{ind}(Q) \). If \( H(P) \cap H(Q) \neq \emptyset \) then \( P \) and \( Q \) are homoclinically related. We will give the proof of the genericity of \( \mathcal{R}_3 \) later.

The following lemma is easy to prove, so we omit the proof.

Lemma 2.2. Let \( P \) and \( Q \) be hyperbolic periodic points and assume there exists a heterodimensional cycle associated to two transitive hyperbolic invariant sets \( \Gamma \) and \( \Sigma \) such that \( \Gamma \) contains \( P \) and \( \Sigma \) contains \( Q \). Then, \( P \) and \( Q \) belong to the same chain recurrence class.

Let us give the proof of Theorem 1.1 using Proposition 3.

Proof of Theorem 1.1. For \( f \in \text{Diff}^1(M) \), let \( \text{Per}^N_0(f) \) be the set of hyperbolic periodic points whose periods are less than \( N \). For every \( f \in \mathcal{R}'_1 \), we take an open neighborhood \( \mathcal{U}_N(f) \subset \text{Diff}^1(M) \) of \( f \) such that every \( g \in \mathcal{U}_N(f) \) satisfies the following conditions:

1. For each \( P_i \in \text{Per}_{\text{ho}}^N(f) \), one can define the continuation \( P_i(g) \) (in particular we require that every \( P_i \) does not exhibit index bifurcation in \( \mathcal{U}_N(f) \)).
2. There is no creation of periodic points with period less than \( N \), more precisely, the set \( \{ P_i(g) \} \) exhausts the periodic points whose periods are less than \( N \).
3. For each \( P_i \in \text{Per}_{\text{ho}}^N(f) \), the signature of \( \log |J(P_i(g))| \) is independent of the choice of \( g \).
Note that since $f$ is Kupka-Smale, $\text{Per}_h^N(f)$ is a finite set. From now on, for each $U_N(f)$, we create an open and dense set $O_N \subset U_N(f)$ as follows. First, for each $g \in U_N(f)$, we define an open set $\mathcal{V}_{i,N}[g] \subset U_N(f)$ as follows:

1. If $P_i$ satisfies one of the following conditions:
   - $\text{ind}(P_i) \neq 1$,
   - volume-contracting (by assumption $P_i$ cannot be conservative), or
   - arbitrarily $C^1$-close to $g$ there exists a $C^1$-diffeomorphism $g'$ such that $H(P_i, g')$ admits a dominated splitting.

   Then $\mathcal{V}_{i,N}[g] := \emptyset$.

2. Otherwise, $\mathcal{V}_{i,N}[g]$ is the open set in $U_N(f)$ obtained by Proposition. Then, put $\mathcal{V}_{i,N}(f) := \bigcup_{g \in U(f)} \mathcal{V}_{i,N}[g]$. Since each $\mathcal{V}_{i,N}[g]$ are open, $\mathcal{V}_{i,N}(f)$ is open. Let us put $W_{i,N}(f) := U_N(f) \setminus \mathcal{V}_{i,N}(f)$ and $O_{i,N}(f) := \mathcal{V}_{i,N}(f) \cup W_{i,N}(f)$. By construction, $O_{i,N}(f)$ is open and dense in $U_N(f)$ and for every $g \in O_{i,N}(f)$, if $P_i(g)$ is a volume-expanding, hyperbolic and index-1 periodic point and $H(P_i, g)$ is wild, then $g$ has a robust cycle associated with transitive hyperbolic sets containing $P_i(g)$ and some hyperbolic periodic point of index 2.

   Now, put $\mathcal{O}_{i,N}(f) := \cap_{g \in U_N(f)} O_{i,N}(f)$. Since each $O_{i,N}(f)$ is open and dense, so is $\mathcal{O}_{i,N}(f)$ in $U_N(f)$ and the set $\mathcal{O}_N := \bigcup_{f \in R} \mathcal{O}_{i,N}(f)$ is the required open and dense subset of $\text{Diff}^1(M)$. Then, put $R_\ast := \cap_N \mathcal{O}_N$. By Baire’s category theorem this is a residual subset of $\text{Diff}^1(M)$ and satisfies the following property: If $f \in R_\ast$ and there exists a hyperbolic, index-1 and volume-expanding periodic point $P_i$ whose homoclinic class is wild, then one can find a robust cycle associated with $P_i$ and some periodic point with index 2.

   Finally, put $R_{\ast \ast} := R_\ast \cap R_2 \cap R_3$. We show that every $f \in R_{\ast \ast}$ satisfies the conclusion of Theorem. Suppose that there is a hyperbolic periodic point $P_i$ of $f$ such that $H(P_i)$ is a wild homoclinic class containing a volume-expanding hyperbolic periodic point $Q$ with $\text{ind}(Q) = 1$. If $\text{ind}(P_i) = 2$, then we have the conclusion. So let us assume $\text{ind}(P_i) = 1$. Since $f \in R_3$, $P_i$ and $Q$ are homoclinically related. For $H(P_i)$ is wild, so is $H(Q)$. Then, by the definition of $R_\ast$, $f$ has a heterodimensional cycle associated to two transitive hyperbolic set such that one of them contains $Q$ and the other contains a hyperbolic periodic point $R$ of index 2. By Lemma 2.2 $Q$ and $R$ belong to the same chain recurrence class. Since $f \in R_2$, we have $R \subset H(Q) = H(P_i)$.

**Remark 1.** Indeed, the proof above says that, for $C^1$-generic diffeomorphisms of a three-dimensional closed smooth manifold, every wild homoclinic class $H(P)$ that contains a index-one volume-expanding hyperbolic periodic point contains a robust heterodimensional cycle.

Now, we give the proof of genericity of $R_3$. For the proof, we need another generic property and one $C^1$-perturbation lemma.

**Lemma 2.3** (Lemma 2.1 in [ABCDW]). There is a residual subset $R_4 \subset \text{Diff}^1(M)$ such that, for every diffeomorphism $f \in R_4$ and every pair of saddles $P(f)$ and $Q(f)$ of $f$, there is a neighborhood $U_f$ of $f$ in $R_4$ such that either $H(P, g) = H(Q, g)$ for all $g \in U_f$, or $H(P, g) \cap H(Q, g) = \emptyset$ for all $g \in U_f$.

**Lemma 2.4** (See Lemma 2.8 in [ABCDW]). Let $P$ be a hyperbolic periodic point of a $C^1$-diffeomorphism $f$. Consider a homoclinic class $H(P, f)$ and any saddle $Q \in H(P, f)$. Then there is $g$ arbitrarily $C^1$-close to $f$ such that $W^s(P, g)$ and $W^u(Q, g)$ (resp. $W^u(P, g)$ and $W^s(Q, g)$) have a non-empty intersection.

**Proof of the genericity of $R_3$.** For every $f \in R_0 \cap R_4$, we take an open neighborhood $U_N'(f) \subset R_0 \cap R_4$ of $f$ such that for every $g \in U_N'(f)$ we have the following conditions: For each $P_i \in \text{Per}_h^N(f)$, one can define the continuation $P_i(g)$ and these continuations exhaust periodic points whose periods are less than $N$ of $g$ (note that $\text{Per}_h^N(f)$ is a finite set since $f \in R_0$). Now, for each $U_N'(f)$, we construct an open and dense set $O_N(f) \subset U_N'(f)$ as follows. First, for each $g \in U_N'(f)$ and $i \neq j$, let us define an open set $\mathcal{V}_{i,j,N}[g] \subset U_N'(f)$ as follows: if $P_i(g)$ and $P_j(g)$ are not homoclinically related, then $\mathcal{V}_{i,j,N}[g]$ is the empty set. Otherwise $\mathcal{V}_{i,j,N}[g] \subset U_N'(f)$ is a non-empty open set satisfying the following: If $h \in \mathcal{V}_{i,j,N}[g]$ then $P_i(h)$ and $P_j(h)$ are homoclinically related. Then, let $\mathcal{V}_{i,j,N}(f) := \bigcup_{g \in U_N'(f)} \mathcal{V}_{i,j,N}[g]$. We put
\[ W_{i,j},N(f) := \mathcal{U}_N'(f) \setminus V_{i,j},N(f) \] and \[ \mathcal{O}_{i,j},N(f) := V_{i,j},N(f) \cup W_{i,j},N(f) \]. By construction, \( \mathcal{O}_{i,j},N(f) \) is open and dense in \( \mathcal{U}_N'(f) \).

Let us see that for \( h \in \mathcal{O}_{i,j},N \) following holds: If \( P_i \in H(P_j, h) \) and \( \text{ind}(P_i) = \text{ind}(P_j) \), then \( P_i \) and \( P_j \) are homoclinically related. If \( h \in V_{i,j},N(f) \), this is clear. We show that for \( h \in W_{i,j},N \), we have \( P_i \notin H(P_j, h) \). Indeed, if \( P_i \in H(P_j, h) \), by applying Lemma 2.3 we can take a neighborhood \( C(h) \subset \mathcal{U}_N'(f) \) of \( h \) such that for all \( h' \in C(h) \), \( H(P_j, h') = H(P_j, h') \). Then, Lemma 2.4 tells us there exists \( h_1 \) arbitrarily \( C^1 \)-close to \( h \) such that \( W^u(P_i, h_1) \) and \( W^s(P_j, h_1) \) have non-empty intersection. By giving arbitrarily small perturbation, we can find a diffeomorphism \( h_2 \) such that \( W^u(P_i, h_2) \) and \( W^s(P_j, h_2) \) have non-empty transversal intersection. Since it is a \( C^1 \)-open property, we can find \( h_3 \) arbitrarily \( C^1 \)-close to \( h_2 \) such that \( W^u(P_i, h_2) \) and \( W^s(P_j, h_2) \) have non-empty transversal intersection (note that \( h_1 \), \( h_2 \) can fail to belong to \( C(h) \)). By a similar argument, we can find \( h_4 \) arbitrarily \( C^1 \)-close to \( h_3 \) such that \( P_i(h_4) \) and \( P_j(h_4) \) are homoclinically related. This is a contradiction, since \( h \in W_{i,j},N(f) = \mathcal{U}_N(f) \setminus V_{i,j},N(f) \) and \( h \) can be found arbitrarily \( C^1 \)-close to \( h \).

We put \( \mathcal{O}_N(f) := \cap_{i \neq j} \mathcal{O}_{i,j},N(f) \). Since \( f_0 \in \mathcal{R}_0 \), the number of the periodic points with period less than \( N \) are finite. So \( \mathcal{O}_N(f) \) is an open and dense subset of \( \mathcal{U}_N'(f) \). Now we define \( \mathcal{U}_N := \cup_{f \in \mathcal{R}_0} \mathcal{O}_N(f) \) and this is an open and dense subset of \( \text{Diff}^1(M) \). Finally, we put \( \mathcal{R}_3 := \cap_N \mathcal{U}_N \). Then, by construction, every diffeomorphism in \( \mathcal{R}_3 \) satisfies the desired condition and by Baire’s category theorem this set is residual in \( \text{Diff}^1(M) \).

In the rest of this section, we discuss the proofs of Proposition 1 and 2. In Section 3 we give the proof of Proposition 1. It is done by three techniques. The first one is the linear algebraic arguments developed in [BDP]. We combine this technique with the second one, the Franks’ lemma that preserves the invariant manifolds [Gou1]. The third one is the result given by Gourmelon [Gou2] about the creation of homoclinic tangency for homoclinic classes that does not admit dominated splittings.

In Section 4 we give the proof of Proposition 2. The proof consists of two steps. The first one is the reduction of the problem to affine dynamics. The second one is the investigation of the reduced dynamics involving some calculations.

## 3 Creation of a homothetic tangency

In this section we prove Proposition 1.

### 3.1 Strategy for the proof of Proposition 1

An important step to Proposition 1 is the following Proposition.

**Proposition 4.** Let \( f \in \text{Diff}^1(M) \) with \( \text{dim} M = 3 \), and let \( P \) be a volume-expanding hyperbolic periodic point of \( f \) such that \( \text{ind}(P) = 1 \) and \( H(P) \) is wild. Then one can find a \( C^1 \)-diffeomorphism \( g \) arbitrarily \( C^1 \)-close to \( f \) such that following holds:

1. There exists a volume-expanding hyperbolic periodic point \( Q \) of index 1.
2. The differential \( dg^{per(Q)}(Q) \) has only positive and real eigenvalues.
3. Two periodic points \( P(g) \) and \( Q \) are homoclinically related.
4. The differential \( dg^{per(Q)}(Q) \) restricted to the stable direction of \( Q \) is a homothety.

By this proposition together with the following result given by Gourmelon, we can prove Proposition 1.

**Lemma 3.1** (Theorem 1.1 in section 6 of [Gou2]). *If the homoclinic class \( H(P, f) \) of a saddle point \( P \) for \( f \) is not trivial and does not admit a dominated splitting of the same index as \( P \), then, there is an arbitrarily small perturbation \( g \) of \( f \), that preserves the dynamics on a neighborhood of \( P \), and such that there is a homoclinic tangency associated to \( P \).*

Let us give the proof of Proposition 1 assuming above two results.
Proof of Proposition 4. Under the hypothesis of Proposition 4, Proposition 3 tells us that we get \( f_1 \) arbitrarily \( C^1 \)-close to \( f \) such that \( f_1 \) has a hyperbolic periodic points \( Q(f_1) \) satisfying all the conclusions of Proposition 3. By taking \( f_1 \) sufficiently close to \( f \), we can assume \( H(P, f_1) = H(Q, f_1) \) does not admit dominated splittings. Then, by applying Lemma 3.1 to \( Q(f_1) \) we get \( f_2 \) arbitrarily close to \( f_1 \) such that \( Q(f_1) = Q(f_2) \) exhibits a homoclinic tangency. Since the perturbation preserves the local dynamics of \( Q(f_1) \), \( d_{f_2}^\text{per}(Q(f_1))(Q(f_2)) = d_{f_1}^\text{per}(Q(f_1))(Q(f_1)) \) and \( Q(f_2) \) is volume-expanding. Thus we have created a homothetic tangency. Furthermore, since \( P(f_1) \) and \( Q(f_1) \) are homoclinically related, if we take \( f_2 \) sufficiently close to \( f_1 \), we know \( P(f_2) \) and \( Q(f_2) \) are homoclinically related, too. Note that we can take \( f_2 \) arbitrarily close to \( f \) because \( f_1 \) can be found arbitrarily close to \( f \). Now the proof is completed.

Thus let us concentrate on the proof of Proposition 4. We divide the proof into two lemmas.

Lemma 3.2. Let \( f \in \text{Diff}^1(M) \) with \( \text{dim} \, M = 3 \) and \( P \) be a volume-expanding hyperbolic periodic point of \( f \) and \( \text{ind}(P) = 1 \). If \( H(P) \) is wild, then \( C^1 \)-arbitrarily close to \( f \) one can find a \( C^1 \)-diffeomorphism \( g \) such that following holds: There exists a volume-expanding hyperbolic index-1 periodic point \( Q \) such that \( P(g) \) and \( Q \) are homoclinically related, and the restriction of \( d_g^\text{per}(Q) \) to the stable direction has two complex eigenvalues.

Lemma 3.3. Let \( f \in \text{Diff}^1(M) \) with \( \text{dim} \, M = 3 \) and \( P \) be an index-1 volume-expanding hyperbolic periodic point of \( f \) and the restriction of \( d_f^\text{per}(P) \) have two contracting complex eigenvalues. If \( H(P) \) is non-trivial, then \( C^1 \)-arbitrarily close to \( f \) one can find a \( C^1 \)-diffeomorphism \( g \) such that following holds: There exists a volume-expanding hyperbolic periodic point \( Q \) whose index is 1 such that \( P(g) \) and \( Q \) are homoclinically related, \( d_g^\text{per}(Q) \) has only positive and real eigenvalues, and the restriction of \( d_g^\text{per}(Q) \) to the stable direction is a homothety.

Let us prove Proposition 4 assuming Lemma 3.2 and 3.3.

Proof of Proposition 4. Suppose \( f, P \) are given as is in the hypothesis of Proposition 4. First, by Lemma 3.2, we get arbitrarily close to \( f \) we can find \( f_1 \) such that there exists a volume-expanding hyperbolic periodic point \( Q(f_1) \) whose index is 1 such that \( P(f_1) \) and \( Q(f_1) \) are homoclinically related, and the restriction of \( d_{f_1}^\text{per}(Q(f_1))(Q(f_1)) \) has two complex eigenvalues. Second, by applying Lemma 3.3 to \( f_1 \) and \( Q(f_1) \), we can find \( f_2 \) in some neighborhood of \( f_1 \) such that there exists a volume-expanding hyperbolic periodic point \( R(f_2) \) whose index is 1, \( Q(f_2) \) and \( R(f_2) \) are homoclinically related and the restriction of \( d_{f_2}^\text{per}(R(f_2))(R(f_2)) \) to the stable direction is a homothety, and a positive eigenvalue for unstable direction. Note that if we take \( f_2 \) sufficiently close to \( f_1 \), then \( P(f_2) \) and \( Q(f_2) \) remain homoclinically related and thus \( P(f_2) \) and \( R(f_2) \) are homoclinically related, too. Since \( f_1 \) can be constructed arbitrarily close to \( f \) and \( f_2 \) can be constructed arbitrarily close to \( f_1 \), \( f_2 \) can be constructed arbitrarily close to \( f \). This ends the proof of our proposition.

In subsection 3.2, we prepare some techniques for the proof of Lemma 3.2. In subsection 3.4, we prove Lemma 3.7 that is needed to prove Lemma 3.2.

3.2 Proof of Lemma 3.2

In this subsection, we prove Lemma 3.2 assuming some results. First, we collect some results for the the proof of Lemma 3.2.

The first one is the Franks’ lemma (see appendix A of [BDV]).

Lemma 3.4 (Franks’ lemma). Let \( f \) be a \( C^1 \)-diffeomorphism defined on a closed manifold \( M \) and consider any \( \delta > 0 \). Then there is \( \varepsilon > 0 \) such that, given any finite set \( \Sigma \subset M \), any neighborhood \( U \) of \( \Sigma \), and any linear maps \( A_x : T_xM \to T_{f(x)}M \) \( (x \in \Sigma) \) such that \( A_x \) is \( \varepsilon \)-close to \( df(x) \), there exists a \( C^1 \)-diffeomorphism \( g \) that is \( \delta \)-close to \( f \) in the \( C^1 \)-topology, coinciding with \( f \) on \( M \setminus U \) and on \( \Sigma \), and \( dg(x) = A_x \) for all \( x \in \Sigma \).

We introduce the Franks’ lemma that preserves invariant manifolds [Gou1]. To state it clearly, we prepare some notations. For a hyperbolic periodic point \( X \) of a diffeomorphism \( f \), we consider the space of linear cocycles over \( O(X) \) (remember that \( O(X) \) is the orbit of \( X \)). Let us denote this space as \( C(X) \), i.e., \( C(X) \) is the set of maps \( \sigma : TM|_{O(X)} \to TM|_{O(X)} \) such that for all \( i \in \mathbb{Z} \), \( \sigma(f^i(X)) = \sigma(f^i(X), \cdot) \) is a
linear isomorphism from $T'_f(x)M$ to $T'_{f^{i+1}}(x)M$. By abuse of notation, we denote the cocycle given by the restriction of $df$ to $TM_{f(x)}$ by $df$.

We define a metric on this space as follows. For $\sigma_1, \sigma_2 \in \mathcal{C}(X)$, the distance between $\sigma_1$ and $\sigma_2$ (denoted by $\text{dist}(\sigma_1, \sigma_2)$) is defined to be the following:

$$\max \left\{ \max_{x \in \mathcal{O}(X)} ||\sigma_1(x) - \sigma_2(x)||, \max_{x \in \mathcal{O}(X)} ||(\sigma_1(x))^{-1} - (\sigma_2(x))^{-1}|| \right\}.$$ 

For $\sigma \in \mathcal{C}(X)$ we denote by $\tilde{\sigma}$ the first return map of $\sigma$, i.e., the linear endomorphism of $T_XM$ given by $\sigma(f_{\text{per}(X)}^{-1}(x)) \circ \cdots \circ \sigma(f(x)) \circ \sigma(X)$. We define the eigenvalues of $\sigma$ by the eigenvalues of $\tilde{\sigma}$. We say that $\sigma$ is hyperbolic if none of the eigenvalues of $\tilde{\sigma}$ has its absolute value equal to one. Note that the set of the hyperbolic cocycles forms an open set in $\mathcal{C}(X)$. Let $\gamma(t)$ be a continuous path in $\mathcal{C}(X)$, i.e., $\gamma(t)$ is a continuous map from $[0,1]$ to $\mathcal{C}(X)$. We define the diameter of $\gamma(t)$ (denoted by $\text{diam}(\gamma(t))$) to be the number $\max_{0 \leq s, t \leq 1} \text{dist}(\gamma(s), \gamma(t))$.

Let $U$ be a neighborhood of $\mathcal{O}(X)$. For $x \in W^s(x) \cap (M \setminus U)$, let $\alpha(x)$ be the least number such that $f^{\alpha(x)}(x) \in U$ holds. We define the stable manifold of $X$ outside $U$ (denoted $W^s_{\text{loc}, U}(X)$) to be the set of points that never leave $U$ once they enter $U$, more precisely,

$$W^s_{\text{loc}, U}(X) := \{ x \in W^s(x) \cap (M \setminus U) | \forall n \geq \alpha(x), f^n(x) \in U \}.$$ 

Let $g$ be a diffeomorphism so close to $f$ that we can define the continuation $X(g)$ of $X$ for $g$. We say that $g$ locally preserves the stable manifold of $f$ outside $U$ if $W^s_{\text{loc}, U}(X,g) \supset W^s_{\text{loc}, U}(X,f)$. Similarly, we can define the unstable manifold outside $U$, etc.

Now let us state the precise statement of the lemma.

**Lemma 3.5** (Gourmelon’s Franks’ lemma, [Gou1]). Let $f$ be a $C^1$-diffeomorphism of $M$ and $X$ be a hyperbolic periodic point of $f$. Suppose that there exists a continuous path $\{ \gamma(t) | 0 \leq t \leq 1 \}$ in $\mathcal{C}(X)$ satisfying the following:

1. For all $i \in \mathbb{Z}$, $\gamma(0)(f^i(X)) = df(f^i(X))$.
2. The diameter $\text{diam}(\gamma(t))$ is less than $\varepsilon > 0$.
3. For all $0 \leq t \leq 1$, $\gamma(t)$ is hyperbolic (hence the dimensions of stable and unstable spaces are constant).

Then, given neighborhood $U$ of $\mathcal{O}(X)$ there exists a $C^1$-diffeomorphism $g$ that is $\varepsilon$-$C^1$-close to $f$ satisfying the following properties:

1. For all $i$, $f^i(X) = g^i(X)$. Especially, $X$ is a periodic point for $g$.
2. As a linear cocycle over $C(X)$, $dg = \gamma(1)$. Especially, $X$ is a hyperbolic periodic point of $g$ and $\text{ind}(X(f)) = \text{ind}(X(g))$.
3. The support of $g$ is contained in $U$, i.e., $\{ x \in M | f(x) \neq g(x) \} \subset U$.
4. $g$ locally preserves the stable and unstable manifolds of $X$ outside $U$.

**Proof.** Apply Theorem 2.1 in [Gou1] putting $I = \{ \dim M - \text{ind}(X) \}$ and $J = \{ \text{ind}(X) \}$. 

As a consequence of Lemma 3.5, we can prove the following lemma.

**Lemma 3.6.** In addition to the hypotheses of Lemma 3.5, suppose that the following property holds:

4. There is a hyperbolic periodic point $Y$ that is homoclinically related to $X$.

Then, there is a (small) neighborhood $V$ of $\mathcal{O}(X)$ and a $C^1$-diffeomorphism $h$ $\varepsilon$-$C^1$-close to $f$ satisfying all the conclusions of Lemma 3.5 and the following property:

5. $X(h)$ is homoclinically related to $Y(h)$.

Let us prove Lemma 3.6 by Lemma 3.5.
Proof. We fix an open neighborhood \( W \) of \( \mathcal{O}(Y) \) that has empty intersection with \( \mathcal{O}(X) \). Take a point \( a \in W^s(X) \cap W^u(Y) \). Since \( a \in W^s(X) \), by replacing \( a \) with \( f^n(a) \) for some \( n > 0 \) if necessary, we can assume \( a \not\in W \) for all \( t \geq 0 \). Furthermore, by shrinking \( W \) if necessary and using the fact \( a \in W^u(Y) \), we can assume the following condition holds: Let \( l_0 > 0 \) be the least number that satisfies \( f^{-l_0}(x) \in W \). Then there exists a small neighborhood \( D^u_Y(a) \) of \( a \) in \( W^u(Y) \) such that \( f^{-l}(D^u_Y(a)) \subset W \) for all \( l \geq l_0 \) and \( f^{-l}(f^l(D^u_Y(a))) \cap W = \emptyset \) for all \( 0 \leq l < l_0 \).

Similarly, we take a point \( b \in W^s(Y) \cap W^u(X) \) such that \( b \not\in W \) and the following holds: Let \( l_1 > 0 \) be the least number that satisfies \( f^{l_1}(b) \in W \). There exists a small neighborhood \( D^s_X(b) \) of \( b \) in \( W^s(Y) \) such that \( f^l(D^s_X(b)) \subset W \) for all \( l \geq l_1 \) and \( f^l(D^s_X(b)) \cap W = \emptyset \) for all \( 0 \leq l < l_1 \).

Similar argument gives us an open neighborhood \( V \) of \( \mathcal{O}(X) \) satisfying the following conditions:

1. For all \( n \geq 0 \), \( f^{-n}(D^u_Y(a)) \cap V = \emptyset \), especially \( a \not\in V \).
2. For all \( n \geq 0 \), \( f^n(D^s_X(b)) \cap V = \emptyset \), especially \( b \not\in V \).
3. Let \( k_0 > 0 \) be the least number that satisfies \( f^{k_0}(a) \in V \). Then there exists a small neighborhood \( D_X^s(a) \) of \( a \) in \( W^s(X) \) such that \( f^{k_0}(D_X^s(a)) \subset V \) for all \( k \geq k_0 \) and \( f^k(D_X^s(b)) \cap V = \emptyset \) for all \( 0 \leq k < k_0 \).
4. Let \( k_1 > 0 \) be the least number that satisfies \( f^{-k_1}(b) \in V \). Then there exists a small neighborhood \( D_Y^u(b) \) of \( b \) in \( W^u(X) \) such that \( f^{-k_1}(D_Y^u(b)) \subset V \) for all \( k \geq k_1 \) and \( f^{-k}(D_Y^u(b)) \cap V = \emptyset \) for all \( 0 \leq k < k_1 \).

By applying Lemma 3.5, we can construct an \( \varepsilon \)-\( C^1 \)-close perturbation \( h \) of \( f \) that satisfies all the conclusions of Lemma 3.5. We show that \( X(h) \) is homoclinically related to \( Y(h) \). To see this, first we check \( a \in W^s(X, h) \cap W^u(Y, h) \). Since \( f^{−n}(D^u_Y(a)) \cap V = \emptyset \) for all \( n \geq 0 \) and the support of \( h \) is contained in \( V \), \( D^u_Y(a) \) is contained in \( W^u(Y, h) \). Second, by the condition of \( D_X^s(a) \), it is contained in the stable manifold outside \( V \). Thus it is contained in \( W^s(X, h) \) and now we know \( a \in W^s(X, h) \cap W^u(Y, h) \). Similarly, we can see \( b \in W^s(Y, h) \cap W^u(X, h) \). Thus the proof is completed.

We collect some results about linear algebra on linear cocycles from [BDP].

The first one roughly says, in dimension two, the absence of the domination implies the creation of complex eigenvalues. The origin of this type of arguments can be found in [M].

**Lemma 3.7.** For any \( K > 0 \) and \( \varepsilon > 0 \), the following holds: Let \( (\Sigma, f, E, A) \) be a two dimensional diagonalizable periodic linear cocycle with positive eigenvalues. If \( (\Sigma, f, E, A) \) is bounded by \( K \) and does not admit dominated splittings, then there exists a periodic point \( X \) and a path \( \{\gamma(t) \mid 0 \leq t \leq 1\} \) in \( C(X) \) such that the following conditions hold:

1. \( \gamma(0) = A|_{\mathcal{O}(X)} \).
2. \( \text{diam}(\gamma(t)) < \varepsilon \).
3. \( \det(\gamma(t)) \) is independent of \( t \).
4. Let \( \lambda_m(t) \leq \lambda_0(t) \) be the absolute value of the eigenvalues of \( \gamma(t) \). Then for \( s < t \) we have \( \lambda_m(s) \leq \lambda_m(t) \) and \( \lambda_0(s) \geq \lambda_0(t) \).
5. \( \gamma(1) \) has two complex eigenvalues.

We give the proof of this lemma in the next subsection.

The next lemma is used to create complex eigenvalues from a linear map which has eigenvalues with multiplicity two.

**Lemma 3.8.** Let \( A \) be a linear endomorphism on a two-dimensional normed linear space such that the eigenvalues of \( A \) are positive real number \( \lambda \) with multiplicity 2. Then there exists \( B \) arbitrarily close to \( A \) such that \( B \) has two complex eigenvalues.
Proof. By taking the Jordan canonical form of $A$, we can assume that $A$ has the following form:

\[
\begin{pmatrix}
\lambda & t \\
0 & \lambda
\end{pmatrix}.
\]

When $t = 0$, the matrix

\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\lambda & t \\
0 & \lambda
\end{pmatrix}
\]

has two complex eigenvalues for sufficiently small $\alpha > 0$ and $\alpha$ can be taken arbitrarily small. If $t \neq 0$, the matrix

\[
\begin{pmatrix}
\lambda & t \\
-\alpha/t & \lambda
\end{pmatrix}
\]

has complex eigenvalue for all $\alpha > 0$.

The following lemmas enables us to "lift up" the perturbation on some sub cocycle or quotient cocycle to the perturbation on the original cocycle.

**Lemma 3.9.** Given $\varepsilon > 0$, a linear cocyle $(\Sigma, f, E, A)$ and its invariant subcocycle $(\Sigma, f, F, A|_F)$, where $A|_F$ denotes the restriction of $A$ to $F$, the following hold:

1. Let $(\Sigma, f, F, B)$ be a cocycle with $\text{dist}(A|_F, B) < \varepsilon$. Then there exists a linear cocycle $(\Sigma, f, E, \tilde{B})$ such that $\text{dist}(A, \tilde{B}) < \varepsilon$ and $A/F = \tilde{B}/F$, where $A/F$ denotes the bundle map derives from $A$ on $E/F$.

2. If $(\Sigma, f, E/F, B)$ is a cocycle with $\text{dist}(A/F, B) < \varepsilon$. Then there exists a linear cocycle $(\Sigma, f, E, \hat{B})$ such that $\text{dist}(A, \hat{B}) < \varepsilon$, $\hat{B}$ leaves $F$ invariant and $A|_F = \hat{B}|_F$.

Proof. See Lemma 4.1 in [BDP].

**Lemma 3.10.** Let $E_1 \oplus E_2 \oplus E_3$ be a splitting of a linear cocycle. If $E_1$ is not dominated by $E_2 \oplus E_3$, then one of the following holds:

1. $E_1$ is not dominated by $E_2$.
2. $E_1/E_2$ is not dominated by $E_3/E_2$.

Proof. See Lemma 4.4 in [BDP].

We prepare some lemmas which enables us to reduce our problem to specific sets. For a homoclinic class $H(P)$, by per$_{+,\mathbb{R}}(H(P))$ we denote the set of the volume-expanding hyperbolic periodic points that have the differentials with distinct positive real eigenvalues and are homoclinically related to $P$.

**Lemma 3.11.** Given $f \in \text{Diff}^1(M)$ and a volume-expanding hyperbolic periodic point $P$, if $H(P)$ is non-trivial, one can find $g \in \text{Diff}^1(M)$ arbitrarily $C^1$-close to $f$ such that per$_{+,\mathbb{R}}(H(P, g))$ is dense in $H(P,g)$.

Proof. See Proposition 2.3 in [ABCDW] and Remark 4.17 in [BDP].

**Lemma 3.12.** Let $f \in \text{Diff}^1(M)$ and $P$ be a hyperbolic periodic point of $f$. Suppose $H(P)$ does not admit dominated splittings. Then any dense $f$-invariant subset $\Sigma \subset H(P)$ does not admit dominated splittings.

Proof. See Lemma 1.4 in [BDP].

Let us start from the proof of Lemma 3.2.

**Proof of Lemma 3.2.** By the quotations we prepared, large part of our proof is already finished. Let us see how to combine them to give the proof.

Suppose that $f$ and $P$ are given as in the hypothesis of Lemma 3.2. Lemma 3.11 implies there exists $f_1$ arbitrarily $C^1$-close to $f$ such that per$_{+,\mathbb{R}}(H(P, f_1))$ is dense in $H(P, f_1)$. Let us consider the periodic linear cocycle derives from $df_i$ on per$_{+,\mathbb{R}}(H(P, f_i))$. Since each periodic point has only real and positive distinct eigenvalues, we can find a splitting $E_1 \oplus E_2 \oplus E_3$ on this cocycle such that corresponding eigenvalues are in the increasing order at each point. Note that since $H(P, f_1)$ does not admit dominated splitting, by Lemma
$E_1$ is not dominated by $(E_2 \oplus E_3)$. Then Lemma \ref{lem:linear-cocycle} says that either $E_1$ is not dominated by $E_2$ or $E_1/E_2$ is not dominated by $E_3/E_2$. We show that we can create the periodic point we claimed in both cases.

Let us consider the first case, where $E_1$ is not dominated by $E_2$. We fix $\epsilon > 0$. Since $M$ is compact, the linear cocycle $df_1$ restricted to $E_1 \oplus E_2$ is bounded. By applying Lemma \ref{lem:bounded-cocycle} to this cocycle, we get a periodic point $Q \in \text{per}_+ \left(H(P,f_1)\right)$ and a path $\gamma(t)$ satisfying the conclusions in Lemma \ref{lem:bounded-cocycle}. We can assume $Q \neq P$ by letting $\epsilon$ sufficiently small. Since $\lambda_m(t) \leq \lambda_b(t) \leq \lambda_0(0) < 1$, there is no index bifurcation during the perturbation.

Then Lemma \ref{lem:bounded-cocycle} tells us there exists a path $\Gamma(t) \subset C(Q)$ such that $X = Q$, $Y = P$ and $\Gamma(t)$ satisfies all the hypotheses of Lemma \ref{lem:linear-cocycle}. Hence by applying Lemma \ref{lem:linear-cocycle} we get a $C^1$-diffeomorphism $f_2$ that is $\epsilon$-$C^1$-close to $f_1$ such that $P(f_2)$ and $Q(f_2)$ satisfy all the conditions we need. Since $f_1$ can be arbitrarily close to $f$ and $\epsilon$ can be taken arbitrarily small, the proof is completed in this case.

Let us see the case where $E_1/E_2$ is not dominated by $E_3/E_2$. Take $\epsilon > 0$. Since $df$ is bounded, the cocycle induced on the quotient bundle $E_1/E_2 \oplus E_3/E_2$ is also bounded. By applying Lemma \ref{lem:bounded-cocycle} to this cocycle, we obtain a periodic point $Q \in \text{per}_+ \left(H(P,f_1)\right)$, a path $\gamma(t)$ with $\text{diam}(\gamma(t)) < \epsilon$ such that they satisfy all conclusions of the Lemma \ref{lem:bounded-cocycle}. We claim that there is $t_0 \in (0,1)$ such that $\lambda_m(t_0)$ (not bigger eigenvalue of $\tilde{\gamma}(t_0)$) is equal to the eigenvalue of $\left(df^\text{per}(Q)\right)_{E_2(Q)}$. Let us denote the eigenvalues of $df^\text{per}(Q)\left(Q\right)$ for the $E_1$-direction by $\mu_i$ ($i = 1, 2, 3$). Since $Q$ is volume-expanding, $\mu_1 \mu_2 \mu_3 > 1$, and since $Q$ has index 1, $\mu_2 < 1$. Hence, we have $\mu_1 \mu_3 > \mu_1 \mu_2 \mu_3 > 1$.

Note that $\lambda_m(0) = \mu_1 < \mu_2$. When $t = 1$, by Lemma \ref{lem:bounded-cocycle} we have $\lambda_m(1) = \lambda_b(1)$. Since $\text{det}(\tilde{\gamma}(t))$ is independent of $t$ and $\mu_1 \mu_3 > 1$, we get $1 < \mu_1 \mu_3 = \lambda_m(1) \lambda_b(1) = (\lambda_m(1))^2$. Then we can see $\lambda_m(1) > 1$ because $\lambda_m(1)$ is positive. Finally, by $\lambda_m(0) = \mu_1 < \mu_2 < 1 < \lambda_m(1)$ and the continuity of $\gamma(t)$, there is $t_0 \in (0,1)$ such that $\lambda_m(t_0) = \mu_2$. Then, we redefine $\gamma(t)$ as follows: $\gamma(t)$ is equal to the original $\gamma(t)$ when $t \in [0,t_0]$, otherwise $\gamma(t) = \gamma(t_0)$. For this modified path $\gamma(t)$, we have $\lambda_b(t) \geq \lambda_b(t_0) = \mu_1 \mu_3 / \lambda_m(t_0) = \mu_1 \mu_3 / \mu_2 > 1$ for all $t$.

By Lemma \ref{lem:bounded-cocycle} we can find a path $\Gamma(t) \subset C(X)$ such that $X = Q$, $Y = P$ and $\Gamma(t)$ satisfy all the hypotheses of Lemma \ref{lem:linear-cocycle}. Applying Lemma \ref{lem:linear-cocycle} we get a $C^1$-diffeomorphism $f_2$ that is $\epsilon$-$C^1$-close to $f_1$ such that $Q(f_2)$ are homoclinically related to $P(f_2)$ and the differential $df_2^\text{per}(Q(f_2))(Q(f_2))$ restricted to the stable direction of $T_{Q(f_2)}M$ has the eigenvalue $\mu_2$ with multiplicity 2. Now Lemma \ref{lem:linear-cocycle} gives $f_3$ which is arbitrarily $C^1$-close to $f_2$ such that $P(f_3)$ and $Q(f_3)$ satisfy all the conditions we need.

Remark 2. We point out two mistakable arguments in the proof of second case.

1. The argument of Lemma \ref{lem:volume-expanding} is necessary. In general, $df_3^\text{per}(Q(f_3))(Q(f_3))$ restricted to the stable direction of $T_{Q(f_3)}M$ is not diagonalizable. We only know there are two eigenvectors one in $E_2$ and the other in $E_1/E_2$. These facts do not guarantee that we have two linearly independent eigenvectors in $E_1 \oplus E_2$.

2. One may wonder why we can get homoclinic relation between $P(f_3)$ and $Q(f_3)$ just by assuring the closeness of $f_2$ and $f_3$. It is because we fix the periodic points and never change till the end. On the contrary, for example, in the proof of Lemma \ref{lem:bounded-cocycle} the situation is more subtle because we need to change the periodic point we choose to decrease the size of the perturbation.

3.3 Proof of Lemma \ref{lem:linear-cocycle}

Here we give the proof of Lemma \ref{lem:linear-cocycle}. Before going into the detail, we see the naive idea of the proof. We start from a non-trivial homoclinic class $H(P)$ such that $\text{ind}(P) = 1$ and the differential restricted to the stable direction has two complex eigenvalues. First, by a small perturbation, we create a hyperbolic periodic point $Q$ which is homoclinically related to $P$ and the differential at $T_QM$ has positive and distinct eigenvalues (this step is carried out inside the proof of Lemma \ref{lem:volume-expanding}). Then we pick up a periodic point $R_n$ such that the differential to the stable direction is “mixed up” under the influence of dynamics around $P$. Now we perturb the diffeomorphism along the orbit of $R_n$ using the Franks’ lemma such that the restriction of differential at $R_n$ to the stable direction is the homothety. We pick up $R_n$ sufficiently close to $Q$ so that the resulted periodic point has the homoclinic relation with $Q$.

To demonstrate this naive idea rigorously and clearly, we need the techniques developed in \cite{BDP} about linear cocycles admitting transitions. We do not give the detailed review of \cite{BDP} here. Instead, we give
Remark 3. The existence of the transition from $\tilde{T}$ is close to a matrix which we want to create. Furthermore, if there exists $Y \in \Sigma$ such that $J(A_{\text{per}}(Y))$ is bigger than 1, then we can choose the point $Q$ and the perturbation $\tilde{T}$ in the lemma such that $J(A_{\text{per}}^\epsilon(Q)) > 1$.

Remark 3. The existence of the transition from $Q$ to itself tells us that given any matrix that is obtained as the product of some $\tilde{T}$ and some $\tilde{T}$, we can find a periodic orbit whose differential is close to that matrix. In the proof, we use the existence of transition to find the periodic point $R_n$ whose differential is close to a matrix which we want to create.

Let us begin the proof of Lemma 3.3.

Proof of Lemma 3.3. Given $f$ and $P$ as in the hypothesis of the Lemma 3.3, we fix $\varepsilon > 0$ and a point $P' \in W^s(P) \setminus W^u(P) \setminus \mathcal{O}(P)$ (we can take such $P'$ because $H(P)$ is non-trivial) and fix a neighborhood $V$ of $\mathcal{O}(P) \cup \mathcal{O}(P')$ satisfying the following: For every $g$ that is $\varepsilon$-close to $f$, if $Z$ is a periodic point such that $\mathcal{O}(Z, g)$ is contained in $V$, then $\text{ind}(Z) = \text{ind}(P)$ and $P$ and $Z$ are homoclinically related. We can take such $V$ because the set $\mathcal{O}(P) \cup \mathcal{O}(P')$ is uniformly hyperbolic.

Let us take a uniformly hyperbolic set $\Sigma_1$ contained in $V$ and contains $P$ and $P'$ such that $\Sigma_1$ admits transitions (in practice, $\Sigma_1$ is a generalized horseshoe containing $P$ and $P'$ as $\Sigma_1$). We fix a basis of tangent space of each point in $\Sigma_1$ and we identify the differential maps between tangent spaces with some matrices. For the detail, see section 1 of [BDP].

We denote the set of hyperbolic periodic points in $\Sigma_1$ by $\Sigma_2$. Let us apply Lemma 3.13 to the periodic linear cocycle $(\Sigma_2, f, TM|_{\Sigma_2}, df)$ with $\varepsilon_0 = \varepsilon/2$ and $\varepsilon_1 = \varepsilon/4$. Then we get a periodic point $Q \in \Sigma_2$ such that the following property holds for $df \in C(Q)$ (remember that $C(Q)$ denotes the set of cocycles on $Q$): $df$ is $\varepsilon_1$-close to a cocycle $\sigma \in C(Q)$ whose first return map $\sigma$ has only real, positive and distinct eigenvalues.

Note that we can choose $Q$ so that $\det(\sigma) > 1$ since $P$ is volume-expanding. We denote the eigenspaces of $\sigma$ by $E_1$, $E_2$ and $E_3$ so that the corresponding eigenvalues are in the increasing order. We also know the transition with matrix $T$ from $Q$ to itself has the property as is described in Lemma 3.13 more precisely, there exists a matrix $\tilde{T}$ that is $(\varepsilon_0 + \varepsilon_1)$-close to $T$ such that the following holds:

- $\tilde{T}(E_3) = E_3$.
- $\tilde{T}(E_1) = E_2$ and $\tilde{T}(E_2) = E_1$.

Let us consider a matrix given as follows:

$$D_n := \tilde{\sigma}^{2n} \circ \tilde{T} \circ \tilde{\sigma}^n \circ \tilde{T} \circ \tilde{\sigma}^n \circ \tilde{T} \circ \tilde{\sigma}^{2n} \circ \tilde{T}.$$
Then, since $Q$ admits the transition with matrix $T$, there exists a periodic point $R_n$ such that the cocycle $df \in C(R_n)$ is $\varepsilon_1 + (\varepsilon_0 + \varepsilon_1) = \varepsilon$-close to the cocycle $\tau$ such that $\tau$ is given by $D_n$.

Now, by applying the Franks’ lemma (see Lemma 3.4), we get a diffeomorphism $f_n$ that is $\varepsilon$-$C^1$-close to $f$ such that $R_n$ is a hyperbolic periodic point of $f_n$ and the differential $df_n^{per}(R_n)/(R_n)$ is equal to $D_n$.

We show that, for each $n$, the linear map $df_n^{per}(R_n)/(R_n)$ leaves $E_1 \oplus E_2$ invariant and acts as a homothetic endomorphism, leaves $E_3$ invariant and the eigenvalue of $E_3$ direction is positive. To see this, let us denote the eigenvalue of $\tilde{\sigma}$ for the $E_i$-direction by $\lambda_i$ ($i = 1, 2, 3$), where $\lambda_i$ is some positive real number and put $\tilde{T}(e_1) = \mu_1 e_1$, $\tilde{T}(e_2) = \mu_2 e_2$ and $\tilde{T}(e_3) = \mu_3 e_3$, where $\mu_1$, $\mu_2$ and $\mu_3$ are non-zero real numbers. Then direct calculations show that $df_n^{per}(R_n)/(R_n)(e_i) = (\lambda_1 \lambda_2) e_i$ for $i = 1, 2$ and $df_n^{per}(R_n)/(R_n)(e_3) = \mu_3 \lambda_3^{\alpha} e_3$. If $n$ is sufficiently large, then $\det(df_n^{per}(R_n))/(R_n))$ is greater than one since $\det \tilde{\sigma}$ is greater than one.

Hence we finished the proof.

Remark 4. The form of the matrix $D_n$ may look bizarre. Let us see why we need to pick up this matrix. We want to choose the matrix whose restriction is a homothety to the stable direction. For instance, the matrix $\tilde{\sigma} = \tilde{T} \circ \tilde{T}$ leaves $E_2$ invariant and acts as a homothetic endomorphism, leaves $E_3$ invariant and the eigenvalue of $E_3$ direction is positive. To see this, let us denote the eigenvalue of $\tilde{\sigma}$ for the $E_i$-direction by $\lambda_i$ ($i = 1, 2, 3$), where $\lambda_i$ is some positive real number and put $\tilde{T}(e_1) = \mu_1 e_1$, $\tilde{T}(e_2) = \mu_2 e_2$ and $\tilde{T}(e_3) = \mu_3 e_3$, where $\mu_1$, $\mu_2$ and $\mu_3$ are non-zero real numbers. Then direct calculations show that $df_n^{per}(R_n)/(R_n)(e_i) = (\lambda_1 \lambda_2) e_i$ for $i = 1, 2$ and $df_n^{per}(R_n)/(R_n)(e_3) = \mu_3 \lambda_3^{\alpha} e_3$. If $n$ is sufficiently large, then $\det(df_n^{per}(R_n))/(R_n))$ is greater than one since $\det \tilde{\sigma}$ is greater than one.

Hence we finished the proof.

\section{Proof of Lemma 3.7}

Finally, we give the proof of Lemma 3.7. The argument of the proof is similar to that of Proposition 3.1 in \cite{BDP}. In addition to the original proof, we need to check two things. The first one is that during the perturbation there is no index bifurcation. The second one is the perturbation is uniformly small.

We divide the proof of Lemma 3.7 into three steps. The first step (Lemma 3.14) tells us if there is a periodic point at which eigenspaces forms a small angle, then one can create complex eigenvalues by a small perturbation. In the second step (Lemma 3.15), we prove that if a periodic linear system does not admit dominated splittings, then one can construct a periodic point with a small angle by a small perturbation. Finally, we combine previous two techniques to prove Lemma 3.7.

We prepare some notations. Given two one-dimensional subspaces $V_1, V_2$ in a two-dimensional Euclidean space, the angle between $V_1, V_2$ is the unique real number $0 \leq \alpha \leq \pi/2$ that satisfies $\cos \alpha = |(v_1, v_2)|/(|v_1||v_2|)$, where $v_i$ is any non-zero vector in $V_i$ ($i = 1, 2$), $(\cdot, \cdot)$ is the inner product and $|\cdot|$ is the norm derives from the inner product.

Let us state the first and the second steps.

\textbf{Lemma 3.14.} For any $K > 0$ and $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon, K) > 0$ such that the following holds: Let $(\Sigma, f, E, A)$ a two-dimensional diagonalizable periodic linear cocycle with real, positive distinct eigenvalues, bounded by $K$. Suppose there is a point $X$ at which the angles between two eigenspaces are less than $\alpha$. Then there exists a path $\{\gamma(t) \mid 0 \leq t \leq 1\}$ in $C(X)$ such that the following conditions hold:

1. $\gamma(0) = A|_{O(X)}$.
2. $\text{diam}(\gamma(t)) < \varepsilon$.
3. $\tilde{\gamma}(t)$ is independent of $t$.
4. $\lambda_m(t) \leq \lambda_0(t)$ be the absolute value of the eigenvalues of $\tilde{\gamma}(t)$. Then for $s < t$ we have $\lambda_m(s) \leq \lambda_m(t)$ and $\lambda_0(s) \geq \lambda_0(t)$.
5. $\tilde{\gamma}(t)$ has two complex eigenvalues.

\textbf{Lemma 3.15.} For any $K > 0$, $\varepsilon > 0$ and $\alpha > 0$, the following holds: Let $(\Sigma, f, E, A)$ a two-dimensional diagonalizable periodic linear cocycle with real positive distinct eigenvalues, bounded by $K$ and does not admit dominated splittings. Then there exists a hyperbolic periodic point $X \in \Sigma$ and a path $\{\gamma(t) \mid 0 \leq t \leq 1\}$ in $C(X)$ such that the following conditions hold:

1. $\gamma(0) = A|_{O(X)}$.
2. \( \text{diam}(\gamma(t)) < \varepsilon. \)

3. \( \det \gamma(t) \) is independent of \( t. \)

4. Let \( \lambda_m(t) \leq \lambda_0(t) \) be the absolute values of the eigenvalues of \( \gamma(t). \) Then for \( s < t \) we have \( \lambda_m(s) \leq \lambda_m(t) \) and \( \lambda_0(s) \geq \lambda_0(t). \)

5. \( \gamma(1) \) has two eigenspaces with angle less than \( \alpha. \)

We need an auxiliary lemma about the effect of a perturbation on the constant of the boundedness of a cocycle.

**Lemma 3.16.** Let \( \sigma \) be a cocycle in \( C(X) \) where \( X \) is a periodic point, and let \( \gamma(t) \) be a path with \( \text{diam}(\gamma(t)) < \varepsilon. \) Then \( \gamma(1) \) is bounded by \( \varepsilon + \| \sigma \|. \)

**Proof.** For every unit vector \( v \) in some \( T_{f^n(X)}M, \) we have

\[
\| (\gamma(1)(f^n(X)))(v) \| \\
\leq \| (\gamma(1)(f^n(X)))(v) - (\gamma(0)(f^n(X)))(v) \| + \| (\gamma(0)(f^n(X)))(v) \| \\
\leq \varepsilon + \| \sigma \|.
\]

We can prove a similar inequality for the inverse of \( \gamma(1). \) \qed

We give the proof of Lemma 3.17 using these three lemmas.

**Proof of Lemma 3.17.** Let \( (\Sigma, f, E, A) \) be a two-dimensional diagonalizable periodic linear cocycle with positive distinct eigenvalues, bounded by \( K \) and suppose that it does not admit dominated splittings.

First, let us assume the angles between two eigenspaces are not bounded below, more precisely, there exists a sequence of periodic points \( (X_n) \) such that the sequence of the angles \( (\alpha_n) \), where \( \alpha_n \) is the angle of the eigenspaces of \( A^{\text{per}}(X_n)/(X_n), \) converges to zero. Fix \( \varepsilon > 0 \) and the constant \( \alpha_0 = \alpha(K, \varepsilon) \) in Lemma 3.14. Since \( \alpha_n \to 0 \) as \( n \to \infty, \) we can take \( n_0 \) such that \( \alpha_{n_0} < \alpha_0. \) Then Lemma 3.14 enable us to find a path \( \gamma(t) \) in \( C(X) \) as we claimed.

Hence, we can assume that the angles between two eigenspaces point are uniformly bounded at any point. By taking appropriate basis at each point, we can assume that the corresponding eigenspaces are orthogonal at each periodic point. Fix \( \varepsilon > 0 \) and the constant \( \alpha_1 = \alpha(K + \varepsilon/2, \varepsilon/2) \) in Lemma 3.14. Applying Lemma 3.15 to \( (\Sigma, f, E, A) \) with constants \( K, \varepsilon/2, \) and \( \alpha_1, \) we get a periodic point \( X \in \Sigma \) and a path \( \gamma_1(t) \) in \( C(X) \) satisfying all the properties in the conclusion of Lemma 3.14 for \( \gamma_1(t), \varepsilon/2 \) and \( \alpha_1. \) Then Lemma 3.14 implies that the cocycle \( \gamma_1(1) \) is bounded by \( K + \varepsilon/2. \) Since the angle between eigenspaces of \( \gamma_1(1) \) is less than \( \alpha_1, \) we can apply Lemma 3.14 (replacing \( K, \alpha \) and \( \varepsilon \) with \( K + \varepsilon/2, \alpha_1 \) and \( \varepsilon/2 \) respectively) in order to obtain a path \( \gamma_2(t). \)

Now we construct a continuous path \( \gamma(t) \) as follows: If \( 0 \leq t \leq 1/2 \) then \( \gamma(t) = \gamma_1(2t). \) If \( 1/2 \leq t \leq 1 \) then \( \gamma(t) = \gamma_2(2t - 1). \) We have \( \text{diam}(\gamma(t)) \) is less than \( \varepsilon \) because \( \text{diam}(\gamma_1(t)), \text{diam}(\gamma_2(t)) \) are less than \( \varepsilon/2. \) Thus we constructed the path we claimed and the proof is completed. \qed

Let us give the proof of Lemma 3.14. For the proof, we need an elementary lemma.

**Lemma 3.17.** Let \( R(x) \) denote the rotation matrix of angle \( x \) on a two-dimensional Euclidian space. Then there exists a positive constant \( C \) such that the following inequality holds:

\[
\| R(s) - R(t) \| \leq C |s - t| \text{ for all } -\pi/4 \leq s, t \leq \pi/4.
\]

**Proof.** We introduce a norm \( \| \cdot \|_2 \) on the space of linear maps as follows. Fix an orthonormal basis of the Euclidean space and for a linear map \( A, \) define

\[
\| A \|_2 = \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_2 = \sqrt{a^2 + b^2 + c^2 + d^2}.
\]
where the matrix denotes the matrix representation of $A$ with respect to the basis. We fix a constant $N$ such that for every $A$ the inequality $\|A\| \leq N\|A\|_2$ holds. Then,

$$\|R(s) - R(t)\|^2 \leq N^2\|R(s) - R(t)\|_2^2 = 2N^2((\cos(s) - \cos(t))^2 + (\sin(s) - \sin(t))^2).$$

Since $-\pi/4 \leq s, t \leq \pi/4$, we get

$$|\cos(s) - \cos(t)| \leq |s - t|/\sqrt{2}, \quad |\sin(s) - \sin(t)| \leq |s - t|.$$

So we have

$$2N^2((\cos(s) - \cos(t))^2 + (\sin(s) - \sin(t))^2) \leq 3N^2|s - t|^2.$$

Hence for $C = \sqrt{3N}$ we have the inequality.

\[\text{Proof of Lemma 3.14.}\] Let $X$ be a point at which the angle of corresponding eigenspaces is less than $\alpha$. By taking an orthonormal basis, we can take a matrix representation of $A^{\per(X)}(X)$ with

$$A^{\per(X)}(X) = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 < \lambda_2$ is positive real numbers and $\mu$ is some non-zero real number. By a direct calculation, we get $(\lambda_2 - \lambda_1)/|\mu| < \tan(\alpha)$ (left hand side is the tangent of the angle between two eigenspaces at $X$).

We fix $\varepsilon > 0$ and put $\alpha = \alpha(K, \varepsilon) := \varepsilon/(CK)$, where $C$ is the constant given in Lemma 3.17. Let us define a path $\gamma(t)$ as follows:

$$[\gamma(t)](f^i(X)) = \begin{cases} A|f^i(X), & \text{if } f^i(X) \neq f^{\per(X)-1}(X), \\
R(-\text{sign}(\mu)\alpha t)A|f^i(X), & \text{if } f^i(X) = f^{\per(X)-1}(X), \end{cases}$$

where $\text{sign}(\mu)$ is the signature of $\mu$.

Let us see this path enjoys all the conditions we claimed in Lemma 3.14. In the following, we only treat the case when $\mu > 0$. The proof for the case $\mu < 0$ can be done similar way. We first examine the diameter of this path. To see this, we only need to check the distance of $\gamma(t)$. Let us check the items above. First, observe that

$$\theta(t) = \sqrt{(\lambda_1 + \lambda_2)^2 + \mu^2 \sin(\beta - \alpha)},$$

where $\theta(t) = \theta(\pm \sqrt{\theta^2 - 4d})/2$ when $\theta > 0$. Therefore if $\theta$ is monotone decreasing, so is $\lambda_0(t)$. Moreover, it implies that $\lambda_m(t)$ is monotone increasing, since $\lambda_m(t)\lambda_0(t)$ is a positive constant.
where $0 < \beta < \pi/2$ is a real number satisfying $\tan \beta = (\lambda_1 + \lambda_2)/\mu$. This shows that $\beta > \alpha$. Thus $\theta(t)$ is positive and monotone decreasing.

Finally, we see $\theta^2 - 4d < 0$. When $t = 1$, we obtain that
\[
\theta(1) = \frac{\mu(\lambda_1 + \lambda_2) - \mu(\lambda_2 - \lambda_1)}{\sqrt{\lambda_2 - \lambda_1} + \mu} = \frac{2\lambda_1}{\sqrt{1 + \left(\frac{\lambda_2 - \lambda_1}{\mu}\right)^2}} < 2\sqrt{\lambda_1^2} < 2\sqrt{\lambda_1\lambda_2} = 2d.
\]

So, we finished the proof. \hfill \Box

Finally, let us give the proof of the Lemma 3.15.

**Proof of Lemma 3.15.** The proof of Lemma 3.15 is just the repetition of Lemma 3.4 in [BDP]. We can easily check the behavior of the eigenvalues during the perturbation. So the proof is left to the reader. \hfill \Box

### 4 Bifurcation of degenerate tangency

In this section we prove Proposition 2.

#### 4.1 Strategy of the proof of Proposition 2

We divide the proof into two steps. The first step is to reduce the dynamics into affine dynamics. The second step is to investigate the bifurcation of the dynamics.

To state our proof clearly, let us give the following definition.

**Definition 4.1.** Let $f \in \text{Diff}^{1}(M)$ with $\dim M = 3$ and $X$ be a hyperbolic index-1 fixed point with a homoclinic tangency and eigenvalues of $df(X)$ are positive and distinct. A one-parameter family of the $C^1$-diffeomorphism $(f_t)_{|t|<\delta} \subset \text{Diff}^{1}(M)$ is said to be an affine unfolding of the degenerate tangency of $f$ with respect to $X$ if the following holds (see Figure 1):

\[(DT1) \quad f_0 = f.\]

\[(DT2) \quad \text{There exists a coordinate chart } \phi : U \to \mathbb{R}^3 \text{ around } X \text{ such that } \phi(U) = (-1,1)^3 \text{ and } \phi(X) \text{ is the origin of } \mathbb{R}^3.\]

\[(DT3) \quad \text{We put } F_t := \phi \circ f_t \circ \phi^{-1}. \text{ For } (x,y,z) \in (-1,1) \times (-1,1) \times (-\mu^{-1},\mu), \text{ we have } F_t(x,y,z) = (\lambda x, \lambda y, \lambda z), \text{ where } 0 < \lambda < \lambda_1 < 1 < \mu. \text{ We also require } X, \lambda, \lambda_1 \text{ and } \mu \text{ are independent of } t. \text{ Thus locally } z\text{-axis is the unstable manifold of } \phi(X) \text{ and } xy\text{-plane is the stable manifold of } \phi(X).\]

\[(DT4) \quad \text{There exist two points } P, Q \in U \text{ with } \phi(P) = (0,0,p) \text{ and } \phi(Q) = (0,q,0), \text{ where } 0 < p,q < 1, \text{ such that the following holds: For some positive integer } N \geq 2, f_0^N(P) = Q \text{ and } f_t^N(P) \not\in U \text{ for } 0 < i < N. \text{ This means } P, Q \in W^s(X) \cap W^u(X).\]

\[(DT5) \quad \text{By the abuse of notation, we denote } \phi \circ f_t^N \circ \phi^{-1} \text{ by } F_t^N. \text{ Then, there exists a small neighborhood } W_0 \text{ of } P \text{ with } \phi(W_0) = [-\varepsilon,\varepsilon] \times [-\varepsilon,\varepsilon] \times [p-\varepsilon,p+\varepsilon] \text{ and } F_t^N(W_0) \subset V \text{ such that for every } (x,y,z+p) \in W_0, F_t^N(x,y,z+p) = (a_1,by+q,cx+t), \text{ where } a_1, b \text{ and } c \text{ are non-zero real numbers. We put } W_1 := F_0^N(W_0) \text{ and call } F_t^N \text{ return map.}\]

Our first step is stated as follows:

**Proposition 5.** If $X$ has a homothetic tangency, then one can find a diffeomorphism $g$ $C^1$-arbitrarily close to $f$ such that there exists a one-parameter family of $C^1$-diffeomorphisms $(g_t)_{|t|<\delta}$ that is an affine unfolding of the degenerate tangency of $g$ with respect to $X(g)$.

The second step is the following:

**Proposition 6.** Let $X$ be a volume-expanding hyperbolic periodic point with an affine unfolding of the degenerate tangency $(f_t)_{|t|<\delta}$ with respect to $X$. Then, for arbitrarily small $\varepsilon > 0$, there exists $0 < \tau \leq \varepsilon$ such that $f_\tau$ has an index-two periodic point $Y$ and there exists a heterodimensional cycle associated to $X(f_\tau)$ and $Y$.

It is clear that these two propositions imply Proposition 2.
4.2 Proof of Proposition 5

We give the proof of Proposition 5. For the proof, we prepare two lemmas.

The first one is a variant of the Franks’ lemma. We omit the proof, since it is easily obtained from the proof of the Franks’ lemma.

Lemma 4.2 (Local linearization). Let \( f \in \text{Diff}^1(M) \) with \( \dim M = m \). Consider \( x \in M \) and a coordinate neighborhoods \( \phi : U \to \mathbb{R}^m \) and \( \psi : V \to \mathbb{R}^m \) of \( x \) and \( f(x) \) respectively such that \( \phi(x) = 0 \) and \( \psi(f(x)) = 0 \) are the origin of \( \mathbb{R}^m \). Then, for any \( \varepsilon > 0 \) and any neighborhood \( U' \) of \( x \), there exist a neighborhood \( U \) of \( x \) contained in \( U' \) and \( \tilde{f} \in \text{Diff}^1(M) \) such that \( \tilde{f} \) is \( \varepsilon \)-\( C^1 \)-close to \( f \), \( \tilde{f} \) coincides with \( f \) on \( M \setminus U' \) and the map \( (\psi \circ \tilde{f} \circ \phi^{-1}) \) coincides with a linear map given by \( d(\psi \circ f \circ \phi^{-1})(0) \) on \( U' \) (where \( 0 \) denotes the origin of \( \mathbb{R}^m \)).

The following lemma is a version of Gourmelon’s Franks’ lemma:

Lemma 4.3 (Lemma 4.1 in \([\text{Gou1}]\)). Let \( f \in \text{Diff}^1(M) \) with \( \dim M = m \). Consider a periodic point \( x \in M \) and a coordinate neighborhood \( \phi : U \to \mathbb{R}^m \) of \( x \) with \( \phi(x) = 0 \). Then, for any \( \varepsilon > 0 \) and any neighborhood \( V \subseteq U \) of \( x \), one can find a \( C^1 \)-diffeomorphism \( \tilde{f} \) that is \( \varepsilon \)-\( C^1 \)-close to \( f \) and a small neighborhood \( \tilde{V} \subset V \) such that \( f(x) = \tilde{f}(x) \) for any \( x \in M \setminus V \), the restriction of \( (\phi \circ f \circ \phi^{-1}) \) to \( \tilde{V} \) is equal to the linear map given by \( d(\phi \circ f \circ \phi^{-1})(0) \) and \( \tilde{f} \) locally preserves the invariant manifolds of \( x \) outside \( \tilde{V} \) (see Lemma 3.9 for definition).

Let us begin the proof of Proposition 5.

Proof of Proposition 5 Let \( f \) be given as is in the hypothesis of Proposition 5. In the following, we assume that \( X \) is a fixed point. The general cases can be reduced to this case by considering the power of \( f \).

First, by using Lemma 4.3 in a sufficiently small neighborhood and taking appropriate coordinate neighborhood, we can take \( f_1 \) that has the following properties:

- There exists a (smooth) coordinate neighborhood \( \phi : U \to \mathbb{R}^3 \) around \( X \) such that \( \phi(U) = (-1,1)^3 \)
- We put \( F_1 := \phi \circ f_1 \circ \phi^{-1} \). For \((x,y,z) \in (-1,1) \times (-1,1) \times (-\mu^{-1},\mu^{-1}) \), \( F_1(x,y,z) = (\lambda x, \lambda y, \mu z) \), where \( 0 < \lambda < 1 < \mu \).
- There exist two points \( P,Q \in U \) with \( \phi(P) = (0,0,p) \) and \( \phi(Q) = (0,0,q) \), where \( 0 < p,q < 1 \) such that following holds: For some positive integer \( N \geq 2 \), \( f_i^N(Q) = P \) and \( f_i^I(P) \notin U \) for \( 0 < i < N \).
- \( W^s(X) \) and \( W^u(X) \) are tangent at \( Q \), in particular, \( T_QW^u(X) \) is contained in \( T_QW^s(X) \).
Let us give a perturbation so that the point of tangency is on the strong stable manifold of $X$. First, we take an interval $J := [a, b]$ that is contained in $(\mu^{-2}p, \mu^{-1}p)$ such that $J$ and the set $(\lambda^n q)_{n\geq 0}$ have the empty intersection. Let $\rho(t)$ be a $C^\infty$-function on $\mathbb{R}$ such that the following holds:

- $\rho(t) = 1$ if $|t| < a$.
- $\rho(t) = 0$ if $|t| > b$.

We modify $F_1$ to $F_2$ as follows:

$$F_2(x, y, z) := (1 - R(X))F_1(X) + R(X)(\lambda x, \tilde{\lambda} y, \mu z),$$

where $R(x, y, z) := \rho(x)\rho(y)\rho(z)$ and $\tilde{\lambda}$ is a real number satisfying $0 < \tilde{\lambda} < \lambda < 1$ and sufficiently close to $\lambda$.

Let us define the map $f_2$ as follows: $f_2(x) = f_1(x)$ when $x \notin U$. Otherwise $f_2(x) = \phi \circ F_2 \circ \phi^{-1}(x)$. If we take $\tilde{\lambda}$ sufficiently close to $\lambda$, then $f_2$ is a diffeomorphism of $M$. Note that $P$ and $Q$ are still the points of tangency of the invariant manifolds of $X$, $P$ is on the strong stable manifold of $X$ and $f_2$ converges to $f_1$ when $\tilde{\lambda} \to \lambda$ in the $C^1$-topology. We fix $\tilde{\lambda}$ that is very close to $\lambda$ and give more perturbations.

Throughout the proof, we often change the coordinate in the following way: Given a real number $x$, for each $0 < \mu < 1$ such that the support of perturbation $\rho$ is equal to zero if and only if the corresponding $N_x$ is contained in $(x, \mu x)$ be a $C^\infty$-function on $\mathbb{R}$ such that the smallest number that satisfies $\rho(\mu x) = \rho(x)$ for any $\mu > 0$.

Let us consider the differential of the return map. Given a diffeomorphism $f$ and a coordinate chart $\phi$, we have the differential of the return map $dF^N(P)$. If we take the renormalization, the differential of the return map is given by $L^{n_e}dF^N(P)L^{n_r}$, where $L$ is a matrix given as follows:

$$L := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$  

Note that from this calculation we can see a component of $dF^N$ is equal to zero if and only if the corresponding component of the differential of the return of the renormalized diffeomorphism is equal to zero.

Let us resume the proof. By taking appropriate renormalization, we can assume $(DT2), (DT3), (DT4)$ hold for $f_2$. Let us consider the differential of the return map $F_2^N$. Since $W^s(X)$ and $W^u(X)$ are tangent at $Q$, we can put

$$dF_2^N(Q) = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & 0 \end{pmatrix}. $$

By applying Lemma 3.3 at $P$, we can find a diffeomorphism $f_3$ arbitrarily close to $f_2$ such that the $c, f, g$ are not equal to zero and $(3, 3)$-component remains to be zero. Note that the use of the Franks’ lemma does not disturb the condition that $P, Q$ are contained in $W^s(X) \cap W^u(X)$ (by letting the support of perturbation sufficiently small). Let us take sufficiently large $t$ and consider the differential $dF_2^N = dF_3^N(Q)$. This is equal to the following:

$$\begin{pmatrix} \lambda^t & 0 & 0 \\ 0 & \tilde{\lambda}^t & 0 \\ 0 & 0 & \mu^t \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & 0 \end{pmatrix} = \begin{pmatrix} \lambda^t a & \lambda^t d & \lambda^t g \\ \lambda^t b & \lambda^t e & \lambda^t h \\ \mu^t c & \mu^t f & 0 \end{pmatrix}. $$

By applying Lemma 3.3 at $f_3^{-1}(Q)$, we perturb $f_3$ to $f_4$ to make the differential $dF_4(f^{-1}(P))$ into the following form:

$$\begin{pmatrix} 1 & 0 & z_l \\ x_l & 1 & y_l \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix}, $$

where

$$x_l := -\tilde{\lambda}/\lambda^t(h/g), \quad y_l := -\tilde{\lambda}/\mu^t(ah/cg) - (\tilde{\lambda}/\mu)^t(b/c), \quad z_l := -\lambda/\mu^t(a/c).$$
Then, by a direct calculation we have $(1, 1)$, $(2, 1)$, $(2, 3)$, and $(3, 3)$-component of $dF_4^N(P)$ is equal to zero. Note that the support of perturbation can be taken arbitrarily small, and the distance between $f_3$ and $f_4$ can be arbitrarily small if we take large $l$, since $x_1,y_1,z_1 \to 0$ when $l \to \infty$.

Let us proceed our perturbation. By taking renormalization, we can put

$$dF_4^N(P) := \begin{pmatrix} 0 & b & e \\ 0 & c & 0 \\ a & d & 0 \end{pmatrix},$$

where $a, b, c, d$ are not necessarily equal to the corresponding numbers appeared in $dF_3$. Since $f_4$ is a diffeomorphism, we know $a, c, e \neq 0$.

Let us take a large integer $l$ and consider the differential $dF_4^N \circ dF_4^l(f_4^{-l}(P))$. This matrix is given as follows:

$$\begin{pmatrix} 0 & b & e \\ 0 & c & 0 \\ a & d & 0 \end{pmatrix} \begin{pmatrix} \lambda^l & 0 & 0 \\ 0 & \tilde{\lambda}^l & 0 \\ 0 & 0 & \mu^l \end{pmatrix} \begin{pmatrix} 0 & b\tilde{\lambda}^l & e\mu^l \\ 0 & c\tilde{\lambda}^l & 0 \\ a\lambda^l & d\lambda^l & 0 \end{pmatrix}.$$

Again by using Lemma 3.3 at $f_4^{-l}(P)$, we perturb $f_4$ to $f_5$ to change the differential $dF_5(f_4^{-l}(P))$ into

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & x_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y \end{pmatrix},$$

where $x_i := -(\tilde{\lambda}/\lambda)^l(d/a)$ and $y := -(\tilde{\lambda}/\mu)^l(b/e)$. Then a direct calculation shows the non-zero components of $dF_5^N(f_5^{-l}(P))$ are only $(1, 3), (2, 2), (3, 1)$.

Then, by taking renormalization, we can put

$$dF_5^N(P) := \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

Now, using Lemma 4.2 at $F_6(N)$, we can construct $F_6$ such that $F_6(N)$ is locally affine map around $P$ such that its differential coincides with $dF_5^N(P)$.

Let us review the properties of $f_6$.

- There exists a coordinate chart $\phi : U \to \mathbb{R}^3$ around $X$ such that $\phi(U) = (-1, 1)^3$ and $\phi(X)$ is the origin of $\mathbb{R}^3$.
- Let us put $F_0 := \phi \circ f_6 \circ \phi^{-1}$. For $(x,y,z) \in (-1,1) \times (-1,1) \times (-\mu^{-1}, \mu^{-1})$, $F_0(x,y,z) = (\lambda x, \tilde{\lambda} y, \mu z)$, where $0 < \lambda < \lambda < 1 < \mu$.
- There exist two points $P, Q \in U$ with $\phi(P) = (0,0,p)$ and $\phi(Q) = (0,0,q)$, where $0 < p, q < 1$ such that following holds: For some positive integer $N \geq 2$, $f_6^N(Q) = P$ and $f_6^N(P) \not\in U$ for $0 < i < N$.
- There exists a small neighborhood $W_0$ of $P$ with $\phi(W_0) = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times [p-\varepsilon, p+\varepsilon]$ and $F_6^N(W_0) \subset \phi(U)$ such that for every $(x,y,z+p) \in W_0$, $F_6^N(x,y,z+p) = (ar+bq+q, cx)$, where $a,b,c$ are non-zero real numbers.

Let $\rho_2(s)$ be a $C^\infty$-function on $\mathbb{R}$ satisfying the following properties: $\rho_2(s) = 0$ if $|s| \leq \varepsilon/3$ and $\rho_2(s) = 1$ if $|s| > 2\varepsilon/3$, where $\varepsilon$ is a positive real number. Then define $R : V \to \mathbb{R}$ by $R(x,y,z) := \rho_2(x)p_2(y)p_2(z-p)$, and construct a one-parameter family of the diffeomorphisms $f_{7,t}$ as follows: For $X \in W_0$, $f_{7,t}(x,y,z) = F_0(x + tR(x,y,z), y, z)$. Otherwise, $f_{7,t}(X) = f_6(X)$. Note that for $t$ sufficiently close to $0$, $f_{7,t}$ is a diffeomorphism of $M$.

Then, by taking $W_0$ sufficiently small if necessary, we can see that $(f_{7,t})_{|t|<\delta}, P, Q, \phi, W_0$ satisfy $(DT1)-(DT5)$ for small $\delta$ and $f_7$ can be taken arbitrarily close to $f$. Hence the proof is completed. \qed
Remark 5. The geometric idea behind the perturbations from $f_2$ to $f_5$ is simple. Let us see that.

Since $Q$ is the point of tangency, the differential of $df^N(P)$ sends the tangent vector $v \in T_{P}(\phi(V))$ in the direction of $z$-axis parallel to $xy$-plane. By a small perturbation, we can assume $df^N(P)(v)$ is not parallel to $y$-direction ($f_1$ to $f_2$).

Let us consider the differential $dF^{N+i}_2(P)$. Take the two-dimensional plane passing $P$ and parallel to the $xz$-plane, and let us consider its image under $dF^{N+i}_2(P)$. If $l$ is sufficiently large, under some generic assumption ($f_2$ to $f_3$), this image turns to be almost parallel to the $xz$-plane. Thus, by a small perturbation, we can assume that the image is parallel to $xz$-plane ($f_3$ to $f_4$).

By a similar argument for $f_4^{-1}$, we see that $y$ direction turns to be almost invariant under long time transition. Hence by giving small perturbation, we can find a diffeomorphisms that preserves the $y$-direction ($f_4$ to $f_5$).

### 4.3 Proof of Proposition 6

Finally, let us give the proof of Proposition 6.

**Proof of Proposition 6** Put $I_n := [-\varepsilon, \varepsilon] \times [q - \varepsilon, q + \varepsilon] \times [(p - \varepsilon)/\mu^n, (p + \varepsilon)/\mu^n]$ and $t_n = p/\mu^n$. For $n$ sufficiently large, we show that $f_{t_n}$ has a hyperbolic periodic point $R_n \in I_n$, with period $n + N$ such that $f_{t_n}$ has a heterodimensional cycle associated to $X$ and $R_n$.

Let us take a point $A \in I_n$ and put $A = (x, y, z)$. Then $F^n(A) = (\lambda^n x, \bar{\lambda}^n y, \mu^n z)$. Note that $F^n(A)$ belongs to $W_0$ if $n$ is sufficiently large. In the following, we assume this condition holds. By the definition of the return map, we can see $F_{n+N}(A) = (c(\mu^n \lambda - p), \bar{\lambda}^n y + q, a\lambda^n x + t)$. Suppose that this is a periodic point of period $n + N$ for $t = t_n$. Then the following equalities hold:

$$c(\mu^n \lambda - p) = x, \quad \bar{\lambda}^n y + q = y, \quad a\lambda^n x + p/\mu^n = z.$$  

By a direct calculation, we get

$$x = 0, \quad y = q/(1 - \bar{\lambda}^n), \quad z = p/\mu^n.$$  

We put $y_n := q/(1 - \bar{\lambda}^n)$ and $z_n := p/\mu^n$. Note that if we take sufficiently large $n$, the point $R_n := (0, y_n, z_n)$ is in $I_n$, since $R_n \to Q$ when $n \to +\infty$. Thus $R_n$ is indeed a periodic point of period $n + N$ for $n$ sufficiently large.

Let us check that $R_n$ is a hyperbolic periodic point of index 2. The derivative of $f^n_{t_n}$ at $R_n$ is given by the matrix below:

$$\begin{pmatrix}
\lambda^n & 0 & 0 \\
0 & \bar{\lambda}^n & 0 \\
0 & 0 & \mu^n
\end{pmatrix}.$$  

The derivative of $f^n_{t_n}$ at $f^n_{t_n}(R_n)$ is given as follows:

$$\begin{pmatrix}
0 & 0 & c \\
0 & b & 0 \\
a & 0 & 0
\end{pmatrix}.$$  

Hence the Jacobian at $R_n$ is equal to

$$\begin{pmatrix}
\lambda^n & 0 & 0 \\
0 & \bar{\lambda}^n & 0 \\
0 & 0 & \mu^n
\end{pmatrix} \begin{pmatrix}
0 & 0 & c \\
0 & b & 0 \\
a & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \lambda^n c \\
0 & \bar{\lambda}^n b & 0 \\
a\mu^n & 0 & 0
\end{pmatrix}.$$  

Now, a direct calculation shows that the eigenvalues of this matrix are given by $\bar{\lambda}^n b$ and $\pm \sqrt{ac\lambda^n \mu^n}$. The absolute value of $\lambda^n b$ is less than one when $n$ is sufficiently large, and the absolute value of $\pm \sqrt{ac\lambda^n \mu^n}$ are greater than one when $n$ is sufficiently large, since $|\lambda| < 1$ and $|\mu\lambda| > 1$ (the second inequality is the consequence of the fact that $X$ is volume expanding).

Let us check that $f_{t_n}$ has a heterodimensional cycle associated to $X$ and $R_n$ (see figure 2). First, we show $W^u(X)$ and $W^s(R_n)$ have non-empty intersection. It is easy to see that $W^u(X)$ contains $z$-axis in
\(\phi(U)\). Hence we only need to check that \(W^s(R_n)\) has an intersection with \(z\)-axis. To see this, we focus on the segment
\[
\ell_n := \{(0, \lambda^n y_n + s, \mu^n z_n) \mid |s| \leq 2\tilde{\lambda}^n y_n\}.
\]
The segment \(\ell_n\) passes \(f^n_{t_n}(R_n)\) and \(z\)-axis. We can see that \(\ell_n\) is contained in \(W_0\) if \(n\) is sufficiently large, since \(f^n_{t_n}(R_n) \to P\) and the length of \(\ell_n\) converges to zero when \(n \to +\infty\). The image of this segment under \(f^N_{t_n}\) is given as follows:
\[
f^N_{t_n}(\ell_n) = \{(0, y_n + s, z_n) \mid |s| \leq 2|b|\tilde{\lambda}^n y_n\}.
\]
So the image of \(\ell_n\) under \(f^{n+k}_{t_n}\) is given as follows:
\[
f^{n+k}_{t_n}(\ell_n) = \{(0, \tilde{\lambda}^n y_n + s, \mu^n z_n) \mid |s| \leq (2|b|\tilde{\lambda}^n)\tilde{\lambda}^n y_n\}.
\]
Thus the restriction of \(f^{n+k}_{t_n}\) to \(\ell_n\) is well defined and \(\ell_n\) is uniformly contracted with the factor \(|b|\tilde{\lambda}^n\). This shows that for sufficiently large \(n\), \(\ell_n\) is contained in the stable manifold of \(R_n\).

Second, let us check that the two invariant manifolds \(W^s(X)\) and \(W^u(R_n)\) have non-empty intersection. As is in the previous case, we only need to check that \(W^u(R_n)\) has an intersection with \(xy\)-plane in \(U\). To see this, we focus on the segment
\[
\pi_n := \{(0, y_n, z_n + s) \mid |s| \leq 2z_n\}.
\]
This is a segment that passes \(R_n\) and have non-empty intersection with \(W^s(X)\). We show that for sufficiently large \(n\), \(\pi_n\) is contained in \(W^u(R_n)\). To see that, let us calculate the inverse image of \(\pi_n\) under \(f_{t_n}^{-2(n+N)}\).

The inverse image of \(\pi_n\) under \(f_{t_n}^{-N}\) is given as follows:
\[
f_{t_n}^{-N}(\pi_n) = \{(s, \tilde{\lambda}^n y_n, \mu^n z_n) \mid |s| \leq 2z_n/c\}.
\]
This set is contained in \(W_0\) if \(n\) is sufficiently large. The inverse image of \(f_{t_n}^{-N}(\pi_n)\) under \(f^{-n}\) is given as follows:
\[
f_{t_n}^{-n-N}(\pi_n) = \{(s, y_n, z_n) \mid |s| \leq 2z_n/(|c|\lambda^n)\}.
\]
Since \(2z_n/(|c|\lambda^n) = 2p/(|c|\mu^n\lambda^n)\) and \(\mu\lambda > 1\), this segment is also contained \(W_1\) when \(n\) is sufficiently large.

The inverse image of \(f_{t_n}^{-n-N}(\pi_n)\) under \(f^{-N}\) is given as follows:
\[
f_{t_n}^{-n-2N}(\pi_n) = \{(0, \tilde{\lambda}^n y_n, \mu^n z_n + s) \mid |s| \leq 2z_n/(|ac|\lambda^n)\}.
\]
If \(n\) is sufficiently large, this segment is also contained in \(W_0\).
Finally, the inverse image of \( f_{-n}^{-2N}(\pi_n) \) under \( f^{-n} \) is given as follows.

\[
f_{-n}^{-2N}(\pi_n) = \{(0, y_n, z_n + s) \mid |s| \leq 2z_n/(|ac|\mu^n\lambda^n)\}.
\]

These calculations show that \( f_{-n}^{-2N} \) uniformly contracts \( \pi_n \) by the factor \((|ac|\mu^n\lambda^n)^{-1}\), and its absolute value is less than 1 if \( n \) is sufficiently large. Hence we know \( \pi_n \) belongs to the unstable manifold of \( R_n \). Thus the proof is completed.

\[\square\]

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